

**From the holdings of the Department of Special Collections , Princeton University Library,
Princeton, NJ**

These documents are either in the public domain, or can only be used for educational and research purposes (Fair use) as per U.S. Copyright law (text below). By accessing this file, all users agree that their use falls in the public domain or within fair use as defined by the copyright law. We do not charge any permission or usage fees for the publication of images of material in our collections, including those provided by Princeton University Library, via our website, catalog, or directly from staff. For more information regarding copyright, credit, and citation please review the information on the Copyright, Credit and Citation Guidelines page of our website.

<https://library.princeton.edu/special-collections/policies/copyright-credit-and-citation-guidelines>. If you have any questions, including how to order higher-quality images from the collection, contact Special Collections Public Services via our Ask Us! Form. <https://library.princeton.edu/special-collections/ask-us>

Doctoral Dissertations, 2011-2026

Mathematics

2016

Khovanov-Rozansky Complexes in the Knot Floer Cube of Resolutions

Dowlin, Nathan P.

Princeton, NJ : Princeton University

<http://arks.princeton.edu/ark:/88435/dsp01pg15bh304>

U.S. Copyright law: The copyright law of the United States (Title 17, §108, United States Code) governs the making of photocopies or other reproductions of copyrighted material. Under certain conditions specified in the law, libraries and archives are authorized to furnish a photocopy or other reproduction. One of these specified conditions is that the photocopy or reproduction is not to be used for any purpose other than private study, scholarship, or research. If a user makes a request for, or later uses, a photocopy or reproduction for purposes in excess of fair use that user may be liable for copyright infringement. This institution reserves the right to refuse to accept a copying order if, in its judgment, fulfillment of the order would involve a violation of copyright law. <https://www.copyright.gov/title17/title17.pdf>

KHOVANOV-ROZANSKY COMPLEXES IN THE KNOT FLOER CUBE OF RESOLUTIONS

NATHAN DOWLIN

A DISSERTATION

PRESENTED TO THE FACULTY
OF PRINCETON UNIVERSITY
IN CANDIDACY FOR THE DEGREE
OF DOCTOR OF PHILOSOPHY

RECOMMENDED FOR ACCEPTANCE

BY THE DEPARTMENT OF
MATHEMATICS

ADVISER: ZOLTÁN SZABÓ

MAY 2016

© Copyright by Nathan Dowlin, 2016.

All Rights Reserved

Abstract

The (untwisted) oriented cube of resolutions for knot Floer homology assigns a complex $C_F(S)$ to a singular resolution S of a knot K . Manolescu conjectured that when S is in braid position, the homology $H_*(C_F(S))$ is isomorphic to the HOMFLY-PT homology of S . Together with a naturality condition on the induced edge maps, this conjecture would prove the spectral sequence from HOMFLY-PT homology to knot Floer homology. Using a basepoint filtration on $C_F(S)$, a recursion formula for HOMFLY-PT homology, and additional sl_n -like differentials on $C_F(S)$, we prove this conjecture. Since the isomorphism is not explicitly defined, the naturality of the induced edge maps remains open.

Acknowledgements

I would like to thank my advisor Zoltán Szabó for his patient support throughout graduate school. His mathematical insights, calm optimism, and willingness to meet nearly every week for four years went above and beyond what is required of an advisor and made this period much less stressful than it could have been. For their valuable advice and discussions, I would like to thank Peter Ozsváth, Sucharit Sarkar, Adam Levine, Ciprian Manolescu, Hao Wu, and Andy Manion.

I would also like to thank my parents and my brother Izaak for their unwavering support in all of my pursuits, for making home feel like home no matter how long I am away, and for making time to come visit throughout graduate school. I would also like to thank Tatiana for her hospitality and for helping to make Christmas Eve, 2015 the most perfect day. Finally, I would like to thank Irina, for always reminding me of what's most important.

For Clyde

Contents

Abstract	iii
Acknowledgements	iv
1 Introduction	1
1.1 Known and Conjectured Spectral Sequences	1
1.2 The Oriented Cube of Resolutions and Manolescu’s Conjecture	5
1.2.1 The Oriented Cube of Resolutions	5
1.2.2 The Homology Theories and Manolescu’s Conjecture	8
1.3 Statement of Results	11
2 The Khovanov-Rozansky Homology of Singular Links	13
2.1 Singular Resolutions and the Ground Ring	13
2.2 HOMFLY-PT Homology and sl_n Homology	13
2.3 Rasmussen’s Spectral Sequences	17
3 A Recursion Formula for the Khovanov-Rozansky Homology of Singular Links	19
3.1 The Composition Product	19
3.1.1 Jaeger’s Formula	19
3.1.2 The Destabilized Composition Product	21
3.1.3 The Composition Product for Singular Graphs	24

3.2	The Categorification for sl_n Homology	25
3.3	A Categorification for HOMFLY-PT Homology	27
4	The Knot Floer Complex at a Vertex in the Cube of Resolutions	32
4.1	Definition of the Complex	32
4.2	The Filtered Complex and the Spectral Sequence from HOMFLY-PT Homology to HFK	35
4.2.1	A Filtration on $CFK^-(S)$	35
4.2.2	Homology of a Cycle	36
4.2.3	Gradings	41
4.3	Additional Differentials and the Spectral Sequences from HFK to sl_n	45
4.4	Proof of Manolescu's Conjecture	54
5	Generalities and Future Work	59
5.1	A Family of Cube Complexes for HOMFLY-PT Homology	59
5.2	The Non-local Ideal and sl_n Homology	61

Chapter 1

Introduction

1.1 Known and Conjectured Spectral Sequences

The last few decades have seen tremendous growth within the field of knot theory. Many new knot invariants have been constructed, including the categorifications of several classical knot polynomials. These categorifications typically take the form of a multi-graded homology theory, whose graded Euler characteristic returns the polynomial in question.

Some of the most notable categorifications include HOMFLY-PT homology, sl_n homology, and knot Floer homology, whose graded Euler characteristics return the HOMFLY-PT polynomial, the sl_n polynomial, and the Alexander polynomial, respectively. The HOMFLY-PT polynomial of a link L is a two variable polynomial $P_H(a, q)(L)$, and it is determined by the following skein relation:

$$aP_H(a, q)(L_+) - a^{-1}P_H(a, q)(L_-) = (q - q^{-1})P_H(a, q)(L_0)$$

where L_+ , L_- , and L_0 are identical except at one crossing, where L_+ has a positive crossing, L_- has a negative crossing, and L_0 has the oriented smoothing (see [3] and [14]). Together with the normalization $P_H(a, q)(unknot) = 1$, this relation uniquely

determines the HOMFLY-PT polynomial.

We can obtain a single-variable polynomial invariant $P_n(q)(L)$ by setting $a = q^n$ in the HOMFLY-PT polynomial:

$$P_n(q)(L) = P_H(q^n, q)(L)$$

For $n \geq 1$, $P_n(q)(L)$ gives the sl_n polynomial, and setting $n = 0$ gives the Alexander polynomial.

The most popular of the sl_n polynomials is the Jones polynomial, which is obtained by setting $n = 2$, and sl_2 homology is isomorphic Khovanov's original categorification of the Jones polynomial, known as Khovanov homology ([7]). See [5] for an explicit description of the isomorphism.

The fact that the sl_n and Alexander polynomials are specializations of the HOMFLY-PT polynomial led to the following conjecture of Dunfield, Gukov, and Rasmussen:

Conjecture 1.1.1 ([2]). *For all $n \geq 1$, there is a spectral sequence from HOMFLY-PT homology to sl_n homology, and there is a spectral sequence from HOMFLY-PT homology to knot Floer homology.*

At the time of this conjecture, HOMFLY-PT homology had not actually been constructed yet, but they conjectured its existence based on certain properties they expected it to have, including the spectral sequences.

Although all of these polynomials have this simple algebraic definition, the Alexander polynomial is somewhat unusual. It was defined in 1923, 60 years before the first non-trivial sl_n polynomial, and its original definition was geometric, based on a Seifert surface for the knot.

This difference in polynomials has an even stronger manifestation on the level of categorifications. The HOMFLY-PT homology and sl_n homology of Khovanov and

Rozansky are defined purely algebraically with roots in representation theory, and they are so similar in construction that the entire package is often referred to as Khovanov-Rozansky homology (see [8] and [9]). Knot Floer homology, on the other hand, is defined using symplectic techniques in an analog of Lagrangian Floer homology ([11]), and since the differentials require the counting of pseudo-holomorphic curves, the complex is often not even combinatorial. A combinatorial formulation was discovered by Sarkar and Wang ([17]), but it requires increasing the number of generators, so the homology is still difficult to compute.

The fact that HOMFLY-PT homology and sl_n homology have such similar constructions was used by Rasmussen to prove the first part of the conjecture.

Theorem 1.1.2 ([16]). *There are a family of spectral sequences $E_k(n)$ for $n \geq 1$ such that the E_2 page is HOMFLY-PT homology and the E_∞ page is sl_n homology.*

However, due to the fundamental differences between Khovanov-Rozansky homology and knot Floer homology, the second part of Conjecture 1.1.1 has remained unsolved for the last decade. This conjectured spectral sequence from HOMFLY-PT homology to knot Floer homology will be the focus of my thesis.

The ideas put forward in the Dunfield, Gukov, and Rasmussen paper were actually somewhat stronger than what we listed in the conjecture, but there has been little progress on the other topics. Instead of a family of spectral sequences, they actually conjectured a differential d_n on HOMFLY-PT homology for all $n \in \mathbb{Z}$ such that

$$H_*(\overline{H}_H(K), d_n) = \begin{cases} \overline{H}_n(K) & \text{if } n \neq 0 \\ \widehat{HFK}(K) & \text{if } n = 0 \end{cases} \quad (1.1)$$

where $\overline{H}_H(K)$ denotes the HOMFLY-PT homology of K , $\overline{H}_n(K)$ for $n \geq 1$ denotes the sl_n homology of K , and $\widehat{HFK}(K)$ denotes the knot Floer homology of K . The homology theories $\overline{H}_n$ for $n \leq -1$ were only conjectured to exist, and were based on

an expected involution ϕ on HOMFLY-PT homology satisfying

$$\phi \circ d_n = d_{-n}$$

from which there would be an isomorphism

$$\overline{H}_{-n}(K) \cong \overline{H}_n(K)$$

The conjectured involution ϕ would give a categorified analog of a symmetry of the HOMFLY-PT polynomial, namely $P_H(a, q)(K) = P_H(a, q^{-1})(K)$.

While this involution has yet to be defined, Rasmussen did succeed in constructing a homology theory $\overline{H}_{-1}$ such that there is a spectral sequence $E_k(-1)$ whose E_2 page is HOMFLY-PT homology and whose E_∞ page is $\overline{H}_{-1}$. Moreover, it satisfies the expected grading conditions, and is isomorphic to sl_1 homology.

$$\overline{H}_{-1}(K) \cong \overline{H}_1(K)$$

The $E_k(-1)$ spectral sequence will play a major role in our work towards the spectral sequence from HOMFLY-PT homology to knot Floer homology. This is because, unlike $E_k(n)$ for $n \geq 1$, which are induced by polynomial gradings, this spectral sequence is induced by the cube of resolutions filtration, which is the context in which we expect our spectral sequence to appear.

1.2 The Oriented Cube of Resolutions and Manolescu's Conjecture

1.2.1 The Oriented Cube of Resolutions

All three of our homology theories can be constructed via a type of complex known as an oriented cube of resolutions. Any knot in the 3-sphere can be represented by a planar projection, or an oriented 4-valent graph with each vertex decorated with a $+$ or a $-$ to specify whether one strand goes over or under the other. We will be using the conventions from Figure 1.1 for positive and negative crossings.

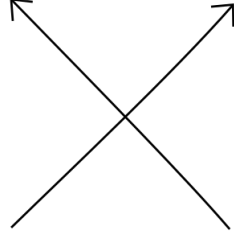


Figure 1.1: Sign Convention for Crossings

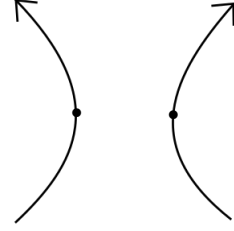
We require that our diagrams be in braid position - this restriction is not necessary for the theories to be defined, but Khovanov-Rozansky homology has only been proved to be invariant under braid-like Reidemeister moves. The naive definition of the complex is not expected to be invariant under non-braid-like Reidemeister moves, as the local bimodules are not isomorphic for unoriented Reidemeister II. However, the author is not aware of any explicit counterexamples for closed diagrams in the current literature.

We will first describe the cube of resolutions in general, then we will define our three homology theories. The cube of resolutions construction depends on the diagram D , so we will call this abstract complex $C(D)$. Each crossing has two oriented

resolutions - the singularization, and the oriented smoothing. We will refer to these as the oriented resolutions of the crossing.



(a) The Singularization of a Crossing



(b) The Oriented Smoothing of a Crossing

Note that if we replace each crossing with one of the oriented resolutions, we are left with an oriented planar graph with the following properties: each vertex is either 2-valent or 4-valent, and at each vertex the number of incoming edges is equal to the number of outgoing edges. A graph obtained in this way from a knot projection D is called a *complete resolution* of D .

The main idea in the oriented cube of resolutions is that each crossing corresponds to a filtration on the complex, with respect to which there are exactly two filtration levels. These two filtration levels correspond to the diagrams obtained by singularizing the crossing and smoothing the crossing. In particular, let D_+ denote a diagram with a positive crossing c , and let D_x and D_s denote the diagram obtained by replacing c with a singularization and a smoothing, respectively. Then the chain complex $C(D_+)$ has the following decomposition:

$$C(D_+) = \begin{array}{ccc} & \begin{array}{c} \text{\scriptsize d} \\ \curvearrowright \end{array} & \\ C(D_+) = & C(D_x) \xrightarrow{\quad d \quad} & C(D_s) \\ & \begin{array}{c} \curvearrowleft \end{array} & \end{array}$$

This complex has a natural filtration, which corresponds to a grading $gr(c)$. The grading of $C(D_x)$ is defined to be 0, and the grading of $C(D_s)$ is defined to be 1. The differentials from $C(D_x)$ to itself and from $C(D_s)$ to itself are both homogeneous with

grading 0, and the differential from $C(D_x)$ to $C(D_s)$ is homogeneous with grading 1.

The decomposition for a negative crossing is similar, but with the roles of the smoothing and singularization reversed. Let D_- denote a diagram with a negative crossing c , and let D_x and D_s denote the diagram obtained by replacing c with a singularization and a smoothing, respectively. Then the chain complex $C(D_-)$ has the following decomposition:

$$C(D_-) = \begin{array}{ccc} & \begin{array}{c} \text{\scriptsize d} \\ \curvearrowright \\ \end{array} & \\ & C(D_s) & \\ & \begin{array}{c} \curvearrowleft \\ \text{\scriptsize d} \end{array} & \end{array} \longrightarrow \begin{array}{ccc} & \begin{array}{c} \text{\scriptsize d} \\ \curvearrowright \\ \end{array} & \\ & C(D_x) & \\ & \begin{array}{c} \curvearrowleft \\ \text{\scriptsize d} \end{array} & \end{array}$$

Here, the grading $gr(c)$ of $C(D_s)$ is defined to be 0 and the grading of $C(D_x)$ is defined to be 1.

