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Elliptic Involution in Bordered Heegaard Floer Homology

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ELLIPTIC INVOLUTION IN BORDERED HEEGAARD FLOER HOMOLOGY

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Abstract

In this thesis, we study the elliptic involution from the point of view of the bordered Heegaard Floer homology.

We show that the bordered Heegaard Floer invariant $\widehat{CFD}$ of a knot complement in S^3 is invariant under the elliptic involution on its boundary. As a computational result, we demonstrate how the bordered Heegaard Floer invariant $\widehat{CFDA}$ associated to the elliptic involution affects such $\widehat{CFD}$ complexes via $\mathcal{A}_\infty$ tensor product. Globally, it “flips” the entire complex, while locally, it modifies the complex nicely in an arrow-wise fashion. The latter local description can be extended to quarter boundary Dehn twists as well.

Additionally, the Heegaard Floer homology of the associated open book decomposition to the elliptic involution is computed.

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To my wife He Yang. Life makes sense when I am with you.

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Chapter 1

Introduction

Heegaard Floer homology is a 3-manifold invariant defined by Peter S. Ozsváth and Zoltán Szabó in [10], which has proved to be very powerful. A knot version is later defined independently by Jacob Rasmussen in [13] and by Ozsváth and Szabó in [9].

Bordered Heegaard Floer homology developed by Lipshitz, Ozsváth, and Thurston in [5] is a great tool to compute Heegaard Floer homology by decomposing a 3-manifold into smaller pieces with parametrized boundaries, computing the “bordered” invariant on each piece, and finally taking an appropriate tensor product to recombine them.

Specifically, suppose Y is a bordered three-manifold with a orientation-preserving boundary parametrization $h : F \rightarrow \partial Y$, with some extra data. A differential graded algebra $\mathcal{A}(F)$ is associated to F . Furthermore, the theory associates to Y a right $\mathcal{A}_\infty$ -module $\widehat{CFA}(Y)$ over $\mathcal{A}(F)$, as well as a left differential graded module $\widehat{CFD}(Y)$ over $\mathcal{A}(-F)$. The two modules are called the *type A module* and *type D module* of Y .

To retrieve the Heegaard Floer homology of a manifold without boundary, the following pairing theorem is proved in [5].

Theorem 1 (Theorem 1.3, Proposition 2.34, [5]). *Suppose that F is a closed oriented surface and*

Y_1 and Y_2 are two 3-manifolds with parametrized boundary F and $-F$, then

$$\widehat{CF}(Y_1 \cup_F Y_2) \cong \widehat{CFA}(Y_1) \tilde{\otimes}_{\mathcal{A}(F)} \widehat{CFD}(Y_2) \cong \widehat{CFA}(Y_1) \boxtimes_{\mathcal{A}(F)} \widehat{CFD}(Y_2),$$

where $\tilde{\otimes}$ and $\boxtimes$ denote the $\mathcal{A}_\infty$ tensor product and a smaller but homotopy equivalent model, respectively.

The question of how $\widehat{CFD}$ modules change when the boundary parametrization of the bordered manifold changes is studied in [4], where a bimodule $\widehat{CFDA}$ is introduced to encode this action.

Specifically, suppose $h : F_1 \rightarrow F_2$ is a diffeomorphism between surfaces F_1 and F_2 , respecting some extra data on F_1 and F_2 , to which a bimodule ${}_{\mathcal{A}(F_1)}\widehat{CFDA}(\psi)_{\mathcal{A}(F_2)}$ is associated.

Theorem 2 (Theorem 2, [4]). *Suppose $(Y, \psi : F_2 \rightarrow -\partial Y)$ is a bordered three-manifold, then:*

$$\widehat{CFDA}(h) \tilde{\otimes}_{\mathcal{A}(F_2)} \widehat{CFD}(Y, \psi) \simeq \widehat{CFD}(Y_2, \psi \circ h)$$

In this thesis, we first study a special case of this question: when the bordered manifold is a knot complement in S^3 and h is the elliptic involution on the torus.

We prove the following result:

Theorem 3. *Given a knot $K \subset S^3$, let X be the knot complement with boundary parametrization $\psi : T^2 \rightarrow \partial X$. Let $h : T^2 \rightarrow T^2$ be the elliptic involution on the torus. We have*

$$\widehat{CFD}(X, \psi) \simeq \widehat{CFD}(X, \psi \circ h).$$

We prove Theorem 3 on the level of bordered Heegaard diagrams in Section 4.1 and also algebraically with the bimodule associated to the elliptic involution in Section 4.2. Even though the latter proof is more tedious, it reveals that the elliptic involution h 's effect on $\widehat{CFD}$ modules can be described not only as a global “flipping”, but also as a mapping of each of its individual arrows. In other words, there exists a map χ from the set of all possible arrows to itself, so that if we replace every arrow Q in $\widehat{CFD}$ with $\chi(Q)$, the result is $\widehat{CFDA}(h) \boxtimes \widehat{CFD}$.

Obviously, Theorem 3 is trivial in the case that the elliptic involution can be extended to $\tilde{h} : X \rightarrow X$, as $\tilde{h}$ is a homeomorphism from (X, ψ) to $(X, \psi \circ h)$, respecting the boundary parametrization. In that case, $\widehat{CFD}$ is trivially invariant. A knot is called *invertible* if there is an orientation-preserving homeomorphism of the 3-sphere which takes the knot to itself but reverses the orientation along the knot.¹ If h can indeed be extended to $X = S^3 \setminus K$, then it can be further extended to S^3 in the obvious fashion and satisfies the condition of K being invertible. It follows that in the cases of K being non-invertible, (X, ψ) and $(X, \psi \circ h)$ are distinct boundary-parametrized three-manifolds. A infinite family of pretzel knots were proved to be non-invertible by H.F. Trotter in 1964 [14]. It is now known that only a tiny proportion of knots up to 16 crossings are invertible [15].

As explained in Section 4.1, an elliptic involution can be viewed as a half boundary Dehn twist in terms of bordered Heegaard diagrams. Its “half”, a quarter boundary Dehn twist q , has a similar “mapping of arrows” effect on $\widehat{CFD}$ modules. We summarize both cases in the following theorem. Refer to Chapter 5 for the exact statements.

Theorem 4. *Given a knot $K \subset S^3$, let X be the knot complement with boundary parametrization $\phi : T^2 \rightarrow \partial X$. Suppose that there exists a reduced and simultaneously horizontally and vertically simplified model for $CFK^-(K)$. We obtain $\widehat{CFD}(X, \phi)$ with the algorithm described in Theorem 6.*

If $\psi = h$ or q , there exists a mapping χ from the set of all arrows allowed by the torus algebra to itself. $\widehat{CFDA}(\psi) \boxtimes \widehat{CFD}(X, \phi)$ can then be obtained by replacing each arrow Q of $\widehat{CFD}(X, \phi)$ with $\chi(Q)$.

The main results of this thesis are included in preprint [16].

One application of this result is to study 3-manifold mutations. It trivially follows that

Proposition 1. *A mutation using an elliptic involution is not detected by $\widehat{HF}$ when either one of the two manifolds with boundary is a knot complement.*

¹The author would like to thank Amitesh Datta for pointing him to this concept.

The author conjectures that the condition of X being a knot complement is not necessary in Theorem 3 and an approach different than the ones used in this thesis will be required to prove it.

It is also interesting to compare with the result in [1], where mutations using genus-2 hyperelliptic involution are studied. It is shown there that mutating by the genus-2 hyperelliptic involution can change the rank of the non-torsion summands of $\widehat{HF}$.

Chapter 2

Background

2.1 Knot Floer Homology

We first review the setup of knot Floer Homology, especially the aspects important to the purpose of this thesis, following overviews in Chapter 11 of [5] and Lecture 4 of [6]. For a comprehensive overview of the knot Floer homology, see [7].

Knot Floer Homology comes in different flavors and associates different chain complexes to a knot in a three-manifold. We primarily focus on the “minus” version denoted by CFK^- . Suppose that a knot K in S^3 is specified by a doubly-pointed Heegaard diagram $\mathcal{H} = (\Sigma, \alpha, \beta, w, z)$, where the base points are w and z . $CFK^-(K)$ is defined as a free, $\mathbb{Z}$ -graded, $\mathbb{Z}$ -filtered chain complex generated over $\mathbb{F}_2[U]$ by $\mathfrak{S}_K$, which are the Heegaard Floer homology generators from diagram $\mathcal{H}$. $CFK^-(K)$ is equipped with a differential

$$\partial^-(x) := \sum_{y \in \mathfrak{S}_K} \sum_{\substack{B \in \tilde{\pi}_2(x,y) \\ \text{ind}(B)=1}} \#(\mathcal{M}^B(x,y)) U^{n_w(B)} \cdot y,$$

where $\tilde{\pi}_2(x,y)$ is the set of homology classes of curves connecting x and y and $n_w(B)$ denotes the intersection number between B and w .

The chain complex $CFK^-(K)$ is endowed with an Alexander filtration A and a Maslov grading M . Both of them are characterized, up to an overall translation, by

$$\begin{aligned}
A(x) - A(y) &= n_z(B) - n_w(B) \\
A(U \cdot x) &= A(x) - 1 \\
M(x) - M(y) &= \text{ind}(B) - 2n_w(B) \\
M(U \cdot x) &= M(x) - 2
\end{aligned} \tag{2.1}$$

For example, if $B \in \tilde{\pi}_2(x, y)$ with $\text{ind}(B) = 1$, $n_z(B) = a$ and $n_w(B) = b$, $U^b y$ will be a term in $\partial^-(x)$. It follows that B drops the Alexander filtration level by $n_z(B)$:

$$A(x) = A(y) + a - b = A(U^b y) + a.$$

B also drops the Maslov grading by 1:

$$M(x) = M(y) + 1 - 2b = M(U^b y) + 1.$$

The associated graded complex to $(CFK^-(K), A)$ is denoted by $gCFK^-(K)$. It can be simply viewed as $CFK^-(K)$ itself, with the disks crossing basepoint z disallowed. A becomes Alexander grading on $gCFK^-(K)$, as the differential preserves A now.

Another version of the knot Floer homology can be obtained by specializing to $U = 0$ in the construction above. The filtered chain complex and its associated graded complex are denoted by $\widehat{CFK}(K)$ and $\widehat{gCFK}(K)$ respectively.

There exists a planar representation of $CFK^-(K)$ that turns out to be very convenient for our purpose. We choose a basis $\{\xi_i\}$ for $CFK^-(K)$ and plot its generators over $\mathbb{F}_2$ on a plane according to the Alexander filtration and the U -power filtration. Specifically a generator of form $U^a \xi_i$ with Alexander filtration level b is plotted at coordinate $(-a, b)$. Thus, the differential of a generator $U^a \xi_i$ can be represented as arrows each starting from $(-a, b)$ and ending at the positions of each terms in $\partial^-(U^a \xi_i)$. From here on, we will use the words “arrow” and “differential” interchangeably

in this sense.

Note, as differential must not increase the level of either filtration, the arrows always point to the left and down.

Definition 1. We call $C := CFK^-(K)$ reduced if ∂^- drops either the Alexander or U power filtration; i.e. each arrow in the planar representation has length at least one.

We call a basis $\{\xi_i\}$ of $C = CFK^-(K)$ over $\mathbb{F}_2[U]$ vertically simplified if for each ξ_i , $\partial^-(\xi_i) \in U \cdot C$ or $\partial^-(\xi_i) \equiv \xi_{i+1} \pmod{U \cdot C}$; i.e. each generator has at most one vertical outgoing arrow. We say the length of the vertical arrow is $A(\xi_i) - A(\xi_{i+1})$, coinciding with the length of the arrow in the planar representation.

Similarly, we call a basis $\{\eta_i\}$ of $C = CFK^-(K)$ over $\mathbb{F}_2[U]$ horizontally simplified if for each η_i , $A(\partial^-(\eta_i)) < A(\eta_i)$ or $\partial^-(\eta_i) = U^m \eta_{i+1} + \epsilon$, where $A(\eta_i) = A(U^m \eta_{i+1})$ and $A(\epsilon) < A(\eta_i)$; i.e. each generator has at most one horizontal outgoing arrow. We say the length of the horizontal arrow is m , again coinciding with the length of the arrow in the planar representation.

Note, in a vertically (horizontal, respectively) simplified basis, $(\partial^-)^2 = 0$ implies that a generator cannot have both incoming and outgoing vertical (horizontal, respectively) arrows.

It is shown that every CFK^- is homotopy equivalent to a chain complex that is reduced and for a reduced chain complex, we can always find a vertically or horizontally simplified basis. See Proposition 11.57 of [5].

Next, we argue that in a vertically (horizontal, respectively) simplified basis, there exists exactly one special generator ξ_0 (η_0 , respectively) that does not have any incoming or outgoing vertical (horizontal, respectively) arrow. Let $C = CFK^-(K)$. Then $C^v := C/(U \cdot C)$ is a new complex that can be viewed as being generated by the same generators over $\mathbb{F}_2$ and disallowing disks going through w . The homology of C^v computes $\widehat{HF}(S^3) \cong \mathbb{F}_2$, as z does not record any information any more. So there must be a generator that does not have any incoming or outgoing vertical arrow and all other generators have exactly one incoming or outgoing vertical arrow. Let C^h be the degree 0 part of the associated graded object of $C^\infty := C \otimes_{F_2[U]} \mathbb{F}_2[U, U^{-1}]$. The argument works similarly for C^h with roles of z and w exchanged. There is also a generator that does not

have any incoming or outgoing horizontal arrow and all other generators have exactly one incoming or outgoing horizontal arrow.

It is also important to note that interchanging the basepoints z and w in a Heegaard diagram affects CFK^- nicely: all the vertical and horizontal arrows are interchanged, or the whole chain complex could be viewed as “flipped.” We formalize this observation in the following Definition 2 and Lemma 1.

Definition 2. *Given a model C for CFK^- of a knot $K \subset S^3$ endowed with Alexander filtration A and Maslov grading M , we defined the flipped complex C^{flip} resulted from switching horizontal and vertical arrows of C as follows:*

We take $C^{flip} \cong C$ as a $\mathbb{F}_2[U]$ -module, i.e. there is an isomorphism $C \rightarrow C^{flip}$ taking a generator x of C to a generator x^{flip} of C^{flip} .

Define a Maslov grading on C^{flip} by $M(x^{flip}) = M(x) - 2A(x)$

Define an Alexander filtration on C^{flip} by $A(x^{flip}) = -A(x)$.

For each arrow in C of the form $x \rightarrow U^r y$ with $A(x) - A(y) = s$, we assign an arrow in C^{flip} of the form $x^{flip} \rightarrow U^{r+s} y^{flip}$.

Lemma 1. *C^{flip} constructed in Definition 2 is indeed a chain complex and all horizontal and vertical arrows are switched. Furthermore, if C is computed directly from a Heegaard diagram $(\Sigma, \alpha, \beta, w, z)$, then C^{flip} is isomorphic to the knot Floer complex C' computed from $(\Sigma, \alpha, \beta, z, w)$ up to overall Maslov grading and Alexander filtration shifts.*

Proof. We begin by observing that arrow $x \rightarrow U^r y$ with $A(x) - A(y) = s$ drops Alexander filtration by $A(x) - A(U^r y) = r + s$ and U -filtration by r , by Equation (2.1). The new arrow in C^{flip} of form $x^{flip} \rightarrow U^{r+s} y^{flip}$ drops Alexander filtration by $A(x^{flip}) - A(U^{r+s} y^{flip}) = -(A(x) - A(y)) + r + s = r$, and the U -filtration by $r + s$. Graphically, the arrow in C represented by a vector $(-r - s, -r)$ is now assigned an arrow in C^{flip} represented by $(-r, -r - s)$. For the Maslov grading, the new

arrow drops it by

$$\begin{aligned}
M(x^{flip}) - M(U^{r+s}y^{flip}) &= M(x^{flip}) - M(y^{flip}) + 2r + 2s \\
&= (M(x) - M(y)) - 2(A(x) - A(y)) + 2r + 2s \\
&= M(x) - M(y) + 2r = M(x) - M(U^r y) = 1,
\end{aligned}$$

which means the Maslov grading on C^{flip} is respected by the differential. $\partial^2 = 0$ is obvious as it is equivalent to there being even number of two-step paths from any given generator to any other generator in C^{flip} . This is true for C and graphically arrows in C^{flip} are just those in C flipped across the diagonal.

Now suppose C is computed directly from a Heegaard diagram $(\Sigma, \alpha, \beta, w, z)$. We denote the knot Floer complex computed from $(\Sigma, \alpha, \beta, z, w)$ by C' . Generators of C' , which we denote by x', y' , etc., are in one-to-one correspondence with generators x, y , etc. of C , hence matching those of C^{flip} .

For each arrow in C of form $x \rightarrow U^r y$ with $A(x) - A(y) = s$, represented by $B \in \tilde{\pi}_2(x, y)$, it follows from Equation (2.1) that $s = A(x) - A(y) = n_z(B) - n_w(B)$ and $M(x) - M(y) = \text{ind}(B) - 2n_w(B)$. As C' is computed from the same Heegaard diagram except with interchanged basepoints, the same holomorphic disk also contributes in C' , denoted by $B' \in \tilde{\pi}_2(x', y')$ satisfying $n_w(B') = n_z(B)$ and $n_z(B') = n_w(B)$. Now we have

$$\begin{aligned}
A(x') - A(y') &= n_z(B') - n_w(B') = n_w(B) - n_z(B) = -(A(x) - A(y)) \\
M(x') - M(y') &= \text{ind}(B') - 2n_w(B') = \text{ind}(B) - 2n_z(B) \\
&= \text{ind}(B) - 2n_w(B) - 2(n_z(B) - n_w(B)) \\
&= M(x) - M(y) - 2(A(x) - A(y))
\end{aligned} \tag{2.2}$$

Note $n_w(B') = n_z(B) = n_w(B) + s = r + s$, so there exists an arrow in C' of form $x' \rightarrow U^{r+s} y'$ with $A(x') - A(y') = -s$, which agrees with our construction of C^{flip} .

It is known that the values of $A(x') - A(y')$ and $M(x') - M(y')$ for all x', y' and $B' \in \tilde{\pi}_2(x', y')$

completely determine the Maslov grading and Alexander filtration of C' up to overall translations. So the fact that our construction of A and M on C^{flip} satisfies Equation (2.2) let us conclude that C^{flip} and C' are indeed isomorphic. $\square$

2.2 Bordered Heegaard Floer Theory

In this section, we give an overview of the bordered Heegaard Floer theory, focusing on the most relevant aspects to this thesis. More details can be found in [5] and [4].

In the bordered Floer theory, boundaries of three-manifolds are 2-surfaces and are specified by *pointed matched circles*.

Definition 3. A pointed matched circle $\mathcal{Z}$ is a tuple $(Z, \mathbf{a}, M, z)$ of an oriented circle Z , $4k$ points $\mathbf{a} = \{a_1, \dots, a_{4k}\}$ in Z , a basepoint $z \in Z \setminus \mathbf{a}$ and a matching: a two-to-one map $M : \mathbf{a} \rightarrow \{1, 2, \dots, 2k\}$. We require that if we thicken Z into $Z \times I$ and attach a 1-handle to each pair of points in $\mathbf{a} \times \{1\} \subset Z \times \{1\}$, the new boundary must be a single S^1 .

Note there is only one pointed matched circle with 4 points.

A pointed matched circle $\mathcal{Z}$ specifies a punctured 2-surface $F^\circ(\mathcal{Z})$ as follows: thicken Z into $Z \times I$, attach a 1-handle to each pair of points in $\mathbf{a} \times \{1\} \subset Z \times \{1\}$, and glue a 2-disk to the new boundary. $F^\circ(\mathcal{Z})$ has a basepoint z on its boundary. Let the closed surface $F(\mathcal{Z})$ be result of gluing a 2-disk to the boundary of $F^\circ(\mathcal{Z})$. We sometimes call this 2-disk the preferred disk in $F(\mathcal{Z})$ and denote it by Δ .

$\mathcal{Z}$ specifies the boundary of a three-manifold as in the following definition.

Definition 4. A bordered 3-manifold is a triple $(Y, \mathcal{Z}, \phi)$, where Y is a compact, oriented 3-manifold with connected boundary ∂Y , $\mathcal{Z}$ is a pointed matched circle, and $\phi : F(\mathcal{Z}) \rightarrow \partial Y$ is a orientation-preserving homeomorphism. We say $\mathcal{Z}$ or $F(\mathcal{Z})$ parametrizes the boundary of Y .

The bordered theory associates a differential graded algebra $\mathcal{A}(\mathcal{Z})$ to each pointed matched circle $\mathcal{Z}$. The algebra can be described in terms of strand diagrams. Since we focus on the genus-1

case of the bordered Floer theory, we only present the algebra associated to the unique pointed matched circle of genus 1. We call it the *torus algebra*.

Definition 5. *The torus algebra $\mathcal{A}$ is a $\mathbb{F}_2$ -vector space generated by elements*

$$\iota_0, \iota_1, \rho_1, \rho_2, \rho_3, \rho_{12}, \rho_{23}, \rho_{123}$$

with the following multiplication rules:

- $\rho_1\rho_2 = \rho_{12}, \rho_2\rho_3 = \rho_{23}, \rho_1\rho_{23} = \rho_{12}\rho_3 = \rho_{123},$
- *all other products of form $\rho_I\rho_J$ are 0,*
- $\iota_0\rho_1 = \rho_1\iota_1 = \rho_1, \iota_1\rho_2 = \rho_2\iota_0 = \rho_2, \iota_0\rho_3 = \rho_3\iota_1 = \rho_3.$

As discussed in the introduction, Bordered Heegaard Floer theory associates two modules to a bordered three-manifold $(Y, \mathcal{Z}, \phi)$: a type A module $\widehat{CFA}(Y)_{\mathcal{A}(\mathcal{Z})}$, which is a right A_∞ -module over $\mathcal{A}(\mathcal{Z})$, and a type D module ${}^{\mathcal{A}(-\mathcal{Z})}\widehat{CFD}(Y)$, which is a left type D module over $\mathcal{A}(-\mathcal{Z})$. The pairing theorem can now be stated as follows. Suppose that Y_1 and Y_2 are two bordered 3-manifolds boundaries parametrized by $\mathcal{Z}$ and $-\mathcal{Z}$, then

$$\widehat{CF}(Y_1 \cup Y_2) \cong \widehat{CFA}(Y_1) \tilde{\otimes}_{\mathcal{A}(\mathcal{Z})} \widehat{CFD}(Y_2) \cong \widehat{CFA}(Y_1) \boxtimes_{\mathcal{A}(\mathcal{Z})} \widehat{CFD}(Y_2),$$

where $\tilde{\otimes}$ and $\boxtimes$ denote the $\mathcal{A}_\infty$ tensor product and a smaller but homotopy equivalent model, respectively.

The bordered theory is extended to manifolds with two boundary components in [4].

Definition 6. *A strongly bordered three-manifold with two boundary components $\mathcal{Y}$ is a tuple*

$$\mathcal{Y} = (Y, \mathcal{Z}_L, \Delta_L, z_L, \phi_L, \mathcal{Z}_R, \Delta_R, z_R, \phi_R, \gamma_z, \nu_z)$$

where:

- Y is a compact, oriented three-manifold with two boundary components $\partial_L Y$ and $\partial_R Y$.

- Δ_L and Δ_R are preferred disks.
- z_L and z_R are basepoints on the boundaries of the disks.
- ϕ_L and ϕ_R are homeomorphisms preserving the preferred disks and the basepoints.

$$\phi_L : (F(\mathcal{Z}_L), z_L) \rightarrow (\partial_L Y, z_L)$$

$$\phi_R : (F(\mathcal{Z}_R), z_R) \rightarrow (\partial_R Y, z_R)$$

- γ_z is a path in Y connecting z_L and z_R .
- ν_z is a framing on γ_z ; or ν_z is an isotopy class of nowhere vanishing normal vector fields to γ_z pointing into Δ_L at z_L and into Δ_R at z_R .

To a strongly bordered three-manifold with two boundary components $\mathcal{Y}$, the bordered Floer theory associates a type DA invariant $\widehat{CFDA}(Y)$, a left type D , right A_∞ -bimodule over $\mathcal{A}(-\mathcal{Z}_L)$ and $\mathcal{A}(\mathcal{Z}_R)$. A few other invariants are associated as well, depending each boundary being A or D typed, see [4] for details.

We say a diffeomorphism $\psi : F(\mathcal{Z}) \rightarrow F(\mathcal{Z})$ is a strongly based if it preserves the preferred disk and basepoint in $F(\mathcal{Z})$. Given such a strongly based diffeomorphism, we can assign extra data to its mapping cylinder and form a corresponding strongly bordered three-manifold M_ψ (Construction 5.27 [4]). M_ψ is constructed such that $\partial_L M_\psi = -F(\mathcal{Z})$. and $\phi_L = -\psi$. Gluing M_ψ to another bordered manifold obviously change the boundary parametrization in the expected way. We denote the type- DA invariant associated to M_ψ by $\widehat{CFDA}(\psi)$.

There are a set of pairing theorems concerning these invariants. The following one is of particular importance to the purpose of this thesis (Theorem 2 and 11, [4]).

Theorem 5. *Suppose $(Y, \phi : F(\mathcal{Z}) \rightarrow -\partial Y)$ is a bordered three-manifold, then:*

$$\widehat{CFDA}(\psi) \boxtimes_{\mathcal{A}(-\mathcal{Z})} \widehat{CFD}(Y, \phi) \simeq \widehat{CFD}(Y_2, \phi \circ \psi)$$

2.3 From CFK^- to $\widehat{CFD}$

In [5], it is demonstrated how to obtain $\widehat{CFD}$ of a knot complement from the knot Floer invariant CFK^- of the knot.

Theorem 6 is the such a procedure described in terms of bases of $CFK^-(K)$. When $CFK^-(K)$ has a reduced model for the knot Floer complex admitting a basis $\{\xi_i\}$ which is simultaneously vertically and horizontally simplified, the procedure is fairly straightforward.

Note that there exists free, $\mathbb{Z}$ -graded, $\mathbb{Z}$ -filtered chain complex over $\mathbb{F}_2[U]$ that does not admit a simultaneously vertically and horizontally simplified basis (due to Jen Hom), but the author is not aware of any CFK^- complex of a knot that does not. A general procedure that does not rely on this mild condition on the knot is also proved in [5], see Theorem 11 in Appendix A.

Theorem 6 (11.27, A.11 [5]). *Given a knot $K \subset S^3$, let X be the knot complement with boundary parametrization $\phi : T^2 \rightarrow \partial X$ corresponding to an integral framing n . Let $CFK^-(K)$ be a reduced model for the knot Floer complex of K admitting a basis $\{\xi_i\}$ which is simultaneously vertically and horizontally simplified. Suppose ξ_v (respectively ξ_h) is the generator of $CFK^-(K)$ which has no incoming or outgoing vertical (respectively horizontal) arrows. Apply the following procedure to get $\widehat{CFD}(X, \phi)$:*

$\iota_0 \widehat{CFD}(X, \phi)$ is identified with $CFK^-(K)$ as an $\mathbb{F}_2$ -module. Denote the corresponding generators of $\iota_0 \widehat{CFD}(X, \phi)$ by $\xi'_1, \xi'_2, \dots$

For each vertical arrow of length ℓ from ξ_i to ξ_{i+1} , we associate a string of basis elements $\kappa_1^i, \kappa_2^i, \dots, \kappa_\ell^i$ for $\iota_1 \widehat{CFD}(X, \phi)$ and differentials among them:

$$\xi'_i \xrightarrow{\rho_1} \kappa_1^i \xleftarrow{\rho_{23}} \dots \xleftarrow{\rho_{23}} \kappa_k^i \xleftarrow{\rho_{23}} \kappa_{k+1}^i \xleftarrow{\rho_{23}} \dots \xleftarrow{\rho_{23}} \kappa_\ell^i \xleftarrow{\rho_{123}} \xi'_{i+1}$$

For each horizontal arrow of length ℓ from ξ_j to ξ_{j+1} , we associate a string of basis elements $\lambda_1^j, \lambda_2^j, \dots, \lambda_\ell^j$ for $\iota_1 \widehat{CFD}(X, \phi)$ and differentials among them:

$$\xi'_j \xrightarrow{\rho_3} \lambda_1^j \xrightarrow{\rho_{23}} \dots \xrightarrow{\rho_{23}} \lambda_k^j \xrightarrow{\rho_{23}} \lambda_{k+1}^j \xrightarrow{\rho_{23}} \dots \xrightarrow{\rho_{23}} \lambda_\ell^j \xrightarrow{\rho_2} \xi'_{j+1}$$

We include one more string of generators and differentials called the unstable chain, depending on framing n . When $n < 2\tau(K)$, it takes form:

$$\xi'_v \xrightarrow{\rho_1} \mu_1 \xleftarrow{\rho_{23}} \cdots \xleftarrow{\rho_{23}} \mu_k \xleftarrow{\rho_{23}} \mu_{k+1} \xleftarrow{\rho_{23}} \cdots \xleftarrow{\rho_{23}} \mu_m \xleftarrow{\rho_3} \xi'_h,$$

where $m = 2\tau(K) - n$. When $n = 2\tau(K)$, it takes form:

$$\xi'_v \xrightarrow{\rho_{12}} \xi'_h.$$

When $n > 2\tau(K)$, it takes form,

$$\xi'_v \xrightarrow{\rho_{123}} \mu_1 \xrightarrow{\rho_{23}} \cdots \xrightarrow{\rho_{23}} \mu_k \xrightarrow{\rho_{23}} \mu_{k+1} \xrightarrow{\rho_{23}} \cdots \xrightarrow{\rho_{23}} \mu_m \xrightarrow{\rho_2} \xi'_h,$$

where $m = n - 2\tau(K)$.

For notational convenience, we introduce the following definition.

Definition 7. Denote the type D module resulted from applying the procedures in Theorem 11 or Theorem 6 to a complex C by $KtD(C)$, standing for CFK^- to $\widehat{CFD}$.

We give an example of this algorithm.

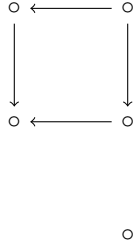


Figure 2.1: CFK^- of a figure eight knot with framing 0.

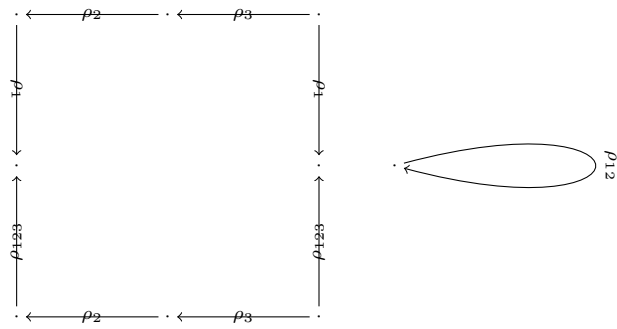


Figure 2.2: The corresponding $\widehat{CFD}$ module.

Chapter 3

$\widehat{CFDA}$ Bimodule of the Elliptic Involution

The elliptic involution on a torus can be decomposed into $(\tau_\mu \circ \tau_\lambda)^3$, where τ_μ and τ_λ are Dehn twists along a meridian and a longitude, respectively [2]. $\widehat{CFDA}(\tau_\mu)$ is generated by p, q, r and its non-trivial algebra actions are given as follows [5]:

$$\begin{aligned}
 m_{0,1,1}(p, \rho_1) &= \rho_1 \otimes q & m_{0,1,1}(p, \rho_{12}) &= \rho_{123} \otimes r \\
 m_{0,1,1}(p, \rho_{123}) &= \rho_{123} \otimes q & m_{0,1,2}(p, \rho_3, \rho_2) &= \rho_3 \otimes r \\
 m_{0,1,2}(p, \rho_3, \rho_{23}) &= \rho_3 \otimes q & m_{0,1,1}(q, \rho_2) &= \rho_{23} \otimes r \\
 m_{0,1,1}(q, \rho_{23}) &= \rho_{23} \otimes q & m_{0,1,0}(r) &= \rho_2 \otimes p \\
 m_{0,1,1}(r, \rho_3) &= q & &
 \end{aligned}$$

$\widehat{CFDA}(\tau_\lambda)$ is generated by p, q, s and its non-trivial algebra actions are given as follows [5]:

$$\begin{aligned}
m_{0,1,2}(q, \rho_2, \rho_1) &= \rho_2 \otimes s & m_{0,1,2}(q, \rho_2, \rho_{12}) &= \rho_2 \otimes p \\
m_{0,1,2}(q, \rho_2, \rho_{123}) &= \rho_{23} \otimes q & m_{0,1,1}(p, \rho_1, \rho_2) &= \rho_{12} \otimes s \\
m_{0,1,1}(p, \rho_{12}) &= \rho_{12} \otimes p & m_{0,1,1}(p, \rho_{123}) &= \rho_{123} \otimes q \\
m_{0,1,1}(p, \rho_3) &= \rho_3 \otimes q & m_{0,1,0}(s) &= \rho_1 \otimes q \\
m_{0,1,1}(s, \rho_2) &= p & m_{0,1,1}(s, \rho_{23}) &= \rho_3 \otimes q
\end{aligned}$$

Generators in both modules satisfy the following equations regarding idempotents.

$$\iota_0 \cdot p \cdot \iota_0 = p \quad \iota_1 \cdot q \cdot \iota_1 = q \quad \iota_1 \cdot r \cdot \iota_0 = r \quad \iota_0 \cdot s \cdot \iota_1 = s$$

By Theorem 12 in [4], $H := \widehat{CFDA}(h)$ can be computed as $\widehat{CFDA}(\tau_\mu) \boxtimes \widehat{CFDA}(\tau_\lambda) \boxtimes \widehat{CFDA}(\tau_\mu) \boxtimes \widehat{CFDA}(\tau_\lambda) \boxtimes \widehat{CFDA}(\tau_\mu) \boxtimes \widehat{CFDA}(\tau_\lambda)$.

