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On Thurston's Euler class one conjecture

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ON THURSTON'S EULER CLASS ONE
CONJECTURE

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Abstract

In 1976, Thurston proved that taut foliations on closed hyperbolic 3-manifolds have Euler class of norm at most one, and conjectured that, conversely, any Euler class with norm equal to one is Euler class of a taut foliation. We construct the first counterexamples to this conjecture, infinitely many indeed. The counterexamples are constructed by Dehn surgeries on certain fibered hyperbolic 3-manifolds. Moreover, they are constructive in the sense that the monodromy of the fibration map is given in terms of Dehn twists and the surgery coefficient is specified. We also suggest an alternative conjecture in terms of faithful representations of the fundamental group of the 3-manifold into certain group of homeomorphisms.

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Chapter 1

Introduction

A two-dimensional foliation of a closed, orientable 3-manifold M is called *taut* if every leaf has a closed transversal, where a transversal is a closed loop that intersects the leaves of the foliation transversely. Manifolds admitting taut foliations, have properties similar to hyperbolic 3-manifolds. In particular they are *irreducible*, i.e., every embedded sphere in M bounds a solid ball. From now on we assume that M is irreducible as well. A connected, compact and orientable surface $S \subset M$ is called *incompressible* if it has no compressing disk, where a disk $(D, \partial D) \subset (M, S)$ is compressing if $D \cap S = \partial D$ and ∂D is homotopically non-trivial in S . In other words an incompressible surface can not be simplified in any obvious way, i.e., surgery along a compressing disk. Roussarie and Thurston showed that taut foliations and connected incompressible surfaces inside 3-manifold M , have an ‘efficient intersection property’ [27] [29] [30]. More precisely, the surface, S , can be isotoped to be a leaf or transverse to the foliation except at finitely many points, where in the latter case, the number of tangencies is exactly equal to $|\chi(S)|$. One can put this into the algebraic language of the Thurston inequality. Let $e(\mathcal{F}) \in H^2(M)$ ¹ be the Euler class of the tangent bundle to the foliation. For each embedded surface S we have the following

¹All (co)homologies have coefficients in $\mathbb{R}$, unless otherwise stated.

inequality for the pairing between homology and cohomology:

$$\langle e(\mathcal{F}), [S] \rangle \leq |\chi(S)| \quad (1.1)$$

Thurston defined a natural (semi)norm on the second homology of 3-manifolds, called *Thurston norm* nowadays. Define the complexity of a connected, embedded and oriented surface S as $c(S) = \max\{|\chi(S)|, 0\}$. If S has multiple components, its complexity is defined as sum of the complexities of its components. For $a \in H_2(M)$ the norm of a , $x(a)$, is defined as:

$$x(a) := \min\{c(S) \mid [S] = a, \text{ } S \text{ is embedded and oriented}\}$$

Up to scaling, Thurston's norm is the same as Gromov simplicial norm [8]. Thurston's norm on $H_2(M)$ naturally defines a dual norm on the vector space dual to $H_2(M)$. The dual vector space is $H^2(M)$, that is where $e(\mathcal{F})$ lives in. Therefore it makes sense to talk about the dual norm of Euler class, $x^*(e(\mathcal{F}))$. In fact, inequality (1.1) can be written in the following compact form.

$$x^*(e(\mathcal{F})) \leq 1 \quad (1.2)$$

In other words, the Euler class has dual norm at most one. This puts extreme bounds on the Euler class of a taut foliation. In particular, if M is hyperbolic, the number of second cohomology classes that can arise as the Euler class of some taut foliation on M is finite. Thurston conjectured that conversely the following happens [30]. We call it ‘the Euler class one conjecture.’ M is called *atoroidal* if there is no embedded, incompressible torus inside M .

Euler class one conjecture 1.0.1 (Thurston). *Let M be an atoroidal closed 3-manifold and let $a \in H^2(M; \mathbb{Z})$ be any integral class with $x^*(a) = 1$. Then there is a taut foliation $\mathcal{F}$ on M whose Euler class is equal to a .*

An irreducible 3-manifold is called *Haken* if it contains an embedded, two-sided, incompressible surface. From Thurston's hyperbolization theorem we know that every closed, atoroidal, Haken 3-manifold is hyperbolic. Since the manifolds that we consider here have positive first Betti number, they are automatically Haken. Hence M being atoroidal in the statement of the above conjecture is equivalent to M being hyperbolic [18]. There are index sum formulae for both sides of inequality (1.1), which are applications of the Poincaré-Hopf formula. It immediately follows from comparing those sums that both sides of (1.1) have the same parity when M is closed [30]. So we always assume having the same parity as a necessary condition, and we call it 'the parity condition'. Gabai gave a positive answer to the Euler class one conjecture for a subset of points in dual unit ball. Note that dual unit ball is a convex polyhedron with integral vertices [30].

