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Knot concordance and matrix factorizations

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Knot concordance and matrix factorizations

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A Dissertation
Presented to the Faculty
of Princeton University
in Candidacy for the Degree
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Abstract

In this thesis, I prove that for the $E(-1)$ spectral sequence constructed by Rasmussen, beginning at the Khovanov-Rozansky $\mathfrak{sl}(n)$ homology of a knot and converging to the homology of the unknot, all higher pages are knot invariants. This is then used to construct a number of numerical knot invariants, each of which is a concordance homomorphism, and these new invariants are applied to obstruct nonorientable surfaces or surfaces in connected sums of $\mathbb{C}P^2$ from bounding a knot, as well as to bounds on the smooth slice genus.

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Chapter 1

Introduction

In [8], Khovanov and Rozansky use the homological algebra of matrix factorizations to construct a link invariant categorifying the $\mathfrak{sl}(n)$ specialization of the HOMFLY-PT polynomial. Later work extended this to other link homologies based on a similar idea: one categorifying the full HOMFLY-PT polynomial [10] or its colored versions ([21] among others), and a conjectural theory categoryifying specializations of the two-variable Kauffman polynomial.

All of these constructions take the same two steps to arrive at the final homology of a link. First, using the theory of matrix factorizations one constructs a functor from a category of appropriately decorated planar graphs and cobordisms between them to the category of chain complexes, and by taking homology then to the category of vector spaces. Then, one resolves each crossing of a link diagram to arrive at a planar graph, and builds a "cube of resolutions" shaped chain complex whose underlying vector space is the direct sum of the spaces assigned to the resolutions in the first step. The homology of this complex is then shown to be an invariant of the link.

Note that the homology of a chain complex is taken twice in this process: first to find the vector space assigned to a planar graph, and second to find the homology of a link. Rasmussen [19] observed (in the context of the full HOMFLY-PT homology, but his ideas work more generally) that in fact the differentials in these two steps are the vertical and horizontal differentials of a double complex, and the link homology is simply the E_2 page of the associated spectral sequence, which he calls $E(-1)$. Rasmussen also calculates that the page $E_\infty(-1)$ page of this spectral sequence will always be simply the homology of the unknot, so all of the higher differentials on $E(-1)$ are a "cancelling differential" on the link homology.

At this point, we may recall the very similar story of Rasmussen's s invariant [18], based on the

Lee spectral sequence from Khovanov homology [12]. The Lee spectral sequence begins at Khovanov homology and converges to the homology of the unknot, and the s invariant is definied in terms of the gradings of surviving generators. Applying this idea to the $E(-1)$ spectral sequences from the $\mathfrak{sl}(n)$ homology for varying n, we arrive at a sequence of knot invarints $t_n(K)$. Just like s and its subsequent extensions to Lee-like spectral sequences from the $\mathfrak{sl}(n)$ homologies, the t_n are concordance invariants and bound the smooth slice genus:

Theorem 1.1. *Each t_n is a homomorphism from the smooth concordance group to the even integers.*

The t_n also give bounds on the smooth slice genus g_4:

Theorem 1.2. $|t_n(K)| \leq 2g_4(K)(n-1)$

This is almost certainly not the best possible bound. In fact, it seems to hold that for all n, $|t_n(K)| \leq 2g_4(K)$, and this can likely be proven via a more careful study of the relationship between the $\mathfrak{sl}(n)$ and HOMFLY homologies. Even with this stronger bound, however, these invariants are not "slice-torus" [15]- unlike $s(K)$, this bound on g_4 is not sharp for torus knots. Only in the case $n = 2$, corresponding to the original Khovanov homology, is the value of t_n known for all torus knots:

Theorem 1.3. *The value $t_2(T_{p,q})$ is determined by the recurrence relation*

$$t_2(T_{p,q}) = -\left\lfloor \frac{q^2}{2} \right\rfloor + t_2(T_{p-q,q})$$

whenever $p > q$, and initial condition $t_2(T_{1,q}) = 0$ since each of these knots is the unknot.

Limited calculations may be performed for $n = 3$, using software developed by Lewark for computing $\mathfrak{sl}(3)$ link homology [14]. For example, $t_3(T_{3,4}) = -2g_4(T_{3,4}) = -6$, but $t_3(T_{4,5}) = -10$ and $t_3(T_{5,6}) = -16$, and both of these values are smaller than twice the corresponding four-genus. All calculations I have done here are consistent with Theorem 1 holding as well for $n = 3$, but with $\lfloor 2n^2/3 \rfloor$ replacing $\lfloor n^2/2 \rfloor$ in the recurrence.

The Khovanov-Rozansky homology is an invariant of oriented links, and for $n > 2$ this restriction is necessary: changing the orientation of some components of a link may fundamentally change the homology. However, for $n = 2$, reorienting some components of a link leaves the homology unchanged up to a grading shift. Representation-theoretically, this may be understood to come from the fact that the vector representation of $\mathfrak{sl}(n)$ is self-dual for $n = 2$ but not for larger n. This insensitivity to orientation may be used to show that, in the case $n = 2$, the invariant t_2 also controls nonorientable surfaces bounding a knot:

Theorem 1.4. *Suppose that a knot $K \subset S^3$ is the boundary of a smooth, possibly nonorientable surface $\Sigma \subset B^4$. Then*

$$\left| t_2(K) - \frac{e(\Sigma)}{2} \right| \leq b_1(\Sigma),$$

where $e(\Sigma)$ is the normal Euler number of Σ, which measures the framing induced on K by a nonvanishing vector field normal to Σ.

Very similar results with t_2 replaced by certain invariants coming from knot Floer homology were proven earlier by Batson [3] and Ozsváth, Stipsicz, and Szabó [17]. Since t_2 turns out to equal these earlier invariants for torus knots, so in particular this gives a new, Khovanov homology-based proof of Batson's result that the nonorientable slice genus $\gamma_4(T_{2k+1,2k+2}) = k$ and in particular that γ_4 may be arbitrarily large.

One final family of results comes out of our better understanding of t_2 compared to other t_n: the calculation of t_2 for torus knots gives also a method of understanding how t_2 changes under cobordisms in connected sums $\#^k \mathbb{C}P^2$, which may even be taken nonorientable.

Theorem 1.5. *Let $\Sigma \subset \#^k \mathbb{C}P^2 - B^4$ be a properly embedded possibly nonorientable connected surface with $\partial \Sigma = K \subset S^3$ a knot, and let ℓ be the minimal self-intersection among all integral lifts of $[\Sigma] \in H_2(\#^k \mathbb{C}P^2 - B^4, S^3; \mathbb{Z}/2)$. Then*

$$t_2(K) \leq b_1(\Sigma) + \frac{e(\Sigma)}{2} - \frac{\ell}{2}$$

Interestingly, unlike most other relative results controlling surfaces with boundary in terms of invariants of the boundary knot, this has nontrivial content even for closed surfaces:

Corollary 1.6. *If $\Sigma \subset \#^k \mathbb{C}P^2$ is a closed, connected surface,*

$$b_1(\Sigma) \geq -\frac{e(\Sigma)}{2} + \frac{\ell}{2}$$

with ℓ as above.

This inequality, however, is not new: it was first proven by Levine, Ruberman, and Strle [13] using the Heegaard Floer d invariant.

The remainder of this thesis is structured as follows: Chapter 2 reviews the relevant portions of the theory of matrix factorizations, working in slightly greater generality than is strictly needed here to allow for future generalizations. Chapter 3 describes the Khovanov-Rozansky $\mathfrak{sl}(n)$ link homology along with a version of Rasmussen's $E(-1)$ spectral sequence from it, and contains a proof of the

technical heart of this thesis: that all pages of this spectral sequence past the second are invariants of a knot. Finally, Chapter 4 (which contains some material previously published in [1] and [2]) applies this spectral sequence to the topological invariants and bounds just discussed.

Chapter 2

Matrix factorizations

2.1 Definitions

In the construction of the Khovanov-Rozansky link homologies, matrix factorizations are assigned to planar objects occuring in resolutions of tangles (to the tangle itself, the original construction assigned a complex of matrix factorizations; we will remember some higher differentials in the form of a so-called Postnikov system). In this chapter, we will give the definitions and basic properties of these objects, as well as prove some lemmas that will greatly simplify future calculations. Similar expositions may be found in [8], [9], or [21].

To fix conventions on gradings and signs, in what follows our rings are graded and concentrated in nonnegative, even degrees with degree 0 piece a characteristic zero field. Typically, they will be polynomial rings $\mathbb{Q}[x_1, \ldots, x_n]$ graded by giving each variable a positive even degree (often but not always 2). If M is a graded module, we will use $q^i M$ to denote M with gradings shifted up by i (so that the degree i piece of $q^i M$ is the degree 0 piece of M); this notation is consistent with taking q to be a one-dimensional $\mathbb{Q}$-vector space concentrated in grading 1 and using juxtaposition for tensor product over $\mathbb{Q}$. Often, our modules will carry an additional $\mathbb{Z}/2\mathbb{Z}$ grading and we will similarly use sM to denote a parity-shifted copy of M. For a homogeneous $x \in M$, we will write $\deg x$ for its $\mathbb{Z}$-degree and $|x|$ for its $\mathbb{Z}/2\mathbb{Z}$ degree.

Tensor products in $\mathbb{Z}/2\mathbb{Z}$ graded contexts will always use Koszul signs for the symmetry

$$x \otimes y \mapsto (-1)^{|x||y|} y \otimes x.$$

Interpreting, as before, s as a one-dimensional odd vector space will then give the correct sign

conventions for grading shifts: for example, if $f : M \to N$ is an odd map, the induced map $sM \to sN$ is $-f$ since before one can apply f to an element of m it must commute past the odd s.

Definition 2.1. *If $w \in R$ is homogeneous of degree $2n$, a matrix factorization of w is a free ($\mathbb{Z}$- and $\mathbb{Z}/2\mathbb{Z}$-)graded R module M together with an odd, degree n differential $d : M \to M$ satisfying $d^2 = w \cdot \mathrm{id}_M$.*

Matrix factorizations of a fixed potential form a dg-category: if (M_1, d_1) and (M_2, d_2) are matrix factorizations of w, then the operation $f \mapsto f \circ d_1 - (-1)^{|f|} d_2 \circ f$ turns the space of R-linear maps $\mathrm{Hom}_R(M_1, M_2)$ into a chain complex:

$$f \mapsto f\circ d_1 - (-1)^{|f|} d_2 \circ f \mapsto f\circ d_1 \circ d_1 - (-1)^{|f|+1} d_2 \circ f \circ d_1 - (-1)^{|f|} d_2 \circ f \circ d_1 - f \circ d_2 \circ d_2 = wf - wf = 0,$$

and a similar computation shows that composition of homomorphisms is a chain map. We will write $\mathrm{CHom}_w(M_1, M_2)$ for this chain complex, $\mathrm{Hom}_w(M_1, M_2)$ for its homology, and $\mathrm{Hom}^0_w(M_1, M_2)$ for the subspace of even, degree 0 elements. In general, $\mathrm{CHom}_w(M_1, M_2)$ and $\mathrm{Hom}^0_w(M_1, M_2)$ will be the objects of primary interest with $\mathrm{Hom}_w(M_1, M_2)$ serving mainly to pass in-between these spaces. In particular, absent further context when we speak of a homomorphism $M_1 \to M_2$ we will always mean an element of $\mathrm{Hom}^0_w(M_1, M_2)$, a homotopy class of degree-0 chain maps, and will refer to chain maps of other degrees as homomorphisms $M_1 \to q^i s^j M_2$ for appropriate i, j.

