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Non-Archimedean Relative Characters and The Orbit Method

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Non-Archimedean Relative Characters and the Orbit Method

Trajan Hammonds

A Dissertation
Presented to the Faculty
of Princeton University
in Candidacy for the Degree
of Doctor of Philosophy

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Abstract

The asymptotics of relative characters for real Lie groups were studied for Gan-Gross-Prasad pairs (G, H) by Nelson and Venkatesh. To do so, they use the orbit method to develop a theory of microlocal analysis of Lie group representations. This involves characterizing vectors by the property that they are approximate eigenvectors under the action of small subgroups. They successfully compute the asymptotics of relative characters whenever the conductor of the associated Rankin-Selberg L-function $L(\pi \boxtimes \sigma^\vee)$ lies in a stable locus, i.e. away from conductor dropping. In the non-archimedean case, although a systematic theory of coadjoint orbits remains unclear, the resulting non-archimedean microlocal analysis is much simpler. In this thesis, we focus on relative characters and present a method which overcomes the stability hypothesis and allows for significant conductor dropping. Namely, for $(G, H) = (\mathrm{PGL}_2, \mathrm{GL}_1)$ over a non-archimedean field, this thesis computes the asymptotics of the relative character and expresses them as the integral over an appropriate geometric orbit.

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to my parents and dog

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Chapter 1

General Introduction

1.1 Main Philosophy

The central motivation for this thesis revolves around two seemingly different areas of study: the orbit method from representation theory and microlocal analysis. These methods were fruitfully tied together in the work of Nelson and Venkatesh [NV21]. Before explaining this relationship, which forms the backbone of this thesis, we will informally state two theorems which will be helpful to keep in mind for the reading of this thesis.

Theorem 1.1.1. Let Y_0^ℓ be the restriction to the sphere of the weight zero vector under the action of $H = \mathrm{SO}_2(\mathbf{R})$ living in the irreducible representation V_ℓ of $G = \mathrm{SO}_3(\mathbf{R})$. Then as long as $g \in \mathrm{SO}_3(\mathbf{R})$ is not a z-axis rotation, we have as $\ell \to \infty$,

$$\langle gY_0^\ell, Y_0^\ell \rangle \sim \text{F.T.(equator on } S_\ell^2),$$

where S_ℓ^2 is the two-sphere in $\mathbf{R}^3$ of radius $\ell + 1/2$ attached to the representation V_ℓ. The right-hand side means we are taking the Fourier transform of the surface measure on S_ℓ^2 restricted to the equator.

Remark 1.1.2. We think of the equator as the $z = 0$ slice of the sphere S_ℓ^2, corresponding to the fact that Y_0^ℓ has weight zero.

The main result of this thesis is proving an analogue of Theorem 1.1.1 in the case $(G, H) = (\mathrm{PGL}_2, \mathrm{GL}_1)$ over a non-archimedean field F. Instead of considering the action of a single g, we consider more generally averaging operators $\mathrm{Op}(a) \in \mathrm{End}(\pi)$. Moreover, instead of considering the zero weight vector, we consider, for characters χ of H, any 'weight vector'; that is, we work with functionals $\ell_\chi \in \mathrm{Hom}_H(\pi, \chi)$ that transform

by χ under the action of H. We prove the following under mild support conditions on the function $a : \mathrm{Lie}(G)^* \to \mathbf{C}$:

Theorem 1.1.3 (See Theorem 2.1.1)**.** Let π be a generic irreducible unitary representation of $G(F)$ and χ be a character on $H(F)$. Then for functionals ℓ_χ, which we can think of as 'weight vectors', we have

$$\langle \mathrm{Op}(a)\ell_\chi, \ell_\chi \rangle = \int_{\mathcal{O}_{\pi,\chi}} a(\xi)\, d\xi.$$

Remark 1.1.4. In the real archimedean case, $\mathcal{O}_{\pi,\chi}$ is the $x = c(\chi)$ slice of the hyperboloid in $\mathbf{R}^3$ associated to π.

For the rest of this section, we will elucidate why one might expect these theorems by explaining the relationship between the orbit method and microlocal analysis, and we will give a proof of Theorem 1.1.1.

The orbit method provides a correspondence between irreducible representations of a Lie group G and geometric objects called coadjoint orbits, living in the dual Lie algebra $\mathrm{Lie}(G)^*$ of G. Kirillov developed much of the orbit method in the 1960s, including his formula

Theorem 1.1.5 (Kirillov Formula)**.** Let G be a compact Lie group and V be an irreducible representation of G. Let $\mathcal{O}_V \subseteq \mathrm{Lie}(G)^*$ be the coadjoint orbit associated to V. Then the trace χ_V of the representation V can be expressed as the Fourier transform of a normalized surface measure on $\mathcal{O}_V$. Namely, for all X near the origin in $\mathrm{Lie}(G)$,

$$\chi_V(\exp X) = \sqrt{j(X)^{-1}} \int_{\mathcal{O}_V} e^{i\langle X,\xi\rangle}\, d\mu(\xi),$$

where $j(X)$ is the Jacobian of the exponential map $\exp : \mathrm{Lie}(G) \to G$.

For a reference see [NV21, Sec 6]. The best way to get a feel for this formula is through an example.

Example 1.1.6 (Spherical Harmonics for SO_3)**.** Let $G = \mathrm{SO}_3(\mathbf{R})$. There's one irreducible representation of each odd dimension $(2\ell + 1)$ given the vector space V_ℓ of homogeneous harmonic polynomials of degree ℓ with natural action $g.P(x, y, z) = P((x, y, z)g)$ for $P \in V_\ell$. For example, when $\ell = 1$, $V_\ell = \mathrm{span}\{x, y, z\}$ and when $\ell = 2$, $V_\ell = \mathrm{span}\{xy, yz, zx, x^2 - y^2, y^2 - z^2\}$. We can diagonalize the action of z-axis rotations by angle θ, denoted R_θ. We can then list this eigenbasis with the associated eigenvectors or *weights* as

follows:

$$V_1 = \begin{cases} x+iy & e^{i\theta} \\ z & 1 \\ x-iy & e^{-i\theta} \end{cases}$$

and

$$V_2 = \begin{cases} (x+iy)^2 & e^{-2i\theta} \\ (x+iy)z & e^{i\theta} \\ x^2+y^2-2z^2 & 1 \\ \vdots & \vdots \end{cases}.$$

The Lie algebra $\mathrm{Lie}(\mathrm{SO}_3)$ is given by

$$\mathrm{Lie}(\mathrm{SO}_3) = \{X \in M_3 : X + X^T = 0\} = \left\{ \begin{bmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{bmatrix} : x, y, z \in \mathbf{R} \right\}.$$

We can identify the Lie algebra with $\mathbf{R}^3$ by identifying matrices of the form above with vectors $v = (x, y, z)$. Then the coadjoint action is the same as the adjoint action, i.e. $\mathrm{Ad}_g X_v = X_{gv}$ where gv is the natural action of G on v. The coadjoint orbit $\mathcal{O}$ of v is just the set $\{gv : g \in G\}$ which are all rotations of the point v on the sphere of radius $|v|$. Therefore, the coadjoint orbits of SO_3 are spheres. To figure out which coadjoint orbit is associated to the representation V_ℓ, we compute the trace. Since the trace is a class function, and every element g is conjugate to a z-axis rotation, it suffices to compute $\chi_{V_\ell}(R_\theta)$. The trace is the sum of the eigenvalues so we have

$$\chi_{V_\ell}(R_\theta) = e^{-i\ell\theta} + \cdots + e^{i\ell\theta} = \frac{e^{i(\ell+1/2)\theta} - e^{-i(\ell+1/2)\theta}}{e^{i\theta/2} - e^{-i\theta/2}}.$$

The Jacobian of the exponential map is given by

$$j(\theta) = \left(\frac{e^{i\theta/2} - e^{-i\theta/2}}{i\theta} \right)^2.$$

. Therefore

$$\sqrt{j(x)}\chi_{V_\ell}(\exp x) = \frac{e^{i(\ell+1/2)\theta} - e^{-i(\ell+1/2)\theta}}{i\theta},$$

where $x = (\theta, 0, 0)$ as a vector in $\mathbf{R}^3$. We won't compute it here, but one may recognize the right-hand side as the Fourier transform of a sphere in $\mathbf{R}^3$ with radius $\ell + 1/2$ with measure normalized by $\frac{1}{2\pi(\ell+1/2)}$.

On the other hand, microlocal analysis originated in the 1970s. In the context of Fourier analysis on $\mathbf{R}^n$, it refers to the process of decomposing $f \in L^2(\mathbf{R}^n)$ into pieces $f = \sum_i f_i$ where both f_i and $\hat{f}_i$ are localized, that is, both the support of f_i and $\hat{f}_i$ is concentrated in a small region of $\mathbf{R}^n$. The Uncertainty Principle ensures that these regions can't be too small. In particular,

Theorem 1.1.7 (Uncertainty Principle)**.** For $f \in L^2(\mathbf{R}^n)$ we have

$$\left(\int_{\mathbf{R}^n} x^2 |f(x)|^2 \, dx\right)\left(\int_{\mathbf{R}^n} \xi^2 |\hat{f}(\xi)|^2 \, d\xi\right) \gtrsim ||f||_2^4.$$

Definition 1.1.8. Following [Fef83], we say a function $f \in L^2(\mathbf{R}^n)$ is *microlocalized* in $M \times N \subset \mathbf{R}^{2n}$ if $f(x)$ is mostly concentrated in M and $\hat{f}(\xi)$ is mostly concetrated in N.

The uncertainty principle then demands that $M \times N$ has volume $\gtrsim 1$. Microlocalization in $M \times N$ also has a physical interpretation of being localized in position x in M and in momentum ξ in N. For a general manifold M, the position-frequency space $\mathbf{R}^{2n}$ is replaced by the cotangent bundle T^*M. This space is often referred to as 'phase space'.

We need to answer the question of how one finds microlocalized functions, or how do we create them? To do this one uses the theory of pseudodifferential operators. Rather than a formal definition, we can give a simple example using localization operators on $L^2(\mathbf{R})$. For instance, let χ_A be the (smoothened) indicator function of a set A. Then we have space localization given by

$$f \mapsto \chi_I f$$

and frequency localization given by

$$f \mapsto (\chi_J \hat{f})^\vee = \chi_J^\vee * f.$$

Together they form an operator

$$\chi_I(x)(\chi_J \hat{f})^\vee(x) = \chi_I(x) \int_{\mathbf{R}} \chi_J(\xi) \hat{f}(\xi) e^{-ix\xi}\, d\xi = \int_{\mathbf{R}} \alpha_{I,J}(x,\xi) \hat{f}(\xi) e^{-ix\xi}\, d\xi.$$

Then, replacing $\alpha_{I,J}$ with a function $a \in C_c^\infty(\mathbf{R}^2) \cong C_c^\infty(T^*\mathbf{R})$ gives rise to pseudodifferential operators $\mathrm{Op}(a)$. When a is a function of ξ only, $\mathrm{Op}(a)$ is a Fourier multiplier, i.e. $\widehat{\mathrm{Op}(a)f}(\xi) = a(\xi)\hat{f}(\xi)$. When a is a function of x only, $\mathrm{Op}(a)$ is a position multiplier, i.e. $(\mathrm{Op}(a)f)(x) = a(x)f(x)$.

Now we can explain how this might relate to the orbit method. For a vector v in a representation π of G, our goal is to decompose v into pieces $v = \sum_i v_i$ such that each v_i is *microlocalized*. Before we say what it means for a vector to be microlocalized, we make a trivial observation about the Kirillov formula. Evaluating at $X = 0$ tells us that $\dim \pi = \mathrm{vol}(\mathcal{O}_\pi)$. We could then try to associate to each partition $\mathcal{O}_\pi = \bigsqcup P_j$ into pieces of unit volume, an orthonormal basis of $\{v_j\}$ of π such that

$$\pi(\exp X) v_j \approx e^{i\langle x, P_j\rangle} v_j.$$

Indeed, this would explain the Kirillov formula in the following sense: The left-hand side of the formula is the trace over an orthonormal basis $\mathcal{B}$ of π given by

$$\chi_V(\exp X) = \sum_{v \in \mathcal{B}} \langle \pi(\exp X) v, v\rangle \approx \sum_{1 \le j \le \dim V} e^{i\langle x, P_j\rangle}.$$

Ignoring the Jacobian factor for a moment, splitting up the integral on the RHS according to the partition gives

$$\int_{\mathcal{O}_\pi} e^{i\langle x,\xi\rangle}\, d\mu(\xi) = \sum_j \int_{P_j} e^{i\langle x,\xi\rangle}\, d\mu(\xi).$$

These are equal as long as each integral on the right-hand side is roughly constant and equal to $e^{i\langle x, P_j\rangle}$. It remains then to make sense of the pairing $\langle x, P_j\rangle$ and to show that the oscillatory integrals $\int_{P_j} e^{i\langle x,\xi\rangle}\, d\mu(\xi)$ don't vanish. Solving both issues at once, we can take P_j to be sufficiently concentrated around some element $\tau \in \mathcal{O}_\pi$. That way, the phase $\langle x, P_j\rangle$ is roughly constant and is equal to $\langle x, \tau\rangle$. This gives us the following definition.

Definition 1.1.9. For a representation π of G, a vector $v \in \pi$ is *microlocalized* at $\tau \in \mathrm{Lie}(G)^*$ if

$$\pi(\exp x) v \approx e^{i\langle x,\tau\rangle} v,$$

for all $x \in \mathrm{Lie}(G)$ sufficiently close to zero. We will sometimes denote vectors microlocalized at τ by v_τ.

We call the corresponding region $P_j := P_\tau$ its *microlocal support*. When π is a representation $\pi = \pi_T$ of highest weight T, for example a principal series representation corresponding to a Maass form φ with eigenvalue $1/4 + T^2$, we can take P_τ to be

$$P_\tau = \mathcal{O}_\pi \cap \left(\tau + O\left(T^{1/2}\right)\right).$$

To create localized vectors, we will construct integral operators $\mathrm{Op}(a)$ associated to Schwartz functions a on $\mathrm{Lie}(G)^*$. Their essential property is that

$$\mathrm{Op}(a)v_\tau \approx a(P_\tau)v_\tau \approx a(\tau)v_\tau,$$

for v_τ localized at τ and a is essentially constant on P_τ. When P_τ is a $T^{1/2}$-neighborhood around τ, this requires that a is roughly constant at scales $T^{1/2}$. To this end, we define $\mathrm{Op}(a)$ (see [NV21, Sec 2] for a rigorous definition) as

$$\mathrm{Op}(a)v = \int_{X \in \mathrm{Lie}(G), |X| \ll 1} a^\vee(X)\pi(\exp X)v \, dX.$$

When a is a bump function around τ, its inverse Fourier transform decays very quickly, so that the truncation in the integral is redundant. Roughly, if v is localized at τ, then we expect

$$\mathrm{Op}(a)v \approx \int_{\mathrm{Lie}(G)} a^\vee(x)e^{i\langle \tau, x\rangle} v \, dx = a(\tau)v.$$

In our first example, we will show why vectors that are approximate eigenfunctions under small group actions deserve the name *microlocalized vectors*. To do this, we make a more quantitative definition of localization.

Definition 1.1.10. We say $v \in \pi$ is R-localized at $\tau \in \mathrm{Lie}(G)^*$ if

$$||Xv - i\langle \tau, X\rangle v|| \leq R||v||,$$

for all X in a basis for $\mathrm{Lie}(G)$.

Example 1.1.11 (The Heisenberg Group)**.** Take $L^2(\mathbf{R})$ and define the operators $T_x : f(t) \mapsto f(t+x)$ and $M_y : f(t) \mapsto e^{ity}f(t)$. They satisfy $T_xT_x = T_{x+x}$ and $M_yM_y = M_{y+y}$. Also $(T_xM_yf) = e^{ixy}(M_yT_xf)$.

So these two groups generate a 3 dimensional group and gives a rep of $\begin{bmatrix} 1 & x & z \\ & 1 & y \\ & & 1 \end{bmatrix}$ on $L^2(\mathbf{R})$ via

$\begin{bmatrix} 1 & x & z \\ & 1 & y \\ & & 1 \end{bmatrix}$ acts on $L^2(\mathbf{R})$ as $e^{iz}T_xM_y$. This is the Heisenberg group. The Lie algebra action is

$$Xf(t) = \frac{df}{dt},\ Yf(t) = itf(t),\ Zf(t) = if(t).$$

To be localized, we should have $Xf \approx i\tau_x f$ and $Yf \approx i\tau_y f$. So to be R-localized, we demand

$$||itf(t) - \tau_y f(t)||_2 \leq R,\ ||f||_2 = 1,$$

i.e. $\int (t-\tau_y)^2|f|^2\,dt \leq R^2$, i.e. most of the mass of $|f|^2$ is within $O(R)$ of $t = \tau_y$. For Xf to be R-localized, we look instead at the Fourier transform so that the condition reads

$$||\xi\hat{f}(\xi) - \tau_x\hat{f}(\xi)||_2 \leq R,\ ||\hat{f}||_2 = 1,$$

i.e. $\int (\xi-\tau_x)^2|\hat{f}|^2\,d\xi \leq R^2$, i.e. most of the mass of $|\hat{f}|^2$ is within $O(R)$ of τ_x. Therefore both f and its Fourier transform are concentrated in some small region, so the localized 'vector' f is indeed microlocalized in the usual sense as a function on $L^2(\mathbf{R})$.