Theorem 1.0.2 (Gabai). *Let M be a compact oriented 3-manifold, possibly with tori boundary, and let $a \in H^2(M, \partial M)$ be a vertex of the dual unit ball. Then there is a taut foliation on M whose Euler class is equal to a .*

Our first aim is to construct counterexamples to the Euler class one conjecture in the general setting of $C^{\infty,0}$ taut foliations, where a foliation is called $C^{\infty,0}$ if the leaves are smoothly immersed. The difficulty is that the conjecture is about manifolds with infinite first homology and so Gabai's theorem [8] guarantees the existence of many taut foliations on M ; therefore the usual methods for ruling out the existence of taut foliations do not work here and one needs a more delicate argument for ruling out taut foliations with a certain Euler class.

Theorem 1.0.3. *There are infinitely many closed hyperbolic 3-manifolds M for which the Euler class one conjecture does not hold; i.e., there is an integral point in the unit dual ball, satisfying the necessary parity condition, which is not realized by any taut $C^{\infty,0}$ -foliation.*

Our counterexamples are obtained by a suitable Dehn surgery on certain fibered hyperbolic 3-manifolds. Dehn surgery is the operation of removing a solid torus from a 3-manifold and gluing it back differently. The constructed counterexamples are explicit in the sense that the monodromy of the fibration map is given in term of Dehn twists. Moreover, the surgery coefficient is given. These manifolds are fairly simple from homological point of view, i.e., their first betti number is equal to two and the unit ball of their Thurston's norm has a simple shape (Figure 4.3). This is the simplest that one can hope for, since Gabai's theorem implies the truth of the conjecture for 3-manifolds whose first Betti number is equal to one.

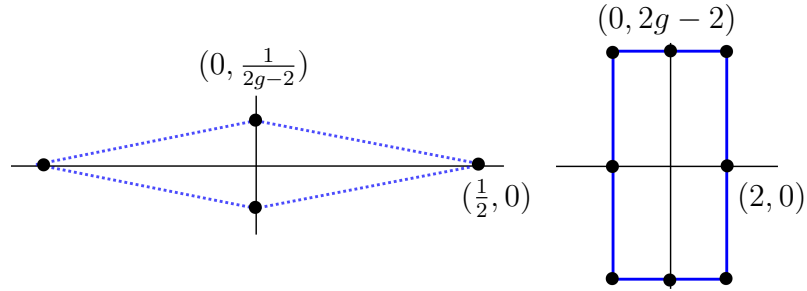


Figure 1.1: Left: Unit ball of Thurston norm, Right: Unit ball of dual Thurston norm

We say a connected, embedded, incompressible surface S is ‘fully-marked’ when in (1.1) the equality holds. The crucial but elementary observation is that any compact leaf is fully-marked [30] (see section 2.0.2). The converse, however, cannot be true since one can homotope $\mathcal{F}$ to a new taut foliation without changing the Euler class but with a drastic change in the leaves, so that there is no compact leaf anymore. We prove a converse to this statement up to homotopy of plane fields of foliations under some assumptions. This is a main ingredient in the proof of theorem 1.0.3. Here is the statement for closed manifolds.

Theorem 1.0.4. *(Gabai-Yazdi) Let M be a closed hyperbolic 3-manifold, $\mathcal{F}$ be a transversely oriented $C^{\infty,0}$ taut foliation on M , and S be a fully-marked surface in M . Assume that S is the unique norm-minimizing surface in its homology class. There is a new $C^{\infty,0}$ taut foliation $\mathcal{G}$ that has S as a leaf and the plane fields tangent to $\mathcal{F}$ and $\mathcal{G}$ are homotopic.*