If we apply these decompositions at all k crossings of a knot projection D , then the complex has components corresponding to the 2^k complete resolutions of D . Using the k gradings (from each gr_c) as a coordinate system, each complete resolution S lies at a vertex in the k -dimensional cube, and will have a graded group $C(S)$ assigned to it. Note that the graded groups lying at each vertex in the cube, as well as the differentials, have not been defined, as they will depend on which of the three homology theories we are discussing.

Since the differential is monotone non-decreasing in each gr_c , we define the *cube grading* to be the sum of them over all k crossings.

$$gr_{cube} = \sum_{i=1}^k gr(c_i)$$

This grading is exactly the height of a vertex in the cube. The sum of filtrations is again a filtration, so the differential is monotone non-decreasing with respect to gr_{cube} . The differential d can therefore be decomposed as

$$d = d_0 + d_1 + \dots + d_k$$

where each d_i is homogeneous of degree i with respect to the cube grading. The differentials d_0 map each vertex in the cube to themselves, the d_1 differentials map along edges in the cube, the d_2 differentials map along 2-dimensional faces in the cube, etc. Since d_0 splits over the vertices of the cube, there is a well-defined complex $C(S)$ obtained by restricting d_0 to the vertex in the cube corresponding to S . These complexes are the building blocks of the cube of resolutions.

We define this filtered complex $(C(D), d_0 + d_1 + \dots + d_k)$ to be an oriented cube of resolutions. Note that there are many algebraic objects that can be constructed from this complex - one could take the total homology $H_*(C(D), d_0 + d_1 + \dots + d_k)$, or using the cube filtration, one could examine various pages of the induced spectral sequence. Since the homology exists in finitely many filtration levels, this spectral sequence is strongly convergent to the total homology. It turns out the HOMFLY-PT homology and knot Floer homology are defined in different ways from the cube of resolutions, and Manolescu conjectured that the spectral sequence from HOMFLY-PT homology to HFK can be constructed in this way.

1.2.2 The Homology Theories and Manolescu's Conjecture

In this section we will describe the general framework of the three homology theories, with the precise definitions coming in Chapters 2 and 4. In particular, we describe here how they come from the oriented cube of resolutions, which will allow us to state Manolescu's conjecture.

Let $C_H(D, d_0^H + d_1^H + \dots + d_k^H)$ denote the cube of resolutions for HOMFLY-PT homology. It turns out that $d_i^H = 0$ for $i \geq 2$, but that isn't relevant for this discussion. The HOMFLY-PT homology $H_H(K)$ is defined to be

$$H_H(K) = H_*(H_*(C_H(D), d_0^H), (d_1^H)^*)$$

where D is a braid diagram of the knot K and $(d_1^H)^*$ is the induced map on homology. HOMFLY-PT homology is a knot invariant, which means that the homology does not depend on the choice of braid diagram for K . Note that this homology can be equivalently viewed as the E_2 page of the spectral sequence induced by the cube filtration on the HOMFLY-PT complex.

The sl_n homology $H_n(K)$ is defined in the same way. If $C_n(D, d_0^{(n)} + d_1^{(n)} + \dots + d_k^{(n)})$ denotes the cube of resolutions for sl_n homology, then $H_n(K)$ is given by

$$H_n(K) = H_*(H_*(C_n(D), d_0^{(n)}), (d_1^{(n)})^*)$$

where again, D is a braid diagram for K , but the homology is independent of the choice of D .

Knot Floer homology is a completely different story. There is not a standard way to define the complex $CFK^-(K)$, as it depends on a choice of Heegaard diagram for K , and there are many different ways to do so for a knot K . However, the chain homotopy type of $CFK^-(K)$ does not depend on the choice of diagram.

Although there isn't a known diagram for which $CFK^-(K)$ decomposes as an oriented cube of resolutions, there is a construction which is chain homotopy equivalent to $CFK^-(K)$ which does come from a cube of resolutions. We will denote this complex $(C_F(D), d_0^F + d_1^F + \dots + d_k^F)$. Unlike the HOMFLY-PT and sl_n homology, the knot Floer homology $HFK^-(K)$ is the *total* homology of this complex.

$$HFK^-(K) \cong H_*(C_F(D), d_0^F + d_1^F + \dots + d_k^F)$$

Recall that for all of these complexes, each vertex in the cube of resolutions can be viewed as a complex corresponding to the complete resolution S at that vertex.

We will denote these complexes by $C_H(S)$, $C_n(S)$, and $C_F(S)$, with the corresponding homologies given by $H_H(S)$, $H_n(S)$, and $H_F(S)$.

Manolescu made the following conjecture:

Conjecture 1.2.1 ([10]). *Let S denote a complete resolution of a diagram D in braid position. Then $H_H(S) \cong H_F(S)$ as bigraded vector spaces.*

An immediate consequence of this conjecture is an isomorphism

$$H_*(C_H(D), d_0^H) \cong H_*(C_F(D), d_0^F)$$

which maps each vertex in the HOMFLY-PT cube of resolutions to the same vertex in the knot Floer cube of resolutions. The spectral sequence from HOMFLY-PT homology to knot Floer homology would then follow from the induced edge maps being the same. In other words, if f is the isomorphism between them, then the square below being commutative would prove the spectral sequence.

$$\begin{array}{ccc} H_*(C_H(D), d_0^H) & \xrightarrow{f} & H_*(C_F(D), d_0^F) \\ \downarrow (d_1^H)^* & & \downarrow (d_1^F)^* \\ H_*(C_H(D), d_0^H) & \xrightarrow{f} & H_*(C_F(D), d_0^F) \end{array}$$

This idea can also be explained in terms of the spectral sequences induced by the cube filtrations on $C_H(D)$ and $C_F(D)$. Letting $E_k^H(D)$ and $E_k^F(D)$ denote the two spectral sequences, we see that the HOMFLY-PT homology is given by $E_2^H(D)$, and the knot Floer homology is given by $E_\infty^F(D)$. Manolescu's conjecture would imply an isomorphism $E_1^H(D) \cong E_1^F(D)$, and the induced edge maps commuting with this isomorphism would imply that $E_2^H(D) \cong E_2^F(D)$. This gives a spectral sequence whose E_2 page is isomorphic to HOMFLY-PT homology and whose E_∞ page is $HFK^-(K)$.

1.3 Statement of Results

Manolescu showed in [10] that for a singular braid S , both $H_H(S)$ and $H_F(S)$ have a purely algebraic formulation in terms of Tor groups. Letting R denote the polynomial ring $\mathbb{Q}[U_1, \dots, U_k]$, where k is the number of edges in the singular braid S , he showed that there are ideals L , Q , and N in R such that

$$H_H(S) \cong \operatorname{Tor}_R(R/L, R/Q) \quad \text{and} \quad H_F(S) \cong \operatorname{Tor}_R(R/L, R/N)$$

These Tor groups can naturally be viewed as bigraded vector spaces, where the dimension in each bigrading is finite. Thus, Conjecture 1.2.1 is equivalent to an isomorphism bigraded vector spaces

$$\operatorname{Tor}_R(R/L, R/Q) \cong \operatorname{Tor}_R(R/L, R/N)$$

Unfortunately, these Tor groups turned out to be difficult to compare due to the non-local nature of the ideal N .

In this thesis, we will prove Conjecture 1.2.1 using a very different approach. First, we will use a filtration on the complex $C_F(S)$ induced by adding basepoints to the Heegaard diagram for S to prove the following:

Theorem 1.3.1. *If d^{filtered} denotes the basepoint-filtered differential on $C_F(S)$, then there is an isomorphism of bigraded vector spaces*

$$H_*(C_F(S), d^{\text{filtered}}) \cong H_H(S)$$

Corollary 1.3.2. *There is a spectral sequence which starts at $H_H(S)$ and converges to $H_F(S)$.*

The second tool that we will use is an additional family of differentials on $C_F(S)$ for all $n \geq 1$. We will denote this complex by $C_{F(n)}$.

Theorem 1.3.3. *For all $n \geq 1$, there is an isomorphism*

$$H_*(C_{F(n)}(S)) \cong H_{n+1}(S)$$

The differential on $C_{F(n)}$ can be filtered by the Alexander grading, and when we only consider those differentials which preserve the Alexander grading, we get back the complex $C_F(S)$. Thus, using the Alexander filtration on $C_{F(n)}$, we get the following corollary:

Corollary 1.3.4. *For all $n \geq 2$, there is a spectral sequence which starts at $H_F(S)$ and converges to $H_n(S)$.*

For all $n \geq 1$, there is also a known spectral sequence which starts at $H_H(S)$ and converges to $H_n(S)$ ([16]). Thus, $H_H(S)$ and $H_F(S)$ are both ‘limits’ of sl_n homology (in a suitable sense).

We are able to use these additional differentials to prove that the spectral sequence induced by the basepoint filtration on $C_F(S)$ collapses at the E_1 page. But the E_1 page is given by $H_*(C_F(S), d^{filtered}) \cong H_H(S)$, and the spectral sequence converges to the total homology $H_F(S)$, so this proves Conjecture 1.2.1.

Theorem 1.3.5. *Let S denote a complete resolution of a diagram D in braid position. Then $H_H(S) \cong H_F(S)$ as bigraded vector spaces.*

The isomorphisms $H_*(C_F(S), d^{filtered}) \cong H_H(S)$ and $H_*(C_{F(n)}(S)) \cong H_{n+1}(S)$ are not canonical - they rely on a categorification of a relationship among knot polynomials known as the composition product. Some work in this area has been done by Wagner ([18]), but we need the results in greater generality to prove the above theorems as bigraded vector spaces. These results are discussed in Chapter 3.

Chapter 2

The Khovanov-Rozansky Homology of Singular Links

2.1 Singular Resolutions and the Ground Ring

A complete resolution S of a knot K in braid position can be viewed as an oriented planar graph with the following properties:

1. All vertices are either 2-valent or 4-valent.
2. The number of incoming edges is equal to the number of outgoing edges at each vertex.
3. If Z is an oriented cycle in S , then the unique disc $D \subset \mathbb{R}^2$ with boundary Z intersects the center of the braid.

Let $e_1, \dots, e_k$ denote the edges of S . To each edge e_i , we assign an indeterminant U_i . All three homology theories will be defined over the ground ring $R = \mathbb{Q}[U_1, \dots, U_k]$.

2.2 HOMFLY-PT Homology and sl_n Homology

This section will give a brief description of the HOMFLY-PT and sl_n as defined in [8] and [9]. We will use the grading conventions from [16], though we will leave out

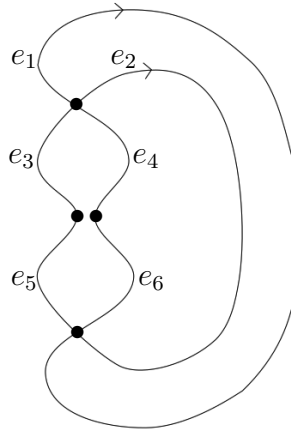


Figure 2.1: An example of a singular braid diagram

the overall grading shifts coming from the braid number. The reader can refer to these resources for further background. The HOMFLY-PT and sl_n complexes have the same generators, with the sl_n complex having strictly more differentials than the HOMFLY-PT complex. For this reason, we will start by defining the HOMFLY-PT complex, then we will describe the additional differentials to make the sl_n complex.

The HOMFLY-PT complex for links comes equipped with a triple-grading, and the sl_n complex with a bigrading. One of the gradings in both theories, however, comes from the height in the cube of resolutions, so it will be fixed for a single resolution. The HOMFLY-PT complex will therefore come with a bigrading, and the sl_n complex with a single grading. For the HOMFLY-PT complex, these gradings are called the quantum grading, denoted gr_q , and the horizontal grading, denoted gr_h . Multiplication by the U_i increases the quantum grading by 2 and preserves the horizontal grading.

Let $V_2(S)$ denote the 2-valent vertices in S and $V_4(S)$ the 4-valent vertices of S . For vertices v in $V_2(S)$, there is a unique outgoing edge e_i and a unique incoming edge e_j . Define $L(v)$ to be the linear term $U_i - U_j$. Similarly, for vertices v in $V_4(S)$ there are two outgoing edges e_i and e_j and two incoming edges e_k and e_l . We define $L(v)$ to

be the linear term $U_i + U_j - U_k - U_l$ and $Q(v)$ to be the quadratic term $U_i U_j - U_k U_l$.

The HOMFLY-PT complex is a tensor product of complexes $C_H(v)$ for each vertex v . For v in $V_2(S)$, $C_H(v)$ is defined as

$$R\{0, -2\} \xrightarrow{L(v)} R\{0, 0\}$$

where $R\{i, j\}$ refers to the thing R shifted by i in gr_q and by j in gr_h . For v in $V_4(S)$, $C_H(v)$ is defined as

$$\begin{array}{ccc} R\{1, -4\} & \xrightarrow{L(v)} & R\{1, -2\} \\ \downarrow Q(v) & & \downarrow Q(v) \\ R\{-1, -2\} & \xrightarrow{L(v)} & R\{-1, 0\} \end{array}$$

Note that the differential is homogeneous of degree $\{2, 2\}$. The HOMFLY-PT complex for the singular diagram S is given by

$$C_H(S) = \bigotimes_{v \in S} C_H(v)$$

where the tensor product is taken over R , and the HOMFLY-PT homology $H_H(S)$ is the homology of $C_H(S)$.

We will now define the additional differentials which give sl_n homology. For a vertex v in S with outgoing edges E_{out} and incoming edges E_{in} , let the potential w_n be given by

$$w_n(v) = \sum_{e_i \in E_{out}} U_i^{n+1} - \sum_{e_j \in E_{in}} U_j^{n+1}$$

For v in $V_2(S)$, let $u_1(v)$ be the unique element in R such that $u_1(v)L(v) = w_n(v)$. For v in $V_4(S)$, we can choose $u_1(v)$ and $u_2(v)$ such that $u_1(v)L(v) + u_2(v)Q(v) = w_n(v)$.

Unlike the 2-valent case, the choice is not unique, but the reader can refer to ([8], p. 6) for the precise choice. (It is not relevant for our discussion.)

For each vertex v , we will add new differentials to $C_H(v)$ to make a new complex $C_n(V)$. For v in $V_2(S)$, $C_n(v)$ is given by

$$\begin{array}{ccc} & L(v) & \\ R\{0, -2\} & \xrightarrow{\quad} & R\{0, 0\} \\ & \xleftarrow{\quad u_1(v) \quad} & \end{array}$$

and for v in $V_4(S)$, $C_n(v)$ is given by

$$\begin{array}{ccccc} & & L(v) & & \\ & & \xrightarrow{\quad} & & \\ R\{1, -4\} & & & & R\{1, -2\} \\ & & \xleftarrow{\quad u_1(v) \quad} & & \\ & & & & \\ Q(v) & & & & Q(v) \\ & & u_2(v) & & u_2(v) \\ & & \xrightarrow{\quad} & & \\ R\{-1, -2\} & & L(v) & & R\{-1, 0\} \\ & & \xleftarrow{\quad u_1(v) \quad} & & \end{array}$$

Observe that for both types of vertices, the differential on $C_n(v)$ satisfies $d^2 = w_n(v)I$. Such a complex is called a matrix factorization with potential w_n . Since d^2 is non-zero, its homology is not well-defined. However, we are interested in the tensor product of $C_n(v)$ over all vertices v in S . Define the sl_n complex $C_n(S)$ by

$$C_n(S) = \bigotimes_{v \in S} C_n(v)$$

where again the tensor product is taken over R .

