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A Study in the Asymptotic Behavior of Nonlinear Evolution Equations with Nonlocal Operators

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A Study in the Asymptotic Behavior of Nonlinear Evolution Equations with Nonlocal Operators

Andrei Tarfulea

A Dissertation
Presented to the Faculty
of Princeton University
in Candidacy for the Degree
of Doctor of Philosophy

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Abstract

Nonlinear evolution equations appear in a very wide variety of physical, economical, and numerical models. Many exotic phenomena demand the use of nonlocal operators in these models. This thesis focuses on investigating the asymptotic behavior of two such classes of equations: the surface quasigeostrophic (SQG) equation, of hydrodynamic origin, and the fractional Fisher-KPP equation, a reaction-diffusion equation with a non-standard diffusion process. We first prove the absence of anomalous dissipation of energy for the forced critical SQG equation with vanishing hyperviscosity through the analysis of stationary statistical solutions. Then we use precise nonlinear lower bounds on the fractional Laplacian to prove global regularity for the forced critical SQG equation (bootstrapping directly from L^∞ to Hölder continuity) and use this to further prove the existence of a compact global attractor (of finite fractal dimension) for the associated dynamics. Lastly, we prove a comparative exponential decay estimate on the derivatives of the solution to the fractional Fisher-KPP equation (starting from decaying initial data), which then proves a flattening/symmetrization result for the reaction fronts.

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To my undergraduate mentor Paul J. Sally Jr.

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Chapter 1

Introduction

1.1 Nonlocal operators in evolution equations

Nonlinear evolution equations are mathematical models describing a variety of dynamical processes. Though traditionally the evolution can be determined locally, nonlocal interactions are sometimes necessary to encompass a broader range of phenomena. A simple example could be

$$\partial_t \varphi(t,x) = (\mathcal{D}\varphi)(x) := \int (\varphi(t,x+y) - \varphi(t,x))\, k(x,y) dy$$

for some kernel k. Such equations emerge from simplified models of statistical mechanics, such as the (space-homogeneous) Boltzmann equation.

The study of nonlocal equations is motivated by a variety of real world applications and may have several sources. Hydrodynamic models that include advection and conservative flows are implicitly nonlocal due to the nature of the pressure term.

A common origin for exotic nonlocal effects is perhaps the study of non-standard diffusions, often appearing in the equation as a linear integro-differential operator $\mathcal{D}$.

These are of great interest in the context of reaction-diffusion equations

$$\partial_t \varphi + \mathcal{D}\varphi = f(\varphi),$$

where φ acts as a concentration with a prescribed nonlinear chemical interaction term f.

Such operators may also arise from analogues to the Biot-Savart law appearing in active scalar equations of the form

$$\partial_t \varphi + u \cdot \nabla \varphi + \mathcal{D}\varphi = 0, \quad u(t,x) = \int \varphi(t,y) K(x-y) dy.$$

It is easy to see that the Navier-Stokes equations in two dimensions are of the above type when expressed in vorticity form.

1.1.1 Exotic diffusion from jump processes

A Lévy process is a stochastic process (a random trajectory in $\mathbb{R}^d$) that generalizes the concept of Brownian motion. Specifically, it may contain jump discontinuities. In the context of integro-differential equations, they model certain exotic types of fast diffusion seen in nuclear physics, biological models, dilute gases (i.e., simplified kinetic models derived from the Boltzmann equation), and certain chemical reactions. Jump processes are particularly important in financial mathematics and stochastic control problems. Several stochastic models, notably particle systems, can be used to derive a nonlocal version of the porous medium equation as well as Hamilton-Jacobi equations and conservation laws with fractional diffusion. Nonlocal operators arising from Lévy processes have also found applications in medicine, denoising algorithms, and meteorology.

More specifically, the process $X = \{X(t)\}_{t \geq 0}$ with values in $\mathbb{R}^d$ is said to be Lévy if it has independent increments, if $X(t-s)$ and $X(t) - X(s)$ have the same probability

law for all $t > s$, and if the trajectory $X(t)$ is almost surely continuous from the right. It follows that a Lévy process X with $X(0) = 0$ is infinitely divisible; for any n, X can be expressed as the sum of n independent identically distributed random variables. By a classical theorem of Lévy and Khintchine, we have that for any $\xi \in \mathbb{R}^d$

$$\mathbb{E}\left(e^{i\xi \cdot X(t)}\right) = e^{t\eta(\xi)},$$

with

$$\eta(\xi) = i\xi \cdot b - \frac{1}{2}\langle A\xi, \xi\rangle + \int_{\mathbb{R}^d} \left(e^{i\xi \cdot y} - 1 - i\xi \cdot y\chi_B(y)\right) d\mu(y),$$

where b is a vector (an inherent drift in the random walk), A is a positive matrix (the "Brownian part"), χ_B is a smooth cutoff of the unit ball, and μ is a Borel measure that gives finite mass to the function $|y|^2/(|y|^2 + 1)$ (the "jumping part").

To make this into a deterministic model for diffusion, we can define a linear semigroup $\{U_t\}_{t\geq 0}$ on smooth functions φ via $(U_t f)(x) = \mathbb{E}(\varphi(x + X(t)))$. This semigroup has an infinitesimal generator $\mathcal{D}$ (that is, $\partial_t \varphi(x,t) = \mathcal{D}\varphi(x,t)$ for all $(x,t) \in \mathbb{R}^d \times \mathbb{R}$) given by

$$\mathcal{D}\varphi(x) = b \cdot \nabla\varphi(x) + \operatorname{tr}(A\nabla^2\varphi)(x) + \int_{\mathbb{R}^d} \left[\varphi(x + y) - \varphi(x) - y \cdot \nabla\varphi(x)\chi_B(y)\right] d\mu(y). \tag{1.1}$$

The first two terms constitute a standard (local) elliptic operator. The integral term incorporates the exotic diffusion arising from the Lévy process. As such, we may assume without loss of generality that A and b are zero; should the model demand drift or Brownian diffusion on top of the nonlocal effects of $\mathcal{D}$, these may be written as separate terms in the equation.

1.1.2 The fractional Laplacian

We now introduce the class of operators Λ^α (for $\alpha \in (0,2)$) collectively known as the fractional Laplacian of order $\alpha/2$:

$$\Lambda^\alpha \varphi(x) = (-\Delta)^{\alpha/2}\varphi(x) = \mathcal{F}^{-1}(|\xi|^\alpha \widehat{\varphi}(\xi))(x) = \lim_{\epsilon \to 0} c_{N,\alpha} \int_{|y|>\epsilon} \frac{\varphi(x) - \varphi(x+y)}{|y|^{d+\alpha}} dy. \tag{1.2}$$

We will also simply write Λ in place of Λ^1. Evidently, Λ^α has the same form as equation (1.1), and is therefore a generator for a Lévy process. The Lévy measure μ is a radially symmetric algebraic kernel, homogeneous of degree $d+\alpha$; the integral above is in the sense of principal values. Moreover, from the definition using Fourier transforms, it is apparent that $\Lambda(\Lambda\varphi) = -\Delta\varphi$.

The fractional Laplacian also makes sense on the torus $\mathbb{T}^d$ as a Fourier multiplier with symbol $|k|^\alpha$. Note that the eigenvalues of $\Lambda = (-\Delta)^{1/2}$ are given by $|k|$, with $k \in \mathbb{Z}^d_* = \mathbb{Z}^d \setminus \{0\}$. We label them in increasing order (counting multiplicity) as

$$0 < \lambda_1 \leq \ldots \leq \lambda_n \leq \ldots.$$

and denote the eigenfunction associated to λ_j by e_j. Then $\{e_j\}_{j\geq 1}$ is an orthonormal basis of L^2, and the sets $\{\lambda_j\}_{j\geq 1}$ and $\{|k|\}_{k\in\mathbb{Z}^d_*}$ are equal. We adopt the convention that $\mathbb{T}^d = [-\pi,\pi]^d$, so that $\lambda_1 = 1$.

To obtain strong pointwise in x estimates on the fractional Laplacian, we will make use of the kernel representation of (1.2). To maintain these estimates on $\mathbb{T}^d$, we recall the kernel representation of the periodic fractional Laplacian [44] (see also [61, 123]). Note that other very useful characterizations are available, see e.g. [12] to obtain monotonicity formulae.

For $\alpha \in (0,2)$ and $\varphi \in C^\infty(\mathbb{T}^d)$ we have the pointwise definition

$$\Lambda^\alpha \varphi(x) = P.V. \int_{\mathbb{T}^d} (\varphi(x) - \varphi(x+y))\, K_{\alpha,d}(y) dy,$$

where kernel $K_{\alpha,d}$ is defined on $\mathbb{T}^d \setminus \{0\}$ as

$$K_{\alpha,d}(y) = c_{\alpha,d} \sum_{k \in \mathbb{Z}^d} \frac{1}{|y - 2\pi k|^{d+\alpha}} \tag{1.3}$$

and the normalization constant is

$$c_\alpha = \frac{2^\alpha \Gamma(1+\alpha/2)}{|\Gamma(-\alpha/2)|\pi}. \tag{1.4}$$

Under this normalization one has that $\lim_{\alpha \to 2-} \Lambda^\alpha \varphi = -\Delta \varphi$, and $\lim_{\alpha \to 0+} \Lambda^\alpha \varphi = \varphi - \bar{\varphi}$, pointwise in $x \in \mathbb{T}^d$, where $\bar{\varphi}$ is the mean of φ over $\mathbb{T}^d$. When $\alpha \in (0,1)$ the above definition is valid for $\varphi \in C^{\alpha+\varepsilon}$, while for $\alpha \in [1,2)$ we need that $\varphi \in C^{1,\alpha-1+\varepsilon}$, for some $\varepsilon > 0$.

Outline of the thesis

We present here the main results of the work focusing on the asymptotic behavior of the surface quasigeostrophic (SQG) equation and the fractional Fisher-KPP equation as case studies in the analysis of nonlinear nonlocal PDE. Chapter 1 continues with an introduction of the SQG equation in Section 1.3 as well as a summary of the three main results (and their methods of proof) concerning this model. In like fashion, Section 1.4 introduces the fractional Fisher-KPP equation and summarizes the two main results there. The remainder of the thesis is divided into three chapters presenting the proofs and technical details.

In Chapter 2 we prove the absence of anomalous dissipation of energy for the

forced, critical SQG equation with vanishing hyperviscosity. In Section 2.1 we present the forced viscous SQG equations and prove some properties of solutions, including the relative compactness of positive semiorbits. In Section 2.2 we introduce the notion of stationary statistical solutions of the viscous equations. In Section 2.3 we prove that inviscid limits of stationary statistical solutions are stationary statistical solutions of the forced critical SQG equations which preserve the energy dissipation balance. In Section 2.4 we construct stationary statistical solutions using time averages and in Section 2.5 we present the argument by contradiction and concluding remarks on this subject. The results of this chapter may be seen as an independent publication in [38].

Chapter 3 presents the new proof of global regularity for the forced, critical SQG equation on the torus (making use of precise nonlinear lower bounds on the fractional Laplacian to bootstrap directly from L^∞ to Hölder continuity) as well as a proof of existence (and finite-dimensionality) of a compact global attractor for the induced dynamics on H^1. Section 3.1 deals with preliminaries, including the periodic setting, and a statement of the uniform nonlinear Poincaré inequality, whose proof appeared in [33, Appendix A]. Section 3.2 is devoted to an example of our method of proof, in the simpler setting of the forced critical Burgers equation. Section 3.3 is devoted to the proof of global regularity of forced SQG, and Sections 3.4 and 3.5 to the existence and finite dimensionality of the attractor. We present some of the technical aspects in Section 3.6, including a proof of backward uniqueness (for marginally smoother forces) and a lemma of general interest about switching the order of derivatives and supremum. The results of this chapter may be seen as an independent publication in [39].

In Chapter 4 we prove a comparative exponential decay of the derivatives of the solution to the fractional Fisher-KPP equation and subsequently a symmetrization result for the level sets of the solution. In Section 4.1, we gather some known (but

useful) facts that will be used throughout the proofs of the main theorems. Section 4.2 is devoted to the proof of the comparative exponential decay of derivatives while Section 4.3 is devoted to estimating the fractional Laplacian. The symmetrization result for the renormalized level sets is then proved in Section 4.4 via an ODE argument. The results of this chapter may be seen as an independent submission [124] that is, at this time, under consideration for publication.

1.2 Asymptotic dynamics for the forced, critical SQG equation

The forced, critically dissipative SQG equation is given by

$$\partial_t \theta + u \cdot \nabla \theta + \kappa \Lambda \theta = f \tag{1.5}$$

$$u = \mathcal{R}^{\perp} \theta = (-\mathcal{R}_2 \theta, \mathcal{R}_1 \theta) \tag{1.6}$$

$$\theta(\cdot, 0) = \theta_0 \tag{1.7}$$

where $\Lambda = (-\Delta)^{1/2}$, $\mathcal{R}_j = \partial_j \Lambda^{-1}$ is the jth Riesz transform, and the equations are set on the periodic domain $\mathbb{T}^2 = [-\pi, \pi]^2$. Here $\kappa > 0$ is a positive constant, θ_0 is the initial condition, and $f = f(x)$ is a time-independent forcing term. The force is assumed to belong to $L^\infty(\mathbb{T}^2) \cap H^1(\mathbb{T}^2)$, while the initial data is assumed to belong to $H^1(\mathbb{T}^2)$. We consider forcing and initial data of zero average, that is $\int_{\mathbb{T}^2} f(x)dx = \int_{\mathbb{T}^2} \theta_0(x)dx = 0$, which immediately implies that a solution θ of (1.5)–(1.7) obeys $\int_{\mathbb{T}^2} \theta(x,t)dx = 0$ for any $t \geq 0$. In Chapter 3 we work exclusively with solutions of zero average.

1.2.1 Physical origins

The SQG equation describes the evolution of a surface temperature field θ in a rapidly rotating, stably stratified fluid with potential vorticity [35, 92]. The model is technically three-dimensional. The atmospheric temperature field at "higher altitudes" satisfies the Laplace equation on the upper half-space with θ as Dirichlet boundary data. From the mathematical point of view, the non-dissipative SQG equations (the system (1.5)–(1.7) with $\kappa = 0$) have properties that are similar to those of the 3D Euler equations in vorticity form [35], and yet one may for instance prove the global existence of finite energy weak solutions [121], albeit for completely different reasons than for 2D Euler. Initial numerical simulations [35, 36, 119] have furthermore exhibited highly nonlinear features in the inviscid evolution such as the formation of sharp fronts. The latter issue has been studied analytically [16, 45, 46, 47, 72] and also in more recent numerical simulations [34, 48, 60, 118]. Whether solutions of the inviscid SQG equations can develop singularities in finite time remains an open problem.

If in the three-dimensional quasigeostrophic equations [120] we take into account damping (Ekman pumping at the boundary proportional to the vertical derivative of the 3D atmospheric temperature field at the surface) and external sources we arrive at the system (1.5)–(1.7). The square root of the Laplacian naturally arises as the Dirichlet-to-Neumann map for the surface temperature θ; this is in contrast to the more common origin of Lévy processes for nonlocal diffusive operators. See e.g. [24] and references therein.

Among the dissipative operators Λ^α with $0 < \alpha \leq 2$, the exponent $\alpha = 1$ appearing in (1.5) is not just physically motivated but also mathematically challenging. In view of the underlying scaling invariance associated with (1.5) (see Remark 3.1.3 below) and the available conservation laws (the L^∞ maximum principle), it is customary to refer to the dissipation power $\alpha = 1$ as *critical*, although the question of whether a dramatic change in the behavior of the solution occurs for $\alpha < 1$ (the *supercritical*

case) remains an open problem.

1.2.2 Properties

From the mathematical point of view, the issue of *global regularity* vs *finite-time blowup* for the fractionally dissipative SQG equation has been extensively studied over the past two decades. The subcritical case ($\alpha > 1$) has been essentially resolved in [41, 121]. See also [66, 98, 114, 125, 139] and references therein for further qualitative properties of weak and strong solutions in the subcritical case.

The issue of global regularity for the critical SQG equation is more challenging since the balance between the nonlinearity and the dissipative term in (1.5)–(1.7) remains the same no matter at which scales one zooms in. This is why treating the equation as a perturbation of the fractional heat equation fails to be useful for large initial data (and nonlinearity). In order to obtain bounds on the solution one has to appeal to finer methods. Two different such sophisticated tools have been introduced independently in [13, 106] to show that solutions of (1.5)–(1.7) do not develop singularities in finite time, for arbitrarily large initial data. Before these works, the global regularity was only known for initial data which is small in the L^∞ norm [26] and other scaling critical spaces [17, 18, 44, 93, 100, 117, 140].

The proof of global regularity in [106] is based on constructing a family of Lipschitz moduli of continuity with the property that if the initial data obeys such a modulus so will the solution of the critical SQG equation for all later times, and so that this family behaves nicely under rescaling. See also [65] for applications of the modulus of continuity method to the case of the whole space, [86] for global regularity in the presence of a Lipschitz forcing term, [105] in the presence of a linear dispersive force, and [129] for regularity of critical linear drift-diffusion equations.

The proof of [13] on the other hand employs the ideas of De Giorgi iteration to the case of a nonlocal parabolic equation, and shows that bounded weak solutions

must in fact instantly become Hölder continuous (and hence classical). We also refer to [11, 14] and references therein for applications of the De Giorgi ideas to more degenerate nonlocal parabolic equations, and [43, 128] for linear fractional advection-diffusion equations (the latter results for linear equations are in fact sharp [130]).

In an attempt to find a bridge between these two proofs, in [104] a completely different proof of global regularity for critical SQG was obtained. The proof of [104] relies on keeping track of the action of a dual SQG evolution on a suitable family of test functions (Hardy molecules), in order to control the evolution of a Hölder norm.

In the three proofs of global regularity for (1.5)–(1.6) mentioned above, the quantitative way in which the dissipation is dominating the nonlinear term is not immediately transparent. In order to shed more light on this issue [40] introduced *nonlinear* lower bounds for *linear* nonlocal operators such as the fractional Laplacian. The nonlinear nature of these lower bounds appears because of the conservation laws which are available a priori for solutions. The inequalities bound the dissipative term from below and the nonlinear term from above in a way which makes it transparent why the *size* of the dissipation dominates the *size* of the nonlinearity. In [40] it was shown that the nonlinear lower bound for Λ is almost sufficient: it implies the global regularity for the critical SQG equation, but only if we additionally know that the solution has the *only small shocks* property (there exits $0 < \delta < 1$ and $L > 0$ such that $|\theta(x,t) - \theta(y,t)| \leq \delta$ whenever $|x - y| \leq L$). It was proved that if the initial data has only small shocks, the size of these shocks can at most double as time evolves. This extra step is however quite technical as it once more uses the nonlinear lower bound for the fractional Laplacian, and it depends strongly on the nature of the nonlinearity.

At this stage we note that to date the global regularity for the supercritical SQG equation with arbitrarily large initial data remains open. The type of known results are: small data global well-posedness (cf. [17, 18, 26, 44, 93, 100, 117, 140, 142] and references therein), conditional regularity (cf. [42, 43, 67, 68]), eventual regular-

ity (cf. [50, 103, 127]), or global regularity for dissipative operators which are only logarithmically supercritical (cf. [51, 52, 141]). The nonlinear lower bound for the fractional Laplacian (cf. [39, 40] and this work) may be employed to recover most of these results. A recent result [137] improves the eventual regularity theory through the use of similar nonlinear lower bounds, obtaining global regularity for genuinely supercritical SQG equations (i.e., dissipation like Λ^{1-s} for $s > 0$) and initial data bounded in H^2 by a constant $M(s)$ with the property that $M(s) \to \infty$ as $s \to 0$. See also [73] for an analysis of almost sharp fronts in the dynamics of fully inviscid SQG.

1.2.3 The issue of anomalous dissipation

One of the basic statements of the physical theory of hydrodynamics concerns the anomalous dissipation of energy in three dimensional turbulence [87]. It has largely been verified experimentally [131], but not mathematically. The statement concerns the average behavior of the rate of energy dissipation

$$\epsilon := \nu \langle |\nabla u^{(\nu)}|^2 \rangle$$

as $\nu \to 0$. Here ν is the kinematic viscosity, $u^{(\nu)}$ is the velocity (satisfying a hydrodynamic equation with viscosity parameter ν, and so divergence-free and usually mean zero), and $\langle \dots \rangle$ represents an ensemble average or space-time average. The assertion of turbulence theory is that ϵ is a *positive number*, and that it does not vanish in the limit of zero viscosity. The term "anomalous dissipation" comes from field theory in physics and refers to the fact that, in the limit of vanishing viscosity, there is still remnant dissipation, even though the limit equation conserves energy.

There are two distinct approaches to the question of anomalous dissipation. In the first, the limit of zero viscosity is taken on solutions of the initial value problem with fixed initial data. Under appropriate conditions this leads to a solution of the corre-

sponding initial value problem of the inviscid equation. This equation conserves energy if solutions are smooth, but might dissipate energy if solutions are not sufficiently smooth. This circle of ideas, and specifically the precise degree of smoothness needed, goes by the name of "Onsager conjecture" [20, 27, 55, 56, 57, 58, 59, 69, 70, 71, 94]. This approach is therefore about the initial value problem for the limit equations and it requires a lack of smoothness from the solutions. The blow-up problem is open for the 3D incompressible Euler equations, and this allows for the possibility of dissipative solutions arising from smooth initial data. Anomalous dissipation of energy is known for incompressible the 2D Euler equations, though only for very rough solutions; in two dimensions non-smooth solutions cannot arise spontaneously from smooth ones. The class of dissipative solutions of the inviscid equations is very large.

The second way of looking at the anomalous dissipation issue is to take long time averages first, in order to achieve a "permanent regime" of the viscous equations, and only then send the viscosity to zero. This second approach is espoused in Chapter 2 for the critical, forced SQG equation with vanishing hyperviscosity. Denoting by $S^{(\nu)}(t, u_0)$ the solution of the viscous equation at time $t \geq 0$ which started at time $t = 0$ from the initial data u_0, the second approach looks at

$$\langle |\nabla u|^2 \rangle = \lim_{T\to\infty} \frac{1}{T} \int_0^T \int_{\mathbb{R}^d} \left| \nabla (S^{(\nu)}(t, u_0)) \right|^2 dx dt$$

and asks if $\lim_{\nu\to 0} \nu \langle |\nabla u|^2 \rangle = \epsilon$ is positive and (if so) how ϵ depends on the forcing and initial data. In the absence of forcing the long time limit vanishes even for Leray weak solutions of 3D Navier-Stokes equations: the "permanent regime" is trivial, and turbulence is decaying. One must then take finite time averages, with durations that diverge with vanishing viscosity at an appropriate rate. This unforced case is perhaps the only instance where a general, global, viscosity-independent, a priori upper bound on ϵ is rigorously known.

The long time averaged dissipation may be non-trivial if the flow is forced, either at the boundary or with body forces. A conceptual difficulty arises then because there exist situations in which the rate of dissipation, as defined, is infinite. There are rigorous studies [53, 54, 62] where bounds for ϵ are expressed in terms of the average kinetic energy of the solutions in the case of forced Navier-Stokes equations; however, there are no a priori bounds on the average kinetic energy that are independent of viscosity or Reynolds number. The question of obtaining examples and a characterization of flows with uniform upper bounds for ϵ is open.

In fact, the opposite situation can be easily found: ϵ can be unbounded when we consider spatially periodic 2D forces that are eigenfunctions of the Stokes operator. We write the Navier-Stokes equation symbolically as

$$\partial_t u + \nu A u + B(u,u) = f,$$

where $A = -\mathbb{P}\Delta$ is the Stokes operator with $\mathbb{P}$ the projector on divergence-free vectors and $B(u,u) = \mathbb{P}(u \cdot \nabla u)$ is the quadratic nonlinearity. We take $Af = \lambda f$ with f and u periodic, divergence-free functions of two spatial variables. We consider $u_f = \frac{1}{\nu\lambda} f$. This is a smooth, time independent solution. Indeed u_f satisfies the time-independent, unforced Euler equation $B(u_f, u_f) = 0$. If $u_0 = u_f$ then $\langle |\nabla u_f|^2 \rangle = \nu^{-2}\lambda^{-1}\|f\|^2_{L^2}$, where f does not depend on ν. This makes the limit of ϵ infinite as viscosity vanishes. The steady solutions u_f are perfectly admissible as 3D periodic flows as well. They are unstable if λ is not the first eigenvalue of the Stokes operator, but stable in 2D if λ is the first eigenvalue. Incidentally, the upper bound $\epsilon \leq C(U^3\sqrt{\lambda} + \nu\lambda U)$ (with $U = \|u_f\|_{L^2}$) of [62] is true in this case as well; it is imprecise, however, with both left and right hand sides diverging as $\nu \to 0$, but at different rates.

An interesting recent asymptotics and numerical study [88] reports finding solutions of the 2D Navier-Stokes equation that "settle" to a condensate that has a

nontrivial component in the first eigendirection of the corresponding Stokes operator and has bounded amplitude as viscosity vanishes. This of course is impossible for all initial data, as demonstrated above, but it is an intriguing possibility for parts of the phase space. Assuming smooth initial data and forcing (even allowing f to not be an eigenfunction of the Stokes operator), the solution of the Navier-Stokes equations converges to the corresponding solution of the forced Euler equation $u(t) = S^{(0)}(t, u_0)$ on $[0, T]$, for any T, a function that solves

$$\partial_t u + B(u, u) = f$$

with initial data u_0. For Kolmogorov forcing (forces which are eigenfunctions of the Stokes operator), the putative existence of time independent solutions $u^{(\nu)}$ which are uniformly bounded in ν in energy norm, implies the convergence of (a subsequence of) $u^{(\nu)}$ to a time independent solution u^0 of the forced incompressible Euler equations $B(u^0, u^0) = f$. If the solutions $S^{(\nu)}(t, u_0)$ are at a bounded distance from $u^{(\nu)}$ uniformly in time, for large time, one can prove that $S^{(0)}(t, u_0)$ are the same bounded distance from u^0 for large time. In particular, if $S^{(\nu)}(t, u_0)$ converges uniformly in time to $u^{(\nu)}$, then $S^{(0)}(t, u_0)$ converges in time to $u^{(0)}$. Smooth steady solutions of the forced Euler equations $B(u^{(0)}, u^{(0)}) = f$ with Kolmogorov forcing can be easily constructed, but for which initial data solutions converge to them is another matter. Such behavior, if it exists at all, must be rather special.

The dynamics of the forced Euler equation, the existence of bounded sequences of stationary solutions of the periodic, forced Navier-Stokes equations, and even of solutions with bounded average dissipation of energy remain open problems.

Bounds can be obtained for 2D forced Navier-Stokes equations with bottom drag (friction)

$$\partial_t u + \gamma u + \nu A u + B(u, u) = f,$$

where the friction coefficient $\gamma > 0$ is kept fixed. Then the energy is bounded (in terms of γ) uniformly for small ν and indeed so is the enstrophy (H^1 norm) uniformly in ν. Consequently, there is no anomalous dissipation of energy. The absence of anomalous dissipation of enstrophy is more subtle because there are no upper bounds for the average H^2 norm for arbitrary forces. The paper [37] proves nevertheless that the dissipation of enstrophy vanishes in the limit of zero viscosity, for arbitrary time-independent forces.

1.2.4 Absence of anomalous dissipation for forced, viscous, critical SQG

In Chapter 2, we consider the equation

$$\partial_t \theta + (\mathcal{R}^\perp \theta) \cdot \nabla \theta + \kappa \mathcal{D} \theta - \nu \Delta \theta = f$$

in $\mathbb{R}^2$, where $\mathcal{D} = \mathbb{I} + \Lambda$ is the damping operator, $\mathcal{R}^\perp = (-\mathcal{R}_2, \mathcal{R}_1)$ are Riesz transforms as before, $f \in L^\infty(\mathbb{R}^2) \cap L^1(\mathbb{R}^2)$ is time independent deterministic forcing, $\kappa > 0$ is fixed, and $\nu > 0$. We prove that there is no dissipative anomaly,

$$\lim_{\nu \to 0} \nu \langle |\nabla \theta|^2 \rangle = 0,$$

where $\langle \cdots \rangle$ is space-time average on solutions. The proof of absence of anomalous dissipation follows the same blueprint as the proof in [37]. We establish first (Proposition 2.1.1) that the viscous semi-orbits are relatively compact in the phase space. Then we introduce the adequate statistical solutions for both viscous ($\nu > 0$) and inviscid ($\nu = 0$) equations. These are measures in phase space, shown in Theorem 2.4.2 to arise naturally as long time limits on solutions. Next Theorems 2.2.5 and 2.3.1 prove that the zero viscosity limits of statistical solutions of the viscous equa-

tions are themselves statistical solutions of the inviscid equations, and that these preserve the energy balance. Once this is achieved, the absence of anomalous dissipation (Theorem 2.5.1) follows from an argument by contradiction. There are a number of technical difficulties encountered in the proof for SQG that are not present in the case of 2D Navier-Stokes. In order to obtain the uniform integrability property on positive semiorbits we use nonlocal calculus identities. The weak continuity of the nonlinear term, the most technical part of the crucial Lemma 2.2.3, is proved using a commutator structure of the nonlinearity; a structure that was used already in [121]. The energy balance is proved using a formula for nonlinear fluxes [27] and a bound in $H^{\frac{1}{2}}$ that is available for critical SQG, and that replaces the Besov space argument of [20, 27].

1.2.5 The nonlinear lower bounds imply global regularity

In Chapter 3, we give a fully transparent, new and final proof of global regularity for the critical SQG equation, which relies *solely* on the nonlinear lower bound for the fractional Laplacian (Theorem 3.3.5) established in [40]. One of the main new ideas over [40] is to replace the smallness coming from the "only small shocks" property (for which time dependence is difficult to control) with the smallness of the α at the exponent of the Hölder space C^α. This simple idea is also implicitly present in the De Giorgi's methodology: the size of the Hölder exponent obtained in the bootstrap step depends on the size of the data in L^∞, the ellipticity constant, and the size of the force. The advantage of the method outlined below is that the propagation of C^α regularity, for some *sufficiently small* α, is achieved directly, without appealing to the complex De Giorgi iteration technique (Theorem 3.3.3). Once the solution is Hölder continuous, the nonlinear lower bound may also be used to immediately obtain the higher regularity of solutions. To better outline the method introduced here, we first apply it to the critical Burgers equation in Section 3.2, and then to the critical SQG

equation in Section 3.3.

We note that the proof of propagation of Hölder continuity also works in the presence of a force which lies merely in $L^\infty \cap H^1$ (the modulus of continuity proof given in [86] requires Lipschitz forcing). Moreover, the argument is *dynamic* rather than in the spirit of a maximum principle, in the sense that the size of the Hölder norm of the solution is not just bounded in terms of the initial data and force, but as time evolves its size is estimated solely in terms of the force (Theorem 3.4.2 and Lemma 3.4.3). In the unforced case this amounts to proving the decay of the Höder norm and higher Sobolev norms.

1.2.6 Long time dynamics: the existence of a finite dimensional global attractor

Armed with a proof of regularity that is dynamic, we continue Chapter 3 with a study on the long time dynamics of solutions of the forced SQG equation. The bounds we obtain on the solution incorporate an explicit decay from the initial data in both weak and *strong topologies* (Theorem 3.4.2). Using this crucial ingredient we prove the existence of a compact global attractor for the forced SQG dynamics (Theorem 3.4.1); the decay in the unforced case has been addressed in [64]. We also prove it has finite box-counting dimension, and a forteriori finite Hausdorff dimension (Theorem 3.5.4). The space-periodic setting is needed for this purpose. The critical SQG equation is *quasilinear*. To the best of our knowledge, until now all proofs of finite dimensionality have been done either for dissipative or damped *semilinear* equations, or *subcritical* quasilinear equations. See e.g. the works [21, 23, 28, 30, 31, 32, 63, 77, 80, 81, 84, 82, 83, 85, 90, 96, 97, 109, 143] for the 2D periodic Navier-Stokes equations and related systems, the books [3, 19, 29, 79, 91, 113, 122, 135], and references therein. In the context of the dissipative SQG equation, the global attractor has been previously addressed only for the subcritical regime: [99] proved the existence of a compact

global attractor and [138] showed it has finite fractal dimension (see also [6] for weak attractors).

In order to prove finite dimensionality we establish the existence of a compact absorbing set in phase space. We work in the phase space H^1, which is the largest Hilbert space in which uniqueness of weak solutions of SQG is currently available. Weak solutions are known to exist for initial data in L^2 but their uniqueness is not known [121]. It is known that solutions of the unforced SQG with initial data in H^1 exist for short time [100], as a result of weak-strong stability of the equation in $H^{1+\epsilon}$. The time of existence however depends on the initial function θ_0 and *not only on its norm*. There is no lower bound on the time of existence based solely on the H^1 norm. Nevertheless, the solution is unique, and becomes instantly smooth [64, 100, 117]. The same result can be proved for smooth forced SQG. We need only a limited amount of smoothness, in particular C^α, for small $\alpha > 0$. We use the nonlinear maximum principle [40] (estimate (3.29) below) to show the global persistence of a C^α norm, with α small compared to the L^∞ norm of the solution (Theorem 3.3.3). We use the L^p Poincaré inequality for the fractional Laplacian (Proposition 3.1.2 below) to show that solutions become bounded in L^∞ with a bound that depends only on the norm of the force and not on the initial data, after a time that depends on the initial H^1 data. We apply again the new proof of global existence to show that the solution becomes bounded in a C^α space with both α and the solution bound depending only on norms of the forces (Lemma 3.4.3). Since we now have supercritical information, we use the nonlinear maximum principle again in its C^α variant to show that the solution becomes bounded in $H^{3/2}$. The upshot is that there exists a compact absorbing set $\mathcal{B}$ for the evolution of SQG in H^1 which is a bounded set in $H^{3/2}$ (Theorem 3.4.2). This means that for any initial data $\theta_0 \in H^1$ there exists a time $t(\theta_0)$ after which the unique solution $S(t)\theta_0$ with initial datum θ_0 belongs to this absorbing set $\mathcal{B}$, which in turn depends on the forces alone and not on the initial data. We show additional properties

of the solution after this transient time: higher regularity, specifically $S(t)\theta_0 \in H^2$ for almost all $t > t(\theta_0)$, continuity in phase space (Proposition 3.4.5), and backward uniqueness (Proposition 3.4.7), i.e., the injectivity of $S(t)$ on the absorbing set. These properties are used to show that the the set $\mathcal{A} = \cap_{t>0} S(t)\mathcal{B}$ is the global attractor for the evolution in H^1 (Theorem 3.4.1). For any $\theta_0 \in H^1$ the solution tends to the global attractor. The convergence is uniform on bounded sets in $H^{1+\epsilon}$. We also establish compactness of the linearization of the solution map and local uniform approximation results (Proposition 3.5.2). The finite dimensionality of the attractor (Theorem 3.5.4) is then obtained by applying classical tools [28, 29].

1.3 Asymptotic behavior for the fractional Fisher-KPP equation

In Chapter 4 we investigate a strong symmetrization phenomenon, observed in [49], for the level sets of reaction-diffusion equations of Fisher-KPP type with fractional diffusion. The model under consideration is

$$\partial_t u + \Lambda^\alpha u = f(u) \tag{1.8}$$

on $\mathbb{R}^d$. Here, $\alpha \in (0, 2)$ and Λ^α is again the fractional Laplacian of order $\alpha/2$. The nonlinear term f is assumed to be of KPP-type [108]: $f(0) = f(1) = 0$, $f(y) > 0$ for $y \in (0, 1)$, and $f'(y) < f(y)/y$. Notation-wise, let $K = f'(0)$ and $\lambda = K/(d + \alpha)$.

1.3.1 Physical origins

The Kolmogorov-Petrovsky-Piskounov (KPP) equation (in dimensionless form) [108],

$$\partial_t u - \Delta u = f(u),$$

was introduced for the rigorous analytical study of reaction-diffusion models. In [74], the special nonlinearity $f(u) = u(1-u)$ was introduced, originally focusing on the spatial spread of an advantageous allele in a given population. The prototypical nonlinearity above has since been generalized to a broader class of local reaction terms of KPP-type (see below) that maintain the key properties required to reproduce the analysis for the simpler case.

Such equations appear in the sciences as models of diverse physical, chemical, and biological phenomena. The above equation has been used to study flame propagation, crystallization, plasma physics, and nuclear reactors. Since f may be nonlinear, explicit solutions usually cannot be found. Much attention has been given to the propagation of reaction fronts in these equations. The nature of the equation may isolate certain wave forms and allow only these to propagate without distortion, or it may propagate different profiles at different speeds.

1.3.2 Properties

With the assumptions above, and for a compactly supported initial datum $u(.,0) := u_0$, it is well-known that the level sets of u will spread exponentially fast in time (Cabré-Roquejoffre [10]). Under the above assumptions, for all $c > \lambda$, we have

$$\lim_{t\to+\infty} \sup_{|x|\geq e^{ct}} u(x,t) = 0$$

and for all $c < \lambda$, we have

$$\lim_{t\to+\infty} \inf_{|x|\leq e^{ct}} u(x,t) = 1.$$

Thus, when renormalized by the exponential, the level sets of u asymptotically look round. It is then natural to ask whether this property holds in a more precise fashion, and a first answer is that given in Cabré-Coulon-Roquejoffre [9]: for a given $h \in (0,1)$

we have

$$\{u = h\} \subseteq \{C^{-1}e^{\lambda t} \leq |x| \leq Ce^{\lambda t}\},$$

for some constant $C > 0$. The next question is whether one can make this constant more precise. Given the rapid growth of the level sets, one could expect the possibility for very erratic behavior. To confirm this, the case was investigated numerically by A.-C. Coulon in her PhD thesis. We reproduce here a sample of her simulations. The initial datum (Fig. 1) is pictured below,

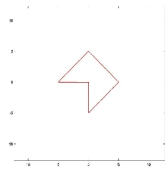


Figure 1.1: Level sets of u_0

and the time evolution is shown in Fig. 2.