We omit the computation of the tensor product, but instead describe the type DA bimodule H that results from canceling the following arrows in the following order:

$$\begin{aligned}
p \otimes p \otimes p \otimes s \otimes r \otimes p &\rightarrow p \otimes p \otimes p \otimes p \otimes p \otimes p, \\
p \otimes p \otimes p \otimes s \otimes r \otimes s &\rightarrow p \otimes p \otimes p \otimes p \otimes p \otimes s, \\
p \otimes s \otimes r \otimes s \otimes q \otimes q &\rightarrow p \otimes p \otimes p \otimes s \otimes q \otimes q, \\
p \otimes s \otimes r \otimes s \otimes r \otimes p &\rightarrow p \otimes s \otimes r \otimes p \otimes p \otimes p, \\
p \otimes s \otimes r \otimes s \otimes r \otimes s &\rightarrow p \otimes s \otimes r \otimes p \otimes p \otimes s, \\
q \otimes q \otimes r \otimes s \otimes r \otimes p &\rightarrow q \otimes q \otimes r \otimes p \otimes p \otimes p, \\
q \otimes q \otimes r \otimes s \otimes r \otimes s &\rightarrow q \otimes q \otimes r \otimes p \otimes p \otimes s, \\
r \otimes p \otimes p \otimes s \otimes r \otimes p &\rightarrow r \otimes p \otimes p \otimes p \otimes p \otimes p, \\
r \otimes p \otimes p \otimes s \otimes r \otimes s &\rightarrow r \otimes p \otimes p \otimes p \otimes p \otimes s,
\end{aligned}$$

$$r \otimes s \otimes q \otimes q \otimes r \otimes s \rightarrow q \otimes q \otimes r \otimes s \otimes q \otimes q,$$

$$r \otimes s \otimes r \otimes s \otimes q \otimes q \rightarrow r \otimes p \otimes p \otimes s \otimes q \otimes q,$$

$$r \otimes s \otimes r \otimes s \otimes r \otimes p \rightarrow r \otimes s \otimes r \otimes p \otimes p \otimes p,$$

$$r \otimes s \otimes r \otimes s \otimes r \otimes s \rightarrow r \otimes s \otimes r \otimes p \otimes p \otimes s.$$

For simplicity, we rename the remaining generators $r \otimes s \otimes q \otimes q \otimes r \otimes p, q \otimes q \otimes q \otimes q \otimes r \otimes p, p \otimes s \otimes q \otimes q \otimes r \otimes p, q \otimes q \otimes q \otimes q \otimes q \otimes q, p \otimes s \otimes q \otimes q \otimes r \otimes s, p \otimes s \otimes q \otimes q \otimes q \otimes q, r \otimes s \otimes q \otimes q \otimes q \otimes q, q \otimes q \otimes q \otimes q \otimes r \otimes s$ with $x_3, x_1, x_2, v, u, y_3, y_2, y_1$, respectively. Now H can be described as in Figure 3.1.

The following relations with idempotents hold.

$$\iota_1 \cdot x_3 \cdot \iota_0 = x_3$$

$$\iota_1 \cdot x_1 \cdot \iota_0 = x_1$$

$$\iota_0 \cdot x_2 \cdot \iota_0 = x_2$$

$$\iota_1 \cdot v \cdot \iota_1 = v$$

$$\iota_0 \cdot u \cdot \iota_1 = u$$

$$\iota_0 \cdot y_3 \cdot \iota_1 = y_3$$

$$\iota_1 \cdot y_2 \cdot \iota_1 = y_2$$

$$\iota_1 \cdot y_1 \cdot \iota_1 = y_1$$

It is easy to verify that H is operationally bounded.

Throughout this thesis, we adapt the following color code for arrows: We use color blue, green, blue, red, and pink for D -side arrows with $\rho_1, \rho_2, \rho_3, \rho_{23}$, and ρ_{123} respectively, and for A -side arrows with A^∞ action of $\rho_1, \rho_2, \rho_3, \rho_{23}$, and ρ_{123} respectively. Finally, when tensoring arrows with the same color from both sides, we generally keep the color for the resulted arrow in the tensor product.

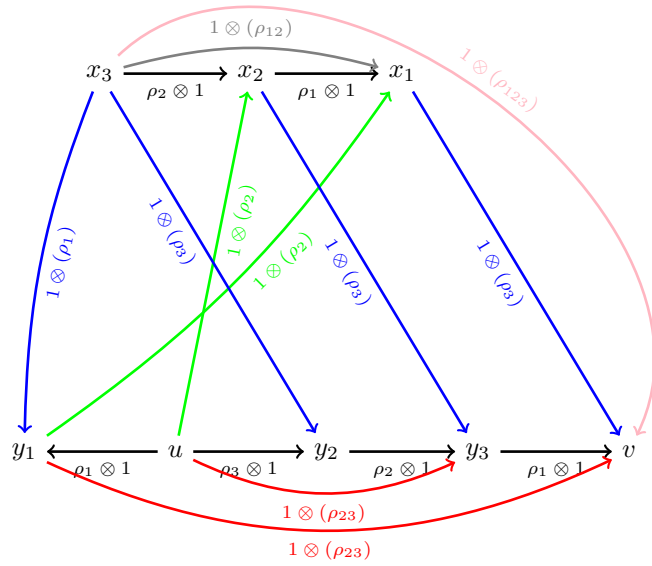


Figure 3.1: Arrows are color coded according to their A^∞ actions.

Chapter 4

Invariance of $\widehat{CFD}$ under the Elliptic Involution

In this chapter, we give two proofs of Theorem 3.

By Theorem 11 in [4], $\widehat{CFD}(X, \phi \circ h) = \widehat{CFDA}(h) \boxtimes \widehat{CFD}(X, \phi)$. In order to prove the invariance of $\widehat{CFD}$ under elliptic involution, we only need to compute the box tensor product above and show that it is homotopy equivalent to $\widehat{CFD}(X)$. To do this, we prove the following proposition, which implies Theorem 3 immediately:

Proposition 2. *Given a knot $K \subset S^3$, let X be the knot complement with boundary parametrization $\phi : T^2 \rightarrow \partial X$. Let C be a reduced and horizontally simplified model for the knot Floer complex $CFK^-(K)$.*

Then, $\widehat{CFDA}(h) \boxtimes \widehat{CFD}(X, \phi)$ is homotopy equivalent to the type D module $KtD(C^{flip})$ resulted from applying the procedure in Theorem 6 to C^{flip} , i.e.

$$\widehat{CFDA}(h) \boxtimes \widehat{CFD}(X, \phi) \simeq KtD(C^{flip}).$$

Proposition 2 implies Theorem 3 as follows.

Proof of Theorem 3 assuming Proposition 2. We fix a doubly pointed Heegaard diagram $(\Sigma, \alpha, \beta, w, z)$ for knot $K \subset S^3$ and denote by $CFK^-(K)$ the knot Floer complex computed from the diagram. $(\Sigma, \alpha, \beta, z, w)$ is a doubly pointed Heegaard diagram for knot $-K \subset S^3$. By Lemma 1, switching basepoints has exactly the same effect as switching the horizontal and vertical arrows, i.e. $CFK^-(-K) = CFK^-(K)^{flip}$. It is known that $CFK^-(K) \simeq CFK^-(-K)$ from [10], so $CFK^-(K) \simeq CFK^-(K)^{flip}$. Next, we find a reduced and horizontally simplified model C of $CFK^-(K)$. The process of reducing and horizontally simplifying $CFK^-(K)$ can be flipped to reduce and vertically simplify $CFK^-(K)^{flip}$ to C^{flip} . As a result, $CFK^-(K) \simeq CFK^-(K)^{flip} \simeq C^{flip}$, which means C^{flip} is another model for the knot Floer complex $CFK^-(K)$ of knot K , so Theorem 6 implies that the type D module $KtD(C^{flip})$ that results from applying the procedure to C^{flip} is homotopy equivalent to $\widehat{CFD}(X, \phi)$, i.e. $KtD(C^{flip}) \simeq \widehat{CFD}(X, \phi)$. Hence, Proposition 2 implies Theorem 3. □

Now all there is left to do is to prove Proposition 2.

4.1 Heegaard Diagram Proof

We first prove Proposition 2 in terms of bordered Heegaard diagrams. The author would like to thank Adam Levine for suggesting this proof.

Proof. Recall from the statement of Proposition 2, C is a reduced and horizontally simplified model for the knot Floer complex $CFK^-(K)$. By Lemma 1, C^{flip} in Proposition 2 is simply a model for $CFK^-(-K)$. Thus, $KtD(C^{flip})$ is a model of $\widehat{CFD}(X')$, where X' is $S^3 \setminus -K$ with the same framing. All there is left to prove is:

$$\widehat{CFDA}(h) \boxtimes \widehat{CFD}(X, \phi) \simeq \widehat{CFD}(X'). \quad (4.1)$$

Now we prove Equation (4.1) by showing that the two sides correspond to equivalent bordered Heegaard diagrams.

Suppose that knot K is specified by a doubly-pointed Heegaard diagram $(\Sigma, \alpha, \beta, w, z)$ and $-K$ is then specified by $(\Sigma, \alpha, \beta, z, w)$. Recall that in the procedure to transform a doubly-pointed Heegaard diagram for K into a bordered Heegaard diagram for $S^3 \setminus K$ with some specified framing, the base points only determines the location of base point z in the bordered diagram, see Section 11.7 in [5]. If we interchange w and z in the doubly-pointed Heegaard diagram for K , we get exactly the same bordered diagram except that base point z and the numbering of the boundary regions move counter-clockwise by two regions. See Figure 4.1.

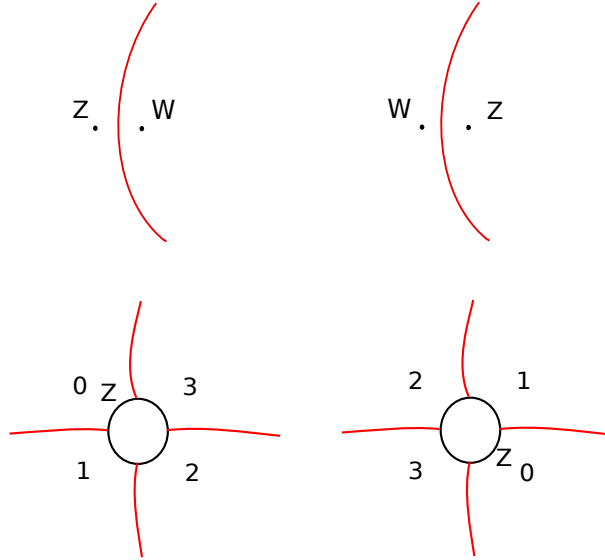


Figure 4.1: Doubly-pointed Heegaard diagram $(\Sigma, \alpha, \beta, w, z)$ for K and $(\Sigma, \alpha, \beta, z, w)$ for $-K$, on the top. Resulted bordered Heegaard diagram for $X = S^3 \setminus K$ and $X' = S^3 \setminus -K$ on the bottom.

A doubly bordered Heegaard diagram for the half boundary Dehn twist is shown in Figure 4.2. We demonstrate that this diagram shifts any bordered Heegaard diagram's basepoint z by two regions, hence transforming a diagram of any X to that of the corresponding X' .

Without loss of generality, we can assume the two red arcs in the bordered diagram for X take the standard position, as shown on the left side of Figure 4.3a. (A homeomorphism could bring all red circles and arcs in any Heegaard diagram to the standard position, while moving the blue circles too.) Note, any additional genus, red circles, and blue circles are omitted here, as they will not be altered.

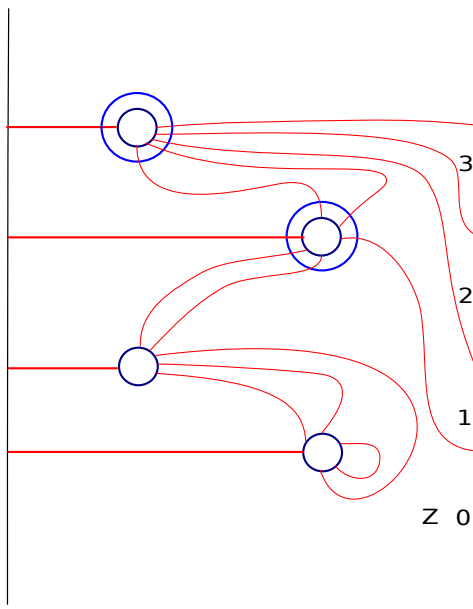


Figure 4.2: A bordered Heegaard diagram for the half boundary Dehn twist.

After combining it with Figure 4.2, two new red circles are formed, see Figure 4.3a. Each of them intersects with one blue circle once. Hence we destabilize the two pairs of intersecting red and blue circles, see Figure 4.3b.

Figure 4.4a is the result of simply pulling some parts of the red arcs straight. We then perform a boundary half-Dehn twist on the boundary of the bordered surface, which is essentially a rotation by 180 degrees, see Figure 4.4b.

Next, we handle slide the two alpha arcs over all the remaining omitted red circles in the diagram of X . Finally, we arrive at Figure 4.5, which is what we set out to obtain.

□

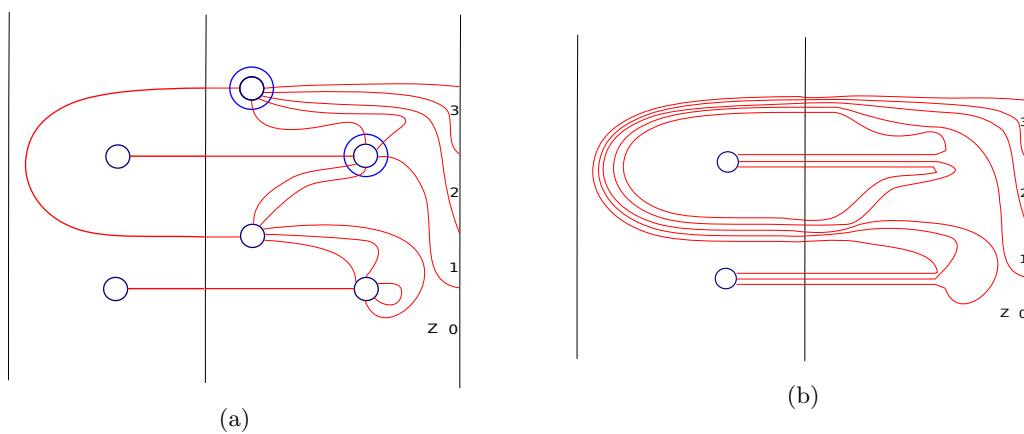


Figure 4.3

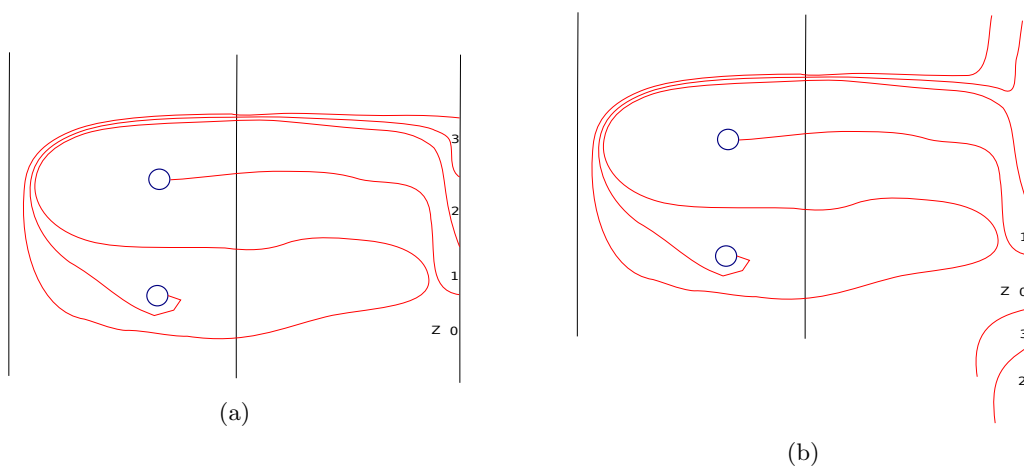


Figure 4.4

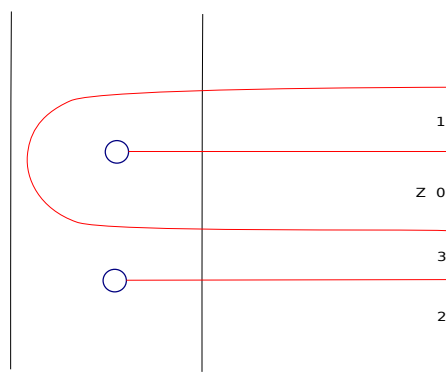


Figure 4.5

4.2 Algebraic Proof

Even though the proof in Section 4.1 is straightforward, it does not provide any insight on how tensoring with $H = \widehat{CFDA}(h)$ affects $\widehat{CFD}$ modules. In this section, we attempt to prove Proposition 2 algebraically by examining the tensor product closely. It will also yield a local description of the effect of H , see Theorem 7.

For computational convenience, we prove Proposition 2 with an extra assumption. We assume that we have a model C of $CFK^-(K)$ which is simultaneously horizontally and vertically simplified. We again remark that even though there exist examples of filtered chain complexes that can not be simultaneously horizontally and vertically simplified (due to Jen Hom), we are not aware of any examples that can be realized as CFK^- of a knot.

A proof of the general statement can be obtained by an argument of similar flavor. As it is of less conceptual interest, we include it in the appendix. Please refer to Appendix A.

Recall our notation, $H = \widehat{CFDA}(h)$, where h is the elliptic involution and $KtD(C)$ (respectively, $KtD(C^{flip})$) is the type D module resulted from applying the procedure in Theorem 6 to C (respectively, C^{flip}).

Recall that in the procedure, $\iota_0 \widehat{CFD}(X, \phi)$ is identified with $CFK^-(K)$ as an $\mathbb{F}_2$ -module. To avoid confusion, we still denote the corresponding generator to ξ_i (ξ_v, ξ_h , respectively) in $\iota_0 \widehat{CFD}(X, \phi)$ by ξ'_i (ξ'_v, ξ'_h , respectively).

Each generator in $\iota_0 \widehat{CFD}(X, \phi)$ corresponds to three generators in $H \boxtimes KtD(C)$ and each one in $\iota_1 \widehat{CFD}(X, \phi)$ corresponds to five generators in $H \boxtimes \widehat{CFD}(X, \phi)$, see Figure 4.6a. An arrow with ρ_1 ($\rho_3, \rho_2, \rho_{12}, \rho_{23}, \rho_{123}$, respectively) corresponds to arrows in $H \boxtimes \widehat{CFD}(X, \phi)$ in Figure 4.6b (Figure 4.6c, Figure 4.6d, Figure 4.6e, Figure 4.6f, Figure 4.6g respectively.)

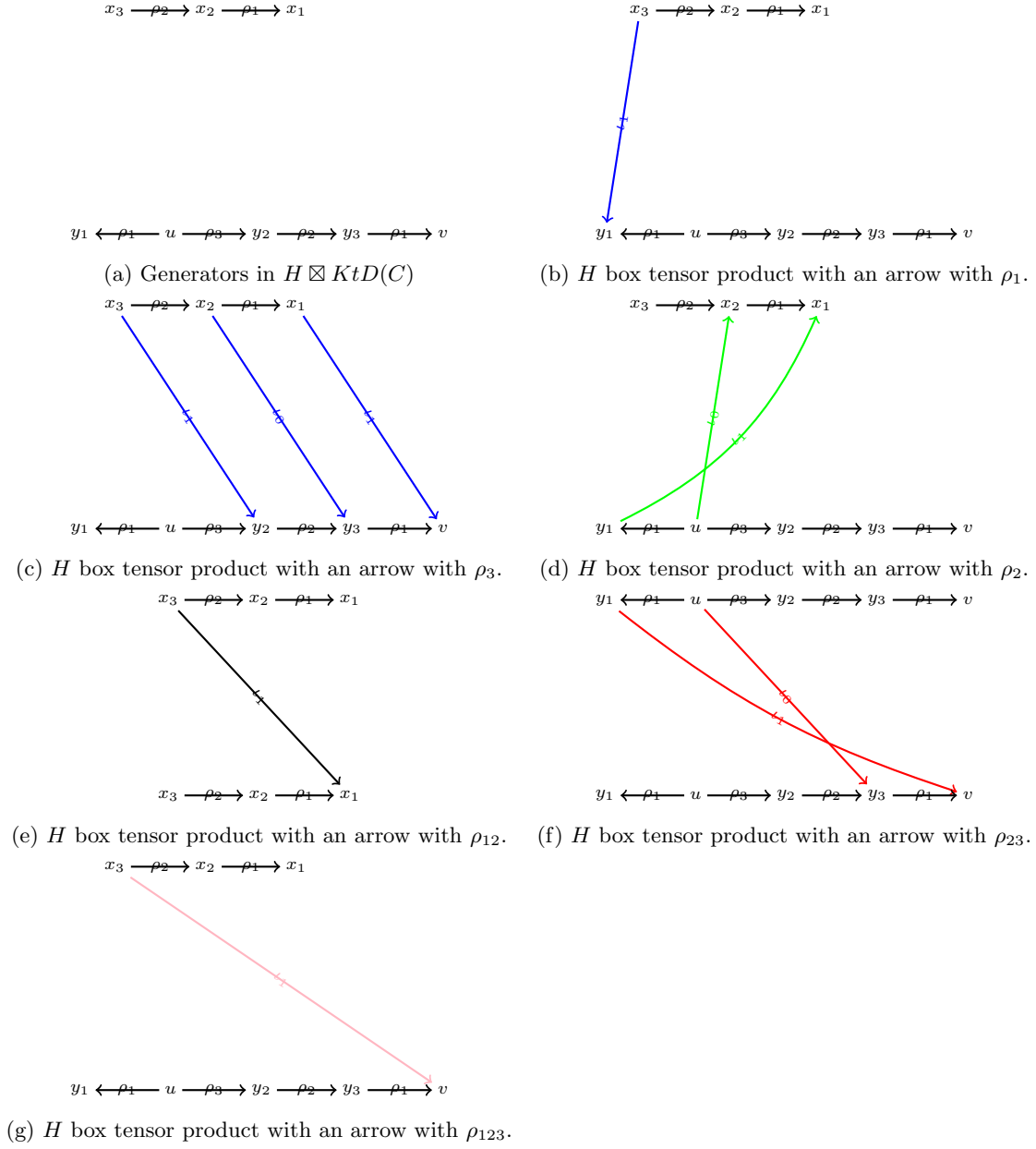


Figure 4.6: Tensoring for generators and arrows

4.2.1 A String of ρ_{23} 's

According to Theorem 6, the middle part of each chain of arrows in $KtD(C)$ is a (possibly empty) string of ρ_{23} 's. We first investigate how tensoring with H affects a string of ρ_{23} 's,

$$\dots \xrightarrow{\rho_{23}} * \xrightarrow{\rho_{23}} * \xrightarrow{\rho_{23}} * \xrightarrow{\rho_{23}} * \dots$$

The result of the tensor product is shown in Figure 4.7b, and after we cancel all the red arrows, the result is a string of ρ_{23} 's going the opposite direction, shown in Figure 4.7c.

We observe that for each vertical arrow $\xi_i \rightarrow \xi_j$, the corresponding chain in $KtD(C)$ has a string of ρ_{23} 's in the *opposite* direction as the original arrow, where by *opposite* we mean pointing toward ξ'_i and away from ξ'_j . For each horizontal arrow $\xi_i \rightarrow \xi_j$ in C^{flip} , the corresponding chain in $KtD(C^{flip})$ has a string of ρ_{23} 's in the *same* direction as the original arrow, where by *same* we mean pointing away from ξ'_i and toward from ξ'_j . The box tensor product of H with a string of ρ_{23} 's gives exactly the correspondence between the “middle” parts of the two corresponding chains.

Now we investigate how the joints between these strings of ρ_{23} 's are also “flipped” by tensoring with H .

Since C is both horizontally and vertically simplified, at any generator $\xi_i \neq \xi_v$ or ξ_h , we have the following four kinds of joints in Figure 4.8.

For each kind of joint at ξ_i , we argue the corresponding joint at ξ'_i in $KtD(C)$ can be identified with a joint in $KtD(C^{flip})$ by tensoring with H . Our way of doing this is simply through cancellation in $H \boxtimes KtD(C)$ and match the result to the corresponding part of $KtD(C^{flip})$. Then what is left to show $H \boxtimes KtD(C) = KtD(C^{flip})$ is how these cancellation at different parts of $H \boxtimes KtD(C)$ agree with each other.

We organize the proof that all “local” cancellations agree as follows: if an arrow at ξ_i in C has length more than 1, then the chain of arrows at ξ'_i in $KtD(C)$ is connected to a string of ρ_{23} 's. We observe how the cancellation at this joint in $H \boxtimes KtD(C)$ agree with the cancellation we perform for a string of ρ_{23} 's, so the chain of ρ_{23} 's along with the end point ξ'_i is matched to a flipped chain in $KtD(C^{flip})$. If an arrow at ξ_i in C has length exactly 1, ξ'_i in $KtD(C)$ is connected to another

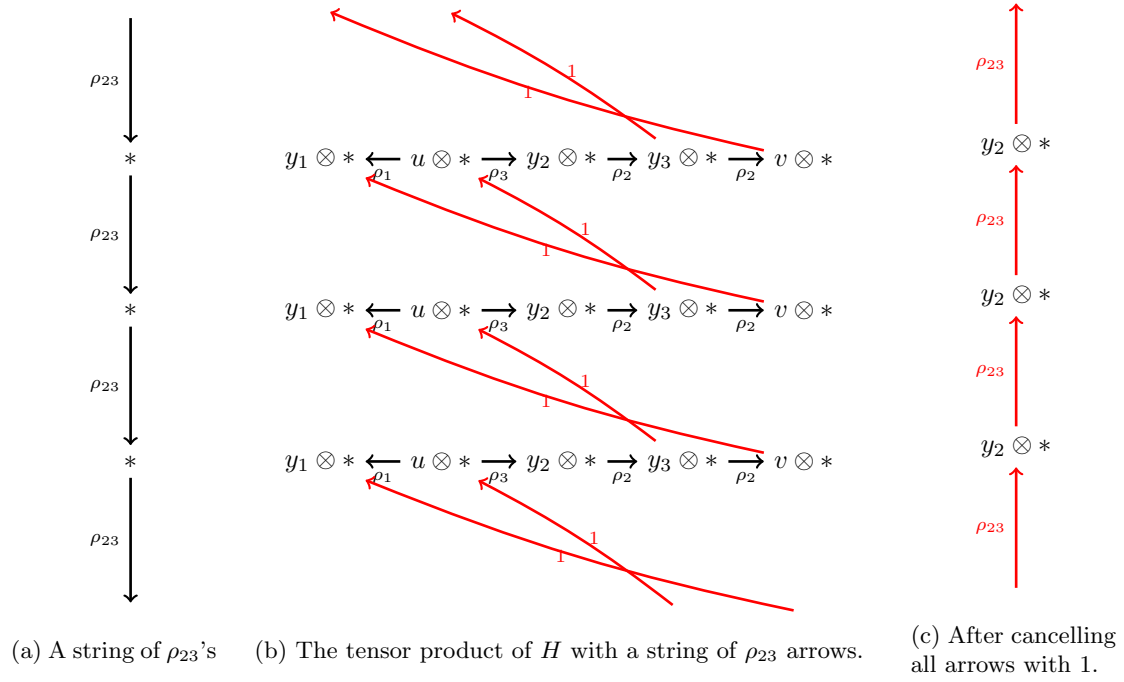


Figure 4.7: H tensoring with a string of ρ_{23} 's.

generator ξ'_j through a short chain of two arrows. After we discuss the cancellation *at both* of these joint, we show it agrees and gives the desired result when the two joint are put together.

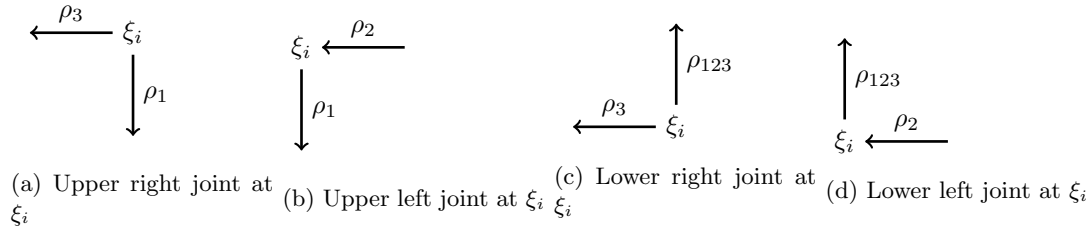


Figure 4.8: Four kinds of joints at $\xi_i \neq \xi_v$ or ξ_h

4.2.2 Upper Right Joint

For a generator ξ_i with the upper right joint in Figure 4.8, it appears in $KtD(C)$ as Figure 4.9. For either of the two arrows not immediately connecting to ξ'_i , we have two possibilities, corresponding to original arrows in C having length 1 or having length more than 1.

To be concise, the corresponding part of the box tensor product $H \boxtimes KtD(C)$ for all four cases are shown in Figure 4.10, where all four cases share the same generators and arrows in blue. On the left side, the two red arrows (two green arrows, respectively) come from cases where the arrow out of λ_1 is ρ_{23} (ρ_2 , respectively). Similarly, on the bottom right, the two red arrows (one pink arrow, respectively) come from cases where the arrow into κ_1 is ρ_{23} (ρ_{123} , respectively).

Now cancel the three blue arrows that are pointing leftward and rearrange generators to arrive at Figure 4.11. Note the joint at $u \otimes \kappa_1$ in $H \boxtimes KtD(C)$, which is boxed in Figure 4.11, can now be identified with the corresponding part of $KtD(C^{flip})$.

If either of the two possibilities is ρ_{23} , we can continue to cancel the set of two red arrows in Figure 4.11 in the same way we canceled arrows in the case of a string of ρ_{23} 's, then arrive at Figure 4.12. In these cases, cancellation done here agree with those in the case of tensoring with a string of ρ_{23} 's, so chains containing ρ_{23} 's at $u \otimes \kappa_1$ in $H \boxtimes KtD(C)$ can be identified with chains in $KtD(C^{flip})$, compare with Figure 4.9. Arrows in cases where either arrow at ξ_i have length 1 will be brought into shape after we deal with the joints ξ_i is connected to.

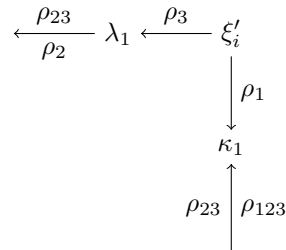


Figure 4.9: Upper right joint at ξ_i

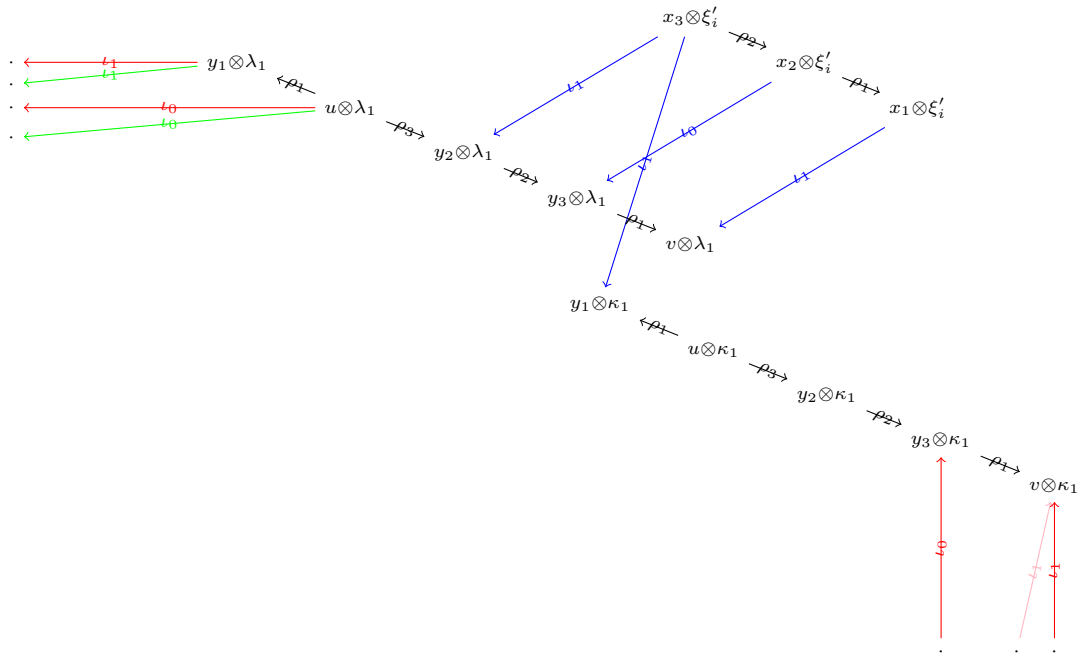


Figure 4.10: Upper right joint; H box tensor product with the joint.