As mentioned above, the HOMFLY-PT differentials are homogeneous of degree $\{2, 2\}$. These differentials are denoted by d_+ . The new differentials, those with coefficients $u_1(v)$ and $u_2(v)$, are homogeneous of degree $\{2n, -2\}$. These are denoted

d_- . The total differential $d_{tot} = d_+ + d_-$ is not homogeneous in this bigrading. However, if we look at the grading $gr_n = gr_q + (n-1)gr_h/2$, then d_{tot} is homogeneous of degree $n+1$.

Additionally, $d_{tot}^2 = 0$. This can be seen from the fact that the potential is additive under tensor product, so $d_{tot}^2 = \sum_{v \in S} w_n(v)$. The sum must be zero because each edge e_i is an outgoing edge for one vertex, which will contribute U_i^{n+1} , and an incoming edge for another vertex, which will contribute $-U_i^{n+1}$.

We have shown that $C_n(S)$ is a well-defined chain complex which is homogeneous with respect to the grading gr_n . We define the sl_n homology $H_n(S)$ to be the homology of this complex.

Remark 2.2.1. The definitions given here correspond to the unreduced theories in [16] as opposed to the middle or reduced homologies. This choice will make our arguments involving the composition product easier, although they would still work for the reduced versions using the destabilized composition product, which is described in Section 3.1.2.

2.3 Rasmussen's Spectral Sequences

Rasmussen showed in [16] that there are a family of spectral sequences $E_k(n)$ which start at HOMFLY-PT homology and converge to sl_n homology. These spectral sequences are somewhat difficult to prove for the case of knots and links, but they are much simpler for fully singular diagrams.

The differential d_{tot} on $C_n(S)$ admits a filtration induced by the grading $(gr_q - gr_h)/(2n+2)$. With respect to this grading, d_+ is homogeneous of degree 0, and d_- is homogeneous of degree 1. Note that this filtration is not bounded above, so *a priori* it does not induce a well-defined spectral sequence to the total homology. However, if we look at the induced differentials which change this grading by k , we see that

they also decrease the horizontal grading by $2 - 4k$. Since the complex is bounded in horizontal grading, this bounds the length of induced differentials, so the filtration does induce a spectral sequence to the total homology.

The E_1 page of this spectral sequence is $H(C_n(S), d_+)$, which is exactly the definition of HOMFLY-PT homology. The E_∞ page is the homology with respect to d_{tot} , or sl_n homology.

It turns out that this spectral sequence collapses at the E_2 page, given by

$$H_*(H_*(C_n(S), d_+), d_-^*)$$

We will denote this page by $H^\pm(C_n(S))$. Note that $H^\pm(C_n(S))$ is bigraded, as both d_+ and d_- are homogeneous. The fact that all higher differentials are trivial follows from the following lemma.

Lemma 2.3.1 ([16]). *The homology $H^\pm(C_n(S))$ lies in a single horizontal grading, namely $gr_h = 2r(S)$, where $r(S)$ is the rotation number of S . Since S is a singular braid oriented clockwise, $r(S)$ is the negative of the number of strands in S .*

Since none of the higher differentials preserve the horizontal grading, they must all be trivial, causing the spectral sequence to collapse.

Corollary 2.3.2. *Viewing $H^\pm(C_n(S))$ as singly graded with grading gr_n , there is a graded isomorphism $H^\pm(C_n(S)) \cong H_n(S)$.*

Remark 2.3.3. The reader familiar with [16] may note that our homology lies in $gr_h = -2r(S)$, while Rasmussen's lies in $gr_h = 1 - r(S)$. This difference comes from the fact that our homology is unreduced, which changes the grading by 2, and because we are leaving out the overall grading shift of $r(S) - 1$.

Chapter 3

A Recursion Formula for the Khovanov-Rozansky Homology of Singular Links

3.1 The Composition Product

3.1.1 Jaeger's Formula

The first composition product formula was defined by Jaeger in [6]. In order to talk about the composition product, we must first define labelings of a diagram. Let K be a knot with corresponding diagram D . Viewing D as an oriented 4-valent graph, we say that a subset S of the edges of D is a homological cycle if at each vertex in D the number of incoming edges in S is equal to the number of outgoing edges in S . A *labeling* f of the diagram D is a function from the set of edges in D to the set $\{1, 2\}$ such that $f^{-1}(1)$ is a homological cycle. (Note that $f^{-1}(1)$ is a homological cycle iff $f^{-1}(2)$ is a homological cycle.)

We will place a restriction on which homological cycles are allowed. A homological cycle is said to make a turn at a crossing c if it has one incoming edge at c and one

outgoing edge at c , and those edges are not diagonal from one another. Let $T(f)$ denote the number of turns of the labeling f . A labeling f is *admissible* if $f^{-1}(1)$ doesn't make any left turns at positive crossings or right turns at negative crossings.



Figure 3.1: Non-Admissible Labelings

Since the homological cycle $f^{-1}(1)$ uniquely determines the labeling f , we will say that a homological cycle Z is admissible if the unique labeling f with $f^{-1}(1) = Z$ is admissible. The two cycles $f^{-1}(1)$ and $f^{-1}(2)$ can both be viewed as diagrams of links if we retain the crossing information whenever one of them contains all four edges at a crossing, and forget it otherwise. We will refer to these diagrams as $D_{f,1}$ and $D_{f,2}$, respectively.

Finally, we will need some combinatorial data about the diagrams $D_{f,1}$ to define the composition product. Given a diagram D , consider the diagram obtained by changing each crossing in D to the oriented smoothing. The resulting diagram must be a collection of oriented circles - these are known as the Seifert circles of D . We define the *rotation number* $r(D)$ to be sum of the signs of the Seifert circles, with a circle contributing a $+1$ if it is oriented counterclockwise and -1 if it is oriented clockwise. Note that when D is a braid, $r(D)$ is simply the negative of the number of strands in the braid, i.e. $r(D) = -b(D)$.

With the HOMFLY-PT polynomial P_H as defined in the introduction, define

$$P'_H(a, q, D) = \left(\frac{a - a^{-1}}{q - q^{-1}} \right) (a^{w(D)}) P_H(a, q, D)$$

Note that P'_H is invariant under Reidemeister II and III moves, but performing a

Reidemeister I move changes the writhe, so one picks up a factor of a or a^{-1} depending on the sign of the crossing. With this normalization, $P'_H(unknot) = \frac{a-a^{-1}}{q-q^{-1}}$, and $P'_H(\emptyset) = 1$, where $\emptyset$ denotes the empty diagram. Jaeger's composition product can be stated as follows:

$$\begin{aligned} \sum_{f \text{ admissible}} (q - q^{-1})^{T(f)} a_1^{r(D_{f,2})} a_2^{-r(D_{f,1})} P'_H(a_1, q, D_{f,1}) P'_H(a_2, q, D_{f,2}) \\ = P'_H(a_1 a_2, q, D) \end{aligned} \quad (3.1)$$

The proof of this formula is combinatorial in nature - one can show that it behaves properly under Reidemeister moves and that it satisfies the necessary skein relation via local computations. In fact, Jaeger showed in [6] that this formula is invariant under all Reidemeister moves, so the formula holds for arbitrary diagrams D instead of just braid diagrams.

To complete the proof, we just have to check that it works on the unknot, which is calculated below.

$$a_2^{-1} \frac{a_1 - a_1^{-1}}{q - q^{-1}} + a_1 \frac{a_2 - a_2^{-1}}{q - q^{-1}} = \frac{a_1 a_2 - a_1^{-1} a_2^{-1}}{q - q^{-1}} = P'_H(a_1 a_2, q, unknot)$$

3.1.2 The Destabilized Composition Product

We develop an adaptation of Jaeger's composition product for a decorated braid diagram D , i.e. a diagram in braid position that has one marked edge e_0 , in the special case where $a_1 = q$. (Again, the condition that our diagram is a braid is not necessary for this definition, but it will be necessary on the level of homology.) The diagram D now has a special Seifert circle, the one containing the marked edge e_0 - we will call this circle S_0 . We will define the sign of a Seifert circle S as follows:

$$\text{sign}(S) = \begin{cases} +1 & \text{if } S \text{ is oriented CCW and } S \text{ does not contain } S_0 \\ -1 & \text{if } S \text{ is oriented CCW and } S \text{ contains } S_0 \\ -1 & \text{if } S \text{ is oriented CW and } S \text{ does not contain } S_0 \\ +1 & \text{if } S \text{ is oriented CW and } S \text{ contains } S_0 \\ 0 & \text{if } S = S_0 \end{cases} \quad (3.2)$$

An alternative way to view these signs is to imagine our diagram is in S^2 instead of the plane, so that each Seifert circle bounds two discs. To determine the sign of the Seifert circle, we view it as the boundary of the disc that does not contain the edge e_0 . Then, as before, we say that it is +1 if it is oriented CCW and negative if it is oriented CW. The special circle containing e_0 has no contribution.

With these sign conventions, let $r(D)$ denote the sum of the signs of the Seifert circles. We can define our reduced version of the composition product by

$$\sum_{\substack{f \text{ admissible} \\ f(e_0)=2}} \left[(q - q^{-1})^{T(f)} q^{r(D_{f,2})-s(D_{f,2})} a^{-r(D_{f,1})-s(D_{f,1})} \cdot P_H(q, q, D_{f,1}) P_H(a, q, D_{f,2}) \right] = P_H(qa, q, D) \quad (3.3)$$

where the signs of the Seifert circles in both $D_{f,1}$ and $D_{f,2}$ are given by (3.2) relative to the marked edge e_0 , even though e_0 always belongs to $D_{f,2}$. The quantities $s(D_{f,i})$, defined to be the sum of the signs of the crossings in D with at least one adjacent edge labeled i , stems from the fact that Jaeger's composition product came with factors of $a^{w(D)}$. (Note that $s(D_{f,1}) = w(D) - w(D_{f,2})$ and $s(D_{f,2}) = w(D) - w(D_{f,1})$.)

The proof of this equality is identical to the proof for Jaeger's, since everything is the same locally. The only differences (aside from the notational difference of

removing the shifts by $w(D)$) are that our labelings require that $f(e_0) = 2$ and we don't have the factor of $\frac{a-a^{-1}}{q-q^{-1}}$. Jaeger's calculations show that our construction satisfies the correct skein relation, and that it is invariant under Reidemeister moves that take place away from the marked edge e_0 . By the equivalence of knots and (1,1) tangles, these are the only Reidemeister moves we need to show invariance. Thus, to complete the proof, we just need to check that the formula holds on the base case of the unknot.

Remark 3.1.1. It is important to note that the equivalence of knots and (1,1) tangles is not true when restricted to braids - one must sometimes isotope through non-braid diagrams in order to connect two equivalent tangles. Fortunately, Jaeger showed in [6] that Equation 3.1 is invariant under all Reidemeister moves regardless of orientation, giving an invariant for arbitrary diagrams.

There is only one labeling that contributes for the unknot. Since there is only one edge, it must be the marked edge e_0 , and $f(e_0) = 2$. For this labeling f , $r(D_{f,1}) = r(D_{f,2}) = s(D_{f,1}) = s(D_{f,2}) = 0$, $P_1(\phi) = \frac{q-q^{-1}}{q-q^{-1}} = 1$, and $P_H(unknot) = 1$, so (3.3) becomes $1 \cdot 1 = 1$. This establishes the base case, which proves the formula.

An Example: The Right Handed Trefoil

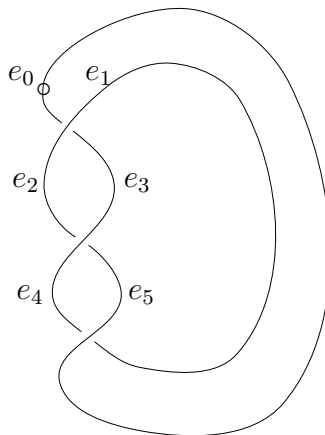


Figure 3.2: Braid Diagram for the Right Handed Trefoil

This diagram D has four local homological cycles which we will describe in terms

of the edges in $f^{-1}(1)$, since this set uniquely characterizes f . These four sets are $\emptyset$, $e_1e_2e_5$, $e_1e_3e_4$, and $e_1e_3e_5$. Their contributions are listed in the table below. The sum of these contributions is $a^{-2} + a^{-2}q^{-4} - a^{-4}q^{-4}$, which is equal to $P_H(qa, q, D)$. We can see this equality by substituting $a \mapsto aq$ in the contribution from $\emptyset$.

Cycle	Contribution
$\emptyset$	$a^{-2}q^2 + a^{-2}q^{-2} - a^{-4}$
$e_1e_2e_5$	$(q - q^{-1})q^{-3}a^{-2}$
$e_1e_3e_4$	$(q - q^{-1})q^{-3}a^{-2}$
$e_1e_3e_5$	$(q - q^{-1})^3q^{-3}a^{-2}$
Total	$a^{-2} + a^{-2}q^{-4} - a^{-4}q^{-4}$

3.1.3 The Composition Product for Singular Graphs

The composition product formula can be specialized to the sl_n polynomials to give

$$\sum_{f \text{ admissible}} (q - q^{-1})^{T(f)} q^{mr(D_{f,1}) - nr(D_{f,2})} P'_n(q, D_{f,1}) P'_m(q, D_{f,2}) = P'_{m+n}(q, D) \quad (3.4)$$

This formula was extended by Wagner to singular braids in the following way. If S is a singular braid, we can define labelings of S in the same way as labelings for knots. We will drop the admissibility condition at 4-valent vertices since they no longer correspond to positive or negative crossings. Finally, given a labelling f of S , let $T_1(S_{f,1})$ denote the number of vertices $v \in V_4(S)$ at which $f^{-1}(1)$ contains the edges e_1 and e_3 in Figure 3.3.

Similarly, let $T_2(S_{f,1})$ denote the number of vertices $v \in V_4(S)$ at which $f^{-1}(1)$ contains the edges e_2 and e_4 . With this terminology, the composition product for singular braids can be stated as

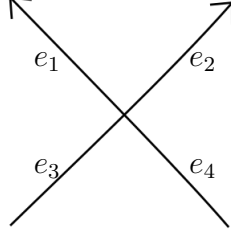


Figure 3.3: A labeled 4-valent vertex

$$P'_{m+n}(S) = \sum_{f \in L(S)} q^{\sigma_{m,n}(f)} P'_n(S_{f,1}) P'_m(S_{f,2})$$

where $P'_k(S)$ is the unreduced sl_k polynomial of S , $\sigma_{m,n}(f) = T_2(S_{f,1}) - T_1(S_{f,1}) + mr(S_{f,1}) - nr(S_{f,2})$, and $r(S_{f,i})$ is the negative of the number of strands in the singular braid $S_{f,i}$.