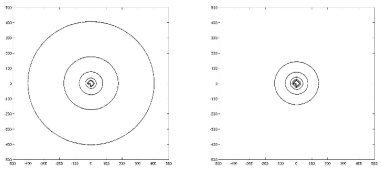


Figure 1.2: Level sets of u: $\alpha = 1$ (left) and $\alpha = 1.6$ (right)

Surprisingly, this strongly suggests asymptotic symmetrization.

1.3.3 Symmetrization for the standard F-KPP equation

For the standard F-KPP equation, we have a very general theorem of Jones [95]: for the solutions of (1.8) with $\alpha = 2$:

$$\partial_t u - \Delta u = f(u) \tag{1.9}$$

with any nonlinearity f (not limited to KPP) and compactly supported u_0, the theorem asserts that for any later time, and for any regular value h of u, and for any x on the level set $\{u(.,t) = h\}$, the normal line $\Delta_{x,t}$ to the level set passing through x intersects the convex hull of $\operatorname{supp} u_0$. If the level sets of u expand, this has very important consequences: the level sets of u symmetrize asymptotically, and have bounded oscillation. This is a very remarkable result, notably because it does not depend on the precise expansion rate of the level sets of u. In particular, if the nonlinearity f is of KPP type, then

$$\{u = h\} = \{|x| = 2\sqrt{K}t - \frac{d+2}{2}\mathrm{Log}t + O(1)\}. \tag{1.10}$$

The term $2\sqrt{K}t$ is due to Aronson-Weinberger [2], and the logarithmic correction to Bramson [8] ($d = 1$), and Gärtner [89] ($d > 1$). We note here the sharp qualitative change in behavior between the exponentially advancing level sets of the fractional F-KPP equation and the *linearly* advancing level sets of the standard F-KPP equation. The nonlocal diffusion draws from values of u on all of $\mathbb{R}^d$, so that as the invasion proceeds Λu becomes stronger along the reaction front.

We briefly describe a proof, due to H. Berestycki [5], of Jones's theorem. Assuming the contrary, there exists a hyperplane H containing $\Delta_{x,t}$ and such that the convex hull of $\operatorname{supp} u_0$ lies strictly in H^-, the lower half space bounded by H. If $\widehat{u}$ is the reflection of $u(.,t)$ about H, and $v = u - \widehat{u}$, then v satisfies a linear equation and

is positive in H^-, since it is positive at $t = 0$ and vanishes on H. At time t, any derivative of v in a direction normal to H is nonzero, contradicting that $\Delta_{x,t} \subset H$. We immediately see that this argument cannot be applied to our case, simply because v would need to be nonnegative in H^+ instead of H^-, which is impossible.

1.3.4 Decay of derivatives and renormalized symmetrization

In Chapter 4 we show that, nevertheless, a rather strong form of symmetrization occurs. The main ingredient is the comparative decay estimate for derivatives of u of the form

$$|\partial u(x,t)| \leq ce^{-\delta t}u(x,t) \quad , \quad |\partial^2 u(x,t)| \leq ce^{-\delta t}u(x,t).$$

This is shown in Theorem 4.2.1; we believe the result is of general interest. The proof is divided into three stages corresponding to three different ranges of $|x|$. In the long range, we employ a Duhamel formula and make use of known estimates for the heat kernel of the fractional Laplacian. In the medium range, we construct a radial supersolution, treating the fractional Laplacian as a perturbation term (this step is the most technical and necessitates this three-stage approach since the analysis does not cover the short and long ranges). In the short range, we have that $f'(u)$ is negative, and can obtain a relative maximum principle for the derivatives of u. The proof is completed with an argument by contradiction.

Theorem 4.2.1 allows us to prove an analogous estimate for the fractional Laplacian. This permits us in turn to reduce the problem to the ODE $\dot{u} = (1+O(e^{-\delta t}))f(u)$, which is much simpler to study. The main result of Chapter 4 is Theorem 4.4.1, establishing symmetrization of the level sets of u in an exponentially renormalized sense. This provides an explanation for the observed behavior of u in Figures 1 and 2. However we note that our results could be improved in two ways.

- The assumptions on u_0 seem slightly non optimal: indeed, it would certainly

suffice to have $u_0(x) = O(|x|^{d+\alpha+\epsilon})$ with $\epsilon > 0$. Our theorem would remain valid under that assumption, at the cost of heavier computations. However, when u_0 decays like (or slower than) $|x|^{d+\alpha}$ different phenomena occur, as was observed in [49].

- We do not go as far as proving a Jones type theorem. Indeed what we obtain is a precise estimate of the normalized level sets instead of the true level sets. In other words, lower order (but still exponentially growing) terms may prevent a full symmetrization.

These issues will be the subjects of future investigations.

Chapter 2

Absence of anomalous dissipation of energy in forced critical SQG

2.1 Forced, viscous, critical SQG

For convenience, we recall the setting. We consider the equation

$$\partial_t \theta + u \cdot \nabla \theta + \gamma \mathcal{D}\theta - \nu \Delta \theta = f \tag{2.1}$$

for a scalar valued $\theta : \mathbb{R}^2 \times [0, \infty) \to \mathbb{R}$. Here

$$u = \mathcal{R}^{\perp}\theta \tag{2.2}$$

with $\mathcal{R}^{\perp} = (-\mathcal{R}_2, \mathcal{R}_1)$, and $\mathcal{R} = \nabla(-\Delta)^{-\frac{1}{2}}$ the Riesz transforms. The damping operator $\mathcal{D}$ is given by

$$\mathcal{D} = \Lambda + 1 \tag{2.3}$$

with $\Lambda = (-\Delta)^{\frac{1}{2}}$. The coefficient $\gamma > 0$ is fixed throughout the chapter and the coefficient $\nu > 0$ is a parameter that we will let vary. The force $f \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$

is fixed and time independent. We will use the pointwise identity [25, 44]

$$2\varphi(x)\cdot\Lambda\varphi(x)=\Lambda(|\varphi|^2)(x)+D[\varphi](x) \tag{2.4}$$

with

$$D[\varphi](x)=c\int_{\mathbb{R}^2}\frac{(\varphi(x)-\varphi(y))^2}{|x-y|^3}dy. \tag{2.5}$$

This is a special case of Proposition 3.1.1 from Chapter 3.

Proposition 2.1.1. *Let $\nu>0$, $f\in L^1(\mathbb{R}^2)\cap L^\infty(\mathbb{R}^2)$, $\theta_0\in L^1(\mathbb{R}^2)\cap L^\infty(\mathbb{R}^2)$. The solution $\theta(x,t)=S^{(\nu)}(t,\theta_0)$ of (2.1) exists for all time, is unique, satisfies the energy equation*

$$\frac{d}{2dt}\|\theta\|^2_{L^2(\mathbb{R}^2)}+\gamma\|\theta\|^2_{H^{\frac{1}{2}}(\mathbb{R}^2)}+\nu\|\nabla\theta\|^2_{L^2(\mathbb{R}^2)}=(f,\theta)_{L^2(\mathbb{R}^2)} \tag{2.6}$$

and the bounds

$$\|\theta(\cdot,t)\|_{L^p(\mathbb{R}^2)}\le e^{-\gamma t}\left\{\|\theta_0\|_{L^p(\mathbb{R}^2)}-\frac{1}{\gamma}\|f\|_{L^p(\mathbb{R}^2)}\right\}+\frac{1}{\gamma}\|f\|_{L^p(\mathbb{R}^2)} \tag{2.7}$$

for $1\le p\le\infty$. Moreover the positive semi-orbit

$$O_+(\theta_0):=\{\theta=\theta(\cdot,t)\,|\;\;t\ge 0\}\subset L^2(\mathbb{R}^2)$$

is uniformly integrable: for every $\epsilon>0$, there exists $R>0$ such that

$$\int_{|x|\ge R}|\theta(x,t)|^2dx\le\epsilon \tag{2.8}$$

holds for all $t\ge 0$.

We used the notation

$$(f,g)_{L^2(\mathbb{R}^2)}=\int_{\mathbb{R}^2}f(x)g(x)dx$$

and we note that

$$\|\theta\|^2_{H^{\frac{1}{2}}(\mathbb{R}^2)} = (\mathcal{D}\theta, \theta)_{L^2(\mathbb{R}^2)}.$$

Proof of Proposition 2.1.1. Existence, uniqueness and regularity follows along well established lines and will not be presented here. The bounds (2.7) follow from the maximum principles and nonlocal calculus identities of which (2.4) is the quadratic example (see [44]) and which imply that

$$\int_{\mathbb{R}^2} \varphi^{p-1} \Lambda \varphi dx \geq 0$$

if p is even or if φ is nonnegative. A formal proof can be found in Chapter 3, Section 3.3, for Proposition 3.3.1. The uniform integrability property (or "no-travel" property [37]) is all that remains. We consider the function

$$Y_R(t) = \int_{\mathbb{R}^2} \chi\left(\frac{x}{R}\right) \theta^2(x,t) dx$$

where χ is a nonnegative smooth function supported in $\{x \in \mathbb{R}^2 \mid \ |x| \geq \frac{1}{2}\}$ and identically equal to 1 for $|x| \geq 1$. We take (2.1), multiply by $2\chi(\frac{x}{R})\theta(x)$ and integrate. The more challenging term we encounter is

$$2\gamma \int_{\mathbb{R}^2} (\Lambda\theta(x))\chi\left(\frac{x}{R}\right) \theta(x,t) dx.$$

Using (2.4) we have

$$\begin{aligned}
&2\gamma \textstyle\int_{\mathbb{R}^2} (\Lambda\theta(x))\chi\left(\frac{x}{R}\right) \theta(x,t) dx \\
&\geq \gamma \textstyle\int_{\mathbb{R}^2} \Lambda(\theta(x)^2) \left(1 - \left(1 - \chi\left(\frac{x}{R}\right)\right)\right) dx \\
&= -\gamma \textstyle\int_{\mathbb{R}^2} (\theta(x)^2) \Lambda \left(1 - \chi\left(\frac{x}{R}\right)\right) dx
\end{aligned}$$

where $\Lambda(1-\chi)$ is well defined because $1 - \chi \in C_0^\infty$. Moreover $1 - \chi\left(\frac{x}{R}\right) = \varphi\left(\frac{x}{R}\right)$ and

therefore, in view of the fact that $\Lambda(\varphi(\frac{x}{R})) = \frac{1}{R}(\Lambda\varphi)(\frac{x}{R})$ and $|\Lambda\varphi(x)| \leq C$,

$$2\gamma \int_{\mathbb{R}^2} (\Lambda\theta(x))\chi\left(\frac{x}{R}\right)\theta(x,t)dx \geq -\frac{C\gamma}{R}\|\theta(\cdot,t)\|^2_{L^2(\mathbb{R}^2)}.$$

The contribution of the nonlinear term $u \cdot \nabla\theta$ is bounded by integrating by parts and using

$$\|u\|_{L^3(\mathbb{R}^2)} \leq C\|\theta\|_{L^3(\mathbb{R}^2)}.$$

The contribution of the forcing term is bounded by

$$2\left|\int_{\mathbb{R}^2} f\chi\theta dx\right| \leq C\|\theta(\cdot,t)\|_{L^2(\mathbb{R}^2)}\sqrt{\int_{|x|\geq\frac{R}{2}} |f(x)|^2 dx}.$$

We obtain

$$\begin{aligned}
&\tfrac{d}{dt}Y_R(t) + 2\gamma Y_R(t) \\
&\leq \tfrac{C}{R}\left[\|\theta(\cdot,t)\|^3_{L^3(\mathbb{R}^2)} + \gamma\|\theta(\cdot,t)\|^2_{L^2(\mathbb{R}^2)}\right] + \tfrac{C\nu}{R^2}\|\theta(\cdot,t)\|^2_{L^2(\mathbb{R}^2)} \\
&+C\|\theta(\cdot,t)\|_{L^2(\mathbb{R}^2)}\sqrt{\textstyle\int_{|x|\geq\frac{R}{2}} |f(x)|^2 dx}.
\end{aligned}$$

Because of (2.7) and the fact that f^2 is integrable, the right hand side is as small as we wish, uniformly in time, provided R is chosen large enough. The choice of R depends only on γ, f and on norms of θ_0 in L^2 and L^3, and can be made uniformly in ν for bounded ν, although we do not need this. Once we choose R so that the right-hand side is less than $\gamma\epsilon$ we have the inequality

$$Y_R(t) \leq e^{-2\gamma t}Y_R(0) + \frac{\epsilon}{2}$$

and the uniform integrability follows from the fact that $Y_R(0)$ is small for large R. $\square$

2.2 Stationary statistical solutions

We introduce first the notion of stationary statistical solution for forced viscous SQG, in the spirit of [75, 76] and [37]

Definition 2.2.1 (**Viscous Stationary Statistical Solution**). *A stationary statistical solution of (2.1) is a Borel probability measure $\mu^{(\nu)}$ on $L^2(\mathbb{R}^2)$ such that*

$$\int_{L^2(\mathbb{R}^2)} \|\theta\|_{H^1}^2 d\mu^{(\nu)}(\theta) < \infty \qquad ((a))$$

$$\int_{L^2(\mathbb{R}^2)} (N^{(\nu)}(\theta), \Psi'(\theta))_{L^2(\mathbb{R}^2)} d\mu^{(\nu)}(\theta) = 0 \qquad ((b))$$

for all $\Psi \in \mathcal{T}$, and

$$\int_{E_1 \le \|\theta\|_{H^{\frac{1}{2}}} \le E_2} \left(\gamma \|\theta\|_{H^{\frac{1}{2}}}^2 + \nu \|\nabla\theta\|_{L^2(\mathbb{R}^2)}^2 - (f,\theta)_{L^2(\mathbb{R}^2)} \right) d\mu^{(\nu)}(\theta) \le 0 \qquad ((c))$$

for all $E_1 \le E_2$.

Here

$$N^{(\nu)}(\theta) = R^{\perp}\theta \cdot \nabla\theta + \gamma \mathcal{D}\theta - \nu\Delta\theta - f \tag{2.9}$$

and the class of cylindrical test functions $\mathcal{T}$ is defined by

Definition 2.2.2 (**Cylindrical Test Functions**). *$\Psi \in \mathcal{T}$ if there exist N, $w_1, \ldots, w_N$ $\in C_0^\infty(\mathbb{R}^2)$, $\epsilon \ge 0$ and $\psi : \mathbb{R}^N \to \mathbb{R}$, smooth, such that*

$$\Psi(\theta) = \psi((J_\epsilon(\theta), w_1)_{L^2(\mathbb{R}^2)}, \ldots, (J_\epsilon(\theta), w_N)_{L^2(\mathbb{R}^2)})$$

with J_ϵ a standard mollifier, i.e. convolution with $\epsilon^{-2} j(\frac{x}{\epsilon})$, $j \in C_0^\infty(\mathbb{R}^2)$, $j \ge 0$, $j(-x) = j(x)$, $\int_{\mathbb{R}^2} j(x)dx = 1$, if $\epsilon > 0$, and $J_\epsilon = \mathbb{I}$ if $\epsilon = 0$.

We note that the test functions are locally bounded and sequentially weakly continuous in $L^2(\mathbb{R}^2)$. We recall the elementary but important fact that weak continuity of real valued functions implies strong continuity, but in general continuity does not imply weak continuity. We identify $\Psi'(\theta)$ as an element of $L^2(\mathbb{R}^2)$ defined by

$$(\varphi, \Psi'(\theta))_{L^2(\mathbb{R}^2)} = \sum_{k=1}^{N} \left(\frac{\partial \psi}{\partial y_k}(y(\theta)) \right) (J_\epsilon(\varphi), w_k)_{L^2(\mathbb{R}^2)} \tag{2.10}$$

with

$$y(\theta) = \left((J_\epsilon(\theta), w_1)_{L^2(\mathbb{R}^2)}, \ldots, (J_\epsilon(\theta), w_N)_{L^2(\mathbb{R}^2)} \right), \tag{2.11}$$

that is

$$\Psi'(\theta)(x) = \sum_{k=1}^{N} \left(\frac{\partial \psi}{\partial y_k}(y(\theta)) \right) (J_\epsilon w_k)(x). \tag{2.12}$$

We extend the definition (2.10) to more general φ; that is,

$$(N^{(\nu)}(\theta), \Psi'(\theta))_{L^2(\mathbb{R}^2)} = F_1(\theta) + \nu F_2(\theta) + F_3(\theta) \tag{2.13}$$

with

$$F_1(\theta) = \gamma(\theta, \mathcal{D}\Psi'(\theta))_{L^2(\mathbb{R}^2)} - (f, \Psi'(\theta))_{L^2(\mathbb{R}^2)}, \tag{2.14}$$

$$F_2(\theta) = (\theta, (-\Delta)\Psi'(\theta))_{L^2(\mathbb{R}^2)} \tag{2.15}$$

and

$$F_3(\theta) = -(\theta R^\perp \theta, \nabla \Psi'(\theta))_{L^2(\mathbb{R}^2)}. \tag{2.16}$$

Let us note that the Borel σ-algebra associated to the strong topology in $L^2(\mathbb{R}^2)$ is the same as the Borel σ algebra associated to the weak topology because any open ball is a countable union of closed balls, which are convex, hence weakly closed. The function $\theta \mapsto \|\theta\|^2_{H^1(\mathbb{R}^2)}$ is a Borel measurable function in $L^2(\mathbb{R}^2)$ because it is everywhere the limit of a sequence of continuous functions $\theta \mapsto \|J_\epsilon \theta\|^2_{H^1(\mathbb{R}^2)}$. The same

of course applies to $\|\theta\|^2_{H^{\frac{1}{2}}(\mathbb{R}^2)}$. Therefore conditions (a) and (c) in Definition 2.2.1 make mathematical sense. Moreover, condition (c) implies that $\mu^{(\nu)}$ is supported in the ball

$$\|\theta\|_{H^{\frac{1}{2}}(\mathbb{R}^2)} \leq \frac{1}{\gamma}\|f\|_{L^2(\mathbb{R}^2)} \tag{2.17}$$

as it is easily seen by taking $E_1 \geq \gamma^{-1}\|f\|_{L^2(\mathbb{R}^2)}$. The integrand in condition (b) is locally bounded and weakly continuous:

Lemma 2.2.3 (Weak Continuity). *For any fixed $\Psi \in \mathcal{T}$ the maps*

$$\theta \mapsto F_i(\theta)$$

$i = 1, 2, 3$ are locally bounded in $L^2(\mathbb{R}^2)$ and weakly continuous in $L^2(\mathbb{R}^2)$ on bounded sets of $L^2(\mathbb{R}^2) \cap L^p(\mathbb{R}^2)$, $1 \leq p < 2$. In particular

$$\theta \mapsto (N^{(\nu)}(\theta), \Psi'(\theta))_{L^2(\mathbb{R}^2)}$$

is locally bounded and weakly continuous on bounded sets of $L^2(\mathbb{R}^2) \cap L^p(\mathbb{R}^2)$, $1 \leq p < 2$.

Proof of Lemma 2.2.3. After inspection of the definitions (2.14, 2.15, 2.16) it is clear that the only nontrivial statement is about F_3. It is only for F_3 that we also need the $L^p(\mathbb{R}^2)$ bound for $p < 2$. If θ_n is weakly converging to θ in $L^2(\mathbb{R}^2)$, then the vectors $y(\theta_n)$ defined in (2.11) converge to the vector $y(\theta)$ and because the sequence θ_n is bounded in $L^2(\mathbb{R}^2)$, it follows that $y(\theta_n)$ belong to a fixed compact set in $\mathbb{R}^N$. The functions $\frac{\partial \psi}{\partial y_k}$ are continuous, so all we need to check is the convergence

$$(\theta_n R^\perp \theta_n, \nabla J_\epsilon w_k)_{L^2(\mathbb{R}^2)} \to (\theta R^\perp \theta, \nabla J_\epsilon w_k)_{L^2(\mathbb{R}^2)}.$$

In order to do this we make use of the assumed bound

$$\sup_n \|\theta_n\|_{L^p(\mathbb{R}^2)} \leq A_p.$$

We note first that θ, the weak limit in $L^2(\mathbb{R}^2)$, also obeys

$$\|\theta\|_{L^p(\mathbb{R}^2)} \leq A_p.$$

This follows by essentially restricting θ^{p-1} (or $\operatorname{sign}\theta$ if $p = 1$) on large compacts, integrating against θ_n, passing to the limit in n and then letting the compacts grow to the whole space. The weak continuity of the nonlinearity in SQG is proved using the representation [121]

$$(\theta R^\perp \theta, \nabla\varphi)_{L^2(\mathbb{R}^2)} = \frac{1}{2}\int_{\mathbb{R}^2} (\Lambda^{-1}\theta)(x)\,[\Lambda, \nabla\varphi]\,(R^\perp(\theta)(x)dx \tag{2.18}$$

for smooth compactly supported φ, with $[\Lambda, a]b = \Lambda(ab) - a\Lambda b$, the commutator of the operators Λ and of multiplication by a. In our case $\varphi = J_\epsilon w_k$.

In order to make sense of the terms in (2.18) let us recall that the Riesz potential is given by [132]

$$\Lambda^{-1}\theta(x) = c\int_{\mathbb{R}^2} \frac{\theta(x-y)}{|y|}dy$$

for an appropriate constant c. If $\theta \in L^p(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)$ then $\Lambda^{-1}\theta \in L^2(\mathbb{R}^2) + L^\infty(\mathbb{R}^2)$. Indeed,

$$\sup_x \left| \int_{|y|\geq 1} \frac{1}{|y|}\theta(x-y)dy \right| \leq C\|\theta\|_{L^p(\mathbb{R}^2)}$$

because $|y|^{-1} \in L^q(|y| \geq 1)$, $q > 2$, $q^{-1} + p^{-1} = 1$, and

$$\left\| \int_{|y|\leq 1} \frac{1}{|y|}\theta(x-y)dz \right\|_{L^2(dx)} \leq C\|\theta\|_{L^2(\mathbb{R}^2)}$$

as it is easily seen by duality or by Fourier transform. Then, we note that

$$\Lambda(ab)(x) - a(x)\Lambda b(x) = cP.V. \int_{\mathbb{R}^2} b(y) \frac{a(x) - a(y)}{|x-y|^3} dy$$

and therefore, if a is compactly supported in a ball of radius L and if $|x| \geq 2L$, then, pointwise

$$|\Lambda(ab)(x) - a(x)\Lambda b(x)| \leq C|x|^{-3} \|a\|_{L^2(\mathbb{R}^2)} \|b\|_{L^2(\mathbb{R}^2)}.$$

Thus, if L is the radius of a ball in $\mathbb{R}^2$ containing the support of $\varphi = J_\epsilon w_k$ and we denote

$$C_\varphi(\theta)(x) = [\Lambda, \nabla\varphi] \cdot R^\perp \theta$$

we have, for $\rho \geq 2L$,

$$\left| \int_{|x|\geq\rho} (\Lambda^{-1}\theta)(x) C_\varphi(\theta)(x) dx \right| \leq C\rho^{-1} \|\theta\|_{L^p(\mathbb{R}^2)} \|\theta\|_{L^2(\mathbb{R}^2)} + C\rho^{-2} \|\theta\|^2_{L^2(\mathbb{R}^2)}.$$

We have thus

$$\left| \int_{|x|\geq\rho} (\Lambda^{-1}\theta_n)(x) C_\varphi(\theta_n)(x) dx \right| \leq \epsilon$$

uniformly for all n and also for θ, provided ρ is large enough so that $\rho \geq 2L$ and

$$C\rho^{-1} A_p A_2 + C\rho^{-2} A_2^2 \leq \epsilon$$

where A_2 is the bound on $\|\theta_n\|_{L^2(\mathbb{R}^2)}$. It is well-known and easy to show that

$$\theta \mapsto C_\varphi(\theta) = [\Lambda, \nabla\varphi]\, R^\perp \theta$$

is a bounded linear operator in $L^2(\mathbb{R}^2)$ for fixed $\varphi \in C_0^\infty(\mathbb{R}^2)$. Thus,

$$\left| \int_{|x|\leq\rho} (\Lambda^{-1}\theta)(x) C_\varphi(\theta)(x) dx \right| \leq C \|\Lambda^{-1}\theta\|_{L^2(B(0,\rho))} \|\theta\|_{L^2(\mathbb{R}^2)}.$$

The proof of the identity (2.18) is best explained by denoting $\psi = \Lambda^{-1}\theta$ and $u = R^{\perp}\theta = \nabla^{\perp}\psi$. Then we have

$$\begin{aligned}
&(\theta R^{\perp}\theta, \nabla\varphi)_{L^2(\mathbb{R}^2)} = \textstyle\int_{\mathbb{R}^2}(\Lambda\psi)(u\cdot\nabla\varphi)dx \\
&= \textstyle\int_{\mathbb{R}^2}\psi\left([\Lambda,\nabla\varphi]\cdot u\right)dx + \int_{\mathbb{R}^2}\psi\nabla\varphi\cdot\Lambda u dx \\
&= \textstyle\int_{\mathbb{R}^2}\psi\left([\Lambda,\nabla\varphi]\cdot u\right)dx + \int_{\mathbb{R}^2}\psi\nabla\varphi\cdot\nabla^{\perp}\theta dx \\
&= \textstyle\int_{\mathbb{R}^2}\psi\left([\Lambda,\nabla\varphi]\cdot u\right)dx - (\theta R^{\perp}\theta, \nabla\varphi)_{L^2(\mathbb{R}^2)}.
\end{aligned}$$

In order to obtain the last term we integrated by parts and used $\nabla^{\perp}\cdot\nabla\varphi = 0$. Now the weak continuity follows by writing

$$\begin{aligned}
&\textstyle\int_{\mathbb{R}^2}(\Lambda^{-1}\theta_n)(x)C_{\varphi}(\theta_n)(x)dx - \int_{\mathbb{R}^2}(\Lambda^{-1}\theta)(x)C_{\varphi}\theta(x)dx = \\
&\textstyle\int_{B(0,\rho)}(\Lambda^{-1}(\theta_n-\theta))(x)C_{\varphi}(\theta_n)(x)dx + \int_{B(0,\rho)}(\Lambda^{-1}\theta)(x)C_{\varphi}(\theta_n-\theta)(x)dx \\
&+ \textstyle\int_{|x|\geq\rho}(\Lambda^{-1}\theta_n)(x)C_{\varphi}(\theta_n)(x)dx - \int_{|x|\geq\rho}(\Lambda^{-1}\theta)(x)C_{\varphi}\theta(x)dx.
\end{aligned}$$

For any fixed $\epsilon > 0$ we take $\rho > 0$ large enough that the last two terms are each less than ϵ. Having fixed rho, $(\Lambda^{-1}\theta)\chi_{B(0,\rho)}$ is a fixed function in $L^2(\mathbb{R}^2)$ (here $\chi_{B(0,\rho)}$ is the characteristic function), and, because C_{φ} is a bounded linear operator in $L^2(\mathbb{R}^2)$ the sequence $C_{\varphi}(\theta_n - \theta)$ converges weakly to zero in $L^2(\mathbb{R}^2)$. Thus, letting $n \to \infty$ the ante-penultimate term converges to zero. Finally, for the first term

$$\left|\int_{B(0,\rho)}(\Lambda^{-1}(\theta_n-\theta))(x)C_{\varphi}(\theta_n)(x)dx\right| \leq CA_2\|\Lambda^{-1}(\theta_n-\theta)\|_{L^2(B(0,\rho))}$$

and this converges to zero because $\theta_n - \theta$ converges weakly to zero in $L^2(\mathbb{R}^2)$ and is bounded in $L^p(\mathbb{R}^2)$, $p < 2$. Indeed, by the previous considerations about Λ^{-1}, for $\chi \in C_0^{\infty}(\mathbb{R}^2)$ we have that $\chi\Lambda^{-1}(\theta_n - \theta)$ is bounded in $H^1(\mathbb{R}^2)$ and converges weakly to 0 in $L^2(\mathbb{R}^2)$. Thus $\Lambda^{-1}(\theta_n - \theta)$ converges strongly to zero in $L^2(B(0,\rho))$ implying that the first term converges to zero. We conclude that the limit difference is in absolute value less than 2ϵ and ϵ was arbitrary. □

Definition 2.2.4 (**Inviscid Stationary Statistical Solution**). *A stationary statistical solution of the forced critical SQG equation*

$$\partial_t \theta + R^\perp \theta \cdot \nabla \theta + \gamma \mathcal{D}\theta = f \tag{2.19}$$

is a Borel probability measure μ on $L^2(\mathbb{R}^2)$ such that

$$\int_{L^2(\mathbb{R}^2)} \|\theta\|^2_{H^{\frac{1}{2}}(\mathbb{R}^2)} d\mu(\theta) < \infty, \tag{2.20}$$

and the equation

$$\int_{L^2(\mathbb{R}^2)} (N(\theta), \Psi'(\theta))_{L^2(\mathbb{R}^2)} d\mu(\theta) = 0 \tag{2.21}$$

holds for all cylindrical test functions $\Psi \in \mathcal{T}$, where

$$N(\theta) = R^\perp \theta \cdot \nabla \theta + \gamma \mathcal{D}\theta - f. \tag{2.22}$$

We say that the stationary statistical solution satisfies the energy dissipation balance if

$$\int_{L^2(\mathbb{R}^2)} \left\{ \gamma \|\theta\|^2_{H^{\frac{1}{2}}(\mathbb{R}^2)} - (f, \theta)_{L^2(\mathbb{R}^2)} \right\} d\mu(\theta) = 0. \tag{2.23}$$

Theorem 2.2.5 (**Weak Limit is a Statistical Solution**). *Let $\mu^{(\nu)}$ be a sequence of stationary statistical solutions of the viscous forced critical SQG equation (2.1) with $f \in L^1(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)$, with $\nu \to 0$. Assume that there exists $1 \leq p < 2$ and A_p such that the supports of the measures $\mu^{(\nu)}$ are included in*

$$B_p = \left\{ \theta \in L^p(\mathbb{R}^2) \cap L^2(\mathbb{R}^2) \mid \ \|\theta\|_{L^p(\mathbb{R}^2)} \leq A_p \right\}. \tag{2.24}$$

Then there exists a subsequence, denoted also $\mu^{(\nu)}$ and a stationary statistical solution

μ of the forced critical SQG equation (2.19) such that

$$\lim_{\nu\to 0}\int_{L^2(\mathbb{R}^2)}\Phi(\theta)d\mu^{(\nu)}(\theta)=\int_{L^2(\mathbb{R}^2)}\Phi(\theta)d\mu(\theta) \tag{2.25}$$

holds for all weakly continuous, locally bounded real valued functions Φ.

Proof of Theorem 2.2.5. As we saw in (2.17) the support of the measures $\mu^{(\nu)}$ is included in

$$B=\left\{\theta\in H^{\frac{1}{2}}(\mathbb{R}^2)\,|\;\;\|\theta\|_{H^{\frac{1}{2}}(\mathbb{R}^2)}\leq\frac{\|f\|_{L^2(\mathbb{R}^2)}}{\gamma}\right\}. \tag{2.26}$$

The set $A_p=B\cap B_p$ is weakly closed in $L^2(\mathbb{R}^2)$ and it is a separable metrizable compact space with the weak $L^2(\mathbb{R}^2)$ topology. By Prokhorov's theorem the sequence $\mu^{(\nu)}$ is tight and therefore has a weakly convergent subsequence. The limit μ is a Borel probability on A_p. The extension of μ to $L^2(\mathbb{R}^2)$, denoted again by μ and given by $\mu(X)=\mu(X\cap A_p)$ is a Borel measure because A_p is weakly closed. The measure μ satisfies (2.20) because it is supported in B. The equation (2.21) is satisfied because we may pass to the limit in (b) of Definition 2.2.1 in view of Lemma 2.2.3. □

2.3 Inviscid limit and energy dissipation balance

Theorem 2.3.1 (Weak Limit Satisfies Energy Balance). *Let $\mu^{(\nu)}$ be a sequence $\nu\to 0$ of stationary statistical solutions of the forced viscous SQG equation (2.1) supported in*

$$A=\left\{\theta\,|\;\;\|\theta\|_{L^p(\mathbb{R}^2)}\leq A_p,\;\|\theta\|_{L^\infty(\mathbb{R}^2)}\leq A_\infty,\;\|\theta\|_{H^{\frac{1}{2}}(\mathbb{R}^2)}\leq\frac{\|f\|_{L^2(\mathbb{R}^2)}}{\gamma}\right\}. \tag{2.27}$$

Let μ be any weak limit of $\mu^{(\nu)}$ in $L^2(\mathbb{R}^2)$. Then μ is a stationary statistical solution of the forced critical SQG equation (2.19) that satisfies the energy dissipation balance (2.23).

Proof of Theorem 2.3.1. By Theorem 2.2.5, we know that any weak limit is a stationary statistical solution of the forced critical SQG equation. We check that it is supported on A. The set A is weakly closed in $L^2(\mathbb{R}^2)$, and because its complement U is weakly open and

$$\mu(U) \leq \liminf_{\nu \to 0} \mu^{(\nu)}(U) = 0$$

it follows that μ is supported in the set A. The rest of the proof is done by showing that (2.21) and the fact that μ is supported in A imply (2.23).