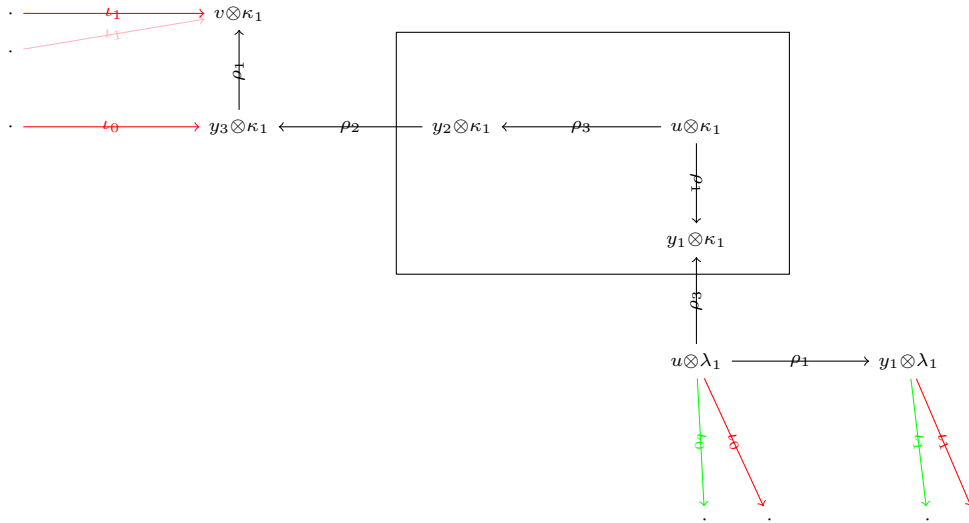


Figure 4.11: Upper right joint; after canceling the three blue arrows pointing left

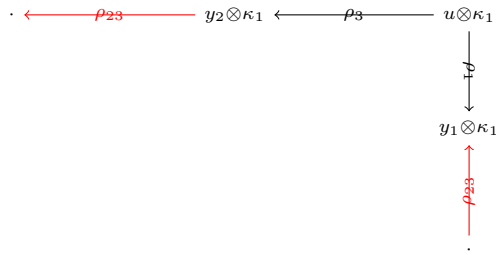


Figure 4.12: Upper right joint; cases where either arrow at ξ'_i have length more than 1.

4.2.3 Upper Left Joint

For a generator ξ_i with the upper left joint in Figure 4.8, it appears in $KtD(C)$ as Figure 4.13. For either of the two arrows not immediately connecting to ξ'_i , we have two possibilities, corresponding to original arrows in C having length 1 or having length more than 1. To be concise, the corresponding part of the box tensor product $H \boxtimes KtD(C)$ for all four cases are shown in Figure 4.14, where all four cases share the same generators and arrows in the middle. On the right side, the two red arrows (three blue arrows, respectively) come from cases where in Figure 4.13 the arrow going into λ_ℓ has a ρ_{23} (ρ_3 , respectively). Similarly, on bottom left, the two red arrows (one pink arrow, respectively) come from cases where the arrow going into κ_1 has a ρ_{23} (ρ_{123} , respectively).

Then we cancel the blue arrow and the two green arrows in the middle and arrive at Figure 4.15. Note the joint at $u \otimes \kappa_1$, which is boxed in Figure 4.15, can now be matched exactly to the corresponding part of $KtD(C^{flip})$, which would look like Figure 4.19.

If either of the two possibilities is ρ_{23} , we observe that cancellation of either set of two red arrows in Figure 4.15 agree with the cancellations for a string of ρ_{23} 's in Section 4.2.1, which shows chains containing ρ_{23} 's at $u \otimes \kappa_1$ in $H \boxtimes KtD(C)$ (Figure 4.16) exactly match corresponding chains in $KtD(C^{flip})$, again see Figure 4.19.

If the horizontal arrow at λ_ℓ has a ρ_3 coming in, see Figure 4.13, meaning original arrow out of ξ_i has length 1, the ρ_3 must come from another ξ'_j . ξ'_j belongs to either a upper right joint or a lower right joint. We defer the latter case to the lower right joint section, Section 4.2.4. In the former case, we combine the tensor product here and the tensor product Figure 4.10 in Section 4.2.2. Now $\lambda_\ell = \lambda_1$ and the green arrows in Figure 4.14 and Figure 4.10 are identified, so are the three horizontal blue arrows in Figure 4.14 and Figure 4.10. Reviewing the cancellation in this section and the upper right joint section, we find they combine together without conflicting and result in Figure 4.17, which is what the corresponding part in $KtD(C^{flip})$ look like.

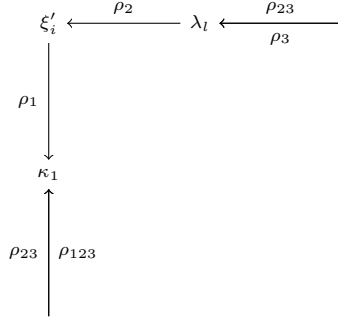


Figure 4.13: Upper left joint at ξ'_i

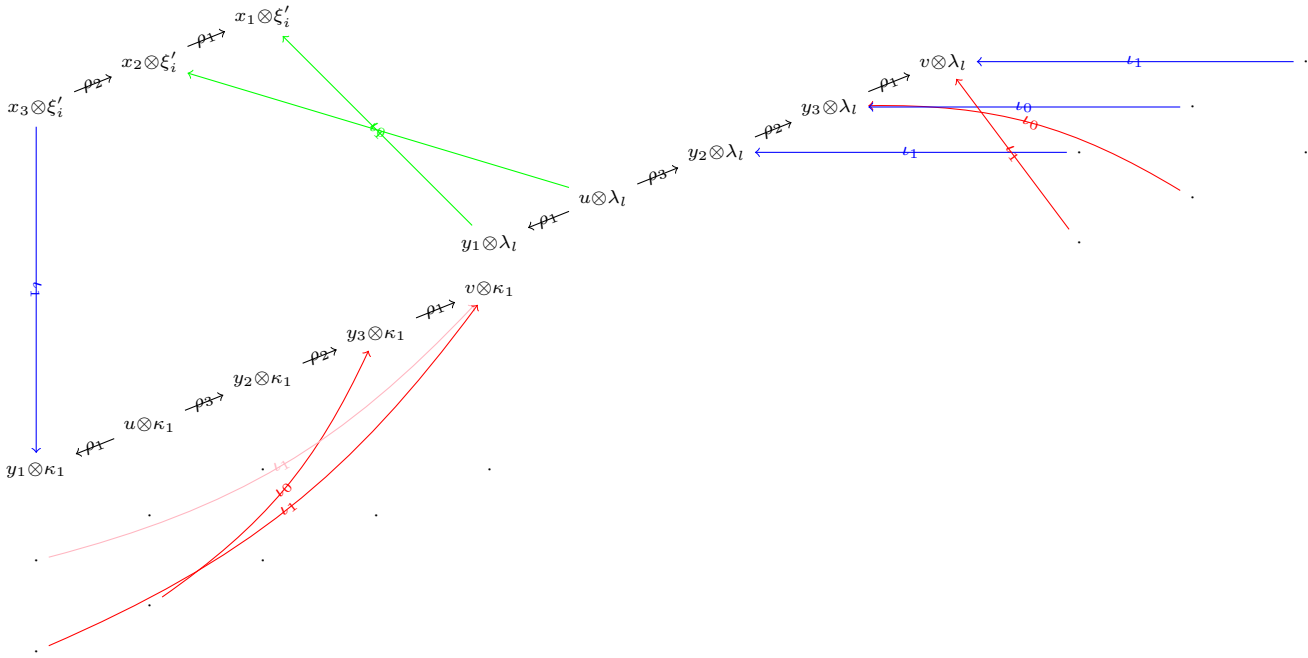


Figure 4.14: Upper left joint; tensor product with H .

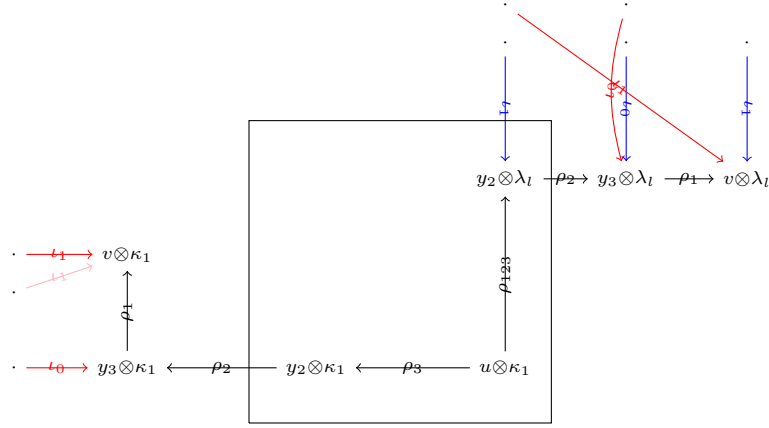


Figure 4.15: Upper left joint; after cancellation.

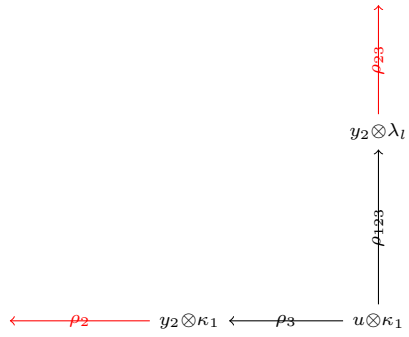


Figure 4.16: Upper left joint; cases where either arrow at ξ'_i have length more than 1

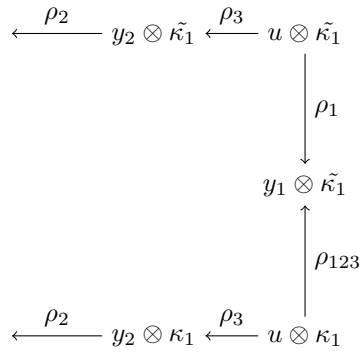


Figure 4.17: Upper left joint connected to an upper right joint.

4.2.4 Lower Right Joint

For a generator ξ_i with the lower right joint in Figure 4.8, it appears in $KtD(C)$ as Figure 4.19. Again, for either of the two arrows not immediately connecting to ξ'_i , we have two possibilities, corresponding to original arrows in C having length 1 or having length more than 1, and the corresponding part of the box tensor product $H \boxtimes KtD(C)$ for all four cases are shown in Figure 4.20, where all four cases share the same generators and arrows in the middle. On the top right, the two red arrows (one blue arrow, respectively) come from cases where in Figure 4.19 the arrow going out of κ_ℓ (into κ_ℓ , respectively) has a ρ_{23} (ρ_1 , respectively). Similarly, on the left side, the two red arrows (two green arrows, respectively) come from cases where the arrow going into λ_1 has a ρ_{23} (ρ_2 , respectively).

Then we cancel the three blue arrows and the pink arrow in the middle and arrive at Figure 4.21. Note the joint at $y_3 \otimes \kappa_\ell$, which is boxed in Figure 4.21, can now be identified with the corresponding part of $KtD(C^{flip})$, which would look like the joint in Figure 4.13.

If either of the two possibilities in Figure 4.19 is ρ_{23} , as above, we observe that cancellation of either set of two red arrows in Figure 4.21 agree with the cancellations we performed for a string of ρ_{23} 's in Section 4.2.1. This shows chains containing ρ_{23} 's at $y_3 \otimes \kappa_\ell$ in $H \boxtimes KtD(C)$ (Figure 4.22) exactly match corresponding chains in $KtD(C^{flip})$, again see Figure 4.13.

Now we deal with the possibilities other than ρ_{23} in Figure 4.19. Suppose arrows at ξ_i takes form as in Figure 4.18.

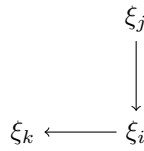


Figure 4.18: Lower right joint at ξ_i in C

If the vertical arrow $\xi_j \rightarrow \xi_i$, has length 1 in C , the joint at ξ_j could be either a upper right joint or a upper left joint. We argue that the latter case can not happen. If it did happen, denote the generator connecting to ξ_j through a horizontal arrow by ξ_r , then $\partial^2 \xi_r$ has a term $U \xi_i$. Because

the C is simultaneously horizontally and vertically simplified, there can not be any arrow going into ξ_i besides $\xi_j \rightarrow \xi_i$, leaving $\partial^2 \xi_r$ nonzero.

For the case where ξ_j belongs to a upper right joint, we combine the corresponding parts of $H \boxtimes KtD(C)$ in Figure 4.10 and Figure 4.20. Now generators has κ_ℓ are identified those with κ_1 . The downward point blue arrow in Figure 4.10 is identified with the blue arrow on top of Figure 4.20, so are the pink arrow in each of the figures. The cancellation we performed for each type of joints can be both applied and result in Figure 4.23. It can be identified with the corresponding part in $KtD(C^{flip})$.

If the horizontal arrow $\xi_i \rightarrow \xi_k$ has length 1 in C , the joint at ξ_k could be either a upper left joint or a lower left joint. We postpone the latter case to the lower left joint section, Section 4.2.5. For the former case, we argue that it could not happen. $\partial^2 \xi_j$ has term $U\xi_k$, but there can not be any arrow going into ξ_k besides $\xi_i \rightarrow \xi_k$, leaving $\partial^2 \xi_j$ nonzero.

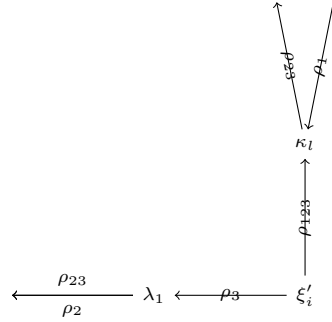


Figure 4.19: Lower right joint at ξ'_i

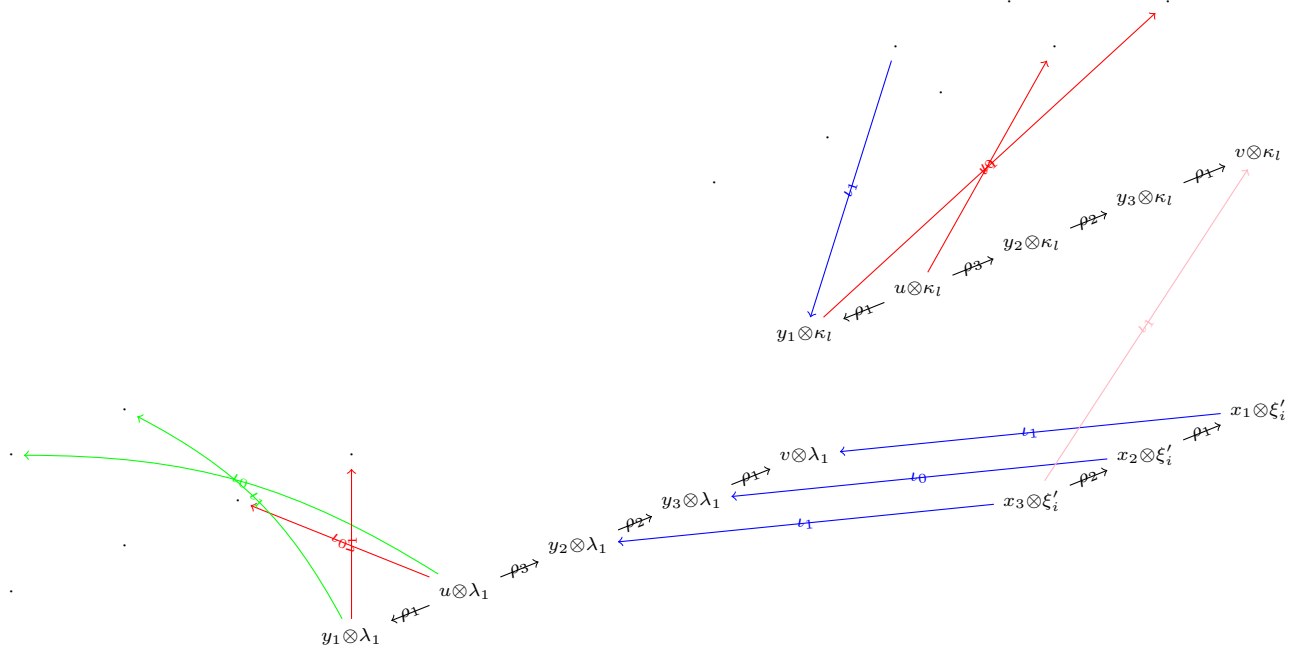


Figure 4.20: Lower right joint; tensor product with H .

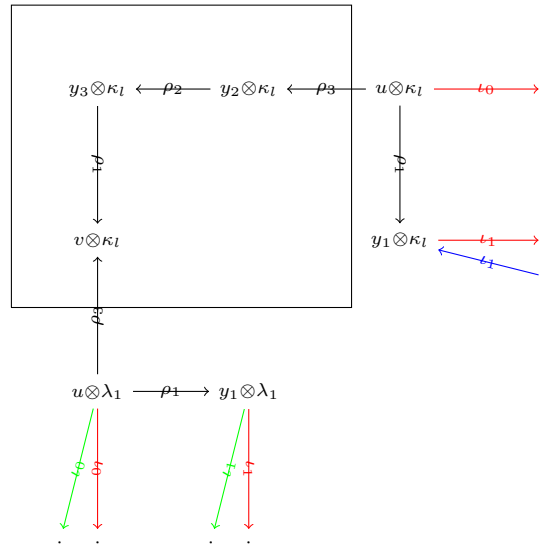


Figure 4.21: Lower right joint; after cancellation.

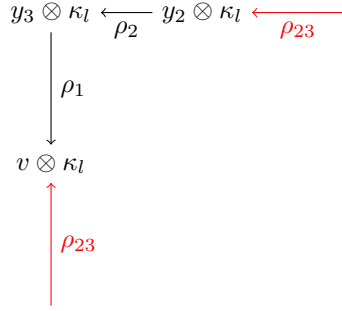


Figure 4.22: Lower right joint; cases where either arrow at ξ'_i have length more than 1

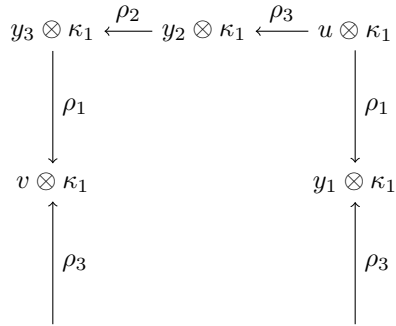


Figure 4.23: A lower right joint connected to a upper right joint

We include a summary of all eight possible ways two joints could overlap, see Figure 4.24.

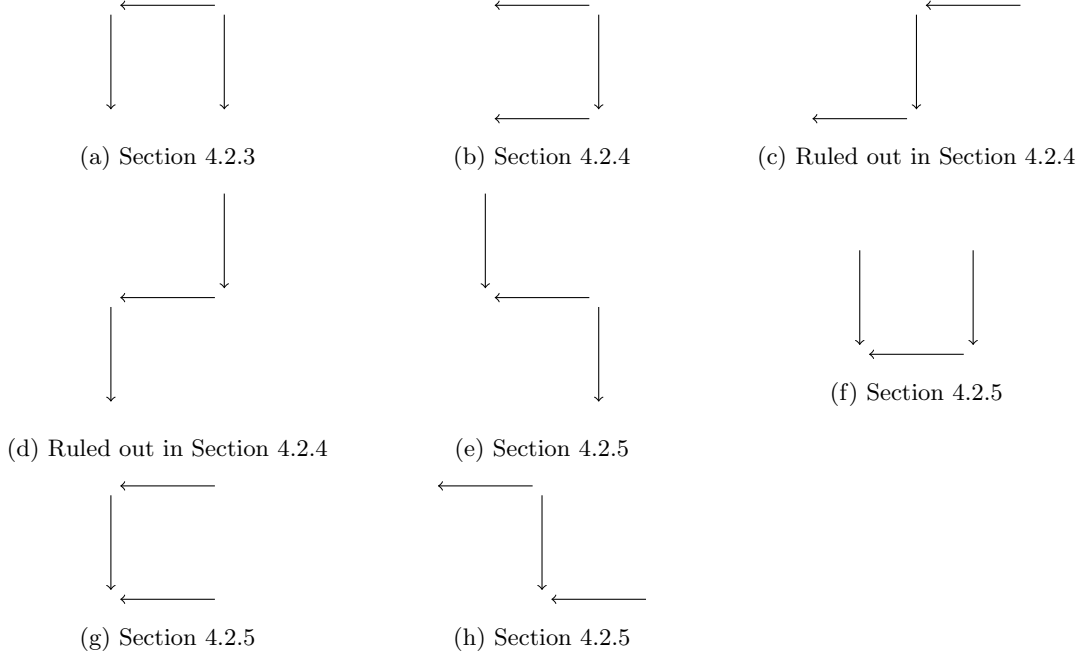


Figure 4.24: A summary of eight possible overlapping joints

4.2.5 Lower Left Joint

For a generator ξ_i with the lower left joint in Figure 4.8, it appears in $KtD(C)$ as Figure 4.25. Again, for either of the two arrows not immediately connecting to ξ'_i , we have two possibilities, corresponding to original arrows in C having length 1 or having length more than 1, and the corresponding part of the box tensor product $H \boxtimes KtD(C)$ for all four cases are shown in Figure 4.26, where all four cases share the same generators and arrows in the middle. On the top, the two red arrows (one blue arrow, respectively) come from cases where in Figure 4.25 the arrow going out of κ_ℓ (into κ_ℓ , respectively) has a ρ_{23} (ρ_1 , respectively). Similarly, on the right side, the two red arrows (three blue arrows, respectively) come from cases where in Figure 4.25 the arrow going into λ_ℓ has a ρ_{23} (ρ_3 , respectively).

Then we cancel the two green arrows and the pink arrow in the middle and arrive at Figure 4.27.

Note the joint at $y_3 \otimes \kappa_\ell$, which is boxed in Figure 4.27, can now be identified with the corresponding part of $KtD(C^{flip})$, which would look like the joint itself.

If either of the two possibilities in Figure 4.25 is ρ_{23} , as above, we observe that cancellation of either set of two red arrows in Figure 4.27 agree with the cancellations we performed for a string of ρ_{23} 's in Section 4.2.1. This shows chains containing ρ_{23} 's at $y_3 \otimes \kappa_\ell$ in $H \boxtimes KtD(C)$ (Figure 4.28) exactly match corresponding chains in $KtD(C^{flip})$, which would again look like the joint itself.

Now we deal with the four cases of overlapping joints involving a lower left joint, see the last four diagrams of Figure 4.24. First, for the case in Figure 4.24e, we combine Figure 4.26 with Figure 4.10. All generators involving λ_ℓ are identified with those with λ_1 , and the two sets of two green arrows and two sets of three blue arrows are respectively identified. We perform the same two sets of cancellations and arrive at Figure 4.29. It can be identified with the corresponding part in $KtD(C^{flip})$.

The remaining three cases in Figure 4.24f, Figure 4.24g, and Figure 4.24h can be proved in a similar fashion: we combine two tensor products of H with corresponding joints shown in this and above sections, then identify overlapping generators and arrows, and finally observe that cancellations done separately for each joint can be combined without conflicting. We provide the results in Figure 4.30, Figure 4.31, and Figure 4.32, and omit further details.

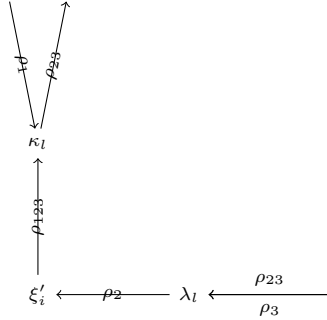


Figure 4.25: Lower left joint at ξ'_i

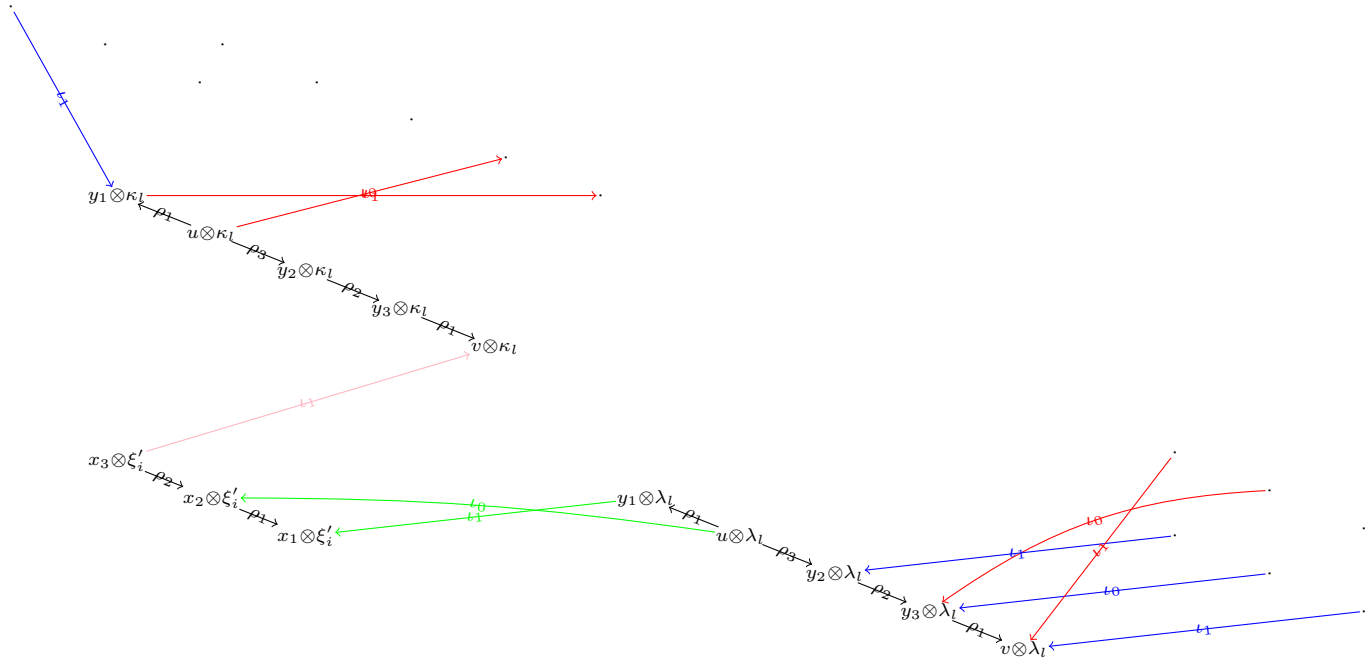


Figure 4.26: Lower left joint; tensor product with H .

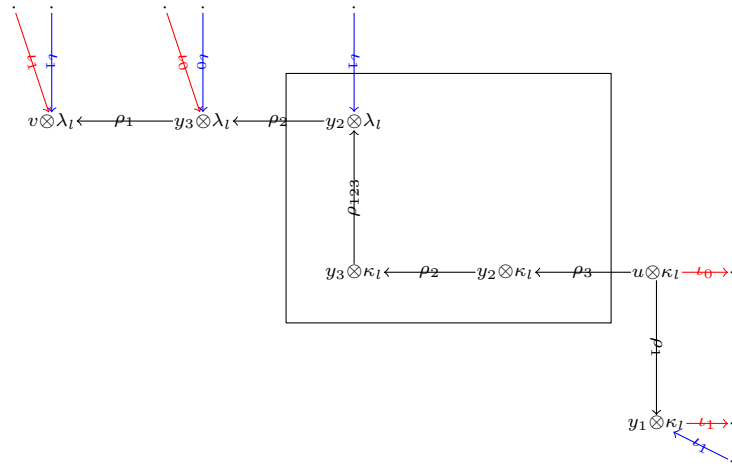


Figure 4.27: Lower left joint; after cancellation.

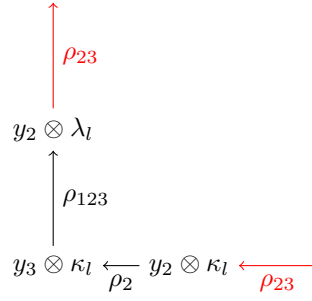


Figure 4.28: Lower left joint; cases where either arrow at ξ'_i have length more than 1

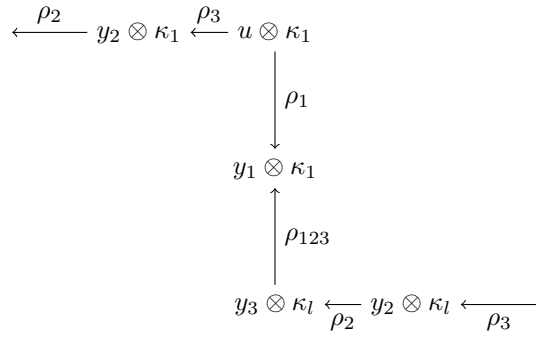


Figure 4.29: A lower left joint connected to a upper right joint

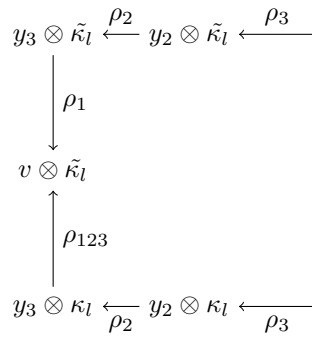


Figure 4.30: A lower left joint connected to a lower right joint. $\tilde{\kappa}_\ell$ refers to the κ_ℓ from the lower right joint.

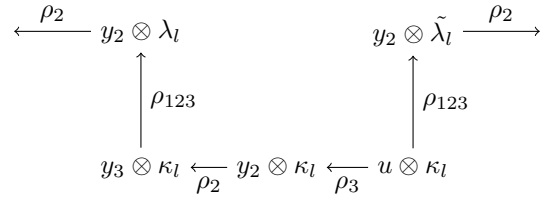


Figure 4.31: A lower left joint connected to a upper left joint. $\tilde{\lambda}_\ell$ refers to the λ_ℓ from the upper left joint.

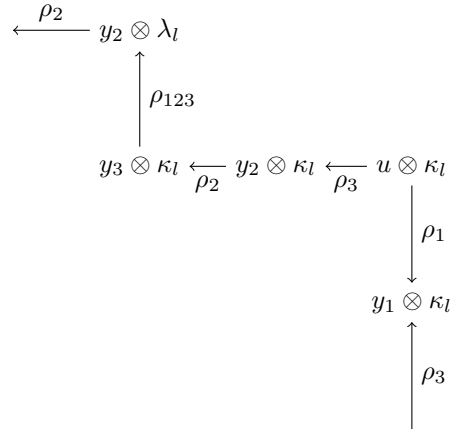


Figure 4.32: A lower left joint connected to a upper right joint in a different way

4.2.6 The Unstable Chain

So far we have avoided ξ_v , ξ_h , and the unstable chain between them in our discussion. We now argue that the proof for the unstable chain can be mostly reduced to the cases discussed above.

ξ_v has no connecting vertical arrow, so its corresponding generator ξ_v^{flip} in C^{flip} has no horizontal arrow. In other words, ξ_v^{flip} is the preferred generator in the horizontally simplified basis, or the “ ξ_h ” of C^{flip} . We still denotes it by ξ_v^{flip} to avoid confusion. A similar remark applies to ξ_h as well.

The rest of the proof is divided into three cases according to the sign of $2\tau(K) - n$.

Case 1: $2\tau(K) - n \geq 1$

We first look at the case where $m = 2\tau(K) - n \geq 1$.

Recall when $n < 2\tau(K)$, the unstable chain takes form:

$$\xi_v' \xrightarrow{\rho_1} \mu_1 \xleftarrow{\rho_{23}} \dots \xleftarrow{\rho_{23}} \mu_k \xleftarrow{\rho_{23}} \mu_{k+1} \xleftarrow{\rho_{23}} \dots \xleftarrow{\rho_{23}} \mu_m \xleftarrow{\rho_3} \xi_h',$$

If $\xi_v = \xi_h$, the unstable chain appears as in Figure 4.33. Now, H has the effect on the string of ρ_{23} 's of flipping them into the opposite direction. At generator ξ_v' , the arrows are exactly like a upper right joint in Figure 4.9 in Section 4.2.2, so the arrows are also flipped. The proof there carries over to show that H tensoring with the unstable chain in $KtD(C)$ matches the unstable chain in $KtD(C^{flip})$.

A corner case that deserves a bit of extra attention is when $m = 2\tau(K) - n = 1$. In this case, a chain of ρ_{23} 's is missing and the joint is next to itself. This scenario is not covered by previous discussion, but they are only two isolated arrows. A simple computation of tensor product confirms they are flipped as well.

Now suppose $\xi_v \neq \xi_h$. We observe that we can interpret arrows at ξ_v' and ξ_h' as joints we have already discussed.

For example, arrows at ξ_v' match exactly to a upper left joint as in Figure 4.13 if ξ_v has an

incoming arrow, and a upper right joint as in Figure 4.9 if ξ_v has an outgoing arrow, see Figure 4.34.

