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Integrodifferential equations for fluids in two dimensions

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Princeton, NJ : Princeton University

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Integrodifferential equations for fluids in two dimensions

Jia Shi

A Dissertation
Presented to the Faculty
of Princeton University
in Candidacy for the Degree
of Doctor of Philosophy

Recommended for Acceptance
by the Department of
Mathematics
Adviser: Charles Fefferman, Javier Gomez-Serrano

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Abstract

In this thesis, we study two problems. The first one is about the regularity of the solutions to the Muskat equation. The Muskat equation describes the interface of two liquids in a porous medium. We show that if a solution to the Muskat problem is sufficiently smooth, then it must be analytic except at the points where a turnover of the fluids happens. We also show analyticity in a region that degenerates at the turnover points provided some additional conditions are satisfied.

The other problem concerns the radial symmetry properties of stationary and uniformly rotating solutions of the 2D Euler/g-SQG equations. We show some rigidity results giving conditions under which the solutions must be radial. We also show some flexibility results: the existence of non-radial solutions. The results on this second problem are joint work with Javier Gomez-Serrano, Jaemin Park and Yao Yao. They are based on the papers [42], [44], [45], and [43].

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To my family and friends

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Chapter 1

The regularity of the solutions to the Muskat equation

1.1 Introduction

The Muskat problem is a free boundary problem studying the interface between fluids in porous media [76]. It can also describe the Hele-Shaw cell [87]. We focus on the problem where two fluids have different density ρ_1, ρ_2 and the same viscosity. The equation in the periodic setting reads

$$\frac{\partial z_i}{\partial t}(\alpha,t) = \frac{\rho_2-\rho_1}{2}\int_{-\pi}^{\pi} \frac{\sin(z_1(\alpha)-z_1(\alpha-\beta))(\partial_\alpha z_i(\alpha)-\partial_\alpha z_i(\alpha-\beta))}{\cosh(z_2(\alpha)-z_2(\alpha-\beta))-\cos(z_1(\alpha)-z_1(\alpha-\beta))}d\beta, \tag{1.1}$$

for $i=1,2$. Here $z(\alpha,t)=(z_1(\alpha,t),z_2(\alpha,t))$ is a parameterization of the boundary curve. $z(\alpha,t)-(\alpha,0)$ is periodic in α.

Depending on the position of the heavier fluid, the Muskat problem is divided into three regimes. When the interface is a graph and the heavier fluid is on the bottom, it is a stable regime. When the interface is a graph and heavier one is above the boundary, it is a stable regime when goes backward in time. The third regime is an unstable regime with turnover points in the curve, in which the heavier liquid lies below at some place and above at other place. The Rayleigh-Taylor coefficient σ is used to characterize these three regimes and is defined as

$$\sigma = \frac{\rho_2-\rho_1}{2}\frac{\partial_\alpha z_1(\alpha,t)}{(\partial_\alpha z_1(\alpha,t))^2+(\partial_{\alpha,t} z_2(\alpha,t))^2}. \tag{1.2}$$

$\sigma \geq 0$ is corresponding to the stable regime and $\sigma \leq 0$ the backward stable regime. When σ changes

sign, it is in the unstable regime.

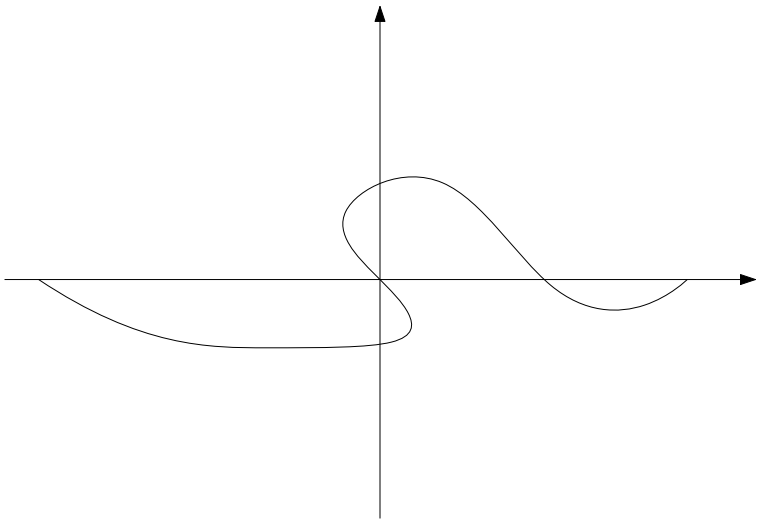


Figure 1.1: An interface with turnover points.

In the unstable regime, the solution which has turnover points was discovered in [13], [14], and [24], [25] .

However uniqueness is not known and is a hard unsolved problem. The trouble is that the known solutions are real analytic, and uniqueness can be proven only within the class of real analytic solutions. For the analytic solutions, the shrinking of analycitity region leads to a good term which can help us to get the uniqueness. So our current goal is to get a complete understanding of the analyticity behavior.

We show the sufficiently smooth solution is analytic except at the turnover points, and the degenerated analyticity near the turnover points. The existence and uniqueness are crucially related to the way the real-analyticity degenerates at those points.

Theorem 1.1.1. *Let* $f(\alpha,t) = (f_1(\alpha,t), f_2(\alpha,t)) \in C^1([-t_0,t_0], H^6[-\pi.\pi])$ *be a solution of the Muskat equation satisfying the Arc-chord condition. If* $\partial_\alpha f_1(\alpha_0,t) \neq 0$*, and* $-t_0 < t < t_0$*, then* $f(\cdot,t)$ *is analytic at* α_0*.*

1.1.1 The outline of the proof of Theorem 1.1.1

In Section 2.2, we define a cut off function $\lambda(\alpha)$ and focus on $f^c(\alpha,t) = \lambda(\alpha)f(\alpha,t)$. We then localize the equation such that the modified $R-T$ condition has a fixed sign. In order to make use of the sign, if the sign is positive, we let the time go forward, if the sign is negative, we let the time go backward.

In Section 2.3, we introduce $c(\alpha)$ with $\operatorname{supp} c(\alpha) \subset \{\alpha|\lambda(\alpha) = 1\}$. With the assumption that $f^c(\alpha, t)$ is analytic in domain $D_A = \{\alpha + iy| - c(\alpha)t \leq y \leq c(\alpha)t\}$, we derive the equation on the curve $\{(\alpha + ic(\alpha)\gamma t)|\alpha \in [-\pi, \pi]\}$ for fixed $\gamma \in [-1, 1]$. Then we obtain the equation

$$\frac{d}{dt} z(\alpha, \gamma, t) = T(z(\alpha, \gamma, t), t), \tag{1.3}$$

with $z(\alpha, \gamma, 0) = f^c(\alpha, 0)$. The analyticity assumption on f^c is dropped after we get (1.3).

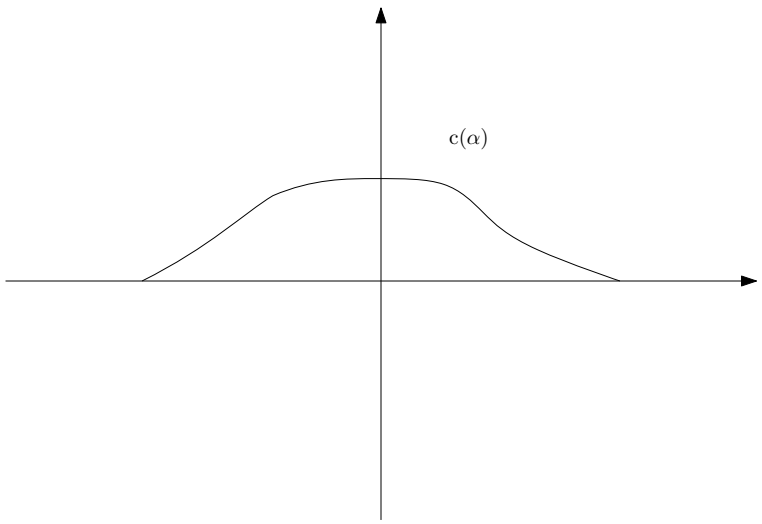


Figure 1.2: The curve $c(\alpha)$ for Theorem 1.1.1.

In section 2.4, for each fixed γ, we use the energy estimate and the Galerkin method to show the existence of the solution $z(\alpha, \gamma, t)$. The main term is controlled by Gårding's inequality, where we use a lemma from [14]. This part is similar as in [12] and [14].

In Section 2.5, Section 2.6, and Section 2.7, we verify that the $z(\alpha, 0, t)$ coincides with the $f^c(\alpha, t)$ and that $z(\alpha, \gamma, t)$ is also smooth enough with respect to γ.

In Section 2.8, we derive some lemmas about the Cauchy-Riemann operator and use those lemmas to show analyticity of $z(\alpha, \frac{y}{c(\alpha)t}\gamma, t)$ by checking it satisfies the Cauchy-Riemann equations.

Remark 1.1.2. *In [12], the analyticity domain can be chosen as a strip, and the analyticity follows directly from existence. Since our $c(\alpha)$ supported in a small region, we do not have such good behavior.*

Remark 1.1.3. *Our method concerning the analytcity is not limited to the Muskat problem. See Section 11.3 in Chapter 11.*

At turnover point α_0, $\partial_\alpha z_1(\alpha_0, t) = 0$. If we assume the second order derivative not vanishing

at this point, the Rayleigh-Taylor condition has different sign at each side of the α_0. Therefore we cannot expect the same instant analyticity result and break down happens. Given an extra assumption, we have the following theorem:

Theorem 1.1.4. *Let $f(\alpha,t)=(f_1(\alpha,t),f_2(\alpha,t))\in C^1([-t_0,t_0],C^{100}([-\pi.\pi])$ be a solution of the Muskat equation with two turnover points. $Z_1(t)$, $Z_2(t)$ are values of α of these two turnover points. If we assume the solution satisfying the following three conditions:*

$$\partial_\alpha^2 f_1(Z_1(t),t)\neq 0, \tag{1.4}$$

$$\partial_\alpha f_1(\alpha,t)\neq 0 \text{ except at } Z_1(t),\ Z_2(t), \tag{1.5}$$

and

$$(\frac{dZ_1}{dt}(t)+p.v.\int_{-\pi}^{\pi}K(f(Z_1(t),t)-f(\beta,t))d\beta)\partial_\alpha^2 f_1(Z_1(t),t)<0, \tag{1.6}$$

then when $-t_0<t<t_0$, $f(\cdot,t)$ can be analytically extended to region $\Omega=\{x+iy|-\epsilon_1(t)+Z_\mu(t)\leq x\leq Z_1(t)+\epsilon_1(t),|y|\leq\epsilon_2(t)(x-Z_1(t))^2\}$.

Remark 1.1.5. *For generic initial condition with two turnover points, the quantity* (1.6) *is not 0. Here we treat case where this quantity is negative. It would be very interesting to understand also the greater than or equal to 0 case.*

1.1.2 The outline of the proof of Theorem 1.1.4

In Section 3.2, we discuss the extra assumption (1.6). In Section 3.3, we change the variable such that the turnover happens at two fixed α. Without loss of generality, we let one of the points be $\alpha=0$ and focus on the behavior around 0.

In Section 3.4, we localize the equation and split the solution in to right hand side and left hand side at 0. In order to avoid the singularity, we subtract a polynomial of high order before we do the split. Without loss of generality, we only study the solution on the right hand side $\tilde{f}^+(\alpha,t)$.

In Section 3.5 and 3.6, we introduce a translation by a function $\tau(t)$ with $\tau(0)=0$, a modified

equation

$$\frac{d}{dt}h(\alpha,\gamma,t) = T(h(\alpha,\gamma,t),\gamma,t), \tag{1.7}$$

with $h(\alpha,\gamma,0) = \tilde{f}^{+}(\alpha,0)$, and a desingularized version

$$\frac{d}{dt}h^{\epsilon}(\alpha,\gamma,t) = T^{\epsilon}(h(\alpha,\gamma,t),\gamma,t), \tag{1.8}$$

with $h^{\epsilon}(\alpha,\gamma,0) = \tilde{f}^{+}(\alpha,0)$. They satisfies the following conditions:

1. When $\gamma = 0$, $\tilde{f}^{+(20)}(\alpha+\tau(t),t)$ is a solution of (1.7).

2. If $\tilde{f}^{+(20)}(\alpha,t)$ is analytic in $\tilde{D}_A = \{\alpha+iy | \alpha \geq 0, -c(\alpha)\gamma t \leq y \leq c(\alpha)\gamma t\}$, then $\tilde{f}^{+(20)}(\alpha+\tau(t)+ic(\alpha+\tau(t))\gamma t,t)$ is a solution of (1.7). (1.3).

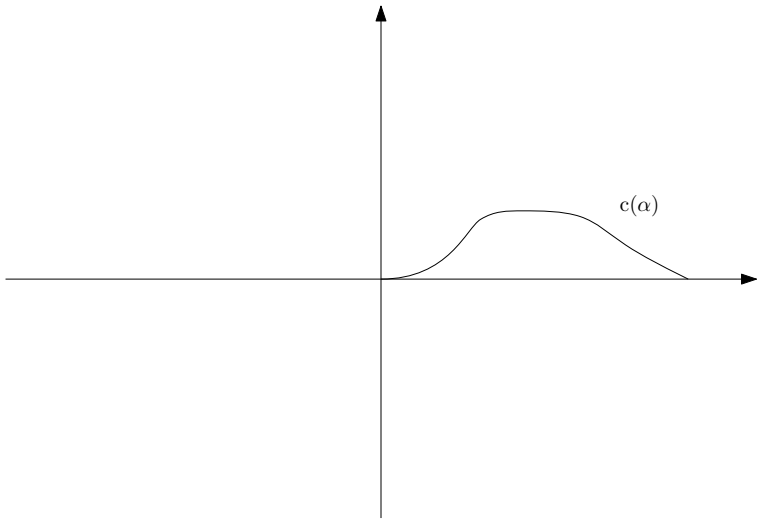


Figure 1.3: The curve $c(\alpha)$ for Theorem 1.1.4.

3. If $h^{\epsilon}(\alpha,\gamma,t) \sim (\alpha+\tau(t))^k$ near $\alpha = -\tau(t)$, then $T^{\epsilon}(h^{\epsilon}(\alpha,\gamma,t)) \sim (\alpha+\tau(t))^k$ near $\alpha = -\tau(t)$.

4.$\lim_{\epsilon\to 0} T^{\epsilon}(h(\alpha,\gamma,t)) = T(h(\alpha,\gamma,t))$ if $h(\alpha,\gamma,t)$ is sufficiently smooth.

In Section 3.7 and 3.8, we prove that the T^{ϵ} satisfies the properties of the Picard theorem.

In Section 3.9, we prove an energy estimate of (1.8) which is independent of ϵ. A modified version of Gårding's inequality lemma 3.9.4 is introduced . In the end of the subsection, we get the existence of solution $h(\alpha,\gamma,t) \in C^1_t([0,t_0], C^1_{\gamma}([-1,1], H^5_{\alpha}[-\pi,\pi]))$, for sufficently small t_0.

In Section 3.10, we show $h(\alpha,0,t) = \tilde{f}^{+(20)}(\alpha+\tau(t),t)$. Therefore $h(\alpha,\gamma,t)$ is an extension of $\tilde{f}^{+(20)}(\alpha+\tau(t),t)$.

In Section 3.11, we again introduce some lemmas of the Cauchy-Riemann operator and use the

them to show the analyticity.

Remark 1.1.6. *The modified Gårding's inequality in Section 3.9 is non-trivial and not only a perturbation of the previous version lemma from [14]. That is because* $\Lambda_{\mathcal{T}_\pi}(\max(\alpha, 0))$ *is not in* L^∞*, where* $\Lambda_{\mathcal{T}_\pi}$ *is* $(-\Delta)^{\frac{1}{2}}$ *on the Torus* $[0, \pi]$*.*

Remark 1.1.7. *The additional assumption* (1.6) *is used to show the translation function* $\tau(t)$ *increasing, which is crucial to our proof when we show the Picard theorem.*

Chapter 2

The regularity far away from the turnover points

2.1 Notation

In the chapter we use the following notation:

δ: a sufficiently small number.

$\lambda(\alpha)$: $\lambda(\alpha) \geq 0$ and in $C^{100}(-\infty, \infty)$, satisfying

$$\lambda(\alpha) = \begin{cases} 1 & |\alpha| \leq \delta, \\ 0 & |\alpha| \geq 2\delta, \end{cases}$$

δ_c: sufficiently small number depending on δ.

$c(\alpha)$:

$$\begin{cases} c(\alpha) = \delta_c, \text{ when } \alpha \leq \frac{\delta}{32}, \\ \operatorname{supp} c(\alpha) \subset [-\frac{\delta}{8}, \frac{\delta}{8}] \\ c(\alpha) \in C^{100}(-\infty, \infty), \|c(\alpha)\|_{C^{100}(-\infty,\infty)} \leq \delta. \end{cases}$$

$f(\alpha, t) = (f_1(\alpha, t), f_2(\alpha, t))$: the original solution of the Muskat equation.

$f^c(\alpha, t)$, $\tilde{f}(\alpha, t)$:

$$f^c(\alpha, t) = \lambda(\alpha) f(\alpha, t),$$

$$\tilde{f}(\alpha, t) = (1 - \lambda(\alpha)) f(\alpha, t).$$

t_0: the original solution exists when $t \in [-t_0, t_0]$.

t_1: the energy estimate holds in $[0, t_1]$.

2.2 Localize the equation

This step is to localize the equation such that the $R - T$ coefficient has a fixed sign. Without loss of generality, we study the behavior at origin. Let $\lambda(\alpha) \in C^{100}(-\infty, \infty)$ satisfying $\lambda(\alpha) \geq 0$ and

$$\lambda(\alpha) = \begin{cases} 1 & |\alpha| \leq \delta, \\ 0 & |\alpha| \geq 2\delta, \end{cases}$$

and $f^c(\alpha, t) = f(\alpha, t)\lambda(\alpha)$, $\tilde{f}(\alpha, t) = f(\alpha, t)(1 - \lambda(\alpha))$. Here δ is a sufficiently small number such that when $\alpha \in [-2\delta, 2\delta]$, $\partial_\alpha f_1(\alpha, 0)$ has a fixed sign. Without loss of generality, we assume

$$\partial_\alpha f_1(\alpha, 0) > 0. \tag{2.1}$$

Then we have

$$\frac{\partial f_\mu^c}{\partial t} = \lambda(\alpha) \int_{-\pi}^{\pi} \frac{\sin(f_1^c(\alpha) - f_1^c(\beta) + \tilde{f}_1(\alpha) - \tilde{f}_1(\beta))(\partial_\alpha f_\mu^c(\alpha) - \partial_\alpha f_\mu^c(\beta))}{\cosh(f_2^c(\alpha) - f_2^c(\beta) + \tilde{f}_2(\alpha) - \tilde{f}_2(\beta)) - \cos(f_1^c(\alpha) - f_1^c(\beta) + \tilde{f}_1(\alpha) - \tilde{f}_1(\beta)))} d\beta + \tag{2.2}$$

$$\lambda(\alpha) \int_{-\pi}^{\pi} \frac{\sin(f_1^c(\alpha) - f_1^c(\beta) + \tilde{f}_1(\alpha) - \tilde{f}_1(\beta))(\partial_\alpha \tilde{f}_\mu(\alpha) - \partial_\alpha \tilde{f}_\mu(\beta))}{\cosh(f_2^c(\alpha) - f_2^c(\beta) + \tilde{f}_2(\alpha) - \tilde{f}_2(\beta)) - \cos(f_1^c(\alpha) - f_1^c(\beta) + \tilde{f}_1(\alpha) - \tilde{f}_1(\beta)))} d\beta. \tag{2.3}$$

We have $f^c \in C^1([0, t_0], H^6(\mathcal{T}))$, $\tilde{f} \in C^1([0, t_0], H^6[-\pi, \pi])$.

2.3 The equation on the complex plane

2.3.1 Change the contour

Let $c(\alpha)$ satisfy

$$\begin{cases} c(\alpha) = \delta_c, \text{ when } \alpha \leq \frac{\delta}{32}, \\ \operatorname{supp} c(\alpha) \subset [-\frac{\delta}{8}, \frac{\delta}{8}] \\ c(\alpha) \in C^{100}(-\infty, \infty), \|c(\alpha)\|_{C^{100}(-\infty,\infty)} \leq \delta. \end{cases} \tag{2.4}$$

Here $c(\alpha)$ is defined such that $\tilde{f}, \lambda$ can be analytically extended to the complex domain $D_A = \{(x+iy)| -\infty \le x \le \infty, -c(x)t \le y \le c(x)t\}$ and satisfy

$$\tilde{f}(\alpha + ic(\alpha)t) = \tilde{f}(\alpha), \tag{2.5}$$

and

$$\lambda(\alpha + ic(\alpha)t) = \lambda(\alpha). \tag{2.6}$$

Now we assume f^c is also analytic in this complex domain D_A. For any fixed $\gamma \in [-1, 1]$, we want to find the new equation on the contour $\{\alpha + ic(\alpha)\gamma t | \alpha \in [-\pi, \pi]\}$. Let $\alpha^t_\gamma = \alpha + ic(\alpha)\gamma t$. We have

$$\frac{df^c_\mu(\alpha^t_\gamma, t)}{dt} = ic(\alpha)\gamma(\partial_\alpha f^c_\mu)(\alpha^t_\gamma, t) + (\partial_t f^c_\mu)(\alpha^t_\gamma, t) \tag{2.7}$$
$$= ic(\alpha)\gamma(\partial_\alpha f^c_\mu)(\alpha^t_\gamma, t)$$
$$+ \lambda(\alpha^t_\gamma)\int_{-\pi}^{\pi} \frac{\sin(f^c_1(\alpha^t_\gamma, t) - f^c_1(\beta^t_\gamma, t) + \tilde{f}_1(\alpha^t_\gamma, t) - \tilde{f}_1(\beta^t_\gamma, t))((\partial_\alpha f^c_\mu)(\alpha^t_\gamma, t) - (\partial_\beta f^c_\mu)(\beta^t_\gamma, t))}{\cosh(f^c_2(\alpha^t_\gamma, t) - f^c_2(\beta^t_\gamma, t) + \tilde{f}_2(\alpha^t_\gamma, t) - \tilde{f}_2(\beta^t_\gamma, t)) - \cos(f^c_1(\alpha^t_\gamma, t) - f^c_1(\beta^t_\gamma, t) + \tilde{f}_1(\alpha^t_\gamma, t) - \tilde{f}_1(\beta^t_\gamma, t))}$$
$$(1 + ic'(\beta)\gamma t)d\beta$$
$$+ \lambda(\alpha^t_\gamma)\int_{-\pi}^{\pi} \frac{\sin(f^c_1(\alpha^t_\gamma, t) - f^c_1(\beta^t_\gamma, t) + \tilde{f}_1(\alpha^t_\gamma, t) - \tilde{f}_1(\beta^t_\gamma, t))((\partial_\alpha \tilde{f}_\mu)(\alpha^t_\gamma, t) - (\partial_\beta \tilde{f}_\mu)(\beta^t_\gamma, t))}{\cosh(f^c_2(\alpha^t_\gamma, t) - f^c_2(\beta^t_\gamma, t) + \tilde{f}_2(\alpha^t_\gamma, t) - \tilde{f}_2(\beta^t_\gamma, t)) - \cos(f^c_1(\alpha^t_\gamma, t) - f^c_1(\beta^t_\gamma, t) + \tilde{f}_1(\alpha^t_\gamma, t) - \tilde{f}_1(\beta^t_\gamma, t))}$$
$$(1 + ic'(\beta)\gamma t)d\beta.$$

2.3.2 The equation for z

Let $z(\alpha, \gamma, t)$ be the solution of the equation (2.7) with initial data $z(\alpha, \gamma, 0) = f^c(\alpha, 0)$. Our motivation is to set $z(\alpha, \gamma, t) = f^c(\alpha + ic(\alpha)\gamma t, t)$. Since

$$(\partial_\alpha f^c_\mu)(\alpha^t_\gamma) = \frac{\partial_\alpha(f^c_\mu(\alpha^t_\gamma))}{1 + ic'(\alpha)\gamma t},$$

we have

$$\frac{dz_\mu(\alpha, \gamma, t)}{dt} = \frac{ic(\alpha)\gamma}{1 + ic'(\alpha)\gamma t}\partial_\alpha z_\mu(\alpha, \gamma, t) + \tag{2.8}$$
$$\lambda(\alpha^t_\gamma)\int_{-\pi}^{\pi} \frac{\sin(z_1(\alpha, \gamma, t) - z_1(\beta, \gamma, t) + \tilde{f}_1(\alpha^t_\gamma, t) - \tilde{f}_1(\beta^t_\gamma, t))(\frac{\partial_\alpha \mu z(\alpha, \gamma, t)}{1 + ic'(\alpha)\gamma t} - \frac{\partial_\beta z_\mu(\beta, \gamma, t)}{1 + ic'(\beta)\gamma t})(1 + ic'(\beta)\gamma t)d\beta}{\cosh(z_2(\alpha, \gamma, t) - z_2(\beta, \gamma, t) + \tilde{f}_2(\alpha^t_\gamma, t) - \tilde{f}_2(\beta^t_\gamma, t)) - \cos(z_1(\alpha, \gamma, t) - z_1(\beta, \gamma, t) + \tilde{f}_1(\alpha^t_\gamma, t) - \tilde{f}_1(\beta^t_\gamma, t))}$$
$$+ \lambda(\alpha^t_\gamma)\int_{-\pi}^{\pi} \frac{\sin(z_1(\alpha, \gamma, t) - z_1(\beta, \gamma, t) + \tilde{f}_1(\alpha^t_\gamma) - \tilde{f}_1(\beta^t_\gamma))((\partial_\alpha \tilde{f}_\mu)(\alpha^t_\gamma, t) - (\partial_\beta \tilde{f}_\mu)(\beta^t_\gamma, t))(1 + ic'(\beta)\gamma t)d\beta}{\cosh(z_2(\alpha, \gamma, t) - z_2(\beta, \gamma, t) + \tilde{f}_2(\alpha^t_\gamma, t) - \tilde{f}_2(\beta^t_\gamma, t)) - \cos(z_1(\alpha, \gamma, t) - z_1(\beta, \gamma, t) + \tilde{f}_1(\alpha^t_\gamma, t) - \tilde{f}_1(\beta^t_\gamma, t))}$$

with

$$z(\alpha,\gamma,0)=f(\alpha,0).$$

We drop the analyticity assumption of f^c from now. Notice that $\tilde{f}$ and λ can still be analytically extended to D_A as in (2.5) and (2.6).

2.4 The existence of z for fixed γ

We first do the energy estimate. The idea of the energy estimate is similar as in [14] and [12]. We assume z is of finite Fourier mode here.

Since $\tilde{f}(\alpha+ic(\alpha)t,t)=\tilde{f}(\alpha,t)$, $\lambda(\alpha+ic(\alpha)t)=\lambda(\alpha)$, we have

$$\begin{aligned}
&\frac{dz_\mu(\alpha,\gamma,t)}{dt}=T(z) \qquad (2.9)\\
&=\frac{ic(\alpha)\gamma}{1+ic'(\alpha)\gamma t}\partial_\alpha z_\mu(\alpha)+\\
&\lambda(\alpha)\int_{-\pi}^{\pi}\frac{\sin(z_1(\alpha)-z_1(\beta)+\tilde{f}_1(\alpha)-\tilde{f}_1(\beta))(\frac{\partial_\alpha z_\mu(\alpha)}{1+ic'(\alpha)\gamma t}-\frac{\partial_\beta z_\mu(\beta)}{1+ic'(\beta)\gamma t})(1+ic'(\beta)\gamma t)d\beta}{\cosh(z_2(\alpha)-z_2(\beta)+\tilde{f}_2(\alpha)-\tilde{f}_2(\beta))-\cos(z_1(\alpha)-z_1(\beta)+\tilde{f}_1(\alpha)-\tilde{f}_1(\beta))}\\
&+\lambda(\alpha)\int_{-\pi}^{\pi}\frac{\sin(z_1(\alpha)-z_1(\beta)+\tilde{f}_1(\alpha)-\tilde{f}_1(\beta))((\partial_\alpha\tilde{f}_\mu)(\alpha)-(\partial_\beta\tilde{f}_\mu)(\beta))(1+ic'(\beta)\gamma t)d\beta}{\cosh(z_2(\alpha)-z_2(\beta)+\tilde{f}_2(\alpha)-\tilde{f}_2(\beta))-\cos(z_1(\alpha)-z_1(\beta)+\tilde{f}_1(\alpha)-\tilde{f}_1(\beta))}.
\end{aligned}$$

Here we omit the dependency of z on γ and t, and the dependency of $\tilde{f}$ on t for the sake of simplicity. Let

$$U=H^5[\mathcal{T}],$$

where $\mathcal{T}$ is Torus of length 2π and

$$\|z\|_{Arc}=\sup_{\alpha\in[-2\delta,2\delta],\beta\in[-\pi,\pi]}\frac{(\alpha-\beta)^2}{|\cosh(z_2(\alpha)-z_2(\beta)+\tilde{f}_1(\alpha)-\tilde{f}_1(\beta))-\cos(z_1(\alpha)-z_1(\beta)+\tilde{f}_1(\alpha)-\tilde{f}_1(\beta))|}.$$

For the L^2 norm, we have

$$\|T(z)\|_{L^2[-\pi,\pi]}\lesssim\|T(z)\|_{L^\infty[-\pi,\pi]}\lesssim C(\|z\|_U)\|z\|_{Arc}.$$

Here C is a bounded function depending on δ, δ_c and $\|f\|_{C^1([0,t],H^6[-\pi,\pi])}$, $\|\tilde{f}\|_{C^1([0,t_0],H^6[-\pi,\pi])}$. We will keep using the same notation C in the following proof.

Now we take 5th derivative and have

$$
\begin{aligned}
&\partial_\alpha^5 T(z) = \\
&= \frac{ic(\alpha)\gamma}{1+ic'(\alpha)\gamma t}\partial_\alpha^6 z_\mu(\alpha) + \\
&\lambda(\alpha)\int_{-\pi}^{\pi} \frac{\sin(z_1(\alpha)-z_1(\beta)+\tilde{f}_1(\alpha)-\tilde{f}_1(\beta))(\frac{\partial_\alpha^6 z_\mu(\alpha)}{1+ic'(\alpha)\gamma t}-\frac{\partial_\beta^6 z_\mu(\beta)}{1+ic'(\beta)\gamma t})(1+ic'(\beta)\gamma t)d\beta}{\cosh(z_2(\alpha)-z_2(\beta)+\tilde{f}_1(\alpha)-\tilde{f}_1(\beta))-\cos(z_1(\alpha)-z_1(\beta)+\tilde{f}_1(\alpha)-\tilde{f}_1(\beta))} \\
&+\lambda(\alpha)\int_{-\pi}^{\pi} \frac{\sin(z_1(\alpha)-z_1(\beta)+\tilde{f}_1(\alpha)-\tilde{f}_1(\beta))((\partial_\alpha^6\tilde{f}_\mu)(\alpha)-(\partial_\beta^6\tilde{f}_\mu)(\beta))(1+ic'(\beta)\gamma t)d\beta}{\cosh(z_2(\alpha)-z_2(\beta)+\tilde{f}_1(\alpha)-\tilde{f}_1(\beta))-\cos(z_1(\alpha)-z_1(\beta)+\tilde{f}_1(\alpha)-\tilde{f}_1(\beta))} \\
&+\sum_i O^i \\
&= T_1(z)+T_2(z)+T_3(z)+\sum_i O^i.
\end{aligned}
$$

Here O_i terms contain at most 5th derivative on both z and $\tilde{f}$.

Before we show the explicit form of O^i, we introduce some notations. Let

$$V_g^k(\alpha) = (g_1(\alpha,t), g_2(\alpha,t), \partial_\alpha g_1(\alpha,t), \partial_\alpha g_2(\alpha,t), ..., \partial_\alpha^k g_1(\alpha,t), \partial_\alpha^k g_2(\alpha,t)), \tag{2.10}$$

$$\tilde{V}_g^k(\alpha) = (\frac{g_1(\alpha,t)}{1+ic'(\alpha)\gamma t}, \frac{g_2(\alpha,t)}{1+ic'(\alpha)\gamma t}, ..., \partial_\alpha^k g_1(\alpha,t)\partial_\alpha^5(\frac{1}{1+ic'(\alpha)\gamma t}), \partial_\alpha^k g_2(\alpha,t)\partial_\alpha^5(\frac{1}{1+ic'(\alpha)\gamma t})).$$

$V_{g,i}^k(\alpha)$ is the ith component in $V_{g,i}^k(\alpha)$ and $\tilde{V}_{g,i}^k(\alpha)$ the ith component in $\tilde{V}_{g,i}^k(\alpha)$.

When we write $X_i(\alpha,t)$, we mean

$$X_i(\alpha,t) = (\partial_\alpha^{l_i}\frac{1}{1+ic'(\alpha)\gamma t}), \tag{2.11}$$

with $l_i \le 5$.

A function $K_{-\sigma}^j(A,B)$, $K_{-\sigma}^j(A,B,C)$ is of $-\sigma$ type if for A, B, C in R^n, it has the form

$$
\begin{aligned}
&c_j\frac{\sin(A_1+B_1)^{m_1}\cos(A_1+B_1)^{m_2}}{(\cosh(A_2+B_2)-\cos(A_1+B_1))^{m_0}} \\
&\times(\sinh(A_2+B_2))^{m_3}(\cosh(A_2+B_2))^{m_4}\Pi_{j=1}^{m_5}(A_{\lambda_j})\Pi_{j=1}^{m_6}(B_{\lambda_{j,2}})\Pi_{j=1}^{m_7}(C_{\lambda_{j,3}}),
\end{aligned} \tag{2.12}
$$

with $m_1+m_3+m_5+m_6+m_7-2m_0 \ge -\sigma$. c_j is a constant.

We claim that we can write other terms as following three types, by separating the highest order

term in the derivative. Here we omit the dependency on γ and t.

$$O^{1,i} = \partial_\alpha^{b_i}\lambda(\alpha)\int_{-\pi}^{\pi} K_{-1}^i(V_z^3(\alpha)-V_z^3(\beta), V_{\tilde{f}}^3(\alpha)-V_{\tilde{f}}^3(\beta), \tilde{V}_z^3(\alpha)-\tilde{V}_z^3(\beta))X_{i'}(\beta)(\tilde{z}^3(\alpha)-\tilde{z}^3(\beta))d\beta,$$

where $\tilde{z}^3 \in V_z^3 \cup \tilde{V}_z^3 \cup V_{\tilde{f}}^3$, $1 \le b_i \le 5$.

$$O^{2,i} = \partial_\alpha^{b_i}\lambda(\alpha)\int_{-\pi}^{\pi} K_{-1}^i(V_z^2(\alpha)-V_z^2(\beta), V_{\tilde{f}}^2(\alpha)-V_{\tilde{f}}^2(\beta), \tilde{V}_z^2(\alpha)-\tilde{V}_z^2(\beta))X_{i'}(\beta)(\tilde{z}^5(\alpha)-\tilde{z}^5(\beta))d\beta,$$

where $\tilde{z}^5 \in V_z^5 \cup \tilde{V}_z^5 \cup V_{\tilde{f}}^5$, $1 \le b_i \le 5$.

$$O^{3,i} = \partial_\alpha^{b_i}(\frac{ic(\alpha)\gamma}{1+ic'(\alpha)\gamma t})\partial_\alpha^{b_i'} z_\mu(\alpha),$$

where $1 \le b_i, b_i' \le 5$.

Then we have the following lemmas

Lemma 2.4.1. *We have*

$$\|O_i\|_{L^2[-\pi,\pi]} \lesssim C(\|z\|_U, \|z\|_{Arc}).$$

Proof. Since K_{-1}^i is of -1 type, we have

$$\|K_{-1}^i(V_z^3(\alpha)-V_z^3(\beta), V_{\tilde{f}}^3(\alpha)-V_{\tilde{f}}^3(\beta), \tilde{V}_z^3(\alpha)-\tilde{V}_z^3(\beta))(\alpha-\beta)\|_{C^0[-2\delta,2\delta]\times[-\pi,\pi]} \lesssim C(\|z\|_U, \|z\|_{Arc}),$$

we could use lemma 2.6.1 to get the result for $O_{1,i}$. Moreover, we have

$$\|K_{-1}^i(V_z^2(\alpha)-V_z^2(\beta), V_{\tilde{f}}^2(\alpha)-V_{\tilde{f}}^2(\beta), \tilde{V}_z^2(\alpha)-\tilde{V}_z^2(\beta))(\alpha-\beta)\|_{C^1[-2\delta,2\delta]\times[-\pi,\pi]} \lesssim C(\|z\|_U, \|z\|_{Arc}),$$

we then use lemma 2.5.3 to get the estimate for $O^{2,i}$. $O^{3,i}$ can be bounded easily. □

Lemma 2.4.2. *We have*

$$\|T_3(z)\|_{L^2[-\pi,\pi]} \lesssim C(\|z\|_U, \|z\|_{Arc}).$$

Proof. Let

$$K(z(\alpha)-z(\beta), \tilde{f}(\alpha)-\tilde{f}(\beta)) = \frac{\sin(z_1(\alpha)-z_1(\beta)+\tilde{f}_1(\alpha)-\tilde{f}_1(\beta))}{\cosh(z_2(\alpha)-z_2(\beta)+\tilde{f}_2(\alpha)-\tilde{f}_2(\beta))-\cos(z_1(\alpha)-z_1(\beta)+\tilde{f}_1(\alpha)-\tilde{f}_1(\beta))}.$$

It is also of K_{-1} type. We have

$$\|K(z(\alpha)-z(\beta),\tilde{f}(\alpha)-\tilde{f}(\beta))(\alpha-\beta)\|_{C^1[-2\delta,2\delta]\times[-\pi,\pi]}\lesssim C(\|z\|_U,\|z\|_{Arc}).$$

Then the result follows from lemma 2.5.3. □

Then we are left to deal with T_1+T_2.

We have

$$\begin{aligned}
&T_2(z(\alpha))=\lambda(\alpha)p.v.\int_{-\pi}^{\pi}K(z(\alpha)-z(\beta),\tilde{f}(\alpha)-\tilde{f}(\beta))(1+ic'(\beta)\gamma t)d\beta\frac{\partial_\alpha^6 z_\mu(\alpha)}{(1+ic'(\alpha)\gamma t)}\\
&-\lambda(\alpha)p.v.\int_{-\pi}^{\pi}K(z(\alpha)-z(\beta),\tilde{f}(\alpha)-\tilde{f}(\beta))\partial_\beta^6 z_\mu(\beta)d\beta\\
&=T_{2,1}(z)(\alpha)+T_{2,2}(z)(\alpha).
\end{aligned}$$

Moreover, we could further split the $T_{2,2}$ and have

$$\begin{aligned}
&T_{2,2}(z)(\alpha)=-\lambda(\alpha)p.v.\int_{-\pi}^{\pi}K(z(\alpha)-z(\beta),\tilde{f}(\alpha)-\tilde{f}(\beta))\partial_\beta^6 z_\mu(\beta)d\beta\\
&=-\lambda(\alpha)\lim_{\beta\to\alpha}(K(z(\alpha)-z(\beta),\tilde{f}(\alpha)-\tilde{f}(\beta))\tan(\frac{\alpha-\beta}{2}))p.v.\int_{-\pi}^{\pi}\cot(\frac{\alpha-\beta}{2})\partial_\beta^6 z(\beta)d\beta\\
&-\lambda(\alpha)\int_{-\pi}^{\pi}(K(z(\alpha)-z(\beta),\tilde{f}(\alpha)-\tilde{f}(\beta))\tan(\frac{\alpha-\beta}{2})-\lim_{\beta\to\alpha}(K(z(\alpha)-z(\beta),\tilde{f}(\alpha)-\tilde{f}(\beta))\tan(\frac{\alpha-\beta}{2})))\\
&\cot(\frac{\alpha-\beta}{2})\partial_\beta^6 z(\beta)d\beta\\
&=T_{2,2,1}(z)+T_{2,2,2}(z).
\end{aligned}$$

Since $K(z(\alpha)-z(\beta),\tilde{f}(\alpha)-\tilde{f}(\beta))$ is of -1 type, we have,

$$\|K(z(\alpha)-z(\beta),\tilde{f}(\alpha)-\tilde{f}(\beta))\tan(\frac{\alpha-\beta}{2})\|_{C^3_{\alpha,\beta}([-2\delta,2\delta]\times[-\pi,\pi])}\lesssim C(\|z\|_{Arc}+\|z\|_U).\tag{2.13}$$

Let

$$\begin{aligned}
&\tilde{K}(z(\alpha)-z(\beta),\tilde{f}(\alpha)-\tilde{f}(\beta))\\
&=\frac{K(z(\alpha)-z(\beta),\tilde{f}(\alpha)-\tilde{f}(\beta))\tan(\frac{\alpha-\beta}{2})-\lim_{\beta\to\alpha}(K(z(\alpha)-z(\beta),\tilde{f}(\alpha)-\tilde{f}(\beta))\tan(\frac{\alpha-\beta}{2}))}{\tan(\frac{\alpha-\beta}{2})}.
\end{aligned}$$

Then

$$\begin{aligned}&\|\tilde{K}(z(\alpha)-z(\beta),\tilde{f}(\alpha)-\tilde{f}(\beta))\|_{C^2_{\alpha,\beta}([-2\delta,2\delta]\times[-\pi,\pi])}\\&\lesssim C(\|z\|_{Arc}+\|z\|_U).\end{aligned}$$

We can do the integration by parts in the $T_{2,2,2}(z)$ and have

$$\begin{aligned}T_{2,2,2}(z)(\alpha)&=-\lambda(\alpha)\int_{-\pi}^{\pi}\tilde{K}(z(\alpha)-z(\beta),\tilde{f}(\alpha)-\tilde{f}(\beta))\partial_\beta^6 z(\beta)d\beta\\&=\lambda(\alpha)\int_{-\pi}^{\pi}\partial_\beta\tilde{K}(z(\alpha)-z(\beta),\tilde{f}(\alpha)-\tilde{f}(\beta))\partial_\beta^5 z(\beta)d\beta.\end{aligned}$$

Therefore we have

$$\|T_{2,2,2}(z)(\alpha)\|_{L^2[-\pi,\pi]}\lesssim C(\|z\|_{Arc}+\|z\|_U)\|\partial_\alpha^5 z(\alpha)\|_{L^2[-\pi,\pi]}.$$

In conclusion, we have

$$\begin{aligned}&\frac{d\partial_\alpha^5 z_\mu(\alpha,\gamma,t)}{dt}=T_1(z)+T_{2,1}(z)+T_{2,2,1}(z)+(T_{2,2,2}(z)+T_3(z)+\sum_i O^i(z))\qquad(2.14)\\&=(\frac{ic(\alpha)\gamma}{1+ic'(\alpha)\gamma t}+\frac{\lambda(\alpha)}{1+ic'(\alpha)\gamma t}p.v.\int_{-\pi}^{\pi}K(z(\alpha)-z(\beta),\tilde{f}(\alpha)-\tilde{f}(\beta))(1+ic'(\beta)\gamma t)d\beta)\partial_\alpha^6 z_\mu(\alpha)\\&-\lambda(\alpha)\lim_{\beta\to\alpha}(K(z(\alpha)-z(\beta),\tilde{f}(\alpha)-\tilde{f}(\beta))\tan(\frac{\alpha-\beta}{2}))2\pi\Lambda(\partial_\alpha^5 z)(\alpha)\\&+(T_{2,2,2}(z)+T_3(z)+\sum_i O^i(z))\end{aligned}$$

where Λ is $(-\Delta)^{\frac{1}{2}}$ on the Torus $\mathbb{T}$ of length 2π and

$$\|(T_{2,2,2}(z)+T_3(z)+\sum_i O^i)\|_{L^2}\lesssim C(\|z\|_{Arc}+\|z\|_U).\qquad(2.15)$$

Then we have

$$\begin{aligned}&\frac{d}{dt}\|\partial_\alpha^5 z(\alpha,\gamma,t)\|^2_{L^2_\alpha[-\pi,\pi]}=2\Re\int_{-\pi}^{\pi}\partial_\alpha^5 z(\alpha,\gamma,t)\cdot\overline{\partial_\alpha^5\frac{d}{dt}z(\alpha,\gamma,t)}d\alpha\qquad(2.16)\\&=\sum_{\mu=1,2}\underbrace{(2\Re\int_{-\pi}^{\pi}\partial_\alpha^5 z_\mu(\alpha)\overline{(\frac{ic(\alpha)\gamma}{1+ic'(\alpha)\gamma t}+\frac{\lambda(\alpha)}{1+ic'(\alpha)\gamma t}p.v.\int_{-\pi}^{\pi}K(z(\alpha)-z(\beta),\tilde{f}(\alpha)-\tilde{f}(\beta))(1+ic'(\beta)\gamma t)d\beta)}}_{\text{main term}}\\&\underbrace{\cdot\partial_\alpha^6 z_\mu(\alpha)d\alpha}_{\text{main term}}\end{aligned}$$

$$\underbrace{-2\Re\int_{-\pi}^{\pi}\partial_\alpha^5 z_\mu(\alpha)\overline{\lambda(\alpha)\lim_{\beta\to\alpha}(K(z(\alpha)-z(\beta),\tilde{f}(\alpha)-\tilde{f}(\beta))\tan(\frac{\alpha-\beta}{2}))2\pi\Lambda(\partial_\alpha^5 z_\mu)(\alpha)}d\alpha)}_{\text{main term}}$$

$$+ B.T.$$

where $B.T. \leq C(\|z\|_{Arc} + \|z\|_U)$. Next we show the following lemma to control the main terms.

Lemma 2.4.3. *If $L_1(\alpha)$, $L_2(\alpha) \in C^2(\mathcal{T})$, $-\Re L_1(\alpha) \geq |\Im L_2(\alpha)|$, $h \in H^1$, then we have*

$$\int_{-\pi}^{\pi} h(\alpha)\overline{L_1(\alpha)(\Lambda h)(\alpha)}d\alpha + \int_{-\pi}^{\pi} h(\alpha)\overline{L_2(\alpha)\partial_\alpha h(\alpha)}d\alpha \leq C(\|L_1\|_{C^2(\mathcal{T})} + \|L_2\|_{C^2(\mathcal{T})})\|h(\alpha)\|_{L^2}^2.$$

Proof. First, we have

$$\begin{aligned}
I_2 &= \int_{-\pi}^{\pi} h(\alpha)\overline{L_2(\alpha)\partial_\alpha h(\alpha)}d\alpha \\
&= \int_{-\pi}^{\pi} \Re L_2(\alpha)(\Re h(\alpha)\Re h'(\alpha) + \Im h(\alpha)\Im h'(\alpha))d\alpha \\
&- \int_{-\pi}^{\pi} \Im L_2(\alpha)(\Re h(\alpha)\Im h'(\alpha) - \Im h(\alpha)\Re h'(\alpha))d\alpha \\
&= I_{2,1} + I_{2,s}.
\end{aligned}$$

We could do the integration by parts to $I_{2,1}$ and have

$$|I_{2,1}| = |\frac{1}{2}\int_{-\pi}^{\pi} \frac{d\Re L_2(\alpha)}{d\alpha}((\Re h(\alpha))^2 + (\Im h(\alpha))^2)d\alpha| \lesssim C(\|L_2(\alpha)\|_{C^1(\mathcal{T})})\|h(\alpha)\|_{L^2}^2.$$

Moreover,

$$\begin{aligned}
I_1 &= \int_{-\pi}^{\pi} h(\alpha)\overline{L_1(\alpha)(\Lambda h)(\alpha)}d\alpha \\
&= \int_{-\pi}^{\pi} \Re L_1(\alpha)(\Re h(\alpha)\Lambda\Re h(\alpha) + \Im h(\alpha)\Lambda\Im h(\alpha))d\alpha \\
&- \int_{-\pi}^{\pi} \Im L_1(\alpha)(\Re h(\alpha)\Lambda\Im h(\alpha) - \Im h(\alpha)\Lambda\Re h(\alpha))d\alpha \\
&= I_{1,M} + I_{1,2}.
\end{aligned}$$

We can still do the integration by parts to the $I_{1,2}$ and have

$$\int_{-\pi}^{\pi} \Im L_1(\alpha)(\Re h(\alpha)\Lambda\Im h(\alpha) - \Im h(\alpha)\Lambda\Re h(\alpha))d\alpha$$

$$= \int_{-\pi}^{\pi} (\Lambda(\Im L_1 \Re h)(\alpha) - \Im L_1(\alpha)\Lambda\Re h(\alpha))\Im h(\alpha) d\alpha.$$

Moreover, for any $g_1 \in H^2(\mathcal{T})$, $g_2 \in H^1(\mathcal{T})$, we have

$$\|\Lambda(g_1 g_2) - g_1 \Lambda g_2\|_{L^2(\mathcal{T})} \lesssim \|g_2\|_{L^2(\mathcal{T})} \|g_1\|_{H^2(\mathcal{T})}.$$

Hence

$$\begin{aligned} I_{1,2} &\lesssim \|\Im L_1(\alpha)\|_{H^2(\mathcal{T})} \|\Im h(\alpha)\|_{L^2(\mathcal{T})} \|\Re h(\alpha)\|_{L^2(\mathcal{T})} \\ &\lesssim \|\Im L_1(\alpha)\|_{H^2(\mathcal{T})} \|h(\alpha)\|^2_{L^2(\mathcal{T})}. \end{aligned}$$

Now we are left to control $I_{1,M} + I_{2,s}$. We have

$$\begin{aligned} I_{1,M} + I_{2,s} &= \int_{-\pi}^{\pi} \Re L_1(\alpha)(\Re h(\alpha)\Lambda\Re h(\alpha) + \Im h(\alpha)\Lambda\Im h(\alpha)) d\alpha \\ &+ \int_{-\pi}^{\pi} \Im L_2(\alpha)(\Re h(\alpha)\Im h'(\alpha) - \Im h(\alpha)\Re h'(\alpha)) d\alpha. \end{aligned}$$

Now we use a lemma from [12, Section 2.4].

Lemma 2.4.4. *Let a, b be real valued functions on $\mathcal{T}$, $a(\alpha) \geq |b(\alpha)|$ and satisfying $a, b \in C^2(\mathcal{T})$. Then we have*

$$\Re \int_T \overline{f(x)}(a(x)\Lambda f(x) + b(x) i f'(x)) dx \geq -C(\|a\|_{C^2(\mathcal{T})} + \|b\|_{C^2(\mathcal{T})}) \int_T |f(x)|^2 dx.$$

Then from lemma 3.9.4, we have

$$I_{1,M} + I_{2,s} \lesssim C(\|L_1\|_{C^2(\mathcal{T})} + \|L_2\|_{C^2(\mathcal{T})}) \|h(\alpha)\|^2_{L^2}.$$

Then we get the result.

□

Now let

$$\begin{aligned} L^1_z(\alpha, \gamma, t) &= -2\pi \lim_{\beta \to 0} (K(z(\alpha, \gamma, t) - z(\beta, \gamma, t), \tilde{f}(\alpha, t) - \tilde{f}(\beta, t)) \tan(\frac{\alpha - \beta}{2})) \qquad (2.17) \\ &= -2\pi \frac{\partial_\alpha z_1(\alpha, \gamma, t) + \partial_\alpha \tilde{f}_1(\alpha, t)}{(\partial_\alpha z_1(\alpha, \gamma, t) + \partial_\alpha \tilde{f}_1(\alpha, t))^2 + (\partial_\alpha z_2(\alpha, \gamma, t) + \partial_\alpha \tilde{f}_2(\alpha, t))^2}, \end{aligned}$$

and

$$L_z^2(\alpha,\gamma,t)=(\frac{ic(\alpha)\gamma}{1+ic'(\alpha)\gamma t}+\frac{1}{1+ic'(\alpha)\gamma t}p.v.\int_{-\pi}^{\pi}K(z(\alpha,\gamma,t)-z(\beta,\gamma,t),\tilde{f}(\alpha,t)-\tilde{f}(\beta,t))(1+ic'(\beta)\gamma t)d\beta). \tag{2.18}$$

Since $\operatorname{supp} c(\alpha)\subset\{\alpha|\lambda(\alpha)=1\}$, from (2.16), we have

$$\begin{aligned}&\frac{d}{dt}\|\partial_\alpha^5 z(\alpha,\gamma,t)\|^2_{L^2_\alpha[-\pi,\pi]}\\&=\sum_{\mu=1,2}\underbrace{2\Re\int_{-\pi}^{\pi}\partial_\alpha^5 z_\mu(\alpha)\overline{\lambda(\alpha)L_z^2(\alpha)\partial_\alpha^6 z_\mu(\alpha)}d\alpha}_{\text{main term}}\underbrace{+2\Re\int_{-\pi}^{\pi}\partial_\alpha^5 z_\mu(\alpha)\overline{\lambda(\alpha)L_z^1(\alpha)\Lambda(\partial_\alpha^5 z_\mu)(\alpha)}d\alpha}_{\text{main term}}\\&+B.T.\end{aligned}$$

From lemma (2.4.3), if $-\Re L_z^1(\alpha)\geq|\Im L_z^2(\alpha)|$ for $\alpha\in[-2\delta,2\delta]$, since $\operatorname{supp}\lambda\subset[-2\delta,2\delta]$, we have

$$\frac{d}{dt}\|\partial_\alpha^5 z(\alpha,\gamma,t)\|^2_{L^2_\alpha[-\pi,\pi]}\lesssim B.T.$$

Moreover, when $t=0$, $\alpha\in[-2\delta,2\delta]$, from (2.17) and (2.18) we have

$$\begin{aligned}&-\Re L_z^1(\alpha,\gamma,0)=2\pi\frac{\partial_\alpha f_1^c(\alpha,0)+\partial_\alpha\tilde{f}_1(\alpha,0)}{(\partial_\alpha f_1^c(\alpha,0)+\partial_\alpha\tilde{f}_1(\alpha,0))^2+(\partial_\alpha f_2^c(\alpha,0)+\partial_\alpha\tilde{f}_2(\alpha,0))^2}\\&=2\pi\frac{\partial_\alpha f_1(\alpha,0)}{(\partial_\alpha f_1(\alpha,0))^2+(\partial_\alpha f_2(\alpha,0))^2}\end{aligned}$$

and

$$\begin{aligned}&\Im L_z^2(\alpha,0)=ic(\alpha)\gamma+\Im(p.v.\int_{-\pi}^{\pi}K(f^c(\alpha,0)-f^c(\beta,0),\tilde{f}(\alpha,0)-\tilde{f}(\beta,0))d\beta)\\&=ic(\alpha)\gamma.\end{aligned}$$

From (2.1), we could choose δ_c in (2.4) to be sufficiently small and have

$$\inf_{\alpha\in[-2\delta,2\delta]}(-\Re L_z^1(\alpha,\gamma,0)-|\Im L_z^2(\alpha,\gamma,0)|)>0.$$

Then let

$$\|z\|_{RT}(t)=\sup_{\alpha\in[-2\delta,2\delta]}\frac{1}{|\Re L_z^1(\alpha,\gamma,t)+|\Im L_z^2(\alpha,\gamma,t)||}.$$

If $\|z\|_{RT}(t) < \infty$ for all $0 \le t \le t_0$, we have

$$\frac{d}{dt}\|z\|_U^2 \lesssim C(\|z\|_U + \|z\|_{Arc}), \tag{2.19}$$

for $0 \le t \le t_0$.

Therefore, we could let $\|z\|_{\tilde{U}} = \|z\|_U + \|z\|_{Arc} + \|z\|_{RT}$. From (2.19), and the following lemma 2.4.5, we have

$$\frac{d}{dt}\|z\|_{\tilde{U}}^2 \lesssim C(\|z\|_{\tilde{U}}).$$

Then $\|z(\alpha,\gamma,t)\|_{\tilde{U}}$ is bounded for sufficiently small time t_1. We claim that the bound and the time can be chosen such that it holds for all $\gamma \in [-1,1]$.

Lemma 2.4.5. *We have the following two estimates:*

$$\frac{d}{dt}\|z\|_{Arc} \lesssim C(\|z\|_{\tilde{U}}),$$

and

$$\frac{d}{dt}\|z\|_{RT} \lesssim C(\|z\|_{\tilde{U}}).$$

Proof. For the $\|z\|_{RT}$, we have

$$\frac{d}{dt}\sup_{\alpha\in[-2\delta,2\delta]}\frac{1}{|\Re L_z^1(\alpha)+|\Im L_z^2(\alpha)||} \le \sup_{\alpha\in[-2\delta,2\delta]}\frac{1}{|\Re L_z^1(\alpha)+|\Im L_z^2(\alpha)||^2}(\|\frac{d}{dt}L_z^1(\alpha)\|_{L_\alpha^\infty[-2\delta,2\delta]}+\|\frac{d}{dt}L_z^2(\alpha)\|_{L_\alpha^\infty[-2\delta,2\delta]}).$$

From (2.17), we have

$$\begin{aligned}\|\frac{d}{dt}L_z^1(\alpha)\|_{L_\alpha^\infty[-2\delta,2\delta]} &\lesssim C(\|z\|_{\tilde{U}})(\|\frac{d}{dt}z(\alpha,\gamma,t)\|_{C^1(\mathcal{T})}+\|\frac{d}{dt}\tilde{f}(\alpha,t)\|_{C^1(\mathcal{T})})\\ &\lesssim C(\|z\|_{\tilde{U}})(\|T(z)\|_{C^1(\mathcal{T})}+C).\end{aligned}$$

From (2.9), it is easy to get

$$\|T(z)\|_{C^1(\mathcal{T})} \lesssim C(\|z\|_{\tilde{U}}).$$

Then

$$\|\frac{d}{dt}L_z^1(\alpha)\|_{L_\alpha^\infty[-2\delta,2\delta]} \lesssim C(\|z\|_{\tilde{U}}).$$

From (2.18), we have

$$\begin{aligned}
&\|\frac{d}{dt}L_z^2(\alpha,\gamma,t)\|_{L_\alpha^\infty[-2\delta,2\delta]} \lesssim C + C\|\frac{d}{dt}p.v.\int_{-\pi}^{\pi} K(z(\alpha)-z(\beta),\tilde{f}(\alpha)-\tilde{f}(\beta))(1+ic'(\beta)\gamma t)d\beta\|_{L_\alpha^\infty[-2\delta,2\delta]}\\
&+\|p.v.\int_{-\pi}^{\pi} K(z(\alpha)-z(\beta),\tilde{f}(\alpha)-\tilde{f}(\beta))(1+ic'(\beta)\gamma t)d\beta\|_{L_\alpha^\infty[-2\delta,2\delta]}\\
&\le \|p.v.\int_{-\pi}^{\pi} \nabla K(z(\alpha)-z(\beta),\tilde{f}(\alpha)-\tilde{f}(\beta))\cdot(\frac{dz}{dt}(\alpha)-\frac{dz}{dt}(\beta)+\frac{d\tilde{f}}{dt}(\alpha)-\frac{d\tilde{f}}{dt}(\beta))(1+ic'(\beta)\gamma t)d\beta\|_{L_\alpha^\infty[-2\delta,2\delta]}\\
&+\|p.v.\int_{-\pi}^{\pi} K(z(\alpha)-z(\beta),\tilde{f}(\alpha)-\tilde{f}(\beta))(ic'(\beta)\gamma)d\beta\|_{L_\alpha^\infty[-2\delta,2\delta]}\\
&+\|p.v.\int_{-\pi}^{\pi} K(z(\alpha)-z(\beta),\tilde{f}(\alpha)-\tilde{f}(\beta))(1+ic'(\beta)\gamma t)d\beta\|_{L_\alpha^\infty[-2\delta,2\delta]} + C\\
&= Term_{2,1}+Term_{2,2}+Term_{2,3}+C.
\end{aligned}$$

From condition (2.13) and lemma 2.5.2, we have

$$Term_{2,2}+Term_{2,3} \lesssim C(\|z\|_{\tilde{U}}).$$

Moreover, $\nabla K(z(\alpha)-z(\beta),\tilde{f}(\alpha)-\tilde{f}(\beta))$ is of -2 type, and

$$\begin{aligned}
&\|\nabla K(z(\alpha)-z(\beta),\tilde{f}(\alpha)-\tilde{f}(\beta))\cdot(\frac{dz}{dt}(\alpha)-\frac{dz}{dt}(\beta)+\frac{d\tilde{f}}{dt}(\alpha)-\frac{d\tilde{f}}{dt}(\beta))(\alpha-\beta)\|_{C^1([-2\delta,2\delta]\times[-\pi,\pi])}\\
&\lesssim (\|T(z)\|_{C^2(\mathcal{T})}+C)C(\|z\|_{\tilde{U}}).
\end{aligned}$$

From (2.9), it is easy to get

$$\|T(z)\|_{C^2(\mathcal{T})} \lesssim C(\|z\|_{\tilde{U}}).$$

Then $Term_{2,1} \lesssim C(\|z\|_{\tilde{U}})$. Hence

$$\|\frac{d}{dt}L_z^2(\alpha)\|_{L_\alpha^\infty[-2\delta,2\delta]} \lesssim C(\|z\|_{\tilde{U}}).$$

Then we have the estimate

$$\frac{d}{dt}\|z\|_{RT} = \frac{d}{dt}\sup_{\alpha\in[-2\delta,2\delta]}\frac{1}{|\Re L_z^1(\alpha)+|\Im L_z^2(\alpha)||} \lesssim C(\|z\|_{\tilde{U}}).$$

Moreover, we have

$$\begin{aligned}
&\frac{d}{dt}\|z\|_{Arc} = \frac{d}{dt}\sup_{\alpha\in[-2\delta,2\delta],\beta\in[-\pi,\pi]}|\frac{1}{(\frac{\cosh(z_2(\alpha)-z_2(\beta)+\tilde{f}_1(\alpha)-\tilde{f}_1(\beta))-\cos(z_1(\alpha,\gamma,t)-z_1(\beta,\gamma,t)+\tilde{f}_1(\alpha)-\tilde{f}_1(\beta))}{(\alpha-\beta)^2})}|\\
&\leq \sup_{\alpha\in[-2\delta,2\delta],\beta\in[-\pi,\pi]}|\frac{1}{(\frac{\cosh(z_2(\alpha)-z_2(\beta)+\tilde{f}_1(\alpha)-\tilde{f}_1(\beta))-\cos(z_1(\alpha,\gamma,t)-z_1(\beta,\gamma,t)+\tilde{f}_1(\alpha)-\tilde{f}_1(\beta))}{(\alpha-\beta)^2})}|^2\\
&\cdot\|\frac{d}{dt}(\frac{\cosh(z_2(\alpha)-z_2(\beta)+\tilde{f}_1(\alpha)-\tilde{f}_1(\beta))-\cos(z_1(\alpha,\gamma,t)-z_1(\beta,\gamma,t)+\tilde{f}_1(\alpha)-\tilde{f}_1(\beta))}{(\alpha-\beta)^2})\|_{L^\infty_{\alpha,\beta}[-\pi,\pi]\times[-\pi,\pi]}\\
&\leq \|z\|_{Arc}^2\\
&(\|\frac{\sinh(z_2(\alpha)-z_2(\beta)+\tilde{f}_2(\alpha)-\tilde{f}_2(\beta))}{\alpha-\beta}\frac{\frac{d}{dt}(z_2(\alpha)-z_2(\beta))+\frac{d}{dt}(\tilde{f}_2(\alpha)-\tilde{f}_2(\beta))}{(\alpha-\beta)}\|_{L^\infty_{\alpha,\beta}[-\pi,\pi]\times[-\pi,\pi]}\\
&+\|\frac{\sin(z_1(\alpha)-z_1(\beta)+\tilde{f}_1(\alpha)-\tilde{f}_1(\beta))}{\alpha-\beta}\frac{\frac{d}{dt}(z_1(\alpha)-z_1(\beta))+\frac{d}{dt}(\tilde{f}_1(\alpha)-\tilde{f}_2(\beta))}{(\alpha-\beta)}\|_{L^\infty_{\alpha,\beta}[-\pi,\pi]\times[-\pi,\pi]})\\
&\lesssim \|z\|_{Arc}^2\|z\|_U(\|T(z)\|_{C^1_\alpha[-\pi,\pi]}+C)\\
&\lesssim C(\|z\|_{\tilde{U}}).
\end{aligned}$$

We also introduce a corollary here to be used in the later section.

Corollary 2.4.6. *For $g(\alpha)\in H^1(\mathcal{T})$, if $z\in H^5(\mathcal{T})$, $\|z\|_{Arc}\leq\infty$ and $-\Re L_z^1(\alpha)-|\Im L_z^2(\alpha)|>0$ when $\alpha\in[-2\delta,2\delta]$ for $\gamma\in[-1,1]$, then we have*

$$\begin{aligned}
&\sup_{\gamma\in[-1,1]}(\int_{-\pi}^{\pi}g(\alpha)\overline{\lambda(\alpha)\int_{-\pi}^{\pi}K(z(\alpha)-z(\beta),\tilde{f}(\alpha)-\tilde{f}(\beta))(\frac{\partial_\alpha g(\alpha)}{1+ic'(\alpha)\gamma t}-\frac{\partial_\alpha g(\beta)}{1+ic'(\beta)\gamma t})d\beta}d\alpha\\
&+\int_{-\pi}^{\pi}g(\alpha)\overline{\partial_\alpha g(\alpha)\frac{ic(\alpha)\gamma}{1+ic'(\alpha)\gamma t}}d\alpha)\lesssim\|g\|_{L^2}^2.
\end{aligned}$$

□

2.4.1 The approximation of the equation

Now we approximate the problem and have the following equations,

$$\frac{dz^n(\alpha,\gamma,t)}{dt} = \varphi_n * T^n(\varphi_n * z^n) = \varphi_n * \left(\frac{ic(\alpha)\gamma}{1+ic'(\alpha)\gamma t}\partial_\alpha(\varphi_n * z^n)(\alpha)\right)+$$

$$\varphi_n * \bigg(\lambda(\alpha)\int_{-\pi}^{\pi} K^n((\varphi_n * z^n)(\alpha) - (\varphi_n * z^n)(\beta), \tilde{f}(\alpha) - \tilde{f}(\beta))(\frac{\partial_\alpha(\varphi_n * z_\mu^n)(\alpha)}{1 + ic'(\alpha)\gamma t} - \frac{\partial_\beta(\varphi_n * z_\mu^n)(\beta)}{1 + ic'(\beta)\gamma t})(1 + ic'(\beta)\gamma t)d\beta\bigg)$$
$$+ \varphi_n * \bigg(\lambda(\alpha)\int_{-\pi}^{\pi} K^n((\varphi_n * z^n)(\alpha) - (\varphi_n * z^n)(\beta), \tilde{f}(\alpha) - \tilde{f}(\beta))(\tilde{f}_\mu(\alpha) - \tilde{f}_\mu(\beta))(1 + ic'(\beta)\gamma t)d\beta\bigg),$$

where

$$\begin{aligned}
&K^n((\varphi_n * z^n)(\alpha) - (\varphi_n * z^n)(\beta), \tilde{f}(\alpha) - \tilde{f}(\beta)) \\
&= \sin(\Delta((\varphi_n * z_1^n)(\alpha) + \tilde{f}_1(\alpha))) \\
&\cdot \frac{1}{|\cosh(\Delta((\varphi_n * z_2^n)(\alpha) + \tilde{f}_2(\alpha))) - \cos(\Delta((\varphi_n * z_1^n)(\alpha) + \tilde{f}_1(\alpha)))|^2 + \frac{1}{n}\sin(\frac{\alpha-\beta}{2})^2} \\
&\cdot \overline{\cosh(\Delta((\varphi_n * z_2^n)(\alpha) + \tilde{f}_2(\alpha))) - \cos(\Delta((\varphi_n * z_1^n)(\alpha) + \tilde{f}_1(\alpha)))},
\end{aligned}$$

with

$$\Delta((\varphi_n * z_2^n)(\alpha) + \tilde{f}_2(\alpha)) = (\varphi_n * z_2^n)(\alpha) - (\varphi_n * z_2^n)(\beta) + \tilde{f}_2(\alpha) - \tilde{f}_2(\beta), \tag{2.20}$$

$$\Delta((\varphi_n * z_1^n)(\alpha) + \tilde{f}_1(\alpha)) = (\varphi_n * z_1^n)(\alpha) - (\varphi_n * z_1^n)(\beta) + \tilde{f}_1(\alpha) - \tilde{f}_1(\beta), \tag{2.21}$$

and initial value $z^n(\alpha, \gamma, 0) = \varphi_n * f(\alpha, 0)$. Here the convolution of φ_n is the projection to the finite Fourier mode of α.

By the Picard theorem, for any fixed $\gamma \in [-1, 1]$, there exists solutions in $C^1([0, t_n], H^5_\alpha(\mathcal{T}))$. Moreover, by the structure of our approximation, we have $z^n = \varphi_n * z^n$, and for $1 \le j \le 5$,

$$\begin{aligned}
&\frac{d}{dt}\int_{-\pi}^{\pi} |\partial_\alpha^j z^n(\alpha, \gamma, t)|^2 d\alpha \\
&= 2\Re\int_{-\pi}^{\pi} \partial_\alpha^j z^n(\alpha, \gamma, t)\overline{\partial_\alpha^j(\varphi_n * T^n(\varphi_n * z^n))}d\alpha \\
&= 2\Re\int_{-\pi}^{\pi} \partial_\alpha^j(\varphi_n * z^n)(\alpha, \gamma, t)\overline{\partial_\alpha^j(T^n(\varphi_n * z^n))}d\alpha.
\end{aligned}$$

Then we can do the similar energy estimate as the previous section by letting

$$\|z^n\|_{\tilde{U}^n} = \|z^n\|_{H^5[-\pi,\pi]} + \|z^n\|_{RT^n} + \|z^n\|_{Arc^n},$$

where

$$\|z^n\|_{Arc^n} = \sup_{\alpha\in[-2\delta,2\delta],\beta\in[-\pi,\pi]} \frac{|\cosh(\Delta((\varphi_n * z_2^n)(\alpha) + \tilde{f}_2(\alpha))) - \cos(\Delta((\varphi_n * z_1^n)(\alpha) + \tilde{f}_1(\alpha)))|(\alpha-\beta)^2}{|\cosh(\Delta((\varphi_n * z_2^n)(\alpha) + \tilde{f}_2(\alpha))) - \cos(\Delta((\varphi_n * z_1^n)(\alpha) + \tilde{f}_1(\alpha)))|^2 + \frac{1}{n}\sin(\frac{\alpha-\beta}{2})^2},$$

with $\Delta((\varphi_n * z_2^n)(\alpha) + \tilde{f}_2(\alpha))$ and $\Delta((\varphi_n * z_1^n)(\alpha) + \tilde{f}_1(\alpha))$ from (2.20), (2.21), and letting

$$\|z^n\|_{RT^n} = \sup_{\alpha\in[-2\delta,2\delta]} \frac{1}{|\Re L_z^{1,n}(\alpha) + |\Im L_z^{2,n}(\alpha)||},$$

with

$$L_{z^n}^{1,n}(\alpha,\gamma,t) = -2\pi \lim_{\beta\to 0}(K^n((\varphi_n * z^n)(\alpha) - (\varphi_n * z^n)(\beta), \tilde{f}(\alpha) - \tilde{f}(\beta)) \tan(\frac{\alpha-\beta}{2})),$$

$$L_{z^n}^{2,n}(\alpha,\gamma,t) = (\frac{ic(\alpha)\gamma}{1+ic'(\alpha)\gamma t} + \frac{1}{1+ic'(\alpha)\gamma t} p.v. \int_{-\pi}^{\pi} K^n((\varphi_n * z^n)(\alpha) - (\varphi_n * z^n)(\beta), \tilde{f}(\alpha) - \tilde{f}(\beta))(1+ic'(\beta)\gamma t)d\beta).$$

Then we can use the similar energy estimate and the compactness argument to show there exist a solution

$$z(\alpha,\gamma,t) \in L^\infty([0,t_1], H_\alpha^5(\mathcal{T})). \tag{2.22}$$

satisfying

$$z(\alpha,\gamma,t) = \int_0^t T(z)d\tau + f^c(\alpha,0), \tag{2.23}$$

for sufficiently small time t_1. Moreover,

$$\|z\|_{Arc} < \infty, \tag{2.24}$$

and

$$-\Re L_z^1(\alpha) - |\Im L_z^2(\alpha)| > 0, \text{ when } \alpha \in [-2\delta, 2\delta]. \tag{2.25}$$

Since the energy estimate has a bound for all $\gamma \in [-1,1]$, we have a existence time t_1 that holds for γ.

Now we abuse the notation and write $T(z)$ as $T(z(\alpha,\gamma,t),\gamma,t)$. We have the following lemma:

Lemma 2.4.7. *For any $g(\alpha), h(\alpha) \in H^{j+1}(\mathcal{T})$, $j = 3,4$, $\|g\|_{Arc} \le \infty$ and $\|h\|_{Arc} \le \infty$, we have*

$$\|T(g(\alpha),\gamma,t)\|_{H^j(\mathcal{T})} \lesssim 1. \tag{2.26}$$

$$\|T(g(\alpha),\gamma,t)-T(h(\alpha),\gamma,t)\|_{H^j(\mathcal{T})}\lesssim\|g(\alpha)-g(\alpha)\|_{H^{j+1}(\mathcal{T})}. \tag{2.27}$$

$$\|T(g(\alpha),\gamma,t)-T(g(\alpha),\gamma,t')\|_{H^j(\mathcal{T})}\lesssim|t-t'|. \tag{2.28}$$

Proof. We only show (2.26) and the left can be shown in the same way. From (2.9), we have

$$\begin{aligned}
&T(g(\alpha),\gamma,t)=\frac{ic(\alpha)\gamma}{1+ic'(\alpha)\gamma t}\partial_\alpha g(\alpha)+\\
&\lambda(\alpha)\int_{-\pi}^{\pi}K(g(\alpha)-g(\beta),\tilde{f}(\alpha)-\tilde{f}(\beta))(\frac{\partial_\alpha g(\alpha)}{1+ic'(\alpha)\gamma t}-\frac{\partial_\beta g(\beta)}{1+ic'(\beta)\gamma t})(1+ic'(\beta)\gamma t)d\beta\\
&+\lambda(\alpha)\int_{-\pi}^{\pi}K(g(\alpha)-g(\beta),\tilde{f}(\alpha)-\tilde{f}(\beta))(\partial_\alpha\tilde{f}(\alpha)-\partial_\beta\tilde{f}(\beta))(1+ic'(\beta)\gamma t)d\beta\\
&=T_1(g(\alpha),\gamma,t)+T_2(g(\alpha),\gamma,t)+T_3(g(\alpha),\gamma,t).
\end{aligned}$$

It is trivial that T_1 satisfying the (2.26) since $c(\alpha)$ is sufficiently smooth.

Moreover, the $\partial_\alpha\tilde{f}\in H^5(\mathcal{T})$ and is more regular than $\frac{\partial_\alpha g(\alpha)}{1+ic'(\alpha)\gamma t}$. Hence we only consider T_2. For T_2, we have

$$\|K(g(\alpha)-g(\beta),\tilde{f}(\alpha)-\tilde{f}(\beta))(\alpha-\beta)\|_{C^{j-2}([-2\delta,2\delta]\times\mathcal{T})}\lesssim 1. \tag{2.29}$$

Then

$$\|T_2\|_{L^2(\mathcal{T})}\lesssim\|T_2\|_{L^\infty(\mathcal{T})}\lesssim 1.$$

Moreover, we can use the notation from (2.12), (2.10), and get

$$\begin{aligned}
&\partial_\alpha^j T_2(g)=\\
&\lambda(\alpha)\int_{-\pi}^{\pi}K(g(\alpha)-g(\beta),\tilde{f}(\alpha)-\tilde{f}(\beta))(\frac{\partial_\alpha^{j+1}g(\alpha)}{1+ic'(\alpha)\gamma t}-\frac{\partial_\beta^{j+1}g(\beta)}{1+ic'(\beta)\gamma t})(1+ic'(\beta)\gamma t)d\beta\\
&+\sum_{j'}\partial_\alpha^{b_{j'}}\lambda(\alpha)\int_{-\pi}^{\pi}K_{-1}^{j'}(V_g^{[\frac{j+1}{2}]}(\alpha)-V_g^{[\frac{j+1}{2}]}(\beta),V_f^{[\frac{j+1}{2}]}(\alpha)-V_f^{[\frac{j+1}{2}]}(\beta),\tilde{V}_g^{[\frac{j+1}{2}]}(\alpha)-\tilde{V}_g^{[\frac{j+1}{2}]}(\beta))\\
&\cdot X_i(\beta)(\tilde{z}^j(\alpha)-\tilde{z}^j(\beta))d\beta\\
&=Term_{2,1}+Term_{2,2}.
\end{aligned}$$

Here $\tilde{z}^j \in V_g^j \cup \tilde{V}_g^j \cup V_f^j$. $[\frac{j+1}{2}]$ is the biggest integer less than $\frac{j+1}{2}$. Then from (2.29), we could use lemma 2.5.3 to bound $Term_1$. Moreover, since $j+1-[\frac{j+1}{2}] \geq [\frac{j+1}{2}] \geq 2$. We have

$$\|K_{-1}^{j'}(V_g^{[\frac{j+1}{2}]}(\alpha) - V_g^{[\frac{j+1}{2}]}(\beta), V_f^{[\frac{j+1}{2}]}(\alpha) - V_f^{[\frac{j+1}{2}]}(\beta), \tilde{V}_g^{[\frac{j+1}{2}]}(\alpha) - \tilde{V}_g^{[\frac{j+1}{2}]}(\beta))(\alpha-\beta)\|_{C^0([-2\delta,2\delta]\times[-\pi,\pi])}$$
$$\lesssim C(\|g\|_{H^{j+1}}\|\tilde{f}\|_{H^{j+1}}).$$

Then we could use lemma 2.6.1 to bound $Term_2$. □

Then from lemma 2.4.7, (2.22) and (2.23), we have

$$z(\alpha,\gamma,t) \in L^\infty([0,t_1], H_\alpha^5(\mathcal{T})) \cap C^0([0,t_1], H_\alpha^4(\mathcal{T})) \cap C^1([0,t_1], H_\alpha^3(\mathcal{T})). \tag{2.30}$$

2.5 The uniqueness

In this section we show there exists sufficiently $0 < t_2 \leq t_1$ such that for $0 \leq t \leq t_2$, we have $z(\alpha,0,t) = f^c(\alpha,t)$.

Let $z^0(\alpha,t) = z(\alpha,0,t)$. From (2.9), we have

$$\frac{dz_\mu^0(\alpha,t)}{dt} = \lambda(\alpha)\int_{-\pi}^{\pi} K(z^0(\alpha)-z^0(\beta), \tilde{f}(\alpha)-\tilde{f}(\beta))(\partial_\alpha z_\mu^0(\alpha) - \partial_\beta z_\mu^0(\beta))d\beta$$
$$+\lambda(\alpha)\int_{-\pi}^{\pi} K(z^0(\alpha)-z^0(\beta), \tilde{f}(\alpha)-\tilde{f}(\beta))(\partial_\alpha \tilde{f}_\mu(\alpha) - \partial_\beta \tilde{f}_\mu(\beta))d\beta.$$

Moreover, from (2.2), we have

$$\frac{df_\mu^c(\alpha,t)}{dt} = \lambda(\alpha)\int_{-\pi}^{\pi} K(f^c(\alpha)-f^c(\beta), \tilde{f}(\alpha)-\tilde{f}(\beta))(\partial_\alpha f_\mu^c(\alpha) - \partial_\beta f_\mu^c(\beta))d\beta$$
$$+\lambda(\alpha)\int_{-\pi}^{\pi} K(f^c(\alpha)-f^c(\beta), \tilde{f}(\alpha)-\tilde{f}(\beta))(\partial_\alpha \tilde{f}_\mu(\alpha) - \partial_\beta \tilde{f}_\mu(\beta))d\beta.$$

Then we have the equation for the difference:

$$\frac{d(z_\mu^0(\alpha,t)-f^c(\alpha,t))}{dt} = \lambda(\alpha)\int_{-\pi}^{\pi} K(z^0(\alpha)-z^0(\beta), \tilde{f}(\alpha)-\tilde{f}(\beta))(\partial_\alpha(z_\mu^0-f_\mu^c)(\alpha) - \partial_\beta(z_\mu^0-f_\mu^c)(\beta))d\beta$$
$$+\lambda(\alpha)\int_{-\pi}^{\pi}(K(z^0(\alpha)-z^0(\beta), \tilde{f}(\alpha)-\tilde{f}(\beta)) - K(f^c(\alpha)-f^c(\beta), \tilde{f}(\alpha)-\tilde{f}(\beta)))$$
$$\cdot(\partial_\alpha \tilde{f}_\mu(\alpha) + \partial_\alpha f_\mu^c(\alpha) - \partial_\beta \tilde{f}_\mu(\beta) - \partial_\beta f_\mu^c(\beta))d\beta$$
$$= Term_1 + Term_2.$$

We first control $Term_2$, we have

$$
\begin{aligned}
&Term_2 = \\
&\lambda(\alpha)\int_{-\pi}^{\pi}\int_0^1 \frac{d}{d\tau}(K(f^c(\alpha)-f^c(\beta)-\tau(f^c(\alpha)-z^0(\alpha)-(f^c(\beta)-z^0(\beta))),\tilde{f}(\alpha)-\tilde{f}(\beta))) \\
&\cdot(\partial_\alpha\tilde{f}_\mu(\alpha)+\partial_\alpha f^c_\mu(\alpha)-\partial_\beta\tilde{f}_\mu(\beta)-\partial_\beta f^c_\mu(\beta))d\tau d\beta \\
&=-\lambda(\alpha)\int_{-\pi}^{\pi}\int_0^1 \nabla K(f^c(\alpha)-f^c(\beta)-\tau(f^c(\alpha)-z^0(\alpha)-(f^c(\beta)-z^0(\beta))),\tilde{f}(\alpha)-\tilde{f}(\beta))) \\
&\cdot(f^c(\alpha)-z^0(\alpha)-(f^c(\beta)-z^0(\beta)))(\partial_\alpha\tilde{f}_\mu(\alpha)+\partial_\alpha f^c_\mu(\alpha)-\partial_\beta\tilde{f}_\mu(\beta)-\partial_\beta f^c_\mu(\beta))d\tau d\beta.
\end{aligned}
\tag{2.31}
$$

Since the component of ∇K is of -2 type, we have

$$
\begin{aligned}
&\|\nabla K(f^c(\alpha)-f^c(\beta)-\tau(f^c(\alpha)-z^0(\alpha)-(f^c(\beta)-z^0(\beta))),\tilde{f}(\alpha)-\tilde{f}(\beta)))(\alpha-\beta)^2\|_{C^1_{\alpha,\beta}([-2\delta,2\delta]\times[-\pi,\pi])} \\
&\lesssim(\|f^c(\alpha)\|_{C^2[-\pi,\pi]}+\|z^0(\alpha)\|_{C^2[-\pi,\pi]}+\|\tilde{f}(\alpha)\|_{C^2[-\pi,\pi]})C(\|f^c-\tau(f^c-z^0)\|_{Arc}).
\end{aligned}
\tag{2.32}
$$

When $t=0$, we have $z^0=f^c$, then

$$
\|f^c-\tau(f^c-z^0)\|_{Arc}=\|f^c\|_{Arc}\lesssim 1. \tag{2.33}
$$

Moreover, we have the following lemma

Lemma 2.5.1. *For $g,h\in C^1(\mathcal{T})$, $\|h\|_{Arc}<\infty$, there exists δ depending on $\|h\|_{Arc}$ and $\|h\|_{C^1(\mathcal{T})}$ such that when $\|g-h\|_{C^1(\mathcal{T})}\le\delta$, $\|g\|_{Arc}<\infty$.*

Proof. We have

$$
\begin{aligned}
&|\cosh(g_2(\alpha)-g_2(\beta)+\tilde{f}_2(\alpha)-\tilde{f}_2(\beta))-\cosh(h_2(\alpha)-h_2(\beta)+\tilde{f}_2(\alpha)-\tilde{f}_2(\beta))| \\
&\le|g_2(\alpha)-g_2(\beta)-(h_2(\alpha)-h_2(\beta))|\int_0^1|\sinh((1-\tau)(h_2(\alpha)-h_2(\beta))+\tau(g_2(\alpha)-g_2(\beta)))|d\tau \\
&\le(\alpha-\beta)^2\|g-h\|_{C^1(T)}C(\|h\|_{C^1(T)}+\|g-h\|_{C^1(T)}).
\end{aligned}
$$

and

$$
\begin{aligned}
&|\cos(g_1(\alpha)-g_1(\beta)+\tilde{f}_1(\alpha)-\tilde{f}_1(\beta))-\cos(h_1(\alpha)-h_1(\beta)+\tilde{f}_1(\alpha)-\tilde{f}_1(\beta))| \\
&\le|g_1(\alpha)-g_1(\beta)-(h_1(\alpha)-h_1(\beta))|\int_0^1|\sin((1-\tau)(h_1(\alpha)-h_1(\beta))+\tau(g_1(\alpha)-g_1(\beta)))|d\tau
\end{aligned}
$$

$$\leq (\alpha-\beta)^2\|g-h\|_{C^1(T)}C(\|h\|_{C^1(T)}+\|g-h\|_{C^1(T)}).$$

Since

$$|\cosh(h_2(\alpha)-h_2(\beta)+\tilde{f}_2(\alpha)-\tilde{f}_2(\beta))-\cos(h_1(\alpha)-h_1(\beta)+\tilde{f}_1(\alpha)-\tilde{f}_1(\beta))|\geq\|h\|_{Arc}|\alpha-\beta|^2.$$

We have

$$\begin{aligned}&|\cosh(g_2(\alpha)-g_2(\beta)+\tilde{f}_2(\alpha)-\tilde{f}_2(\beta))-\cos(g_1(\alpha)-g_1(\beta)+\tilde{f}_1(\alpha)-\tilde{f}_1(\beta))|\\&\geq(\|h\|_{Arc}-\|g-h\|_{C^1(T)}C(\|h\|_{C^1(T)}+\|g-h\|_{C^1(T)}))|\alpha-\beta|^2.\end{aligned}$$

Then we have the result. □

Since we have $z^0(\alpha,t)\in C^1([0,t_1],H^3(\mathcal{T}))$, $f^c(\alpha,t)\in C^1([0,t_1],H^6(\mathcal{T}))$, then from (2.33), and lemma 2.5.1, there exists t_2, satisfying $0\leq t_2\leq t_1$, such that for $0\leq t\leq t_2$,

$$\|f^c-\tau(f^c-z^0)\|_{Arc}\lesssim 1. \tag{2.34}$$

Then from corollary 2.5.4, (2.31) ,(2.32), and (2.34), we have

$$\|Term_2\|^2_{L^2(\mathcal{T})}\leq\|z^0-f^c\|^2_{L^2(\mathcal{T})}.$$

Then

$$\begin{aligned}&\frac{d}{dt}\int_{-\pi}^{\pi}|z^0(\alpha,t)-f^c(\alpha,t)|^2d\alpha\\&=2\Re\int_{-\pi}^{\pi}(z^0(\alpha,t)-f^c(\alpha,t))\overline{\frac{d}{dt}(z^0(\alpha,t)-f^c(\alpha,t))d\alpha}\\&=\sum_{\mu=1,2}2\Re\int_{-\pi}^{\pi}(z^0_\mu(\alpha,t)-f^c_\mu(\alpha,t))\overline{\lambda(\alpha)\int_{-\pi}^{\pi}K(z^0(\alpha)-z^0(\beta),\tilde{f}(\alpha)-\tilde{f}(\beta))(\partial_\alpha(z^0_\mu-f^c_\mu)(\alpha)-\partial_\beta(z^0_\mu-f^c_\mu)(\beta))d\beta}d\alpha\\&+B.T^0\end{aligned}$$

where $B.T^0\lesssim\|z^0(\alpha,t)-f^c(\alpha,t)\|^2_{L^2[-\pi,\pi]}$. Then from corollary 2.4.6 when $\gamma=0$, conditions (2.25), (2.22), (2.24), we have

$$\frac{d}{dt}\int_{-\pi}^{\pi}|z^0(\alpha,t)-f^c(\alpha,t)|^2d\alpha\lesssim\|z^0(\alpha,t)-f^c(\alpha,t)\|^2_{L^2(\mathcal{T})}.$$

Moreover, we have $z^0(\alpha,0)=f^c(\alpha,0)$. Therefore we have

$$z^0(\alpha,t)=f^c(\alpha,t). \tag{2.35}$$

for $0\le t\le t_2$.

Lemma 2.5.2. *For $G(\alpha,\beta)\in C^1([-2\delta,2\delta]\times[-\pi,\pi])$, we have*

$$\|p.v.\int_{-\pi}^{\pi}\frac{G(\alpha,\beta)}{(\alpha-\beta)}d\beta\|_{L^\infty[-2\delta,2\delta]}\lesssim\|G(\alpha,\beta)\|_{C^1([-2\delta,2\delta]\times[-\pi,\pi])}.$$

Proof. We have

$$\begin{aligned}&|p.v.\int_{-\pi}^{\pi}\frac{G(\alpha,\beta)}{(\alpha-\beta)}d\beta|\\&=|\int_{-\pi}^{\pi}\frac{G(\alpha,\beta)-G(\alpha,\alpha)}{(\alpha-\beta)}d\beta|+|p.v.\int_{-\pi}^{\pi}\frac{1}{\alpha-\beta}d\beta G(\alpha,\alpha)|\lesssim\|G(\alpha,\beta)\|_{C^1([-2\delta,2\delta]\times[-\pi,\pi])}.\end{aligned}$$

□

Lemma 2.5.3. *For $g(\alpha)\in L^2[-\pi,\pi]$, $G(\alpha,\beta)\in C^1([-2\delta,2\delta]\times[-\pi,\pi])$, we have*

$$\|\int_{-\pi}^{\pi}G(\alpha,\beta)\frac{g(\alpha)-g(\beta)}{(\alpha-\beta)}d\beta\|_{L^2[-2\delta,2\delta]}\lesssim\|G(\alpha,\beta)\|_{C^1([-2\delta,2\delta]\times[-\pi,\pi])}\|g\|_{L^2[-\pi,\pi]}.$$

Proof. We have

$$\begin{aligned}&\int_{-\pi}^{\pi}G(\alpha,\beta)\frac{g(\alpha)-g(\beta)}{(\alpha-\beta)}d\beta\\&=\int_{-\pi}^{\pi}\frac{G(\alpha,\beta)-G(\alpha,\alpha)}{\alpha-\beta}(g(\alpha)-g(\beta))d\beta+G(\alpha,\alpha)p.v.\int_{-\pi}^{\pi}\frac{g(\alpha)-g(\beta)}{\alpha-\beta}d\beta\\&=Term_1+Term_2.\end{aligned}$$

Here

$$\|Term_1\|_{L^2[-2\delta,2\delta]}\lesssim\|G(\alpha,\beta)\|_{C^1([-2\delta,2\delta]\times[-\pi,\pi])}\|g(\alpha)\|_{L^2[-\pi,\pi]},$$

and

$$\|Term_2\|_{L^2[-2\delta,2\delta]}\lesssim\|G(\alpha,\alpha)\|_{C^0([-2\delta,2\delta])}\|g(\alpha)\|_{L^2[-\pi,\pi]}.$$

□

Corollary 2.5.4. *For $g(\alpha) \in L^2[-\pi,\pi]$,$h(\alpha) \in C^2[-\pi,\pi]$, $G(\alpha,\beta) \in C^1([-2\delta,2\delta]\times[-\pi,\pi])$, we have*

$$\|\int_{-\pi}^{\pi} G(\alpha,\beta)\frac{(g(\alpha)-g(\beta))(h(\alpha)-h(\beta))}{(\alpha-\beta)^2}d\beta\|_{L^2[-2\delta,2\delta]} \lesssim \|G(\alpha,\beta)\|_{C^1([-2\delta,2\delta]\times[-\pi,\pi])}\|g\|_{L^2[-2\delta,2\delta]}\|h\|_{C^2[-\pi,\pi]}.$$

Proof. We can use lemma 2.5.3 and let $\tilde{G}(\alpha,\beta) = G(\alpha,\beta)\frac{h(\alpha)-h(\beta)}{(\alpha-\beta)}$. □

2.6 The continuity of z with respect to γ

We first show $\|z(\alpha,\gamma,t) - z(\alpha,\gamma',t)\|_{H^3(\alpha)} \lesssim |\gamma-\gamma'|$.

For the sake of simplicity, we further shrink the time t_1 to $\tilde{t}_1$ such that for all $0 \le t \le \tilde{t}_1$, $\gamma,\gamma' \in [-1,1]$, $\tau \in [0,1]$, we have $\|\tau z(\alpha,\gamma,t)+(1-\tau)z(\alpha,\gamma',t)-\tau z(\alpha,\gamma,0)-(1-\tau)z(\alpha,\gamma',0)\|_{C^1(\mathcal{T})} = \|\tau z(\alpha,\gamma,t)+(1-\tau)z(\alpha,\gamma',t)-f^c(\alpha,0)\|_{C^1(\mathcal{T})}$ is sufficiently small. Then from lemma 2.5.1, we have

$$\|\tau z(\alpha,\gamma,t)+(1-\tau)z(\alpha,\gamma',t)\|_{Arc} < \infty.$$

This is not necessary but helps to simplify our estimate in this section.

Now we estimate the difference, we have

$$\begin{aligned}
&\frac{dz(\alpha,\gamma)}{dt} - \frac{dz(\alpha,\gamma')}{dt} = T(z(\alpha,\gamma,t),\gamma,t) - T(z(\alpha,\gamma',t),\gamma',t) \qquad (2.36)\\
&= (T(z(\alpha,\gamma,t),\gamma,t) - T(z(\alpha,\gamma',t),\gamma,t)) + (\int_{\gamma}^{\gamma'}(\partial_\eta T)(z(\alpha,\gamma',t),\eta,t)d\eta)\\
&= Term_1 + Term_2.
\end{aligned}$$

For $Term_2$, we have

$$\begin{aligned}
&(\partial_\gamma T)(z(\alpha),\gamma,t) = \frac{d}{d\gamma}(\frac{ic(\alpha)\gamma}{1+ic'(\alpha)\gamma t})\partial_\alpha z(\alpha) \qquad (2.37)\\
&+\lambda(\alpha)\int_{-\pi}^{\pi} K(z(\alpha)-z(\beta),\tilde{f}(\alpha)-\tilde{f}(\beta))\frac{d}{d\gamma}(\frac{1+ic'(\beta)\gamma t}{1+ic'(\alpha)\gamma t}-1)d\beta\partial_\alpha z(\alpha)\\
&+\lambda(\alpha)\int_{-\pi}^{\pi} K(z(\alpha)-z(\beta),\tilde{f}(\alpha)-\tilde{f}(\beta))(\partial_\alpha\tilde{f}(\alpha)-\partial_\beta\tilde{f}(\beta))ic'(\beta)td\beta\\
&= Term_{2,1}+Term_{2,2}+Term_{2,3}.
\end{aligned}$$

Since $z(\alpha,\gamma,t)\in L_t^\infty([0,t_0],H^5(\mathcal{T}))$, we have

$$\|Term_{2,1}\|_{H^3(\mathcal{T})}\lesssim 1.$$

Moreover,

$$Term_{2,2}=\lambda(\alpha)\int_{-\pi}^{\pi}K(z(\alpha)-z(\beta),\tilde{f}(\alpha)-\tilde{f}(\beta))[\frac{(ic(\beta)-ic(\alpha))t}{1+ic'(\alpha)\gamma t}-\frac{(ic(\beta)-ic(\alpha))\gamma tic'(\alpha)t}{(1+c'(\alpha)\gamma t)^2}]d\beta\partial_\alpha z(\alpha).$$

Since K is of -1 type, we have

$$K(z(\alpha)-z(\beta),\tilde{f}(\alpha)-\tilde{f}(\beta))(\alpha-\beta)\in C^3([-2\delta,2\delta]\times[-\pi,\pi]). \tag{2.38}$$

Therefore $\|Term_{2,2}\|_{H^3(\mathcal{T})}\lesssim 1$. Moreover, $Term_{2,3}$ can be bounded in the similar way since $|\frac{\partial_\alpha\tilde{f}(\alpha)-\partial_\beta\tilde{f}(\beta)}{\alpha-\beta}|\in C^3([-2\delta,2\delta]\times[-\pi,\pi])$ and we get

$$\|Term_{2,3}\|_{H^3(\mathcal{T})}\lesssim 1.$$

Then we have

$$\|(\partial_\gamma T)(z(\alpha),\gamma,t)\|_{H^3(\mathcal{T})}\lesssim 1. \tag{2.39}$$

and

$$\|Term_2\|_{H^3(\mathcal{T})}\lesssim|\gamma-\gamma'|. \tag{2.40}$$

For $Term_1$, we have

$$Term_1=\frac{ic(\alpha)\gamma}{1+ic'(\alpha)\gamma t}\partial_\alpha(z(\alpha,\gamma)-z(\alpha,\gamma')) \tag{2.41}$$

$$+\lambda(\alpha)\int_{-\pi}^{\pi}K(z(\alpha,\gamma)-z(\beta,\gamma),\tilde{f}(\alpha)-\tilde{f}(\beta))(\frac{\partial_\alpha z(\alpha,\gamma)-\partial_\alpha z(\alpha,\gamma')}{1+ic'(\alpha)\gamma t}-(\frac{\partial_\beta z(\beta,\gamma)-\partial_\beta z(\beta,\gamma')}{1+ic'(\beta)\gamma t}))$$

$$\cdot(1+ic'(\beta)\gamma t)d\beta \tag{2.42}$$

$$+\lambda(\alpha)\int_{-\pi}^{\pi}(K(z(\alpha,\gamma)-z(\beta,\gamma),\tilde{f}(\alpha)-\tilde{f}(\beta))-K(z(\alpha,\gamma')-z(\beta,\gamma'),\tilde{f}(\alpha)-\tilde{f}(\beta)))$$

$$(\frac{\partial_\alpha z(\alpha,\gamma')}{1+ic'(\alpha)\gamma t}-\frac{\partial_\beta z(\beta,\gamma')}{1+ic'(\beta)\gamma t}+\partial_\alpha\tilde{f}(\alpha)-\partial_\beta\tilde{f}(\beta))(1+ic'(\beta)\gamma t)d\beta$$

$$=Term_{1,1}+Term_{1,2}+Term_{1,3}. \tag{2.43}$$

It is easy to get

$$\|Term_{1,1}\|_{L^2(T)} + \|Term_{1,2}\|_{L^2(T)} \lesssim \|z(\alpha,\gamma) - z(\alpha,\gamma')\|_{H^3(\mathcal{T})}. \tag{2.44}$$

Moreover,

$$Term_{1,3} = \lambda(\alpha)\int_{-\pi}^{\pi}\int_0^1 \nabla K(\tau(z(\alpha,\gamma)-z(\beta,\gamma)) + (1-\tau)(z(\alpha,\gamma')-z(\beta,\gamma')), \tilde{f}(\alpha)-\tilde{f}(\beta))d\tau \tag{2.45}$$

$$\cdot (z(\alpha,\gamma) - z(\beta,\gamma) - (z(\alpha,\gamma') - z(\beta,\gamma')))$$
$$(\frac{\partial_\alpha z(\alpha,\gamma')}{1+ic'(\alpha)\gamma t} - \frac{\partial_\beta z(\beta,\gamma')}{1+ic'(\beta)\gamma t} + \partial_\alpha \tilde{f}(\alpha) - \partial_\beta \tilde{f}(\beta))(1+ic'(\beta)\gamma t)d\beta.$$

Since the component of ∇K is of -2 type, we have

$$\sup_\tau \|\nabla K(\tau(z(\alpha,\gamma)-z(\beta,\gamma)+(1-\tau)(z(\alpha,\gamma')-z(\beta,\gamma')), \tilde{f}(\alpha)-\tilde{f}(\beta))(\alpha-\beta)^2\|_{C^3([-2\delta,2\delta]\times[-\pi,\pi])} \lesssim 1,$$

then we have

$$\|Term_{1,3}\|_{L^2(T)} \lesssim \|z(\alpha,\gamma) - z(\alpha,\gamma')\|_{H^3(T)}. \tag{2.46}$$

Now we control $\partial_\alpha^3 Term_1$. For $\partial_\alpha^3 Term_{1,1}$, we have

$$\partial_\alpha^3 Term_{1,1} = \frac{ic(\alpha)\gamma}{1+ic'(\alpha)\gamma t}\partial_\alpha^4(z(\alpha,\gamma)-z(\alpha,\gamma')) + \sum_{1\le j\le 3} C_{1,j}\partial_\alpha^j(\frac{ic(\alpha)\gamma}{1+ic'(\alpha)\gamma t})\partial_\alpha^{4-j}(z(\alpha,\gamma)-z(\alpha,\gamma')) \tag{2.47}$$

$$= Term^3_{1,1,M} + Term^3_{1,1,2}.$$

Here $\|Term^3_{1,1,2}\|_{L^2(\mathcal{T})} \lesssim \|z(\alpha,\gamma) - z(\alpha,\gamma')\|_{H^3(\mathcal{T})}$. For $\partial_\alpha^3 Term_{1,2}$, we first introduce several lemmas.

Lemma 2.6.1. *For $g(\alpha) \in H^1[-\pi,\pi]$, $G(\alpha,\beta) \in C^0([-2\delta,2\delta]\times[-\pi,\pi])$, we have*

$$\|\int_{-\pi}^{\pi} G(\alpha,\beta)\frac{g(\alpha)-g(\beta)}{(\alpha-\beta)}d\beta\|_{L^2[-2\delta,2\delta]} \lesssim \|G(\alpha,\beta)\|_{C^0([-2\delta,2\delta]\times[-\pi,\pi])}\|g\|_{H^1[-\pi,\pi]}.$$

Proof. We have

$$
\begin{aligned}
&|\int_{-\pi}^{\pi} G(\alpha,\beta)\frac{g(\alpha)-g(\beta)}{(\alpha-\beta)}d\beta| \\
&\leq \|G(\alpha,\beta)\|_{C^0([-2\delta,2\delta]\times[-\pi,\pi])}\int_{-\pi}^{\pi}\frac{|g(\alpha)-g(\beta)|}{(\alpha-\beta)}d\beta \\
&\leq \|G(\alpha,\beta)\|_{C^0([-2\delta,2\delta]\times[-\pi,\pi])}\int_{-\pi}^{\pi}\int_0^1 |g'(\tau(\alpha)+(1-\tau)(\beta))|d\tau d\beta \\
&\lesssim \|G(\alpha,\beta)\|_{C^0([-2\delta,2\delta]\times[-\pi,\pi])}\|g\|_{H^1[-\pi,\pi]}.
\end{aligned}
$$

□

Lemma 2.6.2. *For $g(\alpha)\in H^k(\mathcal{T})$, $G(\alpha,\beta)\in C^1([-2\delta,2\delta]\times\mathcal{T})\cap C^k([-2\delta,2\delta]\times\mathcal{T})$, for $k\geq 0$, we have*

$$
\|\int_{-\pi}^{\pi} G(\alpha,\beta)\frac{g(\alpha)-g(\beta)}{\alpha-\beta}d\beta\|_{H^k[-2\delta,2\delta]}\lesssim(\|G(\alpha,\beta)\|_{C^1([-2\delta,2\delta]\times\mathcal{T})}+\|G(\alpha,\beta)\|_{C^k([-2\delta,2\delta]\times\mathcal{T})})\|g\|_{H^k(\mathcal{T})}.
$$

Proof. From lemma 2.5.3, when $k=0$, we have the L^2 norm. Moreover, when $k\geq 1$, we have

$$
\begin{aligned}
&\partial_\alpha^k\int_{-\pi}^{\pi} G(\alpha,\beta)\frac{g(\alpha)-g(\beta)}{\alpha-\beta}d\beta \\
&=\partial_\alpha^k\int_{-\pi}^{\pi} G(\alpha,\alpha-\beta)\frac{g(\alpha)-g(\alpha-\beta)}{\beta}d\beta \\
&=\sum_{0\leq j\leq k} C_j\int_{-\pi}^{\pi}\partial_\alpha^j G(\alpha,\alpha-\beta)\frac{\partial_\alpha^{k-j}g(\alpha)-\partial_\alpha^{k-j}g(\alpha-\beta)}{\beta}d\beta.
\end{aligned}
$$

For $j\leq k-1$, we could use lemma 2.5.3 to get the estimate. For $j=k$, we could use lemma 2.6.1 to get the estimate. □

Lemma 2.6.3. *For $k\geq 2$, $g(\alpha)\in H^k(\mathcal{T})$, $G(\alpha,\beta)\in C^k([-2\delta,2\delta]\times\mathcal{T})$, we have*

$$
\begin{aligned}
&\|\partial_\alpha^k(\int_{-\pi}^{\pi} G(\alpha,\beta)\frac{g(\alpha)-g(\beta)}{\alpha-\beta}d\beta)-\int_{-\pi}^{\pi} G(\alpha,\beta)\frac{\partial_\alpha^k g(\alpha)-\partial_\beta^k g(\beta)}{\alpha-\beta}d\beta\|_{L^2[-2\delta,2\delta]} \\
&\lesssim \|G(\alpha,\beta)\|_{C^k([-2\delta,2\delta]\times\mathcal{T})}\|g\|_{H^{k-1}(\mathcal{T})}
\end{aligned}
$$

Proof. We have

$$
\begin{aligned}
&\partial_\alpha^k(\int_{-\pi}^{\pi} G(\alpha,\beta)\frac{g(\alpha)-g(\beta)}{\alpha-\beta}d\beta)-\int_{-\pi}^{\pi} G(\alpha,\beta)\frac{\partial_\alpha^k g(\alpha)-\partial_\beta^k g(\beta)}{\alpha-\beta}d\beta \\
&=\sum_{1\leq j\leq k} C_j\int_{-\pi}^{\pi}\partial_\alpha^j(G(\alpha,\alpha-\beta))\frac{\partial_\alpha^{k-j}g(\alpha)-\partial_\beta^{k-j}g(\alpha-\beta)}{\beta}d\beta.
\end{aligned}
$$

Then for $j \leq k-1$. we could use lemma 2.5.3 to get the estimate. Then for $j = k$. we could use lemma 2.6.1 to get the estimate. □

Lemma 2.6.4. *For $g(\alpha) \in H^3(\mathcal{T})$, $h(\alpha) \in H^3(\mathcal{T})$, $G(\alpha,\beta) \in C^3([-2\delta, 2\delta] \times \mathcal{T})$, we have*

$$\begin{aligned}&\|\partial_\alpha^3(\int_{-\pi}^{\pi} G(\alpha,\beta)\frac{g(\alpha)-g(\beta)}{\alpha-\beta}\frac{h(\alpha)-h(\beta)}{\alpha-\beta}d\beta)\|_{L^2[-2\delta,2\delta]}\\ &\lesssim \|G(\alpha,\beta)\|_{C^3([-2\delta,2\delta]\times\mathcal{T})}\|g\|_{H^3(\mathcal{T})}\|h\|_{H^3(\mathcal{T})}.\end{aligned}$$

Proof. We take the derivative and have

$$\begin{aligned}&\partial_\alpha^3(\int_{-\pi}^{\pi} G(\alpha,\alpha-\beta)\frac{g(\alpha)-g(\alpha-\beta)}{\beta}\frac{h(\alpha)-h(\alpha-\beta)}{\beta}d\beta)\\ &=\sum_{j_1+j_2+j_3=3}\int_{-\pi}^{\pi}\partial_\alpha^{j_1}(G(\alpha,\alpha-\beta))\frac{\partial_\alpha^{j_2}g(\alpha)-\partial_\alpha^{j_2}g(\alpha-\beta)}{\beta}\frac{\partial_\alpha^{j_3}h(\alpha)-\partial_\alpha^{j_3}h(\alpha-\beta)}{\beta}d\beta.\end{aligned}$$

If $j_2 \leq 1$, $j_3 \leq 1$, then

$$\begin{aligned}&\|\int_{-\pi}^{\pi}\partial_\alpha^{j_1}(G(\alpha,\alpha-\beta))\frac{\partial_\alpha^{j_2}g(\alpha)-\partial_\alpha^{j_2}g(\alpha-\beta)}{\beta}\frac{\partial_\alpha^{j_3}h(\alpha)-\partial_\alpha^{j_3}h(\alpha-\beta)}{\beta}d\beta\|_{L^2[-2\delta,2\delta]}\\ &\lesssim \|\partial_\alpha^{j_1}(G(\alpha,\alpha-\beta))\|_{C^0([-2\delta,2\delta]\times\mathcal{T})}\|g\|_{C^2(\mathcal{T})}\|h\|_{C^2(\mathcal{T})}\\ &\lesssim \|G(\alpha,\beta)\|_{C^3([-2\delta,2\delta]\times\mathcal{T})}\|g\|_{H^3(\mathcal{T})}\|h\|_{H^3(\mathcal{T})}.\end{aligned}$$

If $j_2 = 3$, then $j_1 = j_3 = 0$,by lemma 2.5.4, we have

$$\begin{aligned}&\|\int_{-\pi}^{\pi}\partial_\alpha^{j_1}(G(\alpha,\alpha-\beta))\frac{\partial_\alpha^{j_2}g(\alpha)-\partial_\alpha^{j_2}g(\alpha-\beta)}{\beta}\frac{\partial_\alpha^{j_3}h(\alpha)-\partial_\alpha^{j_3}h(\alpha-\beta)}{\beta}d\beta\|_{L^2[-2\delta,2\delta]}\\ &\lesssim \|(G(\alpha,\alpha-\beta))\|_{C^1([-2\delta,2\delta]\times\mathcal{T})}\|g\|_{H^3(\mathcal{T})}\|h\|_{C^2(\mathcal{T})}\\ &\lesssim \|G(\alpha,\beta)\|_{C^3([-2\delta,2\delta]\times\mathcal{T})}\|g\|_{H^3(\mathcal{T})}\|h\|_{H^3(\mathcal{T})}.\end{aligned}$$

If $j_2 = 2$, then $j_1 = 1$ or $j_3 = 1$, by lemma 2.6.1, we have

$$\begin{aligned}&\|\int_{-\pi}^{\pi}\partial_\alpha^{j_1}(G(\alpha,\alpha-\beta))\frac{\partial_\alpha^{j_2}g(\alpha)-\partial_\alpha^{j_2}g(\alpha-\beta)}{\beta}\frac{\partial_\alpha^{j_3}h(\alpha)-\partial_\alpha^{j_3}h(\alpha-\beta)}{\beta}d\beta\|_{L^2[-2\delta,2\delta]}\\ &\lesssim \|\partial_\alpha^{j_1}(G(\alpha,\alpha-\beta))\|_{C^0([-2\delta,2\delta]\times\mathcal{T})}\|g\|_{H^3(\mathcal{T})}\|\partial_\alpha^{j_3}h\|_{C^1(\mathcal{T})}\\ &\lesssim \|G(\alpha,\beta)\|_{C^3([-2\delta,2\delta]\times\mathcal{T})}\|g\|_{H^3(\mathcal{T})}\|h\|_{H^3(\mathcal{T})}.\end{aligned}$$

When $j_3 = 3$ or $j_3 = 2$, it can treated similarly as $j_2 = 3$ and $j_2 = 2$. □

Then from lemma 2.6.3 and (2.38), we have

$$\begin{aligned}
&\partial_\alpha^3 Term_{1,2} = \lambda(\alpha)\int_{-\pi}^{\pi} K(z(\alpha,\gamma)-z((\alpha-\beta,\gamma)),\tilde{f}(\alpha)-\tilde{f}(\alpha-\beta)) \qquad (2.48)\\
&\cdot(\partial_\alpha^3(\frac{\partial_\alpha(z(\alpha,\gamma)-z(\alpha,\gamma'))}{1+ic'(\alpha)\gamma t})-\partial_\alpha^3(\frac{\partial_\alpha(z(\alpha-\beta,\gamma)-z(\alpha-\beta,\gamma'))}{1+ic'(\alpha-\beta)\gamma t}))(1+ic'(\alpha-\beta)\gamma t)d\beta\\
&+Term_{1,2,1}^3,
\end{aligned}$$

where $\|Term_{1,2,1}^3\|_{L^2(\mathcal{T})} \lesssim \|z(\alpha,\gamma)-z(\alpha,\gamma')\|_{H^3(\mathcal{T})}^2$. Moreover, we have

$$\begin{aligned}
&\lambda(\alpha)\int_{-\pi}^{\pi} K(z(\alpha,\gamma)-z((\alpha-\beta,\gamma)),\tilde{f}(\alpha)-\tilde{f}(\alpha-\beta)) \qquad (2.49)\\
&\cdot(\partial_\alpha^3(\frac{\partial_\alpha(z(\alpha,\gamma)-z(\alpha,\gamma'))}{1+ic'(\alpha)\gamma t})-\partial_\alpha^3(\frac{\partial_\alpha(z(\alpha-\beta,\gamma)-z(\alpha-\beta,\gamma'))}{1+ic'(\alpha-\beta)\gamma t}))(1+ic'(\alpha-\beta)\gamma t)d\beta\\
&=\lambda(\alpha)\int_{-\pi}^{\pi} K(z(\alpha,\gamma)-z((\alpha-\beta,\gamma)),\tilde{f}(\alpha)-\tilde{f}(\alpha-\beta))\\
&\cdot(\frac{\partial_\alpha^4(z(\alpha,\gamma)-z(\alpha,\gamma'))}{1+ic'(\alpha)\gamma t}-\frac{\partial_\alpha^4(z(\alpha-\beta,\gamma)-z(\alpha-\beta,\gamma'))}{1+ic'(\alpha-\beta)\gamma t}))(1+ic'(\alpha-\beta)\gamma t)d\beta\\
&+\sum_{0\le j\le 2} C_j\lambda(\alpha)\int_{-\pi}^{\pi} K(z(\alpha,\gamma)-z((\alpha-\beta,\gamma)),\tilde{f}(\alpha)-\tilde{f}(\alpha-\beta))(1+ic'(\alpha-\beta)\gamma t)\\
&\cdot(\partial_\alpha^{1+j}(z(\alpha,\gamma)-z(\alpha,\gamma'))\partial_\alpha^{3-j}(\frac{1}{1+ic'(\alpha)\gamma t})-\partial_\alpha^{1+j}(z(\alpha-\beta,\gamma)-z(\alpha-\beta,\gamma'))\partial_\alpha^{3-j}(\frac{1}{1+ic'(\alpha-\beta)\gamma t}))d\beta\\
&=Term_{1,2,M}^3+Term_{1,2,3}^3.
\end{aligned}$$

Then from lemma 2.5.3, we have

$$\|Term_{1,2,3}^3\|_{L^2(\mathcal{T})} \lesssim \|z(\alpha,\gamma)-z(\alpha,\gamma')\|_{H^3(\mathcal{T})}. \qquad (2.50)$$

For $\partial_\alpha^3 Term_{1,3}$, we use equation (2.45). Since

$$\|\nabla K(\tau(z(\alpha,\gamma)-z(\beta,\gamma))+(1-\tau)(z(\alpha,\gamma')-z(\beta,\gamma')),\tilde{f}(\alpha)-\tilde{f}(\beta))(\alpha-\beta)^2\|_{C^3([-2\delta,2\delta]\times[-\pi,\pi])} \lesssim 1,$$

and

$$\frac{\partial_\alpha z(\alpha,\gamma')}{1+ic'(\alpha)\gamma t}+\partial_\alpha\tilde{f}(\alpha)\in H^4(\mathcal{T}).$$

from lemma 2.6.4, we have

$$\|\partial_\alpha^3 Term_{1,3}\|_{L^2(\mathcal{T})} \lesssim \|z(\alpha,\gamma)-z(\alpha,\gamma')\|_{H^3(\mathcal{T})}. \qquad (2.51)$$

In conclusion, from (2.44), (2.46), and (2.40), we have

$$\frac{d}{dt}\int_{-\pi}^{\pi}|z(\alpha,\gamma)-z(\alpha,\gamma')|^2d\alpha = 2\Re\int_{-\pi}^{\pi}(z(\alpha,\gamma)-z(\alpha,\gamma'))\overline{Term_1+Term_2}d\alpha$$
$$\lesssim |\gamma-\gamma'|^2+\|z(\alpha,\gamma)-z(\alpha,\gamma')\|^2_{H^3(\mathcal{T})}.$$

From (2.47) ,(2.48), (2.49), (2.50), (2.51) and (2.40), we have

$$\begin{aligned}
&\frac{d}{dt}\int_{-\pi}^{\pi}|\partial_\alpha^3(z(\alpha,\gamma)-z(\alpha,\gamma'))|^2d\alpha\\
&=2\Re\int_{-\pi}^{\pi}\partial_\alpha^3(z(\alpha,\gamma)-z(\alpha,\gamma'))\cdot\overline{\partial_\alpha^3Term_1+\partial_\alpha^3Term_2}d\alpha\\
&=2\Re\int_{-\pi}^{\pi}\partial_\alpha^3(z(\alpha,\gamma)-z(\alpha,\gamma'))\cdot\overline{Term^3_{1,1,M}+Term^3_{1,2,M}}d\alpha+B.T^0\\
&=\sum_{\mu=1,2}2\Re\int_{-\pi}^{\pi}\partial_\alpha^3(z_\mu(\alpha,\gamma)-z_\mu(\alpha,\gamma'))\overline{\frac{ic(\alpha)\gamma}{1+ic'(\alpha)\gamma t}\partial_\alpha^4(z_\mu(\alpha,\gamma)-z(\alpha,\gamma'))d\alpha}\\
&+\sum_{\mu=1,2}2\Re\int_{-\pi}^{\pi}\partial_\alpha^3(z_\mu(\alpha,\gamma)-z_\mu(\alpha,\gamma'))\overline{\lambda(\alpha)\int_{-\pi}^{\pi}K(z(\alpha,\gamma)-z(\alpha-\beta,\gamma),\tilde f(\alpha)-\tilde f(\alpha-\beta))}\\
&\cdot(\frac{\partial_\alpha^4(z(\alpha,\gamma)-z(\alpha,\gamma'))}{1+ic'(\alpha)\gamma t}-\frac{\partial_\alpha^4(z(\alpha-\beta,\gamma)-z(\alpha-\beta,\gamma'))}{1+ic'(\alpha-\beta)\gamma t}))(1+ic'(\alpha-\beta)\gamma t)d\beta d\alpha\\
&+B.T^0,
\end{aligned}$$

where

$$|B.T^0|\lesssim\|z(\alpha,\gamma)-z(\alpha,\gamma')\|^2_{H^3(\mathcal{T})}+|\gamma-\gamma'|^2.$$

Then from corollary 2.4.6, we have

$$\frac{d}{dt}\int_{-\pi}^{\pi}|\partial_\alpha^3(z(\alpha,\gamma)-z(\alpha,\gamma'))|^2d\alpha\lesssim\|z(\alpha,\gamma)-z(\alpha,\gamma')\|^2_{H^3(\mathcal{T})}+|\gamma-\gamma'|^2.$$

Moreover, the initial date $\|z(\alpha,\gamma)-z(\alpha,\gamma')\|^2_{H^3(\mathcal{T})}|_{t=0}=0$. Therefore we have

$$\|\frac{z(\alpha,\gamma)-z(\alpha,\gamma')}{\gamma-\gamma'}\|_{H^3(\mathcal{T})}\lesssim 1. \tag{2.52}$$

2.7 The differentiability of z with respect to γ

Now we show the differentiability. We define a new function $w(\alpha,\gamma,t)$. It satisfies the equation that $\frac{dz}{d\gamma}$ would satisfy if it is differentiable.

Let w be the solution of the following equation:

$$\frac{dw(\alpha,\gamma,t)}{dt} = \tilde{T}(w) = D_z T(z(\alpha,\gamma,t),\gamma,t)[w] + \partial_\gamma T(\alpha,\gamma,t), \tag{2.53}$$

with initial value $w(\alpha,\gamma,0) = 0$. Here $D_z T(z(\alpha,\gamma,t),\gamma,t)[w]$ is the Gateaux derivative.

As in the existence of $z(\alpha,\gamma,t)$, we first show the energy estimate. First, from (2.39), we have

$$\|(\partial_\gamma T)(z(\alpha),\gamma,t)\|_{H^3(\mathcal{T})} \lesssim 1.$$

Moreover,

$$\begin{aligned}
&D_z T(z(\alpha,\gamma,t),\gamma,t)[w] = \frac{ic(\alpha)\gamma}{1+ic'(\alpha)\gamma t}\partial_\alpha w(\alpha,\gamma) + \\
&\lambda(\alpha)\int_{-\pi}^{\pi} K(z(\alpha,\gamma)-z(\beta,\gamma),\tilde{f}(\alpha,\gamma)-\tilde{f}(\beta,\gamma))(\frac{\partial_\alpha w(\alpha,\gamma)}{1+ic'(\alpha)\gamma t} - \frac{\partial_\beta w(\beta,\gamma)}{1+ic'(\beta)\gamma t})(1+ic'(\beta)\gamma t)d\beta \\
&+\lambda(\alpha)\int_{-\pi}^{\pi} \nabla K(z(\alpha,\gamma)-z(\beta,\gamma),\tilde{f}(\alpha,\gamma)-\tilde{f}(\beta,\gamma))\cdot(w(\alpha,\gamma)-w(\beta,\gamma)) \\
&\cdot(\partial_\alpha\tilde{f}(\alpha,\gamma)-\partial_\beta\tilde{f}(\beta,\gamma)+\frac{\partial_\alpha z(\alpha,\gamma)}{1+ic'(\alpha)\gamma t} - \frac{\partial_\beta z(\beta,\gamma)}{1+ic'(\beta)\gamma t})(1+ic'(\beta)\gamma t)d\beta.
\end{aligned} \tag{2.54}$$

It has the similar structure as (2.41) and (2.45). The only difference in the first two terms in (2.41) and (2.54) is that $\partial_\alpha w(\alpha,\gamma)$ takes the place of $\frac{z(\alpha,\gamma)-z(\alpha,\gamma')}{\gamma-\gamma'}$. In (2.45), and the third term of (2.41), $w(\alpha,\gamma)$ takes the place of $\frac{z(\alpha,\gamma)-z(\alpha,\gamma')}{\gamma-\gamma'}$ and $\int_0^1 \nabla K(\tau(z(\alpha,\gamma)-z(\beta,\gamma))+(1-\tau)(z(\alpha,\gamma')-z(\beta,\gamma')),\tilde{f}(\alpha)-\tilde{f}(\beta))d\tau$ is replaced by $\nabla K(z(\alpha,\gamma)-z(\beta,\gamma))$. Therefore we could use the similar estimate and have

$$\frac{d}{dt}\|w\|^2_{H^3(\mathcal{T})} \lesssim C(\|w\|^2_{H^3(\mathcal{T})}).$$

As in the existence of $z(\alpha,\gamma,t)$, we could do the similar energy estimate to the approximation of the equation

$$\frac{dw^n(\alpha,\gamma,t)}{dt} = \tilde{T}(w) = \varphi_n * (D_z T(z(\alpha,\gamma,t),\gamma,t)[\varphi_n * w^n]) + \varphi_n * (\partial_\gamma T(\alpha,\gamma,t)), \tag{2.55}$$

with initial value $w^n(\alpha,\gamma,0) = 0$. Then from the Picard theorem and compactness argument, there exists $0 \le t_3 \le t_1$, such that

$$w(\alpha,\gamma,t) = \int_0^t \tilde{T}(w(\alpha,\gamma,\tau))d\tau,$$

and

$$\|w(\alpha,\gamma,t)\|_{L^\infty([0,t_3],H^3_\alpha(\mathcal{T}))} \lesssim 1. \tag{2.56}$$

We claim there is an uniform t_3 holds for all $\gamma \in [-1,1]$. Moreover, we have the following lemma

Lemma 2.7.1. *For any $g(\alpha), h(\alpha) \in H^{j+1}(\mathcal{T})$, $j \leq 2$, we have*

$$\|\tilde{T}(g(\alpha),\gamma,t)\|_{H^j(\mathcal{T})} \lesssim 1. \tag{2.57}$$

$$\|\tilde{T}(g(\alpha),\gamma,t) - \tilde{T}(h(\alpha),\gamma,t)\|_{H^j(\mathcal{T})} \lesssim \|g(\alpha) - h(\alpha)\|_{H^{j+1}(\mathcal{T})}. \tag{2.58}$$

$$\lim_{t\to t'} \|\tilde{T}(g(\alpha),\gamma,t) - \tilde{T}(g(\alpha),\gamma,t')\|_{H^j(\mathcal{T})} = 0. \tag{2.59}$$

Proof. It is easy to get these bounds since $z(\alpha,\gamma,t) \in L^\infty([0,t_0],H^5_\alpha(\mathcal{T})) \cap C^0([0,t_0],H^4_\alpha(\mathcal{T}))$. □

Then we have $w(\alpha,\gamma,t) \in L^\infty([0,t_3],H^3_\alpha(\mathcal{T})) \cap C^0([0,t_3],H^2_\alpha(\mathcal{T})) \cap C^1([0,t_3],H^1_\alpha(\mathcal{T}))$.

We claim that we could do the similar argument as in the estimate of $\|z(\alpha,\gamma) - z(\alpha,\gamma')\|_{H^3_\alpha(\mathcal{T})} \lesssim |\gamma - \gamma'|$ to get

$$\|w(\alpha,\gamma) - w(\alpha,\gamma)\|_{H^1_\alpha(\mathcal{T})} \lesssim |\gamma - \gamma'|.$$

Then from (2.56), we have

$$\lim_{\gamma'\to\gamma} \|w(\alpha,\gamma') - w(\alpha,\gamma)\|_{H^2_\alpha(\mathcal{T})} = 0. \tag{2.60}$$

Now we show w is the derivative of z with respect of γ. Let

$$v(\alpha,\gamma,\gamma',t) = \frac{z(\alpha,\gamma,t) - z(\alpha,\gamma',t)}{\gamma - \gamma'} - w(\alpha,\gamma,t).$$

From (2.53) and (2.36), we have

$$\frac{dv(\alpha,\gamma,\gamma',t)}{dt} = \frac{T(z(\alpha,\gamma,t),\gamma,t) - T(z(\alpha,\gamma',t),\gamma,t)}{\gamma - \gamma'} + \frac{\int_\gamma^{\gamma'} (\partial_\eta T)(z(\alpha,\gamma',t),\eta,t)d\eta}{\gamma - \gamma'}$$

$$
\begin{aligned}
&- D_z T(z(\alpha,\gamma,t),\gamma,t)[w(\alpha,\gamma,t)] - (\partial_\gamma T)(z(\alpha,\gamma,t),\gamma,t) \\
&= (\frac{T(z(\alpha,\gamma,t),\gamma,t) - T(z(\alpha,\gamma',t),\gamma,t)}{\gamma-\gamma'} - D_z T(z(\alpha,\gamma,t),\gamma,t)[w(\alpha,\gamma,t)]) \\
&+ (\frac{\int_\gamma^{\gamma'} (\partial_\eta T)(z(\alpha,\gamma',t),\eta,t)d\eta}{\gamma-\gamma'} - (\partial_\gamma T)(z(\alpha,\gamma,t),\gamma,t)) \\
&= Term_1 + Term_2.
\end{aligned}
$$

We have

$$
\begin{aligned}
Term_2 &= \frac{1}{\gamma-\gamma'}\int_\gamma^{\gamma'} (\partial_\eta T)(z(\alpha,\gamma',t),\eta,t)d\eta - (\partial_\gamma T)(z(\alpha,\gamma,t),\gamma,t) \\
&= \int_\gamma^{\gamma'} \frac{(\partial_\eta T)(z(\alpha,\gamma',t),\eta,t) - (\partial_\gamma T)(z(\alpha,\gamma',t),\gamma,t)}{\gamma-\gamma'}d\eta + ((\partial_\gamma T)(z(\alpha,\gamma',t),\gamma,t) - (\partial_\gamma T)(z(\alpha,\gamma,t),\gamma,t)).
\end{aligned}
$$

From (2.37) and (2.52), we have

$$
\|((\partial_\gamma T)(z(\alpha,\gamma',t),\gamma,t) - (\partial_\gamma T)(z(\alpha,\gamma,t),\gamma,t))\|_{L^2(\mathcal{T})} \lesssim \|z(\alpha,\gamma,t) - z(\alpha,\gamma',t)\|_{H^2(\mathcal{T})} \lesssim |\gamma-\gamma'|.
$$

and

$$
\|(\partial_\eta T)(z(\alpha,\gamma',t),\eta,t) - (\partial_\gamma T)(z(\alpha,\gamma',t),\gamma,t)\|_{L^2(\mathcal{T})} \lesssim |\eta-\gamma|.
$$

Then

$$
\|Term_2\|_{L^2(\mathcal{T})} \lesssim |\gamma-\gamma'|.
$$

Moreover, for $Term_1$, from (2.54), (2.36), (2.41) and (2.45) we have

$$
\begin{aligned}
&\frac{T(z(\alpha,\gamma,t),\gamma,t) - T(z(\alpha,\gamma',t),\gamma,t)}{\gamma-\gamma'} - D_z T(z(\alpha,\gamma,t),\gamma,t)[w(\alpha,\gamma,t)] \\
&= \frac{ic(\alpha)\gamma}{1+ic'(\alpha)\gamma t}\partial_\alpha(v(\alpha,\gamma,\gamma')) \\
&+ \lambda(\alpha)\int_{-\pi}^{\pi} K(z(\alpha,\gamma)-z(\beta,\gamma),\tilde{f}(\alpha)-\tilde{f}(\beta))(\frac{\partial_\alpha v(\alpha,\gamma,\gamma')}{1+ic'(\alpha)\gamma t} - \frac{\partial_\beta v(\beta,\gamma,\gamma')}{1+ic'(\beta)\gamma t})(1+ic'(\beta)\gamma t)d\beta \\
&+ \lambda(\alpha)\int_{-\pi}^{\pi}\int_0^1 \nabla K(\tau(z(\alpha,\gamma)-z(\beta,\gamma)) + (1-\tau)(z(\alpha,\gamma')-z(\beta,\gamma')),\tilde{f}(\alpha)-\tilde{f}(\beta)) \\
&\cdot (\frac{z(\alpha,\gamma)-z(\beta,\gamma)-(z(\alpha,\gamma')-z(\beta,\gamma'))}{\gamma-\gamma'}) \\
&(\frac{\partial_\alpha z(\alpha,\gamma')}{1+ic'(\alpha)\gamma t} - \frac{\partial_\beta z(\beta,\gamma')}{1+ic'(\beta)\gamma t} + \partial_\alpha\tilde{f}(\alpha) - \partial_\beta\tilde{f}(\beta))(1+ic'(\beta)\gamma t)d\tau d\beta \\
&- \lambda(\alpha)\int_{-\pi}^{\pi} \nabla K(z(\alpha,\gamma)-z(\beta,\gamma),\tilde{f}(\alpha,\gamma)-\tilde{f}(\beta,\gamma))\cdot(w(\alpha,\gamma)-w(\beta,\gamma)) \\
&\cdot (\partial_\alpha\tilde{f}(\alpha) - \partial_\beta\tilde{f}(\beta) + \frac{\partial_\alpha z(\alpha,\gamma)}{1+ic'(\alpha)\gamma t} - \frac{\partial_\beta z(\beta,\gamma)}{1+ic'(\beta)\gamma t})(1+ic'(\beta)\gamma t)d\beta
\end{aligned}
$$

$$= Term_{1,1} + Term_{1,2} + Term_{1,3} + Term_{1,4}.$$

Here

$$\begin{aligned}
&Term_{1,3} + Term_{1,4}\\
&= \lambda(\alpha)\int_{-\pi}^{\pi}\int_0^1 \nabla K(\tau(z(\alpha,\gamma)-z(\beta,\gamma))+(1-\tau)(z(\alpha,\gamma')-z(\beta,\gamma')),\tilde f(\alpha)-\tilde f(\beta))d\tau\cdot(v(\alpha,\gamma,\gamma')-v(\beta,\gamma,\gamma'))\\
&\cdot(\partial_\alpha\tilde f(\alpha)-\partial_\beta\tilde f(\beta)+\frac{\partial_\alpha z(\alpha,\gamma')}{1+ic'(\alpha)\gamma t}-\frac{\partial_\beta z(\beta,\gamma')}{1+ic'(\beta)\gamma t})(1+ic'(\beta)\gamma t)d\beta\\
&+\lambda(\alpha)\int_{-\pi}^{\pi}[\int_0^1 \nabla K(\tau(z(\alpha,\gamma)-z(\beta,\gamma))+(1-\tau)(z(\alpha,\gamma')-z(\beta,\gamma')),\tilde f(\alpha)-\tilde f(\beta))d\tau\\
&-\nabla K(z(\alpha,\gamma)-z(\beta,\gamma),\tilde f(\alpha)-\tilde f(\beta))]\cdot(w(\alpha,\gamma)-w(\beta,\gamma))\\
&(\frac{\partial_\alpha z(\alpha,\gamma')}{1+ic'(\alpha)\gamma t}-\frac{\partial_\beta z(\beta,\gamma')}{1+ic'(\beta)\gamma t}+\partial_\alpha\tilde f(\alpha)-\partial_\beta\tilde f(\beta))(1+ic'(\beta)\gamma t)d\beta\\
&+\lambda(\alpha)\int_{-\pi}^{\pi}\nabla K(z(\alpha,\gamma)-z(\beta,\gamma),\tilde f(\alpha)-\tilde f(\beta))\cdot(w(\alpha,\gamma)-w(\beta,\gamma))\\
&(\frac{\partial_\alpha z(\alpha,\gamma')}{1+ic'(\alpha)\gamma t}-\frac{\partial_\beta z(\beta,\gamma')}{1+ic'(\beta)\gamma t}-(\frac{\partial_\alpha z(\alpha,\gamma)}{1+ic'(\alpha)\gamma t}-\frac{\partial_\beta z(\beta,\gamma)}{1+ic'(\beta)\gamma t}))d\beta\\
&= Term_{1,5} + Term_{1,6} + Term_{1,7}.
\end{aligned}$$

Since the component of ∇K is of -2 type, we could use lemma 2.5.3 to bound $Term_{1,5}$ and have

$$\|Term_{1,5}\|^2_{L^2(\mathcal{T})} \lesssim \|v(\alpha,\gamma,\gamma')\|^2_{L^2(\mathcal{T})}.$$

For $Term_{1,6}$, we have

$$\begin{aligned}
&\|[\int_0^1 \nabla K(\tau(z(\alpha,\gamma)-z(\beta,\gamma))+(1-\tau)(z(\alpha,\gamma')-z(\beta,\gamma')),\tilde f(\alpha)-\tilde f(\beta))d\tau\\
&-\nabla K(z(\alpha,\gamma)-z(\beta,\gamma),\tilde f(\alpha)-\tilde f(\beta))](\alpha-\beta)^2\|_{C^1([-2\delta,\delta]\times[-\pi,\pi])}\\
&\lesssim \|z(\alpha,\gamma)-z(\alpha,\gamma')\|_{H^3(\mathcal{T})}.
\end{aligned}$$

Then from lemma 2.5.4, and (2.52), we have

$$\|Term_{1,6}\|_{L^2(\mathcal{T})} \lesssim |\gamma-\gamma'|.$$

From lemma (2.5.4), we again have

$$
\begin{aligned}
&\|Term_{1,7}\|_{L^2(\mathcal{T})} \\
&\lesssim \|\nabla K(z(\alpha,\gamma)-z(\beta,\gamma),\tilde{f}(\alpha)-\tilde{f}(\beta))(\alpha-\beta)^2\|_{C^1([-2\delta,\delta]\times[-\pi,\pi])}\|w(\alpha,\gamma)\|_{C^2(\mathcal{T})}\|\frac{\partial_\alpha z(\alpha,\gamma')}{1+ic'(\alpha)\gamma t}-\frac{\partial_\alpha z(\alpha,\gamma)}{1+ic'(\alpha)\gamma t}\|_{L^2(\mathcal{T})} \\
&\lesssim \|z(\alpha,\gamma)-z(\alpha,\gamma')\|_{H^1(T)} \lesssim |\gamma-\gamma'|,
\end{aligned}
$$

where we use (2.56) and (2.52). Therefore we have

$$
\begin{aligned}
&\frac{d}{dt}\int_{-\pi}^{\pi}|v(\alpha,\gamma,\gamma')|^2 d\alpha \\
&= 2\Re\int_{-\pi}^{\pi} v(\alpha,\gamma,\gamma')\overline{\frac{ic(\alpha)\gamma}{1+ic'(\alpha)\gamma t}\partial_\alpha(v(\alpha,\gamma,\gamma'))}d\alpha \\
&+ 2\Re\int_{-\pi}^{\pi} v(\alpha,\gamma,\gamma')\overline{\lambda(\alpha)\int_{-\pi}^{\pi}K(z(\alpha,\gamma)-z(\beta,\gamma),\tilde{f}(\alpha)-\tilde{f}(\beta))(\frac{\partial_\alpha v(\alpha,\gamma,\gamma')}{1+ic'(\alpha)\gamma t}-\frac{\partial_\beta v(\beta,\gamma,\gamma')}{1+ic'(\beta)\gamma t})(1+ic'(\beta)\gamma t)d\beta}d\alpha \\
&+ B.T^0.
\end{aligned}
$$

where

$$B.T^0 \lesssim |\gamma-\gamma'|^2 + \|v(\alpha,\gamma,\gamma')\|^2_{L^2(\mathcal{T})}.$$

Then by corollary 2.4.6, and the initial value $v(\alpha,\gamma,\gamma',0)=0$, from the Gronwall's inequality we have $\lim_{\gamma'\to\gamma}\|v(\alpha,\gamma,\gamma',t)\|_{L^2(\mathcal{T})}=0$ when $t\le t_3$.

Form (2.56), and (2.52), we have

$$\|v(\alpha,\gamma,\gamma',t)\|_{H^3(\mathcal{T})}\lesssim 1.$$

Then from the interpolation theorem, we have

$$\lim_{\gamma'\to\gamma}\|v(\alpha,\gamma,\gamma',t)\|_{H^2(\mathcal{T})}=0.$$

Then from (2.60), we have

$$z(\alpha,\gamma)\in C^1_\gamma([-1,1],H^2_\alpha(\mathcal{T})),$$

with

$$\frac{d}{d\gamma}z(\alpha,\gamma,t)=w(\alpha,\gamma).$$

From (2.53), we also have

$$\frac{d}{dt}\frac{d}{d\gamma}z = \frac{dw}{dt} = \frac{dT(z)}{d\gamma} = \frac{d}{d\gamma}\frac{d}{dt}z.$$

In conclusion we have $\begin{cases} z(\alpha,\gamma,t) \in C^1_t([0,t_1], H^3_\alpha(\mathcal{T})), \\ z(\alpha,\gamma,t) \in C^1_\gamma([-1,1], H^2_\alpha(\mathcal{T})), \\ \frac{dz}{d\gamma} \in C^0_t([0,t_3], H^2_\alpha(\mathcal{T})) \cap C^1_t([0,t_3], H^1_\alpha(\mathcal{T})) \\ \frac{d}{dt}\frac{d}{d\gamma}z = \frac{d}{d\gamma}\frac{d}{dt}z \\ z(\alpha,0,t) = f^c(\alpha,t), \text{ when } 0 \le t \le t_2. \end{cases}$

Then we will show the analyticity in the next section.

2.8 The analyticity

In this section, we want to show $f(\alpha,t)$ is an real analytic function near 0 for each fixed t, $0 < t < t_3$. We first show that it is enough to prove

$$\frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t}\frac{d}{d\alpha}z(\alpha,\gamma,t) - \frac{d}{d\gamma}z(\alpha,\gamma,t) = 0. \tag{2.61}$$

Lemma 2.8.1. *If z satisfies (2.61), then $f(x)$ can be analytically extended to $D_A = \{\alpha+iy | -\infty < \alpha < \infty, -c(\alpha)t \le y \le c(\alpha)t\}$.*

Proof. From the uniqueness (3.166), we have $z(\alpha,0,t) = f^c(\alpha,t)$. Then $f^c(\alpha+iy,t) = z(\alpha, \frac{y}{c(\alpha)t}, t)$ is a extension of $f^c(\alpha,t)$ on D_A. Moreover, when $c(\alpha) \neq 0$, we have

$$\frac{d}{d\alpha}(z(\alpha,\frac{y}{c(\alpha)t},t)) = (\partial_\alpha z)(\alpha,\frac{y}{c(\alpha)t},t) - (\partial_\gamma z)(\alpha,\frac{y}{c(\alpha)t},t)(\frac{yc'(\alpha)}{c(\alpha)^2 t}).$$

$$\frac{d}{dy}(z(\alpha,\frac{y}{c(\alpha)t},t)) = (\partial_\gamma z)(\alpha,\frac{y}{c(\alpha)t},t)(\frac{1}{c(\alpha)t}).$$

Then we have

$$\begin{aligned} &\frac{d}{d\alpha}f^c(\alpha+iy,t) + i\frac{d}{dy}f^c(\alpha+iy,t) \\ &= (\partial_\alpha z)(\alpha,\frac{y}{c(\alpha)t},t) - (\partial_\gamma z)(\alpha,\frac{y}{c(\alpha)t},t)(\frac{yc'(\alpha)}{c(\alpha)^2 t}) + i(\partial_\gamma z)(\alpha,\frac{y}{c(\alpha)t},t)(\frac{1}{c(\alpha)t}). \end{aligned}$$

Now let $\frac{y}{c(\alpha)t} = \gamma$. Then

$$\frac{d}{d\alpha} f^c(\alpha + iy, t) + i\frac{d}{dy} f^c(\alpha + iy, t) = \partial_\alpha z(\alpha, \gamma, t) - (\frac{c'(\alpha)\gamma}{c(\alpha)} - \frac{i}{c(\alpha)t})\partial_\gamma z(\alpha, \gamma, t)$$
$$= \partial_\alpha z(\alpha, \gamma, t) - (\frac{ic'(\alpha)\gamma t + 1}{ic(\alpha)t})\partial_\gamma z(\alpha, \gamma, t)$$
$$= 0.$$

Moreover, $z(\alpha, \gamma, t) \in C^1_\gamma([-1, 1], H^2_\alpha(\mathcal{T}))$. Then $\partial_\alpha f^c$, $\partial_\gamma f^c$ are continuous. Therefore we have the analyticity of f^c near 0. We also have $f(\alpha, t) = f^c(\alpha, t)$ when $|\alpha| \leq \delta$. Then we have the result. □

Let

$$A_0(h)(\alpha, \gamma, t) = (\frac{ic(\alpha)t}{1 + ic'(\alpha)\gamma t}\partial_\alpha - \partial_\gamma)h(\alpha, \gamma, t).$$

Before we prove $A_0(z) = 0$, we first introduce some general lemmas.

Lemma 2.8.2. *If all the derivatives are well-defined and $\partial_\alpha\partial_\gamma h(\alpha, \gamma, t) = \partial_\gamma\partial_\alpha h(\alpha, \gamma, t)$, we have*

$$(\frac{ic(\alpha)t}{1 + ic'(\alpha)\gamma t}\partial_\alpha - \partial_\gamma)\frac{\partial_\alpha h(\alpha, \gamma)}{1 + ic'(\alpha)\gamma t} = \frac{\partial_\alpha}{1 + ic'(\alpha)\gamma t}(\frac{ic(\alpha)t}{1 + ic'(\alpha)\gamma t}\partial_\alpha - \partial_\gamma)h(\alpha, \gamma),$$

Proof. First, for the right hand side, we have

$$(RHS) = \frac{\frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t}\partial^2_\alpha h(\alpha, \gamma)}{1 + ic'(\alpha)\gamma t} + \frac{ic(\alpha)t}{1 + ic'(\alpha)\gamma t}(\frac{-ic''(\alpha)\gamma t}{(1 + ic'(\alpha)\gamma t)^2}\partial_\alpha h(\alpha, \gamma)) - \frac{\partial_\alpha\partial_\gamma h(\alpha, \gamma)}{1 + ic'(\alpha)\gamma t} + \frac{ic'(\alpha)t\partial_\alpha h(\alpha, \gamma)}{(1 + ic'(\alpha)\gamma t)^2}.$$

Also

$$(LHS) = [\frac{ic'(\alpha)t}{(1 + ic'(\alpha)\gamma t)^2} - \frac{ic(\alpha)tic''(\alpha)\gamma t}{(1 + ic'(\alpha)\gamma t)^3}]\partial_\alpha h(\alpha, \gamma) + \frac{ic(\alpha)t}{(1 + ic'(\alpha)\gamma t)^2}\partial^2_\alpha h(\alpha, \gamma) - \frac{1}{1 + ic'(\alpha)\gamma t}\partial_\alpha\partial_\gamma h(\alpha, \gamma).$$

From the two equalities above, we have the result. □

Lemma 2.8.3. *If all the derivatives are well-defined and we have*

$$\frac{d}{dt}h = \frac{ic(\alpha)\gamma}{1 + ic'(\alpha)\gamma t}\partial_\alpha h + \tilde{T}(h),$$

and $\frac{d}{d\alpha}\frac{d}{dt}h = \frac{d}{dt}\frac{d}{d\alpha}h$, $\frac{d}{dt}\frac{d}{d\gamma}h = \frac{d}{d\gamma}\frac{d}{dt}h$, $\frac{d}{d\alpha}\frac{d}{d\gamma}h = \frac{d}{d\gamma}\frac{d}{d\alpha}h$, then we have

$$\frac{d}{dt}A_0(h) = \frac{ic(\alpha)\gamma}{1 + ic'(\alpha)\gamma t}\partial_\alpha A_0(h) + A_0(\tilde{T}(h))$$

Proof.

$$
\begin{aligned}
&\frac{d}{dt}A_0(h)(\alpha,\gamma,t)\\
&=\frac{d}{dt}(\frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t})\partial_\alpha h+\frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t}\partial_\alpha\frac{d}{dt}h-\partial_\gamma\frac{d}{dt}h\\
&=\frac{d}{dt}(\frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t})\partial_\alpha h+(\frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t}\partial_\alpha-\partial_\gamma)(\frac{ic(\alpha)\gamma}{1+ic'(\alpha)\gamma t}\partial_\alpha h)+A_0(\tilde{T}(h))\\
&=\frac{d}{dt}(\frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t})\partial_\alpha h+\frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t}\partial_\alpha(\frac{ic(\alpha)\gamma}{1+ic'(\alpha)\gamma t})\partial_\alpha h+\frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t}\frac{ic(\alpha)\gamma}{1+ic'(\alpha)\gamma t}\partial_\alpha^2 h\\
&-\partial_\gamma(\frac{ic(\alpha)\gamma}{1+ic'(\alpha)\gamma t})\partial_\alpha h-(\frac{ic(\alpha)\gamma}{1+ic'(\alpha)\gamma t}\partial_\gamma\partial_\alpha h)+A_0(\tilde{T}(h))\\
&=(\underbrace{\frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t}\partial_\alpha(\frac{ic(\alpha)\gamma}{1+ic'(\alpha)\gamma t})}_{Term_1}+\underbrace{\frac{d}{dt}(\frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t})-\partial_\gamma(\frac{ic(\alpha)\gamma}{1+ic'(\alpha)\gamma t})}_{Term_2})\partial_\alpha h\\
&+\underbrace{\frac{ic(\alpha)\gamma}{1+ic'(\alpha)\gamma t}(\frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t}\partial_\alpha-\partial_\gamma)\partial_\alpha h}_{Term_3}+A_0(\tilde{T}(h)).
\end{aligned}
$$

Moreover, we have

$$
\underbrace{\frac{d}{dt}(\frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t})-\partial_\gamma(\frac{ic(\alpha)\gamma}{1+ic'(\alpha)\gamma t})}_{Term_2}=0,
$$

$$
\underbrace{\frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t}\partial_\alpha(\frac{ic(\alpha)\gamma}{1+ic'(\alpha)\gamma t})}_{Term_1}=\frac{ic(\alpha)\gamma}{1+ic'(\alpha)\gamma t}\partial_\alpha(\frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t}),
$$

and

$$
\begin{aligned}
&\underbrace{\frac{ic(\alpha)\gamma}{1+ic'(\alpha)\gamma t}(\frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t}\partial_\alpha-\partial_\gamma)\partial_\alpha h}_{Term_3}\\
&=\frac{ic(\alpha)\gamma}{1+ic'(\alpha)\gamma t}\partial_\alpha((\frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t}\partial_\alpha-\partial_\gamma)h)-\frac{ic(\alpha)\gamma}{1+ic'(\alpha)\gamma t}\partial_\alpha(\frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t})\partial_\alpha h.
\end{aligned}
$$

Therefore

$$
\frac{d}{dt}A_0(h)(\alpha,\gamma,t)=\frac{ic(\alpha)\gamma}{1+ic'(\alpha)\gamma t}\partial_\alpha((\frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t}\partial_\alpha-\partial_\gamma)h)+A_0(\tilde{T}(h))=\frac{ic(\alpha)\gamma}{1+ic'(\alpha)\gamma t}\partial_\alpha A_0(h)+A_0(\tilde{T}(h)).
$$

□

Lemma 2.8.4. *Let $\tilde{K}$ be meromorphic. $\partial_\alpha X(\alpha,\gamma)$, $\partial_\gamma X(\alpha,\gamma)$ are well-defined and in $C^0_\alpha[-\pi,\pi]$, $\partial_\alpha h(\alpha,\gamma)$ and $\partial_\gamma h(\alpha,\gamma)$ are well-defined vector functions with components in $C^0_\alpha[-\pi,\pi]$. If for fixed α, there is no singular point in the integrals below and $c(\pi)=c(-\pi)=0$, $c(\alpha)\in W^{2,\infty}$, then we have*

$$
\begin{aligned}
&A_0(\int_{-\pi}^{\pi}\tilde{K}(h(\alpha,\gamma)-h(\beta,\gamma))X(\beta,\gamma)(1+ic'(\beta)\gamma t)d\beta)=\\
&\int_{-\pi}^{\pi}\nabla\tilde{K}(h(\alpha,\gamma)-h(\beta,\gamma))\cdot(A_0(h)(\alpha,\gamma,t)-A_0(h)(\beta,\gamma,t))X(\beta,\gamma)(1+ic'(\beta)\gamma t)d\beta\\
&+\int_{-\pi}^{\pi}\tilde{K}(h(\alpha,\gamma)-h(\beta,\gamma))A_0(X)(\beta,\gamma)(1+ic'(\beta)\gamma t)d\beta\\
&=D_h(\int_{-\pi}^{\pi}\tilde{K}(h(\alpha,\gamma)-h(\beta,\gamma))X(\beta,\gamma)(1+ic'(\beta)\gamma t)d\beta)[A_0(h)]\\
&+\int_{-\pi}^{\pi}\tilde{K}(h(\alpha,\gamma)-h(\beta,\gamma,t))A_0(X)(\beta,\gamma)(1+ic'(\beta)\gamma t)d\beta.
\end{aligned}
$$

Here D_h is the Gateaux derivative.

Proof. We have

$$
\begin{aligned}
&A_0(\int_{\alpha-\pi}^{\alpha+\pi}\tilde{K}(h(\alpha,\gamma)-h(\alpha-\beta,\gamma))X(\alpha-\beta,\gamma)(1+ic'(\alpha-\beta)\gamma t)d\beta)=\\
&\underbrace{\frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t}(\tilde{K}(h(\alpha,\gamma)-h(-\pi,\gamma))X(-\pi,\gamma)(1+ic'(-\pi)\gamma t)-\tilde{K}(h(\alpha,\gamma)-h(\pi,\gamma))X(\pi,\gamma)(1+ic'(\pi)\gamma t))}_{Term_1}\\
&+\underbrace{\int_{\alpha-\pi}^{\alpha+\pi}\nabla\tilde{K}(h(\alpha,\gamma)-h(\alpha-\beta,\gamma))\cdot((\frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t}\partial_\alpha-\partial_\gamma)h(\alpha,\gamma)-(\frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t}\partial_\alpha-\partial_\gamma)h(\alpha-\beta,\gamma,t))}_{Term_2}\\
&\underbrace{X(\alpha-\beta,\gamma)(1+ic'(\alpha-\beta)\gamma t)d\beta}_{Term_2}\\
&+\underbrace{\int_{\alpha-\pi}^{\alpha+\pi}\tilde{K}(h(\alpha,\gamma)-h(\alpha-\beta,\gamma))(\frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t}\partial_\alpha-\partial_\gamma)X(\alpha-\beta,\gamma)(1+ic'(\alpha-\beta)\gamma t)d\beta}_{Term_3}\\
&+\underbrace{\int_{\alpha-\pi}^{\alpha+\pi}\tilde{K}(h(\alpha,\gamma)-h(\alpha-\beta,\gamma))X(\alpha-\beta,\gamma)(\frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t}ic''(\alpha-\beta)\gamma t-ic'(\alpha-\beta)t)d\beta}_{Term_4}\\
&=Term1+\\
&+\underbrace{\int_{\alpha-\pi}^{\alpha+\pi}\nabla\tilde{K}(h(\alpha,\gamma)-h(\alpha-\beta,\gamma))\cdot((\frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t}\partial_\alpha-\partial_\gamma)h(\alpha,\gamma)-(\frac{ic(\alpha-\beta)t}{1+ic'(\alpha-\beta)\gamma t}\partial_\alpha-\partial_\gamma)h(\alpha-\beta,\gamma,t))}_{Term_{2,1}}\\
&\underbrace{X(\alpha-\beta,\gamma)(1+ic'(\alpha-\beta)\gamma t)d\beta}_{Term_{2,1}}
\end{aligned}
$$

$$+\underbrace{\int_{\alpha-\pi}^{\alpha+\pi}\nabla\tilde{K}(h(\alpha,\gamma)-h(\alpha-\beta,\gamma))\cdot((\frac{ic(\alpha-\beta)t}{1+ic'(\alpha-\beta)\gamma t}\partial_\alpha-\frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t}\partial_\alpha)h(\alpha-\beta,\gamma,t))}_{Term_{2,2}}$$

$$\underbrace{X(\alpha-\beta,\gamma)(1+ic'(\alpha-\beta)\gamma t)d\beta}_{Term_{2,2}}$$

$$+\underbrace{\int_{\alpha-\pi}^{\alpha+\pi}\tilde{K}(h(\alpha,\gamma)-h(\alpha-\beta,\gamma))(\frac{ic(\alpha-\beta)t}{1+ic'(\alpha-\beta)\gamma t}\partial_\alpha-\partial_\gamma)X(\alpha-\beta,\gamma)(1+ic'(\alpha-\beta)\gamma t)d\beta}_{Term_{3,1}}$$

$$+\underbrace{\int_{\alpha-\pi}^{\alpha+\pi}\tilde{K}(h(\alpha,\gamma)-h(\alpha-\beta,\gamma))(\frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t}\partial_\alpha-\frac{ic(\alpha-\beta)t}{1+ic'(\alpha-\beta)\gamma t}\partial_\alpha)X(\alpha-\beta,\gamma)(1+ic'(\alpha-\beta)\gamma t)d\beta}_{Term_{3,2}}$$

$$+\underbrace{\int_{\alpha-\pi}^{\alpha+\pi}\tilde{K}(h(\alpha,\gamma)-h(\alpha-\beta,\gamma))X(\alpha-\beta,\gamma)(\frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t}ic''(\alpha-\beta)\gamma t-ic'(\alpha-\beta)t)d\beta}_{Term_4}$$

$$=Term_1+Term_2+Term_{2,2}+Term_{3,1}+Term_{3,2}+Term_4.$$

Moreover,

$$Term_{2,2}+Term_{3,2}+Term_4=$$
$$=\int_{\alpha-\pi}^{\alpha+\pi}\frac{d}{d\beta}(\tilde{K}(h(\alpha,\gamma)-h(\alpha-\beta,\gamma))X(\alpha-\beta,\gamma)(ic(\alpha-\beta)t-\frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t}(1+ic'(\alpha-\beta)\gamma t)))d\beta$$
$$=\tilde{K}(h(\alpha,\gamma)-h(-\pi,\gamma))X(-\pi,\gamma)(ic(-\pi)t-\frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t}(1+ic'(-\pi)\gamma t))$$
$$-\tilde{K}(h(\alpha,\gamma)-h(\pi,\gamma))X(\pi,\gamma)(ic(\pi)t-\frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t}(1+ic'(\pi)\gamma t))$$

We use the condition that $c(-\pi)=c(\pi)=0$ and we could get the $Term_1+Term_{2,2}+Term_{3,2}+Term4=0$. Then we have the result. □

Now we use lemma 2.8.2, 2.8.3, 2.8.4, to show the result. From (2.8), we have

$$\frac{dz_\mu(\alpha,\gamma,t)}{dt}=\frac{ic(\alpha)\gamma}{1+ic'(\alpha)\gamma t}\partial_\alpha z_\mu(\alpha,\gamma,t)+$$
$$+\lambda(\alpha_\gamma^t)\int_{-\pi}^{\pi}K(z(\alpha,\gamma,t)-z(\beta,\gamma,t)+\tilde{f}(\alpha_\gamma^t,t)-\tilde{f}(\beta_\gamma^t,t))(\frac{\partial_\alpha z_\mu(\alpha,\gamma,t)}{1+ic'(\alpha)\gamma t}-\frac{\partial_\beta z_\mu(\beta,\gamma,t)}{1+ic'(\beta)\gamma t})(1+ic'(\beta)\gamma t)d\beta$$
$$+\lambda(\alpha_\gamma^t)\int_{-\pi}^{\pi}K(z(\alpha,\gamma,t)-z(\beta,\gamma,t)+\tilde{f}(\alpha_\gamma^t,t)-\tilde{f}(\beta_\gamma^t,t))((\partial_\alpha\tilde{f}_\mu)(\alpha_\gamma^t,t)-(\partial_\beta\tilde{f}_\mu)(\beta_\gamma^t,t))(1+ic'(\beta)\gamma t)d\beta.$$

Since we have $A_0(\lambda(\alpha_\gamma^t))=0$, from lemma 2.8.3, we get

$$\frac{dA_0(z_\mu)}{dt}=\frac{ic(\alpha)\gamma}{1+ic'(\alpha)\gamma t}\partial_\alpha A_0(z_\mu)(\alpha,\gamma,t)$$
$$+\lambda(\alpha_\gamma^t)A_0(\int_{-\pi}^{\pi}K(z(\alpha,\gamma,t)-z(\beta,\gamma,t)+\tilde{f}(\alpha_\gamma^t,t)-\tilde{f}(\beta_\gamma^t,t))$$
$$(\frac{\partial_\alpha z_\mu(\alpha,\gamma,t)}{1+ic'(\alpha)\gamma t}-\frac{\partial_\beta z_\mu(\beta,\gamma,t)}{1+ic'(\beta)\gamma t}+(\partial_\alpha\tilde{f}_\mu)(\alpha_\gamma^t,t)-(\partial_\beta\tilde{f}_\mu)(\beta_\gamma^t,t))(1+ic'(\beta)\gamma t)d\beta).$$

Let

$$h(\alpha,\gamma)=(z(\alpha,\gamma,t)+\tilde{f}(\alpha_\gamma^t,t),\frac{\partial_\alpha z_\mu(\alpha,\gamma,t)}{1+ic'(\alpha)\gamma t}+(\partial_\alpha\tilde{f}_\mu)(\alpha_\gamma^t,t)),$$

$$\tilde{K}(h(\alpha,\gamma)-h(\beta,\gamma))=K(h_1(\alpha,\gamma)-h_1(\beta,\gamma))(h_2(\alpha,\gamma)-h_2(\beta,\gamma)),$$

and

$$X(\alpha,\gamma)=1.$$

then by lemma 2.8.4 and lemma 2.8.2 and $A_0((\partial_\alpha\tilde{f}_\mu)(\alpha_\gamma^t,t))=0$, and $A_0(\tilde{f}_\mu(\alpha_\gamma^t,t))=0$, we have

$$\frac{dA_0(z_\mu)}{dt}=\frac{ic(\alpha)\gamma}{1+ic'(\alpha)\gamma t}\partial_\alpha A_0(z_\mu)(\alpha,\gamma,t)$$
$$+\lambda(\alpha_\gamma^t)\int_{-\pi}^{\pi}K(z(\alpha,\gamma,t)-z(\beta,\gamma,t)+\tilde{f}(\alpha_\gamma^t,t)-\tilde{f}(\beta_\gamma^t,t))(\frac{\partial_\alpha A_0(z_\mu)(\alpha,\gamma,t)}{1+ic'(\alpha)\gamma t}-\frac{\partial_\beta A_0(z_\mu)(\beta,\gamma,t)}{1+ic'(\beta)\gamma t})$$
$$\cdot(1+ic'(\beta)\gamma t)d\beta$$
$$+\lambda(\alpha_\gamma^t)\int_{-\pi}^{\pi}\nabla K(z(\alpha,\gamma,t)-z(\beta,\gamma,t)+\tilde{f}(\alpha_\gamma^t,t)-\tilde{f}(\beta_\gamma^t,t))\cdot(A_0(z)(\alpha,\gamma,t)-A_0(z)(\beta,\gamma,t))$$
$$\cdot(\frac{\partial_\alpha z_\mu(\alpha,\gamma,t)}{1+ic'(\alpha)\gamma t}-\frac{\partial_\beta z_\mu(\beta,\gamma,t)}{1+ic'(\beta)\gamma t}+(\partial_\alpha\tilde{f}_\mu)(\alpha_\gamma^t,t)-(\partial_\beta\tilde{f}_\mu)(\beta_\gamma^t,t))(1+ic'(\beta)\gamma t)d\beta$$
$$=Term_1+Term_2+Term_3.$$

Since

$$\|\nabla K(z(\alpha,\gamma,t)-z(\beta,\gamma,t)+\tilde{f}(\alpha_\gamma^t,t)-\tilde{f}(\beta_\gamma^t,t))(\alpha-\beta)^2\|_{C^1([-2\delta,2\delta]\times[-\pi,\pi])}\lesssim 1,$$

$$\|\frac{\partial_\alpha z_\mu(\alpha,\gamma,t)}{1+ic'(\alpha)\gamma t}+(\partial_\alpha\tilde{f}_\mu)(\alpha_\gamma^t,t)\|_{C^2([-2\delta,2\delta]\times[-\pi,\pi])}\lesssim 1,$$

by lemma 2.5.4, we have

$$\|Term_3\|_{L^2(\mathcal{T})}\lesssim\|A_0(z)\|_{L^2(\mathcal{T})}.$$

Then by corollary 2.4.6, we have

$$\frac{d}{dt}\|A_0(z)\|^2_{L^2(\mathcal{T})} \lesssim \|A_0(z)\|^2_{L^2(\mathcal{T})}.$$

Moreover when $t = 0$, $A_0(z) = -\partial_\gamma z(\alpha, \gamma, 0) = -\partial_\gamma f(\alpha, 0) = 0$. Therefore $A_0(z) = 0$, when $t \leq t_3$.

Chapter 3

The degenerated regularity near the turnover points

3.1 Notation

In this chapter we use the following notation:

$Z_1(t)$, $Z_2(t)$: They are values of α of the turnover points.

δ: a sufficiently small number.

$\lambda(\alpha)$: $\lambda(\alpha) \geq 0$ and in $C^{100}(-\infty, \infty)$, satisfying

$$\lambda(\alpha) = \begin{cases} 1 & |\alpha| \leq \delta, \\ 0 & |\alpha| \geq 2\delta. \end{cases}$$

δ_c: sufficiently small number depending on δ.

$c(\alpha)$:

$$\begin{cases} c(\alpha) = \delta_c \alpha^2, \text{ when } \alpha \leq \frac{\delta}{32} \\ \operatorname{supp} c(\alpha) \subset [0, \frac{\delta}{8}] \\ c(\alpha) \in C^{100}([0,\infty]) \cap C^{1,1}([-\infty,\infty]),\ \|c(\alpha)\|_{C^{100}([0,\infty])\cap C^{1,1}(-\infty,\infty)} \leq \delta. \end{cases}$$

$x(\alpha, t)$: The new variable such that $-\frac{\pi}{2}$, 0 be two fixed turnover points.

$f(\alpha, t) = (f_1(\alpha, t), f_2(\alpha, t))$: the original solution of the Muskat equation.

$\tilde{f}(\alpha,t)$:

$$\tilde{f}(\alpha,t)=f(x(\alpha,t),t).$$

$\tilde{f}^{+}(\alpha,t)$:

$$\tilde{f}^{+}(\alpha,t)=(\tilde{f}(\alpha,t)-\sum_{i=1}^{60}\frac{\tilde{f}^{<i>}(0)}{i!}\alpha^{i})\lambda(\alpha)1_{\alpha\geq 0}.$$

$\tilde{f}^{L}(\alpha,t)$:

$$\tilde{f}^{L}(\alpha,t)=\tilde{f}(\alpha,t)-\tilde{f}^{+}(\alpha,t).$$

$\tilde{\lambda}(\alpha)$: $\tilde{\lambda}(\alpha)\geq 0$ and in $C^{100}(-\infty,\infty)$, satisfying

$$\tilde{\lambda}(\alpha)=\begin{cases}1 & |\alpha|\leq 10\delta,\\ 0 & |\alpha|\geq 20\delta,\end{cases}$$

$X_i(\alpha,t)$: functions that are products of $\partial_{\alpha}^{l_i}\tilde{f}^{L}(\alpha,t)$,$\partial_{\alpha}^{l_i'}\frac{1}{x(\alpha,t)}$, $\partial_{\alpha}^{l_i''}x(\alpha,t)$, with $l_i,l_i',l_i''\leq 21$.

$\tilde{X}_i(\alpha,t)$:

$$\tilde{X}_i(\alpha,t)=\partial_{\alpha}^{b_i}(\frac{\frac{dx(\alpha,t)}{dt}}{\frac{dx(\alpha,t)}{d\alpha}}),$$

with $b_i\leq 20$.

$\tilde{D}_A$: $\tilde{D}_A=\{\alpha+iy|0<\alpha<\frac{\delta}{4},|y|<c(\alpha)t\}$.

t_0: the original solution exists when $t\in[-t_0,t_0]$.

δ_1: the bound of the C^2 norm of the solution to control the arc-chord condition.

t_{RT}: the Refined R-T holds in $[0,t_{RT}]$.

t_1: the energy estimate holds in $[0,t_1]$.

V^k: we set

$$V^k(\alpha)=(\tilde{f}_1^{+}(\alpha,t),\tilde{f}_2^{+}(\alpha,t),\partial_{\alpha}\tilde{f}_1^{+}(\alpha,t),\partial_{\alpha}\tilde{f}_2^{+}(\alpha,t),...,\partial_{\alpha}^{k}\tilde{f}_1^{+}(\alpha,t),\partial_{\alpha}^{k}\tilde{f}_2^{+}(\alpha,t)).$$

V_h^k: we set

$$V_h^k(\alpha,\gamma,t)=(D^{-20}h_1(\alpha,\gamma,t),D^{-20}h_2(\alpha,\gamma,t),...,D^{-20+k}h_1(\alpha,\gamma,t),D^{-20+k}h_2(\alpha,\gamma,t)).$$

$V^k_{\tilde{f}^L}$: we set

$$V^k_{\tilde{f}^L}(\alpha) = (\tilde{f}^L_1(\alpha,t), \tilde{f}^L_2(\alpha,t), \partial_\alpha \tilde{f}^L_1(\alpha,t), \partial_\alpha \tilde{f}^L_2(\alpha,t), ..., \partial^k_\alpha \tilde{f}^L_1(\alpha,t), \partial^k_\alpha \tilde{f}^L_2(\alpha,t)).$$

S:

$$S = \{h(\alpha,\gamma,t) = 0, \text{when } \alpha \le -\tau(t) \text{ and } \alpha \ge \frac{\pi}{8}\}.$$

S_{δ_1}:

$$S_{\delta_1} = \{h(\alpha,\gamma,t) = 0, \text{when } \alpha \le -\tau(t) \text{ and } \alpha \ge \frac{\pi}{8}, \sup_{\gamma,t} \|h\|_{C^2_\alpha[-\tau(t),\pi]} \lesssim \delta_1\}.$$

$V^{i_o,k}$:

$$V^{i_o,k} = C^{i_0}_\gamma([-1,1], H^k_\alpha[-\pi,\pi]).$$

U:

$$U = V^{0,k} \cap V^{1,k-1} ... \cap V^{3,k-3} \cap S_{\delta_1}.$$

3.2 Discussion of the assumption

Let $Z_1(t)$, $Z_2(t)$ be values of α of these two turnover points. By this definition of $Z_1(t)$, we have $\frac{d}{dt}\partial_\alpha f(Z_1(t),t) = 0$. Then

$$(\partial_t \partial_\alpha f_1)(Z_1(t),t) + \frac{dZ_1}{dt}(t)(\partial^2_\alpha f_1)(Z_1(t),t) = 0. \tag{3.1}$$

Moreover,

$$\begin{aligned}
&(\partial_t \partial_\alpha f_1)(\alpha,t) = \partial_\alpha \partial_t f_1(\alpha,t) \\
=&p.v \int_{-\pi}^{\pi} \Big(\frac{\cos(f_1(\alpha,t) - f_1(\beta,t))}{\cosh(f_2(\alpha,t) - f_2(\beta,t)) - \cos(f_1(\alpha,t) - f_1(\beta,t))} - \frac{(\sin(f_1(\alpha,t) - f_1(\beta,t)))^2}{(\cosh(f_2(\alpha,t) - f_2(\beta,t)) - \cos(f_1(\alpha,t) - f_1(\beta,t)))^2}\Big) \\
&\cdot (\partial_\alpha f_1(\alpha,t))(\partial_\alpha f_1(\alpha,t) - \partial_\beta f_1(\beta,t)) d\beta \\
&- p.v \int_{-\pi}^{\pi} \frac{\sin(f_1(\alpha,t) - f_1(\beta,t))(\partial_\alpha f_1(\alpha,t) - \partial_\beta f_1(\beta,t)) \sinh(f_2(\alpha,t) - f_2(\beta,t))}{(\cosh(f_2(\alpha,t) - f_2(\beta,t)) - \cos(f_1(\alpha,t) - f_1(\beta,t)))^2} \partial_\alpha f_2(\alpha,t) d\beta \\
&+ p.v \int_{-\pi}^{\pi} \frac{\sin(f_1(\alpha,t) - f_1(\beta,t))}{\cosh(f_2(\alpha,t) - f_2(\beta,t)) - \cos(f_1(\alpha,t) - f_1(\beta,t))} d\beta \partial^2_\alpha f_1(\alpha,t).
\end{aligned}$$

We can plug in $\alpha = Z_1(t)$ and get

$(\partial_t \partial_\alpha f_1)(Z_1(t),t)$

$$= -p.v \int_{-\pi}^{\pi} \frac{\sin(f_1(Z_1(t),t) - f_1(\beta,t))(\partial_\alpha f_1 Z_1(t),t) - \partial_\beta f_1(\beta,t))\sinh(f_2(Z_1(t),t) - f_2(\beta,t))}{(\cosh(f_2(Z_1(t),t) - f_2(\beta,t)) - \cos(f_1(Z_1(t),t) - f_1(\beta,t)))^2} \partial_\alpha f_2(Z_1(t),t) d\beta$$
$$+ p.v \int_{-\pi}^{\pi} \frac{\sin(f_1(Z_1(t),t) - f_1(\beta,t))}{\cosh(f_2(Z_1(t),t) - f_2(\beta,t)) - \cos(f_1(Z_1(t),t) - f_1(\beta,t))} d\beta \partial_\alpha^2 f_1(Z_1(t),t).$$

Then from (3.1), we have

$$p.v. \int_{-\pi}^{\pi} \frac{\sin(f_1(Z_1(t),t) - f_1(\beta,t))(\partial_\alpha f_1(Z_1(t),t) - \partial_\beta f_1(\beta,t))\sinh(f_2(Z_1(t),t) - f_2(\beta,t))}{(\cosh(f_2(Z_1(t),t) - f_2(\beta,t)) - \cos(f_1(Z_1(t),t) - f_1(\beta,t)))^2} \partial_\alpha f_2(Z_1(t),t) d\beta$$
$$= (p.v. \int_{-\pi}^{\pi} \frac{\sin(f_1(Z_1(t),t) - f_1(\beta,t))}{\cosh(f_2(Z_1(t),t) - f_2(\beta,t)) - \cos(f_1(Z_1(t),t) - f_1(\beta,t))} d\beta + \frac{dZ_1(t)}{dt}) \partial_\alpha^2 f_1(Z_1(t),t).$$

Then our assumption is equivalent to

$$p.v \int_{-\pi}^{\pi} \frac{\sin(f_1(Z_1(t),t) - f_1(\beta,t))(\partial_\alpha f_1(Z_1(t),t) - \partial_\beta f_1(\beta,t))\sinh(f_2(Z_1(t),t) - f_2(\beta,t))}{(\cosh(f_2(Z_1(t),t) - f_2(\beta,t)) - \cos(f_1(Z_1(t),t) - f_1(\beta,t)))^2} \partial_\alpha f_2(Z_1(t),t) d\beta < 0. \tag{3.2}$$

The next lemma shows there exists curve(though may not be a solution) satisfying the 3.2.

Lemma 3.2.1. *There exists curve $(f_1^0(\alpha), f_2^0(\alpha))$ with two turnover points satisfying*

$$p.v. \int_{-\pi}^{\pi} \frac{\sin(f_1(\alpha) - f_1(\beta))(\partial_\alpha f_1(\alpha) - \partial_\beta f_1(\beta))(\sinh(f_2(\alpha) - f_2(\beta)))f_2'(\alpha) d\beta}{(\cosh(f_2(\alpha) - f_2(\beta)) - \cos(f_1(\alpha) - f_1(\beta)))^2} < 0 \tag{3.3}$$

at both two points α_0, α_1.

Proof. The $f_1(\alpha), f_2(\alpha)$ we choose satisfy

$$|f_1'(\alpha)|_{L^\infty} \leq 2,$$
$$|f_1''(\alpha)|_{L^\infty} \leq \frac{4}{c},$$
$$|f_1(\alpha)| \leq c \text{ when } x \in [-c, c],$$
$$f_1(\alpha) = \alpha, \text{ when } x \in [-\pi, c] \cup [c, \pi],$$
$$f_2(\alpha) = \tilde{f}_2(\frac{\alpha}{c}), x \in [-c, c],$$
$$\inf_{\alpha \in [-1,1]} \tilde{f}_2'(\alpha) \geq \frac{1}{2} \sup_{\alpha \in [-1,1]} \tilde{f}_2'(\alpha) = 1,$$
$$f_2(\alpha) \text{ is increasing when } \alpha \in [-\pi + \eta, \pi - \eta],$$
$$f_2(-\frac{\pi}{2}) = -3, f_2(-c) = -1, f_2(c) = 1, f_2(\frac{\pi}{2}) = 3,$$
$$\sup |f_2(\alpha)| = 6.$$

$$\partial_\alpha f_1(\alpha_0) = \partial_\alpha f_1(\alpha_1) = 0,$$

$$-c < \alpha_0 < \alpha_1 < c,$$

$$|c| < \frac{\pi}{16}, |\eta| < \frac{\pi}{16}$$

$$|c| \le |\eta|.$$

Now we show (3.3). From now on we take $\alpha = \alpha_\mu$, $\mu = 0, 1$.

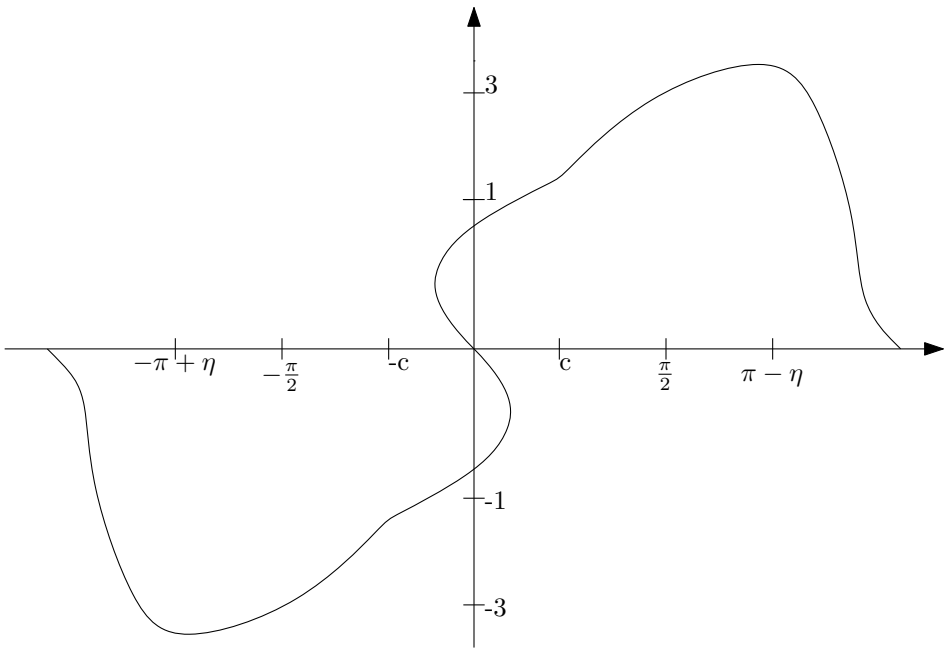


Figure 3.1: A graph of the interface satisfying (3.3).

For $\beta \in [-c, c]$, we have

$$\begin{aligned}
&|p.v. \int_{-c}^{c} \frac{\sin(f_1(\alpha) - f_1(\beta))(\partial_\alpha f_1(\alpha) - \partial_\beta f_1(\beta))(\sinh(f_2(\alpha) - f_2(\beta)))d\beta}{(\cosh(f_2(\alpha) - f_2(\beta)) - \cos(f_1(\alpha) - f_1(\beta)))^2}| \\
&\le 4e^{12}|c| \frac{\sup_{\tau\in[-c,c]} |f_1''(\tau)|^2(\alpha-\beta)^3 \sup_{\tau\in[-c,c]} |f_2'(\tau)||\alpha-\beta|}{\inf_{\tau\in[-c,c]} |f_2'(\tau)|^4(\alpha-\beta)^4} \\
&\lesssim c^2,
\end{aligned}$$

where we use $\partial_\alpha f_1(\alpha) = 0$, and the following two inequalities

$$|\sinh(f_2(\alpha) - f_2(\beta))| \le e^{2\sup|f_2|}|f_2(\alpha) - f_2(\beta)|,$$

and

$$|\cosh(f_2(\alpha)-f_2(\beta))-\cos(f_1(\alpha)-f_1(\beta))|\geq|\cosh(f_2(\alpha)-f_2(\beta))-1|\geq\frac{1}{2}(f_2(\alpha)-f_2(\beta))^2.$$

When $\beta\in[-\pi+\eta,\frac{-3}{4}\pi]\cup[-\frac{\pi}{2},-c]$, we have $\pi>f_1(\alpha)-f_1(\beta)>0$, $\partial_\alpha f_1(\alpha)-\partial_\beta f_1(\beta)=0-\partial_\beta f_1(\beta)<0$, $\sinh(f_2(\alpha)-f_2(\beta))>0$, hence corresponding integral is negative.

For $[c,\frac{\pi}{2}]\cup[\frac{3}{4}\pi,\pi-\eta]$, similarly we have $-\pi<f_1(\alpha)-f_1(\beta)<0$, $\partial_\alpha f_1(\alpha)-\partial_\beta f_1(\beta)=0-\partial_\beta f_1(\beta)<0$, $\sinh(f_2(\alpha)-f_2(\beta))<0$, hence corresponding integral is still negative.

For $[-\frac{3}{4}\pi,-\frac{1}{2}\pi]\cup[\frac{1}{2}\pi,\frac{3}{4}\pi]$, we still have the negative sign and moreover

$$|\frac{\sin(f_1(\alpha)-f_1(\beta))(\partial_\alpha f_1(\alpha)-\partial_\beta f_1(\beta))(\sinh(f_2(\alpha)-f_2(\beta)))d\beta}{(\cosh(f_2(\alpha)-f_2(\beta))-\cos(f_1(\alpha)-f_1(\beta)))^2}|\geq|\frac{\sin(\frac{\pi}{6})1\sinh(1)}{(1+\cosh(12))^2}| \tag{3.4}$$

Here we use $\frac{5}{6}\pi>f_1(\alpha)-f_1(\beta)>\frac{1}{4}\pi$ and $|f_2(\alpha)-f_2(\beta)|>1$, $\sup|f_2|\leq 6$.

For $[-\pi,-\pi+\eta],[\pi-\eta,\pi]$, we have

$$\begin{aligned}&|p.v.\int_{-\pi}^{-\pi+\eta}\frac{\sin(f_1(\alpha)-f_1(\beta))(\partial_\alpha f_1(\alpha)-\partial_\beta f_1(\beta))(\sinh(f_2(\alpha)-f_2(\beta)))d\beta}{(\cosh(f_2(\alpha)-f_2(\beta))-\cos(f_1(\alpha)-f_1(\beta)))^2}|\\ \leq&\eta\frac{e^{2*6}}{(1-(\cos\frac{\pi}{4}))^2}\end{aligned}$$

Here we use

$$\frac{3}{2}\pi>f_1(\alpha)-f_1(\beta)>\frac{1}{4}\pi,$$

and

$$|\cosh(f_2(\alpha)-f_2(\beta))-\cos(f_1(\alpha)-f_1(\beta))|\geq|\cos(f_1(\alpha)-f_1(\beta))-1|.$$

Then let η and c be sufficiently small, and we can get the result. □

3.3 Fix the turnover points

Without loss of generality, we set $Z_1(0)=0$, $Z_2(0)=-\frac{\pi}{2}$.

We now change the variable so that the turning points happen at 0 and $\frac{\pi}{2}$ for all the time. Let

$$x(\alpha,t)=\alpha-\sin(\alpha)(Z_2(t)+\frac{\pi}{2}-Z_1(t))+Z_1(t). \tag{3.5}$$

Due to the smallness of $Z_1(t)$ and $Z_2(t) - \frac{\pi}{2}$ when t is sufficiently small, $x(\cdot, t)$ is a diffeomorphism from $\mathcal{R} \to \mathcal{R}$ and $\mathcal{T} \to \mathcal{T}$. Let $\tilde{f}(\alpha, t) = f(x(\alpha, t), t)$. Then we have

$$\frac{d\tilde{f}_\mu(\alpha,t)}{dt} = \frac{d\tilde{f}_\mu}{d\alpha}(\alpha,t)(\frac{\frac{dx(\alpha,t)}{dt}}{\frac{dx(\alpha,t)}{d\alpha}}) + \int_{-\pi}^{\pi} K(\tilde{f}(\alpha,t) - \tilde{f}(\beta,t))(\frac{\partial_\alpha \tilde{f}_\mu(\alpha,t)}{\frac{dx(\alpha,t)}{d\alpha}} - \frac{\partial_\alpha \tilde{f}_\mu(\beta,t)}{\frac{dx(\beta,t)}{d\beta}})(\frac{dx(\beta,t)}{d\beta})d\beta. \tag{3.6}$$

with

$$\partial_\alpha \tilde{f}_1(0,t) = 0,\ \partial_\alpha \tilde{f}_1(-\frac{\pi}{2}, t) = 0.$$

3.4 Localization

Next we separate $\tilde{f}$ and let

$$\tilde{f} = \tilde{f}^+ + \tilde{f}^L \tag{3.7}$$

Here

$$\tilde{f}^+(\alpha,t) = (\tilde{f}(\alpha,t) - \sum_{i=1}^{60} \frac{\tilde{f}^{<i>}(0)}{i!}\alpha^i)\lambda(\alpha)1_{\alpha\geq 0}, \tag{3.8}$$

where $\lambda(\alpha) \geq 0$ and in $C^{100}(-\infty, \infty)$, satisfying

$$\lambda(\alpha) = \begin{cases} 1 & |\alpha| \leq \delta, \\ 0 & |\alpha| \geq 2\delta. \end{cases}$$

We will focus on $\tilde{f}^+$ and study the behavior when $\alpha > 0$. Then $\alpha < 0$ part can be done by the same way letting t go to the other direction. Notice that from the condition in Theorem **??**, it is now in $C^{59}(-\infty, \infty)$. Then we have

$$\begin{aligned} \frac{d\tilde{f_\mu}^+(\alpha,t)}{dt} &= \underbrace{\frac{d\tilde{f_\mu}^+}{d\alpha}(\alpha,t)(\frac{\frac{dx(\alpha,t)}{dt}}{\frac{dx(\alpha,t)}{d\alpha}})}_{Term_1} \\ &\underbrace{-\frac{d\tilde{f_\mu}^L(\alpha,t)}{dt} + \frac{d\tilde{f_\mu}^L}{d\alpha}(\alpha,t)(\frac{\frac{dx(\alpha,t)}{dt}}{\frac{dx(\alpha,t)}{d\alpha}})}_{Term_2} \\ &\underbrace{+\int_{-\pi}^{\pi} K(\tilde{f}^+(\alpha,t) - \tilde{f}^+(\beta,t) + \tilde{f}^L(\alpha,t) - \tilde{f}^L(\beta,t))(\frac{\partial_\alpha \tilde{f_\mu}^+(\alpha,t)}{\frac{dx(\alpha,t)}{d\alpha}} - \frac{\partial_\beta \tilde{f_\mu}^+(\beta,t)}{\frac{dx(\beta,t)}{d\beta}})(\frac{dx(\beta,t)}{d\beta})d\beta}_{Term_3} \end{aligned} \tag{3.9}$$

$$\underbrace{+\int_{-\pi}^{\pi} K(\tilde{f}^+(\alpha,t)-\tilde{f}^+(\beta,t)+\tilde{f}^L(\alpha,t)-\tilde{f}^L(\beta,t))(\frac{\partial_\alpha \tilde{f_\mu}^L(\alpha,t)}{\frac{dx(\alpha,t)}{d\alpha}}-\frac{\partial_\beta \tilde{f_\mu}^L(\beta,t)}{\frac{dx(\beta,t)}{d\beta}})(\frac{dx(\beta,t)}{d\beta})d\beta}_{Term_4}$$

$$= Term_1 + Term_2 + Term_3 +$$

$$\underbrace{+\int_{-\pi}^{\pi}\int_0^1 \nabla K(\zeta\tilde{f}^+(\alpha,t)-\zeta\tilde{f}^+(\beta,t)+\tilde{f}^L(\alpha,t)-\tilde{f}^L(\beta,t))\cdot(\tilde{f}^+(\alpha,t)-\tilde{f}^+(\beta,t))}_{Term_{4,1}}$$

$$\underbrace{(\frac{\partial_\alpha \tilde{f_\mu}^L(\alpha,t)}{\frac{dx(\alpha,t)}{d\alpha}}-\frac{\partial_\beta \tilde{f_\mu}^L(\beta,t)}{\frac{dx(\beta,t)}{d\beta}})(\frac{dx(\beta,t)}{d\beta})d\zeta d\beta}_{Term_{4,1}}$$

$$\underbrace{+\int_{-\pi}^{\pi} K(\tilde{f}^L(\alpha,t)-\tilde{f}^L(\beta,t))(\frac{\partial_\alpha \tilde{f_\mu}^L(\alpha,t)}{\frac{dx(\alpha,t)}{d\alpha}}-\frac{\partial_\beta \tilde{f_\mu}^L(\beta,t)}{\frac{dx(\beta,t)}{d\beta}})(\frac{dx(\beta,t)}{d\beta})d\beta}_{Term_{4,2}}.$$

where we separate the last term into the part depending on $\tilde{f}^+$ and the other part.

3.4.1 Take the derivative

We take 20th derivative and get

$$\frac{d\tilde{f_\mu}^{+(20)}(\alpha,t)}{dt} = \tilde{f_\mu}^{+(21)}(\alpha)(\frac{\frac{dx(\alpha,t)}{dt}}{\frac{dx(\alpha,t)}{d\alpha}})$$

$$+\int_{-\pi}^{\pi} K(\tilde{f}^+(\alpha,t)-\tilde{f}^+(\beta,t)+\tilde{f}^L(\alpha,t)-\tilde{f}^L(\beta,t))(\frac{\tilde{f_\mu}^{+(21)}(\alpha,t)}{\frac{dx(\alpha,t)}{d\alpha}}-\frac{\tilde{f_\mu}^{+(21)}(\beta,t)}{\frac{dx(\beta,t)}{d\beta}})(\frac{dx(\beta,t)}{d\beta})d\beta$$

$$+\sum_i O^i + T_{fixed}(\alpha)$$

$$= \tilde{f_\mu}^{+(21)}(\alpha)(\frac{\frac{dx(\alpha,t)}{dt}}{\frac{dx(\alpha,t)}{d\alpha}}+\frac{1}{\frac{dx(\alpha,t)}{d\alpha}}p.v.\int_{-\pi}^{\pi} K(\tilde{f}^+(\alpha,t)-\tilde{f}^+(\beta,t)+\tilde{f}^L(\alpha,t)-\tilde{f}^L(\beta,t))(\frac{dx(\beta,t)}{d\beta})d\beta)$$

$$-p.v.\int_{-\pi}^{\pi} K(\tilde{f}^+(\alpha,t)-\tilde{f}^+(\beta,t)+\tilde{f}^L(\alpha,t)-\tilde{f}^L(\beta,t))(\frac{d\tilde{f_\mu}^{+(20)}(\beta,t)}{d\beta})d\beta$$

$$+\sum_i O^i + T_{fixed}(\alpha)$$

Here $T_{fixed}(\alpha)$ comes from $Term_2 + Term_{4,2}$. We have

$$T_{fixed}(\alpha) = \partial_\alpha^{20}(-\frac{d\tilde{f_\mu}^L(\alpha,t)}{dt}+\frac{d\tilde{f_\mu}^L}{d\alpha}(\alpha)(\frac{\frac{dx(\alpha,t)}{dt}}{\frac{dx(\alpha,t)}{d\alpha}}) \tag{3.10}$$

$$+\int_{-\pi}^{\pi} K(\tilde{f}^L(\alpha,t)-\tilde{f}^L(\beta,t))(\frac{\partial_\alpha \tilde{f}_\mu^{\ L}(\alpha,t)}{\frac{dx(\alpha,t)}{d\alpha}}-\frac{\partial_\beta \tilde{f}_\mu^{\ L}(\beta,t)}{\frac{dx(\beta,t)}{d\beta}})(\frac{dx(\beta,t)}{d\beta})d\beta).$$

and it depends on $\tilde{f}^L$ but not $\tilde{f}^+$. The O^i terms are the terms with at most 20th derivative hit on $\tilde{f}^+$.

Before we show the explicit form of O^i, we introduce some notations. Let

$$V^k(\alpha)=(\tilde{f}_1^+(\alpha,t),\tilde{f}_2^+(\alpha,t),\partial_\alpha\tilde{f}_1^+(\alpha,t),\partial_\alpha\tilde{f}_2^+(\alpha,t),...,\partial_\alpha^k\tilde{f}_1^+(\alpha,t),\partial_\alpha^k\tilde{f}_2^+(\alpha,t)). \tag{3.11}$$

$$V^k_{\tilde{f}^L}(\alpha)=(\tilde{f}_1^L(\alpha,t),\tilde{f}_2^L(\alpha,t),\partial_\alpha\tilde{f}_1^L(\alpha,t),\partial_\alpha\tilde{f}_2^L(\alpha,t),...,\partial_\alpha^k\tilde{f}_1^L(\alpha,t),\partial_\alpha^k\tilde{f}_2^L(\alpha,t)). \tag{3.12}$$

When we write $X_i(\alpha,t)$, we mean

$$X_i(\alpha,t)=(\partial_\alpha^{l_i}\tilde{f}^L(\alpha,t))^{\phi_i}(\partial_\alpha^{l_i'}(\frac{1}{\frac{dx}{d\alpha}(\alpha,t)}))^{\phi_i'}(\partial_\alpha^{l_i''}\frac{d}{d\alpha}x(\alpha,t))^{\phi_{i''}}, \tag{3.13}$$

with $l_i,l_i',l_1''\le 21$, $\phi_i,\phi_i',\phi_{i''}\in\{0,1\}$. $V_X(\alpha,t)$ is the vector with all component of $X_i(\alpha,t)$.