Remark 2.2. *The original construction of Khovanov-Rozansky link homology involves chain complexes over the homotopy category of matrix factorizations (morphisms are $\mathrm{Hom}_w(M_1, M_2)$); the invariants constructed in this thesis come from working instead homotopy-coherently with $\mathrm{CHom}_w(M_1, M_2)$.*

2.1.1 Fusion

If M and M' are matrix factorizations of two potentials w and w' over rings R and R' with a common subring S, then giving $M \otimes_S M'$ the differential

$$d(m \otimes m') = dm \otimes m' + (-1)^{|m|} m \otimes dm'$$

turns $M \otimes_R M'$ into a factorization of $w + w'$ over the ring $R \otimes_S R'$:

$$d^2(m \otimes m') = d^2 m \otimes m' + (-1)^{|m|} dm \otimes dm' + (-1)^{|m|+1} dm \otimes dm' + m \otimes d^2 m' = (w + w')m \otimes m'.$$

The special case of this that will be used most directly in constructing the link homologies

occurs when R and R' are polynomial rings $\mathbb{Q}[x_1, \ldots, x_p, y_1, \ldots, y_q]$ and $\mathbb{Q}[y_1, \ldots, y_q, z_1, \ldots, z_r]$, respectively, S is the polynomials in the common variables $\mathbb{Q}[y_1, \ldots, y_q]$, and the potentials are of the form

$$w = w_0(x_1, \ldots, x_p) - w_1(y_1, \ldots, y_q)$$
$$w' = w_1(y_1, \ldots, y_q) - w_2(z_1, \ldots, z_r).$$

In this case, the potential of $M \otimes M'$

$$w_0(x_1, \ldots, x_p) - w_2(z_1, \ldots, z_r)$$

lies in the "boundary" subring

$$\mathbb{Q}[x_1, \ldots, x_p, z_1, \ldots, z_r]$$

, so we may restrict scalars of the tensor product $M \otimes SM'$ to this subring and still have a matrix factorization, which we will call the fusion of M and M' along the y_i. Initially, the fusion has infinite rank over $\mathbb{Q}[x_1, \ldots, x_p, z_1, \ldots, z_r]$. However, under a hypothesis on w_1, it will always be homotopy equivalent to a finite rank factorization:

Definition 2.3. *An element w of a polynomial ring $\mathbb{Q}[x_1, \ldots, x_n]$ is a nondegenerate potential if the partial derivatives $\partial_{x_i} w$ form a regular sequence, or equivalently if the quotient $\mathbb{Q}[x_1, \ldots, x_n]/(\partial_{x_i} w)$ is finite dimensional.*

Proposition 2.4 (Proposition 13 from [8] or the main Theorem in [6])**.** *If w_1 is a nondegenerate potential, then the fusion of M and M' along the y_i is homotopy equivalent to a finite rank matrix factorization of $w_0 - w_2$ over the ring*

$$\mathbb{Q}[x_1, \ldots, x_p, z_1, \ldots, z_r].$$

In practice, we can see this finiteness through more explicit calculations of the tensor product without needing this general result, but this is perhaps the easiest way to see, for example that the Khovanov-Rozansky link homologies are always finite dimensional (which is certainly of interest when we take an Euler characteristic to decategorify).

The heart of the Khovanov-Rozansky link homology construction comes from a graphical calculus for this fusion operation. Continuing the notation from the previous paragraph, we may draw both

M as a disc with an incoming arc labelled by w_0 and the variables $x_1, \ldots, x_p$, and an outoing arc labelled by w_1 and the variables $y_1, \ldots, y_q$, and similarly draw M' with an incoming arc labelled by w_1 and the variables $y_1, \ldots, y_q$ and an outgoing arc labelled by w_2 and the variables $z_1, \ldots, z_r$. Then the fusion of M and M' along the y_i is represented by joining the matching arcs, leaving unconnected an incoming and outgoing arc corresponding to the ground ring and potential of the fusion.

More generally, if the potential of a matrix factorization decomposes into a sum of terms, each a polynomial in a set of variables disjoint from the others, then we may represent this factorization by a picture with one boundary arc per term, labelling the arc with the corresponding variables and term and orienting the arc outwards if the term occurs in the potential with the opposite sign. With these conventions, any diagram built out of pictures of this type by joining arcs with matching potentials and consistent orientations corresponds to a valid fusion, resulting in a matrix factorization over the polynomial ring in the variables labelling unpaired edges and with potential given by adding up the terms labelling those edges.

2.2 Koszul factorizations and maps between them

The simplest examples of matrix factorizations come just from factoring w in R: if $w = ab$, then taking $M_{a,b} = q^{|a|/2}R \oplus sq^{|b|/2}R$ and the differential

$$d = \begin{pmatrix} 0 & b \\ a & 0 \end{pmatrix}$$

gives a matrix factorization of w. Taking tensor products of these, from any expression

$$w = a_1 b_1 + \cdots + a_n b_n,$$

we get a matrix factorization of w with rank 2^n. Following [8], we will write this as a double column:

$$\begin{pmatrix} a_1, & b_1 \\ \vdots & \vdots \\ a_n, & b_n \end{pmatrix} = M_{a_1,b_1} \otimes \cdots \otimes M_{a_n,b_n} \tag{2.1}$$

The one-row matrix factorization $M_{a,b}$ is symmetric up to parity under switching a and b, so in

general we have isomorphisms

$$\begin{pmatrix} a_1, & b_1 \\ \vdots & \vdots \\ a_i, & b_i \\ \vdots & \vdots \\ a_n, & b_n \end{pmatrix} \cong s \begin{pmatrix} a_1, & b_1 \\ \vdots & \vdots \\ b_i, & a_i \\ \vdots & \vdots \\ a_n, & b_n \end{pmatrix} \tag{2.2}$$

Another perspective on the Koszul factorizations, which breaks this symmetry but makes others more apparent, involves exterior algebras: Let F be a graded free R module with basis $e_1, \ldots, e_n$ and e_i odd and in degree $\deg w/2 - \deg a_i = (\deg b_i - \deg a_i)/2$, and let $f_1, \ldots, f_n$ be the dual basis of F^*. Then the underlying module of the Koszul factorization is isomorphic to the exterior algebra of F with degrees shifted:

$$\begin{pmatrix} a_1, & b_1 \\ \vdots & \vdots \\ a_n, & b_n \end{pmatrix} \cong q^{-\frac{1}{2}(|a_1|+\cdots|a_n|)} \bigwedge\nolimits^{\bullet} F$$

and the differential is given by

$$d\omega = (a_1 e_1 + \cdots + a_n e_n) \wedge \omega + (b_1 f_1 + \cdots + b_n f_n) \neg \omega \tag{2.3}$$

Khovanov and Rozansky use this presentation to prove that particular modifications of the a_i and b_i, corresponding to a change of basis of F, lead to an isomorphic matrix factorization. Specifically, for homogeneous λ of the appropriate degree, there are isomorphisms

$$\begin{pmatrix} \vdots & \vdots \\ a_i, & b_i \\ \vdots & \vdots \\ a_j, & b_j \\ \vdots & \vdots \end{pmatrix} \cong \begin{pmatrix} \vdots & \vdots \\ a_i, & b_i - \lambda b_j \\ \vdots & \vdots \\ a_j + \lambda a_i, & b_j \\ \vdots & \vdots \end{pmatrix} \qquad \text{((2.34) in [8])}$$

and (conjugating the above with the isomorphism switching a_j with b_j)

$$\begin{pmatrix} \vdots & \vdots \\ a_i, & b_i \\ \vdots & \vdots \\ a_j, & b_j \\ \vdots & \vdots \end{pmatrix} \cong \begin{pmatrix} \vdots & \vdots \\ a_i, & b_i - \lambda a_j \\ \vdots & \vdots \\ a_j, & b_j + \lambda a_i \\ \vdots & \vdots \end{pmatrix} \qquad \text{((2.34) in [8])}$$

Repeated uses of this second isomorphism gives the following theorem, which will dramatically simplify many calculations:

Theorem 2.5 (Theorem 2.1 from [9]). *If $a_1, \ldots, a_n$ is a regular sequence and*

$$a_1 b_1 + \cdots + a_n b_n = a_1 b_1' + \cdots + a_n b_n',$$

then

$$\begin{pmatrix} a_1, & b_1 \\ \vdots & \vdots \\ a_n, & b_n \end{pmatrix} \cong \begin{pmatrix} a_1, & b_1' \\ \vdots & \vdots \\ a_n, & b_n' \end{pmatrix}.$$

Using this theorem, we can often avoid specifying the b_i: if w is contained in the ideal of R generated by $a_1, \ldots, a_n$, and the a_i form a regular sequence, write

$$\begin{pmatrix} a_1, & * \\ \vdots & \vdots \\ a_n, & * \end{pmatrix}$$

for one of the isomorphic Koszul factorizations corresponding to a choice of b_i. This may also be applied partially - if some subsequence of the a_i form a regular sequence, leaving just the corresponding b_i unspecified gives an unambiguous matrix factorization.

Thinking of Koszul factorizations in terms of the exterior algebra can also be used to find chain maps between nonisomorphic factorizations:

Proposition 2.6 (Proposition 8.13 from [21]). *With $\vec{a} = a_1 e_1 + \cdots + a_n e_n$ and $\vec{b} = b_1 f_1 + \cdots + b_n f_n$, suppose that $\vec{a} = T\vec{c}$ for some graded free module F', $\vec{c} \in F'$, and $T : F' \to F$ homogeneous of degree*

0. *Then from this data one may construct a homomorphism*

$$\begin{pmatrix} \vec{a}, & \vec{b} \end{pmatrix} \to q^{-(n \cdot deg w - \sum_i \deg c_i - \sum_i \deg b_i)/2} \begin{pmatrix} \vec{c}, & T^* \vec{b} \end{pmatrix},$$

and the composite

$$\begin{pmatrix} \vec{a}, & \vec{b} \end{pmatrix} \to q^k \begin{pmatrix} \vec{c}, & T^* \vec{b} \end{pmatrix} \to q^k s^n \begin{pmatrix} T^* \vec{b}, & \vec{c} \end{pmatrix} \to q^{2k} s^n \begin{pmatrix} \vec{b}, & \vec{a} \end{pmatrix} \to q^{2k} \begin{pmatrix} \vec{a}, & \vec{b} \end{pmatrix}$$

is equal to multiplication by $\det T$*, where the first and third maps are the ones given by this proposition and the second and fourth come from interchanging the columns.*

In favorable situations, these give all of the chain maps between a two given matrix factorizations:

Theorem 2.7. *Suppose that*

$$M = \begin{pmatrix} a_1, & b_1 \\ \vdots & \vdots \\ a_n, & b_n \end{pmatrix}$$

and

$$M' = \begin{pmatrix} a'_1, & b'_1 \\ \vdots & \vdots \\ a'_n, & b'_n \end{pmatrix}$$

are two Koszul factorizations of the same potential $w \in R$*, and that* $(a_1, \ldots, a_n, a'_1, \ldots, a'_n)$ *is a regular sequence. Then there are homomorphisms*

$$f : M \to q^{-(n \deg w - \sum_i \deg a_i - \sum_i \deg a'_i)/2} s^n M'$$

and

$$g : M' \to q^{-(n \deg w - \sum_i \deg a_i - \sum_i \deg a'_i)/2} s^n M$$

that generate $\mathrm{Hom}_w(M, M')$ *and* $\mathrm{Hom}_w(M', M)$*, respectively, as* R*-modules. Furthermore,* f *and* g *are composites of change-of-basis isomorphisms modifying the* b_i *and* b'_i *with a map coming from Proposition 2.6, and in particular* $f \circ g$ *and* $g \circ f$ *are both equal to scalar multiplication by the same element of* R*.*

Proof. First, the complex $\mathrm{CHom}_w(M, M')$ is given by the Koszul factorization

$$q^{-(\sum_i \deg a_i + \sum_i \deg a_i')/2} \begin{pmatrix} a_1 & b_1 \\ \vdots & \vdots \\ a_n & b_n \\ a_1', & -b_1' \\ \vdots & \vdots \\ a_n', & -b_n' \end{pmatrix}.$$

(This is a factorization of 0, hence a chain complex). Since $a_1, \ldots, a_n, a_1', \ldots, a_n'$ is a regular sequence, we may use Theorem 2.5 to say that $\mathrm{CHom}_w(M, M')$ is also isomorphic to

$$q^{-(\sum_i \deg a_i + \sum_i \deg a_i')/2} \begin{pmatrix} a_1 & 0 \\ \vdots & \vdots \\ a_n & 0 \\ a_1', & 0 \\ \vdots & \vdots \\ a_n', & 0 \end{pmatrix},$$

or in other words to the Koszul complex associated to the sequence $a_1, \ldots, a_n, a_1', \ldots, a_n'$. Since this sequence is regular, the homology of this Koszul complex is isomorphic to $R/(a_1, \ldots, a_n, a_1', \ldots, a_n')$, and we may take f to be the image of a generator. Dually, g may be found by running the same argument with M and M' reversed.

