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NONLINEAR WAVES IN GENERAL RELATIVITY AND FLUID DYNAMICS

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A DISSERTATION
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OF PRINCETON UNIVERSITY
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Abstract

This thesis deals with the analysis of partial differential equations describing nonlinear wave-like phenomena in three different settings: general relativity, the compressible Navier–Stokes equations, and magnetohydrodynamics.

Many results on the global dynamics of hyperbolic equations, particularly in the case of the Einstein equations, rely on the use of the *vector field method*. This method requires the initial data to be highly localized around a single point in space. In this first part of the dissertation, we extend the classical vector field method of Klainerman to deal with initial data localized around several points whose pairwise distances are assumed to be large. We are therefore able to prove global stability for solutions to quasilinear wave equations satisfying the null condition when the initial data are not required to be localized around a single point. This probes a regime which was not accessible by previous physical-space methods. This part is based on joint work with John Anderson.

The second chapter of this thesis deals with the global dynamics of the compressible Navier–Stokes equations in one and two space dimensions. A particular case of these equations arises in geophysical fluid dynamics as the viscous *shallow water equations*. Concerning the one-dimensional model, we introduce a quantity, called the *active potential*, which allows us to control the dynamics for a large range of pressure and viscosity laws. As a byproduct, we are able to prove a conjecture formulated in 1994 by Peter Constantin. This part is based on joint work with Peter Constantin, Theodore Drivas, and Huy Nguyen.

Finally, the third chapter deals with the equations of magnetohydrodynamics (MHD). We prove that a suitably regularized Voigt—MHD model admits a global-in-time solution, which moreover converges, in the infinite time limit, to a solution of the steady three-dimensional incompressible Euler equations. This can be regarded as a rigorous construction of an MHD equilibrium by means of a process known in the physics literature as *magnetic relaxation*: the magnetic field is drawn towards equilibrium by an MHD-type system. This part is based on joint work with Peter Constantin.

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To my parents, and to my brother.

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Introduction

The purpose of this thesis is the rigorous mathematical study of several phenomena which arise in physics, and which can be modeled through *partial differential equations* (PDEs). Some of the questions studied here arise naturally in the context of physical theory, others are the result of pure mathematical speculation. We will always try to connect the mathematical questions analyzed here to the corresponding physical motivation.

The main theme that underlies this exploration is the ubiquitous presence in nature of *nonlinear undulatory phenomena*. Their mathematical study is of great importance in understanding very basic mechanisms in physics whose inner workings we still need to explain. There will be many examples of such basic questions that I will point out in the following. Some of these fundamental problems include the long time dynamics of localized objects in general relativity, the dynamical behavior of compressible and viscous fluids, the modeling of waves in the ocean, the issues of relaxation and reconnection in plasmas, the Parker problem and the solar corona problem in magnetohydrodynamics. Although these areas can seem to be very far from each other, the techniques used to study them from the mathematical viewpoint provide several connections.

In this thesis, we mainly deal with nonlinear waves which draw their motivation from three distinct areas of physics: general relativity, compressible viscous fluid dynamics, and magnetohydrodynamics. We now proceed to describe in more detail the results of this thesis. The manuscript is divided into three chapters, which can be read independently of each other.

0.1 Chapter 1: general relativity and nonlinear wave equations

In this chapter, we are going to explore some questions connected to *Einstein's theory of general relativity* and, more generally, to *nonlinear wave equations*.

Einstein's remarkable theory describes gravitation, and, in vacuum, it can be very compactly expressed as the vanishing of the Ricci tensor on a $3 + 1$ Lorentzian manifold (M, g) :

$$\text{Ric}(g)_{\mu\nu} = 0. \tag{0.1.1}$$

One of the many remarkable properties of (0.1.1) is that it is secretly a *system of nonlinear hyperbolic equations*. Indeed, system (0.1.1) can be realized as a system of quasilinear hyperbolic equations after choosing an appropriate gauge, see e.g. [38].

In this chapter, we will start from equation (0.1.1) as a motivating example, and we will be lead to consider a class of nonlinear hyperbolic equations, which we will describe later in more detail. Although our result will be about a specific class of hyperbolic equations, the reader should keep in mind that we are always starting from the Einstein equations as a main motivation.

In addition to the motivation arising from general relativity, hyperbolic equations are of independent interest because of the connections with other branches of physics. Two examples of nonlinear hyperbolic equations are the irrotational compressible Euler equations, which describe the dynamics of a compressible gas (see [27]) and the equations governing the motion of elastic materials (see [82]).

In the following sections of this part of the introduction, we will first introduce the issue of global stability in general relativity (Section 0.1.1). We will then show its connections with the fundamental *final state conjecture*, which we will touch upon in Section 0.1.2. In Section 0.1.3, we will review existing stability results for hyperbolic equations, showing that

much of the existing analysis is based on having initial data which is *highly localized*. In the following section (Section 0.1.4), we will describe the main result of Chapter 1 of this thesis, obtained in collaboration with John Anderson. We show that, for a specific class of nonlinear wave equations, the localization requirement can be relaxed, and global stability can be achieved for initial data which is localized around several distant points. Finally, in Section 0.1.5, we describe how these techniques can be applied to the case of the Einstein equations (work in progress with John Anderson), which is the main starting motivation of our study. This analysis probes a novel regime of the *final state conjecture*.

0.1.1 The issue of global stability in general relativity

The Einstein field equations (whose form in the absence of matter is given in (0.1.1)) were formulated by Einstein in 1915. The intrinsic geometric nature of these equations is perhaps one of the reasons why their study as an evolution problem came much later.

The study of (0.1.1) as an initial value problem was indeed initiated in the work of Choquet-Bruhat in 1952 [38]. In this fundamental work, Choquet-Bruhat proved that the Einstein equations are locally well posed if the initial data are assumed to be sufficiently regular. The work is of utmost importance since it initiated the rigorous analysis of the Einstein equations from a purely *PDE perspective*, moreover it gave Einstein’s theory solid mathematical footing, and it laid the foundations for much of the subsequent analysis.

An important take-home message from [38] is that the Einstein equations, which *a priori* do not have an “evolution equation” structure, can be recast into an initial value problem in a way that enables one to show local existence. In this discussion, we restrict to the special case in which the initial data is posed on $\mathbb{R}^3$. Consider a metric $\bar{g}$ and a symmetric two-covariant tensor $\bar{k}$ on $\mathbb{R}^3$ (the second fundamental form). We call $(\mathbb{R}^3, \bar{g}, \bar{k})$ an *initial data set* for the Einstein equations if $\bar{g}$ and $\bar{k}$ satisfy the *Einstein constraint equations*. We won’t formulate the constraint equations here in full generality, but let us just remark that,

in the vacuum case and if $\bar{k} = 0$, they reduce to imposing that the metric $\bar{g}$ be scalar-flat:

$$R_{\bar{g}} = 0,$$

where $R_{\bar{g}}$ is the scalar curvature of the metric $\bar{g}$.

In this context, the result [38] asserts that, if $\bar{g}$ and $\bar{k}$ are sufficiently smooth, there exists a spacetime neighborhood $\mathcal{N} \subset \mathbb{R}^{1+3}$, $\mathcal{N} \supset \{0\} \times \mathbb{R}^3$, and a sufficiently regular metric g on $\mathcal{N}$ which restricts to $\bar{g}$ on $\{0\} \times \mathbb{R}^3$, such that $\bar{k}$ is the second fundamental form of the embedding $(\mathbb{R}^3, \bar{g}) \hookrightarrow (\mathbb{R}^{3+1}, g)$, and finally such that the Einstein vacuum equations are satisfied by g on $\mathcal{N}$:

$$\text{Ric}(g)_{\mu\nu} = 0.$$

Having settled, in a sufficiently regular space, the issue of local well-posedness, the immediate natural question to ask is: what are the global dynamics of the Einstein equations? This is an incredibly complicated question, because of the nonlinear and geometric nature of the Einstein equations themselves, as well as the complex and varied range of phenomena that they display.

A more restricted question that one can ask is: what is the behavior of the Einstein equations in the neighborhood of known solutions? Perhaps the simplest such solution is the so-called *trivial solution*, i.e., the Minkowski spacetime. This is the manifold $M = \mathbb{R}^{1+3}$ endowed with the flat Lorentzian metric which has the following representation, in the standard coordinate system: $g = \text{diag}(-1, 1, 1, 1)$. Here, “diag” indicates the diagonal matrix with elements listed in the parenthesis.

The first result which truly explored this scenario was the monumental work of Christodoulou and Klainerman [21], which appeared in 1993. In this foundational work, the authors show the following result:

Theorem 0.1.1 (Christodoulou and Klainerman, 1993). *Consider the trivial initial data set for the Einstein equations: $(\mathbb{R}^3, \bar{g} = \delta_{ij}, \bar{k} = 0)$, where δ_{ij} is the trivial metric on $\mathbb{R}^3$.*

Consider moreover a small (in a suitable Sobolev space) and localized perturbation of $\bar{g}$ and $\bar{k}$. Then, this perturbation launches a global solution g of (0.1.1) defined on $[0, \infty) \times \mathbb{R}^3$, and moreover the solution asymptotes to the trivial solution with quantitative rates for very large positive times.

This monumental result was the first of its kind in mathematical general relativity, as it dealt with the global dynamics of the Einstein equations in the absence of symmetries. This was a remarkable advance from the point of view of the mathematical theory. Moreover, let us mention that the result requires the initial data to be small in a high-order Sobolev space, and it also requires such “perturbed” initial data to be localized around a single point. In practice, in [21], this requirement is encoded in the fact that the perturbation is required to be small in a *weighted* Sobolev space, with radial weights.

After this result, the subject of long time dynamics in general relativity has been widely studied, and research efforts have been focused on the global dynamics of the Einstein equations on perturbations of other known special solutions. First, there were results of a linear nature, which considered the linear wave equation on black hole spacetimes (this is the easiest model problem for the Einstein equations). On a fixed Schwarzschild background, we cite the fundamental results of Blue–Sterbenz and Dafermos–Rodnianski (see resp. [9, 30]). After that, some works established proofs of decay of the linear wave equation on a fixed Kerr background with parameter $|a| \ll M$. We refer to the work of Andersson–Blue, Dafermos–Rodnianski and Tataru–Tohaneanu (see resp. [4, 31, 86]). Finally, for the full subextremal range of parameters of the Kerr black hole $|a| < M$, the problem presents additional difficulties. A complete proof of decay of solutions to the wave equation in this case has been given by Dafermos–Rodnianski–Shlapentokh–Rothman in [32].

After analyzing the wave equation on black hole spacetimes, research efforts were devoted to understanding the full linearized picture of black hole stability. In this context, Dafermos, Holzegel and Rodnianski [29] proved the linear stability of the Schwarzschild black hole as a first step in the program to solve the Kerr stability conjecture. More recently, Dafermos–

Holzegel–Rodnianski in [28] proved decay for the spin ± 2 Teukolsky equations on slowly rotating Kerr ($|a| \ll M$), a crucial step to address the nonlinear stability of Kerr in the slowly rotating case. Finally, in the context of electrovacuum, the linearized stability of Reissner–Nordström under the Einstein–Maxwell system has been established in the Ph.D. thesis of Giorgi [41].

Concerning results on the full nonlinear stability of the Einstein equations, a seminal result in the nonlinear case was obtained by Holzegel in his Ph.D. thesis [48]. More recently, a solution to a restricted case of the problem of stability of Schwarzschild as a solution to the Einstein vacuum equations has appeared. In the work [62], Klainerman and Szeftel established the nonlinear stability of the Schwarzschild black hole under axially symmetric polarized perturbations. In particular, these perturbations ensure that the resulting evolution will converge to the Schwarzschild black hole. The full, finite codimension nonlinear stability of Schwarzschild, without symmetry assumptions, has been announced in [88].

0.1.2 The final state conjecture

All the research efforts outlined above can be connected to different regimes of one crucial open question in general relativity, the *final state conjecture*, a preliminary version of which was formulated by Penrose in the 1970s. In rough terms, the conjecture can be formulated as follows.

Conjecture 0.1.1. *Sufficiently smooth and asymptotically flat initial data for the Einstein vacuum equations (0.1.1) will launch a solution to the Einstein vacuum equations which, in general, will exhibit gravitational collapse and formation of black holes. Nevertheless, generically, the solution will be asymptotic to a finite number of Kerr black holes traveling away from each other plus residual gravitational radiation which exhibits decay.*

The results mentioned in Section 0.1.1 all probe different regimes of this conjecture. For instance, the monumental result on the global nonlinear stability of Minkowski space [21]

shows that the final state conjecture is true in a neighborhood of Minkowski space: there is no black hole formation and moreover the spacetime just asymptotes to the flat one with residual gravitational radiation. All the results on black hole stability probe the regime in which one black hole is already present, and they show that perturbations of such an object, asymptotically, will also look like a black hole with residual gravitational radiation.

All the above regimes, though, focus on the case in which the perturbation is localized around a point (or around a black hole). However, there are cases of the *final state conjecture* which clearly depart from this picture. For example, think of the scattering problem of two black holes coming from infinity. One would like to show that, in some regimes, the two black holes interact without merging and “scatter”. Clearly, in this case, the initial data cannot be localized and, in fact, it is highly non-localized. Another such example is the merger of two black holes, which is also a particular case of the final state conjecture, and draws physical motivation from the gravitational wave observations of LIGO [1].

The merger problem and full scattering of two black holes do not seem to be within reach of current mathematical techniques. However, in this first chapter, we wish to depart from the usual “localized” initial data under which the Einstein equations are studied, and to focus instead on initial data which are localized around several points located at a large distance away from each other. We wish to probe the final state conjecture in the regime close to Minkowski space, although allowing our perturbations to be non-localized.

Our study, although it concerns a model problem and not the Einstein equations themselves, represents the first attempt at modeling a situation displaying many localized perturbations and, as such, the techniques introduced here could be applicable to probe new regimes of the final state conjecture.

We will now introduce the model problem on which we carry out our study (quasilinear wave equations satisfying the classical null condition), and we will then give a rough description of our result.

0.1.3 Model problems: nonlinear wave equations and the null condition

The Einstein equations belong to a class of hyperbolic wave equations with a special structure, which essentially allows for small data global existence. In our work, we will focus on quasilinear hyperbolic equations satisfying the *classical null condition*, which is a somewhat stronger assumption than the one satisfied by the Einstein equations. Nevertheless, advances in understanding the classical null condition were absolutely fundamental in understanding the global dynamical properties of the Einstein equations. This is the main motivation why we consider quasilinear wave equations as a model problem to study the Einstein equations. In a forthcoming work, we plan to extend our results to the Einstein equations themselves (see Section 0.1.5).

The story of the classical null condition is rich and interesting. Here, we describe how this condition was formulated, its central role in the analysis of nonlinear hyperbolic PDEs, and its deep connections with the story of the global properties of nonlinear hyperbolic equations. Small data long time existence results for wave equations originate with the works of John and Klainerman. In [57], Klainerman was able to establish small data global existence results for nonlinear wave equations with quadratic nonlinearity in sufficiently high space dimensions (see also [61]).

In [59], Klainerman introduced the vector field method, which is a way to prove decay of solutions to wave equations by differentiating the equations themselves with vector fields that commute with the symbol, and then using weighted Sobolev estimates combined with energy estimates to deduce pointwise decay. This enabled to bypass the use of the fundamental solution which was widely used in previous work on the subject, and it led to a sharp global existence result for nonlinear wave equations in dimensions $n + 1$ for $n \geq 3$. Indeed, using these methods, Klainerman proved that global existence for small data holds for nonlinear waves with quadratic nonlinearity in $\mathbb{R}^{1+n}$, with $n \geq 4$, and for cubic nonlinearities in the physical case $\mathbb{R}^{1+3}$. This result is sharp in the sense that certain nonlinear wave equations

with quadratic nonlinearities do not admit global-in-time solutions in $3 + 1$ dimensions for general arbitrarily small initial data (see the work of John in [52] and [53]).

It is therefore natural to investigate whether there exists a condition which can distinguish between nonlinearities that lead to small data global existence and nonlinearities that do not. This led to the discovery of the *classical null condition*, which was first introduced by Klainerman in [58]. Essentially, a quadratic nonlinearity satisfies the null condition if a derivative transversal to the light cone is always multiplied by a derivative tangent to it. See equation (0.1.3) for the exact definition.

Remarkably, the trivial solution to a nonlinear wave equation satisfying the null condition (0.1.3) is globally stable in $3 + 1$ dimensions. This was first shown by Klainerman using the vector field method [60], and Christodolou gave a different proof using conformal compactification in [20]. Since then, there have been other proofs, as well as many generalizations.

The Einstein vacuum equations do not satisfy the classical null condition when written in wave coordinates, making this case more subtle. The monumental work of Christodoulou and Klainerman [21] shows that nevertheless, in this case, one has global stability for initial data which is small and localized. This follows from a closer analysis of the Einstein equations themselves and from the identification of a geometric form of the null condition.

In a paper which appeared in 2010, Lindblad and Rodnianski identified a generalization of the classical null condition which is referred to as the *weak null condition*. The weak null condition is satisfied by the Einstein vacuum equations in wave coordinates, and this observation allowed Lindblad and Rodnianski to prove the global nonlinear stability of Minkowski space using wave coordinates in [67]. We will discuss these issues more in detail in Section 0.1.5, in which we outline the extension of our results to the Einstein case.

Before introducing our main result, let us note that many of the results described in this section and in Section 0.1.1 depend essentially on the use of the vector field method in different forms. This method, in particular, requires the use of vector field commutators which carry radial weights, as the Sobolev estimates attached to it require weights to deduce point-

wise decay. It therefore does not come as a surprise that the works discussed in the present section all require the initial data to be localized. Indeed, all work on the stability of *quasilinear hyperbolic equations* requires the initial data to be somehow localized around a point. Moreover, even in the semilinear case, previous results which allow for non-localized data often rely on low regularity techniques, which fail to provide a good asymptotic description of the solution.

In our work, as we have already mentioned, we wish to depart from this regime and to introduce a robust and geometric method to deal with quasilinear wave equations in the case in which the initial data are not required to be localized around a point.¹ This work therefore constitutes the first example of a global stability result for quasilinear wave equations in which the initial data is not localized. We describe our result in the next section.

0.1.4 Global existence for nonlinear wave equations with multi-localized initial data

In this section, we state a rough version of the main theorem of Chapter 1 of this thesis, restricted to the case in which the initial data are localized around two points. For the statement in full generality, see Theorem 1.4.1. This part is based on joint work with John Anderson [3].^{2,3}

We first introduce precisely the class of equations we are going to study. These are *systems of quasilinear wave equations satisfying the classical null condition*. Precisely, we

¹The issue of global well-posedness for wave equations in translation-invariant spaces has been a very active area of research, due also to its connections with low-regularity issues. Even though the author is not aware of examples in the quasilinear case, in the case of semilinear wave equations there are many results showing global existence using scale-invariant spaces. As we already mentioned, these results often fail to provide a good quantitative description of the dynamics.

²This work has been presented at the following seminars and conferences: Junior analysis seminar, Imperial College London, July 2019; EquaDiff conference, Universität Leiden, July 2019; AMS sectional meeting, UC Riverside, November 2019; analysis seminar at Princeton University, December 2019.

³We are indebted to Yakov Shlapentokh-Rothman for making us aware of this problem.

require our equations to be of the following form:

$$\square\phi_A + \sum_{B,C=1}^M F_{BC}^A(d\phi_B, d^2\phi_C) = \sum_{B,C=1}^M G_{BC}^A(d\phi_B, d\phi_C), \quad A = 1, \dots, M. \quad (0.1.2)$$

Here, $\phi_A : [0, T] \times \mathbb{R}^3 \rightarrow \mathbb{R}^M$ is the unknown, and F_{BC}^A and G_{BC}^A are collections of respectively trilinear and bilinear forms satisfying the null condition:

$$F_{BC}^A(N, N, N) = 0, \quad G_{BC}^A(N, N) = 0 \quad \text{for all } N \in \mathbb{R}^4 \text{ such that } m(N, N) = 0. \quad (0.1.3)$$

Here, m is the Minkowski metric, and $\square$ denotes the wave operator induced by m on $\mathbb{R}^{3+1}$. See Section 1.3.3 for additional details.

This class of systems was the one introduced by Klainerman in his classical paper [58], and can be viewed as a model problem for the vacuum Einstein equations. Under these conditions, we have the following theorem:

Theorem 0.1.2 (Rough version of Theorem 1.4.1 for $N = 2$). *For all sufficiently small data supported in two Euclidean balls of unit radius, nonlinear wave equations of the form (0.1.2) satisfying the classical null condition (0.1.3) admit global-in-time solutions. Moreover, the smallness of initial data is measured in a norm which does not depend on the distance between the two balls. For instance, requiring that the $H^{20}(\mathbb{R}^3)$ norm of the data is small enough, independently of the distance between the two balls, is sufficient to ensure global existence.*

A few remarks are in order.

Remark 0.1.3. Note that, since the distance between the two Euclidean balls can be chosen to be arbitrarily large, this regime is not covered by the classical results described above, which require the data to be highly localized. Indeed, from the point of view of the classical theory, this is a *large data regime*, as the norm which we require to be small (the $H^{20}(\mathbb{R}^3)$ norm, for instance), does not carry radial weights.

Remark 0.1.4. Our theorem also holds for data which are supported in a finite number $N > 2$

of balls of radius one, although we require a condition on the ratio of their mutual distances. See the statement of Theorem 1.4.1 for further details. Moreover, our theorem holds in the case of non-compactly supported data, as long as the data decays at a sufficient fast rate away from a number of points. In this thesis, we specialize to the case of compact support, but the paper [3] deals with the more general case of non-compactly supported data. This is relevant to the extension to the Einstein case, which we are going to discuss briefly in the next section.

0.1.5 Extension to the Einstein equations

In the previous sections we have discussed how our result deals with global existence for nonlinear wave equations when the initial data are localized around several points. The results presented here can be viewed as a model problem to study the more complicated case of the Einstein equations, and we plan, in joint work with John Anderson, to address the case of the Einstein equations in a forthcoming paper.

The first issue that needs to be dealt with in the case of the Einstein equations is the existence of suitable initial data. It is unknown to the author whether initial data localized to several points in space have been constructed in the literature. Although this is not the main focus of our analysis, we expect to be able to prove the existence of such initial data by using a gluing construction à la Corvino–Schoen, essentially by gluing regular vacuum initial data sets to a Brill–Lindquist type initial data set.

After having dealt with the issue of constraints, a natural model problem to consider is the classical example of the weak null condition. This is the semilinear coupled system

$$\begin{aligned}\square\phi &= m(d\psi, d\psi), \\ \square\psi &= (\partial_t\phi)^2.\end{aligned}\tag{0.1.4}$$

Here, m again is the Minkowski metric. Note that this system does not satisfy the classical null condition, since in the nonlinearity in the second equation contains two derivatives

transversal to the light cone.

Nevertheless, the system (0.1.4) satisfies the so-called *weak null condition*, and it can be shown that the initial value problem for system (0.1.4) with small and localized initial data admits global-in-time solutions. Note that the energy of these solutions will, in general, grow logarithmically.

In the main part of Chapter 1 we will argue that a key element of our approach is to show estimates independent of a parameter $R > 0$, which corresponds to the distance between the balls where the initial data are supported (in the case $N = 2$). It is then conceivable that, if the estimates which we will show are not “borderline” with respect to the parameter R , there is potentially an avenue to deal with the logarithmic divergences intrinsic to problem (0.1.4). In reality, the strategy has to involve more elements than this naïve outline. For a treatment of the multi-localized case in a slightly different model problem satisfying the weak null condition, see the senior thesis of Cicortas [22].

Finally, the full Einstein equations present additional difficulties. First, the constraint equations force the initial data to be of noncompact support. Moreover, the initial data only satisfy weak decay in terms of the distance from the center of each ball. It is a well known fact that this will imply that also the u -decay, then, will be weak. One needs a strategy to deal with this issue. See the work of Keir [55], in which global existence for a general class of nonlinear wave equations satisfying the weak null condition is obtained when the decay of initial data is very weak. Moreover, another issue that one encounters when dealing with the Einstein equations is the presence of *modified scattering*, which is the absence of linear scattering. Linear scattering is, roughly speaking, the fact that the “main contribution” to the solution essentially solves an associated linear asymptotic system, which in the case of the classical null condition reduces to a system of linear wave equations. Since the Einstein equations in wave coordinates do not display linear scattering, the resulting solution will not be approximated well by the linear flow, and it will display logarithmic divergences in the energies. One needs to take this fact into account when dealing with the global stability

problem for the Einstein equations in our case.

0.1.6 Outline of the chapter

We will start by motivating the problem in Section 1.1. We will then proceed to a discussion of the main ingredients of our approach in Section 1.2. In the sections which follow (sections 1.3 through 1.10) we will then introduce the necessary notation, state the precise form of our main theorem (Theorem 1.4.1) and finally provide a proof of the main result.

0.2 Chapter 2: compressible and viscous fluids

In this chapter, we are going to deal with equations arising from several physical situations in fluid dynamics. The main goal of the present chapter is to deal with well-known descriptions of a fluid which is both compressible and viscous. First and foremost, these models arise in the description of the motion of a gas. Moreover, they also arise in geophysics and in the theory of lubrication as approximate descriptions of the motion of the free surface of an incompressible fluid. To be concrete, in the latter case we think of the fluid as being water. Starting from the fact that the fluid is governed (in the “bulk”) either by the *incompressible* Euler equations or by the *incompressible* Navier–Stokes equations, under certain assumptions, we will see that the motion of the free surface of an *incompressible fluid* can be approximately modeled by an initial value problem involving a *compressible fluid*. At a first glance, this can seem to be quite a bizarre feature of the models considered. We will deal with the issues of global well-posedness and long-time dynamics of some of these compressible models, which are questions that arise naturally from the underlying physical theory.

The plan of this part of the introduction is as follows. We now introduce some of the physical situations which motivate our interest in these issues, mostly arising in geophysical fluid mechanics and in the theory of lubrication, providing a heuristic explanation why compressible models can arise, under some assumptions, from incompressible free boundary

problems (Section 0.2.1), and we comment on some of the issues intrinsic to the compressible and inviscid description. We then introduce the main models which are going to be our focus for the rest of the chapter, in one and two space dimensions (Section 0.2.2). We finally discuss the main results contained in this chapter: some results on global dynamics for the one-dimensional model, and uniform bounds for certain two-dimensional models under symmetry assumptions.

0.2.1 Physical motivation

We start our discussion from the well-known incompressible Euler equations on $\mathbb{R}^n$, with $n = 2$ or 3 :

$$\begin{aligned}\partial_t u + u \cdot \nabla u + \nabla p &= F, \\ \operatorname{div} u &= 0.\end{aligned}\tag{0.2.1}$$

Here, $u = u(t, x)$ is a time-dependent vector field with values in $\mathbb{R}^n$ (the velocity of the fluid), $p(t, x)$ is a time-dependent real valued function (the pressure), div is the divergence of a vector field, and $F = F(t, x)$ is a given vector field (the forcing). These equations were formulated by Euler in 1757, and have played a fundamental role in fluid dynamics since then, as they can model the flow of water in certain regimes. Global well-posedness in two space dimensions is known, although the long time behavior is still relatively poorly understood in that case. In three space dimensions, for reasonable classes of large initial data, the issue of global well-posedness is still not understood for smooth initial data of finite energy.

In many physical applications, one wishes to model the motion of a mass of water with free boundary. See, for example, the motion of the surface of the sea or a lake, the motion of a drop of liquid, or the flow of a thin thread of liquid. We focus here on one specific example, the motion of a mass of water in a basin.

Here is a standard mathematical formulation of the problem. We consider a mass of water in two space dimensions (in the xy plane, y being the “vertical” direction) which is

modeled by the 2D Euler equations having velocity $u = (u_1, u_2)$ and pressure p . We also assume that gravity is acting on the fluid. We suppose that the mass is bounded below by the “bottom topography”, which is represented by the set $y = -b(x)$, where b is a smooth (and positive) real function. Moreover, we assume that the water is bounded above by the “free surface” $y = \eta(t, x)$. The motion of the free surface is governed by the equation:

$$\partial_t \eta(t, x) + u_1(t, x, \eta(t, x)) \partial_x \eta(t, x) = u_2(t, x, \eta(t, x)).$$

This equation encodes the fact that the free surface is transported by the fluid. Moreover, we assume that, on the top surface, the pressure is identically zero. We finally need to impose boundary conditions at $y = -b(x)$, and we suppose that, there, the fluid velocity is always tangent to the boundary $y = -b(x)$. All in all, we obtain the following free boundary problem for the 2D incompressible Euler equations:

$$\partial_t u + u \cdot \nabla u + \nabla p + g \hat{e}_2 = 0 \quad \text{on } \Omega(t), \quad (0.2.2)$$

$$\operatorname{div} u = 0 \quad \text{on } \Omega(t), \quad (0.2.3)$$

$$p = 0 \quad \text{on } y = \eta(t, x), \quad (0.2.4)$$

$$\partial_t \eta(t, x) + u_1(t, x, \eta(t, x)) \partial_x \eta(t, x) = u_2(x, \eta(t, x)), \quad (0.2.5)$$

$$u \cdot \nabla(y + b(x)) = 0 \quad \text{on } y = -b(x). \quad (0.2.6)$$

Here, $\hat{e}_2$ is the unit vector pointing upwards along the y -axis, g is a constant (the gravitational acceleration), and $\Omega(t)$ is the set of all (x, y) such that $-b(x) \leq y \leq \eta(t, x)$.

We now integrate the mass conservation (equation (0.2.3)) in the y -direction to get:

$$\begin{aligned} & u_2(t, x, \eta) - u_2(t, x, -b) + \int_{-b}^{\eta} \partial_x(u_1) dy \\ &= \partial_t \eta(t, x) + u_1(t, x, \eta(t, x)) \partial_x \eta(t, x) + u_1(t, x, -b) b(x) + \int_{-b}^{\eta} \partial_x(u_1) dy \\ &= \partial_t \eta(t, x) + \partial_x \left(\int_{-b}^{\eta} u_1 dy \right) = 0. \end{aligned}$$

The y -component of the momentum equation (0.2.2) gives, neglecting vertical acceleration (this is the so-called *long wave approximation*), that the pressure satisfies:

$$p(t, x, y) = g(\eta(t, x) - y).$$

Going back to the x -component of the momentum equation, neglecting again vertical acceleration, and substituting, we get

$$\partial_t u_1(t, x) + u_1(t, x) \partial_x u_1(t, x) + g \partial_x \eta(t, x) = 0.$$

Moreover, neglecting vertical acceleration, we also get

$$\partial_t \eta(t, x) + \partial_x ((b + \eta(t, x)) u_1(t, x)) = 0$$

Letting now $h = b + \eta$, and $v(t, x) = u_1(t, x)$, we obtain the *inviscid shallow water system*

$$\begin{aligned} \partial_t v + v \partial_x v + g \partial_x h &= F, \\ \partial_t h + \partial_x (hv) &= 0 \end{aligned} \tag{0.2.7}$$

for some constant forcing F .

Remark 0.2.1. Note that we started from an inviscid free boundary problem and we obtained a closed, *compressible* system in one space dimension. This heuristic reasoning also holds in three space dimensions. Moreover, note that, setting $h = \rho$, this system is analogous to the system describing the motion of a polytropic compressible gas in one space dimension with pressure law $p(\rho) = \rho^2$. Hence, in this picture, the density plays the role of the height of the water.

Having introduced a model for shallow water, we proceed to outline a few questions arising from physics which are relevant in this context.

- **Storm surge.** This physical phenomenon appears in oceanic physics. One would like to understand the effect of a storm on the water height far away from the “eye” of the storm itself. This problem is relevant in engineering and physics, as predicting water height is essential for several applications. The (two-dimensional) shallow water equations (in particular, their viscous counterpart) are a suitable model to address this situation, as in this case the *long-wave approximation* holds, although the basin is not “shallow” *per se*. A suitable mathematical formulation of this problem could be as follows.

Consider the shallow water equations (0.2.7) on the real line, and impose initial data $v_0 = 0$, and $h_0 = 1$. Moreover, suppose that the forcing F is either constant or time periodic, and it is localized around the origin. Does the height $h(t, x)$ satisfy uniform-in-time bounds for points x far away from the origin?

In reality, although physically motivated, the inviscid shallow water equations (0.2.7) are not a very good model to address this issue from the mathematical viewpoint, as they will develop shocks in finite time. Moreover, even in one space dimension, a large data theory for entropy solutions is not available, hence to probe the regime of interest we would have to address a much more complicated problem. On the other hand, if one considers viscous models (in the physical situation of height-dependent viscosity), the well-posedness theory is available, and moreover results in the direction of the storm surge problem can be achieved. Two of the results presented in what follows have this flavor (although they do not address the storm surge problem *per se*), and we are going to discuss about these results in more detail in sections 0.2.3 and 0.2.4 of this introduction.

- **Drop pinch-off and the theory of lubrication.** Compressible approximations appear also in the theory of lubrication. One such example (a viscous and compressible model) is the so-called *slender jet model* [35]. We are going to describe this model in more detail later (in Section 0.2.2), let us just mention now that it is a one-dimensional compressible

system which models the dynamics of drop formation close to drop pinch-off. There are various mathematical questions that can be posed in this direction, such as the following.

Consider the slender jet model (0.2.15)–(0.2.16), and impose periodic initial conditions, with the initial density (or height) bounded from below by a positive number. Do solutions to the corresponding initial value problem break down in finite time? If they do, is it possible to give an asymptotic description of the dynamics close to drop pinch-off?

In this chapter, we are going to partially address this issue, giving a continuation criterion which depends only on the density (or the height) for the slender jet model. This answers a 1994 conjecture of Peter Constantin, recorded in [34]. This result is described in more detail in Section 0.2.3 of this introduction.

• **Rogue waves.** Finally, let us mention another physical situation from geophysical fluid dynamics in which viscous and compressible models are relevant. The main issue at hand concerns modeling the height distribution of waves in the open ocean, i.e., understanding the likelihood of measuring waves of a certain height. The statistics of waves are fairly well understood, except for the fact that there are extreme events in the tails of the distribution, the so-called *rogue waves*, which have been measured in nature and are difficult to model.

The height distribution of waves in the sea can be described by a simple Gaussian model, which we briefly outline. First, let us note that η is a plane wave solution to the linearized water wave equation if it has the form

$$A \cos(k \cdot x - \omega t), \quad \text{with} \quad \omega = \sqrt{g|k| \tanh(|k|H)}, \quad (0.2.8)$$

Here, $x \in \mathbb{R}^2$, $-H$ is the height of the sea bottom, g is the gravitational acceleration (a constant), k is the wavenumber (a vector in $\mathbb{R}^2$), ω is the frequency of the wave, and A is the amplitude of the wave. This approximation is valid in case we have $A|k| \ll 1$. One standard way of representing the sea surface is a random superposition of waves of the form (0.2.8).

Let ω_i be a sequence of i.i.d. random variables, with some known distribution, and let α_i be a sequence of i.i.d. random variables, uniformly distributed in $[-\pi, \pi]$.

Then, the Gaussian model is obtained considering the average

$$\eta(x, t) = \frac{1}{N} \sum_{n=1}^N a(|k_n|) \cos(k_n \cdot x - \omega_n t + \alpha_n),$$

where ω_n and $|k_n|$ are connected by the dispersion relation above. The amplitude a is assumed to be a known function, and the directions of the k_n are chosen uniformly in $[-\pi, \pi]$.

From this model, one can extract predictions on the probability distribution of the height of the water. For example, at $x = 0$ and assuming that the frequencies are concentrated around a reference frequency ω_r , the probability density for the wave height η is, in the limit $N \rightarrow \infty$,

$$f(z) = \frac{z}{\sigma_S^2} \exp\left(-\frac{z^2}{2\sigma_S^2}\right),$$

where σ_S is a parameter. This is the *Rayleigh distribution*.

Here we have presented a simple model for random waves in the sea. It is seen experimentally that models of this form do not make good predictions *in the tails*, and as we have mentioned before, this is due to the appearance of *rogue waves*, which are thought of being a chiefly *nonlinear phenomenon*.

Typically, a rogue wave is measured when

$$\frac{H}{H_s} > 2, \quad \text{or} \quad \frac{\eta_c}{H_s} > 1.25.$$

Here, H_s is the significant wave height (4 times the standard deviation of the sea surface height), H is the wave height from trough to crest, and η_c is the height of the crest above the sea level.

There are many interesting questions from the mathematical viewpoint in this area, and many models for the appearance of rogue waves have emerged. We summarize some here.

1. **Interaction with the bottom topography.** It has been suggested that the effect of the bottom topography in equations of the kind (0.2.7) could be responsible for the appearance of rogue waves in the ocean.
2. **Spatial focusing.** Superposition of waves creates unusually high crests. In compressible models of fluids, which include equations (0.2.7), recently there has been a proof of the existence of solutions which start from finite energy and whose density diverges in finite time [74]. This is a proof that spatial focusing can lead to blow up in compressible and viscous models of fluids.
3. **Modulational instability.** This line of thought has its roots in the work of Benjamin–Feir on water waves [7] and of Zakharov on the cubic nonlinear Schrödinger equation in one dimension [93]. The instability is induced by small sideband perturbations of a stationary wave, which interact nonlinearly with the stationary wave in a constructive way, and grow exponentially. Although this is an interesting subject, due to the different nature of the models considered (they are dispersive models) it goes beyond the scope of this manuscript.

Having outlined some of the physical motivation associated with the models at hand, we now introduce the equations in a more precise and systematic way. We will then describe the results achieved in this context.

0.2.2 Models of compressible and viscous fluids

The first model under consideration describes a compressible, one dimensional fluid with pressure law $p(\rho) = c_p \rho^\gamma$, and viscosity law $\mu(\rho) = c_\mu \rho^\alpha$, where $\alpha, \gamma, c_\mu, c_p$ are constants. Moreover, we pose the problem with periodic boundary conditions on the one-dimensional torus, which we denote by $\mathbb{T}$. More precisely, the system is formulated as follows.

$$\partial_t \rho + \partial_x(u\rho) = 0, \tag{0.2.9}$$

$$\partial_t(\rho u) + \partial_x(\rho u^2) = -\partial_x p(\rho) + \partial_x(\mu(\rho)\partial_x u) + \rho f, \quad (0.2.10)$$

$$(\rho, u)|_{t=0} = (\rho_0, u_0) \quad (0.2.11)$$

with constitutive laws given by

$$p(\rho) = c_p \rho^\gamma, \quad \mu(\rho) = c_\mu \rho^\alpha, \quad c_p \neq 0, \quad c_\mu > 0. \quad (0.2.12)$$

A particular example of this model encodes the one-dimensional barotropic compressible Navier–Stokes equations (when both c_p and c_μ are positive). Note that, when $\alpha = 0$, this model is simply referred to as compressible Navier–Stokes, and when $\alpha > 0$ we say that the viscosity is *degenerate*.

Moreover, the inviscid shallow water model falls in the category of systems described above with the choices $\gamma = 2$, $c_p > 0$ and $c_\mu = 0$ (compare with equation (0.2.7)).

However, we will be interested in *viscous* models of shallow water, such as those appearing in [40, 71]. In this case, the viscosity is degenerate (we assume $\alpha = 1$). The system under consideration can be formulated as follows:

$$\partial_t h + \partial_x(uh) = 0, \quad (0.2.13)$$

$$\partial_t(hu) + \partial_x(hu^2) + \frac{g}{2}\partial_x h^2 = 4\nu\partial_x(h\partial_x u) + hf, \quad (0.2.14)$$

where h is the surface height and u is the fluid velocity, $g > 0$ is the gravitational acceleration at sea level, $\nu > 0$ is a viscosity coefficient, and f is a forcing term. Choosing $\rho = h$ and the appropriate constants we see that these equations reduce to the system (0.2.9)–(0.2.11).

The system (0.2.9)–(0.2.11) is also relevant to the theory of lubrication. In this context, the above equations can encode the so-called *slender jet model*. It has been argued that the slender jet model provides a valid asymptotic description of drop formation in a thin thread of fluid immediately before the drop pinch-off time. For an introduction to the subject and a derivation of this model, see the book by Eggers and Fontelos [35]. See also [34].

We consider a three-dimensional thin thread of fluid axisymmetric about the x -axis, and we assume in the following that all quantities involved only depend on the coordinate x and on the time coordinate t . We also postulate that surface tension effects are present in this fluid. Upon defining $h(t, x) : [0, T] \times \mathbb{T} \rightarrow \mathbb{R}$ to be the neck radius of the thread of fluid (the distance from the surface of the fluid to the x axis at a given point x), and $u(t, x) : [0, T] \times \mathbb{T} \rightarrow \mathbb{R}$ to be the Eulerian velocity of all points with a given x -coordinate, we obtain (close to drop pinch-off, and under heuristic approximations) the following system, also called the *slender jet system*:

$$\partial_t h + u \partial_x h + \frac{1}{2} \partial_x u h = 0, \quad (0.2.15)$$

$$\partial_t u + u \partial_x u + \gamma \partial_x \left(\frac{1}{h} \right) = \nu \frac{\partial_x (h^2 \partial_x u)}{h^2}. \quad (0.2.16)$$

Here, $\gamma > 0$ is a coefficient which encodes surface tension, and ν is a viscosity coefficient.

Upon defining $\rho = h^2$, equations (0.2.15)–(0.2.16) become equations (0.2.9)–(0.2.10) with the following pressure and viscosity laws:

$$p(\rho) = -\gamma \sqrt{\rho} \quad \text{and} \quad \mu(\rho) = \nu \rho.$$

Note that here the pressure term is non-positive, in sharp contrast with the compressible Navier–Stokes system. In fact, the corresponding “inviscid” model (setting $\nu = 0$ in the slender jet system) is badly ill-posed.

Remark 0.2.2. Local well-posedness of (0.2.9)–(0.2.11) in an appropriate regularity class is established in Proposition 2.1.2 for arbitrary smooth $p(\rho)$ and smooth non-negative $\mu(\rho)$. This covers the special case of power law equations of state (0.2.12) in the entire parameters range relevant to our results.

Let us now turn to the two dimensional case. Consider now $x \in \mathbb{R}^2$, let $u(t, x) : [0, T] \times \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be a time-dependent vector field, and let $\rho(t, x) : [0, T] \times \mathbb{R}^2 \rightarrow \mathbb{R}$ be a time-

dependent positive density profile. We are going to be considering 2D compressible and viscous models of the following form:

$$\begin{aligned}\partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla p(\rho) &= \operatorname{div}(\mu(\rho)\nabla u) + \rho f, \\ \partial_t \rho + \operatorname{div}(\rho u) &= 0.\end{aligned}\tag{0.2.17}$$

Here, the pressure is $p(\rho) = c_p \rho^\gamma$, the viscosity coefficient is $\mu(\rho) = c_\mu \rho^\alpha$, with $c_p, c_\mu, \alpha, \gamma$ positive constants, and $\otimes$ denotes the tensor product.

Models of this form appear in the theory of shallow water, as in the one-dimensional case. The 2D shallow water system is a quasilinear system of PDEs (Saint-Venant, 1871), governing the motion of a free surface of water, under the assumption that the aspect ratio of the basin is small (the aspect ratio is the depth of the basin over its width). For a derivation of this system, see the book by Pedlosky [81].

We formulate the 2D shallow water in general terms as in [15]. We consider the two-dimensional system:

$$\begin{aligned}\partial_t(hu) + \operatorname{div}(hu \otimes u) + gh\nabla(h-b) + c_r hu^\perp + c_s \nabla \Delta h + r_0 u + r_1 u|u| \\ = \nu \operatorname{div}(h\nabla u) + hf, \\ \partial_t h + \operatorname{div}(hu) = 0,\end{aligned}\tag{0.2.18}$$

where the unknowns are a time dependent vector field $u(t, x)$ and the water height $h(t, x)$. Furthermore, $b(x) : \mathbb{R}^2 \rightarrow \mathbb{R}$ is the bottom topography. We assume that $b \leq h$. We also have that g (a constant) is the gravitational acceleration, and c_r, c_s, r_0, r_1, ν are also non-negative constants. The term $c_r hu^\perp$ corresponds to the Coriolis force (where $u^\perp = (-u_2, u_1)$ if $u = (u_1, u_2)$), the term $c_s \nabla \Delta h$ corresponds to surface tension, and the terms $-r_0 u - r_1 u|u|$ encode respectively bottom friction in the laminar and in the turbulent regime.

Remark 0.2.3. For the local well-posedness theory of this system in both the inviscid and viscous case, see Section 2.3.

Upon setting c_r, c_s, r_0, r_1 and ν to be zero we obtain the inviscid 2D shallow water system, which is analogous to system (0.2.17) upon setting $h = \rho$, $p(\rho) = \rho^2$, and $c_\mu = 0$. A key feature of the inviscid case is the conservation of potential vorticity. Indeed, if the forcing f is a gradient, we have that the quantity

$$\varpi := \frac{\omega}{\rho}$$

satisfies the transport equation

$$\partial_t \varpi + u \cdot \nabla \varpi = 0.$$

Here, ω is the vorticity (a scalar) $\omega = \partial_2 u_1 - \partial_1 u_2$.

Having introduced the models under consideration, we proceed to a rough description of our results.

0.2.3 Results in the one-dimensional case: continuation criterion, global solutions and bounds on time averages

In this section, we collect rough versions of the results in the one-dimensional case. For the precise statements and the relevant proofs, we refer to Section 2.1. This part is based on joint work with Constantin, Drivas and Nguyen [23].⁴

The first result we obtain is a continuation criterion (Theorem 2.1.1):

Theorem 0.2.4 (Rough version of Theorem 2.1.1). *Consider the one-dimensional compressible model (0.2.9)–(0.2.11). For a wide range of the constants c_p , α and γ which includes the one-dimensional viscous shallow water model (0.2.13)–(0.2.14) and the slender jet model (0.2.15)–(0.2.16), the following holds. If the initial data (u_0, ρ_0) are sufficiently*

⁴This work was presented at the following seminars and conferences: Current trends in kinetic theory, University of Maryland, October 2017 (presentation by Theodore Drivas); Nonlinear and Stochastic PDE and Applications, AMS Sectional Meeting, Boston, April 2018 (presentation by Theodore Drivas).

regular and ρ_0 is strictly positive (no vacuum initially), as long as there holds

$$\inf_{t \in [0, T^*)} \min_{x \in \mathbb{T}} \rho(t, x) > 0,$$

the solution (u, ρ) can be continued past the time T^* .

This theorem shows that solutions to system (0.2.9)–(0.2.11) can become singular only if vacuum formation occurs. In a specific regime (corresponding to the slender jet equations) this settles a conjecture of Peter Constantin formulated in 1994 and recorded in [34].

After having established the continuation criterion, the next two results are focused on the issues of global existence and persistence of regularity in the case $c_p > 0$.

Theorem 0.2.5 (Rough version of Theorem 2.1.2). *Assume*

$$c_p > 0, \quad \alpha \in \left(\frac{1}{2}, 1\right], \quad \text{and} \quad \gamma \geq 2\alpha,$$

that the initial data (u_0, ρ_0) for system (0.2.9)–(0.2.11) are sufficiently regular, and that ρ_0 is strictly positive initially. Then, the solution launched by (u_0, ρ_0) is global.

This result in particular applies to the one-dimensional viscous shallow water equations (0.2.13)–(0.2.14), and provides a global existence proof alternative to that in [44]. Moreover, the result allows for a more degenerate viscosity coefficient than in [73], which considers the case of $\alpha < \frac{1}{2}$ and $\gamma > 1$.

In the case $c_p > 0$ and when the viscosity is more degenerate ($\alpha > 1$) we prove global existence for a specific class of large initial data.

Theorem 0.2.6 (Rough version of Theorem 2.1.3). *Assume that $c_p > 0$ and that*

$$\alpha \geq 0, \quad \gamma \in [\alpha, \alpha + 1], \quad \gamma > 1. \tag{0.2.19}$$

Then, for sufficiently regular initial data (u_0, ρ_0) satisfying the inequality

$$\partial_x u_0(x) \leq \frac{c_p}{c_\mu} \rho_0(x)^{\gamma-\alpha} \quad \forall x \in \mathbb{T}, \quad (0.2.20)$$

the solution with initial data (u_0, ρ_0) is global.

We note that the condition imposed on initial data is not a smallness condition. Moreover, such condition arises naturally from the analysis of the so-called *active potential*, which is a crucial element of our strategy and which we are going to describe in more detail later (see equation (0.2.22)).

The last result in this section concerns long-time averages.

Theorem 0.2.7 (Rough version of Theorem 2.1.5). *Assume that the forcing in the system (0.2.9)–(0.2.11) is sufficiently smooth and has mean zero. Then, for $c_p > 0$ and a for range of constants α, γ which includes the one-dimensional viscous shallow water system, we have that the following quantity is uniformly bounded in terms of initial data for all times $T > 0$:*

$$\frac{1}{T} \int_0^T \|\rho(t, \cdot)\|_{L^\infty(\mathbb{T})} dt \leq C(u_0, \rho_0, f). \quad (0.2.21)$$

Here, $C(u_0, \rho_0, f)$ is a positive constant which depends on initial data and the forcing f .

This theorem in particular applies for the viscous shallow water system (0.2.13)–(0.2.14) which admits global-in-time solutions. The interpretation of the bound (0.2.21) (when ρ is interpreted as the surface height) is that long-time average of the maximum surface height remains bounded, showing that, for very long time, the water height cannot grow indefinitely in an average sense, and moreover proving that, on average, no extreme events can develop. The result therefore provides a partial answer to some questions posed in Section 0.2.1 in the context of the *storm surge* and *rogue wave* problems.

To conclude this overview, let us describe two central elements of our method of proof. The first is the *active potential*, and the second is the so-called *Bresch–Desjardins entropy*.

These are both “good” quantities which provide precious information about the dynamics of system (0.2.9)–(0.2.11).

The active potential

The active potential arises in the momentum equation (0.2.10) and it is the potential corresponding to the force on the right hand side of such equation:

$$\rho D_t u = \partial_x w. \quad (0.2.22)$$

Here, $D_t = \partial_t + u \partial_x$ denotes the material derivative. The potential

$$w = -p(\rho) + \mu(\rho) \partial_x u.$$

combines the viscous stress with the pressure. As w depends on the unknowns and in turn determines their evolution, we refer to it as an *active potential*. Remarkably, w satisfies a *forced quadratic heat equation with linear drift and less degenerate diffusion* with the new dissipation term $\frac{\mu(\rho)}{\rho} \partial_x^2 w$. Moreover, w contains one derivative of u and no derivative of ρ . On one hand, energy estimates for the coupled system of ρ and w allow us to control all the high Sobolev regularity of ρ and u as long as ρ is positive, leading to the proof of the continuation criterion. On the other hand, the heat equation for w satisfies a maximum principle which enables us to obtain global regular solutions for a class of large data when the viscosity is strongly degenerate (see Theorem 2.1.3).

The fact that the active potential solves a nondegenerate evolution with a maximum principle was previously observed in [24] in the context of a 1D Hele-Shaw model, where it played a similar role. In addition, let us mention that the effective viscous flux used in [47] and [68] is also an active potential.

The Bresch–Desjardins entropy

Another important ingredient of our approach is the use of a coercive quantity which is a priori bounded in the evolution for systems of the type (0.2.9)–(0.2.11). This quantity was introduced in the paper [12] in the two-dimensional case for the shallow water equations. For simplicity, let us restrict to the case of the viscous shallow water equations ($\mu(\rho) = \rho$ and $p(\rho) = \rho^2$). In this case, it can be verified that the following quantity can be bounded uniformly in $T > 0$ in terms of initial data:

$$\int_{\mathbb{T}} \left(\frac{\rho}{2} |u + \partial_x(\log \rho)|^2 + \rho^2 \right) (T, x) dx + 2 \int_0^T \int_{\mathbb{T}} |\partial_x \rho|^2 dx dt$$

It is easy to see that, combined with the standard energy estimate, this entropy gives more information about spatial derivatives of ρ . This quantity will play an essential role especially in the proof of the averaged uniform bounds (Theorem 2.1.5).

Having described the results and the relevant techniques in the one dimensional case, we now turn to the analysis of the two-dimensional equations.

0.2.4 Results in two space dimensions: uniform bounds

This part of the manuscript concerns two-dimensional models of the form (0.2.17). After having reviewed the existing literature on the subject, as well as some elementary facts, we are going to consider the case of non-degenerate viscosity ($\alpha = 0$). Restricting our attention to axially symmetric solutions, we shall show an extension of the Lagrangian approach in [51], thereby proving uniform bounds for the density when the equations are posed on a circular annulus with Dirichlet boundary conditions. Here is a rough version of the relevant theorem.

Theorem 0.2.8 (Rough version of Theorem 2.4.12). *Consider the two-dimensional compressible model (0.2.17), where we set $c_p > 0$, $c_\mu > 0$, $\gamma = 2$, $\alpha = 0$. Let us pose the problem on an annulus $\mathcal{A} := \{x \in \mathbb{R}^2 : R_1 \leq |x| \leq R_2\}$, for $R_1 < R_2$ positive constants, and let us impose Dirichlet boundary conditions of the following type: we require u to vanish on*

$\partial\mathcal{A}$ for all non-negative times. Let us pose initial data (u_0, ρ_0) which are axially symmetric, sufficiently smooth, and such that ρ_0 is uniformly bounded away from zero and from above. Let us moreover suppose that the problem admits a solution up to time $T > 0$. Then, the solution satisfies the uniform bound:

$$\rho(t, x) \leq C \quad \text{for } t \in [0, T], \quad x \in \mathcal{A}.$$

Here, C is a positive constant which can be computed explicitly from initial data and is moreover independent of T . In this context, axially symmetric means solutions such that ρ is a radial function for all times, and $u = \nabla\phi(t, |x|) + \nabla^\perp\psi(t, |x|)$, for functions ϕ and $\psi : [0, T] \times [R_1, R_2] \rightarrow \mathbb{R}$.

We can view this result again in the context of the *storm surge* problem outlined in Section 0.2.1, although it is not a result for the viscous shallow water model *per se*, since here the viscosity is non-degenerate ($\alpha = 0$). Note moreover that the analogue of this result is false for the same equation on the whole plane $\mathbb{R}^2$ without boundary conditions, as shown in the recent work [74]. In this work, the authors show in particular that there exist smooth and finite energy solutions to the system (0.2.17) for which the density goes to infinity in finite time at a given point.

0.2.5 Outline of the chapter

For the reader's convenience, we outline the contents of Chapter 2. First, in Section 2.1, we are going to give the precise statements and the proofs of the theorems in the one-dimensional case: rough versions of these results have been stated in Section 0.2.3. We will then proceed review the relevant literature on the shallow water equations in Section 2.2, and we will describe some classical results in Section 2.3, both in the inviscid and viscous case. We will finally discuss the issue of uniform bounds in two space dimensions in the case of non-degenerate viscosity in Section 2.4.

0.3 Chapter 3: the equations of magnetohydrodynamics

This chapter is concerned with the mathematical description of plasmas. A plasma is an *ionized gas*, which roughly means that the gas is in a state in which all the atoms that compose it have some of their electrons “free to move”, and in particular the free electrons are not tied to a specific nucleus. One can think of it as a number of positively charged particles (the nuclei) immersed in a negatively charged “bath” (the electrons). Physically, this situation occurs when a gas is heated to very high temperatures. For instance, this situation occurs in electrical discharges, as well as in astrophysical situations (in the interior of stars and in their vicinity). Moreover, from the physical point of view, researchers have long been interested in the control and stability of plasmas in order to obtain controlled nuclear fusion reactions.

Among the many models of plasmas, two kinds have emerged as the most widely used. The first class of models is of a kinetic nature. In these models, the gas satisfies a Vlasov (or Boltzmann) type equation, and different species (the protons and the electrons) interact via the electromagnetic field. From the mathematical point of view, there is a broad literature on this kind of models (see, for instance, the work [42]).

The second category of models is hydrodynamical, and it comprises the so-called models of *magnetohydrodynamics*: the present chapter will focus on the analysis of these types of models. These equations, just like the Euler equations, arise imposing the standard conservation of momentum, energy and mass to the fluid, in addition to the electromagnetic effects. They were introduced by Alfvén in the 1940s [2]. The most general form of such equations models the physical situation of a compressible fluid. The field of compressible magnetohydrodynamics is an extremely rich one, although in this chapter we are going to focus on the idealized situation in which the fluid is incompressible. We will moreover specify our discussion to the case in which the magnetic effects are dominant, thereby neglecting the contribution of the electric field. Finally, we will restrict our discussion to the non-relativistic

case.

Let us introduce the class of models we will be interested in. We will specify our discussion to the two and three dimensional cases, in the case of the whole $\mathbb{R}^n$ or in the case of periodic boundary conditions.

Let $n \in \{2, 3\}$ and consider $\Omega \in \{\mathbb{T}^n, \mathbb{R}^n\}$, and let $u(t, x), B(t, x) : [0, T] \times \Omega \rightarrow \mathbb{R}^n$ be time-dependent vector fields, respectively the velocity of the plasma and the magnetic field. We say that u and B satisfy the *incompressible, viscous and resistive magnetohydrodynamics* (MHD) system if the following relations hold true:

$$\begin{aligned}
\partial_t u + u \cdot \nabla u + \nabla p &= B \cdot \nabla B + \nu \Delta u, \\
\partial_t B + u \cdot \nabla B - B \cdot \nabla u &= \mu \Delta B, \\
\operatorname{div} u &= 0, \\
\operatorname{div} B &= 0, \\
(u, B)|_{t=0} &= (u_0, B_0).
\end{aligned} \tag{0.3.1}$$

Here, $\nu \geq 0$ is a coefficient encoding the viscosity of the fluid, $\mu \geq 0$ encodes the resistivity, and $p = p(t, x)$ is the so-called hydrodynamical pressure (a scalar function). The first equation in display (0.3.1) is the conservation of momentum, where the $B \cdot \nabla B$ term arises from the contribution of the Lorentz force exerted by the magnetic field on the moving fluid (see equation 0.3.2). The second equation in display (0.3.1) is the induction equation on B (arising from the Maxwell equations). If $\nu = \mu = 0$, this system is called *ideal magnetohydrodynamics*.

Let us now describe some key features of system (0.3.1). We are going to first make a distinction between *hydrodynamical pressure* and *plasma pressure*. Let us focus for the moment on the three-dimensional case. Let us define the current J to be the curl of B :

$J = \nabla \times B$. Note that we have the following identity:

$$J \times B = (\nabla \times B) \times B = B \cdot \nabla B - \frac{1}{2} \nabla |B|^2.$$

Hence, rewriting the first equation of (0.3.1) in this case, we have:

$$\partial_t u + u \cdot \nabla u + \nabla(p + \frac{1}{2}|B|^2) = J \times B + \nu \Delta u \quad (0.3.2)$$

We define the *plasma pressure* $P := -p - \frac{1}{2}|B|^2$. We make this definition so that, in the case in which $u = 0$ identically, B satisfies the equation

$$J \times B + \nabla P = 0.$$

The same definition of plasma pressure holds in two space dimensions. Moreover, note that the term $J \times B$ is precisely the Lorentz force acting on the fluid.

We will now describe how the *topology* of B behaves under the evolution provided by the induction equation. Let us examine the induction equation on B in the non-resistive case ($\mu = 0$). Let us suppose that we have a sufficiently regular solution to the induction equation (u, B) . Then, let us consider the Lagrangian flow $\Psi(t, a)$ induced by u , where $a \in \mathbb{R}^n$ is the Lagrangian label:

$$\partial_t \Psi(t, a) = u(t, \Psi(t, a)), \quad \Psi(0, a) = a.$$

Under these conditions, consider the push-forward of the vector field B_0 under the flow Ψ :

$$\tilde{B} := \Psi_* B_0.$$

This object is defined as follows. Define $\eta(t, x)$ to be the inverse of $\Psi(t, a)$: $\eta(t, \Psi(t, a)) = a$. Then,

$$\Psi_* B_0(t, x) := D_a \Psi(t, \eta(t, x)) B_0(\eta(t, x)).$$

It can be easily checked that, if B is a sufficiently regular solution to the induction equation with initial datum B_0 and vector field u , then B coincides with $\tilde{B}$.

This fact is what in the physics literature is sometimes referred to as the *frozen field line property*: the magnetic field lines (the integral lines of B) are “frozen in the fluid”, which means that they get advected by the flow of u . This can be easily checked to be equivalent to the fact that $B = \Psi_* B_0$. In particular, this means that, under the evolution provided by the induction equation, the magnetic field does not “change topology”.

On the other hand, when $\mu > 0$, the topology of the field lines changes under the evolution, a process known as *magnetic reconnection*.

An important 3D invariant in the case $\mu = 0$ is the so-called *magnetic helicity*. Restricting to the case of the torus $\mathbb{T}^3$, we consider the *magnetic potential* A , which is a vector potential for B :

$$B = \nabla \times A.$$

Then, the magnetic helicity is defined as

$$\mathcal{H}[B] := \int_{\mathbb{T}^3} B \cdot A \, dx.$$

It is easy to verify that, if B satisfies the induction equation for a sufficiently smooth u with zero resistivity, then $\mathcal{H}[B]$ is conserved along the evolution.

A remarkable property of the magnetic helicity is that it bounds from below the energy, up to a multiplicative constant. In other words, there exists $C > 0$ such that the following inequality holds:

$$|\mathcal{H}[B]| \leq C \int_{\mathbb{T}^3} |B|^2 \, dx.$$

Indeed, we have, by the Cauchy–Schwarz inequality and the Poincaré inequality, on $\mathbb{T}^3$,

$$|\mathcal{H}[B]|^2 \leq \int_{\mathbb{T}^3} |B|^2 \, dx \int_{\mathbb{T}^3} |A|^2 \, dx \leq C \left(\int_{\mathbb{T}^3} |B|^2 \, dx \right)^2.$$

From the physical point of view, many issues connected to the MHD equations have been explored. The rigorous treatment of many of such questions from the mathematical point of view is still lacking and, in what follows, we make an incomplete list of interesting questions.

The existence of MHD equilibria in the 3D case

Seeking steady solutions to the system (0.3.1) with $u = 0$ in the non-viscous case, we are lead to the following equations:

$$\begin{aligned} B \cdot \nabla B - \nabla p &= 0, \\ \operatorname{div} B &= 0. \end{aligned} \tag{0.3.3}$$

Solutions B to this equation are called *ideal MHD equilibria*. Moreover, equation (0.3.1) can be immediately seen to be equivalent to the steady Euler system (both in three and two space dimensions).

The construction of MHD equilibria has attracted strong interest from the early days of plasma physics, due to its connections to *nuclear fusion*. Indeed, several of the proposed models of a fusion reactor aim to confine a plasma in a bounded region of space with high enough temperature and for a sufficiently long time so that fusion reactions can occur. It is widely believed that a magnetic field configuration around which this process can occur needs to be a MHD equilibrium, which is moreover required to satisfy some stability requirements. Let us not go into the details here, let us just mention that there are mathematical results which ensure the existence of such equilibria in specific settings (see the discussion in Section 0.3.1), however all the mathematical results so far fail to construct viable equilibria that satisfy the requirements imposed by the physics. This continues to be an active area of research.

Long time dynamics: magnetic relaxation and the Parker problem

A mechanism to construct MHD equilibria involves the analysis of a system of the type (0.3.1) in the infinite time limit. This approach is classical and has been proposed, among others, by Moffatt [76]. It is sometimes referred to as *magnetic relaxation*.

Let us restrict to the non-resistive and viscous case, in which $\mu = 0$ and $\nu > 0$. Then, heuristically, from the momentum equation (since it is a heat-type equation) we expect that $u, \partial_t u$ converge to zero in a Sobolev space in the infinite time limit. Plugging this ansatz into the induction equation, we see that heuristically $\partial_t B = 0$ in the limit, and the momentum equation just reduces to the steady 3D Euler equations. The resulting limit B_∞ , if it exists, can be shown to be non-trivial if the initial configuration B_0 had nontrivial helicity (due to helicity conservation) and moreover, heuristically, it is the push-forward of B_0 under a diffeomorphism.

The first issue with the heuristic outlined above is that, unfortunately, even in the case of two space dimensions, for large general initial data, it is not known whether the system (0.3.1) in the non-resistive and viscous case is globally well-posed. This raises several issues connected to the approach outlined above.

Moreover, there is some evidence in the physics literature which suggests that, in the infinite time limit, the magnetic field B will develop infinite gradients. It is moreover conjectured that the discontinuities in the infinite time limit are “tangential”, i.e. they are heuristically located on curves which are streamlines of the limiting B field. The rigorous treatment of the formation of tangential discontinuities is known as the *Parker problem*, and it is widely open from the mathematical viewpoint. See the book by Parker [80] for some physical motivation and heuristics.

Magnetic reconnection

Related to the issue of discontinuity formation is the issue of *magnetic reconnection*. This is an effect which is manifest in the presence of positive resistivity ($\mu > 0$). In rough terms, this

effect predicts that the magnetic field lines change topology (or reconnect) in the presence of strong gradients of B , and this process releases the energy stored in the magnetic field B . There are several physical models which address this situation, the most widely known is the Sweet–Parker model (see [8]), which arises from basic energy balance considerations. A general PDE description of magnetic reconnection, in both the non-relativistic and the relativistic case, is still lacking. For a rigorous construction of solutions which display reconnection in the hydrodynamical case (in a particular scenario), see the paper by Enciso, Lucà, and Peralta-Salas [36].

It is believed, from the physical point of view, that the rate at which magnetic reconnection (a resistive effect) occurs is very much tied to the speed at which the non-resistive MHD system drives the magnetic field towards high-gradient configurations: the faster high gradients appear in the non-resistive system, the higher the rate of reconnection. It is therefore conjectured that fast magnetic reconnection driven by the formation of high gradients is responsible for high release of energy in astrophysical situations. For instance, it is empirically verified that the plasma surrounding the sun (the so-called *solar corona*) is much hotter than the core of the sun itself, a fact referred to as the *solar corona problem*. A proposed explanation for this elusive fact is that fast magnetic reconnection happens in the solar corona which in turn releases high amounts of energy [91].

Having introduced some of the physical motivation connected to the MHD equations, we will now present some mathematical results pertinent to the questions described above. We are first going to review some of the existing literature concerning existence of ideal MHD equilibria, as well as the magnetic relaxation problem. We will then introduce a result, obtained jointly with P. Constantin, which shows that, for a suitable regularization of the MHD equations, the infinite time limiting procedure is rigorously justified and yields the existence of nontrivial MHD equilibria. Finally, we are going to deal with the issue of discontinuity formation in the infinite time limit, and we are going to provide two examples

of gradient growth, respectively in two and three space dimensions.

0.3.1 Previous results on the existence of 3D MHD equilibria and magnetic relaxation

In this section, we shall review (in a non-exhaustive way) some results on existence of three-dimensional MHD equilibria. We refer to the lecture notes [45], and the review [50] for a comprehensive introduction to the subject.

A classical paper in the physics literature is the 1958 paper by Kruskal and Kulsrud [65], in which the authors introduce the problem of magnetostatic equilibria, and moreover argue that solutions to such problem can be obtained by minimizing a certain functional. The paper [16] constructs discontinuous axisymmetric equilibria of this kind, and uses a KAM-type argument to show that there exist non-axisymmetric configurations which are analytic away from the surfaces of discontinuities and moreover satisfy the static ideal MHD equations in a weak sense everywhere.

Recently, remarkable solutions to the 3D steady Euler equations have been found by Gavrilov [39]. These solutions have the surprising property of being *smooth* and *compactly supported*, and moreover they are axisymmetric. This is in striking contrast with the fact that, in some settings, the 3D steady Euler equations are rigid and satisfy *Liouville-type theorems*. See the recent paper by Constantin, La and Vicol [26] in which these compactly supported solutions are constructed in a more general framework.

In the direction of *Liouville-type theorems*, let us cite the work by Chae and Constantin for Beltrami flows [17] and the work by Hamel and Nadirashvili [43] in the case of $\mathbb{R}^2$ under fairly general conditions.

On the subject of discontinuous weak solutions, a classically known solution to the 3D steady Euler equations is *Hill's spherical vortex*, a solution that can be written explicitly and which is smooth everywhere except at a given sphere, across which its derivatives are discontinuous [46]. Moreover, in a recent paper, Kraus and Hudson [64] argue numerically

that, by a limiting procedure, one can obtain steady MHD equilibria with pressure that is continuous and has a fractal (Cantor-like) profile.

On the issue of magnetic relaxation there is also extensive work. For an overview, we refer to the expository article [6]. This circle of ideas was developed by Moffatt in the foundational papers [76, 77], in which also stability considerations are addressed. For the issue of discontinuity formation in the relaxation process, see the classical work of Taylor [87]. See also the work of Brenier on an MHD-type system [11] and on connections with optimal transport [10].

After having given a quick overview of the literature on the subject, we describe the results obtained in this part of the manuscript. The first result is in the direction of magnetic relaxation. We will then proceed to provide two examples of gradient growth in the MHD system.

0.3.2 A result on long time dynamics and magnetic relaxation

As we have pointed out previously, both the 3D and 2D MHD system are not known to be globally well-posed for sufficiently smooth and large initial data. Therefore, in order to address the issue of long-time dynamics from the mathematical point of view, we work using a suitable regularization of the MHD system itself. Here, we are going to focus our attention on a modified MHD-type system, given by the so-called *Voigt regularization*. Voigt regularized models have been widely studied by Titi and other authors in the context of the analysis of the Euler equations, see [66].

Let us briefly describe this type of regularization in the context of the Euler equations in three dimensions (the MHD case is completely analogous). Recall the incompressible 3D Euler equations:

$$\begin{aligned}\partial_t u + u \cdot \nabla u + \nabla p &= 0, \\ \operatorname{div} u &= 0.\end{aligned}\tag{0.3.4}$$

Now, we essentially replace the momentum equation in (0.3.4) by the following equation:

$$\partial_t u + \mathfrak{L}^{-1}(u \cdot \nabla u + \nabla p) = 0,$$

Here, $\mathfrak{L}^{-1}$ is a smoothing operator. In our case, we will restrict our attention to operators of the type $\mathfrak{L} = I + (-\Delta)^\alpha$, for some positive number α , where I is the identity operator. If α is sufficiently high, then, the corresponding incompressible system can be seen to have global solutions for large and sufficiently smooth initial data.

The same global well-posedness statement holds for the corresponding regularization of an MHD-type system. The system we will be focusing on can be formulated as follows:

$$\begin{aligned} \partial_t \mathfrak{L}u + u \cdot \nabla u + \nabla q &= B \cdot \nabla B + \nu \Delta u, \\ \partial_t \mathfrak{L}B + u \cdot \nabla B &= B \cdot \nabla u, \\ \operatorname{div} u &= 0, \quad \operatorname{div} B = 0, \\ (u, B)|_{t=0} &= (u_0, B_0). \end{aligned} \tag{0.3.5}$$

Note in particular that the regularization is applied to both the velocity field u and to the magnetic field B . In particular, this means that the induction equation no longer holds exactly, and therefore, along the evolution, our magnetic field will “change its topology”.

We obtain the following result (joint work with P. Constantin).

Theorem 0.3.1 (Rough version of Theorem 3.1.9). *Consider sufficiently regular initial data for the system (0.3.5) on the three dimensional torus $\mathbb{T}^3$, which moreover satisfy an integral mean zero condition. Then, the system (0.3.5) launches a unique global solution (u, B) which, on a suitable sequence of times $t_k \rightarrow \infty$ as $k \rightarrow \infty$, satisfies that $B(t_k) \rightarrow B_\infty$ in $H^1(\mathbb{T}^3)$, for some regular vector field B_∞ . Moreover, there exists a sufficiently regular function q_∞ such that the steady 3D Euler equations hold for B_∞ :*

$$B_\infty \cdot \nabla B_\infty + \nabla q_\infty = 0, \quad \operatorname{div} B_\infty = 0.$$

Finally, system (0.3.5) admits a quantity that can be interpreted as a “modified magnetic helicity”, which is conserved along the flow and in particular implies that B_∞ is non-trivial.

We note that the limiting object B_∞ does not necessarily need to be the push-forward under a diffeomorphism of the initial vector field B_0 . It is an interesting research direction whether, without imposing any regularization on the induction equation, one can obtain the same result, or whether there is formation of discontinuities in the limit.

In what follows, we are going to describe two specific scenarios for the *ideal MHD* equations in which there is localized formation of infinite gradients, albeit in a periodic domain or in an infinite energy situation.

0.3.3 Some examples of discontinuity formation and growth in MHD

In this section, we provide two simple examples of gradient growth (and discontinuity formation) for the MHD system in the infinite time limit.

The first example, in the context of the 3D ideal MHD system, is a classical one, and perhaps not very well known.⁵ It can be formulated as follows.

Proposition 0.3.1 (Rough version of Proposition 3.2.1). *Consider the ideal MHD system (0.3.1) on the three-dimensional torus ($\mu = \nu = 0$). There exist initial data (u_0, B_0) for this system which launch a global solution (u, B) , such that the following holds:*

$$\|\nabla B\|_{L^\infty(\mathbb{T}^3)} \rightarrow \infty \quad \text{as} \quad t \rightarrow \infty. \quad (0.3.6)$$

More precisely, the growth is exponential: $\|\nabla B\|_{L^\infty(\mathbb{T}^3)} \geq Ce^t$ for some positive constant C .

A few remarks are in order.

Remark 0.3.2. Note that the initial data in this example can be chosen to be arbitrarily small in a high-order Sobolev norm.

⁵We thank Tarek Elgindi for making us aware of this example.

Remark 0.3.3. Note that the subject of gradient growth is an extensively studied topic in hydrodynamics, especially in the context of the 2D incompressible Euler equations. Let us mention that, in the Euler case, the sharp growth rate of gradients is expected to be double exponential, and there are some results in this direction in the case of domains with boundary [56].

We now describe a second growth example in the context of the 2D MHD system. This example is self-similar, and moreover it has the feature that the corresponding initial datum has infinite energy.

Proposition 0.3.2 (Rough version of Proposition 3.2.2). *Let us consider the ideal MHD system (0.3.1) ($\mu = \nu = 0$) on the whole plane $\mathbb{R}^2$. There exist initial data (u_0, B_0) with infinite energy which launch a global solution to the ideal MHD system with the following property: the L^∞ norm of B diverges to infinity as time goes to infinity at an exponential rate on all open subsets of $\mathbb{R}^2$.*

We see that, in this case, the supremum of B is growing exponentially. An interesting related question is whether this construction can be localized to obtain growth for finite energy solutions of 2D MHD on the whole plane. We refer to Section 3.2 for the precise construction.

0.3.4 Outline of the chapter

The chapter is structured as follows. First, we are going to introduce the result on magnetic relaxation in Section 3.1. For completeness, we are going to include a proof of large data global existence of solution to the Voigt-regularized system (Proposition 3.1.8), and we will subsequently provide a proof of the magnetic relaxation theorem (Theorem 3.1.9). Then, in Section 3.2, we will provide two examples of growth in the context of ideal MHD.

Chapter 1

Nonlinear waves: multi-localized data

In this chapter, we are going to deal with the issue of non-localized initial data for nonlinear wave equations. Our main goal will be to prove Theorem 1.4.1 (a rough version of which was given in the introduction as Theorem 0.1.2). We will therefore first motivate the problem in Section 1.1, and we will discuss the main ingredients of our approach in Section 1.2. The following sections of this chapter (sections 1.3 through 1.10) will deal with the precise statement and proof of the main theorem (Theorem 1.4.1). The content of this chapter is based on work in collaboration with John Anderson, and it is taken from the paper [3].

1.1 Multi-localized data: introduction and motivation

We will now proceed to motivate the problem and our strategy in more detail. We first provide a precise description of the reason why the classical theory does not apply to our case (Section 1.1.1). To gain more intuition, we will then prove the main theorem in the simplified special case of the so-called *Nirenberg trick* in Section 1.1.2. Note that this is a very specific example of a *semilinear* equation, which is much easier than the *quasilinear* case we deal with in our main result. After that, we will state a second rough version of our main theorem in the case in which the initial data are localized around $N > 2$ points

(Section 1.1.3). Finally, in Section 1.1.4, we will state a consequence of our main theorem on global solutions to quasilinear wave equations which applies to a class of initial data with arbitrarily large energy.

1.1.1 The classical theory and the localization requirement

As we mentioned in the introduction, previous results concerning global existence for nonlinear wave equations require the initial data to be localized around a single point in a quantitative sense. This essentially arises from the fact that, in order to prove decay for a linear problem using a physical space approach, we need to use appropriate versions of weighted Sobolev embedding theorems.

For example, in the foundational paper of Klainerman [60], the requirement for global existence of solutions to a nonlinear wave equation in $\mathbb{R}^{3+1}$ (satisfying the classical null condition) is that, schematically, the following inequality holds true:

$$\|\partial\Gamma^\alpha\phi_0\|_{L^2(\Sigma_0)} \leq \varepsilon. \quad (1.1.1)$$

Here, α is a multi-index, Γ^α denotes derivation with respect to a number of Lorentz vector fields (plus the scaling vector field) indexed by α , Σ_0 is the surface whose t coordinate is equal to 0 (on which we impose the initial condition ϕ_0), and finally ∂ denotes the spacetime gradient.

Let us briefly recall the form of the Lorentz vector fields (i.e., the Killing vector fields relative to the Minkowski spacetime), plus the scaling vector field, as expressed in the standard coordinate system:

$$x_i\partial_j - x_j\partial_i, \quad t\partial_i + x_i\partial_t, \quad t\partial_t + r\partial_r. \quad (1.1.2)$$

It is clear from the definition that, on the initial surface Σ_0 , these vector fields will carry weights which grow linearly away from the origin. It is then clear that the norm in inequality (1.1.1) is a weighted Sobolev norm with weights growing away from the origin.

Let us now describe a scenario which departs from the classical one, namely a case in which the data are not localized around the origin, and let us compute the size of the norm appearing in inequality (1.1.1). Consider the points $(R, 0, 0)$ and $(-R, 0, 0)$ on the x -axis in $\mathbb{R}^3$. We then fix two pairs of real valued functions (ϕ_0^R, ϕ_1^R) and $(\phi_0^{-R}, \phi_1^{-R})$ which satisfy the following conditions:

$$\text{Supp}(\phi_0^R), \text{Supp}(\phi_1^R) \subset B((R, 0, 0), 1), \quad \text{Supp}(\phi_0^{-R}), \text{Supp}(\phi_1^{-R}) \subset B((-R, 0, 0), 1).$$

Here, $B(a, r)$ is the Euclidean (3-dimensional) ball of radius r centered at a . Let us moreover suppose that all these functions have $H^k(\mathbb{R}^3)$ norm of size ε , for some large integer $k > 0$.

We then study the initial value problem for a nonlinear wave equation with $R > 0$ as a parameter and $(\phi, \partial_t \phi)|_{t=0} = (\phi_0, \phi_1)$ as initial data, where $\phi_i = \phi_i^R + \phi_i^{-R}$ for $i = 0, 1$. To fix ideas, let us suppose that the nonlinear wave equation satisfies the classical null condition, as in [60]. For $R = 1$, we know that, for ε sufficiently small, we will reduce to the case in which inequality (1.1.1) holds true, and therefore the resulting initial value problem will admit a global solution by the classical theory.

In the case in which R is allowed to be large (and ε does not depend on R), the norm which appeared in inequality (1.1.1) will grow at the following rate:

$$\|\partial \Gamma^\alpha \phi\|_{L^2(\Sigma_0)} \gtrsim \varepsilon R^{|\alpha|}.$$

This holds because the Lorentz vector fields grow linearly in size away from the origin in $\mathbb{R}^3$. This hinders in a fundamental way the application of the classical theory. To prove our result, we will have to circumvent this difficulty. We touch upon this issue again in Section 1.2.1 below. We now proceed to prove our main theorem in the specific case of a *semilinear* wave equation introduced by Nirenberg.

1.1.2 A motivating example: the Nirenberg equation

As a motivating example, we now prove a version of Theorem 0.1.2 for a specific equation, the *Nirenberg example*, which is a *semilinear* wave equation satisfying the classical null condition which moreover admits an *explicit* representation formula for solutions, thereby rendering the proof almost trivial. Nevertheless, studying this particular example serves as a preliminary check before trying to prove a more general result. The inapplicability of this reasoning to a general *quasilinear* wave equation indicates that it is significantly more difficult to prove our result in the general case.

Let us consider the initial value problem for the Nirenberg example:

$$\begin{aligned}\square\phi &= m(d\phi, d\phi), \\ \phi|_{t=0} &= \phi_0^R + \phi_0^{-R}, \\ \partial_t\phi|_{t=0} &= 0.\end{aligned}\tag{1.1.3}$$

We recall that m is the Minkowski metric, $\square$ is the standard wave operator induced by m , and we take ϕ_0^R to be a smooth function supported in the unit ball centered at the point $(x, y, z) = (R, 0, 0)$, while taking ϕ_0^{-R} to be a smooth function supported in the unit ball centered at the point $(x, y, z) = (-R, 0, 0)$. Furthermore, we assume that both ϕ_0^R and ϕ_0^{-R} have small $H^{N_0}(\mathbb{R}^3)$ norm, where N_0 is assumed to be a large positive integer.

Remark 1.1.1. This norm does not depend on the distance between the supports of ϕ_0^R and ϕ_0^{-R} .

It is then straightforward to note that, upon defining

$$u := \exp(-\phi),$$

the initial value problem (1.1.3) is equivalent to the following linear problem:

$$\begin{aligned}\square u &= 0, \\ u|_{t=0} &= \exp(-\phi_0^R - \phi_0^{-R}) =: u_0, \\ \partial_t u|_{t=0} &= 0.\end{aligned}\tag{1.1.4}$$

The solution ϕ to the original nonlinear wave equation can be recovered by taking $\phi = \log u$. It is clear that u_0 is positive everywhere. Furthermore, it is evident that, as long as u solving (1.1.4) remains positive, the solution ϕ to problem (1.1.3) will not develop singularities.

The function u_0 , in addition to being positive everywhere, is identically 1 outside of the union of the supports of ϕ_0^R and ϕ_0^{-R} . This set is contained, by our choice, in the union of two unit balls centered resp. at $(R, 0, 0)$ and $(-R, 0, 0)$.