3.2 The Categorification for sl_n Homology

Wagner further showed that this relationship is true on the level of categorifications

$$H_{m+n}(S) = \bigoplus_{f \in L(S)} H_n(S_{f,1}) \otimes H_m(S_{f,2}) \{\sigma_{m,n}(f)\} \quad (\text{with polynomial grading})$$

where the sl_n homology groups are singly graded, using the polynomial grading gr_n . In order for us to generalize this theorem, it will be useful to have a bigraded version of it using our (gr_q, gr_h) gradings. Since the bigraded $H_n(S)$ lies in a single horizontal grading, namely $gr_h = 2r(S)$, and $gr_n = gr_q + (n-1)gr_h/2$, the q -grading is uniquely determined by gr_n :

$$gr_n = gr_q + (n-1)r(S)$$

Thus, if we view each $H_n(S)$ as singly graded where the grading is gr_q instead, the formula becomes

$$H_{m+n}(S)\{(n+m-1)r(S)\} = \quad (\text{with quantum grading})$$

$$\bigoplus_{f \in L(S)} H_n(S_{f,1}) \otimes H_m(S_{f,2}) \{\sigma_{m,n}(f) + (n-1)r(S_{f,1}) + (m-1)r(S_{f,2})\}$$

This can be simplified to

$$H_{m+n}(S) = \bigoplus_{f \in L(S)} H_n(S_{f,1}) \otimes H_m(S_{f,2}) \{T_2(S_{f,1}) - T_1(S_{f,1}) - 2nr(S_{f,2})\}$$

Finally, to make this bigraded, we need to add in the horizontal grading. Since $H_{m+n}(S)$ lies in horizontal grading $2r(S)$, $H_n(S_{f,1})$ lies in horizontal grading $r(S_{f,1})$, and $H_m(S_{f,2})$ lies in horizontal grading $r(S_{f,2})$, we see that the tensor product

$$H_n(S_{f,1}) \otimes H_m(S_{f,2})$$

always lies in horizontal grading $2r(S_{f,1}) + 2r(S_{f,2})$. Since $r(S_{f,1}) + r(S_{f,2}) = r(S)$, it follows that the tensor product lies in grading $2r(S)$ - the same grading as $H_{m+n}(S)$. Thus, we can add the horizontal grading to the composition product formula, with no grading shift required for each labeling f . The bigraded formula is then

$$H_{m+n}(S) = \bigoplus_{f \in L(S)} H_n(S_{f,1}) \otimes H_m(S_{f,2}) \{T_2(S_{f,1}) - T_1(S_{f,1}) - 2nr(S_{f,2}), 0\}$$

where the bigrading is given by (gr_q, gr_h) .

3.3 A Categorification for HOMFLY-PT Homology

In relating these formulas to knot Floer homology, we will be most interested in the case when $n = 1$. In this case, the previous formula becomes

$$H_{m+1}(S) = \bigoplus_{f \in L(S)} H_1(S_{f,1}) \otimes H_m(S_{f,2}) \{T_2(S_{f,1}) - T_1(S_{f,1}) - 2r(S_{f,2}), 0\} \quad (3.5)$$

In this section we will prove a generalization of this formula to HOMFLY-PT homology. Letting $H_H(S)$ denote the HOMFLY-PT homology with the standard bigrading (gr_q, gr_h) , define $H_H(S) < k >$ to be HOMFLY-PT homology with a new grading $(gr_q + kgr_h, gr_h)$.

Theorem 3.3.1. *There is an isomorphism of bigraded vector spaces*

$$\bigoplus_f H_1(S_{f,1}) \otimes H_H(S_{f,2}) \{T_2(S_{f,1}) - T_1(S_{f,1}) - 2r(S_{f,2}), 0\} \cong H_H(S) < 1 >$$

The proof of this theorem will rely heavily the fact that for all $n \geq 1$, there is a differential $d_-(n)$ on $H_H(S)$ which is homogeneous with bigrading $\{2n, -2\}$, and

$$H_*(H_H(S), d_-(n)) \cong H_n(S)$$

where we are viewing $H_n(S)$ as a bigraded vector space. (This is the homology $H_n^\pm(S)$ from Section 2.3.) Recall that as a bigraded vector space, $H_n(S)$ lies in a single horizontal grading, namely $gr_h = 2r(S)$.

The theorem will be proved in two parts. First we will show that any bigraded vector space with certain algebraic properties must be isomorphic to $H_H(S) < k >$, and then we will show that our construction satisfies those properties for $k = 1$.

Lemma 3.3.2. *Let $H(S)$ denote a bigraded vector space with the following properties:*

- 1) $H(S)$ is bounded above and below in horizontal grading.
- 2) $H(S)$ is bounded below in q -grading.
- 3) $H(S)$ is finite dimensional in each bigrading.
- 4) For all $n \geq n_0$, there is a differential d_n on $H(S)$ which is homogeneous with bigrading $\{2n, -2\}$ such that $H_*(H(S), d_n) \cong H_n(S)$, where the isomorphism is as bigraded vector spaces.

Then $H(S) \cong H_H(S)$.

Note that HOMFLY-PT homology itself satisfies these conditions, so they are not vacuous.

Proof. Let $H^{i,j}$ denote the homology in bigrading $\{i, j\}$, and similarly for $H_H^{i,j}(S)$. The lemma states that for any integers i and j , $\dim(H^{i,j}(S)) = \dim(H_H^{i,j}(S))$.

Suppose for some i, j , the dimensions do not agree. Since $H(S)$ is bounded in horizontal grading, we can take choose i_0 and j_0 so that j_0 is minimized subject to the constraint that $\dim(H^{i_0, j_0}(S)) \neq \dim(H_H^{i_0, j_0}(S))$. Note that the choice for i_0 may not be unique.

Since both $H(S)$ and $H_H(S)$ are bounded below in q -grading, there exists a constant $a(S)$ such that for all $i \leq a(S)$, $\dim(H^{i,j}(S)) = \dim(H_H^{i,j}(S)) = 0$

Choose $n > \max(i_0 - a, n_0)$. We will now use the fact that $H_*(H(S), d_n) \cong H_n(S)$ as bigraded vector spaces. We can rewrite sl_n homology as the homology of $H_H(S)$ with respect to $d_-(n)$.

$$H_*(H(S), d_n) \cong H_*(H_H(S), d_-(n))$$

The differentials on both of these complexes have bigrading $\{2n, -2\}$. We can put an equivalence relation on $\mathbb{Z}^2$ where $(i, j) \sim (i', j')$ if $(i - i', j - j') = k(2n, -2)$, and

both complexes must split according to this equivalence relation. In other words, for a fixed equivalence class A , the sum

$$\bigoplus_{(i,j) \in A} H^{i,j}(S)$$

gives a subcomplex of $(H(S), d_n)$, and similarly for $(H_H(S), d_-(n))$.

Consider the summand corresponding to equivalence class of (i_0, j_0) . For the complex $(H(S), d_n)$, this summand looks like

$$\dots \xrightarrow{d_n} H^{i_0-2n, j_0+2}(S) \xrightarrow{d_n} H^{i_0, j_0}(S) \xrightarrow{d_n} H^{i_0+2n, j_0-2}(S) \xrightarrow{d_n} \dots$$

and for $(H_H(S), d_-(n))$,

$$\dots \xrightarrow{d_-(n)} H_H^{i_0-2n, j_0+2}(S) \xrightarrow{d_-(n)} H_H^{i_0, j_0}(S) \xrightarrow{d_-(n)} H_H^{i_0+2n, j_0-2}(S) \xrightarrow{d_-(n)} \dots$$

Now, since the complex is bounded below in q -grading and we've chosen n sufficiently large, all of the chain groups before the (i_0, j_0) summand are trivial, so the complexes become

$$\dots 0 \xrightarrow{d_n} 0 \xrightarrow{d_n} H^{i_0, j_0}(S) \xrightarrow{d_n} H^{i_0+2n, j_0-2}(S) \xrightarrow{d_n} \dots$$

and

$$\dots 0 \xrightarrow{d_-(n)} 0 \xrightarrow{d_-(n)} H_H^{i_0, j_0}(S) \xrightarrow{d_-(n)} H_H^{i_0+2n, j_0-2}(S) \xrightarrow{d_-(n)} \dots$$

Since both complexes are bounded in horizontal grading, the two chain complexes are both finitely generated. They therefore have a well-defined Euler characteristic, and since the homologies are isomorphic, the Euler characteristics must be the same.

Since the alternating sums of the dimension of homology is the same as the alternating sum of the dimension of the chain groups themselves, we have the following:

$$\sum_{k=0}^N (-1)^k \dim(H^{i_0+2nk, j_0-2k}(S)) = \sum_{k=0}^N (-1)^k \dim(H_H^{i_0+2nk, j_0-2k}(S))$$

Furthermore, we know that for $j < j_0$, there is an equality $\dim(H^{i,j}(S)) = \dim(H_H^{i,j}(S))$. Thus, the two subcomplexes above have the same dimension in all of the bigradings except bigrading (i_0, j_0) . But this means that for $k \geq 1$, the terms in the two sums are equal, which forces the $k = 0$ terms to be equal. This contradicts our assumption that $\dim(H^{i_0, j_0}(S)) \neq \dim(H_H^{i_0, j_0}(S))$, proving the isomorphism. $\square$

Corollary 3.3.3. *Let $H(S)$ be a bigraded vector space that satisfies conditions 1-4 in Lemma 3.3.2 with one difference - instead of d_n having bigrading $\{2n, -2\}$, it has bigrading $\{2n - 2k, -2\}$. Then $H(S) \cong H_H(S) < k >$.*

Proof. After the change in grading $(gr_q, gr_h) \mapsto (gr_q + kgr_h, gr_h)$, the conditions of Lemma 3.3.2 are satisfied, so this homology is isomorphic to $H_H(S)$. Shifting back to the original gradings proves the corollary. $\square$

To prove our main theorem, we now need to show that our homology

$$\bigoplus_f H_1(S_{f,1}) \otimes H_H(S_{f,2}) \{T_2(S_{f,1}) - T_1(S_{f,1}) - 2r(S_{f,2}), 0\}$$

satisfies the conditions of this corollary for $k = 1$. It clearly satisfies conditions 1, 2, and 3 from Lemma 3.3.2 since each of the summands do, so we just need to define the d_n differentials.

We will define d_n for $n \geq 2$ as follows. It will preserve the direct sum decomposition, and it will act on each $H_1(S_{f,1}) \otimes H_H(S_{f,2})$ summand by $1 \otimes d_-(n-1)$, where $d_-(n-1)$ is the standard sl_{n-1} differential on the HOMFLY-PT homology of $S_{f,2}$.

Since $H_*(H_H(S_{f,2}), d_-(n-1)) \cong H_{n-1}(S_{f,2})$, the homology with respect to d_n is

$$\bigoplus_f H_1(S_{f,1}) \otimes H_{n-1}(S_{f,2}) \{T_2(S_{f,1}) - T_1(S_{f,1}) - 2r(S_{f,2}), 0\}$$

From Section 3.2, we know that this sum is isomorphic as a bigraded vector space to $H_n(S)$. The differential d_n has bigrading $\{2n-2, -2\}$, so applying Corollary 3.3.3, this proves Theorem 3.3.1.

Chapter 4

The Knot Floer Complex at a Vertex in the Cube of Resolutions

4.1 Definition of the Complex

We will assume that the reader is familiar with Heegaard diagrams and knot Floer homology. For background on the subject, refer to [11], [12], [15]. The oriented cube of resolutions for HFK was originally defined with twisted coefficients by Ozsváth and Szabó, and they noted some similarities between their complex and HOMFLY-PT homology ([13]). The complex was further studied by Gilmore, who reframed the relationships in terms of framed trivalent graphs ([4]).

However, we will be dealing with the untwisted version defined by Manolescu in [10]. This is in some ways the most natural version, as it doesn't involve twisted coefficients, and the total homology of the complex is the usual knot Floer homology. The oriented cube of resolutions uses the Heegaard diagram shown in Figure 4.1.

Note that unlike in [10], we do not have a marked edge at which an α and a β circle are removed. Instead, we place an additional X and O outside of our braid - we will denote this special X by X_0 . Since discs are not allowed to pass through X_0 , this can

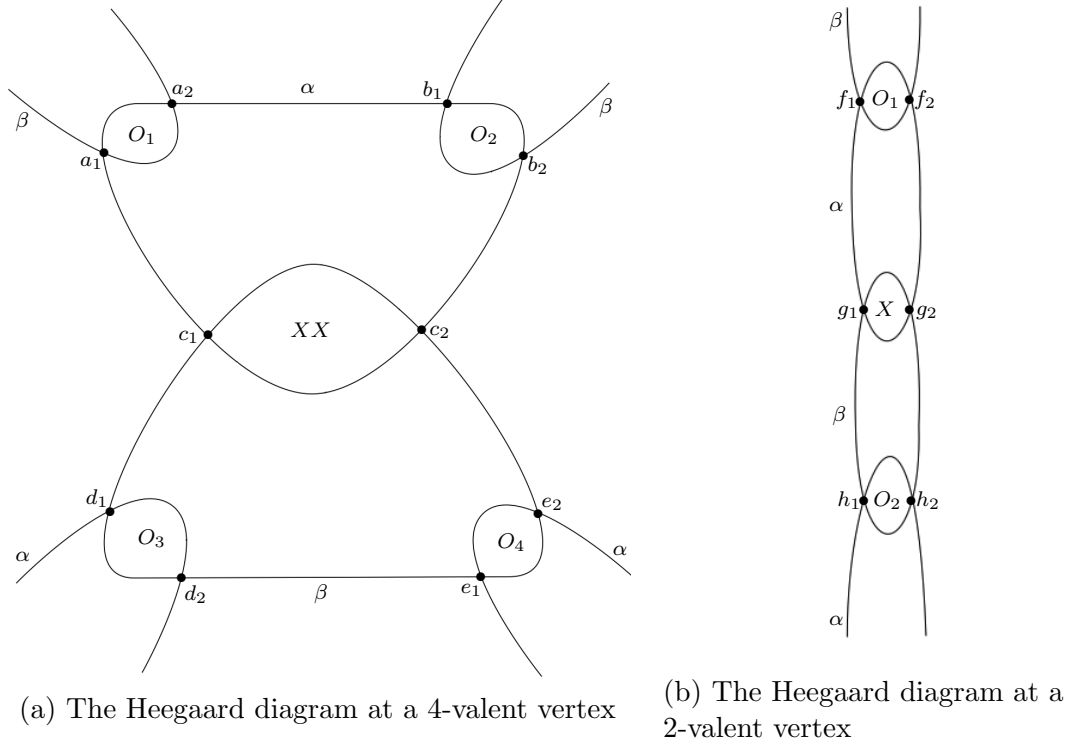


Figure 4.1: The local Heegaard diagram for a singular link

be viewed as puncturing the sphere, making our diagram a truly planar diagram. We will also set the U corresponding to the O equal to zero to avoid increasing the ground ring. The net effect of this change is that we have added an unlinked component and reduced it, so the homology is twice that of [10].

The knot Floer complex corresponding to this Heegaard diagram is denoted $CFK^-(S)$. There are several versions of knot Floer homology, and the minus sign refers to the fact that none of the U_i in the ground ring will be set to 0.