Now that we understand microlocalized vectors, we might ask "What are they good for?". There is a very important principle that underpins the motivation for this thesis, which is the following: if v is localized at τ, then

$$\langle gv_\tau, v_\tau\rangle \approx 0 \text{ unless } g.\tau \approx \tau,$$

that is, the matrix coefficient is negligible unless g is near the centralizer G_τ of τ. Moreover, following the heuristics of the Kirillov formula, we expect that after writing $g = \exp X$ (and ignoring Jacobian factors)

$$\langle(\exp X)v_\tau, v_\tau\rangle \approx \int_{P_\tau} e^{i\langle X,\xi}\,d\xi.$$

We can also phrase this in terms of linear functionals. If $\ell \in \mathrm{Hom}_H(\pi, \mathbf{C})$ and $v_\tau \in \pi$ is microlocalized at

τ, then

$$\ell(v_\tau) \approx 0 \text{ unless } \tau \text{ is close to } \mathrm{Lie}(H)^\perp,$$

where $\mathrm{Lie}(H)^\perp$ is the region in $\mathrm{Lie}(G)^*$ dual to $\mathrm{Lie}(H)$. In the examples we consider in this thesis $(G, H) = (\mathrm{SO}_3, \mathrm{SO}_2)$ and $(\mathrm{PGL}_2(\mathbf{R}), \mathrm{GL}_1(\mathbf{R}))$, $\mathrm{Lie}(H)$ is a one-dimensional real line and $\mathrm{Lie}(H)^\perp$ is the plane orthogonal to that line and passing through zero. This principle also extends to functionals that transform under H by a character. Namely, if σ is a character and $\ell : \pi \to \mathbf{C}$ transforms under H by σ, then

$$\ell(v_\tau) \approx 0 \text{ unless } \tau \text{ is close to } \mathbf{C}(\sigma) + \mathrm{Lie}(H)^\perp,$$

where $\mathbf{C}(\sigma)$ is the conductor of σ. So functionals $\ell \in \mathrm{Hom}_H(\pi, \sigma)$ are negligible on microlocalized vectors v_τ unless τ is near the shifted plane $\mathbf{C}(\sigma) + \mathrm{Lie}(H)^\perp$. This picture will be made more clear by working through an example.

1.1.1 Spherical Harmonics : A Motivating Example

We return to the case of $G = \mathrm{SO}_3(\mathbf{R})$ and the representation V_ℓ of G given by harmonic homogeneous polynomials of degree ℓ in three variables x, y and z. Each V_ℓ has dimension $2\ell + 1$. The name "spherical harmonic" comes from the fact that these representations form the building blocks of the L^2 space of the sphere. Namely, the restrictions of these polynomials to the sphere gives rise to the decomposition

$$L^2(S^2) = \bigoplus_{\ell \in \mathbf{N}} V_\ell.$$

We can further break up V_ℓ into exact eigenfunctions for z-axis rotations, yielding vectors Y_m^ℓ. For instance, if $g \in G$ is rotation around the z-axis by angle θ, $gY_m^\ell = e^{im\theta} Y_m^\ell$. We can define H to be the subgroup of z-axis rotations and identify H with $\mathrm{SO}_2(\mathbf{R})$. Hence we recognize the Y_m^ℓ as weight vectors under the action of H, and we see for instance that the highest weight vector is given by $Y_\ell^\ell = (x + iy)^\ell$. The coadjoint orbit is given by the sphere S_ℓ^2 of radius $\ell + 1/2$ and we identify $\mathrm{Lie}(H)^*$ with the vertical z-axis (see Figure 1.1).

Although these Y_m^ℓ give an explicit basis, the action of $\mathrm{SO}(3)$ is generally complicated; only the z-axis rotations are easy. One thing in particular we'd like to understand are the matrix coefficients $\langle gY_m^\ell, Y_m^\ell \rangle$. One can show that for fixed $g \in \mathrm{SO}(3)$, $g \neq 1$ and g not a z-axis rotation, the matrix coefficient for the highest weight vector $\langle gY_\ell^\ell, Y_\ell^\ell \rangle$ decays exponentially in ℓ but that this is not the case for the zero-weight vector Y_0^ℓ.

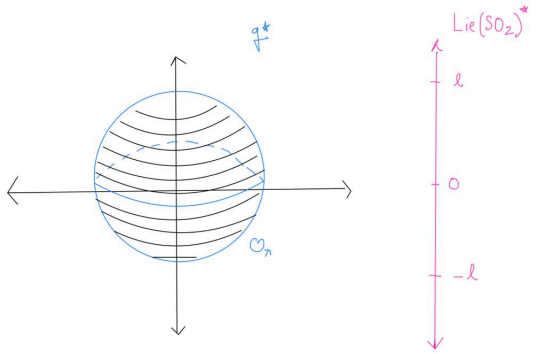


Figure 1.1: Partition of Sphere into Level Sets

The asymptotic behavior of the matrix coefficients of Y_0^ℓ are related to the asymptotics of 6j-symbols. The asymptotics for $6j$-symbols were conjectured by Ponzano and Regge [PR68] but a rigorous proof, due to Roberts, did not appear until 2001. The novelty of Roberts' paper [Rob02] is the use of ideas coming from geometric quantization. Along the way of proving $6j$-symbol asymptotics, he also proves the following well-known matrix coefficient asymptotic using geometric quantization.

Theorem 1.1.12 (Roberts, 2002, Szegö 1933)**.** As $\ell \to \infty$, we have

$$\langle gY_0^\ell, Y_0^\ell \rangle \sim \sqrt{\frac{2}{\pi \ell \sin\theta}} \cos\left((2\ell+1)\frac{\theta}{2} + \frac{\pi}{4}\right),$$

where θ is the angle through which g rotates the z-axis.

Under the orbit method philosophy, since Y_0^ℓ is invariant under the action of H, we expect that the matrix coefficient $\langle gY_0^\ell, Y_0^\ell \rangle$ should be obtained by taking the Fourier transform of the region of the sphere S_ℓ^2 that is close to the xy-plane $\mathrm{Lie}(H)^\perp$. This corresponds to the equatorial strip of width 1 (see Figure 1.1).

More generally, the basis Y_m^ℓ should correspond to regions of unit symplectic volume obtained by disintegrating the coadjoint orbit S_ℓ^2 along the projection map $\pi : \mathrm{Lie}(G)^* \to \mathrm{Lie}(H)^*$, as in the Figure 1.1.

Our strategy for proving Theorem 1.1.1 is to prove that the right-hand side of Roberts' asymptotic is given by the Fourier transform of the equator, or 'z = 0 slice' of the sphere S_ℓ^2.

Proof of Theorem 1.1.1. First we compute the Fourier transform of the equator E itself, that is,

$$\hat{\mu}(x) = \int_E e^{ix\cdot\xi}\, d\xi.$$

Since the surface measure is radial, so is its Fourier transform, so it suffices to compute it at an element in the Lie algebra viewed in $\mathbb{R}^3$ as $(\theta, 0, 0)$. Here θ is the angle through which $g = \exp x$ rotates the z-axis. (We

could have also picked $(0, \theta, 0)$ and the calculation would be the same). Then we can write

$$\hat{\mu}(x) = \hat{\mu}(\theta e_1) = \int_E e^{i\theta\xi_1}\, d\xi. \tag{1.1}$$

We know that since $\xi \in S^2_\ell$, we can write $\xi_1 = \pm\sqrt{(\ell+1/2)^2 - \xi_2^2}$. After a change of variables and partitioning the equator into two halves, we obtain

$$\hat{\mu}(\theta) = \int_{-1}^{1} e^{i(\ell+1/2)\theta\sqrt{1-y^2}} \frac{dy}{\sqrt{1-y^2}} + \int_{1}^{-1} e^{-i(\ell+1/2)\theta\sqrt{1-y^2}} \frac{dy}{\sqrt{1-y^2}} \tag{1.2}$$

Here the change of variables uses the arclength measure $\sqrt{1+|\nabla f(x)|^2}\, dx$. So we obtain that $\hat{\mu}(\theta) = 2\Re\left(\int_{-1}^{1} e^{i(\ell+1/2)\theta\sqrt{1-y^2}} \frac{dy}{\sqrt{1-y^2}}\right)$. We can evaluate this integral using stationary phase. Denoting the phase function $\Phi_\theta(y) = \theta\sqrt{1-y^2}$ we see that it's critical points are where $y = 0$ which represent the 'poles' of the equator. We can evaluate the integral as

$$\begin{aligned} I(\ell) &= \int e^{i(\ell+1/2)\Phi_\theta(y)} \frac{dy}{\sqrt{1-y^2}} \\ &= e^{i(\ell+1/2)\Phi(0) + i\frac{\pi}{4}\mathrm{sgn}(\Phi''(0))} \frac{1}{\sqrt{1-0^2}} \sqrt{\frac{2\pi}{\ell|\Phi''(0)|}} + O(\ell^{-3/2}) \\ &= e^{i\theta(\ell+1/2) - i\frac{\pi}{4}} \sqrt{\frac{2\pi}{\ell\theta}} + O(\ell^{-3/2}). \end{aligned}$$

Taking twice the real part we obtain

$$\sqrt{\frac{2\pi}{\ell\theta}} \cos\left((2\ell+1)\frac{\theta}{2} - \frac{\pi}{4}\right) + O(\ell^{-3/2}). \tag{1.3}$$

We are almost done, but we must remember to multiply by the Jacobian factor. The exponential map $\exp : \mathfrak{g}/\mathfrak{h} \to G/H$ has Jacobian $j(X) = \frac{\sin|X|}{|X|}$, so for $X = (\theta, 0, 0)$ as in our case, we see that the Jacobian satisfies $\sqrt{j(X)}^{-1} = \sqrt{\frac{\theta}{\sin\theta}}$. So we obtain

$$\sqrt{j(\theta)}^{-1} \widehat{\mu_E}(\theta) = \sqrt{\frac{2\pi}{\ell\sin\theta}} \cos\left((2\ell+1)\frac{\theta}{2} - \frac{\pi}{4}\right) + O(\ell^{-3/2}), \tag{1.4}$$

as desired. □

1.1.2 Literature and This Thesis

In [NV21], their union of microlocal analysis with the orbit method is used profitably to deduce asymptotics for relative characters for GGP pairs (G, H) equal to $(\mathrm{SO}_{n+1}, \mathrm{SO}_n)$ or (U_{n+1}, U_n). Such relative characters are related to the matrix coefficient integrals $\mathcal{H}_\sigma : \pi \otimes \bar{\pi} \to \mathbf{C}$ given by

$$\mathcal{H}_\sigma(v \otimes v) = \sum_{u \in \mathcal{B}(\sigma)} \int_{s \in H} \langle \pi(s)v, v \rangle \langle u, \sigma(s)u \rangle \, ds,$$

for $v \in \pi$ a representation of G and $\mathcal{B}(\sigma)$ a basis for σ, a representation of H. These matrix coefficient integrals are essentially the local integrals appearing in the Ichino-Ikeda formula for the Rankin-Selberg L-functions $L(s, \pi \times \sigma)$. The relative character is then defined to be

$$\mathcal{H}_{\pi,\sigma}(a) := \sum_{v \in \mathcal{B}(\pi)} \mathcal{H}_\sigma(\mathrm{Op}(a)v \otimes v).$$

Their main theorem for relative character asymptotics proves that

$$\mathcal{H}_{\pi,\sigma}(a) \sim \int_{\mathcal{O}_{\pi,\sigma}} a,$$

where $\mathcal{O}_{\pi,\sigma}$ is the analogue of the slices of the sphere (from the SO_3 example) in higher dimensions (See Section 2.1 for more details).

The main theorem of this thesis is proving an analogue of the above theorem for $(G, H) = (\mathrm{PGL}_2, \mathrm{GL}_1)$ over non-archimedean fields (see Theorem 2.1.1). Since microlocal analysis over non-archimedean fields is much simpler, the advantage of our theorem is that it allows for certain "degenerate" slices $\mathcal{O}_{\pi,\sigma}$ which corresponds to conductor dropping of the Rankin-Selberg L-function (see Section 2.1).

The strategy of Nelson and Venkatesh has most notably led to many breakthroughs on the subconvexity problem including [N23], [N21], [M23], [HN18] to name a few. We don't pursue any applications to subconvexity in this thesis and entirely avoid any relative trace formula techniques. For a great and more thorough introduction to the subject we recommend the original paper [NV21] of Nelson and Venkatesh, as well as the thorough exposition in the recent preprint [AT25], which also considers non-archimedean microlocal analysis.

Chapter 2

Asymptotics for Non-Archimedean Relative Characters

2.1 Introduction

Let $G = \mathrm{PGL}_2(\mathbf{F})$ and $H = \mathrm{GL}_1(\mathbf{F})$ embedded diagonally in G, with $\mathbf{F}$ a non-archimedean local field like $\mathbf{Q}_p$. For a representation π of G and a character χ of H, our two main players will be the integral operator $\mathrm{Op}(a)$ and the linear functional ℓ_χ. For a function $a \in \mathrm{Fun}(\mathrm{Lie}(G)^*)$, we define

$$\mathrm{Op}(a) := \int_{x \in \mathrm{Lie}(G)} a^\vee(x)\pi(\exp x)\, dx.$$

(See Section 2.3 for more details). Meanwhile, the linear functional $\ell_\chi : \pi \to \mathbf{C}$ is given by

$$\ell_\chi(v) := \int_{\mathbf{F}^\times} W_v \begin{pmatrix} h & \\ & 1 \end{pmatrix} \overline{\chi}(h)\, d^\times h,$$

where W_v is the image of v in its Whittaker model (See Section 2.4 for more details). Our main theorem is about the *relative character* which we define as

$$\mathcal{H}_{\pi,\chi}(a) = \sum_{v \in \mathcal{B}(\pi)} \ell_\chi(\mathrm{Op}(a)v)\overline{\ell_\chi(v)} \tag{2.1}$$

for any orthonormal basis $\mathcal{B}(\pi)$ of π. In the theory of Lie groups, Fourier transforms of characters have a nice description in terms of the wave front set (see [How81]). There is a similar picture for relative characters,

which, in the relative Langlands program, play a similar role as characters do in the Langlands program. We can now state the main theorem.

Theorem 2.1.1. Let π be a principal series or supercuspidal unitary representation of G and χ a unitary character of H. If π is principal series $\pi = \chi_0 \boxplus \chi_0^{-1}$, assume that $\chi \neq \chi_0, \chi_0^{-1}$ or an unramified twist. Let $a \in \mathrm{Fun}(\mathrm{Lie}(G)^*)$ be a function constant at scales $|T| = q^N$ with $N \geq c(\chi)/2$ and supported on elements $T\tau$ with y and z coordinates τ_y and τ_z satisfying $|\tau_y|, |\tau_z| \leq \min\left(q^{c(\chi_0\chi^{-1})/2}, q^{c(\chi_0^{-1}\chi^{-1})/2}\right)$ in the principal series case and $|\tau_y|, |\tau_z| \leq (q^{c(\pi\otimes\overline{\chi})/2})$ in the supercuspidal case. Then, there exists a hyperbola (see 2.7) $\mathrm{Hyp}(\pi, \chi) \subset \mathrm{Lie}(G)^*$ so that

$$\mathcal{H}_{\pi,\chi}(a) = \int_{\mathrm{Hyp}(\pi,\chi)} a(\xi)\, d\xi. \tag{2.2}$$

When $F = \mathbf{R}$, the analogue of this theorem was proven by Nelson and Venkatesh in [NV21] for pairs of representations (π, σ) of (G, H) where (G, H) are the Gan-Gross-Prasad pairs $(\mathrm{SO}_{n+1}, \mathrm{SO}_n)$ or (U_{n+1}, U_n). In this case, the hyperbola in Theorem 2.1.1 has the following interpretation in terms of coadjoint orbits. Let $\mathfrak{g}$ and $\mathfrak{h}$ be the Lie algebras of G and H respectively. Dual to the inclusion $\mathfrak{h} \hookrightarrow \mathfrak{g}$ we have the projection $\mathrm{pr} : \mathfrak{g}^* \twoheadrightarrow \mathfrak{h}^*$. Let $(G, H) = (\mathrm{PGL}_2(\mathbf{R}), \mathrm{GL}_1(\mathbf{R}))$ and π is principal series and σ is a character, and denote by $\mathbf{C}(\pi)$ and $\mathbf{C}(\sigma)$ the conductors of π and σ. The coadjoint orbit for π is given by

$$\mathcal{O}_\pi := \left\{ \begin{pmatrix} x & y+z \\ y-z & -x \end{pmatrix} : x^2 + y^2 - z^2 = \mathbf{C}(\pi)^2 \right\} \subseteq \mathfrak{g}^* \cong \mathbf{R}^3, \tag{2.3}$$

and the coadjoint orbit for σ is given by

$$\mathcal{O}_\sigma := \{x = \mathbf{C}(\sigma)\} \subseteq \mathfrak{h}^* \cong \mathbf{R}. \tag{2.4}$$

When $\mathbf{C}(\sigma) \neq \mathbf{C}(\pi)$, the relative coadjoint orbit is

$$\mathcal{O}_{\pi,\sigma} := \mathcal{O}_\pi \cap \mathrm{pr}^{-1}(\mathcal{O}_\sigma) = \{\mathrm{y}^2 - \mathrm{z}^2 = \mathbf{C}(\pi)^2 - \mathbf{C}(\sigma)^2\},$$

which carves out a hyperbola, which is the analogue of our $\mathrm{Hyp}(\pi, \chi)$ (see 2.1). However, when $\mathbf{C}(\sigma) = \mathbf{C}(\pi)$ the resulting locus is a cross which has a singularity at the origin. In [NV21], they have to rule out this degeneracy by working with *stable* pairs of representations (π, σ), that is, those representations for which H acts simply transitively on $\mathcal{O}_{\pi,\sigma}$. This notion of stability amounts to a condition on the infinitesimal characters λ_π and λ_σ of π and σ: namely that $\{$ eigenvalues of $\lambda_\pi\} \cap \{$ eigenvalues of $\lambda_\sigma\} = \emptyset$. Moreover, we say a pair

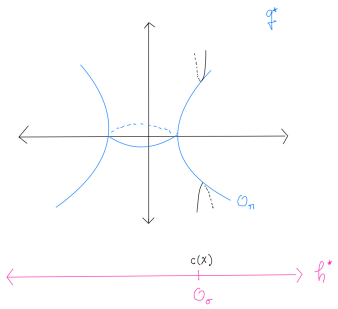


Figure 2.1: Relative Coadjoint Orbit

$(\lambda, \mu) \in \mathfrak{g}^* \times \mathfrak{h}^*$ is stable if $\{$ eigenvalues of $\lambda\} \cap \{$ eigenvalues of $\mu\} = \emptyset$. For a stable pair of representations (π_T, σ_T), indexed by their highest weight, Nelson and Venkatesh study the asymptotic behavior of $\mathcal{H}_{\pi,\sigma}(a_T)$ for a T-dependent function a. They prove for any a supported in a fixed compact subset $U \subset \mathfrak{g}^*$ of stable elements, as $T \to \infty$

$$\mathcal{H}_{\pi,\sigma}(a_T) = \int_{\mathcal{O}_{\pi,\sigma}} a + O\left(T^{2\delta-1}\right)$$

for some fixed $0 \leq \delta < 1/2$.

