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Some Results on a Fully Nonlinear Equation in Conformal Geometry

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SOME RESULTS ON A FULLY NONLINEAR EQUATION IN CONFORMAL GEOMETRY

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Abstract

In this thesis we study the question of solving an equation arising in conformal geometry. Specifically, we obtain information about the equation $\sigma_k(g^{-1}A_g) = \text{constant}$. Here g is a metric in a given conformal class $[g_0]$ and A_g is the Schouten tensor of g .

First, we generalize a result of Chang, Gursky, and Yang to show that under certain geometric conditions, entire solutions of this equation on $\mathbb{R}^n$ must necessarily arise via pullback from the sphere under stereographic projection. This “Obata” type theorem provides an alternate proof of a result of Li and Li. However, our approach uses integral estimates which may be more amenable to application on manifolds which are not locally conformally flat.

Second, we study the question of solving $\sigma_k = \text{constant}$ on n -manifolds M , the “ σ_k -Yamabe problem”, when $k = 2$ and $n = 3$. The σ_k -Yamabe problem has been studied for many values of n and k before (by Li and Li, by Guan and Wang, by Gursky and Viaclovsky, by Sheng, Trudinger, and Wang, and by Trudinger and Wang, among others); however, some cases remain open. The case where $n > 2k$ is unresolved in general. A perhaps more natural question to consider in this case is solving $v_{2k} = \text{constant}$, for $n > 2k$, where v_{2k} are the renormalized volume coefficients studied by Graham and by Chang and Fang. Unfortunately, the approach used for other values of n and k does not apply in this final case—for v_{2k} , one cannot use the local estimates of Guan and Wang and of Chen. This is because the local estimates use algebraic properties of σ_k which do not hold for v_{2k} . Though the σ_2 -Yamabe problem in dimension 3 has been studied before, we study an alternative approach which avoids the use of the local estimates. We hope this approach may prove useful for resolving the question of solving $v_{2k} = \text{constant}$.

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To my parents Bill and Geness, and to my brother and best friend, Chris

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Chapter 1

Introduction

For a given Riemannian manifold (M, g_0) of dimension n , *conformal geometry* seeks to understand geometric behavior under a conformal change of metric $g_0 \rightarrow g = e^{2w}g_0$. Here w is a real valued function on M . To this end, the Schouten tensor A_g is defined as

$$(A^g)_{ij} = Ric_{ij}^g - \frac{R^g}{2(n-1)}g$$

(Depending on the author the right hand side of this equation may be multiplied by $\frac{1}{n-2}$) The Schouten tensor is an important object of study in conformal geometry because it determines the behavior of the Riemann curvature tensor under a conformal change. Specifically, we have the Ricci decomposition for the Riemann tensor

$$Riem_g = W_g \oplus \frac{1}{n-2}A_g \circ g$$

Here $\circ$ is the Kulkarni-Nomizu product and W_g is the Weyl tensor. The important property of the decomposition above is that the $(1, 3)$ Weyl tensor (that is, one upper index and three lower indices) is pointwise invariant under a conformal change of the metric g . Thus the Schouten tensor encodes the behavior of the Riemann tensor under a conformal change.

One of the most well known problems in conformal geometry is the *Yamabe Problem*. The Yamabe Problem asks if it is possible to find a metric of constant scalar curvature among all metrics conformal to a fixed background metric. The problem has been solved in the affirmative in the case when M is compact (see Yamabe [33], Trudinger [33], Aubin [1], and Schoen [30]).

The eigenvalues $\lambda_1, \dots, \lambda_n$ of the Schouten tensor are the eigenvalues of the matrix $g^{-1}A_g$.

Because the trace of the Schouten tensor is a multiple of the scalar curvature, we can interpret the Yamabe Problem as asking for solutions to the equation

$$\sigma_1(g^{-1}A_g) = \text{constant},$$

where σ_1 denotes the trace the matrix $g^{-1}A_g$. A natural generalization of the Yamabe Problem is then to find solutions of

$$\sigma_k(g^{-1}A_g) = \text{constant}.$$

This is the main equation of interest for this thesis. Here $\sigma_k(A_g)$ is the k -th elementary symmetric function of the eigenvalues $\lambda_1, \dots, \lambda_n$

$$\sigma_k(g^{-1}A_g) = \sum_{1 \leq i_1 < \dots < i_k \leq n} \lambda_{i_1} \dots \lambda_{i_k}$$

Occasionally we will simply write $\sigma_k = \sigma_k(A_g) = \sigma_k(g^{-1}A_g)$ for convenience.

The main result of Chapter 3, Theorem 3.1 is a generalization of Theorem 0.1 of [8] to the case $k > 2$. In particular we show that finite volume metrics on $\mathbb{R}^n$ conformal to the standard metric and satisfying $\sigma_k = \text{constant}$ must necessarily arise by pulling back the round metric on $\mathbb{S}^n$ to $\mathbb{R}^n$ via stereographic projection. This can be thought of as a fully nonlinear generalization of Obata's Theorem (see Section 3.1 for a discussion) that any metric conformal to the standard metric on the sphere $\mathbb{S}^n$ with constant scalar curvature must necessarily be isometric to the standard metric. Theorem 3.1 has also been proved by A. Li and Y. Li for all n and k [25]; however, the approach given here is different and has the advantage that its integral estimates may be more applicable to general Einstein manifolds. In particular, [25] uses the method of moving planes (which depends on the geometry of $\mathbb{R}^n$) whereas our approach (adapted from [8]) uses integral estimates. We note that Theorem 3.1 is useful to characterize limiting metrics that may arise when using a blow-up technique—see, for example, [6].

The main results of Chapter 4 are concerned with the question of solving this $\sigma_k = \text{constant}$, where $k = 2$ and $n = 3$. For general n and k this is known as the σ_k -Yamabe Problem. Viaclovsky introduced this problem in [36]. It was originally addressed in the case $k = 2$, $n = 4$ by Chang, Gursky, and Yang in 2002 [7]. Later it was answered for locally conformally flat manifolds by Li and Li, 2003 [25] and by Guan and Wang, 2003 [22]. The case $k > n/2$ was resolved by Gursky and Viaclovsky, 2007 [20] and in the case $2 \leq k \leq n/2$, provided the equation $\sigma_k = \text{constant}$ is

variational, by Sheng, Trudinger, and Wang 2010 [31]. The case $k < n/2$ remains open in the case when M is not locally conformally flat and when the equation $\sigma_k = \text{constant}$ is not variational.

The “renormalized volume coefficients” v_{2k} are an interesting quantity related to the σ_k . They have been studied by Graham [15],[16], Chang and Fang [4], and Chang, Fang, and Graham [5] among others. In order to describe the renormalized volume coefficients, we will first need some background about Poincare-Einstein spaces.

Poincare-Einstein (PE) manifolds were originally developed by Fefferman and Graham [11] for use in conformal geometry. They have since become a topic of intense study due to their connection with the anti de Sitter/conformal field theory correspondence in string theory introduced by Maldacena [26]. One may think of Poincare-Einstein spaces as natural generalizations of the Poincare ball model of hyperbolic space. Specifically, if M^{n+1} is a compact manifold with boundary $\partial M = X^n$, and g is a metric on the interior of M , then g is said to be *conformally compact* if $g = \rho^{-2}\bar{g}$ for a metric $\bar{g}$ defined on all of M and a non-negative function ρ satisfying $\rho^{-1}(0) = X$ and $d\rho \neq 0$ on X (such a function ρ is called a *defining function*). If in addition $|\nabla \rho|_{\bar{g}} = 1$ on a neighborhood of $X = \partial M$, then we say that ρ is a *geodesic defining function*. A manifold which is both conformally compact and Einstein is called *Poincare Einstein*. For a fixed g with $g = \rho^{-2}\bar{g}$, different choices of ρ will correspond to different choices of $\bar{g}$. However, for all such $\bar{g}$, $\gamma = \bar{g}|_X$ must lie in a single conformal class $[\gamma]$. This class is called the *asymptotic infinity* of M . In this way, there is a correspondence between PE metrics on M and conformal classes on ∂X .

An important fact following from the above definitions is that if we write $g = \rho^{-2}\bar{g}$ with ρ a geodesic defining function, then near X we have $\bar{g} = d\rho^2 + g_\rho$, where

$$\begin{aligned} g_\rho &\sim g_{(0)} + \rho^2 g_{(2)} + \cdots + \rho^{n-1} g_{(n-1)} + \rho^n g_{(n)} + \rho^{n+1} g_{(n+1)} + \cdots, \text{ n odd} \\ g_\rho &\sim g_{(0)} + \rho^2 g_{(2)} + \cdots + \rho^{n-2} g_{(n-2)} + \rho^n g_{(n)} + \rho^n \log \rho h + \rho^{n+1} g_{(n+1)} + \cdots, \text{ n even} \end{aligned} \tag{1.1}$$

where the g_k and h are symmetric 2-tensors defined on X . The terms g_k are determined by $g_0 = \gamma$ for $k < n$, as is h when n is even. However, though g_n satisfies some simple constraints, it is essentially undetermined by γ (g_j , $j > n$, is determined by g_n).

The renormalized volume coefficients v_{2k} arise when computing the volume form of a PE metric g using the (1.1). Specifically, with the notation as above, where $g = e^{-2\rho}\bar{g}$ is a PE metric and ρ a

geodesic defining function, we see

$$dv_g = -\rho^{-n-1} \left(\frac{det g_\rho}{det \gamma} \right)^{1/2} dv_\gamma d\rho$$

$$\left(\frac{det g_\rho}{det \gamma} \right)^{1/2} = 1 + v_2 \rho^2 + v_4 \rho^4 + \dots$$

where in the second line we have an expansion in even powers of ρ . These coefficients have the following important properties:

1. v_{2k} is defined for $2k \leq n$ for n even, and for all k when n is odd, and in each case is determined by γ
2. When M is locally conformally flat or when $k = 1, 2$, $v_{2k} = \sigma_k(A_{g_\rho})$
3. v_{2k} is variational when defined, in the sense that the Euler-Lagrange equation for $\int_X v_{2k}(\gamma) dv_\gamma$, as γ varies over a conformal class, is $v_{2k} = \text{constant}$.
4. Under a conformal change $\hat{\gamma} = e^{2w}\gamma$, the expression for the conformal transformation of v_{2k} involves at most second derivatives of w

We can think of the last property as saying that $v_{2k} = \sigma_k + \text{lower order terms}$, in a sense that can be made precise.

To provide some context, recall the state of the Yamabe problem as discussed earlier in this section. In particular, the unresolved case of the σ_k -Yamabe problem is when $k < \frac{n}{2}$ and σ_k is not variational (or when the underlying space is not locally conformally flat). The four properties listed above strongly suggest that we should instead look to solve $v_{2k} = \text{constant}$, rather than $\sigma_k = \text{constant}$, in the unresolved cases of the σ_k -Yamabe problem.

Previously, a major difficulty in such an approach arose from the fact that σ_k has some useful algebraic properties which do not hold in general for v_{2k} . Our approach avoids using algebraic properties of the σ_k (which are crucial in the local estimates of [23] or [10], upon which previous results rely), and instead uses only the continuity method. In particular, we adapt pieces of the approach of [8] from dimension 4 to dimension 3. Though our results are not as strong as in the four dimensional case, they provide some progress in adapting to non-critical dimensions. Since $v_4 = \sigma_2(A_\gamma)$ for any dimension, we have an approach to consider $v_4 = \sigma_2 = \text{constant}$ in dimension 3. Hopefully, we can generalize our result to other v_{2k} to address the last remaining open case in the σ_k Yamabe problem.

In dimension 4, [8] shows that there is a one parameter family of equations, denoted $(*)_\delta$, connecting $\sigma_2 = \text{constant}$ at $\delta = 0$ and another equation at $\delta = 1$ (similar to $Q = \text{constant}$) whose solution is known to exist (see Corollary 2.2 and ensuing discussion in [7]). The authors then use the continuity method to show that, starting with a solution of $(*)_1$, the set of $\delta \in [0, 1]$ such that $(*)_\delta$ has a solution is both open and closed. Hence in particular $\sigma_2 = \text{constant}$ has a solution. Openness (Proposition 4.1, [7]) is proved by showing that the linearization of $(*)_\delta$ has no zero eigenvalues. Closedness (Theorem 3.1, Theorem 5.1, [7]) is proved by showing a-priori estimates for solutions of $(*)_\delta$.

In dimension 3 we use a similar approach. In our case, the one parameter family of equations (4.6) connects the equation $Q = \text{constant}$ and $\sigma_2 = \text{constant}$. In Chapter 4 we address what aspects of the analogous continuity method fail and succeed. First we mention some known results regarding when the equation $Q = \text{constant}$ has solutions. Next, we remark that in general it is not the case that the linearization of (4.6) has no non-zero eigenvalues. Hence openness does not hold in general. However, we do still have a-priori estimates (see Theorems 4.2 and 4.3) so closedness holds.

In Chapter 2 we provide some basic facts and notation used throughout the paper. In Chapter 3 we prove Theorem 3.1. Theorems 4.2 and 4.3 are shown in Chapter 4. Chapter 5 provides some closing discussion. In the appendix we adapt [35] to dimension 3 (this is very simple purely due to the Sobolev inequality being “stronger” in dimension 3 than dimension 4, however the argument is included for completeness).

Chapter 2

Background and Notation

In this section we will introduce necessary background and notation for the remainder of this thesis.

Notation used in only one section will be addressed therein.

We say that two metrics g and g_0 are *conformal* provided $g = \lambda g_0$ for some positive function λ . We will denote the Ricci curvature of a metric g by Ric_g and the scalar curvature by R_g .

The Schouten Tensor of a metric g is given by

$$A_g = Ric_g - \frac{R}{2(n-1)}g$$

We will also write

$$E_g = Ric_g - \frac{R_g}{n}g \tag{2.1}$$

for the traceless Ricci tensor.

If a matrix M has eigenvalues $\lambda_1, \dots, \lambda_n$, then for $1 \leq k \leq n$ we define

$$\sigma_k(M) = \sum_{1 \leq i_1 < \dots < i_n \leq n} \lambda_{i_1} \cdots \lambda_{i_n}$$

and we also denote $\sigma_k(A_g) = \sigma_k(g^{-1}A_g)$. Also, the traceless part of M is written

$$\overset{\circ}{M} = M - \frac{\sigma_1(M)}{n}I$$

where I is the identity matrix. Occasionally we will use the algebraic fact

$$2\sigma_2(M) = \sigma_1(M)^2 - \sigma_1(M^2) \tag{2.2}$$

We write

$$M \in \Gamma_k^+ \text{ provided } \sigma_1(M), \dots, \sigma_k(M) > 0$$

Sometimes we may write $g \in \Gamma_k^+$ as shorthand for $g^{-1}A_g \in \Gamma_k^+$.

The Newton tensor T_k is defined by

$$T_k(M) = \sigma_k(M)I - \sigma_{k-1}(M)M + \dots + (-1)^{k-1}\sigma_1(M)M^{k-1} + (-1)^k M^k$$

The Newton tensor is divergence free on locally conformally flat manifolds and when $k = 1$ (see [36]):

$$\nabla_i T_{ij} = 0 \tag{2.3}$$

When $M \in \Gamma_k^+$, $T_j(M)$ is positive definite for $j < k$ (see [12]). Also, the Newton tensor satisfies

$$\text{Tr}(T_k(M)) = (n - k)\sigma_k(M)$$

As is convenient we may write $T^k(M)$ or $T_k(M)$ to mean the k -th Newton tensor.

Chapter 3

Entire Solutions of $\sigma_k = \text{constant}$

3.1 Introduction and Notation

Throughout this section $g = v^{-2}|dx|^2$ will denote a metric conformal to the Euclidean metric on $\mathbb{R}^n$. Here $v > 0$. **Note: here we write the conformal factor as v^{-2} . We will use a different convention in Chapter 4.**

As mentioned in the introduction, we seek to generalize Theorem 0.1 in [8] to the following:

Theorem 3.1. *Let $g = v^{-2}(x)|dx|^2$ be a conformal metric on $\mathbb{R}^n$, $n \geq 2k$, with $g^{-1}A_g \in \Gamma_k^+$, satisfying*

$$\sigma_k(g^{-1}A_g) = \text{constant} \quad (3.1)$$

Assume in addition that

$$\text{vol}(g) = \int_{\mathbb{R}^n} v^{-n} dx < \infty \quad (3.2)$$

Then $v = a|x|^2 + b_i x^i + c$ for constants a, b_i, c . In particular, g is obtained by pulling back the round metric on $\mathbb{S}^n$ to $\mathbb{R}^n$.

To give context for the result we prove, we first recall the following result of Obata [27]:

Theorem 3.2. *Suppose g_0 is the round on the sphere $\mathbb{S}^n$. Then for any metric $g = v^{-2}g_0$ conformal to the round metric with $R_g = \text{constant}$, g is necessarily isometric to the round metric on $\mathbb{S}^n$.*

Proof. Suppose g is a metric as given in the statement of the theorem. Then the trace-free Ricci tensor E may be written in terms of v as

$$E_g = -(n-2)v\nabla_g^2(v^{-1}) + \frac{n-2}{n}v\Delta_g(v^{-1})g \quad (3.3)$$

Take the g inner product of (3.3) with $v^{-1}E_g$ and integrate over $\mathbb{S}^n$ to find

$$\int_{\mathbb{S}^n} |E_g|^2 v^{-1} dv_g = -(n-2) \int_{\mathbb{S}^n} g(E_g, \nabla_g^2(v^{-1}) + \frac{n-2}{n} v \text{Tr}_g(E_g) \Delta_g(v^{-1})) dv_g$$

As E is trace free, the second term on the right side is zero. Hence by the Divergence Theorem,

$$\int_{\mathbb{S}^n} |E_g|^2 v^{-1} dv_g = (n-2) \int_{\mathbb{S}^n} g(\delta E_g, \nabla(v^{-1})) dv_g$$

where δ is the divergence operator. However, the second Bianchi identity tells us that $\delta E_g = \frac{n-2}{2n} \nabla R_g$. Since R_g is constant, we see that E_g is divergence free. Thus

$$\int_{\mathbb{S}^n} |E_g|^2 v^{-1} dv_g = 0$$

Hence $E_g \equiv 0$ and so g is Einstein. □

We will modify Obata's result so as to reduce it (in the non-compact case we consider here) to a tail term estimate (see Proposition 3.8). The outline of this chapter is as follows. In Section 3.2 we prove some basic algebraic facts about the traceless Newton tensor $\tilde{T}^k$. In Section 3.3 we obtain an equation for $\int \sigma_k$ in terms of lower order expressions and “tail terms” (terms involving derivatives of a cutoff function). Next we estimate the tail term in Sections 3.4 and 3.5. Finally, we conclude with the proof of the Theorem 3.1.

The main differences between the argument here and the one in [8] is the necessity of the recursive formula in Section 3.3 (to deal with values of k larger than 2) as well as the Taylor expansion in tensor estimate in Section 3.2. The tensor estimate involves an indirect approach, different from the Lagrange multiplier technique in [8]. The remainder of the argument is essentially a direct adaptation of their proof.

In this chapter we denote the k -th power sum of $g^{-1}A_g$, p_k , to be the sum of the k -th powers of the eigenvalues of $g^{-1}(A_g)$,

$$p_k = \text{Tr}((g^{-1}A_g)^k)$$

Note that for the remainder of this chapter (except where explicitly indicated in Section 3.6) all terms are written with respect to the $|dx|^2$ metric—for example, ∇_i is simply a Euclidean derivative, $\nabla_i v = \partial_i v = v_i$, and $|\nabla v|$ denotes the norm of the Euclidean gradient with respect to the $|dx|^2$ metric.

3.2 Tensor Estimates

This section generalizes the results of Proposition 2.2 in [8]. Note that $\hat{T}^1(A_g) = -E$.

Proposition 3.3. *Suppose $g^{-1}A_g \in \Gamma_k^+$. Then*

- *The following holds with equality if and only if $\hat{T}^1 = 0$.*

$$g(\hat{T}^k, \hat{T}^1) \geq 0 \quad (3.4)$$

- *For some positive constant $C = C(n, k)$,*

$$g(\hat{T}^k, \hat{T}^k) \leq C\sigma_{k-1}g(\hat{T}^k, \hat{T}^1) \quad (3.5)$$

Remark: We note that (3.4) is a trivial consequence of (3.5). However, (3.4) (which is simple to show) is used in the proof of (3.5) (which has a much more difficult proof) and (3.4) is used often enough that we feel it is convenient to state the two results separately.

Proof. Note that the above statements do not depend on the geometric properties of A_g . Instead they only use the algebraic property that $A_g \in \Gamma_k^+$.

Proof of (3.4): We will imitate the proof given in [8] using Lemma 23 from [36]. First, note that $\hat{T}^1 = \frac{\sigma_1}{n}g - A$. Since $\hat{T}^k$ is traceless,

$$\begin{aligned} g(\hat{T}^k, \hat{T}^1) &= -g(\hat{T}^k, A_g) \\ &= -g(T^k - \frac{n-k}{n}\sigma_k g, A_g) \\ &= -g(T^k, A_g) - \frac{n-k}{n}\sigma_k \sigma_1 \\ &= -(k+1)\sigma_{k+1} + \frac{n-k}{n}\sigma_k \sigma_1 \end{aligned}$$

In the last line we have used Proposition 1.2 of [29]. Hence we must simply show that

$$(k+1)\sigma_{k+1} \leq \frac{n-k}{n}\sigma_k \sigma_1$$

holds on Γ_k^+ with equality if and only if $\hat{T}^1 = 0$. This is exactly the content of Lemma 23 in [36].

Proof of (3.5): We will take a different approach from [8] to prove the statement for arbitrary k . Our proof only shows the existence of a constant $C = C(n, k)$ but does not find its value.

Because T^k , σ_k only depend on the vector Λ of eigenvalues of $g^{-1}A_g$, for the purpose of this proof we will think of T^k and σ_k as functions defined on points $\Lambda \in \mathbb{R}^n$

First, notice that by (3.4) the right side of (3.5) is non-negative for $\Lambda \in \Gamma_k^+$, and by continuity the same holds on $\overline{\Gamma_k^+}$. Next, notice that both sides of (3.5) are homogeneous in Λ of the same degree. Hence it suffices to prove (3.5) on

$$K := \mathbb{S}^{n-1} \cap \overline{\Gamma_k^+}$$

a compact set. So if we can show that (3.5) holds in K near the zeroes of the right side of (3.5), then (3.5) holds on all of K and the result follows from the extreme value theorem. In other words, we want to show that

$$\frac{h(\Lambda)}{f(\Lambda)} := \frac{g(\overset{\circ}{T}_k, \overset{\circ}{T}_k)}{-\sigma_{k-1}g(\overset{\circ}{T}_k, \overset{\circ}{T}_1)}$$

is bounded near the zeroes of f . To do so, we write

$$K = \partial K \cup \overset{\circ}{K} \quad \text{where}$$

$$\partial K = \partial \Gamma_k^+ \cap K$$

$$\overset{\circ}{K} = \Gamma_k^+ \cap K$$

The proof proceeds in several steps

- Step 1: We compute the zeroes of f on K . Specifically, we obtain that the only zero of f on $\overset{\circ}{K}$ is when $\Lambda = \frac{1}{\sqrt{n}}(1, \dots, 1)$. The only zeroes of f on ∂K occur at the intersection of K with certain coordinate planes, Z_{n-k+1} , defined below.
- Step 2: We obtain a useful expression for h .
- Step 3: We check that h vanishes to much faster than f at the zeroes of f on ∂K ; that is, if Λ is a zero of f on ∂K , we show that $h = o(f)$ near Λ . We prove this by comparing Taylor expansions.
- Step 4: We show that both h and f vanish to the second order at $\frac{1}{\sqrt{n}}(1, \dots, 1) \in \overset{\circ}{K}$.

Step 1:

As computed in the proof of (3.4),

$$f = \frac{n-k}{k} \sigma_k \sigma_{k-1} \sigma_1 - (k+1) \sigma_{k+1} \sigma_{k-1}$$

and, moreover, we saw that the only zero of f on $\overset{\circ}{K}$ is when $\Lambda = \frac{1}{\sqrt{n}}(1, \dots, 1) = x_0$. So it remains to compute the zeroes of f on ∂K —we show that these are the intersections of K with the coordinate $n - k + 1$ planes.

Now, since Γ_k^+ is the component of $\sigma_k > 0$ containing the set $\{x \in \mathbb{R}^n : x_1, \dots, x_n > 0\}$ (see equation (2.2) in [23]), we have $\sigma_k = 0$ on ∂K . So we have $f = -(k+1)\sigma_{k+1}\sigma_{k-1}$ on ∂K . Hence $f = 0$ on ∂K if and only if $\sigma_{k-1} = 0$ or $\sigma_{k+1} = 0$. Assume for the moment that the first case holds, that is $\sigma_{k+1} = 0$ (we will see shortly that $\sigma_{k-1} = 0$ is a weaker condition under the assumption $\sigma_k = 0$). So, we now check that

$$\begin{aligned} & \{\sigma_{k+1} = 0\} \cap \partial\Gamma_+^k \\ & \subset \{\Lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n : \lambda_{i_1} = \dots = \lambda_{i_{n-k+1}} = 0 \text{ for some } 1 \leq i_1 < \dots < i_{n-k+1} \leq n\} \\ & =: Z_{n-k+1} \end{aligned} \tag{3.6}$$

We in fact prove the more general statement that $\{\sigma_{j+1} = 0\} \cap \{\sigma_j = 0\}$ in $\mathbb{R}^n$ is Z_{n-j+1} (any j).