Using now the Kirchhoff formula for solutions to the linear wave equation in $\mathbb{R}^{3+1}$, we obtain that

$$u(t, x) = \frac{1}{4\pi t^2} \int_{\partial B(x, t)} \underbrace{(u_0(y))}_{(a)} + \underbrace{(y - x) \cdot \nabla u_0(y)}_{(b)} dS(y) \tag{1.1.5}$$

Here, $\partial B(x, t)$ is the sphere of radius t centered at the point $x \in \mathbb{R}^3$, whereas $dS(y)$ denotes the induced area form on $\partial B(x, t)$.

We then note that, if R is chosen to be large enough, the term corresponding to (a) in (1.1.5) is, upon integration, positive and of size $\sim t^2$, regardless of the choice of x and t . This is because the initial data for u_0 is 1 on the whole sphere $\partial B(x, t)$, except at most on two sets of unit measure. On the other hand, the term corresponding to (b) is, upon integration, at most of size t . This is because the expression $(y - x) \cdot \nabla u_0(y)$, restricted to $y \in \partial B(x, t)$, vanishes outside of a set of measure comparable to one (the union of supports of ϕ_0^R and ϕ_0^{-R}). On the same set, on the other hand, the expression (b) is of size at most t .

In conclusion, for large enough values of $R > 0$, u stays positive for all times. Thus the solution ϕ does not develop a singularity. This reasoning proves Theorem 0.1.2 in the very

specific case of equation (1.1.3), and it suggests that the result may hold true for a more general class of nonlinear wave equations satisfying the null condition (0.1.3).

We now proceed to state the main theorem in rough form for the general case of $N \geq 2$ localized pieces of initial data.

1.1.3 Rough version of the main theorem for $N \geq 2$ localized pieces of data

Before describing the ideas of our proof, we state a more precise version of our global stability theorem which is applicable to the case of $N \geq 2$ localized pieces of initial data.

Let us consider the following system of quasilinear wave equations satisfying the classical null condition:

$$\square \phi_A + \sum_{B,C=1}^M F_{BC}^A(d\phi_B, d^2\phi_C) = \sum_{B,C=1}^M G_{BC}^A(d\phi_B, d\phi_C), \quad A = 1, \dots, M. \quad (1.1.6)$$

Here, $\phi_A : [0, T] \times \mathbb{R}^3 \rightarrow \mathbb{R}^M$ is the unknown, and F_{BC}^A and G_{BC}^A are collections of respectively trilinear and bilinear forms satisfying the null condition:

$$F_{BC}^A(N, N, N) = 0, \quad G_{BC}^A(N, N) = 0 \quad \text{for all } N \in \mathbb{R}^4 \text{ such that } m(N, N) = 0. \quad (1.1.7)$$

We moreover impose initial data localized around a configuration of N points as follows. Consider an arbitrary configuration of points $\{p_i\}_{i \in \{1, \dots, N\}}$. We now scale the configuration by a number $R \geq 1$ about the origin, i.e. we consider the set of points $\{Rp_i\}_{i \in \{1, \dots, N\}}$. We then consider the union U_R of all the unit balls centered at the points Rp_i , and we consider initial data supported in U_R . See Figure 1.1 to visualize the initial data in these conditions.

Then, we have the following rough version of the global stability theorem (Theorem 1.4.1).

Theorem 1.1.2 (Rough version of Theorem 1.4.1, case $N \geq 2$). *For all sufficiently small initial data which are localized in the set U_R defined above, i.e. whose support is localized*

around N points as in Figure 1.1, the system of nonlinear wave equations (1.1.6) admits global-in-time solutions. Moreover, the smallness of initial data is measured in a norm which does not depend on the scaling parameter R , but it depends only on N and d_{Π} , with d_{Π} being the ratio of the largest and the smallest pairwise distances between the N points.

We believe that the dependence of ε on the ratio d_{Π} of the largest and smallest pairwise distances between the points is purely technical. We conjecture that the result is true with ε independent of the parameter d_{Π} , but the proof in this thesis cannot directly be used to establish this result.

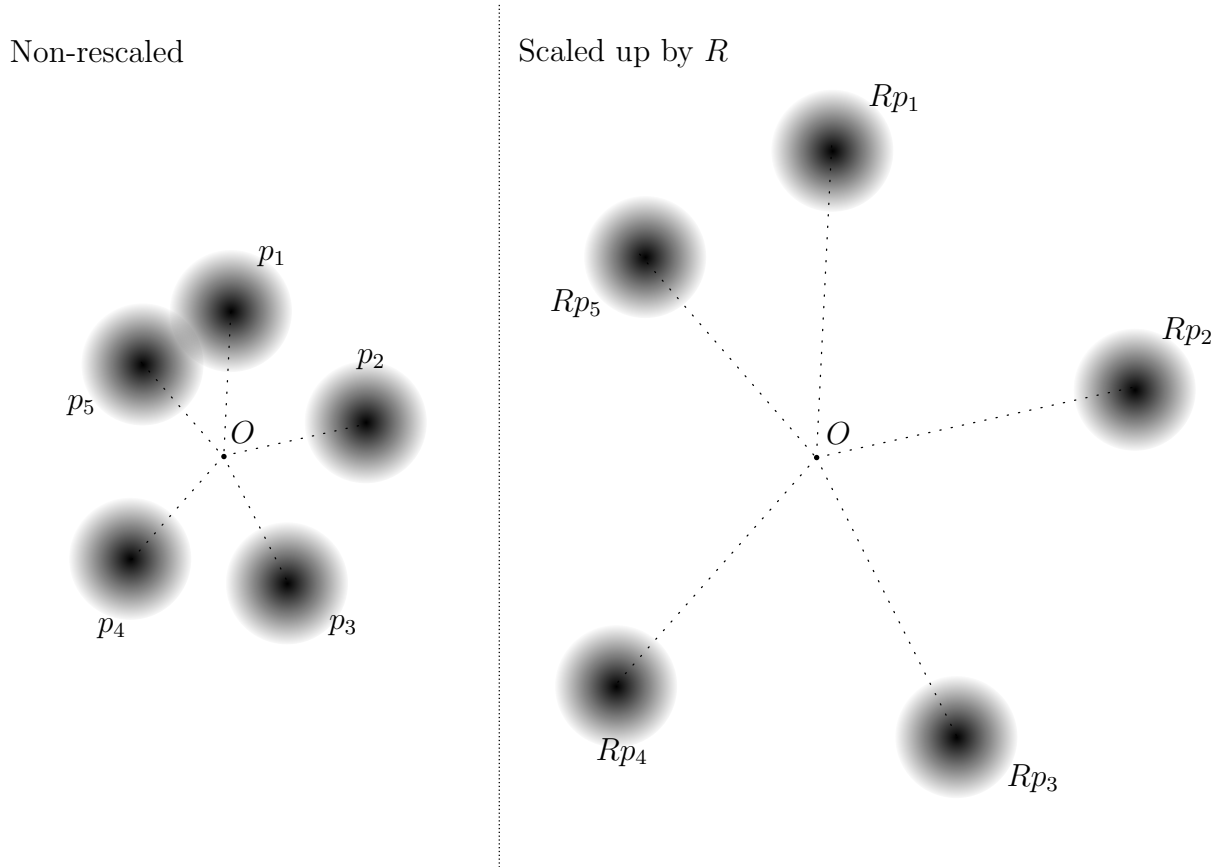


Figure 1.1: Initial data configuration and the parameter R .

1.1.4 A large data global existence result

As a corollary of the main theorem described above, we are able to prove global existence for a class of initial data whose H^1 norm is arbitrarily large. We have the following result.

Theorem 1.1.3 (Rough version of Theorem 1.4.8). *Given any $L > 0$, there exist initial data with H^1 norm of size L supported in $N(L)$ distinct balls of unit radius for which the system of nonlinear wave equations (1.1.6) admits a global-in-time solution.*

In this case, we note that the freedom of choosing the distance between each individual piece of initial data allows us to have initial data of arbitrarily large energy. The interaction between two parts of the solution will therefore be small in view of the decay before such interaction occurs, since the time of first interaction can be made arbitrarily large.

Ours is not the first result on global existence of solutions to nonlinear wave equations with large initial data. See for instance the works [92] and [75] in which the authors are able to construct solutions with large initial data by concentrating most of the energy in outgoing waves. In our case, we require the data to be localized around a number of points, and the data is still mostly outgoing as the interaction between individual waves is controlled.

We now proceed to describe the main ingredients of the proof.

1.2 Overview of the main ideas and outline of the proof

In this section, we will describe the main ideas of the proof. We start by commenting on the dependence of the Klainerman–Sobolev inequality on the parameter R in Section 1.2.1. We then explain the general strategy to improve the L^2 estimates in terms of the parameter R in Section 1.2.2. We will subsequently proceed to outline how we can obtain improved L^2 estimates in the case of a wave equation with quadratic nonlinearity satisfying the null condition in Section 1.2.3. Subsequently, in Section 1.2.4, we will describe how some important geometric facts will be instrumental in showing the improved energy bounds and the

associated trilinear estimates. Finally, we will explain the intuition behind our improved R -weighted Sobolev embeddings in Section 1.2.5.

1.2.1 R -dependence of the Klainerman–Sobolev inequality

It is well known that, in order to prove global stability for nonlinear wave equations, the two main ingredients are energy estimates and pointwise estimates. The pointwise estimates are usually deduced from energy estimates via appropriate weighted Sobolev embedding inequalities, and they allow to recover the energy estimates themselves in a bootstrap argument.

An example of a weighted Sobolev embedding is the Klainerman–Sobolev inequality (see [59]), which is one of the main ingredients to prove global stability of the trivial solutions to certain nonlinear wave equations. The inequality reads:

$$|f|(t, r, \omega) \leq \frac{C}{(1+t+r)(1+|t-r|)^{\frac{1}{2}}} \sum_{|\alpha| \leq 2} \|\Gamma^\alpha f\|_{L^2(\Sigma_t)}. \quad (1.2.1)$$

Here, α is a multi-index with $|\alpha|$ its length, and Γ^α corresponds to a product of Lorentz fields along with the scaling vector field. Moreover, f is a smooth function, Σ_t is the set of points whose time coordinate is equal to t , and (r, ω) are polar coordinates.

To fix ideas, let us restrict our discussion here to the *linear wave equation*. Choosing now $f = \partial\phi$, with ϕ a solution to the linear wave equation, the expression appearing in the RHS of the above Sobolev inequality is an energy term (recall that the Lorentz vector fields commute with the wave operator). In this case, we obtain the following decay estimate:

$$|\partial\phi|(t, r, \omega) \leq \frac{C}{(1+t+r)(1+|t-r|)^{\frac{1}{2}}} \sum_{|\alpha| \leq 2} \|\partial\Gamma^\alpha \phi\|_{L^2(\Sigma_0)}. \quad (1.2.2)$$

Here, we have used the energy bound: $\|\partial\Gamma^\alpha \phi\|_{L^2(\Sigma_t)} \leq \|\partial\Gamma^\alpha \phi\|_{L^2(\Sigma_0)}$, which holds for the linear wave equation.

In the multi-localized case, we note that the initial norm $\|\partial\Gamma^\alpha \phi\|_{L^2(\Sigma_0)}$ will be large as a function of R . This is because the Lorentz fields carry weights which grow linearly away

from a fixed point. As a result, the estimate will inherit large R -weights in the RHS:

$$|\partial\phi|(t, r, \omega) \leq \frac{CR^2}{(1+t+r)(1+|t-r|)^{\frac{1}{2}}} \|\partial\phi\|_{H^2(\Sigma_0)}, \quad (1.2.3)$$

The dependence on R in the above estimate is a serious obstruction to proving global stability independently of R . In the next sections, we will describe our strategy to close an argument which is uniform in the parameter R . We start by outlining our general strategy.

1.2.2 Improved estimates in terms of R : general strategy

In order to overcome the obstruction sketched in the previous section, the first observation is that the solution of the full nonlinear problem will be given by the superposition of the individual solutions arising from each piece of data considered individually, plus a residual “interaction term”. It is heuristically intuitive that this interaction term will be small as a function of the parameter R . More precisely, subtracting off the N solutions arising from each localized piece of data, we will obtain an inhomogeneous equation for the “remainder”, and our goal will be to show that solutions to this equation have energy which is small in terms of the parameter R . The derivation of equations for the remainder is carried out in Sections 1.7.2 and 1.7.3, and the improved energy estimates on the remainder are carried out in Section 1.7.4. This constitutes an improvement in terms of L^2 norms.

In addition to this L^2 improvement, we will also exploit the specific geometric configuration of the initial data in order to show an improvement in the pointwise estimates (Section 1.7.5). This relies on appropriately modified Klainerman–Sobolev inequalities which take into account the specific geometry and the dependence on the parameter R (Section 1.6). Finally, the proof of global stability will follow from these considerations (Section 1.8).

We now proceed to describe the main ingredients of the approach outlined above: the L^2 improvement and the L^∞ improvement.

1.2.3 Improved L^2 estimates: trilinear estimates on null forms

As we mentioned in the previous section, an important element of our approach is the observation that the solution to the full problem can be decomposed as the sum of solutions arising from each individual piece of initial data plus a small remainder.

More precisely, we will denote by ϕ_i the solution to the system (1.1.6) arising from the piece of data localized near the point Rp_i . We then consider the “remainder”

$$\psi := \phi - \sum_{i=1}^N \phi_i.$$

The equation for ψ is a nonlinear wave equation with vanishing initial data and inhomogeneous terms containing ϕ_i . The inhomogeneity in the equation for ψ is composed of terms encoding the interaction between ϕ_i and ϕ_j , for $i \neq j$, which have the following schematic form: $m(d\phi_i, d\phi_j)$. It is therefore natural to consider the linear inhomogeneous equation with vanishing initial data:

$$\square \psi_{ij} = 2m(d\phi_i, d\phi_j). \quad (1.2.4)$$

In order to prove that the energy of ψ_{ij} satisfies a bound in terms of R , we apply the standard energy estimate on (1.2.4), which leads to estimating the *trilinear term*

$$\int \int m(d\phi_i, d\phi_j) \partial_t \psi_{ij} dx dt, \quad (1.2.5)$$

In the course of our argument, we will be able to prove *trilinear estimates* on the above expression, which will show an improved control of ψ_{ij} of the type:

$$\|\partial \psi_{ij}\|_{L^2(\Sigma_t)} \leq C \frac{\varepsilon^2}{R}. \quad (1.2.6)$$

Note the favorable weight of R in the RHS of the previous estimate.

The argument will then be closed in terms of an unknown Ψ which satisfies a *nonlinear*

equation. The function Ψ is obtained by going “one step further” in the approximation of ϕ subtracting off the interaction terms, and it is defined as follows:

$$\Psi := \phi - \sum_{i=1}^N \phi_i - \sum_{i \neq j} \psi_{ij}. \quad (1.2.7)$$

The resulting equation for Ψ is a sourced nonlinear equation with vanishing initial data. In addition, Ψ satisfies better estimates (than ψ_{ij}) in terms of the parameter R . The estimates on ψ_{ij} will allow us to control Ψ in a sufficiently strong way to close a bootstrap argument.

We now turn to describing in more detail the trilinear estimates used in the course of our argument, as well as the strategy we follow to prove them. The procedure is formalized in Section 1.7.4.

1.2.4 The trilinear estimates and the geometry of two interacting waves

In this section, we comment on some elements of the proof of the trilinear estimates which allow us to show the L^2 improvement (inequality (1.2.6)).

Our goal will be to show the following inequality (without loss of generality we set $i = 1$ and $j = 2$):

$$\left| \int \int m(d\phi_1, d\phi_2) \partial_t \psi_{12} dx dt \right| \leq C \frac{\varepsilon^3}{R}. \quad (1.2.8)$$

One can easily check that the naïve pointwise estimates for ϕ_1 and ϕ_2 :

$$|\partial \phi_1| \leq C \frac{\varepsilon}{t}, \quad |\partial \phi_2| \leq C \frac{\varepsilon}{t},$$

are not sufficient to prove the above trilinear estimate. Indeed, we need to exploit the additional null structure in a crucial way.

Note moreover that there is a spacetime region in which interaction occurs between $\partial_{u_1} \phi_1$ and $\partial_{u_2} \phi_2$. Here, $\partial_{u_i} = \partial_t - \partial_{r_i}$, and r_i is the radial coordinate with Rp_i as its center (see

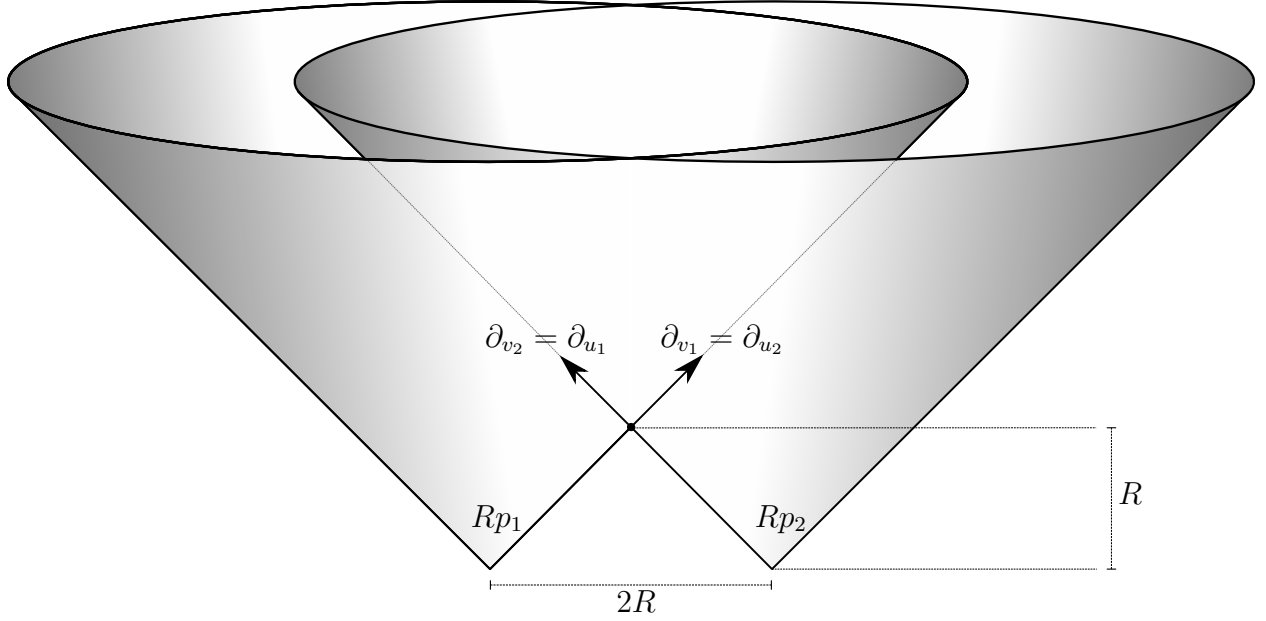


Figure 1.2: Interaction of two nonlinear waves.

Figure 1.2). This in particular implies that there will be interaction between the “slowest decaying” derivatives relative to the left cone, and the “slowest decaying” derivatives relative to the right cone. In the case of the null condition with localized initial data, this situation never occurs as derivatives transversal to the light cone are always multiplied by derivatives tangent to it. Recall that, for a wave equation, derivatives tangent to the light cone decay faster than derivatives transversal to it.

In the spacetime region where the above interaction occurs (namely, where the light cones first intersect), we expect the situation to be analogous to the corresponding situation of a general quadratic nonlinear wave equation, and note that such equations *exhibit blowup for small initial data* (see, for instance, [52]). Therefore, one needs to take additional care in dealing with this spacetime region.

To show the interaction estimate (1.2.8), we have to take advantage of several geometric facts which we record here.

- The two solutions ϕ_1 and ϕ_2 will not interact before time proportional to R .

- The measure of the set of interaction between $\partial_{u_1}\phi_1$ and $\partial_{u_2}\phi_2$ is small in terms of R . This is encoded in the change of coordinates introduced in Lemma 1.5.2.
- The “good” derivatives with respect to ϕ_1 and ϕ_2 become asymptotically aligned. This means that the light cones adapted to ϕ_1 and to ϕ_2 become very similar for very large times. This fact is formalized in Proposition 1.5.4.
- The vector field ∂_t can be written in terms of the good derivatives of ϕ_1 and ϕ_2 in the region in which the interaction between ϕ_1 and ϕ_2 is largest.

The estimates arising from the observations above and the associated trilinear estimates are established in Section 1.7.4.

We now turn to describing how one can make use of the geometry to show an improvement in the pointwise estimates.

1.2.5 Improved L^∞ estimates: R -weighted inequalities

In this section, we describe how we can take advantage of the special geometry of the problem in order to improve the Klainerman–Sobolev estimates in terms of the parameter R .

We first restrict to the case $N = 2$. The fundamental observation here is that there are exactly three Lorentz vector fields whose flow fixes two points in spacetime. To make ideas more concrete, if the two points are located at time $t = 0$ on the x -axis, the flow generated by the following three Lorentz vector fields:

$$z\partial_y - y\partial_z, \quad t\partial_y + y\partial_t, \quad t\partial_z + z\partial_t \tag{1.2.9}$$

will fix the two points. As a consequence, these three vector fields will not introduce R -weights on initial data that is localized around two points located on the x -axis.

This in particular implies that each of those vector fields can be used in a weighted Klainerman–Sobolev inequality without introducing R -weights for a configuration of initial

data which is localized around two points. This observation will allow us, for instance, to obtain an estimate of the form

$$|f(t, r, \omega)| \leq \frac{CR^{\frac{1}{2}}}{(1+t+r)} \sum_{|\alpha| \leq 3} \|\bar{\Gamma}^\alpha f\|. \quad (1.2.10)$$

Here, $\bar{\Gamma}^\alpha$ denotes a string of vector fields chosen among the set composed of the vector fields appearing in display (1.2.9), in addition to every vector field of the form Γ/R , where Γ is either a Lorentz vector field or the scaling vector field. Note that none of these vector fields introduce R -weights on initial data. The precise version of estimate (1.2.10) is proven in Lemma 1.6.1 below. Note the improvement in terms of R when compared with inequality (1.2.3). This is only one example of improvement in the L^∞ estimates, and the full argument will require additional improved Klainerman–Sobolev inequalities. See Section 1.6.

The above inequality is important in the case $N = 2$, where it can be used crucially to deduce improved L^∞ estimates on the functions ψ_{ij} . In addition, the inequality is useful even when $N > 2$, as we always consider pairwise interaction between parts of the solution which arise from different localized pieces of initial data. This fact allows us to take advantage of (1.2.10) even when the data are localized around more than two different points. This procedure will be carried out in detail in Section 1.7.

The above L^2 improvement arising from the trilinear estimates, together with the improvement in the pointwise estimates, will enable us to close a bootstrap argument in the unknown Ψ .

Having provided a rough description of some elements of our proof, we proceed with the actual setup of the machinery we are going to use.

1.3 Setup

In this section, we introduce some notation and recall several crucial facts about the equations under consideration.

1.3.1 Coordinate systems

We consider $(\mathbb{R}^{3+1}, m)$ parametrized by the usual coordinates (t, x, y, z) , where m is the Minkowski metric. We shall always use Σ_s to denote the affine hyperplane with t -coordinate equal to s .

Let $\Pi := \{p_1, \dots, p_N\}$ be a collection of points in $\mathbb{R}^3$ satisfying the following properties:

$$\min_{\substack{i \neq j \\ i, j \in \{1, \dots, N\}}} |p_i - p_j| \geq 2.$$

We also define

$$d_\Pi := \left(\max_{i, j} |p_i - p_j| \right) / \left(\min_{\substack{i \neq j \\ i, j \in \{1, \dots, N\}}} |p_i - p_j| \right). \quad (1.3.1)$$

Let $R \geq 1$ be a parameter. We will consider initial data centered around the points $R \cdot p_i$, on the initial surface Σ_0 . Here, $|w|$ denotes the Euclidean length of $w \in \mathbb{R}^3$ as measured in $\mathbb{R}^3$. We recall that we prove global existence for initial data whose size (measured in a suitable higher-order Sobolev norm) depends only on N and d_Π , and not on R .

For every $i \in \{0, 1, \dots, N\}$, we consider polar coordinates adapted to the point Rp_i . Let us furthermore assume that $p_0 = 0$. The polar coordinates $(r_i, \theta_i, \varphi_i)$ satisfy the following relations, for all $w \in \mathbb{R}^3$:

$$w - Rp_i = r_i \begin{pmatrix} \cos \theta_i \\ \cos \varphi_i \sin \theta_i \\ \sin \varphi_i \sin \theta_i \end{pmatrix}. \quad (1.3.2)$$

We adopt the following convention:

$$r := r_0, \quad \theta := \theta_0, \quad \varphi := \varphi_0. \quad (1.3.3)$$

In particular, in coordinates (r, θ, φ) , the set $\theta = 0$ corresponds to the x -axis (this is an abuse of notation as such coordinates break down at $\theta = 0$).

These polar coordinates induce the usual null coordinates with outgoing and incoming

hypersurfaces.

Definition 1.3.1. Consider the usual coordinate systems (t, x, y, z) and $(t, r_i, \theta_i, \varphi_i)$ on $\mathbb{R}^{1+3}$, where $(r_i, \theta_i, \varphi_i)$ have been introduced above. Then, the following defines a set of null coordinates for all $i \in \{0, \dots, N\}$:

$$u_i := t - r_i, \quad v_i := t + r_i, \quad \theta_i, \quad \varphi_i. \quad (1.3.4)$$

In the case $i = 0$, these reduce to the usual null coordinates centered at the origin. Furthermore, in this case the set corresponding to $\theta = 0$ in (r, θ, ϕ) coordinates is the positive half of the x -axis.

For every $i, j \in \{0, \dots, N\}$, $i \neq j$, we know that $Rp_i \neq Rp_j$. Therefore, up to a translation and a rotation, we can always assume that the point Rp_i has (x, y, z) -coordinates $(-R_{\frac{1}{2}}|p_i - p_j|, 0, 0)$, and that the point Rp_j has (x, y, z) -coordinates $(R_{\frac{1}{2}}|p_i - p_j|, 0, 0)$.

We then define cylindrical coordinates on $\mathbb{R}^3$ (x, ρ, φ) adapted to the x -axis, satisfying the following relations:

$$x, \quad y = \rho \cos \varphi, \quad z = \rho \sin \varphi. \quad (1.3.5)$$

Remark 1.3.2. Note that the coordinate φ introduced here coincides with the φ introduced before in (1.3.3).

We now take coordinates on all of $\mathbb{R}^{1+3}$ that are adapted to hypersurfaces which are hyperboloids in two directions and flat in the third direction. These coordinates will be denoted by $(\tau, \alpha, x, \varphi)$, and they satisfy the relations:

$$t = \tau \cosh(\alpha), \quad x = x, \quad y = \tau \sinh(\alpha) \cos(\varphi), \quad z = \tau \sinh(\alpha) \sin(\varphi). \quad (1.3.6)$$

Remark 1.3.3. Again, the coordinate φ defined here coincides with the φ introduced in (1.3.3).

Definition 1.3.4. For some $\bar{\tau} \geq 0$, we let $H_{\bar{\tau}}$ be the hypersurface which, in the coordinates defined by (1.3.6), is defined as

$$H_{\bar{\tau}} := \{\tau = \bar{\tau}\}. \quad (1.3.7)$$

Remark 1.3.5. Geometrically, these surfaces are just the unit hypersurface (i.e., the hypersurface defined by $\tau = 1$) scaled by a factor of τ with respect to the origin. They can also be seen to be the hypersurfaces found by intersecting the light cone $t = r$ with the hyperplane $x = \tau$ and translating the resulting two dimensional surface in the x -direction.

1.3.2 Vector fields and commutation

We shall now introduce notation for the vector fields we shall use. This includes notation for the vector fields adapted to each point p_i , adapted to pairs of points p_i and p_j , and the appropriate rescalings by the parameter R .

Let $i \neq j$, $i, j \in \{1, \dots, N\}$. Let O_{ij} be the unique isometry of $\mathbb{R}^3$ (composition of a translation and a rotation in $\mathbb{R}^3$) such that the following holds:

$$O_{ij}\left(-\frac{|p_i - p_j|}{2}, 0, 0\right) = p_i, \quad O_{ij}\left(\frac{|p_i - p_j|}{2}, 0, 0\right) = p_j,$$

Using this transformation, we can without loss of generality assume that

$$p_i = \left(-\frac{|p_i - p_j|}{2}, 0, 0\right), \quad \text{and} \quad p_j = \left(\frac{|p_i - p_j|}{2}, 0, 0\right).$$

We now let $\Gamma_{(w_1 w_2)}^{(h)}$, $h \in \{i, j\}$ be the following vector fields:

$$\begin{aligned}
& \text{Rotation vector fields} \quad \left\{ \begin{array}{l} \Gamma_{(xw_1)}^{(0)} := x\partial_{w_1} - w_1\partial_x, \quad \text{if } w_1 \in \{y, z\}, \\ \Gamma_{(xw_1)}^{(i)} := \left(x + R\frac{|p_i - p_j|}{2}\right)\partial_{w_1} - w_1\partial_x, \quad \text{if } w_1 \in \{y, z\}, \\ \Gamma_{(xw_1)}^{(j)} := \left(x - R\frac{|p_i - p_j|}{2}\right)\partial_{w_1} - w_1\partial_x, \quad \text{if } w_1 \in \{y, z\}, \\ \Gamma_{(yz)}^{(i)} = \Gamma_{(yz)}^{(j)} = \Gamma_{(yz)}^{(0)} := y\partial_z - z\partial_y, \end{array} \right. \\
& \text{Lorentz boosts} \quad \left\{ \begin{array}{l} \Gamma_{(tx)}^{(0)} := x\partial_t + t\partial_x, \\ \Gamma_{(tx)}^{(i)} := \left(x + R\frac{|p_i - p_j|}{2}\right)\partial_t + t\partial_x, \\ \Gamma_{(tx)}^{(j)} := \left(x - R\frac{|p_i - p_j|}{2}\right)\partial_t + t\partial_x, \\ \Gamma_{(tw_1)}^{(0)} = \Gamma_{(tw_1)}^{(i)} = \Gamma_{(tw_1)}^{(j)} := w_1\partial_t + t\partial_{w_1}, \text{ if } w_1 \in \{y, z\} \end{array} \right. \quad (1.3.8)
\end{aligned}$$

$$\text{Scaling vector fields} \quad \left\{ \begin{array}{l} S^{(0)} := x\partial_x + y\partial_y + z\partial_z + t\partial_t, \\ S^{(i)} := \left(x + R\frac{|p_i - p_j|}{2}\right)\partial_x + y\partial_y + z\partial_z + t\partial_t, \\ S^{(j)} := \left(x - R\frac{|p_i - p_j|}{2}\right)\partial_x + y\partial_y + z\partial_z + t\partial_t. \end{array} \right.$$

We also rename the rotation vector fields as follows:

$$\Omega_{(ab)}^{(h)} := \Gamma_{(ab)}^{(h)}, \quad \text{for all } a, b \in \{x, y, z\}, \quad h \in \{0, i, j\}. \quad (1.3.9)$$

We furthermore define the set of all translations and spatial translations as follows:

$$\mathbf{T} := \{\partial_x, \partial_y, \partial_z, \partial_t\}, \quad \mathbf{T}_s := \{\partial_x, \partial_y, \partial_z\}. \quad (1.3.10)$$

We also define the set

$$\mathbf{K} := \mathbf{T} \cup \{S^{(0)}\} \cup \bigcup_{\substack{w_1, w_2 \in \{t, x, y, z\} \\ w_1 \neq w_2}} \{\Gamma_{(w_1 w_2)}^{(0)}\}. \quad (1.3.11)$$

Note that this is the usual set of all Killing vector fields of Minkowski space along with the scaling vector field.

Similarly, we have the set of all Killing vector fields based at Rp_h , for $h \in \{i, j\}$:

$$\mathbf{K}^{(h)} := \mathbf{T} \cup \{S^{(h)}\} \cup \bigcup_{w_1, w_2 \in \{x, y, z\}} \{\Gamma_{(w_1 w_2)}^{(h)}\}. \quad (1.3.12)$$

Let us furthermore define the “good” vector fields

$$\mathbf{\Gamma} := \{\Gamma_{(yz)}^{(0)}, \Gamma_{(tz)}^{(0)}, \Gamma_{(ty)}^{(0)}\}. \quad (1.3.13)$$

This is the set of vector fields which do not introduce weights on initial data which is localized around the two points Rp_i and Rp_j . Let us further define the R -weighted Lorentz fields adapted to either the piece of data localized around Rp_i or Rp_j as follows:

$$\mathbf{\Gamma}^{(h)} := \mathbf{\Gamma} \cup \mathbf{T} \cup \bigcup_{w_1, w_2 \in \{t, x, y, z\}} \{R^{-1}\Gamma_{(w_1 w_2)}^{(h)}\} \cup \{R^{-1}S^{(h)}\}, \quad \text{for } h \in \{0, i, j\}. \quad (1.3.14)$$

Also, we define the good derivatives adapted to the h -th light cone:

$$\mathbf{G}^{(h)} := \left\{ \partial_{v_h}, \frac{1}{1+r_h}\Gamma_{(xy)}^{(h)}, \frac{1}{1+r_h}\Gamma_{(xz)}^{(h)}, \frac{1}{1+r_h}\Gamma_{(yz)}^{(h)} \right\}, \quad \text{for } h \in \{0, i, j\}. \quad (1.3.15)$$

In the above, we used the vector field ∂_{v_h} induced by null coordinates $(u_h, v_h, \theta_h, \varphi_h)$ adapted to Rp_h , and defined as in equation (1.3.2). Note that $r_h = \frac{1}{2}(v_h + u_h)$. We also note that the last three vector fields in (1.3.15) are rotations, and their span is two-dimensional everywhere (they are tangent to the spheres of constant r_h -coordinate).

We then define the sets of R -normalized rotations with respect to the h -th light cone as follows:

$$\mathbf{\Omega}^{(h)} := \{\Omega_{(xy)}^{(h)}, \Omega_{(xz)}^{(h)}, \Omega_{(yz)}^{(h)}\} \quad \mathbf{\Omega}_R^{(h)} := \left\{ \Omega_{(yz)}^{(h)}, \frac{\Omega_{(xy)}^{(h)}}{R}, \frac{\Omega_{(xz)}^{(h)}}{R} \right\}, \quad \text{with } h \in \{0, i, j\}. \quad (1.3.16)$$

We define the set of all R -renormalized Minkowski Killing fields (plus scaling) based at the origin as follows:

$$\mathbf{K}_R := \mathbf{T} \cup \bigcup_{w_1, w_2 \in \{t, x, y, z\}} \{R^{-1}\Gamma_{(w_1 w_2)}^{(0)}\} \cup \{R^{-1}S^{(0)}\}. \quad (1.3.17)$$

And the similar set for the h -th light cone, with $h \in \{i, j\}$:

$$\mathbf{K}_R^{(h)} := \mathbf{T} \cup \bigcup_{w_1, w_2 \in \{t, x, y, z\}} \{R^{-1}\Gamma_{(w_1 w_2)}^{(h)}\} \cup \{R^{-1}S^{(h)}\}. \quad (1.3.18)$$

We now define a shorthand notation for “good” and “bad” derivatives.

Definition 1.3.6. Let $\zeta \in \mathcal{C}^\infty(\mathbb{R}^4)$. We define

$$|\partial\zeta| := \sum_{B \in \mathbf{T}} |B\zeta|, \quad |\bar{\partial}^{(h)}\zeta| := \sum_{G \in \mathbf{G}^{(h)}} |G\zeta|, \quad (1.3.19)$$

so that the former notation indicates bad derivatives (all unit derivatives are included in this formula), and the latter comprises *only* good derivatives adapted to the h -th light cone.

We now define multi-indices.

Definition 1.3.7 (Multi-index). Let $\mathbf{A}$ be a set of vector fields, and $m \in \mathbb{N}$, $m \geq 0$. We let $I_{\mathbf{A}}^m$ to be the set of ordered lists of m elements of $\mathbf{A}$. We furthermore let

$$I_{\mathbf{A}}^{\leq m} := \bigcup_{m_1 \in \{0, \dots, m\}} I_{\mathbf{A}}^{m_1}.$$

Definition 1.3.8. Let $m \in \mathbb{N}$, $m \geq 0$, and let $\mathbf{A}$ be a set of vector fields. Let $f \in \mathcal{C}^\infty(\mathbb{R}^4)$, and let $I \in I_{\mathbf{A}}^m$, such that

$$I = (V_1, V_2, \dots, V_m), \text{ with } V_i \in \mathbf{A} \ \forall i \in \{1, \dots, m\}.$$

We then define the derivative of f by the multi-index I as follows:

$$V^I f := V_1 V_2 \cdots V_m f. \quad (1.3.20)$$

Definition 1.3.9. Let $m_1, m_2 \in \mathbb{N}$, and let $I \in I_{\mathbf{A}}^{m_1}$, $J \in I_{\mathbf{A}}^{m_2}$. We say that $K \in I_{\mathbf{A}}^{m_1+m_2}$ satisfies

$$K = I + J,$$

if K can be decomposed into two disjoint ordered lists K_1 and K_2 , such that $K = K_1 \cup K_2$ as sets (counting multiplicity) and, in addition, $I = K_1$, $J = K_2$, where the last two equations are understood in the sense of ordered lists, i. e. taking into account multiplicity and ordering.

Remark 1.3.10. Note that, with this definition, the sum of I and J is not unique.

Definition 1.3.11. We also define inclusion between multi-indices in the following way. We say that $I \subset J$ if $I \in I_{\mathbf{A}}^{m_1}$, $J \in I_{\mathbf{A}}^{m_2}$, with $m_1, m_2 \in \mathbb{N}$, $0 \leq m_1 \leq m_2$, and if I , as a list, is obtained from J by removing $m_2 - m_1$ elements of J (and preserving the ordering).

Lemma 1.3.12 (Leibniz rule). *Let $\mathbf{A}$ be a set of vector fields, and let f, g be smooth functions, $m \in \mathbb{N}$, $m \geq 0$. Let $K \in I_{\mathbf{A}}^m$. We then have*

$$V^K(fg) = \sum_{I+J=K} V^I f V^J g, \quad (1.3.21)$$

where the sum is over all $I \in I_{\mathbf{A}}^{m_1}$, $J \in I_{\mathbf{A}}^{m_2}$ such that $m_1, m_2 \in \mathbb{N}$, $m_1, m_2 \geq 0$, $m_1 + m_2 = m$, and furthermore $I + J = K$ in the sense of Definition 1.3.9.

Definition 1.3.13 (Shorthand notation for iterated derivatives). Let $\eta \in \mathcal{C}^\infty(\mathbb{R}^4)$, and let m, k be non-negative integers. We then define

$$|\partial^k f| := \sum_{I \in I_T^k} |V^I f|, \quad \text{where} \quad \mathbf{T} := \{\partial_x, \partial_y, \partial_z, \partial_t\}. \quad (1.3.22)$$

Also,

$$|\partial^{\leq m} f| := \sum_{k=0}^m |\partial^k f|. \quad (1.3.23)$$

Similarly, for spatial derivatives only, we define:

$$|\widehat{\partial}^k f| := \sum_{I \in I_{T_s}^k} |V^I f|, \quad \text{where} \quad \mathbf{T}_s := \{\partial_x, \partial_y, \partial_z\}. \quad (1.3.24)$$

1.3.3 Classical null forms

Definition 1.3.14. Let k be a positive integer. We say that a k -linear form with constant coefficients $F : T\mathbb{R}^4 \times \dots \times T\mathbb{R}^4 \rightarrow \mathbb{R}$ is a classical null form if, for every null vector ξ with respect to the Minkowski metric, we have that

$$F(\xi, \dots, \xi) = 0. \quad (1.3.25)$$

We recall the following lemma about null forms and commutation.

Lemma 1.3.15 ([49], Lemma 6.6.5). *Let*

$$\Gamma \in \mathbf{K} \cup \mathbf{K}^{(i)} \cup \mathbf{K}^{(j)}.$$

Let also F be a classical trilinear null form with components $F_{\alpha\beta\gamma}$, and G be a classical bilinear null form with components $G_{\alpha\beta}$, both understood as per Definition 1.3.14. Then, we

have, for functions $\eta, \zeta \in \mathcal{C}^\infty(\mathbb{R}^4)$,

$$\begin{aligned}\Gamma(F_{\alpha\beta\gamma}\partial^\alpha\eta\partial^\beta\partial^\gamma\zeta) &= F_{\alpha\beta\gamma}\partial^\alpha\Gamma\eta\partial^\beta\partial^\gamma\zeta + F_{\alpha\beta\gamma}\partial^\alpha\eta\partial^\beta\partial^\gamma\Gamma\zeta + F'_{\alpha\beta\gamma}\partial^\alpha\eta\partial^\beta\partial^\gamma\zeta, \\ \Gamma(g_{\alpha\beta}\partial^\alpha\eta\partial^\beta\zeta) &= G_{\alpha\beta}\partial^\alpha\Gamma\eta\partial^\beta\zeta + G_{\alpha\beta}\partial^\alpha\eta\partial^\beta\Gamma\zeta + G'_{\alpha\beta}\partial^\alpha\eta\partial^\beta\zeta.\end{aligned}\tag{1.3.26}$$

where F', G' are also classical null forms in the sense of Definition 1.3.14.

Remark 1.3.16. In what follows, to simplify notation, we will denote a classical trilinear null form F with components $F_{\alpha\beta\gamma}$ acting on two functions η, ζ by

$$F(d\eta, d^2\zeta) := F_{\alpha\beta\gamma}\partial^\alpha\eta\partial^\beta\partial^\gamma\zeta$$

(d^2 indicates that the null form is acting on the hessian of ζ). Similarly, we will denote the action of a bilinear null form G with components $G_{\alpha\beta}$ on two functions η, ζ by

$$G(d\eta, d\zeta) := G_{\alpha\beta}\partial^\alpha\eta\partial^\beta\zeta.$$

Iterating Lemma 1.3.15, we can prove the following

Lemma 1.3.17 (Null forms and commutation). *Let $K \in I_{\mathbf{K} \cup \mathbf{K}^{(i)} \cup \mathbf{K}^{(j)}}^m$, with $m \in \mathbb{N}$, $m \geq 0$. Let also F be a trilinear null form, and G be a bilinear null form. In these conditions, for all smooth functions $\eta, \zeta \in \mathcal{C}^\infty(\mathbb{R}^4)$, we have*

$$V^K F(d\eta, d^2\zeta) = \sum_{I+J \subset K} F_{IJ}(dV^I\eta, d^2V^J\zeta),\tag{1.3.27}$$

$$V^K G(d\eta, d\zeta) = \sum_{I+J \subset K} G_{IJ}(dV^I\eta, dV^J\zeta).\tag{1.3.28}$$

Here, every F_{IJ} is a trilinear null form as in Definition 1.3.14, and every G_{IJ} is a bilinear null form as in Definition 1.3.14. The sum is taken over all I, J such that $I \in I_{\mathbf{\Gamma}^{(i)}}^{m_1}$, $J \in I_{\mathbf{\Gamma}^{(i)}}^{m_1}$, $m_1 + m_2 \leq m$, $I + J \subset K$.

Proof of Lemma 1.3.17. The proof follows from Lemma 1.3.15 and an induction argument.

□

We now recall a lemma on the structure of null forms.

Lemma 1.3.18 (Structure of null forms). *Let F, G be resp. a classical trilinear null form and a classical bilinear null form in the sense of Definition 1.3.14. Recall the definition of the good derivatives adapted to the h -th light cone in equation (1.3.15). Then, there exists a positive constant $C > 0$ such that the following holds. Let $\zeta, \eta \in \mathcal{C}^\infty(\mathbb{R}^4)$. We have the pointwise inequality*

$$|G(d\zeta, d\eta)| \leq C(|\bar{\partial}^{(h)}\zeta||\partial\eta| + |\partial\zeta||\bar{\partial}^{(h)}\eta|), \quad (1.3.29)$$

$$|F(d\zeta, d^2\eta)| \leq C(|\bar{\partial}^{(h)}\zeta||\partial^2\eta| + |\partial\zeta||\bar{\partial}^{(h)}\partial\eta|). \quad (1.3.30)$$

Recall the expressions $|\bar{\partial}^{(h)}\zeta|$ and $|\partial\zeta|$ from Definition 1.3.6, and that $h \in \{i, j\}$.

Proof. The proof is straightforward writing the expressions for F and G in the null frame and making use of condition (1.3.25) with $\xi = \partial_{u_h}$. □

Combining Lemma 1.3.17 and Lemma 1.3.18, we obtain the following:

Lemma 1.3.19 (Fundamental null form inequality). *Let F, G be resp. a classical trilinear null form and a classical bilinear null form in the sense of Definition 1.3.14. Let also $R \geq 1$. We define the set of vector fields*

$$\mathbf{Z} := \mathbf{K} \cup \mathbf{K}_R \cup \bigcup_{h \in \{i, j\}} (\Gamma^{(h)} \cup \mathbf{K}^{(h)} \cup \mathbf{K}_R^{(h)}).$$

Let $m \in \mathbb{N}$, $m \geq 0$. Let furthermore I be a multi-index in $I_{\mathbf{Z}}^{\leq m}$. Then, there exists a positive constant $C > 0$ such that the following holds. Let $\zeta, \eta \in \mathcal{C}^\infty(\mathbb{R}^4)$. We have the pointwise inequalities

$$|V^I G(d\zeta, d\eta)| \leq C \sum_{H+K \subset I} (|\bar{\partial}^{(i)} V^H \zeta| |\partial V^K \eta| + |\partial V^H \zeta| |\bar{\partial}^{(i)} V^K \eta|), \quad (1.3.31)$$

$$|V^I F(d\zeta, d^2\eta)| \leq C \sum_{H+K \subset I} (|\bar{\partial}^{(i)} V^H \zeta| |\partial^2 V^K \eta| + |\partial V^H \zeta| |\partial \bar{\partial}^{(i)} V^K \eta|). \quad (1.3.32)$$

Here, we used the expressions $|\bar{\partial}^{(i)} \zeta|$ and $|\partial \zeta|$ from Definition 1.3.6.

Proof of Lemma 1.3.19. The proof for elements of $\mathbf{K}$ and $\mathbf{K}^{(h)}$ is evident. Regarding the remaining elements in the set $\mathbf{Z}$, we note that the condition $R \geq 1$ suffices to bound the terms F' and G' arising from the application of equation (1.3.26), as these terms will acquire an R^{-1} factor in the commutation. $\square$

1.4 Precise statement of the main theorems

1.4.1 Main theorem

We proceed to state the main theorem of this chapter.

Theorem 1.4.1 (Nonlinear waves with null condition and multi-localized data). *Let M, N, n_0 be non-negative integers, such that $n_0 \geq 19$. Let $\{F_{BC}^A\}_{A,B,C=1,\dots,M}$ and $\{G_{BC}^A\}_{A,B,C=1,\dots,M}$ be resp. a collection of trilinear and bilinear classical null forms, i. e. each of the F_{BC}^A and G_{BC}^A is as in Definition 1.3.14 and satisfies:*

$$F_{BC}^A : T\mathbb{R}^4 \times T\mathbb{R}^4 \times T\mathbb{R}^4 \rightarrow \mathbb{R}, \quad G_{BC}^A : T\mathbb{R}^4 \times T\mathbb{R}^4 \rightarrow \mathbb{R}.$$

Let furthermore $\Pi := \{p_1, \dots, p_N\}$ be a collection of N points in $\mathbb{R}^3$, with d_Π the ratio of the largest and smallest distances between them as defined in display (1.3.1).

Then, there exists $\varepsilon_0 = \varepsilon_0(d_\Pi, \{F_{BC}^A\}_{A,B,C=1,\dots,M}, \{G_{BC}^A\}_{A,B,C=1,\dots,M}, N) > 0$ such that the following holds true for all $R \geq 0$ and for all $0 < \varepsilon < \varepsilon_0$.

Consider a collection of functions

$$(\bar{\phi}_{A,i}^{(0)}, \bar{\phi}_{A,i}^{(1)})_{A \in \{1,\dots,M\}, i \in \{1,\dots,N\}} \quad (1.4.1)$$

which satisfy the support and smallness conditions:

$$\begin{aligned} \text{supp}(\bar{\phi}_{A,i}^{(0)}) &\subset B(0,1), & \text{supp}(\bar{\phi}_{A,i}^{(1)}) &\subset B(0,1), \\ \|\bar{\phi}_{A,i}^{(0)}\|_{H^{n_0+1}(\mathbb{R}^3)} + \|\bar{\phi}_{A,i}^{(1)}\|_{H^{n_0}(\mathbb{R}^3)} &\leq \varepsilon, \end{aligned} \tag{1.4.2}$$

for all $i \in \{1, \dots, N\}$, and $A \in \{1, \dots, M\}$. Here, $B(0,1)$ denotes the Euclidean ball of unit radius centered at the origin.

Let us then construct initial data as follows. Let w_i be the point $R \cdot p_i$. Let us define, for all $A \in \{1, \dots, M\}$,

$$\phi_A^{(0)} := \sum_{i=1}^N \bar{\phi}_{A,i}^{(0)}(x - w_i), \quad \phi_A^{(1)} := \sum_{i=1}^N \bar{\phi}_{A,i}^{(1)}(x - w_i). \tag{1.4.3}$$

Let us consider the initial value problem given by the following system of quasilinear wave equations with the specified data:

$$\begin{aligned} \square \phi_A + \sum_{B,C=1}^M F_{BC}^A(d\phi_B, d^2\phi_C) &= \sum_{B,C=1}^M G_{BC}^A(d\phi_B, d\phi_C), \quad A = 1, \dots, M, \\ \phi_A|_{t=0} &= \phi_A^{(0)}, \quad A = 1, \dots, M, \\ \partial_t \phi_A|_{t=0} &= \phi_A^{(1)}, \quad A = 1, \dots, M. \end{aligned} \tag{1.4.4}$$

Then, the initial value problem (1.4.4) admits a global-in-time solution ϕ_A , which furthermore decays with quantitative rates. Thus, the trivial solution to (1.4.4) is asymptotically stable under this class of non-localized perturbations uniformly in the scale R .

Remark 1.4.2. We note that the global solution ϕ_A constructed in the previous theorem moreover satisfies uniform (in R) energy estimates and quantitative decay estimates obtained by combining the inequalities in the statements of Lemma 1.5.1, Theorem 1.4.4 and Theorem 1.4.7.

Remark 1.4.3. We note that we require L^2 bounds on 19 derivatives of the initial data. The proof of Theorem 1.4.1 could be optimized in terms of the number of derivatives, but we are

not interested in such issues here.

From now on, to simplify notation, we will specialize our discussion to the case in which we have one single equation, instead of a system of equations, as no conceptual element is introduced in the proof for systems. We will therefore restrict our attention to a single equation of the type

$$\square\phi + F(d\phi, d^2\phi) = G(d\phi, d\phi), \quad (1.4.5)$$

where F is a trilinear null form as in Definition 1.3.14 and G is a bilinear null form as in Definition 1.3.14. In the case of a single equation, G is necessarily a constant multiple of the Minkowski metric m , although the theorem holds for more general G (arising from systems of equations).

We now proceed to state the theorems used in the proof of the main Theorem 1.4.1.

1.4.2 Statement of the auxiliary theorems

Theorem 1.4.4 (Interaction of two localized pieces of initial data). *There exists a constant $C > 0$ such that the following holds. Let $0 < \varepsilon < \varepsilon_0$, where ε_0 is as in the statement of Theorem 1.4.1. Let ψ_{ij} be a solution to the equation*

$$\begin{aligned} \square\psi_{ij} + F(d\phi_i, d^2\phi_j) &= G(d\phi_i, d\phi_j), \\ \psi_{ij}|_{t=0} &= 0, \\ \partial_t\psi_{ij}|_{t=0} &= 0. \end{aligned} \quad (1.4.6)$$

Here, ϕ_i , $i \in \{1, \dots, N\}$, have been defined in Section 1.7.1, and ϕ_i corresponds to the solution arising from the i^{th} piece of initial data.

For simplicity, let us suppose that $i = 1$, $j = 2$, the initial data for ϕ_1 is centered at the point $w_1 = (-\frac{1}{2}|p_1 - p_2|R, 0, 0)$, and the initial data for ϕ_2 is centered at the point $w_2 = (\frac{1}{2}|p_1 - p_2|R, 0, 0)$ (we are assuming, without loss of generality, that $|p_i - p_j| \geq 2$).

Then, for all $K_1 \in I_{\Gamma^{(h)} \cup \mathbf{K}_R}^{\leq n-1}$ a multiindex of length at most $n - 1$ composed of elements

of $\Gamma^{(h)} \cup \mathbf{K}_R$, $h \in \{1, 2\}$, for all $K_2 \in I_{\Gamma^{(h)} \cup \mathbf{K}_R}^{\leq n-4}$, and for all $t \geq 0$, the following estimates hold for ψ_{12} :

$$\|\partial V^{K_1} \psi_{12}\|_{L^2(\Sigma_t)} \leq \frac{C\varepsilon^2}{R}, \quad (1.4.7)$$

$$\|\partial V^{K_1} \psi_{12}\|_{L^2(H_t)} \leq \frac{C\varepsilon^2}{R}, \quad (1.4.8)$$

$$\|(1 + |u_h|)^{-\frac{1}{2} - \frac{\delta}{2}} \bar{\partial}^{(h)} V^{K_1} \psi_{12}\|_{L^2(\mathbb{R}^{3+1})} \leq \frac{C\varepsilon^2}{R}, \quad (1.4.9)$$

$$\|\partial V^{K_2} \psi_{12}\|_{L^\infty(\Sigma_t)} \leq \frac{C\varepsilon^2}{\sqrt{Rt}}, \quad (1.4.10)$$

$$\|\partial V^{K_2} \psi_{12}\|_{L^\infty(\Sigma_t)} \leq \frac{C\varepsilon^2 R^2}{t(1 + |u_i|)^{\frac{1}{2}}}, \quad (1.4.11)$$

$$\|\bar{\partial}^{(h)} V^{K_2} \psi_{12}\|_{L^\infty(\Sigma_t)} \leq \frac{C\varepsilon^2 R^3}{t^{\frac{3}{2}}}. \quad (1.4.12)$$

Here, $h \in \{1, 2\}$. The analogous estimates hold true for ψ_{ij} , with straightforward changes for the vector fields $\Gamma^{(h)}$.

Remark 1.4.5. We note that the result is more generally true with $d(p_i, p_j)$ replacing R . We are using that all of the distances are comparable to R up to a factor of d_Π which we suppress because ε is allowed to depend on d_Π . In fact, the factor of d_Π in these estimates would go in the denominator, which would in fact improve the estimates.

Remark 1.4.6. Note that the function ψ_{12} is supported only at times $t \geq \frac{R}{2}$. Indeed, the equation has vanishing initial data, and we know that the inhomogeneous terms are 0 before $t = \frac{R}{2}$ by comparing the domains of influence of ϕ_1 and ϕ_2 . Furthermore, we have that the intersection of the supports of ϕ_1 and ϕ_2 is contained in the set

$$\left\{t \geq \left(1 - \frac{1}{|p_1 - p_2|}\right)r_1\right\} \cap \left\{t \geq \left(1 - \frac{1}{|p_1 - p_2|}\right)r_2\right\}.$$

Theorem 1.4.7 (Existence of solutions to the nonlinear equation). *There exists a constant $C > 0$ such that, for every $0 < \varepsilon < \varepsilon_0$, where ε_0 is as in the statement of Theorem 1.4.1, we*

have the following. We consider the initial value problem

$$\begin{aligned}
& \square \Psi + F(d\Psi, d^2\Psi) \\
& + \sum_{\substack{i,j=1,\dots,N \\ i \neq j}} (F(d\psi_{ij}, d^2\Psi) + F(d\Psi, d^2\psi_{ij})) + \sum_{\substack{g,h,i,j=1,\dots,N \\ g \neq h, i \neq j}} F(d\psi_{gh}, d^2\psi_{ij}) \\
& + \sum_{i=1}^N \sum_{\substack{g,h=1,\dots,N \\ g \neq h}} (F(d\phi_i, d^2\psi_{gh}) + F(d\psi_{gh}, d^2\phi_i)) \\
& + \sum_{i=1}^N (F(d\phi_i, d^2\Psi) + F(d\Psi, d^2\phi_i)) \\
& = \sum_{\substack{i,j=1,\dots,N \\ i \neq j}} (G(d\psi_{ij}, d\Psi) + G(d\Psi, d\psi_{ij})) + \sum_{\substack{g,h,i,j=1,\dots,N \\ g \neq h, i \neq j}} G(d\psi_{gh}, d\psi_{ij}) \quad (1.4.13)
\end{aligned}$$

$$\begin{aligned}
& + \sum_{i=1}^N \sum_{\substack{g,h=1,\dots,N \\ g \neq h}} (G(d\phi_i, d\psi_{gh}) + G(d\psi_{gh}, d\phi_i)) \\
& + \sum_{i=1}^N (G(d\phi_i, d\Psi) + G(d\Psi, d\phi_i)) + G(d\Psi, d\Psi), \quad (1.4.14)
\end{aligned}$$

$$\Psi|_{t=0} = 0,$$

$$\partial_t \Psi|_{t=0} = 0.$$

Here, ϕ_i are as in the statement of Lemma 1.5.1, and ψ_{ij} is as in the statement of Theorem 1.4.4. Under these hypotheses, we have that equation (1.4.13) admits a global solution Ψ . Moreover, we have the following quantitative decay estimates, valid for all $I \in I_{\mathbf{K}_R}^{\leq N_0}$, with $N_0 \geq 7$, and $i \in \{1, \dots, N\}$:

$$\begin{aligned}
& \|\partial V^I \Psi\|_{L^2(\Sigma_t)} \leq \varepsilon^{3-\delta} R^{-\frac{3}{2}+\delta} \quad \text{for } t \geq 0, \\
& \|(1 + |u_i|)^{-\frac{1}{2}-\frac{\delta}{2}} \bar{\partial}^{(i)} V^I \Psi\|_{L^2([0,T] \times \mathbb{R}^3)} \leq \varepsilon^{3-\delta} R^{-\frac{3}{2}+\delta} \quad \text{for } t \geq 0, \\
& |\partial V^J \Psi(t, r, \theta, \varphi)| \leq \varepsilon^{3-\delta} (1+t)^{-1} R^{-\frac{1}{2}+\delta} \quad \text{for } t \geq 0, \\
& |\bar{\partial}^{(i)} V^J \Psi(t, r, \theta, \varphi)| \leq \varepsilon^{3-\delta} (1+t)^{-\frac{3}{2}} R^{\frac{3}{2}+\delta} \quad \text{for } t \geq R^{20},
\end{aligned}$$

$$|\partial V^J \Psi(t, r, \theta, \varphi)| \leq \varepsilon^{3-\delta} (1 + v_i)^{-1} (1 + |u_i|)^{-\frac{1}{2}} R^{\frac{1}{2}+\delta} \quad \text{for } t \geq R^{20}.$$

1.4.3 Proof of the main theorem given the auxiliary theorems

Proof of Theorem 1.4.1 given Theorems 1.4.4 and 1.4.7. We note that Lemma 1.5.1 gives the existence of global-in-time solutions ϕ_i to the initial value problem (1.5.1). Moreover, we have that each ψ_{ij} exists globally because it solves a linear wave equation. Then, by Theorem 1.4.7, we have the global-in-time existence of Ψ satisfying the initial value problem (1.4.13). We then let

$$\phi := \Psi + \sum_{i=1}^N \phi_i + \sum_{\substack{i,j=1,\dots,N \\ i \neq j}} \psi_{ij},$$

and note that ϕ is a smooth global solution to the initial value problem:

$$\begin{aligned} \square \phi + F(d\phi, d^2\phi) &= G(d\phi, d\phi), \\ \phi|_{t=0} &= \phi^{(0)}, \\ \partial_t \phi|_{t=0} &= \phi^{(1)}. \end{aligned} \tag{1.4.15}$$

The calculations showing this are carried out in more detail in Sections 1.7.2 and 1.7.3. Furthermore, one can clearly reconstruct the decay estimates for ϕ from the known decay estimates for Ψ, ψ_{ij}, ϕ_i . This concludes the proof of the main theorem (Theorem 1.4.1). $\square$

1.4.4 A large data global existence result

We are also able to prove a large data global existence result.

Theorem 1.4.8. *Let $L \in \mathbb{R}$, $L \geq 0$, and $M \in \mathbb{N}$, $M \geq 0$ be given. There exist a real number ε_0 , an integer $N(L)$ and points $w_i \in \mathbb{R}^3$, $i \in \{1, \dots, N(L)\}$, such that the following holds. There exists a collection*

$$(\bar{\phi}_{A,i}^{(0)}, \bar{\phi}_{A,i}^{(1)})_{A \in \{1, \dots, M\}, i \in \{1, \dots, N(L)\}}$$

of pairs of smooth vector valued functions on $\mathbb{R}^3$ satisfying the following properties:

$$\text{supp}(\bar{\phi}_{A,i}^{(0)}) \subset B(0,1), \quad \text{supp}(\bar{\phi}_{A,i}^{(1)}) \subset B(0,1), \quad (1.4.16)$$

$$\|\bar{\phi}_{A,i}^{(0)}\|_{H^{N_1}(B(0,1))} \leq \varepsilon_0, \quad \|\bar{\phi}_{A,i}^{(1)}\|_{H^{N_1-1}(B(0,1))} \leq \varepsilon_0, \quad (1.4.17)$$

for all $i \in \{1, \dots, N(L)\}$ and all $A \in \{1, \dots, M\}$, where $N_1 \geq 19$ (here, $B(0,1)$ is the ball of unit radius centered at the origin). Moreover, under these conditions, the trivial solution to the initial value problem (1.4.4) is asymptotically stable under perturbations of the form

$$\phi_A^{(0)} = \sum_{i=1}^{N(L)} \bar{\phi}_{A,i}^{(0)}(x - w_i), \quad \phi_A^{(1)} = \sum_{i=1}^{N(L)} \bar{\phi}_{A,i}^{(1)}(x - w_i).$$

Moreover, we have that

$$\|\phi_A^{(0)}\|_{H^1(\mathbb{R}^3)} \geq L, \quad \|\phi_A^{(1)}\|_{L^2(\mathbb{R}^3)} \geq L,$$

meaning that the initial data is allowed to have arbitrarily large H^1 norm.

The configuration of the data is further described in Section 1.9. Moreover, we shall once again restrict ourselves to studying single equations of the form (1.4.5) to simplify notation, although the same proof would establish the Theorem for quasilinear systems as in (1.4.4).

1.5 Geometry of interacting wave fronts

In this section, we prove three important technical lemmas which we will need in the rest of the chapter. They are a statement concerning improved u -decay for solution to quasilinear wave equations satisfying the null condition and with localized initial data (Lemma 1.5.1), an important change of coordinates (Lemma 1.5.2, which is used to formalize the fact that the measure of the interaction region of two nonlinear waves originating from sources located far away from each other is small), and finally a lemma concerning the asymptotic comparison

of null derivatives with respect to two different light cones (Lemma 1.5.4).

1.5.1 Decay properties for localized initial data

We require the following technical result about improved decay rates of waves originating from a localized source.

Lemma 1.5.1 (Improved u -decay for solutions to quasilinear wave equations). *Let $R > 0$, $n \in \mathbb{N}$, $n \geq 0$. Let also F, G be a trilinear resp. bilinear classical null form as in Definition 1.3.14. There exist a universal constant $C > 0$ and a $\delta > 0$ such that the following holds. Let ζ be a smooth solution to the following initial value problem:*

$$\begin{aligned}\square\zeta + F(d\zeta, d^2\zeta) &= G(d\zeta, d\zeta), \\ \zeta|_{t=0} &= \zeta^{(0)}, \\ \partial_t\zeta|_{t=0} &= \zeta^{(1)}.\end{aligned}\tag{1.5.1}$$

We further suppose the following bounds on the initial data $\zeta^{(0)} : \mathbb{R}^3 \rightarrow \mathbb{R}$ and $\zeta^{(1)} : \mathbb{R}^3 \rightarrow \mathbb{R}$:

$$\begin{aligned}\text{supp}(\zeta^{(0)}) &\subset B(0, 1), \quad \text{supp}(\zeta^{(1)}) \subset B(0, 1), \\ \|\zeta^{(0)}\|_{H^{n+7}} + \|\zeta^{(1)}\|_{H^{n+6}} &\leq \varepsilon.\end{aligned}\tag{1.5.2}$$

Here, $B(0, 1)$ denotes the unit ball centered at the origin.

In these conditions, for every multi-index $I \in I_{\mathbf{K}}^{\leq n}$, we have the following decay properties:

$$|\bar{\partial}^{(0)} V^I \zeta| \leq C\varepsilon \frac{1}{(1+r^2)(1+|u|)^\delta}, \quad |\partial V^I \zeta| \leq C\varepsilon \frac{1}{(1+v)(1+|u|)^{1+\delta}}.\tag{1.5.3}$$

Here, we used the definition of good derivatives adapted to the light cone emanating from the origin in equation (1.3.15).

Furthermore, we have the following uniform L^2 estimates, valid for all $I \in I_{\mathbf{K}}^{\leq n+4}$, and all $t \geq 0$:

$$\|\partial V^I \zeta\|_{L^2(\Sigma_t)} \leq C\varepsilon.\tag{1.5.4}$$

For the proof of this statement, we refer to the paper [3].

1.5.2 An important change of coordinates

We shall now define a coordinate system which will be useful when computing the interaction between waves originating from different points in space, and we prove a change of variables formula for this coordinate system. We suppose without loss of generality here that $i = 1$, $j = 2$, and that

$$p_1 = (-1, 0, 0), \quad p_2 = (1, 0, 0),$$

so that we focus on interaction of waves emanating from resp. the points $(-R, 0, 0)$ and $(R, 0, 0)$. This Lemma is used in proving the improved energy estimates in Section 1.7.4.

Lemma 1.5.2. *On $\mathbb{R}^4$, parametrized by cartesian coordinates (t, x, y, z) we consider the coordinate system (t, r_1, r_2, φ) defined by the following relations:*

$$\begin{aligned} t &= t, \\ (x + R)^2 + \rho^2 &= r_1^2, \\ (x - R)^2 + \rho^2 &= r_2^2, \\ \varphi &= \arctan(z/y). \end{aligned} \tag{1.5.5}$$

Here, $\rho^2 := y^2 + z^2$. We also consider null coordinates defined by:

$$u_h := t - r_h, \quad v_h := t + r_h, \quad \text{where } h \in \{1, 2\}. \tag{1.5.6}$$

Then, for every smooth and integrable function $f : \mathbb{R}^4 \rightarrow \mathbb{R}$, the following change of variable formulas hold:

$$\begin{aligned} \int_{\mathbb{R}^4} f(t, x, y, z) dt dx dy dz &= \int_{\mathbb{R} \times \mathcal{R} \times [0, 2\pi)} f(t, r_1, r_2, \varphi) \frac{r_1 r_2}{2R} dt dr_1 dr_2 d\varphi, \\ \int_{\mathbb{R}^4} f(t, x, y, z) dt dx dy dz &= \int_{\mathcal{S} \times [0, 2\pi)} f(u_1, v_1, u_2, \varphi) \frac{r_1 r_2}{4R} du_1 dv_1 du_2 d\varphi, \end{aligned} \tag{1.5.7}$$

where $\mathcal{R}$ is the set of those values of r_1 and r_2 which satisfy:

$$r_1 + r_2 \geq 2R, \quad |r_1 - r_2| \leq 2R.$$

Furthermore, $\mathcal{S}$ is the set of those values of u_1, v_1 and u_2 such that

$$v_1 - u_2 \geq 2R, \quad |u_2 - u_1| \leq 2R.$$

Finally, the following formulas for integrals on null cones hold true:

$$\begin{aligned} \int_{C_{\bar{u}_1}^{(1)}} f \, dS_{C_{u_1}^{(1)}} &= \int_{\mathcal{S}_{\bar{u}_1} \times [0, 2\pi)} f(v_1, u_2, \varphi) \frac{r_1 r_2}{4R} dv_1 du_2 d\varphi, \\ \int_{C_{\bar{u}_2}^{(2)}} f \, dS_{C_{u_2}^{(2)}} &= \int_{\mathcal{S}_{\bar{u}_2} \times [0, 2\pi)} f(v_2, u_1, \varphi) \frac{r_1 r_2}{4R} dv_2 du_1 d\varphi. \end{aligned} \tag{1.5.8}$$

Here, $C_{\bar{u}_1}^{(1)}$ is the cone $\{u_1 = \bar{u}_1\} \cap \{t \geq 0\}$, and similarly $C_{\bar{u}_2}^{(2)}$ is the cone $\{u_2 = \bar{u}_2\} \cap \{t \geq 0\}$.

Furthermore, $dS_{C_{u_h}^{(h)}}$, $h = 1, 2$ is the volume form induced on cones of constant u_h coordinate, with $h = 1, 2$. Moreover, $\mathcal{S}_{\bar{u}_1}$ is the set of those values of v_1, u_2 such that

$$v_1 - u_2 \geq 2R, \quad v_1 \geq -\bar{u}_1, \quad |u_2 - \bar{u}_1| \leq 2R,$$

and, similarly, $\mathcal{S}_{\bar{u}_2}$ is the set of those values of v_2, u_1 such that

$$v_2 - u_1 \geq 2R, \quad v_2 \geq -\bar{u}_2, \quad |u_1 - \bar{u}_2| \leq 2R.$$

Remark 1.5.3. We note that r_1 is the Euclidean distance in $\mathbb{R}^3$ to the point $Rp_1 = (-R, 0, 0)$, and similarly r_2 is the Euclidean distance in $\mathbb{R}^3$ to the point $Rp_2 = (R, 0, 0)$.

Proof of Lemma 1.5.2. Let $\rho^2 := y^2 + z^2$. On $\mathbb{R}^3$, we consider the following coordinates

(r_1, r_2, φ) :

$$\begin{aligned} r_1^2 &= (x + R)^2 + \rho^2, \\ r_2^2 &= (x - R)^2 + \rho^2, \\ \varphi &= \arctan(z/y). \end{aligned} \tag{1.5.9}$$

so that the relations hold:

$$x = \frac{r_1^2 - r_2^2}{4R}, \quad y = \rho \cos \varphi, \quad z = \rho \sin \varphi. \tag{1.5.10}$$

where $\rho = \sqrt{\frac{1}{2}(r_1^2 + r_2^2) - R^2 - x^2}$. Let us now compute the Jacobian determinant of such change of variables. We have

$$J_1 := |\det(J_{(r_1, r_2, \varphi)}(x, y, z))| = \begin{vmatrix} \frac{r_1}{2R} & -\frac{r_2}{2R} & 0 \\ \partial_{r_1} \rho \cos \varphi & \partial_{r_2} \rho \cos \varphi & -\rho \sin \varphi \\ \partial_{r_1} \rho \sin \varphi & \partial_{r_2} \rho \sin \varphi & \rho \cos \varphi \end{vmatrix} \tag{1.5.11}$$

Expanding with respect to the first row,

$$J_1 = \left| \frac{r_1}{4R} \partial_{r_2} \rho^2 + \frac{r_2}{4R} \partial_{r_1} \rho^2 \right| = \frac{1}{4R} |2r_1 r_2| = \frac{r_1 r_2}{2R},$$

since we have

$$\partial_{r_1} \rho^2 = r_1 - \frac{r_1(r_1^2 - r_2^2)}{4R^2}, \quad \partial_{r_2} \rho^2 = r_2 + \frac{r_2(r_1^2 - r_2^2)}{4R^2}$$

Therefore, we get the following expression for the corresponding volume forms:

$$dx \wedge dy \wedge dz = \frac{r_1 r_2}{2R} dr_1 \wedge dr_2 \wedge d\varphi. \tag{1.5.12}$$

Let us now consider the product manifold $\mathbb{R} \times \mathbb{R}^3$, and let the first variable to be time (t).

Now, we shall find the range of admissible r_1 and r_2 . It is clear that $r_1 \geq 0$ and $r_2 \geq 0$. Moreover, because the distance between Rp_1 and Rp_2 is $2R$ and r_1 is the distance from Rp_1

while r_2 is the distance from Rp_2 , we have that $r_1 + r_2 \geq 2R$ from the triangle inequality. Similarly, it follows that $|r_1 - r_2| \leq 2R$ from the triangle inequality, as desired (note that these inequalities alone ensure that both r_1 and r_2 are non-negative).

We now define (u_1, v_1, u_2, φ) as follows:

$$u_1 := t - r_1, \quad v_1 := t + r_1, \quad u_2 := t - r_2.$$

This implies

$$t = \frac{1}{2}(u_1 + v_1), \quad r_1 = \frac{1}{2}(v_1 - u_1), \quad r_2 = \frac{1}{2}(u_1 + v_1) - u_2.$$

The Jacobian determinant is now

$$J_2 := |\det(J_{(u_1, v_1, u_2, \varphi)}(t, r_1, r_2, \varphi))| = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & -1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} = \frac{1}{2}.$$

All in all, we obtain that the following formulas hold. We then have, if f is a smooth function,

$$\begin{aligned} \int_{\mathbb{R}^4} f(t, x, y, z) dt dx dy dz &= \int_{\mathbb{R} \times \mathcal{R} \times [0, 2\pi)} f(t, r_1, r_2, \varphi) \frac{r_1 r_2}{2R} dt dr_1 dr_2 d\varphi \\ \int_{\mathbb{R}^4} f(t, x, y, z) dt dx dy dz &= \int_{\mathcal{S} \times [0, 2\pi)} f(u_1, v_1, u_2, \varphi) \frac{r_1 r_2}{4R} du_1 dv_1 du_2 d\varphi \end{aligned}$$

where $\mathcal{R}$ is the subset of the values of r_1 and r_2 such that

$$r_1 + r_2 \geq 2R, \quad |r_1 - r_2| \leq 2R.$$

Translating these bounds into (u_1, v_1, u_2, φ) coordinates we obtain that the second integral in the display above is over the set $\mathcal{S} \times [0, 2\pi)$, where $\mathcal{S}$ is the set of those values of u_1, v_1, u_2

which satisfy:

$$v_1 - u_2 \geq 2R, \quad |u_2 - u_1| \leq 2R.$$

This concludes the proof of the change of variables in display (1.5.7).

The proof of (1.5.8) follows in a straightforward way. $\square$

1.5.3 Asymptotic comparison of derivatives intrinsic to two distinct light cones

Here, we wish to formalize the fact that, as time increases, “good” derivatives for both cones become aligned on the interaction set, thus giving rise to improved estimates. This Lemma is used in the proof of the improved energy estimates in Section 1.7.4.

Lemma 1.5.4. *Let $c \in (0, 1/10)$, let $\eta \in \mathcal{C}^\infty(\mathbb{R}^4)$, and recall the coordinates r_1 and r_2 . There exists a constant $C > 0$ such that the following pointwise inequalities hold true:*

$$|\bar{\partial}^{(1)}\eta| \leq C \frac{R}{r_1} |\partial\eta| + C |\bar{\partial}^{(2)}\eta|, \quad |\bar{\partial}^{(2)}\eta| \leq C \frac{R}{r_2} |\partial\eta| + C |\bar{\partial}^{(1)}\eta|. \quad (1.5.13)$$

Moreover, let the spacetime region $\mathcal{I}_{12}$ be defined as the region where both $|u_1| \leq cR$ and $|u_2| \leq cR$. Restricting to the region $\mathcal{I}_{12}$, we have the following estimate, valid for every smooth function η :

$$|\partial\eta| \leq C \frac{t}{R} (|\bar{\partial}^{(1)}\eta| + |\bar{\partial}^{(2)}\eta|). \quad (1.5.14)$$

Proof of Lemma 1.5.4. Let us focus on proving the first inequality in display (1.5.13), the second being analogous. Without loss of generality, let us furthermore assume that $r_1 \geq cR$, as the claim is clear in case $r_1 \leq cR$.

We will derive an expression for good derivatives adapted to one of the cones in terms of the other. We have the following relation:

$$r_1^2 - r_2^2 = 4xR,$$

hence, taking the gradient of both sides, we have

$$r_1 \partial_{r_1} - r_2 \partial_{r_2} = 2R \partial_x.$$

Here, ∂_{r_h} is the coordinate vector field induced by $(t, r_h, \theta_h, \varphi_h)$ (defined in display (1.3.2)), and $h \in \{1, 2\}$. Recalling that $\partial_{v_h} = \partial_t + \partial_{r_h}$, $h = 1, 2$:

$$r_1 \partial_{v_1} = 2R \partial_x + r_2 \partial_{v_2} - (r_2 - r_1) \partial_t = 2R \partial_x + r_2 \partial_{v_2} + \frac{4xR}{r_1 + r_2} \partial_t.$$

We then have, from the triangle inequality, $|r_1 - r_2| \leq 2R$, which implies $\frac{r_2}{r_1} \leq 1 + \frac{2R}{r_1}$, which, by the fact that we have $r_1 \geq cR$, implies $\frac{r_2}{r_1} \leq 1 + \frac{2}{c}$, so that

$$\frac{r_2}{r_1} \leq C,$$

for some positive constant C . Similarly, we have

$$\left| \frac{4xR}{r_1 + r_2} \right| \leq CR,$$

due to the fact that $r_1 + r_2 \geq |x|$. All in all, we obtain the following, if $\eta \in \mathcal{C}^\infty(\mathbb{R}^4)$:

$$|\partial_{v_1} \eta| \lesssim \frac{R}{r_1} |\partial \eta| + |\bar{\partial}^{(2)} \eta|. \quad (1.5.15)$$

We now turn to the proof for angular derivatives. We have that

$$\Omega_{(xy)}^{(1)} = (x + R) \partial_y - y \partial_x = \Omega_{(xy)}^{(2)} + 2R \partial_y.$$

Hence,

$$\frac{1}{r_1} \Omega_{(xy)}^{(1)} = \frac{(x + R)}{r_1} \partial_y - \frac{y}{r_1} \partial_x = \frac{1}{r_1} \Omega_{(xy)}^{(2)} + \frac{2R}{r_1} \partial_y, \quad (1.5.16)$$

giving us the desired result for $\frac{1}{r_1} \Omega_{(xy)}^{(1)}$. A similar argument gives us the desired result for the

other rotation vector fields adapted to r_1 , which completes the proof of the first inequality in display (1.5.13). The proof of the second inequality is identical.

We now turn to the proof of bound (1.5.14). Let us first restrict to the case $t \geq 2(1 - c)^{-1}R$. Recall the definition of the region $\mathcal{I}_{12} := \{|u_1| \leq cR\} \cap \{|u_2| \leq cR\}$. We note that

$$\partial_y = \frac{1}{2R}(\Omega_{(xy)}^{(1)} - \Omega_{(xy)}^{(2)}), \quad \partial_z = \frac{1}{2R}(\Omega_{(xz)}^{(1)} - \Omega_{(xz)}^{(2)}).$$

From these, for any smooth function η , we obtain, in the region $\mathcal{I}_{12}$,

$$|\partial_z \eta| + |\partial_y \eta| \leq C \frac{t}{R} (|\bar{\partial}^{(1)} \eta| + |\bar{\partial}^{(2)} \eta|). \quad (1.5.17)$$

This is because we are assuming $R \leq \frac{1}{2}(1 - c)t$, which, together with the fact that $r_1, r_2 \leq cR + t$ implies $r_1, r_2 \leq Ct$, for some positive constant C . Now, we also have that

$$y\partial_x = -\Omega_{(xy)}^{(2)} + (x - R)\partial_y, \quad z\partial_x = -\Omega_{(xz)}^{(2)} + (x - R)\partial_z.$$

This implies:

$$(y^2 + z^2)\partial_x = -y\Omega_{(xy)}^{(2)} + y(x - R)\partial_y - z\Omega_{(xz)}^{(2)} + z(x - R)\partial_z.$$

We now note that

$$4xR = r_1^2 - r_2^2 \leq (t + cR)^2 - (t - cR)^2 = 4ctR,$$

(and similarly for $-x$) hence we have that $|x| \leq ct$. We also have that, as $r_1^2 + r_2^2 = 2x^2 + 2y^2 + 2z^2 + 2R^2$, and $r_1, r_2 \geq t - cR$,

$$y^2 + z^2 \geq (t - cR)^2 - R^2 - c^2t^2 = (1 - c^2)t^2 - 2cRt + (c^2 - 1)R^2. \quad (1.5.18)$$

Now, one can verify that, for $t \geq 2(1 - c)^{-1}R$,

$$\frac{1}{2}(1 - c^2)t^2 - 2cRt + (c^2 - 1)R^2 \geq 0,$$

which implies, by (1.5.18), that $y^2 + z^2 \geq \frac{1}{2}(1 - c^2)t^2$. Using this, the fact that

$$|x|, |y|, |z|, r_1, r_2 \leq t + cR \leq Ct$$

(as $t \geq 2(1 - c)^{-1}R$), and the bounds (1.5.17), we can conclude that

$$|\partial_x \eta| \leq C \frac{t}{R} (|\bar{\partial}^{(1)} \eta| + |\bar{\partial}^{(2)} \eta|)$$

when $t \geq 2(1 - c)^{-1}R$.

Now, we consider the identity, which follows from the definition of r_1 and r_2 ,

$$r_1^2 + r_2^2 = 2(x^2 + y^2 + z^2 + R^2).$$

Taking the gradient of this expression, and adding $(r_1 + r_2)\partial_t$ on both sides, we obtain

$$\partial_t = \frac{1}{r_1 + r_2} (r_1 \partial_{v_1} + r_2 \partial_{v_2} - 2(x \partial_x + y \partial_y + z \partial_z)).$$

We conclude using the previous bounds obtained on ∂_x , ∂_y and ∂_z :

$$|\partial_t \eta| \leq C \frac{t}{R} (|\bar{\partial}^{(1)} \eta| + |\bar{\partial}^{(2)} \eta|),$$

always restricting to the case $t \geq 2(1 - c)^{-1}R$. This proves claim (1.5.14), when restricting to the case $t \geq 2(1 - c)^{-1}R$

Let us now turn to the case $t \leq 2(1 - c)^{-1}R$. The bounds (1.5.17) for ∂_y and ∂_z are still valid (note that every point in the region $\mathcal{I}_{12}$ satisfies $t \geq c'R$, for some positive constant c').