Remark 2.1.2. Away from stable pairs is where the conductor $C(\pi \times \overline{\sigma})$ of the Rankin-Selberg L-function $L(s, \pi \times \overline{\sigma})$ drops. Recent subconvexity results such as [N23] rely on these relative character asymptotics away from the stable range and thus require the *non-conductor dropping* assumption

$$\forall i, j \; |\lambda_{\pi,i} - \lambda_{\sigma,j}| \gg T.$$

The advantage of our main theorem is that it provides a relative character asymptotic in the case where the conductor is allowed to drop significantly. Indeed, the assumption on π and χ in our theorem is analogous to the non-conductor dropping requirement

$$\forall i, j \; |\lambda_{\pi,i} - \lambda_{\sigma,j}| \gg 1,$$

but only in the non-archimedean case for $(G, H) = (\mathrm{PGL}_2, \mathrm{GL}_1)$. In this case $\lambda_{\pi,i}$ is just the conductor exponent of χ_0 or χ_0^{-1} and $\lambda_{\sigma,j}$ is the conductor exponent of χ.

2.1.1 Sketch of the Proof

By formally applying Parseval to Equation 2.1, it suffices to compute

$$\langle \mathrm{Op}(a)\ell_\chi, \ell_\chi \rangle,$$

for a as in Theorem 2.1.1. Such functions have a decomposition into wave packets (Proposition 2.3.4), so by linearity we can compute the relative character for a single wave packet 1_τ^T. The function 1_τ^T is a bump function around some fixed element $\tau \in \mathfrak{g}^*$. In fact, 1_τ^T is the Fourier transform of a wave packet supported on the congruence subgroup $K(N)$ oscillating at fixed frequency τ (See Section 2.3). To compute the relative character for $a = 1_\tau^T$, we will factor $\mathrm{Op}(a)$ into directions, corresponding to the Iwahori factorization $K(N) = K_- K_0 K_+$. For any function $b : F \to \mathbf{C}$ and for $\star \in \{-, 0, +\}$, $\mathrm{Op}^\star(b)$ averages the action of $K_\star$, weighted by b (see Equation 2.10 for a precise definition). Therefore, we factor $a = a_- a_0 a_+$ into such functions, each of which is a one-variable function on $\mathrm{Lie}(G_\star)^*$. The function a_+ (resp. a_-) most naturally pairs with the operator Op^- (resp. Op^+) due to the duality between $\mathfrak{g}$ and $\mathfrak{g}^*$ (see Section 2.3.2). We will use the Weyl element $w = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and the $\mathrm{GL}_2 \times \mathrm{GL}_1$ local functional equation to go between Op^- and Op^+ (see Section 2.5.2). In particular, we have the relation $\mathrm{Op}^+ = \pi(w)\,\mathrm{Op}^-\,\pi(w)$. These decompositions and factorizations allow us to express the relative character as

$$\begin{aligned}
\langle \mathrm{Op}(a)\ell_\chi, \ell_\chi \rangle &= \langle \mathrm{Op}^-(a_+)\,\mathrm{Op}^0(a_0)\,\mathrm{Op}^+(a_-)\ell_\chi, \ell_\chi \rangle \\
&= \langle \pi(w)\,\mathrm{Op}^0(a_0)\,\mathrm{Op}^+(a_-)\ell_\chi, \mathrm{Op}^+(a_+)\pi(w)\ell_\chi \rangle \\
&= \gamma(\pi \otimes \overline{\chi}) \langle \pi(w)\,\mathrm{Op}^+(a_-)\,\mathrm{Op}^0(a_0)\ell_\chi, \mathrm{Op}^+(a_+)\ell_{\chi^{-1}} \rangle \\
&= \gamma(\pi \otimes \overline{\chi}) a_0(\tau_x) \langle \pi(w)\,\mathrm{Op}^+(a_-)\ell_\chi, \mathrm{Op}^+(a_+)\ell_{\chi^{-1}} \rangle.
\end{aligned}$$

Denote this last inner product as

$$I = \langle \pi(w)\,\mathrm{Op}^+(a_+)\ell_\chi, \mathrm{Op}^+(a_-)\ell_{\chi^{-1}} \rangle.$$

From here, we work in the Kirillov model $\mathbf{K}(\pi, \psi)$ for π, since this model diagonalizes the action of the unipotent subgroup G_+ in G. Indeed, after identifying ℓ_χ with a suitable vector in the Kirillov model, we essentially have

$$\mathrm{Op}^+(a_+)\ell_\chi = \{h \mapsto a_+(h)\chi(h)\} \in \mathbf{K}(\pi, \psi)$$

and

$$\mathrm{Op}^+(a_-)\ell_{\overline{\chi}} = \{h \mapsto a_-(h)\overline{\chi}(h)\} \in \mathbf{K}(\pi, \psi).$$

Then we can expand into a basis of characters and applying Mellin inversion leads us to study

$$I = \int_\eta \langle a_-\overline{\chi}, \eta\rangle\langle \eta, \pi(w)a_+\chi\rangle \, d\eta = \int_\eta \gamma(\pi\otimes\overline{\eta})\langle a_-\overline{\chi}, \eta\rangle\langle \overline{\eta}, a_+\overline{\chi}\rangle \, d\eta.$$

At this point we crucially use that we are working with a specific choice of test function a, that is $a = 1_\tau^T$. The technical bulk of this paper is spent calculating I, which we carry out in Section 2.5.

On the other hand, evaluating the integral on the right hand side of Theorem 2.1.1 is relatively straightforward. We conclude the proof by carrying out the computation of the right-hand side and comparing it with the previous computation of the left-hand side.

2.1.2 Organization of the Paper

In Section 2.2 we give some important definitions and set-up for the rest of the paper. In Section 2.3 , we explicitly describe the functions we work with, and examine some of their properties, along with proving the desired properties of the operators $\mathrm{Op}(a)$. In Section 2.4, we prove some of the desired properties of the linear functionals and show how they are related to the relative character. In Section 2.5, we carry out the calculations necessary to complete the proof sketch from the previous section. Finally, in Section 2.7, we compute the corresponding integral in phase space and show that it agrees with the relative character.

2.2 Preliminaries

Let v be a finite place of a global field $\mathbf{F}$ and let $\mathbf{F}_v$ denote the corresponding non-archimedean local field. We only work locally so, by abuse of notation, we'll refer to $\mathbf{F}_v$ as just F. Let $\mathcal{O}$ be its ring of integers with maximal ideal $\mathfrak{p}$. We denote by ϖ its uniformizer. The absolute value on F will be denoted $|\cdot|$ (with the subscript sometimes omitted) so that $|\varpi| = q^{-1}$, where q is the size of the residue field $\mathcal{O}/\mathfrak{p}$. Let $\mathrm{val}(x) = v(x) : F \to \mathbf{Z}$ be such that $|x| = q^{-v(x)}$. For the purposes of this thesis, we will assume that q is odd.

We will often use ψ to denote an additive character of F, i.e. $\psi(x+y) = \psi(x)\psi(y)$. We let $\mathrm{cond}(\psi)$ denote the largest ideal of $\mathcal{O}_F$ on which ψ is trivial and denote its conductor exponent $\mathrm{c}(\psi)$ so that $\mathrm{cond}(\psi) = \mathfrak{p}^{\mathrm{c}(\psi)}$. If $\mathrm{cond}(\psi) = \mathcal{O}_\mathrm{F}$, we say ψ is unramified, and we have $\mathrm{c}(\psi) = 0$. Letters like χ, ν, or η will denote

multiplicative characters of $F^\times$. We denote the multiplicative subgroup $\mathcal{O}^\times \cap (1+\mathfrak{p}^n)$ by $U(n)$. We let $\mathrm{cond}(\chi)$ denote the largest subgroup of $\mathcal{O}_F^\times$ on which χ is trivial and denote its conductor exponent $\mathrm{c}(\chi)$ so that $\mathrm{cond}(\chi) = \mathrm{U}(\mathrm{c}(\chi))$. If $\mathrm{cond}(\chi) = \mathcal{O}_\mathrm{F}^\times$, then $\mathrm{c}(\chi) = 0$ and we say that χ is unramified. Every unramified character η can be written as $\eta(x) = |x|^s$ for some $s \in \mathbf{C}$. Let $X(F^\times)$ denote the group of multiplicative characters of F. The unramified characters of $F^\times$ form a subgroup of $X(F^\times)$ isomorphic to $\mathbf{C}/(\mathbf{Z} \cdot 2\pi i/\log q)$.

There is a unique Haar measure du on the additive group F normalized so that $\mathrm{vol}(\mathcal{O}, du) = 1$. The multiplicative Haar measure $d^\times u$ on $F^\times$ is normalized so that $\mathrm{vol}(\mathcal{O}^\times, d^\times u) = 1$. The measures satisfy $d^\times u = \zeta_F(1)\frac{du}{|\cdot|}$, where $\zeta_F(1) = (1-q^{-1})^{-1}$ is the local zeta factor. With these measures, we have $\mathrm{vol}(\mathfrak{p}^n, du) = q^{-n}$ and $\mathrm{vol}(U(n), d^\times u) = \zeta_F(1)q^{-n}$. Let $\mathcal{X}_n$ denote the set of characters with conductor at most n. Then $|\mathcal{X}_n| = \zeta_F(1)^{-1}q^n$.

We define the congruence subgroups

$$K^0[m_v] := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G(\mathcal{O}_v) \mid c \equiv 0 \bmod \varpi_v^{m_v} \right\}.$$

If $\pi_v^{K^0[m-1]}$ is empty but $\pi_v^{K^0[m]}$ is nonzero, then we call m the conductor exponent of π and write $C(\pi) = \mathfrak{p}^{v(\pi)}$. The main subgroup of interest for us is the principal congruence subgroup, defined as

$$K(n) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G(\mathcal{O}_v) \mid a-1, b, c, d-1 \equiv 0 \bmod \varpi_v^n \right\}.$$

We have a decomposition of $K_v(n)$ into its Iwahori factorization given by

$$K(n) = N(\varpi^n)A(1+\varpi^n)N_-(\varpi^n),$$

where for $\mathfrak{a} \subset \mathcal{O}$, $N(\mathfrak{a})$ has upper right entries in $\mathfrak{a}$, $A(\mathfrak{a})$ has diagonal entries in $1+\mathfrak{a}$ and $N_-(\mathfrak{a})$ has bottom left entries in $\mathfrak{a}$.

We denote

$$N := \left\{ n(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} : x \in F \right\} \text{ and } A := \left\{ a(y) = \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} y \in F^\times \right\}.$$

2.2.1 Whittaker and Kirillov Models

For a generic irreducible representation π and additive character ψ of F, we embed π into its smooth Whittaker model $\mathcal{W}(\pi, \psi)$ given by the space of functions (see [JS83] or [BH06, Sec 37])

$$\mathcal{W}(\pi, \psi) = \{W \in C^\infty(G) : n(x)W(g) = \psi(x)W(g) \text{ for all } (x, g) \in F \times G\}.$$

The restriction map $\mathcal{W}(\pi, \psi) \to \{$ functions $A_F \to \mathbf{C}\}$ is injective and its image *contains* $C_c^\infty(A_F)$ but does not consist solely of such functions. Therefore each $W \in \mathcal{W}(\pi, \psi)$ is determined by a function $W : F^\times \to \mathbf{C}$ so that

$$W(y) := W(a(y)).$$

Moreover, every compactly supported smooth function on A_F arises this way. This realization of π as a space of functions on $F^\times$ is the Kirillov model of π. For unitary π, an invariant inner product is given by

$$\langle W_1, W_2 \rangle = \int_{F^\times} W_1(y)\overline{W_2(y)}\, d^\times y.$$

2.2.2 GL_1 Iwasawa-Tate Theory

If χ is a multiplicative character, then its twisted dual is $\chi^\vee := \chi^{-1}|\cdot|$. Let $d^\times x$ be the Haar measure on $F^\times$, which we take to be $\frac{dx}{|x|}$ where dx is the Haar measure on F. Define local zeta functionals $Z(f, \chi)$ for f a Schwartz function on F by

$$Z(f, \chi) = \int_{F^\times} f(x)\chi(x)\, d^\times x.$$

We next recall the local L-factors; $L(\chi) = (1 - \chi(\varpi))^{-1}$ if χ is unramified and $L(\chi) = 1$ if χ is ramified. Then the GL_1 local functional equation states that

$$\frac{Z(f, \chi)}{L(\chi)}\epsilon(\chi, \psi) = \frac{Z(\hat{f}, \chi^\vee)}{L(\chi^\vee)},$$

where $\hat{f}$ denotes the Fourier transform $\hat{f}(y) = \int_F f(x)\psi(xy)\, dx$. It is important to understand the epsilon factor which will turn out to play a privileged role in our analysis. To do this we first define the Gauss sum (see [GH11])

$$g(\nu, \psi) = \int_{\mathcal{O}^\times} \nu(x)\psi(x)\, d^\times x.$$

It has the property that if the conductors of ν and ψ are different, then the Gauss sum vanishes. If the conductors are say, both $\mathfrak{p}^n$, then $|g(\nu,\psi)|^2 = q^{-n}$. When χ is replaced by $\chi|\cdot|^{s-1/2}$, we write $\epsilon(s,\chi,\psi)$ for the epsilon factor.

Lemma 1. Suppose χ is a multiplicative character of the form $\eta|\cdot|^s$ with conductor $\mathrm{c}(\chi) = n$. Let ψ be an unramified additive character. Then

$$\epsilon(s,\chi,\psi) = q^{n(1-s)}\eta(\varpi^n)g(\eta^{-1},\psi_{\varpi^{-n}}), \tag{2.5}$$

where $\psi_t(x) := \psi(tx)$.

Proof. See [GH11] Thm 2.3.8. □

We will use the following corollary often, whose proof is immediate.

Corollary 2.2.1. For χ unitary, the epsilon factor satisfies

$$|\epsilon(1/2,\chi,\psi)|^2 = 1.$$

We will also use the fact that $\epsilon(s,\chi|\cdot|^a) = q^{-ac(\chi)}\epsilon(s,\chi)$. Lastly, we need two propositions to understand the interaction between multiplicative characters and additive characters.

Proposition 2.2.2. [BH06, Sec 1.8] Let χ be a multiplicative character with conductor exponent $c(\chi) = n$. Then for a fixed unramified additive character ψ, there is $\alpha_\chi \in F$ such that

$$\chi(1+x) = \psi(\alpha_\chi x),$$

for $x \in \mathfrak{p}^{n/2+1}$.

We remark that such an α_χ is not unique and is only defined up to $\mathfrak{p}^{-n/2-1}$.

Proposition 2.2.3. [Tat75, Sec 3] Let χ and ω be two characters with $m = c(\omega) \leq c(\chi)/2 = n/2$. Assume ψ is unramified and n is even. Let α_χ be such that $\chi(\exp a) = \psi(\alpha_\chi a)$ for $a \in \mathfrak{p}^{n/2}/\mathfrak{p}^n$. Then

$$\epsilon(1/2,\chi\omega,\psi) = \omega(\alpha_\chi)\epsilon(1/2,\chi,\psi).$$

2.2.3 Local Representations

In order to carry out the calculations of the relative character we need to associate some data to our representations. Namely, we need to understand the conductor $\mathrm{c}(\pi\otimes\chi)$, and the elements $\alpha_{\pi,\chi} \in F$ which will

appear in the definition of $\mathrm{Hyp}(\pi, \chi)$. We have already seen the definition of α_χ which appeared at the end of the previous subsection.