When we write $\tilde{X}_i(\alpha,t)$, we mean

$$\tilde{X}_i(\alpha,t)=\partial_\alpha^{b_i}(\frac{\frac{dx(\alpha,t)}{dt}}{\frac{dx(\alpha,t)}{d\alpha}}), \tag{3.14}$$

with $b_i\le 21$.

We have

$$X_i(\alpha,t),\tilde{X}_i(\alpha,t),\tilde{f}^L(\alpha,t)\in C^1([0,t_0],C^{38}(-\infty,\infty)) \tag{3.15}$$

and they can be analytically extended to the set $\{\alpha+iy|0<\alpha<\frac{\delta}{2}\}$.

We claim that we can write other terms as following five types, by separating the highest order term in the derivative. When we write $\sum_i O^i$, we always mean $\sum_{1\le i'\le 5}\sum_i O^{1,i}$.

$$O^{1,i}=\int_{-\pi}^{\pi} K_o^i(V^{10}(\alpha)-V^{10}(\beta),V^{21}_{\tilde{f}^L}(\alpha)-V^{21}_{\tilde{f}^L}(\beta),V_X(\alpha)-V_X(\beta))X_{i'}(\beta)(\partial_\alpha^{b_i}\tilde{f}_v^+(\alpha,t)-\partial_\alpha^{b_i}\tilde{f}_v^+(\beta,t))d\beta.$$

$$O^{2,i} = \int_{-\pi}^{\pi} K_o^i(V^{10}(\alpha) - V^{10}(\beta), V_{\tilde{f}^L}^{21}(\alpha) - V_{\tilde{f}^L}^{21}(\beta), V_X(\alpha) - V_X(\beta)) X_{i'}(\beta) \partial_\alpha^{s_i} \tilde{f}_\mu^+(\alpha) (\partial_\alpha^{b_i} \tilde{f}_v^+(\alpha, t) - \partial_\alpha^{b_i} \tilde{f}_v^+(\beta, t)) d\beta.$$

$$O^{3,i} = \int_{-\pi}^{\pi} \tilde{K}_o^i(V^{10}(\alpha) - V^{10}(\beta), V_{\tilde{f}^L}^{21}(\alpha) - V_{\tilde{f}^L}^{21}(\beta), V_X(\alpha) - V_X(\beta)) X_{i'}(\beta) \partial_\alpha^{b_i} \tilde{f}_\mu^+(\alpha, t) d\beta.$$

$$O^{4,i} = \int_{-\pi}^{\pi} \int_0^1 K_o^i(\zeta V^{10}(\alpha) - \zeta V^{10}(\beta), V_{\tilde{f}^L}^{21}(\alpha) - V_{\tilde{f}^L}^{21}(\beta), V_X(\alpha) - V_X(\beta)) X_{i'}(\beta) \zeta^{q_i} (\partial_\alpha^{b_i} \tilde{f}_v^+(\alpha, t) - \partial_\alpha^{b_i} \tilde{f}_v^+(\beta, t)) d\zeta d\beta.$$

$$O^{5,i} = \partial_\alpha^{b_i} \tilde{f}_u^+(\alpha, t) \tilde{X}_i(\alpha)$$

with $0 \le b_i \le 20$, $s_i \le 10$. Here we have K_o^i, $\tilde{K}_o^i$ are the kernels of the form

$$\begin{aligned}
&\sin(\tilde{f}_1^+(\alpha,t) - \tilde{f}_1^+(\beta,t) + \tilde{f}_1^L(\alpha,t) - \tilde{f}_1^L(\beta,t))^{m_1} \cos(\tilde{f}_1^+(\alpha,t) - \tilde{f}_1^+(\beta,t) + \tilde{f}_1^L(\alpha,t) - \tilde{f}_1^L(\beta,t))^{m_2} \\
&\times \sinh(\tilde{f}_2^+(\alpha,t) - \tilde{f}_2^+(\beta,t) + \tilde{f}_2^L(\alpha,t) - \tilde{f}_2^L(\beta,t))^{m_3} \cosh(\tilde{f}_2^+(\alpha,t) - \tilde{f}_2^+(\beta,t) + \tilde{f}_2^L(\alpha,t) - \tilde{f}_2^L(\beta,t))^{m_4} \\
&\times \Pi_{j=1}^{m_5} [\partial_\alpha^{\lambda_j} \tilde{f}_1^+(\alpha,t) - \partial_{\alpha,t}^{\lambda_j} \tilde{f}_1^+(\beta,t)] \times \Pi_{j=1}^{m_6} [\partial_\alpha^{\lambda_{j,2}} \tilde{f}_2^+(\alpha,t) - \partial_\alpha^{\lambda_{j,2}} \tilde{f}_2^+(\beta,t)] \\
&\times \Pi_{j=1}^{m_7} [\partial_\alpha^{\lambda_{j,3}} \tilde{f}_1^L(\alpha,t) - \partial_\alpha^{\lambda_{j,3}} \tilde{f}_1^L(\beta,t)] \times \Pi_{j=1}^{m_8} [\partial_\alpha^{\lambda_{j,4}} \tilde{f}_2^L(\alpha,t) - \partial_\alpha^{\lambda_{j,4}} \tilde{f}_2^L(\beta,t)] \\
&\times \Pi_{j=1}^{m_9} [X_{i,j}(\alpha) - X_{i,j}(\beta)] \\
&\times (\cosh(\tilde{f}_2^+(\alpha,t) - \tilde{f}_2^+(\beta,t) + \tilde{f}_2^L(\alpha,t) - \tilde{f}_2^L(\beta,t)) - \cos(\tilde{f}_1^+(\alpha,t) - \tilde{f}_1^+(\beta,t) + \tilde{f}_1^+(\alpha,t) - \tilde{f}_1^+(\beta,t)))^{-m_0}.
\end{aligned}$$

with $1 \le \lambda_{j,1}, \lambda_{j,2} \le 10$, $1 \le \lambda_{j,3}, \lambda_{j,4} \le 21$, and for all K_o^i,

$$m_1 + m_3 + m_5 + m_6 + m_7 + m_8 + m_9 - 2m_0 \ge -1, \tag{3.16}$$

for all $\tilde{K}_o^i$,

$$m_1 + m_3 + m_5 + m_6 + m_7 + m_8 + m_9 - 2m_0 \ge 0. \tag{3.17}$$

$O_{1,i}$, $O_{2,i}$ and $O_{3,i}$ come from $Term_3$. There are three terms because $Term_3$ can be written as

$$\int_{-\pi}^{\pi} K(\tilde{f}^+(\alpha,t)-\tilde{f}^+(\beta,t)+\tilde{f}^L(\alpha,t)-\tilde{f}^L(\beta,t))(\frac{\partial_\alpha \tilde{f_\mu}^+(\alpha,t)}{\frac{dx(\alpha,t)}{d\alpha}}-\frac{\partial_\alpha \tilde{f_\mu}^+(\beta,t)}{\frac{dx(\beta,t)}{d\beta}})(\frac{dx(\beta,t)}{d\beta})d\beta$$
$$=\int_{-\pi}^{\pi} K(\tilde{f}^+(\alpha,t)-\tilde{f}^+(\beta,t)+\tilde{f}^L(\alpha,t)-\tilde{f}^L(\beta,t))(\partial_\alpha \tilde{f_\mu}^+(\alpha,t)-\partial_\alpha \tilde{f_\mu}^+(\beta,t))d\beta$$
$$+\int_{-\pi}^{\pi} K(\tilde{f}^+(\alpha,t)-\tilde{f}^+(\beta,t)+\tilde{f}^L(\alpha,t)-\tilde{f}^L(\beta,t))(\frac{\frac{dx(\beta,t)}{d\beta}}{\frac{dx(\alpha,t)}{d\alpha}}-1)\partial_\alpha \tilde{f_\mu}^+(\alpha,t)d\beta.$$

$O^{4,i}$ comes from $Term_{4,1}$.

$O^{5,i}$ comes from $Term_1$.

Notice that $O^{2,i}$ can be written as a product of $O^{1,i}$ type times $\partial_\alpha^{s_i}\tilde{f}_\mu^+$ with $s_i \le 10$. And for each ζ in the inner integral, $O^{4,i}$ is similar as $O_{1,i}$ except that we change $V^{10}(\alpha)$ to $\zeta V^{10}(\alpha)$ and times ζ^{q_i}. These two terms will behave similarly as $O_{1,i}$.

3.4.2 Change the contour

Let $c(\alpha)$ satisfy

$$\begin{cases} c(\alpha)=\delta_c\alpha^2, \text{ when } \alpha \le \frac{\delta}{32} \\ \text{supp}\, c(\alpha) \subset [0,\frac{\delta}{8}] \\ c(\alpha)\in C^{100}([0,\infty])\cap C^{1,1}([-\infty,\infty]),\ \|c(\alpha)\|_{C^{100}([0,\infty])\cap C^{1,1}(-\infty,\infty)} \le \delta. \end{cases}$$

Then we assume that $\tilde{f}(\cdot,t)$ can be extended to an analytic function in the region $\tilde{D}_A=\{\alpha+iy|0<\alpha<\frac{\delta}{4},|y|<c(\alpha)t\}$. We remark that $\tilde{f}^L(\alpha,t)$, $x(\alpha,t)$ can be analytically extended to the region $\tilde{D}_A$ without any assumption.

With this assumption we change the contour. From now on ,we will omit the dependency of t in $\tilde{f}$ for the simplicity. Let $\alpha_{\gamma,0}^t=\alpha+ic(\alpha)\gamma t$. For $-1\le\gamma\le 1$, we have

$$\frac{d\tilde{f}_\mu^{+(20)}(\alpha_{\gamma,0}^t)}{dt}=\frac{d\tilde{f}_\mu^{+(20)}(\alpha_{\gamma,0}^t)}{d\alpha}\frac{1}{1+ic'(\alpha)\gamma t}(ic(\alpha)\gamma+\frac{\partial_t x}{\partial_\alpha x}(\alpha_{\gamma,0}^t)+$$
$$\frac{1}{\frac{dx}{d\alpha}(\alpha_{\gamma,0}^t)}p.v.\int_{-\pi}^{\pi}K(\tilde{f}^+(\alpha_{\gamma,0}^t)-\tilde{f}^+(\beta_{\gamma,0}^t)+\tilde{f}^L(\alpha_{\gamma,0}^t)-\tilde{f}^L(\beta_{\gamma,0}^t))x'(\beta_{\gamma,0}^t)(1+ic'(\beta)\gamma t)d\beta)$$
$$-p.v.\int_{-\pi}^{\pi}K(\tilde{f}^+(\alpha_{\gamma,0}^t)-\tilde{f}^+(\beta_{\gamma,0}^t)+\tilde{f}^L(\alpha_{\gamma,0}^t)-\tilde{f}^L(\beta_{\gamma,0}^t))\frac{d\tilde{f}_\mu^{+(20)}(\beta_{\gamma,0}^t)}{d\beta}d\beta$$
$$+\sum_i O^i+T_{fixed}(\alpha_{\gamma,0}^t).$$

The "p.v." shows up because we write $\frac{d\tilde{f}^{+(20)}(\beta^t_{\gamma,0})}{d\beta}$ and $\frac{d\tilde{f}^{+(20)}(\alpha^t_{\gamma,0})}{d\alpha}$ into two terms. There is in fact no singularity, and then we could change the contour safely. Now O^i becomes

$$O^{1,i} = \int_{-\pi}^{\pi} K_o^i(V^{10}(\alpha^t_{\gamma,0}) - V^{10}(\beta^t_{\gamma,0}), V^{21}_{\tilde{f}^L}(\alpha^t_{\gamma,0}) - V^{21}_{\tilde{f}^L}(\beta^t_{\gamma,0}), V_X(\alpha^t_{\gamma,0}) - V_X(\beta^t_{\gamma,0}))X_{i'}(\beta^t_{\gamma,0})$$
$$((\partial_\alpha^{b_i}\tilde{f}_v^+)(\alpha^t_{\gamma,0}) - (\partial_\alpha^{b_i}\tilde{f}_v^+)(\beta^t_{\gamma,0}))(1 + ic'(\beta)\gamma t)d\beta.$$

$$O^{2,i} = \int_{-\pi}^{\pi} K_o^i(V^{10}(\alpha^t_{\gamma,0}) - V^{10}(\beta^t_{\gamma,0}), V^{21}_{\tilde{f}^L}(\alpha^t_{\gamma,0}) - V^{21}_{\tilde{f}^L}(\beta^t_{\gamma,0}), V_X(\alpha^t_{\gamma,0}) - V_X(\beta^t_{\gamma,0}))X_{i'}(\beta^t_{\gamma,0})$$
$$(\partial_\alpha^{s_i}\tilde{f}_\mu^+)(\alpha^t_{\gamma,0})((\partial_\alpha^{b_i}\tilde{f}_v^+)(\alpha^t_{\gamma,0}) - (\partial_\alpha^{b_i}\tilde{f}_v^+)(\beta^t_{\gamma,0}))(1 + ic'(\beta)\gamma t)d\beta.$$

$$O^{3,i} = \int_{-\pi}^{\pi} \tilde{K}_o^i(V^{10}(\alpha^t_{\gamma,0}) - V^{10}(\beta^t_{\gamma,0}), V^{21}_{\tilde{f}^L}(\alpha^t_{\gamma,0}) - V^{21}_{\tilde{f}^L}(\beta^t_{\gamma,0}), V_X(\alpha^t_{\gamma,0}) - V_X(\beta^t_{\gamma,0}))X_{i'}(\beta^t_{\gamma,0})$$
$$(\partial_\alpha^{b_i}\tilde{f}_\mu^+)(\alpha^t_{\gamma,0})(1 + ic'(\beta)\gamma t)d\beta.$$

$$O^{4,i} = \int_{-\pi}^{\pi}\int_0^1 K_o^i(\zeta V^{10}(\alpha^t_{\gamma,0}) - \zeta V^{10}(\beta^t_{\gamma,0}), V^{21}_{\tilde{f}^L}(\alpha^t_{\gamma,0}) - V^{21}_{\tilde{f}^L}(\beta^t_{\gamma,0}), V_X(\alpha^t_{\gamma,0}) - V_X(\beta^t_{\gamma,0}))$$
$$X_{i'}(\beta^t_{\gamma,0})\zeta^{q_i}((\partial_\alpha^{b_i}\tilde{f}_v^+)(\alpha^t_{\gamma,0}) - (\partial_\alpha^{b_i}\tilde{f}_v^+)(\beta^t_{\gamma,0}))d\zeta(1 + ic'(\beta)\gamma t)d\beta.$$

$$O^{5,i} = \partial_\alpha^{b_i}\tilde{f}_u^+(\alpha^t_{\gamma,0})\tilde{X}_i(\alpha^t_{\gamma,0})$$

3.5 The modified equation

3.5.1 Change the variable again

Here the coefficient before $\frac{d}{d\alpha}\tilde{f}^{+(20)}(\alpha^t_{\gamma,0})$ may not vanish at zero. We again change variable in order to make it vanish at zero, which is crucial for us in the energy estimate after perturbation. We have

$$\frac{1}{1 + ic'(\alpha)\gamma t}(ic(\alpha)\gamma + \frac{\partial_t x}{\partial_\alpha x}(\alpha^t_{\gamma,0}) + \frac{1}{\frac{dx}{d\alpha}(\alpha^t_{\gamma,0})}$$

$$
\begin{aligned}
&\cdot p.v.\int_{-\pi}^{\pi} K(\tilde{f}^{+}(\alpha_{\gamma,0}^{t})-\tilde{f}^{+}(\beta_{\gamma,0}^{t})+\tilde{f}^{L}(\alpha_{\gamma,0}^{t})-\tilde{f}^{L}(\beta_{\gamma,0}^{t}))x'(\beta_{\gamma,0}^{t})(1+ic'(\beta)\gamma t)d\beta)\\
&=\frac{1}{1+ic'(\alpha)\gamma t}(ic(\alpha)\gamma+\frac{\partial_t x}{\partial_\alpha x}(\alpha_{\gamma,0}^{t})+\frac{1}{\frac{dx}{d\alpha}(\alpha_{\gamma,0}^{t})}\\
&\cdot p.v.\int_{-\pi}^{\pi} K(\tilde{f}^{+}(\alpha_{\gamma,0}^{t})-\tilde{f}^{+}(\beta_{\gamma,0}^{t})+\tilde{f}^{L}(\alpha_{\gamma,0}^{t})-\tilde{f}^{L}(\beta_{\gamma,0}^{t}))x'(\beta_{\gamma,0}^{t})(1+ic'(\beta)\gamma t)d\beta\\
&-\frac{\partial_t x}{\partial_\alpha x}(0)-\frac{1}{\frac{dx}{d\alpha}(0)}p.v.\int_{-\pi}^{\pi} K(\tilde{f}^{+}(0)-\tilde{f}_0^{+}(\beta_{\gamma,0}^{t})+\tilde{f}^{L}(0)-\tilde{f}^{L}(\beta_{\gamma}^{t}))x'(\beta_{\gamma}^{t})(1+ic'(\beta)\gamma t)d\beta\\
&+\frac{\partial_t x}{\partial_\alpha x}(0)+\frac{1}{\frac{dx}{d\alpha}(0)}p.v.\int_{-\pi}^{\pi} K(\tilde{f}^{+}(0)-\tilde{f}_0^{+}(\beta)+\tilde{f}^{L}(0)-\tilde{f}^{L}(\beta))x'(\beta)d\beta).
\end{aligned}
$$

Here we use the analyticity assumption of $\tilde{f}^{+}$ and the property of $c(\alpha)$. Since $c(\alpha)=\delta_c\alpha^2$, we could change the contour in the last step. We have the following lemma

Lemma 3.5.1. *For sufficiently small t, the inequality below is true:*

$$
(\frac{\partial_t x}{\partial_\alpha x}(0,t)+\frac{1}{\frac{dx}{d\alpha}(0,t)}\int_{-\pi}^{\pi} K(\tilde{f}^{+}(0,t)-\tilde{f}^{+}(\beta,t)+\tilde{f}^{L}(0,t)-\tilde{f}^{L}(\beta,t))\frac{dx(\beta,t)}{d\beta}d\beta)\partial_\alpha^2\tilde{f}_1^{L}(0,t)<0. \tag{3.18}
$$

Proof. From (1.6), we have

$$
(\frac{dZ_1}{dt}(t)+p.v.\int_{-\pi}^{\pi} K(f(Z_1(t),t)-f(\beta,t))d\beta)(\partial_\alpha^2 f_1)(Z_1(t),t)<0.
$$

(3.5) gives:

$$
\frac{dx(0,t)}{dt}=\frac{dZ_1(t)}{dt},
$$

and

$$
\frac{dx(\alpha,t)}{d\alpha}=1-\cos(\alpha)(Z_2(t)-\frac{\pi}{2}-Z_1(t))>0,
$$

when t is sufficiently small. Moreover, we can change the variable and get

$$
\begin{aligned}
&p.v.\int_{-\pi}^{\pi} K(f(Z_1(t),t)-f(\beta,t))d\beta\\
&=p.v.\int_{-\pi}^{\pi} K(\tilde{f}^{+}(0,t)-\tilde{f}^{+}(\beta,t)+\tilde{f}^{L}(0,t)-\tilde{f}^{L}(\beta,t))\frac{dx(\beta,t)}{d\beta}d\beta.
\end{aligned}
$$

We also have

$$
\partial_\alpha^2\tilde{f}_1(\alpha,t)=(\partial_x^2 f_1)(x(\alpha,t),t)(\frac{dx}{d\alpha})^2(\alpha,t)+(\partial_x f_1)(x(\alpha,t),t)(\frac{d^2x}{d\alpha^2}(\alpha,t)).
$$

Then at $\alpha = 0$, we get

$$\begin{aligned}(\partial_\alpha^2 \tilde{f}_1)(0,t) &= (\partial_x^2 f_1)(Z_1(t),t)(\frac{dx}{d\alpha})^2(\alpha,0) + (\partial_x f_1)(Z_1(t),t)\frac{d^2x}{d\alpha^2}(\alpha,t)\\ &= (\partial_x^2 f_1)(Z_1(t),t)(\frac{dx}{d\alpha})^2(\alpha,0),\end{aligned}$$

where we use the fact that $Z_1(t)$ is a turnover point. By definition of $\tilde{f}^+, \tilde{f}^L$ in (3.7), we get $\partial_\alpha^2 \tilde{f}_1^L(0,t) = \partial_\alpha^2 \tilde{f}_1(0,t)$. Therefore, we have

$$\begin{aligned}&\frac{\partial_t x}{\partial_\alpha x}(0,t) + \frac{1}{\frac{dx}{d\alpha}(0,t)}\int_{-\pi}^{\pi} K(\tilde{f}^+(0,t) - \tilde{f}^+(\beta) + \tilde{f}^L(0,t) - \tilde{f}^L(\beta,t))\frac{dx(\beta,t)}{d\beta}d\beta)\partial_\alpha^2 \tilde{f}_1^L(0,t)\\ &= \frac{1}{\frac{dx}{d\alpha}(\alpha,t)}(\frac{dZ_1}{dt}(t) + p.v.\int_{-\pi}^{\pi} K(f(Z_1(t),t) - f(\beta,t))d\beta)(\partial_x^2 f_1)(Z_1(t),t)\frac{1}{(\frac{dx}{d\alpha})^2(\alpha,t)}\\ &< 0,\end{aligned}$$

when t is sufficiently small. □

Without loss of generality, we assume

$$\partial_\alpha^2 \tilde{f}_1^L(0,t) > 0. \tag{3.19}$$

We claim if $\partial_\alpha^2 \tilde{f}_1^L(0,t) < 0$, we could let the time go backward for $\alpha > 0$ and let the time go forward when $\alpha < 0$ and the same proof follows. Then

$$(\frac{\partial_t x}{\partial_\alpha x}(0) + \frac{1}{\frac{dx}{d\alpha}(0)}\int_{-\pi}^{\pi} K(\tilde{f}^+(0) - \tilde{f}_0^+(\beta) + \tilde{f}^L(0) - \tilde{f}^L(\beta))\frac{dx(\beta)}{d\beta}d\beta) < 0.$$

Hence there exists $\tau(t) \geq 0$ with $\tau'(t) > 0, \tau(0) = 0$ such that

$$-\tau'(t) = (\frac{\partial_t x}{\partial_\alpha x}(0) + \frac{1}{\frac{dx}{d\alpha}(0)}\int_{-\pi}^{\pi} K(\tilde{f}^+(0) - \tilde{f}_0^+(\beta) + \tilde{f}^L(0) - \tilde{f}^L(\beta))\frac{dx(\beta)}{d\beta}d\beta).$$

Therefore

$$\begin{aligned}&\frac{d\tilde{f}_\mu^{+(20)}(\alpha_{\gamma,0}^t)}{dt}\\ =&\frac{d\tilde{f}_\mu^{+(20)}(\alpha_{\gamma,0}^t)}{d\alpha}[\frac{1}{1+ic'(\alpha)\gamma t}(ic(\alpha)\gamma + \frac{\partial_t x}{\partial_\alpha x}(\alpha_{\gamma,0}^t) + \frac{1}{(\frac{dx}{d\alpha}(\alpha_{\gamma,0}^t))}\\ &\cdot p.v.\int_{-\pi}^{\pi} K(\tilde{f}^+(\alpha_{\gamma,0}^t) - \tilde{f}^+(\beta_{\gamma,0}^t) + \tilde{f}^L(\alpha_{\gamma,0}^t) - \tilde{f}^L(\beta_{\gamma,0}^t))x'(\beta_{\gamma,0}^t)(1+c'(\beta)\gamma t)d\beta\end{aligned}$$

$$
\begin{aligned}
&- \frac{\partial_t x}{\partial_\alpha x}(0) - \frac{1}{\frac{dx}{d\alpha}(0)} p.v. \int_{-\pi}^{\pi} K(\tilde{f}^+(0) - \tilde{f}_0^+(\beta_{\gamma,0}^t) + \tilde{f}^L(0) - \tilde{f}^L(\beta_{\gamma,0}^t)) x'(\beta_{\gamma,0}^t)(1 + c'(\beta)\gamma t) d\beta)] \\
&+ \frac{d\tilde{f}_\mu^{+(20)}(\alpha_{\gamma,0}^t)}{d\alpha} \frac{1}{1 + ic'(\alpha)\gamma t} \cdot (-\tau'(t)) \\
&- p.v. \int_{-\pi}^{\pi} K(\tilde{f}^+(\alpha_{\gamma,0}^t) - \tilde{f}^+(\beta_{\gamma,0}^t) + \tilde{f}^L(\alpha_{\gamma,0}^t) - \tilde{f}^L(\beta_{\gamma,0}^t)) \frac{d\tilde{f}_\mu^{+(20)}(\beta_{\gamma,0}^t)}{d\beta} d\beta \\
&+ \sum_i O_i + T_{fixed}(\alpha_{\gamma,0}^t).
\end{aligned}
$$

Then we change variable again. Let $\alpha_\gamma^t = \alpha + \tau(t) + ic(\alpha + \tau(t))\gamma t$. Since

$$
\frac{d\tilde{f}_\mu^{+(20)}(\alpha_{\gamma,0}^t)}{d\alpha} \frac{1}{1 + ic'(\alpha)\gamma t} \cdot (-\tau'(t)) = \frac{d\tilde{f}_\mu^{+(20)}(\alpha_{\gamma,0}^t)}{d\alpha} \cdot (-\tau'(t)) + \frac{d\tilde{f}_\mu^{+(20)}(\alpha_{\gamma,0}^t)}{d\alpha} \frac{ic'(\alpha)\gamma t \tau'(t)}{1 + ic'(\alpha)\gamma t}
$$

We get

$$
\begin{aligned}
&\frac{d\tilde{f}_\mu^{+(20)}(\alpha_\gamma^t)}{dt} \\
=&\frac{d\tilde{f}_\mu^{+(20)}(\alpha_\gamma^t)}{d\alpha} B_{\tilde{f}}(\alpha, \gamma, t) \\
&- p.v. \int_{-\pi}^{\pi} K(\tilde{f}^+(\alpha_\gamma^t) - \tilde{f}^+(\beta_\gamma^t) + \tilde{f}^L(\alpha_\gamma^t) - \tilde{f}^L(\beta_\gamma^t)) \frac{d\tilde{f}_\mu^{+(20)}(\beta_\gamma^t)}{d\beta} d\beta \\
&+ \sum_i O^i + T_{fixed}(\alpha_\gamma^t)
\end{aligned}
$$

where

$$
\begin{aligned}
B_{\tilde{f}}(\alpha, \gamma, t) = [&\frac{1}{1 + ic'(\alpha + \tau(t))\gamma t}(ic(\alpha + \tau(t))\gamma + ic'(\alpha + \tau(t))\gamma t \tau'(t) + \frac{\partial_t x}{\partial_\alpha x}(\alpha_\gamma^t) \\
&+ \frac{1}{(\frac{dx}{d\alpha}(\alpha_\gamma^t))} \cdot p.v. \int_{-\pi}^{\pi} K(\tilde{f}^+(\alpha_\gamma^t) - \tilde{f}^+(\beta_\gamma^t) + \tilde{f}^L(\alpha_\gamma^t) - \tilde{f}^L(\beta_\gamma^t)) \\
&\frac{dx(\beta_\gamma^t)}{d\beta} d\beta - \frac{\partial_t x}{\partial_\alpha x}(0) - \frac{1}{\frac{dx}{d\alpha}(0)} p.v. \int_{-\pi}^{\pi} K(\tilde{f}^+(0) - \tilde{f}_0^+(\beta_\gamma^t) + \tilde{f}^L(0) - \tilde{f}^L(\beta_\gamma^t)) \frac{dx(\beta_\gamma^t)}{d\beta} d\beta)]
\end{aligned}
$$

And other terms become

$$
\begin{aligned}
O^{1,i} = &\int_{-\pi}^{\pi} K_o^i(V^{10}(\alpha_\gamma^t) - V^{10}(\beta_\gamma^t), V_{\tilde{f}^L}^{21}(\alpha_\gamma^t) - V_{\tilde{f}^L}^{21}(\beta_\gamma^t), V_X(\alpha_\gamma^t) - V_X(\beta_\gamma^t)) X_{i'}(\beta_\gamma^t) \\
&((\partial_\alpha^{b_i} \tilde{f}_v^+)(\alpha_\gamma^t) - (\partial_\alpha^{b_i} \tilde{f}_v^+)(\beta_\gamma^t))(1 + ic'(\beta + \tau(t))\gamma t) d\beta.
\end{aligned}
$$

$$O^{2,i} = \int_{-\pi}^{\pi} K_o^i(V^{10}(\alpha_\gamma^t) - V^{10}(\beta_\gamma^t), V_{\tilde{f}^L}^{21}(\alpha_\gamma^t) - V_{\tilde{f}^L}^{21}(\beta_\gamma^t), V_X(\alpha_\gamma^t) - V_X(\beta_\gamma^t))X_{i'}(\beta_\gamma^t)$$
$$(\partial_\alpha^{s_i}\tilde{f}_\mu^+)(\alpha_\gamma^t)((\partial_\alpha^{b_i}\tilde{f}_v^+)(\alpha_\gamma^t) - (\partial_\alpha^{b_i}\tilde{f}_v^+)(\beta_\gamma^t))(1 + ic'(\beta + \tau(t))\gamma t)d\beta.$$

$$O^{3,i} = \int_{-\pi}^{\pi} \tilde{K}_o^i(V^{10}(\alpha_{\gamma,}^t) - V^{10}(\beta_\gamma^t), V_{\tilde{f}^L}^{21}(\alpha_\gamma^t) - V_{\tilde{f}^L}^{21}(\beta_\gamma^t), V_X(\alpha_\gamma^t) - V_X(\beta_\gamma^t))X_{i'}(\beta_\gamma^t)$$
$$(\partial_\alpha^{b_i}\tilde{f}_\mu^+)(\alpha_\gamma^t)(1 + ic'(\beta + \tau(t))\gamma t)d\beta.$$

$$O^{4,i} = \int_{-\pi}^{\pi}\int_0^1 K_o^i(\zeta V^{10}(\alpha_\gamma^t) - \zeta V^{10}(\beta_\gamma^t), V_{\tilde{f}^L}^{21}(\alpha_\gamma^t) - V_{\tilde{f}^L}^{21}(\beta_{\gamma,0}^t), V_X(\alpha_\gamma^t) - V_X(\beta_\gamma^t))$$
$$X_{i'}(\beta_\gamma^t)\zeta^{q_i}((\partial_\alpha^{b_i}\tilde{f}_v^+)(\alpha_\gamma^t) - (\partial_\alpha^{b_i}\tilde{f}_v^+)(\beta_\gamma^t))d\zeta(1 + ic'(\beta + \tau(t))\gamma t)d\beta.$$

$$O^{5,i} = \partial_\alpha^{b_i}\tilde{f}_u^+(\alpha_\gamma^t)\tilde{X}_i(\alpha_\gamma^t)$$

3.5.2 Times a cut off function

Now we want to make sure the solution of our new equation is supported in a small region to focus on the local behavior. Let $\tilde{\lambda}$ defined on the union of a real line and a disk $D_\delta = \{|\alpha + iy| \le \delta\}$. On the real line, let $\tilde{\lambda} \in C^{100}(-\infty, \infty)$ satisfying

$$\text{supp}\,\tilde{\lambda} \subset [-20\delta, 20\delta],\ \tilde{\lambda} = 1 \text{ when } x \in [-10\delta, 10\delta]. \tag{3.20}$$

On the disk $\tilde{\lambda} = 1$. Therefore we have $\tilde{\lambda}\lambda = \lambda$, and then

$$\tilde{\lambda}\partial_\alpha^i\tilde{f}^+(\alpha, t) = \partial_\alpha^i\tilde{f}^+(\alpha, t).$$

We get

$$\frac{d\tilde{f}_\mu^{+(20)}(\alpha_\gamma^t)}{dt}$$

$$
\begin{aligned}
=&\frac{d\tilde{f}_{\mu}^{+(20)}(\alpha_{\gamma}^{t})\tilde{\lambda}(\alpha_{\gamma}^{t})}{dt}\\
=&\tilde{\lambda}^{'}(\alpha_{\gamma}^{t})(1+ic'(\alpha+\tau(t))\gamma t)\tilde{f}_{\mu}^{+(21)}(\alpha_{\gamma}^{t})+\tilde{\lambda}(\alpha_{\gamma}^{t})\frac{d}{dt}(\tilde{f}_{\mu}^{+(20)}(\alpha_{\gamma}^{t}))\\
=&\tilde{\lambda}(\alpha_{\gamma}^{t})\underbrace{\frac{\tilde{f}_{\mu}^{+(20)}(\alpha_{\gamma}^{t})}{d\alpha}B_{\tilde{f}}(\alpha,\gamma,t)}_{M_1}\\
&-\tilde{\lambda}(\alpha_{\gamma}^{t})\underbrace{p.v.\int_{-\pi}^{\pi}K(\tilde{f}^{+}(\alpha_{\gamma}^{t})-\tilde{f}^{+}(\beta_{\gamma}^{t}))+\tilde{f}^{L}(\alpha_{\gamma}^{t})-\tilde{f}^{L}(\beta_{\gamma}^{t}))\frac{d\tilde{f}_{\mu}^{+(20)}(\beta_{\gamma}^{t}))}{d\beta}d\beta}_{M_2}\\
&+\tilde{\lambda}(\alpha_{\gamma}^{t})\cdot\sum_{i}O^{i}+\tilde{\lambda}(\alpha_{\gamma}^{t})T_{fixed}(\alpha_{\gamma}^{t})\\
&=\tilde{\lambda}(\alpha_{\gamma}^{t})M_1(\tilde{f}^{+(20)})-\tilde{\lambda}(\alpha_{\gamma}^{t})M_2(\tilde{f}^{+(20)})+\sum_{i}\tilde{\lambda}(\alpha_{\gamma}^{t})O^{i}(\tilde{f}^{+(20)})+\tilde{\lambda}(\alpha_{\gamma}^{t})T_{fixed}(\alpha_{\gamma}^{t})
\end{aligned}
$$

Here we use the fact that the support of $\tilde{\lambda}^{'}$ and $\tilde{f}^{+}$ is empty.

3.5.3 Modify the terms to preserve the behavior near $\alpha=-\tau(t)$

In order to preserve the good behavior near $\alpha=-\tau(t)$, we need to do further changes to M_2 and O_i.

We do not change M_1 and let

$$
M_1^{c}(\tilde{f}^{+(20)})=M_1(\tilde{f}^{+(20)})=\frac{\tilde{f}_{\mu}^{+(20)}(\alpha_{\gamma}^{t})}{d\alpha}B_{\tilde{f}}(\alpha,\gamma,t). \tag{3.21}
$$

.

For M_2, we separate the integral into the "near $-\tau(t)$ part" and the "far away from $-\tau(t)$" part and do integration by parts. We have

$$
\begin{aligned}
&M_2(\tilde{f}^{+(20)})=p.v.\int_{-\pi}^{\pi}K(\tilde{f}^{+}(\alpha_{\gamma}^{t})-\tilde{f}^{+}(\beta_{\gamma}^{t})+\tilde{f}^{L}(\alpha_{\gamma}^{t})-\tilde{f}^{L}(\beta_{\gamma}^{t}))\frac{d\tilde{f}_{\mu}^{+(20)}(\beta_{\gamma}^{t})}{d\beta}d\beta\\
&=p.v.\int_{-\tau(t)}^{2\alpha+\tau(t)}K(\tilde{f}^{+}(\alpha_{\gamma}^{t})-\tilde{f}^{+}(\beta_{\gamma}^{t})+\tilde{f}^{L}(\alpha_{\gamma}^{t})-\tilde{f}^{L}(\beta_{\gamma}^{t}))d(\tilde{f}_{\mu}^{+(20)}(\beta_{\gamma}^{t})-\tilde{f}_{\mu}{}^{+(20)}(\alpha_{\gamma}^{t}))\\
&+\int_{2\alpha+\tau(t)}^{\pi}K(\tilde{f}^{+}(\alpha_{\gamma}^{t})-\tilde{f}^{+}(\beta_{\gamma}^{t})+\tilde{f}^{L}(\alpha_{\gamma}^{t})-\tilde{f}^{L}(\beta_{\gamma}^{t}))d(\tilde{f}_{\mu}{}^{+(20)}(\beta_{\gamma}^{t}))\\
&=K(\tilde{f}^{+}(\alpha_{\gamma}^{t})-\tilde{f}^{+}((2\alpha+\tau(t))_{\gamma}^{t})+\tilde{f}^{L}(\alpha_{\gamma}^{t})-\tilde{f}^{L}((2\alpha+\tau(t))_{\gamma}^{t}))(\tilde{f}_{\mu}^{+(20)}((2\alpha+\tau(t))_{\gamma}^{t})-\tilde{f}_{\mu}^{+(20)}(\alpha_{\gamma}^{t}))\\
&-K(\tilde{f}^{+}(\alpha_{\gamma}^{t})-\tilde{f}^{+}(0)+\tilde{f}^{L}(\alpha_{\gamma}^{t})-\tilde{f}^{L}(0))(-\tilde{f}_{\mu}^{+(20)}(\alpha_{\gamma}^{t}))\\
&-p.v.\int_{-\tau(t)}^{2\alpha+\tau(t)}\frac{d}{d\beta}(K(\tilde{f}^{+}(\alpha_{\gamma}^{t})-\tilde{f}^{+}(\beta_{\gamma}^{t})+\tilde{f}^{L}(\alpha_{\gamma}^{t})-\tilde{f}^{L}(\beta_{\gamma}^{t}))(\tilde{f}_{\mu}^{+(20)}(\beta_{\gamma}^{t})-\tilde{f}_{\mu}^{+(20)}(\alpha_{\gamma}^{t}))d\beta
\end{aligned}
$$

$$
\begin{aligned}
&- K(\tilde{f}^{+}(\alpha_\gamma^t) - \tilde{f}^{+}((2\alpha+\tau(t))_\gamma^t) + \tilde{f}^{L}(\alpha_\gamma^t) - \tilde{f}^{L}((2\alpha+\tau(t))_\gamma^t))\tilde{f}_\mu^{+(20)}((2\alpha+\tau(t))_\gamma^t) \\
&- \int_{2\alpha+\tau(t)}^{\pi} \frac{\frac{d}{d\beta}(K(\tilde{f}^{+}(\alpha_\gamma^t) - \tilde{f}^{+}(\beta_\gamma^t) + \tilde{f}^{L}(\alpha_\gamma^t) - \tilde{f}^{L}(\beta_\gamma^t))}{1+ic'(\beta+\tau(t))\gamma t}\tilde{f}_\mu^{+(20)}(\beta_\gamma^t)(1+ic'(\beta+\tau(t))\gamma t)d\beta \\
&= Term_1 + Term_2 + Term_3 + Term_4 + Term_5.
\end{aligned}
$$

We further do the change to the $Term_5$. We change the contour from the curve to a vertical line and a horizontal line starting from $2\alpha+\tau(t)$. Here $1+ic'(2\alpha+\tau(t))\eta t$ in the denominator of $Term_{5,2}$ coming from $1+ic'(\beta+\tau)\gamma t$ by changing β to $2\alpha+\tau(t)$ and γ to η.

$$
\begin{aligned}
Term\ 5 &= \int_0^{\gamma} \frac{\frac{d}{d\beta}(K(\tilde{f}^{+}(\alpha_\gamma^t) - \tilde{f}^{+}(\beta_\eta^t) + \tilde{f}^{L}(\alpha_\gamma^t) - \tilde{f}^{L}(\beta_\eta^t))|_{\beta=2\alpha+\tau(t)}}{1+ic'(2\alpha+2\tau(t))\eta t}\tilde{f}_\mu^{+(20)}((2\alpha+\tau(t))_\eta^t)ic(2\alpha+2\tau(t))td\eta \\
&- \int_{2\alpha+\tau(t)}^{\pi} \frac{d}{d\beta}(K(\tilde{f}^{+}(\alpha_\gamma^t) - \tilde{f}^{+}(\beta_0^t) + \tilde{f}^{L}(\alpha_\gamma^t) - \tilde{f}^{L}(\beta_0^t))\tilde{f}_\mu^{+(20)}(\beta_0^t)d\beta \\
&= Term_{5,1} + Term_{5,2}.
\end{aligned}
$$

Here after change the contour, we change the $\tilde{f}^{+(20)}(\beta_\gamma^t)$ to $\tilde{f}^{+(20)}(\beta_0^t) = \tilde{f}^{+(20)}(\beta+\tau(t))$ when $\beta > 2\alpha+\tau(t)$ in the integral. Notice that the $\tilde{f}^{+(20)}(\beta+\tau(t))$ is known and well-defined even without the analyticity assumption. In the latter section, we change $\tilde{f}(\alpha_\gamma^t)$ to an unknown solution $h(\alpha,\gamma,t)$, and all $\tilde{f}^{+(20)}(\beta+\tau(t))$ will be kept same. In conclusion, let

$$
M_2^c = Term_1 + Term_2 + Term_3 + Term_4 + Term_{5,1} + Term_{5,2}. \tag{3.22}
$$

For $O^{1,i}$, we change it in the following way. First we separate the terms in the bracket $((\partial_\alpha^{b_i}\tilde{f}_v^{+})(\alpha_\gamma^t) - (\partial_\alpha^{b_i}\tilde{f}_v^{+})(\beta_\gamma^t))$ and use the fact that $\tilde{f}^{+}=0$ when $\alpha < -\tau(t)$. We then have

$$
\begin{aligned}
O^{1,i} =& p.v.\int_{-\pi}^{\pi} K_o^i(V^{10}(\alpha_\gamma^t) - V^{10}(\beta_\gamma^t), V_{\tilde{f}^L}^{21}(\alpha_\gamma^t) - V_{\tilde{f}^L}^{21}(\beta_\gamma^t), V_X(\alpha_\gamma^t) - V_X(\beta_\gamma^t))X_{i'}(\beta_\gamma^t) \\
&(\partial_\alpha^{b_i}\tilde{f}_v^{+})(\alpha_\gamma^t)(1+ic'(\beta+\tau(t))\gamma t)d\beta \\
&- p.v.\int_{-\tau(t)}^{\pi} K_o^i(V^{10}(\alpha_\gamma^t) - V^{10}(\beta_\gamma^t), V_{\tilde{f}^L}^{21}(\alpha_\gamma^t) - V_{\tilde{f}^L}^{21}(\beta_\gamma^t), V_X(\alpha_\gamma^t) - V_X(\beta_\gamma^t))X_{i'}(\beta_\gamma^t) \\
&(\partial_\alpha^{b_i}\tilde{f}_v^{+})(\beta_\gamma^t)(1+ic'(\beta+\tau(t))\gamma t)d\beta \\
&= Term_1 + Term_2.
\end{aligned}
$$

We do not change the first term (like in M_1). For the term 2 we separate the integral into the "near $-\tau(t)$ part" and the "far away from $-\tau(t)$" part like in M_2. But we do not do the integration by parts and change the contour directly. Because this term has less singularity than M_2. Here we

have

$$
\begin{aligned}
Term_2 = &- p.v. \int_{-\tau(t)}^{\pi} K_o^i(V^{10}(\alpha_\gamma^t) - V^{10}(\beta_\gamma^t), V_{\tilde{f}^L}^{21}(\alpha_\gamma^t) - V_{\tilde{f}^L}^{21}(\beta_\gamma^t), V_X(\alpha_\gamma^t) - V_X(\beta_\gamma^t)) X_{i'}(\beta_\gamma^t) \\
&(\partial_\alpha^{b_i} \tilde{f}_v^+)(\beta_\gamma^t)(1 + ic'(\beta + \tau(t))\gamma t) d\beta \\
&= -p.v. \int_{-\tau(t)}^{2\alpha+\tau(t)} K_o^i(V^{10}(\alpha_\gamma^t) - V^{10}(\beta_\gamma^t), V_{\tilde{f}^L}^{21}(\alpha_\gamma^t) - V_{\tilde{f}^L}^{21}(\beta_\gamma^t), V_X(\alpha_\gamma^t) - V_X(\beta_\gamma^t)) X_{i'}(\beta_\gamma^t) \\
&(\partial_\alpha^{b_i} \tilde{f}_v^+)(\beta_\gamma^t)(1 + ic'(\beta + \tau(t))\gamma t) d\beta \\
&+ \int_0^{\gamma} K_o^i(V^{10}(\alpha_\gamma^t) - V^{10}((2\alpha + \tau(t))_\eta^t), V_{\tilde{f}^L}^{21}(\alpha_\gamma^t) - V_{\tilde{f}^L}^{21}((2\alpha + \tau(t))_\eta^t), V_X(\alpha_\gamma^t) - V_X((2\alpha + \tau(t))_\eta^t) \\
&X_{i'}((2\alpha + \tau(t))_\eta^t)(\partial_\alpha^{b_i} \tilde{f}_v^+)((2\alpha + \tau(t))_\eta^t)(ic(2\alpha + 2\tau(t))t) d\eta \\
&- \int_{2\alpha+\tau(t)}^{\pi} K_o^i(V^{10}(\alpha_\gamma^t) - V^{10}(\beta_0^t), V_{\tilde{f}^L}^{21}(\alpha_\gamma^t) - V_{\tilde{f}^L}^{21}(\beta_0^t), V_X(\alpha_\gamma^t) - V_X(\beta_0^t)) X_{i'}(\beta_0^t) \\
&(\partial_\alpha^{b_i} \tilde{f}_v^+)(\beta_0^t) d\beta \\
&= Term_{2,1} + Term_{2,2} + Term_{2,3}.
\end{aligned}
$$

Hence we let

$$
O^{1,i,c} = Term_1 + Term_{2,1} + Term_{2,2} + Term_{2,3}. \tag{3.23}
$$

For the $O^{2,i}$ terms, notice that it is a $O^{1,i}$ type terms times an extra factor $\partial_\alpha^{s_i} \tilde{f}_\mu^+$. we do the same change as $O^{1,i}$ and do not change the factor $\partial_\alpha^{s_i} \tilde{f}_\mu^+$. For the type $O^{4,i,c}$, for any fixed ζ in the integral, it is $O^{1,i}$ type term except there is coefficient ζ. We do the same change for each fixed ζ. We do not change $O^{3,i}$(like $M_1(h)$) and $O^{5,i}$ and let

$$
O^{3,i,c} = O^{3,i}, O^{5,i,c} = O^{5,i}.
$$

In conclusion, we have

$$
\frac{d\tilde{f}_\mu^{+(20)}(\alpha_\gamma^t)}{dt} = \tilde{\lambda}(\alpha_\gamma^t) M_1^c(\tilde{f}^{+(20)}) - \tilde{\lambda}(\alpha_\gamma^t) M_2^c(\tilde{f}^{+(20)}) + \sum_i \tilde{\lambda}(\alpha_\gamma^t) O_i^c(\tilde{f}^{+(20)}) + \tilde{\lambda}(\alpha_\gamma^t) T_{fixed}(\alpha_\gamma^t).
$$

We claim $M_1 = M_1^c$, $M_2 = M_1^c$, $O_i = O_i^c$ with the analyticity assumption on $\tilde{f}^{+(20)}$. They are same on $\gamma = 0$ even without the analyticity condition.

3.5.4 Conclusion: the modified equation T in the complex plane

We first let $\tilde{f}^{+(20)}(\alpha_\gamma^t) = h(\alpha, \lambda, t)$ and write the equation in h. The first step is to get the lower order derivative of $\tilde{f}^+$ from h. Here we use the property that up to 59 order of derivative of $\tilde{f}^+$ at $-\tau(t)$ is 0. We claim that for all $0 \le i \le 20$,

$$\tilde{f}^{+(i)}(\alpha_\gamma^t) = D^{-(20-i)}(h)(\alpha, \gamma, t),$$

where

$$D^{-(20-i)}h(\alpha,\gamma,t)$$
$$= \begin{cases} \int_0^\alpha (1+ic'(\alpha_1+\tau(t))\gamma t)...\int_0^{\alpha_{i-1}}(1+ic'(\alpha_i+\tau(t))\gamma t)h(\alpha_i,\gamma,t)d\alpha_i d\alpha_{i-1}...d\alpha_1 & -\tau(t)\le\alpha\le\pi \\ 0 & -\pi\le\alpha<-(\tau(t)) \end{cases}$$

Then we have the equation for h by changing the corresponding $\tilde{f}^{+(i)}(\alpha_\gamma^t)$ to $D^{-(20-i)}(h)(\alpha,\gamma,t)$ whenever $\gamma \neq 0$. We will keep $\tilde{f}^+$ if $\gamma = 0$ (on the real line). When $\alpha > -\tau(t)$, we have

$$\begin{aligned} &\frac{dh_\mu(\alpha,\gamma,t)}{dt} = T(h) \\ &= \tilde{\lambda}(\alpha_\gamma^t)M_1^c(h) - \tilde{\lambda}(\alpha_\gamma^t)M_2^c(h) + \tilde{\lambda}(\alpha_\gamma^t)\sum_i O^{1,i,c}(h) + \tilde{\lambda}(\alpha_\gamma^t)T_{fixed}(\alpha_\gamma^t). \end{aligned} \tag{3.24}$$

From (3.21),

$$M_1^c(h) = \frac{dh_\mu(\alpha,\gamma,t)}{d\alpha}B_{D^{-20}(h)}(\alpha,\gamma,t), \tag{3.25}$$

with

$$\begin{aligned} &B_{D^{-20}(h)}(\alpha,\gamma,t) = [\frac{1}{1+ic'(\alpha+\tau(t))\gamma t}(ic(\alpha+\tau(t))\gamma + ic'(\alpha+\tau(t))\gamma t\tau'(t) + \frac{\partial_t x}{\partial_\alpha x}(\alpha_\gamma^t) \\ &+ \frac{1}{(\frac{dx}{d\alpha}(\alpha_\gamma^t))}\cdot p.v.\int_{-\pi}^{\pi} K(D^{-20}(h)(\alpha,\gamma,t) - D^{-20}(h)(\beta,\gamma,t) + \tilde{f}^L(\alpha_\gamma^t) - \tilde{f}^L(\beta_\gamma^t)) \\ &\frac{dx(\beta_\gamma^t)}{d\beta}d\beta - \frac{\partial_t x}{\partial_\alpha x}(0) - \frac{1}{\frac{dx}{d\alpha}(0)}p.v\int_{-\pi}^{\pi} K(D^{-20}(h)(-\tau(t),\gamma,t) - D^{-20}(h)(\beta,\gamma,t) + \tilde{f}^L(0) - \tilde{f}^L(\beta_\gamma^t))\frac{dx(\beta_\gamma^t)}{d\beta}d\beta)]. \end{aligned}$$

From (3.22), the next term is

$$M_2^c(h) = Term_1 + Term_2 + Term_3 + Term_4 + Term_{5,1} + Term_{5,2} \tag{3.26}$$

$$= \underbrace{K(D^{-20}(h)(\alpha,\gamma,t) - D^{-20}(h)(2\alpha+\tau(t),\gamma,t) + \tilde{f}^L(\alpha_\gamma^t) - \tilde{f}^L((2\alpha+\tau(t))_\gamma^t))(h_\mu(2\alpha+\tau(t)),\gamma,t) - h_\mu(\alpha,\gamma,t))}_{Term_1}$$

$$\underbrace{-K(D^{-20}(h)(\alpha,\gamma,t) - D^{-20}(h)(-\tau(t),\gamma,t) + \tilde{f}^L(\alpha_\gamma^t) - \tilde{f}^L(0))(-h_\mu(\alpha,\gamma,t))}_{Term_2}$$

$$\underbrace{-p.v.\int_{-\tau(t)}^{2\alpha+\tau(t)} \frac{d}{d\beta}(K(D^{-20}(h)(\alpha,\gamma,t) - D^{-20}(h)(\beta,\gamma,t) + \tilde{f}^L(\alpha_\gamma^t) - \tilde{f}^L(\beta_\gamma^t))(h_\mu(\beta,\gamma,t) - h_\mu(\alpha,\gamma,t))d\beta}_{Term_3}$$

$$\underbrace{-K(D^{-20}(h)(\alpha,\gamma,t) - D^{-20}(h)(2\alpha+\tau(t),\gamma,t) + \tilde{f}^L(\alpha_\gamma^t) - \tilde{f}^L((2\alpha+\tau(t))_\gamma^t))h_\mu(2\alpha+\tau(t),\gamma,t)}_{Term_4}$$

$$\underbrace{\int_0^\gamma \frac{\frac{d}{d\beta}(K(D^{-20}(h)(\alpha,\gamma,t) - D^{-20}(h)(\beta,\eta,t) + \tilde{f}^L(\alpha_\gamma^t) - \tilde{f}^L(\beta_\eta^t))|_{\beta=2\alpha+\tau(t)}}{1+ic'(2\alpha+2\tau(t))\eta t} h_\mu(2\alpha+\tau(t),\eta,t)ic(2\alpha+2\tau(t))td\eta}_{Term_{5,1}}$$

$$\underbrace{-\int_{2\alpha+\tau(t)}^{\pi} \frac{d}{d\beta}(K(D^{-20}(h)(\alpha,\gamma,t) - \tilde{f}^+(\beta_0^t)) + \tilde{f}^L(\alpha_\gamma^t) - \tilde{f}^L(\beta_0^t)))\tilde{f}_\mu^{+(20)}(\beta_0^t)d\beta}_{Term_{5,2}}.$$

For the sake of later proof, we separate the $Term_3$ into the singular part and the non-singular part.

We also separate $Term_{5,2}$ into the part depending on h and the part not depending on h, which is

$$\begin{aligned}
Term_3 &= \int_{-\tau(t)}^{2\alpha+\tau(t)} \frac{d}{d\beta}(K(D^{-20}(h)(\alpha,\gamma,t) - D^{-20}(h)(\beta,\gamma,t) + \tilde{f}^L(\alpha_\gamma^t) - \tilde{f}^L(\beta_\gamma^t)))(h_\mu(\alpha,\gamma,t) - h_\mu(\beta,\gamma,t))d\beta \\
&= \lim_{\beta\to\alpha}(\frac{d}{d\beta}(K(D^{-20}(h)(\alpha,\gamma,t) - D^{-20}(h)(\beta,\gamma,t) + \tilde{f}^L(\alpha_\gamma^t) - \tilde{f}^L(\beta_\gamma^t)))(\alpha-\beta)^2) \\
&\cdot p.v.\int_{-\tau(t)}^{2\alpha+\tau(t)} \frac{h_\mu(\alpha,\gamma,t) - h_\mu(\beta,\gamma,t)}{(\alpha-\beta)^2}d\beta \\
&+ \int_{-\tau(t)}^{2\alpha+\tau(t)} \frac{1}{\alpha-\beta}[\frac{d}{d\beta}(K(D^{-20}(h)(\alpha,\gamma,t) - D^{-20}(h)(\beta,\gamma,t) + \tilde{f}^L(\alpha_\gamma^t) - \tilde{f}^L(\beta_\gamma^t)))(\alpha-\beta)^2 \\
&- \lim_{\beta\to\alpha}(\frac{d}{d\beta}(K(D^{-20}(h)(\alpha,\gamma,t) - D^{-20}(h)(\beta,\gamma,t) + \tilde{f}^L(\alpha_\gamma^t) - \tilde{f}^L(\beta_\gamma^t)))(\alpha-\beta)^2)] \\
&\cdot \frac{h_\mu(\alpha,\gamma,t) - h_\mu(\beta,\gamma,t)}{(\alpha-\beta)}d\beta \\
&= \underbrace{Term_{3,1}}_{\text{singular part}} + Term_{3,2}.
\end{aligned}$$

$$\begin{aligned}
Term_{5,2} = &-\int_{2\alpha+\tau(t)}^{\pi} \frac{d}{d\beta}(K(-\tilde{f}^+(\beta_0^t) + \tilde{f}^L(\alpha_\gamma^t) - \tilde{f}^L(\beta_0^t)))\tilde{f}_\mu^{+(20)}(\beta_0^t)d\beta \\
&- \int_0^1 d\zeta_1 \int_{2\alpha+\tau(t)}^{\pi} \frac{d}{d\zeta_1}\frac{d}{d\beta}(K(\zeta_1 D^{-20}(h)(\alpha,\gamma,t) - \tilde{f}^+(\beta_0^t) + \tilde{f}^L(\alpha_\gamma^t) - \tilde{f}^L(\beta_0^t)))\tilde{f}_\mu^{+(20)}(\beta_0^t)d\beta
\end{aligned}$$

$$= \underbrace{Term_{5,2,1}}_{\text{does not depend on h}} + Term_{5,2,2}.$$

For the $O^{1,i,c}$, we further denote

$$V_h^k(\alpha,\gamma,t) = (D^{-20}h_1(\alpha,\gamma,t), D^{-20}h_2(\alpha,\gamma,t), ..., D^{-20+k}h_1(\alpha,\gamma,t), D^{-20+k}h_2(\alpha,\gamma,t)). \quad (3.27)$$

from (3.23), we then have

$$\begin{aligned}
&O^{1,i,c}(h) = Term_1 + Term_{2,1} + Term_{2,2} + Term_{2,3}.\\
=&p.v.\int_{-\pi}^{\pi} K_o^i(V_h^{10}(\alpha,\gamma,t) - V_h^{10}(\beta,\gamma,t), V_{\tilde{f}^L}^{21}(\alpha_\gamma^t) - V_{\tilde{f}^L}^{21}(\beta_\gamma^t), V_X(\alpha_\gamma^t) - V_X(\beta_\gamma^t))X_{i'}(\beta_\gamma^t)\\
&D^{-20+b_i}(h)(\alpha,\gamma,t)(1+ic'(\beta+\tau(t))\gamma t)d\beta\\
&- p.v.\int_{-\tau(t)}^{2\alpha+\tau(t)} K_o^i(V_h^{10}(\alpha,\gamma,t) - V_h^{10}(\beta,\gamma,t), V_{\tilde{f}^L}^{21}(\alpha_\gamma^t) - V_{\tilde{f}^L}^{21}(\beta_\gamma^t), V_X(\alpha_\gamma^t) - V_X(\beta_\gamma^t))X_{i'}(\beta_\gamma^t)\\
&(D^{-20+b_i}(h)(\beta,\gamma,t))(1+ic'(\beta+\tau(t))\gamma t)d\beta\\
&+ \int_0^{\gamma} K_o^i(V_h^{10}(\alpha,\gamma,t) - V_h^{10}(2\alpha+\tau(t),\eta,t), V_{\tilde{f}^L}^{21}(\alpha_\gamma^t) - V_{\tilde{f}^L}^{21}((2\alpha+\tau(t))_\eta^t), V_X(\alpha_\gamma^t) - V_X((2\alpha+\tau(t))_\eta^t)\\
&X_{i'}((2\alpha+\tau(t))_\eta^t)(D^{-20+b_i}(h)(2\alpha+\tau(t),\gamma,t))(ic(2\alpha+2\tau(t))t)d\eta\\
&- \int_{2\alpha+\tau(t)}^{\pi} K_o^i(V_h^{10}(\alpha,\gamma,t) - V^{10}(\beta_0^t), V_{\tilde{f}^L}^{21}(\alpha_\gamma^t) - V_{\tilde{f}^L}^{21}(\beta_0^t), V_X(\alpha_\gamma^t) - V_X(\beta_0^t))X_{i'}(\beta_0^t)\\
&(\partial_\alpha^{b_i}\tilde{f}_v^+)(\beta_0^t)d\beta.
\end{aligned}$$

We also separate the $Term_{2,3}$ as in the M_2. We have

$$\begin{aligned}
&Term_{2,3} = -\int_{2\alpha+\tau(t)}^{\pi} K_o^i(-V^{10}(\beta_0^t)), V_{\tilde{f}^L}^{21}(\alpha_\gamma^t) - V_{\tilde{f}^L}^{21}(\beta_0^t), V_X(\alpha_\gamma^t) - V_X(\beta_0^t))X_{i'}(\beta_0^t)(\partial_\alpha^{b_i}\tilde{f}_v^+)(\beta_0^t)d\beta\\
&- \int_0^1 d\zeta_0 \int_{2\alpha+\tau(t)}^{\pi} \frac{d}{d\zeta_0} K_o^i(\zeta_0 V_h^{10}(\alpha,\gamma,t) - V^{10}(\beta_0^t), V_{\tilde{f}^L}^{21}(\alpha_\gamma^t) - V_{\tilde{f}^L}^{21}(\beta_0^t), V_X(\alpha_\gamma^t) - V_X(\beta_0^t))X_{i'}(\beta_0^t)\\
&(\partial_\alpha^{b_i}\tilde{f}_v^+)(\beta_0^t)d\beta\\
&= Term_{2,3,1} + Term_{2,3,2}.
\end{aligned}$$

Hence we get

$$O^{1,i,c}(h) = Term_1 + Term_{2,1} + Term_{2,2} + Term_{2,3,1} + Term_{2,3,2}.$$

For $O^{2,i,c}$ type terms, we have

$$O^{2,i,c}(h) = O^{1,\tilde{i},c}(h) D^{-20+s_i} h_\mu(\alpha,\gamma,t). \tag{3.28}$$

For $O^{4,i,c}$ type terms, we have

$$O^{4,i,c}(h) = \int_0^1 O_\zeta^{1,\tilde{i},c}(h)\zeta^{q_0} d\zeta. \tag{3.29}$$

Here $O^{1,\tilde{i},c}$ means a $O^{1,i,c}$ type term and $O_\zeta^{1,\tilde{i},c}(h)$ means $O^{1,i,c}$ type term except changing some $h(\alpha)-h(\beta)$ to $\zeta(h(\alpha)-h(\beta))$.

For $O^{3,i,c}$ type terms, we have

$$O^{3,i,c}(h) = \int_{-\pi}^{\pi} \tilde{K}_o^i(V_h^{10}(\alpha_\gamma^t) - V_h^{10}(\beta_\gamma^t), V_{\tilde{f}^L}^{21}(\alpha_\gamma^t) - V_{\tilde{f}^L}^{21}(\beta_\gamma^t), V_X(\alpha_\gamma^t) - V_X(\beta_\gamma^t)) X_{i'}(\beta_\gamma^t) \tag{3.30}$$
$$(1 + ic'(\beta+\tau(t))\gamma t) d\beta D^{-20+b_i}(h_\mu)(\alpha,\gamma,t).$$

For $O^{5,i,c}$ type terms, we have

$$O^{5,i,c}(h) = D^{-20+b_i}(h_\mu)(\alpha,\gamma,t)\tilde{X}_i(\alpha_\gamma^t). \tag{3.31}$$

When $\alpha \le -\tau(t)$, we let $h(\alpha,\gamma,t) = 0$, which is also compatible with $\tilde{f}^{+(20)}$.

In conclusion, we have

$$\frac{dh_\mu(\alpha,\gamma,t)}{dt} = T(h) = \begin{cases} \tilde{\lambda}(\alpha_\gamma^t)M_1^c(h) - \tilde{\lambda}(\alpha_\gamma^t)M_2^c(h) + \tilde{\lambda}(\alpha_\gamma^t)\sum_i O^{i,c}(h) + \tilde{\lambda}(\alpha_\gamma^t)T_{fixed}(\alpha_\gamma^t) & -\tau(t) \le \alpha \le \pi \\ 0 & -\pi \le \alpha < -(\tau(t)), \end{cases} \tag{3.32}$$

with the initial value $h_\mu(\alpha,\gamma,0) = \tilde{f}^{+(20)}(\alpha,0)$. Notice that all the terms which do not depend on h are well defined even without the analyticity assumption on f.

The equation at $\gamma = 0$: T^0

From the modified equation (3.32), we have for any fixed γ, $\frac{d}{dt}h(\alpha,\gamma,t)$ only depends on $h(\alpha,\eta,t)$ with $|\eta| \le |\gamma|$. Therefore we have

$$\frac{dh_\mu(\alpha,0,t)}{dt} = T^0(h(\alpha,0,t)) \tag{3.33}$$

$$=\begin{cases} \tilde{\lambda}(\alpha_0^t)(M_1^{c,0}(h(\alpha,0,t)) - M_2^{c,0}(h(\alpha,0,t)) + \sum_i O^{i,c,0}(h(\alpha,0,t)) + T_{fixed}(\alpha_0^t)) & -\tau(t) \le \alpha \le \pi \\ 0 & -\pi \le \alpha < -(\tau(t)), \end{cases}$$

with the initial value $h_\mu(\alpha,0,0) = \tilde{f}^{+(20)}(\alpha,0)$. Let $h_0(\alpha,t) = h(\alpha,0,t)$,

$$D_0^{-(20-i)}h_0(\alpha,t) = \begin{cases} \int_0^\alpha \int_0^{\alpha_1} ... \int_0^{\alpha_{i-1}} h_0(\alpha_i,t) d\alpha_i d\alpha_{i-1} ... d\alpha_1 & -\tau(t) \le \alpha \le \pi \\ 0 & -\pi \le \alpha < -(\tau(t)). \end{cases} \tag{3.34}$$

we have

$$M_1^{c,0}(h_0) = \underbrace{\frac{h_{0,\mu}(\alpha,t)}{d\alpha} B_{D_0^{-20}(h_0)}(\alpha,t)}_{\text{main term}},$$

with

$$\begin{aligned} &B_{D_0^{-20}(h_0)}(\alpha,t) = [\frac{\partial_t x}{\partial_\alpha x}(\alpha_0^t) \\ &+ \frac{1}{(\frac{dx}{d\alpha}(\alpha_0^t))} \cdot p.v. \int_{-\pi}^{\pi} K(D_0^{-20}(h_0)(\alpha,t) - D_0^{-20}(h_0)(\beta,t) + \tilde{f}^L(\alpha_0^t) - \tilde{f}^L(\beta_0^t)) \\ &\frac{dx(\beta_0^t))}{d\beta} d\beta - \frac{\partial_t x}{\partial_\alpha x}(0) - \frac{1}{\frac{dx}{d\alpha}(0)} p.v \int_{-\pi}^{\pi} K(D_0^{-20}(h_0)(-\tau(t),t) - D_0^{-20}(h_0)(\beta,t) + \tilde{f}^L(0) - \tilde{f}^L(\beta_0^t)) \frac{dx(\beta_0^t)}{d\beta} d\beta)]. \end{aligned}$$

$$\begin{aligned} &M_2^{c,0}(h) = Term_1 + Term_2 + Term_{3,1} + Term_{3,2} + Term_4 + Term_{5,2,1} + Term_{5,2,2} \\ &= \underbrace{K(D_0^{-20}(h_0)(\alpha,t) - D_0^{-20}(h_0)(2\alpha+\tau(t),t) + \tilde{f}^L(\alpha_0^t) - \tilde{f}^L((2\alpha+\tau(t))_0^t))(h_{0,\mu}(2\alpha+\tau(t)),t) - h_{0,\mu}(\alpha,t))}_{Term_1} \\ &\underbrace{-K(D_0^{-20}(h_0)(\alpha,t) - D_0^{-20}(h_0)(-\tau(t),t) + \tilde{f}^L(\alpha_0^t) - \tilde{f}^L(0))(-h_{0,\mu}(\alpha,t))}_{Term_2} \\ &+ \underbrace{\lim_{\beta\to\alpha}(\frac{d}{d\beta}(K(D_0^{-20}(h_0)(\alpha,t) - D_0^{-20}(h_0)(\beta,t) + \tilde{f}^L(\alpha_0^t) - \tilde{f}^L(\beta_0^t)))(\alpha-\beta)^2) p.v. \int_{-\tau(t)}^{2\alpha+\tau(t)} \frac{h_{0,\mu}(\alpha,t) - h_{0,\mu}(\beta,t)}{(\alpha-\beta)^2} d\beta}_{\text{main term } Term_{3,1}} \\ &+ \underbrace{\int_{-\tau(t)}^{2\alpha+\tau(t)} \frac{1}{\alpha-\beta}[\frac{d}{d\beta}(K(D_0^{-20}(h_0)(\alpha,t) - D_0^{-20}(h_0)(\beta,t) + \tilde{f}^L(\alpha_0^t) - \tilde{f}^L(\beta_0^t))(\alpha-\beta)^2 -}_{Term_{3,2}} \\ &\underbrace{\lim_{\beta\to\alpha}(\frac{d}{d\beta}(K(D_0^{-20}(h_0)(\alpha,t) - D_0^{-20}(h_0)(\beta,t) + \tilde{f}^L(\alpha_0^t) - \tilde{f}^L(\beta_0^t)))(\alpha-\beta)^2)] \frac{h_{0,\mu}(\alpha,t) - h_{0,\mu}(\beta,t)}{(\alpha-\beta)} d\beta}_{Term_{3,2}} \end{aligned}$$

$$\underbrace{-K(D_0^{-20}(h_0)(\alpha,t)-D_0^{-20}(h_0)(2\alpha+\tau(t),t)+\tilde{f}^L(\alpha_0^t)-\tilde{f}^L((2\alpha+\tau(t))_0^t))h_{0,\mu}(2\alpha+\tau(t)),t)}_{Term_4}$$

$$\underbrace{-\int_{2\alpha+\tau(t)}^{\pi}\frac{d}{d\beta}(K(-\tilde{f}^+(\beta_0^t)+\tilde{f}^L(\alpha_0^t)-\tilde{f}^L(\beta_0^t)))\tilde{f}_\mu^{+(20)}(\beta_0^t)d\beta}_{Term_{5,2,1}}$$

$$\underbrace{-\int_0^1 d\zeta_1\int_{2\alpha+\tau(t)}^{\pi}\frac{d}{d\zeta_1}\frac{d}{d\beta}(K(\zeta_1 D_0^{-20}(h_0)(\alpha,t)-\tilde{f}^+(\beta_0^t)+\tilde{f}^L(\alpha_0^t)-\tilde{f}^L(\beta_0^t)))\tilde{f}_\mu^{+(20)}(\beta_0^t)d\beta}_{Term_{5,2,2}}.$$

The equation does not have $Term_{5,1}$ because here $\gamma=0$. For the $O^{1,i,c}$, we still denote

$$V_{h_0}^k(\alpha,t)=(D_0^{-20}h_{0,1}(\alpha,t),D_0^{-20}h_{0,2}(\alpha,t),D^{-19}h_{0,1}(\alpha,t),D^{-19}h_{0,2}(\alpha,t),...,D^{-20+k}h_{0,1}(\alpha,t),D^{-20+k}h_{0,2}(\alpha,t)).$$

we then have

$$\begin{aligned}
&O^{1,i,c,0}(h_0)=Term\ 1+Term\ 2,1+Term\ 2,3,1+Term\ 2,3,2.\\
=&p.v.\int_{-\pi}^{\pi}K_o^i(V_{h_0}^{10}(\alpha,t)-V_{h_0}^{10}(\beta,t),V_{\tilde{f}^L}^{21}(\alpha_0^t)-V_{\tilde{f}^L}^{21}(\beta_0^t),V_X(\alpha_0^t)-V_X(\beta_0^t))X_{i'}(\beta_0^t)\\
&D_0^{-20+b_i}(h_0)(\alpha,t)d\beta\\
&-p.v.\int_{-\tau(t)}^{2\alpha+\tau(t)}K_o^i(V_{h_0}^{10}(\alpha,t)-V_{h_0}^{10}(\beta,t),V_{\tilde{f}^L}^{21}(\alpha_0^t)-V_{\tilde{f}^L}^{21}(\beta_0^t),V_X(\alpha_0^t)-V_X(\beta_0^t))X_{i'}(\beta_0^t)\\
&(D_0^{-20+b_i}(h_0)(\beta,t))d\beta\\
&-\int_{2\alpha+\tau(t)}^{\pi}K_o^i(-V^{10}(\beta_0^t),V_{\tilde{f}^L}^{21}(\alpha_0^t)-V_{\tilde{f}^L}^{21}(\beta_0^t),V_X(\alpha_0^t)-V_X(\beta_0^t))X_{i'}(\beta_0^t)(\partial_\alpha^{b_i}\tilde{f}_v^+)(\beta_0^t)d\beta\\
&-\int_0^1 d\zeta_0\int_{2\alpha+\tau(t)}^{\pi}\frac{d}{d\zeta_0}K_o^i(\zeta_0 V_{h_0}^{10}(\alpha,t)-V^{10}(\beta_0^t),V_{\tilde{f}^L}^{21}(\alpha_0^t)-V_{\tilde{f}^L}^{21}(\beta_0^t),V_X(\alpha_0^t)-V_X(\beta_0^t))X_{i'}(\beta_0^t)\\
&(\partial_\alpha^{b_i}\tilde{f}_v^+)(\beta_0^t)d\beta.
\end{aligned}$$

Similarly,

$$O^{2,i,c,0}(h_0)=O^{1,\tilde{i},c,0}(h_0)D_0^{-20+s_i}h_{0,\mu}(\alpha,t),$$

and

$$O^{4,i,c,0}(h_0)=\int_0^1 O_\zeta^{1,\tilde{i},c,0}(h_0)\zeta^{q_0}d\zeta.$$

Here $O^{1,\tilde{i},c,0}$ means a $O^{1,i,c,0}$ type term and $O_\zeta^{1,\tilde{i},c,0}(h)$ means $O^{1,i,c,0}$ type term except changing some $h(\alpha)-h(\beta)$ to $\zeta(h(\alpha)-h(\beta))$.

For $O^{3,i,c,0}$ type terms, we have

$$O^{3,i,c,0}(h_0) = \int_{-\pi}^{\pi} \tilde{K}_o^i(V_{h_0}^{10}(\alpha_0^t) - V_{h_0}^{10}(\beta_0^t)), V_{\tilde{f}^L}^{21}(\alpha_0^t) - V_{\tilde{f}^L}^{21}(\beta)_0^t, V_X(\alpha_0^t) - V_X(\beta_0^t))X_{i'}(\beta_0^t)$$
$$d\beta D_0^{-20+b_i}(h_{0,\mu})(\alpha, t).$$

For $O^{5,i,c}$ type terms, we have

$$O^{5,i,c,0}(h_0) = D_0^{-20+b_i}(h_{0,\mu})(\alpha, t)\tilde{X}_i(\alpha_0^t).$$

We claim that $\tilde{f}^{+(20)}(\alpha + \tau(t), t)$ is a solution of this equation. We will show that it is the only solution in the later Section 3.10.

3.6 Perturb the modified equation

In order to use the Picard theorem to get the existence, we first desingularize the equation and let

$$\frac{dh_\mu(\alpha,\gamma,t)}{dt} = T^\epsilon(h) = \begin{cases} \tilde{\lambda}(\alpha_\gamma^t)M_1^\epsilon(h) - \tilde{\lambda}(\alpha_\gamma^t)M_2^\epsilon(h) + \tilde{\lambda}(\alpha_\gamma^t)\sum_i O^{i,\epsilon}(h) + \tilde{\lambda}(\alpha_\gamma^t)T_{fixed}(\alpha_\gamma^t) & -\tau(t) \le \alpha \le \pi \\ 0 & -\pi \le \alpha < -(\tau(t)) \end{cases}$$

We claim that when $h \in H^5[-\tau(t), \pi]$, then $\|T^\epsilon(h) - T(h)\|_{L^\infty[-\tau(t),\pi]} = 0$..

3.6.1 Perturb M_1

For the first term, we desingularize the term $\frac{dh}{d\alpha}$. Let

$$M_1^\epsilon(h) = B_{D^{-20}(h)}(\alpha,\gamma,t)b_l \int_{-\tau(t)}^{2\alpha+\tau(t)} \frac{(h(\alpha,\gamma,t) - h(\beta,\gamma,t))(\alpha-\beta)\epsilon}{((\alpha-\beta)^2+\epsilon^2)^2} d\beta, \tag{3.35}$$

with

$$B_{D^{-20}(h)}(\alpha,\gamma,t) = [\frac{1}{1+ic'(\alpha+\tau(t))\gamma t}(ic(\alpha+\tau(t))\gamma + ic'(\alpha+\tau(t))\gamma t\tau'(t) + \frac{\partial_t x}{\partial_\alpha x}(\alpha_\gamma^t, t)$$
$$+ \frac{1}{(\partial_\alpha x(\alpha_\gamma^t, t))} \cdot p.v. \int_{-\pi}^{\pi} K(D^{-20}(h)(\alpha,\gamma,t) - D^{-20}(h)(\beta,\gamma,t) + \tilde{f}^L(\alpha_\gamma^t, t) - \tilde{f}^L(\beta_\gamma^t, t))\frac{dx(\beta_\gamma^t, t)}{d\beta} d\beta$$
$$- \frac{\partial_t x}{\partial_\alpha x}(0,t) - \frac{1}{\partial_\alpha x(0,t)} p.v \int_{-\pi}^{\pi} K(D^{-20}(h)(-\tau(t),\gamma,t) - D^{-20}(h)(\beta,\gamma,t) + \tilde{f}^L(0,t) - \tilde{f}^L(\beta_\gamma^t, t))\frac{dx(\beta_\gamma^t, t)}{d\beta} d\beta)].$$

Here

$$b_l = \frac{1}{\lim_{\epsilon\to 0}\int_{-\tau(t)}^{\alpha+\tau(t)} \frac{(\alpha-\beta)^2\epsilon}{((\alpha-\beta)^2+\epsilon^2)^2}d\beta} = \frac{2}{\pi}. \tag{3.36}$$

Note that b_l is a constant when $\alpha > -\tau(t)$.

Due to the property of Poisson Integral, we claim when $h \in C^2_\alpha[-\pi,\pi]$,

$$\lim_{\epsilon\to 0}(M_1^\epsilon(h) - B_{D^{-20}(h)}\partial_\alpha h) = 0,$$

as $\epsilon \to 0$ when $\alpha > -\tau(t)$.

3.6.2 Perturb M_2

For the M_2, we desingularize the $Term_{3,1}$ and $Term_{3,2}$. Also notice that only $Term_{3,1}$ is the main term in the energy estimate. The perturbation of $Term_{3,2}$ helps to show the continuity in t but does not affect the energy estimate.

$$\begin{aligned}
&M_2^\epsilon(h) = Term_1 + Term_2 + Term_{3,1}^\epsilon + Term_{3,2}^\epsilon + Term_4 + Term_{5,1} + Term_{5,2,1} + Term_{5,2,2}\\
&= \underbrace{K(D^{-20}(h)(\alpha,\gamma,t) - D^{-20}(h)(2\alpha+\tau(t),\gamma,t) + \tilde{f}^L(\alpha_\gamma^t) - \tilde{f}^L((2\alpha+\tau(t))_\gamma^t))(h_\mu(2\alpha+\tau(t),\gamma,t) - h_\mu(\alpha,\gamma,t))}_{Term_1}\\
&\underbrace{-K(D^{-20}(h)(\alpha,\gamma,t) - D^{-20}(h)(-\tau(t),\gamma,t) + \tilde{f}^L(\alpha_\gamma^t,t) - \tilde{f}^L(0,t))(-h_\mu)(\alpha,\gamma,t)}_{Term_2}\\
&+ Term_{3,1}^\epsilon + Term_{3,2,1}^\epsilon + Term_{3,2,2}\\
&\underbrace{-K(D^{-20}(h)(\alpha,\gamma,t) - D^{-20}(h)(2\alpha+\tau(t),\gamma,t) + \tilde{f}^L(\alpha_\gamma^t,t) - \tilde{f}^L((2\alpha+\tau(t))_\gamma^t,t))h_\mu(2\alpha+\tau(t),\gamma,t)}_{Term_4}\\
&+ \underbrace{\int_0^\gamma \frac{\frac{d}{d\beta}(K(D^{-20}(h)(\alpha,\gamma,t) - D^{-20}(h)(\beta,\eta,t) + \tilde{f}^L(\alpha_\gamma^t,t) - \tilde{f}^L(\beta_\eta^t,t)))|_{\beta=2\alpha+\tau(t)}}{1+ic'(2\alpha+2\tau(t))\eta t}h_\mu((2\alpha+\tau(t)),\eta,t)}_{Term_{5,1}}\\
&\underbrace{ic(2\alpha+2\tau(t))td\eta}_{Term_{5,1}}\\
&\underbrace{-\int_{2\alpha+\tau(t)}^\pi \frac{d}{d\beta}(K(-\tilde{f}^+(\beta_0^t,t) + \tilde{f}^L(\alpha_\gamma^t,t) - \tilde{f}^L(\beta_0^t,t)))\tilde{f}^{+(20)}(\beta_0^t,t)d\beta}_{Term_{5,2,1}}\\
&\underbrace{-\int_0^1 d\zeta_1\int_{2\alpha+\tau(t)}^\pi \frac{d}{d\zeta_1}\frac{d}{d\beta}(K(\zeta_1 D^{-20}(h)(\alpha,\gamma,t) - \tilde{f}^+(\beta_0^t,t) + \tilde{f}^L(\alpha_\gamma^t,t) - \tilde{f}^L(\beta_0^t,t)))\tilde{f}^{+(20)}(\beta_0^t,t)d\beta}_{Term_{5,2,2}}.
\end{aligned}$$

Let

$$K_{M_{2,3}}(\alpha,\beta)=\frac{d}{d\beta}(K(D^{-20}(h)(\alpha,\gamma,t)-D^{-20}(h)(\beta,\gamma,t)+\tilde{f}^L(\alpha^t_\gamma,t)-\tilde{f}^L(\beta^t_\gamma,t))),$$

and

$$Term^\epsilon_{3,1}=\lim_{\beta\to\alpha}(K_{M_{2,3}}(\alpha,\beta)(\alpha-\beta)^2)\int_{-\tau(t)}^{2\alpha+\tau(t)}\frac{h^\epsilon_\mu(\alpha,\gamma,t)-h^\epsilon_\mu(\beta,\gamma,t)}{(\alpha-\beta)^2+\epsilon^2}d\beta$$

$$\begin{aligned}Term^\epsilon_{3,2}&=(\int_{-\tau(t)}^{2\alpha+\tau(t)}\frac{K_{M_{2,3}}(\alpha,\beta)(\alpha-\beta)^2-\lim_{\beta\to\alpha}(K_{M_{2,3}}(\alpha,\beta)(\alpha-\beta)^2)}{\alpha-\beta}\frac{(h^\epsilon_\mu(\beta,\gamma,t))(\alpha-\beta)}{(\alpha-\beta)^2+\epsilon^2}d\beta\\&+p.v.\int_{-\tau(t)}^{2\alpha+\tau(t)}\frac{K_{M_{2,3}}(\alpha,\beta)(\alpha-\beta)^2-\lim_{\beta\to\alpha}(K_{M_{2,3}}(\alpha,\beta)(\alpha-\beta)^2)}{(\alpha-\beta)^2}d\beta h^\epsilon_\mu(\alpha,\gamma,t)\\&=Term^\epsilon_{3,2,1}+Term_{3,2,2}.\end{aligned}$$

In conclusion, we have

$$\begin{aligned}M^\epsilon_2(h)&=Term_1+Term_2+Term^\epsilon_{3,1}+Term^\epsilon_{3,2,1}+Term_{3,2,2}\\&+Term_4+Term_{5,1}+Term_{5,2,1}+Term_{5,2,2}.\end{aligned}\tag{3.37}$$

3.6.3 Perturb $\mathbf{O}^i$

Similarly, for the $O^{1,i}$ terms, we also perturb the the term $Term_{2,1}$, which is an integral from $-\tau(t)$ to $2\alpha+\tau(t)$. Still, this perturbation helps to get the continuity and the energy estimate holds even without the perturbation.

$$\begin{aligned}&O^{1,i,\epsilon}(h)=Term_1+Term^\epsilon_{2,1}+Term_{2,2}+Term_{2,3,1}+Term_{2,3,2.}\\=&p.v.\int_{-\pi}^{\pi}K^i_o(V^{10}_h(\alpha,\gamma,t)-V^{10}_h(\beta,\gamma,t),V^{21}_{\tilde{f}^L}(\alpha^t_\gamma,t)-V^{21}_{\tilde{f}^L}(\beta^t_\gamma,t),V_X(\alpha^t_\gamma,t)-V_X(\beta^t_\gamma,t))X_{i'}(\beta^t_\gamma)\\&D^{-20+b_i}(h)(\alpha,\gamma,t)(1+ic'(\beta+\tau(t))\gamma t)d\beta\\&+Term^\epsilon_{2,1}\\&+\int_0^\gamma K^i_o(V^{10}_h(\alpha,\gamma,t)-V^{10}_h(2\alpha+\tau(t),\eta,t)),V^{21}_{\tilde{f}^L}(\alpha^t_\gamma,t)-V^{21}_{\tilde{f}^L}((2\alpha+\tau(t),t)^t_\eta),V_X(\alpha^t_\gamma,t)-V_X((2\alpha+\tau(t),t)^t_\eta)\\&X_{i'}((2\alpha+\tau(t))^t_\eta,t)(D^{-20+b_i}(h)(2\alpha+\tau(t),\gamma,t))(ic(2\alpha+2\tau(t))t)d\eta\\&-\int_{2\alpha+\tau(t)}^{\pi}K^i_o(-V^{10}(\beta^t_0,t),V^{21}_{\tilde{f}^L}(\alpha^t_\gamma,t)-V^{21}_{\tilde{f}^L}(\beta^t_0,t),V_X(\alpha^t_\gamma,t)-V_X(\beta^t_0,t))X_{i'}(\beta^t_0,t)(\partial^{b_i}_\alpha\tilde{f}^+_v)(\beta^t_0,t)d\beta\end{aligned}$$

$$-\int_0^1 d\zeta_0 \int_{2\alpha+\tau(t)}^{\pi} \frac{d}{d\zeta_0} K_o^i(\zeta_0 V_h^{10}(\alpha,\gamma,t) - V^{10}(\beta_0^t,t), V_{\tilde{f}^L}^{21}(\alpha_\gamma^t,t) - V_{\tilde{f}^L}^{21}(\beta_0^t,t), V_X(\alpha_\gamma^t,t) - V_X(\beta_0^t,t))$$
$$X_{i'}(\beta_0^t,t)(\partial_\alpha^{b_i}\tilde{f}_v^+)(\beta_0^t,t)d\beta.$$

Let

$$K_{O^{1,i,3}}(\alpha,\beta) = K_o^i(V_h^{10}(\alpha,\gamma,t) - V_h^{10}(\beta,\gamma,t), V_{\tilde{f}^L}^{21}(\alpha_\gamma^t,t) - V_{\tilde{f}^L}^{21}(\beta_\gamma^t,t), V_X(\alpha_\gamma^t,t) - V_X(\beta_\gamma^t,t))$$
$$X_{i'}(\beta_\gamma^t,t)(1+ic'(\beta+\tau(t))\gamma t).$$

and

$$Term_{2,1}^{\epsilon} = -\int_{-\tau(t)}^{2\alpha+\tau(t)} K_{O^{1,i,3}}(\alpha,\beta)(\alpha-\beta)\frac{(\alpha-\beta)D^{-20+b_i}(h)(\beta,\gamma,t)}{(\alpha-\beta)^2+\epsilon^2}d\beta.$$

In conclusion, we have

$$O^{1,i,\epsilon}(h) = Term_1 + Term_{2,1}^{\epsilon} + Term_{2,2} + Term_{2,3,1} + Term_{2,3,2}. \tag{3.38}$$

For $O^{2,i,\epsilon}$ type terms, we have

$$O^{2,i,\epsilon}(h) = O^{1,\tilde{i},\epsilon}(h)D^{-20+s_i}h_\mu(\alpha,\gamma,t). \tag{3.39}$$

For $O^{4,i,\epsilon}$ type terms, we have

$$O^{4,i,\epsilon}(h) = \int_0^1 O_\zeta^{1,\tilde{i},\epsilon}(h)\zeta^{q_0}d\zeta. \tag{3.40}$$

For $O^{3,i,\epsilon}$ and $O^{5,i,\epsilon}$ type terms, we do not change them. We have

$$O^{3,i,\epsilon}(h) = \int_{-\pi}^{\pi} \tilde{K}_o^i(V_h^{10}(\alpha_{\gamma,}^t) - V_h^{10}(\beta_\gamma^t), V_{\tilde{f}^L}^{21}(\alpha_\gamma^t) - V_{\tilde{f}^L}^{21}(\beta_\gamma^t), V_X(\alpha_\gamma^t) - V_X(\beta_\gamma^t))X_{i'}(\beta_\gamma^t) \tag{3.41}$$
$$(1+ic'(\beta+\tau(t))\gamma t)d\beta D^{-20+b_i}(h_\mu)(\alpha,\gamma,t),$$

and

$$O^{5,i,\epsilon}(h) = D^{-20+b_i}(h_\mu)(\alpha,\gamma,t)\tilde{X}_i(\alpha_\gamma^t,t). \tag{3.42}$$

3.7 The good behavior of T^ϵ

Lemma 3.7.1. *Let*

$$h, g \in C_t^0([0, t*], C_\gamma^{i_0}([-1, 1], H^k[-\pi, \pi])) \cap S_{\delta_1} = W^{i_0,k}. \tag{3.43}$$

Here

$$S = \{h(\alpha, \gamma, t) = 0, \textit{when } \alpha \le -\tau(t) \textit{ and } \alpha \ge \frac{\pi}{8}\},$$

$$S_{\delta_1} = \{h(\alpha, \gamma, t) = 0, \textit{when } \alpha \le -\tau(t) \textit{ and } \alpha \ge \frac{\pi}{8}, \sup_{\gamma,t} \|h\|_{C_\alpha^2[-\tau(t),\pi]} \lesssim \delta_1\},$$

δ_1, t^* are fixed sufficiently small numbers and $i_0, k \in N$, $i_0 + k \le 40$, $i_0 \le 4$. The bound of δ_1 is used in the Arc-Chord condition lemma 3.7.19.

Then

$$T^\epsilon(h) \in C_t^0([0, t_*], C_\gamma^{i_0}([-1, 1], H^k[-\pi, \pi])) \cap S. \tag{3.44}$$

Moreover, the Lipschitz condition holds;

$$\|T^\epsilon(h) - T^\epsilon(g)\|_{W^{i_0,k}} \lesssim \|h - g\|_{W^{i_0,k}}, \tag{3.45}$$

Here $\lesssim$ depends on ϵ, δ, δ_1, t_* and δ_c, $\|h\|_{W^{i_0,k}}$, $\|g\|_{W^{i_0,k}}$.

Proof. From lemma 3.7.2, lemma 3.7.11, and lemma 3.7.17, we have the result. □

3.7.1 Different types of T_i^ϵ

Since we have done the desingularization, it should be easy to show it is in $H^k[-\tau(t), \pi]$. The main difficulty here is to show the good behavior around $-\tau(t)$ ($\partial_\alpha^j T^\epsilon(h)(-\tau(t), \gamma, t) = 0$, for $j \le k - 1$). The idea is

1. the terms depending on h preserves this behavior,
2. the sum of terms which does not depend on h also satisfies this behavior.

The part one is true because either the term is a product of h(or lower derivative of h) with some good functions, or the integral with h only lives from $-\tau(t)$ to $2\alpha + \tau(t)$.

The second part is true because when on the real line($\gamma = 0$), the sum automatically satisfies this condition because the good property of $\tilde{f}^+$. And changing from the real line to the complex plane($\gamma \neq 0$) does not affect the behavior of the sum around $-\tau(t)$.

Lemma 3.7.2. *All terms in T^ϵ can be written as following types: (The conditions hold for all $i_0, k \in \mathbb{N}$ with $i_0 + k \leq 30, i_0 \leq 4$).*

The first type (only one term):

$$T_1^\epsilon(h) = \begin{cases} \tilde{\lambda}(\alpha_\gamma^t) L_{h^s}^1(\alpha,\gamma,t) \int_{-\tau(t)}^{2\alpha+\tau(t)} \frac{h(\alpha,\gamma,t)-h(\beta,\gamma,t)}{(\alpha-\beta)^2+\epsilon^2} d\beta & -\tau(t) < \alpha \leq \pi, \\ 0 & -\pi \leq \alpha \leq -(\tau(t)), \end{cases}$$

with $L_{h^s}^1$ satisfying the following condition:

$$\lim_{\alpha \to (-\tau(t)^+)} L_{h^s}^1(-\alpha,\gamma,t) = 0, \tag{3.46}$$

L_h can be extended to a function in $C_\gamma^{i_0}([-1,1], C_\alpha^{k+2}[-\tau(t), \frac{\pi}{4}])$, with

$$\sup_t \|L_{h^s}^1(\alpha,\gamma,t)\|_{C_\gamma^{i_0}([-1,1],C_\alpha^{k+2}[-\tau(t),\frac{\pi}{4}])} \lesssim 1, \tag{3.47}$$

$$\sup_t \|L_{h^s}^1(\alpha,\gamma,t) - L_{g^s}^1(\alpha,\gamma,t)\|_{C_\gamma^{i_0}([-1,1],C_\alpha^{k+2}[-\tau(t),\frac{\pi}{4}])} \lesssim \|h-g\|_{W^{i_0,k}}, \tag{3.48}$$

$$\sup_t \|D_h L_{h^s}^1(\alpha,\gamma,t)[g]\|_{C_\gamma^{i_0}([-1,1],C_\alpha^{k+2}[-\tau(t),\frac{\pi}{4}])} \lesssim \|g\|_{W^{i_0,k}}, \tag{3.49}$$

and for any $0 \leq t' < t \leq t*$,

$$\|L_{h^s(\alpha,\gamma,t)}^1(\alpha,\gamma,t) - L_{h^s(\alpha,\gamma,t')}^1(\alpha,\gamma,t')\|_{C_\gamma^{i_0}([-1,1],C^{k+1}[-\tau(t'),\frac{\pi}{4}])} \tag{3.50}$$

$$\lesssim \mathcal{O}(t-t') + \|h(\alpha,\gamma,t) - h(\alpha,\gamma,t')\|_{C_\gamma^{i_0}([-1,1],H^k[-\pi,\pi])}, \tag{3.51}$$

with $|\mathcal{O}(t-t')| \to 0$ as $t \to t'$ independent of h. Here D_h is the Gateaux derivative.

The second type (only one term):

$$T_2^\epsilon(h) = \begin{cases} \tilde{\lambda}(\alpha_\gamma^t) L_{h^s}^2(\alpha,\gamma,t) \int_{-\tau(t)}^{2\alpha+\tau(t)} \frac{(h(\alpha,\gamma,t)-h(\beta,\gamma,t))(\alpha-\beta)\epsilon}{((\alpha-\beta)^2+\epsilon^2)^2} d\beta. & -\tau(t) \leq \alpha \leq \pi \\ 0 & -\pi \leq \alpha < -(\tau(t)) \end{cases}$$

with $L_{h^s}^2$ satisfying the same conditions as $L_{h^s}^1$.

The third type:

$$T_3^{\epsilon,i}(h)=\begin{cases}\tilde{\lambda}(\alpha_\gamma^t)\int_{-\tau(t)}^{2\alpha+\tau(t)}L_{h^s}^{3,i}(\alpha,\gamma,\beta,t)\frac{D^{-20+b_i}h(\beta,\gamma,t)(\alpha-\beta)}{(\alpha-\beta)^2+\epsilon^2}d\beta. & -\tau(t)<\alpha\leq\pi\\ 0 & -\pi\leq\alpha\leq-(\tau(t))\end{cases}$$

satisfies the following condition:

$L_{h^s}^{3,i}$ *can be extended to a function in* $C_\gamma^{i_0}([-1,1],C_{\alpha,\beta}^{k+2}([-\tau(t),\frac{\pi}{4}]\times[-\tau(t),\frac{2}{3}\pi]))$*, and satisfying*

$$\sup_t\|L_{h^s}^{3,i}(\alpha,\beta,\gamma,t)\|_{C_\gamma^{i_0}([-1,1],C_{\alpha,\beta}^{k+2}([-\tau(t),\frac{\pi}{4}]\times[-\tau(t),\frac{2}{3}\pi]))}\lesssim 1, \tag{3.52}$$

$$\sup_t\|L_{h^s}^{3,i}(\alpha,\beta,\gamma,t)-L_{g^s}^{3,i}(\alpha,\beta,\gamma,t)\|_{C_\gamma^{i_0}[-1,1],C_{\alpha,\beta}^{k+2}([-\tau(t),\frac{\pi}{4}]\times[-\tau(t),\frac{2}{3}\pi))}\lesssim\|h-g\|_{W^{i_0,k}}, \tag{3.53}$$

$$\sup_t\|D_hL_{h^s}^{3,i}(\alpha,\beta,\gamma,t)[g]\|_{C_\gamma^{i_0}([-1,1],C_{\alpha,\beta}^{k+2}([-\tau(t),\frac{\pi}{4}]\times[-\tau(t),\frac{2}{3}\pi]))}\lesssim\|g\|_{W^{i_0,k}}, \tag{3.54}$$

and for any $0\leq t'<t\leq t_0$*,*

$$\begin{aligned}&\|L_{h^s(\alpha,\gamma,t)}^{3,i}(\alpha,\beta,\gamma,t)-L_{h^s(\alpha,\gamma,t')}^{3,i}(\alpha,\beta,\gamma,t')\|_{C_\gamma^{i_0}([-1,1],C_{\alpha,\beta}^{k+1}([-\tau(t'),\frac{\pi}{4}]\times[-\tau(t'),\frac{2\pi}{3}]))}\\&\lesssim\mathcal{O}(t-t')+\|h(\alpha,\gamma,t)-h(\alpha,\gamma,t')\|_{C_\gamma^{i_0}([-1,1],H^k[-\pi,\pi])},\end{aligned} \tag{3.55}$$

with $\lim_{x\to0}\mathcal{O}(x)=0$*. Here* $b_i\leq20$*.*

The fourth type:

$$T_4^i(h)=\begin{cases}\tilde{\lambda}(\alpha_\gamma^t)L_{h^s}^{4,i}(\alpha,\gamma,t)D^{-20+b_i}h(\alpha,\gamma,t)\ \textit{or}\ \tilde{\lambda}(\alpha_\gamma^t)L_h^{4,i}(\alpha,\gamma,t)D^{-20+b_i}h(2\alpha+\tau(t),\gamma,t) & -\tau(t)<\alpha\leq\pi\\ 0 & -\pi\leq\alpha\leq-(\tau(t))\end{cases}$$

with $L_{h^s}^{4,i}$ *satisfying the following conditions. Here* $b_i\leq20$*.*

$$\sup_t\|L_{h^s}^{4,i}(\alpha,\gamma,t)\|_{C_\gamma^{i_0}([-1,1],C_\alpha^{k+2}[-\tau(t),\frac{\pi}{4}])}\lesssim1, \tag{3.56}$$

$$\sup_t \|L^{4,i}_{h^s}(\alpha,\gamma,t) - L^{4,i}_{g^s}(\alpha,\gamma,t)\|_{C^{i_0}_\gamma([-1,1],C^{k+2}_\alpha[-\tau(t),\frac{\pi}{4}])} \lesssim \|h-g\|_{W^{i_0,k}}, \tag{3.57}$$

$$\sup_t \|D_h L^1_{h^s}(\alpha,\gamma,t)[g]\|_{C^{i_0}_\gamma([-1,1],C^{k+2}_\alpha[-\tau(t),\frac{\pi}{4}])} \lesssim \|g\|_{W^{i_0,k}}, \tag{3.58}$$

and for any $0 \le t' < t \le t*$,

$$\begin{aligned} &\|L^{4,i}_{h^s(\alpha,\gamma,t)}(\alpha,\gamma,t) - L^{4,i}_{h^s(\alpha,\gamma,t')}(\alpha,\gamma,t')\|_{C^{i_0}_\gamma([-1,1],C^{k+1}_\alpha[-\tau(t'),\frac{\pi}{4}])} \\ &\lesssim \mathcal{O}(t-t') + \|h(\alpha,\gamma,t) - h(\alpha,\gamma,t')\|_{C^{i_0}_\gamma([-1,1],H^k[-\pi,\pi])}. \end{aligned} \tag{3.59}$$

The fifth type:

$$T^i_5(h) = \begin{cases} \tilde{\lambda}(\alpha^t_\gamma)\int_0^\gamma D^{-20+b_i}h(2\alpha+\tau(t),\eta,t)L^{5,i}_{h^s}(\alpha,\gamma,\eta,t)d\eta & -\tau(t) < \alpha \le \pi \\ 0 & -\pi \le \alpha \le -\tau(t) \end{cases}$$

Here $b_i \le 20$. $L^{5,i}_{h^s}$ *satisfies the following condition:*

$$\sup_t \|L^{5,i}_{h^s}(\alpha,\gamma,\eta,t)\|_{C^{i_0}_{\gamma,\eta}([-1,1]\times[-1,1],C^{k+2}_\alpha[-\tau(t),\frac{\pi}{4}])} \lesssim 1, \tag{3.60}$$

and

$$\sup_t \|L^{5,i}_{h^s}(\alpha,\gamma,\eta,t) - L^{5,i}_{g^s}(\alpha,\gamma,\eta,t)\|_{C^{i_0}_{\gamma,\eta}([-1,1]\times[-1,1],C^{k+2}_\alpha[-\tau(t),\frac{\pi}{4}])} \lesssim \|h-g\|_{W^{i_0,k}}, \tag{3.61}$$

$$\sup_t \|D_h L^{5,i}_{h^s}(\alpha,\gamma,\eta,t)[g]\|_{C^{i_0}_{\gamma,\eta}([-1,1]\times[-1,1],C^{k+2}_\alpha[-\tau(t),\frac{\pi}{4}])} \lesssim \|g\|_{W^{i_0,k}}, \tag{3.62}$$

, and for any $0 \le t' < t \le t_*$,

$$\begin{aligned} &\|L^{5,i}_{h^s(\alpha,\gamma,t)}(\alpha,\gamma,\eta,t) - L^{5,i}_{h^s(\alpha,\gamma,t')}(\alpha,\gamma,\eta,t')\|_{C^{i_0}_{\gamma,\eta}([-1,1]\times[-1,1],C^{k+1}_\alpha[-\tau(t'),\frac{\pi}{4}])} \\ &\lesssim \mathcal{O}(t-t') + \|h(\alpha,\gamma,t) - h(\alpha,\gamma,t')\|_{C^{i_0}_\gamma([-1,1],H^k[-\pi,\pi])}. \end{aligned} \tag{3.63}$$

The fixed terms are

$$FT = \begin{cases} \tilde{\lambda}(\alpha_\gamma^t)\sum_j \int_{2\alpha+\tau(t)}^{\pi} L_j(\alpha_\gamma^t,\beta,t)d\beta + \tilde{\lambda}(\alpha_\gamma^t)T_{fixed}(\alpha_\gamma^t,t), & -\tau(t) < \alpha \le \pi \\ 0 & -\pi \le \alpha \le -\tau(t) \end{cases}$$

with $L_j(\alpha_\gamma^t,\beta,t) \in C_\gamma^{i_0}([-1,1], C_{\alpha,\beta}^{k+2}((-\tau(t),\frac{\pi}{4}]\times(2\alpha+\tau(t),\pi]))$*, and for* $0 \le t' \le t < t_0$*,*

$$\sup_t \|L_j(\alpha_\gamma^t,\beta,t)\|_{C_\gamma^{i_0}([-1,1],C_{\alpha,\beta}^{k+2}((-\tau(t),\frac{\pi}{4}]\times(2\alpha+\tau(t),\pi]))} \lesssim 1, \tag{3.64}$$

$$\|L_j(\alpha_\gamma^t,\beta,t) - L_j(\alpha_\gamma^{t'},\beta,t')\|_{C_\gamma^{i_0}([-1,1],C_{\alpha,\beta}^{k+2}((-\tau(t'),\frac{\pi}{4}]\times(2\alpha+\tau(t),\pi]))} \lesssim \mathcal{O}(t-t'). \tag{3.65}$$

$$\sup_t \|T_{fixed}(\alpha_\gamma^t,t)\|_{C_\gamma^{i_0}([-1,1],C_\alpha^{k+2}(-\tau(t),\frac{\pi}{4}])} \lesssim 1, \tag{3.66}$$

$$\|T_{fixed}(\alpha_\gamma^t,t) - T_{fixed}(\alpha_\gamma^{t'},t')\|_{C_\gamma^{i_0}([-1,1],C_\alpha^{k+1}(-\tau(t'),\frac{\pi}{4}])} \lesssim \mathcal{O}(t-t'). \tag{3.67}$$

For $l'+l'' \le k+2$*,* $l \le i_0$*, we have*

$$\lim_{\alpha\to-\tau(t)} (\frac{d}{d\alpha})^{l''}((\partial_\alpha^{l'}\partial_\gamma^{l})L_j(\alpha_\gamma^t, 2\alpha+\tau(t),t)) = 0. \tag{3.68}$$

Here fixed terms not only include $T_{fixed}(\alpha_\gamma^t,t)$ *which only depends on* $\tilde{f}^L$*, but also include the terms depending on* $\tilde{f}^+$ *on the real line.*

Proof. We will only show $\lambda(\alpha_\gamma^t)M_1^\epsilon$, $-\lambda(\alpha_\gamma^t)M_2^\epsilon$, $\lambda(\alpha_\gamma^t)O^{1,i,\epsilon}$, $\lambda(\alpha_\gamma^t)O^{3,i,\epsilon}$, and $\lambda(\alpha_\gamma^t)O^{5,i,\epsilon}$. From (3.38), (3.39), (3.40), we claim that $O^{2,i,\epsilon}$ and $O^{4,i,\epsilon}$ have the similar form of $O^{1,i,\epsilon}$. The sub-types of L_{h^s} used below and why they satisfy the conditions are explained in the lemma 3.7.3, 3.7.4, 3.7.6, 3.7.7, 3.7.8, 3.7.9 and 3.7.10 .

From (3.35), $\lambda(\alpha_\gamma^t)M_1^\epsilon$ is of the type T_2^ϵ. From (3.41), $\lambda(\alpha_\gamma^t)O^{3,i,\epsilon}$ is in T_4 with sub-type 1. From

(3.42), $\lambda(\alpha_\gamma^t)O^{5,i,\epsilon}$ is in T_4 with sub type 6. From (3.37), we also have

$$-\lambda(\alpha_\gamma^t)M_2^\epsilon(h) = \underbrace{(-\lambda(\alpha_\gamma^t)Term_1)}_{T_4,\text{ sub-type 2}} + \underbrace{(-\lambda(\alpha_\gamma^t)Term_2)}_{\text{a sum of terms of } T_4,\text{ sub-type 2}} + \underbrace{(-\lambda(\alpha_\gamma^t)Term_{3,1}^\epsilon)}_{\lambda(\alpha_\gamma^t)T_1^\epsilon} + \underbrace{(-\lambda(\alpha_\gamma^t)Term_{3,2,1}^\epsilon)}_{T_3^\epsilon\text{ sub-type 1}}$$

$$+ \underbrace{-(\lambda(\alpha_\gamma^t)Term_{3,2,2})}_{T_4\text{ sub-type 5}} + \underbrace{(-(\lambda(\alpha_\gamma^t)Term_4)}_{T_4,\text{ sub-type 2}} + \underbrace{(-(\lambda(\alpha_\gamma^t)Term_{5,1})}_{T_5,\text{ sub-type 1}} + \underbrace{(-(\lambda(\alpha_\gamma^t)Term_{5,2,1})}_{\text{fixed terms, sub-type 1}} + \underbrace{(-(\lambda(\alpha_\gamma^t)Term_{5,2,2})}_{T_4,\text{ sub-type 3}}.$$

and

$$\lambda(\alpha_\gamma^t)O^{1,i,\epsilon} = \underbrace{\lambda(\alpha_\gamma^t)Term_1}_{T_4,\text{ sub-type 1}} + \underbrace{\lambda(\alpha_\gamma^t)Term_{2,1}^\epsilon}_{T_3^\epsilon,\text{ sub-type 2}} + \underbrace{\lambda(\alpha_\gamma^t)Term_{2,2}}_{T_5,\text{ sub-type 2}} + \underbrace{\lambda(\alpha_\gamma^t)Term_{2,3,1}}_{\text{fixed terms, sub-type 2}} + \underbrace{\lambda(\alpha_\gamma^t)Term_{2,3,2}}_{T_4^\epsilon,\text{ sub-type 4}}.$$

Before we show the sub-types, we introduce some notations. A function $K_{-\sigma}^j(A,B,C)$ is of $-\sigma$ type if for A, B, C in R^n, it has the form

$$c_j \frac{\sin(A_1+B_1)^{m_1}\cos(A_1+B_1)^{m_2}}{(\cosh(A_2+B_2)-\cos(A_1+B_1))^{m_0}} \tag{3.69}$$

$$\times(\sinh(A_2+B_2))^{m_3}(\cosh(A_2+B_2))^{m_4}\Pi_{j=1}^{m_5}(A_{\lambda_j})\Pi_{j=1}^{m_6}(B_{\lambda_{j,2}})\Pi_{j=1}^{m_7}(C_{\lambda_{j,3}}), \tag{3.70}$$

with $m_1+m_3+m_5+m_6+m_7-2m_0 \geq -\sigma$. c_j is a constant.

We use $\Delta g(\alpha,\gamma,t)$ to show $g(\alpha,\gamma,t)-g(\beta,\gamma,t)$ and notation in (3.11), (3.12), (3.13), and (3.27). Here we take $A=\Delta V_h^{10}(\alpha,\gamma,t)$, $B=\Delta V_{\tilde{f}^L}^{21}(\alpha_\gamma^t,t)$, $C=\Delta V_X(\alpha_\gamma^t,t)$. For example,

$$K_{-\sigma}^j(\Delta V_h^{10}(\alpha,\gamma,t),\Delta V_{\tilde{f}^L}^{21}(\alpha_\gamma^t,t),\Delta V_X(\alpha_\gamma^t,t)) =$$
$$K_{-\sigma}^j(V_h^{10}(\alpha,\gamma,t)-V_h^{10}(\beta,\gamma,t),V_{\tilde{f}^L}^{21}(\alpha_\gamma^t,t)-V_{\tilde{f}^L}^{21}(\beta_\gamma^t,t),V_X(\alpha_\gamma^t,t)-V_X(\beta_\gamma^t,t)).$$

We then use this notation to write L_{h^s} into different sub-forms and show they satisfy the properties in lemma 3.7.2.

Lemma 3.7.3. *L_{h^s} in T_1^ϵ is*

$$L_{h^s}^1(\alpha,\gamma,t) = -\lim_{\beta\to\alpha}(\frac{d}{d\beta}K_{-1}^j(\Delta V_h^{10}(\alpha,\gamma,t),\Delta V_{\tilde{f}^L}^{21}(\alpha_\gamma^t,t),\Delta V_X(\alpha_\gamma^t,t))(\alpha-\beta)^2)$$

with particular $K_{-1}^j(A,B,C)=\frac{\sin(A_1+B_1)}{\cosh(A_2+B_2)-\cos(A_1+B_1)}$. Moreover, it satisfies the conditions (3.46), (3.47), (3.48), (3.49), *and* (3.50).

Proof. Since we have

$$\frac{d}{d\beta}K_{-\sigma}(V_h^{10}(\alpha,\gamma,t)-V_h^{10}(\beta,\gamma,t),V_{\tilde{f}^L}^{21}(\alpha_\gamma^t,t)-V_{\tilde{f}^L}^{21}(\beta_\gamma^t,t),V_X(\alpha_\gamma^t,t)-V_X(\beta_\gamma^t,t)) \tag{3.71}$$
$$=\sum_{l'}K_{-(\sigma+1)}^{l'}(V_h^{10}(\alpha,\gamma,t)-V_h^{10}(\beta,\gamma,t),V_{\tilde{f}^L}^{21}(\alpha_\gamma^t,t)-V_{\tilde{f}^L}^{21}(\beta_\gamma^t,t),V_X(\alpha_\gamma^t,t)-V_X(\beta_\gamma^t,t))$$
$$G_{l'}(\partial_\beta(V_h^{10}(\beta,\gamma,t)),\partial_\beta(V_{\tilde{f}^L}^{21}(\beta_\gamma^t,t)),\partial_\beta(V_X(\beta_\gamma^t,t))),$$

with

$$\|G_{l'}(\partial_\beta(V_h^{10}(\beta,\gamma,t)),\partial_\beta(V_{\tilde{f}^L}^{21}(\beta_\gamma^t,t)),\partial_\beta(V_X(\beta_\gamma^t,t)))\|_{C_\gamma^{i_0}([-1,1],C^{k+3}[-\tau(t),\pi])}\lesssim 1.$$

The conditions (3.47), (3.48), (3.49), (3.50) follows from lemma 3.7.21 (the behavior of $K^{-\sigma}$ when $\beta>-\tau(t)$). The (3.46) relies on the particular kernel. We have

$$L_{h^s}^1(\alpha,\gamma,t)=-\lim_{\beta\to\alpha}(\frac{d}{d\beta}(\frac{\sin(A_1+B_1)}{\cosh(A_2+B_2)-\cos(A_1+B_1)})(\alpha-\beta)^2),$$

with $A_\mu=D^{-20}h_\mu(\alpha,\gamma,t)-D^{-20}h_\mu(\beta,\gamma,t)$, $B_\mu=\tilde{f}_\mu^L(\alpha,\gamma,t)-\tilde{f}_\mu^L(\beta,\gamma,t)$. Then

$$\begin{aligned}&\frac{d}{d\beta}(\frac{\sin(A_1+B_1)}{\cosh(A_2+B_2)-\cos(A_1+B_1)})(\alpha-\beta)^2\\&=\frac{\cos(A_1+B_1)(\alpha-\beta)^2}{\cosh(A_1+B_1)-\cos(A_1+B_1)}\frac{d}{d\beta}(A_1+B_1)\\&-\frac{\sin(A_1+B_1)(\sinh(A_2+B_2)\frac{d}{d\beta}(A_2+B_2)+\sin(A_1+B_1)\frac{d}{d\beta}(A_1+B_1))}{(\cosh(A_1+B_1)-\cos(A_2+B_2))^2}(\alpha-\beta)^2.\end{aligned}$$

From $\partial_\alpha\tilde{f}(0,t)=0$, $\partial_\alpha D^{-20}h(-\tau(t),\gamma,t)=0$, we have

$$\lim_{\alpha\to-\tau(t)^+}\lim_{\alpha\to\beta}\frac{\sin(A_1+B_1)}{(\alpha-\beta)}=\lim_{\alpha\to-\tau(t)^+}\lim_{\alpha\to\beta}\frac{A_1+B_1}{(\alpha-\beta)}=\frac{d}{d\alpha}\tilde{f}(-\tau(t)_\gamma^t,t)=0.$$

Then we get the condition (3.46). □

Lemma 3.7.4. *$L_{h^s}^2$ in T_2^ϵ is*

$$\begin{aligned}&L_{h^s}^2(\alpha,\gamma,t)=\frac{2}{\pi}B_{D^{-20}(h)}(\alpha,\gamma,t)=\frac{2}{\pi}[\frac{1}{1+ic'(\alpha+\tau(t))\gamma t}(ic(\alpha+\tau(t))\gamma+ic'(\alpha+\tau(t))\gamma t\tau'(t)+\frac{\partial_t x}{\partial_\alpha x}(\alpha_\gamma^t)\\&+\frac{1}{(\partial_\alpha x(\alpha_\gamma^t))}\cdot p.v.\int_{-\pi}^{\pi}K(D^{-20}(h)(\alpha,\gamma,t)-D^{-20}(h)(\beta,\gamma,t)+\tilde{f}^L(\alpha_\gamma^t)-\tilde{f}^L(\beta_\gamma^t))\\&\frac{dx(\beta_\gamma^t)}{d\beta}d\beta-\frac{\partial_t x}{\partial_\alpha x}(0)-\frac{1}{\partial_\alpha x(0)}p.v\int_{-\pi}^{\pi}K(D^{-20}(h)(-\tau(t),\gamma,t)-D^{-20}(h)(\beta,\gamma,t)+\tilde{f}^L(0)-\tilde{f}^L(\beta_\gamma^t))\frac{dx(\beta_\gamma^t)}{d\beta}d\beta)].\end{aligned}$$

and satisfies the conditions (3.46), (3.47), (3.48), (3.49).

Proof. Because $c(\alpha) \in C^{100}[0,\infty)$ with $c(0) = c'(0) = 0$, $x(\alpha_\gamma^t)$ is sufficiently smooth. Then we only need to show the next lemma 3.7.5. □

Lemma 3.7.5. *If*

$$L_{h^s}(\alpha,\gamma,t) = p.v.\int_{-\pi}^{\pi} K_{-1}^i(\Delta V_h^{10}(\alpha,\gamma,t), \Delta V_{\tilde{f}^L}^{21}(\alpha_\gamma^t,t), \Delta V_X(\alpha_\gamma^t,t))X_i(\beta_\gamma^t,t)(1+ic'(\beta+\tau(t))\gamma t)d\beta,$$

for some K_{-1} type kernel, then it satisfies (3.47), (3.48), (3.49), (3.50) *and*

$$lim_{\alpha\to-\tau(t)^+} L_{h^s}(\alpha,\gamma,t) = L_{h^s}(-\tau(t),\gamma,t). \tag{3.72}$$

Proof. First, we have

$$\begin{aligned}
&L_{h^s}(\alpha,\gamma,t) = p.v.\int_{-\pi}^{\pi} K_{-1}^i(0, \Delta V_{\tilde{f}^L}^{21}(\alpha_\gamma^t,t), \Delta V_X(\alpha_\gamma^t,t))X_i(\beta_\gamma^t,t)(1+ic'(\beta+\tau(t))\gamma t)d\beta\\
&+ p.v.\int_{-\pi}^{\pi}\int_0^1 \frac{d}{d\zeta}K_{-1}^i(\zeta\Delta V_h^{10}(\alpha,\gamma,t), \Delta V_{\tilde{f}^L}^{21}(\alpha_\gamma^t,t), \Delta V_X(\alpha_\gamma^t,t))X_i(\beta_\gamma^t,t)(1+ic'(\beta+\tau(t))\gamma t)d\beta\\
&= Term_1 + Term_2.
\end{aligned}$$

For the $Term_1$, we first show

$$lim_{\alpha\to-\tau(t)^+} Term_1(\alpha,\gamma,t) = Term_1(-\tau(t),\gamma,t). \tag{3.73}$$

Since

$$\begin{aligned}
Term_1 &= p.v.\int_{-\pi}^{\pi} K_{-1}^i(0, \Delta V_{\tilde{f}^L}^{21}(\alpha_\gamma^t,t), \Delta V_X(\alpha_\gamma^t,t))X_i(\beta_\gamma^t,t)(1+ic'(\beta+\tau(t))\gamma t)d\beta\\
&= \lim_{\beta\to\alpha}(K_{-1}^i(0, \Delta V_{\tilde{f}^L}^{21}(\alpha_\gamma^t,t), \Delta V_X(\alpha_\gamma^t,t))(\alpha-\beta))X_i(\alpha_\gamma^t,t)(1+ic'(\alpha+\tau(t))\gamma t)p.v.\int_{-\pi}^{\pi}\frac{1}{\alpha-\beta}d\beta\\
&+ \int_{-\pi}^{\pi}(K_{-1}^i(0, \Delta V_{\tilde{f}^L}^{21}(\alpha_\gamma^t,t), \Delta V_X(\alpha_\gamma^t,t))(\alpha-\beta)X_i(\beta_\gamma^t,t)(1+ic'(\beta+\tau(t))\gamma t)\\
&- \lim_{\beta\to\alpha}(K_{-1}^i(0, \Delta V_{\tilde{f}^L}^{21}(\alpha_\gamma^t,t), \Delta V_X(\alpha_\gamma^t,t))(\alpha-\beta))X_i(\alpha_\gamma^t,t)(1+ic'(\alpha+\tau(t))\gamma t))\frac{1}{\alpha-\beta}d\beta
\end{aligned}$$

From lemma 3.7.20, we could use the Lebesgue convergence theorem to get (3.73). Now we show $Term_1$ satisfying condition (3.47) and (3.50). Since $Term_1$ dose not depend on h, the other condition are satisfied automatically. From (3.8), we further separate $\tilde{f}^L(\alpha,t)$ into the part that is polynomial

in $[-\frac{\delta}{2}, -\frac{\delta}{2}]$ and the part that vanishes up to 59th order at 0, which is

$$\tilde{f}^{-}(\alpha,t) = (\tilde{f}(\alpha,t) - \sum_{i=1}^{60} \frac{\tilde{f}^{<i>}(0)}{i!}\alpha^i)\lambda(\alpha)1_{\alpha\leq 0}, \tag{3.74}$$

and

$$\tilde{f}^{L}(\alpha,t) = \tilde{f}^{L,L}(\alpha,t) + \tilde{f}^{-}(\alpha,t).$$

Here $\tilde{f}^{-}(\alpha,t) \in C^{59}(-\infty,\infty)$ and when $\alpha > 0$,

$$\tilde{f}^{-}(\alpha,t) = 0,$$

also for $j \leq 59$,

$$\partial_\alpha^j \tilde{f}^{-}(0,t) = 0.$$

$\tilde{f}^{L,L}(\alpha,t)$ is a polynomial in $[-\frac{\delta}{2}, -\frac{\delta}{2}]$. Moreover, from (3.13), we have

$$X_i(\alpha,t) = (\partial_\alpha^{l_i}\tilde{f}^{L}(\alpha,t))^{\phi_i}(\partial_\alpha^{l_i'}(\frac{1}{\frac{dx}{d\alpha}(\alpha,t)}))^{\phi_i'}(\partial_\alpha^{l_i''}\frac{d}{d\alpha}x(\alpha,t))^{\phi_{i''}}, \tag{3.75}$$

with $l_i, l_i', l_1'' \leq 21$, $\phi_i, \phi_i', \phi_{i''} \in 0, 1$. Then we can also separate X_i as $X_i^L + X_i^-$, with

$$X_i^L = \begin{cases} \partial_\alpha^{l_i}\tilde{f}^{L,L}(\alpha,t)(\partial_\alpha^{l_i'}(\frac{1}{\frac{dx}{d\alpha}(\alpha,t)}))^{\phi_i'}(\partial_\alpha^{l_i''}\frac{d}{d\alpha}x(\alpha,t))^{\phi_{i''}} & \phi_i = 1, \\ (\frac{1}{\frac{dx}{d\alpha}(\alpha,t)})^{\phi_i'}(\partial_\alpha^{l_i''}\frac{d}{d\alpha}x(\alpha,t))^{\phi_{i''}} & \phi_i = 0, \end{cases} \tag{3.76}$$

$$X_i^- = \begin{cases} \partial_\alpha^{l_i}\tilde{f}^{-}(\alpha,t)(\partial_\alpha^{l_i'}(\frac{1}{\frac{dx}{d\alpha}(\alpha,t)}))^{\phi_i'}(\partial_\alpha^{l_i''}\frac{d}{d\alpha}x(\alpha,t))^{\phi_{i''}} & \phi_i = 1, \\ 0 & \phi_i = 0. \end{cases} \tag{3.77}$$

Then we rewrite $Term_1$, we have

$$\begin{aligned} Term_1 &= p.v.\int_{-\pi}^{\pi} K_{-1}^i(0, \Delta V_{\tilde{f}^{L,L}}(\alpha_\gamma^t,t) + \Delta V_{\tilde{f}^{-}}(\alpha_\gamma^t,t), \Delta V_{X^L}(\alpha_\gamma^t,t) + \Delta V_{X^-}(\alpha_\gamma^t,t)) \\ &\cdot (X_i^L(\beta_\gamma^t,t) + X_i^-(\beta_\gamma^t,t))(1 + ic'(\beta+\tau(t))\gamma t)d\beta \\ &= p.v.\int_{-\pi}^{\pi} K_{-1}^i(0, \Delta V_{\tilde{f}^{L,L}}(\alpha_\gamma^t,t), \Delta V_{X^L}(\alpha_\gamma^t,t)) \cdot (X_i^L(\beta_\gamma^t,t))(1 + ic'(\beta+\tau(t))\gamma t)d\beta \\ &+ \int_0^1 d\zeta p.v.\int_{-\pi}^{\pi} \frac{d}{d\zeta}(K_{-1}^i(0, \Delta V_{\tilde{f}^{L,L}}(\alpha_\gamma^t,t) + \zeta\Delta V_{\tilde{f}^{-}}(\alpha_\gamma^t,t), \Delta V_{X^L}(\alpha_\gamma^t,t) + \zeta\Delta V_{X^-}(\alpha_\gamma^t,t)) \end{aligned}$$

$$\cdot (X_i^L(\beta_\gamma^t,t)+\zeta X_i^-(\beta_\gamma^t,t)))(1+ic'(\beta+\tau(t))\gamma t)d\beta$$
$$= Term_{1,1}+Term_{1,2}.$$

Since the component of $V_{\tilde{f}^{L,L}}(\alpha_\gamma^t,t)$ and $V_{X^L}(\alpha_\gamma^t,t)$ are analytic in $\tilde{D}=\{\alpha+iy||\alpha|\le\frac{\delta}{4}\}$. We want to change the contour of $Term_{1,1}$ when $\beta\le-\tau(t)$. Let

$$\tilde{c}(\alpha)=\begin{cases} c(\alpha) & \alpha\ge 0,\\ c(-\alpha) & \alpha\le 0,\end{cases} \tag{3.78}$$

and

$$\tilde{\alpha}_\gamma^t=\alpha+\tau(t)+i\tilde{c}(\alpha+\tau(t))\gamma t. \tag{3.79}$$

Since by the definition $c(\alpha)=\alpha^2$ when α is sufficiently small, we have $\tilde{c}(\alpha)\in C^{60}(-\infty,\infty)$. By the arc-chord condition (3.93), there is no zero point of denominator when $\beta\le-\tau(t)$, $\alpha>-\tau(t)$, therefore we could change the contour, and have

$$Term_{1,1}=p.v.\int_{-\pi}^{\pi}K_{-1}^i(0,\Delta V_{\tilde{f}^{L,L}}(\tilde{\alpha}_\gamma^t,t),\Delta V_{X^L}(\tilde{\alpha}_\gamma^t,t))X_i^L(\tilde{\beta}_\gamma^t,t)(1+i\tilde{c}'(\beta+\tau(t))\gamma t)d\beta.$$

Since from (3.15) the components in $V_{\tilde{f}^{L,L}}(\alpha,t)$ and $V_{X^L}(\alpha,t)$ are also in $C_t^0([0,1],C^{38}(-\infty,\infty))$, then when $k+i_0\le 30$, $Term_{1,1}$ satisfies the smoothness condition (3.47) and the continuity in t condition (3.50). For $Term_{1,2}$, we have

$$\frac{d}{d\zeta}(K_{-1}^i(0,\Delta V_{\tilde{f}^{L,L}}(\alpha_\gamma^t,t)+\zeta\Delta V_{\tilde{f}^-}(\alpha_\gamma^t,t),\Delta V_{X^L}(\alpha_\gamma^t,t)+\zeta\Delta V_{X^-}(\alpha_\gamma^t,t))(X_i^L(\beta_\gamma^t,t)+\zeta X_i^-(\beta_\gamma^t,t)))$$
$$= K_{-1}^i(0,\Delta V_{\tilde{f}^{L,L}}(\alpha_\gamma^t,t)+\zeta\Delta V_{\tilde{f}^-}(\alpha_\gamma^t,t),\Delta V_{X^L}(\alpha_\gamma^t,t)+\zeta\Delta V_{X^-}(\alpha_\gamma^t,t))\partial_\beta^{l_i}\tilde{f}^-(\beta_\gamma^t,t)$$
$$\cdot\phi_i\cdot((\partial_\beta^{l_i'}\frac{1}{\partial_\beta x})(\beta_\gamma^t,t))^{\phi_i'}((\partial_\beta^{l_i''}\partial_\beta x)(\beta_\gamma^t,t))^{\phi_i''}$$
$$+\sum_{j'}K_{-2}^{j'}(0,\Delta V_{\tilde{f}^{L,L}}(\alpha_\gamma^t,t)+\zeta\Delta V_{\tilde{f}^-}(\alpha_\gamma^t,t),\Delta V_{X^L}(\alpha_\gamma^t,t)+\zeta\Delta V_{X^-}(\alpha_\gamma^t,t))(\partial_\alpha^{l_j'}\tilde{f}^-(\alpha_\gamma^t,t)-\partial_\beta^{l_j'}\tilde{f}^-(\beta_\gamma^t,t))$$
$$\cdot(X_i^L(\beta_\gamma^t,t)+\zeta X_i^-(\beta_\gamma^t,t))$$
$$+\sum_{n}K_{-2}^{n}(0,\Delta V_{\tilde{f}^{L,L}}(\alpha_\gamma^t,t)+\zeta\Delta V_{\tilde{f}^-}(\alpha_\gamma^t,t),\Delta V_{X^L}(\alpha_\gamma^t,t)+\zeta\Delta V_{X^-}(\alpha_\gamma^t,t))$$
$$(\phi_n\cdot\partial_\alpha^{l_n}\tilde{f}^-(\alpha_\gamma^t,t)((\partial_\alpha^{l_n'}\frac{1}{\partial_\alpha x})(\alpha_\gamma^t,t))^{\phi_n'}((\partial_\alpha^{l_n''}\partial_\alpha x)(\alpha_\gamma^t,t))^{\phi_n''}-\phi_n\cdot\partial_\beta^{l_i}\tilde{f}^-(\beta_\gamma^t,t)((\partial_\beta^{l_n'}\frac{1}{\partial_\beta x})(\beta_\gamma^t,t))^{\phi_n'}((\partial_\beta^{l_n''}\partial_\beta x)(\beta_\gamma^t,t))^{\phi_n''})$$
$$\cdot(X_i^L(\beta_\gamma^t,t)+\zeta X_i^-(\beta_\gamma^t,t))$$

$$= KTerm_{1,2,1} + KTerm_{1,2,2} + KTerm_{1,2,3},$$

for some -2 type functions $K^{j'}_{-2}$ and K^{n}_{-2}, with all index of derivative less than 21, $\phi, \phi'_i, \phi'_i, \phi_n, \phi'_n, \phi''_n \in \{0,1\}$. Here the first term shows up when the derivative hits on $(X^L_i(\beta^t_\gamma, t) + \zeta X^-_i(\beta^t_\gamma, t))$, the second one shows up when it hits on $\zeta \Delta V_{\tilde{f}^-}(\alpha^t_\gamma, t)$, the last one shows up when it hits on $\zeta \Delta V_{X^-}(\alpha^t_\gamma, t)$. Then the term becomes

$$\begin{aligned}
&Term_{1,2}\\
&= Term_{1,2,1} + Term_{1,2,2} + Term_{1,2,3}\\
&= \int_0^1 d\zeta p.v. \int_{-\pi}^{\pi} KTerm_{1,2,1} d\beta + \int_0^1 d\zeta p.v. \int_{-\pi}^{\pi} KTerm_{1,2,2} d\beta + \int_0^1 d\zeta p.v. \int_{-\pi}^{\pi} KTerm_{1,2,3} d\beta.
\end{aligned}$$

For $Term_{1,2,1}$, we have

$$\begin{aligned}
&Term_{1,2,1}\\
&= \int_0^1 d\zeta p.v. \int_{-\pi}^{\pi} K^i_{-1}(0, \Delta V_{\tilde{f}^{L,L}}(\alpha^t_\gamma, t) + \zeta \Delta V_{\tilde{f}^-}(\alpha^t_\gamma, t), \Delta V_{X^L}(\alpha^t_\gamma, t) + \zeta \Delta V_{X^-}(\alpha^t_\gamma, t))\\
&\cdot \partial^{l_i}_\beta \tilde{f}^-(\beta^t_\gamma, t) \cdot \phi_i \cdot ((\partial^{l'_i}_\beta \frac{1}{\partial_\beta x})(\beta^t_\gamma, t))^{\phi'_i} ((\partial^{l''_i}_\beta \partial_\beta x)(\beta^t_\gamma, t))^{\phi''_i} d\beta\\
&= \int_0^1 d\zeta p.v. \int_{-\pi}^{-\tau(t)} K^i_{-1}(0, \Delta V_{\tilde{f}^{L,L}}(\alpha^t_\gamma, t) + \zeta \Delta V_{\tilde{f}^-}(\alpha^t_\gamma, t), \Delta V_{X^L}(\alpha^t_\gamma, t) + \zeta \Delta V_{X^-}(\alpha^t_\gamma, t))\\
&\cdot \partial^{l_i}_\beta \tilde{f}^-(\beta^t_\gamma, t) \cdot \phi_i \cdot ((\partial^{l'_i}_\beta \frac{1}{\partial_\beta x})(\beta^t_\gamma, t))^{\phi'_i} ((\partial^{l''_i}_\beta \partial_\beta x)(\beta^t_\gamma, t))^{\phi''_i} d\beta,
\end{aligned}$$

where we use $\tilde{f}^-(\beta^t_\gamma, t) = 0$ when $\beta > -\tau(t)$ and $c(\beta + \tau(t)) = 0$ when $\beta < -\tau(t)$.

Since

$$(\partial^j_\beta \tilde{f}^-)(\beta + \tau(t), t))|_{\beta = -\tau(t)} = 0$$

when $j \leq 38$, from lemma 3.9.19, we have

$$\| \frac{\partial^{l_i}_\beta \tilde{f}^-(\beta + \tau(t), t)}{(\beta + \tau(t))^l} \|_{H^{38-l'}} \lesssim 1.$$

Then by $k + i_0 \leq 30$, from lemma 3.7.28, we have (3.47) and (3.50) for $Term_{1,2,1}$.

Since from (3.74), $\tilde{f}^-(\alpha^t_\gamma, t) = 0$ when $\alpha \geq -\tau(t)$, then $\partial^{l'_i}_\alpha \tilde{f}^-(\alpha^t_\gamma, t)$ and $\partial^{l'_n}_\alpha \tilde{f}^-(\alpha^t_\gamma, t)$ disappear in $Term_{1,2,2}$ and $Term_{1,2,3}$. Moreover, from (3.77), (3.76), and (3.15) we have

$$(X^L_i(\beta^t_\gamma, t) + \zeta X^-_i(\beta^t_\gamma, t)) \in C^{i_0}_\gamma([-1,1], H^{30}[-\pi, \tau(t)])$$

when $i_0 \leq 4$. Hence this two terms can be done in the same way by lemma 3.7.28.

For $Term_2$, we first claim that

$$\begin{aligned}
&\frac{d}{d\zeta}K^i_{-1}(\zeta\Delta V_h^{10}(\alpha,\gamma,t),\Delta V^{21}_{\tilde{f}^L}(\alpha^t_\gamma,t),\Delta V_X(\alpha^t_\gamma,t))\\
&=\sum_l K^l_{-2}(\zeta\Delta V_h^{10}(\alpha,\gamma,t),\Delta V^{21}_{\tilde{f}^L}(\alpha^t_\gamma,t),\Delta V_X(\alpha^t_\gamma,t))(D^{-20+s_l}h(\alpha,\gamma,t)-D^{-20+s_l}h(\beta,\gamma,t)),
\end{aligned}$$

with $s_l \leq 10$ for some -2 type K^l_{-2}. Therefore we have

$$\begin{aligned}
&p.v.\int_{-\pi}^{\pi}\int_0^1 \frac{d}{d\zeta}K^i_{-1}(\Delta\zeta V_h^{10}(\alpha,\gamma,t),\Delta V^{21}_{\tilde{f}^L}(\alpha^t_\gamma,t),\Delta V_X(\alpha^t_\gamma,t))X^i(\beta^t_\gamma,t)(1+ic'(\beta+\tau(t))\gamma t)d\beta\\
&=\sum_l\int_{-\pi}^{-\tau(t)} K^l_{-2}(\zeta\Delta V_h^{10}(\alpha,\gamma,t),\Delta V^{21}_{\tilde{f}^L}(\alpha^t_\gamma,t),\Delta V_X(\alpha^t_\gamma,t))(1+ic'(\beta+\tau(t))\gamma t)X^i(\beta^t_\gamma,t)d\beta D^{-20+s_l}h(\alpha,\gamma,t)\\
&+\sum_l p.v.\int_{-\tau(t)}^{\pi} K^l_{-2}(\zeta\Delta V_h^{10}(\alpha,\gamma,t),\Delta V^{21}_{\tilde{f}^L}(\alpha^t_\gamma,t),\Delta V_X(\alpha^t_\gamma,t))X^i(\beta^t_\gamma,t)(D^{-20+s_l}h(\alpha,\gamma,t)-D^{-20+s_l}h(\beta,\gamma,t))\\
&(1+ic'(\beta+\tau(t))\gamma t)d\beta\\
&=Term_{2,1}+\int_{-\tau(t)}^{\pi}(K^l_{-2}(\zeta\Delta V_h^{10}(\alpha,\gamma,t),\Delta V^{21}_{\tilde{f}^L}(\alpha^t_\gamma,t),\Delta V_X(\alpha^t_\gamma,t))(\alpha-\beta)^2)\\
&\frac{(D^{-20+s_l}h(\alpha,\gamma,t)-D^{-20+s_l}h(\beta,\gamma,t)-(\alpha-\beta)\frac{d}{d\alpha}D^{-20+s_l}h(\alpha,\gamma,t))(1+ic'(\beta+\tau(t))\gamma t)}{(\alpha-\beta)^2}X^i(\beta^t_\gamma,t)d\beta\\
&+\frac{d}{d\alpha}D^{-20+s_l}h(\alpha,\gamma,t)p.v.\int_{-\tau(t)}^{\pi}(K^l_{-2}(\zeta\Delta V_h^{10}(\alpha,\gamma,t),\Delta V^{21}_{\tilde{f}^L}(\alpha^t_\gamma,t),\Delta V_X(\alpha^t_\gamma,t))(\alpha-\beta)^2)(1+ic'(\beta+\tau(t))\gamma t)\\
&X^i(\beta^t_\gamma,t)\frac{1}{\alpha-\beta}d\beta\\
&=Term_{2,1}+Term_{2,2}+Term_{2,3}.
\end{aligned}$$

From lemma 3.9.19, lemma 3.7.14, we have $\|\frac{\partial^l_\alpha D^{-20+s_l}h(\alpha,\gamma,t)}{(\alpha+\tau(t))^{k-l+20-s_l}}\|_{L^2_\alpha[-\tau(t),\pi]} \lesssim \|h\|_{H^k_\alpha[-\tau(t),\pi]}$. Since $s_l \leq 10$, by corollary 3.7.26 (the behavior of $K_{-\sigma}$ when $\beta < -\tau(t)$) and $i_0 \leq 4$, then we have $Term_{2,1}$ satisfies the conditions (3.47), (3.48), (3.49), (3.50) and

$$\lim_{\alpha\to-\tau(t)^+} Term_{2,1} = Term_{2,1}|_{\alpha=-\tau(t)}.$$

From corollary 3.7.14, we have $D^{-20+s_l}h \in H^{k+20-s_l}_\alpha[-\tau(t),\pi]$. Since $s_l \leq 10$, from lemma 3.7.21 (the behavior of $K^{-\sigma}$ when $\beta > -\tau(t)$), $Term_{2,2}$ satisfies the conditions (3.47), (3.48), (3.49), (3.50), and

$$\lim_{\alpha\to-\tau(t)^+} Term_{2,2} = Term_{2,2}|_{\alpha=\tau(t)}.$$

For the last term, we further take the most singular term and get

$$
\begin{aligned}
T_{2,3} &= \frac{d}{d\alpha} D^{-20+s_l} h(\alpha,\gamma,t) p.v. \int_{-\tau(t)}^{\pi} [(K^l_{-2}(\zeta \Delta V_h^{10}(\alpha,\gamma,t), \Delta V_{\tilde{f}^L}^{21}(\alpha^t_\gamma,t), \Delta V_X(\alpha^t_\gamma,t))(\alpha-\beta)^2) \\
&\cdot (1+ic'(\beta+\tau(t))\gamma t) X^i(\beta^t_\gamma,t) \\
&- \lim_{\beta\to\alpha}((K^l_{-2}(\zeta \Delta V_h^{10}(\alpha,\gamma,t), \Delta V_{\tilde{f}^L}^{21}(\alpha^t_\gamma,t), \Delta V_X(\alpha^t_\gamma,t))(\alpha-\beta)^2)) X^i(\alpha^t_\gamma,t)(1+ic'(\alpha+\tau(t))\gamma t)] \\
&\cdot \frac{1}{\alpha-\beta} d\beta \\
&+ \frac{d}{d\alpha} D^{-20+s_l} h(\alpha,\gamma,t) \lim_{\beta\to\alpha} (K^l_{-2}(\zeta \Delta V_h^{10}(\alpha,\gamma,t), \Delta V_{\tilde{f}^L}^{21}(\alpha^t_\gamma,t), \Delta V_X(\alpha^t_\gamma,t))(\alpha-\beta)^2) X^i(\alpha^t_\gamma,t) \\
&\cdot (1+ic'(\alpha+\tau(t))\gamma t) p.v. \int_{-\tau(t)}^{\pi} \frac{1}{\alpha-\beta} d\beta \\
&= T_{2,3,1} + T_{2,3,2}.
\end{aligned}
$$

$T_{2,3,1}$ can be bounded in the similar way as in $T_{2,2}$ and then it satisfies the conditions (3.47), (3.48), (3.49), (3.50), and

$$\lim_{\alpha\to-\tau(t)^+} Term_{2,3,1} = Term_{2,3,1}|_{\alpha=\tau(t)}.$$

From lemma 3.9.19, lemma 3.7.14, we have $\|\frac{\partial_\alpha^l \partial_\alpha D^{-20+s_l} h(\alpha,\gamma,t)}{(\alpha+\tau(t))^{k-l+20-s_l-1}}\|_{L^2_\alpha[-\tau(t),\pi]} \lesssim \|h\|_{H^k_\alpha[-\tau(t),\pi]}$. Since $s_l \le 10$, the log singularity in $p.v. \int_{-\tau(t)}^{\pi} \frac{1}{\alpha-\beta} d\beta$ is cancelled. Hence from lemma 3.7.21 (the behavior of $K^{-\sigma}$ when $\beta > -\tau(t)$), $Term_{2,3,2}$ also satisfies the conditions (3.47), (3.48), (3.49), (3.50), and

$$\lim_{\alpha\to-\tau(t)^+} Term_{2,3.3} = Term_{2,3,2}|_{\alpha=\tau(t)}.$$

□

Lemma 3.7.6. *$L^{3,i}_{h^s}$ in T^ϵ_3 has the following two forms: sub-type 1:*

$$
\begin{aligned}
L^{3,i}_{h^s}(\alpha,\beta,\gamma,t) &= ((\alpha-\beta)^2 \frac{d}{d\beta}(K^i_{-1}(\Delta V_h^{10}(\alpha,\gamma,t), \Delta V_{\tilde{f}^L}^{21}(\alpha^t_\gamma,t), \Delta V_X(\alpha^t_\gamma,t))) X_i(\beta^t_\gamma,t) \\
&- \lim_{\beta\to\alpha}((\alpha-\beta)^2 \frac{d}{d\beta}(K^i_{-1}(\Delta V_h^{10}(\alpha,\gamma,t), \Delta V_{\tilde{f}^L}^{21}(\alpha^t_\gamma,t), \Delta V_X(\alpha^t_\gamma,t))) X_i(\alpha^t_\gamma,t)) \frac{1}{\alpha-\beta}
\end{aligned}
$$

or the form sub-type 2

$$L^{3,i}_{h^s}(\alpha,\beta,\gamma,t) = ((\alpha-\beta)(K^i_{-1}(\Delta V_h^{10}(\alpha,\gamma,t), \Delta V_{\tilde{f}^L}^{21}(\alpha^t_\gamma,t), \Delta V_X(\alpha^t_\gamma,t))) X_i(\beta^t_\gamma,t)(1+ic'(\beta+\tau(t))\gamma t)$$

and all functions of these forms satisfy the conditions (3.52), (3.53), (3.54), (3.55).

Proof. For the first sub-type, we use (3.71), (3.15) and lemma 3.7.21 (the behavior of $K^{-\sigma}$ when $\beta > -\tau(t)$. For the sub-type 2, we use (3.15) and lemma 3.7.21 directly. □

Lemma 3.7.7. *$L_{h^s}^{4,i}$ in T_4 has the following six forms: sub-type 1:*

$$L_{h^s}^{4,i}(\alpha,\gamma,t) = p.v.\int_{-\pi}^{\pi} K_{-1}^{i}(\Delta V_h^{10}(\alpha,\gamma,t), \Delta V_{\tilde{f}^L}^{21}(\alpha_\gamma^t,t), \Delta V_X(\alpha_\gamma^t,t)) X_i(\beta_\gamma^t,t)(1+ic'(\beta+\tau(t))\gamma t)d\beta$$

sub-type 2

$$L_{h^s}^{4,i}(\alpha,\gamma,t) = K_{-1}^{i}(\Delta V_h^{10}(\alpha,\gamma,t), \Delta V_{\tilde{f}^L}^{21}(\alpha_\gamma^t,t), \Delta V_X(\alpha_\gamma^t,t)|_{\beta=2\alpha+\tau(t)},$$

or

$$L_{h^s}^{4,i}(\alpha,\gamma,t) = K_{-1}^{i}(\Delta V_h^{10}(\alpha,\gamma,t), \Delta V_{\tilde{f}^L}^{21}(\alpha_\gamma^t,t), \Delta V_X(\alpha_\gamma^t,t)|_{\beta=-\tau(t)},$$

with particular $K_{-1}^{j}(A,B,C) = \frac{\sin(A_1+B_1)}{\cosh(A_2+B_2)-\cos(A_1+B_1)}$.

sub-type 3

$$L_{h^s}^{4,i}(\alpha,\gamma,t) = \int_0^1 d\zeta \int_{2\alpha+\tau(t)}^{\pi} (\frac{d}{d\beta}(K_{-1}^{i}(\zeta V_h^{10}(\alpha,\gamma,t) - V^{10}(\beta_0^t), \Delta V_{\tilde{f}^L}^{21}(\alpha_\gamma^t,t), \Delta V_X(\alpha_\gamma^t,t))) X_i(\beta_0^t,t)\tilde{f}^{+(b_i)}(\beta_0^t)d\beta.$$

sub-type 4

$$L_{h^s}^{4,i}(\alpha,\gamma,t) = \int_0^1 d\zeta \int_{2\alpha+\tau(t)}^{\pi} (K_{-1}^{i}(\zeta V_h^{10}(\alpha,\gamma,t) - V^{10}(\beta_0^t), \Delta V_{\tilde{f}^L}^{21}(\alpha_\gamma^t,t), \Delta V_X(\alpha_\gamma^t,t)) X_i(\beta_0^t,t)\tilde{f}^{+(b_i)}(\beta_0^t)d\beta.$$

sub-type 5

$$\begin{aligned} L_{h^s}^{4,i}(\alpha,\gamma,t) = p.v.\int_{-\tau(t)}^{2\alpha+\tau(t)} ((\alpha-\beta)^2 \frac{d}{d\beta}(K_{-1}^{i}(\Delta V_h^{10}(\alpha,\gamma,t), \Delta V_{\tilde{f}^L}^{21}(\alpha_\gamma^t,t), \Delta V_X(\alpha_\gamma^t,t))) \\ - \lim_{\beta\to\alpha}((\alpha-\beta)^2 \frac{d}{d\beta}(K_{-1}^{i}(\Delta V_h^{10}(\alpha,\gamma,t), \Delta V_{\tilde{f}^L}^{21}(\alpha_\gamma^t,t), \Delta V_X(\alpha_\gamma^t,t))))\frac{1}{(\alpha-\beta)^2}d\beta \end{aligned}$$

sub-type 6

$$L_{h^s}^{4,i}(\alpha,\gamma,t) = \tilde{X}_i(\alpha_\gamma^t,t),$$

with $b_i \leq 20$. Moreover, all functions of these forms satisfy the conditions of $L_{h^s}^{4,i}$ in T_4 (3.56), (3.57), (3.58) and (3.59).

Proof. The sub-type 1 has the same form as in lemma 3.7.5 when we control the $L^1_{h^s}$. Therefore we can use lemma 3.7.5 to get the result. The sub-type 2 relies on the particular $K^j_{-1}(A,B,C) = \frac{\sin(A_1+B_1)}{\cosh(A_2+B_2)-\cos(A_1+B_1)}$ and we use corollary 3.7.23 and corollary 3.7.24. For sub-type 3, we have

$$\begin{aligned}
&\frac{d}{d\beta}K_{-\sigma}(\zeta V_h^{10}(\alpha,\gamma,t)-V^{10}(\beta_0^t),\Delta V^{21}_{\tilde{f}^L}(\alpha^t_\gamma,t),\Delta V_X(\alpha^t_\gamma,t))\\
&=\sum_{l'}K^{l'}_{-(\sigma+1)}(\zeta V_h^{10}(\alpha,\gamma,t)-V^{10}(\beta_0^t),\Delta V^{21}_{\tilde{f}^L}(\alpha^t_\gamma,t),\Delta V_X(\alpha^t_\gamma,t))\\
&\cdot G_{l'}(\partial_\beta(V^{10}(\beta_0^t,t)),\partial_\beta(V^{21}_{\tilde{f}^L}(\beta^t_\gamma,t)),\partial_\beta(V_X(\beta^t_\gamma,t))),
\end{aligned}$$

with

$$\|G_{l'}(\partial_\beta(V^{10}(\beta_0^t,t)),\partial_\beta(V^{21}_{\tilde{f}^L}(\beta^t_\gamma,t)),\partial_\beta(V_X(\beta^t_\gamma,t)))\|_{C^{i_0}_\gamma([-1,1],C^{k+3}[-\tau(t),\pi])}\lesssim 1.$$

From lemma 3.9.19, we have

$$\|\frac{\tilde{f}^{+(b_i)}(\beta_0^t)}{(\beta+\tau(t))^l}\|_{H^{38-l}}\lesssim 1. \tag{3.80}$$

Then we can use (3.15) and corollary 3.7.27 to get the estimate for sub-type 3. For sub-type 4 we can use (3.80), (3.15), and corollary 3.7.27 directly. For sub-type 5, we could use (3.71) and lemma 3.7.21 with -2 type kernel , then the corresponding conditions follows. Sub-type 6 is trivial from (3.15). □

Lemma 3.7.8. *$L^{5,i}_{h^s}$ in T_5 has the following two forms: sub-type 1*

$$\begin{aligned}
&L^{5,i}_{h^s}(\alpha,\gamma,\eta,t)=\frac{d}{d\beta}(K^i_{-1}(V_h^{10}(\alpha,\gamma,t)-V_h^{10}(\beta,\eta,t),V^{21}_{\tilde{f}^L}(\alpha^t_\gamma,t)-V^{21}_{\tilde{f}^L}(\beta^t_\eta,t),V_X(\alpha^t_\gamma,t)-V_X(\alpha^t_\eta,t))|_{\beta=2\alpha+\tau(t)}\\
&\times\frac{ic(2\alpha+2\tau(t))t}{1+ic'(2\alpha+2\tau(t))\eta t}.
\end{aligned}$$

and sub-type 2

$$\begin{aligned}
&L^{5,i}_{h^s}(\alpha,\gamma,\eta,t)=K^i_{-1}(V_h^{10}(\alpha,\gamma,t)-V_h^{10}(\beta,\eta,t),V^{21}_{\tilde{f}^L}(\alpha^t_\gamma,t)-V^{21}_{\tilde{f}^L}(\beta^t_\eta,t),V_X(\alpha^t_\gamma,t)-V_X(\beta^t_\eta,t))|_{\beta=2\alpha+\tau(t)}\\
&X^j((2\alpha+\tau(t))^t_\eta)\times ic(2\alpha+2\tau(t))t.
\end{aligned}$$

and all functions of these forms satisfy the conditions of $L^{5,j}_{h^s}$ in T_5 (3.60), (3.61), (3.62), and (3.63).

Proof. Since

$$\frac{ic(2\alpha+2\tau(t))}{(\alpha+\tau(t))^2}\in H^{30}_{\alpha}[-\tau(t),\pi], \tag{3.81}$$

the estimates for sub-type 2 follow from (3.15) and lemma 3.7.22. For sub-type 1, we have

$$\begin{aligned}
&\frac{d}{d\beta}K_{-\sigma}(V_h^{10}(\alpha,\gamma,t)-V_h^{10}(\beta,\eta,t),V_{\tilde{f}^L}^{21}(\alpha^t_\gamma,t)-V_{\tilde{f}^L}^{21}(\beta^t_\eta,t),V_X(\alpha^t_\gamma,t)-V_X(\alpha^t_\eta,t))\\
&=\sum_{l'}K^{l'}_{-(\sigma+1)}(V_h^{10}(\alpha,\gamma,t)-V_h^{10}(\beta,\eta,t),V_{\tilde{f}^L}^{21}(\alpha^t_\gamma,t)-V_{\tilde{f}^L}^{21}(\beta^t_\eta,t),V_X(\alpha^t_\gamma,t)-V_X(\alpha^t_\eta,t))\\
&G_{l'}(\partial_\beta(V_h^{10}(\beta,\gamma,t)),\partial_\beta(V_{\tilde{f}^L}^{21}(\beta^t_\eta,t)),\partial_\beta(V_X(\beta^t_\eta,t))),
\end{aligned}$$

with

$$\|G_{l'}(\partial_\beta(V_h^{10}(\beta,\gamma,t)),\partial_\beta(V_{\tilde{f}^L}^{21}(\beta^t_\eta,t)),\partial_\beta(V_X(\beta^t_\eta,t)))\|_{C^{i_0}_{\gamma,\eta}([-1,1]\times[-1,1],C^{k+3}[-\tau(t),\pi])}\lesssim 1.$$

Then we could again use (3.81), (3.15) and lemma 3.7.22 to get the result. □

Lemma 3.7.9. *L^j in the fixed terms have the following two forms: sub-type 1*

$$L^j(\alpha,\beta,\gamma,t)=(\frac{d}{d\beta}(K^j_{-1}(-V^{10}(\beta^t_0,t),\Delta V_{\tilde{f}^L}^{21}(\alpha^t_\gamma,t),\Delta V_X(\alpha^t_\gamma,t)))X_j(\beta^t_0,t)\tilde{f}^{+(b_i)}(\beta^t_0,t).$$

sub-type 2

$$L^j_{h^s}(\alpha,\beta,\gamma,t)=(K^j_{-1}(-V^{10}(\beta^t_0,t),\Delta V_{\tilde{f}^L}^{21}(\alpha^t_\gamma,t),\Delta V_X(\alpha^t_\gamma,t))X_j(\beta^t_0,t)\tilde{f}^{+(b_i)}(\beta^t_0,t).$$

and all functions of these forms satisfy the conditions (3.64)*,* (3.65) *and* (3.68)*.*

Proof. For sub-type 2, we again could use corollary 3.7.27, (3.15) and (3.80). For sub-type 1, we use additionally,

$$\begin{aligned}
&\frac{d}{d\beta}K_{-\sigma}(-V^{10}(\beta^t_0),\Delta V_{\tilde{f}^L}^{21}(\alpha^t_\gamma,t),\Delta V_X(\alpha^t_\gamma,t)))\\
&=\sum_{l'}K^{l'}_{-(\sigma+1)}(-V^{10}(\beta^t_0,t),\Delta V_{\tilde{f}^L}^{21}(\alpha^t_\gamma,t),\Delta V_X(\alpha^t_\gamma,t)))\\
&\cdot G_{l'}(\partial_\beta(V^{10}(\beta^t_0,t)),\partial_\beta(V_{\tilde{f}^L}^{21}(\beta^t_\gamma,t)),\partial_\beta(V_X(\beta^t_\gamma,t))),
\end{aligned}$$

with

$$\|G_{l'}(\partial_\beta(V^{10}(\beta_0^t,t)),\partial_\beta(V^{21}_{\tilde{f}^L}(\beta_\gamma^t,t)),\partial_\beta(V_X(\beta_\gamma^t,t)))\|_{C^{i_0}_\gamma([-1,1],C^{k+3}[-\tau(t),\pi])}\lesssim 1.$$

□

Lemma 3.7.10. *When $i_0+k\le 30$, $T_{fixed}(\alpha_\gamma^t,t)$ satisfies condition* (3.66)*, and* (3.67)*.*

Proof. From (3.10), we have

$$T_{fixed}(\alpha,t)=\partial_\alpha^{20}(-\frac{d\tilde{f_\mu}^L(\alpha,t)}{dt}+\frac{d\tilde{f_\mu}^L}{d\alpha}(\alpha,t)(\frac{\frac{dx(\alpha,t)}{dt}}{\frac{dx(\alpha,t)}{d\alpha}}) \tag{3.82}$$

$$+\int_{-\pi}^{\pi}K(\tilde{f}^L(\alpha,t)-\tilde{f}^L(\beta,t))(\frac{\partial_\alpha\tilde{f_\mu}^L(\alpha,t)}{\frac{dx(\alpha,t)}{d\alpha}}-\frac{\partial_\beta\tilde{f_\mu}^L(\beta,t)}{\frac{dx(\beta,t)}{d\beta}})(\frac{dx(\beta,t)}{d\beta})d\beta)$$

$$=T^1_{fixed}(\alpha,t)+T^2_{fixed}(\alpha,t)+T^3_{fixed}(\alpha,t). \tag{3.83}$$

We have $\tilde{f}^L(\alpha,t)\in C^1([0,t],C^{59}(-\infty,\infty))$ and analytic in $\tilde{D}=\{\alpha+iy|0<\alpha<\frac{\pi}{4}\}$, then $T^1_{fixed}(\alpha^t_\gamma,t)$, $T^2_{fixed}(\alpha^t_\gamma,t)$ satisfy (3.66) and (3.67). For $T^3_{fixed}(\alpha,t)$, first from the arc-chord condition (3.89), it is well defined. Moreover, we have

$$T^3_{fixed}(\alpha,t)=\sum_j\int_{-\pi}^{\pi}K_0^j(0,V^{21}_{\tilde{f}^L}(\alpha,t)-V^{21}_{\tilde{f}^L}(\beta,t),V_X(\alpha,t)-V_X(\beta,t))X_j(\beta,t)d\beta, \tag{3.84}$$

for some K_0 type kernels. Then

$$T^3_{fixed}(\alpha^t_\gamma,t)=\sum_j\int_{-\pi}^{\pi}K_0^j(0,V^{21}_{\tilde{f}^L}(\alpha^t_\gamma,t)-V^{21}_{\tilde{f}^L}(\beta^t_\gamma,t),V_X(\alpha^t_\gamma,t)-V_X(\beta^t_\gamma,t))X_j(\beta^t_\gamma,t)(1+ic'(\beta+\tau(t))\gamma t)d\beta,$$

for some K_0 type kernel K_0^j. Since K_0 type is also a K_{-1} type, we could use lemma 3.7.5 with $h=0$. Then we have the result. □

□

3.7.2 The behaviour of T^ϵ

Lemma 3.7.11. *When T^ϵ is of the types one to five in previous lemma 3.7.2, then it satisfies the condition* (3.44)*,* (3.45)*.*

Proof. We will deal with $T^\epsilon_1(h)$, and other types can be shown in the same way with lemma 3.7.14.

From our construction, since there is no singularity in the denominator, we claim the following two bound is true

$$\sup_t \|T_1^\epsilon(h)\|_{C_\gamma^{i_0}([-1,1],H^k[-\tau(t),\pi])} \lesssim 1,$$

$$\sup_t \|T_1^\epsilon(h) - T_1^\epsilon(g)\|_{C_\gamma^{i_0}([-1,1],H^k[-\tau(t),\pi])} \lesssim \|h-g\|_{W^{i_0,k}}.$$

Moreover, by the definition of $\tilde{\lambda}$ (3.20), we have $\tilde{\lambda}(\alpha_\gamma^t) = 0$ when $\alpha > \frac{\pi}{8}$ if t and δ is sufficiently small. Also we know $T_1^\epsilon(h) = 0$ when $\alpha \leq -\tau(t)$. Hence $\operatorname{supp} T_1^\epsilon(h) \in C_\gamma^{i_0}([-1,1], H^k[-\tau(t),\pi])$ and $T_1^\epsilon(h) \in S$.

Then we consider the behavior of $T_1^\epsilon(h)$ near $-\tau(t)$. We have

$$\begin{aligned}
&\frac{d}{d\alpha^j}(L_{h^s}^1(\alpha,\gamma,t)\int_{-\tau(t)}^{2\alpha+\tau(t)} \frac{h(\alpha,\gamma,t)-h(\beta,\gamma,t)}{(\alpha-\beta)^2+\epsilon^2}d\beta)\\
&= \sum_{j'\leq j} C_{j,j'}\partial_\alpha^{j-j'}(L_{h^s}^1(\alpha,\gamma,t))\partial_\alpha^{j'}(\int_{-\alpha-\tau(t)}^{\alpha+\tau(t)} \frac{h(\alpha,\gamma,t)-h(\alpha-\beta,\gamma,t)}{(\beta)^2+\epsilon^2}d\beta)\\
&= \sum_{j'\leq j} C_{j,j'}\partial_\alpha^{j-j'}(L_{h^s}^1(\alpha,\gamma,t))(\int_{-\tau(t)}^{2\alpha+\tau(t)} \frac{\partial_\alpha^{j'}h(\alpha,\gamma,t)-\partial_\alpha^{j'}h(\beta,\gamma,t)}{(\alpha-\beta)^2+\epsilon^2}d\beta+\\
&\sum_{j''\leq j'} \partial_\alpha^{j'-j''-1}(\frac{\partial_\alpha^{j''}h(\alpha,\gamma,t)-\partial_\alpha^{j''}h(-\tau(t),\gamma,t)}{(\alpha+\tau(t))^2+\epsilon^2}) + \partial_\alpha^{j'-j''-1}(\frac{\partial_\alpha^{j''}h(\alpha,\gamma,t)-\partial_\alpha^{j''}h(2\alpha+\tau(t),\gamma,t)}{(\alpha+\tau(t))^2+\epsilon^2})).
\end{aligned}$$

Since for any γ, t, we have $h \in H^k[-\pi,\pi] \cap S$. We get

$$\lim_{\alpha\to-\tau(t)^+} \partial_\alpha^j h(\alpha,\gamma,t) = 0,$$

when $j \leq k-1$. Then by the boundedness conditions of L_h^1, we have

$$\lim_{\alpha\to-\tau^+} \frac{d}{d\alpha^j}(L_{h^s}^1(\alpha,\gamma,t)\int_{-\tau(t)}^{2\alpha+\tau(t)} \frac{h(\alpha,\gamma,t)-h(\beta,\gamma,t)}{(\alpha-\beta)^2+\epsilon^2}d\beta) = 0.$$

Similarly, when $j \leq k-1$, $l \leq i_0$, we have

$$\lim_{\alpha\to-\tau^+} \frac{d}{d\alpha^j}\frac{d}{d\gamma^l}(L_{h^s}^1(\alpha,\gamma,t)\int_{-\tau(t)}^{2\alpha+\tau(t)} \frac{h(\alpha,\gamma,t)-h(\beta,\gamma,t)}{(\alpha-\beta)^2+\epsilon^2}d\beta) = 0.$$

Then by the fact that $T_1^\epsilon(h) = 0$ when $\alpha \leq -\tau(t)$, we have

$$\sup_t \|T_1^\epsilon(h)\|_{C_\gamma^{i_0}([-1,1],H^k[-\pi,\pi])} \lesssim 1,$$

and

$$\sup_t \|T_1^\epsilon(h) - T_1^\epsilon(g)\|_{C_\gamma^{i_0}([-1,1],H^k[-\pi,\pi])} \lesssim \|h-g\|_{C_t^0(C_\gamma^{i_0}([-1,1],H^k[-\pi,\pi]))}.$$

Finally, we show the continuity of $T_1^\epsilon(h)$ in t. When $0 \le t < t' \le t_0$, $\alpha > -\tau(t')$, we have

$$\begin{aligned}
&T_1^\epsilon(h)(t) - T_1^\epsilon(h)(t') = \tilde{\lambda}(\alpha_\gamma^t) L_{h^s}^1(\alpha,\gamma,t) \int_{-\tau(t)}^{2\alpha+\tau(t)} \frac{h(\alpha,\gamma,t)-h(\beta,\gamma,t)}{(\alpha-\beta)^2+\epsilon^2} d\beta \\
&- \tilde{\lambda}(\alpha_\gamma^{t'}) L_{h^s}^1(\alpha,\gamma,t') \int_{-\tau(t')}^{2\alpha+\tau(t')} \frac{h(\alpha,\gamma,t')-h(\beta,\gamma,t')}{(\alpha-\beta)^2+\epsilon^2} d\beta \\
&= (\tilde{\lambda}(\alpha_\gamma^t) - \tilde{\lambda}(\alpha_\gamma^{t'})) L_{h^s}^1(\alpha,\gamma,t) \int_{-\tau(t)}^{2\alpha+\tau(t)} \frac{h(\alpha,\gamma,t)-h(\beta,\gamma,t)}{(\alpha-\beta)^2+\epsilon^2} d\beta \\
&+ \tilde{\lambda}(\alpha_\gamma^{t'}) (L_{h^s}^1(\alpha,\gamma,t) - L_h^1(\alpha,\gamma,t')) \int_{-\tau(t)}^{2\alpha+\tau(t)} \frac{h(\alpha,\gamma,t)-h(\beta,\gamma,t)}{(\alpha-\beta)^2+\epsilon^2} d\beta \\
&+ \tilde{\lambda}(\alpha_\gamma^{t'}) L_{h^s}^1(\alpha,\gamma,t') (\int_{-\tau(t)}^{2\alpha+\tau(t)} \frac{h(\alpha,\gamma,t)-h(\beta,\gamma,t)}{(\alpha-\beta)^2+\epsilon^2} d\beta - \int_{-\tau(t')}^{2\alpha+\tau(t')} \frac{h(\alpha,\gamma,t')-h(\beta,\gamma,t')}{(\alpha-\beta)^2+\epsilon^2} d\beta) \\
&= Term_1 + Term_2 + Term_3.
\end{aligned}$$

Since $\tilde{\lambda}(\alpha) \in C^{100}[-\pi,\pi]$, from the condition of $L_h^1(\alpha,\gamma,t)$ (3.47) we have

$$\|Term_1\|_{C_\gamma^{i_0}([-1,1],H^k[-\tau(t'),\pi])} \lesssim \mathcal{O}(t-t').$$

From the Lipschitz condition of $L_h^1(\alpha,\gamma,t)$ (3.48), we have

$$\|Term_2\|_{C_\gamma^{i_0}([-1,1],H^k[-\tau(t'),\pi])} \lesssim \mathcal{O}(t-t').$$

For the last term, we have

$$\begin{aligned}
Term_3 &= \tilde{\lambda}(\alpha_\gamma^{t'}) L_{h^s}^1(\alpha,\gamma,t') (\int_{-\tau(t')}^{2\alpha+\tau(t')} \frac{h(\alpha,\gamma,t)-h(\alpha,\gamma,t')-(h(\beta,\gamma,t)-h(\beta,\gamma,t'))}{(\alpha-\beta)^2+\epsilon^2} d\beta \\
&+ \tilde{\lambda}(\alpha_\gamma^{t'}) L_{h^s}^1(\alpha,\gamma,t') (\int_{-\tau(t)}^{-\tau(t')} + \int_{2\alpha+\tau(t')}^{2\alpha+\tau(t)}) \frac{h(\alpha,\gamma,t)-h(\beta,\gamma,t)}{(\alpha-\beta)^2+\epsilon^2} d\beta.
\end{aligned}$$

Then from (3.47), (3.50), we have

$$\begin{aligned}
&\|Term_3\|_{C_\gamma^{i_0}([-1,1],H^k[-\tau(t'),\pi])} \lesssim \|h(\alpha,\gamma,t) - h(\alpha,\gamma,t')\|_{C_\gamma^{i_0}([-1,1],H^k[-\pi,\pi])} \\
&+ \sqrt{\tau(t)-\tau(t')} \|h(\alpha,\gamma,t)\|_{C_\gamma^{i_0}([-1,1],H^k[-\pi,\pi])}.
\end{aligned}$$

For $-\tau(t) \leq \alpha \leq -\tau(t')$, we have

$$T_1^\epsilon(h)(t) - T_1^\epsilon(h)(t') = T_1^\epsilon(h)(t).$$

Moreover,

$$\begin{aligned}
&\|T_1^\epsilon(h)(t)\|_{C_\gamma^{i_0}([-1,1],H^k[-\tau(t),-\tau(t')])} \\
&= \|\tilde{\lambda}(\alpha_\gamma^t)L_{h^s}^1(\alpha,\gamma,t)\int_{-\tau(t)}^{2\alpha+\tau(t)} \frac{h(\alpha,\gamma,t)-h(\beta,\gamma,t)}{(\alpha-\beta)^2+\epsilon^2}d\beta\|_{C_\gamma^{i_0}([-1,1],H^k[-\tau(t),-\tau(t')])} \\
&\lesssim \|h(\alpha,\gamma,t)\|_{C_\gamma^{i_0}([-1,1],H^k[-\tau(t),-\tau(t)+2(\tau(t)-\tau(t'))])}.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
&\|T_1^\epsilon(h)(t) - T_1^\epsilon(h)(t')\|_{C_\gamma^{i_0}([-1,1],H^k[-\pi,\pi])} \lesssim \mathcal{O}(t-t') + \sqrt{\tau(t)-\tau(t')}+ \\
&\|h(\alpha,\gamma,t)-h(\alpha,\gamma,t')\|_{C_\gamma^{i_0}([-1,1],H^k[-\pi,\pi])} + \|h(\alpha,\gamma,t)\|_{C_\gamma^{i_0}([-1,1],H^k[-\tau(t),-\tau(t)+2(\tau(t)-\tau(t'))])}.
\end{aligned}$$

From lemma 3.7.12 and corollary 3.7.13, it tends to 0 as $t \to t'$ or $t' \to t$. □

Lemma 3.7.12. *If $h(\alpha,\gamma,t) \in C_t^0([0,t*],C_\gamma^0([-1,1],L^2[-\pi,\pi])$, then for $0 \leq t,t' \leq t_0$, we have*

$$\lim_{t'\to t}\|h(\alpha,\gamma,t)\|_{C_\gamma^0([-1,1],L^2[-\tau(t),\tau(t')])} = 0.$$

Proof. For any $\epsilon > 0$, there exists δ such that when $|\gamma-\gamma'| \leq \delta$, we have

$$\|h(\alpha,\gamma,t)-h(\alpha,\gamma',t)\|_{L^2[-\pi,\pi]} \leq \epsilon.$$

Let $-1 \leq \gamma_1 < \gamma_2 < ... < \gamma_n \leq 1$, with $|\gamma_i - \gamma_j| < \delta$. By the property of L^2 integral, there exists δ_t such that when $|t-t'| \leq \delta_t$, we have

$$\sup_i \|h(\alpha,\gamma_i,t)\|_{L^2[-\tau(t),\tau(t')]} \leq \epsilon.$$

Then when $|t-t'| \leq \delta_t$, $\|h(\alpha,\gamma,t)\|_{L^2[-\tau(t),\tau(t')]} \leq 2\epsilon$. □

Corollary 3.7.13. *If $h(\alpha,\gamma,t) \in C_t^0([0,t*],C_\gamma^0([-1,1],L^2[-\pi,\pi])$, then for $0 \leq t,t' \leq t_0$, we have*

$$\lim_{t\to t'}\|h(\alpha,\gamma,t)\|_{C_\gamma^0([-1,1],L^2[-\tau(t),\tau(t')])} = 0.$$

Proof. For any $\epsilon > 0$, there exists $\tilde{\delta} > 0$, such that when $|t - t'| \leq \tilde{\delta}$, we have

$$\sup_{\gamma} \|h(\alpha, \gamma, t) - h(\alpha, \gamma, t')\|_{L^2[-\pi,\pi]} < \epsilon.$$

Since from the lemma 3.7.12, $\lim_{t\to t'} \|h(\alpha, \gamma, t')\|_{C^0_\gamma([-1,1], L^2[-\tau(t),\tau(t')])} = 0$, then

$$\overline{\lim_{t\to t'}} \sup_{\gamma} \|h(\alpha, \gamma, t)\|_{L^2[-\tau(t),\tau(t')]} \leq \lim_{t\to t'} \|h(\alpha, \gamma, t')\|_{C^0_\gamma([-1,1], L^2[-\tau(t),\tau(t')])} + \epsilon = \epsilon.$$

□

Lemma 3.7.14. *If*

$$h(\alpha, \gamma, t) \in W^{i_0,k},$$

and when $\alpha \leq -\tau(t)$, $h(\alpha, \gamma, t) = 0$. *we have*

$$D^{-j}(h(\alpha, \gamma, t)) \in C^0_t([0, t_0], C^{i_0}_\gamma([-1, 1], H^{k+j}[-\pi, \pi])) \tag{3.85}$$

with $D^{-j}(h) = 0$ *when* $\alpha \leq -\tau(t)$. *Moreover,*

$$D^{-j}(h(2\alpha + \tau(t), \gamma, t)) \in C^0_t([0, t_0], C^{i_0}_\gamma([-1, 1], H^{k+j}[-\pi, \frac{\pi}{3}])). \tag{3.86}$$

Proof. We prove the (3.85), and the other one follows the same way. For the (3.85), when $j = 1$, we have

$$D^{-1}h(\alpha, \gamma, t) = \int_{-\tau(t)}^{\alpha} h(\alpha_0, \gamma, t)(1 + ic'(\alpha_0 + \tau(t))\gamma t)d\alpha_0. \tag{3.87}$$

Therefore

$$\|\partial_\alpha D^{-1}h(\alpha, \gamma, t)\|_{C^{i_0}_\gamma([-1,1],H^k[-\tau(t),\pi])} \lesssim \|h(\alpha, \gamma, t)\|_{C^{i_0}_\gamma([-1,1],H^k[-\tau(t),\pi])}.$$

Moreover, for $j \leq k - 1$, we have

$$\begin{aligned}
&\lim_{\alpha\to-\tau(t)} \partial^j_\alpha \partial_\alpha D^{-1}h(\alpha, \gamma, t) \\
&= \lim_{\alpha\to-\tau(t)} \sum_{j,j'} C_{j,j'} \partial^{j'}_\alpha h(\alpha, \gamma, t)\partial^{j-j'}_\alpha (1 + ic'(\alpha + \tau(t))\gamma t) = 0.
\end{aligned}$$

By the definition, $D^{-1}(h) = 0$ when $\alpha \leq -\tau(t)$. Then $D^{-1}(h) \in C^{i_0}_\gamma([-1, 1], H^{k+1}[-\pi, \pi])$.

For $0 \le t' < t \le t_0$, $\alpha > -\tau(t')$, we have

$$
\begin{aligned}
&D^{-1}(h)(t) - D^{-1}(h)(t') \\
&= \int_{-\tau(t)}^{\alpha} h(\alpha_0,\gamma,t)(1+ic'(\alpha_0+\tau(t))\gamma t)d\alpha_0 - \int_{-\tau(t')}^{\alpha} h(\alpha_0,\gamma,t')(1+ic'(\alpha_0+\tau(t'))\gamma t')d\alpha_0 \\
&= \int_{-\tau(t')}^{\alpha} (h(\alpha_0,\gamma,t)-h(\alpha_0,\gamma,t'))(1+ic'(\alpha_0+\tau(t))\gamma t) + (h(\alpha_0,\gamma,t')(ic'(\alpha_0+\tau(t))\gamma t - ic'(\alpha_0+\tau(t'))\gamma t')d\alpha_0 \\
&+ \int_{-\tau(t)}^{-\tau(t')} h(\alpha_0,\gamma,t)(1+c'(\alpha_0+\tau(t))\gamma t)d\alpha_0.
\end{aligned}
$$

Therefore we have

$$
\begin{aligned}
&\|D^{-1}(h)(t) - D^{-1}(h)(t')\|_{C_\gamma^{i_0}([-1,1],H^{k+1}[-\pi,\pi])} \lesssim \|h(\alpha_0,\gamma,t)-h(\alpha_0,\gamma,t')\|_{C_\gamma^{i_0}([-1,1],H^{k+1}[-\tau(t),\pi])} + \mathcal{O}(t-t') \\
&+ \sqrt{\tau(t)-\tau(t')}.
\end{aligned}
$$

For $-\tau(t) < \alpha < -\tau(t')$, we have

$$
\begin{aligned}
&\|D^{-1}(h)(t) - D^{-1}(h)(t')\|_{C_\gamma^{i_0}([-1,1],H^{k+1}[-\tau(t),\tau(t')])} \lesssim \|D^{-1}(h)(t)\|_{C_\gamma^{i_0}([-1,1],H^{k+1}[-\tau(t),-\tau(t')])} \\
&\lesssim \|h\|_{C_\gamma^{i_0}([-1,1],H^{k}[-\tau(t),-\tau(t')])},
\end{aligned}
$$

where we use (3.87) in the last inequality. In conclusion, from lemma 3.7.12 and corollary 3.7.13, we have

$$
\|D^{-1}(h)(t) - D^{-1}(h)(t')\|_{C_\gamma^{i_0}([-1,1],H^{k+1}[-\pi,\pi])}
$$

tends to 0 as $t \to t'$ or $t' \to t$. We can then use induction to show the $j > 1$ case. □

Before we show estimate the FT terms in lemma 3.7.2, we first go back to study the original equation on the real line:

3.7.3 The loose bound for $\epsilon = 0$ case on the real line

Let

$$
S^0 = \{h(\alpha,t) = 0, \text{when } \alpha \le -\tau(t) \text{ and } \alpha \ge \frac{\pi}{8}\},
$$

$$
S^0_{\delta_1} = \{h(\alpha,t) = 0, \text{when } \alpha \le -\tau(t) \text{ and } \alpha \ge \frac{\pi}{8}, \sup_t \|h\|_{C^2_\alpha[-\pi,\pi]} \lesssim \delta_1\},
$$

Corollary 3.7.15. *When $\gamma = 0$, T^0 in previous section 1.8.1 (3.33) can also be written as :*

$$T^0(h)(\alpha,0,t) = T_1(h)(\alpha,0,t) + T_2(h)(\alpha,0,t) + \sum_i T_3^i(h)(\alpha,0,t) + \sum_i T_4^i(h)(\alpha,0,t) + FT(\alpha,0,t),$$

with

$$T_1(h)(\alpha,0,t) = \lim_{\epsilon\to 0} T_1^\epsilon(h)(\alpha,0,t) = \begin{cases} \tilde{\lambda}(\alpha_0^t) L^1_{h^s}(\alpha,0,t) p.v. \int_{-\tau(t)}^{2\alpha+\tau(t)} \frac{h(\alpha,0,t)-h(\beta,0,t)}{(\alpha-\beta)^2} d\beta & -\tau(t) < \alpha \le \pi, \\ 0 & -\pi \le \alpha \le -(\tau(t)), \end{cases}$$

$$T_2(h) = \lim_{\epsilon\to 0} T_2^\epsilon(h)(\alpha,0,t) = \begin{cases} \tilde{\lambda}(\alpha_0^t) L^2_{h^s}(\alpha,0,t) \frac{\pi}{2} \partial_\alpha h(\alpha,0,t) & -\tau(t) \le \alpha \le \pi \\ 0 & -\pi \le \alpha < -(\tau(t)) \end{cases},$$

and

$$T_3(h) = \lim_{\epsilon\to 0} T_3^\epsilon(h)(\alpha,0,t) = \begin{cases} \tilde{\lambda}(\alpha_0^t) p.v. \int_{-\tau(t)}^{2\alpha+\tau(t)} L^{3,i}_{h^s}(\alpha,0,\beta,t) \frac{D^{-20+b_i} h(\beta,0,t)}{\alpha-\beta} d\beta & -\tau(t) < \alpha \le \pi \\ 0 & -\pi \le \alpha \le -(\tau(t)) \end{cases}.$$

Moreover, $T_i^0(h(\alpha,0,t)) = T_i(h)(\alpha,0,t)$ maps $C_t^0([0,t_0], H^k[-\pi,\pi]) \cap S^0_{\delta_1}$ to $C_t^0([0,t_0], H^{k-3}[-\pi,\pi]) \cap S^0$ and satisfying the Lipchitz condition for $i = 1,2,3$.

$T_i^0(h(\alpha,0,t)) = T_i(h)(\alpha,0,t)$ maps $C_t^0([0,t_0], H^k[-\pi,\pi]) \cap S^0_{\delta_1}$ to $C_t^0([0,t_0], H^k[-\pi,\pi]) \cap S^0$ satisfying the Lipchitz condition for $i = 4$.

Proof. The similar proof follows as in lemma 3.7.2, lemma 3.7.11. Here we point out several differences. First we take $\gamma = 0$. Since $T^i(h)(\alpha,0,t)$ only depends on $h(\alpha,\gamma,t)$ when $\gamma = 0$, this difference does not affect the proof. Second we take $\epsilon = 0$ in T_1^ϵ, T_2^ϵ, and T_3^ϵ, and lose several orders of derivative. Therefore we use $k-3$ instead of k and the similar proof follows. Also, when $\gamma = 0$, $T_5^i(h) = 0$, hence the fifth type does not show up. □

Lemma 3.7.16. *The sum of all fixed terms $FT^0 \in C^0([0,t], H^{35}[-\pi,\pi]) \cap S^0$.*

Proof. Notice that $\tilde{f}^{+(20)}$ is a solution of

$$\frac{d\tilde{f}^{+(20)}(\alpha+\tau(t))}{dt} = T^0(\tilde{f}^{+(20)}(\alpha+\tau(t))).$$

Since $\tilde{f}^{+(20)}(\alpha+\tau(t)) \in C_t^1([0,t_0], H^{38}[-\pi,\pi])$, and $\tilde{f}^{+(20)}(\alpha+\tau(t)) = 0$ when $\alpha \le -\tau(t)$ or $\alpha \ge \frac{\pi}{8}$, we have $T^0(\tilde{f}^{+(20)}(\alpha+\tau(t)) \in C^0([0,t], H^{38}[-\pi,\pi]) \cap S^0$. Moreover, from corollary 3.7.15,

$T_i^0(\tilde{f}^{+(20)}(\alpha+\tau(t))) \in C^0([0,t], H^{35}[-\pi,\pi]) \cap S^0$ for $1 \le i \le 5$. Then fixed terms FT^0 also satisfies

$$FT^0 \in C^0([0,t], H^{35}[-\pi,\pi]) \cap S^0.$$

□

Then we deal with FT (when $gamma \neq 0$) in lemma 3.7.2.

Lemma 3.7.17. *The fixed terms FT in lemma 3.7.2 satisfies the conditions* (3.44). (3.45).

Proof. Since they do not depend on h, we only need to show (3.44). By (3.64), (3.65), (3.66), and (3.67), we have $FT \in C_\gamma^{i_0}([-1,1], C^{k+2}(-\tau(t),\pi])$. Then we show for any $\gamma \in [-1,1]$, $l \le k-1$, we have

$$\lim_{\alpha \to -\tau(t)} \partial_\alpha^l FT = 0. \tag{3.88}$$

From lemma 3.7.16, we have (3.88) is true when $\gamma = 0$. Moreover, for $\gamma \neq 0$, We have

$$\begin{aligned}
&\lim_{\alpha\to-\tau(t)} \partial_\alpha^l (\sum_j \int_{2\alpha+\tau(t)}^{\pi} L_j(\alpha_\gamma^t, \beta, t) d\beta + T_{fixed}(\alpha_\gamma^t, t)) \\
&=_{(3.68)} \lim_{\alpha\to-\tau(t)} \sum_j \int_{2\alpha+\tau(t)}^{\pi} (\partial_\alpha^l L_j(\alpha_\gamma^t, \beta, t)) d\beta + (\partial_\alpha^l T_{fixed}(\alpha_\gamma^t, t)) \\
&= \lim_{\alpha\to-\tau(t)} \sum_j \int_{2\alpha+\tau(t)}^{\pi} \sum_{l'} C_{l,l'} (\partial_\alpha^{l'} L_j)(\alpha_\gamma^t, \beta, t) \Pi_{n=1}^{l'} \partial_\alpha^{\sigma(l,l',n)} (1 + ic'(\alpha+\tau(t))\gamma t) d\beta \\
&+ \sum_{l'} C_{l,l'} (\partial_\alpha^{l'} T_{fixed})(\alpha_\gamma^t, t) \Pi_{n=1}^{l'} \partial_\alpha^{\sigma(l,l',n)} (1 + ic'(\alpha+\tau(t))\gamma t) \\
&= \lim_{\alpha\to-\tau(t)} \sum_{l'} C_{l,l'} \Pi_{n=1}^{l'} \partial_\alpha^{\sigma(l,l',n)} (1 + ic'(\alpha+\tau(t))\gamma t) (\sum_j \int_{2\alpha+\tau(t)}^{\pi} (\partial_\alpha^{l'} L_j)(\alpha_\gamma^t, \beta, t) d\beta + (\partial_\alpha^{l'} T_{fixed})(\alpha_\gamma^t, t)) \\
&= \sum_{l'} C_{l,l'} \lim_{\alpha\to-\tau(t)} (\Pi_{n=1}^{l'} \partial_\alpha^{\sigma(l,l',n)} (1 + ic'(\alpha+\tau(t))\gamma t)) \lim_{\alpha\to-\tau(t)} (\sum_j \int_{2\alpha+\tau(t)}^{\pi} (\partial_\alpha^{l'} L_j)(\alpha_\gamma^t, \beta, t) d\beta + (\partial_\alpha^{l'} T_{fixed})(\alpha_\gamma^t, t)) \\
&= 0,
\end{aligned}$$

where we use condition (3.68) in the second equality and the $\gamma = 0$ case and (3.64) in the last equality. Then $FT \in C_\gamma^{i_0}([-1,1], H^k[-\pi,\pi])$.

Then we show continuity in t. When $0 < t < t' < t*$, $\alpha > -\tau(t')$ we have

$$FT(t) - FT(t') = \tilde{\lambda}(\alpha_\gamma^t) \sum_j \int_{2\alpha+\tau(t)}^{\pi} L_j(\alpha_\gamma^t, \beta, t) d\beta + \tilde{\lambda}(\alpha_\gamma^t) T_{fixed}(\alpha_\gamma^t, t) -$$

$$
\begin{aligned}
&(\tilde{\lambda}(\alpha_\gamma^{t'})\sum_j\int_{2\alpha+\tau(t')}^{\pi} L_j(\alpha_\gamma^{t'},\beta,t')d\beta+\tilde{\lambda}(\alpha_\gamma^{t'})T_{fixed}(\alpha_\gamma^{t'},t'))\\
&=(\tilde{\lambda}(\alpha_\gamma^{t})-\tilde{\lambda}(\alpha_\gamma^{t'}))(\sum_j\int_{2\alpha+\tau(t)}^{\pi} L_j(\alpha_\gamma^{t},\beta,t)d\beta+\tilde{\lambda}(\alpha_\gamma^{t})T_{fixed}(\alpha_\gamma^{t},t))\\
&+\tilde{\lambda}(\alpha_\gamma^{t'})\sum_j\int_{2\alpha+\tau(t)}^{\pi}(L_j(\alpha_\gamma^{t},\beta,t)-L_j(\alpha_\gamma^{t'},\beta,t'))d\beta-\tilde{\lambda}(\alpha_\gamma^{t'})\int_{2\alpha+\tau(t')}^{2\alpha+\tau(t')} L_j(\alpha_\gamma^{t'},\beta,t')d\beta\\
&+\tilde{\lambda}(\alpha_\gamma^{t'})(T_{fixed}(\alpha_\gamma^{t},t)-T_{fixed}(\alpha_\gamma^{t'},t')).
\end{aligned}
$$

Then by the conditions of L_j (3.64), (3.65) and the conditions of T_{fixed} (3.66), (3.67), we have

$$
\|FT(t)-FT(t')\|_{C_\gamma^{i_0}([-1,1],H^k(-\tau(t'),\pi])}\lesssim\mathcal{O}(t-t')+\mathcal{O}(t-t')+|\tau(t)-\tau(t')|+\mathcal{O}(t-t').
$$

Moreover, from (3.64), (3.66),

$$
\|FT(t)-FT(t')\|_{C_\gamma^{i_0}([-1,1],H^k(-\tau(t),\tau(t')))}=\|FT(t)\|_{C_\gamma^{i_0}([-1,1],H^k(-\tau(t),\tau(t')))}\lesssim\mathcal{O}(t-t').
$$

When $\alpha<-\tau(t)$, $FT(t)=FT(t')=0$. Then we have the result. □

3.7.4 The loose bound for $\epsilon=0$ case

As in the lemma 3.7.15, we can also control the limit case when $\gamma\neq 0$, and have:

Corollary 3.7.18. *T in previous section* (3.32) *can also be written as the*

$$
T(h)=T_1(h)+T_2(h)+\sum_i T_3^i(h)+\sum_i T_4^i(h)+\sum_i T_5^i(h)+FT.
$$

with

$$
T_1(h)(\alpha,\gamma,t)=\lim_{\epsilon\to 0}T_1^\epsilon(h)(\alpha,\gamma,t)=\begin{cases}\tilde{\lambda}(\alpha_\gamma^t)L_{h^s}^1(\alpha,\gamma,t)p.v.\int_{-\tau(t)}^{2\alpha+\tau(t)}\frac{h(\alpha,\gamma,t)-h(\beta,\gamma,t)}{(\alpha-\beta)^2}d\beta & -\tau(t)<\alpha\leq\pi,\\ 0 & -\pi\leq\alpha\leq-(\tau(t)),\end{cases}
$$

$$
T_2(h)=\lim_{\epsilon\to 0}T_2^\epsilon(h)(\alpha,\gamma,t)=\begin{cases}\tilde{\lambda}(\alpha_\gamma^t)L_{h^s}^2(\alpha,\gamma,t)\frac{\pi}{2}\partial_\alpha h(\alpha,\gamma,t) & -\tau(t)\leq\alpha\leq\pi\\ 0 & -\pi\leq\alpha<-(\tau(t))\end{cases},
$$

and

$$T_3(h)=\lim_{\epsilon\to 0}T_3^{\epsilon}(h)(\alpha,\gamma,t)=\begin{cases}\tilde{\lambda}(\alpha_\gamma^t)p.v.\int_{-\tau(t)}^{2\alpha+\tau(t)}L_{h^s}^{3,i}(\alpha,\gamma,\beta,t)\frac{D^{-20+b_i}h(\beta,\gamma,t)}{(\alpha-\beta)}d\beta & -\tau(t)<\alpha\le\pi\\ 0 & -\pi\le\alpha\le-(\tau(t))\end{cases}.$$

T_4^i, T_5^i and FT are unchanged from lemma 3.7.2.

Moreover, The unperturbed version $T_1(h),T_2(h),T_3^i(h)$ maps $C_t^0([0,t_0],C_\gamma^{i_0}([-1,1],H^k[-\pi,\pi]))\cap S_{\delta_1}$ to $C_t^0([0,t_0],C_\gamma^{i_0}([-1,1],H^{k-3}[-\pi,\pi]))\cap S$.

Proof. Everything is same except we take $\epsilon=0$ and lose several orders of derivative. Here k to $k-3$ is not the best bound but it is enough here. □

3.7.5 Lemmas

In the following lemmas, we always assume $i_0\le 4$, $k+i_0\le 30$, and $\lim_{x\to 0}\mathcal{O}(x)=0$. We first introduce a lemma to guarantee the Arc-chord condition, when $\|h\|_{C_\alpha^2}\lesssim\delta_1$, with δ_1 sufficiently small.