Moreover, we have, as before,

$$\begin{aligned}\partial_t &= \frac{1}{r_1 + r_2} (r_1 \partial_{v_1} + r_2 \partial_{v_2} - 2(x \partial_x + y \partial_y + z \partial_z)), \\ \partial_x &= \frac{1}{2R} (r_1 \partial_{v_1} - r_2 \partial_{v_2} + (r_2 - r_1) \partial_t).\end{aligned}$$

Substituting, we have

$$\left(1 + \frac{x(r_2 - r_1)}{R(r_1 + r_2)}\right) \partial_t = \frac{1}{r_1 + r_2} (r_1 \partial_{v_1} + r_2 \partial_{v_2} - 2(y \partial_y + z \partial_z)) - \frac{x}{(r_1 + r_2)R} (r_1 \partial_{v_1} - r_2 \partial_{v_2}). \quad (1.5.19)$$

We then note that, since $r_1^2 - r_2^2 = 4xR$,

$$1 + \frac{x(r_2 - r_1)}{R(r_1 + r_2)} = 1 - \frac{4x^2}{(r_1 + r_2)^2} \geq 1 - \frac{2x^2}{x^2 + R^2}.$$

Now, if we can prove that that $|x| \leq \frac{1}{2}R$, $1 - \frac{2x^2}{x^2 + R^2}$ would be bounded below by $1/2$, and we would use equation (1.5.19) to conclude. Now, we have

$$|x| = \frac{1}{4R} |r_1^2 - r_2^2| \leq ct \leq 2c(1 - c)^{-1}R < \frac{1}{2}R,$$

since $c < \frac{1}{10}$. Hence equation (1.5.19) implies:

$$|\partial_t \eta| \leq C(|\bar{\partial}^{(1)} \eta| + |\bar{\partial}^{(2)} \eta|),$$

which is the claim for the ∂_t derivative restricted to the region $t \leq 2(1 - c)^{-1}R$. We finally use the relation

$$\partial_x = \frac{1}{2R} (r_1 \partial_{v_1} - r_2 \partial_{v_2} + (r_2 - r_1) \partial_t)$$

to deduce the claim for the ∂_x derivative. This concludes the proof of the lemma. $\square$

1.6 R -weighted Klainerman–Sobolev inequalities

We require two different sets of global Sobolev inequalities depending on whether we are seeking to obtain estimates on the linear equations for ψ_{ij} or the nonlinear equation. In the estimates for ψ_{ij} , because the main interactions come from the two functions ϕ_i and ϕ_j , there is a rotation that does not introduce R weights for ψ_{ij} . Indeed, the centers of the supports of ϕ_i and ϕ_j lie on a line, and rotations about this line do not introduce R -weights in the initial data for ϕ_i and ϕ_j . Similarly, the Lorentz boosts tangent to certain two-dimensional hyperboloids which are translation invariant along this line do not produce R -weights on the initial data either. On the other hand, in the estimates for the nonlinear problem, all Killing vector fields introduce R -weights in the initial data. Therefore, we need to use the usual Killing vector fields divided by the parameter R (see also the discussion in Section 1.2.5).

1.6.1 Sobolev inequalities for the linear equations

We list here the modified Klainerman–Sobolev inequalities we need to use in this setting. We begin with the estimates that are used for ψ_{ij} . Without loss of generality (by changing coordinates), we can assume that the line connecting the centers of the supports of ϕ_i and ϕ_j is just the x axis, as was done in the above, so that the two pieces of initial data are localized around the point $Rp_1 = (-R, 0, 0)$ and $Rp_2 = (R, 0, 0)$. These estimates are used in Section 1.7.5.

Lemma 1.6.1. *Let $f \in C^\infty(\mathbb{R}^3)$. Consider polar coordinates (r, θ, φ) adapted to the x -axis, so that $r > 0$, $\theta \in [0, \pi)$, $\varphi \in [0, 2\pi)$, and so that $\partial_\varphi = y\partial_z - z\partial_y$. There exists a positive constant C such that the following holds. We have that, for $\theta \in [\pi/8, 7\pi/8]$,*

$$|f(r, \theta, \varphi)|^2 \leq C \frac{R}{r^2} \sum_{I \in I_{\Gamma^{(0)}}^{\leq 3}} \|V^I f\|_{L^2(\mathbb{R}^3)}^2. \quad (1.6.1)$$

Here, recall the definition of the vector fields $\Gamma^{(0)}$ in display (1.3.14).

This implies that, for $h \in \{0, i, j\}$, we have that, restricting to the region where $\rho = \sqrt{y^2 + z^2} \geq \frac{1}{10}t$,

$$|f(t, x, y, z)|^2 \leq C \frac{R}{t^2} \sum_{I \in I_{\mathbf{r}(h)}^{\leq 3}} \|V^I f\|_{L^2(\Sigma_t)}^2. \quad (1.6.2)$$

Proof of Lemma 1.6.1. Let us consider the sphere $\mathbb{S}^2$ with coordinates (θ, φ) , so that $\theta = 0$ corresponds to a point lying on the positive x -axis. Let $\chi(\theta)$ be a smooth cutoff function such that

$$\chi(\theta) = \begin{cases} 1 & \text{if } \theta \in [\pi/8, 7\pi/8], \\ 0 & \text{if } \theta \in [0, \pi/16] \cup (15\pi/16, \pi]. \end{cases}$$

We shall use a localized Sobolev embedding on the unit sphere. We have that $\partial_\varphi = \Omega_{(yz)} = y\partial_z - z\partial_y$. With $\partial_\theta^R := R^{-1}\partial_\theta$, we now estimate

$$\begin{aligned} |\chi(\theta)f(r, \theta, \varphi)|^2 &\leq 2 \int_0^\theta |f\partial_\theta \chi| |\chi f| + \chi |\partial_\theta f| |\chi f| d\vartheta \\ &\leq 2 \frac{R}{r} \int_0^\theta \left(\left| f \frac{1}{R} \partial_\theta \chi \right| |\chi f| + \left| \chi \frac{1}{R} \partial_\theta f \right| |\chi f| \right) r d\vartheta \\ &\leq 2 \frac{R}{r} \int_0^\theta ((f\partial_\theta^R \chi)^2 + 2(\chi f)^2 + (\chi \partial_\theta^R f)^2) r d\vartheta. \end{aligned}$$

Multiplying and dividing by r and integrating over $\mathbb{S}^1$ in φ , we have that

$$\int_{\mathbb{S}^1} |\chi(\theta)f(r, \theta, \varphi)|^2 d\varphi \leq C \frac{R}{r^2} \int_0^\theta \int_{\mathbb{S}^1} ((f\partial_\theta^R \chi)^2 + 2(\chi f)^2 + (\chi \partial_\theta^R f)^2) r^2 d\varphi d\vartheta \quad (1.6.3)$$

Now, taking ∂_φ , we have that

$$\begin{aligned} \int_{\mathbb{S}^1} |\partial_\varphi(\chi(\theta)f(r, \theta, \varphi))|^2 d\varphi &= \int_{\mathbb{S}^1} |\chi(\theta)\partial_\varphi f(r, \theta, \varphi)|^2 d\varphi \\ &\leq C \frac{R}{r^2} \int_0^\theta \int_{\mathbb{S}^1} ((\partial_\varphi f \partial_\theta^R \chi)^2 + 2(\chi \partial_\varphi f)^2 + (\chi \partial_\theta^R \partial_\varphi f)^2) r^2 d\varphi d\vartheta. \end{aligned} \quad (1.6.4)$$

Now, we recall the Sobolev inequality on $\mathbb{S}^1$ given by

$$\|f\|_{L^\infty(\mathbb{S}^1)}^2 \leq C(\|f\|_{L^2(\mathbb{S}^1)}^2 + \|\partial_\varphi f\|_{L^2(\mathbb{S}^1)}^2).$$

Treating χf as a function on $\mathbb{S}^1$ in φ and r, θ fixed along with using both (1.6.3) and (1.6.4) gives us that

$$\|\chi f\|_{L^\infty}^2 \leq C \frac{R}{r^2} \left(\|f\|_{L^2(S_r)}^2 + \|\partial_\theta^R f\|_{L^2(S_r)}^2 + \|\partial_\varphi f\|_{L^2(S_r)}^2 + \|\partial_\theta^R \partial_\varphi f\|_{L^2(S_r)}^2 \right) \quad (1.6.5)$$

with C depending on χ . We note that we have used the fact that $0 < c \leq \sin(\theta)$ on the support of χ , as $\chi(\theta) = 0$ for $\theta \leq \frac{\pi}{16}$ and for $\theta \geq \frac{15\pi}{16}$. Here, S_r is the sphere of radius r centered at the origin in $\mathbb{R}^3$.

We now have that

$$\partial_\theta^R = a_1(x, y, z)\Omega_{(xy)} + a_2(x, y, z)\Omega_{(xz)} + a_3(x, y, z)\Omega_{(yz)},$$

with a_1, a_2 , and a_3 smooth functions on $\mathbb{R}^3$ which are all pointwise controlled by $\frac{C}{R}$. Thus, using Hölder's inequality, we have that

$$\|\partial_\theta^R f\|_{L^2(S_r)}^2 \leq C \left(\|\Omega_{(xy)}^R f\|_{L^2(S_r)}^2 + \|\Omega_{(xz)}^R f\|_{L^2(S_r)}^2 + \|\Omega_{(yz)}^R f\|_{L^2(S_r)}^2 \right).$$

for any $f \in \mathcal{C}^\infty(S_r)$. Here, we denoted $\Omega_{(xy)}^R := R^{-1}\Omega_{(xy)}$, and $\Omega_{(xz)}^R := R^{-1}\Omega_{(xz)}$. The claim is then easily obtained by the trace lemma in the r -direction (Lemma 1.10.1), noting that, for a smooth function f , we have the pointwise inequality $|\partial_r f| \leq C|\partial f|$. $\square$

Now, we must also get estimates using the x translation invariant hyperboloids. Recall the hyperboloidal coordinates $(\tau, \alpha, x, \varphi)$ introduced in display (1.3.6). Recall furthermore that we defined $\rho := \sqrt{y^2 + z^2}$. Then, we have the following lemma, which is used in Section 1.7.5.

Lemma 1.6.2. *There is a positive constant C such that the following inequality holds, for*

all (t, x, y, z) such that $t \geq 1$ and $\rho = \sqrt{y^2 + z^2} \leq \frac{1}{10}t$:

$$|f(t, x, y, z)| \leq \frac{C}{t} \sum_{I \in I_{\mathbf{r}(h)}^{\leq 3}} \|V^I f\|_{L^2(H_\tau \cap \{\rho \leq \frac{1}{10}t\})}. \quad (1.6.6)$$

Here, $h \in \{0, i, j\}$, and τ satisfies $\tau = \sqrt{t^2 - \rho^2}$.

Proof of Lemma 1.6.2. Consider coordinates $(\tau, \bar{y}, \bar{z})$ such that

$$t = \tau \sqrt{1 + \bar{y}^2 + \bar{z}^2}, \quad y = \tau \bar{y}, \quad z = \tau \bar{z}.$$

We then note that the coordinate vector fields $\partial_{\bar{y}}$ and $\partial_{\bar{z}}$ are parallel to the Lorentz boosts $\Gamma_{(ty)}$ and $\Gamma_{(tz)}$:

$$\partial_{\bar{y}} = \frac{\tau}{t}(y\partial_t + t\partial_y), \quad \partial_{\bar{z}} = \frac{\tau}{t}(z\partial_t + t\partial_z). \quad (1.6.7)$$

We then consider the function

$$\tilde{f}(\bar{y}, \bar{z}) := f(\tau \sqrt{1 + \bar{y}^2 + \bar{z}^2}, \tau \bar{y}, \tau \bar{z}).$$

We then use the following version of the Sobolev embedding, valid for all $(\bar{y}, \bar{z})$ in the ball $\tilde{B}$, defined as the set where $\bar{y}^2 + \bar{z}^2 \leq 1/99$:

$$|\tilde{f}(\bar{y}, \bar{z})|^2 \leq C(\|\tilde{f}\|_{L^2(\tilde{B})}^2 + \|\partial_{\bar{y}}^2 \tilde{f}\|_{L^2(\tilde{B})}^2 + \|\partial_{\bar{z}}^2 \tilde{f}\|_{L^2(\tilde{B})}^2). \quad (1.6.8)$$

The claim then follows by changing variables in the integrals appearing in display (1.6.8), using the expression (1.6.7), and applying the trace lemma in the x -direction (Lemma 1.10.2).

Finally, the restriction $\bar{y}^2 + \bar{z}^2 \leq 1/99$ translates to $t^2 \leq 100/99 \tau^2$, which implies $\rho \leq \frac{1}{10}t$. $\square$

1.6.2 Sobolev inequalities for the nonlinear equation

Finally, we turn to the L^∞ estimates needed to close the bootstrap argument for the nonlinear equation in the proof of Theorem 1.4.7. In this case, we note that all the Lorentz vector fields will introduce R -weights on initial data. For this purpose, we are going to be using the vector fields $\mathbf{K}_R$, as defined in display (1.3.17). All of these vector fields have $\frac{1}{R}$ weights. However, following the discussion above in Section 1.2.5, it is too wasteful to naively use the classical Klainerman–Sobolev inequality introducing these weights. Therefore, we shall now reprove the Klainerman–Sobolev inequality in this modified setting, being careful to keep track of the R weights. The first result we prove is suitable to gain additional weights in a region of large r -coordinate. This estimate is used in Section 1.7.5.

Lemma 1.6.3. *Let $f : \mathbb{R}^{1+3} \rightarrow \mathbb{R}$ be a smooth function. Then, we have the estimate*

$$|f(t, r, \theta, \varphi)|^2 \leq C \frac{R^2}{(1+r)^2} \sum_{I \in I_{\mathbf{K}_R}^{\leq 3}} \|V^I f\|_{L^2(\Sigma_t)}^2. \quad (1.6.9)$$

Here, we used the definition of the set $\mathbf{K}_R$ in display (1.3.17).

Sketch of proof of Lemma 1.6.3. We note that this can be proven in exactly the same way as Lemma 1.6.1 above by appropriately replacing every occurrence of ∂_φ with ∂_φ/R (we note that in this case, we also think of ∂_φ as introducing a bad R -weight), upon dividing all inequalities by a factor of R . When we take the square root, the final inequality we obtain is thus worse by a factor of R . $\square$

For r very small, we need the following estimate, whose proof follows the proof of the Klainerman–Sobolev inequality in the analogous region (see Section 9 of [69] and also [83]). This estimate is used in Section 1.8, when we prove the main theorem.

Lemma 1.6.4. *Let $f : \mathbb{R}^{1+3} \rightarrow \mathbb{R}$ be a smooth function. We fix a smooth, positive, even function $\chi : \mathbb{R} \rightarrow \mathbb{R}$ with $\chi = 1$ for $|x| \leq \frac{4}{5}$ and with $\chi = 0$ for $|x| \geq \frac{9}{10}$. Then, there exists*

a constant $C > 0$ such that the following estimate holds:

$$\left| \chi\left(\frac{r}{t}\right) f(t, r, \theta, \varphi) \right| \leq \frac{CR^{\frac{3}{2}}}{t^{\frac{3}{2}}} \sum_{I \in I_{\mathbf{K}_R}^{\leq 3}} \|V^I f\|_{L^2(\Sigma_t)}. \quad (1.6.10)$$

Moreover, as a result of this, in the region where $t \geq cR$ where c is some constant, we have that

$$\left| \chi\left(\frac{r}{t}\right) f(t, r, \theta, \varphi) \right| \leq \frac{CR}{t} \sum_{I \in I_{\mathbf{K}_R}^{\leq 3}} \|V^I f\|_{L^2(\Sigma_t)}. \quad (1.6.11)$$

Proof. We recall the following Sobolev inequality on $\mathbb{R}^3$, for smooth and compactly supported functions g :

$$\|g\|_{L^\infty(\mathbb{R}^3)} \leq C \|\partial^2 g\|_{L^2(\mathbb{R}^3)}^{\frac{3}{4}} \|g\|_{L^2(\mathbb{R}^3)}^{\frac{1}{4}}.$$

Now, by Lemma 9.10 in the lecture notes [69], we have that

$$|\partial^2 g| \leq \frac{C}{|t-r|^2} \sum_{I \in I_{\mathbf{K}_R}^{\leq 2}} |V^I g|.$$

We then combine the previous two displays, choosing $g(t, r, \theta, \varphi) = \chi\left(\frac{r}{t}\right) f(t, r, \theta, \varphi)$. We then apply the chain rule, and use the fact that in the region considered we have $u \geq ct$ for some positive constant c . Upon an application of Hölder's inequality, we conclude. $\square$

We finally recall the classical estimate by Klainerman, appropriately modified to be used with R -weighted vector fields:

Lemma 1.6.5 (Classical Klainerman–Sobolev inequality with R -weights). *There exists a constant $C > 0$ such that, for all f smooth functions on $\mathbb{R}^{1+3}$, parametrized by coordinates (t, r, θ, φ) , and for all $R \geq 1$, we have the following inequality:*

$$(1+r+t)(1+|t-r|)^{\frac{1}{2}} |f(t, r, \theta, \varphi)| \leq CR^2 \sum_{I \in \mathbf{K}_R^{\leq 2}} \|V^I f\|_{L^2(\Sigma_t)}. \quad (1.6.12)$$

Proof of Lemma 1.6.5. The result without R -weights is classical, a proof can be found for

example in [69] and also in [83]:

$$(1 + r + t)(1 + |t - r|)^{\frac{1}{2}} |f(t, r, \theta, \varphi)| \leq C \sum_{I \in \mathbf{K}^{\leq 2}} \|V^I f\|_{\Sigma_t}.$$

Recall now that every element in $\mathbf{K}$ can be written as R multiplying an element in $\mathbf{K}_R$. This concludes the proof, since $R \geq 1$. $\square$

1.7 Main Estimates

1.7.1 Structure of the initial data

In this section, we record a few properties of the initial data of Theorem 1.4.1.

Let us first suppose, without loss of generality, that the configuration of points Π introduced in the statement of Theorem 1.4.1 is such that the following condition holds:

$$\min_{\substack{i, j \in \{1, \dots, N\} \\ i \neq j}} |p_i - p_j| \geq 2. \quad (1.7.1)$$

We then recall the initial data as defined in the statement of Theorem 1.4.1, and we set:

$$\tilde{\phi}_i^{(0)} := \bar{\phi}_i^{(0)}(x - w_i), \quad \tilde{\phi}_i^{(1)} := \bar{\phi}_i^{(1)}(x - w_i). \quad (1.7.2)$$

Recall that, here, $w_i = R \cdot p_i$, that $x \in \mathbb{R}^3$, and that $i \in \{1, \dots, N\}$.

Condition (1.7.1) now ensures that, for $i \neq j$, with $i, j \in \{1, \dots, N\}$, the support of $\tilde{\phi}_i^{(0)}$ is disjoint from the support of $\tilde{\phi}_j^{(0)}$, and similarly the support of $\tilde{\phi}_i^{(1)}$ is disjoint from that of $\tilde{\phi}_j^{(1)}$.

1.7.2 Derivation of the equation for the first iterate

In this section, we derive the system satisfied by the difference

$$\psi := \phi - \sum_{i=0}^N \phi_i, \quad (1.7.3)$$

where each of the ϕ_i is the global solution to the following initial value problem:

$$\begin{aligned} \square \phi_i + F(d\phi_i, d^2\phi_i) &= G(d\phi_i, d\phi_i), \\ \phi_i|_{t=0} &= \tilde{\phi}_i^{(0)}, \\ \partial_t \phi_i|_{t=0} &= \tilde{\phi}_i^{(1)}. \end{aligned} \quad (1.7.4)$$

Remark 1.7.1. Note that the initial data for these $N + 1$ auxiliary problems is $(\tilde{\phi}_i^{(0)}, \tilde{\phi}_i^{(1)})$ and it was defined in Section 1.7.1. It is not to be confused with the data in the statement of Theorem 1.4.1.

We have the following lemma:

Lemma 1.7.2. *Let N be a non-negative integer. Let furthermore $(\tilde{\phi}_i^{(0)}, \tilde{\phi}_i^{(1)})$ be a collection of smooth functions in $\mathbb{R}^3$, for $i \in \{1, \dots, N\}$, as defined in formula (1.7.2). Recall that, by construction,*

$$\phi^{(0)} := \sum_{i=0}^N \tilde{\phi}_i^{(0)}, \quad \phi^{(1)} := \sum_{i=0}^N \tilde{\phi}_i^{(1)}.$$

Suppose that ϕ is the smooth solution to the following initial value problem:

$$\begin{aligned} \square \phi + F(d\phi, d^2\phi) &= G(d\phi, d\phi), \\ \phi|_{t=0} &= \phi^{(0)}, \\ \partial_t \phi|_{t=0} &= \phi^{(1)}. \end{aligned} \quad (1.7.5)$$

Suppose that ϕ_i , $i \in \{1, \dots, N\}$ is the smooth solution to the initial value problem (1.7.4).

Then, letting

$$\psi := \phi - \sum_{i=1}^N \phi_i, \quad (1.7.6)$$

we have that ψ satisfies the following system:

$$\begin{aligned} \square\psi + F(d\psi, d^2\psi) + \sum_{\substack{i,j=1,\dots,N \\ i \neq j}} F(d\phi_i, d^2\phi_j) + \sum_{i=1}^N (F(d\phi_i, d^2\psi) + F(d\psi, d^2\phi_i)) \\ = G(d\psi, d\psi) + \sum_{\substack{i,j=1,\dots,N \\ i \neq j}} G(d\phi_i, d\phi_j) + \sum_{i=1}^N (G(d\phi_i, d\psi) + G(d\psi, d\phi_i)), \end{aligned} \quad (1.7.7)$$

$$\psi|_{t=0} = 0,$$

$$\partial_t \psi|_{t=0} = 0.$$

Proof of Lemma 1.7.2. The proof follows from a straightforward calculation. For simplicity, let's assume $F = 0$ (the proof in the other case being totally analogous). We start by calculating:

$$\begin{aligned} G(d\psi, d\psi) &= G(d\phi - \sum_{i=1}^N d\phi_i, d\phi - \sum_{i=1}^N d\phi_i) = \\ &= G(d\phi, d\phi) - \sum_{i=1}^N G(d\psi, d\phi_i) - \sum_{i=1}^N G(d\phi_i, d\psi) - \sum_{\substack{i,j=1,\dots,N \\ i \neq j}} G(d\phi_i, d\phi_j) - \sum_{i=1}^N G(d\phi_i, d\phi_i) \\ &= \square \left(\phi - \sum_{i=1}^N \phi_i \right) - \sum_{i=1}^N G(d\psi, d\phi_i) - \sum_{i=1}^N G(d\phi_i, d\psi) - \sum_{\substack{i,j=1,\dots,N \\ i \neq j}} G(d\phi_i, d\phi_j). \end{aligned}$$

From the fact that $\psi = \phi - \sum_{i=1}^N \phi_i$ the claim then follows readily. It is also evident that $\psi|_{t=0} = 0$, and that $\partial_t \psi|_{t=0} = 0$. $\square$

1.7.3 Derivation of the equation for the second iterate

In this section, we derive the system satisfied by the difference

$$\Psi := \psi - \sum_{\substack{i,j=1 \\ i \neq j}}^N \psi_{ij}, \quad (1.7.8)$$

where each of the ψ_{ij} 's is the global solution to the following initial value problem:

$$\begin{aligned} \square \psi_{ij} + F(d\phi_i, d^2\phi_j) + F(d\phi_j, d^2\phi_i) &= 2G(d\phi_i, d\phi_j), \\ \psi_{ij}|_{t=0} &= 0, \\ \partial_t \psi_{ij}|_{t=0} &= 0, \end{aligned} \quad (1.7.9)$$

valid for all $i, j \in \{1, \dots, N\}$.

Remark 1.7.3. Note that the equation satisfied by ψ_{ij} is formed taking equation (1.7.7) and considering only the inhomogeneous contributions from ϕ_i and ϕ_j .

We have the following lemma.

Lemma 1.7.4. *Let N be a positive integer and let ψ , ϕ_i , $i \in \{1, \dots, N\}$ be as in the statement of Lemma 1.7.2. Define furthermore ψ_{ij} as in equation (1.7.9), and let Ψ be as in equation (1.7.8). Under these conditions, Ψ satisfies the following initial value problem:*

$$\begin{aligned} &\square \Psi + F(d\Psi, d^2\Psi) \\ &+ \sum_{\substack{i,j=1,\dots,N \\ i \neq j}} (F(d\psi_{ij}, d^2\Psi) + F(d\Psi, d^2\psi_{ij})) + \sum_{\substack{g,h,i,j=1,\dots,N \\ g \neq h, i \neq j}} F(d\psi_{gh}, d^2\psi_{ij}) \\ &+ \sum_{i=1}^N \sum_{\substack{g,h=1,\dots,N \\ g \neq h}} (F(d\phi_i, d^2\psi_{gh}) + F(d\psi_{gh}, d^2\phi_i)) \\ &+ \sum_{i=1}^N (F(d\phi_i, d^2\Psi) + F(d\Psi, d^2\phi_i)) \end{aligned}$$

$$\begin{aligned}
&= \sum_{\substack{i,j=1,\dots,N \\ i \neq j}} (G(d\psi_{ij}, d\Psi) + G(d\Psi, d\psi_{ij})) + \sum_{\substack{g,h,i,j=1,\dots,N \\ g \neq h, i \neq j}} G(d\psi_{gh}, d\psi_{ij}) \\
&+ \sum_{i=1}^N \sum_{\substack{g,h=1,\dots,N \\ g \neq h}} (G(d\phi_i, d\psi_{gh}) + G(d\psi_{gh}, d\phi_i)) \\
&+ \sum_{i=1}^N (G(d\phi_i, d\Psi) + G(d\Psi, d\phi_i)) + G(d\Psi, d\Psi),
\end{aligned} \tag{1.7.10}$$

$$\Psi|_{t=0} = 0,$$

$$\partial_t \Psi|_{t=0} = 0.$$

Proof of Lemma 1.7.4. We start from equation (1.7.7):

$$\begin{aligned}
&\square\psi + F(d\psi, d^2\psi) + \sum_{\substack{i,j=1,\dots,N \\ i \neq j}} F(d\phi_i, d^2\phi_j) \\
&+ \sum_{i=1}^N (F(d\phi_i, d^2\psi) + F(d\psi, d^2\phi_i)) \\
&= G(d\psi, d\psi) + \sum_{\substack{i,j=1,\dots,N \\ i \neq j}} G(d\phi_i, d\phi_j) \\
&+ \sum_{i=1}^N (G(d\phi_i, d\psi) + G(d\psi, d\phi_i)).
\end{aligned}$$

This implies:

$$\begin{aligned}
&\square\left(\psi - \sum_{\substack{i,j=1,\dots,N \\ i \neq j}} \psi_{ij}\right) + F(d\psi, d^2\psi) \\
&+ \sum_{i=1}^N (F(d\phi_i, d^2\psi) + F(d\psi, d^2\phi_i)) \\
&= G(d\psi, d\psi) \\
&+ \sum_{i=1}^N (G(d\phi_i, d\psi) + G(d\psi, d\phi_i)).
\end{aligned} \tag{1.7.11}$$

The conclusion follows readily using equation (1.7.8). $\square$

1.7.4 The improved energy estimates and the trilinear estimates

We shall now prove the trilinear estimates described in Section 1.2.3. More precisely, we shall obtain improved energy estimates on solutions to the first iterates ψ_{ij} , which require us to prove these trilinear estimates. Indeed, we must control a trilinear spacetime integral in order to control the ∂_t energy of the functions ψ_{ij} in (1.7.9). These trilinear estimates will control the bilinear interaction between the solutions ϕ_i and ϕ_j . The trilinear estimates which result are explicitly stated in Proposition 1.7.2.

The functions ψ_{ij} satisfy linear equations with fixed inhomogeneities, so we already know that solutions exist globally. The improvements introduced by these estimates will be strong enough to prove global existence for the nonlinear equation arising from the second iterate.

We have the following lemma:

Proposition 1.7.1. *Let ψ_{ij} be a solution to the initial value problem (1.7.9). For simplicity, let us suppose that the initial data for ϕ_i is centered at the point $w_i = (-R, 0, 0)$, and the initial data for ϕ_j is centered at the point $w_j = (R, 0, 0)$ (we are assuming, without loss of generality, that $|p_i - p_j| = 2$). Then, we have that, for all multi-indices $I \in I_{\mathbf{r}(h)}^{\leq n-1}$, $h \in \{1, 2\}$, the following inequality holds true:*

$$\sup_{t \geq 0} \|\partial V^I \psi_{ij}\|_{L^2(\Sigma_t)} \leq C \frac{\varepsilon^2}{R}, \quad (1.7.12)$$

where the constant C can depend on the distance between p_i and p_j , which we recall are the points where the initial data for ϕ_i and ϕ_j , respectively, are centered when $R = 1$.

We note that the following trilinear estimates are established as a result of the proof of Proposition 1.7.1.

Proposition 1.7.2. *Let η_h , with $h \in \{1, 2\}$, be solutions to the following initial value problem:*

$$\begin{aligned} \square \eta_h &= 0, \\ (\eta_h|_{t=0}, \partial_t \eta_h|_{t=0}) &= (\eta_h^{(0)}, \eta_h^{(1)}). \end{aligned} \quad (1.7.13)$$

Let us furthermore suppose that the support of the initial data for η_1 is localized around the point $(-R, 0, 0)$ and that the support of initial data for η_2 is localized around the point $(R, 0, 0)$:

$$\text{supp } (\eta_1^{(0)}), \text{ supp } (\eta_1^{(1)}) \subset B(Rp_1, 1), \quad \text{supp } (\eta_2^{(0)}), \text{ supp } (\eta_2^{(1)}) \subset B(Rp_2, 1), \quad (1.7.14)$$

Here, $p_1 = (-1, 0, 0)$, $p_2 = (1, 0, 0)$, and $B(x_1, a)$ is the Euclidean three-dimensional ball of radius a centered at the point x_1 .

Moreover, let $\tilde{F}$ and $\tilde{G}$ be resp. a trilinear null form and a bilinear null form, according to Definition 1.3.14. For $f(t, x)$ an arbitrary smooth function in spacetime, we have that the following estimates hold true:

$$\begin{aligned} & \int_0^\infty \int_{\Sigma_s} \left(|\tilde{F}(d\eta_1, d^2\eta_2)| |f(s, x)| + |\tilde{G}(d\eta_1, d\eta_2)| |f(s, x)| \right) dx ds \\ & \leq C \frac{(\|\eta_1^{(0)}\|_{H^8(\Sigma_0)} + \|\eta_1^{(1)}\|_{H^7(\Sigma_0)}) (\|\eta_2^{(0)}\|_{H^8(\Sigma_0)} + \|\eta_2^{(1)}\|_{H^7(\Sigma_0)})}{R} \sup_{\bar{u}_1 \in \mathbb{R}} \|f\|_{L^2(C_{\bar{u}_1}^{(1)})}. \end{aligned} \quad (1.7.15)$$

Furthermore, the same inequality would hold if one were to replace $\sup_{\bar{u}_1 \in \mathbb{R}} \|f\|_{L^2(C_{\bar{u}_1}^{(1)})}$ with $\sup_{\bar{u}_2 \in \mathbb{R}} \|f\|_{L^2(C_{\bar{u}_2}^{(2)})}$.

Recall that, here, the notation $C_{\bar{u}_h}^{(h)}$ for $h \in \{1, 2\}$ denotes the outgoing null cone $\{u_h = \bar{u}_h\} \cap \{t \geq 0\}$, where the coordinate u_h has been introduced in Definition 1.3.1.

It is possible to deduce Proposition 1.7.2 from the the proof of Proposition 1.7.1, noting that the null forms $\tilde{F}$ and $\tilde{G}$ correspond to the inhomogeneities in the linear equation for ψ_{ij} , while the function $f(t, x)$ corresponds to the multiplier we are using in the proof of Proposition 1.7.1.

The trilinear estimates (1.7.15) are useful because they gain a power of R^{-1} . However, we note that they are very wasteful in terms of derivatives required on the functions η_1 and η_2 .

Remark 1.7.5. In Proposition 1.7.2, we require control on 8 derivatives of $\eta_1^{(0)}$ and $\eta_2^{(0)}$ as we

need to estimate 2 derivatives in L^∞ of both η_1 and η_2 , and we require the improved decay of Lemma 1.5.1.

We now turn to the proof of Proposition 1.7.1.

Proof of Proposition 1.7.1. Let us first commute equation (1.7.9) with $I \in I_{\mathbf{r}(h)}^{\leq n-1}$. We obtain:

$$\begin{aligned} \square V^I \psi_{ij} + \sum_{H+K \subset I} (F_{HK}(dV^H \phi_i, d^2 V^K \phi_j) + F_{HK}(dV^H \phi_j, d^2 V^K \phi_i)) \\ = \sum_{H+K \subset I} G_{HK}(dV^H \phi_i, dV^K \phi_j) \end{aligned} \quad (1.7.16)$$

Here, every F_{HK} , G_{HK} is a trilinear (resp. bilinear) null form as in Definition 1.3.14. We now multiply the evolution equation in (1.7.16) by $\partial_t V^I \psi_{ij}$ and integrate by parts in a spacetime region bounded by two slabs Σ_0 and Σ_t with $t \geq R/10$. This gives us that

$$\begin{aligned} \frac{1}{2} \int_{\Sigma_t} |\partial V^I \psi_{ij}|^2 dx = \int_{\frac{R}{10}}^t \int_{\Sigma_s} \left(\sum_{H+K \subset I} (F_{HK}(dV^H \phi_i, d^2 V^K \phi_j) + F_{HK}(dV^H \phi_j, d^2 V^K \phi_i)) \right. \\ \left. - \sum_{H+K \subset I} G_{HK}(dV^H \phi_i, dV^K \phi_j) \right) \partial_t V^I \psi_{ij} dx ds =: \mathfrak{B}, \end{aligned} \quad (1.7.17)$$

where we have used the fact that, by domain of dependence, any product involving ϕ_i and ϕ_j will vanish for $t \leq \frac{R}{10}$. Indeed, along with the fact that ψ_{ij} has vanishing initial data, this implies that $\int_{\Sigma_{R/10}} |\partial V^I \psi_{ij}|^2 dx = 0$. Now, we decompose the spacetime integration region into two regions. The first region is where $|u_i| \leq \frac{R}{10}$ and $|u_j| \leq \frac{R}{10}$, and the second piece is where at least one of $|u_i|$ or $|u_j|$ is larger than $\frac{R}{10}$. With $\mathcal{I}_{ij}$ the set of all points where $|u_i| \leq \frac{R}{10}$ and $|u_j| \leq \frac{R}{10}$, we have that $\mathfrak{B} = \mathfrak{B}_1 + \mathfrak{B}_2$, where

$$\begin{aligned} \mathfrak{B}_1 = \int_{\{R/10 \leq s \leq t\} \cap \mathcal{I}_{ij}} \left(\sum_{H+K \subset I} (F_{HK}(dV^H \phi_i, d^2 V^K \phi_j) + F_{HK}(dV^H \phi_j, d^2 V^K \phi_i)) \right. \\ \left. - \sum_{H+K \subset I} G_{HK}(dV^H \phi_i, dV^K \phi_j) \right) \partial_t V^I \psi_{ij} dx ds, \end{aligned} \quad (1.7.18)$$

and

$$\begin{aligned} \mathfrak{B}_2 = & \int_{\{R/10 \leq s \leq t\} \cap \mathcal{I}_{ij}^c} \left(\sum_{H+K \subset I} (F_{HK}(dV^H \phi_i, d^2 V^K \phi_j) + F_{HK}(dV^H \phi_j, d^2 V^K \phi_i)) \right. \\ & \left. - \sum_{H+K \subset I} G_{HK}(dV^H \phi_i, dV^K \phi_j) \right) \partial_t V^I \psi_{ij} dx ds. \end{aligned} \quad (1.7.19)$$

We can further decompose $\mathcal{I}_{ij}^c$ (the set of points “far away” from at least one of the light cones) into the set where $|u_i| \geq \frac{R}{10}$, which we shall call $\mathcal{E}_i^*$, and the remainder, $\mathcal{E}_i' = \mathcal{I}_{ij}^c \setminus \mathcal{E}_i^*$. We note that we must have that $|u_j| \geq \frac{R}{10}$ in $\mathcal{E}_i'$. Now, for $\mathfrak{B}_2$, we have that

$$\begin{aligned} \mathfrak{B}_2 = & \int_{\{R/10 \leq s \leq t\} \cap \mathcal{E}_i^*} \left(\sum_{H+K \subset I} (F_{HK}(dV^H \phi_i, d^2 V^K \phi_j) + F_{HK}(dV^H \phi_j, d^2 V^K \phi_i)) \right. \\ & \left. - \sum_{H+K \subset I} G_{HK}(dV^H \phi_i, dV^K \phi_j) \right) \partial_t V^I \psi_{ij} dx ds \\ & + \int_{\{R/10 \leq s \leq t\} \cap \mathcal{E}_i'} \left(\sum_{H+K \subset I} (F_{HK}(dV^H \phi_i, d^2 V^K \phi_j) + F_{HK}(dV^H \phi_j, d^2 V^K \phi_i)) \right. \\ & \left. - \sum_{H+K \subset I} G_{HK}(dV^H \phi_i, dV^K \phi_j) \right) \partial_t V^I \psi_{ij} dx ds =: \mathfrak{B}_{21} + \mathfrak{B}_{22}. \end{aligned}$$

We shall now get estimates for these integrals over $\mathcal{E}_i^*$ and $\mathcal{E}_i'$, terms $\mathfrak{B}_{21}$ and $\mathfrak{B}_{22}$. Thus, in the following, it suffices to get estimates for the first integral over $\mathcal{E}_i^*$ where $|u_i| \geq \frac{R}{10}$, as the estimates for the integral over $\mathcal{E}_i'$ follow in the same way after replacing i with j in the following argument.

Now, we have that

$$\begin{aligned} \mathfrak{B}_{21} \leq & \sup_{R/10 \leq s \leq t} \|\partial_t V^I \psi_{ij}\|_{L^2(\Sigma_s \cap \mathcal{E}_i^*)} \\ & \times \int_{R/10}^t \left(\sum_{H+K \subset I} \left\| (F_{HK}(dV^H \phi_i, d^2 V^K \phi_j) + F_{HK}(dV^H \phi_j, d^2 V^K \phi_i)) \right\|_{L^2(\Sigma_s \cap \mathcal{E}_i^*)} \right. \\ & \left. + \sum_{H+K \subset I} \left\| G_{HK}(dV^H \phi_i, dV^K \phi_j) \right\|_{L^2(\Sigma_s \cap \mathcal{E}_i^*)} \right) ds. \end{aligned}$$

We now divide the region $\mathcal{E}_i^*$ in two further subregions:

$$\hat{\mathcal{E}}_i^* := \mathcal{E}_i^* \cap \{r_i \leq \frac{1}{10}t\}, \quad \check{\mathcal{E}}_i^* := \mathcal{E}_i^* \cap \{r_i \geq \frac{1}{10}t\}$$

Let us first focus on the region $\hat{\mathcal{E}}_i^*$. We have that, in this region, $u_i \geq Ct$, and hence, using the decay properties of ϕ which follow from Proposition 1.5.1, we have

$$\begin{aligned} & \|G_{HK}(dV^H\phi_i, dV^K\phi_j)\|_{L^2(\Sigma_s \cap \hat{\mathcal{E}}_i^*)} \\ & \leq C \|\partial V^H\phi_i\|_{L^\infty(\Sigma_s \cap \hat{\mathcal{E}}_i^*)} \|\partial V^K\phi_j\|_{L^2(\Sigma_s \cap \hat{\mathcal{E}}_i^*)} \leq C \|\partial V^H\phi_i\|_{L^\infty(\Sigma_s \cap \hat{\mathcal{E}}_i^*)} \|\partial V^K\phi_j\|_{L^2(\Sigma_s \cap \hat{\mathcal{E}}_i^*)} \quad (1.7.20) \\ & \leq C \frac{\varepsilon^2}{s^{2+\delta}}. \end{aligned}$$

Furthermore, a similar reasoning holds for the terms in F so that, overall, we obtain

$$\begin{aligned} & \|F_{HK}(dV^H\phi_i, d^2V^K\phi_j)\|_{L^2(\Sigma_s \cap \hat{\mathcal{E}}_i^*)} + \|F_{HK}(dV^H\phi_j, d^2V^K\phi_i)\|_{L^2(\Sigma_s \cap \hat{\mathcal{E}}_i^*)} \\ & + \|G_{HK}(dV^H\phi_i, dV^K\phi_j)\|_{L^2(\Sigma_s \cap \hat{\mathcal{E}}_i^*)} \leq C \frac{\varepsilon^2}{s^{2+\delta}}. \end{aligned} \quad (1.7.21)$$

Let us now focus on the region $\check{\mathcal{E}}_i^*$. Using Lemma (1.3.19) on the structure of classical null forms, combined with the estimates in Proposition 1.5.3, plus the bound (1.5.13), along with the fact that $|u_i| \geq \frac{R}{10}$ in $\mathcal{E}_i^*$ and the fact that $r_i \geq \frac{1}{10}t$ in the region $\check{\mathcal{E}}_i^*$, we have that

$$\begin{aligned} & \|G_{HK}(dV^H\phi_i, dV^K\phi_j)\|_{L^2(\Sigma_s \cap \check{\mathcal{E}}_i^*)} \\ & \leq C \|\bar{\partial}^{(i)}V^H\phi_i\|_{L^2(\Sigma_s \cap \check{\mathcal{E}}_i^*)} \|\partial V^K\phi_j\|_{L^2(\Sigma_s \cap \check{\mathcal{E}}_i^*)} + C \|\partial V^H\phi_i\|_{L^2(\Sigma_s \cap \check{\mathcal{E}}_i^*)} \|\bar{\partial}^{(i)}V^K\phi_j\|_{L^2(\Sigma_s \cap \check{\mathcal{E}}_i^*)} \\ & \leq C\varepsilon^2 \frac{1}{R^\delta s^2} + C \left\| \frac{R}{r_i} \partial V^H\phi_i \right\|_{L^2(\Sigma_s \cap \check{\mathcal{E}}_i^*)} \|\partial V^K\phi_j\|_{L^2(\Sigma_s \cap \check{\mathcal{E}}_i^*)} + C \|\partial V^H\phi_i\|_{L^2(\Sigma_s \cap \check{\mathcal{E}}_i^*)} \|\bar{\partial}^{(j)}V^K\phi_j\|_{L^2(\Sigma_s \cap \check{\mathcal{E}}_i^*)} \\ & \leq C\varepsilon^2 \frac{1}{R^\delta s^2} + C \frac{R}{s} \|\partial V^H\phi_i\|_{L^2(\Sigma_s \cap \check{\mathcal{E}}_i^*)} \|\partial V^K\phi_j\|_{L^2(\Sigma_s \cap \check{\mathcal{E}}_i^*)} + C\varepsilon^2 \frac{1}{s^2} \\ & \leq C\varepsilon^2 \frac{1}{R^\delta s^2} + C\varepsilon^2 \frac{R}{s} \frac{1}{sR^{1+\delta}} + C\varepsilon^2 \frac{1}{s^2} \leq C\varepsilon^2 \frac{1}{s^2}. \end{aligned} \quad (1.7.22)$$

Using again Proposition 1.5.1, we can get the same estimates for the two terms involving

F in the region $\check{\mathcal{E}}_i^*$. We obtain:

$$\begin{aligned} & \|F_{HK}(dV^H\phi_i, d^2V^K\phi_j)\|_{L^2(\Sigma_s\cap\check{\mathcal{E}}_i^*)} + \|F_{HK}(dV^H\phi_j, d^2V^K\phi_i)\|_{L^2(\Sigma_s\cap\check{\mathcal{E}}_i^*)} \\ & + \|G_{HK}(dV^H\phi_i, dV^K\phi_j)\|_{L^2(\Sigma_s\cap\check{\mathcal{E}}_i^*)} \leq C\frac{\varepsilon^2}{s^2}. \end{aligned} \quad (1.7.23)$$

The reasoning is totally analogous for the term $\mathfrak{B}_{22}$, restricting to the region $\mathcal{E}'_i$. Thus, altogether, combining estimates (1.7.21) and (1.7.23) and the estimates for $\mathfrak{B}_{22}$, we have that

$$\begin{aligned} |\mathfrak{B}_2| & \leq C\varepsilon^2 \sup_{R/10 \leq s \leq t} \|\partial_t V^I \psi_{ij}\|_{L^2(\Sigma_s \cap \mathcal{I}_{ij}^c)} \int_{R/10}^t \frac{1}{s^2} ds \\ & \leq C\varepsilon^2 \sup_{R/10 \leq s \leq t} \|\partial_t V^I \psi_{ij}\|_{L^2(\Sigma_s \cap \mathcal{I}_{ij}^c)} \frac{1}{R}. \end{aligned} \quad (1.7.24)$$

Now, for $\mathfrak{B}_1$, we begin by decomposing with respect to a null frame adapted to ϕ_i . The same argument works using a null frame for ϕ_j instead. Now, using Lemma 1.3.19 on the structure of null forms on G_{HK} and F_{HK} , we have that

$$\begin{aligned} |\mathfrak{B}_1| & \leq C \int_{\{R/10 \leq s \leq t\} \cap \mathcal{I}_{ij}} \sum_{H+K \subset I} \left(|\bar{\partial}^{(i)} V^H \phi_i| |\partial^2 V^K \phi_j| + |\partial V^H \phi_i| |\partial \bar{\partial}^{(i)} V^K \phi_j| \right. \\ & \quad + |\bar{\partial}^{(i)} V^H \phi_j| |\partial^2 V^K \phi_i| + |\partial V^H \phi_j| |\partial \bar{\partial}^{(i)} V^K \phi_i| \\ & \quad \left. + |\bar{\partial}^{(i)} V^H \phi_i| |\partial V^K \phi_j| + |\partial V^H \phi_i| |\bar{\partial}^{(i)} V^K \phi_j| \right) |\partial_t V^I \psi_{ij}| dx ds \end{aligned} \quad (1.7.25)$$

For the terms in which $\bar{\partial}^{(i)}$ falls on ϕ_i or $\partial\phi_i$ (i. e. the first, fourth, and fifth term in the previous display), we can use the pointwise estimate (1.5.3) to bound the $\bar{\partial}^{(i)}$ term in L^∞ , along with the energy estimate in Proposition 1.5.1 to bound the term containing $\partial\phi_j$ in

$L^2(\Sigma_s)$. This gives us that

$$\begin{aligned}
& \int_{\{R/10 \leq s \leq t\} \cap \mathcal{I}_{ij}} \sum_{H+K \leq I} \left(|\bar{\partial}^{(i)} V^H \phi_i| |\partial^2 V^K \phi_j| \right. \\
& \quad \left. + |\partial V^H \phi_j| |\partial \bar{\partial}^{(i)} V^K \phi_i| + |\bar{\partial}^{(i)} V^H \phi_i| |\partial V^K \phi_j| \right) |\partial_t V^I \psi_{ij}| dx ds \\
& \leq C \varepsilon^2 \sup_{R/10 \leq s \leq t} \|\partial_t V^I \psi_{ij}\|_{L^2(\Sigma_s \cap \mathcal{I}_{ij})} \int_{R/10}^t \frac{1}{s^2} ds \\
& \leq C \sup_{R/10 \leq s \leq t} \|\partial_t \psi_{ij}\|_{L^2(\Sigma_s \cap \mathcal{I}_{ij})} \frac{\varepsilon^2}{R}.
\end{aligned} \tag{1.7.26}$$

To bound the remaining terms in the RHS of (1.7.25), we must now get an estimate for the terms

$$\begin{aligned}
& \int_{\{R/10 \leq s \leq t\} \cap \mathcal{I}_{ij}} \sum_{H+K \leq I} \left(|\partial V^H \phi_i| |\partial \bar{\partial}^{(i)} V^K \phi_j| \right. \\
& \quad \left. + |\bar{\partial}^{(i)} V^H \phi_j| |\partial^2 V^K \phi_i| + |\partial V^H \phi_i| |\bar{\partial}^{(i)} V^K \phi_j| \right) |\partial_t V^I \psi_{ij}| dx ds =: \mathfrak{B}_3.
\end{aligned}$$

We have that $|\bar{\partial}^{(i)} f| \leq C \left(\frac{R}{s} |\partial f| + |\bar{\partial}^{(j)} f| \right)$ in $\{R/10 \leq s \leq t\} \cap \mathcal{I}_{ij}$, which implies

$$\begin{aligned}
|\mathfrak{B}_3| & \leq \int_{\{R/10 \leq s \leq t\} \cap \mathcal{I}_{ij}} \sum_{H+K \leq I} \left(|\partial V^H \phi_i| |\partial \bar{\partial}^{(j)} V^K \phi_j| \right. \\
& \quad \left. + |\bar{\partial}^{(j)} V^H \phi_j| |\partial^2 V^K \phi_i| + |\partial V^H \phi_i| |\bar{\partial}^{(j)} V^K \phi_j| \right) |\partial_t V^I \psi_{ij}| dx ds \\
& + C \int_{\{R/10 \leq s \leq t\} \cap \mathcal{I}_{ij}} \sum_{H+K \leq I} \left(|\partial V^H \phi_i| |\partial^2 V^K \phi_j| \right. \\
& \quad \left. + |\partial V^H \phi_j| |\partial^2 V^K \phi_i| + |\partial V^H \phi_i| |\partial V^K \phi_j| \right) |\partial_t V^I \psi_{ij}| \frac{R}{s} dx ds =: \mathfrak{B}_{31} + \mathfrak{B}_{32}.
\end{aligned}$$

Now, the term $\mathfrak{B}_{31}$ can be controlled exactly in the same way as in estimate (1.7.26), as terms like $\bar{\partial}^{(j)} V^K \phi_j$ have very strong pointwise decay. Thus, we must only control terms $\mathfrak{B}_{32}$.

We begin by noting that we can decompose:

$$\partial_t V^I \psi_{ij} = \frac{1}{2}(\partial_{v_j} V^I \psi_{ij} + \partial_{u_j} V^I \psi_{ij}). \quad (1.7.27)$$

Moreover, using bound (1.5.14) from Lemma (1.5.4), we can write

$$|\partial_{u_j} \psi_{ij}| \leq C \frac{s}{R} (|\bar{\partial}^{(i)} \psi_{ij}| + |\bar{\partial}^{(j)} \psi_{ij}|),$$

since we are restricting to the region $\{R/10 \leq s \leq t\} \cap \mathcal{I}_{ij}$. Thus, we have that

$$\begin{aligned} |\mathfrak{B}_{32}| \leq C \int_{\{R/10 \leq s \leq t\} \cap \mathcal{I}_{ij}} \sum_{H+K \subset I} & \left(|\partial V^H \phi_i| |\partial^2 V^K \phi_j| \right. \\ & \left. + |\partial V^H \phi_j| |\partial^2 V^K \phi_i| + |\partial V^H \phi_i| |\partial V^K \phi_j| \right) (|\bar{\partial}^{(i)} V^I \psi_{ij}| + |\bar{\partial}^{(j)} V^I \psi_{ij}|) dx ds. \end{aligned} \quad (1.7.28)$$

Now, we note that the terms $\bar{\partial}^{(i)} V^I \psi_{ij}$ and $\bar{\partial}^{(j)} V^I \psi_{ij}$ all appear in the characteristic ∂_t energy for the wave equation satisfied by ψ_{ij} (equation (1.7.9)). Indeed, the terms with $\bar{\partial}^{(i)}$ appear in the characteristic ∂_t -energy through outgoing cones adapted to the i -th piece of initial data (the cones $C_{\bar{u}_i}^{(i)}$, defined by $\{u_i = \bar{u}_i\} \cap \{t \geq 0\}$, where the coordinate u_i has been introduced in Definition 1.3.1), whereas the terms with $\bar{\partial}^{(j)}$ appear in the characteristic ∂_t -energy through outgoing cones $C_{\bar{u}_j}^{(j)}$ adapted to ϕ_j . Thus, we write the RHS of inequality (1.7.28) as two separate terms, and we shall consider different foliations of outgoing null cones to control each of those terms.

We begin with the term involving $\bar{\partial}^{(i)} V^I \psi_{ij}$. We define

$$\begin{aligned} \mathfrak{B}_4 := \int_{\{R/10 \leq s \leq t\} \cap \mathcal{I}_{ij}} \sum_{H+K \subset I} & \left(|\partial V^H \phi_i| |\partial^2 V^K \phi_j| \right. \\ & \left. + |\partial V^H \phi_j| |\partial^2 V^K \phi_i| + |\partial V^H \phi_i| |\partial V^K \phi_j| \right) |\bar{\partial}^{(i)} V^I \psi_{ij}| dx ds. \end{aligned} \quad (1.7.29)$$

Using the Hölder inequality in mixed Lebesgue spaces, we now estimate $\bar{\partial}^{(i)} V^I \psi_{ij}$ in L^2 of the outgoing cones $C_{\bar{u}_i}^{(i)}$ and L^∞ in the u_i -direction, while estimating the multiplying factor

in L^2 of the outgoing cones $C_{u_i}^{(i)}$ and L^1 in the u_i -direction. This gives us that

$$\begin{aligned} \mathfrak{B}_4 \leq & \sup_{-R/10 \leq u_i \leq R/10} \|\bar{\partial}^{(i)} V^I \psi_{ij}\|_{L^2(\tilde{C}_{u_i}^{(i),t,R})} \int_{-R/10}^{R/10} \left(\|\partial V^H \phi_i\| \|\partial^2 V^K \phi_j\|_{L^2(\tilde{C}_{u_i}^{(i),t,R})} \right. \\ & \left. + \|\partial V^H \phi_j\| \|\partial^2 V^K \phi_i\|_{L^2(\tilde{C}_{u_i}^{(i),t,R})} + \|\partial V^H \phi_i\| \|\partial V^K \phi_j\|_{L^2(\tilde{C}_{u_i}^{(i),t,R})} \right) du_i, \end{aligned} \quad (1.7.30)$$

where, if $\bar{u}_i \in \mathbb{R}$, we are denoting by $\tilde{C}_{\bar{u}_i}^{(i),R,t}$ the usual outgoing cone $C_{\bar{u}_i}^{(i)}$ defined by $\{u_i = \bar{u}_i\} \cap \{t \geq 0\}$ intersected with the set $\{R/10 \leq s \leq t\} \cap \mathcal{I}_{ij}$.

We now wish to calculate the innermost integral (the integrals in the L^2 norms) using the coordinates (u_i, v_i, u_j, φ) described in Lemma 1.5.2, adapted to ϕ_i and ϕ_j . We note that, in the region $\mathcal{I}_{ij} \cap \{t \geq R/10\}$, r_i and r_j are comparable, as the triangle inequality implies $r_i \leq r_j + 2R$, and since $r_j \geq cR$ for some positive constant c , we have $r_i \leq c'R_j$, for some positive constant c' . This also implies that v_i is comparable to v_j .

Now, using the pointwise bounds on ϕ_i and ϕ_j given in Lemma 1.5.1 (bounds (1.5.3)), we have that, since we are always differentiating at most $n - 1$ times, and the estimates (1.5.3) give control over $n + 1$ derivatives,

$$\begin{aligned} \mathfrak{B}_4 & \leq C \sup_{-R/10 \leq u_i \leq R/10} \|\bar{\partial}^{(i)} V^I \psi_{ij}\|_{L^2(\tilde{C}_{u_i}^{(i),t,R})} \\ & \times \varepsilon^2 \int_{u_i \in [-R/10, R/10]} \left(\int_{v_i \in [R/10, \infty)} \int_{u_j \in \mathbb{R}} \frac{r_i r_j R^{-1}}{v_i^2 (1 + |u_i|)^{2+2\delta} v_j^2 (1 + |u_j|)^{2+2\delta}} du_j dv_i \right)^{\frac{1}{2}} du_i \\ & \leq C \sup_{-R/10 \leq u_i \leq R/10} \|\bar{\partial}^{(i)} V^I \psi_{ij}\|_{L^2(\tilde{C}_{u_i}^{(i),t,R})} \\ & \times \varepsilon^2 \int_{u_i \in [-R/10, R/10]} \left(\int_{v_i \in [R/10, \infty)} \int_{u_j \in \mathbb{R}} \frac{1}{v_i^2 (1 + |u_i|)^{2+2\delta} (1 + |u_j|)^{2+2\delta}} \frac{1}{R} du_j dv_i \right)^{\frac{1}{2}} du_i \\ & \leq C \varepsilon^2 \frac{1}{R} \sup_{-R/10 \leq u_i \leq R/10} \|\bar{\partial}^{(i)} V^I \psi_{ij}\|_{L^2(\tilde{C}_{u_i}^{(i),t,R})}. \end{aligned} \quad (1.7.31)$$

Remark 1.7.6. This is the most important estimate in this proposition. Here, we used the crucial fact that we have improved u -decay for ϕ_i and ϕ_j . Note in particular that, if the u_i -decay for ϕ_i were not integrable in u_i , we would not be able to close the argument (as the

integral in (1.7.31) would diverge).

Using the same argument for terms involving $\bar{\partial}^{(j)}\psi_{ij}$ in display (1.7.28), and putting the resulting estimates together, we finally obtain

$$\begin{aligned} & \|\partial V^I \psi_{ij}\|_{L^2(\Sigma_t)}^2 \\ & \leq C\varepsilon^2 \frac{1}{R} \left(\sup_{R/10 \leq s \leq t} \|\partial V^I \psi_{ij}\|_{L^2(\Sigma_s)} \right. \\ & \quad \left. + \sup_{-R/10 \leq u_i \leq R/10} \|\bar{\partial}^{(i)} V^I \psi_{ij}\|_{L^2(\tilde{C}_{u_i}^{(i), R, t})} + \sup_{-R/10 \leq u_j \leq R/10} \|\bar{\partial}^{(j)} V^I \psi_{ij}\|_{L^2(\tilde{C}_{u_j}^{(j), R, t})} \right). \end{aligned}$$

The inequality in the last display holds for all $t \geq 0$. Thus, we can take the supremum in t on both sides. The LHS becomes $\sup_{t \geq 0} \|\partial V^I \psi_{ij}\|_{L^2(\Sigma_t)}^2$. Furthermore, we obtain (note that ψ_{ij} is zero whenever $u_i \leq -3R$ or $u_j \leq -3R$, by domain of dependence):

$$\begin{aligned} & \sup_{t \geq 0} \|\partial V^I \psi_{ij}\|_{L^2(\Sigma_t)}^2 \\ & \leq C\varepsilon^2 \frac{1}{R} \left(\sup_{t \geq 0} \|\partial V^I \psi_{ij}\|_{L^2(\Sigma_t)} + \sup_{u_i \in \mathbb{R}} \|\bar{\partial}^{(i)} V^I \psi_{ij}\|_{L^2(C_{u_i}^{(i)})} + \sup_{u_j \in \mathbb{R}} \|\bar{\partial}^{(j)} V^I \psi_{ij}\|_{L^2(C_{u_j}^{(j)})} \right). \end{aligned} \tag{1.7.32}$$

We proceed by multiplying again the evolution equation in (1.7.16) by $\partial_t V^I \psi_{ij}$ and integrating by parts in the spacetime region $\mathcal{S}_0$ to the future of the hypersurface Σ_0 and to the past of both the hypersurface Σ_t and the outgoing cone $C_{u_i}^{(i)}$, for $u_i \geq -3R$. In other words,

$$\mathcal{S}_0 := \{(\bar{t}, \bar{r}_i, \bar{\theta}, \bar{\varphi}) : 0 \leq \bar{t} \leq t, \quad \bar{t} - \bar{r}_i \leq u_i\}.$$

It's straightforward to see that the error term appearing in the RHS of the resulting estimate

can be controlled in the same fashion as terms $\mathfrak{B}_1$ through $\mathfrak{B}_4$. This gives us that

$$\begin{aligned}
& \|\bar{\partial}^{(i)} V^I \psi_{ij}\|_{L^2(C_{u_i}^{(i)} \cap \{0 \leq s \leq t\})}^2 \\
& \leq C\varepsilon^2 \frac{1}{R} \left(\sup_{R/10 \leq s \leq t} \|\partial V^I \psi_{ij}\|_{L^2(\Sigma_s)} \right. \\
& \quad \left. + \sup_{-R/10 \leq u_i \leq R/10} \|\bar{\partial}^{(i)} V^I \psi_{ij}\|_{L^2(C_{u_i}^{(i),R,t})} + \sup_{-R/10 \leq u_j \leq R/10} \|\bar{\partial}^{(j)} V^I \psi_{ij}\|_{L^2(C_{u_j}^{(j),R,t})} \right).
\end{aligned} \tag{1.7.33}$$

Here, we denoted by $C_{u_i}^{(i),R,t}$ the usual outgoing cone $C_{u_i}^{(i)}$ of constant u_i to the future of the initial hypersurface Σ_0 , intersected with the set $\{R/10 \leq s \leq t\}$:

$$C_{u_i}^{(i),R,t} := \{(\bar{t}, \bar{r}_i, \bar{\theta}, \bar{\varphi}) : R/10 \leq \bar{t} \leq t, \quad \bar{t} - \bar{r}_i = u_i\}.$$

The analogous definition holds for $C_{u_j}^{(j),R,t}$. We can now send t to ∞ and use the monotone convergence theorem, giving us that the LHS of display (1.7.33) becomes $\|\bar{\partial}^{(i)} V^I \psi_{ij}\|_{L^2(C_{u_i}^{(i)})}^2$.

We bound the RHS by the trivial estimate and we obtain, for all $u_i \in \mathbb{R}$,

$$\begin{aligned}
& \|\bar{\partial}^{(i)} V^I \psi_{ij}\|_{L^2(C_{u_i}^{(i)})}^2 \\
& \leq C\varepsilon^2 \frac{1}{R} \left(\sup_{t \geq 0} \|\partial V^I \psi_{ij}\|_{L^2(\Sigma_t)} + \sup_{u_i \geq -3R} \|\bar{\partial}^{(i)} V^I \psi_{ij}\|_{L^2(C_{u_i}^{(i)})} \right. \\
& \quad \left. + \sup_{u_j \geq -3R} \|\bar{\partial}^{(j)} V^I \psi_{ij}\|_{L^2(C_{u_j}^{(j)})} \right).
\end{aligned}$$

Then, taking the supremum in u_i gives us that

$$\begin{aligned}
& \sup_{u_i \in \mathbb{R}} \|\bar{\partial}^{(i)} V^I \psi_{ij}\|_{L^2(C_{u_i}^{(i)})}^2 \\
& \leq C\varepsilon^2 \frac{1}{R} \left(\sup_{t \geq 0} \|\partial V^I \psi_{ij}\|_{L^2(\Sigma_t)} + \sup_{u_i \in \mathbb{R}} \|\bar{\partial}^{(i)} V^I \psi_{ij}\|_{L^2(C_{u_i}^{(i)})} \right. \\
& \quad \left. + \sup_{u_j \in \mathbb{R}} \|\bar{\partial}^{(j)} V^I \psi_{ij}\|_{L^2(C_{u_j}^{(j)})} \right).
\end{aligned}$$

Similarly, using the same argument with respect to outgoing cones adapted to ϕ_j gives

us that

$$\begin{aligned} & \sup_{u_j \in \mathbb{R}} \|\bar{\partial}^{(j)} V^I \psi_{ij}\|_{L^2(C_{u_j}^{(j)})}^2 \\ & \leq C\varepsilon^2 \frac{1}{R} \left(\sup_{t \geq 0} \|\partial V^I \psi_{ij}\|_{L^2(\Sigma_t)} + \sup_{u_i \in \mathbb{R}} \|\bar{\partial}^{(i)} V^I \psi_{ij}\|_{L^2(C_{u_i}^{(i)})} + \sup_{u_j \in \mathbb{R}} \|\bar{\partial}^{(j)} V^I \psi_{ij}\|_{L^2(C_{u_j}^{(j)})} \right). \end{aligned}$$

Thus, we have that

$$\begin{aligned} & \sup_{t \geq 0} \|\partial V^I \psi_{ij}\|_{L^2(\Sigma_t)}^2 + \sup_{u_i \in \mathbb{R}} \|\bar{\partial}^{(i)} V^I \psi_{ij}\|_{L^2(C_{u_i}^{(i)})}^2 + \sup_{u_j \in \mathbb{R}} \|\bar{\partial}^{(j)} V^I \psi_{ij}\|_{L^2(C_{u_j}^{(j)})}^2 \\ & \leq C\varepsilon^2 \frac{1}{R} \left(\sup_{t \geq 0} \|\partial V^I \psi_{ij}\|_{L^2(\Sigma_t)} + \sup_{u_i \in \mathbb{R}} \|\bar{\partial}^{(i)} V^I \psi_{ij}\|_{L^2(C_{u_i}^{(i)})} + \sup_{u_j \in \mathbb{R}} \|\bar{\partial}^{(j)} V^I \psi_{ij}\|_{L^2(C_{u_j}^{(j)})} \right). \end{aligned}$$

This implies that

$$\sup_{t \geq 0} \|\partial V^I \psi_{ij}\|_{L^2(\Sigma_t)} + \sup_{u_i \in \mathbb{R}} \|\bar{\partial}^{(i)} V^I \psi_{ij}\|_{L^2(C_{u_i}^{(i)})} + \sup_{u_j \in \mathbb{R}} \|\bar{\partial}^{(j)} V^I \psi_{ij}\|_{L^2(C_{u_j}^{(j)})} \leq C\varepsilon^2 \frac{1}{R}, \quad (1.7.34)$$

as desired. $\square$

We also note that we can prove an energy estimate for the ∂_t energy of ψ_{ij} through the outgoing cone adapted to any of the ϕ_k . The argument is the same as in the above, as it just involves controlling the same spacetime integral. We record the result here, as it will be important later.

Lemma 1.7.7. *Let us assume the hypotheses of Lemma 1.7.1. Then, we have that, for all multi-indices $I \in I_{\Gamma^{(h)}}^{\leq n-1}$, $h \in \{i, j\}$, the following inequality holds true:*

$$\sup_{u_k \in \mathbb{R}} \|\bar{\partial}^{(k)} V^I \psi_{ij}\|_{L^2(C_{u_k}^{(k)})} \leq C\varepsilon^2 \frac{1}{R}, \quad (1.7.35)$$

where the constant C is allowed to depend on the distance between p_i and p_j , which we recall are the points where the initial data for ϕ_i and ϕ_j , respectively, are centered when $R = 1$. Here, the integration is over outgoing cones $C_{u_k}^{(k)}$ adapted to any of the ϕ_k . Recall that, for $\bar{u}_k \in \mathbb{R}$, $C_{\bar{u}_k}^{(k)}$ is defined as the set $\{u_k = \bar{u}_k\} \cap \{t \geq 0\}$.

1.7.5 L^∞ estimates on the linear equation

Having proved suitable L^2 estimates for ψ_{ij} , our goal is now to use them to deduce L^∞ estimates on the solution ψ_{ij} to the linear equation (1.7.9). With this in mind, we will employ the L^2 estimates just derived in Proposition 1.7.1, together with the R -weighted Klainerman–Sobolev estimates of Section 1.6 (we recall that these are modifications of estimates first showed in [59]). These estimates account for the fact that some of the vector fields carry $\frac{1}{R}$ weights. We have the following proposition.

Proposition 1.7.3. *Let $n \geq 4$, $n \in \mathbb{N}$. Let the smooth function ψ_{ij} arise as a solution to the initial value problem (1.7.9). In particular, ψ_{ij} satisfies estimate (1.7.12) from Proposition 1.7.1. In these conditions, there exists a constant $C > 0$ such that ψ_{ij} satisfies the following L^∞ estimates, for all $t \geq 0$, and for all $K \in I_{\bar{K}_R}^{\leq n-4}$:*

$$\|\partial V^K \psi_{ij}\|_{L^\infty(\Sigma_t)} \leq \frac{C\varepsilon^2}{R^{\frac{1}{2}}(1+t)}, \quad \text{for } i \neq j \text{ and } (i, j) \in \{1, \dots, N\} \times \{1, \dots, N\}, \quad (1.7.36)$$

$$\|\bar{\partial}^{(i)} V^K \psi_{ij}\|_{L^\infty(\Sigma_t)} \leq \frac{C\varepsilon^2 R^3}{(1+t)^{\frac{3}{2}}}, \quad \text{for } i \neq j \text{ and } (i, j) \in \{1, \dots, N\} \times \{1, \dots, N\}, \quad (1.7.37)$$

$$|\partial V^K \psi_{ij}|(t, u_i, \omega) \leq \frac{C\varepsilon^2 R^2}{t(1+|u_i|)^{\frac{1}{2}}}, \quad \text{for } i \neq j \text{ and } (i, j) \in \{1, \dots, N\} \times \{1, \dots, N\}. \quad (1.7.38)$$

Here, we considered $(u_i, v_i, \theta_i, \varphi_i)$ coordinates introduced in Definition 1.3.1 (recall that $t = \frac{1}{2}(u_i + v_i)$).

Proof. Let us first focus on the bound (1.7.36). Let us initially restrict to the region $\mathcal{A}$ for which $\rho = \sqrt{y^2 + z^2} \geq \frac{t}{10}$. The modified Sobolev inequality of Lemma 1.6.1 (inequality (1.6.2)) now implies:

$$|f(t, x, y, z)|^2 \leq C \frac{R}{t^2} \sum_{I \in I_{\bar{\Gamma}(h)}^{\leq 3}} \|V^I f\|_{L^2(\Sigma_t)}^2.$$

Here, $h \in \{i, j\}$. We set $f := \partial_a V^K \psi_{ij}$, with $K \in I_{\mathbf{K}_R}^{\leq n-4}$, and $a \in \{t, x, y, z\}$. We then have

$$|\partial V^K \psi_{ij}|^2 \leq C \frac{R}{t^2} \sum_{a \in \{t, x, y, z\}} \sum_{I \in I_{\mathbf{\Gamma}^{(h)}}^{\leq 3}} \|V^I \partial_a V^K \psi_{ij}\|_{L^2(\Sigma_t)}^2 \leq C \frac{R}{t^2} \sum_{J \in I_{\mathbf{\Gamma}^{(h)}}^{\leq 3}} \|\partial V^J V^K \psi_{ij}\|_{L^2(\Sigma_t)}^2.$$

The last inequality is obtained by “commuting out” the ∂_a derivative, keeping in mind that Lie brackets of elements of $\mathbf{T}$ and $\mathbf{\Gamma}^{(h)}$ are in $\mathbf{T}$. We finally employ the spacelike estimate (1.7.12) to deduce the claim (1.7.36) restricted to the region $\mathcal{A}$ (recall that, in particular, we have $\mathbf{K}_R \subset \mathbf{\Gamma}^{(h)}$).

Let us now focus on the region $\mathcal{A}^c$. Recall the hyperboloidal coordinates $(\tau, \alpha, x, \varphi)$ defined in display (1.3.6). Furthermore, recall the hyperboloids $H_{\bar{\tau}}$ defined by $H_{\bar{\tau}} := \{\tau = \bar{\tau}\}$. We now commute equation (1.7.9) with V^I , where $I \in I_{\mathbf{\Gamma}^{(h)}}^{\leq n-1}$, with $h \in \{i, j\}$. We then multiply the commuted equation (1.7.9) by $\partial_t V^I \psi_{ij}$ and integrate in the spacetime region between Σ_0 and $H_{\bar{\tau}}$. The inhomogeneous error terms which arise from this estimate are treated exactly as in the proof of Proposition 1.7.1. We now use Lemma 1.10.3 to deduce that the future boundary term on $H_{\bar{\tau}}$ controls all derivatives of $V^I \phi$ in a non-degenerate manner (the lemma follows from the fact that every hypersurface $H_{\bar{\tau}} \cap \mathcal{A}^c$ is uniformly spacelike). We thus arrive at the estimate:

$$\|\partial V^I \psi_{ij}\|_{L^2(H_{\bar{\tau}} \cap \mathcal{A}^c)} \leq C \frac{\epsilon^2}{R}. \quad (1.7.39)$$

Now, we use the Sobolev embedding on hyperboloids (Lemma 1.6.2) to deduce:

$$|f(t, x, y, z)| \leq \frac{C}{t} \sum_{I \in I_{\mathbf{\Gamma}^{(h)}}^{\leq 3}} \|V^I f\|_{L^2(H_{\bar{\tau}} \cap \mathcal{A}^c)} \quad (1.7.40)$$

where $t^2 = y^2 + z^2 + \tau^2$, and (t, x, y, z) belongs to $H_{\bar{\tau}} \cap \mathcal{A}^c$. Setting $f = \partial_a V^K \psi_{ij}$, with

$a \in \{t, x, y, z\}$ and $K \in I_{\mathbf{K}_R}^{\leq n-4}$, we have

$$\begin{aligned} |\partial V^K \psi_{ij}| &\leq C \frac{1}{t} \sum_{a \in \{t, x, y, z\}} \sum_{I \in I_{\mathbf{\Gamma}(h)}^{\leq 3}} \|V^I \partial_a V^K \psi_{ij}\|_{L^2(H_\tau \cap \mathcal{A}^c)} \\ &\leq C \frac{1}{t} \sum_{J \in I_{\mathbf{\Gamma}(h)}^{\leq 3}} \|\partial V^J V^K \psi_{ij}\|_{L^2(H_\tau \cap \mathcal{A}^c)}. \end{aligned}$$

The last inequality again follows from the fact that Lie brackets of elements of $\mathbf{\Gamma}^{(h)}$ and $\mathbf{T}$ are in $\mathbf{T}$. We now use estimate (1.7.39) to conclude the proof of claim (1.7.36).

Finally, claims (1.7.37) and (1.7.38) follow directly from the L^2 estimate (1.7.12), combined with the classical Klainerman–Sobolev inequality with R -weights of Lemma 1.6.5 (and the standard argument involving integration along a line of constant v_i -coordinate, cf. **Step 3** of the proof of Theorem 1.4.7 in Section 1.8). $\square$

1.8 Proof of Theorem 1.4.7

In this section, we will close the argument and use all the results obtained so far to conclude global existence to the nonlinear equation (1.1.3).

Proof of Theorem 1.4.7. We begin by noting that it suffices to prove uniform estimates assuming that $R \geq R_0$, for some positive number R_0 . Indeed, if $R \leq R_0$, we can restrict ε to a smaller value, depending on R_0 , and use the classical theory to conclude global stability. Thus, in the following, we shall without loss of generality use the fact that, restricting to the case $R \geq R_0$, we have the inequality $\log(R) \leq CR^\delta$ for some uniform positive constant C .

We start from the equation satisfied by Ψ :

$$\begin{aligned} &\square \Psi + F(d\Psi, d^2\Psi) \\ &+ \sum_{\substack{i, j=1, \dots, N \\ i \neq j}} (F(d\psi_{ij}, d^2\Psi) + F(d\Psi, d^2\psi_{ij})) + \sum_{\substack{g, h, i, j=1, \dots, N \\ g \neq h, i \neq j}} F(d\psi_{gh}, d^2\psi_{ij}) \end{aligned}$$

$$\begin{aligned}
& + \sum_{i=1}^N \sum_{\substack{g,h=1,\dots,N \\ g \neq h}} (F(d\phi_i, d^2\psi_{gh}) + F(d\psi_{gh}, d^2\phi_i)) \\
& + \sum_{i=1}^N (F(d\phi_i, d^2\Psi) + F(d\Psi, d^2\phi_i)) \\
& = G(d\Psi, d\Psi) \tag{1.8.1} \\
& + \sum_{\substack{i,j=1,\dots,N \\ i \neq j}} (G(d\psi_{ij}, d\Psi) + G(d\Psi, d\psi_{ij})) + \sum_{\substack{g,h,i,j=1,\dots,N \\ g \neq h, i \neq j}} G(d\psi_{gh}, d\psi_{ij}) \\
& + \sum_{i=1}^N \sum_{\substack{g,h=1,\dots,N \\ g \neq h}} (G(d\phi_i, d\psi_{gh}) + G(d\psi_{gh}, d\phi_i)) \\
& + \sum_{i=1}^N (G(d\phi_i, d\Psi) + G(d\Psi, d\phi_i)), \\
& \Psi|_{t=0} = 0, \\
& \partial_t \Psi|_{t=0} = 0.
\end{aligned}$$

We shall use the estimates obtained for ϕ_i and ψ_{ij} above to prove estimates and global existence for Ψ . We will thus have global existence and estimates for the long time behavior of ϕ , as we recall the definition

$$\Psi := \phi - \sum_{i=1}^N \phi_i - \sum_{\substack{i,j \in \{1,\dots,N\} \\ i \neq j}} \psi_{ij}, \tag{1.8.2}$$

and all the ϕ_i 's and ψ_{ij} 's are global.

For ease of notation, let us suppose for the remainder of the proof that G is identically 0, as estimating the terms in G is exactly analogous to estimating the terms in F , and it requires fewer derivatives.

In order to solve this equation, we shall now set up a continuity argument in the parameter T , which we define to be the maximal time for which the initial value problem (1.8.1) admits

a solution on the set $[0, T] \times \mathbb{R}^3$ and satisfies the following bootstrap assumptions¹ on $[0, T]$, for all $I \in I_{\mathbf{K}_R}^{\leq N_0}$, all $J \in I_{\mathbf{K}_R}^{\leq [N_0/2]+1}$, and all $i \in \{1, \dots, N\}$:

$$\|\partial V^I \Psi\|_{L^2(\Sigma_t)} \leq \varepsilon^{3-\delta} R^{-\frac{3}{2}+\delta} \quad \text{for } t \in [0, T], \quad (1.8.3)$$

$$\|(1 + |u_i|)^{-\frac{1}{2}-\frac{\delta}{2}} \bar{\partial}^{(i)} V^I \Psi\|_{L^2([0, T] \times \mathbb{R}^3)} \leq \varepsilon^{3-\delta} R^{-\frac{3}{2}+\delta} \quad \text{for } t \in [0, T], \quad (1.8.4)$$

$$|\partial V^J \Psi(t, r, \theta, \varphi)| \leq \varepsilon^{3-\delta} (1+t)^{-1} R^{-\frac{1}{2}+\delta} \quad \text{for } t \in [0, T], \quad (1.8.5)$$

$$|\bar{\partial}^{(i)} V^J \Psi(t, r, \theta, \varphi)| \leq \varepsilon^{3-\delta} (1+t)^{-\frac{3}{2}} R^{\frac{3}{2}+\delta} \quad \text{for } t \in [R^{20}, T], \quad (1.8.6)$$

$$|\partial V^J \Psi(t, r, \theta, \varphi)| \leq \varepsilon^{3-\delta} (1+v_i)^{-1} (1+|u_i|)^{-\frac{1}{2}} R^{\frac{1}{2}+\delta} \quad \text{for } t \in [R^{20}, T]. \quad (1.8.7)$$

Here, we adopted the convention that the interval $[a, b]$, with $a > b$, is the empty set. The estimates below will imply that this holds at $T = 0$ with a better power in ε . By the local existence theory for a single quasilinear equation, we know that in fact $T > 0$ (note that, for this first non-emptiness step, the precise value of T here is allowed to depend on R).

We will then proceed to improve these bootstrap estimates. Namely, we will prove, under the bootstrap assumptions (1.8.3)–(1.8.7), the following bounds for all $I \in I_{\mathbf{K}_R}^{\leq N_0}$, all $J \in I_{\mathbf{K}_R}^{\leq N_0-3}$ and all $i \in \{1, \dots, N\}$:

$$\|\partial V^I \Psi\|_{L^2(\Sigma_t)} \leq \varepsilon^{3-\frac{3\delta}{4}} R^{-\frac{3}{2}+\delta} \quad \text{for } t \in [0, T], \quad (1.8.8)$$

$$\|(1 + |u_i|)^{-\frac{1}{2}-\frac{\delta}{2}} \bar{\partial}^{(i)} V^I \Psi\|_{L^2([0, T] \times \mathbb{R}^3)} \leq \varepsilon^{3-\frac{3\delta}{4}} R^{-\frac{3}{2}+\delta} \quad \text{for } t \in [0, T], \quad (1.8.9)$$

$$|\partial V^J \Psi(t, r, \theta, \varphi)| \leq \varepsilon^{3-\frac{7\delta}{8}} (1+t)^{-1} R^{-\frac{1}{2}+\delta} \quad \text{for } t \in [0, T], \quad (1.8.10)$$

$$|\bar{\partial}^{(i)} V^J \Psi(t, r, \theta, \varphi)| \leq \varepsilon^{3-\frac{7\delta}{8}} (1+t)^{-\frac{3}{2}} R^{\frac{3}{2}+\delta} \quad \text{for } t \in [R^{20}, T], \quad (1.8.11)$$

$$|\partial V^J \Psi(t, r, \theta, \varphi)| \leq \varepsilon^{3-\frac{7\delta}{8}} (1+v_i)^{-1} (1+|u_i|)^{-\frac{1}{2}} R^{\frac{1}{2}+\delta} \quad \text{for } t \in [R^{20}, T]. \quad (1.8.12)$$

¹Note that, in the course of our argument, we will need to be able to control at most $[N_0/2]+2$ derivatives of Ψ in L^∞ . This suggests that we should require $|J| \leq [N_0/2] + 1$ in the bootstrap assumptions.

This will imply that the initial value problem (1.8.1) admits a global-in-time solution, upon choosing $N_0 \geq 7$ (this choice ensures that $N_0 - 3 \geq \lfloor \frac{N_0}{2} \rfloor + 1$).