Let π be a generic supercuspidal representation of $\mathrm{GL}_2(F)$ with trivial central character. It corresponds to a two-dimensional representation $\rho = \mathrm{Ind}_{E/F}\,\xi := \mathrm{Ind}_{\mathcal{W}_E}^{\mathcal{W}_F}\,\xi$ under the Local Langlands correspondence, where $\mathbf{E}$ is a quadratic extension of F and ξ a character of $\mathbf{E}^\times$. Note that since we work with PGL_2, we require $\det\rho = 1$. We can view ξ as a character of the Weil group $\mathcal{W}_E$ by class field theory. Explicitly, if $\alpha_{\mathbf{E}}$ is the Artin reciprocity map

$$\alpha_{\mathbf{E}} : \mathcal{W}_{\mathbf{E}}^{\mathrm{ab}} \to \mathbf{E}^\times,$$

then $\xi \circ \alpha_{\mathbf{E}}$ is a character of $\mathcal{W}_{\mathbf{E}}$. For any character η of $F^\times$ or ψ of F, let $\eta_{\mathbf{E}} := \eta \circ \mathrm{Nm}_{\mathbf{E}/F}$ and $\psi_{\mathbf{E}} := \psi \circ \mathrm{Tr}_{\mathbf{E}/F}$. Lastly let $e := e(\mathbf{E}|F)$ be the ramification index and $f := f(\mathbf{E}|F)$ be the inertial degree. The LLC works well with gamma and epsilon factors in the following sense (see [BH06, Sec 34]): for any character ν on $F^\times$,

$$\gamma(\pi \otimes \nu) = \gamma(\rho \otimes \nu) = \gamma(\xi\nu_E)$$

and

$$\epsilon(1/2, \pi \otimes \nu, \psi) = \epsilon(1/2, \xi\nu_E, \psi_E).$$

In particular, the conductor satisfies $\mathrm{c}(\pi \otimes \overline{\chi}) = f\mathrm{c}(\xi\chi_E^{-1})$.

Definition 2.2.4. We define $\alpha_{\pi,\chi}$ for π supercuspidal as $\mathrm{Nm}(\alpha_{\xi\chi_E^{-1}})$ where $\alpha_{\xi\chi_E^1}$ is such that

$$\xi\chi_E^{-1}(1+x) = \psi(\alpha_{\xi\chi_E^{-1}}x),$$

for $x \in \mathfrak{p}_{\mathbf{E}}^{c(\xi\chi_E^{-1})/2}$.

The other case is that π is principal series. In this case we write $\pi = \chi_0 \boxplus \chi_0^{-1} = \mathrm{Ind}_{B(F)}^{G(F)}(\chi_0, \chi_0^{-1})$. The gamma factor and epsilon factor both factor into the product of GL_1 gamma and epsilon factors. Namely,

$$\gamma(\pi \otimes \nu) = \gamma(\chi_0\nu)\gamma(\chi_0^{-1}\nu)$$

and

$$\epsilon(1/2, \pi \otimes \nu, \psi) = \epsilon(1/2, \chi_0\nu, \psi)\epsilon(1/2, \chi_0^{-1}\nu, \psi).$$

To understand $\mathrm{c}(\pi \otimes \overline{\chi})$, let $\chi_\flat = \chi_0\chi^{-1}$ and $\chi_\sharp = \chi_0^{-1}\chi^{-1}$. Then $\mathbf{C}(\pi \otimes \overline{\chi})$ is the sum of the conductors of $\chi_\flat$ and $\chi_\sharp$, i.e.

$$\mathbf{C}(\pi \otimes \overline{\chi}) = c(\chi_\flat) + c(\chi_\sharp).$$

Definition 2.2.5. For π principal series we define $\alpha_{\pi,\chi} = \alpha_{\chi_\sharp}\alpha_{\chi_\flat} \in F$, where $\alpha_{\chi_\flat}$ and $\alpha_{\chi_\sharp}$ are defined analogously as Proposition 2.2.2.

2.2.4 GL_2 Hecke-Jacquet-Langlands Theory

First we address the global theory. Let $\mathbf{F}$ be the global field and $F = \mathbf{F}_v$ be the corresponding local field. Let φ be a cusp form on GL_2, i.e. $\varphi \in L^2_{\mathrm{cusp}}(\mathrm{GL}_2(\mathbf{F})\backslash\mathrm{GL}_2(\mathbf{A}))$. For χ a character of $\mathbf{F}^\times\backslash\mathbf{A}^\times$ we define the global zeta integral

$$Z(s, \varphi, \chi) = \int_{\mathbf{F}^\times\backslash\mathbf{A}^\times} \varphi(a(y))\chi(y)|y|^{s-1/2}\, d^\times y.$$

By the left-invariance of w, we have

$$\varphi(a(y)) = \varphi(wa(y)) = \varphi(wa(y)w^{-1}w) = \omega(y)\varphi(a(y^{-1})w),$$

where ω denotes the central character of π. Note that this relationship also forces φ to decay at 0, as well as at infinity (since it is a cusp form). Hence the global functional equation says that

$$Z(s, \varphi, \chi) = Z(1 - s, \pi(w)\varphi, \chi^{-1}\omega^{-1}).$$

From the global functional equation, we can deduce the local functional equation by using the factorization φ in its Whittaker model. We have

$$Z(s, \varphi, \chi) = \prod_v Z(s, W_{\varphi,v}, \chi_v),$$

where for $\mathrm{Re}(\chi_v) + \mathrm{Re}(s) > \frac{1}{2}$

$$Z(s, W_v, \chi_v) = \int_{\mathbf{F}_v^\times} W_v \begin{pmatrix} a & \\ & 1 \end{pmatrix} \chi_v(a)|a|^{s-1/2}\, d^\times a.$$

When v is unramified,

$$Z(s, W_v, \chi_v) = L(s, \pi_v \otimes \chi_v) = (1 - \mu_v\chi_v(\varpi)q_v^{-s})^{-1}(1 - \nu_v\chi_v(\varpi)q_v^{-s})^{-1},$$

where $\pi_v = \mathrm{Ind}_{\mathrm{B}(\mathrm{F_v})}^{\mathrm{G}(\mathrm{F_v})}(\mu_\mathrm{v}, \nu_\mathrm{v})$. The identity

$$Z(s, \varphi, \chi) = L(s, \pi \otimes \chi) \prod_v \frac{Z(s, W_v, \chi_v)}{L(s, \pi_v \otimes \chi_v)}$$

leads to

Proposition 2.2.6. Local Functional Equation (see [JL70, Thm 2.18]) We have

$$\frac{Z(s, W_v, \chi_v)}{L(s, \pi_v \otimes \chi_v)}\epsilon(s, \pi_v, \chi_v, \psi_v) = \frac{Z(1 - s, \pi(w)W_v, \omega_v^{-1}\chi_v^{-1})}{L(1 - s, \pi_v \otimes \omega_v^{-1}\chi_v^{-1})},$$

where $\epsilon(s, \pi_v, \chi_v, \psi_v)$ is of the form Ae^{Bs}. More succinctly, we may write

$$Z(s, W_v, \chi_v)\gamma(s, \pi_v \otimes \chi_v, \psi_v) = Z(1 - s, \pi(w)W_v, \omega_v^{-1}\chi_v^{-1}),$$

where $\gamma(s, \pi_v \otimes \chi_v, \psi_v) = \epsilon(s, \pi_v, \chi_v, \psi_v)\frac{L(1-s, \pi_v \otimes \omega_v^{-1}\chi_v^{-1})}{L(s, \pi_v \otimes \chi_v)}$.

Throughout, we will specialize the local functional equation to $s = 1/2$, reserving s for a variable name later. The local functional equation provides a way to evaluate $\pi(w)W(a(y))$ via Mellin inversion. Explicitly, (see [MV10]), we have an equality of the form

$$\pi(w)W \begin{pmatrix} y & \\ & 1 \end{pmatrix} = \int_{\widehat{F^\times}} \gamma(1/2, \pi \otimes \chi)\omega\chi(y) \int_{F^\times} W \begin{pmatrix} a & \\ & 1 \end{pmatrix} \chi(a)\, d^\times a\, d\chi,$$

where ω is the central character of π, which for us is trivial.

2.3 Test Functions and Op Calculus

In this section we define $\mathrm{Op}(a)$ for Schwartz functions a on $\mathrm{Lie}(G)^*$. For our class of functions a, we write a decomposition of a as $a = \sum a_\tau$ with each a_τ a bump function around $T\tau$. We then show for these functions we can write $\mathrm{Op}(a_\tau) = \mathrm{Op}^-(a_{\tau,+})\,\mathrm{Op}^0(a_{\tau,0})\,\mathrm{Op}^+(a_{\tau,-})$ in any order, where each $a_{\tau,\star}$ is a one-variable function on F.

2.3.1 Test Functions

Let $\mathfrak{g}$ be the Lie algebra of G with coordinates

$$\begin{pmatrix} p_x & p_y \\ p_z & -p_x \end{pmatrix} = p \in \mathfrak{g}, \tag{2.6}$$

and the dual Lie algebra $\mathfrak{g}^*$ with coordinates

$$\begin{pmatrix} \xi_x & \xi_z \\ \xi_y & -\xi_x \end{pmatrix} = \xi \in \mathfrak{g}^*. \tag{2.7}$$

For the Lie algebra, we will use the basis $e_0 = \begin{pmatrix} 1/2 & \\ & -1/2 \end{pmatrix}$, $e_+ = \begin{pmatrix} & 1 \\ & \end{pmatrix}$ and $e_- = \begin{pmatrix} & \\ 1 & \end{pmatrix}$. For a fixed unramified unitary character ψ of $\mathcal{O}_F$, there is a natural bilinear pairing $\mathfrak{g} \times \mathfrak{g}^* \to \mathbf{C}^*$ given by the trace pairing

$$\langle x, \xi \rangle = \psi(\mathrm{Tr}(x\xi)).$$

Note that the coordinates y and z in Equations 2.6 and 2.7 get switched when going from the Lie algebra to the dual Lie algebra or vice versa; this is due to the duality induced by the trace pairing defined above. In the archimedean case, one would construct operators that project onto *microlocalized* vectors, i.e. those satisfying

$$\pi(\exp x)v \approx e^{ix\tau/h}v, \tag{2.8}$$

for all $x \in \mathfrak{g}$ of size $|x| = O(h)$. When Equation 2.8 holds, we say v is microlocalized at τ. We think of h as an infinitesimal scaling parameter that goes to zero, so that we are picking up vectors with symmetry properties under a shrinking neighborhood of the identity. We will attempt to mimic this construction in the

non-archimedean case.

Remark 2.3.1. In other recent work involving microlocal analysis in non-archimedean setting, authors have worked with *minimal vectors* in the case of supercuspidal representations (c.f. [HN18]), and *microlocal lift vectors* in the case of principal series representations (c.f. [N18]). Here we opt to work with vectors that exactly match the notion of microlocalisation in Equation 2.8.

Let $k(N)$ be the set of matrices in $\mathfrak{g} \cap \mathfrak{p}^N \mathrm{Mat}_2(\mathcal{O})$. Let $K(N)$ be the image of $k(N)$ under the exponential map.

Definition 2.3.2. A non-archimedean microlocalized vector is a vector satisfying

$$\pi(\exp x)v = \langle -x, \tau \rangle v$$

for all $x \in k(N)$ and some $\tau \in \mathfrak{g}^\star$.

Remark 2.3.3. In comparison to the archimedean notion in Equation 2.8, we should think of h as $h = 1/T$ where $T = \varpi^{-N}$. Then as $N \to \infty$, $|T| \to \infty$, and microlocalization becomes an eigenvector property of an increasingly small neighborhood of the origin.

We want to find an a so that $\mathrm{Op}(a)$ projects onto non-archimedean microlocalized vectors. To that end, we pick $a = a_\tau$ so that its inverse Fourier transform is given by

$$a_\tau^\vee(x) = \mathrm{vol}(k(N))^{-1} 1_{k(N)}(x) \langle x, T\tau \rangle.$$

The phase $\langle x, T\tau \rangle$ is playing the role of the archimedean phase $e^{ix\xi/h}$ with $h = 1/T$. Note that a_τ is a T-dependent function, but we will often omit this dependence to avoid cumbersome notation.

These functions form the essential building blocks of the test functions we consider in this paper. The operators $\mathrm{Op}(a_\tau)$ have the property that their image consists of vectors that are microlocalized at τ in the sense of Definition 2.3.2. The main theorem applies to any linear combination of functions a_τ. We will sometimes refer to them as "non-archimedean wavepackets". If we think of them as being supported on matrices whose elements are in $T^{-1}\mathcal{O}$, then the dual region is given by

$$\{\xi \in \mathfrak{g}^* : |x \cdot \xi| \leq 1 \ \forall x \in T^{-1}k(0)\} =: Tk(0)^\perp.$$

Lemma 2. The Fourier transform $a_\tau(\xi)$ is a bump function on $T\tau + Tk(0)^\perp$.

Proof. We have

$$\begin{aligned} a_\tau(\xi) = \widehat{a_\tau^\vee}(\xi) &= \frac{1}{\mathrm{vol}(k(N))} \int_{x\in\mathfrak{g}} \psi(xT\tau) 1_{T^{-1}k(0)}(x)\psi(-x\xi)\,dx = \frac{1}{\mathrm{vol}(k(N))} \int_{x\in T^{-1}k(0)} \psi(x(T\tau-\xi))\,dx \\ &= \int_{u\in k(0)} \psi(u(\tau - T^{-1}\xi))\,du = 1_{k(0)^\perp}(\tau - T^{-1}\xi) \\ &= 1_{T(\tau+k(0)^\perp)}(\xi). \end{aligned}$$

As an abuse of notation, we will sometimes write this characteristic function as $1_{T(\tau+\mathcal{O})}$ or even more succinctly as just 1_τ^T. □

In other words, $a_\tau(\xi)$ is the characteristic function of the ball $B(T\tau, T)$ where

$$B(x,R) = \{y \in \mathfrak{g}^* : |y-x| \le R\}.$$

Then we can imagine tiling the space $\mathfrak{g}^*$ by translates of $Tk(0)^\perp$, one for each $\tau \in \mathfrak{g}^*/Tk(0)^\perp$. This gives rise to the following decomposition (c.f. [L23, Lemma 3.4])

Proposition 2.3.4 (Wavepacket Decomposition)**.** Let $T = \varpi^{-N}$ and suppose a is a Schwartz function on $\mathfrak{g}^*$ that is constant on cosets $Tk(0)^\perp$. Let $a_\tau = 1_\tau^T := 1_{T(\tau+\mathcal{O})}$ be the characteristic function of the ball $B(T\tau, T)$. Then we may write

$$a = \sum_{\tau\in\mathfrak{g}^*/Tk(0)^\perp} a(\tau) 1_\tau^T. \tag{2.9}$$

Each term $a1_\tau^T$ is Fourier supported on $T^{-1}k(0)^\perp$ and has constant size on each ball $B(T\tau, T)$.

The wavepacket decomposition gives a description of a everywhere on $\mathfrak{g}^*$, but in practice, the relative character will only "see" the $a1_\tau^T$ supported near a $T\tau$ that lives within $Tk(0)^\perp$ of the relative coadjoint orbit $\mathcal{O}_{\pi,\sigma}$.

2.3.2 Op Calculus

We can now define a class of operators

$$\mathrm{Op}(a) := \int_{x\in\mathfrak{g}} a^\vee(x)\pi(\exp x)\,dx.$$

Fix an element $\tau \in \mathfrak{g}^*$, once and for all. Then $a_\tau^\vee(x)$ will be the T-dependent function as before that oscillates at frequency $T\tau$ on $k(N)$ and $a_\tau(\xi)$ its Fourier transform which is the bump 1_τ^T on $T\tau$.

We have the Iwahori factorization $K(N) = K_- K_0 K_+$, where K_- is the negative unipotent, K_0 is the diagonal, and K_+ is the unipotent subgroup. We denote the preimages of $K_\star$ under the exponential map by $k_\star$ for $\star \in \{0, +, -\}$. They correspond to the decomposition in the Lie algebra $\mathfrak{g} = \mathfrak{g}_0 + \mathfrak{g}_+ + \mathfrak{g}_-$ where each space is given by

$$\mathfrak{g}_0 := \begin{pmatrix} * & \\ & -* \end{pmatrix}, \ \mathfrak{g}_+ := \begin{pmatrix} & * \\ & \end{pmatrix}, \ \mathfrak{g}_- := \begin{pmatrix} & \\ * & \end{pmatrix}.$$

The trace pairing induces dualities $\mathfrak{g}_0^* \cong \mathfrak{g}_0$, $\mathfrak{g}_\pm^* \cong \mathfrak{g}_\mp$. Each K_* is one-dimensional, so we can parametrize them by elements of F satisfying appropriate congruence conditions. This means we can decompose $\tau \in \mathfrak{g}^*$ by $\tau = \tau_0 + \tau_- + \tau_+$ where $\tau_0 \in k(N) \cap \mathfrak{g}_0$ and $\tau_\pm \in k(N) \cap \mathfrak{g}_\mp$. For $\star \in \{-, 0, +\}$, we define one-dimensional functions $a_{\tau_\star} \colon F \to \mathbf{C}$ so that

$$a_{\tau_\star}^\vee(x) = \frac{1}{\mathrm{vol}(k_\star)} \mathbb{1}_{T^{-1}\mathcal{O}}(x) \psi(xT\tau_\star).$$

Now we can define the operators $\mathrm{Op}^\star$ for $\star \in \{0, \pm\}$ and $a \colon F \to \mathbf{C}$ by

$$\mathrm{Op}^\pm(a) = \int_{x \in \mathfrak{g}} a^\vee(x) \pi(\exp(x e_\pm))\, dx$$

and

$$\mathrm{Op}^0(a) = \int_{x \in \mathfrak{g}} a^\vee(x) \pi(\exp(x e_0))\, dx$$

where $e_\star$ is the basis element in $k_\star$, for instance $e_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

On the functions $a_{\tau_\star}$ we have, for instance,

$$\begin{aligned} \mathrm{Op}^\pm(a_{\tau_\mp}) &= \int_{x \in k_\star} a_{\tau_\mp}^\vee(x) \pi(\exp(x e_\pm))\, dx \\ &= \int_{u \in \mathcal{O}} \psi(u\tau_\mp) \pi(\exp(T^{-1} u e_\pm))\, du, \end{aligned}$$

and similarly for $\mathrm{Op}^\pm(a_{\tau_0})$. Therefore, for $\dagger \in \{0, \pm\}$, function $a_{\tau_\dagger}$ has inverse Fourier transform that oscillates in the $\dagger$-direction at frequency $\tau_\dagger$, while $\mathrm{Op}^\star$ averages the action of π restricted to $k_\star$.