Note that, under our assumptions on the grading of R, the spaces of maps $M \to q^k M'$ and $M' \to q^k M$ are both one-dimensional and spanned by f and g respectively for $k = -(n \deg w - \sum_i \deg a_i - \sum_i \deg a_i')/2$, and vanish for any larger k. Most applications of this theorem in what follows will use only this fact, since it frequently allows us to determine the differentials in a chain complex without explicit calculation.

To see that f and g come from the construction of Proposition 2.6, we can more carefully use the exactness of the Koszul complex. Since

$$a_1 b_1 + \cdots + a_n b_n - a_1' b_1' - \cdots - a_n' b_n',$$

the vector

$$(b_1, \dots, b_n, -b'_1, \dots, -b'_n)$$

is a cycle representing a class in the first cohomology of the Koszul complex for the a_i and a'_i. Since this cohomology vanishes, this cycle is a boundary, so we may write

$$\begin{pmatrix} b_1 \\ \vdots \\ b_n \\ b'_1 \\ \vdots \\ b'_n \end{pmatrix} = \left(\begin{array}{c|c} P & T \\ \hline -T & Q \end{array}\right) \begin{pmatrix} a_1 \\ \vdots \\ a_n \\ a'_1 \\ \vdots \\ a'_n \end{pmatrix}$$

with P and Q antisymmetric. By modifying the b_i and b'_i using the change-of-basis isomorphisms, we may take P and Q to be zero, and then T expresses the b_i in terms of a'_i and b'_i in terms of a_i. This means that Proposition 2.6 may be applied to produce maps $M \to q^k M'$ and $M' \to q^k M$ for $k = -(n \deg w - \sum_i \deg a_i - \sum_i \deg a'_i)/2$, which must equal f and g up to scale by the observation in the previous paragraph. □

Corollary 2.8. *If w is a nondegenerate potential in a polynomial ring on $2n$ variables and*

$$M = \begin{pmatrix} a_1, & b_1 \\ \vdots & \vdots \\ a_n, & b_n \end{pmatrix}$$

is an n-row Koszul matrix factorization (half as many rows as variables), then $\mathrm{Hom}_w(M, M)$ is isomorphic as an R-module to $R/(a_i, b_i)$. In particular, $\mathrm{Hom}_0(M, M)$ is one-dimensional and spanned by the identity and $\mathrm{Hom}_0(M, q^k M)$ vanishes for any $k > 0$.

Proof. By Theorem 2.7, it suffices to show that $a_1, \dots, a_n, b_1, \dots, b_n$ is a regular sequence, or equivalently that the variety cut out by the a_i and b_i has codimension $2n$ and so dimension 0. However, the partial derivatives $\partial_i w$ cut out a zero-dimensional variety, and since

$$\partial_i w = a_1 \partial_i b_1 + \partial_i a_1 b_1 + \cdots + a_n \partial_i b_n + \partial_i a_n b_n$$

the a_i and b_i cut out a subvariety of this, completing the proof. □

A particularly important matrix factorization in this theory is the arc factorization, also called the stabilized diagonal [5], which exists for any potential of the form $w(x_1, \ldots, x_n) - w(y_1, \ldots, y_n)$ for w a polynomial in n variables. If, for each i, we set $x_i = y_i$, then also $w(x_1, \ldots, x_n) = w(y_1, \ldots, y_n)$, so $w(x_1, \ldots, x_n) - w(y_1, \ldots, y_n)$ is contained in the ideal generated by $x_i - y_i$. Since the $x_i - y_i$ certainly form a regular sequence, we may form the Koszul factorization

$$q^{-(n \deg w - \sum_i \deg x_i)/2} \begin{pmatrix} x_1 - y_1 & * \\ \vdots & \vdots \\ x_n - y_n & * \end{pmatrix}.$$

In the graphical calculus used for fusions, we will represent this as simply an oriented arc, with incoming boundary labelled by w and the x_i and outgoing boundary labelled by w and the y_i. This choice of notation is justified by the fact that this factorization is an identity for fusion:

Proposition 2.9 (Proposition 15 in [8])**.** *If M is any matrix factorization of a potential $w'(z_i) - w(x_i)$, then the fusion of M with the one-arc factorization described above along the x_i is homotopy equivalent to the result of renaming the x_i to y_i in M.*

A key lemma here, that may be used to prove the above proposition in the case where M is Koszul and more generally is critical to computing fusions of Koszul factorizations, is Khovanov and Rozansky's "excluding a variable" lemma:

Lemma 2.10. *Suppose*

$$M = \begin{pmatrix} a_1 & b_1 \\ \vdots & \vdots \\ a_n & b_n \end{pmatrix}$$

is a Koszul factorization of $w \in R[x]$, and that the variable x is not present in w (so in fact $w \in R$) If $a_1 = x - a$ for some $a \in R$, then the restriction of scalars of M to R is homotopy equivalent to the Koszul factorization

$$\begin{pmatrix} \overline{a}_2 & \overline{b}_2 \\ \vdots & \vdots \\ \overline{a}_n & \overline{b}_n \end{pmatrix}$$

where $\bar{a_i}$ and $\bar{b_i}$ are the images of a_i and b_i, respectively, under the quotient $R[x] \to R$ sending x to a.

For certain calculations involving the one-arc factorization, we will need a more explicit version

of its second column. For this, we may take

$$\begin{pmatrix} x_1 - y_1 & (w(x_1, x_2, \ldots, x_n) - w(y_1, x_2, \ldots, x_n))/(x_1 - y_1) \\ x_2 - y_2 & (w(y_1, x_2, \ldots, x_n) - w(y_1, y_2, \ldots, x_n))/(x_2 - y_2) \\ \vdots & \vdots \\ x_n - y_n & (w(y_1, \ldots, y_{n-1}, x_n) - w_(y_1, \ldots, y_{n-1}, y_n))/(x_n - y_n) \end{pmatrix}.$$

Most importantly, this allows us to compute the fusion of the two one-arc factorizations of $w(x_1, \ldots, x_n) - w(y_1, \ldots, y_n)$ and of $w(y_1, \ldots, y_n) - w(x_1, \ldots, x_n)$ along both the x variables and the y variables. Graphically, this is a closed circle, and is a factorization of 0, so a chain complex. After applying Lemma 2.10 on each of the y_i, we are left with the Koszul factorization

$$\begin{pmatrix} 0 & \partial_1 w(x_1, \ldots, x_n) \\ \vdots & \vdots \\ 0 & \partial_n w(x_1, \ldots, x_n) \end{pmatrix},$$

or simply the Koszul complex of the partial derivatives of w. In particular, if w is a nondegenerate potential, the homology of this chain complex is just the quotient $R/(\partial_i w)$. In the link homologies to be constructed, this will be the homology of the unknot.

Since there are as many relations $\partial_i w$ as there are variables in R and the relation form a regular sequence, the quotient $R/(\partial_i w)$ is naturally a Frobenius algebra and gives a 2-dimensional TQFT. In fact, this TQFT structure is visible before closing the arc to a circle: this one-arc factorization is the image of the interval under an extended TQFT. To be fully precise, one must either allow the extended TQFT axioms to hold only up to sign or to work with framed cobordisms, for more (and for the proof that the extended TQFT axioms hold) see [4], or see section 9 of [8] for a discussion in the case which will be used here.

Giving this structure requires three maps: the "birth" $\mathbb{Q} \to R/(\partial_i w)$ which is simply the inclusion of 1, the "death" $R/(\partial_i w) \to \mathbb{Q}$ which is the Froebenius trace, and finally a saddle map which can be presented using Proposition 2.6: in the case of a one-variable potential (the only one needed in what follows), the two sides of the saddle are Koszul factorizations

$$\begin{pmatrix} x_1 - y_1 & (w(x_1) - w(y_1))/(x_1 - y_1) \\ x_2 - y_2 & (w(x_2) - w(y_2))/(x_2 - y_2) \end{pmatrix}$$

and

$$\begin{pmatrix} x_1 - y_2 & (w(x_1) - w(y_2))/(x_1 - y_2) \\ x_2 - y_1 & (w(x_2) - w(y_1))/(x_2 - y_1) \end{pmatrix}$$

or, after applying a change-of-basis isomorphism and writing $w(a, b) = (w(a) - w(b))/(a - b)$ for simplicity,

$$\begin{pmatrix} x_1 + x_2 - y_1 - y_2 & * \\ x_2 - y_2 & w(x_2, y_2) - w(x_1, y_1) \end{pmatrix}$$

and

$$\begin{pmatrix} x_1 + x_2 - y_1 - y_2 & * \\ x_2 - y_1 & w(x_2, y_1) - w(x_2, y_1) \end{pmatrix},$$

and $w(x_2, y_1) - w(x_2, y_1)$ and $x_1 + x_2 - y_1 - y_2$ are both in the ideal generated by $x_1 + x_2 - y_1 - y_2$ and $x_2 - y_2$.

The result we need from this discussion, which will be critical to showing that our knot invariants are in fact concordance invariants, is that if D is a one-manifold with boundary and Γ is a cobordism $D \to D$ build from births, deaths, and saddles such that Γ is diffeomorphic to $D \times I$, then if we take M the be the matrix factorization build via fusions of a one-arc matrix factorization patterned on D, and $f : D \to D$ the composition of birth, death, and saddle maps corresponding to Γ, then f is a homotopy equivalence. This is simply a special case of the topological invariance of this TQFT, and is not harmed by TQFT axioms holding only up to sign.