Lemma 3.7.19. *If $\|g\|_{C_\alpha^2}$, $\|\tilde{f}^+\|_{C_\alpha^2}$, is sufficiently small, $g(\alpha)$ satisfying $\partial_\alpha^i g(-\tau(t))=0$ for $0\le i\le 1$ and $g(\beta)=0$ when $\beta\le-\tau(t)$, we have the following inequalities. Here $\tilde{f}^+,\tilde{f}^L,\tilde{f}^{L,L}$ are fixed and defined in* (3.7) (3.74)*. When $-1\le\gamma\le 1$, $|\eta|\le|\gamma|$, $\frac{\pi}{4}\ge\alpha>-\tau(t)$,*

$$\cosh(\tilde{f}_2^L(\alpha+\tau(t))-\tilde{f}_2^L(\beta+\tau(t)))-\cos(\tilde{f}_1^L(\alpha+\tau(t))-\tilde{f}_1^L(\beta+\tau(t)))\gtrsim\|\alpha-\beta\|_{\mathbb{R}/2\pi\mathbb{Z}}^2. \tag{3.89}$$

$$\cosh(\tilde{f}_2^{L,L}(\alpha+\tau(t))-\tilde{f}_2^{L,L}(\beta+\tau(t)))-\cos(\tilde{f}_1^{L,L}(\alpha+\tau(t))-\tilde{f}_1^{L,L}(\beta+\tau(t)))\gtrsim\|\alpha-\beta\|_{\mathbb{R}/2\pi\mathbb{Z}}^2. \tag{3.90}$$

$$\cosh(\tilde{f}_2^L(\alpha_\gamma^t)-\tilde{f}_2^L(\beta_\gamma^t))-\cos(\tilde{f}_1^L(\alpha_\gamma^t)-\tilde{f}_1^L(\beta_\gamma^t))\gtrsim\|\alpha-\beta\|_{\mathbb{R}/2\pi\mathbb{Z}}^2. \tag{3.91}$$

$$\cosh(\tilde{f}_2^{L,L}(\alpha_\gamma^t)-\tilde{f}_2^{L,L}(\beta_\gamma^t)))-\cos(\tilde{f}_1^{L,L}(\alpha_\gamma^t)-\tilde{f}_1^{L,L}(\beta_\gamma^t))\gtrsim\|\alpha-\beta\|_{\mathbb{R}/2\pi\mathbb{Z}}^2. \tag{3.92}$$

$$\cosh(\tilde{f}_2^{L,L}(\tilde{\alpha}_\gamma^t)-\tilde{f}_2^{L,L}(\tilde{\beta}_\gamma^t))-\cos(\tilde{f}_1^{L,L}(\tilde{\alpha}_\gamma^t)-\tilde{f}_1^{L,L}(\tilde{\beta}_\gamma^t))\gtrsim\|\alpha-\beta\|_{\mathbb{R}/2\pi\mathbb{Z}}^2. \tag{3.93}$$

$$\cosh(g_2(\alpha)-g_2(\beta)+\tilde{f}_2^L(\alpha_\gamma^t)-\tilde{f}_2^L(\beta_\gamma^t))-\cos(g_1(\alpha)-g_1(\beta)+\tilde{f}_1^L(\alpha_\gamma^t)-\tilde{f}_1^L(\beta_\gamma^t))\gtrsim\|\alpha-\beta\|_{\mathbb{R}/2\pi\mathbb{Z}}^2. \tag{3.94}$$

$$\cosh(\tilde{f}_2^L(\alpha_\gamma^t)-\tilde{f}_2^L((2\alpha+\tau(t))_\eta^t))-\cos(\tilde{f}_1^L(\alpha_\gamma^t)-\tilde{f}_1^L((2\alpha+\tau(t))_\eta^t)))\gtrsim(\alpha+\tau(t))^2. \tag{3.95}$$

$$\cosh(g_2(\alpha)+\tilde{f}_2^L(\alpha_\gamma^t)-\tilde{f}_2^L((2\alpha+\tau(t))_\eta^t))-\cos(g_1(\alpha)+\tilde{f}_1^L(\alpha_\gamma^t)-\tilde{f}_1^L((2\alpha+\tau(t))_\eta^t))\gtrsim(\alpha+\tau(t))^2. \tag{3.96}$$

$$\cosh(\tilde{f}_2^L(\alpha_\gamma^t)-\tilde{f}_2^L((-\tau(t))_\eta^t))-\cos(\tilde{f}_1^L(\alpha_\gamma^t)-\tilde{f}_1^L((-\tau(t))_\eta^t)))\gtrsim(\alpha+\tau(t))^2. \tag{3.97}$$

$$\cosh(g_2(\alpha)+\tilde{f}_2^L(\alpha_\gamma^t)-\tilde{f}_2^L(-\tau(t))_\eta^t))-\cos(g_1(\alpha)+\tilde{f}_1^L(\alpha_\gamma^t)-\tilde{f}_1^L(-\tau(t)_\eta^t))\gtrsim(\alpha+\tau(t))^2. \quad (3.98)$$

Moreover, when additionally $\beta \geq 2\alpha+\tau(t)$ we have

$$\cosh(\tilde{f}_2^L(\alpha_\gamma^t)-\tilde{f}_2^L(\beta_0^t))-\cos(\tilde{f}_1^L(\alpha_\gamma^t)-\tilde{f}_1^L(\beta_0^t))\gtrsim\|\alpha-\beta\|^2_{\mathbb{R}/2\pi\mathbb{Z}}. \quad (3.99)$$

$$\cosh(-\tilde{f_2^+}(\beta_0^t)+\tilde{f}_2^L(\alpha_\gamma^t)-\tilde{f}_2^L(\beta_0^t))-\cos(-\tilde{f_1^+}(\beta_0^t)+\tilde{f}_1^L(\alpha_\gamma^t)-\tilde{f}_1^L(\beta_0^t))\gtrsim\|\alpha-\beta\|^2_{\mathbb{R}/2\pi\mathbb{Z}}. \quad (3.100)$$

$$\cosh(g_2(\alpha)-\tilde{f_2^+}(\beta_0^t)+\tilde{f}_2^L(\alpha_\gamma^t)-\tilde{f}_2^L(\beta_0^t))-\cos(g_1(\alpha)-\tilde{f_1^+}(\beta_0^t)+\tilde{f}_1^L(\alpha_\gamma^t)-\tilde{f}_1^L(\beta_0^t))\gtrsim\|\alpha-\beta\|^2_{\mathbb{R}/2\pi\mathbb{Z}}. \quad (3.101)$$

When additionally $\beta \leq -\tau(t)$, $0 \leq \zeta \leq 1$ we have

$$\cosh(g_2(\alpha)-g_2(\beta)+\tilde{f}_2^L(\alpha_\gamma^t)-\tilde{f}_2^L(\beta_\gamma^t))-\cos(g_1(\alpha)-g_1(\beta)+\tilde{f}_1^L(\alpha_\gamma^t)-\tilde{f}_1^L(\beta_\gamma^t))\gtrsim\|\alpha-\beta\|^2_{\mathbb{R}/2\pi\mathbb{Z}}. \quad (3.102)$$

$$\cosh(\tilde{f}_2^L(\alpha_\gamma^t)-\tilde{f}_2^L(\beta_0^t))-\cos(\tilde{f}_1^L(\alpha_\gamma^t)-\tilde{f}_1^L(\beta_0^t))\gtrsim\|\alpha-\beta\|^2_{\mathbb{R}/2\pi\mathbb{Z}}. \quad (3.103)$$

$$\cosh(\tilde{f}_2^{L,L}(\alpha_\gamma^t)-\tilde{f}_2^{L,L}(\beta_0^t))-\cos(\tilde{f}_1^{L,L}(\alpha_\gamma^t)-\tilde{f}_1^{L,L}(\beta_0^t))\gtrsim\|\alpha-\beta\|^2_{\mathbb{R}/2\pi\mathbb{Z}}, \quad (3.104)$$

$$\cosh(-\zeta\tilde{f}_2^-(\beta_0^t)+\tilde{f}_2^{L,L}(\alpha_\gamma^t)-\tilde{f}_2^{L,L}(\beta_0^t))-\cos(-\zeta\tilde{f}_1^-(\beta_0^t)+\tilde{f}_1^{L,L}(\alpha_\gamma^t)-\tilde{f}_1^{L,L}(\beta_0^t))\gtrsim\|\alpha-\beta\|^2_{\mathbb{R}/2\pi\mathbb{Z}}. \quad (3.105)$$

Proof. For $A, \tilde{A}, B, \tilde{B} \in \mathbb{R}$, we have

$$\begin{aligned}
&|\cosh(A)-\cos(\tilde{A})-(\cosh(B)-\cos(\tilde{B}))| \quad (3.106)\\
&\leq|\sinh(A+\tau(B-A))||A-B|+|\sin(\tilde{A}+\tau(\tilde{B}-\tilde{A}))||\tilde{A}-\tilde{B}|\\
&\leq C(|A|+|B|)(|A|+|B|)(|A-B|)+C(|\tilde{A}|+|\tilde{B}|)(|\tilde{A}|+|\tilde{B}|)(|\tilde{A}-\tilde{B}|).
\end{aligned}$$

with some increasing function $C(x)$. Since $f(\alpha,t)$ satisfies the arc-chord condition, we have

$$\cosh(f_2(\alpha)-f_2(\beta))-\cos(f_1(\alpha)-f_1(\beta))\gtrsim(\alpha-\beta)^2.$$

Because $\tilde{f}(\alpha,t)=f(X(\alpha,t),t)$, we have

$$\begin{aligned}
&\cosh(\tilde{f}_2(\alpha)-\tilde{f}_2(\beta))-\cos(\tilde{f}_1(\alpha)-\tilde{f}_1(\beta))\\
&=\cosh(f_2(x(\alpha,t))-f_2(x(\beta,t))-\cos(f_1(x(\alpha,t))-f_1(x(\beta),t))\gtrsim(x(\alpha,t)-x(\beta,t))^2\gtrsim(\alpha-\beta)^2
\end{aligned}$$

Then we only need to bound the difference, by (3.106), we have

$$
\begin{aligned}
&|\cosh(\tilde{f}_2^L(\alpha)-\tilde{f}_2^L(\beta))-\cos(\tilde{f}_1^L(\alpha)-\tilde{f}_2^L(\beta))|\\
&\geq |\cosh(\tilde{f}_2(\alpha)-\tilde{f}_2(\beta))-\cos(\tilde{f}_1(\alpha)-\tilde{f}_2(\beta))|-(\alpha-\beta)^2C(\|\tilde{f}(\alpha)\|_{C^1_\alpha}+\|\tilde{f}^+(\alpha)\|)_{C^1_\alpha}\|\tilde{f}^+(\alpha)\|_{C^1_\alpha}\\
&\gtrsim (\alpha-\beta)^2-(\alpha-\beta)^2C(\|\tilde{f}(\alpha)\|_{C^1_\alpha}+\|\tilde{f}^+(\alpha)\|_{C^1_\alpha})\|\tilde{f}^+(\alpha)\|_{C^1_\alpha},
\end{aligned}
$$

where we use $f=\tilde{f}^L+\tilde{f}^+$. f is fixed and by (3.8), we can take $\|\tilde{f}^+(\alpha)\|_{C^1_\alpha}$ to be sufficiently small if we let δ sufficiently small. Therefore we have (3.89). Similarly, from (3.74), $\|\tilde{f}^-(\alpha)\|_{C^1_\alpha}$ can also be controlled to be sufficiently small by choosing δ. Therefore we have (3.90).

We also have

$$
|\tilde{f}_i^L(\alpha)-\tilde{f}_i^L(\beta)-(\tilde{f}_i^L(\alpha+ic(\alpha)\gamma t)-\tilde{f}_i^L(\beta+ic(\beta)\gamma t))|\leq|\alpha-\beta|\|\tilde{f}_i^L(\alpha)-\tilde{f}_i^L(\alpha+ic(\alpha)\gamma t)\|_{C^1_\alpha}.
$$

As $t\to 0$, $\|\tilde{f}_i^L(\alpha)-\tilde{f}_i^L(\alpha+ic(\alpha)\gamma t)\|_{C^1_\alpha}$ can be taken sufficiently small. Hence we get (3.91). (3.92) follows the same way as (3.91). From (3.79), $\|\tilde{f}_1^{L,L}(\tilde{\alpha_\gamma^t})-\tilde{f}_1^{L,L}(\tilde{\beta_\gamma^t})\|_{C^1_\alpha}$ tends to 0 as $t\to 0$, then from (3.92) we have (3.93). Since $\|g\|_{C^2_\alpha}$ is sufficiently small, we have (3.94).

We take $\beta=2\alpha+\tau(t)$ in (3.91) to get

$$
\cosh(\tilde{f}_2^L(\alpha_\gamma^t)-\tilde{f}_2^L((2\alpha+\tau(t))_\gamma^t))-\cos(\tilde{f}_1^L(\alpha_\gamma^t)-\tilde{f}_1^L((2\alpha+\tau(t))_\gamma^t)))\gtrsim(\alpha+\tau(t))^2.
$$

By $|\tilde{f}^L(\alpha_\gamma^t)-\tilde{f}^L(\alpha_\eta^t)|\lesssim t(\alpha+\tau(t))^2$ and $|g(\alpha)|\leq\|g\|_{C^2}(\alpha+\tau(t))^2$ we get (3.96). (3.97) and (3.98) are similar by taking $\beta=-\tau(t)$ instead of $2\alpha+\tau(t)$.

When $\beta\geq 2\alpha+\tau(t)$, from (3.91), we have

$$
\cosh(\tilde{f}_2^L(\alpha_\gamma^t)-\tilde{f}_2^L(\beta_\gamma^t))-\cos(\tilde{f}_1^L(\alpha_\gamma^t)-\tilde{f}_1^L(\beta_\gamma^t))\gtrsim\|\alpha-\beta\|^2_{\mathbb{R}/2\pi\mathbb{Z}}.
$$

By $|\tilde{f}^L(\beta_\gamma^t)-\tilde{f}^L(\beta_0^t)|\lesssim t(\beta+\tau(t))^2$, $|\tilde{f}^+(\beta)|\leq\|\tilde{f}^+\|_{C^2}(\beta+\tau(t))^2$, $|g(\alpha)|\leq\|\tilde{f}^+\|_{C^2}(\beta+\tau(t))^2$, and $(\beta+\tau(t))^2\lesssim\|\alpha-\beta\|^2_{\mathbb{R}/2\pi\mathbb{Z}}$, we have (3.100) and (3.101).

When $\beta\leq-\tau(t)$, we still use (3.91) to get

$$
\cosh(\tilde{f}_2^L(\alpha_\gamma^t)-\tilde{f}_2^L(\beta_\gamma^t))-\cos(\tilde{f}_1^L(\alpha_\gamma^t)-\tilde{f}_1^L(\beta_\gamma^t))\gtrsim\|\alpha-\beta\|^2_{\mathbb{R}/2\pi\mathbb{Z}}
$$

Since when $\beta\leq-\tau(t)$, we have $\beta_\gamma^t=\beta_0^t$. Then we have (3.103). Moreover, $g(\beta)=0$ when $\beta\leq-\tau(t)$ and $|g(\alpha)|\leq\|g\|_{C^2}(\alpha+\tau(t))^2$. Hence we get (3.102).

Similarly, since $|\tilde{f}^-(\beta_0^t)| \le \|\tilde{f}^-\|_{C^2}(\beta+\tau(t))^2$, we can use (3.92) to get (3.104) and(3.105). $\square$

Lemma 3.7.20. *For any $\sigma \in \mathcal{N}$, $t \le t_0$, we have*

$$\|K_{-\sigma}(0, V_{\tilde{f}^L}^{21}(\alpha_\gamma^t, t) - V_{\tilde{f}^L}^{21}(\beta_\gamma^t, t), V_X(\alpha_\gamma^t, t) - V_X(\beta_\gamma^t, t))(\alpha-\beta)^\sigma\|_{C_\gamma^{i_0}([-1,1], C^{0,1}([-\tau(t),\pi]\times[-\pi,\pi]))} \lesssim 1 \tag{3.107}$$

$$\|K_{-\sigma}(0, V_{\tilde{f}^{L,L}}(\tilde{\alpha}_\gamma^t, t) - V_{\tilde{f}^{L,L}}(\tilde{\beta}_\gamma^t, t), V_X(\tilde{\alpha}_\gamma^t, t) - V_X(\tilde{\beta}_\gamma^t, t))(\alpha-\beta)^\sigma\|_{C_\gamma^{i_0}([-1,1], C^{K+4}([-\tau(t),\pi]\times[-\pi,\pi]))} \lesssim 1, \tag{3.108}$$

and for $0 \le t' < t \le t_0$,

$$\begin{aligned}&\|K_{-\sigma}(0, V_{\tilde{f}^{L,L}}(\tilde{\alpha}_\gamma^t, t) - V_{\tilde{f}^{L,L}}(\tilde{\beta}_\gamma^t, t), V_X(\tilde{\alpha}_\gamma^t, t) - V_X(\tilde{\beta}_\gamma^t, t))(\alpha-\beta)^\sigma \\ &- K_{-\sigma}(0, V_{\tilde{f}^{L,L}}(\tilde{\alpha}_\gamma^{t'}, t') - V_{\tilde{f}^{L,L}}(\tilde{\beta}_\gamma^{t'}, t'), V_X(\tilde{\alpha}_\gamma^{t'}, t') - V_X(\tilde{\beta}_\gamma^{t'}, t'))(\alpha-\beta)^\sigma\|_{C_\gamma^{i_0}([-1,1], C^{K+3}([-\tau(t'),\pi]\times[-\pi,\pi]))} \\ &\lesssim \mathcal{O}(t-t').\end{aligned} \tag{3.109}$$

Proof. For (3.107), from the arc-chord condition (3.91), , and the structure of $K^{-\sigma}$ we only need to show

$$\|\frac{(\partial_\alpha^{d_i}\tilde{f}^L)(\alpha_\gamma^t, t) - (\partial_\alpha^{d_i}\tilde{f}^L)^L(\beta_\gamma^t, t)}{(\alpha-\beta)}\|_{C_\gamma^{i_0}([-1,1], C^{0,1}([-\tau(t),\pi]\times[-\pi,\pi]))} \lesssim 1, \tag{3.110}$$

with $d_i \le 21$, and

$$\|\frac{X(\alpha_\gamma^t, t) - X(\beta_\gamma^t, t)}{(\alpha-\beta)}\|_{C_\gamma^{i_0}([-1,1], C^{0,1}([-\tau(t),\pi]\times[-\pi,\pi]))} \lesssim 1, \tag{3.111}$$

From (3.15), $X(\alpha,t)$, $\partial_\alpha^{d_i}\tilde{f}(\alpha,t) \in C^1([0,t], C^{38}(-\infty,\infty))$. We also have $c(\alpha) \in C^{1,1}(-\infty,\infty)$. Therefore

$$X(\alpha_\gamma^t, t),\ \tilde{f}^L(\alpha_\gamma^t, t) \in L^\infty([0,t], C_\gamma^{i_0}([-1,1], C^{1,1}(-\infty,\infty))).$$

Then (3.110) and (3.111) follows and we have (3.107).

Similarly, since $\tilde{c}(\alpha) \in C^{60}(-\infty,\infty)$ and $i_0 + k \le 30$, we have

$$X(\tilde{\alpha}_\gamma^t, t),\ \tilde{f}^L(\tilde{\alpha}_\gamma^t, t) \in C^0([0,t], C_\gamma^{i_0}([-1,1], C^{k+4}(-\infty,\infty))), \tag{3.112}$$

Then from arc-chord condition (3.93), we have (3.108).

For the difference in t, we claim

$$
\begin{aligned}
&K_{-\sigma}(0,\Delta V_{\tilde{f}^{L,L}}(\tilde{\alpha}^t_\gamma,t),\Delta V_X(\tilde{\alpha}^t_\gamma,t))(\alpha-\beta)^\sigma - K_{-\sigma}(0,\Delta V_{\tilde{f}^{L,L}}(\tilde{\alpha}^{t'}_\gamma,t'),\Delta V_X(\tilde{\alpha}^{t'}_\gamma,t'))(\alpha-\beta)^\sigma\\
&=\sum_l \int_0^1 K^l_{-(\sigma+1)}(0,\Delta V_{\tilde{f}^{L,L}}(\tilde{\alpha}^t_\gamma,t)+\zeta(\Delta V_{\tilde{f}^{L,L}}(\tilde{\alpha}^{t'}_\gamma,t')-\Delta V_{\tilde{f}^{L,L}}(\tilde{\alpha}^t_\gamma,t))\\
&,\Delta V_X(\tilde{\alpha}^t_\gamma,t)+\zeta(\Delta V_X(\tilde{\alpha}^{t'}_\gamma,t')-\Delta V_X(\tilde{\alpha}^t_\gamma,t)),t))d\zeta(\alpha-\beta)^{\sigma+1}\frac{(\Delta V_{\tilde{f}^{L,L}}(\tilde{\alpha}^{t'}_\gamma,t')-\Delta V_{\tilde{f}^{L,L}}(\tilde{\alpha}^t_\gamma,t))}{(\alpha-\beta)}\\
&+\sum_{l'} \int_0^1 K^{l'}_{-(\sigma+1)}(0,\Delta V_{\tilde{f}^{L,L}}(\tilde{\alpha}^t_\gamma,t)+\zeta(\Delta V_{\tilde{f}^{L,L}}(\tilde{\alpha}^{t'}_\gamma,t')-\Delta V_{\tilde{f}^{L,L}}(\tilde{\alpha}^t_\gamma,t))\\
&,\Delta V_X(\tilde{\alpha}^t_\gamma,t)+\zeta(\Delta V_X(\tilde{\alpha}^{t'}_\gamma,t')-\Delta V_X(\tilde{\alpha}^t_\gamma,t)),t))d\zeta(\alpha-\beta)^{\sigma+1}\frac{(\Delta V_X(\tilde{\alpha}^{t'}_\gamma,t')-\Delta V_X(\tilde{\alpha}^t_\gamma,t))}{(\alpha-\beta)},
\end{aligned}
$$

for some $K_{-(\sigma+1)}$ kernel. Then from (3.108) (we claim the extra ζ term will not affect the result) and (3.112), we get (3.109). □

Lemma 3.7.21. *Let* $Y^{i_0,k}(t)=C^{i_0}_\gamma([-1,1],C^k([-\tau(t),\frac{\pi}{4}]\times[-\tau(t),\pi))$. *For* h, g *in*

$$W^{i_0,k}=C^0_t([0,t_0],C^{i_0}_\gamma([-1,1],H^k([-\pi,\pi])))\cap S_{\delta_1},$$

we have the following four inequalities for $t\le t*$*:*

$$
\begin{aligned}
&\|K_{-\sigma}(V^{10}_h(\alpha,\gamma,t)-V^{10}_h(\beta,\gamma,t),V^{21}_{\tilde{f}^L}(\alpha^t_\gamma,t)-V^{21}_{\tilde{f}^L}(\beta^t_\gamma,t),V_X(\alpha^t_\gamma,t)-V_X(\beta^t_\gamma,t))(\alpha-\beta)^\sigma\|_{Y^{i_0,k+4}(t)} \qquad (3.113)\\
&\lesssim 1,
\end{aligned}
$$

$$
\begin{aligned}
&\|K_{-\sigma}(V^{10}_h(\alpha,\gamma,t)-V^{10}_h(\beta,\gamma,t),V^{21}_{\tilde{f}^L}(\alpha^t_\gamma,t)-V^{21}_{\tilde{f}^L}(\beta^t_\gamma,t),V_X(\alpha^t_\gamma,t)-V_X(\beta^t_\gamma,t)) \qquad (3.114)\\
&-K_{-\sigma}(V^{10}_g(\alpha,\gamma,t)-V^{10}_g(\beta,\gamma,t),V^{21}_{\tilde{f}^L}(\alpha^t_\gamma,t)-V^{21}_{\tilde{f}^L}(\beta^t_\gamma,t),V_X(\alpha^t_\gamma,t)-V_X(\beta^t_\gamma,t))(\alpha-\beta)^\sigma\|_{Y^{i_0,k+4}(t)}\\
&\lesssim \|h-g\|_{W^{i_0,k}},
\end{aligned}
$$

$$
\sup_t \|D_hK_{-\sigma}(V^{10}_h(\alpha,\gamma,t)-V^{10}_h(\beta,\gamma,t),V^{21}_{\tilde{f}^L}(\alpha^t_\gamma,t)-V^{21}_{\tilde{f}^L}(\beta^t_\gamma,t),V_X(\alpha^t_\gamma,t)-V_X(\beta^t_\gamma,t))[g](\alpha-\beta)^\sigma\|_{Y^{i_0,k+4}(t)} \qquad (3.115)
$$

$\lesssim \|g\|_{W^{i_0,k}}$

and for $0 \le t' < t \le t_0$,

$$\|K_{-\sigma}(V_h^{10}(\alpha,\gamma,t) - V_h^{10}(\beta,\gamma,t), V_{\tilde{f}^L}^{21}(\alpha_\gamma^t,t) - V_{\tilde{f}^L}^{21}(\beta_\gamma^t,t), V_X(\alpha_\gamma^t,t) - V_X(\beta_\gamma^t,t)) \tag{3.116}$$
$$- K_{-\sigma}(V_h^{10}(\alpha,\gamma,t') - V_h^{10}(\beta,\gamma,t'), V_{\tilde{f}^L}^{21}(\alpha_\gamma^{t'},t') - V_{\tilde{f}^L}^{21}(\beta_\gamma^{t'},t'), V_X(\alpha_\gamma^{t'},t') - V_X(\beta_\gamma^{t'},t'))(\alpha-\beta)^\sigma\|_{Y^{i_0,k+3}(t')}$$
$$\lesssim \mathcal{O}(t-t') + \|h(\alpha,\gamma,t) - h(\alpha,\gamma,t')\|_{C_\gamma^{i_0}([-1,1],H^k[-\pi,\pi])}.$$

Proof. We have

$$\|\frac{V_h^{10}(\alpha,\gamma,t) - V_h^{10}(\beta,\gamma,t)}{\alpha-\beta}\|_{Y^{i_0,k+4}(t)} \lesssim \|h\|_{W^{i_0,k}},$$

$$\|\frac{V_{\tilde{f}^L}^{21}(\alpha_\gamma^t,t) - V_{\tilde{f}^L}^{21}(\beta_\gamma^t,t)}{\alpha-\beta}\|_{Y^{i_0,k+4}(t)} \lesssim \|\tilde{f}^L(\alpha)\|_{C_t^0([0,t_0],C^{38}(-\infty,\infty))},$$

$$\|\frac{V_X(\alpha_\gamma^t,t) - V_X(\beta_\gamma^t,t)}{\alpha-\beta}\|_{Y^{i_0,k+4}(t)} \lesssim \|X\|_{C_t^0([0,t_0],C^{38}(-\infty,\infty))},$$

From the arc-chord condition (3.94), by taking $g(\alpha) = D^{-20}h(\alpha,\gamma,t)$, we have

$$\cosh(h_2(\alpha) - h_2(\beta) + \tilde{f}_2^L(\alpha_\gamma^t) - \tilde{f}_2^L(\beta_\gamma^t)) - \cos(h_1(\alpha) - h_1(\beta) + \tilde{f}_1^L(\alpha_\gamma^t) - \tilde{f}_1^L(\beta_\gamma^t)) \gtrsim \|\alpha-\beta\|_{\mathbb{R}/2\pi\mathbb{Z}}^2.$$

Then from the structure of $K^{-\sigma}$ and inequalities above, we have (3.113). Moreover, since

$$\frac{d}{d\zeta}K_{-\sigma}(\Delta V_h^{10}(\alpha,\gamma,t) + \zeta(\Delta V_g^{10}(\alpha,\gamma,t) - \Delta V_h^{10}(\alpha,\gamma,t)), \Delta V_{\tilde{f}^L}^{21}(\alpha_\gamma^t,t), \Delta V_X(\alpha_\gamma^t,t))(\alpha-\beta)^\sigma$$
$$= \sum_l K_{-(\sigma+1)}^l(\Delta V_h^{10}(\alpha,\gamma,t) + \zeta(\Delta V_g^{10}(\alpha,\gamma,t) - \Delta V_h^{10}(\alpha,\gamma,t)), \Delta V_{\tilde{f}^L}^{21}(\alpha_\gamma^t,t), \Delta V_X(\alpha_\gamma^t,t))$$
$$\cdot (\alpha-\beta)^{\sigma+1}\frac{D^{-20+s_l}h(\alpha,\gamma,t) - D^{-20+s_l}g(\alpha,\gamma,t)}{(\alpha-\beta)},$$

with $s_l \le 10$, we have (3.114) and (3.115) by using the (3.113) for $\sigma+1$ type. For the difference in

t with $0 \le t' < t \le t_0$, we have for $\alpha, \beta \ge -\tau(t')$

$$
\begin{aligned}
&K_{-\sigma}(\Delta V_h^{10}(\tilde{\alpha}_\gamma^t, t), \Delta V_{\tilde{f}^L}^{21}(\alpha_\gamma^t, t), \Delta V_X(\alpha_\gamma^t, t))(\alpha-\beta)^\sigma - K_{-\sigma}(\Delta V_h^{10}(\alpha_\gamma^{t'}, t'), \Delta V_{\tilde{f}^L}^{21}(\alpha_\gamma^{t'}, t'), \Delta V_X(\alpha_\gamma^{t'}, t'))(\alpha-\beta)^\sigma \\
&= \sum_{l_1} \int_0^1 K_{-(\sigma+1)}^{l_1}(\Delta V_h^{10}(\alpha_\gamma^t, t) + \zeta(\Delta V_h^{10}(\alpha_\gamma^{t'}, t') - \Delta V_h^{10}(\alpha_\gamma^t, t)), \Delta V_{\tilde{f}^L}^{21}(\alpha_\gamma^t, t) + \zeta(\Delta V_{\tilde{f}^L}^{21}(\alpha_\gamma^{t'}, t') - \Delta V_{\tilde{f}^L}^{21}(\alpha_\gamma^t, t)) \\
&, \Delta V_X(\alpha_\gamma^t, t) + \zeta(\Delta V_X(\alpha_\gamma^{t'}, t') - \Delta V_X(\alpha_\gamma^t, t)), t)) d\zeta \cdot (\alpha-\beta)^{\sigma+1} \frac{(\Delta D^{-20+s_{l_1}} h(\alpha, \gamma, t') - \Delta D^{-20+s_{l_1}} h(\alpha, \gamma, t))}{(\alpha-\beta)} \\
&\sum_{l_2} \int_0^1 K_{-(\sigma+1)}^{l_1}(\Delta V_h^{10}(\alpha_\gamma^t, t) + \zeta(\Delta V_h^{10}(\alpha_\gamma^{t'}, t') - \Delta V_h^{10}(\alpha_\gamma^t, t)), \Delta V_{\tilde{f}^L}^{21}(\alpha_\gamma^t, t) + \zeta(\Delta V_{\tilde{f}^L}^{21}(\alpha_\gamma^{t'}, t') - \Delta V_{\tilde{f}^L}^{21}(\alpha_\gamma^t, t)) \\
&, \Delta V_X(\alpha_\gamma^t, t) + \zeta(\Delta V_X(\alpha_\gamma^{t'}, t') - \Delta V_X(\alpha_\gamma^t, t)), t)) d\zeta (\alpha-\beta)^{\sigma+1} \frac{(\Delta \partial_\alpha^{n_{l_2}} \tilde{f}^L(\alpha_\gamma^{t'}, t') - \Delta \partial_\alpha^{n_{l_2}} \tilde{f}^L(\alpha_\gamma^t, t))}{(\alpha-\beta)} \\
&+ \sum_{l_3} \int_0^1 K_{-(\sigma+1)}^{l_2}(\Delta V_h^{10}(\alpha_\gamma^{t'}, t') - \Delta V_h^{10}(\alpha_\gamma^t, t)), \Delta V_{\tilde{f}^L}^{21}(\alpha_\gamma^t, t) + \zeta(\Delta V_{\tilde{f}^L}^{21}(\alpha_\gamma^{t'}, t') - \Delta V_{\tilde{f}^L}^{21}(\alpha_\gamma^t, t)) \\
&, \Delta V_X(\alpha_\gamma^t, t) + \zeta(\Delta V_X(\alpha_\gamma^{t'}, t') - \Delta V_X(\alpha_\gamma^t, t)), t)) d\zeta \cdot (\alpha-\beta)^{\sigma+1} \frac{(\Delta X_{n_{l_3}}(\alpha_\gamma^{t'}, t') - \Delta X_{n_{l_3}}(\alpha_\gamma^t, t))}{(\alpha-\beta)},
\end{aligned}
$$

with $s_{l_1} \le 10$, $n_{l_2} \le 21$. Then we can get (3.116). □

Lemma 3.7.22. *Let $\tilde{Y}^{i_0,k}(t) = C_\gamma^{i_0}([-1,1]\times[-1,1], C^k([-\tau(t), \frac{\pi}{4}])$. For h, g in $W^{i_0,k}$, we have the following four inequalities:*

$$
\begin{aligned}
&\|K_{-\sigma}(V_h^{10}(\alpha,\gamma,t) - V_h^{10}(2\alpha+\tau(t),\eta,t), V_{\tilde{f}^L}^{21}(\alpha_\gamma^t, t) - V_{\tilde{f}^L}^{21}((2\alpha+\tau(t))_\eta^t, t), V_X(\alpha_\gamma^t, t) - V_X((2\alpha+\tau(t))_\eta^t, t)) \\
&(\alpha+\tau(t))^\sigma\|_{\tilde{Y}^{i_0,k+4}(t)} \lesssim 1,
\end{aligned} \tag{3.117}
$$

$$
\begin{aligned}
&\|K_{-\sigma}(V_h^{10}(\alpha,\gamma,t) - V_h^{10}(2\alpha+\tau(t),\eta,t), V_{\tilde{f}^L}^{21}(\alpha_\gamma^t, t) - V_{\tilde{f}^L}^{21}((2\alpha+\tau(t))_\eta^t, t), V_X(\alpha_\gamma^t, t) - V_X((2\alpha+\tau(t))_\eta^t, t)) \\
&- K_{-\sigma}(V_g^{10}(\alpha,\gamma,t) - V_g^{10}(2\alpha+\tau(t),\eta,t), V_{\tilde{f}^L}^{21}(\alpha_\gamma^t, t) - V_{\tilde{f}^L}^{21}((2\alpha+\tau(t))_\eta^t, t), V_X(\alpha_\gamma^t, t) - V_X((2\alpha+\tau(t))_\eta^t, t)) \\
&(\alpha+\tau(t))^\sigma\|_{\tilde{Y}^{i_0,k+4}(t)} \lesssim \|h-g\|_{W^{i_0,k}},
\end{aligned} \tag{3.118}
$$

$$
\begin{aligned}
&\|D_h K_{-\sigma}(V_h^{10}(\alpha,\gamma,t) - V_h^{10}(2\alpha+\tau(t),\eta,t), V_{\tilde{f}^L}^{21}(\alpha_\gamma^t, t) - V_{\tilde{f}^L}^{21}((2\alpha+\tau(t))_\eta^t, t), V_X(\alpha_\gamma^t, t) - V_X((2\alpha+\tau(t))_\eta^t, t))[g]
\end{aligned} \tag{3.119}
$$

$(\alpha+\tau(t))^{\sigma}\|_{\tilde{Y}^{i_0,k+4}(t)} \lesssim \|h-g\|_{W^{i_0,k}}$,

and for $0 \le t' < t \le t_0$,

$$\|K_{-\sigma}(V_h^{10}(\alpha,\gamma,t) - V_h^{10}(2\alpha+\tau(t),\eta,t), V_{\tilde{f}^L}^{21}(\alpha_\gamma^t,t) - V_{\tilde{f}^L}^{21}((2\alpha+\tau(t))_\eta^t,t), V_X(\alpha_\gamma^t,t) - V_X((2\alpha+\tau(t))_\eta^t,t)) \tag{3.120}$$

$$- K_{-\sigma}(V_h^{10}(\alpha,\gamma,t') - V_h^{10}(2\alpha+\tau(t'),\eta,t'), V_{\tilde{f}^L}^{21}(\alpha_\gamma^{t'},t') - V_{\tilde{f}^L}^{21}((2\alpha+\tau(t'))_\eta^{t'},t'), V_X(\alpha_\gamma^{t'},t') - V_X((2\alpha+\tau(t'))_\eta^{t'},t'))$$

$(\alpha+\tau(t))^{\sigma}\|_{\tilde{Y}^{i_0,k+3}(t')} \lesssim \mathcal{O}(t-t') + \|h(\alpha,\gamma,t) - h(\alpha,\gamma,t')\|_{C_\gamma^{i_0}([-1,1],H^k[-\pi,\pi])}$.

Proof. We can use the similar proof as in lemma 3.7.21, by using arc-chord condition (3.96) and by the following inequalities. For (3.117), (3.118), (3.119), we have

$$\|\frac{V_h^{10}(\alpha,\gamma,t)}{(\alpha+\tau(t))}\|_{Y^{i_0,k+4}(t)} \lesssim \|h\|_{W^{i_0,k}},$$

$$\|\frac{V_h^{10}(2\alpha+\tau(t),\gamma,t)}{(\alpha+\tau(t))}\|_{Y^{i_0,k+4}(t)} \lesssim \|h\|_{W^{i_0,k}},$$

$$\|\frac{V_{\tilde{f}^L}^{21}(\alpha_\gamma^t,t) - V_{\tilde{f}^L}^{21}((2\alpha+\tau(t))_\eta^t,t)}{(\alpha+\tau(t))}\|_{Y^{i_0,k+4}(t)} \lesssim \|\tilde{f}^L(\alpha)\|_{C_t^0([0,t_0],C^{38}(-\infty,\infty))},$$

$$\|\frac{V_X(\alpha_\gamma^t,t) - V_X((2\alpha+\tau(t))_\eta^t,t)}{\alpha-\beta}\|_{Y^{i_0,k+4}(t)} \lesssim \|X\|_{C_t^0([0,t_0],C^{38}(-\infty,\infty))}.$$

For (3.120), we have

$$\|\frac{V_h^{10}(\alpha,\gamma,t)}{(\alpha+\tau(t))} - \frac{V_h^{10}(\alpha,\gamma,t')}{(\alpha+\tau(t'))}\|_{Y^{i_0,k+4}(t')} \lesssim \|h(\alpha,\gamma,t) - h(\alpha,\gamma,t')\|_{C_\gamma^{i_0}([-1,1],H^k[-\pi,\pi])}$$

$$\|\frac{V_h^{10}(2\alpha+\tau(t),\gamma,t)}{(\alpha+\tau(t))} - \frac{V_h^{10}(2\alpha+\tau(t'),\gamma,t')}{(\alpha+\tau(t'))}\|_{Y^{i_0,k+4}(t)} \lesssim \mathcal{O}(t-t'),$$

$$\|\frac{V^{21}_{\tilde{f}^L}(\alpha^t_\gamma,t)-V^{21}_{\tilde{f}^L}((2\alpha+\tau(t))^t_\eta,t)}{(\alpha+\tau(t))}-\frac{V^{21}_{\tilde{f}^L}(\alpha^{t'}_\gamma,t')-V^{21}_{\tilde{f}^L}((2\alpha+\tau(t'))^{t'}_\eta,t')}{(\alpha+\tau(t'))}\|_{Y^{i_0,k+3}(t')}\lesssim\mathcal{O}(t-t'),$$

$$\|\frac{V_X(\alpha^t_\gamma,t)-V_X((2\alpha+\tau(t))^t_\eta,t)}{\alpha+\tau(t)}-\frac{V_X(\alpha^{t'}_\gamma,t')-V_X((2\alpha+\tau(t))^{t'}_\eta,t')}{\alpha+\tau(t')}\|_{Y^{i_0,k+3}(t)}\lesssim\mathcal{O}(t-t').$$

We can control the $\frac{V^{21}_{\tilde{f}^L}(\alpha^t_\gamma,t)-V^{21}_{\tilde{f}^L}((2\alpha+\tau(t))^t_\eta,t)}{\alpha+\tau(t)}$ because

$$\begin{aligned}&V^{21}_{\tilde{f}^L}(\alpha^t_\gamma,t)-V^{21}_{\tilde{f}^L}((2\alpha+\tau(t))^t_\eta,t)=\\&V^{21}_{\tilde{f}^L}(\alpha^t_\gamma,t)-V^{21}_{\tilde{f}^L}((2\alpha+\tau(t))^t_\gamma,t)+V^{21}_{\tilde{f}^L}((2\alpha+\tau(t))^t_\gamma,t)-V^{21}_{\tilde{f}^L}((2\alpha+\tau(t))^t_\eta,t),\end{aligned}$$

and $\alpha^t_\gamma-\alpha^t_\eta=ic(\alpha+\tau(t))(\gamma-\eta)t$ with $\frac{c(\alpha+\tau(t))}{(\alpha+\tau(t))^2}\in C^\infty[-\tau(t),\pi]$. □

Corollary 3.7.23. *Let $\overline{Y}^{i_0,k}=C^{i_0}_\gamma([-1,1],C^k[-\tau(t),\frac{\pi}{4}])$. For h, g in $W^{i_0,k}$, with particular*

$$K_{-1}(A,B,C)=\frac{\sin(A_1+B_1)}{\cosh(A_2+B_2)-\cos(A_1+B_1)},$$

we have the following four inequalities:

$$\|K_{-1}(\Delta V^{10}_h(\alpha,\gamma,t),\Delta V^{21}_{\tilde{f}^L}(\alpha^t_\gamma,t),\Delta V_X(\alpha^t_\gamma,t))|_{\beta=2\alpha+\tau(t)}\|_{\overline{Y}^{i_0,k+3}}\lesssim 1,$$

$$\begin{aligned}&\|K_{-1}(\Delta V^{10}_h(\alpha,\gamma,t),\Delta V^{21}_{\tilde{f}^L}(\alpha^t_\gamma,t),\Delta V_X(\alpha^t_\gamma,t))\\&-K_{-1}(\Delta V^{10}_g(\alpha,\gamma,t),\Delta V^{21}_{\tilde{f}^L}(\alpha^t_\gamma,t),\Delta V_X(\alpha^t_\gamma,t))|_{\beta=2\alpha+\tau(t)}\|_{\overline{Y}^{i_0,k+3}}\\&\lesssim\|h-g\|_{W^{i_0,k}},\end{aligned}$$

$$\begin{aligned}&\|D_hK_{-1}(\Delta V^{10}_h(\alpha,\gamma,t),\Delta V^{21}_{\tilde{f}^L}(\alpha^t_\gamma,t),\Delta V_X(\alpha^t_\gamma,t))[g]|_{\beta=2\alpha+\tau(t)}\|_{\overline{Y}^{i_0,k+3}}\\&\lesssim\|g\|_{W^{i_0,k}},\end{aligned}$$

and for $0 \le t' < t \le t_0$,

$$\begin{aligned}
&\|K_{-1}(\Delta V_h^{10}(\alpha,\gamma,t), \Delta V_{\tilde{f}^L}^{21}(\alpha_\gamma^t, t), \Delta V_X(\alpha_\gamma^t, t)) \\
&- K_{-1}(\Delta V_h^{10}(\alpha,\gamma,t'), \Delta V_{\tilde{f}^L}^{21}(\alpha_\gamma^{t'}, t'), \Delta V_X(\alpha_\gamma^{t'}, t'))|_{\beta=2\alpha+\tau(t)}\|_{\overline{Y}^{i_0,k+2}} \\
&\lesssim \mathcal{O}(t-t') + \|h(\alpha,\gamma,t) - h(\alpha,\gamma,t')\|_{C_\gamma^{i_0}([-1,1],H^k[-\pi,\pi])}.
\end{aligned}$$

Proof. We use the similar proof as the previous lemma 3.7.22, except that here we have

$$\|\frac{\sin(A_1+B_1)|_{\beta=2\alpha+\tau(t)}}{(\alpha+\tau(t))^2}\|_{C^0([0,t*],\overline{Y}^{i_0,k+3})} \lesssim 1,$$

because the $\partial_\alpha \tilde{f}_1^L(0,t) = 0$. This helps us to get one more $(\alpha+\tau(t))$ in the numerator. In fact, we have

$$\begin{aligned}
&A_1 + B_1|_{\beta=2\alpha+\tau(t)} = D^{-20}h_1(\alpha,\gamma,t) - D^{-20}h_1(2\alpha+\tau(t),\gamma,t) + \tilde{f}_1^L((\alpha)_\gamma^t, t) - \tilde{f}_1^L((2\alpha+\tau(t))_\gamma^t, t) \\
&= -\int_0^1 \frac{D^{-19}h_1(\alpha+\zeta(\alpha+\tau(t)),\gamma,t)}{\alpha+\tau(t)}(1+ic'((\alpha+\tau(t)(1+\zeta))\gamma t)d\zeta(\alpha+\tau(t))^2 \\
&- \int_0^1 \frac{\tilde{f}_1^{L'}((\alpha+\zeta(\alpha+\tau(t)))_\gamma^t, t)}{\alpha+\tau(t)}(1+ic'((\alpha+\tau(t)(1+\zeta))\gamma t)d\zeta(\alpha+\tau(t))^2.
\end{aligned}$$

Since $D^{-19}h_1(\alpha+\zeta(\alpha+\tau(t)),\gamma,t)|_{\alpha=-\tau(t)} = 0$ and $\tilde{f}_1^{L'}((\alpha+\zeta(\alpha+\tau(t)))_\gamma^t, t)_{\alpha=-\tau(t)} = 0$, we get the result. □

Corollary 3.7.24. *Let* $\overline{Y}^{i_0,k} = C_\gamma^{i_0}([-1,1], C^{k+2}[-\tau(t), \frac{\pi}{4}])$. *For* h, g *in* $W^{i_0,k}$, *with particular*

$$K_{-1}(A,B,C) = \frac{\sin(A_1+B_1)}{\cosh(A_2+B_2) - \cos(A_1+B_1)},$$

we have the following four inequalities:

$$\|K_{-1}(\Delta V_h^{10}(\alpha,\gamma,t), \Delta V_{\tilde{f}^L}^{21}(\alpha_\gamma^t, t), \Delta V_X(\alpha_\gamma^t, t))|_{\beta=-\tau(t)}\|_{\overline{Y}^{i_0,k+3}} \lesssim 1,$$

$$\begin{aligned}
&\|K_{-1}(\Delta V_h^{10}(\alpha,\gamma,t), \Delta V_{\tilde{f}^L}^{21}(\alpha_\gamma^t, t), \Delta V_X(\alpha_\gamma^t, t)) \\
&- K_{-1}(\Delta V_g^{10}(\alpha,\gamma,t), \Delta V_{\tilde{f}^L}^{21}(\alpha_\gamma^t, t), \Delta V_X(\alpha_\gamma^t, t))|_{\beta=-\tau(t)}\|_{\overline{Y}^{i_0,k+3}} \\
&\lesssim \|h-g\|_{W^{i_0,k}},
\end{aligned}$$

$$\|D_h K_{-1}(\Delta V_h^{10}(\alpha,\gamma,t),\Delta V_{\tilde{f}^L}^{21}(\alpha_\gamma^t,t),\Delta V_X(\alpha_\gamma^t,t))[g]|_{\beta=-\tau(t)}\|_{\overline{Y}^{i_0,k+3}}$$
$$\lesssim \|g\|_{W^{i_0,k}},$$

and for $0 \le t' < t \le t_0$,

$$\|K_{-1}(\Delta V_h^{10}(\alpha,\gamma,t),\Delta V_{\tilde{f}^L}^{21}(\alpha_\gamma^t,t),\Delta V_X(\alpha_\gamma^t,t))$$
$$- K_{-1}(\Delta V_h^{10}(\alpha,\gamma,t'),\Delta V_{\tilde{f}^L}^{21}(\alpha_\gamma^{t'},t'),\Delta V_X(\alpha_\gamma^{t'},t'))|_{\beta=-\tau(t)}\|_{\overline{Y}^{i_0,k+2}}$$
$$\lesssim \mathcal{O}(t-t') + \|h(\alpha,\gamma,t)-h(\alpha,\gamma,t')\|_{C_\gamma^{i_0}([-1,1],H^k[-\pi,\pi])}.$$

Proof. We use the same proof as the previous corollary 3.7.23 by the arc-chord condition (3.98). □

Lemma 3.7.25. *For* h, g *in* $W^{i_0,k}$, $0 \le \zeta_0 \le 1$, $l+l' \le k+3$, $j \le i_0$, *when* $-\tau(t) < \alpha \le \frac{\pi}{4}$, $\zeta \in [0,1]$, *we have the following four inequalities:*

$$\sup_{\beta\in[-\pi,-\tau(t)]\cup[2\alpha+\tau(t),\pi]} |(\partial_\alpha^l\partial_\beta^{l'}\partial_\gamma^j)K_{-\sigma}(\zeta V_h^{10}(\alpha,\gamma,t)-V(\beta_0^t,t),V_{\tilde{f}^L}^{21}(\alpha_\gamma^t,t)-V_{\tilde{f}^L}^{21}(\beta_0^t,t),V_X(\alpha_\gamma^t,t)-V_X(\beta_0^t,t))| \tag{3.121}$$
$$\lesssim (\frac{1}{|\beta-\alpha|})^{l+l'+j+\sigma}$$

$$\sup_{\beta\in[-\pi,-\tau(t)]\cup[2\alpha+\tau(t),\pi]} |(\partial_\alpha^l\partial_\beta^{l'}\partial_\gamma^j)(K_{-\sigma}(\zeta V_h^{10}(\alpha,\gamma,t)-V(\beta_0^t,t),V_{\tilde{f}^L}^{21}(\alpha_\gamma^t,t)-V_{\tilde{f}^L}^{21}(\beta_0^t,t),V_X(\alpha_\gamma^t,t)-V_X(\beta_0^t,t)) \tag{3.122}$$
$$- K_{-\sigma}(\zeta V_g^{10}(\alpha,\gamma,t)-V(\beta_0^t,t),V_{\tilde{f}^L}^{21}(\alpha_\gamma^t,t)-V_{\tilde{f}^L}^{21}(\beta_0^t,t),V_X(\alpha_\gamma^t,t)-V_X(\beta_0^t,t)))|$$
$$\lesssim \|h-g\|_{W^{i_0,k}}(\frac{1}{|\beta-\alpha|})^{l+l'+1+j+\sigma}$$

$$\sup_{\beta\in[-\pi,-\tau(t)]\cup[2\alpha+\tau(t),\pi]} |(\partial_\alpha^l\partial_\beta^{l'}\partial_\gamma^j)D_h(K_{-\sigma}(\zeta V_h^{10}(\alpha,\gamma,t)-V(\beta_0^t,t),V_{\tilde{f}^L}^{21}(\alpha_\gamma^t,t)-V_{\tilde{f}^L}^{21}(\beta_0^t,t),V_X(\alpha_\gamma^t,t)-V_X(\beta_0^t,t))[g]| \tag{3.123}$$

$$\lesssim \|g\|_{W^{i_0,k}}(\frac{1}{|\beta-\alpha|})^{l+l'+1+j+\sigma}$$

and for $0 \le t' < t \le t*$, $\alpha > -\tau(t')$

$$\sup_{\beta\in[-\pi,-\tau(t)]\cup[2\alpha+\tau(t),\pi]} |(\partial_\alpha^l\partial_\beta^{l'}\partial_\gamma^j)(K_{-\sigma}(\zeta V_h^{10}(\alpha,\gamma,t)-V(\beta_0^t,t),V_{\tilde{f}^L}^{21}(\alpha_\gamma^t,t)-V_{\tilde{f}^L}^{21}(\beta_0^t,t),V_X(\alpha_\gamma^t,t)-V_X(\beta_0^t,t)) \tag{3.124}$$

$$-K_{-\sigma}(\zeta V_h^{10}(\alpha,\gamma,t')-V(\beta_0^{t'},t'),V_{\tilde{f}^L}^{21}(\alpha_\gamma^{t'},t')-V_{\tilde{f}^L}^{21}(\beta_0^{t'},t'),V_X(\alpha_\gamma^{t'},t')-V_X(\beta_0^{t'},t')))|$$
$$\lesssim (\mathcal{O}(t-t')+\|h(\alpha,\gamma,t)-h(\alpha,\gamma,t')\|_{C_\gamma^{i_0}([-1,1],H^k[-\pi,\pi])})(\frac{1}{|\beta-\alpha|})^{l+l'+1+j+\sigma}.$$

Proof. For the first bound (3.121), we have

$$\partial_\alpha^l\partial_\beta^{l'}\partial_\gamma^j K_{-\sigma}(\zeta V_h^{10}(\alpha,\gamma,t)-V(\beta_0^t,t),V_{\tilde{f}^L}^{21}(\alpha_\gamma^t,t)-V_{\tilde{f}^L}^{21}(\beta_0^t,t),V_X(\alpha_\gamma^t,t)-V_X(\beta_0^t,t))$$
$$=\sum_l K_{-(\sigma+l+l'+j)}^l(\zeta V_h^{10}(\alpha,\gamma,t)-V(\beta_0^t,t),V_{\tilde{f}^L}^{21}(\alpha_\gamma^t,t)-V_{\tilde{f}^L}^{21}(\beta_0^t,t),V_X(\alpha_\gamma^t,t)-V_X(\beta_0^t,t))G^l(\alpha,\beta,\gamma,t)$$

with $|G^l(\alpha,\beta,\gamma,t)| \lesssim 1$. Then we can only consider $\tilde{\sigma} = \sigma + l + l' + j$ without derivative. Recall that $K_{\tilde{\sigma}}$ has the form (3.69), by arc-chord conditions (3.101) and (3.102), we have

$$|\frac{1}{denominator}| \lesssim \frac{1}{(\alpha-\beta)^{2m_0}}.$$

Moreover, we have

$$|\zeta V_h^{10}(\alpha,\gamma,t)-V(\beta_0^t,t)| \le |V_h^{10}(\alpha,\gamma,t)|+|V^{10}(\alpha_0^t,t)|+|V(\alpha_0^t,t)-V(\beta_0^t,t)|$$
$$\lesssim |\alpha+\tau(t)|+|\alpha+\tau(t)|+|\alpha-\beta|,$$

$$|V_{\tilde{f}^L}^{21}(\alpha_\gamma^t,t)-V_{\tilde{f}^L}^{21}(\beta_0^t,t)| \le |V_{\tilde{f}^L}^{21}(\alpha_0^t,t)-V_{\tilde{f}^L}^{21}(\beta_0^t,t)|+|V_{\tilde{f}^L}^{21}(\alpha_0^t,t)-V_{\tilde{f}^L}^{21}(\alpha_\gamma^t,t)| \lesssim |\alpha-\beta|+|\alpha+\tau(t)| \lesssim |\alpha-\beta|,$$

and

$$|V_X(\alpha_\gamma^t,t)-V_X(\beta_0^t,t)| \le |V_X(\alpha_0^t,t)-V_X(\beta_0^t,t)|+|V_X(\alpha_0^t,t)-V_X(\alpha_\gamma^t,t)| \lesssim |\alpha-\beta|+|\alpha+\tau(t)| \lesssim |\alpha-\beta|.$$

Hence we have

$$|numerator| \lesssim |\alpha-\beta|^{m_1+m_3+m_5+m_6}.$$

Then we get the first inequality (3.121). For the (3.122),(3.123) and (3.124), as in the previous lemma, we could write

$$\frac{d}{d\zeta_1}K_{-\sigma}(\zeta_1\zeta V_h^{10}(\alpha,\gamma,t)-V(\beta_0^t,t),V_{\tilde{f}^L}^{21}(\alpha_\gamma^t,t)-V_{\tilde{f}^L}^{21}(\beta_0^t,t),V_X(\alpha_\gamma^t,t)-V_X(\beta_0^t,t)),$$

and

$$\frac{d}{d\zeta_1}K_{-\sigma}(\zeta_1\zeta V_h^{10}(\alpha,\gamma,t)-\zeta_1 V(\beta_0^t,t),\zeta_1 V_{\tilde{f}^L}^{21}(\alpha_\gamma^t,t)-\zeta_1 V_{\tilde{f}^L}^{21}(\beta_0^t,t),\zeta_1 V_X(\alpha_\gamma^t,t)-\zeta_1 V_X(\beta_0^t,t)).$$

as sums of the product of $K_{-(\sigma+1)}$ terms and other non-singular terms to bound the difference. □

Corollary 3.7.26. *When $\beta<-\tau(t)$ in the previous lemma, the $\frac{1}{|\alpha-\beta|}$ bound can be replaced by $\frac{1}{\alpha+\tau(t)}$. Moreover, here*

$$\begin{aligned}
&K_{-\sigma}(\zeta V_h^{10}(\alpha,\gamma,t)-V_{\tilde{f}^L}^{21}(\beta_0^t,t),V_{\tilde{f}^L}^{21}(\alpha_\gamma^t,t)-V_{\tilde{f}^L}^{21}(\beta_0^t,t),V_X(\alpha_\gamma^t,t)-V_X(\beta_0^t,t))\\
&=K_{-\sigma}(\zeta V_h^{10}(\alpha,\gamma,t),V_{\tilde{f}^L}^{21}(\alpha_\gamma^t,t)-V_{\tilde{f}^L}^{21}(\beta_\gamma^t,t),V_X(\alpha_\gamma^t,t)-V_X(\beta_\gamma^t,t))\\
&=K_{-\sigma}(\zeta(V_h^{10}(\alpha,\gamma,t)-V_h^{10}(\beta,\gamma,t)),V_{\tilde{f}^L}^{21}(\alpha_\gamma^t,t)-V_{\tilde{f}^L}^{21}(\beta_\gamma^t,t),V_X(\alpha_\gamma^t,t)-V_X(\beta_\gamma^t,t))
\end{aligned}$$

Proof. We use $\frac{1}{|\alpha-\beta|}\lesssim\frac{1}{\alpha+\tau(t)}$ when $\beta<-\tau(t)$. In the equation we use $\beta_\gamma^t=\beta_0^t$ and $V(\beta_0^t,t)=V_{\tilde{f}^L}^{21}(\beta_\gamma^t,t)=V_h^{10}(\beta_0^t,t)=V_h^{10}(\beta_\gamma^t,t)=0$. □

Corollary 3.7.27. *When $\beta>2\alpha+\tau(t)$ in the previous lemma, the $\frac{1}{|\alpha-\beta|}$ bound can be replaced by $\frac{1}{\beta+\tau(t)}$.*

Lemma 3.7.28. *For $0\le\zeta\le1$, $l+l'\le k+3$, $j\le i_0$, when $-\tau(t)<\alpha\le\frac{\pi}{4}$, we have the following two inequalities:*

$$\begin{aligned}
&\sup_{\beta\in[-\pi,-\tau(t)]}|(\partial_\alpha^l\partial_\beta^{l'}\partial_\gamma^j)K_{-\sigma}^i(0,\Delta V_{\tilde{f}^{L,L}}(\alpha_\gamma^t,t)+\zeta\Delta V_{\tilde{f}^-}(\alpha_\gamma^t,t),\Delta V_{X^L}(\alpha_\gamma^t,t)+\zeta\Delta V_{X^-}(\alpha_\gamma^t,t)) \quad (3.125)\\
&\lesssim(\frac{1}{|\beta-\alpha|})^{l+l'+j+\sigma}\lesssim(\frac{1}{|\beta+\tau(t)|})^{l+l'+j+\sigma},
\end{aligned}$$

and for $0 \le t' < t \le t_0,\ \alpha > -\tau(t')$

$$\sup_{\beta\in[-\pi,-\tau(t)]} |(\partial_\alpha^l \partial_\beta^{l'} \partial_\gamma^j)(K_{-1}^i(0, \Delta V_{\tilde{f}^{L,L}}(\alpha_\gamma^t, t) + \zeta \Delta V_{\tilde{f}^-}(\alpha_\gamma^t, t), \Delta V_{X^L}(\alpha_\gamma^t, t) + \zeta \Delta V_{X^-}(\alpha_\gamma^t, t)) \tag{3.126}$$

$$\begin{aligned}
&- K_{-\sigma}^i(0, \Delta V_{\tilde{f}^{L,L}}(\alpha_\gamma^{t'}, t') + \zeta \Delta V_{\tilde{f}^-}(\alpha_\gamma^t, t'), \Delta V_{X^L}(\alpha_\gamma^{t'}, t') + \zeta \Delta V_{X^-}(\alpha_\gamma^{t'}, t')))| \\
&\lesssim (\mathcal{O}(t-t') + \mathcal{O}(t-t') + \|h(\alpha,\gamma,t) - h(\alpha,\gamma,t')\|_{C_\gamma^{i_0}([-1,1],H^k[-\pi,\pi])})(\frac{1}{|\beta+\alpha|})^{l+l'+1+j+\sigma} \\
&\lesssim (\mathcal{O}(t-t'))(\frac{1}{|\beta+\tau(t)|})^{l+l'+1+j+\sigma}.
\end{aligned} \tag{3.127}$$

Proof. We have

$$\begin{aligned}
&|V_{\tilde{f}^{L,L}}(\alpha_\gamma^t, t) - V_{\tilde{f}^{L,L}}(\beta_0^t, t)| \le |V_{\tilde{f}^{L,L}}(\alpha_0^t, t) - V_{\tilde{f}^{L,L}}(\beta_0^t, t)| + |V_{\tilde{f}^{L,L}}(\alpha_0^t, t) - V_{\tilde{f}^{L,L}}(\alpha_\gamma^t, t)| \\
&\lesssim |\alpha-\beta| + |\alpha+\tau(t)| \lesssim |\alpha-\beta|,
\end{aligned}$$

$$|V_{\tilde{f}^-}(\beta_0^t, t)| \lesssim |\beta+\tau(t)| \lesssim |\alpha-\beta|,$$

$$|V_{X^L}(\alpha_\gamma^t, t) - V_{X^L}(\beta_0^t, t)| \le |V_{X^L}(\alpha_0^t, t) - V_{X^L}(\beta_0^t, t)| + |V_{X^L}(\alpha_0^t, t) - V_{X^L}(\alpha_\gamma^t, t)| \lesssim |\alpha-\beta| + |\alpha+\tau(t)|$$
$$\lesssim |\alpha-\beta|,$$

$$|V_{X^-}(\beta_0^t, t)| \lesssim |\beta+\tau(t)| \lesssim |\alpha-\beta|.$$

Then we can use the similar proof as in lemma 3.7.25 by arc-chord condition (3.105) instead of (3.102). □

3.8 Picard theorem

Lemma 3.8.1. *If take δ sufficiently small, there exists t^ϵ, and $h^\epsilon \in C^1([0,t_\epsilon], C^{i_0}_\gamma([-1,1], H^k_\alpha[-\pi,\pi]))\cap S_{\delta_1}$ such that*

$$\begin{cases} \frac{d}{dt}h^\epsilon(\alpha,\gamma,t) = T^\epsilon(h^\epsilon)(\alpha,\gamma,t), \\ h^\epsilon(\alpha,\gamma,0) = \tilde{f}^{+(20)}(\alpha,0). \end{cases}$$

Proof. Let

$$\begin{cases} h^\epsilon_{n+1}(\alpha,\gamma,t) = \int_0^t T^\epsilon(h^\epsilon_n)(\alpha,\gamma,\tau)d\tau + \tilde{f}^{+(20)}(\alpha,0), \\ h^\epsilon_0(\alpha,\gamma,t) = 0. \end{cases}$$

From lemma 3.7.2, there exists sufficiently small t_ϵ such that for all n, $h^\epsilon_n \in C^1([0,t_\epsilon], C^{i_0}_\gamma([-1,1], H^k_\alpha[-\pi,\pi]))\cap S_{\delta_1}$ and

$$\begin{aligned} &\|h_{n+1}(\alpha,\gamma,t) - h_n(\alpha,\gamma,t)\|_{C^0([0,t_\epsilon],C^{i_0}_\gamma([-1,1],H^k_\alpha[-\pi,\pi]))\cap S_{\delta_1}} \\ &\le \frac{1}{2}\|h_n(\alpha,\gamma,t) - h_{n-1}(\alpha,\gamma,t)\|_{C^0([0,t_1],C^{i_0}_\gamma([-1,1],H^k_\alpha[-\pi,\pi]))\cap S_{\delta_1}}. \end{aligned}$$

Therefore, we can take limit and there exists $h^\epsilon(\alpha,\gamma,t) \in C^0([0,t_\epsilon], C^{i_0}_\gamma([-1,1], H^k_\alpha[-\pi,\pi]))\cap S_{\delta_1}$ satisfying

$$h^\epsilon(\alpha,\gamma,t) = \int_0^t T^\epsilon(h^\epsilon)(\alpha,\gamma,\tau)d\tau + \tilde{f}^{+(20)}(\alpha,0).$$

Moreover, by lemma 3.7.2, we have $h^\epsilon \in C^1([0,t_\epsilon], C^{i_0}_\gamma([-1,1], H^k_\alpha[-\pi,\pi]))\cap S_{\delta_1}$ with

$$\frac{d}{dt}h^\epsilon(\alpha,\gamma,t) = T^\epsilon(h^\epsilon)(\alpha,\gamma,t).$$

□

We remark that if $\tau(t) < 0$, h_n may not be 0 when $\alpha < -\tau(t)$ and then the iteration breaks. That is the reason we need the extra assumption (1.6).

3.9 Energy estimate independent of ϵ

Let $V^{i_o,k} = C^{i_0}_{\gamma}([-1,1], H^k_{\alpha}[-\pi,\pi])$, we want to show the energy estimate in

$$U = V^{0,k} \cap V^{1,k-1} ... \cap V^{3,k-3} \cap S_{\delta_1}. \tag{3.128}$$

3.9.1 Control the main term-first step

From the structure of T^{ϵ}, only T^{ϵ}_1 T^{ϵ}_2 and $T^{i,\epsilon}_3$ depending on ϵ. From Lemma 3.9.12, $\|T^{i,\epsilon}_3(h^{\epsilon})\|_U \lesssim 1$ independent of ϵ. Therefore we only need to consider T^{ϵ}_1 T^{ϵ}_2 .Also notice that there is only one $T^{\epsilon}_1(h^{\epsilon})$ and $T_2(h^{\epsilon})$.

Therefore, for L^2 norm, we have

$$\begin{aligned}
&\frac{d}{dt}\int_{-\pi}^{\pi} |h^{\epsilon}(\alpha,\gamma,t)|^2 d\alpha \\
&= 2\Re \int_{-\pi}^{\pi} h^{\epsilon}(\alpha,\gamma,t)\overline{T^{\epsilon}_1(h)(\alpha,\gamma,t)}d\alpha + 2\Re\int_{-\pi}^{\pi} h^{\epsilon}(\alpha,\gamma,t)\overline{T^{\epsilon}_2(h)(\alpha,\gamma,t)}d\alpha + B.T \\
&= E^0_1 + E^0_2 + B.T.
\end{aligned}$$

Here $B.T. \leq B(\|h\|_U)$ with B a bounded function. From lemma 3.9.15, with $k=0$, and the property of $L^1_{h^{\epsilon,s}}$ (3.46) and (3.47), we have

$$\begin{aligned}
E^0_1 &= 2\Re\int_{-\tau(t)}^{\pi} h^{\epsilon}(\alpha,\gamma,t)\overline{\int_{-\tau(t)}^{2\alpha+\tau(t)} \frac{h^{\epsilon}(\alpha,\gamma,t)-h^{\epsilon}(\beta,\gamma,t)}{(\alpha-\beta)^2+\epsilon^2} d\beta \tilde{\lambda}(\alpha^t_{\gamma})\frac{L^1_{h^{\epsilon,s}}(\alpha,\gamma,t)}{\alpha+\tau(t)}(\alpha+\tau(t))}d\alpha \\
&= 2\Re\int_{-\tau(t)}^{\pi} h^{\epsilon}(\alpha,\gamma,t)\overline{\int_{-\tau(t)}^{\pi} \frac{h^{\epsilon}(\alpha,\gamma,t)-h^{\epsilon}(\beta,\gamma,t)}{(\alpha-\beta)^2+\epsilon^2} d\beta \tilde{\lambda}(\alpha^t_{\gamma})L^1_{h^{\epsilon,s}}(\alpha,\gamma,t)}d\alpha + B.T^0 \\
&= E^0_{1,M} + B.T.^0
\end{aligned}$$

Here

$$B.T.^0 \leq C(\|h\|_{C^0_{\gamma}([-1,1],L^2_{\alpha}[-\tau(t),\pi])}, \|L^1_{h^{\epsilon,s}}\|_{C^0_{\gamma}([-1,1],C^1_{\alpha}[-\tau(t),\pi])}, \|L^2_{h^{\epsilon,s}}\|_{C^0_{\gamma}([-1,1],C^1_{\alpha}[-\tau(t),\pi])})\|h(\alpha,\gamma,t)\|^2_{L^2_{\alpha}[-\tau(t),\pi]}.$$

For the first term, we have

$$\begin{aligned}
E^0_{1,M} &= 2\int_{-\tau(t)}^{\pi} \tilde{\lambda}(\alpha^t_{\gamma})\Re L^1_{h^{\epsilon,s}}(\alpha,\gamma,t)\Re h^{\epsilon}(\alpha,\gamma,t)\int_{-\tau(t)}^{\pi}\frac{\Re h^{\epsilon}(\alpha,\gamma,t)-\Re h^{\epsilon}(\beta,\gamma,t)}{(\alpha-\beta)^2+\epsilon^2}d\beta d\alpha \\
&+ 2\int_{-\tau(t)}^{\pi} \tilde{\lambda}(\alpha^t_{\gamma})\Re L^1_{h^{\epsilon,s}}(\alpha,\gamma,t)\Im h^{\epsilon}(\alpha,\gamma,t)\int_{-\tau(t)}^{\pi}\frac{\Im h^{\epsilon}(\alpha,\gamma,t)-\Im h^{\epsilon}(\beta,\gamma,t)}{(\alpha-\beta)^2+\epsilon^2}d\beta d\alpha
\end{aligned}$$

$$- 2\int_{-\tau(t)}^{\pi} \tilde{\lambda}(\alpha_\gamma^t)\Im L^1_{h^{\epsilon,s}}(\alpha,\gamma,t)\Re h^\epsilon(\alpha,\gamma,t)\int_{-\tau(t)}^{\pi} \frac{\Im h^\epsilon(\alpha,\gamma,t)-\Im h^\epsilon(\beta,\gamma,t)}{(\alpha-\beta)^2+\epsilon^2}d\beta d\alpha$$
$$+ 2\int_{-\tau(t)}^{\pi} \tilde{\lambda}(\alpha_\gamma^t)\Im L^1_{h^{\epsilon,s}}(\alpha,\gamma,t)\Im h^\epsilon(\alpha,\gamma,t)\int_{-\tau(t)}^{\pi} \frac{\Re h^\epsilon(\alpha,\gamma,t)-\Re h^\epsilon(\beta,\gamma,t)}{(\alpha-\beta)^2+\epsilon^2}d\beta d\alpha.$$

We call the first two terms $E^0_{1,M,M}$ and the last two terms $E^0_{1,M,s}$. We leave it as this form for now and move on to control E^0_2. Similarly, from (3.46), (3.47), and lemma 3.9.18, we have

$$E^0_2 = 2\Re\int_{-\tau(t)}^{\pi} h^\epsilon(\alpha,\gamma,t)\overline{\int_{-\tau(t)}^{2\alpha+\tau(t)} \frac{(h^\epsilon(\alpha,\gamma,t)-h^\epsilon(\beta,\gamma,t))(\alpha-\beta)\epsilon}{((\alpha-\beta)^2+\epsilon^2)^2}d\beta\tilde{\lambda}(\alpha_\gamma^t)\frac{L^2_{h^{\epsilon,s}}(\alpha,\gamma,t)}{\alpha+\tau(t)}(\alpha+\tau(t))d\alpha}$$
$$= 2\Re\int_{-\tau(t)}^{\pi} h^\epsilon(\alpha,\gamma,t)\overline{\int_{-\tau(t)}^{\pi} \frac{(h^\epsilon(\alpha,\gamma,t)-h^\epsilon(\beta,\gamma,t))(\alpha-\beta)\epsilon}{((\alpha-\beta)^2+\epsilon^2)^2}d\beta\tilde{\lambda}(\alpha_\gamma^t)\frac{L^2_{h^{\epsilon,s}}(\alpha,\gamma,t)}{\alpha+\tau(t)}(\alpha+\tau(t))d\alpha} + B.T^0$$
$$= E^0_{2,M} + B.T^0$$

For $E^0_{2,M}$, we have

$$E^0_{2,M} = 2\int_{-\tau(t)}^{\pi} \tilde{\lambda}(\alpha_\gamma^t)\Re L^2_{h^{\epsilon,s}}(\alpha,\gamma,t)\Re h^\epsilon(\alpha,\gamma,t)\int_{-\tau(t)}^{\pi} \frac{\Re h^\epsilon(\alpha,\gamma,t)-\Re h^\epsilon(\beta,\gamma,t)(\alpha-\beta)\epsilon}{((\alpha-\beta)^2+\epsilon^2)^2}d\beta d\alpha$$
$$+ 2\int_{-\tau(t)}^{\pi} \tilde{\lambda}(\alpha_\gamma^t)\Re L^2_{h^{\epsilon,s}}(\alpha,\gamma,t)\Im h^\epsilon(\alpha,\gamma,t)\int_{-\tau(t)}^{\pi} \frac{\Im h^\epsilon(\alpha,\gamma,t)-\Im h^\epsilon(\beta,\gamma,t)(\alpha-\beta)\epsilon}{((\alpha-\beta)^2+\epsilon^2)^2}d\beta d\alpha$$
$$- 2\int_{-\tau(t)}^{\pi} \tilde{\lambda}(\alpha_\gamma^t)\Im L^2_{h^{\epsilon,s}}(\alpha,\gamma,t)\Re h^\epsilon(\alpha,\gamma,t)\int_{-\tau(t)}^{\pi} \frac{\Im h^\epsilon(\alpha,\gamma,t)-\Im h^\epsilon(\beta,\gamma,t)(\alpha-\beta)\epsilon}{((\alpha-\beta)^2+\epsilon^2)^2}d\beta d\alpha$$
$$+ 2\int_{-\tau(t)}^{\pi} \tilde{\lambda}(\alpha_\gamma^t)\Im L^2_{h^{\epsilon,s}}(\alpha,\gamma,t)\Im h^\epsilon(\alpha,\gamma,t)\int_{-\tau(t)}^{\pi} \frac{\Re h^\epsilon(\alpha,\gamma,t)-\Re h^\epsilon(\beta,\gamma,t)(\alpha-\beta)\epsilon}{((\alpha-\beta)^2+\epsilon^2)^2}d\beta d\alpha$$
$$= B.T^0 + E^0_{2,M,s},$$

where we used lemma 3.9.21 to show the first two terms are $B.T^0$ Now we are left with

$$E^0_{1,M,M} + E^0_{1,M,s} + E^0_{2,M,s}$$
$$= 2\int_{-\tau(t)}^{\pi} \tilde{\lambda}(\alpha_\gamma^t)\Re L^1_{h^{\epsilon,s}}(\alpha,\gamma,t)\Re h^\epsilon(\alpha,\gamma,t)\int_{-\tau(t)}^{\pi} \frac{\Re h^\epsilon(\alpha,\gamma,t)-\Re h^\epsilon(\beta,\gamma,t)}{(\alpha-\beta)^2+\epsilon^2}d\beta d\alpha$$
$$+ 2\int_{-\tau(t)}^{\pi} \tilde{\lambda}(\alpha_\gamma^t)\Re L^1_{h^{\epsilon,s}}(\alpha,\gamma,t)\Im h^\epsilon(\alpha,\gamma,t)\int_{-\tau(t)}^{\pi} \frac{\Im h^\epsilon(\alpha,\gamma,t)-\Im h^\epsilon(\beta,\gamma,t)}{(\alpha-\beta)^2+\epsilon^2}d\beta d\alpha$$
$$- 2\int_{-\tau(t)}^{\pi} \tilde{\lambda}(\alpha_\gamma^t)\Im L^1_{h^{\epsilon,s}}(\alpha,\gamma,t)\Re h^\epsilon(\alpha,\gamma,t)\int_{-\tau(t)}^{\pi} \frac{\Im h^\epsilon(\alpha,\gamma,t)-\Im h^\epsilon(\beta,\gamma,t)}{(\alpha-\beta)^2+\epsilon^2}d\beta d\alpha$$
$$+ 2\int_{-\tau(t)}^{\pi} \tilde{\lambda}(\alpha_\gamma^t)\Im L^1_{h^{\epsilon,s}}(\alpha,\gamma,t)\Im h^\epsilon(\alpha,\gamma,t)\int_{-\tau(t)}^{\pi} \frac{\Re h^\epsilon(\alpha,\gamma,t)-\Re h^\epsilon(\beta,\gamma,t)}{(\alpha-\beta)^2+\epsilon^2}d\beta d\alpha$$
$$- 2\int_{-\tau(t)}^{\pi} \tilde{\lambda}(\alpha_\gamma^t)\Im L^2_{h^{\epsilon,s}}(\alpha,\gamma,t)\Re h^\epsilon(\alpha,\gamma,t)\int_{-\tau(t)}^{\pi} \frac{\Im h^\epsilon(\alpha,\gamma,t)-\Im h^\epsilon(\beta,\gamma,t)(\alpha-\beta)\epsilon}{((\alpha-\beta)^2+\epsilon^2)^2}d\beta d\alpha$$
$$+ 2\int_{-\tau(t)}^{\pi} \tilde{\lambda}(\alpha_\gamma^t)\Im L^2_{h^{\epsilon,s}}(\alpha,\gamma,t)\Im h^\epsilon(\alpha,\gamma,t)\int_{-\tau(t)}^{\pi} \frac{\Re h^\epsilon(\alpha,\gamma,t)-\Re h^\epsilon(\beta,\gamma,t)(\alpha-\beta)\epsilon}{((\alpha-\beta)^2+\epsilon^2)^2}d\beta d\alpha.$$

Let $d^\epsilon(\alpha,\gamma,t) = 2\Re L^1_{h^{\epsilon,s}}(\alpha,\gamma,t)$, $d^\epsilon_2(\alpha,\gamma,t) = 2\Im L^1_{h^{\epsilon,s}}(\alpha,\gamma,t)$, $d^\epsilon_3(\alpha,\gamma,t) = 2\Im L^2_{h^{\epsilon,s}}(\alpha,\gamma,t)$. From the (3.148), we have

$$-d^\epsilon(\alpha,\gamma,t) \geq 2C|d^\epsilon_2(\alpha,\gamma,t)| + 2C|d^\epsilon_3(\alpha,\gamma,t)|, \tag{3.129}$$

with $C = 9$, $\alpha \in (-\tau(t), \frac{\pi}{4}) \cap \operatorname{supp}\tilde{\lambda}(\alpha^t_\gamma)$. Then by the refined Gårding's inequality 3.9.4, we have

$$E^0_{1,M,M} + E^0_{1,M,s} + E^0_{2,M,s} \leq B.T^0$$

Then we finish the L^2 case. We also introduce a corollary here for in the later sections. The proof is same as the L^2 case of the energy estimate.

Corollary 3.9.1. *If* $\operatorname{supp} h \subset [-1,1] \times [-\tau(t), \frac{\pi}{4}]$,

$$h \in C^0_\gamma([-1,1], L^2_\alpha[-\tau(t),\pi]),$$

for i=1,2, for any $\tilde{L}_1$, $\tilde{L}_2$ *satisfying the following properties:*

$$\tilde{L}_i(\alpha,\gamma,t) \in C^0_\gamma([-1,1], C^1_\alpha[-\tau(t),\pi]),$$

$$\tilde{L}_i(-\tau(t),\gamma,t) = 0,$$

and when $\alpha \in (-\tau(t), \frac{\pi}{4}) \cap \operatorname{supp}\tilde{\lambda}(\alpha^t_\gamma)$,

$$-\Re\tilde{L}_1(\alpha,\gamma,t) \geq 2C|\Im\tilde{L}_1(\alpha,\gamma,t)| + 2C|\Im\tilde{L}_2(\alpha,\gamma,t)|,$$

then we have

$$\begin{aligned}
&\sup_\gamma[2\Re\int_{-\tau(t)}^{\pi} h(\alpha,\gamma,t)\overline{\tilde{\lambda}(\alpha^t_\gamma)\int_{-\tau(t)}^{2\alpha+\tau(t)} \frac{h(\alpha,\gamma,t)-h(\beta,\gamma,t)}{(\alpha-\beta)^2+\epsilon^2} d\beta\tilde{L}_1(\alpha,\gamma,t)}d\alpha \\
&\quad + 2\Re\int_{-\tau(t)}^{\pi} h(\alpha,\gamma,t)\overline{\tilde{\lambda}(\alpha^t_\gamma)\int_{-\tau(t)}^{2\alpha+\tau(t)} \frac{(h(\alpha,\gamma,t)-h(\beta,\gamma,t))(\alpha-\beta)\epsilon}{(\alpha-\beta)^2+\epsilon^2} d\beta\tilde{L}_2(\alpha,\gamma,t)}d\alpha] \\
&\quad \lesssim C(\|h\|_{C^0_\gamma([-1,1],L^2_\alpha[-\tau(t),\pi])}, \|\tilde{L}_1\|_{C^0_\gamma([-1,1],C^1_\alpha[-\tau(t),\pi])}, \|\tilde{L}_2\|_{C^0_\gamma([-1,1],C^1_\alpha[-\tau(t),\pi])})\|h\|^2_{L^2_\alpha[-\tau(t),\pi])}
\end{aligned} \tag{3.130}$$

When h is sufficiently smooth, we can also take limit to the previous corollary 3.130 and get:

Corollary 3.9.2. *If we also have $h \in C^0_\gamma([-1,1], H^3_\alpha[-\tau(t),\pi])$, the following inequality holds*

$$\begin{aligned}
&\sup_\gamma [2\Re \int_{-\tau(t)}^{\pi} h(\alpha,\gamma,t)\overline{\tilde{\lambda}(\alpha^t_\gamma) p.v. \int_{-\tau(t)}^{2\alpha+\tau(t)} \frac{h(\alpha,\gamma,t)-h(\beta,\gamma,t)}{(\alpha-\beta)^2} d\beta \tilde{L}_1(\alpha,\gamma,t)} d\alpha \\
&\quad + 2\Re \int_{-\tau(t)}^{\pi} h(\alpha,\gamma,t)\overline{\tilde{\lambda}(\alpha^t_\gamma)\frac{\pi}{2}\partial_\alpha h(\alpha,\gamma,t)\tilde{L}_2(\alpha,\gamma,t)} d\alpha] \\
&\quad \lesssim C(\|h\|_{C^0_\gamma([-1,1],L^2_\alpha[-\tau(t),\pi])}, \|\tilde{L}_1\|_{C^0_\gamma([-1,1],C^1_\alpha[-\tau(t),\pi])}, \|\tilde{L}_2\|_{C^0_\gamma([-1,1],C^1_\alpha[-\tau(t),\pi])}) \|h\|^2_{L^2_\alpha[-\tau(t),\pi])}
\end{aligned}$$

Proof. Since $h \in C^0_\gamma([-1,1], H^3_\alpha[-\tau(t),\pi])$, we have

$$\lim_{\epsilon\to 0} \| \int_{-\tau(t)}^{2\alpha+\tau(t)} \frac{h(\alpha,\gamma,t)-h(\beta,\gamma,t)}{(\alpha-\beta)^2+\epsilon^2} d\beta - p.v. \int_{-\tau(t)}^{2\alpha+\tau(t)} \frac{h(\alpha,\gamma,t)-h(\beta,\gamma,t)}{(\alpha-\beta)^2} d\beta \|_{L^\infty_\alpha[-\tau(t),\pi]} \to 0,$$

and

$$\lim_{\epsilon\to 0} \| \int_{-\tau(t)}^{2\alpha+\tau(t)} \frac{(h(\alpha,\gamma,t)-h(\beta,\gamma,t))(\alpha-\beta)\epsilon}{(\alpha-\beta)^2+\epsilon^2} d\beta - \frac{\pi}{2}\partial_\alpha h(\alpha,\gamma,t) \|_{L^\infty_\alpha[-\tau(t),\pi]} \to 0.$$

Then we can take the limit of (RHS) of (3.130) and get the result. □

We also claim the following corollary is true by fixing $\gamma = 0$ in the proof of corollary 3.9.2.

Corollary 3.9.3. *If* $\operatorname{supp} h \subset [-\tau(t), \frac{\pi}{4}]$,

$$h(\alpha,t) \in H^3_\alpha[-\tau(t),\pi]),$$

for i=1,2, for any $\tilde{L}_1$, $\tilde{L}_2$ satisfying the following properties:

$$\tilde{L}_i(\alpha,t) \in C^1_\alpha[-\tau(t),\pi],$$

$$\tilde{L}_i(-\tau(t),t) = 0,$$

and

$$-\Re\tilde{L}_1(\alpha,t) \geq 2C|\Im\tilde{L}_1(\alpha,t)| + 2C|\Im\tilde{L}_2(\alpha,t)|,$$

then we have

$$\begin{aligned}
&2\Re \int_{-\tau(t)}^{\pi} h(\alpha,t)\overline{\tilde{\lambda}(\alpha^t_0) \int_{-\tau(t)}^{2\alpha+\tau(t)} \frac{h(\alpha,t)-h(\beta,t)}{(\alpha-\beta)^2} d\beta \tilde{L}_1(\alpha,t)} d\alpha \\
&+ 2\Re \int_{-\tau(t)}^{\pi} h(\alpha,t)\overline{\tilde{\lambda}(\alpha^t_0) \int_{-\tau(t)}^{2\alpha+\tau(t)} \frac{(h(\alpha,t)-h(\beta,t))(\alpha-\beta)\epsilon}{(\alpha-\beta)^2} d\beta \tilde{L}_2(\alpha,t)} d\alpha]
\end{aligned} \tag{3.131}$$

$$\lesssim C(\|h\|_{L^2_\alpha[-\tau(t),\pi]}, \|\tilde{L}_1\|_{C^1_\alpha[-\tau(t),\pi]}, \|\tilde{L}_2\|_{C^1_\alpha[-\tau(t),\pi]})\|h\|^2_{L^2_\alpha[-\tau(t),\pi]}$$

3.9.2 Refined Gårding's inequality

Lemma 3.9.4. *Let $c(\alpha), \tilde{c}(\alpha) \in C([0,\pi])$,$f, g \in L^2([0,\pi])$, satisfying*

$$|c(\alpha)| \lesssim \alpha, |\tilde{c}(\alpha)| \lesssim \alpha, \operatorname{supp} c, \operatorname{supp} \tilde{c}, \operatorname{supp} f, \operatorname{supp} g \subset [0, \frac{\pi}{4}],$$

there exists $C > 0$, such that when $C|c(\alpha)| \leq \tilde{c}(\alpha)$, we have

$$\begin{aligned} &\int_0^\pi c(\alpha) f(\alpha) \int_0^\pi \frac{(\alpha-\beta)\epsilon}{((\alpha-\beta)^2+\epsilon^2)^2}(g(\alpha)-g(\beta))d\beta \\ &\leq \int_0^\pi \tilde{c}(\alpha) f(\alpha) \int_0^\pi \frac{1}{((\alpha-\beta)^2+\epsilon^2}(f(\alpha)-f(\beta))d\beta d\alpha \\ &+ \int_0^\pi \tilde{c}(\alpha) g(\alpha) \int_0^\pi \frac{1}{(\alpha-\beta)^2+\epsilon^2}(g(\alpha)-g(\beta))d\beta d\alpha + B.T, \end{aligned} \tag{3.132}$$

and

$$\begin{aligned} &\int_0^\pi c(\alpha) f(\alpha) \int_0^\pi \frac{1}{(\alpha-\beta)^2+\epsilon^2}(g(\alpha)-g(\beta))d\beta d\alpha \\ &\leq \int_0^\pi \tilde{c}(\alpha) f(\alpha) \int_0^\pi \frac{1}{(\alpha-\beta)^2+\epsilon^2}(f(\alpha)-f(\beta))d\beta d\alpha \\ &+ \int_0^\pi \tilde{c}(\alpha) g(\alpha) \int_0^\pi \frac{1}{(\alpha-\beta)^2+\epsilon^2}(g(\alpha)-g(\beta))d\beta d\alpha + B.T. \end{aligned} \tag{3.133}$$

where

$$B.T \lesssim (\|f\|^2_{L^2} + \|g\|^2_{L^2})$$

independent of ϵ. We can take $C = 9$.

Corollary 3.9.5. *If we also assume $f, g \in H^3$, we have*

$$\begin{aligned} &\int_0^\pi c(\alpha) f(\alpha) \frac{1}{b_l} g'(\alpha) d\alpha \\ &\leq \int_0^\pi \tilde{c}(\alpha) f(\alpha) \int_0^\pi \frac{1}{(\alpha-\beta)^2}(f(\alpha)-f(\beta))d\beta d\alpha \\ &+ \int_0^\pi \tilde{c}(\alpha) g(\alpha) \int_0^\pi \frac{1}{(\alpha-\beta)^2}(g(\alpha)-g(\beta))d\beta d\alpha + B.T, \end{aligned}$$

$$\int_0^\pi c(\alpha)f(\alpha)p.v.\int_0^\pi \frac{1}{(\alpha-\beta)^2}(g(\alpha)-g(\beta))d\beta$$
$$\leq \int_0^\pi \tilde{c}(\alpha)f(\alpha)p.v.\int_0^\pi \frac{1}{(\alpha-\beta)^2}(f(\alpha)-f(\beta))d\beta d\alpha$$
$$+\int_0^\pi \tilde{c}(\alpha)g(\alpha)p.v.\int_0^\pi \frac{1}{(\alpha-\beta)^2}(g(\alpha)-g(\beta))d\beta d\alpha + B.T.$$

with $b_l = \frac{2}{\pi}$ from (3.36).

Proof. We can take the limit of the previous lemma and get the result. □

Proof. Now we start to show the lemma 3.9.4. First we introduce some lemmas.

Lemma 3.9.6. *Let H_R be the Hilbert transform on the real line. We have*

$$\int_0^\pi c(\alpha)f(\alpha)\int_0^\pi \frac{(\alpha-\beta)\epsilon}{((\alpha-\beta)^2+\epsilon^2)^2}g(\beta)d\beta d\alpha$$
$$=\int_0^\pi c(\alpha)f(\alpha)\int_0^\pi (H_{\mathcal{R}}g(\beta)-H_{\mathcal{R}}g(\alpha))(\frac{1}{((\alpha-\beta)^2+\epsilon^2)^2}-\frac{2\epsilon^2}{((\alpha-\beta)^2+\epsilon^2)^2})d\beta d\alpha + B.T.$$

Proof. Since

$$H_{\mathcal{R}}(\frac{\epsilon}{\beta^2+\epsilon^2}) = \frac{\beta}{\beta^2+\epsilon^2}, \tag{3.134}$$

by changing the order of the Hilbert transform and the derivative, we have

$$H_{\mathcal{R}}(\frac{\beta\epsilon}{(\beta^2+\epsilon^2)^2}) = \frac{1}{2}\frac{\beta^2-\epsilon^2}{(\beta^2+\epsilon^2)^2} = \frac{1}{2}[\frac{1}{\beta^2+\epsilon^2}-\frac{2\epsilon^2}{(\beta^2+\epsilon^2)^2}].$$

Now we extend $g(\beta)$ to be a function g on $\mathcal{R}$ with $g(\beta)=0$ on $[0,\pi]^c$. By the property of Hilbert transform, we have

$$\int_0^\pi c(\alpha)f(\alpha)\int_0^\pi \frac{(\alpha-\beta)\epsilon}{((\alpha-\beta)^2+\epsilon^2)^2}g(\beta)d\beta d\alpha$$
$$=\int_0^\pi c(\alpha)f(\alpha)\int_{-\infty}^\infty \frac{(\alpha-\beta)\epsilon}{((\alpha-\beta)^2+\epsilon^2)^2}g(\beta)d\beta d\alpha$$
$$=\int_0^\pi c(\alpha)f(\alpha)\int_{-\infty}^\infty \frac{1}{2}[\frac{1}{(\alpha-\beta)^2+\epsilon^2}-\frac{2\epsilon^2}{((\alpha-\beta)^2+\epsilon^2)^2}]H_{\mathcal{R}}g(\beta)d\beta d\alpha$$

Now we claim

$$\int_0^\pi c(\alpha)f(\alpha)\int_{-\infty}^{\infty}[\frac{1}{(\alpha-\beta)^2+\epsilon^2}-\frac{2\epsilon^2}{((\alpha-\beta)^2+\epsilon^2)^2}]H_{\mathcal{R}}g(\beta)d\beta d\alpha \tag{3.135}$$
$$=\int_0^\pi c(\alpha)f(\alpha)\int_0^{\pi}[\frac{1}{(\alpha-\beta)^2+\epsilon^2}-\frac{2\epsilon^2}{((\alpha-\beta)^2+\epsilon^2)^2}]H_{\mathcal{R}}g(\beta)d\beta d\alpha+B.T.$$

Let us verify it. Since $|c(\alpha)|\lesssim|\alpha|$, we can use the Hilbert inequality and have

$$\|f(\alpha)c(\alpha)\int_{-\infty}^{0}\frac{1}{(\alpha-\beta)^2+\epsilon^2}H_{\mathcal{R}}g(\beta)d\beta\|_{L^1}\lesssim\|\int_0^\infty\frac{1}{\alpha+\beta}H_{\mathcal{R}}g(-\beta)d\beta\|_{L^2[0,\infty)}\|f\|_{L^2}$$
$$\lesssim\|H_{\mathcal{R}}(g)\|_{L^2}\|f\|_{L^2}=B.T.$$

For the other terms, we use $\operatorname{supp}c(\alpha)\subset[0,\frac{\pi}{4}]$ and we have (3.135). Moreover

$$\int_0^\pi(\frac{1}{(\alpha-\beta)^2+\epsilon^2}-\frac{2\epsilon^2}{(\alpha-\beta)^2+\epsilon^2})d\beta=\int_0^\pi\frac{d}{d\beta}\frac{\alpha-\beta}{(\alpha-\beta)^2+\epsilon^2}d\beta=\frac{\alpha-\pi}{(\alpha-\pi)^2+\epsilon^2}-\frac{\alpha}{\alpha^2+\epsilon^2}. \tag{3.136}$$

Since $|\frac{\alpha-\pi}{(\alpha-\pi)^2+\epsilon^2}-\frac{\alpha}{\alpha^2+\epsilon^2}|\cdot|c(\alpha)|\lesssim1$ on $[0,\frac{\pi}{4}]$, we have

$$\int_0^\pi c(\alpha)f(\alpha)\int_0^{\pi}[\frac{1}{(\alpha-\beta)^2+\epsilon^2}-\frac{2\epsilon^2}{((\alpha-\beta)^2+\epsilon^2)^2}]H_{\mathcal{R}}g(\beta)d\beta d\alpha \tag{3.137}$$
$$=\int_0^\pi c(\alpha)f(\alpha)\int_0^{\pi}[\frac{1}{(\alpha-\beta)^2+\epsilon^2}-\frac{2\epsilon^2}{((\alpha-\beta)^2+\epsilon^2)^2}](H_{\mathcal{R}}g(\beta)-H_{\mathcal{R}}g(\alpha))d\beta d\alpha+B.T.$$

□

Lemma 3.9.7. *We have*

$$\int_0^\pi\int_0^\pi c(\alpha)f(\alpha)(g(\alpha)-g(\beta))\frac{1}{(\alpha-\beta)^2+\epsilon^2}d\beta d\alpha$$
$$=\frac{1}{2}\int_0^\pi\int_0^\pi c(\alpha)(f(\alpha)-f(\beta)))(g(\alpha)-g(\beta))(\frac{1}{(\alpha-\beta)^2+\epsilon^2}-\frac{1}{(\alpha+\beta)^2+\epsilon^2})d\beta d\alpha+B.T.$$

$$\int_0^\pi\int_0^\pi c(\alpha)f(\alpha)(g(\alpha)-g(\beta))\frac{\epsilon^2}{((\alpha-\beta)^2+\epsilon^2)^2}d\beta d\alpha$$
$$=\frac{1}{2}\int_0^\pi\int_0^\pi c(\alpha)(f(\alpha)-f(\beta)))(g(\alpha)-g(\beta))(\frac{\epsilon^2}{((\alpha-\beta)^2+\epsilon^2)^2}-\frac{\epsilon^2}{((\alpha+\beta)^2+\epsilon^2)^2})d\beta d\alpha+B.T.$$

We remark that we do not need the $\operatorname{supp}f,\operatorname{supp}g$ *condition here.*

Proof. We only show the first one and the second one follows the same way.

$$\begin{aligned}
&\int_0^\pi \int_0^\pi c(\alpha)f(\alpha)(g(\alpha)-g(\beta))\frac{1}{(\alpha-\beta)^2+\epsilon^2}d\beta d\alpha \\
=&\frac{1}{2}\int_0^\pi \int_0^\pi c(\alpha)(f(\alpha)-f(\beta))(g(\alpha)-g(\beta))(\frac{1}{(\alpha-\beta)^2+\epsilon^2}-\frac{1}{(\alpha+\beta)^2+\epsilon^2})d\beta d\alpha \\
&+\frac{1}{2}\int_0^\pi \int_0^\pi (c(\alpha)-c(\beta))f(\beta)(g(\alpha)-g(\beta))\frac{1}{(\alpha-\beta)^2+\epsilon^2}d\beta d\alpha \\
&+\frac{1}{2}\int_0^\pi \int_0^\pi c(\alpha)(f(\alpha)-f(\beta))(g(\alpha)-g(\beta)(\frac{1}{(\alpha+\beta)^2+\epsilon^2})d\beta d\alpha \\
=&I_1+I_2+I_3
\end{aligned}$$

We now show I_2+I_3 are good terms.

$$\begin{aligned}
&I_2+I_3 \\
=&\frac{1}{2}\int_0^\pi \int_0^\pi (c(\alpha)-c(\beta))f(\beta)(g(\alpha)-g(\beta))\frac{1}{(\alpha-\beta)^2+\epsilon^2}d\beta d\alpha \\
&+\frac{1}{2}\int_0^\pi \int_0^\pi c(\alpha)(f(\alpha)-f(\beta))(g(\alpha)-g(\beta)(\frac{1}{(\alpha+\beta)^2+\epsilon^2})d\beta d\alpha \\
=&\frac{1}{2}\int_0^\pi (-\int_0^\pi \frac{(c(\alpha)-c(\beta))}{(\alpha-\beta)^2+\epsilon^2}d\alpha+\int_0^\pi \frac{c(\alpha)}{(\alpha+\beta)^2+\epsilon^2}d\alpha)f(\beta)g(\beta)d\beta \\
&+\frac{1}{2}\int_0^\pi \int_0^\pi \frac{(c(\alpha)-c(\beta))g(\alpha)}{(\alpha-\beta)^2+\epsilon^2}d\alpha f(\beta)d\beta \\
&+\frac{1}{2}\int_0^\pi \int_0^\pi \frac{1}{(\alpha+\beta)^2+\epsilon^2}d\beta c(\alpha)f(\alpha)g(\alpha)d\alpha \\
&-\frac{1}{2}\int_0^\pi \int_0^\pi \frac{(c(\alpha))f(\beta)}{(\alpha+\beta)^2+\epsilon^2})d\beta g(\alpha)d\alpha \\
&-\frac{1}{2}\int_0^\pi \int_0^\pi \frac{(c(\alpha))g(\beta)}{(\alpha+\beta)^2+\epsilon^2})d\beta f(\alpha)d\alpha \\
=&J_1+J_2+J_3+J_4+J_5
\end{aligned}$$

For the first term, from lemma 3.9.8, we have that $-\int_0^\pi \frac{(c(\alpha)-c(\beta))}{(\alpha-\beta)^2+\epsilon^2}d\alpha+\int_0^\pi \frac{c(\alpha)}{(\alpha+\beta)^2+\epsilon^2}d\alpha$ is bounded. Then $J_1=B.T.$

For J_2, we have

$$\begin{aligned}
&\int_0^\pi \frac{(c'(\alpha)(\alpha-\beta))g(\alpha)}{(\alpha-\beta)^2+\epsilon^2}d\alpha \\
=&\int_{-\infty}^\infty \frac{(c'(\alpha)(\alpha-\beta))g(\alpha)}{(\alpha-\beta)^2+\epsilon^2}d\alpha \\
=&\int_{-\infty}^\infty \frac{\epsilon}{(\alpha-\beta)^2+\epsilon^2}H_{\mathcal{R}}(c'(\alpha)g(\alpha))d\alpha
\end{aligned}$$

So

$$\| \int_0^\pi \frac{(c(\alpha)-c(\beta))g(\alpha)}{(\alpha-\beta)^2+\epsilon^2} d\alpha \|_{L^2} \lesssim \|g\|_{L^2} + \| \int_0^\pi \frac{(c(\alpha)-c(\beta)-c'(\alpha)(\alpha-\beta))g(\alpha)}{(\alpha-\beta)^2+\epsilon^2} d\alpha \|_{L^2}$$
$$\lesssim \|g\|_{L^2}.$$

Then J_2 can be controlled by Holder inequality.

J_3 is bounded because the boundness of $\int_0^\pi \frac{1}{(\alpha+\beta)^2+\epsilon^2} d\beta c(\alpha)$, and J_4, J_5 can be controlled by the Hilbert's inequality and $|\frac{c(\alpha)}{(\alpha+\beta)^2+\epsilon^2}| \lesssim |\frac{1}{\alpha+\beta}|$. □

Lemma 3.9.8. *We have* $\sup_{0\le\beta\le\pi} | - \int_0^\pi \frac{c(\alpha)-c(\beta)}{(\alpha-\beta)^2+\epsilon^2} d\alpha + \int_0^\pi \frac{c(\alpha)}{(\alpha+\beta)^2+\epsilon^2} d\alpha| \lesssim 1.$

Proof. Since $\operatorname{supp} c$ and $\operatorname{supp} \tilde{c} \subset [0, \frac{\pi}{4}]$, when $\beta \ge \frac{\pi}{3}$, we have

$$\begin{aligned} & | - \int_0^\pi \frac{c(\alpha)-c(\beta)}{(\alpha-\beta)^2+\epsilon^2} d\alpha + \int_0^\pi \frac{c(\alpha)}{(\alpha+\beta)^2+\epsilon^2} d\alpha| \\ &= | - \int_0^{\frac{\pi}{4}} \frac{(c(\alpha))}{(\alpha-\beta)^2+\epsilon^2} d\alpha + \int_0^\pi \frac{c(\alpha)}{(\alpha+\beta)^2+\epsilon^2} d\alpha| \\ &\lesssim 1. \end{aligned}$$

When $\beta \le \frac{\pi}{3}$, we have

$$\begin{aligned} & - \int_0^\pi \frac{(c(\alpha)-c(\beta))}{(\alpha-\beta)^2+\epsilon^2} d\alpha + \int_0^\pi \frac{c(\alpha)}{(\alpha+\beta)^2+\epsilon^2} d\alpha \\ &= - \int_{2\beta}^\pi \frac{(c(\alpha)-c(\beta))}{(\alpha-\beta)^2+\epsilon^2} d\alpha + \int_0^\pi \frac{c(\alpha)}{(\alpha+\beta)^2+\epsilon^2} d\alpha - \int_0^{2\beta} \frac{(c(\alpha)-c(\beta)-c'(\beta)(\alpha-\beta))}{(\alpha-\beta)^2+\epsilon^2} d\alpha \\ &= - \int_0^{\pi-2\beta} \frac{c(\alpha+2\beta)}{(\alpha+\beta)^2+\epsilon^2} d\alpha + \int_0^{\pi-2\beta} \frac{c(\beta)}{(\alpha+\beta)^2+\epsilon^2} d\alpha + \int_0^{\pi-2\beta} \frac{c(\alpha)}{(\alpha+\beta)^2+\epsilon^2} d\alpha + \int_{\pi-2\beta}^\pi \frac{c(\alpha)}{(\alpha+\beta)^2+\epsilon^2} d\alpha \\ & - \int_0^{2\beta} \frac{(c(\alpha)-c(\beta)-c'(\beta)(\alpha-\beta))}{(\alpha-\beta)^2+\epsilon^2}) d\alpha \\ &= Term_1 + Term_2 + Term_3 + Term_4 + Term_5. \end{aligned}$$

Here

$$|Term_1 + Term_3| = | \int_0^{\pi-2\beta} \frac{c(\alpha+2\beta)-c(\alpha)}{(\alpha+\beta)^2+\epsilon^2} d\alpha| \lesssim 2\beta \int_0^{\pi-2\beta} \frac{1}{(\alpha+\beta)^2} d\alpha \lesssim 1,$$

$$|Term_2| \lesssim \beta \int_0^\pi \frac{1}{(\alpha+\beta)^2} d\alpha \lesssim 1,$$

and since $\beta \le \frac{\pi}{3}$,

$$|Term_4| \lesssim 1,$$

$$|Term_5| \lesssim 2\beta \lesssim 1.$$

Then we have the result. □

Lemma 3.9.9. *We have*

$$\int_0^\pi \int_0^\pi c(\alpha) f(\alpha) \frac{f(\alpha) - f(\beta)}{(\alpha-\beta)^2 + \epsilon^2} d\alpha \tag{3.138}$$
$$= \int_0^\pi \int_0^\pi c(\alpha) H_{\mathcal{R}} f(\alpha) (H_{\mathcal{R}} f(\alpha) - H_{\mathcal{R}} f(\beta)) (\frac{1}{(\alpha-\beta)^2 + \epsilon^2}) d\beta d\alpha + B.T. \tag{3.139}$$

Proof. We first show

$$\int_0^\pi \int_0^\pi c(\alpha) f(\alpha) (f(\alpha) - f(\beta)) (\frac{1}{(\alpha-\beta)^2 + \epsilon^2} - \frac{2\epsilon^2}{((\alpha-\beta)^2 + \epsilon^2)^2}) d\beta d\alpha \tag{3.140}$$
$$= \int_0^\pi \int_0^\pi c(\alpha) H_{\mathcal{R}} f(\alpha) (H_{\mathcal{R}} f(\alpha) - H_{\mathcal{R}} f(\beta)) (\frac{1}{(\alpha-\beta)^2 + \epsilon^2} - \frac{2\epsilon^2}{((\alpha-\beta)^2 + \epsilon^2)^2}) d\beta d\alpha + B.T.$$

From (3.134), and (3.136), We have

$$\begin{aligned}
&\int_0^\pi \int_0^\pi c(\alpha) f(\alpha) (f(\alpha) - f(\beta)) (\frac{1}{(\alpha-\beta)^2 + \epsilon^2} - \frac{2\epsilon^2}{((\alpha-\beta)^2 + \epsilon^2)^2}) d\beta d\alpha \\
=& - \int_0^\pi c(\alpha) f(\alpha) \int_{-\infty}^\infty f(\beta) (\frac{1}{(\alpha-\beta)^2 + \epsilon^2} - \frac{2\epsilon^2}{((\alpha-\beta)^2 + \epsilon^2)^2}) d\beta d\alpha + B.T. \\
=& - 2 \int_0^\pi c(\alpha) f(\alpha) \int_{-\infty}^\infty H_{\mathcal{R}} f(\beta) \frac{(\alpha-\beta)\epsilon}{((\alpha-\beta)^2 + \epsilon^2)^2} d\beta d\alpha \\
=& - 2 \int_0^\pi f(\alpha) \int_{-\infty}^\infty c(\beta) H_{\mathcal{R}} f(\beta) \frac{(\alpha-\beta)\epsilon}{((\alpha-\beta)^2 + \epsilon^2)^2} d\beta d\alpha \\
&+ 2 \int_0^\pi f(\alpha) \int_{-\infty}^\infty H_{\mathcal{R}} f(\beta) \frac{(c(\beta) - c(\alpha))(\alpha-\beta)\epsilon}{((\alpha-\beta)^2 + \epsilon^2)^2} d\beta d\alpha \\
=& - 2 \int_0^\pi c(\beta) H_{\mathcal{R}} f(\beta) \int_{-\infty}^\infty f(\alpha) \frac{(\alpha-\beta)\epsilon}{((\alpha-\beta)^2 + \epsilon^2)^2} d\alpha d\beta + B.T. \\
=& - \int_0^\pi c(\beta) H_{\mathcal{R}} f(\beta) \int_{-\infty}^\infty H_{\mathcal{R}} f(\alpha) (\frac{1}{(\alpha-\beta)^2 + \epsilon^2} - \frac{2\epsilon^2}{((\alpha-\beta)^2 + \epsilon^2)^2}) d\alpha d\beta + B.T. \\
=& - \int_0^\pi c(\beta) H_{\mathcal{R}} f(\beta) \int_{-\infty}^\infty (H_{\mathcal{R}} f(\alpha) - H_{\mathcal{R}} f(\beta)) (\frac{1}{(\alpha-\beta)^2 + \epsilon^2} - \frac{2\epsilon^2}{((\alpha-\beta)^2 + \epsilon^2)^2}) d\alpha d\beta + B.T. \\
=& - \int_0^\pi c(\alpha) H_{\mathcal{R}} f(\alpha) \int_{-\infty}^\infty (H_{\mathcal{R}} f(\beta) - H_{\mathcal{R}} f(\alpha)) (\frac{1}{(\alpha-\beta)^2 + \epsilon^2} - \frac{2\epsilon^2}{((\alpha-\beta)^2 + \epsilon^2)^2}) d\beta d\alpha + B.T. \\
=& - \int_0^\pi c(\alpha) H_{\mathcal{R}} f(\alpha) \int_0^\pi (H_{\mathcal{R}} f(\beta) - H_{\mathcal{R}} f(\alpha)) (\frac{1}{(\alpha-\beta)^2 + \epsilon^2} - \frac{2\epsilon^2}{((\alpha-\beta)^2 + \epsilon^2)^2}) d\beta d\alpha + B.T.
\end{aligned}$$

where we use the Hilbert's inequality and the fact that $|c(\alpha)| \lesssim |\alpha|$.

Then we show

$$\int_0^\pi \int_0^\pi c(\alpha)f(\alpha)(f(\alpha)-f(\beta))(\frac{\epsilon^2}{((\alpha-\beta)^2+\epsilon^2)^2})d\beta d\alpha \tag{3.141}$$
$$= \int_0^\pi \int_0^\pi c(\alpha)H_{\mathcal{R}}f(\alpha)(H_{\mathcal{R}}f(\alpha)-H_{\mathcal{R}}f(\beta))(\frac{\epsilon^2}{((\alpha-\beta)^2+\epsilon^2)^2})d\beta d\alpha + B.T.$$

From (3.141) and (3.140), we can get (3.138).

For the LHS of (3.141), we have

$$\begin{aligned}
&\int_0^\pi \int_0^\pi c(\alpha)f(\alpha)(f(\alpha)-f(\beta))(\frac{\epsilon^2}{((\alpha-\beta)^2+\epsilon^2)^2})d\beta d\alpha \\
=&\int_{-\infty}^\infty c(\alpha)f(\alpha)\int_{-\infty}^\infty (f(\alpha)-f(\beta))\frac{\epsilon^2}{((\alpha-\beta)^2+\epsilon^2)^2}d\beta d\alpha + B.T. \\
=&\int_{-\infty}^\infty H_{\mathcal{R}}(c(\alpha)f(\alpha))\int_{-\infty}^\infty (H_{\mathcal{R}}f(\alpha)-H_{\mathcal{R}}f(\beta))\frac{\epsilon^2}{((\alpha-\beta)^2+\epsilon^2)^2}d\beta d\alpha + B.T. \\
=&\int_{-\infty}^\infty c(\alpha)H_{\mathcal{R}}f(\alpha)\int_{-\infty}^\infty (H_{\mathcal{R}}f(\alpha)-H_{\mathcal{R}}f(\beta))\frac{\epsilon^2}{((\alpha-\beta)^2+\epsilon^2)^2}d\beta d\alpha \\
&+\int_{-\infty}^\infty \int_0^\pi \frac{(c(\alpha)-c(\gamma))f(\gamma)}{\alpha-\gamma}d\gamma \int_{-\infty}^\infty (H_{\mathcal{R}}f(\alpha)-H_{\mathcal{R}}f(\beta))\frac{\epsilon^2}{((\alpha-\beta)^2+\epsilon^2)^2}d\beta d\alpha + B.T.
\end{aligned}$$

Here we take $c(\alpha)=0$ when $\alpha \le 0$ and $\alpha \ge \pi$.

Then we are left to show

$$\int_{-\infty}^\infty \int_0^\pi \frac{(c(\alpha)-c(\gamma))f(\gamma)}{\alpha-\gamma}d\gamma \int_{-\infty}^\infty (H_{\mathcal{R}}f(\alpha)-H_{\mathcal{R}}f(\beta))\frac{\epsilon^2}{((\alpha-\beta)^2+\epsilon^2)^2}d\beta d\alpha = B.T. \tag{3.142}$$

Here

$$\begin{aligned}
&\int_{-\infty}^\infty \int_0^\pi \frac{(c(\alpha)-c(\gamma))f(\gamma)}{\alpha-\gamma}d\gamma \int_{-\infty}^\infty (H_{\mathcal{R}}f(\alpha)-H_{\mathcal{R}}f(\beta))\frac{\epsilon^2}{((\alpha-\beta)^2+\epsilon^2)^2}d\beta d\alpha \\
=&\int_{-\infty}^\infty \int_0^\pi \frac{(c(\alpha)-c(\gamma))f(\gamma)}{\alpha-\gamma}d\gamma \int_{-\infty}^\infty (H_{\mathcal{R}}f(\alpha))\frac{\epsilon^2}{((\alpha-\beta)^2+\epsilon^2)^2}d\beta d\alpha \\
&-\int_{-\infty}^\infty \int_0^\pi \frac{(c(\alpha)-c(\gamma))f(\gamma)}{\alpha-\gamma}d\gamma \int_{-\infty}^\infty (H_{\mathcal{R}}f(\beta))\frac{\epsilon^2}{((\alpha-\beta)^2+\epsilon^2)^2}d\beta d\alpha \\
=&\int_{-\infty}^\infty \int_0^\pi \frac{(c(\alpha)-c(\gamma))f(\gamma)}{\alpha-\gamma}d\gamma \int_{-\infty}^\infty (H_{\mathcal{R}}f(\alpha))\frac{\epsilon^2}{((\alpha-\beta)^2+\epsilon^2)^2}d\beta d\alpha \\
&-\int_{-\infty}^\infty \int_{-\infty}^\infty \int_0^\pi \frac{(c(\beta)-c(\gamma))f(\gamma)}{\alpha-\gamma}d\gamma (H_{\mathcal{R}}f(\alpha))\frac{\epsilon^2}{((\alpha-\beta)^2+\epsilon^2)^2}d\beta d\alpha \\
=&\int_{-\infty}^\infty \int_{-\infty}^\infty \int_0^\pi (\frac{(c(\alpha)-c(\gamma))}{\alpha-\gamma}-\frac{(c(\beta)-c(\gamma))}{\beta-\gamma})f(\gamma)d\gamma \frac{\epsilon^2}{((\alpha-\beta)^2+\epsilon^2)^2}d\beta (H_{\mathcal{R}}f(\alpha))d\alpha
\end{aligned}$$

For the integral on $\alpha < 0$, and $\beta < 0$,

$$
\begin{aligned}
&|\int_{-\infty}^{0}\int_{-\infty}^{0}\int_{0}^{\pi}(\frac{(c(\alpha)-c(\gamma))}{\alpha-\gamma}-\frac{(c(\beta)-c(\gamma))}{\beta-\gamma})f(\gamma)d\gamma\frac{\epsilon^2}{((\alpha-\beta)^2+\epsilon^2)^2}d\beta(H_{\mathcal{R}}f(\alpha))d\alpha|\\
=&|\int_{0}^{\infty}\int_{0}^{\infty}\int_{0}^{\pi}(\frac{c(\gamma)}{\alpha+\gamma}-\frac{c(\gamma)}{\beta+\gamma})f(\gamma)d\gamma\frac{\epsilon^2}{((\alpha-\beta)^2+\epsilon^2)^2}d\beta H_{\mathcal{R}}f(-\alpha)d\alpha|\\
\leq&|\int_{0}^{\infty}\int_{0}^{\infty}\frac{\int_{0}^{\pi}\frac{c(\gamma)}{(\alpha+\gamma)(\beta+\gamma)}f(\gamma)d\gamma\epsilon^2|\alpha-\beta|}{((\alpha-\beta)^2+\epsilon^2)^2}d\beta|H_{\mathcal{R}}f(-\alpha)|d\alpha|\\
\leq&|\int_{0}^{\infty}\int_{0}^{\infty}\frac{\int_{0}^{\pi}\frac{|f(\gamma)|}{|(\alpha+\gamma)|}d\gamma\epsilon^2|\alpha-\beta|}{((\alpha-\beta)^2+\epsilon^2)^2}d\beta|H_{\mathcal{R}}f(-\alpha)|d\alpha|\\
\lesssim&|\int_{0}^{\infty}\int_{0}^{\pi}(\frac{|f(\gamma)|}{|(\alpha+\gamma)|}d\gamma|H_{\mathcal{R}}f(-\alpha)|d\alpha|\\
&=B.T,
\end{aligned}
$$

where we use the Holder inequality and the Hilbert's inequality in the last step.

For the integral over $\alpha < 0$, $\beta > 0$,

$$
\frac{(c(\alpha)-c(\gamma))}{\alpha-\gamma}-\frac{(c(\beta)-c(\gamma))}{\beta-\gamma}=\frac{c(\alpha)-c(\beta)}{\alpha-\gamma}-\frac{(c(\beta)-c(\gamma))(\alpha-\beta)}{(\alpha-\gamma)(\beta-\gamma)}.
$$

We have

$$
\begin{aligned}
&|\int_{-\infty}^{0}\int_{0}^{\infty}\int_{0}^{\pi}\frac{c(\alpha)-c(\beta)}{\alpha-\gamma}f(\gamma)d\gamma\frac{\epsilon^2}{((\alpha-\beta)^2+\epsilon^2)^2}d\beta(H_{\mathcal{R}}f(\alpha))d\alpha|\\
\leq&|\int_{0}^{\infty}\int_{0}^{\infty}\int_{0}^{\pi}\frac{1}{\alpha+\gamma}|f(\gamma)|d\gamma\frac{\epsilon^2|c(-\alpha)-c(\beta)|}{((\alpha+\beta)^2+\epsilon^2)^2}d\beta|H_{\mathcal{R}}f(-\alpha)|d\alpha|\\
\lesssim&|\int_{0}^{\infty}\int_{0}^{\pi}(\frac{|f(\gamma)|}{|(\alpha+\gamma)|}d\gamma|H_{\mathcal{R}}f(-\alpha)|d\alpha|\\
&=B.T.
\end{aligned}
$$

Here we use $|c(-\alpha)-c(\beta)| \lesssim |\alpha+\beta|$. Moreover,

$$
\begin{aligned}
&|\int_{-\infty}^{0}\int_{0}^{\infty}\int_{0}^{\pi}\frac{(c(\beta)-c(\gamma))(\alpha-\beta)}{(\alpha-\gamma)(\beta-\gamma)}f(\gamma)d\gamma\frac{\epsilon^2}{((\alpha-\beta)^2+\epsilon^2)^2}d\beta(H_{\mathcal{R}}f(\alpha))d\alpha|\\
\lesssim&|\int_{0}^{\infty}\int_{0}^{\infty}\int_{0}^{\pi}\frac{1}{\alpha+\gamma}|f(\gamma)|d\gamma\frac{\epsilon^2|\alpha+\beta|}{((\alpha+\beta)^2+\epsilon^2)^2}d\beta|H_{\mathcal{R}}f(-\alpha)|d\alpha|\\
\lesssim&|\int_{0}^{\infty}\int_{0}^{\pi}(\frac{|f(\gamma)|}{|(\alpha+\gamma)|}d\gamma|H_{\mathcal{R}}f(-\alpha)|d\alpha|\\
&=B.T,
\end{aligned}
$$

where we use $|\frac{(c(\beta)-c(\gamma))}{\beta-\gamma}| \lesssim 1$. For the integral over$\alpha > 0$, $\beta < 0$,

$$\frac{(c(\alpha)-c(\gamma))}{\alpha-\gamma} - \frac{(c(\beta)-c(\gamma))}{\beta-\gamma} = \frac{(c(\alpha)-c(\gamma))(\beta-\alpha)}{(\alpha-\gamma)(\beta-\gamma)} + \frac{c(\alpha)-c(\beta)}{\beta-\gamma}.$$

We have

$$\begin{aligned}
&|\int_0^\infty \int_{-\infty}^0 \int_0^\pi \frac{(c(\alpha)-c(\beta))}{(\beta-\gamma)} f(\gamma)d\gamma \frac{\epsilon^2}{((\alpha-\beta)^2+\epsilon^2)^2} d\beta (H_{\mathcal{R}} f(\alpha))d\alpha| \\
=&|\int_0^\infty \int_0^\infty \int_0^\pi \frac{1}{(\beta+\gamma)} f(\gamma)d\gamma \frac{\epsilon^2(c(\alpha)-c(-\beta))}{((\alpha+\beta)^2+\epsilon^2)^2} d\beta (H_{\mathcal{R}} f(\alpha))d\alpha| \\
\lesssim&|\int_0^\infty \int_0^\infty \int_0^\pi \frac{1}{(\beta+\gamma)} |f(\gamma)|d\gamma \frac{\epsilon^2|\alpha+\beta|}{((\alpha+\beta)^2+\epsilon^2)^2} d\beta |H_{\mathcal{R}} f(\alpha)|d\alpha| \\
\lesssim&\|H_{\mathcal{R}} f\|_{L^2}\| \int_{-\infty}^\infty \int_0^\pi \frac{1}{(\beta+\gamma)} |f(\gamma)|1_{\beta\geq 0} d\gamma \frac{\epsilon^2|\alpha+\beta|}{((\alpha+\beta)^2+\epsilon^2)^2} d\beta\|_{L^2} \\
\lesssim&\|H_{\mathcal{R}}(f)\|_{L^2}\| \int_0^\pi \frac{1}{(\beta+\gamma)} |f(\gamma)|d\gamma\|_{L^2} \int_{-\infty}^\infty \frac{\epsilon^2|\beta|}{((\beta)^2+\epsilon^2)^2} d\beta \\
= B.T.&
\end{aligned}$$

Moreover,

$$\begin{aligned}
&|\int_0^\infty \int_{-\infty}^0 \int_0^\pi \frac{(c(\alpha)-c(\gamma))(\alpha-\beta)}{(\beta-\gamma)(\alpha-\gamma)} f(\gamma)d\gamma \frac{\epsilon^2}{((\alpha-\beta)^2+\epsilon^2)^2} d\beta H_{\mathcal{R}} f(\alpha)d\alpha| \\
\leq&|\int_0^\infty \int_0^\infty \int_0^\pi \frac{1}{(\beta+\gamma)} |f(\gamma)|d\gamma \frac{\epsilon^2|\alpha+\beta|}{((\alpha+\beta)^2+\epsilon^2)^2} d\beta |H_{\mathcal{R}} f(\alpha)|d\alpha|,
\end{aligned}$$

Then we can use the same way as in the previous inequality to bound it.

For the integral over $2\pi > \alpha > 0$, $\beta > 0$, we have

$$|\frac{(c(\alpha)-c(\gamma))}{\alpha-\gamma} - \frac{(c(\beta)-c(\gamma))}{\beta-\gamma}| \lesssim |\alpha-\beta|.$$

Hence

$$\begin{aligned}
&|\int_0^{2\pi} \int_0^\infty \int_0^\pi \frac{(c(\alpha)-c(\gamma))}{\alpha-\gamma} - \frac{(c(\beta)-c(\gamma))}{\beta-\gamma} f(\gamma)d\gamma \frac{\epsilon^2}{((\alpha-\beta)^2+\epsilon^2)^2} d\beta (H_{\mathcal{R}} f(\alpha))d\alpha| \\
\lesssim&|\int_0^{2\pi} \int_0^\infty \int_0^\pi |f(\gamma)|d\gamma \frac{\epsilon^2|\alpha-\beta|}{((\alpha-\beta)^2+\epsilon^2)^2} d\beta |H_{\mathcal{R}} f(\alpha)|d\alpha| \\
\lesssim&\int_0^{2\pi} |H_{\mathcal{R}} f(\alpha)|d\alpha \int_0^\pi |f(\gamma)|d\gamma \\
\lesssim&\|f\|_{L^2}^2
\end{aligned}$$

$= B.T.$

For the integral over $\beta > 0, \alpha > 2\pi$,

$$\frac{(c(\alpha)-c(\gamma))}{\alpha-\gamma} - \frac{(c(\beta)-c(\gamma))}{\beta-\gamma} = \frac{c(\alpha)-c(\beta)}{\alpha-\gamma} - \frac{(c(\beta)-c(\gamma))(\alpha-\beta)}{(\alpha-\gamma)(\beta-\gamma)}.$$

$$\begin{aligned}
&|\int_{2\pi}^{\infty}\int_0^{\infty}\int_0^{\pi} \frac{c(\alpha)-c(\beta)}{\alpha-\gamma} f(\gamma)d\gamma \frac{\epsilon^2}{((\alpha-\beta)^2+\epsilon^2)^2} d\beta (H_{\mathcal{R}}f(\alpha))d\alpha| \\
\lesssim &|\int_{2\pi}^{\infty}\int_0^{\infty}\int_0^{\pi} |f(\gamma)|d\gamma \frac{\epsilon^2|\alpha-\beta|}{((\alpha-\beta)^2+\epsilon^2)^2} d\beta \frac{|H_{\mathcal{R}}f(\alpha)|}{\alpha-\pi} d\alpha| \\
\lesssim &\int_{2\pi}^{\infty} |H_{\mathcal{R}}f(\alpha)| \frac{1}{\alpha-\pi} d\alpha \int_0^{\pi} |f(\gamma)|d\gamma \\
\lesssim &(\int_{2\pi}^{\infty} |H_{\mathcal{R}}f(\alpha)|^2 d\alpha)^{\frac{1}{2}} \int_0^{\pi} |f(\gamma)|d\gamma \\
= &B.T.
\end{aligned}$$

And

$$\begin{aligned}
&|\int_{2\pi}^{\infty}\int_0^{\infty}\int_0^{\pi} \frac{(c(\beta)-c(\gamma))(\alpha-\beta)}{(\alpha-\gamma)(\beta-\gamma)} f(\gamma)d\gamma \frac{\epsilon^2}{((\alpha-\beta)^2+\epsilon^2)^2} d\beta (H_{\mathcal{R}}f(\alpha))d\alpha| \\
\lesssim &|\int_{2\pi}^{\infty}\int_0^{\infty}\int_0^{\pi} |f(\gamma)|d\gamma \frac{\epsilon^2|\alpha-\beta|}{((\alpha-\beta)^2+\epsilon^2)^2} d\beta \frac{|H_{\mathcal{R}}f(\alpha)|}{\alpha-\pi} d\alpha|
\end{aligned}$$

Then we can use the same method as before to show the result. By (3.140) and (3.141), we can show the lemma. □

Now we use those lemmas to show the theorem. We first show (3.132). By lemma 3.9.6,

$$(LHS) = \int_0^{\pi} c(\alpha)f(\alpha) \int_0^{\pi} (H_{\mathcal{R}}g(\beta) - H_{\mathcal{R}}g(\alpha))(\frac{-1}{((\alpha-\beta)^2+\epsilon^2)} + \frac{2\epsilon^2}{((\alpha-\beta)^2+\epsilon^2)^2})d\beta d\alpha + B.T. \tag{3.143}$$

$$= L_1 + L_2 + B.T.$$

Next, we use lemma 3.9.7 and get

$$L_1 = -\frac{1}{2}\int_0^{\pi}\int_0^{\pi} c(\alpha)(f(\alpha)-f(\beta)))(H_{\mathcal{R}}g(\beta)-H_{\mathcal{R}}g(\alpha))(\frac{1}{(\alpha-\beta)^2+\epsilon^2} - \frac{1}{(\alpha+\beta)^2+\epsilon^2})d\beta d\alpha + B.T. \tag{3.144}$$

$$L_2 = \int_0^\pi \int_0^\pi c(\alpha)(f(\alpha)-f(\beta)))(H_{\mathcal{R}}g(\beta)-H_{\mathcal{R}}g(\alpha))(\frac{\epsilon^2}{((\alpha-\beta)^2+\epsilon^2)^2}-\frac{\epsilon^2}{((\alpha+\beta)^2+\epsilon^2)^2})d\beta d\alpha+B.T. \tag{3.145}$$

Since we have when $0 \le \alpha \le \pi, 0 \le \beta \le \pi$,

$$\frac{1}{(\alpha-\beta)^2+\epsilon^2} - \frac{1}{(\alpha+\beta)^2+\epsilon^2} > 0,$$

$$\frac{\epsilon^2}{((\alpha-\beta)^2+\epsilon^2)^2} - \frac{\epsilon^2}{((\alpha+\beta)^2+\epsilon^2)^2} > 0,$$

$$\frac{\epsilon^2}{((\alpha-\beta)^2+\epsilon^2)^2} - \frac{\epsilon^2}{((\alpha+\beta)^2+\epsilon^2)^2} \le 4(\frac{1}{(\alpha-\beta)^2+\epsilon^2} - \frac{1}{(\alpha+\beta)^2+\epsilon^2},)$$

By Holder inequality, we get

$$\begin{aligned} |(LHS)| \le & \frac{9}{4}\int_0^\pi \int_0^\pi c(\alpha)(f(\alpha)-f(\beta))^2(\frac{1}{(\alpha-\beta)^2+\epsilon^2} - \frac{1}{(\alpha+\beta)^2+\epsilon^2})d\beta d\alpha \\ & + \frac{9}{4}\int_0^\pi \int_0^\pi c(\alpha)(H_{\mathcal{R}}g(\beta)-H_{\mathcal{R}}g(\alpha))^2(\frac{1}{(\alpha-\beta)^2+\epsilon^2} - \frac{1}{(\alpha+\beta)^2+\epsilon^2})d\beta d\alpha + B.T. \end{aligned} \tag{3.146}$$

Moreover, in (3.133), by lemma 3.9.7, and 3.9.9,

$$\begin{aligned} (RHS) = & \frac{1}{4}\int_0^\pi \int_0^\pi \tilde{c}(\alpha)(f(\alpha)-f(\beta))^2(\frac{1}{(\alpha-\beta)^2+\epsilon^2} - \frac{1}{(\alpha+\beta)^2+\epsilon^2})d\beta d\alpha \\ & + \frac{1}{4}\int_0^\pi \int_0^\pi \tilde{c}(\alpha)(H_{\mathcal{R}}f(\beta)-H_{\mathcal{R}}f(\alpha))^2(\frac{1}{(\alpha-\beta)^2+\epsilon^2} - \frac{1}{(\alpha+\beta)^2+\epsilon^2})d\beta d\alpha \\ & \frac{1}{4}\int_0^\pi \int_0^\pi \tilde{c}(\alpha)(g(\alpha)-g(\beta))^2(\frac{1}{(\alpha-\beta)^2+\epsilon^2} - \frac{1}{(\alpha+\beta)^2+\epsilon^2})d\beta d\alpha \\ & + \frac{1}{4}\int_0^\pi \int_0^\pi \tilde{c}(\alpha)(H_{\mathcal{R}}g(\beta)-H_{\mathcal{R}}g(\alpha))^2(\frac{1}{(\alpha-\beta)^2+\epsilon^2} - \frac{1}{(\alpha+\beta)^2+\epsilon^2})d\beta d\alpha \\ & + B.T. \end{aligned} \tag{3.147}$$

By 3.146 and 3.147, we can take $C = 9$ and get (3.132). For (3.133), by lemma 3.9.7, we have

$$\begin{aligned} & (LHS) = \frac{1}{2}\int_0^\pi \int_0^\pi c(\alpha)(f(\alpha)-f(\beta))(g(\alpha)-g(\beta))(\frac{1}{(\alpha-\beta)^2+\epsilon^2} - \frac{1}{(\alpha+\beta)^2+\epsilon^2})d\beta d\alpha + B.T. \\ & \le \frac{1}{4}\int_0^\pi \int_0^\pi c(\alpha)(f(\alpha)-f(\beta))^2(\frac{1}{(\alpha-\beta)^2+\epsilon^2} - \frac{1}{(\alpha+\beta)^2+\epsilon^2})d\beta d\alpha \\ & + \frac{1}{4}\int_0^\pi \int_0^\pi c(\alpha)(g(\alpha)-g(\beta))^2(\frac{1}{(\alpha-\beta)^2+\epsilon^2} - \frac{1}{(\alpha+\beta)^2+\epsilon^2})d\beta d\alpha + B.T. \end{aligned}$$

The (RHS) of (3.133) is same as (3.147), since (3.132) and (3.133) have same RHS. Then we can again take $C = 9$. □

3.9.3 The Refined R-T condition

In this section we show when $\alpha \in (-\tau(t), \frac{\pi}{4}) \cap \operatorname{supp} \tilde{\lambda}(\alpha_\gamma^t)$ the following inequality is true, for $t < t_{RT}$ with t_{RT} sufficiently small independent of ϵ:

$$-d^\epsilon(\alpha, \gamma, t) \geq 2C|d_2^\epsilon(\alpha, \gamma, t)| + 2C|d_3^\epsilon(\alpha, \gamma, t)|. \tag{3.148}$$

From (3.46), we have

$$d^\epsilon(-\tau(t), \gamma, t) = d_2^\epsilon(-\tau(t), \gamma, t) = d_3^\epsilon(-\tau(t), \gamma, t) = 0.$$

Therefore it is enough to show the following refined R-T condition for $\alpha \in (-\tau(t), \frac{\pi}{4}) \cap \operatorname{supp} \tilde{\lambda}(\alpha_\gamma^t)$.

-

$$\partial_\alpha d^\epsilon(\alpha, \gamma, t) - 2C|\partial_\alpha d_2^\epsilon(-\tau(t), \gamma, t)| - 2C|\partial_\alpha d_3^\epsilon(-\tau(t), \gamma, t)| > 0. \tag{3.149}$$

Here we have $\operatorname{supp} \tilde{\lambda}(\alpha_\gamma^t) \subset [-20\delta, 20\delta]$. Therefore we only need to show 3.149 holds when $\alpha \in (-\tau(t), 20\delta)$.

Lemma 3.9.10. (3.149) *holds for small $t \leq t_{RT}$ independent of ϵ.*

Proof. From lemma 3.9.11, we know at $t = 0$,

$$-\partial_\alpha d^\epsilon(\alpha, \gamma, 0) = -\partial_\alpha \Re L^1_{h^{\epsilon,s}}(\alpha, \gamma, 0) > C_0.$$

when $\alpha \in [0, 20\delta]$. $\partial_\alpha d_2(\alpha, \gamma, 0) = \partial_\alpha d_3(\alpha, \gamma, 0) = 0$ because everything is real at $t = 0$. From (3.50), we also have when $0 \leq t' \leq t$,

$$\begin{aligned} &\|L^j_{h^{\epsilon,s}}(\alpha, \gamma, t) - L^j_{h^\epsilon,s}(\alpha, \gamma, t')\|_{C^0_\gamma([-1,1], C^1_\alpha[\tau(t'), \frac{\pi}{4}])} \\ &\lesssim \|h^{\epsilon,s}(\alpha, \gamma, t) - h^{\epsilon,s}(\alpha, \gamma, t')\|_U + \mathcal{O}(t - t'). \end{aligned}$$

Moreover,

$$\begin{aligned} &\|L^j_{h^{\epsilon,s}}(\alpha, \gamma, t) - L^j_{h^{\epsilon,s}}(-\tau(t'), \gamma, t)\|_{C^0_\gamma([-1,1], C^1_\alpha[-\tau(t), -\tau(t')])} \\ &\leq (\tau(t) - \tau(t'))\|L^i_{h^{\epsilon,s}}(\alpha, \gamma, t)\|_{C^0_\gamma([-1,1], C^2_\alpha[-\tau(t), -\tau(t')]} \lesssim B(\|h\|_U)(\tau(t) - \tau(t')), \end{aligned}$$

when $k \geq 2$. Hence there exists small t_{RT} which is independent of ϵ, such that(3.149) holds when $t \leq t_{RT}$.

Lemma 3.9.11. *When $t = 0$, if δ is sufficiently small, there exists C_0 depending on δ such that*

$$-\partial_\alpha d^\epsilon(\alpha,\gamma,0) = -\partial_\alpha \Re L^1_{h^{\epsilon,s}}(\alpha,\gamma,0) > C_0.$$

when $\alpha \in [0, 20\delta]$.

Proof. Recall that

$$-L^1_{h^{\epsilon,s}}(\alpha,\gamma,t) = \lim_{\beta\to\alpha}(\frac{d}{d\beta}(\frac{\sin(A_1+B_2)}{\cosh(A_2+B_2)-\cos(A_1+B_1)})(\alpha-\beta)^2),$$

where $A + B = D^{-20}h^\epsilon(\alpha,\gamma,t) - D^{-20}h^\epsilon(\beta,\gamma,t) + \tilde{f}^L(\alpha^t_\gamma,t) - \tilde{f}^L(\beta^t_\gamma,t)$. □

When $t = 0$, we have

$$A + B = \tilde{f}^+(\alpha,0) - \tilde{f}^+(\beta,0) + \tilde{f}^L(\alpha,0) - \tilde{f}^L(\beta,0) = \tilde{f}(\alpha,0) - \tilde{f}(\beta,0).$$

Then

$$\begin{aligned}
&-L^1_{h^{\epsilon,s}}(\alpha,\gamma,0) = \lim_{\beta\to\alpha}(\frac{d}{d\beta}(\frac{\sin(\tilde{f}_1(\alpha,0)-\tilde{f}_1(\beta,0))}{\cosh(\tilde{f}_2(\alpha,0)-\tilde{f}_2(\beta,0))-\cos(\tilde{f}_1(\alpha,0)-\tilde{f}_1(\beta,0))})(\alpha-\beta)^2)\\
&= \lim_{\beta\to\alpha}((\frac{\cos(\tilde{f}_1(\alpha,0)-\tilde{f}_1(\beta,0))\tilde{f}_1'(\beta,0)}{\cosh(\tilde{f}_2(\alpha,0)-\tilde{f}_2(\beta,0))-\cos(\tilde{f}_1(\alpha,0)-\tilde{f}_1(\beta,0))}-\\
&\frac{\sin(\tilde{f}_1(\alpha,0)-\tilde{f}_1(\beta,0))(-\sinh(\tilde{f}_2(\alpha,0)-\tilde{f}_2(\beta,0))\tilde{f}_2'(\beta,0)-\sin(\tilde{f}_1(\alpha,0)-\tilde{f}_1(\beta,0))\tilde{f}_1'(\beta,0))}{(\cosh(\tilde{f}_2(\alpha,0)-\tilde{f}_2(\beta,0))-\cos(\tilde{f}_1(\alpha,0)-\tilde{f}_1(\beta,0)))^2})(\alpha-\beta)^2)\\
&= \frac{2\tilde{f}_1'(\alpha,0)}{(\tilde{f}_1'(\alpha,0))^2+(\tilde{f}_2'(\alpha,0))^2} + \frac{4\tilde{f}_1'(\alpha,0)}{(\tilde{f}_1'(\alpha,0))^2+(\tilde{f}_2'(\alpha,0))^2}\\
&= 6\frac{\tilde{f}_1'(\alpha,0)}{(\tilde{f}_1'(\alpha,0))^2+(\tilde{f}_2'(\alpha,0))^2}
\end{aligned}$$

By the assumption (3.19) we have $\frac{d^2}{d\alpha^2}\tilde{f}_1(\alpha,0)|_{\alpha=-\tau(0)} = \frac{d^2}{d\alpha^2}\tilde{f}_1(\alpha,0)|_{\alpha=0} > 0$, moreover, by the condition of turnover points, we have $\frac{d}{d\alpha}\tilde{f}_1(\alpha,0)|_{\alpha=0} = 0$. □

Then by letting δ sufficiently small, we get the result.

3.9.4 Control the higher order term

Now we control

$$E^k = \frac{d}{dt}\int_{-\pi}^{\pi} |\partial^k_\alpha h^\epsilon(\alpha,\gamma,t)|^2 d\alpha.$$

Like in E^0 case, similarly, we have

$$\begin{aligned}
&\frac{d}{dt}\int_{-\pi}^{\pi}|\partial_\alpha^k h^\epsilon(\alpha,\gamma,t)|^2 d\alpha \\
&= 2\Re\int_{-\pi}^{\pi}\partial_\alpha^k h^\epsilon(\alpha,\gamma,t)\overline{\partial_\alpha^k T_1^\epsilon(h^\epsilon)(\alpha,\gamma,t)}d\alpha + 2\Re\int_{-\pi}^{\pi}\partial_\alpha^k h^\epsilon(\alpha,\gamma,t)\overline{\partial_\alpha^k T_2^\epsilon(h^\epsilon)(\alpha,\gamma,t)}d\alpha + B.T \\
&= E_1^k + E_2^k + B.T.
\end{aligned}$$

For the first term, we have

$$\begin{aligned}
E_1^k &= 2\Re\int_{-\tau(t)}^{\pi}\partial_\alpha^k h^\epsilon(\alpha,\gamma,t)\overline{\partial_\alpha^k(\tilde{\lambda}(\alpha_\gamma^t)L^1_{h^{\epsilon,s}}(\alpha,\gamma,t)\int_{-\tau(t)}^{2\alpha+\tau(t)}\frac{(h^\epsilon(\alpha,\gamma,t)-h^\epsilon(\beta,\gamma,t))}{(\alpha-\beta)^2+\epsilon^2}d\beta)}d\alpha \\
&= 2\Re\int_{-\tau(t)}^{\pi}\partial_\alpha^k h^\epsilon(\alpha,\gamma,t)\overline{\tilde{\lambda}(\alpha_\gamma^t)L^1_{h^{\epsilon,s}}(\alpha,\gamma,t)\partial_\alpha^k\int_{-\tau(t)}^{2\alpha+\tau(t)}\frac{(h^\epsilon(\alpha,\gamma,t)-h^\epsilon(\beta,\gamma,t))}{(\alpha-\beta)^2+\epsilon^2}d\beta)}d\alpha + B.T.
\end{aligned}$$

where we use lemma 3.9.25 and condition of $L^1_{h^{\epsilon,s}}$ (3.47). Moreover, by lemma 3.9.22, and condition of $L^1_{h^{\epsilon,s}}$ (3.47), we also have

$$\begin{aligned}
E_1^k &= \Re\int_{-\tau(t)}^{\pi}\partial_\alpha^k h^\epsilon(\alpha,\gamma,t)\overline{\frac{L^1_{h^{\epsilon,s}}(\alpha,\gamma,t)}{\alpha+\tau(t)}(\alpha+\tau(t))\int_{-\tau(t)}^{2\alpha+\tau(t)}\frac{(\partial_\alpha^k h^\epsilon(\alpha,\gamma,t)-\partial_\alpha^k h^\epsilon(\beta,\gamma,t))}{(\alpha-\beta)^2+\epsilon^2}d\beta)}d\alpha + B.T \\
&= E_{1,M}^k + B.T.
\end{aligned}$$

Similarly, for the next term, we have

$$\begin{aligned}
E_2^k &= 2\Re\int_{-\tau(t)}^{\pi}\partial_\alpha^k h^\epsilon(\alpha,\gamma,t)\overline{\partial_\alpha^k(\tilde{\lambda}(\alpha_\gamma^t)L^2_{h^{\epsilon,s}}(\alpha,\gamma,t)\int_{-\tau(t)}^{2\alpha+\tau(t)}\frac{(h^\epsilon(\alpha,\gamma,t)-h^\epsilon(\beta,\gamma,t))(\alpha-\beta)\epsilon}{((\alpha-\beta)^2+\epsilon^2)^2}d\beta)}d\alpha \\
&= 2\Re\int_{-\tau(t)}^{2\alpha+\tau(t)}\partial_\alpha^k h^\epsilon(\alpha,\gamma,t)\overline{(\tilde{\lambda}(\alpha_\gamma^t)L^2_{h^{\epsilon,s}}(\alpha,\gamma,t)\partial_\alpha^k(\int_{-\tau(t)}^{2\alpha+\tau(t)}\frac{(h^\epsilon(\alpha,\gamma,t)-h^\epsilon(\beta,\gamma,t))(\alpha-\beta)\epsilon}{((\alpha-\beta)^2+\epsilon^2)^2}d\beta))}d\alpha + B.T \\
&= E_{2,M}^0 + B.T.
\end{aligned}$$

where we use corollary 3.9.27 and condition of $L^2_{h^{\epsilon,s}}$ (3.47). Moreover, by lemma 3.9.23, and condition of $L^2_{h^{\epsilon,s}}$ (3.46), we again also have

$$\begin{aligned}
E_2^k &= \Re\int_{-\tau(t)}^{\pi}\partial_\alpha^k h^\epsilon(\alpha,\gamma,t)\overline{\tilde{\lambda}(\alpha_\gamma^t)\frac{L^2_{h^{\epsilon,s}}(\alpha,\gamma,t)}{\alpha+\tau(t)}(\alpha+\tau(t))\int_{-\tau(t)}^{2\alpha+\tau(t)}\frac{(\partial_\alpha^k h^\epsilon(\alpha,\gamma,t)-\partial_\alpha^k h^\epsilon(\beta,\gamma,t))(\alpha-\beta)\epsilon}{((\alpha-\beta)^2+\epsilon^2)^2}d\beta d\alpha} \\
&+ B.T = E_{2,M}^k + B.T.
\end{aligned}$$

Notice that $E^k_{1,M}+E^k_{2,M}$ has the same form of $E^0_{1,M}+E^0_{2,M}$ except that $\partial^k_\alpha h^\epsilon(\alpha,\gamma,t)$ takes the place of $h^\epsilon(\alpha,\gamma,t)$. Hence corollary 3.9.1 leads to

$$E^k_{1,M}+E^k_{2,M}\lesssim B(\|h\|_U).$$

3.9.5 The derivative with respect to γ

For $V^{i_0,k}$, still from the structure of T^ϵ, only T^ϵ_1 T^ϵ_2 and $T^{i,\epsilon}_3$ depending on ϵ. From Lemma 3.9.12, $\|T^{i,\epsilon}_3(h^\epsilon)\|_U\lesssim 1$ independent of ϵ. We have

$$\begin{aligned}
&\frac{d}{dt}\int_{-\pi}^{\pi}|\partial^k_\alpha\partial^{i_0}_\gamma h^\epsilon(\alpha,\gamma,t)|^2d\alpha\\
&=2\Re\int_{-\pi}^{\pi}\partial^k_\alpha\partial^{i_0}_\gamma h^\epsilon(\alpha,\gamma,t)\overline{\partial^k_\alpha\partial^{i_0}_\gamma T^\epsilon_1(h^\epsilon)(\alpha,\gamma,t)}d\alpha+2\Re\int_{-\pi}^{\pi}\partial^k_\alpha\partial^{i_0}_\gamma h^\epsilon(\alpha,\gamma,t)\overline{\partial^k_\alpha\partial^{i_0}_\gamma T^\epsilon_2(h^\epsilon)(\alpha,\gamma,t)}d\alpha+B.T\\
&=E^{k,i_0}_1+E^{k,i_0}_2+B.T.
\end{aligned}$$

We consider the first term and have

$$\begin{aligned}
&E^{k,i_0}_1\\
&=2\Re\int_{-\tau(t)}^{\pi}\partial^k_\alpha\partial^{i_0}_\gamma h^\epsilon(\alpha,\gamma,t)\overline{\partial^k_\alpha(L^1_{h^{\epsilon,s}}(\alpha,\gamma,t)\int_{-\tau(t)}^{2\alpha+\tau(t)}\frac{\partial^{i_0}_\gamma(h)(\alpha,\gamma,t)-\partial^{i_0}_\gamma(h)(\beta,\gamma,t)}{(\alpha-\beta)^2+\epsilon^2}d\beta)}d\alpha\\
&+2\Re\int_{-\tau(t)}^{\pi}\partial^k_\alpha\partial^{i_0}_\gamma h^\epsilon(\alpha,\gamma,t)\overline{\sum_{j\le i_0-1,l\le k}C_{j,i_0}\partial^{k-l}_\alpha\partial^{i_0-j}_\gamma L^1_{h^{\epsilon,s}}(\alpha,\gamma,t)\partial^l_\alpha\int_{-\tau(t)}^{2\alpha+\tau(t)}\frac{\partial^j_\gamma(h^\epsilon)(\alpha,\gamma,t)-\partial^j_\gamma(h^\epsilon)(\beta,\gamma,t)}{(\alpha-\beta)^2+\epsilon^2}d\beta}d\alpha.
\end{aligned}$$

From corollary 3.9.25, for $l\le k$, $j\le i_0-1$, we have

$$\begin{aligned}
&\|\partial^l_\alpha\int_{-\tau(t)}^{2\alpha+\tau(t)}\frac{\partial^j_\gamma(h^\epsilon)(\alpha,\gamma,t)-\partial^j_\gamma(h^\epsilon)(\beta,\gamma,t)}{(\alpha-\beta)^2+\epsilon^2}d\beta\|_{L^2[-\tau(t),\frac{\pi}{4}]}\lesssim\|h^\epsilon(\alpha,\gamma,t)\|_{V^{i_0-1,k+1}}\\
&\lesssim B(\|h\|_U).
\end{aligned}$$

Here we have the last inequality because the structure of U (3.128). Similarly,

$$\begin{aligned}
&E^{k,i_0}_2\\
&=2\Re\int_{-\tau(t)}^{\pi}\partial^k_\alpha\partial^{i_0}_\gamma h^\epsilon(\alpha,\gamma,t)\overline{\partial^k_\alpha(L^2_{h^{\epsilon,s}}(\alpha,\gamma,t)\int_{-\tau(t)}^{2\alpha+\tau(t)}\frac{(\partial^{i_0}_\gamma(h^\epsilon)(\alpha,\gamma,t)-\partial^{i_0}_\gamma(h^\epsilon)(\beta,\gamma,t))(\alpha-\beta)\epsilon}{((\alpha-\beta)^2+\epsilon^2)^2}d\beta)}d\alpha\\
&+2\Re\int_{-\tau(t)}^{\pi}\partial^k_\alpha\partial^{i_0}_\gamma h^\epsilon(\alpha,\gamma,t)
\end{aligned}$$

$$\cdot \sum_{j\le i_0-1,l\le k} C_{j,i_0}\partial_\alpha^{k-l}\partial_\gamma^{i_0-j}L^2_{h^{\epsilon,s}}(\alpha,\gamma,t)\partial_\alpha^l\int_{-\tau(t)}^{2\alpha+\tau(t)}\frac{(\partial_\gamma^j(h^\epsilon)(\alpha,\gamma,t)-\partial_\gamma^j(h^\epsilon)(\beta,\gamma,t))(\alpha-\beta)\epsilon}{((\alpha-\beta)^2+\epsilon^2)^2}d\beta d\alpha.$$

From corollary 3.9.26, for $l\le k$, $j\le i_0-1$, we have

$$\|\partial_\alpha^l\int_{-\tau(t)}^{2\alpha+\tau(t)}\frac{(\partial_\gamma^j(h^\epsilon)(\alpha,\gamma,t)-\partial_\gamma^j(h^\epsilon)(\beta,\gamma,t))(\alpha-\beta)\epsilon}{(\alpha-\beta)^2+\epsilon^2}d\beta\|_{L^2[-\tau(t),\frac{\pi}{4}]}\lesssim\|h^\epsilon(\alpha,\gamma,t)\|_{V^{i_0-1,k+1}}$$
$$\lesssim B(\|h\|_U).$$

Then we have

$$E_1^{k,i_0}+E_2^{k,i_0}=2\Re\int_{-\tau(t)}^{\pi}\partial_\alpha^k\partial_\gamma^{i_0}h^\epsilon(\alpha,\gamma,t)\overline{\partial_\alpha^k(L^1_{h^{\epsilon,s}}(\alpha,\gamma,t)\int_{-\tau(t)}^{2\alpha+\tau(t)}\frac{\partial_\gamma^{i_0}(h^\epsilon)(\alpha,\gamma,t)-\partial_\gamma^{i_0}(h^\epsilon)(\beta,\gamma,t)}{(\alpha-\beta)^2+\epsilon^2}d\beta)}d\alpha$$
$$+2\Re\int_{-\tau(t)}^{\pi}\partial_\alpha^k\partial_\gamma^{i_0}h^\epsilon(\alpha,\gamma,t)\overline{\partial_\alpha^k(L^2_{h^{\epsilon,s}}(\alpha,\gamma,t)\int_{-\tau(t)}^{2\alpha+\tau(t)}\frac{(\partial_\gamma^{i_0}(h^\epsilon)(\alpha,\gamma,t)-\partial_\gamma^{i_0}(h^\epsilon)(\beta,\gamma,t))(\alpha-\beta)\epsilon}{((\alpha-\beta)^2+\epsilon^2)^2}d\beta)}d\alpha+B.T.$$

Then it can be controlled in the same way as $E_1^k+E_2^k$ because here $\partial_\gamma^{i_0}h$ takes the place of h and other parts remain same. In conclusion, we have

$$\frac{d}{dt}\|h^\epsilon\|_U^2\lesssim B(\|h^\epsilon\|_U).$$

3.9.6 Take the limit

Now we have a uniform bound in U for the $h^\epsilon(\alpha,\gamma,t)$. Let $k=20$. We claim that we can use the compactness argument to take the limit and show there exists sufficiently small time t_1 such that there exists $h\in C^1([0,t_1],C^1([-1,1],H^5[-\pi,\pi]))$ satisfying

$$h(\alpha,\gamma,t)=0,\ \text{when}\ \alpha\le-\tau(t),\tag{3.150}$$

and

$$\frac{dh(\alpha,\gamma,t)}{dt}=T(h),$$

where $T(h)$ is from (3.32). Moreover, when $\gamma=0$, $h_0(\alpha,t)=h(\alpha,0,t)$ is a solution of

$$\frac{dh_0(\alpha,t)}{dt}=T^0(h_0),$$

where $T^0(h_0)$ is from (3.33).

Like in (3.129), let $d(\alpha,\gamma,t)=2\Re L^1_{h^s}(\alpha,\gamma,t)$, $d_2^\epsilon(\alpha,\gamma,t)=2\Im L^1_{h^s}(\alpha,\gamma,t)$, $d_3^\epsilon(\alpha,\gamma,t)=2\Im L^2_{h^s}(\alpha,\gamma,t)$.

We claim the corresponding refined R-T condition (3.129) also holds for the limit:

$$-d(\alpha,\gamma,t) \geq 2C|d_2(\alpha,\gamma,t)| + 2C|d_3(\alpha,\gamma,t)|. \tag{3.151}$$

3.9.7 Lemmas for the good terms

We introduce some lemmas. "$\lesssim$" in this section does not depend on ϵ. If we do not specify, here $h(\alpha,\gamma,t) \in C^{i_0}_{\gamma}([-1,1], H^k_{\alpha}[-\pi,\pi]) \cap S_{\delta}$.

Lemma 3.9.12. *The inequality for T^{ϵ}_3 in* (3.44) *and* (3.45) *is independent of ϵ_1.*

Proof. We only show the $b_i = 20$ case. By lemma 3.85, we get the same result for $b_i < 20$.

First, we show (3.44). For the L_2 case, we have

$$\begin{aligned}
&\int_{-\tau(t)}^{2\alpha+\tau(t)} L^{3,i}_{h^s}(\alpha,\beta,\gamma,t)\frac{\tilde{h}(\beta,\gamma,t)(\alpha-\beta)}{(\alpha-\beta)^2+\epsilon^2}d\beta \\
&= \int_{-\tau(t)}^{2\alpha+\tau(t)} (L^{3,i}_{h^s}(\alpha,\beta,\gamma,t) - L^{3,i}_{h}(\alpha,\alpha,\gamma,t))\frac{h(\beta,\gamma,t)(\alpha-\beta)}{(\alpha-\beta)^2+\epsilon^2}d\beta \\
&+ L^{3,i}_{h^s}(\alpha,\alpha,\gamma,t)\int_{-\tau(t)}^{2\alpha+\tau(t)} \frac{h(\beta,\gamma,t)(\alpha-\beta)}{(\alpha-\beta)^2+\epsilon^2}d\beta \\
&= Term_1 + Term_2.
\end{aligned}$$

By the smoothness of $L^{3,i}_{h^s}$ (3.52), we claim

$$\|\frac{(L^{3,i}_{h^s}(\alpha,\beta,\gamma,t) - L^{3,i}_{h^s}(\alpha,\alpha,\gamma,t))(\alpha-\beta)}{(\alpha-\beta)^2+\epsilon^2}\|_{C^{i_0}_{\gamma}([-1,1],C^0((-\tau(t),\frac{\pi}{4})\times(-\tau(t),\frac{2\pi}{3})))} \lesssim 1.$$

Then we have

$$\|Term_1\|_{C^{i_0}_{\gamma}([-1,1],L^2[-\tau(t),\frac{\pi}{4}])} \lesssim 1.$$

By the following lemma 3.9.14, we also get the similar bound for $Term_2$:

$$\|Term_2\|_{C^{i_0}_{\gamma}([-1,1],L^2[-\tau(t),\frac{\pi}{4}])} \lesssim 1.$$

Then we take kth derivative and have

$$\partial^k_{\alpha}\int_{-\tau(t)}^{2\alpha+\tau(t)} L^{3,i}_{h^s}(\alpha,\beta,\gamma,t)\frac{h(\beta,\gamma,t)(\alpha-\beta)}{(\alpha-\beta)^2+\epsilon^2}d\beta$$

$$= \sum_{k' \le k} \int_{-\alpha-\tau(t)}^{\alpha+\tau(t)} \partial_\alpha^{k'}(L_{h^s}^{3,i}(\alpha, \alpha-\beta, \gamma, t)) \frac{\partial_\alpha^{k-k'} h(\alpha-\beta, \gamma, t)(\beta)}{(\beta)^2+\epsilon^2} d\beta$$

$$+ \sum_{k_1+k_2+k_3 \le k-1} C_{k_1,k_2,k_3} \partial_\alpha^{k_3}(\partial_\alpha^{k_1}(L_{h^s}^{3,i}(\alpha, \alpha-\beta, \gamma, t))|_{\beta=\alpha+\tau(t)}) \partial_\alpha^{k_2}(\frac{(\partial_\alpha^{k-1-k_1-k_2-k_3} h)(-\tau(t), \gamma, t)(\alpha+\tau(t))}{(\alpha+\tau(t))^2+\epsilon^2})$$

$$- \sum_{k_1+k_2+k_3 \le k-1} C'_{k_1,k_2,k_3} \partial_\alpha^{k_3}(\partial_\alpha^{k_1}(L_{h^s}^{3,i}(\alpha, \alpha-\beta, \gamma, t))|_{\beta=-\alpha-\tau(t)}) \partial_\alpha^{k_2}(\frac{(\partial_\alpha^{k-1-k_1-k_2-k_3} h)(2\alpha+\tau(t), \gamma, t)(\alpha+\tau(t))}{(\alpha+\tau(t))^2+\epsilon^2})$$

$$= Term_1^k + Term_2^k + Term_3^k.$$

Since from (3.52), we have for $j + j' \le k$,

$$\|\frac{(\partial_\alpha^j \partial_\beta^{j'} L_{h^s}^{3,i}(\alpha, \beta, \gamma, t) - \partial_\alpha^j \partial_\beta^{j'} L_{h^s}^{3,i}(\alpha, \alpha, \gamma, t))(\alpha-\beta)}{(\alpha-\beta)^2+\epsilon^2}\|_{C_\gamma^{i_0}([-1,1], C^0((-\tau(t), \frac{\pi}{4}) \times (-\tau(t), \frac{2\pi}{3})))} \lesssim 1.$$

then $Term_1^k$ can be controlled in the same way in L^2. $Term_2^k$ and $Term_3^k$ can be controlled by (3.52) and lemma 3.9.19.

For (3.45), we have

$$\int_{-\tau(t)}^{2\alpha+\tau(t)} L_{h^s}^{3,i}(\alpha, \beta, \gamma, t) \frac{h(\beta, \gamma, t)(\alpha-\beta)}{(\alpha-\beta)^2+\epsilon^2} d\beta - \int_{-\tau(t)}^{2\alpha+\tau(t)} L_{g^s}^{3,i}(\alpha, \beta, \gamma, t) \frac{g(\beta, \gamma, t)(\alpha-\beta)}{(\alpha-\beta)^2+\epsilon^2} d\beta$$

$$= \int_{-\tau(t)}^{2\alpha+\tau(t)} (L_{h^s}^{3,i}(\alpha, \beta, \gamma, t) - L_{g^s}^{3,i}(\alpha, \beta, \gamma, t)) \frac{h(\beta, \gamma, t)(\alpha-\beta)}{(\alpha-\beta)^2+\epsilon^2} d\beta$$

$$+ \int_{-\tau(t)}^{2\alpha+\tau(t)} L_{g^s}^{3,i}(\alpha, \beta, \gamma, t) \frac{(h(\beta, \gamma, t) - g(\beta, \gamma, t))(\alpha-\beta)}{(\alpha-\beta)^2+\epsilon^2} d\beta$$

$$= Term_1^D + Term_2^D.$$

$Term_2^D$ can be bounded similarly as (3.44) by changing h to $h - g$. $Term_1^D$ can also be bounded in the same way by using (3.53) instead of (3.52). □

Lemma 3.9.13. *We have*

$$\|p.v. \int_{-\tau(t)}^{2\alpha+\tau(t)} \frac{L_{h^s}^{3,i}(\alpha, \beta, \gamma, t) h(\beta, \gamma, t)}{(\alpha-\beta)} d\beta\|_{C_\gamma^{i_0}([-1,1], L^2[-\tau(t), \frac{\pi}{4}])}$$

$$\lesssim \|L_{h^s}^{3,i}(\alpha, \beta, \gamma, t)\|_{C_\gamma^{i_0}([-1,1], C^1((-\tau(t), \frac{\pi}{4}) \times (-\tau(t), \frac{2\pi}{3})))} \|h(\alpha, \gamma, t)\|_{C_\gamma^{i_0}([-1,1], L^2[-\tau(t), \pi])}.$$

Proof. We have

$$\|p.v. \int_{-\tau(t)}^{2\alpha+\tau(t)} \frac{L_{h^s}^{3,i}(\alpha, \beta, \gamma, t) h(\beta, \gamma, t)}{(\alpha-\beta)} d\beta\|$$

$$
\begin{aligned}
&= p.v. \int_{-\tau(t)}^{2\alpha+\tau(t)} \frac{(L_{h^s}^{3,i}(\alpha,\beta,\gamma,t) - L_{h^s}^{3,i}(\alpha,\alpha,\gamma,t))h(\beta,\gamma,t)}{(\alpha-\beta)} d\beta \\
&+ L_{h^s}^{3,i}(\alpha,\alpha,\gamma,t) p.v. \int_{-\tau(t)}^{2\alpha+\tau(t)} \frac{h(\beta,\gamma,t)}{(\alpha-\beta)} d\beta \\
&= Term_1 + Term_2.
\end{aligned}
$$

Moreover, we have

$$
|Term_1|_{C_\gamma^{i_0}([-1,1],L^2[-\tau(t),\frac{\pi}{4}])} \lesssim \|L_{h^s}^{3,i}(\alpha,\beta,\gamma,t)\|_{C_\gamma^{i_0}([-1,1],C^1((-\tau(t),\frac{\pi}{4})\times(-\tau(t),\frac{2\pi}{3})))} \|h(\alpha,\gamma,t)\|_{C_\gamma^{i_0}([-1,1],L^2[-\tau(t),\pi])}.
$$

Let

$$
h_{ex}(\alpha) = \begin{cases} h(\alpha) & -\pi \le \alpha \le \pi, \\ 0 & \alpha < -\pi \text{ or } \alpha > \pi. \end{cases}
$$

Then

$$
\begin{aligned}
&p.v. \int_{-\tau(t)}^{2\alpha+\tau(t)} \frac{h(\beta,\gamma,t)}{(\alpha-\beta)} d\beta \\
&= \int_{-\infty}^{\infty} \frac{h(\beta,\gamma,t)}{(\alpha-\beta)} d\beta - \int_{2\alpha+\tau(t)}^{\infty} \frac{h_{ex}(\beta,\gamma,t)}{(\alpha-\beta)} d\beta - \int_{-\infty}^{-\tau(t)} \frac{h_{ex}(\beta,\gamma,t)}{(\alpha-\beta)} d\beta \\
&= Term_{2,1} + Term_{2,2} + Term_{2,3}.
\end{aligned}
$$

Then we could use the property of Hilbert transform to control $Term_{2,1}$ and Hilbert's Inequality to control $Term_{2,1} + Term_{2,2}$. □

Lemma 3.9.14. *If $h(\alpha) \in H^k[-\pi,\pi]$ with $\partial_\alpha^j h(-\tau(t)) = 0$ when $j \le k-1$, we have*

$$
\| \int_{-\tau(t)}^{2\alpha+\tau(t)} \frac{h(\beta)(\alpha-\beta)}{(\alpha-\beta)^2+\epsilon^2} d\beta \|_{H^k[-\tau(t),\frac{\pi}{4}]} \lesssim \|h(\alpha,\gamma,t)\|_{H^k[-\tau(t),\pi]}.
$$

Proof. We first consider the L^2 case. Let

$$
h_{ex}(\alpha) = \begin{cases} h(\alpha) & -\pi \le \alpha \le \pi, \\ 0 & \alpha < -\pi \text{ or } \alpha > \pi. \end{cases}
$$

$$
G(\alpha) = \int_{2\alpha+\tau(t)}^{\infty} \frac{h(\beta)(\alpha-\beta)}{(\alpha-\beta)^2+\epsilon^2} d\beta.
$$

Then

$$|G(\alpha-\tau(t))| = |\int_{2\alpha}^{\infty} \frac{h(\beta-\tau(t))(\alpha-\beta)}{(\alpha-\beta)^2+\epsilon^2}d\beta| \lesssim \int_{2\alpha}^{\infty} |\frac{h_{ex}(\beta-\tau(t))}{(\alpha+\beta)^2}|d\beta.$$

By the Hilbert's inequality, we have

$$\|G(\alpha-\tau(t))\|_{L^2[0,\infty]} \lesssim \|h_{ex}(\alpha)\|_{L^2[-\tau(t),\infty]}.$$

Moreover,

$$\int_{-\infty}^{\infty} \frac{h_{ex}(\beta)((\alpha-\beta)^2+\epsilon^2)}{(\alpha-\beta)^2+\epsilon^2}d\beta = C_H \int_{-\infty}^{\infty} H(h_{ex})(\beta)\frac{\epsilon}{(\alpha-\beta)^2+\epsilon^2}d\beta.$$

Here H is the Hilbert transform on the real line. Hence

$$\|\int_{-\infty}^{\infty} \frac{h_{ex}(\beta)}{(\alpha-\beta)^2+\epsilon)}d\beta\|_{L^2[-\tau(t),\frac{\pi}{4}]} \lesssim \|h_{ex}\|_{L^2[-\infty,\infty]} \lesssim \|h\|_{L^2[-\tau(t),\pi]}.$$

Since

$$\int_{-\infty}^{\infty} \frac{h_{ex}(\beta)(\alpha-\beta)}{(\alpha-\beta)^2+\epsilon^2)} - G(\alpha) = \int_{-\tau(t)}^{2\alpha+\tau(t)} \frac{h(\beta)(\alpha-\beta)}{(\alpha-\beta)^2+\epsilon^2}d\beta,$$

we get the result. For $K \leq 1$, there are extra boundary terms. We have

$$\begin{aligned}
&|\partial_\alpha \int_{-\alpha-\tau(t)}^{\alpha+\tau(t)} \frac{h(\alpha-\beta)(\beta)}{(\beta)^2+\epsilon^2}d\beta| \\
&\lesssim |\int_{-\tau(t)}^{2\alpha+\tau(t)} \frac{\partial_\alpha^k h(\beta)(\alpha-\beta)}{(\alpha-\beta)^2+\epsilon^2}d\beta| \\
&+ |\sum_{k'k-1} C_{k,k'}\partial_\alpha^{k-1-k'}(\frac{\partial_\alpha^{k'}h(-\tau(t))(\alpha+\tau(t))}{(\alpha+\tau(t))^2+\epsilon^2}) - \partial_\alpha^{k-1-k'}(\frac{\partial_\alpha^{k'}h(2\alpha+\tau(t))(\alpha+\tau(t))}{(\alpha+\tau(t))^2+\epsilon^2})|.
\end{aligned}$$

Since $\partial_\alpha^j h(\alpha,\gamma,t) = 0$ when $j \leq k-1$, we have

$$|Term_1| \lesssim \sum_{k'\leq k-1} (|\frac{\partial_\alpha^{k'}h(\alpha)}{(\alpha+\tau(t))^{k-k'+1}}| + |\frac{\partial_\alpha^{k'}h(2\alpha+\tau(t))}{(\alpha+\tau(t))^{k-k'+1}}|).$$

By lemma 3.9.19, we have,

$$\|Term_1\|_{L^2[-\tau(t),\frac{\pi}{4}]} \lesssim \|h\|_{H^k[-\pi,\pi]}.$$

For the $Term_2$, it can be controlled in the same way as in L^2 case. □

Lemma 3.9.15. *We have*

$$\|\partial_\alpha^j((\alpha+\tau(t))\int_{2\alpha+\tau(t)}^{\pi}\frac{h(\alpha)-h(\beta)}{(\alpha-\beta)^2+\epsilon^2}d\beta)\|_{L^2[-\tau(t),\frac{\pi}{4}]}\lesssim\|h(\alpha)\|_{H^k[-\pi,\pi]},\tag{3.152}$$

and

$$\|\partial_\alpha^{j'}(\int_{2\alpha+\tau(t)}^{\pi}\frac{h(\alpha)-h(\beta)}{(\alpha-\beta)^2+\epsilon^2}d\beta)\|_{L^2[-\tau(t),\frac{\pi}{4}]}\lesssim\|h(\alpha)\|_{H^k[-\pi,\pi]},\tag{3.153}$$

for all $j\leq k$, $j'\leq k-1$.

Proof. We have

$$\begin{aligned}
&\partial_\alpha^j((\alpha+\tau(t))\int_{2\alpha+\tau(t)}^{\pi}\frac{h(\alpha)-h(\beta)}{(\alpha-\beta)^2+\epsilon^2}d\beta)\\
&=\partial_\alpha^j((\alpha+\tau(t))\int_{\alpha-(\pi)}^{-\alpha-\tau(t)}\frac{h(\alpha)-h(\alpha-\beta)}{(\beta)^2+\epsilon^2}d\beta)\\
&=(\alpha+\tau(t))\partial_\alpha^j\int_{\alpha-(\pi)}^{-\alpha-\tau(t)}\frac{h(\alpha)-h(\alpha-\beta)}{(\beta)^2+\epsilon^2}d\beta+j\partial_\alpha^{j-1}\int_{\alpha-(\pi)}^{-\alpha-\tau(t)}\frac{h(\alpha)-h(\alpha-\beta)}{(\beta)^2+\epsilon^2}d\beta\\
&=(\alpha+\tau(t))\int_{\alpha-(\pi)}^{-\alpha-\tau(t)}\frac{\partial_\alpha^j h(\alpha)-\partial_\alpha^j h(\alpha-\beta)}{(\beta)^2+\epsilon^2}d\beta\\
&+\sum_{j'\leq j-1}C_{j,j'}(\alpha+\tau(t))(\partial_\alpha^{j'}(\frac{\partial_\alpha^{j-j'-1}h(\alpha)-\partial_\alpha^{j-j'-1}h(-\tau(t))}{(\alpha+\tau(t))^2+\epsilon^2})+\partial_\alpha^{j'}(\frac{\partial_\alpha^{j-j'-1}h(\alpha)-\partial_\alpha^{j-j'-1}h(\pi)}{(\alpha-\pi)^2+\epsilon^2}))\\
&+j\sum_{j'\leq j-2}\tilde{C}_{j,j'}(\partial_\alpha^{j'}(\frac{\partial_\alpha^{j-j'-2}h(\alpha)-\partial_\alpha^{j-j'-2}h(-\tau(t))}{(\alpha+\tau(t))^2+\epsilon^2})+\partial_\alpha^{j'}(\frac{\partial_\alpha^{j-j'-2}h(\alpha)-\partial_\alpha^{j-j'-2}h(\pi)}{(\alpha-\pi)^2+\epsilon^2}))\\
&+j\int_{\alpha-(\pi)}^{-\alpha-\tau(t)}\frac{\partial_\alpha^{j-1}h(\alpha)-\partial_\alpha^{j-1}h(\alpha-\beta)}{(\beta)^2+\epsilon^2}d\beta\\
&=Term1+\sum_{j'\leq j-1}C_{j,j'}(\alpha+\tau(t))(\partial_\alpha^{j'}(\frac{\partial_\alpha^{j-j'-1}h(\alpha)}{(\alpha+\tau(t))^2+\epsilon^2}))+\sum_{j'\leq j-1}C_{j,j'}(\alpha+\tau(t))(\partial_\alpha^{j'}(\frac{\partial_\alpha^{j-j'-1}h(\alpha))}{(\alpha-\pi)^2+\epsilon^2}))\\
&+j\sum_{j'\leq j-2}\tilde{C}_{j,j'}(\partial_\alpha^{j'}(\frac{\partial_\alpha^{j-j'-2}h(\alpha))}{(\alpha+\tau(t))^2+\epsilon^2})+j\sum_{j'\leq j-2}\tilde{C}_{j,j'}(\partial_\alpha^{j'}(\frac{\partial_\alpha^{j-j'-2}h(\alpha)}{(\alpha-\pi)^2+\epsilon^2}))\\
&+j\int_{\alpha-(\pi)}^{-\alpha-\tau(t)}\frac{\partial_\alpha^{j-1}h(\alpha)-\partial_\alpha^{j}h(\alpha-\beta)}{(\beta)^2+\epsilon^2}d\beta\\
&=Term1+Term2+Term3+Term4+Term5+Term6.
\end{aligned}$$

The first term can be controlled by lemma 3.9.16. Moreover,

$$|Term2|\lesssim\sum_{j'\leq j-1,j''\leq j'}\frac{|\partial_\alpha^{j-j'-1+j''}h(\alpha)|}{(\alpha+\tau(t))^{2+j'-j''}}||\alpha+\tau(t)|$$

$$\lesssim \sum_{j''\leq j-1} \frac{|\partial_\alpha^{j-1-j''}h(\alpha)|}{(\alpha+\tau(t))^{2+j''}}||\alpha+\tau(t)|$$
$$\lesssim \sum_{j''\leq j-1} \frac{|\partial_\alpha^{j-1-j''}h(\alpha)|}{(\alpha+\tau(t))^{1+j''}}.$$

Hence it can be controlled by lemma 3.9.19. Term 3 can be bounded easily because there is no singularity in the denominator. Term 4 is similar as Term 2 and Term 5 is similar as Term 3. Term 6 can be controlled by lemma 3.9.17. Then we get (3.152).

(3.153) can be shown in the same way. It is the part of (3.152) when one derivative hits on $(\alpha+\tau(t))$. □

Lemma 3.9.16. *We have*

$$\|(\alpha+\tau(t))\int_{2\alpha+\tau(t)}^{\pi}\frac{h(\alpha)-h(\beta)}{(\alpha-\beta)^2+\epsilon^2}d\beta\|_{L^2[-\tau(t),\frac{\pi}{4}]}\lesssim\|h(\alpha)\|_{L^2[-\tau(t),\pi]}.$$

Proof. We have

$$\begin{aligned}
&|(\alpha+\tau(t))\int_{\alpha-(\pi)}^{-\alpha-\tau(t)}\frac{h(\alpha)-h(\alpha-\beta)}{(\beta)^2+\epsilon^2}d\beta|\\
&\leq|(\alpha+\tau(t))\int_{\alpha-(\pi)}^{-\alpha-\tau(t)}\frac{1}{(\beta)^2+\epsilon^2}d\beta||h(\alpha)|\\
&+|(\alpha+\tau(t))\int_{2\alpha+\tau(t)}^{\pi}\frac{h(\beta)}{(\alpha-\beta)^2+\epsilon^2}d\beta|\\
&\lesssim|h(\alpha)|+Term_2(\alpha).
\end{aligned}$$

Moreover, we have

$$\begin{aligned}
&Term_2(\alpha-\tau(t))=|(\alpha)\int_{2\alpha-\tau(t)}^{\pi}\frac{h(\beta)}{(\alpha-\tau(t)-\beta)^2+\epsilon^2}d\beta| \qquad (3.154)\\
&=|(\alpha)\int_{2\alpha}^{\pi+\tau(t)}\frac{h(\beta-\tau(t))}{(\alpha-\beta)^2+\epsilon^2}d\beta|\\
&\lesssim\int_0^{2\pi}\frac{|h(\beta-\tau(t))|}{(\alpha+\beta)}d\beta.
\end{aligned}$$

Hence by Hilbert inequality, we have

$$\|Term_2(\alpha-\tau(t))\|_{L^2[0,\frac{\pi}{4}+\tau(t)]}\lesssim\|h(\alpha-\tau(t)\|_{L^2[0,\pi-\tau(t)]}.$$

Then we have the result. □

Lemma 3.9.17. *When $h \in H^1[-\tau(t), \pi]$, with $h(-\tau(t)) = 0$, we have*

$$\| \int_{2\alpha+\tau(t)}^{\pi} \frac{h(\alpha) - h(\beta)}{(\alpha-\beta)^2 + \epsilon^2} d\beta \|_{L^2[-\tau(t), \frac{\pi}{4}]} \lesssim \|h(\alpha)\|_{H^1[-\tau(t),\pi]}.$$

Proof. First,

$$\begin{aligned}
&\int_{2\alpha+\tau(t)}^{\pi} \frac{h(\alpha) - h(\beta)}{(\alpha-\beta)^2 + \epsilon^2} d\beta \\
&= (\alpha + \tau(t)) \int_{2\alpha+\tau(t)}^{\pi} \frac{\frac{h(\alpha)}{\alpha+\tau(t)} - \frac{h(\beta)}{\beta+\tau(t)}}{(\alpha-\beta)^2 + \epsilon^2} d\beta + \int_{2\alpha+\tau(t)}^{\pi} \frac{(\alpha-\beta)\frac{h(\beta)}{\beta+\tau(t)}}{(\alpha-\beta)^2 + \epsilon^2} d\beta \\
&= Term_1 + Term_2.
\end{aligned}$$

$Term_1$ can be bounded by lemma 3.9.16 and lemma 3.9.19 with $l = 1$. We also have

$$|Term_2| \leq \int_{2\alpha+\tau(t)}^{\pi} \frac{\frac{h(\beta)}{\beta+\tau(t)}}{|\alpha-\beta|} d\beta.$$

Then it can be bounded by Hilbert's inequality as in (3.154), and we have

$$\|Term_2\|_{L^2[-\tau(t), \frac{\pi}{4}]} \lesssim \| \frac{h(\beta)}{\beta + \tau(t)} \|_{L^2[-\tau(t),\pi]}.$$

Then by lemma 3.9.19 with $l = 1$, we have the result. □

Lemma 3.9.18. *We have*

$$\|\partial_\alpha^j ((\alpha + \tau(t)) \int_{2\alpha+\tau(t)}^{\pi} \frac{(h(\alpha) - h(\beta))(\alpha-\beta)\epsilon}{((\alpha-\beta)^2 + \epsilon^2)^2} d\beta)\|_{L^2[-\tau(t), \frac{\pi}{4}]} \lesssim \|h(\alpha)\|_{H^k[-\pi,\pi]}, \tag{3.155}$$

$$\|\partial_\alpha^{j'} \int_{2\alpha+\tau(t)}^{\pi} \frac{(h(\alpha) - h(\beta))(\alpha-\beta)\epsilon}{((\alpha-\beta)^2 + \epsilon^2)^2} d\beta\|_{L^2[-\tau(t), \frac{\pi}{4}]} \lesssim \|h(\alpha)\|_{H^k[-\pi,\pi]}, \tag{3.156}$$

for all $j \leq k, j' \leq k-1$. Here $\lesssim$ doe not depend on ϵ.

Proof. It has the similar structure as in the previous lemma. For (3.155), we have

$$\begin{aligned}
&\partial_\alpha^j ((\alpha + \tau(t)) \int_{2\alpha+\tau(t)}^{\pi} \frac{(h(\alpha) - h(\beta))(\alpha-\beta)\epsilon}{((\alpha-\beta)^2 + \epsilon^2)^2} d\beta) \\
&= \partial_\alpha^j ((\alpha + \tau(t)) \int_{\alpha-(\pi)}^{-\alpha-\tau(t)} \frac{(h(\alpha) - h(\alpha-\beta))\beta\epsilon}{((\beta)^2 + \epsilon^2)^2} d\beta)
\end{aligned}$$

$$
\begin{aligned}
&= (\alpha+\tau(t))\partial_\alpha^j \int_{\alpha-(\pi)}^{-\alpha-\tau(t)} \frac{(h(\alpha)-h(\alpha-\beta))\beta\epsilon}{((\beta)^2+\epsilon^2)^2}d\beta + j\partial_\alpha^{j-1}\int_{\alpha-(\pi)}^{-\alpha-\tau(t)} \frac{(h(\alpha)-h(\alpha-\beta))\beta\epsilon}{((\beta)^2+\epsilon^2)^2}d\beta \\
&= (\alpha+\tau(t))\int_{\alpha-(\pi)}^{-\alpha-\tau(t)} \frac{(\partial_\alpha^j h(\alpha)-\partial_\alpha^j h(\alpha-\beta))(\beta)\epsilon}{((\beta)^2+\epsilon^2)^2}d\beta \\
&+ \sum_{j'\le j-1} C_{j,j'}(\alpha+\tau(t))\partial_\alpha^{j'}\Big(\frac{(\partial_\alpha^{j-j'-1}h(\alpha)-\partial_\alpha^{j-j'-1}h(-\tau(t)))(\alpha+\tau(t))\epsilon}{((\alpha+\tau(t))^2+\epsilon^2)^2}\Big) \\
&+ \sum_{j'\le j-1} C_{j,j'}(\alpha+\tau(t))\partial_\alpha^{j'}\Big(\frac{(\partial_\alpha^{j-j'-1}h(\alpha)-\partial_\alpha^{j-j'-1}h(\pi))(\alpha-\pi)\epsilon}{((\alpha-\pi)^2+\epsilon^2)^2}\Big) \\
&+ j\sum_{j'\le j-2} \tilde{C}_{j,j'}\partial_\alpha^{j'}\Big(\frac{(\partial_\alpha^{j-j'-2}h(\alpha)-\partial_\alpha^{j-j'-2}h(-\tau(t)))(\alpha+\tau(t))\epsilon}{((\alpha+\tau(t))^2+\epsilon^2)^2}\Big) \\
&+ j\sum_{j'\le j-2} \tilde{C}_{j,j'}\partial_\alpha^{j'}\Big(\frac{(\partial_\alpha^{j-j'-2}h(\alpha)-\partial_\alpha^{j-j'-2}h(\pi))(\alpha-\pi)\epsilon}{((\alpha-\pi)^2+\epsilon^2)^2}\Big) \\
&+ j\int_{\alpha-\pi}^{-\alpha-\tau(t)} \frac{(\partial_\alpha^{j-1}h(\alpha)-\partial_\alpha^{j-1}h(\alpha-\beta))\beta\epsilon}{((\beta)^2+\epsilon^2)^2}d\beta \\
&= Term1 + \sum_{j'\le j-1} C_{j,j'}(\alpha+\tau(t))\partial_\alpha^{j'}\Big(\frac{(\partial_\alpha^{j-j'-1}h(\alpha))(\alpha+\tau(t))\epsilon}{((\alpha+\tau(t))^2+\epsilon^2)^2}\Big) \\
&+ \sum_{j'\le j-1} C_{j,j'}(\alpha+\tau(t))\partial_\alpha^{j'}\Big(\frac{\partial_\alpha^{j-j'-1}h(\alpha))(\alpha-\pi)\epsilon}{((\alpha-\pi)^2+\epsilon^2)^2}\Big) \\
&+ j\sum_{j'\le j-2} \tilde{C}_{j,j'}\Big(\partial_\alpha^{j'}\Big(\frac{\partial_\alpha^{j-j'-2}h(\alpha))(\alpha+\tau(t))\epsilon}{((\alpha+\tau(t))^2+\epsilon^2)^2}\Big) \\
&+ j\sum_{j'\le j-2} \tilde{C}_{j,j'}\partial_\alpha^{j'}\Big(\frac{\partial_\alpha^{j-j'-2}h(\alpha)(\alpha-\pi)\epsilon}{((\alpha-\pi)^2+\epsilon^2)^2}\Big) + Term6 \\
&= Term1 + Term2 + Term3 + Term4 + Term5 + Term6.
\end{aligned}
$$

Each term can be controlled in the same way as in the corresponding lemma 3.9.15 by using

$$\frac{\alpha\epsilon}{(\alpha^2+\epsilon^2)^2} \lesssim \frac{1}{(\alpha^2+\epsilon^2)}.$$

(3.156) can also be shown in the same way. It is part of (3.155) when one derivative hits on $(\alpha+\tau(t))$. □

Lemma 3.9.19. *If $h(\alpha) \in H^l[-\tau(t),\pi]$ with $\partial_\alpha^{l'}h(-\tau(t)) = 0$ for $l' \le l-1$, then for $j, j' \le l$ we have*

$$\|\frac{\partial_\alpha^j h(\alpha)}{(\alpha+\tau(t))^{l-j}}\|_{L^2[-\tau(t),\pi]} \lesssim \|h(\alpha)\|_{H^l[-\tau(t),\pi]}.$$

Proof. We have

$$\begin{aligned}
&\|\frac{\partial_\alpha^j h(\alpha)}{(\alpha+\tau(t))^{l-j}}\|_{L^2} = \|\frac{\int_0^1 \partial_\alpha^{j+1} h(\alpha-\zeta(\alpha+\tau(t)))d\zeta}{(\alpha+\tau(t))^{l-j-1}}\|_{L^2} \\
&\leq \int_0^1 \|\frac{\partial_\alpha^{j+1} h(\tilde{\alpha})}{(\tilde{\alpha}+\tau(t))^{l-j-1}(\frac{1}{1-\zeta})^{l-j-1}}\|_{L^2}\frac{1}{\sqrt{1-\zeta}}d\zeta \\
&\leq \int_0^1 \|\frac{\partial_\alpha^{j+1} h(\tilde{\alpha})}{(\tilde{\alpha}+\tau(t))^{l-j-1}}\|_{L^2}(1-\zeta)^{l-j-1}\frac{1}{\sqrt{1-\zeta}}d\zeta \\
&\lesssim \|\frac{\partial_\alpha^{j+1} h(\tilde{\alpha})}{(\tilde{\alpha}+\tau(t))^{l-j-1}}\|_{L^2}.
\end{aligned}$$

We could repeat this process and get the result. □

Corollary 3.9.20. *If $h(\alpha) \in H^l[-\tau(t),\pi]$ with $\partial_\alpha^{l'} h(-\tau(t)) = 0$ for $l' \leq l-1$, then for $j \leq l$ we have*

$$\|\frac{\partial_\alpha^j h(\alpha)}{(\alpha+\tau(t))^{l-j-j'}}\|_{H^{j'}[-\tau(t),\pi]} \lesssim \|h(\alpha)\|_{H^l[-\tau(t),\pi]}.$$

Lemma 3.9.21. *If $L_{h^2}(-\tau(t)) = 0$, $L_{h^s}(\alpha) \in C^1[-\tau(t),\pi]$, we have*

$$|\int_{-\tau(t)}^{\pi} L_{h^s}(\alpha)h(\alpha)\int_{-\tau(t)}^{\pi}\frac{(h(\alpha)-h(\beta))(\alpha-\beta)\epsilon}{((\alpha-\beta)^2+\epsilon^2)^2}d\beta d\alpha| \lesssim \|L_{h^s}(\alpha)\|_{C^1[-\tau(t),\pi]}\|h\|^2_{L^2[-\tau(t),\pi]}.$$

Proof. We separate $h(\alpha)$, $h(\beta)$ and get

$$\begin{aligned}
&|\int_{-\tau(t)}^{\pi} L_{h^s}(\alpha)h(\alpha)\int_{-\tau(t)}^{\pi}\frac{(h(\alpha)-h(\beta))(\alpha-\beta)\epsilon}{((\alpha-\beta)^2+\epsilon^2)^2}d\beta d\alpha| \\
&\lesssim \int_{-\tau(t)}^{\pi} h^2(\alpha)\frac{L_{h^s}(\alpha)}{\alpha+\tau(t)}\int_{2\alpha+\tau(t)}^{\pi}\frac{(\alpha-\beta)\epsilon}{((\alpha-\beta)^2+\epsilon^2)^2}d\beta d\alpha||\alpha+\tau(t)| \\
&+|\int_{-\tau(t)}^{\pi} L_{h^s}(\alpha)h(\alpha)\int_{-\tau(t)}^{\pi}\frac{(h(\beta))(\alpha-\beta)\epsilon}{((\alpha-\beta)^2+\epsilon^2)^2}d\beta d\alpha| \\
&\lesssim \int_{-\tau(t)}^{\pi}|h(\alpha)|^2 d\alpha\|L_{h^s}(\alpha)\|_{C^1[-\tau(t),\pi]} \\
&+|\frac{1}{2}\int_{-\tau(t)}^{\pi} h(\alpha)\int_{-\tau(t)}^{\pi}\frac{(L_{h^s}(\alpha)-L_{h^s}(\beta))}{\alpha-\beta}\frac{(h(\beta))(\alpha-\beta)^2\epsilon}{((\alpha-\beta)^2+\epsilon^2)^2}d\beta d\alpha| \\
&\lesssim \|L_{h^s}(\alpha)\|_{C^1[-\tau(t),\pi]}\|h\|^2_{L^2[-\tau(t),\pi]} \\
&+|\int_{-\tau(t)}^{\pi}|h(\alpha)|\int_{-\tau(t)}^{\pi}\|L_{h^s}(\alpha)\|_{C^1[-\tau(t),\pi]}\frac{|h(\beta)|\epsilon}{(\alpha-\beta)^2+\epsilon^2}d\beta d\alpha| \\
&\lesssim \|L_{h^s}(\alpha)\|_{C^1[-\tau(t),\pi]}\|h\|^2_{L^2[-\tau(t),\pi]},
\end{aligned}$$

where we use the Young's inequality in the last step. □

Lemma 3.9.22. *We have*

$$\|(\alpha+\tau(t))(\partial_\alpha^j\int_{-\tau(t)}^{2\alpha+\tau(t)}\frac{h(\alpha)-h(\beta)}{(\alpha-\beta)^2+\epsilon^2}d\beta-\int_{-\tau(t)}^{2\alpha+\tau(t)}\frac{\partial_\alpha^j h(\alpha)-\partial_\alpha^j h(\beta)}{(\alpha-\beta)^2+\epsilon^2}d\beta)\|_{L^2[-\tau(t),\frac{\pi}{4}]}\lesssim\|h\|_{H^k[-\tau(t),\pi]},$$

when $j\leq k$. For any $j'\leq k-1$, we have

$$\|(\partial_\alpha^{j'}\int_{-\tau(t)}^{2\alpha+\tau(t)}\frac{h(\alpha)-h(\beta)}{(\alpha-\beta)^2+\epsilon^2}d\beta-\int_{-\tau(t)}^{2\alpha+\tau(t)}\frac{\partial_\alpha^{j'} h(\alpha)-\partial_\alpha^j h(\beta)}{(\alpha-\beta)^2+\epsilon^2}d\beta)\|_{L^2[-\tau(t),\frac{\pi}{4}]}\lesssim\|h\|_{H^k[-\tau(t),\pi]}.$$

Proof. First we have

$$\begin{aligned}
&\partial_\alpha^j\int_{-\tau(t)}^{2\alpha+\tau(t)}\frac{h(\alpha)-h(\beta)}{(\alpha-\beta)^2+\epsilon^2}d\beta\\
&=\partial_\alpha^j\int_{-(\alpha+\tau(t))}^{\alpha+\tau(t)}\frac{h(\alpha)-h(\alpha-\beta)}{(\beta)^2+\epsilon^2}d\beta\\
&=\int_{-(\alpha+\tau(t))}^{\alpha+\tau(t)}\frac{\partial_\alpha^j h(\alpha)-\partial_\alpha^j h(\alpha-\beta)}{(\beta)^2+\epsilon^2}d\beta+\sum_{l+l'=j-1}C^{l,l'}\partial_\alpha^{l-l'}\frac{\partial_\alpha^{l'}h(\alpha)-\partial_\alpha^{l'}h(-\tau(t))}{(\alpha+\tau(t))^2+\epsilon^2}\\
&-\sum_{l+l'=j-1}C^{l,l'}\partial_\alpha^{l-l'}\frac{\partial_\alpha^{l'}h(\alpha)-\partial_\alpha^{l'}h(2\alpha+\tau(t))}{(\alpha+\tau(t)))^2+\epsilon^2}\\
&=\int_{-(\alpha+\tau(t))}^{\alpha+\tau(t)}\frac{\partial_\alpha^j h(\alpha)-\partial_\alpha^j h(\alpha-\beta)}{(\beta)^2+\epsilon^2}d\beta+Term\ 2+Term\ 3.
\end{aligned}$$

We have

$$|Term\ 2|+|Term\ 3|\lesssim\sum_{l\leq j-1,l'\leq l}\frac{|\partial_\alpha^{l'}h(2\alpha+\tau(t))|}{(\alpha+(\tau(t)))^{l-l'+2}}+\sum_{l\leq j-1,l'\leq l}\frac{|\partial_\alpha^{l'}h(\alpha)|}{(\alpha+\tau(t)))^{l-l+2}},$$

and

$$(\alpha+\tau(t))(|Term\ 2|+|Term\ 3|)\lesssim\sum_{l\leq j-1,l'\leq l}\frac{|\partial_\alpha^{l'}h(2\alpha+\tau(t))|}{(\alpha+\tau(t))^{l-l'+1}}+\sum_{l\leq j-1,l'\leq l}\frac{|\partial_\alpha^{l'}h(\alpha)|}{(\alpha+\tau(t)))^{l-l'+1}}.$$

They can be bounded by lemma 3.9.19. Here we also use $h(2\alpha+\tau(t))$ can be extended to $H^k[-\tau(t),\pi]$ with $\partial_\alpha^{j'}h(2\alpha+\tau(t))|_{\alpha=-\tau(t)}=0$ when $j\leq k-1$. Then we get first inequality. The second inequality can be done in the same way. □

Lemma 3.9.23. *We have*

$$\|(\alpha+\tau(t))(\partial_\alpha^j\int_{-\tau(t)}^{2\alpha+\tau(t)}\frac{(h(\alpha)-h(\beta))(\alpha-\beta)\epsilon}{((\alpha-\beta)^2+\epsilon^2)^2}d\beta-\int_{-\tau(t)}^{2\alpha+\tau(t)}\frac{(\partial_\alpha^j h(\alpha)-\partial_\alpha^k h(\beta))(\alpha-\beta)\epsilon}{((\alpha-\beta)^2+\epsilon^2)^2}d\beta)\|_{L^2[-\tau(t),\frac{\pi}{4}]}$$

$$\lesssim\|h\|_{H^k[-\tau(t),\pi]},$$

for $j \le k$. For any $j' \le k-1$, we have

$$\|(\partial_\alpha^{j'} \int_{-\tau(t)}^{2\alpha+\tau(t)} \frac{(h(\alpha)-h(\beta))(\alpha-\beta)\epsilon}{((\alpha-\beta)^2+\epsilon^2)^2} d\beta - \int_{-\tau(t)}^{2\alpha+\tau(t)} \frac{(\partial_\alpha^{j'} h(\alpha) - \partial_\alpha^{j'} h(\beta))(\alpha-\beta)\epsilon}{((\alpha-\beta)^2+\epsilon^2)^2} d\beta)\|_{L^2[-\tau(t),\frac{\pi}{4}]} \lesssim \|h\|_{H^k[-\tau(t),\pi]}.$$

Proof. The proof is same as the previous lemma (3.9.22). □

Lemma 3.9.24. *If $h(-\tau(t)) = 0$, $h = 0$ when $\alpha > \frac{\pi}{4}$ and $h \in H^1[-\tau(t),\pi]$, $\tau(t) < \frac{\pi}{2}$, we have*

$$\| \int_{-\tau(t)}^{\pi} \frac{h(\alpha)-h(\beta)}{(\alpha-\beta)^2+\epsilon^2} d\beta\|_{L^2[-\tau(t),\frac{\pi}{4}]} \lesssim \|h\|_{H^1[-\tau(t),\pi]}.$$

Proof. First, the following estimate holds:

$$\| \int_{-\tau(t)}^{\pi} \frac{h(\alpha)-h(\beta)}{(\alpha-\beta)^2+\epsilon^2} d\beta - \int_{-\tau(t)}^{\pi-\tau(t)} \frac{h(\alpha)-h(\beta)}{\sin^2(\alpha-\beta)+\epsilon^2} d\beta\|_{L^2[-\tau(t),\frac{\pi}{4}]} \lesssim \|h\|_{H^1[-\tau(t),\pi]}. \tag{3.157}$$

In fact, we have

$$\| \int_{\frac{\pi}{2}}^{\pi} \frac{h(\alpha)-h(\beta)}{(\alpha-\beta)^2+\epsilon^2} d\beta\|_{L^2[-\tau(t),\frac{\pi}{4}]} \lesssim \|h(\alpha)\|_{L^2[-\tau(t),\pi]}.$$

Moreover,

$$\begin{aligned}
&\| \int_{-\tau(t)}^{\frac{\pi}{2}} \frac{h(\alpha)-h(\beta)}{(\alpha-\beta)^2+\epsilon^2} d\beta - \int_{-\tau(t)}^{\frac{\pi}{2}} \frac{h(\alpha)-h(\beta)}{\sin^2(\alpha-\beta)+\epsilon^2} d\beta\|_{L^2[-\tau(t),\frac{\pi}{4}]} \\
&\lesssim \| \int_{-\tau(t)}^{\frac{\pi}{2}} |h(\alpha)-h(\beta)| |\frac{\sin(\alpha-\beta)^2-(\alpha-\beta)^2}{(\sin^2(\alpha-\beta)+\epsilon^2)((\alpha-\beta)^2+\epsilon^2)}| d\beta\| \\
&\lesssim \|h(\alpha)\|_{L^2[-\tau(t),\pi]}.
\end{aligned}$$

We also have

$$\begin{aligned}
&\| \int_{\frac{\pi}{2}}^{\pi} \frac{h(\alpha)-h(\beta)}{\sin^2(\alpha-\beta)+\epsilon^2} d\beta\|_{L^2[-\tau(t),\frac{\pi}{4}]} \\
&= \| \int_{\frac{\pi}{2}}^{\pi-\tau(t)} \frac{h(\alpha)}{\sin^2(\alpha-\beta)+\epsilon^2} \|_{L^2[-\tau(t),\frac{\pi}{4}]} \\
&\lesssim \|h(\alpha) \frac{1}{(\alpha+\tau(t))} \|_{L^2[-\tau(t),\frac{\pi}{4}]} \\
&\lesssim \|h(\alpha)\|_{H^1[-\tau(t),\pi]}.
\end{aligned}$$

Here we use lemma 3.9.19 in the last inequality. Therefore (3.157) is true.