The next thing to understand is how these operators compose. We aim for something like

$$\mathrm{Op}(a)\,\mathrm{Op}(b) = \mathrm{Op}(ab),$$

so that the operators commute. It will turn out that for our choice of a and b, and for our purposes, we can essentially get this property. Before we state the proposition, we define a product

$$\star : k(n) \times k(n) \to k(n)$$

by requiring that

$$e^x e^y = e^{x \star y}.$$

The Baker-Campbell-Hausdorff formula then implies that $x \star y = x + y + \frac{1}{2}[x, y] + O(x^2 y, y^2 x)$.

Proposition 2.3.5. Let $\xi \in k(-m)$, that is each entry of ξ lies in $\varpi^{-m}\mathcal{O}$. If $N \geq \frac{m}{2}$, then for $x, y \in k(N)$,

$$\langle x, \xi\rangle\langle y, \xi\rangle = \langle (x \star y), \xi\rangle. \tag{2.10}$$

In other words, the map

$$\psi_\xi : K(N) \underset{\log}{\longrightarrow} k(N) \xrightarrow{\langle\ ,\ \xi\rangle} \mathbf{C}^\times, \tag{2.11}$$

is a character.

Proof. It suffices to prove that the difference

$$\{x, y\} = x \star y - x - y$$

pairs trivially with ξ. This is easy to verify. First note that $\{x, y\} = \frac{1}{2}[x, y] + O(x^2 y) + O(y^2 x)$. For $x, y \in k(N)$, we have $\{x, y\} \in k(2N)$. Therefore, every entry a_{ij} of the matrix $[x, y]$ satisfies $a_{ij} \in \mathfrak{p}^{2N}$. So

$$\langle \{x, y\}, \xi\rangle \equiv \langle [x, y], \xi\rangle = \psi(a_{11}\xi_x + a_{12}\xi_y + a_{21}\xi_z - a_{11}\xi_x) = 1$$

since each term $a_{ij}\xi_\star \in \mathcal{O}$ by the condition $2N \geq m$.

□

To see how this helps the Op-calculus, define

$$(a \star b)(\xi) = \int_{x,y} a^\vee(x) b^\vee(y) \langle x \star y, \xi \rangle \, dx \, dy.$$

Then by Fourier inversion

$$\begin{aligned} \mathrm{Op}(a)\,\mathrm{Op}(b) &= \int_{x,y} a^\vee(x) b^\vee(y) \pi(e^x) \pi(e^y) \, dx \, dy \\ &= \int_{x,y} a^\vee(x) b^\vee(y) \pi(e^{x \star y}) \, dx \, dy = \mathrm{Op}(a \star b). \end{aligned}$$

If $x \star y = x + y$, then this would be the standard convolution and we would get $a \star b = \widehat{(a^\vee * b^\vee)} = ab$, and hence $\mathrm{Op}(a \star b) = \mathrm{Op}(ab)$. It suffices to show that the phase $\psi(\cdot\{x, y\}) = 1$. At this point we will need to make a specific choice of a and b.

Proposition 2.3.6. Recall $T = \varpi^{-N}$ is our scaling parameter and let $a = a_\tau = 1_{T(\tau + \mathcal{O})}$ with $|T\tau| \leq q^{2N}$. Then

$$\mathrm{Op}(a_\tau) = \mathrm{Op}^-(a_{\tau_+})\,\mathrm{Op}^0(a_{\tau_0})\,\mathrm{Op}^+(a_{\tau_-}), \tag{2.12}$$

and moreover the operators on the right commute.

Proof. Recall we have the Iwahori factorization $K(N) = K_-(N) K_0(N) K_+(N)$. Let

$$\psi_\tau : K(N) \to \mathbf{C}^\times,$$

be the character given by $\psi_\tau(x) = \langle x, T\tau \rangle = \psi(\mathrm{trace}(xT\tau)$. We want to show that that the equation

$$\psi_\tau(k_- k_0 k_+) = \psi_\tau^-(k_-) \psi_\tau^0(k_0) \psi_\tau^+(k_+).$$

hold for specific choices of characters ψ_τ^-, ψ_τ^0, and ψ_τ^+ of K_-, K_0, and K_+ associated to the coordinates τ_-, τ_+, τ_0. This will imply Equation 2.12 because we can also factor $\pi(\exp k_- k_0 k_+)$ as $\pi(\exp k_-)\pi(\exp k_0)\pi(\exp k_+)$. By Proposition 2.3.5, the character ψ_τ is trivial on $K(2N)$ and moreover $[K(N), K(N)] \subset K(2N)$ so ψ_τ defines a character on $K(N)/K(2N)$. Therefore, it suffices to show that $\psi_\tau(k_- k_0 k_+)$ and $\psi_\tau^-(k_-)\psi_\tau^0(k_0)\psi_\tau^+(k_+)$

agree up to terms in $K(2N)$. To that end, consider the matrix product

$$\begin{pmatrix} 1 & \\ n & 1 \end{pmatrix} \begin{pmatrix} t & \\ & t^{-1} \end{pmatrix} \begin{pmatrix} 1 & u \\ & 1 \end{pmatrix} \in K_-(n)K_0(n)K_+(n).$$

Expanding we obtain

$$\begin{pmatrix} t & tu \\ nt & ntu + t^{-1} \end{pmatrix}.$$

We know that $t \equiv 1 \bmod \mathfrak{p}^N$ and $u, n \equiv 0 \bmod \mathfrak{p}^N$ so $tu \equiv u \bmod \mathfrak{p}^{2N}$ and $tn \equiv n \bmod \mathfrak{p}^{2N}$. Moreover $ntu + t^{-1} \equiv t^{-1} \bmod \mathfrak{p}^{2N}$. So this matrix product satisfies

$$\begin{pmatrix} 1 & \\ n & 1 \end{pmatrix} \begin{pmatrix} t & \\ & t^{-1} \end{pmatrix} \begin{pmatrix} 1 & u \\ & 1 \end{pmatrix} \equiv \begin{pmatrix} t & u \\ n & t^{-1} \end{pmatrix} \bmod \mathfrak{p}^{2N}.$$

It's easy to verify that every ordering of the matrix product above is congruent to $\begin{pmatrix} t & u \\ n & t^{-1} \end{pmatrix} \bmod \mathfrak{p}^{2N}$. Since ψ_τ is trivial on $K(2N)$, this concludes the proof of (2.12). Next we prove the commutativity. We prove it for Op^+ and Op^-, the other cases are identical. We have

$$\mathrm{Op}^+(a_{\tau_-})\,\mathrm{Op}^-(a_{\tau_+}) = \int_{x,y \in \varpi^N \mathcal{O}} \psi(xT\tau_-)\psi(xT\tau_+)\pi(\exp(ye_+ \star xe_-))\,dx\,dy.$$

We can write $ye_+ \star xe_- = ye_+ + xe_- + \{ye_+, xe_-\} = xe_- + ye_+ + \{ye_+, xe_-\}$ and make the change of variables $ze_+ = ye_+ + \{ye_+, xe_-\}$. But note that $ze_+ \equiv ye_+ \bmod k(2N)$, so we obtain

$$\begin{aligned} \mathrm{Op}^+(a_{\tau_-})\,\mathrm{Op}^-(a_{\tau_+}) &= \int_{x,z \in \varpi^N \mathcal{O}} \psi(xT\tau_-)\psi(zT\tau_+)\pi(\exp(xe_-))\pi(\exp(ze_+))\,dx\,dz \\ &= \mathrm{Op}^-(a_{\tau_+})\,\mathrm{Op}^+(a_{\tau_-}). \end{aligned}$$

□

2.4 Invariant Functionals and Relative Characters

Recall from Equation 2.1 the relative character for our case is given by

$$\mathcal{H}_\sigma(a) = \sum_{v \in \mathcal{B}(\pi)} \ell_\chi(\mathrm{Op}(a)v)\overline{\ell_\chi(v)},$$

for π unitary, where

$$\ell_\chi(v) = \int_H W_v \begin{pmatrix} h & \\ & 1 \end{pmatrix} \overline{\chi(h)}\, dh.$$

Here W_v is the image of v in the Whittaker model of π and σ is one dimensional so it is a character. One can think of the relative character as follows: Take the inner product $\langle g^{-1}\ell_\chi, \ell_\chi\rangle$. This is a distribution on G. One can expand the inner product formally, applying Parseval, to obtain the function on G

$$\sum_{v \in \mathcal{B}(\pi)} \langle \ell_\chi, gv\rangle\langle v, \ell_\chi\rangle.$$

Now to obtain a real number out of this, we integrate against g with an integral operator $\pi(f)$ or, equivalently, an operator $\mathrm{Op}(a)$. This gives the relative character.

In order to make use of this viewpoint, it will be beneficial to identify ℓ_χ with a vector. Such a vector does not exist, in general, but in fact we only need to find a vector that agrees with ℓ_χ on the image of $\mathrm{Op}(a_\tau)$, as that is all we will use the functional for. To this end, we define the following vector:

Definition 2.4.1. For $R \in q^{\mathbf{Z}}$, let $v_\chi^R \in \pi$ be a vector in the Kirillov model given by

$$v_\chi^R(h) = \chi(h)\mathbb{1}_{[-R,R]}(v(h)). \tag{2.13}$$

Proposition 2.4.2. Let v be in π. Then for χ a unitary character of $F^\times$, we have

$$\ell_\chi(v) = \lim_{R\to\infty} \langle v, v_\chi^R\rangle. \tag{2.14}$$

Proof. This follows from the behavior of the Whittaker function W_v. $W_v(y)$ behaves like $|y|^{1/2-\epsilon}$ as $|y| \to 0$ and $W_v(y)$ decays at infinity (see the discussion in [MV10, Sec 3.2]). Hence $\ell_\chi(v)$ is absolutely convergent and so we can write it as $\lim_{R\to\infty}\langle v, v_\chi^R\rangle$. □

2.5 Analysis of the Relative Character

We will now start the analysis of computing the image of $\mathrm{Op}(a_\tau)$ following the proof sketch in Section 2.1.1. We won't have to use that π is supercuspidal or principal series until the calculation of Op^- where an analysis of the $\mathrm{GL}_2 \times \mathrm{GL}_1$ functional equation will force a choice of representation π. The main result of this section is Proposition 2.5.8 which computes the relative character. We will prove Proposition 2.5.8 in Section 2.6.

2.5.1 The Image of $\mathrm{Op}(a_\tau)$

We now fix a τ for the rest of the paper and choose $a = a_\tau = 1_\tau^T = \mathbb{1}_{B(T\tau,T)}(\xi)$. As a short-hand, $a_\star$ will mean $a_{\tau_\star}$. We first start with $\mathrm{Op}^0(a_0)v_\chi^R$. For any vector v, we have

$$\begin{aligned}
\ell_\chi(\mathrm{Op}^0(a_0)v) &= \int_{h\in H} \left(\mathrm{Op}^0(a)W_v\right) \begin{pmatrix} h & \\ & 1 \end{pmatrix} \overline{\chi}(h)\, d^\times h \\
&= \int_{h\in F^\times} \int_{u\in\mathcal{O}} \psi(u\tau_x) \left(\pi\left(\exp(T^{-1}ue_0)\right) W_v\right) \begin{pmatrix} h & \\ & 1 \end{pmatrix} \overline{\chi}(h)\, du\, d^\times h \\
&= \int_{h\in F^\times} \overline{\chi}(h) \int_{u\in\mathcal{O}} \psi(u\tau_x) W_v \begin{pmatrix} h\exp(T^{-1}u) & \\ & 1 \end{pmatrix} d^\times h\, du \\
&= \int_{u\in\mathcal{O}} \psi(u\tau_x)\chi(\exp(T^{-1}u))\, du \int_{h\in F^\times} W_v \begin{pmatrix} y & \\ & 1 \end{pmatrix} \overline{\chi}(y)\, d^\times h \\
&= \left(\int_{u\in\mathcal{O}} \psi(u\tau_x)\chi(\exp(T^{-1}u))\, du\right) \ell_\chi(v).
\end{aligned}$$

where e_0 is the basis element $\begin{pmatrix} 1/2 & \\ & -1/2 \end{pmatrix}$. It remains to evaluate the integral. Take N large enough so that $2N \geq c(\chi)$, as in the theorem statement. Since q is odd, $\chi(\exp x) = \chi(1+x)$ for $x \in \mathfrak{p}^{c(\chi)/2}$. Therefore by Proposition 2.2.2 we can write

$$\begin{aligned}
\ell_\chi(\mathrm{Op}^0(a)v) &= \left(\int_{u\in\mathcal{O}} \psi(u(\tau_x + \alpha_\chi T^{-1}))\, du\right) \ell_\chi(v) = 1_\mathcal{O}(\tau_x + T^{-1}\alpha_\chi)\ell_\chi(v) \\
&= a_0(\alpha_\chi)\ell_\chi(v)
\end{aligned}$$

We now see what the H-direction operator $\mathrm{Op}^0(a_0)$ is really doing. It wants to project onto vectors that are microlocalized at elements in the dual Lie algebra whose rescaled x-coordinate is near the pre-image under the map $\mathrm{Lie}(G)^* \to \mathrm{Lie}(H)^*$ of the conductor f of χ. In other words, it's detecting the 'slice'

$\mathcal{O}_\pi \cap \big(\mathrm{c}(\chi) + \mathrm{Lie}(H)^\perp\big)$.

We next calculate $\mathrm{Op}^+(a_-)v_\chi^R$. For any vector W_v in the Kirillov model, we have

$$\begin{aligned}(\mathrm{Op}^+(a_-)W_v)(h) &= \int_{u\in\mathcal{O}} \psi(u\tau_y)\pi(\exp(T^{-1}ue_+)W_v)\begin{pmatrix} h & \\ & 1\end{pmatrix} du \\ &= \left(\int_{u\in\mathcal{O}} \psi(u(\tau_y + T^{-1}h))\, du\right) W_v \begin{pmatrix} h & \\ & 1\end{pmatrix}\end{aligned}$$

The exact behavior of the integral depends on whether $\tau_y \in \mathcal{O}$, but the general effect is to restrict the support of h. Indeed the dichotomy is as follows:

$$\int_{u\in\mathcal{O}} \psi(u(\tau_y + \varpi^N h))\, du = \begin{cases} 1_{v(h)\geq -N}(h) & \text{if } \tau_y \in \mathcal{O} \\ 1_{\varpi^{-N-r}(\mathfrak{p}^r + \varpi^r\tau_y)}(h) & \text{if } \tau_y \in \varpi^{-r}\mathcal{O}^\times,\ r>0.\end{cases}$$

Therefore

$$\mathrm{Op}^+(a_-)v_\chi^R = \begin{cases} v_\chi^R(h)1_{v(h)\geq -N} & \text{if } \tau_y \in \mathcal{O} \\ v_\chi^R(h)1_{\varpi^{-N-r}(\mathfrak{p}^r + \varpi^r\tau_y)}(h) & \text{if } \tau_y \in \varpi^{-r}\mathcal{O}^\times,\ r>0.\end{cases}$$

To understand this better, first we'd like to make sense of the characteristic function $\mathbb{1}_{\varpi^{-N-r}(\mathfrak{p}^r + \varpi^r\tau_y)}(h)$. Notice that

$$\mathfrak{p}^r + \varpi^r\tau_y \subset \mathcal{O}^\times,$$

the characteristic function $\mathbb{1}_{\varpi^{-N-r}(\mathfrak{p}^r + \varpi^r\tau_y)}(h)$ defines a character of $\mathcal{O}^\times$.

Lemma 3. Let $u_0 \in \mathcal{O}^\times$ and $\mathcal{X}_m$ denote the group of characters of conductor at most m. Then for $u \in \mathcal{O}^\times$,

$$1_{u_0+\mathfrak{p}^m}(u) = \frac{1}{|\mathcal{X}_m|}\sum_{\omega\in\mathcal{X}_m} \omega^{-1}(u_0)\omega(u) =: \mathbb{E}_{\omega\in\mathcal{X}_m}\, \omega^{-1}(u_0)\omega(u) \tag{2.15}$$

Proof. The proof follows from Fourier analysis on the finite group $(\mathcal{O}/\mathfrak{p}^m)^\times$. □

With this, we can update our calculation of $\mathrm{Op}^+(a_+)$ to read

$$\mathrm{Op}^+(a_-)v_\chi^R = \begin{cases} v_\chi^R(h)1_{v(h)\geq -N} & \text{if } \tau_y \in \mathcal{O} \\ v_\chi^R(h)1_{v(h)=-N-r}\,\mathbb{E}_{\omega\in\mathcal{X}_r}\,\omega^{-1}(\varpi^r\tau_y)\omega(\varpi^{N+r}h) & \text{if } \tau_y \in \varpi^{-r}\mathcal{O}^\times,\ r>0. \end{cases}$$

Note that both $\varpi^r\tau_y$ and $\varpi^{N+r}h$ are units.

2.5.2 Negative Unipotent & Functional Equations

We next need to compute the contribution from the negative unipotent direction, that is, we need to understand $\mathrm{Op}^-(a)v$. First we record a simple lemma.