Chapter 3

Khovanov-Rozansky's $\mathfrak{sl}(n)$ homology

3.1 The fundamental matrix factorizations

In addition to the arc, there are two fundamental matrix factorizations from which the others appearing here will be generated by fusion: one used to define the complex associated to a crossing and a third appearing in the proof of the third Reidemeister move. Each of these is a Koszul factorization, constructed in a uniform way via the following lemma:

Lemma 3.1. *Suppose R is a ring, and S is a subring generated by finitely many elements $s_1, \ldots, s_n$. Then for any $f \in S$, the element $f \otimes 1 - 1 \otimes f \in R \otimes R$ is contained in the $R \otimes R$-ideal generaged by the $s_i \otimes 1 - 1 \otimes s_i$. If furthermore S is a polynomial ring on the s_i and R is free as and S-module, then the elements $s_i \otimes 1 - 1 \otimes s_i$ are a regular sequence.*

Proof. The set of $f \in R$ such that $f \otimes 1 - 1 \otimes f$ is contained in the ideal generated by $s_i \otimes 1 - 1 \otimes s_i$ is closed under addition and, since

$$f_1 f_2 \otimes 1 - 1 \otimes f_1 f_2 = (f_1 \otimes 1 - 1 \otimes f_1) \cdot (f_2 \otimes 1) + (1 \otimes f_1) \cdot (f_2 \otimes 1 - 1 \otimes f_2),$$

also closed under multiplication. Since it certainly contains the s_i, it contains all of s. For the additional statement, if S is a polynomial ring on the s_i then the elements $s_i \otimes 1 - 1 \otimes s_i$ form an $S \otimes S$-regular sequence, and if R is free over S then tensoring up from $S \otimes S$ to $R \otimes R$ preserves exactness of the Koszul complex so they will also form an $R \otimes R$-regular sequence. □

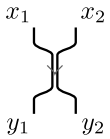


Figure 3.1: The graphical representation of the "double edge" matrix factorization occuring in the complex assigned to a crossing.

We will use this with R a polynomial ring and S the subring of symmetric polynomials, generated by the elementary symmetric polynomials.

Specifically, the factorization that appears in the complex for a crossing has potential $x_1^{n+1} + x_2^{n+1} - y_1^{n+1} - y_2^{n+1}$, corresponding to a graph with two incoming edges labelled x_1 and x_2 and two outgoing edges labelled y_1 and y_2. The polynomial $x_1^{n+1} + x_2^{n+1}$ is symmetric, so is a polynomial in $e_1(x_1, x_2) = x_1 + x_2$ and $e_2(x_1, x_2) = x_1 x_2$. Therefore, identifying $\mathbb{Q}[x_1, x_2, y_1, y_2] \cong \mathbb{Q}[x_1, x_2]^{\otimes 2}$, the lemma applies and the polynomials

$$e_1(x_1, x_2) - e_1(y_1, y_2)$$

and

$$e_2(x_1, x_2) - e_2(y_1, y_2)$$

form a regular sequence and generate an ideal containing $x_1^{n+1} + x_2^{n+1} - y_1^{n+1} - y_2^{n+1}$, so we may form the Koszul matrix factorization

$$\begin{pmatrix} e_1(x_1, x_2) - e_1(y_1, y_2) & * \\ e_2(x_1, x_2) - e_2(y_1, y_2) & * \end{pmatrix}$$

of $x_1^{n+1} + x_2^{n+1} - y_1^{n+1} - y_2^{n+1}$. Graphically, we will represent this factorization by the image in figure 3.1.

We may perform this same construction in more variables, although only the three variable case is needed here: In $\mathbb{Q}[x_1, x_2, x_3, y_1, y_2, y_3]$, the potential $x_1^{n+1} + x_2^{n+1} + x_3^{n+1} - y_1^{n+1} - y_2^{n+1} - y_3^{n+1}$ is contained in the ideal generated by $e_i(x_1, x_2, x_3) - e_i(y_1, y_2, y_3)$ for $i = 1, 2, 3$, and these generators form a regular sequence so we may form a Koszul factorization

$$\begin{pmatrix} e_1(x_1, x_2, x_3) - e_1(y_1, y_2, y_3) & * \\ e_2(x_1, x_2, x_3) - e_2(y_1, y_2, y_3) & * \\ e_3(x_1, x_2, x_3) - e_3(y_1, y_2, y_3) & * \end{pmatrix}$$

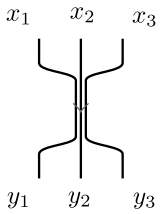


Figure 3.2: The graphical representation of the "triple edge" matrix factorization occuring in the simplified complex for the third Reidemeister move.

of $x_1^{n+1} + x_2^{n+1} + x_3^{n+1} - y_1^{n+1} - y_2^{n+1} - y_3^{n+1}$, and will draw this as shown in figure 3.1.

3.2 Postnikov systems and crossing complexes

To a tangle, Khovanov and Rozansky assign a complex of matrix factorizations built from the above fundamental factorizations. In order to track the additional information from $E(-1)$, we will record some additional homotopies, replacing chain complexes over the homotopy category of matrix factorizations with Postnikov systems (name following [9], this is essentially identical to the multifactorizations from [1] with only a difference in grading conventions) over the dg-category with morphism spaces $\mathrm{CHom}(M, M')$.

Definition 3.2. *A Postnikov system of matrix factorizations of w is a sequence $\{M_i\}_{i\in\mathbb{Z}}$ of matrix factorizations of w, together with a sequence of elements $d_{i,k} \in \mathrm{CHom}(M_i, M_{i+k})$ for $i \in \mathbb{Z}$ and $k > 0$, such that $d_{i,k}$ has degree $(1-k)\cdot(\deg w)/2$ and odd $\mathbb{Z}/2$ grading, and*

$$D(d_{i,k}) + d_{i+1,k-1} \circ d_{i,1} + d_{i+2,k-2} \circ d_{i,2} \cdots + d_{i+k-1,1} \circ d_{i,k-1} = 0 \tag{3.1}$$

where D is the differential of $\mathrm{CHom}(M_i, M_{i+k})$.

In particular, the $d_{i,1}$ are cycles of $\mathrm{CHom}(M_i, M_{i+1})$ representing classes in $\mathrm{Hom}_0(M_i, sM_{i+1})$, the $d_{i,2}$ are nullhomotopies of $d_{i+1,1} \circ d_{i,1}$, and the higher $d_{i,k}$ represent higher coherences. If we remember only the classes $[d_{i,1}] \in \mathrm{Hom}_0(M_i, sM_{i+1})$, then, we are left with a chain complex over the homotopy category of chain complexes that we will call the associated graded complex.

Note that the equation satisfied by the $d_{i,k}$ implies that, if we give the direct sum $\oplus_i q^{-i(\deg w)/2} M_i$ the differential $d + \sum_{i,k} d_{i,k}$ where d is the sum of the differentials of the individual M_i, it becomes a matrix factorization of w. This operation is what Khovanov and Rozansky call the convolution of the Postnikov system in [9], in which it plays a key role in their construction of a possible $\mathfrak{so}(2n)$ link homology. Here, the convolution will be used to determine that the $E(-1)$ spectral sequence

converges to the homology of an unknot. A Postnikov system may be thought of as a filtration on its convolution, motivating the definition of a homomorphism or homotopy as a filtered homomorphism or homotopy between the convolutions:

Definition 3.3. *If $\{M_i\}$ and $\{M'_i\}$ are two Postnikov systems, a homomorphism $f : M \to M'$ is a sequence of maps $f_{i,k} : M_i \to M'_{i+k}$ for all i and all $k \geq 0$ of degree $-k \cdot (\deg w)/2$ and even $\mathbb{Z}/2$ grading, such that*

$$Df_{i,k} + f_{i+1,k-1} \circ d_{i,1} - d'_{i+k-1,1} \circ f_{i,k-1} + \cdots + f_{i+k,0} \circ d_{i,k} - d'_{i,k} \circ f_{i,0} = 0.$$

The homomorphism $f_{i,k}$ is nullhomotopic if there exists a sequence of maps $h_{i,k} : M_i \to M'_{i+k}$ for all i and all $k \geq -1$ of degree $(-1-k) \cdot (\deg w)/2$ and odd $\mathbb{Z}/2$ grading such that

$$Dh_{i,k} + h_{i+1,k-1} \circ d_{i,1} + d'_{i+k-1,1} \circ h_{i,k-1} + \cdots + h_{i+k+1,-1} \circ d_{i,k+1} + d'_{i-1,k+1} \circ h_{i,-1} = f_{i,k}$$

where we take $f_{i,-1} = 0$.

Note that we must begin indexing the maps comprising a homotopy at -1 since a homotopy of chain complexes involves maps backwards in the complex, so this cannot be stricly analagous to the usual notion of a filtered homotopy.

While a full Postnikov system in general carries much more information than the associated graded complex, in favorable situations all of the higher differentials are in fact uniquely determined:

Proposition 3.4 (Corollary 5.4 in [9])**.** *Suppose that $\{M_i\}_{i\in\mathbb{Z}}$ is a bounded sequence of matrix factorizations (in the sense that $M_i = 0$ for all but finitely many i), and $d_{i,k}$ and $d'_{i,k}$ are two systems of differentials making $\{M_i\}$ into a Postnikov system. If $d_{i,1}$ and $d'_{i,1}$ are homotopic for all i and $\mathrm{Hom}_0(M_i, sq^{k(\deg w)/2} M_{i+k}) = 0$ for all i and all $k > 1$, then the two Postikov systems are isomorphic.*

Proof. By the boundedness assumption, any map $\{f_{i,k}\}$ with $f_{i,0} = 1$ for all i will be an isomorphism, so we will begin by taking all $f_{i,0} = 1$ and choose the later $f_{i,k}$ inductively. Suppose that we have found $f_{i,k}$ for all $k < k_0$, each satisfying Equation 3.3. Then Equation 3.3 gives elements $\alpha_i \in \mathrm{CHom}(M_i, M_{i+k_0})$ such that in order to continue we need to find f_{i,k_0} with $Df_{i,k_0} = \alpha_i$. Using Equatios 3.3 and 3.1, we may compute that $D\alpha_i = 0$, and then the assumption that $\mathrm{Hom}_0(M_i, sq^{k_0(\deg w)/2} M_{i+k_0}) = 0$ shows that the desired f_{i,k_0} exist. □

In the next section, we will follow Khovanov and Rozansky's proof that $\mathfrak{sl}(n)$ homology is Rei-

demeister invariant to show that it works equally well for Postnikov systems as for complexes over the homotopy category of matrix factorizations. This will require adapting a few key tools to this setting. First, Postnikov systems admit a notion of fusion just as complexes do:

Definition 3.5. *Suppose that $\{M_i\}$ and $\{M'_i\}$ are two Postnikov systems over potentials w and w' that admit a fusion along a common set of variables $y_1, \dots, y_q$. Then the fusion of $\{M_i\}$ and $\{M'_i\}$ along $y_1, \dots, y_q$ is the Postnikov system with underlying matrix factorizations*

$$(M \otimes M')_i = \bigoplus_{j+j'=i} M_j \otimes M'_{j'}$$

where $M_j \otimes M'_{j'}$ is the correspnding fusion of the underlying matrix factorizations, and differentials given by

$$d^{\otimes}_{i,k}|_{M_j \otimes M'_{j'}} = d_{j,k} \otimes 1 + 1 \otimes d'_{j',k}.$$

We will use an identical graphical calculus for fusions of Postnikov systems as for fusions of matrix factorizations.