Moreover, if $h \in H^{\frac{3}{2}}[\mathcal{T}]$, where $\mathcal{T}$ is the torus of length π, for any $f \in C^\infty(\mathcal{T})$, we have

$$\begin{aligned}
&\int_{-\tau(t)}^{\pi-\tau(t)} f(\alpha) \int_{-\tau(t)}^{\pi-\tau(t)} \frac{h(\alpha)-h(\beta)}{\sin^2(\alpha-\beta)+\epsilon^2} d\beta d\alpha \\
=&\int_{-\tau(t)}^{\pi-\tau(t)} \Lambda_{\mathcal{T}}^{-\frac{1}{2}}(f-\hat{f}(0))(\alpha) \int_{-\tau(t)}^{\pi-\tau(t)} \frac{\Lambda_{\mathcal{T}}^{\frac{1}{2}}h(\alpha)-\Lambda_{\mathcal{T}}^{\frac{1}{2}}h(\beta)}{\sin^2(\alpha-\beta)+\epsilon^2} d\beta d\alpha \\
\leq&\frac{1}{2}(\int_{-\tau(t)}^{\pi-\tau(t)} \int_{-\tau(t)}^{\pi-\tau(t)} \frac{(\Lambda_{\mathcal{T}}^{-\frac{1}{2}}(f(\alpha)-\hat{f}(0))-\Lambda_{\mathcal{T}}^{-\frac{1}{2}}(f(\beta)-\hat{f}(0)))^2}{\sin^2(\alpha-\beta)} d\beta d\alpha)^{\frac{1}{2}} \\
&\cdot(\int_{-\tau(t)}^{\pi-\tau(t)} \int_{-\tau(t)}^{\pi-\tau(t)} \frac{(\Lambda_{\mathcal{T}}^{\frac{1}{2}}h(\alpha)-\Lambda_{\mathcal{T}}^{\frac{1}{2}}h(\beta))^2}{\sin^2(\alpha-\beta)}) d\beta d\alpha^{\frac{1}{2}} \\
\lesssim&(\int_{-\tau(t)}^{\pi-\tau(t)} \Lambda_{\mathcal{T}}^{-\frac{1}{2}}(f(\alpha)-\hat{f}(0))\Lambda_{\mathcal{T}}^{\frac{1}{2}}(f(\alpha)-\hat{f}(0))d\alpha)^{\frac{1}{2}}(\int_{-\tau(t)}^{\pi-\tau(t)} \Lambda_{\mathcal{T}}^{\frac{1}{2}}(h(\alpha))\Lambda_{\mathcal{T}}^{\frac{3}{2}}(h(\alpha))d\alpha)^{\frac{1}{2}} \\
\lesssim&\frac{1}{2}\|f\|_{L^2}\|\Lambda_{\mathcal{T}}h(\alpha)\|_{L^2} \lesssim \|f\|_{L^2}\|h(\alpha)\|_{H^1}
\end{aligned}$$

Since $h(-\tau(t))=0$, $h=0$ when $\alpha > \frac{\pi}{4}$ and $h \in H^1[-\tau(t),\pi]$, we have $h \in H^1[\mathcal{T}]$. Then we could choose a sequence in $H^{\frac{3}{2}}$ to approximate it and get the result. Therefore

$$\|\int_{-\tau(t)}^{\pi} \frac{h(\alpha)-h(\beta)}{\sin^2(\alpha-\beta)+\epsilon^2} d\beta\|_{L^2[-\tau(t),\frac{\pi}{4}]} \lesssim \|h\|_{H^1[-\tau(t),\pi]}. \tag{3.158}$$

□

Corollary 3.9.25. *we have*

$$\|\partial_\alpha^j \int_{-\tau(t)}^{2\alpha+\tau(t)} \frac{h(\alpha)-h(\beta)}{(\alpha-\beta)^2+\epsilon^2} d\beta\|_{L^2[-\tau(t),\frac{\pi}{4}]} \lesssim \|h\|_{H^k[-\tau(t),\pi]}.$$

for $j \leq k-1$.

Proof. From lemma 3.9.22, we have

$$\|\partial_\alpha^j \int_{-\tau(t)}^{2\alpha+\tau(t)} \frac{h(\alpha)-h(\beta)}{(\alpha-\beta)^2+\epsilon^2} d\beta - \int_{-\tau(t)}^{2\alpha+\tau(t)} \frac{\partial_\alpha^j h(\alpha)-\partial_\beta^j h(\beta)}{(\alpha-\beta)^2+\epsilon^2} d\beta\|_{L^2[-\tau(t),\frac{\pi}{4}]} \lesssim \|h\|_{H^k[-\tau(t),\pi]},$$

Moreover, from lemma 3.9.17, we have

$$\|\int_{2\alpha+\tau(t)}^{\pi} \frac{\partial_\alpha^j h(\alpha)-\partial_\beta^j h(\beta)}{(\alpha-\beta)^2+\epsilon^2} d\beta\|_{L^2[-\tau(t),\frac{\pi}{4}]} \lesssim \|h\|_{H^k[-\tau(t),\pi]},$$

From lemma 3.9.24, we have

$$\| \int_{-\tau(t)}^{\pi} \frac{\partial_\alpha^j h(\alpha) - \partial_\beta^j h(\beta)}{(\alpha-\beta)^2+\epsilon^2} d\beta \|_{L^2[-\tau(t),\frac{\pi}{4}]} \lesssim \|h\|_{H^k[-\tau(t),\pi]}.$$

Then the three inequalities lead to the result. □

Lemma 3.9.26. *If $h \in H^1[-\tau(t),\pi]$, $h(-\tau(t))=0$, $h=0$ when $\alpha > \frac{\pi}{4}$, we have*

$$\| \int_{-\tau(t)}^{\pi} \frac{(h(\alpha)-h(\beta))(\alpha-\beta)\epsilon}{((\alpha-\beta)^2+\epsilon^2)^2} d\beta \|_{L^2_\alpha[-\tau(t),\frac{\pi}{4}]} \lesssim \|h\|_{H^1[-\tau(t),\pi]}.$$

Proof. We have

$$\begin{aligned}
&\| \int_{-\tau(t)}^{\pi} \frac{(h(\alpha)-h(\beta))(\alpha-\beta)\epsilon}{((\alpha-\beta)^2+\epsilon^2)^2} d\beta \|_{L^2[-\tau(t),\frac{\pi}{4}]} \\
&= \| \int_{\alpha+\tau(t)}^{-\pi+\tau(t)+\alpha} \frac{\int_0^1 h'(\alpha-\zeta(\beta))d\zeta \beta^2\epsilon}{(\beta^2+\epsilon^2)^2} d\beta \|_{L^2[-\tau(t),\frac{\pi}{4}]} \\
&\lesssim \int_{-\pi}^{\pi} \frac{\| \int_0^1 h'(\alpha-\zeta(\beta))d\zeta \|_{L^2[-\tau(t),\frac{\pi}{4}]} \beta^2\epsilon}{(\beta^2+\epsilon^2)^2} d\beta \| \\
&\lesssim \|h\|_{H^1[-\tau(t),\pi]} \int_{-\pi}^{\pi} \frac{\beta^2\epsilon}{(\beta^2+\epsilon^2)^2} d\beta \\
&\lesssim \|h\|_{H^1[-\tau(t),\pi]}.
\end{aligned}$$

□

Corollary 3.9.27. *We have*

$$\|\partial_\alpha^j \int_{-\tau(t)}^{2\alpha+\tau(t)} \frac{(h(\alpha)-h(\beta))(\alpha-\beta)\epsilon}{((\alpha-\beta)^2+\epsilon^2)^2} d\beta \|_{L^2[-\tau(t),\frac{\pi}{4}]} \lesssim \|h\|_{H^k[-\tau(t),\pi]},$$

for $j \le k-1$.

Proof. From lemma 3.9.23, we have

$$\|\partial_\alpha^j \int_{-\tau(t)}^{2\alpha+\tau(t)} \frac{(h(\alpha)-h(\beta))(\alpha-\beta)\epsilon}{((\alpha-\beta)^2+\epsilon^2)^2} d\beta - \int_{-\tau(t)}^{2\alpha+\tau(t)} \frac{(\partial_\alpha^j h(\alpha)-\partial_\beta^j h(\beta))(\alpha-\beta)\epsilon}{((\alpha-\beta)^2+\epsilon^2)^2} d\beta \|_{L^2[-\tau(t),\frac{\pi}{4}]} \lesssim \|h\|_{H^k[-\tau(t),\pi]}.$$

Moreover, from lemma 3.9.18, we have

$$\| \int_{2\alpha+\tau(t)}^{\pi} \frac{(\partial_\alpha^j h(\alpha)-\partial_\beta^j h(\beta))(\alpha-\beta)\epsilon}{((\alpha-\beta)^2+\epsilon^2)^2} d\beta \|_{L^2[-\tau(t),\frac{\pi}{4}]} \lesssim \|h\|_{H^k[-\tau(t),\pi]},$$

By lemma 3.9.26, we get

$$\| \int_{-\tau(t)}^{\pi} \frac{(\partial_\alpha^j h(\alpha) - \partial_\beta^j h(\beta))(\alpha-\beta)\epsilon}{((\alpha-\beta)^2+\epsilon^2)^2} d\beta \|_{L^2[-\tau(t),\frac{\pi}{4}]} \lesssim \|h\|_{H^k[-\tau(t),\pi]}.$$

By the three inequalities above, we get the result. □

3.10 The uniqueness of T^0

From corollary 3.7.15, let $h_0(\alpha,t) = h(\alpha,0,t)$, $g(\alpha,t) = \tilde{f}^{+(20)}(\alpha_0^t,t)$, we have

$$\begin{aligned} &\frac{d}{dt}h_0(\alpha,t) = T^0(h_0(\alpha,t)) = \\ &= T_1^0(h) + T_2^0(h) + \sum_i T_3^{0,i}(h) + \sum_i T_3^{0,i}(h) + \sum_i T_4^{0,i}(h) + FT^0. \end{aligned} \tag{3.159}$$

and

$$\begin{aligned} &\frac{d}{dt}g(\alpha,t) = T^0(g(\alpha,t)) = \\ &= T_1^0(g(\alpha,t)) + T_2^0(g(\alpha,t)) + \sum_i T_3^{0,i}(g(\alpha,t)) + \sum_i T_3^{0,i}(g(\alpha,t)) + \sum_i T_4^{0,i}(g(\alpha,t)) + FT^0. \end{aligned} \tag{3.160}$$

Here $T_5^{0,i}$ disappear because we let $\gamma = 0$. Corollary 3.7.15 give

$$\|T_4^{0,i}(h_0) - T_4^{0,i}(g)\|_{L^2_\alpha[-\pi,\pi]} \lesssim \|h_0(\alpha,t) - g(\alpha,t)\|_{L^2_\alpha[-\pi,\pi]}. \tag{3.161}$$

Here $\lesssim$ depends on $\|h_0\|_{H^5_\alpha[-\pi,\pi]}$ and $\|g\|_{H^5_\alpha[-\pi,\pi]}$. Let $L^1_{h^{0,s}} = L^1_{h^s}(\alpha,0,t)$, $L^2_{h^{0,s}} = L^2_{h^s}(\alpha,0,t)$ and $L^{3,i}_{h^{0,s}} = L^{3,i}_{h^s}(\alpha,0,t)$, from similar estimate as in lemma 3.7.2 by taking $\gamma = 0$, we have

$$\begin{aligned} &\|L^1_{h^{0,s}}(\alpha,t) - L^1_{g^s}(\alpha,t)\|_{C^1_\alpha[-\tau(t),\frac{\pi}{4}]} + \|L^2_{h^{0,s}}(\alpha,t) - L^2_{g^s}(\alpha,t)\|_{C^1_\alpha[-\tau(t),\frac{\pi}{4}]} \\ &+ \sum_i \|L^{3,i}_{h^{0,s}}(\alpha,\beta,t) - L^{3,i}_{g^s}(\alpha,\beta,t)\|_{C^1_\alpha([-\tau(t),\frac{\pi}{4}]\times[-\tau(t),\frac{2\pi}{3}])} \lesssim \|h_0(\alpha,t) - g(\alpha,t)\|_{L^2_\alpha[-\pi,\pi]}, \end{aligned} \tag{3.162}$$

$$\begin{aligned} &\|L^1_{h^{0,s}}(\alpha,t)\|_{C^1_\alpha[-\tau(t),\frac{\pi}{4}]} + \|L^2_{h^{0,s}}(\alpha,t)\|_{C^1_\alpha[-\tau(t),\frac{\pi}{4}]} \\ &+ \sum_i \|L^{3,i}_{h^{0,s}}(\alpha,\beta,t)\|_{C^1_\alpha([-\tau(t),\frac{\pi}{4}]\times[-\tau(t),\frac{2\pi}{3}])} \lesssim 1. \end{aligned} \tag{3.163}$$

and

$$L^1_{h^{0,s}}(\alpha,t)|_{\alpha=-\tau(t)},\ L^2_{h^{0,s}}(\alpha,t)|_{\alpha=-\tau(t)}=0. \tag{3.164}$$

Then

$$\begin{aligned}
&T_3^{0,i}(h_0)-T_3^{0,i}(g)\\
&=\tilde{\lambda}(\alpha_0^t)p.v.\int_{-\tau(t)}^{\pi}(L^{3,i}_{h^{0,s}}(\alpha,\beta,t)-L^{3,i}_{g^s}(\alpha,\beta,t))\frac{D_0^{-20+b_i}g(\beta,t)}{(\alpha-\beta)^2}(\alpha-\beta)d\beta\\
&+\tilde{\lambda}(\alpha_0^t)p.v.\int_{-\tau(t)}^{\pi}L^{3,i}_{h^{0,s}}(\alpha,\beta,t)\frac{D_0^{-20+b_i}h_0(\beta,t)-D_0^{-20+b_i}g(\beta,t)}{(\alpha-\beta)^2}(\alpha-\beta)d\beta\\
&=Term_1+Term_2.
\end{aligned}$$

Here D_0 is the same as (3.34). From (3.162), we have

$$\|Term_1\|_{L^2_\alpha[-\tau(t),\frac{\pi}{4}]}\lesssim\|h_0(\alpha,t)-g(\alpha,t)\|_{L^2_\alpha[-\pi,\pi]}.$$

From (3.163) and lemma 3.9.13, we have the same bound for $Term_2$. Therefore we have

$$\|T_3^{0,i}(h_0)-T_3^{0,i}(g)\|_{L^2_\alpha[-\tau(t),\frac{\pi}{4}]}\lesssim\|h_0(\alpha,t)-g(\alpha,t)\|_{L^2_\alpha[-\pi,\pi]}. \tag{3.165}$$

Now we control $\|h_0-g\|_{L^2_\alpha[-\pi,\pi]}$. By (3.159), (3.160), (3.165), and (3.161), we have

$$\begin{aligned}
&\frac{d}{dt}\int_{-\pi}^{\pi}|h_0-g|^2d\alpha=2\Re\int(h_0-g)\overline{T_1(h_0)-T_1(g)}d\alpha+2\Re\int(h_0-g)\overline{T_2(h_0)-T_2(g)}+B.T^0\\
&=DE_1+DE_2+B.T^0.
\end{aligned}$$

Here $B.T^0\lesssim B(\|h_0\|_{H^5[-\pi,\pi]}+\|g\|_{H^5[-\pi,\pi]})\|h_0-g\|_{L^2[-\pi,\pi]}$. Moreover, we have

$$\begin{aligned}
DE_1+DE_2&=2\Re\int_{-\pi}^{\pi}(h_0(\alpha)-g(\alpha))\overline{(L^{1,0}_{h^s}(\alpha)-L^{1,0}_{g^s}(\alpha))p.v.\int_{-\tau(t)}^{2\alpha+\tau(t)}\frac{g(\alpha)-g(\beta)}{(\alpha-\beta)^2}d\beta\tilde{\lambda}(\alpha_0^t)}d\alpha\\
&+2\Re\int_{-\pi}^{\pi}(h_0(\alpha)-g(\alpha))\overline{L^{1,0}_{h^s}(\alpha)p.v\int_{-\tau(t)}^{2\alpha+\tau(t)}\frac{(h_0(\alpha)-g(\alpha))-(h_0(\beta)-g(\beta))}{(\alpha-\beta)^2}d\beta\tilde{\lambda}(\alpha_0^t)}d\alpha\\
&+2\Re\int_{-\pi}^{\pi}(h_0(\alpha)-g(\alpha))\overline{(L^{2,0}_{h^s}(\alpha)-L^{2,0}_{g^s}(\alpha))\frac{\pi}{2}\partial_\alpha g(\alpha)\tilde{\lambda}(\alpha_0^t)}d\alpha\\
&+2\Re\int_{-\pi}^{\pi}(h_0(\alpha)-g(\alpha))\overline{L^{2,0}_{h^s}(\alpha)\frac{\pi}{2}\partial_\alpha(h_0(\alpha)-g(\alpha))\tilde{\lambda}(\alpha_0^t)}d\alpha
\end{aligned}$$

$$= DE_{1,1} + DE_{1,M} + DE_{2,1} + DE_{2,M}.$$

Since

$$\|p.v.\int_{-\tau(t)}^{2\alpha+\tau(t)} \frac{g(\alpha)-g(\beta)}{(\alpha-\beta)^2}d\beta\|_{L^\infty[-\tau(t),\frac{\pi}{4}]} = \|p.v.\int_{-\tau(t)}^{2\alpha+\tau(t)} \frac{g(\alpha)-g(\beta)-(\alpha-\beta)\partial_\alpha g(\alpha)}{(\alpha-\beta)^2}d\beta\|_{L^\infty[-\tau(t),\frac{\pi}{4}]}$$
$$\lesssim \|g\|_{H^5[-\pi,\pi]},$$

from (3.162), we have $|DE_{1,1}| \lesssim B.T^0$. We also have $|DE_{2,1}| \lesssim B.T^0$ from (3.162). Finally, from (3.163), (3.164), corollary 3.9.3 and the refined R-T condition (3.151)(with $\gamma = 0$), we get

$$DE_{1,M} + DE_{2,M} \lesssim B.T^0.$$

Then

$$\frac{d}{dt}\int_{-\pi}^{\pi} |h_0 - g|^2 d\alpha \lesssim B.T^0.$$

At $t = 0$, $h_0(\alpha, 0) = \tilde{f}^{+(20)}(\alpha, 0)$ by definition. Therefore we have

$$h_0(\alpha, t) = g(\alpha, t). \tag{3.166}$$

3.11 Analyticity

Let

$$A(h)(\alpha,\gamma,t) = (\frac{ic(\alpha+\tau(t))t}{1+ic'(\alpha+\tau(t))\gamma t}\frac{d}{d\alpha} - \frac{d}{d\gamma})h(\alpha,\gamma,t).$$

We have

$$\frac{dh}{dt} = T(h).$$

The idea is to show that $\frac{dA(h)}{dt}$ is almost $D_hT(A(h))$, with D_h the Gateaux derivative. Then by the fact that the main term T_1 and T_2 is quasilinear, we could still use the refined Gårding's inequality to show

$$\frac{d}{dt}\|A(h)\|_{L^2}^2 \lesssim \|A(h)\|_{L^2}^2,$$

and get the result.

Now we begin the proof. From (3.32), we have when $\alpha > -\tau(t)$,

$$\frac{dh_\mu(\alpha,\gamma,t)}{dt} = \tilde{\lambda}(\alpha_\gamma^t)M_1^c(h) - \tilde{\lambda}(\alpha_\gamma^t)M_2^c(h)$$
$$+ \tilde{\lambda}(\alpha_\gamma^t)\sum_i O^{1,i,c}(h) + \tilde{\lambda}(\alpha_\gamma^t)\sum_i O^{2,i,c}(h) + \tilde{\lambda}(\alpha_\gamma^t)\sum_i O^{3,i,c}(h) + \tilde{\lambda}(\alpha_\gamma^t)\sum_i O^{4,i,c}(h) + \tilde{\lambda}(\alpha_\gamma^t)\sum_i O^{5,i,c}(h)$$
$$+\tilde{\lambda}(\alpha_\gamma^t)T_{fixed}(\alpha_\gamma^t).$$

First, from (3.25), $\tilde{\lambda}(\alpha_\gamma^t)M_1^c(h)$ can be separated into two parts.

$$\begin{aligned} M_1^c(h) &= B^1_{D^{-20}(h)}(\alpha,\gamma,t)\frac{dh_\mu}{d\alpha} + B^2_{D^{-20}(h)}(\alpha,\gamma,t)\frac{dh_\mu}{d\alpha} \\ &= M_{1,1}^c(h) + M_{1,2}^c(h), \end{aligned} \tag{3.167}$$

with $B^1_{D^{-20}(h)}(\alpha,\gamma,t) = \frac{ic(\alpha+\tau(t))\gamma}{1+ic'(\alpha+\tau(t))\gamma t} + ic'(\alpha+\tau(t))\gamma t\tau'(t)$ and

$$B^2_{D^{-20}(h)}(\alpha,\gamma,t) = \frac{1}{1+ic'(\alpha+\tau(t))\gamma t}\Big(\frac{\partial_t x}{\partial_\alpha x}(\alpha_\gamma^t) - \frac{\partial_t x}{\partial_\alpha x}(0)$$
$$+ \frac{1}{(\partial_\alpha x)(\alpha_\gamma^t)}\cdot p.v.\int_{-\pi}^{\pi} K(D^{-20}(h)(\alpha,\gamma,t) - D^{-20}(h)(\beta,\gamma,t) + \tilde{f}^L(\alpha_\gamma^t) - \tilde{f}^L(\beta_\gamma^t))$$
$$\frac{dx(\beta_\gamma^t)}{d\beta}d\beta - \frac{1}{(\partial_\alpha x)(0)}p.v\int_{-\pi}^{\pi} K(D^{-20}(h)(-\tau(t),\gamma,t) - D^{-20}(h)(\beta,\gamma,t) + \tilde{f}^L(0) - \tilde{f}^L(\beta_\gamma^t))\frac{dx(\beta_\gamma^t)}{d\beta}d\beta).$$

By the definition of $c(\alpha+\tau(t))$, it is supported in where $\tilde{\lambda}(\alpha+\tau(t)) = 1$, therefore

$$\begin{aligned} \tilde{\lambda}(\alpha_\gamma^t)M_1^c(h) &= \Big(\frac{ic(\alpha+\tau(t))\gamma}{1+ic'(\alpha+\tau(t))\gamma t} + ic'(\alpha+\tau(t))\gamma t\tau'(t)\Big)\frac{dh_\mu}{d\alpha} + \tilde{\lambda}(\alpha_\gamma^t)B^2_{D^{-20}(h)}(\alpha,\gamma,t)\frac{dh_\mu}{d\alpha} \\ &= \Big(\frac{ic(\alpha+\tau(t))\gamma}{1+ic'(\alpha+\tau(t))\gamma t} + ic'(\alpha+\tau(t)\gamma t)\gamma\tau'(t)\Big)\frac{dh_\mu}{d\alpha} + \tilde{\lambda}(\alpha_\gamma^t)M_{1,2}^c(h). \end{aligned}$$

The reason we do this separation is that the first term comes from the derivative when it hits on $\alpha+\tau(t)+ic(\alpha+\tau(t))\gamma t$, which makes it play a different role from the other terms.

Therefore, from lemma 3.11.6, and the trivial term $A(\tilde{\lambda}(\alpha_\gamma^t)) = 0$, we have when $\alpha > -\tau(t)$,

$$\frac{dA(h_\mu)(\alpha,\gamma,t)}{dt} = \Big(\frac{ic(\alpha+\tau(t))\gamma}{1+ic'(\alpha+\tau(t))\gamma t} + ic'(\alpha+\tau(t))\gamma t\tau'(t)\Big)\frac{dA(h_\mu)}{d\alpha} + \tilde{\lambda}(\alpha_\gamma^t)A(M_{1,2}^c(h)) - \tilde{\lambda}(\alpha_\gamma^t)A(M_2^c(h))$$
$$+ \tilde{\lambda}(\alpha_\gamma^t)\sum_i A(O^{1,i,c}(h)) + \tilde{\lambda}(\alpha_\gamma^t)\sum_i A(O^{2,i,c}(h)) + \tilde{\lambda}(\alpha_\gamma^t)\sum_i A(O^{3,i,c}(h)) + \tilde{\lambda}(\alpha_\gamma^t)\sum_i A(O^{4,i,c}(h))$$
$$+ \tilde{\lambda}(\alpha_\gamma^t)\sum_i A(O^{5,i,c}(h)) + \tilde{\lambda}(\alpha_\gamma^t)A(T_{fixed}(\alpha_\gamma^t)).$$

We then calculate $A(M^c_{1,2}(h))$. First, we further separate $B^2_{D^{-20}}(h)$ and have

$$B^2_{D^{-20}(h)}(1+ic'(\alpha+\tau(t))\gamma t) = B^{2,1}_{D^{-20}(h)}(1+ic'(\alpha+\tau(t))\gamma t) + B^{2,2}_{D^{-20}(h)}(1+ic'(\alpha+\tau(t))\gamma t) \tag{3.168}$$

$$\begin{aligned}
&+ B^{2,3}_{D^{-20}(h)}(1+ic'(\alpha+\tau(t))\gamma t) + B^{2,4}_{D^{-20}(h)}(1+ic'(\alpha+\tau(t))\gamma t)\\
&= \frac{\partial_t x}{\partial_\alpha x}(\alpha^t_\gamma) - \frac{\partial_t x}{\partial_\alpha x}(0)\\
&+ \int_{-\pi}^{\pi} K(D^{-20}(h)(\alpha,\gamma,t) - D^{-20}(h)(\beta,\gamma,t) + \tilde{f}^L(\alpha^t_\gamma) - \tilde{f}^L(\beta^t_\gamma))(\frac{1}{(\partial_\alpha x)(\alpha^t_\gamma)} - \frac{1}{(\partial_\beta x)(\beta^t_\gamma)})\frac{dx(\beta^t_\gamma)}{d\beta}d\beta\\
&+ p.v.\int_{-\pi}^{\pi} K(D^{-20}(h)(\alpha,\gamma,t) - D^{-20}(h)(\beta,\gamma,t) + \tilde{f}^L(\alpha^t_\gamma) - \tilde{f}^L(\beta^t_\gamma))(1+ic'(\beta+\tau(t))\gamma t)d\beta\\
&- \frac{1}{(\partial_\alpha x)(0)}\int_{-\pi}^{\pi} K(D^{-20}(h)(-\tau(t),\gamma,t) - D^{-20}(h)(\beta,\gamma,t) + \tilde{f}^L(0) - \tilde{f}^L(\beta^t_\gamma))\frac{dx(\beta^t_\gamma)}{d\beta}d\beta.
\end{aligned}$$

Here we remove the "p.v." in the last term by corollary 3.7.24. Since $\tilde{f}^L(\alpha,t)$, $\partial_t x(\alpha,t)$, and $x(\alpha,t)$ are analytic in $\tilde{D} = \{\alpha+iy||\alpha| \le \frac{\delta}{4}\}$, we get

$$A(B^{2,1}_{D^{-20}(h)}(\alpha,\gamma,t)(1+ic'(\alpha+\tau(t))\gamma t)) = 0.$$

Moreover, by letting

$$\ddot{h}(\alpha+\tau(t)) = (D^{-20}(h)(\alpha,\gamma,t), \tilde{f}^L(\alpha^t_\gamma), \frac{1}{(\partial_\alpha x)(\alpha^t_\gamma)}),$$

$$\breve{K}(\ddot{h}(\alpha+\tau(t))-\ddot{h}(\beta+\tau(t))) = K(D^{-20}(h)(\alpha,\gamma,t)-D^{-20}(h)(\beta,\gamma,t)+\tilde{f}^L(\alpha^t_\gamma)-\tilde{f}^L(\beta^t_\gamma))(\frac{1}{(\partial_\alpha x)(\alpha^t_\gamma)}-\frac{1}{(\partial_\beta x)(\beta^t_\gamma)}),$$

and

$$X(\beta+\tau(t),\gamma) = (\partial_\beta x)(\beta^t_\gamma),$$

from lemma 3.11.7, and corollary 3.11.18, we have ,

$$A(B^{2,2}_{D^{-20}(h)}(\alpha,\gamma,t)(1+ic'(\alpha+\tau(t))\gamma t)) = (D_h(B^{2,2}_{D^{-20}(h)}(\alpha,\gamma,t)(1+ic'(\alpha+\tau(t))\gamma t))[A(h)].$$

Similarly, from corollary 3.11.16 and corollary 3.11.18, we have

$$A(B^{2,4}_{D^{-20}(h)}(\alpha,\gamma,t)(1+ic'(\alpha+\tau(t))\gamma t)) = (D_h(B^{2,4}_{D^{-20}(h)}(\alpha,\gamma,t)(1+ic'(\alpha+\tau(t))\gamma t))[A(h)].$$

Therefore, by using lemma 3.11.5, we get

$$A(M_{1,2}^c(h)) = A(B^2_{D^{-20}(h)}(\alpha,\gamma,t)(1+ic'(\alpha+\tau(t))\gamma t)\frac{\frac{d}{d\alpha}(h_\mu)(\alpha,\gamma,t)}{1+ic'(\alpha+\tau(t))\gamma t}) \tag{3.169}$$

$$= \sum_{j=1}^{4} A(B^{2,j}_{D^{-20}(h)}(\alpha,\gamma,t)(1+ic'(\alpha+\tau(t))\gamma t)\frac{\frac{d}{d\alpha}h_\mu(\alpha,\gamma,t)}{1+ic'(\alpha+\tau(t))\gamma t})$$

$$= \sum_{j=1,2,4} (D_h(B^{2,j}_{D^{-20}(h)}(\alpha,\gamma,t)(1+ic'(\alpha+\tau(t))\gamma t))[A(h)]\frac{\frac{d}{d\alpha}h_\mu(\alpha,\gamma,t)}{1+ic'(\alpha+\tau(t))\gamma t})$$

$$+ \sum_{j=1,2,4} (B^{2,j}_{D^{-20}(h)}(\alpha,\gamma,t)(1+ic'(\alpha+\tau(t))\gamma t)\frac{\frac{d}{d\alpha}A(h_\mu)(\alpha,\gamma,t)}{1+ic'(\alpha+\tau(t))\gamma t})$$

$$+ A(p.v.\int_{-\pi}^{\pi} K(D^{-20}(h)(\alpha,\gamma,t) - D^{-20}(h)(\beta,\gamma,t) + \tilde{f}^L(\alpha^t_\gamma) - \tilde{f}^L(\beta^t_\gamma)(1+ic'(\beta+\tau(t))\gamma t)d\beta$$

$$(\frac{\frac{d}{d\alpha}h_\mu(\alpha,\gamma,t)}{1+ic'(\alpha+\tau(t))\gamma t}))$$

$$= A_{1,2,1} + A_{1,2,2} + A_{1,2,3}.$$

The last term will be calculated together with M_2^c to get rid of the principal value.

For the next term $M_2^c(h)$, we again rewrite this term. From (3.26), by integration by parts, we have

$$M_2^c(h) = p.v.\int_{-\tau(t)}^{2\alpha+\tau(t)} (K(D^{-20}(h)(\alpha,\gamma,t) - D^{-20}(h)(\beta,\gamma,t) + \tilde{f}^L(\alpha^t_\gamma) - \tilde{f}^L(\beta^t_\gamma))\frac{d}{d\beta}(h_\mu(\beta,\gamma,t))d\beta$$

$$- K(D^{-20}(h)(\alpha,\gamma,t) - D^{-20}(h)(2\alpha+\tau(t),\gamma,t) + \tilde{f}^L(\alpha^t_\gamma) - \tilde{f}^L((2\alpha+\tau(t))^t_\gamma))h_\mu(2\alpha+\tau(t),\gamma,t)$$

$$+ \int_0^\gamma \frac{\frac{d}{d\beta}(K(D^{-20}(h)(\alpha,\gamma,t) - D^{-20}(h)(\beta,\eta,t) + \tilde{f}^L(\alpha^t_\gamma) - \tilde{f}^L(\beta^t_\eta))|_{\beta=2\alpha+\tau(t)}}{1+ic'(2\alpha+2\tau(t))\eta t}h_\mu((2\alpha+\tau(t)),\eta,t)$$

$$ic(2\alpha+2\tau(t))td\eta$$

$$- \int_{2\alpha+\tau(t)}^{\pi} \frac{d}{d\beta}(K(D^{-20}(h)(\alpha,\gamma,t) - \tilde{f}^+(\beta^t_0)) + \tilde{f}^L(\alpha^t_\gamma) - \tilde{f}^L(\beta^t_0))\tilde{f}^+_\mu(\beta^t_0)d\beta.$$

Then by (3.166), we can replace the $\tilde{f}^{+(20)}(\beta^t_0)$ by $h(\beta,0,t)$, and get

$$M_2^c(h) = p.v.\int_{-\tau(t)}^{2\alpha+\tau(t)} (K(D^{-20}(h)(\alpha,\gamma,t) - D^{-20}(h)(\beta,\gamma,t) + \tilde{f}^L(\alpha^t_\gamma) - \tilde{f}^L(\beta^t_\gamma))\frac{d}{d\beta}(h_\mu(\beta,\gamma,t))d\beta$$

$$- K(D^{-20}(h)(\alpha,\gamma,t) - D^{-20}(h)(2\alpha+\tau(t),\gamma,t) + \tilde{f}^L(\alpha^t_\gamma) - \tilde{f}^L((2\alpha+\tau(t))^t_\gamma))h_\mu(2\alpha+\tau(t),\gamma,t)$$

$$- \int_0^\gamma \nabla K(D^{-20}(h)(\alpha,\gamma,t) - D^{-20}(h)(2\alpha+\tau(t),\eta,t) + \tilde{f}^L(\alpha^t_\gamma) - \tilde{f}^L((2\alpha+\tau(t))^t_\eta))$$

$$\cdot (D^{-19}(h)(2\alpha+\tau(t),\eta,t) + (\tilde{f}^L)'((2\alpha+\tau(t))^t_\eta))h_\mu((2\alpha+\tau(t)),\eta,t)ic(2\alpha+2\tau(t))td\eta$$

$$+ \int_{2\alpha+\tau(t)}^{\pi} \nabla K(D^{-20}(h)(\alpha,\gamma,t) - D^{-20}(h)(\beta,0,t) + \tilde{f}^L(\alpha^t_\gamma) - \tilde{f}^L(\beta^t_0))$$

$$\cdot (D^{-19}(h)(\beta,0,t)+(\tilde{f}^L)'(\beta_0^t))h_\mu(\beta,0,t)d\beta.$$

Then from (3.169), we have

$$\begin{aligned}
&A_{1,2,3}-A(M_2^c(h))=A(\int_{-\tau(t)}^{2\alpha+\tau(t)}(K(D^{-20}(h)(\alpha,\gamma,t)-D^{-20}(h)(\beta,\gamma,t)+\tilde{f}^L(\alpha_\gamma^t)-\tilde{f}^L(\beta_\gamma^t))\\
&(\frac{\frac{d}{d\alpha}(h_\mu)(\alpha,\gamma,t)}{1+ic'(\alpha+\tau(t))\gamma t}-\frac{\frac{d}{d\beta}(h_\mu)(\beta,\gamma,t)}{1+ic'(\beta+\tau(t))\gamma t})(1+ic'(\beta+\tau(t))\gamma t)d\beta\\
&+K(D^{-20}(h)(\alpha,\gamma,t)-D^{-20}(h)(2\alpha+\tau(t),\gamma,t)+\tilde{f}^L(\alpha_\gamma^t)-\tilde{f}^L((2\alpha+\tau(t))_\gamma^t))h_\mu(2\alpha+\tau(t)),\gamma,t)\\
&+\int_0^\gamma\nabla K(D^{-20}(h)(\alpha,\gamma,t)-D^{-20}(h)(2\alpha+\tau(t),\eta,t)+\tilde{f}^L(\alpha_\gamma^t)-\tilde{f}^L((2\alpha+\tau(t))_\eta^t))\\
&\cdot(D^{-19}(h)(2\alpha+\tau(t),\eta,t)+(\tilde{f}^L)'((2\alpha+\tau(t))_\eta^t))h_\mu((2\alpha+\tau(t)),\eta,t)ic(2\alpha+2\tau(t))td\eta\\
&-\int_{2\alpha+\tau(t)}^{\pi}\nabla K(D^{-20}(h)(\alpha,\gamma,t)-D^{-20}(h)(\beta,0,t)+\tilde{f}^L(\alpha_\gamma^t)-\tilde{f}^L(\beta_0^t))\\
&\cdot(D^{-19}(h)(\beta,0,t)+(\tilde{f}^L)'(\beta_0^t))h_\mu(\beta,0,t)d\beta\\
&+\int_{[-\pi,-\tau(t)]\cup[2\alpha+\tau(t),\pi]}K(D^{-20}(h)(\alpha,\gamma,t)-D^{-20}(h)(\beta,\gamma,t)+\tilde{f}^L(\alpha_\gamma^t)-\tilde{f}^L(\beta_\gamma^t))(1+ic'(\beta+\tau(t))\gamma t)d\beta\\
&\cdot\frac{\frac{d}{d\alpha}(h_\mu)(\alpha,\gamma,t)}{1+ic'(\alpha+\tau(t))\gamma t}).
\end{aligned}$$

From corollary 3.11.13 and 3.11.18, we have

$$\begin{aligned}
&A_{1,2,3}-A(M_2^c(h))=\int_{-\tau(t)}^{2\alpha+\tau(t)}(K(D^{-20}(h)(\alpha,\gamma,t)-D^{-20}(h)(\beta,\gamma,t)+\tilde{f}^L(\alpha_\gamma^t)-\tilde{f}^L(\beta_\gamma^t))\\
&\underbrace{(\underbrace{\frac{\frac{d}{d\alpha}A(h_\mu)(\alpha,\gamma,t)}{1+ic'(\alpha+\tau(t))\gamma t}}_{A_{2,0}}\underbrace{-\frac{\frac{d}{d\beta}A(h_\mu)(\beta,\gamma,t)}{1+ic'(\beta+\tau(t))\gamma t}(1+ic'(\beta+\tau(t))\gamma t)}_{A_{2,1}}\,d\beta}\\
&+\int_{-\tau(t)}^{2\alpha+\tau(t)}(D_hK(D^{-20}(h)(\alpha,\gamma,t)-D^{-20}(h)(\beta,\gamma,t)+\tilde{f}^L(\alpha_\gamma^t)-\tilde{f}^L(\beta_\gamma^t))[A(h)])\\
&\underbrace{(\underbrace{\frac{\frac{d}{d\alpha}h_\mu(\alpha,\gamma,t)}{1+ic'(\alpha+\tau(t))\gamma t}}_{A_{2,2}}\underbrace{-\frac{\frac{d}{d\beta}h_\mu(\beta,\gamma,t)}{1+ic'(\beta+\tau(t))\gamma t}(1+ic'(\beta+\tau(t))\gamma t)d\beta}_{A_{2,3}}}\\
&\underbrace{+K(D^{-20}(h)(\alpha,\gamma,t)-D^{-20}(h)(2\alpha+\tau(t),\gamma,t)+\tilde{f}^L(\alpha_\gamma^t)-\tilde{f}^L((2\alpha+\tau(t))_\gamma^t))(A(h))_\mu(2\alpha+\tau(t),\gamma,t)}_{A_{2,4}}\\
&\underbrace{+D_hK(D^{-20}(h)(\alpha,\gamma,t)-D^{-20}(h)(2\alpha+\tau(t),\gamma,t)+\tilde{f}^L(\alpha_\gamma^t)-\tilde{f}^L((2\alpha+\tau(t))_\gamma^t))[A(h)](h)_\mu(2\alpha+\tau(t),\gamma,t)}_{A_{2,5}}
\end{aligned}\tag{3.170}$$

$$\underbrace{+\int_0^{\gamma}\nabla K(D^{-20}(h)(\alpha,\gamma,t)-D^{-20}(h)(2\alpha+\tau(t),\eta,t)+\tilde{f}^L(\alpha_\gamma^t)-\tilde{f}^L((2\alpha+\tau(t))_\eta^t))}_{A_{2,6}}$$

$$\underbrace{\cdot(D^{-19}(h)(2\alpha+\tau(t),\eta,t)+(\tilde{f}^L)'((2\alpha+\tau(t))_\eta^t))\frac{2ic(\alpha+\tau(t))t(1+ic'(2\alpha+2\tau(t))\eta t)}{(1+ic'(\alpha+\tau(t))\gamma t)(ic(2\alpha+2\tau(t))t)}A(h_\mu)((2\alpha+\tau(t)),\eta,t)}_{A_{2,6}}$$

$$\underbrace{ic(2\alpha+2\tau(t))td\eta}_{A_{2,6}}$$

$$+\underbrace{\int_0^{\gamma}\nabla K(D^{-20}(h)(\alpha,\gamma,t)-D^{-20}(h)(2\alpha+\tau(t),\eta,t)+\tilde{f}^L(\alpha_\gamma^t)-\tilde{f}^L((2\alpha+\tau(t))_\eta^t))}_{A_{2,7}}$$

$$\underbrace{\cdot D^{-19}(A(h))(2\alpha+\tau(t),\eta,t)h_\mu((2\alpha+\tau(t)),\eta,t)\frac{2ic(\alpha+\tau(t))t(1+ic'(2\alpha+2\tau(t))\eta t)}{(1+ic'(\alpha+\tau(t))\gamma t)(ic(2\alpha+2\tau(t))t)}ic(2\alpha+2\tau(t))td\eta)}_{A_{2,7}}$$

$$+\underbrace{\int_0^{\gamma}\nabla^2 K(D^{-20}(h)(\alpha,\gamma,t)-D^{-20}(h)(2\alpha+\tau(t),\eta,t)+\tilde{f}^L(\alpha_\gamma^t)-\tilde{f}^L((2\alpha+\tau(t))_\eta^t))}_{A_{2,8}}$$

$$\underbrace{[(D^{-20}(A(h))(\alpha,\gamma,t)-\frac{2ic(\alpha+\tau(t))t(1+ic'(2\alpha+2\tau(t))\eta t)}{(1+ic'(\alpha+\tau(t))\gamma t)(ic(2\alpha+2\tau(t))t)}D^{-20}(A(h))(2\alpha+\tau(t),\eta,t)),}_{A_{2,8}}$$

$$\underbrace{(D^{-19}(h)(2\alpha+\tau(t),\eta,t)+(\tilde{f}^L)'((2\alpha+\tau(t))_\eta^t))]h_\mu((2\alpha+\tau(t)),\eta,t)ic(2\alpha+2\tau(t))td\eta,}_{A_{2,8}}$$

$$\underbrace{-\int_{2\alpha+\tau(t)}^{\pi}\nabla^2 K(D^{-20}(h)(\alpha,\gamma,t)-D^{-20}(h)(\beta,0,t)+\tilde{f}^L(\alpha_\gamma^t)-\tilde{f}^L(\beta_0^t))[D^{-20}(A(h))(\alpha,\gamma,t),}_{A_{2,9}}$$

$$\underbrace{(D^{-19}(h)(\beta,0,t)+(\tilde{f}^L)'(\beta_0^t))]h_\mu(\beta,0,t)d\beta}_{A_{2,9}}$$

$$\underbrace{+\int_{[-\pi,-\tau(t)]\cup[2\alpha+\tau(t),\pi]}D_h(K(D^{-20}(h)(\alpha,\gamma,t)-D^{-20}(h)(\beta,\gamma,t)+\tilde{f}^L(\alpha_\gamma^t)-\tilde{f}^L(\beta_\gamma^t)))[A(h)]}_{A_{2,10}}$$

$$\underbrace{\cdot(1+ic'(\beta+\tau(t))\gamma t)d\beta\frac{\frac{d}{d\alpha}(h_\mu)(\alpha,\gamma,t)}{1+ic'(\alpha+\tau(t))\gamma t}}_{A_{2,10}}$$

$$\underbrace{+\int_{[-\pi,-\tau(t)]\cup[2\alpha+\tau(t),\pi]}K(D^{-20}(h)(\alpha,\gamma,t)-D^{-20}(h)(\beta,\gamma,t)+\tilde{f}^L(\alpha_\gamma^t)-\tilde{f}^L(\beta_\gamma^t))(1+ic'(\beta+\tau(t))\gamma t)d\beta}_{A_{2,11}}$$

$$\underbrace{\cdot\frac{\frac{d}{d\alpha}A(h_\mu)(\alpha,\gamma,t)}{1+ic'(\alpha+\tau(t))\gamma t}}_{A_{2,11}}$$

$$=A_{2,0}+A_{2,1}+A_{2,2}+A_{2,3}+A_{2,4}+A_{2,5}+A_{2,6}+A_{2,7}+A_{2,8}+A_{2,9}+A_{2,10}+A_{2,11}.$$

Here some terms disappear because $A(\tilde{f}^L(\alpha_\gamma^t)) = 0$. We have

$$\begin{aligned}
&A_{2,0} + A_{2,2} + A_{2,10} + A_{2,11} \\
&= D_h(B^{2,3}_{D^{-20}(h)}(1 + ic'(\alpha + \tau(t))\gamma t))[A(h)]\frac{\frac{d}{d\alpha}h_\mu(\alpha,\gamma,t)}{1 + ic'(\alpha + \tau(t))\gamma t} \\
&+ B^{2,3}_{D^{-20}(h)}(1 + ic'(\alpha + \tau(t))\gamma t)\frac{\frac{d}{d\alpha}A(h_\mu)(\alpha,\gamma,t)}{1 + ic'(\alpha + \tau(t))\gamma t}.
\end{aligned}$$

Therefore, from (3.169), we get

$$\begin{aligned}
&A_{1,2,1} + A_{1,2,2} + A_{2,0} + A_{2,2} + A_{2,10} + A_{2,11} \qquad (3.171)\\
&= \sum_{j=1}^{4}(D_h(B^{2,j}_{D^{-20}(h)}(\alpha,\gamma,t))(1 + ic'(\alpha + \tau(t))\gamma t))[A(h)]\frac{\frac{d}{d\alpha}h_\mu(\alpha,\gamma,t)}{1 + ic'(\alpha + \tau(t))\gamma t}) \\
&+ \sum_{j=1}^{4}(B^{2,j}_{D^{-20}(h)}(\alpha,\gamma,t))(1 + ic'(\alpha + \tau(t))\gamma t)\frac{\frac{d}{d\alpha}A(h_\mu)(\alpha,\gamma,t)}{1 + ic'(\alpha + \tau(t))\gamma t}) \\
&= D_h(B^{2}_{D^{-20}(h)}(1 + ic'(\alpha + \tau(t))\gamma t))[A(h)]\frac{\frac{d}{d\alpha}h_\mu(\alpha,\gamma,t)}{1 + ic'(\alpha + \tau(t))\gamma t} \\
&+ B^{2}_{D^{-20}(h)}(1 + ic'(\alpha + \tau(t))\gamma t)\frac{\frac{d}{d\alpha}A(h_\mu)(\alpha,\gamma,t)}{1 + ic'(\alpha + \tau(t))\gamma t} \\
&= Term_1 + Term_2.
\end{aligned}$$

From lemma 3.7.4, we have $B_{D^{-20}(h)}(\alpha,\gamma,t) = \frac{\pi}{2}L^2_{h^s}(\alpha,\gamma,t)$. Since $B^1_{D^{-20}(h)}(\alpha,\gamma,t)$ in (3.167) does not depend on h, from (3.49), the following inequality holds:

$$\|D_h(B^2_{D^{-20}(h)}(\alpha,\gamma,t)(1 + ic'(\alpha + \tau(t))\gamma t))[A(h)]\|_{C^0_\gamma([-1,1],L^2_\alpha[-\tau(t),\frac{\pi}{4}])} \lesssim \|A(h)\|_{C^0_\gamma([-1,1],L^2_\alpha[-\tau(t),\pi])}.$$

Moreover, we have $h \in C^1([0,t], C^1([-1,1], H^5[-\pi,\pi]))$. Hence $Term_1$ in (3.169) is bounded and from (3.170), (3.169), (3.171), we have

$$A(M^c_{1,2}(h) - M^c_2(h)) = B^2_{D^{-20}(h)}(\alpha,\gamma,t)\frac{dA(h_\mu)}{d\alpha} + \sum_{j=1,3\le j\le 9} A_{2,j} + B.T^0,$$

with

$$\|B.T^0\|_{C^0_\gamma([-1,1],L^2_\alpha[-\tau(t),\frac{\pi}{4}])} \lesssim \|A(h)\|_{C^0_\gamma([-1,1],L^2_\alpha[-\tau(t),\pi])}.$$

In the next lemma 3.11.1, we show that $C^0_\gamma([-1,1], L^2_\alpha[-\tau(t),\frac{\pi}{4}])$ norms of all terms from $A_{2,3}$ to $A_{2,9}$ are bounded by $\|A(h)\|_{C^0_\gamma([-1,1],L^2_\alpha[-\tau(t),\pi])}$.

Lemma 3.11.1. *All $C^0_\gamma([-1,1], L^2_\alpha[-\tau(t), \frac{\pi}{4}])$ norms of $A_{2,3}$ to $A_{2,9}$ are bounded by $\|A(h)\|_{C^0_\gamma([-1,1],L^2_\alpha[-\tau(t),\pi])}$.*

Proof. For $A_{2,3}$, we have $K(D^{-20}(h)(\alpha,\gamma,t) - D^{-20}(h)(\beta,\gamma,t) + \tilde{f}^L(\alpha^t_\gamma) - \tilde{f}^L(\beta^t_\gamma))$ is a kernel of type $\sigma = -1$. Therefore from lemma 3.7.21, we have the bound.

For $A_{2,4}$ and $A_{2,5}$, we have $K(D^{-20}(h)(\alpha,\gamma,t) - D^{-20}(h)(2\alpha+\tau(t),\gamma,t) + \tilde{f}^L(\alpha^t_\gamma) - \tilde{f}^L((2\alpha+\tau(t))^t_\gamma))$ is a term of sub-type 2 of $L^{4,i}_{h^s}$ in lemma 3.7.7 , then we can also control these two terms from (3.56) and (3.58).

For $A_{2,6}$, and $A_{2,7}$, we have the component of $\nabla K(D^{-20}(h)(\alpha,\gamma,t) - D^{-20}(h)(2\alpha+\tau(t),\eta,t) + \tilde{f}^L(\alpha^t_\gamma) - \tilde{f}^L((2\alpha+\tau(t))^t_\eta))$ has the form K^j_{-2} in lemma 3.7.22. Moreover,

$$|\frac{ic(\alpha+\tau(t))t}{(\alpha+\tau(t))^2}| \lesssim 1.$$

Then from lemma 3.7.22, we have the result. For $A_{2,8}$, we have the component of $\nabla^2 K(D^{-20}(h)(\alpha,\gamma,t) - D^{-20}(h)(2\alpha+\tau(t),\eta,t) + \tilde{f}^L(\alpha^t_\gamma) - \tilde{f}^L((2\alpha+\tau(t))^t_\eta))$ has the form K^j_{-3} in lemma 3.7.22. Moreover,

$$|\frac{ic(\alpha+\tau(t))t}{(\alpha+\tau(t))^2}| \lesssim 1,$$

$$\|\frac{D^{-20}(A(h))(\alpha,\gamma,t)}{\alpha+\tau(t)}\|_{C^0_\gamma([-1,1],L^2_\alpha(-\tau(t),\frac{\pi}{4}))} \lesssim \|A(h)\|_{C^0_\gamma([-1,1],L^2_\alpha(-\tau(t),\pi))},$$

and

$$\|\frac{D^{-20}(A(h))(2\alpha+\tau(t),\eta,t)}{\alpha+\tau(t)}\|_{C^0_\gamma([-1,1],L^2_\alpha(-\tau(t),\frac{\pi}{4}))} \lesssim \|A(h)\|_{C^0_\gamma([-1,1],L^2_\alpha(-\tau(t),\pi))}.$$

Then from lemma 3.7.22, we have the bound.

For $A_{2,9}$, we still use the component of $\nabla^2 K(D^{-20}(h)(\alpha,\gamma,t) - D^{-20}(h)(\beta,0,t) + \tilde{f}^L(\alpha^t_\gamma) - \tilde{f}^L(\beta^t_0))$ has the form K^j_{-3} in lemma 3.7.27. Moreover,

$$\|\frac{h(\beta,0,t)}{(\beta+\tau(t))^3}\|_{L^\infty_\beta[-\tau(t),\pi]} \lesssim 1.$$

Then from lemma 3.7.27, we can bound this term. □

For the term $A_{2,1}$, we again do the integration by parts and have

$$-A_{2,1} = p.v.\int_{-\tau(t)}^{2\alpha+\tau(t)} (K(D^{-20}(h)(\alpha,\gamma,t) - D^{-20}(h)(\beta,\gamma,t) + \tilde{f}^L(\alpha^t_\gamma) - \tilde{f}^L(\beta^t_\gamma))\frac{d}{d\beta}(A(h_\mu)(\beta,\gamma,t))d\beta$$
$$= K(D^{-20}(h)(\alpha,\gamma,t) - D^{-20}(h)(2\alpha+\tau(t),\gamma,t)) + \tilde{f}^L(\alpha^t_\gamma) - \tilde{f}^L((2\alpha+\tau(t))^t_\gamma))$$

$$
\begin{aligned}
&(A(h_\mu)(2\alpha+\tau(t),\gamma,t)-A(h_\mu)(\alpha,\gamma,t))\\
&-K(D^{-20}(h)(\alpha,\gamma,t)-D^{-20}(h)(-\tau(t),\gamma,t))+\tilde{f}^L(\alpha^t_\gamma)-\tilde{f}^L((2\alpha+\tau(t))^t_\gamma))(A(h_\mu)(-\tau(t),\gamma,t)-A(h_\mu)(\alpha,\gamma,t))\\
&-p.v.\int_{-\tau(t)}^{2\alpha+\tau(t)}\frac{d}{d\beta}(K(D^{-20}(h)(\alpha,\gamma,t)-D^{-20}(h)(\beta,\gamma,t)+\tilde{f}^L(\alpha^t_\gamma)-\tilde{f}^L(\beta^t_\gamma))(A(h_\mu)(\beta,\gamma,t)-A(h_\mu)(\alpha,\gamma,t))d\beta\\
&=A_{2,1,1}+A_{2,1,2}\\
&-\lim_{\beta\to\alpha}(\frac{d}{d\beta}(K(D^{-20}(h)(\alpha,\gamma,t)-D^{-20}(h)(\beta,\gamma,t)+\tilde{f}^L(\alpha^t_\gamma)-\tilde{f}^L(\beta^t_\gamma))(\alpha-\beta)^2)\\
&p.v.\int_{-\tau(t)}^{2\alpha+\tau(t)}\frac{(A(h_\mu)(\beta,\gamma,t)-A(h_\mu)(\alpha,\gamma,t))}{(\alpha-\beta)^2}d\beta\\
&-\int_{-\tau(t)}^{2\alpha+\tau(t)}[\lim_{\beta\to\alpha}(\frac{d}{d\beta}(K(D^{-20}(h)(\alpha,\gamma,t)-D^{-20}(h)(\beta,\gamma,t)+\tilde{f}^L(\alpha^t_\gamma)-\tilde{f}^L(\beta^t_\gamma)))(\alpha-\beta)^2)\\
&-\frac{d}{d\beta}(K(D^{-20}(h)(\alpha,\gamma,t)-D^{-20}(h)(\beta,\gamma,t)+\tilde{f}^L(\alpha^t_\gamma)-\tilde{f}^L(\beta^t_\gamma)))(\alpha-\beta)^2]\frac{(A(h_\mu)(\beta,\gamma,t)-A(h_\mu)(\alpha,\gamma,t))}{(\alpha-\beta)^2}d\beta\\
&=A_{2,1,1}+A_{2,1,2}+A_{2,1,M}\\
&-p.v.\int_{-\tau(t)}^{2\alpha+\tau(t)}[\lim_{\beta\to\alpha}(\frac{d}{d\beta}(K(D^{-20}(h)(\alpha,\gamma,t)-D^{-20}(h)(\beta,\gamma,t)+\tilde{f}^L(\alpha^t_\gamma)-\tilde{f}^L(\beta^t_\gamma)))(\alpha-\beta)^2)\\
&-\frac{d}{d\beta}(K(D^{-20}(h)(\alpha,\gamma,t)-D^{-20}(h)(\beta,\gamma,t)+\tilde{f}^L(\alpha^t_\gamma)-\tilde{f}^L(\beta^t_\gamma)))(\alpha-\beta)^2]\frac{(A(h_\mu)(\beta,\gamma,t))}{(\alpha-\beta)^2}d\beta\\
&+p.v.\int_{-\tau(t)}^{2\alpha+\tau(t)}[\lim_{\beta\to\alpha}(\frac{d}{d\beta}(K(D^{-20}(h)(\alpha,\gamma,t)-D^{-20}(h)(\beta,\gamma,t)+\tilde{f}^L(\alpha^t_\gamma)-\tilde{f}^L(\beta^t_\gamma)))(\alpha-\beta)^2)\\
&-\frac{d}{d\beta}(K(D^{-20}(h)(\alpha,\gamma,t)-D^{-20}(h)(\beta,\gamma,t)+\tilde{f}^L(\alpha^t_\gamma)-\tilde{f}^L(\beta^t_\gamma)))(\alpha-\beta)^2]\frac{1}{(\alpha-\beta)^2}d\beta(A(h_\mu)(\alpha,\gamma,t))\\
&=A_{2,1,1}+A_{2,1,2}+A_{2,1,M}+A_{2,1,4}+A_{2,1,5}.
\end{aligned}
$$

(3.150) gives $A(h)(-\tau(t),\gamma,t)=0$. For $A_{2,1,1}$, $A_{2,1,2}$ and $A_{2,1,5}$ from lemma 3.7.2, we have

$$A_{2,1,1}=L^{4,j_1}_{h^s}(\alpha,\gamma,t)(A(h)(2\alpha+\tau(t),\gamma,t)-A(h)(\alpha,\gamma,t)),$$

$$A_{2,1,1}+A_{2,1,2}=L^{4,j_2}_{h^s}(\alpha,\gamma,t)A(h)(\alpha,\gamma,t),$$

and

$$A_{2,1,5}=L^{4,j_3}_{h^s}(\alpha,\gamma,t)A(h)(\alpha,\gamma,t).$$

The first two are of sub-type 2, the last one is of sub-type 5 in lemma 3.7.7. Therefore, by lemma 3.7.7, we have

$$\|A_{2,1,1}+A_{2,1,2}+A_{2,1,5}\|_{C^0_\gamma([-1,1],L^2_\alpha[-\tau(t),\frac{\pi}{4}])}\lesssim\|A(h)\|_{C^0_\gamma([-1,1],L^2_\alpha[-\tau(t),\pi])}.$$

Similarly,

$$A_{2,1,4} = -p.v. \int_{-\tau(t)}^{2\alpha+\tau(t)} L_h^{3,j_4}(\alpha,\beta,\gamma,t)\frac{A(h)(\beta,\gamma,t)}{(\alpha-\beta)}d\beta,$$

with $L_h^{3,j_4}(\alpha,\gamma,t)$ of sub-type 1 in lemma 3.7.6. Then from (3.52), and lemma 3.9.13, we have

$$\|A_{2,1,4}\|_{C_\gamma^0([-1,1],L_\alpha^2[-\tau(t),\frac{\pi}{4})]} \lesssim \|A(h)\|_{C_\gamma^0([-1,1],L_\alpha^2[-\tau(t),\pi])}.$$

For the main term $A_{2,1,M}$, it is of the form

$$A_{2,1,M} = L_{h^s}^1(\alpha,\gamma,t)\int_{-\tau(t)}^{2\alpha+\tau(t)} \frac{(A(h_\mu)(\beta,\gamma,t) - A(h_\mu)(\alpha,\gamma,t))}{(\alpha-\beta)^2}d\beta.$$

with $L_{h^s}^1$ in lemma 3.7.3. Therefore

$$\begin{aligned} A(M_{1,2}^c(h) - M_2^c(h)) &= B_{D^{-20}(h)}^2(\alpha,\gamma,t)\frac{dA(h_\mu)}{d\alpha} \\ &+ L_{h^s}^1(\alpha,\gamma,t)\int_{-\tau(t)}^{2\alpha+\tau(t)} \frac{(A(h_\mu)(\alpha,\gamma,t) - A(h_\mu)(\beta,\gamma,t))}{(\alpha-\beta)^2}d\beta + B.T^0, \end{aligned} \tag{3.172}$$

Now we show some lemmas control the O^i terms and $T_{fixed}(\alpha_\gamma^t,t)$.

Lemma 3.11.2. *We have*

$$\|A(O^{1,i,c})\|_{C_\gamma^0([-1,1],L_\alpha^2[-\tau(t),\frac{\pi}{4}])} \lesssim \|A(h)\|_{C_\gamma^0([-1,1],L_\alpha^2[-\tau(t),\pi])}$$

Proof. First, we still rewrite the $O^{1,i,c}$ by changing the $\tilde{f}^{+(20)}(\beta+\tau(t),t)$ by $h(\beta,0,t)$ and also write singular terms together to cancel the $p.v.$. We have

$$\begin{aligned} &O^{1,i,c}(h) = p.v.\int_{-\pi}^{\pi} K_o^i(V_h^{10}(\alpha,\gamma,t) - V_h^{10}(\beta,\gamma,t), V_{\tilde{f}^L}^{21}(\alpha_\gamma^t) - V_{\tilde{f}^L}^{21}(\beta_\gamma^t), V_X(\alpha_\gamma^t) - V_X(\beta_\gamma^t))X_{i'}(\beta_\gamma^t) \\ &D^{-20+b_i}(h)(\alpha,\gamma,t)(1+ic'(\beta+\tau(t))\gamma t)d\beta \\ &- p.v.\int_{-\tau(t)}^{2\alpha+\tau(t)} K_o^i(V_h^{10}(\alpha,\gamma,t) - V_h^{10}(\beta,\gamma,t), V_{\tilde{f}^L}^{21}(\alpha_\gamma^t) - V_{\tilde{f}^L}^{21}(\beta_\gamma^t), V_X(\alpha_\gamma^t) - V_X(\beta_\gamma^t))X_{i'}(\beta_\gamma^t) \\ &(D^{-20+b_i}(h)(\beta,\gamma,t))(1+ic'(\beta+\tau(t))\gamma t)d\beta \\ &+ \int_0^\gamma K_o^i(V_h^{10}(\alpha,\gamma,t) - V_h^{10}(2\alpha+\tau(t),\eta,t), V_{\tilde{f}^L}^{21}(\alpha_\gamma^t) - V_{\tilde{f}^L}^{21}((2\alpha+\tau(t))_\eta^t), V_X(\alpha_\gamma^t) - V_X((2\alpha+\tau(t))_\eta^t) \\ &X_{i'}((2\alpha+\tau(t))_\eta^t)(D^{-20+b_i}(h)(2\alpha+\tau(t),\eta,t))(ic(2\alpha+2\tau(t))t)d\eta \\ &- \int_{2\alpha+\tau(t)}^{\pi} K_o^i(V_h^{10}(\alpha,\gamma,t) - V_h^{10}(\beta,0,t)), V_{\tilde{f}^L}^{21}(\alpha_\gamma^t) - V_{\tilde{f}^L}^{21}(\beta_0^t), V_X(\alpha_\gamma^t) - V_X(\beta_0^t))X_{i'}(\beta_0^t) \end{aligned}$$

$$(D^{-20+b_i}h)(\beta,0,t)d\beta$$

$$= \int_{-\tau(t)}^{2\alpha+\tau(t)} K_o^i(V_h^{10}(\alpha,\gamma,t) - V_h^{10}(\beta,\gamma,t), V_{\tilde{f}^L}^{21}(\alpha_\gamma^t) - V_{\tilde{f}^L}^{21}(\beta_\gamma^t), V_X(\alpha_\gamma^t) - V_X(\beta_\gamma^t))X_{i'}(\beta_\gamma^t)$$

$$(D^{-20+b_i}(h)(\alpha,\gamma,t) - D^{-20+b_i}(h)(\beta,\gamma,t))(1+ic'(\beta+\tau(t))\gamma t)d\beta$$

$$+ \int_0^\gamma K_o^i(V_h^{10}(\alpha,\gamma,t) - V_h^{10}(2\alpha+\tau(t),\eta,t), V_{\tilde{f}^L}^{21}(\alpha_\gamma^t) - V_{\tilde{f}^L}^{21}((2\alpha+\tau(t))_\eta^t), V_X(\alpha_\gamma^t) - V_X((2\alpha+\tau(t))_\eta^t)$$

$$X_{i'}((2\alpha+\tau(t))_\eta^t)(D^{-20+b_i}(h)(2\alpha+\tau(t),\eta,t))(ic(2\alpha+2\tau(t))t)d\eta$$

$$- \int_{2\alpha+\tau(t)}^{\pi} K_o^i(V_h^{10}(\alpha,\gamma,t) - V_h^{10}(\beta,0,t)), V_{\tilde{f}^L}^{21}(\alpha_\gamma^t) - V_{\tilde{f}^L}^{21}(\beta_0^t), V_X(\alpha_\gamma^t) - V_X(\beta_0^t))X_{i'}(\beta_0^t)$$

$$(D^{-20+b_i}h)(\beta,0,t)d\beta$$

$$+ \int_{[-\pi,-\tau(t)]\cup[2\alpha+\tau(t),\pi]} K_o^i(V_h^{10}(\alpha,\gamma,t) - V_h^{10}(\beta,\gamma,t), V_{\tilde{f}^L}^{21}(\alpha_\gamma^t) - V_{\tilde{f}^L}^{21}(\beta_\gamma^t), V_X(\alpha_\gamma^t) - V_X(\beta_\gamma^t))X_{i'}(\beta_\gamma^t)$$

$$D^{-20+b_i}(h)(\alpha,\gamma,t)(1+ic'(\beta+\tau(t))\gamma t)d\beta$$

Then by corollary 3.11.10, we have

$$A(O^{1,i,c}(h)) = p.v.\int_{-\pi}^{\pi} D_h K_o^i(V_h^{10}(\alpha,\gamma,t) - V_h^{10}(\beta,\gamma,t), V_{\tilde{f}^L}^{21}(\alpha_\gamma^t) - V_{\tilde{f}^L}^{21}(\beta_\gamma^t), V_X(\alpha_\gamma^t) - V_X(\beta_\gamma^t))[A(h)]X_{i'}(\beta_\gamma^t)$$

$$(1+ic'(\beta+\tau(t))\gamma t)d\beta D^{-20+b_i}(h)(\alpha,\gamma,t)$$

$$+ p.v.\int_{-\pi}^{\pi} K_o^i(V_h^{10}(\alpha,\gamma,t) - V_h^{10}(\beta,\gamma,t), V_{\tilde{f}^L}^{21}(\alpha_\gamma^t) - V_{\tilde{f}^L}^{21}(\beta_\gamma^t), V_X(\alpha_\gamma^t) - V_X(\beta_\gamma^t))X_{i'}(\beta_\gamma^t)$$

$$D^{-20+b_i}(A(h))(\alpha,\gamma,t)(1+ic'(\beta+\tau(t))\gamma t)d\beta$$

$$- p.v.\int_{-\tau(t)}^{2\alpha+\tau(t)} D_h K_o^i(V_h^{10}(\alpha,\gamma,t) - V_h^{10}(\beta,\gamma,t), V_{\tilde{f}^L}^{21}(\alpha_\gamma^t) - V_{\tilde{f}^L}^{21}(\beta_\gamma^t), V_X(\alpha_\gamma^t) - V_X(\beta_\gamma^t))[A(h)]X_{i'}(\beta_\gamma^t)$$

$$(D^{-20+b_i}(h)(\beta,\gamma,t))(1+ic'(\beta+\tau(t))\gamma t)d\beta$$

$$- p.v.\int_{-\tau(t)}^{2\alpha+\tau(t)} K_o^i(V_h^{10}(\alpha,\gamma,t) - V_h^{10}(\beta,\gamma,t), V_{\tilde{f}^L}^{21}(\alpha_\gamma^t) - V_{\tilde{f}^L}^{21}(\beta_\gamma^t), V_X(\alpha_\gamma^t) - V_X(\beta_\gamma^t))X_{i'}(\beta_\gamma^t)$$

$$(D^{-20+b_i}(A(h))(\beta,\gamma,t))(1+ic'(\beta+\tau(t))\gamma t)d\beta$$

$$+ \int_0^\gamma \nabla_1 K_o^i(V_h^{10}(\alpha,\gamma,t) - V_h^{10}(2\alpha+\tau(t),\eta,t), V_{\tilde{f}^L}^{21}(\alpha_\gamma^t) - V_{\tilde{f}^L}^{21}((2\alpha+\tau(t))_\eta^t), V_X(\alpha_\gamma^t) - V_X((2\alpha+\tau(t))_\eta^t)$$

$$\cdot (V_{A(h)}^{10}(\alpha,\gamma,t) - \frac{2ic(\alpha+\tau(t))t(1+ic'(2\alpha+2\tau(t))\eta t)}{(1+ic'(\alpha+\tau(t))\gamma t)(ic(2\alpha+2\tau(t))t)} V_{A(h)}^{10}(2\alpha+\tau(t),\eta,t))$$

$$X_{i'}((2\alpha+\tau(t))_\eta^t)(D^{-20+b_i}(h)(2\alpha+\tau(t),\eta,t))(ic(2\alpha+2\tau(t))t)d\eta$$

$$+ \int_0^\gamma K_o^i(V_h^{10}(\alpha,\gamma,t) - V_h^{10}(2\alpha+\tau(t),\eta,t), V_{\tilde{f}^L}^{21}(\alpha_\gamma^t) - V_{\tilde{f}^L}^{21}((2\alpha+\tau(t))_\eta^t), V_X(\alpha_\gamma^t) - V_X((2\alpha+\tau(t))_\eta^t)$$

$$X_{i'}((2\alpha+\tau(t))_\eta^t)\frac{2ic(\alpha+\tau(t))t(1+ic'(2\alpha+2\tau(t))\eta t)}{(1+ic'(\alpha+\tau(t))\gamma t)(ic(2\alpha+2\tau(t))t)}(D^{-20+b_i}(A(h))(2\alpha+\tau(t),\eta,t))(ic(2\alpha+2\tau(t))t)d\eta$$

$$-\int_{2\alpha+\tau(t)}^{\pi}\nabla_1 K_o^i(V_h^{10}(\alpha,\gamma,t)-V_h^{10}(\beta,0,t)),V_{\tilde{f}^L}^{21}(\alpha_\gamma^t)-V_{\tilde{f}^L}^{21}(\beta_0^t),V_X(\alpha_\gamma^t)-V_X(\beta_0^t))\cdot V_{A(h)}^{10}(\alpha,\gamma,t)X_{i'}(\beta_0^t)$$

$$(D^{-20+b_i}h)(\beta,0,t)d\beta$$

$$= AO_{1,1}+AO_{1,2}+AO_{1,3}+AO_{1,4}+AO_{1,5}+AO_{1,6}+AO_{1,7}.$$

Here some terms disappear because $A(V_{\tilde{f}^L}(\alpha_\gamma^t))=A(V_X(\alpha_\gamma^t))=0$ and $A(X_{i'}(\alpha_\gamma^t))=0$.

For $AO_{1,1}$ and $AO_{1,2}$, $p.v.\int_{-\pi}^{\pi}K_o^i(V_h^{10}(\alpha,\gamma,t)-V_h^{10}(\beta,\gamma,t),V_{\tilde{f}^L}^{21}(\alpha_\gamma^t)-V_{\tilde{f}^L}^{21}(\beta_\gamma^t),V_X(\alpha_\gamma^t)-V_X(\beta_\gamma^t))X_{i'}(\beta_\gamma^t)(1+ic'(\beta+\tau(t))\gamma t)d\beta$ is of the same form as $L_{h^s}^{4,i}$ as the sub-type 1 in lemma 3.7.7. Then from (3.58) we can control $AO_{1,1}$. Similarly, from (3.56), we can control $AO_{1,2}$.

For $AO_{1,3}$ and $AO_{1,4}$, $K_o^i(V_h^{10}(\alpha,\gamma,t)-V_h^{10}(\beta,\gamma,t),V_{\tilde{f}^L}^{21}(\alpha_\gamma^t)-V_{\tilde{f}^L}^{21}(\beta_\gamma^t),V_X(\alpha_\gamma^t)-V_X(\beta_\gamma^t))X_{i'}(\beta_\gamma^t)(1+ic'(\beta+\tau(t))\gamma t)(\alpha-\beta)$ is of the form sub-type 2 of $L_h^{3,i}$ in the lemma (3.7.6). Then $AO_{1,4}$ can be bounded by (3.54) and lemma 3.9.13. $AO_{1,3}$ can be controlled similarly by using (3.52) instead of (3.54).

For $AO_{1,5}$, the component of $\nabla_1 K_o^i(V_h^{10}(\alpha,\gamma,t)-V_h^{10}(2\alpha+\tau(t),\eta,t),V_{\tilde{f}^L}^{21}(\alpha_\gamma^t)-V_{\tilde{f}^L}^{21}((2\alpha+\tau(t))_\eta^t),V_X(\alpha_\gamma^t)-V_X((2\alpha+\tau(t))_\eta^t)$ has the form K_{-2}^j in lemma 3.7.22. Moreover,

$$|\frac{ic(\alpha+\tau(t))t}{(\alpha+\tau(t))^2}|\lesssim 1.$$

Then from lemma 3.7.22, we have the result.

For $AO_{1,6}$, the component of $K_o^i(V_h^{10}(\alpha,\gamma,t)-V_h^{10}(2\alpha+\tau(t),\eta,t),V_{\tilde{f}^L}^{21}(\alpha_\gamma^t)-V_{\tilde{f}^L}^{21}((2\alpha+\tau(t))_\eta^t),V_X(\alpha_\gamma^t)-V_X((2\alpha+\tau(t))_\eta^t)$ has the form K_{-1}^j in lemma 3.7.22. Then it can be bounded in the same way as in $AO_{1,5}$.

For $AO_{1,7}$, we have the component of $\nabla_1 K_o^i(V_h^{10}(\alpha,\gamma,t)-V_h^{10}(\beta,0,t)),V_{\tilde{f}^L}^{21}(\alpha_\gamma^t)-V_{\tilde{f}^L}^{21}(\beta_0^t),V_X(\alpha_\gamma^t)-V_X(\beta_0^t))$ has the form K_{-2}^j in lemma 3.7.27. Moreover,

$$\|\frac{h(\beta,0,t)}{(\beta+\tau(t))^2}\|_{L_\beta^\infty[-\tau(t),\pi]}\lesssim 1,$$

Then from lemma 3.7.27, we can bound this term. □

Corollary 3.11.3. *We have*

$$\|A(O^{2,i,c})\|_{C_\gamma^0([-1,1],L_\alpha^2[-\tau(t),\frac{\pi}{4}])}\lesssim\|A(h)\|_{C_\gamma^0([-1,1],L_\alpha^2[-\tau(t),\frac{\pi}{4}])},$$

and

$$\|A(O^{4,i,c})\|_{C^0_\gamma([-1,1],L^2_\alpha[-\tau(t),\frac{\pi}{4}])} \lesssim \|A(h)\|_{C^0_\gamma([-1,1],L^2_\alpha[-\tau(t),\frac{\pi}{4}])},$$

Proof. From (3.28) and (3.29), they can be bounded in the same way as $A(O^{1,i,c})$. □

Lemma 3.11.4. *We have*

$$\|A(O^{3,i,c})\|_{C^0_\gamma([-1,1],L^2_\alpha[-\tau(t),\frac{\pi}{4}])} \lesssim \|A(h)\|_{C^0_\gamma([-1,1],L^2_\alpha[-\tau(t),\pi])}.$$

Proof. From (3.30), we have

$$O^{3,i,c}(h) = \int_{-\pi}^{\pi} \tilde{K}^i_o(V^{10}_h(\alpha^t_\gamma) - V^{10}_h(\beta^t_\gamma), V^{21}_{\tilde{f}^L}(\alpha^t_\gamma) - V^{21}_{\tilde{f}^L}(\beta^t_\gamma), V_X(\alpha^t_\gamma) - V_X(\beta^t_\gamma))X_{i'}(\beta^t_\gamma) \tag{3.173}$$
$$(1 + ic'(\beta + \tau(t))\gamma t)d\beta D^{-20+b_i}(h_\mu)(\alpha,\gamma,t).$$

Then

$$A(O^{3,i,c}(h)) = A(\int_{-\pi}^{\pi} \tilde{K}^i_o(V^{10}_h(\alpha^t_\gamma) - V^{10}_h(\beta^t_\gamma), V^{21}_{\tilde{f}^L}(\alpha^t_\gamma) - V^{21}_{\tilde{f}^L}(\beta^t_\gamma), V_X(\alpha^t_\gamma) - V_X(\beta^t_\gamma))X_{i'}(\beta^t_\gamma)$$
$$(1 + ic'(\beta + \tau(t))\gamma t)d\beta)D^{-20+b_i}(h_\mu)(\alpha,\gamma,t)$$
$$+ \int_{-\pi}^{\pi} \tilde{K}^i_o(V^{10}_h(\alpha^t_\gamma) - V^{10}_h(\beta^t_\gamma), V^{21}_{\tilde{f}^L}(\alpha^t_\gamma) - V^{21}_{\tilde{f}^L}(\beta^t_\gamma), V_X(\alpha^t_\gamma) - V_X(\beta^t_\gamma))X_{i'}(\beta^t_\gamma)(1 + ic'(\beta + \tau(t))\gamma t)d\beta$$
$$D^{-20+b_i}A(h_\mu)(\alpha,\gamma,t)$$
$$= AO_{3,1} + AO_{3,2}.$$

Since from (3.17), there is no singularity in $\tilde{K}^i_o$, from lemma 2.8.4,, we have

$$AO_{3,1} = D_h(\int_{-\pi}^{\pi} \tilde{K}^i_o(V^{10}_h(\alpha^t_\gamma) - V^{10}_h(\beta^t_\gamma), V^{21}_{\tilde{f}^L}(\alpha^t_\gamma) - V^{21}_{\tilde{f}^L}(\beta^t_\gamma), V_X(\alpha^t_\gamma) - V_X(\beta^t_\gamma))X_{i'}(\beta^t_\gamma)$$
$$(1 + ic'(\beta + \tau(t))\gamma t)d\beta)[A(h)]D^{-20+b_i}(h_\mu)(\alpha,\gamma,t),$$

where we use $A(\tilde{f}^L(\alpha^t_\gamma, t)) = 0$ and $A(X_i(\alpha^t_\gamma, t)) = 0$.

The $\int_{-\pi}^{\pi} \tilde{K}^i_o(V^{10}_h(\alpha^t_\gamma) - V^{10}_h(\beta^t_\gamma), V^{21}_{\tilde{f}^L}(\alpha^t_\gamma) - V^{21}_{\tilde{f}^L}(\beta^t_\gamma), V_X(\alpha^t_\gamma) - V_X(\beta^t_\gamma))X_{i'}(\beta^t_\gamma)(1 + ic'(\beta + \tau(t))\gamma t)d\beta$ is of the same form as $L^{4,i}_{h^s}$ as the sub-type 1 in lemma 3.7.7. Then from (3.58) we can control $AO_{3,1}$. Simlilarly, from (3.56) we can control $AO_{3,2}$. □

From (3.31), and lemma 3.11.17, it is easy to get

$$\|A(O^{5,i,c})\|_{C^0_\gamma([-1,1],L^2_\alpha[-\tau(t),\frac{\pi}{4}])} \lesssim \|A(h)\|_{C^0_\gamma([-1,1],L^2_\alpha[-\tau(t),\frac{\pi}{4}])}.$$

From lemma 3.7.10, since $\tilde{f}^L(\alpha,t)$ is analytic in the region D_A, we have $A(T^1_{fixed}(\alpha^t_\gamma,t))=0$ and $A(T^2_{fixed}(\alpha^t_\gamma,t))=0$. From lemma 2.8.4, we get $A(T^3_{fixed}(\alpha^t_\gamma,t))=0$. Therefore $A(T_{fixed}(\alpha^t_\gamma,t))=0$.

Now we only left with the main terms

$$\frac{dA(h_\mu)(\alpha,\gamma,t)}{dt} = (\frac{ic(\alpha+\tau(t))}{1+ic'(\alpha+\tau(t)\gamma t)\tau'(t))}\frac{dA(h_\mu)}{d\alpha} + \tilde{\lambda}(\alpha^t_\gamma)B^2_{D^{-20}(h)}(\alpha,\gamma,t)\frac{dA(h_\mu)}{d\alpha}$$
$$+\tilde{\lambda}(\alpha^t_\gamma)L^1_{h^s}(\alpha,\gamma,t)p.v.\int_{-\tau(t)}^{2\alpha+\tau(t)}\frac{(A(h_\mu)(\alpha,\gamma,t)-A(h_\mu)(\beta,\gamma,t))}{(\alpha-\beta)^2}d\beta + B.T.^0$$

By (3.167), the first two terms can combine and from lemma 3.7.4, we have

$$\frac{dA(h_\mu)(\alpha,\gamma,t)}{dt} = \tilde{\lambda}(\alpha^t_\gamma)\frac{\pi}{2}L^2_{h^s}(\alpha,\gamma,t)\frac{dA(h_\mu)}{d\alpha}$$
$$+\tilde{\lambda}(\alpha^t_\gamma)L^1_{h^s}(\alpha,\gamma,t)p.v.\int_{-\tau(t)}^{2\alpha+\tau(t)}\frac{(A(h_\mu)(\alpha,\gamma,t)-A(h_\mu)(\beta,\gamma,t))}{(\alpha-\beta)^2}d\beta + B.T^0.$$

Now we have

$$\begin{aligned}&\frac{d}{dt}\|A(h_\mu)\|^2_{L^2}\\ &=2\Re\int_{-\pi}^{\pi}A(h_\mu)(\alpha,\gamma,t)\overline{\tilde{\lambda}(\alpha^t_\gamma)\frac{\pi}{2}L^2_{h^s}(\alpha,\gamma,t)\frac{dA(h_\mu)}{d\alpha}}\\ &\overline{+\tilde{\lambda}(\alpha^t_\gamma)L^1_{h^s}(\alpha,\gamma,t)\int_{-\tau(t)}^{2\alpha+\tau(t)}\frac{(A(h_\mu)(\alpha,\gamma,t)-A(h_\mu)(\beta,\gamma,t))}{(\alpha-\beta)^2}d\beta d\alpha}\\ &+B.T.^0\end{aligned}$$

Then we can again use corollary in the energy estimate 3.9.2 and (3.151) to get

$$\frac{d}{dt}\|A(h_\mu)\|^2_{L^2}\lesssim\|A(h_\mu)\|^2_{L^2}.$$

. The initial value $A(h_\mu)(\alpha,\gamma,0)=0$ by the definition $h(\alpha,\gamma,0)=\tilde{f}^+(\alpha,0)$. Then $A(h_\mu)(\alpha,\gamma,t)=0$ when t is sufficiently small. Then we have the analyticity.

3.11.1 Lemmas

Lemma 3.11.5. *If all the derivatives are well-defined and $\partial_\alpha\partial_\gamma h(\alpha,\gamma,t)=\partial_\gamma\partial_\alpha h(\alpha,\gamma,t)$, we have*

$$(\frac{ic(\alpha+\tau(t))t}{1+ic'(\alpha+\tau(t))\gamma t}\partial_\alpha-\partial_\gamma)\frac{\partial_\alpha h(\alpha,\gamma)}{1+ic'(\alpha+\tau(t))\gamma t}=\frac{\partial_\alpha}{1+ic'(\alpha+\tau(t))\gamma t}(\frac{ic(\alpha+\tau(t))t}{1+ic'(\alpha+\tau(t))\gamma t}\partial_\alpha-\partial_\gamma)h(\alpha,\gamma),$$

Proof. We can get the result from lemma 2.8.2. □

Lemma 3.11.6. *If all the derivatives are well-defined, $\frac{d}{d\alpha}\frac{d}{dt}h=\frac{d}{dt}\frac{d}{d\alpha}h$, $\frac{d}{dt}\frac{d}{d\gamma}h=\frac{d}{d\gamma}\frac{d}{dt}h$, $\frac{d}{d\alpha}\frac{d}{d\gamma}h=\frac{d}{d\gamma}\frac{d}{d\alpha}h$, and h satisfies*

$$\frac{d}{dt}h=\frac{ic(\alpha+\tau(t))\gamma+ic'(\alpha+\tau(t))\gamma t\tau'(t)}{1+ic'(\alpha+\tau(t))\gamma t}\partial_\alpha h+T(h),$$

then we have

$$\frac{d}{dt}A(h)=\frac{ic(\alpha+\tau(t))\gamma+ic'(\alpha+\tau(t))\gamma t\tau'(t)}{1+ic'(\alpha+\tau(t))\gamma t}\partial_\alpha A(h)+AT(h).$$

Proof.

$$\begin{aligned}
&\frac{d}{dt}A(h)(\alpha,\gamma,t)\\
&=\frac{d}{dt}(\frac{ic(\alpha+\tau(t))t}{1+ic'(\alpha+\tau(t))\gamma t})\partial_\alpha h+\frac{ic(\alpha+\tau(t))t}{1+ic'(\alpha+\tau(t))\gamma t}\partial_\alpha\frac{d}{dt}h-\partial_\gamma\frac{d}{dt}h\\
&=\underbrace{\partial_t(\frac{ic(\alpha+\tau(t))t}{1+ic'(\alpha+\tau(t))\gamma t})\partial_\alpha h}_{Term_1}+\underbrace{(\frac{ic(\alpha+\tau(t))t}{1+ic'(\alpha+\tau(t))\gamma t}\partial_\alpha-\partial_\gamma)(\frac{ic(\alpha+\tau(t))\gamma+ic'(\alpha+\tau(t))\gamma t\tau'(t)}{1+ic'(\alpha+\tau(t))\gamma t}\partial_\alpha h)}_{Term_2}+AT(h)
\end{aligned}$$

We have

$$\begin{aligned}
Term_2&=(\frac{ic(\alpha+\tau(t))\gamma+ic'(\alpha+\tau(t))\gamma t\tau'(t)}{1+ic'(\alpha+\tau(t))\gamma t})(\frac{ic(\alpha+\tau(t))t}{1+ic'(\alpha+\tau(t))\gamma t}\partial_\alpha-\partial_\gamma)\partial_\alpha h\\
&+(\frac{ic(\alpha+\tau(t))t}{1+ic'(\alpha+\tau(t))\gamma t})\partial_\alpha(\frac{ic(\alpha+\tau(t))\gamma+ic'(\alpha+\tau(t))\gamma t\tau'(t)}{1+ic'(\alpha+\tau(t))\gamma t})\partial_\alpha h-\partial_\gamma(\frac{ic(\alpha+\tau(t))\gamma+ic'(\alpha+\tau(t))\gamma t\tau'(t)}{1+ic'(\alpha+\tau(t))\gamma t})\partial_\alpha h\\
&=\underbrace{(\frac{ic(\alpha+\tau(t))\gamma+ic'(\alpha+\tau(t))\gamma t\tau'(t)}{1+ic'(\alpha+\tau(t))\gamma t})\partial_\alpha((\frac{ic(\alpha+\tau(t))t}{1+ic'(\alpha+\tau(t))\gamma t}\partial_\alpha-\partial_\gamma)h)}_{Term_{2,1}}\\
&\underbrace{-(\frac{ic(\alpha+\tau(t))\gamma+ic'(\alpha+\tau(t))\gamma t\tau'(t)}{1+ic'(\alpha+\tau(t))\gamma t})\partial_\alpha(\frac{ic(\alpha+\tau(t))t}{1+ic'(\alpha+\tau(t))\gamma t})}_{Term_{2,2}}\partial_\alpha h\\
&+\underbrace{(\frac{ic(\alpha+\tau(t))t}{1+ic'(\alpha+\tau(t))\gamma t})\partial_\alpha(\frac{ic(\alpha+\tau(t))\gamma+ic'(\alpha+\tau(t))\gamma t\tau'(t)}{1+ic'(\alpha+\tau(t))\gamma t})\partial_\alpha h}_{Term_{2,3}}\underbrace{-\partial_\gamma(\frac{ic(\alpha+\tau(t))\gamma+ic'(\alpha+\tau(t))\gamma t\tau'(t)}{1+ic'(\alpha+\tau(t))\gamma t})\partial_\alpha h}_{Term_{2,4}}
\end{aligned}$$

Moreover, we have

$$
\begin{aligned}
&Term_1 + Term_{2,2} + Term_{2,3} + Term_{2,4} \\
&= [\partial_t(\frac{ic(\alpha+\tau(t))t}{1+ic'(\alpha+\tau(t))\gamma t}) \\
&- (\frac{ic(\alpha+\tau(t))\gamma + ic'(\alpha+\tau(t))\gamma t\tau'(t)}{1+ic'(\alpha+\tau(t))\gamma t})\partial_\alpha(\frac{ic(\alpha+\tau(t))t}{1+ic'(\alpha+\tau(t))\gamma t}) \\
&+ \frac{ic(\alpha+\tau(t))t}{1+ic'(\alpha+\tau(t))\gamma t}\partial_\alpha(\frac{ic(\alpha+\tau(t))\gamma + ic'(\alpha+\tau(t))\gamma t\tau'(t)}{1+ic'(\alpha+\tau(t))\gamma t}) \\
&- \partial_\gamma(\frac{ic(\alpha+\tau(t))\gamma + ic'(\alpha+\tau(t))\gamma t\tau'(t)}{1+ic'(\alpha+\tau(t))\gamma t})]\partial_\alpha h \\
&= [\underbrace{(\frac{ic(\alpha+\tau(t))}{1+ic'(\alpha+\tau(t))\gamma t}) + (\frac{ic'(\alpha+\tau(t))\tau'(t)t}{1+ic'(\alpha+\tau(t))\gamma t})}_{C_1} - \underbrace{\frac{ic(\alpha+\tau(t))t(ic'(\alpha+\tau(t))\gamma + ic''(\alpha+\tau(t))\gamma\tau'(t)t)}{(1+ic'(\alpha+\tau(t))\gamma t)^2}}_{C_3} \\
&- (\frac{ic(\alpha+\tau(t))\gamma + ic'(\alpha+\tau(t))\gamma t\tau'(t)}{1+ic'(\alpha+\tau(t))\gamma t})(\underbrace{\frac{ic'(\alpha+\tau(t))t}{1+ic'(\alpha+\tau(t))\gamma t}}_{C_4} - \underbrace{\frac{ic(\alpha+\tau(t))t(ic''(\alpha+\tau(t))\gamma t))}{(1+ic'(\alpha+\tau(t))\gamma t)^2}}_{C_2}) \\
&+ \frac{ic(\alpha+\tau(t))t}{1+ic'(\alpha+\tau(t))\gamma t}(\underbrace{\frac{ic'(\alpha+\tau(t))\gamma + ic''(\alpha+\tau(t))\gamma t\tau'(t)}{1+ic'(\alpha+\tau(t))\gamma t}}_{C_3} \\
&- \underbrace{\frac{(ic(\alpha+\tau(t))\gamma + ic'(\alpha+\tau(t))\gamma t\tau'(t))(ic''(\alpha+\tau(t))\gamma t)}{(1+ic'(\alpha+\tau(t))\gamma t)^2}}_{C_2}) \\
&- \underbrace{(\frac{ic(\alpha+\tau(t)) + ic'(\alpha+\tau(t))t\tau'(t)}{1+ic'(\alpha+\tau(t))\gamma t})}_{C_1} + \underbrace{(\frac{(ic(\alpha+\tau(t))\gamma + ic'(\alpha+\tau(t))\gamma t\tau'(t))(ic'(\alpha+\tau(t))t)}{(1+ic'(\alpha+\tau(t))\gamma t)^2})}_{C_4}]\partial_\alpha h \\
&= 0.
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
&\frac{d}{dt}A(h)(\alpha,\gamma,t) = Term_{2,1} + A(T(h)) \\
&= (\frac{ic(\alpha+\tau(t))\gamma + ic'(\alpha+\tau(t))\gamma t\tau'(t)}{1+ic'(\alpha+\tau(t))\gamma t})\partial_\alpha((\frac{ic(\alpha+\tau(t))t}{1+ic'(\alpha+\tau(t))\gamma t}\partial_\alpha - \partial_\gamma)h) + A(T(h)) \\
&= (\frac{ic(\alpha+\tau(t))\gamma + ic'(\alpha+\tau(t))\gamma t\tau'(t)}{1+ic'(\alpha+\tau(t))\gamma t})\partial_\alpha A(h) + A(T(h)).
\end{aligned}
$$

Hence we have the result. □

Let

$$
A_0(h)(\alpha,\gamma) = (\frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t}\frac{d}{d\alpha} - \frac{d}{d\gamma})h(\alpha,\gamma).
$$

$$A(h)(\alpha+\tau(t),\gamma)=(\frac{ic(\alpha+\tau(t))t}{1+ic'(\alpha+\tau(t))\gamma t}\frac{d}{d\alpha}-\frac{d}{d\gamma})h(\alpha+\tau(t),\gamma).$$

Now we do the following lemmas from A_0 and A. In those lemmas, the $\ddot{h}$, $\breve{h}$ we use are vector functions satisfying the conditions in the lemmas correspondingly.

Lemma 3.11.7. *Let $\tilde{K}$ be meromorphic, $\partial_\alpha h(\alpha,\gamma)$ and $\partial_\gamma h(\alpha,\gamma)$ is well-defined and in $C^0_\alpha[-\pi+\tau(t),\pi+\tau(t)]$. If for fixed $\alpha>-\tau(t)$, there is no singular point in the integrals below and $c(\pi+\tau(t))=c(-\pi+\tau(t))=0$, $c(\alpha)\in W^{2,\infty}$, then we have*

$$\begin{aligned}
&A(\int_{-\pi}^{\pi}\tilde{K}(h(\alpha+\tau(t),\gamma)-h(\beta+\tau(t),\gamma))X(\beta+\tau(t),\gamma)(1+ic'(\beta+\tau(t))\gamma t)d\beta)\\
&=\int_{-\pi}^{\pi}\nabla\tilde{K}(h(\alpha+\tau(t),\gamma)-h(\beta+\tau(t),\gamma))\cdot(A(h)(\alpha+\tau(t),\gamma,t)-A(h)(\beta+\tau(t),\gamma,t))\\
&X(\beta+\tau(t),\gamma)(1+ic'(\beta+\tau(t))\gamma t)d\beta\\
&+\int_{-\pi}^{\pi}\tilde{K}(h(\alpha+\tau(t),\gamma)-h(\beta+\tau(t),\gamma))A(X)(\beta+\tau(t),\gamma)(1+ic'(\beta+\tau(t))\gamma t)d\beta\\
&=D_h(\int_{-\pi}^{\pi}\tilde{K}(h(\alpha+\tau(t),\gamma)-h(\beta+\tau(t),\gamma))X(\beta+\tau(t),\gamma)(1+ic'(\beta+\tau(t))\gamma t)d\beta)[A_0(h)]\\
&+\int_{-\pi}^{\pi}\tilde{K}(h(\alpha+\tau(t),\gamma)-h(\beta+\tau(t),\gamma,t))A_0(X)(\beta+\tau(t),\gamma)(1+ic'(\beta+\tau(t))\gamma t)d\beta
\end{aligned}$$

Here D_h is the Gateaux derivative.

Proof. The proof is similar as in lemma 2.8.4. □

Lemma 3.11.8. *Let $\breve{K}$ be meromorphic, $\breve{h}(\alpha,\gamma),\ddot{h}(\alpha,\gamma),X(\alpha,\gamma)\in C^1_\gamma([-1,1],C^1_\alpha[-\pi,\pi])$. If for fixed $\alpha>0$, there is no singular point in the integrals of $O(\breve{h},\ddot{h})$ below and $c(0)=c(\pi)=c(-\pi)=0$, $c(\alpha)\in W^{2,\infty}$, then when $0<\alpha<\frac{\pi}{2}$, for*

$$\begin{aligned}
O(\breve{h},\ddot{h})&=\int_0^{2\alpha}\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(\beta,\gamma))(\breve{h}(\alpha,\gamma)-\breve{h}(\beta,\gamma))(X(\beta,\gamma))(1+ic'(\beta)\gamma t)d\beta\\
&+\int_0^{\gamma}\breve{h}(2\alpha,\eta)\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(2\alpha,\eta))ic(2\alpha)t(X(2\alpha,\eta))d\eta\\
&-\int_{2\alpha}^{\pi}\breve{h}(\beta,0)\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(\beta,0))(X(\beta,0))d\beta\\
&+\int_{[-\pi,0]\cup[2\alpha,\pi]}\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(\beta,\gamma))(X(\beta,\gamma))(1+ic'(\beta)\gamma t)d\beta\breve{h}(\alpha,\gamma)\\
&=O_1+O_2+O_3+O_4,
\end{aligned}$$

we have

$$A_0(O(\breve{h},\ddot{h})) = \underbrace{\int_0^{2\alpha} \nabla\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(\beta,\gamma))\cdot([A_0(\ddot{h})(\alpha,\gamma)]-[A_0(\ddot{h})(\beta,\gamma)])(1+ic'(\beta)\gamma t)}_{FA_1}$$

$$\underbrace{(\breve{h}(\alpha,\gamma)-\breve{h}(\beta,\gamma))X(\beta,\gamma)d\beta}_{FA_1}$$

$$+\underbrace{\int_0^{2\alpha} \breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(\beta,\gamma))([A_0(\breve{h})(\alpha,\gamma)]-[A_0(\breve{h})(\beta,\gamma)])(X(\beta,\gamma))(1+ic'(\beta)\gamma t)d\beta}_{FA_2}$$

$$+\underbrace{\int_0^{2\alpha} \breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(\beta,\gamma))(\breve{h}(\alpha,\gamma)-\breve{h}(\beta,\gamma))[A_0(X)(\beta,\gamma)](1+ic'(\beta)\gamma t)d\beta}_{FA_3}$$

$$\underbrace{+\int_0^{\gamma} \breve{h}(2\alpha,\eta)\nabla\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(2\alpha,\eta))\cdot[A_0(\ddot{h})(\alpha,\gamma)]X(2\alpha,\eta)ic(2\alpha)td\eta}_{FA_4}$$

$$\underbrace{-\int_0^{\gamma} \breve{h}(2\alpha,\eta)\nabla\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(2\alpha,\eta))\cdot(\frac{2ic(\alpha)t}{1+ic'(\alpha)\gamma t})(\frac{1+ic'(2\alpha)\eta t}{ic(2\alpha)t})[A_0(\ddot{h})(2\alpha,\eta)]X(2\alpha,\eta)ic(2\alpha)td\eta}_{FA_5}$$

$$+\underbrace{\int_0^{\gamma} \breve{h}(2\alpha,\eta)\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(2\alpha,\eta))(\frac{2ic(\alpha)t}{1+ic'(\alpha)\gamma t})(\frac{1+ic'(2\alpha)\eta t}{ic(2\alpha)t})[A_0(X)(2\alpha,\eta)]ic(2\alpha)td\eta}_{FA_6}$$

$$+\underbrace{\int_0^{\gamma} (\frac{2ic(\alpha)t}{1+ic'(\alpha)\gamma t})(\frac{1+ic'(2\alpha)\eta t}{ic(2\alpha)t})[A_0(\breve{h})(2\alpha,\eta)]\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(2\alpha,\eta))X(2\alpha,\eta)ic(2\alpha)td\eta}_{FA_7}$$

$$\underbrace{-\int_{2\alpha}^{\pi} \breve{h}(\beta,0)\nabla\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(\beta,0))\cdot[A_0(\ddot{h})(\alpha,\gamma)](X(\beta,0))d\beta}_{FA_8}$$

$$+\underbrace{\int_{[-\pi,0]\cup[2\alpha,\pi]} \nabla\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(\beta,\gamma))\cdot([A_0(\ddot{h})(\alpha,\gamma)]-[A_0(\ddot{h})(\beta,\gamma)])(X(\beta,\gamma))(1+ic'(\beta)\gamma t))d\beta\breve{h}(\alpha,\gamma)}_{FA_9}$$

$$+\underbrace{\int_{[-\pi,0]\cup[2\alpha,\pi]} \breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(\beta,\gamma))[A_0(X)(\beta,\gamma)](1+ic'(\beta)\gamma t))d\beta\breve{h}(\alpha,\gamma)}_{FA_{10}}$$

$$+\underbrace{\int_{[-\pi,0]\cup[2\alpha,\pi]} \breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(\beta,\gamma))X(\beta,\gamma)(1+ic'(\beta)\gamma t))d\beta[A_0(\breve{h})(\alpha,\gamma)]}_{FA_{11}}$$

Proof. First, consider O_1,

$A_0(O_1) =$

$$
\begin{aligned}
&A(\int_{-\alpha}^{\alpha} \breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(\alpha-\beta,\gamma))(\breve{h}(\alpha,\gamma)-\breve{h}(\alpha-\beta,\gamma))X(\alpha-\beta,\gamma)(1+ic'(\alpha-\beta)\gamma t)d\beta)\\
&=\underbrace{\frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t}[\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(0,\gamma))(\breve{h}(\alpha,\gamma)-\breve{h}(0,\gamma))X(0,\gamma)(1+ic'(0)\gamma t)}_{Term_1}\\
&\underbrace{+\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(2\alpha,\gamma))(\breve{h}(\alpha,\gamma)-\breve{h}(2\alpha,\gamma))X(2\alpha,\gamma)(1+ic'(2\alpha)\gamma t)]}_{Term_1}\\
&\underbrace{+\int_{-\alpha}^{\alpha}\nabla\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(\alpha-\beta,\gamma))\cdot([(\frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t}\partial_\alpha-\partial_\gamma)\ddot{h}(\alpha,\gamma)]-[(\frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t}\partial_\alpha-\partial_\gamma)\ddot{h}(\alpha-\beta,\gamma)])}_{Term_2}\\
&\underbrace{(1+ic'(\alpha-\beta)\gamma t)(\breve{h}(\alpha,\gamma)-\breve{h}(\alpha-\beta,\gamma))X(\alpha-\beta,\gamma)d\beta}_{Term_2}\\
&\underbrace{+\int_{-\alpha}^{\alpha}\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(\alpha-\beta,\gamma))([(\frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t}\partial_\alpha-\partial_\gamma)\breve{h}(\alpha,\gamma)]-[(\frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t}\partial_\alpha-\partial_\gamma)\breve{h}(\alpha-\beta,\gamma)])X(\alpha-\beta,\gamma)}_{Term_3}\\
&\underbrace{(1+ic'(\alpha-\beta)\gamma t)d\beta}_{Term_3}\\
&\underbrace{+\int_{-\alpha}^{\alpha}\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(\alpha-\beta,\gamma))(\breve{h}(\alpha,\gamma)-\breve{h}(\alpha-\beta,\gamma))[(\frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t)}\partial_\alpha-\partial_\gamma)X(\alpha-\beta,\gamma)]}_{Term_4}\\
&\underbrace{(1+ic'(\alpha-\beta)\gamma t)d\beta}_{Term_4}\\
&\underbrace{+\int_{-\alpha}^{\alpha}\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(\alpha-\beta,\gamma))(\breve{h}(\alpha,\gamma)-\breve{h}(\alpha-\beta,\gamma))X(\alpha-\beta,\gamma)[\frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t}(ic''(\alpha-\beta)\gamma t)-ic'(\alpha-\beta)t]d\beta}_{Term_5}\\
&=Term_1+Term_2+Term_3+Term_4+Term_5.
\end{aligned}
$$

So we have

$$
\begin{aligned}
&Term_2+Term_3+Term_4+Term_5\\
&=\underbrace{\int_{-\alpha}^{\alpha}\nabla\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(\alpha-\beta,\gamma))\cdot([(\frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t}\partial_\alpha-\partial_\gamma)\ddot{h}(\alpha,\gamma)]-[(\frac{ic(\alpha-\beta)t}{1+ic'(\alpha-\beta)\gamma t}\partial_\alpha-\partial_\gamma)\ddot{h}(\alpha-\beta,\gamma)])}_{Term_{2,1}}\\
&\underbrace{(1+ic'(\alpha-\beta)\gamma t)X(\alpha-\beta,\gamma)(\breve{h}(\alpha,\gamma)-\breve{h}(\alpha-\beta,\gamma))d\beta}_{Term_{2,1}}\\
&\underbrace{+\int_{-\alpha}^{\alpha}\nabla\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(\alpha-\beta,\gamma))\cdot[(\frac{ic(\alpha-\beta)t}{1+ic'(\alpha-\beta)\gamma t}-\frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t})\partial_\alpha\ddot{h}(\alpha-\beta,\gamma)]}_{Term_{2,2}}
\end{aligned}
$$

$$\underbrace{(1+ic'(\alpha-\beta)\gamma t)X(\alpha-\beta,\gamma)(\breve{h}(\alpha,\gamma)-\breve{h}(\alpha-\beta,\gamma))d\beta}_{Term_{2,2}}$$

$$\underbrace{+\int_{-\alpha}^{\alpha}\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(\alpha-\beta,\gamma))([(\frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t}\partial_\alpha-\partial_\gamma)\breve{h}(\alpha,\gamma)]-[(\frac{ic(\alpha-\beta)t}{1+ic'(\alpha-\beta)\gamma t}\partial_\alpha-\partial_\gamma)\breve{h}(\alpha-\beta,\gamma)])}_{Term_{3,1}}$$

$$\underbrace{X(\alpha-\beta,\gamma)(1+ic'(\alpha-\beta)\gamma t)d\beta}_{Term_{3,1}}$$

$$\underbrace{+\int_{-\alpha}^{\alpha}\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(\alpha-\beta,\gamma))[(\frac{ic(\alpha-\beta)t}{1+ic'(\alpha-\beta)\gamma t}-\frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t})\partial_\alpha)\breve{h}(\alpha-\beta,\gamma)]}_{Term_{3,2}}$$

$$\underbrace{X(\alpha-\beta,\gamma)(1+ic'(\alpha-\beta)\gamma t)d\beta}_{Term_{3,2}}$$

$$\underbrace{+\int_{-\alpha}^{\alpha}\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(\alpha-\beta,\gamma))(\breve{h}(\alpha,\gamma)-\breve{h}(\alpha-\beta,\gamma))[(\frac{ic(\alpha-\beta)t}{1+ic'(\alpha-\beta)\gamma t}\partial_\alpha-\partial_\gamma)X(\alpha-\beta,\gamma)]}_{Term_{4,1}}$$

$$(1+ic'(\alpha-\beta)\gamma t)d\beta_{Term_{4,1}}$$

$$\underbrace{+\int_{-\alpha}^{\alpha}\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(\alpha-\beta,\gamma))(\breve{h}(\alpha,\gamma)-\breve{h}(\alpha-\beta,\gamma))[(\frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t}\partial_\alpha-\frac{ic(\alpha-\beta)t}{1+ic'(\alpha-\beta)\gamma t}\partial_\alpha)X(\alpha-\beta,\gamma)]}_{Term_{4,2}}$$

$$\underbrace{(1+ic'(\alpha-\beta)\gamma t)d\beta}_{Term_{4,2}}$$

$$\underbrace{+\int_{-\alpha}^{\alpha}\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(\alpha-\beta,\gamma))(\breve{h}(\alpha,\gamma)-\breve{h}(\alpha-\beta,\gamma))X(\alpha-\beta,\gamma)}_{Term_5}$$

$$\underbrace{[\frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t}(ic''(\alpha-\beta)\gamma t)-ic'(\alpha-\beta)t]d\beta}_{Term_5}$$

$$=Term_{2,1}+Term_{2,2}+Term_{3,1}+Term_{3,2}+Term_{4,1}+Term_{4,2}+Term_5.$$

We have

$$Term_{2,2}+Term_{3,2}+Term_{4_2}+Term_5$$

$$=-\int_{-\alpha}^{\alpha}\frac{d}{d\beta}[\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(\alpha-\beta,\gamma))(\frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t}-\frac{ic(\alpha-\beta)t}{1+ic'(\alpha-\beta)\gamma t})(1+ic'(\alpha-\beta)\gamma t)$$

$$X(\alpha-\beta,\gamma)(\breve{h}(\alpha,\gamma)-\breve{h}(\alpha-\beta,\gamma))]d\beta$$

$$=-\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(0,\gamma))(\frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t}-\frac{ic(0)t}{1+ic'(0)\gamma t})(1+ic'(0)\gamma t)(\breve{h}(\alpha,\gamma)-\breve{h}(0,\gamma))X(0,\gamma)$$

$$+\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(2\alpha,\gamma))(\frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t}-\frac{ic(2\alpha)t}{1+ic'(2\alpha)\gamma t})(\breve{h}(\alpha,\gamma)-\breve{h}(2\alpha,\gamma))X(2\alpha,\gamma)(1+ic'(2\alpha)\gamma t).$$

Therefore we have

$$\begin{aligned}
A_0(O_1) =& \underbrace{\frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t}[\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(0,\gamma))(\breve{h}(\alpha,\gamma)-\breve{h}(0,\gamma))X(0,\gamma)(1+ic'(0)\gamma t)}_{B_1}\\
&\underbrace{+\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(2\alpha,\gamma))(\breve{h}(\alpha,\gamma)-\breve{h}(2\alpha,\gamma))X(2\alpha,\gamma)(1+ic'(2\alpha)\gamma t)]}_{B_1}\\
&+\underbrace{Term_{2,1}}_{FA_1}+\underbrace{Term_{3,1}}_{FA_2}++\underbrace{Term_{4,1}}_{FA_3}\\
&\underbrace{-\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(0,\gamma))(\frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t}-\frac{ic(0)t}{1+ic'(0)\gamma t})(1+ic'(0)\gamma t)(\breve{h}(\alpha,\gamma)-\breve{h}(0,\gamma))X(0,\gamma)}_{B_2}\\
&\underbrace{+\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(2\alpha,\gamma))(\frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t}-\frac{ic(2\alpha)t}{1+ic'(2\alpha)\gamma t})(\breve{h}(\alpha,\gamma)-\breve{h}(2\alpha,\gamma))X(2\alpha,\gamma)(1+ic'(2\alpha)\gamma t)}_{B_3}\\
&=\underbrace{Term_{2,1}}_{FA_1}+\underbrace{Term_{3,1}}_{FA_2}+\underbrace{Term_{4,1}}_{FA_3}\\
&\underbrace{+\frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t}[\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(0,\gamma))(\breve{h}(\alpha,\gamma))X(0,\gamma)(1+ic'(0)\gamma t)}_{B_{1,1}}\\
&\underbrace{+\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(2\alpha,\gamma))(\breve{h}(\alpha,\gamma))X(2\alpha,\gamma)(1+ic'(2\alpha)\gamma t)]}_{B_{1,1}}\\
&\underbrace{+\frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t}[\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(0,\gamma))(-\breve{h}(0,\gamma))X(0,\gamma)(1+ic'(0)\gamma t)}_{B_{1,2}}\\
&\underbrace{+\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(2\alpha,\gamma))(-\breve{h}(2\alpha,\gamma))X(2\alpha,\gamma)(1+ic'(2\alpha)\gamma t)]}_{B_{1,2}}\\
&\underbrace{-\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(0,\gamma))(\frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t}-\frac{ic(0)t}{1+ic'(0)\gamma t})(1+ic'(0)\gamma t)(\breve{h}(\alpha,\gamma))X(0,\gamma)}_{B_{2,1}}\\
&\underbrace{-\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(0,\gamma))(\frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t}-\frac{ic(0)t}{1+ic'(0)\gamma t})(1+ic'(0)\gamma t)(-\breve{h}(0,\gamma))X(0,\gamma)}_{B_{2,2}}\\
&\underbrace{+\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(2\alpha,\gamma))(\frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t}-\frac{ic(2\alpha)t}{1+ic'(2\alpha)\gamma t})(\breve{h}(\alpha,\gamma))X(2\alpha,\gamma)(1+ic'(2\alpha)\gamma t)}_{B_{3,1}}\\
&\underbrace{+\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(2\alpha,\gamma))(\frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t}-\frac{ic(2\alpha)t}{1+ic'(2\alpha)\gamma t})(-\breve{h}(2\alpha,\gamma))X(2\alpha,\gamma)(1+ic'(2\alpha)\gamma t)}_{B_{3,2}}.
\end{aligned}$$

Here $B_1 = B_{1,1} + B_{1,2}$, $B_2 = B_{2,1} + B_{2,2}$, $B_3 = B_{3,1} + B_{3,2}$.

Then we consider $A_0(O_2)$, we have

$$
\begin{aligned}
&A_0(O_2) = A(\int_0^\gamma \breve{h}(2\alpha,\eta)\breve{K}(\ddot{h}(\alpha,\gamma) - \ddot{h}(2\alpha,\eta))X(2\alpha,\eta)ic(2\alpha)td\eta)\\
&= \underbrace{\int_0^\gamma (\frac{2ic(\alpha)t}{1+ic'(\alpha)\gamma t}(\partial_\alpha h)(2\alpha,\eta))X(2\alpha,\eta)\breve{K}(\ddot{h}(\alpha,\gamma) - \ddot{h}(2\alpha,\eta))ic(2\alpha)td\eta}_{Term_1}\\
&+ \underbrace{\int_0^\gamma \breve{h}(2\alpha,\eta)(\frac{2ic(\alpha)t}{1+ic'(\alpha)\gamma t}(\partial_\alpha X)(2\alpha,\eta))\breve{K}(\ddot{h}(\alpha,\gamma) - \ddot{h}(2\alpha,\eta))ic(2\alpha)td\eta}_{Term_2}\\
&\underbrace{-\breve{h}(2\alpha,\gamma)X(2\alpha,\gamma)\breve{K}(\ddot{h}(\alpha,\gamma) - \ddot{h}(2\alpha,\gamma))ic(2\alpha)t}_{Term_3}\\
&\underbrace{+\int_0^\gamma \breve{h}(2\alpha,\eta)X(2\alpha,\eta)\nabla\breve{K}(\ddot{h}(\alpha,\gamma) - \ddot{h}(2\alpha,\eta))\cdot(\frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t}(\partial_\alpha\ddot{h})(\alpha,\gamma) - (\partial_\gamma\ddot{h})(\alpha,\gamma))ic(2\alpha)td\eta}_{Term_4 = FA_4}\\
&\underbrace{-\int_0^\gamma \breve{h}(2\alpha,\eta)X(2\alpha,\eta)\nabla\breve{K}(\ddot{h}(\alpha,\gamma) - \ddot{h}(2\alpha,\eta))\cdot(\frac{2ic(\alpha)t}{1+ic'(\alpha)\gamma t}(\partial_\alpha h)(2\alpha,\eta))ic(2\alpha)td\eta}_{Term_5}\\
&\underbrace{+\int_0^\gamma \breve{h}(2\alpha,\eta)X(2\alpha,\eta)\breve{K}(\ddot{h}(\alpha,\gamma) - \ddot{h}(2\alpha,\eta))\frac{i2c(\alpha)t}{1+ic'(\alpha)\gamma t}ic'(2\alpha)td\eta}_{Term_6}\\
&= Term_1 + Term_2 + Term_3 + Term_4 + Term_5 + Term_6
\end{aligned}
$$

Now we do the change to the terms 1,2,5 by writing the terms as $A_0(\breve{h})$.

$$
\begin{aligned}
&Term_1 + Term_2 + Term_5 =\\
&\underbrace{\int_0^\gamma ic(2\alpha)t(\frac{2ic(\alpha)t}{1+ic'(\alpha)\gamma t})[(\frac{1}{2}\frac{d}{d\alpha} - \frac{1+ic'(2\alpha)\eta t}{ic(2\alpha)t}\frac{d}{d\eta})\breve{h}(2\alpha,\eta)]X(2\alpha,\eta)\breve{K}(\ddot{h}(\alpha,\gamma) - \ddot{h}(2\alpha,\eta))d\eta}_{Term_{1,1}}\\
&\underbrace{+\int_0^\gamma ic(2\alpha)t(\frac{2ic(\alpha)t}{1+ic'(\alpha)\gamma t})[(\frac{1+ic'(2\alpha)\eta t}{ic(2\alpha)t}\frac{d}{d\eta})\breve{h}(2\alpha,\eta)]X(2\alpha,\eta)\breve{K}(\ddot{h}(\alpha,\gamma) - \ddot{h}(2\alpha,\eta))d\eta}_{Term_{1,2}}\\
&\underbrace{+\int_0^\gamma ic(2\alpha)t(\frac{2ic(\alpha)t}{1+ic'(\alpha)\gamma t})[(\frac{1}{2}\frac{d}{d\alpha} - \frac{1+ic'(2\alpha)\eta t}{ic(2\alpha)t}\frac{d}{d\eta})X(2\alpha,\eta)]\breve{h}(2\alpha,\eta)\breve{K}(\ddot{h}(\alpha,\gamma) - \ddot{h}(2\alpha,\eta))d\eta}_{Term_{2,1}}\\
&\underbrace{+\int_0^\gamma ic(2\alpha)t(\frac{2ic(\alpha)t}{1+ic'(\alpha)\gamma t})[(\frac{1+ic'(2\alpha)\eta t}{ic(2\alpha)t}\frac{d}{d\eta})X(2\alpha,\eta)]\breve{h}(2\alpha,\eta)\breve{K}(\ddot{h}(\alpha,\gamma) - \ddot{h}(2\alpha,\eta))d\eta}_{Term_{2,2}}\\
&\underbrace{-\int_0^\gamma \breve{h}(2\alpha,\eta)X(2\alpha,\eta)\nabla\breve{K}(\ddot{h}(\alpha,\gamma) - \ddot{h}(2\alpha,\eta))\cdot(\frac{2ic(\alpha)t}{1+ic'(\alpha)\gamma t})[(\frac{1}{2}\frac{d}{d\alpha} - \frac{1+ic'(2\alpha)\eta t}{ic(2\alpha)t}\frac{d}{d\eta})\ddot{h}(2\alpha,\eta)]ic(2\alpha)td\eta}_{Term_{5,1}}
\end{aligned}
$$

$$\underbrace{-\int_0^\gamma \breve{h}(2\alpha,\eta)X(2\alpha,\eta)\nabla\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(2\alpha,\eta))\cdot(\frac{2ic(\alpha)t}{1+ic'(\alpha)\gamma t})[(\frac{1+ic'(2\alpha)\eta t}{ic(2\alpha)t}\frac{d}{d\eta})\ddot{h}(2\alpha,\eta)]ic(2\alpha)td\eta}_{Term_{5,2}}$$

$$= Term_{1,1}+Term_{1,2}+Term_{2,1}+Term_{2,2}+Term_{5,1}+Term_{5,2}.$$

Then, we have

$$\begin{aligned}
&Term_{1,2}+Term_{2,2}+Term_{5,2}+Term_6\\
&=\int_0^\gamma ic(2\alpha)t(\frac{2ic(\alpha)t}{1+ic'(\alpha)\gamma t})[(\frac{1+ic'(2\alpha)\eta t}{ic(2\alpha)t}\frac{d}{d\eta})\breve{h}(2\alpha,\eta)]X(2\alpha,\eta)\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(2\alpha,\eta))d\eta\\
&+\int_0^\gamma ic(2\alpha)t(\frac{2ic(\alpha)t}{1+ic'(\alpha)\gamma t})[(\frac{1+ic'(2\alpha)\eta t}{ic(2\alpha)t}\frac{d}{d\eta})X(2\alpha,\eta)]\breve{h}(2\alpha,\eta)\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(2\alpha,\eta))d\eta\\
&-\int_0^\gamma \breve{h}(2\alpha,\eta)X(2\alpha,\eta)\nabla\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(2\alpha,\eta))\cdot(\frac{2ic(\alpha)t}{1+ic'(\alpha)\gamma t})[(\frac{1+ic'(2\alpha)\eta t}{ic(2\alpha)t}\frac{d}{d\eta})\ddot{h}(2\alpha,\eta)]ic(2\alpha)td\eta\\
&+\int_0^\gamma \breve{h}(2\alpha,\eta)X(2\alpha,\eta)\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(2\alpha,\eta))\frac{i2c(\alpha)t}{1+ic'(\alpha)\gamma t}ic'(2\alpha)td\eta\\
&=\int_0^\gamma \frac{d}{d\eta}[\breve{h}(2\alpha,\eta)X(2\alpha,\eta)\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(2\alpha,\eta))\frac{i2c(\alpha)t}{1+ic'(\alpha)\gamma t}(ic'(2\alpha)\eta t+1)]d\eta\\
&=(\breve{h}(2\alpha,\gamma)X(2\alpha,\gamma)\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(2\alpha,\gamma))\frac{i2c(\alpha)t}{1+ic'(\alpha)\gamma t}(ic'(2\alpha)\gamma t+1))\\
&-\breve{h}(2\alpha,0)X(2\alpha,0)\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(2\alpha,0))\frac{i2c(\alpha)t}{1+ic'(\alpha)\gamma t}.
\end{aligned}$$

Moreover, we have

$$\begin{aligned}
Term_{1,1}&=\int_0^\gamma ic(2\alpha)t(\frac{2ic(\alpha)t}{1+ic'(\alpha)\gamma t})[(\frac{1}{2}\frac{d}{d\alpha}-\frac{1+ic'(2\alpha)\eta t}{ic(2\alpha)t}\frac{d}{d\eta})\breve{h}(2\alpha,\eta)]X(2\alpha,\eta)\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(2\alpha,\eta))d\eta\\
&=\int_0^\gamma ic(2\alpha)t(\frac{2ic(\alpha)t}{1+ic'(\alpha)\gamma t})(\frac{1+ic'(2\alpha)\eta t}{ic(2\alpha)t})[(\frac{ic(2\alpha)t}{2(1+ic'(2\alpha)\eta t)}\frac{d}{d\alpha}-\frac{d}{d\eta})\breve{h}(2\alpha,\eta)]X(2\alpha,\eta)\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(2\alpha,\eta))d\eta\\
&=\underbrace{\int_0^\gamma ic(2\alpha)t(\frac{2ic(\alpha)t}{1+ic'(\alpha)\gamma t})(\frac{1+ic'(2\alpha)\eta t}{ic(2\alpha)t})[A_0(\breve{h})(2\alpha,\eta)]\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(2\alpha,\eta))X(2\alpha,\eta)d\eta,}_{FA_7}
\end{aligned}$$

$$\begin{aligned}
Term_{2,1}&=\int_0^\gamma ic(2\alpha)t(\frac{2ic(\alpha)t}{1+ic'(\alpha)\gamma t})[(\frac{1}{2}\frac{d}{d\alpha}-\frac{1+ic'(2\alpha)\eta t}{ic(2\alpha)t}\frac{d}{d\eta})X(2\alpha,\eta)]\breve{h}(2\alpha,\eta)\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(2\alpha,\eta))d\eta\\
&=\int_0^\gamma ic(2\alpha)t(\frac{2ic(\alpha)t}{1+ic'(\alpha)\gamma t})(\frac{1+ic'(2\alpha)\eta t}{ic(2\alpha)t})[(\frac{ic(2\alpha)t}{2(1+ic'(2\alpha)\eta t)}\frac{d}{d\alpha}-\frac{d}{d\eta})X(2\alpha,\eta)]\breve{h}(2\alpha,\eta)\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(2\alpha,\eta))d\eta\\
&=\underbrace{\int_0^\gamma ic(2\alpha)t(\frac{2ic(\alpha)t}{1+ic'(\alpha)\gamma t})(\frac{1+ic'(2\alpha)\eta t}{ic(2\alpha)t})[A_0(X)(2\alpha,\eta)]\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(2\alpha,\eta))\breve{h}(2\alpha,\eta)d\eta,}_{FA_6}
\end{aligned}$$

and

$$Term_{5,1} = -\int_0^\gamma \breve{h}(2\alpha,\eta)X(2\alpha,\eta)\nabla\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(2\alpha,\eta))\cdot(\frac{2ic(\alpha)t}{1+ic'(\alpha)\gamma t})[(\frac{1}{2}\frac{d}{d\alpha}-\frac{1+ic'(2\alpha)\eta t}{ic(2\alpha)t}\frac{d}{d\eta})\ddot{h}(2\alpha,\eta)]$$

$$ic(2\alpha)td\eta$$

$$= -\int_0^\gamma \breve{h}(2\alpha,\eta)X(2\alpha,\eta)\nabla\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(2\alpha,\eta))\cdot(\frac{2ic(\alpha)t}{1+ic'(\alpha)\gamma t})(\frac{1+ic'(2\alpha)\eta t}{ic(2\alpha)t})$$

$$\cdot[(\frac{ic(2\alpha)t}{2(1+ic'(2\alpha)\eta t)}\frac{d}{d\alpha}-\frac{d}{d\eta})\ddot{h}(2\alpha,\eta)]ic(2\alpha)td\eta$$

$$= \underbrace{-\int_0^\gamma \breve{h}(2\alpha,\eta)X(2\alpha,\eta)\nabla\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(2\alpha,\eta))\cdot(\frac{2ic(\alpha)t}{1+ic'(\alpha)\gamma t})(\frac{1+ic'(2\alpha)\eta t}{ic(2\alpha)t})[(A_0\ddot{h})(2\alpha,\eta)]ic(2\alpha)td\eta}_{FA_5}.$$

Therefore we have

$$A_0(O_2) = Term_{1,1}+Term_{2,1}+Term_{5,1}+Term_4+Term_3+(Term_{1,2}+Term_{2,2}+Term_{5,2}+Term_6)$$

$$= \underbrace{\int_0^\gamma ic(2\alpha)t(\frac{2ic(\alpha)t}{1+ic'(\alpha)\gamma t})(\frac{1+ic'(2\alpha)\eta t}{ic(2\alpha)t})[A_0(\breve{h})(2\alpha,\eta)]\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(2\alpha,\eta))X(2\alpha,\eta)d\eta}_{FA_7}$$

$$+\underbrace{\int_0^\gamma ic(2\alpha)t(\frac{2ic(\alpha)t}{1+ic'(\alpha)\gamma t})(\frac{1+ic'(2\alpha)\eta t}{ic(2\alpha)t})[A_0(X)(2\alpha,\eta)]\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(2\alpha,\eta))\breve{h}(2\alpha,\eta)d\eta}_{FA_6}$$

$$\underbrace{-\int_0^\gamma \breve{h}(2\alpha,\eta)X(2\alpha,\eta)\nabla\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(2\alpha,\eta))\cdot(\frac{2ic(\alpha)t}{1+ic'(\alpha)\gamma t})(\frac{1+ic'(2\alpha)\eta t}{ic(2\alpha)t})[A_0(\ddot{h})(2\alpha,\eta)]ic(2\alpha)td\eta}_{FA_5}$$

$$\underbrace{+\int_0^\gamma \breve{h}(2\alpha,\eta)X(2\alpha,\eta)\nabla\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(2\alpha,\eta))\cdot[(A_0(\ddot{h})(\alpha,\gamma)]ic(2\alpha)td\eta}_{FA_4}$$

$$\underbrace{-\breve{h}(2\alpha,\gamma)X(2\alpha,\gamma)\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(2\alpha,\gamma))ic(2\alpha)t}_{B_4}$$

$$\underbrace{+\breve{h}(2\alpha,\gamma)X(2\alpha,\gamma)\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(2\alpha,\gamma))\frac{i2c(\alpha)t}{1+ic'(\alpha)\gamma t}(ic'(2\alpha)\gamma t+1)}_{B_5}$$

$$\underbrace{-\breve{h}(2\alpha,0)X(2\alpha,0)\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(2\alpha,0))\frac{i2c(\alpha)t}{1+ic'(\alpha)\gamma t}}_{B_6}$$

For O_3, we have

$$A_0(O_3) = A_0(-\int_{2\alpha}^\pi X(\beta,0)\breve{h}(\beta,0)\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(\beta,0))d\beta)$$

$$= \underbrace{\frac{2ic(\alpha)t}{1+ic'(\alpha)\gamma t}\breve{h}(2\alpha,0)X(2\alpha,0)\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(2\alpha,0))}_{B_7}$$
$$\underbrace{-\int_{2\alpha}^{\pi} X(\beta,0)\breve{h}(\beta,0)\nabla\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(\beta,0))\cdot(\frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t}\partial_\alpha-\partial_\gamma)\ddot{h}(\alpha,\gamma)d\beta}_{FA_8}$$

Finally, for O_4, we have

$$A(O_4)=A_0(\int_{[\alpha-\pi,-\alpha]\cup[\alpha,\alpha+\pi]}\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(\alpha-\beta,\gamma))(X(\alpha-\beta,\gamma))(1+ic'(\alpha-\beta)\gamma t)d\beta\breve{h}(\alpha,\gamma))$$
$$=\underbrace{\int_{[-\pi,0]\cup[2\alpha,\pi]}\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(\beta,\gamma))(X(\beta,\gamma))(1+ic'(\beta)\gamma t)d\beta A_0(\breve{h})(\alpha,\gamma)}_{FA_{11}}$$
$$\underbrace{+\frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t}\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(-\pi,\gamma))X(-\pi,\gamma)(1+ic'(-\pi)\gamma t)\breve{h}(\alpha,\gamma)}_{Term_{1,1},B_{8,1}}$$
$$\underbrace{-\frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t}\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(0,\gamma))X(0,\gamma)(1+ic'(0)\gamma t)\breve{h}(\alpha,\gamma)}_{Term_{1,2},B_{8,2}}$$
$$\underbrace{-\frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t}\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(2\alpha,\gamma))X(2\alpha,\gamma)(1+ic'(2\alpha)\gamma t)\breve{h}(\alpha,\gamma)}_{Term_{1,3},B_{8,3}}$$
$$\underbrace{-\frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t}\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(\pi,\gamma))X(\pi,\gamma)(1+ic'(\pi)\gamma t)\breve{h}(\alpha,\gamma)}_{Term_{1,4},B_{8,4}}$$
$$+\underbrace{\int_{[\alpha-\pi,-\alpha]\cup[\alpha,\alpha+\pi]}X(\alpha-\beta,\gamma)\nabla\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(\alpha-\beta,\gamma))}_{Term_2}$$
$$\underbrace{\cdot([(\frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t}\partial_\alpha-\partial_\gamma)\ddot{h}(\alpha,\gamma)]-[(\frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t}\partial_\alpha-\partial_\gamma)\ddot{h}(\alpha-\beta,\gamma,t)])(1+ic'(\alpha-\beta)\gamma t)d\beta\breve{h}(\alpha,\gamma)}_{Term_2}$$
$$+\underbrace{\int_{[\alpha-\pi,-\alpha]\cup[\alpha,\alpha+\pi]}\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(\alpha-\beta,\gamma))[(\frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t}\partial_\alpha-\partial_\gamma)X(\alpha-\beta,\gamma)](1+ic'(\alpha-\beta)\gamma t)d\beta\breve{h}(\alpha,\gamma)}_{Term_3}$$
$$+\underbrace{\int_{[\alpha-\pi,-\alpha]\cup[\alpha,\alpha+\pi]}\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(\alpha-\beta,\gamma))X(\alpha-\beta,\gamma)(\frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t}ic''(\alpha-\beta)\gamma t-ic'(\alpha-\beta)t)d\beta\breve{h}(\alpha,\gamma)}_{Term_4}$$
$$=FA_{11}+Term_{1,1}+Term_{1,2}+Term_{1,3}+Term_{1,4}+$$
$$+\underbrace{\int_{[\alpha-\pi,-\alpha]\cup[\alpha,\alpha+\pi]}X(\alpha-\beta,\gamma)\nabla\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(\alpha-\beta,\gamma))}_{Term_{2,1}}$$

$$\underbrace{\cdot([(\frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t}\partial_\alpha-\partial_\gamma)\ddot{h}(\alpha,\gamma)]-[(\frac{ic(\alpha-\beta)t}{1+ic'(\alpha-\beta)\gamma t}\partial_\alpha-\partial_\gamma)\ddot{h}(\alpha-\beta,\gamma)])(1+ic'(\alpha-\beta)\gamma t)d\beta\breve{h}(\alpha,\gamma)}_{Term_{2,1}}$$

$$+\underbrace{\int_{[\alpha-\pi,-\alpha]\cup[\alpha,\alpha+\pi]}\nabla\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(\alpha-\beta,\gamma))\cdot[(\frac{ic(\alpha-\beta)t}{1+ic'(\alpha-\beta)\gamma t}\partial_\alpha-\frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t}\partial_\alpha)\ddot{h}(\alpha-\beta,\gamma)]}_{Term_{2,2}}$$

$$\underbrace{X(\alpha-\beta,\gamma)(1+ic'(\alpha-\beta)\gamma t)d\beta\breve{h}(\alpha,\gamma)}_{Term_{2,2}}$$

$$+\underbrace{\int_{[\alpha-\pi,-\alpha]\cup[\alpha,\alpha+\pi]}\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(\alpha-\beta,\gamma))([\frac{ic(\alpha-\beta)t}{1+ic'(\alpha-\beta)\gamma t}\partial_\alpha-\partial_\gamma)X(\alpha-\beta,\gamma)](1+ic'(\alpha-\beta)\gamma t)d\beta\breve{h}(\alpha,\gamma)}_{Term_{3,1}}$$

$$+\underbrace{\int_{[\alpha-\pi,-\alpha]\cup[\alpha,\alpha+\pi]}\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(\alpha-\beta,\gamma))[(\frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t}\partial_\alpha-\frac{ic(\alpha-\beta)t}{1+ic'(\alpha-\beta)\gamma t}\partial_\alpha)X(\alpha-\beta,\gamma)]}_{Term_{3,2}}$$

$$\underbrace{(1+ic'(\alpha-\beta)\gamma t)d\beta\breve{h}(\alpha,\gamma)}_{Term_{3,2}}$$

$$+\underbrace{\int_{[\alpha-\pi,-\alpha]\cup[\alpha,\alpha+\pi]}\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(\alpha-\beta,\gamma))X(\alpha-\beta,\gamma)(\frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t}ic''(\alpha-\beta)\gamma t-ic'(\alpha-\beta)t)d\beta\breve{h}(\alpha,\gamma)}_{Term_4}$$

$$=FA_{11}+Term_{1,1}+Term_{1,2}+Term_{1,3}+Term_{1,4}+Term_{2,1}+Term_{2,2}+Term_{3,1}+Term_{3,2}+Term_4.$$

Then we have

$$Term_{2,2}+Term_{3,2}+Term_4=$$

$$=\int_{[\alpha-\pi,-\alpha]\cup[\alpha,\alpha+\pi]}\frac{d}{d\beta}[\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(\alpha-\beta,\gamma))X(\alpha-\beta,\gamma)(ic(\alpha-\beta)t-\frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t}(1+ic'(\alpha-\beta)\gamma t))]d\beta$$

$$\cdot\breve{h}(\alpha,\gamma)$$

$$=\underbrace{\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(-\pi,\gamma))X(-\pi,\gamma)(ic(-\pi)t-\frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t}(1+ic'(-\pi)\gamma t))\breve{h}(\alpha,\gamma)}_{B_{9,1}}$$

$$\underbrace{-\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(0,\gamma))X(0,\gamma)(ic(0)t-\frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t}(1+ic'(0)\gamma t))\breve{h}(\alpha,\gamma)}_{B_{9,2}}$$

$$\underbrace{+\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(2\alpha,\gamma))X(2\alpha,\gamma)(ic(2\alpha)t-\frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t}(1+ic'(2\alpha)\gamma t))\breve{h}(\alpha,\gamma)}_{B_{9,3}}$$

$$\underbrace{-\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(\pi,\gamma))X(\pi,\gamma)(ic(\pi)t-\frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t}(1+ic'(\pi)\gamma t))\breve{h}(\alpha,\gamma).}_{B_{9,4}}$$

Then

$$A_0(O_4) = FA_{11} + \underbrace{Term_{1,1} + Term_{1,2} + Term_{1,3} + Term_{1,4}}_{B_8} + \underbrace{Term_{2,1}}_{FA_9}$$
$$+ \underbrace{Term_{3,1}}_{FA_{10}} + \underbrace{Term_{2,2} + Term_{3,2} + Term_{4}}_{B_9}.$$

Here we use B_8 to show the sum $B_{8,1} + B_{8,2} + B_{8,3} + B_{8,4}$, and B_9 to show the sum $B_{9,1} + B_{9,2} + B_{9,3} + B_{9,4} = B_9$.

Then we are left to show all B terms are cancelled.

$$\begin{aligned}
&B_1 + B_2 + B_3 + B_4 + B_5 + B_6 + B_7 + B_8 + B_9 \\
&= \underbrace{\frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t}\breve{K}(\ddot{h}(\alpha,\gamma) - \ddot{h}(0,\gamma))(\breve{h}(\alpha,\gamma))X(0,\gamma)(1+ic'(0)\gamma t)}_{B_{1,1}} \\
&+ \underbrace{\frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t}\breve{K}(\ddot{h}(\alpha,\gamma) - \ddot{h}(2\alpha,\gamma))(\breve{h}(\alpha,\gamma))X(2\alpha,\gamma)(1+ic'(2\alpha)\gamma t)}_{B_{1,1}} \\
&\underbrace{+\frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t}\breve{K}(\ddot{h}(\alpha,\gamma) - \ddot{h}(0,\gamma))(-\breve{h}(0,\gamma))X(0,\gamma)(1+ic'(0)\gamma t)}_{B_{1,2}} \\
&+ \underbrace{\frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t}\breve{K}(\ddot{h}(\alpha,\gamma) - \ddot{h}(2\alpha,\gamma))(-\breve{h}(2\alpha,\gamma))X(2\alpha,\gamma)(1+ic'(2\alpha)\gamma t)}_{B_{1,2}} \\
&\underbrace{-\breve{K}(\ddot{h}(\alpha,\gamma) - \ddot{h}(0,\gamma))(\frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t} - \frac{ic(0)t}{1+ic'(0)\gamma t})(1+ic'(0)\gamma t)(\breve{h}(\alpha,\gamma))X(0,\gamma)}_{B_{2,1}} \\
&\underbrace{-\breve{K}(\ddot{h}(\alpha,\gamma) - \ddot{h}(0,\gamma))(\frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t} - \frac{ic(0)t}{1+ic'(0)\gamma t})(1+ic'(0)\gamma t)(-\breve{h}(0,\gamma))X(0,\gamma)}_{B_{2,2}} \\
&+ \underbrace{\breve{K}(\ddot{h}(\alpha,\gamma) - \ddot{h}(2\alpha,\gamma))(\frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t} - \frac{ic(2\alpha)t}{1+ic'(2\alpha)\gamma t})(\breve{h}(\alpha,\gamma))X(2\alpha,\gamma)(1+ic'(2\alpha)\gamma t)}_{B_{3,1}} \\
&+ \underbrace{\breve{K}(\ddot{h}(\alpha,\gamma) - \ddot{h}(2\alpha,\gamma))(\frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t} - \frac{ic(2\alpha)t}{1+ic'(2\alpha)\gamma t})(-\breve{h}(2\alpha,\gamma))X(2\alpha,\gamma)(1+ic'(2\alpha)\gamma t)}_{B_{3,2}} \\
&\underbrace{-\breve{h}(2\alpha,\gamma)X(2\alpha,\gamma)\breve{K}(\ddot{h}(\alpha,\gamma) - \ddot{h}(2\alpha,\gamma))ic(2\alpha)t}_{B_4} \\
&\underbrace{+\breve{h}(2\alpha,\gamma)X(2\alpha,\gamma)\breve{K}(\ddot{h}(\alpha,\gamma) - \ddot{h}(2\alpha,\gamma))\frac{i2c(\alpha)t}{1+ic'(\alpha)\gamma t}(ic'(2\alpha)\gamma t + 1)}_{B_5}
\end{aligned}$$

$$\underbrace{-\breve{h}(2\alpha,0)X(2\alpha,0)\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(2\alpha,0))\frac{i2c(\alpha)t}{1+ic'(\alpha)\gamma t}}_{B_6}$$

$$\underbrace{+\frac{2ic(\alpha)t}{1+ic'(\alpha)\gamma t}\breve{h}(2\alpha,0)X(2\alpha,0)\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(2\alpha,0))}_{B_7}$$

$$\underbrace{+\frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t}\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(-\pi,\gamma))X(-\pi,\gamma)(1+ic'(-\pi)\gamma t)\breve{h}(\alpha,\gamma)}_{B_{8,1}}$$

$$\underbrace{-\frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t}\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(0,\gamma))X(0,\gamma)(1+ic'(0)\gamma t)\breve{h}(\alpha,\gamma)}_{B_{8,2}}$$

$$\underbrace{-\frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t}\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(2\alpha,\gamma))X(2\alpha,\gamma)(1+ic'(2\alpha)\gamma t)\breve{h}(\alpha,\gamma)}_{B_{8,3}}$$

$$\underbrace{-\frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t}\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(\pi,\gamma))X(\pi,\gamma)(1+ic'(\pi)\gamma t)\breve{h}(\alpha,\gamma)}_{B_{8,4}}$$

$$\underbrace{+\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(-\pi,\gamma))X(-\pi,\gamma)(ic(-\pi)t-\frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t}(1+ic'(-\pi)\gamma t))\breve{h}(\alpha,\gamma)}_{B_{9,1}}$$

$$\underbrace{-\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(0,\gamma))X(0,\gamma)(ic(0)t-\frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t}(1+ic'(0)\gamma t))\breve{h}(\alpha,\gamma)}_{B_{9,2}}$$

$$\underbrace{+\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(2\alpha,\gamma))X(2\alpha,\gamma)(ic(2\alpha)t-\frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t}(1+ic'(2\alpha)\gamma t))\breve{h}(\alpha,\gamma)}_{B_{9,3}}$$

$$\underbrace{-\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(\pi,\gamma))X(\pi,\gamma)(ic(\pi)t-\frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t}(1+ic'(\pi)\gamma t))\breve{h}(\alpha,\gamma)}_{B_{9,4}}$$

$$=\underbrace{\frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t}\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(0,\gamma))(\breve{h}(\alpha,\gamma))X(0,\gamma)(1+ic'(0)\gamma t)}_{C_1}$$

$$\underbrace{+\frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t}\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(2\alpha,\gamma))(\breve{h}(\alpha,\gamma))X(2\alpha,\gamma)(1+ic'(2\alpha)\gamma t)}_{C_2}$$

$$\underbrace{+\frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t}\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(0,\gamma))(-\breve{h}(0,\gamma))X(0,\gamma)(1+ic'(0)\gamma t)}_{C_3}$$

$$\underbrace{+\frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t}\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(2\alpha,\gamma))(-\breve{h}(2\alpha,\gamma))X(2\alpha,\gamma)(1+ic'(2\alpha)\gamma t)}_{C_4}$$

$$\underbrace{-\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(0,\gamma))(\frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t}-\frac{ic(0)t}{1+ic'(0)\gamma t})(1+ic'(0)\gamma t)(\breve{h}(\alpha,\gamma))X(0,\gamma)}_{C_5}$$

$$\underbrace{-\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(0,\gamma))(\frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t}-\frac{ic(0)t}{1+ic'(0)\gamma t})(1+ic'(0)\gamma t)(-\breve{h}(0,\gamma))X(0,\gamma)}_{C_3}$$

$$\underbrace{+\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(2\alpha,\gamma))(\frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t}-\frac{ic(2\alpha)t}{1+ic'(2\alpha)\gamma t})(\breve{h}(\alpha,\gamma))X(2\alpha,\gamma)(1+ic'(2\alpha)\gamma t)}_{C_6}$$

$$+\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(2\alpha,\gamma))(\underbrace{\frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t}}_{C_4}\underbrace{-\frac{ic(2\alpha)t}{1+ic'(2\alpha)\gamma t}}_{C_7})(-\breve{h}(2\alpha,\gamma))X(2\alpha,\gamma)(1+ic'(2\alpha)\gamma t)$$

$$\underbrace{-\breve{h}(2\alpha,\gamma)X(2\alpha,\gamma)\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(2\alpha,\gamma))ic(2\alpha)t}_{C_7}$$

$$\underbrace{+\breve{h}(2\alpha,\gamma)X(2\alpha,\gamma)\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(2\alpha,\gamma))\frac{i2c(\alpha)t}{1+ic'(\alpha)\gamma t}(ic'(2\alpha)\gamma t+1)}_{C_4}$$

$$\underbrace{-\breve{h}(2\alpha,0)X(2\alpha,0)\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(2\alpha,0))\frac{i2c(\alpha)t}{1+ic'(\alpha)\gamma t}}_{C_8}$$

$$\underbrace{+\frac{2ic(\alpha)t}{1+ic'(\alpha)\gamma t}\breve{h}(2\alpha,0)X(2\alpha,0)\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(2\alpha,0))}_{C_8}$$

$$\underbrace{+\frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t}\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(-\pi,\gamma))X(-\pi,\gamma)(1+ic'(-\pi)\gamma t)\breve{h}(\alpha,\gamma)}_{C_9}$$

$$\underbrace{-\frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t}\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(0,\gamma))X(0,\gamma)(1+ic'(0)\gamma t)\breve{h}(\alpha,\gamma)}_{C_1}$$

$$\underbrace{-\frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t}\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(2\alpha,\gamma))X(2\alpha,\gamma)(1+ic'(2\alpha)\gamma t)\breve{h}(\alpha,\gamma)}_{C_2}$$

$$\underbrace{-\frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t}\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(\pi,\gamma))X(\pi,\gamma)(1+ic'(\pi)\gamma t)\breve{h}(\alpha,\gamma)}_{C_{10}}$$

$$\underbrace{+\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(-\pi,\gamma))X(-\pi,\gamma)(ic(-\pi)t-\frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t}(1+ic'(-\pi)\gamma t))\breve{h}(\alpha,\gamma)}_{C_9}$$

$$\underbrace{-\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(0,\gamma))X(0,\gamma)(ic(0)t-\frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t}(1+ic'(0)\gamma t))\breve{h}(\alpha,\gamma)}_{C_5}$$

$$\underbrace{+\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(2\alpha,\gamma))X(2\alpha,\gamma)(ic(2\alpha)t-\frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t}(1+ic'(2\alpha)\gamma t))\breve{h}(\alpha,\gamma)}_{C_6}$$

$$\underbrace{-\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(\pi,\gamma))X(\pi,\gamma)(ic(\pi)t-\frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t}(1+ic'(\pi)\gamma t))\breve{h}(\alpha,\gamma)}_{C_{10}}$$

$= 0.$

Here we write the same C_i under terms that cancel with each other. $\square$

Remark 3.11.9. *We could change* $[-\pi, \pi]$ *in lemma 3.11.8 to any interval* $[b_L, b_R]$ *satisfying* $b_L < 0 < b_R$. *The corresponding result holds for* $0 < \alpha < \frac{b_R}{2}$.

Corollary 3.11.10. *Let* $\breve{K}$ *be meromorphic,* $0 \leq \tau(t) \leq \frac{\pi}{4}$, $\breve{h}(\alpha,\gamma), \ddot{h}(\alpha,\gamma), X(\alpha,\gamma) \in C^1_\gamma([-1,1], C^1_\alpha[-\pi + \tau(t), \pi + \tau(t)])$. *If for fixed* $\alpha > -\tau(t)$, *there is no singular point in the integrals of* $O(h)$ *below and* $c(0) = c(\pi + \tau(t)) = c(-\pi + \tau(t)) = 0$, $c(\alpha) \in W^{2,\infty}$, *then when* $-\tau(t) < \alpha < \frac{\pi}{4}$, *for*

$$\tilde{O}(\breve{h}, \ddot{h}) = \int_{-\tau(t)}^{2\alpha+\tau(t)} \breve{K}(\ddot{h}(\alpha+\tau(t),\gamma) - \ddot{h}(\beta+\tau(t),\gamma))(\breve{h}(\alpha+\tau(t),\gamma) - \breve{h}(\beta+\tau(t),\gamma))X(\beta+\tau(t),\gamma)$$
$$(1 + ic'(\beta+\tau(t))\gamma t)d\beta$$
$$+ \int_0^\gamma \breve{h}(2\alpha+2\tau(t),\eta)\breve{K}(\ddot{h}(\alpha+\tau(t),\gamma) - \ddot{h}(2\alpha+2\tau(t),\eta))ic(2\alpha+2\tau(t))t(X(2\alpha+2\tau(t),\eta))d\eta$$
$$- \int_{2\alpha+\tau(t)}^{\pi} \breve{h}(\beta+\tau(t),0)\breve{K}(\ddot{h}(\alpha+\tau(t),\gamma) - \ddot{h}(\beta+\tau(t),0))(X(\beta+\tau(t),0))d\beta$$
$$+ \int_{[-\pi,0]\cup[2\alpha+2\tau(t),\pi]} \breve{K}(\ddot{h}(\alpha+\tau(t),\gamma) - \ddot{h}(\beta+\tau(t),\gamma))(X(\beta+\tau(t),\gamma))(1+ic'(\beta+\tau(t))\gamma t)d\beta\breve{h}(\alpha+\tau(t),\gamma)$$

we have

$$A(\tilde{O}(\breve{h}, \ddot{h})) = \underbrace{\int_{-\tau(t)}^{2\alpha+\tau(t)} \nabla\breve{K}(\ddot{h}(\alpha+\tau(t),\gamma) - \ddot{h}(\beta+\tau(t),\gamma)) \cdot ([A(\ddot{h})(\alpha+\tau(t),\gamma)] - [A(\ddot{h})(\beta+\tau(t),\gamma)])}_{FA_1}$$
$$\underbrace{(1+ic'(\beta+\tau(t))\gamma t)(\breve{h}(\alpha+\tau(t),\gamma) - \breve{h}(\beta+\tau(t),\gamma))X(\beta+\tau(t),\gamma)d\beta}_{FA_1}$$
$$+ \underbrace{\int_{-\tau(t)}^{2\alpha+\tau(t)} \breve{K}(\ddot{h}(\alpha+\tau(t),\gamma) - \ddot{h}(\beta+\tau(t),\gamma))([A(\breve{h})(\alpha+\tau(t),\gamma)] - [A(\breve{h})(\beta+\tau(t),\gamma)])}_{FA_2}$$
$$\underbrace{(X(\beta+\tau(t),\gamma))(1+ic'(\beta+\tau(t))\gamma t)d\beta}_{FA_2}$$
$$+ \underbrace{\int_{-\tau(t)}^{2\alpha+\tau(t)} \breve{K}(\ddot{h}(\alpha+\tau(t),\gamma) - \ddot{h}(\beta+\tau(t),\gamma))(\breve{h}(\alpha+\tau(t),\gamma) - \breve{h}(\beta+\tau(t),\gamma))[A(X)(\beta+\tau(t),\gamma)]}_{FA_3}$$
$$\underbrace{(1+ic'(\beta+\tau(t))\gamma t)d\beta}_{FA_3}$$

$$\underbrace{+\int_0^{\gamma}\breve{h}(2\alpha+2\tau(t),\eta)\nabla\breve{K}(\ddot{h}(\alpha+\tau(t),\gamma)-\ddot{h}(2\alpha+2\tau(t),\eta))\cdot[A(\ddot{h})(\alpha+\tau(t),\gamma)]X(2\alpha+2\tau(t),\eta)}_{FA_4}$$

$$\underbrace{ic(2\alpha+2\tau(t))td\eta}_{FA_4}$$

$$\underbrace{-\int_0^{\gamma}\breve{h}(2\alpha+2\tau(t),\eta)\nabla\breve{K}(\ddot{h}(\alpha+\tau(t),\gamma)-\ddot{h}(2\alpha+2\tau(t),\eta))\cdot(\frac{2ic(\alpha+\tau(t))t}{1+ic'(\alpha+\tau(t))\gamma t})(\frac{1+ic'(2\alpha+2\tau(t))\eta t}{ic(2\alpha+2\tau(t))t})}_{FA_5}$$

$$\underbrace{[A(\ddot{h})(2\alpha+2\tau(t),\eta)]X(2\alpha+2\tau(t),\eta)ic(2\alpha+2\tau(t))td\eta}_{FA_5}$$

$$\underbrace{+\int_0^{\gamma}\breve{h}(2\alpha+2\tau(t),\eta)\breve{K}(\ddot{h}(\alpha+\tau(t),\gamma)-\ddot{h}(2\alpha+2\tau(t),\eta))(\frac{2ic(\alpha+\tau(t))t}{1+ic'(\alpha+\tau(t))\gamma t})(\frac{1+ic'(2\alpha+2\tau(t))\eta t}{ic(2\alpha+2\tau(t))t})}_{FA_6}$$

$$\underbrace{[A(X)(2\alpha+2\tau(t),\eta)]ic(2\alpha+2\tau(t))td\eta}_{FA_6}$$

$$\underbrace{+\int_0^{\gamma}(\frac{2ic(\alpha+\tau(t))t}{1+ic'(\alpha+\tau(t))\gamma t})(\frac{1+ic'(2\alpha+2\tau(t))\eta t}{ic(2\alpha+2\tau(t))t})[A(\breve{h})(2\alpha+2\tau(t),\eta)]\breve{K}(\ddot{h}(\alpha+\tau(t),\gamma)-\ddot{h}(2\alpha+2\tau(t),\eta))}_{FA_7}$$

$$\underbrace{X(2\alpha+2\tau(t),\eta)ic(2\alpha+2\tau(t))td\eta}_{FA_7}$$

$$\underbrace{-\int_{2\alpha+\tau(t)}^{\pi}\breve{h}(\beta+\tau(t),0)\nabla\breve{K}(\ddot{h}(\alpha+\tau(t),\gamma)-\ddot{h}(\beta+\tau(t),0))\cdot[A(\ddot{h})(\alpha+\tau(t),\gamma)](X(\beta+\tau(t),0))d\beta}_{FA_8}$$

$$\underbrace{+\int_{[-\pi,0]\cup[2\alpha,\pi]}\nabla\breve{K}(\ddot{h}(\alpha+\tau(t),\gamma)-\ddot{h}(\beta+\tau(t),\gamma))\cdot([A(\ddot{h})(\alpha+\tau(t),\gamma)]-[A(\ddot{h})(\beta+\tau(t),\gamma)])}_{FA_9}$$

$$\underbrace{(X(\beta+\tau(t),\gamma))(1+ic'(\beta+\tau(t))\gamma t))d\beta\breve{h}(\alpha+\tau(t),\gamma)}_{FA_9}$$

$$\underbrace{+\int_{[-\pi,-\tau(t)]\cup[2\alpha+\tau(t),\pi]}\breve{K}(\ddot{h}(\alpha+\tau(t),\gamma)-\ddot{h}(\beta+\tau(t),\gamma))[A(X)(\beta+\tau(t),\gamma)]}_{FA_{10}}$$

$$\underbrace{(1+ic'(\beta+\tau(t))\gamma t))d\beta\breve{h}(\alpha+\tau(t),\gamma)}_{FA_{10}}$$

$$\underbrace{+\int_{[-\pi,-\tau(t)]\cup[2\alpha+\tau(t),\pi]}\breve{K}(\ddot{h}(\alpha+\tau(t),\gamma)-\ddot{h}(\beta+\tau(t),\gamma))X(\beta+\tau(t),\gamma)(1+ic'(\beta+\tau(t))\gamma t))d\beta}_{FA_{11}}$$

$$\underbrace{[A(\breve{h})(\alpha+\tau(t),\gamma)]}_{FA_{11}}.$$

Proof. By letting $[b_L, b_R] = [-\pi + \tau(t), \pi + \tau(t)]$ in Remark 3.11.9, we have

$$\tilde{O}(\breve{h}, \ddot{h})(\alpha) = O(\breve{h}, \ddot{h})(\alpha + \tau(t)),$$

and

$$A(\tilde{O}(\breve{h}, \ddot{h})(\alpha)) = A_0(O(\breve{h}, \ddot{h}))(\alpha + \tau(t)).$$

Then we have the result. □

Lemma 3.11.11. *Let K be meromorphic, $\breve{h}(\alpha,\gamma), \ddot{h}(\alpha,\gamma) \in C^1_\gamma([-1,1], C^1_\alpha[-\pi,\pi])$. If for fixed $\alpha > 0$, there is no singular point in the integrals in $M(\breve{h}\ddot{h})$ below and $c(0) = c(\pi) = c(-\pi) = 0$, $c(\alpha) \in W^{2,\infty}$, then when $0 < \alpha < \frac{\pi}{2}$, for*

$$\begin{aligned}
M(\breve{h}, \ddot{h}) = &\int_0^{2\alpha} \breve{K}(\ddot{h}(\alpha,\gamma) - \ddot{h}(\beta,\gamma))(\frac{(\frac{d}{d\alpha}\breve{h})(\alpha,\gamma)}{1+ic'(\alpha)\gamma t} - \frac{\frac{d}{d\beta}\breve{h}(\beta,\gamma)}{1+ic'(\beta)\gamma t})(1+ic'(\beta)\gamma t)d\beta \\
&+ \breve{K}(\ddot{h}(\alpha,\gamma) - \ddot{h}(2\alpha,\gamma))\breve{h}(2\alpha,\gamma) \\
&+ \int_0^{\gamma} \breve{h}(2\alpha,\eta)\nabla\breve{K}(\ddot{h}(\alpha,\gamma) - \ddot{h}(2\alpha,\eta)) \cdot (\frac{(\frac{d}{d\alpha}\ddot{h})(2\alpha,\eta)}{1+ic'(2\alpha)\eta t})ic(2\alpha)td\eta \\
&- \int_{2\alpha}^{\pi} \breve{h}(\beta,0)\nabla\breve{K}(\ddot{h}(\alpha,\gamma) - \ddot{h}(\beta,0)) \cdot \frac{d\ddot{h}}{d\beta}(\beta,0)d\beta \\
&+ \int_{[-\pi,0]\cup[2\alpha,\pi]} \breve{K}(\ddot{h}(\alpha,\gamma) - \ddot{h}(\beta,\gamma))(1+ic'(\beta)\gamma t)d\beta \frac{(\frac{d}{d\alpha}\breve{h})(\alpha,\gamma)}{1+ic'(\alpha)\gamma t} \\
&= M_1 + M_2 + M_3 + M_4 + M_5.
\end{aligned}$$

we have

$$\begin{aligned}
A_0(M) = &\underbrace{\int_0^{2\alpha} D_{\ddot{h}}(\breve{K}(\ddot{h}(\alpha,\gamma) - \ddot{h}(\beta,\gamma)))[A_0(\ddot{h})](\frac{(\frac{d}{d\alpha}\breve{h})(\alpha,\gamma)}{1+ic'(\alpha)\gamma t} - \frac{\frac{d}{d\beta}\breve{h}(\beta,\gamma)}{1+ic'(\beta)\gamma t})(1+ic'(\beta)\gamma t)d\beta}_{FM_{1,1}} \\
&\underbrace{+ \int_0^{2\alpha} \breve{K}(\ddot{h}(\alpha,\gamma) - \ddot{h}(\beta,\gamma))(\frac{\frac{d}{d\alpha}A_0(\breve{h})(\alpha,\gamma)}{1+ic'(\alpha)\gamma t} - \frac{\frac{d}{d\beta}A_0(\breve{h})(\beta,\gamma)}{1+ic'(\beta)\gamma t})(1+ic'(\beta)\gamma t)d\beta}_{FM_{1,2}} \\
&\underbrace{+D_{\ddot{h}}\breve{K}(\ddot{h}(\alpha,\gamma) - \ddot{h}(2\alpha,\gamma))[A_0(\ddot{h})]\breve{h}(2\alpha,\gamma)}_{FM_{2,1}} \\
&\underbrace{+\breve{K}(\ddot{h}(\alpha,\gamma) - \ddot{h}(2\alpha,\gamma))A_0(\breve{h})(2\alpha,\gamma)}_{FM_{2,2}}
\end{aligned}$$

$$
\begin{aligned}
&\underbrace{+\int_0^{\gamma}\breve{h}(2\alpha,\eta)\nabla^2\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(2\alpha,\eta))[(A_0[\ddot{h}](\alpha,\gamma)),\frac{(\frac{d}{d\alpha}\ddot{h})(2\alpha,\eta)}{1+ic'(2\alpha)\eta t}]ic(2\alpha)td\eta}_{FM_{3,1}}\\
&\underbrace{-\int_0^{\gamma}\breve{h}(2\alpha,\eta)\nabla^2\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(2\alpha,\eta))[(\frac{2ic(\alpha)t(1+ic'(2\alpha)\eta t)}{(1+ic'(\alpha)\gamma t)(ic(2\alpha)t)}A_0(\ddot{h})(2\alpha,\eta)),\frac{(\frac{d}{d\alpha}\ddot{h})(2\alpha,\eta)}{1+ic'(2\alpha)\eta t}]ic(2\alpha)td\eta}_{FM_{3,2}}\\
&\underbrace{+\int_0^{\gamma}\breve{h}(2\alpha,\eta)\nabla\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(2\alpha,\eta))\cdot\frac{2ic(\alpha+\tau(t))t(1+ic'(2\alpha)\eta t)}{(1+ic'(\alpha)\gamma t)(ic(2\alpha)t)}(\frac{(\frac{d}{d\alpha}A_0(\ddot{h}))(2\alpha,\eta)}{1+ic'(2\alpha)\eta t})ic(2\alpha)td\eta}_{FM_{3,3}}\\
&\underbrace{+\int_0^{\gamma}A_0(\breve{h})(2\alpha,\eta)\frac{2ic(\alpha)t(1+ic'(2\alpha)\eta t)}{(1+ic'(\alpha)\gamma t)(ic(2\alpha)t)}\nabla\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(2\alpha,\eta))\cdot(\frac{(\frac{d}{d\alpha}\ddot{h})(2\alpha,\eta)}{1+ic'(2\alpha)\eta t})ic(2\alpha)td\eta}_{FM_{3,4}}\\
&\underbrace{-\int_{2\alpha}^{\pi}\breve{h}(\beta,0)\nabla^2\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(\beta,0))[A_0(\ddot{h})(\alpha,\gamma),\frac{d\ddot{h}}{d\beta}(\beta,0)]d\beta}_{FM_4}\\
&\underbrace{+\int_{[-\pi,0]\cup[2\alpha,\pi]}\nabla\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(\beta,\gamma))\cdot(A_0(\ddot{h})(\alpha,\gamma)-A_0(\ddot{h})(\beta,\gamma))(1+ic'(\beta)\gamma t))d\beta\frac{(\frac{d}{d\alpha}\breve{h})(\alpha,\gamma)}{1+ic'(\alpha)\gamma t}}_{FM_{5,1}}\\
&\underbrace{+\int_{[-\pi,0]\cup[2\alpha,\pi]}\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(\beta,\gamma))(1+ic'(\beta)\gamma t))d\beta\frac{\frac{d}{d\alpha}A_0(\breve{h})(\alpha,\gamma)}{1+ic'(\alpha)\gamma t}}_{FM_{5,2}}.
\end{aligned}
$$

Proof. First, we do the M_1

First, we compare O_1 with M_1. Since in M_1, we only change $\breve{h}(\alpha,\gamma)-\breve{h}(\beta,\gamma)$ by $\frac{\frac{d\breve{h}(\alpha,\gamma)}{d\alpha}}{1+ic'(\alpha)\gamma t}-\frac{\frac{d\breve{h}(\beta,\gamma)}{d\beta}}{1+ic'(\beta)\gamma t}$ and letting $X(\beta,\gamma)=1$, we have

$$
\begin{aligned}
&A_0(M_1)=\\
&\quad A(\int_{-\alpha}^{\alpha}\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(\alpha-\beta,\gamma))(\frac{(\frac{d}{d\alpha}\breve{h})(\alpha,\gamma)}{1+ic'(\alpha)\gamma t}-\frac{\frac{d}{d\alpha}\breve{h}(\alpha-\beta,\gamma)}{1+ic'(\alpha-\beta)\gamma t})(1+ic'(\alpha-\beta)\gamma t)d\beta)\\
&=\underbrace{\frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t}[\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(0,\gamma))(\frac{(\frac{d}{d\alpha}\breve{h})(\alpha,\gamma)}{1+ic'(\alpha)\gamma t}-\frac{\frac{d}{d\alpha}\breve{h}(0,\gamma)}{1+ic'(0)\gamma t})(1+ic'(0)\gamma t)}_{B_1}\\
&\quad\underbrace{+\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(2\alpha,\gamma))(\frac{(\frac{d}{d\alpha}\breve{h})(\alpha,\gamma)}{1+ic'(\alpha)\gamma t}-\frac{(\frac{d}{d\alpha}\breve{h})(2\alpha,\gamma)}{1+ic'(2\alpha)\gamma t})(1+ic'(2\alpha)\gamma t)]}_{B_1}\\
&\quad\underbrace{+\int_0^{2\alpha}\nabla\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(\beta,\gamma))\cdot(A_0(\ddot{h})(\alpha,\gamma)-A_0(\ddot{h})(\beta,\gamma))(1+ic'(\beta)\gamma t)(\frac{(\frac{d}{d\alpha}\breve{h})(\alpha,\gamma)}{1+ic'(\alpha)\gamma t}-\frac{\frac{d}{d\beta}\breve{h}(\beta,\gamma)}{1+ic'(\beta)\gamma t})d\beta}_{FM_{1,1}}\\
&\quad\underbrace{+\int_0^{2\alpha}\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(\beta,\gamma))(A_0(\frac{(\frac{d}{d\alpha}\breve{h})(\alpha,\gamma)}{1+ic'(\alpha)\gamma t})(\alpha,\gamma)-A_0(\frac{(\frac{d}{d\alpha}\breve{h})(\alpha,\gamma)}{1+ic'(\alpha)\gamma t})(\beta,\gamma))(1+ic'(\beta)\gamma t)d\beta}_{FM_{1,2}}
\end{aligned}
$$

$$\underbrace{-\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(0,\gamma))(\frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t}-\frac{ic(0)t}{1+ic'(0)\gamma t})(1+ic'(0)\gamma t)(\frac{(\frac{d}{d\alpha}\breve{h})(\alpha,\gamma)}{1+ic'(\alpha)\gamma t}-\frac{\frac{d}{d\alpha}\breve{h}(0,\gamma)}{1+ic'(0)\gamma t})}_{B_2}$$

$$+\underbrace{\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(2\alpha,\gamma))(\frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t}-\frac{ic(2\alpha)t}{1+ic'(2\alpha)\gamma t})(\frac{(\frac{d}{d\alpha}\breve{h})(\alpha,\gamma)}{1+ic'(\alpha)\gamma t}-\frac{(\frac{d}{d\alpha}\breve{h})(2\alpha,\gamma)}{1+ic'(2\alpha)\gamma t})(1+ic'(2\alpha)\gamma t)}_{B_3}$$

$$=FM_{1,1}+FM_{1,2}$$

$$\underbrace{+\frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t}[\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(0,\gamma))(\frac{(\frac{d}{d\alpha}\breve{h})(\alpha,\gamma)}{1+ic'(\alpha)\gamma t})(1+ic'(0)\gamma t)}_{B_{1,1}}$$

$$\underbrace{+\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(2\alpha,\gamma))(\frac{(\frac{d}{d\alpha}\breve{h})(\alpha,\gamma)}{1+ic'(\alpha)\gamma t})(1+ic'(2\alpha)\gamma t)]}_{B_{1,1}}$$

$$\underbrace{+\frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t}[\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(0,\gamma))(-\frac{\frac{d}{d\alpha}\breve{h}(0,\gamma)}{1+ic'(0)\gamma t})(1+ic'(0)\gamma t)}_{B_{1,2}}$$

$$\underbrace{+\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(2\alpha,\gamma))(-\frac{(\frac{d}{d\alpha}\breve{h})(2\alpha,\gamma)}{1+ic'(2\alpha)\gamma t})(1+ic'(2\alpha)\gamma t)]}_{B_{1,2}}$$

$$\underbrace{-\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(0,\gamma))(\frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t}-\frac{ic(0)t}{1+ic'(0)\gamma t})(1+ic'(0)\gamma t)(\frac{(\frac{d}{d\alpha}\breve{h})(\alpha,\gamma)}{1+ic'(\alpha)\gamma t})}_{B_{2,1}}$$

$$\underbrace{-\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(0,\gamma))(\frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t}-\frac{ic(0)t}{1+ic'(0)\gamma t})(1+ic'(0)\gamma t)(-\frac{\frac{d}{d\alpha}\breve{h}(0,\gamma)}{1+ic'(0)\gamma t})}_{B_{2,2}}$$

$$+\underbrace{\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(2\alpha,\gamma))(\frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t}-\frac{ic(2\alpha)t}{1+ic'(2\alpha)\gamma t})(\frac{(\frac{d}{d\alpha}\breve{h})(\alpha,\gamma)}{1+ic'(\alpha)\gamma t})(1+ic'(2\alpha)\gamma t)}_{B_{3,1}}$$

$$+\underbrace{\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(2\alpha,\gamma))(\frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t}-\frac{ic(2\alpha)t}{1+ic'(2\alpha)\gamma t})(-\frac{(\frac{d}{d\alpha}\breve{h})(2\alpha,\gamma)}{1+ic'(2\alpha)\gamma t})(1+ic'(2\alpha)\gamma t)}_{B_{3,2}}.$$

Here we use lemma 2.8.2 in the $FM_{1,2}$ term. For M_2, we have

$$A_0(M_2)=A_0(\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(2\alpha,\gamma))\breve{h}(2\alpha,\gamma))$$

$$=\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(2\alpha,\gamma))(\frac{2ic(\alpha)t}{1+ic'(\alpha)\gamma t}(\partial_\alpha h)(2\alpha,\gamma)-(\partial_\gamma h)(2\alpha,\gamma))$$

$$+\nabla\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(2\alpha,\gamma))\cdot(\frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t}(\partial_\alpha\ddot{h})(\alpha,\gamma)-(\partial_\gamma\ddot{h})(\alpha,\gamma)-(\frac{2ic(\alpha)t}{1+ic'(\alpha)\gamma t}(\partial_\alpha\ddot{h})(2\alpha,\gamma)-(\partial_\gamma\ddot{h})(2\alpha,\gamma)))$$

$$\breve{h}(2\alpha,\gamma).$$

Then we can write each term into $A_0(\breve{h})$ and the remaining terms. We get

$$A_0(M_2) = \underbrace{\breve{K}(\ddot{h}(\alpha,\gamma) - \ddot{h}(2\alpha,\gamma))(\frac{2ic(\alpha)t}{1+ic'(\alpha)\gamma t} - \frac{ic(2\alpha)t}{1+ic'(2\alpha)\gamma t})(\partial_\alpha h)(2\alpha,\gamma)}_{B_4}$$

$$\underbrace{+\breve{K}(\ddot{h}(\alpha,\gamma) - \ddot{h}(2\alpha,\gamma))(\frac{ic(2\alpha)t}{1+ic'(2\alpha)\gamma t}(\partial_\alpha h)(2\alpha,\gamma) - (\partial_\gamma h)(2\alpha,\gamma))}_{FM_{2,2}}$$

$$\underbrace{-\nabla\breve{K}(\ddot{h}(\alpha,\gamma) - \ddot{h}(2\alpha,\gamma)) \cdot [(\frac{2ic(\alpha)t}{1+ic'(\alpha)\gamma t} - \frac{ic(2\alpha)t}{1+ic'(2\alpha)\gamma t})(\partial_\alpha \ddot{h})(2\alpha,\gamma)]\breve{h}(2\alpha,\gamma)}_{B_5}$$

$$\underbrace{+\nabla\breve{K}(\ddot{h}(\alpha,\gamma) - \ddot{h}(2\alpha,\gamma)) \cdot [(\frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t})(\partial_\alpha \ddot{h})(\alpha,\gamma) - (\partial_\gamma \ddot{h})(\alpha,\gamma) - ((\frac{ic(2\alpha)t}{1+ic'(2\alpha)\gamma t})(\partial_\alpha \ddot{h})(2\alpha,\gamma) - (\partial_\gamma \ddot{h})(2\alpha,\gamma))]}_{FM_{2,1}}$$

$$\underbrace{\breve{h}(2\alpha,\gamma)}_{FM_{2,1}}.$$

For the third term, we have

$$A_0(M_3) = A_0(\int_0^\gamma \breve{h}(2\alpha,\eta)\nabla\breve{K}(\ddot{h}(\alpha,\gamma) - \ddot{h}(2\alpha,\eta)) \cdot (\frac{(\frac{d}{d\alpha}\ddot{h})(2\alpha,\eta)}{1+ic'(2\alpha)\eta t})ic(2\alpha)td\eta)$$

$$= \underbrace{-\breve{h}(2\alpha,\gamma)\nabla\breve{K}(\ddot{h}(\alpha,\gamma) - \ddot{h}(2\alpha,\gamma)) \cdot (\frac{(\frac{d}{d\alpha}\ddot{h})(2\alpha,\gamma)}{1+ic'(2\alpha)\gamma t})ic(2\alpha)t}_{Term_1}$$

$$\underbrace{+\int_0^\gamma (\frac{2ic(\alpha)t}{1+ic'(\alpha)\gamma t}(\partial_\alpha h)(2\alpha,\eta))\nabla\breve{K}(\ddot{h}(\alpha,\gamma) - \ddot{h}(2\alpha,\eta)) \cdot (\frac{(\frac{d}{d\alpha}\ddot{h})(2\alpha,\eta)}{1+ic'(2\alpha)\eta t})ic(2\alpha)td\eta}_{Term_2}$$

$$\underbrace{+\int_0^\gamma \breve{h}(2\alpha,\eta)\nabla^2\breve{K}(\ddot{h}(\alpha,\gamma) - \ddot{h}(2\alpha,\eta))[(\frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t}(\partial_\alpha \ddot{h})(\alpha,\gamma) - (\partial_\gamma \ddot{h})(\alpha,\gamma)), (\frac{(\frac{d}{d\alpha}\ddot{h})(2\alpha,\eta)}{1+ic'(2\alpha)\eta t})]ic(2\alpha)td\eta}_{Term_3, FM_{3,1}}$$

$$\underbrace{-\int_0^\gamma \breve{h}(2\alpha,\eta)\nabla^2\breve{K}(\ddot{h}(\alpha,\gamma) - \ddot{h}(2\alpha,\eta))[(\frac{2ic(\alpha)t}{1+ic'(\alpha)\gamma t}(\partial_\alpha \ddot{h})(2\alpha,\eta)), \frac{(\frac{d}{d\alpha}\ddot{h})(2\alpha,\eta)}{1+ic'(2\alpha)\eta t}]ic(2\alpha)td\eta}_{Term_4}$$

$$\underbrace{+\int_0^\gamma \breve{h}(2\alpha,\eta)\nabla\breve{K}(\ddot{h}(\alpha,\gamma) - \ddot{h}(2\alpha,\eta)) \cdot [(\frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t}\frac{d}{d\alpha})(\frac{(\frac{d}{d\alpha}\ddot{h})(2\alpha,\eta)}{1+ic'(2\alpha)\eta t})]ic(2\alpha)td\eta}_{Term_5}$$

$$\underbrace{+\int_0^\gamma \breve{h}(2\alpha,\eta)\nabla\breve{K}(\ddot{h}(\alpha,\gamma) - \ddot{h}(2\alpha,\eta)) \cdot (\frac{(\frac{d}{d\alpha}\ddot{h})(2\alpha,\eta)}{1+ic'(2\alpha)\eta t})\frac{i2c(\alpha)t}{1+ic'(\alpha)\gamma t}ic'(2\alpha)td\eta}_{Term_6}$$

$$= Term_1 + Term_2 + FM_{3,1} + Term_4 + Term_5 + Term_6.$$

Now we do the change to the terms by writing the terms in $Term_2 + Term_4 + Term_5$ as $A_0(\breve{h})$. We

have

$$
\begin{aligned}
&Term_2 + Term_4 + Term_5 \\
&= \underbrace{\int_0^\gamma ic(2\alpha)t(\frac{2ic(\alpha)t}{1+ic'(\alpha)\gamma t})[(\frac{1}{2}\frac{d}{d\alpha} - \frac{1+ic'(2\alpha)\eta t}{ic(2\alpha)t}\frac{d}{d\eta})\breve{h}(2\alpha,\eta)]\nabla\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(2\alpha,\eta))\cdot(\frac{(\frac{d}{d\alpha}\ddot{h})(2\alpha,\eta)}{1+ic'(2\alpha)\eta t})d\eta}_{Term_{2,1}=FM_{3,4}} \\
&\underbrace{+\int_0^\gamma ic(2\alpha)t(\frac{2ic(\alpha)t}{1+ic'(\alpha)\gamma t})[(\frac{1+ic'(2\alpha)\eta t}{ic(2\alpha)t}\frac{d}{d\eta})\breve{h}(2\alpha,\eta)]\nabla\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(2\alpha,\eta))\cdot(\frac{(\frac{d}{d\alpha}\ddot{h})(2\alpha,\eta)}{1+ic'(2\alpha)\eta t})d\eta}_{Term_{2,2}} \\
&\underbrace{-\int_0^\gamma \breve{h}(2\alpha,\eta)\nabla^2\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(2\alpha,\eta))[(\frac{2ic(\alpha)t}{1+ic'(\alpha)\gamma t})(\frac{1}{2}\frac{d}{d\alpha} - \frac{1+ic'(2\alpha)\eta t}{ic(2\alpha)t}\frac{d}{d\eta})\ddot{h}(2\alpha,\eta),(\frac{(\frac{d}{d\alpha}\ddot{h})(2\alpha,\eta)}{1+ic'(2\alpha)\eta t})]}_{Term_{4,1}=FM_{3,2}} \\
&\underbrace{ic(2\alpha)td\eta}_{Term_{4,1}=FM_{3,2}} \\
&\underbrace{-\int_0^\gamma \breve{h}(2\alpha,\eta)\nabla^2\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(2\alpha,\eta))[(\frac{2ic(\alpha)t}{1+ic'(\alpha)\gamma t})(\frac{1+ic'(2\alpha)\eta t}{ic(2\alpha)t}\frac{d}{d\eta})\ddot{h}(2\alpha,\eta),(\frac{(\frac{d}{d\alpha}\ddot{h})(2\alpha,\eta)}{1+ic'(2\alpha)\eta t})]ic(2\alpha)td\eta}_{Term_{4,2}} \\
&\underbrace{+\int_0^\gamma \breve{h}(2\alpha,\eta)\nabla\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(2\alpha,\eta))\cdot(\frac{2ic(\alpha)t}{1+ic'(\alpha)\gamma t})[(\frac{1}{2}\frac{d}{d\alpha} - \frac{1+ic'(2\alpha)\eta t}{ic(2\alpha)t}\frac{d}{d\eta})(\frac{(\frac{d}{d\alpha}\ddot{h})(2\alpha,\eta)}{1+ic'(2\alpha)\eta t})]ic(2\alpha)td\eta}_{Term_{5,1}=FM_{3,3}} \\
&\underbrace{+\int_0^\gamma \breve{h}(2\alpha,\eta)\nabla\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(2\alpha,\eta))\cdot(\frac{2ic(\alpha)t}{1+ic'(\alpha)\gamma t})[(\frac{1+ic'(2\alpha)\eta t}{ic(2\alpha)t}\frac{d}{d\eta})(\frac{(\frac{d}{d\alpha}\ddot{h})(2\alpha,\eta)}{1+ic'(2\alpha)\eta t})]ic(2\alpha)td\eta}_{Term_{5,2}}.
\end{aligned}
$$

Here we used

$$(\frac{\frac{1}{2}ic(2\alpha)t}{1+ic'(2\alpha)\gamma t}\partial_\alpha - \partial_\gamma)(\frac{(\partial_\alpha\ddot{h})(2\alpha,\eta)}{1+ic'(2\alpha)\eta t}) = A_0(\frac{(\partial_\alpha)\ddot{h}(\alpha,\eta)}{1+ic'(\alpha)\eta t})(2\alpha,\eta) = \frac{(\partial_\alpha A_0(\ddot{h}))(2\alpha,\eta)}{1+ic'(2\alpha)\eta t},$$

when showing $Term_{5,1} = FM_{3,3}$. It follows from lemma 2.8.2:

$$(\frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t}\partial_\alpha - \partial_\gamma)(\frac{(\partial_\alpha)\ddot{h}(\alpha,\eta)}{1+ic'(\alpha)\eta t}) = \frac{(\partial_\alpha)}{1+ic'(\alpha)\eta t}(\frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t}\partial_\alpha - \partial_\gamma)\ddot{h}(\alpha,\eta).$$

We also have

$$
\begin{aligned}
&Term_{2,2} + Term_{4,2} + Term_{5,2} + Term_6 \\
&= \underbrace{\int_0^\gamma ic(2\alpha)t(\frac{2ic(\alpha)t}{1+ic'(\alpha)\gamma t})[(\frac{1+ic'(2\alpha)\eta t}{ic(2\alpha)t}\frac{d}{d\eta})\breve{h}(2\alpha,\eta)]\nabla\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(2\alpha,\eta))\cdot(\frac{(\frac{d}{d\alpha}\ddot{h})(2\alpha,\eta)}{1+ic'(2\alpha)\eta t})d\eta}_{Term_{2,2}}
\end{aligned}
$$

$$\underbrace{-\int_0^\gamma \breve{h}(2\alpha,\eta)\nabla^2\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(2\alpha,\eta))[(\frac{2ic(\alpha)t}{1+ic'(\alpha)\gamma t})(\frac{1+ic'(2\alpha)\eta t}{ic(2\alpha)t}\frac{d}{d\eta})\ddot{h}(2\alpha,\eta),(\frac{(\frac{d}{d\alpha}\ddot{h})(2\alpha,\eta)}{1+ic'(2\alpha)\eta t})]ic(2\alpha)td\eta}_{Term_{4,2}}$$

$$\underbrace{+\int_0^\gamma \breve{h}(2\alpha,\eta)\nabla\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(2\alpha,\eta))\cdot(\frac{2ic(\alpha)t}{1+ic'(\alpha)\gamma t})[(\frac{1+ic'(2\alpha)\eta t}{ic(2\alpha)t}\frac{d}{d\eta})(\frac{(\frac{d}{d\alpha}\ddot{h})(2\alpha,\eta)}{1+ic'(2\alpha)\eta t})]ic(2\alpha)td\eta}_{Term_{5,2}}$$

$$\underbrace{+\int_0^\gamma \breve{h}(2\alpha,\eta)\nabla\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(2\alpha,\eta))\cdot(\frac{(\frac{d}{d\alpha}\ddot{h})(2\alpha,\eta)}{1+ic'(2\alpha)\eta t})\frac{i2c(\alpha)t}{1+ic'(\alpha)\gamma t}ic'(2\alpha)td\eta}_{Term_6}$$

$$=\int_0^\gamma \frac{d}{d\eta}[\breve{h}(2\alpha,\eta)\nabla\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(2\alpha,\eta))\cdot(\frac{(\frac{d}{d\alpha}\ddot{h})(2\alpha,\eta)}{1+ic'(2\alpha)\eta t})\frac{i2c(\alpha)t}{1+ic'(\alpha)\gamma t}(ic'(2\alpha)\eta t+1)]d\eta$$

$$=(\breve{h}(2\alpha,\gamma)\nabla\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(2\alpha,\gamma))\cdot(\frac{(\frac{d}{d\alpha}\ddot{h})(2\alpha,\gamma)}{1+ic'(2\alpha)\gamma t})\frac{i2c(\alpha)t}{1+ic'(\alpha)\gamma t}(ic'(2\alpha)\gamma t+1))$$

$$-(\breve{h}(2\alpha,0)\nabla\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(2\alpha,0))\cdot(\frac{d}{d\alpha}\ddot{h})(2\alpha,0)\frac{i2c(\alpha)t}{1+ic'(\alpha)\gamma t}(1)$$

Finally, we have

$$\begin{aligned}A_0(M_3)&=Term_1+Term_{2,1}+FM_{3,1}+Term_{4,1}+Term_{5,1}+(Term_{2,2}+Term_{4,2}+Term_{5,2}+Term_6)\\&=Term_1+FM_{3,4}+FM_{3,1}+FM_{3,2}+FM_{3,3}+(Term_{2,2}+Term_{4,2}+Term_{5,2}+Term_6)\\&=\underbrace{-\breve{h}(2\alpha,\gamma)\nabla\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(2\alpha,\gamma))\cdot(\frac{(\frac{d}{d\alpha}\ddot{h})(2\alpha,\gamma)}{1+ic'(2\alpha)\gamma t})ic(2\alpha)t}_{B_6}\\&\quad+FM_{3,4}+FM_{3,1}+FM_{3,2}+FM_{3,3}\\&\quad\underbrace{+\breve{h}(2\alpha,\gamma)\nabla\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(2\alpha,\gamma))\cdot(\frac{(\frac{d}{d\alpha}\ddot{h})(2\alpha,\gamma)}{1+ic'(2\alpha)\gamma t})\frac{i2c(\alpha)t}{1+ic'(\alpha)\gamma t}(ic'(2\alpha)\gamma t+1)}_{B_7}\\&\quad\underbrace{-\breve{h}(2\alpha,0)\nabla\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(2\alpha,0))\cdot(\frac{d}{d\alpha}\ddot{h})(2\alpha,0)\frac{i2c(\alpha)t}{1+ic'(\alpha)\gamma t}(1)}_{B_8}\end{aligned}$$

Then we consider M_4

$$\begin{aligned}A_0(M_4)&=A(-\int_{2\alpha}^{\pi}\breve{h}(\beta,0)\nabla\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(\beta,0))\cdot\frac{d\ddot{h}}{d\beta}(\beta,0)d\beta)\\&=\underbrace{\frac{2ic(\alpha)t}{1+ic'(\alpha)\gamma t}\breve{h}(2\alpha,0)\nabla\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(2\alpha,0))\cdot(\frac{d\ddot{h}}{d\alpha})(2\alpha,0)}_{B_9}\end{aligned}$$

$$\underbrace{-\int_{2\alpha}^{\pi}\breve{h}(\beta,0)\nabla^2\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(\beta,0))[(\frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t}\partial_\alpha-\partial_\gamma)\ddot{h}(\alpha,\gamma),\frac{d\ddot{h}}{d\beta}(\beta,0)]d\beta}_{FM_4}.$$

For M_5, we still compare it with O_4. We only change $\breve{h}(\alpha,\gamma)$ to $\frac{(\frac{d}{d\alpha}\breve{h})(\alpha,\gamma)}{1+ic'(\alpha)\gamma t}$ and set $X(\beta,\gamma)=1$. Recall that we have

$$\begin{aligned}
&A_0(O_4)=\\
&\int_{[-\pi,0]\cup[2\alpha,\pi]}\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(\beta,\gamma))(X(\beta,\gamma))(1+ic'(\beta)\gamma t)d\beta A_0(\breve{h})(\alpha,\gamma)\\
&+\int_{[\alpha-\pi,-\alpha]\cup[\alpha,\alpha+\pi]}X(\alpha-\beta,\gamma)\nabla\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(\alpha-\beta,\gamma))\\
&\cdot((\frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t}\partial_\alpha-\partial_\gamma)\ddot{h}(\alpha,\gamma)-(\frac{ic(\alpha-\beta)t}{1+ic'(\alpha-\beta)\gamma t}\partial_\alpha-\partial_\gamma)\ddot{h}(\alpha-\beta,\gamma,t))(1+ic'(\alpha-\beta)\gamma t)d\beta\breve{h}(\alpha,\gamma)\\
&+\int_{[\alpha-\pi,-\alpha]\cup[\alpha,\alpha+\pi]}\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(\alpha-\beta,\gamma))(\frac{ic(\alpha-\beta)t}{1+ic'(\alpha-\beta)\gamma t}\partial_\alpha-\partial_\gamma)X(\alpha-\beta,\gamma)(1+ic'(\alpha-\beta)\gamma t)d\beta\breve{h}(\alpha,\gamma)\\
&+\frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t}\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(-\pi,\gamma))X(-\pi,\gamma)(1+ic'(-\pi)\gamma t)\breve{h}(\alpha,\gamma)\\
&-\frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t}\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(0,\gamma))X(0,\gamma)(1+ic'(0)\gamma t)\breve{h}(\alpha,\gamma)\\
&-\frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t}\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(2\alpha,\gamma))X(2\alpha,\gamma)(1+ic'(2\alpha)\gamma t)\breve{h}(\alpha,\gamma)\\
&-\frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t}\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(\pi,\gamma))X(\pi,\gamma)(1+ic'(\pi)\gamma t)\breve{h}(\alpha,\gamma)\\
&+\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(-\pi,\gamma))X(-\pi,\gamma)(ic(-\pi)t-\frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t}(1+ic'(-\pi)\gamma t))\breve{h}(\alpha,\gamma)\\
&-\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(0,\gamma))X(0,\gamma)(ic(0)t-\frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t}(1+ic'(0)\gamma t))\breve{h}(\alpha,\gamma)\\
&+\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(2\alpha,\gamma))X(2\alpha,\gamma)(ic(2\alpha)t-\frac{ic(\alpha)t}{1+ic'(2\alpha)\gamma t}(1+ic'(2\alpha)\gamma t))\breve{h}(\alpha,\gamma)\\
&-\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(\pi,\gamma))X(\pi,\gamma)(ic(\pi)t-\frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t}(1+ic'(\pi)\gamma t))\breve{h}(\alpha,\gamma).
\end{aligned}$$

Then, we have

$$\begin{aligned}
&A_0(M_5)=\\
&\underbrace{\int_{[-\pi,0]\cup[2\alpha,\pi]}\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(\beta,\gamma))(1+ic'(\beta)\gamma t)d\beta\frac{\frac{d}{d\alpha}A_0(\breve{h})(\alpha,\gamma)}{1+ic'(\alpha)\gamma t}}_{FM_{5,2}}\\
&\underbrace{+\int_{[\alpha-\pi,-\alpha]\cup[\alpha,\alpha+\pi]}\nabla\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(\alpha-\beta,\gamma))}_{FM_{5,1}}
\end{aligned}$$

$$\underbrace{\cdot((\frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t}\partial_\alpha-\partial_\gamma)\ddot{h}(\alpha,\gamma)-(\frac{ic(\alpha-\beta)t}{1+ic'(\alpha-\beta)\gamma t}\partial_\alpha-\partial_\gamma)\ddot{h}(\alpha-\beta,\gamma,t))(1+ic'(\alpha-\beta)\gamma t)d\beta\frac{(\frac{d}{d\alpha}\breve{h})(\alpha,\gamma)}{1+ic'(\alpha)\gamma t}}_{FM_{5,1}}$$

$$\underbrace{+\frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t}\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(-\pi,\gamma))(1+ic'(-\pi)\gamma t)\frac{(\frac{d}{d\alpha}\breve{h})(\alpha,\gamma)}{1+ic'(\alpha)\gamma t}}_{B_{10}}$$

$$\underbrace{-\frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t}\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(0,\gamma))(1+ic'(0)\gamma t)\frac{(\frac{d}{d\alpha}\breve{h})(\alpha,\gamma)}{1+ic'(\alpha)\gamma t}}_{B_{11}}$$

$$\underbrace{-\frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t}\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(2\alpha,\gamma))(1+ic'(2\alpha)\gamma t)\frac{(\frac{d}{d\alpha}\breve{h})(\alpha,\gamma)}{1+ic'(\alpha)\gamma t}}_{B_{12}}$$

$$\underbrace{-\frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t}\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(\pi,\gamma))(1+ic'(\pi)\gamma t)\frac{(\frac{d}{d\alpha}\breve{h})(\alpha,\gamma)}{1+ic'(\alpha)\gamma t}}_{B_{13}}$$

$$\underbrace{+\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(-\pi,\gamma))(ic(-\pi)t-\frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t}(1+ic'(-\pi)\gamma t))\frac{(\frac{d}{d\alpha}\breve{h})(\alpha,\gamma)}{1+ic'(\alpha)\gamma t}}_{B_{14}}$$

$$\underbrace{-\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(0,\gamma))(ic(0)t-\frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t}(1+ic'(0)\gamma t))\frac{(\frac{d}{d\alpha}\breve{h})(\alpha,\gamma)}{1+ic'(\alpha)\gamma t}}_{B_{15}}$$

$$\underbrace{+\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(2\alpha,\gamma))(ic(2\alpha)t-\frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t}(1+ic'(2\alpha)\gamma t))\frac{(\frac{d}{d\alpha}\breve{h})(\alpha,\gamma)}{1+ic'(\alpha)\gamma t}}_{B_{16}}$$

$$\underbrace{-\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(\pi,\gamma))(ic(\pi)t-\frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t}(1+ic'(\pi)\gamma t))\frac{(\frac{d}{d\alpha}\breve{h})(\alpha,\gamma)}{1+ic'(\alpha)\gamma t}}_{B_{17}}.$$

In conclusion, we only need to show the sum of B terms are 0. We have

$$\begin{aligned}
&\sum_{i=1}^{17}B_i\\
&=\underbrace{\frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t}\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(0,\gamma))(\frac{(\frac{d}{d\alpha}\breve{h})(\alpha,\gamma)}{1+ic'(\alpha)\gamma t})(1+ic'(0)\gamma t)}_{B_{1,1}}\\
&\underbrace{+\frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t}\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(2\alpha,\gamma))(\frac{(\frac{d}{d\alpha}\breve{h})(\alpha,\gamma)}{1+ic'(\alpha)\gamma t})(1+ic'(2\alpha)\gamma t)}_{B_{1,1}}\\
&\underbrace{+\frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t}\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(0,\gamma))(-\frac{\frac{d}{d\alpha}\breve{h}(0,\gamma)}{1+ic'(0)\gamma t})(1+ic'(0)\gamma t)}_{B_{1,2}}
\end{aligned}$$

$$\underbrace{+\frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t}\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(2\alpha,\gamma))(-\frac{(\frac{d}{d\alpha}\breve{h})(2\alpha,\gamma)}{1+ic'(2\alpha)\gamma t})(1+ic'(2\alpha)\gamma t)}_{B_{1,2}}$$

$$\underbrace{-\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(0,\gamma))(\frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t}-\frac{ic(0)t}{1+ic'(0)\gamma t})(1+ic'(0)\gamma t)(\frac{(\frac{d}{d\alpha}\breve{h})(\alpha,\gamma)}{1+ic'(\alpha)\gamma t})}_{B_{2,1}}$$

$$\underbrace{-\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(0,\gamma))(\frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t}-\frac{ic(0)t}{1+ic'(0)\gamma t})(1+ic'(0)\gamma t)(-\frac{\frac{d}{d\alpha}\breve{h}(0,\gamma)}{1+ic'(0)\gamma t})}_{B_{2,2}}$$

$$\underbrace{+\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(2\alpha,\gamma))(\frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t}-\frac{ic(2\alpha)t}{1+ic'(2\alpha)\gamma t})(\frac{(\frac{d}{d\alpha}\breve{h})(\alpha,\gamma)}{1+ic'(\alpha)\gamma t})(1+ic'(2\alpha)\gamma t)}_{B_{3,1}}$$

$$\underbrace{+\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(2\alpha,\gamma))(\frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t}-\frac{ic(2\alpha)t}{1+ic'(2\alpha)\gamma t})(-\frac{(\frac{d}{d\alpha}\breve{h})(2\alpha,\gamma)}{1+ic'(2\alpha)\gamma t})(1+ic'(2\alpha)\gamma t)}_{B_{3,2}}$$

$$\underbrace{+\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(2\alpha,\gamma))(\frac{2ic(\alpha)t}{1+ic'(\alpha)\gamma t}-\frac{ic(2\alpha)t}{1+ic'(2\alpha)\gamma t})(\partial_\alpha h)(2\alpha,\gamma)}_{B_4}$$

$$\underbrace{-\nabla\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(2\alpha,\gamma))\cdot[(\frac{2ic(\alpha)t}{1+ic'(\alpha)\gamma t}-\frac{ic(2\alpha)t}{1+ic'(2\alpha)\gamma t})(\partial_\alpha\ddot{h})(2\alpha,\gamma)]\breve{h}(2\alpha,\gamma)}_{B_5}$$

$$\underbrace{-\breve{h}(2\alpha,\gamma)\nabla\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(2\alpha,\gamma))\cdot(\frac{(\frac{d}{d\alpha}\ddot{h})(2\alpha,\gamma)}{1+ic'(2\alpha)\gamma t})ic(2\alpha)t}_{B_6}$$

$$\underbrace{+(\breve{h}(2\alpha,\gamma)\nabla\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(2\alpha,\gamma))\cdot(\frac{(\frac{d}{d\alpha}\ddot{h})(2\alpha,\gamma)}{1+ic'(2\alpha)\gamma t})\frac{i2c(\alpha)t}{1+ic'(\alpha)\gamma t}(ic'(2\alpha)\gamma t+1))}_{B_7}$$

$$\underbrace{-\breve{h}(2\alpha,0)\nabla\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(2\alpha,0))\cdot(\frac{d}{d\alpha}\ddot{h})(2\alpha,0)\frac{i2c(\alpha)t}{1+ic'(\alpha)\gamma t}(1)}_{B_8}$$

$$\underbrace{+\frac{2ic(\alpha)t}{1+ic'(\alpha)\gamma t}\breve{h}(2\alpha,0)\nabla\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(2\alpha,0))\cdot(\frac{d\ddot{h}}{d\alpha})(2\alpha,0)}_{B_9}$$

$$\underbrace{+\frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t}\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(-\pi,\gamma))(1+ic'(-\pi)\gamma t)\frac{(\frac{d}{d\alpha}\breve{h})(\alpha,\gamma)}{1+ic'(\alpha)\gamma t}}_{B_{10}}$$

$$\underbrace{-\frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t}\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(0,\gamma))(1+ic'(0)\gamma t)\frac{(\frac{d}{d\alpha}\breve{h})(\alpha,\gamma)}{1+ic'(\alpha)\gamma t}}_{B_{11}}$$

$$\underbrace{-\frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t}\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(2\alpha,\gamma))(1+ic'(2\alpha)\gamma t)\frac{(\frac{d}{d\alpha}\breve{h})(\alpha,\gamma)}{1+ic'(\alpha)\gamma t}}_{B_{12}}$$

$$\underbrace{-\frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t}\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(\pi,\gamma))(1+ic'(\pi)\gamma t)\frac{(\frac{d}{d\alpha}\breve{h})(\alpha,\gamma)}{1+ic'(\alpha)\gamma t}}_{B_{13}}$$

$$\underbrace{+\breve{K}(\breve{h}(\alpha,\gamma)-\breve{h}(-\pi,\gamma))(ic(-\pi)t-\frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t}(1+ic'(-\pi)\gamma t))\frac{(\frac{d}{d\alpha}\breve{h})(\alpha,\gamma)}{1+ic'(\alpha)\gamma t}}_{B_{14}}$$

$$\underbrace{-\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(0,\gamma))(ic(0)t-\frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t}(1+ic'(0)\gamma t))\frac{(\frac{d}{d\alpha}\breve{h})(\alpha,\gamma)}{1+ic'(\alpha)\gamma t}}_{B_{15}}$$

$$\underbrace{+\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(2\alpha,\gamma))(ic(2\alpha)t-\frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t}(1+ic'(2\alpha)\gamma t))\frac{(\frac{d}{d\alpha}\breve{h})(\alpha,\gamma)}{1+ic'(\alpha)\gamma t}}_{B_{16}}$$

$$\underbrace{-\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(\pi,\gamma))(ic(\pi)t-\frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t}(1+ic'(\pi)\gamma t))\frac{(\frac{d}{d\alpha}\breve{h})(\alpha,\gamma)}{1+ic'(\alpha)\gamma t}}_{B_{17}}$$

$$=\underbrace{\frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t}\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(0,\gamma))(\frac{(\frac{d}{d\alpha}\breve{h})(\alpha,\gamma)}{1+ic'(\alpha)\gamma t})(1+ic'(0)\gamma t)}_{C_1}$$

$$\underbrace{+\frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t}\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(2\alpha,\gamma))(\frac{(\frac{d}{d\alpha}\breve{h})(\alpha,\gamma)}{1+ic'(\alpha)\gamma t})(1+ic'(2\alpha)\gamma t)}_{C_2}$$

$$\underbrace{+\frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t}\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(0,\gamma))(-\frac{\frac{d}{d\alpha}\breve{h}(0,\gamma)}{1+ic'(0)\gamma t})(1+ic'(0)\gamma t)}_{C_3}$$

$$\underbrace{+\frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t}\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(2\alpha,\gamma))(-\frac{(\frac{d}{d\alpha}\breve{h})(2\alpha,\gamma)}{1+ic'(2\alpha)\gamma t})(1+ic'(2\alpha)\gamma t)}_{C_4}$$

$$\underbrace{-\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(0,\gamma))(\frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t}-\frac{ic(0)t}{1+ic'(0)\gamma t})(1+ic'(0)\gamma t)(\frac{(\frac{d}{d\alpha}\breve{h})(\alpha,\gamma)}{1+ic'(\alpha)\gamma t})}_{C_5}$$

$$\underbrace{-\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(0,\gamma))(\frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t}-\frac{ic(0)t}{1+ic'(0)\gamma t})(1+ic'(0)\gamma t)(-\frac{\frac{d}{d\alpha}\breve{h}(0,\gamma)}{1+ic'(0)\gamma t})}_{C_3}$$

$$\underbrace{+\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(2\alpha,\gamma))(\frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t}-\frac{ic(2\alpha)t}{1+ic'(2\alpha)\gamma t})(\frac{(\frac{d}{d\alpha}\breve{h})(\alpha,\gamma)}{1+ic'(\alpha)\gamma t})(1+ic'(2\alpha)\gamma t)}_{C_6}$$

$$+\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(2\alpha,\gamma))(\underbrace{\frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t}}_{C_4}\underbrace{-\frac{ic(2\alpha)t}{1+ic'(2\alpha)\gamma t}}_{C_7})(-\frac{(\frac{d}{d\alpha}\breve{h})(2\alpha,\gamma)}{1+ic'(2\alpha)\gamma t})(1+ic'(2\alpha)\gamma t)$$

$$+\breve{K}(\ddot{h}(\alpha,\gamma)-\ddot{h}(2\alpha,\gamma))(\underbrace{\frac{2ic(\alpha)t}{1+ic'(\alpha)\gamma t}}_{C_4}\underbrace{-\frac{ic(2\alpha)t}{1+ic'(2\alpha)\gamma t}}_{C_7})(\partial_\alpha h)(2\alpha,\gamma)$$

$$
\begin{aligned}
&- \nabla \breve{K}(\ddot{h}(\alpha,\gamma) - \ddot{h}(2\alpha,\gamma)) \cdot [(\underbrace{\frac{2ic(\alpha)t}{1+ic'(\alpha)\gamma t}}_{C_8} \underbrace{- \frac{ic(2\alpha)t}{1+ic'(2\alpha)\gamma t}}_{C_9})(\partial_\alpha \ddot{h})(2\alpha,\gamma)] \breve{h}(2\alpha,\gamma) \\
&\underbrace{-\breve{h}(2\alpha,\gamma)\nabla \breve{K}(\ddot{h}(\alpha,\gamma) - \ddot{h}(2\alpha,\gamma)) \cdot (\frac{(\frac{d}{d\alpha}\ddot{h})(2\alpha,\gamma)}{1+ic'(2\alpha)\gamma t}) ic(2\alpha)t}_{C_9} \\
&\underbrace{+ (\breve{h}(2\alpha,\gamma)\nabla \breve{K}(\ddot{h}(\alpha,\gamma) - \ddot{h}(2\alpha,\gamma)) \cdot (\frac{(\frac{d}{d\alpha}\ddot{h})(2\alpha,\gamma)}{1+ic'(2\alpha)\gamma t}) \frac{i2c(\alpha)t}{1+ic'(\alpha)\gamma t}(ic'(2\alpha)\gamma t + 1))}_{C_8} \\
&\underbrace{-\breve{h}(2\alpha,0)\nabla \breve{K}(\ddot{h}(\alpha,\gamma) - \ddot{h}(2\alpha,0)) \cdot (\frac{d}{d\alpha}\ddot{h})(2\alpha,0) \frac{i2c(\alpha)t}{1+ic'(\alpha)\gamma t}(1)}_{C_{10}} \\
&\underbrace{+ \frac{2ic(\alpha)t}{1+ic'(\alpha)\gamma t}\breve{h}(2\alpha,0)\nabla \breve{K}(\ddot{h}(\alpha,\gamma) - \ddot{h}(2\alpha,0)) \cdot (\frac{d\ddot{h}}{d\alpha})(2\alpha,0)}_{C_{10}} \\
&\underbrace{+ \frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t}\breve{K}(\ddot{h}(\alpha,\gamma) - \ddot{h}(-\pi,\gamma))(1+ic'(-\pi)\gamma t)\frac{(\frac{d}{d\alpha}\breve{h})(\alpha,\gamma)}{1+ic'(\alpha)\gamma t}}_{C_{11}} \\
&\underbrace{- \frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t}\breve{K}(\ddot{h}(\alpha,\gamma) - \ddot{h}(0,\gamma))(1+ic'(0)\gamma t)\frac{(\frac{d}{d\alpha}\breve{h})(\alpha,\gamma)}{1+ic'(\alpha)\gamma t}}_{C_1} \\
&\underbrace{- \frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t}\breve{K}(\ddot{h}(\alpha,\gamma) - \ddot{h}(2\alpha,\gamma))(1+ic'(2\alpha)\gamma t)\frac{(\frac{d}{d\alpha}\breve{h})(\alpha,\gamma)}{1+ic'(\alpha)\gamma t}}_{C_2} \\
&\underbrace{- \frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t}\breve{K}(\ddot{h}(\alpha,\gamma) - \ddot{h}(\pi,\gamma))(1+ic'(\pi)\gamma t)\frac{(\frac{d}{d\alpha}\breve{h})(\alpha,\gamma)}{1+ic'(\alpha)\gamma t}}_{C_{12}} \\
&\underbrace{+ \breve{K}(\ddot{h}(\alpha,\gamma) - \ddot{h}(-\pi,\gamma))(ic(-\pi)t - \frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t}(1+ic'(-\pi)\gamma t))\frac{(\frac{d}{d\alpha}\breve{h})(\alpha,\gamma)}{1+ic'(\alpha)\gamma t}}_{C_{11}} \\
&\underbrace{- \breve{K}(\ddot{h}(\alpha,\gamma) - \ddot{h}(0,\gamma))(ic(0)t - \frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t}(1+ic'(0)\gamma t))\frac{(\frac{d}{d\alpha}\breve{h})(\alpha,\gamma)}{1+ic'(\alpha)\gamma t}}_{C_5} \\
&\underbrace{+ \breve{K}(\ddot{h}(\alpha,\gamma) - \ddot{h}(2\alpha,\gamma))(ic(2\alpha)t - \frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t}(1+ic'(2\alpha)\gamma t))\frac{(\frac{d}{d\alpha}\breve{h})(\alpha,\gamma)}{1+ic'(\alpha)\gamma t}}_{C_6} \\
&\underbrace{- \breve{K}(\ddot{h}(\alpha,\gamma) - \ddot{h}(\pi,\gamma))(ic(\pi)t - \frac{ic(\alpha)t}{1+ic'(\alpha)\gamma t}(1+ic'(\pi)\gamma t))\frac{(\frac{d}{d\alpha}\breve{h})(\alpha,\gamma)}{1+ic'(\alpha)\gamma t}}_{C_{12}} \\
&= 0.
\end{aligned}
$$

Here we use $c(\pi) = c(-\pi) = c(0) = 0$. □

Remark 3.11.12. *We could change* $[-\pi,\pi]$ *in lemma 3.11.11 to any interval* $[b_L,b_R]$ *satisfying* $b_L<0<b_R$. *The corresponding result holds for* $0<\alpha<\frac{b_R}{2}$.