Lemma 4. We have

$$\ell_\chi(\pi(w)W_v) = \gamma(\pi\otimes\chi)\ell_{\chi^{-1}}(W_v). \tag{2.16}$$

Proof. This is just a reformulation of the $\mathrm{GL}_2\times\mathrm{GL}_1$ local functional equation (see Prop 2.2.6). Indeed by the local functional equation, the LHS is

$$\int_{h\in F^\times} W_v\left(\begin{pmatrix} h & \\ & 1\end{pmatrix} w\right)\chi^{-1}(h)\,d^\times h = \gamma(\pi\otimes\chi)\int_{h\in F^\times} W_v\begin{pmatrix} h & \\ & 1\end{pmatrix}\chi(h)\,d^\times h.$$

□

To compute the negative unipotent case, we will work with the functionals ℓ_χ until it is more convenient to switch to the vectors v_χ^R. For instance, we can start on the functional side by writing for any $v\in\pi$ with W_v in the Kirillov model of π,

$$\begin{aligned}\ell_\chi(\mathrm{Op}^-(a_+)v) &= \ell_\chi\left(\int_{u\in\mathcal{O}}\psi(u\tau_z)(\pi(\exp(T^{-1}ue_-))W_v)\,du\right)\\ &= \ell_\chi\left(\int_{u\in\mathcal{O}}\psi(u\tau_z)(\pi(w)\pi(\exp(T^{-1}ue_+))\pi(w)W_v)\,du\right)\\ &= \gamma(\pi\otimes\chi)\ell_{\overline{\chi}}\left(\int_{u\in\mathcal{O}}\psi(u\tau_z)\pi(\exp(T^{-1}ue_+))\pi(w)W_v\right).\end{aligned}$$

We should think of the integral inside as doing approximately the same thing as Op^+, but on the vector $\pi(w)W_v$ instead of W_v. However, the resulting dichotomy will depend on whether $\tau_z\in\mathcal{O}$ rather than τ_y.

Explicitly,

$$\ell_{\overline{\chi}}\left(\int_{u\in\mathcal{O}}\psi(u\tau_z)\pi(\exp(T^{-1}ue_+))\pi(w)W_v\right)=\int_{h\in H}\int_{u\in\mathcal{O}}\psi(u(\tau_z+T^{-1}h))\,du(\pi(w)W_v)(h)\chi(h)\,d^\times h.$$

Therefore for $\tau_z\in\mathcal{O}$,

$$\ell_\chi(\mathrm{Op}^-(a_+)v)=\gamma(\pi\otimes\chi)\ell_{\overline{\chi}}\left((\pi(w)v)1_{v(\cdot)\geq -N}\right).$$

When $\tau_z\notin\mathcal{O}$, the resulting analysis is quite complicated. Therefore, we start with the $\tau_z\in\mathcal{O}$ case, which conveys the central ideas. Since there is a restriction to the support of $\pi(w)v$, we cannot straightforwardly apply the local functional equation again. We need to understand the action of the Weyl element explicitly. To that end, we switch to the vector viewpoint. Namely,

$$\begin{aligned}\ell_\chi\left(\mathrm{Op}^-(a_+)W_v\right)&=\gamma(\pi\otimes\chi)\ell_{\overline{\chi}}\left(\pi(w)(W_v1_{v(\cdot)\geq -N})\right) &(2.17)\\ &=\gamma(\pi\otimes\chi)\lim_{R\to\infty}\langle\pi(w)(W_v1_{v(\cdot)\geq -N}),v_\chi^R\rangle &(2.18)\\ &=\gamma(\pi\otimes\chi)\lim_{R\to\infty}\langle W_v,\pi(w)(v_\chi^R1_{v(\cdot)\geq -N})\rangle. &(2.19)\end{aligned}$$

The next two sections will be spent computing $\pi(w)(v_\chi^R1_{v(\cdot)\geq -N})$ in the Kirillov model in the supercuspidal and principal series case.

2.5.3 Supercuspidal Case

To calculate Equation 2.19 we need to understand $\pi(w)(v_\chi^R1_{v(\cdot)\geq -N})$ in the Kirillov model. To do that, we will first calculate $\pi(w)(v_\chi^R1_{\varpi^m\mathcal{O}^\times})$ and sum the result over a range. We now need to use that we are working with a specific representation, and will carry out the resulting analysis separately for the supercuspidal case and the principal series case. Recall that under the Local Langlands Correspondence there is a representation $\rho=\mathrm{Ind}_{E/F}\xi:=\mathrm{Ind}_{\mathcal{W}_E}^{\mathcal{W}_F}\xi$ corresponding to π. This correspondence is compatible with gamma and epsilon factors. For any character η of $F^\times$ or ψ of F, let $\eta_E:=\eta\circ\mathrm{Nm}_{E/F}$ and $\psi_E:=\psi\circ\mathrm{Tr}_{E/F}$. Let $f:=f(E|F)$ be the inertial degree.

Proposition 2.5.1. In the Kirillov model, for R sufficiently large we have

$$\pi(w)(v_\chi^R1_{\varpi^m\mathcal{O}^\times})(y)=\epsilon(1/2,\xi\chi_E^{-1},\psi_E)v_{\chi^{-1}}^R(y)1_{v(y)=-m-fc(\xi\chi_E^{-1})}. \tag{2.20}$$

Corollary 2.5.2. Then

$$\pi(w)(v_\chi^R 1_{v(\cdot)\geq -N}) = \epsilon(1/2, \xi\chi_E^{-1}, \psi_E) v_{\chi^{-1}}^R 1_{v(\cdot)\leq N - fc(\xi\chi_E^{-1})} \tag{2.21}$$

Proof of Proposition 2.5.1. First, we note that for R sufficiently large, $v_\chi^R(h)1_{\varpi^m\mathcal{O}^\times}(h) = \chi(h)1_{\varpi^m\mathcal{O}^\times}$. Therefore, by Mellin inversion,

$$\begin{aligned}
\pi(w)(v_\chi^R 1_{\varpi^m\mathcal{O}^\times})(y) &= \int_{\widehat{F^\times}} \gamma(\pi\otimes\nu)\nu(y) \int_{F^\times} \chi(h)1_{\varpi^m\mathcal{O}^\times}(h)\nu(h)\, d^\times h\, d\nu \\
&= \int_{\widehat{F^\times}} \gamma(\pi\otimes\nu)\nu(y)\nu(\varpi^m)\chi(\varpi^m) \int_{u\in\mathcal{O}^\times} \chi(u)\nu(u)\, d^\times u\, d\nu \\
&= \int_{\widehat{F^\times}} \gamma(\pi\otimes\nu)\nu(\varpi^m y)\chi(\varpi^m) 1_{\chi=\nu^{-1} \text{ on } \mathcal{O}^\times}.
\end{aligned}$$

The last step is given by orthogonality of characters: χ must agree with ν^{-1} on $\mathcal{O}^\times$, but if this is the case, the product $\nu\chi$ is unramified, so we may write it as $\nu\chi = |\cdot|^s$ for some $s \in \mathbf{C}$. If $x \in F^\times$ then $\nu\chi(x) = z^{\mathrm{val}(x)}$. With this change of variables we can write

$$\pi(w)(v_\chi^R(y)1_{v(y)=m}) = \frac{\log q}{2\pi i} \int_{|z|=1} \gamma(\pi\otimes\nu)\overline{\chi(y)} z^{v(y)+m} \frac{dz}{z}.$$

Since π is supercuspidal, the L-factors are equal to 1, so $\gamma(\pi\otimes\nu) = \epsilon(\pi\otimes\nu)$. Then

$$\epsilon(1/2, \pi\otimes\nu, \psi) = \epsilon(1/2, \xi\nu_E, \psi_E).$$

Since $\chi\nu$ is unramified, $\chi_E\nu_E$ is unramified, and for $x = \varpi^k u \in F^\times$, $\chi_E\nu_E(x) = z^{fk}$, where f is the inertial degree $f(E|F)$. So

$$\epsilon(1/2, \xi\nu_E, \psi_E) = \epsilon(1/2, \xi\chi_E^{-1}\chi_E\nu_E, \psi_E) = \epsilon(1/2, \xi\chi_E^{-1}, \psi_E) z^{fc(\xi\chi_E^{-1})}.$$

Therefore,

$$\pi(w)(v_\chi^R(y)1_{v(y)=m}) = \epsilon(1/2, \xi\chi_E^{-1}, \psi_E) 1_{v(y)=-m-fc(\xi\chi_E^{-1})}.$$

□

When $\tau_z \in \mathcal{O}$, we can write

$$\begin{aligned}\ell_\chi(\mathrm{Op}^-(a_+)W_v) &= \gamma(\pi\otimes\chi)\lim_{R\to\infty}\langle W_v, \pi(w)(v_\chi^R 1_{v(\cdot)\geq -N})\rangle \\ &= \epsilon(1/2,\xi\chi_E,\psi_E)\lim_{R\to\infty}\langle W_v, \epsilon(1/2,\xi\chi_E^{-1},\psi_E)v_{\chi^{-1}}^R(y)1_{v(y)\leq N-fc(\xi\chi_E^{-1})}\rangle \\ &= \epsilon(\xi\chi_E)\overline{\epsilon(\xi\chi_E^{-1})}\ell_{\overline{\chi}}\left(W_v 1_{v(\cdot)\leq N-fc\left(\xi\chi_E^{-1}\right)}\right).\end{aligned}$$

This completes the case where $\tau_z \in \mathcal{O}$. The case where $\tau_z \notin \mathcal{O}$ is not conceptually more difficult, only notationally cumbersome. Recall that we are calculating

$$\ell_\chi(\mathrm{Op}^-(a_+)W_v) = \gamma(\pi\otimes\chi)\lim_{R\to\infty}\langle W_v, \pi(w)(v_\chi^R 1_{v(\cdot)=-N-s}\,\mathbb{E}_{\omega\in\mathcal{X}_s}\,\omega^{-1}(\varpi^s\tau_z)\omega(\varpi^{N+s})\omega)\rangle$$

for $\tau_z \in \varpi^{-s}\mathcal{O}^\times$ with s positive. To do this, we again use Mellin inversion.

Proposition 2.5.3. In the Kirillov model, for R sufficiently large, we have

$$\mathbb{E}_{\omega\in\mathcal{X}_s}\,\pi(w)(v_\chi^R 1_{v(\cdot)=-N-s}\omega)\omega^{-1}(\varpi^s\tau_z)\omega(\varpi^{N+s}) \tag{2.22}$$

is equal to

$$v_{\chi^{-1}}^R 1_{v(\cdot)=N+s-fc(\xi\chi_E^{-1}\omega_E^{-1})}\,\mathbb{E}_{\omega\in\mathcal{X}_s}\,\epsilon(\xi\chi_E^{-1}\omega_E^{-1})\omega^{-1}(\tau_z)\omega(\varpi^N)\omega^{-1}. \tag{2.23}$$

Proof. As in the first proof, we first remark that for R large, $v_\chi^R(h)1_{\varpi^m\mathcal{O}^\times}(h) = \chi(h)1_{\varpi^m\mathcal{O}^\times}$ for any fixed m. By Mellin inversion,

$$\begin{aligned}&\pi(w)(v_\chi^R 1_{v(\cdot)=-N-s}\,\mathbb{E}_{\omega\in\mathcal{X}_s}\,\omega^{-1}(\varpi^s\tau_z)\omega(\varpi^{N+s})\omega) \\ &= \mathbb{E}_{\omega\in\mathcal{X}_s}\,\omega^{-1}(\varpi^s\tau_z)\omega(\varpi^{N+s})\int_{\widehat{F^\times}}\gamma(\pi\otimes\nu)\nu(y)\int_{F^\times} v_\chi^R(h)1_{v(h)=-N-s}\omega(h)\nu(h)\,d^\times h\,d\nu \\ &= \mathbb{E}_{\omega\in\mathcal{X}_s}\,\omega^{-1}(\tau_z)\omega(\varpi^N)\int_{\widehat{F^\times}}\gamma(\pi\otimes\nu)\nu(y)\int_{\mathcal{O}^\times}\chi(\varpi^{-N-s}u)\omega(\varpi^{-N-s}u)\nu(\varpi^{-N-s}u)\,d^\times u\,d\nu \\ &= \mathbb{E}_{\omega\in\mathcal{X}_s}\,\omega^{-1}(\tau_z)\omega(\varpi^N)\int_{\widehat{F^\times}}\gamma(\pi\otimes\nu)\nu(y)\chi\omega\nu(\varpi^{-N-s})1_{\nu=\chi^{-1}\omega^{-1}\text{ on }\mathcal{O}^\times}\,d\nu.\end{aligned}$$

Once again, $\nu\mid_{\mathcal{O}^\times} = \chi^{-1}\omega^{-1}$ implies that $\nu\chi\omega = |\cdot|^s$ for some $s \in \mathbf{C}$. So for $x = \varpi^k u \in F$, we can write $\nu\chi\omega(x) = q^{-ks} := z^k$ so that $z = q^{-s}$. So we can write this all as

$$\mathbb{E}_{\omega\in\mathcal{X}_s}\,\omega^{-1}(\tau_z)\omega(\varpi^N)\frac{\log q}{2\pi i}\int_{|z|=1}\gamma(\pi\otimes\nu)\chi^{-1}(y)\omega^{-1}(y)z^{v(y)-N-s}\frac{dz}{z}.$$

As before, the gamma factor can be simplified to

$$\gamma(\pi \otimes \nu) = \epsilon(1/2, \xi\nu_E, \psi_E) = \epsilon(1/2, \xi\chi_E^{-1}\omega_E^{-1}\chi_E\omega_E\nu_E, \psi_E)$$
$$= \epsilon(1/2, \xi\chi_E^{-1}\omega_E^{-1}, \psi_E) z^{fc(\xi\chi_E^{-1}\omega_E^{-1})}$$

where $f := f(E|F)$ is the inertial degree. So our expression becomes

$$\mathbb{E}_{\omega \in \mathcal{X}_s} \omega^{-1}(\tau_z)\omega(\varpi^N)\frac{\log q}{2\pi i}\int_{|z|=1} \epsilon(\xi\chi_E^{-1}\omega_E^{-1})\chi^{-1}(y)\omega^{-1}(y) z^{v(y)-N-s+fc(\xi\chi_E^{-1}\omega_E^{-1})}\frac{dz}{z}$$
$$= \mathbb{E}_{\omega \in \mathcal{X}_s} \epsilon(\xi\chi_E^{-1}\omega_E^{-1})\omega^{-1}(\tau_z)\omega(\varpi^N)\chi^{-1}(y)\omega^{-1}(y) 1_{v(y)=N+s-fc(\xi\chi_E^{-1}\omega_E^{-1})}.$$

For R large, $v^R_{\chi^{-1}} 1_{v(y)=N+s-fc(\xi\chi_E^{-1}\omega_E^{-1})}$ agrees with $\chi^{-1}(y)1_{v(y)=N+s-fc(\xi\chi_E^{-1}\omega_E^{-1})}$, which yields the result. □

2.5.4 Principal Series Case

As a reminder, in analogy to our work in the supercuspidal case our aim is to compute $\pi(w)(v^R_\chi 1_{v(\cdot)\geq -N})$ in the Kirillov model. To do this, we will compute $\pi(w)(v^R_\chi 1_{v(\cdot)=m})$ and then sum the result over $m \geq -N$. Recall that $\pi = \chi_0 \boxplus \chi_0^{-1}$, where χ_0 is some ramified character of $F^\times$. We will assume that neither $\chi_0\chi^{-1}$ nor $\chi_0^{-1}\chi^{-1}$ is unramified, in which case we say (χ_0, χ^{-1}) is a generic pair. We will denote the characters $\chi_0\chi^{-1}$ and $\chi_0^{-1}\chi^{-1}$ by $\chi_\flat$ and $\chi_\sharp$ with conductors $f_\flat$ and $f_\sharp$.

Proposition 2.5.4. Let (χ_0, χ^{-1}) be a generic pair. Then in the Kirillov model for R sufficiently large we have,

$$\pi(w)(v^R_\chi(y)(y)1_{v(y)=m}) = \epsilon(1/2, \chi_\flat)\epsilon(1/2, \chi_\sharp) v^R_{\chi^{-1}}(y) 1_{v(y)=-m-f_\sharp-f_\flat}.$$

Corollary 2.5.5. When $\tau_z \in \mathcal{O}$, then we must have

$$\pi(w)(v^R_\chi(y)1_{v(y)\geq -N}) = \epsilon(1/2, \chi_\flat)\epsilon(1/2, \chi_\sharp) v^R_{\chi^{-1}}(y) 1_{v(y)\leq N-f_\sharp-f_\flat}.$$

Proof of Proposition 2.5.4. First note that for R sufficiently large, $v^R_\chi(h)1_{\varpi^m\mathcal{O}^\times}(h) = \chi(h)1_{\varpi^m\mathcal{O}^\times}$. By

Mellin inversion,

$$\begin{aligned}\pi(w)(v_\chi^R 1_{\varpi^m\mathcal{O}^\times})(y) &= \int_{\widehat{F^\times}} \gamma(\pi\otimes\nu)\nu(y)\int_{F^\times} v_\chi^R 1_{v(h)=m}\nu(h)\,d^\times h\,d\nu \\ &= \int_{\widehat{F^\times}} \gamma(\pi\otimes\nu)\nu(y)\int_{\varpi^m\mathcal{O}^\times} \chi(h)\nu(h)\,d^\times h\,d\nu \\ &= \int_{\widehat{F^\times}} \gamma(\pi\otimes\nu)\nu(y)\nu(\varpi^m)\chi(\varpi^m)\int_{u\in\mathcal{O}^\times} \chi(u)\nu(u)\,d^\times u\,d\nu \\ &= \int_{\widehat{F^\times}} \gamma(\pi\otimes\nu)\nu(y)\chi(\varpi^m)\nu(\varpi^m)1_{\chi=\nu_{-1}\text{ on }\mathcal{O}^\times}.\end{aligned}$$

Note that if $\nu\mid_{\mathcal{O}^\times}=\chi^{-1}\mid_{\mathcal{O}^\times}$ then $\nu\chi$ is unramified, which implies $\nu\chi=|\cdot|^s$ for some $s\in\mathbf{C}$. If $x\in F^\times$ then $\nu\chi(x)=z^{\mathrm{val}(x)}$. So we can write this as

$$(\pi(w)v_\chi^R)(y)1_{v(y)=m}=\frac{\log q}{2\pi i}\int_{|z|=1}\gamma(\pi\otimes\nu)\overline{\chi}(y)z^{v(y)+m}\frac{dz}{z}.$$

Next we use the factorization of the gamma factor as

$$\gamma(1/2,\pi\otimes\nu)=\epsilon(1/2,\pi\otimes\nu)\frac{L(1/2,\tilde{\pi}\otimes\nu^{-1})}{L(1/2,\pi\otimes\nu)}=\epsilon(1/2,\chi_0\nu)\epsilon(1/2,\chi_0^{-1}\nu)\frac{L(1/2,\chi_0^{-1}\nu^{-1})L(1/2,\chi_0\nu^{-1})}{L(1/2,\chi_0\nu)L(1/2,\chi_0^{-1}\nu)}.$$

Recall $\chi_\flat=\chi_0\chi^{-1}$ and $\chi_\sharp=\chi_0^{-1}\chi^{-1}$ with conductor exponents equal to $f_\flat$ and $f_\sharp$ respectively. Then we can write the epsilon factors as as $\epsilon(1/2,\chi_\flat z^{\mathrm{val}})=\epsilon(1/2,\chi_\flat)z^{f_\flat}$ and $\epsilon(1/2,\chi_\sharp z^{\mathrm{val}})=\epsilon(1/2,\chi_\sharp)z^{f_\sharp}$.