Second, we will need to know that certain simplifications (the well-loved "Gaussian elimination" operations) of the associated graded complex to a Postnikov system also correspond to simplifications of the Postnikov system:

Proposition 3.6. *Suppose that, in a Postnikov system with matrix factorizations $\{M_i\}$ and differentials $d_{i,k}$, we have direct sum decompositions $M_n = A \oplus B$ and $M_{n+1} = C \oplus D$ for some fixed n, and that the component of $d_{n,1}$ going from B to D is an isomorphism ϕ. Then this Postnikov system is homotopy equivalent to one with matrix factorizations*

$$\dots M_{n-1}, A, C, M_{n+2}, \dots$$

and differentials $d'_{i,k}$ chosen by the following rules, where g and g' are the components of $d_{n,1}$ going from A to D and B to C, respectively:

- *If neither i nor $i+k$ is equal to n or $n+1$, then simply $d'_{i,k} = d_{i,k}$.*
- *If $i + k = n$, then $d'_{i,k} = \pi_C \circ d_{i,k}$ is the component of $d_{i,k}$ going into C.*
- *If $i = n + 1$, then $d'_{i,k} = d_{i,k} \circ \iota_A$ is the component of $d_{i,k}$ coming out of A.*
- *If $i + k = n + 1$ and $k > 1$, then $d'_{i,k} = \pi_C \circ d_{i,k} - g' \circ \phi^{-1} \circ \pi_D \circ d_{i,k}$.*

- *If $i = n$ and $k > 1$, then $d'_{i,k} = d_{i,k} \circ \iota_A - d_{i,k} \circ \iota_B \circ \phi^{-1} \circ g$.*
- $d'_{n,1} = \pi_C \circ d_{n,1} \circ \iota_A - g' \circ \phi^{-1} \circ g$.

Note that this last entry is the standard Gaussian elimination formula for a chain complex, showing that if one performs a Gaussian elimination on the associated graded complex of a Postnikov system, the higher differentials may all be carried forwards as well. This should be seen as a form of homological perturbation.

Another very relevant form of homological perturbation is that one can always carry forwards the higher differentials of a Postnikov system after replacing the constituent matrix factorizations by homotopy equivalent ones:

Proposition 3.7 (Theorem 5.2 in [9])**.** *Suppose that, for a Postnikov system with bounded sequence of matrix factorizations $\{M_i\}$ and differentials $d_{i,k}$, we have another sequence matrix factorizations $\{M'_i\}$ with each M'_i homotopy equivalent to M_i. Then there exist differentials $d'_{i,k} : M'_i \to M'_{i+k}$ turning the M'_i into a Postnikov system homotopy equivalent to the original one.*

3.2.1 The Postnikov system assigned to a crossing

To construct the Khovanov-Rozansky link homology (as a Postnikov system), we must first assign a Postnikov system to the positive and negative crossings, and then extend to more general tangles via fusion. These crossings will have only two nonzero matrix factorizations, so only the $d_{i,1}$ can be nonzero (higher differentials come in as one takes larger fusions). Therefore, essentially nothing new needs to be done here not already present in [8].

The matrix factorizations involved in the crossing complexes are shown in Figure 3.2.1. To give the differential explicitly, we will use Proposition 2.6. Specifically, the two-arc matrix factorization M_{res} is the Koszul factorization

$$\begin{pmatrix} x_1 - y_1 & * \\ x_2 - y_2 & * \end{pmatrix}$$

, and the double-arc factorization M_{sing} is the Koszul factorization

$$\begin{pmatrix} x_1 + x_2 - y_1 - y_2 & * \\ x_1 x_2 - y_1 y_2 & * \end{pmatrix}$$

. If $x_1 = y_1$ and $x_2 = y_2$, then certainly $x_1 + x_2 = y_1 + y_2$ and $x_1 x_2 = y_1 y_2$, so we may express the entries in the first column of M_{sing} as linear combinations of the entries in the first column of M_{res},

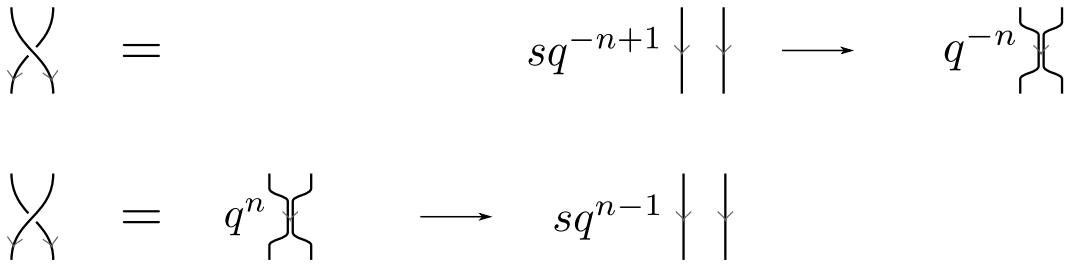


Figure 3.3: The Postnikov systems assigned to a negative or positive crossing. In each case, the center column showing the crossing resolved represented M_0 of the Postnikov system, the column to the right in the first row represents M_1, and the column to the left in the second row represents M_{-1}. All other matrix factorizations in the system are zero.

and Proposition 2.6 then gives the homomorphisms

$$M_{\text{res}} \to q^{-1} M_{\text{sing}}$$

and

$$M_{\text{sing}} \to q^{-1} M_{\text{res}}$$

needed to form $d_{0,1}$ for the negative crossing and $d_{-1,1}$ for the positive crossing, respectively.

At this point, we can already state the definition of the Postnikov system associated to any tangle diagram: simply decompose it into crossings glued along arcs, begin with the Postnikov system just described for each crossing, and then perform fusions for each arc. We will see in the next section that, using a minimal addition to Khovanov and Rozansky's proof, this is in fact an invariant of the tangle up to homotopy equivalence.

Later, when we calculate the E_∞ page of the $E(-1)$ spectral sequence, we will need to know the convolutions of these crossing complexes. These convolutions are calculated by Khovanov and Rozansky in Appendix A of [9], essentially in Theorem A.5, and for both the positive and negative crossing the result is the Koszul factorization

$$\begin{pmatrix} x_1 - y_2 & * \\ x_2 - y_1 & * \end{pmatrix},$$

which may be understood as the factorization associated to a virtual crossing of two arcs, just as

$$\begin{pmatrix} x_1 - y_1 & * \\ x_2 - y_2 & * \end{pmatrix}$$

is associated to two parallel arcs. This fact is essential to many of our future arguments, but we record here one simple consequence:

Proposition 3.8. *If C_+ and C_- are the Postnikov systems associated to the positive and negative crossings, then there is a homomorphism of Posnikov systems $C_+ \to C_-$ inducing a homotopy equivalence on the convolutions, and a similar homomorphism $C_- \to t^2 C_+$ (the result of shifting the indexing on the constituent matrix factorizations of C_+ forwards by 2).*

Proof. Thinking of C_+ and C_- as filtered matrix factorizations, and the convolutions as the result of forgetting the filtration, what needs to be checked here is that the two directions of the homotopy equivalence between the convolutions (coming from the fact that both are isomorphic to the factorization associated to a virtual crossing) give filtered maps $C_+ \to C_-$ and $C_- \to t^2 C_+$. Since $(C_+)_i = 0$ for $i > 0$ and $(C_-)_i = 0$ for $i < 0$, and similarly $(C_-)_i = 0$ for $i > 1$ and $(t^2 C_+)_i = 0$ for $i < 1$, however, this is immediate: there is no room for a nonzero component of these maps to go against the filtration. □

3.3 Reidemeister invariance

Khovanov and Rozansky's proof the their $\mathfrak{sl}(n)$ link homology is invariant under each of the Reidemeister moves proceeds in three steps:

1. Write out the complex of matrix factorizations correspnding to each side of the desired Reidemeister move. Calculate the resulting fusions, to express each matrix factorization involved as a direct sum of Koszul factorizations.

2. Simplify the complexes involved by a sequence of Gaussian eliminations. At the end of this step, the complexes on both sides of the move will have the same underlying matrix factorizations.

3. Verify that the differentials in the two complexes are equal, perhaps up to an isomorphism rescaling the different parts.

The last step is only required in the third Reidemeister move, since in the other two cases the complexes remaining after the second step are concentrated in a single homological degree.

Each of these steps may still be carried out using Postnikov systems instead of complexes over the homotopy category of matrix factorizations. The first two require no additional checking, since replacing the constituent matrix factorizations by direct sum decompositions homotopy equivalent

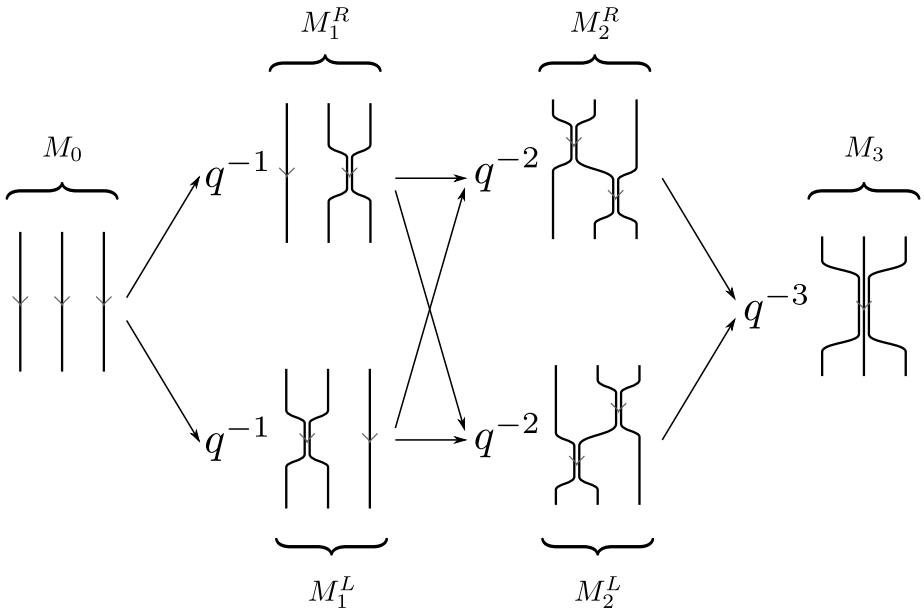


Figure 3.4: The complex to which both sides of the third Reidemeister move simplify, with names for the matrix factorizations involved. We omit an overall grading shift of s^3q^{-3n+3} to reduce clutter.

to the original can be accomplishied via Proposition 3.7, and the simplifying Gaussian eliminations may be performed via Proposition 3.6.

For the last step needed in the third Reidemeister move, Khovanov and Rozansky's arguments show that, after appropriately rescaling the isomorphisms involved, both sides of the third Reidemeister move simplify to a complex as shown in Figure 3.3, with differentials agreeing in the homotopy category of matrix factorizations. To show that the two sides of the third Reidemeister move in fact agree on the level of Postnikov systems, we will use Lemma 3.4: we have two Postnikov systems, with the same underlying matrix factorizations and homotopic first differentials, so the lemma says that they will be isomorphic provided that certain Hom spaces between the constituents vanish. Specifically, to prove that our assignment of Postnikov systems to tangles assigns homotopy equivalent Posntikov systems to isotopic tangles, we need