Remark 1.8.1. Looking ahead to the proof of the large data theorem (Theorem 1.4.8), we note that the estimates we will show are actually better in terms of the parameter R , and read as follows, for all $I \in I_{\mathbf{K}_R}^{\leq N_0}$, all $J \in I_{\mathbf{K}_R}^{\leq N_0-3}$ and all $i \in \{0, \dots, N\}$:

$$\|\partial V^I \Psi\|_{L^2(\Sigma_t)} \leq \varepsilon^{3-\delta} R^{-\frac{3}{2} + \frac{3}{4}\delta} \quad \text{for } t \in [0, T], \quad (1.8.13)$$

$$\|(1 + |u_i|)^{-\frac{1}{2} - \frac{\delta}{2}} \bar{\partial}^{(i)} V^I \Psi\|_{L^2([0, T] \times \mathbb{R}^3)} \leq \varepsilon^{3-\delta} R^{-\frac{3}{2} + \frac{3}{4}\delta} \quad \text{for } t \in [0, T], \quad (1.8.14)$$

$$|\partial V^J \Psi(t, r, \theta, \varphi)| \leq \varepsilon^{3-\delta} (1+t)^{-1} R^{-\frac{1}{2} + \frac{7}{8}\delta} \quad \text{for } t \in [0, T], \quad (1.8.15)$$

$$|\bar{\partial}^{(i)} V^J \Psi(t, r, \theta, \varphi)| \leq \varepsilon^{3-\delta} (1+t)^{-\frac{3}{2}} R^{\frac{3}{2} + \frac{7}{8}\delta} \quad \text{for } t \in [R^{20}, T], \quad (1.8.16)$$

$$|\partial V^J \Psi(t, r, \theta, \varphi)| \leq \varepsilon^{3-\delta} (1+v_i)^{-1} (1+|u_i|)^{-\frac{1}{2}} R^{\frac{1}{2} + \frac{7}{8}\delta} \quad \text{for } t \in [R^{20}, T]. \quad (1.8.17)$$

We divide the proof in several **Steps**. In **Step 0**, we will introduce some preliminary calculations. In **Step 1**, we will prove estimate (1.8.8) by a ∂_t -energy estimate, whereas, in **Step 2**, we will prove estimates (1.8.9), again by a ∂_t -energy estimate. As we shall see, the only difference between **Step 1** and **Step 2** is in how the boundary terms are treated, as the bulk terms in the respective estimates will be bounded in essentially the same way. **Step 2** will be moreover divided in several parts. We will first estimate the “nonlinear” term $F(d\Psi, d^2\Psi)$, we will then proceed to estimate the “mixed” terms

$$\sum_{\substack{i,j=1,\dots,N \\ i \neq j}} (F(d\psi_{ij}, d^2\Psi) + F(d\Psi, d^2\psi_{ij})) + \sum_{i=1}^N (F(d\phi_i, d^2\Psi) + F(d\Psi, d^2\phi_i)).$$

Finally, we will estimate the “inhomogenous” terms

$$\sum_{i=1}^N \sum_{\substack{g,h=1,\dots,N \\ g \neq h}} (F(d\phi_i, d^2\psi_{gh}) + F(d\psi_{gh}, d^2\phi_i)) + \sum_{\substack{g,h,i,j=1,\dots,N \\ g \neq h, i \neq j}} F(d\psi_{gh}, d^2\psi_{ij}).$$

To conclude the proof, in **Step 3** an easy application of the Sobolev lemmas in Section 1.6 will be sufficient to show estimates (1.8.10)–(1.8.12) from (1.8.8) and (1.8.9).

Step 0. Let $n \in \mathbb{N}$, $n \geq 0$. We now define $E_n[f](\Sigma_t)$ for any smooth function f to be the following energy integral:

$$E_{n+1}^2[f](\Sigma_t) := \sum_{I \in I_{\bar{K}_R}^{\leq n}} \|\partial V^I f\|_{L^2(\Sigma_t)}^2. \quad (1.8.18)$$

We now consider the family of (truncated) the null cones $\hat{C}_{\bar{u}_i, T}^{(i)}$: these are defined as follows, for $\bar{u}_i \in \mathbb{R}$:

$$\hat{C}_{\bar{u}_i, T}^{(i)} := C_{\bar{u}_i}^{(i)} \cap \{0 \leq t \leq T\},$$

where the usual outgoing cone $C_{\bar{u}_i}^{(i)}$ is defined as $\{u_i = \bar{u}_i\} \cap \{t \geq 0\}$.

We define $E_n[f](\hat{C}_{\bar{u}_i, T}^{(i)})$ for a smooth function f as follows:

$$\begin{aligned} E_1^2[f](\hat{C}_{\bar{u}_i}^{(i)}) &:= \int_{\hat{C}_{\bar{u}_i, T}^{(i)}} Q(\partial_{v_i}, \partial_t) r_i^2 dv_i d\theta_i d\varphi_i, \\ E_{n+1}^2[f](\hat{C}_{\bar{u}_i, T}^{(i)}) &:= \sum_{I \in I_{\bar{K}_R}^{\leq n}} E_1^2[V^I f](\hat{C}_{\bar{u}_i, T}^{(i)}). \end{aligned} \quad (1.8.19)$$

Here, $Q(\cdot, \cdot)$ is the stress–energy–momentum tensor associated to the linear wave equation as defined in Section 1.10.3, and ∂_{v_i} is defined as a coordinate vector field arising from the coordinates $(u_i, v_i, \theta_i, \varphi_i)$ defined in Definition 1.3.1. It is moreover a properly normalized Lorentzian normal to the cones $\hat{C}_{\bar{u}_i, T}^{(i)}$.

We also recall that the energy integrals in display (1.8.19) give control over good derivatives, i.e. there exists a positive constant C such that the inequality holds true:

$$E_{n+1}^2[f](\hat{C}_{\bar{u}_i, T}^{(i)}) \geq C \sum_{I \in I_{\bar{K}_R}^{\leq n}} \int_{\hat{C}_{\bar{u}_i, T}^{(i)}} |\bar{\partial}^{(i)} V^I f|^2 r_i^2 d\hat{v}_i d\theta_i d\varphi_i. \quad (1.8.20)$$

We now commute equation (1.8.1) with V^I , where $I \in I_{\tilde{\mathbf{K}}_R}^{\leq N_0}$. We obtain (recall that we set $G = 0$ for ease of argument):

$$\begin{aligned}
& \square V^I \Psi + \sum_{H+K \subset I} \left(F_{HK}(dV^H \Psi, d^2 V^K \Psi) \right) \\
& + \sum_{H+K \subset I} \left(\sum_{\substack{i,j=1,\dots,N \\ i \neq j}} (F_{HK}(dV^H \psi_{ij}, d^2 V^K \Psi) + F_{HK}(dV^H \Psi, d^2 V^K \psi_{ij})) \right. \\
& \quad \left. + \sum_{i=1}^N (F_{HK}(dV^H \phi_i, d^2 V^K \Psi) + F_{HK}(dV^H \Psi, d^2 V^K \phi_i)) \right) \quad (1.8.21) \\
& + \sum_{H+K \subset I} \left(\sum_{i=1}^N \sum_{\substack{g,h=1,\dots,N \\ g \neq h}} (F_{HK}(dV^H \phi_i, d^2 V^K \psi_{gh}) + F_{HK}(dV^H \psi_{gh}, d^2 V^K \phi_i)) \right. \\
& \quad \left. + \sum_{\substack{g,h,i,j=1,\dots,N \\ g \neq h, i \neq j}} F_{HK}(dV^H \psi_{gh}, d^2 V^K \psi_{ij}) \right) = 0.
\end{aligned}$$

We can now proceed to **Step 1** of the proof.

Step 1. We now wish to perform a ∂_t -energy estimate on equation (1.8.21). To this end, we note that the following lemma holds true.

Lemma 1.8.2 (Main lemma on spacelike L^2 estimates). *Let F be a trilinear null form as in Definition 1.3.14. There exists a positive ε_0 and a positive constant $C > 0$ such that the following holds. For any smooth function $\tilde{\Psi}$ satisfying the following initial value problem on the set $[0, T] \times \mathbb{R}^3$:*

$$\begin{aligned}
& \square \tilde{\Psi} + F(dg, d^2 \tilde{\Psi}) = h, \\
& \tilde{\Psi}|_{t=0} = 0, \\
& \partial_t \tilde{\Psi}|_{t=0} = 0,
\end{aligned} \quad (1.8.22)$$

with g smooth satisfying the bound $|\partial g_j| \leq \varepsilon_0$, and with h smooth, we have that the inequality

holds, for all $t \in [0, T]$:

$$\begin{aligned} E_1^2[\tilde{\Psi}](\Sigma_t) &\leq C \int_0^t \int_{\mathbb{R}^3} (|F^{\alpha\beta\gamma} \partial_\beta \partial_\alpha g \partial_t \tilde{\Psi} \partial_\gamma \tilde{\Psi}| + |F^{\alpha\beta\gamma} \partial_t \partial_\alpha g \partial_\beta \tilde{\Psi} \partial_\gamma \tilde{\Psi}|) dx ds \\ &+ C \int_0^t \int_{\mathbb{R}^3} |\partial_t \tilde{\Psi}| |h| dx ds + C E_1^2[\tilde{\Psi}](\Sigma_0). \end{aligned} \quad (1.8.23)$$

Proof of Lemma 1.8.2. First of all, let us multiply equation (1.8.22) by $\partial_t \tilde{\Psi}$. Let us write the tensor F in components as $F^{\alpha\beta\gamma}$. Recall that, without loss of generality, we can assume that F is symmetric in the last two indices: $F^{\alpha\beta\gamma} = F^{\alpha\gamma\beta}$. Then, we have,

$$\begin{aligned} &\partial_t \tilde{\Psi} F^{\alpha\beta\gamma} \partial_\alpha g \partial_\beta \partial_\gamma \tilde{\Psi} \\ &= \partial_\beta (F^{\alpha\beta\gamma} \partial_\alpha g \partial_t \tilde{\Psi} \partial_\gamma \tilde{\Psi}) - F^{\alpha\beta\gamma} \partial_\beta \partial_\alpha g \partial_t \tilde{\Psi} \partial_\gamma \tilde{\Psi} \\ &\quad - \frac{1}{2} \partial_t (\partial_\alpha g F^{\alpha\beta\gamma} \partial_\beta \tilde{\Psi} \partial_\gamma \tilde{\Psi}) + \frac{1}{2} \partial_t \partial_\alpha g F^{\alpha\beta\gamma} \partial_\beta \tilde{\Psi} \partial_\gamma \tilde{\Psi}. \end{aligned}$$

We then let $t_1 \geq 0$, and integrate the resulting equation on the region $\{0 \leq t \leq t_1\} \cap \{t+r/2 \leq A\}$, for $A > 0$ large. Note that the boundary of the region $\{t+r/2 \leq A\}$ is strictly spacelike. This means that, possibly restricting ε_0 to be smaller, we have the following inequality:

$$\begin{aligned} &\frac{1}{2} \int_{\Sigma_{t_1} \cap \{r \leq 2(A-t_1)\}} (|\partial \tilde{\Psi}|^2 - |F^{\alpha\beta\gamma} \partial_\alpha g \partial_\beta \tilde{\Psi} \partial_\gamma \tilde{\Psi}|) dx \\ &\leq \int_0^{t_1} \int_{\mathbb{R}^3} (|F^{\alpha\beta\gamma} \partial_\beta \partial_\alpha g \partial_t \tilde{\Psi} \partial_\gamma \tilde{\Psi}| + \frac{1}{2} |F^{\alpha\beta\gamma} \partial_t \partial_\alpha g \partial_\beta \tilde{\Psi} \partial_\gamma \tilde{\Psi}|) dx dt \\ &+ \int_0^{t_1} \int_{\mathbb{R}^3} |\partial_t \tilde{\Psi}| |h| dx dt + C E_1[\tilde{\Psi}](\Sigma_0). \end{aligned} \quad (1.8.24)$$

Using now the fact that $|\partial g| \leq \varepsilon_0$, we conclude that:

$$\int_{\Sigma_{t_1} \cap \{r \leq 2(A-t_1)\}} |\partial \tilde{\Psi}|^2 dx \quad (1.8.25)$$

$$\leq C \int_0^{t_1} \int_{\mathbb{R}^3} (|F^{\alpha\beta\gamma} \partial_\beta \partial_\alpha g \partial_t \tilde{\Psi} \partial_\gamma \tilde{\Psi}| + \frac{1}{2} |F^{\alpha\beta\gamma} \partial_t \partial_\alpha g \partial_\beta \tilde{\Psi} \partial_\gamma \tilde{\Psi}|) dx dt \quad (1.8.26)$$

$$+ C \int_0^{t_1} \int_{\mathbb{R}^3} |\partial_t \tilde{\Psi}| |h| dx dt + C E_1[\tilde{\Psi}](\Sigma_0). \quad (1.8.27)$$

We now conclude by the monotone convergence theorem, upon sending $A \rightarrow \infty$. $\square$

Let us now rewrite the commuted equation (1.8.21) highlighting the top-order terms:

$$\begin{aligned}
& \square V^I \Psi + F \left(\sum_{i=1}^N d\phi_i + \sum_{\substack{i,j=\{1,\dots,N\} \\ i \neq j}} d\psi_{ij} + d\Psi, d^2 V^I \Psi \right) \\
& + \sum_{\substack{H+K \subset I \\ K \neq I}} F_{HK}(dV^H \Psi, d^2 V^K \Psi) \\
& + \sum_{\substack{H+K \subset I \\ K \neq I}} \left(\sum_{\substack{i,j=1,\dots,N \\ i \neq j}} (F_{HK}(dV^H \psi_{ij}, d^2 V^K \Psi) + F_{HK}(dV^H \Psi, d^2 V^K \psi_{ij})) \right. \\
& \quad \left. + \sum_{i=1}^N (F_{HK}(dV^H \phi_i, d^2 V^K \Psi) + F_{HK}(dV^H \Psi, d^2 V^K \phi_i)) \right) \tag{1.8.28} \\
& + \sum_{\substack{i,j=1,\dots,N \\ i \neq j}} F(d\Psi, d^2 V^I \psi_{ij}) + \sum_{i=1}^N F(d\Psi, d^2 V^I \phi_i) \\
& + \sum_{H+K \subset I} \left(\sum_{i=1}^N \sum_{\substack{g,h=1,\dots,N \\ g \neq h}} (F_{HK}(dV^H \phi_i, d^2 V^K \psi_{gh}) + F_{HK}(dV^H \psi_{gh}, d^2 V^K \phi_i)) \right. \\
& \quad \left. + \sum_{\substack{g,h,i,j=1,\dots,N \\ g \neq h, i \neq j}} F_{HK}(dV^H \psi_{gh}, d^2 V^K \psi_{ij}) \right) = 0.
\end{aligned}$$

We now apply Lemma 1.8.2, with $\tilde{\Psi} := V^I \Psi$, and

$$g := \sum_{i=1}^N \phi_i + \sum_{\substack{i,j=\{1,\dots,N\} \\ i \neq j}} \psi_{ij} + \Psi.$$

Note that, by the linear estimates of Proposition 1.7.3 and by the bootstrap assumptions, we can assume that this g is in the conditions of the above lemma, i. e. $|\partial g| \leq \varepsilon_0$.

We then have the following estimate:

$$\begin{aligned}
& E_1^2[V^I\Psi](\Sigma_t) \\
& \leq C \int_0^t \int_{\mathbb{R}^3} (|F^{\alpha\beta\gamma} \partial_\beta \partial_\alpha g \partial_t V^I \Psi \partial_\gamma V^I \Psi| + |F^{\alpha\beta\gamma} \partial_t \partial_\alpha g \partial_\beta V^I \Psi \partial_\gamma V^I \Psi|) dx ds \quad (1.8.29) \\
& \quad + C \int_0^t \int_{\mathbb{R}^3} |\partial_t V^I \Psi| |h| dx ds + C E_1^2[V^I \Psi](\Sigma_0).
\end{aligned}$$

Here, h is composed of the terms contained in lines 2 to 7 of display (1.8.28). Expanding all the terms in display (1.8.29) now gives:

$$\begin{aligned}
& E_1^2[V^I \Psi](\Sigma_t) - C E_1^2[V^I \Psi](\Sigma_0) \\
& \leq C \int_0^t \int_{\mathbb{R}^3} \left(\underbrace{|F^{\alpha\beta\gamma} \partial_\beta \partial_\alpha \Psi \partial_t V^I \Psi \partial_\gamma V^I \Psi|}_{(a_1)} + \underbrace{|F^{\alpha\beta\gamma} \partial_t \partial_\alpha \Psi \partial_\beta V^I \Psi \partial_\gamma V^I \Psi|}_{(a_2)} \right) dx ds \\
& + C \sum_{i=1}^N \int_0^t \int_{\mathbb{R}^3} \left(\underbrace{|F^{\alpha\beta\gamma} \partial_\beta \partial_\alpha \phi_i \partial_t V^I \Psi \partial_\gamma V^I \Psi|}_{(a_3)} + \underbrace{|F^{\alpha\beta\gamma} \partial_t \partial_\alpha \phi_i \partial_\beta V^I \Psi \partial_\gamma V^I \Psi|}_{(a_4)} \right) dx ds \\
& + C \sum_{\substack{i,j \in \{1, \dots, N\} \\ i \neq j}} \int_0^t \int_{\mathbb{R}^3} \left(\underbrace{|F^{\alpha\beta\gamma} \partial_\beta \partial_\alpha \psi_{ij} \partial_t V^I \Psi \partial_\gamma V^I \Psi|}_{(a_5)} \right. \\
& \quad \left. + \underbrace{|F^{\alpha\beta\gamma} \partial_t \partial_\alpha \psi_{ij} \partial_\beta V^I \Psi \partial_\gamma V^I \Psi|}_{(a_6)} \right) dx ds \\
& + C \sum_{\substack{H+K \subset I \\ K \neq I}} \int_0^t \int_{\mathbb{R}^3} \underbrace{|\partial_t V^I \Psi| |F_{HK}(dV^H \Psi, d^2 V^K \Psi)|}_{(a_7)} dx ds \\
& + C \sum_{\substack{H+K \subset I \\ K \neq I}} \int_0^t \int_{\mathbb{R}^3} \left(\sum_{\substack{i,j=1, \dots, N \\ i \neq j}} \left(\underbrace{|F_{HK}(dV^H \psi_{ij}, d^2 V^K \Psi)|}_{(a_8)} + \underbrace{|F_{HK}(dV^H \Psi, d^2 V^K \psi_{ij})|}_{(a_9)} \right) \right. \\
& \quad \left. + \sum_{i=1}^N \left(\underbrace{|F_{HK}(dV^H \phi_i, d^2 V^K \Psi)|}_{(a_{10})} + \underbrace{|F_{HK}(dV^H \Psi, d^2 V^K \phi_i)|}_{(a_{11})} \right) \right) |\partial_t V^I \Psi| dx ds \\
& + C \sum_{\substack{i,j=1, \dots, N \\ i \neq j}} \int_0^t \int_{\mathbb{R}^3} \underbrace{|\partial_t V^I \Psi| |F(d\Psi, d^2 V^I \psi_{ij})|}_{(a_{12})} dx ds \\
& + C \sum_{i=1}^N \int_0^t \int_{\mathbb{R}^3} \underbrace{|\partial_t V^I \Psi| |F(d\Psi, d^2 V^I \phi_i)|}_{(a_{13})} dx ds
\end{aligned} \quad (1.8.30)$$

$$\begin{aligned}
& + C \sum_{H+K \leq I} \int_0^t \int_{\mathbb{R}^3} \left(\sum_{i=1}^N \sum_{\substack{g,h=1,\dots,N \\ g \neq h}} \underbrace{(|F_{HK}(dV^H \phi_i, d^2 V^K \psi_{gh})|)}_{(a_{14})} \right. \\
& \quad \left. + \underbrace{|F_{HK}(dV^H \psi_{gh}, d^2 V^K \phi_i)|}_{(a_{15})} \right) \\
& \quad + \sum_{\substack{g,h,i,j=1,\dots,N \\ g \neq h, i \neq j}} \underbrace{|F_{HK}(dV^H \psi_{gh}, d^2 V^K \psi_{ij})|}_{(a_{16})} \Big) |\partial_t V^I \Psi| dx ds.
\end{aligned}$$

We now proceed to estimate the terms in the previous display one by one. Because inequality (1.8.30) controls the square of the energy, we must recover the square of the bootstrap assumption (1.8.8). We will be wasteful when deriving our estimates in terms of the parameter R . Indeed, to close the bootstrap argument we must only recover $R^{-3+2\delta}$ (this is what we need for (1.8.8)), but we shall keep track of which estimates “have room in R ”. We shall do this by bounding these terms by a factor of $R^{-3+\frac{3}{2}\delta}$ instead of $R^{-3+2\delta}$. This will be needed in the proof of Theorem 1.4.8 in Section 1.9. See also Remark 1.8.1. We shall also be wasteful in terms of the parameter ε . Terms which gain an improvement in powers of ε will be bounded by ε^6 .

• **(a₁) + (a₂).** We have the following estimates (as usual, we assume that the interval $[a, b]$, with $a > b$, is the empty set):

$$\begin{aligned}
& \int_0^t \int_{\mathbb{R}^3} (|F^{\alpha\beta\gamma} \partial_\beta \partial_\alpha \Psi \partial_t V^I \Psi \partial_\gamma V^I \Psi| + |F^{\alpha\beta\gamma} \partial_t \partial_\alpha \Psi \partial_\beta V^I \Psi \partial_\gamma V^I \Psi|) dx ds \\
& \leq C \left(\int_{[0, R^{20}]} \|\partial^2 \Psi\|_{L^\infty(\Sigma_s)} ds \right) \sup_{s \in [0, R^{20}]} \|\partial V^I \Psi\|_{L^2(\Sigma_s)}^2 \\
& \quad + C \int_{[R^{20}, t]} \int_{\mathbb{R}^3} |\partial^2 \Psi| |\bar{\partial}^{(0)} V^I \Psi| |\partial V^I \Psi| dx ds \\
& \quad + C \left(\int_{[R^{20}, t]} \|\bar{\partial}^{(0)} \partial \Psi\|_{L^\infty(\Sigma_s)} ds \right) \sup_{s \in [R^{20}, t]} \|\partial V^I \Psi\|_{L^2(\Sigma_s)}^2 \\
& \leq C \varepsilon^{9-3\delta} R^{-3+2\delta} R^{-\frac{1}{2}+\delta} \log R
\end{aligned}$$

$$\begin{aligned}
& + C\varepsilon^{3-\delta} R^{-\frac{3}{2}+\delta} \left(\int_{[R^{20}, t]} \int_{\mathbb{R}^3} (1+s)^{1+\delta} |\partial^2 \Psi|^2 |\bar{\partial}^{(0)} V^I \Psi|^2 dx ds \right)^{\frac{1}{2}} \\
& + C\varepsilon^{9-3\delta} R^{-3+2\delta} \int_{R^{20}}^t R^{\frac{3}{2}+\delta} (1+s)^{-\frac{3}{2}} ds \\
& \leq C\varepsilon^7 R^{-3}.
\end{aligned} \tag{1.8.31}$$

Here, in the first inequality we used the lemma on the structure of null forms (Lemma 1.3.19), in the second inequality we used the bootstrap assumptions (1.8.3)–(1.8.7), plus the Cauchy–Schwarz inequality on the second term (multiplying and dividing by $(1+s)^{\frac{1+\delta}{2}}$), and finally in the last inequality we used estimate (1.8.54) from Lemma 1.8.4.

• **(a₃) + (a₄).** We need to estimate the following expression:

$$\sum_{i=1}^N \int_0^t \int_{\mathbb{R}^3} (|F^{\alpha\beta\gamma} \partial_\beta \partial_\alpha \phi_i \partial_t V^I \Psi \partial_\gamma V^I \Psi| + |F^{\alpha\beta\gamma} \partial_t \partial_\alpha \phi_i \partial_\beta V^I \Psi \partial_\gamma V^I \Psi|) dx ds.$$

For all $i \in \{1, \dots, N\}$, we have, by Lemma 1.3.19,

$$\begin{aligned}
& \int_0^t \int_{\mathbb{R}^3} (|F^{\alpha\beta\gamma} \partial_\beta \partial_\alpha \phi_i \partial_t V^I \Psi \partial_\gamma V^I \Psi| + |F^{\alpha\beta\gamma} \partial_t \partial_\alpha \phi_i \partial_\beta V^I \Psi \partial_\gamma V^I \Psi|) dx ds \\
& \leq C \int_0^t \int_{\mathbb{R}^3} (|\partial \bar{\partial}^{(i)} \phi_i| |\partial V^I \Psi| |\partial V^I \Psi| + |\partial \partial \phi_i| |\bar{\partial}^{(i)} V^I \Psi| |\partial V^I \Psi|) dx ds.
\end{aligned}$$

Now, by Proposition 1.5.1, we know that $|\partial \bar{\partial}^{(i)} \phi_i| \leq C(N, d_\Pi) \varepsilon (1+t)^{-2}$. This, together with the bootstrap assumption (1.8.3), implies that

$$\begin{aligned}
& \int_0^t \int_{\mathbb{R}^3} |\partial \bar{\partial}^{(i)} \phi_i| |\partial V^I \Psi| |\partial V^I \Psi| dx ds \\
& \leq C(N, d_\Pi) \varepsilon^{7-2\delta} R^{-3+2\delta} \int_{R/10}^t \frac{1}{(1+s)^2} ds \leq C(N, d_\Pi) \varepsilon^{7-2\delta} R^{-3}.
\end{aligned}$$

We also explicitly marked the dependence of the constant C on the quantity d_Π (which is defined in equation (1.3.1)), as well as on the number N .

Now, for the remaining term, we have, applying Hölder's inequality, the bootstrap assumption (1.8.3), and the estimate (1.8.54) from Lemma 1.8.4,

$$\begin{aligned} \int_0^t \int_{\mathbb{R}^3} |\partial \partial \phi_i| |\bar{\partial}^{(i)} V^I \Psi| |\partial V^I \Psi| dx ds &\leq C \left(\int_{R/10}^t \frac{1}{(1+s)^{1+\delta}} \|\partial V^I \Psi\|_{L^2(\Sigma_s)}^2 ds \right)^{\frac{1}{2}} \\ &\times \left(\int_{R/10}^t \int_{\mathbb{R}^3} (1+s)^{1+\delta} |\partial \partial \phi_i|^2 |\bar{\partial}^{(i)} V^I \Psi|^2 dx ds \right)^{\frac{1}{2}} \leq C(N, d_\Pi) \varepsilon^{7-2\delta} R^{-3+2\delta}. \end{aligned}$$

Adding these terms gives us that

$$(\mathbf{a}_3) + (\mathbf{a}_4) \leq C(N, d_\Pi) \varepsilon^{7-2\delta} R^{-3+2\delta} \leq \varepsilon^6 R^{-3}, \quad (1.8.32)$$

where we have used the fact that ε can depend on N and on d_Π . This suffices to bound terms $(\mathbf{a}_3)$ and $(\mathbf{a}_4)$.

• $(\mathbf{a}_5) + (\mathbf{a}_6)$. In this case, we have the estimates:

$$\begin{aligned} &\int_0^t \int_{\mathbb{R}^3} (|F^{\alpha\beta\gamma} \partial_\beta \partial_\alpha \psi_{ij} \partial_t V^I \Psi \partial_\gamma V^I \Psi| + |F^{\alpha\beta\gamma} \partial_t \partial_\alpha \psi_{ij} \partial_\beta V^I \Psi \partial_\gamma V^I \Psi|) dx ds \\ &\leq \int_0^{R^{20}} \int_{\mathbb{R}^3} |\partial^2 \psi_{ij}| |\partial V^I \Psi|^2 dx ds + \int_{R^{20}}^t \int_{\mathbb{R}^3} |\partial \bar{\partial}^{(i)} \psi_{ij}| |\partial V^I \Psi|^2 dx ds \\ &\quad + \int_{R^{20}}^t \int_{\mathbb{R}^3} |\partial^2 \psi_{ij}| |\bar{\partial}^{(i)} V^I \Psi| |\partial V^I \Psi| dx ds \\ &\leq C(N, d_\Pi) \varepsilon^7 R^{-3} + C \varepsilon^{3-\delta} R^{-\frac{3}{2}+\delta} \left(\int_{R^{20}}^t \int_{\mathbb{R}^3} (1+s)^{1+\delta} |\partial^2 \psi_{ij}|^2 |\bar{\partial}^{(i)} V^I \Psi|^2 dx ds \right)^{\frac{1}{2}}. \end{aligned}$$

Here, we used the fundamental lemma on null forms (Lemma 1.3.19), plus the bounds in Proposition 1.7.3, together with the bootstrap assumptions and the Hölder inequality in the last line. We then bound:

$$\left(\int_{R^{20}}^t \int_{\mathbb{R}^3} (1+s)^{1+\delta} |\partial^2 \psi_{ij}|^2 |\bar{\partial}^{(i)} V^I \Psi|^2 dx ds \right)^{\frac{1}{2}} \leq C(N, d_\Pi) \varepsilon^7 R^{-3},$$

where again we used the estimates for ψ_{ij} contained in Proposition 1.7.3, together with

estimate (1.8.54) from Lemma 1.8.4. Summing gives us

$$(\mathbf{a}_5) + (\mathbf{a}_6) \leq C(N, d_{\Pi})\varepsilon^7 R^{-3} \leq \varepsilon^6 R^{-3}, \quad (1.8.33)$$

where we have once again used that ε can depend on N and d_{Π} .

- $(\mathbf{a}_7)$. We have to estimate the term

$$\sum_{\substack{H+K \subset I \\ K \neq I}} \int_0^t \int_{\mathbb{R}^3} |\partial_t V^I \Psi| |F_{HK}(dV^H \Psi, d^2 V^K \Psi)| dx ds.$$

This can be controlled in the same way as $(\mathbf{a}_1) + (\mathbf{a}_2)$. We obtain:

$$(\mathbf{a}_7) \leq C\varepsilon^7 R^{-3}. \quad (1.8.34)$$

- $(\mathbf{a}_8) + (\mathbf{a}_9)$. We need to bound the terms

$$\begin{aligned} & \sum_{\substack{H+K \subset I \\ K \neq I}} \int_0^t \int_{\mathbb{R}^3} \left(\sum_{\substack{i,j=1,\dots,N \\ i \neq j}} (|F_{HK}(dV^H \psi_{ij}, d^2 V^K \Psi)| + |F_{HK}(dV^H \Psi, d^2 V^K \psi_{ij})|) \right) \\ & \quad \times |\partial_t V^I \Psi| dx ds. \end{aligned}$$

These terms can be dealt with exactly as in the case of terms $(\mathbf{a}_5) + (\mathbf{a}_6)$, always estimating ψ_{ij} in L^∞ . We note that we need to bound at most $N_0 + 1$ derivatives of ψ_{ij} in L^∞ . We obtain:

$$(\mathbf{a}_8) + (\mathbf{a}_9) \leq C(N, d_{\Pi})\varepsilon^7 R^{-3}. \quad (1.8.35)$$

- $(\mathbf{a}_{10}) + (\mathbf{a}_{11})$. In this case, we need to bound the terms

$$\sum_{\substack{H+K \subset I \\ K \neq I}} \int_0^t \int_{\mathbb{R}^3} \left(\sum_{i=1}^N (|F_{KH}(dV^H \phi_i, d^2 V^K \Psi)| + |F_{KH}(dV^H \Psi, d^2 V^K \phi_i)|) \right) \\ \times |\partial_t V^I \Psi| dx ds.$$

The same reasoning as the one for terms $(\mathbf{a}_3) + (\mathbf{a}_4)$ will give the required bound. Again, we need to be careful as we always estimate ϕ_i in L^∞ , and in the worst case we need to be able to estimate $N_0 + 1$ derivatives of ϕ_i . We obtain:

$$(\mathbf{a}_{10}) + (\mathbf{a}_{11}) \leq C(N, d_\Pi) \varepsilon^{7-2\delta} R^{-3}. \quad (1.8.36)$$

- $(\mathbf{a}_{12})$. This is the term

$$\sum_{\substack{i,j=1,\dots,N \\ i \neq j}} \int_0^t \int_{\mathbb{R}^3} |\partial_t V^I \Psi| |F(d\Psi, d^2 V^I \psi_{ij})| dx ds.$$

This can be dealt with exactly in the same way as $(\mathbf{a}_5) + (\mathbf{a}_6)$, but note that we need to estimate $N_0 + 2$ derivatives of ψ_{ij} in L^∞ . We obtain:

$$(\mathbf{a}_{12}) \leq C(N, d_\Pi) \varepsilon^7 R^{-3} \leq \varepsilon^6 R^{-3}, \quad (1.8.37)$$

since we are allowed to choose ε small in terms of N and d_Π .

- $(\mathbf{a}_{13})$. We need to bound the term

$$\sum_{i=1}^N \int_0^t \int_{\mathbb{R}^3} |\partial_t V^I \Psi| |F(d\Psi, d^2 V^I \phi_i)| dx ds.$$

This can be dealt with exactly in the same way as $(\mathbf{a}_3) + (\mathbf{a}_4)$, but note that we need to estimate $N_0 + 2$ derivatives of ϕ_i in L^∞ . We obtain:

$$(\mathbf{a}_{13}) \leq C(N, d_\Pi) \varepsilon^{7-2\delta} R^{-3}. \quad (1.8.38)$$

• $(\mathbf{a}_{14}) + (\mathbf{a}_{15})$. In this case, we need to estimate the terms:

$$\begin{aligned} & \sum_{H+K \subset I} \int_0^t \int_{\mathbb{R}^3} \left(\sum_{i=1}^N \sum_{\substack{g,h=1,\dots,N \\ g \neq h}} (|F_{HK}(dV^H \phi_i, d^2 V^K \psi_{gh})| \right. \\ & \quad \left. + |F_{HK}(dV^H \psi_{gh}, d^2 V^K \phi_i)| \right) |\partial_t V^I \Psi| dx ds. \end{aligned}$$

We know that, by an easy domain of dependence argument, ψ_{gh} is supported in the set $\{t \geq R/10\}$. Then, we proceed to estimate, using also Lemma (1.3.19),

$$\begin{aligned} & \int_0^t \int_{\mathbb{R}^3} \left(|F_{HK}(dV^H \phi_i, d^2 V^K \psi_{gh})| + |F_{HK}(dV^H \psi_{gh}, d^2 V^K \phi_i)| \right) |\partial_t V^I \Psi| dx ds \\ &= \int_{R/10}^t \int_{\mathbb{R}^3} \left(|F_{HK}(dV^H \phi_i, d^2 V^K \psi_{gh})| + |F_{HK}(dV^H \psi_{gh}, d^2 V^K \phi_i)| \right) |\partial_t V^I \Psi| dx ds \\ &\leq C \sum_{J_1, J_2 \in I_{\mathbf{K}_R}^{\leq N_0}} \int_{R/10}^t \int_{\mathbb{R}^3} |\bar{\partial}^{(i)} \partial^{\leq 1} V^{J_1} \phi_i| |\partial^{\leq 2} V^{J_2} \psi_{gh}| |\partial V^I \Psi| dx ds \\ &\quad + C \sum_{J_1, J_2 \in I_{\mathbf{K}_R}^{\leq N_0}} \int_{R/10}^t \int_{\mathbb{R}^3} |\partial^{\leq 2} V^{J_1} \phi_i| |\bar{\partial}^{(i)} \partial^{\leq 1} V^{J_2} \psi_{gh}| |\partial V^I \Psi| dx ds \\ &\leq C(N, d_\Pi) \sum_{J_2 \in I_{\mathbf{K}_R}^{\leq N_0}} \int_{R/10}^t \int_{\mathbb{R}^3} \frac{\varepsilon}{s^2} |\partial^{\leq 2} V^{J_2} \psi_{gh}| |\partial V^I \Psi| dx ds \\ &\quad + C(N, d_\Pi) \sum_{J_1, J_2 \in I_{\mathbf{K}_R}^{\leq N_0}} \sup_{s \geq 0} \|\partial V^I \Psi\|_{L^2(\Sigma_s)} \int_{R/10}^t \|\partial^{\leq 2} V^{J_1} \phi_i\| |\bar{\partial}^{(i)} \partial^{\leq 1} V^{J_2} \psi_{gh}| \| \cdot \|_{L^2(\Sigma_s)} ds \\ &\leq C(N, d_\Pi) \varepsilon^{6-\delta} R^{-3} + C(N, d_\Pi) \varepsilon^{6-\delta} R^{-3+\delta}. \end{aligned}$$

The last inequality follows from Lemma 1.8.4, estimate (1.8.53). We note that this term also does not have much “room” in the parameters R and ε . It does, however, have “room” of size $\frac{\delta}{2}$ in the parameter R . Once again, because ε is allowed to depend on N and d_{Π} , we have that, upon possibly restricting ε to a smaller value,

$$C(N, d_{\Pi})\varepsilon^{6-\delta}R^{-3} + C(N, d_{\Pi})\varepsilon^{6-\delta}R^{-3+\delta} \leq \varepsilon^{6-\frac{3\delta}{2}}R^{-3+\delta}. \quad (1.8.39)$$

This concludes the bounds on $(\mathbf{a}_{14}) + (\mathbf{a}_{15})$:

$$(\mathbf{a}_{14}) + (\mathbf{a}_{15}) \leq \varepsilon^{6-\frac{3\delta}{2}}R^{-3+\delta}. \quad (1.8.40)$$

- $(\mathbf{a}_{16})$. We have to estimate the following expression:

$$\sum_{H+K \subset I} \int_1^t \int_{\mathbb{R}^3} \left(\sum_{\substack{g,h,i,j=1,\dots,N \\ g \neq h, i \neq j}} |F_{HK}(dV^H \psi_{gh}, d^2V^K \psi_{ij})| \right) |\partial_t V^I \Psi| dx ds.$$

Let us break up the integral in two pieces, as usual:

$$\begin{aligned} & \int_0^t \int_{\mathbb{R}^3} \left(\sum_{\substack{g,h,i,j=1,\dots,N \\ g \neq h, i \neq j}} |F_{HK}(dV^H \psi_{gh}, d^2V^K \psi_{ij})| \right) |\partial_t V^I \Psi| dx ds \\ & \leq \int_0^{R^{20}} \int_{\mathbb{R}^3} \left(\sum_{\substack{g,h,i,j=1,\dots,N \\ g \neq h, i \neq j}} \underbrace{|F_{HK}(dV^H \psi_{gh}, d^2V^K \psi_{ij})|}_{(a)} \right) |\partial_t V^I \Psi| dx ds \\ & \quad + \int_{[R^{20}, t]} \int_{\mathbb{R}^3} \left(\sum_{\substack{g,h,i,j=1,\dots,N \\ g \neq h, i \neq j}} \underbrace{|F_{HK}(dV^H \psi_{gh}, d^2V^K \psi_{ij})|}_{(b)} \right) |\partial_t V^I \Psi| dx ds. \end{aligned}$$

Here, as usual, we adopt the convention that the interval $[a, b]$, with $a > b$, is the empty set.

We focus first on term (a). We have, by the linear estimates in Lemma 1.7.3,

$$\begin{aligned}
(a) &\leq C(N, d_\Pi) \sum_{J_1 \in I_{\mathbf{K}_R}^{\leq N_0}} \int_0^{R^{20}} \frac{\varepsilon^2}{(1+t)R^{\frac{1}{2}}} \|\partial^2 V^{J_1} \psi_{ij}\|_{L^2(\Sigma_s)} \|\partial V^I \Psi\|_{L^2(\Sigma_s)} ds \\
&\leq C(N, d_\Pi) \varepsilon^{7-\delta} R^{-1} R^{-\frac{1}{2}} \log(R) R^{-\frac{3}{2}+\delta} \leq C(N, d_\Pi) \varepsilon^{7-\delta} R^{-3+\delta} \log(R).
\end{aligned}$$

Using the fact that ε can depend on N and d_Π , we get that

$$C(N, d_\Pi) \varepsilon^{7-\delta} R^{-3+\delta} \log(R) \leq \varepsilon^6 R^{-3+2\delta}. \quad (1.8.41)$$

Focusing now on term (b), we have, again using Lemma 1.3.19,

$$\begin{aligned}
&\int_{[R^{20}, t]} \int_{\mathbb{R}^3} |F_{HK}(dV^H \psi_{gh}, d^2 V^K \psi_{ij})| |\partial_t V^I \Psi| dx ds \\
&\leq C \int_{[R^{20}, t]} \int_{\mathbb{R}^3} |\partial V^H \psi_{gh}| |\partial \bar{\partial}^{(0)} V^K \psi_{ij}| |\partial_t V^I \Psi| dx ds \\
&\quad + C \int_{[R^{20}, t]} \int_{\mathbb{R}^3} |\bar{\partial}^{(0)} V^H \psi_{gh}| |\partial^2 V^K \psi_{ij}| |\partial_t V^I \Psi| dx ds \\
&\leq C(N, d_\Pi) \int_{[R^{20}, t]} \frac{\varepsilon^2}{s^{\frac{3}{2}}} R^3 \|\partial V^K \psi_{gh}\|_{L^2(\Sigma_s)} \|\partial V^I \Psi\|_{L^2(\Sigma_s)} ds \\
&\quad + C(N, d_\Pi) \int_{[R^{20}, t]} \frac{\varepsilon^2}{s^{\frac{3}{2}}} R^3 \|\partial V^K \psi_{gh}\|_{L^2(\Sigma_s)} \|\partial V^I \Psi\|_{L^2(\Sigma_s)} ds \\
&\leq C(N, d_\Pi) \varepsilon^{7-\delta} R^{-3}.
\end{aligned}$$

Here, we used the classical Klainerman–Sobolev inequality in Lemma 1.6.5 (which is wasteful in terms of R -weights), the bounds on energies of ψ_{gh} and ψ_{ij} in Proposition 1.7.1, and finally the bootstrap assumption 1.8.3. Summing gives us that

$$(\mathbf{a}_{16}) \leq C(N, d_\Pi) \varepsilon^{7-\delta} R^{-3} \leq \varepsilon^6 R^{-3+2\delta}, \quad (1.8.42)$$

since we can choose ε to be small, depending on d_Π and N .

Having completed **Step 1** of the proof, we now proceed to **Step 2**.

Step 2. We now proceed to recover the averaged characteristic energy estimate (1.8.9). Averaging the outgoing characteristic energy in the u -direction allows us to control nonlinear interactions in which it is more convenient to estimate “good derivatives” of the solution in L^2 of the outgoing null cones (this is the case when, for example, all of the commutation vector fields are applied to the “good derivative” in the nonlinear error terms). When deriving estimates for quantities which are differentiated at the top order (in our case, when they are differentiated N_0 times), however, this will cause problems, as the background Minkowski structure differs from the causal structure induced by the quasilinear wave equation at hand. Indeed, because the light cones associated to the metric defined by the solution only asymptotically become the Minkowskian light cones, we obtain error terms involving the incoming derivatives on the outgoing cones that we must control in L^2 . Note that the incoming derivative is not intrinsic to the outgoing light cone, and as such it cannot be controlled by the naïve energy estimate. However, since we are seeking to prove an *averaged* estimate, the error term produced in this way can be controlled in the same way as other errors. Note that the fact that the “true” causal structure asymptotically approaches the Minkowski causal structure is encoded in the fact that these error terms will gain “good weights”. In the following discussion, we will treat the situation for arbitrary solutions to appropriately perturbed wave equations, and later specify to the equation at hand.

Recall that, for $\bar{u}_i \in \mathbb{R}$, and $i \in \{0, \dots, N\}$, we defined the cones $C_{\bar{u}_i}^{(i)} := \{u_i = \bar{u}_i\} \cap \{t \geq 0\}$, and the associated “good” derivatives $\bar{\partial}^{(i)}$ in Definition 1.3.6.

Lemma 1.8.3. *Let $\tilde{\Psi}$ be a sufficiently smooth solution, which is moreover decaying at infinity, to the equation $\square \tilde{\Psi} = F^{\alpha\beta\gamma} \partial_\alpha h \partial_\beta \partial_\gamma \tilde{\Psi} + H$, where F satisfies the classical null condition, and h , as well as H , are sufficiently smooth functions. Then, there exists some $\varepsilon_0 > 0$ such*

that, for all $i \in \{1, \dots, N\}$, we have the estimates

$$\begin{aligned} \int_0^t \int_{\Sigma_s} |\bar{\partial}^{(i)} \tilde{\Psi}|^2 \frac{1}{(1 + |u_i|)^{1+\delta}} dx ds &\leq C \|\partial \tilde{\Psi}\|_{L^2(\Sigma_0)}^2 + C \int_0^t \int_{\Sigma_s} |H| |\partial_t \tilde{\Psi}| dx ds \\ &+ C \int_0^t \int_{\Sigma_s} |\bar{\partial} \partial^{\leq 1} h| |\partial \tilde{\Psi}|^2 dx ds + C \int_0^t \int_{\Sigma_s} |\partial^{\leq 1} \partial h| |\bar{\partial}^{(i)} \tilde{\Psi}| |\partial \tilde{\Psi}| dx ds, \end{aligned} \quad (1.8.43)$$

as long as $|\partial h| \leq \varepsilon_0$, and where C can only depend on δ .

Proof. We restrict to the case in which $\bar{\partial} := \bar{\partial}^{(0)}$, where $\bar{\partial}^{(0)}$ was defined in Definition 1.3.6. We also recall the usual functions t , u , v , and r . Furthermore, we denote: $C_u := C_u^{(0)}$. We shall finally denote by $C_{\bar{u}}^{\bar{t}}$ the portion of the cone $C_{\bar{u}}$ between Σ_0 and $\Sigma_{\bar{t}}$: $C_{\bar{u}}^{\bar{t}} := C_{\bar{u}} \cap \{0 \leq t \leq \bar{t}\}$.

We shall do an energy estimate using ∂_t as a multiplier, integrating on the region bounded between Σ_0 , C_u^t , and Σ_t . We shall move the boundary term on the C_u^t cone (arising from integration of $\square \tilde{\Psi}$) on the LHS of the estimate thus obtained and we shall move all the other terms on the RHS. Just as in Lemma 1.8.2, the boundary flux through Σ_0 and the error integral arising from H can be controlled. More precisely, letting $\mathcal{R}_{\bar{u}, \bar{t}} := \{u \leq \bar{u}\} \cap \{0 \leq t \leq \bar{t}\}$, we have

$$\begin{aligned} \int_{C_u^t} |\bar{\partial}^{(0)} \tilde{\Psi}|^2 dv d\omega + \int_{\Sigma_t \cap \mathcal{R}_{u,t}} |\partial \tilde{\Psi}|^2 dx &\leq C \left| \int_{\mathcal{R}_{u,t}} \partial_t \tilde{\Psi} \square \tilde{\Psi} dx ds \right| + C \int_{\Sigma_0} |\partial \tilde{\Psi}|^2 dx \\ &\leq C \underbrace{\left| \int_{\mathcal{R}_{u,t}} F^{\alpha\beta\gamma} \partial_\alpha h \partial_\beta \partial_\gamma \tilde{\Psi} \partial_t \tilde{\Psi} dx ds \right|}_{(a)} + C \int_{\mathcal{R}_{u,t}} |H| |\partial_t \tilde{\Psi}| dx ds + C \|\partial \tilde{\Psi}\|_{L^2(\Sigma_0)}^2 \end{aligned} \quad (1.8.44)$$

We then wish to estimate term (a) in the previous display. To that end, we note the following identity:

$$\begin{aligned} &2F^{\alpha\beta\gamma} \partial_\alpha h \partial_\beta \partial_\gamma \tilde{\Psi} \partial_t \tilde{\Psi} \\ &= F^{\alpha\beta\gamma} \partial_\beta (\partial_\alpha h \partial_t \tilde{\Psi} \partial_\gamma \tilde{\Psi}) - F^{\alpha\beta\gamma} \partial_\alpha \partial_\beta h \partial_t \tilde{\Psi} \partial_\gamma \tilde{\Psi} - \partial_t (F^{\alpha\beta\gamma} \partial_\alpha h \partial_\beta \tilde{\Psi} \partial_\gamma \tilde{\Psi}) \\ &+ F^{\alpha\beta\gamma} (\partial_t \partial_\alpha h \partial_\beta \tilde{\Psi} \partial_\gamma \tilde{\Psi}) + F^{\alpha\beta\gamma} \partial_\gamma (\partial_\alpha h \partial_\beta \tilde{\Psi} \partial_t \tilde{\Psi}) - F^{\alpha\beta\gamma} (\partial_\gamma \partial_\alpha h \partial_\beta \tilde{\Psi} \partial_t \tilde{\Psi}). \end{aligned}$$

This implies, using the fundamental lemma on the structure of null forms (Lemma 1.3.19):

$$\begin{aligned}
(a) &\leq C \int_0^t \int_{\Sigma_s} |\bar{\partial}\partial h| |\partial\tilde{\Psi}|^2 dx ds + C \int_0^t \int_{\Sigma_s} |\partial\partial h| |\bar{\partial}\tilde{\Psi}| |\partial\tilde{\Psi}| dx ds \\
&+ \left| \int_{\mathcal{R}_{u,t}} \underbrace{F^{\alpha\beta\gamma} \partial_\beta (\partial_\alpha h \partial_t \tilde{\Psi} \partial_\gamma \tilde{\Psi}) - \partial_t (F^{\alpha\beta\gamma} \partial_\alpha h \partial_\beta \tilde{\Psi} \partial_\gamma \tilde{\Psi}) + F^{\alpha\beta\gamma} \partial_\gamma (\partial_\alpha h \partial_\beta \tilde{\Psi} \partial_t \tilde{\Psi})}_{(b)} dx ds \right|.
\end{aligned} \tag{1.8.45}$$

We now note that the term (b) is in divergence form, and we now wish to integrate it by parts. For ε_0 sufficiently small, we can once again absorb the error integrals through Σ_t coming from term (b) (after integration by parts) in the LHS, just as in Lemma 1.8.2. Moreover, because the resulting integral over Σ_t has a good sign, we will simply drop it.

All that remains is to control the error terms arising from (b) that are fluxes through C_u^t (after integration by parts). These are the terms:

$$\left| \underbrace{\int_{C_u^t} (F^{\alpha\beta\gamma} N_\beta (\partial_\alpha h \partial_t \tilde{\Psi} \partial_\gamma \tilde{\Psi}) - N_0 F^{\alpha\beta\gamma} (\partial_\alpha h \partial_\beta \tilde{\Psi} \partial_\gamma \tilde{\Psi}) + F^{\alpha\beta\gamma} N_\gamma (\partial_\alpha h \partial_\beta \tilde{\Psi} \partial_t \tilde{\Psi})) r^2 dv d\omega}_{(c)} \right|. \tag{1.8.46}$$

Here, N_β is defined as follows. Note that the Euclidean unit normal to the outgoing cone $u = \text{const}$ is given by $\frac{1}{\sqrt{2}}\partial_u = \frac{1}{\sqrt{2}}(\partial_t - \partial_r) = \frac{1}{\sqrt{2}}\partial_t - \frac{x^i}{\sqrt{2}r}\partial_i$. N_β is then defined as the one-form arising from lowering the index of $\frac{1}{\sqrt{2}}\partial_u$ by means of the *Minkowski* metric.

We then have that, using the null condition on term (c),

$$(c) \leq \int_{C_u^t} \left(|\bar{\partial}h| |\partial\tilde{\Psi}|^2 + |\partial h| |\bar{\partial}\tilde{\Psi}| |\partial\tilde{\Psi}| \right) r^2 dv d\omega. \tag{1.8.47}$$

We then combine estimates (1.8.44), (1.8.45) and (1.8.47), multiply by $\frac{1}{(1+|u|)^{1+\delta}}$ and integrate in u for $u \in \mathbb{R}$. This yields the claim. $\square$

We now turn to the main content of **Step 2**, which is to recover the integrated esti-

mate (1.8.9). Let $i \in \{1, \dots, N\}$, and apply Lemma 1.8.3 to equation (1.8.21), choosing

$$h := \sum_{i=1}^N \phi_i + \sum_{\substack{i,j=\{1,\dots,N\} \\ i \neq j}} \psi_{ij} + \Psi, \quad \tilde{\Psi} := V^I \Psi$$

(with $I \in I_{\mathbf{K}_R}^{\leq N_0}$). Furthermore, we apply the lemma to bound the averaged characteristic energy adapted to the light cone associated with the i -th piece of data. We obtain:

$$\begin{aligned} \int_0^t \int_{\Sigma_s} \frac{|\bar{\partial}^{(i)} V^I \Psi|^2}{(1 + |u_i|)^{1+\delta}} dx ds &\leq C \|\partial V^I \Psi\|_{L^2(\Sigma_0)}^2 + C \int_0^t \int_{\Sigma_s} |H| |\partial_t V^I \Psi| dx ds \\ &+ C \int_0^t \int_{\Sigma_s} |\bar{\partial} \partial^{\leq 1} h| |\partial V^I \Psi|^2 dx ds + C \int_0^t \int_{\Sigma_s} |\partial^{\leq 1} \partial h| |\bar{\partial} V^I \Psi| |\partial V^I \Psi| dx ds, \end{aligned} \quad (1.8.48)$$

with the appropriate choice of h and H arising from equation (1.8.28). The error terms corresponding to H and h are identical to the ones we dealt with in **Step 1**. Hence, these error integrals can be handled in the same way as in **Step 1**

We have the following conclusion:

$$\|(1 + |u_i|)^{-\frac{1}{2}-\frac{\delta}{2}} \bar{\partial}^{(i)} V^I \Psi\|_{L^2([0,T] \times \mathbb{R}^3)} \leq \varepsilon^{3-\delta} R^{-\frac{3}{2}+\frac{3}{4}\delta} \quad \text{for } t \in [0, T], \quad (1.8.49)$$

valid for all $i \in \{1, \dots, N\}$.

This concludes **Step 2** of the proof. We now turn to **Step 3**, in which we show the L^∞ estimates.

Step 3. In this step, we are going to deduce the improved pointwise estimates (1.8.10)–(1.8.12). From **Step 1** and **Step 2**, we know that, for all $I \in I_{\mathbf{K}_R}^{\leq N_0}$,

$$\|\partial V^I \Psi\|_{L^2(\Sigma_t)} \leq \varepsilon^{3-\frac{3\delta}{4}} R^{-\frac{3}{2}+\delta} \quad \text{for } t \in [0, T].$$

Now, using the Sobolev Lemma 1.6.3 and Lemma 1.6.4, we obtain immediately, for all

$$J \in I_{\mathbf{K}_R}^{\leq N_0-3}.$$

$$|\partial V^J \Psi(t, r, \theta, \varphi)| \leq C(d_\Pi) \varepsilon^{3-\frac{3\delta}{4}} (1+t)^{-1} R^{-\frac{1}{2}+\delta} \quad \text{for } t \in [R, T].$$

The constant depends only on d_Π because the rescaled vector fields introduce bad weights that depend only on d_Π . Because ε is allowed to depend on d_Π , we can possibly restrict to a smaller value of ε and obtain

$$|\partial V^J \Psi(t, r, \theta, \varphi)| \leq \varepsilon^{3-\frac{7\delta}{8}} (1+t)^{-1} R^{-\frac{1}{2}+\delta} \quad \text{for } t \in [R, T].$$

The same estimate when for $t \in [0, R]$ follows from commuting the equation with translation vector fields, using the usual Sobolev embedding theorem, and noting that $t \leq R$ in this region. This proves the improved bound (1.8.10).

As for bound (1.8.12), we use the classical Klainerman–Sobolev estimates 1.6.5, and we obtain, as an application of inequality (1.6.12), for all multi-indices $J \in I_{\mathbf{K}_R \cup \mathbf{K}_R^{(i)}}^{\leq N_0-2}$:

$$|\partial V^J \Psi(t, r, \theta, \varphi)| \leq C(N, d_\Pi) \varepsilon^{3-\frac{3\delta}{4}} (1+v_i)^{-1} (1+|u_i|)^{-\frac{1}{2}} R^{\frac{1}{2}+\delta} \quad \text{for } t \in [0, T], \quad (1.8.50)$$

valid for all $i \in \{1, \dots, N\}$.

Upon possibly restricting ε to a smaller value, we infer the bound (1.8.12). Moreover, this implies, in particular, the claim (1.8.11) in the region $|u_i| \geq ct$, for some $c \in (0, 1)$.

Note now that, if $|u_i| \leq ct$, then also $|u_i| \leq cv_i$. Integrating the above display (1.8.50) on a line of constant v_i coordinate in the u_i direction to initial data, we then have, for all multi-indices $J \in I_{\mathbf{K}_R \cup \mathbf{K}_R^{(i)}}^{\leq N_0-2}$:

$$|V^J \Psi(t, r, \theta, \varphi)| \leq C(N, d_\Pi) \varepsilon^{3-\frac{3\delta}{4}} (1+v_i)^{-\frac{1}{2}} R^{\frac{1}{2}+\delta} \quad \text{for } t \in [0, T], \quad i \in \{1, \dots, N\}.$$

Let us now recall the inequality:

$$|\bar{\partial}^{(i)} f| \leq C \frac{R}{1+v_i} \sum_{H \in \mathbf{K}_R^{(i)}} |V^H f|.$$

Using this estimate, we finally have, for all $J \in I_{\mathbf{K}_R}^{\leq N_0-3}$:

$$|\bar{\partial}^{(i)} V^J \Psi(t, r, \theta, \varphi)| \leq C(N, d_{\Pi}) \varepsilon^{3-\frac{3\delta}{4}} (1+t)^{-\frac{3}{2}} R^{\frac{3}{2}+\delta} \quad \text{for } t \in [R^{20}, T],$$

valid for all $i \in \{0, \dots, N\}$.

Upon possibly restricting to a smaller value of ε , we deduce bound (1.8.11). This concludes the proof of the Theorem. $\square$

We now record the calculations that involve using the averaged characteristic energy estimates in controlling the terms in the proof of Theorem 1.4.7.

Lemma 1.8.4. *Let $f, h : \mathbb{R}^{3+1} \rightarrow \mathbb{R}$ be smooth functions, and let $i \in \{1, \dots, N\}$. Let us consider the usual coordinates $(u_i, v_i, \theta_i, \varphi_i)$ and $(t, r_i, \theta_i, \varphi_i)$ (see Definition 1.3.1). Moreover, with $R \geq 10$ and $\alpha, \beta, \gamma, \mu > 0$ parameters, let f satisfy the bulk bound*

$$\int_{\{t \geq 0\}} |\bar{\partial}^{(i)} f|^2 (1 + |u_i|)^{-1-\frac{\delta}{2}} dx ds \leq C_1^2 R^{2\alpha}, \quad (1.8.51)$$

and let h satisfy the pointwise bound

$$|\partial h| \leq \frac{C_2 R^\beta}{(1+v_i)(1+|u_i|)^\gamma} \quad (1.8.52)$$

with $\gamma > \delta$. Here, as usual, we used the notation for $\bar{\partial}^{(i)}$ (the “good derivatives”) introduced in Definition 1.3.6.

Then, the following inequality holds true:

$$\int_{R^\mu}^\infty |||\bar{\partial}^{(i)} f| |\partial h|||_{L^2(\Sigma_t)} dt \leq CC_1 C_2 R^{\alpha+\beta+\mu(\frac{\delta}{2}-2\gamma)}. \quad (1.8.53)$$

Here, C is some constant that does not depend on C_1 , C_2 , R , α , β , γ , μ , f , or h .

Moreover, if we also assume that $\gamma \leq \frac{1}{2} + \frac{\delta}{2}$, we have the following inequality:

$$\left(\int_{R^\mu}^\infty \int_{\Sigma_t} (1+t)^{1+\delta} |\bar{\partial}^{(i)} f|^2 |\partial h|^2 dx dt \right)^{\frac{1}{2}} \leq CC_1 C_2 R^{\alpha+\beta+\mu(\delta-\gamma)}. \quad (1.8.54)$$

Here, again, C is some constant that does not depend on C_1 , C_2 , R , α , β , γ , μ , f , or h .

Proof of Lemma 1.8.4. Let us restrict to the case in which we consider derivatives adapted to the origin (without loss of generality), recalling that we adopt the convention $|\bar{\partial} f| := |\bar{\partial}^{(0)} f|$, and $u := u_0, v := v_0, r := r_0$. Let us first focus on the proof of estimate (1.8.53). We have the following pointwise bound:

$$|\partial h|^2 (1 + |u|)^{1+\frac{\delta}{2}} \leq C_2^2 \frac{R^{2\beta}}{(1+t)^{1+2\gamma-\frac{\delta}{2}}}.$$

This implies that

$$\begin{aligned} \int_{R^\mu}^\infty \left(\int_{\Sigma_t} |\bar{\partial} f|^2 |\partial h|^2 dx \right)^{\frac{1}{2}} dt &= \int_{R^\mu}^\infty \left(\int_{\Sigma_t} |\bar{\partial} f|^2 (1 + |u|)^{-1-\frac{\delta}{2}} (1 + |u|)^{1+\frac{\delta}{2}} |\partial h|^2 dx \right)^{\frac{1}{2}} dt \\ &\leq \int_{R^\mu}^\infty \frac{C_2 R^\beta}{(1+t)^{\frac{1}{2}+\gamma-\frac{\delta}{4}}} \left(\int_{\Sigma_t} |\bar{\partial} f|^2 (1 + |u|)^{-1-\frac{\delta}{2}} dx \right)^{\frac{1}{2}} dt \\ &\leq CC_2 R^\beta \left(\int_{R^\mu}^\infty \frac{1}{(1+t)^{1+2\gamma-\frac{\delta}{2}}} dt \right)^{\frac{1}{2}} \left(\int_{R^\mu}^\infty \int_{\Sigma_t} |\bar{\partial} f|^2 (1 + |u|)^{-1-\frac{\delta}{2}} dx dt \right)^{\frac{1}{2}} \\ &\leq CC_1 C_2 R^{\alpha+\beta+\mu(\frac{\delta}{2}-2\gamma)}, \end{aligned}$$

as desired.

Let us then focus on the proof of bound (1.8.54). We have the following pointwise bound,

under the additional assumption that $\gamma \leq \frac{1}{2} + \frac{\delta}{2}$:

$$|\partial h|^2(1 + |u|)^{1+\delta}(1 + t)^{1+\delta} \leq C_2^2 \frac{R^{2\beta}}{(1 + t)^{2\gamma-2\delta}}.$$

This implies that

$$\begin{aligned} & \left(\int_{R^\mu} \int_{\Sigma_t} (1 + t)^{1+\delta} |\bar{\partial} f|^2 |\partial h|^2 dx dt \right)^{\frac{1}{2}} \\ &= \left(\int_{R^\mu} \int_{\Sigma_t} |\bar{\partial} f|^2 (1 + t)^{1+\delta} (1 + |u|)^{-1-\delta} (1 + |u|)^{1+\delta} |\partial h|^2 dx dt \right)^{\frac{1}{2}} \\ &\leq \frac{C_2 R^\beta}{R^{\mu(\gamma-\delta)}} \left(\int_{R^\mu} \int_{\Sigma_t} |\bar{\partial} f|^2 (1 + |u|)^{-1-\delta} dx dt \right)^{\frac{1}{2}} \\ &\leq CC_1 C_2 R^{\alpha+\beta+\mu(\delta-\gamma)}. \end{aligned}$$

This proves inequality (1.8.54) and concludes the proof of the lemma. $\square$

1.9 Proof of Theorem 1.4.8

Proof of Theorem 1.4.8. Let $L > 0$ be given (this number corresponds to the total energy of the initial data we are going to focus on). Consider $N \in \mathbb{N}$ to be determined later, and let $N_1 \in \mathbb{N}$, $N_1 \geq 13$ (N_1 is the number of derivatives we require on the initial data). Consider moreover a collection of functions $(\bar{\phi}_i^{(0)}, \bar{\phi}_i^{(1)})$, $i \in \{1, \dots, N\}$ which satisfies the following properties:

$$\text{supp}(\bar{\phi}_i^{(0)}) \subset B(0, 1), \quad \text{supp}(\bar{\phi}_i^{(1)}) \subset B(0, 1), \quad (1.9.1)$$

$$\|\bar{\phi}_i^{(0)}\|_{H^{N_1}(B(0,1))} \leq \varepsilon_0, \quad \|\bar{\phi}_i^{(1)}\|_{H^{N_1-1}(B(0,1))} \leq \varepsilon_0, \quad (1.9.2)$$

$$\varepsilon_1 \leq \|\bar{\phi}_i^{(0)}\|_{H^1(B(0,1))}, \quad \varepsilon_1 \leq \|\bar{\phi}_i^{(1)}\|_{L^2(B(0,1))}, \quad \text{for all } i \in \{1, \dots, N\}. \quad (1.9.3)$$

Here, $B(0, 1)$ is the three-dimensional Euclidean ball of radius one centered at the origin.

Moreover, $\varepsilon_0 > 0$ is such that the global stability results of Lemma 1.5.1 hold true for

compactly supported data in the unit ball whose H^{N_1} norms are of size at most $2\varepsilon_0$, with $N_1 \geq 19$ (the number 19 is chosen so that the bootstrap argument for the nonlinear equation in Section 1.8 carries over). Such initial data can easily be seen to exist. Importantly, we note that ε_1 is independent of N .

Let now $N = \lfloor \frac{100L}{\varepsilon_1} \rfloor$. We then take N points p_i on the unit sphere in Σ_0 which are roughly equidistributed. We could take, for example, N points equidistributed on the unit circle $\{x = 0\} \cap \{y^2 + z^2 = 1\}$. In this case, the largest distance between two such points is bounded above by 2, and the smallest pairwise distance between any two distinct such points is bounded below by $2\sin(\pi/N) \geq \frac{\pi}{N}$ (if N is sufficiently large). We note that the ratio between the largest and smallest pairwise distance between the points p_i , which we recall is denoted by d_Π , is a function of N alone.

Now, we scale up by a factor $R > 1$ to be chosen momentarily in terms of N and d_Π . We emphasize that, since N is a function of L and since d_Π is a function of N , the value of R will only depend on L .

For R sufficiently large, we note that the unit balls around each point $w_i := Rp_i$ will be pairwise disjoint. We define the collection of translated functions $(\tilde{\phi}_i^{(0)}, \tilde{\phi}_i^{(1)})$ for $i \in \{1, \dots, N\}$, as follows:

$$\tilde{\phi}_i^{(0)}(x) := \bar{\phi}_i^{(0)}(x - w_i), \quad \tilde{\phi}_i^{(1)}(x) := \bar{\phi}_i^{(1)}(x - w_i), \quad \text{for all } i \in \{1, \dots, N\}. \quad (1.9.4)$$

This implies that both $\tilde{\phi}_i^{(0)}$ and $\tilde{\phi}_i^{(1)}$ are supported in the unit ball centered at w_i , for all $i \in \{1, \dots, N\}$. We also note that, letting

$$\phi^{(0)} := \sum_{i=1}^N \tilde{\phi}_i^{(0)}, \quad \phi^{(1)} := \sum_{i=1}^N \tilde{\phi}_i^{(1)},$$

we have trivially, for R sufficiently large, that $\|\phi^{(0)}\|_{H^1(\Sigma_0)} \geq L$, and that $\|\phi^{(1)}\|_{L^2(\Sigma_0)} \geq L$.

We then wish to show that, for R sufficiently large, there exists a global-in-time solution

to the initial value problem:

$$\begin{aligned}\square\phi + F(d\phi, d^2\phi) &= G(d\phi, d\phi), \\ \phi|_{t=0} &= \phi^{(0)}, \\ \partial_t\phi|_{t=0} &= \phi^{(1)}.\end{aligned}\tag{1.9.5}$$

Now, because of how ε_0 was chosen, we note that the following initial value problem admits a global-in-time solution ϕ_i , for all $i \in \{1, \dots, N\}$:

$$\begin{aligned}\square\phi_i + F(d\phi_i, d^2\phi_i) &= G(d\phi_i, d\phi_i), \\ \phi_i|_{t=0} &= \tilde{\phi}_i^{(0)}, \\ \partial_t\phi_i|_{t=0} &= \tilde{\phi}_i^{(1)},\end{aligned}\tag{1.9.6}$$

Furthermore, every ϕ_i falls off according to the decay rates described in Lemma 1.5.1:

$$|\bar{\partial}^{(i)}V^{J_1}\phi_i| \leq C\varepsilon_0 \frac{1}{(1+r_i^2)(1+|u_i|)^\delta}, \quad |\partial V^{J_1}\phi_i| \leq C\varepsilon_0 \frac{1}{(1+v_i)(1+|u_i|)^{1+\delta}}.$$

Here, $J_1 \in I_{\mathbf{K}_R}^{\leq N_1-7}$.

Thus, with ψ_{ij} defined as in Section 1.7.3, the trilinear estimates in Section 1.7.4 give us that every function ψ_{ij} satisfies the estimates (1.7.12) with ε_0 in place of ε . The energy estimates satisfied by the functions ψ_{ij} are

$$\sup_{t \geq 0} \|\partial V^{J_2}\psi_{ij}\|_{L^2(\Sigma_t)} \leq C(N, d_\Pi) \frac{\varepsilon_0^2}{R},\tag{1.9.7}$$

for all $J_2 \in I_{\mathbf{K}_R}^{\leq N_1-8}$.

Meanwhile, the pointwise estimates satisfied by the functions ψ_{ij} are, for all $J_3 \in I_{\mathbf{K}_R}^{\leq N_1-11}$, and all $(i, j) \in \{1, \dots, N\} \times \{1, \dots, N\}$ such that $i \neq j$:

$$\|\partial V^{J_3}\psi_{ij}\|_{L^\infty(\Sigma_t)} \leq \frac{C(N, d_\Pi)\varepsilon_0^2}{R^{\frac{1}{2}}(1+t)},\tag{1.9.8}$$

$$\|\bar{\partial}^{(i)} V^{J_3} \psi_{ij}\|_{L^\infty(\Sigma_t)} \leq \frac{C(N, d_\Pi) \varepsilon_0^2 R^3}{(1+t)^{\frac{3}{2}}}, \quad (1.9.9)$$

$$|\partial V^{J_3} \psi_{ij}|(t, u_i, \omega) \leq \frac{C(N, d_\Pi) \varepsilon_0^2 R^2}{t(1+|u_i|)^{\frac{1}{2}}}. \quad (1.9.10)$$

We note that, in the above display, neither i nor j can be 0, because the data are compactly supported in the N balls, meaning that the remainder (what we called ϕ_0) arising from the parts of initial data which are located far away from all the centers w_i is not present.

We now repeat the proof of Theorem 1.4.7. This may not be possible unless R is large enough. However, for R large enough, we can use the fact that all the estimates in Section 1.8 (in particular, those concerning terms $(\mathbf{a}_1)$ through $(\mathbf{a}_{16})$) have “room” in the parameter R .

More precisely, we start from the following bootstrap assumptions, valid for all $I \in I_{\mathbf{K}_R}^{\leq N_0}$ and all $J \in I_{\mathbf{K}_R}^{\leq [N_0/2]+1}$, where $N_0 \in \mathbb{N}$, $N_0 \geq 7$, and $i \in \{1, \dots, N\}$:

$$\|\partial V^I \Psi\|_{L^2(\Sigma_t)} \leq \varepsilon_0^{3-\delta} R^{-\frac{3}{2}+\delta} \quad \text{for } t \in [0, T], \quad (1.9.11)$$

$$\|(1+|u_i|)^{-\frac{1}{2}-\frac{\delta}{2}} \bar{\partial}^{(i)} V^I \Psi\|_{L^2([0, T] \times \mathbb{R}^3)} \leq \varepsilon_0^{3-\delta} R^{-\frac{3}{2}+\delta} \quad \text{for } t \in [0, T], \quad (1.9.12)$$

$$|\partial V^J \Psi(t, r, \theta, \varphi)| \leq \varepsilon_0^{3-\delta} (1+t)^{-1} R^{-\frac{1}{2}+\delta} \quad \text{for } t \in [0, T], \quad (1.9.13)$$

$$|\bar{\partial}^{(i)} V^J \Psi(t, r, \theta, \varphi)| \leq \varepsilon_0^{3-\delta} (1+t)^{-\frac{3}{2}} R^{\frac{3}{2}+\delta} \quad \text{for } t \in [R^{20}, T], \quad (1.9.14)$$

$$|\partial V^J \Psi(t, r, \theta, \varphi)| \leq \varepsilon_0^{3-\delta} (1+v_i)^{-1} (1+|u_i|)^{-\frac{1}{2}} R^{\frac{1}{2}+\delta} \quad \text{for } t \in [R^{20}, T]. \quad (1.9.15)$$

These bootstrap assumptions are the same as those in Section 1.8, the only difference being the presence of ε_0 in place of ε . We then seek to improve these bootstrap assumptions, proving the following, for all $I \in I_{\mathbf{K}_R}^{\leq N_0}$ and all $J \in I_{\mathbf{K}_R}^{\leq N_0-3}$, and all $\{i \in 1, \dots, N\}$:

$$\|\partial V^I \Psi\|_{L^2(\Sigma_t)} \leq \varepsilon_0^{3-\delta} R^{-\frac{3}{2}+\frac{3}{4}\delta} \quad \text{for } t \in [0, T], \quad (1.9.16)$$

$$\|(1+|u_i|)^{-\frac{1}{2}-\frac{\delta}{2}} \bar{\partial}^{(i)} V^I \Psi\|_{L^2([0, T] \times \mathbb{R}^3)} \leq \varepsilon_0^{3-\delta} R^{-\frac{3}{2}+\frac{3}{4}\delta} \quad \text{for } t \in [0, T], \quad (1.9.17)$$

$$|\partial V^J \Psi(t, r, \theta, \varphi)| \leq \varepsilon_0^{3-\delta} (1+t)^{-1} R^{-\frac{1}{2}+\frac{7}{8}\delta} \quad \text{for } t \in [0, T], \quad (1.9.18)$$

$$|\bar{\partial}^{(i)} V^J \Psi(t, r, \theta, \varphi)| \leq \varepsilon_0^{3-\delta} (1+t)^{-\frac{3}{2}} R^{\frac{3}{2}+\frac{7}{8}\delta} \quad \text{for } t \in [R^{20}, T], \quad (1.9.19)$$

$$|\partial V^J \Psi(t, r, \theta, \varphi)| \leq \varepsilon_0^{3-\delta} (1+v_i)^{-1} (1+|u_i|)^{-\frac{1}{2}} R^{\frac{1}{2}+\frac{7}{8}\delta} \quad \text{for } t \in [R^{20}, T]. \quad (1.9.20)$$

Looking at how the terms in (1.8.30) were controlled, we note that they all have “room” in the parameter R (by a positive power of R). Indeed, the inequalities in (1.8.31), (1.8.32), (1.8.33), (1.8.34), (1.8.35), (1.8.36), (1.8.37), (1.8.38), (1.8.39), and (1.8.42) still hold with ε_0 instead of ε . In those inequalities, we now use that we can absorb the constant $C(N, d_\Pi)$ by negative powers of R (instead of using positive powers of ε_0). The worst case bound in such inequalities is therefore replaced by

$$C(N, d_\Pi) \varepsilon_0^{6-2\delta} R^{-3+\delta} \log(R) \leq \varepsilon_0^{6-2\delta} R^{-3+\frac{3\delta}{2}}, \quad (1.9.21)$$

for R sufficiently large. This recovers the improved bootstrap assumption (1.9.16) for R sufficiently large, and concludes the proof of Theorem 1.4.8. $\square$

1.10 Technical lemmas

We record the following technical lemmas that were needed in the course of our argument.

1.10.1 Trace lemma on Σ_t

Lemma 1.10.1. *There exists a positive constant C such that the following holds. Let $f : \mathbb{R}^3 \rightarrow \mathbb{R}$ be a smooth function that decays sufficiently rapidly at infinity. We take polar coordinates (r, ω) with $\omega \in \mathbb{S}^2$. Then, we have that*

$$\|f\|_{L^2(S_r)} \leq C \|f\|_{L^2(\mathbb{R}^3)} + C \|\partial f\|_{L^2(\mathbb{R}^3)}, \quad (1.10.1)$$

where S_r is the sphere of radius r , with $r \geq 1$.

Proof of Lemma 1.10.1. We integrate in the r direction using the fundamental theorem of

calculus. We have that

$$h^2(r, \omega) = \int_r^\infty 2h(s, \omega) \partial_r h(s, \omega) ds. \quad (1.10.2)$$

Integrating the previous display over $\mathbb{S}^2$ gives us that

$$\int_{\mathbb{S}^2} h^2(r, \omega) d\omega = \int_r^\infty 2 \int_{\mathbb{S}^2} h(s, \omega) \partial_r h(s, \omega) d\omega ds. \quad (1.10.3)$$

Using the Cauchy–Schwarz inequality, we obtain

$$\int_{\mathbb{S}^2} h^2(r, \omega) d\omega \leq 2 \left(\int_r^\infty \int_{\mathbb{S}^2} h^2(s, \omega) d\omega ds \right)^{\frac{1}{2}} \left(\int_r^\infty \int_{\mathbb{S}^2} (\partial_r h)^2(s, \omega) d\omega ds \right)^{\frac{1}{2}}. \quad (1.10.4)$$

Applying this to the function $h(r, \omega) = rf(r, \omega)$ gives us the desired result. $\square$

1.10.2 Trace lemma on H_t

Lemma 1.10.2. *Recall the coordinates $(\tau, \alpha, x, \varphi)$ introduced in display (1.3.6). Recall moreover the definition of the hyperboloids H_τ (from Definition 1.3.4). There exists a positive constant C such that the following holds. Let $\bar{x} \in \mathbb{R}$. Moreover, let $f : H_\tau \rightarrow \mathbb{R}$ be a smooth, compactly supported function. Then, we have that*

$$\|f\|_{L^2(H_\tau \cap \{x=\bar{x}\})}^2 \leq C \|f\|_{L^2(H_\tau)}^2 + C \|\partial_x f\|_{L^2(H_\tau)}^2. \quad (1.10.5)$$

Here, the L^2 spaces are defined with respect to the induced volume form on the submanifolds considered.

Proof of Lemma 1.10.2. We have that

$$f^2(\tau, \alpha, \bar{x}, \varphi) \leq 2 \int_{\bar{x}}^\infty |f(\tau, \alpha, x, \varphi)| |\partial_x f(\tau, \alpha, x, \varphi)| dx. \quad (1.10.6)$$

Integrating along the set $H_\tau \cap \{x = \bar{x}\}$ (i.e., integrating in the α and φ variables), and using

the Cauchy–Schwarz inequality implies that

$$\|f\|_{L^2(H_\tau \cap \{x=\bar{x}\})}^2 \leq C\|f\|_{L^2(H_\tau)}^2 + C\|\partial_x f\|_{L^2(H_\tau)}^2, \quad (1.10.7)$$

as desired. $\square$

1.10.3 Energy estimates and the hyperboloidal foliation

We recall the stress–energy–momentum tensor associated to the wave equation on Minkowski space (here, m denotes the Minkowski metric):

$$Q_{\mu\nu}[\phi] = \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} m_{\mu\nu} \partial^\gamma \phi \partial_\gamma \phi. \quad (1.10.8)$$

Let now $D \subset \mathbb{R}^4$ be a bounded, open domain with piecewise smooth boundary ∂D , such that every smooth piece of ∂D is spacelike. In particular, this implies that the Lorentzian unit outer normal to ∂D , denoted by $N_{\partial D}$, is well defined. Let now X be a smooth vector field. Using the fact that $\nabla_\mu Q^{\mu\nu} = (\square\phi)\partial^\nu \phi$ and the divergence theorem, we now have

$$\int_D (\square\phi) X \phi \, dxdt + \int_D (\nabla_\mu X_\nu) Q^{\mu\nu} \, dxdt = \int_{\partial D} Q(X, N_{\partial D}) \, d\sigma(\partial D). \quad (1.10.9)$$

Here, $d\sigma(\partial D)$ is the volume form associated to the induced Riemannian metric on the boundary ∂D . When X is a Killing field, the second term in the previous display vanishes, and we are left with

$$\int_D (\square\phi) X \phi \, dxdt = \int_{\partial D} Q(X, N_{\partial D}) \, d\sigma(\partial D). \quad (1.10.10)$$

When $X = T = \partial_t$, we obtain the usual ∂_t energy.

We require the following fact on the ∂_t energy flux through the hyperboloidal foliation H_τ .

Lemma 1.10.3. *Recall the coordinates $(\tau, \alpha, x, \varphi)$ introduced in display (1.3.6). Recall moreover the definition of the hyperboloids H_τ (from Definition 1.3.4). There exists a positive constant C such that the following inequality holds true:*

$$\int_{H_\tau \cap \{\rho \leq t/10\}} Q(X, N_{H_\tau}) d\sigma(H_\tau) \geq C \|\partial\phi\|_{L^2(H_\tau \cap \{\rho \leq t/10\})}^2. \quad (1.10.11)$$

Here, $d\sigma(H_\tau)$ is the volume form associated to the induced Riemannian metric on H_τ , and the L^2 norm on the RHS is defined with respect to the induced volume form (pullback of the ambient volume form) on the hypersurface H_τ .

Sketch of proof. The proof follows in a straightforward manner from the fact that the hypersurface $H_\tau \cap \{\rho \leq t/10\}$ is uniformly spacelike, and from expanding the stress–energy–momentum tensor Q in components. □

Chapter 2

Compressible and viscous fluids

In this chapter, we are going to direct our attention to compressible and viscous models arising in fluid mechanics. We will first present several results concerning a one dimensional compressible and viscous model, obtained in a joint paper with Constantin, Drivas and Nguyen [23]. This is the content of Section 2.1.

We will then focus on two-dimensional models, in particular on the shallow water equations. We will first review some of the literature on the shallow water equations, with an emphasis on the case of degenerate viscosity coefficient (Section 2.2). We will then present (with proofs) a number of classical results on local and global existence in Section 2.3, both in the inviscid and in the viscous case.

Finally, we will direct our attention to the issue of uniform bounds in two space dimensions for the compressible NS system with non-degenerate viscosity. We will show, in Section 2.4, a result on uniform bounds for such system in axial symmetry on a two-dimensional annulus.

2.1 Results on one dimensional compressible and viscous NS

In this section, we are going to focus on one-dimensional models of the form (0.2.9)–(0.2.11). The main theorems are stated in Section 2.1.1. We first prove standard *a priori* estimates in Section 2.1.2. We will then introduce the “active potential” in Section 2.1.3, and there we show that it satisfies a heat equation with *non-degenerate viscosity*. We will then prove our main theorems in sections 2.1.4–2.1.7, and finally Section 2.1.8 deals with the proof of a local existence statement and the conservation of the Bresch–Desjardins entropy.