Corollary 3.11.13. *Let* K *be meromorphic,* $0\le\tau(t)\le\frac{\pi}{4}$ $\breve{h}(\alpha,\gamma),\ddot{h}(\alpha,\gamma)\in C^1_\gamma([-1,1],C^1_\alpha[-\pi+\tau(t),\pi+\tau(t)])$. *If for fixed* $\alpha>0$, *there is no singular point in the integrals in* $M(h)$ *below and* $c(0)=c(\pi+\tau(t))=c(-\pi+\tau(t))=0$, $c(\alpha)\in W^{2,\infty}$, *then when* $0<\alpha<\frac{\pi}{4}$, *for*

$$\begin{aligned}\tilde{M}(\breve{h},\ddot{h})=&\int_{-\tau(t)}^{2\alpha+\tau(t)}\breve{K}(\ddot{h}(\alpha+\tau(t),\gamma)-\ddot{h}(\beta+\tau(t),\gamma))(\frac{(\frac{d}{d\alpha}\breve{h})(\alpha+\tau(t),\gamma)}{1+ic'(\alpha+\tau(t))\gamma t}-\frac{\frac{d}{d\beta}\breve{h}(\beta+\tau(t),\gamma)}{1+ic'(\beta+\tau(t))\gamma t})\\&(1+ic'(\beta+\tau(t))\gamma t)d\beta\\&+\breve{K}(\ddot{h}(\alpha+\tau(t),\gamma)-\ddot{h}(2\alpha+2\tau(t),\gamma))\breve{h}(2\alpha+2\tau(t),\gamma)\\&+\int_0^\gamma\breve{h}(2\alpha+2\tau(t),\eta)\nabla\breve{K}(\ddot{h}(\alpha+\tau(t),\gamma)-\ddot{h}(2\alpha+2\tau(t),\eta))\cdot(\frac{(\frac{d}{d\alpha}\ddot{h})(2\alpha+2\tau(t),\eta)}{1+ic'(2\alpha+2\tau(t))\eta t})ic(2\alpha+2\tau(t))td\eta\\&-\int_{2\alpha+\tau(t)}^{\pi}\breve{h}(\beta+\tau(t),0)\nabla\breve{K}(\ddot{h}(\alpha+\tau(t),\gamma)-\ddot{h}(\beta+\tau(t),0))\cdot\frac{d\ddot{h}}{d\beta}(\beta+\tau(t),0)d\beta\\&+\int_{[-\pi,0]\cup[2\alpha+\tau(t),\pi]}\breve{K}(\ddot{h}(\alpha+\tau(t),\gamma)-\ddot{h}(\beta+\tau(t),\gamma))(1+ic'(\beta+\tau(t))\gamma t)d\beta\frac{(\frac{d}{d\alpha}\breve{h})(\alpha+\tau(t),\gamma)}{1+ic'(\alpha+\tau(t))\gamma t},\end{aligned}$$

we have

$$\begin{aligned}A(\tilde{M})=&\underbrace{\int_{-\tau(t)}^{2\alpha+\tau(t)}D_{\ddot{h}}(\breve{K}(\ddot{h}(\alpha+\tau(t),\gamma)-\ddot{h}(\beta+\tau(t),\gamma)))[A(\ddot{h})](\frac{(\frac{d}{d\alpha}\breve{h})(\alpha+\tau(t),\gamma)}{1+ic'(\alpha+\tau(t))\gamma t}-\frac{\frac{d}{d\beta}\breve{h}(\beta+\tau(t),\gamma)}{1+ic'(\beta+\tau(t))\gamma t})}_{FM_{1,1}}\\&\underbrace{(1+ic'(\beta+\tau(t))\gamma t)d\beta}_{FM_{1,1}}\\&\underbrace{+\int_{-\tau(t)}^{2\alpha+\tau(t)}\breve{K}(\ddot{h}(\alpha+\tau(t),\gamma)-\ddot{h}(\beta+\tau(t),\gamma))(\frac{\frac{d}{d\alpha}A(\breve{h})(\alpha+\tau(t),\gamma)}{1+ic'(\alpha+\tau(t))\gamma t}-\frac{\frac{d}{d\beta}A(\breve{h})(\beta+\tau(t),\gamma)}{1+ic'(\beta+\tau(t))\gamma t})}_{FM_{1,2}}\\&\underbrace{(1+ic'(\beta+\tau(t))\gamma t)d\beta}_{FM_{1,2}}\\&\underbrace{+D_{\ddot{h}}\breve{K}(\ddot{h}(\alpha+\tau(t),\gamma)-\ddot{h}(2\alpha+2\tau(t),\gamma))[A(\ddot{h})]\breve{h}(2\alpha+2\tau(t),\gamma)}_{FM_{2,1}}\\&\underbrace{+\breve{K}(\ddot{h}(\alpha+\tau(t),\gamma)-\ddot{h}(2\alpha+2\tau(t),\gamma))A(\breve{h})(2\alpha+2\tau(t),\gamma)}_{FM_{2,2}}\\&\underbrace{+\int_0^\gamma\breve{h}(2\alpha+2\tau(t),\eta)\nabla^2\breve{K}(\ddot{h}(\alpha+\tau(t),\gamma)-\ddot{h}(2\alpha+2\tau(t),\eta))[(A[\ddot{h}](\alpha+\tau(t),\gamma)),\frac{(\frac{d}{d\alpha}\ddot{h})(2\alpha+2\tau(t),\eta)}{1+ic'(2\alpha+2\tau(t))\eta t}]}_{FM_{3,1}}\end{aligned}$$

$$
\begin{aligned}
&\underbrace{ic(2\alpha+2\tau(t))td\eta}_{FM_{3,1}}\\
&\underbrace{-\int_0^\gamma \breve{h}(2\alpha+2\tau(t),\eta)\nabla^2\breve{K}(\ddot{h}(\alpha+\tau(t),\gamma)-\ddot{h}(2\alpha+2\tau(t),\eta))[(\frac{2ic(\alpha+\tau(t))t(1+ic'(2\alpha+2\tau(t))\eta t)}{(1+ic'(\alpha+\tau(t))\gamma t)(ic(2\alpha+2\tau(t))t)}}_{FM_{3,2}}\\
&\underbrace{A(\ddot{h})(2\alpha+2\tau(t),\eta)),\frac{(\frac{d}{d\alpha}\ddot{h})(2\alpha+2\tau(t),\eta)}{1+ic'(2\alpha+2\tau(t))\eta t}]ic(2\alpha+2\tau(t))td\eta}_{FM_{3,2}}\\
&\underbrace{+\int_0^\gamma \breve{h}(2\alpha+2\tau(t),\eta)\nabla\breve{K}(\ddot{h}(\alpha+\tau(t),\gamma)-\ddot{h}(2\alpha+2\tau(t),\eta))\cdot\frac{2ic(\alpha+\tau(t))t(1+ic'(2\alpha+2\tau(t))\eta t)}{(1+ic'(\alpha+\tau(t))\gamma t)(ic(2\alpha+2\tau(t))t)}}_{FM_{3,3}}\\
&\underbrace{(\frac{(\frac{d}{d\alpha}A_0(\ddot{h}))(2\alpha,\eta)}{1+ic'(2\alpha)\eta t})ic(2\alpha)td\eta}_{FM_{3,3}}\\
&\underbrace{+\int_0^\gamma A(\breve{h})(2\alpha+2\tau(t),\eta)\frac{2ic(\alpha+\tau(t))t(1+ic'(2\alpha+2\tau(t))\eta t)}{(1+ic'(\alpha+\tau(t))\gamma t)(ic(2\alpha+2\tau(t))t)}\nabla\breve{K}(\ddot{h}(\alpha+\tau(t),\gamma)-\ddot{h}(2\alpha+2\tau(t),\eta))}_{FM_{3,4}}\\
&\underbrace{\cdot(\frac{(\frac{d}{d\alpha}\ddot{h})(2\alpha+2\tau(t),\eta)}{1+ic'(2\alpha+2\tau(t))\eta t})ic(2\alpha+2\tau(t))td\eta}_{FM_{3,4}}\\
&\underbrace{-\int_{2\alpha+\tau(t)}^\pi \breve{h}(\beta+\tau(t),0)\nabla^2\breve{K}(\ddot{h}(\alpha+\tau(t),\gamma)-\ddot{h}(\beta+\tau(t),0))[A(\ddot{h})(\alpha+\tau(t),\gamma),\frac{d\ddot{h}}{d\beta}(\beta+\tau(t),0)]d\beta}_{FM_4}\\
&\underbrace{+\int_{[-\pi,-\tau(t)]\cup[2\alpha+\tau(t),\pi]}\nabla\breve{K}(\ddot{h}(\alpha+\tau(t),\gamma)-\ddot{h}(\beta+\tau(t),\gamma))\cdot(A(\ddot{h})(\alpha+\tau(t),\gamma)-A(\ddot{h})(\beta+\tau(t),\gamma))}_{FM_{5,1}}\\
&\underbrace{(1+ic'(\beta+\tau(t))\gamma t))d\beta\frac{(\frac{d}{d\alpha}\breve{h})(\alpha+\tau(t),\gamma)}{1+ic'(\alpha+\tau(t))\gamma t}}_{FM_{5,1}}\\
&\underbrace{+\int_{[-\pi,-\tau(t)]\cup[2\alpha+\tau(t),\pi]}\breve{K}(\ddot{h}(\alpha+\tau(t),\gamma)-\ddot{h}(\beta+\tau(t),\gamma))(1+ic'(\beta+\tau(t))\gamma t))d\beta\frac{\frac{d}{d\alpha}A(\breve{h})(\alpha+\tau(t),\gamma)}{1+ic'(\alpha+\tau(t))\gamma t}}_{FM_{5,2}}.
\end{aligned}
$$

Proof. By letting $[b_L,b_R]=[-\pi+\tau(t),\pi+\tau(t)]$ in Remark 3.11.12, we have

$$\tilde{M}(\breve{h},\ddot{h})(\alpha)=M(\breve{h},\ddot{h})(\alpha+\tau(t)),$$

and

$$A(\tilde{M}(\breve{h},\ddot{h})(\alpha))=A_0(M(\breve{h},\ddot{h}))(\alpha+\tau(t)).$$

Then we have the result. □

Lemma 3.11.14. *Let K be meromorphic, $\breve{h}(\alpha,\gamma) \in C^1_\gamma([-1,1], C^1_\alpha[-\pi,\pi])$. If there is no singular point in the integral below and $c(\pi) = c(-\pi) = 0$, $c(\alpha) \in W^{2,\infty}$, then we have*

$$A_0(\int_{-\pi}^{\pi} \breve{K}(\breve{h}(\beta,\gamma))(1+ic'(\beta)\gamma t)d\beta) = \int_{-\pi}^{\pi} D_h\breve{K}(\breve{h}(\beta,\gamma))[A_0(\breve{h})(\beta,\gamma)](1+ic'(\beta)\gamma t)d\beta$$

Proof.

$$\begin{aligned}
&A_0(\int_{-\pi}^{\pi} \breve{K}(\breve{h}(\beta,\gamma))(1+ic'(\beta)\gamma t))d\beta \\
&= -\int_{-\pi}^{\pi} \nabla\breve{K}(\breve{h}(\beta,\gamma)) \cdot \frac{d\breve{h}(\beta,\gamma)}{d\gamma}(1+ic'(\beta)\gamma t)d\beta\partial_\alpha\breve{h}(\alpha) - \int_{-\pi}^{\pi} \breve{K}(\breve{h}(\beta,\gamma))ic'(\beta)\gamma t d\beta \\
&= \int_{-\pi}^{\pi} \nabla\breve{K}(\breve{h}(\beta,\gamma)) \cdot [(\frac{ic(\beta)t}{1+ic'(\beta)\gamma t}\partial_\beta - \partial_\gamma)\breve{h}(\beta,\gamma)](1+ic'(\beta)t)d\beta \\
&- \int_{-\pi}^{\pi} \nabla\breve{K}(\breve{h}(\beta,\gamma)) \cdot \partial_\beta\breve{h}(\beta,\gamma)ic(\beta)td\beta - \int_{-\pi}^{\pi} \breve{K}(\breve{h}(\beta,\gamma))ic'(\beta)td\beta \\
&= \int_{-\pi}^{\pi} D_h\breve{K}(\breve{h}(\beta))A_0(\breve{h})(\beta,\gamma)(1+ic'(\beta)\gamma t)d\beta.
\end{aligned}$$

□

Corollary 3.11.15. *Let K be meromorphic, $\breve{h}(\alpha,\gamma), X(\alpha,\gamma) \in C^1_\gamma([-1,1], C^1_\alpha[-\pi,\pi])$. If there is no singular point in the integrals below and $c(\pi) = c(-\pi) = 0$, $c(\alpha) \in W^{2,\infty}$, then we have*

$$\begin{aligned}
&A_0(\int_{-\pi}^{\pi} \breve{K}(\breve{h}(\beta,\gamma))X(\beta,\gamma)(1+ic'(\beta)\gamma t)d\beta) \\
&= \int_{-\pi}^{\pi} D_h\breve{K}(\breve{h}(\beta,\gamma))[A_0(\breve{h})(\beta,\gamma)]X(\beta,\gamma)(1+ic'(\beta)\gamma t)d\beta \\
&+ \int_{-\pi}^{\pi} \breve{K}(\breve{h}(\beta,\gamma))A_0(X)(\beta,\gamma)(1+ic'(\beta)\gamma t))d\beta.
\end{aligned}$$

Proof. We can use lemma 3.11.14, by letting the new $\tilde{K}(\breve{h}(\beta,\gamma), X) = \breve{K}(\breve{h}(\beta,\gamma), X(\beta,\gamma))$. □

By using the similar method, we also have

Corollary 3.11.16. *Let K be meromorphic, $\breve{h}(\alpha,\gamma), X(\alpha,\gamma) \in C^1_\gamma([-1,1], C^1_\alpha[-\pi,\pi])$. If there is no singular point in the integrals below and $c(\pi+\tau(t)) = c(-\pi+\tau(t)) = 0$, $c(\alpha+\tau(t)) \in W^{2,\infty}$, then we have*

$$A(\int_{-\pi}^{\pi} \breve{K}(\breve{h}(\beta,\gamma))X(\beta,\gamma)(1+ic'(\beta+\tau(t))\gamma t)d\beta)$$

$$= \int_{-\pi}^{\pi} D_h \breve{K}(\breve{h}(\beta,\gamma))[A(\breve{h})(\beta,\gamma)]X(\beta,\gamma)(1+ic'(\beta+\tau(t))\gamma t)d\beta$$
$$+ \int_{-\pi}^{\pi} \breve{K}(\breve{h}(\beta,\gamma))A(X)(\beta,\gamma)(1+ic'(\beta+\tau(t))\gamma t))d\beta.$$

Lemma 3.11.17. *If* $\breve{h} \in C^1([-1,1],H^1[-\pi,\pi])$ *with* $\breve{h}(\alpha)=0$ *when* $\alpha \le 0$, $c(\alpha) \in W^{2,\infty}$ *and* $c(0)=0$, *then we have*

$$A_0(D_0^{-i}(\breve{h})) = D_0^{-i}(A_0(\breve{h})), \tag{3.174}$$

where

$$D_0^{-1}(h) = \int_0^{\alpha} \breve{h}(\alpha_1,\gamma)(1+ic'(\alpha_1)\gamma t)d\alpha_1.$$

Proof. When i=1, we have

$$A_0(D^{-1}(\breve{h})) = A_0(\int_0^{\alpha} \breve{h}(\alpha_1,\gamma)(1+ic'(\alpha_1)\gamma t)d\alpha_1)$$
$$= ic(\alpha)t\breve{h}(\alpha,\gamma) - \int_0^{\alpha} \partial_\gamma \breve{h}(\alpha_1,\gamma)(1+ic'(\alpha_1)\gamma t)d\alpha_1 - \int_0^{\alpha} \breve{h}(\alpha_1,\gamma)ic'(\alpha_1)td\alpha_1$$
$$= ic(\alpha)t\breve{h}(\alpha,\gamma) + \int_0^{\alpha} (\frac{ic(\alpha_1)t}{1+ic'(\alpha_1)\gamma t}\partial_{\alpha_1} - \partial_\gamma)\breve{h}(\alpha_1,\gamma)(1+ic'(\alpha_1)\gamma t))d\alpha_1$$
$$- \int_0^{\alpha} ic(\alpha_1)t\partial_{\alpha_1}\breve{h}(\alpha_1,\gamma) + \breve{h}(\alpha_1,\gamma)ic'(\alpha_1)td\alpha_1$$

Then we can do integration by parts to the second term and have

$$A_0(D^{-1}(h)) = ic(\alpha)t\breve{h}(\alpha,\gamma) + \int_0^{\alpha} (\frac{ic(\alpha_1)t}{1+ic'(\alpha_1)\gamma t}\partial_{\alpha_1} - \partial_\gamma)\breve{h}(\alpha_1,\gamma)(1+ic'(\alpha_1)\gamma t))d\alpha_1$$
$$- ic(\alpha_1)t\partial_\alpha \breve{h}(\alpha_1,\gamma)|_0^{\alpha}$$
$$= D^{-1}(A_0(\breve{h})).$$

Since $h \in C^1([-1,1],H^1[-\pi,\pi])$, and $h=0$ when $\alpha \le 0$, we have $D^{-i}(h) \in C^1([-1,1],H^{1+i}[-\pi,\pi])$ and $D^{-i}(h)=0$ when $\alpha \le 0$. Then for $j \ge 1$, we could do this process j times and get the result. □

By using the similar method, we have

Corollary 3.11.18. *If* $h \in C^1([-1,1],H^1[-\pi,\pi])$ *with* $\breve{h}(\alpha)=0$ *when* $\alpha \le -\tau(t)$, $c(\alpha) \in W^{2,\infty}$ *and* $c(0)=0$, *then we have*

$$A(D^{-i}(h)) = D^{-i}(A(h)). \tag{3.175}$$

where

$$D^{-1}(h) = \int_{-\tau(t)}^{\alpha} \breve{h}(\alpha_1,\gamma)(1+ic'(\alpha_1+\tau(t))\gamma t)d\alpha_1.$$

Chapter 4

The radial symmetry properties of stationary and uniformly rotating solutions of the 2D Euler/g-SQG equations

4.1 Introduction

Let us start by considering the initial value problem for the two-dimensional incompressible Euler equation in vorticity form. Here the evolution of the vorticity ω is given by

$$\begin{cases} \partial_t \omega + u \cdot \nabla \omega = 0 & \text{in } \mathbb{R}^2 \times \mathbb{R}_+, \\ u(\cdot, t) = -\nabla^\perp (-\Delta)^{-1} \omega(\cdot, t) & \text{in } \mathbb{R}^2, \\ \omega(\cdot, 0) = \omega_0 & \text{in } \mathbb{R}^2, \end{cases} \tag{4.1}$$

where $\nabla^\perp := (-\partial_{x_2}, \partial_{x_1})$. Note that we can express u as $u(\cdot, t) = \nabla^\perp(\omega(\cdot, t) * \mathcal{N})$, where $\mathcal{N}(x) := \frac{1}{2\pi} \ln |x|$ is the Newtonian potential in two dimensions. More generally, the 2D Euler equation belongs to the following family of active scalar equations indexed by a parameter α, $(0 \leq \alpha < 2)$,

known as the generalized Surface Quasi-Geostrophic (gSQG) equations:

$$\begin{cases} \partial_t \omega + u \cdot \nabla \omega = 0 & \text{in } \mathbb{R}^2 \times \mathbb{R}_+, \\ u(\cdot,t) = -\nabla^\perp (-\Delta)^{-1+\frac{\alpha}{2}} \omega(\cdot,t) & \text{in } \mathbb{R}^2, \\ \omega(\cdot,0) = \omega_0 & \text{in } \mathbb{R}^2. \end{cases} \tag{4.2}$$

Here we can also express the Biot–Savart law as

$$u(\cdot,t) = \nabla^\perp (\omega(\cdot,t) * K_\alpha), \tag{4.3}$$

where K_α is the fundamental solution for $-(-\Delta)^{-1+\frac{\alpha}{2}}$, that is,

$$K_\alpha(x) = \begin{cases} \frac{1}{2\pi} \ln|x| & \text{for } \alpha = 0, \\ -C_\alpha |x|^{-\alpha} & \text{for } \alpha \in (0,2), \end{cases} \tag{4.4}$$

where $C_\alpha = \frac{1}{2\pi} \frac{\Gamma(\frac{\alpha}{2})}{2^{1-\alpha}\Gamma(1-\frac{\alpha}{2})}$ is a positive constant only depending on α.

We will be focusing on establishing radial symmetry properties for stationary and uniformly-rotating solutions to equations (4.1) and (4.2). We either work with the *patch* setting, where $\omega(\cdot,t) = 1_{D(t)}$ is an indicator function of a bounded set that moves with the fluid, or the smooth setting, where $\omega(\cdot,t)$ is smooth and compactly-supported in x. For well-posedness results for patch solutions, see the global well-posedness results [6, 20] for (4.1), and local well-posedness results [84, 36, 64, 23] for (4.2) with $\alpha \in (0,2)$.

Let us begin with the definition of a stationary/uniformly-rotating solution in the patch setting. For a bounded open set $D \subset \mathbb{R}^2$, we say $\omega = 1_D$ is a *stationary patch* solution to (4.2) for some $\alpha \in [0,2)$ if $u(x) \cdot \vec{n}(x) = 0$ on ∂D, with u given by (4.3). This leads to the integral equation

$$1_D * \mathcal{K}_\alpha \equiv C_i \quad \text{on } \partial D, \tag{4.5}$$

where the constant C_i can differ on different connected components of ∂D. And if $\omega(x,t) = 1_D(R_{\Omega t}x)$ is a *uniformly-rotating patch* solution with angular velocity Ω (where $R_{\Omega t}x$ rotates a vector $x \in \mathbb{R}^2$ counter-clockwise by angle Ωt about the origin), then 1_D becomes stationary in the rotating frame with angular velocity Ω, that is, $\left(\nabla^\perp (1_D * \mathcal{K}_\alpha) - \Omega x^\perp\right) \cdot \vec{n}(x) = 0$ on ∂D. This is equivalent to

$\nabla^\perp(1_D * \mathcal{K}_\alpha - \frac{\Omega}{2}|x|^2) \cdot \vec{n}(x) = 0$ on ∂D, and as a result we have

$$1_D * \mathcal{K}_\alpha - \frac{\Omega}{2}|x|^2 \equiv C_i \quad \text{on } \partial D, \tag{4.6}$$

where C_i again can take different values along different connected components of ∂D. Note that a stationary patch D also satisfies (4.6) with $\Omega = 0$, and it can be considered as a special case of uniformly-rotating patch with zero angular velocity.

Likewise, in the smooth setting, if $\omega(x,t) = \omega_0(R_{\Omega t}x)$ is a uniformly-rotating solution of (4.2) with angular velocity Ω (which becomes a stationary solution in the $\Omega = 0$ case), then we have $(\nabla^\perp(\omega_0 * K_\alpha) - \Omega x^\perp) \cdot \nabla\omega_0 = 0$. As a result, ω_0 satisfies

$$\omega_0 * \mathcal{K}_\alpha - \frac{\Omega}{2}|x|^2 \equiv C_i \quad \text{on each connected component of a regular level set of } \omega_0, \tag{4.7}$$

where C_i can be different if a regular level set $\{\omega_0 = c\}$ has multiple connected components.

Clearly, every radially symmetric patch/smooth function automatically satisfies (4.6)/(4.7) for all $\Omega \in \mathbb{R}$. The goal of this section is to address the complementary question, which can be roughly stated as following:

Question 4.1.1. *In the patch or smooth setting, under what condition must a stationary/uniformly-rotating solution be radially symmetric?*

4.1.1 The rigidity results for 2D Euler in the patch setting

Let us deal with the patch setting first. In the previous literature, affirmative answers to Question 4.1.1 have only been only obtained for simply-connected patches. Fraenkel [34, Chapter 4] proved that if D satisfies (4.6) with the *same* constant C on the whole ∂D, then D must be a disk. Hmidi [50] proved the case of $\Omega < 0$ with some additional convexity assumption. In the special case $\Omega = \frac{1}{2}$, Hmidi [50] also showed that a simply-connected uniformly-rotating patch D must be a disk.

On the other hand, it is known that there can be non-radial uniformly-rotating patches for $\Omega \in (0, \frac{1}{2})$. Kirchhoff proved [63], any ellipse D with semiaxes a, b is a uniformly-rotating patch with angular velocity $\frac{ab}{(a+b)^2}$. Deem–Zabusky [32] numerically found families of patch solution with m-fold symmetry by bifurcating from a disk at explicit angular velocities $\Omega_m^0 = \frac{m-1}{2m}$. Burbea gave the first rigorous proof of their existence [9]. De la Hoz–Hmidi–Mateu–Verdera showed the existence of m-fold doubly-connected non-radial patches bifurcating at any angular velocity $\Omega \in \left(0, \frac{1}{2}\right)$ from some annulus of radii $b \in (0,1)$ and 1 [31].

Our first main result gives a complete answer to Question 4.1.1 for 2D Euler in the patch setting. Note that D is allowed to be disconnected, and each connected component can be non-simply-connected. Figure 4.1 illustrates a comparison of our result (in red color) with the previous results (in black color).

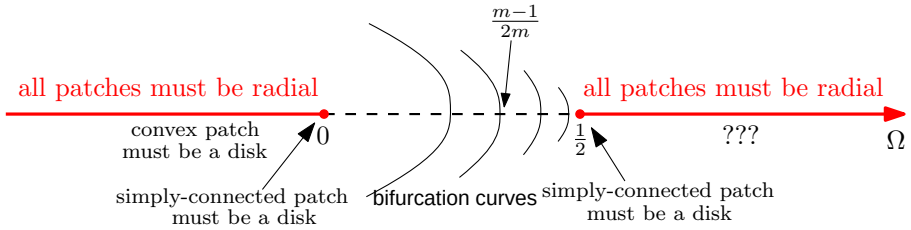


Figure 4.1: For 2D Euler in the patch setting, previous results on Question 4.1.1 are summarized in black color. Our results in Theorem A are colored in red.

Theorem 4.1.2. *Let $D \subset \mathbb{R}^2$ be a bounded open set with $C^{1,\gamma}$ boundary. We assume D is a stationary/uniformly-rotating patch of (4.2) when $\alpha = 0$. Then D must be radially symmetric if $\Omega \in (-\infty, 0) \cup [\frac{1}{2}, \infty)$, and radially symmetric up to a translation if $\Omega = 0$.*

4.1.2 The rigidity results for 2D Euler in the smooth setting

One of our main motivations is to find sufficient rigidity conditions in terms of the vorticity, such that the only stationary/uniformly-rotating solutions are radial ones. Heuristically speaking, this belongs to the broader class of "Liouville Theorem" type of results, which show that solutions satisfying certain conditions must have a simpler geometric structure, such as being constant (in one direction, or all directions) or being radial. In the literature we could not find any conditions on 2D Euler that leads to radial symmetry, although several other Liouville-type results have been established for 2D fluid equations: For 2D Euler, Hamel–Nadirashvili [47, 46] proved that any stationary solution without a stagnation point must be a shear flow. (But note that this result does not apply to our setting (4.7), since the velocity field u associated with any compactly-supported ω_0 must have a stagnation point). See also the Liouville theorem by Koch–Nadirashvili–Seregin–Šverák for the 2D Navier–Stokes equations [65].

Regarding uniformly-rotating smooth solutions ($\Omega \neq 0$) for 2D Euler, Castro–Córdoba–Gómez-Serrano [19] were able to desingularize a vortex patch to produce a smooth m-fold V-state with $\Omega \sim \frac{m-1}{2m} > 0$ for $m \geq 2$. Recently García–Hmidi–Soler [37] studied the construction of V-states bifurcating from other radial profiles (Gaussians and piecewise quadratic functions).

Our second main result is the following theorem, which gives radial symmetry of compactly supported stationary/uniformly-rotating solutions in the smooth setting for $\Omega \leq 0$, under the additional assumption $\omega_0 \geq 0$:

Theorem 4.1.3. *Let $\omega_0 \geq 0$ be smooth and compactly-supported. Then ω_0 must be radially symmetric if $\Omega < 0$, and radially symmetric up to a translation if $\Omega = 0$.*

Although the extra assumption $\omega_0 \geq 0$ might seem unnatural at the first glance, it is indeed necessary (see the subsection 4.1.9).

4.1.3 The rigidity results for g-SQG equations

Even though the kernels K_α are qualitatively similar for all $\alpha \in [0, 2)$, there is a key difference on the symmetry v.s.non-symmetry results between the cases $\alpha = 0$ and $\alpha > 0$: For the 2D Euler equation ($\alpha = 0$), we proved in Theorem A that any rotating patch D with $\Omega \leq 0$ must be radial, even if D is not simply-connected. However, this result is not true for any $\alpha \in (0, 2)$: de la Hoz–Hassainia–Hmidi–Mateu [29] showed that there exist non-radial patches bifurcating from annuli at $\Omega < 0$ and Gómez-Serrano [41] constructed non-radial, doubly connected stationary patches ($\Omega = 0$). Therefore we cannot expect a non-simply-connected rotating patch D with $\Omega \leq 0$ to be radial for $\alpha \in (0, 2)$.

Positive answers to Question 4.1.1 were obtained in the $\Omega = 0$, simply-connected patch case for $\alpha < \frac{5}{3}$ in the following works. Reichel [82, Theorem 2], Lu–Zhu [70] and Han–Lu–Zhu [48] proved this result for $\alpha \in [0, 1)$. Here [70] also covered generic radially increasing potentials not too singular at the origin (which include all Riesz potentials $K_{\alpha,d}$ with $\alpha \in [0, 1)$). Recently, Choksi–Neumayer–Topaloglu [21, Theorem 1.3] further pushed the range to $\alpha \in [0, \frac{5}{3})$, leaving the range $\alpha \in [\frac{5}{3}, 2)$ an open problem. In all these results for $\alpha > 0$, ∂D was assumed to be at least C^1. All above results were obtained using the moving plane method.

In our result, we use a completely different approach to give an affirmative answer to Question 4.1.8 for all $\Omega \leq 0$ and $\alpha \in [0, 2)$, under a weaker assumption on the regularity of ∂D.

Theorem 4.1.4. *Let D be a simply-connected bounded open set with rectifiable boundary. If D is a stationary/uniformly-rotating patch of (4.2) with $\Omega \leq 0$, then it must be a disk.*

In addition, in the smooth setting (4.7), we prove a similar result in Corollary 7.1.4 for uniformly-rotating solutions with $\Omega \leq 0$ for all $\alpha \in [0, 2)$: if the super level-sets $\{\omega_0 > h\}$ are all simply-connected for all $h > 0$, then ω_0 must be radially decreasing.

In g-SQG there also exists non-radial uniformly rotating patches. Hassainia–Hmidi [49] showed

the existence of V-states with C^k boundary regularity in the case $0 < \alpha < 1$, and in [16], Castro–Córdoba–Gómez-Serrano upgraded the result to show existence and C^∞ boundary regularity in the remaining open cases: $\alpha \in [1, 2)$ for the existence, $\alpha \in (0, 2)$ for the regularity. In that case, the solutions bifurcate at angular velocities given by $\Omega_m^\alpha := 2^{\alpha-1}\frac{\Gamma(1-\alpha)}{\Gamma\left(1-\frac{\alpha}{2}\right)^2}\left(\frac{\Gamma\left(1+\frac{\alpha}{2}\right)}{\Gamma\left(2-\frac{\alpha}{2}\right)} - \frac{\Gamma\left(m+\frac{\alpha}{2}\right)}{\Gamma\left(m+1-\frac{\alpha}{2}\right)}\right)$. This boundary regularity was subsequently improved to analytic in [17]. See also [52] for another family of rotating solutions, [29, 83] for the doubly connected case and [18] for a construction in the smooth setting.

One can check that Ω_m^α are increasing functions of m for any α, whose limit is a finite number $\Omega^\alpha := 2^{\alpha-1}\frac{\Gamma(1-\alpha)}{\Gamma\left(1-\frac{\alpha}{2}\right)^2}\frac{\Gamma\left(1+\frac{\alpha}{2}\right)}{\Gamma\left(2-\frac{\alpha}{2}\right)}$ for $\alpha \in [0, 1)$, and $+\infty$ if $\alpha \geq 1$. It is then a natural question to ask whether there exist V-states (with area π) that rotate with angular velocity faster than Ω_α for $\alpha \in (0, 1)$. Our fourth main theorem answers this question among all simply-connected patches:

Theorem 4.1.5 (= Theorem 7.2.1). *For $\alpha \in (0, 1)$, let 1_D be a simply connected V-state of area π and let its angular velocity be $\Omega \geq \Omega^\alpha$. Then D must be the unit disk.*

We illustrate a comparison of our results with the previous results (in black color) in Figure 4.2.

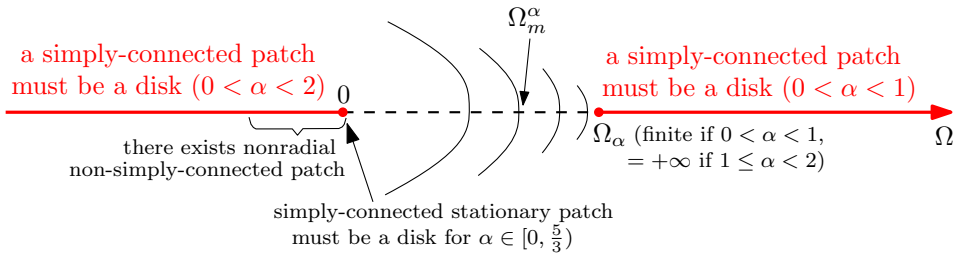


Figure 4.2: For gSQG in the patch setting, previous results on Question 4.1.1 are summarized in black color, with our results in Theorem 4.1.6 and 4.1.5 colored in red.

4.1.4 The outline of the proof for theorem 4.1.2, 4.1.3, 4.1.4 and 4.1.5

While all the previous symmetry results on Question 4.1.1 and 4.1.8 [34, 70, 48, 50, 82, 21] are done by moving plane methods, our approaches are completely different, which have a variational flavor.

Chapter 5 is devoted to show Theorem 4.1.2. It is based on computing the first variation of the energy functional

$$\mathcal{E}[1_D] = -\frac{1}{2}\int_{\mathbb{R}^2} 1_D(x)(1_D * \mathcal{N})(x) - \frac{\Omega}{2}|x|^2 1_D(x)dx$$

in two different ways, as we deform D along a carefully chosen vector field that is divergence-free

in D. On the one hand, we show the first variation should be 0 if D is a stationary/rotating patch with angular velocity Ω; on the other hand, we show that the first variation must be non-zero if $\Omega \leq 0$ or $\Omega \geq \frac{1}{2}$, leading to a contradiction. If D is simply-connected, we give a very short proof in Section 5.2.1, where a rearrangement inequality due to Talenti [89] is crucial to get a sign condition. For a non-simply-connected patch D, the choice of the right vector field is more involved. Since $1_D * \mathcal{N} - \frac{\Omega}{2}|x|^2$ now takes different constant values on different connected components of ∂D, in order to keep the first variation to be 0, we have to modify our perturbation vector field such that it also preserves the area of each hole. We then prove a new version of a rearrangement inequality for this modified vector field in a similar spirit as Talenti's result, leading to a non-zero first variation if D is non-radial and $\Omega \leq 0$ or $\Omega \geq \frac{1}{2}$.

In Chapter 6, we show the smooth setting in Theorem 4.1.3, which is based on a similar idea, but technically more difficult. The point of view is to approximate a smooth function by step functions and consider the above perturbation in each set where the step function is constant. To do this we need to obtain some quantitative (stability) estimates on our version of Talenti's rearrangement inequality, in particular in terms of the Fraenkel asymmetry of the domain in the spirit of Fusco–Maggi–Pratelli [35].

In Chapter 7 Section 7.1, we prove Theorem 4.1.4. It is still based on a variational approach, but we need a different perturbation from the vector field in Theorem 4.1.2, which heavily relies on the Newtonian potential, and fails for general Riesz potential K_α. The key ingredient to prove Theorem 4.1.4 is to perturb D using the continuous Steiner symmetrization [7], which has been successfully applied in other contexts by Carrillo–Hittmeir–Volzone–Yao [10] (nonlinear aggregation models) or Morgan [75] (minimizers of the gravitational energy). This method is much more flexible and allows to treat more singular kernels than in the existing papers using moving plane methods. Due to the low regularity of the kernels, instead of computing the derivative of the energy under the perturbation, we work with finite differences instead.

In Chapter 7 Section 7.2, we prove Theorem 4.1.5 by using maximum principles and monotonicity formulas for nonlocal equations. The idea is to find the smallest disk $B(0,R)$ containing D (which intersects ∂D at some x_0), then use two different ways to compute $\nabla(1_{B(0,R)\setminus D} * K_\alpha)$ at x_0, and obtain a contradiction if $\Omega \geq \Omega_\alpha$ and D is not a disk. The proof works for the full range of $\alpha \in [0,2)$, thus closing the problem raised by Hmidi [50] and de la Hoz–Hassainia–Hmidi–Mateu [30] among all simply-connected patches.

4.1.5 The rigidity results for vortex sheets

A vortex sheet is a weak solution of the 2D Euler equations, whose vorticity $\omega = \mathrm{curl}(v)$ is a delta function supported on a curve or a finite number of curves $\Gamma_i = z_i(\alpha, t)$, i.e.

$$\omega(x,t) = \sum_i \varpi_i(\alpha,t)\delta(x - z_i(\alpha,t)). \tag{4.8}$$

Here $\varpi_i(\alpha,t)$ is the vorticity strength on Γ_i with respect to the parametrization z_i, and the above equation is understood in the sense that

$$\int_{\mathbb{R}^2} \varphi(x) d\omega(x,t) = \sum_i \int \varphi(z_i(\alpha,t))\varpi_i(\alpha,t)d\alpha$$

for all test functions $\varphi(x) \in C_0^\infty(\mathbb{R}^2)$.

The motivation of the study of the equation (4.1) with vortex sheet initial data comes from the fact that in fluids with small viscosity, flows separate from rigid walls and corners [71, 85]. To model it, one may think of a solution to (4.1) with one incompressible fluid where the velocity changes sign in a discontinuous (tangential) way across a streamline z. This discontinuity induces vorticity in z.

The equations of motion of ϖ_i and z_i can be derived by means of the Birkhoff-Rott operator ([15, 69, 71, 88]), namely:

$$BR(z,\varpi)(x,t) = \frac{1}{2\pi} PV \int \frac{(x - z(\beta,t))^\perp}{|x - z(\beta,t)|^2} \varpi(\beta,t)d\beta, \tag{4.9}$$

yielding

$$\partial_t z_i(\alpha,t) = \sum_j BR(z_j,\varpi_i)(z_i(\alpha,t)) + c_i(\alpha,t)\partial_\alpha z_i(\alpha,t) \tag{4.10}$$

$$\partial_t \varpi_i(\alpha,t) = \partial_\alpha(c_i(\alpha,t)\varpi_i(\alpha,t)), \tag{4.11}$$

where the term $c_i(\alpha,t)$ accounts for the reparametrization freedom of the curves. See the paper [55] by Izosimov–Khesin where they propose geodesic, group-theoretic, and Hamiltonian frameworks for their description.

Inspired by the rigidity result in the 2D Euler patch case, we have the following rigidity result:

Theorem 4.1.6. *Let ω be a stationary/uniformly-rotating vortex sheet with angular velocity Ω. Assume that ω is concentrated on Γ, which is a finite union of smooth curves, and ω_0 has positive vorticity strength. With some regularity and positivity assumptions, if $\Omega \leq 0$, Γ must be a union of*

concentric circles, and ω_0 must have constant strength along each circle. In addition, if $\Omega < 0$, all circles must be centered at the origin.

4.1.6 The outline of the proof for 4.1.6

In Section 8.2 the equations for the stationary/rotating vortex sheet are derived.

In Section 8.3 we perform the desingularization procedure by considering patches of thickness $\sim \varepsilon$ which are tubular neighborhoods of the sheet.

Section 8.4 is devoted to construct the aforementioned divergence free vector-field along which the first variation is non-zero. Due to the approximation, we lose the property that any stationary solution has to be a critical point if $\varepsilon > 0$ and very careful, quantitative estimates need to be done. This setup is also reminiscent of the numerical work by Baker–Shelley [3], where they approximate the motion of a vortex sheet by a vortex patch of very small width.

In Section 8.5 we conclude the quantitative estimates and prove the symmetry result from Theorem 4.1.6.

4.1.7 The flexibility results for vortex sheets

On the other hand, we also study the flexibility result for the vortex sheet. In the literature, there are very few known examples of nontrivial steady solutions, and in fact, other than the circle or the line, the list only comprises the segment of length $2a$ and density

$$\gamma(x) = \Omega\sqrt{a^2 - x^2}, \qquad x \in [-a, a], \tag{4.12}$$

which is a rotating solution with angular velocity Ω [4] and the family found by Protas–Sakajo [81], made out of segments rotating about a common center of rotation with endpoints at the vertices of a regular polygon. We remark that none of these are supported on a closed curve.

Numerically, O'Neil [77, 78] used point vortices to approximate the vortex sheet and compute uniformly rotating solutions and Elling [33] constructed numerically self-similar vortex sheets forming cusps. O'Neil [79, 80] also found numerically steady solutions which are combinations of point vortices and vortex sheets.

The main goal of next theorem is to find a class of uniformly-rotating vortex sheets, concentrated on a closed curve which is not a circle. This is the first proof of existence of a family of solutions of such kind.

Theorem 4.1.7. *There exists non-radial uniformly-rotating vortex sheets with angular velocity* $\Omega = 1$ *bifurcating from circles.*

4.1.8 The outline of the proof for theorem 4.1.7

In Section 9.1 we write down the equations and describe the main spaces used in the bifurcation.

Section 9.2 is devoted to the bulk of the proof of Theorem 9.1.2. The main idea is to employ bifurcation theory. However, the standard methods (Crandall-Rabinowitz [28]) fail since the linearized operator around the circle does not satisfy the transversality condition. This phenomenon is usually known in the literature as a *degenerate bifurcation* [60, 59]. In this degenerate situation, we no longer have a single branch emanating from the disk, but two, and therefore the linearized operator fails to describe the local behaviour at the bifurcation point. To overcome this issue, we use a technique from Hmidi–Mateu [51] (in the hyperbolic case) and Hmidi–Renault [54] (in the elliptic case). We first reduce the nonlinear problem to a suitable finite dimensional space through a Lyapunov-Schmidt reduction. In the derived finite dimensional system we perform a higher order expansion and show the structure of the zero sets around the bifurcation point. The explicit calculations details are shown in Section 9.4.

In Section 9.3, numerical calculations of the bifurcation branch and a brief discussion of the numerical methods employed are presented.

4.1.9 Non-trivial, compactly supported stationary solutions for 2D Euler

We consider vorticity of the form $\omega = \sum_{i=1}^{n} \theta_i 1_{D_i}$, where $\theta_i \in \mathbb{R}$ and D_i is a simply-connected domain for $1 \leq i \leq n$. Let $z_i(x) = (b_i + R_i(x))(\cos(x), \sin(x))$, $x \in \mathcal{T}$ be a parameterization of ∂D_i. For the stationary solutions, the equation (4.6) is equivalent to

$$\frac{1}{4\pi} \sum_{i=1}^{n} \theta_i \int_0^{2\pi} \log(|z_i(x) - z_i(y)|^2) z_i^{'}(y) dy \cdot (z_i^{'}(x))^{\perp} = 0. \tag{4.13}$$

4.1.10 Non-radial stationary solutions with infinite energy

Notice that in the smooth case Theorem 4.1.3, we have an additional condition on the positive vorticity, which seems unnatural at the first glimpse. We want to address the necessity of the hypothesis $\omega \geq 0$ by answering (on the positive) the following question:

Question 4.1.8. *Do there exist non-trivial, compactly supported stationary solutions for which* ω *changes sign?*

Our first main result immediately gives the answer to the above question.

Theorem 4.1.9. *There exist non-radial, sign-changing two-layered vortex patch solutions* $\sum_{i=1}^{2} \theta_i 1_{D_i}$*, with analytic boundary to the 2D Euler equation whose kinetic energy is infinite, that is,* $\int_{\mathbb{R}^2} |\nabla^\perp \omega * \mathcal{N}|^2 dx = \infty$.

4.1.11 Non-radial stationary solutions with finite energy

Although the solution in Theorem 4.1.9 has compactly supported vorticity, the velocity field may not be compactly supported. It is an interesting problem to study whether nonradial stationary solution with compactly supported velocity exists. Lately, Gavrilov [38, 39] managed to construct nontrivial stationary solutions of 3D Euler with compactly supported velocity, which was further simplified and extended to other equations by Constantin–La–Vicol [22]. We study the same problem in 2D Euler.

Our second main theorem concerns the solutions with *finite* kinetic energy. For the radial vorticity ω with zero-average, $\int_{\mathbb{R}} \omega(x) dx = 0$, its velocity vanishes outside the support of ω. With this observation, one can easily produce globally non-radial solutions by placing multiple copies of such vorticity so that their supports do not overlap. See Figure 4.3. However the flow on each connected component of the support of the velocity is still circular (around different points). From now on, we say that such a solution is locally radial. More precisely a solution to the 2D Euler equation is locally radial if each connected component of ω is radial up to a translation and $\int_{\mathcal{C}} \omega dx = 0$ for each connected component $\mathcal{C}$ of $\mathrm{supp}(\omega)$. The next theorem states that there exist more non-trivial stationary solutions.

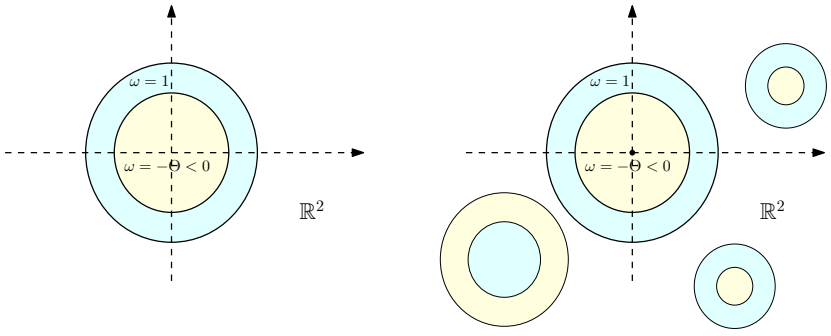


Figure 4.3: One can choose Θ so that the radial vorticity has zero average (left). Since the support of such velocity and vorticity coincide, one can place such solutions to produce non-radial solutions (right).

Theorem 4.1.10. *There exist three-layered vortex patch solutions* $\sum_{i=1}^{3} \theta_i 1_{D_i}$ *to the 2D Euler equation that are not locally radial, with finite kinetic energy and analytic boundary. Furthermore, the solutions have compactly supported velocity.*

4.1.12 The outline of the proof for theorem 4.1.9 and theorem 4.1.10

Section 10.1 sets up the functional framework and the equation that a stationary solution has to satisfy.

Section 10.2 proves Theorem 4.1.9, in the easier case where the finite energy hypothesis is dropped. We pick a two-layered patch and employ the bifurcation theory by checking the hypotheses of the Crandall-Rabinowitz Theorem.

Section 11.1 proves Theorem 4.1.10 with finite energy hypotheses. When we add this hypotheses, the linearized operator degenerates at the starting point of the bifurcation curve, and the standard bifurcation theory does not work. We then work on three-layer patches to get more freedom. We prove theorem by using the Nash-Moser iteration and bounding a Dirichlet to Neumann operator .

The Appendix contains some technical background results used throughout the proof, as well as basic bifurcation theory and some integrals needed for the spectral study.

Chapter 5

The rigidity results for 2D-Euler patch case

5.1 Notations

Through this chapter 5 and the next chapter 6 , we use the following notations.

For a simple closed curve Γ, denote int(Γ) by its interior, which is the bounded connected component of $\mathbb{R}^2$ separated by the curve Γ. Note that the Jordan–Schoenflies theorem guarantees that int(Γ) is open and simply connected.

We say that two disjoint simple closed curves Γ_1 and Γ_2 are nested if $\Gamma_1 \subset \text{int}(\Gamma_2)$ or vice versa. We say that two domains D_1, D_2 are nested if one is contained in a hole of the other one.

For a bounded domain $D \subset \mathbb{R}^2$, we denote by $\partial_{\text{out}} D$ its outer boundary. And if D is doubly-connected, we denote by $\partial_{\text{in}} D$ its inner boundary,

For a set D, we use $1_D(x)$ to denote its indicator function. And for a statement S, we let $\mathbb{1}_S = \begin{cases} 1 & \text{if } S \text{ is true} \\ 0 & \text{if } S \text{ is false} \end{cases}$. (e.g. $\mathbb{1}_{\pi<3} = 0$).

For an open set $U \subset \mathbb{R}^2$, in the boundary integral $\int_{\partial U} \vec{n} \cdot \vec{f} d\sigma$, the vector $\vec{n}$ is taken as the outer normal of the open set U in that integral.

For a countable number of disjoint sets $U_i \subset \mathbb{R}^2$, we denote their union by $\dot{\cup}_i U_i$ to emphasize the disjointedness.

5.2 Radial symmetry of steady/rotating patches for 2D Euler equation

Throughout this section we work with the 2D Euler equation (4.1) in the patch setting. For a stationary or uniformly-rotating patch D with angular velocity $\Omega \in \mathbb{R}$, let

$$f_\Omega(x) := (1_D * \mathcal{N})(x) - \frac{\Omega}{2}|x|^2.$$

Recall that in (4.6) we have shown that $f_\Omega \equiv C_i$ on each connected component of ∂D, where the constants can be different on different connected components.

Our goal in this section is to prove Theorem 4.1.2, which completely answers Question 4.1.1 for 2D Euler patches. As we described in the introduction, our proof has a variational flavor, which is done by perturbing D by a carefully chosen vector field, and compute the first variation of an associated energy functional in two different ways. In Section 5.2.1, we will define the energy functional and the perturbation vector field, and give a one-page proof in Theorem 5.2.1 that answers Question 4.1.1 among simply-connected patches. (Note that even among simply-connected patches, it is an open question whether every rotating patch with $\Omega > \frac{1}{2}$ or $\Omega < 0$ must be a disk.) In the following subsections, we further develop this method, and modify our perturbation vector field to cover non-simply-connected patches.

5.2.1 Warm-up: radial symmetry of simply-connected rotating patches

We begin by providing a sketch and some motivations of our approach, and then give a rigorous proof afterwards in Theorem 5.2.1. Suppose that D is a $C^{1,\gamma}$ simply-connected rotating patch with angular velocity Ω that is *not* a disk. We perturb D in "time" (here the "time" t is just a name for our perturbation parameter, and is irrelevant with the actual time in the Euler equation) with a velocity field $\vec{v}(x) \in C^1(D) \cap C(\overline{D})$ that is divergence-free in D, which we will fix later. That is, consider the transport equation

$$\rho_t + \nabla \cdot (\rho \vec{v}) = 0$$

with $\rho(\cdot, 0) = 1_D$. We then investigate how the "energy functional"

$$\mathcal{E}[\rho] := -\int_{\mathbb{R}^2} \frac{1}{2}\rho(x)(\rho * \mathcal{N})(x) - \frac{\Omega}{2}|x|^2\rho(x)dx$$

changes in time under the perturbation. Formally, we have

$$\begin{aligned}\frac{d}{dt}\mathcal{E}[\rho]\Big|_{t=0} &= -\int_{\mathbb{R}^2} \rho_t(x,0)\Big((\rho(\cdot,0)*\mathcal{N})(x) - \frac{\Omega}{2}|x|^2\Big)dx \\ &= -\int_D \vec{v}(x)\cdot\nabla\Big((1_D*\mathcal{N})(x) - \frac{\Omega}{2}|x|^2\Big)dx.\end{aligned} \tag{5.1}$$

The above transport equation and the energy functional only serve as our motivation, and will not appear in the proof. In the actual proof we only focus on the right hand side of (5.1), which is an integral that is well-defined by itself:

$$\mathcal{I} := -\int_D \vec{v}(x)\cdot\nabla\Big((1_D*\mathcal{N})(x) - \frac{\Omega}{2}|x|^2\Big)dx = -\int_D \vec{v}\cdot\nabla f_\Omega\, dx. \tag{5.2}$$

We will use two different ways to compute $\mathcal{I}$, and show that if D is not a disk, the two ways lead to a contradiction for $\Omega \le 0$ or $\Omega \ge \frac{1}{2}$.

On the one hand, since f_Ω is a constant on ∂D (denote it by c), the divergence theorem yields the following for *every* $\vec{v} \in C^1(D)\cap C(\overline{D})$ that is divergence-free in D:

$$\mathcal{I} = -c\int_{\partial D} \vec{n}\cdot\vec{v}d\sigma + \int_D (\nabla\cdot\vec{v})f_\Omega\, dx = -c\int_D \nabla\cdot\vec{v}dx + \int_D (\nabla\cdot\vec{v})f_\Omega\, dx = 0. \tag{5.3}$$

On the other hand, we fix $\vec{v}$ as follows, which is at the heart of our proof. Let $\vec{v}(x) := -\nabla\varphi(x)$ in D, where

$$\varphi(x) := \frac{|x|^2}{2} + p(x) \quad \text{in } D, \tag{5.4}$$

with $p(x)$ being the solution to Poisson's equation

$$\begin{cases}\Delta p(x) = -2 & \text{in } D \\ p(x) = 0 & \text{on } \partial D.\end{cases} \tag{5.5}$$

Note that φ is harmonic in D, thus $\vec{v}$ is indeed divergence-free in D. This definition of $\vec{v}$ is motivated by the fact that among all divergence-free vector fields in D, such $\vec{v}$ is the closest one to $-\vec{x}$ in the $L^2(D)$ distance. (In fact, such $\vec{v}$ is connected to the gradient flow of $\int_D \frac{|x|^2}{2}dx$ in the metric space endowed by 2-Wasserstein distance, under the constraint that $|D(t)|$ must remain constant [73, 74, 2].) Formally, one expects that D becomes "more symmetric" as we perturb it by $\vec{v}$, which inspires us to consider the first variation of $\mathcal{E}$ under such perturbation.

In the proof we will show that with such choice of $\vec{v}$, we can compute $\mathcal{I}$ in another way and

obtain that $\mathcal{I} > 0$ for $\Omega \leq 0$ and $\mathcal{I} < 0$ for $\Omega \geq \frac{1}{2}$. Therefore in both cases, we obtain a contradiction with $\mathcal{I} = 0$ in (5.3).

Our proof makes use of a rearrangement inequality for solutions to elliptic equations, which is due to Talenti [89]. Below is the form that we will use; the original theorem works for a more general class of elliptic equations.

Proposition 5.2.1 ([89]). *Let $D \subset \mathbb{R}^2$ be a bounded open set, and let p be defined as in (5.5). Let B be an open disk centered at the origin with $|B| = |D|$, and let p_B solve (5.5) in B. Then we have $p^* \leq p_B$ pointwise in B, where p^* is the radial decreasing rearrangement of p. This leads to*

$$\int_D p(x)dx \leq \int_B p_B(x)dx.$$

Using that $p_B(x) = \frac{1}{2}(r_B^2 - |x|^2)$, where $\pi r_B^2 = |B| = |D|$ and $\int_B p_B(x) = \frac{\pi}{4} r_B^4$ it follows that

$$\int_D p(x)dx \leq \frac{1}{4\pi}|D|^2,$$

where the equality is achieved if and only if D is an open disk (cf. [58, Theorem 3.1 and Remark 3.1]).

Now we are ready to prove the following theorem, saying that any simply-connected stationary/rotating patch with $\Omega \leq 0$ or $\Omega \geq \frac{1}{2}$ must be a disk. Interestingly, the same proof can treat the two disjoint intervals $\Omega \leq 0$ and $\Omega \geq \frac{1}{2}$ all at once.

Theorem 5.2.1. *Let D be a simply-connected bounded domain with $C^{1,\gamma}$ boundary. If D is a rotating patch solution with angular velocity Ω, where $\Omega \leq 0$ or $\Omega \geq \frac{1}{2}$, then D must be a disk, and it must be centered at the origin unless $\Omega = 0$.*

Proof. Let D be a rotating patch with $\Omega \in (-\infty, 0] \cup [\frac{1}{2}, \infty)$. As we described above, in this theorem we will use two different ways to compute the integral $\mathcal{I}$ defined in (5.2), where we fix $\vec{v}(x) := -\nabla\varphi(x)$, with φ and p defined as in (5.4) and (5.5).

On the one hand, we have that $\vec{v}$ is divergence free in D, and elliptic regularity theory immediately yields that $\vec{v} \in C^1(D) \cap C(\overline{D})$. Using the assumption that D is a rotating patch, we know f_Ω is a constant on ∂D. (Note that ∂D is a connected closed curve since we assume D is simply-connected). Thus the computation in (5.3) directly gives that $\mathcal{I} = 0$.

On the other hand, we compute $\mathcal{I}$ as follows:

$$\mathcal{I} = -\int_D \vec{v}\cdot\nabla f_\Omega dx = \int_D \nabla\varphi\cdot\nabla f_\Omega dx = \underbrace{\int_D x\cdot\nabla f_\Omega dx}_{=:\mathcal{I}_1} + \underbrace{\int_D \nabla p\cdot\nabla f_\Omega dx}_{=:\mathcal{I}_2}. \tag{5.6}$$

For $\mathcal{I}_1$, we have

$$\begin{aligned}
\mathcal{I}_1 &= \int_D x\cdot\nabla(1_D * \mathcal{N})dx - \int_D x\cdot\Omega x dx \\
&= \frac{1}{2\pi}\int_D\int_D \frac{x\cdot(x-y)}{|x-y|^2}dydx - \Omega\int_D |x|^2 dx \\
&= \frac{1}{4\pi}\int_D\int_D \frac{x\cdot(x-y)-y\cdot(x-y)}{|x-y|^2}dydx - \Omega\int_D |x|^2 dx \\
&= \frac{1}{4\pi}|D|^2 - \Omega\int_D |x|^2 dx,
\end{aligned} \tag{5.7}$$

where the third equality is obtained by exchanging x with y in the first integral, then taking average with the original integral. To compute $\mathcal{I}_2$, using the divergence theorem (and the fact that $p = 0$ on ∂D), we have

$$\mathcal{I}_2 = -\int_D p\Delta f_\Omega dx = (2\Omega - 1)\int_D p dx. \tag{5.8}$$

Plugging (5.7) and (5.8) into (5.6) gives

$$\mathcal{I} = \frac{1}{4\pi}|D|^2 - \Omega\int_D |x|^2 dx + (2\Omega-1)\int_D p dx. \tag{5.9}$$

When $\Omega = 0$, Proposition 5.2.1 directly gives that $\mathcal{I} > 0$ if D is not a disk, contradicting $\mathcal{I} = 0$.

When $\Omega \in (-\infty, 0)\cup[\frac{1}{2}, \infty)$, let B be a disk centered at the origin with the same area as D. Towards a contradiction, assume $D \neq B$. Among all sets with the same area as D, the disk B is the unique one that minimizes the second moment. To see this, denoting by r_B the radius of B, we have

$$\int_D |x|^2 dx - \int_B |x|^2 dx = \int_{D\setminus B} |x|^2 dx - \int_{B\setminus D} |x|^2 dx \geq \int_{D\setminus B} r_B^2 dx - \int_{B\setminus D} r_B^2 dx = 0,$$

where the last equality follows from $|D\setminus B| = |B\setminus D|$, which is due to $|D| = |B|$; and the inequality

is strict whenever $D \neq B$. Thus if $D \neq B$, we have

$$\int_D |x|^2 dx > \int_B |x|^2 dx = \frac{1}{2\pi}|D|^2,$$

where the last step follows from an elementary computation. Plugging this into (5.9) gives the following inequality for $\Omega \in [\frac{1}{2}, \infty)$:

$$\mathcal{I} < \frac{1}{4\pi}|D|^2 - \frac{\Omega}{2\pi}|D|^2 + (2\Omega - 1)\int_D p dx = (1 - 2\Omega)\Big(\frac{1}{4\pi}|D|^2 - \int_D p dx\Big) \leq 0.$$

On the other hand, for $\Omega \in (-\infty, 0)$, we have

$$\mathcal{I} > \frac{1}{4\pi}|D|^2 - \frac{\Omega}{2\pi}|D|^2 + (2\Omega - 1)\int_D p dx = (1 - 2\Omega)\Big(\frac{1}{4\pi}|D|^2 - \int_D p dx\Big) > 0,$$

and we get a contradiction to $\mathcal{I} = 0$ in all the cases, thus the proof is finished. □

Remark 5.2.2. *In fact, one can easily check that the proof of Theorem 5.2.1 applies to a bounded disconnected patch $D = \dot{\cup}_{i=1}^N D_i$ with $C^{1,\gamma}$ boundary, as long as each connected component D_i is simply-connected, which yields that such a patch cannot be a rotating solution. Here the proof remains the same, except a small change in it: on the one hand, since now we have $f_\Omega = c_i$ on ∂D_i,* (5.3) *should be replaced by*

$$\mathcal{I} = -\sum_{i=1}^N \left(c_i \int_{\partial D_i} \vec{n} \cdot \vec{v} d\sigma + \int_{D_i} (\nabla \cdot \vec{v}) f_\Omega dx \right) = 0.$$

On the other hand, the same argument as in the proof of Theorem 5.2.1 shows $\mathcal{I} \neq 0$ whenever D contains more than one connected component.

Remark 5.2.3. *When the domain D is Lipschitz, we no longer have $\vec{v} = -x - \nabla p \in C^1(\overline{D})$. Nevertheless, the divergence theorem in the proof of Theorem 5.2.1 can still be justified, thanks to the fact that $\nabla p \in L^2(\partial D)$ for a Lipschitz domain D, where $\nabla p|_{\partial D}$ is taken in the nontangential limit sense, and this follows from [56, Theorem 5.6].*

Even in the regime $\Omega \in (0, \frac{1}{2})$, where non-radial rotating patches are known to exist (recall that there exist patches bifurcating from a disk at $\Omega_m = \frac{m-1}{2m}$ for all $m \geq 2$), our approach still allows us to obtain the following quantitative estimate, saying that if a simply-connected patch D rotates with angular velocity $\Omega \in (0, \frac{1}{2})$ that is very close to $\frac{1}{2}$, then D must be very close to a disk, in the sense that their symmetric difference must be small.

Corollary 5.2.4. *Let D be a simply-connected bounded domain with $C^{1,\gamma}$ boundary. Assume D is a rotating patch solution with angular velocity Ω, where $\Omega \in (\frac{1}{4}, \frac{1}{2})$. Let $\delta := \frac{1}{2} - \Omega$. Then we have*

$$|D \triangle B| \leq 2\sqrt{2\delta}|D|,$$

where B is the disk centered at the origin with the same area as D.

Proof. In the proof of Theorem 5.2.1, combining the equation $\mathcal{I} = 0$ and (5.9) together, we have that

$$\frac{1}{4\pi}|D|^2 - \Omega \int_D |x|^2 dx - (1 - 2\Omega) \int_D p dx = 0.$$

Dividing both sides by Ω and rearranging the terms, we obtain

$$\int_D |x|^2 dx - \frac{1}{2\pi}|D|^2 = \frac{1 - 2\Omega}{\Omega} \left(\frac{1}{4\pi}|D|^2 - \int_D p dx \right) \leq \frac{2\delta |D|^2}{\pi},$$

where in the inequality we used that $2\delta := 1 - 2\Omega$, $\Omega > \frac{1}{4}$, and $\int_D p dx \geq 0$.

Since $\int_B |x|^2 dx = \frac{1}{2\pi}|D|^2$, the above inequality implies that

$$\int_{D \setminus B} |x|^2 dx - \int_{B \setminus D} |x|^2 dx \leq \frac{2\delta |D|^2}{\pi}. \tag{5.10}$$

Since D and B has the same area, let us denote $\beta := |D \setminus B| = |B \setminus D|$. Among all sets $U \subset B^c$ with area β, $\int_U |x|^2 dx$ is minimized when U is an annulus with area β and inner circle coinciding with ∂B. To see this, let U_0 be such an annulus, and denote by r_{in} and r_{out} its inner and outer radius (note that r_{in} is also the radius of B). Then we have

$$\int_U |x|^2 dx - \int_{U_0} |x|^2 dx = \int_{U \setminus U_0} |x|^2 dx - \int_{U_0 \setminus U} |x|^2 dx \geq \int_{U \setminus U_0} r_{out}^2 dx - \int_{U_0 \setminus U} r_{out}^2 dx = 0,$$

where the inequality follows from that $U \setminus U_0 \subset B^c \setminus U_0 \subset B(0, r_{out})^c$ (recall that $U \subset B^c$), and the last equality follows from $|U \setminus U_0| = |U_0 \setminus U|$, which is due to $|U| = |U_0|$.

Thus an elementary computation gives

$$\int_{D \setminus B} |x|^2 dx \geq \inf_{U \subset B^c, |U| = \beta} \int_U |x|^2 dx = \frac{\beta(2|B| + \beta)}{2\pi}.$$

Likewise, among all sets $V \subset B$ with area β, $\int_V |x|^2 dx$ is maximized when V is an annulus with

area β and outer circle coinciding with ∂B, thus

$$\int_{B\setminus D}|x|^2dx\leq\sup_{V\subset B,|V|=\beta}\int_V|x|^2dx=\frac{\beta(2|B|-\beta)}{2\pi}.$$

Subtracting these two inequalities yields

$$\int_{D\setminus B}|x|^2dx-\int_{B\setminus D}|x|^2dx\geq\frac{\beta^2}{\pi},$$

and combining this with (5.10) immediately gives

$$\beta^2\leq 2\delta|D|^2,$$

thus $|D\triangle B|=2\beta\leq 2\sqrt{2\delta}|D|$. □

5.2.2 Radial symmetry of non-simply-connected stationary patches

In this subsection, we aim to prove radial symmetry of a connected rotating patch D with $\Omega\leq 0$, where D is allowed to be non-simply-connected. Let $D\subset\mathbb{R}^2$ be a bounded domain with $C^{1,\gamma}$ boundary. Assume D has n holes with $n\geq 0$, and then let $h_1,\cdots,h_n\subset\mathbb{R}^2$ denote the n holes of D (each h_i is a bounded open set). Note that ∂D has $n+1$ connected components: they include the outer boundary of D, which we denote by ∂D_0, and the inner boundaries ∂h_i for $i=1,...,n$. We orient ∂h_i the opposite way as ∂D_0.

To begin with, we point out that even for the steady patch case $\Omega=0$, the proof of Theorem 5.2.1 cannot be directly adapted to the non-simply-connected patch. If we define $\vec{v}$ in the same way, then the second way to compute $\mathcal{I}$ still goes through (since Proposition 5.2.1 still holds for non-simply-connected D), and leads to $\mathcal{I}>0$ if D is not a disk. But the first way to compute $\mathcal{I}$ no longer gives $\mathcal{I}=0$: if D is stationary and not simply-connected, $f(x):=(1_D*\mathcal{N})(x)$ may take different constant values on different connected components of ∂D, thus the identity (5.3) no longer holds.

In order to fix this issue, we still define $\vec{v}=-\nabla\varphi=-\nabla(\frac{|x|^2}{2}+p)$, but modify the definition of p in the following lemma. Compared to the previous definition (5.5), the difference is that p now takes different values $0,c_1,\ldots,c_n$ on each connected component of ∂D. The lemma shows that there exist values of $\{c_i\}_{i=1}^n$, such that $\int_{\partial h_i}\nabla p\cdot\vec{n}d\sigma=-2|h_i|$ along the boundary of each hole. As we will see later, this leads to $\int_{\partial h_i}\vec{v}\cdot\vec{n}d\sigma=0$ for $i=1,\ldots,n$, which ensures $\mathcal{I}=0$. (Of course, with p defined in the new way, the second way of computing $\mathcal{I}$ no longer follows from Proposition 5.2.1, and we

will take care of this later in Proposition 5.2.7.)

Lemma 5.2.5. *Let D, h_i and ∂D_0 be given as in the first paragraph of Section 5.2.2. Then there exist positive constants $\{c_i\}_{i=1}^n$, such that the solution $p : \overline{D} \to \mathbb{R}$ to the Poisson equation*

$$\begin{cases} \Delta p = -2 & \text{in } D, \\ p = c_i & \text{on } \partial h_i \quad \text{for } i = 1, \dots, n, \\ p = 0 & \text{on } \partial D_0. \end{cases} \tag{5.11}$$

satisfies

$$\int_{\partial h_i} \nabla p \cdot \vec{n} \, d\sigma = -2|h_i| \quad \text{for } i = 1, \dots, n. \tag{5.12}$$

Here $|h_i|$ is the area of the domain $h_i \subset \mathbb{R}^2$.

Proof. Let u satisfy that

$$\begin{cases} \Delta u = -2 & \text{in } D \\ u = 0 & \text{on } \partial D. \end{cases}$$

Furthermore let the function v_j for $j = 1, ..., n$ be the solution to

$$\begin{cases} \Delta v_j = 0 & \text{in } D \\ v_j = 0 & \text{on } \partial D \setminus \partial h_j \\ v_j = 1 & \text{on } \partial h_j, \end{cases}$$

where $v_j \in C^2(D) \cap C^1(\overline{D})$ by elliptic regularity. Now we consider the following linear equation,

$$Ax = b, \tag{5.13}$$

where $A_{i,j} = \int_{\partial h_i} \nabla v_j \cdot \vec{n} \, d\sigma$ and $b_i = -2|h_i| - \int_{\partial h_i} \nabla u \cdot \vec{n} \, d\sigma$. We argue that (5.13) has a unique solution. Thanks to the divergence theorem, we have

$$0 = \int_D \Delta v_j dx = \int_{\partial D_0} \nabla v_j \cdot \vec{n} d\sigma - \sum_{i=1}^n \int_{\partial h_i} \nabla v_j \cdot \vec{n} d\sigma.$$

Therefore,

$$\sum_{i=1}^{n} A_{i,j} = \int_{\partial D_0} \nabla v_j \cdot \vec{n} d\sigma < 0, \tag{5.14}$$

where the last inequality follows from the Hopf Lemma [57] since v_j attains its minimum value 0 on ∂D_0, and $v_j \not\equiv 0$ on ∂D. A similar argument gives that $A_{i,j} > 0$ for $i \neq j$ and $A_{j,j} < 0$. Thus A is invertible by Gershgorin circle theorem [40], leading to a unique solution of (5.13). Let us denote the solution by $x = (c_1, ..., c_n)^t$. Then the function p defined by

$$p := u + \sum_{i=1}^{n} c_i v_i$$

satisfies the desired properties (5.12).

Now we prove that $c_i > 0$ for $i \geq 1$. Suppose that $c_{i*} := \min_i c_i \leq 0$. Then by the minimum principle, p attains its minimum on ∂h_{i^*}. Therefore,

$$0 \leq \int_{\partial h_{i^*}} \nabla p \cdot \vec{n} d\sigma = -2|h_{i^*}| < 0,$$

which is a contradiction. □

Remark 5.2.6. *In order to push the regularity of the domain down to Lipschitz, to prove* (5.14)*, instead of the Hopf Lemma, we may use the following observation in 2 dimensions [1, p.935-p.936]: let $D \subset \mathbb{R}^2$ be a Lipschitz domain and let $\Sigma \subset \partial D$ be relatively open with respect to ∂D. Let v_j be a harmonic function in D and continuous in $\overline{D}$. If v_j vanishes on Σ and its normal derivative vanishes in a subset of Σ with positive surface measure, then $v_j \equiv 0$ in D. In addition, the use of the divergence theorem is justified by the same argument as in Remark 5.2.3.*

Next we prove a parallel version of Talenti's theorem for the function p constructed in Lemma 5.2.5. We will use this result throughout Section 8.2–8.3.

Proposition 5.2.7. *Let $D \subset \mathbb{R}^2$ be a bounded domain with $C^{1,\gamma}$ boundary. Assume D has n holes with $n \geq 0$, and denote by $h_1, \cdots, h_n \subset \mathbb{R}^2$ the holes of D (each h_i is a bounded open set). Let $p : \overline{D} \to \mathbb{R}$ be the function constructed in Lemma 5.2.5. Then the following two estimates hold:*

$$\sup_{\overline{D}} p \leq \frac{|D|}{2\pi} \tag{5.15}$$

and

$$\int_D p(x)dx \leq \frac{|D|^2}{4\pi} \tag{5.16}$$

Furthermore, for each of the two inequalities above, the equality is achieved if and only if D is either a disk or an annulus.

Proof. The proof is divided into two parts: In step 1 we prove the two inequalities (5.15) and (5.16), and in step 2 we show that equality can be achieved if and only if D is a disk or an annulus.

Step 1. When D is simply-connected, (5.15) and (5.16) directly follow from Talenti's theorem Proposition 5.2.1. Next we consider a non-simply-connected domain D, and prove that these inequalities also hold when $p : \overline{D} \to \mathbb{R}$ is defined as in Lemma 5.2.5.

For $k \in \mathbb{R}^+$, let us denote $D_k := \{x \in D : \ p(x) > k\}$, $g(k) := |D_k|$ and $\tilde{D}_k := D_k \dot{\cup} (\dot{\cup}_{\{i:c_i>k\}} \overline{h_i})$. Elliptic regularity theory gives that $p \in C^\infty(D)$, thus by Sard's theorem, k is a regular value for almost every $k \in (0, \sup_D p)$, that is, $|\nabla p(x)| > 0$ on $\{x \in D : \ p(x) = k\}$. Thus $\{x \in D : \ p(x) = k\}$ is a union of smooth simple closed curves and equal to $\partial \tilde{D}_k$ for almost every $k \in (0, \sup_D p)$.

Since $\partial D_k = \partial \tilde{D}_k \dot{\cup} (\dot{\cup}_{\{i:c_i>k\}} \partial h_i)$ for $k \notin \{c_1, ..., c_n\}$, we compute

$$\begin{aligned} g(k) = -\frac{1}{2}\int_{D_k} \Delta p(x)dx \ &= -\frac{1}{2}\int_{\partial D_k} \nabla p \cdot \vec{n} d\sigma \\ &= -\frac{1}{2}\int_{\partial \tilde{D}_k} \nabla p \cdot \vec{n} d\sigma + \frac{1}{2}\sum_{\{i:c_i>k\}} \int_{\partial h_i} \nabla p \cdot \vec{n} d\sigma \\ &= -\frac{1}{2}\int_{\partial \tilde{D}_k} \nabla p \cdot \vec{n} d\sigma - \sum_{\{i:c_i>k\}} |h_i|, \end{aligned}$$

where the last identity is due to (5.12). Therefore, it follows that

$$g(k) + \sum_{\{i:c_i>k\}} |h_i| = -\frac{1}{2}\int_{\partial \tilde{D}_k} \nabla p \cdot \vec{n} d\sigma = \frac{1}{2}\int_{\partial \tilde{D}_k} |\nabla p| d\sigma, \tag{5.17}$$

where the last equality follows from the fact that ∇p is perpendicular to the tangent vector on the level set.

On the other hand, the coarea formula yields that

$$g(k) = \int_{\mathbb{R}} \int_{\partial \tilde{D}_s} 1_{D_k} \frac{1}{|\nabla p|} d\sigma ds = \int_k^\infty \int_{\partial \tilde{D}_s} \frac{1}{|\nabla p|} d\sigma ds.$$

Therefore, it follows that for almost every $k \in (0, \sup_D p)$,

$$g'(k) = -\int_{\partial \tilde{D}_k} \frac{1}{|\nabla p|} d\sigma. \tag{5.18}$$

Thus it follows from (5.17) and (5.18) that

$$g'(k)\left(g(k) + \sum_{\{i:c_i>k\}} |h_i|\right) = -\frac{1}{2}\left(\int_{\partial \tilde{D}_k} |\nabla p| d\sigma\right)\left(\int_{\partial \tilde{D}_k} \frac{1}{|\nabla p|} d\sigma\right) \leq -\frac{1}{2} P(\tilde{D}_k)^2, \tag{5.19}$$

where $P(E)$ denotes the perimeter of a rectifiable curve ∂E. Note that the last inequality becomes equality if and only if $|\nabla p|$ is a constant on $\partial \tilde{D}_k$. Also, the isoperimetric inequality gives that

$$P(\tilde{D}_k)^2 \geq 4\pi |\tilde{D}_k|, \tag{5.20}$$

where equality holds if and only if $\tilde{D}_k$ is a disk. This yields that

$$g'(k)\left(g(k) + \sum_{\{i:c_i>k\}} |h_i|\right) \leq -2\pi |\tilde{D}_k| = -2\pi \left(g(k) + \sum_{\{i:c_i>k\}} |h_i|\right). \tag{5.21}$$

Therefore, $g'(k) \leq -2\pi$ for almost every $k \in (0, \sup_D p)$. Combining it with the fact that $g(0) = |D|$, we have

$$g(k) \leq (g(0) - 2\pi k)_+ = (|D| - 2\pi k)_+ \text{ for almost every } k \geq 0.$$

This proves that $\sup_{\overline{D}} p \leq \frac{|D|}{2\pi}$. It follows that

$$\int_D p(x)dx = \int_D \int_0^{\frac{|D|}{2\pi}} 1_{\{k<p(x)\}} dk dx = \int_0^{\frac{|D|}{2\pi}} g(k) dk \leq \int_0^{\frac{|D|}{2\pi}} (|D| - 2\pi k)_+ dx = \frac{|D|^2}{4\pi}.$$

Step 2. Now we show that for the two inequalities (5.15) and (5.16), the equality is achieved if and only if D is either a disk or an annulus. First, if D is either a disk or annulus centered at some $x_0 \in \mathbb{R}^2$, then uniqueness of solution to Poisson's equation gives that p is radially symmetric about x_0. Since we have $\Delta p = -2$ in D and $p = 0$ on the outer boundary of D, this gives an explicit formula $p(x) = -\frac{|x-x_0|^2}{2} + \frac{R^2}{2}$ for $x \in D$, where R is the outer radius of D. For either a disk or an annulus, one can explicitly compute $\sup_D p$ and $\int_D p dx$ to check that equalities in (5.15) and (5.16) are achieved.

To prove the converse, assume that either (5.15) or (5.16) achieves equality, and we aim to show that D is either a disk or an annulus. In order for either equality to be achieved, (5.21) needs to

achieve equality at almost every $k \in (0, \sup_D p)$. In addition, $g(k)$ needs to be continuous in k since $g(k)$ is decreasing. Since (5.21) follows from a combination of the Cauchy-Schwarz inequality in (5.19) and the isoperimetric inequality in (5.20), we need to have all the three conditions below in order for either (5.15) or (5.16) to achieve equality:

(1) $|\nabla p|$ is a constant on each level set $\partial \tilde{D}_k$ for almost every $k \in (0, \sup_D p)$;

(2) $\tilde{D}_k$ is a disk for almost every $k \in (0, \sup_D p)$.

(3) $g(k) = |D_k|$ is continuous in k. As a result, $|\tilde{D}_k|$ is continuous in k at all $k \neq c_i$, with $c_i > 0$ defined in (5.11).

Next we will show that if all these three conditions are satisfied, then D must be an annulus or disk. First, note that by sending $k \searrow 0$ in condition (2), and combining it with the continuity of $|\tilde{D}_k|$ as $k \searrow 0$, it already gives that the outer boundary of D must be a circle. Therefore if D is simply-connected, it must be a disk.

If D is non-simply-connected, using condition (2) and (3), we claim that D can have only one hole, which must be a disk, and p must achieve its maximum value in $\overline{D}$ on the boundary of the hole. To see this, let h_i be any hole of D, and recall that $p|_{\partial h_i} = c_i$. As we consider the set limit of $\tilde{D}_k$ as k approaches c_i from below and above, by definition of $\tilde{D}_k$ we have

$$\lim_{k \nearrow c_i} \tilde{D}_k = \lim_{k \searrow c_i} \tilde{D}_k \dot{\cup} (\dot{\cup}_{\{j: c_j = c_i\}} \overline{h_j}).$$

By (2) and (3), the left hand side $\lim_{k \nearrow c_i} \tilde{D}_k$ is a disk, and the set $\lim_{k \searrow c_i} \tilde{D}_k$ on the right hand side is also a disk (if the limit is non-empty). But after taking union with the holes $\{h_j : c_j = c_i\}$ (each is a simply-connected set), the right hand side will be a disk if and only if $\lim_{k \searrow c_i} \tilde{D}_k$ is empty, $\dot{\cup}_{\{j: c_j = c_i\}} \overline{h_j} = \overline{h_i}$, and h_i is a disk. This implies $c_i = \sup_D p$ and $c_j < c_i$ for all $j \neq i$. But since h_i is chosen to be any hole of D, we know D can have only one hole (call it h), which is a disk, and $\sup_D p = p|_{\partial h}$. Finally, note that condition (1) gives that all the disks $\{\tilde{D}_k\}$ are concentric, and as a result we have D is an annulus, finishing the proof. □

Finally, we are ready to show that every connected stationary patch D with $C^{1,\gamma}$ boundary must be either a disk or an annulus.

Theorem 5.2.2. *Let $D \subset \mathbb{R}^2$ be a bounded domain with $C^{1,\gamma}$ boundary. Suppose that $\omega(x) := 1_D(x)$ is a stationary patch solution to the 2D Euler equation in the sense of* (4.5)*. Then D is either a disk or an annulus.*

Proof. If D has n holes (where $n \geq 0$), denote them by $h_1, \ldots, h_n$. By (4.5), the function $f := 1_D * \mathcal{N}$

is constant on each of connected component of ∂D, and let us denote

$$f(x) = \begin{cases} a_i & \text{on } \partial h_i \\ a_0 & \text{on } \partial D_0. \end{cases} \tag{5.22}$$

Let $p : \overline{D} \to \mathbb{R}$ be defined as in Lemma 5.2.5, and let $\varphi := \frac{|x|^2}{2} + p$. Similar to the proof of Theorem 5.2.1, we calculate $\mathcal{I} := \int_D \nabla\varphi \cdot \nabla f dx$ in two different ways. Note that $\nabla f = \nabla(f - a_0)$ in D. Applying the divergence theorem to $\mathcal{I}$ and using (5.22) and $\Delta\varphi = 0$ in D, it follows that

$$\mathcal{I} = \int_{\partial D} (\nabla\varphi \cdot \vec{n})(f - a_0) d\sigma - \int_D \Delta\varphi (f - a_0) dx = \sum_{i=1}^{n} (a_i - a_0) \int_{\partial h_i} \nabla\varphi \cdot \vec{n} d\sigma. \tag{5.23}$$

By definition of φ, and combining it with the property of p in (5.12), we have

$$\begin{aligned} \int_{\partial h_i} \nabla\varphi \cdot \vec{n} d\sigma &= \int_{\partial h_i} \nabla \left(\frac{|x|^2}{2} \right) \cdot \vec{n} d\sigma + \int_{\partial h_i} \nabla p \cdot \vec{n} d\sigma \\ &= \int_{h_i} 2 dx + \int_{\partial h_i} \nabla p \cdot \vec{n} d\sigma = 0. \end{aligned} \tag{5.24}$$

Plugging this into (5.23) gives $\mathcal{I} = 0$. On the other hand, we also have

$$\mathcal{I} = \int_D x \cdot \nabla f dx + \int_D \nabla p \cdot \nabla f dx =: E_1 + E_2.$$

We compute

$$E_1 = \int_D x \cdot (1_D * \nabla\mathcal{N}) dx = \int_D \int_D \frac{1}{2\pi} \frac{x \cdot (x - y)}{|x - y|^2} dy dx = \frac{|D|^2}{4\pi}, \tag{5.25}$$

where the last equality is obtained by exchanging x with y and taking the average with the original integral. For E_2, the divergence theorem yields that

$$E_2 = \int_{\partial D} p \nabla f \cdot \vec{n} d\sigma - \int_D p \Delta f dx = \int_{\partial D} p \nabla f \cdot \vec{n} d\sigma - \int_D p dx.$$

Using the property of p in (5.11) and the fact that $\Delta f = 0$ in h_i, the divergence theorem yields

$$\int_{\partial D} p \nabla f \cdot \vec{n} d\sigma = -\sum_{i=1}^{n} \int_{\partial h_i} p \nabla f \cdot \vec{n} d\sigma = -\sum_{i=1}^{n} c_i \int_{h_i} \Delta f dx = 0, \tag{5.26}$$

As a result, we have $E_2 = -\int_D p dx$. If D is neither a disk nor an annulus, Proposition 5.2.7 gives

$$\mathcal{I} = E_1 + E_2 = \frac{|D|^2}{4\pi} - \int_D p dx > 0,$$

contradicting $\mathcal{I} = 0$. □

In the next corollary, we generalize the above result to a nonnegative stationary patch with multiple (disjoint) patches.

Corollary 5.2.8. *Let $\omega(x) := \sum_{i=1}^n \alpha_i 1_{D_i}$, where $\alpha_i > 0$, each D_i is a bounded domain with $C^{1,\gamma}$ boundary, and $D_i \cap D_j = \emptyset$ if $i \neq j$. Assume that ω is a stationary patch solution, that is, the function $f(x) := \omega * \mathcal{N}$ satisfies $\nabla^\perp f \cdot \vec{n} = 0$ on ∂D_i for all $i = 1, \dots, n$. Then ω is radially symmetric up to a translation.*

Proof. Following similar notations as the beginning of Section 5.2.2, we denote the outer boundary of D_i by ∂D_{i0}, and the holes of each D_i (if any) by h_{ik} for $k = 1, \dots, N_i$. Let $p_i : \overline{D_i} \to \mathbb{R}$ be defined as in Lemma 5.2.5, that is, p_i satisfies

$$\begin{cases} \Delta p_i = -2 & \text{in } D_i \\ p_i = c_{ik} & \text{on } \partial h_{ik} \\ p_i = 0 & \text{on } \partial D_{i0}, \end{cases}$$

where c_{ik} is chosen such that $\int_{\partial h_{ik}} \nabla p_i \cdot \vec{n} d\sigma = -2|h_{ik}|$. We then define $\varphi : \cup_{i=1}^n \overline{D}_i \to \mathbb{R}$, such that in each $\overline{D}_i$ we have $\varphi = \varphi_i := \frac{|x|^2}{2} + p_i$.

Similar to Theorem 11.1.7, we compute

$$\mathcal{I} := \int_{\mathbb{R}^2} \omega \nabla\varphi \cdot \nabla f dx = \sum_{i=1}^n \int_{D_i} \alpha_i \nabla\varphi_i \cdot \nabla f dx$$

in two different ways. On the one hand, since $f = \omega * \mathcal{N}$ is a constant on each connected component of ∂D_i, the same computation of Theorem 11.1.7 yields that $\int_{D_i} \nabla\varphi_i \cdot \nabla f dx = 0$, therefore $\mathcal{I} = 0$. On the other hand, since $\nabla\varphi = x + \nabla p_i$ in each D_i, we break $\mathcal{I}$ into

$$\mathcal{I} = \sum_{i,j=1}^n \alpha_i\alpha_j \int_{D_i} x \cdot \nabla(1_{D_j} * \mathcal{N}) dx + \sum_{i,j=1}^n \alpha_i\alpha_j \int_{D_i} \nabla p_i \cdot \nabla(1_{D_j} * \mathcal{N}) dx =: \mathcal{I}_1 + \mathcal{I}_2.$$

For $\mathcal{I}_1$, we compute

$$\begin{aligned}\mathcal{I}_1 &= \sum_{i,j=1}^{n} \frac{\alpha_i\alpha_j}{2}\left(\int_{D_i} x\cdot\nabla(1_{D_j}*\mathcal{N})dx + \int_{D_j} x\cdot\nabla(1_{D_i}*\mathcal{N})dx\right)\\ &= \sum_{i,j=1}^{n} \frac{\alpha_i\alpha_j}{2}\left(\int_{D_i}\int_{D_j} \frac{x\cdot(x-y)}{2\pi|x-y|^2}dydx + \int_{D_j}\int_{D_i} \frac{x\cdot(x-y)}{2\pi|x-y|^2}dydx\right)\\ &= \sum_{i,j=1}^{n} \frac{\alpha_i\alpha_j}{4\pi}|D_i||D_j|, \end{aligned} \tag{5.27}$$

where we exchanged i with j to get the first equality. For $\mathcal{I}_2$, we have

$$\mathcal{I}_2 = \sum_{i=1}^{n}\alpha_i^2\int_{D_i}\nabla p_i\cdot\nabla(1_{D_i}*\mathcal{N})dx + \sum_{i\neq j}\alpha_i\alpha_j\int_{D_i}\nabla p_i\cdot\nabla(1_{D_j}*\mathcal{N})dx =: I_{21}+I_{22}.$$

By the same computation for E_2 in the proof of Theorem 11.1.7, we have

$$I_{21} = -\sum_{i=1}^{n}\alpha_i^2\int_{D_i} p_i dx. \tag{5.28}$$

For $i\neq j$, we denote $j\prec i$ if D_j is contained in a hole of D_i. (And if D_j is not contained in any hole of D_i, we say $j\not\prec i$.) Using this notation, the divergence theorem directly yields that

$$\int_{\partial D_i} p_i\nabla(1_{D_j}*\mathcal{N})\cdot\vec{n}d\sigma = -\sum_{k=1}^{N_i}\int_{\partial h_{ik}} p_i\nabla(1_{D_j}*\mathcal{N})\cdot\vec{n}d\sigma = 0 \quad \text{if } j\not\prec i. \tag{5.29}$$

And if $j\prec i$, then the divergence theorem and (5.15) in Proposition 5.2.7 yield

$$\int_{\partial D_i} p_i\nabla(1_{D_j}*\mathcal{N})\cdot\vec{n}d\sigma \geq -\sup_{\partial D_i} p_i|D_j| \geq -\frac{1}{2\pi}|D_i||D_j| \quad \text{if } j\prec i. \tag{5.30}$$

Hence it follows that

$$I_{22} \geq -\sum_{i\neq j}\mathbb{1}_{j\prec i}\frac{\alpha_i\alpha_j}{2\pi}|D_i||D_j| = -\sum_{i\neq j}(\mathbb{1}_{j\prec i}+\mathbb{1}_{i\prec j})\frac{\alpha_i\alpha_j}{4\pi}|D_i||D_j|, \tag{5.31}$$

where the last step is obtained by exchanging i,j and taking average with the original sum. Note that we have $\mathbb{1}_{j\prec i}+\mathbb{1}_{i\prec j}\leq 1$ for any $i\neq j$. From (5.27), (5.28) and (5.31), we obtain

$$\mathcal{I} \geq \sum_{i=1}^{n}\alpha_i^2\left(\frac{|D_i|^2}{4\pi} - \int_{D_i} p_i dx\right) + \sum_{\substack{j\not\prec i \\ \text{and} \\ i\neq j \\ i\not\prec j}}\alpha_i\alpha_j\frac{|D_i||D_j|}{4\pi}. \tag{5.32}$$

Since we already know that $\mathcal{I} = 0$ and all the summands in (5.32) are nonnegative, it follows that

$$\frac{|D_i|^2}{4\pi} = \int_{D_i} p_i dx \text{ for all } i = 1, \dots, n \text{ and } \{(i,j) : i \neq j, i \not\prec j \text{ and } j \not\prec i\} = \emptyset.$$

Therefore every D_i is either a disk or an annulus by Proposition 5.2.7 and they are nested. By relabeling the indices, we can assume that $i \prec i+1$ for $i = 1, \dots, n-1$.

Next we prove that all D_i's are concentric by induction. For $k \geq 1$, suppose $D_1, \dots, D_k$ are known to be concentric about some $o \in \mathbb{R}^2$. To show D_{k+1} is also centered at o, we break f into

$$f = \sum_{i=1}^{k} (\alpha_i 1_{D_i}) * \mathcal{N} + \sum_{i=k+1}^{n} (\alpha_i 1_{D_i}) * \mathcal{N}.$$

In the first sum, each D_i is centered at o for $i \leq k$, thus Lemma 5.2.9(a) (which we prove right after this theorem) yields that $\sum_{i=1}^{k} (\alpha_i 1_{D_i}) * \mathcal{N} = \frac{C}{2\pi} \ln|x-o|$ on $\partial_{\text{in}} D_{k+1}$, where $C = \sum_{i=1}^{k} \alpha_i |D_i| > 0$. In the second sum, for each $i \geq k+1$, since each D_i is an annulus with $\partial_{\text{in}} D_{k+1}$ in its hole, Lemma 5.2.9(b) gives that $1_{D_i} * \mathcal{N} \equiv \text{const}$ on $\partial_{\text{in}} D_{k+1}$ for all $i \geq k+1$. Thus overall we have $f = \frac{C}{2\pi} \ln|x-o| + C_2$ on $\partial_{\text{in}} D_{k+1}$ for $C > 0$. Combining it with the assumption that f is a constant on $\partial_{\text{in}} D_{k+1}$, we know D_{k+1} must also be centered at o, finishing the induction step. □

Now we state and prove the lemma used in the proof of Corollary 5.2.8, which follows from standard properties of the Newtonian potential.

Lemma 5.2.9. *Assume $g \in L^\infty(\mathbb{R}^2)$ is radially symmetric about some $o \in \mathbb{R}^2$, and is compactly supported in $B(o,R)$. Then $\eta := g * \mathcal{N}$ satisfies the following:*

(a) $\eta(x) = \dfrac{\int_{\mathbb{R}^2} g dx}{2\pi} \ln|x-o|$ for all $x \in B(0,R)^c$.

(b) If in addition we have $g \equiv 0$ in $B(o,r)$ for some $r \in (0,R)$, then $\eta = \text{const}$ in $B(o,r)$.

Proof. To show (a), we take any $x \in B(o,R)^c$ and consider the circle $\Gamma \ni x$ centered at o. By radial symmetry of η about o and the divergence theorem, we have

$$\nabla\eta \cdot \frac{x}{|x|} = \frac{1}{|\Gamma|} \int_\Gamma \nabla\eta \cdot \vec{n} d\sigma = \frac{1}{|\Gamma|} \int_{\text{int}(\Gamma)} \Delta\eta dx = \frac{\int_{\mathbb{R}^2} g(x) dx}{2\pi|x-o|},$$

which implies $\eta(x) = \frac{\int g dx}{2\pi} \ln|x-o| + C$. To show that $C = 0$, for $|x|$ sufficiently large we have

$$|C| = \Big| \int_{B(o,R)} g(x)(\mathcal{N}(x-y) - \mathcal{N}(x-o)) dy \Big| \leq \|g\|_{L^\infty(\mathbb{R}^2)} \sup_{y \in B(o,R)} |\mathcal{N}(x-o) - \mathcal{N}(x-y)|,$$

and by sending $|x| \to \infty$ we have $C = 0$, which gives (a). To show (b), it suffices to prove that $\nabla\eta \equiv 0$ in $B(o,r)$. Take any $x \in B(o,r)$, and consider the circle $\Gamma_2 \ni x$ centered at o. Again, symmetry and the divergence theorem yield that

$$|\nabla\eta(x)| = \frac{1}{|\Gamma_2|}\int_{\Gamma_2} \nabla\eta \cdot \vec{n} d\sigma = \frac{1}{|\Gamma_2|}\int_{\mathrm{int}(\Gamma_2)} \Delta\eta dx = \frac{\int_{\mathrm{int}(\Gamma)} g(x)dx}{|\Gamma|} = 0,$$

finishing the proof of (b). □

5.2.3 Radial symmetry of non-simply-connected rotating patches with $\Omega < 0$

In this subsection, we show that a nonnegative uniformly rotating patch solution (with multiple disjoint patches) must be radially symmetric if the angular velocity $\Omega < 0$.

Theorem 5.2.3. *For $i = 1, \dots, n$, let D_i be a bounded domain with $C^{1,\gamma}$ boundary, and assume $D_i \cap D_j = \emptyset$ for $i \neq j$. If $\omega = \sum_{i=1}^n \alpha_i 1_{D_i}$ is a nonnegative rotating patch solution with $\alpha_i > 0$ and angular velocity $\Omega < 0$, then ω must be radially symmetric.*

Proof. In this proof, let

$$f_\Omega(x) := \omega * \mathcal{N} - \frac{\Omega}{2}|x|^2.$$

In each D_i, let us define p_i as in Lemma 5.2.5. Let $\varphi_i := \frac{|x|^2}{2} + p_i$ in each D_i. As in Theorem 5.2.3, we compute $\mathcal{I} := \sum_{i=1}^n \alpha_i \int_{D_i} \nabla\varphi_i \cdot \nabla f_\Omega dx$ in two different ways. Since f_Ω is a constant on each connected component of ∂D_i and $\nabla\varphi_i$ is divergence free in D_i, we still have $\mathcal{I} = 0$ as in the proof of Theorem 11.1.7.

On the other hand, we have

$$\begin{aligned}\mathcal{I} &= \sum_{i=1}^n \alpha_i \int_{D_i} (x + \nabla p_i) \cdot \nabla (\omega * \mathcal{N})\, dx + \underbrace{(-\Omega)}_{\geq 0} \sum_{i=1}^n \alpha_i \int_{D_i} (x + \nabla p_i) \cdot x dx \\ &=: \mathcal{I}_1 + (-\Omega)\mathcal{I}_2.\end{aligned}$$

As in the proof of Corollary 5.2.8, we have

$$\mathcal{I}_1 = \sum_{i=1}^n \alpha_i^2 \left(\frac{|D_i|^2}{4\pi} - \int_{D_i} p_i dx \right) + \sum_{j\neq i \text{ and } i\neq j} \alpha_i\alpha_j \frac{|D_i||D_j|}{4\pi} \geq 0. \tag{5.33}$$

Note that $\mathcal{I}_1 = 0$ as long as all D_i's are nested annuli/disk, even if they are not concentric. For $\mathcal{I}_2$,

using Cauchy-Schwarz inequality in the second step, and Lemma 5.2.10 in the third step (which we will prove right after this theorem), we have

$$\begin{aligned}\mathcal{I}_2 &= \sum_{i=1}^{n} \alpha_i \left(\int_{D_i} |x|^2 dx + \int_{D_i} \nabla p_i \cdot x dx \right) \\ &\geq \sum_{i=1}^{n} \alpha_i \left(\int_{D_i} |x|^2 dx - \Big(\int_{D_i} |\nabla p_i|^2 dx \Big)^{1/2} \Big(\int_{D_i} |x|^2 dx \Big)^{1/2} \right) \geq 0.\end{aligned} \tag{5.34}$$

Combining (5.33) and (5.34) gives us $\mathcal{I} \geq 0$. If there is any D_i that is not a disk or annulus centered at the origin, Lemma 5.2.10 would give a strict inequality in the last step of (5.34), which leads to $\mathcal{I} > 0$ and thus contradicts with $\mathcal{I} = 0$. □

Now we state and prove the lemma that is used in the proof of Theorem 5.2.3.

Lemma 5.2.10. *Let D be a bounded domain with $C^{1,\gamma}$ boundary, and let p be as in Lemma 5.2.5. Then we have*

$$-\int_D \nabla p \cdot x dx = \int_D |\nabla p|^2 dx \leq \int_D |x|^2 dx. \tag{5.35}$$

Furthermore, in the inequality, "=" is achieved if and only if D is a disk or annulus centered at the origin.

Proof. We compute

$$\begin{aligned}\int_D |\nabla p|^2 dx &= \int_{\partial D} p \nabla p \cdot \vec{n} d\sigma + \int_D 2p dx \\ &= -\int_{\partial D} px \cdot \vec{n} d\sigma + \int_D 2p dx,\end{aligned}$$

where in the last equality we use that p is constant along each ∂h_i, as well as the following identity due to (5.12) and the divergence theorem (here $\vec{n}$ is the outer normal of h_i):

$$\int_{\partial h_i} \nabla p \cdot \vec{n} d\sigma = -2|h_i| = -\int_{h_i} \Delta \frac{|x|^2}{2} dx = -\int_{\partial h_i} x \cdot \vec{n} d\sigma.$$

On the other hand, the divergence theorem yields

$$-\int_D \nabla p \cdot x dx = -\int_{\partial D} px \cdot \vec{n} d\sigma + \int_D 2p dx.$$

Therefore using Young's inequality $-\nabla p \cdot x \leq \frac{1}{2}|\nabla p|^2 + \frac{1}{2}|x|^2$ (where the equality is achieved if and

only if $-\nabla p = x$), we have

$$\int_D |\nabla p|^2 dx = -\int_D \nabla p \cdot x dx \leq \frac{1}{2}\int_D |\nabla p|^2 dx + \frac{1}{2}\int_D |x|^2 dx,$$

which proves (5.35). Here the equality is achieved if and only if $-\nabla p = x$ in D, which is equivalent with $p + \frac{|x|^2}{2}$ being a constant in D, and it can be extended to $\overline{D}$ due to continuity of p. By our construction of p in Lemma 5.2.5, p is already a constant on each connected component of ∂D, implying $\frac{|x|^2}{2}$ is constant on each piece of ∂D, hence ∂D must be a family of circles centered at the origin. By the assumption that D is connected, it must be either a disk or annulus centered at the origin. □

5.2.4 Radial symmetry of non-simply-connected rotating patches with $\Omega \geq \frac{1}{2}$

In this final subsection for patches, we consider a bounded open set D with $C^{1,\gamma}$ boundary. D can have multiple connected components, and each connected component can be non-simply-connected. If 1_D is a rotating patch solution to the Euler equation with angular velocity $\Omega \geq \frac{1}{2}$, we will show D must be radially symmetric and centered at the origin.

To do this, one might be tempted to proceed as in Theorem 5.2.1 and replace $p : D \to \mathbb{R}$ by the function defined in Lemma 5.2.5. Here the first way of computing $\mathcal{I} = \int_D (x + \nabla p) \cdot \nabla f_\Omega dx$ still yields $\mathcal{I} = 0$, but the second way gives some undesired terms caused by the holes h_i:

$$\mathcal{I} = \frac{1}{4\pi}|D|^2 - \Omega \int_D |x|^2 dx + (2\Omega - 1)\int_D p dx + 2\Omega \sum_{i=1}^{n} p|_{\partial h_i}|h_i|.$$

Due to the last term on the right hand side, we are unable to show $\mathcal{I} \leq 0$ when $\Omega \geq \frac{1}{2}$ as we did before in Theorem 5.2.1. For this reason, we take a different approach in the next theorem. Instead of defining p as a function in D and $\mathcal{I}$ as an integral in D, we want to define them in D^c. But since D^c is unbounded, we define p^R and $\mathcal{I}_R$ in a truncated set $B(0,R) \setminus D^c$, and then use two different ways to compute $\mathcal{I}_R$. By sending $R \to \infty$, we will show that the two ways give a contradiction unless D is radially symmetric.

Theorem 5.2.4. *For a bounded open set D with $C^{1,\gamma}$ boundary, assume that 1_D is a rotating patch solution to the Euler equation with angular velocity $\Omega \geq \frac{1}{2}$. Then D is radially symmetric and centered at the origin.*

Proof. Since D is bounded, let us choose $R_0 > 0$ such that $B_{R_0} \supset D$. For any $R > R_0$, consider the open set $B_R \setminus \overline{D}$, which may have multiple connected components. We call the component touching ∂B_R as $D_{0,R}$, and name the other connected components by $U_1, \ldots, U_n$. Throughout this proof we assume that $n \geq 1$: if not, then each connected component of D is simply connected, which has been already treated in Theorem 5.2.1 and Remark 5.2.2. We also define $V := B_R \setminus D_{0,R}$, which is the union of D and all its holes. Note that V may have multiple connected components, but each must be simply-connected. See Figure 5.1 for an illustration of $D_{0,R}$, $\{U_i\}_{i=1}^n$ and V.

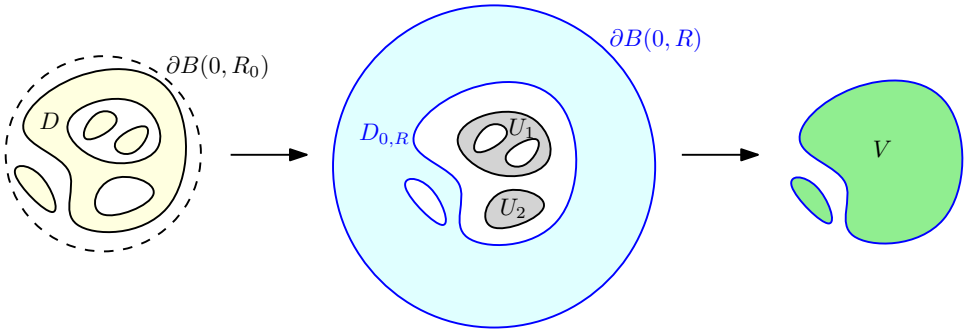


Figure 5.1: For a set $D \subset B(0, R_0)$ (the whole yellow region on the left), the middle figure illustrates the definition of $D_{0,R}$ (the blue region), $\{U_i\}$ (the gray regions), and right right figure illustrate $V = B_R \setminus D_{0,R}$ (the green region).

To prove the theorem, the key idea is to define p^R and $\mathcal{I}_R$ in $B_R \setminus D$, instead of in D. Let $p_{0,R}$ and p_i be defined as in Lemma 5.2.5 in $D_{0,R}$ and U_i respectively, then set $\varphi_{0,R} := p_{0,R} + \frac{|x|^2}{2}$ in $D_{0,R}$, and $\varphi_i := p_i + \frac{|x|^2}{2}$ in U_i for $i = 1, \ldots, n$. Finally, define p^R and $\varphi^R : \mathbb{R}^2 \to \mathbb{R}$ as

$$p^R := p_{0,R} 1_{D_{0,R}} + \sum_{i=1}^n p_i 1_{U_i}, \qquad \varphi^R := \varphi_{0,R} 1_{D_{0,R}} + \sum_{i=1}^n \varphi_i 1_{U_i}.$$

Since 1_D rotates with angular velocity $\Omega \geq \frac{1}{2}$, we know $f_\Omega := 1_D * \mathcal{N} - \frac{\Omega}{2}|x|^2$ is constant on each connected component of ∂D. Next we will compute

$$\mathcal{I}_R := \int_{B_R \setminus \overline{D}} \nabla f_\Omega \cdot \nabla \varphi^R dx \tag{5.36}$$

in two different ways. If some connected component of ∂D is not a circle, we will derive a contradiction by sending $R \to \infty$. We point out that as we increase R, the domain $D_{0,R}$ will change, but the sets $\{U_i\}_{i=1}^n$ and V all remain unchanged.

On the one hand, we break $\mathcal{I}_R$ into

$$\mathcal{I}_R = \int_{D_{0,R}} \nabla f_\Omega \cdot \nabla \varphi_{0,R} dx + \sum_{i=1}^n \int_{U_i} \nabla f_\Omega \cdot \nabla \varphi_i dx =: \mathcal{I}_R^1 + \mathcal{I}_R^2.$$

Since f_Ω is constant on each connected component of ∂U_i, the same computation as (5.23)–(5.24) gives $\mathcal{I}_R^2 = 0$. For $\mathcal{I}_R^1$, note that although f_Ω is a constant along the boundary of each hole of $D_{0,R}$, it is *not* a constant along $\partial_{\text{out}} D_{0,R} = \partial B_R$. Thus similar computations as (5.23)–(5.24) now give

$$\begin{aligned} \mathcal{I}_R^1 &= \int_{\partial B_R} \big(1_D * \mathcal{N} - \frac{\Omega}{2} R^2\big) \nabla \varphi_{0,R} \cdot \vec{n} d\sigma \\ &= \int_{\partial B_R} \left((1_D * \mathcal{N})(x) - |D| \mathcal{N}(x)\right) \nabla \varphi_{0,R} \cdot \vec{n} d\sigma(x), \end{aligned} \tag{5.37}$$

where in the second equality we used $\int_{\partial B_R} \nabla \varphi_{0,R} \cdot \vec{n} d\sigma = 0$ and the fact that $\mathcal{N}(x)$ is constant on ∂B_R. For any $x \in \partial B_R$, since $D \subset B_{R_0}$ and $R > R_0$, we can control $(1_D * \mathcal{N})(x) - |D|\mathcal{N}(x)$ as

$$\left|(1_D * \mathcal{N})(x) - |D|\mathcal{N}(x)\right| \le \frac{1}{2\pi} \int_D \Big| \log|x-y| - \log|x| \Big| dy \le \frac{|D|}{2\pi} \left| \log\left(1 - \frac{R_0}{R}\right) \right| \text{ on } \partial B_R. \tag{5.38}$$

We introduce the following lemma to control $|\nabla \varphi_{0,R} \cdot \vec{n}|$ on ∂B_R, whose proof is postponed to the end of this subsection.

Lemma 5.2.11. *Let $D \subset B_{R_0}$ be an open set with $C^{1,\gamma}$ boundary. For any $R > R_0$, let $D_{0,R}$, V, $p_{0,R}$ and $\varphi_{0,R}$ be defined as in the proof of Theorem 5.2.4. Then we have*

$$|\nabla \varphi_{0,R} \cdot \vec{n}| \le \frac{N R_0^2}{2R \log(R/R_0)} \quad \text{on } \partial B_R, \tag{5.39}$$

where $N > 0$ is the number of connected components of V (and is independent of R).

Once we have this lemma, plugging (5.39) and (5.38) into (5.37) yields

$$|\mathcal{I}_R^1| \le \frac{N|D|R_0^2}{2} \left| \log\left(1 - \frac{R_0}{R}\right) \right| (\log(R/R_0))^{-1} \le |D| \frac{C(D, R_0)}{R \log R} \to 0 \quad \text{as } R \to \infty.$$

Combining this with $\mathcal{I}_R^2 = 0$ gives

$$\lim_{R \to \infty} \mathcal{I}_R = 0. \tag{5.40}$$

Next we compute $\mathcal{I}_R$ in another way. Note that $1_{B_R} * \mathcal{N} - \frac{|x|^2}{4}$ is a radial harmonic function in B_R,

thus is equal to some constant C_R in B_R. Using this fact, we can rewrite f_Ω as

$$f_\Omega = 1_D * \mathcal{N} - \frac{\Omega}{2}|x|^2 = (1_D - 1_{B_R}) * \mathcal{N} - \Big(\frac{\Omega}{2} - \frac{1}{4}\Big)|x|^2 + C_R.$$

As a result, $\mathcal{I}_R$ can be rewritten as

$$\begin{aligned}\mathcal{I}_R &= -\int_{B_R\setminus\overline{D}} \nabla\left(1_{B_R\setminus\overline{D}} * \mathcal{N}\right)\cdot\nabla\varphi^R dx - \frac{(2\Omega-1)}{2}\int_{B_R\setminus\overline{D}} x\cdot\nabla\varphi^R dx \\ &=: -\mathcal{J}_R^1 - \frac{(2\Omega-1)}{2}\mathcal{J}_R^2.\end{aligned} \tag{5.41}$$

Next we will show $\mathcal{J}_R^1, \mathcal{J}_R^2 \geq 0$, leading to $\mathcal{I}_R \leq 0$. Let us start with $\mathcal{J}_R^2$. Applying Lemma 5.2.10 to each of $D_{0,R}$ and $\{U_i\}_{i=1}^n$ immediately gives

$$\mathcal{J}_R^2 \geq \int_{D_{0,R}} |x|^2 + \nabla p_{0,R}\cdot x dx + \sum_{i=1}^n \int_{U_i} |x|^2 + \nabla p_i \cdot x dx =: T_{0,R} + \sum_{i=1}^n T_i \geq 0. \tag{5.42}$$

Note that the T_i's are independent of R for $i = 1,\dots,n$, and we know $T_i \geq 0$ with equality achieved if and only if U_i is an annulus or a disk centered at the origin. This will be used later to show all $\{U_i\}_{i=1}^n$ are centered at the origin in the $\Omega > \frac{1}{2}$ case. (When $\Omega = \frac{1}{2}$, the coefficient of $\mathcal{J}_R^2$ becomes 0 in (5.41), thus a different argument is needed in this case.)

We now move on to $\mathcal{J}_R^1$. We first break it as

$$\mathcal{J}_R^1 = \int_{B_R\setminus\overline{D}} \nabla(1_{B_R\setminus\overline{D}} * \mathcal{N})\cdot x dx + \int_{B_R\setminus\overline{D}} \nabla(1_{B_R\setminus\overline{D}} * \mathcal{N})\cdot\nabla p^R dx =: J_{11} + J_{12}.$$

An identical computation as (5.25) gives $J_{11} = \dfrac{1}{4\pi}\Big(|D_{0,R}| + \displaystyle\sum_{i=1}^n |U_i|\Big)^2$. For J_{12}, the same computation as (5.28)–(5.30) gives the following (where we used that each U_i lies in a hole of $D_{0,R}$ for $i = 1,\dots,n$):

$$J_{12} \geq -\int_{D_{0,R}} p_{0,R} dx - \sum_{i=1}^n \int_{U_i} p_i dx - \sum_{1\leq i\leq n} |U_i| \sup_{D_{0,R}} p_{0,R} - \sum_{\substack{1\leq i,j\leq n \\ j\prec i}} \frac{|U_i||U_j|}{2\pi}.$$

Adding up the estimates for J_{11} and J_{12}, we get

$$\begin{aligned}\mathcal{J}_R^1 \geq & \left(\frac{1}{4\pi}|D_{0,R}|^2 - \int_{D_{0,R}} p_{0,R}dx\right) + \left(\sum_{1\leq i\leq n}|U_i|\right)\left(\frac{1}{2\pi}|D_{0,R}| - \sup_{D_{0,R}} p_{0,R}\right) \\ & + \sum_{i=1}^n \left(\frac{1}{4\pi}|U_i|^2 - \int_{U_i} p_i dx\right) + \sum_{\substack{j\neq i \text{ and } i\neq j \\ i\neq j}} \frac{1}{4\pi}|U_i||U_j|.\end{aligned} \tag{5.43}$$

By Proposition 5.2.7, all terms on the right hand side are nonnegative. But note that only the two terms in the second line are independent of R. Plugging (5.43) and (5.42) into (5.41) gives the following (where we only keep the terms independent of R on the right hand side):

$$\liminf_{R\to\infty}(-\mathcal{I}_R) \geq \sum_{i=1}^n \left(\frac{1}{4\pi}|U_i|^2 - \int_{U_i} p_i dx\right) + \sum_{\substack{j\neq i \text{ and } i\neq j \\ i\neq j}} \frac{1}{4\pi}|U_i||U_j| + \frac{2\Omega-1}{2}\sum_{i=1}^n T_i \geq 0.$$

Combining this with the previous limit (5.40), we know U_i must be an annulus or a disk for $i = 1,\ldots,n$, and they must be nested in each other. In addition, if $\Omega > \frac{1}{2}$, we have $T_i = 0$ for $i = 1,\ldots,n$, implying that each U_i is centered at the origin.

The radial symmetry of $D_{0,R}$ is more difficult to obtain. Although the first two terms on the right hand side of (5.43) are both strictly positive if $D_{0,R}$ is not an annulus, we need some uniform-in-R lower bound to get a contradiction in the $R\to\infty$ limit. Between these two terms, it turns out the second term is easier to control. This is done in the next lemma, whose proof we postpone to the end of this subsection.

Lemma 5.2.12. *Let $D \subset B_{R_0}$ be an open set with $C^{1,\gamma}$ boundary. For any $R > R_0$, let $D_{0,R}$, V and $p_{0,R}$ be given as in the proof of Theorem 5.2.4. If V is not a single disk, there exists some constant $C(V) > 0$ only depending on V, such that*

$$\liminf_{R\to\infty}\left(\frac{1}{2\pi}|D_{0,R}| - \sup_{D_{0,R}} p_{0,R}\right) \geq C(V) > 0.$$

If V is not a disk, Lemma 5.2.12 gives $\liminf_{R\to\infty}\mathcal{J}_R^1 > \left(\sum_{1\leq i\leq n}|U_i|\right)C(V) > 0$. (Recall that in the beginning of this proof we assume $\sum_{1\leq i\leq n}|U_i| > 0$, and it is independent of R.) This implies $\liminf_{R\to\infty}(-\mathcal{I}_R) \geq C(V) > 0$, contradicting (5.40).

So far we have shown that ∂D is a union of nested circles, and it remains to show that they are all centered at 0. For the $\Omega > \frac{1}{2}$ case, we already showed all $\{U_i\}_{i=1}^n$ are centered at 0, so it suffices to show the outmost circle ∂V (denote by $B(\tilde{o},\tilde{r})$) is also centered at 0. By definition of $\{U_i\}_{i=1}^n$,

we have $D = B(\tilde{o},\tilde{r}) \setminus \big(\dot{\cup}_{i=1}^n U_i\big)$. Note that $1_{B(\tilde{o},\tilde{r})} * \mathcal{N} = \frac{|x-\tilde{o}|^2}{4} + C$ for some constant C, and $1_{\cup_{i=1}^n U_i} * \mathcal{N}$ is radially increasing. Therefore f_Ω can be written as

$$f_\Omega = 1_{B(\tilde{o},\tilde{r})} * \mathcal{N} - 1_{\cup_{i=1}^n U_i} * \mathcal{N} - \frac{\Omega}{2}|x|^2 = \frac{|x-\tilde{o}|^2}{4} - g(x),$$

where g is radially symmetric, and strictly increasing in the radial variable. Since both f_Ω and $\frac{|x-\tilde{o}|^2}{4}$ are known to take constant values on $\partial B(\tilde{o},\tilde{r})$, it implies g must be constant on $\partial B(\tilde{o},\tilde{r})$ too, and the fact that g is a radially increasing function gives that $\tilde{o} = 0$. This finishes the proof for $\Omega > \frac{1}{2}$.

For $\Omega = \frac{1}{2}$, we do not know whether $\{U_i\}_{i=1}^n$ are centered at 0 yet. Denote by U_1 be the innermost one. Then we have

$$f_\Omega(x) = \frac{|x-\tilde{o}|^2}{4} - 1_{\cup_{i=1}^n U_i} * \mathcal{N} - \frac{|x|^2}{4} = \frac{\tilde{o}\cdot x}{2} + \text{const} \quad \text{for } x \in \partial_{\text{out}} U_1, \tag{5.44}$$

where the second equality follows from Lemma 5.2.9(b), where we used that $1 \prec j$ for all $2 \le j \le n$. Combining (5.44) with the fact that $f_\Omega = \text{const}$ on $\partial_{\text{out}} U_1$ gives $\tilde{o} = 0$, that is, the outmost circle must be centered at 0. This leads to $f_\Omega = -\sum_{i=1}^n 1_{U_i} * \mathcal{N}$. Since $f_\Omega = \text{const}$ on each connected component of ∂U_i, we can apply the last part in the proof of Corollary 5.2.8 to show that all $\{U_i\}_{i=1}^n$ are all concentric. Denoting their center by o_1, we can show that $o_1 = 0$: Lemma 5.2.9(a) gives $f_\Omega(x) = C \ln|x - o_1|$ for some $C < 0$ on $\partial B(\tilde{o},\tilde{r})$, and since we have $f = \text{const}$ on $\partial B(\tilde{o},\tilde{r})$ and $\tilde{o} = 0$, it implies that $o_1 = 0$, finishing the proof. □

Proof of Lemma 5.2.11. For notational simplicity, we shorten $p_{0,R}$, $D_{0,R}$ and $\varphi_{0,R}$ into p_R, D_R and φ_R thoughout this proof. Recall that $\partial D_R = \partial B_R \cup \partial V$. Clearly we have $\varphi_R = \frac{R^2}{2}$ on ∂B_R, due to $p_R = 0$ on $\partial_{\text{out}} D_R = \partial B_R$. We claim that

$$-\frac{NR_0^2}{2} \le \varphi_R - \frac{R^2}{2} \le \frac{R_0^2}{2} \quad \text{on } \partial V, \tag{5.45}$$

where $N \ge 1$ is the number of connected components of V. Once it is proved, we apply the comparison principle to the functions $\varphi_R - \frac{R^2}{2}$ and $\pm g$, where

$$g(x) := \frac{NR_0^2}{2\log(R/R_0)} \log\frac{R}{|x|}.$$

Note that $g \equiv 0$ on ∂B_R, and $g \ge \frac{NR_0^2}{2}$ on ∂V since $\partial V \subset B_{R_0}$. If $0 \notin \overline{D_R}$, then the functions $\varphi_R - \frac{R^2}{2}$ and $\pm g$ are all harmonic in D_R, their values on ∂B_R are all 0, and their boundary values

on ∂V are ordered due to (5.45). The comparison principle in D_R then yields

$$-g(x) \le \varphi_R(x) - \frac{R^2}{2} \le g(x) \quad \text{ in } D_R. \tag{5.46}$$

Since $\varphi_R - \frac{R^2}{2} \equiv g \equiv 0$ on ∂B_R, (5.46) gives $|\nabla \varphi_R \cdot \vec{n}| \le |\nabla g \cdot \vec{n}| = \frac{N R_0^2}{2R \log(R/R_0)}$ on ∂B_R, which is the desired estimate (5.39). And if $0 \in \overline{D_R}$, then (5.46) still holds in $D_R \setminus B_\epsilon$ for all sufficiently small $\epsilon > 0$ by applying the comparison principle in this set, and (5.39) again follows as a consequence.

In the rest of the proof we will show (5.45). Its second inequality is straightforward:

$$\varphi_R - \frac{R^2}{2} \le \frac{R_0^2}{2} + \sup_{\overline{D_R}} p_R - \frac{R^2}{2} \le \frac{R_0^2}{2} \quad \text{ on } \partial V,$$

here the first inequality follows from the definition of φ_R and the fact that $V \subset B_{R_0}$, and the second inequality is due to $\sup_{\overline{D_R}} p_R \le \frac{|D_R|}{2\pi} \le \frac{R^2}{2}$ in Proposition 5.2.7.

It remains to prove the first inequality of (5.45). Let us fix any $R > R_0$. Denote the N connected components of ∂V by $\{\Gamma_i\}_{i=1}^N$, and let $\Gamma_0 := \partial B_R$. These notations lead to $\partial D_R = \cup_{i=0}^N \Gamma_i$. For $i = 0, \dots, N$, let $L_i \subset \mathbb{R}$ be the range of $\varphi_R - \frac{R^2}{2}$ on Γ_i. By continuity of φ_R, each L_i is a closed bounded interval, which can be a single point. Clearly, $L_0 = \{0\}$ due to $\varphi_R|_{\partial B_R} = \frac{R^2}{2}$. Towards a contradiction, suppose

$$v_{\min} := \min_{1 \le i \le N} \inf L_i = \inf_{\partial V} \Big(\varphi_R - \frac{R^2}{2}\Big) =: -\frac{N|R_0|^2}{2} - \delta \quad \text{for some } \delta > 0. \tag{5.47}$$

As for the maximum value, since $L_0 = \{0\}$ we have

$$v_{\max} := \max_{0 \le i \le N} \sup L_i \ge 0. \tag{5.48}$$

For $i = 1, \dots, N$, using $p_R|_{\Gamma_i} = \text{const}$, $\varphi_R = p_R + \frac{|x|^2}{2}$ and $\Gamma_i \subset B_{R_0}$, the length of each interval L_i satisfies

$$\big|L_i\big| = \operatorname{osc}_{\Gamma_i} \frac{|x|^2}{2} \le \frac{R_0^2}{2} \quad \text{ for } i = 1, \dots, N. \tag{5.49}$$

Comparing (5.49) with (5.47)–(5.48), we know the union of $\{L_i\}_{i=0}^N$ cannot fully cover the interval $[v_{\min}, v_{\max}]$, thus they can be separated in the following sense: there exists a nonempty proper subset $S \subset \{0, \dots, N\}$, such that the range of L_i for indices in S and $S^c := \{0, \dots, N\} \setminus S$ are strictly

separated by at least δ, i.e. $\min_{i\in S}\inf L_i \geq \max_{i\in S^c}\sup L_i + \delta$. In terms of φ_R, we have

$$\min_{i\in S}\inf_{\Gamma_i}\varphi_R \geq \max_{i\in S^c}\max_{\Gamma_i}\varphi_R + \delta. \tag{5.50}$$

Since φ_R is harmonic in D_R, whose boundary is $\cup_{i=0}^N\Gamma_i$, it is a standard comparison principle exercise to show that (5.50) implies

$$\sum_{i\in S}\int_{\Gamma_i}\nabla\varphi_R\cdot\vec{n}d\sigma > 0, \tag{5.51}$$

where $\vec{n}$ denotes the outer normal of D_R. But on the other hand, we have

$$\int_{\Gamma_i}\nabla\varphi_R\cdot\vec{n}d\sigma = 0 \quad \text{for } i = 0,\dots,N. \tag{5.52}$$

To see this, the cases $i = 1,\dots,N$ can be done by an identical computation as (5.24), and the $i = 0$ case follows from $\int_{\partial D_R}\varphi_R\cdot\vec{n}d\sigma = \int_{D_R}\Delta\varphi_R dx = 0$ and the fact that $\partial D_R = \cup_{i=0}^N\Gamma_i$. Thus we have obtained a contradiction between (5.51) and (5.52), completing the proof. □

Proof of Lemma 5.2.12. Assume V has N connected components $\{V_j\}_{j=1}^N$ for $N \geq 1$. For notational simplicity, we shorten $D_{0,R}$, $p_{0,R}$ and $\varphi_{0,R}$ into D_R, p_R and φ_R in this proof. Let $\epsilon_R := \frac{1}{2\pi}|D_R| - \sup_{D_R}p_R$, which is nonnegative by Proposition 5.2.7. Towards a contradiction, assume there exists a diverging subsequence $\{R_i\}_{i=1}^\infty$ such that $\lim_{i\to 0}\epsilon_{R_i} = 0$.

Define $\tilde{\varphi}_{R_i} := \varphi_{R_i} - \frac{R_i^2}{2}$. We claim that $\{\tilde{\varphi}_{R_i}\}_{i=1}^\infty$ has a subsequence that converges locally uniformly to some bounded harmonic function φ_∞ in $\mathbb{R}^2\setminus V$.

To show this, we will first obtain a uniform bound of $\{\tilde{\varphi}_{R_i}\}_{i=1}^\infty$. Note that (5.45) gives that $\sup_{\partial V}|\tilde{\varphi}_{R_i}| \leq \frac{NR_0^2}{2}$ for all $i\in\mathbb{N}^+$. Since $\tilde{\varphi}_{R_i}\equiv 0$ on ∂B_{R_i} for all $i\in\mathbb{N}^+$, the maximum principle for harmonic function gives $\sup_{D_{R_i}}|\tilde{\varphi}_{R_i}| \leq \frac{NR_0^2}{2}$ for all $i\in\mathbb{N}^+$.

For any $R > 2R_0$, we will obtain a uniform gradient estimate for $\{\tilde{\varphi}_{R_i}\}$ in D_R for all $R_i > 2R$. First note that since ∂B_R is in the interior of D_{R_i} (due to $R_i > 2R$), interior estimate for harmonic function (together with the above uniform bound) gives that $\|\tilde{\varphi}_{R_i}\|_{C^2(\partial B_R)} \leq C(N, R_0)$. On the other boundary ∂V, recall that $\tilde{\varphi}_{R_i}|_{\partial V_j} = \frac{|x|^2}{2} + c_{i,j}$, with $|c_{i,j}| \leq \frac{(N+1)R_0^2}{2}$. Thus $\|\tilde{\varphi}_{R_i}\|_{C^2(\partial D_R)} \leq C(N, R_0)$ for all $R_i > 2R$, and the standard elliptic regularity theory gives the uniform gradient estimate $\sup_{D_R}|\nabla\tilde{\varphi}_{R_i}| \leq C(V)$. This allows us to take a further subsequence (which we still denote by $\{\tilde{\varphi}_{R_i}\}$) that converges uniformly in $\overline{D_R}$ to some harmonic function $\tilde{\varphi}_\infty \in C(\overline{D_R})$. Since $R > 2R_0$ is arbitrary, we can repeat this procedure (for countably many times) to obtain a subsequence that converges locally uniformly to a harmonic function $\tilde{\varphi}_\infty$ in $\mathbb{R}^2\setminus V$, where $\tilde{\varphi}_\infty|_{\partial V_j} = \frac{|x|^2}{2} + c_j$ with

$|c_j| \leq \frac{(N+1)R_0^2}{2}$. This finishes the proof of the claim.

Now define

$$\tilde{p}_{R_i} := p_{R_i} - \frac{R_i^2}{2} = \tilde{\varphi}_{R_i} - \frac{|x|^2}{2},$$

which is known to converge locally uniformly to $\tilde{p}_\infty := \tilde{\varphi}_\infty - \frac{|x|^2}{2}$ in $\mathbb{R}^2 \setminus V$. Note that $\tilde{p}_\infty$ is *not* radially symmetric up to any translation: To see this, recall that $\tilde{p}_\infty|_{\partial V_j} \equiv c_j$. If $\tilde{p}_\infty$ is radial about some x_0, it must be of the form $-\frac{|x-x_0|^2}{2} + c$ due to $\Delta\tilde{p}_\infty = -2$. As a result, the level sets of $\tilde{p}_\infty$ are all nested circles, thus V must be a single disk (where we used that each connected component of V is simply-connected).

Next we will show that $\lim_{i\to 0} \epsilon_{R_i} = 0$ implies $\tilde{p}_\infty$ is radial up to a translation, leading to a contradiction. For $k \in \mathbb{R}$, let $g_i(k) := |\{x \in D_{R_i} : p_{R_i}(x) > k\}|$. In the proof of Proposition 5.2.7, we have shown that $g_i(0) = |D_{R_i}|$, g_i is decreasing in k, with $g_i'(k) \leq -2\pi$ for almost every $k \in (0, \sup_{D_{R_i}} p_{R_i})$. Since $\sup_{D_{R_i}} p_{R_i} = \frac{1}{2\pi}|D_{R_i}| - \epsilon_{R_i}$, we can control $g_i(k)$ from below and above as follows:

$$(|D_{R_i}| - 2\pi k - 2\pi\epsilon_{R_i})_+ \leq g_i(k) \leq (|D_{R_i}| - 2\pi k)_+ \quad \text{for all } k \geq 0. \tag{5.53}$$

Likewise, define $\tilde{g}_i(k) := |\{x \in D_{R_i} : \tilde{p}_{R_i}(x) > k\}|$, and $\tilde{g}_\infty(k) := |\{x \in D_{R_i} : \tilde{p}_\infty(x) > k\}|$. Since $\tilde{p}_{R_i} = p_{R_i} - \frac{R_i^2}{2}$, we have $\tilde{g}_i(k) = g_i(k + \frac{R_i^2}{2})$ for all $k \geq -\frac{R_i^2}{2}$, thus (5.53) is equivalent to

$$(-|V| - 2\pi k - 2\pi\epsilon_{R_i})_+ \leq \tilde{g}_i(k) \leq (-|V| - 2\pi k)_+ \quad \text{for all } k \geq -\frac{R_i^2}{2}.$$

The locally uniform convergence of p_{R_i} gives $\lim_{i\to\infty} \tilde{g}_i = \tilde{g}_0$, and since we assume $\lim_{i\to\infty} \epsilon_{R_i} = 0$, we take the $i \to \infty$ limit in the above inequality and obtain

$$\tilde{g}_\infty(k) = (-2\pi k - |V|)_+ \quad \text{for all } k \in \mathbb{R},$$

which implies

$$\tilde{g}_\infty'(k) = -2\pi \quad \text{for all } k \in (-\infty, \sup_{\mathbb{R}^2\setminus V} \tilde{p}_\infty). \tag{5.54}$$

Applying the proof of Proposition 5.2.7 to $\tilde{p}_\infty$ (note that the proof still goes through even though $\tilde{p}_\infty$ takes negative values, and is defined in an unbounded domain), we have that (5.54) can happen only if $\tilde{D}_k := \{\tilde{p}_\infty > k\} \dot\cup (\dot\cup_{\{j:c_j>k\}} \overline{V_j})$ is a disk for almost every $k \in (-\infty, \sup_{\mathbb{R}^2\setminus V} \tilde{p}_\infty)$, and $|\nabla\tilde{p}_\infty|$ is a constant on almost every $\partial\tilde{D}_k$. These two conditions imply that all regular sets of $\tilde{p}_\infty$ are concentric circles, thus $\tilde{p}_\infty$ is radial up to a translation, and we have obtained a contradiction. $\square$

Chapter 6

The rigidity results for 2D-Euler smooth case

6.1 Notation

We will use the same notation as in section 5.1.

6.2 Radial symmetry of nonnegative smooth stationary/rotating solutions for 2D Euler with $\Omega \leq 0$

Let $\omega(x,t) = \omega_0(R_{\Omega t}x)$ be a nonnegative compactly-supported stationary/rotating solution of 2D Euler with angular velocity $\Omega \leq 0$. Recall that by (4.7), $f_\Omega := \omega_0 * \mathcal{N} - \frac{\Omega}{2}|x|^2$ is a constant along each connected component of a regular level set of ω_0. In this section, we prove that ω_0 is radial up to a translation for $\Omega = 0$, and radial for $\Omega < 0$. As we discussed in the introduction, the $\omega \geq 0$ condition is necessary: in [43] we show that there exists a compactly-supported, sign-changing smooth stationary vorticity ω_0 that is non-radial, and the construction also works for $\Omega < 0$ that is close to 0.

Most of this section is devoted to the proof of Theorem 6.2.1 in the $\Omega = 0$ case (the $\Omega < 0$ case is done in Corollary 6.2.5 as a simple extension). In the proof, the two key steps are to show that every connected component of a regular level set of ω is a circle, and these circles are concentric. These are done by approximating ω by a step function $\omega_n = \sum_{i=1}^{M_n} \alpha_i 1_{D_i}$ such that the sets $\{D_i\}$ are disjoint, and $\|\omega - \omega_n\|_{L^\infty} = O(1/n)$. We then define $\varphi^n = \frac{|x|^2}{2} + \sum_{i=1}^{M_n} 1_{D_i} p_i$ corresponding to

this step function ω_n, and compute $\mathcal{I}_n = \int \omega_n \nabla\varphi^n \cdot \nabla f dx$ in two ways.

Due to the $O(1/n)$ error in the approximation, the qualitative statement in Proposition 5.2.7 that "the equality is achieved if and only if D is a disk or annulus" is no longer good enough for us. We need to obtain various quantitative versions of (5.15) for doubly-connected domains, and three such versions are stated below.

In Lemma 6.2.2, the quantitative constant $c_0 > 0$ depends on the Fraenkel asymmetry of the outer boundary defined in Definition 6.2.1.

Definition 6.2.1 (c.f. [35, Section 1.2])**.** *For a bounded open set $E \subseteq \mathbb{R}^2$, we define the* Fraenkel asymmetry $\mathcal{A}(E) \in [0,1)$ *as*

$$\mathcal{A}(E) := \inf_{x_0} \left\{ \frac{|E\Delta(x_0 + rB)|}{|E|} : x_0 \in \mathbb{R}^2, \pi r^2 = |E| \right\},$$

where B is a unit disk in $\mathbb{R}^2$.

Lemma 6.2.2. *Let D be a doubly connected set. Let us denote the hole of D by an open set h, and let $\tilde{D} := D \cup \overline{h}$. We define p in D as in Lemma 5.2.5. Then if $\mathcal{A}(\tilde{D}) > 0$, there is a constant $c_0 > 0$ that only depends on $\mathcal{A}(\tilde{D})$, such that*

$$p|_{\partial h} \leq \frac{|D|}{2\pi}(1 - c_0).$$

Lemma 6.2.2 will be used in the proof of Theorem 6.2.1 to show that all level sets of ω are circles. To obtain radial symmetry of ω, we also need to show all these level sets are concentric. To do this, we need to obtain some quantitative lemmas for a region between two non-concentric disks. In Lemma 6.2.3 we consider a thin tubular region between two non-concentric disks whose radii are close to each other, and obtain a quantitative version of (5.15) for such domain.

Lemma 6.2.3. *For $\epsilon > 0$, consider two open disks $B_1 := B(o_1, 1)$ and $B_2 = B(o_2, 1+\epsilon)$ such that $B_1 \subset B_2$. Suppose $|o_1 - o_2| = a\epsilon$ with $a \in (0,1)$, and let p be defined as in Lemma 5.2.5 in $D := \overline{B_2} \setminus B_1$. Then if ϵ and a satisfy that $0 < \epsilon < \frac{a^2}{64}$, we have*

$$p\big|_{\partial B_1} \leq \frac{|D|}{2\pi}\left(1 - \frac{a^2}{16}\right). \tag{6.1}$$

In Lemma 6.2.4 we consider a region between two non-concentric disks (that is not necessarily a thin tubular region), and obtain a quantitative version of (5.15) for such domain.

Lemma 6.2.4. *Consider two open disks $B_r := B(o_1, r)$ and $B_R = B(o_2, R)$ such that $B_r \subset B_R$. Let p be defined as in (5.2.5) in $D := B_R \backslash \overline{B_r}$. Suppose $l := |o_1 - o_2| > 0$ and there exist $\delta_1 > 0$, and $\delta_2 > 0$ such that $\delta_1 < r < R < \delta_2$. Then there exist a constant c_0 that only depends on δ_1, δ_2 and l such that*

$$p|_{\partial B_r} \leq \frac{|D|}{2\pi}(1 - c_0).$$

The proofs of the above quantitative lemmas will be postponed to Section 6.2.1. Now we are ready to prove the main theorem.

Theorem 6.2.1. *Let ω be a smooth[1] compactly supported nonnegative stationary solution to the 2D Euler equation. Then ω is radially symmetric up to a translation.*

Proof. Note that as mentioned in step 1 of Proposition 5.2.7, by Sard's theorem, we have that for almost every $k \in (0, \|\omega\|_{L^\infty})$, $\omega^{-1}(\{k\})$ is a smooth 1-manifold. Furthermore, since ω is compactly supported, each such level set is a disjoint union of finite number of simple closed curves. For any such closed curve, we call it a "*level set component*" in this proof.

We split the proof into several steps. Throughout step 1, 2 and 3, we prove that all level set components of ω must be circles. In step 4, we will prove that any two level set components are nested, i.e. one is contained in the other. Lastly we present the proof that all level set components are concentric in step 5 and 6.

Step 1. Towards a contradiction, suppose there is $k > 0$ that is a regular value of ω, and $\omega^{-1}(\{k\})$ has a connected component Γ that differs from a circle. Recall that $\mathrm{int}(\Gamma)$ denotes the interior of Γ, which is open and simply connected. Since Γ is not a circle, we have $\mathcal{A}(\mathrm{int}(\Gamma)) > 0$, with $\mathcal{A}$ as in Definition 6.2.1.

In this step, we investigate level set components near Γ. Since k is a regular value, we can find an open neighborhood U of Γ and a constant $\eta > 0$ such that $|\nabla\omega| > \eta$ in U. For any $x \in \Gamma$, consider the flow map $\Phi_t(x)$ originating from x, given by

$$\frac{d}{dt}\Phi_t(x) = \frac{\nabla\omega(\Phi_t(x))}{|\nabla\omega(\Phi_t(x))|^2}$$

with initial condition $\Phi_0(x) = x$. Since $\frac{\nabla\omega}{|\nabla\omega|^2}$ is smooth and bounded in U, we can choose $\delta_1 > 0$ so that $\Phi_t(\Gamma) := \{\Phi_t(x) : x \in \Gamma\}$ lies in U for any $t \in (-\delta_1, \delta_1)$. Note that Φ_t's are diffeomorphisms,

[1] In fact it is sufficient to assume $\omega_0 \in C^2(\mathbb{R}^2)$. In the proof, the application of Sard's theorem in Step 1 is the step that requires the highest regularity and it is applicable for functions in $C^2(\mathbb{R}^2)$. (See [86, Theorem 6.1]). Therefore $\omega_0 \in C^2(\mathbb{R}^2)$ ensures that $\omega^{-1}(\{k\})$ is a C^2 1-manifold for almost every $k \in (0, \|\omega\|_{L^\infty})$, whose regularity is sufficient for all the following steps.

thus $\Phi_t(\Gamma)$ is also a smooth simple closed curve for $t \in (-\delta_1, \delta_1)$. Then we observe that

$$\frac{d}{dt}\omega(\Phi_t(x)) = \nabla\omega(\Phi_t(x)) \cdot \frac{\nabla\omega(\Phi_t(x))}{|\nabla\omega(\Phi_t(x))|^2} = 1 \quad \text{for } (t,x) \in (-\delta_1,\delta_1)\times\Gamma. \tag{6.2}$$

Hence for each $t \in (-\delta_1, \delta_1)$, $\Phi_t(\Gamma)$ is a level set component and

$$\omega(\Phi_{t_1}(\Gamma)) \neq \omega(\Phi_{t_2}(\Gamma)) \text{ if } t_1 \neq t_2. \tag{6.3}$$

By continuity of the map $(t,x) \mapsto \Phi_t(x)$, we can find $\delta_2 \in (0, \delta_1)$ such that

$$\mathcal{A}(\text{int}(\Phi_t(\Gamma))) > \frac{1}{2}\mathcal{A}(\text{int}(\Gamma)) \text{ for any } t \in (-\delta_2, \delta_2). \tag{6.4}$$

Since two different level sets cannot intersect, we can assume without loss of generality that $\text{int}(\Phi_{-\delta_2}(\Gamma)) \subset \text{int}(\Phi_{\delta_2}(\Gamma))$. Then it follows from the intermediate value theorem and (6.2) that

$$\text{int}(\Phi_{-\delta_2}(\Gamma)) \subset \Phi_t(\Gamma) \subset \text{int}(\Phi_{\delta_2}(\Gamma)), \text{ for any } t \in (-\delta_2, \delta_2). \tag{6.5}$$

Lastly we denote $V := \text{int}(\Phi_{\delta_2}(\Gamma)) \backslash \overline{\text{int}(\Phi_{-\delta_2}(\Gamma))}$ which is a nonempty open doubly-connected set, therefore $|V| > 0$.

Step 2. For any integer $n > 1$, we claim that we can approximate ω by a step function ω_n of the form $\omega_n(x) = \sum_{i=1}^{M_n} \alpha_i 1_{D_i}(x)$, which satisfies all the following properties.

(a) Each D_i is a domain with smooth boundary and possibly has a finite number of holes.

(b) Each connected component of ∂D_i is a level set component of ω.

(c) $D_i \cap D_j = \emptyset$ if $i \neq j$.

(d) $\|\omega_n - \omega\|_{L^\infty(\mathbb{R}^2)} \leq \frac{2}{n}\|\omega\|_{L^\infty(\mathbb{R}^n)}$.

To construct such ω_n for a fixed $n > 1$, let $r_0 = 0$ and $r_{n+1} = \|\omega\|_{L^\infty}$. We pick $r_1, \ldots, r_n$ to be regular values of ω, such that $0 < r_1 < \cdots < r_n < \|\omega\|_{L^\infty}$, and $r_{i+1} - r_i < \frac{2}{n}\|\omega\|_{L^\infty}$ for $i = 0, \ldots, n$. We denote $D_i := \left\{x \in \mathbb{R}^2 : r_i < \omega(x) < r_{i+1}\right\}$ for $i = 1, \ldots, n-1$, and let $D_n := \left\{x \in \mathbb{R}^2 : \omega(x) > r_n\right\}$. Thus for each $i = 1, \cdots, n$, D_i is a bounded open set with smooth boundary. We can then write it as $D_i = \dot{\cup}_{l=1}^{m_i} D_i^l$ for some $m_i \in \mathbb{N}$ where D_i^l's are connected components of D_i. Then let $\omega_n(x) := \sum_{i=1}^{n} r_i \sum_{l=1}^{m_i} 1_{D_i^l}$. By relabeling the indices, we rewrite $\omega_n(x) = \sum_{i=1}^{M_n} \alpha_i 1_{D_i}$,

where $M_n = \sum_{i=1}^n m_i$, and each $\alpha_i \in \{r_1, \dots, r_n\}$. One can easily check that such ω_n satisfies properties (a)–(d).

Of course, there are many ways to choose the values $r_1, \cdots, r_n$, with each choice leading to a different ω_n. From now on, for any $n > 1$, we fix $\omega_n(x) := \sum_{i=1}^{M_n} \alpha_i 1_{D_i}(x)$ as any function constructed in the above way. (Note that α_i and D_i all depend on n, but we omit their n dependence for notational simplicity.)

Finally, let us point out that for $\omega_n(x)$ constructed above, if $D_i \subset V$ for some i, then D_i must be doubly connected, since step 1 shows that all level set components in V are nested curves. We will use this in step 3 and 5.

Step 3. For any $n > 1$, let $\omega_n(x) = \sum_{i=1}^{M_n} \alpha_i 1_{D_i}(x)$ be constructed in Step 2. For each D_i, let we define p_i^n in D_i as in Lemma 5.2.5. We set

$$\begin{cases} p^n := \sum_{i=1}^{M_n} p_i^n 1_{D_i} \\ \varphi_i^n := p_i + \frac{|x|^2}{2} \text{ in } D_i \\ \varphi^n := \sum_{i=1}^{M_n} \varphi_i^n 1_{D_i}. \end{cases} \tag{6.6}$$

As in Theorem 11.1.7, let $f := \omega * \mathcal{N}$, and we will compute

$$\mathcal{I}^n := \int_{\mathbb{R}^2} \omega_n(x) \nabla \varphi^n(x) \cdot \nabla f(x) dx = \sum_{i=1}^{M_n} \alpha_i \int_{D_i} \nabla \varphi^n(x) \cdot \nabla f(x) dx \tag{6.7}$$

in two different ways and derive a contradiction by taking the $n \to \infty$ limit.

On the one hand, the same computation as in (5.23)–(5.24) yields that

$$\mathcal{I}^n = \sum_{i=1}^{M_n} \alpha_i \left(\int_{\partial D_i} f(x) \nabla \varphi_i^n(x) \cdot \vec{n} d\sigma - \int_{D_i} f(x) \Delta \varphi_i^n(x) dx \right) = 0. \tag{6.8}$$

On the other hand,

$$\begin{aligned} \mathcal{I}^n &= \int_{\mathbb{R}^2} \omega_n(x) x \cdot \nabla f(x) dx + \int_{\mathbb{R}^2} \omega_n(x) \nabla p^n(x) \cdot \nabla f(x) dx \\ &=: \mathcal{I}_1^n + \mathcal{I}_2^n. \end{aligned}$$

Since ω_n satisfies property (d) in step 2, it follows that

$$\lim_{n\to\infty} \mathcal{I}_1^n = \int_{\mathbb{R}^2} \omega(x) x \cdot \nabla f(x) dx.$$

A similar computation as in (5.25) yields that

$$\begin{aligned}
\int_{\mathbb{R}^2} \omega(x) x \cdot \nabla f(x) dx &= \frac{1}{2\pi} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \omega(x)\omega(y) \frac{x \cdot (x-y)}{|x-y|^2} dxdy \\
&= \frac{1}{4\pi} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \omega(x)\omega(y) dxdy \\
&= \frac{1}{4\pi} \left(\int_{\mathbb{R}^2} \omega(x) dx \right)^2, \qquad (6.9)
\end{aligned}$$

where we used the symmetry of the integration domain to get the second equality.

Now we estimate the limit of $\mathcal{I}_2^n$. By Lemma 5.2.10, we have $\int_{D_i} |\nabla p_i^n|^2 dx \leq \int_{D_i} |x|^2 dx$, hence $\|\omega_n \nabla p\|_{L^2(\mathbb{R}^2)}$ is uniformly bounded. Since $\omega_n \to \omega$ in L^∞, the bounded convergence theorem yields that

$$\lim_{n\to\infty} \int_{\mathbb{R}^2} \omega_n \nabla p^n \cdot \nabla \left((\omega_n - \omega) * \mathcal{N} \right)(x) dx = 0,$$

therefore

$$\liminf_{n\to\infty} \mathcal{I}_2^n = \liminf_{n\to\infty} \int_{\mathbb{R}^2} \omega_n(x) \nabla p^n(x) \cdot \nabla(\omega_n * \mathcal{N}) dx.$$

From now on, we will omit the n dependence in p_i^n for notational simplicity. Let us break the integral in the right hand side as

$$\begin{aligned}
\int_{\mathbb{R}^2} \omega_n(x) \nabla p^n(x) \cdot \nabla(\omega_n * \mathcal{N}) dx &= \sum_{i,j=1}^{M_n} \alpha_i \alpha_j \int_{D_i} \nabla p_i \cdot \nabla(1_{D_j} * \mathcal{N}) dx \\
&= \sum_{i=1}^{M_n} \alpha_i^2 \int_{D_i} \nabla p_i \cdot \nabla(1_{D_i} * \mathcal{N}) dx + \sum_{i\neq j} \alpha_i \alpha_j \int_{D_i} \nabla p_i \cdot \nabla(1_{D_j} * \mathcal{N}) dx \\
&=: F_1 + F_2. \qquad (6.10)
\end{aligned}$$

For F_1, the divergence theorem yields

$$F_1 = \sum_{i=1}^{M_n} \alpha_i^2 \left(\int_{\partial D_i} p_i \nabla(1_{D_i} * \mathcal{N}) \cdot \vec{n} d\sigma - \int_{D_i} p_i dx \right) = -\sum_{i=1}^{M_n} \alpha_i^2 \int_{D_i} p_i dx, \qquad (6.11)$$

where the second equality follows from an identical computation as in (5.26). Then by Proposition 5.2.7, we have

$$F_1 \geq -\frac{1}{4\pi}\sum_{i=1}^{M_n}\alpha_i^2|D_i|^2. \tag{6.12}$$

For F_2, the divergence theorem yields

$$F_2 = \sum_{i\neq j}\alpha_i\alpha_j\left(\int_{\partial D_i} p_i\nabla(1_{D_j}*\mathcal{N})\cdot\vec{n}d\sigma - \int_{D_i} p_i 1_{D_j}dx\right) = \sum_{i\neq j}\alpha_i\alpha_j\int_{\partial D_i} p_i\nabla(1_{D_j}*\mathcal{N})\cdot\vec{n}d\sigma,$$

where we use property (c) in step 2 to get the last equality.

For $i\neq j$, recall that as in the proof of Corollary 5.2.8, we denote $j\prec i$ if D_j is contained in a hole of D_i. Then divergence theorem gives

$$\int_{\partial D_i} p_i\nabla(1_{D_j}*\mathcal{N})\cdot\vec{n}d\sigma\begin{cases}=0 & \text{if } j\not\prec i,\\ \geq -\sup_{\partial D_i} p_i|D_j| & \text{if } j\prec i.\end{cases} \tag{6.13}$$

Next we will improve this inequality for $j\prec i$ and $i\in L$, where $L:=\{i: D_i\subset V\}$. (Note that L depends on ω_n, where we omit this dependence for notational simplicity.) From the discussion at the end of step 2, we know that D_i has exactly one hole for all $i\in L$. Using the divergence theorem together with this observation, (6.13) becomes

$$\int_{\partial D_i} p_i\nabla(1_{D_j}*\mathcal{N})\cdot\vec{n}d\sigma\begin{cases}=0 & \text{if } j\not\prec i,\\ \geq -\sup_{D_i} p_i|D_j| & \text{if } j\prec i \text{ and } i\notin L,\\ =-p_i|_{\partial_{\text{in}}D_i}|D_j| & \text{if } j\prec i \text{ and } i\in L.\end{cases} \tag{6.14}$$

For the second case on the right hand side, we simply use the crude bound $\sup_{D_i} p_i\leq\frac{|D_i|}{2\pi}$ from Proposition 5.2.7. For the third case we can have a better bound: for any $i\in L$, by Lemma 6.2.2 and (6.4), there exists an $\epsilon>0$ that only depends on $\mathcal{A}(\text{int}(\Gamma))$ (and in particular is independent of n), such that $p_i|_{\partial_{\text{in}}D_i}\leq(\frac{1}{2\pi}-\epsilon)|D_i|$. Thus (6.14) now becomes

$$\int_{\partial D_i} p_i\nabla(1_{D_j}*\mathcal{N})\cdot\vec{n}d\sigma\begin{cases}=0 & \text{if } j\not\prec i,\\ \geq -\frac{1}{2\pi}|D_i||D_j| & \text{if } j\prec i \text{ and } i\notin L,\\ \geq -(\frac{1}{2\pi}-\epsilon)|D_i||D_j| & \text{if } j\prec i \text{ and } i\in L.\end{cases} \tag{6.15}$$

Now we are ready to estimate F_2. Let us break it into

$$\begin{aligned} F_2 &= \sum_{\substack{j\prec i\\(i,j)\notin L\times L}} \alpha_i\alpha_j \int_{\partial D_i} p_i \nabla(1_{D_j} * \mathcal{N})\cdot \vec{n} d\sigma + \sum_{\substack{j\prec i\\(i,j)\in L\times L}} \alpha_i\alpha_j \int_{\partial D_i} p_i \nabla(1_{D_j} * \mathcal{N})\cdot \vec{n} d\sigma \\ &\geq -\sum_{\substack{j\prec i\\(i,j)\notin L\times L}} \alpha_i\alpha_j \frac{1}{2\pi}|D_i||D_j| - \sum_{\substack{j\prec i\\(i,j)\in L\times L}} \alpha_i\alpha_j\Big(\frac{1}{2\pi}-\epsilon\Big)|D_i||D_j| \end{aligned}$$

where the first equality follows from case 1 of (6.15), and the second inequality follows from case 2,3 of (6.15). Finally, by exchanging i with j and taking average with the original inequality, we have

$$\begin{aligned} F_2 &\geq -\frac{1}{4\pi}\sum_{\substack{i\neq j\\(i,j)\notin L\times L}} (\mathbb{1}_{i\prec j}+\mathbb{1}_{j\prec i})\alpha_i\alpha_j|D_i||D_j| - \frac{1}{2}\sum_{\substack{i\neq j\\(i,j)\in L\times L}} (\mathbb{1}_{i\prec j}+\mathbb{1}_{j\prec i})\alpha_i\alpha_j\Big(\frac{1}{2\pi}-\epsilon\Big)|D_i||D_j| \\ &\geq -\frac{1}{4\pi}\sum_{\substack{i\neq j\\(i,j)\notin L\times L}} \alpha_i\alpha_j|D_i||D_j| - \frac{1}{2}\sum_{\substack{i\neq j\\(i,j)\in L\times L}} \alpha_i\alpha_j\Big(\frac{1}{2\pi}-\epsilon\Big)|D_i||D_j| \\ &= -\frac{1}{4\pi}\sum_{i\neq j}\alpha_i\alpha_j|D_i||D_j| + \frac{\epsilon}{2}\sum_{\substack{i\neq j\\(i,j)\in L\times L}} \alpha_i\alpha_j|D_i||D_j|, \end{aligned} \tag{6.16}$$

where the second inequality is due to the fact that for any $i\neq j$, at most one of $i\prec j$ and $j\prec i$ can be true, thus we always have $\mathbb{1}_{i\prec j}+\mathbb{1}_{j\prec i}\leq 1$.

Therefore, from (6.12) and (6.16) it follows that

$$\begin{aligned} F_1+F_2 &\geq -\frac{1}{4\pi}\sum_{i,j=1}^{M_n}\alpha_i\alpha_j|D_i||D_j| + \frac{\epsilon}{2}\sum_{(i,j)\in L\times L}\alpha_i\alpha_j|D_i||D_j| - \frac{\epsilon}{2}\sum_{i\in L}\alpha_i^2|D_i|^2 \\ &= -\frac{1}{4\pi}\left(\sum_{i=1}^{M_n}\alpha_i|D_i|\right)^2 + \frac{\epsilon}{2}\left(\sum_{i\in L}\alpha_i|D_i|\right)^2 - \frac{\epsilon}{2}\sum_{i\in L}\alpha_i^2|D_i|^2. \end{aligned} \tag{6.17}$$

Since we will send $n\to\infty$, in the rest of step 3 we will denote L by L^n to emphasize that L depends on ω_n. (In fact α_i and D_i depend on n as well, and we omit the n dependence for them to avoid overcomplicating the notations.)

Note that $\sum_{i\in L^n}\alpha_i 1_{D_i}$ converges to $\omega 1_V$ in $L^1(\mathbb{R}^2)$. Also if $i\in L^n$, then the nondegeneracy of $|\nabla\omega|$ on V yields that $\lim_{n\to\infty}\sup_{i\in L^n}|D_i|=0$, consequently

$$\lim_{n\to\infty}\sum_{i\in L^n}\alpha_i^2|D_i|^2 \leq \|\omega\|_{L^\infty}\lim_{n\to\infty}\sup_{i\in L^n}|D_i|\int_{\mathbb{R}^2}\omega dx = 0.$$

Therefore it follows that

$$\begin{aligned}\liminf_{n\to\infty} \mathcal{I}_2^n &= \liminf_{n\to\infty} (F_1 + F_2) \\ &\geq -\lim_{n\to\infty} \frac{1}{4\pi} \left(\sum_{i=1}^{M_n} \alpha_i |D_i| \right)^2 + \lim_{n\to\infty} \frac{\epsilon}{2} \left(\sum_{i\in L^n} \alpha_i |D_i| \right)^2 \\ &= -\frac{1}{4\pi} \left(\int_{\mathbb{R}^2} \omega(x) dx \right)^2 + \frac{\epsilon}{2} \left(\int_V \omega(x) dx \right)^2. \end{aligned} \tag{6.18}$$

Note that ω is strictly positive in V, due to $|\nabla\omega| > 0$ in V and $\omega \geq 0$ in $\mathbb{R}^2$. Thus from (6.8), (6.9) and (6.18), it follows that

$$0 = \lim_{n\to\infty} \mathcal{I}^n \geq \lim_{n\to\infty} \mathcal{I}_1^n + \liminf_{n\to\infty} \mathcal{I}_2^n \geq \frac{\epsilon}{2} \left(\int_V \omega(x) dx \right)^2 > 0, \tag{6.19}$$

which is a contradiction and we have proved that any level set component is a circle.

Step 4. In this step we show that every pair of level set components are nested. Towards a contradiction, assume that there exist two level set components Γ_1 and Γ_2 that are not nested.

From step 3, we know that Γ_1 and Γ_2 are circles. Then the disks $\text{int}(\Gamma_1)$ and $\text{int}(\Gamma_2)$ are disjoint, and they must be separated by a positive distance since Γ_1 and Γ_2 are level sets of regular values of ω. As in step 1, using the flow map Φ_t originating from the two circles, we can find disjoint open annuli V_1 and V_2 such that $\Gamma_i \subset V_i$ for $i = 1, 2$, and both $\partial_{\text{out}} V_i$ and $\partial_{\text{in}} V_i$ are level set components of ω.

For any $n > 1$, let $\omega_n(x) = \sum_{i=1}^{M_n} \alpha_i 1_{D_i}(x)$ be constructed in step 2, and let

$$L_1^n := \{i : D_i \subset V_1\} \quad \text{and} \quad L_2^n := \{i : D_i \subset V_2\}.$$

Let p_i be defined in (6.6) of step 3, and $\mathcal{I}^n$ defined in (6.7). Then on the one hand, the same computations in step 3 give

$$\lim_{n\to\infty} \mathcal{I}^n = 0 \quad \text{and} \quad \lim_{n\to\infty} \mathcal{I}_1^n = \frac{1}{4\pi} \left(\int_{\mathbb{R}^2} \omega(x) dx \right)^2. \tag{6.20}$$

Let F_1 and F_2 be given by (6.10). For F_1, the estimate (6.12) still holds. For F_2, using (6.13)

and Proposition 5.2.7, we have

$$F_2 \geq -\frac{1}{4\pi} \sum_{i \prec j \text{ or } j \prec i} \alpha_i \alpha_j |D_i||D_j|.$$

Since V_1 and V_2 are assumed to be not nested, if $(i,j) \in L_1^n \times L_2^n$ then neither $i \prec j$ nor $j \prec i$. Therefore it follows that

$$F_2 \geq -\frac{1}{4\pi} \sum_{i \neq j} \alpha_i \alpha_j |D_i||D_j| + \frac{1}{4\pi} \sum_{(i,j) \in L_1 \times L_2} \alpha_i \alpha_j |D_i||D_j| + \frac{1}{4\pi} \sum_{(j,i) \in L_1 \times L_2} \alpha_i \alpha_j |D_i||D_j|.$$

Combining the estimates for F_1 and F_2 yields

$$F_1 + F_2 \geq -\frac{1}{4\pi} \sum_{i,j=1}^{M_n} \alpha_i \alpha_j |D_i||D_i| + \frac{1}{2\pi} \left(\sum_{i \in L_1^n} \alpha_i |D_i| \right) \left(\sum_{i \in L_2^n} \alpha_i |D_i| \right).$$

As $n \to \infty$, since $\sum_{i \in L_1^n} \alpha_i 1_{D_i}$ and $\sum_{i \in L_2^n} \alpha_i 1_{D_i}$ converge to $\omega 1_{V_1}$ and $\omega 1_{V_2}$ respectively in $L^1(\mathbb{R}^2)$, we have

$$\liminf_{n \to \infty} \mathcal{I}_2^n \geq \lim_{n \to \infty} (F_1 + F_2) = -\frac{1}{4\pi} \left(\int_{\mathbb{R}^2} \omega(x) dx \right)^2 + \frac{1}{2\pi} \left(\int_{V_1} \omega(x) dx \right) \left(\int_{V_2} \omega(x) dx \right). \quad (6.21)$$

Combining (6.20) and (6.21) gives us a similar contradiction as in (6.19), except that $\frac{\epsilon}{2} \left(\int_V \omega(x) dx \right)^2$ is now replaced by $\frac{1}{2\pi} \left(\int_{V_1} \omega(x) dx \right) \left(\int_{V_2} \omega(x) dx \right)$. Thus we complete the proof that level sets are nested.

Step 5. In this step, we aim to show that all level set components are concentric within the *same connected component* of $\{\omega > 0\} := \left\{ x \in \mathbb{R}^2 : \omega > 0 \right\}$. This immediately implies that each connected component of $\{\omega > 0\}$ is an annulus or a disk, and ω is radially symmetric about its center.

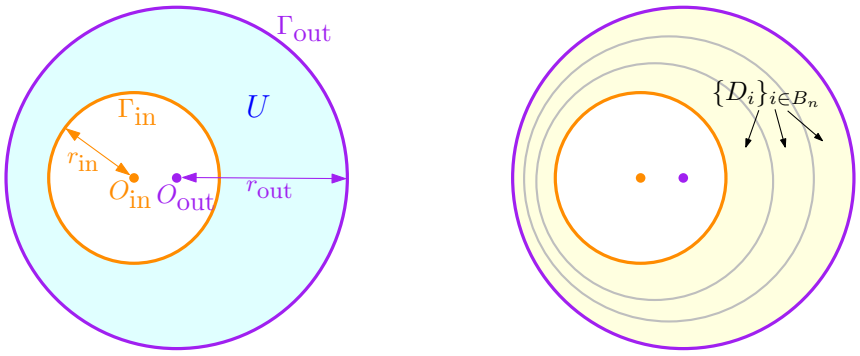
Towards a contradiction, suppose that there are two level set components Γ_{in} and Γ_{out} in the same connected component of $\{\omega > 0\}$, such that they are nested circles, but their centers O_{in} and O_{out} do not coincide. We denote their radii by r_{in} and r_{out}, and define

$$U := \overline{\text{int}(\Gamma_{\text{out}})} \setminus \text{int}(\Gamma_{\text{in}}).$$

For an illustration of Γ_{in} and Γ_{out} and U, see Figure 6.1(a).

We claim that ω is uniformly positive in U. Recall that all level set components of ω are nested

by step 4. Thus if ω achieves zero in U, the zero level set must be also nested between Γ_{in} and Γ_{out}, since it can be taken as a limit of level set components whose value approaches 0; but this contradicts with the assumption that Γ_{in} and Γ_{out} lie in the same connected component of $\{\omega > 0\}$. As a result, we have $\omega_{\min} := \inf_U \omega > 0$.



(a) (b)

Figure 6.1: (a) Illustration of the circles Γ_{in} and Γ_{out}, whose centers are O_{in} and O_{out}. The set U is colored in blue. (b) For a fixed n, each open set $\{D_i\}_{i\in B_n}$ is colored in yellow. Note that their union gives exactly the set U.

For a sufficiently large n, let $\omega_n = \sum_{i=1}^{M_n} \alpha_i 1_{D_i}(x)$ be given in step 2, where we further require both Γ_{in} and Γ_{out} coincide with some boundary of D_i. (This is allowed in our construction of ω_n in step 2, since ω is regular along both Γ_{in} and Γ_{out}.) Let us denote

$$B_n := \{1 \le i \le M_n : D_i \subset U\},$$

and note that $U := \cup_{i\in B_n} \overline{D}_i$. See Figure 6.1(b) for an illustration of $\{D_i\}_{i\in B_n}$.

As before, we denote $i \prec j$ if D_i is nested in D_j. For the integral $\mathcal{I}^n$ in (6.7), on the one hand, we have $\mathcal{I}^n = 0$ for all $n > 1$ by (6.8). On the other hand, following the same argument as in step 3 up to (6.13) (where we also use that each D_i is already known to be doubly-connected, thus $\int_{\partial D_i} p_i \nabla(1_{D_j} * \mathcal{N}) \cdot \vec{n} d\sigma = -p_i|_{\partial_{\text{in}} D_i} |D_j|$ if $j \prec i$), we have

$$\begin{aligned}
\liminf_{n\to\infty} \mathcal{I}^n &= \liminf_{n\to\infty} \left(\frac{1}{4\pi} \Big(\sum_{i=1}^{M_n} \alpha_i |D_i| \Big)^2 - \sum_{i=1}^{M_n} \alpha_i^2 \int_{D_i} p_i dx - \sum_{1\le i,j\le M_n, j\prec i} \alpha_i \alpha_j p_i\big|_{\partial_{\text{in}} D_i} |D_j| \right) \\
&\ge \liminf_{n\to\infty} \Bigg(\underbrace{\sum_{1\le i,j\le M_n,\, j\prec i} \alpha_i \alpha_j \Big(\frac{1}{2\pi} |D_i| - p_i\big|_{\partial_{\text{in}} D_i} \Big) |D_j|}_{=:T_n} \Bigg),
\end{aligned}$$

where in the last step we used Proposition 5.2.7.

Note that Proposition 5.2.7 gives $T_n > 0$, where we have strict positivity, since $O_{\text{in}} \neq O_{\text{out}}$

implies that some $\{D_i\}_{i\in B_n}$ must be non-radial. But since the area of these D_i's may approach 0 as $n\to\infty$, in order to derive a contradiction after taking $\liminf_{n\to\infty}$, we need to obtain a quantitative estimate for Proposition 5.2.7 for a thin tubular region D_i between two circles, which is done in Lemma 6.2.3.