Here is the idea of the proof for theorem 1.0.3. After computing the unit ball of the dual Thurston's norm, we pick a specific element of dual norm equal to one. We assume that a taut foliation with the prescribed Euler class exists. Therefore, we want to get a contradiction at the end. We pick a fully-marked surface S . By theorem 1.0.4 we can assume it is a leaf (otherwise, replace the foliation by a homotopic one that has S as a leaf). Then we can cut the manifold and foliation along S to get a foliation on a simpler manifold, $M \setminus S$. The way we constructed the manifold M imposes that the manifold $M \setminus S$ consists of two parts: a product part and a 'twisted part'. The twisted part is a solid torus with two sutures on its boundary. Each suture goes three times in the longitude direction and once in the meridian direction. Sutured manifolds were introduced by Gabai for studying taut foliations on 3-manifolds [8]. Here one should think about sutures as some extra data on the boundary of 3-manifold, which tells us how the foliation intersects the boundary. After some work, we show that the restriction of the foliation to the twisted part is taut and has Euler class zero. Then we directly prove that this twisted part does not have such a taut foliation to get a contradiction. In order to prove that the twisted part does not have such a taut foliation, we introduce the *transversal set of a leaf*. It consists of free homotopy classes of all positive closed transversals to a given leaf. Using some basic properties of this set, we show that if such a taut foliation exists then there should exist a transversal that is homotopically trivial. However this is not possible for taut foliations by a celebrated theorem of Novikov.

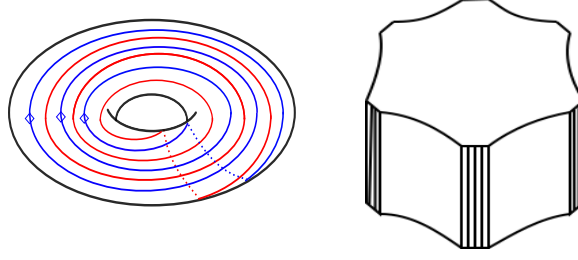


Figure 1.2: Left: Sutured solid torus , Right: its cross section

Now we sketch the proof of theorem 1.0.4. We should note that most of the ideas of the proof of this theorem are given generously by David Gabai. By Thurston's general position, S can be isotoped such that the induced singular foliation on S has only saddle singularities. However, there might be two-dimensional Reeb components on S . Since S is fully-marked, all saddle singularities have the same sign, i.e., the normal vector to the surface and the normal vector to the foliation always agree or always disagree. First we show that there is a metric on M such that the angle between S and the foliation is small at every point of S . Here small means less than ϵ for any fixed choice of positive ϵ . Cut M along S to get the manifold $M \setminus S$. The boundary of $M \setminus S$ consist of two copies of S . We want to 'smooth' the foliation along S by adding leaves to the boundary of $M \setminus S$, i.e., obtaining a foliation on $M \setminus S$ that is tangential to the boundary. Then the desired foliation $\mathcal{G}$ can be obtained by gluing two copies of S in the boundary of $M \setminus S$. Moreover, we need to do this in a way that the resulting foliation $\mathcal{G}$ on M is taut and is homotopic to $\mathcal{F}$. It turns out if we just do the smoothing process, the resulting foliation might not be taut, i.e., can have Reeb components. The issue comes from certain 'bad annuli' and 'bad solid tori' inside the induced foliation on $M \setminus S$. We show that one can avoid the unpleasant situation after isotoping S . This is done by defining a complexity function and using an infinite descent. The existence of the metric mentioned above is also used in this part.

Chapter 2

Preliminaries

2.0.1 Taut Foliations

A (two dimensional) *foliation* $\mathcal{F}$ of 3-manifold M is a partition of M into immersed surfaces (called *leaves*) such that locally they fit together nicely, like a stack of papers. $\mathcal{F}$ is called *transversely oriented* if there is a compatible choice of transverse orientation for each leaf. Since the manifold M is orientable, a transverse orientation induces an orientation on each leaf as well. Every closed orientable 3-manifold has a foliation, so the existence of a foliation does not give much information about the ambient manifold. However, there is a class of foliations called taut foliations that reflect many topological and geometric properties of their ambient manifold. $\mathcal{F}$ is called *taut* if for every leaf there is a closed transversal intersecting that leaf. For technical reasons, we need to specify the degree of smoothness that we consider here.

Definition 2.0.5. A foliation $\mathcal{F}$ is called $C^{1,0}$ if the tangent plane field $T\mathcal{F}$ exists and is continuous. If $\mathcal{F}$ is $C^{1,0}$ and all leaves are smoothly immersed, $\mathcal{F}$ is called $C^{\infty,0}$.

A foundational theorem of Calegari states that every topological (two dimensional) foliation of a 3-manifold is isotopic to a $C^{\infty,0}$ foliation [1].