2.1.1 Precise statement of the results in the one-dimensional case

Let us recall the precise form of the system under study:

$$\partial_t \rho + \partial_x(u\rho) = 0, \tag{2.1.1}$$

$$\partial_t(\rho u) + \partial_x(\rho u^2) = -\partial_x p(\rho) + \partial_x(\mu(\rho)\partial_x u) + \rho f, \tag{2.1.2}$$

$$(\rho, u)|_{t=0} = (\rho_0, u_0). \tag{2.1.3}$$

The constitutive laws for pressure and viscosity are given by:

$$p(\rho) = c_p \rho^\gamma, \quad \mu(\rho) = c_\mu \rho^\alpha, \quad c_p \neq 0, \quad c_\mu > 0. \tag{2.1.4}$$

We are going to prove four theorems in this part of the thesis. The first result, Theorem 2.1.1, provides a blowup criterion for equations (2.1.1)–(2.1.3) with a wide range of constitutive pressure and viscosity laws (2.1.4). In what follows, we denote by $\mathbb{T}$ the interval $(0, 1]$ with periodic boundary conditions.

Theorem 2.1.1. *Assume any of the following three conditions*

$$(i) \quad c_p > 0 \text{ and } \alpha > \frac{1}{2}, \quad \gamma \neq 1, \quad \gamma \geq \alpha - \frac{1}{2},$$

(ii) $c_p < 0$ and $\frac{1}{2} < \alpha \leq \frac{3}{2}$, $\gamma < 1$, $0 < \gamma \leq \alpha$,

(iii) $c_p > 0$ and $\gamma > 1$, $\alpha \geq 0$.

Let $k \geq 3$ and assume further that

$$f \in L^2(0, T; H^{k-1}(\mathbb{T})) \quad \text{for all } T > 0.$$

If (ρ, u) is a solution of (2.1.1)-(2.1.3) on $[0, T^*)$ such that

$$\rho \in \mathcal{C}(0, T; H^k(\mathbb{T})), \quad u \in \mathcal{C}(0, T; H^k(\mathbb{T})) \cap L^2(0, T; H^{k+1}(\mathbb{T})), \quad \forall T \in (0, T^*) \quad (2.1.5)$$

and

$$\inf_{t \in [0, T^*)} \min_{x \in \mathbb{T}} \rho(x, t) > 0,$$

then (ρ, u) satisfies

$$\sup_{T \in [0, T^*)} \|\rho\|_{L^\infty(0, T; H^k)} + \sup_{T \in [0, T^*)} \|u\|_{L^\infty(0, T; H^k)} + \sup_{T \in [0, T^*)} \|u\|_{L^2(0, T; H^{k+1})} < \infty \quad (2.1.6)$$

and can be continued in the class (2.1.5) past T^* .

Theorem 2.1.1 says that the only possible way for a singularity to form starting from smooth data is if the density becomes zero. This applies in particular to the viscous shallow water wave equations (0.2.13)-(0.2.14). In the slender jet equations (0.2.15)-(0.2.16) which model incompressible fluid drop formation, this says that singularities can only form at the onset of drop break-off. This answers a conjecture of Peter Constantin recorded in [34].

Our next two theorems concern the long-time existence and persistence of regularity. Theorem 2.1.2 establishes global existence for arbitrarily large data, within a range of pressure and viscosity of the form (2.1.4).

Theorem 2.1.2. *Assume*

$$c_p > 0, \quad \alpha \in \left(\frac{1}{2}, 1\right], \quad \text{and} \quad \gamma \geq 2\alpha.$$

Let $k \geq 3$ be an integer and let ρ_0 and u_0 belong to $H^k(\mathbb{T})$ such that $\rho_0(x) > 0$ for all $x \in \mathbb{T}$.

Assume further that

$$f \in L^2(0, T; H^{k-1}(\mathbb{T})) \quad \text{for all } T > 0.$$

Then there exists a unique global solution (ρ, u) to (2.1.1)-(2.1.3) such that

$$\rho \in \mathcal{C}(0, T; H^k(\mathbb{T})), \quad u \in \mathcal{C}(0, T; H^k(\mathbb{T})) \cap L^2(0, T; H^{k+1}(\mathbb{T}))$$

for all $T > 0$, and $\rho(x, t) > 0$ for all $(x, t) \in \mathbb{T} \times \mathbb{R}^+$.

This result applies to the viscous shallow water equations (0.2.13)-(0.2.14), giving an alternative proof to that of [44]. Let us note that [44] assumes only H^1 regularity of initial data.

For more degenerate viscosity $\mu(\rho) = c_\mu \rho^\alpha$ with $\alpha > 1$, we prove global existence for a class of large initial data.

Theorem 2.1.3. *Assume that $c_p > 0$ and either*

$$\alpha > \frac{1}{2}, \quad \gamma \in [\alpha, \alpha + 1], \quad \gamma \neq 1 \quad \text{or} \tag{2.1.7}$$

$$\alpha \geq 0, \quad \gamma \in [\alpha, \alpha + 1], \quad \gamma > 1. \tag{2.1.8}$$

Assume further that

$$f(x, t) = f(t) \in L^2((0, T)) \quad \forall T > 0.$$

Let $k \geq 4$ be an integer and let u_0 and ρ_0 belong to $H^k(\mathbb{T})$ such that $\rho_0(x) > 0$ for all $x \in \mathbb{T}$

and

$$\partial_x u_0(x) \leq \frac{c_p}{c_\mu} \rho_0(x)^{\gamma-\alpha} \quad \forall x \in \mathbb{T}. \quad (2.1.9)$$

Then there exists a unique global solution (ρ, u) to (2.1.1)-(2.1.3) such that

$$\rho \in \mathcal{C}(0, T; H^k(\mathbb{T})), \quad u \in \mathcal{C}(0, T; H^k(\mathbb{T})) \cap L^2(0, T; H^{k+1}(\mathbb{T}))$$

for all $T > 0$, and $\rho(x, t) > 0$ for all $(x, t) \in \mathbb{T} \times \mathbb{R}^+$.

Remark 2.1.4. We note that (2.1.9) does not impose any smallness conditions on the initial data. The unique global solution in Theorem 2.1.2 satisfies

$$\partial_x u(x, t) \leq \frac{c_p}{c_\mu} \rho(x, t)^{\gamma-\alpha}$$

for all $(x, t) \in \mathbb{T} \times \mathbb{R}^+$. Moreover, the proof provides a lower bound for the minimum of density ρ , see (2.1.94) and (2.1.97),

$$\min_{x \in \mathbb{T}} \rho(x, t) \geq \begin{cases} \left(\rho_m(0)^{\alpha-\gamma} + t \frac{c_p}{c_\mu} (\gamma - \alpha) \right)^{\frac{-1}{\gamma-\alpha}} & \text{when } \gamma > \alpha, \\ \rho_m(0) \exp \left(-t \frac{c_p}{c_\mu} \right) & \text{when } \gamma = \alpha. \end{cases}$$

Our last theorem establishes a bound on the time-averaged maximum density for a certain range of parameters assuming mean zero forcing.

Theorem 2.1.5. *Assume that (ρ, u) is a sufficiently smooth solution to the system (2.1.1)–(2.1.3) on $[0, T^*)$. Assume that*

$$f = \partial_x g \quad (2.1.10)$$

for some periodic function g satisfying

$$g \in L^\infty(0, T^*; L^\infty(\mathbb{T})), \quad \text{and} \quad \partial_x g, \partial_t g \in L^\infty(0, T^*; L^\infty(\mathbb{T})).$$

Let us also assume that

$$\alpha \geq 1/2, \quad \gamma \in [\max\{2 - \alpha, \alpha\}, \alpha + 1], \quad \text{and} \quad c_p, c_\mu > 0.$$

Then, we have the following bound

$$\frac{1}{T} \int_0^T \|\rho(\cdot, t)\|_{L^\infty(\mathbb{T})} dt \leq C_1 + \frac{1}{T} C_2, \quad (2.1.11)$$

where C_1 and C_2 are defined in equation (2.1.103). In particular, C_1 depends only on c_μ , c_p , α , γ , $\|\rho_0\|_{L^1}$, $\|\partial_x g\|_{L^\infty(0,T;L^\infty)}$, and $\|\partial_t g\|_{L^\infty(0,T;L^\infty)}$, whereas C_2 depends only on c_μ , c_p , γ , α , $\|\rho_0\|_{L^\infty}$, $\|\rho_0^{-1}\|_{L^\infty}$, $\|u_0\|_{L^2}$, $\|\partial_x \rho_0\|_{L^2}$, and $\|g\|_{L^\infty(0,T;L^\infty)}$. Consequently, if $T^* = \infty$ then

$$\limsup_{T \rightarrow \infty} \frac{1}{T} \int_0^T \|\rho(\cdot, t)\|_{L^\infty(\mathbb{T})} dt \leq C_3 \quad (2.1.12)$$

where C_3 depends only on c_μ , c_p , α , γ , $\|\rho_0\|_{L^1}$, $\|\partial_x g\|_{L^\infty(0,\infty;L^\infty)}$, and $\|\partial_t g\|_{L^\infty(0,\infty;L^\infty)}$.

Theorem 2.1.5 applies for the viscous shallow water wave system (0.2.13),(0.2.14) for which global existence is established by Theorem 2.1.2. The interpretation of the bound (2.1.12) with $h \equiv \rho$ is that long-time average of the maximum surface height remains bounded, showing that, on average, no extreme events can develop.

2.1.2 A priori estimates: mass, energy and Bresch–Desjardins's entropy

Assume that (ρ, u) is a solution of (2.1.1)-(2.1.3) on the time interval $[0, T^*)$ such that

$$\rho \in \mathcal{C}(0, T; H^3), \quad u \in \mathcal{C}(0, T; H^3) \cap L^2(0, T; H^4)$$

for any $T < T^*$ and

$$\underline{\rho} := \inf_{t \in [0, T^*)} \min_{x \in \mathbb{T}} \rho(x, t) > 0. \quad (2.1.13)$$

In what follows we denote by $M(\cdot, \dots, \cdot)$ a positive function that is increasing in each argument.

First, from the continuity equation (2.1.1), total mass is conserved:

$$\|\rho(\cdot, t)\|_{L^1(\mathbb{T})} = \|\rho_0\|_{L^1(\mathbb{T})}. \quad (2.1.14)$$

We have the following standard energy balance:

Lemma 2.1.6 (Energy Balance). *Let $\bar{\rho} \geq 0$, and*

$$e := \frac{1}{2}\rho|u|^2 + \pi(\rho), \quad \pi(\rho) = \rho \int_{\bar{\rho}}^{\rho} \frac{p(s)}{s^2} ds. \quad (2.1.15)$$

Then, the balance

$$\frac{d}{dt} \int_{\mathbb{T}} e(x, t) dx = - \int_{\mathbb{T}} \mu(\rho) |\partial_x u|^2 dx + \int_{\mathbb{T}} f \rho u dx \quad (2.1.16)$$

holds for any $t \in [0, T^*)$.

Using the equation of state for the density (2.1.4) and recalling that $\bar{\rho} \geq 0$ is an arbitrary constant that we are free to fix, we have an explicit formula for $\pi(\rho)$ from (2.1.15)

$$\pi(\rho) = c_p \rho \int_{\bar{\rho}}^{\rho} s^{\gamma-2} ds = \begin{cases} \frac{c_p}{\gamma-1} \rho^{\gamma} & \gamma > 1, \bar{\rho} = 0 \quad \text{or} \quad \gamma \in (0, 1), \bar{\rho} = \infty, \\ c_p \rho \log(\rho) & \gamma = 1, \bar{\rho} = 1. \end{cases} \quad (2.1.17)$$

Note that the function π satisfies

$$\pi''(\rho) = \frac{p'(\rho)}{\rho}.$$

Lemma 2.1.7. *The following two assertions hold true.*

1. If $\gamma \in (1, \infty)$ and $c_p > 0$, then $\pi(\rho) \geq 0$ and

$$\begin{aligned} & \|e\|_{L^\infty(0,T;L^1)} + \|\mu(\rho)|\partial_x \rho|^2\|_{L^1(0,T;L^1)} \\ & \leq \left(\|e(\cdot, 0)\|_{L^1} + \|f\|_{L^2(0,T;L^\infty)}^2 \|\rho_0\|_{L^1(\mathbb{T})} \right) \exp(2T). \end{aligned} \quad (2.1.18)$$

2. If $\gamma \in (0, 1)$ and $c_p \neq 0$, then

$$\int_{\mathbb{T}} |\pi(\rho)| dx \leq \left| \frac{c_p}{\gamma - 1} \right| \int (\rho_0 + 1) dx \quad (2.1.19)$$

and there exists a positive constant $C = C(\gamma, \alpha, c_p, c_\mu)$ such that

$$\begin{aligned} & \|\rho u^2\|_{L^\infty(0,T;L^1)} + \|\mu(\rho)|\partial_x \rho|^2\|_{L^1(0,T;L^1)} \\ & \leq \left(\|\rho_0 u_0^2\|_{L^1(\mathbb{T})} + C(1 + \|f\|_{L^2(0,T;L^\infty)}^2)(1 + \|\rho_0\|_{L^1(\mathbb{T})}) \right) \exp(T). \end{aligned} \quad (2.1.20)$$

Proof. First, using the mass conservation (2.1.14) we bound

$$\begin{aligned} \int_{\mathbb{T}} f \rho u dx & \leq \frac{1}{2} \int_{\mathbb{T}} f^2 \rho + \int_{\mathbb{T}} \frac{1}{2} \rho u^2 \\ & \leq \|f\|_{L^\infty(\mathbb{T})}^2 \int_{\mathbb{T}} \rho + \int_{\mathbb{T}} \frac{1}{2} \rho u^2 \\ & \leq \|f\|_{L^\infty(\mathbb{T})}^2 \|\rho_0\|_{L^1(\mathbb{T})} + \int_{\mathbb{T}} \frac{1}{2} \rho u^2. \end{aligned} \quad (2.1.21)$$

Step 1. If $\gamma \in (1, \infty)$ and $c_p > 0$, then we have $\pi(\rho) \geq 0$. It then follows from (2.1.21) that

$$\int_{\mathbb{T}} f \rho u dx \leq \|f\|_{L^\infty(\mathbb{T})}^2 \|\rho_0\|_{L^1(\mathbb{T})} + \int_{\mathbb{T}} e(x, t) dx. \quad (2.1.22)$$

Ignoring the first term on the right hand side of (2.1.16), then using (2.1.22) and Grönwall's lemma we obtain

$$\|e\|_{L^\infty(0,T;L^1)} \leq \left(\|e(\cdot, 0)\|_{L^1} + \|f\|_{L^2(0,T;L^\infty)}^2 \|\rho_0\|_{L^1(\mathbb{T})} \right) \exp(T). \quad (2.1.23)$$

Next, we integrate (2.1.16) in time and use (2.1.22), (2.1.23) together with the fact that $e(x, t) \geq 0$ to get

$$\begin{aligned}
\|\mu(\rho)|\partial_x \rho|^2\|_{L^1(0,T;L^1)} &\leq \|e(\cdot, 0)\|_{L^1} + \|f\|_{L^2(0,T;L^\infty)}^2 \|\rho_0\|_{L^1(\mathbb{T})} + T\|e\|_{L^\infty(0,T;L^1)} \\
&\leq \left(\|e(\cdot, 0)\|_{L^1} + \|f\|_{L^2(0,T;L^\infty)}^2 \|\rho_0\|_{L^1(\mathbb{T})} \right) (1 + T) \exp(T) \\
&\leq \left(\|e(\cdot, 0)\|_{L^1} + \|f\|_{L^2(0,T;L^\infty)}^2 \|\rho_0\|_{L^1(\mathbb{T})} \right) \exp(2T).
\end{aligned}$$

Step 2. If $\gamma \in (0, 1)$ then

$$\int_{\mathbb{T}} |\pi(\rho)| dx \leq \left| \frac{c_p}{\gamma - 1} \right| \int (\rho(t) + 1) dx \leq \left| \frac{c_p}{\gamma - 1} \right| \int (\rho_0 + 1) dx \quad (2.1.24)$$

where we used the fact that $\rho^\gamma \leq \max\{1, \rho\}$ together with the mass conservation (2.1.1). Ignoring the first term on the right hand side of (2.1.16) and using (2.1.24), (2.1.21) we find

$$\begin{aligned}
&\int_{\mathbb{T}} \frac{1}{2} \rho u^2(x, t) dx \\
&\leq \int_{\mathbb{T}} \frac{1}{2} \rho_0 u_0^2 dx + \int_{\mathbb{T}} \pi(\rho_0(x)) dx - \int_{\mathbb{T}} \pi(\rho(x, t)) dx + \int_0^t \int_{\mathbb{T}} f \rho u(x, s) dx ds \\
&\leq \int_{\mathbb{T}} \frac{1}{2} \rho_0 u_0^2 dx + C(\|\rho_0\|_{L^1(\mathbb{T})} + 1) + \|f(t)\|_{L^\infty(\mathbb{T})}^2 \|\rho_0\|_{L^1(\mathbb{T})} + \int_0^t \int_{\mathbb{T}} \frac{1}{2} \rho u^2(x, s) dx ds
\end{aligned}$$

for some positive constant $C = C(\gamma, \alpha, c_p, c_\mu)$. Grönwall's lemma then yields

$$\|\rho u^2\|_{L^\infty(0,T;L^1)} \leq \left(\|\rho_0 u_0^2\|_{L^1(\mathbb{T})} + C(1 + \|f\|_{L^2(0,T;L^\infty)}^2)(1 + \|\rho_0\|_{L^1(\mathbb{T})}) \right) \exp(T). \quad (2.1.25)$$

Again, we integrate (2.1.16) in time and use (2.1.21), (2.1.25), (2.1.24) to arrive at

$$\begin{aligned}
&\|\mu(\rho)|\partial_x \rho|^2\|_{L^1(0,T;L^1)} \\
&\leq \left(\|\rho_0 u_0^2\|_{L^1(\mathbb{T})} + C(1 + \|f\|_{L^2(0,T;L^\infty)}^2)(1 + \|\rho_0\|_{L^1(\mathbb{T})}) \right) \exp(2T).
\end{aligned}$$

□

If either $\gamma \in (1, \infty)$ and $c_p > 0$ or $\gamma \in (0, 1)$ and $c_p \neq 0$, it follows from (2.1.17)-(2.1.20) that

$$\|\sqrt{\rho}u\|_{L^\infty(0,T;L^2)} \leq M(E_0, \|f\|_{L^2(0,T;L^\infty)}, T), \quad (2.1.26)$$

$$\|\rho^{\frac{\alpha}{2}} \partial_x u\|_{L^2(0,T;L^2)} \leq M(E_0, \|f\|_{L^2(0,T;L^\infty)}, T), \quad (2.1.27)$$

$$\|\rho\|_{L^\infty(0,T;L^{\max\{1,\gamma\}})} \leq M(E_0, \|f\|_{L^2(0,T;L^\infty)}, T) \quad (2.1.28)$$

where

$$E_0 := \|\rho_0 u_0^2\|_{L^1(\mathbb{T})} + \|\rho_0 \tilde{\gamma}\|_{L^1(\mathbb{T})} + \|\rho_0\|_{L^1(\mathbb{T})}. \quad (2.1.29)$$

Lemma 2.1.8 (Bresch–Desjardins’s Entropy [12]). *Let*

$$s := \frac{\rho}{2} \left| u + \frac{\partial_x \rho}{\rho^2} \mu(\rho) \right|^2 + \pi(\rho). \quad (2.1.30)$$

Then, the balance

$$\frac{d}{dt} \int_{\mathbb{T}} s(x, t) dx = - \int_{\mathbb{T}} |\partial_x \rho|^2 \mu(\rho) \frac{p'(\rho)}{\rho^2} dx + \int_{\mathbb{T}} f \rho \left(u + \frac{\partial_x \rho}{\rho^2} \mu(\rho) \right) dx \quad (2.1.31)$$

holds for any $t \in [0, T^*)$.

A proof of Lemma 2.1.8 can be found in [12, 14, 15] and is given for completeness in Section 2.1.8. The first term on the right hand side of (2.1.31) is negative whenever $c_p > 0$ and positive whenever $c_p < 0$.

Lemma 2.1.9. *Define*

$$E_1 := E_0 + \|\partial_x(\rho_0^{\alpha-\frac{1}{2}})\|_{L^2(\mathbb{T})}. \quad (2.1.32)$$

1. *If $c_p > 0$ and $\gamma \neq 1$, $\gamma \geq \alpha - \frac{1}{2}$, $\alpha > \frac{1}{2}$, then*

$$\|\rho\|_{L^\infty(0,T;L^\infty)} \leq M(E_1, \|f\|_{L^2(0,T;L^\infty)}, T). \quad (2.1.33)$$

2. If $c_p < 0$ and $0 < \gamma \leq \alpha$, $\gamma < 1$, $\alpha \in (\frac{1}{2}, \frac{3}{2}]$, then

$$\|\rho\|_{L^\infty(0,T;L^\infty)} \leq M(E_1, \|f\|_{L^2(0,T;L^\infty)}, \frac{1}{\underline{\rho}}, T). \quad (2.1.34)$$

3. Under the conditions of 1. or 2., we have

$$\|\partial_x \rho\|_{L^\infty(0,T;L^2)} \leq M(E_1, \|f\|_{L^2(0,T;L^\infty)}, \frac{1}{\underline{\rho}}, T). \quad (2.1.35)$$

4. If $c_p > 0$, $\gamma > 1$ and $\alpha \geq 0$ then (2.1.34) and (2.1.35) hold.

Remark 2.1.10. The bound for (2.1.33) is independent of $\underline{\rho}$. This fact will be important in the proof of Theorem 2.1.2.

Proof. Step 1. Since $c_p > 0$, the first term on the right hand side of (2.1.31) is negative, and thus

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{T}} s(x, t) dx &\leq \int_{\mathbb{T}} f \rho \left(u + \frac{\partial_x \rho}{\rho^2} \mu(\rho) \right) dx \\ &\leq \frac{1}{2} \int_{\mathbb{T}} f^2 \rho dx + \int_{\mathbb{T}} \frac{1}{2} \rho \left(u + \frac{\partial_x \rho}{\rho^2} \mu(\rho) \right)^2 dx \\ &\leq \frac{1}{2} \|f(t)\|_{L^\infty(\mathbb{T})}^2 \|\rho_0\|_{L^1(\mathbb{T})} + \int_{\mathbb{T}} \frac{1}{2} \rho \left(u + \frac{\partial_x \rho}{\rho^2} \mu(\rho) \right)^2 dx. \end{aligned} \quad (2.1.36)$$

When $\gamma > 1$ we have $\pi(\rho) \geq 0$, hence $s > 0$ and

$$\frac{d}{dt} \int_{\mathbb{T}} s(x, t) dx \leq \frac{1}{2} \|f(t)\|_{L^\infty(\mathbb{T})}^2 \|\rho_0\|_{L^1(\mathbb{T})} + \int_{\mathbb{T}} s(x, t) dx.$$

Grönwall's lemma then yields

$$\|s\|_{L^\infty(0,T;L^1)} \leq \left(\|s(0, \cdot)\|_{L^1(\mathbb{T})} + \|f\|_{L^2(0,T;L^\infty)}^2 \|\rho_0\|_{L^1(\mathbb{T})} \right) \exp(T). \quad (2.1.37)$$

We combine (2.1.37) with (2.1.26) and the fact that

$$\|s(0, \cdot)\|_{L^1(\mathbb{T})} \leq \|\rho_0 u_0^2\|_{L^1(\mathbb{T})} + \|\partial_x(\rho_0^{\alpha-\frac{1}{2}})\|_{L^2(\mathbb{T})}^2. \quad (2.1.38)$$

In view of (2.1.27), this implies

$$\|\partial_x(\rho^{\alpha-\frac{1}{2}})\|_{L^\infty(0,T;L^2(\mathbb{T}))} \leq M(E_1, \|f\|_{L^2(0,T;L^\infty)}, T) \quad (2.1.39)$$

with

$$E_1 = E_0 + \|\partial_x(\rho_0^{\alpha-\frac{1}{2}})\|_{L^2(\mathbb{T})}.$$

On the other hand, when $\gamma \in (0, 1)$ we write

$$\begin{aligned} & \frac{d}{dt} \int_{\mathbb{T}} \frac{1}{2} \rho \left(u + \frac{\partial_x \rho}{\rho^2} \mu(\rho) \right)^2 dx \\ & \leq \frac{d}{dt} \int_{\mathbb{T}} \pi(\rho(x, t)) dx + \frac{1}{2} \|f(t)\|_{L^\infty(\mathbb{T})}^2 \|\rho_0\|_{L^1(\mathbb{T})} + \int_{\mathbb{T}} \frac{1}{2} \rho \left(u + \frac{\partial_x \rho}{\rho^2} \mu(\rho) \right)^2 dx \end{aligned}$$

where we recall from (2.1.19)

$$\int_{\mathbb{T}} |\pi(\rho)| dx \leq \left| \frac{c_p}{\gamma - 1} \right| \int (\rho_0 + 1) dx. \quad (2.1.40)$$

It follows from Grönwall's lemma that

$$\begin{aligned} & \sup_{t \in [0, T]} \int_{\mathbb{T}} \frac{1}{2} \rho \left(u + \frac{\partial_x \rho}{\rho^2} \mu(\rho) \right)^2(x, t) dx \\ & \leq \left(\int_{\mathbb{T}} \frac{1}{2} \rho \left(u + \frac{\partial_x \rho}{\rho^2} \mu(\rho) \right)^2(x, 0) dx + C(1 + \|f\|_{L^2(0, T; L^\infty)}^2) (1 + \|\rho_0\|_{L^1(\mathbb{T})}) \right) \exp(T) \\ & \leq M(E_1, \|f\|_{L^2(0, T; L^\infty)}, T). \end{aligned}$$

Combined with (2.1.26), this implies the bound (2.1.39) when $\gamma \in (0, 1)$.

Next, we recall from (2.1.28) the bound for $\|\rho^\gamma\|_{L^1(\mathbb{T})}$. By the assumption that $\gamma \geq \alpha - \frac{1}{2}$,

we obtain

$$\|\rho^{\alpha-\frac{1}{2}}\|_{L^\infty(0,T;L^1)} \leq C(1 + \|\rho^\gamma\|_{L^\infty(0,T;L^1)} + \|\rho\|_{L^\infty(0,T;L^1)}) \leq M(E_0, \|f\|_{L^2(0,T;L^\infty)}, T).$$

This combined with (2.1.39) and Nash's inequality

$$\|\rho^{\alpha-\frac{1}{2}}\|_{L^\infty(0,T;L^2)} \leq C\|\rho^{\alpha-\frac{1}{2}}\|_{L^\infty(0,T;L^1)}^{2/3} \|\partial_x(\rho^{\alpha-\frac{1}{2}})\|_{L^\infty(0,T;L^2)}^{1/3} + C\|\rho^{\alpha-\frac{1}{2}}\|_{L^\infty(0,T;L^1)}$$

leads to

$$\|\rho^{\alpha-\frac{1}{2}}\|_{L^\infty(0,T;H^1)} \leq M(E_1, \|f\|_{L^2(0,T;L^\infty)}, T).$$

The stated bound (2.1.33) then follows by Sobolev embedding $H^1 \subseteq L^\infty$.

Step 2. In this case, $c_p < 0$ and thus the first term on the right hand side of (2.1.31) is positive and is equal to

$$\begin{aligned} -\gamma c_p c_\mu \int_{\mathbb{T}} |\rho^{(\gamma+\alpha-3)/2} \partial_x \rho|^2 dx &\leq -2\gamma \frac{c_p}{c_\mu} \int_{\mathbb{T}} \rho^{\gamma-\alpha+1} (|u + c_\mu \rho^{\alpha-2} \partial_x \rho|^2 + |u|^2) dx \\ &= -2\gamma \frac{c_p}{c_\mu} \int_{\mathbb{T}} \rho^{\gamma-\alpha} (s(x, t) - \pi(\rho) + \rho |u|^2) dx. \end{aligned}$$

Note that (2.1.36) provides the bound

$$\int_{\mathbb{T}} f \rho (u + \frac{\partial_x \rho}{\rho^2} \mu(\rho)) dx \leq \frac{1}{2} \|f(t)\|_{L^\infty(\mathbb{T})}^2 \|\rho_0\|_{L^1(\mathbb{T})} + \int_{\mathbb{T}} \frac{1}{2} \rho (u + \frac{\partial_x \rho}{\rho^2} \mu(\rho))^2 dx.$$

In addition, since $\gamma \in (0, 1)$, part 2 of Lemma 2.1.7 provides a bound for $\pi(\rho)$ and ρu^2 . Moreover, note that when $c_p < 0$ and $\gamma \in (0, 1)$ we have $\pi(\rho), s \geq 0$. Using these together with the assumption that $\gamma \leq \alpha$ we have

$$\begin{aligned} &\frac{d}{dt} \int_{\mathbb{T}} s(x, t) dx \\ &\leq -2\gamma \frac{c_p}{c_\mu} \int_{\mathbb{T}} \rho^{\gamma-\alpha} (s(x, t) - \pi(\rho) + \rho |u|^2) dx + \|f(t)\|_{L^\infty(\mathbb{T})}^2 \|\rho_0\|_{L^1(\mathbb{T})} + \int_{\mathbb{T}} s(x, t) dx. \end{aligned}$$

$$\begin{aligned}
&\leq -2\gamma \frac{c_p}{c_\mu} \left(\frac{1}{\underline{\rho}}\right)^{\gamma-\alpha} \int_{\mathbb{T}} (s(x, t) - \pi(\rho) + \rho|u|^2) dx + \|f(t)\|_{L^\infty(\mathbb{T})}^2 \|\rho_0\|_{L^1(\mathbb{T})} + \int_{\mathbb{T}} s(x, t) dx. \\
&\leq \left(-2\gamma \frac{c_p}{c_\mu} \left(\frac{1}{\underline{\rho}}\right)^{\gamma-\alpha} + 1 \right) \int_{\mathbb{T}} s(x, t) dx - 2\gamma \frac{c_p}{c_\mu} \left(\frac{1}{\underline{\rho}}\right)^{\gamma-\alpha} \int_{\mathbb{T}} (-\pi(\rho) + \rho|u|^2) dx \\
&\quad + \|f(t)\|_{L^\infty(\mathbb{T})}^2 \|\rho_0\|_{L^1(\mathbb{T})} \\
&\leq \left(-2\gamma \frac{c_p}{c_\mu} \left(\frac{1}{\underline{\rho}}\right)^{\gamma-\alpha} + 1 \right) \int_{\mathbb{T}} s(x, t) dx + M(E_0, \|f\|_{L^2(0, T; L^\infty)}, \frac{1}{\underline{\rho}}, T) \\
&\quad + \|f(t)\|_{L^\infty(\mathbb{T})}^2 \|\rho_0\|_{L^1(\mathbb{T})}
\end{aligned}$$

for $t \leq T$. By Grönwall's lemma and (2.1.38), we deduce that

$$\begin{aligned}
\|s\|_{L^\infty(0, T; L^1)} &\leq M(E_0 + \|s(\cdot, 0)\|_{L^1(\mathbb{T})}, \|f\|_{L^2(0, T; L^\infty)}, \frac{1}{\underline{\rho}}, T) \\
&\leq M(E_1, \|f\|_{L^2(0, T; L^\infty)}, \frac{1}{\underline{\rho}}, T).
\end{aligned}$$

Combining this with (2.1.26) gives

$$\|\partial_x(\rho^{\alpha-\frac{1}{2}})\|_{L^\infty(0, T; L^2)} \leq M(E_1, \|f\|_{L^2(0, T; L^\infty)}, \frac{1}{\underline{\rho}}, T). \quad (2.1.41)$$

Since $\alpha - \frac{1}{2} \in (0, 1]$, the mass conservation (2.1.28) implies

$$\|\rho^{\alpha-\frac{1}{2}}\|_{L^\infty(0, T; L^1)} \leq C(1 + \|\rho_0\|_{L^1(\mathbb{T})}). \quad (2.1.42)$$

Combined with (2.1.41), this yields

$$\|\rho^{\alpha-\frac{1}{2}}\|_{L^\infty(0, T; H^1)} \leq M(E_1, \|f\|_{L^2(0, T; L^\infty)}, \frac{1}{\underline{\rho}}, T) \quad (2.1.43)$$

from which (2.1.34) follows.

Step 3. The bound (2.1.35) follows from (2.1.33) & (2.1.39) and (2.1.34) & (2.1.41) respectively.

Step 4. In case $c_p > 0$, $\gamma > 1$ and $\alpha \geq 0$, the bounds (2.1.34) and (2.1.35) follow from

Propositions 4.5 and 4.6 in [73]. □

2.1.3 The active potential

We introduce in this section the *active potential* $w := -p(\rho) + \mu(\rho)\partial_x u$. This is a good unknown upon which much of the analysis is based. We first show that w satisfies a *forced quadratic heat equation with linear drift*.

Proposition 2.1.1. *Let*

$$w := -p(\rho) + \mu(\rho)\partial_x u. \quad (2.1.44)$$

Then w satisfies

$$\begin{aligned} \partial_t w = & \rho^{-1}\mu(\rho)\partial_x^2 w - (u + \mu(\rho)\frac{\partial_x \rho}{\rho^2})\partial_x w + \left(\rho\frac{p'(\rho)}{\mu(\rho)} - 2\frac{(\rho\mu'(\rho) + \mu(\rho))}{\mu(\rho)^2}p(\rho) \right) w \\ & - \frac{(\rho\mu'(\rho) + \mu(\rho))}{\mu(\rho)^2}w^2 + \left(\rho\frac{p'(\rho)}{\mu(\rho)} - \frac{(\rho\mu'(\rho) + \mu(\rho))}{\mu(\rho)^2}p(\rho) \right) p(\rho) + \mu(\rho)\partial_x f. \end{aligned} \quad (2.1.45)$$

Moreover, the following balance holds

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{T}} \frac{1}{2} |w|^2(x, t) dx = & - \int_{\mathbb{T}} \rho^{-1}\mu(\rho) |\partial_x w|^2 dx - \int_{\mathbb{T}} \left(u + \frac{\mu'(\rho)}{\rho} \partial_x \rho \right) w \partial_x w dx \\ & + \int_{\mathbb{T}} \left(\rho\frac{p'(\rho)}{\mu(\rho)} - 2\frac{(\rho\mu'(\rho) + \mu(\rho))}{\mu(\rho)^2}p(\rho) \right) |w|^2 dx - \int_{\mathbb{T}} \frac{(\rho\mu'(\rho) + \mu(\rho))}{\mu(\rho)^2} w^3 dx \\ & + \int_{\mathbb{T}} \left(\rho\frac{p'(\rho)}{\mu(\rho)} - \frac{(\rho\mu'(\rho) + \mu(\rho))}{\mu(\rho)^2}p(\rho) \right) p(\rho) w dx + \int_{\mathbb{T}} \mu(\rho) \partial_x f w dx. \end{aligned} \quad (2.1.46)$$

Proof. From the definition of $w := -p(\rho) + \mu(\rho)\partial_x u$ given by (2.1.44), we compute

$$\partial_x w = (\partial_x \rho)(-p'(\rho) + \mu'(\rho)\partial_x u) + \mu(\rho)\partial_x^2 u. \quad (2.1.47)$$

Thus, we have

$$\partial_t w = (\partial_t \rho)(-p'(\rho) + \mu'(\rho)\partial_x u) + \mu(\rho)\partial_t \partial_x u$$

$$\begin{aligned}
&= -\partial_x(u\rho)(-p'(\rho) + \mu'(\rho)\partial_x u) + \mu(\rho)\partial_t\partial_x u \\
&= -\rho\partial_x u(-p'(\rho) + \mu'(\rho)\partial_x u) - u(\partial_x w - \mu(\rho)\partial_x^2 u) + \mu(\rho)\partial_t\partial_x u.
\end{aligned} \tag{2.1.48}$$

The momentum equation (2.1.2) gives

$$\begin{aligned}
\partial_t u &= -u\partial_x u + \rho^{-1}\partial_x w + f, \\
\partial_t\partial_x u &= -\partial_x u\partial_x u - u\partial_x^2 u - \frac{\partial_x \rho}{\rho^2}\partial_x w + \rho^{-1}\partial_x^2 w + \partial_x f.
\end{aligned}$$

Combining the above results, we find

$$\begin{aligned}
\partial_t w &= -\rho\partial_x u(-p'(\rho) + \mu'(\rho)\partial_x u) - u\partial_x w + u\mu(\rho)\partial_x^2 u \\
&\quad - \mu(\rho)(|\partial_x u|^2 + u\partial_x^2 u) - \mu(\rho)\frac{\partial_x \rho}{\rho^2}\partial_x w + \rho^{-1}\mu(\rho)\partial_x^2 w + \mu(\rho)\partial_x f \\
&= \rho^{-1}\mu(\rho)\partial_x^2 w + \rho(\partial_x u)p'(\rho) - (\rho\mu'(\rho) + \mu(\rho))|\partial_x u|^2 \\
&\quad - (u + \mu(\rho)\frac{\partial_x \rho}{\rho^2})\partial_x w + \mu(\rho)\partial_x f \\
&= \rho^{-1}\mu(\rho)\partial_x^2 w + \rho(w + p(\rho))\frac{p'(\rho)}{\mu(\rho)} - \frac{(\rho\mu'(\rho) + \mu(\rho))}{\mu(\rho)^2}(w + p(\rho))^2 \\
&\quad - (u + \mu(\rho)\frac{\partial_x \rho}{\rho^2})\partial_x w + \mu(\rho)\partial_x f
\end{aligned}$$

which, after rearrangement, establishes equation (2.1.45). For the energy, multiplying the equation (2.1.45) by w yields

$$\begin{aligned}
&\partial_t \left(\frac{1}{2}|w|^2 \right) \\
&= \partial_x \left(\frac{\mu(\rho)}{\rho} w \partial_x w \right) - \frac{\mu(\rho)}{\rho} |\partial_x w|^2 - \partial_x \left(\frac{\mu(\rho)}{\rho} \right) w \partial_x w - \left(u + \frac{\mu(\rho)}{\rho^2} \partial_x \rho \right) w \partial_x w \\
&\quad + \left(\rho \frac{p'(\rho)}{\mu(\rho)} - 2 \frac{(\rho\mu'(\rho) + \mu(\rho))}{\mu(\rho)^2} p(\rho) \right) |w|^2 - \frac{(\rho\mu'(\rho) + \mu(\rho))}{\mu(\rho)^2} w^3 \\
&\quad + \left(\rho \frac{p'(\rho)}{\mu(\rho)} - \frac{(\rho\mu'(\rho) + \mu(\rho))}{\mu(\rho)^2} p(\rho) \right) p(\rho) w + \mu(\rho) \partial_x f w.
\end{aligned}$$

Integrating in space yields the balance. □

Let us remark that in (2.1.45) the new viscosity coefficient is $\frac{\mu(\rho)}{\rho}$ which is less degenerate than the original viscosity $\mu(\rho)$ for the momentum equation. In particular, when $\mu(\rho) = c_\mu \rho^\alpha$ with $\alpha \leq 1$, $\frac{\mu(\rho)}{\rho}$ is not degenerate when ρ goes to 0. Energy estimates for the coupled system of ρ and w will allow us to control all the high Sobolev regularity of ρ and w as long as ρ is positive. This leads to the proof of our continuation criterion in Theorem 2.1.1: no singularity occurs before vacuum formation.

Furthermore, (2.1.45) can be regarded as a nonlinear heat equation with variable coefficients. Note that the zero-order term in (2.1.45) has the form $\lambda \rho^{2\gamma-\alpha}$ where λ depends only on c_μ and c_p . It can be readily seen that when the zero-order term and the forcing term in (2.1.45) are nonpositive, w remains nonpositive if it is nonpositive initially. This fact will be exploited as the key ingredient in proving the existence of global solutions in Theorem 2.1.3 when the viscosity is strongly degenerate.

2.1.4 Proof of Theorem 2.1.1

Throughout this section, we suppose that

$$0 < \underline{\rho} \leq \rho(x, t) \quad t \in [0, T^*), \quad x \in \mathbb{T}. \quad (2.1.49)$$

and assume any of the following three conditions

- (i) $c_p > 0$ and $\alpha > \frac{1}{2}$, $\gamma \geq \alpha - \frac{1}{2}$, $\gamma \neq 1$
- (ii) $c_p < 0$ and $\alpha \in (\frac{1}{2}, \frac{3}{2}]$, $0 < \gamma \leq \alpha$, $\gamma < 1$
- (iii) $c_p > 0$ and $\alpha \geq 0$, $\gamma > 1$.

Under these assumptions, by Lemma 2.1.9, we have

$$\|\rho\|_{L^\infty(0, T; L^\infty(\mathbb{T}))} \leq M(E_1, \|f\|_{L^2(0, T; L^\infty)}, \frac{1}{\underline{\rho}}, T), \quad (2.1.50)$$

and

$$\|\partial_x \rho\|_{L^\infty(0,T;L^2(\mathbb{T}))} \leq M(E_1, \|f\|_{L^2(0,T;L^\infty)}, \frac{1}{\underline{\rho}}, T). \quad (2.1.51)$$

Lemma 2.1.11.

$$\begin{aligned} \|w\|_{L^\infty(0,T;L^2)} + \|\partial_x w\|_{L^2(0,T;L^2)} + \|\partial_x u\|_{L^\infty(0,T;L^2)} + \|\partial_x^2 u\|_{L^2(0,T;L^2)} \\ \leq M(E_2, \|f\|_{L^2(0,T;H^1)}, \frac{1}{\underline{\rho}}, T), \end{aligned} \quad (2.1.52)$$

where $E_2 = E_1 + \|\partial_x u_0\|_{L^2}$.

Proof. As a consequence of (2.1.49), (2.1.50), and (2.1.46), there exist

$$c := c(E_1, \|f\|_{L^2(0,T;L^\infty)}, \frac{1}{\underline{\rho}}, T) > 0,$$

and $C := C(E_1, \|f\|_{L^2(0,T;L^\infty)}, \frac{1}{\underline{\rho}}, T) > 0$ such that

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{T}} \frac{1}{2} |w|^2(x, t) dx &\leq -\frac{1}{c} \int_{\mathbb{T}} |\partial_x w|^2 dx + \int_{\mathbb{T}} (|u| + C|\partial_x \rho|) |w \partial_x w| dx \\ &\quad + C \left(\int_{\mathbb{T}} |w|^2 dx + \int_{\mathbb{T}} |w|^3 dx + \int_{\mathbb{T}} |\partial_x f|^2 dx + 1 \right). \end{aligned} \quad (2.1.53)$$

We bound

$$\begin{aligned} \int_{\mathbb{T}} |\partial_x w w u| dx &\leq \|\partial_x w\|_{L^2} \|w\|_{L^2} \|u\|_{L^\infty} \\ &\leq C_1 \|\partial_x w\|_{L^2} \|w\|_{L^2} \|u\|_{H^1} \leq \frac{1}{4c} \|\partial_x w\|_{L^2}^2 + C \|w\|_{L^2}^2 \|u\|_{H^1}^2 \end{aligned}$$

where C_1 denotes absolute constants in this proof. Next, applying Gagliardo–Nirenberg’s inequality and Young’s inequality implies

$$\int_{\mathbb{T}} |w|^3 dx \leq \|w\|_{L^3}^3 \leq C_1 (\|\partial_x w\|_{L^2}^{\frac{1}{2}} \|w\|_{L^2}^{\frac{5}{2}} + \|w\|_{L^2}^3) \leq \frac{1}{4c} \|\partial_x w\|_{L^2}^2 + C \|w\|_{L^2}^{\frac{10}{3}} + C \|w\|_{L^2}^3$$

and

$$\begin{aligned}
\int_{\mathbb{T}} |\partial_x w w \partial_x \rho| \, dx &\leq \|\partial_x w\|_{L^2} \|w\|_{L^\infty} \|\partial_x \rho\|_{L^2} \\
&\leq C_1 \|\partial_x w\|_{L^2} (\|\partial_x w\|_{L^2}^{\frac{1}{2}} \|w\|_{L^2}^{\frac{1}{2}} + \|w\|_{L^2}) \|\partial_x \rho\|_{L^2} \\
&\leq C_1 \|\partial_x w\|_{L^2}^{\frac{3}{2}} \|w\|_{L^2}^{\frac{1}{2}} \|\partial_x \rho\|_{L^2} + C_1 \|\partial_x w\|_{L^2} \|w\|_{L^2} \|\partial_x \rho\|_{L^2} \\
&\leq \frac{1}{4c} \|\partial_x w\|_{L^2}^2 + C \|w\|_{L^2}^2 \|\partial_x \rho\|_{L^2}^4 + C \|w\|_{L^2}^2 \|\partial_x \rho\|_{L^2}^2.
\end{aligned}$$

Putting together the above bounds, and interpolating, yields the following inequality

$$\frac{1}{2} \frac{d}{dt} \|w\|_{L^2}^2 + \frac{1}{4c} \|\partial_x w\|_{L^2}^2 \leq C \|w\|_{L^2}^2 (\|w\|_{L^2}^2 + \|\partial_x \rho\|_{L^2}^4 + 1) + C \|\partial_x f\|_{L^2}^2 + C. \quad (2.1.54)$$

In view of (2.1.51), we have

$$\int_0^T \|\partial_x \rho(\cdot, t)\|_{L^2}^4 \, dt \leq M(E_1, \|f\|_{L^2(0,T;L^\infty)}, \frac{1}{\underline{\rho}}, T).$$

Furthermore, using the definition of w together with bounds (2.1.50) & (2.1.27), we have

$$\|w\|_{L^2(0,T;L^2)} \leq M(E_1, \|f\|_{L^2(0,T;L^\infty)}, \frac{1}{\underline{\rho}}, T).$$

The last two displays, together with Grönwall's lemma applied to (2.1.54), yields the bound

$$\begin{aligned}
&\|w\|_{L^\infty(0,T;L^2(\mathbb{T}))} + \|\partial_x w\|_{L^2(0,T;L^2(\mathbb{T}))} \\
&\leq M(\|w_0\|_{L^2}, c, C, E_1, \|f\|_{L^1(0,T;H^1)}, \frac{1}{\underline{\rho}}, T) \leq M(E_1, \|f\|_{L^1(0,T;H^1)}, \frac{1}{\underline{\rho}}, T).
\end{aligned}$$

Here, we used the fact that

$$\|w_0\|_{L^2}^2 \leq 2c_p^2 \|\rho_0\|_{L^\infty}^{2\gamma} + 2c_\mu^2 \|\rho_0\|_{L^\infty}^{2\alpha} \|\partial_x u_0\|_{L^2}^2.$$

The above bound can be used to obtain similar estimates for $\|\partial_x u\|_{L^\infty(0,T;L^2)}$ as well as

for $\|\partial_x^2 u\|_{L^2(0,T;L^2)}$ directly from the definition of w (2.1.44). $\square$

Lemma 2.1.12.

$$\begin{aligned} & \|\partial_x^2 \rho\|_{L^\infty(0,T;L^2)} + \|\partial_x w\|_{L^\infty(0,T;L^2)} + \|\partial_x^2 w\|_{L^2(0,T;L^2)} \\ & + \|\partial_x^2 u\|_{L^\infty(0,T;L^2)} + \|\partial_x^3 u\|_{L^2(0,T;L^2)} \leq M(E_3, \|f\|_{L^1(0,T;H^1)}, \frac{1}{\underline{\rho}}, T) \end{aligned} \quad (2.1.55)$$

where

$$E_3 = E_2 + \|\partial_x^2 \rho_0\|_{L^2} + \|\partial_x^2 u_0\|_{L^2}.$$

Proof. To prove this lemma, we obtain energy estimates for the mass equation (2.1.1) and the w -equation (2.1.45) simultaneously. The proof proceeds in 4 steps.

Step 1. Let $m \geq 2$ be an arbitrary integer. Differentiating equation (2.1.1) m times, then multiplying the resulting equation by $\partial_x^m \rho$ and integrating in space we get

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_{\mathbb{T}} |\partial_x^m \rho|^2 &= - \int_{\mathbb{T}} \partial_x^m (u \partial_x \rho) \partial_x^m \rho - \int_{\mathbb{T}} \partial_x^m (\rho \partial_x u) \partial_x^m \rho \\ &= - \int_{\mathbb{T}} u \partial_x \partial_x^m \rho \partial_x^m \rho - \int_{\mathbb{T}} ([\partial_x^m, u] \partial_x \rho) \partial_x^m \rho \\ &\quad - \int_{\mathbb{T}} ([\partial_x^m, \rho] \partial_x u) \partial_x^m \rho - \int_{\mathbb{T}} \rho \partial_x^{m+1} u \partial_x^m \rho. \end{aligned}$$

Using the Kato–Ponce commutator estimate [54] and the inequality

$$\|\partial_x g\|_{L^\infty(\mathbb{T})} \leq C \|\partial_x^2 g\|_{L^2(\mathbb{T})} \leq C_n \|\partial_x^n g\|_{L^2(\mathbb{T})} \quad \forall n \geq 3,$$

we have

$$\|[\partial_x^m, u] \partial_x \rho\|_{L^2} \leq C \|\partial_x u\|_{L^\infty} \|\partial_x^{m-1} \partial_x \rho\|_{L^2} + C \|\partial_x^m u\|_{L^2} \|\partial_x \rho\|_{L^\infty} \leq C \|\partial_x^m u\|_{L^2} \|\partial_x^m \rho\|_{L^2}$$

and

$$\|[\partial_x^m, \rho] \partial_x u\|_{L^2} \leq C \|\partial_x \rho\|_{L^\infty} \|\partial_x^{m-1} \partial_x u\|_{L^2} + C \|\partial_x^m \rho\|_{L^2} \|\partial_x u\|_{L^\infty} \leq C \|\partial_x^m u\|_{L^2} \|\partial_x^m \rho\|_{L^2}.$$

In addition,

$$\left| \int_{\mathbb{T}} u \partial_x \partial_x^m \rho \partial_x^m \rho \right| = \frac{1}{2} \left| \int_{\mathbb{T}} \partial_x u |\partial_x^m \rho|^2 \right| \leq \frac{1}{2} \|\partial_x u\|_{L^\infty} \|\partial_x^m \rho\|_{L^2}^2 \leq C \|\partial_x^m u\|_{L^2} \|\partial_x^m \rho\|_{L^2}^2.$$

We thus obtain

$$\frac{d}{dt} \|\partial_x^m \rho\|_{L^2}^2 \leq C \|\partial_x^m u\|_{L^2} \|\partial_x^m \rho\|_{L^2}^2 + \|\rho\|_{L^\infty} \|\partial_x^{m+1} u\|_{L^2} \|\partial_x^m \rho\|_{L^2}. \quad (2.1.56)$$

Step 2. Recall equation (2.1.45) with power-law pressure and viscosity

$$\begin{aligned} \partial_t w &= c_\mu \rho^{\alpha-1} \partial_x^2 w - (u + c_\mu \rho^{\alpha-2} \partial_x \rho) \partial_x w + \frac{c_p}{c_\mu} (\gamma - 2(\alpha + 1)) \rho^{\gamma-\alpha} w \\ &\quad - \frac{1}{c_\mu} (\alpha + 1) \rho^{-\alpha} w^2 + \frac{c_p^2}{c_\mu} (\gamma - (\alpha + 1)) \rho^{2\gamma-\alpha} + c_\mu \rho^\alpha \partial_x f. \end{aligned} \quad (2.1.57)$$

Differentiating in space, multiplying the resulting equation by $\partial_x w$ and integrating by parts in x leads to

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_{\mathbb{T}} |\partial_x w|^2 &= -c_\mu \int_{\mathbb{T}} \rho^{\alpha-1} |\partial_x^2 w|^2 + \int_{\mathbb{T}} (u + c_\mu \rho^{\alpha-2} \partial_x \rho) \partial_x w \partial_x^2 w \\ &\quad + \frac{c_p}{c_\mu} (\gamma - 2(\alpha + 1)) \int_{\mathbb{T}} |\partial_x w|^2 \rho^{\gamma-\alpha} \\ &\quad + \frac{c_p}{c_\mu} (\gamma - \alpha) (\gamma - 2(\alpha + 1)) \int_{\mathbb{T}} w \rho^{\gamma-\alpha-1} \partial_x w \partial_x \rho \\ &\quad - \frac{2}{c_\mu} (\alpha + 1) \int_{\mathbb{T}} \rho^{-\alpha} w |\partial_x w|^2 + \frac{\alpha}{c_\mu} (\alpha + 1) \int_{\mathbb{T}} w^2 \partial_x w \partial_x \rho \rho^{-\alpha-1} \\ &\quad + \frac{c_p^2}{c_\mu} (2\gamma - \alpha) (\gamma - (\alpha + 1)) \int_{\mathbb{T}} \rho^{2\gamma-\alpha-1} \partial_x w \partial_x \rho - c_\mu \int_{\mathbb{T}} \rho^\alpha \partial_x^2 w \partial_x f \\ &=: -c_\mu \int_{\mathbb{T}} \rho^{\alpha-1} |\partial_x^2 w|^2 + \sum_{j=1}^7 H_j. \end{aligned}$$

after integrating by parts. By virtue of (2.1.49) and (2.1.50), there exists

$$c := c(E_1, \|f\|_{L^2(0,T;L^\infty)}, \frac{1}{\underline{\rho}}, T) > 0$$

such that

$$c_\mu \int_{\mathbb{T}} \rho^{\alpha-1} |\partial_x^2 w|^2 \geq \frac{1}{c} \int_{\mathbb{T}} |\partial_x^2 w|^2.$$

Note that, under our assumptions, ρ and $1/\rho$ are bounded (see (2.1.49) and (2.1.50)). Therefore all coefficients involving L^∞ norms of ρ to some power can be bounded by some constant $C = M(E_1, \|f\|_{L^2(0,T;L^\infty)}, \frac{1}{\underline{\rho}}, T, \gamma, \alpha)$. The constant may change line by line.

- Estimate for H_1 :

$$\begin{aligned} & \left| \int_{\mathbb{T}} (u + c_\mu \rho^{\alpha-2} \partial_x \rho) \partial_x w \partial_x^2 w \right| \\ & \leq \|\partial_x^2 w\|_{L^2} \|\partial_x w\|_{L^2} \|u\|_{L^\infty} + C \|\partial_x^2 w\|_{L^2} \|\partial_x w\|_{L^2} \|\partial_x \rho\|_{L^\infty} \\ & \leq \frac{1}{10c} \|\partial_x^2 w\|_{L^2}^2 + C \|\partial_x w\|_{L^2}^2 \|u\|_{H^1}^2 + C \|\partial_x w\|_{L^2}^2 \|\partial_x^2 \rho\|_{L^2}^2. \end{aligned}$$

- Estimate for H_2 :

$$\left| \int_{\mathbb{T}} |\partial_x w|^2 \rho^{\gamma-\alpha} \right| \leq C \|\partial_x w\|_{L^2}^2.$$

- Estimate for H_3 :

$$\begin{aligned} \left| \int_{\mathbb{T}} w \partial_x w \partial_x \rho \rho^{\gamma-\alpha-1} \right| & \leq \|\rho^{\gamma-\alpha-1}\|_\infty \|w\|_{L^\infty} \|\partial_x w\|_{L^2} \|\partial_x \rho\|_{L^2} \\ & \leq C \|w\|_{L^2} \|\partial_x w\|_{L^2} \|\partial_x \rho\|_{L^2} + C \|\partial_x w\|_{L^2}^2 \|\partial_x \rho\|_{L^2}. \end{aligned}$$

- Estimate for H_4 :

$$\begin{aligned} \left| \int_{\mathbb{T}} \rho^{-\alpha} w |\partial_x w|^2 \right| & \leq \frac{1}{\underline{\rho}^\alpha} \|w\|_{L^\infty} \|\partial_x w\|_{L^2}^2 \leq \frac{1}{4\underline{\rho}^{2\alpha}} \|w\|_{L^\infty}^2 + C \|\partial_x w\|_{L^2}^4 \\ & \leq C \|w\|_{H^1}^2 + C \|\partial_x w\|_{L^2}^4. \end{aligned}$$

- Estimate for H_5 :

$$\begin{aligned}
\left| \int_{\mathbb{T}} w^2 \partial_x w \partial_x \rho \rho^{-\alpha-1} \right| &\leq \frac{1}{\underline{\rho}^{1+\alpha}} \|\partial_x w\|_{L^2} \|w\|_{L^\infty}^2 \|\partial_x \rho\|_{L^2} \\
&\leq C \|\partial_x w\|_{L^2} \|w\|_{H^1}^2 \|\partial_x \rho\|_{L^2} \\
&\leq C \|\partial_x w\|_{L^2} \|w\|_{L^2}^2 \|\partial_x \rho\|_{L^2} + C \|\partial_x w\|_{L^2}^3 \|\partial_x \rho\|_{L^2}.
\end{aligned}$$

- Estimate for H_6 :

$$\left| \int_{\mathbb{T}} \rho^{\gamma-\alpha-1} \partial_x w \partial_x \rho \right| \leq C \|\partial_x w\|_{L^2} \|\partial_x \rho\|_{L^2}.$$

- Estimate for H_7 :

$$\left| \int_{\mathbb{T}} \rho^\alpha \partial_x^2 w \partial_x f \right| \leq \frac{1}{10c} \|\partial_x^2 w\|_{L^2}^2 + C \|\partial_x f\|_{L^2}^2.$$

Putting together the above estimates gives

$$\begin{aligned}
&\frac{d}{dt} \|\partial_x w\|_{L^2}^2 + \frac{1}{2c} \|\partial_x^2 w\|_{L^2}^2 \\
&\leq C \left(\|\partial_x w\|_{L^2}^2 \|u\|_{H^1}^2 + \|\partial_x w\|_{L^2}^2 \|\partial_x^2 \rho\|_{L^2}^2 + \|\partial_x w\|_{L^2}^4 + \|\partial_x w\|_{L^2}^3 \|\partial_x \rho\|_{L^2} \right) + G
\end{aligned} \tag{2.1.58}$$

with

$$\begin{aligned}
G = C \left(\|\rho\|_{L^\infty} \|\partial_x w\|_{L^2}^2 + \|w\|_{L^2} \|\partial_x w\|_{L^2} \|\partial_x \rho\|_{L^2} + \|\partial_x w\|_{L^2}^2 \|\partial_x \rho\|_{L^2} \right. \\
\left. + \|w\|_{H^1}^2 + \|\partial_x w\|_{L^2} \|w\|_{L^2}^2 \|\partial_x \rho\|_{L^2} + \|\partial_x w\|_{L^2} \|\partial_x \rho\|_{L^2} + \|\partial_x f\|_{L^2}^2 \right).
\end{aligned}$$

By virtue of the estimates (2.1.50), (2.1.51) and (2.1.52) we deduce that

$$\|G\|_{L^1((0,T))} \leq M(E_2, \|f\|_{L^2(0,T;H^1)}, \frac{1}{\underline{\rho}}, T).$$

Step 3. Letting $m = 2$ in (2.1.56) and using the embedding $H^1(\mathbb{T}) \subset L^\infty(\mathbb{T})$ we get

$$\frac{d}{dt} \|\partial_x^2 \rho\|_{L^2}^2 \leq C \|\partial_x^2 u\|_{L^2} \|\partial_x^2 \rho\|_{L^2}^2 + C \|\rho\|_{H^1} \|\partial_x^3 u\|_{L^2} \|\partial_x^2 \rho\|_{L^2}.$$

Recalling the definition (2.1.44) $w = -c_p \rho^\gamma + c_\mu \rho^\alpha \partial_x u$ we have

$$\begin{aligned}
\partial_x^3 u &= \partial_x^2 \left(\frac{w}{c_\mu \rho^\alpha} + \frac{c_p}{c_\mu} \rho^{\gamma-\alpha} \right) \\
&= \frac{\partial_x^2 w}{c_\mu \rho^\alpha} - 2\alpha \frac{\partial_x w \partial_x \rho}{c_\mu \rho^{\alpha+1}} - \alpha \frac{w \partial_x^2 \rho}{c_\mu \rho^{\alpha+1}} + \alpha(\alpha+1) \frac{w |\partial_x \rho|^2}{c_\mu \rho^{\alpha+2}} \\
&\quad + \frac{c_p}{c_\mu} (\gamma - \alpha) \partial_x^2 \rho \rho^{\gamma-\alpha-1} + \frac{c_p}{c_\mu} (\gamma - \alpha)(\gamma - \alpha - 1) |\partial_x \rho|^2 \rho^{\gamma-\alpha-2}.
\end{aligned} \tag{2.1.59}$$

Consequently

$$\begin{aligned}
\|\partial_x^3 u\|_{L^2} &\leq C \left(\|\partial_x^2 w\|_{L^2} + \|\partial_x w\|_{L^2} \|\partial_x \rho\|_{L^\infty} + \|w\|_{H^1} \|\partial_x^2 \rho\|_{L^2} \right. \\
&\quad \left. + \|w\|_{L^\infty} \|\partial_x \rho\|_{L^2} \|\partial_x \rho\|_{L^\infty} + \|\rho^{\gamma-\alpha-1}\|_\infty \|\partial_x^2 \rho\|_{L^2} + \|\rho^{\gamma-\alpha-2}\|_\infty \|\partial_x \rho\|_{L^2} \|\partial_x \rho\|_{L^\infty} \right).
\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}
&\frac{d}{dt} \|\partial_x^2 \rho\|_{L^2}^2 \\
&\leq C \left(\|\partial_x^2 u\|_{L^2} \|\partial_x^2 \rho\|_{L^2}^2 + \|\rho\|_{H^1} \|\partial_x^2 w\|_{L^2} \|\partial_x^2 \rho\|_{L^2} + \|\rho\|_{H^1} \|\partial_x w\|_{L^2} \|\partial_x^2 \rho\|_{L^2} \|\partial_x \rho\|_{L^\infty} \right. \\
&\quad \left. + \|w\|_{H^1} \|\rho\|_{H^1} \|\partial_x^2 \rho\|_{L^2}^2 + \|w\|_{L^\infty} \|\rho\|_{H^1}^2 \|\partial_x \rho\|_{L^\infty} \|\partial_x^2 \rho\|_{L^2} \right. \\
&\quad \left. + \|\rho\|_{H^1} \|\partial_x^2 \rho\|_{L^2}^2 + \|\rho\|_{H^1}^2 \|\partial_x^2 \rho\|_{L^2}^2 \right) \\
&\leq \frac{1}{10c} \|\partial_x^2 w\|_{L^2}^2 + C \left(\|\partial_x^2 u\|_{L^2} \|\partial_x^2 \rho\|_{L^2}^2 + \|\rho\|_{H^1}^2 \|\partial_x^2 \rho\|_{L^2}^2 + \|\rho\|_{H^1} \|\partial_x w\|_{L^2} \|\partial_x^2 \rho\|_{L^2}^2 \right. \\
&\quad \left. + \|w\|_{H^1} \|\rho\|_{H^1} \|\partial_x^2 \rho\|_{L^2}^2 + \|w\|_{H^1} \|\rho\|_{H^1}^2 \|\partial_x^2 \rho\|_{L^2}^2 + \|\rho\|_{H^1} \|\partial_x^2 \rho\|_{L^2}^2 + \|\rho\|_{H^1}^2 \|\partial_x^2 \rho\|_{L^2}^2 \right) \\
&\leq \frac{1}{10c} \|\partial_x^2 w\|_{L^2}^2 + F \|\partial_x^2 \rho\|_{L^2}^2,
\end{aligned} \tag{2.1.60}$$

with

$$\begin{aligned}
F &= C \left(\|\partial_x^2 u\|_{L^2} + \|\rho\|_{H^1}^2 + \|\rho\|_{H^1} \|\partial_x w\|_{L^2} \right. \\
&\quad \left. + \|w\|_{H^1} \|\rho\|_{H^1} + \|w\|_{H^1} \|\rho\|_{H^1}^2 + \|\rho\|_{H^1} + \|\rho\|_{H^1}^2 \right).
\end{aligned}$$

Combining the estimates (2.1.50), (2.1.51) and (2.1.52) yields

$$\|F\|_{L^1((0,T))} \leq M(E_2, \|f\|_{L^2(0,T;H^1(\mathbb{T}))}, \frac{1}{\underline{\rho}}, T).$$

Step 4. Adding (2.1.60) to (2.1.58) leads to

$$\begin{aligned} & \frac{d}{dt}(\|\partial_x^2 \rho\|_{L^2}^2 + \|\partial_x w\|_{L^2}^2) + \frac{1}{4c} \|\partial_x^2 w\|_{L^2}^2 \\ & \leq \|\partial_x w\|_{L^2}^2 H + \|\partial_x^2 \rho\|_{L^2}^2 (F + C \|\partial_x w\|_{L^2}^2) + G \\ & \leq (\|\partial_x w\|_{L^2}^2 + \|\partial_x^2 \rho\|_{L^2}^2) (H + F + C \|\partial_x w\|_{L^2}^2) + G \end{aligned} \tag{2.1.61}$$

with

$$H = C (\|u\|_{H^1}^2 + \|\partial_x w\|_{L^2}^2 + \|\partial_x w\|_{L^2} \|\partial_x \rho\|_{L^2})$$

satisfying, in virtue of (2.1.50), (2.1.51) and (2.1.52),

$$\|H\|_{L^1((0,T))} \leq M(E_2, \|f\|_{L^2(0,T;H^1)}, \frac{1}{\underline{\rho}}, T).$$

Finally, we integrate (2.1.61) in time, then apply Grönwall's lemma, the estimates for F , G and H , and the estimate (2.1.52) on $\|\partial_x w\|_{L^2(0,T;L^2)}$ to obtain

$$\begin{aligned} & \|\partial_x^2 \rho\|_{L^\infty(0,T;L^2)} + \|\partial_x w\|_{L^\infty(0,T;L^2)} + \frac{1}{c} \|\partial_x^2 w\|_{L^2(0,T;L^2)} \\ & \leq M(E_2, \|f\|_{L^2(0,T;H^1)}, \frac{1}{\underline{\rho}}, T, \|\partial_x^2 \rho_0\|_{L^2}, \|\partial_x w_0\|_{L^2}) \\ & \leq M(E_3, \|f\|_{L^2(0,T;H^1)}, \frac{1}{\underline{\rho}}, T), \end{aligned}$$

where

$$E_3 = E_2 + \|\partial_x^2 \rho_0\|_{L^2} + \|\partial_x^2 u_0\|_{L^2}.$$

It then follows easily that

$$\|\partial_x^2 u\|_{L^\infty(0,T;L^2)} + \|\partial_x^3 u\|_{L^2(0,T;L^2)} \leq M(E_3, \|f\|_{L^2(0,T;H^1)}, \underline{\frac{1}{\rho}}, T).$$

□

Lemma 2.1.13. *For any $k \geq 2$ there exists M_k depending only on k such that*

$$\begin{aligned} & \|\partial_x^k \rho\|_{L^\infty(0,T;L^2)} + \|\partial_x^{k-1} w\|_{L^\infty(0,T;L^2)} + \|\partial_x^k w\|_{L^2(0,T;L^2)} \\ & + \|\partial_x^k u\|_{L^\infty(0,T;L^2)} + \|\partial_x^{k+1} u\|_{L^2(0,T;L^2)} \leq M_k(E_{k+1}, \|f\|_{L^2(0,T;H^{k-1})}, \underline{\frac{1}{\rho}}, T) \end{aligned} \quad (2.1.62)$$

where

$$E_{k+1} = E_k + \|\partial_x^k \rho_0\|_{L^2} + \|\partial_x^k u_0\|_{L^2}.$$

Proof. The proof proceeds by induction in k . According to Lemma 2.1.12, (2.1.62) holds for $k = 2$. Assuming that (2.1.62) holds for $k - 1$ with $k \geq 3$, to obtain it for k we perform H^k energy estimate for ρ and H^{k-1} energy estimate for w . This follows along the same lines as that of Lemma 2.1.12. We first apply (2.1.56) with $m = k$ to have

$$\begin{aligned} \frac{d}{dt} \|\partial_x^k \rho\|_{L^2}^2 & \leq C \|\partial_x^k u\|_{L^2} \|\partial_x^k \rho\|_{L^2}^2 + \|\rho\|_{L^\infty} \|\partial_x^{k+1} u\|_{L^2} \|\partial_x^k \rho\|_{L^2} \\ & \leq M(E_k, \|f\|_{L^2(0,T;H^{k-2})}, \underline{\frac{1}{\rho}}, T) \left(\|\partial_x^k u\|_{L^2} \|\partial_x^k \rho\|_{L^2}^2 + \|\partial_x^{k+1} u\|_{L^2} \|\partial_x^k \rho\|_{L^2} \right). \end{aligned} \quad (2.1.63)$$

By differentiating k times the formula

$$\partial_x u = \frac{1}{c_\mu} w \rho^{-\alpha} + c_p \rho^{\gamma-\alpha}$$

and using the induction hypothesis together with the fact that $k \geq 3$ we obtain

$$\begin{aligned} \|\partial_x^{k+1} u\|_{L^2} & \leq C \|[\partial_x^k, \rho^{-\alpha}] w\|_{L^2} + C \|\rho^{-\alpha} \partial_x^k w\|_{L^2} + \|\partial_x^k \rho^{\gamma-\alpha}\|_{L^2} \\ & \leq C \|\partial_x \rho^{-\alpha}\|_{L^\infty} \|w\|_{H^{k-1}} + C \|\rho^{-\alpha}\|_{H^k} \|w\|_{L^\infty} \end{aligned}$$

$$\begin{aligned}
& + C\|\rho^{-\alpha}\|_{L^\infty}\|\partial_x^k w\|_{L^2} + \|\partial_x^k \rho^{\gamma-\alpha}\|_{L^2} \\
& \leq C\|\rho^{-\alpha}\|_{H^2}\|w\|_{H^{k-1}} + C\|\rho^{-\alpha}\|_{H^k}\|w\|_{H^1} + \\
& \quad + C\|\rho^{-\alpha}\|_{H^1}\|\partial_x^k w\|_{L^2} + \|\rho^{\gamma-\alpha}\|_{H^k} \\
& \leq M(E_k, \|f\|_{L^2(0,T;H^{k-2})}, \frac{1}{\underline{\rho}}, T) (\|\partial_x^k w\|_{L^2} + \|\partial_x^k \rho\|_{L^2} + 1).
\end{aligned}$$

It then follows from (2.1.63) that

$$\begin{aligned}
& \frac{d}{dt} \|\partial_x^k \rho\|_{L^2}^2 \\
& \leq M(E_k, \|f\|_{L^2(0,T;H^{k-2})}, \frac{1}{\underline{\rho}}, T) \left[\|\partial_x^k \rho\|_{L^2}^2 (\|\partial_x^k u\|_{L^2} + 1) + \|\partial_x^k w\|_{L^2} \|\partial_x^k \rho\|_{L^2} + 1 \right] \quad (2.1.64) \\
& \leq \frac{1}{10c} \|\partial_x^k w\|_{L^2}^2 + M(E_k, \|f\|_{L^2(0,T;H^{k-2})}, \frac{1}{\underline{\rho}}, T) \left[\|\partial_x^k \rho\|_{L^2}^2 (\|\partial_x^k u\|_{L^2} + 1) + 1 \right]
\end{aligned}$$

where $c = c(E_1, \|f\|_{L^2(0,T;L^\infty)}, \frac{1}{\underline{\rho}}, T) > 0$ is a positive number such that

$$\rho^{\alpha-1} \geq \frac{1}{c} \quad \forall (x, t) \in \mathbb{T} \times [0, T^*).$$

Next, we differentiate equation (2.1.57) $k-1$ times in x , multiply the resulting equation by $\partial_x^{k-1} w$ and integrate over $\mathbb{T}$. We estimate successively each resulting term on the right hand side of (2.1.57).

1. The dissipation term:

$$\begin{aligned}
& \int_{\mathbb{T}} \partial_x^{k-1} (\rho^{\alpha-1} \partial_x^2 w) \partial_x^{k-1} w = - \int_{\mathbb{T}} \partial_x^{k-2} (\rho^{\alpha-1} \partial_x^2 w) \partial_x^k w \\
& = - \int_{\mathbb{T}} \rho^{\alpha-1} |\partial_x^k w|^2 - \int_{\mathbb{T}} \partial_x^k w \sum_{\ell=1}^{k-2} C_\ell \partial_x^\ell \rho^{\alpha-1} \partial_x^{k-\ell} w \\
& \leq -\frac{1}{c} \|\partial_x^k w\|_{L^2}^2 + C \|\partial_x^k w\|_{L^2} \sum_{\ell=1}^{k-2} C_\ell \|\partial_x^\ell \rho^{\alpha-1}\|_{L^\infty} \|\partial_x^{k-\ell} w\|_{L^2} \\
& \leq -\frac{1}{c} \|\partial_x^k w\|_{L^2}^2 + C \|\partial_x^k w\|_{L^2} \|\rho\|_{H^{k-1}} (\|\partial_x^{k-1} w\|_{L^2} + \|w\|_{L^2}) \\
& \leq -\frac{1}{2c} \|\partial_x^k w\|_{L^2}^2 + C' \|\rho\|_{H^{k-1}}^2 (\|\partial_x^{k-1} w\|_{L^2}^2 + \|w\|_{L^2}^2)
\end{aligned}$$

$$\leq -\frac{1}{2c} \|\partial_x^k w\|_{L^2}^2 + M(E_k, \|f\|_{L^2(0,T;H^{k-2})}, \frac{1}{\underline{\rho}}, T) (\|\partial_x^{k-1} w\|_{L^2}^2 + 1).$$

2. The drift term. We have

$$\begin{aligned} & \int_{\mathbb{T}} \partial_x^{k-1} (u \partial_x w + c_\mu \rho^{\alpha-2} \partial_x \rho \partial_x w) \partial_x^{k-1} w \\ &= - \int_{\mathbb{T}} \partial_x^{k-2} (u \partial_x w) \partial_x^k w - c_\mu \int_{\mathbb{T}} \partial_x^{k-2} (\partial_x \frac{\rho^{\alpha-1}}{\alpha-1} \partial_x w) \partial_x^k w \end{aligned}$$

where we adopted the convention $\frac{\rho^{\alpha-1}}{\alpha-1} = \ln \rho$ when $\alpha = 1$. Noting that $H^{k-2}(\mathbb{T})$ is an algebra for $k \geq 3$, we then bound

$$\begin{aligned} & \left| \int_{\mathbb{T}} \partial_x^{k-1} (u \partial_x w + c_\mu \rho^{\alpha-2} \partial_x \rho \partial_x w) \partial_x^{k-1} w \right| \\ & \leq C \|\partial_x^k w\|_{L^2} \|u\|_{H^{k-2}} \|w\|_{H^{k-1}} + C \|\partial_x^k w\|_{L^2} \left\| \frac{\rho^{\alpha-1}}{\alpha-1} \right\|_{H^{k-1}} \|w\|_{H^{k-1}} \\ & \leq \frac{1}{20c} \|\partial_x^k w\|_{L^2}^2 + C' \|u\|_{H^{k-2}}^2 \|w\|_{H^{k-1}}^2 + C' \left\| \frac{\rho^{\alpha-1}}{\alpha-1} \right\|_{H^{k-1}}^2 \|w\|_{H^{k-1}}^2 \\ & \leq \frac{1}{20c} \|\partial_x^k w\|_{L^2}^2 + M(E_k, \|f\|_{L^2(0,T;H^{k-2})}, \frac{1}{\underline{\rho}}, T) (\|\partial_x^{k-1} w\|_{L^2}^2 + 1) \end{aligned}$$

3. The nonlinear term:

$$\begin{aligned} & \left| \int_{\mathbb{T}} \partial_x^{k-1} (\rho^{-\alpha} w^2) \partial_x^{k-1} w \right| = \left| \int_{\mathbb{T}} \partial_x^{k-2} (\rho^{-\alpha} w^2) \partial_x^k w \right| \\ & \leq C \|\rho^{-\alpha}\|_{H^{k-2}} \|w\|_{H^{k-2}}^2 \|\partial_x^k w\|_{L^2} \\ & \leq \frac{1}{20c} \|\partial_x^k w\|_{L^2}^2 + C' \|\rho^{-\alpha}\|_{H^{k-2}}^2 \|w\|_{H^{k-2}}^4 \\ & \leq \frac{1}{20c} \|\partial_x^k w\|_{L^2}^2 + M(E_k, \|f\|_{L^2(0,T;H^{k-2})}, \frac{1}{\underline{\rho}}, T). \end{aligned}$$

4. The zero-order term:

$$\begin{aligned} & \left| \int_{\mathbb{T}} \partial_x^{k-1} (\rho^{2\gamma-\alpha}) \partial_x^{k-1} w \right| \leq C \|\rho^{2\gamma-\alpha}\|_{H^{k-1}} \|\partial_x^{k-1} w\|_{L^2} \\ & \leq M(E_k, \|f\|_{L^2(0,T;H^{k-2})}, \frac{1}{\underline{\rho}}, T) \|\partial_x^{k-1} w\|_{L^2}. \end{aligned}$$

5. The forcing term:

$$\begin{aligned}
\left| \int_{\mathbb{T}} \partial_x^{k-1} (\rho^\alpha \partial_x f) \partial_x^{k-1} w \right| &= \left| \int_{\mathbb{T}} \partial_x^{k-2} (\rho^\alpha \partial_x f) \partial_x^k w \right| \\
&\leq C \|\rho^\alpha\|_{H^{k-2}} \|\partial_x f\|_{H^{k-2}} \|\partial_x^k w\|_{L^2} \\
&\leq \frac{1}{20c} \|\partial_x^k w\|_{L^2}^2 + M(E_k, \|f\|_{L^2(0,T;H^{k-2})}, \frac{1}{\underline{\rho}}, T) \|f\|_{H^{k-1}}^2.
\end{aligned}$$

Putting the estimates 1. through 5. together, we obtain

$$\begin{aligned}
\frac{1}{2} \frac{d}{dt} \|\partial_x^{k-1} w\|_{L^2}^2 &\leq \frac{-2}{5c} \|\partial_x^k w\|_{L^2}^2 + M(E_k, \|f\|_{L^2(0,T;H^{k-2})}, \frac{1}{\underline{\rho}}, T) \|\partial_x^{k-1} w\|_{L^2}^2 \\
&\quad + M(E_k, \|f\|_{L^2(0,T;H^{k-2})}, \frac{1}{\underline{\rho}}, T) (\|f\|_{H^{k-1}}^2 + 1).
\end{aligned}$$

Combining this with (2.1.64) and Grönwall's lemma leads to

$$\begin{aligned}
&\|\partial_x^k \rho\|_{L^\infty(0,T;L^2)}^2 + \|\partial_x^{k-1} w\|_{L^\infty(0,T;L^2)}^2 + \|\partial_x^k w\|_{L^2(0,T;L^2)}^2 \\
&\leq M \left(\|\partial_x^k \rho_0\|_{L^2}^2 + \|\partial_x^{k-1} w_0\|_{L^2}^2 + \|f\|_{L^2(0,T;H^{k-1})}^2 + T \right) \exp \left(M (\|\partial_x^k u\|_{L^1(0,T;L^2)} + T) \right)
\end{aligned}$$

where we denoted

$$M := M(E_k, \|f\|_{L^2(0,T;H^{k-2})}, \frac{1}{\underline{\rho}}, T)$$

and used the fact that the $L^2(0, T; H^k)$ norm of u is controlled by M .

It follows easily from this that $\|\partial_x^k u\|_{L^\infty(0,T;L^2)}$ and $\|\partial_x^{k+1} u\|_{L^2(0,T;L^2)}$ can be controlled by the same bound. This finishes the proof of (2.1.62). $\square$

In view of Lemmas 2.1.11, 2.1.12 and 2.1.13 we have proved that

$$\begin{aligned}
&\sup_{T \in [0, T^*)} \|\rho\|_{L^\infty(0,T;H^k)} + \sup_{T \in [0, T^*)} \|u\|_{L^\infty(0,T;H^k)} + \sup_{T \in [0, T^*)} \|u\|_{L^2(0,T;H^{k+1})} \\
&\leq M_k \left(\|(\rho_0, u_0)\|_{H^k \times H^k}, \|f\|_{L^2(0,T^*;H^{\max\{k-1,1\}})}, \frac{1}{\underline{\rho}}, T^* \right) < \infty
\end{aligned} \tag{2.1.65}$$

for $k \geq 1$. Appealing to local existence, established by Prop. 2.1.2, the solution can be

extended past T^* .