We take a sequence $w_j \in C_0^\infty(\mathbb{R}^2)$ that is an orthonormal basis of $L^2(\mathbb{R}^2)$. We fix $\epsilon > 0$ and consider the sequence of test functions

$$\Psi_m(\theta) = \frac{1}{2} \sum_{k=1}^{m} (J_\epsilon(\theta), w_j)^2_{L^2(\mathbb{R}^2)}$$

i.e. we take $\psi(y) = \frac{1}{2} \sum_{k=1}^{m} y_k^2$ in Definition 2.2.2. We note that

$$(N(\theta), \Psi_m'(\theta))_{L^2(\mathbb{R}^2)} = \sum_{k=1}^{m} (J_\epsilon(\theta), w_j)_{L^2(\mathbb{R}^2)} (J_\epsilon(N(\theta)), w_j)_{L^2(\mathbb{R}^2)}.$$

This is uniformly bounded in m because

$$\left|(N(\theta), \Psi_m'(\theta))_{L^2(\mathbb{R}^2)}\right| \leq \|J_\epsilon \theta\|_{L^2(\mathbb{R}^2)} \|J_\epsilon N(\theta)\|_{L^2(\mathbb{R}^2)}.$$

On A, the right hand side is bounded uniformly in θ

$$\left|(N(\theta), \Psi_m'(\theta))_{L^2(\mathbb{R}^2)}\right| \leq A_2((2 + \epsilon^{-\frac{1}{2}})\|f\|_{L^2(\mathbb{R}^2)} + \epsilon^{-1} A_2 A_\infty)$$

and, by Parseval, it converges to $(J_\epsilon(\theta), J_\epsilon(N(\theta)))_{L^2(\mathbb{R}^2)}$ pointwise, as $m \to \infty$. So, we deduce from (2.21) and the Lebesgue dominated convergence theorem that

$$\int_{L^2(\mathbb{R}^2)} (J_\epsilon(\theta), J_\epsilon(N(\theta)))_{L^2(\mathbb{R}^2)} d\mu(\theta) = 0$$

for any $\epsilon > 0$. This can be written as

$$I_\epsilon + K_\epsilon = 0 \tag{2.28}$$

where the two terms are

$$I_\epsilon = \int_{L^2(\mathbb{R}^2)} (J_\epsilon(\theta), J_\epsilon(\gamma \mathcal{D}\theta - f))_{L^2(\mathbb{R}^2)} d\mu(\theta) \tag{2.29}$$

and

$$K_\epsilon = \int_{L^2(\mathbb{R}^2)} (J_\epsilon(\theta), J_\epsilon((R^\perp\theta) \cdot \nabla\theta))_{L^2(\mathbb{R}^2)} d\mu(\theta). \tag{2.30}$$

Now

$$(J_\epsilon(\theta), J_\epsilon(u) \cdot \nabla J_\epsilon(\theta))_{L^2(\mathbb{R}^2)} = 0$$

in view of the incompressibility of $u = R^\perp\theta$, so

$$K_\epsilon = \int_{L^2(\mathbb{R}^2)} (J_\epsilon(\theta), \nabla \cdot \rho_\epsilon(u, \theta))_{L^2(\mathbb{R}^2)} d\mu(\theta), \tag{2.31}$$

where

$$\rho_\epsilon(u, \theta) = J_\epsilon(u\theta) - (J_\epsilon(u))(J_\epsilon(\theta)). \tag{2.32}$$

We show that $\lim_{\epsilon \to 0} K_\epsilon = 0$. We use the identity [27]

$$\rho_\epsilon(u, \theta) = r_\epsilon(u, \theta) - (u - J_\epsilon(u))(\theta - J_\epsilon(\theta))$$

with

$$r_\epsilon(u, \theta)(x) = \int_{\mathbb{R}^2} j(z)(\delta_{\epsilon z}(u)(x))(\delta_{\epsilon z}(\theta)(x)) dz,$$

$$\delta_{\epsilon z}(u)(x) = u(x - \epsilon z) - u(x),$$

and

$$\delta_{\epsilon z}(\theta)(x) = \theta(x - \epsilon z) - \theta(x).$$

Clearly also

$$(J_\epsilon u - u)(J_\epsilon \theta - \theta) = \int_{\mathbb{R}^4} j(z_1)j(z_2)(\delta_{\epsilon z_1} u)(\delta_{\epsilon z_2}\theta)dz_1 dz_2.$$

The inequality

$$\|\delta_{\epsilon z}\theta\|^2_{L^4(\mathbb{R}^2)} \leq C(\epsilon|z|)^{\frac{1}{2}} \|\theta\|_{L^\infty(\mathbb{R}^2)} \|\theta\|_{H^{\frac{1}{2}}(\mathbb{R}^2)} \tag{2.33}$$

and its consequence (because of the boundeness of $R^\perp$ in $L^4(\mathbb{R}^2)$)

$$\|\delta_{\epsilon z} R^\perp\theta\|^2_{L^4(\mathbb{R}^2)} \leq C(\epsilon|z|)^{\frac{1}{2}} \|\theta\|_{L^\infty(\mathbb{R}^2)} \|\theta\|_{H^{\frac{1}{2}}(\mathbb{R}^2)} \tag{2.34}$$

follow from the elementary inequality

$$\|\delta_{\epsilon z}\theta\|_{L^2(\mathbb{R}^2)} \leq C(\epsilon|z|)^{\frac{1}{2}} \|\theta\|_{H^{\frac{1}{2}}(\mathbb{R}^2)} \tag{2.35}$$

which is proved by Fourier transform. Consequently,

$$\|\rho_\epsilon(R^\perp\theta_1, \theta_2)\|_{L^2(\mathbb{R}^2)} \leq C\epsilon^{\frac{1}{2}} \|\theta_1\|^{\frac{1}{2}}_{L^\infty(\mathbb{R}^2)} \|\theta_2\|^{\frac{1}{2}}_{L^\infty(\mathbb{R}^2)} \|\theta_1\|^{\frac{1}{2}}_{H^{\frac{1}{2}}(\mathbb{R}^2)} \|\theta_2\|^{\frac{1}{2}}_{H^{\frac{1}{2}}(\mathbb{R}^2)}. \tag{2.36}$$

The integrand in K_ϵ is bounded

$$\left|(\nabla J_\epsilon(\theta), \rho_\epsilon(u, \theta))_{L^2(\mathbb{R}^2)}\right| \leq C\|\theta\|_{L^\infty(\mathbb{R}^2)} \|\theta\|^2_{H^{\frac{1}{2}}(\mathbb{R}^2)} \tag{2.37}$$

and converges to 0 as $\epsilon \to 0$, for fixed $\theta \in H^{\frac{1}{2}}(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$. Indeed, the trilinear map

$$(\theta_1, \theta_2, \theta_3) \mapsto (\nabla J_\epsilon(\theta_1), \rho_\epsilon(R^\perp\theta_1, \theta_2))_{L^2(\mathbb{R}^2)}$$

is separately continuous from $H^{\frac{1}{2}}(\mathbb{R}^2)$ to $\mathbb{R}$ uniformly on A and uniformly in ϵ. This can be seen from the expression

$$(\nabla J_\epsilon(\theta_3), \rho_\epsilon(R^\perp\theta_1, \theta_2))_{L^2(\mathbb{R}^2)} = -\frac{1}{\epsilon}\int \nabla_z j(z)(\delta_{\epsilon z}\theta_3, \rho_\epsilon(R^\perp\theta_1, \theta_2))_{L^2(\mathbb{R}^2)}dz$$

and the bound obtained from (2.35) and (2.36)

$$\begin{aligned}&\left|(\nabla J_\epsilon(\theta_3), \rho_\epsilon(R^\perp\theta_1, \theta_2))_{L^2(\mathbb{R}^2)}\right| \\ &\leq C\|\theta_3\|_{H^{\frac{1}{2}}(\mathbb{R}^2)}\|\theta_1\|^{\frac{1}{2}}_{L^\infty(\mathbb{R}^2)}\|\theta_2\|^{\frac{1}{2}}_{L^\infty(\mathbb{R}^2)}\|\theta_1\|^{\frac{1}{2}}_{H^{\frac{1}{2}}(\mathbb{R}^2)}\|\theta_2\|^{\frac{1}{2}}_{H^{\frac{1}{2}}(\mathbb{R}^2)}.\end{aligned} \tag{2.38}$$

This explains (2.37) and also shows the pointwise vanishing of the integrand in K_ϵ as $\epsilon \to 0$: the integrand in (2.31) obviously tends to zero for smooth θ, and θ in A can be approximated in the norm of $H^{\frac{1}{2}}(\mathbb{R}^2)$ by smooth functions. Therefore, from the Lebegue dominated convergence theorem

$$\lim_{\epsilon\to 0} K_\epsilon = 0.$$

It remains to consider the limit of I_ϵ, but this is quite straightforward,

$$\lim_{\epsilon\to 0} I_\epsilon = \int_{L^2(\mathbb{R}^2)} (\gamma\|\theta\|^2_{H^{\frac{1}{2}}(\mathbb{R}^2)} - (\theta, f)_{L^2(\mathbb{R}^2)})d\mu(\theta)$$

and thus the proof is complete. □

2.4 Long time averages

In this section we consider long time averages of solutions and the stationary statistical solutions they generate. We start by recalling the concept of generalized (Banach) limit:

Definition 2.4.1 (Banach Limit). *A generalized limit (Banach limit) is a bounded*

linear functional

$$Lim_{t\to\infty} : BC([0,\infty)) \to \mathbb{R}$$

such that

1. $Lim_{t\to\infty}(g) \geq 0, \quad \forall g \in BC([0,\infty)),\ g \geq 0.$
2. $Lim_{t\to\infty}(g) = \lim_{t\to\infty} g(t)$ *whenever the usual limit exists.*

The space $BC([0,\infty))$ is the Banach space of all bounded continuous real valued functions defined on $[0,\infty)$ endowed with the sup norm. It can be shown that the generalized limit always saitisfies

$$\liminf_{t\to\infty} g(t) \leq Lim_{t\to\infty}(g) \leq \limsup_{t\to\infty} g$$

for all $g \in BC([0,\infty))$. Moreover, given a fixed $g_0 \in BC([0,\infty))$ and a sequence $t_j \to \infty$ for which $\lim_{j\to\infty} g_0(t_j) = l$ exists, a generalized limit can be found which satisfies $Lim_{t\to\infty}(g_0) = l$. This implies that one can choose a generalized limit that obeys $Lim_{t\to\infty}(g_0) = \limsup_{t\to\infty} g_0(t)$. With this language at our disposal, we now state the result about long time averages of viscous forced critical SQG.

Theorem 2.4.2 (Construction of Statistical Solutions). *Let $f \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ and $\theta_0 \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$. Let $Lim_{t\to\infty}$ be a Banach limit. Then the map*

$$\Phi \mapsto Lim_{t\to\infty} \frac{1}{t} \int_0^t \Phi(S^{(\nu)}(s,\theta_0))ds \tag{2.39}$$

for $\Phi \in C(L^2(\mathbb{R}^2))$ (strongly continuous, real valued) defines a a stationary statistical solution $\mu^{(\nu)}$ of the viscous forced SQG equation (2.1):

$$\int_{L^2(\mathbb{R}^2)} \Phi(\theta) d\mu^{(\nu)}(\theta) = Lim_{t\to\infty} \frac{1}{t} \int_0^t \Phi(S^{(\nu)}(s,\theta_0))ds. \tag{2.40}$$

The measure is supported in the set

$$A = \left\{ \theta \mid \|\theta\|_{H^{\frac{1}{2}}(\mathbb{R}^2)} \leq \frac{\|f\|_{L^2(\mathbb{R}^2)}}{\gamma},\ \|\theta\|_{L^p(\mathbb{R}^2)} \leq A_p,\ 1 \leq p \leq \infty \right\} \tag{2.41}$$

with

$$A_p = \|\theta_0\|_{L^p(\mathbb{R}^2)} + \frac{\|f\|_{L^p(\mathbb{R}^2)}}{\gamma},\ 1 \leq p \leq \infty.$$

The inequality

$$\int_{L^2(\mathbb{R}^2)} \left[\nu \|\nabla\theta\|^2_{L^2(\mathbb{R}^2)} + \gamma \|\theta\|^2_{H^{\frac{1}{2}}(\mathbb{R}^2)} - (f,\theta)_{L^2(\mathbb{R}^2)} \right] d\mu^{(\nu)}(\theta) \leq 0 \tag{2.42}$$

holds.

Proof of Theorem 2.4.2. The positive semiorbit

$$O_+(\theta_0) = \{\theta \mid\ \exists s \geq 0,\ \theta = S^{(\nu)}(s,\theta_0)\}$$

is relatively compact in $L^2(\mathbb{R}^2)$ because it is bounded in $H^1(\mathbb{R}^2)$ and uniformly integrable by Proposition 2.1.1, (2.8). For any $\Phi \in C\left(\overline{O_+(\theta_0)}\right)$ the function $t \mapsto \frac{1}{t}\int_0^t \Phi(S^{(\nu)}(s,\theta_0))ds$ is a bounded continuous function on $[0,\infty)$ so we may apply $Lim_{t\to\infty}$ to it. (Of course, $C(L^2(\mathbb{R}^2)) \subset C\left(\overline{O_+(t_0,\theta_0)}\right)$.) The map

$$\Phi \mapsto Lim_{t\to\infty}\frac{1}{t}\int_0^t \Phi(S^{(\nu)}(s,\theta_0))ds$$

is a positive functional on $C\left(\overline{O_+(\theta_0)}\right)$. Because of the Riesz representation theorem on compact spaces, it follows that there exists a Borel measure representing it, i.e. (2.40) holds. The measure is supported on $\overline{O_+(\theta_0)}$. We take a test function $\Psi \in \mathcal{T}$.

Then

$$\int_{L^2(\mathbb{R}^2)} (N^{(\nu)}(\theta), \Psi'(\theta))_{L^2(\mathbb{R}^2)} d\mu^{(\nu)}(\theta) = Lim_{t\to\infty} \frac{1}{t} \int_0^t \frac{d}{ds} \Psi(S^{(\nu)}(s, \theta_0)) ds$$

holds and the right hand side vanishes, verifying (b) of Definition (2.2.1). Because of (2.7) the semiorbit is included in the set

$$\{\theta \mid \ \|\theta\|_{L^p(\mathbb{R}^2)} \leq A_p, \ \ 1 \leq p \leq \infty\}.$$

The fact that the support of $\mu^{(\nu)}$ is included in A follows as shown before from property (c) of Definition 2.2.1. In order to check (a), (c) of Definition 2.2.1 we would like to take long time averages in the energy balance (2.6). In order to do so, we first mollify the equation. This is due to the fact that $\|\nabla\theta\|^2_{L^2(\mathbb{R}^2)}$ is not continuous in $L^2(\mathbb{R}^2)$. We put

$$\theta_\epsilon(x,t) = J_\epsilon(S^{(\nu)}(t,\theta_0))(x), \quad u_\epsilon(x,t) = J_\epsilon R^\perp(S^{(\nu)}(t,\theta_0)),$$

and applying J_ϵ to (2.1), multiplying by θ_ϵ and integrating we deduce

$$\begin{aligned} &\tfrac{1}{t}\int_0^t \left[\gamma\|\theta_\epsilon(s)\|^2_{H^{\frac{1}{2}}(\mathbb{R}^2)} - (J_\epsilon f, \theta_\epsilon(s))_{L^2(\mathbb{R}^2)} + \nu\|\nabla\theta_\epsilon(s)\|^2_{L^2(\mathbb{R}^2)}\right] ds \\ &= \tfrac{1}{2t}\left[\|\theta_\epsilon(0)\|^2_{L^2(\mathbb{R}^2)} - \|\theta_\epsilon(t)\|^2_{L^2(\mathbb{R}^2)}\right] + \tfrac{1}{t}\int_0^t (\rho(u_\epsilon(s),\theta_\epsilon(s)), \nabla\theta_\epsilon(s))_{L^2(\mathbb{R}^2)} ds. \end{aligned}$$

We obtain

$$\begin{aligned} &\int \left[\gamma\|J_\epsilon\theta\|^2_{H^{\frac{1}{2}}(\mathbb{R}^2)} - (J_\epsilon f, J_\epsilon\theta)_{L^2(\mathbb{R}^2)} + \nu\|\nabla J_\epsilon\theta\|^2_{L^2(\mathbb{R}^2)}\right] d\mu^{(\nu)}(\theta) \\ &= Lim_{t\to\infty}\tfrac{1}{t}\int_0^t (\rho(u_\epsilon(s),\theta_\epsilon(s)), \nabla\theta_\epsilon(s))_{L^2(\mathbb{R}^2)} ds. \end{aligned} \tag{2.43}$$

Because of (2.6) and (2.7)

$$\begin{aligned} &\limsup_{t\to\infty} \tfrac{1}{t}\int_0^t \left[\gamma\|S^{(\nu)}(s,\theta_0)\|^2_{H^{\frac{1}{2}}(\mathbb{R}^2)} + \nu\|\nabla S^{(\nu)}(s,\theta_0)\|^2_{L^2(\mathbb{R}^2)} ds\right] \\ &\leq \tfrac{1}{\gamma}\|f\|^2_{L^2(\mathbb{R}^2)} \end{aligned} \tag{2.44}$$

and because J_ϵ does not increase L^2 norms, we deduce from (2.44) that

$$\sup_{0<\epsilon} \int_{L^2(\mathbb{R}^2)} \left[\gamma \|J_\epsilon \theta\|^2_{H^{\frac{1}{2}}(\mathbb{R}^2)} + \nu \|\nabla J_\epsilon \theta\|^2_{L^2(\mathbb{R}^2)} \right] d\mu^{(\nu)}(\theta) \leq \frac{1}{\gamma} \|f\|^2_{L^2(\mathbb{R}^2)}.$$

The functions $\|\theta\|^2_{H^{\frac{1}{2}}(\mathbb{R}^2)}$ and $\|\nabla\theta\|^2_{L^2(\mathbb{R}^2)}$ are Borel measurable and so, by Fatou, we obtain (a) of Definition 2.2.1. Using the $H^1 \cap L^\infty$ information we have

$$\|\rho_\epsilon(R^\perp\theta, \theta)\|_{L^2(\mathbb{R}^2)} \leq C\sqrt{\epsilon} \|\theta\|_{L^\infty(\mathbb{R}^2)} \|\nabla\theta\|_{L^2(\mathbb{R}^2)}$$

and thus

$$\begin{aligned} &Lim_{t\to\infty} \tfrac{1}{t} \textstyle\int_0^t \rho(u_\epsilon(s)\theta_\epsilon(s)), \nabla\theta_\epsilon(s))_{L^2(\mathbb{R}^2)} ds \\ &\leq C\epsilon \left[\|\theta_0\|_{L^\infty(\mathbb{R}^2)} + \tfrac{1}{\gamma} \|f\|_{L^\infty(\mathbb{R}^2)} \right] \tfrac{1}{\nu\gamma} \|f\|^2_{L^2(\mathbb{R}^2)}. \end{aligned}$$

This implies that the right hand side of (2.43) converges to zero as $\epsilon \to 0$. This proves (2.42) by Fatou. In order to prove (c) of Definition 2.2.1 we take $\chi'(y)$, a smooth, nonnegative, compactly supported function defined for $y \geq 0$. Then $\chi(y) = \int_0^y \chi'(e)de$ is bounded on $\mathbb{R}_+$ and

$$\frac{d}{dt} \chi(\|\theta_\epsilon(t)\|^2_{L^2(\mathbb{R}^2)}) = \chi'(\|\theta_\epsilon(t)\|^2_{L^2(\mathbb{R}^2)}) \frac{d}{dt} \|\theta_\epsilon(t)\|^2_{L^2(\mathbb{R}^2)}.$$

We proceed as above and obtain

$$\begin{aligned} &\textstyle\int_{L^2(\mathbb{R}^2)} \chi'(\|\theta\|^2_{L^2(\mathbb{R}^2)}) \left\{ \nu \|\nabla\theta\|^2_{L^2(\mathbb{R}^2)} + \gamma \|\theta\|^2_{H^{\frac{1}{2}}(\mathbb{R}^2)} - (f, \theta)_{L^2(\mathbb{R}^2)} \right\} d\mu^{(\nu)}(\theta) \\ &\leq 0. \end{aligned}$$

We let $\chi'(y)$ converge pointwise to the characteristic function of the interval $[E_1^2, E_2^2]$ with $0 \leq \chi'(y) \leq 2$. This proves (c) of Definition 2.2.1 and concludes the proof of this theorem. □

2.5 Conclusion

Theorem 2.5.1 (Absence of Anomalous Dissipation). *Let* $\theta_0, f \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$. *Then*

$$\lim_{\nu\to 0} \nu \left(\limsup_{t\to\infty} \frac{1}{t} \int_0^t \|\nabla S^{(\nu)}(s,\theta_0)\|^2_{L^2(\mathbb{R}^2)} ds \right) = 0.$$

Proof of Theorem 2.5.1. We argue by contradiction. If the conclusion would be false, then there would exist $\delta > 0$, a sequence $\nu_k \to 0$, and, for each ν_k, a sequence of times $t_j \to \infty$ such that

$$\frac{\nu_k}{t_j} \int_0^{t_j} \|\nabla S^{(\nu_k)}(s,\theta_0)\|^2_{L^2(\mathbb{R}^2)} ds \geq \delta$$

holds for all t_j. Because of (2.6)

$$\begin{aligned}
&\delta \leq \tfrac{\nu_k}{t_j} \textstyle\int_0^{t_j} \|\nabla S^{(\nu_k)}(s,\theta_0)\|^2_{L^2(\mathbb{R}^2)} ds = \\
&\tfrac{1}{t_j} \textstyle\int_0^{t_j} \left[-\gamma \|S^{(\nu_k)}(s,\theta_0)\|^2_{H^{\frac{1}{2}}(\mathbb{R}^2)} + (f, S^{(\nu_k)}(s,\theta_0))_{L^2(\mathbb{R}^2)} \right] ds \\
&+ \tfrac{1}{2t_j} \left[\|\theta_0\|^2_{L^2(\mathbb{R}^2)} - \|S^{(\nu_k)}(t,\theta_0)\|^2_{L^2(\mathbb{R}^2)} \right].
\end{aligned}$$

It follows that

$$\limsup_{t\to\infty} \frac{1}{t} \int_0^t \left[-\gamma \|S^{(\nu_k)}(s,\theta_0)\|^2_{H^{\frac{1}{2}}(\mathbb{R}^2)} + (f, S^{(\nu_k)}(s,\theta_0))_{L^2(\mathbb{R}^2)} \right] ds \geq \delta. \tag{2.45}$$

By Theorem 2.4.2 there exists a stationary statistical solution of the forced viscous SQG equation, $\mu^{(\nu_k)}$ supported in A such that

$$\int_{L^2(\mathbb{R}^2)} \left\{ -\gamma \|\theta\|^2_{H^{\frac{1}{2}}(\mathbb{R}^2)} + (f,\theta)_{L^2(\mathbb{R}^2)} \right\} d\mu^{(\nu_k)}(\theta) \geq \delta > 0. \tag{2.46}$$

Passing to a weakly convergent subsequence (denoted again $\mu^{(\nu_k)}$), we find using Theorem 2.2.5 and Theorem 2.3.1 a stationary statistical solution μ of the forced critical SQG equation that satisfies the energy dissipation balance (2.23). The function

$\theta \mapsto (f,\theta)_{L^2(\mathbb{R}^2)}$ is weakly continuous, so

$$\lim_{k\to\infty} \int_{L^2(\mathbb{R}^2)} (f,\theta) d\mu^{(\nu_k)}(\theta) = \int_{L^2(\mathbb{R}^2)} (f,\theta) d\mu.$$

On the other hand, by Fatou

$$\int_{L^2(\mathbb{R}^2)} \|\theta\|^2_{H^{\frac{1}{2}}} d\mu(\theta) \leq \liminf_{k\to\infty} \int_{L^2(\mathbb{R}^2)} \|\theta\|^2_{H^{\frac{1}{2}}} d\mu^{(\nu_k)}(\theta).$$

Using (2.46) we obtain

$$\int_{L^2(\mathbb{R}^2)} \left[\gamma \|\theta\|^2_{H^{\frac{1}{2}}} - (f,\theta)_{L^2(\mathbb{R}^2)}\right] d\mu(\theta) \leq -\delta < 0$$

contradicting (2.23). This concludes the proof of the theorem. □

The forced critical SQG equation is dissipative, and the main result here shows that additional viscous dissipation does not leave any anomalous remnant dissipation. The same result is true for spatially periodic boundary conditions, and for additional dissipation of the type $\nu(-\Delta)^\alpha$. The problem of absence of anomalous dissipation is open for the forced SQG equation without the Λ term in $\mathcal{D}$, i.e. with friction that does not grow like $|k|$ for high wave-numbers k.

The method of proof of [37] and of this chapter is quite general, and is applicable for a large class of equations where no uniform bound on the dissipation is readily available. The main ingredients necessary for the success of the method are: an energy dissipation balance for viscous solutions, relative compactness of viscous semiorbits, weak continuity of the nonlinearity, and enough bounds to control the nonlinear fluxes. The forced SQG equation with wave-number independent friction and the supercritical forced SQG equation have all the mentioned ingredients, except the last one, so what is missing is proving the energy dissipation balance for the long time averages of solutions of the inviscid equation.

Chapter 3

Global regularity and finite dimensional asymptotic dynamics for forced critical SQG

3.1 Preliminary material

We abuse notation and denote in the same way spaces of vector functions and scalar functions. We do not write the subindex "per", to emphasize that we work with $\mathbb{T}^2$-periodic functions in this chapter, i.e. L^2_{per} is simply written as L^2. We also overload notation to denote by $\varphi\colon \mathbb{R}^2 \to \mathbb{R}$ the periodic extension to the whole space of a $\mathbb{T}^2$ periodic function φ.

We use the following convention regarding constants: C shall denote a positive, sufficiently large constant, whose value may change from line to line; C is allowed to depend on the size of the box and other universal constants which are fixed throughout this chapter; to emphasize the dependence of a constant on a certain quantity Q we write C_Q or $C(Q)$; $c, c_0, c_1, ...$ shall denote fixed constants appearing in the estimates, that have to be referred to specifically; again, to emphasize dependence on a certain

quantity Q, we write c_Q or $c(Q)$.

Throughout this chapter, we will work on the torus $\mathbb{T}^2$ and with mean-free functions. Consequently, for $s \in \mathbb{R}$ we may identify the homogeneous Sobolev spaces $\dot{H}^s(\mathbb{T}^2)$ and the inhomogeneous Sobolev spaces $H^s(\mathbb{T}^2)$, and we simply denote these by H^s (without "dots"). As usual these are the closure of (mean-free) $C^\infty(\mathbb{T}^2)$ under the norm

$$\|\varphi\|_{H^s} = \|\Lambda^s \varphi\|_{L^2}.$$

Moreover, for $p \in [1, \infty]$ we denote by $H^{s,p} = H^{s,p}(\mathbb{T}^2)$ the space of mean-free $L^p(\mathbb{T}^2)$ functions φ, which can be written as $\varphi = \Lambda^{-s}\psi$, with $\psi \in L^p$. This is normed by $\|\varphi\|_{H^{s,p}} = \|\Lambda^s \varphi\|_{L^p}$. Lastly, Hölder spaces are denoted as usual by C^α for $\alpha \in (0, 1)$, with seminorm given by

$$[\varphi]_{C^\alpha} = \sup_{x \neq y \in \mathbb{T}^2} \frac{|\varphi(x) - \varphi(y)|}{|x - y|^\alpha}$$

and norm $\|\varphi\|_{C^\alpha} = \|\varphi\|_{L^\infty} + [\varphi]_{C^\alpha}$.

Recall cf. [15, 126, 133] that for $\varphi \in C^\infty(\mathbb{T}^2)$ the periodic Riesz transforms $\mathcal{R}_j$ may be defined in terms of their Fourier multiplier symbol $\widehat{\mathcal{R}_j \varphi}_k = ik_j|k|^{-1}\widehat{\varphi}_k$, for all $k \in \mathbb{Z}^2_*$. Alternatively this is defined as the singular integral

$$\mathcal{R}_j \varphi(x) = P.V. \int_{\mathbb{T}^2} \varphi(x + y) R_j^*(y) dy \tag{3.1}$$

where the periodic Riesz transform kernel R_j^* is given by

$$R_j^*(y) = R_j(y) + \sum_{k \in \mathbb{Z}^2_*} \left(R_j(x + 2\pi k) - R_j(2\pi k)\right) \tag{3.2}$$

for $y \neq 2\pi\mathbb{Z}^2$, and R_j is the whole space Riesz-transform kernel given by

$$R_j(y) = \frac{\Omega_j(y/|y|)}{|y|^2} = \frac{y_j}{2\pi|y|^3} \tag{3.3}$$

for $y \neq 0$. The explicit form of the Ω_j is not important: it is a *smooth* function which has *zero mean on* $\mathbb{S}^1$. Note that if we extend φ periodically to $\mathbb{R}^2$, we may rewrite (3.1) as

$$\mathcal{R}_j\varphi(x) = P.V. \int_{\mathbb{R}^2} \varphi(x+y)R_j(y)dy \tag{3.4}$$

where the principal value is both as $|y| \to 0$ and $|y| \to \infty$. See e.g. [15, pp. 256–261], or [133, Chapter VII] for a proof.

We recall the following two statements, which we use frequently throughout this work.

Proposition 3.1.1 (**Pointwise identity**). *Let $\alpha \in (0,2)$ and $\varphi \in C^\infty(\mathbb{T}^2)$. Then we have that*

$$2\varphi(x)\Lambda^\alpha\varphi(x) = \Lambda^\alpha(\varphi(x)^2) + D_\alpha[\varphi](x) \tag{3.5}$$

holds, where

$$\begin{aligned} D_\alpha[\varphi](x) &= P.V. \int_{\mathbb{T}^2} (\varphi(x) - \varphi(x+y))^2 K_\alpha(y)dy \\ &= P.V. \int_{\mathbb{R}^2} (\varphi(x) - \varphi(x+y))^2 \frac{c_\alpha}{|y|^{2+\alpha}}dy \end{aligned} \tag{3.6}$$

pointwise for $x \in \mathbb{T}^2$, and we denote by φ the periodic extension of φ to all of $\mathbb{R}^2$.

Identity (3.5) was proven in [44] (see also [25, 40]). The same identity was used in [136, Lemma 3.1] for the periodic case in one dimension, in the context of Stokes waves. In [44] the pointwise estimate $f'(\varphi)\Lambda^\alpha\varphi - \Lambda^\alpha f(\varphi) \geq 0$ was established for

functions f which are non-decreasing and convex. The second equality in (3.6) follows from Fubini's theorem, a change of variables and the Dominated Convergence theorem. We remark here that equation (3.5) holds for $\varphi \in C^\infty(\mathbb{R}^2)$ analogously. A special case of this fact was used in Chapter 2.

Proposition 3.1.2 (**Fractional Laplacian in L^p**). *Let $p = 4q$, $q \geq 1$, $0 \leq \alpha \leq 2$, and let $\varphi \in C^\infty$ have zero mean on $\mathbb{T}^d$. Then*

$$\int_{\mathbb{T}^d} \theta^{p-1}(x)\Lambda^\alpha\theta(x)dx \geq \frac{1}{p}\|\Lambda^{\alpha/2}(\theta^{p/2})\|_{L^2}^2 + \frac{1}{C_{\alpha,d}}\|\theta\|_{L^p}^p \tag{3.7}$$

holds, with an explicit constant $C_{\alpha,d} \geq 1$, which is independent of p.

When the second term on the right of (3.7) is absent, the above statement was proven in [44]. Since for $p = 4q$, $q \geq 1$, $\theta^{p/2}$ is not of zero mean, it is not immediately clear that the first term on the right of (3.7) dominates the pth power of the L^p norm (in contradistinction with the case $p = 2$, which is the classical Poincaré inequality). The proof of Proposition 3.1.2 to our knowledge was first given in [33, Appendix A]. For the sake of convenience we include a sketch of the proof in Section 3.6.1 below.

Remark 3.1.3 (**Scaling**). Choosing to work on the periodic box $\mathbb{T}^2 = [-\pi, \pi]^2$ is just a matter of convenience for the presentation, so that the group of characters is $\mathbb{Z}^2$. All the results in this chapter may be translated to the case of a general periodic box $\mathbb{T}^2_L = [-L/2, L/2]^2$ as follows. The critical SQG equation has a natural scaling invariance associated to it. If $\theta(x,t)$ is a solution of (1.5)–(1.7) on $[0,T] \times \mathbb{T}^2$, with force $f(x)$ and initial data $\theta_0(x)$, then

$$\theta_\lambda(x,t) = \theta(x/\lambda, t/\lambda)$$

is also a solution of the equations, but on the space-time domain $[0, \lambda T] \times [-\lambda\pi, \lambda\pi]^2$, with force $f_\lambda(x) = (1/\lambda)f(x/\lambda)$ and initial condition $\theta_{0\lambda}(x) = \theta_0(x/\lambda)$. The value

of κ remains unchanged. Thus, in order to work on the box $\mathbb{T}^2_L$, one merely has to set $\lambda = L/(2\pi)$ in the below argument. Note also that $\|\theta\|_{L^\infty} = \|\theta_\lambda\|_{L^\infty}$ for any $\lambda > 0$ and that the L^∞ norm is non-increasing along solution paths, which is why it is customary to refer to (1.5)–(1.7) as the *critical* SQG equations.

3.2 Global regularity for the forced critical Burgers equation

In order to present the main idea of the proof of global regularity for the critical forced SQG equation, we first consider the one dimensional critical forced Burgers equation

$$\partial_t \theta + \theta \partial_x \theta + \Lambda \theta = f \tag{3.8}$$

on a periodic domain $\mathbb{T} = [-\pi, \pi]$, with smooth force, and smooth initial data θ_0. We assume that the data and the force have zero mean on $\mathbb{T}$, so that the same holds for the solution θ at later times.

First notice that we have a global in time control on L^p norms of the solution. Since

$$\int_{\mathbb{T}} \theta \partial_x \theta \theta^{p-1} dx = \frac{1}{p+1} \int_{\mathbb{T}} \partial_x (\theta^{p+1}) dx = 0$$

we multiply (3.8) with θ^{p-1}, integrate over $\mathbb{T}$, and use the the lower bound in Proposition 3.1.2 to obtain

$$\frac{d}{dt} \|\theta\|_{L^p} + c\|\theta\|_{L^p} \leq \|f\|_{L^p}$$

for all $p \geq 2$ even, where the constant c is independent of p. Integrating in time and

then passing $p \to \infty$ thus yields

$$\|\theta(t)\|_{L^\infty} \leq \|\theta_0\|_{L^\infty} + \frac{1}{c}\|f\|_{L^\infty} =: B_\infty \tag{3.9}$$

for all $t \geq 0$. See also Proposition 3.3.1 below for similar estimates for the SQG equation.

Our goal is to give a global in (positive) time bound for the C^α norm of the solution, for some $\alpha \in (0,1)$. It is well-known that due to the critical power of the dissipation, such a bound would in turn imply that the solution cannot develop singularities in finite time.

Let $\alpha \in (0, \alpha_0]$, where $\alpha_0 \in (0,1)$ is to be determined later in terms of the initial data and the force. In order to study the propagation of Hölder continuity in (3.8) we study the evolution of

$$v(x,t;h) = \frac{|\delta_h \theta(x)|}{|h|^\alpha}$$

where $\delta_h \theta(x) = \theta(x+h) - \theta(x)$ is the usual finite difference. Note that a bound on $\sup_{x,h\in\mathbb{T}^2} |v(t,x;h)|$ is in fact equivalent to a bound on $[\theta(t)]_{C^\alpha}$. Evaluating (3.8) at $x+h$ and x and taking the difference one obtains

$$(\partial_t + \theta\partial_x + (\delta_h\theta)\partial_h + \Lambda_x)(\delta_h\theta) = (\delta_h f).$$

Multiplying by $|h|^{-2\alpha}(\delta_h\theta)$ and appealing to Proposition 3.1.1 we thus arrive at the pointwise inequality

$$\begin{aligned}(\partial_t + \theta\partial_x + (\delta_h\theta)\partial_h + \Lambda_x)\, v^2 + \frac{D[\delta_h\theta]}{|h|^{2\alpha}} &= \frac{4\alpha}{|h|^{2\alpha+1}}(\delta_h\theta)^3 + \frac{2(\delta_h f)}{|h|^\alpha} v \\ &\leq \frac{4\alpha}{|h|^{1-\alpha}} v^3 + \frac{4\|f\|_{L^\infty}}{|h|^\alpha} v \end{aligned} \tag{3.10}$$

where

$$D[\delta_h\theta] = cP.V. \int (\delta_h\theta(x) - \delta_h\theta(x+y))^2 \frac{1}{|y|^2} dy.$$

The main idea is to combine a nonlinear lower bound for $D[\delta_h\theta]$, with the smallness of α to obtain an ODE for $\|v(t)\|_{L^\infty_{t,x}}$, which has global bounded solutions.

Using the argument in [40], see also (3.29) below, we have that

$$D[\delta_h\theta](x) \geq \frac{|\delta_h\theta(x)|^3}{C\|\theta\|_{L^\infty}|h|}$$

for some $C > 0$. Therefore, using (3.9) we arrive at

$$\frac{D[\delta_h\theta]}{|h|^{2\alpha}} \geq \frac{|\delta_h\theta(x)|^3}{C\|\theta\|_{L^\infty}|h|^{1+2\alpha}} = \frac{v^3}{CB_\infty|h|^{1-\alpha}}.$$

Inserting the above bound in (3.10) yields

$$(\partial_t + \theta\partial_x + (\delta_h\theta)\partial_h + \Lambda_x)\, v^2 + \frac{v^3}{C_0 B_\infty |h|^{1-\alpha}} \leq \frac{4\alpha v^3}{|h|^{1-\alpha}} + \frac{4\|f\|_{L^\infty} v}{|h|^\alpha}. \tag{3.11}$$

Therefore if we choose α_0 such that

$$\alpha_0 \leq \frac{1}{8C_0B_\infty}$$

the nonlinear term on the right side of (3.11) can be absorbed into the left side of the inequality, and moreover, once we appeal to the ε-Young inequality in order to hide the forcing term in the dissipation, we arrive at

$$(\partial_t + \theta\partial_x + (\delta_h\theta)\partial_h + \Lambda_x)\, v^2 + \frac{v^3}{4C_0 B_\infty |h|^{1-\alpha}} \leq C_1 B_\infty^{1/2} \|f\|_{L^\infty}^{3/2} |h|^{\frac{1-4\alpha}{2}} \tag{3.12}$$

for some $C_1 > 0$ which depends only on C_0. Formally evaluating (3.12) at a point

$(\bar{x}, \bar{h})$ where v^2 attains its maximal value, and noting that such a point must necessarily obey $|\bar{h}| \leq \pi$ (the latter is due to the periodicity in h of $\delta_h\theta$, and the strictly decaying nature of $|h|^{-\alpha}$), we obtain

$$\begin{aligned}(\partial_t v^2)(\bar{x}, \bar{h}) + \frac{v^3(\bar{x}, \bar{h})}{4C_0 B_\infty \pi^{1-\alpha}} &\leq (\partial_t + \theta\partial_x + (\delta_h\theta)\partial_h + \Lambda_x)\, v^2(\bar{x}, \bar{h}) + \frac{v^3(\bar{x}, \bar{h})}{4C_0 B_\infty \pi^{1-\alpha}} \\ &\leq C_1 B_\infty^{1/2} \|f\|_{L^\infty}^{3/2} \pi^{\frac{1-4\alpha}{2}}, \end{aligned} \tag{3.13}$$

as long as we impose

$$\alpha_0 \leq \frac{1}{4}.$$

In (3.13) we used that at the maximum (in joint x and h) of v^2 we must have $\partial_x v^2 = \partial_h v^2 = 0$, and $\Lambda v^2 \geq 0$. One can then rigorously show that for almost every t in $[0, T_*)$, the maximal time of existence of a smooth solution to the initial value problem associated to (3.8), we have

$$\frac{d}{dt}\|v(t)\|_{L^\infty_{x,h}}^2 \leq (\partial_t v^2)(\bar{x}, \bar{h})$$

which combined with (3.13) yields that for any $\alpha \in (0, \alpha_0]$, with $\alpha_0 = \min\{1/(8C_0B_\infty), 1/4\}$, we have

$$[\theta(t)]_{C^\alpha} = \|v(t)\|_{L^\infty_{x,h}} \leq \max\left\{[\theta_0]_{C^\alpha}, (4C_0C_1)^{1/2}\pi^{\frac{1-2\alpha}{2}} B_\infty^{1/2}\|f\|_{L^\infty}^{1/2}\right\}$$

for all $t \in [0, T_*)$, and thus a posteriori for all $t \geq 0$, thereby completing the proof of global regularity for critical Burgers.

Notice that the key ingredients were the nonlinear lower bound on Λ and the smallness of α. This argument carries over to the SQG case modulo some technical issues having to do with the fact that the velocity depends linearly but in a nonlocal

fashion on θ. We give details in Section 3.3 below, where we also fully justify the arguments presented here only formally for clarity of the exposition.