First note that the following fact: if a non-constant polynomial $p(t)$ has all real roots, then so does its derivative $p'(t)$. In addition, the roots of p' are the multiple roots of p , plus a (necessarily simple, by degree considerations) root between each pair of consecutive, distinct roots of p . So if $t = t_0$ is a root of p' of degree $j \geq 2$, then t_0 is a root of p of degree $j + 1$.

Take $\Lambda = (\lambda_1, \dots, \lambda_n)$ in $\mathbb{R}^n$ with $\sigma_k(\Lambda) = \sigma_{k+1}(\Lambda) = 0$ and consider the polynomial $p_\Lambda(t) := \prod_{i=1}^n (t - \lambda_i) = t^n - \sigma_1(\Lambda)t^{n-1} + \dots + (-1)^n \sigma_n(\Lambda)$. Then $p^{(n-k+1)}(t)$ has a derivative of order two at $t = 0$. Hence p has a zero of order $n - k + 1$ at $t = 0$. That is, $n - k + 1$ of the λ_i are zero. Hence $\Lambda \in Z_{n-k+1}$. This completes the case when $\sigma_{k+1} = 0$.

If instead we have $\sigma_{k-1} = 0$, the same argument shows that $\Lambda \in Z_{n-k+3}$. However, since $Z_{n-k+3} \subset Z_{n-k+1}$, we again must have $\Lambda \in Z_{n-k+1}$. This shows that the only zeroes of f on ∂K occur on Z_{n-k+1} , completing Step 1.

Step 2:

Before we compute the Taylor expansion of h , we first need a more useful expression for it. We

compute

$$\begin{aligned}
h(\Lambda) &= g(\mathring{T}^k, \mathring{T}^k) = g(T^k - \frac{1}{n}Tr(T^k)g, T^k - \frac{1}{n}Tr(T^k)g) \\
&= g(T^k, T^k - \frac{1}{n}Tr(T^k)g) \\
&= g(T^k, T^k) - \frac{1}{n}(Tr(T^k))^2 \\
&= g(T^k, T^k) - \frac{1}{n}(n-k)^2\sigma_k^2
\end{aligned}$$

Interpreting T_k as a map from vectors (of eigenvalues) to vectors (of eigenvalues), recall

$$\begin{aligned}
T_i^k(\Lambda) &= \frac{\partial \sigma_{k+1}(\Lambda)}{\partial \lambda_i} \text{ so that} \\
T_i^k(\Lambda) &= \sigma_k(\check{\Lambda}_i)
\end{aligned}$$

Hence we have

$$\begin{aligned}
g(T^k, T^k) &= \sigma_k(\check{\Lambda}_1)^2 + \cdots + \sigma_k(\check{\Lambda}_n)^2 \quad \text{and so} \\
h(\Lambda) &= g(\mathring{T}^k, \mathring{T}^k) = \sigma_k(\check{\Lambda}_1)^2 + \cdots + \sigma_k(\check{\Lambda}_n)^2 - \frac{(n-k)^2}{n}\sigma_k^2
\end{aligned} \tag{3.7}$$

where $\check{\Lambda}_i$ is Λ with the i -th eigenvalue omitted. This completes step 2.

Step 3:

Now that we have determined the zero set of f on K , we check the boundedness of $\frac{h}{f}$. Here we check the boundedness near ∂K (boundedness on $\mathring{K}$ is proven in Step 4). As we will see, $h(\Lambda) = 0$ at a point $\Lambda \in K$ if $f(\Lambda) = 0$ also. So we will compare the Taylor expansions of h and f .

Consider a point $\Lambda = (\lambda_1, \dots, \lambda_n) \in \partial K$ such that $f(\Lambda) = 0$. Then by Step 1 we know $\Lambda \in Z_{n-k+1}$. As we will see, the behavior of h and f at Λ will be determined by the number of coordinates of Λ which are equal to zero. We will need to keep track of this number. For exactly one j with $1 \leq j \leq k-1$, we have $\Lambda \in (Z_{n-j} - Z_{n-j+1}) \cap K$. This means that exactly j coordinates of Λ are nonzero.

Assume without loss of generality that $\lambda_1 = \cdots = \lambda_{n-j} = 0$. Thus, we will write $\Lambda = (\alpha|\beta) = (\alpha_1, \dots, \alpha_{n-j}, \beta_1, \dots, \beta_j)$ where α is a zero vector and β has no coordinates equal to zero. For a general point $x \in \mathbb{R}^n$, we will write $x = (y|z) = (y_1, \dots, y_{n-j}, z_1, \dots, z_j)$, where any of the coordinates of x may or may not be zero.

Now, since $k > j$,

$$\begin{aligned}
h(\Lambda) &= \sigma_k(\check{\Lambda}_1)^2 + \cdots + \sigma_k(\check{\Lambda}_n)^2 - \frac{(n-k)^2}{n} \sigma_k(\Lambda)^2 \\
&= \sigma_k(\check{\alpha}_1|\beta)^2 + \cdots + \sigma_k(\check{\alpha}_{n-j}|\beta)^2 + \sigma_k(\alpha|\check{\beta}_1)^2 + \cdots + \sigma_k(\alpha|\check{\beta}_j)^2 - \frac{(n-k)^2}{n} \sigma_k(\alpha|\beta)^2 \\
&= (n-j)\sigma_k(0|\beta)^2 + \sigma_k(0|\check{\beta}_1)^2 + \cdots + \sigma_k(0|\check{\beta}_j)^2 - \frac{(n-k)^2}{n} \sigma_k(0|\beta)^2 \\
&= 0
\end{aligned}$$

where $\check{\Lambda}_i$ is Λ with the i -th coordinate removed. So at points in ∂K , if $f(\Lambda) = 0$, then $h(\Lambda) = 0$ as well.

We will compute the Taylor series of f and h at points in ∂K where $f = 0$. The strategy is as follows. First, by homogeneity, $\frac{h}{f}$ will be bounded on a neighborhood in K of a point $\Lambda \in \partial K$ if and only if $\frac{h}{f}$ is bounded on a neighborhood in the tangent plane T to K at Λ . Thus it suffices to compare the Taylor series of the functions f and h restricted to T —that is, restricted to the plane through Λ which is orthogonal to Λ .

To do so, we will use a few simple facts.

1. If p is an integer with $1 \leq p \leq n$, then $\sigma_p(\Lambda) = \sigma_p(\beta)$ is zero for $p > j$, and the largest p for which $\sigma_p(\Lambda) \neq 0$ is $p = j$. Moreover, $\sigma_j(\Lambda) \geq 0$ since $\Lambda \in \partial\Gamma_k^+$, and so since $\sigma_j(\Lambda) = \beta_1 \cdots \beta_j \neq 0$ we have $\sigma_j(\Lambda) > 0$. Also, one can easily check that $\sigma_1(\Lambda) > 0$.
2. Because we will only be interested in the Taylor series restricted to T , we will only take derivatives of f and h in directions orthogonal to Λ . In other words, if we extend the collection coordinate directions $\{y_1, \dots, y_{n-j}\}$ to a basis for the orthogonal complement of Λ ,

$$\{b_1, \dots, b_{n-1}\} = \{y_1, \dots, y_{n-j}, w_1, \dots, w_{j-1}\}$$

then we may compute the Taylor series with respect to this basis. However, note that since both f and h are zero on Z_{n-j} , any derivative of f or h in a direction $w_1, \dots, w_{j-1}$ will necessarily vanish. This is because any linear combination of $w_1, \dots, w_{j-1}$ added to Λ will still lie in Z_{n-j} . Thus to compute the Taylor series for f and h restricted to T , it suffices to compute only the terms with derivatives in the directions $y_1, \dots, y_{n-j}$.

3. If $l = (l_1, \dots, l_{n-j})$ is a multi-index of length $n-j$ with $|l| = m$ and all entries non-negative, let $\partial^l \sigma_p$ denote $\partial_{y_1}^{l_1} \cdots \partial_{y_{n-j}}^{l_{n-j}} \sigma_p$. Then $\partial^l \sigma_p$ evaluated at Λ will be 0 if any entries of l are greater than one or if $m < p-j$.

4. For $m \geq p - j$ and $0 \leq l_1, \dots, l_{n-j} \leq 1$, we have $(\partial^l \sigma_p)(\Lambda) = \sigma_{p-m}(\beta)$.

The last two statements are simple to check because Λ has exactly j entries equal to zero and no variable in σ_p occurs to a power higher than one.

Now, let's compute the Taylor series for f of order $2k - 1 - 2j$. By Fact (3) above, any derivative of $f = \frac{n-k}{k} \sigma_k \sigma_{k-1} \sigma_1 - (k+1) \sigma_{k+1} \sigma_{k-1}$ of order less than $2k - 1 - 2j$ will necessarily vanish at Λ . Moreover, by the same logic, any derivative of $\sigma_{k+1} \sigma_{k-1}$ of order $2k - 1 - 2j$ will necessarily vanish at Λ . Thus to compute the Taylor polynomial for f of order $2k - 1 - 2j$ at Λ , it suffices to compute the Taylor polynomial for $\sigma_k \sigma_{k-1} \sigma_1$ of order $2k - 1 - 2j$ at Λ .

From Facts (2), (3), and (4) above, we see that the Taylor polynomial of order $2k - 1 - 2j$ for f at Λ on T is given by

$$\sum_{|\delta| \leq 2k-1-2j} \frac{(y-\alpha)^\delta}{\delta!} (\partial^\delta f)(\Lambda) = \frac{n-k}{k} \sum_{(\eta, \zeta) \in \Delta_1} \frac{y^{\eta+\zeta}}{(\eta+\zeta)!} \sigma_j(\beta) \sigma_j(\beta) \sigma_1(\beta) \quad (3.8)$$

where δ is a multi-index of length $n - j$ and Δ_1 is the collection of pairs of multi-indices of length $n - j$ such that

$$0 \leq \eta_1, \dots, \eta_{n-j}, \zeta_1, \dots, \zeta_n \leq 1$$

$$|\eta| = k - j$$

$$|\zeta| = k - 1 - j$$

By Fact (1), each coefficient $\sigma_j(\beta) \sigma_j(\beta) \sigma_1(\beta)$ in this sum is positive and hence each term in this sum is positive on a neighborhood of Λ in Γ_k^+ . To show that $\frac{h}{f}$ is bounded, we show that every term in the Taylor series for h is a term in (3.8) multiplied by something $o(|y|)$. This implies that $h = f \cdot o(|y|)$ and hence $\frac{h}{f}$ is bounded near Λ .

To this end, we can argue as for f that the Taylor series of h vanishes up to order $2k - 2j$. Hence by Facts (2), (3), and (4) we know that every term in the Taylor series of h is a constant multiple of a term of the form

$$\begin{aligned} \sigma_{k-|\eta|}(\beta) \sigma_{k-|\zeta|}(\beta) \frac{y^{\eta+\zeta}}{(\eta+\zeta)!}, & \quad (\eta, \zeta) \in \Delta_2, \text{ or} \\ \sigma_{k-|\eta|}(\check{\beta}_i) \sigma_{k-|\zeta|}(\check{\beta}_i) \frac{y^{\eta+\zeta}}{(\eta+\zeta)!}, & \quad (\eta, \zeta) \in \Delta_2 \end{aligned}$$

where Δ_2 is the collection of all pairs of multi-indices of length $n - j$ such that

$$\begin{aligned} 0 &\leq \eta_1, \dots, \eta_{n-j}, \zeta_1, \dots, \zeta_n \leq 1 \\ |\eta|, |\zeta| &\geq k - j \end{aligned}$$

However, comparing such terms with terms in (3.8), we see that every term τ_h in the Taylor series for h is a multiple of a term τ_f in the Taylor polynomial of order $2k - 1 - 2j$ for f , with $|\tau_h| \leq o(|y|)\tau_f$. Hence $\frac{h}{f}$ vanishes at Λ , and so $\frac{h}{f}$ is bounded on ∂K as Λ was arbitrary.

Step 4:

Finally, it remains only to address the zero of $\frac{h}{f}$ on $\overset{\circ}{K}$. That is, let $\Lambda = \frac{1}{\sqrt{n}}(1, \dots, 1)$. We compute the second order Taylor expansions of h and f at Λ . We then have, for h :

$$\begin{aligned} h(\Lambda) &= \sum_{l=1}^n (\sigma_k(\check{\Lambda}_l))^2 - \frac{(n-k)^2}{n} \sigma_k(\Lambda)^2 \\ &= \frac{1}{n^{k/2}} \left(n \binom{n-1}{k}^2 - \frac{(n-k)^2}{n} \binom{n}{k}^2 \right) \\ &= 0 \end{aligned}$$

$$\begin{aligned} \nabla_i h(\Lambda) &= \sum_{l=1, l \neq i}^n 2\sigma_k(\check{\Lambda}_l) \sigma_{k-1}(\check{\Lambda}_{il}) - \frac{(n-k)^2}{n} 2\sigma_k(\Lambda) \sigma_{k-1}(\check{\Lambda}_i) \\ &= \frac{1}{n^{\frac{k-1}{2}}} \left((n-1) 2 \binom{n-1}{k} \binom{n-2}{k-1} - \frac{(n-k)^2}{n} 2 \binom{n}{k} \binom{n-1}{k-1} \right) \\ &= 0 \end{aligned}$$

$$\begin{aligned} \nabla_{ii}^2 h(\Lambda) &= \sum_{l=1, l \neq i}^n 2(\sigma_{k-1}(\check{\Lambda}_{il}) \sigma_{k-1}(\check{\Lambda}_{il})) - \frac{(n-k)^2}{n} 2\sigma_{k-1}^2(\check{\Lambda}_i) \\ &= \frac{1}{n^{\frac{k-2}{2}}} \left(2(n-1) \binom{n-2}{k-2}^2 - 2 \frac{(n-k)^2}{n} \binom{n-1}{k-1}^2 \right) \\ &= \frac{1}{n^{\frac{k-2}{2}}} 2(n-k)^2 \binom{n-1}{k-1}^2 \frac{1}{n(n-1)} \end{aligned}$$

$$\begin{aligned}
\nabla_{ij}^2 h(\Lambda) &= \sum_{l=1, l \neq i, j}^n 2(\sigma_{k-1}(\check{\Lambda}_{jl})\sigma_{k-1}(\check{\Lambda}_{il}) + \sigma_k(\check{\Lambda}_l)\sigma_{k-2}(\check{\Lambda}_{ijl})) \\
&\quad - \frac{(n-k)^2}{n} 2(\sigma_{k-1}(\check{\Lambda}_j)\sigma_{k-1}(\check{\Lambda}_i) + \sigma_k(\Lambda)\sigma_{k-2}(\check{\Lambda}_{ij})) \quad \text{for } i \neq j \\
&= \frac{1}{n^{\frac{k-2}{2}}} ((n-2) \binom{n-2}{k-1}^2 + \binom{n-1}{k} \binom{n-3}{k-2}) \\
&\quad - \frac{(n-k)^2}{n} (\binom{n-1}{k-1}^2 + \binom{n}{k} \binom{n-2}{k-2}) \\
&= -\frac{1}{n^{(k-2)/2}} 2(n-k)^2 \binom{n-1}{k-1}^2 \frac{1}{n(n-1)^2}
\end{aligned}$$

For f we have

$$\begin{aligned}
f(\Lambda) &= \frac{n-k}{n} \sigma_k(\Lambda)\sigma_{k-1}(\Lambda)\sigma_1(\Lambda) - (k+1)\sigma_{k-1}(\Lambda)\sigma_{k+1}(\Lambda) \\
&= \frac{1}{n^{k/2}} (\frac{n-k}{n} \binom{n}{k} \binom{n}{k-1} n - (k+1) \binom{n}{k-1} \binom{n}{k+1}) \\
&= 0
\end{aligned}$$

$$\begin{aligned}
\nabla_i f(\Lambda) &= \frac{1}{n^{(k-1)/2}} (\frac{n-q}{n} (\sigma_{k-1}(\check{\Lambda}_i)\sigma_{k-1}(\Lambda)\sigma_1(\Lambda) + \sigma_k(\Lambda)\sigma_{k-2}(\Lambda)\sigma_1(\Lambda) + \sigma_q(\Lambda)\sigma_{k-1}(\Lambda)) \\
&\quad - (k+1)(\sigma_{k-2}(\check{\Lambda}_i\sigma_{k+1}(\Lambda) + \sigma_{k-1}(\Lambda)\sigma_k(\check{\Lambda}_i))) \\
&= \frac{1}{n^{(k-1)/2}} (\frac{n-k}{n} (\binom{n-1}{k-1} \binom{n}{k-1} n + \binom{n}{k} \binom{n-1}{k-2} n + \binom{n}{k} \binom{n}{k-1}) \\
&\quad - (k+1)(\binom{n-1}{k-2} \binom{n}{k+1} + \binom{n}{k-1} \binom{n-1}{k})) \\
&= 0
\end{aligned}$$

$$\begin{aligned}
\nabla_{ii}^2 f(\Lambda) &= \frac{n-k}{n} (\sigma_{k-1}(\check{\Lambda}_i) (\sigma_{k-2}(\check{\Lambda}_i) \sigma_1(\Lambda) + \sigma_{k-1}(\Lambda)) + \sigma_{k-2}(\check{\Lambda}_i) (\sigma_{k-1}(\check{\Lambda}_i) \sigma_1(\Lambda) + \sigma_k(\Lambda)) \\
&\quad + \sigma_{k-1}(\check{\Lambda}_i) \sigma_{k-1}(\Lambda) + \sigma_k(\Lambda) \sigma_{k-2}(\check{\Lambda}_i)) - (q+1) (\sigma_{k-2}(\check{\Lambda}_i) \sigma_k(\check{\Lambda}_i) + \sigma_{k-2}(\check{\Lambda}_i) \sigma_k(\check{\Lambda}_i)) \\
&= \frac{1}{n^{(k-2)/2}} 2 \left(\frac{n-q}{n} \left(n \binom{n-1}{k-1} \binom{n-1}{k-2} + \binom{n}{k-1} \binom{n-1}{k-1} \right) \right. \\
&\quad \left. + \binom{n}{k} \binom{n-1}{k-2} \right) - (k+1) \binom{n-1}{k} \binom{n-1}{k-2} \\
&= \frac{1}{n^{(k-2)/2}} 2 \binom{n-1}{k-1} \binom{n-1}{k} \frac{k}{n-k+1}
\end{aligned}$$

$$\begin{aligned}
\nabla_{ij}^2 f(\Lambda) &= \frac{n-k}{n} (\sigma_{k-2}(\check{\Lambda}_{ij}) \sigma_{k-1}(\Lambda) \sigma_1(\Lambda) + \sigma_{k-1}(\check{\Lambda}_i) \sigma_{k-2}(\check{\Lambda}_j) \sigma_1(\Lambda) + \sigma_{k-1}(\check{\Lambda}_i) \sigma_{k-1}(\Lambda) \\
&\quad + \sigma_{k-1}(\check{\Lambda}_j) \sigma_{k-2}(\check{\Lambda}_i) \sigma_1(\Lambda) + \sigma_k(\Lambda) \sigma_{k-3}(\check{\Lambda}_{ij}) \sigma_1(\Lambda) + \sigma_k(\Lambda) \sigma_{k-2}(\check{\Lambda}_i) \\
&\quad + \sigma_{k-1}(\check{\Lambda}_j) \sigma_{k-1}(\Lambda) + \sigma_k(\Lambda) \sigma_{k-2}(\check{\Lambda}_{ij})) - (k+1) (\sigma_{k-3}(\check{\Lambda}_{ij}) \sigma_{k+1}(\Lambda) \\
&\quad + \sigma_{k-2}(\check{\Lambda}_i) \sigma_k(\check{\Lambda}_j) + \sigma_{k-2}(\check{\Lambda}_j) \sigma_k(\check{\Lambda}_i) + \sigma_{k-1}(\Lambda) \sigma_{k-1}(\check{\Lambda}_{ij})) \quad \text{for } i \neq j \\
&= \frac{1}{n^{(k-2)/2}} \left(\frac{n-k}{n} \left(\binom{n-2}{k-2} \binom{n}{k-1} n + \binom{n-1}{k-1} \binom{n-1}{k-2} n \right. \right. \\
&\quad \left. \left. + \binom{n-1}{k-1} \binom{n}{k-1} + \binom{n-1}{k-1} \binom{n-1}{k-2} n + \binom{n}{k} \binom{n-2}{k-3} n \right. \right. \\
&\quad \left. \left. + \binom{n}{k} \binom{n-1}{k-2} + \binom{n-1}{k-1} \binom{n}{k-1} + \binom{n}{k} \binom{n-1}{k-2} \right) \right. \\
&\quad \left. - (k+1) \left(\binom{n-2}{k-3} \binom{n}{k+1} + \binom{n-1}{k-2} \binom{n-1}{k} \right) \right. \\
&\quad \left. + \binom{n-1}{k-2} \binom{n-1}{k} + \binom{n}{k-1} \binom{n-2}{k-2} \right) \\
&= \frac{1}{n^{(k-2)/2}} 2 \binom{n}{k}^2 \frac{n-k}{n^2(n-1)} \frac{k^2}{n-k+1}
\end{aligned}$$

Summarizing, we have that

$$0 = h(\Lambda) = f(\Lambda)$$

$$0 = \nabla_i h(\Lambda) = \nabla_i f(\Lambda)$$

$$\frac{1}{n^{(k-2)/2}} \frac{2(n-k)^2}{n(n-1)^2} \binom{n-1}{k-1}^2 (n-1)^{\delta_{ij}} = \nabla_{ij}^2 h(\Lambda) = \frac{(n-q)(n-q+1)}{n(n-1)} \nabla_{ij}^2 f(\Lambda)$$

Hence $\frac{h}{f}$ is well defined at Λ . This completes the argument that $\frac{h}{f}$ is bounded on $\mathring{K}$, finishing step 4.

Combining all the steps, we see that h/f is bounded on K and our proof is complete. $\square$

3.3 Expansion of Integrals of σ_k

Next we will show that, in some sense, we can “reduce” integrals of σ_k and v to integrals in σ_s ($s < k$), v , ∇v , and “tail terms” involving $\nabla \eta$.

Proposition 3.4. *Let η be as above. Then*

$$\begin{aligned} k \int \sigma_k v^{-\gamma} \eta dx &= \sum_{s=1}^k a_{k-s} \int \sigma_{k-s} |\nabla v|^{2s} v^{-\gamma} \eta dx \\ &\quad + \sum_{s=1}^k b_{k-s} \int T^{k-s}(\nabla v, \nabla \eta) v^{1-\gamma} |\nabla v|^{2(s-1)} dx \end{aligned} \quad (3.9)$$

where

$$\begin{aligned} a_{k-s} &= \frac{\gamma k - nk + \gamma s - 2ks}{s2^s} \binom{n - \gamma + s - 1}{s - 1} \\ b_{k-1} &= -1 \\ b_{k-s} &= \frac{1}{2^{s-1}} \binom{n - \gamma + s - 1}{s - 1} \end{aligned}$$

Remark: Note that when $n = \gamma$, we have $a_{k-s} = \frac{n-2k}{2^s}$, and for $s \geq 2$, $b_{k-s} = \frac{1}{2^{s-1}}$. In particular, both are positive when $2k < n$. When $\gamma = n - 1$, $2k = n$, we have $a_{k-s} = \frac{-k-s}{2^s} < 0$ for all s .

We will need a few lemmas before we can prove the above proposition.

Lemma 3.5.

$$k\sigma_k(A_g) = v \sum_{i,j} \partial_j (\nabla_i v T_{ij}^{k-1}) - nT^{k-1}(\nabla v, \nabla v) + \frac{n-k+1}{2} \sigma_{k-1} |\nabla v|^2 \quad (3.10)$$

Proof. See equation (3.8) in [14] $\square$

The following lemma is similar to Lemma 3.6 in [14]. However, in [14] the integration is taken over an open set U , whereas in our computation we will be integrating of $\mathbb{R}^n$ and the integrand will contain a cutoff function. Hence we reproduce the argument here.