2.1.5 Proof of Theorem 2.1.2

We assume here that $c_p > 0$ and that $\alpha \in (\frac{1}{2}, 1]$, $\gamma \geq 2\alpha$. By Prop. 2.1.2, there exists a positive time T_0 such that problem (2.1.1)-(2.1.3) has a unique solution (ρ, u) on $[0, T_0]$ such that

$$\rho \in \mathcal{C}(0, T_0; H^k), \quad u \in \mathcal{C}(0, T_0; H^k) \cap L^2(0, T_0; H^{k+1}), \quad k \geq 3, \quad (2.1.66)$$

and $\rho > 0$ on $[0, T_0]$. Let T^* be the maximal lifetime of the classical solution (ρ, u) , so that, by Theorem 2.1.1,

$$\inf_{t \in (0, T^*)} \min_{x \in \mathbb{T}} \rho(x, t) = 0. \quad (2.1.67)$$

We claim that $T^* = \infty$. We will argue by contradiction. Let us note that the H^k regularity, $k \geq 3$, of (ρ, u) suffices to justify all the calculations below. Recall from the proof of Lemma 2.1.8 in Section 2.1.8, that

$$X = u + c_\mu \rho^{\alpha-2} \partial_x \rho, \quad (2.1.68)$$

defined also in equation (2.1.107), satisfies

$$\partial_t X + u \partial_x X = -\gamma \frac{c_p}{c_\mu} \rho^{\gamma-\alpha} (X - u) + f = -\gamma \frac{c_p}{c_\mu} \rho^{\gamma-\alpha} X + \gamma \frac{c_p}{c_\mu} \rho^{\gamma-\alpha} u + f. \quad (2.1.69)$$

By Lemma 2.1.9 1., we have

$$\|\rho\|_{L^\infty(0, T; L^\infty(\mathbb{T}))} \leq M(E_1, \|f\|_{L^2(0, T; L^\infty)}, T). \quad (2.1.70)$$

Since $\gamma \geq 2\alpha \geq \alpha + \frac{1}{2}$ for $\alpha \in (\frac{1}{2}, 1]$, combining the above estimate with (2.1.26), we have

$$\|\rho^{\gamma-\alpha} u\|_{L^\infty(0, T; L^2(\mathbb{T}))} \leq M(E_1, \|f\|_{L^2(0, T; L^\infty)}, T). \quad (2.1.71)$$

Note also

$$\partial_x(\rho^{\gamma-\alpha}u) = (\sqrt{\rho}\partial_x u)\rho^{\gamma-\alpha-\frac{1}{2}} + (\gamma - \alpha)\rho^{\gamma-2\alpha}(\rho^{\alpha-\frac{3}{2}}\partial_x \rho)(\sqrt{\rho}u)$$

Now, estimate (2.1.39) implies

$$\|(\rho^{\alpha-\frac{3}{2}}\partial_x \rho)\|_{L^2(0,T;L^2(\mathbb{T}))} \leq M(E_1, \|f\|_{L^2(0,T;L^\infty)}, T).$$

Putting together this, (2.1.26), (2.1.27), (2.1.70), and the assumption that $\gamma \geq 2\alpha$ we deduce that

$$\|\partial_x(\rho^{\gamma-\alpha}u)\|_{L^2(0,T;L^1(\mathbb{T}))} \leq M(E_1, \|f\|_{L^2(0,T;L^\infty)}, T).$$

which combined with (2.1.71) yields

$$\|\rho^{\gamma-\alpha}u\|_{L^2(0,T;W^{1,1})} \leq M(E_1, \|f\|_{L^2(0,T;L^\infty)}, T). \quad (2.1.72)$$

Since (2.1.69) is a transport equation we then have

$$\begin{aligned} & \|X\|_{L^\infty(0,T;L^\infty)} \\ & \leq (\|X_0\|_{L^\infty} + \gamma \frac{c_p}{c_\mu} \|\rho^{\gamma-\alpha}u\|_{L^1(0,T;L^\infty)} + \|f\|_{L^1(0,T;L^\infty)}) \exp\left(\gamma \frac{c_p}{c_\mu} \|\rho^{\gamma-\alpha}\|_{L^1(0,T;L^\infty)}\right) \quad (2.1.73) \\ & \leq M(E_1, \|X_0\|_{L^\infty}, \|f\|_{L^2(0,T;L^\infty)}, T). \end{aligned}$$

Recall that $X = u + \frac{\partial_x \rho}{\rho^2} \mu(\rho) = u + c_\mu \rho^{\alpha-2} \partial_x \rho$, hence $X \rho^{\gamma-\alpha} = u \rho^{\gamma-\alpha} + c_\mu \rho^{\gamma-2} \partial_x \rho$. It then follows from (2.1.70), (2.1.72) and (2.1.73) that

$$\|\rho^{\gamma-2} \partial_x \rho\|_{L^2(0,T;L^\infty)} \leq M(E_1, \|X_0\|_{L^\infty}, \|f\|_{L^2(0,T;L^\infty)}, T). \quad (2.1.74)$$

Using (2.1.1) and (2.1.2) we obtain

$$\partial_t u + (u - \frac{\mu'(\rho)\partial_x \rho}{\rho})\partial_x u = \frac{\mu(\rho)}{\rho}\partial_x^2 u - \frac{p'(\rho)\partial_x \rho}{\rho} + f = c_\mu \rho^{\alpha-1}\partial_x^2 u - c_p \gamma \rho^{\gamma-2}\partial_x \rho + f. \quad (2.1.75)$$

Using the maximum principle (see the argument leading to (2.1.89) below and a similar argument for the minimum) and the bound (2.1.74) gives

$$\begin{aligned} \|u\|_{L^\infty(0,T;L^\infty)} &\leq \|u_0\|_{L^\infty} + c_p \gamma \|\rho^{\gamma-2}\partial_x \rho\|_{L^1(0,T;L^\infty)} + \|f\|_{L^1(0,T;L^\infty)} \\ &\leq M(E_1, \|(X_0, u_0)\|_{L^\infty}, \|f\|_{L^2(0,T;L^\infty)}, T). \end{aligned} \quad (2.1.76)$$

From the definition of X and (2.1.73), this yields

$$\|\partial_x \rho^{\alpha-1}\|_{L^\infty(0,T;L^\infty)} \leq M(E_1, \|(X_0, u_0)\|_{L^\infty}, \|f\|_{L^2(0,T;L^\infty)}, T) \quad (2.1.77)$$

when $\alpha < 1$, and

$$\|\partial_x \ln \rho\|_{L^\infty(0,T;L^\infty)} \leq M(E_1, \|(X_0, u_0)\|_{L^\infty}, \|f\|_{L^2(0,T;L^\infty)}, T) \quad (2.1.78)$$

when $\alpha = 1$.

When $\alpha < 1$, the continuity equation implies

$$\partial_t(\rho^{\alpha-1}) = -(\alpha-1)\partial_x(u\rho)\rho^{\alpha-2}. \quad (2.1.79)$$

Integrating this in space and time and using the definition of X leads to

$$\begin{aligned} \int_{\mathbb{T}} \rho^{\alpha-1}(x, T) dx &= \int_{\mathbb{T}} \rho_0^{\alpha-1} dx + (\alpha-1)(\alpha-2) \int_0^t \int_{\mathbb{T}} (u\rho\rho^{\alpha-3}\partial_x \rho)(x, z) dx dz \\ &= \int_{\mathbb{T}} \rho_0^{\alpha-1} dx + \frac{1}{c_\mu}(\alpha-2)(\alpha-1) \int_0^t \int_{\mathbb{T}} (uc_\mu \rho^{\alpha-2}\partial_x \rho)(x, z) dx dz \\ &\leq \int_{\mathbb{T}} \rho_0^{\alpha-1} dx + C \int_0^t \int_{\mathbb{T}} X^2(x, z) dx dz, \end{aligned} \quad (2.1.80)$$

valid for $0 \leq t \leq T$.

Similarly, when $\alpha = 1$ we have

$$\left| \int_{\mathbb{T}} \ln \rho(x, t) dx \right| \leq \left| \int_{\mathbb{T}} \ln \rho_0 dx \right| + C \int_0^t \int_{\mathbb{T}} X^2(x, z) dx dz, \quad 0 \leq t \leq T. \quad (2.1.81)$$

Then by virtue of (2.1.73), (2.1.76), (2.1.77), (2.1.80), Poincaré–Wirtinger’s inequality and Sobolev embedding we deduce that

$$\|\rho^{\alpha-1}\|_{L^\infty(0,T;L^\infty)} \leq M(E_1, \|(X_0, u_0)\|_{L^\infty}, \|\rho_0^{\alpha-1}\|_{L^1}, \|f\|_{L^2(0,T;L^\infty)}, T)$$

if $\alpha < 1$.

On the other hand, if $\alpha = 1$, (2.1.70) combined with (2.1.81), Poincaré–Wirtinger’s inequality and Sobolev embedding, yields

$$\|\ln \rho\|_{L^\infty(0,T;L^\infty)} \leq M(E_1, \|(X_0, u_0)\|_{L^\infty}, \|\ln \rho_0\|_{L^1}, \|f\|_{L^2(0,T;L^\infty)}, T).$$

Consequently

$$\inf_{(x,t) \in \mathbb{T} \times [0,T]} \rho(x, t) \geq \mathcal{F} \left(M(E_0, \|(X_0, u_0)\|_{L^\infty}, \|\rho_0^{\alpha-1}\|_{L^1} + \|\ln \rho_0\|_{L^1}, \|f\|_{L^2(0,T;L^\infty)}, T) \right)$$

where

$$\mathcal{F}(z) = \begin{cases} z^{\frac{1}{\alpha-1}} & \text{if } \alpha < 1, \\ e^{-z} & \text{if } \alpha = 1. \end{cases} \quad (2.1.82)$$

Therefore,

$$\begin{aligned} & \inf_{(x,t) \in \mathbb{T} \times [0,T^*)} \rho(x, t) \\ & \geq \mathcal{F} \left(M(E_0, \|(X_0, u_0)\|_{L^\infty}, \|\rho_0^{\alpha-1}\|_{L^1}, \|\ln \rho_0\|_{L^1}, \|f\|_{L^2(0,T^*;L^\infty)}, T^*) \right) > 0 \end{aligned}$$

which contradicts (2.1.67).

2.1.6 Proof of Theorem 2.1.3

Recall the assumptions (2.1.7) and (2.1.8) Assume that $c_p > 0$ and either

$$\alpha > \frac{1}{2}, \quad \gamma \in [\alpha, \alpha + 1], \quad \gamma \neq 1 \quad \text{or} \quad (2.1.83)$$

$$\alpha \geq 0, \quad \gamma \in [\alpha, \alpha + 1], \quad \gamma > 1. \quad (2.1.84)$$

By Prop. 2.1.2, there exists a positive time T_0 such that problem (2.1.1)-(2.1.3) has a unique solution (ρ, u) on $[0, T_0]$ such that

$$\rho \in \mathcal{C}(0, T_0; H^k), \quad u \in \mathcal{C}(0, T_0; H^k) \cap L^2(0, T_0; H^{k+1}), \quad k \geq 4, \quad (2.1.85)$$

and $\rho > 0$ on $[0, T_0]$. Let T^* be the maximal existence time. We claim that $T^* = \infty$. Assume by contradiction that T^* is finite. By Theorem 2.1.1 we have

$$\inf_{t \in [0, T^*)} \min_{x \in \mathbb{T}} \rho(x, t) = 0. \quad (2.1.86)$$

From Lemma 2.1.1, the w equation (2.1.45) is

$$\begin{aligned} \partial_t w &= c_\mu \rho^{\alpha-1} \partial_x^2 w - (u + c_\mu \rho^{\alpha-2} \partial_x \rho) \partial_x w + \frac{c_p}{c_\mu} (\gamma - 2(\alpha + 1)) \rho^{\gamma-\alpha} w \\ &\quad - \frac{1}{c_\mu} (\alpha + 1) \rho^{-\alpha} w^2 + \frac{c_p^2}{c_\mu} (\gamma - (\alpha + 1)) \rho^{2\gamma-\alpha}. \end{aligned} \quad (2.1.87)$$

Note that the assumption $f(x, t) = f(t)$ was used to have $\partial_x f = 0$. It follows from (2.1.85) and the equation (2.1.87) that

$$w \in \mathcal{C}(0, T; H^3) \cap L^2(0, T; H^4), \quad \partial_t w \in \mathcal{C}(0, T; H^1) \subset \mathcal{C}(\mathbb{T} \times [0, T])$$

Thus, $w \in \mathcal{C}^1(\mathbb{T} \times [0, T])$ and thus the function

$$w_M(t) := \max_{x \in \mathbb{T}} w(x, t) \quad (2.1.88)$$

is Lipschitz continuous on $[0, T]$. According to the Rademacher theorem, w_M is differentiable almost everywhere on $[0, T]$. There exists for each $t \in [0, T^*)$ a point x_t such that

$$w_M(t) = w(x_t, t).$$

Let $t \in (0, T)$ be a point at which w_M is differentiable. We have

$$\begin{aligned} w'_M(t) &= \lim_{h \rightarrow 0^+} \frac{w_M(t+h) - w_M(t)}{h} \\ &= \lim_{h \rightarrow 0^+} \frac{w(x_{t+h}, t+h) - w(x_t, t)}{h} \\ &\geq \lim_{h \rightarrow 0^+} \frac{w(x_t, t+h) - w(x_t, t)}{h} = \partial_t w(x_t, t). \end{aligned}$$

On the other hand,

$$\begin{aligned} w'_M(t) &= \lim_{h \rightarrow 0^+} \frac{w_M(t) - w_M(t-h)}{h} \\ &= \lim_{h \rightarrow 0^+} \frac{w(x_t, t) - w(x_{t-h}, t-h)}{h} \\ &\leq \lim_{h \rightarrow 0^+} \frac{w(x_t, t) - w(x_t, t-h)}{h} = \partial_t w(x_t, t). \end{aligned}$$

Thus, $w'_M(t) = \partial_t w(x_t, t)$ if w_M is differentiable at t . We deduce from this and equation (2.1.87) that for almost every $t \in (0, T)$,

$$\partial_t w_M \leq A(t)w_M + B(t)w_M^2 + C(t) \quad (2.1.89)$$

with

$$\begin{aligned} A(t) &:= \frac{c_p}{c_\mu} (\gamma - 2(\alpha + 1)) \rho(x_t)^{\gamma-\alpha} \\ B(t) &:= -\frac{1}{c_\mu} (\alpha + 1) \rho(x_t)^{-\alpha} \\ C(t) &:= \frac{c_p^2}{c_\mu} (\gamma - (\alpha + 1)) \rho(x_t)^{2\gamma-\alpha}. \end{aligned}$$

where we used the facts that $\partial_x^2 w(x_t, t) \leq 0$ and $\partial_x w(x_t, t) = 0$. Note that $B(t) \leq 0$. In addition, the function C is nonpositive under the conditions (2.1.7). The condition on the initial data (2.1.9) is equivalent to $w_M(0) \leq 0$. We deduce that

$$w(t) \leq 0, \quad \forall t < T^*. \quad (2.1.90)$$

At the point y_t where the density attains its minimum value $\rho_m := \rho(y_t, t)$, ρ_m satisfies

$$\partial_t \rho_m = -\partial_x u(y_t) \rho_m = -\frac{w(y_t)}{c_\mu} \rho_m^{1-\alpha} - \frac{c_p}{c_\mu} \rho_m^{\gamma-\alpha+1} \geq -\frac{c_p}{c_\mu} \rho_m^{\gamma-\alpha+1} \quad (2.1.91)$$

where we used (2.1.90). Provided that $\gamma \neq \alpha$, this implies the differential inequality

$$\frac{1}{(\alpha - \gamma)} \partial_t (\rho_m^{\alpha-\gamma}) \geq -\frac{c_p}{c_\mu}. \quad (2.1.92)$$

Since $\alpha < \gamma$, we find

$$\partial_t (\rho_m^{\alpha-\gamma}) \leq \frac{c_p}{c_\mu} (\gamma - \alpha) \quad (2.1.93)$$

which implies

$$\rho_m(t) \geq \left(\rho_m(0)^{\alpha-\gamma} + t \frac{c_p}{c_\mu} (\gamma - \alpha) \right)^{\frac{1}{\alpha-\gamma}}, \quad \forall t < T^* \quad (2.1.94)$$

Since $c_p/c_\mu > 0$, this implies that

$$\inf_{t \in [0, T^*)} \min_{x \in \mathbb{T}} \rho(x, t) \geq \left(\rho_m(0)^{\alpha-\gamma} + T^* \frac{c_p}{c_\mu} (\gamma - \alpha) \right)^{\frac{1}{\alpha-\gamma}} > 0 \quad (2.1.95)$$

which contradicts the assumption (2.1.86). We conclude that the solution (ρ, u) is global in time.

On the other hand, when $\alpha = \gamma$ we have

$$\partial_t \ln \rho_m \geq -\frac{c_p}{c_\mu} \quad (2.1.96)$$

and thus

$$\rho_m(t) \geq \rho_m(0) \exp \left(-t \frac{c_p}{c_\mu} \right) > 0 \quad (2.1.97)$$

which again leads to a contradiction with (2.1.86).

Remark 2.1.14. With a more refined maximum principle argument, one can relax the regularity requirement of $k \geq 4$ which we used to conclude that (2.1.88) is Lipschitz continuous on $[0, T]$.

2.1.7 Proof of Theorem 2.1.5

In this section, we give an upper bound for the long-time average maximum density, assuming that the forcing has zero mean in space. This follows by an application of the Bresch–Desjardins’s entropy and the following elementary lemma.

Lemma 2.1.15. *Let $m \geq \frac{1}{2}$. If $h^m \in W^{1,1}(\mathbb{T})$ then we have*

$$\|h\|_{L^\infty(\mathbb{T})} \leq 2 \|\partial_x(h^m)\|_{L^1(\mathbb{T})}^{\frac{1}{m}} + 4 \|h\|_{L^1(\mathbb{T})}. \quad (2.1.98)$$

Proof of Lemma 2.1.15. Since $h \in W^{1,1}(\mathbb{T}) \subset \mathcal{C}^0(\mathbb{T})$, we have $h \in \mathcal{C}^0(\mathbb{T})$. In particular, there

exists a point $x_0 \in \mathbb{T}$ such that $|h(x_0)| \leq \sqrt{2}\|h\|_{L^1(\mathbb{T})}$. For all $x \in \mathbb{T}$ we have

$$h^m(x) = \int_{x_0}^x \partial_y(h^m(y))dy + h^m(x_0),$$

hence

$$|h(x)|^m \leq \|\partial_x h^m\|_{L^1(\mathbb{T})} + |h(x_0)|^m \leq \|\partial_x(h^m)\|_{L^1(\mathbb{T})} + \sqrt{2}\|h\|_{L^1(\mathbb{T})}^m.$$

In view of the elementary inequality

$$(a+b)^{\frac{1}{m}} \leq 2a^{\frac{1}{m}} + 2b^{\frac{1}{m}}, \quad a, b, m > 0,$$

we thus obtain (2.1.98). □

Proof of Theorem 2.1.5. Recall our assumptions

$$\gamma \in [\max\{2 - \alpha, \alpha\}, \alpha + 1], \quad \alpha \geq 1/2, \quad \text{and} \quad c_p, c_\mu > 0. \quad (2.1.99)$$

Next, by Lemma 2.1.8, the entropy

$$s = \frac{\rho}{2} \left| u + \frac{\partial_x \rho}{\rho^2} \mu(\rho) \right|^2 + \pi(\rho). \quad (2.1.100)$$

satisfies

$$\frac{d}{dt} \int_{\mathbb{T}} s(x, t) dx = - \int_{\mathbb{T}} |\partial_x \rho|^2 \mu(\rho) \frac{p'(\rho)}{\rho^2} dx + \int_{\mathbb{T}} f \rho \left(u + \frac{\partial_x \rho}{\rho^2} \mu(\rho) \right) dx. \quad (2.1.101)$$

Integrating this in time yields

$$\begin{aligned} \int_{\mathbb{T}} s(x, T) dx - \int_{\mathbb{T}} s(x, 0) dx &+ c_p c_\mu \gamma \int_0^T \int_{\mathbb{T}} \rho^{\alpha+\gamma-3} |\partial_x \rho|^2 dx dt \\ &= \int_0^T \int_{\mathbb{T}} f \rho u dx dt + c_\mu \int_0^T \int_{\mathbb{T}} f \rho^{\alpha-1} \partial_x \rho dx dt. \end{aligned}$$

Using the assumption (2.1.10) we calculate

$$\begin{aligned}\int_0^T \int_{\mathbb{T}} f \rho u \, dx dt &= - \int_0^T \int_{\mathbb{T}} g \partial_x(\rho u) \, dx dt = \int_0^T \int_{\mathbb{T}} g \partial_t \rho \, dx dt \\ &= \int_{\mathbb{T}} (g \rho)(x, T) \, dx - \int_{\mathbb{T}} (g \rho)(x, 0) \, dx - \int_0^T \int_{\mathbb{T}} \rho \partial_t g \, dx dt.\end{aligned}$$

This implies

$$\begin{aligned}\left| \int_0^T \int_{\mathbb{T}} f \rho u \, dx dt \right| &\leq 2 \|g\|_{L^\infty(0,T;L^\infty)} \|\rho_0\|_1 + \|\partial_t g\|_{L^1(0,T;L^\infty)} \|\rho_0\|_1 \\ &\leq 2 \|g\|_{L^\infty(0,T;L^\infty)} \|\rho_0\|_1 + T \|\partial_t g\|_{L^\infty(0,T;L^\infty)} \|\rho_0\|_1.\end{aligned}$$

On the other hand, using Cauchy–Schwarz, we have

$$\begin{aligned}\left| c_\mu \int_0^T \int_{\mathbb{T}} f \rho^{\alpha-1} \partial_x \rho \, dx dt \right| &\leq \frac{1}{2} c_p c_\mu \gamma \int_0^T \int_{\mathbb{T}} \rho^{\alpha+\gamma-3} |\partial_x \rho|^2 \, dx dt + C \int_0^T \int_{\mathbb{T}} \rho^{\alpha-\gamma+1} f^2 \, dx dt \\ &\leq \frac{1}{2} c_p c_\mu \gamma \int_0^T \int_{\mathbb{T}} \rho^{\alpha+\gamma-3} |\partial_x \rho|^2 \, dx dt + CT(1 + \|\rho_0\|_1) \|f\|_{L^\infty(0,T;L^\infty)}^2.\end{aligned}$$

Here, C is a constant which depends only on c_γ, c_p and γ . We have used the assumption (2.1.99) that γ belongs to the range $\gamma \in [\max\{2 - \alpha, \alpha\}, \alpha + 1]$ with $\alpha \geq 1/2$ to have $0 \leq \alpha - \gamma + 1 \leq 1$.

Note that the allowed range of γ and α requires that $\gamma \geq 3/2$ always. Since, in particular $\gamma > 1$ we have $\pi(\rho) \geq 0$ and $s \geq 0$. Thus, putting all together, we obtain the bound

$$\begin{aligned}\frac{1}{2} c_p c_\mu \gamma \int_0^T \int_{\mathbb{T}} \rho^{\alpha+\gamma-3} |\partial_x \rho|^2 \, dx dt &\leq 2 \|g\|_{L^\infty(0,T;L^\infty)} \|\rho_0\|_1 + T \|\partial_t g\|_{L^\infty(0,T;L^\infty)} \|\rho_0\|_1 \\ &\quad + CT(1 + \|\rho_0\|_1) \|\partial_x g\|_{L^\infty(0,T;L^\infty)}^2 + \int_{\mathbb{T}} s(x, 0) \, dx.\end{aligned}$$

We thus obtain

$$\frac{1}{2}c_p c_\mu \gamma \int_0^T \int_{\mathbb{T}} \rho^{\alpha+\gamma-3} |\partial_x \rho|^2 dx dt \leq M_1 T + M_0,$$

where M_0 is a constant which depends only on c_μ , c_p , γ , α , $\|\rho_0\|_{L^\infty}$, $\|\rho_0^{-1}\|_{L^\infty}$, $\|u_0\|_{L^2}$, $\|\partial_x \rho_0\|_{L^2}$, $\|g\|_{L^\infty(0,T;L^\infty)}$, and M_1 depends only on c_μ , c_p , γ , $\|\rho_0\|_{L^1}$, $\|\partial_t g\|_{L^\infty(0,T;L^\infty)}$, and $\|\partial_x g\|_{L^\infty(0,T;L^\infty)}$.

In particular,

$$\int_0^T \int_{\mathbb{T}} |\partial_x (\rho^{\frac{1}{2}(\alpha+\gamma-1)})|^2 dx dt \leq M_3 T + M_2,$$

where $M_{i+2} = \frac{(\alpha+\gamma-1)^2}{2c_p c_\mu \gamma} M_i$, for $i = 0, 1$. Here, we used the fact that $\alpha + \gamma - 1 > 0$.

By assumption (2.1.99) we have that $\alpha + \gamma \geq 2 \max\{1, \alpha\} \geq 2$ which implies $\frac{1}{m} \leq 2$. We now apply Lemma 2.1.15 with $m := \frac{1}{2}(\alpha + \gamma - 1)$. Using the embedding $L^2(\mathbb{T}) \subset L^1(\mathbb{T})$, we obtain

$$\int_0^T \|\rho(\cdot, t)\|_{L^\infty} dt \leq 2 \int_0^T \|\partial_x(\rho^m)\|_{L^2}^{\frac{1}{m}} dt + 4T \|\rho_0\|_{L^1}.$$

Consequently,

$$\begin{aligned} & \int_0^T \|\rho(\cdot, t)\|_{L^\infty} dt \\ & \leq 2 \int_0^T (\|\partial_x(\rho^m)\|_{L^2}^2 + 1) dt + 4T \|\rho_0\|_{L^1} \leq 2(M_3 T + M_2) + 2T + 4T \|\rho_0\|_{L^1}. \end{aligned}$$

Hence,

$$\frac{1}{T} \int_0^T \|\rho(\cdot, t)\|_{L^\infty} dt \leq (2M_3 + 2 + 4\|\rho_0\|_{L^1}) + \frac{2}{T} M_2, \quad (2.1.102)$$

and the claim follows, with the definition

$$C_1 = 2M_2, \quad C_2 := 2M_3 + 2 + 4\|\rho_0\|_{L^1}. \quad (2.1.103)$$

□

2.1.8 Technical tools: Bresch-Desjardins's entropy and local well-posedness

Bresch-Desjardins's entropy

For the sake of completeness we present the proof of Lemma 2.1.8 which essentially follows from [12, 14, 15]. From the continuity equation (2.1.1), any smooth $\xi(\rho)$ satisfies

$$\partial_t \xi(\rho) = \partial_t \rho \xi'(\rho) = -\partial_x(u\rho) \xi'(\rho) = -u \partial_x \xi(\rho) - \rho(\partial_x u) \xi'(\rho). \quad (2.1.104)$$

Using equation (2.1.104) applied to the function $\partial_x \xi(\rho)$, we find the evolution of $\rho \partial_x \xi(\rho)$:

$$\begin{aligned} \partial_t(\rho \partial_x \xi(\rho)) &= -\partial_x(\rho u) \partial_x \xi(\rho) + \rho \partial_t \partial_x \xi(\rho) \\ &= -\partial_x(\rho u) \partial_x \xi(\rho) - \rho \partial_x(u \partial_x \xi(\rho) + \rho(\partial_x u) \xi'(\rho)) \\ &= -\partial_x(\rho u) \partial_x \xi(\rho) - \rho \partial_x u \partial_x \xi(\rho) - \rho u \partial_x^2 \xi(\rho) - \rho \partial_x(\rho(\partial_x u) \xi'(\rho)) \\ &= -\partial_x(\rho u \partial_x \xi(\rho)) - \rho \partial_x u \partial_x \xi(\rho) - \rho \partial_x(\rho(\partial_x u) \xi'(\rho)) \\ &= -\partial_x(\rho u \partial_x \xi(\rho)) - \partial_x(\rho^2(\partial_x u) \xi'(\rho)). \end{aligned} \quad (2.1.105)$$

Then, letting $X := u + \partial_x \xi(\rho)$, combining equation (2.1.105) with the momentum equation (2.1.2) yields

$$\partial_t(\rho X) = -\partial_x(\rho u X) - \partial_x p(\rho) + \partial_x(\mu(\rho) \partial_x u) - \partial_x(\rho^2(\partial_x u) \xi'(\rho)) + \rho f. \quad (2.1.106)$$

We now choose $\rho^2 \xi'(\rho) = \mu(\rho)$, so that the final two terms in (2.1.106) cancel. Thus with this choice,

$$X = u + \frac{\partial_x \rho}{\rho^2} \mu(\rho) \quad (2.1.107)$$

and, by (2.1.106), ρX satisfies

$$\partial_t(\rho X) = -\partial_x(\rho u X) - \partial_x p(\rho) + \rho f. \quad (2.1.108)$$

Whence, we obtain

$$\partial_t(\rho X^2) = -\partial_x(\rho u X^2) - 2X\partial_x p(\rho) + 2\rho f X. \quad (2.1.109)$$

Integrating in space

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int_{\mathbb{T}} (\rho X^2)(x, t) dx \\ &= - \int_{\mathbb{T}} \rho u \frac{\partial_x p(\rho)}{\rho} dx - \int_{\mathbb{T}} |\partial_x \rho|^2 \mu(\rho) \frac{p'(\rho)}{\rho^2} dx + \int_{\mathbb{T}} f \rho \left(u + \frac{\partial_x \rho}{\rho^2} \mu(\rho) \right) dx \\ &= - \int_{\mathbb{T}} \rho u \partial_x \pi'(\rho) dx - \int_{\mathbb{T}} |\partial_x \rho|^2 \mu(\rho) \frac{p'(\rho)}{\rho^2} dx + \int_{\mathbb{T}} f \rho \left(u + \frac{\partial_x \rho}{\rho^2} \mu(\rho) \right) dx \\ &= - \frac{d}{dt} \int_{\mathbb{T}} \pi(\rho) dx - \int_{\mathbb{T}} |\partial_x \rho|^2 \mu(\rho) \frac{p'(\rho)}{\rho^2} dx + \int_{\mathbb{T}} f \rho \left(u + \frac{\partial_x \rho}{\rho^2} \mu(\rho) \right) dx. \end{aligned} \quad (2.1.110)$$

The global balance (2.1.31) for entropy $s := \frac{1}{2}\rho X^2 + \pi(\rho)$ follows.

Local well-posedness

Proposition 2.1.2. *Assume that $p : \mathbb{R}^+ \rightarrow \mathbb{R}$ and $\mu : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ are $\mathcal{C}^\infty$ functions away from zero. Let ρ_0 and u_0 belong to $H^k(\mathbb{T})$ for an integer $k \geq 1$, such that $r_0 := \min_{x \in \mathbb{T}} \rho_0 > 0$. Suppose that for all $T > 0$*

$$f \in L^2(0, T; H^{k-1}(\mathbb{T})).$$

Then, there exists a $T_0 > 0$ depending only on $\|(\rho_0, u_0)\|_{H^k(\mathbb{T}) \times H^k(\mathbb{T})}$, r_0 and f , and a unique strong solution (ρ, u) to (2.1.1)-(2.1.3) on $[0, T_0]$ with data (ρ_0, u_0) such that

$$\rho \in \mathcal{C}(0, T_0; H^k(\mathbb{T})), \quad u \in \mathcal{C}(0, T_0; H^k(\mathbb{T})) \cap L^2(0, T_0; H^{k+1}(\mathbb{T}))$$

and $\rho(x, t) > \frac{r_0}{2}$ for all $(x, t) \in \mathbb{T} \times [0, T_0]$.

Proof. Step 0. (Iteration Scheme) We are going to set up an iteration argument and prove that the iterates converge to the desired solution. Let us first suppose that the initial data ρ_0, u_0 are smooth, and let us define $r_0 := \min_{x \in \mathbb{T}} \rho_0$.

Let us initialize our scheme as follows:

$$(\rho_0(x, t), u_0(x, t)) := (\rho_0(x), u_0(x)),$$

$$\rho_1(x, t) = \rho_0(x),$$

and we define $u_1(x, t)$ so that

$$\begin{aligned} \partial_t u_1 - \frac{\mu(\rho_1)}{\rho_1} \partial_x^2 u_1 &= -u_0 \partial_x u_0 - \frac{1}{\rho_0} \partial_x p(\rho_0) + \frac{\partial_x \mu(\rho_0)}{\rho_0} \partial_x u_0 + f, \\ u_1|_{t=0} &= u_0(x, 0). \end{aligned} \tag{2.1.111}$$

Let now $n \geq 2$. Given ρ_{n-1}, u_{n-1} , we iteratively define ρ_n first, and subsequently u_n as follows

$$\partial_t \rho_n + u_{n-1} \partial_x \rho_n = -\rho_{n-1} \partial_x u_{n-1}, \tag{2.1.112}$$

$$\partial_t u_n - \frac{\mu(\rho_n)}{\rho_n} \partial_x^2 u_n = -u_{n-1} \partial_x u_{n-1} - \frac{1}{\rho_{n-1}} \partial_x p(\rho_{n-1}) + \frac{\partial_x \mu(\rho_{n-1})}{\rho_{n-1}} \partial_x u_{n-1} + f, \tag{2.1.113}$$

$$(\rho_n, u_n)|_{t=0} = (\rho_0, u_0). \tag{2.1.114}$$

Let $k \geq 1$ be an integer. We let, for ease of notation,

$$A := \|\rho_0\|_{H^k} + \|u_0\|_{H^k}.$$

We are going to prove, by induction on n , that there exists $T_0 > 0$ such that the following assertions hold.

Step 1: There exists $u_1 \in \mathcal{C}^\infty(\mathbb{T} \times [0, T_0])$ satisfying (2.1.111) and

$$\|u_1\|_{L^\infty(0, T_0; H^k)} \leq 2A, \quad \int_0^{T_0} \int_{\mathbb{T}} \frac{\mu(\rho_1)}{\rho_1} (\partial_x^{k+1} u_1)^2 dx dt \leq 8A. \tag{2.1.115}$$

Step 2: For $n \geq 2$, there exists $\rho_n \in \mathcal{C}^\infty(\mathbb{T} \times [0, T_0])$ satisfying (2.1.112), (2.1.114), and

$$\rho_n(x, t) \geq \frac{r_0}{2} \text{ on } \mathbb{T} \times [0, T_0].$$

Furthermore,

$$\|\rho_n\|_{L^\infty(0, T_0; H^k)} \leq 2A.$$

Step 3: There exists $u_n \in \mathcal{C}^\infty(\mathbb{T} \times [0, T_0])$ satisfying (2.1.113), (2.1.114), and

$$\|u_n\|_{L^\infty(0, T_0; H^k)} \leq 2A, \quad \int_0^{T_0} \int_{\mathbb{T}} \frac{\mu(\rho_n)}{\rho_n} (\partial_x^{k+1} u_n)^2 dx dt \leq 8A.$$

Step 4: The sequence (ρ_n, u_n) is Cauchy in the space

$$L^\infty(0, T_0; L^2) \times (L^\infty(0, T_0; L^2) \cap L^2(0, T_0; H^1)).$$

Step 5: There exist

$$u \in \mathcal{C}(0, T_0; H^k) \cap L^2(0, T_0; H^{k+1})$$

and

$$\rho \in \mathcal{C}(0, T_0; H^k)$$

such that (ρ, u) is a strong solution to the system (2.1.1)–(2.1.2) with initial data (ρ_0, u_0) . In particular, if $k = 3$, said solution is a classical solution.

Step 6: The constructed strong solution is unique.

Let us now turn to the details.

Step 1. This is the base case of the induction. The existence of u_1 in the conditions follows from the general theory of linear parabolic equations, using the fact that ρ_0 is bounded from below by r_0 , and that all functions involved are smooth. The bound (2.1.115) is obtained exactly as in **Step 3**, and we omit the details here.

Step 2. Let $n \geq 2$. Let us adopt the following nomenclature:

$$\rho := \rho_n, \quad \eta := \rho_{n-1}, \quad u := u_n, \quad v := u_{n-1}.$$

We recall the induction hypotheses:

$$\begin{aligned} \|v\|_{L^\infty(0,T_0;H^k)} &\leq 2A, & \|\eta\|_{L^\infty(0,T_0;H^k)} &\leq 2A, \\ \int_0^{T_0} \int_{\mathbb{T}} \frac{\mu(\eta)}{\eta} (\partial_x^{k+1} v)^2 dx dt &\leq 8A, & \inf_{t \in [0,T_0]} \inf_{x \in \mathbb{T}} \eta(x,t) &\geq \frac{r_0}{2}. \end{aligned} \quad (2.1.116)$$

Existence up to time T_0 and smoothness for ρ_n follow from the method of characteristics.

In what follows, $M(\cdot, \dots, \cdot)$ will always denote a positive, continuous function increasing in all its arguments. We first notice that, due to the mass equation (2.1.112) and the maximum principle, for all $k \geq 1$ and $0 \leq t \leq T_0$,

$$\inf_{\mathbb{T}} \rho(\cdot, t) \geq \inf_{\mathbb{T}} \rho_0 - \int_0^t \|\eta(\cdot, s) \partial_x v(\cdot, s)\|_{L^\infty} ds \geq \inf_{\mathbb{T}} \rho_0 - M(A) \sqrt{t} \|\partial_x^2 v\|_{L^2(0,t;L^2)}. \quad (2.1.117)$$

Hence, restricting T_0 to be small only as a function of A and r_0 , we have

$$\inf_{t \in [0,T_0]} \inf_{x \in \mathbb{T}} \rho(x, t) \geq \frac{r_0}{2}.$$

We have therefore recovered the last induction hypothesis in (2.1.116).

Let us now differentiate the mass equation (2.1.112) k -times, multiply it by $\partial_x^k \rho$ and integrate by parts

$$\frac{1}{2} \partial_t \int_{\mathbb{T}} (\partial_x^k \rho)^2 dx + \int_{\mathbb{T}} \partial_x^k \rho \partial_x^k (v \partial_x \rho) dx = - \int_{\mathbb{T}} \partial_x^k \rho \partial_x^k (\eta \partial_x v). \quad (2.1.118)$$

If $k = 1$, we obtain

$$\frac{1}{2} \partial_t \|\rho\|_{L^2}^2 \leq C \|\partial_x^2 v\|_{L^2} \|\rho\|_{L^2}^2 + \|\rho\|_{L^2} \|\eta\|_{L^\infty} \|\partial_x v\|_{L^2}, \quad (2.1.119)$$

$$\frac{1}{2}\partial_t \|\partial_x \rho\|_{L^2}^2 \leq C \|\partial_x^2 v\|_{L^2} \|\partial_x \rho\|_{L^2}^2 + 2 \|\partial_x \rho\|_{L^2} \|\partial_x \eta\|_{L^2} \|\partial_x v\|_{L^\infty} \quad (2.1.120)$$

$$+ \|\partial_x \rho\|_{L^2} \|\eta\|_{L^\infty} \|\partial_x^2 v\|_{L^2}. \quad (2.1.121)$$

Combining (2.1.119) and (2.1.121), integrating and using the induction hypotheses, we obtain, for suitable T_0 (depending only on A and r_0)

$$\|\rho\|_{L^\infty(0, T_0; H^1)} \leq 2A. \quad (2.1.122)$$

If $k \geq 2$, in addition to previous estimate (2.1.119), we also have, for the terms appearing in (2.1.118),

$$\begin{aligned} \left| \int_{\mathbb{T}} \partial_x^k \rho \partial_x^k (v \partial_x \rho) dx \right| &= \left| \frac{1}{2} \int_{\mathbb{T}} v \partial_x (\partial_x^k \rho)^2 dx + \int_{\mathbb{T}} \partial_x^k \rho ([\partial_x^k, v] \partial_x \rho) dx \right| \\ &\leq \frac{1}{2} \|\partial_x v\|_{L^\infty} \|\rho\|_{H^k}^2 + \|\rho\|_{H^k} \|[\partial_x^k, v] \partial_x \rho\|_{L^2} \leq C \|v\|_{H^2} \|\rho\|_{H^k}^2 + C \|\rho\|_{H^k}^2 \|v\|_{H^k}. \end{aligned} \quad (2.1.123)$$

Furthermore,

$$\begin{aligned} \left| \int_{\mathbb{T}} \partial_x^k \rho \partial_x^k (\eta \partial_x v) \right| &\leq \|\rho\|_{H^k} \|\eta \partial_x^{k+1} v\|_{L^2} + \|\rho\|_{H^k} \|[\partial_x^k, \eta] \partial_x v\|_{L^2} \\ &\leq C \|\rho\|_{H^k} \left(\left\| \frac{\eta^3}{\mu(\eta)} \right\|_{L^\infty}^{\frac{1}{2}} \left\| \left(\frac{\mu(\eta)}{\eta} \right)^{\frac{1}{2}} \partial_x^{k+1} v \right\|_{L^2} + \|\eta\|_{H^2} \|v\|_{H^k} + \|v\|_{H^2} \|\eta\|_{H^k} \right). \end{aligned} \quad (2.1.124)$$

Now, due to our assumptions on μ and the induction hypothesis, we have

$$\left\| \frac{\eta^3}{\mu(\eta)} \right\|_{L^\infty}^{\frac{1}{2}} \leq M(A, r_0^{-1}),$$

where M depends on μ and is an increasing function of its arguments.

Upon summation of (2.1.119) and (2.1.118), using (2.1.119) and (2.1.124),

$$\frac{1}{2} \partial_t \|\rho\|_{H^k}^2$$

$$\leq C\|v\|_{H^k}\|\rho\|_{H^k}^2 + C\|\rho\|_{H^k}\|\eta\|_{H^k}\|v\|_{H^k} + M(A, r_0^{-1})\|\rho\|_{H^k} \left\| \left(\frac{\mu(\eta)}{\eta} \right)^{\frac{1}{2}} \partial_x^{k+1} v \right\|_{L^2}.$$

We now use the induction hypothesis (2.1.116) to obtain, for $0 \leq t \leq T_0$,

$$\partial_t (\|\rho\|_{H^k} \exp(-2CA^2 t)) \leq 4CA^2 + M(A, r_0^{-1}) \left\| \left(\frac{\mu(\eta)}{\eta} \right)^{\frac{1}{2}} \partial_x^{k+1} v \right\|_{L^2}.$$

Upon integration, we obtain the following inequality:

$$\|\rho\|_{H^k} \leq \exp(2CA^2 t) \left(\|\rho_0\|_{H^k} + 4CA^2 t + 8A\sqrt{t}M(A, r_0^{-1}) \right).$$

It is now straightforward to choose T_0 , depending only on A and r_0 , such that the induction hypothesis

$$\|\rho\|_{L^\infty(0, T_0; H^k)} \leq 2A$$

is recovered for ρ , in case $k \geq 2$.

Step 3. We now turn to the estimates on the momentum equation (2.1.113). Multiplying such equation by u and integrating by parts yields

$$\frac{1}{2} \partial_t \int_{\mathbb{T}} u^2 dx - \int_{\mathbb{T}} \frac{\mu(\rho)}{\rho} u \partial_x^2 u dx = \int_{\mathbb{T}} u \cdot G_0 dx, \quad (2.1.125)$$

where $G_0 := -v \partial_x v - \frac{1}{\eta} \partial_x p(\eta) + \frac{\partial_x \mu(\eta)}{\eta} \partial_x v + f$. If $k \geq 1$, this implies

$$\begin{aligned} \frac{1}{2} \partial_t \|u\|_{L^2}^2 + \int_{\mathbb{T}} \frac{\mu(\rho)}{\rho} (\partial_x u)^2 dx &\leq M(A, r_0^{-1}) \|\rho\|_{H^1} \|\partial_x u\|_{L^2} \|u\|_{L^\infty} \\ &+ C \|u\|_{L^2} \|v\|_{H^1}^2 + M(A, r_0^{-1}) (\|\eta\|_{H^1} \|u\|_{L^2} + \|\eta\|_{H^1} \|v\|_{H^1} \|u\|_{H^1} + \|f\|_{L^2} \|u\|_{L^2}). \end{aligned} \quad (2.1.126)$$

Here, we used integration by parts and the following Lemma

Lemma 2.1.16. *Let f be a smooth function away from 0, and k be a positive integer. Let $u \in H^k(\mathbb{T}) \cap L^\infty(\mathbb{T})$, and suppose that there exists $r_0 > 0$ such that $u \geq r_0$ on $\mathbb{T}$. Then, there*

exists a positive and continuous function M which depends only on f , k and is increasing in both its arguments such that the following inequality holds:

$$\|f \circ u\|_{H^k(\mathbb{T})} \leq M(\|u\|_{L^\infty(\mathbb{T})}, r_0^{-1}) \|u\|_{H^k(\mathbb{T})}. \quad (2.1.127)$$

Proof of Lemma 2.1.16. The proof of the lemma follows from Theorem 2.87 in [5], §2.8.2, and a straightforward cutoff argument. $\square$

Remark 2.1.17. In what follows, we will always suppress the dependence of M on k and f , since they are fixed at the beginning of the argument.

Differentiating k -times ($k \geq 1$) equation (2.1.113), multiplying by $\partial_x^k u$, and integrating by parts yields

$$\frac{1}{2} \partial_t \int_{\mathbb{T}} (\partial_x^k u)^2 dx - \int_{\mathbb{T}} (\partial_x^k u) \partial_x^k \left(\frac{\mu(\rho)}{\rho} \partial_x^2 u \right) dx = - \int_{\mathbb{T}} (\partial_x^{k+1} u) \cdot G_k dx. \quad (2.1.128)$$

Here, we defined

$$G_k := \partial_x^{k-1} \left(-v \partial_x v - \frac{1}{\eta} \partial_x p(\eta) + \frac{\partial_x \mu(\eta)}{\eta} \partial_x v + f \right), \quad \text{for } k \geq 1.$$

When $k = 1$, the previous display (2.1.128) implies, upon integration by parts, an application of the Cauchy–Schwarz inequality, the induction hypotheses, Lemma 2.1.16 and the bounds obtained in **Step 2**, that

$$\begin{aligned} \frac{1}{2} \partial_t \|\partial_x u\|_{L^2}^2 + \frac{1}{2} \int_{\mathbb{T}} \frac{\mu(\rho)}{\rho} (\partial_x^2 u)^2 dx &\leq \int_{\mathbb{T}} \frac{\rho}{\mu(\rho)} G_1^2 dx \\ &\leq M(A, r_0^{-1}) (\|v\|_{H^1}^4 + \|\eta\|_{H^1}^2 + \|\eta\|_{H^1}^2 \|\partial_x v\|_{L^2} \|\partial_x^2 v\|_{L^2} + \|f\|_{L^2}^2). \end{aligned} \quad (2.1.129)$$

Integrating (2.1.129) and, subsequently, (2.1.126), upon restricting T_0 to be sufficiently

small only as a function of A and r_0 , we have, in case $k = 1$,

$$\|u\|_{L^\infty(0,T_0;H^1)} \leq 2A, \quad \int_0^{T_0} \frac{\mu(\rho)}{\rho} (\partial_x^2 u)^2 dx dt \leq 8A.$$

Let's focus now on the case $k \geq 2$. We have

$$\begin{aligned} & - \int_{\mathbb{T}} (\partial_x^k u) \partial_x^k \left(\frac{\mu(\rho)}{\rho} \partial_x^2 u \right) dx \\ &= - \int_{\mathbb{T}} (\partial_x^k u) \partial_x^{k+1} \left(\frac{\mu(\rho)}{\rho} \partial_x u \right) dx + \int_{\mathbb{T}} (\partial_x^k u) \partial_x^k \left(\partial_x \left(\frac{\mu(\rho)}{\rho} \right) \partial_x u \right) dx \\ &= \int_{\mathbb{T}} \frac{\mu(\rho)}{\rho} (\partial_x^{k+1} u)^2 dx \\ & \quad + \underbrace{\int_{\mathbb{T}} \partial_x^{k+1} u \left[\partial_x^k, \frac{\mu(\rho)}{\rho} \right] (\partial_x u) dx}_{(a)} - \underbrace{\int_{\mathbb{T}} (\partial_x^{k+1} u) \partial_x^{k-1} \left(\partial_x \left(\frac{\mu(\rho)}{\rho} \right) \partial_x u \right) dx}_{(b)}. \end{aligned}$$

We estimate the last two terms in the previous display:

$$\begin{aligned} |(a)| &\leq \frac{1}{10} \int_{\mathbb{T}} \frac{\mu(\rho)}{\rho} (\partial_x^{k+1} u)^2 dx + C \int_{\mathbb{T}} \frac{\rho}{\mu(\rho)} \left(\left[\partial_x^k, \frac{\mu(\rho)}{\rho} \right] (\partial_x u) \right)^2 dx \\ &\leq \frac{1}{10} \int_{\mathbb{T}} \frac{\mu(\rho)}{\rho} (\partial_x^{k+1} u)^2 dx + M(A, r_0^{-1}) \left\| \left[\partial_x^k, \frac{\mu(\rho)}{\rho} \right] (\partial_x u) \right\|_{L^2} \\ &\leq \frac{1}{10} \int_{\mathbb{T}} \frac{\mu(\rho)}{\rho} (\partial_x^{k+1} u)^2 dx \\ & \quad + M(A, r_0^{-1}) \left(\left\| \partial_x \frac{\mu(\rho)}{\rho} \right\|_{L^\infty} \|\partial_x^k u\|_{L^2} + \|\partial_x u\|_{L^\infty} \left\| \partial_x^k \frac{\mu(\rho)}{\rho} \right\|_{L^2} \right) \\ &\leq \frac{1}{10} \int_{\mathbb{T}} \frac{\mu(\rho)}{\rho} (\partial_x^{k+1} u)^2 dx + M(A, r_0^{-1}) \|u\|_{H^k}. \end{aligned} \tag{2.1.130}$$

Here, M is a continuous and increasing function of its arguments. We used the bounds obtained in **Step 2**, the Kato–Ponce commutator estimate, the fact that $k \geq 2$ and Lemma 2.1.16 quoted below, applied to the function $\frac{\mu(\rho)}{\rho}$.

Similarly, the following estimate holds true, for $k \geq 2$:

$$|(b)| \leq \frac{1}{10} \int_{\mathbb{T}} \frac{\mu(\rho)}{\rho} (\partial_x^{k+1} u)^2 dx + M(A, r_0^{-1}) \|u\|_{H^k}. \tag{2.1.131}$$

Again, M is a positive, continuous and increasing function of its arguments.

We now proceed to estimate the terms contained in the RHS of equation (2.1.128) (the terms named “ G ”), in case $k \geq 2$:

$$\left| \int_{\mathbb{T}} (\partial_x^{k+1} u) \cdot G_k \, dx \right| \leq \frac{1}{10} \int_{\mathbb{T}} \frac{\mu(\rho)}{\rho} (\partial_x^{k+1} u)^2 \, dx + 5 \int_{\mathbb{T}} \frac{\rho}{\mu(\rho)} G_k^2 \, dx.$$

Due to the bounds on ρ , we have

$$\int_{\mathbb{T}} \frac{\rho}{\mu(\rho)} G_k^2 \, dx \leq M(A, r_0^{-1}) \|G_k\|_{L^2}^2.$$

Let us now define two auxiliary functions h (the thermodynamic enthalpy) and ζ in such a way that

$$h'(x) = \frac{p'(x)}{x}, \quad \zeta'(x) = \frac{\mu'(x)}{x}, \quad \text{for } x > 0.$$

We now estimate:

$$\|\partial_x^{k-1}(v \partial_x v)\|_{L^2}^2 \leq C \|v\|_{H^2}^2 \|v\|_{H^k}^2 \leq C A^4.$$

Furthermore,

$$\|\partial_x^{k-1} \left(\frac{\partial_x p(\eta)}{\eta} \right)\|_{L^2}^2 \leq \|h(\eta)\|_{H^k}^2 \leq M(A, r_0^{-1}),$$

where we used Lemma 2.1.16, applied to the function h .

Finally, we have, since $k \geq 2$,

$$\begin{aligned} & \left\| \partial_x^{k-1} \left(\frac{\partial_x \mu(\eta)}{\eta} \partial_x v \right) \right\|_{L^2}^2 \\ &= \|\partial_x \zeta(\eta) \partial_x v\|_{H^{k-1}}^2 \leq C (\|\zeta(\eta)\|_{H^k} \|\partial_x v\|_{L^\infty} + \|v\|_{H^k} \|\partial_x \zeta(\eta)\|_{L^\infty}) \\ &\leq M(A, r_0^{-1}). \end{aligned}$$

Hence, for the term G_k , we have

$$\left| \int_{\mathbb{T}} (\partial_x^{k+1} u) \cdot G_k \, dx \right| \leq \frac{1}{10} \int_{\mathbb{T}} \frac{\mu(\rho)}{\rho} (\partial_x^{k+1} u)^2 \, dx + M(A, r_0^{-1}) (1 + \|f\|_{H^{k-1}}^2). \quad (2.1.132)$$

Putting together estimates (2.1.125), (2.1.128), (2.1.130), (2.1.131), (2.1.132), and ignoring the positive integral term in the LHS, we obtain the inequality

$$\frac{1}{2} \partial_t \|u\|_{H^k}^2 \leq M(A, r_0^{-1}) \|u\|_{H^k} + M(A, r_0^{-1}) (1 + \|f\|_{H^{k-1}}^2).$$

Using Grönwall's inequality, upon restricting T_0 to be small depending only on A , r_0 and f , we deduce that

$$\|u\|_{L^\infty(0, T_0; H^k)} \leq 2A. \quad (2.1.133)$$

We now revisit the same estimates without discarding the positive integral term in the LHS. We obtain, upon restricting T_0 to be smaller, depending only on A and r_0 and f , that

$$\int_0^{T_0} \int_{\mathbb{T}} \frac{\mu(\rho)}{\rho} (\partial_x^{k+1} u)^2 dx dt \leq 8A. \quad (2.1.134)$$

We have therefore recovered the induction hypotheses 2.1.116, and in particular the sequence (ρ_n, u_n) is uniformly bounded in $L^\infty(0, T_0; H^k(\mathbb{T})) \times (L^\infty(0, T_0; H^k(\mathbb{T})) \cap L^2(0, T_0; H^{k+1}(\mathbb{T})))$.

Step 4. We now show that, for some T_0 , depending only on A, r_0 , the sequence (ρ_n, u_n) is Cauchy in the space $L^\infty(0, T_0; L^2) \times (L^\infty(0, T_0; L^2) \cap L^2(0, T_0; L^2))$.

Let's first consider the equation satisfied by $\delta u_n := u_{n+1} - u_n$:

$$\begin{aligned} \partial_t(\delta u_n) - \frac{\mu(\rho_{n+1})}{\rho_{n+1}} \partial_x^2 u_{n+1} + \frac{\mu(\rho_n)}{\rho_n} \partial_x^2 u_n \\ = \frac{1}{2} \partial_x(u_n^2 - u_{n-1}^2) + \partial_x(h(\rho_n) - h(\rho_{n-1})) + \partial_x \zeta(\rho_n) \partial_x u_n - \partial_x \zeta(\rho_{n-1}) \partial_x u_{n-1}. \end{aligned} \quad (2.1.135)$$

Recall that we defined h and ζ so that the following equalities hold true:

$$\partial_x h(\rho) = \frac{\partial_x p(\rho)}{\rho}, \quad \zeta(\rho) = \frac{\partial_x \mu(\rho)}{\rho}.$$

We now multiply equation (2.1.135) by δu_n and integrate by parts. We have:

$$\begin{aligned} & \int_{\mathbb{T}} (\delta u_n) \left(-\frac{\mu(\rho_{n+1})}{\rho_{n+1}} \partial_x^2 u_{n+1} + \frac{\mu(\rho_n)}{\rho_n} \partial_x^2 u_n \right) dx \\ &= \underbrace{- \int_{\mathbb{T}} (\delta u_n) \frac{\mu(\rho_{n+1})}{\rho_{n+1}} \partial_x^2 u_{n+1} dx}_{(a)} + \underbrace{\int_{\mathbb{T}} \left(\frac{\mu(\rho_n)}{\rho_n} - \frac{\mu(\rho_{n+1})}{\rho_{n+1}} \right) \partial_x^2 u_n (\delta u_n) dx}_{(b)}. \end{aligned}$$

Note that, due to **Step 3**, there exists $c = c(A, r_0)$ such that, up to time T_0 , there holds

$$\frac{\mu(\rho_i)}{\rho_i} \geq c \text{ for all integers } i \geq 0.$$

Hence, for the term in (a), upon integration by parts,

$$\begin{aligned} (a) &\geq c \|\partial_x(\delta u_n)\|_{L^2}^2 - \frac{1}{c} \left\| \partial_x \frac{\mu(\rho_n)}{\rho_n} \right\|_{L^2} \|\delta u_n\|_{L^\infty} \|\partial_x(\delta u_n)\|_{L^2} \\ &\geq c \|\partial_x(\delta u_n)\|_{L^2}^2 - M(A, r_0^{-1}) \left(\|\delta u_n\|_{L^2}^{\frac{1}{2}} \|\partial_x(\delta u_n)\|_{L^2}^{\frac{3}{2}} + \|\delta u_n\|_{L^2} \|\partial_x(\delta u_n)\|_{L^2} \right) \\ &\geq \frac{c}{2} \|\partial_x(\delta u_n)\|_{L^2}^2 - M(A, r_0^{-1}) \|\delta u_n\|_{L^2}^2. \end{aligned}$$

Here, we used Lemma 2.1.16, the Gagliardo–Nirenberg–Sobolev inequality and the Young inequality.

We now estimate

$$(b) \geq -M(A, r_0^{-1}) \|\delta \rho_n\|_{L^2} \|\partial_x^2 u_n\|_{L^2} \|\delta u_n\|_{L^2}^{\frac{1}{2}} \|\delta u_n\|_{H^1}^{\frac{1}{2}}.$$

Let us now turn to the terms appearing in the RHS of (2.1.135). We define

$$\begin{aligned} & \underbrace{\int_{\mathbb{T}} \frac{1}{2} \partial_x (u_n^2 - u_{n-1}^2) (\delta u_n) dx}_{(c)} + \underbrace{\int_{\mathbb{T}} (\delta u_n) \partial_x (h(\rho_n) - h(\rho_{n-1})) dx}_{(d)} \\ & \quad + \underbrace{\int_{\mathbb{T}} (\delta u_n) (\partial_x \zeta(\rho_n) \partial_x u_n - \partial_x \zeta(\rho_{n-1}) \partial_x u_{n-1}) dx}_{(e)}. \end{aligned}$$

Then, for (c), we have, after integration by parts,

$$|(c)| \leq M(A) \|\partial_x(\delta u_n)\|_{L^2} \|\delta u_{n-1}\|_{L^2} \leq \frac{1}{10c} \|\partial_x(\delta u_n)\|_{L^2}^2 + M(A) \|\delta u_{n-1}\|_{L^2}^2.$$

Concerning the term (d), instead,

$$|(d)| = \left| \int_{\mathbb{T}} \partial_x(\delta u_n) (h(\rho_n) - h(\rho_{n-1})) dx \right| \leq \frac{1}{10c} \|\partial_x(\delta u_n)\|_{L^2}^2 + M(A, r_0^{-1}) \|\delta \rho_{n-1}\|_{L^2}^2.$$

Again, we used the fact that, due to the uniform bounds on ρ_n , h is Lipschitz of constant depending only on A and r_0 .

Finally, concerning (e),

$$\begin{aligned} |(e)| &\leq \left| \int_{\mathbb{T}} (\delta u_n) \partial_x \zeta(\rho_n) \partial_x(\delta u_{n-1}) dx \right| + \left| \int_{\mathbb{T}} (\delta u_n) \partial_x (\zeta(\rho_n) - \zeta(\rho_{n-1})) \partial_x u_{n-1} dx \right| \\ &\leq \|\delta u_n\|_{L^\infty} \|\partial_x \zeta(\rho_n)\|_{L^2} \|\partial_x(\delta u_{n-1})\|_{L^2} + \left| \int_{\mathbb{T}} (\zeta(\rho_n) - \zeta(\rho_{n-1})) \partial_x((\delta u_n) \partial_x u_{n-1}) dx \right| \\ &\leq M(A, r_0^{-1}) \left(\|\delta u_n\|_{L^2}^{\frac{1}{2}} \|\partial_x(\delta u_n)\|_{L^2}^{\frac{1}{2}} \|\partial_x(\delta u_{n-1})\|_{L^2} + \|\partial_x(\delta u_{n-1})\|_{L^2} \|\delta u_n\|_{L^2} \right) \\ &\quad + M(A, r_0^{-1}) (\|\delta \rho_{n-1}\|_{L^2} \|\partial_x \delta u_n\|_{L^2} \|\partial_x^2 u_n\|_{L^2}^{\frac{1}{2}} + \|\delta \rho_{n-1}\|_{L^2} \|\delta u_n\|_{L^\infty} \|\partial_x^2 u_n\|_{L^2}) \end{aligned}$$

where $\delta \rho_{n-1} := \rho_n - \rho_{n-1}$. Putting together the estimates on the momentum equation, we have

$$\begin{aligned} &\frac{1}{2} \partial_t \|\delta u_n\|_{L^2}^2 + \frac{1}{10c} \|\partial_x(\delta u_n)\|_{L^2}^2 \\ &\leq M(A, r_0^{-1}) (\|\delta u_n\|_{L^2}^2 + \|\delta u_{n-1}\|_{L^2}^2 + \|\delta \rho_{n-1}\|_{L^2}^2) \\ &\quad + M(A, r_0^{-1}) \|\delta \rho_n\|_{L^2} \|\partial_x^2 u_n\|_{L^2} \|\delta u_n\|_{L^2}^{\frac{1}{2}} \|\partial_x(\delta u_n)\|_{L^2}^{\frac{1}{2}} \\ &\quad + M(A, r_0^{-1}) (\|\delta u_n\|_{L^2}^{\frac{1}{2}} \|\partial_x(\delta u_n)\|_{L^2}^{\frac{1}{2}} \|\partial_x(\delta u_{n-1})\|_{L^2} + \|\partial_x(\delta u_{n-1})\|_{L^2} \|\delta u_n\|_{L^2}) \\ &\quad + M(A, r_0^{-1}) (\|\delta \rho_n\|_{L^2} \|\partial_x \delta u_n\|_{L^2} \|\partial_x^2 u_n\|_{L^2}^{\frac{1}{2}} + \|\delta \rho_n\|_{L^2} \|\delta u_n\|_{L^\infty} \|\partial_x^2 u_n\|_{L^2}). \end{aligned}$$

Upon integration between time $s = 0$ and $s = t$, using Hölder's inequality and the bounds

obtained in **Step 1**,

$$\begin{aligned}
& \frac{1}{2} \|(\delta u_n)(\cdot, t)\|_{L^2}^2 + \frac{1}{10c} \|\partial_x(\delta u_n)\|_{L^2(0,t;L^2)}^2 \\
& \leq M(A, r_0^{-1}) (\|\delta u_n\|_{L^2(0,t;L^2)}^2 + \|\delta u_{n-1}\|_{L^2(0,t;L^2)}^2 + \|\delta \rho_{n-1}\|_{L^2(0,t;L^2)}^2) \\
& + M(A, r_0^{-1}) t^{\frac{1}{4}} \|\delta \rho_n\|_{L^\infty(0,t;L^2)} \|\delta u_n\|_{L^\infty(0,t;L^2)}^{\frac{1}{2}} \|\partial_x(\delta u_n)\|_{L^2(0,t;L^2)}^{\frac{1}{2}} \\
& + M(A, r_0^{-1}) t^{\frac{1}{4}} \|\delta u_n\|_{L^\infty(0,t;L^2)}^{\frac{1}{2}} \|\partial_x(\delta u_n)\|_{L^2(0,t;L^2)}^{\frac{1}{2}} \|\partial_x(\delta u_{n-1})\|_{L^2(0,t;L^2)} \\
& + M(A, r_0^{-1}) t^{\frac{1}{2}} \|\partial_x(\delta u_{n-1})\|_{L^2(0,t;L^2)} \|\delta u_n\|_{L^\infty(0,t;L^2)} \\
& + M(A, r_0^{-1}) t^{\frac{1}{4}} \|\delta \rho_{n-1}\|_{L^\infty(0,t;L^2)} \|\partial_x(\delta u_n)\|_{L^2(0,t;L^2)} \\
& + M(A, r_0^{-1}) t^{\frac{1}{4}} \|\delta \rho_{n-1}\|_{L^\infty(0,t;L^2)} \|\partial_x \delta u_n\|_{L^2(0,t;L^2)} \\
& + M(A, r_0^{-1}) t^{\frac{1}{4}} \|\delta \rho_{n-1}\|_{L^\infty(0,t;L^2)} \|\delta u_n\|_{L^\infty(0,t;L^2)}^{\frac{1}{2}} \|\partial_x(\delta u_n)\|_{L^2(0,t;L^2)}^{\frac{1}{2}} \\
& \leq \frac{1}{20c} \|\partial_x(\delta u_n)\|_{L^2(0,t;L^2)}^2 + M(A, r_0^{-1}) t^{\frac{1}{4}} (\|\delta u_n\|_{L^\infty(0,t;L^2)}^2 + \|\delta u_{n-1}\|_{L^\infty(0,t;L^2)}^2 + \\
& \quad \|\delta \rho_{n-1}\|_{L^\infty(0,t;L^2)}^2 + \|\partial_x(\delta u_{n-1})\|_{L^2(0,t;L^2)}^2).
\end{aligned} \tag{2.1.136}$$

Let us now calculate the equation satisfied by differences of ρ_n :

$$\partial_t(\delta \rho_n) = -u_n \partial_x \rho_{n+1} + u_{n-1} \partial_x \rho_n - \rho_n \partial_x u_n + \rho_{n-1} \partial_x u_{n-1}. \tag{2.1.137}$$

Multiplying equation (2.1.137) by $\delta \rho_n$, we obtain

$$\begin{aligned}
& \frac{1}{2} \partial_t \|\delta \rho_n\|_{L^2}^2 = \\
& - \underbrace{\int_{\mathbb{T}} (\delta \rho_n) (u_n \partial_x \rho_{n+1} - u_{n-1} \partial_x \rho_n) dx}_{(a)} - \underbrace{\int_{\mathbb{T}} (\delta \rho_n) (\rho_n \partial_x u_n - \rho_{n-1} \partial_x u_{n-1}) dx}_{(b)}.
\end{aligned}$$

Considering (a), we have, integrating by parts, using Gagliardo–Nirenberg–Sobolev and Hölder’s inequality,

$$|(a)| \leq \left| \int_{\mathbb{T}} (\delta \rho_n) (\delta u_{n-1}) \partial_x \rho_{n+1} dx \right| + \left| \int_{\mathbb{T}} \partial_x (\delta \rho_n) (\delta \rho_n) u_{n-1} dx \right|$$

$$\leq M(A)(\|\delta\rho_n\|_{L^2}\|\delta u_{n-1}\|_{H^1}^{\frac{1}{2}}\|\delta u_{n-1}\|_{L^2}^{\frac{1}{2}} + \|\delta\rho_n\|_{L^2}^2\|\partial_x^2 u_n\|_{L^2}^{\frac{1}{2}}).$$

On the other hand, (b) yields

$$\begin{aligned} |(b)| &\leq \left| \int_{\mathbb{T}} (\delta\rho_n)(\delta\rho_{n-1})\partial_x u_n dx \right| + \left| \int_{\mathbb{T}} (\delta\rho_n)\partial_x(\delta u_{n-1})\rho_{n-1} dx \right| \\ &\leq M(A)(\|\delta\rho_n\|_{L^2}^2 + \|\delta\rho_{n-1}\|_{L^2}^2)\|\partial_x^2 u_n\|_{L^2}^{\frac{1}{2}} + M(A)\|\partial_x(\delta u_{n-1})\|_{L^2}\|\delta\rho_n\|_{L^2}. \end{aligned}$$

Putting together the estimates on the mass equation yields

$$\begin{aligned} &\frac{1}{2}\partial_t\|\delta\rho_n\|_{L^2}^2 \\ &\leq M(A)\left(\|\delta\rho_n\|_{L^2}\|\partial_x(\delta u_{n-1})\|_{L^2}^{\frac{1}{2}}\|\delta u_{n-1}\|_{L^2}^{\frac{1}{2}} + \|\delta\rho_n\|_{L^2}^2\|\partial_x^2 u_n\|_{L^2}^{\frac{1}{2}}\right) \\ &+ M(A)\|\delta\rho_n\|_{L^2}\|\delta u_{n-1}\|_{L^2} \\ &+ M(A)(\|\delta\rho_n\|_{L^2}^2 + \|\delta\rho_{n-1}\|_{L^2}^2)\|\partial_x^2 u_n\|_{L^2}^{\frac{1}{2}} + M(A)\|\partial_x(\delta u_{n-1})\|_{L^2}\|\delta\rho_n\|_{L^2}. \end{aligned}$$

Upon integration, the previous display yields

$$\begin{aligned} \frac{1}{2}\|\delta\rho_n(t, \cdot)\|_{L^2}^2 &\leq M(A)t^{\frac{3}{4}}\|\delta\rho_n\|_{L^\infty(0,t;L^2)}\|\partial_x(\delta u_{n-1})\|_{L^2(0,t;L^2)}^{\frac{1}{2}}\|\delta u_{n-1}\|_{L^\infty(0,t;L^2)}^{\frac{1}{2}} \\ &+ M(A)t^{\frac{3}{4}}\|\delta\rho_n\|_{L^\infty(0,t;L^2)}^2 + M(A)t(\|\delta\rho_n\|_{L^\infty(0,t;L^2)}^2 + \|\delta u_{n-1}\|_{L^\infty(0,t;L^2)}^2) \\ &+ M(A)t^{\frac{3}{4}}(\|\delta\rho_n\|_{L^\infty(0,t;L^2)}^2 + \|\delta\rho_{n-1}\|_{L^\infty(0,t;L^2)}^2) \\ &+ M(A)t^{\frac{1}{2}}\|\partial_x(\delta u_{n-1})\|_{L^2(0,t;L^2)}\|\delta\rho_n\|_{L^\infty(0,t;L^2)} \\ &\leq M(A)t^{\frac{1}{2}}\left(\|\delta\rho_n\|_{L^\infty(0,t;L^2)}^2 + \|\partial_x(\delta u_{n-1})\|_{L^2(0,t;L^2)}^2 + \|\delta u_{n-1}\|_{L^\infty(0,t;L^2)}^2\right. \\ &\quad \left.+ \|\delta\rho_{n-1}\|_{L^\infty(0,t;L^2)}^2\right). \end{aligned} \tag{2.1.138}$$

Combining now (2.1.136) and (2.1.138), we obtain, for suitably small t depending only

on A and r_0 ,

$$\begin{aligned} & \frac{1}{4} \|\delta \rho_n\|_{L^\infty(0,t;L^2)}^2 + \frac{1}{4} \|\delta u_n\|_{L^\infty(0,t;L^2)}^2 + \frac{1}{20c} \|\partial_x(\delta u_n)\|_{L^2(0,t;L^2)}^2 \\ & \leq M(A, r_0^{-1}) t^{\frac{1}{4}} (\|\partial_x(\delta u_{n-1})\|_{L^2(0,t;L^2)}^2 + \|\delta u_{n-1}\|_{L^\infty(0,t;L^2)}^2 + \|\delta \rho_{n-1}\|_{L^\infty(0,t;L^2)}^2). \end{aligned}$$

Upon suitable choice of T_0 , this implies that the sequence (ρ_n, u_n) is Cauchy in the space $L^\infty(0, T_0; L^2) \times (L^\infty(0, T_0; L^2) \cap L^2(0, T_0; H^1))$.

Step 5. Denote

$$X^m = L^\infty(0, T_0; H^m) \times (L^\infty(0, T_0; H^m) \cap L^2(0, T_0; H^{m+1}))$$

a Banach space with its canonical norm. We have proved in the previous steps that (ρ_n, u_n) is bounded in X^k and Cauchy in X^{k-1} . The latter implies that (ρ_n, u_n) converges to some (ρ, u) in X^{k-1} . The former implies that some subsequence (ρ_{n_j}, u_{n_j}) converges weak-* to some (ρ_*, u_*) in X^k . Since both weak-* convergence in X^k and strong convergence in X^{k-1} imply convergence in the sense of distributions we deduce that $(\rho, u) = (\rho_*, u_*) \in X^k$. It can be easily verified that (ρ, u) is a strong solution to the system (2.1.1)–(2.1.2). Moreover, since $\rho_n \rightarrow \rho$ strongly in $L^2(0, T_0; L^2)$ and (ρ_n) is bounded in $L^\infty(0, T_0; H^1)$ it follows by interpolation that $\rho_n \rightarrow \rho$ strongly in $L^\infty(0, T_0; H^{3/4})$, and hence in $L^\infty(0, T_0; L^\infty)$. This combined with the fact that $\rho_n(x, t) \geq \frac{r_0}{2}$ for all $(x, t) \in \mathbb{T} \times [0, T_0]$ (see **Step 2**) yields

$$\rho(x, t) \geq \frac{r_0}{2} \quad \forall (x, t) \in \mathbb{T} \times [0, T_0].$$

Step 6. We now establish uniqueness of strong solutions. Consider solutions (ρ_1, u_1) and (ρ_2, u_2) , such that

$$\rho_i \in \mathcal{C}(0, T_0; H^k(\mathbb{T})), \quad u_i \in \mathcal{C}(0, T_0; H^k(\mathbb{T})) \cap L^2(0, T_0; H^{k+1}(\mathbb{T})), \quad \text{for } i = 1, 2.$$

and let $(\delta\rho, \delta u) = (\rho_1 - \rho_2, u_1 - u_2)$. We have

$$\partial_t \delta u + \delta u \partial_x u_1 + u_2 \partial_x \delta u = -\partial_x((\rho_1) - (\rho_2)) + \rho_1^{-1} \partial_x(\mu(\rho_1) \partial_x u_1) - \rho_2^{-1} \partial_x(\mu(\rho_2) \partial_x u_2), \quad (2.1.139)$$

$$\partial_t \delta \rho + \partial_x(u_1 \delta \rho + \rho_2 \delta u) = 0, \quad (2.1.140)$$

$$(\delta \rho, \delta u)|_{t=0} = (0, 0) \quad (2.1.141)$$

We now notice that equation (2.1.139) is the same as equation (2.1.135), upon formally substituting $n = 1$ in the LHS, and $n = 2$ in the RHS. Similarly, recalling (2.1.137), we have

$$\underbrace{\partial_t(\delta \rho_n)}_{(a)} = \underbrace{-u_n}_{(b)} \underbrace{\partial_x \rho_{n+1}}_{(a)} + \underbrace{u_{n-1}}_{(b)} \underbrace{\partial_x \rho_n}_{(a)} - \underbrace{\rho_n \partial_x u_n + \rho_{n-1} \partial_x u_{n-1}}_{(b)}.$$

Formally substituting $n = 1$ in terms (a), and $n = 2$ in terms (b), we obtain (2.1.140). It is then straightforward to see that the same estimates as in **Step 4** yield uniqueness of strong solutions. $\square$

2.2 The two-dimensional shallow water system: a review of existing literature

In this section and in the following, we are mainly going to be concerned with systems of the type (0.2.18) in two space dimensions. For completeness, let us recall the form of the system:

$$\begin{aligned} \partial_t(hu) + \operatorname{div}(hu \otimes u) + gh\nabla(h - b) + c_r hu^\perp + c_s \nabla \Delta h + r_0 u + r_1 u|u| \\ = \nu \operatorname{div}(h\nabla u) + fh, \end{aligned} \quad (2.2.1)$$

$$\partial_t h + \operatorname{div}(hu) = 0,$$

Here, the unknowns are a time dependent vector field $u(t, x)$ and the water height $h(t, x)$. Furthermore, $b(x) : \mathbb{R}^2 \rightarrow \mathbb{R}$ is the bottom topography. We assume that $b \leq h$. We also have that g is the gravitational acceleration constant, and c_r, c_s, r_0, r_1, ν are non-negative constants. Moreover, $u^\perp = (-u_2, u_1)$.

We will now review the existing literature on the shallow water system.

The inviscid case

This is the case in which we set $c_r = c_s = r_0 = r_1 = \nu = 0$ in system (2.2.1). Local existence can be established by standard techniques if the initial data lies in a sufficiently high-order Sobolev space (H^3 is enough), and the height h is uniformly bounded away from 0 initially. In this case, it is possible to proceed by symmetrization and standard results on symmetric hyperbolic systems (see Section 2 of [19], and for instance the book by Majda [70]). For completeness, we review the proof in Section 2.3.1.

Local existence and global stability of classical solutions in the viscous case

The local existence for the viscous and compressible NS systems with *non-degenerate viscosity* was first established by Matsumura and Nishida in the context of the Euler equations of gas dynamics in [72]. Sundbye and Kloeden subsequently adapted the method to show local existence in the case of degenerate viscosity of the kind present in system (2.2.1).

In addition, Kloeden [63] considered the SW system on the two-dimensional torus (thus, system (2.2.1) with $c_r = c_s = r_0 = r_1 = 0$, and $\nu > 0$) and showed the existence of global-in-time solutions when the initial data is close to a constant state ($u = 0$ and $h = \text{constant}$ initially) and the constant forcing f is conservative and small. Later, there were several generalizations by Sundbye [84, 85] for the same system, in which moreover inverse polynomial decay rates are obtained by means of time-weighted energy estimates:

$$\|(\nabla u, \nabla h)(t)\|_{L^\infty(\mathbb{R}^2)} \leq C(1+t)^{-1}, \quad \|(\nabla^2 u)(t)\|_{L^\infty(\mathbb{R}^2)} \leq C(1+t)^{-3/2}.$$

Remark 2.2.1 (Initial data containing vacuum). A different issue is when the initial data contains “vacuum”, i.e. h is allowed to be 0 somewhere initially. We mention the paper [33], in which the authors consider a 2D NS system with Coriolis force and non-degenerate viscosity of the following form:

$$\begin{aligned}\partial_t(hu) + \operatorname{div}(hu \otimes u) + gh\nabla h + f(hu)^\perp &= \mu\Delta u + (\mu + \lambda)\nabla(\operatorname{div} u), \\ \partial_t h + \operatorname{div}(hu) &= 0.\end{aligned}$$

Here, $\mu > 0$, and $\mu + \lambda \geq 0$. They are able to prove, under mild assumptions on the initial data, that a local *classical* solution exists even in the case in which h vanishes initially. In particular, the vacuum set can have arbitrarily large measure.

Remark 2.2.2 (Lagrangian approach). Let us also note that there is a number of works which prove local existence of solutions to the 2D compressible and viscous NS equations using a Lagrangian approach. The origin of the approach dates back to Nash in his 1962 paper [78]. Moreover, An–Ton adopted a Lagrangian approach to show local existence in Hölder spaces [89].

Remark 2.2.3 (Global existence for small data: minimal regularity). In the paper by Wang and Xu [90], Littlewood–Paley techniques are used to construct local (global for small data) solutions with H^{s+2} initial data, with $s > 0$, in the viscous case.

Weak solutions for degenerate viscosity

Few results are known away from small data, due to the very degenerate nature of the equations when $h \rightarrow 0$. By a result of Bresch and Desjardins [12], we have global existence of weak solutions. Other papers by the same group on related topics are [13] and [14]. See also the paper by Orenca [79].

Theorem 2.2.4 (Existence of weak solutions [12]). *Consider the following SW system on*

the two-dimensional torus $\mathbb{T}^2$:

$$\partial_t(hu) + \operatorname{div}(hu \otimes u) + r_1 h|u|u - \kappa h \nabla \Delta h + h \nabla h - \nu \operatorname{div}(h \nabla u) = 0,$$

$$\partial_t h + \operatorname{div}(hu) = 0.$$

$$h(0, x) = h_0, \quad (hu)(0, x) = m_0.$$

If the initial data satisfy:

$$h_0 \in L^2(\Omega), \quad |m_0|^2/h_0 \in L^2(\Omega), \quad \nabla(\sqrt{h_0}) \in (L^2(\Omega))^2, \quad \kappa \nabla(h_0) \in (L^2(\Omega))^2,$$

and if $\nu > 0$, and either $\kappa > 0$ or $r_1 > 0$ then there is a global weak solution to the above system.

The definition of weak solution in this case is as follows.

Definition 2.2.5. (u, h) is a weak solution on $(0, T)$ of the system (2.2.4) if the following holds true:

- The initial conditions are satisfied in $\mathcal{D}'(\Omega)$, and the system is satisfied in $(\mathcal{D}'((0, T) \times \Omega))^3$.
- The energy inequality (2.2.2) is satisfied a. e. on $[0, T]$.
- The following technical conditions hold true:

$$\begin{aligned} \nabla \sqrt{h} &\in L^\infty(0, T; (L^2(\Omega))^2), \quad \sqrt{h}u \in L^\infty(0, T; (L^2(\Omega))^2), \\ \sqrt{h} \nabla u &\in L^2(0, T; (L^2(\Omega))^4), \quad \nabla h \in L^2(0, T; (L^2(\Omega))^2), \\ r_1 h^{\frac{1}{3}}u &\in L^3(0, T; (L^3(\Omega))^2), \quad \kappa \nabla^2 h \in L^2(0, T; (L^2(\Omega))^4). \end{aligned}$$

The classical energy inequality for the system above is the following:

$$\begin{aligned} & \int_{\Omega} \left(h^2 + h \frac{|u|^2}{2} + \kappa \frac{|\nabla h|^2}{2} \right) + \int_0^t \int_{\Omega} \nu h |\nabla u|^2 + \int_0^t \int_{\Omega} r_1 h |u|^3 \\ & \leq \int_{\Omega} \left(h_0^2 + h_0 \frac{|u_0|^2}{2} + \kappa \frac{|\nabla h_0|^2}{2} \right) \end{aligned} \quad (2.2.2)$$

The main point is that the energy inequality (2.2.2) does not guarantee compactness in a space that would allow to pass to the limit in the equations. The main insight here is an additional estimate which ensures compactness in the space defined above, and which allows sufficient information to pass to the limit.

The coercive quantity found by the authors is called the *Bresch–Desjardins entropy*, and gives in particular the following a priori control:

$$\frac{d}{dt} \int_{\Omega} \left(\frac{1}{2} h |u|^2 + \nu \nabla \log h|^2 + \frac{h^2}{2} + \kappa \frac{|\nabla h|^2}{2} \right) + \nu \kappa \int_{\Omega} |\nabla^2 h|^2 + \nu r_1 \int_{\Omega} h |u|^3 \leq C. \quad (2.2.3)$$

The smoothing effect of rotation in the inviscid case

In the regime where the Coriolis force is strong, it has been observed that the time of existence provided by the classical local existence theorems can be significantly extended. We are going to cite two results here, the first one is valid in a small data regime (and uses dispersive techniques), whereas the second is valid in a large data regime.

Theorem 2.2.6 (Cheng – Xie, 2009 [19]). *Consider the initial value problem for the inviscid, rotating SW system on $\mathbb{R}^2$, about the constant state 1 (here, $\rho = h - 1$):*

$$\begin{aligned} \partial_t u + u \cdot \nabla u + \nabla \rho + u^\perp &= 0, \\ \partial_t \rho + \operatorname{div}(\rho u) + \operatorname{div} u &= 0. \\ h(0, x) = h_0, \quad u(0, x) &= u_0. \end{aligned} \quad (2.2.4)$$

Suppose that both ρ and u_0 are smooth and compactly supported, and suppose that the “relative

vorticity" is zero:

$$\rho_0 = \partial_1(u_0)_2 - \partial_2(u_0)_1.$$

Then, if a higher-order Sobolev norm of the initial data is sufficiently small, the system (2.2.4) admits a global-in-time solution.

The previous result assumed smallness of data. Away from small data, Cheng and Tadmor proved existence for long time when the effect of rotation dominates.