2.0.2 Euler Class

Given any plane bundle $\mathcal{T}$ with base space B (so a continuously varying family of planes parametrized by B) one can associate a characteristic class to it, called the Euler class, which lives in $H^2(B)$ and measures how twisted the bundle is compared to the trivial bundle, in particular the Euler class of the trivial bundle is $0 \in H^2(B)$. It can also be seen as the obstruction for finding a section of $\mathcal{T}$. In this paper we are concerned with the case that $\mathcal{T}$ is the tangent bundle to a foliation $\mathcal{F}$ on the 3-manifold M . There is a geometric description of Euler class in this case as follows: Let $e(\mathcal{F}) \in H^2(M)$ be the Euler class of $\mathcal{F}$. Since second homology and cohomology are dual spaces, in order to understand an element in $H^2(M)$ we need to understand its pairing with every element of $H_2(M)$. So given an embedded oriented surface S inside M , we can put it in general position w.r.t. $\mathcal{F}$ so that they are transverse to each other except at a finite number of tangencies which are of saddle or center type. Then:

$$\langle e(\mathcal{F}), [S] \rangle = \sum_{\substack{\text{tangent} \\ \text{points } p}} \pm i(p) \quad (2.1)$$

Here $i(p)$ is the index of tangency point. Saddle points have index -1 and center points have index $+1$. The $\pm$ sign is chosen according to whether the orientation of the two planes ‘tangent plane to S ’ and ‘tangent plane to foliation’ agree at the point p or not. Note that the same index sum without $\pm$ sign gives the Euler characteristic of S (or negative of that) by the Poincare-Hopf formula. In particular if S is a leaf of $\mathcal{F}$ the one can perturb S so that all tangencies have the same sign and so we have:

$$\langle e(\mathcal{F}), [S] \rangle = \pm \chi(S), \text{ For } S \text{ a compact leaf of } \mathcal{F} \quad (2.2)$$

We have mentioned so far that a surface S can be put in general position w.r.t. the foliation so that there are only saddle or center tangencies. If the foliation is taut and the surface S is incompressible, then Thurston and Roussarie showed that one can actually get rid of center tangencies as well. This is an ‘efficient intersection property’ for surfaces and taut foliations. Note that this has the following interesting consequence:

$$\langle e(\mathcal{F}), [S] \rangle \leq |\chi(S)|, \quad \mathcal{F} \text{ a taut foliation} \quad (2.3)$$

Remark 2.0.6. Since we are working with $C^{\infty,0}$ foliations, we should note that both general positions that we mentioned hold in the $C^{\infty,0}$ case as well [28] [11] although the original argument was for C^2 foliations.

2.0.3 Pseudo-Anosov maps

Let $S := S_g$ be a surface of genus g . A multi-curve on S is a union of disjoint simple closed curves on S . Given a homeomorphism ϕ of S , one can look at the action of ϕ on the set of multi-curves on S . Pseudo-Anosov homeomorphisms can be characterized by the property that they do not fix isotopy class of any multi-curve.

Thurston’s hyperbolization theorem states that the mapping torus of a homeomorphism $\phi : S \rightarrow S$ is a hyperbolic manifold if and only if ϕ is isotopic to a pseudo-Anosov representative.

2.0.4 Penner’s construction of pseudo-Anosov maps

Thurston gave the first hands-on construction of pseudo-Anosov maps in terms of Dehn twists. His construction made use of twists along two curves α and β such that $\alpha \cup \beta$ fills the surface. Penner generalized this construction to opposite twists along multi curves. A *multi curve* is a union of distinct (up to isotopy) and disjoint simple

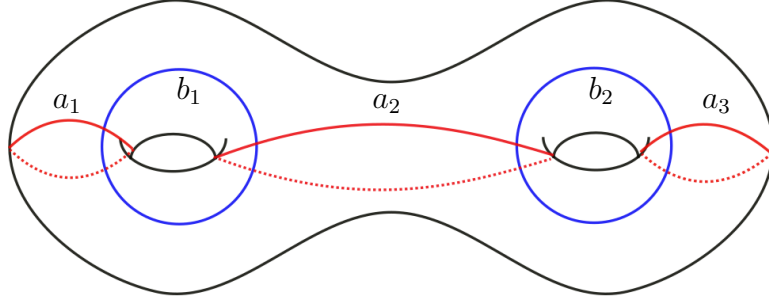


Figure 2.1: Two multi-curves that together fill

closed curves on S . Let $\alpha = a_1 \cup \dots \cup a_m$ and $\beta = b_1 \cup \dots \cup b_n$ be two multi curves on S such that $\alpha \cup \beta$ fills the surface. Let τ_{a_i} be the positive (right handed) Dehn twist along a_i . Define τ_{b_j} similarly. Penner's theorem states that any word in τ_{a_i} and $\tau_{b_j}^{-1}$ is pseudo-Anosov provided that all τ_{a_i} and τ_{b_j} are used at least once [26]. Note that we are doing positive Dehn twist along the curves in one collection (multi curve) and negative twists along the other one.