Lemma 3.9. *The following* Hom *spaces vanish:*

$$\mathrm{Hom}_0(M_0, sq^{2n}M_2^R) = 0 \tag{3.2}$$

$$\mathrm{Hom}_0(M_0, sq^{2n}M_2^L) = 0 \tag{3.3}$$

$$\mathrm{Hom}_0(M_1^R, sq^{2n}M_3) = 0 \tag{3.4}$$

$$\mathrm{Hom}_0(M_1^L, sq^{2n}M_3) = 0 \tag{3.5}$$

$$\mathrm{Hom}_0(M_0, sq^{3n}M_2^R) = 0 \tag{3.6}$$

$$\mathrm{Hom}_0(M_0, sq^{3n}M_2^L) = 0 \tag{3.7}$$

$$\tag{3.8}$$

Proof. By symmetry, only the first of each of these three pairs must be proven. In each diagram, let us label the incoming strands x_1, x_2, and x_3, and the outgoing strands y_1, y_2, and y_3. Then the four matrix factorizations involved are the Koszul factorizations

$$M_0 = \begin{pmatrix} x_1 - y_1 & * \\ x_2 - y_2 & * \\ x_3 - y_3 & * \end{pmatrix},$$

$$M_1^R = \begin{pmatrix} x_1 - y_1 & * \\ e_1(x_2, x_3) - e_1(y_2, y_3) & * \\ e_2(x_2, x_3) - e_2(y_2, y_3) & * \end{pmatrix},$$

$$M_2^R = \begin{pmatrix} e_1(x_1, x_2, x_3) - e_1(y_1, y_2, y_3) & * \\ e_2(x_1, x_2) - e_2(y_1, y_2 + y_3 - x_3) & * \\ e_2(x_1 + x_2 - y_1, x_3) - e_2(y_2, y_3) & * \end{pmatrix},$$

and

$$M_3 = \begin{pmatrix} e_1(x_1, x_2, x_3) - e_1(y_1, y_2, y_3) & * \\ e_2(x_1, x_2, x_3) - e_2(y_1, y_2, y_3) & * \\ e_3(x_1, x_2, x_3) - e_3(y_1, y_2, y_3) & * \end{pmatrix},$$

after excluding the internal variable in the case of M_2^R. By the proof of Corollary 2.8, the entries of any one of these Koszul factorizations form a regular sequence, so the quotient of $R = \mathbb{Q}[x_1, x_2, x_3, y_1, y_2, y_3]$ by the entries of any one of these factorizations will be finite dimesional. Let I_0, I_1^R, I_2^R, and I_3 be the ideals generated by the entries of the left column in each of these Koszul

factorizations, and note that $I_0 \supset I_1^R \supset I_2^R \supset I_3$. Therefore, for any $i \leq j$, the quotient of R by the entries of the left column of $M_i^{(R)}$ and the right column of $M_j^{(R)}$ will be finite dimensional (R in superscript where required by the notation), so these polynomials form a regular sequence as well. This is the last hypothesis needed to apply Lemma 3.4, so (after counting the degrees of the various entries), we compute that

$$\mathrm{Hom}(M_i^{(R)}, M_j^{(R)})$$

is generated over the ring R by a map with parity and degree both equal to $|i-j|$. In particular,

$$\mathrm{Hom}_0(M_i^{(R)}, s^a q^b M_j^{(R)})$$

can be nonzero only when $a \equiv i - j \pmod 2$ and $b \leq -|i-j|$, covering all required cases. □

3.4 Reduced knot homology and the $E(-1)$ spectral sequence

At this point, we have assigned to every tangle a Postnikov system of matrix factorizations. When the tangle is in fact a link, the potential of the factorizations will be zero, so we in fact have a Postnikov system of chain complexes. A Postnikov system of chain complexes is essentially the same as a filtered chain complex: it consists of a sequence of vector spaces $\{C_i\}$, a differential $d_{i,0}$ on each C_i, and maps from the definition of a Postnikov system $d_{i,k} : C_i \to C_{i+k}$, and the identities required of a Postnikov system simply say that the total endomorphism $D = \sum_{i,k} d_{i,k}$ of $\oplus_i C_i$ satisfies $D^2 = 0$. If we think of $\oplus_i C_i$ as filterd by the subspaces $\oplus_{i>n} C_i$, then D preserves this filtration and similarly a homomorphism of Postnikov systems as we have defined it is equivalent to a filtered chain map. However, a homotopy between maps of Postnikov systems is not equivalent to a filtered homotopy: our defionition allows homotopies to reach one step back in the filtration, and this is necessary in order to have Reidemeister invariance. The following lemma, whose proof is not substantially different than the standard fact that the spectral sequence of a filtered complex is filtered homotopy invariant, shows that we do not lose much in allowing this:

Lemma 3.10. *Suppose that C and C' are filtered chain complex, $f : C \to C'$ and $g : C' \to C$ are filtered chain maps, and $f \circ g$ and $g \circ f$ are both homotopic to the identity via homotopies that send the k^{th} filtered piece of C or C' into the $k - i^{th}$. Then for $r > i$, f and g induce mutually inverse isomorphisms on the E^r page of the spectral sequences associated to C and C'.*

Since, for every link, we have a filtered chain complex, we get a spectral sequence assigned to every

link, an $\mathfrak{sl}(N)$ analogue of Rasmussen's $E(-1)$ spectral sequence. We will denote the pages of this sequence $E_k(-1)(L,N)$. Tracing through the definitions, $E_1(-1)(L,N)$ consists of the homologies of each individual matrix factorization in the Postnikov system. This is simply the direct sum of the vector spaces assigned to MOY graphs in the Khovanov-Rozansky construction. To get the the second page, we take homology with respect to the map induced by the $d_{i,1}$, which is exactly the Khovanov-Rozansky differential. Therefore, $E_2(-1)(L,N)$ is simply the Khovanov-Rozansky $\mathfrak{sl}(N)$ homology of N, with the original $\mathbb{Z}$-grading playing the role of the quantum grading and the grading induced by the filtration as the homological grading.

Finally, to calculate the $E_\infty(-1)(L,N)$ page, we simply ignore the filtration and compute the homology of the chain complex $\oplus_i C_i$ with differential D, or equivalently the convolution of the original Postnikov system. Since the convolution of the Postnikov system assigned to a crossing is simply a virtual crossing, this will be the homology of a fusion of virtural crossings patterned after the crossings of L, which is simply the Khovanov-Rozansky $\mathfrak{sl}(N)$ homology of an unlink with the same number of components as L.

However, to extract concordance homomorphisms from this package, we would prefer to have the E_∞ page one dimensional for a knot. This will be accomplished by working instead with a reduced version.

From a diagram of a knot K, break the knot open at a chosen basepoint to produce a tangle with one incoming and one outgoing boundary component, and consider the Postnikov system assigned to this tangle. This will be a Postnikov system of matrix factorizations of the potential $x^{N+1}-y^{N+1}$. However, not all matrix factorizations of this potential occur as constituents in this Postnikov system - only direct sums of grading shifts of the one-arc factorization M. This follows from iterating the same direct-sum decompositions (Propositions 29,30,31, and 32 in [8]) used in the proof of invariance and to show that the Khovanov-Rozansky homology actually categorifies the $\mathfrak{sl}(N)$ specialization of the HOMFLY-PT polynomial.

Therefore, instead of thinking of the Postnikov system assigned to this tangle as taking values in the dg-category of matrix factorizations of $x^{N+1}-y^{N+1}$, we may think of it as taking values in the full subcategory consisting of direct sums of grading shifts of M. Informally, the reduced homology of a knot will come from counting the number of copies of M occuring in each grading, replacing each M summand with a $\mathbb{Q}$ summand and letting the new differential track only identity maps between copies of M in the old differential. The chain complex $\mathrm{CHom}(M,M)$ is the dg-algebra with two commuting generators x,ζ, with x even and in degree 2, ζ odd and in degree $N-1$, and differentials $dx=0$ and $d\zeta=x^N$. There is a quotient dg-homomorphism from this algebra to $\mathbb{Q}$

sending x and ζ to zero, and to construct the reduced chain complex we will push forwards our Postnikov system along this homomorphism: the differentials in the original system were matrices with entries in $\mathrm{CHom}(M, M)$, and we replace these matrices with their images under the quotient map. The result is a $\mathbb{Z}/2 \oplus \mathbb{Z} \oplus \mathbb{Z}$-graded chain complex, with terms in the differential having degree

$$st, st^2q^{-n-1}, st^3q^{-2n-2}, \ldots, st^kq-(k-1)(n+1), \ldots,$$

where s and q are used as before and t refers to the new $\mathbb{Z}$-grading (coming from the indexing of the Postnikov system). However, the identities used to simplify each of the matrix factorizations involved here into copies of M imply that for each term in this reduced complex, the s degree is equal to the parity of the t degree. This has two consequences: first, we may ignore the s degree when working with this complex, using instead the parity of the t degree, and second that the terms in the differential of degree $st^kq-(k-1)(n+1)$ with k even must vanish, leaving only terms in the differential of degree $st^{2k+1}q^{-2k(n+1)}$ for $k \geq 0$.

The spectral sequence associated to this filtered complex is the reduced $E(-1)$ spectral sequence of K. To be specific, let us call the filtered chain complex constructed in the previous paragraph $\overline{\mathrm{CKR}}^{-1}(K, N)$, and the E_r page of the associated spectral sequence $\overline{E_r(-1)}(K, N)$. Then $\overline{E_2(-1)}(K, N)$ is simply the reduced Khovanov-Rozansky $\mathfrak{sl}(N)$ homology of K [1], and the page $\overline{E_\infty(-1)}(K, N)$ will be one dimensional (because the convolution of the original Postnikov system associated to the cut-open tangle will be a single copy of M, which becomes a single copy of $\mathbb{Q}$ upon going to reduced homology).

We may now define the invariants that are the main goal of this thesis:

Definition 3.11. *$t_N(K)$ is the t-grading of the surviving generator of $\overline{E_\infty(-1)}(K, N)$.*

We need only specify the t-grading of the generator, because this determines the q-grading: all of the terms in the differential of $\overline{\mathrm{CKR}}^{-1}(K, N)$ have the same $\mathbb{Z}/2$ degree, and change the linear combination

$$(n+1) \cdot (t \text{ degree}) + (q \text{ degree})$$

by $-n-1$. This means that the surviving generator on E_∞ must have even $\mathbb{Z}/2$ degree and this linear combination must vanish, giving us

[1] as defined e.g. at the end of section 7 of [8] - we arrive at the reduced homology via a slightly different route that makes it easier to see that the higher differentials may be carried along as well, but the effect on the E_2 page is the same. To verify this, we need only check that applying their constructions to the arc matrix factorization M - closing the arc to a circle and then modding out on the E_1 page by the variable x - produce a single copy of $\mathbb{Q}$.

Proposition 3.12. *$t_N(K)$ is an even integer, and the reduced $\mathfrak{sl}(N)$ homology of K is nonzero in degree $q^{-(n+1)t_N(K)}t^{t_N(K)}$.*

Since the reduced complex is defined in terms of the cut-open tangle from a knot, it behaves very well with respect to connected sums:

Proposition 3.13. *The reduced complex $\overline{\mathrm{CKR}}^{-1}(K\#K', N)$ is isomorphic to the tensor product $\overline{\mathrm{CKR}}^{-1}(K, N) \otimes \overline{\mathrm{CKR}}^{-1}(K', N)$ of the constituent knots.*

Proof. The cut-open tangle of $K\#K'$ is obtained by gluing one outgoing endpoint of the cut-open tangle of K to an incoming endpoint of the cut-open tangle of K', so the Postnikov system of matrix factorizations for $K\#K'$ is the fusion of those for K and K'. Since the fusion of the one-arc factorization M with itself is isomorphic to M, and the map $\mathrm{CHom}(M, M) \to \mathrm{CHom}(M\otimes M, M\otimes M)$ coming from tensoring a chain map with the identity on either the left or the right is an isomorphism, this fusion becomes a tensor product (over $\mathbb{Q}$) after passing to reduced homology. □

Corollary 3.14. $t_N(K\#K') \cong t_N(K) + t_N(K')$

At this point, only one part of Theorem 1.1 remains to be showsn: that each t_N is a concordance invariant. This is done via the TQFT structure on matrix factorizations.