Lemma 3.6. *For a given $\rho > 1$, let $\eta \in C^\infty(\mathbb{R}^n)$ satisfy $\eta \equiv 1$ on $B(\rho)$, $\text{supp } \eta \subset B(2\rho)$, and $\nabla \eta \lesssim \rho^{-1}$. For $s \in \mathbb{N}$, we have*

$$\begin{aligned}
& \int \sum_{i,j} T^{k-s}(\nabla v, \nabla v) |\nabla v|^{2(s-1)} v^{-\gamma} \eta dx \\
&= (1 + \frac{k-s}{2s}) \int \sigma_{k-s} |\nabla v|^{2s} v^{-\gamma} \eta dx + \frac{s+n+1-\gamma}{2s} \int T^{k-s-1}(\nabla v, \nabla v) |\nabla v|^{2s} v^{-\gamma} \eta dx \\
&\quad - \frac{n-k+s+1}{4s} \int \sigma_{k-s-1} |\nabla v|^{2(s+1)} v^{-\gamma} \eta dx - \frac{1}{2s} \int T^{k-s-1}(\nabla v, \nabla \eta) |\nabla v|^{2s} v^{1-\gamma} dx
\end{aligned} \tag{3.11}$$

Proof. Recall that when $g = v^{-2}|dx|^2$, we have $g^{-1}A_g = v\nabla_{ij}^2 v - \frac{1}{2}|\nabla v|^2\delta_{ij}$ and $T_{ij}^{k-s} = \sigma_{k-s}g_{ij} - \sum_l A_{il}T_{lj}^{k-s-1}$. We then compute

$$\begin{aligned}
& \int T^{k-s}(\nabla v, \nabla v) |\nabla v|^{2(s-1)} v^{-\gamma} \eta dx \\
&= \int \sigma_{k-s} |\nabla v|^{2s} v^{-\gamma} \eta dx - \int \langle T^{k-s-1}, A_g \cdot \nabla v \otimes \nabla v \rangle_g |\nabla v|^{2(s-1)} v^{-\gamma} \eta dx \\
&= \int \sigma_{k-s} |\nabla v|^{2s} v^{-\gamma} \eta dx - \int \langle T^{k-s-1}, \nabla^2 v \cdot \nabla v \otimes \nabla v \rangle_g |\nabla v|^{2(s-1)} v^{1-\gamma} \eta dx \\
&\quad + \frac{1}{2} \int T^{k-s-1}(\nabla v, \nabla v) |\nabla v|^{2s} v^{-\gamma} \eta dx
\end{aligned} \tag{3.12}$$

Next we integrate by parts the second term in the above expression:

$$\begin{aligned}
& - \int \langle T^{k-s-1}, \nabla^2 v \cdot \nabla v \otimes \nabla v \rangle_g |\nabla v|^{2(s-1)} v^{1-\gamma} \eta dx \\
&= - \frac{1}{2s} \int T^{k-s-1}(\nabla(|\nabla v|^{2s}), \nabla v) v^{1-\gamma} \eta dx \\
&= \frac{1-\gamma}{2s} \int T^{k-s-1}(\nabla v, \nabla v) |\nabla v|^{2s} v^{-\gamma} \eta dx + \frac{1}{2s} \int \sum_{j,l} \partial_l (\nabla_j v T_{lj}^{k-s-1}) |\nabla v|^{2s} v^{1-\gamma} \eta dx \\
&\quad - \frac{1}{2s} \int T^{k-s-1}(\nabla v, \nabla \eta) |\nabla v|^{2s} v^{1-\gamma} dx
\end{aligned}$$

Of course, the boundary term vanishes because of the cutoff function. Plugging in (3.10), we have

$$\begin{aligned}
& \int T^{k-s-1}(\nabla v, \nabla v) |\nabla v|^{2(s-1)} v^{1-\gamma} \eta dx \\
&= \frac{1+n-\gamma}{2s} \int T^{k-s-1}(\nabla v, \nabla v) |\nabla v|^{2s} v^{-\gamma} \eta dx + \frac{k-s}{2s} \int \sigma_{k-s} |\nabla v|^{2s} v^{-\gamma} \eta dx \\
&\quad - \frac{n-k+s+1}{4s} \int \sigma_{k-s-1} |\nabla v|^{2(s+1)} v^{-\gamma} \eta dx - \frac{1}{2s} \int T^{k-s-1}(\nabla v, \nabla \eta) |\nabla v|^{2s} v^{1-\gamma} ds
\end{aligned} \tag{3.13}$$

Now, using (3.13) in (3.12), we are finished. $\square$

Finally, we are ready to prove the main proposition of this section. We imitate the proof of Proposition 5.5 in [14].

Proof. of Proposition 3.4. Let $\gamma \in \mathbb{R}$. Integrate (3.10) over the annulus A_ρ , and we have

$$\begin{aligned}
& k \int \sigma_k v^{-\gamma} \eta dx \\
&= \int \sum_{i,j} \nabla_j (\nabla_i T_{ij}^{k-1}) v^{1-\gamma} \eta dx - n \int T_{ij}^{k-1} (\nabla v, \nabla v) v^{-\gamma} \eta dx \\
&\quad + \frac{n-k+1}{2} \int \sigma_{k-1} |\nabla v|^2 v^{-\gamma} \eta dx \\
&= (-1 + \gamma - n) \int T^{k-1} (\nabla v, \nabla v) v^{-\gamma} \eta dx + \frac{n-k+1}{2} \int \sigma_{k-1} |\nabla v|^2 v^{-\gamma} \eta dx \\
&\quad - \int T^{k-1} (\nabla v, \nabla \eta) v^{1-\gamma} dx
\end{aligned}$$

Following the notation of [14], we write

$$\begin{aligned}
\mathcal{A}_{k-s} &= - \frac{s+n-\gamma}{s} \int T^{k-s} (\nabla v, \nabla v) |\nabla v|^{2(s-1)} v^{-\gamma} \eta dx \\
&\quad + \frac{n-k+s}{2s} \int \sigma_{k-s} |\nabla v|^{2s} v^{-\gamma} \eta dx
\end{aligned}$$

Using (3.11) in the expression for $\mathcal{A}_{k-s}$, we have

$$\begin{aligned}
\mathcal{A}_{k-s} &= \left(\frac{n-k+s}{2s} - \frac{s+n-\gamma}{s} \frac{k+s}{2s} \right) \int \sigma_{k-s} |\nabla v|^{2s} v^{-\gamma} \eta dx \\
&\quad + \frac{s+n-\gamma}{s} \frac{s+1}{2s} \left(- \frac{s+n-\gamma+1}{s+1} \int T^{k-s-1} (\nabla v, \nabla v) |\nabla v|^{2s} v^{-\gamma} \eta dx \right) \\
&\quad + \frac{s+n-\gamma}{s} \frac{s+1}{2s} \left(\frac{n-k+s+1}{2(s+1)} \int \sigma_{k-s-1} |\nabla v|^{2(s+1)} v^{-\gamma} \eta dx \right) \\
&\quad + \frac{s+n-\gamma}{s} \frac{1}{2s} \int T^{k-s-1} (\nabla v, \nabla \eta) |\nabla v|^{2s} v^{1-\gamma} dx \\
&= \frac{\gamma k - nk + \gamma s - 2ks}{2s^2} \int \sigma_{k-s} |\nabla v|^{2s} v^{-\gamma} \eta dx + \frac{(n-\gamma+s)(s+1)}{2s^2} \mathcal{A}_{k-s-1} \\
&\quad + \frac{n-\gamma+s}{2s^2} \int T^{k-s-1} (\nabla v, \nabla \eta) |\nabla v|^{2s} v^{1-\gamma} dx
\end{aligned}$$

Thus we see that

$$\begin{aligned}
k \int \sigma_k v^{-\gamma} \eta dx &= \sum_{s=1}^k a_{k-s} \int \sigma_{k-s} |\nabla v|^{2s} v^{-\gamma} \eta dx \\
&\quad + \sum_{s=1}^k b_{k-s} \int T^{k-s} (\nabla v, \nabla \eta) |\nabla v|^{2(s-1)} v^{1-\gamma} dx
\end{aligned}$$

where

$$\begin{aligned}
a_{k-s} &= \frac{\gamma k - nk + \gamma s - 2ks}{2s^2} \prod_{j=1}^{s-1} \frac{(n - \gamma + j)(j + 1)}{2j^2} \\
&= \frac{\gamma k - nk + \gamma s - 2ks}{2^s s} \prod_{j=1}^{s-1} \frac{n - \gamma + j}{j} \\
&= \frac{\gamma k - nk + \gamma s - 2ks}{2^s s} \binom{n - \gamma + s - 1}{s - 1}
\end{aligned}$$

and

$$\begin{aligned}
b_{k-1} &= -1 \\
b_{k-s} &= \frac{n - \gamma + s - 1}{2(s - 1)^2} \prod_{j=3}^s \frac{(n - \gamma + j - 2)(j - 1)}{2(j - 2)^2} \\
&= \frac{(s - 1)!(n - \gamma + s - 1)!}{2^{s-1}((s - 1)!)^2(n - \gamma)!} \\
&= \frac{1}{2^{s-1}} \binom{n - \gamma + s - 1}{s - 1} \quad (s \geq 2)
\end{aligned}$$

as desired (in the above products we use the convention that the product over an empty set has value 1). $\square$

3.4 Tail Term Estimate

One fact we will need throughout this section is the following: for $A_g \in \Gamma_k$, $s < k$, (so that T^s is positive definite),

$$\begin{aligned}
|T^s| &= \sqrt{|T^s|^2} \\
&= \sqrt{\text{Tr}(T^s \cdot T^s)} \\
&= \sqrt{\text{Tr}(T^s)^2 - 2\sigma_2(T^s)} \\
&\leq \text{Tr}(T^s) \\
&= (n - s)\sigma_s
\end{aligned} \tag{3.14}$$

where we have used $\sigma_2(T^s) \geq 0$ because T^s is positive definite.

Before we can give the main result of this section, we will first need a technical lemma.

Lemma 3.7. *Let $C_0, \dots, C_m$ be given and suppose $m < k$, $2k < n$, $1 \leq s \leq k$, and $j < 2(k - m)$.*

Then there exists a constant $\overline{C}(C_0, \dots, C_m, m, j, k, n)$ such that

$$\begin{aligned} & \rho^{-j} \int \sigma_m |\nabla v|^{2k-j-2m} v^{j-n} \eta dx \\ & \leq \sum_{l=0}^m C_{m-l} \int \sigma_{m-l} |\nabla v|^{2(k-m+l)} v^{-n} \eta dx + \overline{C} \left(\int v^{-n} \eta dx \right)^{\frac{n-2k}{n}} \end{aligned} \quad (3.15)$$

In particular, $C_0, \dots, C_m$ can be taken as small as desired, with the tradeoff that $\overline{C}$ will become very large.

Proof. We induct on m .

Base case: suppose $m = 0$. By definition $\sigma_0 = 1$, so by Holder's and Young's inequaltities we have

$$\begin{aligned} & \rho^{-j} \int |\nabla v|^{2k-j} v^{j-n} \eta dx \\ & \leq \left(\int |\nabla v|^{2k} v^{-n} \eta dx \right)^{\frac{2k-j}{2k}} (\rho^{-2k} \int v^{2k-n} dx)^{\frac{j}{2k}} \\ & \leq \epsilon_1 \int |\nabla v|^{2k} v^{-n} \eta dx + C_{\epsilon_1} \rho^{-2k} \int v^{2k-n} \eta dx \\ & \leq \epsilon_1 \int |\nabla v|^{2k} v^{-n} \eta dx + C_{\epsilon_1} \rho^{-2k} \left(\int v^{-n} \eta dx \right)^{\frac{n-2k}{n}} \left(\int \eta dx \right)^{\frac{2k}{n}} \\ & \leq \epsilon_1 \int |\nabla v|^{2k} v^{-n} \eta dx + C_{\epsilon_1} \left(\int v^{-n} \eta dx \right)^{\frac{n-2k}{n}} \end{aligned}$$

where $\epsilon_1(j, k)$ and $C_{\epsilon_1}(j, k)$ are to be determined. So, if we take $\epsilon_1 < \frac{C_0}{2}$, the proof is complete in the base case.

Induction hypothesis: suppose (3.15) holds for $0 \leq m \leq m_0 \leq k-1$, m_0 fixed.

Induction step: Let $m = m_0$, and compute:

$$\begin{aligned} & \rho^{-j} \int \sigma_{m+1} |\nabla v|^{2k-j-2m-2} v^{j-n} \eta dx \\ & \leq \left(\int \sigma_{m+1} |\nabla v|^{2k-2m-2} v^{-n} \eta dx \right)^{\frac{2k-j-2m-2}{2k-2m-2}} \\ & \quad \cdot (\rho^{-2k+2m+2} \int \sigma_{m+1} v^{2k-2m-2-n} \eta dx)^{\frac{j}{2k-2m-2}} \\ & \leq \epsilon_m \int \sigma_{m+1} |\nabla v|^{2k-2m-2} v^{-n} \eta dx + C_{\epsilon_m} \rho^{-2k+2m+2} \int \sigma_{m+1} v^{2k-2m-2-n} \eta dx \end{aligned}$$

where $\epsilon_m(m, j, k, n)$ and $C_{\epsilon_m}(m, j, k, n)$ are to be determined. We now apply (3.9) and (3.14) to the

second term on the last line:

$$\begin{aligned}
& \rho^{-2k+2m+2} \int \sigma_{m+1} v^{2k-2m-2-n} \eta dx \\
& \lesssim \rho^{-2k+2m+2} \left(\sum_{s=1}^{m+1} \int \sigma_{m+1-s} |\nabla v|^{2s} v^{2k-2m-2-n} \eta dx \right. \\
& \quad \left. + \rho^{-1} \sum_{s=1}^{m+1} \int \sigma_{m+1-s} |\nabla v|^{2s-1} v^{2k-2m-1-n} \eta dx \right)
\end{aligned}$$

and by strong induction, we are finished. $\square$

We now give the main result of this section.

Proposition 3.8. *Suppose $g = v^{-2}|dx|^2$ is a conformal metric on $\mathbb{R}^n$. If $g^{-1}A_g \in \Gamma_k^+$, $\sigma_k(g^{-1}A_g) = \alpha$ is constant, $2k < n$, and $\int v^{-n} dx = \beta < \infty$, then there exists a constant $C(\alpha, \beta, n)$ so that*

$$\int_{A_\rho} \sigma_{k-1} |\nabla v|^2 v^{1-n} dx \leq C \rho^2, \quad \rho \gg 1 \tag{3.16}$$

where $A(\rho) = \{x \in \mathbb{R}^n : \rho \leq |x| \leq 2\rho\}$.

Proof. We will show that $I(\rho) := \int_{A_\rho} \sigma_{k-1} |\nabla v|^2 v^{-n} dx$ is bounded independent of ρ (note the different power of v in $I(\rho)$ compared to (3.16)) is bounded independent of ρ . By Lemma 1.3 in [8], $v(x) \leq C|x|^2$, so this will suffice.

To prove the proposition we will use expansion (3.9) with $\gamma = n$. In this case, the coefficients of all the “main” terms (that is, those not involving derivatives of η) on the right hand side are positive. The left side of (3.9) is bounded above (and below) from the assumption of finite volume and $\sigma_k = \text{constant}$. Also, $I(\rho)$ is a main term on the right hand side. Thus, if we can show that the “tail” terms (those involving derivatives of η) on the right side of (3.9) are bounded in magnitude by the main terms, then by the boundedness of the left side of (3.9) and sign considerations, $I(\rho)$ must also be bounded (we will see this in more detail below).

Now, consider a tail term of the form

$$\begin{aligned}
\tau_{k-s} &:= \int T^{k-s}(\nabla v, \nabla \eta) |\nabla v|^{2(s-1)} v^{-n+1} dx \\
&\lesssim \rho^{-1} \int \sigma_{k-s} |\nabla v|^{2s-1} v^{-n+1} dx
\end{aligned}$$

where we have again used (3.14). This last term is of the form in (3.15) with $m = k - s$ and $j = 1$, so if we choose the constants $C_0, \dots, C_{m-l}$ in the statement of Lemma 3.7 to be sufficiently small,

we obtain

$$\begin{aligned}
\text{constant} &= k \int \sigma_k v^{-n} \eta dx \\
&= \sum_{s=1}^k a_{k-s} \int \sigma_{k-s} |\nabla v|^{2s} v^{-\gamma} \eta dx \\
&\quad + \sum_{s=1}^k b_{k-s} \int T^{k-s}(\nabla v, \nabla \eta) |\nabla v|^{2(s-1)} v^{1-\gamma} dx \\
&\geq \sum_{s=1}^k \frac{a_{k-s}}{2} \int \sigma_{k-s} |\nabla v|^{2s} v^{-\gamma} \eta dx - C \left(\int v^{-n} \eta dx \right)^{\frac{n-2k}{n}}
\end{aligned}$$

The last term on the right is finite, and $a_{k-s} > 0$ for $\gamma = n$. As $I(\rho)$ is a term on the right hand side, is bounded independent of ρ . As mentioned above, $v(x) \leq C|x|^2$, so the proposition holds. $\square$

3.5 Tail Term Estimate, $n = 2k$

We have the following analogue of Lemma 3.7 in the case $n = 2k$.

Lemma 3.9. *Let $C_0, \dots, C_m$ be given and suppose $m < k$, $2k = n$, $1 \leq s \leq k$, $j < n - 2m$. Then there exists a constant $\overline{C}(C_0, \dots, C_m, m, j, n)$ such that*

$$\begin{aligned}
&\rho^{-j} \int \sigma_m |\nabla v|^{n-j-2m} v^{j+1-n} \eta dx \\
&\leq \sum_{l=0}^m C_{m-l} \int \sigma_{m-l} |\nabla v|^{2(k-m+l)} v^{1-n} \eta dx + \overline{C} \rho^2
\end{aligned}$$

In particular, $C_0, \dots, C_m$ may be taken as small as desired, with the tradeoff that $\overline{C}$ will become very large.

Proof. Again, we induct on m . In the base case, we have

$$\begin{aligned}
&\rho^{-j} \int |\nabla v|^{n-j} v^{j+1-n} \eta dx \\
&\leq \left(\int |\nabla v|^n v^{1-n} \eta dx \right)^{n-j} n (\rho^{-n} \int v \eta dx)^{\frac{j}{n}} \\
&\leq \epsilon_1 \int |\nabla v|^n v^{1-n} \eta dx + C_{\epsilon_1} \rho^{-n} \rho^{-2} \int \eta dx \\
&\leq \epsilon_1 \int |\nabla v|^n v^{1-n} \eta dx + C_{\epsilon_1} \rho^2
\end{aligned}$$

so, as before, if we take $\epsilon_1 < C_0/2$, the base case is finished.

The induction step is identical to the proof of Lemma 3.7 except that for each v^{power} appearing in an integrand, $power$ is increased by one. $\square$

We can now extend Proposition 3.8 to the $n = 2k$ case.

Proposition 3.10. *Suppose $g = v^{-2}|dx|^2$ is a conformal metric on $\mathbb{R}^n$. If $g^{-1}A_g \in \Gamma_k^+$, $\sigma_k(g^{-1}A_g) = \alpha$ is a constant, $2k = n$, and $\int_{\mathbb{R}^n} v^{-n} dx = \beta < \infty$, then there exists a constant $C(\alpha, \beta, n)$ so that*

$$\int_{A_\rho} \sigma_{k-1} |\nabla v|^2 v^{1-n} dx \leq C\rho^2, \quad \rho \gg 1 \quad (3.17)$$

Proof. Take $\gamma = n - 1$ in Proposition 3.4. Then from the remark after the statement of the proposition, we see there exist positive constants $D_0, \dots, D_k$ and constants $d_0, \dots, d_{k-1}$ such that

$$\begin{aligned} 0 &\leq D_k \int \sigma_k v^{1-n} \eta dx + D_{k-1} \int \sigma_{k-1} |\nabla v|^2 v^{1-n} \eta dx + \dots + D_0 \int |\nabla v|^{2k} v^{1-n} \eta dx \\ &\leq \rho^{-1} d_{k-1} \int \sigma_{k-1} |\nabla v| v^{2-n} \eta dx + \dots + \rho^{-1} d_0 \int |\nabla v|^{2k-1} v^{2-n} \eta dx \end{aligned}$$

Notice that each term on the first line is positive. By Lemma 3.9, if we take $C_0, \dots, C_{k-1}$ sufficiently small, we then have

$$\begin{aligned} 0 &\leq D_k \int \sigma_k v^{1-n} \eta dx + \frac{D_{k-1}}{2} \int \sigma_{k-1} |\nabla v|^2 v^{1-n} \eta dx + \dots + \frac{D_0}{2} \int |\nabla v|^{2k} v^{1-n} \eta dx \\ &\leq \bar{C} \rho^2 \end{aligned}$$

and we are finished. $\square$

3.6 Proof of Theorem

Proof. of Theorem 3.1 We imitate the proof of Theorem 0.1 in [8]. Except where explicitly indicated, the computations in this section are with respect to $g = v^{-2}|dx|^2$, as are all derivatives and norms. Let $\rho > 1$ and let η be a cut-off function supported in $B(2\rho)$ satisfying $\eta \equiv 1$ on $B(\rho)$, $|\partial_i \eta|_{\mathbb{R}^n} \lesssim \rho^{-1}$. We have the following expression for the trace-free Ricci tensor E of g in terms of v :

$$E = -(n-2)v\nabla_g^2(v^{-1}) + \frac{n-2}{n}v\Delta_g(v^{-1})g \quad (3.18)$$

We take the inner product of both sides with $v^{-1}\eta^2\hat{T}^k$ and integrate over $\mathbb{R}^n$. We will use the fact

that $\dot{T}^k$ is trace free and also the divergence theorem.

$$\begin{aligned}\int -g(\dot{T}^k, E)v^{-1}\eta^2 dv_g &= \int (n-2)g(\dot{T}^k, \nabla_g^2(v^{-1})\eta^2) dv_g \\ &= - \int (n-2)g(\delta\dot{T}^k, \nabla(v^{-1}))\eta^2 dv_g \\ &\quad - \int (n-2)\dot{T}^k(\nabla_g(v^{-1}), \nabla_g(\eta^2)) dv_g\end{aligned}$$

Since $\sigma_k(g^{-1}A_g)$ is constant, $\dot{T}^k$ is divergence free (Proposition 3.3, [29]). Thus the first term on the right side vanishes. Now,

$$\begin{aligned}\int -g(\dot{T}^k, E)v^{-1}\eta dv_g &= - \int (n-2)\dot{T}^k(\nabla_g(v^{-1}), \nabla_g(\eta^2)) dv_g \\ &\lesssim \int |\dot{T}^k| |\nabla_g(v^{-1})| |\nabla_g(\eta^2)| dv_g \\ &\lesssim \int |\dot{T}^k| |\nabla_g v| |\nabla_g \eta| v^{-2} dv_g \\ &\lesssim \int \sigma_{k-1}^{1/2} |g(\dot{T}^k, E)|^{1/2} |\nabla_g v| |\nabla_g \eta| v^{-2} \eta dv_g\end{aligned}$$

In the last step we have used (3.5) (note $E = -\dot{T}^1$). Now, by the Schwartz Inequality,

$$\begin{aligned}\int -g(\dot{T}^k, E)v^{-1}\eta dv_g &\lesssim \left(\int_{\text{supp}\nabla\eta} |g(\dot{T}^k, E)| v^{-1} \eta^2 dv_g \right)^{1/2} \\ &\quad \times \left(\int \sigma_{k-1} |\nabla_g v|^2 |\nabla_g \eta|^2 v^{-3} dv_g \right)^{1/2}\end{aligned}\tag{3.19}$$

By inequality (3.4), $-g(\dot{T}^k, E) \geq 0$. In addition, $\text{supp}\nabla\eta \subset \text{supp}\eta$, and hence (3.19) gives

$$0 \leq \int -g(\dot{T}^k, E)v^{-1}\eta^2 dv_g \lesssim \int \sigma_{k-1} |\nabla_g v|^2 |\nabla_g \eta|^2 v^{-3} dv_g$$

From the definition of η , we have $|\nabla\eta|_g^2 = v^2 |\nabla\eta|_{\mathbb{R}^n}^2 \leq \rho^{-2} v^2$, $|\nabla v|_g^2 = v^2 |\nabla v|^2$, and $dv_g = v^{-n} dx$ so we have

$$\begin{aligned}\int -g(\dot{T}^k, E)v^{-1}\eta dv_g &\lesssim \rho^{-2} \int \sigma_{k-1} |\nabla v|^2 v^{1-n} dx \\ &\leq C\end{aligned}$$

by Propositions 3.8 and 3.10. Hence

$$\int_{\mathbb{R}^n} -g(\dot{T}^k, E)v^{-1} dv_g < \infty$$

and so

$$\int_{\text{supp} \nabla \eta} -g(\mathring{T}^k, E) v^{-1} \eta^2 dv_g = \int_{\text{supp} \nabla \eta} |g(\mathring{T}^k, E)| v^{-1} \eta^2 dv_g \rightarrow 0$$

as $\rho \rightarrow \infty$. In particular, as $\rho \rightarrow \infty$ in equation (3.19), the right hand side is a product of a term tending to zero and a bounded term. Hence both sides of equation (3.19) tend to zero, and

$$\int_{\mathbb{R}^n} -g(\mathring{T}^k, E) v^{-1} dv_g = 0$$

Thus $-g(\mathring{T}^k, E) \equiv 0$ on $\mathbb{R}^n$. Equation (3.4) then gives $E \equiv 0$. Theorem 3.1 quickly follows, as if we write E in terms of the Euclidean metric g_0 ,

$$E_g = -(n-2)v^{-1}\nabla_0^2 v - \frac{n-2}{n}v^{-1}\Delta_0 v g_0$$

or in other words, for each i, j

$$\partial_i \partial_j v = \frac{\delta_{ij}}{n} \sum_{k=1}^n v_{kk}$$

The only solutions of this system are of the required form. □

Chapter 4

Some Results on a Continuity

Method Approach to Solving $\sigma_2 =$ constant on 3-Manifolds

4.1 Introduction

In order to describe the approach we use to study $\sigma_2 = \text{constant}$ in dimension 3, we must first mention two related families of objects: the Paneitz (or more generally, GJMS [17]) operators and the Q curvatures. These families arise in the following way.