Similarly, arrows at ξ'_h match exactly to a lower right joint as in Figure 4.19 if ξ_h has an incoming arrow, and a upper right joint as in Figure 4.9 if ξ_h has an outgoing arrow, see Figure 4.36.

Now the proof is easy. In the case where ξ_v has an incoming horizontal arrow, the corresponding generator ξ_v^{flip} in C^{flip} is a generator with no horizontal arrow and an incoming vertical arrow, so its corresponding generator $\xi_v^{flip'}$ in $KtD(C^{flip})$ has a lower right joint. Now H matches the upper left joint at ξ'_v to the lower right joint at $\xi_v^{flip'}$ as discussed in Section 4.2.3. The case where ξ_h has an incoming vertical arrow works backwards using the H action discussed in Section 4.2.4.

Similarly, the correspondence between the case where ξ_v has an outgoing horizontal arrow and the case where ξ_h has an outgoing vertical arrow can be shown with the H action discussed in Section 4.2.2.

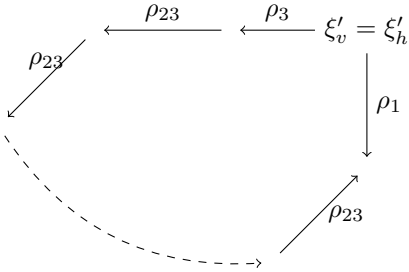


Figure 4.33: Unstable chain in the case where $\xi_v = \xi_h$

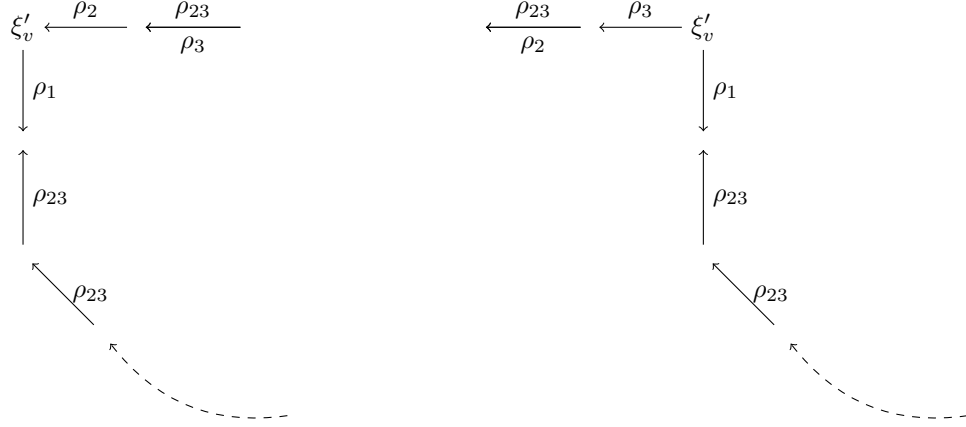


Figure 4.34: Joints at ξ'_v

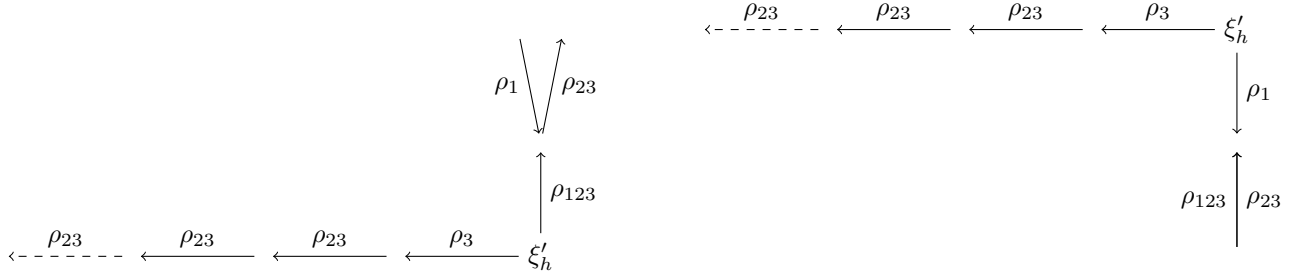


Figure 4.35: Joints at ξ'_h

Case 2: $2\tau(K) - n \leq -1$

This case is very similar to the previous case. Recall that when $n > 2\tau(K)$, the unstable chain takes form,

$$\xi'_v \xrightarrow{\rho_{123}} \mu_1 \xrightarrow{\rho_{23}} \dots \xrightarrow{\rho_{23}} \mu_k \xrightarrow{\rho_{23}} \mu_{k+1} \xrightarrow{\rho_{23}} \dots \xrightarrow{\rho_{23}} \mu_m \xrightarrow{\rho_2} \xi'_h,$$

where $m = n - 2\tau(K)$.

If $\xi_v = \xi_h$, the joint at ξ_v looks exactly like a lower left joint discussed in Section 4.2.5. Tensoring with H fixes it. The corner case of $m = 1$ can also be taken care of with a simple and direct computation.

If $\xi_v \neq \xi_h$, as in the previous case, joints at ξ_v and ξ_h can be identified with various different joint discussed in the previous four sections, see Figure 4.36 and Figure 4.37. Again it follows from

the previous sections that tensoring with H flips the unstable chain.

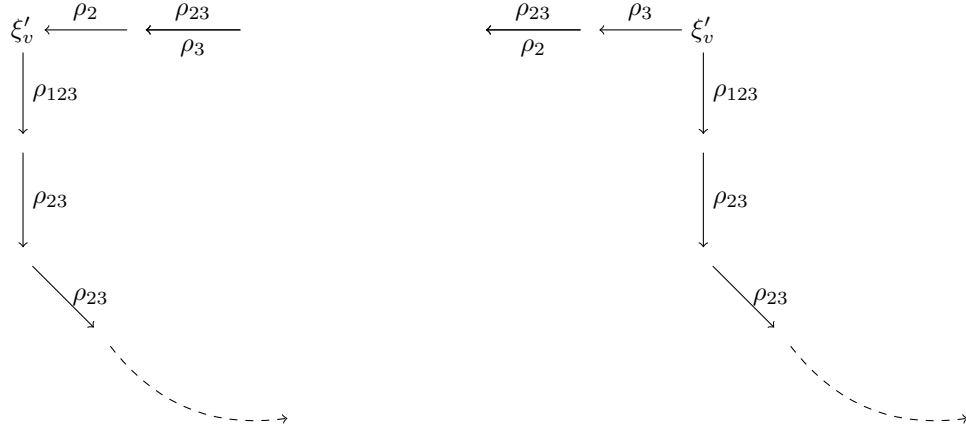


Figure 4.36: Joints at ξ'_v can be identified with a lower left joint and a lower right joint respectively.

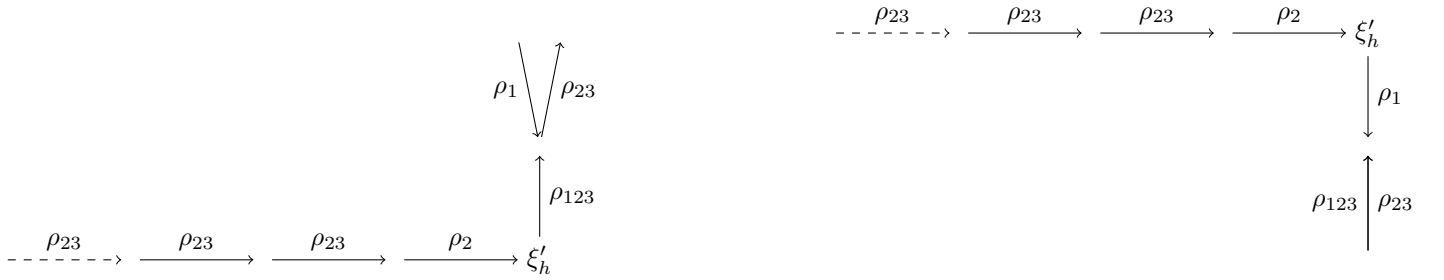


Figure 4.37: Joints at ξ'_h can be identified with a lower left joint or a upper left joint, respectively.

Case 3: $2\tau(K) - n = 0$

Recall when $n = 2\tau(K)$, the unstable chain takes form:

$$\xi'_v \xrightarrow{\rho_{12}} \xi'_h.$$

In the case where $\xi_v = \xi_h$, the joint at ξ_v is an isolated generator with an arrow pointing to itself. Again a direct computation proves our claim.

Now we come to the last piece of the puzzle. If $\xi_v \neq \xi_h$, they are linked up by a single arrow

of ρ_{12} . ξ_v (ξ_h , respectively) has either an incoming or outgoing horizontal (vertical, respectively) arrow. In other words, there are four different configurations around the arrow of ρ_{12} and the end of chains at ξ'_v and ξ'_h .

We need to show that tensoring with H simply reverses the direction of this single arrow of ρ_{12} while converting the end of chain at ξ'_v (ξ'_h , respectively) to the end of chain at $\xi_v^{flip'}$ ($\xi_h^{flip'}$, respectively).

For example, if ξ_v has an incoming horizontal arrow, see Figure 4.38 for the corresponding part in $KtD(C)$, in which we combine two cases, i.e ξ_h has an incoming arrow (relevant parts drawn in red) and ξ_h has an outgoing arrow (relevant parts drawn in blue). Note in the rest of this section, the color coding is different from our previous convention. The tensor product is shown in Figure 4.39 and after cancellation in Figure 4.40.

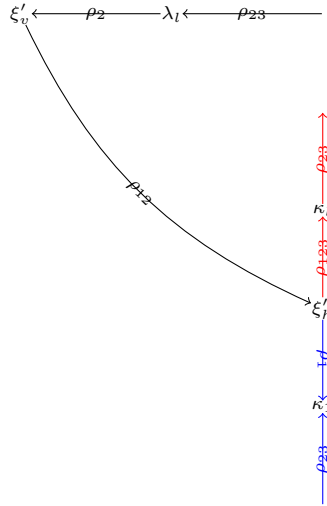


Figure 4.38

The tensor product has an arrow of ρ_{12} in both the red and blue cases, with its direction reversed. They are both followed by a ρ_{123} , which is exactly what we need; indeed, ξ_v^{flip} has an incoming vertical arrow in C^{flip} , so $\xi_v^{flip'} \in KtD(C^{flip})$ has an outgoing ρ_{123} .

In the red case, ξ_h^{flip} has an incoming horizontal arrow in C^{flip} , so $\xi_h^{flip'} \in KtD(C^{flip})$ has an incoming ρ_2 , which is confirmed by the tensor product in Figure 4.40. Similarly, in the blue case,

ξ_h^{flip} has an outgoing horizontal arrow in C^{flip} , so $\xi_h^{flip'} \in KtD(C^{flip})$ has an outgoing ρ_3 , which is again confirmed.

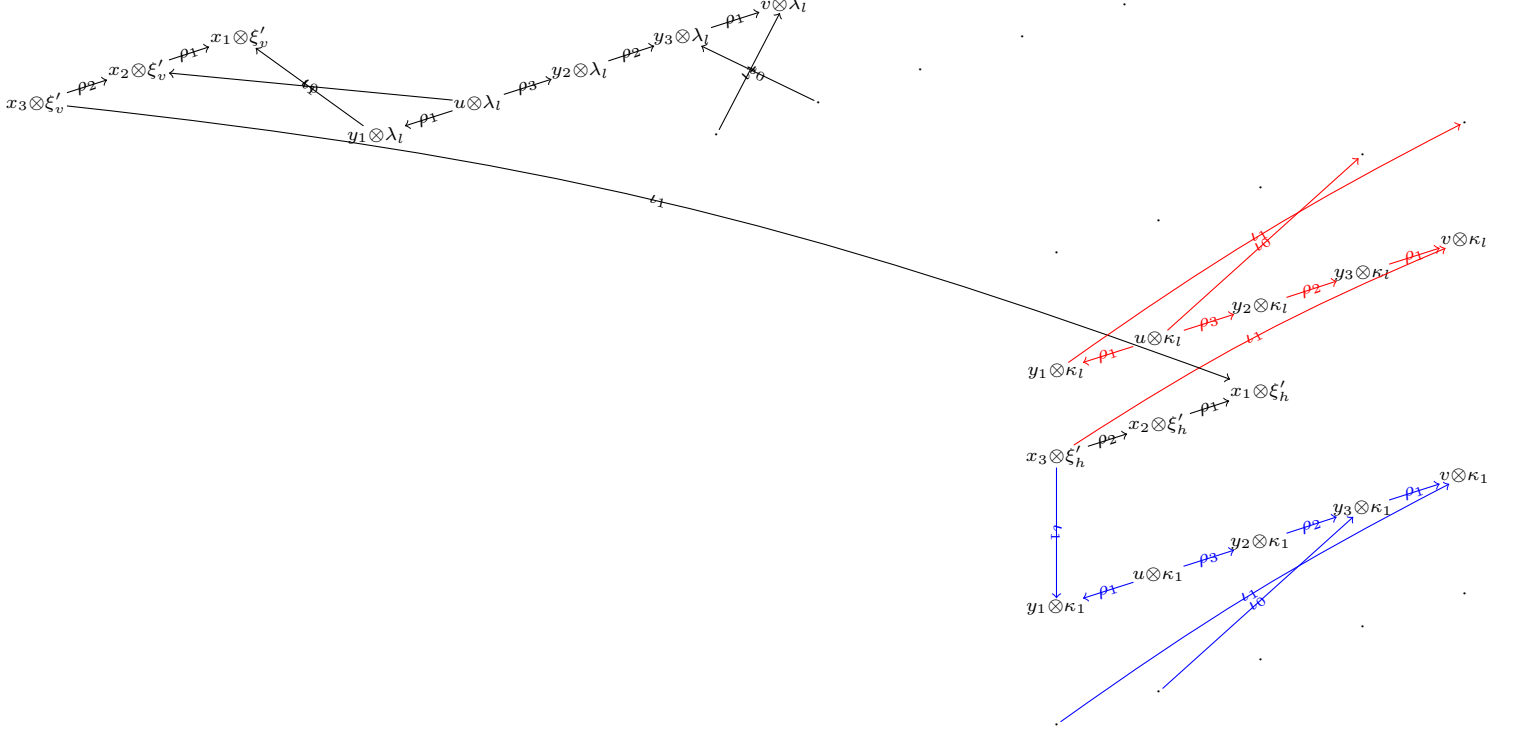


Figure 4.39

In Figure 4.38, we assume that the arrows at ξ_v and ξ_h have lengths larger than 1 and it is obvious that the cancellations there does not conflict with those for a chain of ρ_{23} 's.

It is also necessary to consider the cases where these arrows have length 1, or, in our language of “joints”, the unstable chain is right next to joints. To see no conflict arises, one simply needs to combine the cancellation here with the cancellation performed at each connecting joint discussed above. It is clear that the two sets of cancellations agree nicely. We omit the tedious details.

The only case left to consider is where ξ_v has an outgoing horizontal arrow, see Figure 4.41 for the corresponding part in $KtD(C)$. The argument is almost parallel. We again combine two cases on the ξ_h side, i.e ξ_h has an incoming arrow (relevant parts drawn in red) and ξ_h has an outgoing arrow (relevant parts drawn in blue). The tensor product is shown in Figure 4.42 and

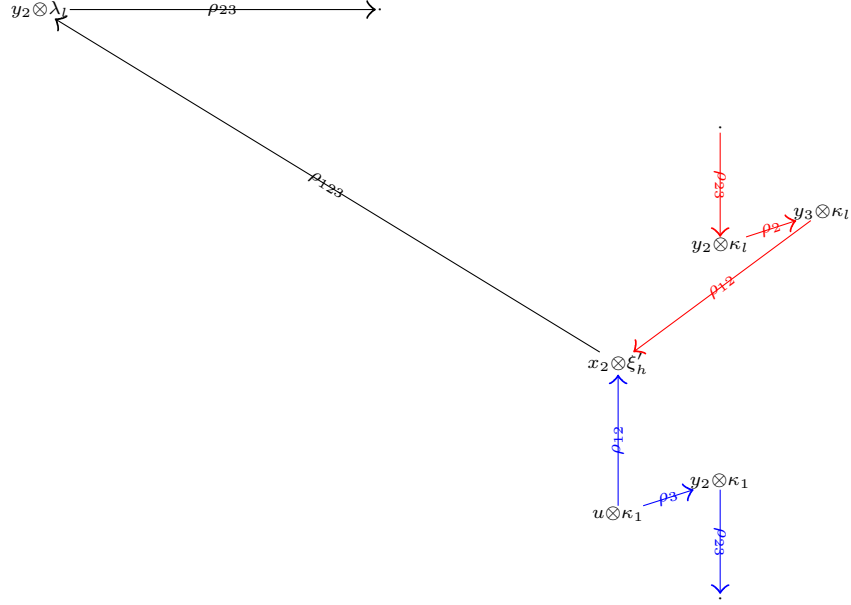


Figure 4.40

after cancellation in Figure 4.43.

The tensor product has an arrow of ρ_{12} in both the red and blue cases, with its direction reversed. They are both followed by a ρ_1 , which is exactly what we need; indeed, ξ_v^{flip} has an outgoing vertical arrow in C^{flip} , so $\xi_v^{flip} \in KtD(C^{flip})$ has an outgoing ρ_1 .

On the ξ_h side, arrows turn out exactly the same as before. The proof that cancellations here agree with those performed for joints is similar to the case where ξ_v has an incoming horizontal arrow. We again omit the details.

Now, we have finally finished the proof.

Jonathan Hanselman and Liam Watson developed a “loop” calculus [3] that is very convenient for computations on $\widehat{CFD}$ modules like the ones appear in this section. In the case where C admit a basis that is simultaneously horizontally and vertically simplified, it is possible to give a more concise proof using their language.

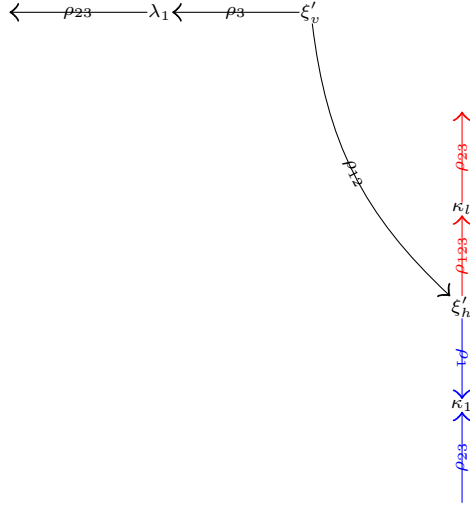


Figure 4.41

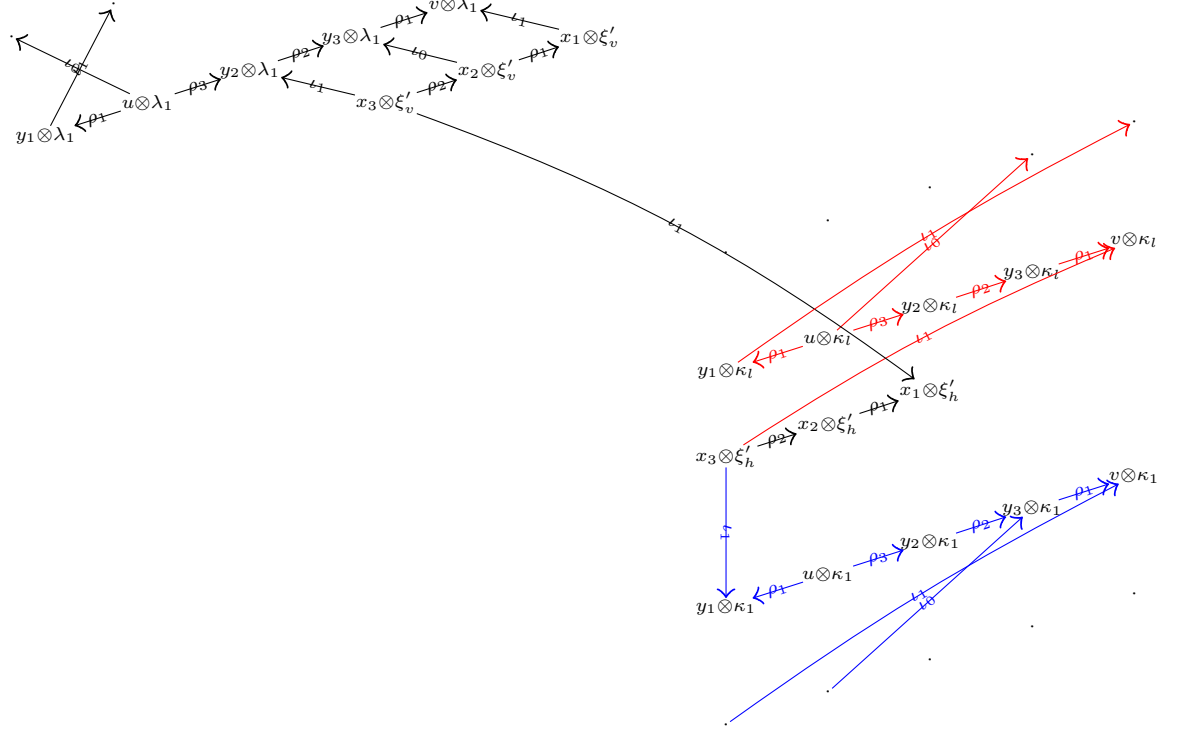


Figure 4.42

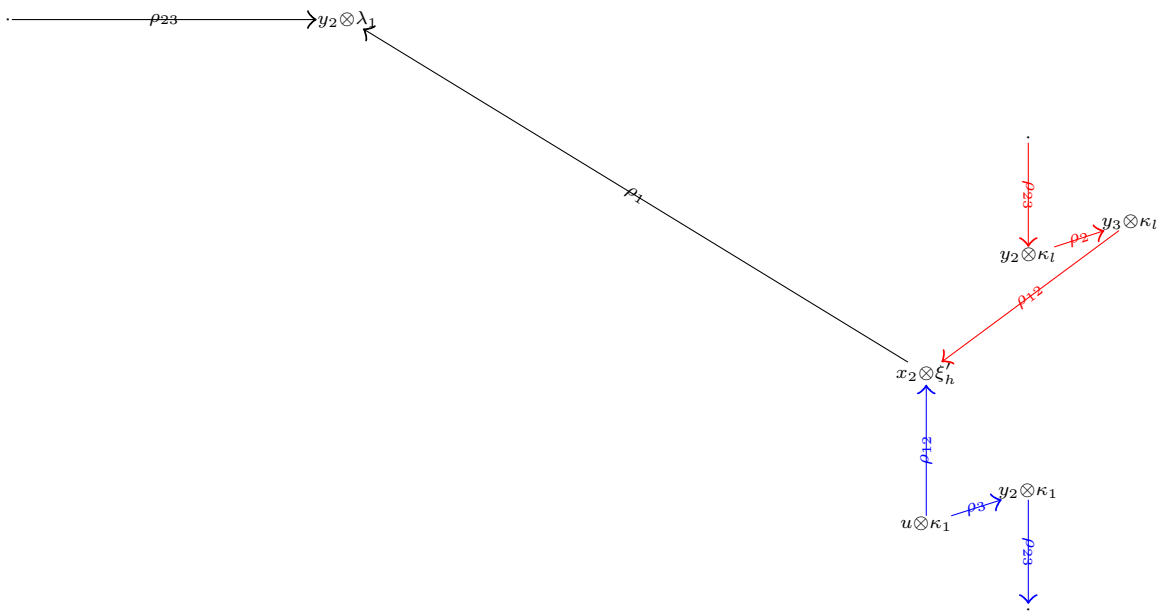


Figure 4.43

Chapter 5

An Algebraic Description of Boundary Dehn Twists' Effects on $\widehat{CFD}$ of Knot Complements

5.1 Half Boundary Dehn Twist

In Section 4.2, we investigate how tensoring with $H := \widehat{CFDA}(h)$ changes $\widehat{CFD}(X)$, where X is a knot complement. We described a global “flipping” behavior, but there is an interesting local pattern at play as well. In fact, there exists a “mapping” of arrows that transforms $\widehat{CFD}(X)$ into $\widehat{CFDA}(h) \boxtimes \widehat{CFD}(X)$. Specifically, we have

Theorem 7. *Given a knot $K \subset S^3$, let X be the knot complement with boundary parametrization $\phi : T^2 \rightarrow \partial X$.*

Suppose that there exists a reduced and simultaneously horizontally and vertically simplified model for $CFK^-(K)$. We obtain $\widehat{CFD}(X, \phi)$ with the algorithm described in Theorem 6. Then, $\widehat{CFDA}(h) \boxtimes \widehat{CFD}(X, \phi)$ can be obtained by locating each arrow of $\widehat{CFD}(X, \phi)$ in one of the following

2-tuples and then replacing it with the other arrow in that 2-tuple.

$$(\xrightarrow{\rho_1}, \xrightarrow{\rho_3}), (\xrightarrow{\rho_2}, \xleftarrow{\rho_{123}}), (\xrightarrow{\rho_{23}}, \xleftarrow{\rho_{23}}), (\xrightarrow{\rho_{12}}, \xleftarrow{\rho_{12}}).$$

Note the possible direction change.

Proof. In the proof in Section 4.2, if one examines the tensor product of $\widehat{CFDA}(h)$ with each joint and chain, it is obvious that such an exchange of arrows does take place. $\square$

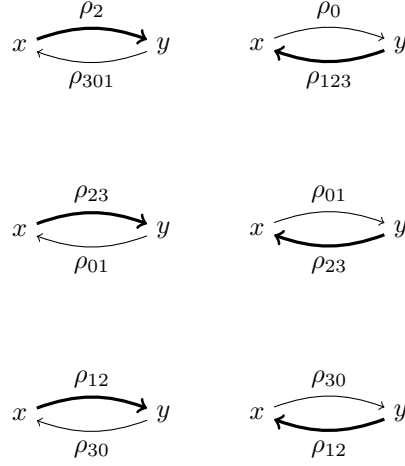
This description can be understood intuitively with the deeper structures of bordered Floer modules. If one considers the border theory of the minus version of Heegaard Floer homology, which is under development by Peter S. Ozsváth and Zoltán Szabó, holomorphic disks covering the basepoint z must be counted in order to recover HF^- and the bordered algebra must be enlarged to include ρ_0 . Heuristically, for a disk $x \xrightarrow{\rho_1} y$ in the current theory, one could conceive another disk going from y back to x except going around the boundary in the other direction and picking up a ρ_{230} . In other words, $x \xrightarrow{\rho_1} y$ in the current theory corresponds to a circle of two arrows in the minus theory.

$$\begin{array}{ccc} & \rho_1 & \\ x & \xrightarrow{\quad} & y \\ & \xleftarrow{\rho_{230}} & \end{array}$$

Now, as we have seen in Section 4.1, H has the effect of a half boundary Dehn twist, which rotates the numbering of the boundary regions by 2. So the new circle of arrows results from the twist is

$$\begin{array}{ccc} & \rho_3 & \\ x & \xrightarrow{\quad} & y \\ & \xleftarrow{\rho_{012}} & \end{array}$$

Coming back to the current theory, the arrow with ρ_0 is discarded and only a ρ_3 is left. Performing the half boundary Dehn twist again will transform the circle of arrows back to its original form. Hence, H appears to have the effect of switching arrows between $\xrightarrow{\rho_1}$ and $\xrightarrow{\rho_3}$.



Similar arguments work for the other three pairs of arrows.

Without the help of the algebraic machinery for the bordered theory of HF^- , this argument is only heuristics, but it nonetheless provides some insights to why H has such interesting effects.

5.2 Quarter Boundary Dehn Twist

Going further along this idea, we can also investigate the effect of quarter Dehn twist on $\widehat{CFD}$ modules of knot complements.¹

In this case, we are instructed by this heuristics to believe that quarter Dehn twist rotates arrows cyclically in the following 4-tuples, also see Figure 5.2.

$$(\xrightarrow{\rho_1}, \xrightarrow{\rho_2}, \xrightarrow{\rho_3}, \xleftarrow{\rho_{123}}), (\xrightarrow{\rho_{12}}, \xrightarrow{\rho_{23}}, \xleftarrow{\rho_{12}}, \xleftarrow{\rho_{23}}). \quad (5.1)$$

¹The author thanks Adam Levine for suggesting this.

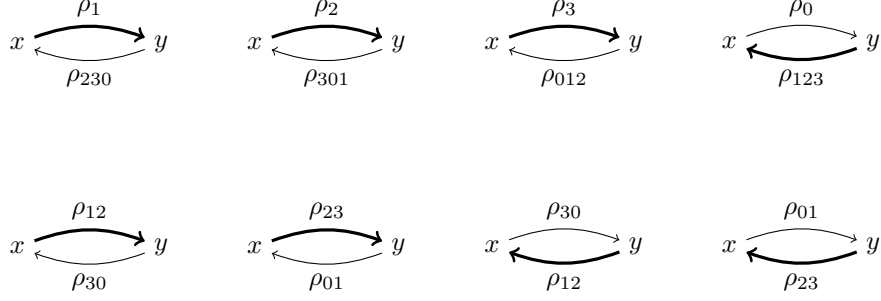


Figure 5.2

We summarize the statement as follows.

Theorem 8. *Given a knot $K \subset S^3$, let X be the knot complement with boundary parametrization $\phi : T^2 \rightarrow \partial X$.*

Suppose that there exists a reduced and simultaneously horizontally and vertically simplified model for $CFK^-(K)$. We obtain $\widehat{CFD}(X, \phi)$ with the algorithm described in Theorem 6. Then, $\widehat{CFDA}(q) \boxtimes \widehat{CFD}(X, \phi)$ can be obtained by locating each arrow of $\widehat{CFD}(X, \phi)$ in one of the following 4-tuples and then replacing it with the cyclically next arrow in that 4-tuple.

$$(\xrightarrow{\rho_1}, \xrightarrow{\rho_2}, \xrightarrow{\rho_3}, \xleftarrow{\rho_{123}}), (\xrightarrow{\rho_{12}}, \xrightarrow{\rho_{23}}, \xleftarrow{\rho_{12}}, \xleftarrow{\rho_{23}}).$$

Note the possible direction change.

Proof. Parallel to what we did in the previous chapters, a quarter boundary Dehn twist q rotates the boundary by one region. q can be decomposed into Dehn twists as $\tau_\mu \circ \tau_\lambda \circ \tau_\mu$. As before, we compute $\widehat{CFDA}(q)$ as the tensor product of the $\widehat{CFDA}$ modules of each Dehn twist and perform cancellations. See Figure 5.3 for a diagram of the bimodule.

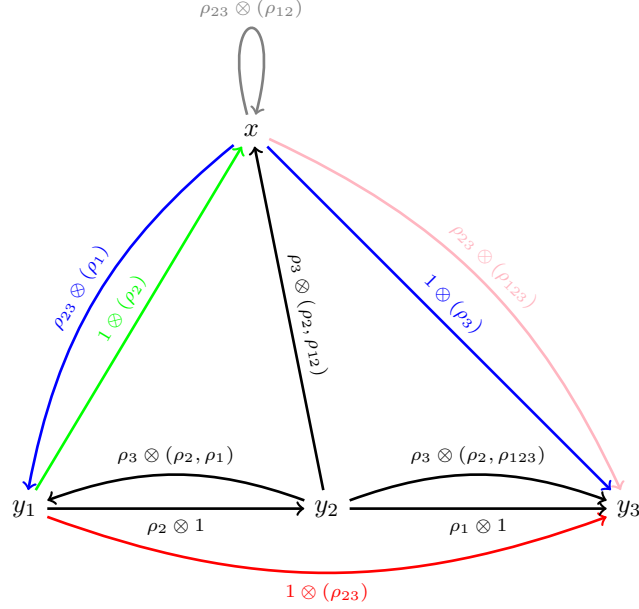


Figure 5.3: Arrows are color coded according to their A^∞ actions.

The following relations with idempotents hold.

$$\iota_1 \cdot x \cdot \iota_0 = x$$

$$\iota_1 \cdot y_1 \cdot \iota_1 = y_1$$

$$\iota_0 \cdot y_2 \cdot \iota_1 = y_2$$

$$\iota_1 \cdot y_3 \cdot \iota_1 = y_3$$

We could go through a similar set of careful combinatorial arguments to prove that $\widehat{CFDA}(q)$ does have the claimed effect in Equation (5.1). However, the following computation might be a more clear and straightforward way to convince oneself.

Note we analyzed the case of half boundary Dehn twist by enumerating its effect on different pieces of a $\widehat{CFD}$ module. We realized that all the effects are completely local, i.e. confined to a joint or a chain. Knowing this, we can construe a set of $\widehat{CFD}$ modules that contain all the joints, chains of different lengths, and all the overlapping joints. Taking tensor products of $\widehat{CFDA}(q)$ and

each of these representative modules will then reveal the effect on all the building blocks. For a $\widehat{CFD}$ module of an arbitrary knot complement, we can then argue that $\widehat{CFDA}(q)$ changes it in the way we believed.

The following is a list of (possibly nonexistent) CFK^- modules that correspond to $\widehat{CFD}$ modules per Theorem 6. It is not hard to see that these $\widehat{CFD}$ modules will contain all the building blocks. What is left for the proof is simply to take tensor products of each of them with $\widehat{CFDA}(q)$ and performed cancellations. The claimed effect indeed takes place on each module and at each building block. The details are omitted here.