Proposition 3.15. *If K is concordant to K', then $t_N(K) = t_N(K')$.*

Proof. Exhibit a concordance between K and K' as a sequence of tangle diagrams $D_1, \ldots, D_k$, with D_1 a diagram for the cut-open tangle of K, D_k a diagram for the cut-open tangle of K', and two successive diagrams differing by a Reidemeister move, birth or death of a circle, or saddle move. Writing $M(D_i)$ for the Postnikov system of matrix factorizations assigned to D_i, in each case we have a map $M(D_i) \to M(D_{i+1})$: for a Reidemeister move this is the isomorphism from the invariance proof, and for a cobordism step we may identify small discs in D_i and D_{i+1} outside of which they are identical and inside of which carry out the birth/death or saddle. Since matrix factorizations form a projective TQFT, we have a homomorphism between the matrix factorizations on either side of portion inside this disc, and tensoring with the identity on the Postnikov system outisde gives a map $M(D_i) \to M(D_{i+1})$. The convolutions of $M(D_i)$ and $M(D_{i+1})$ correspond to virtualizing all crossings, so are isomorphic to the factorizations assigned to an arc and an appropriate number of circles. On these convolutions, the map $M(D_i) \to M(D_{i+1})$ is simply the corresponding TQFT map. Therefore, the composite map $M(D_1) \to M(D_k)$ acts on the convolutions of these Postnikov systems as the TQFT map assigned to the concordance annulus, which is an isomorphism. Passing

to the reduced complexes, we have a filtered chain map $\overline{\mathrm{CKR}}^{-1}(K,N) \to \overline{\mathrm{CKR}}^{-1}(K',N)$ inducing an isomorphism on the overall homology, and a similar map going the other direction. Together these imply that $E_\infty(-1)(K,N) \cong E_\infty(-1)(K',N)$, so $t_N(K) = t_N(K')$. $\square$

Chapter 4

Bounding topological quantities with t_N

4.1 Reformulating the definition and first bounds

Thinking of the quantity $t_N(K)$ in terms of the E_∞ page of the $E(-1)$ spectral sequence is most helpful for calculations - indeed, Proposition 3.12 directly comes from this interpretation and is the basis of almost all of the cases in which t_N can be calculated easily. However, for inequalities, the additional flexibility of working directly with the filtered complex $\overline{\mathrm{CKR}}^{-1}(K,N)$ is essential. The page $E_\infty(-1)(K,N)$ is, by the definition of the spectral sequence of a filtered complex, the associated graded vector space to the overall homology of $\overline{\mathrm{CKR}}^{-1}(K,N)$ with its induced filtration. Therefore, the following is an equivalent defition of $t_N(K)$:

Definition 4.1. *$t_N(K)$ is the largest integer j for which the complex $\overline{\mathrm{CKR}}^{-1}(K,N)$ admits a cycle representing a nonzero homology class supported in the subcomplex of $\overline{\mathrm{CKR}}^{-1}(K,N)$ formed by the generators in t-grading at least j.*

This formulation immediately gives the following inequality:

Lemma 4.2. *Suppose that there is a filtered chain map $\overline{\mathrm{CKR}}^{-1}(K,N) \to \overline{\mathrm{CKR}}^{-1}(K',N)$ inducing an isomorphism on the overall homology. Then $t_N(K) \leq t_N(K')$.*

This will be the basis for all of the bounds in this chapter. For example, combining it with Proposition 3.8 controls how t_N may change under a crossing change in a knot:

Theorem 4.3. *Suppose K_+ and K_- differ only by a crossing change in which K_+ has a positive crossing and K_- a negative one. Then*

$$t_N(K_+) \leq t_N(K_-) \leq t_N(K_+) + 2.$$

Proof. The Postnikov systems associated to cut-open tangles for K_+ and K_- may be expressed as fusions of a common outer portion with the Postnikov systems associated to a positive or negative crossing. Tensoring the identity on the outer portion with the maps from Proposition 3.8 and then passing to the reduced complex gives filtered chain maps

$$\overline{\mathrm{CKR}}^{-1}(K_+, N) \to \overline{\mathrm{CKR}}^{-1}(K_-, N) \to t^2\overline{\mathrm{CKR}}^{-1}(K_+, N),$$

each inducing an isomorphism on homology. The conclusion now follows from Lemma 4.2. □

4.2 Genus bounds

The definition of the invariants $t_N(K)$ is in outline very similar to Rasmussen's $s(K)$ and many similar concordance invariants from knot homologies: one adds higher differentials of some kind to the original complex computing the link homology resulting in a one dimensional homology, and extracts the grading of a surviving generator. For both our constructions and the ones leading to s, these higher differentials are functorial under the cobordism maps for the link homology. For the Lee spectral sequence used in constructing s, one then has that these cobordism maps induce isomorphisms on the overall homology of the filtered complex for surfaces of any genus, leading to the proof that $s(K)$ bounds the slice genus. However, that strategy fails in our context: on the overall homology of our filtered complexes, the cobordism maps agree with those coming from the TQFT for the Frobenius algebra $\mathbb{Q}[x]/x^N$. In the context most relevant to $t_N(K)$, reduced homology of a knot, this means that the cobordism maps vanish as soon as the genus is positive, and there is no way to combine these maps with Lemma 4.2 to get a genus bound.

Luckily, another strategy may be used, adapting arguments I used in [2] in the context of Khovanov homology. First, since $t_N(K)$ is already known to be a concordance invariant, we may restrict attention to ribbon surfaces:

Lemma 4.4. *Suppose K bounds a smooth connected surface in B^4 of genus g. Then K is concordant to a knot which bounds a ribbon surface of genus g.*

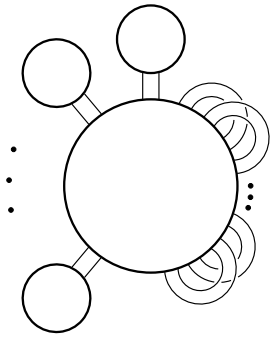


Figure 4.1: The combinatorial template for a ribbon surface.

Proof. Factor the cobordism from K to the empty link into births, saddles, and then deaths of circles. Since our surface is connected, every born circle must be connected back to K by a fusion band. Choose one such fusion band for each born circle, and reorder the saddles so that these come before all others. Let K' be the knot appearing after all births and these chosen fusion bands. Then the first portion of the cobordism is a concordance from K to K', and the remainder is a ribbon surface of genus g bounding K'. □

Therefore, we may assume that a knot K bounds a ribbon surface of genus g, and our goal is to prove that $|t_N(K)| \leq 2g$. After sliding handles, we may assume that combinatorially our surface looks like the one in Figure 4.2, containing $k+1$ discs, k fusion bands connecting peripheral discs to a central one, and $2g$ bands connected to the central disc. Let K_n be the knot obtained from K by insterting n positive full twists into each of the bands of this surface. Adding a positive full twist to a band (consisting of two strands with opposite orientation) may be accomplished by changing a crossing from positive to negative, so for each $n \geq 0$ we have $t_N(K) \leq t_N(K_n)$ by Theorem 4.3. Therefore, to complete the proof of Theorem 1.2, we need only show that

Lemma 4.5. *For sufficiently large n, $t_N(K_n) \leq 2g(N-1)$.*

This in turn will follow purely from a computation of the $\mathfrak{sl}(N)$ homology, combined with Proposition 3.12:

Lemma 4.6. *For sufficiently large n, the reduced $\mathfrak{sl}(N)$ homology of K_n is nonzero in degree $q^{-(N+1)i}t^i$ only if $i \leq 2g(N-1)$.*

Proof. We rely on a calculation due to Krasner [11] of the Khovanov-Rozansky homology of the tangle consisting of n postive full twists on a pair of oppositely oriented strands, as a four-ended

tangle. He shows that this complex is

$$q^{1-N}M_{\text{cut}} \to q^{-1-N}tM_{\text{cut}} \to q^{1-3N}t^2M_{\text{cut}} \to q^{-1-3N}t^3M_{\text{cut}} \to \cdots$$
$$\to q^{1-(2n-1)N}t^{2n-2}M_{\text{cut}} \to q^{-1-(2n-1)N}t^{2n-1}M_{\text{cut}} \to q^{2nN}t^{2n}M_{\text{band}},$$

where M_{band} is the matrix factorization of the untwisted band, M_{cut} is the matrix factorization of the resolution cutting the band (or orientedly resolving any crossing in the twist), and the differentials repeat with period 2 and are independent of n in the initial part of the complex. Therefore, the $\mathfrak{sl}(N)$ homology of K_n may be computed by taking a fusion of a copy of this complex for each band with a fixed outer complex representing the portions of the diagram outside of the bands. This in turn represents the $\mathfrak{sl}(N)$ homology of K_n as the homology of a cube-of-resolutions chain complex out of the $\mathfrak{sl}(N)$ homologies of K_0 and the links obtained by cutting some of the bands in the ribbon surface. However, terms in this complex for which fewer than all of the bands are cut always pick up a grading shift of $q^{2nN}t^{2n}$ for each uncut band, and for sufficiently large n this pushes all of their contribution to the homology off of the line $q^{-(N+1)i}t^i$. After all of the bands are cut, the way in which they are knotted is no longer relevant, so this shows that the $\mathfrak{sl}(N)$ homology of K_n on the line $q^{-(N+1)i}t^i$ for sufficiently large n depends only on the combinatorics of the bands in our ribbon surface. In particular, we may assume that our ribbon surface is exactly the one shown in Figure 4.2, for which we may calculate that for large n the Poincare series of the reduced $\mathfrak{sl}(N)$ homology of K_n approaches the limit

$$(q^{1-N} + q^{-1-N}t + q^{1-3N}t^2 + q^{-1-3N}t^3 + \cdots)^{2g}.$$

On the line $q^{-(N+1)i}t^i$, this contains terms in homological degree $2g + (N-2)k$ for k between 0 and $2g$, so is supported in homological degree at most $2g(N-1)$. □

It seems likely that considering the HOMFLY homology of the K_n and the spectral sequences [19] from HOMFLY homology to the $\mathfrak{sl}(N)$ homologies in the above proof would sharpen the bound to $t_N(K) \leq 2g_4(K)$: since, for a wide range of gradings, we know all of the $\mathfrak{sl}(N)$ homologies of K_n for large n, their HOMFLY homologies in a similar range may be computed using only the existence of a spectral sequence to the $\mathfrak{sl}(N)$ homologies. In the end one finds that in small homological degree, the spectral sequences from HOMFLY to $\mathfrak{sl}(N)$ for the K_n collapse immediately. Of the terms on the line $q^{-(N+1)i}t^i$, only the one in homological degree $2g$ comes from a term in the HOMFLY homology with the correct a degree to be a surviving generator of the $E(-1)$ spectral sequence on

HOMFLY homology, so better understanding the relationship between the $E(-1)$ spectral sequences on the HOMFLY and $\mathfrak{sl}(N)$ homolgies would likely show that only this term may survive the $E(-1)$ spectral sequence in this context.

4.3 Nonorientable genus bounds

While for $N > 2$ the $\mathfrak{sl}(N)$ homology strongly depends on link orientation, because it is needed to decide which resolutions of a crossing appear in the complex, for $N = 2$ the dependence is less essential. This is because the double-edge matrix factorization winds up looking like an unoriented resolution: the double-edge is the Koszul factorization

$$\begin{pmatrix} x_1 + x_2 - y_1 - y_2 & * \\ x_1x_2 - y_1y_2 & 3(x_1 + x_2) \end{pmatrix} \cong \begin{pmatrix} y_1 + y_2 & * \\ \frac{1}{3}(x_1x_2 - y_1y_2) & x_1 + x_2 \end{pmatrix} \cong \begin{pmatrix} x_1 + x_2 & * \\ y_1 + y_2 & * \end{pmatrix}$$

is a tensor product of two horizontal arc factorizations with the signs on x_2 and y_2 changed (corresponding to the changed orientations at those endpoints). This means that reorienting one of the strands in the crossing only changes the assigned complex by an overall grading shift: the crossing changes between negative and positive, but of the two resolutions of the crossing a different one is the oriented one, and these two changes mean that the two components in the crossing complex stay in the same order when the knot is reoriented.