3.3 Global regularity for forced critical SQG

In this section we give a new proof of global existence and uniqueness of solutions for (1.5)–(1.7) which has the advantage that the bounds require merely $f \in L^\infty \cap H^1$. The proof is based on the nonlinear lower bound for the fractional Laplacian discovered in [40]. We first recall some L^p estimates for solutions of the forced critical SQG equation.

Proposition 3.3.1 (Absorbing ball in L^p). *Let θ be a smooth solution of (1.5)–(1.7), and let $p \geq 2$ be even. Then we have*

$$\|\theta(\cdot,t)\|_{L^p} \leq \|\theta_0\|_{L^p} e^{-tc_0\kappa} + \frac{1}{c_0\kappa}\|f\|_{L^p}(1-e^{-tc_0\kappa}) \tag{3.14}$$

for some universal constant c_0. Moreover,

$$\|\theta(\cdot,t)\|_{L^\infty} \leq \|\theta_0\|_{L^\infty} e^{-tc_0\kappa} + \frac{1}{c_0\kappa}\|f\|_{L^\infty}(1-e^{-tc_0\kappa}) \tag{3.15}$$

holds with the same universal constant c_0.

Proof. Multiplying (1.5) by θ^{p-1}, and using Proposition 3.1.2 we arrive at

$$\frac{d}{dt}\|\theta\|_{L^p} + c_0\kappa\|\theta\|_{L^p} \leq \|f\|_{L^p}$$

for some $c_0 > 0$ which is independent of p. Therefore we obtain (3.14) for all $t \geq 0$ and $p \geq 2$ even. Moreover, since the constants appearing in (3.14) are independent

of p, and we are on a periodic domain $\mathbb{T}^2 = [-\pi, \pi]^2$, we have that

$$\|\theta(\cdot,t)\|_{L^p} \leq (2\pi)^{2/p}\|\theta_0\|_{L^\infty}e^{-tc_0\kappa} + \frac{(2\pi)^{2/p}}{c_0\kappa}\|f\|_{L^\infty}(1-e^{-tc_0\kappa})$$

which yields (3.15) upon passing $p \to \infty$. □

In particular, Proposition 3.3.1 shows that for any $p \in [2, \infty)$ even and $p = \infty$ we have that

$$\|\theta(\cdot,t)\|_{L^p} \leq \|\theta_0\|_{L^p} + \frac{1}{c_0\kappa}\|f\|_{L^p} := M_p(\theta_0, f) = M_p \tag{3.16}$$

for any $t \geq 0$. Next, we recall the following local existence [100, 117] and smoothing [64] result.

Proposition 3.3.2 (Local solution). *Assume $\theta_0 \in H^1$ and $f \in H^1 \cap L^\infty$. There exists $T_* = T_*(\theta_0, f) > 0$ and a unique solution θ of the initial value problem (1.5)–(1.7) which obeys the energy inequality and*

$$\theta \in C([0,T_*); H^1) \cap L^2(0,T_*; H^{3/2}). \tag{3.17}$$

Moreover, for any $\beta \geq 0$, if $f \in H^\beta$ we have that

$$\sup_{0<t<T_*} t^\beta \|\theta(\cdot,t)\|_{H^{1+\beta}} \leq c_1\kappa^{-\beta}\|\theta_0\|_{H^1} + c_1\kappa^{-1}T_*^\beta\|f\|_{H^\beta} \tag{3.18}$$

for some positive universal constant c_1. Additionally, we have $\lim_{t\to 0+} t^\beta\|\theta(\cdot,t)\|_{H^{1+\beta}} = 0$, when $\beta > 0$.

We emphasize that the time of existence T_* doesn't depend in a locally uniform way on $\|\theta_0\|_{H^1}$. The above result was proven in the aforementioned works in the absence of a forcing term, but it is not difficult to verify that estimate (3.18) holds assuming f is sufficiently smooth. We omit these details. Note that the solution may be extended

uniquely past T_* assuming that a priori we know e.g. that $\sup_{t\in[T_*/2,T_*)}[\theta(t,\cdot)]_{C^\alpha}<\infty$, for some $\alpha>0$.

The first result we obtain is the propagation of Hölder continuity for smooth solutions, which may then be applied to a sequence of solutions to a regularized problem, in order to obtain in the limit the corresponding result for weak solutions (see Theorem 3.3.4 below).

Theorem 3.3.3 (**Propagation of Hölder regularity**). *Let θ_0 and f be sufficiently smooth, $T>0$ be arbitrary, and let $\theta\in C^{1/2}((0,T);C^{1,1/2})$ be the unique classical solution of the initial value problem* (1.5)–(1.7). *As in* (3.16) *above, define*

$$M_\infty = M_\infty(\theta_0,f) = \|\theta_0\|_{L^\infty} + (c_0\kappa)^{-1}\|f\|_{L^\infty}.$$

There exists a sufficiently small universal constant $\varepsilon_0>0$ such that for any α with

$$0<\alpha\le\alpha_0=\min\left\{\frac{\varepsilon_0\kappa}{M_\infty},\frac{1}{4}\right\}, \tag{3.19}$$

if $\theta_0\in C^\alpha$ we have

$$[\theta(t)]_{C^\alpha}\le M_\alpha(t) \tag{3.20}$$

for all $t\in[0,T]$, where M_α is the solution of the ordinary differential equation (3.49) *below. In particular, we have that*

$$M_\alpha(t)\le\max\left\{[\theta_0]_{C^\alpha},\frac{M_\infty}{\varepsilon_0}\right\} \tag{3.21}$$

for any $t\in[0,T]$, and also

$$M_\alpha(t)\le\frac{2M_\infty}{\varepsilon_0} \tag{3.22}$$

for all $t \geq t_\alpha = t_\alpha(M_\infty, [\theta_0]_{C^\alpha})$*, which is defined explicitly in* (3.50) *below.*

Proof of Theorem 3.3.3. For $\alpha > 0$, we look at the evolution of weighted finite differences

$$v(x,t;h) = \frac{|\delta_h \theta(x)|}{|h|^\alpha}.$$

Since $\theta_0 \in C^\alpha$, we have that

$$\|v_0\|_{L^\infty_{x,h}} \leq [\theta_0]_{C^\alpha} \leq M_\alpha(0). \tag{3.23}$$

Our goal is to find an upper bound $M_\alpha(t)$ such that

$$\|v(\cdot,t;\cdot)\|_{L^\infty_{x,h}} \leq M_\alpha(t) \tag{3.24}$$

for any $t \geq 0$. In view of the periodicity in h of $\delta_h\theta(x)$, we would in turn obtain from (3.24) that

$$[v(\cdot,t)]_{C^\alpha} = \sup_{x,h\in\mathbb{T}^2} \frac{|\delta_h\theta(x,t)|}{|h|^\alpha} \leq \|v(\cdot,t;\cdot)\|_{L^\infty_{x,h}} \leq M_\alpha(t) \tag{3.25}$$

which would then conclude the proof of the theorem.

In order to find $M_\alpha(t)$ for which (3.24) holds, we first write the equation obeyed by $\delta_h\theta$. If follows by taking finite differences in (1.5) that pointwise in x, t, h we have

$$\left(\partial_t + u\cdot\nabla_x + (\delta_h u)\cdot\nabla_h + \kappa\Lambda\right)\delta_h\theta = \delta_h f. \tag{3.26}$$

Upon multiplying (3.26) by

$$\frac{\delta_h\theta(x,t)}{|h|^{2\alpha}}$$

and using Proposition 3.1.1, we obtain that v obeys the equation

$$\begin{aligned}(\partial_t + u\cdot\nabla_x + (\delta_h u)\cdot\nabla_h + \kappa\Lambda)\, v^2 + \frac{\kappa D[\delta_h\theta]}{|h|^{2\alpha}} &= 4\alpha(\delta_h u)\cdot\frac{h}{|h|}\frac{v^2}{|h|} + \frac{2(\delta_h f)v}{|h|^\alpha} \\ &\leq 4\alpha|\delta_h u|\frac{v^2}{|h|} + \frac{4\|f\|_{L^\infty}v}{|h|^\alpha} \qquad (3.27)\end{aligned}$$

where $\delta_h u = \mathcal{R}_x^\perp(\delta_h\theta)$, and

$$D[\delta_h\theta](x) = \frac{1}{2\pi}P.V.\int_{\mathbb{R}^2}\Big(\delta_h\theta(x) - \delta_h\theta(x+y)\Big)^2\frac{1}{|y|^3}dy \qquad (3.28)$$

from (3.6) with $\alpha = 1$, and we have used the explicit formula (1.4) for the normalizing constant. First we give a lower bound on the dissipative term $D[\delta_h\theta]$, and then estimate the velocity increment $|\delta_h u|$ in (3.27).

Throughout the proof we will use a cutoff function $\chi\colon [0,\infty) \to [0,\infty)$ which is smooth, non-increasing, identically 1 on $[0,1]$, vanishes on $[2,\infty)$, and obeys $|\chi'| \leq 2$.

In the spirit of [40], we obtain a nonlinear lower bound for the dissipative term in (3.27). Pointwise in x and h it holds that

$$D[\delta_h\theta](x) \geq \frac{|\delta_h\theta(x)|^3}{c_2\|\theta\|_{L^\infty}|h|} \qquad (3.29)$$

for some universal constant $c_2 > 0$. To prove (3.29), we proceed as follows. For $r \geq 4|h|$ to be determined, we have

$$
\begin{aligned}
D[\delta_h\theta](x) &\geq \frac{1}{2\pi}\int_{\mathbb{R}^2}\frac{(\delta_h\theta(x)-\delta_h\theta(x+y))^2}{|y|^3}\left(1-\chi\left(\frac{|y|}{r}\right)\right)dy \\
&\geq \frac{1}{2\pi}(\delta_h\theta(x))^2\int_{|y|\geq 2r}\frac{1}{|y|^3}dy-\frac{1}{\pi}|\delta_h\theta(x)|\left|\int_{\mathbb{R}^2}\frac{\delta_h\theta(x+y)}{|y|^3}\left(1-\chi\left(\frac{|y|}{r}\right)\right)dy\right| \\
&\geq \frac{1}{2r}|\delta_h\theta(x)|^2-\frac{1}{\pi}|\delta_h\theta(x)|\int_{\mathbb{R}^2}|\theta(x+y)|\left|\delta_{-h}\left(\frac{1}{|y|^3}-\frac{1}{|y|^3}\chi\left(\frac{|y|}{r}\right)\right)\right|dy \\
&\geq \frac{1}{2r}|\delta_h\theta(x)|^2-C|\delta_h\theta(x)|\|\theta\|_{L^\infty}|h|\int_{\mathbb{R}^2}\left(\frac{\mathbf{1}_{|y|\geq 3r/4}}{|y|^4}+\frac{\mathbf{1}_{3r/4\leq|y|\leq 9r/4}}{r^4}\right)dy \\
&\geq \frac{1}{2r}|\delta_h\theta(x)|^2-C|\delta_h\theta(x)|\|\theta\|_{L^\infty}\frac{|h|}{r^2} \qquad (3.30)
\end{aligned}
$$

for some $C \geq 1$. In the above estimate we have used estimate (3.32) below. Setting

$$
r = \frac{4C\|\theta\|_{L^\infty}}{|\delta_h\theta(x)|}|h|
$$

in (3.30), which obeys $r \geq 4|h|$ due to $|\delta_h\theta| \leq 2\|\theta\|_{L^\infty}$, completes the proof of (3.29).

In (3.30) above we have used the following. The mean value theorem gives that for a smooth g we have

$$
|\delta_h g(y)| = |h\cdot\nabla g((1-\lambda)y+\lambda(y+h))| \leq |h|\max_{\lambda\in[0,1]}|\nabla g(y+\lambda h)| \qquad (3.31)
$$

for $h, y \in \mathbb{R}^2$. In particular, for

$$
g(y) = \frac{1}{|y|^3}\left(1-\chi\left(\frac{|y|}{r}\right)\right)
$$

we have that

$$
\max_{\lambda\in[0,1]}|\nabla g(y+\lambda h)| \leq C\left(\frac{\mathbf{1}_{|y|\geq 3r/4}}{|y|^4}+\frac{\mathbf{1}_{3r/4\leq|y|\leq 9r/4}}{r^4}\right) \qquad (3.32)
$$

whenever $|h| \leq r/4$.

Combining estimate (3.29) with the a priori bound (3.16) (with $p = \infty$) we obtain that the positive term on the right side of (3.27) is bounded from below as

$$\frac{\kappa D[\delta_h \theta]}{|h|^{2\alpha}} \geq \frac{\kappa}{|h|^{2\alpha}} \frac{|\delta_h \theta|^3}{c_2 M_\infty |h|} = \frac{\kappa}{c_2 M_\infty} \frac{v^3}{|h|^{1-\alpha}} \tag{3.33}$$

pointwise in x and h.

We now estimate the velocity finite difference $\delta_h u$. For this purpose fix x and h (ignore t dependence) and let $\rho > 0$ be such that

$$\rho \geq 4|h|. \tag{3.34}$$

As before we let χ be a smooth cutoff function, that is 1 on $[0, 1]$, non-increasing, vanishes on $[2, \infty)$, and obeys $|\chi'| \leq 2$. We decompose the singular integral defining $\delta_h u$ as

$$\delta_h u(x) = \mathcal{R}_x^\perp(\delta_h \theta(x)) = \frac{1}{2\pi} P.V. \int_{\mathbb{R}^2} \frac{y^\perp}{|y|^3} \delta_h \theta(x+y) dy = \delta_h u_{in}(x) + \delta_h u_{out}(x),$$

where the inner piece is given by

$$\begin{aligned} \delta_h u_{in}(x) &= \frac{1}{2\pi} P.V. \int_{\mathbb{R}^2} \frac{y^\perp}{|y|^3} \chi\left(\frac{|y|}{\rho}\right) \delta_h \theta(x+y) dy \\ &= \frac{1}{2\pi} P.V. \int_{\mathbb{R}^2} \frac{y^\perp}{|y|^3} \chi\left(\frac{|y|}{\rho}\right) \Big(\delta_h \theta(x+y) - \delta_h \theta(x)\Big) dy \end{aligned}$$

by using that the kernel of $\mathcal{R}^\perp$ has zero average on the unit sphere, and the outer piece is given by

$$\begin{aligned} \delta_h u_{out}(x) &= \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{y^\perp}{|y|^3} \left(1 - \chi\left(\frac{|y|}{\rho}\right)\right) \delta_h \theta(x+y) dy \\ &= \frac{1}{2\pi} \int_{\mathbb{R}^2} \delta_{-h} \left(\frac{y^\perp}{|y|^3} \left(1 - \chi\left(\frac{|y|}{\rho}\right)\right)\right) \theta(x+y) dy \end{aligned}$$

by using a finite-difference-by-parts.

For the inner piece, by appealing to the Cauchy-Schwartz inequality, we obtain

$$|\delta_h u_{in}| \leq C \left(\rho D[\delta_h \theta]\right)^{1/2}. \tag{3.35}$$

In order to bound the outer piece, we recall that the mean value theorem gives

$$|\delta_h g(y)| \leq |h| \max_{\lambda \in [0,1]} |\nabla g(y + \lambda h)| \tag{3.36}$$

for smooth functions g. We apply (3.36) with

$$g(y) = \frac{y^{\perp}}{|y|^3} \left(1 - \chi\left(\frac{|y|}{\rho}\right)\right)$$

which yields

$$|\delta_{-h} g(y)| \leq |h| \max_{\lambda \in [0,1]} |\nabla g(y - \lambda h)| \leq C|h| \frac{\mathbf{1}_{|y| \geq \rho/2}}{|y|^3}$$

when $|h| \leq \rho/4$. This in turn implies that

$$|\delta_h u_{out}| \leq C|h| \int_{|y| \geq \rho/2} \frac{1}{|y|^3} |\theta(x+y)| dy \leq \frac{C M_\infty |h|}{\rho}. \tag{3.37}$$

Combining the inner (3.35) and outer (3.37) velocity estimates, we obtain that

$$|\delta_h u| \leq C \left(\left(\rho D[\delta_h \theta]\right)^{1/2} + \frac{M_\infty |h|}{\rho} \right) \tag{3.38}$$

if ρ is chosen so that $\rho \geq 4|h|$.

Using the Cauchy-Schwartz inequality we obtain from (3.38) that

$$\begin{aligned} 4\alpha|\delta_h u|\frac{v^2}{|h|} &\leq C\alpha\left(\rho D[\delta_h\theta]\right)^{1/2}\frac{v^2}{|h|} + C\alpha\frac{M_\infty v^2}{\rho} \\ &\leq \frac{\kappa D[\delta_h\theta]}{2|h|^{2\alpha}} + \frac{c_3\alpha^2\rho v^4}{\kappa|h|^{2-2\alpha}} + \frac{c_3\alpha M_\infty v^2}{\rho} \end{aligned}$$

for some sufficiently large $c_3 > 0$. Therefore, letting

$$\rho = \frac{\kappa^{1/2}M_\infty^{1/2}|h|^{1-\alpha}}{\alpha^{1/2}v} = \frac{\kappa^{1/2}M_\infty^{1/2}}{\alpha^{1/2}|\delta_h\theta|}|h| \tag{3.39}$$

we obtain

$$4\alpha|\delta_h u|\frac{v^2}{|h|} \leq \frac{\kappa D[\delta_h\theta]}{2|h|^{2\alpha}} + \frac{2c_3\alpha^{3/2}M_\infty^{1/2}}{\kappa^{1/2}}\frac{v^3}{|h|^{1-\alpha}}. \tag{3.40}$$

Note that in order to define ρ as in (3.39), we need to ensure $\rho \geq 4|h|$. By the triangle inequality we have $|\delta_h\theta| \leq 2M_\infty$ and hence indeed

$$\rho \geq \left(\frac{\kappa}{2\alpha M_\infty}\right)^{1/2}|h| \geq 4|h|$$

holds, since by assumption $\alpha \in (0, \alpha_0]$ and

$$\alpha_0 \leq \frac{\kappa}{32M_\infty}. \tag{3.41}$$

Combining (3.27), the lower bound (3.33), estimate (3.40), and recalling the definition of M_∞ in (3.16), we arrive at

$$\left(\partial_t + u\cdot\nabla_x + (\delta_h u)\cdot\nabla_h + \kappa\Lambda\right)v^2 + \left(\frac{\kappa}{2c_2M_\infty} - \alpha^{3/2}\frac{2c_3M_\infty^{1/2}}{\kappa^{1/2}}\right)\frac{v^3}{|h|^{1-\alpha}} \leq \frac{4c_0\kappa M_\infty v}{|h|^\alpha}. \tag{3.42}$$

Since $\alpha \leq \alpha_0$, with

$$\alpha_0 \leq \frac{\kappa}{(8c_2c_3)^{2/3}M_\infty} \tag{3.43}$$

we furthermore obtain from (3.42) that

$$(\partial_t + u \cdot \nabla_x + (\delta_h u) \cdot \nabla_h + \kappa\Lambda)\, v^2 + \frac{\kappa}{4c_2M_\infty}\frac{v^3}{|h|^{1-\alpha}} \leq \frac{4c_0\kappa M_\infty v}{|h|^\alpha}. \tag{3.44}$$

pointwise in x and h. Upon using the ε-Young inequality for the right hand side, we find that

$$(\partial_t + u \cdot \nabla_x + (\delta_h u) \cdot \nabla_h + \kappa\Lambda)\, v^2 + \frac{\kappa}{6c_2M_\infty}\frac{v^3}{|h|^{1-\alpha}} \leq c_4\kappa|h|^{\frac{1-4\alpha}{2}}M_\infty^2 \tag{3.45}$$

for a positive universal constant $c_4 = (4c_0)^{3/2}(4c_2)^{1/2}$.

We now proceed as in [44, Section 4], and refer to Section 3.6.2 below for details. Since θ is sufficiently smooth v^2 is a bounded continuous function in both x and h, which is periodic in x. Moreover, given $x, h \in \mathbb{T}^2$ and $k \in \mathbb{Z}^2_*$ we have that

$$v(x,t;h)^2 \geq v^2(x,t;h+2\pi k),$$

in view of the periodicity in h of $\delta_h\theta$, and the strict monotonicity of $|h|^{-\alpha}$. Therefore there exists *at least* one point $(\bar{x},\bar{h}) = (\bar{x}(t),\bar{h}(t)) \in \mathbb{T}^2 \times \mathbb{T}^2$ where the function $v(\cdot,t;\cdot)^2$ attains its maximum. We define

$$g(t) = \sup_{x,h\in\mathbb{T}^2} v(x,t;h)^2 = v(\bar{x}(t),t;\bar{h}(t))^2. \tag{3.46}$$

In Section 3.6.2 below we show that g is Lipschitz continuous on $[0,T]$, and that for

almost every t there exists $(\bar{x}(t); \bar{h}(t))$ such that

$$g'(t) = (\partial_t v^2)(\bar{x}(t), t; \bar{h}(t))$$

and (3.46) holds. Evaluating (3.45) at the joint x, h-maximum $(\bar{x}, \bar{h})$, using that at the maximum we have

$$\nabla_x v(\bar{x}, t; \bar{h})^2 = 0 = \nabla_h v(\bar{x}, t; \bar{h})^2 \quad \text{and} \quad \Lambda v(\bar{x}, t; \bar{h})^2 \geq 0,$$

and the fact that $|\bar{h}| \leq 4\pi$ we thus obtain

$$g'(t) + \frac{\kappa}{6c_2 M_\infty (4\pi)^{1-\alpha}} g(t)^{3/2} \leq c_4 \kappa M_\infty^2 (4\pi)^{\frac{1-4\alpha}{2}} \tag{3.47}$$

once we additionally assume that

$$\alpha_0 \leq \frac{1}{4}. \tag{3.48}$$

Since we are now dealing with an ordinary differential equation, by the usual comparison principle for ODEs it follows that

$$\|v(t)\|^2_{L^\infty_{t,x}} = g(t) \leq M_\alpha(t)^2,$$

where $M_\alpha(t)$ is the solution of the initial value problem

$$\frac{d}{dt} M_\alpha^2 + \frac{\kappa}{c_5 M_\infty} M_\alpha^3 = c_5^2 \kappa M_\infty^2, \quad M_\alpha(0) = [\theta_0]_{C^\alpha}, \tag{3.49}$$

where $c_5 = c_5(c_2, c_4)$, is a fixed deterministic constant which is *independent* of κ, M_∞, or α.

In particular, this proves that

$$[\theta(t)]_{C^\alpha} \le M_\alpha(t) \le \max\left\{[\theta_0]_{C^\alpha}, c_5 M_\infty\right\}$$

for any $t \ge 0$. Moreover there exists $t_\alpha = t_\alpha(M_\infty, [\theta_0]_{C^\alpha}) \ge 0$ defined as

$$t_\alpha = \begin{cases} 0, & \text{if } [\theta_0]_{C^\alpha} \le 2c_5 M_\infty \\ \frac{1}{7\kappa}\left(\frac{[\theta_0]_{C^\alpha}^2}{4c_5^2 M_\infty^2} - 1\right), & \text{if } [\theta_0]_{C^\alpha} > 2c_5 M_\infty \end{cases} \tag{3.50}$$

such that

$$[\theta(t)]_{C^\alpha} \le 2c_5 M_\infty$$

for any $t \ge t_\alpha$. The above bound shows that the solution forgets the initial data even in the C^α norm. □

Theorem 3.3.3 implies the propagation of Hölder continuity for weak solutions.

Theorem 3.3.4 (Hölder propagation for weak solutions). *Assume $f \in L^\infty \cap H^1$, $\theta_0 \in L^\infty \cap H^1$, $T > 0$ is arbitrary, and let $\theta \in L^\infty(0,T;H^1) \cap L^2(0,T;H^{3/2})$ be the unique weak solution of the critical, forced SQG equation (1.5)–(1.7). Let $\alpha_0 = \alpha_0(\|\theta_0\|_{L^\infty}, \|f\|_{L^\infty}) \le 1/4$ be defined as in (3.19). For any $\alpha \in (0, \alpha_0]$, if $\theta_0 \in C^\alpha$, then $\theta \in L^\infty(0,T;C^\alpha)$. Moreover, we have $[\theta(t)]_{C^\alpha} \le M_\alpha(t)$ for a function $M_\alpha(t)$ which obeys (3.21) and (3.22).*

Proof of Theorem 3.3.4. Let J_ε be a standard mollifier operator. For $\varepsilon \in (0,1]$, we let θ_ε be the solution of

$$\partial_t \theta^\varepsilon + \kappa\Lambda\theta^\varepsilon + u^\varepsilon \cdot \nabla\theta^\varepsilon - \varepsilon\Delta\theta^\varepsilon = J_\varepsilon f, \quad u^\varepsilon = \mathcal{R}^\perp\theta^\varepsilon, \quad \theta_0^\varepsilon = \theta_0. \tag{3.51}$$

As in Proposition 3.3.1 we obtain that

$$\|\theta^\varepsilon(t)\|_{L^\infty} \leq \|\theta_0\|_{L^\infty} + \frac{1}{c_0\kappa}\|J_\varepsilon f\|_{L^\infty} \leq \|\theta_0\|_{L^\infty} + \frac{1}{c_0\kappa}\|f\|_{L^\infty} = M_\infty(\theta_0, f). \quad (3.52)$$

Indeed, the addition of the regularizing term $-\varepsilon\Delta\theta^\varepsilon$ in the equation does not change any part of the argument, and J_ε is given by convolving with an L^1 kernel of mean 1. Once we have that $\theta^\varepsilon \in L^\infty_{t,x}$, a supercritical information for the dissipation given by the Laplacian, the existence of a unique global smooth solution of (3.51) follows from classical arguments (see e.g. [41] for subcritical SQG). Since $J_\varepsilon f \in C^\infty$, in fact a bootstrap shows that $\theta^\varepsilon \in C^\infty((0,T)\times\mathbb{T}^2)$, but with bounds that depend on ε.

At this stage we compute α_0 as in (3.19), which is independent of ε due to (3.52), and then apply Theorem 3.3.3 to θ^ε, which we are allowed to since θ^ε is smooth. We emphasize that the presence of the regularizing term $-\varepsilon\Delta$ does not require *any* modification to the proof of Theorem 3.3.3. The Laplacian (in x) does not affect the finite differences in h, and the negative Laplacian evaluated at the maximum of a function is non-negative. Therefore, for any $\varepsilon \in (0,1]$, we have

$$[\theta^\varepsilon(t)]_{C^\alpha} \leq M_\alpha(t) \quad (3.53)$$

for all $t \geq 0$, where $M_\alpha(t)$ (which is independent of ε) is given by the solution of (3.49), and obeys the bounds (3.21) for all time, and (3.22) for long enough time.

The sequence of solutions $\{\theta^\varepsilon\}_{\varepsilon\in(0,1]}$ is thus uniformly bounded in $L^\infty(0,T;C^\alpha)$. In particular, since $|\mathbb{T}^2| < \infty$ this implies that θ^ε is uniformly bounded in $L^\infty(0,T;H^\alpha)$, and from (3.51) we have that $\partial_t\theta^\varepsilon$ is uniformly bounded in $L^\infty(0,T;H^{\alpha-2})$. Since the injection of $H^\alpha(\mathbb{T}^2)$ into $L^2(\mathbb{T}^2)$ is compact, and the injection of $L^2(\mathbb{T}^2)$ in $H^{\alpha-2}(\mathbb{T}^2)$ is continuous, the Aubin-Lions compactness lemma and the uniform in ε estimates obtained earlier, imply that there exists $\bar{\theta} \in L^\infty(0,T;C^\alpha)$, with bounds inherited (e.g.

by duality) directly from (3.53), such that

$$\theta^\varepsilon \to \bar{\theta} \text{ in } L^2(0,T;L^2).$$

The above strong convergence in $L^2_{t,x}$ is sufficient to pass to the limit in the weak formulation of (3.51), and show that $\bar{\theta}$ is a weak solution of the critical forced SQG equation (1.5)–(1.7) on $[0,T)$. This is seen by writing the nonlinear term in divergence form.

To conclude the proof we notice that in fact $\theta = \bar{\theta}$. This follows in the spirit of weak-strong uniqueness: one writes the equation obeyed by $\theta - \bar{\theta}$ and performs an L^2 energy estimate. The equation for the difference has zero initial data and zero force. Using that $\bar{\theta} \in L^\infty_t C^\alpha_x$ we have $\int \mathcal{R}^\perp \bar{\theta} \cdot \nabla(\theta - \bar{\theta})(\theta - \bar{\theta}) dx = 0$, and since $\theta \in L^\infty_t H^1_x \cap L^2_t H^{3/2}_x$ we have $|\int \mathcal{R}^\perp(\theta - \bar{\theta}) \cdot \nabla \theta (\theta - \bar{\theta}) dx| \leq \kappa \|\theta - \bar{\theta}\|^2_{H^{1/2}} + C\|\theta - \bar{\theta}\|^2_{L^2} \|\theta\|^2_{H^{3/2}}$. The proof of $\theta = \bar{\theta}$ is then concluded via the Grönwall inequality. □

The results obtained in this section may be summarized as follows.

Theorem 3.3.5 (Global regularity). *Let $\theta_0 \in H^1(\mathbb{T}^2)$ and $f \in L^\infty(\mathbb{T}^2) \cap H^1(\mathbb{T}^2)$. There exists a unique global solution $\theta \in L^\infty([0,\infty); H^1) \cap L^2_{loc}((0,\infty); H^{3/2})$ of the initial value problem (1.5)–(1.7). For any $t_1 > 0$ we have $\theta \in L^\infty([t_1,\infty); H^{3/2}) \cap L^2_{loc}([t_1,\infty); H^2)$.*

It is clear that if we would furthermore assume $f \in C^\infty(\mathbb{T}^2)$, then $\theta \in C^\infty([t_1,\infty) \times \mathbb{T}^2)$, for any $t_1 > 0$.

Proof of Theorem 3.3.5. By the local existence result of Proposition 3.3.2, there exists a time $T_* = T_*(\theta_0, f) > 0$, and a unique solution $\theta \in L^\infty_t H^1_x \cap L^2_t H^{3/2}_x$ of (1.5)–(1.7) on $[0,T_*)$. In addition, by the local smoothing estimate (3.18), since $f \in H^1$ we conclude that $\theta(t_1) \in H^2 \supset C^{\alpha_0}$, where $\alpha_0 = \alpha_0(\|\theta_0\|_{L^\infty}, \|f\|_{L^\infty})$ is as in Theorem 3.3.3, and $t_1 \in (0,T_*)$ is arbitrary. The propagation of Hölder continuity result of Theorem 3.3.4,

with initial data $\theta(t_1)$ then yields

$$\sup_{t\in[t_1,T_*-\tau]}[\theta(t)]_{C^{\alpha_0}} \leq C(\theta_0, f)$$

where the constant $C(\theta_0, f)$ is *independent* of τ. Since the Hölder bound is supercritical for the natural scaling of the equation, we use the C^{α_0} version of the nonlinear lower bound for the fractional Laplacian [40, Theorem 2.2] in order to bootstrap in regularity and obtain

$$\sup_{t\in[t_1,T_*-\tau]}\|\theta(t)\|^2_{H^{3/2}} + \int_{t_1}^{T_*-\tau}\|\theta(t)\|^2_{H^2}dt \leq C(\theta_0, f)$$

independently of τ. The proof of this bootstrap procedure is given as part the proof of Theorem 3.4.2 below, and we omit the details here to avoid redundancy. The above estimate in particular shows that the solution θ may be continued uniquely past T_*, since for data in $H^{3/2}$ the time of existence of the $L^\infty_t H^1_x \cap L^2_t H^{3/2}_x$ solution depends only on the size of the *norm* in $H^{3/2}$ (we do not have this fact available if the initial data merely lies in H^1). Having extended the solution past T_*, we repeat the above argument and conclude the proof of global regularity. □

3.4 Existence of a global attractor

In view of the global existence established in Theorem 3.3.5, we define a *solution operator* $S(t)$ for the initial value problem (1.5)–(1.7) via

$$S(t)\colon H^1 \to H^1, \qquad S(t)\theta_0 = \theta(\cdot, t), \tag{3.54}$$

for any $t \geq 0$. In this section we establish (cf. Theorem 3.4.1 below) the existence of a global attractor $\mathcal{A}$ for the long-time dynamics of $S(t)$ on the phase space H^1.

Theorem 3.4.1 (Existence of a global attractor). *The solution map* $S\colon [0,\infty)\times H^1 \to H^1$ *associated to* (1.5)–(1.6) *with* $f \in L^\infty \cap H^1$*, possesses a global attractor* $\mathcal{A}$*, which is a non-empty compact connected invariant set (i.e.* $S(t)\mathcal{A} = \mathcal{A}$ *for all* $t \geq 0$*) such that for every* $\theta_0 \in H^1$ *we have*

$$\lim_{t\to\infty} \operatorname{dist}(S(t)\theta_0, \mathcal{A}) = 0.$$

The set $\mathcal{A}$ *is maximal in the sense that for any bounded subset* $\mathcal{B}_1 \subset H^{1+\delta}$ *with* $\delta > 0$*, which is invariant under* $S(t)$*, obeys* $\mathcal{B}_1 \subset \mathcal{A}$*. Moreover, there exists* $M_{\mathcal{A}}$ *which depends only on* $\kappa, \|f\|_{L^\infty\cap H^1}$*, and universal constants, such that if* $\theta \in \mathcal{A}$*, we have that*

$$\|\theta\|_{H^{3/2}} \leq M_{\mathcal{A}} \tag{3.55}$$

and

$$\frac{1}{T}\int_t^{t+T} \|S(\tau)\theta\|_{H^2}^2 d\tau \leq M_{\mathcal{A}}^2 \tag{3.56}$$

for any $T > 0$ *and* $t \in \mathbb{R}$*. In particular, for* $\theta \in \mathcal{A}$ *we have* $\|S(t)\theta\|_{H^2} \leq M_{\mathcal{A}}$ *for almost every* t*.*

The proof of Theorem 3.4.1, given at the end of this section, follows closely the steps outlined in [29], and relies on the following main ingredients:

(i) There exists a compact absorbing set $\mathcal{B}$ (which is a ball around the origin in $H^{3/2}$) for the dynamics induced by $S(t)$ on the phase space H^1 (cf. Theorem 3.4.2 below).

(ii) For each $\theta_0 \in H^1$ the solution $S(t)\theta_0\colon [0,\infty) \to H^1$ is a continuous function of t, and for fixed $t > 0$, we have that $S(t)\colon \mathcal{B} \to H^1$ is a Lipschitz continuous

function of θ_0 (cf. Remark 3.4.4 and Proposition 3.4.5 below).

Establishing (i), the existence of a compact absorbing ball, turns out to be the most important step. For this we need to use the global regularity *twice*. From the local existence of solutions we pick up a time when a C^α norm of the solution is finite, with α small. Then we guarantee first that the solution satisfies strong bounds for all time, but the bounds depend of the initial data. However, after long enough time the L^∞ norm of the solution obeys a bound that no longer depends on initial data (its size depends solely on force). At that time, because we have guaranteed that the solution remained smooth enough in the meantime, we apply again the C^α persistence result, but this time the size of θ in L^∞ is given by f, which permits a calculation with an α that depends only on f. After an additional time, we obtain a bound of this C^α norm that depends only on f. At this stage, because the bounds make the situation subcritical, with constants which depend on f only, we bootstrap in regularity and obtain that the size of the $H^{3/2}$ norm is determined by f alone.

Theorem 3.4.2 (**Absorbing ball in $H^{3/2}$**). *Let $\theta_0 \in H^1$ and $f \in L^\infty \cap H^1$. There exists a time $t_{H^{3/2}} = t_{H^{3/2}}(\theta_0, f)$ and an $M_{3/2,f} = M_{3/2,f}(\|f\|_{L^\infty \cap H^1})$ such that for all $t \geq t_{H^{3/2}}$ we have*

$$\|S(t)\theta_0\|_{H^{3/2}} \leq M_{3/2,f}. \tag{3.57}$$

That is,

$$\mathcal{B} = \{\theta \in H^{3/2} \colon \|\theta\|_{H^{3/2}} \leq M_{3/2,f}\} \tag{3.58}$$

is an absorbing set. Moreover, there exists an $M_{2,f} = M_{2,f}(\|f\|_{L^\infty \cap H^1})$, such that

$$\frac{1}{T}\int_t^{t+T} \|S(\tau)\theta_0\|_{H^2}^2 d\tau \leq M_{2,f}^2 \tag{3.59}$$

for any $t \geq t_{H^{3/2}}$ *and any* $T > 0$.

As described in the above outline, in order to prove Theorem 3.4.2, we first need to show that after waiting a long enough time, the solution belongs to a Hölder space, with both the Hölder exponent and the Hölder norm independent of the initial data. We achieve this in the following lemma, by combining the estimates established in Proposition 3.3.1 and Theorem 3.3.3 (respectively Theorem 3.3.4).

Lemma 3.4.3 (**Absorbing ball in** C^α)**.** *Let* $\theta_0 \in H^1$, $f \in L^\infty \cap H^1$, *and define the Hölder exponent*

$$\alpha_* = \alpha_*(\|f\|_{L^\infty}) := \min\left\{\frac{\varepsilon_1 \kappa^2}{\|f\|_{L^\infty}}, \frac{1}{4}\right\} \tag{3.60}$$

where $\varepsilon_1 > 0$ *is a universal constant. There exists a time* $t_{\alpha_*} = t_{\alpha_*}(\theta_0, f)$, *such that*

$$\|S(t)\theta_0\|_{C^{\alpha_*}} \leq M_{\infty,f} := \frac{2\|f\|_{L^\infty}}{\varepsilon_1 \kappa} \tag{3.61}$$

for all $t \geq t_{\alpha_*}$.