One interesting fact is that, modulo lower order terms, the Laplacian is conformally invariant. More precisely, if (M, g_0) is an n -dimensional manifold, define the *conformal Laplacian* L via

$$L_0 f = -4 \frac{n-1}{n-2} \Delta_0 f + R_0 f$$

where R is the scalar curvature of g_0 and f is a given function. Then if $g = u^{\frac{4}{n-2}} g_0$ is a conformal metric, we have

$$L_g f = u^{-\frac{n+2}{n-2}} L_0(u f) \tag{4.1}$$

Moreover, setting $f \equiv 1$ gives

$$R_g = 4 \frac{n-1}{n-2} u^{-\frac{n+2}{n-2}} L_0 u \tag{4.2}$$

A natural question, then, is whether there exist other differential operators which transform similarly to (4.1) under conformal scaling, and if there exist corresponding “curvature functions” analogous to (4.2).

Remarkably, in many cases, generalizations exist. In arbitrary dimensions, [3] and [17] provide more information. However, for our purposes, we are interested in the “conformally invariant square of the Laplacian” in dimension 3, which we will denote P , and the corresponding Q -curvature, which we will denote Q . Some information about these quantities can be found in, for example, [24] and [37]. They are given by

$$Q = -\frac{1}{4}\Delta R - 2|Ric|^2 + \frac{23}{32}R^2 \quad (4.3)$$

$$Pf = \Delta^2 f + 4\nabla_i(Ric_{ij}\nabla_j f) - \frac{5}{4}\nabla_i(R\nabla_i f) - \frac{1}{2}Qf \quad (4.4)$$

There are two important properties of Q which will be crucial for our discussion (see also [24]):

- The Q curvature transforms in a simple way under a conformal rescaling. In particular, if $g = u^{-4}g_0$, then

$$Q_g = -2u^7 P_0 u \quad (4.5)$$

- The equation $Q = \text{constant}$ can be connected to the equation $\sigma_2 = \text{constant}$ via a useful one parameter family of metrics. Specifically, the following two equations are (by definition of Q and $\sigma_2(g^{-1}(A_g))$) equivalent

$$\mu_\alpha = Q + \frac{1-\alpha}{32}(R^2 + 8\Delta R) \quad (4.6)$$

$$\mu_\alpha = 4\sigma_2 - \frac{\alpha}{32}(R^2 + 8\Delta R) \quad (4.7)$$

Here μ_α is a constant depending on α . We note that if $\alpha = 1$, then $Q = \mu_1$. If $\alpha = 0$, then $\sigma_2 = \mu_0$.

We also make the following observation to serve as additional motivation for our approach. Whenever w is a function such that $g = e^{2w}g_0$ is a solution of $Q = \text{constant} > 0$, then (4.7) is satisfied with $\alpha = 1$. If g has volume 1 and positive scalar curvature, then integrating both sides of

(4.7) with respect to g , we have

$$0 < \mu_\alpha = \int 4\sigma_2 - \frac{R^2}{32} dv_g < 4 \int \sigma_2 dv_g$$

Relabel g as the background metric g_0 . Provided M has positive Yamabe invariant, a result of Ge, Lin, and Guan states:

Proposition 4.1. *(See Corollary 2 of [13]) Let (M, g_0) be a closed manifold of dimension 3 with positive Yamabe constant $Y_1(g_0) > 0$ and $g_0 \in \Gamma_1^+$ such that*

$$\int_{M^3} \sigma_2(g_0^{-1} A_{g_0}) dv_0 > 0$$

then there is a metric $g \in [g_0]$ with $g \in \Gamma_2^+$, i.e. with

$$R_g > 0 \text{ and } \sigma_2(g^{-1} A_g) > 0$$

In other words, once we have a constant Q curvature metric we can find one with $\sigma_2 > 0$ pointwise. This suggests that by starting with a metric satisfying $Q = \text{constant}$ we may be able to deform to one with $\sigma_2 = \text{constant}$.

The significance of above discussion is that it suggests using a continuity method approach to solve $\sigma_2 = \text{constant}$ by starting with a constant Q -curvature metric (such metrics are known to exist in some cases and have been studied before, see [24],[37]). In other words, let

$$I = \{\alpha \in [0, 1] : (4.6) \text{ has a solution satisfying } (4.10) - (4.14)\}$$

To run the continuity method we must check that that I is open and closed.

We assume that we have $1 \in I$ —that is, we start with a metric with $Q = \text{constant}$. In dimension 3, the existence of solutions of this equation has been studied in several papers; we briefly mention some of the results here. In [37], Xu and Yang show that on 3 dimensional manifolds with positive Paneitz operator, one can solve the equation $Q = \text{constant}$ in the same conformal class. To achieve this they minimize the functional

$$F_p(u) = \left(\int |u|^{-p} dv_0 \right)^{2/p} \int u P u dv_0$$

over $W^{2,2}$. A minimizer u for $p = 6$ satisfies $Pu = \text{constant} * u^{-7}$, which by (4.5) implies $g = u^{-4} g_0$

has constant Q -curvature. However, at $p = 6$ the equation is critical so they minimize the functional for $p > 6$ to obtain a minimizer u_p . They then show that the u_p have a limit as $p \rightarrow 6$. Moreover, they also show that the existence of a conformal metric with positive Paneitz operator is preserved under connected sums.

In [24], Hang and Yang give another condition which guarantees the existence of a constant Q -curvature metric. Specifically, they define the functional

$$E(u) = \int u P u d v_0 \quad (4.8)$$

In general it is not clear that $F_6(u)$ has a global minimum over positive functions $u \in W^{2,2}$. However, under their condition

$$(P^+) := \text{for any } u \in W^{2,2}, \text{ with } u \geq 0, \text{ and } u = 0 \text{ at some point, we have } E(u) > 0$$

they show that $F_6(u)$ does have a global minimum over positive functions $u \in W^{2,2}$ (which in particular satisfies $Pu = \text{constant} * u^{-7}$ so that $g = u^{-4}g_0$ has constant Q -curvature).

Openness of I depends on the linearized equation having all eigenvalues nonzero. This holds in dimension 4 (see section 4 in [7]) but at present we are unsure of a condition to guarantee this holds in dimension 3. One can check that in certain cases, the linearization of (4.6) will not necessarily have all eigenvalues non-zero. A useful class of examples is provided by the *Berger spheres*, defined as follows. The 3-sphere $\mathbb{S}^3$ may be identified with the Lie group

$$SU(2) = \left\{ \begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix} : a, b \in \mathbb{C}, |a|^2 + |b|^2 = 1 \right\}$$

The Lie algebra of $SU(2)$ is

$$\mathfrak{su}(2) = \left\{ \begin{pmatrix} it_1 & it_2 - t_3 \\ t_2 + t_3 & -it_1 \end{pmatrix} : t_1, t_2, t_3 \in \mathbb{R} \right\}$$

A natural orthonormal basis for $\mathfrak{su}(2)$ is

$$X_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad X_3 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

Given $a_1, a_2, a_3 \in \mathbb{R}$, left-invariance uniquely determines a metric g_{a_1, a_2, a_3} on $\mathbb{S}^3$ such that $a_1^{-1}X_1, a_2^{-1}X_2, a_3^{-1}X_3$ is an orthonormal basis. In the case $a_2 = a_3 = 1$, several geometric quantities related to this metric are computed in [24]. They also have a method compute the eigenvalues of

the Paneitz operator for these metrics, and using the same technique one can check that for certain values of a_1 and α , the linearization of (4.6) will have a zero eigenvalue. Hence, unlike the four dimensional case, in three dimensions we do not in general expect I to be open.

Fortunately, closedness of I is easier to check. One must simply obtain priori estimates for solutions of (4.6)—this is the content of the main results of the second section of our paper.

Our first result is analogous to Theorem 3.1 in [7] but adapted to dimension 3. The main difference is that we must assume have one additional constraint, (4.11). The significance of this constraint will be addressed in Chapter 5. Our hypotheses are

$$(M, g_0) \text{ is a manifold of positive Yamabe class} \quad (4.9)$$

$$w_\alpha \text{ is a smooth solution of (4.6) for } \alpha_0 < \alpha \leq 1 \quad (4.10)$$

$$\int e^{3w_\alpha} dv_0 = 1, \text{ that is, } g_\alpha = e^{2w_\alpha} g_0 \text{ has unit volume} \quad (4.11)$$

$$w_\alpha \text{ has positive scalar curvature} \quad (4.12)$$

$$\frac{1}{C_0} \leq \mu_\alpha \leq C_0 \text{ for some fixed } C_0 \quad (4.13)$$

$$|\int w_\alpha dv_0| \leq C_0 \text{ for } \alpha_0 < \alpha \leq 1 \quad (4.14)$$

Note: for $\alpha > 0$, we write $g_\alpha = e^{2w_\alpha} g_0$ to be the metric satisfying (4.6). For the purpose of consistency with standard literature, the metric g_0 always denotes the background metric—we do not mean that g_0 is a metric satisfying (4.6) at $\alpha = 0$.

Then

Theorem 4.2. *Under hypotheses (4.9)-(4.14) with $\alpha_0 > 0$, there exists a smooth function w_{α_0} satisfying (4.6) with $\alpha = \alpha_0$. Moreover, w_{α_0} has positive scalar curvature and satisfies*

$$\int (\Delta_0 e^{-w/2})^2 dv_0 \leq \frac{C}{\alpha} \quad (4.15)$$

$$\int \Delta_0 e^{-w/2} |\nabla e^{-w/2}|_0^2 e^{w/2} dv_0 \leq C \quad (4.16)$$

$$\int |\nabla e^{-w/2}|_0^4 e^w dv_0 \leq C \quad (4.17)$$

$$|w| \leq C \quad (4.18)$$

where C depends only on the constant in (4.9)-(4.14) and (M, g_0) .

Remark: We note that under the hypotheses of Theorem 4.2, using (4.18) and the chain rule

applied to (4.15)-(4.17) gives

$$\begin{aligned}\int (\Delta_0 w)^2 dv_0 &\leq \frac{C}{\alpha} \\ \int -\Delta_0 w |\nabla w|_0^2 dv_0 &\leq C \\ \int |\nabla w|_0^4 dv_0 &\leq C\end{aligned}$$

respectively. Since

$$\int |\nabla_0^2 w|_0^2 dv_0 = \int (\Delta_0 w)^2 - Ric_0(\nabla w, \nabla w) dv_0 \leq \frac{C}{\alpha}$$

we then have

$$\|w\|_{W^{2,2}} \leq \frac{C}{\alpha}$$

The above theorem provides a-priori estimates for solutions of (4.6) for $\alpha > 0$ and allows us to show that I is closed when it is bounded away from 0. However, if we want to take the limit of solutions of (4.6) as $\alpha \rightarrow 0$ we need something stronger than (4.15). We prove the analogue of Theorem 5.1 from [7], again in dimension 3.

Theorem 4.3. *Assume (4.9)-(4.14) with $\alpha_0 = 0$. Then there exists a $w \in W^{2,3}$ such that*

$$\|w\|_{2,3} \leq C \tag{4.19}$$

$$|w| \leq C \tag{4.20}$$

such that $g = e^{2w} g_0$ has positive scalar curvature almost everywhere. Moreover, w is a weak solution of $\sigma_2 = \text{constant}$ in the sense that $w_\alpha \rightharpoonup w$ weakly in $W^{2,3}$ as $\alpha \rightarrow 0$. Here C depends only on (M, g_0) and the constant in (4.9)-(4.14).

We can think of Theorems 4.2 and 4.3 as saying that I is closed (though technically at $\alpha = 0$ we only have a weak solution).

As explained in the first chapter, we seek to adapt the proof from [7] to dimension 3. In Section 4.2, we first provide some background information. In Section 4.3, we adapt a result from [19] to show that solutions of our main equation (4.6) satisfy a Weitzenböck formula. Specifically, we derive (4.43) by using that fact that solutions of (4.6) are actually critical points for a particular functional. In Section 4.4, we show Theorem 4.2 simply by integrating our main equation (4.6) and estimating the terms we obtain. Theorem 4.2 is analogous to Theorem 3.1 from [7]. Note that we

cite a result from [35] which is from dimension 4—we adapt this to dimension 3 in the appendix. Finally in Section 4.5, we show Theorem 4.3. Again, the approach is to imitate the argument of [7], in particular Theorem 5.1 and Proposition 5.2. The idea of the argument is explained in the beginning of Section 4.5.

4.2 Preliminary Formulæ

A straightforward algebraic calculation shows that in dimension 3

$$\sigma_1(A_g) = \frac{R_g}{4} \quad (4.21)$$

$$\sigma_2(A_g) = -\frac{1}{2}|E_g|_g^2 + \frac{1}{48}R_g^2 \quad (4.22)$$

$$= -\frac{1}{2}|Ric_g|_g^2 + \frac{3}{16}R_g^2 \quad (4.23)$$

$$A_g = E_g + \frac{R_g}{12}g_{ij} \quad (4.24)$$

$$T_g = -E_g + \frac{R_g}{6}g_{ij} \quad (4.25)$$

Moreover, if $g = e^{2w}g_0$, then the following expressions give formulæ for curvature quantities under conformal change:

$$R_g = R_0 e^{-2w} - 4\Delta w + 2|\nabla w|^2 \quad (4.26)$$

$$Ric_g = Ric_0 - \nabla^2 w - \Delta w g - dw \otimes dw + |\nabla w|^2 g \quad (4.27)$$

$$A_g = A_0 - \nabla^2 w - dw \otimes dw + \frac{1}{2}|\nabla w|^2 g \quad (4.28)$$

where all derivatives and curvatures are with respect to g . The same formula written with the derivatives and curvatures of g_0 will also be useful.

$$R_g = e^{-2w}(R_0 - 4\Delta_0 w - 2|\nabla_0 w|_0^2) \quad (4.29)$$

$$Ric_g = Ric_0 - \nabla_0^2 w + dw \otimes dw - \Delta_0 w g_0 - |\nabla_0 w|_0^2 g_0 \quad (4.30)$$

$$A_g = A_0 - \nabla_0^2 w + dw \otimes dw - \frac{1}{2}|\nabla_0 w|_0^2 g_0 \quad (4.31)$$

We assume throughout this chapter that $g, g_0 \in \Gamma_1^+$; that is, $\sigma_1 > 0$ for both g and g_0 .

Next we prove some basic estimates about Ric and T , adapted to the dimension 3 case from [7].

Lemma 4.4. *Suppose the scalar curvature R is positive at a fixed point $p \in M$, where M is a 3-dimensional manifold. Then for all $X \in T_p(M)$, we have*

$$T(X, X) \geq \frac{4}{R} \sigma_2(A) g(X, X) \quad (4.32)$$

$$Ric(X, X) \geq \frac{8 + 4\sqrt{3}}{R} \sigma_2(A) g(X, X) \quad (4.33)$$

Proof. By [32] p.234, $|E(X, X)| \leq \sqrt{\frac{2}{3}} |E| |X|^2$. Thus

$$\begin{aligned} Ric(X, X) &= E(X, X) + \frac{R}{3} |X|^2 \\ &\geq -\sqrt{\frac{2}{3}} |E| |X|^2 + \frac{R}{3} |X|^2 \\ &= (-\sqrt{\frac{2}{3}} (\frac{|E|}{\sqrt{aR}}) (\sqrt{aR}) + \frac{R}{3}) |X|^2 \\ &\geq (-\frac{|E|^2}{\sqrt{6}aR} - \frac{aR}{\sqrt{6}} + \frac{R}{3}) |X|^2 \\ &= \frac{-2(2 + \sqrt{3})}{R} |E|^2 + \frac{2 + \sqrt{3}}{12} R |X|^2 \\ &= \frac{8 + 4\sqrt{3}}{R} \sigma_2(A) |X|^2 \end{aligned}$$

if we take $a = \frac{1}{2}(\sqrt{\frac{2}{3}} - \sqrt{\frac{1}{3}})$. The other inequality is similar, as $T = -E + \frac{R}{6}g$. $\square$

Finally, we will need some basic properties of metrics with positive scalar curvature.

Lemma 4.5. *Fix a constant A and suppose $g_0, g \in \Gamma_1^+$ with $g = e^{2w}g_0$ and $|\int w dv_0| \leq A$. Then there exists $C(g_0, A) > 0$ with*

$$w \geq -C \quad (4.34)$$

$$\Delta_0 w \leq C \quad (4.35)$$

$$\int w^2 dv_0 \leq C \quad (4.36)$$

$$\int |\nabla w|_0^2 dv_0 \leq C \quad (4.37)$$

Proof. $g_0^{-1}A_{g_0}, g^{-1}A_g \in \Gamma_1^+$ is equivalent to the condition that $R_{g_0}, R_g > 0$. Suppose the minimum of w occurs at a point $p \in M$. Then at p , equation (4.29) gives

$$\begin{aligned} 4\Delta_0 w &= -e^{2w}R_g + R_0 - 2|\nabla w|^2 \\ &\leq C(g_0) \end{aligned} \quad (4.38)$$

and so (4.35) holds. By Green's formula,

$$-w(x) + \bar{w} = \int G(x, y) \Delta_0 w(y) dv_0(y) \quad (4.39)$$

where $G(x, y)$ is the Green's function for (M, g_0) and $\bar{w} = \int w dv_0$ satisfies $|\bar{w}| \leq A$. As M is compact, we may add a function to G and assume that it is positive. So using (4.39) and (4.35) gives (4.34).

Next, integration of (4.38) gives (4.37). (4.36) follows from the hypothesis and the Poincare inequality. $\square$

4.3 An Identity for Solutions of (4.6)

Interestingly, solutions of (4.6) are actually critical points of a certain Riemannian functional (4.40). Consequently, they satisfy an identity (4.43) which will be useful in a later computation. Note that (4.43) is essentially the dimension 3 version of equation (5.10) in [7]. The proof we give below uses a different technique than the one used in [7] but either approach would suffice.

We first show

Lemma 4.6. *Solutions w of (4.6) are critical points of the functional*

$$\mathcal{F}_\alpha : g \rightarrow \int_M (4\sigma_2(g^{-1}A_g) - \frac{\alpha}{32}R_g^2) dv_g \quad (4.40)$$

over metrics $g = e^{2w}g_0$ satisfying $\text{Vol}(g) = 1$.

Recall that the *gradient* $\nabla \mathcal{F}_\alpha$ of the Riemannian functional $\mathcal{F}_\alpha$ at g is a symmetric 2-tensor such that if $g(t)$ is a one parameter family of metrics with $g(0) = g, g'(0) = h$, then

$$\frac{d}{dt}(\mathcal{F}_\alpha[g(t)])|_{t=0} = \int_M \langle \nabla \mathcal{F}_\alpha, h \rangle_g dv_g$$

(See [2] for more information) Equation (3.12) in [19] says that for $\alpha = 0$

$$\nabla \mathcal{F}_0 = 2\Delta E_{ij} + \frac{1}{6}\Delta R g_{ij} - \frac{1}{2}\nabla_i \nabla_j R - 8E_{im}E_{mj} - \frac{5}{6}R E_{ij} + \frac{1}{9}R^2 g_{ij} - 6\sigma_2(A_g)g_{ij} \quad (4.41)$$

Hence to compute $\nabla \mathcal{F}_\alpha$ we must find $\nabla \mathcal{G}$, where

$$\mathcal{G} : g \rightarrow \int_M R_g^2 dv_g$$

By [2], [19], we know that

$$\begin{aligned}(R^2)' &= 2RR' = 2R(-\Delta(trh) + \delta^2 h - Ric^{ij}h_{ij}) \\ (dv_g)' &= \frac{1}{2}tr(h)\end{aligned}$$

where $'$ denotes $\frac{d}{dt}|_{t=0}$. Here all derivatives, curvatures, etc. are with respect to g . Hence

$$\left(\int_M R_g^2 dv_g\right)' = \int_M (-2R\Delta tr(h) + 2R\delta^2 h - 2RRic^{ij}h_{ij} + \frac{1}{2}R^2 tr(h)) dv_g$$

Integrating by parts and using (2.1) gives

$$\nabla \mathcal{G} = -2\Delta_g R_g + 2\nabla_{ij}^2 R_g - 2RE_{ij} - \frac{2}{3}R^2 g_{ij} + \frac{1}{2}R^2 g_{ij}$$

Combining this with equation (4.41), we find

$$\begin{aligned}(\nabla \mathcal{F}_\alpha)_{ij} &= (\nabla(\mathcal{F}_0 - \frac{\alpha}{32}\mathcal{G}))_{ij} = 2\Delta E_{ij} + \frac{1}{6}\Delta R g_{ij} - \frac{1}{2}\nabla_i \nabla_j R \\ &\quad - 8E_{im}E_{mj} - \frac{5}{6}RE_{ij} + \frac{1}{9}R^2 g_{ij} - 6\sigma_2 g_{ij} \\ &\quad - \frac{\alpha}{32}(-2\Delta R g_{ij} + 2\nabla_i \nabla_j R - 2RE_{ij} - \frac{2}{3}R^2 g_{ij} + \frac{1}{2}R^2 g_{ij})\end{aligned}\tag{4.42}$$

Let $\mathcal{A}_1 := \{w \in C^\infty(M) \mid \int dv_g = \int e^{3w} dv_0 = 1\}$ denote the functions with average equal to zero with respect to g . Occasionally we will write $g \in \mathcal{A}_1$ to mean $g = e^{2w}g_0$ where $w \in \mathcal{A}_1$. We have the following analogue of Proposition 3.1 of [19]:

Proof. of Lemma 4.6

As in the proof in [19], the Euler equation for $\mathcal{F}_\alpha$ restricted to volume 1 metrics (not necessarily conformal to g_0) is

$$\nabla \mathcal{F}_\alpha = \lambda g$$

where $\lambda \in \mathbb{R}$ is a Lagrange multiplier.