- A string of ρ_{23} 's of any length,
- a string of ρ_{12} 's of any length,
- i by j boxes, where $i, j = 1, 2, 3$, see Figure 5.4a
- staircases with varying step width, with ξ_v and ξ_h connected by the unstable chain of various lengths, see Figure 5.4b, ..., Figure 5.4g.

□

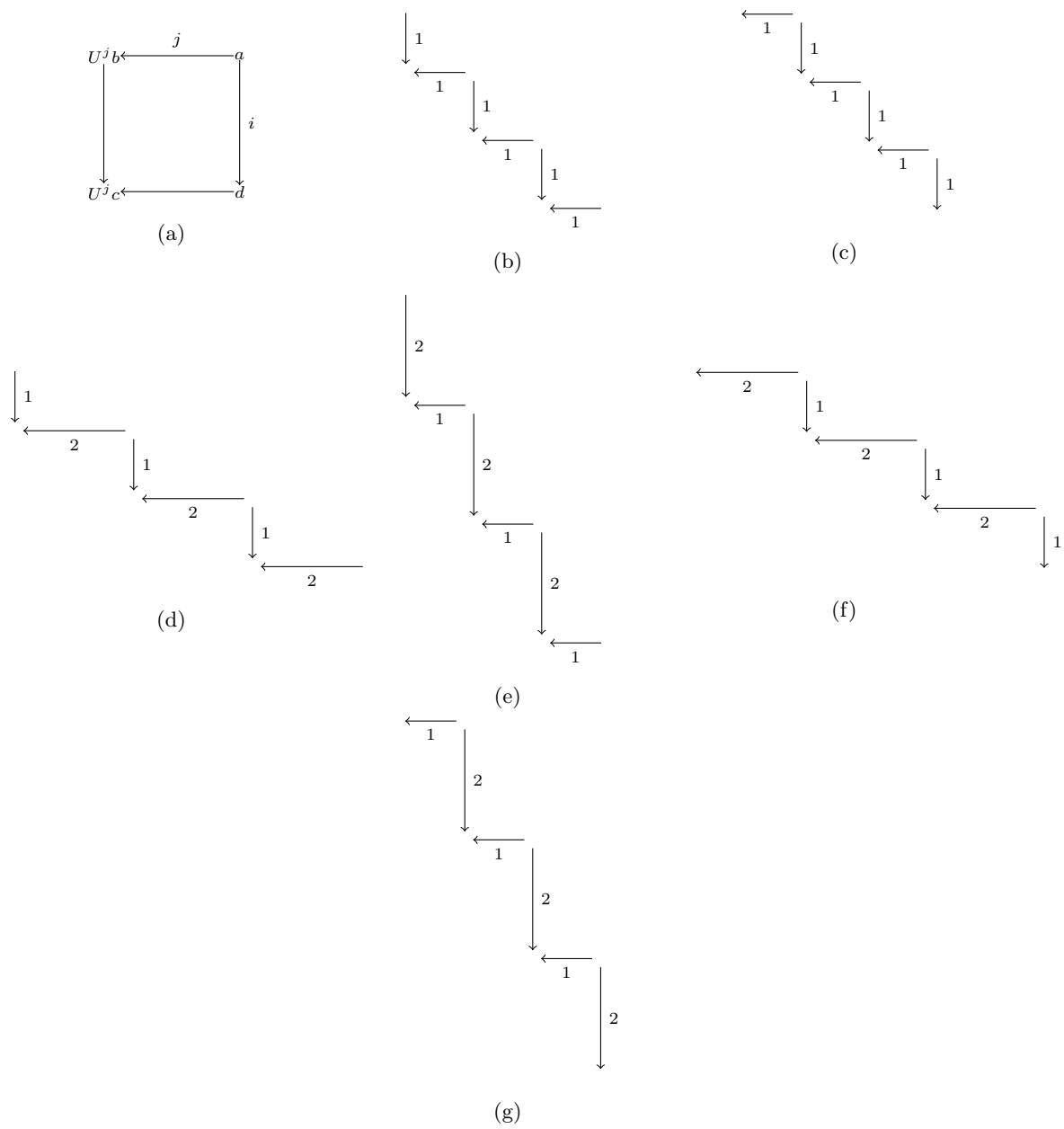


Figure 5.4

Chapter 6

Associated Generalized Open Book

Let M_h be the strongly based bordered 3-manifold associated with elliptic involution h . A naturally related object is its associated generalized open book $(Y, K) := (M_h^\circ, K)$, as described in Chapter 5 of [4]. In this case of bordered three-manifold associated to a surface diffeomorphism, this concept coincides with the classic notion of the open book associated to h .

Recall that $Y = M_h^\circ$ is obtained from identifying the two boundaries of M_h first, and then performing a 0-surgery on the arc γ_z , which is now a framed knot. K is the core of the surgery torus (Construction 5.20, [4]).

We would like to compute $\widehat{HFK}(Y, K)$. First, K is fibered knot in Y with Seifert genus 1, by the construction of the generalized open book. $\widehat{HFK}(Y, K)$ is supported in Alexander grading -1, 0, and 1. It is shown that the topmost group of $\widehat{HFK}(Y, K)$ is a single $\mathbb{Z}$ in [8]. And also by symmetry, we have $\text{rk}(\widehat{HFK}(Y, K, \pm 1))=1$.

Theorem 14 in [4] states that there is an identification between the knot Floer homology of the generalized open book and the Hochschild homology of the bimodule which identifies the Alexander grading on $\widehat{HFK}$ with the weight on the bimodule.

So the Alexander grading 0 component of the knot Floer homology can be computed from the Hochschild homology of the weight 0 part $\widehat{CFDA}(h, 0)$ of the $\widehat{CFDA}$ bimodule. We follow the model $\widetilde{CH}$ introduced in Section 2.3.5 of [4].

x_2, v, y_1 , and y_2 are the only generators that survive in the cyclicization, hence they are the generators of $\widehat{CH}$. v and y_1 have no outgoing arrows with trivial A -side action, so $\tilde{\partial}(v) = \tilde{\partial}(y_1) = 0$. x_2 does have such an arrow to x_1 spitting out a ρ_1 , however x_1 has no outgoing arrows with trivial A -side action or m_2 action taking ρ_1 . Similarly for y_2 , we can go to y_3 first, then to v , but no further. We have $\tilde{\partial} = 0$ and $\widehat{HFK}(Y, K, 0)$ has rank 4.

Now we have $\widehat{HFK}(Y, K)$ rank 1, 4, and 1 in Alexander grading -1, 0, and 1 respectively.

It is also interesting to know the Heegaard Floer homology of the ambient manifold Y . Y is a plumbed three-manifold and can be realized as the boundary of a four-manifold X with plumbing graph G in Figure 6.1. We utilize the methods introduced in [12] to compute the Heegaard Floer homology of Y .

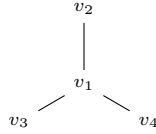


Figure 6.1: Plumbing graph G of four-manifold X , whose boundary is Y . $m(v_i) = -2, \forall i = 1, \dots, 4$.

Following [12], let $\text{Char}(G)$ denote the set of characteristic vectors for the intersection form. For each characteristic vector K and vertex v_i , let

$$2n = \langle K, v_i \rangle + v_i \cdot v_i,$$

and $\mathcal{T}_0^+$ the $\mathbb{Z}[U]$ -module $\mathbb{Z}[U^{-1}, U]/(U \cdot \mathbb{Z}[U])$.

Now let

$$\mathbb{H}^+(G) \subset \text{Hom}(\text{Char}(G), \mathcal{T}_0^+)$$

denote the functions that satisfy the following relations:

$$U^n \cdot \phi(K + \text{PD}[v]) = \phi(K), \text{ if } n \geq 0$$

$$\phi(K + \text{PD}[v]) = U^{-n} \cdot \phi(K), \text{ if } n \leq 0$$

The following is proved in [12].

Theorem 9 (Theorem 1.2 [12]). *Let G be a negative-definite weighted graph with at most one vertex whose weight plus degree is larger than 0. Then, for each Spin^c structure $\mathfrak{t}$ over $-Y$, there is an isomorphism of graded $\mathbb{Z}[U]$ modules,*

$$HF^+(-Y(G), \mathfrak{t}) \cong \mathbb{H}^+(G, \mathfrak{t}).$$

To determine the rank of $\text{Ker}U$ as $\mathbb{Z}$ -module in $HF^+(Y)$, we follow the algorithm described in [12], which proceeds as follows.

Fix a characteristic vector K satisfying inequality

$$m(v) + 2 \leq \langle K, v \rangle \leq -m(v), \quad (6.1)$$

for all vertices v . Now, successively apply the following algorithm to find a path of vectors ($K = K_0, \dots, K_n$) so that the K_i for $i < n$ satisfy the bounds

$$|\langle K_i, v \rangle| \leq -m(v),$$

for all vertices v . Given K_i , choose any vertex v_{i+1} with

$$\langle K_i, v_{i+1} \rangle = -m(v_{i+1}), \text{ then let } K_{i+1} = K_i + 2\text{PD}[v_{i+1}].$$

This algorithm can terminate in one of two ways: either,

- the final vector $L = K_n$ satisfies the inequality

$$m(v) \leq \langle L, v \rangle \leq -m(v) - 2 \quad (6.2)$$

at each vertex v , or

- there is some vertex v for which

$$\langle K_n, v \rangle > -m(v). \quad (6.3)$$

Such a path is called a *full path*.

The following result is proved in [12].

Theorem 10 (Proposition 3.2 [12]). *Fix an equivalence class in $\mathbb{K}^+(G)$ which contains no representatives of the form $U^m \otimes K'$ for $m > 0$. Each such equivalence class has a unique representative K satisfying Equation (6.1). Indeed, a characteristic vector K satisfying these bounds is inequivalent to an element of the form $U^m \otimes K'$ (with $m > 0$) if and only if we can find a full path $(K = K_0, K_1, \dots, K_n = L)$ terminating with a characteristic vector $L = K_n$ which satisfies Inequality 6.2 for each vertex v .*

The intersection form of X can be read off from the plumbing graph as:

$$Q_X = \begin{pmatrix} -2 & 1 & 1 & 1 \\ 1 & -2 & 0 & 0 \\ 1 & 0 & -2 & 0 \\ 1 & 0 & 0 & -2 \end{pmatrix}$$

It is easy to see that $v_i \cdot v_i \equiv 0 \pmod{2} \forall i$, hence, if we write characteristic vectors as tuples

$$(\langle K, v_1 \rangle, \dots, \langle K, v_4 \rangle),$$

it is clear that

$$\text{Char}(G) = \{(x_1, x_2, x_3, x_4) \mid x_i \in 2\mathbb{Z}, \forall i\},$$

Note $PD[v_1] = (-2, 1, 1, 1)$, $PD[v_2] = (1, -2, 0, 0)$, $PD[v_3] = (1, 0, -2, 0)$, $PD[v_4] = (1, 0, 0, -2)$, and the roles of the v_2 , v_3 , and v_4 are completely interchangeable.

The set of initial vectors satisfying Equation (6.1) is $\{(x_1, x_2, x_3, x_4) \mid x_i \in \{0, 2\}, \forall i\}$. We are looking for a path where each intermediate vector falls in $\{(x_1, x_2, x_3, x_4) \mid x_i \in \{-2, 0, 2\}, \forall i\}$ and

each terminal vector falls in $\{(x_1, x_2, x_3, x_4) | x_i \in \{-2, 0\}, \forall i\}$.

It is clear that the following are four full paths that end with characteristic vectors satisfying Equation (6.2).

$$\{(0, 0, 0, 0)\}$$

$$\{(0, 2, 0, 0), (2, -2, 0, 0), (-2, 0, 2, 2), (0, 0, -2, 2), (2, 0, -2, -2), (-2, 2, 0, 0), (0, -2, 0, 0)\}$$

$$\{(0, 0, 2, 0), (2, 0, -2, 0), (-2, 2, 0, 2), (0, -2, 0, 2), (2, -2, 0, -2), (-2, 0, 2, 0), (0, 0, -2, 0)\}$$

$$\{(0, 0, 0, 2), (2, 0, 0, -2), (-2, 2, 2, 0), (0, 2, -2, 0), (2, -2, -2, 0), (-2, 0, 0, 2), (0, 0, 0, -2)\}$$

Now we consider vector $(2, 0, 0, 0)$. The next step in a full path would have to be $(-2, 2, 2, 2)$, and then no matter in what order do we deal with the entries of 2's, we soon arrive at a vector K for which $\langle K, v_i \rangle = 4$ for some i .

If the initial vector takes form $(2, x_2, x_3, x_4)$, where one or more of x_2, x_3, x_4 are 2's, any choice of v_i would lead to a 4 at some other entry.

The only other possible initial vectors take form $(0, x_2, x_3, x_4)$, where at least two of x_2, x_3, x_4 are 2's. Without loss of generality, we assume it is $(0, 2, 2, x_4)$, where $x_4 \in \{0, 2\}$. We fix $\langle K, v_2 \rangle$ first, arriving at $(2, -2, 2, x_4)$. Fixing either $\langle K, v_1 \rangle$ or $\langle K, v_3 \rangle$ in the next step would make the other one a 4.

Now we claim the initial characteristic vectors of the four paths above belong to four different Spin^c structures. Spin^c structures over Y is identified with the set of $2H^2(X, Y)$ -orbits in $\text{Char}(G)$ and the image of $H^2(X, Y)$ in $H^2(X)$ is spanned by the Poincaré duals of the spheres corresponding to the vertices [12], in this case, vectors $PD[v_1] = (-2, 1, 1, 1), PD[v_2] = (1, -2, 0, 0), PD[v_3] = (1, 0, -2, 0), PD[v_4] = (1, 0, 0, -2)$. A linear algebra argument shows that these four vectors belong to different orbits.

The gradings of these vectors can be computed easily as

$$\frac{K^2 + |G|}{4} = \frac{K^T Q_X^{-1} K + 4}{4} = \begin{cases} 1, & \text{if } K = (0, 0, 0, 0) \\ 0, & \text{otherwise.} \end{cases}$$

We conclude $HF^+(-Y(G)) \cong \mathcal{T}_0^+ \oplus \mathcal{T}_0^+ \oplus \mathcal{T}_0^+ \oplus \mathcal{T}_1^+$

Appendix A

General Algebraic Proof

In Section 4.2, we proved Proposition 2 algebraically, but with a mild condition on the knot. In this chapter, a more complicated algebraic argument is given to prove the theorem in general.

We set out to prove Proposition 2 without the extra assumption of $CFK^-(K)$ admitting a basis that is simultaneously horizontally and vertically simplified. Instead, we pick a basis that is only reduced and horizontally simplified as in Proposition 2, which is always possible.

We first argue that it suffices to prove Proposition 2 only in the case where boundary parametrization ϕ corresponds to a integral framing $n \leq 2\tau(K) - 3$. For a general framing, we obtain $\widehat{CFD}(X, \phi')$ for the desired framing by tensoring $\widehat{CFD}(X, \phi)$ with $\widehat{CFDA}(\tau_\mu)$ repeatedly. Theorem 3 is now proved by the fact $\widehat{CFDA}(h) \boxtimes \widehat{CFDA}(\tau_\mu) \cong \widehat{CFDA}(\tau_\mu) \boxtimes \widehat{CFDA}(h)$, since $h \circ \tau_\mu = \tau_\mu \circ h$ as topological maps.

For the purpose of this chapter, we will use Theorem 11, the base-free version of the algorithm from $\widehat{CFD}$ to CFK^- in [5], as follows. We interpret the result in terms of arrows for the convenience of our later proof.

A few definitions are needed first. We denote $\widehat{gCFK}(\mathcal{H}, w, z)$ by C and its summand in Alexander grading r by $C(r)$. Let $\partial^i : C(r) \rightarrow C(r + i)$ be the map counting holomorphic disks ϕ with $n_z(\phi) = -i$ and $n_w(\phi) = 0$ if $i \leq 0$, and holomorphic disks ϕ with $n_z(\phi) = 0$ and $n_w(\phi) = i$ if $i \geq 0$. Note ∂^i drops (or raises) the Alexander grading by r . Also define $\partial_w := \sum_{i \geq 0} \partial^i$ and $\partial_z := \sum_{i \leq 0} \partial^i$.

C has two filtrations, $C(\geq s) := \bigoplus_{r \geq s} C(r)$ and $C(\leq s) := \bigoplus_{r \leq s} C(r)$, preserved by ∂_w and ∂_z respectively. We primarily use $(C(\leq s), \partial_w)$, defined to be the quotient of (C, ∂_w) by subcomplex $(C(\geq s+1), \partial_w)$. Similarly, $(C(\geq s), \partial_z) := (C, \partial_z)/(C(\leq s-1), \partial_z)$.

Theorem 11 (11.35, A.11 [5]). *Let $K \subset S^3$ be a knot with meridian μ and θ -framed longitude λ . Given a large enough positive integer n , $\widehat{CFD}(S^3 \setminus \text{nb}d(K))$, with framing $-n$, denoted by $\widehat{CFD}$ for convenience, can be described by the following.*

$$\iota_0 \widehat{CFD} := \bigoplus_{s \in \mathbb{Z}} V_s^0 \quad \text{and} \quad \iota_1 \widehat{CFD} := \bigoplus_{s \in \mathbb{Z} + \frac{n+1}{2}} V_s^1,$$

where

$$V_s^0 := C(s)$$

$$V_s^1 := \begin{cases} C(\leq s + \frac{n-1}{2}) & \text{if } s \leq -\frac{n}{4} \\ \mathbb{F}_2 & \text{if } |s| < \frac{n}{4} \\ C(\geq s - \frac{n-1}{2}) & \text{if } s \geq \frac{n}{4} \end{cases}$$

The differentials are described below:

- *Arrows with only idempotents:*
 - within $V_s^0 = C(s)$, they are ∂^0 from the knot complex;
 - within $V_s^1 = C(\leq s + \frac{n-1}{2})$, when $s \leq -\frac{n}{4}$, they are the same as ∂_w ;
 - within $V_s^1 = C(\geq s - \frac{n-1}{2})$, when $s \geq \frac{n}{4}$, they are the same as ∂_z ;
 - within $V_s^1 = \mathbb{F}_2$, when $|s| < \frac{n}{4}$, there are none.
- *Arrows with ρ_1 :*
 - from $V_s^0 = C(s)$ to $V_{s+\frac{n-1}{2}}^1 = C(\geq s)$, they are the same as the inclusion of the subcomplex.
- *Arrows with ρ_2 :*

- from $V_s^1 = C(\leq s + \frac{n-1}{2})$ to $V_{s+\frac{n+1}{2}}^0 = C(s + \frac{n+1}{2})$, they are the same as the composition of maps:

$$\pi \circ \partial_w : C(\leq s + \frac{n-1}{2}) \rightarrow C(s + \frac{n+1}{2}),$$

where $\pi : C \rightarrow C(s + \frac{n+1}{2})$ is the projection.

- Arrows with ρ_3 :

- from $V_s^0 = C(s)$ to $V_{s-\frac{n-1}{2}}^1 = C(\leq s)$, they are the same as the inclusion of the subcomplex.

- There are no arrows with ρ_{12} .

- Arrows with ρ_{23} :

- for $s < -\frac{n}{4}$, from $V_s^1 = C(\leq s + \frac{n-1}{2})$ to $V_{s+1}^1 = C(\leq s + \frac{n+1}{2})$, they are the same as the inclusion of the subcomplex;
- for $s \leq -\frac{n}{4} < s+1$, from $V_s^1 = (C, \partial_w)$ to $V_{s+1}^1 = \mathbb{F}_2$, they are the same as a chain map inducing an isomorphism in homology;
- for $|s| < \frac{n-2}{4}$, from $V_s^1 = \mathbb{F}_2$ to $V_{s+1}^1 = \mathbb{F}_2$, it is the unique isomorphism;
- for $s < \frac{n}{4} \leq s+1$, from $V_s^1 = \mathbb{F}_2$ to $V_{s+1}^1 = (C, \partial_z)$, they are the same as a chain map inducing an isomorphism in homology;
- for $s > \frac{n}{4}$, from $V_s^1 = C(\geq s - \frac{n-1}{2})$ to $V_{s+1}^1 = C(\geq s - \frac{n+1}{2})$, they are the same as the projection map.

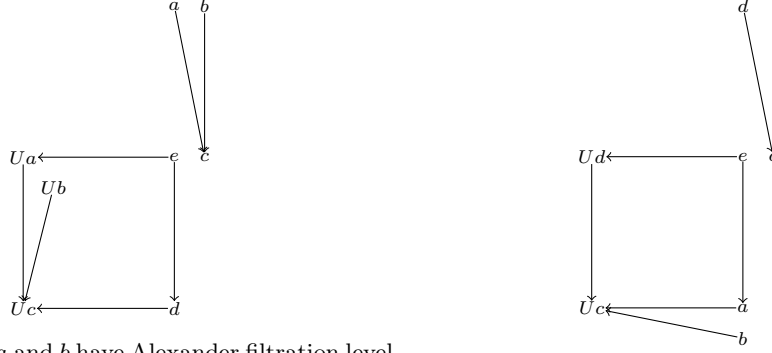
- Arrows with ρ_{123} :

- from $V_s^0 = C(s)$ to $V_{s+\frac{n+1}{2}}^1 =$, they are the same as the composition of maps:

$$i \circ \partial_w^1 : C(s) \rightarrow (C(\geq s+1), \partial_z),$$

where $i : C(s+1) \rightarrow (C(\geq s+1), \partial_z)$ is the inclusion.

First, we give an example of Theorem 11. Suppose we have a knot Floer complex C as in Figure A.1a.



- (a) Generators a and b have Alexander filtration level 1. Alexander filtration level of generators c and e is 0, and that of d is -1.
- (b) The Alexander filtration levels for a, b, c, d and e are -1, -1, 0, 1, and 0, respectively.

Figure A.1

Applying Theorem 11 with $n = 7$ yields $KtD(C)$ as in Figure A.2.

If we apply the procedures in Theorem 11 to C^{flip} , which is shown in Figure A.1b, we get the complex $KtD(C^{flip})$ shown in Figure A.3. We now rearrange the generators by rotating both V^0 and V^1 180 degrees to arrive at Figure A.4.

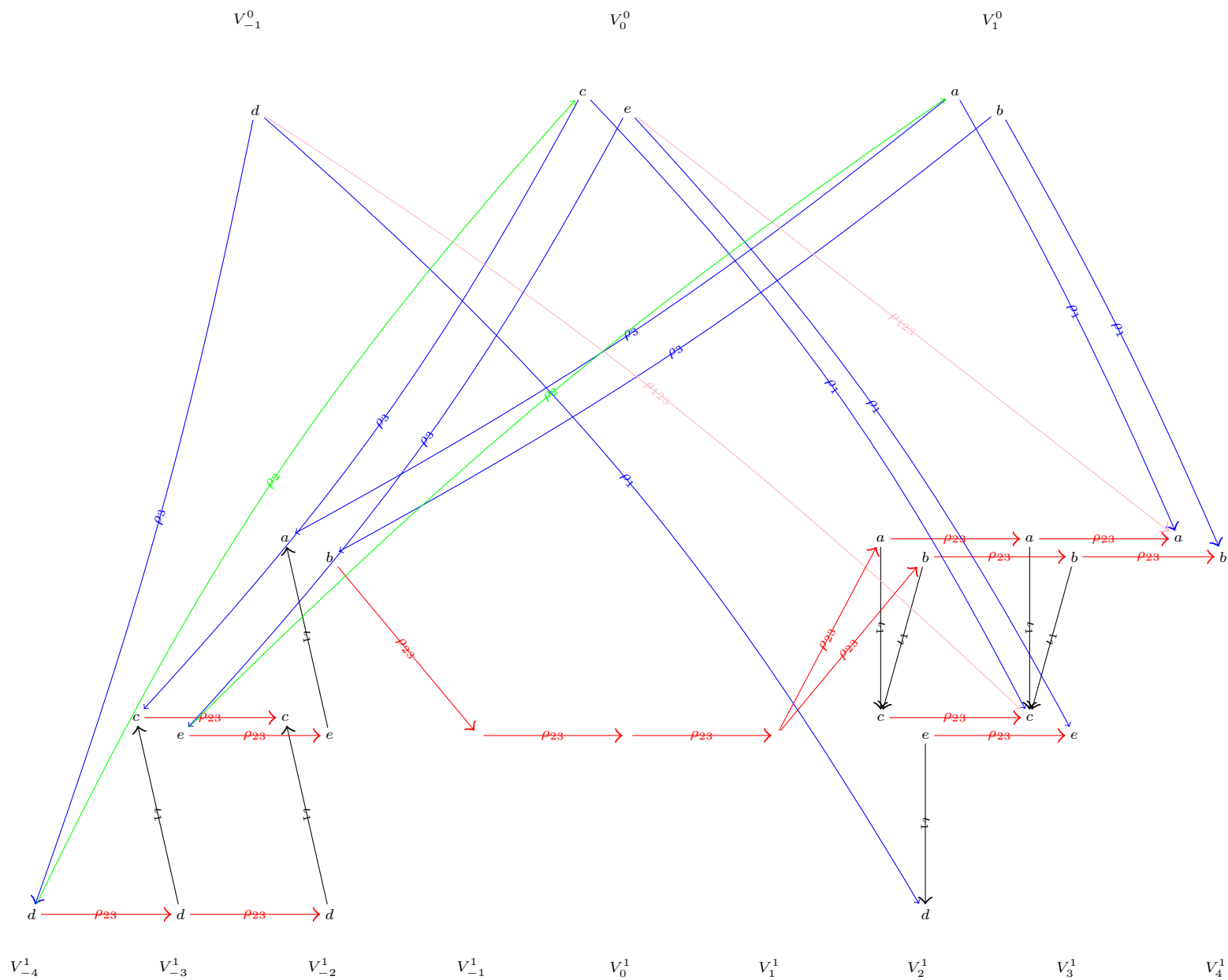


Figure A.2

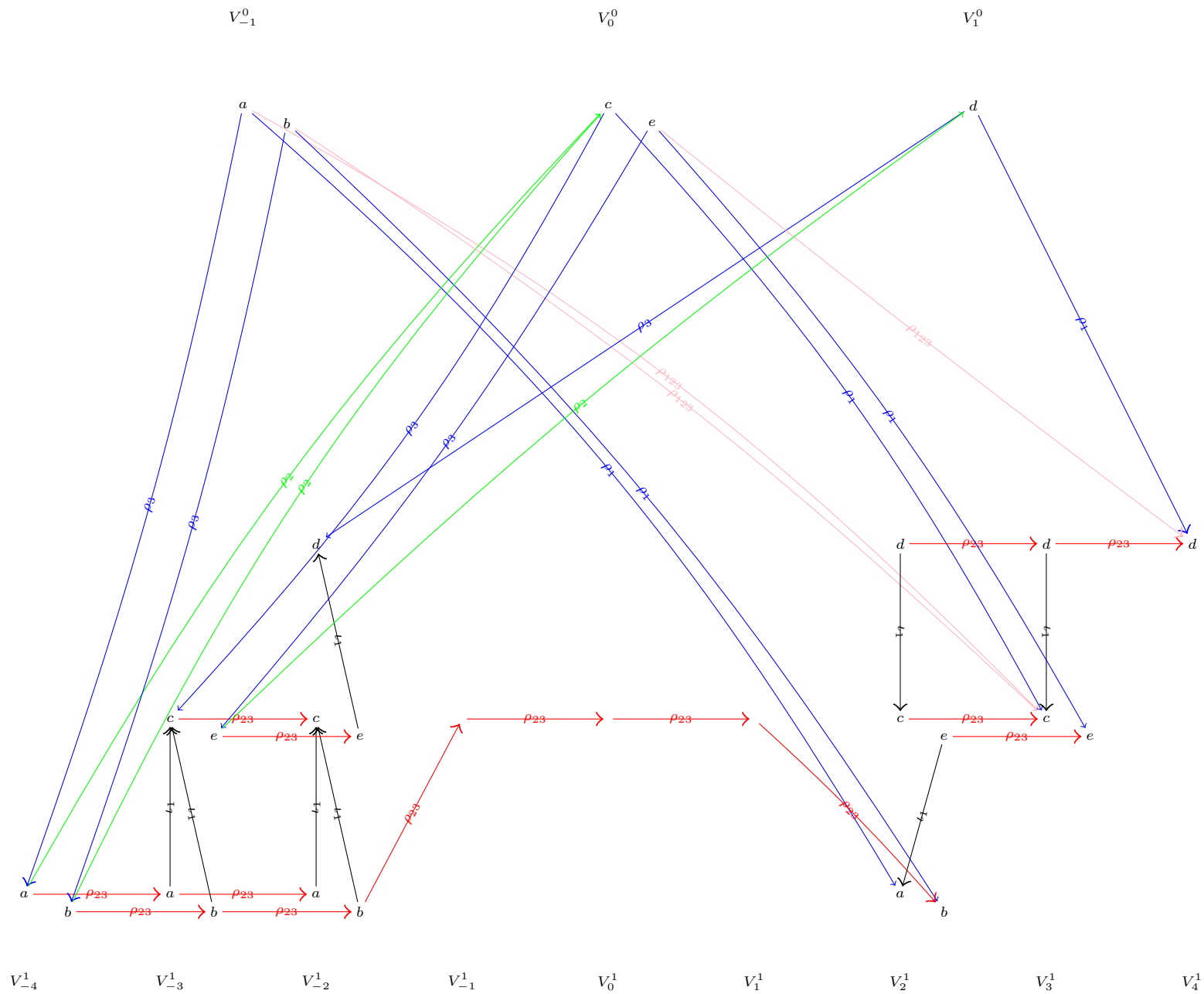


Figure A.3

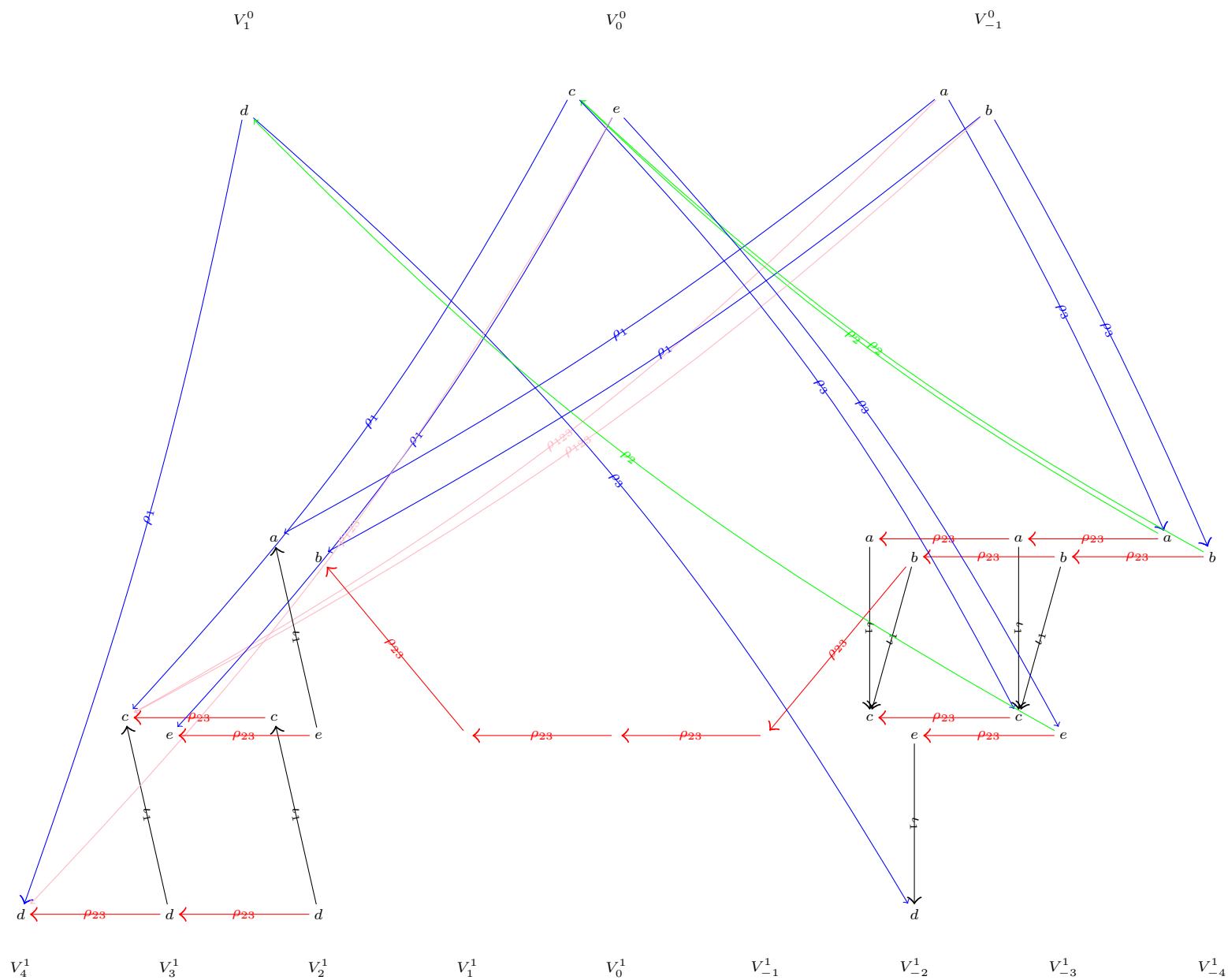


Figure A.4

Comparing the result with $KtD(C)$, we see the generators can be completely identified between them. So are the differentials without non-trivial algebra elements. All arrows with ρ_{23} 's have their directions reversed, except those between the central string of ρ_{23} 's and the two full copies of C immediately next to the string. ρ_1 's and ρ_3 's switch sides. With careful choices of n and the Alexander filtrations on C^{flip} , we can prove this is in general true.