The complex ascribed to a vertex in the cube of resolutions, which we will denote $C_F(S)$, is the tensor product of $CFK^-(S)$ with a certain Koszul complex. Using the terminology from the previous section, we can define $C_F(S)$ as follows:

$$C_F(S) = CFK^-(S) \otimes \bigotimes_{v \in V_4(S)} \left(R \xrightarrow{L(v)} R \right)$$

We will denote the Koszul complex by $K(S)$.

The Generators of $CFK^-(S)$

In order to understand the homology of $C_F(S)$, we are going to need some tools for understanding $CFK^-(S)$. Let $E(S)$ denote the set of edges of S , and let x be a generator of the complex $CFK^-(S)$ (i.e. an n -tuple of intersection points of the α and β curves). We ascribe a subset Z of E to the generator x as follows.

Each O_i in the Heegaard diagram is contained in a unique minimal bigon. The boundary of this bigon contains two intersection points - if either of these intersection points are in the n -tuple x , then e_i is in Z . For example, in Figure 4.1a, there are 5 types of generators: (a, d) , (a, e) , (b, d) , (b, e) , and (c) . The underlying sets of edges of these generators are e_1e_3 , e_1e_4 , e_2e_3 , e_2e_4 , and $\emptyset$, respectively.

As observed in [13], Z must satisfy two conditions. First, for any vertex v in S , the number of incoming edges in Z must equal the number of outgoing edges in Z , and second, Z can not contain all four edges at any 4-valent vertex in S . In other words, Z must be a disjoint union of oriented circles contained in S . We call such a set of edges a *multi-cycle*. Note that multi-cycles differ from the homological cycles in Section 3.1 in that multi-cycles cannot contain all four edges at a vertex.

Let $CFK^-(Z)$ denote the R -module spanned by generators x where the multi-cycle underlying x is Z , and let $C_F(Z)$ be the tensor product of $CFK^-(Z)$ with the Kozsul complex.

$$C_F(Z) = CFK^-(Z) \otimes \bigotimes_{v \in V_4(S)} \left(R \xrightarrow{L(v)} R \right)$$

4.2 The Filtered Complex and the Spectral Sequence from HOMFLY-PT Homology to HFK

4.2.1 A Filtration on $CFK^-(S)$

It turns out that there is a filtration on $CFK^-(S)$ that divides generators according to their underlying cycles. In other words, if there is a filtration-preserving differential d with $d(x) = y$, then x and y have the same underlying cycle.

This filtration is induced by placing additional basepoints p_i in our Heegaard diagram as shown in Figure 3. The markings p_i are in canonical bijection with regions in $\mathbb{R}^2 - S$.

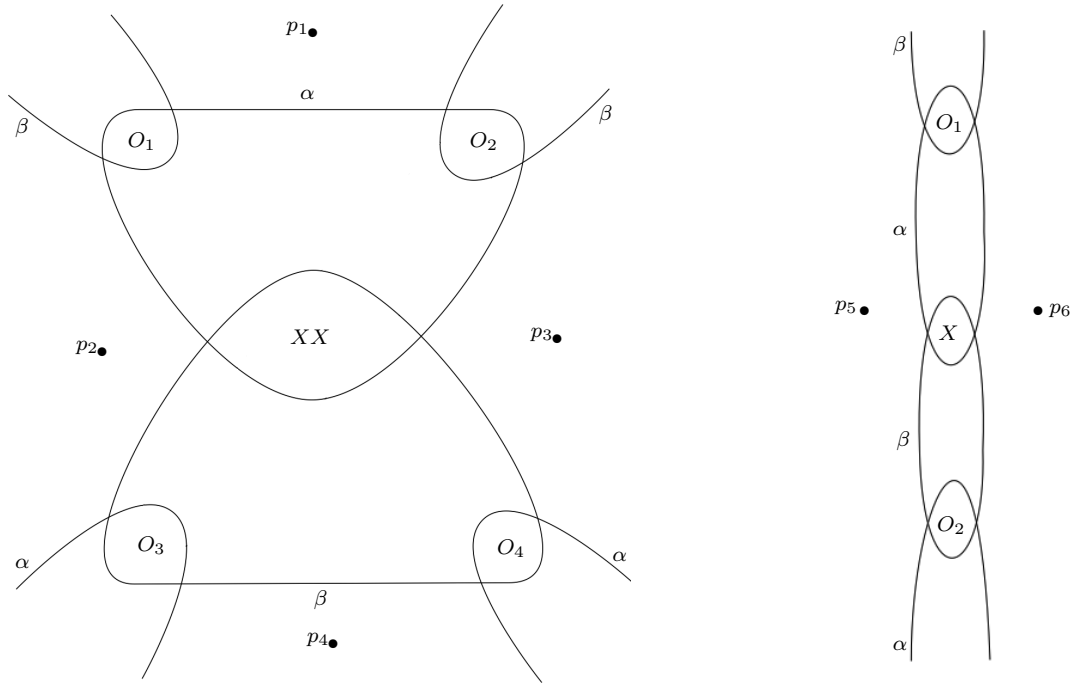


Figure 4.2: Local Diagrams with Additional Markings

Lemma 4.2.1. *These markings define a filtration on the complex $CFK^-(S)$, where the change in filtration level of a differential is given by the sum of the multiplicities of the corresponding holomorphic disc at these basepoints.*

Proof. It is sufficient to show that any periodic domain has multiplicity zero at these markings. This follows from that fact that for any α or β circle, the markings and X_0 lie on the same side. So for any periodic domain, the multiplicity at any of these points is the same as that of X_0 , which is required to be zero. $\square$

We extend this filtration to $C_F(S)$ by placing the Koszul complex in a single filtration level. Let d_k denote the component of the differential on $C_F(S)$ which increases the filtration by k .

Lemma 4.2.2. *The differential d_0 preserves $C_F(Z)$, i.e. it does not change the underlying cycle of a generator.*

Proof. Each basepoint gives a filtration on our complex corresponding to a region in the knot projection. Let x be a generator with multi-cycle Z , and let C be an oriented 2-chain with boundary Z . If we require that C has multiplicity 0 on the outer region (the one corresponding to X_0), it is clear that this 2-chain is unique.

Within the planar Heegaard diagram for K , we can find a disc that connects x to a generator corresponding to the empty cycle (see [13], section 3), and since each region in the knot projection contains a basepoint, the multiplicities of the disc at each basepoint will be equal to the multiplicity of C in that region. Thus, each filtration level uniquely determines a 2-chain C , whose boundary gives the multi-cycle Z . Since no two multi-cycles correspond to the same 2-chain, this completes the proof. $\square$

This means that the filtered homology, also called the homology of the associated graded object, must split over the multi-cycles Z .

4.2.2 Homology of a Cycle

Before computing the homology $H(C_F(Z), d_0)$, we will need a definition. If S is a singular braid and Z is a multi-cycle in S , let $S - Z$ denote the diagram obtained by

removing all edges in Z from S . Note that $S - Z$ is still a singular braid because Z is an oriented cycle in the graph.

Given a cycle Z , the complex $CFK^-(Z)$ is easy to compute. Each intersection point in the Heegaard diagram lies on a unique convex bigon (convex in the traditional planar geometry sense), and this bigon either contains an X , an XX , or a O_i . There are canonical bijections between the O_i bigons and the edges e_i , between the X bigons and $V_2(S)$ and between the XX bigons and $V_4(S)$.

Given a generator x , let $W_2(x)$ denote the set of vertices at which x has an intersection point on one of the X bigons, and let $W_4(x)$ denote the set of vertices at which x has an intersection point on one of the XX bigons. Note that $W_2(x)$ and $W_4(x)$ are uniquely determined by the underlying cycle Z of x . In particular, $W_2(x)$ and $W_4(x)$ are those vertices which are not endpoints of any edges in Z . We can therefore define $W_2(Z)$ and $W_4(Z)$ accordingly.

The complex for a cycle Z can now be described as follows. Each edge e_i in Z corresponds to two intersection points, which are connected by a bigon containing O_i . These are the only filtered differentials involving these two intersection points, so $CFK^-(Z)$ is going to come with a tensor summand of the Koszul complex

$$\bigotimes_{e_i \in Z} R \xrightarrow{U_i} R$$

Each vertex v in $W_2(Z)$ also corresponds to two intersection points. They are connected by two bigons, one which passes through O_i (where e_i is the outgoing edge from v) and one which passes through O_j (where e_j is the incoming edge at v). These two bigons will give a coefficient of $\pm(U_i - U_j)$. Thus, we also get a tensor summand of the Koszul complex

$$\bigotimes_{v \in W_2(Z)} R \xrightarrow{L(v)} R$$

Proving the signs requires slightly more advance machinery, and will be discussed at the end of the section.

Finally, the vertices v in $W_4(Z)$ correspond to two intersection points, also connected by two bigons. One passes through O_i and O_j , where e_i and e_j are the outgoing edges of v , and the other passes through O_k and O_l , where e_k and e_l are the incoming edges at v . These two bigons will contribute a coefficient of $\pm(U_i U_j - U_k U_l)$, giving us the last Koszul complex

$$\bigotimes_{v \in W_4(Z)} R \xrightarrow{Q(v)} R$$

These are all the generators and all the differentials, so the total complex is given by

$$CFK^-(Z) = \left[\bigotimes_{e_i \in Z} R \xrightarrow{U_i} R \right] \otimes \left[\bigotimes_{v \in W_2(Z)} R \xrightarrow{L(v)} R \right] \otimes \left[\bigotimes_{v \in W_4(Z)} R \xrightarrow{Q(v)} R \right]$$

and so the total complex for $C_F(Z)$ is given by

$$C_F(Z) = \left[\bigotimes_{e_i \in Z} R \xrightarrow{U_i} R \right] \otimes \left[\bigotimes_{v \in W_2(Z)} R \xrightarrow{L(v)} R \right] \otimes \left[\bigotimes_{v \in W_4(Z)} R \xrightarrow{Q(v)} R \right] \otimes \left[\bigotimes_{v \in V_4(S)} R \xrightarrow{L(v)} R \right]$$

Lemma 4.2.3. *The filtered homology $H_*(C_F(Z), d_0)$ is isomorphic to $H_H(S - Z)$.*

Proof. The U_i in the first tensor product form a regular sequence in R , so we can cancel all of these differentials. This has the effect of setting U_i equal to zero for all e_i in Z . Let R_Z be the quotient $R/\{U_i = 0 \text{ for } e_i \in Z\}$. Note that this is precisely the ground ring for the singular braid $S - Z$.

We are left with the complex

$$\left[\bigotimes_{v \in W_2(Z)} R_Z \xrightarrow{L(v)} R_Z \right] \otimes \left[\bigotimes_{v \in W_4(Z)} R_Z \xrightarrow{Q(v)} R_Z \right] \otimes \left[\bigotimes_{v \in V_4(S)} R_Z \xrightarrow{L(v)} R_Z \right]$$

For each 4-valent vertex v in $S - Z$, we have tensor summands $R_Z \xrightarrow{L(v)} R_Z$ and $R_Z \xrightarrow{Q(v)} R_Z$, which together give a summand of

$$\begin{array}{ccc} R_Z & \xrightarrow{L(v)} & R_Z \\ Q(v) \downarrow & & \downarrow Q(v) \\ R_Z & \xrightarrow{L(v)} & R_Z \end{array}$$

which is precisely the HOMFLY-PT summand $C_H(v)$. For 2-valent vertices v in $S - Z$, there are two possibilities to consider - v is 2-valent in S ($v \in W_2(Z)$), and v is 4-valent in S ($v \in V_4(S), v \notin W_4(Z)$). When v is 2-valent in S , we get the summand

$$R_Z \xrightarrow{L(v)} R_Z$$

which is again the HOMFLY-PT summand $C_H(v)$ for $S - Z$. For v 4-valent in S , let e_i and e_j be the outgoing edges at v and e_k and e_l the incoming edges at v . Since $S - Z$ is 2-valent at v , we know that Z must include one outgoing edge and one incoming edge. Without loss of generality, assume they are e_i and e_k . To avoid confusion, we will write out the terms of the linear elements, as $L(v)$ refers to $U_i + U_j - U_k - U_l$ in S , while $L(v)$ refers to $U_j - U_l$ in $S - Z$.

In $C_F(Z)$, we have the summand

$$R_Z \xrightarrow{U_i + U_j - U_k - U_l} R_Z$$

In the HOMFLY-PT complex for $S - Z$, on the other hand, we have the summand

$$R_Z \xrightarrow{U_j - U_l} R_Z$$

Fortunately, since e_i and e_k are in Z , U_i and U_k are zero in R_Z , so $U_i + U_j - U_k - U_l = U_j - U_l$, making the above complexes isomorphic.

Thus, after canceling the Koszul complex on the edges in Z , we get exactly the HOMFLY-PT complex for $S - Z$. It follows that $H_*(C_F(Z), d_0) \cong H_H(S - Z)$. $\square$

Corollary 4.2.4. *The filtered homology decomposes as the direct sum*

$$H_*(C_F(S), d_0) \cong \bigoplus_Z H_H(S - Z)$$

Remark 4.2.5. Signs. Since we are discussing Koszul complexes, the $\pm$ in the terms $\pm(U_i - U_j)$ and $\pm(U_i U_j - U_k U_l)$ are not relevant. Some will have to come with positive signs and some with negative to make $d^2 = 0$, but where they are doesn't impact the chain homotopy type. What we need to show is that the two bigons in each case come with *different* signs.

The two-valent vertex corresponds to a specific X marking in the diagram. This X lies within the same α circle as O_i and the same β circle as O_j . In CFK^- , we do not allow discs to pass through the X basepoints. However, if we do allow them to pass through only this X , we get a new complex. In this complex, d^2 is non-zero - instead, it is a multiple of the identity. This multiple is determined by the α and β degenerations, which will correspond to the α and β circles containing X . Since the α circle contains O_i , it gives a coefficient of U_i , and similarly, the β circle gives a coefficient of U_j . It was shown in [1] that the moduli spaces can be oriented such that the α and β degenerations come with opposite signs, so choosing such a sign convention, this gives

$$d^2 = \pm(U_i - U_j)I$$

Moreover, the additional differentials are also subject to the basepoint filtration, so we get

$$d_0^2 = \pm(U_i - U_j)I$$

This X basepoint lies inside a minimal bigon, and this bigon now contributes to the differential with a coefficient of ± 1 . The local contribution therefore must be the the complex in Figure 4.3, so U_i and U_j must come with opposite sign.

$$\begin{array}{ccc} & \pm(U_i - U_j) & \\ R & \xrightarrow{\hspace{1.5cm}} & R \\ & \xleftarrow{\hspace{1.5cm}} & \\ & \pm 1 & \end{array}$$

Figure 4.3: Local complex when allowing discs to pass through the X basepoint.

The argument for the quadratic term is the same, only instead of allowing discs to pass through an X , we are allowing them to pass through an XX . The α degeneration is $U_i U_j$ and the β degeneration is $U_k U_l$, and they must come with opposite sign, so we get the complex in Figure 4.4, which proves that $U_i U_j$ and $U_k U_l$ come with opposite sign.

$$\begin{array}{ccc} & \pm(U_i U_j - U_k U_l) & \\ R & \xrightarrow{\hspace{1.5cm}} & R \\ & \xleftarrow{\hspace{1.5cm}} & \\ & \pm 1 & \end{array}$$

Figure 4.4: Local complex when allowing discs to pass through the XX basepoint.