By the remark following the proof of Proposition 3.1 in [19], the Euler equation for $\mathcal{F}_\alpha|_{\mathcal{A}_1}$ is

obtained by taking the trace. In particular, (4.42) gives

$$\begin{aligned}
3\lambda &= \text{Tr}(\mathcal{F}_\alpha) \\
&= -8|E|^2 + \frac{1}{3}R^2 - 18\sigma_2(A_g) - \frac{\alpha}{32}(-4\Delta R - \frac{1}{2}R^2) \\
&= -2\sigma_2 + \frac{\alpha}{8}\Delta R + \frac{\alpha}{64}R^2
\end{aligned}$$

□

Now we imitate the proof of Proposition 3.2 from [19]. In particular, we have

Proposition 4.7. *Suppose that M is compact and g is critical for $\mathcal{F}_\alpha|_{\mathcal{A}_1}$. Then we have*

$$\frac{1}{2}\Delta|E|^2 = |\nabla E|^2 + \frac{1}{4}\langle E, \nabla^2 R \rangle + 4\text{tr}(E^3) + \frac{5}{12}R|E|^2 + \frac{\alpha}{32}(\langle \nabla^2 R, E \rangle - R|E|^2) \quad (4.43)$$

Proof. At each point of M , we have $\langle \nabla \mathcal{F}_\alpha - \lambda g, E \rangle = 0$. By (4.42), we see that

$$2\langle E, \Delta E \rangle - \frac{1}{2}\langle E, \nabla^2 R \rangle - 8\text{tr}(E^3) - \frac{5}{6}R|E|^2 - \frac{\alpha}{16}(\langle \nabla^2 R, E \rangle - R|E|^2) = 0$$

Now simply use $\frac{1}{2}\Delta|E|^2 = \langle E, \Delta E \rangle + |\nabla E|^2$. □

4.4 A Priori Estimates for Solutions for $\alpha \in (0, 1]$

As mentioned in the first chapter, the Paneitz operator and the Q -curvature are connected via equations involving conformal change of metric. In particular, in 3-dimensions, we have that if $g = u^{-4}g_0$, then

$$\frac{1}{2}Q_g u^{-7} = -P_0 u \quad (4.44)$$

(see [24]). For the purpose of this computation, writing $g = u^{-4}g_0$ may be the most natural. However, for the sake of consistency of notation with the rest of this chapter, we will instead write $g = e^{2w}g_0$. So (4.44) becomes

$$\frac{1}{2}Q_g e^{7w/2} = -P_0 e^{-w/2}$$

From (4.4), we can multiply this equation by $e^{-w/2}$ and integrate to obtain

$$\begin{aligned}
\int \frac{1}{2} Q dv_g &= \int \frac{1}{2} Q e^{3w} dv_0 \\
&= \int -e^{-w/2} P_0 e^{-w/2} dv_0 \\
&= \int -(\Delta_0 e^{-w/2})^2 + 4Ric_0(\nabla e^{-w/2}, \nabla e^{-w/2}) - \frac{5}{4} R_0 |\nabla e^{-w/2}|_0^2 + \frac{1}{2} Q_0 e^{-w} dv_0 \quad (4.45)
\end{aligned}$$

By (4.29) and the chain rule, we also have

$$\begin{aligned}
\int R_g^2 dv_g &= \int R_g^2 e^{3w} dv_0 \\
&= \int (R_0 e^{-2w} + 8e^{-3w/2} \Delta_0 e^{-w/2} - 16e^{-w} |\nabla e^{-w/2}|_0^2) e^{3w} dv_0 \\
&= \int R_0^2 e^{-w} + 16e^{-w/2} R_0 \Delta_0 e^{-w/2} - 32R_0 |\nabla e^{-w/2}|_0^2 \\
&\quad + 64(\Delta_0 e^{-w/2})^2 - 256e^{-w/2} |\nabla e^{-w/2}|_0^2 \Delta_0 e^{w/2} + 256 |\nabla e^{-w/2}|_0^4 e^w dv_0 \quad (4.46)
\end{aligned}$$

Combining (4.45) and (4.46), any solution of (4.6) satisfies

$$\begin{aligned}
\mu_\alpha &= \int Q_g + \frac{1-\alpha}{32} R_g^2 dv_g \quad (4.47) \\
&= \int -2\alpha(\Delta_0 e^{-w/2})^2 - 8(1-\alpha)e^{w/2} |\nabla e^{-w/2}|_0^2 \Delta_0 e^{-w/2} + 8(1-\alpha) |\nabla e^{-w/2}|_0^4 e^w \\
&\quad + 8Ric_0(\nabla e^{-w/2}, \nabla e^{-w/2}) - \left(\frac{7}{2} - \alpha\right) R_0 |\nabla e^{-w/2}|_0^2 + Q_0 e^{-w} \\
&\quad + \frac{1-\alpha}{32} R_0^2 e^{-w} + \frac{1-\alpha}{2} e^{-w/2} R_0 \Delta_0 e^{-w/2} dv_0
\end{aligned}$$

Lemma 4.8. *For any solution w of (4.6) satisfying (4.9)-(4.14), the estimates (4.15)-(4.18) hold.*

Proof. Lemma 4.5 gives

$$\int e^{-w} dv_0 \leq \int e^{-C} \leq C \quad (4.48)$$

$$\int |\nabla e^{-w/2}|_0^2 dv_0 = \frac{1}{4} \int |\nabla w|^2 e^{-w} dv_0 \leq C \int |\nabla w|_0^2 dv_0 \leq C \quad (4.49)$$

where C depends only on D and (M, g_0) .

Applying this to (4.47) we have we have

$$\int 2\alpha(\Delta_0 e^{-w/2})^2 + 8(1-\alpha)e^{w/2}|\nabla e^{-w/2}|_0^2 \Delta_0 e^{-w/2} - 8(1-\alpha)|\nabla e^{-w/2}|_0^4 e^w dv_0 \leq C \quad (4.50)$$

By (4.29) and the chain rule this implies

$$\int 2\alpha(\Delta_0 e^{-w/2})^2 + (1-\alpha)R_g|\nabla e^{-w/2}|_0^2 e^{2w} + 8(1-\alpha)|\nabla e^{-w/2}|_0^4 e^w dv_0 \leq C \quad (4.51)$$

We first show that (4.15)-(4.18) hold for w_α . There are two cases to consider: α bounded away from zero and α near zero.

Case I: Suppose $\alpha \in [1/2, 1]$. Then because R_g is positive, (4.51) gives

$$\int (\Delta_0 e^{-w/2})^2 dv_0 \leq \int 2\alpha(\Delta_0 e^{-w/2})^2 dv_0 \leq C \quad (4.52)$$

and so integrating by parts we obtain

$$\int |\nabla e^{-w/2}|_0^2 dv_0 = \int (\Delta_0 e^{-w/2})^2 - Ric_0(\nabla e^{-w/2}, \nabla e^{-w/2}) dv_0 \leq C \quad (4.53)$$

Combining this with (4.49) and the Sobolev inequality gives

$$\int |\nabla e^{-w/2}|^6 dv_0 \leq C \quad (4.54)$$

and (4.15) is proved. Since the metric has volume 1, (4.54) gives

$$\int |\nabla e^{-w/2}|^4 e^w dv_0 \leq (\int |\nabla e^{-w/2}|^6 dv_0)^{2/3} (\int e^{3w} dv_0)^{1/3} \leq C$$

showing (4.17). Hence (4.16) then follows from (4.4) and (4.52) because

$$\int \Delta_0 e^{-w/2} |\nabla e^{-w/2}|_0^2 e^{w/2} dv_0 \leq (\int (\Delta_0 e^{-w/2})^2 dv_0)^{1/2} (\int |\nabla e^{-w/2}|_0^4 e^w)^{1/2} \leq C$$

This completes case I.

Case II: Suppose $\alpha \in [0, 1/2]$. Then since we have positive scalar curvature, (4.51) gives

$$\alpha \int (\Delta_0 e^{-w/2})^2 dv_0 \leq C$$

so that (4.15) follows as in case I. (4.51) also gives (4.17):

$$\int |\nabla e^{-w/2}|_0^4 e^w dv_0 \leq 8(1-\alpha) \int |\nabla e^{-w/2}|_0^4 e^w dv_0 \leq C$$

Hence (4.50) implies (4.16)

$$\int e^{w/2} |\nabla e^{w/2}|_0^2 \Delta_0 e^{-w/2} dv_0 \leq 8(1-\alpha) \int e^{w/2} |\nabla e^{w/2}|_0^2 \Delta_0 e^{-w/2} dv_0 \leq C$$

and we are finished with case II.

Next, we notice that (4.18) holds in both case I and case II as follows. By the chain rule,

$$\int |\nabla e^{-w/2}|_0^4 e^w dv_0 = \frac{1}{4} \int |\nabla w|_0^4 e^{-w} dv_0$$

Hence we have

$$\int |\nabla w|_0^3 dv_0 \leq \left(\int |\nabla w|_0^4 e^{-w} dv_0 \right)^{3/4} \left(\int e^{3w} dv_0 \right)^{1/4} \leq C$$

By the Moser-Trudinger inequality, w is in the Orlicz class $e^{L^{3/2}}$ and hence $e^{\lambda w}$ is integrable for any λ . So for $\epsilon > 0$, we have

$$\int |\nabla w|_0^{4-\epsilon} dv_0 \leq \left(\int |\nabla w|_0^4 e^{-w} dv_0 \right)^{\frac{4-\epsilon}{4}} \left(\int e^{(\frac{4}{\epsilon}-1)w} dv_0 \right)^{\epsilon/4} \leq C_\epsilon$$

so that $\|w\|_{W^{1,4-\epsilon}} \leq C_\epsilon$. Hence for some $\beta > 0$ we have $\|w\|_{C^\beta} \leq C$. Because assumption, the mean value of w with respect to dv_0 is bounded. Hence $|w| \leq C$.

□

Proof. of Theorem 4.2 By Lemma 4.8 and the remark following Theorem 4.2, we know that as $\alpha \rightarrow \alpha_0$, w_α is bounded in $W^{2,2}$. By Lemma 6.1 for every k we have uniform C^k bounds on w_α . Hence a subsequence of w_α converges in C^∞ to a smooth function w_{α_0} satisfying (4.6). Moreover, w_{α_0} satisfies (4.15)-(4.18). Now, for each $\alpha > \alpha_0$, we have by (4.29)

$$4\Delta_0 w_\alpha + 2|\nabla w_\alpha|_0^2 + R_\alpha e^{2w_\alpha} = R_0$$

As $R_\alpha > 0$, multiplying this equation by a smooth ϕ and integrating we have

$$\int 4\phi \Delta_0 w_\alpha + 2\phi |\nabla w_\alpha|_0^2 dv_0 \leq \int \phi R_0 dv_0$$

Since $w_\alpha \rightarrow w_{\alpha_0}$ in $W^{2,2}$ we have

$$\int 4\phi\Delta_0 w_{\alpha_0} + 2\phi|\nabla w_{\alpha_0}|_0^2 dv_0 \leq \int \phi R_0 dv_0$$

and so $R_{\alpha_0} \geq 0$ almost everywhere. R_{α_0} is smooth because w_{α_0} is smooth, and hence $R_{\alpha_0} \geq 0$ everywhere. Moreover, by (4.22) we may write (4.6) as

$$\mu_\alpha = -2|E|^2 + \left(\frac{1}{12} - \frac{\alpha}{32}\right)R^2 - \frac{\alpha}{8}\Delta R \quad (4.55)$$

for the metric g_{α_0} . As μ_α is bounded below, at the minimum of R we have $R^2 \geq \frac{1}{C}$. Because $R \geq 0$, we must necessarily have $R \geq \frac{1}{C}$.

□

4.5 Estimates for α near 0

We now proceed as in [7], section 5. From this point forward the constant m_α denotes the constant appearing in the (4.6), and assume $g = e^{2w}g_0$ is a solution of (4.6). The integrals in this section will be with respect to the volume form dv_g and all curvatures and norms will be with respect to g except when explicitly specified with a 0 to indicate g_0 . For example $R = R_g$ and $R_0 = R_{g_0}$. Also, throughout this section we will use the letter C to denote a constant (which may vary from line to line) depending only on g_0 and μ_α as well as the constants appearing in (4.9)-(4.14) (and, in particular, independent of α). Throughout this section it may be necessary to take α close to 0, all results in this section apply for α sufficiently small.

The idea is as follows. As a first step we seek to prove

$$\int R^3 \leq C \int \alpha |\nabla w|^6 + C \int R^2 + C \quad (4.56)$$

To do this, we first recall that by (2.3) the first Newton tensor T_{ij} is divergence free. Hence we have that for any function f ,

$$0 = \int \nabla_i (T_{ij} \nabla_j f) = \int T_{ij}^1 \nabla_{ij}^2 f$$

If $f = R$, we will see that

$$0 = \int T_{ij} \nabla_{ij}^2 R \geq \int \frac{5}{3} \left(\frac{1}{24} - \frac{\alpha}{64} \right) R^3 + 16 \text{Tr}(E^3) + \text{lower order terms}$$

If $f = \frac{1}{2} |\nabla w|^2$, we will see that

$$0 = \frac{1}{2} \int T_{ij} \nabla_{ij}^2 (|\nabla w|^2) \geq \int -\text{Tr}(E^3) + \frac{R^3}{864} - \frac{R}{8} |\nabla w|^4 + \text{lower order terms}$$

Taking a linear combination of this two expressions will cancel the $\text{Tr}(E^3)$ terms. From this we (4.56) quickly follows. The steps in this argument consist mainly of using (4.6), integration by parts, and algebra.

Once (4.56) is obtained, Theorem 4.3 follows from a few more computations.

First, we have

Lemma 4.9. *If T_{ij} is the first Newton tensor, then*

$$T_{ij} \nabla_i \nabla_j R = 4\Delta\sigma_2 + 4|\nabla E|^2 + 16\text{Tr}(E^3) + \frac{5}{3}R|E|^2 - \frac{1}{6}|\nabla R|^2 + \frac{\alpha}{8}(\langle \nabla^2 R, E \rangle - R|E|^2) \quad (4.57)$$

Proof. We have

$$\begin{aligned} \Delta(-\sigma_2) &= \frac{1}{2}\Delta|E|^2 - \frac{1}{48}\Delta(R^2) \\ &= \frac{1}{2}\Delta|E|^2 - \frac{1}{24}R\Delta R - \frac{1}{24}|\nabla R|^2 \\ &= |\nabla E|^2 + \frac{1}{4}\langle E, \nabla^2 R \rangle + 4\text{tr}(E^3) + \frac{5}{12}R|E|^2 \\ &\quad + \frac{\alpha}{32}(\langle \nabla^2 R, E \rangle - R|E|^2) - \frac{1}{24}R\Delta R - \frac{1}{24}|\nabla R|^2 \\ &= |\nabla E|^2 - \frac{1}{4}T_{ij} \nabla_i \nabla_j R + 4\text{tr}(E^3) + \frac{5}{12}R|E|^2 \\ &\quad + \frac{\alpha}{32}\alpha(\langle \nabla^2 R, E \rangle - R|E|^2) - \frac{1}{24}|\nabla R|^2 \end{aligned}$$

where in the third equality we used (4.43). □

Write $I = \int T_{ij} \nabla_i \nabla_j R$. So we have

$$I \geq \int 4|\nabla E|^2 - \frac{1}{6}|\nabla R|^2 + 16\text{Tr}(E^3) + \frac{5}{3}R|E|^2 + \frac{\alpha}{8}(\langle \nabla^2 R, E \rangle - R|E|^2) \quad (4.58)$$

First we estimate the term $\int R|E|^2$.

Lemma 4.10. *We have*

1. $\int R|E|^2 \geq \int (\frac{1}{24} - \frac{\alpha}{64})R^3 - 2\mu_\alpha R$
2. $\int R|E|^2 \leq \int \frac{\alpha}{8}|\nabla R|^2 + (\frac{1}{24} - \frac{\alpha}{64})R^3$

Proof. From our equation (4.6),

$$\int \frac{\alpha}{128}R\Delta R = \int \frac{1}{8}R(-\mu_\alpha - \frac{1}{2}|E|^2 + (\frac{1}{48} - \frac{\alpha}{128})R^2)$$

and hence

$$\int R|E|^2 = \int \frac{\alpha}{8}|\nabla R|^2 - 2\mu_\alpha R + (\frac{1}{24} - \frac{\alpha}{64})R^3$$

As μ_α is positive both parts follow. □

Next we estimate more of the terms in (4.58).

Lemma 4.11.

$$\int 4|\nabla E|^2 - \frac{1}{6}|\nabla R|^2 \geq \int \frac{\alpha}{2} \frac{(\Delta R)^2}{R} - \frac{\alpha}{16}|\nabla R|^2 \quad (4.59)$$

Proof. Differentiating our equation (4.6), we obtain

$$0 = \frac{\alpha}{16}\nabla(\Delta R) + |E|\nabla|E| - 2(\frac{1}{48} - \frac{\alpha}{128})R\nabla R$$

Multiply both sides by $\frac{8}{R}\nabla R$ and integrate to obtain

$$0 = \int \frac{\alpha}{2R}\langle \nabla R, \nabla(\Delta R) \rangle + 8\frac{|E|}{R}\langle \nabla R, \nabla|E| \rangle - 16(\frac{1}{48} - \frac{\alpha}{128})|\nabla R|^2 \quad (4.60)$$

By the AM-GM inequality, we have

$$\begin{aligned} \int 8\frac{|E|}{R}\langle \nabla R, \nabla|E| \rangle &\leq \int 4|\nabla|E||^2 + 4\frac{|E|^2}{R^2}|\nabla R|^2 \\ &\leq \int 4|\nabla E|^2 + 4\frac{|E|^2}{R^2}|\nabla R|^2 \end{aligned}$$

Now we use our equation (4.6) to replace the $|E|^2$ piece of the last term above:

$$\begin{aligned}
\int 4 \frac{|E|^2}{R^2} |\nabla R|^2 &= \int \frac{4}{R^2} \left(-\frac{\alpha}{8} \Delta R - 2\mu_\alpha + 2\left(\frac{1}{48} - \frac{\alpha}{128}\right) R^2 \right) |\nabla R|^2 \\
&= \int -\frac{\alpha}{2R^2} |\nabla R|^2 \Delta R - 8 \frac{\mu_\alpha}{R^2} |\nabla R|^2 + 8\left(\frac{1}{48} - \frac{\alpha}{128}\right) |\nabla R|^2
\end{aligned}$$

Substituting this expression into the previous inequality, we have

$$\int 8 \frac{|E|}{R} \langle \nabla R, \nabla |E| \rangle \leq \int 4 |\nabla E|^2 + 8\left(\frac{1}{48} - \frac{\alpha}{128}\right) |\nabla R|^2 - \frac{\alpha}{2R^2} |\nabla R|^2 \Delta R - 8 \frac{\mu_\alpha}{R^2} |\nabla R|^2 \quad (4.61)$$

Next, integrate by parts the first term of (4.60):

$$\int \frac{\alpha}{2R} \langle \nabla R, \nabla (\Delta R) \rangle = \int -\frac{\alpha}{2R} (\Delta R)^2 + \frac{\alpha}{2R^2} |\nabla R|^2 \Delta R \quad (4.62)$$

Using (4.61) and (4.62) in (4.60), we obtain

$$\begin{aligned}
0 &\leq \int -\frac{\alpha}{2R} (\Delta R)^2 + \frac{\alpha}{2R^2} |\nabla R|^2 \Delta R + 4 |\nabla E|^2 + 8\left(\frac{1}{48} - \frac{\alpha}{128}\right) |\nabla R|^2 \\
&\quad - \frac{\alpha}{2R^2} |\nabla R|^2 \Delta R - 8 \frac{\mu_\alpha}{R^2} |\nabla R|^2 - 16\left(\frac{1}{48} - \frac{\alpha}{128}\right) |\nabla R|^2 \\
&= \int -\frac{\alpha}{2R} (\Delta R)^2 + 4 |\nabla E|^2 - 8\left(\frac{1}{48} - \frac{\alpha}{128}\right) |\nabla R|^2 - 8 \frac{\mu_\alpha}{R^2} |\nabla R|^2
\end{aligned}$$

Rearranging terms and using $\mu_\alpha > 0$, we are finished. $\square$

Corollary 4.12.

$$I \geq \int \frac{\alpha}{2R} (\Delta R)^2 - \frac{\alpha}{16} |\nabla R|^2 + 16 \text{Tr}(E^3) + \frac{5}{3} \left(\frac{1}{24} - \frac{\alpha}{64} \right) R^3 - \frac{10}{3} \mu_\alpha R + \frac{\alpha}{8} (\langle \nabla^2 R, E \rangle - R |E|^2)$$

Proof. Use Lemmas 4.10 and 4.11 in (4.58). $\square$

Lemma 4.13. *For α sufficiently small,*

$$\int \alpha |\nabla R|^2 \lesssim \int \alpha R^3 + R^2 + 1$$

Proof. From Lemma 4.2 of [19], we see that

$$\text{Tr}(E^3) \geq -\frac{1}{\sqrt{6}} |E|^3$$

so that

$$\begin{aligned} 16Tr(E^3) + \frac{5}{3}R|E|^2 &\geq -\frac{16}{\sqrt{6}}|E|^3 + \frac{5}{3}R|E|^2 \\ &= |E|^2\left(-\frac{16}{\sqrt{6}}|E| + \frac{5}{3}R\right) \end{aligned}$$

From the AM-GM inequality, we have

$$-\frac{16}{\sqrt{6}}|E| = -\frac{16}{\sqrt{6}}\left(\sqrt{\frac{a|E|^2}{R}}\right)\left(\sqrt{\frac{R}{a}}\right) \geq -\frac{8}{\sqrt{6}}\left(\frac{a|E|^2}{R} + \frac{R}{a}\right)$$

where a is any constant. If we choose $a = 4\sqrt{6}$, then we have

$$\begin{aligned} \int 16tr(E^3) + \frac{5}{3}R|E|^2 &\geq \int \frac{|E|^2}{R}(-32|E|^2 + \frac{4}{3}R^2) \\ &= \int 64\frac{|E|^2}{R}\sigma_2(A) \\ &= \int 64\frac{|E|^2}{R}\left(\frac{\alpha}{16}\Delta R + \frac{\alpha}{128}R^2 + \mu_\alpha\right) \\ &\geq \int 4\alpha\frac{\Delta R}{R}|E|^2 \end{aligned} \tag{4.63}$$

Now, by the AM-GM inequality,

$$\int 4\alpha\frac{\Delta R}{R}|E|^2 \geq 2\alpha \int -\eta\frac{(\Delta R)^2}{R} - \frac{|E|^4}{\eta R}$$

where η is a constant to be chosen later. Next, use our equation (4.6) to replace $|E|^2$ in the second term, above, to obtain

$$\begin{aligned} 4 \int \alpha\frac{\Delta R}{R}|E|^2 &\geq 2\alpha \int -\eta\frac{(\Delta R)^2}{R} - \frac{|E|^2}{\eta R}\left(-\frac{\alpha}{8}\Delta R - 2\mu_\alpha + \left(\frac{1}{24} - \frac{\alpha}{64}\right)R^2\right) \\ &= 128\alpha \int -\eta\frac{(\Delta R)^2}{R} + \frac{\alpha}{8\eta}\frac{\Delta R}{R}|E|^2 + 2\frac{\mu_\alpha}{\eta}\frac{|E|^2}{R} - \frac{\frac{1}{24} - \frac{\alpha}{64}}{\eta}R|E|^2 \end{aligned}$$

Now we combine terms to obtain

$$\int 4\alpha\left(1 - \frac{\alpha}{16\eta}\right)\frac{\Delta R}{R}|E|^2 \geq 2\alpha \int -\eta\frac{(\Delta R)^2}{R} - \frac{\frac{1}{24} - \frac{\alpha}{128}}{\eta}R|E|^2$$

so that

$$\int 512\alpha\frac{\Delta R}{R}|E|^2 \geq \frac{2\alpha}{1 - \frac{\alpha}{16\eta}} \int -\eta\frac{(\Delta R)^2}{R} - \frac{\frac{1}{24} - \frac{\alpha}{128}}{\eta}R|E|^2$$

Substituting this into (4.63) gives

$$\int 16tr(E^3) + \frac{5}{3}R|E|^2 \geq \frac{256\alpha}{1-\alpha\frac{8}{\eta}} \int -\eta \frac{(\Delta R)^2}{R} - \frac{\frac{1}{24}-2\alpha}{\eta} R|E|^2 \quad (4.64)$$

Next, substituting (4.64) into (4.58) and using (4.59), we have

$$\begin{aligned} 0 &= I \\ &\geq \int \frac{\alpha}{2} \frac{(\Delta R)^2}{R} + \frac{2\alpha}{1-\frac{\alpha}{16\eta}} \left(-\eta \frac{(\Delta R)^2}{R} - \frac{\frac{1}{24}-\frac{\alpha}{64}}{\eta} R|E|^2 \right) + \frac{\alpha}{8} (\langle \nabla^2 R, E \rangle - R|E|^2) \\ &= \int \frac{\alpha}{2} \left(1 - \frac{4\eta^2}{\eta - \frac{\alpha}{16}} \right) \frac{(\Delta R)^2}{R} - \frac{\frac{\alpha}{4} - \frac{3}{32}\alpha^2}{3\eta - \frac{3\alpha}{16}} R|E|^2 + \frac{\alpha}{8} (\langle \nabla^2 R, E \rangle - R|E|^2) \end{aligned}$$

provided α sufficiently small. In particular, if we choose η small enough, say $\eta = \frac{1}{1000}$, then we can estimate the terms above by

$$0 \geq \int \frac{\alpha}{4} \frac{(\Delta R)^2}{R} + \frac{\alpha}{8} \langle \nabla^2 R, E \rangle - C\alpha R|E|^2 - CR^2 - C$$

The term of the form $\frac{\alpha}{8} \langle \nabla^2 R, E \rangle$ can be rewritten as $-\frac{\alpha}{48} |\nabla R|^2$ by integrating by parts and using the Bianchi identity $\nabla_i Ric_{ij} = \frac{1}{2} \nabla_j R$. By Lemma 4.10 we have

$$\begin{aligned} 0 &\geq \int \frac{\alpha}{4} \frac{(\Delta R)^2}{R} - \left(\frac{\alpha}{48} + \mu\alpha^2 \right) |\nabla R|^2 - C\alpha R^3 - CR^2 - C \\ &\geq \int 32\alpha \frac{(\Delta R)^2}{R} - 3\frac{\alpha}{128} |\nabla R|^2 - C\alpha R^3 - CR^2 - C \end{aligned} \quad (4.65)$$

Also

$$\int 3\alpha |\nabla R|^2 = \int -3\alpha R \Delta R \leq \int 16\alpha \frac{(\Delta R)^2}{R} + \frac{9\alpha}{64} R^3$$

so plugging this into (4.65)

$$0 \geq \int 16\alpha |\nabla R|^2 - C\alpha R^3 - CR^2 - C$$

and we are finished. □

Now we consider

$$II := \int T_{ij} \nabla_i \nabla_j V = 0$$

where $V = \frac{1}{2} |\nabla w|^2$.