Proof of Lemma 3.4.3. By Proposition 3.3.2 there exists $t_0 = t_0(\theta_0, f) > 0$ such that $S(t_0)\theta_0 \in H^2$ and moreover by (3.18) we have

$$\|S(t_0)\theta_0\|_{H^2} \leq C\kappa^{-1} t_0^{-1} \|\theta_0\|_{H^1} + C\kappa^{-1}\|f\|_{H^1}. \tag{3.62}$$

By the Sobolev embedding it follows from (3.62) that

$$\begin{aligned} \|S(t_0)\theta_0\|_{C^{1/4}} &\leq C\|S(t_0)\theta_0\|_{H^2} \\ &\leq C\kappa^{-1} t_0^{-1}\|\theta_0\|_{H^1} + C\kappa^{-1}\|f\|_{H^1} = C(\kappa, \theta_0, f). \end{aligned} \tag{3.63}$$

In particular, $\|S(t_0)\theta_0\|_{L^\infty} \leq C(\kappa, \theta_0, f)$ holds.

Throughout this proof the value of $C(\kappa, \theta_0, f)$ may change from line to line, since we just want to emphasize the dependence of this bound solely on data and force.

We now apply Theorem 3.3.3 with initial data $S(t_0)\theta_0$. Let ε_0 be the constant in Theorem 3.3.3, and define

$$\alpha_1 = \min\left\{\frac{\varepsilon_0\kappa}{M_\infty(S(t_0)\theta_0, f)}, \frac{1}{4}\right\}$$

where

$$M_\infty(S(t_0)\theta_0, f) = \|S(t_0)\theta_0\|_{L^\infty} + \frac{\|f\|_{L^\infty}}{c_0\kappa}$$

is as defined as in (3.16), with corresponding constant c_0. Since $S(t_0)\theta_0 \in C^{1/4} \subseteq C^{\alpha_1}$ and (3.63) holds, it follows from estimate (3.21), that

$$[S(t)\theta_0]_{C^{\alpha_1}} \leq C(\kappa, \theta_0, f)$$

for all $t \geq t_0$. Since in fact we know $S(t_0)\theta_0 \in H^{3/2}$, we bootstrap the above global in time estimate for the C^{α_1} norm, which is a subcritical quantity, to obtain

$$\|S(t)\theta_0\|_{H^{3/2}} \leq C(\kappa, \theta_0, f) \tag{3.64}$$

for all $t \geq t_0$. We refer to the Proof of Theorem 3.4.2 for the main idea in this bootstrap argument.

We now apply Proposition 3.3.1, with initial data $S(t_0)\theta_0 \in L^\infty$ bounded as in (3.63), to conclude that there exists

$$t_1 = t_1(\theta_0, f) = \frac{1}{c_0\kappa}\log\left(1 + \frac{c_0\kappa\|S(t_0)\theta_0\|_{L^\infty}}{\|f\|_{L^\infty}}\right)$$

such that

$$\|S(t)\theta_0\|_{L^\infty} \leq \frac{2\|f\|_{L^\infty}}{c_0\kappa}$$

for all $t \geq t_1 + t_0$.

Now finally define

$$\alpha_* = \min\left\{\frac{\varepsilon_0 c_0 \kappa^2}{2\|f\|_{L^\infty}}, \frac{1}{4}\right\}$$

and apply the argument in Theorem 3.3.3, with initial data taken to be $S(t_0+t_1)\theta_0 \in H^{3/2} \subseteq C^{\alpha_*}$. We conclude from (3.22) and (3.64) that there exists $t_{\alpha_*} = t_{\alpha_*}(\theta_0, f)$ with $t_1 + t_0 \leq t_{\alpha_*} < \infty$, such that

$$\|S(t)\theta_0\|_{C^{\alpha_*}} \leq \frac{4\|f\|_{L^\infty}}{c_0\varepsilon_0\kappa}$$

holds for all $t \geq t_{\alpha_*}$, which concludes the proof the theorem. □

The proof of Theorem 3.4.2 now follows from Lemma 3.4.3 and a bootstrap procedure, which is based on the sub-criticality of the Hölder norm and the nonlinear lower bound on the fractional Laplacian [40].

Proof of Theorem 3.4.2. Let $\alpha_* = \alpha_*(\|f\|_{L^\infty})$, and $t_{\alpha_*} = t_{\alpha_*}(\theta_0, f)$ be as defined in Lemma 3.4.3. Using estimate (3.61), we have that

$$\|S(t)\theta_0\|_{C^{\alpha_*}} \leq M_{\infty,f} \quad \left(:= \frac{2\|f\|_{L^\infty}}{\varepsilon_1\kappa}\right) \tag{3.65}$$

for all $t \geq t_{\alpha_*}$, and moreover, by (3.64) we know that

$$\|S(t)\theta_0\|^2_{H^{3/2}} \leq C(\kappa, \theta_0, f) < \infty \tag{3.66}$$

for all $t \geq t_{\alpha_*}$.

For the rest of the proof, denote

$$\theta_0^* = S(t_{\alpha_*})\theta_0. \tag{3.67}$$

The first step is to obtain a bound on time averages of the $H^{3/2}$ norm of the solution. We apply ∇ to (1.5) and pointwise in x take inner product with $\nabla\theta$, and apply Proposition 3.1.1 to obtain

$$(\partial_t + u\cdot\nabla + \kappa\Lambda)\,|\nabla\theta|^2 + \kappa D[\nabla\theta] = -2\partial_k u_j \partial_j\theta\partial_k\theta + 2\nabla f\cdot\nabla\theta \tag{3.68}$$

where

$$D[\nabla\theta](x) = \frac{1}{2\pi} P.V. \int_{\mathbb{R}^2} |\nabla\theta(x) - \nabla\theta(x+y)|^2 \frac{1}{|y|^3} dy$$

and we use the same notation for the $\mathbb{T}^2$-periodic function $\nabla\theta$, and its extension to all of $\mathbb{R}^2$ by periodicity. The main observation here is that since we already have from (3.65) a bound for $\sup_{t\geq t_{\alpha_*}} \|S(t)\theta_0\|_{C^{\alpha_*}}$, we have an *improved nonlinear lower bound* on $D[\nabla\theta]$. Indeed, from [40, Theorem 2.2] we have

$$D[\nabla\theta](x) \geq \frac{|\nabla\theta(x)|^{\frac{3-\alpha_*}{1-\alpha_*}}}{C[\theta]_{C^{\alpha_*}}^{\frac{1}{1-\alpha_*}}} \tag{3.69}$$

where the constant C depends only on α_*, and is uniformly bounded for $\alpha_* \in (0, 1/2]$. Combining (3.69) with (3.65), we arrive at

$$D[\nabla\theta](x,t) \geq \frac{|\nabla\theta(x,t)|^{\frac{3-\alpha_*}{1-\alpha_*}}}{c_7 M_{\infty,f}^{\frac{1}{1-\alpha_*}}} \tag{3.70}$$

for all $t \geq t_{\alpha_*}$, where c_7 is a universal constant which is independent of α_*, for $\alpha_* \in (0, 1/2]$.

Next, we estimate the nonlinear term on the right side of (3.68). Let χ be a smooth cutoff function, that is 1 on $[0,1]$, non-increasing, vanishes on $[2,\infty)$, and obeys $|\chi'| \leq 2$. For $\rho > 0$ to be determined, using the same argument which led to (3.38), we obtain

$$\begin{aligned}
|\nabla u(x)| &\leq \frac{1}{2\pi}\left| P.V. \int_{\mathbb{R}^2} \frac{y^\perp}{|y|^3}\chi\left(\frac{|y|}{\rho}\right)\nabla\theta(x+y)dy\right| \\
&\quad + \frac{1}{2\pi}\left|\int_{\mathbb{R}^2}\frac{y^\perp}{|y|^3}\left(1-\chi\left(\frac{|y|}{\rho}\right)\right)\nabla\theta(x+y)dy\right| \\
&\leq C\Big(\rho D[\nabla\theta](x)\Big)^{1/2} + C\frac{\|\theta\|_{L^\infty}}{\rho}
\end{aligned}$$

for some universal constant $C > 0$. Therefore, we have

$$\begin{aligned}
2|\nabla u(x)||\nabla\theta(x)|^2 &\leq C\Big(\rho D[\nabla\theta](x)\Big)^{1/2}|\nabla\theta(x)|^2 + C\frac{\|\theta\|_{L^\infty}}{\rho}|\nabla\theta(x)|^2 \\
&\leq \frac{\kappa}{2}D[\nabla\theta](x) + C\left(\frac{\rho}{\kappa}|\nabla\theta(x)|^4 + \frac{\|\theta\|_{L^\infty}}{\rho}|\nabla\theta(x)|^2\right) \\
&\leq \frac{\kappa}{2}D[\nabla\theta](x) + \frac{C}{\kappa^{1/2}}\|\theta\|_{L^\infty}^{1/2}|\nabla\theta(x)|^3 \qquad (3.71)
\end{aligned}$$

by letting $\rho = \kappa^{1/2}\|\theta\|_{L^\infty}^{1/2}|\nabla\theta(x)|^{-1}$.

Since $t \geq t_{\alpha_*}$, we combine (3.65) with (3.68), (3.70), and (3.71), to arrive at

$$\left(\partial_t + u\cdot\nabla + \kappa\Lambda\right)|\nabla\theta|^2 + \frac{\kappa}{4}D[\nabla\theta] + \frac{\kappa|\nabla\theta(x,t)|^{\frac{3-\alpha_*}{1-\alpha_*}}}{4c_7 M_{\infty,f}^{\frac{1}{1-\alpha_*}}} \leq 2|\nabla f||\nabla\theta| + \frac{c_8 M_{\infty,f}^{1/2}|\nabla\theta|^3}{\kappa^{1/2}} \qquad (3.72)$$

for some universal constant c_8. Using the ε-Young inequality, since $(3-\alpha_*)/(1-\alpha_*) > 3$ we furthermore infer from (3.72) that

$$(\partial_t + u\cdot\nabla + \kappa\Lambda)|\nabla\theta|^2 + \frac{\kappa}{4}D[\nabla\theta] + \frac{\kappa|\nabla\theta(x,t)|^{\frac{3-\alpha_*}{1-\alpha_*}}}{8c_7 M_{\infty,f}^{\frac{1}{1-\alpha_*}}}$$
$$\leq 2|\nabla f||\nabla\theta| + \frac{c_8(8c_7)^{\frac{3-3\alpha_*}{2\alpha_*}}}{\kappa^{\frac{9-7\alpha_*}{4\alpha_*}}} M_{\infty,f}^{\frac{9-\alpha_*}{4\alpha_*}}. \tag{3.73}$$

Next we integrate (3.73) over $\mathbb{T}^2$, and use

$$\frac{1}{2}\int_{\mathbb{T}^2} D[\nabla\theta](x)dx = \int_{\mathbb{T}^2} \nabla\theta\cdot\Lambda\nabla\theta dx = \|\theta\|_{H^{3/2}}^2 \geq \|\theta\|_{H^1}^2$$

to obtain

$$\frac{d}{dt}\|\theta\|_{H^1}^2 + \frac{\kappa}{6}\|\theta\|_{H^1}^2 + \frac{\kappa}{6}\|\theta\|_{H^{3/2}}^2 \leq \frac{6}{\kappa}\|f\|_{H^1}^2 + \frac{c_8(8c_7)^{\frac{3-3\alpha_*}{2\alpha_*}}}{\kappa^{\frac{9-7\alpha_*}{4\alpha_*}}} M_{\infty,f}^{\frac{9-\alpha_*}{4\alpha_*}} \tag{3.74}$$

for times $t \geq t_{\alpha_*}$. Using the Grönwall inequality we obtain from (3.74) that

$$\|S(t+t_{\alpha_*})\theta_0\|_{H^1}^2 = \|S(t)\theta_0^*\|_{H^1}^2$$
$$\leq \|\theta_0^*\|_{H^1}^2 e^{-\frac{t\kappa}{6}} + \left(\frac{36}{\kappa^2}\|f\|_{H^1}^2 + \frac{c_8(8c_7)^{\frac{3-3\alpha_*}{2\alpha_*}}}{6\kappa^{\frac{9-3\alpha_*}{4\alpha_*}}} M_{\infty,f}^{\frac{9-\alpha_*}{4\alpha_*}}\right)(1-e^{-\frac{t\kappa}{6}}). \tag{3.75}$$

Recall cf. (3.66) and (3.67) that $\|\theta_0^*\|_{H^1}^2 = \|S(t_{\alpha_*})\theta_0\|_{H^1}^2 \leq C(\kappa,\theta_0,f)$. We conclude from (3.75) that there exists

$$t_{H^1} = t_{H^1}(\theta_0, f) \geq t_{\alpha_*}$$

such that for all $t \geq t_{H^1}$ we have

$$\|S(t)\theta_0\|_{H^1}^2 \leq \frac{72}{\kappa^2}\|f\|_{H^1}^2 + \frac{c_8(8c_7)^{\frac{3-3\alpha_*}{2\alpha_*}}}{3\kappa^{\frac{9-3\alpha_*}{4\alpha_*}}} M_{\infty,f}^{\frac{9-\alpha_*}{4\alpha_*}} =: M_{1,f}^2. \tag{3.76}$$

Note that cf. (3.60) we have $\alpha_* = \alpha_*(\|f\|_{L^\infty})$ and cf. (3.61) we have $M_{\infty,f} = M_{\infty,f}(\|f\|_{L^\infty})$, so that $M_{1,f} = M_{1,f}(\kappa, \|f\|_{L^\infty \cap H^1})$. The dependence on κ and f may be computed explicitly from (3.60)–(3.61) and (3.76).

Inequality (3.76) not only gives an absorbing ball in H^1, but combined with (3.74), integrated between t and $t+1$, it also gives the bound

$$\int_t^{t+1} \|\theta(s)\|^2_{H^{3/2}} ds \leq \frac{6+\kappa}{\kappa} M^2_{1,f} \tag{3.77}$$

for all $t \geq t_{H^1}$.

Estimate (3.77) now directly implies the existence of an absorbing ball for $S(t)$ in $H^{3/2}$. To see this, we take the L^2 inner product of (1.5) with $\Lambda^3\theta$ and write

$$\begin{aligned}
\frac{d}{dt}\|\theta\|^2_{H^{3/2}} + \kappa\|\theta\|^2_{H^2} &\leq \frac{1}{\kappa}\|f\|^2_{H^1} + 2\left|\int_{\mathbb{T}^2} \left(\Lambda^{3/2}(u\cdot\nabla\theta) - u\cdot\nabla\Lambda^{3/2}\right)\Lambda^{3/2}\theta dx\right| \\
&\leq \frac{1}{\kappa}\|f\|^2_{H^1} + C\|\theta\|_{H^{3/2}}\|\Lambda^{3/2}\theta\|_{L^4}\|\Lambda\theta\|_{L^4} \\
&\leq \frac{1}{\kappa}\|f\|^2_{H^1} + \frac{\kappa}{2}\|\theta\|^2_{H^2} + \frac{c_9}{\kappa}\|\theta\|^4_{H^{3/2}}
\end{aligned} \tag{3.78}$$

for some universal constant $c_9 > 0$. In the above estimate we have appealed to the commutator estimate (3.95) of Lemma 3.6.1, and we used the the Sobolev embedding $H^{1/2} \subset L^4$. We obtain from (3.78) that

$$\frac{d}{dt}\|\theta\|^2_{H^{3/2}} + \frac{\kappa}{2}\|\theta\|^2_{H^2} \leq \frac{1}{\kappa}\|f\|^2_{H^1} + \left(\frac{c_9}{\kappa}\|\theta\|^2_{H^{3/2}}\right)\|\theta\|^2_{H^{3/2}} \tag{3.79}$$

for $t \geq 0$.

At this stage we apply the Uniform Grönwall Lemma 3.6.4, with the functions

$$x = \|\theta\|^2_{H^{3/2}}, \quad a(t) = \frac{c_9}{\kappa}\|\theta\|^2_{H^{3/2}}, \quad b = \frac{1}{\kappa}\|f\|^2_{H^1}$$

that by (3.77) obey the bounds

$$\int_t^{t+1} x(s)ds \le \frac{6+\kappa}{\kappa} M_{1,f}^2, \quad \int_t^{t+1} a(s)ds \le \frac{c_9(6+\kappa)}{\kappa^2} M_{1,f}^2, \quad \int_t^{t+1} b(s)ds = \frac{1}{\kappa}\|f\|_{H^1}^2$$

for any $t \ge t_{H^1}$. We conclude from (3.108) that

$$\|S(t)\theta_0\|_{H^{3/2}}^2 \le \left(\frac{6+\kappa}{\kappa} M_{1,f}^2 + \frac{1}{\kappa}\|f\|_{H^1}^2\right) \exp\left(\frac{c_9(6+\kappa)}{\kappa^2} M_{1,f}^2\right) =: M_{3/2,f}^2 \tag{3.80}$$

for any $t \ge t_{H^{3/2}}$, where

$$t_{H^{3/2}} = t_{H^{3/2}}(\theta_0, f) =: t_{H^1} + 1.$$

We also note that since both α^* and $M_{1,f}$ depend only on κ and $\|f\|_{L^\infty \cap H^1}$ (and universal constants), we have $M_{3/2,f} = M_{3/2,f}(\kappa, \|f\|_{L^\infty \cap H^1})$.

Lastly, we notice that $L_t^2 H_x^2$ bounds are also available from the above argument. By combining (3.79) with (3.80) we obtain

$$\frac{1}{T}\int_t^{t+T} \|S(\tau)\theta_0\|_{H^2}^2 d\tau \le \frac{2}{\kappa^2}\|f\|_{H^1}^2 + \frac{2c_9}{\kappa^2} M_{3/2,f}^4 =: M_{2,f}^2 \tag{3.81}$$

for any $t \ge t_{H^{3/2}}$ and $T > 0$. This concludes the proof of the theorem. □

Remark 3.4.4 (Uniform attraction). Theorem 3.4.2 guarantees that for $\theta_0 \in H^1$, there exists a time $t_{H^{3/2}}$ which depends on θ_0 (and f) so that $S(t)\theta_0 \in \mathcal{B}$ for $t \ge t_{H^{3/2}}$. Note however that $t_{H^{3/2}}(\theta_0, f)$ does not depend solely on $\|\theta_0\|_{H^1}$, and it is not a priori locally uniform with respect to initial data. The sole reason for this is that the time of local existence of the solution arising from initial data $\theta_0 \in H^1$ is not guaranteed to depend only on $\|\theta_0\|_{H^1}$. On the other hand, we would like to emphasize that if $\theta_0 \in H^{1+\delta}$ with $\delta > 0$, it can be shown that $t_{H^{3/2}} = t_{H^{3/2}}(\|\theta_0\|_{H^{1+\delta}}, \|f\|_{L^\infty \cap H^1})$ and the time of entering the absorbing ball is a non-decreasing function of its arguments.

In particular, this implies that given a ball $\mathcal{B}_R = \{\theta \in H^{1+\delta} \colon \|\theta\|_{H^{1+\delta}} \leq R\}$*, there exists a time* $t_R = t_R(R, \|f\|_{L^\infty \cap H^1})$ *such that* $S(t)\mathcal{B}_R \subset \mathcal{B}$ *for all* $t \geq t_R$. The reason for this fact is the following. For this smoother initial data $\theta_0 \in H^{1+\delta}$, we find a local time of existence of a unique $L_t^\infty H_x^1 \cap L_t^2 H_x^{3/2}$ solution, that depends only on $\|\theta_0\|_{H^{1+\delta}}$ and norms of f. Then going line-by-line through the proofs of Lemma 3.4.3 and Theorem 3.4.2 above shows that by waiting long enough, depending only on the $H^{1+\delta}$ norm of θ_0 and on norms of f, the C^{α_*} norm and then the $H^{3/2}$ norm of $S(t)\theta_0$ are under control. To avoid redundancy we omit further details.

As stated in the outline below Theorem 3.4.1, besides having a compact absorbing set $\mathcal{B}$ we need continuity properties of $S(t)$ on $\mathcal{B}$. The next lemma in particular proves the Lipschitz continuity of $S(t) \colon \mathcal{B} \to H^1$ for fixed $t > 0$. The proof is given in Section 3.6.3 below.

Proposition 3.4.5 (Continuity). *For fixed* $\theta_0 \in H^1$ *we have that* $S(\cdot)\theta_0 \colon [0, \infty) \to H^1$ *is continuous. Fix a ball* $\mathcal{B}_0 \in H^{3/2}$*. We have that for* $\theta_0, \widetilde{\theta}_0 \in \mathcal{B}_0$ *with* $\|\theta_0 - \widetilde{\theta}_0\|_{H^1} \leq \varepsilon\kappa$*, for some universal* $0 < \varepsilon \ll 1$*, we have*

$$\|S(t)\theta_0 - S(t)\widetilde{\theta}_0\|_{H^1} \leq e(t)\|\theta_0 - \widetilde{\theta}_0\|_{H^1}$$

for some non-decreasing continuous function of time $e(t)$*. In particular, if* $\{t_n\}_{n\geq 1}$ *is a sequence of times that diverge to* ∞ *as* $n \to \infty$*, and* $\{\theta_{0,n}\}_{n\geq 1} \subset \mathcal{B}_0$ *are a sequence of initial data such that*

$$\|S(t_n)\theta_{0,n} - \theta_0\|_{H^1} \to 0$$

as $n \to \infty$ *for some* $\theta_0 \in H^1$*, then for any fixed* $t > 0$ *we have*

$$\|S(t + t_n)\theta_{0,n} - S(t)\theta_0\|_{H^1} = \|S(t)S(t_n)\theta_{0,n} - S(t)\theta_0\|_{H^1} \to 0$$

as $n \to \infty$.

The proof of the existence of the global attractor on the phase space H^1 now follows. We only outline the main steps, as the proof follows the same argument given in [29, pp. 133–136].

Proof of Theorem 3.4.1. By Theorem 3.4.2 we have a compact absorbing set $\mathcal{B} = \{\theta \in H^{3/2} \colon \|\theta\|_{H^{3/2}} \leq M_{3/2,f}\}$, where $M_{3/2,f}$ can be computed in terms of κ, $\|f\|_{L^\infty \cap H^1}$, and universal constants. The idea is that for any bounded $\mathcal{B}_1 \subset H^{1+\delta}$ with $\delta > 0$ by Remark 3.4.4 we have that the omega limit set obeys $\omega(\mathcal{B}_1) \subset \mathcal{B}$ and thus, by Proposition 3.4.5 we have

$$S(t)(\omega(\mathcal{B}_1)) = \omega(\mathcal{B}_1)$$

for all $t \geq 0$, where the omega limit sets $\omega(\cdot)$ are taken in the H^1 topology. The solution map is continuous with respect to initial data in $\mathcal{B}$, and the global attractor is just

$$\mathcal{A} = \omega(\mathcal{B}).$$

This set is clearly compact, non-empty, and invariant in positive time. We also have that $\lim_{t\to\infty} \operatorname{dist}(S(t)\theta_0, \mathcal{A}) = 0$ for any $\theta_0 \in H^1$ since $S(t)\omega(\theta_0) = \omega(\theta_0) \subset \mathcal{B}$ and the definition of $\mathcal{A}$. Note that one may in fact show that

$$\mathcal{A} = \bigcap_{t>0} S(t)\mathcal{B}.$$

One inclusion follows by the definition of an omega limit set, while the other one follows by the invariance of $\omega(B)$ in positive time and Remark 3.4.4. The bounds (3.55)–(3.56) follow by taking $M_{\mathcal{A}} = \max\{M_{3/2,f}, M_{2,f}\}$, where $M_{3/2,f}$ and $M_{2,f}$ are as defined by (3.80) and (3.81) above. □

Remark 3.4.6 (Higher regularity). If $f \in C^\infty(\mathbb{T}^2)$, it can be shown that in fact $\mathcal{A} \subset C^\infty(\mathbb{T}^2)$ with bounds that depend only on κ and f. Similar statements hold in

the real analytic or Sobolev categories.

We conclude this section by noting that if the force f lies in $H^{1+\varepsilon}$ for some $\varepsilon > 0$, one may show the solution operator $S(t)$ is injective on $\mathcal{A}$.

Proposition 3.4.7 (Backwards uniqueness). *Assume that $f \in H^{1+\varepsilon}(\mathbb{T}^2)$ for some $\varepsilon > 0$. Let $\theta_0^{(1)}, \theta_0^{(2)} \in H^{1+\varepsilon}$ be two initial data, and let*

$$\theta^{(i)}(t) = S(t)\theta_0^{(i)} \in C([0,\infty); H^{3/2}) \cap L^2(0,\infty; H^2)$$

be the corresponding solutions of the initial value problem (1.5)–(1.7) for $i \in \{1,2\}$. If there exists $T > 0$ such that $\theta^{(1)}(T) = \theta^{(2)}(T)$, then $\theta_0^{(1)} = \theta_0^{(2)}$ holds.

The proof uses the classical log-convexity method of Agmon and Nirenberg [1], and is given in Section 3.6.3 below.

3.5 Finite dimensionality of the attractor

In this section we establish a bound on the fractal (and a forteriori Hausdorff) dimension of the global attractor $\mathcal{A}$ for $S(t)$ evolving on H^1. The physical meaning of this bound is that the long-time behavior of solutions to the forced critical SQG equations can be fully described by a finite number of independent degrees of freedom.

We recall that the fractal dimension $d_f(Z)$ of a compact set Z is given by

$$d_f(Z) = \limsup_{r \to 0} \frac{\log n_Z(r)}{\log(1/r)}$$

where $n_Z(r)$ is the minimal number of balls in H^1 of radii r needed to cover Z. Note that fractal dimension gives an upper bound (which may be strict) for the Hausdorff dimension of a compact set Z.

The proof closely follows the outline given in [28, 29], where the connection with global Lyapunov exponents and the Kaplan-Yorke formula is established. The main

idea is as follows. Assume $\mathcal{A}$ is covered by a finite number of balls of radius r. Let the flow $S(t)$ transport a ball of radius r centered at θ_0. Then up to an $o(r)$ error the image of the ball is an ellipsoid centered at $S(t)\theta_0$, with semi-axes on the directions given by the eigenvalues of $M(t,\theta_0)$, of lengths given by r multiplied by the eigenvalues of $M(t,\theta_0)$, where $M(t,\theta_0) = (S'(t,\theta_0)^* S'(t,\theta_0))^{1/2}$, and $S'(t,\theta_0)$ is the Fréchet derivative of $S(t)\theta_0$. A control on the volume of this ellipsoid (given in terms of the product of the eigenvalues of $M(t,\theta_0)$) then gives a bound on the number of balls of radius r needed to (re-)cover the ellipsoid. It then turns out that in order to estimate the fractal dimension of $\mathcal{A}$, it is sufficient to find an integer N with the property that n-dimensional volume elements carried by the flow decay exponentially, for all $n \geq N+1$. We now make these ideas more precise.

Definition 3.5.1 (**Continuous differentiability of** $S(t)$**).** *The solution map $S(t)$ is continuously differentiable on $\mathcal{A}$ if for every $\theta_0 \in \mathcal{A}$ there exists a linear operator*

$$S'(t,\theta_0)\colon H^1 \to H^1$$

and a positive function $e(r,t)$, which is continuos with respect to both variables, such that

$$\sup_{\theta_0,\varphi_0\in\mathcal{A},0<\|\theta_0-\varphi_0\|_{H^1}\leq r} \frac{\|S(t)\varphi_0 - S(t)\theta_0 - S'(t,\theta_0)[\varphi_0-\theta_0]\|_{H^1}^2}{\|\varphi_0-\theta_0\|_{H^1}^2} \leq e(r,t) \tag{3.82}$$

with

$$\lim_{r\to 0^+} e(r,t) = 0 \tag{3.83}$$

and moreover

$$\sup_{\theta_0\in\mathcal{A},\|\xi_0\|_{H^1}=1} \|S'(t,\theta_0)[\xi_0]\|_{H^1} < \infty \tag{3.84}$$

for every $t \geq 0$.

The solution map $S(t)$ induced by the critical SQG equation is indeed continuously differentiable on $\mathcal{A}$. For $\theta_0 \in \mathcal{A}$, write $\theta = \theta(t) = S(t)\theta_0$ and for $\xi \in H^1$ let us introduce the elliptic operator

$$A_{\theta_0}(t)[\xi] = A_\theta[\xi] = -\kappa\Lambda\xi - \mathcal{R}^\perp\theta \cdot \nabla\xi - \mathcal{R}^\perp\xi \cdot \nabla\theta. \tag{3.85}$$

We express $S'(t, \theta_0)$ using $A_{\theta_0}(t)$.

Proposition 3.5.2 (**Linearization about a trajectory on the attractor).** *The solution map* $S(t)$ *associated to* (1.5)–(1.6) *is continuously differentiable on* $\mathcal{A}$. *Moreover, the linear operator* $S'(t, \theta_0)$, *when acting on an element* $\xi_0 \in H^1$ *is given by*

$$S'(t, \theta_0)[\xi_0] = \xi(t)$$

where $\xi(t)$ *is the solution of*

$$\partial_t\xi = A_{\theta_0}(t)[\xi] := -\kappa\Lambda\xi - \mathcal{R}^\perp\theta \cdot \nabla\xi - \mathcal{R}^\perp\xi \cdot \nabla\theta, \quad \xi(0) = \xi_0. \tag{3.86}$$

Also, for any $t > 0$ *the operator* $S'(t, \theta_0)$ *is compact.*

It follows from the proof that the function $e(r, t)$ in Definition 3.5.1 may be taken $\approx r^{2-a}\exp(Ct)$ for some $a \in (0, 1)$ and some $C > 0$, which depends only on κ and $\|f\|_{L^\infty \cap H^1}$. The proof of Proposition 3.5.2 is quite technical, and we defer it to Section 3.6.3.

We next show that there is an N such that volume elements which are carried by the flow of $S(t)\theta_0$, with $\theta_0 \in \mathcal{A}$, decay exponentially for dimensions larger than N. Consider $\theta_0 \in \mathcal{A}$, and an initial orthogonal set of infinitesimal displacements $\{\xi_{1,0}, \ldots, \xi_{n,0}\}$ for some $n \geq 1$. The volume of the parallelepiped they span is given

by

$$V_n(0) = \|\xi_{1,0} \wedge \ldots \wedge \xi_{n,0}\|_{H^1}.$$

The reason we have introduced in Proposition 3.5.2 the linearization $S'(t, \theta_0)$ of the flow near $S(t)\theta_0$ is that these displacements ξ_i evolve exactly under this linearization, that is, we define

$$\xi_i(t) = S'(t, \theta_0)[\xi_{i,0}] \quad \text{for all } i \in \{1, \ldots, n\}, \text{ and } t \geq 0,$$

or equivalently the ξ_i obey the equation

$$\partial_t \xi_i = A_{\theta_0}(t)[\xi_i], \quad \xi_i(0) = \xi_{i,0},$$

where $A_{\theta_0}(t)$ is defined in (3.85) above. Then it follows cf. [28, 29] that the volume elements

$$V_n(t) = \|\xi_1(t) \wedge \ldots \wedge \xi_n(t)\|_{H^1}$$

satisfy

$$V_n(t) = V_n(0) \exp\left(\int_0^t \mathrm{Tr}(P_n(s) A_{\theta_0}(s)) ds\right)$$

where the orthogonal projection $P_n(s)$ is onto the linear span of $\{\xi_1(s), \ldots, \xi_n(s)\}$ in the Hilbert space H^1, and $\mathrm{Tr}(P_n(s)A_\theta)$ is defined by

$$\mathrm{Tr}(P_n(s)A_\theta) = \sum_{j=1}^{n} \int_{\mathbb{T}^2} (-\Delta \varphi_j(s)) A_\theta[\varphi_j(s)] dx \tag{3.87}$$

for $n \geq 1$, with $\{\varphi_1(s), \ldots, \varphi_n(s)\}$ an orthonormal set spanning the linear span of

$\{\xi_1(s), \ldots, \xi_n(s)\}$. The value of $\mathrm{Tr}(P_n(s)A_\theta)$ does not depend on the choice of this orthonormalization. Therefore, letting

$$\langle P_n A_{\theta_0} \rangle := \limsup_{T\to\infty} \frac{1}{T} \int_0^T \mathrm{Tr}(P_n(t)A_{\theta_0}(t))dt \tag{3.88}$$

we obtain

$$V_n(t) \leq V_n(0) \exp\left(t \sup_{\theta_0 \in \mathcal{A}} \sup_{P_n(0)} \langle P_n A_{\theta_0} \rangle \right) \tag{3.89}$$

for all $t \geq 0$, where the supremum over $P_n(0)$ is a supremum over all choices of initial n orthogonal set of infinitesimal displacements that we take around θ_0.

Next, we show that n-dimensional volume elements decay exponentially in time (at a rate that is bounded from below), whenever n is sufficiently large, *independently* of the choice of θ_0 in $\mathcal{A}$, and *independently* of initial set of orthogonal displacements $\{\xi_{i,0}\}_{i=1}^n$ which define $P_n(0)$. The key is to to show that the symmetric part of the operator $A_{\theta_0}(t)$ obeys good quadratic form bounds in the H^1 topology.

Proposition 3.5.3 (Contractivity of large dimensional volume elements). *There exists $N = N(\kappa, M_{\mathcal{A}})$ such that for any $\theta_0 \in \mathcal{A}$ and any set of initial orthogonal displacements $\{\xi_{i,0}\}_{i=1}^n$, we have*

$$\langle P_n A_{\theta_0} \rangle < 0 \tag{3.90}$$

whenever $n \geq N$. In particular, $V_n(t)$ decays exponentially in t for any $n \geq N$.

Proof of Proposition 3.5.3. Let $\xi \in H^1$ be arbitrary. By the definition of $A_\theta[\xi]$ in

(3.85), and the fact that $\mathcal{R}^{\perp}\theta$ is divergence-free, we have

$$\begin{aligned}\int_{\mathbb{T}^2}\Lambda^2\xi A_\theta[\xi]dx &= -\kappa\|\xi\|^2_{H^{3/2}} + \left|\int_{\mathbb{T}^2}\partial_k\mathcal{R}^{\perp}\theta\cdot\nabla\xi\partial_k\xi dx\right| + \left|\int_{\mathbb{T}^2}\partial_k(\mathcal{R}^{\perp}\xi\cdot\nabla\theta)\partial_k\xi dx\right| \\ &\leq -\kappa\|\xi\|^2_{H^{3/2}} + C\|\theta\|_{H^{3/2}}\|\xi\|_{H^{3/2}}\|\xi\|_{H^1} + C\|\xi\|_{H^{1/2}}\|\theta\|_{H^2}\|\xi\|_{H^{3/2}}.\end{aligned}$$

Here we have appealed to the Sobolev embedding $H^{1/2}\subset L^4$. Using the Poincaré inequality it follows that

$$\int_{\mathbb{T}^2}\Lambda^2\xi A_\theta[\xi]dx \leq -\frac{\kappa}{2}\|\xi\|^2_{H^{3/2}} + \frac{c_{10}}{\kappa}\|\theta\|^2_{H^2}\|\xi\|^2_{H^1}. \tag{3.91}$$

for some universal constant $c_10 > 0$. Now, for any θ_0 and any $t \geq 0$ the definition (3.87), the inequality (3.91), the normalization of the φ_j's (recall that $\varphi_1(t),\ldots,\varphi_n(t)$ form an orthonormal set spanning the linear span of $\{\xi_1(t),\ldots,\xi_n(t)\}$), and estimate (3.56) yield

$$\begin{aligned}\frac{1}{T}\int_0^T \mathrm{Tr}(P_n(t)A_{\theta_0}(t))dt &= \frac{1}{T}\int_0^T\sum_{j=1}^n\int_{\mathbb{T}^2}(-\Delta\varphi_j(t))A_\theta[\varphi_j(t)]dxdt \\ &\leq -\frac{\kappa}{4}\frac{1}{T}\int_0^T\sum_{j=1}^n\|\varphi_j(t)\|^2_{H^{3/2}}dt + \frac{c_{10}}{\kappa}\frac{1}{T}\int_0^T\|\theta(t)\|^2_{H^2}\sum_{j=1}^n\|\varphi_j(t)\|^2_{H^1}dt \\ &\leq -\frac{\kappa}{4}\frac{1}{T}\int_0^T\mathrm{Tr}(P_n(t)\Lambda) + n\frac{c_{10}}{\kappa}\frac{1}{T}\int_0^T\|\theta(t)\|^2_{H^2}dt \\ &\leq -\frac{\kappa}{c_{11}}n^{3/2} + n\frac{c_{10}}{\kappa}M^2_{\mathcal{A}}\end{aligned} \tag{3.92}$$

where in the last inequality we have used that the eigenvalues $\{\lambda_j\}_{j\geq 1}$ of $\Lambda^{1/2}$ obey

$$\lambda_j \geq \frac{1}{c_{11}}j^{1/2}$$

for a sufficiently large universal constant $c_{11} > 0$ (see e.g. [29]). Choosing

$$N = N(\kappa, M_{\mathcal{A}}) = \left\lceil \left(c_{10} c_{11} \kappa^{-2} M_{\mathcal{A}}^2\right)^2 \right\rceil \tag{3.93}$$

the lemma now follows directly from (3.92) and the definition (3.88). □

The upshot of Proposition 3.5.3 is that N-dimensional volume elements decay exponentially in time. This also implies that the fractal (box-counting) dimension of $\mathcal{A}$ is finite, and is bounded by this N.