Next we show that the sets $\{D_i\}_{i\in B_n}$ that are "non-radial to some extent" must occupy a certain portion of U. For $i\in B_n$, denote by o^i_{in} and r^i_{in} the center and radius of $\partial_{\text{in}}D_i$, and likewise o^i_{out} and r^i_{out} the center and radius of $\partial_{\text{out}}D_i$. Note that if D_i is the inner-most set in $\{D_i\}_{i\in B_n}$, then we have $o^i_{\text{in}}=O_{\text{in}}$, and the outmost D_i satisfies $o^i_{\text{out}}=O_{\text{out}}$. In addition, if $\partial_{\text{out}}D_i=\partial_{\text{in}}D_j$ for some $i,j\in B_n$, then $o^i_{\text{out}}=o^j_{\text{in}}$. Thus the triangle inequality gives

$$\sum_{i\in B_n}|o^i_{\text{in}}-o^i_{\text{out}}|\geq|O_{\text{in}}-O_{\text{out}}|=:c_0>0. \tag{6.22}$$

In order to apply Lemma 6.2.3 (which requires the region to have inner radius 1), for each $i\in B_n$, consider the scaling

$$\tilde{p}_i(x):=(r^i_{\text{in}})^{-2}p_i(r^i_{\text{in}}x).$$

Then $\tilde{p}_i$ is defined in $\tilde{D}_i:=(r^i_{\text{in}})^{-1}D_i$. Due to the scaling, $\tilde{D}_i$ has inner radius 1 (denote the hole by $\tilde{h}_i$), and outer radius $1+\epsilon_i$, where $\epsilon_i:=\frac{r^i_{\text{out}}-r^i_{\text{in}}}{r^i_{\text{in}}}>0$. In addition, the distance between the centers of $\partial_{\text{in}}\tilde{D}_i$ and $\partial_{\text{out}}\tilde{D}_i$ is $a_i\epsilon_i$, where

$$a_i:=\frac{|o^i_{\text{in}}-o^i_{\text{out}}|}{|r^i_{\text{in}}-r^i_{\text{out}}|}.$$

One can also easily check that $\tilde{p}_i$ satisfies $\Delta\tilde{p}_i=-2$ in $\tilde{D}_i$, and $\int_{\partial\tilde{h}_i}\nabla\tilde{p}\cdot\vec{n}d\sigma=-2|\tilde{h}_i|=-2\pi$. By Lemma 6.2.3, if $0<\epsilon_i<\frac{(a_i)^2}{64}$, then $\tilde{p}_i|_{\partial\tilde{h}_i}\leq\frac{|\tilde{D}_i|}{2\pi}(1-\frac{a_i^2}{16})$. Thus in terms of p_i, we have

$$p_i\big|_{\partial_{\text{in}}D_i}\leq\frac{1}{2\pi}|D_i|-c_1a_i^2|D_i|\quad\text{ if } i\in B_n \text{ satisfies } \ r^i_{\text{out}}-r^i_{\text{in}}\leq c_2a_i^2, \tag{6.23}$$

where $c_1:=\frac{1}{32\pi}$ and $c_2:=\frac{r_{\text{in}}}{64}$ are independent of n and i, due to the fact that $r^i_{\text{in}}\geq r_{\text{in}}>0$ for all $i\in B_n$. Using the definition of a_i, (6.22) can be written as

$$\sum_{i\in B_n}a_i|r^i_{\text{in}}-r^i_{\text{out}}|\geq c_0. \tag{6.24}$$

Note that $\sum_{i\in B_n}|r^i_{\text{in}}-r^i_{\text{out}}|$ satisfies the upper bound

$$\sum_{i\in B_n}|r^i_{\text{in}}-r^i_{\text{out}}|\leq\frac{|U|}{2\pi r_{\text{in}}}=:M, \tag{6.25}$$

which follows from

$$|U| = \sum_{i\in B_n} |D_i| = \pi \sum_{i\in B_n} |r_{\text{in}}^i - r_{\text{out}}^i| \underbrace{(r_{\text{in}}^i + r_{\text{out}}^i)}_{>2r_{\text{in}}}.$$

Combining (6.24) and (6.25) gives

$$\sum_{i\in B_n} \mathbb{1}_{a_i > \frac{c_0}{2M}} |r_{\text{in}}^i - r_{\text{out}}^i| \geq \sum_{i\in B_n} \Big(a_i - \frac{c_0}{2M}\Big)|r_{\text{in}}^i - r_{\text{out}}^i| \geq \frac{c_0}{2}, \tag{6.26}$$

where the first inequality follows from $\mathbb{1}_{a_i > \frac{c_0}{2M}} \geq a_i - \frac{c_0}{2M}$ (recall that $a_i \in (0,1)$), and the second inequality follows from subtracting $\frac{c_0}{2M}$ times (6.25) from (6.24).

Let

$$K_n := \Big\{ i \in B_n : a_i > \frac{c_0}{2M} \Big\}.$$

Using this definition and the fact that $|D_i| > 2\pi r_{\text{in}} |r_{\text{in}}^i - r_{\text{out}}^i|$, (6.26) can be rewritten as

$$\sum_{i\in K_n} |D_i| > 2\pi r_{\text{in}} \sum_{i\in K_n} |r_{\text{in}}^i - r_{\text{out}}^i| \geq \pi r_{\text{in}} c_0. \tag{6.27}$$

Now we take a sufficiently large n, and discuss two cases (note that different n may lead to different cases):

Case 1. Every $i \in K_n$ satisfies $r_{\text{out}}^i - r_{\text{in}}^i \leq \min\{c_2(\frac{c_0}{2M})^2, \frac{r_{\text{in}} c_0}{4 r_{\text{out}}}\}$. By definition of K_n, we have $r_{\text{out}}^i - r_{\text{in}}^i \leq c_2(\frac{c_0}{2M})^2 \leq c_2 a_i^2$ for $i \in K_n$. (The motivation of the second term in the min function will be made clear later.) Then by (6.23), we have

$$\frac{1}{2\pi}|D_i| - p_i\big|_{\partial_{\text{in}} D_i} \geq c_1 a_i^2 |D_i| \geq \frac{c_1 c_0^2}{4M^2}|D_i| \quad \text{ for all } i \in K_n.$$

Since K_n is a subset of B_n (and recall that $\alpha_i \geq \omega_{\min} > 0$ for all $i \in B_n$), we have the following lower bound for T_n:

$$\begin{aligned} T_n &\geq \omega_{\min}^2 \sum_{i,j\in K_n,\, j\prec i} \Big(\frac{1}{2\pi}|D_i| - p_i\big|_{\partial_{\text{in}} D_i}\Big)|D_j| \\ &\geq \omega_{\min}^2 \sum_{i,j\in K_n,\, j\prec i} \frac{c_1 c_0^2}{4M^2}|D_i||D_j| = \omega_{\min}^2 \sum_{i,j\in K_n,\, i\neq j} \frac{c_1 c_0^2}{8M^2}|D_i||D_j| \\ &= \omega_{\min}^2 \frac{c_1 c_0^2}{16M^2} \left(\Big(\sum_{i\in K_n} |D_i| \Big)^2 - \sum_{i\in K_n} |D_i|^2 \right) \end{aligned} \tag{6.28}$$

Note that the second term in the min function in the assumption gives

$$\max_{i\in K_n}|D_i| < 2\pi r_{\text{out}}(r^i_{\text{out}} - r^i_{\text{in}}) \le \frac{\pi r_{\text{in}} c_0}{2} \le \frac{1}{2}\sum_{i\in K_n}|D_i|,$$

where we use (6.27) in the last inequality. Applying this to the right hand side of (6.28) gives

$$T_n \ge \omega^2_{\min}\frac{c_1 c_0^2}{16M^2}\cdot\frac{1}{2}\Big(\sum_{i\in K_n}|D_i|\Big)^2 \ge \omega^2_{\min}\frac{c_1 c_0^2}{32M^2}(\pi r_{\text{in}} c_0)^2.$$

Case 2. If Case 1 is not true, then there must be some $i_0 \in K_n$ satisfying $r^{i_0}_{\text{out}} - r^{i_0}_{\text{in}} > \min\{c_2(\frac{c_0}{2M})^2, \frac{r_{\text{in}}c_0}{4r_{\text{out}}}\} =: c_3$, which leads to

$$|o^{i_0}_{\text{in}} - o^{i_0}_{\text{out}}| = a_{i_0}(r^{i_0}_{\text{out}} - r^{i_0}_{\text{in}}) > \frac{c_0 c_3}{2M} =: l.$$

Although this set D_{i_0} is likely not thin enough for us to apply Lemma 6.2.3, since $|o^{i_0}_{\text{in}} - o^{i_0}_{\text{out}}|$ is bounded below by a positive constant independent of n, we can still use Lemma 6.2.4 to conclude that $\frac{1}{2\pi}|D_{i_0}| - p_{i_0}\big|_{\partial_{\text{in}}D_{i_0}} \ge c_4$ for some $c_4 > 0$ only depending on $r_{\text{in}}, r_{\text{out}}$ and l. This leads to

$$T_n \ge \sum_{i_0\succ j}\omega_{\min}\alpha_j c_4|D_j| \ge \omega_{\min}c_4\sum_{D_j\subset\text{int}(\Gamma_{\text{in}})}\alpha_j|D_j| \ge \omega_{\min}c_4\cdot\frac{1}{2}\int_{\text{int}(\Gamma_{\text{in}})}\omega dx,$$

where the last inequality follows from the fact that for all sufficiently large n, the definition of ω_n gives $\sum_{D_j\subset\text{int}(\Gamma_{\text{in}})}\alpha_j|D_j| = \int_{\text{int}(\Gamma_{\text{in}})}\omega_n dx \ge \frac{1}{2}\int_{\text{int}(\Gamma_{\text{in}})}\omega dx$. Note that the last integral is positive since $\omega > 0$ on Γ_{in}, and it is clearly independent of n.

From the above discussion, for all sufficiently large n, regardless of whether we are in Case 1 or 2 for this n, we always have that T_n is bounded below by some uniformly positive constant independent of n. Therefore taking the $n\to\infty$ limit gives

$$\liminf_{n\to\infty}\mathcal{I}^n \ge \liminf_{n\to\infty}T_n > 0.$$

This contradicts $\mathcal{I}^n = 0$, therefore finishing the proof of step 5.

Step 6. It remains to show that all connected components of $\{\omega > 0\}$ are concentric. If $\{\omega > 0\}$ has finitely many connected components, we could proceed similarly as the end of the proof of Corollary 5.2.8. But since $\{\omega > 0\}$ may have countably many connected components, we need to use a different argument.

Let us denote the connected components of $\{\omega > 0\}$ by $\{U_i\}_{i\in I}$, where I may have countably

many elements. Denote their centers by $\{o_i\}_{i\in I}$, their radii by $\{R_i\}_{i\in I}$, and their outer boundaries by $\{\partial_{\text{out}}U_i\}_{i\in I}$. Without loss of generality, suppose the x-coordinates of their centers $\{o_i^1\}_{i\in I}$ are not all identical.

Among the (possibly infinitely many) collection of circles $\{\partial_{\text{out}}U_i\}_{i\in I}$, let Γ_r be the "circle with rightmost center" among them, in the following sense:

• If there exists some $i_0 \in I$ such that $o_{i_0}^1 = \sup_{i\in I} o_i^1$, we define $\Gamma_r := \partial_{\text{out}}U_{i_0}$. (If the supremum is achieved at more than one indices, we set i_0 to be any of them.)

• Otherwise, take any subsequence $\{i_k\}_{k\in\mathbb{N}} \subset I$ such that $\lim_{k\to\infty} o_{i_k}^1 = \sup_{i\in I} o_i^1$. Since ω has compact support, we can extract a further subsequence (which we still denote by $\{i_k\}_{k\in\mathbb{N}}$), such that both o_{i_k} and r_{i_k} converge as $k \to \infty$, and denote their limit by $O_r \in \mathbb{R}^2$ and $R_r \in \mathbb{R}$. Finally let $\Gamma_r := \partial B(O_r, R_r)$.

With the above definition, we point out that $f := \omega * \mathcal{N} = \text{const}$ on Γ_r. Note that in both cases above, we can find a sequence of level set components of ω that converges to Γ_r, in the sense that the Hausdorff distance between the two sets goes to 0. Since $f = \text{const}$ on each level set component of ω, continuity of f gives that $f = \text{const}$ on Γ_r.

Let $f_i(x) := (\omega 1_{U_i}) * \mathcal{N}$ for $i \in I$; note that by definition we have $f = \sum_{i\in I} f_i$. Lemma 5.2.9 gives the following:

(a) For all $x \in (\text{int}(\partial_{\text{out}}U_i))^c$, we have $f_i(x) = \frac{1}{2\pi}\big(\int_{U_i} \omega dx\big) \ln|x - o_i|$.

(b) If U_i is doubly-connected, then $f_i = \text{const}$ in $\text{int}(\partial_{\text{in}}U_i)$, where the constants are different for different i.

Note that for any $i \in I$, U_i must be either nested inside Γ_r, or have Γ_r nested in its hole. (By a slight abuse of notation, we use $i \prec \Gamma_r$ and $i \succ \Gamma_r$ to denote these two relations.) Let $\Gamma_r^R := (O_r^1 + R_r, O_r^2)$ and $\Gamma_r^L := (O_r^1 - R_r, O_r^2)$ be the rightmost/leftmost point of the circle Γ_r. Note that (b) implies $f_i(\Gamma_r^R) = f_i(\Gamma_r^L)$ for all $i \succ \Gamma_r$, whereas (a) gives the following for all $i \prec \Gamma_r$:

$$f_i(\Gamma_r^R) = \frac{\int_{U_i} \omega dx}{2\pi} \ln|\Gamma_r^R - o_i| \geq \frac{\int_{U_i} \omega dx}{2\pi} \ln|\Gamma_r^L - o_i| = f_i(\Gamma_r^L),$$

where the inequality follows from that $|O_r^1 + R_r - o_i^1| \geq |O_r^1 - R_r - o_i^1|$, which is a consequence of $o_i^1 \leq O_r^1$ due to our choice of O_r. (Also note that Γ_r^R and Γ_r^L have the same y-coordinate.)

As a result, summing over all $i \in I$ gives $f(\Gamma_r^R) \geq f(\Gamma_r^L)$, where the equality is achieved if and only if $o_i^1 = O_r$ for all $i \prec \Gamma_r$. Now we discuss two cases:

Case 1. There is some $i \prec \Gamma_r$ with $o_i^1 < O_r$. In this case the above discussion gives $f(\Gamma_r^R) > f(\Gamma_r^L)$, which directly leads to a contradiction to $f = \text{const}$ on Γ_r.

Case 2. If case 1 does not hold, then let us define Γ_l as a "circle with leftmost center" among $\{\partial_{\text{out}} U_i\}_{i\in I}$ in the same way as Γ_r. Then we must have $O_l^1 < O_r^1$, and since case 1 does not hold (i.e. all $i \prec \Gamma_r$ satisfy that $o_i^1 = O_r$), we must have $\Gamma_l \succ \Gamma_r$. By definition of Γ_r, there exists some U_{i_0} whose outer boundary is sufficiently close to Γ_r and center sufficiently close to O_r. As a result, $i_0 \prec \Gamma_l$ and $o_{i_0}^1 > O_l^1$.

Let Γ_l^L and Γ_l^R be the leftmost/rightmost point of Γ_l. A parallel argument as above then gives that $f_i(\Gamma_l^L) \geq f_i(\Gamma_l^R)$ for all $i \in I$. Since we have found an $i_0 \prec \Gamma_l$ with $o_{i_0}^1 > O_l^1$, we have $f_{i_0}(\Gamma_l^L) > f_{i_0}(\Gamma_l^R)$, thus summing over all $i \in I$ gives the strict inequality $f(\Gamma_l^L) > f(\Gamma_l^R)$, contradicting with $f = \text{const}$ on Γ_l.

In both cases above we have obtained a contradiction, thus $\{o_i\}_{i\in I}$ must have the same x-coordinate. An identical argument shows that their y-coordinate must also be identical, thus $\{U_i\}_{i\in I}$ are concentric. Since ω is known to be radial within each U_i (about its own center) in step 1–5, the proof is now finished. □

In the next corollary, we show that the above proof for stationary smooth solutions can be extended (with some modifications) to show radial symmetry of nonnegative rotating smooth solutions with $\Omega < 0$.

Corollary 6.2.5. *Let $\omega(x,t) = \omega_0(R_{\Omega t}x)$ be a smooth, nonnegative compactly-supported uniformly-rotating solution of 2D Euler with angular velocity $\Omega < 0$. Then ω_0 is radially symmetric about the origin.*

Proof. The proof is very similar to the proof of Theorem 6.2.1, and we only highlight the differences. Let $\{\omega_n\}$ be the same approximation for ω_0 as in step 2 of Theorem 6.2.1. We consider the same setting as in (6.6) and (6.7), except with $f(x)$ replaced by $f_\Omega(x) := \omega_0 * \mathcal{N} - \frac{\Omega}{2}|x|^2$. From the assumption on ω_0, we have that f_Ω is a constant on each level set component of ω_0. Thus the same computations in (6.8) give $\mathcal{I}^n = 0$ for all $n > 1$.

On the other hand, we have

$$\mathcal{I}^n = \int_{\mathbb{R}^2} \omega_n \nabla\varphi^n \cdot \nabla(\omega_0 * \mathcal{N})\, dx + (-\Omega)\int_{\mathbb{R}^2} \omega_n \nabla\varphi^n \cdot x dx =: \mathcal{I}_1^n + \underbrace{(-\Omega)}_{\geq 0}\mathcal{I}_2^n. \tag{6.29}$$

The same argument as in (5.34) of Theorem 5.2.3 gives that $\mathcal{I}_2^n \geq 0$. As for $\mathcal{I}_1^n$, in step 3 – step 5 of the proof of Theorem 6.2.1, we have already shown that $\liminf_{n\to\infty} \mathcal{I}_1^n \geq 0$, and the equality is achieved if and only if each connected component of $\{\omega_0 > 0\}$ is radially symmetric up to a translation, and they are all nested.

Let us decompose supp ω_0 into (possibly infinitely many) connected components $\cup_{i\in I} U_i$, with centers $\{o_i\}_{i\in I}$. Our goal is to show $o_i \equiv (0,0)$ for $i \in I$. Note that it suffices to show that their x-coordinates satisfy $\sup_{i\in I} o_i^1 \leq 0$. Once we prove this, a parallel argument gives $\inf_{i\in I} o_i^1 \geq 0$, which implies $o_i^1 \equiv 0$ for $i \in I$, and the same can be done for the y-coordinate.

Towards a contradiction, suppose $\sup_{i\in I} o_i^1 > 0$. We can then define Γ_r in the same way as step 6 of the proof of Theorem 6.2.1, i.e. it is the "circle with rightmost center" among $\{\partial_{\text{out}} U_i\}_{i\in I}$, and its center O_r satisfies $O_r^1 = \sup_{i\in I} o_i^1 > 0$. Since $f_\Omega = \text{const}$ along each level set component of ω_0, we again have that $f_\Omega = \text{const}$ on Γ_r. Let Γ_r^R and Γ_r^L be the rightmost/leftmost points on Γ_r. Note that their distances to the origin satisfy $|\Gamma_r^R| > |\Gamma_r^L|$, where the strict inequality is due to the assumption $O_r^1 > 0$.

Let us define $f_i(x) = (\omega_0 1_{U_i}) * \mathcal{N}$ for $i \in I$, and note that $f_\Omega = (\sum_{i\in I} f_i) - \Omega|x|^2$. The properties (a,b) in step 6 of Theorem 6.2.1 still hold for f_i, thus we have $f_i(\Gamma_r^R) \geq f_i(\Gamma_r^L)$ for all $i \in I$. This leads to

$$f_\Omega(\Gamma_r^R) = \Big(\sum_{i\in I} f_i(\Gamma_r^R)\Big) + \underbrace{(-\Omega)}_{>0} \big|\Gamma_r^R\big|^2 > \Big(\sum_{i\in I} f_i(\Gamma_r^L)\Big) + (-\Omega)\big|\Gamma_r^L\big|^2 = f_\Omega(\Gamma_r^L),$$

contradicting the fact that $f_\Omega \equiv \text{const}$ on Γ_r. □

6.2.1 Proofs of the quantitative lemmas

Before the proof of Lemma 6.2.2, let us state two lemmas that we will use in the proof. The first one is a quantitative version of the isoperimetric inequality obtained by Fusco, Maggi and Pratelli [35].

Lemma 6.2.6 (c.f. [35, Section 1.2]). *Let $E \subseteq \mathbb{R}^2$ be a bounded open set. Then there is some constant $c \in (0,1)$, such that*

$$P(E) \geq 2\sqrt{\pi}\sqrt{|E|}\left(1 + c\mathcal{A}(E)^2\right),$$

where $P(E) = \mathcal{H}^1(\partial E)$ denotes the perimeter of E.

The second lemma is a simple result relating the Fraenkel asymmetry of a set E with its subsets U.

Lemma 6.2.7 (c.f. [26, Lemma 4.4]). *Let $E \subseteq \mathbb{R}^2$ be a bounded open set. For all $U \subseteq E$ satisfying*

$|U| \geq |E|(1 - \frac{\mathcal{A}(E)}{4})$, *we have*

$$\mathcal{A}(U) \geq \frac{\mathcal{A}(E)}{4}.$$

Proof of Lemma 6.2.2. The proof of the Lemma 6.2.2 is similar to [26, Proposition 4.5] obtained by Craig, Kim and the last author. For the sake of completeness, we give a proof below. Let $g(k)$, D_k and $\tilde{D}_k$ be defined as in Proposition 5.2.7, let $\tilde{D} = D \cup \overline{h}$ and define $p_h := p|_{\partial h}$. We start by following the proof of Proposition 5.2.7, except that after obtaining (5.19), instead of using the isoperimetric inequality, we use the stability version in Lemma 6.2.6 to control $P(\tilde{D}_k)$. This gives

$$\begin{aligned} g'(k)\big(g(k) + |h|\mathbb{1}_{p_h > k}\big) &\leq -\frac{1}{2}P(\tilde{D}_k)^2 \\ &\leq -2\pi|\tilde{D}_k|\left(1 + c\mathcal{A}(\tilde{D}_k)^2\right)^2 \\ &\leq -2\pi\big(g(k) + |h|\mathbb{1}_{p_h > k}\big)\left(1 + c\mathcal{A}(\tilde{D}_k)^2\right). \end{aligned}$$

Hence it follows from Lemma 6.2.7 that

$$g'(k) \leq -2\pi\Big(1 + c\frac{\mathcal{A}(\tilde{D})^2}{16}\Big) \text{ for all } k \text{ such that } |\tilde{D}_k| \geq |\tilde{D}|\Big(1 - \frac{\mathcal{A}(\tilde{D})}{4}\Big). \tag{6.30}$$

We claim that

$$g(k) \leq |D| - 2\pi\Big(1 + c\frac{\mathcal{A}(\tilde{D})^2}{16}\Big)k \quad \text{for } k \leq \min\Big\{p_h, \frac{\mathcal{A}(\tilde{D})|\tilde{D}|}{16\pi}\Big\}. \tag{6.31}$$

Towards a contradiction, suppose there is $k_0 \leq \min\Big\{p_h, \frac{\mathcal{A}(\tilde{D})|\tilde{D}|}{16\pi}\Big\}$ such that (6.31) is violated. Since $1 + c\frac{\mathcal{A}(\tilde{D})^2}{16} \leq 2$, we have

$$g(k_0) > |D| - 4\pi k_0 \geq |D| - \frac{\mathcal{A}(\tilde{D})|\tilde{D}|}{4},$$

therefore

$$\begin{aligned} |\tilde{D}_{k_0}| &= g(k_0) + |h| \\ &> |D| - \frac{\mathcal{A}(\tilde{D})|\tilde{D}|}{4} + |h| \\ &= |\tilde{D}| - \frac{\mathcal{A}(\tilde{D})|\tilde{D}|}{4} = |\tilde{D}|\Big(1 - \frac{\mathcal{A}(\tilde{D})}{4}\Big). \end{aligned}$$

Hence for all $k \in (0, k_0]$, $g'(k)$ satisfies the inequality (6.30). Thus we have

$$\begin{aligned} g(k_0) &\leq \int_0^{k_0} -2\pi\Big(1 + c\frac{\mathcal{A}(\tilde{D})^2}{16}\Big)dk + |D| \\ &= |D| - 2\pi\Big(1 + c\frac{\mathcal{A}(\tilde{D})^2}{16}\Big)k_0, \end{aligned}$$

contradicting our assumption on k_0.

Finally, to control p_h, we discuss two cases below, depending on which one in the minimum function in (6.31) is smaller. For simplicity, we denote $A := \frac{\mathcal{A}(\tilde{D})|\tilde{D}|}{16\pi}$ and $B := c\frac{\mathcal{A}(\tilde{D})^2}{16}$.

Case 1: $p_h \leq A$. In this case (6.31) holds for all $k \leq p_h$, thus

$$0 \leq g(p_h) \leq |D| - 2\pi\,(1 + B)\,p_h,$$

implying

$$p_h \leq \frac{|D|}{2\pi\,(1+B)} \leq \frac{|D|}{2\pi}(1 - c_0),$$

for some constant c_0 which only depends on $\mathcal{A}(\tilde{D})$.

Case 2: $p_h > A$. In this case (6.31) gives $g(A) \leq |D| - 2\pi(1 + B)A$ and we use a crude bound for $k \geq A$ that is $g'(k) \leq -2\pi$. Therefore for $k > A$,

$$\begin{aligned} g(k) &= \int_A^k g'(k)dk + g(A) \leq -2\pi(k - A) + |D| - 2\pi(1 + B)A \\ &= |D| - 2\pi k - 2\pi AB \\ &\leq |D|\Big(1 - \frac{\mathcal{A}(\tilde{D})}{8}B\Big) - 2\pi k, \end{aligned}$$

where the last inequality follows from $A > \frac{\mathcal{A}(\tilde{D})|D|}{16\pi}$. Plugging in $k = p_h$ gives

$$0 \leq g(p_h) \leq |D|\Big(1 - \frac{\mathcal{A}(\tilde{D})}{8}B\Big) - 2\pi p_h,$$

leading to

$$p_h \leq \frac{|D|}{2\pi}(1 - c_0),$$

again c_0 only depends on $\mathcal{A}(\tilde{D})$. □

Next we prove Lemma 6.2.3.

Proof of Lemma 6.2.3. Without loss of generality, we can assume that $o_1 = (0,0)$ and $o_2 = (a\epsilon, 0)$. To estimate $p|_{\partial B_1}$, we decompose p into

$$p = p\big|_{\partial B_1} g + u,$$

where g satisfies

$$\begin{cases} \Delta g = 0 & \text{in } D \\ g = 1 & \text{on } \partial B_1 \\ g = 0 & \text{on } \partial B_2, \end{cases} \tag{6.32}$$

and u satisfies

$$\begin{cases} \Delta u = -2 & \text{in } D \\ u = 0 & \text{on } \partial D. \end{cases} \tag{6.33}$$

Using this decomposition as well as the definition of p, we have

$$-2|B_1| = \int_{\partial B_1} \nabla p \cdot \vec{n} d\sigma = p\big|_{\partial B_1} \int_{\partial B_1} \nabla g \cdot \vec{n} d\sigma + \int_{\partial B_1} \nabla u \cdot \vec{n} d\sigma,$$

where $\vec{n}$ is the outer-normal of B_1 throughout this proof. Thus

$$p\big|_{\partial B_1} = \frac{1}{\int_{\partial B_1} \nabla g \cdot \vec{n} d\sigma} \left(-2\pi - \int_{\partial B_1} \nabla u \cdot \vec{n} d\sigma \right). \tag{6.34}$$

To estimate $p\big|_{\partial B_1}$, it remains to estimate the two integrals in (6.34).

The function g can be explicitly constructed using the conformal mapping from D to a perfect annulus centered at 0. Consider the Möbius map $h : \mathbb{C} \to \mathbb{C}$ given by

$$h(z) := \frac{z+b}{1+bz},$$

where $b \in \mathbb{R}$ will be fixed soon. Note that the unit circle and the real line are both invariant under h, and ∂B_2 is mapped to some circle centered on the real line. In order to make $h(\partial B_2)$ centered at 0, since the left/right endpoints of ∂B_2 are $\pm(1+\epsilon) + a\epsilon$, we look for $b \in \mathbb{R}$ that solves

$$h(1 + \epsilon + a\epsilon) = -h(-1 - \epsilon + a\epsilon). \tag{6.35}$$

Plugging the definition of h into the above equation, we know that b is a root of the quadratic polynomial

$$f(b) := b^2 - \frac{2 + (1-a^2)\epsilon}{a} b + 1.$$

Clearly, for $0 < a < 1$, f has two positive roots whose product is 1, thus one is in $(0,1)$ and the other in $(1, +\infty)$. We define b to be the root in $(0,1)$. One can easily check that $f(a) < 0$, and $f(\frac{a}{2}) > 0$ if $a^2 > 2(1-a^2)\epsilon$, which is true due to our assumption $a^2 > 64\epsilon$. Thus for all $\epsilon \in (0, \frac{a^2}{64})$ we have

$$0 < \frac{a}{2} < b < a < 1.$$

Note that h is holomorphic in $\mathbb{C}$ except at the two singularity points $-b$ and $-\frac{1}{b}$. We have already shown that $-b \in B_1$, thus it is outside of D. Next we will show that $-\frac{1}{b} \in B_2^c$, thus is also outside of D. To see this, note that

$$\frac{-1-\epsilon+a\epsilon+b}{1+b(-1-\epsilon+a\epsilon)} = h(-1-\epsilon+a\epsilon) = -h(1+\epsilon+a\epsilon) = -\frac{1+\epsilon+a\epsilon+b}{1+b(1+\epsilon+a\epsilon)} < 0,$$

where the inequality follows from the fact that $a, b, \epsilon > 0$. Since the numerator of the left hand side is already known to be negative due to $a, b \in (0,1)$, its denominator must be positive, leading to $-\frac{1}{b} < -1-\epsilon+a\epsilon$, i.e. $-\frac{1}{b} \in B_2^c$.

Now we define $g : \mathbb{R}^2 \setminus \{(-b,0) \cup (-1/b, 0)\} \to \mathbb{R}$ as

$$g(x) := -\frac{1}{\log|h(1+\epsilon+a\epsilon)|} \log|h(z)| + 1 \quad \text{for } z = x_1 + ix_2.$$

Let us check that g indeed satisfies (6.32): first note that g satisfies the boundary conditions in (6.32), since h maps D to a perfect annulus centered at the origin, whose inner boundary is ∂B_1. In addition, g is harmonic in $\mathbb{R}^2 \setminus \{(-b,0) \cup (-1/b,0)\}$, thus harmonic in D.

Using the explicit formula of g, we have

$$\Delta g(x) = -\frac{2\pi}{\log|h(1+\epsilon+a\epsilon)|} \left(\delta_{(-b,0)}(x) - \delta_{(-\frac{1}{b},0)}(x) \right)$$

in the distribution sense. We can then apply the divergence theorem to g in B_1, and compute the integral containing g in (6.34) explicitly as

$$\int_{\partial B_1} \nabla g \cdot \vec{n} d\sigma = -\frac{2\pi}{\log|h(1+\epsilon+a\epsilon)|}. \tag{6.36}$$

As for the integral containing u in (6.34), we compare u with a radial barrier function

$$w(x) := -2(|x| - 1)(|x| - 1 - 2\epsilon),$$

which satisfies $w = 0$ on ∂B_1 and $w > 0$ on ∂B_2. Note that

$$\Delta w = \left(\partial_{rr} + \frac{1}{r}\partial_r\right) w = -8 + \frac{4 + 4\epsilon}{r} \leq -2 \quad \text{in } D,$$

where we used that $\epsilon \in (0, \frac{1}{2})$ and $r > 1$ in D in the last inequality. Thus $w - u$ is superharmonic in D and nonnegative on ∂D, which allows us to apply the classical maximum principle to obtain $u \leq w$ in $\overline{D}$. Combining this with the fact that $u = w = 0$ on ∂B_1, we have

$$\nabla u(x) \cdot \vec{n}(x) \leq \nabla w(x) \cdot \vec{n}(x) = \frac{d}{dr} w(r)\Big|_{r=1} = 4\epsilon \quad \text{for all } x \in \partial B_1,$$

hence

$$\int_{\partial B_1} \nabla u \cdot \vec{n} d\sigma \leq 8\pi\epsilon. \tag{6.37}$$

Plugging (6.36) and (6.37) into (6.34), we obtain

$$p\big|_{\partial B_1} \leq \log(|h(1 + \epsilon + a\epsilon)|)(1 + 4\epsilon).$$

Since $\log s \leq s - 1$ for $s > 1$, it follows that

$$\begin{aligned}
\log|h(1 + \epsilon + a\epsilon)| \leq h(1 + \epsilon + a\epsilon) - 1 &= \frac{1 + \epsilon + a\epsilon + b}{1 + b(1 + \epsilon + a\epsilon)} - 1 \\
&= \epsilon\left(1 + \frac{a - 2b - ab - b\epsilon - ab\epsilon}{1 + b(1 + \epsilon + a\epsilon)}\right) \\
&\leq \epsilon\left(1 - \frac{ab}{4}\right) \leq \epsilon\left(1 - \frac{a^2}{8}\right),
\end{aligned}$$

where we used $b > \frac{a}{2}$ to obtain the last two inequalities. Finally, using that $\epsilon < \frac{a^2}{64}$, we have

$$p\big|_{\partial B_1} \leq \epsilon\left(1 - \frac{a^2}{8}\right)\left(1 + \frac{a^2}{16}\right) \leq \epsilon\left(1 - \frac{1}{16}a^2\right) < \frac{|D|}{2\pi}\left(1 - \frac{1}{16}a^2\right),$$

where in the last step we use that $|D| = \pi(1+\epsilon)^2 - \pi > 2\pi\epsilon$. This finishes the proof of the lemma. □

Finally we give the proof of Lemma 6.2.4.

Proof of Lemma 6.2.4. Without loss of generality, we can assume that o_2 is the origin. Let

$\beta := p|_{\partial B_r}$. From the proof of Proposition 5.2.7, we already know that $g'(k) \leq -2\pi$, where $g(k) := |\{x \in D : p(x) > k\}|$. This implies that $g(k) \geq -2\pi(k-\beta)$. Therefore we have

$$\int_D p dx \geq \int_0^\beta g(k)dk \geq \int_0^\beta -2\pi(k-\beta)dk = \pi\beta^2.$$

On the other hand, the same computation in the proof of Lemma 5.2.10 gives

$$\beta|B_r| + \int_D p dx = \frac{1}{2}\int_D |\nabla p|^2 dx \leq \frac{1}{2}\int_D |x|^2 dx.$$

Since

$$\begin{aligned}\frac{1}{2}\int_D |x|^2 dx &= \frac{1}{2}\left(\int_{B_R} |x|^2 dx - \int_{B_r} |x|^2 dx\right) \\ &= \frac{|D|^2}{4\pi} + \frac{|D||B_r|}{2\pi} + \frac{|B_r|^2}{4\pi} - \frac{1}{2}\int_{B_r} |x|^2 dx \\ &= \frac{|D|^2}{4\pi} + \frac{|D||B_r|}{2\pi} - \frac{l^2|B_r|}{2},\end{aligned}$$

it follows that

$$\pi\beta^2 + \beta|B_r| \leq \frac{|D|^2}{4\pi} + \frac{|D||B_r|}{2\pi} - \frac{l^2|B_r|}{2}. \tag{6.38}$$

By solving the quadratic inequality (6.38), we find that

$$\beta \leq \frac{|D|}{2\pi}(1-c_0),$$

for some constant c_0 which only depends on δ_1, δ_2 and l. □

Chapter 7

The rigidity results for g-SQG

7.1 Radial symmetry for stationary/rotating gSQG solutions with $\Omega \leq 0$

In this section, we consider the family of gSQG equations with $0 < \alpha < 2$, and study the symmetry property for rotating patch/smooth solutions with angular velocity $\Omega \leq 0$.

Let us deal with patch solutions first. As we have discussed in the introduction, we cannot expect a non-simply-connected patch D with $\Omega \leq 0$ to be radial, due to the non-radial examples in [29, 41] for $\alpha \in (0, 2)$. For a simply-connected patch D, the constant on the right hand side of (4.6) is the same on ∂D, which motivates us to consider Question 4.1.8 in the introduction. The goal of this section is to prove Theorem 4.1.4, which gives an affirmative answer to Question 4.1.8 for the whole range $\alpha \in [0, 2)$.

Our results are not limited to the Riesz potentials $K_{\alpha,d}$ in (4.4); in fact, we only need the potential being radially increasing and not too singular at the origin. Below we state our assumption on the potential K, which covers the whole range of $K_{\alpha,d}$ with $\alpha \in [0, 2)$.

(HK) Let $K \in C^1(\mathbb{R}^d \setminus \{0\})$ be radially symmetric with $K'(r) > 0$ for all $r > 0$. (Here we denote $K(x) = K(r)$ by a slight abuse of notation.) Also assume there is some $C > 0$, $\delta > 0$ such that $K'(r) \leq Cr^{-d-1+\delta}$ for all $0 < r \leq 1$.

Our proof is done by a variational approach, which relies on a continuous Steiner symmetrization argument in a similar spirit as [10].

7.1.1 Definition and properties of continuous Steiner symmetrization

Below we define the continuous Steiner symmetrization for a bounded open set $D \subset \mathbb{R}^d$ with respect to the direction $e_1 = (1, 0, \dots, 0)$, which can be easily adapted to any other direction in $\mathbb{R}^d$. The definition is the same as [10, Section 2.2.1], which we briefly outline below for completeness.

For a *one-dimensional* open set $U \subset \mathbb{R}$, we define its continuous Steiner symmetrization $M^\tau[U]$ as follows. If $U = (a, b)$ is an open interval, then $M^\tau[U]$ shifts the midpoint of this interval towards the origin with velocity 1, while preserving the length of interval. That is,

$$M^\tau[U] := \begin{cases} \big(a - \tau \operatorname{sgn}(\frac{a+b}{2}), b - \tau \operatorname{sgn}(\frac{a+b}{2})\big) & \text{for } 0 \le \tau < \frac{|a+b|}{2}, \\ (-\frac{b-a}{2}, \frac{b-a}{2}) & \text{for } \tau \ge \frac{|a+b|}{2}. \end{cases}$$

If $U = \cup_{i=1}^N U_i$ is a finite union of open intervals, then $M^\tau[U]$ is defined by $\cup_{i=1}^N M^\tau[U_i]$, and as soon as two intervals touch each other, we merge them into one interval as in [10, Definition 2.10(2)]. Finally, if $U = \cup_{i=1}^\infty U_i$ is a countable union of open intervals, we define $M^\tau[U]$ as a limit of $M^\tau[\cup_{i=1}^N U_i]$ as $N \to \infty$ as in [10, Definition 2.10(3)]. See [10, Figure 1] for an illustration of $M^\tau[U]$.

Next we move on to higher dimensions. We denote a point $x \in \mathbb{R}^d$ by (x_1, x'), where $x' = (x_2, \dots, x_d) \in \mathbb{R}^{d-1}$. For a bounded open set $D \subset \mathbb{R}^d$ and any $x' \in \mathbb{R}^{d-1}$, we define the *section* of D with respect to the direction x_1 as

$$D_{x'} := \{x_1 \in \mathbb{R} : (x_1, x') \in D\},$$

which is an open set in $\mathbb{R}$. If the section $D_{x'}$ is a single open interval centered at 0 for all $x' \in \mathbb{R}^{d-1}$, then we say the set D is *Steiner symmetric* about the hyperplane $\{x_1 = 0\}$. Note that this definition is stronger than being symmetric about $\{x_1 = 0\}$. For example, an annulus in $\mathbb{R}^2$ is symmetric about $\{x_1 = 0\}$, but not Steiner symmetric about it.

Finally, for any $\tau > 0$, the *continuous Steiner symmetrization* of $D \subset \mathbb{R}^d$ is defined as

$$S^\tau[D] := \{(x_1, x') \in \mathbb{R}^d : x_1 \in M^\tau[D_{x'}]\},$$

with M^τ given above being the continuous Steiner symmetrization for one-dimensional open sets. See Figure 7.1 for a comparison of the sets D and $S^\tau[D]$ for small $\tau > 0$.

One can easily check that $S^\tau[D]$ satisfies the following properties.

Lemma 7.1.1. *For any bounded open set $D \subset \mathbb{R}^d$, its continuous Steiner symmetrization $S^\tau[D]$*

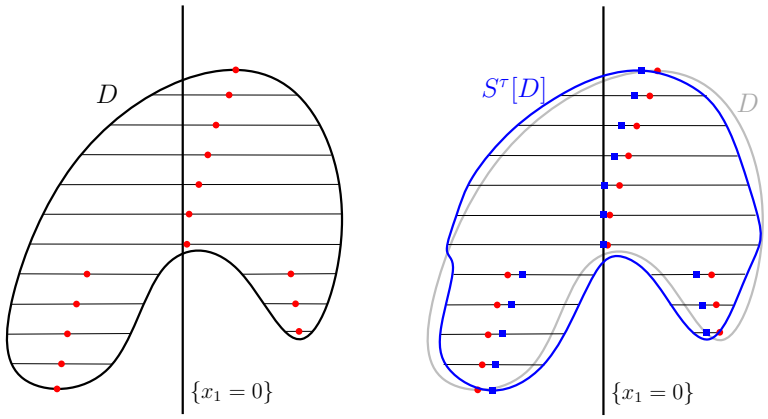


Figure 7.1: Illustration of the continuous Steiner symmetrization $S^\tau[D]$ for a set $D \subset \mathbb{R}^2$. The left figure is the set D, with the midpoints of all subintervals of its 1D section highlighted in red circles. The right figure shows the set $S^\tau[D]$ for some small $\tau > 0$, with the new midpoints denoted by blue squares.

satisfies the following properties:

(a) $|S^\tau[D]| = |D|$ *for any* $\tau > 0$*, where* $|\cdot|$ *denotes the Lebesgue measure in* $\mathbb{R}^d$*.*

(b) $(S^\tau[D])\triangle D \subset B^\tau[D]$ *for any* $\tau > 0$*, where* $\triangle$ *is the symmetric difference between the two sets, and* $B^\tau[D]$ *is the* τ*-neighborhood of* ∂D*, given by*

$$B^\tau[D] := \{x \in \mathbb{R}^d : \operatorname{dist}(x, \partial D) \leq \tau\}. \tag{7.1}$$

Proof. (a) is a direct consequence of the fact that $|M^\tau[U]| = |U|$ for any open set $U \subset \mathbb{R}$ and $\tau > 0$ [10, Lemma 2.11(b)]. To prove (b), one can start with the one-dimensional version: For any bounded open set $U \subset \mathbb{R}$, we have $M^\tau[U]\triangle U \subset \{x \in \mathbb{R} : \operatorname{dist}(x, \partial U) \leq \tau\}$, which follows from the fact that the intervals move with velocity at most 1. Thus for any bounded open set $D \subset \mathbb{R}^d$,

$$\begin{aligned} S^\tau[D]\triangle D &= \{(x_1, x') \in \mathbb{R}^d : x_1 \in M^\tau[D_{x'}]\triangle D_{x'}\} \\ &\subset \{(x_1, x') : \operatorname{dist}(x_1, \partial(D_{x'})) \leq \tau\} \\ &\subset B^\tau[D], \end{aligned}$$

finishing the proof. □

7.1.2 Simply-connected patch solutions with $\Omega \leq 0$

We assume that $D \subset \mathbb{R}^d$ satisfies the following condition.

(HD) $D \subset \mathbb{R}^d$ is an open set, and there exists some $M > 0$ depending on D, such that $|B^\tau[D]| \le M\tau$ for all sufficiently small $\tau > 0$, where $B^\tau[D]$ is given in (7.1).

It can be easily checked that for $d \ge 2$, any bounded open set D with Lipschitz continuous boundary satisfies condition **(HD)**. In fact, for $d = 2$, we will show any bounded open set $D \subset \mathbb{R}^2$ with a rectifiable boundary satisfies **(HD)**, with a precise bound

$$|B^\tau[D]| \le 2|\partial D|\tau \quad \text{for all } \tau \ge 0, \tag{7.2}$$

where $|\partial D|$ is the total length of ∂D. Let us first prove (7.2) holds for any polygon $P \subset \mathbb{R}^2$. Erect two polygons at distance τ from P and the transversal sides being bisectors of the inner angles of P (see Figure 7.2). It is clear that $B^\tau[P]$ is contained in the trapezoidal region, which has area no more than $2|\partial P|\tau$. Finally, this can be extended to the general case by approximating any rectifiable curve by polygons.

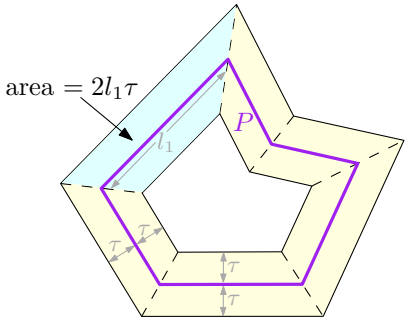


Figure 7.2: Illustration of the polygon P and the underlying trapezoidal region (the whole colored region). Here the blue trapezoid has area $2l_1\tau$ (l_1 is the corresponding side length in P), and summing over all edges gives a total area $2|\partial P|\tau$. Since the trapezoids may intersect for large τ, the whole trapezoidal region has area no more than $2|\partial P|\tau$.

Below we state our main theorem of this section, which is slightly more general than Theorem 4.1.4.

Theorem 7.1.1. *Let $D \subset \mathbb{R}^d$ and $K \in C^1(\mathbb{R}^d \setminus \{0\})$ satisfy the conditions* **(HD)** and **(HK)** *respectively. Let $g \in C^1(\mathbb{R}^d)$ be a radial function with $g'(r) > 0$ for all $r > 0$.*

If D satisfies that

$$1_D * K - \frac{\Omega}{2} g(x) = const \quad \text{on } \partial D \tag{7.3}$$

for some $\Omega \le 0$ (where the constant is the same on all connected components of ∂D), then D is a ball. Moreover, the ball is centered at the origin if $\Omega < 0$.

Remark 7.1.2. *(1) Note that D does not need to be simply-connected in Theorem 7.1.1. However, since the constant on the right hand side of* (7.3) *is assumed to be the same on all connected components of ∂D, comparing with* (4.6)*, Theorem 7.1.1 only implies that all simply-connected patches with $\Omega \leq 0$ must be a disk.*

(2) In the case $\Omega = 0$, the problem is translation invariant, so in the proof we assume without loss of generality that the center of mass of D is at the origin.

Proof. We prove it by contradiction. Without loss of generality, we assume K satisfies the additional assumption that $K(1) = 0$ (note that (7.3) still holds if we add any constant to K), and D is not Steiner symmetric about the hyperplane $\{x_1 = 0\}$. Let $D^\tau := S^\tau[D]$ be the continuous Steiner symmetrization of D at time $\tau > 0$. By Lemma 7.1.1(b), we have

$$D^\tau \triangle D \subset B^\tau[D], \tag{7.4}$$

where B^τ is defined in (7.1). Let us consider the functional

$$\mathcal{E}[D] := \underbrace{\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} 1_D(x)1_D(y)K(x-y)dxdy}_{=:\mathcal{I}[D]} + \underbrace{(-\Omega)\int_{\mathbb{R}^d} g(x)1_D(x)dx}_{=:\mathcal{V}[D]}.$$

We will use two different ways to compute $\frac{d^+}{d\tau}\mathcal{E}[D^\tau]\Big|_{\tau=0}$, where $\frac{d^+}{d\tau}$ denotes the right derivative. On the one hand, using the equation (7.3) and the regularity assumptions on D, K and g, we aim to show that

$$\frac{d^+}{d\tau}\mathcal{E}[D^\tau]\Big|_{\tau=0} = 0. \tag{7.5}$$

Instead of directly taking the derivative, we consider the finite difference

$$\mathcal{E}[D^\tau] - \mathcal{E}[D] = \underbrace{\int_{\mathbb{R}^d} 2(1_{D^\tau} - 1_D)\left(1_D * K - \frac{\Omega}{2}g(x)\right)dx}_{=:I_1} + \underbrace{\int_{\mathbb{R}^d}(1_{D^\tau} - 1_D)((1_{D^\tau} - 1_D) * K)dx}_{=:I_2},$$

where in the equality we used that $\int 1_D(1_{D^\tau} * K)dx = \int 1_{D^\tau}(1_D * K)dx$ for any radial kernel K.

Let us control the term I_1 first. First note that (7.4) implies that the integrand is supported in $B_\tau[D]$. Next we claim that **(HK)** implies $1_D * K - \frac{\Omega}{2}g \in C^{0,\delta'}(\mathbb{R}^d)$ for $\delta' := \min\{\delta, 1\}$, where $C^{0,1}$ stands for Lipschitz continuity. The proof is a simple potential theory estimate, which we provide

below for completeness. For any $x, z \in \mathbb{R}^d$,

$$\begin{aligned}|(1_D * K)(x+z) - (1_D * K)(x)| &= \left| \int_{\mathbb{R}^d} 1_D(x-y)(K(y+z) - K(y))dy \right| \\ &\leq \int_{|y|<2|z|} |K(y+z) - K(y)|dy + \int_{|y|>2|z|} 1_D(x-y)|K(y+z) - K(y)|dy \\ &=: J_1 + J_2.\end{aligned}$$

Since $(1_D * K)(x) \in L^\infty(\mathbb{R}^d)$, one can moreover assume that $|z| < \frac{1}{3}$, then, a crude estimate gives

$$J_1 \leq \int_{|y|<2|z|} |K(y+z)| + |K(y)|dy \leq 2\int_{|y|<3|z|} |K(y)|dy \leq C(d)|z|^\delta,$$

where in the last step we used that **(HK)** and $K(1) = 0$ imply $|K(y)| \leq C|y|^{-d+\delta}$ for $|y| \leq 1$. For J_2, note that **(HK)** and the mean-value theorem gives

$$|K(y+z) - K(y)| \leq C|y|^{-d-1+\delta}|z| \quad \text{for all } |y| > 2|z|,$$

and plugging it into the integral gives $J_2 \leq C(d, |D|)|z|^\delta$. Putting the estimates for J_1 and J_2 together gives that $1_D * K \in C^{0,\delta'}(\mathbb{R}^d)$ for $\delta' = \min\{\delta, 1\}$, and combining this with the assumption $g \in C^1(\mathbb{R}^d)$ gives $1_D * K - \frac{\Omega}{2}g \in C^{0,\delta'}(\mathbb{R}^d)$.

In addition, by (7.3), We have $1_D * K - \frac{\Omega}{2}g(x) \equiv C_0$ on ∂D for some constant C_0. Thus we have

$$\left| 1_D * K - \frac{\Omega}{2}g(x) - C_0 \right| \leq C(\delta', d, |D|)\tau^{\delta'} \quad \text{in } B^\tau[D]$$

for some constant $C > 0$, where we used the Hölder continuity of $1_D * K - \frac{\Omega}{2}g$ and the definition of $B^\tau[D]$. This leads to

$$|I_1| \leq 2|B^\tau[D]| \sup_{x \in B^\tau[D]} \left| 1_D * K - \frac{\Omega}{2}g(x) - C_0 \right| \leq C(\delta', d, |D|)M\tau^{1+\delta'},$$

where in the first inequality we used that $\int_{B_\tau}(1_D - 1_{D^\tau})C_0 dx = 0$, which follows from Lemma 7.1.1(a); and in the second inequality we used **(HD)**.

Next, using (7.4) we control I_2 by the crude bound

$$|I_2| \leq \int_{\mathbb{R}^d} 1_{B^\tau[D]}|(1_{B^\tau[D]} * K)|dx \leq |B^\tau[D]| \, \|1_{B^\tau[D]} * K\|_\infty \leq M\tau \|(1_{B^\tau[D]})^* * K\|_\infty,$$

where the last step follows from the Hardy–Littlewood inequality, where $(1_{B^\tau[D]})^*$ is the radial

decreasing rearrangement of $1_{B^\tau[D]}$. By **(HD)**, $(1_{B^\tau[D]})^*$ is a characteristic function of a ball whose radius is bounded by $C(d)(M\tau)^{1/d}$, thus

$$\|(1_{B^\tau[D]})^* * K\|_\infty \leq \int_0^{C(d)(M\tau)^{1/d}} |K(r)|\, \omega_d r^{d-1} dr \leq \int_0^{C(d)(M\tau)^{1/d}} \omega_d r^{-1+\delta} dx \leq C(d)(M\tau)^{\frac{\delta}{d}},$$

and plugging it into the I_2 estimate gives

$$|I_2| \leq C(d) M^{\frac{d+\delta}{d}} \tau^{1+\frac{\delta}{d}}.$$

Putting the estimates of I_1 and I_2 together directly yields

$$\frac{|\mathcal{E}[D^\tau] - \mathcal{E}[D]|}{\tau} \leq C(\delta', d, M, |D|)\tau^{\min\{\frac{\delta}{d}, \delta'\}},$$

and since $\delta > 0$ we have $\frac{d^+}{d\tau}\mathcal{E}[D^\tau]\Big|_{\tau=0} = 0$.

Now, we use another way to calculate $\frac{d^+}{d\tau}\mathcal{E}[D^\tau]\big|_{\tau=0}$. Let us deal with the $\Omega < 0$ case first. Since K is radial and increasing in r, it has been shown in [7, Corollary 2] and [68, Theorem 3.7] that the interaction energy $\mathcal{I}[D^\tau] = \int_{D^\tau}\int_{D^\tau} K(x-y)dxdy$ is non-increasing along the continuous Steiner symmetrization, leading to

$$\frac{d^+}{d\tau}\mathcal{I}[D^\tau] \leq 0 \quad \text{ for all } \tau \geq 0.$$

For the other term $\mathcal{V}[D^\tau] = (-\Omega)\int_{D^\tau} g(x)dx$, by the assumptions that $g'(r) > 0$ for all $r > 0$ and D is not Steiner symmetric about $\{x_1 = 0\}$, we can use [10, Lemma 2.22] to show, for $\Omega < 0$,

$$\frac{d^+}{d\tau}\mathcal{V}[D^\tau]\Big|_{\tau=0} = (-\Omega)\frac{d^+}{d\tau}\int_{D^\tau} g(x)dx\Big|_{\tau=0} < 0.$$

Adding them together gives

$$\frac{d^+}{d\tau}\mathcal{E}[D^\tau]\Big|_{\tau=0} < 0$$

leading to a contradiction with (7.5).

In the $\Omega = 0$ case, recall that we assume that the center of mass of D is at the origin. Thus if D is not Steiner symmetric about $\{x_1 = 0\}$, the same proof as [10, Proposition 2.15] gives that $\mathcal{I}[D]$ must be decreasing to the first order for a short time, leading to

$$\frac{d^+}{d\tau}\mathcal{E}[D^\tau]\Big|_{\tau=0} = \frac{d^+}{d\tau}\int_{D^\tau}\int_{D^\tau} K(|x-y|)dxdy\Big|_{\tau=0} < 0,$$

again contradicting (7.5). We point out that although the proposition was stated for continuous densities, the same proof works for the patch setting. In addition, although [10] only dealt with the kernels no more singular than Newtonian potential, the proof indeed holds for all kernels K satisfying **(HK)**: see [11, Theorem 6] for an extension to all Riesz potentials $K_{\alpha,d}$ with $\alpha \in (0,2)$. □

The above theorem immediately leads to the following result concerning simply-connected stationary/rotating patch solution with $\Omega \leq 0$.

Theorem 7.1.2. *Let $D \subset \mathbb{R}^2$ be a bounded, simply-connected domain with rectifiable boundary. If 1_D is a V-state for (4.2) for some $\alpha \in [0,2)$ with angular velocity $\Omega \leq 0$, then D must be a disk. In addition, the disk must be centered at the origin if $\Omega < 0$.*

Proof. We have $1_D * K - \frac{\Omega}{2}|x|^2 = C$ for some constant C on ∂D. For the Euler equation, $K = \frac{1}{2\pi}\ln|x|$. For the g-SQG equation. $K = -C_\alpha |x|^{-\alpha}$. In both cases, the proof follows from Theorem 7.1.1. □

Remark 7.1.3. *As we discussed in the beginning of this subsection, in the case of gSQG with $\alpha \in (0,2)$, Theorem 7.1.2 is not true if the simply connected assumption is dropped, due to the non-radial patches in [29, 41] bifurcating from annuli.*

7.1.3 Smooth solutions with simply-connected level sets with $\Omega \leq 0$

The rest of this section is devoted to the smooth setting. We will show that any nonnegative smooth rotating solution of the Euler or gSQG equation with angular velocity $\Omega \leq 0$ must be radial, under the additional assumption that all the super level-sets U^h

$$U^h := \{x \in \mathbb{R}^d : \omega_0(x) > h\} \tag{7.6}$$

are simply-connected for any $h > 0$. We believe that the simply-connected assumption is necessary, since it is likely that the bifurcation arguments from annuli in [29, 41] can be extended to the smooth setting as well, using a similar argument as in [19] or [18].

Theorem 7.1.3. *Let $\omega(x) \in C^1(\mathbb{R}^2)$ be nonnegative and compactly supported. In addition, assume the super level-set U^h as in (7.6) is simply connected for all $h \in (0, \sup\omega)$. Assume K satisfies* **(HK)**. *If for some $\Omega \leq 0$, we have*

$$\omega * K - \frac{\Omega}{2}|x|^2 = C_0(h) \quad \text{on } \partial U^h \quad \text{for all } h \in (0, \sup\omega), \tag{7.7}$$

then ω is radially decreasing up to a translation. Moreover, it is centered at the origin if $\Omega < 0$.

Proof. We prove it by contradiction. For the $\Omega < 0$ case, without loss of generality, we assume ω is not symmetric decreasing about the line $x_1 = 0$. For the $\Omega = 0$ case, similar to Remark 7.1.2, without loss of generality we assume the center of mass is at the origin, and then we assume ω is not symmetric decreasing about the line $x_1 = 0$.

For any $\tau \geq 0$, we define the continuous Steiner symmetrization $\omega^\tau(x)$ in the same way as [10, Definition 2.12]:

$$\omega^\tau(x) := \int_0^{h_0} 1_{S^\tau[U^h]}(x)\, dh,$$

where $h_0 := \sup \omega$, and $S^\tau[U^h]$ is the continuous Steiner symmetrization of the super level set U^h at time $\tau \geq 0$.

Consider the energy functional

$$\mathcal{E}[\omega] := \underbrace{\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \omega(x)\omega(y)K(x-y)dxdy}_{=:\mathcal{I}[\omega]} + \underbrace{(-\Omega)\int_{\mathbb{R}^2}\omega(x)|x|^2dx}_{=:\mathcal{V}[\omega]}.$$

We proceed similarly as in Theorem 7.1.1 to compute $\frac{d^+}{d\tau}\mathcal{E}[\omega^\tau]$ in two different ways. We first rewrite the finite difference $\mathcal{E}[\omega^\tau] - \mathcal{E}[\omega]$ as

$$\begin{aligned} & \mathcal{E}[\omega^\tau] - \mathcal{E}[\omega] \\ = & \int_{\mathbb{R}^2} 2(\omega^\tau(x) - \omega(x))\left(\omega * K - \frac{\Omega}{2}|x|^2\right)dx + \iint_{\mathbb{R}^2\times\mathbb{R}^2} (\omega^\tau(x) - \omega(x))(\omega^\tau(y) - \omega(y))K(x-y)dxdy \\ =: & \ I_1 + I_2. \end{aligned} \tag{7.8}$$

Since $\omega \in C_c^1(\mathbb{R}^2)$ and K satisfies **(HK)** (hence is locally integrable), one can easily check that $\omega * K - \frac{\Omega}{2}|x|^2$ is Lipschitz in $\tilde{D} := \{x \in \mathbb{R}^2 : \operatorname{dist}(x, \operatorname{supp}\omega) \leq 1\}$. Note that we have $\operatorname{supp}\omega^\tau \in \tilde{D}$ for all $\tau \in [0,1]$. Combining this fact with the assumption (7.7), there exists $C_1 > 0$ independent of h, such that

$$\left|(\omega * K)(x) - \frac{\Omega}{2}|x|^2 - C_0(h)\right| \leq C_1\tau \quad \text{on } S^\tau[U^h] \triangle U^h \quad \text{for all } h \in (0, h_0). \tag{7.9}$$

Let us first rewrite I_1 as

$$I_1 = 2\int_0^{h_0}\int_{\mathbb{R}^2}\left(1_{S^\tau[U^h]}(x) - 1_{U^h}(x)\right)\left((\omega * K)(x) - \frac{\Omega}{2}|x|^2\right)dx\, dh.$$

By Lemma 7.1.1(a), we have $\int_{\mathbb{R}^2}(1_{S^\tau[U^h]}(x) - 1_{U^h}(x))dx = 0$ for all $h \in (0, h_0)$. Thus we can control

I_1 as

$$\begin{aligned}
|I_1| &= \left| 2\int_0^{h_0} \int_{\mathbb{R}^2} \Big(1_{S^\tau[U^h]}(x) - 1_{U^h}(x)\Big)\Big((\omega * K)(x) - \frac{\Omega}{2}|x|^2 - C_0(h)\Big)dxdh \right| \\
&\le 2C_1\tau \int_0^{h_0} \Big|(S^\tau[U^h]) \triangle U^h\Big| dh \\
&\le 2C_1\tau \int_0^{h_0} 2|\partial U^h|\tau \, dh \\
&= 4C_1\tau^2 \int_{\text{supp } \omega} |\nabla\omega| dx \le C(\omega)\tau^2.
\end{aligned} \tag{7.10}$$

Here in the second line we used (7.9); in the third line we used Lemma 7.1.1(b) and the property (7.2) in two dimensions; and in the fourth line we used the co-area formula and the fact that $\omega \in C_c^1$.

We next move on to I_2. Since $|\nabla\omega|$ is bounded, Lemma 7.1.1(b) leads to

$$\begin{aligned}
|\omega^\tau(x) - \omega(x)| &= \left| \int_0^\infty 1_{S^\tau[U^h]}(x) - 1_{U^h}(x) dh \right| \\
&\le C\|\nabla\omega\|_{L^\infty}\tau \quad \text{ for all } x \in \mathbb{R}^2,
\end{aligned}$$

and supp $\omega^\tau \in \tilde{D}$ for all $\tau \in [0,1]$. Thus

$$\begin{aligned}
|I_2| &\le \|\omega^\tau - \omega\|_{L^1} \|(\omega^\tau - \omega) * K\|_{L^\infty} \\
&\le \|\omega^\tau - \omega\|_{L^1} \|\omega^\tau - \omega\|_{L^\infty} \int_{\tilde{D}} |K(x)| dx \\
&\le C(\omega)\tau^2.
\end{aligned}$$

Combining the estimates for I_1 and I_2 gives $\mathcal{E}[\omega^\tau] - \mathcal{E}[\omega] \le C(\omega)\tau^2$ for all $\tau \in [0,1]$, thus

$$\frac{d^+}{d\tau}\mathcal{E}[\omega^\tau]\Big|_{\tau=0} = \frac{d^+}{d\tau}(I_1 + I_2)\Big|_{\tau=0} = 0. \tag{7.11}$$

On the other hand, we compute $\frac{d^+}{d\tau}\mathcal{E}[\omega^\tau]\big|_{\tau=0}$ in a different way as

$$\frac{d^+}{d\tau}\mathcal{E}[\omega^\tau]\Big|_{\tau=0} = \frac{d^+}{d\tau}(\mathcal{I}[\omega^\tau] + \mathcal{V}[\omega^\tau])\Big|_{\tau=0}.$$

In the $\Omega < 0$ case, similarly as in Theorem 7.1.1, we have $\mathcal{I}[\omega^\tau]$ is non-increasing along the continuous Steiner symmetrization by [7, Corollary 2] and [68, Theorem 3.7], thus

$$\frac{d^+}{d\tau}\mathcal{I}[\omega^\tau] \le 0 \quad \text{ for all } \tau > 0.$$

For $\mathcal{V}[\omega^\tau]$, by the assumption that ω is not symmetric decreasing about $\{x_1 = 0\}$, we again use [10, Lemma 2.22] to show, for $\Omega < 0$,

$$\frac{d^+}{d\tau}\mathcal{V}[\omega^\tau] = (-\Omega)\frac{d^+}{d\tau}\int_{\mathbb{R}^2}\omega^\tau(x)|x|^2dx\bigg|_{\tau=0} < 0.$$

Adding them together gives $\frac{d^+}{d\tau}\mathcal{E}[D^\tau]\big|_{\tau=0} < 0$, contradicting (7.11).

In the $\Omega = 0$ case, we assume that the center of mass of ω is at the origin. Thus if ω is not symmetric decreasing about $\{x_1 = 0\}$, the same proof as [10, Proposition 2.15] gives that $\mathcal{I}[D]$ must be decreasing to the first order for a short time (again, the proof holds for all kernels K satisfying **(HK)**; see [11, Theorem 6] for extensions to Riesz kernels $K_{\alpha,d}$ with $\alpha \in (0,2)$). This gives $\frac{d^+}{d\tau}\mathcal{E}[D^\tau]\big|_{\tau=0} < 0$, again contradicting (7.11). □

The above theorem immediately gives the following corollary concerning the V-states for the Euler and gSQG equations.

Corollary 7.1.4. *Assume $\omega(x) \in C^1(\mathbb{R}^2)$ is a nonnegative, compactly supported V-state satisfying the Euler equation or the gSQG equation for some $\alpha \in (0,2)$ with $\Omega \leq 0$. In addition, assume the super level-set U^h as in (7.6) is simply connected for all $h \in (0, \sup\omega)$. Then ω must be radially decreasing if $\Omega < 0$, and radially decreasing up to a translation if $\Omega = 0$.*

Proof. For the Euler equation, $K = \frac{1}{2\pi}\ln|x|$. For the gSQG equation. $K = -C_\alpha|x|^{-\alpha}$. In both cases, the proof follows from Theorem 7.1.3. □

7.2 Radial symmetry of rotating gSQG solutions with $\Omega > \Omega_\alpha$

In this section, we focus on rotating gSQG patches with area π and $\alpha \neq 0$. As we discussed in the introduction, for $\alpha \in [0,2)$, there exist rotating patches bifurcating from the unit disk at angular velocities $\Omega_m^\alpha = 2^{\alpha-1}\frac{\Gamma(1-\alpha)}{\Gamma\left(1-\frac{\alpha}{2}\right)^2}\left(\frac{\Gamma\left(1+\frac{\alpha}{2}\right)}{\Gamma\left(2-\frac{\alpha}{2}\right)} - \frac{\Gamma\left(m+\frac{\alpha}{2}\right)}{\Gamma\left(m+1-\frac{\alpha}{2}\right)}\right)$, where Ω_m^α is increasing in m for any fixed $\alpha \in [0,2)$. Let us denote $\Omega_\alpha := \lim_{m\to\infty}\Omega_m^\alpha$. If $\alpha \in (0,1)$ we have that

$$\Omega_\alpha = 2^{\alpha-1}\frac{\Gamma(1-\alpha)}{\Gamma\left(1-\frac{\alpha}{2}\right)^2}\frac{\Gamma\left(1+\frac{\alpha}{2}\right)}{\Gamma\left(2-\frac{\alpha}{2}\right)}. \tag{7.12}$$

Note that Ω_α is a continuous function of α for $\alpha \in (0,1)$, with $\Omega_0 = \frac{1}{2}$, and $\Omega_\alpha = +\infty$ for all $\alpha \in [1,2)$.

A natural question is whether there can be rotating patches with area π with $\Omega \geq \Omega_\alpha$ for $\alpha \in (0,1)$. Note that the area constraint is necessary for all $\alpha > 0$: if D is a rotating gSQG patch

for $\alpha \in (0,2)$ with angular velocity Ω, then one can easily check that its scaling $\lambda D = \{\lambda x : x \in D\}$ is a rotating patch with angular velocity $\lambda^{-\alpha}\Omega$.

In Theorem 5.2.4, we showed that for the 2D Euler case ($\alpha = 0$), all rotating patches with $\Omega \geq \Omega_0 = \frac{1}{2}$ must be a disk. In this section, our goal is to show that all *simply-connected* rotating patches with area π with $\Omega \geq \Omega_\alpha$ for $\alpha \in (0,1)$ must be a disk. Whether there exist non-simply-connected or disconnected rotating patches with $\Omega \geq \Omega_\alpha$ for $\alpha \in (0,1)$ is still an open question.

Below is the main theorem of this section. Recall that for $\alpha \in (0,2)$, $K_\alpha = -C_\alpha |x|^{-\alpha}$ is the fundamental solution for $-(-\Delta)^{-1+\frac{\alpha}{2}}$, where $C_\alpha = \frac{1}{2\pi}\frac{\Gamma(\frac{\alpha}{2})}{2^{1-\alpha}\Gamma(1-\frac{\alpha}{2})}$.

Theorem 7.2.1. *Let $D \subset \mathbb{R}^2$ be a bounded, simply-connected patch with C^1 boundary. Let us denote $R := \max_{x\in D} |x|$. Assume that D is a uniformly rotating patch with angular velocity Ω of the gSQG equation with $\alpha \in (0,1)$, i.e.,*

$$1_D * K_\alpha - \frac{\Omega}{2}|x|^2 = C \quad \text{on } \partial D. \tag{7.13}$$

Let $\Omega_c(R) := R^{-\alpha}\Omega_\alpha$. If $\Omega \geq \Omega_c(R)$, then D must coincide with $B(0,R)$.

Remark 7.2.1. *(a) Note that all sets $D \subset \mathbb{R}^2$ with area π must have $R \geq 1$. In this case we have $\Omega_c(R) \leq \Omega_\alpha$, thus Theorem 7.2.1 immediately implies that all simply-connected rotating patches with area π and $\Omega \geq \Omega_\alpha$ must be a disk.*

(b) Note that the constant Ω_α is sharp, since there exist patches bifurcating from a disk of radius 1 at velocities Ω_m^α, which can get arbitrarily close to Ω_α as $m \to \infty$ [49, Theorem 1.4].

Proof. Towards a contradiction, assume that $D \neq B(0,R)$. Let $x_0 \in \partial D$ be the farthest point from 0. Then we have that $D \subset B(0,R)$, and let $U := B(0,R) \setminus D$. See Figure 7.3 for an illustration of U and x_0. Then (7.13) can be rewritten as

$$1_U * K_\alpha = 1_{B(0,R)} * K_\alpha - \frac{\Omega}{2}|x|^2 - C \quad \text{on } \partial D. \tag{7.14}$$

The key idea of this proof is to use two different ways to compute $\nabla(1_U * K_\alpha)(x_0)\cdot x_0$, and obtain a contradiction if $\Omega \geq \Omega_c(R)$. On the one hand,

$$\nabla(1_U * K_\alpha)(x_0)\cdot x_0 = \alpha C_\alpha \int_U \frac{(x_0 - y)\cdot x_0}{|x_0 - y|^{\alpha+2}} dy > 0, \tag{7.15}$$

where we used the fact that $(x_0 - y)\cdot x_0 > 0$ for all $y \in U \subset B(0,R)$ since the two vectors point to the same halfplane.

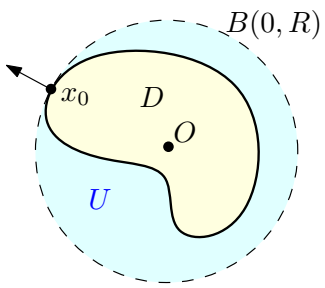


Figure 7.3: Illustration of the set U and the point x_0.

On the other hand, we claim the following properties hold for $1_U * K_\alpha$:

1. $\Delta(1_U * K_\alpha) < 0$ in D.

2. Along ∂D, the minimum of $1_U * K_\alpha$ is achieved at x_0.

To show property 1, using the fact that $K_\alpha = -C_\alpha |x|^{-\alpha}$ is the fundamental solution for $-(-\Delta)^{-1+\frac{\alpha}{2}}$, we have $1_U * K_\alpha = -(-\Delta)^{-1+\frac{\alpha}{2}} 1_U$, thus $\Delta(1_U * K_\alpha) = (-\Delta)^{\alpha/2} 1_U$. Thus for any $x \in D$, using the singular integral definition of the fractional Laplacian [66, Theorem 1.1, Definition (e)] and the fact that $1_U \equiv 0$ in D, we have

$$(-\Delta)^{\alpha/2} 1_U(x) = C_1(\alpha) \int_{\mathbb{R}^2} \frac{1_U(x) - 1_U(y)}{|x-y|^{2+\alpha}} dy = C_1(\alpha) \int_{\mathbb{R}^2} \frac{0 - 1_U(y)}{|x-y|^{2+\alpha}} dy < 0 \quad \text{for } x \in D$$

for some constant $C_1(\alpha) > 0$. Note that despite the denominator being singular, the integral indeed converges for all $x \in D$, due to the fact that D is open and the integrand is identically zero in D which yields

$$\Delta(1_U * K_\alpha)(x) = (-\Delta)^{\alpha/2} 1_U(x) < 0 \text{ in } D. \tag{7.16}$$

We now move on to property 2. Due to (7.14) and the fact that x_0 is the outmost point on ∂D, it suffices to show that the radial function $1_{B(0,R)} * K_\alpha - \frac{\Omega}{2}|x|^2$ is non-increasing in $|x|$ for all $\Omega \geq \Omega_c(R)$. We prove this in Proposition 7.2.2 right after this theorem.

The above claims allow us to apply the maximum principle to $1_U * K_\alpha$ (recall that $1_U * K_\alpha$ is superharmonic in D by property 1), which yields that the minimum of $1_U * K_\alpha$ in $\overline{D}$ is also achieved at x_0, thus

$$\nabla(1_U * K_\alpha)(x_0) \cdot \vec{n}(x_0) \leq 0,$$

where $\vec{n}(x_0)$ is the outer normal of D at x_0. Since $\vec{n}(x_0) = x_0/|x_0|$, the above inequality contradicts with (7.15). As a result, D must coincide with $B(0,R)$. □

Now we prove the proposition that was used in the proof of the above theorem.

Proposition 7.2.2. *For a fixed $\alpha \in (0,1)$ and $R > 0$, let $\Omega_c(R)$ be the smallest number such that*

$$g_R(x) := 1_{B(0,R)} * K_\alpha - \frac{\Omega_c}{2}|x|^2$$

is non-increasing in $|x|$. Then we have $\Omega_c(R) = R^{-\alpha}\Omega_\alpha$, with Ω_α given in (7.12).

Proof. Recall that $K_\alpha = -C_\alpha |x|^{-\alpha}$ with $C_\alpha = \frac{1}{2\pi}\frac{\Gamma(\frac{\alpha}{2})}{2^{1-\alpha}\Gamma(1-\frac{\alpha}{2})}$. Since $|x|^2$ and $1_{B(0,R)} * K_\alpha$ are both radially symmetric and increasing in $|x|$, we have

$$\Omega_c(R) = 2C_\alpha \sup_{|x_1|\neq|x_2|} \frac{\int_{B(0,R)} |x_2 - y|^{-\alpha} dy - \int_{B(0,R)} |x_1 - y|^{-\alpha} dy}{|x_1|^2 - |x_2|^2}.$$

Let us denote the fraction above by $F(x_1, x_2)$. We claim that the $\sup_{|x_1|\neq|x_2|} F(x_1, x_2)$ is attained when $|x_1| = R$, and $|x_2| \to R$.

To prove the claim, we first compute $I(x) := \int_{B(0,R)} |x-y|^{-\alpha} dy$. Taking the Fourier transform:

$$I(x) = CR^{2-\alpha} \int_0^\infty r^{a-2} J_1(r) J_0\left(\frac{|x|r}{R}\right) dr,$$

where C is some positive constant. By Sonine-Schafheitlin's formula [90, p. 401] and by continuity, we obtain

$$I(x) = \begin{cases} CR^{2-\alpha}2^{\alpha-2}\frac{\Gamma(\frac{\alpha}{2})}{\Gamma(2-\frac{\alpha}{2})}\, {}_2F_1\left(\frac{\alpha}{2}-1, \frac{\alpha}{2}, 1, \frac{|x|^2}{R^2}\right) & \text{if } |x| \le R \\ CR^{2-\alpha}2^{\alpha-2}|x|^{-\alpha}R^\alpha \frac{\Gamma(\frac{\alpha}{2})}{\Gamma(1-\frac{\alpha}{2})}\, {}_2F_1\left(\frac{\alpha}{2}, \frac{\alpha}{2}, 2, \frac{R^2}{|x|^2}\right) & \text{if } |x| > R. \end{cases}$$

By the mean value theorem, it is enough to check that $\min J(z) = J(R^2)$, where

$$\begin{aligned} J(z) &= \begin{cases} \frac{d}{dz}\left({}_2F_1\left(\frac{\alpha}{2}-1, \frac{\alpha}{2}, 1, \frac{z}{R^2}\right)\right) & \text{if } z \le R^2 \\ \frac{d}{dz}\left(\left(1-\frac{\alpha}{2}\right) z^{-\frac{\alpha}{2}} R^\alpha\, {}_2F_1\left(\frac{\alpha}{2}, \frac{\alpha}{2}, 2, \frac{R^2}{z}\right)\right) & \text{if } z > R^2 \end{cases} \\ &= \begin{cases} \frac{\alpha(\alpha-2)}{4}\frac{1}{R^2}\, {}_2F_1\left(\frac{\alpha}{2}, 1+\frac{\alpha}{2}, 2, \frac{z}{R^2}\right) & \text{if } z \le R^2 \\ \frac{\alpha(\alpha-2)}{4} z^{-1-\frac{\alpha}{2}} R^\alpha\, {}_2F_1\left(\frac{\alpha}{2}, 1+\frac{\alpha}{2}, 2, \frac{R^2}{z}\right) & \text{if } z > R^2 \end{cases} \end{aligned}$$

Writing the series expansion (respectively at $z = 0$ and $z = \infty$) of the hypergeometric series:

$$\begin{aligned} \frac{\alpha(\alpha-2)}{4}\frac{1}{R^2}\, {}_2F_1\left(\frac{\alpha}{2}, 1+\frac{\alpha}{2}, 2, z\right) &= \frac{1}{R^2}\sum_{n=0}^\infty \frac{\Gamma\left(\frac{\alpha}{2}+n\right)\Gamma\left(n+1+\frac{\alpha}{2}\right)}{\Gamma\left(\frac{\alpha}{2}-1\right)\Gamma\left(\frac{\alpha}{2}\right)\Gamma(1+n)\Gamma(2+n)}\left(\frac{z}{R^2}\right)^n \\ \frac{\alpha(\alpha-2)}{4} z^{-1-\frac{\alpha}{2}} R^\alpha\, {}_2F_1\left(\frac{\alpha}{2}, 1+\frac{\alpha}{2}, 2, \frac{1}{z}\right) &= \left(\frac{1}{z}\right)^{\frac{\alpha}{2}} R^{\alpha-2}\sum_{n=1}^\infty \frac{\Gamma\left(\frac{\alpha}{2}+n\right)\Gamma\left(n-1+\frac{\alpha}{2}\right)}{\Gamma\left(\frac{\alpha}{2}-1\right)\Gamma\left(\frac{\alpha}{2}\right)\Gamma(n)\Gamma(n+1)}\left(\frac{R^2}{z}\right)^n, \end{aligned}$$

which are both minimized at $z = R^2$ since every coefficient is negative. This proves the claim.

The claim immediately implies

$$\Omega_c(R) = -\frac{C_\alpha}{R}\frac{d}{d|x|}\int_{B(0,R)} |x-y|^{-\alpha} dy\Big|_{|x|=R}, \tag{7.17}$$

Where $\frac{d}{d|x|}$ denotes the derivative in the radial variable (recall that $\int_{B(0,R)} |x-y|^{-\alpha} dy$ is radially symmetric). To compute the derivative at $|x| = R$, we can simply compute the partial derivative in the x_1 direction at the point $(R, 0)$:

$$\begin{aligned}
&\frac{\partial}{\partial x_1}\int_{B(0,R)} |x-y|^{-\alpha} dy\Big|_{x=(R,0)} = -\alpha\int_{B(0,R)} \left((R-y_1)^2 + y_2^2\right)^{-\frac{\alpha}{2}-1}(R-y_1)dy_1dy_2 \\
&= -2\int_0^R \left((R-y_1)^2 + {y_2}^2\right)^{-\frac{\alpha}{2}}\Big|_{y_1=-\sqrt{R^2-{y_2}^2}}^{y_1=\sqrt{R^2-{y_2}^2}} dy_2 \\
&= -2^{1-\frac{\alpha}{2}}R^{1-\alpha}\left(\int_0^1 \left(1-\sqrt{1-u^2}\right)^{-\frac{\alpha}{2}} du - \int_0^1 \left(1+\sqrt{1-u^2}\right)^{-\frac{\alpha}{2}} du\right) \\
&= -2^{1-\frac{\alpha}{2}}R^{1-\alpha}\left(\int_0^{\frac{\pi}{2}} (1-\cos\theta)^{-\frac{\alpha}{2}}\cos\theta\, d\theta - \int_0^{\frac{\pi}{2}} (1+\cos\theta)^{-\frac{\alpha}{2}}\cos\theta\, d\theta\right) \\
&=: -2^{1-\frac{\alpha}{2}}R^{1-\alpha}(I_1 - I_2),
\end{aligned} \tag{7.18}$$

where in the third line we used the identity $(R \pm \sqrt{R^2-y_2^2})^2 + y_2^2 = 2R^2(1 \pm \sqrt{1-(R^{-1}y_2)^2})$, as well as the substitution $u = R^{-1}y_2$.

Using a substitution $\theta = 2\beta$, we rewrite I_1 as

$$I_1 = 2\int_0^{\frac{\pi}{4}} (1-\cos(2\beta))^{-\frac{\alpha}{2}}\cos(2\beta)\, d\beta = 2^{1-\frac{\alpha}{2}}\int_0^{\frac{\pi}{4}} (\sin\beta)^{-\alpha}(1-2\sin^2\beta)\, d\beta.$$

Likewise, the substitution $\theta = \pi - 2\beta$ allows us to rewrite $-I_2$ as

$$-I_2 = 2\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (1-\cos(2\beta))^{-\frac{\alpha}{2}}\cos(2\beta)\, d\beta = 2^{1-\frac{\alpha}{2}}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (\sin\beta)^{-\alpha}(1-2\sin^2\beta)\, d\beta.$$

Adding the above two identities for I_1 and $-I_2$ together gives

$$\begin{aligned}
I_1 - I_2 &= 2^{1-\frac{\alpha}{2}}\int_0^{\frac{\pi}{2}} (\sin\beta)^{-\alpha}(1-2\sin^2\beta)\, d\beta \\
&= 2^{1-\frac{\alpha}{2}}\left(\frac{1}{2}B\Big(\frac{1-\alpha}{2},\frac{1}{2}\Big) - B\Big(\frac{3-\alpha}{2},\frac{1}{2}\Big)\right) \\
&= 2^{-\frac{\alpha}{2}}\frac{\Gamma(\frac{1-\alpha}{2})\Gamma(\frac{1}{2})}{\Gamma(1-\frac{\alpha}{2})} - 2^{1-\frac{\alpha}{2}}\frac{\Gamma(\frac{3-\alpha}{2})\Gamma(\frac{1}{2})}{\Gamma(2-\frac{\alpha}{2})},
\end{aligned}$$

where B stands for the beta function. Here the second identity follows from the property that $B(x,y) = 2\int_0^{\pi/2} (\sin\theta)^{2x-1}(\cos\theta)^{2y-1}d\theta$, and the third line follows from the property that $B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$. According to the properties of the gamma function $\Gamma(z+1) = z\Gamma(z)$ and $\Gamma(z)\Gamma(z+\frac{1}{2}) = 2^{1-2z}\sqrt{\pi}\Gamma(2z)$, we have

$$I_1 - I_2 = 2^{-1+\frac{\alpha}{2}} \frac{\alpha}{2-\alpha} \cdot \frac{2\pi\Gamma(1-\alpha)}{\Gamma(1-\frac{\alpha}{2})^2}. \tag{7.19}$$

Finally, plugging this into (7.18) and (7.17) gives

$$\begin{aligned}
\Omega_c(R) &= R^{-\alpha} C_\alpha 2^{1-\frac{\alpha}{2}} (I_1 - I_2) \\
&= R^{-\alpha} \frac{1}{2\pi} \frac{\Gamma(\frac{\alpha}{2})}{2^{1-\alpha}\Gamma(1-\frac{\alpha}{2})} \frac{\alpha}{2-\alpha} \left(\frac{2\pi\Gamma(1-\alpha)}{\Gamma(1-\frac{\alpha}{2})^2} \right) \\
&= R^{-\alpha} \frac{2^{\alpha-1}\Gamma(1-\alpha)\Gamma(\frac{\alpha}{2}+1)}{\Gamma(1-\frac{\alpha}{2})^2\Gamma(2-\frac{\alpha}{2})} = R^{-\alpha}\Omega_\alpha,
\end{aligned}$$

finishing the proof. □

At the end of this section, we point out that Theorem 7.2.1 directly gives the following quantitative estimate: if a simply-connected patch D rotates with angular velocity $\Omega \in (0, \Omega_\alpha)$ that is very close to Ω_α, then D must be very close to a disk in terms of symmetric difference.

Corollary 7.2.3. *Assume $0 < \alpha < 1$. Let D be a rotating patch with area π and angular velocity $\Omega \in (0, \Omega_\alpha)$, and let B be the unit disk. Then we have*

$$|D \triangle B| \leq 2\pi \left(\left(\frac{\Omega_\alpha}{\Omega} \right)^{2/\alpha} - 1 \right).$$

Note that for a fixed $\alpha \in (0,1)$, the right hand side goes to 0 as $\Omega \nearrow \Omega_\alpha$.

Proof. Denote $R := \max_{x\in D} |x|$. If D is a rotating patch with angular velocity Ω and is not a disk, Theorem 7.2.1 gives that $\Omega \leq R^{-\alpha}\Omega_\alpha$, which gives that $R \leq (\frac{\Omega_\alpha}{\Omega})^{1/\alpha}$. Thus $D \subset B(0, (\frac{\Omega_\alpha}{\Omega})^{1/\alpha})$, which implies that the symmetric difference $D \triangle B$ satisfies

$$|D \triangle B| = 2|D \setminus B| \leq 2 \left| B\Big(0, \big(\frac{\Omega_\alpha}{\Omega}\big)^{1/\alpha}\Big) \setminus B \right| = 2\pi \left(\left(\frac{\Omega_\alpha}{\Omega} \right)^{2/\alpha} - 1 \right).$$

□

Chapter 8

The rigidity results for vortex sheets

8.1 Notations

For a bounded domain $D \subset \mathbb{R}^2$, we denote $|D|$ by its area (i.e. its Lebesgue measure). For $x \in \mathbb{R}^2$ and $r > 0$, denote by $B(x, r)$ or $B_r(x)$ the open ball centered at x with radius r.

Through Section 3-5 of this paper, we will desingularize the vortex sheet into a vortex layer with width $\sim \epsilon$, and obtain various quantitative estimates. In all these estimates, we say a term f is $O(g(\epsilon))$ if $|f| \leq Cg(\epsilon)$ for some constant C independent of ϵ.

For a domain $U \subset \mathbb{R}^2$, in the boundary integral $\int_{\partial U} \vec{f} \cdot n d\sigma$, n denotes the outer normal of the domain U.

8.2 Equations for a stationary/rotating vortex sheet

Let $\omega(\cdot, t) = \omega_0(R_{\Omega t})$ be a stationary/rotating vortex sheet solution to the incompressible 2D Euler equation, where $\omega_0 \in \mathcal{M}(\mathbb{R}^2) \cap H^{-1}(\mathbb{R}^2)$ is a Radon measure. Here $\Omega = 0$ corresponds to a stationary solution, and $\Omega \neq 0$ corresponds to a rotating solution. Assume ω_0 is concentrated on Γ, which is a finite disjoint union of curves. Throughout this paper we assume Γ satisfies the following:

(H1) Each connected component of Γ is smooth and with finite length, and it is either a simple closed curve (denote them by $\Gamma_1, \ldots, \Gamma_n$), or a non-self-intersecting curve with two endpoints (denote them by $\Gamma_{n+1}, \ldots, \Gamma_{n+m}$). Here we require $n + m \geq 1$, but allow either n or m to be 0.

Let us denote

$$d_\Gamma := \min_{k \neq i} \operatorname{dist}(\Gamma_i, \Gamma_k), \tag{8.1}$$

which is strictly positive since we assume the curves $\{\Gamma_i\}_{i=1}^{n+m}$ are disjoint. For $i = 1, \ldots, n+m$, denote by L_i the length of Γ_i. Let $z_i : S_i \to \Gamma_i$ denote a constant-speed parameterization of Γ_i (in counter-clockwise direction if Γ_i is a closed curve), where the parameter domain S_i is given by

$$S_i := \begin{cases} \mathbb{R}/\mathbb{Z} & \text{for } i = 1, \ldots, n, \\ [0,1] & \text{for } i = n+1, \ldots, n+m. \end{cases}$$

Note that this gives $|z_i'| \equiv L_i$, and the arc-chord constant

$$F_\Gamma := \max_{i=1,\ldots,n+m} \sup_{\alpha \neq \beta \in S_i} \frac{|\alpha - \beta|}{|z_i(\alpha) - z_i(\beta)|} \tag{8.2}$$

is finite, since Γ is non-self-intersecting. Let $\mathbf{s} : \Gamma \to \mathbb{R}^2$ be the unit tangential vector on Γ, given by $\mathbf{s}(z_i(\alpha)) := \frac{z_i'(\alpha)}{|z_i'(\alpha)|} = \frac{z_i'(\alpha)}{L_i}$, and $\mathbf{n} : \Gamma \to \mathbb{R}^2$ be the unit normal vector, given by $\mathbf{n} = \mathbf{s}^\perp$. See Figure 8.1 for an illustration.

For $i = 1, \ldots, n+m$, let us denote by $\gamma_i(\alpha)$ the vorticity strength at $z_i(\alpha)$ with respect to the arclength parametrization, which is related to $\varpi_i(\alpha)$ by

$$\gamma_i(\alpha) = \frac{\varpi_i(\alpha)}{|z_i'(\alpha)|} \quad \text{for } \alpha \in S_i. \tag{8.3}$$

Throughout this paper we will be working with γ_i, instead of ϖ_i. We impose the following regularity and positivity assumptions on γ_i:

(H2) Assume that $\gamma_i \in C^2(S_i)$ for $i = 1, \ldots, n$ and $\gamma_i \in C^b(S_i) \cap C^1(S_i^\circ)$ for some $b \in (0,1)$ for $i = n+1, \ldots, n+m$.[1]

(H3) For $i = 1, \ldots, n$, assume $\gamma_i > 0$ in S_i. And for $i = n+1, \ldots, n+m$, assume $\gamma_i > 0$ in S_i°, and $\gamma_i(0) = \gamma_i(1) = 0$.

Note that for a closed curve, **(H3)** implies that γ_i is uniformly positive; whereas for an open curve, γ_i is positive in the interior of S_i but vanishes at its endpoints. This is because any stationary/rotating vortex sheet with continuous γ_i must have it vanishing at the two endpoints of any open curve: if not, one can easily check that $|BR(z_i(\alpha)) \cdot \mathbf{n}(z_i(\alpha))| \to \infty$ as α approaches the

[1] For an open curve $i = n+1, \ldots, n+m$, note that **(H2)** does not require γ_i to be C^1 up to the boundary of S_i, and its derivative is allowed to blow up at the endpoints. This is motivated by the fact that in the explicit uniformly-rotating solution (4.12), its strength γ is Hölder continuous in $[-a, a]$ and smooth in the interior, but its derivative blows up at the endpoints.

endpoint, thus such a vortex sheet cannot be stationary in the rotating frame.

With the above notations of z_i and γ_i, the Birkhoff-Rott integral (4.9) along the sheet can now be expressed as

$$BR(z_i(\alpha)) = \sum_{k=1}^{n+m} BR_k(z_i(\alpha)) := \sum_{k=1}^{n+m} PV \int_{S_k} K_2(z_i(\alpha) - z_k(\alpha'))\, \gamma_k(\alpha') |z_k'(\alpha')|\, d\alpha', \tag{8.4}$$

with the kernel K_2 given by

$$K_2(x) := (2\pi)^{-1} \nabla^\perp \log|x| = \frac{x^\perp}{2\pi|x|^2}, \tag{8.5}$$

and the principal value in (8.4) is only needed for the integral with $k = i$.

Let $\mathbf{v} : \mathbb{R}^2 \to \mathbb{R}^2$ be the velocity field generated by ω_0, given by $\mathbf{v} := \nabla^\perp(\omega_0 * \mathcal{N})$. Note that $\mathbf{v} \in C^\infty(\mathbb{R}^2 \setminus \Gamma)$, but $\mathbf{v}$ is discontinuous across Γ. Let $\mathbf{v}^+, \mathbf{v}^- : \Gamma \to \mathbb{R}^2$ denote the two limits of $\mathbf{v}$ on the two sides of Γ (with $\mathbf{v}^+$ being the limit on the side that $\mathbf{n}$ points into – see Figure 8.1 for an illustration), and $[\mathbf{v}] := \mathbf{v}^- - \mathbf{v}^+$ the jump in $\mathbf{v}$ across the sheets. $[\mathbf{v}]$ is related to the vortex-sheet strength γ as follows (see [71, Eq. (9.8)] for a derivation): $[\mathbf{v}] \cdot \mathbf{n} = 0$, and

$$[\mathbf{v}] \times \mathbf{n} = [\mathbf{v}] \cdot \mathbf{s} = \gamma.$$

In addition, the Birkhoff-Rott integral (8.4) is the the average of $\mathbf{v}^+$ and $\mathbf{v}^-$, namely

$$BR(z_i(\alpha)) = \frac{1}{2}(\mathbf{v}^+(z_i(\alpha)) + \mathbf{v}^-(z_i(\alpha))) \quad \text{for all } \alpha \in S_i, i = 1, \ldots, n+m.$$

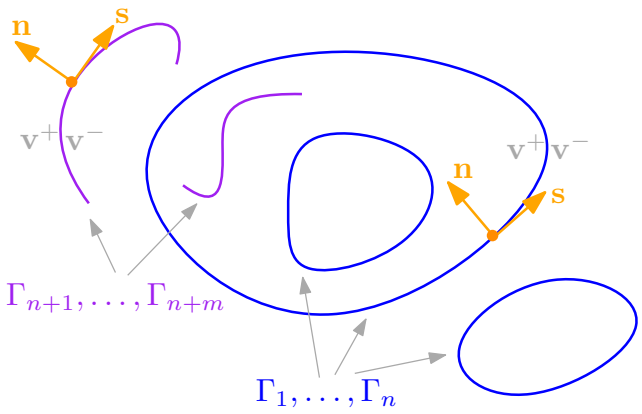


Figure 8.1: Illustration of the closed curves $\Gamma_1, \ldots, \Gamma_n$ and the open curves $\Gamma_{n+1}, \ldots, \Gamma_{n+m}$, and the definitions of $\mathbf{n}$, $\mathbf{s}$, $\mathbf{v}^+$ and $\mathbf{v}^-$.

In the following lemma, we derive the equation that the Birkhoff-Rott integral satisfies for a stationary/rotating vortex sheet.

Lemma 8.2.1. *Assume $\omega(\cdot,t) = \omega_0(R_{\Omega t}x)$ is a stationary/uniformly-rotating vortex sheet with angular velocity $\Omega \in \mathbb{R}$, and ω_0 is concentrated on $\cup_{i=1}^{n+m}\Gamma_i$, with z_i and γ_i defined as above. Then the Birkhoff-Rott integral BR (8.4) and the strength γ_i satisfy the following two equations:*

$$(BR - \Omega x^{\perp}) \cdot \mathbf{n} = \mathbf{v}^{+} \cdot \mathbf{n} = \mathbf{v}^{-} \cdot \mathbf{n} = 0 \quad \text{on } \Gamma, \tag{8.6}$$

and

$$(BR(z_i(\alpha)) - \Omega z_i^{\perp}(\alpha)) \cdot \mathbf{s}(z_i(\alpha))\,\gamma_i(\alpha) = \begin{cases} C_i & \text{on } S_i \text{ for } i = 1,\dots,n, \\ 0 & \text{on } S_i \text{ for } i = n+1,\dots,n+m. \end{cases} \tag{8.7}$$

In particular, the above two equations imply that $BR(z_i(\alpha)) - \Omega z_i^{\perp}(\alpha) \equiv \mathbf{0}$ for $i = n+1,\dots,n+m$.

Proof. By definition of the stationary/uniformly-rotating solutions, ω_0 is a stationary vortex sheet in the rotating frame with angular velocity Ω. In this rotating frame, an extra velocity $-\Omega z_i^{\perp}$ should be added to the right hand side of (4.10). Therefore the evolution equations (4.10)–(4.11) become the following in the rotating frame (where we also use (8.4)):

$$\partial_t z_i(\alpha,t) = BR(z_i(\alpha,t)) - \Omega z_i^{\perp}(\alpha,t) + c_i(\alpha,t)\partial_\alpha z_i(\alpha,t) \tag{8.8}$$

$$\partial_t \varpi_i(\alpha,t) = \partial_\alpha(c_i(\alpha,t)\varpi_i(\alpha,t)), \tag{8.9}$$

where the term $c_i(\alpha,t)$ accounts for the reparametrization freedom of the curves. Since ω_0 is stationary in the rotating frame, $z_i(\cdot,t)$ parametrizes the same curve as $z_i(\cdot,0)$. Therefore $\partial_t z_i(\alpha,t)$ is tangent to the curve Γ_i, and multiplying $\mathbf{n}(z_i(\alpha,t))$ to (8.8) gives

$$0 = \partial_t z_i(\alpha,t) \cdot \mathbf{n}(z_i(\alpha,t)) = (BR(z_i(\alpha,t)) - \Omega z_i^{\perp}(\alpha,t)) \cdot \mathbf{n}(z_i(\alpha,t)), \tag{8.10}$$

where we use that $\mathbf{n}(z_i(\alpha,t)) \cdot \partial_\alpha z_i(\alpha,t) = 0$. This proves (9.1).

Now we prove (9.2). Towards this end, let us choose

$$c_i(\alpha,t) := -\frac{(BR(z_i(\alpha,t)) - \Omega z_i^{\perp}(\alpha,t)) \cdot \mathbf{s}(z_i(\alpha,t))}{|\partial_\alpha z_i(\alpha,t)|},$$

so that multiplying $\mathbf{s}(z_i(\alpha,t))$ to (8.8) gives $\partial_t z_i(\alpha,t) \cdot \mathbf{s}(z_i(\alpha,t)) = 0$, and combining it with (8.10) gives $\partial_t z_i(\alpha,t) = 0$. In other words, with such choice of c_i, the parametrization $z_i(\alpha,t)$ remains fixed

in time. Since ω_0 is stationary in the rotating frame, we know that with a fixed parametrization $z_i(\alpha, t) = z_i(\alpha, 0)$, the strength $\varpi_i(\alpha, t)$ must also remain invariant in time. Thus (8.9) becomes

$$c_i(\alpha, t)\varpi_i(\alpha, t) \equiv C_i.$$

Plugging the definition of c_i into the equation above and using the fact that z_i is invariant in t, we have

$$\frac{(BR(z_i(\alpha)) - \Omega z_i^{\perp}(\alpha)) \cdot \mathbf{s}(z_i(\alpha))\varpi_i(\alpha)}{|\partial_\alpha z_i(\alpha)|} \equiv -C_i \quad \text{for all } \alpha \in S_i,$$

and finally the relationship between γ_i and ϖ_i in (8.3) yields (9.2) for $i = 1, \ldots, n$.

And for the open curves $i = n+1, \ldots, n+m$, note that we do not have any reparametrization freedom at the two endpoints $\alpha = 0, 1$, therefore the endpoint velocity $BR(z_i(0,t)) - \Omega z_i^{\perp}(0,t)$ must be 0 to ensure that ω_0 is stationary in the rotating frame. This immediately leads to $C_i = 0$ for $i = n+1, \ldots, n+m$, finishing the proof of (9.2). □

8.3 Approximation by a thin vortex layer

Our aim in this section is to desingularize the vortex sheet ω_0. Namely, for $0 < \epsilon \ll 1$, we will construct a vorticity $\omega^\epsilon \in L^\infty(\mathbb{R}^2) \cap L^1(\mathbb{R}^2)$ that only takes values 0 and ϵ^{-1}, and is supported in an $O(\epsilon)$ neighborhood of Γ, such that ω^ϵ weakly converges to ω_0 as $\epsilon \to 0^+$.

For each $i = 1, \ldots, n+m$, we will describe a neighborhood of Γ_i using the following change of coordinates: let $R_i^\epsilon : S_i \times \mathbb{R} \to \mathbb{R}^2$ be given by

$$R_i^\epsilon(\alpha, \eta) := z_i(\alpha) + \epsilon\gamma_i(\alpha)\mathbf{n}(z_i(\alpha))\eta, \tag{8.11}$$

and let

$$D_i^\epsilon := \{R_i^\epsilon(\alpha, \eta) : \alpha \in S_i^\circ, \eta \in (-1, 0)\}.$$

Note that each D_i^ϵ is a connected open set, and for all $\epsilon > 0$ sufficiently small, the sets $(D_i^\epsilon)_{i=1}^{n+m}$ are disjoint. For $i = 1, \ldots, n$, the domains D_i^ϵ are doubly-connected with smooth boundary, and its inner boundary coincides with Γ_i; see the left of Figure 8.2 for an illustration. And for $i = n+1, \ldots, n+m$, the domains D_i^ϵ are simply-connected, and its boundary is smooth except at most two points; see the right of Figure 8.2 for an illustration.

In addition, for $\epsilon > 0$ that is sufficiently small, one can check that $R_i^\epsilon : S_i^\circ \times (-1, 0) \to D_i^\epsilon$ is a

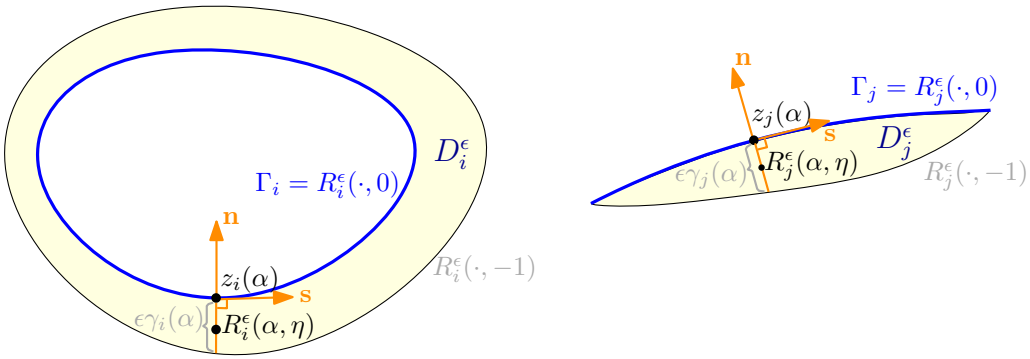


Figure 8.2: Illustration of the definitions of R_i^ϵ and D_i^ϵ for a closed curve (left) and an open curve (right).

diffeomorphism. Since $\gamma_i \in C^1(S_i)$ and $z_i \in C^2(S_i)$, we only need to show $R_i^\epsilon : S_i^\circ \times (-1, 0) \to D_i^\epsilon$ is injective. Below we prove this fact in a stronger quantitative version, which will be used later.

Lemma 8.3.1. *For any $i = 1, \dots, n+m$, assume Γ_i and γ_i satisfy* **(H1)**–**(H2)**. *Then the map $R_i^\epsilon : S_i^\circ \times (-1, 0) \to D_i^\epsilon$ given by* (8.11) *is injective. In addition, there exist some $c_0, \epsilon_0 > 0$ depending on $\|z_i\|_{C^2(S_i)}$, $\|\gamma_i\|_{L^\infty(S_i)}$ and F_Γ, such that for all $\epsilon \in (0, \epsilon_0)$ we have*

$$|R_i^\epsilon(\alpha', \eta') - R_i^\epsilon(\alpha, \eta)| \geq c_0\big(|\alpha' - \alpha| + \epsilon|\gamma_i(\alpha)\eta - \gamma_i(\alpha')\eta'|\big), \tag{8.12}$$

for all $\alpha, \alpha' \in S_i^\circ$, $\eta, \eta' \in (-1, 0)$.[2]

Proof. To begin with, note that (8.12) immediately implies that $R_i^\epsilon : S_i^\circ \times (-1, 0) \to D_i^\epsilon$ is injective, where we used the positivity assumption $\gamma_i > 0$ in S_i° in **(H2)**. Thus it suffices to prove (8.12). Throughout the proof, we fix any $i \in \{1, \dots, n+m\}$, and we will omit the subscript i for notational simplicity. Using the definition (8.11), let us break $R^\epsilon(\alpha', \eta') - R^\epsilon(\alpha, \eta)$ into

$$R^\epsilon(\alpha', \eta') - R^\epsilon(\alpha, \eta) = \underbrace{z(\alpha') - z(\alpha)}_{=:T_1} + \underbrace{\epsilon\left(\gamma(\alpha')\eta' - \gamma(\alpha)\eta\right)\mathbf{n}(z(\alpha'))}_{=:T_2} + \underbrace{\epsilon\gamma(\alpha)\eta\left(\mathbf{n}(z(\alpha')) - \mathbf{n}(z(\alpha))\right)}_{=:T_3}. \tag{8.13}$$

For T_1 and T_3, we have

$$\begin{aligned} |T_1 - z'(\alpha')(\alpha' - \alpha)| \leq& \|z\|_{C^2(S)}|\alpha - \alpha'|^2, \\ |T_3| \leq& \epsilon\gamma(\alpha)\|z\|_{C^2(S)}|\alpha - \alpha'|. \end{aligned} \tag{8.14}$$

[2] In fact, (8.12) also holds (with a slightly smaller ϵ_0 and c_0) for $\eta, \eta' \in (-2, 2)$, even though such R_i^ϵ may not belong to D_i^ϵ. We will use this fact later in the proof of Lemma 8.3.5.

Also, using that $z'(\alpha') = L\mathbf{s}(z(\alpha'))$ is perpendicular to $\mathbf{n}(z(\alpha'))$, we have

$$\begin{aligned}|z'(\alpha')(\alpha'-\alpha)+T_2| &= |L(\alpha'-\alpha)\mathbf{s}(z(\alpha')) + \epsilon\left(\gamma(\alpha')\eta' - \gamma(\alpha)\eta\right)\mathbf{n}(z(\alpha'))| \\ &\geq \frac{1}{2}L|\alpha'-\alpha| + \frac{1}{2}\epsilon\left|\gamma(\alpha')\eta' - \gamma(\alpha)\eta\right|,\end{aligned}$$

where we use that $\sqrt{x^2+y^2} \geq \frac{1}{2}(|x|+|y|)$. Combining this with (8.14) gives

$$|T_1+T_2+T_3| \geq |\alpha-\alpha'|\left(\frac{L}{2} - \|z\|_{C^2(S)}\left(|\alpha-\alpha'| + \epsilon\gamma(\alpha)\right)\right) + \frac{1}{2}\epsilon|\gamma(\alpha')\eta' - \gamma(\alpha)\eta|,$$

thus

$$|R^\epsilon(\alpha',\eta') - R^\epsilon(\alpha,\eta)| \geq \frac{L}{4}|\alpha-\alpha'| + \frac{1}{2}\epsilon|\gamma(\alpha')\eta' - \gamma(\alpha)\eta| \tag{8.15}$$

for all $0 < \epsilon < L(8\|z\|_{C^2}\|\gamma\|_{L^\infty})^{-1}$ and $|\alpha-\alpha'| \leq \frac{L}{8\|z\|_{C^2}}$.

For $|\alpha-\alpha'| > \frac{L}{8\|z\|_{C^2}}$, recall that the definition of F_Γ in (8.2) gives $|z(\alpha')-z(\alpha)| \geq F_\Gamma^{-1}|\alpha'-\alpha|$. Thus a crude estimate gives

$$|R^\epsilon(\alpha',\eta') - R^\epsilon(\alpha,\eta)| \geq |z(\alpha')-z(\alpha)| - 2\epsilon\|\gamma\|_{L^\infty(S)} \geq \frac{1}{2F_\Gamma}|\alpha'-\alpha| + \epsilon|\gamma(\alpha')\eta' - \gamma(\alpha)\eta| \tag{8.16}$$

for $0 < \epsilon < L(64F_\Gamma\|z\|_{C^2}\|\gamma\|_{L^\infty})^{-1}$. (Note that for such ϵ we have $4\epsilon\|\gamma\|_{L^\infty} \leq \frac{1}{2F_\Gamma}|\alpha'-\alpha|$ due to our assumption that $|\alpha-\alpha'| > \frac{L}{8\|z\|_{C^2}}$).

Finally, combining (8.15) and (8.16), it follows that (8.12) holds for $c_0 = \min\{\frac{L}{4}, \frac{1}{2F_\Gamma}, \frac{1}{2}\}$ and $\epsilon_0 = \min\{L(8\|z\|_{C^2}\|\gamma\|_{L^\infty})^{-1}, L(64F_\Gamma\|z\|_{C^2}\|\gamma\|_{L^\infty})^{-1}\}$. This finishes the proof. □

In the next lemma we compute the partial derivatives and Jacobian of $R_i^\epsilon(\alpha,\eta)$, which will be useful later.

Lemma 8.3.2. *For any $i = 1,\ldots,n+m$, let z_i be a constant-speed parameterization of the curve Γ_i (with length L_i), and let R_i^ϵ be given by* (8.11)*. Then its partial derivatives are*

$$\begin{aligned}\partial_\alpha R_i^\epsilon(\alpha,\eta) &= z_i'(\alpha) + \epsilon\left(\gamma_i'(\alpha)\frac{z_i'(\alpha)^\perp}{L_i}\eta + \gamma_i(\alpha)\frac{z_i''(\alpha)^\perp}{L_i}\eta\right), \\ \partial_\eta R_i^\epsilon(\alpha,\eta) &= \epsilon\gamma_i(\alpha)\frac{z_i'(\alpha)^\perp}{L_i}.\end{aligned} \tag{8.17}$$

Moreover, its Jacobian is given by

$$\det(\nabla_{\alpha,\eta}R_i^\epsilon) = \epsilon L_i\gamma_i(\alpha) - \epsilon^2 L_i\gamma_i^2(\alpha)\kappa_i(\alpha)\eta, \tag{8.18}$$

where $\kappa_i(\alpha)$ denotes the signed curvature of Γ_i at $z_i(\alpha)$.

Proof. Since z_i is the constant-speed parameterization of Γ_i (which has length L_i), we have $|z_i'| \equiv L_i$ and $\mathbf{n}(z_i(\alpha)) = z_i'(\alpha)^\perp / L_i$. Taking the α and η partial derivatives of (8.11) directly yields (8.17).

Putting the two partial derivatives into columns of a 2×2 matrix and computing the determinant, we have

$$\begin{aligned}\det(\nabla_{\alpha,\eta} R_i^\epsilon) &= \epsilon\gamma_i(\alpha)\frac{|z_i'(\alpha)|^2}{L_i} + \epsilon^2\gamma_i^2(\alpha)\frac{z_i''(\alpha)^\perp \cdot z_i'(\alpha)}{L_i^2}\eta \\ &= \epsilon L_i\gamma_i(\alpha) - \epsilon^2 L_i\gamma_i^2(\alpha)\kappa_i(\alpha)\eta,\end{aligned}$$

where in the second equality we used that $z_i''(\alpha) = \kappa_i(\alpha)\mathbf{n}(z_i(\alpha))L_i^2$ (recall that z_i has constant speed L_i). This finishes the proof. □

Remark 8.3.3. *We point out that for each $i = 1, \dots, n+m$, the determinant formula* (8.18) *immediately gives the following approximation of $|D_i^\epsilon|$, which will be helpful in the proofs later:*

$$\frac{|D_i^\epsilon|}{\epsilon} = \frac{1}{\epsilon}\int_{D_i^\epsilon} 1 dx = \frac{1}{\epsilon}\int_{S_i}\int_{-1}^{0} \det(\nabla_{\alpha,\eta} R_i^\epsilon(\alpha,\eta))\, d\eta d\alpha = L_i \int_{S_i} \gamma_i(\alpha) d\alpha + O(\epsilon), \tag{8.19}$$

where the $O(\epsilon)$ error term has its absolute value bounded by $C\epsilon$, with C only depending on $\|z_i\|_{C^2(S_i)}$ and $\|\gamma_i\|_{L^\infty(S_i)}$.

Finally, let $D^\epsilon := \cup_{i=1}^{n+m} D_i^\epsilon$, and $\omega^\epsilon : \mathbb{R}^2 \to \mathbb{R}$ is defined as

$$\omega^\epsilon(x) := \epsilon^{-1} 1_{D^\epsilon}(x) = \epsilon^{-1}\sum_{i=1}^{n+m} 1_{D_i^\epsilon}(x),$$

and let

$$\mathbf{v}^\epsilon = \nabla^\perp(\omega^\epsilon * \mathcal{N}) \tag{8.20}$$

be the velocity field generated by ω^ϵ.

In the next lemma we aim to obtain some fine estimate of $\mathbf{v}^\epsilon$ in the thin vortex layer D^ϵ. Our goal is to show that along each cross section of the thin layer (i.e. fix i and α, and let η vary in $[-1, 0]$), the function $\eta \mapsto \mathbf{v}^\epsilon(R_i^\epsilon(\alpha,\eta))$ is almost a linear function in η, with the endpoint values (at $\eta = -1$ and 0) being almost $\mathbf{v}^-(z_i(\alpha))$ and $\mathbf{v}^+(z_i(\alpha))$ respectively.

Lemma 8.3.4. *For $i = 1, \dots, n+m$, assume Γ_i and γ_i satisfy* **(H1)**–**(H3)**. *Let*

$$g_i(\alpha,\eta) := BR(z_i(\alpha)) - \Big(\eta + \frac{1}{2}\Big)[\mathbf{v}](z_i(\alpha)) \quad \text{for } \alpha \in S_i,$$

and note that $g_i(\alpha, 0) = \mathbf{v}^+(z_i(\alpha))$ *and* $g_i(\alpha, -1) = \mathbf{v}^-(z_i(\alpha))$ *(see Figure 8.3 for an illustration of* $g_i(\alpha, \eta)$*). Then for all sufficiently small* $\epsilon > 0$*, for all* $i = 1, \dots, n+m$ *we have*

$$|\mathbf{v}^\epsilon(R_i^\epsilon(\alpha, \eta)) - g_i(\alpha, \eta)| \le C\epsilon^b |\log \epsilon| \quad \text{for all } \alpha \in S_i, \eta \in [-1, 0], \tag{8.21}$$

where $b \in (0, 1)$ *is as in* **(H2)**, *and* C *depends on* b, $\max_i \|z_i\|_{C^2(S_i)}$, $\max_i \|\gamma_i\|_{C^b(S_i)}$, d_Γ *and* F_Γ.

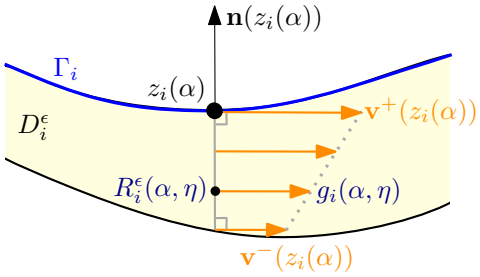


Figure 8.3: Illustration of the definition of $g_i(\alpha, \cdot)$ (the orange arrows).

Proof. Let i be any fixed index in $1, \dots, n+m$. We begin with breaking $\mathbf{v}^\epsilon$ into contributions from different components $\{D_k^\epsilon\}_{k=1}^{n+m}$, namely

$$\mathbf{v}^\epsilon(x) = \sum_{k=1}^{n+m} \mathbf{v}_k^\epsilon(x) := \sum_{k=1}^{n+m} \epsilon^{-1} \int_{D_i^\epsilon} K_2(x - y) dy,$$

where the kernel K_2 is given by (8.5). Similarly, we can break $BR(z_i(\alpha))$ into $BR(z_i(\alpha)) = \sum_{k=1}^{n+m} BR_k(z_i(\alpha))$, where BR_k is the contribution from the k-th integral in (8.4), and note that the PV symbol is only needed for $k = i$.

- *Estimates for* $k \neq i$ *terms.* For any $k \neq i$, we aim to show that

$$|\mathbf{v}_k^\epsilon(R_i^\epsilon(\alpha, \eta)) - BR_k(z_i(\alpha))| \le C\epsilon, \tag{8.22}$$

where C depends on $d_\Gamma, \max_k \|z_k\|_{C^2}$ and $\max_k \|\gamma_k\|_{L^\infty}$. Applying a change of variable $y = R_k^\epsilon(\alpha', \eta')$, we can rewrite $\mathbf{v}_k^\epsilon$ as

$$\begin{aligned} \mathbf{v}_k^\epsilon(R_i^\epsilon(\alpha, \eta)) &= \epsilon^{-1} \int_{D_k^\epsilon} K_2(R_i^\epsilon(\alpha, \eta) - y)\, dy \\ &= \int_{S_k} \int_{-1}^{0} \underbrace{K_2(R_i^\epsilon(\alpha, \eta) - R_k^\epsilon(\alpha', \eta'))}_{=:T_1} \underbrace{\epsilon^{-1} \det(\nabla_{\alpha', \eta'} R_k^\epsilon(\alpha', \eta'))}_{=:T_2} d\eta' d\alpha'. \end{aligned} \tag{8.23}$$

Using the facts that $R_i^\epsilon(\alpha,\eta) - R_k^\epsilon(\alpha',\eta') = z_i(\alpha) - z_k(\alpha') + O(\epsilon)$ as well as $|z_i(\alpha) - z_k(\alpha')| \geq d_\Gamma > 0$ (recall that d_Γ is as given in (8.1)), for all sufficiently small $\epsilon > 0$ we have $T_1 = K_2(z_i(\alpha) - z_k(\alpha')) + O(\epsilon)$. For T_2, the explicit formula (8.18) for the determinant gives $T_2 = L_k\gamma_k(\alpha') + O(\epsilon)$. Plugging these into the above integral yields

$$\mathbf{v}_k^\epsilon(R_i^\epsilon(\alpha,\eta)) = \int_{S_k} K_2(z_i(\alpha) - z_k(\alpha'))L_k\gamma_k(\alpha')\,d\alpha' + O(\epsilon) = BR_k(z_i(\alpha)) + O(\epsilon),$$

finishing the proof of (8.22).

• *Estimates for the $k = i$ term.* It will be more involved to control the $k = i$ term, and our goal is to show that

$$\left|\mathbf{v}_i^\epsilon(R_i^\epsilon(\alpha,\eta)) - BR_i(z_i(\alpha)) + \Big(\eta + \frac{1}{2}\Big)[\mathbf{v}](z_i(\alpha))\right| \leq C\epsilon^b|\log\epsilon|. \tag{8.24}$$

To begin with, we again rewrite $\mathbf{v}_i^\epsilon$ as in (8.23) with $k = i$, and plug in the formula (8.18) for the determinant. This leads to

$$\begin{aligned}\mathbf{v}_i^\epsilon(R_i^\epsilon(\alpha,\eta)) &= \int_{S_k}\int_{-1}^0 K_2(R_i^\epsilon(\alpha,\eta) - R_i^\epsilon(\alpha',\eta'))\left(L_i\gamma_i(\alpha') - \epsilon L_i\gamma_i^2(\alpha')\kappa_i(\alpha')\eta'\right)\,d\eta' d\alpha' \\ &=: I_1 + I_2,\end{aligned}$$

where I_1, I_2 are the contributions from the two terms in the last parenthesis respectively. Let us control I_2 first, and we claim that

$$|I_2| \leq C\epsilon|\log\epsilon|. \tag{8.25}$$

Using (8.12) of Lemma 8.3.1 and the fact that $|K_2(x)| \leq |x|^{-1}$, we can bound I_2 as

$$\begin{aligned}|I_2| &= \left|\int_{S_k}\int_{-1}^0 K_2(R_i^\epsilon(\alpha,\eta) - R_i^\epsilon(\alpha',\eta'))\,\epsilon L_i\gamma_i^2(\alpha')\kappa_i(\alpha')\eta'\,d\eta' d\alpha'\right| \\ &\leq C\epsilon\int_{S_k}\int_{-1}^0 \frac{\gamma_i(\alpha')}{|\alpha' - \alpha| + \epsilon|\gamma_i(\alpha')\eta' - \gamma_i(\alpha)\eta|}\,d\eta' d\alpha' \\ &\leq C\epsilon\int_{S_k}\int_{-\|\gamma_i\|_\infty}^{\|\gamma_i\|_\infty} \frac{1}{|\alpha' - \alpha| + \epsilon|\theta'|}\,d\theta' d\alpha' \quad (\theta' := \gamma_i(\alpha')\eta' - \gamma_i(\alpha)\eta) \\ &\leq C\epsilon\int_{-1/\epsilon}^{1/\epsilon}\int_{-\|\gamma_i\|_\infty}^{\|\gamma_i\|_\infty} \frac{1}{|\beta'| + |\theta'|}\,d\theta' d\beta' \quad (\beta' := \epsilon^{-1}(\alpha' - \alpha)) \\ &\leq C\epsilon|\log\epsilon|\end{aligned} \tag{8.26}$$

where C depends on $\|z_i\|_{C^2}$ and $\|\gamma_i\|_{L^\infty}$.

In the rest of the proof we focus on estimating $I_1 = \int_{S_k}\int_{-1}^{0} K_2(R_i^\epsilon(\alpha,\eta)-R_i^\epsilon(\alpha',\eta'))L_i\gamma_i(\alpha')\,d\eta' d\alpha'$. For $t\in[0,1]$, let us define

$$\begin{aligned} f(\alpha,\alpha',\eta,\eta';t) &:= R_i^\epsilon(\alpha,\eta-t\eta') - R_i^\epsilon(\alpha',\eta'-t\eta'),\\ J(t) &:= \int_{S_k}\int_{-1}^{0} K_2(f(\alpha,\alpha',\eta,\eta';t))L_i\gamma_i(\alpha')\,d\eta' d\alpha'. \end{aligned} \tag{8.27}$$

Note that in the definition of f, the argument $\eta - t\eta'$ of R_i^ϵ belongs to $[-1,1]$, instead of $[-1,0]$ as in the original definition of (8.11). Here $R_i^\epsilon(\alpha,\eta-t\eta')$ is defined as in the formula (8.11), even though it might not belong to D_i^ϵ. Clearly, $J(0) = I_1$. The motivation for us to define such f and $J(t)$ is that at $t=1$, we have

$$J(1) = \int_{S_k}\int_{-1}^{0} K_2(R_i^\epsilon(\alpha,\eta-\eta')-z_i(\alpha'))L_i\gamma_i(\alpha')\,d\eta' d\alpha' = \int_{-1}^{0}\mathbf{v}_i(R_i^\epsilon(\alpha,\eta-\eta'))\,d\eta', \tag{8.28}$$

where $\mathbf{v}_i$ is the velocity field generated by the sheet Γ_i. Recall that $\mathbf{v}_i$ has a jump across Γ_i, where we denote its limits on two sides by $\mathbf{v}_i^\pm$. Using Lemma 8.3.5, which we will prove momentarily, we have

$$\mathbf{v}_i(R_i^\epsilon(\alpha,\eta-\eta')) = \begin{cases} \mathbf{v}_i^+(z_i(\alpha)) + O(\epsilon^b|\log\epsilon|) & \text{if } \eta-\eta'\in(0,2),\\ \mathbf{v}_i^-(z_i(\alpha)) + O(\epsilon^b|\log\epsilon|) & \text{if } \eta-\eta'\in(-2,0). \end{cases} \tag{8.29}$$

We can then split the integration domain on the right hand side of (8.28) into $\eta'\in(-1,\eta)$ and $\eta'\in(\eta,0)$, and use (8.29) to approximate the integrand in each interval. This gives

$$\begin{aligned} J(1) &= (\eta+1)\mathbf{v}_i^+(z_i(\alpha)) - \eta\mathbf{v}_i^-(z_i(\alpha)) + O(\epsilon^b|\log\epsilon|)\\ &= BR_i(z_i(\alpha)) - \Big(\eta+\frac{1}{2}\Big)[\mathbf{v}](z_i(\alpha)) + O(\epsilon^b|\log\epsilon|), \end{aligned} \tag{8.30}$$

where in the last step we used that $[\mathbf{v}](z_i(\alpha)) = [\mathbf{v}_i](z_i(\alpha))$, since all other $\mathbf{v}_k$ with $k\neq i$ are continuous across Γ_i.

Finally, it remains to control $|J(0)-J(1)|$. Note that by (8.12), we have

$$f(\alpha,\alpha',\eta,\eta';t) \geq c_0\big(|\alpha-\alpha'| + \epsilon|\gamma_i(\alpha')\eta' - \gamma_i(\alpha)\eta|\big).$$

In addition, we have

$$\left|\frac{\partial}{\partial t}f(\alpha,\alpha',\eta,\eta';t)\right| = \big|\epsilon\big(\gamma_i(\alpha)\mathbf{n}(z_i(\alpha)) - \gamma_i(\alpha')\mathbf{n}(z_i(\alpha'))\big)\eta'\big| \leq C\epsilon|\alpha-\alpha'|^b,$$

where the last inequality follows from **(H2)** and the fact that $\mathbf{n}(z_i(\alpha)) \in C^1(S_i)$. Therefore, for any $t \in (0,1)$, taking the t derivative of (8.27) and using that $|\nabla K_2(x)| \le |x|^{-2}$, we have

$$\begin{aligned}
|J'(t)| &\le C \int_{S_k} \int_{-1}^{0} \frac{\epsilon |\alpha - \alpha'|^b \gamma_i(\alpha')}{\big(|\alpha - \alpha'| + \epsilon |\gamma_i(\alpha')\eta' - \gamma_i(\alpha)\eta|\big)^2} d\eta' d\alpha' \\
&\le C\epsilon \int_{S_k} \int_{-1}^{0} \frac{\gamma_i(\alpha')}{|\alpha - \alpha'|^{1-b} \left(|\alpha - \alpha'| + \epsilon |\gamma_i(\alpha')\eta' - \gamma_i(\alpha)\eta|\right)} d\eta' d\alpha' \\
&\le C\epsilon^b \int_{-1/\epsilon}^{1/\epsilon} \int_{-\|\gamma_i\|_\infty}^{\|\gamma_i\|_\infty} \frac{1}{|\beta'|^{1-b}(|\beta'| + |\theta'|)} \, d\theta' d\beta' \quad (\theta' := \gamma_i(\alpha')\eta' - \gamma_i(\alpha)\eta,\, \beta' := \epsilon^{-1}(\alpha' - \alpha)) \\
&\le C\epsilon^b \int_{-1/\epsilon}^{1/\epsilon} |\beta'|^{b-1} \log\left(1 + \frac{\|\gamma_i\|_{L^\infty}}{|\beta'|}\right) d\beta' \\
&\le C\epsilon^b,
\end{aligned}$$

where C depends on b, $\|\gamma_i\|_{C^b(S_i)}$, $\|z_i\|_{C^2(S_i)}$ and F_Γ. This leads to

$$|J(1) - I_1| = |J(1) - J(0)| \le C\epsilon^b |\log \epsilon|.$$

Finally, combining this with (8.30) and (8.25) yields (8.24), finishing the proof of the $k = i$ case. We can then conclude the proof by taking the sum of this estimate with all the $k \neq i$ estimates in (8.22). □

The following lemma proves (8.29). Let $\mathbf{v}_i$ be the velocity field generated by the sheet Γ_i, which is smooth in $\mathbb{R}^2 \setminus \Gamma_i$, and has a discontinuity across Γ_i. It is known that $\mathbf{v}_i$ converges to $\mathbf{v}_i^{\pm}$ respectively on the two sides of Γ_i [71]. However, we were unable to find a quantitative convergence rate (in terms of the distance from the point to Γ_i) in the literature, especially under the assumption that γ_i is only in $C^b(S_i)$ for the open curves. Below we prove such an estimate.

Lemma 8.3.5. *For $i = 1, \dots, n+m$, let $\mathbf{v}_i$ be the velocity field generated by the sheet Γ_i, given by*

$$\mathbf{v}_i(x) := \int_{S_i} K_2(x - z_i(\alpha'))\, \gamma_i(\alpha') |z_i'(\alpha')|\, d\alpha' \quad \text{for } x \in \mathbb{R}^2 \setminus \Gamma_i.$$

Then there exist constants $C, \epsilon_0 > 0$ depending on on b (as in **(H2)***), $\|z_i\|_{C^2(S_i)}$, $\|\gamma_i\|_{C^b(S_i)}$ and F_Γ, such that for all $\epsilon \in (0, \epsilon_0)$ and $\eta \in (-2, 2)$ we have*

$$\left|\mathbf{v}_i(R_i^\epsilon(\alpha, \eta)) - \mathbf{v}_i^+(z_i(\alpha))\right| \le C\epsilon^b |\log \epsilon| \quad \text{if } \eta \in (0, 2), \tag{8.31}$$

$$\left|\mathbf{v}_i(R_i^\epsilon(\alpha, \eta)) - \mathbf{v}_i^-(z_i(\alpha))\right| \le C\epsilon^b |\log \epsilon| \quad \text{if } \eta \in (-2, 0), \tag{8.32}$$

where

$$\mathbf{v}_i^+ = BR_i(z_i(\alpha)) + \frac{\mathbf{n}(z_i(\alpha))^\perp \gamma_i(\alpha)}{2}, \quad \mathbf{v}_i^- = BR_i(z_i(\alpha)) - \frac{\mathbf{n}(z_i(\alpha))^\perp \gamma_i(\alpha)}{2},$$

and BR_i is the contribution from the i-th integral in (8.4).

Proof. We will show (8.31) only since (8.32) can be treated in the same way. From the definition of R_i^ϵ in (8.11), we have

$$\begin{aligned}\mathbf{v}_i(R_i^\epsilon(\alpha,\eta)) &= \frac{L_i}{2\pi}\int_{S_i} \frac{(z_i(\alpha)-z_i(\alpha'))^\perp \gamma_i(\alpha')}{|z_i(\alpha)-z_i(\alpha')+\epsilon\eta\mathbf{n}(z_i(\alpha))\gamma_i(\alpha)|^2}\,d\alpha' + \frac{L_i}{2\pi}\int_{S_i} \frac{\epsilon\eta\mathbf{n}(z_i(\alpha))^\perp\gamma_i(\alpha)\gamma_i(\alpha')}{|z_i(\alpha)-z_i(\alpha')+\epsilon\eta\mathbf{n}(z_i(\alpha))\gamma_i(\alpha)|^2}\,d\alpha' \\ &=: A_1 + A_2.\end{aligned}$$

We claim that for all $\epsilon > 0$ sufficiently small and $\eta \in [0,2)$, we have

$$|A_1 - BR_i(z(\alpha))| \le C\epsilon^b |\log\epsilon|, \tag{8.33}$$

$$\left| A_2 - \frac{\mathbf{n}(z(\alpha))^\perp\gamma(\alpha)}{2}\right| \le C\epsilon^b, \tag{8.34}$$

and note that these two claims immediately yield (8.31). From now on, let us fix $i \in \{1,\dots,n+m\}$ and omit it in the notation for simplicity. Throughout this proof, let us denote

$$\mathbf{y}(\alpha,\alpha') := z(\alpha) - z(\alpha') \quad \text{and} \quad \mathbf{c}(\alpha) := \epsilon\eta\mathbf{n}(z(\alpha))\gamma(\alpha),$$

so that

$$A_1 = \frac{L}{2\pi}\int_S \frac{\mathbf{y}^\perp(\alpha,\alpha')\gamma(\alpha')}{|\mathbf{y}(\alpha,\alpha')+\mathbf{c}(\alpha)|^2}d\alpha', \quad A_2 = \frac{L}{2\pi}\int_S \frac{\mathbf{c}^\perp(\alpha)\gamma(\alpha')}{|\mathbf{y}(\alpha,\alpha')+\mathbf{c}(\alpha)|^2}d\alpha'.$$

Note that

$$F_\Gamma^{-1}|\alpha-\alpha'| \le |\mathbf{y}(\alpha,\alpha')| \le \|z\|_{C^1}|\alpha-\alpha'|. \tag{8.35}$$

For the closed curves with $i = 1,\dots,n$, since z has period 1, we can always set $\alpha - \alpha' \in [-\frac{1}{2},\frac{1}{2})$ in this proof.

Applying (8.12) (with $\eta' = 0$), we have

$$|\mathbf{y}(\alpha,\alpha')+\mathbf{c}(\alpha)|^2 \ge c_0(|\alpha-\alpha'|^2 + \epsilon^2\eta^2\gamma^2(\alpha)) = c_0(|\alpha-\alpha'|^2 + |\mathbf{c}(\alpha)|^2). \tag{8.36}$$

Since $z'(\alpha) = L\mathbf{s}(z(\alpha))$, let us define

$$\tilde{\mathbf{y}}(\alpha,\alpha') := L\mathbf{s}(z(\alpha))(\alpha-\alpha'),$$

which is a close approximation of $\mathbf{y}$ in the sense that

$$|\mathbf{y}(\alpha,\alpha') - \tilde{\mathbf{y}}(\alpha,\alpha')| \le \|z\|_{C^2}(\alpha-\alpha')^2. \tag{8.37}$$

Using $\mathbf{s}(z(\alpha)) \perp \mathbf{n}(z(\alpha))$, we have

$$|\tilde{\mathbf{y}}(\alpha,\alpha') + \mathbf{c}(\alpha)|^2 = L^2|\alpha-\alpha'|^2 + \epsilon^2\eta^2\gamma^2(\alpha) = L^2|\alpha-\alpha'|^2 + |\mathbf{c}(\alpha)|^2. \tag{8.38}$$

From now on, for notational simplicity, we compress the dependence of $\mathbf{y}(\alpha,\alpha'), \tilde{\mathbf{y}}(\alpha,\alpha'), \mathbf{c}(\alpha)$ on α and α' in the rest of the proof.

- *Estimate* (8.33). Note that $BR_i(z(\alpha))$ can also be written using the above notations as

$$BR_i(z(\alpha)) = \frac{L}{2\pi} PV \int_S \frac{\mathbf{y}^\perp}{|\mathbf{y}|^2}\gamma(\alpha')d\alpha,$$

thus $A_1 - BR_i(z(\alpha))$ can be written as follows:

$$\begin{aligned} A_1 - BR_i(z(\alpha)) &= \frac{L}{2\pi} PV \int_S \underbrace{\left(\frac{\mathbf{y}^\perp}{|\mathbf{y}+\mathbf{c}|^2} - \frac{\mathbf{y}^\perp}{|\mathbf{y}|^2}\right)}_{=:\mathbf{f}(\mathbf{y},\mathbf{c})}\gamma(\alpha')d\alpha' \\ &= \frac{L}{2\pi}\int_S \mathbf{f}(\mathbf{y},\mathbf{c})(\gamma(\alpha')-\gamma(\alpha))d\alpha' + \frac{L\gamma(\alpha)}{2\pi} PV \int_S \mathbf{f}(\mathbf{y},\mathbf{c})d\alpha' \\ &=: A_{11} + A_{12}. \end{aligned}$$

A direct computation gives

$$\mathbf{f}(\mathbf{y},\mathbf{c}) = -\frac{\mathbf{y}^\perp}{|\mathbf{y}|^2}\,\frac{2\mathbf{y}\cdot\mathbf{c}+|\mathbf{c}|^2}{|\mathbf{y}+\mathbf{c}|^2}. \tag{8.39}$$

Since $\mathbf{y}\cdot\mathbf{c} = (\mathbf{y}-\tilde{\mathbf{y}})\cdot\mathbf{c} \le C|\alpha-\alpha'|^2|\mathbf{c}|$, (where we use $\tilde{\mathbf{y}} \perp \mathbf{n}(z(\alpha))$ and (8.37)), combining this with (8.35) and (8.36) gives a crude bound

$$|\mathbf{f}(\mathbf{y},\mathbf{c})| \lesssim \frac{|\alpha-\alpha'|^2|\mathbf{c}| + |\mathbf{c}|^2}{|\alpha-\alpha'|(|\alpha-\alpha'|^2+|\mathbf{c}|^2)}.$$

Plugging this into A_{11} and using the Hölder continuity of γ, we have

$$\begin{aligned} |A_{11}| &\lesssim \int_S \frac{|\alpha-\alpha'|^2|\mathbf{c}| + |\mathbf{c}|^2}{|\alpha-\alpha'|(|\alpha-\alpha'|^2+|\mathbf{c}|^2)}|\alpha-\alpha'|^b d\alpha' \\ &\lesssim \int_{|\theta|<|\mathbf{c}|}(|\theta|^{1+b}|\mathbf{c}|^{-1} + |\theta|^{b-1})d\theta + \int_{|\mathbf{c}|\le|\theta|\le 1}(|\mathbf{c}||\theta|^{b-1} + |\mathbf{c}|^2|\theta|^{b-3})d\theta \quad (\theta := \alpha'-\alpha) \\ &\lesssim |\mathbf{c}|^b \le C\epsilon^b, \end{aligned}$$

where the last step follows from the fact that $|\mathbf{c}| \leq 2\epsilon\|\gamma\|_\infty$. Now let us turn to A_{12}, which requires a more delicate estimate of $\mathbf{f}(\mathbf{y}, \mathbf{c})$. Let us break A_{12} as

$$A_{12} = \frac{L\gamma(\alpha)}{2\pi}\int_S (\mathbf{f}(\mathbf{y},\mathbf{c}) - \mathbf{f}(\tilde{\mathbf{y}},\mathbf{c}))d\alpha' + \frac{L\gamma(\alpha)}{2\pi} PV \int_S \mathbf{f}(\tilde{\mathbf{y}},\mathbf{c})d\alpha' =: B_1 + B_2.$$

For B_1, let us take the gradient of $\mathbf{f}(\mathbf{y}, \mathbf{c})$ (as in (8.39)) in the first variable. An elementary computation yields that

$$|\nabla_{\mathbf{x}}\mathbf{f}(\mathbf{x},\mathbf{c})| \leq C|\mathbf{x}|^{-2}\min\left\{1, \frac{|\mathbf{c}|}{|\mathbf{x}|}\right\} \tag{8.40}$$

as long as $\mathbf{x}$ satisfies

$$|\mathbf{x}+\mathbf{c}|^2 \geq c_0(|\mathbf{x}|^2 + |\mathbf{c}|^2). \tag{8.41}$$

We point out that $\mathbf{x} = \xi\mathbf{y} + (1-\xi)\tilde{\mathbf{y}}$ indeed satisfies (8.41) for all $\xi \in [0,1]$: to see this, in the proof of Lemma 8.3.1, if we replace T_1 in (8.13) by $\xi\mathbf{y} + (1-\xi)\tilde{\mathbf{y}}$, one can easily check the proof still goes through for $\xi \in [0,1]$. In addition, for any $\xi \in [0,1]$ we also have

$$|\xi\mathbf{y} + (1-\xi)\tilde{\mathbf{y}}| \geq c_0|\alpha - \alpha'|. \tag{8.42}$$

Thus the gradient estimate (8.40) together with (8.37) and (8.42) yields

$$|f(\mathbf{y},\mathbf{c}) - f(\tilde{\mathbf{y}},\mathbf{c})| \lesssim \min\{1, |\mathbf{c}||\alpha-\alpha'|^{-1}\} \lesssim \min\{1, \epsilon|\alpha-\alpha'|^{-1}\},$$

and plugging this into B_1 gives

$$|B_1| \lesssim \epsilon + \int_{\epsilon<|\alpha-\alpha'|<1} \epsilon|\alpha-\alpha'|^{-1}d\alpha' \lesssim \epsilon|\log\epsilon|.$$

As for B_2, using the definition of $\tilde{\mathbf{y}}$, the identity (8.38) and the fact that $\tilde{\mathbf{y}}\cdot\mathbf{c} = 0$, we have

$$\begin{aligned} B_2 &= \frac{L\gamma(\alpha)}{2\pi} PV \int_S -\frac{\tilde{\mathbf{y}}^\perp}{|\tilde{\mathbf{y}}|^2}\frac{|\mathbf{c}|^2}{|\tilde{\mathbf{y}}+\mathbf{c}|^2}d\alpha' \\ &= \frac{L\gamma(\alpha)|\mathbf{c}|^2\mathbf{n}(z(\alpha))}{2\pi L} PV \int_S \frac{\alpha'-\alpha}{|\alpha'-\alpha|^2(L^2|\alpha'-\alpha|^2 + |\mathbf{c}|^2)}d\alpha'. \end{aligned}$$

For the closed curves $i = 1, \ldots, n$, we immediately have $B_2 = 0$ since $\alpha - \alpha' \in [-\frac{1}{2}, \frac{1}{2})$, and the integrand is an odd function of $\alpha' - \alpha$.

For the open curves $i = n+1, \dots, n+m$, the above integral becomes

$$\begin{aligned} B_2 &= \frac{L\gamma(\alpha)|\mathbf{c}|^2\mathbf{n}(z(\alpha))}{2\pi L} PV \int_{-\alpha}^{1-\alpha} \frac{\theta}{|\theta|^2(L^2|\theta|^2+|\mathbf{c}|^2)} d\theta \quad (\theta := \alpha' - \alpha) \\ &= \frac{L\gamma(\alpha)|\mathbf{c}|^2\mathbf{n}(z(\alpha))}{2\pi L} \int_{\alpha}^{1-\alpha} \frac{\theta}{\theta^2(L^2\theta^2+|\mathbf{c}|^2)} d\theta, \end{aligned}$$

where in the second inequality we used that the integral in $[-\alpha, \alpha]$ gives zero contribution to the principal value, since the integrand is odd.

Next we discuss two cases. If $\alpha > |\mathbf{c}|$, we bound the integrand by $C\theta^{-3}$, which gives

$$|B_2| \le C\gamma(\alpha)|\mathbf{c}|^2\alpha^{-2} \le C|\mathbf{c}|^2\alpha^{b-2} \le C|\mathbf{c}|^b \le C\epsilon^b.$$

where the second inequality follows from the assumption $\gamma(0) = 0$ for an open curve in **(H3)**, as well as the Hölder continuity of γ. And if $0 < \alpha \le |\mathbf{c}|$, the integrand can be bounded above by $\theta^{-1}|\mathbf{c}|^{-2}$, which immediately leads to

$$|B_2| \le C\gamma(\alpha)|\log\alpha| \le C|\mathbf{c}|^b|\log|\mathbf{c}|| \le C\epsilon^b|\log\epsilon|.$$

In both cases we have $|B_2| \le C\epsilon^b|\log\epsilon|$, and combining it with the B_1 and A_{11} estimates gives (8.33).

• *Estimate* (8.34). We break A_2 into

$$\begin{aligned} A_2 &= \frac{L\mathbf{c}^\perp}{2\pi} \int_S \frac{\gamma(\alpha') - \gamma(\alpha)}{|\mathbf{y}+\mathbf{c}|^2} d\alpha' + \frac{L\mathbf{c}^\perp\gamma(\alpha)}{2\pi} \int_S \left(\frac{1}{|\mathbf{y}+\mathbf{c}|^2} - \frac{1}{|\tilde{\mathbf{y}}+\mathbf{c}|^2} \right) d\alpha' + \frac{L\mathbf{c}^\perp\gamma(\alpha)}{2\pi} \int_S \frac{1}{|\tilde{\mathbf{y}}+\mathbf{c}|^2} d\alpha' \\ &=: A_{21} + A_{22} + A_{23}. \end{aligned}$$

For A_{21}, (8.36) and the Hölder continuity of γ immediately lead to

$$|A_{21}| \le C|\mathbf{c}| \int_S \frac{|\alpha-\alpha'|^b}{|\alpha-\alpha'|^2+|\mathbf{c}|^2} d\alpha' \le |\mathbf{c}|^b \le C\epsilon^b. \tag{8.43}$$

For A_{22}, its integrand can be controlled as

$$\left| \frac{1}{|\mathbf{y}+\mathbf{c}|^2} - \frac{1}{|\tilde{\mathbf{y}}+\mathbf{c}|^2} \right| \le \frac{|\mathbf{y}-\tilde{\mathbf{y}}|(|\mathbf{y}+\mathbf{c}|+|\tilde{\mathbf{y}}+\mathbf{c}|)}{|\mathbf{y}+\mathbf{c}|^2|\tilde{\mathbf{y}}+\mathbf{c}|^2} \le \frac{C|\alpha-\alpha'|^2}{(|\alpha-\alpha'|^2+|\mathbf{c}|^2)^{3/2}},$$

where the last step follows from (8.36), (8.37) and (8.38). This allows us to control A_{22} as

$$|A_{22}| \leq C|\mathbf{c}| \int_{-1}^{1} \frac{\theta^2}{(\theta^2 + |\mathbf{c}|^2)^{3/2}} d\theta \leq C|\mathbf{c}| \left| \log |\mathbf{c}| \right| \leq C\epsilon |\log \epsilon|. \tag{8.44}$$

Finally, for the A_{23} term, (8.38) gives

$$A_{23} = \frac{L\mathbf{c}^{\perp}\gamma(\alpha)}{2\pi} \int_S \frac{1}{L^2|\alpha' - \alpha|^2 + |\mathbf{c}|^2} d\alpha' = \frac{\mathbf{n}^{\perp}(\alpha)\gamma(\alpha)}{2\pi} \int_I \frac{1}{\theta^2 + 1} d\theta \quad (\text{set } \theta := \frac{L(\alpha' - \alpha)}{|\mathbf{c}|}),$$

where the integration interval $I = (-\frac{L}{2|\mathbf{c}|}, \frac{L}{2|\mathbf{c}|})$ for $i = 1, \ldots, n$, and $I = (-\frac{L\alpha}{|\mathbf{c}|}, \frac{L(1-\alpha)}{|\mathbf{c}|})$ for $i = n+1, \ldots, n+m$, and in the last equality we also used that $\frac{\mathbf{c}^{\perp}}{|\mathbf{c}|} = \mathbf{n}^{\perp}$. For $i = 1, \ldots, n$, one can easily check that

$$\left| \int_I \frac{1}{\theta^2 + 1} d\theta - \pi \right| = 2 \int_{\frac{L}{2|\mathbf{c}|}}^{\infty} \frac{1}{\theta^2 + 1} d\theta \leq C|\mathbf{c}| \leq C\epsilon,$$

which immediately leads to

$$\left| A_{23} - \frac{\mathbf{n}(z(\alpha))^{\perp}\gamma(\alpha)}{2} \right| = \left| \frac{\mathbf{n}^{\perp}(\alpha)\gamma(\alpha)}{2\pi} \left(\int_I \frac{1}{\theta^2 + 1} d\theta - \pi \right) \right| \leq C\epsilon$$

for $i = 1, \ldots, n$. Next we turn to the open curves $i = n+1, \ldots, n+m$, and let us assume $\alpha \in [0, \frac{1}{2}]$ without loss of generality. In this case we have

$$\left| \int_I \frac{1}{\theta^2 + 1} d\theta - \pi \right| = \int_{-\infty}^{-\frac{L\alpha}{|\mathbf{c}|}} \frac{1}{\theta^2 + 1} d\theta + \int_{\frac{L(1-\alpha)}{|\mathbf{c}|}}^{\infty} \frac{1}{\theta^2 + 1} d\theta \leq \min \left\{ C\frac{|\mathbf{c}|}{\alpha}, \frac{\pi}{2} \right\} + C\epsilon.$$

where we used $1 - \alpha > \frac{1}{2}$ to control the second integral by $C\epsilon$. Using the above inequality as well as the fact that $\gamma(\alpha) \leq C\alpha^b$ due to **(H3)**, we have

$$\left| A_{23} - \frac{\mathbf{n}(z(\alpha))^{\perp}\gamma(\alpha)}{2} \right| = \frac{\gamma(\alpha)}{2\pi} \left| \int_I \frac{1}{\theta^2 + 1} d\theta - \pi \right| \leq C\alpha^b \min \left\{ \frac{|\mathbf{c}|}{\alpha}, 1 \right\} + C\epsilon \leq C(|\mathbf{c}|^b + \epsilon) \leq C\epsilon^b$$

for $i = n+1, \ldots, n+m$. Finally, combining the A_{23} estimates together with (8.43) and (8.44) yields (8.34). □

8.4 Constructing a divergence-free perturbation

In this section, we aim to construct a divergence-free velocity field $\mathbf{u}^\epsilon : D^\epsilon \to \mathbb{R}^2$, such that $-\mathbf{u}^\epsilon$ tends to make each D_i^ϵ "more symmetric". Let $\mathbf{u}^\epsilon : D^\epsilon \to \mathbb{R}^2$ be given by

$$\mathbf{u}^\epsilon := x + \nabla p^\epsilon \quad \text{in } D^\epsilon, \tag{8.45}$$

where the function $p^\epsilon : \overline{D^\epsilon} \to \mathbb{R}$ is chosen such that

$$\nabla \cdot \mathbf{u}^\epsilon = 0 \quad \text{in } D^\epsilon, \tag{8.46}$$

and on each connected component l of ∂D^ϵ, u^ϵ satisfies

$$\int_l \mathbf{u}^\epsilon \cdot n \, d\sigma = 0, \tag{8.47}$$

where n is the unit normal of l pointing outwards of D^ϵ. Note that ∂D^ϵ has a total of $2n+m$ connected components: D_i^ϵ is doubly-connected for $i = 1, \dots, n$ (denote its outer and inner boundaries by $\partial D_{i,\text{out}}^\epsilon$ and $\partial D_{i,\text{in}}^\epsilon$; note that $\partial D_{i,\text{in}}^\epsilon$ coincides with Γ_i), whereas it is simply-connected for $i = n+1, \dots, n+m$ (denote its boundary by ∂D_i^ϵ).

Next we show that there indeed exists a function p^ϵ so that $\mathbf{u}^\epsilon$ satisfies (8.46)–(8.47). Clearly, (8.46) requires that p^ϵ satisfies

$$\Delta p^\epsilon = -2 \quad \text{in } D^\epsilon. \tag{8.48}$$

Once (8.46) is satisfied, the divergence theorem yields that $\mathbf{u}^\epsilon$ satisfies (8.47) for each $l = \partial D_i^\epsilon$ for $i = n+1, \dots, n+m$.

Next let us set the boundary conditions as

$$p^\epsilon|_{\partial D_i^\epsilon} = 0 \quad \text{for } i = n+1, \dots, n+m. \tag{8.49}$$

For $i = 1, \dots, n$, let

$$p^\epsilon = \begin{cases} 0 & \text{on } \partial D_{i,\text{out}}^\epsilon \\ c_i^\epsilon & \text{on } \partial D_{i,\text{in}}^\epsilon = \Gamma_i \end{cases} \qquad \text{for } i = 1, \dots, n, \tag{8.50}$$

where $c_i^\epsilon > 0$ is the unique constant such that

$$\int_{\partial U_i} \nabla p^\epsilon \cdot n d\sigma = -2|U_i| \quad \text{for } i = 1, \dots, n, \tag{8.51}$$

where U_i is the domain enclosed by $\partial D^\epsilon_{i,\text{in}} = \Gamma_i$ (thus U_i is independent of ϵ), and n is the outer normal of U_i (thus the inner normal of D_i^ϵ). The existence of c_i^ϵ is guaranteed by [42, Lemma 2.5]. One can then check that $\int_{\partial U_i} \mathbf{u}^\epsilon \cdot n d\sigma = 0$. Applying the divergence theorem in D_i^ϵ then gives us that $\int_{\partial D^\epsilon_{i,\text{out}}} \mathbf{u}^\epsilon \cdot n d\sigma = 0$ as well, thus $\mathbf{u}^\epsilon$ satisfies (8.47) for $i = 1, \dots, n$.

In [42] we proved a rearrangement inequality for such p^ϵ in a similar spirit of Talenti's rearrangement inequality for elliptic equations [89], which we state below.

Lemma 8.4.1 ([42, Proposition 2.6])**.** *The function $p^\epsilon : \overline{D^\epsilon} \to \mathbb{R}$ defined in* (8.48)–(8.51) *satisfies the following in each D_i^ϵ for $i = 1, \dots, n+m$:*

$$\sup_{D_i^\epsilon} p^\epsilon \le \frac{|D_i^\epsilon|}{2\pi}, \tag{8.52}$$

and

$$\int_{D_i^\epsilon} p^\epsilon(x) dx \le \frac{|D_i^\epsilon|^2}{4\pi}. \tag{8.53}$$

Moreover, each inequality above achieves equality if and only D_i^ϵ is either a disk or an annulus.

Note that the inequalities (8.52)–(8.53) hold for any domain with $C^{1,\alpha}$ boundary. Even though the inequalities are strict when D_i^ϵ is non-radial, they are not strong enough to rule out non-radial vortex sheets, as we need quantitative versions of strict inequalities that are still valid in the $\epsilon \to 0^+$ limit. As we will see in the proof of Proposition 8.5.2, the key step is to show that if some Γ_i is either not a circle or does not have a constant γ_i, then the following quantitative version of (8.53) holds: $\epsilon^{-2}\left(\frac{|D_i^\epsilon|^2}{4\pi} - \int_{D_i^\epsilon} p^\epsilon(x)dx\right) \ge c_0 > 0$, where c_0 is independent of ϵ.

In order to upgrade (8.53) into a quantitative version, we need to obtain some fine estimates for p^ϵ that take into account the shape of the thin domains D_i^ϵ. For $i = n+1, \dots, n+m$, since $p^\epsilon = 0$ on ∂D_i^ϵ, and the domain D_i^ϵ is a thin simply-connected domain with width $\epsilon \ll 1$, intuitively one would expect that $|p^\epsilon| \le C\epsilon^2$. The next proposition shows that this crude estimate is indeed true, and its proof is postponed to Section 8.4.1.

Proposition 8.4.2. *For any $i = n+1, \dots, n+m$, let $p^\epsilon : \overline{D_i^\epsilon} \to \mathbb{R}$ be given by* (8.48)–(8.49)*. Then*

there exist ϵ_1 and C only depending on $\|z_i\|_{C^2(S_i)}, \|\gamma_i\|_{L^\infty(S_i)}$ and F_Γ, such that

$$|p^\epsilon| \le C\epsilon^2 \quad \text{in } D_i^\epsilon$$

for all $\epsilon \in (0, \epsilon_1)$.

For $i = 1, \dots, n$, the estimate is more involved, since p^ϵ takes different values c_i^ϵ and 0 on the inner and outer boundaries of D_i^ϵ. Heuristically speaking, since D_i^ϵ is a doubly-connected thin tubular domain with width $\sim \epsilon$, we would expect that p_i^ϵ (in α, η coordinate) changes almost linearly from 0 to c_i^ϵ as η goes from -1 (outer boundary) to 0 (inner boundary). Next we will show that the error between $p^\epsilon(R_i^\epsilon(\alpha, \eta))$ and the linear-in-η function $c_i^\epsilon(1+\eta)$ is indeed controlled by $O(\epsilon^2)$. We will also obtain fine estimates of the gradient of the function $c_i^\epsilon(1+\eta)$, as well as the boundary value c_i^ϵ. Again, its proof is postponed to Section 8.4.1.

Proposition 8.4.3. *For any $i = 1, \dots, n$, let $p^\epsilon : \overline{D_i^\epsilon} \to \mathbb{R}$ and $c_i^\epsilon \in \mathbb{R}$ be given by* (8.48) *and* (8.50)–(8.51)*. For such p^ϵ, let us define $\tilde{p}^\epsilon$,$q^\epsilon : \overline{D_i^\epsilon} \mapsto \mathbb{R}$ as follows:*

$$\begin{aligned} \tilde{p}^\epsilon(R_i^\epsilon(\alpha, \eta)) &:= c_i^\epsilon(1+\eta) \quad \text{for } \alpha \in S_i, \eta \in [0, -1], \\ q^\epsilon &:= p^\epsilon - \tilde{p}^\epsilon \qquad \text{in } \overline{D_i^\epsilon}. \end{aligned} \tag{8.54}$$

Also let

$$\beta_i := \frac{2|U_i|}{L_i \int_{S_i} \gamma_i^{-1}(\alpha) d\alpha}. \tag{8.55}$$

Then there exist ϵ_1 and C only depending on $\|z_i\|_{C^3(S_i)}, \|\gamma_i\|_{C^2(S_i)}$ and F_Γ, such that for all $\epsilon \in (0, \epsilon_1)$ we have the following:

$$\begin{cases} |q^\epsilon| \le C\epsilon^2 & \text{in } D_i^\epsilon, \\ q^\epsilon = 0 & \text{on } \partial D_i^\epsilon, \end{cases} \tag{8.56}$$

$$\left| \frac{c_i^\epsilon}{\epsilon} - \beta_i \right| \le C\epsilon, \tag{8.57}$$

$$\left| \nabla \tilde{p}^\epsilon(R_i^\epsilon(\alpha, \eta)) - \frac{\beta_i}{\gamma_i(\alpha)} \mathbf{n}(z_i(\alpha)) \right| \le C\epsilon \quad \text{for } \alpha \in S_i, \eta \in [0, -1]. \tag{8.58}$$

8.4.1 Proof of the quantitative lemmas for p^ϵ

In this subsection we aim to prove Propositions 8.4.2 and 8.4.3. We start with a technical lemma on estimating the solution of Poisson's equation (with zero boundary condition) in the domain D_i^ϵ.

Lemma 8.4.4. *For any $i = 1, \dots, n+m$, assume Γ_i and γ_i satisfy* **(H1)**–**(H3)**. *Let $v^\epsilon \in C^2(D_i^\epsilon) \cap C(\overline{D_i^\epsilon})$ solve the Poisson's equation with zero boundary condition:*

$$\begin{cases} \Delta v^\epsilon = -1 & \text{in } D_i^\epsilon, \\ v^\epsilon = 0 & \text{on } \partial D_i^\epsilon. \end{cases} \tag{8.59}$$

Then there exist positive constants $\epsilon_0 = C(\|z_i\|_{C^2(S_i)}, \|\gamma_i\|_{L^\infty(S_i)}, F_\Gamma)$ and $C_1, C_2 = C(\|\gamma_i\|_{L^\infty(S_i)})$, such that for all $\epsilon \in (0, \epsilon_0)$ we have

$$0 \le v^\epsilon \le C_1 \epsilon^2 \quad \text{in } D_i^\epsilon \tag{8.60}$$

and

$$\|\nabla v^\epsilon\|_{L^\infty(\Gamma_i)} \le C_2 \epsilon \quad \text{for } i = 1, \dots, n. \tag{8.61}$$

Proof. Throughout the proof, let $i \in \{1, \dots, n+m\}$ be fixed. For notational simplicity, in the rest of the proof we omit the subscript i in R_i^ϵ, D_i^ϵ, S_i, z_i and γ_i.

Step 1. We start with a simple geometric result that D^ϵ is "flat" in a small neighborhood of any interior $z(\alpha)$. For any $\alpha \in S^\circ$, let $V^\epsilon(\alpha) := D^\epsilon \cap B_{6\epsilon\|\gamma\|_\infty}(z(\alpha))$, where $\|\cdot\|_\infty$ denotes $\|\cdot\|_{L^\infty(S)}$. We will show that any $y \in V^\epsilon(\alpha)$ satisfies

$$\big|(z(\alpha) - y) \cdot \mathbf{n}(z(\alpha))\big| \le 2\epsilon\|\gamma\|_\infty \tag{8.62}$$

for all sufficiently small $\epsilon > 0$ (to be quantified in (8.67)). See Figure 8.4(a) for an illustration.

Since $y \in V^\epsilon(\alpha) \subset D^\epsilon$, there exist $\beta \in S$ and $\eta \in (-1, 0)$ such that $y = R^\epsilon(\beta, \eta) = z(\beta) + \epsilon\gamma(\beta)\mathbf{n}(z(\beta))\eta$. It follows that

$$\begin{aligned} \big|(z(\alpha) - y) \cdot \mathbf{n}(z(\alpha))\big| &\le |(z(\alpha) - z(\beta)) \cdot \mathbf{n}(z(\alpha))| + \epsilon\|\gamma\|_\infty \\ &\le \|z''\|_\infty(\alpha - \beta)^2 + \epsilon\|\gamma\|_\infty, \end{aligned} \tag{8.63}$$

where in the second inequality we used

$$|(z(\alpha) - z(\beta)) - z'(\alpha)(\alpha - \beta)| \le \|z''\|_\infty(\alpha - \beta)^2 \tag{8.64}$$

and $z'(\alpha) \cdot \mathbf{n}(z(\alpha)) = 0$. To bound $\alpha - \beta$ on the right hand side of (8.63), the fact that $y \in$

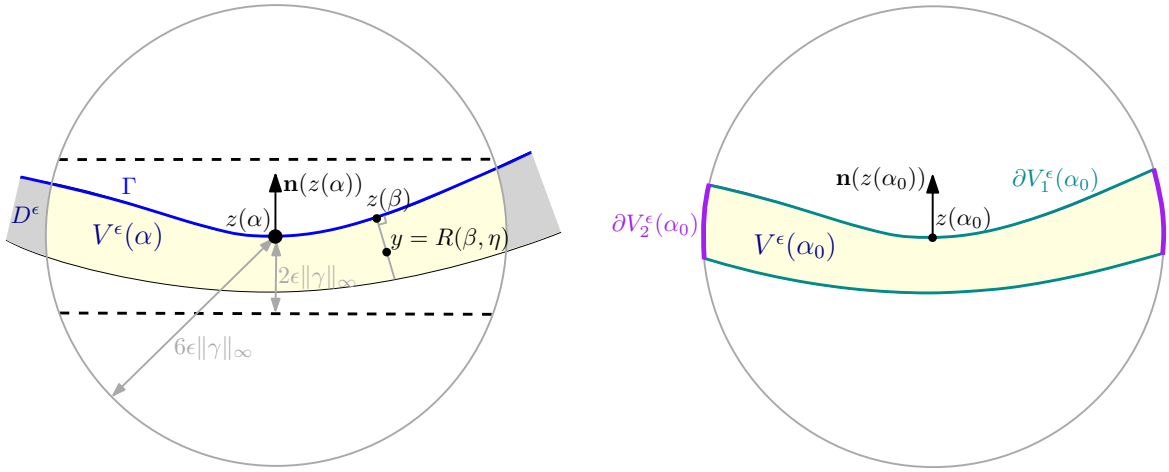


Figure 8.4: (a) In Step 1, $V^\epsilon(\alpha)$ (the yellow set) must lie between the two dashed lines for small ϵ. (b) In Step 2, $\partial V^\epsilon(\alpha_0)$ is decomposed into $\partial V_1^\epsilon(\alpha_0)$ (in dark green) and $\partial V_2^\epsilon(\alpha_0)$ (in purple).

$B_{6\epsilon\|\gamma\|_\infty}(z(\alpha))$ gives

$$6\epsilon\|\gamma\|_\infty \geq |z(\alpha) - y| \geq |z(\alpha) - z(\beta)| - \epsilon\gamma(\beta), \tag{8.65}$$

which implies $|z(\alpha) - z(\beta)| \leq 7\epsilon\|\gamma\|_\infty$. Since the arc-chord constant F_Γ given in (8.2) is finite, this implies

$$|\alpha - \beta| \leq 7F_\Gamma\|\gamma\|_\infty\epsilon. \tag{8.66}$$

Plugging this into the right hand side of (8.63), we know (8.62) holds for all

$$0 < \epsilon \leq (49\|z''\|_\infty F_\Gamma^2\|\gamma\|_\infty)^{-1}. \tag{8.67}$$

Step 2. Next we prove (8.60). Note that v^ϵ is superharmonic in D^ϵ and vanishes on the boundary, thus it follows from the maximum principle that $v^\epsilon \geq 0$ in D^ϵ. Denote $M := \max_{x\in D^\epsilon} v^\epsilon(x)$, and pick $x_0 = R(\alpha_0, \eta_0) \in D^\epsilon$ such that $v^\epsilon(x_0) = M$. Note that $\alpha_0 \in S^\circ$. Without loss of generality, we can assume that $z(\alpha_0) = (0, 0)$ and $\mathbf{s}(z(\alpha_0)) = \mathbf{e}_1 := (1, 0)$, so that $\mathbf{n}(z(\alpha_0)) = (0, 1)$ and $x_0 = (0, \epsilon\gamma(\alpha_0)\eta_0)$. Let us consider a barrier function $b_1 : \mathbb{R}^2 \mapsto \mathbb{R}$ given by

$$b_1(x_1, x_2) = x_2^2 - \frac{x_1^2}{2}.$$

Clearly $\Delta b_1 = 1$, so $v^\epsilon + b_1$ is harmonic in D^ϵ. It then follows from the maximum principle that $\max_{\overline{V^\epsilon(\alpha_0)}}(v^\epsilon + b_1)$ is achieved at some boundary point $\tilde{x}_0 \in \partial V^\epsilon(\alpha_0)$. Let us break $\partial V^\epsilon(\alpha_0)$ into

$\partial V_1^\epsilon(\alpha_0) \cup \partial V_2^\epsilon(\alpha_0)$ (see Figure 8.4(b) for an illustration), given by

$$\partial V_1^\epsilon(\alpha_0) := \partial D^\epsilon \cap B_{6\epsilon\|\gamma\|_\infty}(z(\alpha_0)), \quad \partial V_2^\epsilon(\alpha_0) := \overline{D^\epsilon} \cap \partial B_{6\epsilon\|\gamma\|_\infty}(z(\alpha_0)). \tag{8.68}$$

We claim that $\tilde{x}_0 \in \partial V_1^\epsilon(\alpha_0)$. To see this, note that any $y = (y_1, y_2) \in \partial V_2^\epsilon(\alpha_0)$ satisfies $|y| = 6\epsilon\|\gamma\|_\infty$ and $|y_2| \le 2\epsilon\|\gamma\|_\infty$, where the latter follows from (8.62) and our assumptions that $\mathbf{s}(z(\alpha_0)) = \mathbf{e}_1$ and $z(\alpha_0) = (0,0)$. This implies that $|y_1| \ge 4\epsilon\|\gamma\|_\infty > |y_2|$, thus $b_1(y) < 0$. Using that $v^\epsilon(x_0) = M \ge v^\epsilon(y)$ and $b_1(x_0) = b_1(0, \epsilon\gamma(\alpha_0)\eta_0) \ge 0$, we have $(v^\epsilon + b_1)(y) < (v^\epsilon + b_1)(x_0)$. This shows that $\max_{\overline{V^\epsilon(\alpha_0)}}(v^\epsilon + b_1)$ cannot be achieved on $\partial V_2^\epsilon(\alpha_0)$, finishing the proof of the claim.

Since $\tilde{x}_0 \in \partial V_1^\epsilon(\alpha_0) \subset \partial D^\epsilon$, the boundary condition in (8.59) yields that $v^\epsilon(\tilde{x}_0) = 0$. Thus

$$M + b_1(x_0) = v^\epsilon(x_0) + b_1(x_0) \le v^\epsilon(\tilde{x}_0) + b_1(\tilde{x}_0) = b_1(\tilde{x}_0).$$

Using $b_1(x_0) = b_1(0, \epsilon\gamma(\alpha_0)\eta_0) \ge 0$, the above inequality becomes

$$M \le b_1(\tilde{x}_0) \le |\tilde{x}_0|^2 \le 36\|\gamma\|_\infty^2 \epsilon^2, \tag{8.69}$$

where the second inequality follows from the definition of b_1. This proves (8.60) for $C_1 = 36\|\gamma\|_\infty^2$.

Step 3. It remains to prove (8.61). First note that for $i \in \{1, \dots, n\}$, the assumptions **(H1)**–**(H3)** yield that D_i^ϵ has C^2 boundary, therefore $v^\epsilon \in C^2(D_i^\epsilon) \cap C^1(\overline{D_i^\epsilon})$. Let us fix $i \in \{1, \dots, n\}$ and any $\alpha \in S$, and we aim to show that $|\nabla v^\epsilon(z(\alpha))| \le C_2\epsilon$. Again, without loss of generality we can assume that $z(\alpha) = (0,0)$ and $\mathbf{s}(z(\alpha)) = \mathbf{e}_1$. Let us consider a new barrier function $b_2 : \mathbb{R}^2 \to \mathbb{R}$

$$b_2(x_1, x_2) := x_2^2 + 4\epsilon\|\gamma\|_\infty x_2 - \frac{x_1^2}{2}, \tag{8.70}$$

which satisfies $b_2(0,0) = 0$, and one can easily check that its zero level set has horizontal tangent at $(0,0)$ (thus tangent to ∂D^ϵ at $z(\alpha)$).

Again, let us decompose $\partial V^\epsilon(\alpha)$ as $\partial V_1^\epsilon(\alpha) \cup \partial V_2^\epsilon(\alpha)$ as in (8.68) (except that α_0 now becomes α). We claim that for all sufficiently small $\epsilon > 0$, the new barrier function b_2 satisfies

$$\Delta b_2 = 1 \quad \text{in } V^\epsilon(\alpha), \tag{8.71}$$

$$b_2 \le 0 \quad \text{on } \partial V_1^\epsilon, \tag{8.72}$$

$$b_2 \le -\epsilon^2 \quad \text{on } \partial V_2^\epsilon. \tag{8.73}$$

Let us assume for a moment that (8.71)–(8.73) are true. Then it follows that

$$v^\epsilon + C_2 b_2 \le 0 \text{ in } V^\epsilon(\alpha), \tag{8.74}$$

where $C_2 := \max\{1, C_1\}$ and C_1 is as in (8.60) (in the end of step 2 we have $C_1 = 36\|\gamma\|_\infty^2$). To show (8.74), note that $v^\epsilon + C_2 b_2$ is subharmonic in $V^\epsilon(\alpha)$ due to (8.71) and the definition of C_2, thus its maximum is attained on its boundary. The boundary conditions in (8.59) and (8.72) yield that $v^\epsilon + C_2 b_2 \le 0$ on $\partial V_1^\epsilon(\alpha)$; whereas (8.60), (8.73) and the definition of C_2 yield that $v^\epsilon + C_2 b_2 \le 0$ on $\partial V_2^\epsilon(\alpha)$. Thus $v^\epsilon + C_2 b_2 \le 0$ on $\partial V_1^\epsilon(\alpha) \cup \partial V_2^\epsilon(\alpha)$, implying (8.74).

However, $v^\epsilon + C_2 b_2$ is actually zero at $z(\alpha) \in \partial V^\epsilon(\alpha)$, therefore Hopf's Lemma implies that $\nabla\left(v^\epsilon + C_2 b_2\right)(z(\alpha)) \cdot \vec{n}(z(\alpha)) > 0$, where $\vec{n}(z(\alpha))$ is the outer normal of ∂D^ϵ at $z(\alpha)$. Hence

$$|\nabla v^\epsilon(z(\alpha))| = -\nabla v^\epsilon(z(\alpha)) \cdot \vec{n}(z(\alpha)) < C_2 \nabla b_2(z(\alpha)) \cdot \vec{n}(z(\alpha)) = 4C_2\|\gamma\|_\infty \epsilon, \tag{8.75}$$

where the first equality follows from the fact that v^ϵ is superharmonic in D^ϵ and constant on ∂D^ϵ, and the second equality is a direct computation of ∇b_2. Thus (8.75) proves (8.61).

To complete the proof, we only need to prove (8.71)–(8.73) for small $\epsilon > 0$. Note that (8.71) follows immediately from computing the Laplacian of b_2. For (8.72), let us pick $y \in \partial V_1^\epsilon(\alpha)$, and we aim to show that $b_2(y) \le 0$. Note that $y = R^\epsilon(\beta, 0)$ or $R^\epsilon(\beta, -1)$ for some $\beta \in S$. We first deal with the first case.

Let us denote $y = (y_1, y_2)$. Rewriting (8.64) into two inequalities for the two components, and using that $z(\alpha) = (0,0)$ and $z'(\alpha) = L\mathbf{e}_1$ (L is the length of the curve Γ_i), we have

$$|0 - y_1 - L(\alpha - \beta)| \le \|z''\|_\infty (\alpha - \beta)^2 \tag{8.76}$$

$$|y_2| = |0 - y_2| \le \|z''\|_\infty (\alpha - \beta)^2. \tag{8.77}$$

Also, (8.66) gives $|\alpha - \beta| \le 7F_\Gamma \|\gamma\|_\infty \epsilon$. Applying it to (8.76), for all $\epsilon > 0$ sufficiently small we have that

$$|y_1| \ge \frac{L}{2}|\alpha - \beta|. \tag{8.78}$$

Plugging (8.78) and (8.77) into $b_2(y) = -\frac{1}{2}y_1^2 + y_2^2 + 4\epsilon\|\gamma\|_\infty y_2$, we have

$$\begin{aligned} b_2(y) &\le -\frac{L^2}{8}(\alpha - \beta)^2 + \|z''\|_\infty^2 (\alpha - \beta)^4 + 4\epsilon\|\gamma\|_\infty \|z''\|_\infty (\alpha - \beta)^2 \\ &\le \left(-\frac{L^2}{8} + C\epsilon^2 + C\epsilon\right)(\alpha - \beta)^2 \le 0, \end{aligned}$$

for all $\epsilon > 0$ sufficiently small, where the second inequality follows from (8.66). This finishes the proof of (8.72) for the case $y = R^\epsilon(\beta, 0)$.

Before we deal with the case $y = R^\epsilon(\beta, -1)$, let us prove (8.73) first. For any $y = (y_1, y_2) \in \partial V_2^\epsilon(\alpha)$, (8.62) gives $|y_2| \le 2\epsilon\|\gamma\|_\infty$. Combining this with $|y| = 6\epsilon\|\gamma\|_\infty$ yields $|y_1| \ge \sqrt{32}\epsilon\|\gamma\|_\infty$. Thus

$$b_2(y) \le (2\epsilon\|\gamma\|_\infty)^2 + 4\epsilon\|\gamma\|_\infty(2\epsilon\|\gamma\|_\infty) - \frac{(\sqrt{32}\epsilon\|\gamma\|_\infty)^2}{2} \le -4\epsilon^2\|\gamma\|_\infty^2.$$

Finally we turn to the proof of (8.72) for the case $y = R^\epsilon(\beta, -1)$. Note that the curve $\{R^\epsilon(\beta, -1) : \beta \in S\} \cap B_{6\epsilon\|\gamma\|_\infty}(z(\alpha))$ lies in the interior of the region bounded by $\Gamma \cap B_{6\epsilon\|\gamma\|_\infty}(z(\alpha))$ on the top, $\partial B_{6\epsilon\|\gamma\|_\infty}(z(\alpha))$ on the sides, and $y_2 = -2\epsilon\|\gamma\|_\infty$ on the bottom. (The last one follows from (8.62) and our assumption that $\mathbf{s}(z(\alpha)) = \mathbf{e}_1$). We have already shown $b_2 \le 0$ on $\Gamma \cap B_{6\epsilon\|\gamma\|_\infty}(z(\alpha))$ and the lateral boundaries, and it is easy to check that $b_2 \le 0$ on $y_2 = -2\epsilon\|\gamma\|_\infty$. Since the set $\{b_2 \le 0\}$ is simply-connected, it implies that $b_2 \le 0$ in the interior of this region, finishing the proof. □

Note that (8.60) of Lemma 8.4.4 immediately implies Proposition 8.4.2. (The only difference is that $\Delta v^\epsilon = -1$ in Lemma 8.4.4 whereas $\Delta p^\epsilon = -2$ in Proposition 8.4.2, so the constant C in Proposition 8.4.2 is twice of that in (8.60)). The lemma also implies the following corollary, which will be helpful in the proof of Proposition 8.4.3.

Corollary 8.4.5. *For any $i = 1, \dots, n+m$, assume Γ_i and γ_i satisfy* **(H1)**–**(H3)**. *Assume $v^\epsilon \in C^2(D_i^\epsilon) \cap C(\overline{D_i^\epsilon})$ satisfies that*

$$\begin{cases} |\Delta v^\epsilon| \le C_0 & \text{in } D_i^\epsilon, \\ v^\epsilon = 0 & \text{on } \partial D_i^\epsilon, \end{cases}$$

for some constant $C_0 > 0$. Then for the same constants ϵ_0, C_1, C_2 as in Lemma 8.4.4, the following holds for all $\epsilon \in (0, \epsilon_0)$:

$$|v^\epsilon| \le C_0 C_1 \epsilon^2 \quad \text{in } D_i^\epsilon, \tag{8.79}$$

and if $v^\epsilon \in C^2(D_i^\epsilon) \cap C^1(\overline{D_i^\epsilon})$, we have

$$\|\nabla v^\epsilon\|_{L^\infty(\Gamma_i)} \le C_0 C_2 \epsilon \quad \text{for} \quad i = 1, \dots, n. \tag{8.80}$$

Proof. Let $\tilde{v}$ be a solution to

$$\begin{cases} \Delta \tilde{v} = -C_0 & \text{in } D_i^\epsilon, \\ \tilde{v} = 0 & \text{on } \partial D_i^\epsilon. \end{cases}$$

It is clear that $v^\epsilon + \tilde{v}$ is super-harmonic and $v^\epsilon - \tilde{v}$ is sub-harmonic in D_i^ϵ, and they both vanish on the boundary. Thus the maximum principle implies that

$$-\tilde{v} \le v^\epsilon \le \tilde{v} \quad \text{in } D_i^\epsilon. \tag{8.81}$$

Applying (8.60) of Lemma 8.4.4 to $\frac{\tilde{v}}{C_0}$, we obtain $0 \le \tilde{v} \le C_0 C_1 \epsilon^2$ in D_i^ϵ for all $\epsilon \in (0, \epsilon_0)$, leading to (8.79). Furthermore, (8.81) and the fact that v^ϵ and v both have zero boundary condition imply that

$$|\nabla v^\epsilon| \le |\nabla \tilde{v}| \quad \text{on } \partial D_i^\epsilon.$$

We then apply (8.61) of Lemma 8.4.4 to $\frac{\tilde{v}}{C_0}$ and obtain $\|\nabla v^\epsilon\|_{L^\infty(\Gamma_i)} \le C_0 C_2 \epsilon$, which proves (8.80). □

Now we are ready to prove Proposition 8.4.3.

Proof of Proposition 8.4.3. Throughout the proof, let $i \in \{1, \dots, n\}$ be fixed. For notational simplicity, in the rest of the proof we omit the subscript i from all terms.

We claim that

$$\left| \nabla \tilde{p}^\epsilon(R^\epsilon(\alpha, \eta)) - \frac{c^\epsilon}{\epsilon \gamma(\alpha)} \mathbf{n}(z(\alpha)) \right| \le C\epsilon \quad \text{for all } \alpha \in S, \eta \in [0, -1], \tag{8.82}$$

$$\|\Delta q^\epsilon\|_{L^\infty(D^\epsilon)} \le C \tag{8.83}$$

for some constant $C > 0$ only depending on $\|z_i\|_{C^3(S_i)}, \|\gamma_i\|_{C^2(S_i)}$ and F_Γ. Assuming these are true, let us explain how they lead to (8.56)–(8.58). By (8.50) and (8.54), p^ϵ and $\tilde{p}^\epsilon$ have the same boundary condition, thus $q^\epsilon = 0$ on ∂D^ϵ. This and (8.83) allow us to apply Corollary 8.4.5 to q^ϵ to obtain the estimate (8.79), implying (8.56).

Due to (8.80) of Corollary 8.4.5, we also have

$$\|\nabla q^\epsilon\|_{L^\infty(\Gamma)} \le C\epsilon. \tag{8.84}$$

Using (8.51) and $p^\epsilon = \tilde{p}^\epsilon + q^\epsilon$, we have

$$\begin{aligned} -2|U| &= \int_{\partial U} \nabla \tilde{p}^\epsilon \cdot n d\sigma + \int_{\partial U} \nabla q^\epsilon \cdot n d\sigma \\ &= -\frac{c^\epsilon L}{\epsilon} \int_S \gamma^{-1}(\alpha) d\alpha + O(\epsilon), \end{aligned}$$

where the second equality follows from (8.82) for $\eta = 0$, $n(z(\alpha)) = -\mathbf{n}(z(\alpha))$ and $d\sigma = Ld\alpha$, as well as (8.84). Rearranging the terms and using the definition of β in (8.55) yields (8.57).

Finally, note that (8.57) and (8.82) directly lead to (8.58), where we are using the fact that γ_i is uniformly positive for $i = 1, \dots, n$, due to **(H3)**.

The rest of the proof is devoted to proving the claims (8.82) and (8.83). For (8.82), we compute the gradient of $\tilde{p}^\epsilon$. Differentiating (8.54) with respect to α and η, we obtain

$$(\nabla_{\alpha,\eta} R^\epsilon(\alpha,\eta))^t \nabla \tilde{p}(R^\epsilon(\alpha,\eta)) = \begin{pmatrix} 0 \\ c^\epsilon \end{pmatrix}, \tag{8.85}$$

where $(\nabla_{\alpha,\eta} R^\epsilon)^t$ denotes the transpose of the Jacobian matrix of R^ϵ. Since $\nabla_{\alpha,\eta} R^\epsilon = (\partial_\alpha R^\epsilon, \partial_\eta R^\epsilon)$, using the formula for inverses of 2×2 matrices, we have

$$\left((\nabla_{\alpha,\eta} R^\epsilon)^t\right)^{-1} = \frac{1}{J(\alpha,\eta)} \big(-(\partial_\eta R^\epsilon)^\perp, (\partial_\alpha R^\epsilon)^\perp\big). \tag{8.86}$$

where $J(\alpha,\eta) := \det(\nabla_{\alpha,\eta} R^\epsilon)$. Multiplying the inverse matrix on both sides of (8.85), we have

$$\nabla \tilde{p}^\epsilon(R^\epsilon(\alpha,\eta)) = \frac{1}{J}\big(-(\partial_\eta R^\epsilon)^\perp, (\partial_\alpha R^\epsilon)^\perp\big) \begin{pmatrix} 0 \\ c^\epsilon \end{pmatrix} = \frac{c^\epsilon}{J} (\partial_\alpha R^\epsilon)^\perp. \tag{8.87}$$

Recall that Lemma 8.3.2 gives $(\partial_\alpha R^\epsilon)^\perp = z'(\alpha)^\perp + O(\epsilon) = L\mathbf{n}(z(\alpha)) + O(\epsilon)$, and $J = \epsilon L\gamma + O(\epsilon^2)$. Plugging these into (8.87) gives

$$\nabla \tilde{p}^\epsilon(R(\alpha,\eta)) = \frac{c^\epsilon}{\epsilon} \left(\frac{\mathbf{n}(z(\alpha))}{\gamma} + O(\epsilon) \right). \tag{8.88}$$

Note that it follows from (8.52) that $c^\epsilon \le \frac{|D^\epsilon|}{2\pi}$, where $|D^\epsilon| \le C\epsilon$ due to (8.19). These imply

$$\frac{c_\epsilon}{\epsilon} \le C, \tag{8.89}$$

and applying it to (8.88) yields (8.82).

To prove (8.83), since $q^\epsilon = p^\epsilon - \tilde{p}^\epsilon$ and $\Delta p^\epsilon = -2$ in D^ϵ, it suffices to show that

$$|\Delta \tilde{p}^\epsilon| \le C \quad \text{in } D^\epsilon, \tag{8.90}$$

and we will begin with an explicit computation of $\partial_{x_1 x_1} \tilde{p}^\epsilon$ and $\partial_{x_2 x_2} \tilde{p}^\epsilon$. Let us denote $R^\epsilon =: (R^1, R^2)$.

For notational simplicity, in the rest of the proof we will use subscripts on R^ϵ, R^1, R^2 and J to denote their partial derivative, e.g. $R^1_\alpha := \partial_\alpha R^1$.

From (8.87), it follows that

$$\partial_{x_1}\tilde{p}^\epsilon(R^\epsilon(\alpha,\eta)) = -\frac{c^\epsilon}{J}R^2_\alpha.$$

Differentiating in α and η, we get

$$\begin{aligned}\nabla\left(\partial_{x_1}\tilde{p}^\epsilon\right)\left(R^\epsilon(\alpha,\eta)\right) &= \left(\left(\nabla_{\alpha,\eta}R^\epsilon\right)^t\right)^{-1}\nabla_{\alpha,\eta}\left(-\frac{c^\epsilon}{J}R^2_\alpha\right)\\ &= \frac{c^\epsilon}{J}\begin{pmatrix} R^2_\eta & -R^2_\alpha \\ -R^1_\eta & R^1_\alpha\end{pmatrix}\begin{pmatrix}\frac{J_\alpha}{J^2}R^2_\alpha - \frac{1}{J}R^2_{\alpha\alpha}\\ \frac{J_\eta}{J^2}R^2_\alpha - \frac{1}{J}R^2_{\alpha\eta}\end{pmatrix},\end{aligned}$$

thus

$$\partial_{x_1x_1}\tilde{p}^\epsilon(R(\alpha,\eta)) = \frac{c^\epsilon}{J}\left(\frac{J_\alpha}{J^2}R^2_\eta R^2_\alpha - \frac{1}{J}R^2_\eta R^2_{\alpha\alpha} - \frac{J_\eta}{J^2}(R^2_\alpha)^2 + \frac{1}{J}R^2_\alpha R^2_{\alpha\eta}\right).$$

Likewise, $\partial_{x_2x_2}\tilde{p}(R(\alpha,\eta))$ takes the same expression except every R^2 is changed into R^1. Adding them together gives

$$\Delta\tilde{p}^\epsilon(R(\alpha,\eta)) = \frac{c^\epsilon}{J}\left(\frac{J_\alpha}{J^2}R^\epsilon_\eta\cdot R^\epsilon_\alpha - \frac{1}{J}R^\epsilon_\eta\cdot R^\epsilon_{\alpha\alpha} - \frac{J_\eta}{J^2}R^\epsilon_\alpha\cdot R^\epsilon_\alpha + \frac{1}{J}R^\epsilon_\alpha\cdot R^\epsilon_{\alpha\eta}\right). \tag{8.91}$$

Using the explicit formulae of R_α, R_η and J in Lemma 8.3.2, we directly obtain $|R^\epsilon_\alpha|, |R^\epsilon_{\alpha\alpha}| \le C$; $|R^\epsilon_\eta|, |R^\epsilon_{\alpha\eta}|, |J_\alpha| \le C\epsilon$; $|J_\eta| \le C\epsilon^2$; and $J^{-1} \le C\epsilon^{-1}$ when ϵ is sufficiently small, where C depends on $\|z_i\|_{C^3(S_i)}$ and $\|\gamma_i\|_{C^2(S_i)}$. As a result, all the four terms in the parenthesis of (8.91) are bounded by some constant C independent of ϵ. Finally, (8.89) yields $\frac{c_\epsilon}{J} \le C$ as well, thus $|\Delta\tilde{p}^\epsilon| \le C$, and this proves the second claim (8.83). □

8.5 Proof of the symmetry result

In this section we prove that a stationary vortex sheet with positive vorticity must be radially symmetric up to a translation, and a rotating vortex sheet with positive vorticity and angular velocity $\Omega < 0$ must be radially symmetric. The key idea of the proof is to define the integral

$$\begin{aligned}I^\epsilon &:= \int_{D^\epsilon}\epsilon^{-1}\mathbf{u}^\epsilon\cdot\nabla\left(\omega^\epsilon * \mathcal{N} - \frac{\Omega}{2}|x|^2\right)dx\\ &= \int_{D^\epsilon}\epsilon^{-1}(x+\nabla p^\epsilon)\cdot\nabla\left(\omega^\epsilon * \mathcal{N} - \frac{\Omega}{2}|x|^2\right)dx,\end{aligned} \tag{8.92}$$

and compute it in two different ways. The motivation of the definition is as follows. As discussed in [42, Section 2.1], I^ϵ can be thought of as a first variation of an "energy functional"

$$\mathcal{E}[\omega^\epsilon] := \int \frac{1}{2}\omega^\epsilon(\omega^\epsilon * \mathcal{N}) - \frac{\Omega}{2}\omega^\epsilon|x|^2\,dx$$

when we perturb ω^ϵ by a divergence free vector $\mathbf{u}^\epsilon$ in D^ϵ. (This functional $\mathcal{E}$ only serves as our motivation, and will not appear in the proof.) On the one hand, using that ω_0 is stationary in the rotating frame with angular velocity Ω and ω^ϵ is a close approximation of ω_0, we will show in Proposition 8.5.1 that I^ϵ is of order $O(\epsilon|\log\epsilon|)$, thus goes to zero as $\epsilon \to 0$. On the other hand, using the particular $\mathbf{u}^\epsilon$ that we constructed in Section 8.4, we will prove in Proposition 8.5.2 that if $\Omega = 0$, I^ϵ is strictly positive independently of ϵ unless all the vortex sheets are nested circles with constant density; and also prove a similar result in Corollary 8.5.3 for $\Omega < 0$.

Proposition 8.5.1. *Assume $\omega(\cdot,t) = \omega_0(R_{\Omega t}\cdot)$ is a stationary/uniformly-rotating vortex sheet with angular velocity $\Omega \in \mathbb{R}$, where ω_0 satisfies* **(H1)**–**(H3)**. *Then there exists some $C > 0$ only depending on b (as in* **(H2)**), $\max_i \|z_i\|_{C^3(S_i)}$, $\max_{i\le n}\|\gamma_i\|_{C^2(S_i)}$, $\max_{i>n}\|\gamma_i\|_{C^b(S_i)}$, d_Γ *and* F_Γ, *such that $|I^\epsilon| < C\epsilon^b|\log\epsilon|$ for all sufficiently small $\epsilon > 0$.*

Proof. Let us decompose $I^\epsilon =: \sum_{i=1}^{n+m} I_i^\epsilon$, where $I_i^\epsilon := \int_{D_i^\epsilon} \epsilon^{-1}(x + \nabla p^\epsilon)\cdot\nabla(\omega^\epsilon * \mathcal{N} - \frac{\Omega}{2}|x|^2)dx$.

We start with showing that $|I_i^\epsilon| \le C\epsilon^b|\log\epsilon|$ for $i = n+1,\dots,n+m$. For such i, $p^\epsilon = 0$ on ∂D_i^ϵ, thus the divergence theorem (and the fact that $\omega^\epsilon = \epsilon^{-1}$ in D_i^ϵ) gives

$$I_i^\epsilon = \underbrace{\int_{D_i^\epsilon} \epsilon^{-1} x\cdot\nabla\left(\omega^\epsilon * \mathcal{N} - \frac{\Omega}{2}|x|^2\right)dx}_{=:T_i^\epsilon} - \int_{D_i^\epsilon}\epsilon^{-1}(\epsilon^{-1} - 2\Omega)p^\epsilon(x)dx.$$

Using the estimate $|p^\epsilon| \le C\epsilon^2$ in Proposition 8.4.2 and the fact that $|D_i^\epsilon| \le C\epsilon$ from (8.19), we easily bound the second integral by $C\epsilon$. To control the first integral T_i^ϵ, we rewrite it using the change of variables $x = R_i^\epsilon(\alpha,\eta)$ and the definition $\mathbf{v}^\epsilon := \nabla^\perp(\omega^\epsilon * \mathcal{N})$ in (8.20): (also note that on the right hand side we group ϵ^{-1} with the determinant)

$$T_i^\epsilon = \int_{S_i}\int_{-1}^{0} R_i^\epsilon(\alpha,\eta)\cdot\underbrace{\left(-(\mathbf{v}^\epsilon)^\perp(R_i^\epsilon(\alpha,\eta)) - \Omega R_i^\epsilon(\alpha,\eta)\right)}_{=:J_i^\epsilon}\underbrace{\epsilon^{-1}\det(\nabla_{\alpha,\eta}R_i^\epsilon(\alpha,\eta))}_{=:K_i^\epsilon}\,d\eta d\alpha,$$

and recall that the exact expression of the determinant was given in (8.18). Let us take a closer look at the integrand, which is a product of 3 terms. Clearly, the definition of R_i^ϵ gives $R_i^\epsilon(\alpha,\eta) =$

$z_i(\alpha) + O(\epsilon)$. As for the middle term J_i^ϵ, Lemma 8.3.4 yields

$$J_i^\epsilon(\alpha,\eta) = -BR^\perp(z_i(\alpha)) + \left(\eta + \frac{1}{2}\right)[\mathbf{v}]^\perp(z_i(\alpha)) - \Omega z_i(\alpha) + O(\epsilon^b|\log\epsilon|). \tag{8.93}$$

Using the fact that $BR(z_i(\alpha)) = \Omega z_i^\perp(\alpha)$ for $i = n+1,\dots,n+m$ (which follows from (9.1) and (9.2)), it becomes

$$J_i^\epsilon(\alpha,\eta) = \left(\eta + \frac{1}{2}\right)[\mathbf{v}]^\perp(z_i(\alpha)) + O(\epsilon^b|\log\epsilon|). \tag{8.94}$$

Also it follows from (8.18) that $K_i^\epsilon(\alpha,\eta) = L_i\gamma_i(\alpha) + O(\epsilon)$. Plugging these three estimates into the above integral gives

$$T_i^\epsilon = \int_{S_i}\int_{-1}^{0} z_i(\alpha)\cdot\left(\eta+\frac{1}{2}\right)[\mathbf{v}]^\perp(z_i(\alpha))L_i\gamma_i(\alpha)d\eta d\alpha + O(\epsilon^b|\log\epsilon|) = O(\epsilon^b|\log\epsilon|),$$

where the last step follows from the fact that $\int_{-1}^{0}(\eta+\frac{1}{2})d\eta = 0$. This finishes the proof that $|I_i^\epsilon| \le C\epsilon^b|\log\epsilon|$ for $i = n+1,\dots,n+m$, where C depends on b, $\max_i \|z_i\|_{C^2(S_i)}$, $\max_i\|\gamma_i\|_{C^b(S_i)}$, d_Γ and F_Γ.