Lemma 4.14.

$$\begin{aligned}
T_{ij}\nabla_i\nabla_jV &= T_{ij}\nabla_i\nabla_kw\nabla_j\nabla_kw - \nabla_kw\nabla_kA_{ij}T_{ij} \\
&\quad + \nabla_kw\nabla_kA_{ij}^0T_{ij} - T_{ij}\nabla_i|\nabla w|^2\nabla_jw \\
&\quad + \frac{1}{4}R\langle\nabla w, \nabla|\nabla w|^2\rangle + R_{ikjm}T_{ij}\nabla_mw\nabla_kw
\end{aligned} \tag{4.66}$$

Proof. First, $\nabla_jV = \nabla_j(\frac{1}{2}|\nabla w|^2) = \nabla_j\nabla_kw\nabla_kw$. Hence $\nabla_i\nabla_jV = \nabla_i\nabla_kw\nabla_j\nabla_kw + \nabla_i\nabla_j\nabla_kw\nabla_kw$. Moreover, $\nabla_i\nabla_j\nabla_kw = \nabla_i\nabla_k\nabla_jw$. Hence we have

$$\nabla_i\nabla_k\nabla_jw = \nabla_k\nabla_i\nabla_jw + R_{ikjm}\nabla_mw$$

and so

$$\nabla_i\nabla_jV = \nabla_i\nabla_kw\nabla_j\nabla_kw + \nabla_k\nabla_i\nabla_jw\nabla_kw + R_{ikjm}\nabla_mw\nabla_kw \tag{4.67}$$

Rearranging (4.28),

$$\nabla^2w = A_0 - A_g - dw \otimes dw + \frac{1}{2}|\nabla w|^2g$$

and so we have

$$\nabla_k\nabla_i\nabla_jw = -\nabla_kA_{ij} + \nabla_kA_{ij}^0 - \nabla_k\nabla_iw\nabla_jw - \nabla_iw\nabla_k\nabla_jw + \frac{1}{2}\nabla_k|\nabla w|^2g_{ij}$$

Plug this equation into (4.67) to obtain

$$\begin{aligned}
\nabla_i\nabla_jV &= \nabla_i\nabla_kw\nabla_j\nabla_kw - \nabla_kw\nabla_kA_{ij} + \nabla_kw\nabla_kA_{ij}^0 \\
&\quad - \nabla_i\nabla_kw\nabla_jw\nabla_kw - \nabla_j\nabla_kw\nabla_iw\nabla_kw \\
&\quad + \frac{1}{2}\nabla_kw\nabla_k|\nabla w|^2g_{ij} + R_{ikjm}\nabla_mw\nabla_kw
\end{aligned}$$

Take the inner product of both sides with T_{ij} and use $T_{ij}g_{ij} = 2\sigma_1(A)$ with $\sigma_1(A) = \frac{1}{4}R$ to obtain the desired result. □

Next we will further expand (4.66). For notational simplicity write

$$T_{ij}\nabla_i\nabla_jV = II_1 + \cdots + II_6$$

Lemma 4.15.

$$\begin{aligned}
II_1 &\equiv T_{ij} \nabla_i \nabla_k w \nabla_j \nabla_k w \\
&= -Tr(E^3) + \frac{R^3}{864} + 2T_{ij} A_{jk} \nabla_i w \nabla_k w - 2|\nabla w|^2 \sigma_2(A) \\
&\quad + \frac{R}{8} |\nabla w|^4 - 2T_{ij} A_{ik} A_{jk}^0 + T_{ij} A_{ik}^0 A_{jk}^0 \\
&\quad - 2T_{ij} A_{ik}^0 \nabla_j w \nabla_k w + T_{ij} A_{ij}^0 |\nabla w|^2
\end{aligned}$$

Proof. Since all the tensors under consideration are symmetric, (4.28) gives

$$\begin{aligned}
II_1 &= T_{ij} \nabla_i \nabla_k w \nabla_j \nabla_k w \\
&= T_{ij} (-A_{ik} + A_{ik}^0 - \nabla_i w \nabla_k w + \frac{1}{2} |\nabla w|^2 g_{ik}) \\
&\quad \cdot (-A_{jk} + A_{jk}^0 - \nabla_j w \nabla_k w + \frac{1}{2} |\nabla w|^2 g_{jk}) \\
&= T_{ij} A_{ik} A_{jk} - 2T_{ij} A_{ik} A_{jk}^0 + 2T_{ij} A_{ik} \nabla_j w \nabla_k w - T_{ij} A_{ij} |\nabla w|^2 \\
&\quad + T_{ij} A_{ik}^0 A_{jk}^0 - 2T_{ij} A_{ik}^0 \nabla_j w \nabla_k w + T_{ij} A_{ij}^0 |\nabla w|^2 \\
&\quad + T_{ij} \nabla_i w \nabla_j w |\nabla w|^2 - T_{ij} \nabla_i w \nabla_j w |\nabla w|^2 + \frac{1}{4} T_{ii} |\nabla w|^4 \\
&= -Tr(E^3) + \frac{R^3}{864} - 2T_{ij} A_{ik} A_{jk}^0 + 2T_{ij} A_{ik} \nabla_j w \nabla_k w - 2\sigma_2(A) |\nabla w|^2 \\
&\quad + T_{ij} A_{ik}^0 A_{jk}^0 - 2T_{ij} A_{ik}^0 \nabla_j w \nabla_k w + T_{ij} A_{ij}^0 |\nabla w|^2 + \frac{R}{8} |\nabla w|^4
\end{aligned}$$

In the last equality we have used (4.21)-(4.25) to find

$$T_{ii} = \frac{R}{2} \tag{4.68}$$

$$T_{ij} A_{ij} = \sigma_1^2(A) - |A|^2 = 2\sigma_2(A) \tag{4.69}$$

$$\begin{aligned}
T_{ij} A_{ik} A_{kj} &= (-E + \frac{R}{6} g)_{ij} (E + \frac{R}{12} g)_{ik} (E + \frac{R}{12} g)_{kj} \\
&= -Tr(E^3) + \frac{R^3}{864}
\end{aligned}$$

□

Lemma 4.16.

$$\begin{aligned}
II_2 &\equiv -\nabla_k w \nabla_k A_{ij} T_{ij} \\
&= -\langle \nabla w, \nabla \sigma_2(A) \rangle
\end{aligned}$$

Proof. From the basic properties of the Newton tensor, $\nabla_k \sigma_2(A) = T_{ij} \nabla_k A_{ij}$. □

Lemma 4.17.

$$\begin{aligned} II_5 &\equiv \frac{1}{4} R \langle \nabla w, \nabla |\nabla w|^2 \rangle \\ &= -\frac{1}{2} R A_{ij} \nabla_i w \nabla_j w + \frac{1}{2} R A_{ij}^0 \nabla_i w \nabla_j w - \frac{1}{4} R |\nabla w|^4 \end{aligned}$$

Proof. From (4.28), we have

$$\begin{aligned} II_5 &= \frac{1}{4} R \langle \nabla w, \nabla |\nabla w|^2 \rangle \\ &= \frac{1}{2} R \nabla_i \nabla_j w \nabla_i w \nabla_j w \\ &= \frac{1}{2} R (-A_{ij} + A_{ij}^0 - \nabla_i w \nabla_j w + \frac{1}{2} |\nabla w|^2 g_{ij}) \nabla_i w \nabla_j w \\ &= -\frac{1}{2} R A_{ij} \nabla_i w \nabla_j w + \frac{1}{2} R A_{ij}^0 \nabla_i w \nabla_j w - \frac{1}{4} R |\nabla w|^4 \end{aligned}$$

□

Lemma 4.18.

$$\begin{aligned} II_6 &= R_{ikjm} S_{ij} \nabla_m w \nabla_k w \\ &= -2T_{ik} A_{jk} \nabla_i w \nabla_j w + \frac{R}{2} A_{ij} \nabla_i w \nabla_j w + 2\sigma_2(A) |\nabla w|^2 \end{aligned}$$

Proof. The Weyl Tensor vanishes in dimension 3, so by the Ricci Decomposition $Rm = A_g \circ g$.

Hence

$$\begin{aligned} R_{ikjm} T_{ij} \nabla_m w \nabla_k w &= (g_{ij} A_{km} - g_{im} A_{jk} - g_{jk} A_{im} + g_{km} A_{ij}) T_{ij} \nabla_m w \nabla_k w \\ &= T_{kk} A_{ij} \nabla_i w \nabla_j w - 2T_{ik} A_{jk} \nabla_i w \nabla_j w + A_{ij} T_{ij} |\nabla w|^2 \end{aligned}$$

Applying (4.68) and (4.69) to the last term gives the result. □

Combining Lemmas 4.14-4.18, we have

Corollary 4.19.

$$\begin{aligned}
T_{ij}\nabla_i\nabla_jV &= -Tr(E^3) + \frac{R^3}{864} - \frac{R}{8}|\nabla w|^4 - \langle \nabla w, \nabla\sigma_2(A) \rangle - T_{ij}\nabla_i|\nabla w|^2\nabla_jw \\
&\quad - 2T_{ij}A_{ik}A_{jk}^0 + T_{ij}A_{ik}^0A_{jk}^0 - 2T_{ij}A_{ik}^0\nabla_iw\nabla_kw + T_{ij}A_{ij}^0|\nabla w|^2 \\
&\quad + \nabla_kw\nabla_kA_{ij}^0T_{ij} + \frac{1}{2}RA_{ij}^0\nabla_iw\nabla_jw
\end{aligned}$$

Proof. We have

$$\begin{aligned}
T_{ij}\nabla_i\nabla_jV &= II_1 + II_2 + II_3 + II_4 + II_5 + II_6 \\
&= (-Tr(E^3) + \frac{R^3}{864} + 2T_{ij}A_{jk}\nabla_iw\nabla_kw - 2|\nabla w|^2\sigma_2(A) + \frac{R}{8}|\nabla w|^4 \\
&\quad - 2T_{ij}A_{ik}A_{jk}^0 + T_{ij}A_{ik}^0A_{jk}^0 - 2T_{ij}A_{ik}^0\nabla_jw\nabla_kw + T_{ij}A_{ij}^0|\nabla w|^2) \\
&\quad + (-\langle \nabla w, \nabla\sigma_2(A) \rangle) + (\nabla_kw\nabla_kA_{ij}^0T_{ij}) + (-T_{ij}\nabla_i|\nabla w|^2\nabla_jw) \\
&\quad + (-\frac{1}{2}RA_{ij}\nabla_iw\nabla_jw + \frac{1}{2}RA_{ij}^0\nabla_iw\nabla_jw - \frac{1}{4}R|\nabla w|^4) \\
&\quad + (-2T_{ik}A_{jk}\nabla_iw\nabla_jw + \frac{R}{2}A_{ij}\nabla_iw\nabla_jw + 2\sigma_2(A)|\nabla w|^2) \\
&= -Tr(E^3) + \frac{R^3}{864} - \frac{R}{8}|\nabla w|^4 - 2T_{ij}A_{ik}A_{jk}^0 \\
&\quad + T_{ij}A_{ik}^0A_{jk}^0 - 2T_{ij}A_{ik}^0\nabla_jw\nabla_kw + T_{ij}A_{ij}^0|\nabla w|^2 - \langle \nabla w, \nabla\sigma_2(A) \rangle \\
&\quad + \nabla_kw\nabla_kA_{ij}^0T_{ij} - T_{ij}\nabla_i|\nabla w|^2\nabla_jw + \frac{1}{2}RA_{ij}^0\nabla_iw\nabla_jw
\end{aligned}$$

□

Before estimating $T_{ij}\nabla_i\nabla_jV$, we first do some basic calculations.

Lemma 4.20.

$$\begin{aligned}
|A^0| &\leq C \\
|\nabla A^0| &\leq C + C|\nabla w|
\end{aligned}$$

Proof. The proof given for Claim 5.17 in [7] does not depend on dimension and hence applies here without modification. □

Corollary 4.21. *We have*

$$\begin{aligned}
-T_{ij}A_{ik}A_{jk}^0 &\geq -C|Ric|^2 \\
T_{ij}A_{ik}^0A_{jk}^0 &\geq -C|Ric| \\
-2T_{ij}A_{ik}^0\nabla_j w \nabla_k w &\geq -C|Ric||\nabla w|^2 \\
T_{ij}A_{ij}^0|\nabla w|^2 &\geq -C|Ric||\nabla w|^2 \\
\nabla_k w \nabla_k A_{ij}^0 T_{ij} &\geq -C|Ric||\nabla w| - C|Ric||\nabla w|^2 \\
\frac{1}{2}RA_{ij}^0\nabla_i w \nabla_j w &\geq -C|Ric||\nabla w|^2
\end{aligned}$$

Proof. By definition, $|T|, |A| \leq C|Ric|$. Combining this with the Lemma 4.20 gives the results. $\square$

Corollary 4.22.

$$\begin{aligned}
T_{ij}\nabla_i\nabla_j V &\geq -Tr(E^3) + \frac{R^3}{864} - \frac{R}{8}|\nabla w|^4 - \langle \nabla w, \nabla \sigma_2(A) \rangle - T_{ij}\nabla_i|\nabla w|^2\nabla_j w \\
&\quad - C|Ric|^2 - C|Ric||\nabla w|^2 - C
\end{aligned} \tag{4.70}$$

Proof. Apply Corollary 4.21 to Corollary 4.19. $\square$

Before we estimate this expression, we will need the following technical lemma.

Lemma 4.23.

$$\int -\alpha|\nabla^2 w|^2|\nabla w|^2 \gtrsim \int -\alpha R^3 - \alpha|\nabla w|^6 - \alpha R^2 - \alpha \tag{4.71}$$

Proof. By (4.28) we have

$$\begin{aligned}
|\nabla^2 w|^2 &= |-A_{ij} + A_{ij}^0 - \nabla_i w \nabla_j w + \frac{1}{2}|\nabla w|^2 g_{ij}|^2 \\
&\lesssim |A|^2 + |\nabla w|^4 + 1
\end{aligned}$$

Consequently

$$\int -\alpha|\nabla^2 w|^2|\nabla w|^2 \geq \int -C\alpha|A|^2|\nabla w|^2 - C\alpha|\nabla w|^6 - C\alpha$$

Now, $\sigma_2(A) = \frac{1}{2}(Tr^2(A) - |A|^2) = -\frac{1}{2}|A|^2 + \frac{1}{32}R^2$, and so for α small

$$\begin{aligned}
\int -\alpha|\nabla^2 w|^2|\nabla w|^2 &\geq \int -C\alpha|\nabla w|^2(-2\sigma_2(A) + \frac{1}{16}R^2) - C\alpha|\nabla w|^6 - C\alpha \\
&\geq \int -C\alpha|\nabla w|^2\sigma_2(A) - C\alpha R^2|\nabla w|^2 - C\alpha|\nabla w|^6 - C\alpha \\
&= \int -C\alpha|\nabla w|^2(\frac{\alpha}{128}R^2 + \frac{\alpha}{16}\Delta R + \mu_\alpha) - C\alpha R^2|\nabla w|^2 - C\alpha|\nabla w|^6 - C\alpha \\
&\geq \int -C\alpha^2|\nabla w|^2\Delta R - C\alpha R^2|\nabla w|^2 - C\alpha|\nabla w|^6 - C\alpha \\
&\geq \int C\alpha^2\langle \nabla|\nabla w|^2, \nabla R \rangle - C\alpha R^2|\nabla w|^2 - C\alpha|\nabla w|^6 - C\alpha \\
&\geq \int -C\alpha^2|\nabla^2 w||\nabla w||\nabla R| - C\alpha R^2|\nabla w|^2 - C\alpha|\nabla w|^6 - C\alpha \\
&\geq \int -C\alpha^2|\nabla R|^2 - C\alpha^2|\nabla^2 w|^2|\nabla w|^2 - C\alpha R^2|\nabla w|^2 - C\alpha|\nabla w|^6 - C\alpha
\end{aligned}$$

Hence for α small,

$$\int -\alpha|\nabla^2 w|^2|\nabla w|^2 \geq \int -C\alpha^2|\nabla R|^2 - C\alpha R^2|\nabla w|^2 - C\alpha|\nabla w|^6 - C\alpha$$

By Lemma 4.13, we have

$$\int -C\alpha^2|\nabla R|^2 \geq \int -C\alpha^2 R^3 - C\alpha R^2 - C\alpha$$

By Young's Inequality, $R^2|\nabla w|^2 \lesssim R^3 + |\nabla w|^6$ and so

$$\int -\alpha|\nabla^2 w|^2|\nabla w|^2 \gtrsim \int -\alpha R^3 - \alpha R^2 - \alpha|\nabla w|^6 - \alpha$$

□

We now estimate II .

Proposition 4.24. *For all $\alpha > 0$ sufficiently small, we have*

$$\begin{aligned}
0 &= II \tag{4.72} \\
&\geq \int -Tr(E^3) + \frac{R^3}{864} - \frac{R}{8}|\nabla w|^4 - C\alpha R^3 - C\alpha|\nabla w|^6 \\
&\quad - CR^2 - C
\end{aligned}$$

Proof. Using (4.26) and integrating by parts, we obtain

$$\begin{aligned} - \int \langle \nabla w, \nabla \sigma_2(A) \rangle &= \int \Delta w \sigma_2(A) \\ &= \int -\frac{R}{4} \sigma_2(A) + \frac{R_0}{4} e^{-2w} \sigma_2(A) + \frac{1}{2} |\nabla w|^2 \sigma_2(A) \end{aligned} \quad (4.73)$$

Again using integration by parts and using (4.28), (4.69), and $\nabla_i T_{ij} = 0$, we have

$$\begin{aligned} \int -T_{ij} \nabla_i |\nabla w|^2 \nabla_j w &= \int |\nabla w|^2 \nabla_i T_{ij} \nabla_j w + |\nabla w|^2 T_{ij} \nabla_i \nabla_j w \\ &= \int |\nabla w|^2 T_{ij} (-A_{ij} + A_{ij}^0 - \nabla_i w \nabla_j w + \frac{1}{2} |\nabla w|^2 g_{ij}) \\ &= \int -|\nabla w|^2 T_{ij} A_{ij} + T_{ij} A_{ij}^0 |\nabla w|^2 - |\nabla w|^2 T_{ij} \nabla_i w \nabla_j w + \frac{R}{4} |\nabla w|^4 \\ &= \int -2|\nabla w|^2 \sigma_2(A) + T_{ij} A_{ij}^0 |\nabla w|^2 + |\nabla w|^2 Ric_{ij} \nabla_i w \nabla_j w + \frac{R}{4} |\nabla w|^4 \end{aligned} \quad (4.74)$$

Substituting (4.73) and (4.74) into (4.70) and combining terms,

$$\begin{aligned} 0 &= II \\ &\geq \int -Tr(E^3) + \frac{R^3}{864} - \frac{R}{8} |\nabla w|^4 - \frac{3}{2} |\nabla w|^2 \sigma_2(A) \\ &\quad - \frac{R}{4} \sigma_2(A) + \frac{R_0}{4} R_0 e^{-2w} \sigma_2(A) + |\nabla w|^2 Ric_{ij} \nabla_i w \nabla_j w \\ &\quad + T_{ij} A_{ij}^0 |\nabla w|^2 - C |Ric|^2 - C |Ric| |\nabla w|^2 - C \end{aligned} \quad (4.75)$$

Now, by our equation (4.6)

$$\begin{aligned} \int -\frac{R}{4} \sigma_2(A) &= \int -\frac{R}{4} \left(\frac{\alpha}{128} R^2 + \frac{\alpha}{16} \Delta R + \mu_\alpha \right) \\ &= \int -\frac{\alpha}{64} R \Delta R - \frac{\alpha}{512} R^3 - \frac{\mu_\alpha}{4} R \\ &= \int \frac{\alpha}{64} |\nabla R|^2 - \frac{\alpha}{512} R^3 - \frac{\mu_\alpha}{4} R \end{aligned} \quad (4.76)$$

Similarly

$$\begin{aligned}
\int \frac{R_0}{4} e^{-2w} \sigma_2(A) &= \int \frac{R_0}{4} e^{-2w} \left(\frac{\alpha}{16} \Delta R + \frac{\alpha}{128} R^2 + \mu_\alpha \right) \\
&= \int \frac{\alpha}{64} R_0 e^{-2w} \Delta R + \frac{\alpha}{512} R_0 e^{-2w} R^2 + \frac{\mu_\alpha}{4} R_0 e^{-2w} \\
&= \int -\frac{\alpha}{64} \langle \nabla R_0, \nabla R \rangle e^{-2w} - \frac{\alpha}{64} \langle \nabla e^{-2w}, \nabla R \rangle R_0 + \frac{\alpha}{512} R_0 e^{-2w} R^2 + \frac{\mu_\alpha}{512} R_0 e^{2w} \\
&\geq \int -C\alpha |\nabla R| - C\alpha |\nabla w| |\nabla R| - C \\
&\geq \int -\alpha |\nabla R|^2 - C |\nabla w|^2 - C \\
&\geq \int -\alpha |\nabla R|^2 - C
\end{aligned}$$

by Young's Inequality. Combining this with (4.76), we find

$$\int -\frac{R}{4} \sigma_2(A) + \frac{R_0}{4} e^{-2w} \sigma_2(A) \geq \int \frac{\alpha}{128} |\nabla R|^2 - \frac{\alpha}{512} R^3 - \frac{\mu_\alpha}{512} R - C \quad (4.77)$$

In addition, we have

$$\begin{aligned}
\int -\frac{3}{2} |\nabla w|^2 \sigma_2(A) &= -\frac{3}{2} \int |\nabla w|^2 \left(\frac{\alpha}{16} \Delta R + \frac{\alpha}{128} R^2 + \mu_\alpha \right) \\
&\geq \int -\frac{3\alpha}{32} |\nabla w|^2 \Delta R - C |\nabla w|^2 - C\alpha R^2 |\nabla w|^2 \\
&\geq \int \frac{3\alpha}{32} \langle \nabla |\nabla w|^2, \nabla R \rangle - C - C\alpha R^2 |\nabla w|^2 \\
&\geq \int -\frac{3\alpha}{16} |\nabla^2 w| |\nabla w| |\nabla R| - C - C\alpha R^2 |\nabla w|^2 \\
&\geq \int -\frac{\alpha}{128} |\nabla R|^2 - \frac{9\alpha}{8} |\nabla^2 w|^2 |\nabla w|^2 - C - C\alpha R^2 |\nabla w|^2 \\
&\geq \int -\frac{\alpha}{128} |\nabla R|^2 - C\alpha R^3 - C\alpha |\nabla w|^6 - CR^2 - C
\end{aligned} \quad (4.78)$$

using Young's Inequality again, as well as (4.71). Using (4.77) and (4.78) in (4.75), we have

$$0 = II \quad (4.79)$$

$$\begin{aligned}
&\geq \int -Tr(E^3) + \frac{R^3}{864} - \frac{R}{8} |\nabla w|^4 + |\nabla w|^2 Ric_{ij} \nabla_i w \nabla_j w - C\alpha R^3 \\
&\quad - C\alpha |\nabla w|^6 - CR^2 - C + |\nabla w|^2 T_{ij} A_{ij}^0 - C|Ric|^2 - C|Ric| |\nabla w|^2
\end{aligned}$$

The last three terms in (4.79) can be controlled easily:

$$\begin{aligned}
\int \frac{1}{2} |\nabla w|^2 T_{ij} A_{ij}^0 &\geq \int -C |Ric| |\nabla w|^2 \\
\int -C |Ric|^2 &= \int C(2\sigma_2(A) - \frac{3}{8} R^2) \\
&\geq \int C - CR^2 \\
\int -|Ric|^2 |\nabla w|^2 &\geq \int -|Ric|^2 - |\nabla w|^4 \\
&\geq \int -CR^2 - C
\end{aligned}$$

where in the last line we have used Theorem 4.2 for w . Hence

$$\begin{aligned}
0 &= II \\
&\geq \int -Tr(E^3) + \frac{R^3}{864} - \frac{R}{8} |\nabla w|^4 + |\nabla w|^2 Ric_{ij} \nabla_i w \nabla_j w \\
&\quad - C\alpha R^3 - C\alpha |\nabla w|^6 - CR^2 - C
\end{aligned} \tag{4.80}$$

Next we use (4.33) and integrate by parts to estimate the Ricci term in (4.80):

$$\begin{aligned}
\int |\nabla w|^2 Ric_{ij} \nabla_i w \nabla_j w &\geq \int \frac{8+4\sqrt{3}}{R} \sigma_2(A) |\nabla w|^4 \\
&\geq \int \frac{8}{R} \left(\frac{\alpha}{16} \Delta R + \frac{\alpha}{128} R^2 + \mu_\alpha \right) |\nabla w|^4 \\
&\geq \int \frac{\alpha}{2} \frac{\Delta R}{R} |\nabla w|^4 \\
&= \int -\frac{\alpha}{2} \nabla R \nabla(R^{-1}) |\nabla w|^4 - \frac{\alpha}{2} \frac{\nabla R}{R} \nabla |\nabla w|^4 \\
&\geq \int \frac{\alpha}{2} \frac{|\nabla R|^2}{R^2} |\nabla w|^4 - 2\alpha |\nabla w|^2 \nabla^2 w \left(\frac{\nabla R}{R}, \nabla w \right) \\
&\geq \int \frac{\alpha}{2} \frac{|\nabla R|^2}{R^2} |\nabla w|^4 - \frac{\alpha}{2} \frac{|\nabla R|^2}{R^2} |\nabla w|^4 - 128\alpha |\nabla^2 w|^2 |\nabla w|^2 \\
&= \int -2\alpha |\nabla^2 w|^2 |\nabla w|^2
\end{aligned}$$