Theorem 2.2.7 (Cheng–Tadmor, 2007 [18]). *Consider the initial value problem for the following inviscid, rotating SW system on $\mathbb{R}^2$:*

$$\begin{aligned} \partial_t u + u \cdot \nabla u + \frac{1}{\sigma} \nabla h + \frac{1}{\tau} u^\perp &= 0, \\ \partial_t h + \operatorname{div}(hu) + \frac{1}{\sigma} \operatorname{div} u &= 0. \\ h(0, x) &= h_0, \quad u(0, x) = u_0. \end{aligned} \tag{2.2.5}$$

Suppose that $h_0, u_0 \in H^m(\mathbb{R}^2)$, $m > 5$, and $\tau > 0$ is chosen so that the following subcritical condition is satisfied pointwise:

$$\tau \omega_0 + \tau^2 |\eta_0|^2 < 1, \forall x \in \mathbb{R}^2, \quad \omega_0 = \partial_2 u_1 - \partial_1 u_2.$$

Here, η_0 is the spectral gap of the matrix ∇u_0 . Assume $\sigma \leq 1$, and that $\alpha_0 := \inf_{\mathbb{R}^2} (1 + \sigma h_0) > 0$ (initial height away from 0). Then, there is a constant C independent of σ and τ such that the local-in-time solution to (2.2.5) exists at least until time

$$T = C \log \left(\frac{\sigma^2}{\tau} \right).$$

The subcritical condition serves the purpose to build periodic solutions to the *pressureless* system:

$$\partial_t u + u \cdot \nabla u + \frac{1}{\tau} u^\perp = 0. \tag{2.2.6}$$

The proof then proceeds by a perturbation argument around such solution. The authors claim that this result is consistent with observed so-called *near-inertial oscillations*. These are phenomena in the oceans exhibiting almost periodic dynamics up to twenty days after a storm.

2.3 Some basic facts about the shallow water equations: well-posedness and global stability

In this section, we are going to review a few fundamental results concerning compressible inviscid and compressible viscous fluids (with degenerate viscosity coefficient). First, in Section 2.3.1, we are going to describe the classical proof of local existence to the inviscid system. Then, in Section 2.3.2, we will deal with the issue of local existence in the viscous case with degenerate viscosity. Finally, in Section 2.3.3, we will prove global stability for small initial data for the shallow water system.

2.3.1 Local existence in the inviscid case

This proof is classical and it is based on reformulating the system in symmetric hyperbolic form. Recall that the inviscid shallow water equations are equivalent to the barotropic compressible Euler equations with exponent $\gamma = 2$. Here is a precise statement of the result.

Theorem 2.3.1. *Consider the inviscid shallow water system (barotropic compressible Euler with $\gamma = 2$) with constant bottom topography:*

$$\begin{cases} \partial_t h + \operatorname{div} (uh) = 0, \\ \partial_t u + u \cdot \nabla u + g \nabla h + u^\perp = 0, \\ (u, h)|_{t=0} = (u_0, h_0). \end{cases} \quad (2.3.1)$$

Suppose that there exists $\bar{h} > 0$ such that the following conditions hold true. Let $u_0 \in$

$(H^3(\mathbb{R}^2))^2$, $h_0 \in H^3(\mathbb{R}^2)$, and suppose that there exists $\bar{h} > 0$ such that $h_0 \geq \frac{\bar{h}}{2}$ on $\mathbb{R}^2$, and furthermore

$$\left\| \sqrt{h_0} - \sqrt{\bar{h}} \right\|_{H^3(\mathbb{R}^2)} < \infty. \quad (2.3.2)$$

Then, there exists a time $T > 0$, $T(\|u_0\|_{H^3(\mathbb{R}^2)}, \left\| \sqrt{h_0} - \sqrt{\bar{h}} \right\|_{H^3(\mathbb{R}^2)}, \bar{h}, g)$ such that a solution to the system (2.3.1) exists on $[0, T) \times \mathbb{R}^2$ and furthermore it is such that $u, \sqrt{h} - \sqrt{\bar{h}} \in \mathcal{C}^0(0, T; H^3) \cap \mathcal{C}^1(0, T; H^2)$.

Proof. By rescaling, we can suppose $\bar{h} = 1$. Indeed, making the following transformations:

$$w := (g\bar{h})^{-\frac{1}{2}}u(t, (g\bar{h})^{\frac{1}{2}}x), \quad z := (\bar{h})^{-1}h(t, (g\bar{h})^{\frac{1}{2}}x), \quad (2.3.3)$$

we obtain the system

$$\begin{cases} \partial_t z + \operatorname{div} (wz) = 0, \\ \partial_t w + w \cdot \nabla w + \nabla z + w^\perp = 0, \\ (w, z)|_{t=0} = (w_0, z_0). \end{cases} \quad (2.3.4)$$

We will be performing estimates in terms of $\rho := z - 1$. With respect to ρ , Equations (2.3.4) can be written as follows:

$$\begin{aligned} \partial_t \rho + \operatorname{div} w + \operatorname{div} (w\rho) &= 0, \\ \partial_t w + w \cdot \nabla w + \nabla \rho + w^\perp &= 0. \end{aligned}$$

We now define m by the following Equation:

$$1 + \frac{m}{2} = \sqrt{1 + \rho}. \quad (2.3.5)$$

The condition (2.3.2) enforces that

$$\|m_0\|_{H^3} < \infty.$$

As long as $\rho > -1$, and if (w, h) is continuously differentiable, the system (2.3.1) is equivalent

to the following system:

$$\begin{cases} \partial_t m + (1 + \frac{m}{2}) \operatorname{div} w + w \cdot \nabla m = 0, \\ \partial_t w + w \cdot \nabla w + (1 + \frac{1}{2}m) \nabla m + w^\perp = 0, \\ (w, m)|_{t=0} = (w_0, m_0). \end{cases} \quad (2.3.6)$$

Here, m_0 is defined as in equation (2.3.5). This system (2.3.6) can now be written in symmetric hyperbolic form. We define W to be the array $W := (w^1, w^2, m)$, we obtain the following system:

$$\partial_t W + \sum_{i=1}^2 A_i \partial_i W + BW = 0, \quad (2.3.7)$$

where the matrices A_i, B are as follows:

$$A_1 = \begin{pmatrix} W^1 & 0 & 1 + \frac{W^3}{2} \\ 0 & W^1 & 0 \\ 1 + \frac{W^3}{2} & 0 & W^1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} W^2 & 0 & 0 \\ 0 & W^2 & 1 + \frac{W^3}{2} \\ 0 & 1 + \frac{W^3}{2} & W^2 \end{pmatrix},$$

$$B = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The local existence and regularity for such a system now follows by Lemma 2.3.1. $\square$

Remark 2.3.2. Note that the matrices A_1 and A_2 are not simultaneously diagonalizable.

We sketch the proof of the classical local existence statement (see [70]).

Proposition 2.3.1. *Consider the initial value problem:*

$$\begin{cases} \partial_t W + \sum_{i=1}^2 A_i(W) \partial_i W + B(W) = 0, \\ W(0, x) = W_0. \end{cases} \quad (2.3.8)$$

We consider W as an unknown vector in $\mathbb{R}^k$. We let $A_i(W)$ be $k \times k$ matrices, linear as

functions of W , and $B(W)$ a linear function (vector in $\mathbb{R}^k$) of W , vanishing at $W = 0$. If $W_0 \in H^3(\mathbb{R}^2)$, there exists $T > 0$, depending on $\|W_0\|_{H^3(\mathbb{R}^2)}$, B , and A such that the system (2.3.8) has a unique solution W . Furthermore,

$$W \in L^\infty(0, T; H^3(\mathbb{R}^2)), \quad W_t \in L^\infty(0, T; H^2(\mathbb{R}^2)) \quad (2.3.9)$$

Proof of Lemma 2.3.1. The proof will be divided in three **Steps**. In **Step 1**, we mollify and linearize the system (2.3.8), in order to solve it by the linear theory. In **Step 2**, we use the linearized system in an iteration process, obtaining uniform estimates (here the fact that the A_i 's are symmetric is crucial). Finally, in **Step 3**, we pass to the limit using the Gagliardo–Nirenberg–Sobolev inequality.

• **Step 1.** We let φ be a smooth function in $\mathbb{R}^2$, nonnegative, supported in the unit ball and such that it integrates to 1 (the Friedrichs mollifier). Given $\varepsilon > 0$, we define $\varphi_{(\varepsilon)}(x) := \varepsilon^{-2}\varphi(\varepsilon^{-1}x)$. Given a L^1_{loc} function (or vector) U , we finally define

$$U_\varepsilon := \varphi_{(\varepsilon)} \star U, \quad (2.3.10)$$

the mollified version of U .

We now set up the linearized and smoothed problem. We construct a sequence W_j , $j \in \mathbb{N}$, as follows. Let $\varepsilon_0 > 0$ be a small number. We let W_{start} be the constant-in-time smooth function $W_{\text{start}}(x, t) := W_0 \star \varphi_{(\varepsilon_0)}$. We initialize by solving the linear problem:

$$\begin{cases} \partial_t W_1 + \sum_{i=1}^2 A_i(W_{\text{start}}) \partial_i W_1 + B(t, x, W_{\text{start}}) = 0, \\ W_1(0, x) = W_0 \star \varphi_{(\varepsilon_0)}. \end{cases} \quad (2.3.11)$$

For $j > 1$, given W_j , we obtain W_{j+1} as the solution of the following linear problem:

$$\begin{cases} \partial_t W_{j+1} + \sum_{i=1}^2 A_i(W_j) \partial_i W_{j+1} + B(t, x, W_j) = 0, \\ W_{j+1}(0, x) = W_0 \star \varphi_{(\varepsilon_0 2^{-j})}. \end{cases} \quad (2.3.12)$$

By the linear theory (see Evans [37] or Lemma 2.3.4), given W_j , a smooth solution W_{j+1} to the problem (2.3.12) exists for all time.

• **Step 2.** By performing energy estimates, we show the following statement about system (2.3.12).

Claim. There is a constant $E > 0$ and a number $T^* > 0$ such that, if $W_j \in \mathcal{C}^\infty([0, T^*] \times \mathbb{R}^2)$, with

$$\|W_0\|_{H^3(\mathbb{R}^2)} \leq E, \quad \sup_{t \in [0, T^*)} \|W_j\|_{H^3(\mathbb{R}^2)} \leq 2E, \quad \sup_{t \in [0, T^*)} \|\partial_t W_j\|_{H^2(\mathbb{R}^2)} \leq 2E, \quad (2.3.13)$$

and W_{j+1} is obtained from W_j via the system (2.3.12), then we have

$$\sup_{t \in [0, T^*)} \|W_{j+1}\|_{H^3(\mathbb{R}^2)} \leq 2E, \quad \sup_{t \in [0, T^*)} \|\partial_t W_{j+1}\|_{H^2(\mathbb{R}^2)} \leq 2E. \quad (2.3.14)$$

Proof of Claim. We let $Y := W_j$, $Z := W_{j+1}$. Let α be a multi-index in the spatial variables. We differentiate system (2.3.12) to get:

$$\partial_t \partial^\alpha Z + \sum_{i=1}^2 A_i(Y) \frac{\partial(\partial^\alpha Z)}{\partial x^i} = F_\alpha, \quad (2.3.15)$$

where

$$F_\alpha = \partial^\alpha B(Y) + \sum_{i=1}^2 \left(A_i(Y) \frac{\partial(\partial^\alpha Z)}{\partial x^i} - \partial^\alpha \left(A_i(Y) \frac{\partial Z}{\partial x^i} \right) \right)$$

Multiplying by $\partial^\alpha Z$ and integrating by parts leads to inequality (2.3.22), which in this case

is rephrased as

$$\begin{aligned}
& \int_{\mathbb{R}^2} |\partial^\alpha Z|^2(t, x) dx - \int_{\mathbb{R}^2} |\partial^\alpha Z|^2(s, x) dx \\
& \leq \int_s^t \int_{\mathbb{R}^2} \left| \sum_{i=1}^2 \partial_{x_i} A_i(Y) \right| \cdot |\partial^\alpha Z|(\tau, x) dx d\tau + \int_s^t \int_{\mathbb{R}^2} |F_\alpha(\tau, x)| \cdot |\partial^\alpha Z|(\tau, x) dx d\tau.
\end{aligned} \tag{2.3.16}$$

The previous inequality, upon summation over α and combined with the estimate on the nonlinear terms in Lemma 2.3.5, gives:

$$\begin{aligned}
& \|Z\|_{H^3(\mathbb{R}^2)}^2(t) - \|Z\|_{H^3(\mathbb{R}^2)}^2(s) \\
& \leq C \int_s^t (\|Z\|_{H^3(\mathbb{R}^2)}^2(\tau) + \|Z\|_{H^3(\mathbb{R}^2)}(\tau)) \|Y\|_{H^3(\mathbb{R}^2)}(\tau) d\tau.
\end{aligned} \tag{2.3.17}$$

We conclude by an application of the Grönwall inequality in τ .

• **Step 3.** By considering the equation satisfied by the difference $W_{j+1} - W_j$, and the energy inequality, plus the bounds derived in the **Claim**, it is easy to deduce that, along the sequence W_j , we have the Cauchy property, possibly by shrinking T^* :

$$\sum_{j \in \mathbb{N}} \sup_{t \in [0, T^*]} \|W_{j+1} - W_j\|_{L^2(\mathbb{R}^2)} < \infty. \tag{2.3.18}$$

By completeness, there exists $W \in \mathcal{C}(0, T^*; L^2(\mathbb{R}^2))$ such that

$$\lim_{j \rightarrow \infty} \sup_{t \in [0, T^*]} \|W_j - W\|_{L^2(\mathbb{R}^2)} = 0.$$

We now use the following Sobolev interpolation inequality (valid for $0 \leq s' \leq s$):

$$\|f\|_{H^{s'}(\mathbb{R}^2)} \leq C \|f\|_{L^2(\mathbb{R}^2)}^{1-s'/s} \|f\|_{H^s(\mathbb{R}^2)}^{s'/s}. \tag{2.3.19}$$

We obtain:

$$\lim_{j \rightarrow \infty} \sup_{t \in [0, T^*]} \|W_j - W\|_{\mathcal{C}^1(\mathbb{R}^2)} = 0.$$

By equation (2.3.12), passing to the limit, we obtain that

$$\frac{\partial W_j}{\partial t} \rightarrow \frac{\partial W}{\partial t} \quad \text{in } \mathcal{C}(0, T^*; \mathcal{C}(\mathbb{R}^2)). \quad (2.3.20)$$

This in particular implies that $W \in \mathcal{C}^1([0, T^*] \times \mathbb{R}^2)$, and that hence W is a classical solution to (2.3.8).

By Banach–Alaoglu, we can obtain weak convergence up to a subsequence so that

$$W_j \rightharpoonup W \text{ in } L^\infty(0, T^*; H^3(\mathbb{R}^2)), \quad \partial_t W_j \rightharpoonup \partial_t W \text{ in } L^\infty(0, T^*; H^2(\mathbb{R}^2)).$$

By uniqueness of the limit, this implies the regularity in the statement. □

Remark 2.3.3. The regularity in this theorem can be improved to

$$W \in \mathcal{C}(0, T^*; H^3(\mathbb{R}^2)), \quad \partial_t W_j \in \mathcal{C}(0, T^*; H^2(\mathbb{R}^2)).$$

We recall here the fundamental Lemma establishing existence and regularity to a linear hyperbolic system.

Lemma 2.3.4. *Consider a positive number $T > 0$, and the following linear symmetric hyperbolic system:*

$$\begin{aligned} \partial_t Z + \sum_{i=1}^2 A_i(t, x) \frac{\partial Z}{\partial x^i} &= F(t, x) \text{ on } [0, T) \times \mathbb{R}^2, \\ Z|_{t=0} &= Z_0. \end{aligned} \quad (2.3.21)$$

Here, the unknown Z is a vector in $\mathbb{R}^2$, the $A_i(t, x)$ are 2×2 matrices with smooth coefficients in t, x , such that $A_i(t, x)$ is symmetric for all $(t, x) \in [0, T] \times \mathbb{R}^2$, and $F(t, x)$ is a smooth

vector, such that they satisfy the following conditions:

- $Z_0 \in (H^3(\mathbb{R}^2))^2 \cap (C^\infty(\mathbb{R}^2))^2$,
- $A_i \in (L^\infty(0, T; H^3(\mathbb{R}^2)))^{2 \times 2}$ for $i = 1, 2$,
- $F \in (L^\infty(0, T; H^3(\mathbb{R}^2)))^2$.

Then, there exists $Z \in (C^\infty([0, T) \times \mathbb{R}^2))^2 \cap (L^\infty(0, T; H^3(\mathbb{R}^2)))^2$ solving the system (2.3.21).

Furthermore, for any multi-index α , the following holds true for all $T \geq t \geq s \geq 0$:

$$\begin{aligned} & \int_{\mathbb{R}^2} |\partial^\alpha Z|^2(t, x) dx - \int_{\mathbb{R}^2} |\partial^\alpha Z|^2(s, x) dx \\ & \leq \int_s^t \int_{\mathbb{R}^2} \left| \sum_{i=1}^2 \partial_{x_i} A_i(\tau, x) \right| \cdot |\partial^\alpha Z|(\tau, x) dx d\tau + \int_s^t \int_{\mathbb{R}^2} |F_\alpha(\tau, x)| \cdot |\partial^\alpha Z|(\tau, x) dx d\tau. \end{aligned} \quad (2.3.22)$$

Here, we have the following definition for F_α :

$$F_\alpha(\tau, x) = \partial^\alpha F(\tau, x) + \sum_{i=1}^2 \left(A_i(Y) \frac{\partial(\partial^\alpha Z)}{\partial x^i} - \partial^\alpha \left(A_i(Y) \frac{\partial Z}{\partial x^i} \right) \right) (\tau, x)$$

Proof (sketch). The constructive element in this proof is the Cauchy–Kowalevskaya Theorem, combined with an approximation argument (Stone–Weierstrass). We let $m \geq 3$, and we approximate the initial data Z_0 , the matrices A_i and the forcing F uniformly in $\mathcal{C}^m$ by analytic functions on a $3d$ cone with large enough aperture (to ensure that the boundary terms in the energy estimates will be positive). By the linearity of the equations, the solution provided by the Cauchy–Kowalevskaya theorem exists on the whole cone. We then employ energy estimates on the cone to obtain uniform bounds in H^m norm along this sequence, and prove that a subsequence is Cauchy in L^2 uniformly in time. We conclude by interpolation and passing to the limit in the equations. $\square$

Lemma 2.3.5 (Estimates on the nonlinear terms). *Consider the matrices*

$$A_1(W) = \begin{pmatrix} W^1 & 0 & 1 + \frac{W^3}{2} \\ 0 & W^1 & 0 \\ 1 + \frac{W^3}{2} & 0 & W^1 \end{pmatrix}, \quad A_2(W) = \begin{pmatrix} W^2 & 0 & 0 \\ 0 & W^2 & 1 + \frac{W^3}{2} \\ 0 & 1 + \frac{W^3}{2} & W^2 \end{pmatrix},$$

$$B = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Here, $W \in (C^\infty(\mathbb{R}^2))^3$. We define the nonlinear term as follows. Given a multi-index α of order at most 3, we set

$$N_\alpha := \partial^\alpha(BY) + \sum_{i=1}^2 \left(A_i(Y) \frac{\partial(\partial^\alpha Z)}{\partial x^i} - \partial^\alpha \left(A_i(Y) \frac{\partial Z}{\partial x^i} \right) \right).$$

Here, Y and Z are both in $(C^\infty(\mathbb{R}^2))^3$. In these conditions, we have the following bound:

$$\|N_\alpha\|_{L^2(\mathbb{R}^2)} \leq C \|Z\|_{H^3(\mathbb{R}^2)} \|Y\|_{H^3(\mathbb{R}^2)}. \quad (2.3.23)$$

Proof. This proof is straightforward using the chain rule, the form of A_i and the Sobolev embedding $H^2(\mathbb{R}^2) \hookrightarrow L^\infty(\mathbb{R}^2)$. $\square$

Notice also that the above proof of the local existence statement provides us immediately with a continuation criterion.

Proposition 2.3.2 (Continuation criterion). *Let $\bar{h} > 0$, and let*

$$(u_0, h_0 - \bar{h}) \in (H^3(\mathbb{R}^2))^2 \times H^3(\mathbb{R}^2)$$

be initial data for the inviscid shallow water system (2.3.1), such that $h_0 \geq \bar{h} > 0$. Then, there exists a maximal time $T > 0$ such that the shallow water system (2.3.1) admits a

solution on $[0, T) \times \mathbb{R}^2$ in the conditions of Theorem 2.3.1. Suppose furthermore that there exists a large number $k > 0$, such that, on $[0, T) \times \mathbb{R}^2$, we have the lower bound on the height: $h \geq \bar{h}/k$. Then, at T it is necessary that the H^3 norm of the solution blows up:

$$\|u(t)\|_{H^3(\mathbb{R}^2)} + \|h(t)\|_{H^3(\mathbb{R}^2)} \rightarrow \infty \text{ as } t \rightarrow T. \quad (2.3.24)$$

2.3.2 Local existence in the viscous case

In this section, we focus on proving a local existence statement for the SW system in the degenerate viscosity case. The degenerate viscosity coefficient gives error terms of high order which need to be dealt with in a careful way. In the case at hand, the iteration scheme needs to be modified, and that was first done by Matsumura and Nishida in [72]. We wish to prove the following Theorem.

Theorem 2.3.6 (Local existence to the viscous equations). *Consider the viscous shallow water system*

$$\begin{cases} \partial_t h + \operatorname{div}(uh) = 0, \\ \partial_t u + u \cdot \nabla u + g \nabla h + u^\perp - \nu h^{-1} \operatorname{div}(h \nabla u) = 0, \\ (u, h)|_{t=0} = (u_0, h_0). \end{cases} \quad (2.3.25)$$

Here, $\nu > 0$ is the viscosity coefficient. Suppose that there exists $\bar{h} > 0$ such that $h_0 \geq \frac{\bar{h}}{2}$ on $\mathbb{R}^2$, and furthermore $u_0 \in H^4(\mathbb{R}^2)$, $h_0 - \bar{h} \in H^4(\mathbb{R}^2)$. Then, there exists a time $T > 0$, $T(\|u_0\|_{H^4(\mathbb{R}^2)}, \|h_0\|_{H^4(\mathbb{R}^2)}, \bar{h}, g, \nu)$ such that a classical solution to the system (2.3.25) exists on $[0, T) \times \mathbb{R}^2$ and furthermore it is such that

$$h - \bar{h} \in L^\infty(0, T; H^4(\mathbb{R}^2)) \cap L^\infty(0, T; H^3(\mathbb{R}^2)),$$

and

$$u \in (L^\infty(0, T; H^4(\mathbb{R}^2)))^2 \cap (L^\infty(0, T; H^2(\mathbb{R}^2)))^2.$$

Remark 2.3.7. The requirement $u, h - \bar{h} \in H^4(\mathbb{R}^2)$ can be relaxed to $u, h - \bar{h} \in H^3(\mathbb{R}^2)$. The

regularity of the solution can be improved to $h - \bar{h} \in \mathcal{C}^0(0, T; H^4(\mathbb{R}^2)) \cap \mathcal{C}^1(0, T; H^3(\mathbb{R}^2))$, and $u \in (\mathcal{C}^0(0, T; H^4(\mathbb{R}^2)))^2 \cap (\mathcal{C}^1(0, T; H^2(\mathbb{R}^2)))^2$

Proof. We divide the proof in four **Steps**. In the first step, we will set up a linearized version of the equations in (2.3.25). Then, in the second step, we will solve such linear equations uniformly in time and prove a priori estimates for the solution. In the third step we will bound the right hand side of the linearized equations. Finally, we will close the argument by setting up the iteration and proving convergence for small time of the iterates.

• **Step 1.** For simplicity, we let $\bar{h} = 1$, and we consider the perturbed linear equations when $\eta > 0$ and v are given, with $\eta \in \mathcal{C}^\infty(\mathbb{R}^2)$, and $v \in (\mathcal{C}^\infty(\mathbb{R}^2))^2$, and we solve for ρ, u . Here is the linearized version:

$$\begin{cases} \partial_t u + v \cdot \nabla v + g \nabla \eta + v^\perp - \nu \Delta u - \nu(1 + \eta)^{-1} \nabla \eta \cdot \nabla v = 0, \\ \partial_t \rho + (1 + \eta) \operatorname{div} v + v \cdot \nabla \rho = 0, \\ (u, \rho)|_{t=0} = (u_0, \rho_0). \end{cases} \quad (2.3.26)$$

We now let $L^i(u) := \partial_t u^i - \nu \Delta u^i$, $L^0(\rho) = \partial_t \rho + v \cdot \nabla \rho$, and rewrite the linearized system in the following fashion, with the appropriate definitions of F and G :

$$\begin{cases} L^0(\rho) = F(x, t, v, \eta), \\ L^i(u) = G^i(x, t, v, \eta), \quad i = 1, 2 \\ (u, \rho)|_{t=0} = (u_0, \rho_0). \end{cases} \quad (2.3.27)$$

This is the linearized system we will be working with.

• **Step 2.** We prove here a priori estimates for the linearized system. These estimates are useful both to prove existence of solutions to the linearized equations themselves (see [37],

Chapter 7), and later to prove existence to the nonlinear problem. We have the following Lemma.

Lemma 2.3.8 (A priori estimates for L^0). *Let $T > 0$. There exists a constant $C > 0$ such that the following holds. Suppose that $\eta \in \mathcal{C}^\infty([0, T] \times \mathbb{R}^2)$, and furthermore that $v \in (\mathcal{C}^\infty([0, T] \times \mathbb{R}^2))^2$, and that F is a smooth function of its arguments. Then there exists a solution $\rho \in \mathcal{C}^\infty([0, T] \times \mathbb{R}^2)$ to the transport equation*

$$L^0(\rho) = F(x, t, v, \eta), \quad \rho(0) = \rho_0, \quad (2.3.28)$$

with smooth initial data ρ_0 . Furthermore, ρ satisfies the following inequality:

$$\sup_{t \in [0, T]} \|\rho(t)\|_{H^4(\mathbb{R}^2)} \leq \exp(CE T) \left(\|\rho_0\|_{H^4(\mathbb{R}^2)} + \int_0^T \exp(-CE\tau) \|F\|_{H^4(\mathbb{R}^2)}(\tau) d\tau \right), \quad (2.3.29)$$

where $E := \sup_{t \in [0, T]} \|v\|_{H^4(\mathbb{R}^2)}$.

Proof of Lemma 2.3.8. We solve equation (2.3.28) until time T by the method of characteristics, as the field v is supposed to be smooth (global Cauchy–Lipschitz applies). Furthermore, by the linearity and hyperbolic character of the equation, we can reduce to the case when both u_0 and F have compact support (by a partition of unity).

By virtue of these reductions, the solution is now smooth and compactly supported, so we can just multiply and integrate in the classical way. Integration by parts and the Grönwall inequality yield the result. $\square$

The following Lemma establishes a priori inequalities for the momentum equation. It relies on its parabolic character.

Lemma 2.3.9 (A priori estimates for L^i). *Let $T, \nu > 0$. There exists a constant $C > 0$ such that the following holds. Suppose that $\eta \in \mathcal{C}^\infty([0, T] \times \mathbb{R}^2)$, and furthermore that $v \in (\mathcal{C}^\infty([0, T] \times \mathbb{R}^2))^2$, and that $\eta > \tilde{\eta} > -1$, and that G^i is a Schwartz function of its*

arguments, for $i = 1, 2$. Suppose the initial data u_0 are smooth and compactly supported. Suppose furthermore that

$$\sup_{t \in [0, T]} \|G^i(x, t, v, \eta)\|_{H^3(\mathbb{R}^2)} < \infty.$$

Then, there exists a classical solution $u \in \mathcal{C}^\infty((0, T) \times \mathbb{R}^2)$ of the inhomogeneous heat equation

$$L^i(u) = G^i(x, t, v, \eta), \text{ for } i = 1, 2, \quad u(0) = u_0, \quad (2.3.30)$$

with initial data u_0 . Also, the following a priori inequality holds true:

$$\begin{aligned} & \sup_{t \in [0, T]} \|u(t)\|_{H^4(\mathbb{R}^2)}^2(\tau) + \nu \int_0^T \|\nabla u\|_{H^4(\mathbb{R}^2)}^2 d\tau \\ & \leq C \|u_0\|_{H^4(\mathbb{R}^2)}^2 + C \int_0^T \sum_{i=1}^2 \|G^i(x, \tau, v, \eta)\|_{H^3(\mathbb{R}^2)}^2 d\tau. \end{aligned} \quad (2.3.31)$$

Proof of Lemma 2.3.9. Let $\nu = 1$ for simplicity. We define the solution by the representation formula:

$$\hat{u}(\xi, t) = \exp(-|\xi|^2 t) \left(\hat{u}_0(\xi) + \int_0^t \exp(\tau |\xi|^2) \hat{G}_i(\xi, \tau) d\tau \right). \quad (2.3.32)$$

In these conditions, it is easy to show that u is well defined, and it satisfies $u \in \mathcal{C}^\infty((0, T) \times \mathbb{R}^2)$. Furthermore, $u \in H^4(\mathbb{R}^2)$ for all $t \in [0, T]$. Let us consider $t > 0$. We are in the conditions to take the ∂^α derivative of equation (2.3.30), multiply it by $\partial^\alpha u$, and integrate over $[t_1, t] \times \mathbb{R}^2$. The construction of u then gives us enough regularity to cancel the boundary terms arising in the integration. We use integration by parts on the term in G_i on the right hand side, use the Young inequality with small parameter, and finally hide the term in $\partial^\alpha u$ inside the following term in the left hand side:

$$\int_{t_1}^t \|\nabla \partial^\alpha u\|_{L^2(\mathbb{R}^2)}^2 d\tau.$$

We then take the limit $t_1 \rightarrow 0$, by continuity in H^4 of the solution given by formula (2.3.32), we conclude. □

• **Step 3.** In this step, we establish important inequalities for the nonlinear terms appearing on the right hand side.

Lemma 2.3.10 (Inequalities for the nonlinear right hand side). *There exists a constant $C > 0$ and a real polynomial $p(x)$ with positive coefficients such that the following holds true. Let $\eta \in \mathcal{C}^\infty(\mathbb{R}^2)$, and $v \in (\mathcal{C}^\infty(\mathbb{R}^2))^2 \cap H^4(\mathbb{R}^2)$, such that furthermore $\eta \geq \tilde{\eta} > -1$. We consider the following terms:*

$$F := -(1 + \eta) \operatorname{div} v, \quad G := -v \cdot \nabla v - v^\perp - g \nabla \eta + \nu(1 + \eta)^{-1} \nabla \eta \cdot \nabla v.$$

Then, we have the following inequalities:

$$\begin{aligned} \|F\|_{H^4(\mathbb{R}^2)}^2 &\leq C \|1 + \eta\|_{L^\infty(\mathbb{R}^2)}^2 \|\nabla v\|_{H^4(\mathbb{R}^2)}^2 + C \|\nabla \eta\|_{H^3(\mathbb{R}^2)}^2 \|v\|_{H^4(\mathbb{R}^2)}^2, \\ \|G\|_{H^3(\mathbb{R}^2)}^2 &\leq C \|v\|_{H^4(\mathbb{R}^2)}^4 + p((1 - \tilde{\eta})^{-1})(1 + \|\eta\|_{H^4(\mathbb{R}^2)}^2) \|\eta\|_{H^4(\mathbb{R}^2)}^2 \|v\|_{H^4(\mathbb{R}^2)}^2 \\ &\quad + C \|\eta\|_{H^4(\mathbb{R}^2)}^2. \end{aligned}$$

Proof. The proof of these inequalities is straightforward using the product rule and the Sobolev embedding $H^2(\mathbb{R}^2) \hookrightarrow L^\infty(\mathbb{R}^2)$. $\square$

Similarly, we can establish inequalities when differences are involved:

Lemma 2.3.11 (Inequalities for the right hand side). *There exists a constant $C > 0$ and a real polynomial of three variables with positive coefficients $p(x_1, x_2, x_3)$ such that the following holds true. Let $\eta, \eta_1 \in \mathcal{C}^\infty(\mathbb{R}^2) \cap H^4(\mathbb{R}^2)$, and $u, u_1, v, v_1 \in (\mathcal{C}^\infty(\mathbb{R}^2))^2 \cap H^4(\mathbb{R}^2)$, such that furthermore $\eta, \eta_1 \geq \tilde{\eta} > -1$. We consider the following terms:*

$$\begin{aligned} \tilde{F} &:= -(1 + \eta) \operatorname{div} v + (1 + \eta_1) \operatorname{div} v_1, \\ \tilde{G} &:= -v \cdot \nabla v - v^\perp - g \nabla \eta + \nu(1 + \eta)^{-1} \nabla \eta \cdot \nabla v \\ &\quad + v_1 \cdot \nabla v_1 + (v_1)^\perp + g \nabla \eta_1 - \nu(1 + \eta_1)^{-1} \nabla \eta_1 \cdot \nabla v_1. \end{aligned}$$

Then, we have the following inequalities:

$$\begin{aligned}
& \left\| \tilde{F} \right\|_{L^2(\mathbb{R}^2)}^2 \leq 2(1 + \|\eta_1\|_{H^3(\mathbb{R}^2)}^2) \|\nabla(v_1 - v)\|_{L^2(\mathbb{R}^2)}^2 + C \|v\|_{H^3(\mathbb{R}^2)}^2 \|\eta - \eta_1\|_{L^2(\mathbb{R}^2)}^2, \\
& \left\| (u - u_1) \cdot \tilde{G} \right\|_{L^1(\mathbb{R}^2)} \\
& \leq C \|v - v_1\|_{L^2(\mathbb{R}^2)} (\|u - u_1\|_{L^2(\mathbb{R}^2)} + \|v - v_1\|_{L^2(\mathbb{R}^2)}) + \|\eta - \eta_1\|_{L^2(\mathbb{R}^2)}^2 \\
& + q(\|v\|_{H^4(\mathbb{R}^2)}, (1 - \tilde{\eta})^{-1}, \|\eta\|_{H^4(\mathbb{R}^2)}) (\|u - u_1\|_{L^2(\mathbb{R}^2)} \|v - v_1\|_{L^2(\mathbb{R}^2)} + \|v - v_1\|_{L^2(\mathbb{R}^2)}^2)
\end{aligned}$$

Proof. Once again, the proof of these inequalities is straightforward using the product rule, integration by parts and the Sobolev embedding $H^2(\mathbb{R}^2) \hookrightarrow L^\infty(\mathbb{R}^2)$. Particular care must be taken to deal with the term

$$\nu(1 + \eta)^{-1} \nabla \eta \cdot \nabla v - \nu(1 + \eta_1)^{-1} \nabla \eta_1 \cdot \nabla v_1.$$

This is why four derivatives are needed in the estimate for $\tilde{G}$. □

• **Step 4.** We follow the same procedure as in **Step 4** of the proof of Proposition 2.3.1.

The previous steps and the Sobolev interpolation inequality in the following form:

$$\|f\|_{H^{s'}(\mathbb{R}^2)} \leq C \|f\|_{L^2(\mathbb{R}^2)}^{1-s'/s} \|f\|_{H^s(\mathbb{R}^2)}^{s'/s} \quad (2.3.33)$$

let us conclude the existence of a solution (u, ρ) such that $u, \rho \in L^\infty(0, T; H^4(\mathbb{R}^2))$, $\partial_t \rho \in L^\infty(0, T; H^3(\mathbb{R}^2))$ and $\partial_t u \in L^\infty(0, T; H^2(\mathbb{R}^2))$. This solution is in particular a classical solution. □

Remark 2.3.12. The regularity of the solution can be improved to

$$u, \rho \in \mathcal{C}(0, T; H^4(\mathbb{R}^2)), \partial_t \rho \in \mathcal{C}(0, T; H^3(\mathbb{R}^2)), \text{ and } \partial_t u \in \mathcal{C}(0, T; H^2(\mathbb{R}^2))$$

Remark 2.3.13. The fact that we have a local existence result in H^4 follows from the form of the nonlinear right hand side (the estimates in Lemma 2.3.11 require uniform boundedness for u and η in $H^4(\mathbb{R}^2)$).

2.3.3 Global stability for small initial data for the SW system with degenerate viscosity

In this section, we sketch the proof of a global stability theorem for small initial data in the context of the SW equations. The treatment is after the works of Kloeden [63] and Sundbye [85].

Theorem 2.3.14 (Global existence for small data for the viscous, unforced shallow water equations). *Let $\bar{h} > 0$, and let $f \in \mathbb{R}$. There exists a small number $\varepsilon > 0$ such that the following holds true. Consider initial data for the shallow water system $u_0 \in (H^4(\mathbb{R}^2))^2$, h_0 such that $h_0 - \bar{h} \in H^4(\mathbb{R}^2)$, and furthermore suppose the smallness condition*

$$\|u_0\|_{H^4(\mathbb{R}^2)}, \|h_0 - \bar{h}\|_{H^3(\mathbb{R}^2)} \leq \varepsilon. \quad (2.3.34)$$

Then, u_0 and h_0 give rise to a global-in-time solution (u, h) to the 2d viscous shallow water system:

$$\begin{cases} \partial_t h + \operatorname{div} (uh) = 0, \\ \partial_t u + u \cdot \nabla u + g \nabla h + u^\perp - \nu h^{-1} \operatorname{div} (h \nabla u) = 0, \\ (u, h)|_{t=0} = (u_0, h_0). \end{cases} \quad (2.3.35)$$

with the following regularity properties:

$$\begin{aligned} u &\in L^\infty(0, \infty; H^4(\mathbb{R}^2)) \cap L^\infty(0, \infty; H^2(\mathbb{R}^2)), \\ h - \bar{h} &\in L^\infty(0, \infty; H^4(\mathbb{R}^2)) \cap L^\infty(0, \infty; H^3(\mathbb{R}^2)). \end{aligned}$$

Furthermore, the solution obtained is classical for $t > 0$.

Remark 2.3.15. The requirement of boundedness in $H^4(\mathbb{R}^2)$ is only technical to be able to use the version of the local existence theorem proved in this chapter. Furthermore, the stated regularity can be improved to

$$u \in \mathcal{C}(0, \infty; H^4(\mathbb{R}^2)) \cap \mathcal{C}^1(0, \infty; H^2(\mathbb{R}^2)), \quad h - \bar{h} \in \mathcal{C}(0, \infty; H^4(\mathbb{R}^2)) \cap \mathcal{C}^1(0, \infty; H^3(\mathbb{R}^2)). \quad (2.3.36)$$

To prove this theorem, we separate the linear part of the shallow water system from the nonlinear part. We define

$$\mathcal{L}^0(u, h) := \partial_t h + \bar{h} \operatorname{div} u, \quad \mathcal{L}^i(u, h) := \partial_t u - \nu \Delta u + \nabla h + f u^\perp, \text{ for } i = 1, 2. \quad (2.3.37)$$

Then, the system (2.3.35), upon renaming h , is equivalent to the initial value problem:

$$\begin{aligned} \mathcal{L}^i(u, h) &= G^i, \text{ for } i = 0, 1, 2, \\ u|_{t=0} &= u_0, \quad h|_{t=0} = h_0 - \bar{h} \end{aligned} \quad (2.3.38)$$

Where we defined the nonlinear parts:

$$G^0 = -\operatorname{div}(hu), \quad \begin{pmatrix} G^1 \\ G^2 \end{pmatrix} := -u \cdot \nabla u + \nu(h + \bar{h})^{-1} \nabla h \cdot \nabla u. \quad (2.3.39)$$

The proof relies on weighted energy estimates with weights in time. Let us define the following weighted in time energy. Let $t_2 \geq t_1 \geq 0$:

$$N^2(t_2, t_1) := \sup_{t_1 \leq t \leq t_2} \left(\|(h, u)(t)\|_{H^4(\mathbb{R}^2)}^2 \right) + \int_{t_1}^{t_2} \left(\|\nabla h(s)\|_{H^3(\mathbb{R}^2)}^2 + \|\nabla u(s)\|_{H^4(\mathbb{R}^2)}^2 \right) ds \quad (2.3.40)$$

The goal is to bootstrap this energy: assuming that on $[0, T]$ we have $N^2(0, T) \leq \varepsilon$, we deduce, upon choosing ε small enough, that $N^2(0, T) \leq \varepsilon^2 \leq \frac{1}{2}\varepsilon$. By the local existence

result, this proves global existence in the space

$$\begin{aligned} u &\in L^\infty(0, \infty; H^4(\mathbb{R}^2)) \cap L^\infty(0, \infty; H^2(\mathbb{R}^2)), \\ h - \bar{h} &\in L^\infty(0, \infty; H^4(\mathbb{R}^2)) \cap L^\infty(0, \infty; H^3(\mathbb{R}^2)). \end{aligned}$$

We furthermore define the following error terms, for $k = 0, 1, 2, 3, 4$:

$$\begin{aligned} A^k(t) &:= \int_{\mathbb{R}^2} (\nabla^k G^0 \mathbin{::} \nabla^k h + \bar{h} \sum_{i=1}^2 (\nabla^k G^i \mathbin{::} \nabla^k u^i)) dx, \\ B^k(t) &:= \int_{\mathbb{R}^2} \left(\sum_{i=1}^2 (\nabla^k \partial_{x_i} G^0 \mathbin{::} \nabla^k \partial_{x_i} h) + \frac{\bar{h}}{\nu} \sum_{i=1}^2 (\nabla^k G^i \mathbin{::} \nabla^k \partial_{x_i} h) \right) dx, \\ C^k(t) &:= \int_{\mathbb{R}^2} \frac{\bar{h}}{\nu} \sum_{i=1}^2 (\nabla^k \partial_{x_i} G^0 \mathbin{::} \nabla^k u^i) dx. \end{aligned} \tag{2.3.41}$$

Here, “ $\mathbin{::}$ ” denotes contraction.

Lemma 2.3.16 (Energy estimates). *There exists a constant $C > 0$ such that, if (u, h) satisfy the Shallow Water system (2.3.35) on $[0, T] \times \mathbb{R}^2$, then we have the following inequality, for $0 \leq m \leq 4$:*

$$\begin{aligned} &\|(\nabla^m h, \nabla^m u)(t)\|_{L^2(\mathbb{R}^2)}^2 + \nu \int_0^t \|\nabla^{m+1} u(s)\|^2 ds \\ &\leq C \left\{ \|\nabla^m h_0, \nabla^m u_0\|_{L^2(\mathbb{R}^2)}^2 + \int_0^t |A^m(s)| ds \right\}. \end{aligned} \tag{2.3.42}$$

Furthermore, for $1 \leq m \leq 4$, we have the second inequality:

$$\begin{aligned} &\|\nabla^m h\|_{L^2}^2 + \frac{\bar{h}}{\nu} \int_0^t \|\nabla^m h(s)\|^2 ds \\ &\leq C \left\{ \|\nabla^m h_0\|_{L^2(\mathbb{R}^2)}^2 + \|\nabla^{m-1} u_0\|_{L^2(\mathbb{R}^2)}^2 + \|\nabla^{m-1} u(t)\|_{L^2(\mathbb{R}^2)}^2 \right\} \\ &+ C \left\{ \int_0^t (\|\nabla^m u(s)\|_{L^2(\mathbb{R}^2)}^2 + |B^{m-1}(s)| + |C^{m-1}(s)|) ds \right\} \end{aligned} \tag{2.3.43}$$

Proof. The regularity in the local existence theorem allows us to integrate by parts. For the

first equation (2.3.42), we have the following:

$$\int_0^t \left(\int_{\mathbb{R}^2} (\nabla^m \mathcal{L}^0(u, h) :: \nabla^m h + \bar{h} \sum_{i=1}^2 (\nabla^m \mathcal{L}^i(u, h) :: \nabla^m u^i)) dx \right) ds = \int_0^t A^m(s) ds$$

Upon integration by parts, the left hand side is equal to

$$(\|\nabla^m h(s)\|_{L^2(\mathbb{R}^2)}^2 + \bar{h} \|\nabla^m u(s)\|_{L^2(\mathbb{R}^2)}^2)|_{s=0}^{s=t} + \bar{h} \nu \int_0^t \|\nabla^{m+1} u\|_{L^2(\mathbb{R}^2)}^2 ds.$$

This proves the first inequality (2.3.42). For the second inequality (2.3.43), we consider the following expression

$$\begin{aligned} & \int_0^t \int_{\mathbb{R}^2} \left(\sum_{i=1}^2 (\nabla^{m-1} \partial_{x_i} \mathcal{L}^0(u, h) :: \nabla^{m-1} \partial_{x_i} h) \right. \\ & \quad \left. + \frac{\bar{h}}{\nu} \sum_{i=1}^2 (\nabla^{m-1} \mathcal{L}^i(u, h) :: \nabla^{m-1} \partial_{x_i} h) \right) dx ds \\ & = \int_0^t B^{m-1}(s) ds \end{aligned}$$

This lets us deduce that

$$\begin{aligned} & \left\{ \frac{1}{2} \|\nabla^m h(s)\|_{L^2(\mathbb{R}^2)}^2 + \frac{\bar{h}}{\nu} \int_{\mathbb{R}^2} \sum_{i=1}^2 \nabla^{m-1} u_i(s) :: \nabla^{m-1} \partial_{x_i} h(s) dx \right\} \Big|_{s=0}^{s=t} \\ & + \frac{\bar{h}}{\nu} \int_0^t \|\nabla^m h\|_{L^2(\mathbb{R}^2)}^2 ds \\ & \leq \int_0^t \frac{\bar{h}^2}{\nu} \|\nabla^m u\|_{L^2(\mathbb{R}^2)}^2 ds + \int_0^t (|B^{m-1}(s)| + |C^{m-1}(s)|) ds \end{aligned}$$

It is now easy to infer inequality (2.3.43). □

Let $\eta > 0$ be a small positive real number, and let us consider the following expression:

$$(2.3.42) + \eta(2.3.43).$$

By possibly restricting the value of η , we obtain the following lemma:

Lemma 2.3.17 (Main bootstrap inequality). *There exists a positive constant C such that the following holds true. If (u, h) solves the shallow water system on $[0, T] \times \mathbb{R}^2$ with initial data (u_0, h_0) we have the following inequality:*

$$\begin{aligned} & \sup_{s \in [0, T]} \|(u(s), h(s))\|_{H^4(\mathbb{R}^2)}^2 + \nu \int_0^T \|\nabla u(s)\|_{H^4(\mathbb{R}^2)}^2 ds + \frac{\bar{h}}{\nu} \int_0^T \|\nabla h(s)\|_{H^3(\mathbb{R}^2)}^2 ds \\ & \leq C \|(u_0, h_0)\|_{H^4(\mathbb{R}^2)}^2 + C \int_0^T \left(\sum_{m=0}^4 |A^m(s)| + \sum_{m=0}^3 (|B^m(s)| + |C^m(s)|) \right) ds. \end{aligned} \quad (2.3.44)$$

We can conclude the argument upon proving the following lemma:

Lemma 2.3.18 (Estimates on the nonlinear terms). *There exists a positive constant C and a small number $\varepsilon > 0$ such that the following holds true. Suppose that (u, h) solves the shallow water system on $[0, T] \times \mathbb{R}^2$ satisfying the smallness condition*

$$N(0, T) \leq \varepsilon.$$

Under these conditions, we have the following inequality:

$$\int_0^T \left(\sum_{m=0}^4 |A^m(s)| + \sum_{m=0}^3 (|B^m(s)| + |C^m(s)|) \right) ds \leq C N^2(0, T) \sum_{j=1}^3 N^j(0, T). \quad (2.3.45)$$

Proof. The proof of this lemma is carried out by direct inspection of the nonlinear terms, the fact that $H^s(\mathbb{R}^2)$ is an algebra for $s > 1$, and the Sobolev embedding. Recall:

$$G^0 = -\operatorname{div}(hu), \quad \begin{pmatrix} G^1 \\ G^2 \end{pmatrix} := \underbrace{-u \cdot \nabla u}_{(a)} + \underbrace{\nu(h + \bar{h})^{-1} \nabla h : \nabla u}_{(b)}. \quad (2.3.46)$$

$$A^4(t) := \int_{\mathbb{R}^2} \underbrace{(\nabla^4 G^0 :: \nabla^4 h)}_{(i)} + \bar{h} \underbrace{\sum_{i=1}^2 (\nabla^4 G^i :: \nabla^4 u^i)}_{(ii)} dx. \quad (2.3.47)$$

We have:

$$\begin{aligned}
\left| \int_0^T \int_{\mathbb{R}^2} (i) dx dt \right| &= \left| \int_0^T \int_{\mathbb{R}^2} \sum_{|\gamma|=4} \sum_{\alpha+\beta=\gamma} \operatorname{div} (\partial^\alpha u \partial^\beta h) \partial^\gamma h dx ds \right| \\
&\leq \left| \int_0^T \int_{\mathbb{R}^2} \sum_{|\gamma|=4} \operatorname{div} ((\partial^\gamma u) h) \cdot \partial^\gamma h dx dt \right| + \left| \int_0^T \int_{\mathbb{R}^2} \sum_{|\gamma|=4} \operatorname{div} (u (\partial^\gamma h)) \cdot \partial^\gamma h dx dt \right| \quad (2.3.48) \\
&+ \left| \int_0^T \int_{\mathbb{R}^2} \sum_{|\gamma|=4} \sum_{\substack{\alpha+\beta=\gamma \\ |\alpha|, |\beta| > 0}} \operatorname{div} (\partial^\alpha u \partial^\beta h) \partial^\gamma h dx ds \right|
\end{aligned}$$

The third term has at most two derivatives on one of the terms in the divergence, hence can be dealt with the Sobolev embedding $H^2(\mathbb{R}^2) \hookrightarrow L^\infty(\mathbb{R}^2)$. Concerning the other terms in (2.3.48), for the first we have

$$\begin{aligned}
&\left| \int_0^T \int_{\mathbb{R}^2} \sum_{|\gamma|=4} \operatorname{div} ((\partial^\gamma u) h) \cdot \partial^\gamma h dx dt \right| \\
&\leq CN^3(0, T) + \left| \int_0^T \int_{\mathbb{R}^2} \sum_{|\gamma|=4} ((\partial^\gamma \nabla u) h) \cdot \partial^\gamma h dx dt \right| \quad (2.3.49)
\end{aligned}$$

and we can use the Cauchy-Schwarz inequality on the last term (we can allow up to 5 derivatives on u by the definition of $N(0, T)$). The second term in the RHS of (2.3.48) can be treated in the following way:

$$\left| \int_0^T \int_{\mathbb{R}^2} \sum_{|\gamma|=4} \operatorname{div} (u (\partial^\gamma h)) \cdot \partial^\gamma h dx dt \right| \leq CN^3(0, T) + \left| \int_0^T \int_{\mathbb{R}^2} \sum_{|\gamma|=4} \frac{1}{2} u \cdot |\partial^\gamma \nabla h|^2 dx dt \right| \quad (2.3.50)$$

and we can integrate by parts.

The term in (ii).(a) can be dealt with in a similar manner as in estimate (2.3.49), and

finally we have for the terms in (ii).(b), integrating by parts:

$$\begin{aligned} & \left| \int_0^T \int_{\mathbb{R}^2} \sum_{|\gamma|=4} \partial^\gamma ((h + \bar{h})^{-1} \nabla h : \nabla u) : \partial^\gamma u \, ds \, dx \right| \\ & \leq \int_0^T \left\| \sum_{|\delta|=3} \partial^\delta ((h + \bar{h})^{-1} \nabla h : \nabla u) \right\|_{L^2(\mathbb{R}^2)} (s) \|\nabla u\|_{H^4(\mathbb{R}^2)}(s) \, ds, \end{aligned}$$

and we conclude by the Sobolev embedding.

The terms in B , C can be dealt with in a similar way. $\square$

This concludes our treatment of classical properties of the 2D compressible NS system with degenerate viscosity. We now turn to the issue of global bounds in axisymmetry in the case of non-degenerate viscosity.

2.4 Conditional global existence and uniform bounds for the compressible NS system in 2D under symmetry assumptions

In this section, we are going to deal with the 2D compressible Navier–Stokes system with $\gamma = 2$ and non-degenerate viscosity ($\alpha = 0$). Hence, we are going to be focusing on the following system:

$$\begin{aligned} \partial_t u + u \cdot \nabla u + \nabla \rho - \rho^{-1} \Delta u &= 0, \\ \partial_t \rho + \operatorname{div} (u \rho) &= 0. \end{aligned}$$

First, in Section 2.4.1 we are going to revisit the proof in [51], in which the author proves that, in radial symmetry (with *no swirl*) on a bounded annular domain, the 2D NS system with non-degenerate viscosity admits global solutions, when Dirichlet boundary conditions are imposed at the boundary. As a byproduct of the proof, we will show that the density ρ admits bounds from above and from below which degenerate exponentially in time as time

goes to infinity. In the following section (Section 2.4.2) we are going to extend the proof of uniform bounds to the case of *axial symmetry*, that is to say, to the case in which the solution is still invariant under rotation, but has nonzero swirl.

Let us start by recalling the local existence result relevant to the case at hand, with the associated continuation criterion.

Theorem 2.4.1 (Local existence to the viscous equations). *Consider the compressible and viscous NS system with non-degenerate viscosity*

$$\begin{aligned}\partial_t u + u \cdot \nabla u + \nabla \rho - \rho^{-1} \Delta u &= 0, \\ \partial_t \rho + \operatorname{div} (u \rho) &= 0, \\ (u, \rho)|_{t=0} &= (u_0, \rho_0).\end{aligned}\tag{2.4.1}$$

Suppose that there exists a number $\bar{\rho} > 0$ such that $\rho_0 \geq \frac{\bar{\rho}}{2}$ on $\mathbb{R}^2$, and furthermore $u_0 \in (H^4(\mathbb{R}^2))^2$, $\rho_0 - \bar{\rho} \in H^4(\mathbb{R}^2)$. Then, there exists a time $T > 0$, which depends on $\|u_0\|_{H^4(\mathbb{R}^2)}$, $\|\rho_0 - \bar{\rho}\|_{H^4(\mathbb{R}^2)}$ and $\bar{\rho}$, such that a classical solution to the system (2.4.1) exists on $[0, T) \times \mathbb{R}^2$. Furthermore, this solution is such that

$$(\rho - \bar{\rho}, \partial_t \rho) \in (L^\infty(0, T; H^4(\mathbb{R}^2)), L^\infty(0, T; H^3(\mathbb{R}^2))),$$

and $(u, \partial_t u) \in ((L^\infty(0, T; H^4(\mathbb{R}^2)))^2, (L^\infty(0, T; H^2(\mathbb{R}^2)))^2)$.

Finally, if T^* is the maximal time of existence, we must have

$$\int_0^{T^*} \|\nabla u(t, \cdot)\|_{L^\infty(\mathbb{R}^2)} dt = \infty.$$

Remark 2.4.2. Note that the same local existence statement holds if the initial value problem is posed on an annulus $\mathcal{A}$ in $\mathbb{R}^2$ with boundary conditions: $u(t, x) = 0$ for all $t \geq 0$ and all $x \in \partial\mathcal{A}$.

Remark 2.4.3. The regularity can be reduced to $H^{2+\delta}$, with $\delta > 0$.

2.4.1 Conditional global existence for radially symmetric solutions on $\mathbb{R}^2$

In this section we are going to review a result contained in [51], and we are going to prove that, in the case of $2D$ compressible NS with non-degenerate viscosity in radial symmetry on the whole plane $\mathbb{R}^2$, the behavior of the solution is controlled a priori by certain quantities in a neighborhood of the origin. In particular, it follows from the proof that, if we pose the same problem on a circular annulus centered at the origin, and we impose Dirichlet boundary conditions of the type $u(t, x) = 0$ for all $t \geq 0$ and all $x \in \partial\mathcal{A}$, the problem admits a global solution (see Remark 2.4.6). Moreover, it follows as a byproduct of the proof (in the case of the annulus) that the density ρ will be bounded from below by a multiple of e^{-t} , and above by a multiple of e^t . We employ the strategy contained in [51], and the continuation criterion above (Proposition 2.3.2).

First of all, let us introduce some notation.

Definition 2.4.4. Consider a pair of functions $(u(t, x), \rho(t, x))$ such that $u(t, x) : [0, T] \times \mathbb{R}^2 \rightarrow \mathbb{R}^2$ and $\rho(t, x) : [0, T] \times \mathbb{R}^2 \rightarrow \mathbb{R}$. Letting $R \in \text{SO}(2)$, a rotation in the plane which fixes the origin, we define the push-forward of $(u(t, x), \rho(t, x))$ via R by

$$\tilde{u} := Rv(t, R^{-1}x), \quad \tilde{\rho} := \rho(t, R^{-1}x). \quad (2.4.2)$$

We say that the pair $(u(t, x), \rho(t, x))$ is *axisymmetric* if, for all $R \in \text{SO}(2)$, we have

$$(\tilde{u}, \tilde{\rho}) = (u, \rho).$$

We say that the pair $(u(t, x), \rho(t, x))$ is *radially symmetric* if there exist functions $v(t, r)$ and $\zeta(t, r)$ such that $v(t, r) : [0, T] \times [0, \infty) \rightarrow \mathbb{R}$, $\zeta(t, r) : [0, T] \times [0, \infty) \rightarrow \mathbb{R}$ and the following holds:

$$(u, \rho) = \left(\frac{x}{|x|} v(t, |x|), \zeta(t, |x|) \right).$$

The definitions are analogous when applied to initial data.

Definition 2.4.5. We define

$$H_w^k(\mathbb{R}_{\geq 0}, \tilde{\mu}) \quad (2.4.3)$$

to be the weighted Sobolev space H^k on the nonnegative real numbers $\mathbb{R}_{\geq 0}$ endowed with the measure $\tilde{\mu}$.

We are now ready to state the main proposition of this section.

Proposition 2.4.1. *Consider the viscous and compressible NS system*

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nabla \rho - \rho^{-1} \Delta u = 0, \\ \partial_t \rho + \operatorname{div} (u \rho) = 0, \\ (u, \rho)|_{t=0} = (u_0, \rho_0). \end{cases} \quad (2.4.4)$$

Suppose that there exists $\bar{\rho} > 0$ such that $\rho_0 \geq \frac{\bar{\rho}}{2}$ on $\mathbb{R}^2$, and furthermore $u_0 \in (H^4(\mathbb{R}^2))^2$, $\rho_0 - \bar{\rho} \in H^4(\mathbb{R}^2)$. Suppose also the initial data (u_0, ρ_0) to be radially symmetric. Then, there exists a local radially symmetric classical solution (u, ρ) to the system (2.4.4) up to time $T > 0$, satisfying the regularity properties

$$(\rho - \bar{\rho}, \partial_t \rho) \in (L^\infty(0, T; H^4(\mathbb{R}^2)), L^\infty(0, T; H^3(\mathbb{R}^2))),$$

and $(u, \partial_t u) \in ((L^\infty(0, T; H^4(\mathbb{R}^2)))^2, (L^\infty(0, T; H^2(\mathbb{R}^2))))^2$.

In addition, if there exist a number $R_1 > 0$ and a positive constant C , such that we have

$$\begin{aligned} \sup_{t \in [0, T]} \sup_{|x| \leq R_1} \max\{\rho, \rho^{-1}\} &\leq C, \\ \sup_{t_1 \in [0, T]} \int_{|x| \leq R_1} \frac{|u(t_1, x)|}{|x|} dx + \int_0^T \int_{|x| \leq R_1} \frac{|u(t, x)|^2}{|x|^2} dx dt &\leq C, \end{aligned} \quad (2.4.5)$$

the aforementioned radially symmetric classical solution to the system (2.4.4) can be extended beyond time T .

Remark 2.4.6. Note that the proof described can be specialized to the case of an annulus $\mathcal{A}$ with Dirichlet boundary conditions to provide a global existence statement. More precisely, the following statement holds. Consider two numbers R_1 and R_2 such that $0 < R_1 < R_2$, and consider the annulus $\mathcal{A} := \{x \in \mathbb{R}^2 : R_1 \leq |x| \leq R_2\}$. Then, let us pose the Cauchy problem (2.4.4) on the annulus $\mathcal{A}$ complemented with the Dirichlet boundary conditions $u(t, x) = 0$ for all $t \geq 0$ and $x \in \partial\mathcal{A}$. Then, if we assume that there exists $\bar{\rho} > 0$ such that $\rho_0 \geq \frac{\bar{\rho}}{2}$ on $\mathcal{A}$, and furthermore $u_0 \in (H^4(\mathcal{A}))^2$, $\rho_0 - \bar{\rho} \in H^4(\mathcal{A})$, and that (u_0, ρ_0) is radially symmetric, then (u_0, ρ_0) launches a global classical solution to the Cauchy–Dirichlet problem. Moreover, there are positive constants C_1 and C_2 such that the following bounds hold true for all $t \geq 0$ and all $x \in \mathcal{A}$:

$$C_1 e^{-t} \leq \rho(t, x) \leq C_2 e^t.$$

We are now going to prove Proposition 2.4.1. We will first state and prove a lemma on some elementary properties of radially symmetric local solutions of the system 2.4.7. We will then reformulate the problem in Lagrangian framework. After that, in Lemma 2.4.9, we will show the classical a priori bounds (energy bounds) for solutions to the system 2.4.4 in radial symmetry. After having shown another technical Lemma 2.4.10 on integrals of the inverse density, we will conclude the proof of proposition 2.4.1.

Lemma 2.4.7. *Assume the conditions of above Proposition 2.4.1, and without loss of generality assume $\bar{\rho} = 1$. Then, the local solution provided by Theorem 2.4.1 is itself radially symmetric. More precisely, there exist functions (v, ζ) such that, setting $r := |x|$, we have*

$$\rho(t, x) = \zeta(t, r), \quad u_i(t, x) = \frac{x_i}{r} v(t, r), \quad i = 1, 2, \quad (2.4.6)$$

for all $(t, x) \in [0, T) \times \mathbb{R}^2$. Furthermore, ζ and v satisfy the following properties:

$$\begin{aligned}
(\zeta - \bar{\zeta}, \partial_t \zeta) &\in (L^\infty(0, T; H_w^4(\mathbb{R}_{\geq 0}, r dr)), L^\infty(0, T; H_w^3(\mathbb{R}_{\geq 0}, r dr))), \\
(u, \partial_t u) &\in ((L^\infty(0, T; H_w^4(\mathbb{R}_{\geq 0}, r dr)))^2, (L^\infty(0, T; H_w^2(\mathbb{R}_{\geq 0}, r dr)))^2), \\
\zeta, v &\in \mathcal{C}([0, T) \times \mathbb{R}_{\geq 0}), \\
\zeta(t, \cdot), v(t, \cdot) &\in \mathcal{C}^2(\mathbb{R}_{\geq 0}) \text{ for all } t \in [0, T).
\end{aligned} \tag{2.4.7}$$

Finally, these functions satisfy the boundary conditions

$$\partial_r \zeta(t, 0) = 0, \quad v(t, 0), \partial_r v(t, 0) = 0. \tag{2.4.8}$$

Proof. First, we can prove that our local solution is axially symmetric. In fact, the solution considered is classical. Furthermore, system (2.4.4) is invariant under the push-forward operation (2.4.2), and finally classical solutions to (2.4.4) are unique. This proves that the local solution is axially symmetric. To prove that the considered solution is moreover radially symmetric, consider the equation for $\text{curl } u$. We multiply such Equation by $\text{curl } u$ and integrate by parts. We use spherical symmetry to cancel terms on the RHS, and finally conclude by Grönwall's inequality. Having $\text{curl } u = 0$ identically now implies radial symmetry.

The rest of the Lemma follows from the regularity properties of u and ρ given by Theorem 2.4.1. Note in particular that, due to the Sobolev embedding, at a fixed time t , $\rho \in \mathcal{C}^2(\mathbb{R}^2)$, $u \in (\mathcal{C}^2(\mathbb{R}^2))^2$. $\square$

It then follows from a straightforward calculation that (ζ, v) satisfy the following Equations, written in (t, r) -coordinates:

$$\begin{aligned}
\partial_t \zeta + \partial_r(\zeta v) + r^{-1} \zeta v &= 0, \\
\zeta(\partial_t v + v \partial_r v) &= (\partial_r^2 v + r^{-1} \partial_r v - r^{-2} v) - \frac{1}{2} \partial_r(\zeta^2), \\
(\zeta, v)|_{t=0} &= (\zeta_0, v_0).
\end{aligned} \tag{2.4.9}$$

It is now convenient to pass to Lagrangian coordinates. If (ξ, t) are Lagrangian coordinates, we have the following expression for the Eulerian coordinate r as a function of ξ :

$$\partial_t r(t, \xi) = v(t, r(t, \xi)). \quad (2.4.10)$$

Furthermore, we set the initial condition for $r(t, \xi)$ to be

$$r(0, \xi) = \eta^{-1}(\xi), \quad \eta(r) = \int_0^r s \zeta_0(s) ds. \quad (2.4.11)$$

Using mass conservation and the fact that v vanishes at the origin, we obtain

$$\int_0^{r(t, \xi)} z \zeta(t, z) dz = \xi. \quad (2.4.12)$$

Hence,

$$\frac{\partial r(t, \xi)}{\partial \xi} = \frac{1}{r(t, \xi) \zeta(t, r(t, \xi))}. \quad (2.4.13)$$

Remark 2.4.8. One needs to be careful that, as $\xi \rightarrow 0$, also $r \rightarrow 0$ uniformly on compact subsets of $[0, T)$. Nevertheless, r is not a $\mathcal{C}^1$ function of ξ , as the previous display shows (given that ζ is a continuous function on compact subsets of $[0, T)$, due to the local existence theorem).

All in all, Equations (2.4.9) become, setting $w := \frac{1}{\zeta}$,

$$\begin{aligned} \partial_t w &= \partial_\xi(rv), \\ \partial_t v &= r \partial_\xi \left(\frac{\partial_\xi(rv)}{w} - \frac{1}{2} w^{-2} \right) \end{aligned} \quad (2.4.14)$$

We now prove the standard conservation law in Lagrangian coordinates.

Lemma 2.4.9. *Assume the hypotheses of Proposition 2.4.1, and without loss of generality*

let $\bar{\rho} = 1$. If we have the L^1 -bound:

$$\int_{\mathbb{R}^2} |\rho_0 - 1| dx \leq C, \quad (2.4.15)$$

then we have the following inequality:

$$\int_0^\infty \frac{1}{2} (v^2 + w + w^{-1} - 2) (T, \xi) d\xi + \int_0^T \int_0^\infty w^{-1} (\partial_\xi(rv))^2 dt d\xi \leq C. \quad (2.4.16)$$

Proof. We have:

$$\partial_t \left(\frac{v^2}{2} + \frac{w}{2} + \frac{w^{-1}}{2} - 1 \right) + w^{-1} (\partial_\xi(rv))^2 = \partial_\xi \left(rv \left(\frac{\partial_\xi(rv)}{w} - \frac{1}{2} w^{-2} \right) + \frac{1}{2} rv \right).$$

Indeed,

$$\begin{aligned} \partial_t \left(\frac{v^2}{2} + \frac{w}{2} + \frac{w^{-1}}{2} - 1 \right) &= rv \partial_\xi \left(\frac{\partial_\xi(rv)}{w} - \frac{1}{2} w^{-2} \right) + \frac{1}{2} \left(\frac{w^2 - 1}{w^2} \right) \partial_t w \\ &= \partial_\xi \left(rv \left(\frac{\partial_\xi(rv)}{w} - \frac{1}{2} w^{-2} \right) \right) - w^{-1} (\partial_\xi(rv))^2 + \frac{1}{2} w^{-2} \partial_\xi(rv) + \frac{1}{2} \left(\frac{w^2 - 1}{w^2} \right) \partial_t w \\ &= -w^{-1} (\partial_\xi(rv))^2 + \partial_\xi \left(rv \left(\frac{\partial_\xi(rv)}{w} - \frac{1}{2} w^{-2} \right) + \frac{1}{2} rv \right). \end{aligned}$$

Integration on $[0, T] \times [0, \infty)$ reveals:

$$\begin{aligned} \int_0^\infty \frac{1}{2} (v^2 + w + w^{-1} - 2) (T, \xi) d\xi + \int_0^T \int_0^\infty w^{-1} (\partial_\xi(rv))^2 dt d\xi \\ = \int_0^\infty \frac{1}{2} (v^2 + w + w^{-1} - 2) (0, \xi) d\xi \leq C. \end{aligned} \quad (2.4.17)$$

This is the claim. □

Lemma 2.4.10. *There exists $\alpha > 0$ such that, for all $t \geq 0$ and $i \in \mathbb{N}_{\geq 0}$, we have*

$$\alpha^{-1} \leq \int_i^{i+1} w(t, \xi) d\xi, \quad \int_i^{i+1} w^{-1}(t, \xi) d\xi \leq \alpha \quad (2.4.18)$$

Also, there exists an upper-semicontinuous (in time) and integrable function $l_i(t) : [0, \infty) \rightarrow [i, i+1]$ such that

$$\alpha^{-1} \leq w(t, l_i(t)) \leq \alpha. \quad (2.4.19)$$

Proof. By the previous lemma, we have a uniform bound on the integral $\int_0^\infty w^{-1}(w-1)^2 d\xi$, which, by the Cauchy–Schwarz inequality, implies that there exists $\alpha > 0$, independent of i , such that the following inequalities hold true:

$$\int_i^{i+1} w d\xi \leq \alpha, \quad \int_i^{i+1} w^{-1} d\xi \leq \alpha.$$

Furthermore, Jensen's inequality and the convexity of the function $x \mapsto \frac{1}{x}$ yield:

$$\left(\int_i^{i+1} w^{-1} d\xi \right)^{-1} \leq \alpha, \quad \left(\int_i^{i+1} w d\xi \right)^{-1} \leq \alpha.$$

The final part of the claim follows by setting

$$l_i(t) := \sup_{[i, i+1]} (w_{(t)}^{-1}(\{\hat{w}(t)\})), \quad (2.4.20)$$

where we defined $w_{(t)}(\xi) := w(t, \xi)$, and $\hat{w}(t) := \int_i^{i+1} w d\xi$, and $w_{(t)}^{-1}$ is the inverse of $w_{(t)}$.

By uniform continuity of $w(t, \xi)$ on compact sets, it is easy to see that $l_i(t)$ is well defined and upper semicontinuous as a function of t . $\square$

Proof of Proposition 2.4.1. First, we notice that

$$r^{-1} \partial_t v + \frac{1}{2} \partial_\xi (w^{-2}) = \partial_\xi \left(\frac{\partial_\xi (rv)}{w} \right) = \partial_t \partial_\xi (\log w).$$

Integrate the previous identity over $[0, t] \times [l_i(t), \xi]$ to obtain

$$\frac{1}{2} \int_0^t w^{-2}(s, \xi) ds - \log w(t, \xi) = \log \left(\frac{w_0(l_i(t))}{w_0(\xi)} \right) - \log w(t, l_i(t))$$

$$+ \frac{1}{2} \int_0^t w^{-2}(s, l_i(t)) ds - \int_{l_i(t)}^\xi \int_0^t r^{-1} \partial_s v(s, y) dy ds.$$

Exponentiating the previous relation yields:

$$\frac{1}{w(t, \xi)} \exp \left\{ \frac{1}{2} \int_0^t w^{-2}(s, \xi) ds \right\} = \frac{1}{w(t, l_i(t))} Y_i(t) B_i(t, \xi). \quad (2.4.21)$$

Here, we have

$$\begin{aligned} Y_i(t) &:= \exp \left\{ \frac{1}{2} \int_0^t w^{-2}(s, l_i(t)) ds \right\} \geq 1, \\ B_i(t, \xi) &:= \frac{w_0(l_i(t))}{w_0(\xi)} \exp \left\{ - \int_{l_i(t)}^\xi \int_0^t r^{-1} \partial_s v(s, y) dy ds \right\}. \end{aligned} \quad (2.4.22)$$

We now multiply equation (2.4.21) by $\frac{1}{2} w^{-1}(t, \xi)$, and obtain

$$\partial_t \exp \left\{ \frac{1}{2} \int_0^t w^{-2}(s, \xi) ds \right\} = \frac{1}{2w(t, \xi)w(t, l_i(t))} Y_i(t) B_i(t, \xi). \quad (2.4.23)$$

Hence,

$$\exp \left\{ \frac{1}{2} \int_0^t w^{-2}(s, \xi) ds \right\} = 1 + \frac{1}{2} \int_0^t \frac{1}{w(s, \xi)w(s, l_i(s))} Y_i(s) B_i(s, \xi) ds. \quad (2.4.24)$$

We then have, from (2.4.21) and (2.4.24)

$$\begin{aligned} w(t, \xi) Y_i(t) B_i(t, \xi) &= w(t, l_i(t)) \left(1 + \frac{1}{2} \int_0^t \frac{Y_i(s) B_i(s, \xi)}{w(s, \xi)w(s, l_i(s))} ds \right) \\ &\leq C \left(1 + \int_0^t \frac{Y_i(s) B_i(s, \xi)}{w(s, \xi)} ds \right). \end{aligned} \quad (2.4.25)$$

We now proceed to estimate the integral in $B_i(t, \xi)$. For fixed t , let $\xi_{(t)}(\cdot) := \xi(t, \cdot)$. Let Ξ_t be the set $\Xi_t := \xi_{(t)}^{-1}([0, R_1])$.

$$\left| \int_{l_i(t)}^\xi \int_0^t r^{-1} \partial_s v(s, y) dy ds \right|$$

$$\begin{aligned}
&\leq \left| \int_{l_i(t)}^{\xi} \int_0^t \partial_s(r^{-1}v(s, y)) dy ds \right| + \left| \int_{l_i(t)}^{\xi} \int_0^t r^{-2}v^2(s, y) dy ds \right| \\
&\leq \left| \int_{l_i(t)}^{\xi} r_0^{-1}v_0(y) dy \right| + \left| \int_{l_i(t)}^{\xi} r^{-1}v(t, y) dy \right| + \left| \int_{l_i(t)}^{\xi} \int_0^t r^{-2}v^2(s, y) dy ds \right| \\
&\leq \int_{[i, i+1]} r_0^{-1}|v_0|(y) dy + \int_{[i, i+1]} r^{-1}|v|(t, y) dy + \int_{[i, i+1]} \int_0^t r^{-2}v^2(s, y) dy ds \\
&\leq \int_{[i, i+1] \cap \Xi_0} r_0^{-1}|v_0|(y) dy + \int_{[i, i+1] \cap \Xi_t} r^{-1}|v|(t, y) dy \\
&\quad + \int_0^t \int_{[i, i+1] \cap \Xi_s} r^{-2}v^2(s, y) dy ds \\
&\quad + C \left(\int_{[i, i+1] \setminus \Xi_0} |v_0|^2(y) dy + \int_{[i, i+1] \setminus \Xi_t} |v|^2(t, y) dy + \int_0^t \int_{[i, i+1] \setminus \Xi_s} v^2(s, y) dy ds \right) \\
&\leq C.
\end{aligned} \tag{2.4.26}$$

Here, we used the bounds in 2.4.9 plus the fact that our assumptions imply

$$\int_{\Xi_0} |v_0|(r) dr + \int_{\Xi_t} |v|(t, r) dr + \int_0^t \int_{\Xi_s} r^{-1}v^2(s, r) dr ds \leq C. \tag{2.4.27}$$

From (2.4.26) it is now easy to deduce that, for some $C > 0$,

$$C^{-1} \leq B_i(t, \xi) \leq C, \text{ for } t \in [0, T), \xi \in \mathbb{R}_{\geq 0}. \tag{2.4.28}$$

We now see that, upon integration on $[i, i+1]$, for some $C_1 > 0$,

$$\begin{aligned}
C_1 Y_i(t) &\leq \int_i^{i+1} w(t, \xi) Y_i(t) B_i(t, \xi) d\xi \leq C \int_i^{i+1} \left(1 + \int_0^t \frac{Y_i(s) B_i(s, \xi)}{w(s, \xi)} ds \right) d\xi \\
&\leq C \left(1 + \int_0^t Y_i(s) ds \right).
\end{aligned} \tag{2.4.29}$$

We now use Grönwall's inequality to deduce that

$$Y_i(t) \leq C.$$

Remark 2.4.11. Note that the constant C here grows exponentially in time.

Here, we used the integral form of Grönwall's inequality, for which Y_i needs only to be a bounded and measurable function. $Y_i(t)$ is bounded on every closed time-subinterval of $[0, T)$ because of the regularity given by the local existence theorem. Clearly now we have

$$w(t, \xi) = \exp \left\{ \frac{1}{2} \int_0^t w^{-2}(s, \xi) ds \right\} \frac{w(t, l_i(t))}{Y_i(t) B_i(t, \xi)} \geq C.$$

On the other hand, from equation (2.4.25), we obtain

$$\begin{aligned} w(t, \xi) &\leq C \left(1 + \int_0^t \frac{Y_i(s) B_i(s, \xi)}{w(s, \xi)} ds \right) \leq C \left(1 + \int_0^t \max_{\xi \in [i, i+1]} \frac{1}{w(s, \xi)} ds \right) \\ &\leq C \left(1 + \int_0^t \frac{1}{\min_{\xi \in [i, i+1]} w(s, \xi)} ds \right) \leq C. \end{aligned} \tag{2.4.30}$$

With these bounds on w (which are however dependent on t), we can then conclude that the Lagrangian map is well defined and moreover that we can continue the solution past time T . This concludes the proof of Proposition 2.4.1. $\square$

2.4.2 Uniform bounds for the density of the 2D compressible NS in axisymmetry on a bounded annular domain

Proposition 2.4.1, which we proved in the previous section, provides exponential in time bounds for the density when restricted to the case of an annular domain with Dirichlet boundary conditions in the radial case (see Remark 2.4.6). In this section, we wish to extend the result in order to show *uniform* bounds for the density on an annular domain, in the case in which the solution is assumed to be *axisymmetric*, and it is thus allowed to have non-zero swirl. Fundamental to our approach is the Lagrangian representation formula in [51].

In what follows, we shall first derive the equations in the axisymmetric setting (equation (2.4.33)), and then we shall rephrase them in the Lagrangian framework (equation (2.4.34)). We shall then state the main theorem of this Section 2.4.12, and we will then

focus on proving a crucial representation formula in Lemma 2.4.13. We will then derive the classical energy estimates in the Lagrangian framework (Lemma 2.4.14), and we will finally prove theorem 2.4.12.

Derivation of the equations

We start by imposing the ansatz for the velocity field u :

$$u(t, |x|) = \frac{x}{|x|} v(t, |x|) + \frac{x^\perp}{|x|} \varpi(t, |x|). \quad (2.4.31)$$

We note that

$$\begin{aligned} \nabla u &= \left(\frac{I}{|x|} - \frac{x \otimes x}{|x|^3} \right) v(|x|) + \left(\frac{x \otimes x}{|x|^2} \right) v'(|x|) \\ &+ \left(\frac{R}{|x|} - \frac{x^\perp \otimes x}{|x|^3} \right) \varpi(|x|) + \left(\frac{x^\perp \otimes x}{|x|^2} \right) \varpi'(|x|) \end{aligned} \quad (2.4.32)$$

Here,

$$R = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

and $x^\perp = Rx$. Also, note that R commutes with the Laplacian.

Under these assumptions, we can derive the equations in axisymmetry, on a bounded annular domain. Here, $r \in [R_1, R_2]$, ζ, ϖ and v are all functions on $[0, T] \times [R_1, R_2]$, for some $T > 0$. ζ is the density, ϖ and v are the components of the velocity as in equation (2.4.31).

$$\begin{aligned} \partial_t \zeta + \partial_r(\zeta v) + r^{-1} \zeta v &= 0, \\ \zeta(\partial_t v + v \partial_r v) - \zeta \frac{\varpi^2}{r} &= (\partial_r^2 v + r^{-1} \partial_r v - r^{-2} v) - \frac{1}{2} \partial_r(\zeta^2), \\ \zeta(\partial_t \varpi + v \partial_r \varpi) + \zeta \frac{v \varpi}{r} &= (\partial_r^2 \varpi + r^{-1} \partial_r \varpi - r^{-2} \varpi), \\ v(t, R_1) = v(t, R_2) = \varpi(t, R_1) &= \varpi(t, R_2) = 0, \\ (\zeta, v, \varpi)|_{t=0} &= (\tilde{\zeta}_0, \tilde{v}_0, \tilde{\varpi}_0). \end{aligned} \quad (2.4.33)$$

We let

$$a := \int_{R_1}^{R_2} z \tilde{\zeta}_0(z) dz.$$

We then pass to Lagrangian coordinates (t, ξ) , which are defined by the following relation, in which ξ ranges from 0 to a :

$$\partial_t r(t, \xi) = v(t, r(t, \xi)), \quad r(0, \xi) = r_0(\xi),$$

and here $r_0(\xi)$ is defined implicitly by the equation

$$\int_{R_1}^{r_0(\xi)} z \tilde{\zeta}_0(z) dz = \xi.$$

We now formulate the system in Lagrangian variables. In what follows, we denote the Lagrangian quantities with the same names as their Eulerian counterparts.

The functions r, v, ϖ, ζ are now defined on $[0, T] \times [0, a]$, and we obtain the following system of PDEs, letting $w := \frac{1}{\zeta(t, r(t, \xi))}$ (the inverse density in Lagrangian coordinates):

$$\begin{aligned} \partial_t w &= \partial_\xi(rv), \\ \partial_t v - r^{-1} \varpi^2 &= r \partial_\xi \left(\frac{\partial_\xi(rv)}{w} - \frac{1}{2} w^{-2} \right), \\ \partial_t \varpi + r^{-1} v \varpi &= r \partial_\xi \left(\frac{\partial_\xi(r\varpi)}{w} \right), \\ v(t, 0) &= v(t, a) = \varpi(t, 0) = \varpi(t, a) = 0, \\ (r, \zeta, v, \varpi)|_{t=0} &= (r_0, \zeta_0, v_0, \varpi_0). \end{aligned} \tag{2.4.34}$$

We now state the main theorem of this section, which provides a priori uniform bounds for the density in this setting. Let us note that the bounds provided are uniform in time, unlike those appearing in the proof of proposition 2.4.1.