When (χ_0,χ) is a generic pair, the L-factors are all ramified and equal to 1. So the gamma factor is

$$\gamma(1/2,\pi\otimes\nu)=\epsilon(1/2,\chi_\flat)\epsilon(1/2,\chi_\sharp)z^{f_\flat+f_\sharp}. \tag{2.24}$$

Then we can calculate as follows:

$$\pi(w)(v_\chi^R(y)1_{v(y)=m})=\epsilon(1/2,\chi_\flat)\epsilon(1/2,\chi_\sharp)\overline{\chi}(y)\frac{\log q}{2\pi i}\int_{|z|=1}z^{f_\flat+f_\sharp+m+v(y)}\frac{dz}{z} \tag{2.25}$$

$$=\epsilon(1/2,\chi_\flat)\epsilon(1/2,\chi_\sharp)\overline{\chi}(y)1_{v(y)=-m-f_\sharp-f_\flat}. \tag{2.26}$$

□

Therefore, when $\tau_z\in\mathcal{O}$, we can write

$$\begin{aligned}\ell_\chi(\mathrm{Op}^-(a_+)W_v) &= \gamma(\pi\otimes\chi)\lim_{R\to\infty}\langle W_v,\pi(w)(v_\chi^R 1_{v(\cdot)\geq -N})\rangle \\ &= \gamma(\pi\otimes\chi)\overline{\epsilon(1/2,\chi_\flat)\epsilon(1/2,\chi_\sharp)}\ell_{\chi^{-1}}(W_v 1_{v(\cdot)\leq N-f_\sharp-f_\flat}).\end{aligned}$$

This completes the case where $\tau_z \in \mathcal{O}$. Next we need to understand what happens when $\tau_z \notin \mathcal{O}$.

Proposition 2.5.6. Let $\tau_z \in \varpi^{-s}\mathcal{O}^\times$ be such that $s < \min(f_\sharp, f_\flat)$. Then in the Kirillov model, for R sufficiently large, we have

$$\pi(w)(v_\chi^R 1_{v(\cdot)=-N-s}\mathbb{E}_{\omega\in\mathcal{X}_s}\omega^{-1}(\varpi^s\tau_z)\omega(\varpi^{N+s})\omega) \tag{2.27}$$

is equal to

$$v_{\chi^{-1}}^R 1_{v(\cdot)=N+s-f_\flat^\omega-f_\sharp^\omega}\mathbb{E}_{\omega\in\mathcal{X}_s}\omega^{-1}(\varpi^s\tau_z)\omega^{-1}\epsilon_{f_\flat^\omega}\epsilon_{f_\sharp^\omega}. \tag{2.28}$$

Proof. We have

$$\begin{aligned}
&\pi(w)(v_\chi^R(y)\mathbb{E}_{c(\omega)\le s}1_{v(y)=-N-s}\overline{\omega}(\varpi^s\tau_z)\omega(y))\\
&= \mathbb{E}_{c(\omega)\le s}\overline{\omega(\varpi^s\tau_z)}\int_{\widehat{F^\times}}\gamma(\pi\otimes\nu)\nu(y)\int_{F^\times}v_\chi^R(u)\omega(u)1_{v(u)=-N-s}\nu(u)\,d^\times u\,d\nu\\
&= \mathbb{E}_{c(\omega)\le s}\overline{\omega(\varpi^s\tau_z)}\frac{\log|q|}{2\pi i}\int_{|z|=1}\gamma(\pi\otimes\omega^{-1}\chi^{-1}|\cdot|^{s'})\omega^{-1}(y)\chi^{-1}(y)z^{v(y)-N-s}\frac{dz}{z}\\
&= \mathbb{E}_{c(\omega)\le s}\overline{\omega(\varpi^s\tau_z)}\epsilon(\chi_0\omega^{-1}\chi^{-1})\epsilon(\chi_0^{-1}\omega^{-1}\chi^{-1})\omega^{-1}\chi^{-1}(y)\frac{\log|q|}{2\pi i}\int_{|z|=1}z^{f_\flat^\omega+f_\sharp^\omega+v(y)-N-s}\frac{dz}{z}\\
&= \mathbb{E}_{c(\omega)\le s}\omega^{-1}(\varpi^s\tau_z)\chi^{-1}(y)\omega^{-1}(y)\epsilon(\chi_\flat\omega^{-1})\epsilon(\chi_\sharp\omega^{-1})1_{v(y)=N+s-f_\flat^\omega-f_\sharp^\omega},
\end{aligned}$$

where $\epsilon_{f_\flat^\omega}$ and $\epsilon_{f_\sharp^\omega}$ are the epsilon factors of the characters $\chi_\flat\omega$ and $\chi_\sharp\omega$. □

Remark 2.5.7. In the above proof, the crucial assumption, $s < \min(f_\sharp, f_\flat)$ ensures that twists by ω don't introduce L-factors so that we can write the gamma factor as

$$\gamma(1/2, \pi\otimes\nu) = \epsilon(1/2, \chi_\flat\omega^{-1})\epsilon(1/2, \chi_\sharp\omega^{-1})z^{f_\flat^\omega+f_\sharp^\omega}, \tag{2.29}$$

where $f_\flat^\omega$ and $f_\sharp^\omega$ are the conductors of $\chi_\flat\omega^{-1}$ and $\chi_\sharp\omega^{-1}$ respectively. This assumption should be automatically satisfied in the relevant ranges since the conductor of ω is small compared to the conductors of $\chi_\flat$ and $\chi_\sharp$.

2.5.5 Main Proposition

Recall, once again, that the strategy to calculate the relative character is to factor it using the Iwahori factorization as

$$\langle \mathrm{Op}(a)v_\chi^R, v_\chi^R\rangle = \langle \mathrm{Op}^+(a_+)\pi(w)v_\chi^R, \pi(w)\,\mathrm{Op}^0(a_0)\,\mathrm{Op}^+(a_-)v_\chi^R\rangle,$$

and compute this as $R \to \infty$.

By the local functional equation, this can be written as

$$\gamma(\pi\otimes\overline{\chi})a_0(\alpha_\chi)\langle \mathrm{Op}^+(a_+)v_{\chi^{-1}}^R, \pi(w)\,\mathrm{Op}^+(a_-)v_\chi^R\rangle.$$

We then case on the positions of τ_y and τ_z. Recall when $\tau_z \notin \mathcal{O}$, we write $\tau_z \in \varpi^{-s}\mathcal{O}^\times$ with $s>0$ and when $\tau_z \in \mathcal{O}$ we take $s=0$, and similarly for τ_y and r. Recall the characters $\chi_\sharp = \chi_0^{-1}\chi^{-1}$ and $\chi_\flat = \chi_0\chi^{-1}$. We have the following proposition:

Proposition 2.5.8. Keep the assumptions of Theorem 2.1.1. Then the value of the inner product $\gamma(\pi\otimes\chi^{-1})\langle \mathrm{Op}^+(a_{\tau_+})v_{\chi^{-1}}^R, \pi(w)\,\mathrm{Op}^+(a_{\tau_-})v_\chi^R\rangle$ is given by the following table: where $c(\pi,\chi) = f_\sharp + f_\flat$ in the

	τ_y	
τ_z	$\tau_y \in \mathcal{O}$	$\tau_y \notin \mathcal{O}$
$\tau_z \in \mathcal{O}$	$(2N - c(\pi,\chi))\,\mathbb{1}_{2N\geq c(\pi,\chi)}$	$\frac{1}{\lvert\mathcal{X}_r\rvert}\mathbb{1}_{2N+r\geq c(\pi,\chi)}$
$\tau_z \notin \mathcal{O}$	$\frac{1}{\lvert\mathcal{X}_s\rvert}\mathbb{1}_{2N+s\geq c(\pi,\chi)}$	$\frac{1}{\lvert\mathcal{X}_{\max(r,s)}\rvert}\mathbb{1}_{U(min(r,s))}\left(\frac{\alpha_{\pi,\chi}}{T\tau_y T\tau_z}\right)$

Table 2.1: Table of τ_y and τ_z relationships

principal series case and $fc(\xi\chi_E^{-1})$ in the supercuspidal case and $\alpha_{\pi,\chi} \in \varpi^{-c(\pi,\chi)}\mathcal{O}^\times$ is equal to $\alpha_{\chi_\sharp}\alpha_{\chi_\flat}$ in the principal series case and $\mathrm{Nm}(\alpha_{\xi\chi_E^{-1}})$ in the supercuspidal case.

2.6 Proof of Proposition 2.5.8

2.6.1 Supercuspidal Case

Proof. We start with the $\tau_y, \tau_z \in \mathcal{O}$ case. Then

$$\begin{aligned}\langle \mathrm{Op}^+(a_-)v_{\chi^{-1}}^R, \pi(w)\,\mathrm{Op}^+(a_+)v_\chi^R\rangle &= \langle \mathrm{Op}^+(a_-)v_{\chi^{-1}}^R, \pi(w)v_\chi^R 1_{v(\cdot)\geq -N}\rangle \\ &= \langle v_{\chi^{-1}}^R 1_{v(\cdot)\geq -N}, \epsilon(1/2,\xi\chi_E^{-1},\psi_E)v_{\chi^{-1}}^R(y)1_{v(y)\leq N - fc(\xi\chi_E^{-1})}\rangle \\ &= \overline{\epsilon(1/2,\xi\chi_E^{-1},\psi_E)}\left(2N - fc(\xi\chi_E^{-1})\right)\end{aligned}$$

The next case is when $\tau_y \in \mathcal{O}$ and $\tau_z \in \varpi^{-s}\mathcal{O}^\times$ for some $s > 0$. Then

$$\langle \mathrm{Op}^+(a_-)v^R_{\chi^{-1}}, \pi(w)\,\mathrm{Op}^+(a_+)v^R_\chi\rangle = \langle \mathrm{Op}^+(a_-)v^R_{\chi^{-1}}, \epsilon(\xi\chi_E^{-1})v^R_{\chi^{-1}}1_{v(\cdot)\le N-fc(\xi\chi_E^{-1})}\rangle$$
$$= \langle v^R_{\chi^{-1}}1_{v(\cdot)=-N-s}\,\mathbb{E}_{\omega\in\mathcal{X}_s}\,\omega^{-1}(\varpi^s\tau_z)\omega(\varpi^{N+s})\omega, v^R_{\chi^{-1}}\epsilon(\xi\chi_E^{-1})1_{v(\cdot)\le N-fc(\xi\chi_E^{-1})}\rangle.$$

By linearity this inner product is the same as

$$\overline{\epsilon(\xi\chi_E^{-1})}\,\mathbb{E}_{\omega\in\mathcal{X}_s}\,\omega^{-1}(\varpi^s\tau_z)\omega(\varpi^{N+s})\int_{F^\times} 1_{[-R,R]}\overline{\chi}(y)\omega(y)\chi(y)1_{v(y)=-N-s}1_{v(y)\le N-fc(\xi\chi_E^{-1})}\,d^\times y$$
$$= \overline{\epsilon(\xi\chi_E^{-1})}\,\mathbb{E}_{\omega\in\mathcal{X}_s}\,\omega^{-1}(\varpi^s\tau_z)1_{\omega=|\cdot|^{s'}}1_{-N-s\le N-fc(\xi\chi_E^{-1})},$$

for some $s' \in \mathbf{C}$ since $\int_{\mathcal{O}^\times}\omega(u)\,d^\times u$ is only nonzero if ω is unramified. Therefore, we have calculated that in this case

$$\langle \mathrm{Op}^+(a_-)v^R_{\chi^{-1}}, \pi(w)\,\mathrm{Op}^+(a_+)v^R_\chi\rangle = \frac{\epsilon(\xi\chi_E^{-1})}{|\mathcal{X}_s|}1_{2N+s\ge fc(\xi\chi_E^{-1})}.$$

Next case is when $\tau_y \in \varpi^{-r}\mathcal{O}^\times$ for some $r > 0$ and $\tau_z \in \mathcal{O}$. This case is similar. We have

$$\langle \mathrm{Op}^+(a_-)v^R_{\chi^{-1}},\pi(w)\,\mathrm{Op}^+(a_+)v^R_\chi\rangle = \langle v^R_{\chi^{-1}}1_{v(\cdot)\ge -N}, \pi(w)v^R_\chi 1_{v(\cdot)=-N-r}\,\mathbb{E}_{\omega\in\mathcal{X}_r}\,\omega^{-1}(\varpi^r\tau_y)\omega(\varpi^{N+r})\omega\rangle$$
$$= \langle v^R_{\chi^{-1}}1_{v(\cdot)\ge -N}, v^R_{\chi^{-1}}1_{v(\cdot)=N+r-fc(\xi\chi_E^{-1}\omega_E^{-1})}\,\mathbb{E}_{\omega\in\mathcal{X}_r}\,\epsilon(\xi\chi_E^{-1}\omega_E^{-1})\omega^{-1}(\tau_y)\omega(\varpi^N)\omega\rangle$$
$$= \frac{\epsilon(\xi\chi_E^{-1})}{|\mathcal{X}_r|}1_{2N+r\ge fc(\xi\chi_E^{-1})},$$

again since the inner product picks out $\omega = 1$.

The last case is when $\tau_y \in \varpi^{-r}\mathcal{O}^\times$ and $\tau_z \in \varpi^{-s}\mathcal{O}^\times$, with $r, s > 0$. This gives

$$\langle \mathrm{Op}^+(a_-)v^R_{\chi^{-1}}, \pi(w)\,\mathrm{Op}^+(a_+)v^R_\chi\rangle$$
$$= \langle v^R_{\chi^{-1}}1_{v(\cdot)=-N-s}\,\mathbb{E}_{\omega\in\mathcal{X}_s}\,\omega^{-1}(\varpi^s\tau_z)\omega(\varpi^{N+s})\omega, v^R_{\chi^{-1}}1_{v(\cdot)=N+r-fc(\xi\chi_E^{-1}\eta_E^{-1})}\,\mathbb{E}_{\eta\in\mathcal{X}_r}\,\epsilon(\xi\chi_E^{-1}\eta_E^{-1})\eta^{-1}(\varpi^r\tau_y)\eta(\varpi^{N+r})\eta\rangle$$
$$= \mathbb{E}_{\omega\in\mathcal{X}_s}\,\mathbb{E}_{\eta\in\mathcal{X}_r}\,\overline{\epsilon(\xi\chi_E^{-1}\eta_E^{-1})}\omega^{-1}(\varpi^s\tau_z)\eta(\varpi^r\tau_y)\int_{F^\times}\omega(y)\overline{\eta}(y)1_{v(y)=-N-s}1_{v(y)=N+r-fc(\xi\chi_E^{-1}\eta_E^{-1})}\,d^\times y$$
$$= \mathbb{E}_{\omega\in\mathcal{X}_s}\,\mathbb{E}_{\eta\in\mathcal{X}_r}\,\overline{\epsilon(\xi\chi_E^{-1}\eta_E^{-1})}\eta(\varpi^{s+r}\tau_y\tau_z)1_{2N+r+s=fc(\xi\chi_E^{-1}\eta_E^{-1})}\mathbb{1}_{\omega^{-1}=\eta}.$$

We need to understand how the epsilon factor $\epsilon(1/2, \xi\chi_E^{-1}, \psi_E)$ behaves under twists of characters η_E with small conductor. How small is the conductor? The character η has conductor at most r, and $r <$

$fc(\xi\chi_E^{-1}\eta_E^{-1}) - 2N$ since $s > 0$. For N large enough, the characteristic function $1_{2N+r+s=fc(\xi\chi_E^{-1}\eta_E^{-1})}$ is zero, so the assumption that r is much smaller than the conductor is not very strict. Therefore, we assume $r \leq c(\xi\chi_E^{-1})/2$. This gives $c(\xi\chi_E^{-1}\eta_E^{-1}) = c(\xi\chi_E^{-1})$. Furthermore, let $\alpha_{\xi\chi_E^{-1}} \in \varpi^{-c(\xi\chi_E^{-1})}\mathcal{O}^\times$ such that $\xi\chi_E^{-1}(1+x) = \psi(\alpha_{\xi\chi_E^{-1}}x)$ for $x \in \mathfrak{p}^{c(\xi\chi_E^{-1})/2}$. Now since $c(\eta_E) \leq r \leq c(\xi\chi_E^{-1})/2$, by Proposition 2.2.3

$$\epsilon(1/2, \xi\chi_E^{-1}\eta_E^{-1}, \psi_E) = \eta_E^{-1}(\alpha_{\xi\chi_E^{-1}})\epsilon(1/2, \xi\chi_E^{-1}, \psi_E).$$