Given this version of $\overline{\mathrm{CKR}}^{-1}(K, 2)$ that is insensitive to local orientations in the knot diagram, a somewhat surprisng idea results in bounds on the nonorientable four-genus of a knot. Since the entries of the two Koszul factorizations

$$M_0 = \begin{pmatrix} x_1 + x_2 & * \\ y_1 + y_2 & * \end{pmatrix}$$

and

$$M_1 \begin{pmatrix} x_1 - y_1 & * \\ x_2 - y_2 & * \end{pmatrix}$$

are equal modulo $x_2 + y_1$, there is an isomorphism ϕ between their underlying modules such that, as an element of $\mathrm{CHom}(M_0, M_1)$, $d\phi = (x_2 + y_1)s$ for some homomorphism $s : M_0 \to M_1$. (In fact, s is the saddle map, but this fact will not be needed.) Now, if these two tangles are situated in a knot diagram D in such a way that the resulting band move does not respect orientation, the exterior of D

must consist of a strand connecting the endpoint labelled x_1 to the endpoint labelled y_2 and a strand connecting the endpoint labelled x_2 to the endpoint labelled y_1. If we take reduced homology with a basepoint on this second strand, then $x_2 + y_1$ will be nullhomotopic via "dot-sliding" homotopies, and this homotopy may be used to modify ϕ into a genuine homomorphism. Accounting for the various grading shifts involved in this process, we have

Proposition 4.7 (Proposition 4.7 in [1])**.** *Suppose D and D' are diagrams of knots K and K' differing by a nonorientable band move. Then there is a chain map $\overline{\mathrm{CKR}}^{-1}(K,2) \to t^{\frac{1}{2}(w(D')-w(D))+1}\overline{\mathrm{CKR}}^{-1}(K',2)$, where $w(D)$ and $w(D')$ are the writhes of D and D', respectively*

Since any nonorientable cobordism factors into concordances and a series of nonorientable band moves, Theorem 1.4 follows immediately. There, the normal euler number $e(\Sigma)$ tracks the change in write through successive band moves.

4.4 Bounds in $\#^k\mathbb{C}P^2$

While the Khovanov homology of torus knots has not been fully calculated, some information about it is known and this is enough to enable certain calculations. For example, Rasmussen's calculation of $s(T_{p,q})$ relies on the fact that, in homological degree 0, the reduced Khovanov homology of $T_{p,q}$ is one dimensional. For some torus knots, we are similarly lucky in the present situation and can calculate $t_N(T_{p,q})$ by calculating the $\mathfrak{sl}(N)$ homology along the relevant line of gradings. However, this is still not possible in full generality. Instead, we start from the following piece of information:

Proposition 4.8 (Theorem 1 and Remark 7 from [20])**.** *The Khovanov homology of the negative torus link $\overline{T}_{n,n}$ is supported in homological degree at most $\left\lfloor \frac{n^2}{2} \right\rfloor$.*

Reorienting a component of a link shifts homological gradings in its Khovanov homology twice the linking number between the component and the rest of the link, so we in fact have

Lemma 4.9. *If $L_{a,b}$ is the link formed by reversing the orientation of b strands of $\overline{T}_{a+b,a+b}$, then the Khovanov homology of $L_{a,b}$ is supported in homological degree at most $\left\lfloor \frac{(a-b)^2}{2} \right\rfloor$.*

Proof. The linking number between the b reversed strands and the a maintained ones is $-ab$, so the homology shifts downwards by $2ab$ and the new maximum degree is

$$\left\lfloor \frac{(a+b)^2}{2} \right\rfloor - 2ab = \left\lfloor \frac{(a+b)^2 - 4ab}{2} \right\rfloor - \left\lfloor \frac{(a-b)^2}{2} \right\rfloor$$

□

Note that the orientation conventions used above mirror those used in Stošić, a consequence of a reversal of sign conventions on the homological grading between the original Khovanov homology and $\mathfrak{sl}(N)$ homology.

The seeming coincidence that this depends only on $a - b$ allows us to use this bound to control discs in $\#^k\mathbb{C}P^2$, and later other surfaces, based only on their homology class and boundary:

Theorem 4.10. *Suppose that $\Sigma \subset (\#^k\mathbb{C}P^2) - B^4$ is a properly embedded disc with boundary $K \subset S^3$, and let a_i be the algebraic intersection number between Σ and the complex line Σ_i in the i^{th} copy of $\mathbb{C}P^2$. Then*

$$t_2(K) \leq \sum_i \left\lfloor \frac{a_i^2}{2} \right\rfloor$$

.

Proof. Let $n_i^{\uparrow}$ be the number of positive intersection points between Σ and Σ_i, and $n_i^{\downarrow}$ the number of negative intersection points. For $1 \leq i < n$, let γ_i be an arc connecting Σ_i and Σ_{i+1} disjoint from the other Σ_j and from Σ; then the complement of tubular neighborhoods of the Σ_i and γ_i is diffeomorphic to $S^3 \times [0,1]$, the intersection Σ° of Σ with $S^3 \times 0$ is the mirror of $L = \sqcup_i L_{n_i^{\uparrow}, n_i^{\downarrow}}$, and the intersection of Σ with $S^3 \times 1$ is K. In other words, Σ° gives a cobordism of genus 0 from L to K. Taking the reflection of this cobordism, we have a cobordism from K to L inducing a cobordism map $f_{\Sigma^\circ} : \overline{\mathrm{CKR}}^{-1}(K,2) \to \overline{\mathrm{CKR}}^{-1}(L,2)$ (picking any component of L at which to reduce). On the overall homology, this corresponds to a split map for the Frobenius algebra $\mathbb{Q}[x]/x^2$, and in particular is injective. In particular, a class generating the homology of $\overline{\mathrm{CKR}}^{-1}(K,2)$ and supported in homological degree at least $t_2(K)$ is sent to a nonzero class in the homology of $\overline{\mathrm{CKR}}^{-1}(L,2)$, which must come from a nonzero class in the Khovanov homology of L. This is supported in homological degree at most the value in the statement by Lemma 4.9, but homological degree at least $t_2(K)$. □

A similar statment to this is known for the s invariant [16], but only in the case of a nullhomologous surface. Generalizing this statement to the other t_N seems possible, requiring only an analogue of Lemma 4.9.

To complete the proof of Theorem 1.5, all that is needed is to factor a surface into a disc and a cobordism and apply Theorem 1.4 to the cobordism:

Proof of Theorem 1.5. As in the previous proof, by puncturing Σ where it meets the complex lines we get a cobordism from a union of torus links to K. This cobordism may be factored into births, saddles, and then deaths of circles, and since Σ is connected we may arrange to have the first saddle moves be fusion bands connecting all components of the original torus links and all newborn circles

into a single knot K'. Then K' bounds a disc $D \subset \#^k \mathbb{C}P^2 - B^4$, and $K\#\overline{K'}$ bounds a surface $\Sigma' \subset B^4$.

Let a_i be the intersection numbers of D with complex lines, so that Theorem 4.10 tells us that

$$t_2(K') \leq \sum_i \left\lfloor \frac{a_i^2}{2} \right\rfloor.$$

Then $b_1(\Sigma') = b_1(\Sigma)$, and since e is a nonorientable analogue of the self intersection number

$$e(\Sigma') = e(\Sigma) - \sum_i a_i^2.$$

Therefore,

$$\begin{aligned} t_2(K) &\leq t_2(K') + b_1(\Sigma) + \frac{1}{2}\left(e(\Sigma) - \sum_i a_i^2\right) \\ &\leq b_1(\Sigma) + \frac{1}{2}e(\Sigma) + \sum_i \left\lfloor \frac{a_i^2}{2} \right\rfloor - \frac{a_i^2}{2} \\ &= b_1(\Sigma) + \frac{1}{2}e(\Sigma) - \frac{\ell}{2} \end{aligned}$$

□

Our final goal is to combine all of what we have shown to compute $t_2(T_{p,q})$. Since $\overline{T_{p,q}}\#T_{p-q,q}$ bounds a disc in $\mathbb{C}P^2$ intersecting the complex line in q points, Theorem 4.10 says that

$$t_2(T_{p,q}) \geq -\left\lfloor \frac{q^2}{2} \right\rfloor + t_2(T_{p-q,q}),$$

one direction of Theorem 1.3.

For the other direction, we will use a particularly nice family of nonorientable cobordisms between torus knots: If we perform a saddle move on a torus knot $T_{p,q}$ along a band embedded in the surface on which $T_{p,q}$ is drawn, the result will still be drawn on the torus so will be another torus knot $T_{r,s}$. In general, there are many such bands, but only one which decreases the parameters of the torus knot. In that situation, we will say that $T_{r,s}$ is obtained from $T_{p,q}$ by a pinch move, a concept explored in more depth in [7]. A more diagrammatic perspective is that the pinch move from $T_{p,q}$ comes from taking the unoriented resolution of some crossing in the standard braid diagram.

Each pinch move is a cobordism with $b_1 = 1$ from $T_{p,q}$ to $T_{r,s}$ for some $0 < r < p$ and $0 < s < q$, but neither the values r, s nor the normal Euler number of the cobordism are entirely straightforward

to compute. However, they all change in a predictable way under a full twist:

Lemma 4.11. *If the pinch move from $T_{p,q}$ is a cobordism to $T_{r,s}$ with normal Euler number e, then the pinch move on $T_{p,q+p}$ is a cobordism to $T_{r,s+r}$ with normal Euler number $e - p^2 + r^2$.*

Proof. Since all crossings in the original diagram of $T_{p,q}$ are positive, the normal Euler number of the pinch move is equal to -2 times the number of negative crossings in the diagram immediately after the pinch move. The rectangle in the diagram of $T_{p,q}$ in which the twisting will be done initially contains p parallel coherently oriented strands, and after the pinch move $k = (p - r)/2$ strands will be oriented the other way. When these parallel strands are replaced by a full twist to form $T_{p,q+p}$, before the pinch move there will be no negative crossings but after the pinch move there will be a negative crossing wherever one of the reversed strands crosses one of the unreversed strands. Each pair of strands crosses twice in a full twist, so the number of new negative crossings created is therefore

$$2k(p-k) = 2 \cdot \frac{1}{2}(p-r) \cdot \frac{1}{2}(p+r) = \frac{1}{2}(p^2 - r^2)$$

and the change in normal Euler number is $-p^2 + r^2$. □

This fact together with the full twist inequality gives an inductive proof of the fact that the two bounds coincide:

Proposition 4.12. *For any $p, q > 0$ such that $T_{p,q}$ reduces to $T_{r,s}$ by a pinch move of Euler number e,*

$$t(T_{p,q}) - \lfloor p^2/2 \rfloor = t(T_{p,q+p}) = t(T_{r,s+r}) + (e - p^2 + r^2)/2 + 1$$

Proof. The left hand side of this equation is an lower bound for $t(T_{p,q+p})$ and the right hand side is a upper bound, so it suffices to show that the bounds are the same. Inductively, we may assume that

$$t(T_{p,q}) = t(T_{r,s}) + e/2 + 1$$

and

$$t(T_{r,s+r}) = t(T_{r,s}) - \lfloor r^2/2 \rfloor,$$

so

$$t(T_{p,q}) = t(T_{r,s+r}) + \lfloor r^2/2 \rfloor + e/2 + 1$$

and the needed equality follows from adding $-\lfloor p^2/2 \rfloor$ to both sides of this equation and the fact that p and r have the same parity. □

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