In the rest of the proof we aim to show $|I_i^\epsilon| \le C\epsilon^b|\log\epsilon|$ for $i = 1,\dots,n$, which is slightly more involved. Recall that in Proposition 8.4.3 we defined $\tilde{p}^\epsilon$ and q^ϵ in D_i^ϵ for $i = 1,\dots,n$, where they satisfy $p^\epsilon = \tilde{p}^\epsilon + q^\epsilon$ in D_i^ϵ, and $q^\epsilon = 0$ on ∂D_i^ϵ. This allows us to apply the divergence theorem (to the q^ϵ term only) and decompose I_i^ϵ as

$$I_i^\epsilon = \int_{D_i^\epsilon}\epsilon^{-1}(x+\nabla\tilde{p}_\epsilon)\cdot\nabla\left(\omega^\epsilon * \mathcal{N} - \frac{\Omega}{2}|x|^2\right)dx - \int_{D_i^\epsilon}\epsilon^{-1}(\epsilon^{-1}-2\Omega)q^\epsilon(x)dx =: I_{i,1}^\epsilon + I_{i,2}^\epsilon.$$

We can easily show that $I_{i,2}^\epsilon = O(\epsilon)$: (8.56) of Proposition 8.4.3 gives $|q^\epsilon| \le C\epsilon^2$, and combining it with $|D_i^\epsilon| \le C\epsilon$ in (8.19) immediately yields the desired estimate.

Next we turn to $I_{i,1}^\epsilon$. Again, the change of variables $x = R_i^\epsilon(\alpha,\eta)$ and the definition $\mathbf{v}^\epsilon := \nabla^\perp(\omega^\epsilon * \mathcal{N})$ gives

$$I_{i,1}^\epsilon = \int_{S_i}\int_{-1}^{0}\big(R_i^\epsilon(\alpha,\eta)+\nabla\tilde{p}^\epsilon(R_i^\epsilon(\alpha,\eta))\big)\cdot\underbrace{\Big(-(\mathbf{v}^\epsilon)^\perp(R_i^\epsilon(\alpha,\eta)) - \Omega R_i^\epsilon(\alpha,\eta)\Big)}_{=:J_i^\epsilon}\underbrace{\epsilon^{-1}\det(\nabla_{\alpha,\eta}R_i^\epsilon(\alpha,\eta))}_{=:K_i^\epsilon}\,d\eta d\alpha.$$

For the three terms in the product of the integrand, we will approximate the first term using the definition of R_i^ϵ and (8.58) of Proposition 8.4.3:

$$R_i^\epsilon(\alpha,\eta) + \nabla\tilde{p}^\epsilon(R_i^\epsilon(\alpha,\eta)) = z_i(\alpha) + \frac{\beta_i}{\gamma_i(\alpha)}\mathbf{n}(z_i(\alpha)) + O(\epsilon),$$

where $\beta_i := \frac{2|U_i|}{L_i \int_{S_i} \gamma_i^{-1}(\alpha) d\alpha}$ is given by (8.55). Lemma 8.3.4 allows us to approximate the middle term J_i^ϵ as (8.93), however (8.94) no longer holds since for $i = 1, \dots, n$ we do not have $BR(z_i(\alpha)) = \Omega z_i^\perp(\alpha)$. As for K_i^ϵ, we again use (8.18) to approximate it by $K_i^\epsilon(\alpha, \eta) = L_i \gamma_i(\alpha) + O(\epsilon)$. Plugging these three estimates into the integrand of $I_{i,1}^\epsilon$ gives

$$I_{i,1}^\epsilon = \int_{S_i} \left(z_i(\alpha) + \frac{\beta_i}{\gamma_i(\alpha)} \mathbf{n}(z_i(\alpha)) \right) \cdot \Big(-BR^\perp(z_i(\alpha)) - \Omega z_i(\alpha) \Big) L_i \gamma_i(\alpha) d\alpha + O(\epsilon^b |\log \epsilon|),$$

where we again use the fact that the $(\eta + \frac{1}{2})$ term gives zero contribution since $\int_{-1}^{0} (\eta + \frac{1}{2}) d\eta = 0$. Next we will show the integral on the right hand side is in fact 0. Since ω is a rotating solution with angular velocity Ω, the conditions (9.1) and (9.2) yield that

$$-BR^\perp(z_i(\alpha)) - \Omega z_i(\alpha) = C_i \gamma_i^{-1}(\alpha) \mathbf{n}(z_i(\alpha)),$$

for some constant C_i. Plugging this into the above integral gives

$$\begin{aligned} I_{i,1}^\epsilon &= C_i L_i \int_{S_i} \left(z_i(\alpha) \cdot \mathbf{n}(z_i(\alpha)) + \frac{\beta_i}{\gamma_i(\alpha)} \right) d\alpha + O(\epsilon^b |\log \epsilon|) \\ &= C_i L_i \left(\int_{S_i} z_i(\alpha) \cdot \mathbf{n}(z_i(\alpha)) d\alpha + \frac{2|U_i|}{L_i} \right) + O(\epsilon^b |\log \epsilon|), \end{aligned}$$

where the second step follows from the definition of β_i in (8.55). Let us compute the integral on the right hand side by changing to arclength parametrization and applying the divergence theorem:

$$\int_{S_i} z_i(\alpha) \cdot \mathbf{n}(z_i(\alpha)) d\alpha = -\frac{1}{L_i} \int_{\partial U_i} x \cdot n d\sigma = -\frac{2|U_i|}{L_i},$$

which yields $I_{i,1}^\epsilon = O(\epsilon^b |\log \epsilon|)$, and finishes the proof that $|I_i^\epsilon| \le C \epsilon^b |\log \epsilon|$ for $i = 1, \dots, n$, where C depends on b, $\|z_i\|_{C^3(S_i)}$, $\|\gamma_i\|_{C^2(S_i)}$, d_Γ and F_Γ.

Finally, summing the I_i^ϵ estimates for $i = 1, \dots, n+m$ gives $|I^\epsilon| \le C \epsilon^b |\log \epsilon|$ for all sufficiently small $\epsilon > 0$, thus we can conclude. □

Now we will use a different way to compute I^ϵ. Let us first define a new integral $\tilde{I}^\epsilon$ that is the same as I^ϵ except with Ω set to zero:

$$\tilde{I}^\epsilon := \int_{D^\epsilon} \epsilon^{-1} (x + \nabla p^\epsilon) \cdot \nabla \left(\omega^\epsilon * \mathcal{N} \right) dx. \tag{8.95}$$

Next we will prove that $\tilde{I}^\epsilon$ is strictly positive independently of ϵ unless all the vortex sheets are nested

circles with constant density. As we will see in the proof, the key step is to show that if some Γ_i is either not a circle or does not have a constant γ_i, then the estimates on p^ϵ in Propositions 8.4.2–8.4.3 lead to the following quantitative version of (8.53): $\epsilon^{-2}\left(\frac{|D_i^\epsilon|^2}{4\pi} - \int_{D_i^\epsilon} p^\epsilon(x)dx\right) \geq c_0 > 0$, where c_0 is independent of ϵ.

Proposition 8.5.2. *Let $\tilde{I}^\epsilon$ be defined as in* (8.95)*. Assume that Γ_i and γ_i satisfy* **(H1)**–**(H3)** *for $i = 1, \ldots, n+m$. Then we have $\tilde{I}^\epsilon \geq 0$ for all sufficiently small $\epsilon > 0$.*

In addition, if Γ is ***not*** *a union of nested circles with constant γ_i's on each connected component, there exists some $c_0 > 0$ independent of ϵ, such that $\tilde{I}^\epsilon > c_0 > 0$ for all sufficiently small $\epsilon > 0$.*

Proof. We start by decomposing $\tilde{I}^\epsilon$ as

$$\tilde{I}^\epsilon = \int_{D^\epsilon} \epsilon^{-1} x \cdot \nabla(\omega^\epsilon * \mathcal{N})dx + \int_{D^\epsilon} \epsilon^{-1}\nabla p^\epsilon \cdot \nabla(\omega^\epsilon * \mathcal{N})dx =: I_1^\epsilon + I_2^\epsilon.$$

I_1^ϵ can be easily computed as

$$I_1^\epsilon = \frac{1}{2\pi\epsilon^2}\int_{D^\epsilon}\int_{D^\epsilon} \frac{x\cdot(x-y)}{|x-y|^2}dxdy = \frac{|D^\epsilon|^2}{4\pi\epsilon^2} = \frac{1}{4\pi\epsilon^2}\left(\sum_{i=1}^{n+m}|D_i^\epsilon|\right)^2 \tag{8.96}$$

where the second equality is obtained by exchanging x with y and taking the average with the original integral. As for I_2^ϵ, we have

$$\begin{aligned} I_2^\epsilon &= \frac{1}{\epsilon}\int_{\partial D^\epsilon} p^\epsilon \nabla(\omega^\epsilon * \mathcal{N})\cdot n d\sigma - \frac{1}{\epsilon}\int_{D^\epsilon} p^\epsilon \omega^\epsilon dx \\ &= -\frac{1}{\epsilon}\sum_{i=1}^{n} c_i^\epsilon \int_{\partial U_i} \nabla(\omega^\epsilon * \mathcal{N})\cdot n d\sigma - \frac{1}{\epsilon^2}\int_{D^\epsilon} p^\epsilon dx \\ &\geq -\frac{1}{\epsilon^2}\sum_{i=1}^{n}\sum_{j=1}^{n+m}\frac{|D_i^\epsilon|}{2\pi}\int_{U_i} 1_{D_j^\epsilon}dx - \frac{1}{\epsilon^2}\sum_{i=1}^{n+m}\int_{D_i^\epsilon} p^\epsilon dx, \end{aligned} \tag{8.97}$$

where the first equality follows from the divergence theorem, the second equality follows from the boundary conditions (8.49) and (8.50) for p^ϵ (as well as the fact that ∂U_i and ∂D_i^ϵ have opposite outer normals), and the last inequality follows from the divergence theorem as well as the inequality $c_i^\epsilon \leq \sup_{D_i^\epsilon} p \leq \frac{|D_i^\epsilon|}{2\pi}$ due to (8.52).

Let us denote $j \prec i$ if $i \in \{1, \ldots, n\}$, $j \in \{1, \ldots, n+m\}$, $j \neq i$ and Γ_j lies in the interior of the domain enclosed by Γ_i (that is, $\Gamma_j \subset U_i$). If not, we denote $j \not\prec i$. Note that for sufficiently small

$\epsilon > 0$, we have

$$\int_{U_i} 1_{D_j^\epsilon} dx = \begin{cases} |D_j^\epsilon| & \text{if } j \prec i, \\ 0 & \text{otherwise.} \end{cases} \tag{8.98}$$

Applying this to (8.97) yields

$$\begin{aligned} I_2^\epsilon &\geq -\frac{1}{2\pi\epsilon^2} \sum_{i,j=1}^{n+m} \mathbb{1}_{j\prec i} |D_i^\epsilon||D_j^\epsilon| - \frac{1}{\epsilon^2} \sum_{i=1}^{n+m} \int_{D_i^\epsilon} p_i^\epsilon dx \\ &= -\frac{1}{4\pi\epsilon^2} \sum_{i,j=1}^{n+m} (\mathbb{1}_{j\prec i} + \mathbb{1}_{i\prec j}) |D_i^\epsilon||D_j^\epsilon| - \frac{1}{\epsilon^2} \sum_{i=1}^{n+m} \int_{D_i^\epsilon} p_i^\epsilon dx \end{aligned} \tag{8.99}$$

where in the first step we used that the $i = n+1, \ldots, n+m$ terms have zero contribution in the first sum, due to the definition of $j \prec i$.

Adding (8.96) and (8.99) together, we obtain

$$\tilde{I}^\epsilon \geq \sum_{i=1}^{n+m} \underbrace{\frac{1}{\epsilon^2} \left(\frac{|D_i^\epsilon|^2}{4\pi} - \int_{D_i^\epsilon} p_i^\epsilon dx \right)}_{=:A_i^\epsilon} + \sum_{i,j=1}^{n+m} \underbrace{\frac{1}{\epsilon^2} (\mathbb{1}_{i\neq j} - (\mathbb{1}_{j\prec i} + \mathbb{1}_{i\prec j})) \frac{|D_i^\epsilon||D_j^\epsilon|}{4\pi}}_{=:B_{i,j}^\epsilon}, \tag{8.100}$$

From (8.53), it follows that $A_i^\epsilon \geq 0$ for all $i = 1, \ldots, n+m$, with equality achieved if and only if each D_i^ϵ is a disk or an annulus. Note that $B_{i,j}^\epsilon \geq 0$ as well for all i and j, since for any $i \neq j$, at most one of $i \prec j$ and $j \prec i$ can hold. Putting these together yields that $\tilde{I}^\epsilon \geq 0$ for any sufficiently small $\epsilon > 0$.

In the rest of the proof, we assume Γ is **not** a union of nested circles with constant γ_i's on each connected component. Therefore at least one of the following 3 cases must be true. In each case we aim to show that $\tilde{I}_\epsilon \geq c_0 > 0$, where c_0 is independent of ϵ for all sufficiently small $\epsilon > 0$.

Case 1. There exists some open curve Γ_i that is not a loop. In this case D_i^ϵ is simply-connected, and $p^\epsilon = 0$ on ∂D_i^ϵ by (8.49). Applying Proposition 8.4.2 to p^ϵ in D_i^ϵ, we have $\sup_{D_i^\epsilon} p^\epsilon \leq C\epsilon^2$, where C is independent of ϵ. This leads to $\int_{D_i^\epsilon} p^\epsilon dx \leq C\epsilon^3$, since $|D_i^\epsilon| = O(\epsilon)$ by (8.19). As a result, for the index i we have

$$A_i^\epsilon = \frac{|D_i^\epsilon|^2}{4\pi\epsilon^2} - \epsilon^{-2} \int_{D_i^\epsilon} p_i^\epsilon dx \geq \frac{L_i^2}{4\pi} \left(\int_{S_i} \gamma_i(\alpha) d\alpha \right)^2 - C\epsilon,$$

where we again used (8.19) in the second inequality. This gives that $A_i^\epsilon \geq \frac{L_i^2}{8\pi} (\int_{S_i} \gamma_i(\alpha) d\alpha)^2 > 0$ for all sufficiently small $\epsilon > 0$.

Case 2. There exists some closed curve Γ_i that is either not a circle, or γ_i is not a constant. In

this case we aim to show that $A_i^\epsilon \geq c_0 > 0$, and this will be done by finding good approximations (independent of ϵ) for both terms in A_i^ϵ. For the first term $\frac{|D_i^\epsilon|^2}{4\pi\epsilon^2}$, using (8.19) we again have

$$\frac{|D_i^\epsilon|^2}{4\pi\epsilon^2} \geq \frac{L_i^2}{4\pi}\left(\int_{S_i}\gamma_i(\alpha)d\alpha\right)^2 - C\epsilon =: J_i - C\epsilon, \tag{8.101}$$

where $J_i > 0$ is independent of ϵ. For the second term $\epsilon^{-2}\int_{D_i^\epsilon} p_i^\epsilon dx$, rewriting the integral using the change of variables $x = R_i^\epsilon(a,\eta)$ gives

$$\epsilon^{-2}\int_{D_i^\epsilon} p_i^\epsilon dx = \int_{S_i}\int_{-1}^{0} \frac{p^\epsilon(R_i^\epsilon(\alpha,\eta))}{\epsilon}\frac{\det(\nabla_{\alpha,\eta}R_i^\epsilon)}{\epsilon}d\eta d\alpha.$$

Recall that in Proposition 8.4.3 we defined $\tilde{p}^\epsilon(R_i^\epsilon(\alpha,\eta)) := c_i^\epsilon(1+\eta)$ and q_ϵ such that $p^\epsilon - \tilde{p}_\epsilon = q_\epsilon$. By (8.56) and (8.57), for all $\alpha \in S_i$ and $\eta \in (-1,0)$ we have

$$\left|\frac{p^\epsilon(R_i^\epsilon(\alpha,\eta))}{\epsilon} - \beta_i(1+\eta)\right| \leq \left|\frac{p^\epsilon(R_i^\epsilon(\alpha,\eta))}{\epsilon} - \frac{c_i^\epsilon}{\epsilon}(1+\eta)\right| + \left|\frac{c_i^\epsilon}{\epsilon} - \beta_i\right| \leq C\epsilon,$$

where $\beta_i := \frac{2|U_i|}{L_i\int_{S_i}\gamma_i^{-1}(\alpha)d\alpha}$ is defined in (8.55). Combining this with the expression of the determinant in (8.18), we have

$$\begin{aligned}\epsilon^{-2}\int_{D_i^\epsilon} p_i^\epsilon dx &= \int_{S_i}\int_{-1}^{0}(\beta_i(1+\eta)+O(\epsilon))(L_i\gamma_i(\alpha)+O(\epsilon))d\eta d\alpha \\ &\leq \frac{|U_i|}{\int_{S_i}\gamma_i^{-1}(\alpha)d\alpha}\int_{S_i}\gamma_i(\alpha)d\alpha + C\epsilon =: K_i + C\epsilon,\end{aligned}$$

where K_i is independent of ϵ. Putting this together with (8.101) yields the following:

$$\begin{aligned}A_i^\epsilon &\geq J_i - K_i - C\epsilon \\ &= \frac{L_i^2}{4\pi}\frac{\int_{S_i}\gamma_i(\alpha)d\alpha}{\int_{S_i}\gamma_i^{-1}(\alpha)d\alpha}\left(\int_{S_i}\gamma_i^{-1}(\alpha)d\alpha\int_{S_i}\gamma_i(\alpha)d\alpha - \frac{4\pi|U_i|}{L_i^2}\right) - C\epsilon.\end{aligned} \tag{8.102}$$

Let us take a closer look at the two terms inside the parenthesis. For the first term, Cauchy-Schwarz inequality gives

$$\int_{S_i}\gamma_i^{-1}(\alpha)d\alpha\int_{S_i}\gamma_i(\alpha)d\alpha \geq 1,$$

with equality achieved if and only if γ_i is a constant. For the second term, the isoperimetric inequality yields

$$\frac{4\pi|U_i|}{L_i^2} \leq 1,$$

(recall that $L_i = |\partial U_i|$), with equality achieved if and only U_i is a disk. By the assumption of Case 2, at least one of the inequalities must be strict, thus the parenthesis on the right hand side of (8.102) is strictly positive (and independent of ϵ). Therefore there exists some constant $c_0 > 0$ such that $\tilde{I}^\epsilon \geq A_i^\epsilon \geq c_0$ for all sufficiently small ϵ.

Case 3. There exist $i \neq j$ such that $i \not\prec j$ and $j \not\prec i$. Then it is clear that for such i, j, $B_{i,j}^\epsilon$ in (8.100) is given by $B_{i,j}^\epsilon = \frac{|D_i^\epsilon||D_j^\epsilon|}{4\pi\epsilon^2}$. Hence (8.19) gives

$$B_{i,j}^\epsilon \geq L_i L_j \bigg(\int_{S_i} \gamma_i(\alpha) d\alpha \bigg) \bigg(\int_{S_j} \gamma_j(\alpha) d\alpha \bigg) - C\epsilon,$$

which yields $\tilde{I}^\epsilon \geq \frac{1}{2} L_i L_j (\int_{S_i} \gamma_i d\alpha)(\int_{S_j} \gamma_j d\alpha) > 0$ for all sufficiently small $\epsilon > 0$.

This finishes our discussion on all 3 cases. To conclude, since Γ is not a union of nested circles with constant γ_i's on each connected component, at least one of the 3 cases must hold, and all of them lead to $\tilde{I}^\epsilon \geq c_0 > 0$. □

The above proposition immediately leads to the following corollary for the $\Omega < 0$ case.

Corollary 8.5.3. *Assume that Γ_i and γ_i satisfy* **(H1)**–**(H3)** *for $i = 1, \ldots, n+m$. Let I^ϵ be defined as in* (8.92)*, and assume $\Omega < 0$. Then we have $I^\epsilon \geq 0$ for all sufficiently small $\epsilon > 0$. In addition, if Γ is* ***not*** *a union of concentric circles all centered at the origin with constant γ_i's, there exists some $c_0 > 0$ independent of ϵ, such that $I^\epsilon > c_0 > 0$ for all sufficiently small $\epsilon > 0$.*

Proof. Let us decompose I^ϵ as follows (recall the definition of $\tilde{I}^\epsilon$ in (8.95))

$$I^\epsilon = \tilde{I}^\epsilon + (-\Omega) \left(\epsilon^{-1} \int_{D^\epsilon} (|x|^2 + \nabla p^\epsilon \cdot x) dx \right) =: \tilde{I}^\epsilon + \underbrace{(-\Omega)}_{>0} J^\epsilon. \tag{8.103}$$

Recall that Proposition 8.5.2 gives $\tilde{I}_\epsilon \geq c_0 > 0$ as long as Γ is not a union of nested circles with constant γ_i's. By [42, Lemma 2.11], we have

$$\int_{D_i^\epsilon} (|x|^2 + \nabla p^\epsilon \cdot x) dx \geq 0 \quad \text{for any } i = 1, \ldots, n+m,$$

thus $J^\epsilon \geq 0$. Putting them together, and using the fact that $\Omega < 0$, we know $I^\epsilon \geq c_0 > 0$ if Γ is not a union of nested circles with constant γ_i's.

To finish the proof, we only need to focus on the case that the Γ_i's are nested circles with constant γ_i's, but not all of them are centered at the origin. Assume that there exists $k \in \{1, \ldots, n\}$ such that Γ_k is a circle with radius r_k centered at $x_k \neq 0$. Since γ_k is a constant, D_k^ϵ is an

annulus given by $B(x_k, r_k + \epsilon\gamma_k) \setminus B(x_k, r_k)$. The symmetry of D_k^ϵ about x_k immediately leads to $p^\epsilon|_{D_k^\epsilon} = -\frac{1}{2}|x - x_k|^2 + \frac{1}{2}(r_k + \epsilon\gamma_k)^2$. An elementary computation gives

$$\epsilon^{-1}\int_{D_k^\epsilon}(|x|^2 + \nabla p^\epsilon \cdot x)dx = \epsilon^{-1}\int_{D_k^\epsilon}|x|^2 - (x - x_k)\cdot x dx = \epsilon^{-1}|x_k|^2|D_k^\epsilon| \geq 2\pi r_k\gamma_k|x_k|^2 > 0,$$

where in the second-to-last step we used that $|D_k^\epsilon| = 2\pi\epsilon r_k\gamma_k + \pi\epsilon^2\gamma_k^2$. Setting $c_0 := 2\pi r_k\gamma_k|x_k|^2$ gives $I^\epsilon \geq c_0 > 0$, thus we can conclude. □

Now we are ready to prove Theorem 4.1.2. Note that for $\Omega < 0$, the symmetry result immediately follows from Proposition 8.5.1 and Corollary 8.5.3. For $\Omega = 0$, Proposition 8.5.1–8.5.2 already imply that a stationary vortex sheet with positive strength must be a union of nested circles with constant strength on each of them. To finish the proof, we only need to show that these nested circles must be concentric.

Proof of Theorem 4.1.2. For a uniformly-rotating vortex sheet with $\Omega < 0$, the symmetry result for $\Omega < 0$ is a direct consequence of Proposition 8.5.1 and Corollary 8.5.3. Next we focus on the stationary (i.e. $\Omega = 0$) case.

Combining Propsitions 8.5.1–8.5.2, we obtain that Γ is a union of nested circles, and γ_i is constant on Γ_i for all $i = 1\ldots, n$. It remains to show that all Γ_i's are concentric. Let us denote by $\mathbf{v}_i$ the contribution to the velocity field by Γ_i. Since Γ_i is a circle with constant strength γ_i, a quick application of the divergence theorem yields that $\mathbf{v}_i \equiv 0$ in the open disk enclosed by Γ_i, whereas $\mathbf{v}_i(x) = \dfrac{\gamma_i L_i(x - x_i^0)^\perp}{2\pi|x - x_i^0|^2}$ in the open set outside Γ_i, where x_i^0 is the center of the circle Γ_i.

Without loss of generality, let us reorder the indices such that Γ_i is nested inside Γ_j for $i < j$. Towards a contradiction, let $k > 1$ be such that Γ_k is the first circle that is not concentric with Γ_1. From the above discussion, we know that $\mathbf{v}_i = 0$ on Γ_k for $i = k+1, \ldots, n$ (since Γ_k is nested inside Γ_i), whereas for $i = 1, \ldots, k-1$ we have $\mathbf{v}_i = \dfrac{\gamma_i L_i(x - x_1^0)^\perp}{2\pi|x - x_1^0|^2}$ on Γ_k, since all these Γ_i's have the same center x_1^0 and are nested inside Γ_k. Summing them up (and also using the fact that Γ_k contributes zero normal velocity on itself, since it is a circle with constant strength), we have

$$BR(x)\cdot\mathbf{n} = \sum_{i=1}^{n}\mathbf{v}_i(x)\cdot\mathbf{n} = \left(\sum_{i=1}^{k-1}\gamma_i L_i\right)\frac{(x - x_1^0)^\perp\cdot\mathbf{n}}{2\pi|x - x_1^0|^2} \quad \text{on } \Gamma_k,$$

where the right hand side is not a zero function since Γ_k has a different center from x_1^0. This causes a contradiction with the fact that $\omega = \omega_0$ is stationary. As a result, all $\Gamma_1, \ldots, \Gamma_n$ must be concentric circles, finishing the proof. □

Chapter 9

The flexibility results for vortex sheets

9.1 The equations and the functional spaces

Let $\omega(x,t) = \omega_0(R_{\Omega t}x)$ be a stationary/rotating vortex sheet solution to the incompressible 2D Euler equation, where $\omega_0 \in \mathcal{M}(\mathbb{R}^2) \cap H^{-1}(\mathbb{R}^2)$ and $R_{\Omega t}x$ is the rotation of x counter-clockwise by angle Ωt about the origin. Here $\Omega = 0$ corresponds to a stationary solution, and $\Omega \neq 0$ corresponds to a rotating solution. Assume that ω_0 is concentrated on Γ. Throughout this paper we will assume that Γ is a simple closed curve and $\Omega > 0$. Following [44, Lemma 2.1], we have that:

Lemma 9.1.1. *Assume $\omega(\cdot,t) = \omega_0(R_{\Omega t}x)$ is a stationary/uniformly-rotating vortex sheet with angular velocity $\Omega \in \mathbb{R}$, and ω_0 is concentrated on Γ, with z and γ defined as in section 8.2. Then the Birkhoff-Rott integral BR (8.4) and the strength γ satisfy the following two equations:*

$$(BR - \Omega x^{\perp}) \cdot \mathbf{n} = \mathbf{v}^{+} \cdot \mathbf{n} = \mathbf{v}^{-} \cdot \mathbf{n} = 0 \quad \text{on } \Gamma, \tag{9.1}$$

and

$$(BR(z(\alpha)) - \Omega z^{\perp}(\alpha)) \cdot \mathbf{s}(z(\alpha)) \, \frac{\gamma(\alpha)}{|z'(\alpha)|} = C. \tag{9.2}$$

Note that (9.2) can be written as

$$(I - P_0)\left[\left(BR(z,\Gamma)(z(\theta)) - \Omega z(\theta)^{\perp} \right) \cdot \frac{z'(\theta)\gamma(\theta)}{|z'(\theta)|^2} \right] = 0, \tag{9.3}$$

where P_0 is a projection to the mean, that is, $P_0 f := \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)d\theta$. For simplicity, we also denote $f(\theta)d\theta := \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)d\theta$. Now plugging $z(\theta) = (1+r(\theta))(\cos(\theta),\sin(\theta))$ and $\gamma(\theta) := b + g(\theta)$ into (9.1), (9.2) and (9.3) yields that

$$\mathcal{F}(b,g,r) := (\mathcal{F}_1,\mathcal{F}_2) = (0,0), \tag{9.4}$$

where

$$\begin{aligned}
\mathcal{F}_1(b,g,r) :=& \fint_{-\pi}^{\pi} (b+g(\eta))\frac{(r'(\theta)\cos(\theta-\eta) - (1+r(\theta))\sin(\theta-\eta))\,(1+r(\eta)) - (1+r(\theta))r'(\theta)}{(1+r(\theta))^2 + (1+r(\eta))^2 - 2(1+r(\theta))(1+r(\eta))\cos(\theta-\eta)}d\eta \\
&+ \Omega r'(\theta)(1+r(\theta)), \\
\mathcal{F}_2(b,g,r) :=& (I-P_0)\,\tilde{\mathcal{F}}_2(b,g,r), \\
\tilde{\mathcal{F}}_2(b,g,r) :=& \fint_{-\pi}^{\pi} (b+g(\eta))\frac{(1+r(\theta))^2 - (r'(\theta)\sin(\theta-\eta) + (1+r(\theta))\cos(\theta-\eta))(1+r(\eta))}{(1+r(\theta))^2 + (1+r(\eta))^2 - 2(1+r(\theta))(1+r(\eta))\cos(\theta-\eta)}d\eta \\
&\times \frac{(b+g(\theta))}{r'(\theta)^2 + (1+r(\theta))^2} - \Omega(1+r(\theta))^2\frac{b+g(\theta)}{r'(\theta)^2+(1+r(\theta))^2}.
\end{aligned}$$

Throughout the paper we will work with the following analytic function spaces. Let $c > 0$ be a sufficiently small parameter and let $\mathcal{C}_w(c)$ be the space of analytic functions in the strip $|\mathrm{Im}(z)| \le c$. For $k \in \mathbb{N}$, denote

$$\begin{aligned}
X_c^k &:= \left\{ f(\theta) \in \mathcal{C}_w(c), \quad f(\theta) = \sum_{n=1}^{\infty} a_n \cos(2n\theta), \int_{-\pi}^{\pi} |f(\theta+ic)|^2 + |\partial^k f(\theta+ic)|^2 + |f(\theta-ic)|^2 + |\partial^k f(\theta-ic)|^2 d\theta < \infty \right\} \\
Y_c^k &:= \left\{ f(\theta) \in \mathcal{C}_w(c), \quad f(\theta) = \sum_{n=1}^{\infty} a_n \sin(2n\theta), \int_{-\pi}^{\pi} |f(\theta+ic)|^2 + |\partial^k f(\theta+ic)|^2 + f(\theta-ic)|^2 + |\partial^k f(\theta-ic)|^2 d\theta < \infty \right\},
\end{aligned}$$

From now on, due to scaling considerations, we will fix $\Omega = 1$ and b will play the role as bifurcation parameter. It is clear that $\mathcal{F}(b,0,0) = (0,0)$ for all $b \in \mathbb{R}$ since $\mathcal{F}_1(b,0,0) = 0$ and $\tilde{\mathcal{F}}_2(b,0,0)$ is constant. Our main theorem in this paper is the following:

Theorem 9.1.2. *Let $k \ge 3$, and let $c > 0$ be sufficiently small. Then, there exists a curve of solutions (b,g,r) of $\mathcal{F} = (0,0)$, belonging to $\mathbb{R}\times X_c^k \times X_c^{k+1}$ and a neighbourhood of $(b,g,r) = (2,0,0)$, bifurcating from $(b,g,r) = (2,0,0)$ such that $(g,r) \neq (0,0)$.*

9.2 Proof of the Main Theorem

The goal of this section is to prove the existence of non-radial uniformly-rotating vortex sheets. To do so, we will split the proof into the following steps: first we will prove that the functional $\mathcal{F}$ is C^3, next we will study $D\mathcal{F}$ to show that, as mentioned in the introduction, it is a Fredholm operator of index 0, with $\dim(\mathrm{Ker}(D\mathcal{F})) = 1$. The next step is to apply Lyapunov-Schmidt theory and reduce the problem to a finite (2) dimensional one. In those coordinates, linear expansions fail to be conclusive (all the linear terms vanish) since 2 nontrivial branches emanate from the bifurcation point (as opposed to 1). Instead, we perform a quadratic expansion to determine that locally the bifurcation branches look like two pairs of straight lines (specifically as $x^2 - y^2 = 0$ in some well-chosen coordinates) and hence the bifurcation does not trivialize (as if it had been of the type $x^2 + y^2 = 0$). We conclude the proof by handling the higher order terms and showing that they don't alter the quadratic behaviour in a sufficiently small neighbourhood of the bifurcation point.

9.2.1 Continuity of the functional

In this subsection, we will check the regularity of $\mathcal{F}$. As explained above, we will reduce the infinite dimensional problem to a finite dimensional problem and investigate its Taylor expansion up to quadratic order. Hence, we need to check if the functional is regular enough to do so. To this end, we have the following proposition:

Proposition 9.2.1. *Let $k \geq 3$. Then there exists a neighborhood U of $(2, 0, 0) \in \mathbb{R} \times X_c^k \times X_c^{k+1}$ such that $\mathcal{F} \in C^3\left(U; Y_c^k \times X_c^k\right)$.*

Proof. Since the stream function, $\omega * \mathcal{N}(x) := \frac{1}{2\pi}\int_{\mathbb{R}^2} \log|x-y| d\omega$ is invariant under rotations, it follows immediately that $\mathcal{F}$ is also invariant under rotation by π-radians, hence $\mathcal{F}$ has only even Fourier modes. Also the oddness of $\mathcal{F}_1$ and evenness of $\mathcal{F}_2$ follow from the invariance under reflection.

To prove the regularity, we briefly sketch the idea. We impose $k \geq 3$ to ensure that H^k is a Banach algebra. It is clear that $\mathcal{F}$ is smooth in b. It is also straightforward that, for example, for all (g, r) near $(0, 0) \in X_c^k \times X_c^{k+1}$,

$$\begin{aligned}\frac{d}{dt}\mathcal{F}_1(b, g + th_1, r + th_2)\Big|_{t=0} &= PVh_1(\eta)\frac{(r'(\theta)\cos(\theta-\eta) - (1+r(\theta))\sin(\theta-\eta))\,(1+r(\eta)) - (1+r(\theta))r'(\theta)}{(1+r(\theta))^2 + (1+r(\eta))^2 - 2(1+r(\theta))(1+r(\eta))\cos(\theta-\eta)} \\ &+ (b + g(\eta))\left[\frac{(h_2'(\theta)\cos(\theta-\eta) - h_2(\theta)\sin(\theta-\eta))(1+r(\eta)) - h_2(\theta)r'(\theta)}{(1+r(\theta))^2 + (1+r(\eta))^2 - 2(1+r(\theta))(1+r(\eta))\cos(\theta-\eta)}\right. \\ &+ \left.\frac{(r'(\theta)\cos(\theta-\eta) - (1+r(\theta))\sin(\theta-\eta))\,h_2(\eta) - (1+r(\theta))h_2'(\theta)}{(1+r(\theta))^2 + (1+r(\eta))^2 - 2(1+r(\theta))(1+r(\eta))\cos(\theta-\eta)}\right.\end{aligned}$$

$$
\begin{aligned}
&- \left[(r'(\theta)\cos(\theta-\eta) - (1+r(\theta))\sin(\theta-\eta))\,(1+r(\eta)) - (1+r(\theta))r'(\theta) \right] \\
&\times \frac{[2(1+r(\theta))h_2(\theta) + 2(1+r(\eta))h_2(\eta) - 2\cos(\theta-\eta)(h_2(\theta)(1+r(\eta)) + h_2(\eta)(1+r(\theta)))]}{((1+r(\theta))^2 + (1+r(\eta))^2 - 2(1+r(\theta))(1+r(\eta))\cos(\theta-\eta))^2} \Bigg] d\eta + \Omega h_2'(\theta) \\
&=: D\mathcal{F}_1(b,g,r)[h_1,h_2],
\end{aligned}
$$

and $D\mathcal{F}_1 : \mathbb{R} \times X_c^k \times X_c^{k+1} \mapsto \mathcal{L}(X_c^k \times X_c^{k+1}; Y_c^k \times X_c^k)$ is continuous. A similar derivation can be performed for $D\mathcal{F}_2$. For the higher derivatives, we refer to [16, 17, 41, 53, 83] for the method to deal with the singular integrals arising throughout the calculations. □

9.2.2 Fredholm index of the linearized operator $D\mathcal{F}$

This subsection is devoted to show that $D\mathcal{F}$ is Fredholm of index zero. We can make all the calculations explicit, moreover the operator diagonalizes in Fourier modes. We have the following lemmas:

Lemma 9.2.2. *Let $g(\theta) = \sum_{n=1}^{\infty} a_n \cos(2n\theta)$ and $r(\theta) = \sum_{n=1}^{\infty} b_n \cos(2n\theta)$. Then we have that*

$$
D\mathcal{F}(b,0,0)\,[g,r] = \begin{pmatrix} \hat{g}(\theta) \\ \hat{r}(\theta) \end{pmatrix},
$$

where

$$
\hat{g}(\theta) = \sum_{n=1}^{\infty} \hat{a}_n \sin(2n\theta), \quad \hat{r}(\theta) = \sum_{n=1}^{\infty} \hat{b}_n \cos(2n\theta),
$$

and the coefficients satisfy, for any $n \geq 1$:

$$
M_n \begin{pmatrix} a_n \\ b_n \end{pmatrix} := \begin{pmatrix} -\frac{1}{2} & -2n\left(\Omega - \frac{b}{2}\right) \\ \frac{b}{2} - \Omega & b^2(n-1) \end{pmatrix} \begin{pmatrix} a_n \\ b_n \end{pmatrix} = \begin{pmatrix} \hat{a}_n \\ \hat{b}_n \end{pmatrix}.
$$

Proof. We use (9.20) in Lemma 9.4.2 and obtain

$$
\begin{aligned}
\hat{g}(\theta) = \frac{d}{dt}\mathcal{F}_1(b, tg, tr) &= -\frac{g(\eta)\sin(\theta-\eta)}{2-2\cos(\theta-\eta)} d\eta + \left(\Omega - \frac{b}{2}\right) r'(\theta) \\
&= -\sum_{n=1}^{\infty} a_n \frac{\cos(2n\eta)\sin(\theta-\eta)}{2-2\cos(\theta-\eta)} d\eta + \sum_{n=1}^{\infty} (-2n)\left(\Omega - \frac{b}{2}\right) b_n \sin(2n\theta) \\
&= \sum_{n=1}^{\infty} \left(-\frac{a_n}{2} + (-2n)\left(\Omega - \frac{b}{2}\right) b_n \right) \sin(2n\theta),
\end{aligned}
$$

where the last equality follows from (9.46). Similarly, we apply (9.21) in Lemma 9.4.2 and (9.47)

to obtain

$$\begin{aligned}\hat{r}(\theta) &= \left(\frac{b}{2}-\Omega\right) g(\theta) + b^2 \left(\frac{r(\theta)-r(\eta)}{2-2\cos(\theta-\eta)} d\eta - r(\theta)\right) \\ &= \sum_{n=1}^{\infty} \left(\frac{b}{2}-\Omega\right) a_n \cos(2n\theta) + b^2 \sum_{n=1}^{\infty} b_n \left(\frac{\cos(2n\theta)-\cos(2n\eta)}{2-2\cos(\theta-\eta)} d\eta - \cos(2n\theta)\right) \\ &= \sum_{n=1}^{\infty} \left(\left(\frac{b}{2}-\Omega\right) a_n + b^2 (n-1) b_n\right) \cos(2n\theta).\end{aligned}$$

This proves the lemma. □

Remark 9.2.3. *The same calculation shows that for any $b \neq 2$ there exists $\varepsilon(b) > 0$ such that the only solution (b, g, r) such that $||g||, ||r|| \leq \varepsilon(b)$ is $(b, 0, 0)$ by virtue of the Implicit Function Theorem. In other words, that the only solution $\varepsilon(b)$-close to the disk is the disk itself.*

Lemma 9.2.4. *Let us fix $b = 2$ and $\Omega = 1$. We also denote $v := (0, \cos(2\theta)) \in X_c^k \times X_c^{k+1}$ and $w := (0, \cos(2\theta)) \in Y_c^k \times X_c^k$. Then it holds that*

$$\begin{aligned} Ker\,(D\mathcal{F}(2,0,0)) &= span\,\{v\} \subset X_c^k \times X_c^{k+1}, \\ Im\,(D\mathcal{F}(2,0,0))^{\perp} &= span\,\{w\} \subset Y_c^k \times X_c^k. \end{aligned}$$

Proof. From Lemma 9.2.2, we have

$$M_n = \begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & 4(n-1) \end{pmatrix},$$

for all $n \geq 1$. For all $n \geq 2$, M_n is clearly an isomorphism, while $\mathrm{Ker}(M_1) = \mathrm{Im}(M_1)^{\perp} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. By orthogonality of Fourier modes, this proves the lemma. □

9.2.3 Lyapunov-Schmidt reduction

In this subsection, we will aim to derive a finite dimensional system which is equivalent to (9.4). From Lemma 10.2.7, we have the following orthogonal decompositions of the function spaces:

$$\begin{aligned} X := X_c^k \times X_c^{k+1} &= \mathrm{span}\,\{v\} \oplus \mathrm{Ker}\,(D\mathcal{F}(2,0,0))^{\perp} =: \mathrm{span}\,\{v\} \oplus \mathcal{X}, \quad v \in \mathrm{Ker}\,(D\mathcal{F}(2,0,0)), \\ Y := Y_c^k \times X_c^k &= \mathrm{span}\,\{w\} \oplus \mathrm{Im}\,(D\mathcal{F}(2,0,0)) =: \mathrm{span}\,\{w\} \oplus \mathcal{Y}, \quad w \in \mathrm{Im}\,(D\mathcal{F}(2,0,0))^{\perp}, \end{aligned}$$

where v and w are as defined in Lemma 10.2.7. Let us consider the orthogonal projections

$$P : X \to \operatorname{span}\{v\}, \quad Q : Y \to \operatorname{span}\{w\}.$$

More precisely, we have

$$P(g(\theta), r(\theta)) = \left(0, \left(\frac{1}{\pi}\int r(\eta)\cos(2\eta)d\eta\right)\cos(2\theta)\right) \quad \text{for all } (g,r) \in X_c^k \times X_c^{k+1}, \tag{9.5}$$

$$Q(G(\theta), R(\theta)) = \left(0, \left(\frac{1}{\pi}\int R(\eta)\cos(2\eta)d\eta\right)\cos(2\theta)\right) \quad \text{for all } (G,R) \in Y_c^k \times X_c^k. \tag{9.6}$$

We remark that we will sometimes abuse notation and identify $\mathcal{F}(b,g,r)$ with $\mathcal{F}(b,(g,r))$, where $(g,r) \in X$. Let us define $G : \mathbb{R} \times \operatorname{span}\{v\} \times \mathcal{X} \mapsto Y$ as follows:

$$G(b,f,x) := \mathcal{F}(b, f+x), \quad \text{for} \quad b \in \mathbb{R}, \quad f \in \operatorname{span}\{v\}, \quad x \in \mathcal{X}.$$

Then (9.4) is equivalent to (for $(g,r) = f + x$)

$$QG(b,f,x) = 0 \quad \text{and} \quad (I-Q)\,G(b,f,x) = 0. \tag{9.7}$$

However, it follows from Lemma 10.2.7 that

$$D_x\left((I-Q)G\right)(b,0,0) = (I-Q)\,D\mathcal{F}(b,0)P : \mathcal{X} \mapsto \mathcal{Y} \tag{9.8}$$

is an isomorphism, consequently, the implicit function theorem yields that there is an open set $U \subset \mathbb{R} \times \operatorname{span}\{v\}$ near $(b,0)$ and a function $\varphi : U \mapsto \mathcal{X}$ such that

$$(I-Q)\,G(b,f,\varphi(b,f)) = (I-Q)\,\mathcal{F}(b, f+\varphi(b,f)) = 0.$$

Note that from $\mathcal{F}(b,0) = (0,0)$ for any $b \in \mathbb{R}$, we have

$$\varphi(b,0) = 0, \tag{9.9}$$

and thus (9.7) is equivalent to

$$0 = QG(b,f,\varphi(b,f)) = Q\mathcal{F}(b, f+\varphi(b,f)), \quad (b,f) \in U. \tag{9.10}$$

Since span $\{v\}$ is one dimensional, we have $f = tv$ for some $t \in \mathbb{R}$, therefore the system (9.10) can be written in terms of the variables b and t as

$$\begin{aligned} 0 = Q\mathcal{F}(b, tv + \varphi(b, tv)) &= \int_0^1 \frac{d}{ds} \left(Q\mathcal{F}(b, stv + \varphi(b, stv))\right) ds \\ &= \int_0^1 QD\mathcal{F}(b, stv + \varphi(b, stv))(tv + t\partial_f \varphi(b, stv)v) ds, \end{aligned}$$

where we used (9.9) to obtain the second equality. Dividing the right-hand side by t to get rid of the trivial solutions, we are led to solve the following two dimensional problem:

$$0 = F_{red}(b, t) := \int_0^1 QD\mathcal{F}(b, stv + \varphi(b, stv))(v + \partial_f \varphi(b, stv)v) ds, \quad (b, tv) \in U. \tag{9.11}$$

9.2.4 Quadratic expansion of the reduced functional

The main idea is to expand the reduced functional F_{red} up to quadratic terms. To this end, we recall the following proposition for the derivatives of F_{red}.

Proposition 9.2.5. *([51, Proposition 3], [54, Proposition 3.1]) Let F_{red} be defined as in* (9.11)*. Then the following hold:*

(a) First derivatives:

$$\begin{aligned} \partial_b F_{red}(2, 0) &= Q\partial_b D\mathcal{F}(2, 0)v, \\ \partial_t F_{red}(2, 0) &= \frac{1}{2} \frac{d^2}{dt^2} Q\mathcal{F}(2, tv)\Big|_{t=0}. \end{aligned}$$

(b) Second derivatives:

$$\begin{aligned} \partial_{bb} F_{red}(2, 0) &= 2Q\partial_b D\mathcal{F}(2, 0)\tilde{v}, \\ \partial_{tt} F_{red}(2, 0) &= \frac{1}{3} \frac{d^3}{dt^3} \left[Q\mathcal{F}(2, tv)\right]\Big|_{t=0} + Q \frac{d^2}{dtds} \mathcal{F}(2, tv + s\hat{v})\Big|_{t=s=0}, \\ \partial_{tb} F_{red}(2, 0) &= \frac{1}{2} \partial_b Q \frac{d^2}{dt^2} \mathcal{F}(b, tv)\Big|_{b=2, t=0} + \frac{1}{2} Q\partial_b D\mathcal{F}(2, 0)\hat{v} + Q \frac{d^2}{dtds} \mathcal{F}(2, tv + s\tilde{v})\Big|_{t=s=0}, \end{aligned}$$

where

$$\begin{aligned} \hat{v} &:= -\left[D\mathcal{F}(2, 0)\right]^{-1} \frac{d^2}{dt^2} \left[(I - Q)\mathcal{F}(2, tv)\right]\Big|_{t=0}, \\ \tilde{v} &:= -\left[D\mathcal{F}(2, 0)\right]^{-1} (I - Q)\partial_b D\mathcal{F}(2, 0)v. \end{aligned}$$

Now using the values found in Lemma 9.4.7, we can obtain the derivatives of F_{red}.

Proposition 9.2.6. *Let F_{red} be defined as in* (9.11)*. Then it holds that*

$$\partial_b F_{red}(2,0) = 0, \tag{9.12}$$

$$\partial_t F_{red}(2,0) = 0. \tag{9.13}$$

$$\partial_{bb} F_{red}(2,0) = 2w, \tag{9.14}$$

$$\partial_{tt} F_{red}(2,0) = -8w, \tag{9.15}$$

$$\partial_{tb} F_{red}(2,0) = 0. \tag{9.16}$$

Proof. (9.12) follows immediately from (9.6) and (9.31). For (9.13), we use (9.32) and the orthogonality of the Fourier modes. (9.14) follows from (9.40). (9.15) follows from (9.36) and (9.37). Lastly, (9.16) follows from (9.33), (9.38) and (9.39). □

9.2.5 Proof of Theorem 9.1.2

Now we are ready to prove the main theorem of this section.

Proof. From (9.11), it suffices to show that there exist (b,t) such that $t \neq 0$ and $F_{red}(b,t) = 0$. To do so, we expand F_{red} up to quadratic terms and obtain that for all (b,t) near $(2,0)$,

$$\begin{aligned} F_{red}(b,t) = \Big[& F_{red}(2,0) + \partial_b F_{red}(2,0)(b-2) + \partial_t F_{red}(2,0)t \\ & + \frac{1}{2}\partial_{bb} F_{red}(2,0)(b-2)^2 + \frac{1}{2}\partial_{tt} F_{red}(2,0)t^2 + \partial_{tb} F_{red}(2,0)(b-2)t + \left((b-2)^2 + t^2\right)\epsilon(b,t) \Big] w, \end{aligned}$$

where $\epsilon(b,t)$ is a continuous function such that $\lim_{(b,t)\to(2,0)} \epsilon(b,t) = \epsilon(2,0) = 0$. From Proposition 9.2.6, it follows that (we drop w for simplicity)

$$F_{red}(b,t) = (b-2)^2 - 4t^2 + \left((b-2)^2 + t^2\right)\epsilon(b,t).$$

Now we use the change of variables $b := x + 2$ and $t = xy$, so that

$$\hat{F}(x,y) := \frac{F_{red}(x+2,xy)}{x^2} = \left(1 - 4y^2\right) + (1+y^2)\epsilon(x+2,xy). \tag{9.17}$$

Clearly, $\hat{F}\left(0,\frac{1}{2}\right) = 0$ and $\partial_y \hat{F}\left(0,\frac{1}{2}\right) = -4 \neq 0$. Therefore the implicit function theorem implies that there exists a continuous function ϕ near 0 such that $\hat{F}(x,\phi(x)) = 0$ and $\phi(0) = \frac{1}{2}$. Therefore

it follows from (9.17) that there exists a pair (b, t) such that $t \neq 0$ and $F_{red}(b, t) = 0$. This finishes the proof. □

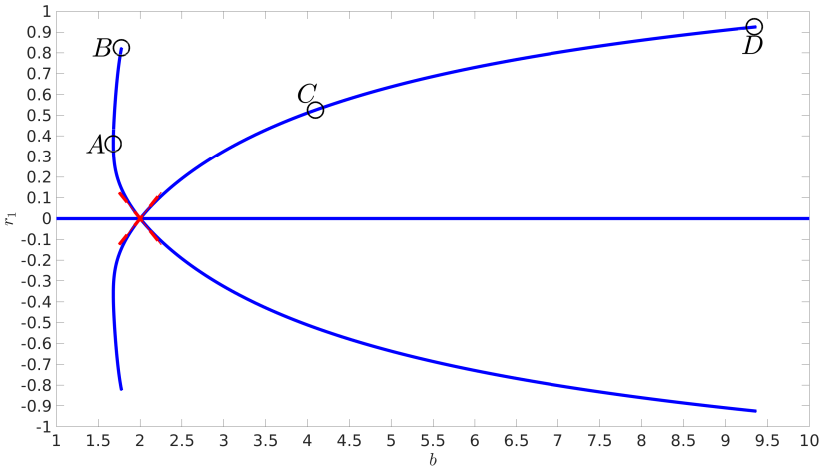


Figure 9.1: Plot of the bifurcation diagram of the solutions given by Theorem 9.1.2. The dashed red lines correspond to the linear expansion (9.17) around the bifurcation point $(2, 0)$. See Figure 9.2 for a numerical plot of the solutions at the points A, B, C, D. The branches continue beyond what is calculated.

9.3 Numerical results

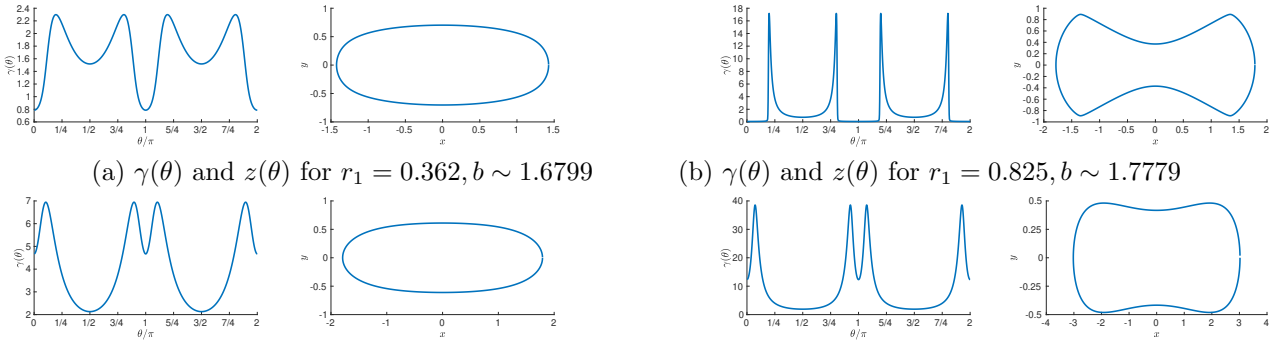
In this section, we describe how to compute numerically the branches of solutions emanating from the disk, previously proved (locally) in Theorem 9.1.2. See Figure 9.1. To do so, we calculate solutions of the form

$$R(\theta) = 1 + \sum_{k=1}^{N} r_k \cos(2k\theta), \quad \gamma(\theta) = \sum_{k=0}^{N} \gamma_k \cos(2k\theta)$$

with $\gamma_0 = b$. We first employed continuation in b, in increments of $\Delta b = 0.001$, starting from $b = 1.8$ and $b = 2.1$ and using as initial guess for the starting b the solution given by the linear theory and for the subsequent b the solution found in the previous iteration. After discovering a fold at approximately $b \sim 1.68$, we switched variables and instead we recalculated using continuation in r_1, which appears to be monotonic along the branches. As before, we start at $r_1 = \pm 0.125$ and take an increment $\Delta r_1 = 0.001$.

To compute a solution for a fixed r_1 we use the Levenberg-Marquardt algorithm [67, 72]. We aim to find a zero of the system of equations $\mathcal{F}(b, g, r)(\theta_j)$, with $\theta_j = \frac{j\pi}{N_\theta}$, $j = 1, \ldots, N_\theta$ and

$N_\theta = 1024$ with variables r_k, $k \neq 1$ (recall that r_1 is fixed at each iteration since it is the continuation parameter) and γ_k. We take $N = 160$. In order to perform the integration in space, we desingularize the principal value at $\eta = \theta$ by subtracting $\frac{1}{2}\mathcal{H}(\gamma)$ to $\mathcal{F}_1$, where $\mathcal{H}$ denotes the Hilbert transform, computed explicitly since we have the Fourier expansion of γ, and perform a trapezoidal integration on the rest (for which the integrand is smooth), with step $h = \frac{2\pi}{N_\theta}$. We remark that the integrand of $\mathcal{F}_2$ has a removable singularity (thus no principal value integration is needed) and can be integrated using the trapezoidal integration if the limit at $\eta = \theta$ is taken properly.



(a) $\gamma(\theta)$ and $z(\theta)$ for $r_1 = 0.362, b \sim 1.6799$

(b) $\gamma(\theta)$ and $z(\theta)$ for $r_1 = 0.825, b \sim 1.7779$

(c) $\gamma(\theta)$ and $z(\theta)$ for $r_1 = 0.525, b \sim 4.0954$

(d) $\gamma(\theta)$ and $z(\theta)$ for $r_1 = 0.925, b \sim 9.3439$

Figure 9.2: Panels (a)-(d): $\gamma(\theta)$ and $z(\theta)$ at the points A–D highlighted in Figure 9.1. In panel (b), γ appears to tend to be concentrated only on the horizontal parts of z, leading to a possible solution consisting only of two symmetric curves (cf. [77, Figure 1]) and a change of topology.

9.4 Appendix

9.4.1 Functional derivatives

Recall that $\mathcal{F}(b, g, r) = (\mathcal{F}_1, \mathcal{F}_2)$ is given in (9.4). For simplicity, we denote

$$
\begin{aligned}
A_1 &:= (b + g(\eta)),\\
A_2 &:= (r'(\theta)\cos(\theta-\eta) - (1+r(\theta))\sin(\theta-\eta))\,(1+r(\eta)) - (1+r(\theta))r'(\theta),\\
A_3 &:= \frac{1}{(1+r(\theta))^2 + (1+r(\eta))^2 - 2(1+r(\theta))(1+r(\eta))\cos(\theta-\eta)},\\
A_4 &:= r'(\theta)(1+r(\theta)),\\
A_5 &:= (b+g(\eta))(b+g(\theta)),\\
A_6 &:= (1+r(\theta))^2 - (r'(\theta)\sin(\theta-\eta) + (1+r(\theta))\cos(\theta-\eta))(1+r(\eta)),\\
A_7 &:= \frac{1}{r'(\theta)^2 + (1+r(\theta))^2},
\end{aligned}
$$

$$A_8 := \frac{(1+r(\theta))^2(b+g(\theta))}{r'(\theta)^2+(1+r(\theta))^2}.$$

We also denote the average integral by $f(\theta)d\theta := \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)d\theta$. Therefore the functional $\mathcal{F}$ can be written as

$$\mathcal{F}_1 = A_1A_2A_3d\eta + \Omega A_4, \tag{9.18}$$

$$\tilde{\mathcal{F}}_2 = A_5A_6A_7A_3d\eta - \Omega A_8. \tag{9.19}$$

We will expand $A_i(g,\theta,\eta)$ and $A_i(r,\theta,\eta)$ up to quadratic/cubic order in g and r.

Lemma 9.4.1. *Let $A_i's$ be as above. We have*

$$\begin{aligned}
A_1 &= b+g(\eta),\\
A_2 &= -\sin(\theta-\eta) + [r'(\theta)(\cos(\theta-\eta)-1) - (r(\theta)+r(\eta))\sin(\theta-\eta)]\\
&\quad + [r'(\theta)r(\eta)(\cos(\theta-\eta)-1) - r'(\theta)(r(\theta)-r(\eta)) - r(\theta)r(\eta)\sin(\theta-\eta)],\\
A_3 &= \frac{1}{2-2\cos(\theta-\eta)} - \frac{r(\theta)+r(\eta)}{2-2\cos(\theta-\eta)}\\
&\quad + \frac{1}{2-2\cos(\theta-\eta)}\left[r(\theta)^2+r(\theta)r(\eta)+r(\eta)^2 - \frac{(r(\theta)-r(\eta))^2}{2-2\cos(\theta-\eta)}\right]\\
&\quad + \frac{1}{2-2\cos(\theta-\eta)}\left[\frac{(r(\theta)+r(\eta))(r(\theta)-r(\eta))^2}{1-\cos(\theta-\eta)} - \left[r(\theta)^3+r(\theta)^2r(\eta)+r(\theta)r(\eta)^2+r(\eta)^3\right]\right] + O(r^4),\\
A_4 &= r'(\theta)+r(\theta)r'(\theta),\\
A_5 &= b^2+b(g(\theta)+g(\eta))+g(\theta)g(\eta),\\
A_6 &= (1-\cos(\theta-\eta)) + [(r(\theta)+r(\eta))(1-\cos(\theta-\eta)) + (r(\theta)-r(\eta)) - r'(\theta)\sin(\theta-\eta)]\\
&\quad + [r(\theta)(r(\theta)-r(\eta)) + r(\theta)r(\eta)(1-\cos(\theta-\eta)) - r'(\theta)r(\eta)\sin(\theta-\eta)],\\
A_7 &= 1-2r(\theta) + \left[3r(\theta)^2 - r'(\theta)^2\right] + \left[4r(\theta)r'(\theta)^2 - 4r(\theta)^3\right] + O(r^4),\\
A_8 &= b+g(\theta) - br'(\theta)^2 + \left[2br(\theta)r'(\theta)^2 - g(\theta)r'(\theta)^2\right] + O(r^4+g^4).
\end{aligned}$$

Proof. Straightforward. □

Linear parts

We denote by A_i^j the jth order term in A_i. For example, $A_3^1 = -\frac{r(\theta)+r(\eta)}{2-2\cos(\theta-\eta)}$.

Lemma 9.4.2. *Let $\mathcal{F}_i$'s and A_i's be defined as before. Then*

$$\left.\frac{d}{dt}\mathcal{F}_1(b,tg,tr)\right|_{t=0} = -\frac{g(\eta)\sin(\theta-\eta)}{2-2\cos(\theta-\eta)}d\eta + \left(\Omega - \frac{b}{2}\right)r'(\theta), \tag{9.20}$$

$$\left.\frac{d}{dt}\tilde{\mathcal{F}}_2(b,tg,tr)\right|_{t=0} = \left(\frac{b}{2}-\Omega\right)g(\theta) + b^2\left(\frac{r(\theta)-r(\eta)}{2-2\cos(\theta-\eta)}d\eta - r(\theta)\right). \tag{9.21}$$

Proof. We compute $\mathcal{F}_1$ first. In view of (9.18), we collect the linear terms in $A_1A_2A_3 + \Omega A_4$ from Lemma 9.4.1. Hence we have

$$\begin{aligned}\mathcal{F}_1(b,tg,tr) &= -\frac{tg(\eta)\sin(\theta-\eta)}{2-2\cos(\theta-\eta)} + bt\frac{r'(\theta)(\cos(\theta-\eta)-1) - (r(\theta)+r(\eta))\sin(\theta-\eta)}{2-2\cos(\theta-\eta)} \\ &\quad + \frac{bt\sin(\theta-\eta)(r(\theta)+r(\eta))}{2-2\cos(\theta-\eta)}d\eta + t\Omega r'(\theta) + O(t^2) \\ &= -\frac{tg(\eta)\sin(\theta-\eta)}{2-2\cos(\theta-\eta)}d\eta + t\left(\Omega - \frac{b}{2}\right)r'(\theta) + O(t^2).\end{aligned}$$

Thus we obtain (9.20) by differentiating with respect to t.

In order to compute the derivative of $\tilde{\mathcal{F}}_2$, we collect the linear terms in (9.19) from Lemma 9.4.1 and obtain

$$\begin{aligned}\tilde{\mathcal{F}}_2(b,tg,tr) &= \frac{bt(g(\theta)+g(\eta))}{2} \\ &\quad + tb^2\left[\frac{(r(\theta)+r(\eta))}{2} + \frac{(r(\theta)-r(\eta))}{2-2\cos(\theta-\eta)} - \frac{r'(\theta)\sin(\theta-\eta)}{2-2\cos(\theta-\eta)}\right] \\ &\quad - tb^2r(\theta) - \frac{tb^2}{2}(r(\theta)+r(\eta)) - t\Omega g(\theta)d\eta + O(t^2) \\ &= t\left(\frac{b}{2}-\Omega\right)g(\theta) + tb^2\left(\frac{r(\theta)-r(\eta)}{2-2\cos(\theta-\eta)}d\eta - r(\theta)\right) + O(t^2),\end{aligned}$$

where we used $g(\eta)d\eta = r(\eta)d\eta = 0$ and $\frac{\sin(\theta-\eta)}{2-2\cos(\theta-\eta)}d\eta = 0$. By differentiating in t, we obtain the desired result (9.21). □

Quadratic parts

Now we compute the quadratic expansion of $\mathcal{F}_1$ and $\tilde{\mathcal{F}}_2$.

Lemma 9.4.3. *Let $\mathcal{F}_i$'s and A_i's be defined as before. Then*

$$\begin{aligned}\mathcal{F}_1 &= \left(\frac{b}{2}+\Omega\right)r(\theta)r'(\theta) - b\frac{r'(\theta)(r(\theta)-r(\eta))}{2-2\cos(\theta-\eta)}d\eta + b\frac{(r(\theta)-r(\eta))^2\sin(\theta-\eta)}{(2-2\cos(\theta-\eta))^2}d\eta \\ &\quad + \textit{linear terms} + O(r^3+g^3),\end{aligned} \tag{9.22}$$

$$\begin{aligned}\tilde{\mathcal{F}}_2 = &\frac{3b^2}{2}r(\theta)^2 - bg(\theta)r(\theta) + b\left(\Omega - \frac{b}{2}\right)r'(\theta)^2 \\ &- \frac{b^2}{2}\frac{(r(\theta)-r(\eta))(5r(\theta)+r(\eta))}{2-2\cos(\theta-\eta)}d\eta + b\frac{(g(\theta)+g(\eta))(r(\theta)-r(\eta))}{2-2\cos(\theta-\eta)}d\eta \\ &- b\frac{g(\eta)r'(\theta)\sin(\theta-\eta)}{2-2\cos(\theta-\eta)}d\eta + \textit{linear terms} + O(r^3+g^3). \end{aligned} \tag{9.23}$$

Proof. We compute $\mathcal{F}_1$ first. By collecting quadratic terms in (9.18) from Lemma 9.4.1, we have (we will have, for example, $A_1^2A_2^0A_3^0 + A_1^0A_2^2A_3^0 + A_1^0A_2^0A_3^2 + A_1^1A_2^1A_3^0 + A_1^1A_2^0A_3^1 + A_1^0A_2^1A_3^1 + \Omega A_4^2$).

$$\begin{aligned}\mathcal{F}_1 = &\, b\left[-\frac{1}{2}r'(\theta)r(\eta) - \frac{r'(\theta)(r(\theta)-r(\eta))}{2-2\cos(\theta-\eta)} - \frac{r(\theta)r(\eta)\sin(\theta-\eta)}{2-2\cos(\theta-\eta)}\right] \\ &- \frac{b\sin(\theta-\eta)}{2-2\cos(\theta-\eta)}\left[r(\theta)^2 + r(\theta)r(\eta) + r(\eta)^2 - \frac{(r(\theta)-r(\eta))^2}{2-2\cos(\theta-\eta)}\right] \\ &+ \left[-\frac{g(\eta)r'(\theta)}{2} - \frac{g(\eta)(r(\theta)+r(\eta))\sin(\theta-\eta)}{2-2\cos(\theta-\eta)}\right] + \frac{g(\eta)(r(\theta)+r(\eta))\sin(\theta-\eta)}{2-2\cos(\theta-\eta)} \\ &+ \left[\frac{br'(\theta)(r(\theta)+r(\eta))}{2} + \frac{b(r(\theta)+r(\eta))^2\sin(\theta-\eta)}{2-2\cos(\theta-\eta)}\right] + \Omega r(\theta)r'(\theta)d\eta + O(r^3+g^3) \\ = &\left(\frac{b}{2}+\Omega\right)r(\theta)r'(\theta) - \frac{br'(\theta)(r(\theta)-r(\eta))}{2-2\cos(\theta-\eta)} + \frac{b(r(\theta)-r(\eta))^2\sin(\theta-\eta)}{(2-2\cos(\theta-\eta))^2}d\eta \\ &+ \text{linear terms} + O(r^3+g^3),\end{aligned}$$

which yields (9.22). Now we will expand $\tilde{\mathcal{F}}_2$ up to the quadratic order. By collecting all quadratic terms in (9.19) from Lemma 9.4.1, we obtain (we will have $A_5^2A_6^0A_7^0A_3^0 + A_5^0A_6^2A_7^0A_3^0 + A_5^0A_6^0A_7^2A_3^0 + A_5^0A_6^0A_7^0A_3^2 + A_5^1A_6^1A_7^0A_3^0 + A_5^1A_6^0A_7^1A_3^0 + A_5^1A_6^0A_7^0A_3^1 + A_5^0A_6^1A_7^1A_3^0 + A_5^0A_6^1A_7^0A_3^1 + A_5^0A_6^0A_7^1A_3^1 - \Omega A_8^2$),

$$\begin{aligned}\tilde{\mathcal{F}}_2 = &\frac{g(\theta)g(\eta)}{2} + \left[b^2\frac{r(\theta)(r(\theta)-r(\eta))}{2-2\cos(\theta-\eta)} + b^2\frac{r(\theta)r(\eta)}{2} - b^2\frac{r'(\theta)r(\eta)\sin(\theta-\eta)}{2-2\cos(\theta-\eta)}\right] \\ &+ b^2\frac{(3r(\theta)^2 - r'(\theta)^2)}{2} + \left[\frac{b^2(r(\theta)^2 + r(\theta)r(\eta) + r(\eta)^2)}{2} - \frac{b^2}{2}\frac{(r(\theta)-r(\eta))^2}{2-2\cos(\theta-\eta)}\right] \\ &+ \left[b\frac{(g(\theta)+g(\eta))(r(\theta)+r(\eta))}{2} + b\frac{(g(\theta)+g(\eta))(r(\theta)-r(\eta))}{2-2\cos(\theta-\eta)} - b\frac{(g(\theta)+g(\eta))r'(\theta)\sin(\theta-\eta)}{2-2\cos(\theta-\eta)}\right] \\ &- b((g(\theta)+g(\eta)))r(\theta) - b\frac{(g(\theta)+g(\eta))(r(\theta)+r(\eta))}{2} \\ &+ \left[-b^2r(\theta)(r(\theta)+r(\eta)) - 2b^2\frac{r(\theta)(r(\theta)-r(\eta))}{2-2\cos(\theta-\eta)} + 2b^2\frac{r(\theta)r'(\theta)\sin(\theta-\eta)}{2-2\cos(\theta-\eta)}\right] \\ &+ \left[-b^2\frac{(r(\theta)+r(\eta))^2}{2} - b^2\frac{(r(\theta)^2-r(\eta)^2)}{2-2\cos(\theta-\eta)} + b^2\frac{r'(\theta)(r(\theta)+r(\eta))\sin(\theta-\eta)}{2-2\cos(\theta-\eta)}\right] \\ &+ b^2r(\theta)(r(\theta)+r(\eta)) + b\Omega r'(\theta)^2d\eta + O(r^3+g^3) \\ = &\, b^2\frac{(3r(\theta)^2 - r'(\theta)^2)}{2} - bg(\theta)r(\theta) + b\Omega r'(\theta)^2 \\ &+ \frac{1}{2-2\cos(\theta-\eta)}\left(-\frac{b^2}{2}(r(\theta)-r(\eta))(5r(\theta)+r(\eta)) + b(g(\theta)+g(\eta))(r(\theta)-r(\eta))\right)\end{aligned}$$

$$- \frac{bg(\eta)r'(\theta)\sin(\theta-\eta)}{2-2\cos(\theta-\eta)}d\eta + \text{linear terms} + O(r^3+g^3),$$

where we used $g(\eta)d\eta = r(\eta)d\eta = \frac{\sin(\theta-\eta)}{2-2\cos(\theta-\eta)}d\eta = 0$. This yields the desired result (9.23). □

Lemma 9.4.4. *Let $\mathcal{F}_i$'s and A_i's be defined as before. Then*

$$\begin{aligned}\frac{d^2}{dtds}\mathcal{F}_1(b,tg+s\tilde{g},tr+s\tilde{r})\Big|_{t=s=0} &= \left(\frac{b}{2}+\Omega\right)(r(\theta)\tilde{r}'(\theta)+\tilde{r}(\theta)r'(\theta)) \\ &\quad - b\frac{r'(\theta)(\tilde{r}(\theta)-\tilde{r}(\eta))}{2-2\cos(\theta-\eta)}d\eta - b\frac{\tilde{r}'(\theta)(r(\theta)-r(\eta))}{2-2\cos(\theta-\eta)}d\eta \\ &\quad + 2b\frac{(r(\theta)-r(\eta))(\tilde{r}(\theta)-\tilde{r}(\eta))\sin(\theta-\eta)}{(2-2\cos(\theta-\eta))^2}, \end{aligned} \tag{9.24}$$

and

$$\begin{aligned}\frac{d^2}{dtds}\tilde{\mathcal{F}}_2(b,tg+s\tilde{g},tr+s\tilde{r})\Big|_{t=s=0} &= 3b^2r(\theta)\tilde{r}(\theta) - b(g(\theta)\tilde{r}(\theta)+\tilde{g}(\theta)r(\theta)) + 2b\left(\Omega-\frac{b}{2}\right)r'(\theta)\tilde{r}'(\theta) \\ &\quad - \frac{b^2}{2}\frac{(r(\theta)-r(\eta))(5\tilde{r}(\theta)+\tilde{r}(\eta))}{2-2\cos(\theta-\eta)}d\eta - \frac{b^2}{2}\frac{(\tilde{r}(\theta)-\tilde{r}(\eta))(5r(\theta)+r(\eta))}{2-2\cos(\theta-\eta)}d\eta \\ &\quad + b\frac{(g(\theta)+g(\eta))(\tilde{r}(\theta)-\tilde{r}(\eta))}{2-2\cos(\theta-\eta)}d\eta + b\frac{(\tilde{g}(\theta)+\tilde{g}(\eta))(r(\theta)-r(\eta))}{2-2\cos(\theta-\eta)}d\eta \\ &\quad - b\frac{g(\eta)\tilde{r}'(\theta)\sin(\theta-\eta)}{2-2\cos(\theta-\eta)}d\eta - b\frac{\tilde{g}(\eta)r'(\theta)\sin(\theta-\eta)}{2-2\cos(\theta-\eta)}d\eta. \end{aligned} \tag{9.25}$$

Proof. We compute $\mathcal{F}_1$ first. From (9.22) in Lemma 9.4.3, we collect all st terms and obtain

$$\begin{aligned}\mathcal{F}_1(b,tg+s\tilde{g},tr+s\tilde{r}) = st\Bigg[&\left(\frac{b}{2}+\Omega\right)(r(\theta)\tilde{r}'(\theta)+\tilde{r}(\theta)r'(\theta)) \\ &-b\frac{r'(\theta)(\tilde{r}(\theta)-\tilde{r}(\eta))}{2-2\cos(\theta-\eta)}d\eta - b\frac{\tilde{r}'(\theta)(r(\theta)-r(\eta))}{2-2\cos(\theta-\eta)}d\eta \\ &+2b\frac{(r(\theta)-r(\eta))(\tilde{r}(\theta)-\tilde{r}(\eta))}{(2-2\cos(\theta-\eta))^2}\Bigg] + \text{linear terms} + O(t^2+s^2).\end{aligned}$$

Once we differentiate the above equation with respect to t and s, the desired result (9.24) follows immediately. Similarly, we collect all st terms from (9.23) and obtain

$$\begin{aligned}\tilde{\mathcal{F}}_2(b,tg+s\tilde{g},tr+s\tilde{r}) = st\Bigg[&3b^2r(\theta)\tilde{r}(\theta) - b(g(\theta)\tilde{r}(\theta)+\tilde{g}(\theta)r(\theta)) + 2b\left(\Omega-\frac{b}{2}\right)r'(\theta)\tilde{r}'(\theta) \\ &-\frac{b^2}{2}\frac{(r(\theta)-r(\eta))(5\tilde{r}(\theta)+\tilde{r}(\eta))}{2-2\cos(\theta-\eta)}d\eta - \frac{b^2}{2}\frac{(\tilde{r}(\theta)-\tilde{r}(\eta))(5r(\theta)+r(\eta))}{2-2\cos(\theta-\eta)}d\eta \\ &+b\frac{(g(\theta)+g(\eta))(\tilde{r}(\theta)-\tilde{r}(\eta))}{2-2\cos(\theta-\eta)}d\eta + b\frac{(\tilde{g}(\theta)+\tilde{g}(\eta))(r(\theta)-r(\eta))}{2-2\cos(\theta-\eta)}d\eta \\ &-b\frac{g(\eta)\tilde{r}'(\theta)\sin(\theta-\eta)}{2-2\cos(\theta-\eta)}d\eta - b\frac{\tilde{g}(\eta)r'(\theta)\sin(\theta-\eta)}{2-2\cos(\theta-\eta)}d\eta\Bigg]\end{aligned}$$

$$+ \text{ linear terms} + O(t^2 + s^2).$$

Once we differentiate the above equation with respect to t and s, the desired result (9.25) follows immediately. □

Cubic parts

We will expand $\tilde{\mathcal{F}}_2$ up to cubic order with respect to the r variable (we fix $g = 0$). We denote $B := A_3 A_6$ so that (9.19) can be written as (with $g = 0$)

$$\tilde{\mathcal{F}}_2 = b^2 B d\eta A_7 - \Omega A_8. \tag{9.26}$$

We will first expand B up to cubic order.

Lemma 9.4.5. *Let $\tilde{\mathcal{F}}_2$, A_i and B be as defined as before. Then*

$$\begin{aligned} B d\eta = &\frac{1}{2} + \frac{r(\theta) - r(\eta)}{2 - 2\cos(\theta - \eta)} d\eta - \frac{1}{2} \frac{r(\theta)^2 - r(\eta)^2}{2 - 2\cos(\theta - \eta)} d\eta \\ &+ \left[\frac{1}{2} \frac{(r(\theta) - r(\eta))(r(\theta)^2 + r(\eta)^2)}{2 - 2\cos(\theta - \eta)} d\eta - \frac{(r(\theta) - r(\eta))^3}{(2 - 2\cos(\theta - \eta))^2} d\eta + \frac{r'(\theta)(r(\theta) - r(\eta))^2 \sin(\theta - \eta)}{(2 - 2\cos(\theta - \eta))^2} d\eta \right] \\ &+ O(r^4). \end{aligned}$$

Proof. Using (9.4.1), we will compute the constant (=: B^0), linear (=: B^1), quadratic (=: B^2) and cubic (=: B^3) terms of $B = A_3 A_6$ separately. It is straightforward that

$$B^0 d\eta = \frac{1}{2}. \tag{9.27}$$

For B^1, we compute $B^1 (= A_3^1 A_6^0 + A_3^0 A_6^1)$,

$$B^1 d\eta = \frac{r(\theta) - r(\eta)}{2 - 2\cos(\theta - \eta)} - \frac{r'(\theta) \sin(\theta - \eta)}{2 - 2\cos(\theta - \eta)} d\eta = \frac{r(\theta) - r(\eta)}{2 - 2\cos(\theta - \eta)} d\eta, \tag{9.28}$$

where we used $\frac{\sin(\theta - \eta)}{2 - 2\cos(\theta - \eta)} d\eta = 0$. For B^2, we compute $A_3^2 A_6^0 + A_3^0 A_6^2 + A_3^1 A_6^1$, hence

$$\begin{aligned} B^2 d\eta = &\left[\frac{r(\theta)^2 + r(\theta) r(\eta) + r(\eta)^2}{2} - \frac{1}{2} \frac{(r(\theta) - r(\eta))^2}{2 - 2\cos(\theta - \eta)} d\eta \right] \\ &+ \left[\frac{r(\theta)(r(\theta) - r(\eta))}{2 - 2\cos(\theta - \eta)} + \frac{r(\theta) r(\eta)}{2} - \frac{r'(\theta) r(\eta) \sin(\theta - \eta)}{2 - 2\cos(\theta - \eta)} d\eta \right] \\ &+ \left[-\frac{1}{2}(r(\theta) + r(\eta))^2 - \frac{r(\theta)^2 - r(\eta)^2}{2 - 2\cos(\theta - \eta)} + \frac{r'(\theta)(r(\theta) + r(\eta)) \sin(\theta - \eta)}{2 - 2\cos(\theta - \eta)} d\eta \right] \end{aligned}$$

$$= -\frac{1}{2}\frac{(r(\theta)^2 - r(\eta)^2)}{2-2\cos(\theta-\eta)}d\eta, \tag{9.29}$$

where we used $\frac{\sin(\theta-\eta)}{2-2\cos(\theta-\eta)}d\eta = 0$. For B^3, we compute $B^3(= A_3^3A_6^0 + A_3^2A_6^1 + A_3^1A_6^2 + A_3^0A_6^3)$,

$$\begin{aligned}
B^3 d\eta &= \left[\frac{(r(\theta)+r(\eta))(r(\theta)-r(\eta))^2}{2-2\cos(\theta-\eta)}d\eta - \frac{1}{2}r(\theta)^3 + r(\theta)^2r(\eta) + r(\theta)r(\eta)^2 + r(\eta)^3 d\eta\right] \\
&\quad + \left[\frac{(r(\theta)+r(\eta))(r(\theta)^2+r(\theta)r(\eta)+r(\eta)^2)}{2}d\eta - \frac{1}{2}\frac{(r(\theta)+r(\eta))(r(\theta)-r(\eta))^2}{2-2\cos(\theta-\eta)}d\eta\right. \\
&\quad + \frac{r(\theta)^3 - r(\eta)^3}{2-2\cos(\theta-\eta)}d\eta - \frac{(r(\theta)-r(\eta))^3}{(2-2\cos(\theta-\eta))^2}d\eta \\
&\quad \left. - \frac{r'(\theta)(r(\theta)^2+r(\theta)r(\eta)+r(\eta)^2)\sin(\theta-\eta)}{2-2\cos(\theta-\eta)}d\eta + \frac{r'(\theta)(r(\theta)-r(\eta))^2\sin(\theta-\eta)}{(2-2\cos(\theta-\eta))^2}d\eta\right] \\
&\quad + \left[-\frac{r(\theta)(r(\theta)^2-r(\eta)^2)}{2-2\cos(\theta-\eta)}d\eta - \frac{r(\theta)r(\eta)(r(\theta)+r(\eta))}{2}d\eta + \frac{r'(\theta)r(\eta)(r(\theta)+r(\eta))\sin(\theta-\eta)}{2-2\cos(\theta-\eta)}d\eta\right] \\
&= \frac{1}{2}\frac{(r(\theta)-r(\eta))(r(\theta)^2+r(\eta)^2)}{2-2\cos(\theta-\eta)}d\eta - \frac{(r(\theta)-r(\eta))^3}{(2-2\cos(\theta-\eta))^2}d\eta + \frac{r'(\theta)(r(\theta)-r(\eta))^2\sin(\theta-\eta)}{(2-2\cos(\theta-\eta))^2}.
\end{aligned} \tag{9.30}$$

Thus the desired result follows from (9.27), (9.28), (9.29) and (9.30). □

Lemma 9.4.6. *Let $\tilde{\mathcal{F}}_2$, A_i's and B be as defined as before. Then*

$$\begin{aligned}
\frac{1}{6}\frac{d^3}{dt^3}\tilde{\mathcal{F}}_2(b,0,tr)\Big|_{t=0} &= \frac{b^2}{2}\frac{(r(\theta)-r(\eta))(3r(\theta)^2+2r(\theta)r(\eta)+r(\eta)^2)}{2-2\cos(\theta-\eta)}d\eta - b^2\frac{(r(\theta)-r(\eta))^3}{(2-2\cos(\theta-\eta))^2}d\eta \\
&\quad + b^2\frac{r'(\theta)(r(\theta)-r(\eta))^2\sin(\theta-\eta)}{(2-2\cos(\theta-\eta))^2}\eta + b^2\frac{(3r(\theta)^2-r'(\theta)^2)(r(\theta)-r(\eta))}{2-2\cos(\theta-\eta)}d\eta \\
&\quad + 2b(b-\Omega)r(\theta)r'(\theta)^2 - 2b^2r(\theta)^3
\end{aligned}$$

Proof. We first collect all cubic terms of $\tilde{\mathcal{F}}_2(b,0,r)$ in r. From (9.26), we have that the cubic terms consist of $b^2(B^3A_7^0 + B^2A_7^1 + B^1A_7^2 + B^0A_7^3) - \Omega A_8^3$. Using Lemma 9.4.1 and 9.4.5 and the fact that A_7 does not depend on η, we obtain

$$\begin{aligned}
\tilde{\mathcal{F}}_2(b,0,r) &= b^2\left[\frac{1}{2}\frac{(r(\theta)-r(\eta))(r(\theta)^2+r(\eta)^2)}{2-2\cos(\theta-\eta)}d\eta - \frac{(r(\theta)-r(\eta))^3}{(2-2\cos(\theta-\eta))^2}d\eta + \frac{r'(\theta)(r(\theta)-r(\eta))^2\sin(\theta-\eta)}{(2-2\cos(\theta-\eta))^2}\eta\right] \\
&\quad + b^2\frac{r(\theta)(r(\theta)^2-r(\eta)^2)}{2-2\cos(\theta-\eta)}d\eta + b^2\frac{(3r(\theta)^2-r'(\theta)^2)(r(\theta)-r(\eta))}{2-2\cos(\theta-\eta)}d\eta + b^2\left[2r(\theta)r'(\theta)^2 - 2r(\theta)^3\right] \\
&\quad - 2b\Omega r(\theta)r'(\theta)^2 + \text{lower order terms} + O(r^4) \\
&= \frac{b^2}{2}\frac{(r(\theta)-r(\eta))(3r(\theta)^2+2r(\theta)r(\eta)+r(\eta)^2)}{2-2\cos(\theta-\eta)}d\eta - b^2\frac{(r(\theta)-r(\eta))^3}{(2-2\cos(\theta-\eta))^2}d\eta \\
&\quad + b^2\frac{r'(\theta)(r(\theta)-r(\eta))^2\sin(\theta-\eta)}{(2-2\cos(\theta-\eta))^2}\eta + b^2\frac{(3r(\theta)^2-r'(\theta)^2)(r(\theta)-r(\eta))}{2-2\cos(\theta-\eta)}d\eta
\end{aligned}$$

$$+ 2b(b-\Omega)r(\theta)r'(\theta)^2 - 2b^2 r(\theta)^3 + \text{lower order terms} + O(r^4).$$

Therefore, the desired result follows immediately. □

9.4.2 Derivatives of the reduced functional

We denote $v := (0, \cos(2\theta))$. Given a pair of functions (g, r), we denote Q be the projection to the second mode of r, that is, $Q(g,r) := \left(\frac{1}{\pi}\int r(\theta)\cos(2\theta)d\theta\right)\cos(2\theta)$.

Lemma 9.4.7. *Let $\mathcal{F}$, v, Q be defined as before. We fix $b = 2$ and $\Omega = 1$. Then,*

$$\partial_b D\mathcal{F}(2,0)v = (\sin(2\theta), 0), \tag{9.31}$$

$$\frac{1}{2}\frac{d^2}{dt^2}\mathcal{F}(2,tv)\Big|_{t=0} = (-2\sin(4\theta), -3\cos(4\theta)), \tag{9.32}$$

$$\partial_b Q\frac{d^2}{dt^2}\mathcal{F}(b,tv)\Big|_{b=2,t=0} = (0,0), \tag{9.33}$$

$$\tilde{v} := -\left[D\mathcal{F}(2,0)\right]^{-1}(I-Q)\partial_b D\mathcal{F}(2,0)v = (2\cos(2\theta), 0), \tag{9.34}$$

$$\hat{v} := -\left[D\mathcal{F}(2,0)\right]^{-1}\frac{d^2}{dt^2}\left[(I-Q)\mathcal{F}(2,tv)\right]\Big|_{t=0} = \left(-8\cos(4\theta), \frac{3}{2}\cos(4\theta)\right), \tag{9.35}$$

$$\frac{d^2}{dtds}Q\mathcal{F}(2,tv+s\hat{v})\Big|_{t=s=0} = (0, -12\cos(2\theta)), \tag{9.36}$$

$$\frac{1}{3}\frac{d^3}{dt^3}Q\mathcal{F}(2,tv)\Big|_{t=0} = (0, 4\cos(2\theta)), \tag{9.37}$$

$$Q\frac{d^2}{dtds}\mathcal{F}(2,tv+s\tilde{v}) = (0,0), \tag{9.38}$$

$$\frac{1}{2}Q\partial_b D\mathcal{F}(2,0)\hat{v} = (0,0), \tag{9.39}$$

$$2Q\partial_b D\mathcal{F}(2,0)\tilde{v} = (0, 2\cos(2\theta)). \tag{9.40}$$

Proof. To prove (9.31), it follows from Lemma 9.4.2 that

$$\partial_b D\mathcal{F}(b,0)v = \left(\sin(2\theta), 2b\left(\frac{\cos(2\theta)-\cos(2\eta)}{2-2\cos(\theta-\eta)}d\eta - \cos(2\theta)\right)\right) = (\sin(2\theta), 0),$$

where the last equality follows from (9.47).

To prove (9.32), note that $\frac{1}{2}\frac{d^2}{dt^2}\mathcal{F}(2,tv)\Big|_{t=0} = \left(\frac{1}{2}\frac{d^2}{dt^2}\mathcal{F}_1(2,tv)\Big|_{t=0}, \frac{1}{2}(I-P_0)\frac{d^2}{dt^2}\tilde{\mathcal{F}}_2(2,tv)\Big|_{t=0}\right)$. Hence it follows from Lemma 9.4.3 that

$$\frac{1}{2}\frac{d^2}{dt^2}\mathcal{F}_1(2,tv)\Big|_{t=0} = -4\cos(2\theta)\sin(2\theta) + 4\sin(2\theta)\frac{\cos(2\theta)-\cos(2\eta)}{2-2\cos(\theta-\eta)}d\eta$$

$$
\begin{aligned}
&\quad + 2\frac{(\cos(2\theta)-\cos(2\eta))^2\sin(\theta-\eta)}{(2-2\cos(\theta-\eta))^2}d\eta \\
&= -4\cos(2\theta)\sin(2\theta) + 4\sin(2\theta)\cos(2\theta) - 2\sin(4\theta) \\
&= -2\sin(4\theta),
\end{aligned}
$$

where the second equality follows from (9.47) and Lemma 9.4.11. Also, Lemma 9.4.3 gives

$$
\begin{aligned}
\frac{1}{2}(I-P_0)\frac{d^2}{dt^2}\tilde{\mathcal{F}}_2(2,tv)\Big|_{t=0} &= (I-P_0)\left[6\cos^2(2\theta)\right] - 2(I-P_0)\frac{(\cos(2\theta)-\cos(2\eta))(5\cos(2\theta)+\cos(2\eta))}{2-2\cos(\theta-\eta)}d\eta \\
&= -4(I-P_0)\cos^2(2\theta) - 2(I-P_0)\frac{(\cos(2\theta)-\cos(2\eta))\cos(2\eta)}{2-2\cos(\theta-\eta)}d\eta \\
&= -3\cos(4\theta),
\end{aligned}
$$

where the second equality follows from (9.47) and the last equality follows from Lemma 9.4.9. Therefore we obtain (9.32).

To prove (9.33), we can repeat the above computation and find that $\frac{d^2}{dt^2}\tilde{\mathcal{F}}_2(b,tv) \in \operatorname{span}\{\cos(4\theta)\}$, independently of b. By projecting it to the space of the second mode, we obtain (9.33).

To prove (9.34), note that $(I-Q)\partial_b D\mathcal{F}(2,0)v = (\sin(2\theta),0)$, which follows from (9.31). Also, it follows from Lemma 9.4.2 and (9.46) that

$$
D\mathcal{F}(2,0)(-2\cos(2\theta),0) = (\sin(2\theta),0) = (I-Q)\partial_b D\mathcal{F}(2,0)v.
$$

This immediately implies (9.34).

To prove (9.35), we use Lemma 9.4.2 and (9.47) and (9.46) to obtain

$$
D\mathcal{F}(2,0)\left(8\cos(4\theta), -\frac{3}{2}\cos(4\theta)\right) = (-4\sin(4\theta), -6\cos(4\theta)) = \frac{d^2}{dt^2}\mathcal{F}(2,tv)\Big|_{t=0},
$$

where the last equality follows from (9.32). Therefore we obtain

$$
-\left[D\mathcal{F}(2,0)\right]^{-1}\frac{d^2}{dt^2}\left[(I-Q)\mathcal{F}(2,tv)\right]\Big|_{t=0} = \left(-8\cos(4\theta), \frac{3}{2}\cos(4\theta)\right),
$$

which proves (9.35).

To prove (9.36), note that $\frac{d^2}{dtds}Q\mathcal{F} = \left(0, P_2\frac{d^2}{dtds}\tilde{\mathcal{F}}_2\right)$. Therefore, it follows from (9.35) and (9.25)

in Lemma 9.4.4 that (plugging $g = 0,\ \tilde{g} = -8\cos(4\theta),\ r = \cos(2\theta)$, and $\tilde{r} = \frac{3}{2}\cos(4\theta)$)

$$\begin{aligned}\frac{d^2}{dtds}P_2\mathcal{F}_2(2, tv + s\tilde{v}) = P_2\,[&(18\cos(2\theta)\cos(4\theta) + 16\cos(2\theta)\cos(4\theta))\\ &-2\frac{(\cos(2\theta) - \cos(2\eta))(\frac{15}{2}\cos(4\theta) + \frac{3}{2}\cos(4\eta))}{2 - 2\cos(\theta - \eta)}d\eta\\ &-3\frac{(\cos(4\theta) - \cos(4\eta))(5\cos(2\theta) + \cos(2\eta))}{2 - 2\cos(\theta - \eta)}d\eta\\ &-16\frac{(\cos(4\theta) + \cos(4\eta))(\cos(2\theta) - \cos(2\eta))}{2 - 2\cos(\theta - \eta)}d\eta\\ &-32\frac{\cos(4\eta)\sin(2\theta)\sin(\theta - \eta)}{2 - 2\cos(\theta - \eta)}d\eta\Big]\\ =:\ & P_2K_1 + P_2K_2 + P_2K_3 + P_2K_4 + P_2K_5.\end{aligned}$$

For K_1, we compute

$$P_2K_1 = P_2(34\cos(2\theta)\cos(4\theta)) = 17P_2(\cos(2\theta) + \cos(6\theta)) = 17\cos(2\theta). \tag{9.41}$$

For K_2 we compute

$$\begin{aligned}P_2K_2 &= P_2\left(-15\cos(4\theta)\frac{\cos(2\theta) - \cos(2\eta)}{2 - 2\cos(\theta - \eta)}d\eta - 3\frac{(\cos(2\theta) - \cos(2\eta))\cos(4\eta)}{2 - 2\cos(\theta - \eta)}d\eta\right)\\ &= P_2\left(-15\cos(2\theta)\cos(4\theta) + \frac{3}{2}(\cos(2\theta) - \cos(6\theta))\right)\\ &= P_2(-6\cos(2\theta) - 9\cos(6\theta))\\ &= -6\cos(2\theta),\end{aligned} \tag{9.42}$$

where the second equality follows from (9.47) and Lemma 9.4.9. For K_3, we compute

$$\begin{aligned}P_2K_3 &= P_2\left(-15\cos(2\theta)\frac{\cos(4\theta) - \cos(4\eta)}{2 - 2\cos(\theta - \eta)}d\eta - 3\frac{(\cos(4\theta) - \cos(4\eta))\cos(2\eta)}{2 - 2\cos(\theta - \eta)}d\eta\right)\\ &= P_2\left(-30\cos(2\theta)\cos(4\theta) - 3\cos(6\theta)\right)\\ &= P_2\left(-15\cos(2\theta) - 18\cos(6\theta)\right)\\ &= -15\cos(2\theta),\end{aligned} \tag{9.43}$$

where the second equality follows from (9.47) and Lemma 9.4.9. For K_4, we compute

$$P_2K_4 = P_2\left(-16\cos(4\theta)\frac{\cos(2\theta) - \cos(2\eta)}{2 - 2\cos(\theta - \eta)}d\eta - 16\frac{(\cos(2\theta) - \cos(2\eta))\cos(4\eta)}{2 - 2 - \cos(\theta - \eta)}d\eta\right)$$

$$
\begin{aligned}
&= P_2\left(-16\cos(2\theta)\cos(4\theta) + 8\cos(2\theta) - 8\cos(6\theta)\right)\\
&= P_2\left(-16\cos(6\theta)\right)\\
&= 0,
\end{aligned}
\tag{9.44}
$$

where the second equality follows from (9.47) and Lemma 9.4.9. For K_5, we compute

$$
\begin{aligned}
P_2K_5 &= -16P_2(\sin(2\theta)\sin(4\theta))\\
&= -8P_2\left(\cos(2\theta) - \cos(6\theta)\right)\\
&= -8\cos(2\theta),
\end{aligned}
\tag{9.45}
$$

where the first equality follows from (9.46). Hence it follows from (9.41), (9.42), (9.43), (9.44) and (9.45) that

$$
\frac{d^2}{dtds}Q\mathcal{F} = \left(0, P_2\frac{d^2}{dtds}\tilde{\mathcal{F}}_2\right) = (0, -12\cos(2\theta)),
$$

which proves (9.36).

To prove (9.37), note that $\frac{1}{3}Q\frac{d^3}{dt^3}\mathcal{F} = \left(0, \frac{1}{3}P_2\frac{d^3}{dt^3}\tilde{\mathcal{F}}_2\right)$. We use Lemma 9.4.6 with $r = \cos(2\theta)$ and obtain

$$
\begin{aligned}
\frac{1}{3}P_2\frac{d^3}{dt^3}\tilde{\mathcal{F}}_2(2, tv) =& P_2\left[4\frac{(\cos(2\theta) - \cos(2\eta))(3\cos^2(2\theta) + 2\cos(2\theta)\cos(2\eta) + \cos^2(2\eta))}{2 - 2\cos(\theta - \eta)}d\eta\right.\\
&-8\frac{(\cos(2\theta) - \cos(2\eta))^3}{(2 - 2\cos(\theta - \eta))^2}d\eta\\
&-16\sin(2\theta)\frac{(\cos(2\theta) - \cos(2\eta))^2\sin(\theta - \eta)}{(2 - 2\cos(\theta - \eta))^2}d\eta\\
&+8\frac{(3\cos^2(2\theta) - 4\sin^2(2\theta))(\cos(2\theta) - \cos(2\eta))}{2 - 2\cos(\theta - \eta)}d\eta\\
&\left.+\left(32\cos(2\theta)\sin^2(2\theta) - 16\cos^3(2\theta)\right)\right]\\
&= P_2L_1 + P_2L_2 + P_2L_3 + P_2L_4 + P_2L_5.
\end{aligned}
$$

For L_1, we compute

$$
\begin{aligned}
P_2L_1 = P_2\Bigg(&12\cos^2(2\theta)\frac{\cos(2\theta) - \cos(2\eta)}{2 - 2\cos(\theta - \eta)}d\eta + 8\cos(2\theta)\frac{(\cos(2\theta) - \cos(2\eta))\cos(2\eta)}{2 - 2\cos(\theta - \eta)}d\eta\\
&+2\frac{\cos(2\theta) - \cos(2\eta)}{2 - 2\cos(\theta - \eta)}d\eta + 2\frac{(\cos(2\theta) - \cos(2\eta))\cos(4\eta)}{2 - 2\cos(\theta - \eta)}d\eta\Bigg)
\end{aligned}
$$

$$= P_2\left(12\cos^3(2\theta) + 8\cos(2\theta)\left(-\frac{1}{2} + \frac{1}{2}\cos(4\theta)\right) + 2\cos(2\theta) + (-\cos(2\theta) + \cos(6\theta))\right)$$
$$= P_2(8\cos(2\theta) + 6\cos(6\theta))$$
$$= 8\cos(2\theta),$$

where the second equality follows from (9.47) and 9.4.9. For L_2, we use Lemma 9.4.10 and obtain

$$P_2L_2 = -8P_2\left(\frac{9}{4}\cos(2\theta) - \cos(6\theta)\right)$$
$$= -18\cos(2\theta).$$

For L_3, we use Lemma 9.4.11 and obtain

$$P_2L_3 = 16P_2\sin(2\theta)\sin(4\theta) = 8P_2(\cos(2\theta) - \cos(6\theta)) = 8\cos(2\theta).$$

For L_4, we compute

$$P_2L_4 = P_2\left(24\cos^2(2\theta)\frac{\cos(2\theta) - \cos(2\eta)}{2 - 2\cos(\theta - \eta)}d\eta - 32\sin^2(2\theta)\frac{\cos(2\theta) - \cos(2\eta)}{2 - 2\cos(\theta - \eta)}d\eta\right)$$
$$= P_2(24\cos^3(2\theta) - 32\sin^2(2\theta)\cos(2\theta))$$
$$= P_2(10\cos(2\theta) + 14\cos(6\theta))$$
$$= 10\cos(2\theta),$$

where the second equality follows from (9.47). For L_5, it follows immediately that

$$P_2L_5 = P_2\left(-4\cos(2\theta) - 12\cos(6\theta)\right) = -4\cos(2\theta).$$

Collecting the above results, we obtain

$$\frac{1}{3}P_2\frac{d^3}{dt^3}\tilde{\mathcal{F}}_2(2, tv) = 4\cos(2\theta),$$

which implies (9.37).

To prove (9.38), we use (9.34) and (9.25) in Lemma 9.4.4 with $g = 0,\ \tilde{g} = 2\cos(2\theta),\ r = \cos(2\theta)$

and $\tilde{r} = 0$ and obtain

$$\begin{aligned} P_2 \frac{d^2}{dtds} \tilde{\mathcal{F}}_2(2, tv + s\tilde{v}) &= P_2 \left(-4\cos^2(2\theta) + 4\frac{(\cos(2\theta) + \cos(2\eta))(\cos(2\theta) - \cos(2\eta))}{2 - 2\cos(\theta - \eta)} d\eta \right. \\ &\quad \left. + 8\frac{\cos(2\eta)\sin(2\theta)\sin(\theta - \eta)}{2 - 2\cos(\theta - \eta)} d\eta \right) \\ &= P_2 \left(-4\cos^2(2\theta) + 4\cos^2(2\theta) + 4\left(-\frac{1}{2} + \frac{1}{2}\cos(4\theta) \right) + 4\sin^2(2\theta) \right) \\ &= 0, \end{aligned}$$

where the second equality follows from (9.47), (9.46) and (9.4.9). This implies (9.38).

To prove (9.39), it follows from (9.35) and Lemma 9.4.2 that

$$\partial_b D\mathcal{F}(2, 0) \left(-8\cos(4\theta), \frac{3}{2}\cos(4\theta) \right) = (3\sin(4\theta), 2\cos(4\theta)).$$

By projecting it to the image of Q, we obtain (9.39).

To prove (9.40), we note that $Q\partial_b D\mathcal{F}(2, 0)\tilde{v} = \left(0, P_2\partial_b \frac{d}{dt}\tilde{\mathcal{F}}_2(2, t\tilde{v})\right)\Big|_{t=0}$. Hence it follows from (9.21) in Lemma 9.4.2 and (9.34) that $P_2\partial_b \frac{d}{dt}\tilde{\mathcal{F}}_2(2, t\tilde{v})\Big|_{t=0} = \cos(2\theta)$. This implies (9.40). □

9.4.3 Basic Integrals

Lemma 9.4.8. *For $\mathbb{N} \ni m \geq 1$, it holds that*

$$\frac{\cos(m\eta)\sin(\theta - \eta)}{2 - 2\cos(\theta - \eta)} d\eta = \frac{1}{2}\sin(m\theta), \tag{9.46}$$

$$\frac{\cos(m\theta) - \cos(m\eta)}{2 - 2\cos(\theta - \eta)} d\eta = \frac{m}{2}\cos(m\theta). \tag{9.47}$$

Proof. For (9.46), it is clear that $\frac{\cos(m\eta)\sin(\theta-\eta)}{2-2\cos(\theta-\eta)} d\eta = \frac{1}{2}\cos(m\eta)\cot\left(\frac{\theta-\eta}{2}\right) d\eta = \frac{1}{2}H(\cos(m\theta))(\theta)$, where H denotes the Hilbert transform in the periodic domain. Therefore the result follows immediately since $H(\cos(m\theta))(\theta) = \sin(m\theta)$.

For (9.47), we recall that $\frac{f(\theta)-f(\eta)}{1-\cos(\theta-\eta)} d\eta = \Lambda f(\theta) =: (-\Delta)^{\frac{1}{2}} f(\theta)$. Thus (9.47) follows immediately.

□

Lemma 9.4.9.

$$\frac{(\cos(2\theta) - \cos(2\eta))\cos(4\eta)}{2 - 2\cos(\theta - \eta)} d\eta = -\frac{1}{2}\cos(2\theta) + \frac{1}{2}\cos(6\theta), \tag{9.48}$$

$$\frac{(\cos(2\theta) - \cos(2\eta))\cos(2\eta)}{2 - 2\cos(\theta - \eta)} d\eta = -\frac{1}{2} + \frac{1}{2}\cos(4\theta), \tag{9.49}$$

$$\frac{(\cos(4\theta)-\cos(4\eta))\cos(2\eta)}{2-2\cos(\theta-\eta)}d\eta = \cos(6\theta), \tag{9.50}$$

$$\frac{(\cos(2\theta)-\cos(2\eta))\sin(2\eta)}{2-2\cos(\theta-\eta)}d\eta = \frac{1}{2}\sin(4\theta). \tag{9.51}$$

Proof. We will show (9.48) only. (9.49), (9.50) and (9.51) can be proved in the same way. For (9.48), one can write

$$\begin{aligned}\frac{(\cos(2\theta)-\cos(2\eta))\cos(4\eta)}{2-2\cos(\theta-\eta)}d\eta &= \frac{1}{2}\Lambda(\cos(2\theta)\cos(4\theta))(\theta) - \frac{1}{2}\cos(2\theta)\Lambda(\cos(4\theta))(\theta)\\ &= -\frac{1}{2}\cos(2\theta)+\frac{1}{2}\cos(6\theta),\end{aligned}$$

where the last equality follows from (9.47). □

Lemma 9.4.10.

$$\frac{(\cos(2\theta)-\cos(2\eta))^3}{(2-2\cos(\theta-\eta))^2}d\eta = \frac{9}{4}\cos(2\theta)-\cos(6\theta).$$

Proof. We compute

$$\begin{aligned}&\frac{(\cos(2\theta)-\cos(2\eta))^3}{(2-2\cos(\theta-\eta))^2}d\eta = \frac{(-2\sin(\theta-\eta)\sin(\theta+\eta))^3}{\left(4\sin^2\left(\frac{\theta-\eta}{2}\right)\right)^2}d\eta = 4-\frac{\cos^3\left(\frac{\theta-\eta}{2}\right)\sin^3(\theta+\eta)}{\sin\left(\frac{\theta-\eta}{2}\right)}d\eta\\ &= 4\frac{\cos^3\frac{\eta}{2}\sin^3(2\theta+\eta)}{\sin\frac{\eta}{2}}d\eta = 4\frac{\cos^3\frac{\eta}{2}}{\sin\frac{\eta}{2}}\left(3\sin^2(2\theta)\cos(2\theta)\cos^2\eta\sin\eta+\cos^3(2\theta)\sin^3\eta\right)d\eta\\ &= 12\sin^2(2\theta)\cos(2\theta)\frac{\cos^3\frac{\eta}{2}\cos^2\eta\sin\eta}{\sin\frac{\eta}{2}}d\eta + 4\cos^3(2\theta)\frac{\cos^3\frac{\eta}{2}\sin^3\eta}{\sin\frac{\eta}{2}}d\eta = \frac{21}{4}\sin^2(2\theta)\cos(2\theta)+\frac{5}{4}\cos^3(2\theta)\\ &= \frac{9}{4}\cos(2\theta)-\cos(6\theta),\end{aligned}$$

which proves the lemma. □

Lemma 9.4.11.

$$\frac{(\cos(2\theta)-\cos(2\eta))^2\sin(\theta-\eta)}{(2-2\cos(\theta-\eta))^2}d\eta = -\sin(4\theta).$$

Proof. Using the integration by parts, we compute

$$\frac{(\cos(2\theta)-\cos(2\eta))^2\sin(\theta-\eta)}{(2-2\cos(\theta-\eta))^2}d\eta = -2\frac{(\cos(2\theta)-\cos(2\eta))\sin(2\eta)}{2-2\cos(\theta-\eta)}d\eta.$$

Therefore the result follows from (9.51). □

Chapter 10

The flexibility results for 2D-Euler-infinite energy

10.1 Functional equations and functional setting for stationary vortex patches

Let us consider vorticity ω of the form $\omega := \sum_{i=1}^{n} \Theta_i 1_{D_i}$ for some $n \in \mathbb{N}$, where $\Theta_i \in \mathbb{R}$ and D_i is a simply-connected domain for each $i = 1, \dots, n$. Suppose the boundary of D_i is parametrized by a time-dependent 2π-periodic curve $z_i(\cdot, t) : \mathbb{T} \mapsto \mathbb{R}^2$. The evolution equation of the boundary can be written as a system of equations,

$$\partial_t z_j(x,t) \cdot \partial_x z_j(x,t)^\perp = u(z_j(x,t),t) \cdot \partial_x z_j(x,t)^\perp. \tag{10.1}$$

where $u(\cdot,t) := \nabla^\perp \left(\omega(\cdot,t) * \frac{1}{2\pi} \log|\cdot|\right)$ is the velocity vector. We will look for stationary solutions to (10.1). Let us assume that each z_i can be written as

$$z_i(x) = (b_i + R_i(x))(\cos(x), \sin(x)), \quad x \in \mathbb{T}, \tag{10.2}$$

where b_i is a positive constant for each i. By plugging these parametrizations into (10.1), we are led to solve the following system for $b := (b_1, \dots, b_n)$, $\Theta := (\Theta_1, \dots, \Theta_n)$ and $R := (R_1, \dots, R_n)$:

$$0 = \mathcal{F}_j(b, \Theta, R) := u(z_j(\cdot)) \cdot z_j'^{\perp} := \sum_{i=1}^{n} \frac{1}{4\pi} \Theta_i u_{i,j}(R) \cdot z_j'^{\perp} = \sum_{i=1}^{n} \frac{1}{4\pi} \Theta_i \underbrace{\left(u_{i,j}^{\theta}(R) R_j' - u_{i,j}^{r}(R)(b_j + R_j) \right)}_{=: S_{ij}(b_i, b_j, R_i, R_j)}. \tag{10.3}$$

where $u_{i,j}(R)(x) := \nabla^{\perp} \left(1_{D_i} * \log |\cdot|^2 \right) (z_j(x))$, which can be thought of as a contribution of the ith patch on the jth curve, and $u_{i,j}^{\theta}$ and $u_{i,j}^{r}$ are the angular and the radial components of $u_{i,j}$. More explicitly, we have

$$\begin{aligned} S_{ij}(b_i, b_j, R_i, R_j) = &\int_0^{2\pi} \cos(x-y)((b_i + R_i(y))R_j'(x) - (b_j + R_j(x))R_i'(y)) \\ &\times \log((b_j + R_j(x))^2 + (b_i + R_i(y))^2 - 2(b_j + R_j(x))(b_i + R_i(y)) \cos(x-y)) dy \\ &- \int_0^{2\pi} \sin(x-y)((b_i + R_i(y))(b_j + R_j(x)) + R_i'(y)R_j'(x)) \\ &\times \log((b_j + R_j(x))^2 + (b_i + R_i(y))^2 - 2(b_j + R_j(x))(b_i + R_i(y)) \cos(x-y)) dy. \end{aligned} \tag{10.4}$$

It is clear that $S_{ij}(b_i, b_j, 0, 0) = 0$, since $u_{i,j}^{r} = 0$ for radial patches. Thus we have

$$\mathcal{F}(b, \Theta, 0) = 0, \text{ for any } b, \Theta. \tag{10.5}$$

In what follows, we will pick one of Θ_i as a bifurcation parameter, while the others are fixed.

We now proceed to discuss the functional spaces that we will use. In Section 10.2 we will work with the following analytic spaces. Following [17], we denote the space of analytic functions in the strip $\{z \in \mathbb{C} : \mathrm{Im}(z) \le c\}$ by $\mathcal{C}_w(c)$. For $k \in \mathbb{Z}$, we will consider the following spaces of $\frac{2\pi}{m}$-periodic functions:

$$X_c^{k,m} := \left\{ f(x) \in \mathcal{C}_w(c), \quad f(x) = \sum_{j=1}^{\infty} a_{jm} \cos(jmx), \quad \|f\|_{X_c^{k,m}} < \infty \right\},$$

$$Y_c^{k,m} := \left\{ f(x) \in \mathcal{C}_w(c), \quad f(x) = \sum_{j=1}^{\infty} a_{jm} \sin(jmx), \quad \|f\|_{Y_c^{k,m}} < \infty \right\},$$

where $\|f\|_{X_c^{k,m}} = \|f\|_{Y_c^{k,m}} := \sum_{j=1}^{\infty} |a_{jm}|^2 (1+jm)^{2k} (\cosh(cjm)^2 + \sinh(cjm)^2)$.

In Section 11.1 we will consider the following spaces of $\frac{2\pi}{m}$-periodic functions:

$$X^{k,m} := \left\{ f(x) \in H^k(\mathbb{T}) : \quad f(x) = \sum_{j=1}^{\infty} a_{jm} \cos(jmx), \quad \|f\|_{X^{k,m}} < \infty \right\},$$
$$Y^{k,m} := \left\{ f(x) \in H^k(\mathbb{T}) : \quad f(x) = \sum_{j=1}^{\infty} a_{jm} \sin(jmx), \quad \|f\|_{Y^{k,m}} < \infty \right\},$$

where $\|f\|_{X^{k,m}} = \|f\|_{Y^{k,m}} := \|f\|_{H^k(\mathbb{T})}$.

10.2 Warm-up: Existence of non-radial stationary vortex patches with infinite energy

As a warm-up, in this section we aim to show that there exists a non-trivial patch solution with infinite kinetic energy, $\frac{1}{2\pi}\int_{\mathbb{R}^2} |\nabla (\omega * \log|\cdot|)|^2 dx = \infty$. Recall that ([71, Proposition 3.3]),

$$\int_{\mathbb{R}^2} |\nabla (\omega * \log|\cdot|)|^2 dx < \infty \iff \int_{\mathbb{R}^2} \omega(x) dx = 0. \tag{10.6}$$

In our proof, we will find a continuous bifurcation curve, emanating from a two-layered vortex patch whose vorticity does not has zero mean.

10.2.1 Main results for infinite energy

Let us consider two-layer vortex patches, that is, $i \in \{1, 2\}$ in the setting in Section 10.1. For $b, \Theta \in (0, 1)$, using the scaling invariance of the equations, we will choose the parameters to be

$$b_1 = 1, \quad b_2 = b, \quad \Theta_1 = \Theta, \quad \Theta_2 = -1, \tag{10.7}$$

so that if $R_1 = R_2 = 0$, then

$$\omega = \begin{cases} \Theta & \text{in } B_1 \backslash B_b \\ \Theta - 1 & \text{in } B_b, \end{cases} \tag{10.8}$$

where B_r denotes the unit disk centered at the origin. Note that the case $\Theta = 1$ corresponds to an annulus. Later, we will fix b as well, and let Θ play the role as a bifurcation parameter. With this

setting, the system (10.3) is equivalent to

$$0 = \mathcal{F}(\Theta, R) = (F^1(\Theta, R), F^2(\Theta, R)), \quad R = (R_1, R_2), \tag{10.9}$$

where we omit the dependence of Θ_2 and b_1, b_2 for notational simplicity.

Theorem 10.2.1. *Let k, m be such that $k \geq 3$, $2 \leq m \in \mathbb{N}$. Let b satisfy the condition in Lemma 10.2.8. Then for some $c > 0$ and $s_0 = s_0(k, m, c, b) > 0$, there exist two bifurcation curves $[0, s_0) \ni s \mapsto (\Theta^{\pm}(s), R^{\pm}(s)) \in \mathbb{R} \times (X_c^{k,m})^2$ such that for each $s \in (0, s_0)$, $(\Theta^{\pm}(s), R^{\pm}(s))$ is a solution of the equation (10.9) and $R^{\pm}(s) \neq 0 \in (X_c^{k,m})^2$. The bifurcation curve emanates from $(\Theta^{\pm}(0), R^{\pm}(0)) = (\Theta_m^{\pm}, 0)$, where $\Theta_m^{\pm}$ are defined in Lemma 10.2.8.*

Theorem 10.2.1 immediately implies the existence of non-radial stationary vortex patches with infinite energy.

Corollary 10.2.2. *Let $2 \leq m \in \mathbb{N}$. Then there is an m-fold symmetric stationary patch solution for the 2D Euler equation with analytic boundary and infinite kinetic energy, that is*

$$\int_{\mathbb{R}^2} |\nabla \left(\omega * \frac{1}{2\pi} \log |\cdot| \right)|^2 dx = \infty.$$

Proof. By Theorem 10.2.1, there are two continuous bifurcation curves $\Psi^{\pm} : [0, s_0) \mapsto \mathbb{R} \times \left(X_c^{k,m}\right)^2$ of solutions of (10.9) such that

$$\Psi^{\pm}(s) = \left(\Theta^{\pm}(s), R(s)\right), \quad F(\Psi^{\pm}(s)) = 0, \quad \text{and} \quad \Psi^{\pm}(0) = \left(\Theta_m^{\pm}, 0\right),$$

for some $s_0 > 0$. From (10.1) and (10.3), it is clear that for each $s \in (0, s_0)$ and for each choice of $\pm$,

$$\omega^s(x) := \begin{cases} \Theta - 1 & \text{if } x \in D_2(s), \\ \Theta & \text{if } x \in D_1(s) \backslash \overline{D_2(s)} \\ 0 & \text{otherwise,} \end{cases} \tag{10.10}$$

is a stationary solution to the Euler equation, where $D_1(s)$ and $D_2(s)$ are the bounded domains determined by (10.2) with $R(s)$. Since $R(s) \in \left(X_c^{k,m}\right)^2$, the boundaries are analytic.

Now we consider the kinetic energy of the solution. From (10.6), it suffices to show that

$\int_{\mathbb{R}^2} \omega^0(x)dx \neq 0$. By the continuity of the bifurcation curve, this immediately implies that $\int_{\mathbb{R}^2} \omega^s(x)dx \neq 0$ for small $s > 0$. Then it follows from Lemma 10.2.11 that $b^2 < \Theta_m^{\pm}$, hence

$$\int_{\mathbb{R}^2} \omega^0(x)dx = \pi\left(-b^2 + \Theta_m^{\pm}\right) > 0.$$

This completes the proof. □

The rest of this section will be devoted to prove Theorem 10.2.1. The proof will be divided into 5 steps. These steps correspond to check the hypotheses of the Crandall-Rabinowitz theorem 11.4.1 for our functional F in (10.9). The hypotheses in Theorem 11.4.1 can be read as follows in our setting:

1. The functional $\mathcal{F}$ satisfies

$$\mathcal{F}(\Theta, R) \,:\, (0,1) \times \{V^\epsilon\} \mapsto (Y_c^{k-1,m})^2,$$

   where V^ϵ is an open neighborhood of 0,

$$V^\epsilon = \left\{(f,g) \in (X_c^{k,m})^2 \,:\, ||f||_{X_c^{k,m}} + ||g||_{X_c^{k,m}} < \epsilon\right\}$$

   for some $\epsilon > 0$ and $k \geq 3$.

2. $\mathcal{F}(\Theta, 0) = 0$ for every $0 < \Theta < 1$.

3. The partial derivatives $\partial_\Theta \mathcal{F}$, $D\mathcal{F}$, $\partial_\Theta D\mathcal{F}$ exist and are continuous, where $D\mathcal{F}$ is Gateaux derivative of $\mathcal{F}$ with respect to the functional variable R.

4. $\mathrm{Ker}(D\mathcal{F}(\Theta_m^{\pm}, 0)) \subset (X_c^{k,m})^2$ and $(Y_c^{k-1,m})^2/\mathrm{Im}(D\mathcal{F}(\Theta_m^{\pm}, 0))$ are one-dimensional (see Proposition 10.2.8 for the definition of $\Theta_m^{\pm}$).

5. $\partial_\Theta D\mathcal{F}(\Theta_m^{\pm}, 0)[v_0] \notin \mathrm{Im}(D\mathcal{F}(\Theta_m^{\pm}, 0))$, where v_0 is a non-zero element in $\mathrm{Ker}(D\mathcal{F}(\Theta_m^{\pm}, 0))$.

Remark 10.2.3. *We remark that if $i \neq j$ then the functions inside the logarithm in $S_{i,j}$ in* (10.4) *are uniformly bounded from below in y for all x by a strictly positive constant depending on the parameters. Then we can analytically extend the integrand in x to the strip $|\Im(z)| \leq c$ in such a way that the real part of this extension stays uniformly bounded away from 0 for a small enough c. The case $i = j$ can be treated similarily as in [17, Remark 2.1].*

10.2.2 Proof of Theorem 10.2.1

Steps 1,2 and 3: Regularity

In order to check the first three steps, it suffices to check if S_{ij} in (10.4) satisfies the hypotheses, since F is a linear combination of S_{ij}. As mentioned in Remark 10.2.3, the case $i \neq j$ is trivial since there is no singularity in the integrand and analytically extended into a strip in $\mathbb{C}$, if (R_1, R_2) is in a sufficiently small neighborhood of $(0,0)$. For $i = j$, the first three steps with slightly different settings were already done in the literature. For example, step 1 can be done in the same way as in [29]. Step 2 follows immediately from (10.5). Existence and continuity of the Gateaux derivatives for the gSQG equation was done in [41] and the same proof can be adapted to our setting straightforwardly.

Step 4: Analysis of the linear part.

In this section, we will focus on the spectral study of the Gateaux derivative $D\mathcal{F}(\Theta, 0) := D_R\mathcal{F}(\Theta, 0)$.

Calculation of $D\mathcal{F}$ We aim to express the Gateaux derivative of $\mathcal{F}$ around $(\Theta, 0)$ in the direction $(H(x), h(x))$ in terms of Fourier series.

Lemma 10.2.4. *Let S_{ij} be defined as in* (10.4)*. Then:*

$$\frac{d}{dt} S_{ij}(b_i, b_j, th_i, th_j)\Big|_{t=0} = \int b_i(h_j'(x)\cos(x-y) - h_i'(y))\log(b_j^2 + b_i^2 - 2b_j b_i \cos(x-y))dy =: \mathcal{L}_1 + \mathcal{L}_2.$$

Proof. Let $V_{ij}^{1,ab}$ (resp. $V_{ij}^{2,ab}$) be the contribution of the first term (resp. second term) of (10.4) where the first factor contributes with a t^a and the second with t^b. We are looking for all combinations such that $a + b = 1$. We start looking at the first summand. We have that:

$$V_{ij}^{1,10} = \int \cos(x-y)(b_i h_j'(x) - b_j h_i'(y))\log(b_j^2 + b_i^2 - 2b_j b_i\cos(x-y))dy.$$

Similarly, for the second one,

$$V_{ij}^{2,01} = -2\int \sin(x-y)(b_i b_j)\frac{h_j(x)(b_j - b_i\cos(x-y)) + h_i(y)(b_i - b_j\cos(x-y))}{b_j^2 + b_i^2 - 2b_j b_i\cos(x-y)}dy,$$
$$V_{ij}^{2,10} = -\int \sin(x-y)(b_i h_j(x) + b_j h_i(y))\log(b_j^2 + b_i^2 - 2b_j b_i\cos(x-y))dy,$$

where we have used Lemma 11.4.2. Integrating by parts in $V_{ij}^{2,01}$:

$$V_{ij}^{2,01} = \int ((b_i h_j(x) + b_j h_i(y))\sin(x-y) - h_i'(y)(b_i - b_j\cos(x-y)))\log(b_j^2 + b_i^2 - 2b_j b_i\cos(x-y))dy,$$

Finally, adding all the log terms and the non-log terms together:

$$V_{ij}^{1,10} + V_{ij}^{2,01} + V_{ij}^{2,10} = \int b_i(h_j'(x)\cos(x-y) - h_i'(y))\log(b_j^2 + b_i^2 - 2b_j b_i \cos(x-y))dy,$$

as we wanted to prove. □

Lemma 10.2.5. *Let $h_i = A_i \cos(mx)$ and $r = \frac{\min\{b_i,b_j\}}{\max\{b_i,b_j\}}$. Then:*

$$\frac{d}{dt} S_{ij}(b_i, b_j, th_i, th_j)\Big|_{t=0} = 2\pi \sin(mx) m b_i \left(A_j r - A_i \frac{r^m}{m}\right).$$

Proof. From Lemma 10.2.4 and Corollary 11.4.4, we have that

$$\mathcal{L}_1 = -m b_i A_j \sin(mx)\mathcal{A}_0(r,1),$$
$$\mathcal{L}_2 = m b_i A_i \sin(mx)\mathcal{A}_0(r,m),$$

Adding the two contributions gives the desired result. □

Note that the functional $\mathcal{F}$ is a linear combination of S_{ij} (see (10.3)). Using the above two lemmas, we obtain the following proposition:

Proposition 10.2.6. *Let $h(x) = \sum_n a_n \cos(nx)$, $H(x) = \sum_n A_n \cos(nx)$, then we have that:*

$$D\mathcal{F}(\Theta, 0)[H, h] = \begin{pmatrix} U(x) \\ u(x) \end{pmatrix},$$

where

$$U(x) = \sum_n U_n \sin(nx), \quad u(x) = \sum_n u_n \sin(nx),$$

and the coefficients satisfy, for any n:

$$\begin{pmatrix} U_n \\ u_n \end{pmatrix} := (-n) M_n(\Theta) \begin{pmatrix} A_n \\ a_n \end{pmatrix} := (-n) \begin{pmatrix} \frac{b^2}{2} - \frac{\Theta}{2} + \frac{\Theta}{2n} & -\frac{b^{n+1}}{2n} \\ \Theta \frac{b^n}{2n} & -\frac{b}{2n} + \frac{b}{2}(1-\Theta) \end{pmatrix} \begin{pmatrix} A_n \\ a_n \end{pmatrix}.$$

Proof. It follows from (10.7), the definition of $\mathcal{F}$ in (10.9), (10.3) and Lemma 10.2.5. □

One dimensionality of the Kernel of the linear operator. Our goal here is to verify the one dimensionality of $\mathrm{Ker}(D\mathcal{F}(\Theta,0))$ for some Θ. More precisely, we will prove the following proposition:

Proposition 10.2.7. *Fix $2 \leq m \in \mathbb{N}$ and take any $b \in (0, b_m)$, where b_m is as in Lemma 10.2.8. Then there exist two $\Theta_m^{\pm} \in (0,1)$, such that $Ker(D\mathcal{F}(\Theta_m^{\pm},0))$ is one-dimensional. Furthermore,*

$$Ker(D\mathcal{F}(\Theta_m^{\pm},0,0)) = span\left\{ v_0(\Theta_m^{\pm})\cos(mx) := \begin{pmatrix} \frac{1}{2m}b - \frac{b}{2}(1-\Theta_m^{\pm}) \\ \frac{\Theta_m^{\pm}}{2m}b^m \end{pmatrix} \cos(mx) \right\} \subset X_c^{k,m} \times X_c^{k,m}.$$

The proof of the above proposition relies on the analysis of the matrix M_n in Lemma 10.2.8 and 10.2.9, which we will prove below.

Lemma 10.2.8. *Let $\Delta_m(\Theta)$ be*

$$\Delta_m(\Theta) := \frac{4m^2}{b} det(M_m(\Theta)) = \big(\Theta b^{2m} + b^2 m(m(1-\Theta)-1) + \Theta(1-m)(m(1-\Theta)-1)\big) \quad (10.11)$$

Then, for any $m \geq 2$, there exists $0 < b_m < 1$ such that for any $0 < b < b_m < 1$, there exists $0 < \Theta_m^- < \Theta_m^+ < 1$ such that $\Delta_m(\Theta_m^{\pm}) = 0$. We also have that $rk(M_m(\Theta_m^{\pm})) = 1$ for those values of Θ_m^+, Θ_m^-, where $rk(A)$ is the rank of a matrix A.

Proof. For fixed m, we study the polynomial $\Delta_m(\Theta)$. We need to solve

$$\Delta_m(\Theta) = m(m-1)\Theta^2 - ((m-1)^2 + b^2m^2 - b^{2m})\Theta + b^2m(m-1) = 0. \quad (10.12)$$

Since $\Delta_m(\Theta)$ is a quadratic function, we only need to show that the discriminant is positive. We have

$$\begin{aligned} D_m &:= ((m-1)^2 + b^2m^2 - b^{2m})^2 - 4m^2(m-1)^2b^2 \\ &= ((m-1)^2 + b^2m^2 - b^{2m} - 2m(m-1)b)((m-1)^2 + b^2m^2 - b^{2m} + 2m(m-1)b) \\ &= (m-1-bm-b^m)(m-1-bm+b^m)((m-1)^2 + b^2m^2 - b^{2m} + 2m(m-1)b) \\ &=: D_{m,1} \cdot D_{m,2} \cdot D_{m,3}. \end{aligned}$$

Since $m \geq 2$, $0 < b < 1$, we have $D_{m,3} > 1 - 1 = 0$. We also have

$$D_{m,2} = m(1-b) + b^m - 1 = (1-b)[m - \frac{1-b^m}{1-b}] = (1-b)(m - (1 + b + b^2 + ... + b^{m-1})) > 0.$$

$D_{m,1}$ is decreasing in b when $b \geq 0$ and $D_{m,1}(0) = m - 1 > 0$, $D_{m,1}(1) = -2$. Let b_m be the only zero point of $D_{m,1}$ in (0,1). If we take $0 < b < b_m$, we have

$$D_{1,m} > 0, D_m > 0. \tag{10.13}$$

Hence $\Delta_m(\Theta)$ has two different solutions $\Theta_m^- < \Theta_m^+$. Moreover, the matrix does not vanish when $b \neq 0$, implying $\mathrm{rk}(M_m(\Theta_m^\pm)) = 1$. Now we are left to show $0 < \Theta_m^\pm < 1$. We have

$$\Theta_m^+ \Theta_m^+ = b^2 < 1,$$

and

$$\Theta_m^+ + \Theta_m^- = \frac{((m-1)^2 + b^2 m^2 - b^{2m})}{m(m-1)} \geq \frac{1-1}{m(m-1)} > 0.$$

Hence, $0 < \Theta_m^- < 1$ and $0 < \Theta_m^+$. If $\Theta_m^+ \geq 1$, then $\Delta_m(1) \leq 0$. However,

$$\begin{aligned} \Delta_m(1) &= b^{2m} - mb^2 - (1-m) \\ &= (1-b^2)(m - \frac{1-b^{2m}}{1-b^2}) \\ &= (1-b^2)(m - 1 - b^2 - ... - b^{2m-2}) > 0. \end{aligned} \tag{10.14}$$

Therefore $0 < \Theta_m^\pm < 1$. □

We now show that $\Delta_{jm}(\Theta_m^\pm) \neq 0$ for any $j \neq 1$.

Lemma 10.2.9. *Let $j > 1$ and let $\Theta_m^\pm$ and $\Theta_{jm}^\pm$ be defined as in the previous Lemma. Then $\Theta_{jm}^+ > \Theta_m^+ > \Theta_m^- > \Theta_{jm}^-$. Hence, $M_{jm}(\Theta_m^\pm)$ is non-singular for all $j > 1$.*

Proof. Since $\Theta_m^\pm$ solves the equation

$$\Theta^2 - \frac{((m-1)^2 + b^2 m^2 - b^{2m})}{m(m-1)}\Theta + b^2 = 0.$$

We only need to show $F(b,m) := \frac{((m-1)^2 + b^2 m^2 - b^{2m})}{m(m-1)}$ is strictly increasing with respect to m. We have

$$\begin{aligned} &F(b, m+1) - F(b,m) \\ &= \frac{(m^2 + b^2(m+1)^2 - b^{2(m+1)})}{(m+1)m} - \frac{((m-1)^2 + b^2 m^2 - b^{2m})}{m(m-1)} \end{aligned}$$

$$= \frac{(m^2 + b^2(m+1)^2 - b^{2(m+1)})(m-1) - ((m-1)^2 + b^2m^2 - b^{2m})(m+1)}{(m+1)m(m-1)}$$
$$= \frac{b^2((m+1)^2(m-1) - m^2(m+1)) + m^2(m-1) - (m+1)(m-1)^2 - b^{2m+2}(m-1) + (m+1)b^{2m}}{(m+1)m(m-1)}$$
$$= \frac{-b^2(m+1) + m - 1 - b^{2m+2}(m-1) + b^{2m}(m+1)}{(m+1)m(m-1)}.$$

Thus

$$F(b, m+1) - F(b, m) > 0 \Leftrightarrow -b^2(m+1) + m - 1 - b^{2m+2}(m-1) + b^{2m}(m+1) > 0$$
$$\Leftrightarrow (1+b^2)(-1+b^{2m}) - (-1+b^2)(1+b^{2m})m > 0 \Leftrightarrow -(1+b^2)(b^{2m-2} + b^{2m-4} ... + b^2 + 1) + (1+b^{2m})m > 0$$
$$\Leftrightarrow -\sum_{k=1}^{m} b^{2m-2k} - \sum_{k=1}^{m} b^{2k} + (1+b^{2m})m > 0 \Leftrightarrow \sum_{k=1}^{m}(1-b^{2k})(1-b^{2m-k}) > 0.$$

It is easy to show the last inequality since $0 < b < 1$. □

Proof of Proposition 10.2.7: Let $m \geq 2$ and let b, $\Theta_m^{\pm}$ be as defined in Lemma 10.2.8. Assume that $H(x) = \sum_j A_{jm} \cos(jmx)$ and $h(x) = \sum_j a_{jm} \cos(jmx)$ satisfy

$$DF(\Theta_m^{\pm}, 0)[H, h] = (0, 0).$$

Then it follows from Proposition 10.2.6 that

$$M_{jm}(\Theta_m^{\pm}) \begin{pmatrix} A_{jm} \\ a_{jm} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

For all $j > 1$, it follows from Lemma 10.2.9 that $M_{jm}(\Theta_m^{\pm})$ is invertible, thus $A_{jm} = a_{jm} = 0$. For $j = 1$, Lemma 10.2.8 tells us that $(A_m, a_m) \in \mathrm{Ker}(M_m(\Theta_m^{\pm})) = \mathrm{span}\left\{ v_0(\Theta_m^{\pm}) := \begin{pmatrix} \frac{1}{2m} b - \frac{b}{2}(1 - \Theta_m^{\pm}) \\ \frac{\Theta_m^{\pm}}{2m} b^m \end{pmatrix} \right\}$. This finishes the proof. □

Codimension of the image of the linear operator. We now characterize the image of $D\mathcal{F}(\Theta_m^{\pm}, 0)$. We have the following proposition:

Proposition 10.2.10. *Let*

$$Z = \left\{ (Q, q) \in Y_c^{k-1,m} \times Y_c^{k-1,m}, Q(x) = \sum_{j=1}^{\infty} Q_{jm} \sin(jmx), q(x) = \sum_{j=1}^{\infty} q_{jm} \sin(jmx), \right.$$

$$\exists \lambda_{Q,q} \in \mathbb{R} \ s.t. \begin{pmatrix} Q_m \\ q_m \end{pmatrix} = \lambda_{Q,q} \begin{pmatrix} -\frac{1}{2m} b^{m+1} \\ -\frac{1}{2m} b + \frac{b}{2}(1-\Theta_m^{\pm}) \end{pmatrix} \Bigg\}.$$

Then $Z = Im\,(D\mathcal{F}(\Theta_m^{\pm}, 0, 0))$.

Proof. In view of Proposition 10.2.6, $\mathrm{Im}\,(D\mathcal{F}(\Theta_m^{\pm}, 0)) \subset Z$ is trivial, since $M_{jm}(\Theta_m^{\pm})$ is non-singular for $j > 1$, and $\mathrm{Im}(M_m(\Theta_m^{\pm})) = \mathrm{span}\left\{ \begin{pmatrix} -\frac{1}{2m} b^{m+1} \\ -\frac{1}{2m} b + \frac{b}{2}(1-\Theta) \end{pmatrix} \right\}$. In order to prove $\mathrm{Im}\,(D\mathcal{F}(\Theta_m^{\pm}, 0)) \supset Z$, we need to check whether the possible preimage satisfies the desired regularity. To do so, we have the following asymptotic lemma:

Lemma 10.2.11. *For fixed $m \geq 2$ and b defined as in Proposition 10.2.7, we have $b^2 - \Theta_m^{\pm} < 0$ and*

$$\frac{b}{4j^2m^2} \Delta_{jm}(\Theta_m^{\pm}) = \frac{b}{4} \left(b^2 - \Theta_m^{\pm}\right)(1 - \Theta_m^{\pm}) + O\left(\frac{1}{jm}\right), \qquad as\ j \to \infty, \tag{10.15}$$

Consequently, we have

$$det(M_{jm}(\Theta_m^{\pm}))^{-1} \lesssim_{m,\Theta} 1, \qquad for\ sufficiently\ large\ j. \tag{10.16}$$

Remark 10.2.12. *As shown in the above lemma, there is no bifurcation curve from the two-layered vortex patch with zero-average. This is due to the fact that the radial vorticity ω determined by b and $\Theta^{\pm}$ as in (10.8) satisfies $\int_{\mathbb{R}^2} \omega dx = \pi\left(-b^2 + \Theta_m^{\pm}\right) > 0$. Note that if we require $\Theta = b^2$ to ensure $\int \omega dx = 0$, it follows from (10.11) that $\Delta_m(b^2) = mb^2(1-b^2) + b^2(b^{2m} - 1)$, which does not vanish for any $m \geq 2$ unless $b = 0$ or $b = 1$. Therefore, for any $0 < b < 1$, the linearized operator is an isomorphism and the implicit function theorem shows that there cannot be a bifurcation.*

bstar$_t hetaFirstweshowthat$b$^2 < \Theta_m^{\pm}$. By Lemma 10.2.8, $\Theta_m^{+}\Theta_m^{-} = b^2$ and $0 < \Theta_m^{\pm} < 1$. Thus $\Theta_m^{\pm} > b^2$. Therefore the first assertion is proved. The second assertion follows directly from (10.11) since

$$\frac{b}{4(jm)^2} \Delta_{jm}(\Theta_m^{\pm}) = \frac{b}{4(jm)^2} \left((b^2 - \Theta_m^{\pm})(1 - \Theta_m^{\pm})(jm)^2 + O\,(m)\right) = \frac{b}{4}\left(b^2 - \Theta_m^{\pm}\right)(1 - \Theta_m^{\pm}) + O\left(\frac{1}{jm}\right).$$

Lastly, by choosing j large so that $|\frac{b}{4(jm)^2} \Delta_{jm}(\Theta_m^{\pm})| > \frac{b}{8}(\Theta_m^{\pm} - b^2)(1 - \Theta_m^{\pm}) > 0$, we have

$$|\det(M_{jm}(\Theta_m^{\pm})^{-1})| = \frac{1}{|\frac{b}{4(jm)^2} \Delta_{jm}(\Theta_m^{\pm})|} \lesssim 1,$$

which proves (10.16). Now for an element $(Q,q) \in Z$, let (H,h) be such that

$$H(x) = \sum_{j=1}^{\infty} A_{jm} \cos(jmx), \quad h(x) = \sum_{j=1}^{\infty} a_{jm} \cos(jmx),$$

with

$$\begin{pmatrix} A_m \\ a_m \end{pmatrix} = -\frac{1}{m} \begin{pmatrix} 0 \\ \lambda_{Q,q} \end{pmatrix}, \quad \begin{pmatrix} A_{jm} \\ a_{jm} \end{pmatrix} = (-jm)^{-1} M_{jm}(\Theta_m^{\pm})^{-1} \begin{pmatrix} Q_{jm} \\ q_{jm} \end{pmatrix} \quad \text{for } j > 1.$$

It is clear from (10.2.6) that $D\mathcal{F}(\Theta_m^{\pm}, 0, 0)(H,h) = (Q,q)$. We will prove that $(H,h) \in X_c^{k,m} \times X_c^{k,m}$. From Lemma 10.2.11 and the fact that $M_{jm}(\Theta_m^{\pm})$ is nonsingular for $j > 1$, it follows that

$$|A_{jm}|^2 + |a_{jm}|^2 \lesssim (jm)^{-2} \left(|Q_{jm}|^2 + |q_{jm}|^2 \right) \quad \text{for all } j > 1. \tag{10.17}$$

Thus, we obtain

$$\begin{aligned} \|H\|^2_{X_c^{k,m}} + \|h\|^2_{X_c^{k,m}} &= \sum_{j=1}^{\infty} \left(|A_{jm}|^2 + |a_{jm}|^2 \right) (1+jm)^{2k} (\cosh(cjm)^2 + \sinh(cjm)^2) \\ &\lesssim \frac{1}{m^2} \lambda_{Q,q}^2 (1+m)^{2k} (\cosh(cm)^2 + \sinh(cm)^2) \\ &\quad + \sum_{j=2}^{\infty} (jm)^{-2} \left(|Q_{jm}|^2 + |q_{jm}|^2 \right) (1+jm)^{2k} (\cosh(cjm)^2 + \sinh(cjm)^2) \\ &\lesssim 1 + (\|Q\|^2_{Y_c^{k-1,m}} + \|q\|^2_{Y_c^{k-1,m}}) \\ &< \infty. \end{aligned}$$

This proves that $(H,h) \in X_c^{k,m} \times X_c^{k,m}$, and therefore $Z \subset \mathrm{Im}\,(D\mathcal{F}(\Theta_m^{\pm}, 0))$. □

Step 5: Transversality

Proposition 10.2.13. *We have that*

$$\partial_{\Theta} D\mathcal{F}(\Theta_m^{\pm}, 0)[v_0] \notin Im(D\mathcal{F}(\Theta_m^{\pm}, 0)), \tag{10.18}$$

where $v_0 = v_0(\Theta_m^{\pm})$ is as given in Proposition 10.2.7.

Proof. For $h(x) = \sum_j a_{jm} \cos(jmx)$, $H(x) = \sum_j A_{jm} \cos(jmx)$, we have that (see Proposition 10.2.6):

$$\partial_\Theta D\mathcal{F}(\Theta_m^\pm, 0)[H,h] = \begin{pmatrix} U(x) \\ u(x) \end{pmatrix},$$

where

$$U(x) = \sum_j U_{jm}\sin(jmx), \quad u(x) = \sum_j u_{jm}\sin(jmx),$$

and the coefficients satisfy, for any j:

$$\begin{pmatrix} U_{jm} \\ u_{jm} \end{pmatrix} := (-jm)\partial_\Theta M_{jm}(\Theta_m^\pm)\begin{pmatrix} A_{jm} \\ a_{jm} \end{pmatrix} := (-jm)\begin{pmatrix} -\frac{1}{2}+\frac{1}{2m} & 0 \\ \frac{b^m}{2m} & -\frac{b}{2} \end{pmatrix}\begin{pmatrix} A_{jm} \\ a_{jm} \end{pmatrix}.$$

Letting

$$v_0(\Theta_m^\pm) = \begin{pmatrix} \frac{b}{2m} - \frac{b}{2}(1-\Theta_m^\pm) \\ \frac{\Theta_m^\pm}{2m}b^m \end{pmatrix}, \quad w(\Theta_m^\pm) = \begin{pmatrix} -\frac{1}{2m}b^{m+1} \\ -\frac{1}{2m}b + \frac{b}{2}(1-\Theta_m^\pm) \end{pmatrix},$$

be the generators of $\mathrm{Ker}(M_m(\Theta_m^\pm))$ and $\mathrm{Im}(M_m(\Theta_m^\pm))$ respectively, the transversality condition is equivalent to prove that $w_1(\Theta_m^\pm)$ and $w(\Theta_m^\pm)$ are not parallel, where

$$w_1(\Theta_m^\pm) = \partial_\Theta M_m(\Theta_m^\pm)v_0(\Theta_m^\pm) = \begin{pmatrix} \frac{b}{4}\left(\frac{1}{m}-1\right)\left(\frac{1}{m}-(1-\Theta_m^\pm)\right) \\ \frac{b^{m+1}}{4m}\left(\frac{1}{m}-1\right) \end{pmatrix}$$

This is equivalent to prove that:

$$0 \neq -\frac{b^2}{8}\left(\frac{1}{m}-1\right)\left(\frac{1}{m}-(1-\Theta_m^\pm)\right)^2 + \frac{b^{2m+2}}{8m^2}\left(\frac{1}{m}-1\right) \Leftrightarrow \left(\Theta_m^\pm - (1-\frac{1+b^m}{m})\right)\left(\Theta_m^\pm - (1-\frac{1-b^m}{m})\right) \neq 0.$$

We prove it by contradiction. If $\Theta_m^\pm = 1 - \frac{1+b^m}{m}$, we have $b^m = m(1-\Theta_m^\pm) - 1$. Moreover, by (10.11),we have

$$\Theta_m^\pm b^{2m} + b^{m+2}m + \Theta_m^\pm(1-m)b^m = 0.$$

Hence,

$$\begin{aligned}
&\Theta_m^{\pm} b^m + b^2 m + \Theta_m^{\pm}(1-m) = 0 \\
&\Rightarrow \Theta_m^{\pm}(m(1-\Theta_m^{\pm}) - 1) + b^2 m + \Theta_m^{\pm}(1-m) = 0 \\
&\Rightarrow -m(\Theta_m^{\pm})^2 + b^2 m = 0.
\end{aligned}$$

Since $\Theta_m^{\pm} > 0, b > 0$, we have

$$\Theta_m^{\pm} = b,$$

implying a contradiction since $\Theta_m^{+}\Theta_m^{-} = b^2$ and $\Theta_m^{+} \neq \Theta_m^{-}$. If $\Theta_m^{\pm} = 1 - \frac{1-b^m}{m}$, we can follow the same way to get $\Theta_m^{\pm} = b$ and get a contradiction. □

Proof of Theorem 10.2.1: All the hypotheses of the Crandall-Rabinowitz theorem were checked in Propositions 10.2.7, 10.2.10 and 10.2.13. Therefore the desired result follows immediately. □

Chapter 11

The flexibility results for 2D-Euler-finite energy

11.1 Existence of non-radial stationary vortex patches with finite energy

In this section, we aim to prove that there exist non-trivial patch solutions with finite kinetic energy, $\frac{1}{2\pi}\int_{\mathbb{R}^2}|\nabla(\omega * \log|\cdot|)|^2 dx < \infty$. As mentioned in (10.6), this property is equivalent to $\int_{\mathbb{R}}\omega dx = 0$. By Remark 10.2.12, we can not use two-layer patches and instead we will consider three-layer patches.

11.1.1 Main results for finite energy

We consider vortex patches with three layers, that is, $i \in \{1,2,3\}$ in the setting in Section 10.1. The total vorticity that we consider is of the form $\omega = \sum_{i=1}^{3}\Theta_i 1_{D_i}$, where D_i is determined by $\partial D_i = \{(b_i + R_i(x))(\cos x, \sin x) : x \in \mathbb{T}\}$. We will look for a bifurcation curve from the radial one, $\sum_{i=1}^{3}\Theta_i 1_{B_{b_i}}$, where B_r denotes the disk with radius r centered at the origin. We have the following parameters and functional variables:

- $b_i \in \mathbb{R}$: the radii of the different layers of the annuli. We will have $1 =: b_1 > b_2 > b_3$.
- $\Theta_i \in \mathbb{R}$: the vorticity at the different layers. We will choose $\Theta_1 := 1$.
- $R := (R_1, R_2, R_3) \in (X^{k,m})^3$, for some $3 \leq k \in \mathbb{N}$: the functional variables that determine the boundaries.

In the rest of this section, we will fix m, b_2 and Θ_2 so that for $2 \le m \in \mathbb{N}$,

$$0 < b_2 < \left(\frac{1}{2}\right)^{\frac{1}{2m}} \quad \text{and} \quad \frac{m({b_2}^2-1)}{(1-{b_2}^{2m})b_2^2} < \Theta_2 < \min\left\{2\frac{{b_2}^{2m-2}(b_2^2-1)m}{1-b_2^{2m}},\ \frac{-1}{b_2^2}\right\}. \tag{11.1}$$

Given R, Θ_i, b_1 and b_2, we choose b_3 so that

$$\int_{\mathbb{R}^2} \omega(x)dx = \sum_{i=1}^{3}\int_0^{2\pi} \Theta_i(b_i + R_i(x))^2 dx = 0. \tag{11.2}$$

Since b_1, b_2 and Θ_1, Θ_2 are fixed constants, (11.2) implies that b_3 is a function of Θ_3 and R, more precisely,

$$\begin{aligned} b_3 &= b_3(\Theta_3, R) \\ &= \sqrt{-\frac{1}{2\pi\Theta_3}\left(\Theta_1\int_0^{2\pi}(b_1+R_1(x))^2dx + \Theta_2\int_0^{2\pi}(b_2+R_2(x))^2dx + \Theta_3\int_0^{2\pi}R_3(x)^2dx\right)} \end{aligned} \tag{11.3}$$

If $\Theta_3, b_3(\Theta_3, R) \neq 0$, then its derivative with respect to R is given by

$$\begin{aligned} Db_3(\Theta_3,R)[h] &:= \frac{d}{dt}b_3(\Theta_3, R+th)\Big|_{t=0} \\ &= -\frac{1}{2\pi\Theta_3 b_3(\Theta_3,R)}\left(\Theta_1\int_{\mathbb{T}}R_1(x)h_1(x)dx + \Theta_2\int_{\mathbb{T}}R_2(x)h_2(x)dx + \Theta_3\int_{\mathbb{T}}R_3(x)h_3(x)dx\right), \end{aligned} \tag{11.4}$$

where we used $\int_{\mathbb{T}} h_i(x)dx = 0$ for $h_i \in X^{k,m}$.

Note that for sufficiently small $\|R_i\|_{L^\infty}$ and $|\Theta_3 - \Theta^*_{3,m}|$, where $\Theta^*_{3,m}$ is as defined in Lemma 11.1.24, we can choose $b_3 = b_3(\Theta_3, R_i)$ so that (11.2) is compatible with $b_2 > b_3 > 0$ (see Lemma 11.1.24). Therefore, a 4-tuple $(\Theta_3, R_1, R_2, R_3) =: (\Theta_3, R)$ uniquely determines $\omega = \sum_{i=1}^3 \Theta_i 1_{D_i}$ such that the boundary of the ith patch surrounds the jth patch if $i < j$ and $\int_{\mathbb{R}}\omega dx = 0$. In the proof, Θ_3 will play the role of the bifurcation parameter and we will look for a bifurcation from $(\Theta^*_{3,m}, 0) \in \mathbb{R}\times(X^{k,m})^3$.

With this setting, the system (10.3) is equivalent to

$$0 = G(\Theta_3, R) := (G_1(\Theta_3,R), G_2(\Theta_3,R), G_3(\Theta_3,R)), \tag{11.5}$$

where

$$G_j(\Theta_3,R) := \mathcal{F}_j(b(\Theta_3,R),\Theta_3,R)), \text{ and } b(\Theta_3,R) := (b_1,b_2,b_3(\Theta_3,R)), \quad j=1,2,3. \tag{11.6}$$

Now, we are ready to state the main theorem of this section:

Theorem 11.1.1. *Let $k \geq 3$ and $2 \leq m \in \mathbb{N}$, $\Theta_1 = b_1 = 1$ and b_2 and Θ_2 as in (11.1). Then for some $s_0 = s_0(m, k, b_2, \Theta_2) > 0$, there exists a bifurcation curve $[0, s_0) \ni s \mapsto (\Theta_3(s), R(s)) \in \mathbb{R} \times (X^{k,m})^3$ such that for each $s \in (0, s_0)$, $(\Theta_3(s), R(s))$ is a solution of the equation (11.5) and $(R(s)) \neq 0 \in (X^{k,m})^3$. The bifurcation curve emanates from $(\Theta_3(0), R(0)) = (\Theta^*_{3,m}, 0)$, where $\Theta^*_{3,m}$ is defined in Lemma 11.1.24.*

Theorem 11.1.1 immediately implies the existence of non-radial stationary vortex patches with finite kinetic energy.

Corollary 11.1.2. *Let $2 \leq m \in \mathbb{N}$ and $k \geq 3$. Then there is an m-fold symmetric stationary patch solution of the 2D Euler equation with H^k-regular boundary and finite kinetic energy, that is*

$$\int_{\mathbb{R}^2} \left| \nabla \left(\omega * \frac{1}{2\pi} \log |\cdot| \right) \right|^2 dx < \infty.$$

Proof. By the definition of b_3 in (11.2), each $\omega = \sum_{i=1}^3 \Theta_i 1_{D_i}$ which is determined by $(\Theta_3(s), R(s))$ for $s \in (0, s_0)$, satisfies $\int_{\mathbb{R}} \omega dx = 0$. This is equivalent to $\int_{\mathbb{R}^2} |\nabla \left(\omega * \frac{1}{2\pi} \log |\cdot| \right)|^2 dx < \infty$, (see (10.6)). □

The existence of the bifurcation curves will be proved by means of a Nash-Moser iteration scheme (Theorem 11.1.7). The proof of Theorem 11.1.1 will be accomplished in Subsection 11.1.5 by checking the hypotheses of Theorem 11.1.7.

11.1.2 Compactly supported velocity

In this subsection, we digress briefly to observe an interesting consequence of Theorem 11.1.1. Thanks to a simple maximum principle lemma, it can be shown that each stationary solution on the bifurcation curves has compactly supported velocity.

Lemma 11.1.3. *(the key Lemma) Assume that $\omega \in L^1 \cap L^\infty(\mathbb{R}^n)$ for $n \geq 2$ is compactly supported and let Ω be the unbounded connected component of $supp(\omega)^c$. We additionally assume that $\int_{\mathbb{R}^n} \omega dx = 0$. Then for $f := \omega * \mathcal{N}$, where*

$$\mathcal{N}(x) = \begin{cases} \frac{1}{2\pi} \log |x| & \text{if } n = 2 \\ \frac{1}{n(2-n)V_n} |x|^{2-n}, \quad V_n := \frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)} & \text{if } n > 2, \end{cases}$$

it holds that

$$\sup_{x\in\Omega} f(x) = \max_{x\in\partial\Omega} f(x) \quad \textit{and} \quad \inf_{x\in\Omega} f(x) = \min_{x\in\partial\Omega} f(x).$$

Consequently, if f is constant on $\partial\Omega$, then f is constant in Ω.

Remark 11.1.4. *The above lemma does not hold without the assumption $\int_{\mathbb{R}^n} \omega dx = 0$. For example, $f := 1_{B_1} * \mathcal{N}$ is harmonic in $\Omega := B_1^c$, while $\max_\Omega f$ is unbounded in Ω^c.*

Proof. It suffices to prove the maximum part since we can apply the argument to $-f$ for the minimum part. The proof is classical but we present a proof for the sake of completeness.

Let $M := \sup_{x\in\Omega} f(x)$. Then $A := \{x \in \Omega : f(x) = M\}$ is relatively closed in Ω, since f is continuous. Furthermore, since f is harmonic in Ω, the mean value property yields that A is open. Therefore A must be either $\emptyset$ or Ω since Ω is connected. If $A = \Omega$, then the result follows trivially. Now, let us suppose that $A = \emptyset$. Towards a contradiction, assume that $M > \max_{x\in\partial\Omega} f(x)$. Since f is bounded in $\mathbb{R}^n$ (this property still holds when $n = 2$, thanks to $\int_{\mathbb{R}^2} \omega dx = 0$), therefore the only possible case is that

$$\lim_{|x|\to\infty} f(x) = M. \tag{11.7}$$

Let us consider $\phi(r) := \frac{1}{|\partial B_r|} \int_{|x|=r} f(x) d\sigma(x)$. For any sufficiently large r such that $\Omega^c \subset B_r$, where B_r is the ball centered at the origin with radius r, it follows that

$$\frac{d}{dr}\phi(r) = \frac{d}{dr}\frac{1}{|S_{n-1}|}\int_{|x|=1} f(rx) d\sigma(x) = \frac{1}{|\partial B_r|}\int_{|x|<r} \omega(x) dx = 0,$$

Furthermore, (11.7) yields that

$$\lim_{r\to\infty} \phi(r) = M,$$

hence, we have $\phi(r) = M$ for all sufficiently large r. For such $r > 0$, we have

$$M = \frac{1}{|\partial B_r|}\int_{|x|=r} f(x) d\sigma \le \max_{x\in\partial B_r} f(x) \le M.$$

This implies A cannot be empty, which is a contradiction. Hence $M = \max_{x\in\partial\Omega} f(x)$. □

Corollary 11.1.5. *Let $2 \le m \in \mathbb{N}$ and $k \ge 3$. There exists an m-fold symmetric stationary patch*

solution of the 2D Euler equation with H^k-regular boundary and compactly supported velocity.

Proof. Let $\omega = \sum_{i=1}^{3} \Theta_i 1_{D_i}$ be the vorticity determined by $(\Theta_3(s), R(s))$ for $s \in (0, s_0)$ in Theorem 11.1.1. From (10.1) and (10.3), it follows that its stream function $f := \frac{1}{2\pi}\omega * \log|\cdot|$ is constant on the outmost boundary ∂D_1. However, Lemma 11.1.3 implies that $\sup_{D_1^c} f = \inf_{D_1^c} f$, therefore, f is constant in D_1^c. This proves that $\mathrm{supp}(\nabla^\perp f) \subset D_1$. □

Remark 11.1.6. *In this paper we focus on patch type solutions $\omega = \sum_{i=1}^{n} \Theta_i 1_{D_i}$. Lemma 11.1.3 is still applicable for smooth ω as well as long as the boundary of $\Omega := supp(\omega)$ can be approximated by regular level curves of ω (See Figure 11.1). This is due to the fact that the stream function of the smooth stationary ω must be constant on each regular level set of ω (see [42, Section 1]), hence the stream function has to be constant on each connected component of $\partial\Omega$. If $\partial\Omega$ has only one connected component, then the velocity vanishes in the unbounded component of Ω^c.*

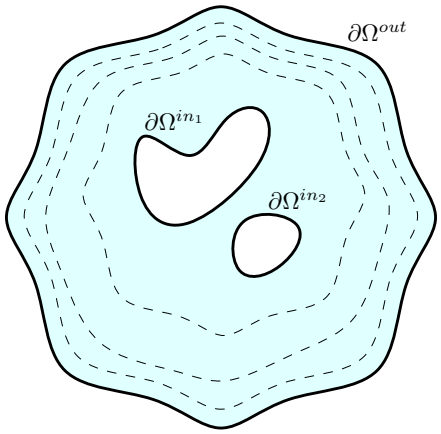


Figure 11.1: Illustration for smooth stationary ω where $\Omega := \mathrm{supp}(\omega)$ is colored in blue, whose boundary is not necessarily connected. The dashed lines are regular level sets of ω, which converges to the outermost boundary. In such case, the stream function is constant on the outermost boundary $\partial\Omega^{out}$, thus Lemma 11.1.3 implies that the velocity vanishes in the unbounded part of Ω^c.

11.1.3 Nash-Moser theorem

We first prove a bifurcation theorem using the Nash-Moser scheme under some assumptions, which will turn out to be satisfied by our nonlinear functional. We follow the ideas from Berti [5].

Let $2 \leq m \in \mathbb{N}$ fixed. We denote

$$X^k := X^{k,m}, \quad Y^k := Y^{k,m}, \quad C^\infty := C^\infty(\mathbb{T}), \quad R := (R_1, R_2, R_3) \in \left(X^{k+1}\right)^3 \cap (C^\infty)^3, \quad \partial_\Theta := \partial_{\Theta_3},$$

for simplicity. Furthermore, for a Banach space X and an element $R \in X$, we denote the norm of R by $|R|_X := \|R\|_X$. In addition, we use the notation $A \lesssim_{a,b} B$ if there exists a constant $C = C(a,b) > 0$ depending on some variables a, b such that $A \leq CB$. We also use $c_0, c_1, ...$ to denote universal constants that may vary from line to line.

Theorem 11.1.7. *Assume that there exists $\Theta_3^* \in \mathbb{R}$ and an open neighborhood $I \times V^3$ of $(\Theta_3^*, 0) \in \mathbb{R} \times \left(X^3\right)^3$ such that for each $2 \leq k \in \mathbb{N}$, $G : I \times \left(X^{k+1}\right)^3 \mapsto \left(Y^k\right)^3$ satisfies the following: For $(\Theta_3, R) \in I \times \left(V^3 \cap (C^\infty)^3\right)$*

(a) (Existence of a curve of trivial solutions) $G(\Theta_3, 0) = 0$ for all $\Theta_3 \in I$.

(b) (Regularity) It holds that

$$|G(\Theta_3, R)|_{(Y^k)^3} \lesssim_k 1 + |R|_{(X^{k+1})^3}, \tag{11.8}$$

$$|\partial_\Theta G(\Theta_3, R)|_{(Y^k)^3} \lesssim_k 1 + |R|_{(X^{k+1})^3} \tag{11.9}$$

$$|D^2 G(\Theta_3, R)[h,h]|_{(Y^k)^3} \lesssim_k (1 + |R|_{(X^{k+3})^3})|h|^2_{(X^{k+1})^3} \tag{11.10}$$

$$|\partial_\Theta DG(\Theta_3, R)[h]|_{(Y^k)^3} \lesssim_k (1 + |R|_{(X^{k+3})^3})|h|_{(X^{k+1})^3} \tag{11.11}$$

$$|\partial_\Theta D^2 G(\Theta_3^*, R)[h,h]|_{(Y^k)^3} \lesssim_k (1 + |R|_{(X^{k+3})^3})|h|^2_{(X^{k+1})^3} \tag{11.12}$$

$$|\partial_{\Theta\Theta} DG(\Theta_3, R)[h]|_{(Y^k)^3} \lesssim_k (1 + |R|_{(X^{k+3})^3})|h|_{(X^{k+1})^3} \tag{11.13}$$

(c) (Decomposition of DG) $DG(\Theta_3, R) \in \mathcal{L}((X^{k+1})^3; (Y^k)^3)$ has the following decomposition:

$$DG(\Theta_3, R)[h] = a(\Theta_3, R)[h] + A(\Theta_3, R)[h],$$

such that $a(\Theta_3, R) \in \mathcal{L}((X^{k+1})^3, (Y^k)^3)$, $A(\Theta_3, R) \in \mathcal{L}((X^{k+1})^3, Y^{k+1} \times (Y^k)^2)$. Also, there exists $0 < \eta < 1$ such that if $\|R\|_{(H^{k+2})^3} \leq \eta$, then

$$|a(\Theta_3, R)[h]|_{(Y^k)^3} \lesssim_k |G(\Theta_3, R)|_{(Y^k)^3} |h|_{(X^{k+1})^3}, \tag{11.14}$$

$$|A(\Theta_3, R)[h]|_{Y^{k+1} \times (Y^k)^2} \lesssim_k (1 + |R|_{(X^{k+3})^3})|h|_{(X^{k+1})^3}, \tag{11.15}$$

($\tilde{c}$-1) $A : \mathbb{R} \times (X^{k+3})^3 \mapsto \mathcal{L}((X^{k+1})^3, Y^{k+1} \times (Y^k)^2)$ is Lipschitz continuous. That is, if

$$(\Theta_3^1, R^1),\ (\Theta_3^2, R^2) \in I \times (V^3 \cap (C^\infty)^3),$$

and $\|R^1\|_{(H^{k+3}(\mathbb{T}))^2}, \|R^2\|_{(H^{k+3}(\mathbb{T}))^2} \leq 1$*, it holds that*

$$|A(\Theta_3^1, R^1)[h] - A(\Theta_3^2, R^2)|_{(Y^k)^3} \lesssim_k \left(|\Theta_3^1 - \Theta_3^2| + |R^1 - R^2|_{(X^{k+3})^3}\right) |h|_{(X^{k+1})^3}. \tag{11.16}$$

(c̃-2) (Tame estimates) There exists $0 < \eta < 1$ *such that if* $\|R\|_{(H^{k+3}(\mathbb{T}))^3} \leq \eta$*, and* $A(\Theta_3, R)[h] = z$ *for some* $z \in (C^\infty)^3$ *and* $h \in Ker(A(\Theta_3^*, 0))^\perp$*, then* $h \in (C^\infty)^3$*. Furthermore, for any even* $\sigma \in \mathbb{N} \cup \{0\}$*, it holds that*

$$|h|_{(X^{k+1+\sigma})^3} \lesssim_{k,\sigma} (1 + |R|_{(X^{k+4+\sigma})^3}) |z|_{Y^{k+1} \times (Y^k)^2} + |z|_{Y^{k+1+\sigma} \times (Y^{k+\sigma})^2}. \tag{11.17}$$

(d) (Fredholm index zero) There exist non-zero vectors v *and* w *such that* v *and* w *are supported on the* m*-th Fourier mode and*

$$Ker(A(\Theta_3^*, 0)) = span\{v\}, \quad Im(A(\Theta_3^*, 0))^\perp = span\{w\},$$

(e) (Transversality) $\partial_\Theta A(\Theta_3^*, 0)[v] \notin Im(A(\Theta_3^*, 0))$*.*

Then, for any $k_0 \geq 2$*, there exist a constant* $s_0 = s_0(k_0) > 0$ *and a curve* $[0, s_0) \ni s \mapsto (\Theta_3(s), R(s)) \in I \times (X^{k_0+1})^3$ *such that* $G(\Theta_3, R) = 0$ *and* $R(s) \neq 0$ *for* $s > 0$*. The curve emanates from* $(\Theta_3(0), R(0)) = (\Theta_3^*, 0)$*.*

Remark 11.1.8. *Note that the evenness of* σ *for the tame estimate in* $(\tilde{c}-2)$ *is simply because* X^k *is a space of even functions and any odd order derivatives of* $h \in X^k$ *are odd.*

The rest of this section is devoted to prove Theorem 11.1.7.

Towards the proof, let $k_0 \geq 2$ be fixed. We define the projections $P : (X^{k_0+1})^3 \mapsto \mathrm{Ker}(A(\Theta_3^*, 0))$ and $Q : (Y^{k_0})^3 \mapsto \mathrm{Im}(A(\Theta_3^*, 0))^\perp$ by

$$PR := (v \cdot R)\, v \quad \text{and} \quad QR := (w \cdot R)w, \tag{11.18}$$

where $(f \cdot g)$ denotes the usual L^2 inner product. Note that from the assumptions (d) and (e), we make an ansatz that for sufficiently small $s > 0$, the bifurcation curve $s \mapsto (\Theta_3(s), R(s))$ can be written as

$$(\Theta_3(s), R(s)) = \left(\Theta_3^* + v \cdot \tilde{R}(s), sv + (I - P)\tilde{R}(s)\right),$$

for some $\tilde{R}(s) \in (X^{k_0+1})^3$ such that $|(I - P)\tilde{R}|_{(X^{k_0+1})^3} = o(s)$. From this ansatz, we define a family

of functionals $\tilde{G}_s : (X^{k_0+1})^3 \mapsto (Y^{k_0})^3$ by

$$\tilde{G}_s(R) := G(\Theta_3^* + v \cdot R, sv + (I-P)R), \tag{11.19}$$

and look for $R \in (X^{k_0+1})^3$ such that $\tilde{G}_s(R) = 0$ and $|(I-P)R|_{(X^{k_0+1})^3} = o(s)$ for sufficiently small $s > 0$. This will be achieved by Newton's method, where the first approximate solution is $R = 0$. To perform Newton's method, we need to study the linearized operator $D\tilde{G}_s$ at each approximate solution $R \neq 0$: For $h \in (X^{k_0+1})^3$, which can be directly computed from (11.19),

$$\begin{aligned} D\tilde{G}_s(R)[h] &:= D_R\tilde{G}_s(R)[h] \\ &= \partial_\Theta G(\Theta_3^* + v \cdot R, sv + (I-P)R)(v \cdot h) + DG(\Theta_3^* + v \cdot R, sv + (I-P)R)[(I-P)h] \\ &=: T_s(R)[h] + a(\Theta_3^* + v \cdot R, sv + (I-P)R)[(I-P)h], \end{aligned} \tag{11.20}$$

where

$$T_s(R)[h] := \partial_\Theta G(\Theta_3^* + v \cdot R, sv + (I-P)R)(v \cdot h) + A(\Theta_3^* + v \cdot R, sv + (I-P)R)[(I-P)h]. \tag{11.21}$$

The above decomposition of $D\tilde{G}_s$ into $T_s + a$ follows from the assumption (c). Recall that Newton's method relies on the invertibility of $T_s(R)$ for small $s > 0$. However, as we will see in the next lemma, T_s is not fully invertible between $(X^{k_0+1})^3$ and $(Y^{k_0})^3$, because of its loss of derivatives. This is a motivation of adapting a Nash-Moser scheme in our proof. However, (11.14) suggests that the inverse of T_s is a good approximate right inverse of $D\tilde{G}_s$. In the next subsection, we will focus on the properties of $T_s(R)$.

Analysis of T_s

We will look for a solution R to $\tilde{G}_s(R) = 0$ in $(X^{k_0+1})^3$ for small $s > 0$. In each step of Newton's method (Nash-Moser iteration), we will regularize the approximate solution R_n (see (11.58)). The theorem will be achieved by proving that R_n converges in $(X^{k_0+1})^3$. However, we will also obtain boundedness of the sequence R_n in the higher norms, which is necessary because of the extra regularity conditions as in (c), $(\tilde{c}-1)$ and $(\tilde{c}-2)$. For this reason, we will establish several lemmas assuming that an approximate solution R is more regular then $(X^{k_0+1})^3$, which will turn out to be true at the end of the proof.

Lemma 11.1.9. *Let $0 < \epsilon < 1$ and $2 \leq k_0 \in \mathbb{N}$ be fixed. There exist positive constants $s_0(\epsilon, k_0)$, $c_0(\epsilon, k_0)$ and $0 < \delta(\epsilon, k_0) < 1$ such that for each $0 < s < s_0$, the following holds: If*

$$|PR|_{(X^{k_0+2})^3} \leq s^{\epsilon}, \tag{11.22}$$

$$|(I-P)R|_{(X^{k_0+2})^3} \leq s^{1+\epsilon}, \tag{11.23}$$

$$|R|_{(H^{k_0+4})^3} \leq \delta, \tag{11.24}$$

then

(A) For all $t \in [0,1]$,

$$(\Theta_3^* + v \cdot R, sv + (I-P)R) \in I \times V^3, \tag{11.25}$$

where V^3 is as in Theorem 11.1.7. Therefore,

$$T_s(R)[h] := \partial_\Theta G(\Theta_3^* + v \cdot R, sv + (I-P)R)(v \cdot h) + A(\Theta_3^* + v \cdot R, sv + (I-P)R)[(I-P)h],$$

is well-defined.

(B) $T_s(R) : (X^{k_0+1})^3 \mapsto Y^{k_0+1} \times (Y^{k_0})^2$ is an isomorphism and

$$|T_s(R)[h]|_{Y^{k_0+1} \times (Y^{k_0})^2} \leq c_0 |h|_{(X^{k_0+1})^3}, \tag{11.26}$$

$$|T_s(R)^{-1}[z]|_{(X^{k_0+1})^3} \leq \frac{c_0}{s} |z|_{Y^{k_0+1} \times (Y^{k_0})^2}, \tag{11.27}$$

$$\left| \left(D\tilde{G}_s(R) \circ T_s(R)^{-1} - I \right)[z] \right|_{(Y^{k_0})^3} \leq c_0 |\tilde{G}_s(R)|_{(Y^{k_0})^3} |T_s(R)^{-1}[z]|_{(X^{k_0+1})^3}, \tag{11.28}$$

for some $c_0 = c_0(\epsilon, k_0) > 0$.

If $R \in C^\infty$, then for any even $\sigma \in \mathbb{N} \cup \{0\}$ and $z \in X^{k_0+1+\sigma} \times (X^{k_0+\sigma})^2$, we have $T_s(R)^{-1}[z] \in \left(X^{k_0+1+\sigma}\right)^3$. Also, we have that

$$\left|T_s(R)^{-1}[z]\right|_{\left(X^{k_0+1+\sigma}\right)^3} \leq \frac{c_0}{s} \left((1 + |R|_{(X^{k_0+4+\sigma})^3}) |z|_{Y^{k_0+1} \times (Y^{k_0})^2} + |z|_{Y^{k_0+1+\sigma} \times (Y^{k_0+\sigma})^2} \right), \tag{11.29}$$

where c_0 may depend on not only ϵ, k_0 but also σ.

(C) Furthermore, we can choose c_0 so that when $R = 0$, the following hold:

$$|P(T_s(0)^{-1}[z])|_{(X^{k_0+1})^3} \leq \frac{c_0}{s}|z|_{Y^{k_0+1}\times(Y^{k_0})^2}, \tag{11.30}$$

$$|(I-P)(T_s(0)^{-1}[z])|_{(X^{k_0+1})^3} \leq c_0|z|_{Y^{k_0+1}\times(Y^{k_0})^2}. \tag{11.31}$$

We will frequently use

$$|T_s(R)^{-1}z|_{\left(X^{k_0+1}\right)^3} \leq \frac{c_0}{s}|z|_{(Y^{k_0+1})^3}, \tag{11.32}$$

which is more crude than (11.27).

Proof. Let us fix $\epsilon > 0$ and $k_0 \geq 2$. We will show that if (11.22)-(11.24) hold for sufficiently small $s > 0$ and for some small $\delta > 0$ depending on ϵ, then (A),(B) and (C) hold for some $c_0 > 0$.

Proof of (A). To see (A), notice that (11.25) is trivial. From the assumptions (b) and (c) in Theorem 11.1.7, the linear operator $T_s(R)$ is well-defined.

Proof of (B). The estimate (11.26), follows from (11.21), (11.15), (11.9) and

$$|sv + (I-P)R|_{\left(H^{k_0+3}\right)^3} \lesssim 1 + \delta \lesssim 1,$$

where the last inequality follows from (11.24). Before proving (11.27), we first prove that

$$|Q\partial_\Theta G(\Theta_3^* + v\cdot R, sv + (I-P)R)|_{X^{k_0+1}\times(X^{k_0})^2} \geq cs, \tag{11.33}$$

for some $c = c(\epsilon) > 0$. The above inequality is a consequence of the transversality condition (e) in Theorem 11.1.7. In what follows, $c, c_1, c_2, \ldots$ denote positive constants that may change from line to line and depend only on ϵ but not on s.

By the regularity assumptions of G in (b) in Theorem 11.1.7, we estimate the quantity in (11.33) using a Taylor expansion up to linear order: For a fixed R, we set $f(s,y) := Q\partial_\Theta G(\Theta_3^* + v\cdot R, sv + (I-P)y)$, so that

$$f(s,R) = Q\partial_\Theta G(\Theta_3^* + v\cdot R, sv + (I-P)R). \tag{11.34}$$

Using the fundamental theorem of calculus, we can write

$$\begin{aligned} f(s,R) &:= \int_0^1 \frac{d}{dt}\left(f(s,tR)\right)dt + f(s,0) \\ &= \int_0^1 \frac{d}{dt}\left(f(s,tR)\right)dt + f(0,0) + \int_0^s \frac{d}{dt} f(t,0)dt. \end{aligned}$$

In terms of G, (11.34) and the above equality give us that

$$\begin{aligned} &|Q\partial_\Theta G(\Theta_3^* + v\cdot R, sv + (I-P)R)|_{Y^{k_0+1}\times(Y^{k_0})^2} = |f(s,R)|_{Y^{k_0+1}\times(Y^{k_0})^2} \\ &\quad \geq -\sup_{0\leq t\leq 1} \|Q\partial_\Theta DG(\Theta_3^* + v\cdot R, sv + t((I-P)R))\|_{\mathcal{L}((X^{k_0+2})^3,(Y^{k_0+1})^3)} |(I-P)R|_{\left(X^{k_0+2}\right)^3} \\ &\quad - |Q\partial_\Theta G(\Theta_3^* + v\cdot R, 0)|_{Y^{k_0+1}\times(Y^{k_0})^2} \\ &\quad + \left|\int_0^s Q\partial_\Theta DG(\Theta_3^* + v\cdot R, tv)[v]dt\right|_{Y^{k_0+1}\times(Y^{k_0})^2} \\ &\quad =: -L_1 - L_2 + L_3, \end{aligned} \tag{11.35}$$

where the first term after the inequality follows from b) in Theorem 11.1.7 and

$$\begin{aligned} \int_0^1 \left|\frac{d}{dt}\left(f(s,tR)\right)\right|_{Y^{k_0+1}\times(Y^{k_0})^2} dt &\leq \int_0^1 \left|\frac{d}{dt}\left(f(s,tR)\right)\right|_{(Y^{k_0+1})^3} dt \\ &\leq \sup_{0\leq t\leq 1} |D_R f(s,tR)[R]|_{(Y^{k_0+1})^3}, \end{aligned}$$

and the rest follows simply from the triangular inequality. Then the regularity assumption (b) in Theorem 11.1.7 yields that (recall $\|R\|_{(X^{k_0+4})^3} < \delta < 1$,)

$$L_1 \leq c_1 |(I-P)R|_{\left(X^{k_0+2}\right)^3} \leq c_1 s^{1+\varepsilon}, \tag{11.36}$$

where the last inequality follows from (11.23). For L_2, we use (a) in Theorem 11.1.7 to obtain

$$L_2 = 0. \tag{11.37}$$

To estimate L_3, we compute

$$L_3 \geq \left|\int_0^s Q\partial_\Theta DG(\Theta_3^*, tv)[v]dt\right|_{Y^{k_0+1}\times(Y^{k_0})^2} - \int_0^s \sup_{u\in(0,|v\cdot R|)} \left|Q\partial_{\Theta\Theta} DG(\Theta_3^* + u, tv)[v]\right|_{Y^{k_0+1}\times(Y^{k_0})^2} |v\cdot R| dt. \tag{11.38}$$

For the first integral, we can write

$$\begin{aligned} Q\partial_\Theta DG(\Theta_3^*, tv)[v] &= Q\partial_\Theta DG(\Theta_3^*, 0)[v] + \int_0^t Q\partial_\Theta D^2 G(\Theta_3^*, uv)[v,v]du \\ &= QA(\Theta_3^*, 0)[v] + \int_0^t Q\partial_\Theta D^2 G(\Theta_3^*, uv)[v,v]du, \end{aligned}$$

where the last equality follows from (a) and (c) in Theorem 11.1.7, which shows $a(\Theta_3^*, 0) = 0$. For the integral term, using (11.12), we have

$$\left|Q\partial_\Theta D^2 G(\Theta_3^*, uv)[v,v]\right|_{Y^{k_0+1}\times(Y^{k_0})^2} \lesssim \left|Q\partial_\Theta D^2 G(\Theta_3^*, uv)[v,v]\right|_{(Y^{k_0+1})^3} \lesssim (1+\|R\|_{(X^{k_0+4})^3})\|v\|_{(X^{k_0+2})^3} \leq c,$$

where the last inequality follows from (11.24). Therefore

$$\begin{aligned} \left|\int_0^s Q\partial_\Theta DG(\Theta_3^*, tv)[v]dt\right|_{X^{k_0+1}\times(X^{k_0})^2} &\geq s\left|QA(\Theta_3^*, 0)[v]\right|_{X^{k_0+1}\times(X^{k_0})^2} - \int_0^s\int_0^t cdudt \\ &\geq s\left|Q\partial_\Theta DG(\Theta_3^*, 0)[v]\right|_{X^{k_0+1}\times(X^{k_0})^2} - c_2 s^2 \\ &\geq c_3 s - c_2 s^2, \end{aligned}$$

where the last inequality follows from the transversality (e). For the second integral in (11.38), we have that for sufficiently small $s > 0$,

$$\sup_{u\in(0,|v\cdot R|),t\in(0,s)} \left|\partial_{\Theta\Theta} DG(\Theta_3^* + u, tv)[v]\right|_{Y^{k_0+1}\times(Y^{k_0})^2} \lesssim \sup_{t\in(0,s)} (1+|tv|_{(X^{k_0+4})^3})|v|_{(X^{k_0+2})^3} \leq c_1,$$

for sufficiently small s, which follows from (11.13). Hence using (11.22), we obtain from (11.38) that

$$L_3 \geq c_3 s - c_2 s^2 - c_1 s^{1+\epsilon} \tag{11.39}$$

Thus the claim (11.33) follows from (11.35) (11.36), (11.37) and (11.39) for small s, depending on ϵ.

Towards the proof of (11.27), we will consider how to invert $A(\Theta_3^* + v\cdot R, sv + (I-P)R)[(I-P)]$. Since the dimension of $\mathrm{Im}(DG(\Theta_3^*, 0)^\perp)$ is one and w is the basis of $\mathrm{Im}(DG(\Theta_3^*, 0))^\perp$, the above claim (11.33) proves that

$$|w\cdot\partial_\Theta G(\Theta_3^* + v\cdot R, sv + (I-P)R)| > cs. \tag{11.40}$$

To simplify the notations, we denote

$$\tilde{A}[h] := A(\Theta_3^* + v \cdot R, sv + (I-P)R)[(I-P)h], \tag{11.41}$$
$$\partial_\Theta G := \partial_\Theta G(\Theta^* + v \cdot R, sv + (I-P)R).$$

We denote by Q_1 the projection from $Y^{k_0+1} \times \left(Y^{k_0}\right)^2$ into $\mathrm{Im}(\tilde{A})^\perp$. Since the norm of R is small, we expect that the functional structure of $\tilde{A}$ should be similar to the structure of $A(\Theta_3^*, 0)$. Indeed, by continuity hypotheses (11.16), (11.24) and Lemma 11.4.15, where we can think of $\tilde{A}$ as a linear map from $\mathrm{Ker}(A(\Theta_3^*, 0))$, we can choose δ small enough so that if (11.24) holds, then there exists $0 \neq w_1 \in Y^{k_0+1} \times (Y^{k_0})^2$ such that

$$\mathrm{Im}(\tilde{A})^\perp = \mathrm{span}\,\{w_1\} \tag{11.42}$$
$$|w - w_1|_{Y^{k_0+1} \times (Y^{k_0})^2} \leq c\delta, \tag{11.43}$$
$$|\tilde{A}|_{\mathcal{L}(\mathrm{Ker}\left(A(\Theta_3^*,0)\right)^\perp, \mathrm{Im}(\tilde{A}))} \leq c, \tag{11.44}$$
$$|\tilde{A}^{-1}|_{\mathcal{L}(\mathrm{Im}(\tilde{A}), \mathrm{Ker}\left(A(\Theta_3^*,0)\right)^\perp)} \leq c. \tag{11.45}$$

Now, we claim that we can further restrict δ if necessary so that

$$|w_1 \cdot \partial_\Theta G| > cs. \tag{11.46}$$

In fact, note that (11.22)-(11.23) imply that $R \in (X^{k_0+2})^3$, thus by (11.9), we have $\partial_\Theta G \in (Y^{k_0+1})^3 \subset Y^{k_0+1} \times (Y^{k_0})^2$. Also we have

$$\begin{aligned}\partial_\Theta G &= \int_0^1 \partial_t \left(\partial_\Theta G(\Theta_3^* + v \cdot R, t(sv + (I-P)R))\right) dt \\ &= \int_0^1 \partial_\Theta DG(\Theta_3^* + v \cdot R, t(sv + (I-P)R))[sv + (I-P)R]dt,\end{aligned}$$

thus

$$|\partial_\Theta G|_{Y^{k_0+1} \times (Y^{k_0})^2} \leq |\partial_\Theta G|_{(Y^{k_0+1})^3} \leq c|sv + (I-P)R|_{(X^{k_0+2})^2} \leq cs, \tag{11.47}$$

where the second inequality follows from (11.9) and the last inequality follows from (11.23). Then

it follows that (recall that $f \cdot g$ is the dot product in L^2 space),

$$\begin{aligned}|\partial_\Theta G \cdot w_1| &\geq |\partial_\Theta G \cdot w| - |\partial_\Theta G \cdot (w_1 - w)| \\ &\geq c_1 s - |\partial_\Theta G \cdot (w_1 - w)| \\ &\geq c_1 s - cs|w - w_1|_{X^{k_0+1} \times (X^{k_0})^2} \\ &\geq c_1 s - c\delta s,\end{aligned}$$

where we used (11.40), (11.47) and (11.43) for the second, third and the fourth inequality, respectively. Hence, assuming δ is small, we have (11.46).

To prove (11.27), pick an arbitrary $z \in Y^{k_0+1} \times (Y^{k_0})^2$. There exists a unique $\eta \in \mathrm{Ker}\,(A(\Theta_3^*, 0))$ and a unique $h \in \mathrm{Ker}\,(A(\Theta_3^*, 0))^\perp$ such that

$$(v \cdot \eta) Q_1 \partial_\Theta G = Q_1 z \tag{11.48}$$

$$(v \cdot \eta)(I - Q_1)\partial_\Theta G + \tilde{A}[h] = (I - Q_1)z. \tag{11.49}$$

In fact, there exists a unique η in (11.48) thanks to (11.46) and that Q_1 is the projection to a one-dimensional space spanned by w_1. Once η is fixed, the existence and the uniqueness of h in (11.49) follows from (11.45). Once η and h are determined, it is clear that

$$T_s(R)[h + \eta] = (v \cdot (h + \eta))\partial_\Theta G + \tilde{A}[\eta + h] = (v \cdot \eta)\partial_\Theta G + \tilde{A}[h] = z, \tag{11.50}$$

where we used $v \cdot h = 0$ and $\tilde{A}[\eta] = \tilde{A}[(I - P)\eta] = 0$, which follows from the definition of $\tilde{A}$ in (11.41). Therefore $T(R)^{-1} z = h + \eta$. Furthermore we have from (11.48) and (11.49) that

$$|\eta|_{(X^{k_0+1})^3} \leq c \frac{|z \cdot w_1|}{|\partial_\Theta G \cdot w_1|} \tag{11.51}$$

$$|h|_{(X^{k_0+1})^3} \leq |\tilde{A}^{-1}(I - Q_1)\partial_\Theta G(v \cdot \eta)|_{(X^{k_0+1})^3} + |\tilde{A}^{-1}(I - Q_1)z|_{(X^{k_0+1})^3}. \tag{11.52}$$

Using (11.46) and (11.45), we obtain

$$|\eta|_{(X^{k_0+1})^3} \leq c \frac{|z|_{Y^{k_0+1} \times (Y^{k_0})^2}}{s}, \tag{11.53}$$

$$|h|_{(X^{k_0+1})^3} \leq c|\partial_\Theta G|_{Y^{k_0+1} \times (Y^{k_0})^2} \frac{|z|_{Y^{k_0+1} \times (Y^{k_0})^2}}{s} + c|z|_{Y^{k_0+1} \times (Y^{k_0})^2} \leq c \frac{|z|_{Y^{k_0+1} \times (Y^{k_0})^2}}{s}, \tag{11.54}$$

where we used (11.47). This proves (11.27).

To show (11.28), we compute

$$\left(D\tilde{G}_s(R)\circ T_s(R)^{-1}-I\right)[z]=a(\Theta_3^*+v\cdot R, sv+(I-P)R)\circ T(R)^{-1}[z],$$

which follows from (11.20). This implies

$$\begin{aligned}|\left(D\tilde{G}_s(R)\circ T_s(R)^{-1}-I\right)[z]|_{\left(X^{k_0}\right)^3} &\le c|a(\Theta_3^*-v\cdot R, sv+(I-P)R)\circ T(R)^{-1}[z]|_{\left(X^{k_0}\right)^3}\\ &\le c|G(\Theta_3^*+v\cdot R, sv+(I-P)R)|_{\left(Y^{k_0}\right)^3}|T_s(R)^{-1}[z]|_{\left(X^{k_0+1}\right)^3}\\ &\le c|\tilde{G}_s(R)|_{\left(Y^{k_0}\right)^3}|T_s(R)^{-1}[z]|_{\left(X^{k_0+1}\right)^3},\end{aligned}$$

where we used (11.14) to get the second inequality and used the definition of $\tilde{G}_s(R)$ to obtain the last inequality.

In order to prove (11.29), we improve on the estimates in (11.51) and (11.52). For (11.51), recall that η is in a one-dimensional space, span$\{v\}$ and v is supported on the m-th Fourier mode. Thus,

$$|\eta|_{(X^{k_0+1+\sigma})^3}\le c|\eta|_{(X^{k_0+1})^3}\le c\frac{|z|_{Y^{k_0+1}\times(Y^{k_0})^2}}{s}, \tag{11.55}$$

where c may depend on σ. Furthermore, ($\tilde{c}$-2) in Theorem 11.1.7 and (11.49) give us that

$$\begin{aligned}|h|_{(X^{k_0+1+\sigma})^3}\le{}& c(1+|R|_{(X^{k_0+4+\sigma})^3})\left(|(v\cdot\eta)\partial_\Theta G|_{Y^{k_0+1}\times(Y^{k_0})^2}+|z|_{Y^{k_0+1}\times(Y^{k_0})^2}\right)\\ &+c\left(|(v\cdot\eta)\partial_\Theta G|_{Y^{k_0+1+\sigma}\times(Y^{k_0+\sigma})^2}+|z|_{Y^{k_0+1+\sigma}\times(Y^{k_0+\sigma})^2}\right).\end{aligned} \tag{11.56}$$

Using (11.47) and (11.53), we have

$$|(v\cdot\eta)\partial_\Theta G|_{Y^{k_0+1}\times(Y^{k_0})^2}\le|\partial_\Theta G|_{Y^{k_0+1}\times(Y^{k_0})^2}\,|v\cdot\eta|\le c\,|z|_{Y^{k_0+1}\times(Y^{k_0})^2}.$$

For the higher norm of $\partial_\Theta G$, we use (11.9) and and (11.53) and obtain

$$\begin{aligned}|(v\cdot\eta)\partial_\Theta G|_{Y^{k_0+1+\sigma}\times(Y^{k_0+\sigma})^2}&\le c\,|\partial_\Theta G|_{(Y^{k_0+1+\sigma})^3}\,|v\cdot\eta|\\ &\le\frac{c}{s}\left(1+|R|_{(X^{k_0+2+\sigma})^3}\right)|z|_{Y^{k_0+1}\times(Y^{k_0})^2}.\end{aligned}$$

Therefore, (11.56) gives us

$$|h|_{(X^{k_0+1+\sigma})^3}\le\frac{c}{s}\left((1+|R|_{(X^{k_0+4+\sigma})^3})\,|z|_{Y^{k_0+1}\times(Y^{k_0})^2}+|z|_{Y^{k_0+1+\sigma}\times(Y^{k_0+\sigma})^2}\right).$$

Combining with (11.55), we obtain (11.29).

Proof of (C). Finally, if $R = 0$, then we have $|\partial_\Theta G|_{Y^{k_0+1}\times(Y^{k_0})^2} = |\partial_\Theta G(\Theta_3, sv)|_{Y^{k_0+1}\times(Y^{k_0})^2} \le cs$. Therefore we can improve (11.54) and obtain

$$|h|_{(X^{k_0+1})^3} \le c|z|_{Y^{k_0+1}\times(Y^{k_0})^2},$$

which implies (11.31). (11.27) follows trivially from (11.30). This completes the proof. □

Lemma 11.1.10. *Let $\tilde{G}_s$ be as in* (11.19). *Then there exists an open set V^3 near $0 \in (X^3)^3$ such that for all $R, \tilde{R} \in V^3$ and $k \ge 2$,*

(a') (Initial value) $|\tilde{G}_s(0)|_{(X^k)^3} \lesssim_k s^2$.