Note that

$$\eta_E^{-1}(\alpha_{\xi\chi_E^{-1}}) = \eta^{-1}(\alpha_{\pi,\chi}).$$

The condition $r \leq c(\xi\chi_E^{-1})/2$ in light of $\omega^{-1} = \eta$ forces $s \leq c(\xi\chi_E^{-1})/2$. Then from the characteristic function, we have

$$fc(\xi\chi_E^{-1}) = 2N + r + s \leq 2N + c(\xi\chi_E^{-1}).$$

Since E is a quadratic extension $f \in \{1, 2\}$. If $f = 1$ we obtain $N \geq 0$ and if $f = 2$, we obtain the "uncertainty principle" $N \geq \frac{c(\xi\chi_E^{-1})}{2}$. So now our inner product is given by

$$\overline{\epsilon(\xi\chi_E^{-1})}\,\mathbb{E}_{\omega\in\mathcal{X}_s}\,\mathbb{E}_{\eta\in\mathcal{X}_r}\,\mathbb{1}_{\eta=\omega^{-1}}\eta(\varpi^{s+r}\tau_y\tau_z\alpha_{\pi,\chi})1_{2N+r+s=fc(\xi\chi_E^{-1})}.$$

Suppose without loss of generality that $s < r$. The above can then be written

$$\overline{\epsilon(\xi\chi_E^{-1})}\,\mathbb{E}_{\eta\in\mathcal{X}_r}\,\mathbb{E}_{\omega\in\mathcal{X}_s}\,1_{\eta=\omega^{-1}}\omega^{-1}(\varpi^{s+r}\tau_y\tau_z\alpha_{\pi,\chi})1_{2N+r+s=fc(\xi\chi_E^{-1})}.$$

The average over ω forces the argument to lie in $U(s)$, and the average over η contributes $\frac{1}{|\mathcal{X}_r|}$, which yields the desired result. □

2.6.2 Principal Series Case

Proof. We start with the $\tau_y, \tau_z \in \mathcal{O}$. Then

$$\begin{aligned}
\langle \mathrm{Op}^+(a_-)v^R_{\chi^{-1}}, \pi(w)\,\mathrm{Op}^+(a_+)v^R_\chi\rangle &= \langle \mathrm{Op}^+(a_-)v^R_{\chi^{-1}}, \pi(w)(v^R_\chi 1_{v(\cdot)\geq -N})\rangle \\
&= \langle v^R_{\chi^{-1}}1_{v(\cdot)\geq -N}, \epsilon_\sharp\epsilon_\flat v^R_{\chi^{-1}}1_{v(\cdot)\leq N-f_\flat-f_\sharp}\rangle \\
&= \overline{\epsilon(1/2,\chi_\sharp)\epsilon(1/2,\chi_\flat)}(2N - f_\flat - f_\sharp)\mathbb{1}_{2N\geq f_\sharp+f_\flat}.
\end{aligned}$$

The next case is when $\tau_y \in \mathcal{O}$ and $\tau_z \in \varpi^{-s}\mathcal{O}^\times$ for some $s > 0$. Then

$$\langle \mathrm{Op}^+(a_-)v^R_{\chi^{-1}}, \pi(w)\,\mathrm{Op}^+(a_+)v^R_\chi\rangle = \langle \mathrm{Op}^+(a_-)v^R_{\chi^{-1}}, \epsilon_\sharp\epsilon_\flat v^R_{\chi^{-1}} 1_{v(\cdot)\le N-f_\sharp-f_\flat}\rangle$$
$$= \langle v^R_{\chi^{-1}} 1_{v(\cdot)=-N-s}\,\mathbb{E}_{\omega\in\mathcal{X}_s}\,\omega^{-1}(\varpi^r\tau_y)\omega(\varpi^{N+s})\omega, \epsilon_\sharp\epsilon_\flat v^R_{\chi^{-1}} 1_{v(\cdot)\le N-f_\sharp-f_\flat}\rangle.$$

By linearity, this inner product is the same as

$$\overline{\epsilon_\sharp\epsilon_\flat}\,\mathbb{E}_{\omega\in\mathcal{X}_s}\,\omega^{-1}(\varpi^s\tau_z)\omega(\varpi^{N+s})\int_{F^\times} 1_{[-R,R]}\overline{\chi}(y)\omega(y)\chi(y)1_{v(y)=-N-s}1_{v(y)\le N-f_\sharp-f_\flat}\,d^\times y$$
$$= \overline{\epsilon_\sharp\epsilon_\flat}\,\mathbb{E}_{\omega\in\mathcal{X}_s}\,\omega^{-1}(\varpi^s\tau_z)1_{\omega=|\cdot|^{s'}}1_{-N-s\le N-f_\flat-f_\sharp},$$

for some $s' \in \mathbf{C}$ since $\int_{\mathcal{O}^\times}\omega(u)\,d^\times u$ is only nonzero if ω is unramified. Therefore, we have calculated that in this case

$$\langle \mathrm{Op}^+(a_-)v^R_{\chi^{-1}}, \pi(w)\,\mathrm{Op}^+(a_+)v^R_\chi\rangle = \frac{\overline{\epsilon(1/2,\chi_\sharp)\epsilon(1/2,\chi_\flat)}}{|\mathcal{X}_s|}1_{2N+s\ge f_\flat+f_\sharp}.$$

Next case is when $\tau_y \in \varpi^{-r}\mathcal{O}^\times$ for some $r > 0$ and $\tau_z \in \mathcal{O}$. This case is very similar. We have

$$\langle \mathrm{Op}^+(a_-)v^R_{\chi^{-1}},\pi(w)\,\mathrm{Op}^+(a_+)v^R_\chi\rangle = \langle v^R_{\chi^{-1}}1_{v(\cdot)\ge -N}, \pi(w)v^R_\chi 1_{v(\cdot)=-N-r}\,\mathbb{E}_{\omega\in\mathcal{X}_r}\,\omega^{-1}(\varpi^r\tau_y)\omega(\varpi^{N+r})\omega\rangle$$
$$= \langle v^R_{\chi^{-1}}1_{v(\cdot)\ge -N}, v^R_{\chi^{-1}}1_{v(\cdot)=N+r-f_\sharp-f_\flat}\,\mathbb{E}_{\omega\in\mathcal{X}_r}\,\epsilon_\flat\epsilon_\sharp\omega^{-1}(\tau_y)\omega(\varpi^N)\omega\rangle$$
$$= \frac{\overline{\epsilon(1/2,\chi_\flat)\epsilon(1/2,\chi_\sharp)}}{|\mathcal{X}_r|}1_{2N+r\ge f_\flat+f_\sharp},$$

again since the inner product picks out $\omega = 1$.

Finally, we tackle the case when $\tau_y \in \varpi^{-r}\mathcal{O}^\times$ and $\tau_z \in \varpi^{-s}\mathcal{O}^\times$, with $r, s > 0$. This gives

$$\langle \mathrm{Op}^+(a_-)v^R_{\chi^{-1}}, \pi(w)\,\mathrm{Op}^+(a_+)v^R_\chi\rangle$$
$$= \langle v^R_{\chi^{-1}}1_{v(\cdot)=-N-s}\mathbb{E}_{\omega\in\mathcal{X}_s}\omega^{-1}(\varpi^s\tau_z)\omega(\varpi^{N+s})\omega, v^R_{\chi^{-1}}1_{v(y)=N+r-f_\flat^\eta-f_\sharp^\eta}\mathbb{E}_{\eta\in\mathcal{X}_r}\eta^{-1}(\varpi^s\tau_{ij})\eta^{-1}(y)\epsilon(\chi_\flat\eta^{-1})\epsilon(\chi_\sharp\eta^{-1})\rangle$$
$$= \mathbb{E}_{\omega\in\mathcal{X}_s}\mathbb{E}_{\eta\in\mathcal{X}_r}\overline{\epsilon(\chi_\flat\eta^{-1})\epsilon(\chi_\sharp\eta^{-1})}\omega^{-1}(\varpi^s\tau_z)\eta(\varpi^r\tau_y)\int_{\mathbf{F}^\times}\omega(y)\overline{\eta}(y)\mathbb{1}_{v(y)=-N-r}\mathbb{1}_{v(y)=N+r-f_\sharp-f_\flat}\,d^\times y$$
$$= \mathbb{E}_{\omega\in\mathcal{X}_s}\mathbb{E}_{\eta\in\mathcal{X}_r}\overline{\epsilon(\chi_\flat\eta^{-1})\epsilon(\chi_\sharp\eta^{-1})}\mathbb{1}_{2N+r+s=f_\flat+f_\sharp}\mathbb{1}_{\omega^{-1}=\eta}.$$

We now need to understand how the epsilon factors $\epsilon(1/2,\chi_\flat\eta^{-1})$ and $\epsilon(1/2,\chi_\sharp\eta^{-1})$ behave under twists by characters η with small conductor. Since η has conductor at most r, and $s > 0$, we have $r < f_\sharp + f_\flat - 2N$. For N large enough, the characteristic function $\mathbb{1}_{2N+r+s=f_\flat+f_\sharp}$ is zero, so we can and do assume r is much

smaller than $f_\flat$ and $f_\sharp$, without losing much. Therefore, $c(\chi_\sharp\eta^{-1}) = c(\chi_\sharp)$ and $c(\chi_\flat\eta^{-1}) = c(\chi_\flat)$. Let $\alpha_\flat := \alpha_{\chi_\flat}$ and $\alpha_\sharp := \alpha_{\chi_\sharp}$. Since $c(\eta) \leq r \leq \min(f_\flat/2, f_\sharp/2)$, we have

$$\epsilon(1/2, \chi_\flat\eta^{-1}, \psi) = \eta^{-1}(\alpha_\flat)\epsilon(1/2, \chi_\flat, \psi)$$
$$\epsilon(1/2, \chi_\sharp\eta^{-1}, \psi) = \eta^{-1}(\alpha_\sharp)\epsilon(1/2, \chi_\sharp, \psi).$$

The condition $r \leq \min(f_\flat/2, f_\sharp/2)$ yields $s \leq \min(f_\flat/2, f_\sharp/2)$ by symmetry. So we have

$$f_\flat + f_\sharp = 2N + r + s \leq 2N + 2\min(f_\flat/2, f_\sharp/2).$$

This implies "uncertainty principle" $N \geq \max(f_\flat/2, f_\sharp/2)$. So the inner product is given by

$$\overline{\epsilon(1/2, \chi_\flat)\epsilon(1/2, \chi_\sharp)}\, \mathbb{E}_{\omega\in\mathcal{X}_s}\, \mathbb{E}_{\eta\in\mathcal{X}_r}\, \mathbb{1}_{\eta=\omega^{-1}}\eta(\varpi^{s+r}\tau_y\tau_z\alpha_\sharp^{-1}\alpha_\flat^{-1})\mathbb{1}_{2N+r+s=f_\flat+f_\sharp}.$$

Suppose without loss of generality that $s < r$. The above can then be written as

$$\overline{\epsilon(1/2, \chi_\flat)\epsilon(1/2, \chi_\sharp)}\, \mathbb{E}_{\omega\in\mathcal{X}_s}\, \mathbb{E}_{\eta\in\mathcal{X}_r}\, \mathbb{1}_{\eta=\omega^{-1}}\omega^{-1}(\varpi^{s+r}\tau_y\tau_z\alpha_\sharp^{-1}\alpha_\flat^{-1})\mathbb{1}_{2N+r+s=f_\flat+f_\sharp}.$$

The average over ω forces the argument of ω^{-1} to lie in $U(s)$ and the average over η contributes $\frac{1}{|\mathcal{X}_r|}$, which yields the desired result. □

2.7 Phase Space Integral

We will now complete the proof of the main theorem 2.1.1. We'd like to relate our calculations of the relative characters to integrals over "phase space" of our functions a. In other words, we plan to complete the proof of the equality

$$\langle \mathrm{Op}(a)v_\chi, v_\chi\rangle = \int_{\mathrm{Hyp}(\pi,\chi)} a(\xi)\, d\xi$$

by calculating the right hand side. It suffices to take $a = a_\tau = 1_\tau^T$ by Proposition 2.3.4. The map

$$\mathrm{Hyp} : \{\text{ Pairs of irreps of } (G, H)\} \to \mathrm{Lie}(G)^*$$
$$(\pi, \sigma) \mapsto \mathrm{Hyp}(\pi, \sigma),$$

is meant to map the data of π and $\sigma = \chi$ to the hyperbola that we integrate over. For us, we have

$$\mathrm{Hyp}(\pi, \chi) = \{(x, y, z) \ : \ x = \alpha_\chi, \ yz = \alpha_{\pi,\chi}\},$$

where $c(\pi, \chi) = f_\flat + f_\sharp$ if π is principal series and $c(\pi, \chi) = fc(\xi\chi_E^{-1})$ if π is supercuspidal. We will take the measure $d\xi$ to be the multiplicative Haar measure on $F^\times$. Now we integrate the characteristic function

$$a_\tau(\xi) = \mathbb{1}_{B(T\tau,T)}(\xi)$$

of the box of size T around the rescaled microlocal frequency $T\tau$. Like before, we first notice that if $T\tau_x$ must be within T of α_χ, otherwise the whole integral is zero (the support of the characteristic function won't overlap $\mathrm{Hyp}(\pi, \chi)$ in that case. Let's again case on when $\tau_y, \tau_z \in \mathcal{O}$.

If $\tau_y, \tau_z \in \mathcal{O}$, then for instance, $\xi_y - T\tau_y \in T\mathcal{O} \implies T^{-1}\xi_y \in \mathcal{O}$. Similarly $T^{-1}\xi_z \in \mathcal{O}$. So we can write $\xi_y = T\xi_y'$ and $\xi_z = T\xi_z'$ with $\xi_y', \xi_z' \in \mathcal{O}$. Then

$$\xi_y\xi_z = \alpha_{\pi,\chi} \implies T\xi_y'T\xi_z' = \alpha_{\pi,\chi} \implies \xi_y'\xi_z' = \varpi^{2N}\alpha_{\pi,\chi}$$

So we have

$$1 \geq |\xi_y'| \geq q^{-2N+c(\pi,\chi)},$$

since $|\xi_z| \leq 1$. So we must have $2N \geq c(\pi, \chi)$, otherwise there is no such ξ_y. Hence, the integral becomes

$$\int_{q^{-2N+c(\pi,\chi)}}^1 \frac{d\xi_y'}{|\xi_y'|} = 2N - c(\pi, \chi),$$

and our contribution in total is

$$(2N - c(\pi, \chi)) \times \mathbb{1}_{2N \geq c(\pi,\chi)}.$$

We next do the case when $\tau_y, \tau_z \in F \backslash \mathcal{O}$. In this case, we have $T^{-1}\xi_y \in \tau_y + \mathcal{O}$ and $T^{-1}\xi_z \in \tau_z + \mathcal{O}$. Let $|\tau_y| = q^r$ and $|\tau_z| = q^s$. Then we can write $\xi_y = T\tau_y u_y$ and $\xi_z = T\tau_z u_z$ with $u_y \in U(r)$ and $u_z \in U(s)$. Then we have

$$\xi_y \xi_z = \alpha_{\pi,\chi} \implies u_y u_z = \frac{\varpi^{2N}}{\tau_y \tau_z} \alpha_{\pi,\chi} = \frac{\alpha_{\pi,\chi}}{T\tau_y T\tau_z}$$

Now suppose $s < r$. Then the right-hand side of the above equation lies in $U(s)$. The stricter congruence is then $u_y \in U(r)$ and for each choice there is exactly one $u_z \in U(s)$ satisfying the above equation. The integral is scaling invariant, because we use the measure $\frac{dy}{y}$, so in this range we need to compute

$$\int_{u_y u_z = \frac{\varpi^{2N - c(\pi,\chi)}}{\tau_y \tau_z \alpha_{\pi,\chi}}} d^\times u_y = \mathrm{vol}(U(r)) = \frac{1}{q^r(1 - q^{-1})}.$$

In total we get

$$\frac{1}{q^r(1 - q^{-1})} \times \mathbb{1}_{U(s)} \left(\frac{\varpi^{2N}}{\tau_y \tau_z} \alpha_{\pi,\chi} \cdot \right)$$

The next case is when $\tau_y \in \mathbf{F} \setminus \mathcal{O}$ and $\tau_z \in \mathcal{O}$. This is just the $s = 0$ version of the previous case. In this case $U(s) = U(0) = \mathcal{O}$ by convention, and this gives rise to the condition that $2N + r - c(\pi, \chi) \geq 0$. So in this case the contribution to the integral is

$$\frac{1}{q^r(1 - q^{-1})} \times \mathbb{1}_{2N + r \geq c(\pi,\chi)} \cdot$$

The final case is when $\tau_y \in \mathcal{O}$ and $\tau_z \in F \setminus \mathcal{O}$. This case is completely symmetric except now $r = 0$ and $s > r$. So by the exact same argument we get

$$\frac{1}{q^s(1 - q^{-1})} \times \mathbb{1}_{2N + s \geq c(\pi,\chi)} \cdot$$

Each of these calculations corresponds to the analogous calculation on the relative character side, so we are done.

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