Theorem 3.5.4 (Finite dimensionality of the attractor). *Let $N = N(\kappa^{-1} M_{\mathcal{A}})$ be as defined in* (3.93) *above. Then the fractal dimension of $\mathcal{A}$ is finite, and we have* $\dim_f(\mathcal{A}) \leq N$.

Proof of Theorem 3.5.4. We follow precisely the lines of the argument in [29, pp. 115–130, and Chapter 14]. The main ingredients are the continuous differentiability of $S(t)$ on $\mathcal{A}$, the compactness of the linearization, and the exponential decay of large-dimensional volume elements which follows from (3.89) and (3.90). We omit further details and refer to [29]. □

3.6 Technical lemmas

3.6.1 Fractional inequalities

We recall the following fractional product (Kato-Ponce), commutator (Kenig-Ponce-Vega), and Sobolev estimates, cf. [98, 101, 102, 125, 134] and references therein.

Lemma 3.6.1 (Fractional calculus). *Lef $f, g \in C^\infty(\mathbb{T}^2)$, $s > 0$, and $p \in (1, \infty)$. Then we have that*

$$\|\Lambda^s (fg)\|_{L^p} \leq C \|g\|_{L^{p_1}} \|\Lambda^s f\|_{L^{p_2}} + C \|\Lambda^s g\|_{L^{p_3}} \|f\|_{L^{p_4}}, \tag{3.94}$$

where $1/p = 1/p_1 + 1/p_2 = 1/p_3 + 1/p_4$, *and* $p_2, p_3 \in (1, \infty)$, *for a sufficiently large constant* C *that depends only on* s, p, p_i. *Moreover,*

$$\|\Lambda^s(fg) - f\Lambda^s g\|_{L^p} \leq C\|\nabla f\|_{L^{p_1}}\|\Lambda^{s-1}g\|_{L^{p_2}} + C\|\Lambda^s f\|_{L^{p_3}}\|g\|_{L^{p_4}} \tag{3.95}$$

where p_i *are as above. For* $q \in [p, \infty)$ *and* f *of zero mean we also have*

$$\|f\|_{L^q} \leq C\|\Lambda^{\frac{2}{p}-\frac{2}{q}} f\|_{L^p} \tag{3.96}$$

for a sufficiently large constant C *that depends only on* p *and* q.

We conclude this section by giving a sketch of the proof of Proposition 3.1.2. The detailed proof can be found in [33], and we give below only the main ideas.

Proof of Proposition 3.1.2. The case $\alpha = 0$ trivially holds, while in the case $\alpha = 2$ estimate (3.7) follows upon integration by parts. Therefore, henceforth consider $\alpha \in (0, 2)$. For $p = 2$ inequality (3.7) holds due to Parseval's identity, and in the rest of the proof we let $p \geq 4$ be even. For $0 < \alpha < 2$ (all integrals are in the sense of principal value) we have

$$\begin{aligned}
&\int \theta^{p-1}(x)\Lambda^\alpha\theta(x)dx \\
&= \frac{1}{2}\iint \left(\theta^{p-1}(x) - \theta^{p-1}(y)\right)(\theta(x) - \theta(y))\, K_\alpha(x-y)dydx \\
&= \frac{1}{2p}\iint f_p(\theta(x), \theta(y))K_\alpha(x-y)dydx + \frac{1}{p}\iint \left(\theta^{p/2}(x) - \theta^{p/2}(y)\right)^2 K_\alpha(x-y)dydx \\
&=: \frac{1}{2p}\mathcal{T} + \frac{1}{p}\|\Lambda^{\alpha/2}(\theta^{p/2})\|_{L^2}^2
\end{aligned} \tag{3.97}$$

where the double integral is over $\mathbb{T}^{2d}$, and we have defined

$$f_p(a, b) = p(a^{p-1} - b^{p-1})(a - b) - 2(a^{p/2} - b^{p/2})^2.$$

it can be easily seen that $f_p(a,b) \geq 0$ on $\mathbb{R}^2$ when p is even, and so the term $\mathcal{T}$ is positive. The main idea is that exactly $\mathcal{T}$ gives the $\|\theta\|_{L^p}^p$ term in the lower bound (3.7).

We next claim that for $p \geq 4$ even, and $a, b \in \mathbb{R}$ we have

$$f_p(a,b) \geq \frac{p-2}{2}(a-b)^2 a^{p-2}. \tag{3.98}$$

This fact may be checked directly using calculus. Using (3.98) we now prove (3.7). Since K_α is positive, letting $e_1 = (1, 0, \ldots, 0)$, we have

$$\begin{aligned}
\mathcal{T} &\geq \frac{p-2}{2} P.V. \iint (\theta(x) - \theta(y))^2 \theta(x)^{p-2} K_\alpha(x-y) dy dx \\
&\geq \frac{p-2}{2} c_{d,\alpha} \iint (\theta(x) - \theta(y))^2 \theta(x)^{p-2} \frac{1}{|x - y - 2\pi e_1|^{d+\alpha}} dy dx \\
&\geq \frac{(p-2)c_{d,\alpha}}{2(2\pi + |\mathrm{diam}(\mathbb{T}^d)|)^{d+\alpha}} \iint (\theta(x) - \theta(y))^2 \theta(x)^{p-2} dy dx \\
&\geq \frac{(p-2)c_{d,\alpha}}{2(2\pi + |\mathrm{diam}(\mathbb{T}^d)|)^{d+\alpha}} \int_{\mathbb{T}^d} \left(\theta^p(x) |\mathbb{T}^d| - 2\theta^{p-1}(x) \int_{\mathbb{T}^d} \theta(y) dy \right) dx.
\end{aligned} \tag{3.99}$$

At this point we use that θ has zero mean. It then follows from (3.99), that

$$\mathcal{T} \geq \frac{(p-2)c_{d,\alpha}|\mathbb{T}^d|}{2(2\pi + |\mathrm{diam}(\mathbb{T}^d)|)^{d+\alpha}} \|\theta\|_{L^p}^p. \tag{3.100}$$

This proves (3.7) with the constant

$$\frac{(p-2)2^\alpha \Gamma((n+\alpha)/2)|\mathbb{T}^d|}{4p(2\pi + |\mathrm{diam}(\mathbb{T}^d)|)^{d+\alpha}|\Gamma(-\alpha/2)|\pi^{d/2}} \geq \frac{2^\alpha \Gamma((n+\alpha)/2)|\mathbb{T}^d|}{8(2\pi + |\mathrm{diam}(\mathbb{T}^d)|)^{d+\alpha}|\Gamma(-\alpha/2)|\pi^{d/2}}$$

which we call $1/C_{d,\alpha}$ for any $p \geq 4$. When $d = 2$ and $\alpha = 1$ the above constant $C_{d,\alpha}$ may be taken to equal $2^9\pi^2$. □

3.6.2 Interchanging spatial suprema with time derivatives

Lemma 3.6.2 (**Switching** d/dt **and** sup)**.** *Let $\mathcal{K} \subset \mathbb{R}^d$ be compact, and let $T > 0$. Consider a function*

$$f\colon (0,T) \times \mathcal{K} \to [0,\infty)$$

and assume that for every $\lambda \in \mathcal{K}$ the functions

$$f_\lambda(\cdot) = f(\cdot,\lambda)\colon (0,T) \to [0,\infty) \qquad \text{and} \qquad \dot{f}_\lambda(\cdot) = (\partial_t f)(\cdot,\lambda)\colon (0,T) \to \mathbb{R}$$

are continuous. Additionally, assume that the following properties hold:

(i) The families $\{f_\lambda\}_{\lambda\in\mathcal{K}}$ and $\{\dot{f}_\lambda\}_{\lambda\in\mathcal{K}}$ are uniformly equicontinuous with respect to t.

(ii) For every $t \in (0,T)$, the functions $f(t,\cdot)\colon \mathcal{K} \to [0,\infty)$ and $(\partial_t f)(t,\cdot)\colon \mathcal{K} \to \mathbb{R}$ are continuous.

Lastly, define

$$F(t) = \sup_{\lambda\in\mathcal{K}} f_\lambda(t)$$

Then, for almost every $t \in (0,T)$ the function F is differentiable at t, and there exists $\lambda_* = \lambda_*(t) \in \mathcal{K}$ such that simultaneously

$$\dot{F}(t) = \dot{f}_{\lambda_*}(t) \quad \text{and} \quad F(t) = f_{\lambda_*}(t) \tag{3.101}$$

hold.

Proof of Lemma 3.6.2. The proof follows along the lines of [44, Theorem 4.1] and [112, Lemma A.3], but for the sake of completeness we present here the full argument.

The uniform equicontinuity of $\{\dot{f}_\lambda(t)\}_{\lambda\in\mathcal{K}}$ implies that there exists $\delta > 0$ such that

$$\sup_{\lambda\in\mathcal{K}} |\dot{f}_\lambda(t) - \dot{f}_\lambda(s)| \le 1 \quad \text{whenever} \quad |t-s| < \delta.$$

Since $(\partial_t f)(T/2, \lambda)$ is a continuous function of λ, it attains its maximum over the compact $\mathcal{K}$ at some λ_0, and we obtain from the above that

$$\sup_{t\in(0,T)} \sup_{\lambda\in\mathcal{K}} |\dot{f}_\lambda(t)| \le |\dot{f}_{\lambda_0}(T/2)| + \frac{T}{\delta} =: M < \infty$$

Therefore, for $t \in (0,T)$, and $|\Delta t| > 0$ sufficiently small so that $t + \Delta t \in (0,T)$, we have

$$\begin{aligned} |F(t) - F(t+\Delta t)| = \left|\sup_{\lambda\in\mathcal{K}} f_\lambda(t) - \sup_{\lambda\in\mathcal{K}} f_\lambda(t+\Delta t)\right| &\le \sup_{\lambda\in\mathcal{K}} |f_\lambda(t) - f_\lambda(t+\Delta t)| \\ &= |\Delta t| \sup_{\lambda\in\mathcal{K}} |\dot{f}_\lambda(\tau_\lambda)| \\ &\quad \text{(for some } \tau_\lambda \in (t, t+\Delta t)) \\ &\le |\Delta t| M. \end{aligned}$$

Therefore, F is Lipschitz continuous on $(0,T)$, and Rademacher's theorem implies that F is differentiable almost everywhere.

Fix $t \in (0,T)$ such that F is differentiable at t, and let $\Delta t > 0$. In view of the continuity in λ of $f(t+\Delta t, \lambda)$ and the compactness of $\mathcal{K}$, there exists $\lambda(t+\Delta t) \in \mathcal{K}$ such that

$$F(t+\Delta t) = f_{\lambda(t+\Delta t)}(t).$$

Since $\mathcal{K}$ is compact, we can find a sequence $\Delta t_n \to 0^+$ and a point $\lambda_* \in \mathcal{K}$ such that

$$\lambda(t+\Delta t_n) \to \lambda_* \quad \text{as} \quad \Delta t_n \to 0^+. \tag{3.102}$$

We now check that

$$F(t) = f_{\lambda_*}(t),$$

i.e., the second statement in (3.101). We write

$$\begin{aligned} |F(t) - f_{\lambda_*}(t)| \le & \lim_{\Delta t_n \to 0^+} |F(t) - F(t + \Delta t_n)| \\ & + \lim_{\Delta t_n \to 0^+} \left| f_{\lambda(t+\Delta t_n)}(t + \Delta t_n) - f_{\lambda(t+\Delta t_n)}(t) \right| \\ & + \lim_{\Delta t_n \to 0^+} \left| f_{\lambda(t+\Delta t_n)}(t) - f_{\lambda_*}(t) \right|. \end{aligned} \tag{3.103}$$

Note that

$$|F(t) - F(t + \Delta t_n)| \le \sup_{\lambda \in \mathcal{K}} |f_\lambda(t) - f_\lambda(t + \Delta t_n)|$$

and

$$\left| f_{\lambda(t+\Delta t_n)}(t + \Delta t_n) - f_{\lambda(t+\Delta t_n)}(t) \right| \le \sup_{\lambda \in \mathcal{K}} |f_\lambda(t) - f_\lambda(t + \Delta t_n)|.$$

In view of the equicontinuity at t of the family $\{f_\lambda\}_{\lambda \in \mathcal{K}}$, we have

$$\lim_{\Delta t_n \to 0^+} \sup_{\lambda \in \mathcal{K}} |f_\lambda(t) - f_\lambda(t + \Delta t_n)| = 0$$

and thus the first two limits on the right side of (3.103) vanish. The third limit vanishes in view of (3.102) and the assumption of continuity with respect to λ of $f(t, \lambda)$, at any given fixed t. This proves the second part of (3.101).

Let $\varepsilon > 0$ and fix $t \in (0, T)$. In view of the uniform equicontinuity of the family $\{\dot{f}_\lambda\}_{\lambda \in \mathcal{K}}$, we have that there exists $\delta_1 = \delta_1(\varepsilon) > 0$, such that

$$\sup_{\lambda \in \mathcal{K}} |\dot{f}_\lambda(t) - \dot{f}_\lambda(\tau)| < \varepsilon \quad \text{whenever} \quad |t - \tau| < \delta_1. \tag{3.104}$$

Also, in view of the continuity with respect to λ of $(\partial_t f)(t,\lambda)$ at a fixed t, there exists $\delta_2 = \delta_2(\varepsilon, t) > 0$ such that

$$|\dot{f}_\lambda(t) - \dot{f}_{\lambda_*}(t)| < \varepsilon \quad \text{whenever} \quad |\lambda - \lambda_*| < \delta_2. \tag{3.105}$$

Let n be sufficiently large, so that $0 < \Delta t_n < \delta_1$ and $|\lambda(t + \Delta t_n) - \lambda_*| < \delta_2$, which is possible in view of (3.102). Using the fundamental theorem of calculus, (3.104), (3.105), and the fact that $\Delta t_n > 0$, we obtain

$$\begin{aligned}
&\frac{F(t+\Delta t_n) - F(t)}{\Delta t_n} \\
&= \frac{f_{\lambda(t+\Delta t_n)}(t+\Delta t_n) - f_{\lambda(t+\Delta t_n)}(t)}{\Delta t_n} + \frac{f_{\lambda(t+\Delta t_n)}(t) - f_{\lambda_*}(t)}{\Delta t_n} \\
&\le \frac{1}{\Delta t_n}\int_t^{t+\Delta t_n} \dot{f}_{\lambda(t+\Delta t_n)}(s)ds \\
&= \frac{1}{\Delta t_n}\int_t^{t+\Delta t_n} \dot{f}_{\lambda_*}(s)ds + \frac{1}{\Delta t_n}\int_t^{t+\Delta t_n} \left(\dot{f}_{\lambda(t+\Delta t_n)}(s) - \dot{f}_{\lambda_*}(s)\right) ds \\
&\quad + \frac{1}{\Delta t_n}\int_t^{t+\Delta t_n} \left(\dot{f}_{\lambda(t+\Delta t_n)}(s) - \dot{f}_{\lambda(t+\Delta t_n)}(t)\right) + \left(\dot{f}_{\lambda_*}(t) - \dot{f}_{\lambda_*}(s)\right) ds \\
&\quad + \frac{1}{\Delta t_n}\int_t^{t+\Delta t_n} \left(\dot{f}_{\lambda(t+\Delta t_n)}(t) - \dot{f}_{\lambda_*}(t)\right) ds \\
&\le \dot{f}_{\lambda_*}(t) + 4\varepsilon,
\end{aligned}$$

which shows that

$$\dot{F}(t) = \lim_{\Delta t_n \to 0+} \frac{F(t+\Delta t_n) - F(t)}{\Delta t_n} \le \dot{f}_{\lambda_*}(t) \tag{3.106}$$

since ε was arbitrary, and we chose t so that $\dot{F}(t)$ exists. Conversely, for $\Delta t_n > 0$ we have

$$\frac{F(t+\Delta t_n)-F(t)}{\Delta t_n} = \frac{f_{\lambda(t+\Delta t_n)}(t+\Delta t_n)-f_{\lambda_*}(t+\Delta t_n)}{\Delta t_n} + \frac{f_{\lambda_*}(t+\Delta t_n)-f_{\lambda_*}(t)}{\Delta t_n}$$
$$\geq \frac{f_{\lambda_*}(t+\Delta t_n)-f_{\lambda_*}(t)}{\Delta t_n}$$

which shows that

$$\dot{F}(t) = \lim_{\Delta t_n \to 0+} \frac{F(t+\Delta t_n)-F(t)}{\Delta t_n} \geq \dot{f}_{\lambda_*}(t) \tag{3.107}$$

Estimates (3.106) and (3.107) prove the first part of (3.101), and hence of the lemma.

□

Corollary 3.6.3. *Assume that $\theta \in C^\beta((0,T);C^{1,\beta}(\mathbb{T}^2))$ is a classical solution of (1.5)–(1.6) on $(0,T)$, for some $\beta \in (0,1)$, with force $f \in C^\beta$. For $0<\alpha<\beta/2$ we define*

$$v(t,x;h) = \frac{|\delta_h\theta(x,t)|}{|h|^\alpha} = \frac{|\theta(x+h,t)-\theta(x,t)|}{|h|^\alpha} \colon \mathbb{T}^2 \times (0,T) \times \mathbb{T}^2 \to \mathbb{R}$$

with the convention that $v(t,x;0)=0$. Then for almost every $t \in (0,T)$, there exists a pair

$$(\bar{x},\bar{h}) = (\bar{x}(t),\bar{h}(t)) \in \mathbb{T}^2 \times \mathbb{T}^2$$

such that

$$v(t,\bar{x};\bar{h})^2 = \sup_{(x,h)\in\mathbb{T}^2\times\mathbb{T}^2} v(t,\cdot;\cdot)^2$$

and moreover

$$\frac{d}{dt}\left(\sup_{(x,h)\in\mathbb{T}^2\times\mathbb{T}^2} v(t,\cdot;\cdot)^2\right) = (\partial_t v^2)(t,\bar{x};\bar{h})$$

holds.

Proof of Corollary 3.6.3. The proof follows by applying Lemma 3.6.2 to the function

$$f(t,\lambda) = v(t,x;h)^2$$

with $\lambda = (x;h) \in \mathbb{T}^2 \times \mathbb{T}^2 = \mathcal{K}$, which is clearly compact. It is clear that f is a non-negative function.

By assumption, for fixed $\lambda = (x;h)$, the function

$$f_\lambda(t) = f(\lambda,t) = v(t,x;h)^2 = \frac{(\theta(x+h,t)-\theta(x,t))^2}{|h|^{2\alpha}}$$

is continuous with respect to t. In fact, since $2\alpha < \beta < 1$, for $t,s \in [0,T]$ and $\lambda = (x;h)$ we have that

$$\begin{aligned}
|f_\lambda(t) - f_\lambda(s)| &= \frac{1}{|h|^{2\alpha}} \left|\delta_h\theta(x,t) + \delta_h\theta(x,s)\right| \left|\delta_h\theta(x,t) - \delta_h\theta(x,s)\right| \\
&\leq \left([\theta(t)]_{C^{2\alpha}} + [\theta(s)]_{C^{2\alpha}}\right) \left(|\theta(x+h,t) - \theta(x+h,s)| + |\theta(x,t) - \theta(x,s)|\right) \\
&\leq 4\|\theta\|_{L^\infty(0,T;C^\beta)} |t-s|^\beta \|\theta\|_{C^\beta(0,T;L^\infty)} \\
&\leq 4|t-s|^\beta \|\theta\|^2_{C^\beta(0,T;C^\beta)}
\end{aligned}$$

which shows that the family $\{f_\lambda\}_{\lambda\in\mathcal{K}}$ is uniformly equicontinuous on $(0,T)$.

Using the equation (1.5) obeyed by θ, we moreover have that

$$\begin{aligned}
\dot{f}_\lambda(t) = (\partial_t f)(t,\lambda) &= \frac{\partial}{\partial t} v(t,x;h)^2 \\
&= \frac{2(\theta(x+h,t)-\theta(x,t))}{|h|^{2\alpha}} (\partial_t\theta(x+h,t) - \partial_t\theta(x,t)) \\
&= \frac{2\delta_h\theta(x,t)}{|h|^{2\alpha}} \left(\delta_h f(x) - \kappa\delta_h\Lambda\theta(x,t) - u(x,t)\cdot\delta_h\nabla\theta(x,t) - \delta_h u(x,t)\cdot\theta(x+h,t)\right)
\end{aligned}$$

is also continuous with respect to t, since by assumption $\nabla\theta, \Lambda\theta \in C^\beta(0,T;C^\beta)$, and

$u \in C^\beta(0,T;C^{1,\beta})$. To verify the equicontinuity of the family $\{\dot{f}_\lambda\}_{\lambda\in\mathcal{K}}$, we note that for $t,s\in[0,T]$ and $(x;h)\in\mathbb{T}^2\times\mathbb{T}^2$ it holds that

$$\begin{aligned}
|\dot{f}_\lambda(t)-\dot{f}_\lambda(s)| &\le C|t-s|^\beta\|\theta\|_{C^\beta(0,T;L^\infty)}\left(\|f\|_{C^{2\alpha}}+\kappa\|\theta\|_{L^\infty(0,T;C^{1,2\alpha})}\right)\\
&\quad + C|t-s|^\beta\|\theta\|_{C^\beta(0,T;L^\infty)}\left(\|\theta\|_{L^\infty(0,T;C^{2\alpha})}\|\theta\|_{L^\infty(0,T;C^{1,2\alpha})}\right)\\
&\quad + C|t-s|^\beta\|\theta\|_{L^\infty(0,T;C^{2\alpha})}\left(\kappa\|\theta\|_{C^\beta(0,T;C^{1,\beta})}+\|\theta\|^2_{C^\beta(0,T;C^{1,\beta})}\right)\\
&\le C|t-s|^\beta\left(\|f\|_{C^\beta}+\kappa\|\theta\|_{C^\beta(0,T;C^{1,\beta})}+\|\theta\|^2_{C^\beta(0,T;C^{1,\beta})}\right)
\end{aligned}$$

which by assumption is a finite number times $|t-s|^\beta$.

It is left to check that for fixed $t\in(0,T)$, the quantities $f(t,\lambda)$ and $\partial_t f(t,\lambda)$ vary continuously with respect to $\lambda=(x;h)$. This can be verified similarly to the equicontinuity of f_λ and $\dot{f}_\lambda$. Note that there is no problem at $h=0$ since by assumption $\beta>2\alpha$. We omit further details. □

3.6.3 Technical details about the existence and size of the attractor

In this section, we present a number of technical lemmas which are needed in order to establish the existence of the global attractor for the solution map $S(t)\colon H^1\to H^1$ associated to the critical SQG equation, and to give an estimate on its fractal dimension.

The following variant of the classical Grönwall lemma is used in the proof of Theorem 3.4.2, in order to bootstrap information about the time average of the $H^{3/2}$ norm, to information about the pointwise in time behavior of the $H^{3/2}$ norm. The lemma is due to Foias and Prodi [81]. See also [29, 122, 135].

Lemma 3.6.4 (Uniform Grönwall Lemma). *Assume $x,a,b\colon[0,\infty)\to[0,\infty)$ are*

functions such that

$$\frac{dx}{dt} \leq ax + b$$

and in addition assume that there exists $r > 0, t_0 > 0$ *such that*

$$\int_t^{t+r} x(s)ds \leq X, \quad \int_t^{t+r} a(s)ds \leq A, \quad \int_t^{t+r} b(s)ds \leq B$$

for all $t \geq t_0$. *Then we have*

$$x(t) \leq (Xr^{-1} + B)e^A \tag{3.108}$$

for all $t \geq t_0 + r$.

Next, we give the proof of the backwards uniqueness property for $S(t)$. The proof uses the classical log-convexity method of Agmon and Nirenberg [1], see also [111, 122, 135].

Proof of Proposition 3.4.7. First, if $\theta \in L^\infty(0,T;H^{3/2}(\mathbb{T}^2)) \cap L^2(0,T;H^2(\mathbb{T}^2))$ is a solution of

$$\partial_t\theta + \mathcal{R}^\perp\theta \cdot \nabla\theta + \kappa\Lambda\theta = f \tag{3.109}$$

with initial data

$$\theta_0 \in H^{1+\varepsilon}(\mathbb{T}^2)$$

and force

$$f \in H^{1+\varepsilon}(\mathbb{T}^2)$$

for some $\varepsilon \in (0,1)$, then in fact

$$\theta \in L^\infty(0,T;H^{3/2+\varepsilon}(\mathbb{T}^2)) \cap L^2(0,T;H^{2+\varepsilon}(\mathbb{T}^2)) \tag{3.110}$$

holds.

To see this, let $s = 3/2 + \varepsilon$. Apply Λ^s to (3.109) and take an L^2 inner product with $\Lambda^s\theta$ to obtain

$$\begin{aligned}
\frac{d}{dt}\|\Lambda^s\theta\|_{L^2}^2 + \kappa\|\Lambda^{s+1/2}\theta\|_{L^2}^2 &\leq \frac{C}{\kappa}\|\Lambda^{s-1/2}f\|_{L^2}^2 + C\|\Lambda^s\theta\|_{L^4}\|[\Lambda^s, \mathcal{R}^\perp\theta\cdot\nabla]\theta\|_{L^{4/3}} \\
&\leq \frac{C}{\kappa}\|\Lambda^{s-1/2}f\|_{L^2}^2 + C|\Lambda^{s+1/2}\theta\|_{L^2}\|\Lambda^s\theta\|_{L^2}\|\nabla\theta\|_{L^4} \\
&\leq \frac{C}{\kappa}\|\Lambda^{s-1/2}f\|_{L^2}^2 + \frac{\kappa}{2}\|\Lambda^{s+1/2}\theta\|_{L^2}^2 + \frac{C}{\kappa}\|\Lambda^s\theta\|_{L^2}^2\|\theta\|_{H^{3/2}}^2.
\end{aligned}$$

The proof of (3.110) is completed using Grönwall's inequality, since $s - 1/2 = 1 + \varepsilon$.

In view of (3.110), it is sufficient to show that if $\theta^{(1)}, \theta^{(2)} \in C(0,T;H^{3/2+\varepsilon}(\mathbb{T}^2)) \cap L^2(0,T;H^{2+\varepsilon}(\mathbb{T}^2))$ are two solutions of (3.109) such that $\theta^{(1)}(T) = \theta^{(2)}(T)$, then we must have that $\theta_0^{(1)} = \theta_0^{(2)}$.

We proceed by contradiction. Without loss of generality we assume that $\theta^{(1)} - \theta^{(2)} \neq 0$ on $[0,T)$, and that the solutions agree at time T. Let $\theta = \theta^{(1)} - \theta^{(2)}$ and $2\bar{\theta} = \theta^{(1)} + \theta^{(2)}$. Similar notation for the velocities. The equation for the difference is

$$\partial_t\theta + \kappa\Lambda\theta + \bar{u}\cdot\nabla\theta + u\cdot\nabla\bar{\theta} = 0,$$

where we know that $\bar{\theta} \in L^\infty(0,T;H^{3/2+\varepsilon}) \cap L^2(0,T;H^{2+\varepsilon})$ for some $\varepsilon > 0$.

We introduce the Hamiltonian

$$H(t) = \|\Lambda^{-1/2}\theta(t)\|_{L^2}^2,$$

the energy

$$E(t) = \|\theta(t)\|_{L^2}^2,$$

the energy dissipation

$$D(t) = \|\Lambda^{1/2}\theta(t)\|_{L^2}^2,$$

the maximum of the Hamiltonian

$$M = 3 \max_{t\in[0,T]} H(t),$$

the Dirichlet quotient

$$Q(t) = \frac{E(t)}{H(t)},$$

and

$$W(t) = \log \frac{M}{H(t)}.$$

By assumption we have that $0 < M < \infty$, $W(t) > 0$ on $[0, T]$, and most importantly

$$\lim_{t\to T^-} W(t) = +\infty$$

which is what we will show is not possible.

As a preliminary, we compute

$$\frac{1}{2}\frac{d}{dt}H = -\kappa\|\theta\|_{L^2}^2 - \int u\cdot\nabla\bar{\theta}\Lambda^{-1}\theta - \int \bar{u}\cdot\nabla\theta\Lambda^{-1}\theta =: -\kappa E - R \tag{3.111}$$

Here we used the two cancellation properties

$$\int u\cdot\nabla\bar{\theta}\Lambda^{-1}\theta = -\int u\cdot\nabla\Lambda^{-1}\theta\ \bar{\theta} = -\int \nabla^{\perp}\Lambda^{-1}\theta\cdot\nabla\Lambda^{-1}\theta\ \bar{\theta} = 0.$$

Also, we shall need

$$\frac{1}{2}\frac{d}{dt}E = -\kappa\|\Lambda^{1/2}\theta\|_{L^2}^2 - \int u\cdot\nabla\bar{\theta}\theta =: -\kappa D - S \tag{3.112}$$

where we used the cancellation property

$$\int \bar{u}\cdot\nabla\theta\ \theta = 0.$$

We now compute the time evolution of W. Using (3.111) we obtain

$$\frac{1}{2}\frac{d}{dt}W = \frac{-\frac{1}{2}\frac{d}{dt}H}{H} = \frac{\kappa\|\theta\|_{L^2}^2 + R}{H}.$$

From the estimate

$$|R| = \left|\int \bar{u}\cdot\nabla\theta\Lambda^{-1}\theta\right| = \left|\int \bar{u}\cdot\nabla\Lambda^{-1}\theta\ \theta\right| \le \|\bar{u}\|_{L^\infty}\|\theta\|_{L^2}^2 = \|\bar{u}\|_{L^\infty}E$$

we obtain

$$\frac{1}{2}\frac{d}{dt}W \le \left(\kappa + \|\mathcal{R}^\perp\bar{\theta}\|_{L^\infty}\right)Q \le \left(\kappa + C\|\bar{\theta}\|_{H^{1+\varepsilon}}\right)Q. \tag{3.113}$$

Next we estimate the evolution of Q. This is the main computation. From (3.111) and (3.112) we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}Q &= \frac{\frac{1}{2}\frac{d}{dt}E}{H} - Q\frac{\frac{1}{2}\frac{d}{dt}H}{H}\\ &= \frac{-\kappa D - S}{H} - Q\frac{(-\kappa E - R)}{H} = -\kappa\frac{D - QE}{H} + \frac{QR - S}{H}.\end{aligned} \tag{3.114}$$

The first observation is that

$$\Omega := D - QE = \|\Lambda^{1/2}\theta\|_{L^2}^2 - \frac{\|\theta\|_{L^2}^4}{\|\Lambda^{-1/2}\theta\|_{L^2}^2} = \|\Lambda^{1/2}\theta - Q\Lambda^{-1/2}\theta\|_{L^2}^2 \ge 0.$$

The second observation is that by appealing to the previous cancellation property we have

$$\begin{aligned} S = \int u \cdot \nabla \bar{\theta} \theta &= \int \Lambda^{-1/2}(u \cdot \nabla \bar{\theta}) \Lambda^{1/2} \theta \\ &= \int \Lambda^{-1/2}(u \cdot \nabla \bar{\theta}) \left(\Lambda^{1/2} \theta - Q \Lambda^{-1/2} \theta \right) + Q \int u \cdot \nabla \bar{\theta} \Lambda^{-1} \theta \\ &= \int \Lambda^{-1/2}(u \cdot \nabla \bar{\theta}) \left(\Lambda^{1/2} \theta - Q \Lambda^{-1/2} \theta \right) \end{aligned}$$

and therefore, using Cauchy-Schwartz and the fractional Sobolev embedding, it follows that

$$\begin{aligned} |S| &\leq \Omega^{1/2} \| \Lambda^{-1/2}(u \cdot \nabla \bar{\theta}) \|_{L^2} \\ &\leq \frac{\kappa}{2} \Omega + \frac{C}{\kappa} \| u \cdot \nabla \bar{\theta} \|_{L^{4/3}}^2 \leq \frac{\kappa}{2} \Omega + \frac{C}{\kappa} E \| \bar{\theta} \|_{H^{3/2}}^2. \end{aligned} \tag{3.115}$$

Next, we write R as a commutator

$$R = \int \bar{u} \cdot \nabla \theta \Lambda^{-1} \theta = \int \left(\Lambda^{-1/2}(\bar{u} \cdot \nabla \theta) - \bar{u} \cdot \nabla \Lambda^{-1/2} \theta \right) \Lambda^{-1/2} \theta.$$

The claim is that

$$|R| \leq C \| \bar{\theta} \|_{H^{2+\varepsilon}} H \tag{3.116}$$

for any $\varepsilon > 0$, and some $C > 0$. To prove (3.116) we write R using Fourier series as

$$R = C \sum_{j+k+l=0, j,k,l \in \mathbb{Z}_*^2} \frac{|l| - |k|}{|l|^{1/2} |k|^{1/2} (|l|^{1/2} + |k|^{1/2})} \widehat{\bar{\theta}}_j \frac{j^\perp \cdot k}{|j|} \widehat{\theta}_k |l|^{-1/2} \widehat{\theta}_l.$$

Taking absolute values in the above we obtain

$$\begin{aligned}
|R| &\leq C \sum_{j+k+l=0, j,k,l\in\mathbb{Z}^2_*} \frac{|j||k|^{1/2}}{|l|(|l|^{1/2}+|k|^{1/2})} |\widehat{\bar{\theta}}_j||\widehat{\theta}_k||\widehat{\theta}_l| \\
&\leq C \sum_{j+k+l=0, j,k,l\in\mathbb{Z}^2_*} \left(1+\frac{|j|^{1/2}}{|l|^{1/2}}\right) \left(|j||\widehat{\bar{\theta}}_j|\right) \left(|k|^{-1/2}|\widehat{\theta}_k|\right) \left(|l|^{-1/2}|\widehat{\theta}_l|\right) \\
&\leq C \| |j||\widehat{\bar{\theta}}_j| \|_{\ell^1(\mathbb{Z}^2)} \| |k|^{-1/2}|\widehat{\theta}_k| \|_{\ell^2(\mathbb{Z}^2_*)} \| |l|^{-1/2}|\widehat{\theta}_l| \|_{\ell^2(\mathbb{Z}^2_*)} \\
&\quad + C \| |j|^{3/2}|\widehat{\bar{\theta}}_j| \|_{\ell^{4/(3+3\delta)}(\mathbb{Z}^2)} \| |k|^{-1/2}|\widehat{\theta}_k| \|_{\ell^2(\mathbb{Z}^2_*)} \| |l|^{-1}|\widehat{\theta}_l| \|_{\ell^{4/(3-3\delta)}(\mathbb{Z}^2_*)} \\
&\leq C \|\bar{\theta}\|_{H^{2+\varepsilon}} \|\Lambda^{-1/2}\theta\|^2_{L^2} \qquad (3.117)
\end{aligned}$$

as long as we choose $\delta \in (0, 2\varepsilon/3)$. This proves inequality (3.116).

Inserting (3.115) and (3.116) into (3.114) yields

$$\frac{d}{dt}Q + \kappa\frac{\Omega}{H} \leq \left(C\|\bar{\theta}\|_{H^{2+\varepsilon}} + \frac{C}{\kappa}\|\bar{\theta}\|^2_{H^{3/2}}\right) Q$$

and by the Grönwall inequality

$$Q(t) \leq Q(0) \exp\left(CT^{1/2}\|\bar{\theta}\|_{L^2(0,T;H^{2+\varepsilon})} + \frac{C}{\kappa}\|\bar{\theta}\|^2_{L^2(0,T;H^{3/2})}\right) \qquad (3.118)$$

for any $t \in [0, T)$.

Estimate (3.118) shows that $Q \in L^\infty(0, T)$, which combined with (3.113) implies that $W \in L^\infty(0, T)$. This contradicts the fact that $H(T) = 0 \Leftrightarrow W(T) = +\infty$, thereby concluding the proof. □

We now give the proof of the continuity property of the solution map with respect to time and with respect to perturbations in the initial data, in the H^1 topology.

Proof of Proposition 3.4.5. The continuity in time for fixed initial data was already given by Proposition 3.3.2, so it remains to check continuity with respect to the initial data which originates from $\mathcal{B}_0$.

Due to Theorem 3.4.2, we know there exists an absorbing ball $\mathcal{B} \subset H^{3/2}$ for the dynamics induced by $S(t)$ on H^1. In particular, by Remark 3.4.4 there exist a time $t_{H^{3/2}}(\mathcal{B}_0)$ such that $S(t)\mathcal{B}_0 \subset \mathcal{B}$ for all $t \geq t_{H^{3/2}}(\mathcal{B}_0)$. Since the sequence t_n in the statement of the proposition diverges as $n \to \infty$, we may assume without loss of generality that $\widetilde{\theta}_{0,n} = S(t_n)\theta_{0,n} \in \mathcal{B}$ for all $n \geq 1$, and even that $S(t)\widetilde{\theta}_{0,n} \in \mathcal{B}$ for all $t \geq 0, n \geq 1$.