4.2.3 Gradings

The knot Floer complex comes equipped with two gradings: the Maslov grading M and the Alexander grading A . The differential decreases the Maslov grading by 1 and preserves the Alexander grading. Multiplication by U_i decreases the Maslov grading by 2 and decreases the Alexander grading by 1.

Certain linear combinations of the Maslov and Alexander gradings return analogs of the quantum and horizontal gradings from the Khovanov-Rozansky complex. Let gr_q be given by $-2M+2A$, and gr_h by $-2M+4A$. Note that the knot Floer differential has bigrading $\{2, 2\}$ with respect to this differential and multiplication by U_i changes the bigrading by $\{2, 0\}$, the same as the Khovanov-Rozansky complex. Instead of the Maslov and Alexander gradings, we will henceforth use the quantum and horizontal gradings.

Before computing gradings, we need to introduce some terminology. Given a multi-cycle Z , let $T_1(Z)$ denote the the number of vertices $v \in V_4(S)$ at which Z contains the edges e_1 and e_3 in Figure 4.5. Similarly, let $D_1(Z)$ denote the number of vertices at which Z contains the edges e_1 and e_4 , $D_2(Z)$ the number of vertices at which Z contains the edges e_2 and e_3 , and $T_2(Z)$ denote the the number of vertices at which Z contains the edges e_2 and e_4 .

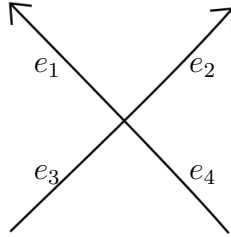


Figure 4.5: A labeled 4-valent vertex

We will now compute the bigrading on the knot Floer complex, up to an overall grading shift. Since the generators corresponding to the empty cycle give a complex which is isomorphic to the HOMFLY-PT complex for S , we can choose our overall shift so that this isomorphism preserves the bigrading.

Let Z denote a multi-cycle in S consisting of k cycles. Viewing Z as a braid diagram for the k -component unlink, we can define $r(Z)$ to be the rotation number of Z . Thus, $r(Z) = -k$. Let x denote the generator corresponding to Z at the bottom of the Koszul complex (i.e. with the largest horizontal grading). Similarly, let y denote

the generator corresponding to the empty cycle with the largest horizontal grading. In [13], Szábo and Ozsváth identify k differentials whose composition takes x to y , whose coefficient in R has degree $T_2(Z) + \frac{1}{2}(D_1(Z) + D_2(Z))$. Using the fact that differentials have bigrading $\{2, 2\}$, we can see that x and y differ in grading by

$$\{2r(Z) + 2T_2(Z) + D_1(Z) + D_2(Z), 2r(Z)\}$$

The bottom generator of the HOMFLY-PT complex has bigrading $\{-|V_4(S)|, 0\}$, so y does as well. Thus, x has bigrading

$$\{-|V_4(S)| + 2r(Z) + 2T_2(Z) + D_1(Z) + D_2(Z), 2r(Z)\}$$

The bottom generator of the HOMFLY-PT complex for $S - Z$ has bigrading $\{-|V_4(S - Z)|, 0\}$, so we get the following:

$$H(C_F(Z), d_0) \cong H_H(S - Z) \{|V_4(S - Z)| - |V_4(S)| + 2r(Z) + 2T_2(Z) + D_1(Z) + D_2(Z), 2r(Z)\}$$

The grading shift in this formula can be simplified somewhat. The quantity $|V_4(S - Z)| - |V_4(S)|$ is the negative of the number of 4-valent vertices in S at which Z contains two edges.

$$|V_4(S - Z)| - |V_4(S)| = -T_1(Z) - T_2(Z) - D_1(Z) - D_2(Z)$$

Thus, the formula becomes

$$H(C_F(Z), d_0) \cong H_H(S - Z) \{2r(Z) + T_2(Z) - T_1(Z), 2r(Z)\}$$

Our choice for the overall grading shift was somewhat arbitrary, so to make it more similar to our composition product formulas, we will add in a grading shift of $\{-2r(S), 0\}$

giving

$$H(C_F(Z), d_0) \cong H_H(S - Z) \{-2r(S - Z) + T_2(Z) - T_1(Z), 2r(Z)\}$$

and we get a graded version of Corollary 4.2.4:

$$H_*(C_F(S), d_0) \cong \bigoplus_Z H_H(S - Z) \{-2r(S - Z) + T_2(Z) - T_1(Z), 2r(Z)\}$$

We will be able to connect this formula to the composition product with the following lemma:

Lemma 4.2.6. *Let f be a labeling of S . The sl_1 homology of $S_{f,1}$ is given by*

$$H_1(S_{f,1}) = \begin{cases} \mathbb{Q}\{0, 2r(S_{f,1})\} & \text{if } S_{f,1} \text{ is a multi-cycle} \\ 0 & \text{otherwise} \end{cases}$$

Applying this lemma, the formula becomes

$$H_*(C_F(S), d_0) \cong \bigoplus_f H_1(S_{f,1}) \otimes H_H(S_{f,2}) \{T_2(S_{f,1}) - T_1(S_{f,1}) - 2r(S_{f,2}), 0\}$$

But by Theorem 3.3.1, this is isomorphic to $H_H(S) < 1 >$. Thus, we have proved the following theorem:

Theorem 4.2.7. *There is an isomorphism of bigraded groups*

$$H_*(C_F(S), d_0) \cong H_H(S) < 1 >$$

Corollary 4.2.8. *There is a spectral sequence whose E_1 page is $H_H(S) < 1 >$ which converges to $H_F(S)$.*

Proof. This is just the spectral sequence induced by the basepoint filtration on $C_F(S)$. □

Manolescu's conjecture is thus equivalent to this spectral sequence collapsing at the E_1 page.

4.3 Additional Differentials and the Spectral Sequences from HFK to sl_n

We are going to add differentials to the complex $C_F(S)$ so that the total homology is isomorphic to $H_{n+1}(S)$ for any $n \geq 1$. These new differentials do not preserve the Alexander grading, so using the Alexander grading as a filtration, this induces a spectral sequence from $H_F(S)$ to $H_{n+1}(S)$.

The complex $C_F(S)$ is constructed as a tensor product of complexes $CFK^-(S)$ and a Koszul complex $K(S)$ on linear elements.

$$C_F(S) = CFK^-(S) \otimes K(S)$$

The complex $CFK^-(S)$ does not count discs which pass through the X or XX markings. For the new differential, we are going to count these discs with certain polynomial coefficients.

Each X marking in the Heegaard diagram corresponds to a 2-valent vertex v in S . Whenever a holomorphic disc passes through this X with multiplicity k , it picks up a coefficient of $u_1(v)^k$. The only exception is the special marking X_0 , at which we still require discs to have multiplicity 0. Similarly, each XX corresponds to a 4-valent vertex v in S . If a holomorphic disc passes through this XX with multiplicity k , it picks up a coefficient of $u_2(v)^k$. We will call this new complex $CFK_n^-(S)$.

Note that there is no guarantee that the differential on this complex squares to

zero - in fact, it doesn't.

To fix this, we will also modify the differential on the Koszul complex. Originally, it was given by

$$K(S) = \bigotimes_{v \in V_4(S)} R \xrightarrow{L(v)} R$$

We are going to add in differentials to make it a matrix factorization:

$$K_n(S) = \bigotimes_{v \in V_4(S)} R \xrightleftharpoons[u_1(v)]{L(v)} R$$

The total complex $C_{F(n)}(S)$ is defined to be the tensor product of $CFK_n^-(S)$ and $K_n(S)$.

$$C_{F(n)}(S) = CFK_n^-(S) \otimes K_n(S)$$

Lemma 4.3.1. *The differential on $C_{F(n)}(S)$ satisfies $d^2 = 0$.*

Proof. At each vertex v in S , d^2 is going to have a local contribution of $w_n(v)$, which is given by

$$\sum_{e_i \in E_{out}} U_i^{n+1} - \sum_{e_j \in E_{in}} U_j^{n+1}$$

The lemma will then follow from the equality $\sum_{v \in S} w_n(v) = 0$.

The quantity d^2 has two contributions, one from $CFK_n^-(S)$ and one from $K_n(S)$. The contribution from $K_n(S)$ can be computed directly to be

$$\sum_{v \in V_4(S)} L(v)u_1(v)$$

The contribution from $CFK_n^-(S)$ can be computed via the α and β degenerations. We orient the moduli spaces so that the α and β degenerations come with opposite signs, with the α degenerations being positive and the β degenerations negative (see [1])

for an explanation of the signs).

At each 2-valent vertex v in S , we have one α circle and one β circle, shown in Figure 4.6 . The α circle contains U_i and X , and the X contributes coefficient $u_1(v)$, so the α degeneration contributes $U_i u_1(v)$. Similarly, the β circle contains U_j and X , so its contribution is $-U_j u_1(v)$. Thus, the net contribution at v is $(U_i - U_j)u_1(v)$. This can be simplified to $L(v)u_1(v) = w_n(v)$.



Figure 4.6: α -degenerations (left) and β -degenerations (right) at a bivalent vertex

At each 4-valent vertex v in S , we also have one α circle and one β circle, shown in Figure 4.7. The α circle contains U_i, U_j , and XX , and the XX contributes coefficient $u_2(v)$, so its contribution is $U_i U_j u_2(v)$. The β circle contains U_k, U_l , and XX , so its contribution is $-U_k U_l u_2(v)$. Thus, the net contribution at v is $(U_i U_j - U_k U_l)u_2(v)$. This can be simplified to $Q(v)u_2(v)$.

Thus, counting the contribution from $K_n(S)$, we see that d^2 is given by

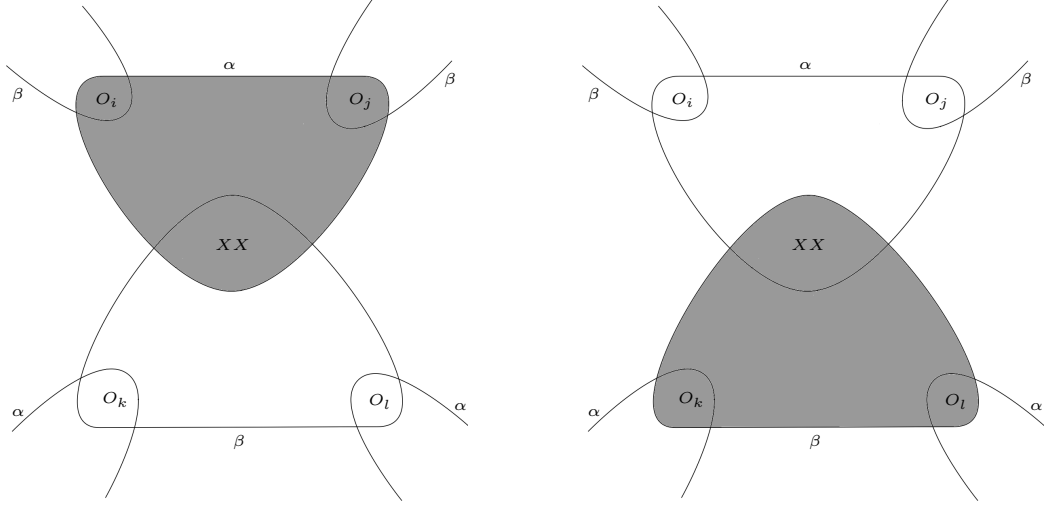


Figure 4.7: α -degenerations (left) and β -degenerations (right) at a four-valent vertex

$$\begin{aligned}
d^2 &= \sum_{v \in V_2(S)} w_n(v) + \sum_{v \in V_4(S)} Q(v)u_2(v) + \sum_{v \in V_4(S)} L(v)u_1(v) \\
&= \sum_{v \in V_2(S)} w_n(v) + \sum_{v \in V_4(S)} L(v)u_1(v) + Q(v)u_2(v) \\
&= \sum_{v \in V_2(S)} w_n(v) + \sum_{v \in V_4(S)} w_n(v) \\
&= \sum_{v \in S} w_n(v) = 0
\end{aligned}$$

□

We can extend the basepoint filtration from Section 4.2.1 to make $C_{F(n)}(S)$ a filtered complex - since we still require discs to have multiplicity 0 at X_0 , the same argument works as in the proof of Lemma 4.2.1. Let d_i denote the differentials which change the filtration level by i . As before, d_0 must preserve multi-cycles, so the homology $H(C_{F(n)}(S), d_0)$ splits over the multi-cycles

$$H(C_{F(n)}(S), d_0) = \bigoplus_Z H(C_{F(n)}(Z), d_0)$$

We want to compute the complex $C_{F(n)}(Z)$. Recall that $C_F(Z)$ was computed to be

$$\left[\bigotimes_{e_i \in Z} R \xrightarrow{U_i} R \right] \otimes \left[\bigotimes_{v \in W_2(Z)} R \xrightarrow{L(v)} R \right] \otimes \left[\bigotimes_{v \in W_4(Z)} R \xrightarrow{Q(v)} R \right] \otimes \left[\bigotimes_{v \in V_4(S)} R \xrightarrow{L(v)} R \right]$$

We can therefore compute $C_{F(n)}(Z)$ by adding the new differentials to this complex. It is not hard to see that the only new discs in $CFK_n^-(Z)$ correspond to bigons containing X or XX basepoints. For example, let e_i be an edge in Z , with x and y the two intersection points corresponding to e_i . When we weren't allowing discs to pass through X or XX , the only disc connecting x and y was the bigon containing U_i . This contributed the tensor summand of

$$R \xrightarrow{U_i} R$$

However, when we allow discs to pass through X and XX , we get two new bigons which map from y to x , shown in Figure 4.8.