We first set the convention to display KtD 's as we did in the example above: V^0 on top of V^1 , with smaller s on the left. Within V_s^1 , we arrange generators vertically according to Alexander level and we call them "in the same column". On the horizontal direction, put generators corresponding to the same generator in C on the same horizontal line. They are denoted by the same letters in the diagrams. Hence they are connected by horizontal arrows with ρ_{23} 's. We say they are on the same "row".

Recall the identifications between various sub/quotient complexes of C and C^{flip} :

$$(C(\leq s), \partial_w) \cong (C^{flip}(\geq -s), \partial_z)$$

$$(C(\geq s), \partial_z) \cong (C^{flip}(\leq -s), \partial_w)$$

Lemma 2. *Let h be the maximal Alexander filtration level and ℓ be the minimal Alexander filtration level of C . Let $t = \max\{h, -\ell\}$. We choose $n = 4t + 3$ for both $KtD(C)$ and $KtD(C^{flip})$ from now on. Note that with the choice of n , we simply have $V_s^1 = \mathbb{F}_2, |s| \leq t$, $V_{t+1}^1 = (C, \partial_z)$, and $V_{-t-1}^1 = (C, \partial_w)$. We call the latter two (and other V_s^1 such that $V_{s \geq t+1}^1 = (C, \partial_z)$ and $V_{s \leq -t-1}^1 = (C, \partial_w)$) the "full copies".*

Now $KtD(C^{flip})$ can be directly constructed from $KtD(C)$ as follows. We describe this construction graphically to facilitate the proof later.

- *Keep all the generators in $KtD(C)$.*
- *Relabel V_s^0 with V_{-s}^0 and V_s^1 with V_{-s}^1 for all s , but still arrange them graphically the same way, i.e. s decreases now from left to right.*
- *Reverse the direction of all ρ_{23} 's, except those between the central sequence of ρ_{23} 's and the*

two full copies on either side of the sequence.

- Replace the rest of ρ_{23} 's (those between the sequence and the two full copies) with isomorphism-inducing chain maps in the opposite direction.
- At every generator in V^0 , switch the arrows ρ_1 and ρ_3 coming out.
- Remove all existing ρ_2 and ρ_{123} .
- For $s < -t$, $V_s^1 = C(\geq -s - \frac{n-1}{2})$ is missing $C(-s - 1 - \frac{n-1}{2})$ if compared to the column on its left $V_{s+1}^1 = C(\geq -s - 1 - \frac{n-1}{2})$. If there exists an downward arrow $x \rightarrow y$ in V_{s+1}^1 , where $y \in C(-s - 1 - \frac{n-1}{2})$, then add an arrow ρ_2 from $x \in V_s^1$ to $y \in V_{s+\frac{n+1}{2}}^0$.
- For every generator $x \in V^0$, if there exists an (downward) outgoing arrow $x \rightarrow y$ of length 1 in V_{-t}^1 , then add an arrow ρ_{123} from $x \in V^0$ to the unique generator y at the leftmost position of its row.

Proof. This proof is nothing more than applying the procedure in Theorem 11 to C^{flip} and comparing with $KtD(C)$.

We call the D-module constructed this way X and argue X can be identified with $KtD(C^{flip})$.

Per our construction, $\forall s$, $(V_s^0 \text{ of } X) \cong (V_{-s}^0 \text{ of } KtD(C)) \cong C(-s) \cong C^{flip}(s) \cong (V_s^0 \text{ of } KtD(C^{flip}))$, as $\mathbb{Z}$ -filtered chain complexes over $\mathbb{F}_2$.

$\forall s \leq -\frac{n}{4} = -t - 0.75$, $(V_s^1 \text{ of } X) \cong (V_{-s}^1 \text{ of } KtD(C)) \cong (C(\geq -s - \frac{n-1}{2} = -s - 2t - 1), \partial_z) \cong (C^{flip}(\leq s + 2t + 1), \partial_w) \cong (V_s^1 \text{ of } KtD(C^{flip}))$, as $\mathbb{Z}$ -filtered chain complexes over $\mathbb{F}_2$.

$\forall |s| \leq \frac{n}{4} = t + 0.75$, $(V_s^1 \text{ of } X) \cong \mathbb{F}_2 \cong (V_s^1 \text{ of } KtD(C^{flip}))$

$\forall s \geq \frac{n}{4} = t + 0.75$, $(V_s^1 \text{ of } X) \cong (V_{-s}^1 \text{ of } KtD(C)) \cong (C(\leq -s + \frac{n-1}{2} = -s + 2t + 1), \partial_w) \cong (C^{flip}(\geq s - 2t - 1), \partial_z) \cong (V_s^1 \text{ of } KtD(C^{flip}))$, as $\mathbb{Z}$ -filtered chain complexes over $\mathbb{F}_2$.

Now we look at arrows with nontrivial algebra elements on them. First, arrows with ρ_1 and ρ_3 . Per our construction, $D_1 : V_s^0 \rightarrow V_{s+\frac{n-1}{2}}^1 \text{ of } KtD(C)$ become $D_3 : V_{-s}^0 \rightarrow V_{-s-\frac{n-1}{2}}^1 \text{ of } X$ and $D_3 : V_s^0 \rightarrow V_{s-\frac{n-1}{2}}^1 \text{ of } KtD(C)$ become $D_1 : V_{-s}^0 \rightarrow V_{-s+\frac{n-1}{2}}^1 \text{ of } X$. So

$$(D_1 : V_{-s}^0 \rightarrow V_{-s+\frac{n-1}{2}}^1 \text{ of } X) \cong (D_3 : V_s^0 = C(s) \rightarrow V_{s-\frac{n-1}{2}}^1 = C(\leq s) \text{ of } KtD(C))$$

$$\begin{aligned}
&\cong (i : C(s) \rightarrow C(\leq s)) \cong (i : C^{flip}(-s) \rightarrow C^{flip}(\geq -s)) \\
&\cong (D_1 : V_{-s}^0 = C^{flip}(-s) \rightarrow V_{-s+\frac{n-1}{2}}^1 = C^{flip}(\geq -s) \text{ of } KtD(C^{flip}))
\end{aligned}$$

Similarly,

$$\begin{aligned}
(D_3 : V_{-s}^0 \rightarrow V_{-s-\frac{n-1}{2}}^1 \text{ of } X) &\cong (D_1 : V_s^0 \rightarrow V_{s+\frac{n-1}{2}}^1 \text{ of } KtD(C)) \\
(i : C(s) \rightarrow C(\geq s)) &\cong (i : C^{flip}(-s) \rightarrow C^{flip}(\leq -s)) \\
(D_3 : V_{-s}^0 = C^{flip}(-s) \rightarrow V_{-s-\frac{n-1}{2}}^1 = C^{flip}(\leq -s))
\end{aligned}$$

The newly added ρ_2 's are exactly done to match those in $KtD(C^{flip})$. They are:

$$\begin{aligned}
(D_2 : V_{-s}^1 = C(\geq -s - \frac{n-1}{2}) \rightarrow^{\partial_z} C \rightarrow^\pi V_{-s-\frac{n+1}{2}}^0 = C(-s - \frac{n+1}{2}) \text{ in } X) \\
\cong (C^{flip}(\leq s + \frac{n-1}{2}) \rightarrow^{\partial_w} C^{flip} \rightarrow^\pi C^{flip}(s + \frac{n+1}{2})),
\end{aligned}$$

which matches D_2 in $KtD(C^{flip})$.

Similar logic goes for the newly added ρ_{123} 's. They are:

$$\begin{aligned}
(D_{123} : V_{-s}^0 = C(s) \rightarrow^{\partial_z} C(s-1) \rightarrow V_{-s+\frac{n+1}{2}}^1 = C(\leq s-1) \text{ in } X) \\
\cong (C^{flip}(-s) \rightarrow^{\partial_w} C(-s+1) \rightarrow C(\geq -s+1)) \cong (D_{123} \text{ in } KtD(C^{flip}))
\end{aligned}$$

D_{23} : The new ρ_{23} 's are: for $s > \frac{n}{4}$,

$$\begin{aligned}
D_{23} : V_s^1 = C(\leq -s + \frac{n-1}{2}) \rightarrow V_{s+1}^1 = C(\leq -s-1 + \frac{n-1}{2}) \\
\cong C^{flip}(\geq s - \frac{n-1}{2}) \rightarrow C^{flip}(\geq s - \frac{n+1}{2}),
\end{aligned}$$

which matches those in $KtD(C^{flip})$. The rest of the ρ_2 's similarly match.

□

Now that we have proved the differences between $KtD(C)$ and $KtD(C^{flip})$ are fairly “local”, we

proceed to show that tensoring with H and canceling carefully transform $KtD(C)$ into $KtD(C^{flip})$.

We first state that cancellations will be done in the order discussed below.

A.1 Right-hand side middle part of rows in V^1

We start from the “right” side of V^1 . For any row of generator x connected by a string of ρ_{23} ’s in $\oplus_{s>t} V_s^1$, we first look at the middle part, i.e., excluding the leftmost one in the full copy and rightmost one where there might be ρ_1, ρ_{123} coming in, see Figure A.5. Black arrows into and out of x ’s represent the potential and possibly multiple arrows of ∂_z within this column.

It looks like Figure A.6 in the tensor product. Each string of y_1, u, u_2, y_3, v corresponds to a single generator before. Black arrows coming in and out of them represent the tensor result of potential arrows of ∂_z .

We first cancel the red arrows from y_1 to v , arriving at Figure A.7. We keep the arrows resulted from cancellation in shape of zigzags to make it easier to see how they were generated.

Then canceling red arrows from u to y_3 yields Figure A.8. Note the new ρ_{23} ’s in the opposite direction between the y_2 ’s. They and the ρ_{23} ’s between them match the generators in $KtD(C^{flip})$ in the way described by Lemma 2. In general, throughout the proof, y_2 ’s in the tensor product are the generators that will survive and match to generators in $KtD(C^{flip})$. We repeat the process above for all such rows in the “right” side. In this process, some undesirable arrows appear as a side effect and we argue that such arrows will not survive at the end. For example, the (potential) arrow (marked by # in Figure A.8) coming from a v_i above in the middle column and going to a y_1 below in the left column. When we follow this process in the row containing the v_i , we will cancel a red arrow going into v_i , meaning any arrow coming out of v_i will be discarded. Similar logic goes to the rest of the undesirable arrows; they either come out v ’s and y_3 ’s, which are the target of arrows to be cancelled, or enter y_1 ’s and u ’s, which are the origin of arrows to be cancelled. One exception will be when the u ’s below are at rightmost positions of their row, where they won’t have red arrowing coming out to be cancelled. We remark that, in such cases, only the (possibly multiple) arrow marked by $*$ could survive, while the others will be gone because of their origins

being targets of arrows to be cancelled. Note in each row, such a surviving arrow can only appear once for each downward arrow in ∂_z . See Figure A.9 for the final product. We emphasize that we perform cancellation described here across all applicable places first before moving on to other parts of the module.

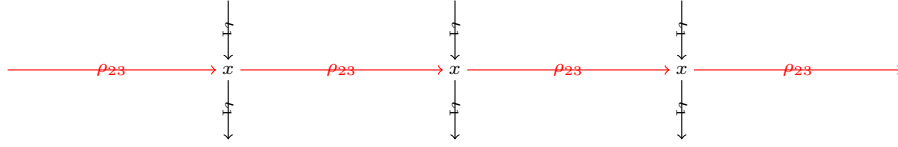


Figure A.5: Middle part of a row.

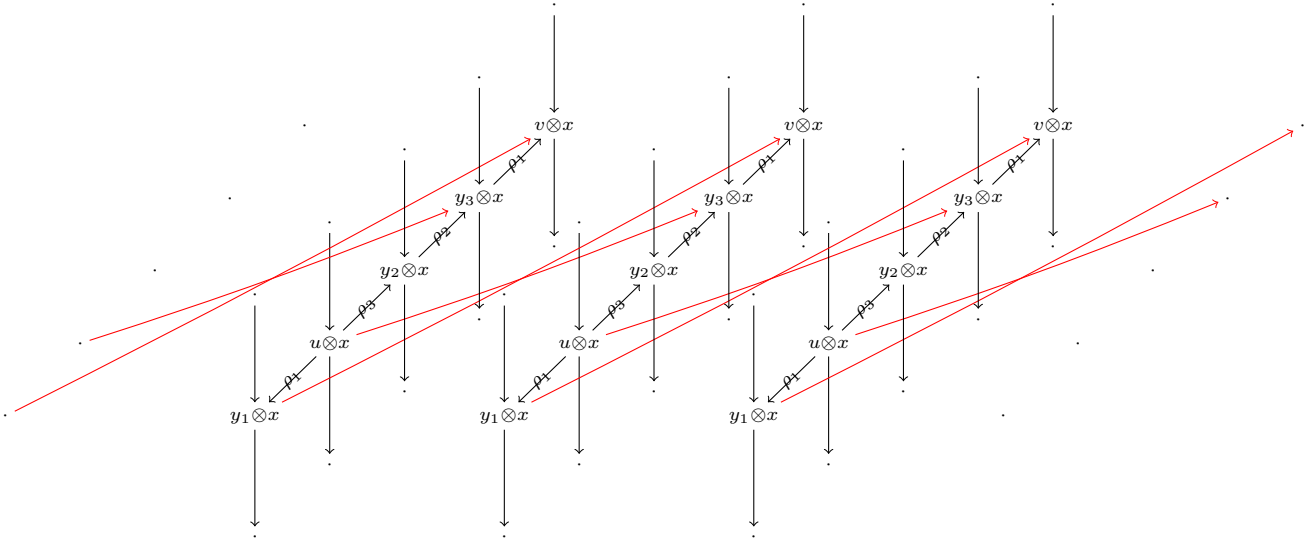


Figure A.6: Middle part of a row.

A.2 Right-hand side rightmost end of rows in V^1

We look at the rightmost positions of rows, see Figure A.10a. The blue, green, and pink arrows represent ρ_1 , ρ_3 , and potential and possibly multiple ρ_2 and ρ_{123} . It corresponds to Figure A.10b in the tensor product. Again the black arrows are potential arrows from tensoring with ∂_z . Note

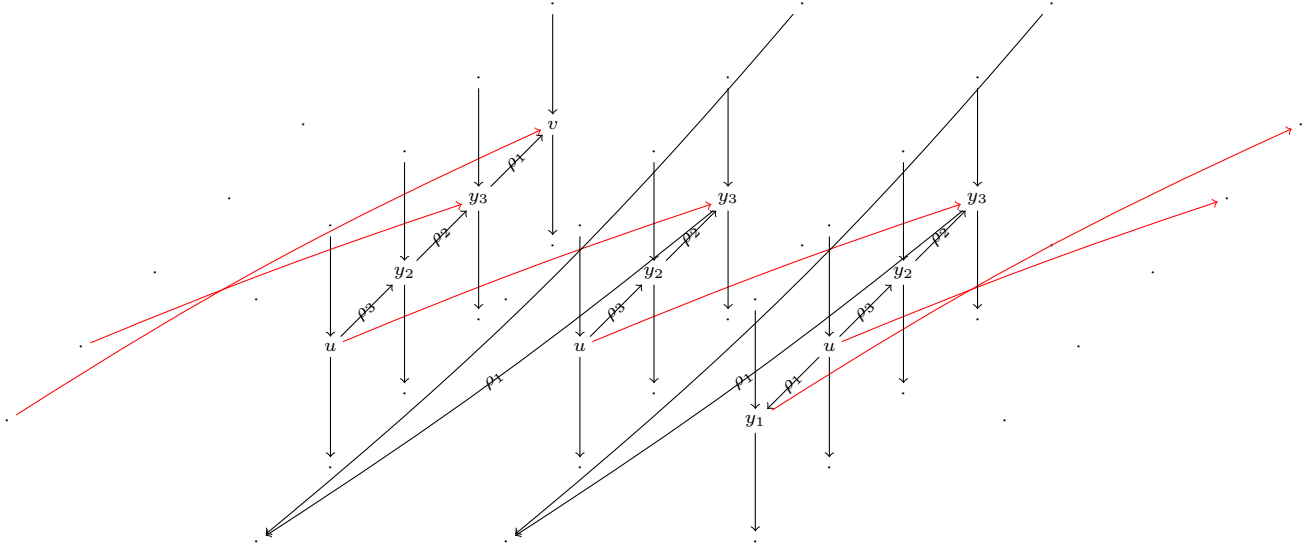


Figure A.7: Middle part of a row.

that there is only one connecting to the right string of generators. There are not any going out because this is the right most position and only y_2 has black arrow(s) coming in because we have performed the cancellation described in Appendix A.1 on rows above. Regarding the left set of generators, there are only three of them and one incoming black arrow for the same reason. The black arrows marked by $*$ are the potential side effect of cancellation performed to rows above, see Appendix A.1.

Next we cancel the two red arrows in the middle successively, see Figure A.11a and Figure A.11b. Now ∂_z into y_2 's and the ρ_{23} among them match those in $KtD(C^{flip})$. As for the messy additional arrows, we keep them in mind and bring them into shape later. We make the observation that the potential side effect ρ_2 arrows going to u 's and the pink arrow going to a y_1 in row below (all marked by $**$ in Figure A.11b) only exist if they enter u 's in the rightmost end of their rows. In the case of the $**\dagger$ arrows, they exist only if x has a length one ∂_z arrow. Even though they enter another right-most position in another row, we clearly see that they do not interfere with the procedure in this section applied there. So we are safe to say that we can perform these cancelations at all right-most ends of rows. An exception is when a row has only one generator, which can only

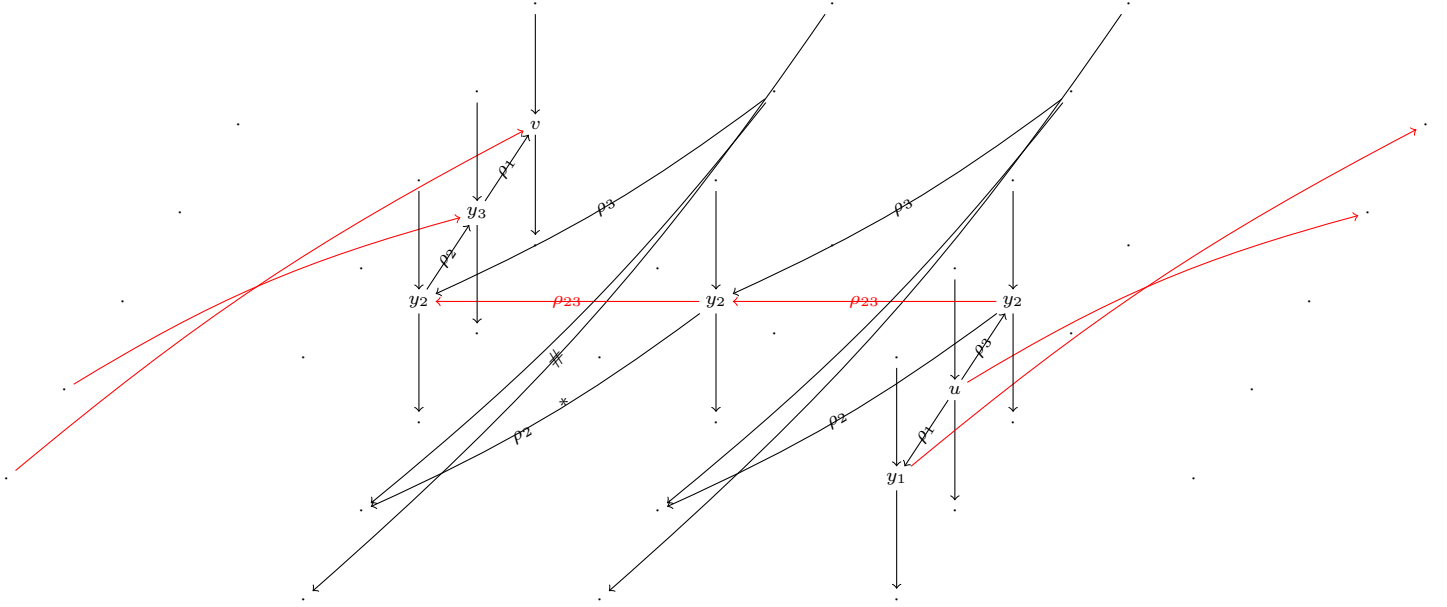


Figure A.8: Middle part of a row.

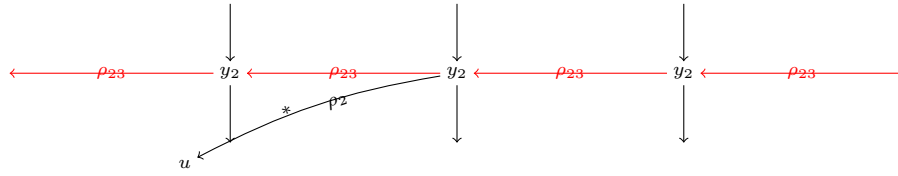
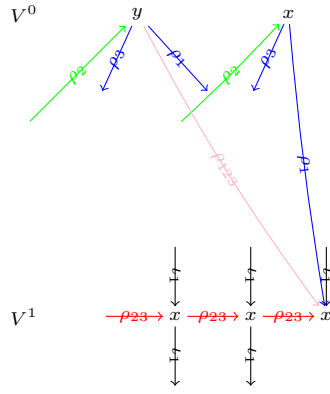
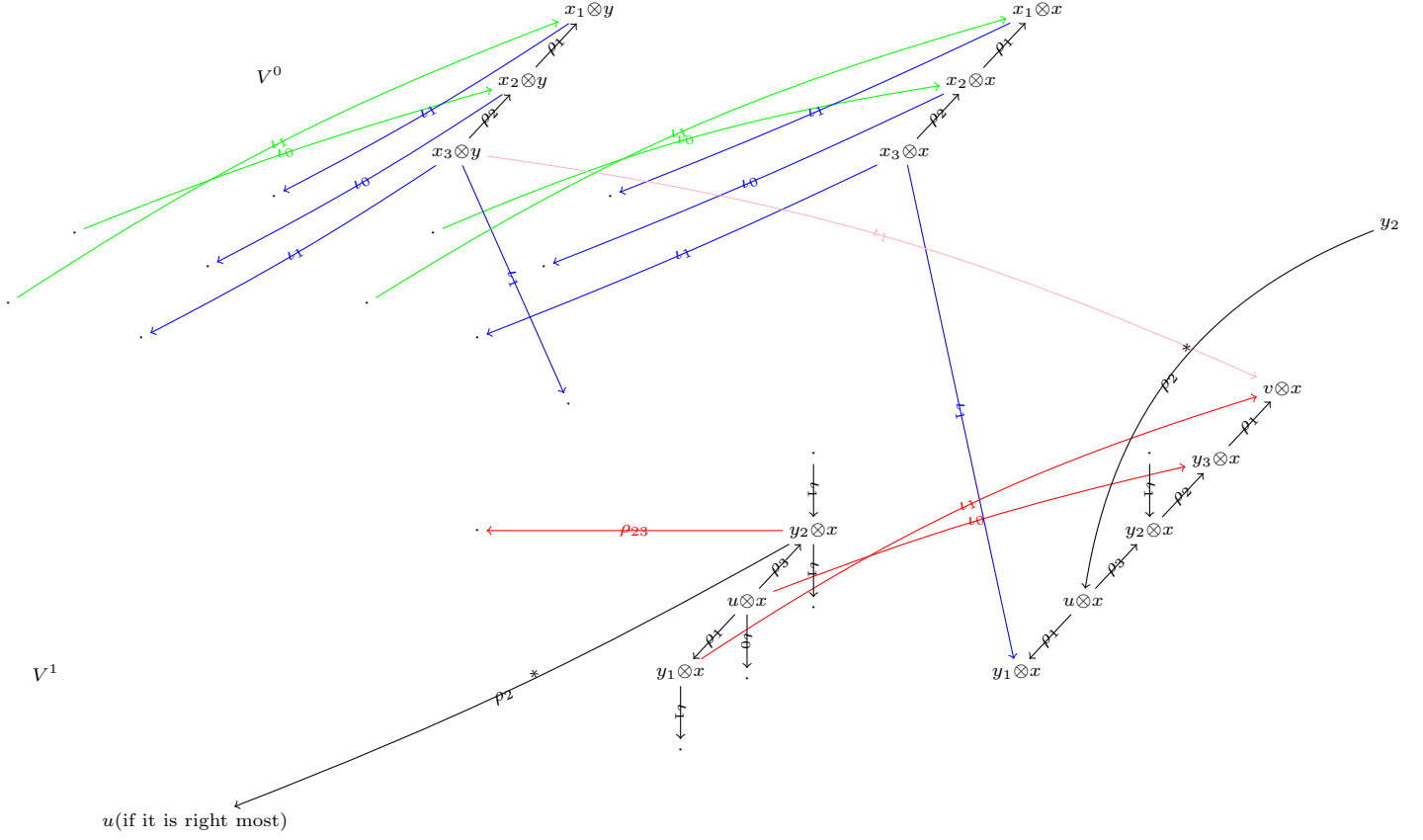


Figure A.9: Middle part of a row.

happen at the bottom row. We discuss that case in Appendix A.3.



(a)



(b)

Figure A.10: Rightmost position of a row

A.3 Right full copy

Figure A.12 shows the part of the full copy in the tensor product. As we have done twice, we cancel the red arrows resulting in Figure A.13. Note the arrows marked with $\#$ will be gone after we apply these cancellations in the rows above. Their existence does not interfere with the cancellation above, as they come out of targets of cancelable arrows. Such arrows coming from rows below would not have interfered here either. Disregarding those arrows, we arrive at Figure A.14. Again the arrow marked with $**$ only exists if it goes into a u in rightmost position, which in this case can only be the bottom row. Figure A.15 depicts the bottom row which has only one generator, with the potential $**$ arrow shown. The blue arrow comes from tensoring with ρ_1 . Note there will not be any pink arrows from tensoring with potential ρ_{123} , because this generator has the lowest Alexander grading.

The only arrows connecting to the full copy are those from the map $i : \mathbb{F}_2 \rightarrow C$, which induces isomorphism in homology. The map i must take the generator e of $\mathbb{F}_2$ to some $\sum_j x_j \in C$. The full picture of the full copy looks like Figure A.16. The set of three generators on the left corresponds to e . Because we can simply treat the middle string of ρ_{23} 's as we did in the simple case, a ρ_{23} coming out of y_2 is obviously the result of those cancellations. Only x_1 and x_2 are shown for simplicity.

A map inducing isomorphism in homology in the opposite direction of i must take $x_j \rightarrow e$ for some j and all other generators of C to 0. So in order to match with $KtD(C^{flip})$, we need one ρ_{23} from one of the x_i 's to e . We first cancel arrow marked by $\#1$ and arrive at Figure A.17. Now cancel the arrow marked by $\#2$ and arrive at Figure A.18. We then look to rid all the v 's in this picture. The set of v 's and the differentials among them make a chain complex C_v over $\mathbb{Z}_2$ slightly different from C . We observe that if we perform the change of basis $x_1 \rightarrow \sum_j x_j$ on C and then the resulting complex minus generator $\sum_j x_j$ is isomorphic to C_v . Following the fact that (C, ∂_z) is homotopy equivalent to $\mathbb{Z}_2$ and $\sum_i x_i$ is the generator of the homology, C_v is homotopy equivalent to $\emptyset$. This means that there is a homotopy equivalence on the current stage of the module that exactly gets rid of all the v 's. A similarly constructed homotopy equivalence gets rid of all the y_3 's. The final result matches the full copy of $KtD(C^{flip})$, see Figure A.19.