Theorem 2.4.12. *Let us suppose that $(v, \varpi, \zeta) : [0, \infty) \times [R_1, R_2]$ is a global classical solution*

to the following system of PDEs:

$$\begin{aligned}
& \partial_t \zeta + \partial_r(\zeta v) + r^{-1} \zeta v = 0, \\
& \zeta(\partial_t v + v \partial_r v) - \zeta \frac{\varpi^2}{r} = (\partial_r^2 v + r^{-1} \partial_r v - r^{-2} v) - \frac{1}{2} \partial_r(\zeta^2), \\
& \zeta(\partial_t \varpi + v \partial_r \varpi) + \zeta \frac{v \varpi}{r} = (\partial_r^2 \varpi + r^{-1} \partial_r \varpi - r^{-2} \varpi), \\
& v(t, R_1) = v(t, R_2) = \varpi(t, R_1) = \varpi(t, R_2) = 0, \\
& (\zeta, v, \varpi)|_{t=0} = (\tilde{\zeta}_0, \tilde{v}_0, \tilde{\varpi}_0).
\end{aligned} \tag{2.4.35}$$

Let us furthermore define:

$$a := \int_{R_1}^{R_2} \tilde{\zeta}_0(r) r dr, \quad E_0^2 := \int_{R_1}^{R_2} \left(\tilde{v}_0^2 + \tilde{\varpi}_0 + (\tilde{\zeta}_0 - 1)^2 \right) r dr. \tag{2.4.36}$$

Then, there exist constants $c_1, c_2 > 0$ such that ζ satisfies the following bound:

$$c_1 \leq \zeta(t, r) \leq c_2, \tag{2.4.37}$$

for all $(t, r) \in [0, \infty) \times [R_1, R_2]$. Furthermore, the constants c_1 and c_2 depend only on the following quantities:

$$c_i = c_i \left(E_0, \quad a, \quad \left\| \tilde{\zeta}_0 \right\|_{L^\infty}, \quad \left\| \tilde{\zeta}_0^{-1} \right\|_{L^\infty}, \quad R_1, \quad R_2, \quad \int_{R_1}^{R_2} |\tilde{v}_0| \tilde{\zeta}_0 dr \right), \text{ for } i = 1, 2. \tag{2.4.38}$$

A representation formula

Let us now derive a representation formula for solutions to the system (2.4.34):

Lemma 2.4.13. *Suppose that system (2.4.34) admits a classical solution. Then, the following representation formula holds true:*

$$w(t, \xi) = \frac{D(t, \xi)}{B(t, \xi)} \left(1 + \frac{1}{2} \int_0^t \frac{B(s, \xi)}{w(s, \xi) D(s, \xi)} ds \right). \tag{2.4.39}$$

Here, the terms are defined as:

$$B(t, \xi) := \exp \left(\frac{1}{2w^*} R_1^2 \int_0^t \int_0^a r^{-2} (v^2 - \varpi^2)(s, y) dy ds \right. \quad (2.4.40)$$

$$\left. + \frac{1}{2w^*} \int_0^t \int_0^a (v^2 + \varpi^2 + w^{-1})(s, y) dy ds + \int_0^t \int_\xi^a r^{-2} (v^2 - \varpi^2)(s, y) dy ds \right),$$

$$D(t, \xi) := w_0(\xi) \exp \left(\frac{1}{w^*} \int_0^a (w_0 \phi_0)(y) dy - \int_0^\xi (r_0^{-1} v_0)(y) dy + \int_{l(t)}^\xi r^{-1} v(t, y) dy \right), \quad (2.4.41)$$

$$\phi_0(\xi) := \int_0^\xi r_0^{-1} v_0(y) dy. \quad (2.4.42)$$

Here, we defined $w^* := \int_0^a w_0(\xi) d\xi$.

Proof. Define σ and ϕ to be

$$\sigma(t, \xi) := \frac{(rv)_\xi}{w} - \frac{1}{2} w^{-2},$$

$$\phi(t, \xi) := \int_0^t \sigma(s, \xi) ds + \int_0^\xi (r_0^{-1} v_0)(y) dy + \int_0^t \int_\xi^a (r^{-2} v^2)(s, y) dy ds \quad (2.4.43)$$

$$- \int_0^t \int_\xi^a (r^{-2} \varpi^2)(s, y) dy ds$$

Then, ϕ obeys the following relation

$$\partial_t(w\phi) - \partial_\xi(rv\phi) = (rv)_\xi - \frac{1}{2} w^{-1} + \frac{1}{2} \left(r^2 \int_\xi^a r^{-2} (v^2 - \varpi^2)(t, y) dy \right)_\xi - \frac{v^2 + \varpi^2}{2}. \quad (2.4.44)$$

Integration of the previous display on the set $[0, t] \times [0, a]$ yields (ϕ_0 is ϕ calculated at time 0)

$$\int_0^a (w\phi)(t, y) dy = \int_0^a (w_0 \phi_0)(y) dy$$

$$- \frac{1}{2} R_1^2 \int_0^t \int_0^a r^{-2} (v^2 - \varpi^2)(s, y) dy ds - \frac{1}{2} \int_0^t \int_0^a (v^2 + \varpi^2 + w^{-1})(s, y) dy ds \quad (2.4.45)$$

We now let w^* be the L^1 norm of w :

$$w^* := \int_0^a w_0(\xi) d\xi = \int_0^a w(s, \xi) d\xi.$$

Due to continuity of ϕ , we can find an upper-semicontinuous function $l(t)$ such that the following holds, for all positive s :

$$\phi(s, l(t)) = \frac{1}{w^*} \int_0^a \phi(s, y) w(s, y) dy. \quad (2.4.46)$$

Now, from (2.4.43) and (2.4.45) we obtain:

$$\begin{aligned} \int_0^t \sigma(s, l(t)) ds &= \phi(t, l(t)) - \int_0^{l(t)} (r_0^{-1} v_0)(y) dy - \int_0^t \int_{l(t)}^a r^{-2} (v^2 - \varpi^2)(s, y) dy ds \\ &= \frac{1}{w^*} \int_0^a (w\phi)(t, y) dy - \int_0^{l(t)} (r_0^{-1} v_0)(y) dy - \int_0^t \int_{l(t)}^a r^{-2} (v^2 - \varpi^2)(s, y) dy ds \\ &= \frac{1}{w^*} \int_0^a (w_0 \phi_0)(y) dy - \frac{1}{2w^*} R_1^2 \int_0^t \int_0^a r^{-2} (v^2 - \varpi^2)(s, y) dy ds \\ &\quad - \frac{1}{2w^*} \int_0^t \int_0^a (v^2 + \varpi^2 + w^{-1})(s, y) dy ds \\ &\quad - \int_0^{l(t)} (r_0^{-1} v_0)(y) dy - \int_0^t \int_{l(t)}^a r^{-2} (v^2 - \varpi^2)(s, y) dy ds. \end{aligned} \quad (2.4.47)$$

We now recall the equation:

$$r^{-1} \partial_t v - \frac{\varpi^2}{r^2} = \partial_\xi \sigma = \partial_\xi \left(\partial_t \log w - \frac{1}{2} w^{-2} \right). \quad (2.4.48)$$

Integrating the previous display on $[l(t), \xi] \times [0, t]$ yields

$$\begin{aligned} &\log w(t, \xi) - \log w(0, \xi) - \frac{1}{2} \int_0^t w^{-2}(s, \xi) ds \\ &= \int_0^t \sigma(s, l(t)) ds + \int_{l(t)}^\xi \int_0^t r^{-1} \partial_t v(s, y) dy ds \\ &\quad - \int_{l(t)}^\xi \int_0^t r^{-2} \varpi^2(s, y) dy ds \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{w^*} \int_0^a (w_0 \phi_0)(y) dy - \frac{1}{2w^*} R_1^2 \int_0^t \int_0^a r^{-2} (v^2 - \varpi^2)(s, y) dy ds \\
&\quad - \frac{1}{2w^*} \int_0^t \int_0^a (v^2 + \varpi^2 + w^{-1})(s, y) dy ds \\
&\quad - \int_0^{l(t)} (r_0^{-1} v_0)(y) dy - \int_0^t \int_{l(t)}^a r^{-2} (v^2 - \varpi^2)(s, y) dy ds \\
&\quad + \int_{l(t)}^\xi r^{-1} v(t, y) dy - \int_{l(t)}^\xi (r_0^{-1} v_0)(y) dy \\
&\quad + \int_{l(t)}^\xi \int_0^t r^{-2} v^2(s, y) dy ds - \int_{l(t)}^\xi \int_0^t r^{-2} \varpi^2(s, y) dy ds \\
&= \frac{1}{w^*} \int_0^a (w_0 \phi_0)(y) dy - \frac{1}{2w^*} R_1^2 \int_0^t \int_0^a r^{-2} (v^2 - \varpi^2)(s, y) dy ds \\
&\quad - \frac{1}{2w^*} \int_0^t \int_0^a (v^2 + \varpi^2 + w^{-1})(s, y) dy ds \\
&\quad - \int_0^\xi (r_0^{-1} v_0)(y) dy - \int_0^t \int_\xi^a r^{-2} (v^2 - \varpi^2)(s, y) dy ds + \int_{l(t)}^\xi r^{-1} v(t, y) dy.
\end{aligned}$$

We now exponentiate the previous display to obtain (we include all the double integrals in the term B and all the single integrals in the term D):

$$\begin{aligned}
&\frac{1}{w(t, \xi)} \exp \left(\frac{1}{2} \int_0^t w^{-2}(s, \xi) ds \right) = \frac{B(t, \xi)}{D(t, \xi)}, \\
&B(t, \xi) := \exp \left(\frac{1}{2w^*} R_1^2 \int_0^t \int_0^a r^{-2} (v^2 - \varpi^2)(s, y) dy ds \right. \\
&\quad \left. + \frac{1}{2w^*} \int_0^t \int_0^a (v^2 + \varpi^2 + w^{-1})(s, y) dy ds + \int_0^t \int_\xi^a r^{-2} (v^2 - \varpi^2)(s, y) dy ds \right), \\
&D(t, \xi) := w_0(\xi) \exp \left(\frac{1}{w^*} \int_0^a (w_0 \phi_0)(y) dy - \int_0^\xi (r_0^{-1} v_0)(y) dy + \int_{l(t)}^\xi r^{-1} v(t, y) dy \right).
\end{aligned} \tag{2.4.49}$$

We now integrate the previous display to obtain

$$\exp \left(\frac{1}{2} \int_0^t w^{-2}(s, \xi) ds \right) = 1 + \frac{1}{2} \int_0^t \frac{B(s, \xi)}{w(s, \xi) D(s, \xi)} ds. \tag{2.4.50}$$

Substituting again, we finally obtain:

$$w(t, \xi) = \frac{D(t, \xi)}{B(t, \xi)} \left(1 + \frac{1}{2} \int_0^t \frac{B(s, \xi)}{w(s, \xi) D(s, \xi)} ds \right). \quad (2.4.51)$$

This proves the representation formula. $\square$

The standard conservation law

We now turn to the proof of the standard conservation law.

Lemma 2.4.14. *If v, ϖ, w satisfy system (2.4.34), then we have the following relations:*

$$\begin{aligned} & \frac{1}{2} \partial_t (v^2 + \varpi^2 + w^{-1} + w - 2) + \frac{1}{w} ((\partial_\xi(rv))^2 + (\partial_\xi(r\varpi))^2) \\ &= \partial_\xi \left(\frac{vr}{w} \partial_\xi(vr) + \frac{\varpi r}{w} \partial_\xi(\varpi r) - \frac{rvw^{-2}}{2} + \frac{rv}{2} \right), \\ & \int_0^a (v^2 + \varpi^2 + w^{-1} + w - 2)(t, y) dy + \int_0^t \int_0^a \frac{1}{w} ((\partial_\xi(rv))^2 + (\partial_\xi(r\varpi))^2)(s, y) ds dy \\ & \leq \int_0^a (v^2 + \varpi^2 + w^{-1} + w - 2)(0, y) dy =: E_0^2. \end{aligned} \quad (2.4.52)$$

Proof of Lemma 2.4.14. Direct computation. $\square$

Proof of the uniform bounds

We shall now use the representation formula we just obtained in order to prove Theorem 2.4.12.

Proof of Theorem 2.4.12. We consider the Lagrangian formulation (2.4.34), being careful that the initial data without tildes are the data in the Lagrangian setting, whereas the data with tildes are the data in the Eulerian setting.

First, we notice that due to the conservation law and Jensen's inequality, we can deduce

that there exists a constant $\alpha > 0$ such that the following holds for all $i \in \mathbb{N}_{\geq 0}$, $i \leq \lfloor a \rfloor - 1$:

$$\alpha^{-1} \leq \int_i^{i+1} w(t, y) dy \leq \alpha, \quad \alpha^{-1} \leq \int_i^{i+1} w^{-1}(t, y) dy \leq \alpha. \quad (2.4.53)$$

This implies the following bound from below for r , if $\xi \geq 1$:

$$r^2(t, \xi) \geq R_1^2 + 2 \int_0^\xi w(t, y) dy \geq R_1^2 + 2\alpha^{-1} \lfloor \xi \rfloor \geq R_1^2 + \alpha^{-1} \xi. \quad (2.4.54)$$

Hence, we obtain the bound for $D(t, \xi)$:

$$\begin{aligned} D(t, \xi) &\leq \\ &\|\phi_0\|_{L^\infty} \exp \left(\frac{1}{w^*} \int_0^a |w_0 \phi_0|(y) dy + \alpha^{\frac{1}{2}} \left(\log \left(1 + \frac{a\alpha^{-1}}{R_1^2} \right) \right)^{\frac{1}{2}} (\|v(t, \cdot)\|_{L^2} + \|v_0\|_{L^2}) \right) \\ &\leq \|w_0\|_{L^\infty} \exp \left(\|w_0\|_{L^\infty} + \alpha^{\frac{1}{2}} \left(\log \left(1 + \frac{a\alpha^{-1}}{R_1^2} \right) \right)^{\frac{1}{2}} (\|v(t, \cdot)\|_{L^2} + \|v_0\|_{L^2}) \right) =: C_1. \end{aligned} \quad (2.4.55)$$

Similarly,

$$\begin{aligned} D(t, \xi) &\geq \underbrace{\left(\inf_{[0, a]} w_0(\xi) \right) \exp \left(-\|w_0\|_{L^\infty} - \alpha^{\frac{1}{2}} \left(\log \left(1 + \frac{a\alpha^{-1}}{R_1^2} \right) \right)^{\frac{1}{2}} (\|v(t, \cdot)\|_{L^2} + \|v_0\|_{L^2}) \right)}_{=: C_2}. \end{aligned} \quad (2.4.56)$$

In view of these inequalities, we have

$$w(t, \xi) \geq \frac{1}{2} \int_0^t \frac{C_2}{C_1} \frac{B(s, \xi)}{B(t, \xi)} \frac{1}{w(s, \xi)} ds. \quad (2.4.57)$$

Let now

$$E_0^2 := \int_0^a (v^2 + \varpi^2 + w^{-1} + w - 2) (0, y) dy.$$

Now, if $t \geq s$, we have the bound

$$\frac{B(s, \xi)}{B(t, \xi)} \geq \exp \left(- \left(\frac{E_0^2 + 2a}{w_0^*} + \frac{E_0^2}{R_1^2} \right) (t - s) \right) = \exp(-C_3(t - s)). \quad (2.4.58)$$

Therefore we get:

$$w(t, \xi) \geq \frac{1}{2} \int_0^t \frac{C_2}{C_1} \frac{\exp(-C_3(t - s))}{w(s, \xi)} ds. \quad (2.4.59)$$

We proceed to integrate the previous inequality:

$$\begin{aligned} 2 \frac{C_2}{C_1} &\geq \frac{C_2}{w(t, \xi) C_1} \int_0^t \frac{C_2}{C_1} \frac{\exp(-C_3(t - s))}{w(s, \xi)} ds = \frac{1}{2} \partial_t \left(\int_0^t \frac{C_2}{C_1} \frac{\exp(-C_3(t - s))}{w(s, \xi)} ds \right)^2 \\ &+ \frac{C_3}{2} \left(\int_0^t \frac{C_2}{C_1} \frac{\exp(-C_3(t - s))}{w(s, \xi)} ds \right)^2. \end{aligned} \quad (2.4.60)$$

The quantity

$$A(t) = \frac{1}{2} \left(\int_0^t \frac{C_2}{C_1} \frac{\exp(-C_3(t - s))}{w(s, \xi)} ds \right)^2$$

then satisfies the inequality, letting $C_2/C_1 =: C_4$:

$$\partial_t A(t) + C_3 A(t) \leq 2C_4.$$

Hence,

$$\partial_t (A(t) \exp(C_3 t)) \leq 2C_4 \exp(C_3 t) \implies A(t) e^{C_3 t} - A(0) \leq 2 \frac{C_4}{C_3} (e^{C_3 t} - 1).$$

This yields: $A(t) \leq 2 \frac{C_4}{C_3} =: C_5$.

In turn, this implies, for all $t \geq 1$,

$$\begin{aligned} \exp(-C_3) \int_{t-1}^t \frac{1}{w(s, \xi)} ds &\leq \int_{t-1}^t \frac{\exp(-C_3(t - s))}{w(s, \xi)} ds \leq \int_0^t \frac{\exp(-C_3(t - s))}{w(s, \xi)} ds \\ &\leq \frac{C_1}{C_2} \sqrt{2C_5} \end{aligned} \quad (2.4.61)$$

By the mean value theorem, there is a point $t_{i, \xi} \in [i - 1, i]$, for all $i \in \mathbb{N}_{\geq 1}$ such that the

following holds:

$$w(t_{i,\xi}, \xi) \geq \frac{C_2}{C_1 \sqrt{2C_5}} \exp(-C_3) =: C_6. \quad (2.4.62)$$

Let us now consider ξ as fixed and let us denote, for simplicity, $t_i := t_{i,\xi}$. From equation (2.4.51), we obtain, if $t \in [t_i, t_{i+1}]$,

$$\begin{aligned} B(t_i, \xi) C_6 &\leq B(t_i, \xi) w(t_i, \xi) = D(t_i, \xi) \left(1 + \frac{1}{2} \int_0^{t_i} \frac{B(s, \xi)}{w(s, \xi) D(s, \xi)} ds \right) \\ &\leq C_1 + \frac{C_1}{2C_2} \int_0^t \frac{B(s, \xi)}{w(s, \xi)} ds. \end{aligned} \quad (2.4.63)$$

Now,

$$\begin{aligned} B(t, \xi) &:= \exp \left(\frac{1}{2w^*} R_1^2 \int_0^t \int_0^a r^{-2} (v^2 - \varpi^2)(s, y) dy ds \right. \\ &\quad \left. + \frac{1}{2w^*} \int_0^t \int_0^a (v^2 + \varpi^2 + w^{-1})(s, y) dy ds + \int_0^t \int_\xi^a r^{-2} (v^2 - \varpi^2)(s, y) dy ds \right) \\ &= \exp \left(\frac{1}{2w^*} R_1^2 \int_{t_i}^t \int_0^a r^{-2} (v^2 - \varpi^2)(s, y) dy ds \right. \\ &\quad \left. + \frac{1}{2w^*} \int_{t_i}^t \int_0^a (v^2 + \varpi^2 + w^{-1})(s, y) dy ds + \int_{t_i}^t \int_\xi^a r^{-2} (v^2 - \varpi^2)(s, y) dy ds \right) B(t_i, \xi) \\ &\leq C_7 B(t_i, \xi). \end{aligned} \quad (2.4.64)$$

Here, C_7 can be chosen as

$$C_7 := \frac{2E_0^2 + 2a}{w^*} + \frac{2E_0^2}{R_1^2}.$$

From the conservation law, we obtain the following inequality:

$$\int_0^t \int_0^a w^{-1} (\partial_\xi (r\varpi))^2(s, y) dy ds \leq C. \quad (2.4.65)$$

Expanding the square and taking into account that $\varpi = 0$ on the boundary, plus the fact that we always have $R_1 \leq r \leq R_2$, we obtain

$$\int_0^t \int_0^a (w\varpi^2 + w^{-1} (\partial_\xi \varpi)^2)(s, y) dy ds \leq C. \quad (2.4.66)$$

Hence,

$$\int_0^t \int_0^a \varpi^2(s, y) dy ds \leq C. \quad (2.4.67)$$

We finally have, due to (2.4.67) and (2.4.53),

$$B(t, \xi) \geq \exp \left(\frac{1}{2w^*} \int_0^t \int_0^a w^{-1}(s, y) dy ds \right) \geq \exp \left(\frac{t}{2w^*} \lfloor a \rfloor \alpha^{-1} - C \right) = C_9 \exp(C_8 t), \quad (2.4.68)$$

where $C_8 := \frac{\lfloor a \rfloor \alpha^{-1}}{2w^*}$. Hence we obtain:

$$B(t, \xi) \leq C_7 B(t_i, \xi) \leq \frac{C_1 C_7}{C_6} + \frac{C_1 C_7}{2C_2 C_6} \int_0^t \frac{B(s, \xi)}{w(s, \xi)} ds. \quad (2.4.69)$$

Therefore,

$$1 \leq \frac{C_1 C_7}{B(t, \xi) C_6} + \frac{C_1 C_7}{2C_2 C_6} \int_0^t \frac{B(s, \xi)}{B(t, \xi) w(s, \xi)} ds. \quad (2.4.70)$$

There exists now $\tau > 0$ (independent of ξ) such that, for $t \geq \tau$,

$$B(t, \xi) \geq 2 \frac{C_1 C_7}{C_6}.$$

Hence, for $t \geq \tau$,

$$w(t, \xi) \geq \frac{C_2}{2C_1} \int_0^t \frac{B(s, \xi)}{w(s, \xi) B(t, \xi)} ds \geq \frac{C_2^2 C_6}{2C_1^2}. \quad (2.4.71)$$

For $t \leq \tau$, it follows straightforwardly from the representation formula that there exists C_{10} such that $w(t, \xi) \geq C_{10}$. It is then immediate, again via the representation formula, to prove the bound from above for w . $\square$

¹This estimate could also be achieved by requiring that $(2w^*)^{-1} \geq R_1^{-2}$, which translates into $R_2 \leq \sqrt{2}R_1$.

Chapter 3

The equations of magnetohydrodynamics

In this final chapter, we are going to consider a class of models arising in plasma physics. We are going to consider the theory of *magnetohydrodynamics* (MHD), both in the ideal case and with viscous and resistive effects.

Here is the structure of this chapter. In Section 3.1, we will present a result on the construction of $3D$ ideal MHD equilibria via infinite time limits of a suitably regularized MHD model. Then in Section 3.2, we will provide two examples of situations (in $3D$ and in $2D$) in which the ideal MHD system exhibits growth of gradients.

3.1 Magnetic relaxation in a Voigt–MHD model

In this section, we are going to formulate and prove a result about convergence to stationary solutions of the $3D$ Euler equations using the Voigt–MHD system, which is a regularized version of the MHD system (we consider the case in which the resistivity is zero and there is positive viscosity).

The plan of this section is as follows. First, in Section 3.1.1, we are going to introduce some notation and some basic facts about the functional framework we will be using. We

will then introduce the system and the notion of solution in Section 3.1.2. After that, we will proceed to show that the system under consideration has global solutions for sufficiently smooth and large initial data (Section 3.1.3). Finally, in Section 3.1.4, we will state the main Theorem of this section (Theorem 3.1.9), and we will provide a proof of it.

3.1.1 Notation and preliminaries

Norms and spaces

Let $\mathbb{T}^3$ be the three-dimensional torus. We will be working with the spaces $H^s(\mathbb{T}^3)$ and $H_0^s(\mathbb{T}^3)$, which we define as follows.

Definition 3.1.1. Let $f \in \mathcal{P}'$, where $\mathcal{P}'$ is the space of periodic distributions on the 3-dimensional torus $\mathbb{T}^3$. Then, consider the Fourier coefficients $\hat{f}(k)$, where $k \in \mathbb{Z}^3$. We say that $f \in H^s(\mathbb{T}^3)$ if the Fourier coefficients of f satisfy:

$$\|f\|_{H^s(\mathbb{T}^3)} := \sum_{k \in \mathbb{Z}^3} (1 + |k|^2)^{\frac{s}{2}} |\hat{f}(k)|^2 < \infty. \quad (3.1.1)$$

We also say that $f \in H_0^s$, if the Fourier coefficients of f satisfy:

$$\hat{f}(0,0,0) = 0, \quad \|f\|_{H_0^s(\mathbb{T}^3)} := \sum_{\substack{k \in \mathbb{Z}^3 \\ k \neq (0,0,0)}} (1 + |k|^2)^{\frac{s}{2}} |\hat{f}(k)|^2 < \infty. \quad (3.1.2)$$

Remark 3.1.2. For conciseness, we will denote $\|f\|_s := \|f\|_{H_0^s}$, the *homogeneous* Sobolev norm.

Remark 3.1.3. We have that $\mathcal{P}' = \cup_{s \in \mathbb{R}} H^s(\mathbb{T}^3)$, and that the Fourier coefficients of $f \in \mathcal{P}'$ determine f uniquely.

We then recall some classical notation (see [25]):

$$H := H_0^0, \quad V := H_0^1. \quad (3.1.3)$$

Then, the dual of V , which we call V' , is naturally identified with H_0^{-1} . If $f, g \in H$ (resp. V), we let

$$(f, g) := \int_{\mathbb{T}^3} f g dx, \quad ((f, g)) := \int_{\mathbb{T}^3} \nabla f \cdot \nabla g dx, \quad (3.1.4)$$

the usual L^2 (resp. H_0^1) inner product. If $f \in V'$ and $g \in V$, we let $\langle f, g \rangle$ be the dual pairing between f and g . Note that, if both f and g are in H , then $\langle f, g \rangle = (f, g)$. We let

$$\|f\| := \|f\|_V, \quad |f| := \|f\|_H. \quad (3.1.5)$$

We finally let, for $f \in V'$, $\|f\|_{V'} := \|f\|_{H_0^{-1}}$. These scalar spaces extend in a straightforward manner to their vectorial counterparts.

Leray projection, the fractional Laplacian, and classical estimates

Let $\mathbb{P} : H \rightarrow H$ be the Leray projector. On the Fourier side, it can be expressed as

$$(\widehat{\mathbb{P}f})_k := \left(\text{Id}_3 - \frac{k \otimes k}{|k|^2} \right) (\hat{f})_k \quad \text{for } k \neq 0, \quad (\widehat{\mathbb{P}f})_0 := 0.$$

Note that it extends naturally as an operator on $\mathcal{P}'$.

If $v, w \in V$, we define

$$\mathbf{B}(v, w) := \mathbb{P}(v \cdot \nabla w). \quad (3.1.6)$$

Let us also recall the following inequality, proved in [25], which follows from Sobolev embedding. There exists a constant $C > 0$ such that, if $u, v, w \in V$,

$$\langle \mathbf{B}(u, v), w \rangle \leq C |u|^{\frac{1}{2}} \|u\|^{\frac{1}{2}} \|v\| \|w\|. \quad (3.1.7)$$

We also recall that elements u of V enjoy the following Poincaré inequality, due to the fact that they are mean zero on $\mathbb{T}^3$:

$$|u| \leq \lambda \|u\|, \quad (3.1.8)$$

where λ is the Poincaré constant.

We now define the fractional Laplacian as the operator $(-\Delta)^\alpha : H_0^s \rightarrow H_0^{s-2\alpha}$ whose action on the Fourier coefficients is as follows:

$$((-\Delta)^\alpha f)_k = |k|^{2\alpha} \hat{f}_k, \quad \text{for } k \neq 0, \quad \hat{f}_0 = 0. \quad (3.1.9)$$

We then let a parameter $a > 0$, and define

$$\mathfrak{L} := \text{Id} + a^2(-\Delta)^\alpha. \quad (3.1.10)$$

We have the following lemma:

Lemma 3.1.4. *Let $\mathfrak{L} := \text{Id} + a^2(-\Delta)^\alpha$, and assume that $\alpha \geq 1$. Then, $\mathfrak{L}$ maps H_0^s onto $H_0^{s-2\alpha}$, is injective and has a well defined inverse $\mathfrak{L}^{-1} : H_0^{s-2\alpha} \rightarrow H_0^s$. Furthermore, the composition operators $\Delta \mathfrak{L}^{-1} = \mathfrak{L}^{-1} \Delta$ are well defined, and are both bounded operators from H_0^s to $H_0^{2\alpha-2+s}$. In particular, there exists a constant $C > 0$ such that, for all $v \in H_0^s$,*

$$\|\mathfrak{L}^{-1} \Delta v\|_{2\alpha-2+s} = \|\Delta \mathfrak{L}^{-1} v\|_{2\alpha-2+s} \leq C \|v\|_s. \quad (3.1.11)$$

Proof of lemma 3.1.4. The proof is evident from the Fourier characterization of the operator $(-\Delta)^\alpha$. □

Remark 3.1.5. By the Fourier characterization of $(-\Delta)^\alpha$ and of $\mathbb{P}$, it is evident that both these operators commute with partial derivatives. Furthermore, we have that

$$[\mathbb{P}, \mathfrak{L}] = 0, \quad [\mathbb{P}, (-\Delta)^\alpha] = 0.$$

We also recall the definition of time-dependent spaces. Let $(X, \|\cdot\|_X)$ be a Banach space, and let $T > 0$. We say that a function $f : [0, T] \rightarrow X$ is such that $f \in L^\infty(0, T; X)$ if the

following holds:

$$\sup_{t \in [0, T]} \|f(t)\|_X < \infty.$$

Let $f : [0, T] \rightarrow X$ be Bochner integrable, and define f' as usual as the (time) weak derivative of f . We then say that $f \in \mathcal{C}^1(0, T; X)$ if there holds

$$\sup_{t \in [0, T]} (\|f(t)\|_X + \|f'(t)\|_X) < \infty.$$

3.1.2 The viscous Voigt–MHD system and notion of solutions

In what follows, we are going to focus our attention on the following system:

$$\begin{aligned} \partial_t u + a^2 \partial_t (-\Delta)^\alpha u + u \cdot \nabla u + \nabla q &= B \cdot \nabla B + \nu \Delta u, \\ \partial_t B + a^2 \partial_t (-\Delta)^\alpha B + u \cdot \nabla B &= B \cdot \nabla u, \\ \nabla \cdot u &= 0, \quad \nabla \cdot B = 0, \\ (u, B)|_{t=0} &= (u_0, B_0). \end{aligned} \tag{3.1.12}$$

Here, $u(t, x)$ and $B(t, x)$ are three-dimensional vector fields depending on position $x \in \mathbb{T}^3$ (the three-dimensional flat torus) and time t . Furthermore $a, \nu > 0$ are fixed parameters, and $q(t, x)$ is the pressure term. We further assume that u_0 and B_0 have mean zero:

$$\int_{\mathbb{T}^3} B_0 dx = 0, \quad \int_{\mathbb{T}^3} u_0 dx = 0.$$

These conditions are propagated by the system (3.1.13), under sufficient regularity assumptions.

In this note, we will use the following reformulation of system (3.1.12) (recall that $\mathfrak{L} = \text{Id} + a^2(-\Delta)^\alpha$):

$$\begin{aligned} \partial_t u &= \mathfrak{L}^{-1} \mathbb{P}(-u \cdot \nabla u + B \cdot \nabla B) + \nu \mathfrak{L}^{-1} \Delta u, \\ \partial_t B &= \mathfrak{L}^{-1} \mathbb{P}(-u \cdot \nabla B + B \cdot \nabla u). \end{aligned} \tag{3.1.13}$$

We set up initial data u_0 and B_0 such that $u_0, B_0 \in H_0^\alpha$, and furthermore u_0, B_0 satisfy:

$$\operatorname{div} u_0 = 0, \quad \operatorname{div} B_0 = 0. \quad (3.1.14)$$

We now define a notion of solution to system (3.1.13):

Definition 3.1.6. Let $\alpha \geq 1$, $T > 0$, and let u, B both in the space $\mathcal{C}^1(0, T; H_0^\alpha)$, such that $u(t = 0) = u_0$, and $B(t = 0) = B_0$ satisfy the divergence-free conditions (3.1.14). Then, we say that (u, B) is a solution to system (3.1.13) if (u, B) solves equations (3.1.13) in the sense of distributions.

Remark 3.1.7. Note that, in particular, if $\alpha > \frac{3}{2}$, the equalities in system (3.1.13) are in the sense of H_0^α functions.

3.1.3 Global existence and regularity

We are first going to prove a local existence statement.

Proposition 3.1.1 (Local existence of solutions to (3.1.13)). *Let (u_0, B_0) in $H_0^\alpha(\mathbb{T}^3)$, and let us suppose that $\alpha \geq 1$, and that u_0 and B_0 satisfy the divergence-free condition (3.1.14). Then, there exists a time $T > 0$ and $u, B \in \mathcal{C}^1(0, T; H_0^\alpha)$ which solve (3.1.13) in the sense of distributions, and such that*

$$(u, B)|_{t=0} = (u_0, B_0).$$

Furthermore, if T_* is the maximal time of existence, we necessarily have either

$$\limsup_{t \rightarrow T_*} \|u(t, \cdot)\|_\alpha \rightarrow \infty, \quad \text{or} \quad \limsup_{t \rightarrow T_*} \|B(t, \cdot)\|_\alpha \rightarrow \infty. \quad (3.1.15)$$

Proof of Proposition 3.1.1. The proof follows by Picard iteration in the following space: $\mathcal{C}^1(0, T; H_0^{-\alpha})$. We follow closely the approach in [66] (Theorem 6.1), the only difference being the α -Laplacian instead of the Laplacian and the presence of the dissipation term

in the momentum equation in display (3.1.13). First of all, we will consider the following evolution equations, which are equivalent to system (3.1.13):

$$\begin{aligned}\partial_t \mathfrak{L}u &= \mathbb{P}(B \cdot \nabla B - u \cdot \nabla u) + \nu \Delta u, \\ \partial_t \mathfrak{L}B &= \mathbb{P}(-u \cdot \nabla B + B \cdot \nabla u).\end{aligned}\tag{3.1.16}$$

We then let $v = \mathfrak{L}u$, $Z = \mathfrak{L}B$, and rewrite the system as follows:

$$\begin{aligned}\partial_t v &= \mathbb{P}(\mathfrak{L}^{-1}Z \cdot \nabla \mathfrak{L}^{-1}Z - \mathfrak{L}^{-1}v \cdot \nabla \mathfrak{L}^{-1}v) + \nu \Delta \mathfrak{L}^{-1}v =: N_1(v, Z), \\ \partial_t Z &= \mathbb{P}(-(\mathfrak{L}^{-1}v) \cdot \nabla (\mathfrak{L}^{-1}Z) + (\mathfrak{L}^{-1}Z) \cdot \nabla (\mathfrak{L}^{-1}v)) =: N_2(v, Z).\end{aligned}\tag{3.1.17}$$

The task is now to show that N_1 and N_2 are Lipschitz mappings from $H_0^{-\alpha}$ to itself, and then the Picard–Lindelöf theorem will apply. We start from N_1 , using the conventions $v_i = \mathfrak{L}u_i$, $Z_i = \mathfrak{L}B_i$:

$$\begin{aligned}& \|N_1(v_1, Z_1) - N_1(v_2, Z_2)\|_{-\alpha} \\ & \leq \|\mathbf{B}(u_1 - u_2, u_2) + \mathbf{B}(u_1, u_1 - u_2)\|_{-\alpha} + \|\mathbf{B}(B_1 - B_2, B_2) + \mathbf{B}(B_1, B_1 - B_2)\|_{-\alpha} \\ & \quad + \nu \|\Delta(u_1 - u_2)\|_{-\alpha} \\ & \leq C|u_1 - u_2| \|u_1 - u_2\| \|u_2\| + C|u_1| \|u_1\| \|u_1 - u_2\| \\ & \quad + C|B_1 - B_2| \|B_1 - B_2\| \|B_2\| + C|B_1| \|B_1\| \|B_1 - B_2\| + \nu \|\Delta(u_1 - u_2)\|_{-\alpha} \\ & \leq C\lambda(\|v_1\|_{1-2\alpha} + \|v_2\|_{1-2\alpha}) \|v_1 - v_2\|_{1-2\alpha} + C\lambda(\|Z_1\|_{1-2\alpha} + \|Z_2\|_{1-2\alpha}) \|Z_1 - Z_2\|_{1-2\alpha} \\ & \quad + C\nu \|v_1 - v_2\|_{-\alpha} \\ & \leq C\lambda(\|v_1\|_{-\alpha} + \|v_2\|_{-\alpha}) \|v_1 - v_2\|_{-\alpha} + C\lambda(\|Z_1\|_{-\alpha} + \|Z_2\|_{-\alpha}) \|Z_1 - Z_2\|_{-\alpha} \\ & \quad + C\nu \|v_1 - v_2\|_{-\alpha}.\end{aligned}$$

Here, we used inequality (3.1.7) to go from the second to the third line, the Poincaré inequality (3.1.8), the fact that, for $\alpha \geq \beta$, we have $\|f\|_\alpha \geq \|f\|_\beta$, and the fact (proved in

Lemma 3.1.4) that, for $z \in H_0^{-\alpha}$,

$$\|\Delta \mathfrak{L}^{-1} z\|_{-\alpha} = \|\mathfrak{L}^{-1} \Delta z\|_{-\alpha} \leq C \|z\|_{-\alpha},$$

since $1 - 2\alpha \leq -\alpha$ if $\alpha \geq 1$. Similarly, we have, for the terms in N_2 ,

$$\begin{aligned} & \|N_2(v_1, Z_1) - N_2(v_2, Z_2)\|_{-\alpha} \\ & \leq \|\mathbf{B}(B_1 - B_2, u_1) + \mathbf{B}(B_2, u_1 - u_2)\|_{-\alpha} + \|\mathbf{B}(u_2 - u_1, B_2) + \mathbf{B}(u_1, B_2 - B_1)\|_{-\alpha} \\ & \leq C|B_1 - B_2| \|B_1 - B_2\| \|u_1\| + C|B_2| \|B_2\| \|u_1 - u_2\| + \\ & \quad + C|u_1 - u_2| \|u_1 - u_2\| \|B_2\| + C|u_1| \|u_1\| \|B_1 - B_2\| \\ & \leq C\lambda(\|v_1\|_{-\alpha} + \|v_2\|_{-\alpha}) \|v_1 - v_2\|_{-\alpha} + C\lambda(\|Z_1\|_{-\alpha} + \|Z_2\|_{-\alpha}) \|Z_1 - Z_2\|_{-\alpha}. \end{aligned}$$

We conclude that the mapping (N_1, N_2) is locally Lipschitz in $H_0^{-\alpha}$, which concludes the existence proof by an application of the Picard–Lindelöf theorem. Finally, the continuation criterion is evident from the fact that

$$\|\mathfrak{L}^{-1} u\|_{-\alpha} \geq C \|u\|_{\alpha}.$$

This concludes the proof. □

Having settled the issue of local existence, we will now prove global existence of solutions to the system (3.1.13).

Theorem 3.1.8 (Global existence of solutions to (3.1.13)). *Let (u_0, B_0) in H_0^{α} , with $\alpha \geq 1$. Then, there exists $u, B \in \mathcal{C}^1(0, \infty; H_0^{\alpha})$ which solve (3.1.13) in the sense of distributions, and such that*

$$(u, B)|_{t=0} = (u_0, B_0).$$

Proof of Theorem 3.1.8. Due to the continuation criterion of Proposition 3.1.1, we only need to show that $\|u\|_{H_0^{\alpha}} + \|B\|_{H_0^{\alpha}}$ is bounded a-priori in terms of initial data. We have the

estimate, taking the L^2 inner product of the momentum equation with u :

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} (|u|^2 + a^2 \|u\|_\alpha^2) &= (u, \partial_t \mathfrak{L}u) = -\nu \|u\|^2 + (\mathbf{B}(B, B), u) - (\mathbf{B}(u, u), u) \\ &= -\nu \|u\|^2 - (\mathbf{B}(B, u), B). \end{aligned}$$

On the other hand, we have, taking the L^2 inner product of the equation for $\partial_t B$ with B ,

$$\frac{1}{2} \frac{d}{dt} (|B|^2 + a^2 \|B\|_\alpha^2) = (B, \partial_t \mathfrak{L}B) = -(\mathbf{B}(u, B), B) + (\mathbf{B}(B, u), B).$$

Recalling that $(\mathbf{B}(u, B), B) = 0$, and summing the two previous displays, we finally get, for all times $t_2 \geq t_1 \geq 0$,

$$\begin{aligned} |u(t_2)|^2 + |B(t_2)|^2 + a^2 \|u(t_2)\|_\alpha^2 + a^2 \|B(t_2)\|_\alpha^2 + \nu \int_{t_1}^{t_2} \|\nabla u(s)\|^2 ds \\ \leq |u(t_1)|^2 + |B(t_1)|^2 + a^2 \|u(t_1)\|_\alpha^2 + a^2 \|B(t_1)\|_\alpha^2, \end{aligned} \tag{3.1.18}$$

which provides the required a-priori control. $\square$

We now show preservation of high regularity for solutions to the system (3.1.13).

Proposition 3.1.2 (Higher regularity of solutions to (3.1.13)). *Let (u_0, B_0) in $H_0^{k+\alpha}$, with $k \in \mathbb{N}$, $k \geq 0$, and $\alpha \in \mathbb{R}$, $\alpha \geq 1$. Then, the solution $u, B \in \mathcal{C}^1(0, \infty; H_0^\alpha)$ to the system (3.1.13) actually satisfies the stronger bounds:*

$$\|(u, B)\|_{L^\infty(0, T; H_0^{k+\alpha})} \leq C(\|u_0\|_{k+\alpha}, \|B_0\|_{k+\alpha}, T). \tag{3.1.19}$$

Proof of Proposition 3.1.2. The proof that follows is formal, as the regularity needed to perform such estimates is not assumed. Nevertheless, these estimates can be made rigorous, for instance, truncating to a finite amount of Fourier modes and then taking a limit. We

first commute the equations by ∂_i , and obtain

$$\begin{aligned}\partial_t \partial_i u + a^2 \partial_t (-\Delta)^\alpha \partial_i u - \Delta(\partial_i u) &= \mathbb{P}(-\partial_i u \cdot \nabla u - u \cdot \nabla \partial_i u + \partial_i B \cdot \nabla B + B \cdot \nabla \partial_i B), \\ \partial_t \partial_i B + a^2 \partial_t (-\Delta)^\alpha \partial_i B &= \mathbb{P}(-\partial_i u \cdot \nabla B - u \cdot \nabla \partial_i B + \partial_i B \cdot \nabla u + B \cdot \nabla \partial_i u).\end{aligned}$$

Multiplying the first equation by $\partial_i u$ and the second equation by $\partial_i B$, summing over i , we get, using the fact that $(\mathbf{B}(a, b), c) = -(\mathbf{B}(a, c), b)$,

$$\begin{aligned}& \partial_t (|u|^2 + a^2 \|u\|_{\alpha+1}^2 + |B|^2 + a^2 \|B\|_{\alpha+1}^2) + \nu \|u\|_2^2 \\ &= -(\mathbf{B}(\partial_i u, u), \partial_i u) + (\mathbf{B}(\partial_i B, B), \partial_i u), \\ &\quad -(\mathbf{B}(\partial_i u, B), \partial_i B) + (\mathbf{B}(\partial_i B, u), \partial_i B).\end{aligned}$$

Now, by Sobolev embedding, it is clear that

$$|(\mathbf{B}(\partial_i u, u), \partial_i u)| \leq C \|u\|_2^{\frac{3}{2}} \|u\|_1^{\frac{3}{2}}, \quad |(\mathbf{B}(\partial_i B, B), \partial_i u)| \leq C \|u\|_2^{\frac{1}{2}} \|u\|_1^{\frac{1}{2}} \|B\|_2 \|B\|_1.$$

Using the a-priori energy estimate (3.1.18), combined with Grönwall's inequality, we obtain

$$(u, B) \in L^\infty(0, T; H_0^{1+\alpha}), \quad (3.1.20)$$

with the required bound.

The proof for larger k is similar, and we omit it here. $\square$

3.1.4 Convergence towards a solution to stationary Euler

In this section, we are going to state and prove the main theorem about convergence in the infinite time limit.

Theorem 3.1.9 (Magnetic relaxation for Voigt–MHD). *Let $u_0, B_0 \in H_0^\alpha(\mathbb{T}^3)$ two divergence-free vector fields, with $\alpha > \frac{3}{2}$. Let (u, B) the global solution to the initial value prob-*

lem (3.1.13) given by Theorem 3.1.8. Then, there is a sequence $\{t_n\}_{n \in \mathbb{N}}$, $t_n \rightarrow \infty$ as $n \rightarrow \infty$, and $B_\infty \in H_0^\alpha$, such that $B(t_n, x)$ tends weakly in H_0^α to $B_\infty(x)$, and B_∞ solves the stationary Euler system:

$$\begin{aligned}\mathbb{P}(B_\infty \cdot \nabla B_\infty) &= 0, \\ \nabla \cdot B_\infty &= 0.\end{aligned}\tag{3.1.21}$$

Furthermore, we have that the following “modified magnetic helicity”:

$$H[B](t) := \int_{\mathbb{T}^3} B \cdot \mathfrak{L}\Psi \, dx \tag{3.1.22}$$

is conserved as a function of time. Here, Ψ is a magnetic potential as defined in Lemma 3.1.12, i. e. a vector field in $\Psi \in \mathcal{C}^1(0, \infty; H_0^\alpha)$ satisfying $\operatorname{div} \Psi = 0$, $\nabla \times \Psi = B$. Therefore, if the initial data is such that $H[B](0) \neq 0$, the resulting B_∞ will be nontrivial.

Remark 3.1.10. As one can see in the statement of the theorem, we are able to prove convergence of B to B_∞ along a sequence $t_n \rightarrow \infty$. It is possible to remove that restriction (so that the limit is no longer on a sequence) by integrating in time the evolution equation for B . Nevertheless, this requires more regularity (a higher value of α).

Proof of Theorem 3.1.9. Step 1. We will start by assuming that there exist constants c_1 , c_2 and c_3 depending on the H_0^α norm of initial data, and the parameters α and a , such the following inequality holds true, for any $t \geq 0$:

$$\|\partial_t u(t, \cdot)\|_\alpha^2 + c_1 \int_0^t |\partial_t \nabla u(s, \cdot)|^2 \, ds \leq c_2 \int_0^t |\nabla u(s, \cdot)|^2 \, ds + c_3. \tag{3.1.23}$$

We will prove inequality (3.1.23) in **Step 2**. We now consider two **Claims**.

Claim 1: Inequality (3.1.23) implies that $\|\partial_t u(t_n, \cdot)\|_\beta^2 \rightarrow 0$ as $n \rightarrow \infty$, for all $\beta < \alpha$, for a sequence $\{t_n\}_{n \in \mathbb{N}}$, $t_n \rightarrow \infty$.

Proof of Claim 1: We can see this easily as (3.1.18) implies

$$\int_0^\infty |\nabla u(s, \cdot)|^2 ds \leq C,$$

and therefore, using (3.1.23), along a dyadic sequence t_n , we obtain $|\partial_t \nabla u(t_n, \cdot)|^2 \leq 1/2^n$. Since $\|\partial_t u(t, \cdot)\|_\alpha^2 \leq C$, we then have by standard arguments that

$$\|\partial_t u(t_n, \cdot)\|_\beta^2 \rightarrow 0$$

as $n \rightarrow \infty$, for all $\beta < \alpha$.

Claim 2: We have $\|u(t, \cdot)\|_\beta \rightarrow 0$ as $t \rightarrow \infty$, for all $\beta < \alpha$.

Proof of Claim 2: Consider the two facts:

$$\int_0^\infty |\nabla u(s, \cdot)|^2 ds \leq C, \quad \int_0^\infty |\partial_t \nabla u(s, \cdot)|^2 ds \leq C.$$

From the first inequality we deduce that, along a dyadic sequence t_n , $|\nabla u(t_n, \cdot)| \rightarrow 0$. Furthermore, for all $t \leq t_n$,

$$|\nabla u(t, \cdot)|^2 \leq \|\nabla u\|_{L^2(t, t_n; H)} \|\partial_t \nabla u\|_{L^2(t, t_n; H)} + |\nabla u(t_n, \cdot)|^2,$$

and the claim follows taking first $n \rightarrow \infty$ and then $t \rightarrow \infty$.

Now, we wish to take the limit of the momentum equation along the sequence $t_n \rightarrow \infty$. We first note that, since we have the uniform bound $\|B(t_n, \cdot)\|_\alpha \leq C$, there will be $B_\infty \in H_0^\alpha$ such that $B_n := B(t_n, \cdot)$ converges weakly in H_0^α to B_∞ . We then see that, along such sequence, due to the bounds proved in **Claim 1** and **Claim 2**, we have that $\partial_t u$ is converging strongly to 0 in H_0^β , and $(-\Delta)^\alpha \partial_t u$ is converging strongly to 0 in $H_0^{\beta-2\alpha}$.

Furthermore, since $\alpha > \frac{3}{2}$, $\mathbb{P}(u_n \cdot \nabla u_n)$ is converging strongly in (say) H to 0, where $u_n := u(t_n, \cdot)$. This allows us to conclude that $\mathbb{P}(B_n \cdot \nabla B_n)$ is converging strongly to 0 in

$H_0^{\beta-2\alpha}$. But since B_n is also converging strongly in H_0^β for $\beta < \alpha$, with $\alpha > \frac{3}{2}$, $\mathbb{P}(B_n \cdot \nabla B_n)$ is actually converging strongly to 0 in H . This in particular implies that the equality $\mathbb{P}(B_\infty \cdot \nabla B_\infty) = 0$ is valid almost everywhere. Furthermore, for $\alpha > \frac{5}{2}$, we obtain a classical solution.

Step 2. We now turn to the proof of inequality (3.1.23). Let us commute ∂_t once through the momentum equation. We let, as usual, $\mathfrak{L} := \text{Id} + a^2(-\Delta)^\alpha$:

$$\partial_t \mathfrak{L} \partial_t u + \mathbb{P}(\partial_t u \cdot \nabla u + u \cdot \nabla \partial_t u) = \mathbb{P}(\partial_t (B \cdot \nabla B)) + \nu \Delta \partial_t u. \quad (3.1.24)$$

Now, the second Equation in display (3.1.13) gives us that (see Remark 3.1.11):

$$\partial_t B = \mathfrak{L}^{-1}(\nabla \times (u \times B)).$$

Hence, upon multiplication of equation (3.1.24) by $\partial_t u$ and integration on $\mathbb{T}^3 \times [0, t]$, we have

$$\begin{aligned} & \frac{a^2}{2} \|\partial_t u(t, \cdot)\|_{H^\alpha}^2 + \int_0^t \int_{\mathbb{T}^3} (\partial_t u_i \mathbb{P}(\partial_t u_j \partial_j u_i))(s, x) dx ds \\ & + \nu \int_0^t \int_{\mathbb{T}^3} |\partial_t \nabla u|^2 dx ds - \int_0^t \int_{\mathbb{T}^3} \partial_t u_i \partial_t \mathbb{P}(\partial_j (B_i B_j)) dx ds \leq C. \end{aligned}$$

Now, we have, integrating ∂_j by parts once in space, using the divergence-free condition of u , plus the Cauchy–Schwarz inequality and the Poincaré inequality (3.1.8),

$$\begin{aligned} & \left| \int_0^t \int_{\mathbb{T}^3} (\partial_t u_i \mathbb{P}(\partial_t u_j \partial_j u_i))(s, x) dx ds \right| \leq \int_0^t \int_{\mathbb{T}^3} |\partial_t \nabla u| |\mathbb{P}(u \partial_t u)| dx ds \\ & \leq \frac{\nu}{10} \|\partial_t \nabla u\|_{L^2(0,t;H)}^2 + C \|\partial_t u\|_{L^\infty(0,t;L^\infty)}^2 \|\nabla u\|_{L^2(0,t;H)}^2. \end{aligned}$$

Let us now estimate $\|\partial_t u\|_{L^\infty(\mathbb{T}^3)}$: since $\alpha > \frac{3}{2}$,

$$\|\partial_t u\|_{L^\infty} \leq C \|\partial_t u\|_\alpha \leq C \|\mathfrak{L}^{-1} \mathbb{P}(u \cdot \nabla u)\|_\alpha + C \|\mathfrak{L}^{-1} \mathbb{P}(B \cdot \nabla B)\|_\alpha + C \|\mathfrak{L}^{-1} \Delta u\|_\alpha$$

$$\begin{aligned}
&\leq C\|u \cdot \nabla u\|_H + C\|B \cdot \nabla B\|_H + C\|u\|_{-\alpha+2} \\
&\leq C\|u\|_\alpha^2 + C\|B\|_\alpha^2 + C\|u\|_\alpha,
\end{aligned}$$

where we used the Sobolev embedding $H^\alpha \hookrightarrow L^\infty$, plus the fact that $-\alpha + 2 \leq \alpha$ for $\alpha \geq 1$.

Hence, by the conservation law (3.1.18),

$$\left| \int_0^t \int_{\mathbb{T}^3} (\partial_t u_i \mathbb{P}(\partial_t u_j \partial_j u_i))(s, x) dx ds \right| \leq \frac{\nu}{10} \|\partial_t \nabla u\|_{L^2(0,t;H)}^2 + C \|\nabla u\|_{L^2(0,t;H)}^2.$$

Now, integrating by parts in space,

$$\begin{aligned}
\left| \int_0^t \int_{\mathbb{T}^3} \partial_t u_i \partial_t \mathbb{P}(\partial_j (B_i B_j)) dx ds \right| &\leq C \int_0^t |\partial_t \nabla u| |\mathbb{P}(B \partial_t B)| ds \\
&\leq C \int_0^t |\partial_t \nabla u| |B \partial_t B| ds \leq C \int_0^t |\partial_t \nabla u| \|B\|_{L^\infty} |\partial_t B| ds.
\end{aligned}$$

Hence:

$$\int_0^t |\partial_t \nabla u| \|B\|_{L^\infty} |\partial_t B| ds \leq \frac{\nu}{10} \|\partial_t \nabla u\|_{L^2(0,t;H)}^2 + C \|\mathfrak{L}^{-1} \nabla \times (u \times B)\|_{L^2(0,t;H)}^2.$$

Now, by the elliptic estimates for $\mathfrak{L}$, Sobolev embedding, and the Poincaré inequality, we have

$$\|\mathfrak{L}^{-1} \nabla \times (u \times B)\|_{L^2(0,t;L^2)}^2 \leq C \|u \times B\|_{L^2(0,t;L^2)}^2 \leq C \|B\|_{L^\infty(0,t;H^\alpha)}^2 \|\nabla u\|_{L^2(0,t;H)}^2.$$

Hence we obtain

$$\left| \int_0^t \int_{\mathbb{T}^3} \partial_t u_i \partial_t \mathbb{P}(\partial_j (B_i B_j)) dx ds \right| \leq \frac{\nu}{10} \|\partial_t \nabla u\|_{L^2(0,t;L^2)}^2 + C \|\nabla u\|_{L^2(0,t;H)}^2.$$

Combining all these estimates together and using the energy conservation statement (3.1.18) we have the claim (3.1.23).

Step 3. Let us compute the time derivative of the modified magnetic helicity:

$$\frac{d}{dt}H[B](t) = (\partial_t B, \mathfrak{L}\Psi) + (B, \partial_t \mathfrak{L}\Psi) = (\partial_t \mathfrak{L}B, \Psi) + (B, \partial_t \mathfrak{L}\Psi).$$

Now, by Lemma 3.1.12, we have that Ψ satisfies the equations $\partial_t \mathfrak{L}\Psi = u \times B$, $\nabla \times \Psi = B$, and since $\mathfrak{L}$ is self-adjoint with respect to the inner product in H , we have

$$\begin{aligned} \frac{d}{dt}H[B](t) &= \int_{\mathbb{T}^3} \partial_t \mathfrak{L}B \cdot \Psi dx + \int_{\mathbb{T}^3} B \cdot \partial_t \mathfrak{L}\Psi dx \\ &= \int_{\mathbb{T}^3} (\nabla \times (u \times B)) \cdot \Psi dx + \int_{\mathbb{T}^3} B \cdot (u \times B) dx \\ &= - \int_{\mathbb{T}^3} (u \times B) \cdot (\nabla \times \Psi) dx = 0. \end{aligned}$$

This concludes the proof of the theorem. □

Remark 3.1.11. Technically, the proof relies on the following identities:

$$\partial_t \mathfrak{L}B = \mathbb{P}(-u \cdot \nabla B + B \cdot \nabla u), \quad \text{and} \quad \partial_t \mathfrak{L}B = \nabla \times (u \times B).$$

If $\alpha > \frac{3}{2}$ we can apply a version of the Leibniz rule to obtain $\nabla \times (u \times B) = \mathbb{P}(\nabla \times (u \times B)) = \mathbb{P}(-u \cdot \nabla B + B \cdot \nabla u)$ almost everywhere.

Finally, we state and prove a lemma on the existence of the magnetic potential.

Lemma 3.1.12 (Existence of the magnetic potential). *Let (u, B) such that*

$$u, B \in \mathcal{C}^1(0, \infty; H_0^\alpha),$$

for $\alpha > \frac{3}{2}$ solving the system (3.1.13), according to Theorem 3.1.8, with divergence-free initial data $(u_0, B_0) \in H_0^\alpha$. Let us consider the following initial value problem, for an unknown vector field Ψ :

$$\begin{aligned} \partial_t \mathfrak{L}\Psi &= u \times B, \\ \Psi|_{t=0} &= \Psi_0. \end{aligned} \tag{3.1.25}$$

Here, Ψ_0 is the unique $H_0^{\alpha+1}(\mathbb{T}^3)$ vector field satisfying the following two properties:

$$\nabla \times \Psi_0 = B_0, \quad \operatorname{div} \Psi_0 = 0. \quad (3.1.26)$$

Under these conditions, if $\alpha > \frac{3}{2}$, we have that the system (3.1.25) has a global solution $\Psi \in \mathcal{C}^1(0, \infty; H^\alpha)$ which satisfies both (3.1.25) and (3.1.26). Furthermore, the following equality holds true for all times $t \geq 0$:

$$\nabla \times \Psi = B. \quad (3.1.27)$$

Proof of Lemma 3.1.12. The existence and regularity part is standard. To prove relation (3.1.27), we take the curl of the evolution equation in display (3.1.25), in order to obtain

$$\partial_t \mathfrak{L}(\nabla \times \Psi) = \nabla \times (u \times B) = \partial_t \mathfrak{L}B. \quad (3.1.28)$$

Integrating in time and using the initial conditions, this gives $\mathfrak{L}(\nabla \times \Psi) = \mathfrak{L}B$ at all times $t \geq 0$. We conclude by the fact that the kernel of $\mathfrak{L}$ in H_0^α is trivial. $\square$

This concludes our treatment of the issue of magnetic relaxation.

3.2 Examples of gradient growth in ideal MHD

In this section, we discuss two examples of gradient growth respectively for 3D MHD and for 2D MHD. Note that the example in 3D (described in Proposition 3.2.1) is on the torus $\mathbb{T}^3$ and is essentially obtained by decoupling the momentum and the induction equation, imposing that the B field is always vertical. The 2D example, on the other hand, is described in Proposition 3.2.2 and concerns a situation in which the energy of the solution necessarily needs to be infinite.

3.2.1 Gradient growth for B in three dimensions

We wish to provide a particular solution to the 3D MHD equation which, in the infinite time limit, creates discontinuities of B .

Proposition 3.2.1. *Consider the 3D ideal MHD system ($\mu = \nu = 0$) on the torus $\mathbb{T}^3$. There exist smooth initial data (u_0, B_0) of arbitrarily small size such that the solution launched by (u_0, B_0) is global and moreover the following inequality holds:*

$$\|\nabla B\|_{L^\infty(\mathbb{T}^3)} \geq Ce^t,$$

for some positive constant C which depends on initial data.

Proof of Proposition 3.2.1. Let us consider the three dimensional torus obtained from the box $[-1, 1] \times [-1, 1] \times [-1, 1]$ with opposite sides identified.

Suppose that u has the following form: $u = (u_1(x, y), u_2(x, y), 0)$, and that B has the following form: $B = (0, 0, B_3(t, x, y))$.

Let us consider the following stream function for u :

$$\Psi := \sin(2\pi x) \sin(2\pi y),$$

and we let $u_1 := \partial_y \Psi$, $u_2 = -\partial_x \Psi$. Under these conditions, u satisfies the steady Euler equations:

$$u \cdot \nabla u + \nabla p = 0, \quad \operatorname{div} u = 0.$$

Let us moreover evolve B_3 according to the following transport equation:

$$\partial_t B_3(t, x, y) + u \cdot \nabla B_3(t, x, y) = 0. \tag{3.2.1}$$

With these choices, we have that the pair (u, B) satisfies the ideal 3D MHD equations:

$$\begin{aligned}\partial_t u + u \cdot \nabla u + \nabla p &= B \cdot \nabla B, \\ \partial_t B + u \cdot \nabla B - B \cdot \nabla u &= 0, \\ \operatorname{div} u &= 0, \quad \operatorname{div} B = 0.\end{aligned}\tag{3.2.2}$$

Now, since u is constant in time, we only need to specify initial data for B . We are going to make the following choice:

$$B_0(x, y) = \chi(y),$$

where χ is a smooth and periodic function, $\chi : [-1, 1] \rightarrow \mathbb{R}$, with the property that $\chi(y) = y$ for $|y| \leq 1/2$. We are going to show that, locally around the origin, the gradient of B_3 grows exponentially in time.

Restricting equation (3.2.1) to the y -axis, we have

$$\partial_t B_3 - \sin(y) \partial_y B_3 = 0.\tag{3.2.3}$$

Let us now define a function $y(t, a)$ by the ODE (a is the Lagrangian label):

$$y'(t, a) = -\sin(y(t, a)), \quad y(0) = a.$$

Then, it can be easily checked that, for all positive t and for all a such that $|a| \leq 1/2$

$$B_3(t, 0, y(t, a), z) = B_0(0, a, z).\tag{3.2.4}$$

Now note that have the following relation satisfied by $y(t, a)$:

$$\tan\left(\frac{y(t, a)}{2}\right) = \tan\left(\frac{a}{2}\right)e^{-t},$$

Differentiating relation (3.2.4) and calculating the result at $a = 0$, we have that

$$|\partial_y B_3(t, 0, 0, z)| = C' e^t,$$

for a positive constant C' , thereby proving the claim. $\square$

Remark 3.2.1. Note that the reasoning above does not change if we multiply the initial data (u_0, B_0) by a small constant $\varepsilon > 0$. This in particular implies that we can consider initial data of arbitrarily small size.

3.2.2 A growing self-similar solution in two dimensions

Here, we describe the two-dimensional example. First, let us note that in the case of ideal MHD, the induction equation can be rewritten as follows:

$$\partial_t B + [u, B] = 0,$$

where $[u, B]$ is the Lie bracket between u and B . To obtain exponential growth, we are led to considering the “eigenvalue problem”

$$[u, B] = \lambda B.$$

If there is a scenario in which u is a steady solution to 2D incompressible Euler, and moreover λ is negative, we can hope to prove exponential growth. This is going to be the strategy behind our example. We now turn to describing the example in detail.

Proposition 3.2.2. *Consider the 2D ideal MHD system ($\mu = \nu = 0$) on the whole plane $\mathbb{R}^2$. There exist smooth initial data (u_0, B_0) which launch a global solution (u, B) with the following property: for any open set $O \subset \mathbb{R}^2$, we have that*

$$\|B\|_{L^\infty(O)} \geq C e^t, \tag{3.2.5}$$

where the constant $C > 0$ depends on O .

Proof of Proposition 3.2.2. Let us start from the ideal 2D MHD system:

$$\begin{aligned}\partial_t u + u \cdot \nabla u + \nabla p &= B \cdot \nabla B, \\ \partial_t B + u \cdot \nabla B - B \cdot \nabla u &= 0, \\ \operatorname{div} u &= 0, \quad \operatorname{div} B = 0.\end{aligned}\tag{3.2.6}$$

We can reformulate this system by using a stream function $\psi(t, x)$ for B . Indeed, it can be easily checked that, if (u, ψ) satisfy the following system:

$$\begin{aligned}\partial_t u + u \cdot \nabla u + \nabla p &= (\nabla^\perp \psi) \cdot \nabla (\nabla^\perp \psi), \\ \partial_t \psi + u \cdot \nabla \psi &= 0, \\ \operatorname{div} u &= 0,\end{aligned}\tag{3.2.7}$$

the pair $(u, \nabla^\perp \psi)$ satisfies the ideal MHD system (3.2.6). Here, as usual, we have defined $\nabla^\perp = (\partial_y, -\partial_x)$.

We now let $\psi = (x - y)^2$ and $\eta = y^2 - x^2$, and moreover we define

$$u = \nabla^\perp \eta \quad \text{and} \quad \tilde{B} = \nabla^\perp \psi.$$

It can be easily checked that both u and $\tilde{B}$ are solution to the steady incompressible 2D Euler equations. Indeed,

$$\Delta \eta = 0, \quad \Delta \psi = 1 = F(\psi),$$

where F is the constant function 1.

Moreover,

$$u \cdot \nabla \psi = -4\psi.$$

It is then easy to check that the vector fields

$$u, \quad B := \exp(4t)\tilde{B}$$

solve the ideal MHD system (3.2.7) with initial data $u_0 = \nabla^\perp \eta$, $B_0 = \nabla^\perp \psi$.

In particular, we easily check that (u_0, B_0) have infinite energy and moreover that the growth property (3.2.5) is satisfied. This concludes the proof of the proposition. $\square$

Bibliography

- [1] B. P. Abbott et al. Observation of gravitational waves from a binary black hole merger. *Phys. Rev. Lett.*, 116(6):061102, 2016.
- [2] H. Alfvén. Existence of electromagnetic-hydrodynamic waves. *Nature*, 150(3805):405–406, 1942.
- [3] J. Anderson and F. Pasqualotto. Global stability for nonlinear wave equations with multi-localized initial data. Preprint: <https://arxiv.org/abs/1906.02617>, 2019.
- [4] L. Andersson and P. Blue. Hidden symmetries and decay for the wave equation on the Kerr spacetime. *Ann. Math. (2)*, 182(3):787–853, 2015.
- [5] H. Bahouri, J.-Y. Chemin, and R. Danchin. *Fourier analysis and nonlinear partial differential equations*, volume 343 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer, Heidelberg, 2011.
- [6] G. K. Batchelor, H. K. Moffatt, and M. G. Worster, editors. *Perspectives in fluid dynamics*. Cambridge University Press, Cambridge, 2000. A collective introduction to current research.
- [7] T. B. Benjamin and J. E. Feir. The disintegration of wave trains on deep water part 1. theory. *Journal of Fluid Mechanics*, 27(3):417–430, 1967.
- [8] D. Biskamp. *Magnetic Reconnection in Plasmas*. Cambridge Monographs on Plasma Physics. Cambridge University Press, 2000.

- [9] P. Blue and J. Sterbenz. Uniform Decay of Local Energy and the Semi-Linear Wave Equation on Schwarzschild Space. *Communications in Mathematical Physics*, 268(2):481–504, 2006.
- [10] Y. Brenier. Optimal transport, convection, magnetic relaxation and generalized Boussinesq equations. *J. Nonlinear Sci.*, 19(5):547–570, 2009.
- [11] Y. Brenier. Topology-preserving diffusion of divergence-free vector fields and magnetic relaxation. *Comm. Math. Phys.*, 330(2):757–770, 2014.
- [12] D. Bresch and B. Desjardins. Existence of Global Weak Solutions for a 2D Viscous Shallow Water Equations and Convergence to the Quasi-Geostrophic Model. *Communications in Mathematical Physics*, 238(1):211–223, July 2003.
- [13] D. Bresch and B. Desjardins. On the construction of approximate solutions for the 2D viscous shallow water model and for compressible Navier-Stokes models. *J. Math. Pures Appl. (9)*, 86(4):362–368, 2006.
- [14] D. Bresch, B. Desjardins, and C. Lin. On some compressible fluid models: Korteweg, lubrication, and shallow water systems. *Comm. Partial Differential Equations*, 28(3-4):843–868, 2003.
- [15] D. Bresch, B. Desjardins, and G. Métivier. Recent mathematical results and open problems about shallow water equations. In *Analysis and simulation of fluid dynamics*, Adv. Math. Fluid Mech., pages 15–31. Birkhäuser, Basel, 2007.
- [16] O. P. Bruno and P. Laurence. Existence of three-dimensional toroidal MHD equilibria with nonconstant pressure. *Comm. Pure Appl. Math.*, 49(7):717–764, 1996.
- [17] D. Chae and P. Constantin. Remarks on a Liouville-Type Theorem for Beltrami Flows. *International Mathematics Research Notices*, 2015(20):10012–10016, 12 2014.

- [18] B. Cheng and E. Tadmor. Long-time existence of smooth solutions for the rapidly rotating shallow-water and Euler equations. *SIAM J. Math. Anal.*, 39(5):1668–1685, 2008.
- [19] B. Cheng and C. Xie. On the classical solutions of two dimensional inviscid rotating shallow water system. *J. Differential Equations*, 250(2):690–709, 2011.
- [20] D. Christodoulou. Global solutions of nonlinear hyperbolic equations for small initial data. *Comm. Pure Appl. Math.*, 39(2):267–282, 1986.
- [21] D. Christodoulou and S. Klainerman. *The global nonlinear stability of the Minkowski space*, volume 41 of *Princeton Mathematical Series*. Princeton University Press, Princeton, NJ, 1993.
- [22] S. Cicortas. *Global stability for weak null wave equations with multi-localized initial data*. Senior Thesis, Princeton University, Princeton, 2020.
- [23] P. Constantin, T. D. Drivas, H. Q. Nguyen, and F. Pasqualotto. Compressible fluids and active potentials. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 37(1):145–180, 2020.
- [24] P. Constantin, T. Elgindi, H. Nguyen, and V. Vicol. On singularity formation in a Hele-Shaw model. *Comm. Math. Phys.*, 363(1):139–171, 2018.
- [25] P. Constantin and C. Foias. *Navier-Stokes equations*. Chicago, IL etc.: University of Chicago Press, 1988.
- [26] P. Constantin, J. La, and V. Vicol. Remarks on a paper by Gavrilov: Grad-Shafranov equations, steady solutions of the three dimensional incompressible Euler equations with compactly supported velocities, and applications. *Geom. Funct. Anal.*, 29(6):1773–1793, 2019.

- [27] R. Courant and K. O. Friedrichs. *Supersonic flow and shock waves*. Springer-Verlag, New York-Heidelberg, 1976. Reprinting of the 1948 original, Applied Mathematical Sciences, Vol. 21.
- [28] M. Dafermos, G. Holzegel, and I. Rodnianski. Boundedness and decay for the Teukolsky equation on Kerr spacetimes I: The case $|a| \ll M$. *Ann. PDE*, 5(1):Paper No. 2, 118, 2019.
- [29] M. Dafermos, G. Holzegel, and I. Rodnianski. The linear stability of the Schwarzschild solution to gravitational perturbations. *Acta Math.*, 222(1):1–214, 2019.
- [30] M. Dafermos and I. Rodnianski. The red-shift effect and radiation decay on black hole spacetimes. *Comm. Pure Appl. Math.*, 62(7):859–919, 2009.
- [31] M. Dafermos and I. Rodnianski. Decay for solutions of the wave equation on Kerr exterior spacetimes I-II: The cases $|a| \ll M$ or axisymmetry. Preprint: <http://arxiv.org/abs/1010.5132>, 2010.
- [32] M. Dafermos, I. Rodnianski, and Y. Shlapentokh-Rothman. Decay for solutions of the wave equation on Kerr exterior spacetimes III: The full subextremal case $|a| < M$. *Ann. of Math. (2)*, 183(3):787–913, 2016.
- [33] B. Duan, Z. Luo, and Y. Zheng. Local existence of classical solutions to shallow water equations with Cauchy data containing vacuum. *SIAM J. Math. Anal.*, 44(2):541–567, 2012.
- [34] J. Eggers and T. F. Dupont. Drop formation in a one-dimensional approximation of the Navier-Stokes equation. *J. Fluid Mech.*, 262:205–221, 1994.
- [35] J. Eggers and M. A. Fontelos. *Singularities: formation, structure, and propagation*. Cambridge Texts in Applied Mathematics. Cambridge University Press, Cambridge, 2015.

- [36] A. Enciso, R. Lucà, and D. Peralta-Salas. Vortex reconnection in the three dimensional Navier-Stokes equations. *Adv. Math.*, 309:452–486, 2017.
- [37] L. C. Evans. *Partial differential equations*, volume 19 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, second edition, 2010.
- [38] Y. Fourès-Bruhat. Théorème d’existence pour certains systèmes d’équations aux dérivées partielles non linéaires. *Acta Math.*, 88:141–225, 1952.
- [39] A. V. Gavrilov. A steady Euler flow with compact support. *Geom. Funct. Anal.*, 29(1):190–197, 2019.
- [40] J.-F. Gerbeau and B. Perthame. Derivation of viscous Saint-Venant system for laminar shallow water; numerical validation. *Discrete Contin. Dyn. Syst. Ser. B*, 1(1):89–102, 2001.
- [41] E. Giorgi. *The linear stability of Reissner-Nordström spacetime for small charge*. PhD thesis, Columbia University, New York, 2019.
- [42] Y. Guo, A. D. Ionescu, and B. Pausader. Global solutions of the Euler-Maxwell two-fluid system in 3D. *Ann. of Math. (2)*, 183(2):377–498, 2016.
- [43] F. Hamel and N. Nadirashvili. A Liouville theorem for the Euler equations in the plane. *Arch. Ration. Mech. Anal.*, 233(2):599–642, 2019.
- [44] B. Haspot. Existence of global strong solution for the compressible Navier-Stokes equations with degenerate viscosity coefficients in 1D. *Math. Nachr.*, 291(14-15):2188–2203, 2018.
- [45] P. Helander. Theory of plasma confinement in non-axisymmetric magnetic fields. *Reports on Progress in Physics*, 77(8):087001, jul 2014.
- [46] M. J. M. Hill. On a spherical vortex. *Philosophical Transactions of the Royal Society of London. A*, 185:213–245, 1894.

- [47] D. Hoff. Global solutions of the Navier-Stokes equations for multidimensional compressible flow with discontinuous initial data. *J. Differential Equations*, 120(1):215–254, 1995.
- [48] G. Holzegel. *The stability problem for black holes*. PhD thesis, University of Cambridge, Cambridge, 2008.
- [49] L. Hörmander. *Lectures on nonlinear hyperbolic differential equations*. Paris: Springer, 1997.
- [50] L.-M. Imbert-Gerard, E. Paul, and A. Wright. An introduction to symmetries in stellarators. ArXiv Preprint: <http://arxiv.org/abs/1310.2664>, 2019.
- [51] S. Jiang. Global spherically symmetric solutions to the equations of a viscous polytropic ideal gas in an exterior domain. *Commun. Math. Phys.*, 178(2):339–374, 1996.
- [52] F. John. Blow-up of solutions of nonlinear wave equations in three space dimensions. *Proceedings of the National Academy of Sciences*, 76(4):1559–1560, 1979.
- [53] F. John. Blow-up for quasilinear wave equations in three space dimensions. *Comm. Pure Appl. Math.*, 34(1):29–51, 1981.
- [54] T. Kato and G. Ponce. Commutator estimates and the Euler and Navier-Stokes equations. *Comm. Pure Appl. Math.*, 41(7):891–907, 1988.
- [55] J. Keir. The weak null condition and global existence using the p -weighted energy method. Preprint: <https://arxiv.org/abs/1808.09982>, 2018.
- [56] A. Kiselev and V. Šverák. Small scale creation for solutions of the incompressible two-dimensional Euler equation. *Ann. of Math. (2)*, 180(3):1205–1220, 2014.
- [57] S. Klainerman. Global existence for nonlinear wave equations. *Communications on Pure and Applied Mathematics*, 33(1):43–101, 1980.

- [58] S. Klainerman. Long time behavior of solutions to nonlinear wave equations. *Proceedings of the International Congress of Mathematicians, Warsaw, (1983)*, pages 1209–1215, 1983.
- [59] S. Klainerman. Uniform decay estimates and the Lorentz invariance of the classical wave equation. *Commun. Pure Appl. Math.*, 38:321–332, 1985.
- [60] S. Klainerman. The null condition and global existence to nonlinear wave equations. In *Nonlinear systems of partial differential equations in applied mathematics, Part 1 (Santa Fe, N.M., 1984)*, volume 23 of *Lectures in Appl. Math.*, pages 293–326. Amer. Math. Soc., Providence, RI, 1986.
- [61] S. Klainerman and G. Ponce. Global, small amplitude solutions to nonlinear evolution equations. *Comm. Pure Appl. Math.*, 36(1):133–141, 1983.
- [62] S. Klainerman and J. Szeftel. Global Nonlinear Stability of Schwarzschild Spacetime under Polarized Perturbations. Preprint: <https://arxiv.org/abs/1711.07597>, 2017.
- [63] P. E. Kloeden. Global existence of classical solutions in the dissipative shallow water equations. *SIAM J. Math. Anal.*, 16(2):301–315, 1985.
- [64] B. F. Kraus and S. R. Hudson. Theory and discretization of ideal magnetohydrodynamic equilibria with fractal pressure profiles. *Physics of Plasmas*, 24(9):092519, 2017.
- [65] M. D. Kruskal and R. M. Kulsrud. Equilibrium of a magnetically confined plasma in a toroid. *Phys. Fluids*, 1:265–274, 1958.
- [66] A. Larios and E. Titi. On the higher-order global regularity of the inviscid voigt-regularization of three-dimensional hydrodynamic models. *Discrete and Continuous Dynamical Systems B*, 14(1531-3492), 2010.
- [67] H. Lindblad and I. Rodnianski. The global stability of Minkowski space-time in harmonic gauge. *Ann. of Math. (2)*, 171(3):1401–1477, 2010.

- [68] P.-L. Lions. *Mathematical topics in fluid mechanics. Vol. 2*, volume 10 of *Oxford Lecture Series in Mathematics and its Applications*. The Clarendon Press, Oxford University Press, New York, 1998. Compressible models, Oxford Science Publications.
- [69] J. Luk. Introduction to nonlinear wave equations. Available online at <https://www.dpmms.cam.ac.uk/~jl845/NWnotes.pdf>, 2014.
- [70] A. Majda. *Compressible fluid flow and systems of conservation laws in several space variables*, volume 53 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 1984.
- [71] F. Marche. Derivation of a new two-dimensional viscous shallow water model with varying topography, bottom friction and capillary effects. *Eur. J. Mech. B Fluids*, 26(1):49–63, 2007.
- [72] A. Matsumura and T. Nishida. The initial value problem for the equations of motion of viscous and heat-conductive gases. *J. Math. Kyoto Univ.*, 20(1):67–104, 1980.
- [73] A. Mellet and A. Vasseur. Existence and uniqueness of global strong solutions for one-dimensional compressible Navier-Stokes equations. *SIAM J. Math. Anal.*, 39(4):1344–1365, 2007/08.
- [74] F. Merle, P. Raphael, I. Rodnianski, and J. Szeftel. On the implosion of a three dimensional compressible fluid. Preprint: <https://arxiv.org/abs/1912.11009>, 2019.
- [75] S. Miao, L. Pei, and P. Yu. On Classical Global Solutions of Nonlinear Wave Equations with Large Data. *International Mathematics Research Notices*, 05 2017.
- [76] H. K. Moffatt. Magnetostatic equilibria and analogous Euler flows of arbitrarily complex topology. I. Fundamentals. *J. Fluid Mech.*, 159:359–378, 1985.

- [77] H. K. Moffatt. Magnetostatic equilibria and analogous Euler flows of arbitrarily complex topology. Part 2. Stability considerations. *Journal of Fluid Mechanics*, 166:359–378, 1986.
- [78] J. Nash. Le problème de cauchy pour les équations différentielles d’un fluide général. *Bulletin de la Société Mathématique de France*, 90:487–497, 1962.
- [79] P. Orenge. Un théorème d’existence de solutions d’un problème de shallow water. *Archive for Rational Mechanics and Analysis*, 130(2):183–204, 1995.
- [80] E. N. Parker. Spontaneous current sheets in magnetic fields : with applications to stellar x-rays. *Spontaneous current sheets in magnetic fields : with applications to stellar x-rays. International Series in Astronomy and Astrophysics*, 1, January 1994.
- [81] J. Pedlosky. *Geophysical Fluid Dynamics*. Springer-Verlag, New York, NY, USA, 2nd edition, 1990.
- [82] T. C. Sideris. Nonresonance and global existence of prestressed nonlinear elastic waves. *Ann. of Math. (2)*, 151(2):849–874, 2000.
- [83] C. D. Sogge. *Lectures on nonlinear wave equations*. Monographs in Analysis, II. International Press, Boston, MA, 1995.
- [84] L. Sundbye. Global existence for the Dirichlet problem for the viscous shallow water equations. *J. Math. Anal. Appl.*, 202(1):236–258, 1996.
- [85] L. Sundbye. Global existence for the Cauchy problem for the viscous shallow water equations. *Rocky Mountain J. Math.*, 28(3):1135–1152, 1998.
- [86] D. Tataru and M. Tohaneanu. A Local Energy Estimate on Kerr Black Hole Backgrounds. *International Mathematics Research Notices*, 2011(2):248, 2011.
- [87] J. B. Taylor. Relaxation and magnetic reconnection in plasmas. *Rev. Mod. Phys.*, 58:741–763, Jul 1986.

- [88] M. Taylor. The nonlinear stability of the Schwarzschild family of black holes (joint work with M. Dafermos, G. Holzegel, I. Rodnianski). *Oberwolfach reports*, 36:74–77, 2018.
- [89] B. A. Ton. Existence and uniqueness of a classical solution of an initial-boundary value problem of the theory of shallow waters. *SIAM J. Math. Anal.*, 12(2):229–241, 1981.
- [90] W. Wang and C. Xu. The Cauchy problem for viscous shallow water equations. *Rev. Mat. Iberoamericana*, 21(1):1–24, 2005.
- [91] K. E. Yang, D. W. Longcope, M. D. Ding, and Y. Guo. Observationally quantified reconnection providing a viable mechanism for active region coronal heating. *Nature Communications*, 9(1):692, 2018.
- [92] S. Yang. Global solutions of nonlinear wave equations with large data. *Selecta Math. (N.S.)*, 21(4):1405–1427, 2015.
- [93] V. E. Zakharov. Stability of periodic waves of finite amplitude on the surface of a deep fluid. *Journal of Applied Mechanics and Technical Physics*, 9(2):190–194, 1968.