So by (4.71), we have

$$\int |\nabla w|^2 Ric_{ij} \nabla_i w \nabla_j w \gtrsim \int -\alpha R^3 - \alpha |\nabla w|^6 - \alpha R^2 - \alpha \tag{4.81}$$

Using (4.81) in (4.80), we obtain (4.72). $\square$

We now have

Theorem 4.25. *Let $g = e^{2w}g_0$ be a solution of (4.6) with with satisfying (4.9)-(4.14). Then provided α is sufficiently small, there exists a constant $C = C(g_0)$ so that*

$$\int R^3 dv \leq C \int \alpha |\nabla w|^6 dv + C \int R^2 dv + C$$

Proof. Note that $|E| \lesssim |Ric| + R \lesssim R$ from the proof of Corollary 4.21. By Corollary 4.12 and Proposition 4.24

$$\begin{aligned} 0 &\geq I + 16II \\ &= \int T_{ij} \nabla_i \nabla_j R + 16T_{ij} \nabla_i \nabla_j V \\ &\geq \int \frac{\alpha}{2} \frac{(\Delta R)^2}{R} - \frac{\alpha}{16} |\nabla R|^2 + 16Tr(E^3) + \frac{5}{3} \left(\frac{1}{24} - \frac{\alpha}{64} \right) R^3 + \frac{\alpha}{8} (\langle \nabla^2 R, E \rangle - R|E|^2) - 16Tr(E^3) \\ &\quad + \frac{R^3}{54} - 2R|\nabla w|^4 - C\alpha R^3 - C\alpha |\nabla w|^6 - CR^2 - C \\ &\geq \int \frac{\alpha}{2} \frac{(\Delta R)^2}{R} - \frac{\alpha}{16} |\nabla R|^2 + \frac{19}{216} R^3 + \frac{\alpha}{8} \langle \nabla^2 R, E \rangle - 2R|\nabla w|^4 - C\alpha R^3 - C\alpha |\nabla w|^6 - CR^2 - C \\ &\geq \int \frac{19}{216} R^3 - \frac{\alpha}{16} |\nabla R|^2 - \frac{\alpha}{8} \langle \nabla R, \nabla (Ric - \frac{R}{3}g) \rangle - 2R|\nabla w|^4 - C\alpha R^3 - C\alpha |\nabla w|^6 - CR^2 - C \\ &\geq \int \frac{19}{216} R^3 - \frac{\alpha}{16} |\nabla R|^2 - \frac{\alpha}{8} \langle \nabla R, \nabla \frac{R}{6} \rangle - 2R|\nabla w|^4 - C\alpha R^3 - C\alpha |\nabla w|^6 - CR^2 - C \\ &\geq \int \frac{R^3}{12} - 2R|\nabla w|^4 - C\alpha R^3 - C\alpha |\nabla w|^6 - CR^2 - C \\ &\geq \int \frac{R^3}{12} - \frac{8\sqrt{6}}{3} |\nabla w|^6 - \frac{R^3}{36} - C\alpha R^3 - C\alpha |\nabla w|^6 - CR^2 - C \\ &= \int \frac{R^3}{18} - \frac{8\sqrt{6}}{3} |\nabla w|^6 - C\alpha R^3 - C\alpha |\nabla w|^6 - CR^2 - C \end{aligned}$$

Here we have used the Bianchi identity and Lemma 4.13 to eliminate the $\langle \nabla^2 R, E \rangle$ term. Rearranging, we obtain the desired inequality for α small:

$$\int \left(\frac{R}{\sqrt{24}} \right)^3 \leq \int (1 + C\alpha) |\nabla w|^6 + CR^2 + C \quad (4.82)$$

□

We have now proved the main estimate necessary for Theorem 4.3. However, a few more results are necessary to obtain the theorem.

First we show the analogue of Lemma 5.21 from [7].

Lemma 4.26.

$$\int \left(\frac{R}{\sqrt{24}}\right)^2 |\nabla w|^2 \leq \int (1 + C\alpha) |\nabla w|^6 + CR^2 + C \quad (4.83)$$

$$\int (\Delta w)^2 |\nabla w|^2 \leq \int \Delta w |\nabla w|^4 + C\alpha R^3 + C\alpha |\nabla w|^6 + CR^2 + C \quad (4.84)$$

$$\int |\nabla w|^6 \leq \int \frac{1}{2} R |\nabla w|^4 + C\alpha R^3 + C\alpha |\nabla w|^6 + CR^2 + C \quad (4.85)$$

$$\int \Delta w |\nabla w|^4 \lesssim \int \alpha R^3 + \alpha |\nabla w|^6 + R^2 + 1 \quad (4.86)$$

Proof. of (4.83) By Young's inequality and Theorem 4.25 we have

$$\begin{aligned} \int \left(\frac{R}{\sqrt{24}}\right)^2 |\nabla w|^2 &\leq \frac{2}{3} \int \left(\frac{R}{\sqrt{24}}\right)^3 + \frac{1}{3} \int |\nabla w|^6 \\ &\leq \int (1 + C\alpha) |\nabla w|^6 + CR^2 + C \end{aligned}$$

of (4.84) By (4.26) we have

$$\begin{aligned} &\int \left(\frac{R}{\sqrt{24}}\right)^2 |\nabla w|^2 - |\nabla w|^6 \\ &= \int \frac{1}{24} (-4\Delta w + 2|\nabla w|^2 + R_0 e^{-2w})^2 |\nabla w|^2 - |\nabla w|^6 \\ &= \int \frac{2}{3} (\Delta w)^2 |\nabla w|^2 + \frac{1}{6} |\nabla w|^6 + \frac{1}{24} R_0^2 e^{-4w} |\nabla w|^2 \\ &\quad - \frac{2}{3} \Delta w |\nabla w|^4 - \frac{1}{3} \Delta w R_0 e^{-2w} |\nabla w|^2 + \frac{1}{6} R_0 e^{-2w} |\nabla w|^4 - |\nabla w|^6 \end{aligned}$$

and so

$$\begin{aligned} &\frac{2}{3} \int (\Delta w)^2 |\nabla w|^2 \\ &= \int \left(\frac{R}{\sqrt{24}}\right)^2 |\nabla w|^2 - |\nabla w|^6 + \frac{2}{3} \Delta w |\nabla w|^4 \\ &\quad - \frac{1}{24} R_0^2 e^{-4w} |\nabla w|^2 + \frac{1}{3} \Delta w R_0 e^{-2w} |\nabla w|^2 - \frac{1}{6} R_0 e^{-2w} |\nabla w|^4 \end{aligned} \quad (4.87)$$

The last two terms on the right hand side can be estimated using (4.26) and Theorem 4.2:

$$\begin{aligned} \int \frac{1}{3} \Delta w R_0 e^{-2w} |\nabla w|^2 &\lesssim \int |\nabla w|^4 + (\Delta w)^2 \\ &\lesssim \int 1 + R^2 \\ \int -\frac{1}{6} R_0 e^{-2w} |\nabla w|^4 &\lesssim \int |\nabla w|^4 \leq C \end{aligned}$$

Hence (4.83) and (4.87) give (4.84).

of (4.85) By (4.26) and (4.28),

$$\begin{aligned}
& \int 2|\nabla w|^2 Ric_{ij} \nabla_i w \nabla_j w \\
&= \int 2|\nabla w|^2 A_{ij} \nabla_i w \nabla_j w + \frac{1}{2} R |\nabla w|^4 \\
&= \int 2|\nabla w|^2 (A_{ij}^0 - \nabla_i \nabla_j w - \nabla_i w \nabla_j w + \frac{1}{2} |\nabla w|^2 g_{ij}) \nabla_i w \nabla_j w + \frac{1}{2} R |\nabla w|^4 \\
&= \int 2|\nabla w|^2 A_{ij}^0 \nabla_i w \nabla_j w - 2|\nabla w|^2 \nabla_i \nabla_j w \nabla_i w \nabla_j w - |\nabla w|^6 + \frac{1}{2} R |\nabla w|^4 \\
&= \int 2|\nabla w|^2 A_{ij}^0 \nabla_i w \nabla_j w - \frac{1}{2} \nabla_i |\nabla w|^4 \nabla_i w - |\nabla w|^6 + \frac{1}{2} R |\nabla w|^4 \\
&= \int 2|\nabla w|^2 A_{ij}^0 \nabla_i w \nabla_j w + \frac{1}{2} \Delta w |\nabla w|^4 - |\nabla w|^6 + \frac{1}{2} R |\nabla w|^4 \\
&= \int 2|\nabla w|^2 A_{ij}^0 + \frac{1}{2} \left(\frac{1}{2} |\nabla w|^2 - \frac{1}{4} R + \frac{1}{4} R_0 e^{-2w} \right) |\nabla w|^4 - |\nabla w|^6 + \frac{1}{2} R |\nabla w|^4 \\
&= \int 2|\nabla w|^2 A_{ij}^0 \nabla_i w \nabla_j w + \frac{1}{8} R_0 e^{-2w} |\nabla w|^4 - \frac{3}{4} |\nabla w|^6 + \frac{3}{8} R |\nabla w|^4 \\
&\leq \int C - \frac{3}{4} |\nabla w|^6 + \frac{3}{8} R |\nabla w|^4
\end{aligned} \tag{4.88}$$

(4.88) and (4.81) give (4.85).

of (4.86) Using (4.26) in (4.85), we have

$$\begin{aligned}
& \int |\nabla w|^6 \\
&\leq \int \frac{1}{2} R |\nabla w|^4 + C \alpha R^3 + C \alpha |\nabla w|^6 + C R^2 + C \\
&= \int \left(\frac{1}{2} R_0 e^{-2w} - 2 \Delta w + |\nabla w|^2 \right) |\nabla w|^4 + C \alpha R^3 + C \alpha |\nabla w|^6 + C R^2 + C \\
&\leq \int -2 \Delta w |\nabla w|^4 + |\nabla w|^6 + C \alpha R^3 + C \alpha |\nabla w|^6 + C R^2 + C
\end{aligned}$$

Rearranging gives (4.86). □

The results of the above Lemma 4.26 allow us to show

Proposition 4.27.

$$\int |\nabla^2 w|^2 |\nabla w|^2 \lesssim \int \alpha |\nabla w|^6 + R^2 + 1 \tag{4.89}$$

Proof. (4.84) and (4.86) give

$$\int (\Delta w)^2 |\nabla w|^2 \lesssim \int \alpha R^3 + \alpha |\nabla w|^6 + R^2 + 1 \tag{4.90}$$

The Bochner formula states

$$\frac{1}{2}\Delta|\nabla w|^2 = |\nabla^2 w|^2 + Ric_{ij}\nabla_i w \nabla_j w + \langle \nabla w, \nabla \Delta w \rangle$$

If we multiply this equation by $|\nabla w|^2$ we obtain

$$\begin{aligned} & \int |\nabla^2 w|^2 |\nabla w|^2 \\ &= \int \frac{1}{2} |\nabla w|^2 \Delta |\nabla w|^2 - |\nabla w|^2 Ric_{ij} \nabla_i w \nabla_j w - |\nabla w|^2 \langle \nabla w, \nabla \Delta w \rangle \\ &= \int -\frac{1}{2} |\nabla |\nabla w|^2|^2 - |\nabla w|^2 Ric_{ij} \nabla_i w \nabla_j w + |\nabla w|^2 (\Delta w)^2 + \Delta w \langle \nabla w, \nabla |\nabla w|^2 \rangle \\ &\leq \int -\frac{1}{2} |\nabla |\nabla w|^2|^2 - |\nabla w|^2 Ric_{ij} \nabla_i w \nabla_j w + |\nabla w|^2 (\Delta w)^2 + \frac{1}{2} |\nabla w|^2 (\Delta w)^2 + \frac{1}{2} |\nabla |\nabla w|^2|^2 \\ &= \int \frac{3}{2} |\nabla w|^2 (\Delta w)^2 - |\nabla w|^2 Ric_{ij} \nabla_i w \nabla_j w \end{aligned} \tag{4.91}$$

Using (4.81) and (4.90) in (4.91) with Theorem 4.25 gives (4.89). $\square$

Two small computations remain before we can complete this section.

Lemma 4.28.

$$\int |A|^3 \lesssim \int R^3 + C \tag{4.92}$$

Proof. We have $|A|^2 = |E|^2 + \frac{1}{48}R^2$ and so

$$\int |A|^3 \lesssim \int |E|^3 + R^3 \tag{4.93}$$

From our given equation (4.6)

$$|E|^2 = \left(\frac{1}{24} - \frac{\alpha}{64}\right)R^2 - \frac{\alpha}{8}\Delta R - \frac{\mu_\alpha}{2}$$

Multiplying by $|E|$ and integrating by parts gives

$$\begin{aligned} \int |E|^3 &= \int \left(\frac{1}{24} - \frac{\alpha}{64}\right)R^2 |E| - \frac{\mu_\alpha}{2} |E| - \frac{\alpha}{8} |E| \Delta R \\ &\leq \int \left(\frac{1}{24} - \frac{\alpha}{64}\right)R^2 |E| + \frac{\alpha}{8} \nabla |E| \nabla R \\ &\leq \int \left(\frac{1}{24} - \frac{\alpha}{64}\right)R^2 |E| + \frac{\alpha}{8} |\nabla E|^2 + \frac{\alpha}{8} |\nabla R|^2 \end{aligned}$$

Since $R^2|E| \leq \frac{2}{3}R^3 + \frac{1}{3}|E|^3$, the previous inequality gives

$$\int |E|^3 \lesssim \int \alpha |\nabla E|^2 + \alpha |\nabla R|^2 + R^3$$

Integrating (4.57) and using the Bianchi identity gives

$$\begin{aligned} \int |\nabla E|^2 &= \int \frac{1}{6} |\nabla R|^2 - 16 \text{Tr}(E^3) - \frac{5}{3} R |E|^2 - \frac{\alpha}{8} (\langle \nabla^2 R, E \rangle - R |E|^2) \\ &= \int \frac{1}{6} |\nabla R|^2 - 16 \text{Tr}(E^3) - \left(\frac{5}{3} - \frac{\alpha}{8}\right) R |E|^2 + \frac{\alpha}{48} |\nabla R|^2 \end{aligned}$$

and so

$$\int |\nabla E|^2 \leq \int |\nabla R|^2 + C |E|^3 + C$$

Substituting into equation implies

$$\int |E|^3 \lesssim \int R^3 + \alpha |\nabla R|^2 + C$$

Using Lemma 4.13 and (4.93) we get (4.92). □

Finally we compute

Lemma 4.29. *For $p > 6$ we have*

$$\left(\int |\nabla w|^p \right)^{\frac{3}{p}} \leq C(p) \int |\nabla w|^6 + C \quad (4.94)$$

Proof. For the purposes of this proof, we will alternate between computations with respect to the metric $g = e^{2w} g_0$ and the background metric g_0 . We will state explicitly which volume form is used in all cases.

First note that the Sobolev embedding theorem implies that in dimension 3, $W^{1,3} \hookrightarrow L^p$ for any p . So for any $f \in W^{1,3}$, we have

$$\left(\int |f|^p dv_0 \right)^{3/p} \lesssim \int |\nabla f|_0^3 dv_0 + \int |f|^3 dv_0$$

Taking $f = |\nabla w|_0 e^{\frac{3-p}{p}w}$ we have

$$\begin{aligned}\int f^p dv_0 &= \int |\nabla w|_0^p e^{(3-p)w} dv_0 \\ &= \int |\nabla w|^p dv_0\end{aligned}$$

Thus

$$\begin{aligned}(\int |\nabla w|^p dv)^{3/p} &\lesssim \int |\nabla(|\nabla w|_0 e^{\frac{3-p}{p}w})|_0^3 dv_0 + \int |\nabla w|_0^3 e^{\frac{9-3p}{p}w} dv_0 \\ &= \int |e^{\frac{3-p}{p}w} \nabla |\nabla w|_0 + |\nabla w|_0 \nabla e^{\frac{3-p}{p}w}|_0^3 dv_0 + \int |\nabla w|_0^3 e^{\frac{9-3p}{p}w} dv_0 \\ &\lesssim \int |\nabla_0^2 w|_0^3 e^{\frac{9-3p}{p}w} + |\nabla w|_0^6 e^{\frac{9-3p}{p}w} + |\nabla w|_0^3 e^{\frac{9-3p}{p}w} dv_0 \\ &\lesssim \int |\nabla_0^2 w|^3 e^{\frac{9-3p}{p}w} + |\nabla w|^6 e^{\frac{9-3p}{p}w} + C dv_0\end{aligned}\tag{4.95}$$

Since w is bounded the expression for the change of ∇^2 under a conformal scaling we have

$$|\nabla_0^2 w|_0^2 \lesssim e^{4w} |\nabla^2 w|^2 + e^{4w} |\nabla w|^4 \lesssim |\nabla^2 w|^2 + |\nabla w|^4\tag{4.96}$$

Using (4.96) in (4.95) gives

$$(\int |\nabla w|^p dv)^{\frac{3}{p}} \lesssim \int |\nabla^2 w|^3 + |\nabla w|^6 + C dv_g$$

By (4.28) we have

$$|\nabla^2 w|^3 \lesssim |A|^3 + |\nabla w|^6 + C\tag{4.97}$$

and so

$$(\int |\nabla w|^p)^{3/p} \lesssim \int |A|^3 + |\nabla w|^6 + C$$

using Theorem 4.25 gives (4.94). □

For the remainder of this section we revert to our convention that all norms, volumes, etc. are with respect to $g = e^{2w} g_0$.

We are now ready to show our main estimate.

Proposition 4.30.

$$\int |\nabla w|^p \leq C(p) \text{ for any } p > 0 \quad (4.98)$$

$$\int R^3 \leq C \quad (4.99)$$

$$\int |A|^3 \leq C \quad (4.100)$$

$$\int |\nabla^2 w|^2 \leq C \quad (4.101)$$

$$\|w\|_{C^\beta} \leq C(\beta) \text{ for any } \beta < 1 \quad (4.102)$$

Proof. of (4.98) Integrating by parts and using Holder's inequality, we have

$$\begin{aligned} \int |\nabla w|^6 &= \int \langle \nabla w, \nabla w \rangle |\nabla w|^4 \\ &= \int -w \Delta w |\nabla w|^4 - w \nabla w \nabla |\nabla w|^4 \\ &\lesssim \int |w| |\nabla^2 w| |\nabla w|^4 \\ &\leq \left(\int |\nabla^2 w|^2 |\nabla w|^2 \right)^{1/2} \left(\int |\nabla w|^6 w^2 \right)^{1/2} \\ &\leq \left(\int |\nabla^2 w|^2 |\nabla w|^2 \right)^{1/2} \left(\int |\nabla w|^{12} \right)^{1/8} \left(\int |\nabla w|^4 |w|^{8/3} \right)^{3/8} \end{aligned}$$

By Theorem 4.2 w is bounded, and so (4.89), (4.94) and Theorem 4.2 give

$$\begin{aligned} \int |\nabla w|^6 &\lesssim \left(\int \alpha |\nabla w|^6 + R^2 + C \right)^{1/2} \left(\int |\nabla w|^6 + C \right)^{1/2} \\ &\lesssim \alpha^{1/2} \int |\nabla w|^6 + \left(\int R^2 \right)^{1/2} \left(\int |\nabla w|^6 \right)^{1/2} + \left(\int R^2 \right)^{1/2} + \left(\int |\nabla w|^6 \right)^{1/2} + C \\ &\lesssim \int R^2 + C \end{aligned}$$

By Theorem 4.25 we have

$$\int R^2 \lesssim \left(\int R^3 \right)^{2/3} \lesssim \left(\int |\nabla w|^6 \right)^{2/3} + C \quad (4.103)$$

Combining the previous two inequalities gives

$$\int |\nabla w|^6 \leq C$$

so that (4.94) implies

$$\int |\nabla w|^p \leq C(p)$$

of (4.99) This follows from (4.98), (4.103) and Theorem 4.25.

of (4.100) This follows from (4.92) and (4.99).

of (4.101) This follows from (4.98), (4.101) and (4.97).

of (4.102) This follows since w is bounded and (4.98).

□

Proof. of Theorem 4.3 The above proposition applies to any solution w_α of (4.6) satisfying the hypotheses of Theorem 4.3. Letting $\alpha \rightarrow 0$ gives a (subsequence of) w_α converges weakly to a function w in $W^{2,2}$. Moreover, w has the properties given in Theorem 4.3. Note that by the same argument given in the proof of Theorem 4.2, w has non-negative scalar curvature almost everywhere.

□

Chapter 5

Final Remarks

5.1 Further Questions from Chapter 3

There are a few directions one might consider to continue the work from Chapter 3:

1. Is the approach in Section 3.2 necessary? Though sufficient, the argument seems surprisingly complicated. Can an alternative argument be given? What is the optimal constant in (3.5)? We note that in [8] the authors use a Lagrange-multiplier argument instead. Such an argument does give the value of an optimal constant. Unfortunately, the argument becomes significantly more complex when adapting to σ_k for $k > 2$. For this reason we instead developed the approach of Section 3.2.
2. Can the argument be extended to the subcritical case $n < 2k$? As a general rule, equations involving σ_k tend to be easier in the subcritical case $n < 2k$, so it is somewhat surprising that the approach seems to work only for $n \geq 2k$ (specifically, for $n < 2k$ the exponent of the volume term in (3.15) has the wrong sign). Moreover, by the result of [25] we know that the appropriate result is in fact true for $n < 2k$, but it would be interesting to find a proof of the result using integral estimates.
3. Is the assumption of finite volume necessary in Theorem 3.1? We note that in [8] finite volume is not a necessary hypothesis for solutions of $\sigma_2 = \text{constant}$ in dimensions 4 and 5, and in fact the same argument adapts to show that finite volume is not necessary in for $\sigma_k = \text{constant}$ in dimensions $2k$ and $2k + 1$. Also, finite volume is not a hypothesis in the work of [25], so perhaps it can be removed in this case.

4. Can Theorem 3.1 be adapted to non locally conformally flat manifolds? Can σ_k be replaced with the renormalized volume coefficients v_{2k} in the statement of Theorem 3.1? There are two obvious difficulties with approaching this problem. First, it is unclear what the analogue of the cone condition $g^{-1}A_g \in \Gamma_k^+$ in Theorem 3.1 should be for v_{2k} . Second, does the linearized form of v_{2k} satisfy the appropriate properties analogous to the Newton tensor T_k ? In particular, as the proof of (3.4) and (3.5) depends strongly on the algebraic structure of σ_k , what should the analogue of the estimates in Section 3.2 be?

5.2 Discussion of Additional Hypotheses Required for Theorems 4.2 and 4.3 in Dimension 3

As mentioned in the introduction, hypotheses (4.11) and (4.13) are required for Theorems 4.2 and 4.3 in dimension 3 but unnecessary in the critical dimension 4 (that is, not necessary in the argument of [7]).

The main difference from dimension 4 is that in dimension 3, (4.11), (4.13) do not follow from (4.9),(4.10),(4.14).

Specifically, note that in dimension 3, (4.17) is equivalent to both of the inequalities

$$\int |\nabla w|_0^4 e^{-w} dv_0 \leq C \tag{5.1}$$

$$\int |\nabla w|_{g_w}^4 dv_{g_w} \leq C \tag{5.2}$$

In dimension 4, the same argument as dimension 3 proves (5.2). However, the e^{-w} factor no longer appears in the the analogue of (5.1) because the e^{4w} term in the volume cancels with the e^{-4w} arising in the expression for the change of $|\nabla w|^4$ under conformal rescaling. Hence under (4.14) (but not (4.11),(4.13)) an argument analogous to the one in Section 4.4 implies that $\int |\nabla w|^4 dv_0$ is bounded in dimension 4. It follows easily that $\int e^{\rho w} dv_0 \leq C(\rho)$. Unfortunately, in dimension 3 we only obtain that $\int |\nabla w|^4 e^{-w} dv_0$ is bounded. Without making some additional assumption it is unclear why the volume must be bounded in this case.

In dimension 4, (4.13) follows from an upper and lower bound on the volume. Specifically, the analogue of (4.7) in dimension 4 is

$$\sigma_2(g^{-1}A_g) - \frac{\alpha}{4}\Delta_g R_g = \mu_\alpha$$

If we integrate this equation over the manifold we find

$$Vol(g)\mu_\alpha = \int \sigma_2(g^{-1}A_g)dv_g \quad (5.3)$$

However, the right hand side of (5.3) is known to be conformally invariant in dimension 4, see equation (0.2) and ensuing discussion in [7]. Hence in dimension 4 we see that (4.13) follows from (4.11). In dimension 3, however, we have

$$Vol(g)\mu_\alpha = \int 4\sigma_2(g^{-1}A_g) - \frac{\alpha}{32}R^2dv_g$$

The right hand side of this equation does not appear to be conformally invariant and so it is unclear if (4.13) would follow from (4.11).