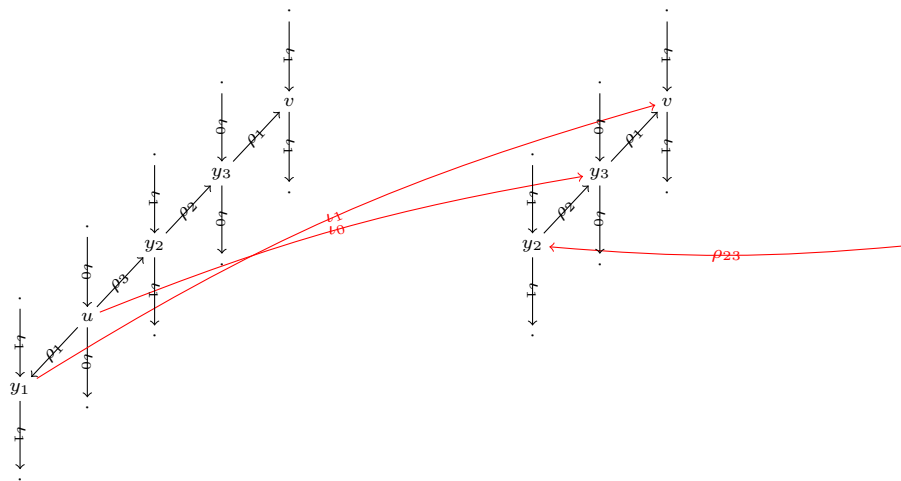


Figure A.12: The right full copy

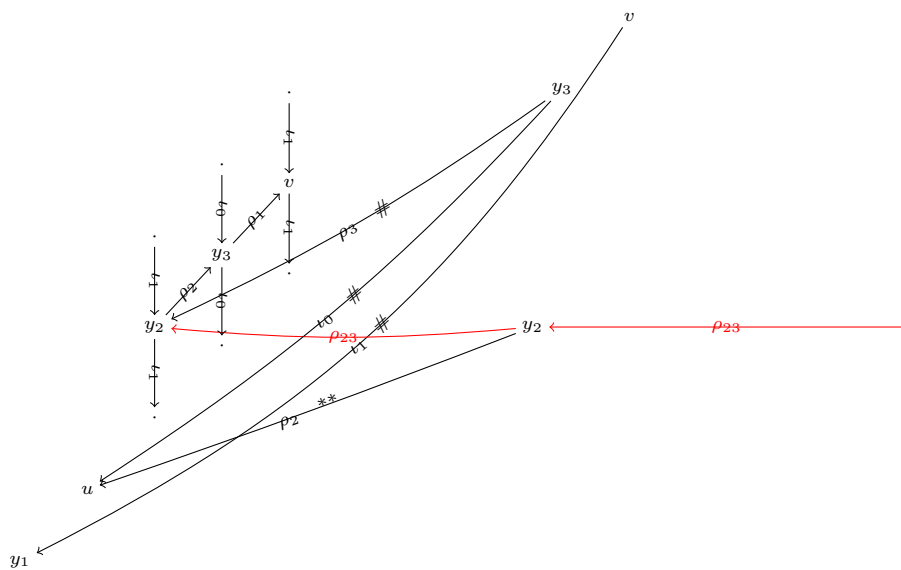


Figure A.13: The right full copy

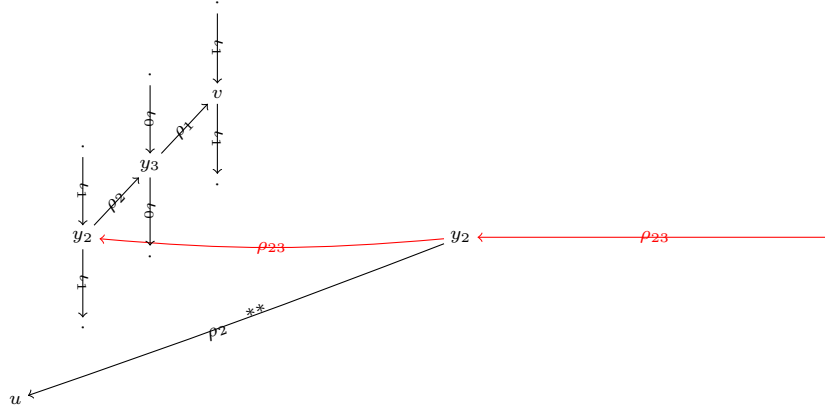


Figure A.14: The right full copy

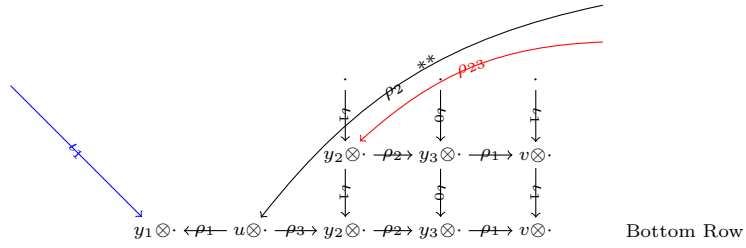


Figure A.15: The right full copy

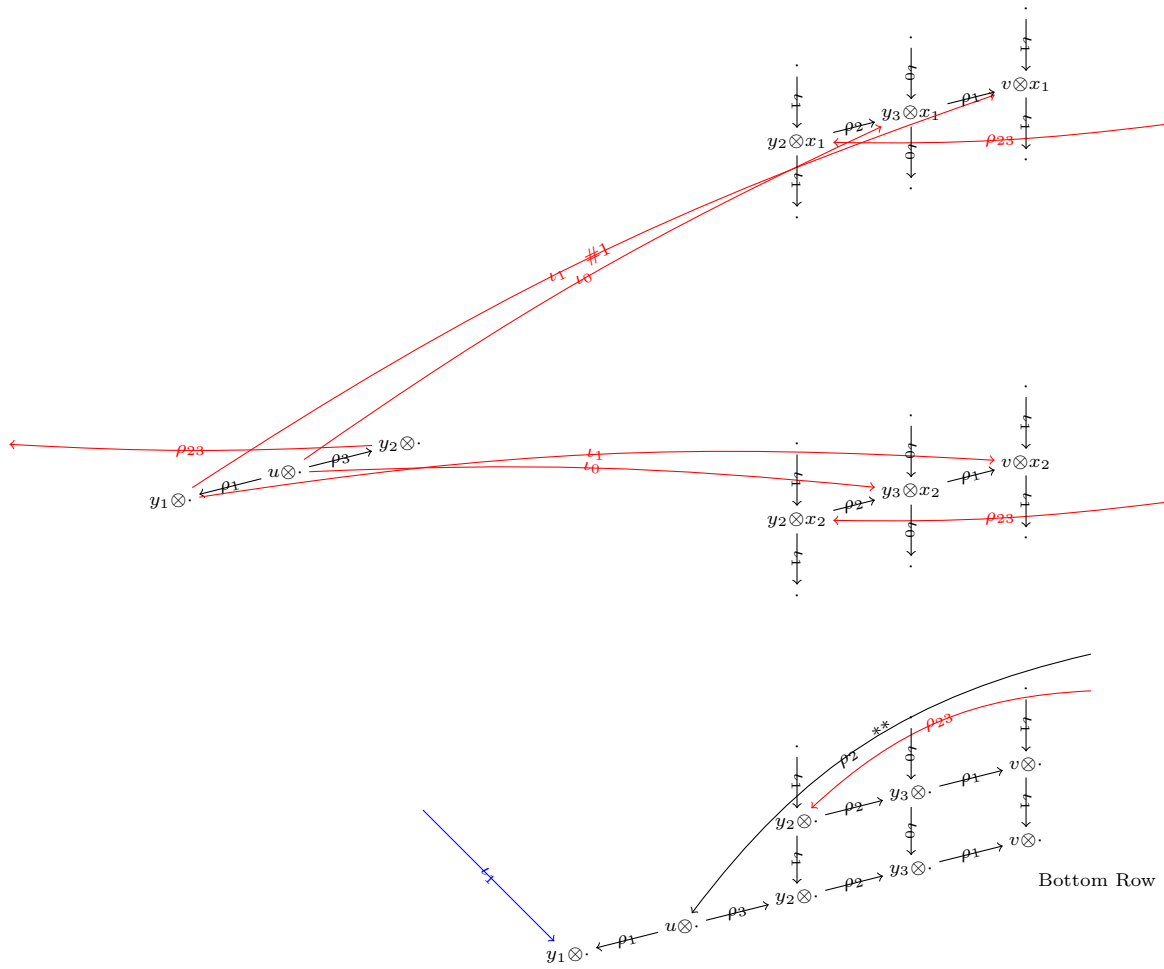


Figure A.16: The right full copy

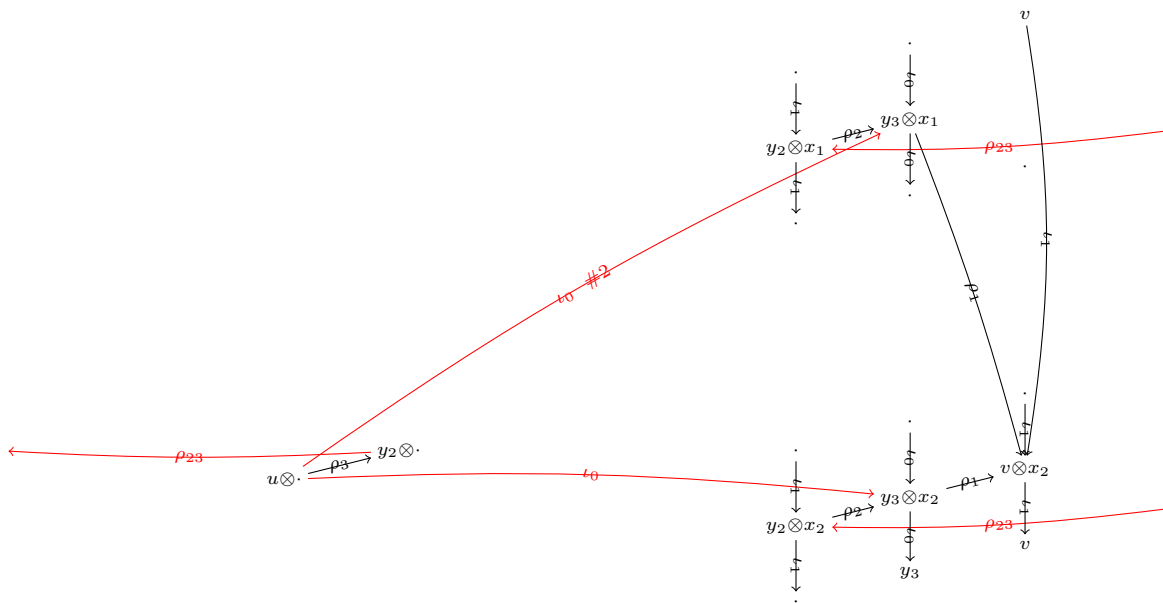
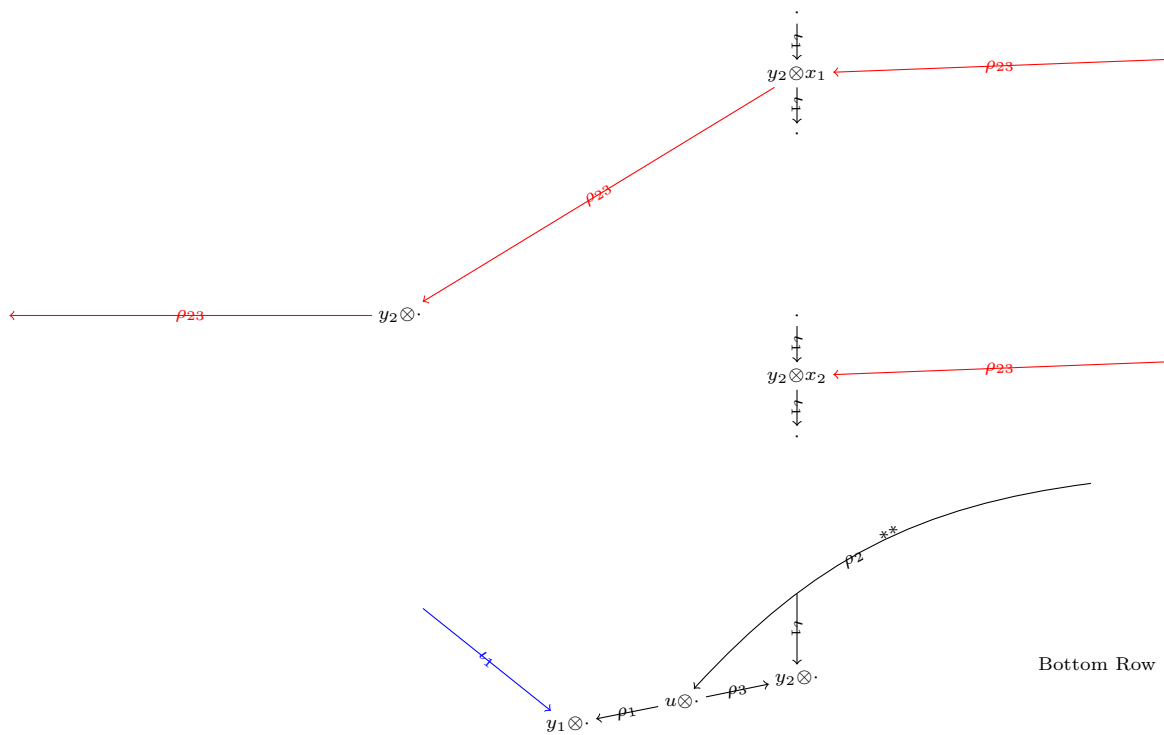


Figure A.17: The right full copy





Bottom Row

Figure A.19: The right full copy

A.4 Left-hand side middle part of rows in V^1

We switch to the left-hand side. Recall that we assumed that C is horizontally-simplified. This means generators in (C, ∂_w) come in pairs except one generator, which we call ξ_h to be consistent with the simplified case. Two generators in each pair are connected by one arrow. A typical pair $a \rightarrow b$ in $KtD(C)$ is displayed in Figure A.20. The green arrow with ρ_2 goes to $b \in V^0$. Per Theorem 11, it always appear one column to the left of the left-most $a \rightarrow b$ arrow in $(C(\leq s), \partial_w)$ for some s . The dots are only place holders to demonstrate that the length of the arrow is 3.

In the “middle” part of this long string, away from the left-most $a \rightarrow b$ arrow and the right-most full copy, the cancellation in the tensor product is just a simplified case of the process described in Appendix A.1, see Figure A.21 and Figure A.22.

The unpaired ξ_h constitutes a single string of ρ_{23} ’s in $KtD(C)$, whose middle part can be dealt with in the same manner. The result is a string of ρ_{23} ’s in the opposite direction, matching $KtD(C^{flip})$.

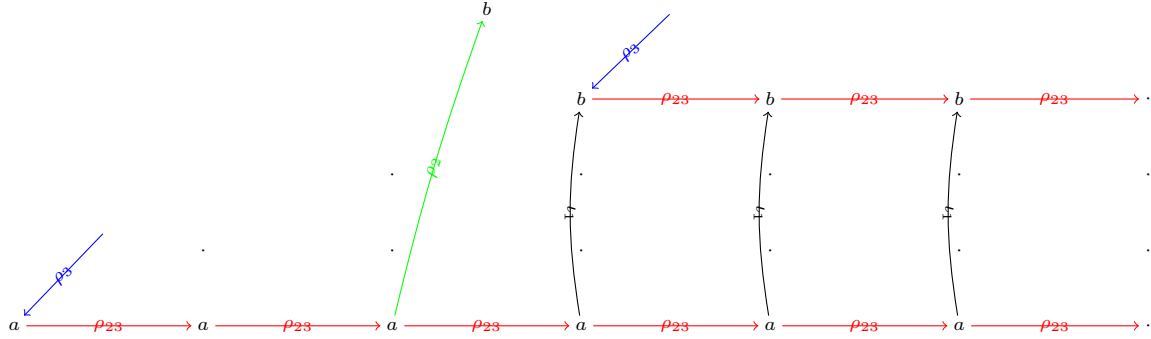


Figure A.20: Left-hand side middle part of rows in V^1

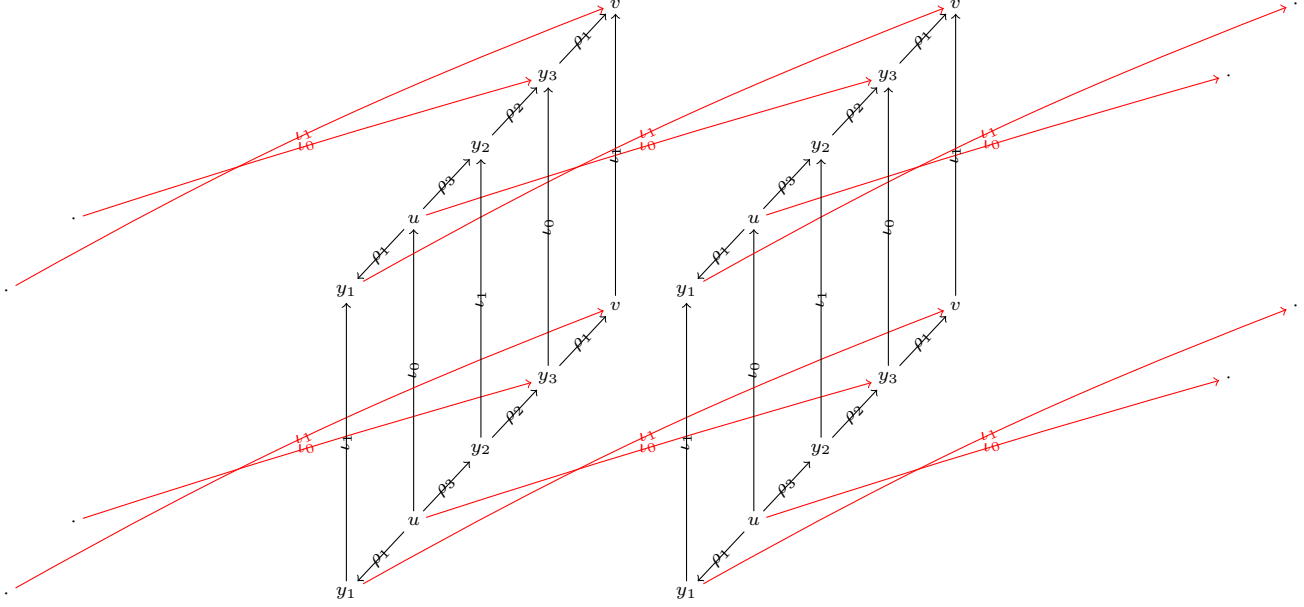


Figure A.21: Left-hand side middle part of rows in V^1

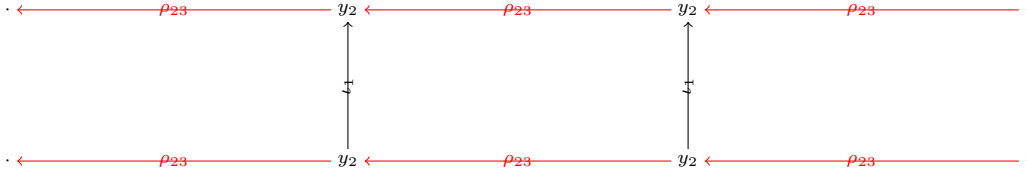


Figure A.22: Left-hand side middle part of rows in V^1

A.5 V^0

In this section we look at tensoring at V^0 where outgoing ρ_1 and ρ_3 connect to the left-most positions of the left-hand side and right-most positions of the right-hand side, see Figure A.23. We first assume the arrow $a \rightarrow b$ has length more than 1. This means that on the left of Figure A.23, the position where the green arrow comes out is not the left most position, where the blue ρ_3 comes in. It also means that there won't be any other length one ∂_w arrow connecting to a or b , so there is not any pink ρ_{123} 's to the right side of this diagram.

Taking the tensor and keeping in mind all the previous cancellation we have done, especially

in Appendix A.2 and Figure A.11b, we arrive at Figure A.24. We first cancel the two pair of red arrows in the same fashion we have always done. We also cancel the two blue arrows from $x_1 \rightarrow v$, then the two blue arrows from $x_2 \rightarrow y_3$, resulting in Figure A.25. The cancelation of the eight arrows so far are simple and generated no new arrows. Note the green arrows are now gone.

Next, we cancel the two pair of red arrows on the left and rearrange the y_1 's and u 's on the right side to the top, see Figure A.26a. Next, we cancel the two blue arrows $x_3 \rightarrow y_1$ on the top, see Figure A.26b.

Now the picture looks very much like $KtD(C^{flip})$ described in Lemma 2; For each generator, its y_2 takes its place. ρ_{23} 's are reversed. ∂_z and ∂_w are intact. For each generator x , its u at the right-most position in the right side takes the place of x in V^0 . ρ_1 and ρ_3 are switched. Let us take a closer look at the green arrows with ρ_2 and marked by $*$ in this section. Say the a has an incoming $c \rightarrow a$ in ∂_z . As seen in Appendix A.1, the green arrow $*$ with ρ_2 comes from c 's y_2 in one column to the right. In other words, if a downward ∂_z arrows arrives at a generator a at a right-most position from a generator c above, then there is a ρ_2 arrow coming out of the y_2 to c 's right and going into the bottom a 's u , which takes the place of the old $a \in V^0$ now. This matches exactly to the green arrows of $KtD(C^{flip})$ described in Lemma 2. The green arrows marked by $**$ exist for the exact same reasons, except from the origin's point of view. They exist only when they are going into a rightmost position, which must be the rightmost position of one row below, as they themselves are at rightmost positions. We again observe that these green arrows never interacted with the cancelation done in this section, so we are safe to say that we the cancellations generalize to all applicable part.

The case depicted in the diagrams is that $a \rightarrow b$ has length 2, the case that it has length more than 2 is similar.

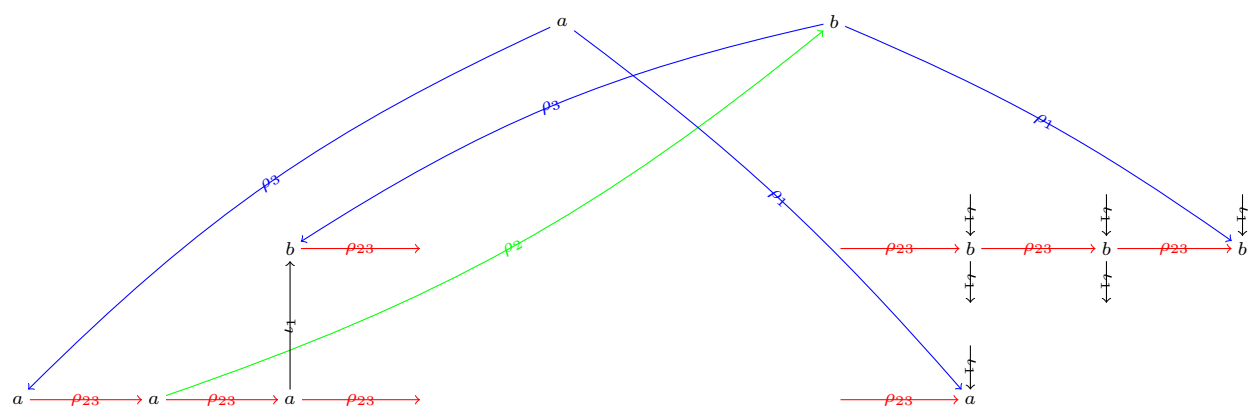
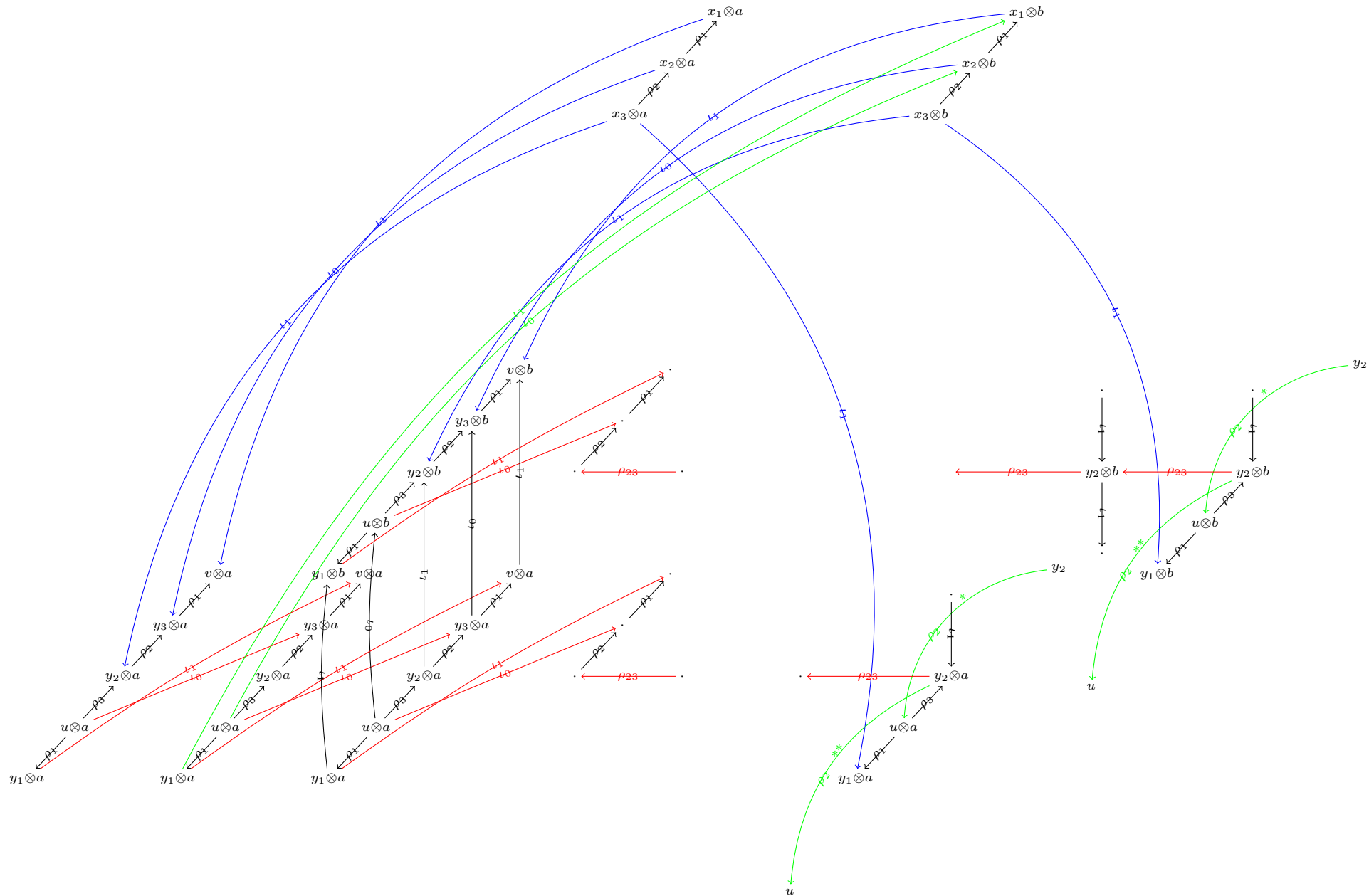
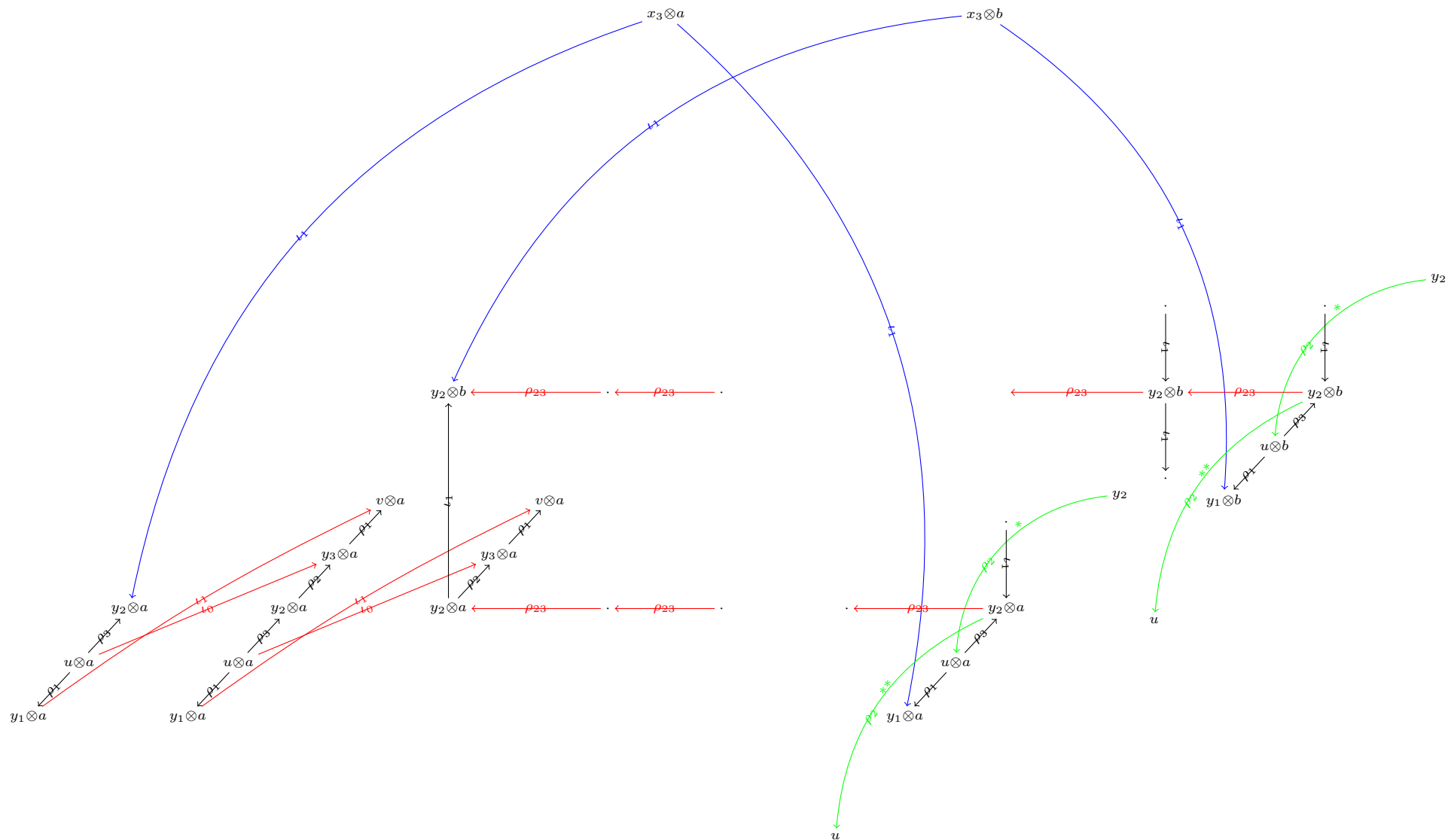
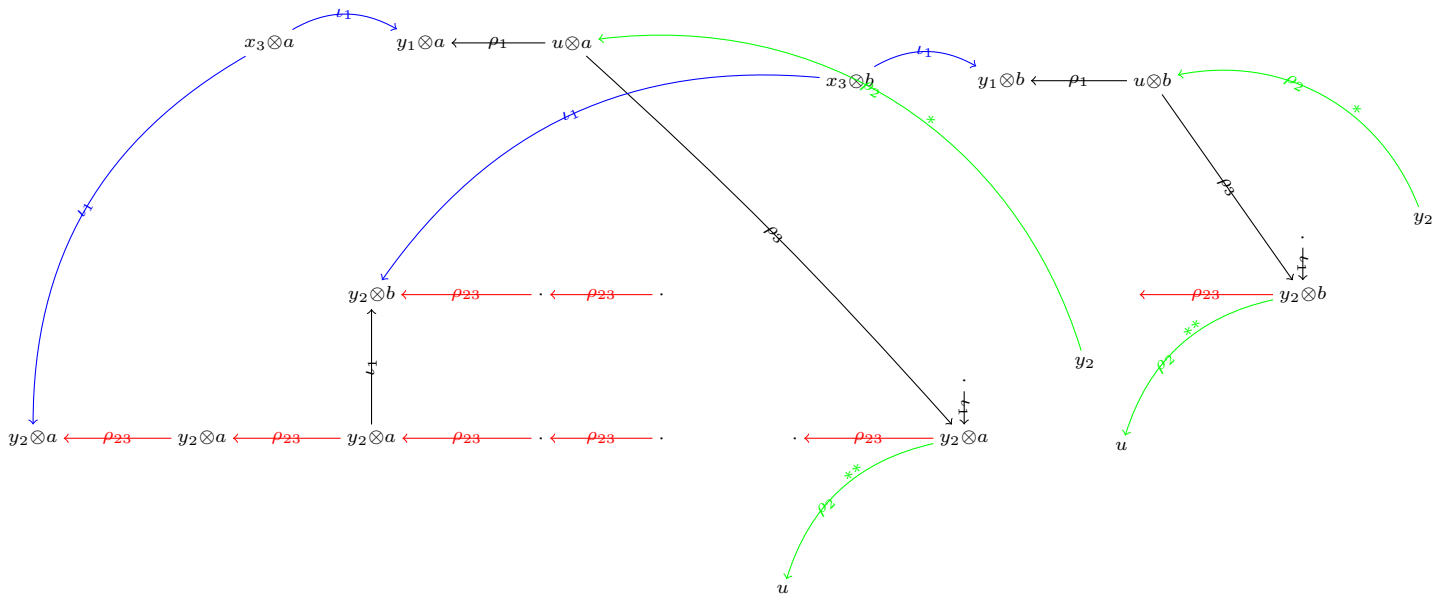


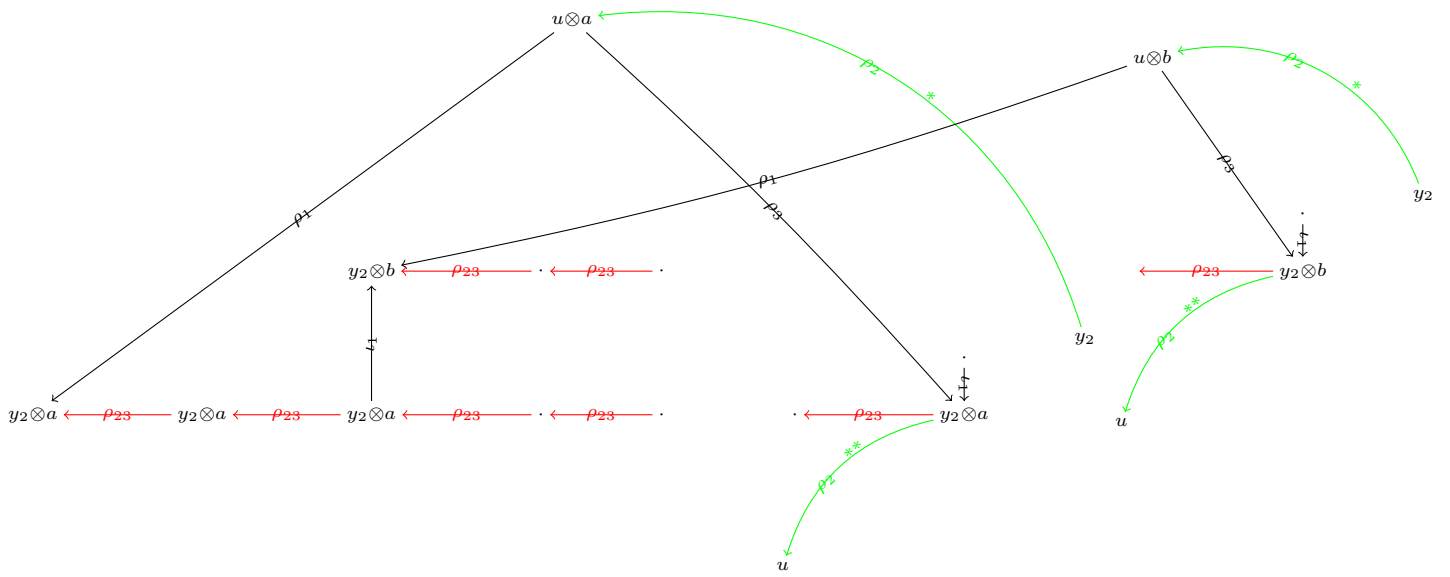
Figure A.23: V^0

Figure A.24: V^0

Figure A.25: V^0



(a)



(b)

Figure A.26: V^0

Now we tackle the most complicated case: a $a \rightarrow b$ arrow in ∂_w with length 1, see Figure A.27. In this case, a at the left-most position has both the incoming ρ_3 and the outgoing ρ_2 . ρ_{123} also appears as the length of the arrow is 1. We first assume b has no outgoing length 1 ∂_z arrows. So no downward going black arrows from the second b from the right.

Now we take the tensor, with Figure A.11b in Appendix A.2 in mind. All potential green arrows marked with $*$ appear again, except b 's y_2 doesn't have one going out as b has no length one ∂_z arrow. The pink ρ_{123} doesn't lead to any new arrows, as argued in Appendix A.2, see Figure A.28. Then, cancel the middle two pairs of reds, yielding only the two desired ρ_{23} 's. Then, cancel the two $x_1 \rightarrow v$'s "for free". Then, cancel the two $x_2 \rightarrow y_3$'s "for free", see Figure A.29. Next, cancel the pair of reds on the left side, again put u 's on top. Then cancel the two $x_3 \rightarrow y_1$'s. Now see Figure A.30. It matches $KtD(C^{flip})$ in a similar fashion as Figure A.26.