Let $\theta_0 \in H^1$ be arbitrary. Fix some $\widetilde{\theta}_0 \in \mathcal{B}$ such that $\widetilde{\theta}(t) = S(t)\widetilde{\theta}_0 \in \mathcal{B}$ for all $t \geq 0$, and such that

$$\|\theta_0 - \widetilde{\theta}_0\|_{H^1} \leq \varepsilon\kappa$$

for $\varepsilon > 0$ to be determined later. Denote $\bar{\theta}(t) = \widetilde{\theta}(t) - \theta(t) = S(t)\widetilde{\theta}_0 - S(t)\theta_0$, for all $t \geq 0$, and let $\bar{u}$ be the corresponding velocity difference. The equation obeyed by $\bar{\theta}$ is

$$\partial_t\bar{\theta} + \kappa\Lambda\bar{\theta} - \bar{u}\cdot\nabla\bar{\theta} + \widetilde{u}\cdot\nabla\bar{\theta} + \bar{u}\cdot\nabla\widetilde{\theta} = 0.$$

Multiplying the above by $-\Delta\theta$, integrating over $\mathbb{T}^2$, and integrating by parts, yields

$$\begin{aligned}\frac{1}{2}\|\bar{\theta}\|_{H^1}^2 + \kappa\|\bar{\theta}\|_{H^{3/2}}^2 &\leq \left|\int(\nabla\bar{u}\cdot\nabla\bar{\theta})\cdot\nabla\bar{\theta} + \int(\nabla\widetilde{u}\cdot\nabla\bar{\theta})\cdot\nabla\bar{\theta} + \int\Lambda^{1/2}(\bar{u}\cdot\nabla\widetilde{\theta})\Lambda^{3/2}\bar{\theta}\right| \\ &\leq C(\|\nabla\bar{u}\|_{L^4} + \|\nabla\widetilde{u}\|_{L^4})\|\bar{\theta}\|_{H^1}\|\nabla\bar{\theta}\|_{L^4} + C\|\Lambda^{1/2}(\bar{u}\cdot\nabla\widetilde{\theta})\|_{L^2}\|\bar{\theta}\|_{H^{3/2}} \\ &\leq C(\|\widetilde{\theta}\|_{H^{3/2}} + \|\bar{\theta}\|_{H^{3/2}})\|\bar{\theta}\|_{H^1}\|\bar{\theta}\|_{H^{3/2}} + C\|\bar{u}\|_{L^\infty}\|\widetilde{\theta}\|_{H^{3/2}}\|\bar{\theta}\|_{H^{3/2}} \quad (3.119)\end{aligned}$$

Here we have also appealed to the fractional Sobolev embedding $H^{1/2} \subset L^4$, and the product estimate from Lemma 3.6.1. We may appeal to the Brezis-Gallouët inequality

$$\|\bar{u}\|_{L^\infty} = \|\mathcal{R}^\perp\bar{\theta}\|_{L^\infty} \leq C\|\bar{\theta}\|_{H^1}\left(1 + \log\frac{\|\bar{\theta}\|_{H^{3/2}}^2}{\|\bar{\theta}\|_{H^1}^2}\right)^{1/2} \quad (3.120)$$

which combined with the inequality

$$a\mu\left(1+\log\frac{\mu^2}{b^2}\right)^{1/2}\leq\varepsilon\mu^2+\frac{a^2}{\varepsilon}\log\frac{2a}{\varepsilon b} \tag{3.121}$$

which holds for any $a,\varepsilon>0$ and $\mu\geq b$ (see [78, 110]), and the estimate (3.119), yields

$$\frac{d}{dt}\|\bar{\theta}\|_{H^1}^2+\frac{\kappa}{2}\|\bar{\theta}\|_{H^{3/2}}^2\leq C_0\|\bar{\theta}\|_{H^1}\|\bar{\theta}\|_{H^{3/2}}^2+\frac{C_0}{\kappa}\|\bar{\theta}\|_{H^1}^2\|\widetilde{\theta}\|_{H^{3/2}}^2\left(1+\log\frac{C_0\|\widetilde{\theta}\|_{H^{3/2}}}{\kappa}\right) \tag{3.122}$$

for some universal constant $C_0>0$. Note that the initial data $\bar{\theta}_0$ obeys

$$\|\bar{\theta}_0\|_{H^1}\leq\varepsilon\kappa$$

where ε is chosen so that $4\varepsilon C_0\leq 1$. Due to continuity in time, we therefore conclude that there exists $T>0$ such that on $[0,T]$ we have $\|\bar{\theta}(t)\|_{H^1}\leq 2\varepsilon\kappa$, and on this time interval from (3.122) we conclude that

$$\frac{d}{dt}\|\bar{\theta}\|_{H^1}^2\leq\|\bar{\theta}\|_{H^1}^2\frac{C_0M^2}{\kappa}\left(1+\log\frac{C_0M}{\kappa}\right). \tag{3.123}$$

Here we used the assumption that $S(t)\widetilde{\theta}_0\in\mathcal{B}$ for all $t\geq 0$ and denoted by M the radius of the ball $\mathcal{B}$. Thus, a posteriori we conclude that we could have chosen

$$T=\frac{\kappa\log 2}{C_0M^2\left(1+\log\frac{C_0M}{\kappa}\right)}=T(\kappa,\mathcal{B})$$

so that

$$\|\bar{\theta}(t)\|_{H^1}\leq 2\varepsilon\kappa$$

for $t\in[0,T]$. It is important that this T is independent of ε.

The proof of the proposition may now be concluded. The above estimates shows that as $\|\widetilde{\theta}_{0,n} - \theta_0\|_{H^1} \to 0$, we have

$$\|S(t)\widetilde{\theta}_{0,n} - S(t)\theta_0\|_{H^1}^2 \to 0 \quad \text{as} \quad n \to \infty$$

for all $t \in [0, T(\kappa, \mathcal{B})]$. We then re-iterate this argument for $t \in [iT(\kappa, \mathcal{B}), (i+1)T(\kappa, \mathcal{B})]$ for all $i \geq 1$, which proves the Proposition. □

We conclude the section by giving the proof of continuous differentiability of the solution map around trajectories on the global attractor.

Proof of Proposition 3.5.2. For $\theta_0, \varphi_0 \in \mathcal{A}$, denote $\xi_0 = \varphi_0 - \theta_0$ and define $\xi(t) = S'(t, \theta_0)[\xi_0]$ via (3.86). We let

$$\eta(t) = \varphi(t) - \theta(t) - \xi(t) = S(t)\varphi_0 - S(t)\theta_0 - S'(t, \theta_0)[\xi_0]$$

and observe that η obeys the equation

$$\partial_t \eta + \kappa\Lambda\eta + \mathcal{R}^\perp\eta \cdot \nabla\theta + \mathcal{R}^\perp\theta \cdot \nabla\eta = -\mathcal{R}^\perp w \cdot \nabla w, \qquad \eta(0) = 0. \tag{3.124}$$

where

$$w(t) = \varphi(t) - \theta(t) = S(t)\varphi_0 - S(t)\theta_0 = S(t)\xi_0. \tag{3.125}$$

In order to estimate $\|\eta(t)\|_{H^1}$, take an L^2 inner product of (3.124) with $-\Delta\eta$ and use Lemma 3.6.1 to obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\eta\|_{H^1}^2+\kappa\|\eta\|_{H^{3/2}}^2\\
&\quad=\int_{\mathbb{T}^2}\mathcal{R}^\perp\eta\cdot\nabla\theta\Delta\eta dx-\int_{\mathbb{T}^2}\partial_k\mathcal{R}^\perp\theta\cdot\nabla\eta\partial_k\eta dx+\int_{\mathbb{T}^2}\mathcal{R}^\perp w\cdot\nabla w\Delta\eta dx\\
&\quad\le\|\Lambda^{3/2}\eta\|_{L^2}\|\Lambda^{1/2}(\mathcal{R}^\perp\eta\cdot\nabla\theta)\|_{L^2}+\|\nabla\mathcal{R}^\perp\theta\|_{L^4}\|\nabla\eta\|_{L^4}\|\nabla\eta\|_{L^2}\\
&\qquad+\|\Lambda^{3/2}\eta\|_{L^2}\|\Lambda^{1/2}(\mathcal{R}^\perp w\cdot\nabla w)\|_{L^2}\\
&\quad\le C\|\eta\|_{H^{3/2}}\left(\|\eta\|_{H^1}\|\theta\|_{H^{3/2}}+\|\eta\|_{H^{1/2}}\|\theta\|_{H^2}\right)\\
&\qquad+C\|\eta\|_{H^{3/2}}\left(\|w\|_{H^1}\|w\|_{H^{3/2}}+\|\mathcal{R}^\perp w\|_{L^\infty}\|w\|_{H^{3/2}}\right)\\
&\quad\le\frac{\kappa}{2}\|\eta\|_{H^{3/2}}^2+\frac{C}{\kappa}\|\eta\|_{H^1}^2\|\theta\|_{H^2}^2+\frac{C}{\kappa}\|w\|_{H^1}^2\|w\|_{H^{3/2}}^2\left(1+\log\frac{\|w\|_{H^{3/2}}^2}{\|w\|_{H^1}^2}\right)
\end{aligned}
\tag{3.126}
$$

for some universal constant $C>0$. In the last inequality we have also appealed to (3.120).

Next we estimate w, as defined in (3.125). It obeys the equation

$$
\partial_t w+\kappa\Lambda w+\mathcal{R}^\perp\varphi\cdot\nabla w+\mathcal{R}^\perp w\cdot\nabla\theta=0,\quad w(0)=\xi_0. \tag{3.127}
$$

We note that in view of Theorem 3.4.1, we a priori have estimates on the $H^{3/2}$ and even H^2 norms of θ and φ, since they are elements of $\mathcal{A}$. Multiplying (3.127) with $-\Delta w$ and integrating, similarly to (3.126) we obtain

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|w\|_{H^1}^2+\kappa\|w\|_{H^{3/2}}^2&\le\|w\|_{H^{3/2}}\left(\|\nabla\mathcal{R}^\perp\varphi\|_{L^4}\|\nabla w\|_{L^2}+C\|\Lambda^{1/2}\mathcal{R}^\perp w\|_{L^4}\|\nabla\theta\|_{L^4}\right)\\
&\quad+C\|w\|_{H^{3/2}}\|\mathcal{R}^\perp w\|_{L^4}\|\Lambda^{1/2}\nabla\theta\|_{L^4}\\
&\le\frac{\kappa}{2}\|w\|_{H^{3/2}}^2+\frac{C}{\kappa}\|w\|_{H^1}^2\left(\|\varphi\|_{H^{3/2}}^2+\|\theta\|_{H^2}^2\right).
\end{aligned}
\tag{3.128}
$$

The Grönwall inequality and the bounds (3.55)–(3.56) for $\theta,\varphi\in\mathcal{A}$ then yield

$$
\|w(t)\|_{H^1}^2\le\|\xi_0\|_{H^1}^2\exp\left(C\kappa^{-1}tM_{\mathcal{A}}^2\right)\le\|\xi_0\|_{H^1}^2K(t,M_{\mathcal{A}}) \tag{3.129}
$$

for all $t \geq 0$. Here and throughout the proof $K(\cdot,\cdot)$ is an increasing continuous function in each variable. This function *may change from line to line.*

Inserting the estimate (3.129) back into (3.128) gives

$$\int_0^t \|w(s)\|_{H^{3/2}}^2 ds \leq \|\xi_0\|_{H^1}^2 C\kappa^{-1}\left(1+\kappa^{-1}tM_{\mathcal{A}}^2\exp(C\kappa^{-1}tM_{\mathcal{A}}^2)\right) \leq \|\xi_0\|_{H^1}^2 K(t,M_{\mathcal{A}}) \tag{3.130}$$

for $t \geq 0$. Estimate (3.130) can be upgraded to a pointwise in time bound, at the cost of losing the dependence of ξ_0 in H^1. Since

$$\|w(0)\|_{H^{3/2}} = \|\xi_0\|_{H^{3/2}} = \|\varphi_0 - \theta_0\|_{H^{3/2}} \leq 2M_{\mathcal{A}} \tag{3.131}$$

we are justified to study the time evolution of $\|w(t)\|_{H^{3/2}}$. Taking an L^2 inner product of (3.127) with $\Lambda^3 w$, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|w\|_{H^{3/2}}^2 + \kappa\|w\|_{H^2}^2 \\
&\quad\leq \|\Lambda^{3/2}w\|_{L^2}\|[\Lambda^{3/2},\mathcal{R}^\perp\varphi\cdot\nabla]w\|_{L^2} + \|\Lambda^2 w\|_{L^2}\|\Lambda(\mathcal{R}^\perp w\cdot\nabla\theta)\|_{L^2} \\
&\quad\leq \|w\|_{H^{3/2}}\left(\|\Lambda^{3/2}\mathcal{R}^\perp\varphi\|_{L^4}\|\nabla w\|_{L^4} + \|\nabla\mathcal{R}^\perp\varphi\|_{L^4}\|\Lambda^{1/2}\nabla w\|_{L^4}\right) \\
&\qquad\qquad + \|w\|_{H^2}\left(\|\nabla^\perp w\|_{L^4}\|\nabla\theta\|_{L^4} + \|\mathcal{R}^\perp w\|_{L^\infty}\|\Lambda\nabla\theta\|_{L^2}\right) \\
&\quad\leq \frac{\kappa}{2}\|w\|_{H^2}^2 + \|w\|_{H^{3/2}}^2\left(\|\varphi\|_{H^2} + \frac{C}{\kappa}\|\theta\|_{H^2}^2\right)
\end{aligned}$$

by using Lemma 3.6.1 and the embedding $H^{1/2}\subset L^4$. It thus follows from the above estimate, the Grönwall inequality, (3.56), and (3.131), that

$$\|w(t)\|_{H^{3/2}}^2 \leq 4M_{\mathcal{A}}^2\exp\left(C\kappa^{-1}t(1+M_{\mathcal{A}}^2)\right) \leq K(t,M_{\mathcal{A}}) \tag{3.132}$$

for some universal $C>0$.

We now combine (3.126) with (3.129) and (3.132) to obtain

$$\begin{aligned}\frac{d}{dt}\|\eta\|_{H^1}^2+\kappa\|\eta\|_{H^{3/2}}^2 &\le \frac{C}{\kappa}\|\eta\|_{H^1}^2\|\theta\|_{H^2}^2+\frac{C}{\kappa}\|w\|_{H^1}^2\|w\|_{H^{3/2}}^2\left(1+\log\frac{\|w\|_{H^{3/2}}^2}{\|w\|_{H^1}^2}\right)\\ &\le \frac{C}{\kappa}\|\eta\|_{H^1}^2\|\theta\|_{H^2}^2+\frac{C}{\kappa}\|w\|_{H^1}^{2-a}\|w\|_{H^{3/2}}^{2+a}\\ &\le \frac{C}{\kappa}\|\eta\|_{H^1}^2\|\theta\|_{H^2}^2+\|\xi_0\|_{H^1}^{2-a}K(t,M_{\mathcal{A}})\|w\|_{H^{3/2}}^2 \qquad (3.133)\end{aligned}$$

where $a\in(0,1)$ is arbitrary. Using the Grönwall inequality combined with (3.56), (3.130), and the fact that $\eta(0)=0$ we conclude from (3.133) that

$$\begin{aligned}\|\eta(t)\|_{H^1}^2 &\le \exp\left(\frac{C}{\kappa}\int_0^t\|\theta(s)\|_{H^2}^2ds\right)\|\xi_0\|_{H^1}^{2-a}K(t,M_{\mathcal{A}})\int_0^t\|w(s)\|_{H^{3/2}}^2ds\\ &\le \|\xi_0\|_{H^1}^{4-a}K(t,M_{\mathcal{A}}) \qquad (3.134)\end{aligned}$$

for a suitable function K as described above, and $a\in(0,1)$. This proves that

$$\lim_{r\to0+}\left(\sup_{\theta_0,\varphi_0\in\mathcal{A},0<\|\xi_0\|_{H^1}\le r}\frac{\|\eta(t)\|_{H^1}^2}{\|\xi_0\|_{H^1}^2}\right)\le\lim_{r\to0+}r^{2-a}K(t,M_{\mathcal{A}})=\lim_{r\to0+}e(r,t)=0$$

with $e(r,t)=r^{2-a}K(t,M_{\mathcal{A}})$, and thus (3.82) holds.

In order to prove (3.84), consider ξ_0 normalized so that $\|\xi_0\|_{H^1}=1$, and let $\theta_0\in\mathcal{A}$ be arbitrary. Then, using similar estimates as above we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\xi\|_{H^1}^2+\kappa\|\xi\|_{H^{3/2}}^2 &\le C\|\xi\|_{H^{3/2}}\|\xi\|_{H^1}\|\theta\|_{H^{3/2}}+\|\mathcal{R}^\perp\xi\|_{L^\infty}\|\xi\|_{H^1}\|\theta\|_{H^2}\\ &\le \frac{\kappa}{2}\|\xi\|_{H^{3/2}}^2+\frac{C}{\kappa}\|\xi\|_{H^1}^2\|\theta\|_{H^2}^2 \qquad (3.135)\end{aligned}$$

which combined with (3.56) yields

$$\|\xi(t)\|_{H^1}^2\le\exp\left(C\kappa^{-1}tM_{\mathcal{A}}^2\right) \qquad (3.136)$$

which indeed proves (3.84).

It remains to prove that for any $t > 0$ and $\theta_0 \in \mathcal{A}$, the operator $S'(t, \theta_0)$ is compact. Without loss of generality we may look at the image under $S'(t, \theta_0)$ of the unit ball in H^1, and show it is precompact. More precisely, we show that the image of this ball is included in ball in $H^{3/2}$. Combine (3.135) and (3.136) to obtain that

$$\int_0^t \|\xi(s)\|^2_{H^{3/2}} ds \leq K(t, M_{\mathcal{A}})$$

for any $t > 0$. By the mean value theorem, there exits $\tau \in (0, t/2)$ such that

$$\|\xi(\tau)\|^2_{H^{3/2}} \leq \frac{1}{t} K(t, M_{\mathcal{A}}). \tag{3.137}$$

Taking the inner product of (3.86) with $\Lambda^3 \xi$, using the usual commutator, Sobolev, and Poincaré inequalities, we obtain

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\xi\|^2_{H^{3/2}} + \kappa\|\xi\|^2_{H^2} &\leq \|[\Lambda^{3/2}, \mathcal{R}^\perp\theta \cdot \nabla]\xi\|_{L^2}\|\Lambda^{3/2}\xi\|_{L^2} + \|\Lambda(\mathcal{R}^\perp\xi \cdot \nabla\theta)\|_{L^2}\|\Lambda^{3/2}\xi\|_{L^2} \\
&\leq C\|\theta\|_{H^2}\|\xi\|^2_{H^{3/2}} + C\|\theta\|_{H^{3/2}}\|\xi\|_{H^2}\|\xi\|_{H^{3/2}} + C\|\mathcal{R}^\perp\xi\|_{L^\infty}\|\theta\|_{H^2}\|\xi\|_{H^2} \\
&\leq \frac{\kappa}{2}\|\xi\|^2_{H^2} + \frac{C}{\kappa}\|\xi\|^2_{H^{3/2}}\|\theta\|^2_{H^2}
\end{aligned} \tag{3.138}$$

at times larger than the τ in (3.137). Integrating (3.138) between τ and t, and using the bound (3.56) and (3.137), we thus obtain

$$\|\xi(t)\|^2_{H^{3/2}} \leq \|\xi(\tau)\|^2_{H^{3/2}} \exp\left(\frac{C}{\kappa}\int_\tau^t \|\theta(s)\|^2_{H^2} ds\right) \leq \frac{1}{t} K(t, M_{\mathcal{A}})$$

for a suitable function K which is continuous and increasing in all its parameters. This concludes the proof of the Proposition. □

Chapter 4

Gradient decay and symmetrization for the fractional Fisher-KPP equation

4.1 Preliminary material

We recall the fractional Fisher-KPP equation

$$\partial_t u + \Lambda^\alpha u = f(u)$$

with f of KPP-type, $\alpha \in (0,2)$, and positive initial data u_0 with $u_0(x) \leq C|x|^{d+\alpha+\epsilon}$ for some $\epsilon > 0$. Also recall that $K = f'(0)$ and $\lambda = K/(d+\alpha)$.

4.1.1 Invariant coordinates

We begin with the following estimate on the growth of u and its level sets, proved in [9].

Theorem 4.1.1. *We have, for u as above and a universal $C > 0$:*

$$\frac{C^{-1}}{1 + e^{-Kt}\,|x|^{d+\alpha}} \leq u(x,t) \leq \frac{C}{1 + e^{-Kt}\,|x|^{d+\alpha}}. \tag{4.1}$$

This motivates the introduction of the invariant coordinates $\xi = xe^{-\lambda t}$:

$$\partial_t u - \lambda \xi \cdot \nabla_\xi u + e^{-\alpha\lambda t}\Lambda^\alpha u - f(u) = 0. \tag{4.2}$$

For any fixed coordinate x_i, set $\varphi = \partial_{x_i} u$. We then have

$$\partial_t \varphi + \Lambda^\alpha \varphi = f'(u)\varphi. \tag{4.3}$$

Letting $v = \partial_{\xi_i} u = e^{\lambda t}\varphi$ yields

$$\partial_t v - \lambda \xi \cdot \nabla v + e^{-\alpha\lambda t}\Lambda^\alpha v - f'(u)v - \lambda v = 0. \tag{4.4}$$

For another fixed coordinate x_j, set $\psi = \partial_{x_j}\varphi = \partial_{x_j}\partial_{x_i} u$ and $\bar{\varphi} = \partial_{x_j} u$, with $\bar{v} = e^{\lambda t}\bar{\varphi}$. This then gives us

$$\partial_t \psi + \Lambda^\alpha \psi = f''(u)\varphi\bar{\varphi} + f'(u)\psi, \tag{4.5}$$

along with the associated equation in exponential coordinates: for $V = \partial_{\xi_j}\partial_{\xi_i} u$, we have

$$\partial_t V - \lambda \xi \cdot \nabla V + e^{-\alpha\lambda t}\Lambda^\alpha V - f'(u)V - f''(u)v\bar{v} - 2\lambda V = 0. \tag{4.6}$$

A consequence of Theorem 4.4.1 will be the large time convergence of the initially compactly supported (or initially exponentially decreasing) solutions of (4.2) to a steady profile. Notice indeed that the steady equation

$$-\lambda \xi \cdot \nabla_\xi u = f(u) \tag{4.7}$$

has a unique one-parameter family of radial solutions $(u_\tau(|\xi|))_{\tau>0}$. Then, assuming Theorem 4.4.1:

Theorem 4.1.2. *There exists a $\tau_\infty(u_0) > 0$ such that*

$$\lim_{t\to+\infty} u(\xi,t) = u_{\tau_\infty(u_0)}(\xi)$$

uniformly on compact subsets of $\mathbb{R}^d$.

4.1.2 Heat kernel

Let $\rho_\alpha(x,t)$ be the heat kernel of Λ^α, in other words the solution of $\partial_t\varphi + \Lambda^\alpha\varphi = 0$ having, as initial datum, the Dirac mass at 0.

Proposition 4.1.3. *There is $C > 0$ such that, for large t and x, we have $\rho_\alpha(x,t) = t^{-d/\alpha}p_\alpha(x/t^{d/\alpha})$, with*

$$\frac{C^{-1}}{|\zeta|^{d+\alpha+1}} \le |\nabla p_\alpha(\zeta)| \le \frac{C}{|\zeta|^{d+\alpha+1}}, \tag{4.8}$$

and there is $c_\alpha > 0$, $\delta > 0$ such that

$$\frac{C^{-1}}{|\zeta|^{d+\alpha+\delta}} \le p_\alpha(\zeta) - \frac{c_\alpha}{|\zeta|^{d+\alpha}} \le \frac{C}{|\zeta|^{d+\alpha+\delta}}. \tag{4.9}$$

It is important to note that the estimates (4.8) and (4.9) are intended for ζ bounded away from 0; ρ_α and its derivatives are bounded functions (in x) for any $t > 0$. A standard way to prove the above proposition is to write

$$\rho_\alpha(x,t) = \mathcal{F}^{-1}\left(e^{-|\xi|^\alpha t}\right),$$

and to evaluate the inverse Fourier transform with the aid of Polya integrals; see for instance [107].

4.1.3 Comparison principles

We will also require the following easy extension of the Maximum Principle:

Theorem 4.1.4 (**S**elective Comparison Principle)**.** *Let $\Omega(t)$ be a time-dependent family of compact domains in $\mathbb{R}^d$ (in the ξ variable) with smooth boundaries and continuous time dependence; that is, $\{(\xi, t) \mid \xi \in \Omega(t)\}$ is an open set in $\mathbb{R}^d \times \mathbb{R}_+$. Let $v(\xi, t)$ be the solution to (4.4) and let $w(\xi, t)$ be a positive function such that $w > |v|$ at $t = 0$. If, for all $t > 0$ we have $w > |v|$ on the closure of the complement of $\Omega(t)$ and*

$$\partial_t w - \lambda\xi \cdot \nabla w + e^{-\alpha\lambda t}\Lambda^\alpha w - f'(u)w - \lambda w > 0$$

everywhere inside $\Omega(t)$, then $w(\xi, t) \geq |v(\xi, t)|$ on all of $\mathbb{R}^d \times \mathbb{R}_+$.

Proof of Theorem 4.1.4. We first show that $w > v$ by looking at the equation for $q = w - v$ on $\Omega(t)$ which, by assumption, starts positive at $t = 0$. Observe that, if q achieves a global minimum value of 0 at time $\bar{t}$ and location $\bar{\xi} \in \Omega(\bar{t})$, then we easily have $\partial_t q(\bar{\xi}, \bar{t}) > 0$ (all other terms on the left are nonpositive); this, however, crucially requires that $w > v$ outside $\Omega(\bar{t})$. So, by continuity in time, q can never become negative inside $\Omega(t)$. To complete the proof, we need to show that $w > -v$. But this is done in precisely the same manner as before, now examining the equation for $\bar{q} = w + v$ and showing that it too remains positive for all time. □

Also note that the above result (with a minor modification) also holds for a solution V of (4.6). That is, if W is a positive function such that $W > |V|$ on the closure of the complement of $\Omega(t)$ for all $t > 0$ and also $W > |V|$ on $\Omega(0)$, and

$$\partial_t W - \lambda\xi \cdot \nabla W + e^{-\alpha\lambda t}\Lambda^\alpha W - f'(u)W - 2\lambda W > |f''(u)v\bar{v}|$$

everywhere inside $\Omega(t)$, then $W(\xi, t) > |V(\xi, t)|$ on all of $\mathbb{R}^d \times \mathbb{R}_+$. The proof is exactly the same since the equations for $W - V$ and $W + V$ have nonnegative inhomogeneities

on the right side of the inequality.

Finally let us mention the following version of Kato's inequality and its well-known consequence for the Fisher-KPP equation.

Lemma 4.1.5. *If $u(x)$ is smooth, then*

$$\Lambda^\alpha |u(x)| \leq \operatorname{sgn}(u)\Lambda^\alpha u(x),$$

in the distributional sense (and in the classical sense if $\alpha < 1$).

Proof of Lemma 4.1.5. Recall the elementary inequality $|a|-|b| \geq \operatorname{sgn}(b)(a-b)$. This implies, for all $(x,h) \in \mathbb{R}^d \times \mathbb{R}^d$:

$$|u(x+h)| - |u(x)| \geq \operatorname{sgn}(u(x))(u(x+h) - u(x)).$$

Integrating both sides in the variable h over $\mathbb{R}^d \backslash (-\varepsilon, \varepsilon)^d$ and letting $\varepsilon \to 0$ yields the result. □

Corollary 4.1.6. *If $\varphi(x,t)$ is either u_t, or $\partial_{x_i} u(x,t)$, ($i \in \{1,...,d\}$), where $u(x,t)$ is the solution of* (1.8)*, then*

$$\partial_t |\varphi| + \Lambda^\alpha |\varphi| \leq K|\varphi|.$$

Additionally, we have

$$\partial_t |\psi| + \Lambda^\alpha |\psi| \leq |f''(u)\varphi\bar{\varphi}| + K|\psi|$$

for $\psi(x,t)$ a solution to (4.5)*.*

This is immediately apparent upon multiplying (4.3) by $\operatorname{sgn}(\varphi)$ (respectively (4.5) by $\operatorname{sgn}(\psi)$) and using Lemma 4.1.5.

4.2 The gradient decay estimate

Throughout this section, all inequalities will be up to a constant independent of the solutions. We also weaken our requirements on the decay of the initial data. Specifically, we insist that

$$0 \leq u_0(x) \leq \frac{C}{1+|x|^{d+\alpha+1}}.$$

With the notations of Section 2, our goal is to prove the comparative exponential decay rate for the gradient and second derivatives given below.

Theorem 4.2.1. *Assume $u(x,0) := u_0(x)$ to be continuous, nonnegative and nonzero, and $u_0(x) = O(e^{-\varepsilon|x|})$ as $|x| \to +\infty$, for some $\varepsilon > 0$. Then we have, for a solution $u(x,t)$ of (1.8), a universal constant C, and a $\delta > 0$ (depending on λ and α):*

$$|\nabla u(x,t)| \leq Ce^{-\delta t}u(x,t), \tag{4.10}$$

and

$$|\nabla^2 u(x,t)| \leq Ce^{-\delta t}u(x,t). \tag{4.11}$$

To do this, we will need to examine both representations for the evolution of the derivative ((4.3) and (4.4)). We have to distinguish three different ranges of $|x|$: long range ($|x| >> e^{\lambda t}$), medium range ($|x| \sim e^{\lambda t}$), and short range ($|x| << e^{\lambda t}$). The proof of (4.10) is stand-alone and presented fully. The proof of (4.11) is nearly identical (and we present it in parallel), but assumes that (4.10) holds a priori.

4.2.1 Long range: kernel estimates

Lemma 4.2.2. *For any fixed $c > 0$ and for any x in the range $|x| \geq ce^{\lambda(\alpha+1)t}$, a solution φ of (4.3) satisfies*

$$|\varphi(x,t)| \leq u(x,t)e^{-\alpha\lambda t}.$$

Later on, we will set the value of c based on the parameters of the problem. For now, c only affects the final constant that appears in estimates (4.10) and (4.11). As such, we suppress its notation in the proof of the lemma. We also draw special attention to the fact that, in the long range, we have an explicit exponent for the decay rate of $\lambda\alpha$. This particular value will later facilitate a compatibility condition between the long and medium ranges.

Proof of Lemma 4.2.2. We first prove the estimate for φ. Set $g(u) = f(u) - Ku$. By Duhamel's formula we have

$$\begin{aligned} \varphi(x,t) = \;& e^{Kt} \int \partial_{y_i}\rho_\alpha(x-y,t)u_0(y)dy \\ + \;& \int_0^t \int e^{K(t-s)}\rho_\alpha(x-y,t-s)\partial_{y_i}g(u(y,s))dyds. \end{aligned} \tag{4.12}$$

Note that $\|\partial_x\rho_\alpha(.,t)\|_{L^1}$ is not integrable near $t = 0$, for $\alpha < 1/2$. Hence the second term in the expression for φ must be split into two pieces; one in a neighborhood of $s = t$ (where $e^{K(t-s)}$ is essentially constant) and the rest where we may integrate by parts. In total, we obtain the following upper bound on $\varphi(x,t)$: $|\varphi(x,t)| \leq I_0 + I_1 + I_2$, where

$$I_0 = e^{Kt} \int_{\mathbb{R}^d} u_0(x-y) \left|\partial_y\rho_\alpha(y,t)\right| dy,$$

$$I_1 = \int_0^{t-1} e^{K(t-s)} \int_{\mathbb{R}^d} \left|f(u(x-y,s)) - Ku(x-y,s)\right| \left|\partial_y\rho_\alpha(y,t-s)\right| dyds,$$

$$I_2 = \int_{t-1}^t \int_{\mathbb{R}^d} \left|f'(u(x-y,s))\varphi(x-y,s) - K\varphi(x-y,s)\right| \rho_\alpha(y,t-s)dyds.$$

So we proceed by estimating each of the three integrals. We begin with I_0 and apply the naive estimate for $\partial_x \rho_\alpha$ (4.8) and a concrete decay rate for the initial data u_0.

$$I_0 \leq e^{Kt} P(t) \int_{\mathbb{R}^d} \frac{1}{1+|x-y|^{d+\alpha+1}} \frac{1}{1+|y|^{d+\alpha+1}} dy, \tag{4.13}$$

where $P(t)$ is a polynomial in t. The specific form and degree of P are not needed since we will be getting a slight exponential decay in time, and that dominates any residual polynomial growth. To handle the main integral in (4.13), we split into two regions: $\{|y| \leq |x|/2\}$ and $\{|y| > |x|/2\}$. Observe that $|x-y| \geq |x| - |y|$ which, in the first case, means $|x-y| \geq |x|/2$. Thus

$$\int_{|y|\leq|x|/2} \frac{1}{1+|x-y|^{d+\alpha+1}} \frac{1}{1+|y|^{d+\alpha+1}} dy \leq \frac{1}{|x|^{d+\alpha+1}} \int_{\mathbb{R}^d} \frac{dy}{1+|y|^{d+\alpha+1}} \leq \frac{1}{|x|^{d+\alpha+1}}$$

for x in the long range. For the second case,

$$\int_{|y|>|x|/2} \frac{1}{1+|x-y|^{d+\alpha+1}} \frac{1}{1+|y|^{d+\alpha+1}} dy \leq \frac{1}{|x|^{d+\alpha+1}} \int_{\mathbb{R}^d} \frac{dy}{1+|x-y|^{d+\alpha+1}} \leq \frac{1}{|x|^{d+\alpha+1}}.$$

Finally, observe that condition (4.1) and the size of $|x|$ imply that $u(x,t) \approx e^{Kt}/|x|^{d+\alpha}$. Since $|x| \geq e^{\lambda(\alpha+1)t}$, we then have

$$I_0 \leq e^{Kt} |x|^{-(d+\alpha+1)} \approx u |x|^{-1} \leq u e^{-\lambda(\alpha+1)t},$$

which gives us our desired comparative decay for I_0.

Continuing, we need to estimate the factors involving f and f'. Since f is smooth enough to have a partial Taylor expansion, we have that $f(u) = f(0) + f'(0)u + O(u^2)$. Therefore $f(u) - Ku = O(u^2)$, and this (along with (4.1)) gives one of the upper

bounds we will use for I_1.

$$I_1 \leq \int_0^{t-1} e^{K(t-s)} Q(t,s) \int_{\mathbb{R}^d} \frac{1}{1+e^{-2Ks}\left|y\right|^{2(d+\alpha)}} \frac{1}{1+\left|x-y\right|^{d+\alpha+1}} dy ds. \tag{4.14}$$

Here $Q(t,s)$ is algebraic in t and s. As discussed earlier, $Q(t,s)$ will not be integrable up to $s=t$, but it is at most polynomial in t on the interval $0 \leq s \leq t-1$. As with I_0, we handle the main integral by splitting into the same two regions: $\{|y| \leq |x|/2\}$ and $\{|y| > |x|/2\}$. In the first case, we get

$$\begin{aligned} &\int_{|y|\leq|x|/2} \frac{1}{1+e^{-2Ks}\left|y\right|^{2(d+\alpha)}} \frac{1}{1+\left|x-y\right|^{d+\alpha+1}} dy \\ &\qquad \leq \frac{1}{\left|x\right|^{d+\alpha+1}} \int_{\mathbb{R}^d} \frac{dy}{1+e^{-2Ks}\left|y\right|^{2(d+\alpha)}} \approx \frac{e^{\lambda ds}}{\left|x\right|^{d+\alpha+1}}. \end{aligned} \tag{4.15}$$

The last estimate follows from a change of variables. In the second case, we get

$$\begin{aligned} &\int_{|y|>|x|/2} \frac{1}{1+e^{-2Ks}\left|y\right|^{2(d+\alpha)}} \frac{1}{1+\left|x-y\right|^{d+\alpha+1}} dy \\ &\leq \frac{e^{2Ks}}{\left|x\right|^{2(d+\alpha)}} \int_{\mathbb{R}^d} \frac{dy}{1+\left|x-y\right|^{d+\alpha+1}} \approx \frac{e^{2Ks}}{\left|x\right|^{2(d+\alpha)}}. \end{aligned} \tag{4.16}$$

Plugging the estimates from (4.15) and (4.16) back into (4.14) shows that

$$I_1 \leq \int_0^{t-1} Q(t,s) e^{K(t-s)} \left(\frac{e^{Ks\frac{d}{d+\alpha}}}{\left|x\right|^{d+\alpha+1}} + \frac{e^{2Ks}}{\left|x\right|^{2(d+\alpha)}} \right) ds \leq P(t) \left(\frac{e^{Kt}}{\left|x\right|^{d+\alpha+1}} + \frac{e^{2Kt}}{\left|x\right|^{2(d+\alpha)}} \right).$$

Remembering that $u \approx e^{Kt}\left|x\right|^{-(d+\alpha)}$ and $|x| \geq e^{\frac{1+\alpha}{d+\alpha}Kt}$, we see that

$$I_1 \leq uP(t) \left(\frac{1}{|x|} + \frac{e^{Kt}}{\left|x\right|^{d+\alpha}} \right) \leq uP(t) \left(e^{-\lambda(1+\alpha)t} + e^{-\alpha Kt} \right),$$

establishing the comparative exponential decay rate for I_1. Observe that our estimates here would not work if we only had $|x| \geq e^{\lambda t}$.