(b') (Taylor estimate) If $\|R\|_{(X^{k+3})^3}, \|\tilde{R}\|_{(X^{k+3})^3} \le \eta$ for some $0 < \eta = \eta(k) < 1$, then

$$\begin{cases} |D\tilde{G}_s(R)[h]|_{(Y^k)^3} \lesssim_k |h|_{(X^{k+1})^3} \\ |\tilde{G}_s(\tilde{R}) - \tilde{G}_s(R) - D\tilde{G}_s(R)[\tilde{R} - R]|_{(Y^k)^3} \lesssim_k |\tilde{R} - R|^2_{(X^{k+1})^3} \\ |\tilde{G}_s(\tilde{R}) - \tilde{G}_s(R) - D\tilde{G}_s(R)[\tilde{R} - R] - \frac{1}{2}D^2\tilde{G}_s[\tilde{R} - R, \tilde{R} - R]|_{(Y^k)^3} \lesssim_k |\tilde{R} - R|^3_{(X^{k+1})^3}. \end{cases}$$

Proof. (a') follows from the fact that $v \in \mathrm{Ker}(DG(\Theta_3^*, 0))$ and $G(\Theta_3^*, 0) = 0$. (b') is due to (b) in Theorem 11.1.7. □

Nash-Moser iteration

For $\beta, N > 0$ and $2 \le k \in \mathbb{N}$, we consider a regularizing operator $S(N) : (X^k)^3 \mapsto (C^\infty)^3$ such that

$$\begin{cases} |S(N)R|_{(X^{k+\beta})^3} \lesssim_{k,\beta} N^\beta |R|_{(X^k)^3} & \forall\ R \in (H^k)^3 \\ |(I - S(N))R|_{(H^k)^3} \lesssim_{k,\beta} N^{-\beta}|R|_{(X^{k+\beta})^3} & \forall\ R \in (H^{k+\beta})^3. \end{cases} \tag{11.57}$$

Note that we can choose S so that $PS(N) = S(N)P$, since v is supported on the m-th Fourier mode (see (d) in Theorem 11.1.7).

For $N_n > 1$ and even integer $\beta > 0$, which will be chosen later, we set

$$\begin{cases} R_{n+1} = R_n - S(N_n)T_s(R_n)^{-1}[\tilde{G}_s(R_n)] & \text{for } n \ge 0 \\ R_0 = 0, \end{cases} \tag{11.58}$$

and

$$\begin{cases} a_n := |R_{n+1} - R_n|_{(X^{k_0+1})^3} \\ a'_n := |R_{n+1} - R_n|_{(X^{k_0+2})^3} \\ a''_n = |R_{n+1} - R_n|_{(X^{k_0+4})^3} \\ b_n := |\tilde{G}_s(R_n)|_{(Y^{k_0})^3} \\ C_n := |T_s(R_n)^{-1}[\tilde{G}_s(R_n)]|_{(X^{k_0+1+\beta})^3}. \end{cases} \tag{11.59}$$

Note that our goal is to show that $\sum_{n=0}^{\infty} a_n < \infty$, which implies that R_n converges in $(X^{k_0+1})^3$. In order to prove the assumptions in (11.22)-(11.24), we will need the boundedness of a'_n and a''_n as well.

Lemma 11.1.11. *Let $\epsilon > 0$ and $k_0 \geq 2$ fixed and let s_0 and δ be defined as in Lemma 11.1.9. We also assume that δ is even smaller if necessary, so that $\delta < \eta(k_0)$ where $\eta(k_0)$ is as in (b') in Lemma 11.1.10. For any $n \geq 0$ such that R_n satisfies* (11.22), (11.23) *and* (11.24) *for some $s \in (0, s_0)$, then we have*

$$a_0 \lesssim_{\varepsilon,k_0} s, \tag{11.60}$$

$$a_n \lesssim_{\varepsilon,k_0} \frac{1}{s} N_n b_n \qquad \text{for } n \geq 1, \tag{11.61}$$

$$b_0 \lesssim_{\varepsilon,k_0,\beta} s^2 \tag{11.62}$$

$$b_1 \lesssim_{\varepsilon,k_0,\beta} (N_0^3 + 1)s^3 + N_0^{-\beta} C_0. \tag{11.63}$$

$$b_{n+1} \lesssim_{\varepsilon,k_0,\beta} a_n^2 + \frac{1}{s} N_n b_n^2 + N_n^{-\beta}(1 + b_n)C_n \qquad \text{for } n \geq 1, \tag{11.64}$$

$$a'_n \lesssim_{\varepsilon,k_0} \frac{1}{s} N_n^2 b_n \qquad \text{for } n \geq 0, \tag{11.65}$$

$$a''_n \lesssim_{\varepsilon,k_0} \frac{1}{s} N_n^4 b_n \qquad \text{for } n \geq 0. \tag{11.66}$$

Furthermore, we have

$$|PR_1|_{(X^{k_0+2})^3} \lesssim_{\varepsilon,k_0} N_0 s \tag{11.67}$$

$$|(I - P)R_1|_{(X^{k_0+2})^3} \lesssim_{\varepsilon,k_0} N_0 s^2. \tag{11.68}$$

Proof. **Proof of** (11.60)**,** (11.61)**,** (11.67) **and** (11.68)**.** We first claim that

$$|PR_1|_{(X^{k_0+1})^3} \lesssim s \tag{11.69}$$

$$|(I-P)R_1|_{(X^{k_0+1})^3} \lesssim s^2, \tag{11.70}$$

Let us prove (11.69) first. Thanks to (a') in Lemma 11.1.10 and (11.30), we have

$$|PR_1|_{(X^{k_0+1})^3} = |S(N_0)PT_s(0)^{-1}[\tilde{G}_s(0)]|_{(X^{k_0+1})^3} \lesssim |PT_s(0)^{-1}[\tilde{G}_s(0)]|_{(X^{k_0+1})^3} \lesssim \frac{1}{s}|\tilde{G}_s(0)|_{(Y^{k_0+1})^3} \lesssim s,$$

where we used that $S(N_0)$ and P commute to obtain the first equality. To prove (11.70), we use (11.31) and compute

$$\begin{aligned}
|(I-P)R_1|_{(X^{k_0+1})^3} &\lesssim |S(N_0)(I-P)T_s(0)^{-1}[\tilde{G}_s(0)]|_{(X^{k_0+1})^3} \\
&\lesssim |(I-P)T_s(0)^{-1}[\tilde{G}_s(0)]|_{(X^{k_0+1})^3} \\
&\lesssim |\tilde{G}_s(0)|_{(Y^{k_0+1})^3} \\
&\lesssim s^2,
\end{aligned}$$

which proves (11.70). With the claim, (11.60) follows immediately. (11.61) (11.67) and (11.68) can be proved in exactly the same way as above using (11.57).

Proof of (11.62)**,** (11.63) **and** (11.64)**.** Note that Lemma 11.1.10 immediately implies (11.62). Now, let us prove (11.63) and (11.64). We have

$$\begin{aligned}
b_{n+1} := |\tilde{G}_s(R_{n+1})|_{(Y^{k_0})^3} &\lesssim |\tilde{G}_s(R_{n+1}) - \tilde{G}_s(R_n) - D\tilde{G}_s(R_n)[R_{n+1}-R_n]|_{(Y^{k_0})^3} \\
&\quad + |\tilde{G}_s(R_n) + D\tilde{G}_s(R_n)[R_{n+1}-R_n]|_{(Y^{k_0})^3} \\
&=: J_1 + J_2,
\end{aligned} \tag{11.71}$$

We claim that

$$\begin{cases}
J_1 \lesssim s^3 & \text{if } n = 0 \\
J_1 \lesssim a_n^2 = |R_{n+1}-R_n|^2_{(X^{k_0+1})^3} & \text{if } n \geq 1.
\end{cases} \tag{11.72}$$

If $n = 0$, we have

$$\begin{aligned}
J_1 &\lesssim \left|\tilde{G}_s(R_1) - \tilde{G}_s(0) - D\tilde{G}_s(0)[R_1] - \frac{1}{2}D^2\tilde{G}_s(0)\,[R_1,R_1]\right|_{(Y^{k_0})^3} + \frac{1}{2}|D^2\tilde{G}_s(0)[R_1,R_1]|_{(Y^{k_0})^3} \\
&\lesssim |R_1|^3_{(X^{k_0+1})^3} + \frac{1}{2}|D^2\tilde{G}_s(0)[R_1,R_1]|_{(Y^{k_0})^3} \\
&\lesssim s^3 + \frac{1}{2}|D^2\tilde{G}_s(0)[R_1,R_1]|_{(Y^{k_0})^3},
\end{aligned}$$

where we used (b') in Lemma 11.1.10 in the second inequality and (11.69) and (11.70) in the last inequality. To estimate $D^2\tilde{G}$, we recall the definition of $\tilde{G}_s$ and compute

$$\begin{aligned} D^2\tilde{G}_s(0)[R_1,R_1] &= \left(\frac{d}{dt}\right)^2 G(\Theta_3^* + tv\cdot R_1, sv + (I-P)tR_1)\Big|_{t=0} \\ &= \partial_{\Theta\Theta}G(\Theta_3^*,sv)(v\cdot R_1)^2 + 2(v\cdot R_1)\partial_\Theta DG(\Theta_3^*,sv)[(I-P)[R_1]] \\ &\quad + D^2G(\Theta_3^*,sv)\left[(I-P)R_1,(I-P)R_1\right]. \end{aligned}$$

Since $\partial_{\Theta\Theta}G(\Theta_3^*,0)=0$, therefore

$$|\partial_{\Theta\Theta}G(\Theta_3^*,sv)|_{(Y^{k_0})^3} = \left|\int_0^s \frac{d}{dt}\left(\partial_{\Theta\Theta}G(\Theta_3^*,tv)\right)dt\right|_{(Y^{k_0})^3} \lesssim s,$$

which implies $|\partial_{\Theta\Theta}G(\Theta_3^*,sv)(v\cdot R_1)^2|_{(Y^{k_0})^3}\lesssim s^3$, since $|v\cdot R_1|\lesssim |PR_1|_{(X^{k_0+1})^3}\lesssim s$, which follows from (11.69). Also it follows from (b), (11.69) and (11.70) that

$$\begin{aligned} |2(v\cdot R_1)\partial_\Theta DG(\Theta_3^*,sv)[(I-P)[R_1]]|_{(Y^{k_0})^3}&\lesssim s^3,\\ |D^2G(\Theta_3^*,sv)[(I-P)R_1,(I-P)R_1]|_{(Y^{k_0})^3}&\lesssim s^4, \end{aligned}$$

therefore, $|D^2\tilde{G}_s(0)[R_1,R_1]|_{(Y^{k_0})^3}\lesssim s^3$. Thus, we have $J_1\lesssim s^3$, which proves (11.72) for $n=0$. If $n\geq 1$, the claim in (11.72) follows immediately from (b') in Lemma 11.1.10.

In order to estimate J_2 in (11.71), we have that for $n\geq 0$,

$$\begin{aligned} J_2 &\lesssim |(I-D\tilde{G}_s(R_n)\circ T_s(R_n)^{-1})[\tilde{G}_s(R_n)]|_{(Y^{k_0})^3} \\ &\quad + |D\tilde{G}_s(R_n)\,(I-S(N_n))\,T_s(R_n)^{-1}[\tilde{G}_s(R_n)]|_{(Y^{k_0})^3} \\ &=: J_{21}+J_{22}. \end{aligned}$$

Then it follows from (11.57) and (11.28) that

$$\begin{aligned} J_{21} &\lesssim |\tilde{G}_s(R_n)|_{(Y^{k_0})^3}|T_s(R_n)^{-1}[\tilde{G}_s(R_n)]|_{(X^{k_0+1})^3} \\ &\lesssim |\tilde{G}_s(R_n)|_{(Y^{k_0})^3}|S(N_n)T_s(R_n)^{-1}[\tilde{G}_s(R_n)]|_{(X^{k_0+1})^3} \\ &\quad + |\tilde{G}_s(R_n)|_{(Y^{k_0})^3}|\,(I-S(N_n))\,T_s(R_n)^{-1}[\tilde{G}_s(R_n)]|_{(X^{k_0+1})^3} \\ &\lesssim N_n|\tilde{G}_s(R_n)|_{(Y^{k_0})^3}|T_s(R_n)^{-1}[\tilde{G}_s(R_n)]|_{(X^{k_0})^3} \end{aligned}$$

$$
\begin{aligned}
&+ N_n^{-\beta}|\tilde{G}_s(R_n)|_{(Y^{k_0})^3}|T_s(R_n)^{-1}[\tilde{G}_s(R_n)]|_{(X^{k_0+1+\beta})^3} \\
&\lesssim \frac{1}{s}N_n|\tilde{G}_s(R_n)|^2_{(Y^{k_0})^3} + N_n^{-\beta}|\tilde{G}_s(R_n)|_{(Y^{k_0})^3}|T_s(R_n)^{-1}[\tilde{G}_s(R_n)]|_{(X^{k_0+1+\beta})^3} \\
&\lesssim \frac{1}{s}N_n b_n^2 + N_n^{-\beta}b_n C_n,
\end{aligned}
$$

where the fourth inequality follows from (11.32). Also we have

$$
J_{22} \lesssim |\,(I - S(N_n))\,T_s(R_n)^{-1}[\tilde{G}_s(R_n)]|_{(X^{k_0+1})^3} \lesssim N_n^{-\beta}C_n.
$$

Hence we have for $n \geq 0$ that

$$
J_2 \lesssim \frac{1}{s}N_n b_n^2 + N_n^{-\beta}(1+b_n)C_n. \tag{11.73}
$$

Thus with (11.71), (11.72), (11.73) and (a') in lemma 11.1.10, we have that

$$
\begin{cases}
b_1 \lesssim (1+N_0)s^3 + N_0^{-\beta}C_0 \\
b_{n+1} \lesssim a_n^2 + \frac{1}{s}N_n b_n^2 + N_n^{-\beta}(1+b_n)C_n \quad \text{for } n \geq 1.
\end{cases} \tag{11.74}
$$

Proof of (11.65) **and** (11.66)**.** Finally, for (11.65) and (11.66), we use (11.32) to compute

$$
|R_{n+1} - R_n|_{(X^{k_0+2})^3} = |S(N_n)T_s(R_n)^{-1}[\tilde{G}_s(R_n)]|_{\left(X^{k_0+2}\right)^3} \lesssim N_n^2\frac{1}{s}b_n, \tag{11.75}
$$

$$
|R_{n+1} - R_n|_{(X^{k_0+4})^3} = |S(N_n)T_s(R_n)^{-1}[\tilde{G}_s(R_n)]|_{\left(X^{k_0+4}\right)^3} \lesssim N_n^4\frac{1}{s}b_n. \tag{11.76}
$$

This finishes the proof. □

Let us now derive a recursive formula for C_n.

Lemma 11.1.12. *For $j = 0, \ldots, n-1$ assume that R_j satisfies the same assumptions as in Lemma 11.1.11, that is, given $\epsilon > 0$ and $k_0 \geq 2$, R_j satisfies* (11.22)*,* (11.23) *and* (11.24) *and the assumptions of Lemma 11.1.10 for $s \in (0, s_0(\epsilon, k_0))$ and $\delta = \delta(\varepsilon, k_0) > 0$. Then,*

$$
C_n \lesssim_{\varepsilon,k_0,\beta} \frac{1}{s}\left(1 + n \sup_{j=0,\ldots n-1} N_j^4 C_j\right), \quad \text{for } n \geq 1 \text{ and } \quad C_0 \lesssim_{\varepsilon,k_0,\beta} s.
$$

Proof. It follows from (a') in Lemma 11.1.10 and (11.29) in Lemma 11.1.9 that

$$
C_0 \lesssim \frac{1}{s}|\tilde{G}_s(0)|_{Y^{k_0+1+\beta}\times(Y^{k_0+\beta})^2} \lesssim \frac{1}{s}|\tilde{G}_s(0)|_{(Y^{k_0+1+\beta})^3} \lesssim s.
$$

Now let us assume that $n \geq 1$. It follows from the definition of C_n in (11.59) and (11.29) that (recall that β is even)

$$C_n \lesssim \frac{1}{s}\left((1+|R_n|_{(X^{k_0+4+\beta})^3})|\tilde{G}_s(R_n)|_{Y^{k_0+1}\times(Y^{k_0})^2} + |\tilde{G}_s(R_n)|_{Y^{k_0+1+\beta}\times(Y^{k_0+\beta})^2}\right).$$

Using (11.8), we have

$$\begin{aligned}
&|\tilde{G}_s(R_n)|_{Y^{k_0+1}\times(Y^{k_0})^2} \lesssim |\tilde{G}_s(R_n)|_{(Y^{k_0+1})^3} \lesssim 1 + |R_n|_{(X^{k_0+2})^3} \lesssim 1,\\
&|\tilde{G}_s(R_n)|_{Y^{k_0+1+\beta}\times(Y^{k_0+\beta})^2} \lesssim 1 + |R_n|_{(X^{k_0+2+\beta})^3}.
\end{aligned}$$

therefore, we have

$$C_n \lesssim \frac{1}{s}(1+|R_n|_{(X^{k_0+4+\beta})^3}).$$

For R_n, we have

$$\begin{aligned}
|R_n|_{(X^{k_0+4+\beta})^3} &\lesssim \sum_{j=0}^{n-1} |R_{j+1}-R_j|_{(X^{k_0+4+\beta})^3}\\
&\lesssim \sum_{j=0}^{n-1} \left|S(N_j)T_s(R_j)^{-1}[\tilde{G}_s(R_j)]\right|_{(X^{k_0+4+\beta})^3}\\
&\lesssim \sum_{j=0}^{n-1} N_j^3 C_j\\
&\lesssim n \sup_{j=0,\dots,n-1} N_j^3 C_j.
\end{aligned}$$

Hence, we obtain the desired result. □

Now we are ready to prove the main theorem of this section.

Proof of Theorem 11.1.7: We fix $k_0 \geq 2$ and pick $\tilde{\epsilon},\ \tilde{s} > 0$ so that

$$\sum_{k=1}^{\infty} s^{-1+2\left(\frac{67}{64}\right)^k} \leq s^{1+\tilde{\epsilon}} \text{ for all } s \in (0,\tilde{s}). \tag{11.77}$$

And we choose

$$\beta := 816 \quad \text{and} \quad \epsilon := \min\left\{\frac{1}{4}, \frac{\tilde{\epsilon}}{2}\right\} > 0, \tag{11.78}$$

and let $s_0 = s_0(\epsilon, k_0)$, $\delta = \delta(\epsilon, k_0)$ be as in Lemma 11.1.9. As before, we can also assume δ is small

enough so that $\delta < \eta(k_0)$ where η is as in (b') in Lemma 11.1.10. Since k_0, ϵ and β are fixed, by Lemma 11.1.11 and 11.1.12, we can find a constant $K > 0$ such that as long as R_n satisfies (11.22),(11.23) and (11.24), for some $s \in (0, s_0)$, it holds that for any sequence of positive numbers N_n, and

$$a_0 \leq Ks \tag{11.79}$$

$$|PR_1|_{(X^{k_0+2})^3} \leq KN_0 s \tag{11.80}$$

$$|(I-P)R_1|_{(X^{k_0+2})^3} \leq KN_0 s^2. \tag{11.81}$$

$$a_n \leq \frac{K}{s} N_n b_n \quad \text{for } n \geq 1, \tag{11.82}$$

$$b_0 \leq Ks^2, \tag{11.83}$$

$$b_1 \leq K\left((N_0^3+1)s^3 + N_0^{-\beta} C_0\right) \tag{11.84}$$

$$b_{n+1} \leq K(a_n^2 + \frac{1}{s} N_n b_n^2 + N_n^{-\beta}(1+b_n)C_n) \quad \text{for } n \geq 1, \tag{11.85}$$

$$a_n' \leq \frac{K}{s} N_n^2 b_n \quad \text{for } n \geq 0, \tag{11.86}$$

$$a_n'' \leq \frac{K}{s} N_n^4 b_n \quad \text{for } n \geq 0, \tag{11.87}$$

$$C_n \leq \frac{K}{s}\left(1 + n \sup_{j=0,\dots n-1} \left|N_j^4 C_j\right|\right) \quad \text{for } n \geq 1, \tag{11.88}$$

$$C_0 \leq Ks. \tag{11.89}$$

In what follows, we assume without loss of generality that

$$K > 1, \quad \text{and} \quad \delta < 1. \tag{11.90}$$

For such K, we can find s^* such that for all $s \in (0, s^*)$ (each of them can be easily verified for small enough $s > 0$),

$$2Kn < s^{-2\left(\frac{67}{64}\right)^{n-1}+1} \quad \text{for all } n \geq 1, \tag{11.91}$$

$$4K^3 s^{\frac{7}{16}} < \frac{1}{2}, \tag{11.92}$$

$$Ks^{\frac{\epsilon}{2}} \leq \frac{1}{2}, \tag{11.93}$$

$$s < 1, \tag{11.94}$$

$$K^2 s^{\frac{3}{4}} \leq \frac{1}{2}\delta, \tag{11.95}$$

$$K\sum_{k=1}^{\infty} s^{-1+\frac{127}{64}\left(\frac{67}{64}\right)^k} \leq \frac{1}{2}\delta, \tag{11.96}$$

$$K(n+2) \leq s^{1-2\left(\frac{67}{64}\right)^n} \text{ for all } n \geq 0, \tag{11.97}$$

$$3K^3 \leq s^{2+\left(-\frac{139}{32}+\frac{143}{64}\cdot\frac{67}{64}\right)\left(\frac{67}{64}\right)^n} \text{ for all } n \geq 1, \tag{11.98}$$

$$3K \leq s^{1+\left(-\frac{141}{32}+\frac{143}{64}\left(\frac{67}{64}\right)\right)\left(\frac{67}{64}\right)^n} \text{ for all } n \geq 1, \tag{11.99}$$

$$6K \leq s^{\left(-\frac{816}{16}+\frac{768}{16}+\frac{143}{64}\frac{67}{64}\right)\left(\frac{67}{64}\right)^n} \text{ for all } n \geq 1. \tag{11.100}$$

Lastly, we fix $s \in (0, \min\{s_0, s^*, \tilde{s}\})$ and

$$N_n := s^{-\frac{1}{16}\left(\frac{67}{64}\right)^n} \quad \text{for } n \geq 0. \tag{11.101}$$

We claim that for all $n \geq 0$,

$(P1)_n$: R_{n+1} satisfies (11.22),(11.23) and (11.24).

$(P2)_n$: $C_n \leq N_n^{768}$.

$(P3)_n$: $b_{n+1} \leq s^{\frac{143}{64}\left(\frac{67}{64}\right)^{n+1}}$.

$(P4)_n$: $a_{n+1} \leq Ks^{-1+\frac{139}{64}\left(\frac{67}{64}\right)^{n+1}}$.

Once we have the above claims, then $(P1)_n$ justifies all the recurrence formulae above for all $n \in \mathbb{N}$, which follows from Lemma 11.1.11 and 11.1.12. Also, it is clear from (11.82) and $(P4)_n$ that

$$\sum_{n=0}^{\infty} a_n \leq KN_0 s + K\sum_{n=1}^{\infty} s^{-1+\frac{139}{64}\left(\frac{67}{64}\right)^n} < \infty.$$

Therefore R_n is a Cauchy sequence in $(X^{k_0+1})^3$, and we can find a limit $R_\infty := \lim_{n\to\infty} R_n$. Then, $(P3)_n$ and the continuity of the functional $R \mapsto \tilde{G}_s$ implies that $\tilde{G}_s(R_\infty) = 0$. From the definition of $\tilde{G}_s$ in (11.19), this implies the existence of a solution $R = R(s) \neq 0$ for each $s \in (0, \min\{s_0, s^*, \tilde{s}\})$ and finishes the proof.

Now we prove the claim $(Pi)_n$ for $i = 1, \ldots, 4$. We will follow the usual induction argument.

Initial step. In the initial case, we assume that $n = 0$. We first prove $(P1)_0$. It follows from (11.80),(11.81) and (11.101) that

$$|PR_1|_{(X^{k_0+2})^3} \leq Ks^{\frac{15}{16}} \leq s^{\frac{1}{4}} \leq s^{\epsilon} \tag{11.102}$$

$$|(I-P)R_1|_{(X^{k_0+2})^3} \leq Ks^{1+\frac{15}{16}} \leq s^{1+\frac{1}{4}} \leq s^{1+\epsilon}, \tag{11.103}$$

where the two second inequalities follow from (11.92) ($Ks^{\frac{15}{16}} \leq 4K^3 s^{\frac{7}{16}} s^{\frac{1}{2}} < s^{\frac{1}{4}}$) and the last inequalities follow from (11.94) and (11.78). Furthermore, it follows from (11.87), (11.83) and (11.101) that

$$|R_1|_{(X^{k_0+4})^3} = a_0'' \leq \frac{K}{s} \cdot \left(s^{-\frac{1}{16}}\right)^4 \cdot \left(Ks^2\right) \leq K^2 s^{\frac{3}{4}} \leq \frac{1}{2}\delta, \tag{11.104}$$

where the last inequality follows from (11.95). Therefore $(P1)_0$ holds for $n = 0$. $(P2)_0$ follows immediately from (11.89) and (11.101). In order to prove $(P3)_0$, note that thanks to (11.84), it is enough to show that

$$K\left((N_0^3+1)s^3 + N_0^{-\beta} C_0\right) \leq s^{\frac{143}{64}\cdot\frac{67}{64}},$$

in other words,

$$K\left(s^{\frac{45}{16}} + s^3 + Ks^{\frac{816}{16}} s\right) \leq s^{\frac{143}{64}\cdot\frac{67}{64}},$$

where we used (11.78), (11.89) and (11.101). By (11.94), $s^{\frac{45}{16}}$ is the largest value among the three in the parentheses, hence it is sufficient to show that $3K^2 < s^{\frac{143}{64}\cdot\frac{67}{64}-\frac{45}{16}}$. Since $\frac{143}{64}\cdot\frac{67}{64} - \frac{45}{16} > -\frac{1}{2}$, it is enough to show that $3K^3 \leq s^{-\frac{1}{2}}$, which follows from (11.92). $(P4)_0$ follows from (11.82), (11.101) and $(P3)_0$.

Induction step. In this step, we assume that $(Pi)_k$ is true for all $0 \leq k \leq n_0$ and aim to prove $(Pi)_{n_0+1}$. Let us prove $(P1)_{n_0+1}$ first. It follows from (11.86), (11.101) and $(P3)_k$ for $k = 0, \ldots, n_0$, that

$$\sum_{k=1}^{n_0+1} a_k' \leq K \sum_{k=1}^{n_0+1} s^{-1}\left(s^{-\frac{1}{16}\left(\frac{67}{64}\right)^k}\right)^2 \cdot s^{\frac{143}{64}\left(\frac{67}{64}\right)^k} \leq K \sum_{k=1}^{n_0+1} s^{-1+2\left(\frac{67}{64}\right)^k} \leq K s^{1+\tilde{\epsilon}},$$

where the last inequality follows from our choice on $\tilde{\epsilon}$ and $s \in \tilde{s}$ in (11.77). Hence, we have

$$\sum_{k=1}^{n_0+1} a_k' \leq \left(Ks^{\frac{\tilde{\epsilon}}{2}}\right) s^{1+\frac{\tilde{\epsilon}}{2}} \leq \frac{1}{2} s^{1+\frac{\tilde{\epsilon}}{2}} \leq \frac{1}{2} s^{1+\epsilon},$$

where the second inequality follows from (11.93) and the last inequality follows from (11.78) and (11.94). Therefore we obtain

$$|PR_{n_0+2}|_{(X^{k_0+2})^3} \leq |PR_1|_{(X^{k_0+2})^3} + \sum_{k=1}^{n_0+1} a_k' \leq Ks^{\frac{15}{16}} + \frac{1}{2} s^{1+\epsilon} \leq \left(Ks^{\frac{7}{16}}\right) s^{\frac{1}{2}} + \left(\frac{1}{2}s\right) s^{\epsilon} \leq s^{\epsilon},$$

where the second inequality follows from (11.102) and the last inequality follows from $\epsilon < \frac{1}{2}$ and $Ks^{\frac{7}{16}} \leq \frac{1}{2}$, which can be deduced from (11.93) and (11.78). Using (11.103), instead of (11.102), one

can easily obtain

$$|(I-P)R_{n_0+2}|_{(X^{k_0+2})^3} \leq s^{1+\epsilon}.$$

To prove (11.24) for R_{n_0+2}, we compute

$$\begin{aligned}|R_{n_0+2}|_{(H^{k_0+4})^3} &\leq a_0'' + \sum_{k=1}^{n_0+1} a_k'' \leq \frac{1}{2}\delta + K\sum_{k=1}^{n_0+1} s^{-1}\left(s^{-\frac{1}{16}\left(\frac{67}{64}\right)^k}\right)^4 \cdot s^{\frac{143}{64}\left(\frac{67}{64}\right)^k} \\ &\leq \frac{1}{2}\delta + K\sum_{k=1}^{\infty} s^{-1+\frac{127}{64}\left(\frac{67}{64}\right)^k} \leq \delta, \end{aligned} \tag{11.105}$$

where the second inequality follows from (11.104), (11.87) and $(P3)_k$, for $0 \leq k \leq n_0$, and the last inequality follows from (11.96). This proves $(P1)_{n_0+1}$.

We turn to $(P2)_{n_0+1}$. Using (11.88) and $(P2)_{n_0}$, we have

$$C_{n_0+1} \leq \frac{K}{s}(2+n_0)N_{n_0}^4 \sup_{j=0,\dots,n_0} C_j \leq \frac{K}{s}(2+n_0)N_{n_0}^{772},$$

where the last inequality follows from $(P2)_{n_0}$. Hence, it suffices to show that

$$\frac{K}{s}(2+n_0)N_{n_0}^{772} \leq (N_{n_0+1})^{768}.$$

Plugging (11.101), this is equivalent to

$$K(n_0+2) \leq s^{1+\left(\frac{67}{64}\right)^{n_0}\left(\frac{772}{16}-\frac{768}{16}\cdot\frac{67}{64}\right)} = s^{1-2\left(\frac{67}{64}\right)^{n_0}},$$

which is true thanks to our choice of s in (11.97). This proves $(P2)_{n_0+1}$.

For $(P3)_{n_0+1}$, thanks to (11.85), it is enough to show that

$$K\left(a_{n_0+1}^2 + \frac{1}{s}N_{n_0+1}b_{n_0+1}^2 + N_{n_0+1}^{-\beta}(1+b_{n_0+1})C_{n_0+1})\right) \leq s^{\frac{143}{64}\left(\frac{67}{64}\right)^{n_0+2}}.$$

Using $(P4)_{n_0}$, (11.101), $(P3)_{n_0}$, (11.78), $(P2)_{n_0+1}$ and (11.94), which implies $b_{n_0+1} \leq 1$, we only need to show that

$$K\left(\underbrace{\left(Ks^{-1+\frac{139}{64}\left(\frac{67}{64}\right)^{n_0+1}}\right)^2}_{=:A_1} + \underbrace{s^{-1}s^{-\frac{1}{16}\left(\frac{67}{64}\right)^{n_0+1}}s^{\frac{143}{32}\left(\frac{67}{64}\right)^{n_0+1}}}_{=:A_2} + \underbrace{2s^{\frac{816}{16}\left(\frac{67}{64}\right)^{n_0+1}}s^{-\frac{768}{16}\left(\frac{67}{64}\right)^{n_0+1}}}_{=:A_3}\right) \leq s^{\frac{143}{64}\left(\frac{67}{64}\right)^{n_0+2}}. \tag{11.106}$$

We will show that $KA_i \le \frac{1}{3} s^{\frac{143}{64}\left(\frac{67}{64}\right)^{n_0+2}}$ for $i = 1, 2, 3$. For A_1, it suffices to show that

$$K^3 s^{-2+\frac{139}{32}\left(\frac{67}{64}\right)^{n_0+1}} \le \frac{1}{3} s^{\frac{143}{64}\left(\frac{67}{64}\right)^{n_0+2}},$$

equivalently,

$$3K^3 \le s^{2+\left(-\frac{139}{32}+\frac{143}{64}\cdot\frac{67}{64}\right)\left(\frac{67}{64}\right)^{n_0+1}},$$

and this follows from our choice of s in (11.98). For A_2, we need to show that

$$Ks^{-1+\frac{141}{32}\left(\frac{67}{64}\right)^{n_0+1}} \le \frac{1}{3} s^{\frac{143}{64}\left(\frac{67}{64}\right)^{n_0+2}},$$

eqvalently,

$$3K \le s^{1+\left(-\frac{141}{32}+\frac{143}{64}\left(\frac{67}{64}\right)\right)\left(\frac{67}{64}\right)^{n_0+1}},$$

and this follows from (11.99). For A_3, it is enough to show that

$$2Ks^{\frac{816}{16}\left(\frac{67}{64}\right)^{n_0+1}} \cdot s^{-\frac{768}{16}\left(\frac{67}{64}\right)^{n_0+1}} \le \frac{1}{3} s^{\frac{143}{64}\left(\frac{67}{64}\right)^{n_0+2}},$$

equivalently,

$$6K \le s^{\left(-\frac{816}{16}+\frac{768}{16}+\frac{143}{64}\frac{67}{64}\right)\left(\frac{67}{64}\right)^{n_0+1}},$$

which follows from (11.100). This proves $(P3)_{n_0+1}$.

Lastly, $(P4)_{n_0+1}$ can be proved by (11.82) and $(P3)_{n_0+1}$, that is,

$$a_{n_0+2} \le Ks^{-1} N_{n_0+2} b_{n_0+2} \le Ks^{-1} s^{\left(-\frac{1}{16}+\frac{143}{64}\right)\left(\frac{67}{64}\right)^{n_0+2}} = Ks^{-1+\frac{139}{64}\left(\frac{67}{64}\right)^{n_0+2}},$$

which finishes the proof. □

11.1.4 Estimates on the velocity

In Subsection 11.1.5, we will check whether our functional G in (11.5) satisfies the hypotheses in Theorem 11.1.7. In this section, we will derive some useful estimates on the velocity vector generated by each patch. Recall that given $b_1, b_2 > 0$, $b_1 \neq b_2$, and $R_1, R_2 \in C^\infty(\mathbb{T})$, we denote for $i, j = 1, 2$,

$$R = (R_1, R_2) \in (C^\infty(\mathbb{T}))^2, \tag{11.107}$$

$$z_i(R)(x) = (b_i + R_i(x))(\cos x, \sin x), \tag{11.108}$$

$$u_{i,j}(R)(x) := \begin{pmatrix} u^1_{i,j}(R) \\ u^2_{i,j}(R) \end{pmatrix} = \int_{\mathbb{T}} \log(|z_j(R)(x) - z_i(R)(y)|^2) z_i(R)'^{\perp}(y) dy. \tag{11.109}$$

Proposition 11.1.13. *For $k \geq 2$, there exists $\epsilon = \epsilon(k, b_1, b_2) > 0$ such that if $\|R\|_{(H^3(\mathbb{T}))^2} \leq \epsilon$, then*

(A) $u_{i,j} : (H^{k+1}(\mathbb{T}))^2 \mapsto (H^k(\mathbb{T}))^2$, and $\|u_{i,j}\|_{(H^k(\mathbb{T}))^2} \lesssim_{k,b_1,b_2} 1 + \|R\|_{(H^{k+1}(\mathbb{T}))^2}$.

(B) The Gateaux derivative $Du_{i,j}(R) : (H^{k+1}(\mathbb{T}))^2 \mapsto (H^{k+1}(\mathbb{T}))^2$ exists and

$$\|Du_{i,j}(R)[h]\|_{(H^{k+1}(\mathbb{T}))^2} \lesssim_{k,b_1,b_2} \|h\|_{(H^{k+1}(\mathbb{T}))^2} + \|R\|_{(H^{k+3}(\mathbb{T}))^2} \|h\|_{(H^1(\mathbb{T}))^2}, \text{ for } h \in (C^\infty(\mathbb{T}))^2.$$

(C) The map $R \mapsto Du_{i,j}(R)$ is Lipschitz continuous, in the sense that for $R, r, h \in (C^\infty(\mathbb{T}))^2$ such that $\|R\|_{(H^3(\mathbb{T}))^2}, \|r\|_{(H^3(\mathbb{T}))^2} \leq \epsilon$, and $\|R\|_{(H^{k+3}(\mathbb{T}))^2}, \|r\|_{(H^{k+3}(\mathbb{T}))^2} \leq 1$,

$$\|Du_{i,j}(R)[h] - Du_{i,j}(r)[h]\|_{(H^{k+1}(\mathbb{T}))^2} \lesssim_{k,b_1,b_2} \|R-r\|_{(H^2(\mathbb{T}))^2} \|h\|_{(H^{k+1}(\mathbb{T}))^2} + \|R-r\|_{(H^{k+3}(\mathbb{T}))^2} \|h\|_{(H^1(\mathbb{T}))^2}.$$

(D) For $\sigma \in \mathbb{N} \cup \{0\}$, there exists a linear operator $T^\sigma_{i,j}(R)$ such that for $h \in (C^\infty(\mathbb{T}))^2$,

$$\left(\frac{d}{dx}\right)^\sigma (Du_{i,j}(R)[h]) = Du_{i,j}(R)[h^{(\sigma)}] + T^\sigma_{i,j}(R)[h],$$

where $T^\sigma_{i,j}(R)$ satisfies

$$\|T^\sigma_{i,j}(R)[h]\|_{(H^{k+1}(\mathbb{T}))^2} \lesssim_k (1 + \|R\|_{(H^{k+4+\sigma}(\mathbb{T}))^2}) \|h\|_{(L^2(\mathbb{T}))^2} + (1 + \|R\|_{(H^4(\mathbb{T}))^2}) \|h^{(k+\sigma)}\|_{(L^2(\mathbb{T}))^2}.$$

Remark 11.1.14. *It is well known that roughly speaking, if ∂D is $C^{k+\alpha}$-regular for some $k \geq 1$, $\alpha > 0$ then the velocity $\nabla^\perp(1_D * \log|x|)$ is also $C^{k+\alpha}$-regular up to the boundary. In (A) in the above Proposition, we do not aim to prove the optimal regularity since it is not necessary in the proof of the main theorem.*

The proof of the proposition will be given after Lemmas 11.1.15 and 11.1.16. We will deal with the case $i = j$ only. If $i \neq j$, then the integrand in $u_{i,j}$ has no singularity thus the result follows straightforwardly in a similar manner. Hence, by abuse of notation, we denote for $b > 0$ and $R \in C^\infty(\mathbb{T})$,

$$z(R)(x) := (b + R(x))(\cos x, \sin x), \tag{11.110}$$

$$u(R)(x) := \begin{pmatrix} u^1(R) \\ u^2(R) \end{pmatrix} := \int_{\mathbb{T}} \log\left(|z(R)(x) - z(R)(y)|^2\right) z(R)'^{\perp}(y)dy. \tag{11.111}$$

Let us write $u(R)$ as

$$\begin{aligned} u(R)(x) &= \int_{\mathbb{T}} \log\left(2 - 2\cos(x-y)\right) z(R)'^{\perp}(y)dy + \int_{\mathbb{T}} \log\left(\frac{|z(R)(x) - z(R)(y)|^2}{2 - 2\cos(x-y)}\right) z(R)'^{\perp}(y)dy \\ &= \int_{\mathbb{T}} \log\left(2 - 2\cos(x-y)\right) z(R)'^{\perp}(y)dy + \int_{\mathbb{T}} K(R)(x,y) z(R)'(y)^{\perp} dy \\ &=: u_L(R)(x) + u_N(R)(x), \end{aligned} \tag{11.112}$$

where

$$\begin{aligned} &K(R)(x,y) = F(R(x), R(y), J(R)(x,y)), \\ &F(u,v,w) := \log(b^2 + b(u+v) + uv + w^2), \\ &J(R)(x,y) = \frac{R(x) - R(y)}{2\sin(\frac{x-y}{2})}. \end{aligned} \tag{11.113}$$

If $\|R\|_{H^3(\mathbb{T})} \leq \epsilon$ for sufficiently small $\epsilon > 0$, it follows from Lemmas 11.4.10 and 11.4.5 that

$$\|J(R)\|_{H^2(\mathbb{T}^2)} \lesssim_b \|R\|_{H^3(\mathbb{T})} \lesssim \epsilon. \tag{11.114}$$

Therefore, for sufficiently small $\epsilon > 0$, Lemmas 11.4.8 and 11.4.10 imply that for $l \geq 2$,

$$\|K(R)\|_{H^l(\mathbb{T}^2)} \lesssim_{l,b} 1 + \|R\|_{H^l(\mathbb{T})} + \|J(R)\|_{H^l(\mathbb{T}^2)} \lesssim_{l,b} 1 + \|R\|_{H^{l+1}(\mathbb{T})},, \tag{11.115}$$

The next lemma will be used to prove the growth condition of the velocity in (A) in Proposition 11.1.13.

Lemma 11.1.15. *For $k \geq 2$, there exists $\epsilon = \epsilon(k,b) > 0$ such that if $R \in C^\infty(\mathbb{T})$ and $\|R\|_{H^3(\mathbb{T})} \leq \epsilon$, then*

$$\|u\|_{(H^k(\mathbb{T}))^2} \lesssim_{k,b} 1 + \|R\|_{H^{k+1}(\mathbb{T})}.$$

Proof. For u_L, then it follows from Lemma 11.4.14 and the fact that $R \mapsto u_L(R)$ is linear that

$$\|u_L(R)\|_{(H^k(\mathbb{T}))^2} \lesssim \|z(R)'\|_{(H^{k-1}(\mathbb{T}))^2} \lesssim 1 + \|R\|_{H^k(\mathbb{T})}.$$

Now, let us consider the nonlinear part. We have

$$\|u_N\|_{(H^k(\mathbb{T}))^2} \lesssim \|K(R)\|_{H^k(\mathbb{T}^2)} \|z(R)'\|_{(L^\infty(\mathbb{T}))^2} \lesssim 1 + \|R\|_{H^{k+1}(\mathbb{T})},$$

where the last inequality follows from (11.115). □

Now we turn to (B), (C) and (D) in Proposition 11.1.13. We will use the following notations:

$$\begin{aligned} DK(R)[h] := \frac{d}{dt} K(R+th)|_{t=0} &= \underbrace{\partial_u F(R(x), R(y), J(R)(x,y))}_{=:\partial_u F(R)(x,y)} h(x) \\ &+ \underbrace{\partial_v F(R(x), R(y), J(R)(x,y))}_{=:\partial_v F(R)(x,y)} h(y) \\ &+ \underbrace{\partial_w F(R(x), R(y), J(R)(x,y))}_{=:\partial_w F(R)(x,y)} J(h)(x,y), \end{aligned} \tag{11.116}$$

where $J(h) := \frac{h(x)-h(y)}{2\sin\left(\frac{x-y}{2}\right)}$ and

$$Dz(R)'^{\perp}[h] := \frac{d}{dt} z(R+th)'^{\perp}|_{t=0} = -h'(x)(\sin x, \cos x) - h(x)(\cos x, \sin x). \tag{11.117}$$

With the above notations, we can write the derivative of the nonlinear term as

$$\begin{aligned} Du_N(R)[h] &= \int_{\mathbb{T}} K(R)(x,y) Dz(R)'^{\perp}[h](y) dy + \int_{\mathbb{T}} DK(R)[h](x,y) z(R)'^{\perp}(y) dy \\ &=: Du_N^1(R)[h] + Du_N^2(R)[h]. \end{aligned} \tag{11.118}$$

Again, Lemma 11.4.8 and (11.114) yield that if $\|R\|_{H^3(\mathbb{T})} \le \epsilon$ for sufficiently small $\epsilon > 0$, then for $l \ge 2$,

$$\begin{aligned} \|\partial_u F(R)\|_{(H^l(\mathbb{T}))^2}, \|\partial_v F(R)\|_{(H^l(\mathbb{T}))^2}, \|\partial_w F(R)\|_{(H^l(\mathbb{T}))^2} &\lesssim_l 1 + \|R\|_{H^l(\mathbb{T})} + \|J(R)\|_{H^l(\mathbb{T}^2)} \\ &\lesssim_l 1 + \|R\|_{H^{l+1}(\mathbb{T})}. \end{aligned} \tag{11.119}$$

Also, if $R_1, R_2 \in C^\infty(\mathbb{T})$ and $\|R_1\|_{H^3(\mathbb{T})}, \|R_2\|_{H^3(\mathbb{T})} \le \epsilon$, then from Lemma 11.4.8, it follows that

$$\|\partial_u F(R_1) - \partial_u F(R_2)\|_{(H^l(\mathbb{T}))^2}, \|\partial_v F(R_1) - \partial_v F(R_2)\|_{(H^l(\mathbb{T}))^2}, \|\partial_w F(R_1) - \partial_w F(R_2)\|_{(H^l(\mathbb{T}))^2} \lesssim_l \|R_1 - R_2\|_{H^{l+1}(\mathbb{T})}. \tag{11.120}$$

For $\partial_w F^{\#}(R)(x) := \lim_{y\to x} \partial_w F(R)(x,y) = \partial_w F(R(x), R(x), R'(x))$, we have

$$\|\partial_w F^{\#}(R)\|_{H^l(\mathbb{T})} \lesssim_l 1+\|R\|_{H^{l+1}(\mathbb{T})}, \tag{11.121}$$

$$\|\partial_w F^{\#}(R)\|_{L^\infty(\mathbb{T})} \lesssim 1+\|R\|_{H^2(\mathbb{T})} \lesssim 1. \tag{11.122}$$

$$\|\partial_w F^{\#}(R_1) - \partial_w F^{\#}(R_2)\|_{H^l(\mathbb{T})} \lesssim_l \|R_1 - R_2\|_{H^{l+1}(\mathbb{T})}, \tag{11.123}$$

$$\|\partial_w F^{\#}(R_1) - \partial_w F^{\#}(R_2)\|_{L^\infty(\mathbb{T})} \lesssim \|R_1 - R_2\|_{H^2(\mathbb{T})}. \tag{11.124}$$

We decompose $Du_N^2(R)[h]$ into

$$\begin{aligned} Du_N^2(R)[h] = &\int_{\mathbb{T}} \underbrace{\partial_u F(R)(x,y) z(R)'^{\perp}}_{=:K_1(R)(x,y)} h(x)dy + \int_{\mathbb{T}} \underbrace{\partial_v F(R)(x,y) z(R)'^{\perp}}_{=:K_2(R)(x,y)} h(y)dy \\ &+ \int_{\mathbb{T}} \underbrace{\partial_w F(R)(x,y) z(R)'^{\perp}}_{=:K_3(R)(x,y)} J(h)(x,y)dy \qquad (11.125) \\ =: &\, Du_{N1}^2(R)[h] + Du_{N2}^2(R)[h] + Du_{N3}^2(R)[h]. \qquad (11.126) \end{aligned}$$

Then it follows from Lemma 11.4.5, (11.119), (11.121) and (11.122) that for $l \geq 2$,

$$\|K_i(R)\|_{H^l(\mathbb{T}^2)} \lesssim_l 1+\|R\|_{H^{l+1}(\mathbb{T})}, \text{ for } i = 1,2,3, \tag{11.127}$$

$$\|{K_3}^{\#}(R)\|_{H^l(\mathbb{T})} \lesssim 1+\|R\|_{H^{l+1}(\mathbb{T})}, \text{ where } {K_3}^{\#}(R)(x) := K_3(R)(x,x), \tag{11.128}$$

$$\|{K_3}^{\#}(R)\|_{L^\infty(\mathbb{T})} \lesssim \|{K_3}^{\#}(R)\|_{H^1(\mathbb{T})} \lesssim \|{K_3}^{\#}(R)\|_{H^2(\mathbb{T})} \lesssim 1+\|R\|_{H^3(\mathbb{T})} \lesssim 1, \tag{11.129}$$

$$\|K_i(R_1) - K_2(R_2)\|_{H^l(\mathbb{T}^2)} \lesssim_l \|R_1 - R_2\|_{H^{l+1}(\mathbb{T})}, \text{ for } i = 1,2,3, \tag{11.130}$$

$$\|{K_3}^{\#}(R_1) - {K_3}^{\#}(R_2)\|_{H^l(\mathbb{T})} \lesssim \|R_1 - R_2\|_{H^{l+1}(\mathbb{T})}, \tag{11.131}$$

$$\|{K_3}^{\#}(R_1) - {K_3}^{\#}(R_2)\|_{L^\infty(\mathbb{T})} \lesssim \|R_1 - R_2\|_{H^2(\mathbb{T})}. \tag{11.132}$$

Furthermore, it follows from the definitions of K_3 and $\partial_w F$ in (11.125), (11.116) and (11.113) that

$$\|\nabla K_3(R)(x,y)\|_{L^\infty(\mathbb{T}^2)} \lesssim 1+\|R''\|_{L^\infty(\mathbb{T})}+\|\nabla J(R)\|_{L^\infty(\mathbb{T}^2)} \lesssim 1+\|R''\|_{L^\infty(\mathbb{T})} \lesssim 1. \tag{11.133}$$

Lemma 11.1.16. *For $k \geq 2$, there exists $\epsilon = \epsilon(k,b) > 0$ such that if $R \in C^\infty(\mathbb{T})$ and $\|R\|_{H^3(\mathbb{T})} \leq \epsilon$, the Gateaux derivative $Du(R) : H^{k+1}(\mathbb{T}) \mapsto (H^{k+1}(\mathbb{T}))^2$ exists and*

$$\|Du(R)[h]\|_{(H^{k+1}(\mathbb{T}))^2} \lesssim_{k,b} \|h\|_{(H^{k+1}(\mathbb{T}))^2}+\|R\|_{H^{k+3}(\mathbb{T})}\|h\|_{H^1(\mathbb{T})}, \text{ for } h \in C^\infty(\mathbb{T}). \tag{11.134}$$

Furthermore, if $R_1, R_2 \in C^\infty(\mathbb{T})$, $\|R_1\|_{H^3(\mathbb{T})}, \|R_2\|_{H^3(\mathbb{T})} \leq \epsilon$, *and* $\|R_1\|_{H^{k+3}(\mathbb{T})}, \|R_2\|_{H^{k+3}(\mathbb{T})} \leq 1$, *then*

$$\|Du(R_1) - Du(R_2)[h]\|_{(H^{k+1}(\mathbb{T}))^2} \lesssim_{k,b} \|R_1 - R_2\|_{H^2(\mathbb{T})}\|h\|_{H^{k+1}(\mathbb{T})} + \|R_1 - R_2\|_{H^{k+3}(\mathbb{T})}\|h\|_{H^1(\mathbb{T})}, \tag{11.135}$$

for $h \in C^\infty(\mathbb{T})$.

Lastly, for $\sigma \in \mathbb{N} \cup \{0\}$, *there exists a linear operator* $T_\sigma(R)$ *such that for* $h \in C^\infty(\mathbb{T})$,

$$\left(\frac{d}{dx}\right)^\sigma (Du(R)[h]) = Du(R)[h^{(\sigma)}] + T^\sigma(R)[h], \tag{11.136}$$

where $T^\sigma(R)$ *satisfies*

$$\|T^\sigma(R)[h]\|_{(H^{k+1}(\mathbb{T}))^2} \lesssim_k ((1+\|R\|_{H^{k+4+\sigma}(\mathbb{T})})\|h\|_{L^2(\mathbb{T})} + (1+\|R\|_{H^4(\mathbb{T})})\|h^{(k+\sigma)}\|_{L^2(\mathbb{T})}. \tag{11.137}$$

Proof. We omit the proof of (11.135) since it can be proved exactly same way as (11.134) with (11.222) in Lemma 11.4.8. In order to prove (11.134), we deal with Du_L. Since it is linear, we have

$$Du_L(R)[h](x) := \int_{\mathbb{T}} \log(2 - 2\cos(x-y)) Dz(R)'^{\perp}[h](y)dy.$$

Hence, it follows from Lemma 11.4.14 that

$$\|Du_L(R)[h]\|_{(H^{k+1}(\mathbb{T}))^2} \lesssim \|Dz(R)'^{\perp}[h]\|_{(H^k(\mathbb{T}))^2} \lesssim \|h\|_{H^{k+1}(\mathbb{T})}. \tag{11.138}$$

Now, we estimate Du_N. Recalling the decomposition in (11.118) we will estimate Du_N^1 and Du_N^2 separately. For Du_N^1, it follows from (11.115) and (11.117) that

$$\|Du_N^1(R)[h]\|_{(H^{k+1}(\mathbb{T}))^2} \lesssim \|K(R)\|_{H^{k+1}(\mathbb{T}^2)}\|Dz(R)'^{\perp}\|_{(L^2(\mathbb{T}))^2} \lesssim (1+\|R\|_{H^{k+2}(\mathbb{T})})\|h\|_{H^1(\mathbb{T})}. \tag{11.139}$$

For Du_N^2, we recall the decomposition in (11.126), and use (11.127) and Lemma 11.4.11 to obtain

$$\|Du_{N1}^2(R)[h]\|_{(H^{k+1}(\mathbb{T}))^2}, \|Du_{N2}^2(R)[h]\|_{(H^{k+1}(\mathbb{T}))^2} \lesssim \|h\|_{H^{k+1}(\mathbb{T})} + \|R\|_{H^{k+2}(\mathbb{T})}\|h\|_{L^\infty(\mathbb{T})}.$$

For $Du^2_{N3}(R)[h]$, we use (11.127), (11.128), (11.129), (11.133) and Lemma 11.4.12 to show that

$$\|Du^2_{N3}(R)[h]\|_{(H^{k+1}(\mathbb{T}))^2} \lesssim \|R\|_{H^{k+3}(\mathbb{T})}\|h\|_{H^1(\mathbb{T})} + \|h\|_{H^{k+1}(\mathbb{T})}.$$

Therefore $\|Du^2_N(R)[h]\|_{(H^{k+1}(\mathbb{T}))^2} \lesssim \|h\|_{H^{k+1}(\mathbb{T})} + \|R\|_{H^{k+3}(\mathbb{T})}\|h\|_{H^1(\mathbb{T})}$. With (11.138), (11.139) and (11.118), the desired result follows.

Now, we turn to (11.136) and (11.137). Recall from (11.112), (11.118) and (11.126) that

$$\begin{aligned} Du(R)[h] &= \int_{\mathbb{T}} \log(2-2\cos(y))h(x-y)dy + \int_{\mathbb{T}} K(R)(x,y)Dz(R)'^{\perp}[h](y)dy \\ &\quad + \int_{\mathbb{T}} K_1(R)(x,y)z(R)'^{\perp}(y)h(x)dy + \int_{\mathbb{T}} K_2(R)(x,y)z(R)'^{\perp}(y)h(y)dy \\ &\quad + \int_{\mathbb{T}} K_3(R)(x,y)z(R)'^{\perp}(y)J(h)(x,y)dy \\ &= I_1(R)[h] + I_2(R)[h] + I_3(R)[h] + I_4(R)[h] + I_5(R)[h] \end{aligned}$$

where K, K_1, K_2, K_3 are of the form $H(R(x), R(y), J(R)(x,y))$ for some smooth function $H : \mathbb{R}^3 \mapsto \mathbb{R}$. It suffices to show that for each $i = 1, \dots, 5$ and $\sigma \in \mathbb{N} \cup \{0\}$,

$$\left(\frac{d}{dx}\right)^{\sigma} I_i(R)[h] = I_i(R)[h^{\sigma}] + T_i^{\sigma}(R)[h],$$

for some $T_i^{\sigma}(R)$ such that

$$\|T_i^{\sigma}(R)[h]\|_{(H^{k+1}(\mathbb{T}))^2} \lesssim (1+\|R\|_{H^{k+4+\sigma}(\mathbb{T})})\|h\|_{L^2(\mathbb{T})} + (1+\|R\|_{H^4(\mathbb{T})})\|h^{(k+\sigma)}\|_{L^2(\mathbb{T})}. \tag{11.140}$$

We only deal with I_5 since the other terms can be done in the same way. We also assume $\sigma \geq 1$ since $\sigma = 0$ follows trivially. Let $\tilde{K}_3(R)(x,y) := K_3(R)(x, x-y)z(R)'^{\perp}(x-y)$. Then it follows from (11.4.5), (11.127), (11.128) and (11.129) that for $l \geq 2$,

$$\|\tilde{K}_3(R)\|_{H^l(\mathbb{T}^2)} \lesssim_l 1+\|R\|_{H^{l+1}(\mathbb{T})}, \tag{11.141}$$

$$\|\tilde{K}_3^{\#}(R)\|_{H^l(\mathbb{T})} \lesssim 1+\|R\|_{H^{l+1}(\mathbb{T})}, \text{ where } \tilde{K}_3^{\#}(R)(x) := \tilde{K}_3(R)(x,x),. \tag{11.142}$$

Using the change of variables, $y \mapsto x - y$, we have

$$\left(\frac{d}{dx}\right)^{\sigma} I_5(R)[h] = \int_{\mathbb{T}} \tilde{K}_3(R)(x,y)J(h^{(\sigma)})(x,x-y)dy + \sum_{p+q=\sigma, q\leq\sigma-1} C_{p,q,\sigma}\int_{\mathbb{T}} (\partial_x)^p \tilde{K}_3(R)(x,y)J(h^{(q)})(x,x-y)dy,$$

$$=: I_5(R)[h^{(\sigma)}](x) + \sum_{p+q=\sigma, q\leq \sigma-1} C_{p,q,\sigma} \underbrace{\int_{\mathbb{T}} (\partial_x)^p \tilde{K}_3(R)(x, x-y) J(h^{(q)})(x,y) dy}_{=: T_5^{\sigma,p,q}(R)[h](x)} \tag{11.143}$$

$$=: I_5(R)[h^{(\sigma)}](x) + T_5^{\sigma}(R)[h](x), \tag{11.144}$$

where $C_{p,q,\sigma}$ is some constant and we used $\partial_x \left(J(h)(x, x-y)\right) = J(h')(x, x-y)$. It suffices to show that $T_5^{\sigma,p,q}(R)[h]$ satisfies (11.140). Let $\tilde{K}_3^*(R)(x,y) := (\partial_x)^p \tilde{K}_3(R)(x, x-y)$. Then it follows from (11.141) and (11.142) that for $l \geq 2$,

$$\|\tilde{K}_3^*(R)\|_{(H^l(\mathbb{T}^2))^2} \lesssim_l 1 + \|R\|_{H^{l+p+1}(\mathbb{T})}, \tag{11.145}$$

$$\|(\tilde{K}_3^*)^{\#}(R)\|_{(H^l(\mathbb{T}))^2} \lesssim 1 + \|R\|_{H^{l+p+1}(\mathbb{T})}, \text{ where } (\tilde{K}_3^*)^{\#}(R)(x) := \tilde{K}_3^*(R)(x,x), \tag{11.146}$$

$$\|(\tilde{K}_3^*)^{\#}(R)\|_{(L^\infty(\mathbb{T}))^2} \lesssim \|\tilde{K}_3(R)\|_{(W^{p,\infty}(\mathbb{T}^2))^2} \lesssim \|\tilde{K}_3(R)\|_{(H^{p+2}(\mathbb{T}^2))^2}. \tag{11.147}$$

Furthermore, it follows similarly as (11.133) that

$$\|\nabla \tilde{K}_3^*(R)\|_{L^\infty(\mathbb{T}^2)} \lesssim \|\nabla^{(p+1)} \tilde{K}_3(R)\|_{L^\infty(\mathbb{T}^2)} \lesssim 1 + \|R^{(p+2)}\|_{L^\infty(\mathbb{T})} \lesssim 1 + \|R\|_{H^{p+3}(\mathbb{T})}. \tag{11.148}$$

Hence it follows from Lemma 11.4.12 that

$$\begin{aligned}
\|T_5^{\sigma,p,q}(R)[h]\|_{H^{k+1}(\mathbb{T})} &\lesssim \left(\|\tilde{K}_3^*(R)\|_{(H^{k+2}(\mathbb{T}^2))^2} + \|(\tilde{K}_3^*)^{\#}(R)\|_{(H^{k+1}(\mathbb{T}^2))^2}\right) \|h^{(q)}\|_{H^1(\mathbb{T})} \\
&\quad + \left(\|(\tilde{K}_3^*)^{\#}(R)\|_{(L^\infty(\mathbb{T}^2))^2} + \|\nabla \tilde{K}_3^*(R)\|_{(L^\infty(\mathbb{T}^2))^2}\right) \|h^{(q)}\|_{H^{k+1}(\mathbb{T})} \\
&\lesssim (1 + \|R\|_{H^{k+3+p}(\mathbb{T})}) \|h^{(q)}\|_{H^1(\mathbb{T})} + (1 + \|R\|_{H^{p+3}(\mathbb{T})}) \|h^{(q)}\|_{H^{k+1}(\mathbb{T})} \\
&\lesssim (1 + \|R\|_{H^{k+3+p}(\mathbb{T})}) \|h\|_{H^{1+q}(\mathbb{T})} + (1 + \|R\|_{H^{p+3}(\mathbb{T})}) \|h\|_{H^{k+1+q}(\mathbb{T})}
\end{aligned}$$

where the second last inequality follows from (11.148). Now, recalling that $\|R\|_{H^3(\mathbb{T})} \leq \epsilon$ and plugging the following inequalities into the above estimate,

$$\|R\|_{H^l(\mathbb{T})} \lesssim \|R\|_{L^2(\mathbb{T})} + \|R^{(l)}\|_{L^2(\mathbb{T})}, \quad \|h\|_{H^l(\mathbb{T})} \lesssim \|h\|_{L^2(\mathbb{T})} + \|h^{(l)}\|_{L^2(\mathbb{T})}, \text{ for } l \geq 0,$$

we obtain

$$\|T_5^{\sigma,p,q}(R)[h]\|_{H^{k+1}(\mathbb{T})} \lesssim (1 + \|R^{(k+p+3)}\|_{L^2(\mathbb{T})}) \left(\|h\|_{L^2(\mathbb{T})} + \|h^{(1+q)}\|_{L^2(\mathbb{T})}\right)$$

$$+ (1+\|R^{(p+3)}\|_{L^2(\mathbb{T})})(\|h\|_{L^2(\mathbb{T})}+\|h^{(k+1+q)}\|_{L^2(\mathbb{T})})$$
$$=: L_1 + L_2. \tag{11.149}$$

For L_1, note that $k+p+3 \le k+3+\sigma$ and $q \le \sigma - 1$, hence we have

$$L_1 \lesssim (1+\|R^{(k+\sigma+3)}\|_{L^2(\mathbb{T})})\|h\|_{L^2(\mathbb{T})}+\|h^{(\sigma)}\|_{L^2(\mathbb{T})}+\|R^{(k+p+3)}\|_{L^2(\mathbb{T})}\|h^{(1+q)}\|_{L^2(\mathbb{T})}. \tag{11.150}$$

Using the interpolation inequality in Lemma 11.4.6, we have

$$\|R^{(k+p+3)}\|_{L^2(\mathbb{T})} = \|(R^{(4)})^{(k+p-1)}\|_{L^2(\mathbb{T})} \lesssim \|R^{(4)}\|_{L^2(\mathbb{T})}^{1-\frac{k+p-1}{k+\sigma}} \|(R^{(4)})^{(k+\sigma)}\|_{L^2(\mathbb{T})}^{\frac{k+p-1}{k+\sigma}}$$
$$\|h^{(1+q)}\|_{L^2(\mathbb{T})} \lesssim \|h\|_{L^2(\mathbb{T})}^{1-\frac{1+q}{k+\sigma}} \|h^{(k+\sigma)}\|_{L^2(\mathbb{T})}^{\frac{1+q}{k+\sigma}}.$$

Thus, we have

$$\|R^{(k+p+3)}\|_{L^2(\mathbb{T})}\|h^{(1+q)}\|_{L^2(\mathbb{T})} \lesssim \left(\|R^{(4)}\|_{L^2(\mathbb{T})}\|h^{(k+\sigma)}\|_{L^2(\mathbb{T})}\right)^{\frac{1+q}{k+\sigma}} \left(\|(R^{(4)})^{(k+\sigma)}\|_{L^2(\mathbb{T})}\|h\|_{L^2(\mathbb{T})}\right)^{\frac{k+p-1}{k+\sigma}}$$
$$\lesssim \|R^{(4)}\|_{L^2(\mathbb{T})}\|h^{(k+\sigma)}\|_{L^2(\mathbb{T})}+\|R^{(k+4+\sigma)}\|_{L^2(\mathbb{T})}\|h\|_{L^2(\mathbb{T})},$$

where we used $p+q=\sigma$ and Young's inequality. Plugging this inequality into (11.150) and using $\|h^{(\sigma)}\|_{L^2(\mathbb{T})} \lesssim \|h\|_{L^2(\mathbb{T})}+\|h^{(k+\sigma)}\|_{L^2(\mathbb{T})}$ to estimate the second term on the right-hand side in (11.150), we obtain

$$L_1 \lesssim (1+\|R\|_{H^{k+4+\sigma}(\mathbb{T})})\|h\|_{L^2(\mathbb{T})} + (1+\|R\|_{H^4(\mathbb{T})})\|h^{(k+\sigma)}\|_{L^2(\mathbb{T})}.$$

Similarly, we can obtain

$$L_2 \lesssim (1+\|R\|_{H^{k+4+\sigma}(\mathbb{T})})\|h\|_{L^2(\mathbb{T})} + (1+\|R\|_{H^4(\mathbb{T})})\|h^{(k+\sigma)}\|_{L^2(\mathbb{T})}.$$

Recalling (11.149), (11.143) and (11.144), we obtain (11.140). This proves (11.137) and finishes the proof. □

Proof of Proposition 11.1.13: If $i=j$, then the results follow immediately from Lemma 11.1.15 and 11.1.16. For $i \neq j$, the same results follow straightforwardly since there is not singularity in the integrands. □

In the next proposition, we estimate the derivative of the velocity with respect to the parameter b in view of (11.162).

Proposition 11.1.17. *Let R, $z_i(R)$, $u_{i,j}$, be as in* (11.107), (11.108) *and* (11.109). *For $k \geq 2$, there exists $\epsilon = \epsilon(k, b_1, b_2) > 0$ such that if $\|R\|_{(H^3(\mathbb{T}))^2} \leq \epsilon$, then*

$$\|\partial_{b_1} u_{i,j}(R)\|_{(H^k(\mathbb{T}))^2} \lesssim 1 + \|R\|_{(H^{k+1}(\mathbb{T}))^2}. \tag{11.151}$$

Also, if $\|R\|_{(H^3(\mathbb{T}))^2}, \|r\|_{(H^3(\mathbb{T}))^2} \leq \epsilon$ and $\|R\|_{(H^{k+1}(\mathbb{T}))^2}, \|r\|_{(H^{k+1}(\mathbb{T}))^2} \leq 1$, then,

$$\|\partial_{b_1} u_{i,j}(R) - \partial_{b_1} u_{i,j}(r)\|_{(H^k(\mathbb{T}))^2} \lesssim \|R - r\|_{(H^{k+1}(\mathbb{T}))^2}. \tag{11.152}$$

If $i, j \neq 1$, then (11.151) and (11.152) follow trivially, since $u_{2,2}$ is independent of b_1. As in Proposition 11.1.13, we only deal with the case where $i = j = 1$.

Lemma 11.1.18. *Let $R \in C^\infty(\mathbb{T})$ and $u(R)$ and $z(R)$ be as in* (11.110) *and* (11.111). *For $k \geq 2$, there exists $\epsilon = \epsilon(k, b) > 0$ such that if $\|R\|_{(H^3(\mathbb{T}))^2} \leq \epsilon$, then*

$$\|\partial_b u(R)\|_{(H^k(\mathbb{T}))^2} \lesssim 1 + \|R\|_{H^{k+1}(\mathbb{T})}. \tag{11.153}$$

Also, if $\|R\|_{H^3(\mathbb{T})}, \|r\|_{H^3(\mathbb{T})} \leq \epsilon$ and $\|R\|_{(H^{k+1}(\mathbb{T}))^2}, \|r\|_{(H^{k+1}(\mathbb{T}))^2} \leq 1$, then,

$$\|\partial_{b_1} u(R) - \partial_{b_1} u(r)\|_{(H^k(\mathbb{T}))^2} \lesssim \|R - r\|_{H^{k+1}(\mathbb{T})}. \tag{11.154}$$

Proof. From (11.112), we have

$$\begin{aligned}
\partial_b u(R)(x) &= \partial_b u_L(R)(x) + \partial_b u_N(R)(x) \\
&= \int_{\mathbb{T}} \log(2 - 2\cos(x - y))(-\cos y, -\sin y) dy + \int_{\mathbb{T}} K_4(R)(x, y) z(R)'^{\perp}(y) dy \\
&\quad + \int_{\mathbb{T}} K_5(R)(x, y)(-\cos y, -\sin y) dy \\
&=: B_1 + B_2(R) + B_3(R)
\end{aligned}$$

where

$$\begin{aligned}
&K_4(R)(x, y) := G(R(x), R(y), J(R)(x, y)), \\
&K_5(R)(x, y) := F(R(x), R(y), J(R)(x, y)), \\
&G(u, v, w) = \partial_b F(u, v, w) = \frac{2b + u + v}{b^2 + b(u + v) + uv + w^2}, \\
&F(u, v, w) = \log(b^2 + b(u + v) + uv + w^2).
\end{aligned}$$

Again, using Lemma 11.4.8, we have that for $\|R\|_{H^3(\mathbb{T})}, \|r\|_{H^3(\mathbb{T})} \leq \epsilon$ for sufficiently small $\epsilon > 0$,

$$\|K_4(R)\|_{H^k(\mathbb{T}^2)}, \|K_5(R)\|_{H^k(\mathbb{T}^2)} \lesssim 1+\|R\|_{H^{k+1}(\mathbb{T})},$$
$$\|K_4(R) - K_4(r)\|_{H^k(\mathbb{T}^2)}, \|K_5(R) - K_5(r)\|_{H^k(\mathbb{T}^2)} \lesssim (1+\|R\|_{H^{k+1}(\mathbb{T})}+\|r\|_{H^{k+1}(\mathbb{T})})\|R - r\|_{H^{k+1}(\mathbb{T})}.$$

Then (11.153) and (11.154) follow immediately from Lemma 11.4.11. □

Proof of Proposition 11.1.17: The case $i = j = 1$ follows immediately from Lemma 11.1.18. If $i \neq j$ then the result follows straightforwardly in a similar manner. □

Regarding (11.10), (11.11) and (11.13) (note that since b_3 in (11.3) depends on Θ_3 we need to estimate the second derivative wit respect to b_3 of G.), we need to estimate the higher derivatives of $u_{i,j}$.

Proposition 11.1.19. *Let R, $z_i(R)$, $u_{i,j}$, be as in* (11.107), (11.108) *and* (11.109). *For $k \geq 2$, there exists $\epsilon = \epsilon(k, b_1, b_2) > 0$ such that if $\|R\|_{(H^3(\mathbb{T}))^2} \leq \epsilon$, then*

$$\|D^2 u_{i,j}(R)[h,h]\|_{(H^k(\mathbb{T}))^2} \lesssim_k (1+\|R\|_{(H^{k+2}(\mathbb{T}))^2})\|h\|^2_{(H^k(\mathbb{T}))^2}, \tag{11.155}$$
$$\|\partial_{b_3} D^2 u_{i,j}(R)[h,h]\|_{(H^k(\mathbb{T}))^2} \lesssim_k (1+\|R\|_{(H^{k+2}(\mathbb{T}))^2})\|h\|^2_{(H^k(\mathbb{T}))^2},$$
$$\|\partial_{b_1} D u_{i,j}(R)[h]\|_{(H^k(\mathbb{T}))^2} \lesssim_k (1+\|R\|_{(H^{k+2}(\mathbb{T}))^2})\|h\|_{(H^k(\mathbb{T}))^2},$$
$$\|\partial_{b_1 b_1} D u_{i,j}(R)[h]\|_{(H^k(\mathbb{T}))^2} \lesssim_k (1+\|R\|_{(H^{k+2}(\mathbb{T}))^2})\|h\|_{(H^k(\mathbb{T}))^2}.$$

Proof. We also consider the $i = j = 1$ case only and briefly sketch the idea for (11.155), since the proof is almost identical to Lemma 11.1.16 and Lemma 11.1.18. Adapting the setting in (11.110) and (11.111), the Gateaux second derivative of u can be written as a linear combination of the integral operators of the form

$$\int_{\mathbb{T}} K(R)(x,y)L(x,y)dy,$$

where $K(R)(x,y) = \tilde{K}(R(x), R(y), J(R)(x,y))$ for some smooth function $\tilde{K} : \mathbb{R}^3 \mapsto \mathbb{R}^2$, and $L(x,y)$ is a product of two of $h(x)$, $h(y)$, $h'(y)$ and $J(h)(x,y)$. As before, K satisfies the estimates in (11.127), (11.128) and (11.129). Let us consider $L(x,y) = J(h)(x,y)^2$ only, that is,

$$T(R)[h,h] := \int_{\mathbb{T}} K(R)(x,y)J(h)(x,y)J(h)(x,y)dy.$$

Therefore, it follows from Lemma 11.4.13 that

$$\|T(R)[h,h]\|_{H^k(\mathbb{T})} \lesssim \|K\|_{H^{k+1}(\mathbb{T})}\|h\|^2_{H^k(\mathbb{T})} \lesssim (1+\|R\|_{H^{k+2}(\mathbb{T})})\|h\|^2_{H^k(\mathbb{T})},$$

where the last inequality follows from the fact that K satisfies (11.127). As mentioned, the other terms can be estimated in a similar manner and we omit the proofs. □

11.1.5 Checking the hypotheses in Theorem 11.1.7

In this subsection we will show the hypotheses in theorem 11.1.7 are satisfied. We will always assume that $2 \leq k \in \mathbb{N}$ is fixed. We denote $R := (R_1, R_2, R_3) \in \left(X^{k+1}(\mathbb{T}) \cap C^\infty(\mathbb{T})\right)^3$. Throughout the rest of the paper, we assume that

$$|\Theta_3^* - \Theta_3| + \|R\|_{(X^3)^3} \leq \epsilon, \tag{11.156}$$

for some ϵ, sufficiently small. Note that from (11.1), (11.3) and Lemma 11.1.24, it follows that

$$0 < b_3(\Theta_3, R) < b_2, \tag{11.157}$$

for sufficiently small ϵ in (11.156).

We recall our functional from (10.3) and (11.6) can be written as (since $b_1, b_2, \Theta_1, \Theta_2$ are fixed, we omit their dependence but mark the dependence of Θ_3 in the notations)

$$G(\Theta_3, R) = \begin{pmatrix} G_1(\Theta_3, R) \\ G_2(\Theta_3, R) \\ G_3(\Theta_3, R) \end{pmatrix} = \frac{1}{4\pi} \begin{pmatrix} \sum_{i=1}^3 \Theta_i \left(R_1' u^\theta_{i,1} - (b_1 + R_1) u^r_{i,1}\right) \\ \sum_{i=1}^3 \Theta_i \left(R_2' u^\theta_{i,2} - (b_2 + R_2) u^r_{i,2}\right) \\ \sum_{i=1}^3 \Theta_i \left(R_3' u^\theta_{i,3} - (b_3 + R_3) u^r_{i,3}\right) \end{pmatrix} =: \begin{pmatrix} R_1' u_1^\theta - (b_1 + R_1) u_1^r \\ R_2' u_2^\theta - (b_2 + R_2) u_2^r \\ R_3' u_3^\theta - (b_3 + R_3) u_3^r \end{pmatrix}, \tag{11.158}$$

where for $i, j, k \in \{1, 2, 3\}$,

$$u_k^\theta := u_k^\theta(\Theta_3, R) = \frac{1}{4\pi} \sum_{i=1}^3 \Theta_i u^\theta_{i,k}, \quad u_k^r := u_k^r(\Theta_3, R) = \frac{1}{4\pi} \sum_{i=1}^3 \Theta_i u^r_{i,k},$$

$$u^\theta_{i,j} = u^\theta_{i,j}(b_3(\Theta_3), \Theta_3, R) = u_{i,j}(b_3(\Theta_3), \Theta_3, R) \cdot \begin{pmatrix} -\sin x \\ \cos x \end{pmatrix},$$

$$u^r_{i,j} = u^r_{i,j}(b_3(\Theta_3), \Theta_3, R) = u_{i,j}(b_3(\Theta_3), \Theta_3, R) \cdot \begin{pmatrix} \cos x \\ \sin x \end{pmatrix},$$

and $u_{i,j}(b_3(\Theta_3), \Theta_3, R)$ is given by (11.108) and (11.109) and $b_3(\Theta_3, R)$ is given by (11.3). We also denote by $\omega(\Theta_3, R), \Psi(\Theta_3, R) : \mathbb{R}^2 \mapsto \mathbb{R}$ the corresponding vorticity and its stream function, more precisely,

$$\omega(\Theta_3, R)(p) := \sum_{i=1}^{3} \Theta_i 1_{D_i}(p), \quad \Psi(\Theta_3, R)(p) := \frac{1}{2\pi} \left(\omega(\Theta_3, R) * \log|\cdot|\right)(p), \quad p \in \mathbb{R}^2, \tag{11.159}$$

$$D_i := \left\{ r(\cos x, \sin x) \in \mathbb{R}^2 : r \le b_i + R_i(x),\ x \in \mathbb{T} \right\}. \tag{11.160}$$

With the stream function, we can write G as

$$G(\Theta_3, R)(x) = -\begin{pmatrix} \partial_x \left(\Psi(\Theta_3, R)(z_1(R)(x))\right) \\ \partial_x \left(\Psi(\Theta_3, R)(z_2(R)(x))\right) \\ \partial_x \left(\Psi(\Theta_3, R)(z_3(R)(x))\right) \end{pmatrix}. \tag{11.161}$$

The derivative of the functional will be given as

$$\begin{aligned} DG(\Theta_3, R)[h] = & \begin{pmatrix} u^\theta_1 & 0 & 0 \\ 0 & u^\theta_2 & 0 \\ 0 & 0 & u^\theta_3 \end{pmatrix} \begin{pmatrix} h'_1 \\ h'_2 \\ h'_3 \end{pmatrix} \\ & + \begin{pmatrix} -h_1 u^r_1(\Theta_3, R) + R'_1 Du^\theta_1(\Theta_3, R)[h] - (b_1 + R_1) Du^r_1(\Theta_3, R)[h] \\ -h_2 u^r_2(\Theta_3, R) + R'_2 Du^\theta_2(\Theta_3, R)[h] - (b_2 + R_2) Du^r_2(\Theta_3, R)[h] \\ -h_3 u^r_3(\Theta_3, R) + R'_3 Du^\theta_3(\Theta_3, R)[h] - (b_3 + R_3) Du^r_3(\Theta_3, R)[h] \end{pmatrix} \\ & + Db_3(\Theta_3, R)[h] \begin{pmatrix} R'_1 \partial_{b_3} u^\theta_1 - (b_1 + R_1) \partial_{b_3} u^r_1 \\ R'_2 \partial_{b_3} u^\theta_2 - (b_2 + R_2) \partial_{b_3} u^r_2 \\ R'_3 \partial_{b_3} u^\theta_3 - (b_3 + R_3) \partial_{b_3} u^r_3 - u^r_3 \end{pmatrix}, \end{aligned} \tag{11.162}$$

where $Db_3(\Theta_3, R)[h]$ is given in (11.4). We define the projection $P_0 : C^\infty(\mathbb{T}) \mapsto C^\infty(\mathbb{T})$ as

$$P_0 f = f - \frac{1}{2\pi} \int_{\mathbb{T}} f(x) dx,$$

and the linear maps,

$$a(\Theta_3, R)[h] := \begin{pmatrix} P_0\left(h_1' u_1^\theta(\Theta_3, R)\right) \\ 0 \\ 0 \end{pmatrix}, \quad A(\Theta_3, R)[h] := DG(\Theta_3, R)[h] - a(\Theta_3, R)[h]. \tag{11.163}$$

Now, we start checking the hypotheses. The hypothesis (a) immediately follows from (10.5), (11.5) and (11.6).

Regularity of the functional G

In this subsection, we use the estimates obtained in Subsection 11.1.4 to prove that the functional G defined in (11.158) satisfies the following proposition:

Proposition 11.1.20. *Let $2 \leq k \in \mathbb{N}$ and Θ_3 and R satisfy* (11.156). *Then, $G : \mathbb{R} \times (X^{k+1})^3 \mapsto (Y^k)^3$ is well-defined and* (11.8)-(11.13) *hold.*

Proof. Note that G_j is the tangential derivative of the stream function on the corresponding boundary D_j (see (11.161)). Thus, $P_0 G_j(\Theta, R) = 0$, that is, each G_j has zero mean. If the boundary of the patch D_i is given by by $R_i \in X^{k+1}$, then $\omega(\Theta_3, R)$ is invariant under $\frac{2\pi}{m}$ rotation about the origin and reflection, therefore, G_j is also $\frac{2\pi}{m}$ periodic and odd. Furthermore, it follows from (A) in Proposition 11.1.13 and Lemma 11.4.5 that

$$\begin{aligned} \|G(\Theta_3, R)\|_{(H^k(\mathbb{T}))^3} &\lesssim \sum_{i,j=1,2,3} \left(\left\| u_{i,j}(\Theta_3, R) \cdot \begin{pmatrix} -\sin x \\ \cos x \end{pmatrix} R_j' \right\|_{H^k(\mathbb{T})} + \left\| u_{i,j}(\Theta_3, R) \cdot \begin{pmatrix} \cos x \\ \sin x \end{pmatrix} (b_j + R_j) \right\|_{H^k(\mathbb{T})} \right) \\ &\lesssim 1 + \|R\|_{(H^{k+1}(\mathbb{T}))^3}, \end{aligned}$$

which proves (11.8) and that $G : \mathbb{R} \times (X^{k+1})^3 \mapsto (Y^k)^3$ is well-defined. For (11.9) - (11.13), the results follow from (A) and (B) in Proposition 11.1.13 and Proposition 11.1.17 and 11.1.19 straightforwardly. □

The Dirichlet-Neumann operator

Here, we aim to prove that the linear operator $a(\Theta_3, R)$ in (11.163) satisfies (11.14). Thanks to Lemma 11.1.3, we know that at a stationary solution, the corresponding velocity outside the support of the vorticity must vanish. In the following proposition, we will prove this quantitatively by using the main idea of [27, 14].

Proposition 11.1.21. *There exists $\eta > 0$ such that if $\|R_1\|_{H^{k+2}(\mathbb{T})} \le \eta$, then*

$$\|u_1^\theta(\Theta_3, R)\|_{H^k(\mathbb{T})} \lesssim_k \|G_1(\Theta_3, R)\|_{H^k(\mathbb{T})}.$$

Proof. We adapt the notations in (11.159), (11.160) and (11.161) for ω, Ψ and D_i. Clearly we have (we omit the dependence of Θ_3 and R for simplicity)

$$u_1^\theta(\alpha) = \nabla^\perp \Psi(z_1(\alpha)) \cdot \begin{pmatrix} -\sin\alpha \\ \cos\alpha \end{pmatrix}, \quad G_1(\alpha) = \nabla\Psi(z_1(\alpha)) \cdot z_1'(\alpha), \quad \alpha \in \mathbb{T}, \tag{11.164}$$

where z_1 is as given in (11.108). Note that Ψ is harmonic in D_1^c. In addition, it follows from (11.2) that $\int_{\mathbb{R}^2} \omega(y)dy = 0$, hence, for $x \in \mathbb{R}^2$,

$$\begin{aligned} |\Psi(x)| &\lesssim \left| \int_{\mathbb{R}^2} \omega(y) \log \frac{|x-y|}{|x|} dy \right| = O\left(\frac{1}{|x|}\right), \\ |\nabla\Psi(x)| &\lesssim \left| \int_{\mathbb{R}^2} \omega(y) \left(\frac{(x-y)}{|x-y|^2} - \frac{x}{|x|^2} \right) dy \right| = O\left(\frac{1}{|x|^2}\right). \end{aligned}$$

With the above decay rates, we can use integration by parts to obtain that for $x \in \partial D_1$,

$$\begin{aligned} 0 &= \int_{D_1^c} \log|x-y| \Delta\Psi(y) dy \\ &= -\frac{1}{2} \int_{\partial D_1} \log|x-y|^2 \nabla\Psi(y) \cdot \vec{n}(y) d\sigma(y) + \int_{\partial D_1} \nabla_y(\log|x-y|) \cdot \vec{n}(y)(\Psi(y) - \Psi(x)) d\sigma(y) \\ &=: -\frac{1}{2} L_1 + L_2, \end{aligned} \tag{11.165}$$

where $\vec{n}$ denotes the outer normal vector on ∂D_1. By the change of variables, $x \mapsto z_1(\alpha)$, and $y \mapsto z_1(\beta)$, we obtain

$$\begin{aligned} L_1 &= \int_{\mathbb{T}} \log|z_1(\alpha) - z_1(\beta)|^2 \nabla\Psi(z_1(\beta)) \cdot (-z_1'(\beta)^\perp) d\beta \\ &= \int_{\mathbb{T}} \log(2 - 2\cos(\alpha-\beta)) \nabla\Psi(z_1(\beta)) \cdot (-z_1'(\beta)^\perp) d\beta \\ &\quad + \int_{\mathbb{T}} \log\left(\frac{|z_1(\alpha) - z_1(\beta)|^2}{2 - 2\cos(\alpha-\beta)} \right) \nabla\Psi(z_1(\beta)) \cdot (-z_1'(\beta)^\perp) d\beta \\ &=: L_{11} + L_{12}. \end{aligned}$$

Similarly, we have

$$L_2 = \int_{\mathbb{T}} \frac{(z_1(\alpha) - z_1(\beta)) \cdot z_1'(\beta)^{\perp}}{|z_1(\alpha) - z_1(\beta)|^2} (\Psi(z_1(\beta)) - \Psi(z_1(\alpha)) d\beta.$$

Hence, we obtain

$$L_{11} = -L_{12} + 2L_2. \tag{11.166}$$

We claim that

$$\|L_{11}\|_{H^{k+1}(\mathbb{T})} \gtrsim \|\nabla\Psi(z_1(\cdot)) \cdot (-z_1'(\cdot)^{\perp})\|_{H^k(\mathbb{T})} \tag{11.167}$$

$$\|L_{12}\|_{H^{k+1}(\mathbb{T})} \lesssim \|R_1\|_{H^{k+2}(\mathbb{T})} \|\nabla\Psi(z_1(\cdot)) \cdot (-z_1'(\cdot)^{\perp})\|_{L^2(\mathbb{T})} \tag{11.168}$$

$$\|L_2\|_{H^{k+1}(\mathbb{T})} \lesssim \|\nabla\Psi(z_1(\cdot)) \cdot (z_1'(\cdot))\|_{H^k(\mathbb{T})}. \tag{11.169}$$

Let us assume the above claims for a moment. Then it follows from the claims, (11.156) and (11.166) that for sufficiently small $\eta > 0$ in the hypothesis of the proposition, we have

$$\|\nabla\Psi(z_1(\cdot)) \cdot (-z_1'(\cdot)^{\perp})\|_{H^k(\mathbb{T})} \lesssim \|\nabla\Psi(z_1(\cdot)) \cdot (z_1'(\cdot))\|_{H^k(\mathbb{T})}.$$

With this inequality, we can obtain

$$\begin{aligned}
\|\nabla\Psi(z_1(\cdot))\|_{H^k(\mathbb{T})} &= \left\| \nabla\Psi(z_1(\cdot)) \cdot \frac{z_1'(\cdot)^{\perp}}{|z_1'(\cdot)|} \right\|_{H^k(\mathbb{T})} + \left\| \nabla\Psi(z_1(\cdot)) \cdot \frac{z_1'(\cdot)}{|z_1'(\cdot)|} \right\|_{H^k(\mathbb{T})} \\
&\lesssim \|z_1'\|_{H^k(\mathbb{T})} \left(\|\nabla\Psi(z_1(\cdot)) \cdot z_1'(\cdot)^{\perp}\|_{H^k(\mathbb{T})} + \|\nabla\Psi(z_1(\cdot)) \cdot z_1'(\cdot)\|_{H^k(\mathbb{T})} \right) \\
&\lesssim \|\nabla\Psi(z_1(\cdot)) \cdot z_1'(\cdot)\|_{H^k(\mathbb{T})},
\end{aligned}$$

where the first inequalities follows from $\frac{1}{c} < |z_1'| < c$ for some $c > 0$ and the second inequality follows from $\|z_1'\|_{H^k(\mathbb{T})} \lesssim \|R_1\|_{H^{k+1}(\mathbb{T})} \leq \eta$ for some small η. under the assumption (11.156). Finally, recalling (11.164), we obtain the desired result.

To finish the proof, we need to prove the claims (11.167)-(11.169). (11.167) follows from Lemma 11.4.14. To see (11.168), we observe that

$$\log\left(\frac{|z_1(\alpha) - z_1(\beta)|^2}{2 - 2\cos(\alpha - \beta)} \right) = \log(1 + \mathcal{K}(\alpha, \beta)),$$

where $\mathcal{K}(\alpha,\beta) := 2(R_1(\alpha)+R_1(\beta)) + R_1(\alpha)R_1(\beta) + \left(\frac{R_1(\alpha)-R_1(\beta)}{2\sin(\frac{\alpha-\beta}{2})}\right)^2$. Thus, it is straightforward that (thanks to (11.156), $1+\mathcal{K} \geq c > 0$ for some $c > 0$)

$$\|\log(1+\mathcal{K})\|_{H^{k+1}(\mathbb{T}^2)} \lesssim \|\mathcal{K}\|_{H^{k+1}(\mathbb{T}^2)} \lesssim \|R_1\|_{H^{k+2}(\mathbb{T})},$$

where the last inequality follows from Lemma 11.4.5 and Lemma 11.4.10. Thus,

$$\begin{aligned}\|L_{12}\|_{H^{k+1}(\mathbb{T})} &\lesssim \left\| \log\left(\frac{|z_1(\alpha)-z_1(\beta)|^2}{2-2\cos(\alpha-\beta)}\right)\right\|_{H^{k+1}(\mathbb{T}^2)} \|\nabla\Psi(z_1(\cdot))\cdot(-z_1'(\cdot)^\perp)\|_{L^2(\mathbb{T})} \\ &\lesssim \|R_1\|_{H^{k+2}(\mathbb{T}^2)} \|\nabla\Psi(z_1(\cdot))\cdot(-z_1'(\cdot)^\perp)\|_{L^2(\mathbb{T})},\end{aligned}$$

which yields (11.168). Lastly, in order to show (11.169), we rewrite L_2 as

$$L_2 = \int_{\mathbb{T}} g(\alpha,\beta)\frac{((\Psi(z_1(\beta))-M_\Psi)-(\Psi(z_1(\alpha))-M_\Psi)}{2\sin(\frac{\alpha-\beta}{2})}d\beta,$$

where

$$g(\alpha,\beta) = \frac{(z_1(\alpha)-z_1(\beta))\cdot z_1'(\beta)^\perp}{2\sin(\frac{\alpha-\beta}{2})}\cdot\frac{2-2\cos(\alpha-\beta)}{|z_1(\alpha)-z_1(\beta)|^2}, \quad M_\Psi := \frac{1}{2\pi}\int_{\mathbb{T}}\Psi(z_1(\alpha))d\alpha.$$

From Lemma 11.4.10, and 11.4.5, we have

$$\|g\|_{H^{k+1}(\mathbb{T}^2)} \lesssim \|R_1\|_{H^{k+2}(\mathbb{T})}.$$

Therefore, it follows from Lemma 11.4.12 that

$$\|L_2\|_{H^{k+1}(\mathbb{T})} \lesssim \|(\Psi(z_1(R,\cdot))-M_\Psi)\|_{H^{k+1}(\mathbb{T})} \lesssim \|\nabla\Psi(z_1(R,\cdot))\cdot(z_1'(R,\cdot))\|_{H^k(\mathbb{T})},$$

where the last inequality follows from Poincaré inequality. Hence (11.169) follows. □

Estimates on the linearized operator

Our goal here is to prove that G satisfies the hypotheses (c), $(\tilde{c}-1)$ and $(\tilde{c}-2)$ in Theorem 11.1.7. More precisely, we will prove the following proposition:

Proposition 11.1.22. *Let $a(\Theta_3,R)$ and $A(\Theta_3,R)$ be as in* (11.163). *Then*

1) $a(\Theta_3,R) \in \mathcal{L}((X^{k+1})^3,(Y^k)^3)$, $A(\Theta_3,R) \in \mathcal{L}((X^{k+1})^3, Y^{k+1}\times(Y^k)^2)$ with the estimates

(11.14) *and* (11.15).

2) *For* $(\Theta_3^1, R^1),\ (\Theta_3^2, R^2) \in I \times (V^3 \cap (C^\infty)^3)$, (11.16) *holds.*

3) *For any even* $\sigma \in \mathbb{N} \cup \{0\}$, *there exists* $0 < \eta < 1$ *such that if* $\|R\|_{(H^{k+3}(\mathbb{T}))^3} \leq \eta$, *and* $A(\Theta_3, R)[h] = z$ *for some* $z \in (C^\infty)^3$ *and* $h \in Ker(A(\Theta_3^*, 0))^\perp$, *then* (11.17) *holds.*

Proof. **Proof of 1).** By definition of $a(\Theta_3, R)$ in (11.163), we have $\int_{\mathbb{T}} a(\Theta_3, R)[h](x)dx = 0$. Furthermore, clearly, h_1' is a $\frac{2\pi}{m}$-periodic function and odd. Using the invariance under rotation/reflection of the stream function, it follows straightforwardly that u_1^θ, which is the radial derivative of the stream function on the outmost boundary, is also $\frac{2\pi}{m}$-periodic and even. Therefore, $a(\Theta_3, R)[h]$ is $\frac{2\pi}{m}$ periodic and odd. (11.14) follows immediately from (11.1.21), and $a(\Theta_3, R) \in \mathcal{L}((X^{k+1})^3, (Y^k)^3)$. Similarly, $A(\Theta_3, R) \in \mathcal{L}((X^{k+1})^3, Y^{k+1} \times (Y^k)^2)$ and (11.15) follows from (B) in Proposition 11.1.13.

Proof of 2). Thanks to (C) in Proposition 11.1.13 and Proposition 11.1.17, each term in $A(\Theta_3, R)$ is Lipschitz continuous with respect to $R \in (H^{k+3}(\mathbb{T})^3)$. Furthermore, $A(\theta_3, R)$ and b_3 depend on Θ_3 smoothly, therefore the result follows immediately.

Proof of 3). In order to prove 3, we first claim that for each even $\sigma \in \mathbb{N} \cup \{0\}$, there exist η and a linear map $T(R) : (C^\infty(\mathbb{T}))^3 \mapsto (C^\infty(\mathbb{T}))^3$ such that if $\|R\|_{(H^{k+3}(\mathbb{T}))^3} \leq \eta$, then

$$\left(\frac{d}{dx}\right)^\sigma A(\Theta_3, R)[h] = A(\Theta_3, R)[h^{(\sigma)}] + L^\sigma(R)[h], \tag{11.170}$$

$$\|L^\sigma(R)[h]\|_{H^{k+1}(\mathbb{T}) \times (H^k(\mathbb{T}))^2} \lesssim_{k,\sigma} \|h\|_{(H^{k+\sigma}(\mathbb{T}))^3} + (1 + \|R\|_{(H^{k+4+\sigma}(\mathbb{T}))^3})\|h\|_{(H^{k+1}(\mathbb{T}))^3}. \tag{11.171}$$

Let us assume that the above claims are true for a moment and let us suppose

$$A(\Theta_3, R)[h] = z, \quad \text{for some } z \in (C^\infty(\mathbb{T}))^3,\ h \in \mathrm{Ker}(A(\Theta_3^*, 0))^\perp \subset (H^{k+1}(\mathbb{T}))^3.$$

From (11.156), 2), Lemma 11.4.15 and the assumption that $\|R\|_{(H^{k+3}(\mathbb{T}))^3} \leq \eta$ for some small enough η, we have that $A(\Theta_3, R) : \mathrm{Ker}(A(\Theta_3^*, 0))^\perp \subset ((H^{k+1}(\mathbb{T}))^3) \mapsto \mathrm{Im}(A(\Theta, R)) \subset H^{k+1}(\mathbb{T}) \times (H^k(\mathbb{T}))^2$ is invertible and

$$\|A(\Theta_3, R)^{-1}\|_{\mathcal{L}(\mathrm{Im}(A(\Theta_3, R)), \mathrm{Ker}(A(\Theta_3^*, 0))^\perp)} \lesssim 1. \tag{11.172}$$

Therefore, we have

$$\|h\|_{(H^{k+1}(\mathbb{T}))^3} \lesssim \|z\|_{H^{k+1}(\mathbb{T}) \times (H^k(\mathbb{T}))^2}. \tag{11.173}$$

Now for each even $\sigma \in \mathbb{N}\{0\}$, it follows from (11.170) that

$$\left(\frac{d}{dx}\right)^{\sigma} A(\Theta_3, R)[h] = A(\Theta_3, R)[h^{(\sigma)}] + L^{\sigma}(R)[h] = z^{(\sigma)}.$$

Thanks to Proposition 11.1.23, we have $h^{(\sigma)} \in \mathrm{Ker}(A(\Theta_3^*, 0))^{\perp}$, thus, it follows from (11.172) that

$$h^{(\sigma)} = A(\Theta_3, R)^{-1}[z^{(\sigma)} - L^{\sigma}(R)[h]],$$

and

$$\begin{aligned}\|h^{(\sigma)}\|_{(H^{k+1}(\mathbb{T}))^3} \lesssim & \|z^{(\sigma)} - L^{\sigma}(R)[h]\|_{H^{k+1}(\mathbb{T})\times(H^k(\mathbb{T}))^2} \\ \lesssim & \|z\|_{H^{k+1+\sigma}(\mathbb{T})\times(H^{k+\sigma}(\mathbb{T}))^2} + \|L^{\sigma}(R)[h]\|_{H^{k+1}(\mathbb{T})\times(H^k(\mathbb{T}))^2}. \end{aligned} \tag{11.174}$$

From (11.171), we also have

$$\|L^{\sigma}(R)[h]\|_{H^{k+1}(\mathbb{T})\times(H^k(\mathbb{T}))^2} \lesssim \|h\|_{(H^{k+\sigma}(\mathbb{T}))^3} + (1+\|R\|_{(H^{k+4+\sigma}(\mathbb{T}))^3})\|h\|_{(H^{k+1}(\mathbb{T}))^3}. \tag{11.175}$$

Using Lemma 11.4.6, we have

$$\begin{aligned}\|h\|_{(H^{k+\sigma}(\mathbb{T}))^3} \lesssim & \|h\|_{(H^{k+1}(\mathbb{T}))^3} + \left\| \left(h^{(k+1)}\right)^{(\sigma-1)} \right\|_{(L^2(\mathbb{T}))^3} \\ \lesssim & \|h\|_{(H^{k+1}(\mathbb{T}))^3} + \left(\|h^{(k+1)}\|_{(L^2(\mathbb{T}))^3}\right)^{\frac{1}{\sigma}} \left\| \left(h^{(k+1)}\right)^{(\sigma)} \right\|_{(L^2(\mathbb{T}))^3}^{\frac{\sigma-1}{\sigma}} \\ \lesssim & (1 + C(\epsilon))\|\|h\|_{(H^{k+1}(\mathbb{T}))^3} + \epsilon\|h^{(\sigma)}\|_{(H^{k+1}(\mathbb{T}))^3}, \end{aligned}$$

for any $\epsilon > 0$, which follows from Young's inequality. Plugging this into (11.175) and using (11.173), we obtain

$$\|L^{\sigma}(R)[h]\|_{H^{k+1}(\mathbb{T})\times(H^k(\mathbb{T}))^2} \lesssim (C(\epsilon)+\|R\|_{(H^{k+4+\sigma}(\mathbb{T}))^3})\|z\|_{H^{k+1}(\mathbb{T})\times(H^k(\mathbb{T}))^2} + \epsilon\|h^{(\sigma)}\|_{(H^{k+1}(\mathbb{T}))^3}$$

Hence we choose sufficiently small ϵ depending on k and σ so that (11.174) yields that

$$\|h^{(\sigma)}\|_{(H^{k+1}(\mathbb{T}))^3} \lesssim \|z\|_{H^{k+1+\sigma}(\mathbb{T})\times(H^{k+\sigma}(\mathbb{T}))^2} + (1+\|R\|_{(H^{k+4+\sigma}(\mathbb{T}))^3})\|z\|_{H^{k+1}(\mathbb{T})\times(H^k(\mathbb{T}))^2}.$$

With this estimate and (11.173), (11.17) follows.

In order to finish the proof, we need to prove the claim (11.170) and (11.171), we note that each

component of $A(\Theta_3, R)[h]$ consists of linear combination of the following forms (see (11.4), (11.162) and (11.163)):

$$u_i^\theta(R)h_i', \text{ for } i=2,3,$$

$$h_i u_i^r(\Theta_3,R),\ R_i' Du_i^\theta(\Theta_3,R)[h],\ (b_i+R_i)Du_1^r(\Theta_3,R)[h],\quad \text{for } i=1,2,3. \tag{11.176}$$

$$\int_{\mathbb{T}} R_i(x)h_i(x)dx\left(R_i'\partial_{b_3}u_i^\theta-(b_i+R_i)\partial_{b_3}u_i^r\right),\quad \text{for i=1,2} \tag{11.177}$$

$$\int_{\mathbb{T}} R_3(x)h_3(x)dx\left(R_3'\partial_{b_3}u_3^\theta-(b_3+R_3)\partial_{b_3}u_3^r-u_3^r\right). \tag{11.178}$$

For $i=2,3$, it is clear that

$$\left(\frac{d}{dx}\right)^\sigma\left(u_i^\theta(R)h_i'\right)=u_i^\theta h_i^{(\sigma+1)}+\sum_{p+q=\sigma,\ q\le\sigma-1}C_{p,q,\sigma}\underbrace{\left(u_i^\theta(R)\right)^{(p)}h_i^{(q+1)}}_{L_1^\sigma(R)[h_i]}, \tag{11.179}$$

for some $C_{p,q,\sigma}$. From Lemma 11.4.7, we have

$$\begin{aligned}\|L_1^\sigma(R)[h_i]\|_{H^k(\mathbb{T})}\lesssim&\|u_i^\theta(R)\|_{H^2(\mathbb{T})}\|h_i\|_{(H^{k+\sigma}(\mathbb{T}))^3}+\|u_i^\theta(R)\|_{H^{k+\sigma}(\mathbb{T})}\|h_i\|_{(H^1(\mathbb{T}))^3}\\ \lesssim&\|h_i\|_{H^{k+\sigma}(\mathbb{T})}+(1+\|R\|_{(H^{k+1+\sigma}(\mathbb{T}))^3})\|h_i\|_{H^1(\mathbb{T})},\end{aligned} \tag{11.180}$$

where the second inequality follows from Proposition 11.1.13. For the other terms, we only deal with $R_1'Du_1^\theta(\Theta_3,R)[h]$, since the other terms can be dealt with in the same way. Thus, we will show that there exists a linear operator $L_2^\sigma(R)[h]$ such that

$$\left(\frac{d}{dx}\right)^\sigma\left(R_1'Du_1^\theta(\Theta_3,R)[h]\right)=R_1'Du_1^\theta(\Theta_3,R)[h^{(\sigma)}]+L_2^\sigma(R)[h], \tag{11.181}$$

$$\|L_2^\sigma(R)[h]\|_{H^{k+1}(\mathbb{T})}\lesssim_{k,\sigma}\|h\|_{(H^{k+\sigma}(\mathbb{T}))^3}+\|R\|_{(H^{k+4+\sigma}(\mathbb{T}))^3}\|h\|_{(H^3(\mathbb{T}))^3}. \tag{11.182}$$

It follows from (D) in Proposition 11.1.13 that there exists $T^\sigma(R)[h]$ such that

$$\begin{aligned}\left(\frac{d}{dx}\right)^\sigma\left(R_1'Du_1^\theta(\Theta_3,R)[h]\right)&=R_1'\left(Du_1^\theta(\Theta_3,R)[h]\right)^{(\sigma)}+\sum_{p+q=\sigma,\ q\le\sigma-1}(R_1')^{(p)}\left(Du_1^\theta(\Theta_3,R)[h]\right)^{(q)}\\ &=R_1'Du_1^\theta(\Theta_3,R)[h^{(\sigma)}]+\underbrace{R_1'T^\sigma(R)[h]+\sum_{p+q=\sigma,\ q\le\sigma-1}C_{p,q,\sigma}(R_1')^{(p)}\left(Du_1^\theta(\Theta_3,R)[h]\right)^{(q)}}_{=:L_2^\sigma(R)[h]},\end{aligned}$$

and

$$\|T^{\sigma}(R)[h]\|_{H^{k+1}(\mathbb{T})} \lesssim (1+\|R\|_{(H^4(\mathbb{T}))^3})\|h^{(k+\sigma)}\|_{(L^2(\mathbb{T}))^3} + (1+\|R\|_{(H^{k+4+\sigma}(\mathbb{T}))^3})\|h\|_{(L^2(\mathbb{T}))^3}.$$

Therefore, from (B) in Proposition 11.1.13 and Lemma 11.4.7, it follows that

$$\begin{aligned}\|R_1' T^{\sigma}(R)[h]\|_{H^{k+1}(\mathbb{T})} \lesssim & \|R_1'\|_{H^{k+1}(\mathbb{T})}\|T^{\sigma}(R)[h]\|_{H^{k+1}(\mathbb{T})} \\ \lesssim & \|h^{(k+\sigma)}\|_{(L^2(\mathbb{T}))^3} + (1+\|R\|_{(H^{k+4+\sigma}(\mathbb{T}))^3})\|h\|_{(L^2(\mathbb{T}))^3}, \end{aligned} \tag{11.183}$$

where we used that $H^{k+1}(\mathbb{T})$ is a Banach algebra and that $\|R\|_{(H^4(\mathbb{T}))^3} \lesssim \|R\|_{(H^{k+3}(\mathbb{T}))^3} \leq \eta$ for some $0 < \eta < 1$. From Lemma 11.4.7, we also have

$$\begin{aligned}\|(R_1')^{(p)} \left(Du_1^{\theta}(\Theta_3, R)[h]\right)^{(q)}\|_{H^{k+1}(\mathbb{T})} \lesssim & \|R_1\|_{H^2(\mathbb{T})}\|Du_1^{\theta}(\Theta_3, R)[h]\|_{H^{k+\sigma}(\mathbb{T})} + \|R_1\|_{H^{k+1+\sigma}(\mathbb{T})}\|Du_1^{\theta}(\Theta_3, R)[h]\|_{H^1(\mathbb{T})} \\ \lesssim & \|Du_1^{\theta}(\Theta_3, R)[h]\|_{H^{k+\sigma}(\mathbb{T})} + \|R_1\|_{H^{k+1+\sigma}(\mathbb{T})}\|Du_1^{\theta}(\Theta_3, R)[h]\|_{H^3(\mathbb{T})} \\ \lesssim & \|h\|_{(H^{k+\sigma}(\mathbb{T}))^3} + \|R\|_{(H^{k+3+\sigma}(\mathbb{T}))^3}\|h\|_{(H^1(\mathbb{T}))^3} \\ & + \|R_1\|_{H^{k+1+\sigma}(\mathbb{T})}\|Du_1^{\theta}(\Theta_3, R)[h]\|_{H^3(\mathbb{T})}\end{aligned}$$

where the last inequality follows from (B) in Proposition 11.1.13,which also tells us that

$$\begin{aligned}\|Du_1^{\theta}(\Theta_3, R)[h]\|_{H^3(\mathbb{T})} \lesssim & (1+\|R\|_{(H^5(\mathbb{T}))^3})\|h\|_{(H^3(\mathbb{T}))^3} \\ \lesssim & (1+\|R\|_{(H^{k+3+\sigma}(\mathbb{T}))^3})\|h\|_{(H^3(\mathbb{T})^3)},\end{aligned}$$

where we used $k \geq 2$. Therefore, we have

$$\|(R_1')^{(p)} \left(Du_1^{\theta}(\Theta_3, R)[h]\right)^{(q)}\|_{H^{k+1}(\mathbb{T})} \lesssim \|h\|_{(H^{k+\sigma}(\mathbb{T}))^3} + (1+\|R\|_{(H^{k+3+\sigma}(\mathbb{T}))^3})\|h\|_{(H^3(\mathbb{T}))^3}.$$

Combining with (11.183), we obtain (11.182). Since every term in (11.176), (11.177) and (11.178) can be shown to satisfy the same property as in (11.181) and (11.182), combining with (11.179) and (11.180), we obtain (11.170) and (11.171).

□

Spectral Study

In this subsection, we will verify the hypotheses (d) and (e). They will be proved in Proposition 11.1.23 and Lemma 11.1.26 respectively.

Proposition 11.1.23. *Let $2 \leq m \in \mathbb{N}$. We can choose $b_1 = \Theta_1 = 1$ and b_2 and Θ_2 be as in (11.1). Also we set b_3 as in (11.3). Then, there exists $\Theta_3^* > 0$ such that $A(\Theta_3^*, 0) \in \mathcal{L}((H^{k+1})^3, H^{k+1} \times (H^k)^2)$ has one-dimensional kernel and codimension one. That is,*

$$Ker(A(\Theta_3^*, 0)) = span\{v\}, \quad Im(A(\Theta_3^*, 0))^\perp = span\{w\}.$$

Furthermore, $0 < b_3(\Theta_3^*, 0) < b_2 < b_1 = 1$ and v and w are supported on the m-th Fourier mode.

The proof of the above proposition will be achieved after proving several lemmas.

Let us first compute the derivative of G at $(\Theta_3, 0)$. Since we have $G(\Theta_3, 0) = 0$ for any Θ_3, 1 in Proposition 11.1.22 yields that $a(\Theta_3, 0) = 0$ for any Θ_3. Moreover, it follows from (11.162), (11.4) and Lemma 10.2.5 that letting $h_1(x) = \sum_{n\geq 1} h_{1,n} \cos(mnx)$, $h_2(x) = \sum_{n\geq 1} h_{2,n} \cos(mnx)$, $h_3(x) = \sum_{n\geq 1} h_{3,n} \cos(mnx)$,

$$A(\Theta_3, 0)[h_1, h_2, h_3] = \begin{pmatrix} Q_1(x) \\ Q_2(x) \\ Q_3(x) \end{pmatrix},$$

where

$$Q_1(x) = \sum_{n\geq 1} q_{1,n} \sin(mnx), \quad Q_2(x) = \sum_{n\geq 1} q_{2,n} \sin(mnx), \quad Q_3(x) = \sum_{n\geq 1} q_{3,n} \sin(mnx)$$

where the coefficients satisfy, for any n:

$$(-mn) M_n(\Theta_3, b_3) \begin{pmatrix} h_{1,n} \\ h_{2,n} \\ h_{3,n} \end{pmatrix} = \begin{pmatrix} q_{1,n} \\ q_{2,n} \\ q_{3,n} \end{pmatrix}, \tag{11.184}$$

with

$$M_n(\Theta_3, b_3) =$$

$$\begin{pmatrix} \left(-\frac{1}{2}+\frac{1}{2mn}\right)-\frac{1}{2}\Theta_2 b_2^2-\frac{1}{2}\Theta_3 b_3^2 & \Theta_2 b_2 \frac{b_2^{mn}}{2mn} & \Theta_3 b_3 \frac{b_3^{mn}}{2mn} \\ \frac{b_2^{mn}}{2mn} & \Theta_2 b_2\left(-\frac{1}{2}+\frac{1}{2mn}\right)+\left(-\frac{1}{2}b_2\right)+\left(-\frac{1}{2}\right)\Theta_3\frac{b_3^2}{b_2} & \Theta_3 b_3 \frac{1}{2mn}\left(\frac{b_3}{b_2}\right)^{mn} \\ \frac{b_3^{mn}}{2mn} & \Theta_2 b_2 \frac{1}{2mn}\left(\frac{b_3}{b_2}\right)^{mn} & \Theta_3 b_3\left(-\frac{1}{2}+\frac{1}{2mn}\right)+\left(-\frac{1}{2}b_3\right)-\frac{\Theta_2 b_3}{2} \end{pmatrix} \tag{11.185}$$

Lemma 11.1.24. *Let $2 \leq m \in \mathbb{N}$, and b_2, Θ_2 satisfy (11.1). Then, there exists $\Theta_3^* := \Theta_{3,m}^* > 0$ and b_3 such that $0 < b_3 < b_2$, and*

$$det(M_1) = 0,\ \Theta_1 b_1^2 + \Theta_2 b_2^2 + \Theta_3^* b_3^2 = 1 + \Theta_2 b_2^2 + \Theta_3^* b_3^2 = 0, \tag{11.186}$$

$$det(M_n(\Theta_3^*, b_3)) \neq 0, \quad \text{if } n \geq 2. \tag{11.187}$$

Proof. We first write $\det(M_1)$ as a polynomial in b_3. Under the constraint $1 + \Theta_2 b_2^2 + \Theta_3 b_3^2 = 0$, we get

$$M_1(\Theta_3, b_3) = \begin{pmatrix} \frac{1}{2m} & \Theta_2 b_2 \frac{b_2^m}{2m} & \Theta_3 b_3 \frac{b_3^m}{2m} \\ \frac{b_2^m}{2m} & \Theta_2 b_2\left(-\frac{1}{2}+\frac{1}{2m}\right)+\left(-\frac{1}{2}b_2\right)+\left(-\frac{1}{2}\right)\Theta_3\frac{b_3^2}{b_2} & \Theta_3 b_3 \frac{1}{2m}\left(\frac{b_3}{b_2}\right)^m \\ \frac{b_3^m}{2m} & \Theta_2 b_2 \frac{1}{2m}\left(\frac{b_3}{b_2}\right)^m & \Theta_3 b_3\left(-\frac{1}{2}+\frac{1}{2m}\right)+\left(-\frac{1}{2}b_3\right)+\left(-\frac{1}{2}\right)\Theta_2 b_3 \end{pmatrix}. \tag{11.188}$$

Therefore, multiplying the first row by $2m$, multiplying the third column by b_3 and dividing the second column by b_2, we get

$$\frac{2mb_3}{b_2}\det(M_1(\Theta_3, b_3)) = \det\begin{pmatrix} 1 & \Theta_2 b_2^m & \Theta_3 b_3^2 b_3^m \\ \frac{b_2^m}{2m} & \Theta_2\left(-\frac{1}{2}+\frac{1}{2m}\right)+\left(-\frac{1}{2}\right)+\left(-\frac{1}{2}\right)\Theta_3\frac{b_3^2}{b_2^2} & \Theta_3 \frac{1}{2m} b_3^2\left(\frac{b_3}{b_2}\right)^m \\ \frac{b_3^m}{2m} & \Theta_2 \frac{1}{2m}\left(\frac{b_3}{b_2}\right)^m & \Theta_3 b_3^2\left(-\frac{1}{2}+\frac{1}{2m}\right)+\left(-\frac{1}{2}b_3^2\right)+\left(-\frac{1}{2}\right)\Theta_2 b_3^2 \end{pmatrix}$$

$$= \det\begin{pmatrix} 1 & \Theta_2 b_2^m & (-1-\Theta_2 b_2^2) b_3^m \\ \frac{b_2^m}{2m} & \Theta_2\left(-\frac{1}{2}+\frac{1}{2m}\right)+\left(-\frac{1}{2}\right)+\left(-\frac{1}{2}\right)\frac{(-1-\Theta_2 b_2^2)}{b_2^2} & (-1-\Theta_2 b_2^2)\frac{1}{2m}\left(\frac{b_3}{b_2}\right)^m \\ \frac{b_3^m}{2m} & \Theta_2 \frac{1}{2m}\left(\frac{b_3}{b_2}\right)^m & (-1-\Theta_2 b_2^2)\left(-\frac{1}{2}+\frac{1}{2m}\right)+\left(-\frac{1}{2}b_3^2\right)+\left(-\frac{1}{2}\right)\Theta_2 b_3^2 \end{pmatrix}$$

$$= \det\begin{pmatrix} 1 & \Theta_2 b_2^m & (-1-\Theta_2 b_2^2) b_3^m \\ \frac{b_2^m}{2m} & \frac{\Theta_2}{2m}-\frac{1}{2}+\frac{1}{2b_2^2} & (-1-\Theta_2 b_2^2)\frac{1}{2m}\left(\frac{b_3}{b_2}\right)^m \\ \frac{b_3^m}{2m} & \Theta_2 \frac{1}{2m}\left(\frac{b_3}{b_2}\right)^m & (-1-\Theta_2 b_2^2)\left(-\frac{1}{2}+\frac{1}{2m}\right)+\left(-\frac{1}{2}-\frac{1}{2}\Theta_2\right) b_3^2 \end{pmatrix}$$

$$=: \det \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix},$$

where we used the condition $1+\Theta_2 b_2^2+\Theta_3 b_3^2=0$ in the second equality to get rid of Θ_3. We further compute

$$\begin{aligned}\frac{2mb_3}{b_2}\det(M_1(\Theta_3,b_3)) &= a_{31}(a_{12}a_{23}-a_{13}a_{22})-a_{32}(a_{11}a_{23}-a_{13}a_{21})+a_{33}(a_{11}a_{22}-a_{12}a_{21}) \\ &= b_3^{2m}\left(\frac{1}{4m}(1-\frac{1}{b_2^2})(-1-\Theta_2 b_2^2)+\frac{\Theta_2(b_2^m-\frac{1}{b_2^m})(-1-\Theta_2 b_2^2)}{4m^2 b_2^m}\right)+A_{3,3}\left(-\frac{1}{2}-\frac{1}{2}\Theta_2\right)b_3^2 \\ &\quad + A_{3,3}(-1-\Theta_2 b_2^2)\left(-\frac{1}{2}+\frac{1}{2m}\right).\end{aligned}$$

where $A_{3,3} := a_{11}a_{22}-a_{12}a_{21} = \frac{\Theta_2}{2m}(1-b_2^{2m})+\frac{1}{2b_2^2}(1-b_2^2)$ is the cofactor. Let us write

$$\frac{2mb_2}{b_3}\det(M_1(\Theta_3,b_3)) = B_0 b_3^{2m}+B_1 b_3^2+B_2 =: f(b_3), \tag{11.189}$$

where

$$B_0 = (-1-\Theta_2 b_2^2)\left(\frac{1}{4m}(1-\frac{1}{b_2^2})+\frac{\Theta_2(b_2^m-\frac{1}{b_2^m})}{4m^2 b_2^m}\right),$$

$$B_1 = A_{3,3}\left(-\frac{1}{2}-\frac{1}{2}\Theta_2\right),$$

$$B_2 = A_{3,3}(-1-\Theta_2 b_2^2)\left(-\frac{1}{2}+\frac{1}{2m}\right).$$

Under the condition in (11.1), we have

$$-1-\Theta_2 b_2^2 > 0, \tag{11.190}$$

$$\frac{1}{4m}(1-\frac{1}{b_2^2})+\frac{\Theta_2(b_2^m-\frac{1}{b_2^m})}{4m^2 b_2^m} > 0, \tag{11.191}$$

$$A_{3,3} > 0, \tag{11.192}$$

$$-\frac{1}{2}-\frac{\Theta_2}{2} > 0, \tag{11.193}$$

$$-\frac{1}{2}+\frac{1}{2m} < 0. \tag{11.194}$$

So $B_0>0$, $B_1>0$, $B_2<0$. We then can use the Descartes rule of signs and show there exists a unique solution $b_3\in(0,\infty)$. Since $B_2<0$, the Descartes rule again tells us that we only need show

that $f(b_2) > 0$ to show the unique solution b_3 such that $f(b_3) = 0$ satisfies $0 < b_3 < b_2$. Once this is done, then Θ_3^* can be uniquely determined by $1 + \Theta_2 b_2^2 + \Theta_3 b_3^2 = 0$ and this proves the existence of b_3 and Θ_3^* satisfying (11.186). Therefore we compute

$$
\begin{aligned}
&f(b_2) = B_0 b_2^{2m} + B_1 b_2^2 + B_2 \\
&= [\frac{(\Theta_2 b_2^2 + 1)(1 - b_2^2)}{4mb_2^2} + \Theta_2 \frac{(\frac{1}{b_2^m} - b_2^m)(1 + \Theta_2 b_2^2)}{4m^2 b_2^m}] b_2^{2m} + (\frac{\Theta_2}{2m}(1 - b_2^{2m}) + \frac{1}{2b_2^2}(1 - b_2^2))(-\frac{b_2^2}{2} + \frac{1}{2} - \frac{1}{2m} - \frac{\Theta_2 b_2^2}{2m}) \\
&= (\frac{b_2^2(1 - b_2^{2m})}{4m^2} - \frac{b_2^2}{2m}\frac{1 - b_2^{2m}}{2m})\Theta_2^2 \\
&+ [\frac{(1 - b_2^2) b_2^{2+2m}}{4m^2 b_2^2} + \frac{(\frac{1}{b_2^m} - b_2^m) b_2^{2m}}{4m^2 b_2^m} + \frac{1 - b_2^{2m}}{2m}(-\frac{b_2^2}{2} + \frac{1}{2} - \frac{1}{2m}) + \frac{1}{2b_2^2}(1 - b_2^2)(-\frac{b_2^2}{2m})]\Theta_2 \\
&+ (b_2^{2m}\frac{1 - b_2^2}{4mb_2^2}) + \frac{1}{2b_2^2}(1 - b_2^2)(-\frac{b_2^2}{2} + \frac{1}{2} - \frac{1}{2m}) \\
&= \frac{1 - b_2^2}{4mb_2^2}(b_2^{2m} + m - mb_2^2 - 1) \\
&= \frac{(1 - b_2^2)^2}{4mb_2^2}(\frac{b_2^{2m} - 1}{1 - b_2^2} + m) \\
&= \frac{(1 - b_2^2)^2}{4mb_2^2}(m - 1 - b_2^2 - b_2^4 - ... - b_2^{2m-2}) > 0.
\end{aligned}
$$

Now we are left to show (11.187), that is, $\det(M_n(\Theta_3^*), b_3) \neq 0$. As before, we have (replacing m by nm)

$$\frac{2nmb_2}{b_3}\det(M_n(\Theta_3^*, b_3)) = B_{0,n} b_3^{2nm} + B_{1,n} b_3^2 + B_{2,n}. \tag{11.195}$$

where

$$B_{0,n} = \frac{1}{4mn}(1 - \frac{1}{b_2^2})(-1 - \Theta_2 b_2^2) + \frac{\Theta_2(b_2^{nm} - \frac{1}{b_2^{nm}})(-1 - \Theta_2 b_2^2)}{4n^2 m^2 b_2^{nm}},$$

$$B_{1,n} = A_{3,3,n}\left(-\frac{1}{2} - \frac{1}{2}\Theta_2\right),$$

$$B_{2,n} = A_{3,3,n}(-1 - \Theta_2 b_2^2)(-\frac{1}{2} + \frac{1}{2nm}),$$

$$A_{3,3,n} = \frac{\Theta_2}{2nm}(1 - b_2^{2nm}) + \frac{1}{2b_2^2}(1 - b_2^2).$$

Since $f(b_3) = 0$, where f is as in (11.189), that is, $b_3^2 = -\frac{B_2 + B_0 b_3^{2m}}{B_1}$. Thus,

$$b_3^{2nm} B_{0,n} + b_3^2 B_{1,n} + B_{2,n} = b_3^{2nm} B_{0,n} - \frac{B_{1,n}}{B_1} B_0 b_3^{2m} - \frac{B_{1,n}}{B_1} B_2 + B_{2,n}.$$

Let $q(z) = z^n B_{0,n} - \frac{B_{1,n}}{B_1} B_0 z - \frac{B_{1,n}}{B_1} B_2 + B_{2,n}$. We will show that

$$q(0) < 0 \text{ and } q'(z) < 0 \quad \text{for } 0 \le z \le b_2^{2m}. \tag{11.196}$$

Once we have the above inequalities, then $q(b_3^{2m}) < 0$, which implies $\det(M_n(\Theta_3^*))$ in (11.195) cannot be zero. This clearly finishes the proof.

To show (11.196), let us first consider $q(0)$. Note that

$$\frac{B_{1,n}}{B_1} = \frac{A_{3,3,n}}{A_{3,3}}, \ \frac{B_{2,n}}{B_2} = \frac{A_{3,3,n}(-\frac{1}{2} + \frac{1}{2mn})}{A_{3,3}(-\frac{1}{2} + \frac{1}{2m})}$$

$$q(0) = -\frac{B_{1,n}}{B_1} B_2 + B_{2,n} < 0 \Leftrightarrow \frac{B_{2,n}}{B_2} > \frac{B_{1,n}}{B_1} \Leftrightarrow \frac{A_{3,3,n}}{A_{3,3}} > 0$$

Thanks to (11.193), it suffices to show that $\frac{A_{3,3,n}}{A_{3,3}} > 1$. Note that

$$\frac{A_{3,3,n}}{A_{3,3}} > 1 \Leftrightarrow \frac{\Theta_2}{2mn}(1 - b_2^{2mn}) > \frac{\Theta_2}{2m}(1 - b_2^{2m}).$$

Clearly, the last inequality holds because $b_2 < 1$, and $\Theta_2 < 0$, therefore we have

$$q(0) < 0.$$

Now, we turn to $q'(z) < 0$. We have

$$q'(z) = nz^{n-1} B_{0,n} - \frac{B_{1,n}}{B_1} B_0.$$

If $B_{0,n} \le 0$, then $z \mapsto q'(z)$ is deacreasing and $q'(0) = -\frac{B_{1,n}}{B_1} B_0 = -\frac{A_{3,3,n}}{A_{3,3}} B_0 < 0$, which yields $q'(z) < 0$ for all $0 \le z \le b_2^{2m}$. If $B_{0,n} > 0$, it is sufficient to show

$$q'(b_2^{2m}) \le n(b_2^{2mn-2m}) B_{0,n} - \frac{B_{1,n}}{B_1} B_0 < 0. \tag{11.197}$$

From (11.1), we have $(1 - \frac{1}{b_2^2}) + \frac{\Theta_2}{m}(1 - \frac{1}{b_2^{2m}}) > \frac{1 - b_2^2}{b_2^2}$. And the lower bound of Θ_2 in (11.1) is equivalent to $\frac{\Theta_2}{m}(1 - b_2^{2m}) + \frac{1}{b_2^2}(1 - b_2^2) > 0$, hence

$$(11.197) \Leftrightarrow \frac{A_{3,3,n}}{A_{3,3}} = \frac{\frac{\Theta_2}{nm}(1 - b_2^{2nm}) + \frac{1}{b_2^2}(1 - b_2^2)}{\frac{\Theta_2}{m}(1 - b_2^{2m}) + \frac{1}{b_2^2}(1 - b_2^2)} > \frac{(1 - \frac{1}{b_2^2}) + \frac{\Theta_2}{mn}(1 - \frac{1}{b_2^{2nm}})}{(1 - \frac{1}{b_2^2}) + \frac{\Theta_2}{m}(1 - \frac{1}{b_2^{2m}})} b_2^{2nm-2m},$$

$$\Leftrightarrow [\frac{\Theta_2}{nm}(1-b_2^{2nm})+\frac{1}{b_2^2}(1-b_2^2)][(1-\frac{1}{b_2^2})+\frac{\Theta_2}{m}(1-\frac{1}{b_2^{2m}})] > [(1-\frac{1}{b_2^2})+\frac{\Theta_2}{mn}(1-\frac{1}{b_2^{2nm}})][\frac{\Theta_2}{m}(1-b_2^{2m})+\frac{1}{b_2^2}(1-b_2^2)]b_2^{2mn-2m},$$

$$\Leftrightarrow \Theta_2[\frac{(1-b_2^{2mn})(1-b_2^{-2})+n(b_2^{-2}-1)(1-b_2^{-2m})}{nm}-\frac{(b_2^{2mn}-1)b_2^{-2m}(b_2^{-2}-1)+n(1-b_2^{2m})b_2^{2mn-2m}(1-b_2^{-2})}{nm}]$$

$$+\frac{1}{b_2^2}(1-b_2^2)(1-\frac{1}{b_2^2})-(1-\frac{1}{b_2^2})\frac{1}{b_2^2}(1-b_2^2)b_2^{2nm-2m}>0$$

$$\Leftrightarrow \frac{\Theta_2(1-b_2^{-2})(1-b_2^{-2m})(1-n)(1-b_2^{2nm})}{mn}-(1-b_2^{-2})^2(1-b_2^{2nm-2m})>0,$$

$$\Leftrightarrow \Theta_2 < \frac{m(b_2^2-1)b_2^{2m}}{b_2^2(1-b_2^{2m})}\frac{1-b_2^{2m(n-1)}}{(1-b_2^{2mn})(1-\frac{1}{n})}.$$

Since $\frac{1-b_2^{2m(n-1)}}{(1-b_2^{2mn})(1-\frac{1}{n})} \leq \frac{n}{n-1} \leq 2$, the condition $\Theta_2 < 2\frac{{b_2}^{2m-2}(b_2^2-1)m}{1-b_2^{2m}}$ in (11.1) leads to the inequality we want. This shows (11.196), and thus (11.187) holds. □

From Lemma 11.1.24, it is clear that the kernel of $A(\Theta_3^*, 0)$ is one dimensional. We denote by $v \in \mathrm{Ker}(A(\Theta_3^*, 0))$ and the one such that

$$v = \cos(mx)\begin{pmatrix} v^1 \\ v^2 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} v^1 \\ v^2 \\ 1 \end{pmatrix} \in Ker(M_1(\Theta_3^*, b_3)). \tag{11.198}$$

Since b_3 and Θ_3^* are tied in the relation $1 + \Theta_2 b_2^2 + \Theta_3^* b_3^2 = 0$, we now drop the dependence on b_3 of $M_n(\Theta_3^*) := M_n(\Theta_3^*, b_3)$.

We now characterize the image of $A(\Theta_3^*, 0)$.

Lemma 11.1.25. *Let b_2, Θ_2 satisfy the conditions in* (11.1) *and let b_3, Θ_3^* be as in Lemma 11.1.24. Then,*

$$Z = \left\{ Q = (Q_1, Q_2, Q_3) \in Y^{k+1,m} \times Y^{k,m} \times Y^{k,m}, \begin{pmatrix} Q_1 \\ Q_2 \\ Q_3 \end{pmatrix} = \sum_{n=1}^{\infty} \begin{pmatrix} q_{1,n} \\ q_{2,n} \\ q_{3,n} \end{pmatrix} \sin(mnx), \tag{11.199}$$

$$\begin{pmatrix} q_{1,1} \\ q_{2,1} \\ q_{3,1} \end{pmatrix} \in span\{c_1, c_2\} \right\}, \tag{11.200}$$

where c_i is the ith column of $M_1(\Theta_3^*, b_3)$. Then $Z = Im(A(\Theta_3^*, 0))$.

Proof. We choose an element $w \in \mathrm{Im}(A(\Theta_3^*, 0))^\perp$ to be the one such that

$$w = \frac{1}{|c_1 \times c_2|} \cos(mx)\,(c_1 \times c_2) \in Im(M_1(\Theta^*))^\perp. \tag{11.201}$$

where c_1 and c_2 are the first two column vectors in $M_1(\Theta_3^*)$, which are linearly independent. Since $\det(M_1(\Theta_3^*)) = 0$, and $\det(M_n(\Theta_3^*, b_3)) \neq 0$ for all $n > 1$, and Proposition 11.1.23, it is clear that $\mathrm{Im}\,(A(\Theta_3^*, 0)) \subset Z$. In order to show the other direction, $Z \subset \mathrm{Im}\,(A(\Theta_3^*, 0))$, we pick an element $Q \in (Q_1, Q_2, Q_3) \in Z$,$Q_i = \sum_{n=1}^\infty q_{1,n} \sin(mnx)$, and we have λ_Q^1, $\lambda_Q^2 \in \mathbb{R}$ such that

$$\begin{pmatrix} q_{1,2} \\ q_{2,2} \\ q_{3,2} \end{pmatrix} = \lambda_Q^1 c_1 + \lambda_Q^2 c_2.$$

Thanks to (11.187), one can find a sequence of vectors $h^n := (h_{1,n}, h_{2,n}, h_{3,n})$ such that

$$(-mn)M_{mn}(\Theta^*)h^n = (-mn)M_{mn}(\Theta^*) \begin{pmatrix} h_{1,n} \\ h_{2,n} \\ h_{3,n} \end{pmatrix} = \begin{pmatrix} q_{1,n} \\ q_{2,n} \\ q_{3,n} \end{pmatrix} =: q^n, \tag{11.202}$$

Therefore it suffices to show that

$$|h_{1,n}| + |h_{2,n}| + |h_{3,n}| \leq C \left(|q_{1.n}| + \frac{|q_{2,n}|}{|mn|} + \frac{|q_{3,n}|}{|mn|} \right), \quad \text{for all sufficiently large } n. \tag{11.203}$$

where C is independent of n. Once we have the above inequality, then it is clear that $Q = A(\Theta_3^*, 0)[h]$ where

$$h_1(x) = \sum_{n=1}^\infty h_{1,n} \cos(mnx), \quad h_2(x) = \sum_{n=1}^\infty h_{2,n} \cos(mnx), \quad h_3(x) = \sum_{n=1}^\infty h_{3,n} \cos(mnx),$$

and $h \in (H^{k+1})^3$, which follows from

$$\begin{aligned} \|h\|_{(H^{k+1})^3}^2 &= \sum_{n \geq 1} |h_{1,n}|^2 |nm|^{2(k+1)} + |h_{2,n}|^2 |nm|^{2(k+1)} + |h_{3,n}|^2 |nm|^{2(k+1)} \\ &\lesssim \sum_{n \geq 1} |mn|^{2(k+1)} \left(|q_{1,n}|^2 + \frac{|q_{2,n}|^2}{|mn|^2} + \frac{|q_{2,n}|^2}{|mn|^2} \right) \\ &\lesssim \|Q_1\|_{H^{k+1}}^2 + \|Q_2\|_{H^k}^2 + \|Q_3\|_{H^k}^2, \end{aligned}$$

$$< \infty.$$

This proves that $Z \subset \mathrm{Im}\,(A(\Theta_3^*, 0))$.

In order to prove (11.203), we denote

$$(-mn)M_n(\Theta_3^*) =: \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{pmatrix} + \begin{pmatrix} 0 & a_{12} & a_{13} \\ a_{21} & 0 & a_{23} \\ a_{31} & a_{32} & 0 \end{pmatrix} =: D_n + S_n.$$

Then (11.202) is equivalent to

$$D_n h^n = q^n - S_n h^n. \tag{11.204}$$

We claim that

$$d_2 := \Theta_2 b_2 + b_2 + \Theta_3 \frac{b_3^2}{b_2} \neq 0, \quad \text{and} \quad d_3 := \Theta_3 b_3 + b_3 + \Theta_2 b_3 \neq 0. \tag{11.205}$$

From (11.185), and $0 < b_3 < b_2 < 1$, it follows that

$$D_n = \begin{pmatrix} -\frac{1}{2} & 0 & 0 \\ 0 & \frac{mn}{2}\left(d_2 - \frac{\Theta_2 b_2}{nm}\right) & 0 \\ 0 & 0 & \frac{nm}{2}\left(d_3 - \frac{\Theta_3^* b_3}{nm}\right) \end{pmatrix} \quad \text{and } |S_n h^n| \lesssim \kappa^{nm} |h^n|, \quad \text{for some } \kappa \in (0,1).$$

Therefore, for sufficiently large n, D_n is a non-singular diagonal matrix, and it follows from (11.204)

$$|h_{1,n}| \lesssim |q_{1,n}| + \kappa^{nm}|h^n|, \quad |h_{2,n}| \lesssim \frac{|q_{2,n}|}{|nm|} + \kappa^{nm}|h^n|, \quad |h_{3,n}| \lesssim \frac{|q_{3,n}|}{|nm|} + \kappa^{nm}|h^n|,$$

which proves (11.203).

We are left to show (11.205). From (11.186),

$$d_2 = \Theta_2 b_2 + b_2 + \Theta_3 \frac{b_3^2}{b_2} = (\Theta_2 b_2^2 + b_2^2 + \Theta_3 b_3^2)\frac{1}{b_2} = \frac{b_2^2 - 1}{b_2} < 0. \tag{11.206}$$

For d_3, let us claim

$$B_0(\frac{1 + \Theta_2 b_2^2}{\Theta_2 + 1})^m + B_1(\frac{1 + \Theta_2 b_2^2}{\Theta_2 + 1}) + B_2 \neq 0, \quad \text{where } B_0, B_1 \text{ and } B_2 \text{ are as in (11.189).} \tag{11.207}$$

Plugging in B_0, B_1, B_2 from (11.189), we can find that (11.207) is equivalent to

$$-[\frac{1}{4m}(1-b_2^{-2})+\frac{\Theta_2(b_2^m-\frac{1}{b_2^m})}{4m^2b_2^m}]\frac{(1+\Theta_2b_2^2)^m}{(\Theta_2+1)^m}-\frac{1}{2}A_{3,3}-A_{3,3}(-\frac{1}{2}+\frac{1}{2m})\neq 0$$

$$\Leftrightarrow -[(1-b_2^{-2})+\frac{\Theta_2}{m}(1-b_2^{-2m})](1+\Theta_2b_2^2)^m\neq[\frac{\Theta_2}{m}(1-b_2^{2m})+\frac{1}{b_2^2}(1-b_2^2)](\Theta_2+1)^m.$$

According to (11.190), (11.191) and (11.192), the RHS is strictly positive and LHS is strictly negative. From Lemma 11.1.24 and (11.189), $B_0b_3^{2m}+B_1b_3^2+B_2=0$. This gives $b_3^2\neq\frac{1+\Theta_2b_2^2}{1+\Theta_2}=\frac{-\Theta_3b_3^2}{1+\Theta_2}$, then

$$d_3=(1+\Theta_2+\Theta_3)b_3\neq 0. \tag{11.208}$$

□

Proof of Proposition 11.1.23: The result follows immediately from Lemma 11.1.24 and 11.1.25. Note that v and w are supported on the mth Fourier mode, thanks to (11.198) and (11.199). □

Lemma 11.1.26. $\partial_{\Theta_3}A(\Theta_3^*,0)[v]\notin Im(A(\Theta_3^*,0))$.

Proof. From (11.184), it follows that

$$\partial_{\Theta_3}A(\Theta_3^*,0)[h]=Q,$$

where

$$Q:=\sum_{n\geq 1}\begin{pmatrix}q_{1,n}\\q_{2,n}\\q_{3,n}\end{pmatrix}\sin(nmx),\quad h=\sum_{n\geq 1}\begin{pmatrix}h_{1,n}\\h_{2,n}\\h_{3,n}\end{pmatrix}\cos(nmx),\quad \begin{pmatrix}q_{1,n}\\q_{2,n}\\q_{3,n}\end{pmatrix}=(-mn)\partial_{\Theta_3}M_n(\Theta_3^*)\begin{pmatrix}h_{1,n}\\h_{2,n}\\h_{3,n}\end{pmatrix}$$

Let us write

$$M_1(\Theta_3^*)=\begin{pmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\a_{31}&a_{32}&a_{33}\end{pmatrix},\quad CM_{33}=\begin{pmatrix}a_{11}&a_{12}\\a_{21}&a_{22}\end{pmatrix},\quad \partial_{\Theta_3}M_1(\Theta_3^*)=\begin{pmatrix}0&0&b_{13}\\0&0&b_{23}\\b_{31}&b_{32}&b_{33}\end{pmatrix}, \tag{11.209}$$

where the vanishing elements in $\partial_{\Theta_3}M_1(\Theta_3^*)$ follows from (11.185). Note that

$$\det(CM_{33})=a_{11}a_{22}-a_{12}a_{21}=\frac{1}{2m}\left(-\frac{1}{2}d_2+\frac{\Theta_2b_2}{2m}(1-b_2^{2m})\right),\quad \text{where } d_2 \text{ is as in (11.205).}$$

From (11.206), we have

$$-\frac{1}{2}d_2+\frac{\Theta_2 b_2}{2m}(1-b_2^{2m})=\frac{1-b_2^2}{2b_2}+\frac{\Theta_2 b_2}{2m}(1-b_2^{2m})>\frac{1-b_2^2}{2b_2}+\frac{b_2^2-1}{2}=\frac{1-b_2^2}{2}\left(\frac{1}{b_2}-1\right)>0,$$

where the first inequality follows from (11.1), and the last inequality follows from $0<b_2<1$. Hence,

$$\det(CM_{33})>0, \tag{11.210}$$

and CM_{33} is invertible. Since $\mathrm{Im}(M_1(\Theta_3^*))^\perp=\mathrm{Ker}(M_1(\Theta_3^*)^T)$, we can choose

$$\tilde{v}=\begin{pmatrix} -(CM_{33})^{-1}\begin{pmatrix} a_{13}\\ a_{23}\end{pmatrix}\\ 1\end{pmatrix},\quad \tilde{w}=\begin{pmatrix} -(CM_{33}^T)^{-1}\begin{pmatrix} a_{31}\\ a_{32}\end{pmatrix}\\ 1\end{pmatrix},$$

so that $\tilde{v}\in \mathrm{Ker}A$, and $\tilde{w}\in(\mathrm{Im}A)^\perp$. Then $\partial_{\Theta_3}A(\Theta_3^*,0)[v]\notin \mathrm{Im}(A(\Theta_3^*,0))$ is equivalent to have

$$\partial_{\Theta_3}M_z(\Theta_3,0)\tilde{v}\cdot\tilde{w}\neq 0.$$

According to (11.209), it is equivalent to have

$$-(b_{13},b_{23})CM_{33}^{T\,-1}\begin{pmatrix} a_{31}\\ a_{32}\end{pmatrix}-(b_{31},b_{32})CM_{33}{}^{-1}\begin{pmatrix} a_{13}\\ a_{23}\end{pmatrix}+b_{33}\neq 0. \tag{11.211}$$

Using $\frac{db_3}{d\Theta_3}=-\frac{b_3}{2\Theta_3}$, which comes from $\Theta_3 b_3^2+\Theta_2 b_2^2+1=0$, we compute

$$\partial_{\Theta_3}M_1(\Theta_3^*)=\begin{pmatrix} 0 & 0 & \frac{-(m-1)b_3^{m+1}}{4m}\\ 0 & 0 & \frac{-b_3^{m+1}(m-1)}{4mb_2^m}\\ -\frac{b_3^m}{4\Theta_3^*} & \frac{-b_3^m\Theta_2}{4\Theta_3^* b_2^{m-1}} & \frac{b_3}{2}(-\frac{1}{2}+\frac{1}{2m})+\frac{b_3}{4\Theta_3^*}+\frac{\Theta_2 b_3}{4\Theta_3^*}\end{pmatrix}.$$

Then we have

$$(b_{13},b_{23})CM_{33}^{T\,-1}\begin{pmatrix} a_{31}\\ a_{32}\end{pmatrix}$$

$$= \frac{1}{\det(CM_{33})}(\frac{-(m-1)}{4m}\frac{1}{2m})b_3^{2m+1}\left(1,\frac{1}{b_2^m}\right)\begin{pmatrix} \frac{1}{2m}\Theta_2 b_2 - \frac{1}{2}b_2 + \frac{1}{2b_2} & -\frac{b_2^m}{2m} \\ -\frac{\Theta_2 b_2^{m+1}}{2m} & \frac{1}{2m}\end{pmatrix}\begin{pmatrix} 1 \\ \frac{\Theta_2}{b_2^{m-1}}\end{pmatrix},$$

and

$$(b_{31}, b_{32})\, {CM_{33}}^{-1}\begin{pmatrix} a_{13} \\ a_{23}\end{pmatrix}$$
$$= \frac{1}{\det(CM_{33})}\frac{\Theta_3^*}{2m}(-\frac{b_3^{2m+1}}{4\Theta_3^*})\left(1,\frac{\Theta_2}{b_2^{m-1}}\right)\begin{pmatrix} \frac{1}{2m}\Theta_2 b_2 - \frac{1}{2}b_2 + \frac{1}{2b_2} & -\frac{\Theta_2 b_2^{m+1}}{2m} \\ -\frac{b_2^m}{2m} & \frac{1}{2m}\end{pmatrix}\begin{pmatrix} 1 \\ \frac{1}{b_2^m}\end{pmatrix}.$$

Hence (11.211) is equivalent to

$$\frac{1}{\det(CM_{33})}(\frac{m-1}{8m^2}+\frac{1}{8m})b_3^{2m+1}\left(1,\frac{1}{b_2^m}\right)\begin{pmatrix} \frac{1}{2m}\Theta_2 b_2 - \frac{1}{2}b_2 + \frac{1}{2b_2} & -\frac{b_2^m}{2m} \\ -\frac{\Theta_2 b_2^{m+1}}{2m} & \frac{1}{2m}\end{pmatrix}\begin{pmatrix} 1 \\ \frac{\Theta_2}{b_2^{m-1}}\end{pmatrix} + b_{33} \neq 0.$$

$$\Leftrightarrow (\frac{m-1}{8m^2}+\frac{1}{8m})b_3^{2m+1}\left(1,\frac{1}{b_2^m}\right)\begin{pmatrix} \frac{1}{2m}\Theta_2 b_2 - \frac{1}{2}b_2 + \frac{1}{2b_2} & -\frac{b_2^m}{2m} \\ -\frac{\Theta_2 b_2^{m+1}}{2m} & \frac{1}{2m}\end{pmatrix}\begin{pmatrix} 1 \\ \frac{\Theta_2}{b_2^{m-1}}\end{pmatrix} + \det(CM_{33})b_{33} \neq 0.$$

$$\Leftrightarrow \det(CM_{33})\left((-\frac{1}{4}+\frac{1}{4m})b_3 + \frac{b_3^3}{4(-1-\Theta_2 b_2^2)} + \frac{\Theta_2 b_3^3}{4(-1-\Theta_2 b_2^2)}\right)$$
$$+ (\frac{m-1}{8m^2}+\frac{1}{8m})b_3^{2m+1}(\frac{1}{2b_2}-\frac{1}{2}b_2+\frac{\Theta_2}{2mb_2^{2m-1}}-\frac{b_2\Theta_2}{2m}) \neq 0,$$

$$\Leftrightarrow (\frac{2m-1}{8m^2})b_3^{2m}(\frac{1}{2b_2}-\frac{1}{2}b_2+\frac{\Theta_2}{2mb_2^{2m-1}}-\frac{b_2\Theta_2}{2m}) + \det(CM_{33})\frac{1+\Theta_2}{4(-1-\Theta_2 b_2^2)}b_3^2 + \det(CM_{33})(-\frac{1}{4}+\frac{1}{4m}) \neq 0. \tag{11.212}$$

To show (11.212), let us write

$$C_0 := (\frac{2m-1}{8m^2})(\frac{1}{2b_2}-\frac{1}{2}b_2+\frac{\Theta_2}{2mb_2^{2m-1}}-\frac{b_2\Theta_2}{2m})$$
$$C_1 := \det(CM_{33})\frac{1+\Theta_2}{4(-1-\Theta_2 b_2^2)}$$
$$C_2 := \det(CM_{33})(-\frac{1}{4}+\frac{1}{4m}),$$

then, it is enough to show that $g(b_3) := C_0 b_3^{2m} + C_1 b_3^2 + C_2 \neq 0$. Note that we have

$$\begin{aligned}
&\det(CM_{33}) > 0, \quad \text{from (11.210)} \\
&(1+\Theta_2) < 0, \quad \text{from (11.1) and } 0 < b_2 < 1 \\
&1 + \Theta_2 b_2^2 < 0, \quad \text{from (11.186) and } \Theta_3^* > 0,
\end{aligned}$$

thus, $C_1 < 0$. we also have $C_2 < 0$ since $m > 1$. For C_0, we have

$$\begin{aligned}
(\frac{1}{2b_2} - \frac{1}{2}b_2 + \frac{\Theta_2}{2m}(\frac{1}{b_2^{2m-1}} - b_2)) &= \frac{b_2}{2}[\frac{1}{b_2^2} - 1 + \frac{\Theta_2}{m}(\frac{1}{b_2^{2m}} - 1)] \\
&= \frac{b_2}{2}(\frac{1}{b_2^2} - 1)[1 + \frac{\Theta_2}{m}(1 + \frac{1}{b_2^2} + ... + \frac{1}{b_2^{2(m-1)}})] \\
&< \frac{b_2}{2}(\frac{1}{b_2^2} - 1)(1 + \frac{\Theta_2}{m}m) \\
&< 0,
\end{aligned}$$

as $b_2 < 1$, $\Theta_2 \leq \frac{-1}{b_2^2} < -1$, which follows from (11.1). This yields that $C_0 < 0$. Therefore the Descartes rule of signs implies that there is no positive root of $g(x) = 0$, which implies $g(b_3) \neq 0$, that is (11.212). □

11.2 Analytic regularity of the boundary

Theorem 11.2.1. *The solutions $R(s) \in (X^{k,m})^3$ constructed in Theorem 11.1.1 are analytic when s is sufficiently small.*

Proof. We use [61, Theorem 1] and [62, Theorem 3.1] to prove the analyticity. Let the stream function

$$\varphi(s) = \sum_{i=1}^{3} \Theta_i 1_{D_i} * \frac{1}{2\pi} \log(\cdot).$$

We first consider the outermost boundary ∂D_1. For any $x_1 \in \partial D_1$, we have

$$\Delta\varphi = \Theta_1, \ in \ D_1 \backslash \overline{D_2},$$

$$\varphi = c_1, \ on \ \partial D_1.$$

From Lemma 11.7, we have

$$\nabla\varphi = 0, \ on \ \partial D_1.$$

Moreover, from Theorem 11.1.1, we know ∂D_1 is in C^2. Thus φ is also in $C^2(\overline{D_1}\backslash\overline{D_2})$. Then by [61, Theorem 1], we have the analyticity of ∂D_1.

Now we consider ∂D_2. We have

$$\Delta\varphi = \begin{cases} \Theta_2 + \Theta_1, & in\ D_2\backslash\overline{D_3}, \\ \Theta_1, & in\ D_1\backslash\overline{D_2} \end{cases}$$

$$\varphi = c_2,\ on\ \partial D_2.$$

From Corollary 11.4.4 and (11.206), at the bifurcation point, we have

$$\begin{aligned} \partial_n\varphi(0) &= \frac{1}{2\pi}\sum_{i=1}^{3}\Theta_i\int_0^{2\pi}\cos(x-y)b_i\log(b_i^2+b_j^2-2b_ib_j\cos(x-y))dy \\ &= \frac{1}{2\pi}(-2\pi)(\Theta_1 b_1\frac{b_2}{b_1}+\Theta_2 b_2+\Theta_3 b_3\frac{b_3}{b_2}) = -(\Theta_1 b_2+\Theta_2 b_2+\Theta_3\frac{b_3^2}{b_2}) \neq 0. \end{aligned}$$

Thus when s is sufficiently small, $\partial_n\varphi(x) \neq 0$.

From Theorem 11.1.1, we have ∂D_2 is in C^2. So φ is also in $C^2((\overline{D_2}\backslash D_3)\cap(\overline{D_1}\backslash D_2))\cap C^1(\overline{D_1})$. Then we can use [62, Theorem 3.1] to get the result. The analyticity of ∂D_3 can be shown in the same way by using (11.208) instead of (11.206). □

11.3 Using the energy estimate to show the analyticity

In this section we introduce an alternative way to get the analyticity, which follows the similar idea from Chapter 1. For the sake of simplicity, here we only show a main general theorem.

Theorem 11.3.1. *Let $T(f)$ be an operator satisfying the following conditions. We assume there exists $\epsilon > 0$, $k \geq 1$, $f_0 \in H^k(\mathcal{T})$, such that when $\|f - f_0\|_{H^k} \lesssim \epsilon$,*

(a) (Boundedness) $T(f) : H^k(\mathbb{T}) \to H^k(\mathbb{T})$ with $\|T(f)\|_{H^k(\mathcal{T})} \lesssim 1$,

(b) (Existence and boundedness of the Fréchet derivative)$\|D_f(T(f))[h]\|_{H^k(\mathcal{T})} \lesssim \|h\|_{H^k(\mathcal{T})}$,

(c) (L^2 boundedness of the Fréchet derivative) $\|D_f(T(f))[h]\|_{L^2(\mathcal{T})} \lesssim \|h\|_{L^2(\mathcal{T})}$,

(d) $\frac{d}{dx}T(f) = D_f(T(f))[\frac{df}{dx}]$,

(e) $iD_f(T(f))[h] = D_f(T(f))[ih]$.

Here $\mathcal{T}$ is the torus of length 2π. If $f_0(x)$ also satisfies the equation

$$f_0^{'}(x) = T(f_0), \tag{11.213}$$

$f_0(x)$ must be real analytic.

Proof. First, we assume f_0 to be an analytic function with analytic continuation $f(x,t)$. Then through the Cauchy-Riemann equation and (11.213) , we have

$$\begin{cases} \frac{d}{dt}f(x,t) = iT(f(x,t)), \\ f(x,0) = f_0(x). \end{cases} \tag{11.214}$$

Our goal is to show this solution $f(x,t)$ does exist and is analytic.

Through (a), (b), we have

$$\|T(f) - T(g)\|_{H^k(\mathcal{T})} \lesssim \|f - g\|_{H^k(\mathcal{T})}.$$

We can use the Picard theorem to show there is a solution satisfying

$$f(x,t) - f(x,0) = \int_0^t iT(f(x,\tau))d\tau.$$

with $|t| < t_0$ for some $t_0 > 0$. Moreover

$$f(x,t) \in W^{2,\infty}((-t_0,t_0), H^k(\mathcal{T})). \tag{11.215}$$

By (11.215), we have

$$\lim_{\Delta t \to 0} \| \frac{f(x,t) - f(x,t+\Delta t)}{\Delta t} - \frac{d}{dt}f(x,t)\|_{H^k(\mathcal{T})} = 0.$$

So

$$\lim_{\Delta t \to 0} \| \frac{\frac{d}{dx}f(x,t) - \frac{d}{dx}f(x,t+\Delta t)}{\Delta t} - \frac{d}{dx}\frac{d}{dt}f(x,t)\|_{H^{k-1}(\mathcal{T})} = 0.$$

Therefore we have

$$\frac{d}{dt}\frac{d}{dx}f(x,t) = \frac{d}{dx}\frac{d}{dt}f(x,t) \in C^0((-t_0,t_0), H^{k-1}(\mathcal{T})) \subset C^0((-t_0,t_0), L^2(\mathcal{T})), \tag{11.216}$$

and

$$\frac{d}{dx}f(x,t)+i\frac{d}{dt}f(x,t)\in W^{1,\infty}((-t_0,t_0),L^2(\mathcal{T})). \tag{11.217}$$

Then we can control $\|\frac{d}{dx}f(x,t)+i\frac{d}{dt}f(x,t)\|_{L^2(\mathcal{T})}$. We have

$$\begin{aligned}
&|\frac{d}{dt}\int_{-\pi}^{\pi}|(\frac{d}{dx}+i\frac{d}{dt})f(x,t)|^2dx| \qquad (11.218)\\
&=|2Re\int_{-\pi}^{\pi}(\frac{d}{dx}+i\frac{d}{dt})f(x,t)\overline{\frac{d}{dt}(\frac{d}{dx}+i\frac{d}{dt})f(x,t)}dx|\\
&=|2Re\int_{-\pi}^{\pi}(\frac{d}{dx}+i\frac{d}{dt})f(x,t)\overline{(\frac{d}{dx}+i\frac{d}{dt})\frac{d}{dt}f(x,t)}dx|\\
&=|2Re\int_{-\pi}^{\pi}(\frac{d}{dx}+i\frac{d}{dt})f(x,t)\overline{(\frac{d}{dx}+i\frac{d}{dt})iT(f(x,t))}dx|\\
&=|2Re\int_{-\pi}^{\pi}(\frac{d}{dx}+i\frac{d}{dt})f(x,t)\overline{iD_fT(f(x,t))[(\frac{d}{dx}+i\frac{d}{dt})[f(x,t)]}dx|\\
&\lesssim\int_{-\pi}^{\pi}|(\frac{d}{dx}+i\frac{d}{dt})f(x,t)|^2dx.
\end{aligned}$$

Here the first equality follows from (11.217), the second from (11.216).

Through (11.213), (11.214), we have

$$\|\frac{d}{dx}f(x,t)+i\frac{d}{dt}f(x,t)\|_{L^2(\mathcal{T})}|_{t=0}=0.$$

Moreover, from(11.217), $\|\frac{d}{dx}f(x,t)+i\frac{d}{dt}f(x,t)\|^2_{L^2(\mathcal{T})}\in W^{1,\infty}(-t_0,t_0)$. Then

$$\|\frac{d}{dx}f(x,t)+i\frac{d}{dt}f(x,t)\|^2_{L^2(\mathcal{T})}=\int_0^t\frac{d}{d\tau}\|\frac{d}{dx}f(x,\tau)+i\frac{d}{d\tau}f(x,\tau)\|^2_{L^2(\mathcal{T})}d\tau.$$

Hence we can use the Gronwall inequality and get

$$\|\frac{d}{dx}f(x,t)+i\frac{d}{dt}f(x,t)\|_{L^2(\mathcal{T})}=0. \tag{11.219}$$

Moreover, since $k\geq 1$, from (11.215), we have $\frac{d}{dt}f(x,t)\in W^{1,\infty}((-t_0,t_0),H^1(\mathcal{T}))$. Then $\frac{d}{dt}f(x,t)$ is continuous in x and t. We also have $f(x,t)=\int_0^x i\frac{d}{dt}f(x,t)dt+f(0,t)$. Therefore $\partial_x f(x,t)$ is continuous in x and t.

Then by the (11.219) and (11.214), we have the analyticity. □

11.4 Appendix

11.4.1 Crandall–Rabinowitz Theorem

We recall the Crandall-Rabinowitz theorem [28], that plays a crucial role throughout this paper.

Theorem 11.4.1. *[28, Theorem 1.7] Let X, Y be Banach spaces, V a neighborhood of 0 in X and*

$$F : (-1, 1) \times V \mapsto Y$$

have the properties:

1. *$F(t, 0) = 0$ for $|t| < 1$,*
2. *The Gateaux derivatives of F, $\partial_t F$ DF, and $\partial_t DF$ exist and are continuous,*
3. *$Ker(DF(0, 0))$ and $Im(DF(0, 0))^{\perp}$ are one-dimensional,*
4. *$\partial_t DF(0, 0)[x_0] \notin Im(DF(0, 0))$, where*

$$Ker(DF(0, 0)) = span\{x_0\}.$$

Then there is a neighborhood U of $(0, 0) \subset \mathbb{R} \times X$, an interval $(-a, a)$ and continuous functions $\varphi : (-a, a) \mapsto \mathbb{R}$, $\psi : (-a, a) \mapsto z$ such that $\varphi(0) = \psi(0)$ and

$$F(\varphi(\alpha), \alpha x_0 + \alpha\psi(\alpha)) = 0, \quad |\alpha| < a.$$

11.4.2 Useful integrals

The following lemmas will deal with the integrals that appear throughout the calculation of the linear operator:

Lemma 11.4.2. *Consider the expansion of the function $\log(A_0 + A_1 t + A_2 t^2)$. We have that:*

$$\log(A_0 + A_1 t + A_2 t^2) \sim \log(A_0) + t\frac{A_1}{A_0} + t^2\left(\frac{2A_0A_2 - A_1^2}{2A_0^2}\right) + t^3\left(\frac{A_1^3 - 3A_0A_1A_2}{3A_0^3}\right)$$

Proof. Follows from the Taylor expansion. □

Lemma 11.4.3. *Let $0 < r < 1$ and $\mathcal{A}_1(r, m)$ be*

$$\mathcal{A}_1(r,m) = \int \frac{\cos(my)}{1+r^2-2r\cos(y)}dy = \frac{2\pi}{1-r^2}r^m$$

Proof. We use the residue theorem. We set γ to be the boundary of the unit disk and let $z = e^{iy}$

$$\begin{aligned}\mathcal{A}_1(r,m) &= \int \frac{\cos(my)}{1+r^2-2r\cos(y)}dy = \frac{1}{ri}\int_\gamma \frac{z^m}{\frac{1+r^2}{r}-z-z^{-1}}\frac{1}{z}dz \\ &= -\frac{1}{ri}\int_\gamma \frac{z^m}{(z-r)(z-\frac{1}{r})}dz = -\frac{2\pi i}{ri}\frac{r^m}{r-\frac{1}{r}} = \frac{2\pi r^m}{1-r^2}.\end{aligned}$$

□

Corollary 11.4.4. *Let $0 < r \leq 1$ and $\mathcal{A}_0(r,m)$ be*

$$\mathcal{A}_0(r,m) = \int \log(1+r^2-2r\cos(y))\cos(my)dy = -\frac{2\pi}{m}r^m$$

Proof. For $r \leq 1$, we integrate by parts and we have

$$\mathcal{A}_0(r,m) = -\frac{r}{m}\int \frac{\cos((m-1)y)-\cos((m+1)y)}{1+r^2-2r\cos(y)}dy = -\frac{2\pi}{m}r^m$$

For $r = 1$, we use the continuity. □

11.4.3 Integral estimates

Lemma 11.4.5. *[71, Lemma 3.4] Let $f, g \in C^\infty(\mathbb{T}^d)$. Then,*

$$\|fg\|_{H^k(\mathbb{T}^d)} \lesssim_k \|f\|_{L^\infty(\mathbb{T}^d)}\|g\|_{H^k(\mathbb{T}^d)} + \|f\|_{H^k(\mathbb{T}^d)}\|g\|_{L^\infty(\mathbb{T}^d)} \quad \text{for } k \in \mathbb{N}\cup\{0\}. \tag{11.220}$$

Lemma 11.4.6. *[?] Let $f \in C^\infty(\mathbb{T}^d)$. Then for $0 \leq k \leq n \in \mathbb{N}\cup\{0\}$, it holds that*

$$\|\nabla^k f\|_{L^2(\mathbb{T}^d)} \lesssim_{k,n} \|f\|_{L^2(\mathbb{T}^d)}^{1-\frac{k}{n}}\|\nabla^n f\|_{L^2(\mathbb{T}^d)}^{\frac{k}{n}}.$$

Lemma 11.4.7. *Let $f, g \in C^\infty(\mathbb{T})$ and $\sigma, p, q \in \mathbb{N}\cup\{0\}$ satisfy $p+q = \sigma$ with $q \leq \sigma - 1$. Then, for each $k \geq 0$, it holds that*

$$\|f^{(p)}g^{(q)}\|_{H^k(\mathbb{T})} \lesssim_{k,\sigma} \|f\|_{H^2(\mathbb{T})}\|g\|_{H^{k+\sigma-1}(\mathbb{T})} + \|f\|_{H^{k+\sigma}(\mathbb{T})}\|g\|_{H^1(\mathbb{T})}.$$

Proof. Using Lemma 11.4.5 and the Sobolev embedding theorem, we have

$$\begin{aligned}\|f^{(p)}g^{(q)}\|_{H^k(\mathbb{T})} &\leq \|f'g\|_{H^{k+\sigma-1}(\mathbb{T})} \\ &\lesssim_{k,\sigma} \|f'\|_{L^\infty(\mathbb{T})}\|g\|_{H^{k+\sigma-1}(\mathbb{T})} + \|f\|_{H^{k+\sigma}(\mathbb{T})}\|g\|_{L^\infty(\mathbb{T})} \\ &\lesssim_{k,\sigma} \|f\|_{H^2(\mathbb{T})}\|g\|_{H^{k+\sigma-1}(\mathbb{T})} + \|f\|_{H^{k+\sigma}(\mathbb{T})}\|g\|_{H^1(\mathbb{T})}.\end{aligned}$$

□

Lemma 11.4.8. *Let $F \in C_0^\infty(\mathbb{R}^n)$ and $u \in (C^\infty(\mathbb{T}^2))^n$. If $\|u\|_{(H^2(\mathbb{T}^2))^n} \lesssim 1$, then for $k \geq 2$,*

$$\|F(u)\|_{H^k(\mathbb{T}^2)} \lesssim_k 1+\|u\|_{(H^k(\mathbb{T}^2))^n}. \tag{11.221}$$

For $u_1, u_2 \in (C^\infty(\mathbb{T}^2))^n$, if $\|u_1\|_{(H^2(\mathbb{T}^2))^n}, \|u_2\|_{(H^2(\mathbb{T}^2))^n} \lesssim 1$ we have that for $k \geq 2$,

$$\|F(u_1) - F(u_2)\|_{H^k(\mathbb{T}^2)} \lesssim_k (1+\|u_1\|_{(H^k(\mathbb{T}^2))^n}+\|u_2\|_{(H^k(\mathbb{T}^2))^n})\|u_1 - u_2\|_{(H^k(\mathbb{T}^2))^n}. \tag{11.222}$$

Proof. Since $\|u\|_{(L^\infty(\mathbb{T}^2))^n} \lesssim \|u\|_{(H^2(\mathbb{T}^2))^n} \lesssim 1$. Then (11.221) follows straightforwardly from Lemma 11.4.5 and Faá di Bruno's formula. For (11.222), we have

$$F(u_1(x))-F(u_2(x)) = \int_0^1 \frac{d}{dt}F((u_1(x)-u_2(x))t+u_2(x))dt = \int_0^1 \nabla F((u_1(x)-u_2(x))t+u_2(x))\cdot(u_1(x)-u_2(x))dt.$$

Thus equation (11.222) follows from Lemma 11.4.5 and (11.221). □

Lemma 11.4.9. *Let $K(x,y) \in C^\infty(\mathbb{T}^2)$ and $\tilde{K}(x,y) := \frac{K(x,y)-K(x,x)}{2\sin\left(\frac{x-y}{2}\right)}$. Then for $k \in \mathbb{N} \cup \{0\}$,*

$$\|\tilde{K}\|_{H^k(\mathbb{T}^2)} \lesssim_k \|K\|_{H^{k+1}(\mathbb{T}^2)}.$$

Proof. We first divide the domain of the integration into three regions:

$$\begin{aligned}D_1 &:= \left\{(x,y) \in \mathbb{T}^2 : |x-y| < 1/2\right\}, \\ D_2 &:= \left\{(x,y) \in \mathbb{T}^2 : |x-y| > 1/2, \quad ||x-y| - 2\pi| < \frac{1}{2}\right\}, \\ D_3 &:= \left\{(x,y) \in \mathbb{T}^2 : |x-y| > 1/2, \quad ||x-y| - 2\pi| > \frac{1}{2}\right\}.\end{aligned}$$

Clearly, $\mathbb{T}^2 = D_1 \cup D_2 \cup D_3$. We will show only that

$$\int_{D_1} \left| D^k \tilde{K}(x,y) \right|^2 dxdy \lesssim_k \|K\|^2_{H^{k+1}(\mathbb{T}^2)}, \tag{11.223}$$

since the other contributions from D_2 and D_3 can be easily proved in the same or easier way.

For $x, y \in D_1$, we can write

$$\tilde{K}(x,y) = \int_0^1 \partial_u \left(K(x,(x-y)u+y) \right) du \frac{1}{\sin\left(\frac{x-y}{2}\right)} = \int_0^1 \partial_y K(x,(x-y)u+y) du \frac{(x-y)}{\sin(\frac{x-y}{2})}.$$

Since $\frac{(x-y)}{\sin\left(\frac{x-y}{2}\right)}$ can be extended to a smooth function in D_1, we use Lemma 11.4.5 and obtain

$$\int_{D_1} \left| D^k \tilde{K}(x,y) \right|^2 dxdy \lesssim \left\| \int_0^1 \partial_y K(x,(x-y)u+y) du \right\|^2_{H^k(\mathbb{T}^2)}.$$

The right-hand side can be estimated as

$$\begin{aligned}
\int_{\mathbb{T}^2} \left| D^k \int_0^1 \partial_y K(x,(x-y)u+y) du \right|^2 dxdy &\lesssim \int_{\mathbb{T}^2} \left(\int_0^1 \left| D^k \partial_y K(x,(x-y)u+y)(1+u^k) \right| du \right)^2 dxdy \\
&= \int_{\mathbb{T}^2} \left| \int_y^x \left| D^k \partial_y K(x,z) \right| \left(1 + \left(\frac{(z-y)}{x-y} \right)^k \right) \frac{1}{x-y} dz \right|^2 dxdy \\
&\lesssim \int_{\mathbb{T}^2} \left| \int_y^x D^k \partial_y K(x,z) \frac{1}{|x-y|} dz \right|^2 dxdy,
\end{aligned}$$

where the equality follows from the change of variables, $(x-y)u+u \mapsto z$. Using the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}
\int_{\mathbb{T}^2} \left| \int_y^x D^k \partial_y K(x,z) \frac{1}{|x-y|} dz \right|^2 dxdy &\lesssim \int_{\mathbb{T}^2} \int_y^x \left| D^k \partial_y K(x,z) \right|^2 dz \frac{1}{|x-y|} dxdy \\
&\lesssim \int_{(x,y,z)\in\mathbb{T}^3, y<z<x} \left| D^k \partial_y K(x,z) \right|^2 \frac{1}{|x-y|} dxdydz \\
&\lesssim \int_{\mathbb{T}^2} \left| D^k \partial_y K(x,z) \right|^2 \int_z^{2\pi} -\log\left(1 - \frac{z}{x}\right) dxdz \\
&\lesssim \int_{\mathbb{T}^2} \left| D^k \partial_y K(x,z) \right|^2 dz \\
&\lesssim \|K\|^2_{H^{k+1}(\mathbb{T}^2)},
\end{aligned}$$

where the second last inequality follows from the fact that $z \mapsto \int_z^{2\pi} -\log\left(1 - \frac{z}{x}\right) dx$ is uniformly bounded. This finishes the proof. □

For $R \in C^\infty(\mathbb{T})$, we denote

$$J(R)(x,y) = \frac{R(x)-R(y)}{2\sin(\frac{x-y}{2})}. \tag{11.224}$$

Lemma 11.4.10. *Let $R \in C^\infty(\mathbb{T})$. Then, for $k \in \mathbb{N} \cup \{0\}$,*

$$\int_{\mathbb{T}^2} \left|D^k J(R)(x,y)\right|^2 dxdy \lesssim_k \|R\|^2_{H^{k+1}(\mathbb{T})}. \tag{11.225}$$

Consequently, $\|J(R)\|_{H^k(\mathbb{T}^2)} \lesssim_k \|R\|_{H^{k+1}(\mathbb{T})}$. Furthermore, we have

$$\left\| \int_{\mathbb{T}} J(R)(\cdot,y)dy \right\|_{H^k(\mathbb{T})} \lesssim \|R\|_{H^k(\mathbb{T})}. \tag{11.226}$$

Proof. For (11.225), it follows from Lemma 11.4.9 by putting $K(x,y) = R(y)$. (11.226) follows from classical singular integral theory, thus we omit the proof. □

Lemma 11.4.11. *Let $f, g \in C^\infty(\mathbb{T}^2)$ and let*

$$T(x) := \int_{\mathbb{T}} f(x,y)g(x,y)dy.$$

Then for $k \in \mathbb{N} \cup \{0\}$,

$$\|T\|_{H^k(\mathbb{T})} \lesssim \|f\|_{L^\infty(\mathbb{T}^2)}\|g\|_{H^k(\mathbb{T}^2)} + \|f\|_{H^k(\mathbb{T}^2)}\|g\|_{L^\infty(\mathbb{T}^2)}.$$

Proof. By Minkowski's integral inequality and lemma 11.4.5, We have

$$\begin{aligned}
\left\| \int_{\mathbb{T}} f(x,y)g(x,y)dy \right\|_{H^k(\mathbb{T})} &\lesssim \int_{\mathbb{T}} \|f(\cdot,y)g(\cdot,y)\|_{H^k(\mathbb{T})}dy \\
&\lesssim \int_{\mathbb{T}} \|f(\cdot,y)\|_{L^\infty(\mathbb{T})}\|g(\cdot,y)\|_{H^k(\mathbb{T})}dy + \int_{\mathbb{T}} \|g(\cdot,y)\|_{L^\infty(\mathbb{T})}\|f(\cdot,y)\|_{H^k(\mathbb{T})}dy \\
&\lesssim \|f\|_{L^\infty(\mathbb{T}^2)} \left(\int_{\mathbb{T}} \|g(\cdot,y)\|^2_{H^k(\mathbb{T})}dy \right)^{\frac{1}{2}} + \|g\|_{L^\infty(\mathbb{T}^2)} \left(\int_{\mathbb{T}} \|f(\cdot,y)\|^2_{H^k(\mathbb{T})}dy \right)^{\frac{1}{2}} \\
&\lesssim \|f\|_{L^\infty(\mathbb{T}^2)}\|g\|_{H^k(\mathbb{T}^2)} + \|g\|_{L^\infty(\mathbb{T}^2)}\|f\|_{H^k(\mathbb{T}^2)}
\end{aligned}$$

□

Lemma 11.4.12. *Let $K \in C^\infty(\mathbb{T}^2)$, $K^\#(x) := K(x,x)$ and $R \in C^\infty(\mathbb{T})$. Let*

$$T(R)(x) := \int_{\mathbb{T}} K(x,y) J(R)(x,y) dy.$$

Then for $k \geq 2$,

$$\|T(R)\|_{H^k(\mathbb{T})} \lesssim_k (\|K\|_{H^{k+1}(\mathbb{T}^2)} + \|K^\#\|_{H^k(\mathbb{T})}) \|R\|_{H^1(\mathbb{T})} + (\|K^\#\|_{L^\infty(\mathbb{T})} + \|\nabla K\|_{L^\infty(\mathbb{T}^2)}) \|R\|_{H^k(\mathbb{T})}.$$

Proof. Let $\tilde{K}(x,y) := \frac{K(x,y) - K^\#(x)}{2\sin\left(\frac{x-y}{2}\right)}$. Clearly, we have

$$\|\tilde{K}\|_{L^\infty(\mathbb{T}^2)} \lesssim \|\nabla K\|_{L^\infty(\mathbb{T})}. \tag{11.227}$$

We write

$$T(R)(x) = K^\#(x) \int_{\mathbb{T}} J(R)(x,y) dy + \int_{\mathbb{T}} \tilde{K}(x,y)(R(x) - R(y)) dy =: T_1(R)(x) + T_2(R)(x).$$

We can then get the L^2 estimate directly by Hölder's inequality. From Lemma 11.4.5, 11.4.10 and $H^1(\mathbb{T}) \hookrightarrow L^\infty(\mathbb{T})$, we have

$$\begin{aligned} \|T_1(R)\|_{H^k(\mathbb{T})} \lesssim & \|K^\#(x)\|_{H^k(\mathbb{T})} \left\| \int_{\mathbb{T}} J(R)(\cdot, y) dy \right\|_{H^1(\mathbb{T})} + \|K^\#\|_{L^\infty(\mathbb{T})} \left\| \int_{\mathbb{T}} J(R)(\cdot, y) dy \right\|_{H^k(\mathbb{T})} \\ \lesssim & \|K^\#(x)\|_{H^k(\mathbb{T})} \|R\|_{H^1(\mathbb{T})} + \|K^\#\|_{L^\infty(\mathbb{T})} \|R\|_{H^k(\mathbb{T})}. \end{aligned} \tag{11.228}$$

For $T_2(R)$, it follows from Lemma 11.4.11 that

$$\begin{aligned} \|T_2(R)\|_{H^k(\mathbb{T})} \lesssim & \|\tilde{K}\|_{H^k(\mathbb{T}^2)} \|R\|_{L^\infty(\mathbb{T})} + \|\tilde{K}\|_{L^\infty(\mathbb{T}^2)} \|R\|_{H^k(\mathbb{T})} \\ \lesssim & \|K\|_{H^{k+1}(\mathbb{T}^2)} \|R\|_{H^1(\mathbb{T})} + \|\nabla K\|_{L^\infty(\mathbb{T})} \|R\|_{H^k(\mathbb{T})}, \end{aligned}$$

where the last inequality follows from (11.227). With (11.228), we obtain the desired result.

□

Lemma 11.4.13. *Let $K \in C^\infty(\mathbb{T}^2)$, $K^\#(x) := K(x,x)$ and $R \in C^\infty(\mathbb{T})$. Let*

$$T(R)(x) := \int_{\mathbb{T}} K(x,y) J(R)(x,y) J(R)(x,y) dy.$$

Then for $k \geq 3$,

$$\|T(R)\|_{H^k(\mathbb{T})} \lesssim_k \|K\|_{H^{k+1}(\mathbb{T}^2)} \|R\|^2_{H^k(\mathbb{T}^2)}$$

Proof. Clearly,

$$\begin{aligned}
\|T(R)\|_{L^2(\mathbb{T})} \lesssim & \|K\|_{L^\infty(\mathbb{T}^2)} \|J(R)\|_{L^\infty(\mathbb{T})} \|J(R)\|_{L^2(\mathbb{T}^2)} \\
\lesssim & \|K\|_{H^2(\mathbb{T}^2)} \|J(R)\|_{H^2(\mathbb{T}^2)} \|R\|_{H^1(\mathbb{T})} \\
\lesssim & \|K\|_{H^2(\mathbb{T}^2)} \|R\|_{H^3(\mathbb{T})} \|R\|_{H^1(\mathbb{T})} \\
\lesssim & \|K\|_{H^2(\mathbb{T}^2)} \|R\|^2_{H^3(\mathbb{T})}, \qquad (11.229)
\end{aligned}$$

where we used $H^2(\mathbb{T}^2) \hookrightarrow L^\infty(\mathbb{T}^2)$ and Lemma 11.4.10.

Now we compute the higher derivatives. Using the change of variables, $y \mapsto x - y$, we have

$$T(R)(x) = \int_{\mathbb{T}} K(x, x-y) J(R)(x, x-y) J(R)(x, x-y) dy.$$

Since $J(R)(x, x-y) = \frac{R(x) - R(x-y)}{2\sin\left(\frac{y}{2}\right)}$, we have

$$\begin{aligned}
\left(\frac{d}{dx}\right)^k T(R) &= \sum_{p_1+p_2+p_3=k} \int_{\mathbb{T}} ((\partial_x)^{p_1} K(x, x-y)) J(R^{(p_2)})(x, x-y) J(R^{(p_3)})(x, x-y) dy \\
&= \sum_{p_1+p_2+p_3=k} \int_{\mathbb{T}} ((\partial_x)^{p_1} \tilde{K}(x, x-y)) J(R^{(p_2)})(x, y) J(R^{(p_3)})(x, y) dy \\
&= \sum_{p_1+p_2+p_3=k} A_{p_1,p_2,p_3},
\end{aligned}$$

where $\tilde{K}(x, y) := K(x, x-y)$. If $p_2 = k$ or $p_3 = k$, then we can write (assuming $p_3 = k$ without loss of generality),

$$\begin{aligned}
A_{0,0,k} &= \int_{\mathbb{T}} \frac{K(x,y)J(R)(x,y) - K(x,x)J(R)(x,x)}{2\sin\left(\frac{x-y}{2}\right)} (R^{(k)}(x) - R^{(k)}(y)) dy + K(x,x)R'(x) \int_{\mathbb{T}} J(R^{(k)})(x,y) dy \\
&= \int_{\mathbb{T}} \frac{K(x,y)J(R)(x,y) - K(x,x)J(R)(x,x)}{2\sin\left(\frac{x-y}{2}\right)} R^{(k)}(x) dy - \int_{\mathbb{T}} \frac{K(x,y)J(R)(x,y) - K(x,x)J(R)(x,x)}{2\sin\left(\frac{x-y}{2}\right)} R^{(k)}(y) dy \\
&\quad + K(x,x)R'(x) \int_{\mathbb{T}} J(R^{(k)})(x,y) dy \\
&= A^1_{0,0,k} + A^2_{0,0,k} + A^3_{0,0,k}.
\end{aligned}$$

Using Lemma 11.4.9 for the first two integrals, Lemma 11.4.10 for the second integral, and Cauchy-

Schwarz inequality, we obtain

$$\begin{aligned}
\|A^1_{0,0,k}\|_{L^2(\mathbb{T})} &\lesssim \left\| \int_{\mathbb{T}} \frac{K(x,y)J(R)(x,y) - K(x,x)J(R)(x,x)}{2\sin\left(\frac{x-y}{2}\right)} dy \right\|_{L^\infty(\mathbb{T})} \|R\|_{H^k(\mathbb{T})} \\
&\lesssim \left\| \int_{\mathbb{T}} \frac{K(x,y)J(R)(x,y) - K(x,x)J(R)(x,x)}{2\sin\left(\frac{x-y}{2}\right)} dy \right\|_{H^1(\mathbb{T})} \|R\|_{H^k(\mathbb{T})} \\
&\lesssim \|KJ(R)\|_{H^2(\mathbb{T}^2)} \|R\|_{H^k(\mathbb{T})} \\
&\lesssim (\|K\|_{L^\infty(\mathbb{T}^2)} \|J(R)\|_{H^2(\mathbb{T}^2)} + \|J(R)\|_{L^\infty(\mathbb{T}^2)} \|K\|_{H^2(\mathbb{T}^2)}) \|R\|_{H^k(\mathbb{T})},
\end{aligned}$$

$$\begin{aligned}
\|A^2_{0,0,k}\|_{L^2(\mathbb{T})} &\lesssim \int_{\mathbb{T}} \left\| \frac{K(\cdot,y)J(R)(\cdot,y) - K(\cdot,\cdot)J(R)(\cdot,\cdot)}{2\sin\left(\frac{\cdot-y}{2}\right)} \right\|_{L^\infty(\mathbb{T})} |R^{(k)}(y)| dy \\
&\lesssim \int_{\mathbb{T}} \left\| \frac{K(\cdot,y)J(R)(\cdot,y) - K(\cdot,\cdot)J(R)(\cdot,\cdot)}{2\sin\left(\frac{\cdot-y}{2}\right)} \right\|_{H^1(\mathbb{T})} |R^{(k)}(y)| dy \\
&\lesssim \left\| \frac{K(x,y)J(R)(x,y) - K(x,x)J(R)(x,x)}{2\sin\left(\frac{x-y}{2}\right)} \right\|_{H^1(\mathbb{T}^2)} \|R\|_{H^k(\mathbb{T})} \\
&\lesssim \|KJ(R)\|_{H^2(\mathbb{T}^2)} \|R\|_{H^k(\mathbb{T})} \\
&\lesssim (\|K\|_{L^\infty(\mathbb{T}^2)} \|J(R)\|_{H^2(\mathbb{T}^2)} + \|J(R)\|_{L^\infty(\mathbb{T}^2)} \|K\|_{H^2(\mathbb{T}^2)}) \|R\|_{H^k(\mathbb{T})},
\end{aligned}$$

and

$$\|A^3_{0,0,k}\|_{L^2(\mathbb{T})} \lesssim \|K\|_{L^\infty(\mathbb{T}^2)} \|R\|_{H^2(\mathbb{T})} \|R\|_{H^k(\mathbb{T})}.$$

Using the Sobolev embedding theorem, we have $\|K\|_{L^\infty(\mathbb{T}^2)} \lesssim \|K\|_{H^2(\mathbb{T}^2)}$ and $\|J(R)\|_{L^\infty(\mathbb{T}^2)} \lesssim \|J(R)\|_{H^2(\mathbb{T}^2)}$, thus

$$\begin{aligned}
\|A_{0,0,k}\|_{L^2(\mathbb{T})} &\lesssim \|K\|_{H^2(\mathbb{T}^2)} \left(\|J(R)\|_{H^2(\mathbb{T}^2)} + \|R\|_{H^2(\mathbb{T})}\right) \|R\|_{H^k(\mathbb{T})} \\
&\lesssim \|K\|_{H^2(\mathbb{T}^2)} \|R\|_{H^3(\mathbb{T})} \|R\|_{H^k(\mathbb{T})} \\
&\lesssim \|K\|_{H^2(\mathbb{T}^2)} \|R\|^2_{H^k(\mathbb{T})}, \qquad (11.230)
\end{aligned}$$

where the second inequality follows from (11.4.10) and the last inequality follows from $k \geq 3$. If $p_1, p_2, p_3 < k$, We use Lemma 11.4.11 to obtain

$$\|A_{p_1,p_2,p_3}\|_{L^2(\mathbb{T})} \lesssim \left\| \int_{\mathbb{T}} \partial_x (K(x,x-y)) J(R)(x,x-y) J(R)(x,x-y) dy \right\|_{H^{k-1}(\mathbb{T}^2)} \qquad (11.231)$$

$$
\begin{aligned}
&= \left\| \int_{\mathbb{T}} ((\partial_x K)(x,y) + (\partial_y K)(x,y)) J(R)(x,y) J(R)(x,y) dy \right\|_{H^{k-1}(\mathbb{T}^2)} \\
&\lesssim \|\nabla K\|_{H^{k-1}(\mathbb{T}^2)} \|J(R)J(R)\|_{L^\infty(\mathbb{T}^2)} + \|J(R)J(R)\|_{H^{k-1}(\mathbb{T}^2)} \|\nabla K\|_{L^\infty(\mathbb{T}^2)} \\
&\lesssim \|K\|_{H^k(\mathbb{T}^2)} \|J(R)\|_{H^2(\mathbb{T}^2)} \|J(R)\|_{H^2(\mathbb{T}^2)} + \|J(R)\|_{H^{k-1}(\mathbb{T}^2)} \|J(R)\|_{L^\infty(\mathbb{T}^2)} \|K\|_{H^3(\mathbb{T}^2)} \\
&\lesssim \|K\|_{H^k(\mathbb{T}^2)} \|R\|^2_{H^k(\mathbb{T}^2)}
\end{aligned}
$$

Similarly, if $p_1 = k$, then

$$
\begin{aligned}
\|A_{k,0,0}\|_{L^2(\mathbb{T})} \lesssim & \|\tilde{K}\|_{H^k(\mathbb{T}^2)} \|J(R)\|^2_{L^\infty(\mathbb{T}^2)} \\
\lesssim & \|K\|_{H^k(\mathbb{T}^2)} \|J(R)\|^2_{H^2(\mathbb{T}^2)} \\
\lesssim & \|K\|_{H^k(\mathbb{T}^2)} \|R\|^2_{H^3(\mathbb{T}^2)} \\
\lesssim & \|K\|_{H^k(\mathbb{T}^2)} \|R\|^2_{H^k(\mathbb{T}^2)}. \qquad (11.232)
\end{aligned}
$$

Therefore it follows from (11.230), (11.231) and (11.232) that

$$
\left\| \left(\frac{d}{dx} \right)^k T(R) \right\|_{H^k(\mathbb{T})} \lesssim \|K\|_{H^{k+1}(\mathbb{T}^2)} \|R\|^2_{H^k(\mathbb{T}^2)}.
$$

With (11.229), the desired result follows □

Lemma 11.4.14. *Define a linear map*

$$
T(R)(x) := \int_{\mathbb{T}} \log(2 - 2\cos(x-y)) R(y) dy.
$$

Then for $k \in \mathbb{N}$, $T : H^{k-1}(\mathbb{T}) \mapsto H^k(\mathbb{T})$ is an isomorphism.

Proof. The result follows immediately from Corollary 11.4.4. □

11.4.4 Stability of linear operators

Lemma 11.4.15. *For Hilbert spaces X, Y, Z, assume that $A : X \mapsto \mathcal{L}(Y, Z)$ satisfies:*

(1) $X \ni x \mapsto A(x) \in \mathcal{L}(Y, Z)$ is Lipschitz continuous near $0 \in X$,

(2) $A(0)$ is injective and has codimension one, that is, there exists $0 \neq z \in Z$ such that

$$
Im(A(0))^\perp = span\{z\}.
$$

Then there exists $\epsilon_0 > 0$ such that if $|x|_X \leq \epsilon_0$, then $A(x)$ is also injective and has codimension one. Furthermore, there exist $c > 0$ and $z_1 \in Z$ such that

$$Im(A(x))^{\perp} = span\{z_1\} \tag{11.233}$$

$$|z - z_1|_Z \leq c|x|_X, \tag{11.234}$$

$$|A(x)|_{\mathcal{L}(Y, Im(A(x)))} \leq c, \tag{11.235}$$

$$|A(x)^{-1}|_{\mathcal{L}(Im(A(x)), Y)} \leq c. \tag{11.236}$$

Proof. Let $Q : Z \mapsto Im(A(0))$ be the projection. Then $QA(0) : Y \mapsto Im(A(0))$ is an isomorphism. Therefore we can choose $\varepsilon > 0$ small enough so that $QA(x) : Y \mapsto Im(A(0))$ is an isomorphism whenever $|x| \leq \varepsilon$. Therefore, if $A(x)[y] = 0$ for some $y \in Y$, then $QA(x)[y] = 0$, which implies $y = 0$. This shows $A(x)$ is injective. In addition, the Fredholm index of $A(0)$ is $\dim(\mathrm{Ker}(A(0))) - \dim(\mathrm{coker}(A(0))) = -1$, therefore the Fredholm index of $A(x)$ is also -1 ([8, Theorem 4.4.2]) for small ϵ, which implies $A(x)$ has codimension one. This proves (11.233). Furthermore, $A(x) : Y \mapsto Im(A(x)) = (\mathrm{span}\{z_1\})^{\perp}$ is an isomorphism, therefore (11.235) and (11.236) follow immediately. We remark that the constant c can be chosen uniformly in x.

In order to prove (11.234), let us assume without loss of generality that $|z|_Z = |z_1|_Z = 1$. For any $y \in Y$, we have

$$\langle A(0)[y], z_1\rangle_Z = \langle (A(0) - A(x))[y], z_1\rangle_Z + \langle A(x)[y], z_1\rangle_Z = \langle (A(0) - A(x))[y], z_1\rangle_Z,$$

where the last equality follows from $A(x)[y] \in \mathrm{Im}(A(x))$ and $z_1 \in \mathrm{Im}(A(x))^{\perp}$. Therefore, we have

$$|\langle A(0)[y], z_1\rangle_Z| \leq |A(0) - A(x)|_{\mathcal{L}(Y;Z)}|y|_Y. \tag{11.237}$$

Note that we can decompose z_1 and find $y \in Y$ such that

$$z_1 = \langle z, z_1\rangle_Z z + \left(\underbrace{z_1 - \langle z, z_1\rangle_Z z}_{=:w}\right) = \langle z, z_1\rangle_Z z + A(0)[y],$$

where the last equality follows from $w \in \mathrm{Im}(A(0))$, hence $y = A(0)^{-1}[w]$. Then, we have

$$\begin{aligned} |w|_Z^2 &= \langle w, w\rangle_Z \\ &= \langle w, z_1\rangle_Z \end{aligned}$$

$$
\begin{aligned}
&= \langle A(0)[y], z_1 \rangle_Z \\
&\le |A(0) - A(x)|_{\mathcal{L}(Y;Z)} |y|_Y \\
&\le |A(0) - A(x)|_{\mathcal{L}(Y;Z)} |A(0)^{-1}|_{\mathcal{L}(\mathrm{Im}(A(0));Y)} |w|_Z,
\end{aligned}
$$

where the second equality follows from $\langle w, z \rangle_Z = 0$ and the first inequality follows from (11.237). Hence

$$
|w|_Z \le |A(0) - A(x)|_{\mathcal{L}(Y;Z)} |A(0)^{-1}|_{\mathcal{L}(\mathrm{Im}(A(0));Y)}. \tag{11.238}
$$

Now, using $z_1 = \langle z, z_1 \rangle_Z z + w$, we have

$$
1 = \langle z_1, z_1 \rangle_Z = \langle z, z_1 \rangle_Z^2 + \langle w, z_1 \rangle_Z = \langle z, z_1 \rangle_Z^2 + \langle w, z_1 - z \rangle_Z,
$$

where the last equality follows from $\langle w, z \rangle = 0$. Hence, we have

$$
\langle z, z_1 \rangle_Z = \sqrt{1 - \langle w, z_1 - z \rangle_Z}.
$$

Therefore we have

$$
|z - z_1|_Z^2 = 2 - 2\langle z, z_1 \rangle_Z = 2 - 2\sqrt{1 - \langle w, z_1 - z \rangle_Z} \le |\langle w, z_1 - z \rangle_z| \le |w|_Z |z_1 - z|_Z,
$$

which implies

$$
|z - z_1| \le |w|_Z \le |A(0) - A(x)|_{\mathcal{L}(Y;Z)} |A(0)^{-1}|_{\mathcal{L}(\mathrm{Im}(A(0));Y)} \lesssim |x|_X,
$$

where the first inequality follows from (11.238) and the last inequality follows from the Lipschitz continuity of $x \mapsto A(x)$. This proves (11.234). □

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