Figure 4.8: Two new bigons from y to x

The type of contribution from these bigons depends on whether the endpoint vertices of e_i are 2-valent or 4-valent, which is why we only showed the local portion of the bigons in Figure 4.8. However, in either case the contribution has a coefficient of degree n . We will denote this coefficient by $p(e_i)$ (the precise polynomial will not be relevant for our computations). The tensor summand then becomes

$$R \begin{array}{c} \xrightarrow{U_i} \\ \xleftarrow{p(e_i)} \end{array} R$$

For a vertex v in $W_2(Z)$, there are two intersection points x and y corresponding to v . In $C_F(Z)$, they contributed a tensor summand of

$$R \xrightarrow{L(v)} R$$

In $C_{F(n)}(Z)$, we have an extra differential corresponding to the bigon from y to x through X (See Figure 4.9a). Since X carries a coefficient of $u_1(v)$, the summand becomes

$$R \begin{array}{c} \xrightarrow{L(v)} \\ \xleftarrow{u_1(v)} \end{array} R$$

Similarly, for 4-valent vertices v in $W_4(Z)$, $C_F(Z)$ contains a tensor summand

$$R \xrightarrow{Q(v)} R$$

In $C_{F(n)}(Z)$, we have an extra differential corresponding to the bigon through XX shown in Figure 4.9b. The XX contributes a coefficient of $u_2(v)$, so the summand becomes

$$R \begin{array}{c} \xrightarrow{Q(v)} \\ \xleftarrow{u_2(v)} \end{array} R$$

Finally, the complex $K(S)$ gets changed to $K_n(S)$, so the whole complex for $C_{F(n)}(Z)$ can be written as

$$\left[\bigotimes_{e_i \in Z} R \begin{array}{c} \xrightarrow{U_i} \\ \xleftarrow{p(e_i)} \end{array} R \right] \otimes \left[\bigotimes_{v \in W_2(Z)} R \begin{array}{c} \xrightarrow{L(v)} \\ \xleftarrow{u_1(v)} \end{array} R \right] \otimes \left[\bigotimes_{v \in W_4(Z)} R \begin{array}{c} \xrightarrow{Q(v)} \\ \xleftarrow{u_2(v)} \end{array} R \right] \otimes \left[\bigotimes_{v \in V_4(S)} R \begin{array}{c} \xrightarrow{L(v)} \\ \xleftarrow{u_1(v)} \end{array} R \right]$$

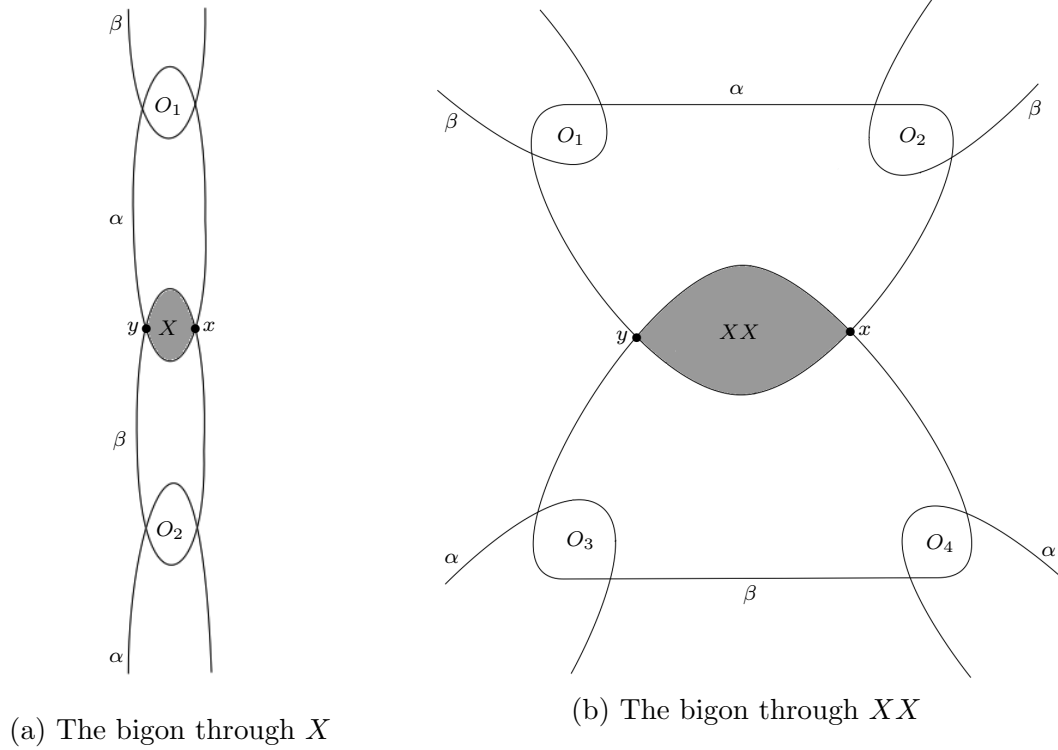


Figure 4.9: New differentials passing through X and XX

Now that we have our complex computed, we want to compare its homology with $H_n(S - Z)$. We will denote the differentials which do not pass through any X or XX basepoints d_{0+} , and the new differentials d_{0-} . Observe that with respect to the bigrading (gr_q, gr_h) introduced in Section 4.2.3, d_{0+} has bigrading $\{2, 2\}$, and d_{0-} has bigrading $\{2n, -2\}$.

Lemma 4.3.2. *Up to an overall grading shift, the homology $H_*(H_*(C_{F(n)}(Z), d_{0+}), d_{0-}^*)$ is isomorphic to $H_*(H_*(C_n(S - Z), d_+), d_-^*)$*

Proof. It follows from Theorem 4.2.3 that $H(C_{F(n)}(Z), d_{0+}) \cong H_H(S - Z)$. To complete the proof, we need to show that d_{0-}^* corresponds to the d_- differential under this isomorphism. For vertices which are 2-valent in both S and $S - Z$ (i.e. $v \in W_2(Z)$), this is obvious. The same is true for vertices which are 4-valent in both S and $S - Z$ (i.e. $v \in W_4(Z)$).

The only identification which is non-trivial is that the d_{0-} differential correspond-

ing to a vertex which is 4-valent in S but 2-valent in $S - Z$ is the same as the d_- differential on the 2-valent vertex in $S - Z$. Let e_i, e_j be the outgoing edges at v and e_k, e_l the incoming edges. The multi-cycle Z must contain one incoming and one outgoing edge - without loss of generality, assume Z contains e_i and e_k . The coefficient of the d_- differential is given by

$$\frac{U_j^{n+1} - U_l^{n+1}}{U_j - U_l}$$

while the coefficient of the d_{0-} differential is given by

$$\frac{U_i^{n+1} + U_j^{n+1} - U_k^{n+1} - U_l^{n+1} - Q(v)u_2(v)}{U_i + U_j - U_k - U_l}$$

Recall that to achieve the isomorphism in Theorem 4.2.3, we first cancelled the Koszul complex on the U_i for e_i in Z , as these elements formed a regular sequence. We therefore want to show that these two coefficients are equal in $R_Z = R/\{U_i = 0 \text{ for } e_i \in Z\}$. Substituting $U_i = U_k = 0$ into the above equation and noting that this causes $Q(v)$ to be zero, we get the desired equality. $\square$

Define $H^\pm(C_{F(n)}(S)) = H_*(H_*(C_{F(n)}(S), d_{0+}), d_{0-}^*)$. Since both d_{0+} and d_{0-} are homogeneous with respect to the bigrading, this homology is bigraded as well. Applying the lemma and adding in the gradings from Section 4.2.3, we see that

$$H^\pm(C_{F(n)}(S)) \cong \bigoplus_Z H^\pm(C_n(S - Z))\{-2r(S - Z) + T_2(Z) - T_1(Z), 2r(Z)\} \quad (4.1)$$

Recall from Lemma 2.3.2 that $H^\pm(C_n(S - Z))$ lies in a single horizontal grading, namely $2r(S - Z)$. Adding in the shift, the homology corresponding to a multi-cycle Z must lie in horizontal grading $2r(S - Z) + 2r(Z) = 2r(S)$. But this does not depend on Z , so we have shown the following:

Lemma 4.3.3. *The homology $H^\pm(C_n(S))$ lies in a single horizontal grading.*

The original differentials on $C_F(S)$ all have bigrading $\{2, 2\}$. The new differentials on $CFK_n^-(S)$ have bigrading $\{2 + 2k(n - 1), 2 - 4k\}$, where k is the sum of the multiplicities of the holomorphic discs at all X and XX markings. The new differentials on $K_n(S)$ all have bigrading $\{2n, -2\}$. Thus, all differentials on $C_n(S)$ change the horizontal grading by 2 (mod 4). This implies that no induced differentials can have horizontal grading 0, which tells us that the remaining differentials on our complex are all trivial, giving us the following:

Lemma 4.3.4. *The total homology $H_*(C_{F(n)}(S), d)$ is isomorphic to $H^\pm(C_{F(n)}(S))$.*

This isomorphism is singly-graded with grading $gr_n = gr_q + (n - 1)gr_h/2$, as the total differential on $C_{F(n)}(S)$ is homogeneous of degree $n + 1$ with respect to this grading.

Going back to (4.1), we know that as bigraded vector spaces, we have the isomorphism

$$H^\pm(C_{F(n)}(S)) \cong \bigoplus_Z H^\pm(C_n(S - Z)) \{-2r(S - Z) + T_2(Z) - T_1(Z), 2r(Z)\}$$

We can use Lemma 4.2.6 in the same way as in the previous section to exchange the cycle notation for labelings by tensoring with sl_1 homology.

$$H^\pm(C_{F(n)}(S)) \cong \bigoplus_f H_1(S_{f,1}) \otimes H_n(S_{f,2}) \{-2r(S_{f,2}) + T_2(S_{f,1}) - T_1(S_{f,1}), 0\}$$

Applying the bigraded composition product formula (3.5), this gives an isomorphism of bigraded vector spaces

$$H^\pm(C_{F(n)}(S)) \cong H_{n+1}(S) \tag{4.2}$$

Since $H_*(C_{F(n)}(S), d) \cong H^\pm(C_{F(n)}(S))$ as graded vector spaces with grading gr_n , this gives an isomorphism

$$H_*(C_{F(n)}(S), d) \cong H_{n+1}(S)$$

where we are viewing $H_{n+1}(S)$ as singly graded, with grading gr_n . Since singly graded sl_{n+1} homology is typically viewed with respect to the grading gr_{n+1} , this isn't quite what we want. Fortunately, since the homology is concentrated in horizontal grading $gr_h = 2r(S)$, we see that $gr_{n+1} = gr_n + r(S)$. Thus, up to an overall grading shift, the formula remains true in the standard grading gr_{n+1} .

Theorem 4.3.5. *Up to an overall grading shift, the total homology $H_*(C_{F(n)}(S), d)$ is isomorphic to $H_{n+1}(S)$.*

Corollary 4.3.6. *For all $n \geq 1$, there is a spectral sequence whose E_1 page is $H_F(S)$ which converges to $H_{n+1}(S)$.*

Proof. All of the original differentials on $C_F(S)$ have Alexander grading 0. The new differentials on CFK_n^- have Alexander grading $k(-n-1)$, where k is the sum of the multiplicities of the disc at the X and XX basepoints, and the new differentials on the Koszul complex have Alexander grading $-n-1$. In particular, all of the new differentials strictly decrease the Alexander grading, so it induces a filtration with respect to which the filtered homology is $H_*(C_F(S))$. Thus, the corresponding spectral sequence has E_1 page $H_F(S)$, and converges to the total homology $H(C_{F(n)}(S)) \cong H_{n+1}(S)$. $\square$

4.4 Proof of Manolescu's Conjecture

At this point, we have three spectral sequences: one from HOMFLY-PT homology to sl_n homology, one from HOMFLY-PT homology to knot Floer homology, and one from knot Floer homology to sl_n homology. Diagrammatically, this looks like

$$\begin{array}{ccc}
H_H(S) < 1 > & \xrightarrow{\quad} & H_F(S) \\
& \searrow & & \swarrow \\
& H_n(S) & &
\end{array}$$

where the arrows correspond to spectral sequences. Since both HOMFLY-PT homology and knot Floer homology have spectral sequences going to sl_n homology for all $n \geq 2$, it is clear that they have a great deal in common. The conjecture of Manolescu is that they are in fact isomorphic. Since we have a spectral sequence from HOMFLY-PT homology to knot Floer homology, this is equivalent to the spectral sequence collapsing at the E_1 page. In this section, we will prove Manolescu's conjecture.

Theorem 4.4.1. *The HOMFLY-PT homology $H_H(S) < 1 >$ and knot Floer homology $H_F(S)$ are isomorphic as bigraded vector spaces.*

Proof. We will start with the complex $C_{F(n)}$ from the previous section. There are two filtrations defined on this complex so far - the one induced by the Alexander grading, and the one induced by the basepoints. Consider the associated bigraded object from these two filtrations.

The differentials always change the Alexander grading by a multiple of $n + 1$, so let d_{ij} denote those differentials which change the Alexander grading by $i(n + 1)$ and change the basepoint grading by j .

Theorem 4.2.7 states that

$$H_*(C_{F(n)}, d_{00}) \cong H_H(S) < 1 >$$

Since $H_F(S) \cong H_*(C_{F(n)}, d_{0*})$, our theorem is equivalent to d_{0k}^* being zero on $H_*(C_{F(n)}, d_{00})$. We will prove this by contradiction. In particular, suppose that some d_{0k}^* is non-zero, and let a denote the smallest such k .

Because a is minimal, it is clear that d_{0a}^* and d_{10}^* anti-commute, as $d_{0a}^* \circ d_{10}^*$ and $d_{10}^* \circ d_{0a}^*$ are the only components of d^2 which change the basepoint filtration by a and the Alexander filtration by $n+1$. These are the differentials that we will be interested in, so we will rename them. We will write d_F instead of d_{0a}^* and d_n instead of d_{10}^* , since d_{10}^* depends on n . The differentials d_F and d_n both act on $H_*(C_{F(n)}, d_{00})$, so using the above isomorphism we will view them as acting on $H_H(S) < 1 >$.

We know from (4.2) that there is an isomorphism of bigraded vector spaces

$$H_*(H_H(S) < 1 >, d_n) \cong H_{n+1}(S)$$

To summarize our setup, we have a family of differentials d_n on $H_H(S) < 1 >$, each having bigrading $(2n, -2)$, such that the homology with respect to each is H_{n+1} , and there is a differential d_F on $H_H(S) < 1 >$ with bigrading $(2, 2)$ which is non-trivial and anti-commutes with each d_n .

We know that the smallest horizontal grading in which $H_H(S) < 1 >$ is non-trivial is $2r(S)$, and that the homology $H_*(H_H(S) < 1 >, d_n)$ lies only in this horizontal grading. Let h_{min} be the minimal horizontal grading on which d_F is non-zero, and let x be an element of $H_H(S) < 1 >$ in bigrading (q, h_{min}) for some q with $d_F(x) \neq 0$. Define $d_F(x) = y$.

We know also that $H_H(S) < 1 >$ is bounded below in quantum grading and d_n changes the quantum grading by $2n$, so choose N sufficiently large that y can not be in the image of d_N . Since y lies in horizontal grading $h_{min} + 2$ (in particular, not h_{min}), it follows that $y \in Ker(d_N)$ iff $y \in Im(d_N)$. Thus, y is not in the kernel of d_N .

But then $d_N \circ d_F(x)$ is non-zero, while $d_F \circ d_N(x)$ must be zero because $d_N(x)$ lies in horizontal grading $h_{min} - 2$, and $d_F = 0$ for all horizontal gradings less than h_{min} , which contradicts the fact that d_N and d_F anticommute. $\square$

We relate this theorem to Manolescu's conjecture with the following corollary.

Corollary 4.4.2. *There is an isomorphism of Tor groups*

$$\mathrm{Tor}_R(R/L, R/N) \cong \mathrm{Tor}_R(R/L, R/Q)$$

as bigraded vector spaces. The bigrading on $\mathrm{Tor}_R(R/L, R/N)$ is given by (q, h) where q is the quantum grading coming from the polynomial ring and h is the homological grading, and the bigrading on $\mathrm{Tor}_R(R/L, R/Q)$ is $(q + h, h)$.

Proof. The difference between our complex $C_F(S)$ and Manolescu's knot Floer complex for S is that we have added an additional unknotted component at infinity, placed the marked edge on that component, and reduced that component (i.e. set that U_i equal to zero).