We handle I_2 in an analogous manner. Note that $f'(u) = f'(0) + O(u)$, so that $f'(u)\varphi - K\varphi = O(u\varphi)$. But we also have, from Corollary 4.1.6,

$$|\varphi(x,t)| \leq e^{Kt}\rho_\alpha *_{x_i} |\partial_x u_0| \leq Ct\frac{e^{Kt}}{|x|^{d+\alpha}},$$

and so $|\varphi(x,t)| \leq Ctu(x,t)$, since Theorem 4.1.1 implies $u(x,t) \geq Ce^{Kt}|x|^{-(d+\alpha)}$. This again implies a $O(tu^2)$ upper bound which reduces the estimate on I_2 to

$$I_2 \leq \int_{t-1}^{t} \bar{Q}(t,s) \int_{\mathbb{R}^d} \frac{1}{1 + e^{-2Ks}\,|y|^{2(d+\alpha)}} \frac{1}{1 + |x-y|^{d+\alpha}} dyds.$$

Here, $\bar{Q}(t,s)$ is again algebraic in t and s, but can now be made integrable on $[t-1,t]$ due to the correct time-homogeneity of $\rho_\alpha(x,t)$. We estimate the convolution integral exactly as we did for I_1 and find that

$$I_2 \leq \int_{t-1}^{t} \bar{Q}(t,s) \left(\frac{e^{Ks\frac{d}{d+\alpha}}}{|x|^{d+\alpha}} + \frac{e^{2Ks}}{|x|^{2(d+\alpha)}} \right) ds \leq P(t) \left(\frac{e^{d\lambda t}}{|x|^{d+\alpha}} + \frac{e^{2Kt}}{|x|^{2(d+\alpha)}} \right).$$

Continuing as we did for I_1, we finally get

$$I_2 \leq uP(t) \left(e^{d\lambda t - Kt} + e^{Kt - K(1+\alpha)t} \right) = uP(t) \left(e^{-\frac{\alpha K t}{d+\alpha}} + e^{-\alpha K t} \right).$$

Finally, we examine the exponents to conclude that, for $|x| \geq e^{\lambda(1+\alpha)t}$, we have $|\varphi(x,t)| \leq u(x,t)e^{-\lambda\alpha t}$. Here, again, the method would not have worked if we merely required $|x| \geq e^{\lambda t}$.

Now we turn our attention to ψ. Looking at (4.5), we employ the Duhamel formula

to obtain

$$\begin{aligned}\psi(x,t) = \;& e^{Kt}\int \partial_{y_i}\partial_{y_i}\rho_\alpha(x-y,t)u_0(y)dy \\ &+ \int_0^t\int e^{K(t-s)}\rho_\alpha(x-y,t-s)\partial_{y_i}g(\varphi(y,s))dyds \\ &+ \int_0^t\int e^{K(t-s)}\rho_\alpha(x-y,t-s)f''(u(y,s))\varphi(y,s)^2dyds.\end{aligned} \tag{4.17}$$

The comparative decay estimates follow analogously to the argument for φ. The first term has even better decay than I_0 since we placed two derivatives on ρ_α. The second term is again split into two parts $\bar{I}_1$ and $\bar{I}_2$ (for the time intervals $[0,t-1]$ and $[t-1,t]$). Since we have already concluded that $|\varphi| \leq ue^{-\lambda\alpha t}$ in this range, the analysis for $\bar{I}_1$ proceeds identically.

Now, $f'(u)\psi - K\psi = O(u\psi)$. The analysis of $\bar{I}_2$ would also be identical to that of I_2 if we knew that $|\psi| = O(u)$ (up to a time-dependent, not exponentially increasing factor). We prove this by appealing to Corollary 4.1.6.

$$|\psi(x,t)| \leq e^{Kt}\rho_\alpha *_x |\partial_{x_j}\partial_{x_i}u_0| + \int_0^t e^{K(t-s)}\rho_\alpha(\cdot,t-s) *_x |f''(u)\varphi\bar{\varphi}|(\cdot,s)ds := J_0 + J_1$$

J_0 is bounded by $Cte^{Kt}/|x|^{d+\alpha}$ as before. J_1 is estimated by the (now proven) comparative exponential bound for φ (and $\bar{\varphi}$) in the long range:

$$\begin{aligned}J_1 \leq \;& Ce^{-2\lambda\alpha t}\textstyle\int_0^t\int e^{K(t-s)}\rho_\alpha(x-y,t-s)u(y,s)^2dyds \\ \leq \;& Ce^{-2\lambda\alpha t}\textstyle\int_0^t e^{K(t-s)}\bar{Q}(t,s)\int \frac{1}{1+e^{-2Ks}|y|^{2(d+\alpha)}}\frac{1}{1+|x-y|^{d+\alpha}}dyds \\ \leq \;& P(t)e^{-2\lambda\alpha t}\left(e^{Kt}|x|^{-(d+\alpha)}\textstyle\int_0^t e^{-\alpha\lambda s}ds + e^{Kt}|x|^{-2(d+\alpha)}\int_0^t e^{Ks}ds\right) \\ \leq \;& P(t)e^{-2\lambda\alpha t}\left(u(x,t)+u(x,t)^2\right).\end{aligned} \tag{4.18}$$

The third inequality was obtained through the same estimates as for I_2. Notice how J_1 inherits an exponential decay from the more rigorous analysis for φ; using a similar approach for the second term of (4.12) would have merely given us a comparative boundedness for φ instead of decay. Hence J_1 is dominated by J_0. Therefore

$\psi = O(tu)$, and we conclude that the second term of (4.17) satisfies a comparative exponential decay rate in the long range.

Lastly, observe that the third term of (4.17) is in fact dominated by J_1. As seen in (4.18), J_1 also satisfies a comparative exponential decay rate (with exponent $-2\lambda\alpha$). This completes the mirrored argument, demonstrating that $|\psi(x,t)| \leq u(x,t)e^{-\lambda\alpha t}$ as long as $|x| \geq e^{\lambda(1+\alpha)t}$. □

4.2.2 Short range: stability of the steady state $u \equiv 1$

The nature of f and condition (4.1) ensures that there exists a fixed $\beta > 0$ such that $\{|\xi| < \frac{3}{2}\beta\} \subseteq \{f'(u) < -\delta_0\}$. That is, we have an exponentially growing ball wherein u is sufficiently close to 1 that $f'(u)$ is negative by at least a fixed amount. Let $\Omega_0(t) = \{|\xi| < \frac{3}{2}\beta\}$. In this region, we use a simple version of the maximum principle. Assume the positive maximum value of φ occurs at time $\bar{t}$ at an interior point $\bar{x} \in \Omega_0(\bar{t})$ (negative minimum values are handled similarly). Then

$$\partial_t\varphi(\bar{x},\bar{t}) + \Lambda^\alpha\varphi(\bar{x},\bar{t}) = f'(u(\bar{x},\bar{t}))\varphi(\bar{x},\bar{t}).$$

However, we have $\Lambda^\alpha\varphi(\bar{x},\bar{t}) \geq 0$. Therefore,

$$\partial_t\varphi(\bar{x},\bar{t}) \leq -\delta_0\varphi(\bar{x},\bar{t}).$$

We conclude that either φ is decaying exponentially (with exponent at least $-\delta_0$) on $\mathbb{R}^d$, *or* the size of φ on $\Omega_0(t)$ is controlled by the size of φ outside $\Omega_0(t)$. Equivalently, v is eventually controlled by a fixed constant times $e^{(\lambda-\delta_0)t}$ whenever $|\xi| < \frac{3}{2}\beta$ so long as v obeys the same bound for (say) $|\xi| > \beta$, which will be established in the next two sections. Certainly, the previous argument shows that this is the case in the long range.

To prove the analogous statement for ψ, we will assume that (4.10) holds with

some uniform $\delta > 0$ in all ranges for φ (the proof of which will, as stated, be stand-alone). Since u is bounded by 1, we have that

$$\partial_t \psi(x,t) + \Lambda^\alpha \psi(x,t) \leq -\delta_0 \psi(x,t) + Ce^{-2\delta t}$$

for all $x \in \Omega_0(t)$. Assume the positive maximum value of ψ occurs at time $\bar{t}$ at an interior point $\bar{x} \in \Omega_0(t)$. Then

$$\partial_t \psi(\bar{x},\bar{t}) \leq -\delta_0 \psi(\bar{x},\bar{t}) + Ce^{-2\delta \bar{t}}$$

again establishing that *either* ψ is decaying exponentially on $\mathbb{R}^d$, *or* the size of ψ on $\Omega_0(t)$ is controlled by the size of ψ outside $\Omega_0(t)$. As before, negative minima are treated similarly.

4.2.3 Medium range: selective comparison principle

For the intermediate range, we appeal to Theorem (4.1.4). Let $\Omega_1(t) = \{\beta \leq |\xi| \leq ce^{\lambda\alpha t}\}$, for $c > 0$ to be determined later. We need to find a w such that $w(\xi,0) \geq |v(\xi,0)|$, and $w > |v|$ outside of $\Omega_1(t)$ for all $t > 0$, and such that

$$\partial_t w - \lambda\xi \cdot \nabla w + e^{-\lambda\alpha t}\Lambda^\alpha w - \lambda w - f'(u)w \geq 0 \tag{4.19}$$

pointwise on $\Omega_1(t)$. It is important that w be defined on all of $\mathbb{R}^d$ because Λ^α is nonlocal; while it can be defined on compact sets, it becomes a different operator than the one in (4.4).

We must therefore provide a candidate supersolution w. For $\xi \in \Omega_1(t)$ and $0 < \nu - d - \alpha \leq 1$, define the family of functions $\widetilde{w}_\nu$ by

$$\widetilde{w}_\nu(\xi,t) = \frac{\beta^\nu w_0}{|\xi|^\nu} + C_\nu e^{-\lambda\alpha t}\left(\frac{\beta^{\nu-d-\alpha}}{|\xi|^\nu} - \frac{1}{|\xi|^{d+\alpha}}\right), \tag{4.20}$$

where $C_\nu, w_0 > 0$ will be determined later. Observe that each such $\widetilde{w}_\nu$ is radial. To simplify notation, let $y = |\xi|$. For fixed ν and t, , $\widetilde{w}_\nu$ is the solution of the following ODE in y (on $\Omega_1(t)$):

$$y\frac{d}{dy}\widetilde{w}_\nu(y,t) + \nu\widetilde{w}_\nu(y,t) = -\bar{C}_\nu e^{-\lambda\alpha t} y^{-(d+\alpha)}. \tag{4.21}$$

Note that $C_\nu = \bar{C}_\nu/(\nu - d + \alpha)$. $\bar{C}_\nu$ will be determined below. Technically, $\bar{C}_\nu$ will also depend on w_0, but the dependence will be linear. More importantly, this dependence is not circular: w_0 is the initial condition at $y = \beta$, and this may be fixed before solving the ODE. w_0 itself will depend on the size of $v(\xi, 0)$.

Our supersolution must be positive, yet $\widetilde{w}_\nu$ is eventually negative for sufficiently large y. We now choose c such that $\widetilde{w}_\nu > 0$ on $\Omega_1(t)$; since $\nu \leq d + \alpha + 1$, such a choice is always possible. Outside of $\Omega_1(t)$, we extend $\widetilde{w}_\nu$ to be smooth, bounded, positive, radial, decreasing in y, and $O(y^{-(d+\alpha)})$. This can obviously be done such that the L^1, C^1, and C^2 norms of $\widetilde{w}$ are uniformly bounded for all $t > 0$. They do, however, depend on ν and w_0.

We now define our candidate supersolution as $w(\xi, t) = e^{\eta t}\widetilde{w}_\nu(|\xi|, t)$ with η and ν fixed; we will determine their values shortly. Observe that $\xi \cdot \nabla w = ye^{\eta t}\partial_y\widetilde{w}$. To prove (4.19), it therefore suffices to show that

$$-e^{\eta t}\partial_t\widetilde{w}_\nu - \eta w + \lambda y\partial_y w(y,t) + (\lambda + f'(u(\xi,t)))\, w(y,t) \leq -e^{-\lambda\alpha t}\left|\Lambda^\alpha w(y,t)\right|$$

on Ω_1(t). Examining (4.20), we see that $\partial_t\widetilde{w}_\nu > 0$ on our domain, so we get a stronger inequality if we ignore the first term above; proving the stronger inequality will be sufficient. Since $f'(u) \leq K = \lambda(d+\alpha)$ and $w > 0$, we can eliminate the ξ dependence entirely and satisfy the above inequality provided that

$$\lambda y\partial_y w(y,t) + \lambda(d + \alpha + 1 - \frac{\eta}{\lambda})w(y,t) \leq -e^{-\lambda\alpha t}\left|\Lambda^\alpha w(y,t)\right|.$$

Treating the diffusion as a perturbation term, we need to bound the size of $|\Lambda^\alpha w(y,t)|$. Our choice of extension for $\widetilde{w}_\nu$ onto $\mathbb{R}^d$ implies (see [7], and estimate (8) of [9]):

$$|\Lambda^\alpha w(y,t)| = e^{\eta t}\,|\Lambda^\alpha \widetilde{w}_\nu(y,t)| \leq e^{\eta t}\frac{\bar{C}_\nu}{y^{d+\alpha}}$$

on Ω_1(t). Our assumptions on the uniform-in-time bounds of the norms of $\widetilde{w}_\nu$ ensure that the constant $\bar{C}_\nu$ can be made time-independent (recall that ν is fixed). It is also linear in w_0, as mentioned before. Thus, proving (4.19) reduces to showing that

$$y\partial_y w(y,t) + (d+\alpha+1-\frac{\eta}{\lambda})w(y,t) = -e^{\eta t-\lambda\alpha t}\frac{\bar{C}_\nu}{y^{d+\alpha}},$$

with $\bar{C}_\nu > 0$ now determined. But this is precisely (4.21), multiplied by $e^{\eta t}$ with $\nu = d+\alpha+1-\frac{\eta}{\lambda}$. Hence, (4.19) holds on Ω_1(t). Clearly, we need $0 \leq \eta < \lambda$. We then choose w_0 sufficiently large that w (that is, the function extended to all of $\mathbb{R}^d$) dominates $|v|$ at $t = 0$.

In order to use Theorem (4.1.4), we need to show that $w - |v| > 0$ outside of $\Omega_1(t)$. It is here that we needed our free parameters ν and η; naively, we would like to take $\eta = 0$ and $\nu = d+\alpha+1$, as this would give the best bounds in the medium range. However, the upper bounds for φ established in the previous two sections are too weak to ensure that $w - |v| > 0$ outside of $\Omega_1(t)$ for all t with that naive choice. In the long range, we only have $|v(\xi,t)| < e^{\lambda(1-\alpha)t}\,|\xi|^{-(d+\alpha)}$. In the short range, we have that $|v(\xi,t)| < e^{(\lambda-\delta_0)t}$ provided the same can be said of v globally.

The added growth rate (in the form of $e^{\eta t}$) allows us to ensure that $w > |v|$ in the long range. Since both w and the upper bound for v are radially decreasing and $O(|\xi|^{-(d+\alpha)})$, it suffices to check the inequality for $|\xi| = ce^{\lambda\alpha t}$. At the boundary between medium range and long range, $w \approx e^{-\nu\lambda\alpha t+\eta t}$ and $|v| < e^{-(d+\alpha)\lambda\alpha t+\lambda t-\lambda\alpha t}$. So it suffices to have $\eta - \lambda\alpha(\nu-d-\alpha) = \eta - \lambda\alpha(1-\frac{\eta}{\lambda}) \geq \lambda(1-\alpha)$, or $\eta \geq \frac{\lambda}{1+\alpha}$. This

leaves enough room for η to be smaller than λ. Fixing constants (and looking at the equation after a transient period to allow the long range decay estimates to come into effect) we have that $w > |v|$ for $|\xi| > ce^{\lambda\alpha t}$.

For the second derivatives, we will similarly employ a supersolution of the form $W(\xi,t) = e^{\zeta t}\widetilde{w}_\nu(|\xi|,t)$ with nearly the same $\widetilde{w}_\nu$ as in (4.20); specifically, we require a larger constant $\bar{C}_\nu$ in equation (4.21). Essentially, we need W to satisfy

$$-\partial_t W + \lambda\xi\cdot\nabla W + 2\lambda W + f'(u)W \leq -e^{-\lambda\alpha t}|\Lambda^\alpha W| - Ce^{2\lambda t - 2\delta t}u^2, \tag{4.22}$$

once again appealing to the full estimate for φ. As before, $|\Lambda^\alpha W(\xi,t)| \leq \bar{C}_\nu e^{\zeta t}|\xi|^{-(d+\alpha)}$. But the last term is dominated by $Ce^{2\lambda-2\delta t}|\xi|^{-2(d+\alpha)}$. If we assume that $\zeta > 2\lambda - 2\delta$, the diffusive term is then a stronger perturbation than the inhomogeneity, and so the latter can be absorbed into the bounds for the former. Our candidate supersolution W will solve (4.22) provided that

$$y\partial_y\widetilde{w}_\nu(y,t) + (d+\alpha+2-\frac{\zeta}{\lambda})\widetilde{w}_\nu(y,t) \leq \frac{\bar{C}_\nu}{y^{d+\alpha}}.$$

This is just (4.21) with $\nu = d+\alpha+2+\zeta/\lambda$, so the equation is satisfied. We want to take ζ as small as possible while still observing the requirement that $0 < \nu - d - \alpha \leq 1$ (this is necessary to ensure W remains positive on $\Omega_1(t)$ with uniform bounds). Thus our arguments impose the constraint that $2\lambda - 2\delta < \zeta < 2\lambda$.

For the final constraint, W must dominate V (that is, $e^{2\lambda t}\psi$ in exponential coordinates) at the interface between medium and long ranges. As before, we have that $W \approx e^{-\nu\lambda\alpha t+\zeta t}$ and $|V| \leq e^{-(d+\alpha)\lambda\alpha t+2\lambda t-\lambda\alpha t}$ when $|\xi| = ce^{\lambda\alpha t}$. We therefore require that $\zeta - \lambda\alpha(\nu-d-\alpha) = \zeta - \lambda\alpha(2-\frac{\zeta}{\lambda}) \geq \lambda(2-\alpha)$, or $\zeta \geq \lambda\frac{2+\alpha}{1+\alpha}$. For $\alpha > 0$, this will always overlap with the admissible range of ζ determined above; and, as expected, the minimum value for ζ decreases as we get better diffusion.

This establishes, as before, that $W > |V|$ in the long range and that W satisfies

the comparison inequality on $\Omega_1(t)$. To satisfy all the conditions for the selective comparison principle, we must show that the supersolutions for v and V also dominate in the short range, $\Omega_0(t)$.

4.2.4 The estimate for φ and ψ

Proof of Theorem 4.2.1. Recall that $\Omega_0(t) = \{|\xi| < \frac{3}{2}\beta\}$ and $\Omega_1(t) = \{\beta < |\xi| < ce^{\lambda\alpha t}\}$. Section 3.2 established that, in $\Omega_0(t)$, $v < C(v_0)e^{(\lambda-\delta_0)t}$ or else v is larger outside of this regime; which, by Lemma 3.1, in fact means v is larger in $\Omega_1(t)$; outside of $\Omega_0(t) \cup \Omega_1(t)$ (a compact set), the comparative exponential bound holds unconditionally.

Fix $\eta = \max\left(\frac{\lambda}{1+\alpha}, \lambda - \delta_0/2\right)$ and w_0 so large that the conditional bound in the short range is stronger than the requirement that $w > v$ in the same regime; that is, $C(v_0)e^{(\lambda-\delta_0)t} < \inf_{\xi\in\Omega_0(t)} w(\xi, t)$ for all $t > 0$ ((4.20) shows that such a choice is always possible). We can also ensure that $w > v$ at $t = 0$.

We then argue by contradiction. Let $\bar{t} > 0$ be the critical time such that $v(\xi, t) < w(\xi, t)$ for all $t < \bar{t}$ and $\xi \in \Omega_0(t)\setminus\Omega_1(t)$, and such that the same upper bound fails for a sequence $t_n \to \bar{t}^+$. This therefore implies a sequence $\{\xi_n\} \in \Omega_0(t_n)\setminus\Omega_1(t_n)$ such that $v(\xi_n, t_n) \geq w(\xi_n, t_n) \geq C(v_0)e^{(\lambda-\delta_0)t}$. By the conditional bound of Section 3.2, there is another sequence $\{\xi_n'\} \in \Omega_1(t_n) \setminus \Omega_0(t_n)$ such that $v(\xi_n', t_n) \geq \inf_{\xi\in\Omega_0(t_n)\setminus\Omega_1(t_n)} w(\xi, t_n)$. By compactness and the continuity of v, we extract a limit point $\bar{\xi}$ with $|\bar{\xi}| \geq \frac{3}{2}\beta$ and $v(\bar{\xi}, \bar{t}) \geq \inf_{|\xi|<\beta} w(\xi, \bar{t})$.

However, for $t < \bar{t}$, the hypothesis of Theorem (4.1.4) was by assumption valid, and therefore $v < w$ up to $\bar{t}$. Moreover, (4.20) immediately shows that there is a continuous, *always positive* $\epsilon(t)$ such that $\sup_{|\xi|\geq\frac{3}{2}\beta} w(\xi, t) < \inf_{|\xi|<\beta} w(\xi, t) - \epsilon(t)$. The continuity of v allows us to extend the inequality to $\bar{t}$. Thus

$$v(\bar{\xi}, \bar{t}) \geq \inf_{|\xi|<\beta} w(\xi, \bar{t}) \geq \sup_{|\xi|\geq\frac{3}{2}\beta} w(\xi, \bar{t}) + \epsilon(\bar{t}) \geq w(\bar{\xi}, \bar{t}) + \epsilon(\bar{t}) \geq v(\bar{\xi}, \bar{t}) + \epsilon(\bar{t}),$$

which is evidently a contradiction. Therefore, $w > v$ (and analogously $w > -v$) in $\Omega_0(t) \setminus \Omega_1(t)$. The hypothesis of Theorem (4.1.4) is always satisfied, and so $w(\xi, t) > |v(\xi, t)|$ for all ξ and t. Precisely the same argument also applies to V with $\zeta = \max\left(\lambda\frac{2+\alpha}{1+\alpha}, 2\lambda - 2\delta, 2\lambda - \delta_0\right)$.

Putting everything together, we have the following three upper bounds (written in terms of φ). Recall that $u \approx |\xi|^{-(d+\alpha)}$ in the medium range.

$$|\varphi(x,t)| \leq C_2 u(x,t) e^{-\lambda\alpha t} \text{ for } |x| > c_1 e^{\lambda(1+\alpha)t}$$

$$|\varphi(x,t)| \leq C_1 u(x,t) e^{\eta t - \lambda t} |\xi|^{-1+\eta/\lambda} \leq C_1 u(x,t) e^{-\delta_1 t} \text{ for } c_0 e^{\lambda t} \leq |x| \leq c_1 e^{\lambda(1+\alpha)t}$$

$$|\varphi(x,t)| \leq C_0 e^{-\delta_0 t} \leq C_0 u(x,t) e^{-\delta_0 t} \text{ for } |x| < c_0 e^{\lambda t}$$

We also get the corresponding three bounds for ψ (recall that $\psi = e^{-2\lambda t} V$).

$$|\psi(x,t)| \leq C_2' u(x,t) e^{-\lambda\alpha t} \text{ for } |x| > c_1 e^{\lambda(1+\alpha)t}$$

$$|\psi(x,t)| \leq C_1' u(x,t) e^{\zeta t - 2\lambda t} |\xi|^{-2+\zeta/\lambda} \leq C_1' u(x,t) e^{-\bar{\delta}_1 t} \text{ for } c_0 e^{\lambda t} \leq |x| \leq c_1 e^{\lambda(1+\alpha)t}$$

$$|\psi(x,t)| \leq C_0 e^{-\delta_0 t} \leq C_0 u(x,t) e^{-\delta_0 t} \text{ for } |x| < c_0 e^{\lambda t}$$

Thus proving Theorem (4.2.1). □

Remark 4.2.3. Theorem (4.2.1) can be extended to a comparative exponential decay for *all* derivatives of u. The dual proof presented above is inductive, essentially repeating the same arguments for ψ while assuming the full result for φ. This can be reworked into a full induction showing the much stronger result that

$$|\nabla^k u(x,t)| \leq C u(x,t) e^{-\delta_k t}$$

for all k. The next section will, however, only require estimates (4.10) and (4.11).

4.3 Bound on the fractional Laplacian

We finally come to the estimate telling us that, at large times, the fractional Laplacian of the solution u is small compared to the size of u.

Lemma 4.3.1. *There exist $C > 0$ and $\delta' > 0$ such that*

$$|\Lambda^\alpha u(x,t)| \leq C u(x,t) e^{-\delta' t}. \tag{4.23}$$

Proof of Lemma 4.3.1. We look at the explicit integral representation for $\Lambda^\alpha u(x,t)$ and initially with $|x| \geq ce^{\lambda t}$. Up to a constant, we have

$$\begin{aligned} \Lambda^\alpha u(x,t) = & \left(\int_{|x-y|<\epsilon e^{\gamma\lambda t}} dy + \int_{|x-y|\geq \epsilon e^{\gamma\lambda t}} dy \right) \frac{u(x,t)-u(y,t)}{|x-y|^{d+\alpha}}, \\ = & \; I_1 + I_2 \end{aligned} \tag{4.24}$$

where $\gamma \in (0,1]$. For the inner piece I_1, we make use of Theorem 4.2.1 and a Taylor expansion: $u(x,t) - u(y,t) = -\nabla u(x,t)\cdot(y-x) + O\left(|\nabla^2 u(cy+(1-c)x,t)||x-y|^2\right)$ for some $c \in (0,1)$. Keeping in mind that

$$\int_A \frac{x-y}{|x-y|^{d+\alpha}} dy = 0$$

on any annulus A centered at x, we then have

$$|I_1| \leq \int_{|x-y|<\epsilon e^{\gamma\lambda t}} \frac{|\nabla^2 u(cx+(1-c)y,t)|}{|x-y|^{d+\alpha-2}} dy \leq e^{-\delta t} \int_{|x-y|<\epsilon e^{\gamma\lambda t}} \frac{u(cx+(1-c)y,t)}{|x-y|^{d+\alpha-2}} dy.$$

Now, in this regime, $u(x,t) \approx e^{Kt}|x|^{-(d+\alpha)}$, and $|cx+(1-c)y| \geq (1-\epsilon)|x|$ because we assumed $|x-y| < \epsilon e^{\gamma\lambda t} \leq \epsilon |x|^\gamma \leq \epsilon |x|$. Therefore,

$$|I_1| \leq (1-\epsilon)^{-(d+\alpha)} u(x,t) e^{-\delta t} \int_0^{\epsilon e^{\gamma\lambda t}} \frac{r^{d-1} dr}{r^{d-\alpha-2}} dr.$$

We then have (up to a constant depending on ϵ)

$$|I_1| \leq u(x,t)e^{-\delta t}e^{\gamma\lambda(2-\alpha)t} \leq u(x,t)e^{-\delta t/2}. \tag{4.25}$$

To obtain the last inequality we must choose γ appropriately. If $(2-\alpha)\lambda \leq \delta/2$, we may take $\gamma = 1$. Otherwise, we take $\gamma = \delta/(2\lambda(2-\alpha)) < 1$. Either way, we obtain our final estimate for I_1.

For the outer piece I_2, we merely use the positivity of u: $(u(x,t)-u(y,t)) < u(x,t)$.

$$|I_2| \leq u(x,t)\int_{|x-y|\geq\epsilon e^{\gamma\lambda t}} \frac{dy}{|x-y|^{d+\alpha}} = u(x,t)\int_{\epsilon e^{\gamma\lambda t}}^{\infty} \frac{r^{d-1}dr}{r^{d+\alpha}} = u(x,t)\epsilon^{-\alpha}e^{-\alpha\gamma\lambda t}.$$

The above inequality and (4.25) show that $|\Lambda^\alpha u(x,t)| \leq u(x,t)e^{-\delta' t}$ for some fixed positive δ' and $|x| > ce^{\lambda t}$. The exact value (in terms of δ, α, and λ) requires an optimization argument for γ.

The case where $|x| < ce^{\lambda t}$, i.e. the short range, is much simpler. In this range, we have that $0 < \bar{c} < u(x,t) \leq 1$. And, moreover, we know from Theorem 4.2.1 that $\|\nabla^2 u\|_{L^\infty} \leq Ce^{-\delta t}$. Using our Taylor expansion again, we can interpolate between these two to obtain:

$$\left|\int_{\mathbb{R}^d} \frac{u(x,t)-u(y,t)}{|x-y|^{d+\alpha}}dy\right| \leq \int_{|x-y|<R} \frac{\|\nabla^2 u(.,t)\|_{L^\infty}dy}{|x-y|^{d+\alpha-2}} + \int_{|x-y|\geq R} \frac{2dy}{|x-y|^{d+\alpha}} \leq Ce^{-\alpha\delta t/2}.$$

The last inequality required us to optimize R to $\|\nabla^2 u(.,t)\|_{L^\infty}^{-1/2}$ and use our uniform estimate. Hence $|\Lambda^\alpha u(x,t)| \leq Ce^{-\delta' t} \leq C\bar{c}^{-1}u(x,t)e^{-\delta' t}$ for all x. □

4.4 Asymptotic symmetrization

We will now look at the explicit (integral) representation for the solution $u(x,t)$ and obtain a better approximate asymptotic behavior than what may be inferred from

(4.1). We will then use this to prove

Theorem 4.4.1. *Let u_0 be as in Theorem 4.2.1. For every $h \in (0,1)$, there is a constant $q_h(u_0) > 0$ and $\delta > 0$ such that*

$$\{u = h\} = \{|x| = q_h(u_0)e^{\lambda t}(1 + O(e^{-\delta t}))\}. \tag{4.26}$$

4.4.1 Behavior of u at infinity for $t = 1$

The first ingredient in the symmetrization proof was presented in Section 4.3. We used Theorem 4.2.1 to establish a similar comparative decay on the asymptotic size of $\Lambda^\alpha u$. The second ingredient is the fact that $u(x,1)$ decays exactly as $|x|^{-(d+\alpha)}$ for large x. This is detailed in the following lemma.

Lemma 4.4.2. *For all $t > 0$, there is $k_t > 0$ and $\delta > 0$ such that*

$$u(x,t) = \frac{k_t}{|x|^{(d+\alpha)}} + O\left(\frac{1}{|x|^{d+\alpha+\delta}}\right). \tag{4.27}$$

Proof of Lemma 4.4.2. Once again it is a matter of applying Duhamel's formula to u. Let

$$\mathcal{D}(s,t,x,y) := (t-s)^{d/\alpha} p_\alpha\left(\frac{x-y}{(t-s)^{1/\alpha}}\right)$$

and

$$D(s,t,x) := \int_{\mathbb{R}^d} \mathcal{D}(s,t,x,y)dy.$$

Then

$$\begin{aligned} u(x,t) =& \frac{e^{Kt}}{t^{d/\alpha}} \int_{\mathbb{R}^d} p_\alpha\left(\frac{x-y}{t^{1/\alpha}}\right) u_0(y)dy + \int_0^t \int_{\mathbb{R}^d} \frac{e^{K(t-s)}}{(t-s)^{d/\alpha}} p_\alpha\left(\frac{x-y}{(t-s)^{1/\alpha}}\right) u^2(y,s)dy \\ =& \frac{e^{Kt}}{t^{d/\alpha}} \int_{\mathbb{R}^d} p_\alpha\left(\frac{x-y}{t^{1/\alpha}}\right) u_0(y)dy + \int_0^t e^{K(t-s)} D(s,t,x)ds. \end{aligned}$$

We have, by Proposition 4.1.3:

$$\begin{aligned}\int_{\mathbb{R}^d} p_\alpha\left(\frac{x-y}{t^{1/\alpha}}\right)u_0(y)dy = &\ c_\alpha t\int_{\mathbb{R}^d}\frac{u_0(y)}{|x-y|^{d+\alpha}}dy + O\left(\frac{1}{|x|^{d+\alpha+\delta}}\right)\\ = &\ \frac{c_\alpha t}{|x|^{d+\alpha}}\int_{\mathbb{R}^d} u_0 dy + O\left(\frac{1}{|x|^{d+\alpha+\delta}}\right)\end{aligned}$$

because u_0 is exponentially decreasing. As for the second term, it is sufficient to estimate $D(s,t,x)$, the integration in time being only between 0 and t. Let us decompose the integral

$$D(s,t,x) = \int_{|x-y|\le|x|/2}\mathcal{D}(s,t,x,y)dy + \int_{|x-y|\ge|x|/2}\mathcal{D}(s,t,x,y)dy := I_1 + I_2.$$

We omit, in order to keep the notations light, the dependence in s, t, x, y. As for I_1, we observe the bound $u(x,t)\le\dfrac{Ce^{Kt}}{|x|^{d+\alpha}}$ simply because we have $\partial_t u + \Lambda^\alpha u \le Ku$. So, we have

$$\begin{aligned}I_1 \le &\ \frac{C_t}{|x|^{2(d+\alpha)}(t-s)^{d/\alpha}}\int_{|x-y|\le|x|/2} p_\alpha\left(\frac{x-y}{(t-s)^{1/\alpha}}\right)dy\\ \le &\ \frac{C_t}{|x|^{2(d+\alpha)}}\|p_\alpha\|_{L^1}.\end{aligned}$$

As for I_2, we use estimate (4.9) in Proposition 4.1.3 and write $I_2 \le I_{21} + CI_{22}$, where

$$I_{22} = (t-s)^{1+\delta/\alpha}\int_{|x-y|\ge|x|/2}\frac{u^2(y,s)}{|x-y|^{d+\alpha+\delta}}dy \le \frac{C_t}{|x|^{d+\alpha+\delta}},$$

and

$$I_{21} = (t-s)\int_{|x-y|\ge|x|/2}\frac{u^2(y,s)}{|x-y|^{d+\alpha}}dy.$$

We further split the integral in I_{21} into two integrals over the regions $\{|x-y|\ge |x|/2\ ,\ |y|\le\varepsilon|x|\}$ and $\{|x-y|\ge|x|/2\ ,\ |y|\ge\varepsilon|x|\}$, denoted (respectively) I_{211} and I_{212}; here, $\varepsilon>0$ is small. We immediately see that $I_{212}\le\dfrac{C_{t,\varepsilon}}{|x|^{2(d+\alpha)}}$,

Now, for $|y|\le\varepsilon|x|$ we have $|x-y| = |x|(1+O(|y|/|x|))$; in other words we have,

setting $C_x := \{|x-y| \geq |x|/2\ ,\ |y| \leq \varepsilon|x|\}$ and using once again estimate (4.9):

$$\begin{aligned} I_{211} = \ & \frac{c_\alpha(t-s)}{|x|^{d+\alpha}} \int_{C_x} u^2(y,s)dy + O\left(\frac{t-s}{|x|^{d+\alpha+1}} \int_{C_x} |y|u^2(y,s)dy\right) \\ & + O\left(\frac{(t-s)^{1+\delta/\alpha}}{|x|^{d+\alpha+\delta}} \int_{C_x} u^2(y,s)dy\right). \end{aligned}$$

Set $l(t) = \displaystyle\int_{\mathbb{R}^d} u^2(y,t)$ and note that $\displaystyle\int_{\mathbb{R}^d} |y|u^2(y,t)dy < +\infty$. Hence there is $l_s > 0$ such that

$$I_{211} = \frac{c_\alpha(t-s)l_s}{|x|^{d+\alpha}} + O(\frac{1}{|x|^{d+\alpha+\delta}}).$$

Gathering everything yields (4.27). □

4.4.2 Analysis of long-time level sets

The proof of Theorem 4.4.1 now reduces to the explicit resolution of a simple ODE.

Proof of Theorem 4.4.1. For all $\lambda \in [0,1)$, there exists $m_\lambda > 0$ such that

$$\forall u \in [0,\lambda], \quad m_\lambda u \leq f(u)/K \leq u.$$

The combination of this remark and Lemma 4.3.1 reduces the full problem (1.8) to

$$\partial_t u = (1 + O(e^{-\delta t}))f(u), \quad u(x,1) = \frac{k_1}{|x|^{d+\alpha}} + O(\frac{1}{|x|^{d+\alpha+\delta}}) \text{ as } |x| \to +\infty, \quad (4.28)$$

keeping in mind that $O(e^{-\delta t})$ might depend on λ. Set

$$G(u) = \int \frac{du}{f(u)},$$

as $u \to 0$ we have

$$G(u) = \frac{\mathrm{Log} u}{K} + g_0 + O(u),$$

where g_0 is a positive constant. Finding the λ-level set of u for large t amounts to

integrating (4.28) with x large, which yields

$$\int_1^t \frac{\partial_t u}{f(u)} = 1 + O(e^{-\delta t}),$$

in other words

$$\mathrm{Log} u + O(u) - \mathrm{Log}\left(\frac{k_1}{|x|^{d+\alpha}} + O(\frac{1}{|x|^{d+\alpha+\delta}})\right) = Kt + q_\infty(u_0) + O(e^{-\delta t}).$$

Specializing $u(x,t) = \lambda$ yields

$$\exp\left(O(\frac{1}{|x|^{d+\alpha}})\right)\left(\frac{k_1}{|x|^{d+\alpha}} + O(\frac{1}{|x|^{d+\alpha+\delta}})\right) = e^{-Kt+q_{\lambda,\infty}(u_0)+O(e^{-\delta t})},$$

for some possibly different constant $q_{\lambda,\infty}(u_0)$. This, after elementary computations, yields (4.26). □

Let us point out again that we have proved, in quite a strong sense, that the large time dynamics of the fractional Fisher-KPP problem (1.8) is the same as that of the ODE $\dot{u} = f(u)$, the fractional diffusion being there only, but this is important to set the initial datum right. This fact was already noticed in [9] and, in an even more evident fashion, in [116].

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