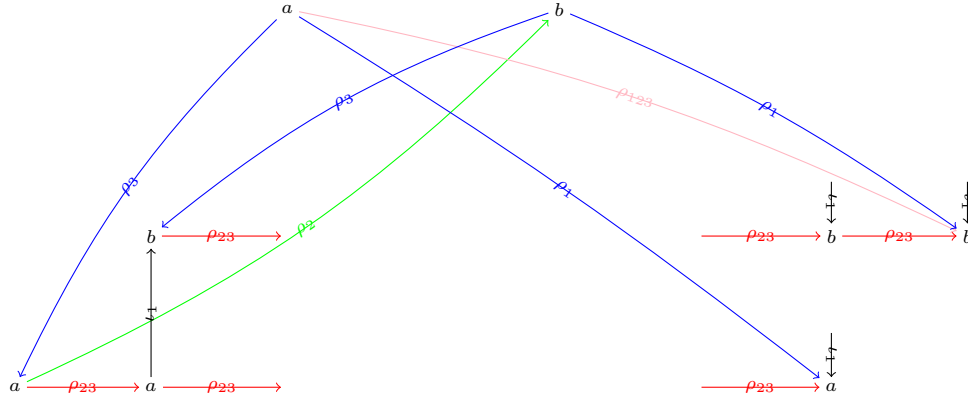


Figure A.27: V^0

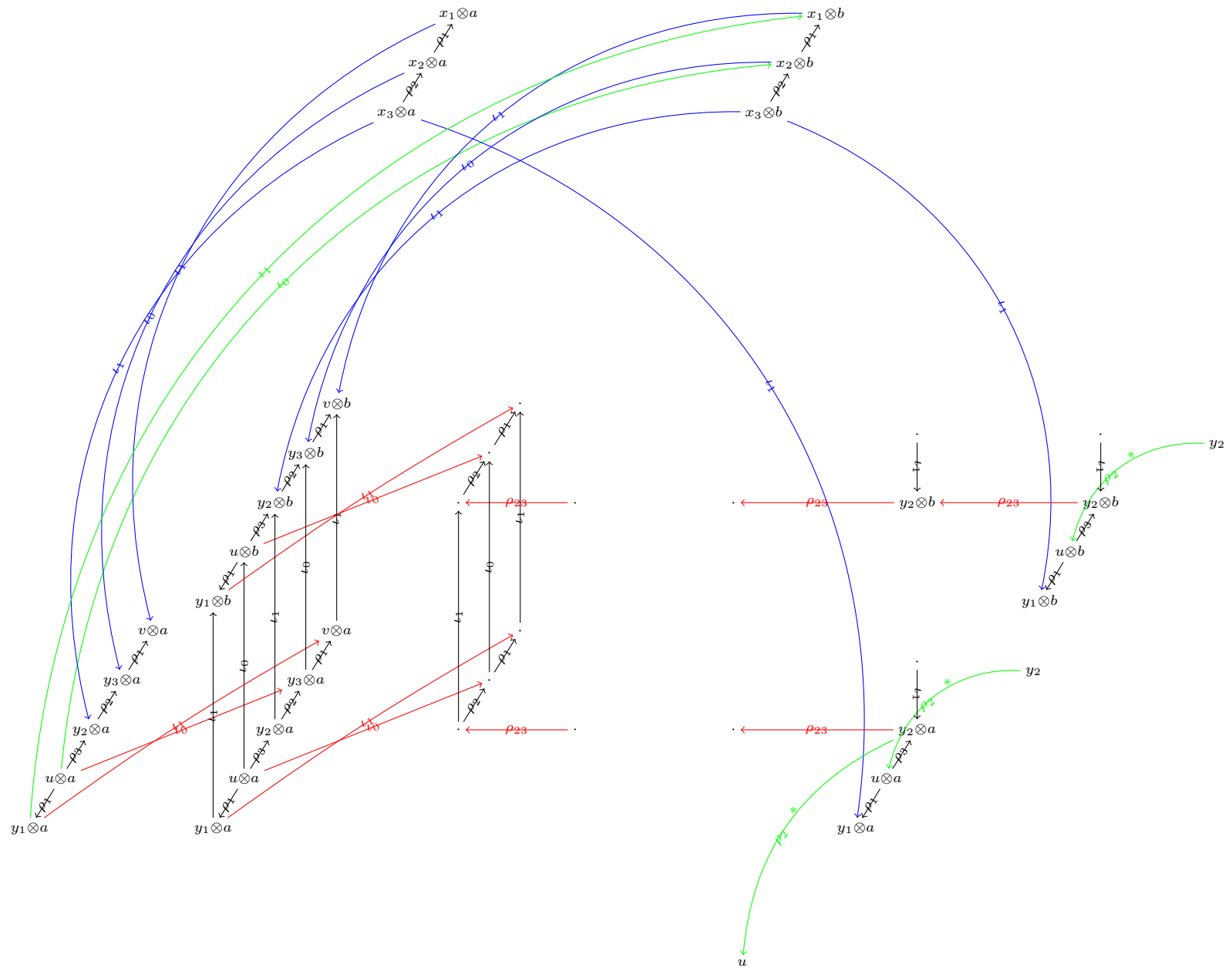


Figure A.28: V^0

Figure A.29: V^0

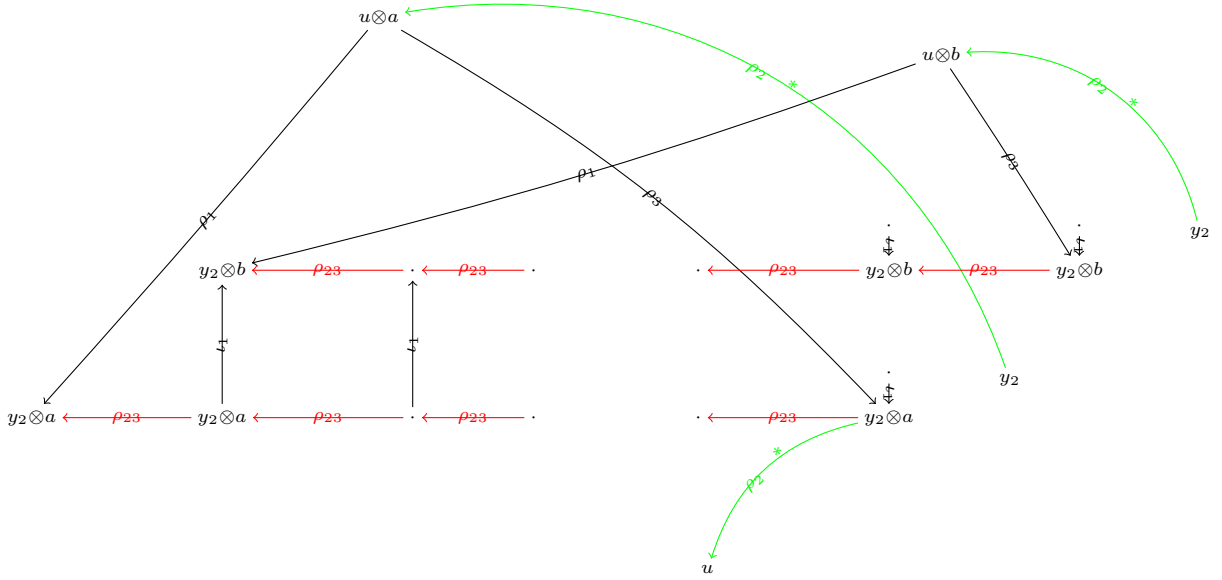


Figure A.30: V^0

Now we consider the possibility of an outgoing length 1 arrows $b \rightarrow c$ in ∂_z . (c can not be the same as a , because that $d^2 = 0$ and horizontal simplifiedness would force C to be infinitely generated.) To ensure $d^2 = 0$, there must be a generator d in one Alexander filtration level below a to form this 1 by 1 box in C , as in Figure A.31. We do not exclude the possibility of other incoming or outgoing vertical arrows at the generators. Figure A.32 shows the corresponding part in $KtD(C)$. Figure A.33 shows the tensor product before any cancellation are applied. The open-ended purple arrows are potential arrows. Now we apply the cancellation described in Appendix A.2 on the right side. Then make some “free” cancelations: cancel all blue $x_1 \rightarrow v$ ’s and then all $x_2 \rightarrow y_3$ ’s, which generate no new arrows. Next, we cancel all the pairs of reds on the left-hand side. All those cancelations have been previously described in details, so we just show the final result in Figure A.34.

A few things to point out here: Arrows marked with $*$ (eight green and one pink) are those dangling arrows in Figure A.11b. They exist because of the potential ∂_z differentials connecting to a, b, c, d (purple ones in Figure A.31). The two green arrows marked by $**$ are the same kind of

dangling arrow whose two ends we finally see in one diagram, as they come from the arrows $b \rightarrow c$ and $a \rightarrow d$ in ∂_z . The pink arrow marked with $**$ is also a manifestation of the pink dangling arrow in Figure A.11b.

Next, we move y_1 's and u 's to the top, and cancel $x_3 \rightarrow y_1$ at b in V^0 , then that arrow at a in V^0 , then that arrow at c in V^0 , then that arrow at d in V^0 , see Figure A.35. Now everything matches to $KtD(C^{flip})$ as described above, except we are missing two ρ_{123} 's. For convenience, we label the u 's in V^0 with the generator it corresponds: u^a, u^b, u^c, u^d and label the y_2 's on the left hand side with the row it is in and the its distance to the leftmost position: $y_2^{d,0}, y_2^{c,1}$, etc. y_2 's on the right hand side are similarly labeled as $y_2^{d,0,r}, y_2^{c,1,r}$, etc. Per Lemma 2, there must be two ρ_{123} 's from u^a to $y_2^{d,0}$ at the left-most position of d 's row on the left, and from u^b to $y_2^{c,0}$. We also have an extra arrow from u^a to $y_2^{c,0}$ (partially pink in the diagram).

To fix this, we do a change of basis: $\langle u^a, u^b, y_2^{d,1}, y_2^{c,1} \rangle \rightarrow \langle u^a + \rho_1 y_2^{d,1}, u^b + \rho_1 y_2^{c,1}, y_2^{d,1}, y_2^{c,1} \rangle$, where d and c are circled and boxed respectively on the diagram. We examine the effect of this change of basis. u^a : $\rho_2 u^a = \rho_2(u^a + \rho_1 y_2^{d,1})$, so its incoming dangling ρ_2 arrow is unaffected. $\partial u^a = \rho_1 y_2^{a,0} + \rho_1 y_2^{c,0} + \rho_3 y_2^{a,0,r}$ and $\partial(u^a + \rho_1 y_2^{d,1}) = \rho_1 y_2^{a,0} + \rho_1 y_2^{c,0} + \rho_3 y_2^{a,0,r} + \rho_1(\rho_{23} y_2^{d,0} + y_2^{c,0}) = \rho_1 y_2^{a,0} + \rho_3 y_2^{a,0,r} + \rho_{123} y_2^{d,0}$, which is exactly what we need. For u^b , its incoming dangling ρ_2 arrow is also unaffected. $\partial u^b = \rho_1 y_2^{b,0} + \rho_3 y_2^{b,0,r}$ and $\partial(u^b + \rho_1 y_2^{c,1}) = \rho_1 y_2^{b,0} + \rho_3 y_2^{b,0,r} + \rho_1 \rho_{23} y_2^{c,0} = \rho_1 y_2^{b,0} + \rho_3 y_2^{b,0,r} + \rho_{123} y_2^{c,0}$, which is also what we need. See Figure A.36 for the final product. Note the dangling partially pink arrow will function as the extra arrow $**$ in Figure A.35 for some other 1 by 1 box starting at d , should there exist one.

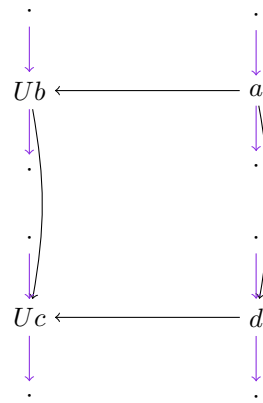


Figure A.31: V^0

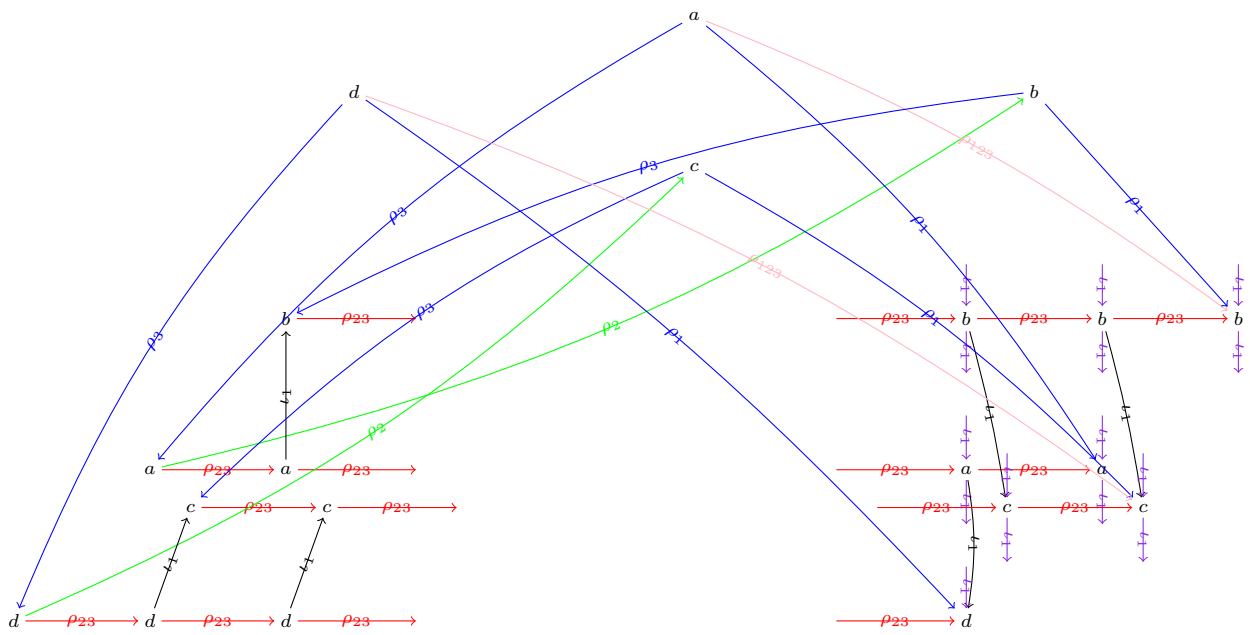
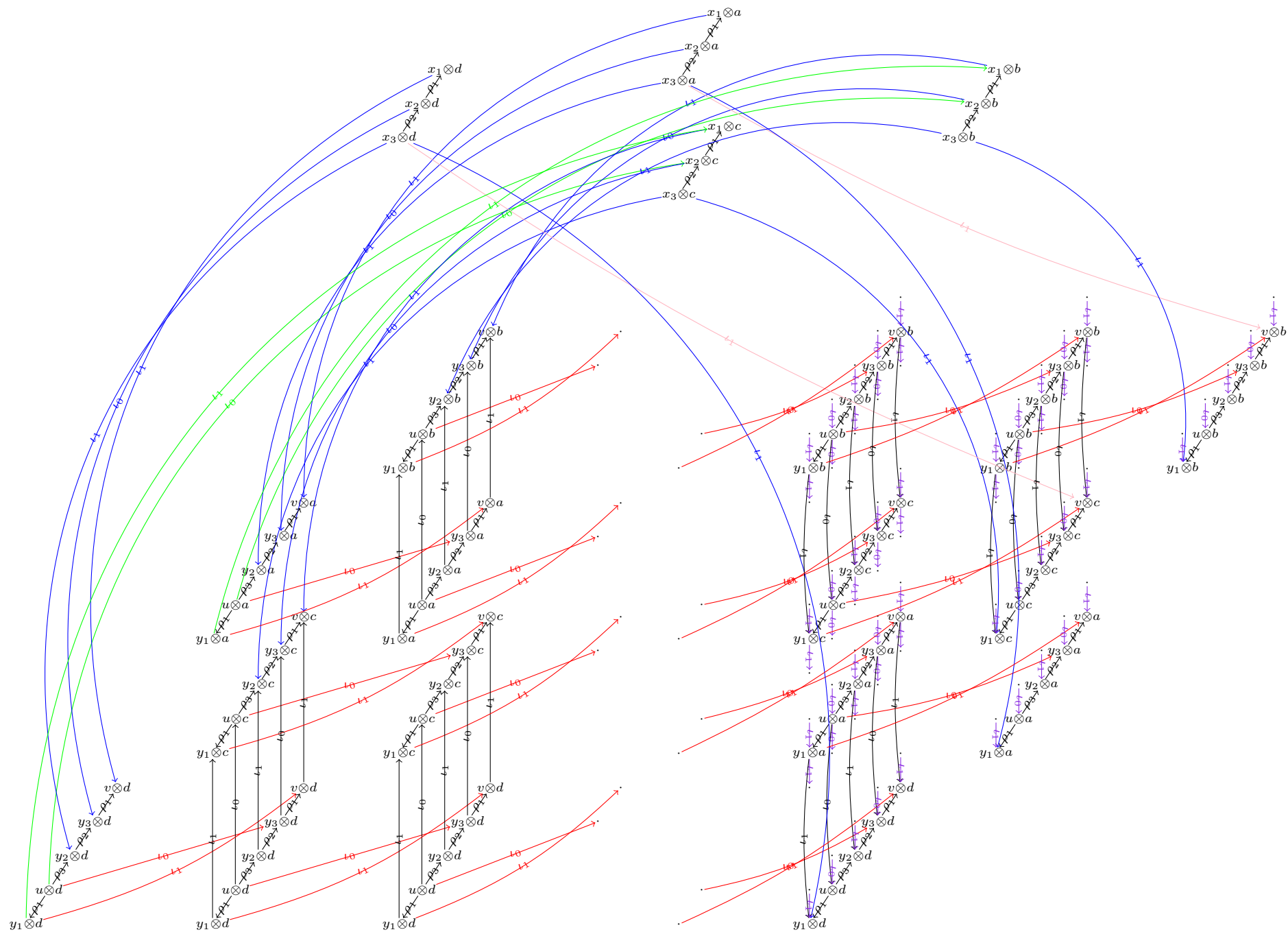
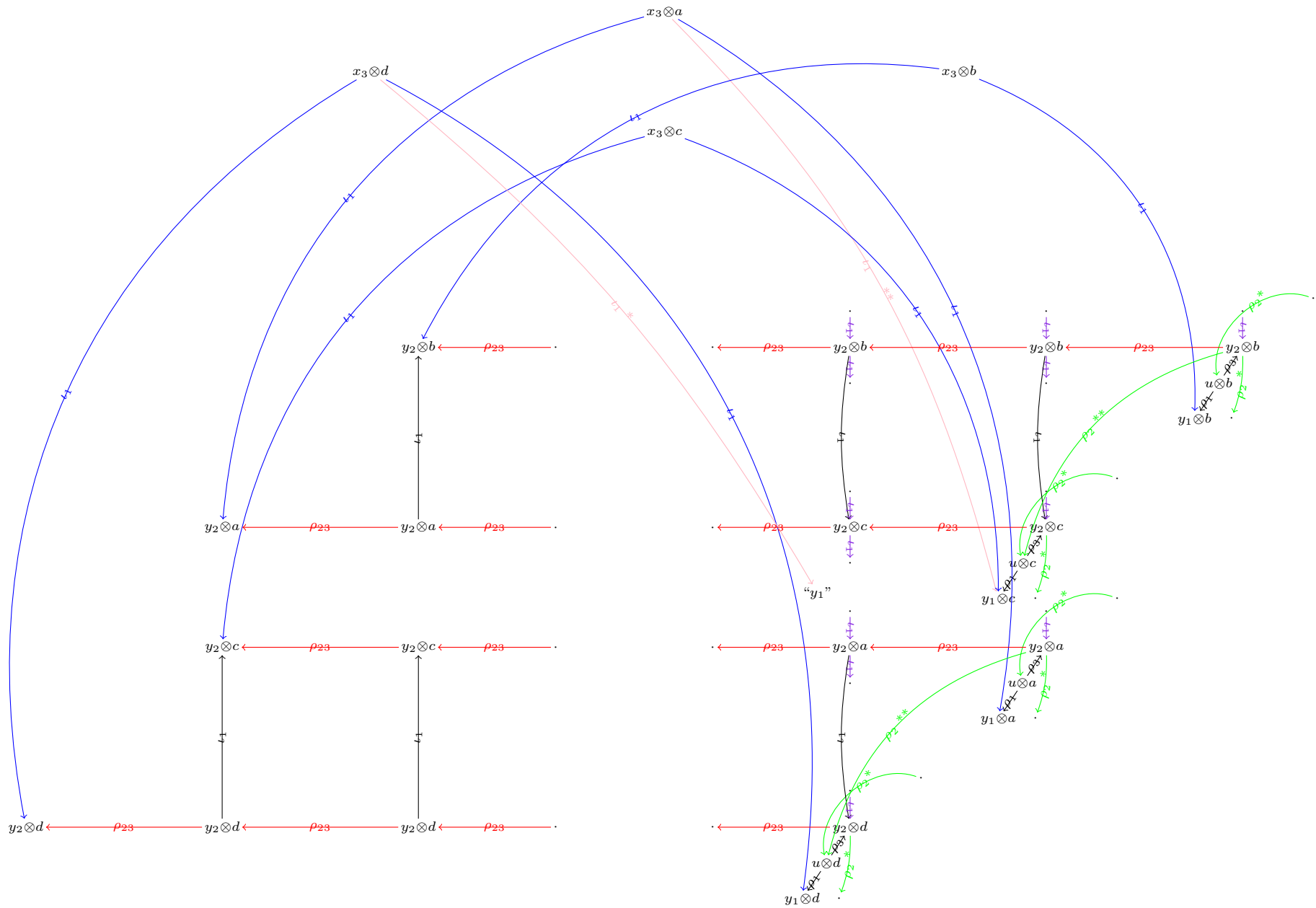
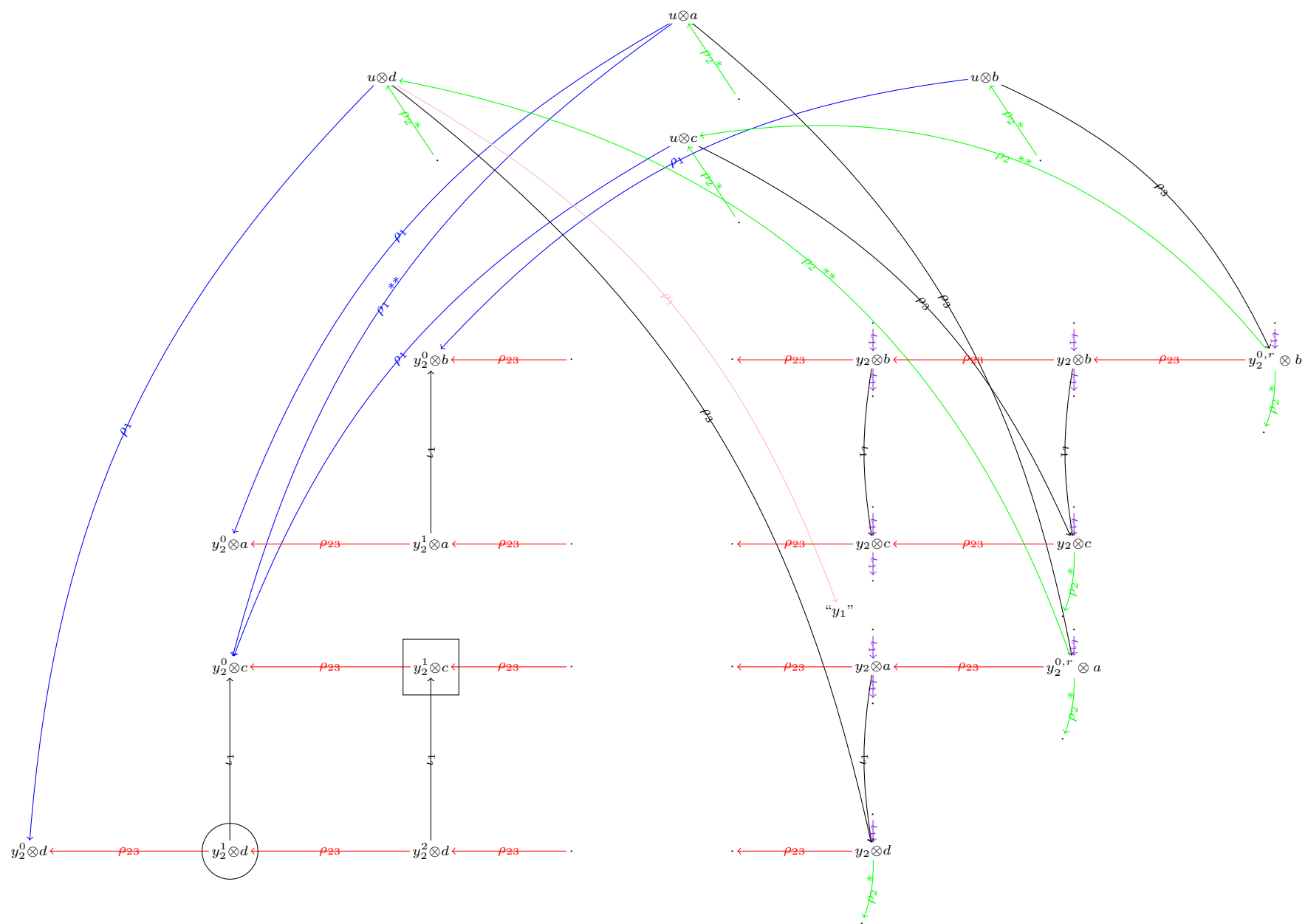
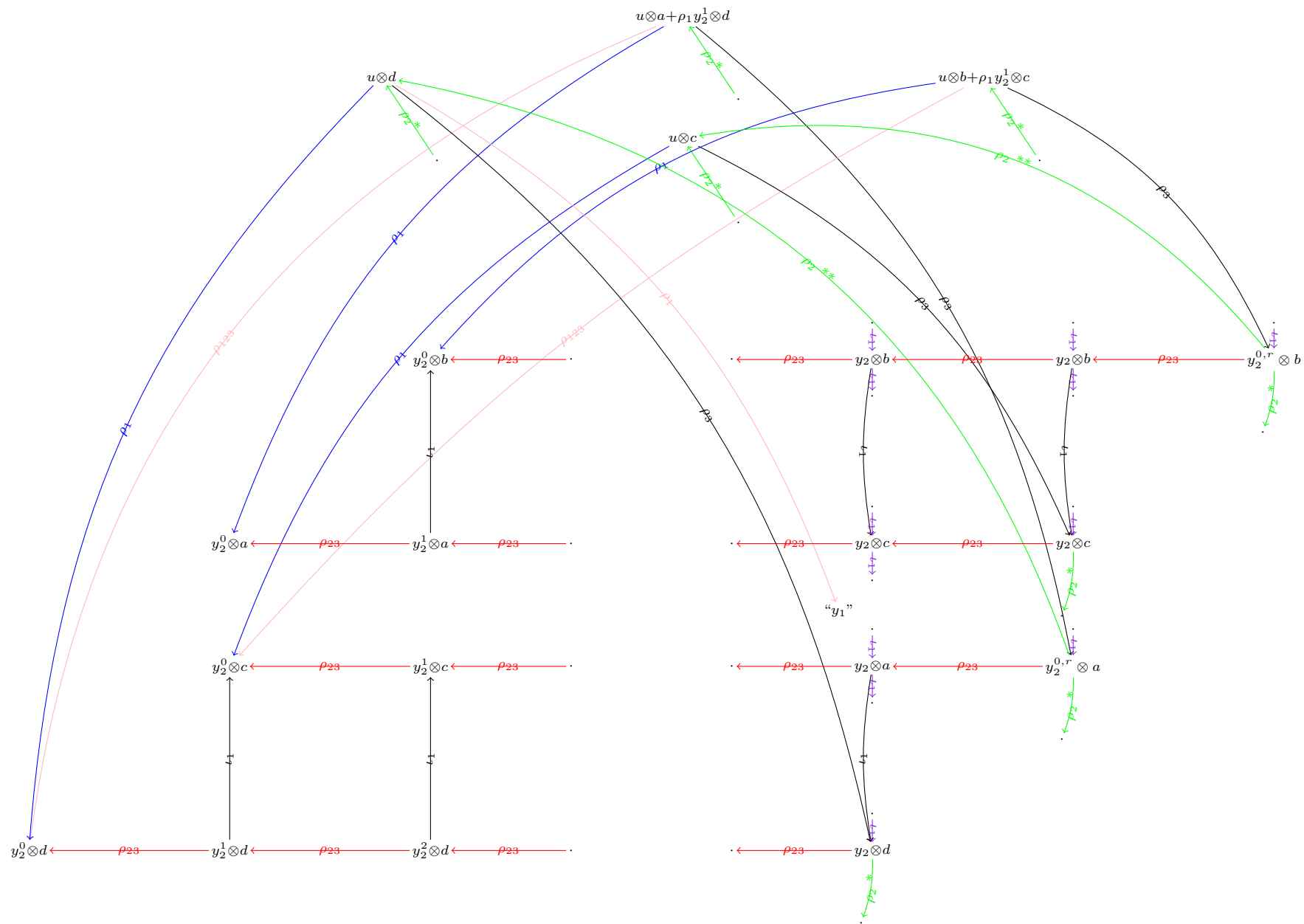


Figure A.32: V^0

Figure A.33: V^0

Figure A.34: V^0

Figure A.35: V^0

Figure A.36: V^0

A.6 Left hand side full copy

The situation at left hand side full copy is simpler than the right hand side, as C is horizontally simplified. Generators come in pairs except ξ_h , which is generator of the homology, see Figure A.37a for a pair $x \rightarrow y$ and ξ_h . The isomorphism-inducing map between the full copy and $\mathbb{F}_2$ must be a single arrow from ξ_h to the generator of $\mathbb{F}_2$.

At a pair $x \rightarrow y$, we cancel the pairs of red arrows, which we have done so many times, see Figure A.37b. Then we cancel the vertical black arrows between y_1 's and u 's, see Figure A.37d. At the row of ξ_h , it is even simpler. It is now a single string of ρ_{23} 's. The usual cancellation bring them into shape.

One last corner case is of the top row, where the full copy meets the left-most position of a row. We need to examine that the cancellation needs to be done in both places are compatible. It is clear that the top row generator has only incoming ρ_3 , but no outgoing ρ_2 . We further observe that the cancellation in Appendix A.5 only concern y_3, v at the top row generator, whereas cancellation in this section remove y_1, u there. This means the two sets of cancellation never interact, hence must be compatible.

A.7 Conclusion of proof

Therefore, we have examined all parts of the modules and can conclude that tensoring with H indeed transforms $KtD(C)$ to $KtD(C^{flip})$. Hence Proposition 2 is proved.

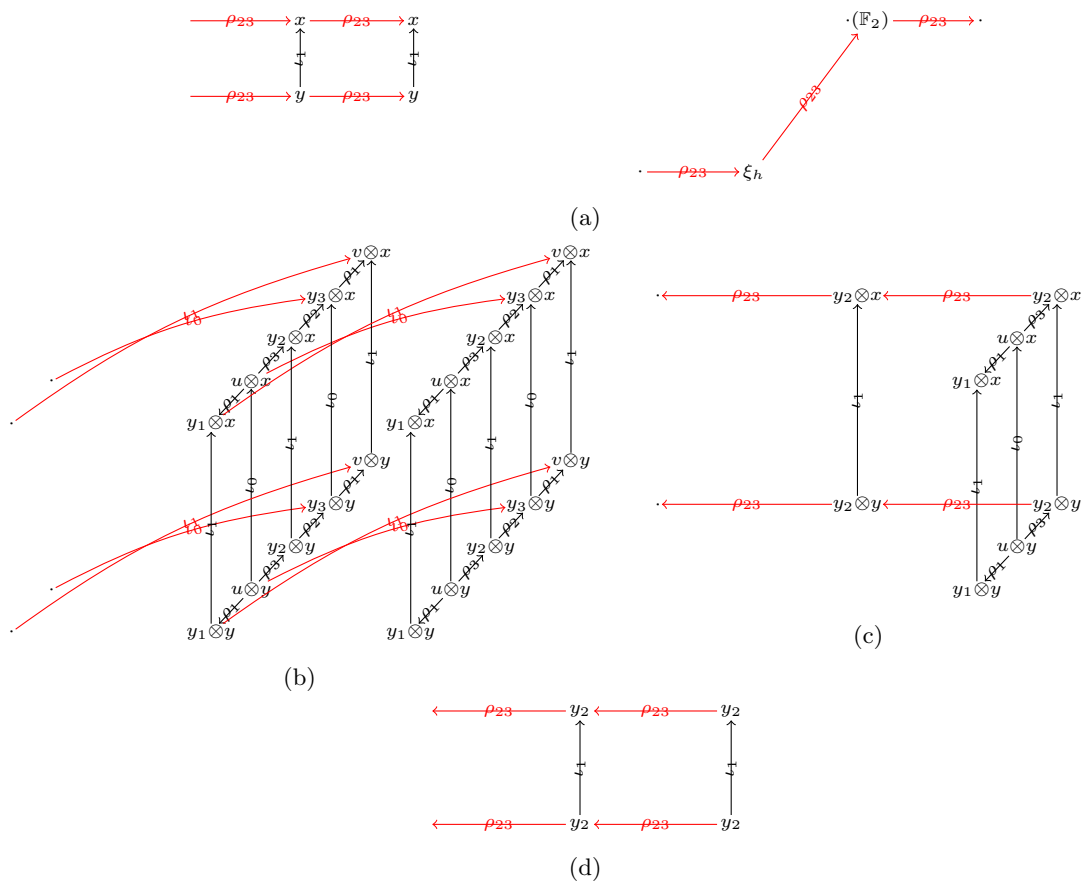


Figure A.37: Left hand side full copy

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