5.3 Further Questions from Chapter 4

There are a few directions one might consider to continue the work from Chapter 4:

1. Are assumptions (4.11),(4.13) strictly necessary? What are the blow-up phenomenon for solutions of (4.6)? Perhaps an approach similar to that of [6] may be useful.
2. When is the linearized version of (4.6) invertible? That is, when do we have “openness” in the continuity approach discussed in the introduction? Unfortunately there do exist cases for which the linearized equation is not necessarily invertible—one can check that for some of the Berger spheres discussed in [24] the linearized version of (4.6) may in fact have a zero eigenvalue. Is there a natural geometric condition which forces the linearization to be invertible?
3. What happens in dimension at least 5? The author suspects that as in dimension 4, the linearization must be invertible in higher dimensions. The tradeoff, however, is that the a-priori estimates in Sections 4.4,4.5 are somehow “less strong” in higher dimensions and will not be enough to show the “closed” part of the continuity method. Perhaps another technique can resolve this issue? Moreover, it is as yet uncertain if the argument of 4.5 adapts to dimension 5 as the estimates are somewhat delicate and depend on constants which vary with dimension.
4. Can this approach be adapted to the renormalized volume coefficients v_{2k} ? At this point it appears that the most natural case to consider would be studying v_n in dimension n . Specifically, the a priori estimates only appear to be sufficient for the “closed” portion of the continuity

method in the subcritical and critical dimensions, while the linearization of (4.6) appears to be guaranteed to be invertible only in the critical and supercritical case. However, even for the next simplest case some significant concerns must be addressed. First, the 6-th order Q curvature does not appear to “connect” to v_6 as naturally as the 4-th order Q curvature “connects” to σ_2 in (4.6). Perhaps one should look for a different equation than $Q = \text{constant}$ as a starting point for the continuity method. Second, in Section 4.5 we added a linear combination of I and II to cancel the $\text{Tr}(E^3)$ term in their respective expressions. Most likely for v_n , $n \geq 6$, there will be more terms to cancel. Hence it will likely be necessary to take a linear combination of more than two summands, complicating the already involved computations from Section 4.5.

5. Perhaps it is worth mentioning that the estimate from Theorem 4.3 can likely be extended to show $W^{2,5-\epsilon}$ estimates for $\epsilon > 0$ (see [8] for the argument in dimension 4). Of course, one would like to see that the function obtained in Theorem 4.3 is in fact smooth. Obtaining higher regularity for σ_2 should not be difficult once $C^{1,\alpha}$ estimates for w are proved (see for example the use of convexity to apply the Evans-Krylov result used in the last section of [6]). Ideally, though, one would prefer to have an argument which avoids using the convexity properties of σ_k (and so, in particular, could be adapted to v_{2k}).
6. Is it possible to solve $\sigma_2 = \text{constant}$ using the variational structure of σ_2 ? That is, critical points of the functional

$$\mathcal{F}_0 = \int \sigma_2(g^{-1}A_g)dv_g$$

among volume 1 metrics in a fixed conformal class must necessarily satisfy $\sigma_2 = \text{constant}$. Before taking the approach discussed in Chapter 4 we originally sought to find a maximizer for $\mathcal{F}_0$. In the case when M is locally conformally flat, $\mathcal{F}_0$ is bounded from above when restricted to Γ_2^+ (see Proposition 11.6, [21]). However, for non-locally conformally flat manifolds it is not clear that $\mathcal{F}_0$ is still bounded. If we take this condition as a hypothesis, it may be possible to argue that a maximizer for $\mathcal{F}_0$ does in fact exist (and hence so would a solution for $\sigma_2 = \text{constant}$).

Chapter 6

Appendix

In this section we translate Lemma 2.1 of [35] to 3-dimensions. Our argument parallels theirs essentially line by line except that we will work in dimension 3 while their paper is for dimension 4. Conceptually, the argument is easier in dimension 3 because, in some sense, the statement $f \in W^{2,2}$ is stronger in dimension 3 than dimension 4. For the sake of completeness, however, we include the argument here.

First we will need to write (4.6) in terms of the conformal factor w and the background metric g_0 . As usual $g = e^{2w}g_0$. By (4.29) we have

$$\begin{aligned} R_g^2 &= e^{-4w} (R_0 - 4\Delta_0 w - 2|\nabla w|_0^2)^2 \\ &= e^{-4w} (R_0^2 - 8R_0\Delta_0 w - 4R_0|\nabla w|_0^2 + 16(\Delta_0 w)^2 + 16\Delta_0 w|\nabla w|_0^2 + 4|\nabla w|_0^4) \end{aligned} \quad (6.1)$$

Also, (4.30) gives

$$\begin{aligned} |Ric_g|_g^2 &= e^{-4w} |Ric_0 - \nabla_0^2 w + dw \otimes dw - \Delta_0 w g_0 - |\nabla w|_0^2 g_0|_0^2 \\ &= e^{-4w} (|Ric_0|_0^2 - 2\langle Ric_0, \nabla_0^2 w \rangle_0 + 2Ric_0(\nabla w, \nabla w) - 2R_0\Delta_0 w - 2R_0|\nabla w|_0^2 + |\nabla_0^2 w|_0^2 \\ &\quad - 2\nabla_0^2 w(\nabla w, \nabla w) + 2(\Delta_0 w)^2 + 2\Delta_0 w|\nabla w|_0^2 + |\nabla w|_0^4 - 2\Delta_0 w|\nabla w|_0^2 - 2|\nabla w|_0^4 \\ &\quad + 3(\Delta_0 w)^2 + 6\Delta_0 w|\nabla w|_0^2 + 3|\nabla w|_0^2) \\ &= e^{-4w} (|Ric_0|_0^2 - 2\langle Ric_0, \nabla_0^2 w \rangle_0 + 2Ric_0(\nabla w, \nabla w) - 2R_0\Delta_0 w - 2R_0|\nabla w|_0^2 + |\nabla_0^2 w|_0^2 \\ &\quad - 2\nabla_0^2 w(\nabla w, \nabla w) + 5(\Delta_0 w)^2 + 6\Delta_0 w|\nabla w|_0^2 + 2|\nabla w|_0^4) \end{aligned} \quad (6.2)$$

Next we note that in dimension 3 we have

$$\Delta_g f = e^{-2w}(\Delta_0 f + \langle \nabla w, \nabla f \rangle_0)$$

for any function f . Hence by (4.29) we have

$$\begin{aligned} \Delta_g R_g &= e^{-2w}(\Delta_0 R_g + \langle \nabla w, \nabla R_g \rangle_0) \\ &= e^{-2w}(\Delta_0(e^{-2w}(R_0 - 4\Delta_0 w - 2|\nabla w|_0^2)) + \langle \nabla w, \nabla(e^{-2w}(R_0 - 4\Delta_0 w - 2|\nabla w|_0^2)) \rangle_0) \\ &= e^{-4w}((-2\Delta_0 w + 4|\nabla w|_0^2)(R_0 - 4\Delta_0 w - 2|\nabla w|_0^2) - 4\langle \nabla w, \nabla R_0 - 4\nabla \Delta_0 w - 2\nabla |\nabla w|_0^2 \rangle_0 \\ &\quad + \Delta_0 R_0 - 4\Delta_0^2 w - 2\Delta_0 |\nabla w|_0^2 \\ &\quad + \langle \nabla w, -2\nabla w(R_0 - 4\Delta_0 w - 2|\nabla w|_0^2) + \nabla R_0 - 4\nabla \Delta_0 w - 2\nabla |\nabla w|_0^2 \rangle_0) \\ &= e^{-4w}(-2R_0 \Delta_0 w + 8(\Delta_0 w)^2 + 4\Delta_0 w |\nabla w|_0^2 + 4R_0 |\nabla w|_0^2 - 16\Delta_0 w |\nabla w|_0^2 - 8|\nabla w|_0^4 \\ &\quad - 4\langle \nabla w, \nabla R_0 \rangle_0 + 16\langle \nabla w, \nabla \Delta_0 w \rangle_0 + 8\langle \nabla w, \nabla |\nabla w|_0^2 \rangle_0 + \Delta_0 R_0 - 4\Delta_0^2 w - 2\Delta_0 |\nabla w|_0^2 \\ &\quad - 2R_0 |\nabla w|_0^2 + 8\Delta_0 w |\nabla w|_0^2 + 4|\nabla w|_0^4 + \langle \nabla w, \nabla R_0 \rangle_0 - 4\langle \nabla w, \nabla \Delta_0 w \rangle_0 - 2\langle \nabla w, \nabla |\nabla w|_0^2 \rangle_0) \\ &= e^{-4w}(-2R_0 \Delta_0 w + 8(\Delta_0 w)^2 - 4\Delta_0 w |\nabla w|_0^2 + 2R_0 |\nabla w|_0^2 - 4|\nabla w|_0^4 - 3\langle \nabla w, \nabla R_0 \rangle_0 \\ &\quad + 12\langle \nabla w, \nabla \Delta_0 w \rangle_0 + 6\langle \nabla w, \nabla |\nabla w|_0^2 \rangle_0 + \Delta_0 R_0 - 4\Delta_0^2 w - 2\Delta_0 |\nabla w|_0^2) \end{aligned} \tag{6.3}$$

Suppose w is a solution of (4.7). Multiplying (4.7) by e^{4w} and using (4.23) and (6.1)-(6.3) we have

$$\begin{aligned}
\mu_\alpha e^{4w} &= e^{4w} (4\sigma_2(g^{-1}A_g) - \frac{\alpha}{32}(R_g^2 + 8\Delta_g R_g)) \\
&= e^{4w} (-2|Ric_g|_g^2 + \frac{24-\alpha}{32}R_g^2 - \frac{\alpha}{4}\Delta_g R_g) \\
&= -2(|Ric_0|_0^2 - 2\langle Ric_0, \nabla_0^2 w \rangle_0 + 2Ric_0(\nabla w, \nabla w) - 2R_0\Delta_0 w - 2R_0|\nabla w|_0^2 \\
&\quad + |\nabla_0^2 w|_0^2 - 2\nabla_0^2 w(\nabla w, \nabla w) + 5(\Delta_0 w)^2 + 6\Delta_0 w|\nabla w|_0^2 + 2|\nabla w|_0^4) \\
&\quad + \frac{24-\alpha}{32}(R_0^2 - 8R_0\Delta_0 w - 4R_0|\nabla w|_0^2 + 16(\Delta_0 w)^2 + 16\Delta_0 w|\nabla w|_0^2 + 4|\nabla w|_0^4) \\
&\quad - \frac{\alpha}{4}(-2R_0\Delta_0 w + 8(\Delta_0 w)^2 - 4\Delta_0 w|\nabla w|_0^2 + 2R_0|\nabla w|_0^2 - 4|\nabla w|_0^4 - 3\langle \nabla w, \nabla R_0 \rangle_0 \\
&\quad + 12\langle \nabla w, \nabla \Delta_0 w \rangle_0 + 6\langle \nabla w, \nabla |\nabla w|_0^2 \rangle_0 + \Delta_0 R_0 - 4\Delta_0^2 w - 2\Delta_0|\nabla w|_0^2) \\
&= -2|Ric_0|_0^2 + 4\langle Ric_0, \nabla_0^2 w \rangle_0 - 4Ric_0(\nabla w, \nabla w) + (-2 + \frac{3\alpha}{4})R_0\Delta_0 w \\
&\quad + (1 - \frac{3\alpha}{8})R_0|\nabla w|_0^2 - 2|\nabla_0^2 w|_0^2 + 4\nabla_0^2 w(\nabla w, \nabla w) + (2 - \frac{5\alpha}{2})(\Delta_0 w)^2 \\
&\quad + \frac{\alpha}{2}\Delta_0 w|\nabla w|_0^2 + (-1 + \frac{7\alpha}{8})|\nabla w|_0^4 + \frac{24-\alpha}{32}R_0^2 + \frac{3\alpha}{4}\langle \nabla w, \nabla R_0 \rangle_0 \\
&\quad - 3\alpha\langle \nabla w, \nabla \Delta_0 w \rangle_0 - \frac{3\alpha}{2}\langle \nabla w, \nabla |\nabla w|_0^2 \rangle_0 - \frac{\alpha}{4}\Delta_0 R_0 + \alpha\Delta_0^2 w + \frac{\alpha}{2}\Delta_0|\nabla w|_0^2
\end{aligned}$$

This can be rewritten as

$$\begin{aligned}
&\alpha\Delta_0^2 w + \frac{\alpha}{2}\Delta_0|\nabla w|_0^2 - 3\alpha\operatorname{div}(\Delta_0 w \nabla w) - \frac{3\alpha}{2}\operatorname{div}(|\nabla w|_0^2 \nabla w) \tag{6.4} \\
&= 2|\nabla_0^2 w|_0^2 - 4\nabla_0^2 w(\nabla w, \nabla w) - (2 + \frac{\alpha}{2})(\Delta_0 w)^2 - 2\alpha\Delta_0 w|\nabla w|_0^2 - (-1 + \frac{7\alpha}{8})|\nabla w|_0^4 \\
&\quad - 4\langle Ric_0, \nabla_0^2 w \rangle_0 + 4Ric_0(\nabla w, \nabla w) - (-2 + \frac{3\alpha}{4})R_0\Delta_0 w - (1 - \frac{3\alpha}{8})R_0|\nabla w|_0^2 \\
&\quad - \frac{3\alpha}{4}\langle \nabla w, \nabla R_0 \rangle_0 + 2|Ric_0|_0^2 - \frac{24-\alpha}{32}R_0^2 + \frac{\alpha}{4}\Delta_0 R_0 + \mu_\alpha e^{4w}
\end{aligned}$$

We remark that (6.4) is analogous to equation (1.3) of [35].

Lemma 6.1. *If w is a smooth solution of (4.6) and $\|w\|_{W^{2,p}} \leq M$ for some $p \geq 2$, then for each k we have*

$$|\partial^\alpha w| \leq C(M, k, \alpha, \frac{1}{\alpha})$$

Here α is a multi-index with $|\alpha| = k$.

Proof. First we recall a few properties of multiplication of functions in Sobolev spaces. We have in

particular for $1 \leq p < 3$, multiplication is a continuous linear map of

$$W^{1,p} \oplus W^{1,p} \rightarrow W^{1,q}, \quad q = \frac{3p}{6-p} \quad (6.5)$$

$$L^p \oplus W^{1,p} \rightarrow L^q, \quad q = \frac{3p}{6-p} \quad (6.6)$$

$$W^{1,p} \oplus W^{1,p} \oplus W^{1,p} \rightarrow L^q, \quad q = \frac{p}{3-p} \quad (6.7)$$

For $3 < p < \infty$, we have

$$W^{1,p} \oplus W^{1,p} \rightarrow W^{1,p} \quad (6.8)$$

$$L^p \oplus W^{1,p} \rightarrow L^p \quad (6.9)$$

$$W^{1,p} \oplus W^{1,p} \oplus W^{1,p} \rightarrow W^{1,p} \quad (6.10)$$

(see [28]). Now, since $w \in W^{2,p}$ it is an easy consequence of Holder's inequality that the right hand side of (6.4) is in $L^{p/2}$. Following the notation of Lemma 2.1 in [35], we can interpret (6.4) as saying that for $2 \leq p < 3$,

$$\Delta_0^2 w + W^{-1, \frac{3p}{6-p}} + W^{-1, \frac{3p}{6-p}} + W^{-1, \frac{p}{3-p}} \in L^{\frac{p}{2}} \quad (6.11)$$

To make sense of this notation, note that the second term on the left hand side of (6.4) is $\frac{\alpha}{2} \Delta_0 |\nabla w|_0^2$. Since $w \in W^{2,p}$, we have $\nabla w \in W^{1,p}$. By (6.5) $|\nabla w|^2 \in W^{1, \frac{3p}{6-p}}$. Hence $\Delta_0 |\nabla w|_0^2 \in W^{-1, \frac{3p}{6-p}}$. The explanation for the other terms on the left hand side of (6.11) is similar.

We remark that here the dependence of C on $\alpha, \frac{1}{\alpha}$ arises due to the coefficients of the terms in the left hand side of (6.4).

Now, for $2 \leq p < 3$ we have $\frac{p}{3-p} \geq \frac{3p}{6-p}$. Also, note that $L^{\frac{p}{2}}$ compactly embeds into $W^{-1, \frac{3p}{6-p}}$. Hence we have

$$\Delta_0^2 w \in W^{-1, \frac{3p}{6-p}}$$

By elliptic regularity and Sobolev embedding,

$$w \in W^{3, \frac{3p}{6-p}} \subset W^{2, \frac{3p}{6-2p}}$$

We may repeat this process as necessary to obtain $w \in W^{2,p'}$ for some $p' > 3$ with $\|w\|_{W^{2,p'}} \leq C(M, \alpha, \frac{1}{\alpha})$.

Now, (6.8)-(6.10) imply that (6.4) can be written

$$\Delta_0^2 w + W^{-1,p'} + W^{-1,p'} + W^{-1,p'} \in L^{\frac{p'}{2}}$$

Arguing as before we have $w \in W^{3,p'}$. As $p' > 3$, we have $w \in C^{2,\alpha}$ for some $0 < \alpha < 1$ and elliptic regularity gives the desired result. $\square$

We remark that the above argument actually works for any $w \in W^{2,p}$ (not necessarily smooth) which is a solution of (4.6) in an appropriate weak sense (again analogous to Lemma 2.1 in [35]). However, such a result is more than is necessary for our purposes.

Bibliography

- [1] Aubin, Thierry. “Equations différentielles non linéaires et problème de Yamabe concernant la courbure scalaire.” *J. Math. Pures Appl.* 55 (1976), 269-296.
- [2] Besse, Arthur L. “Einstein manifolds.” Springer-Verlag, Berlin, 1987.
- [3] Branson, Thomas P. “Differential operators canonically associated to a conformal structure.” *Math. Scand.* 57 (1985), no. 2, 293-345.
- [4] Chang, Sun-Yung Alice; Fang, Hao. “A class of variational functionals in conformal geometry.” *Int. Math. Res. Not. IMRN* 2008, no. 7.
- [5] Chang, Sun-Yung Alice; Fang, Hao; Graham, C. Robin. “A note on renormalized volume functionals.” *Differential Geom. Appl.* Available online 22 November 2013, ISSN 0926-2245, <http://dx.doi.org/10.1016/j.difgeo.2013.10.001>
- [6] Chang, Sun-Yung A.; Gursky, Matthew J.; Yang, Paul C. “An a priori estimate for a fully nonlinear equation on four-manifolds.” Dedicated to the memory of Thomas H. Wolff. *J. Anal. Math.* 87 (2002), 151-186.
- [7] Chang, Sun-Yung A.; Gursky, Matthew J.; Yang, Paul C. “An equation of Monge-Ampère type in conformal geometry, and four-manifolds of positive Ricci curvature.” *Ann. of Math.* (2) 155 (2002), no. 3, 709-787.
- [8] Chang, Sun-Yung A.; Gursky, Matthew J.; Yang, Paul C. “Entire solutions of a fully nonlinear equation.” *Lectures on partial differential equations*, 43-60, *New Stud. Adv. Math.*, 2, Int. Press, Somerville, MA 2003.
- [9] Chang, Sun-Yung A.; Hang, Fengbo. “Integral identities and Minkowski type inequalities involving Schouten tensor.” *Int. Math. Res. Not. IMRN* 2007, no. 11.

- [10] Chen, Szu-Yu Sophie. “Local estimates for some fully nonlinear elliptic equations.” *Int. Math. Res. Not.* 2005, no. 55, 3403-3425.
- [11] Fefferman, Charles; Graham, C. Robin. “Conformal Invariants.” *Elie Cartan et les Mathématiques d’Aujourd’hui, Asterisque* (1985), 95-116.
- [12] Garding, Lars. “An inequality for hyperbolic polynomials.” *J. Math. Mch.* 8 (1959), 957-965.
- [13] Ge, Yuxin; Lin, Chang-Shou; Wang, Guofang. “On the σ_2 -scalar curvature.” *J. Differential Geom.* 84 (2010), no. 1, 45-86.
- [14] Gonzalez Nogueras, Maria del Mar. “Singular sets of a class of fully-nonlinear equations in conformal geometry.” Thesis (Ph.D.)-Princeton University. 2004.
- [15] Graham, C. Robin. “Volume and area renormalizations for conformally compact Einstein metrics.” *The Proceedings of the 19th Winter School “Geometry and Physics” (Srn, 1999).* *Rend. Circ. Mat. Palermo* (2) Suppl. No. 63 (2000), 3142.
- [16] Graham, C. Robin. “Extended obstruction tensors and renormalized volume coefficients.” *Adv. Math.* 220 (2009), no. 6, 1956-1985.
- [17] Graham, C. Robin; Jenne, Ralph; Mason, Lionel J.; Sparling, George A. J. “Conformally invariant powers of the Laplacian. I. Existence.” *J. London Math. Soc.* (2) 46 (1992), no. 3, 557-565.
- [18] Guan, Pengfei; Viaclovsky, Jeff; Wang, Guofang. “Some properties of the Schouten tensor and applications to conformal geometry.” *Trans. Amer. Math. Soc.* 355 (2003), no. 3, 925-933.
- [19] Gursky, Matthew J.; Viaclovsky, Jeff A. “A new variational characterization of three-dimensional space forms.” *Invent. Math.* 145 (2001), no. 2, 251-278.
- [20] Gursky, Matthew J.; Viaclovsky, Jeff A. “Prescribing symmetric functions of the eigenvalues of the Ricci tensor.” *Ann. of Math.* (2) 166 (2007), no. 2, 475-531.
- [21] Guan, Pengfei. “Topics in geometric fully nonlinear equations.” Accessed 9 December 2103. <http://www.math.mcgill.ca/guan/zheda0508.pdf>
- [22] Guan, Pengfei; Wang, Guofang. “A fully nonlinear conformal flow on locally conformally flat manifolds.” *J. Reine Angew. Math.* 557 (2003), 219-238.

- [23] Guan, Pengfei; Wang, Guofang. "Local estimates for a class of fully nonlinear equations arising from conformal geometry." *Int. Math. Res. Not.* 2003, no. 26, 1413-1432.
- [24] Hang, Fengbo; Yang, Paul C. "The Sobolev inequality for Paneitz operator on three manifolds." *Calc. Var. Partial Differential Equations* 21 (2004), no. 1, 57-83.
- [25] Li, Aobing; Li, Yanyan. "On some conformally invariant fully nonlinear equations." *Comm. Pure Appl. Math.* 56 (2003), no. 10, 1416-1464.
- [26] Maldacena, Juan. "The Large N limit of superconformal field theories and supergravity." *Advances in Theoretical and Mathematical Physics* 2 (1998) 231-252.
- [27] Obata, Morio. "Certain conditions for a manifold to be isometric with a sphere." *J. Math. Soc. Japan* 14 (1962), 333-340.
- [28] Palais, Richard S. "Foundations of global non-linear analysis." W. A. Benjamin Inc., New York-Amsterdam, 1968.
- [29] Reilly, Robert C. "On the Hessian of a function and the curvatures of its graph." *Michigan Math. J.* 20 (1973), 373-383.
- [30] Schoen, Richard. "Conformal deformation of a Riemannian metric to constant scalar curvature." *J. Diff. Geom.* 20 (1984), 479-495.
- [31] Sheng, Wei-Min; Trudinger, Neil S.; Wang, Xu-Jia. "The Yamabe problem for higher order curvatures." *J. Differential Geom.* 77 (2007), no. 3, 515-553.
- [32] Stein, Elias M.; Weiss, Guido. "Introduction to fourier analysis on Euclidean spaces," Princeton Math. Series 32, Princeton Univ. Press, Princeton, NJ, 1971.
- [33] Trudinger, Neil S. "Remarks concerning the conformal deformation of Riemannian structures on compact manifolds." *Ann. Scuola Norm. Sup. Pisa* 22 (1968), 265-274.
- [34] Trudinger, Neil S.; Wang, Xu-Jia. "The intermediate case of the Yamabe problem for higher order curvatures." *Int. Math. Res. Not. IMRN* 2010, no. 13, 2437-2458.
- [35] Uhlenbeck, Karen; Viaclovsky, Jeff. "Regularity of weak solutions to critical exponent variational equations." *Math. Res. Lett.* 7 (2000), no. 5-6, 651-656.
- [36] Viaclovsky, Jeff A. "Conformal geometry, contact geometry, and the calculus of variations." *Duke Math. J.* 101 (2000), no. 2, 283-316.

- [37] Xu, Xingwang; Yang, Paul C. “On a fourth order equation in 3-D.” A tribute to J. L. Lions. ESIAM Control Optim. Calc. Var. 8 (2002) 1029-1042.
- [38] Yamabe, Hidehiko. “On a deformation of Riemmanian structures on compact manifolds.” Osaka Math. J. 12 (1960), 21-37.