Manolescu instead placed the marked edge on the leftmost strand of the braid. Let $C_F^M(S)$ denote Manolescu's complex. We know that adding an unknotted component and reducing it doubles the homology. In particular, with respect to the (q, h) bigrading there is an isomorphism

$$H_F(S) \cong H_F^M(S) \otimes V$$

where $V = \mathbb{Q}\{-1, -1\} \oplus \mathbb{Q}\{1, 1\}$.

Similarly, the middle HOMFLY-PT homology can be viewed as the unreduced HOMFLY-PT homology of the 1-1 tangle obtained by breaking an edge in the diagram. Let $C_H^M(S)$ denote the middle HOMFLY-PT homology of S . Then, with respect to the (q, h) grading, we have the following isomorphism

$$H_H(S) \cong H_H^M(S)\{0, -1\} \oplus H_H^M(S)\{0, 1\}$$

But when we switch to the $(q + h, h)$ grading, this becomes

$$H_H(S) \langle 1 \rangle \cong H_H^M(S) \langle 1 \rangle \otimes V$$

The previous theorem states that $H_F(S) \cong H_H(S) < 1 >$. Using the above arguments, this becomes

$$H_F^M(S) \otimes V \cong H_H^M(S) < 1 > \otimes V$$

as bigraded groups.

Since all of our theories are bounded in h -grading and bounded below in q -grading, the above isomorphism implies an isomorphism without tensoring with V .

$$H_F^M(S) \cong H_H^M(S) < 1 >$$

But Manolescu showed that $H_F^M(S) \cong \text{Tor}_R(R/L, R/N)$ and $H_H^M(S) \cong \text{Tor}_R(R/L, R/Q)$, which proves the corollary. $\square$

One interesting aspect of this corollary is that holomorphic geometry was used to prove an algebraic fact about Tor groups. Typically, algebraic frameworks are used to prove things about holomorphic geometry because the discs are too difficult to count - this was the reason for the Tor groups approach. However, in this case we were able to prove the original conjecture using holomorphic geometry and homological algebra. Thus, the Tor groups isomorphism actually followed from $H_H(S) \cong H_F(S)$, instead of the other way around.

Chapter 5

Generalities and Future Work

All of the work so far has been aimed at proving the spectral sequence from HOMFLY-PT homology to knot Floer homology. Since this spectral sequence would follow from the edge maps in the two cubes commuting with our isomorphism, the next steps are to try to understand this isomorphism and make it explicit, and do the same for the edge maps in $C_F(D)$. We plan to continue this line of inquiry in future papers. However, this research has also brought up some questions that could make interesting projects on their own. We will conclude with the discussion of two such problems.

5.1 A Family of Cube Complexes for HOMFLY-PT Homology

The standard construction for HOMFLY-PT homology comes from a cube of resolutions complex. The complex is filtered by the cube filtration, and the HOMFLY-PT homology is defined to be the E_2 page of the spectral sequence induced by this filtration.

$$H_H(D) \cong H_*(H_*(C_H(D), d_0), d_1^*)$$

The E_∞ page of this spectral sequence is sl_{-1} homology, which categorifies $P_{-1}(K)$.

An interesting aspect of the conjectured spectral sequence from HOMFLY-PT homology to HFK is that it is also induced by the cube filtration. In particular, the conjecture states that

$$H_*(H_*(C_F(D), d_0), d_1^*) \cong H_H(D)$$

and that the E_∞ page of this spectral sequence is knot Floer homology, which categorifies $P_0(K)$.

We have shown in the previous sections that if this conjecture is true, it is essentially due to the $n = 1$ categorification of the Jaeger's composition product. On the level of polynomials, this formula replaces $P_H(a, q)$ with $P_H(aq, q)$. The spectral sequence induced by the cube of resolutions corresponds to setting $a = q^{-1}$. Thus, with the standard construction of HOMFLY-PT homology, the total homology (or the E_∞ page) categorifies $P_H(q^{-1}, q) = P_{-1}(K)$, and with the $n = 1$ composition product the total homology categorifies $P_H(1, q) = P_0(K)$, or the Alexander polynomial.

The natural question to ask is whether there is anything special about $n = 1$. If we can categorify the composition product for other values of n , this would give cube of resolutions complexes whose E_2 page is HOMFLY-PT homology and whose total homology is a categorification of $P_{n-1}(K)$.

Since Rasmussen constructed spectral sequences from HOMFLY-PT homology to sl_k homology for all $k \geq 1$, this idea seems reasonable for $n > 1$, as the spectral sequences themselves would already be known to exist. The novelty would be that the spectral sequence would stem from the cube of resolutions as opposed to a filtration on the horizontal grading, and that the *total* homology would give sl_n homology. If these constructions exist, they imply a flexibility in the construction of HOMFLY-PT homology beyond what was conjectured by Dunfield, Gukov, and Rasmussen. Utilizing the composition product formula for each value of n gives something like a

$\mathbb{Z}$ -action on the construction of HOMFLY-PT homology, which causes a shift in the gradings of their conjectured d_n differentials. This may also shed some light on ϕ , their conjectured involution on HOMFLY-PT homology.

5.2 The Non-local Ideal and sl_n Homology

The isomorphism

$$\mathrm{Tor}_R(R/L, R/N) \cong \mathrm{Tor}_R(R/L, R/Q) < 1 >$$

gives us a way to reformulate the HOMFLY-PT homology of singular braids in terms of the non-local ideal N . An interesting idea would be to extend this idea to sl_n homology. We have seen that the total homology of $C_{F(n)}(S)$ is given by

$$H_*(C_{F(n)}(S)) \cong H_1(S_{f,1}) \otimes H_n(S_{f,2})\{\sigma_{n,1}(f)\}$$

which, by Wagner's composition product formula, is isomorphic to $H_{n+1}(S)$. The complex $C_{F(n)}(S)$ is obtained by taking the standard planar Heegaard diagram for S , and adding an additional unknot at infinity. The diagram is required to have a marked edge, which we choose to place on the unknot at infinity. We also reduce the component at infinity.

The Heegaard diagram which gives the non-local ideal is a destabilized version of this. There is no unknot added at infinity, and the marked edge is chosen to lie on the leftmost strand of S . The purpose of the marked edge is to indicate where to add a special bivalent vertex (called an insertion in [10]). The result is a diagram with a marked bivalent vertex, and the marked edge has become the two edges adjacent to this vertex. In the Heegaard diagram, this is the vertex where the α and β circles are missing (see Figure 5.1).

Manolescu showed that the Floer homology of this diagram is isomorphic to R/N , and tensoring with the complex $K(S)$, we get that the homology is isomorphic to $Tor_R(R/L, R/N)$. Note that since R/N lies in a single algebraic grading, when we add in the new differentials from discs passing through the X basepoints, the total homology will remain the same.

In order to connect sl_n homology with the non-local ideal, we need to relate the homology of these two diagrams. Suppose we start with the destabilized diagram, and add an unknot at infinity and reduce it. This adds a new α and β circle, which intersect one another at two points, and have no other intersection points (see Figure 5.1). We will call these points x and y .

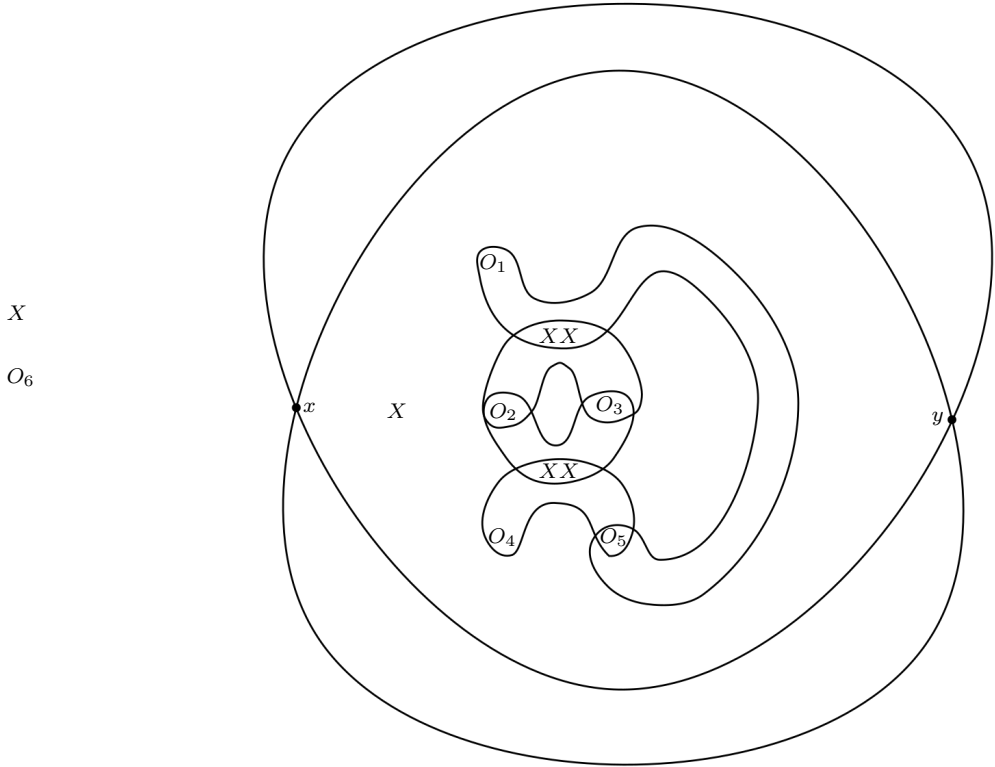


Figure 5.1: Stabilized Diagram with Marked Vertex on S

We know that ignoring any maps between x and y , the homology is precisely R/N .

Thus, we have two copies of R/N with maps between them corresponding to maps between x and y . There are two maps from y to x corresponding to the two empty bigons - these maps come with opposite sign, so the differential is trivial. The outer bigon (the one containing infinity) maps from x to y , but it contains O_6 , and $O_6 = 0$ because we are reducing the unknotted component.

The only remaining possible contributions are the complement of the α circles in the interior bigon, and the complement of the β circles in the interior bigon. Recall that the X counts with coefficient $(U_1^{n+1} - U_4^{n+1})/(U_4 - U_1)$, so the complement of the α circles comes with coefficient $U_4(U_1^{n+1} - U_4^{n+1})/(U_4 - U_1)$ and the complement of the β circles comes with coefficient $U_1(U_1^{n+1} - U_4^{n+1})/(U_4 - U_1)$. We know from holomorphic geometry that exactly one of these must count, and in the ideal quotient R/N we have an equality $U_1 = U_4$, so in either case there is a map from x to y with coefficient $(n+1)U_1^{n+1}$.

We have shown the following: if e_1 is one of the edges incident to marked bivalent vertex on S , then the complex is quasi-isomorphic to

$$R/N \xrightarrow{(n+1)U_1^{n+1}} R/N$$

This gives the homology CFK_n^- for the given Heegaard diagram - we now tensor with $K_n(S)$ to get the object at a vertex in the cube of resolutions. Letting A_n denote the complex $R \xrightarrow{(n+1)U_1^{n+1}} R$, we can rewrite this complex as

$$R/N \otimes K_n(S) \otimes A_n$$

where the tensor products are taken over R .

We would now like to compare this to the homology of $C_{F(n)}(S)$. The only difference in the constructions is that they come from different Heegaard diagrams for the same singular link, and the two diagrams are related by a sequence of isotopies

and handleslides. This is where the argument becomes too technical for us to prove at this time - in order to prove that $C_{F(n)}(S)$ and $R/N \otimes K_n \otimes A_n$ have the same homology, we need to show isotopy and handleslide invariance when we count discs that pass through X and XX with coefficients u_1 and u_2 respectively. This would require going back to the original papers and generalizing the invariance arguments, which we have not done, so we will leave it as a conjecture for now.

Conjecture 5.2.1. *There is an isomorphism $H_*(R/N \otimes K_n(S) \otimes A_n) \cong H_{n+1}(S)$.*

This conjecture is interesting not only because it gives a non-local formulation of sl_n homology, but also because it seems likely that edge maps could be defined to make the cube of resolutions from this complex. The reason for this is that when N is replaced with Q , this gives a version of the standard definition of sl_n homology where the $Q(v)$ maps have been canceled. Since there are edge maps defined for R/Q to give sl_n homology, we could try the same maps over R/N and see if it gives edge maps for sl_{n+1} homology.

Bibliography

- [1] Akram Alishahi and Eaman Eftekhary. A refinement of sutured Floer homology. *Journal of Symplectic Geometry*, 13(3), 2015.
- [2] N. Dunfield, S. Gukov, and J. Rasmussen. The superpolynomial for knot homologies. *Experimental Math*, 15(2):129–160, 2006.
- [3] P. Freyd, D. Yetter, W.B.R. Lickorish J. Hoste, K. Millett, and A. Ocneanu. A new polynomial invariant of knots and links. *Bulletin of the American Mathematical Society*, 12(2):243–298, 1985.
- [4] Allison Gilmore. Framed graphs and the non-local Ideal in the knot Floer cube of resolutions. *Algebraic and Geometric Topology*, 15:1239–1302, 2015.
- [5] Mark C. Hughes. A note on Khovanov-Rozansky sl_2 -homology and ordinary Khovanov homology. *Journal of Knot Theory and its Ramifications*, 23:1387–1425, 2014.
- [6] Francois Jaeger. Composition product and models for the HOMFLY polynomial. *Enseign. Math.*, 35:323–361, 1989.
- [7] Mikhail Khovanov. A categorification of the Jones polynomial. *Duke Mathematical Journal*, 101:359–426, 2000.
- [8] Mikhail Khovanov and Lev Rozansky. Matrix factorizations and link homology. *Fundamenta Mathematicae*, 199:1–99, 2008.

- [9] Mikhail Khovanov and Lev Rozansky. Matrix factorizations and link homology II. *Geometry and Topology*, 12:1387–1425, 2008.
- [10] Ciprian Manolescu. An untwisted cube of resolutions for knot Floer homology. *Quantum Topology*, 5:185–223, 2014.
- [11] Peter Ozsváth and Zoltán Szabó. Holomorphic discs and knot invariants. *Advances in Mathematics*, pages 58–116, 2004.
- [12] Peter Ozsváth and Zoltán Szabó. Holomorphic discs, link invariants, and the multi-variable Alexander polynomial. *Algebraic and Geometric Topology*, 8:615–692, 2008.
- [13] Peter Ozsváth and Zoltán Szabó. A cube of resolutions for knot Floer homology. *Journal of Topology*, 2009.
- [14] J. H. Przytycki and P. Traczyk. Invariants of links of Conway type. *Kobe Journal of Mathematics*, 4:115–139, 1987.
- [15] Jacob Rasmussen. Floer homology and knot complements. *PhD Thesis, Harvard University*, 2003.
- [16] Jacob Rasmussen. Some differentials on Khovanov-Rozansky homology. *Preprint arXiv:0607544v2*, 2006.
- [17] S. Sarkar and J. Wang. An algorithm for computing some Heegaard Floer homologies. *Annals of Mathematics*, 171:1213–1236, 2010.
- [18] Emmanuel Wagner. Khovanov-Rozansky graph homology and composition product. *Preprint arXiv:math/0702230*, 2007.