Example 2.0.7. Let S be a closed surface of genus two and the multi curves $\alpha = a_1 \cup a_2 \cup a_3$ and $\beta = b_1 \cup b_2$ be as in Figure 2.1. It can be easily seen that $\alpha \cup \beta$ fills the surface. By Penner's theorem, the map $f = \tau_{a_1}^2 \circ \tau_{a_2} \circ \tau_{b_2}^{-3} \circ \tau_{a_3} \circ \tau_{b_1}^{-1} \circ \tau_{a_1}$ is pseudo-Anosov.

2.0.5 Thurston Norm

Thurston defined a natural (semi)norm on the second homology of 3-manifolds which measures the minimum complexity between all representatives of a homology class. More precisely for each properly embedded surface S inside M , define its complexity to be

$$c(S) = \max\{0, -\chi(S)\}$$

and define the norm of a homology class $a \in H_2(M, \partial M)$ as follows:

$$x(a) = \min\{c(S) \mid [S] = a\}$$

This is a semi-norm on $H_2(M, \partial M)$ whose unit ball is a convex polyhedron with rational vertices.

Associated to any norm x on a vector space V there is a dual norm x^* on the dual vector space V^* as follows:

$$x^*(u) = \sup\{\langle u, v \rangle \mid x(v) = 1\}$$

This gives a dual norm on $H^2(M, \partial M)$, whose unit ball is a convex polyhedron with integral vertices [30]. The next proposition describes the behavior of the Thurston dual norm under (covering) maps between 3-manifolds.

Proposition 2.0.8. *Let M, N be compact, orientable, irreducible 3-manifolds and $p : N \longrightarrow M$ be a map and let $p^* : H^2(M, \partial M) \longrightarrow H^2(N, \partial N)$ be the induced map on real cohomology equipped with the dual Thurston norm. Then the map p^* is norm decreasing and it preserves the norm if p is a covering map.*

Proof. Let $a \in H^2(M, \partial M)$ with norm equal to k . Then for each $[h] \in H_2(N, \partial N)$ we have:

$$\langle p^*a, [h] \rangle = \langle a, p_*[h] \rangle \leq k.x(p_*[h]) \leq k.x([h])$$

In order to prove the last inequality, we need to introduce Gromov simplicial norm. Let X be a compact manifold possibly with boundary and let $C_*(X)$ be the real (relative) chain complex of X . Each $c \in C_*(X)$ is a finite linear combination of singular simplices in X , i.e. , $c = \sum_i r_i \sigma_i$, where $r_i \in \mathbb{R}$ and σ_i are singular simplices.

Define the ℓ^1 -norm of c as

$$||c|| = \sum_i |r_i|$$

This induces a semi-norm on $H_*(X, \partial X)$ by setting the norm of $\alpha \in H_*(X, \partial X)$ to be

$$g(\alpha) := \inf ||z||,$$

where z varies between all singular cycles representing α [15]. The last inequality comes from the fact that the Thurston norm and the Gromov norm are proportional ($x = \frac{1}{2}g$) [8] and the Gromov norm is decreasing under push-forward (follows from definition). This implies that p^*a has norm at most k , so p^* is norm decreasing.

Now suppose p is a covering map and choose $[F] \in H_2(M, \partial M)$, represented by the embedded surface F , so that the following equality happens:

$$\langle a, [F] \rangle = k.x([F])$$

Let $[f] = p^{-1}(F) \in H_2(N, \partial N)$, then we have:

$$\langle p^*a, [f] \rangle = \langle a, p_*[f] \rangle = \langle a, (\deg(p)).[F] \rangle = k.\deg(p).x([F]) = k.x([f])$$

Here the last equality, $x([f]) = \deg(p).x([F])$, is proved by Gabai (p is a covering map) [8]. Therefore when p is a covering map, it preserves the norm. $\square$

2.0.6 Cornered Manifolds

In our discussion, we need to consider the Euler class of some cornered manifolds as well, whose corners are among three types: convex, concave and two sided (see figure 2.2). Corners are used to specify how leaves of a foliation intersect the boundary of manifold. Note that if there are just convex corners, the cornered manifold is a sutured manifold introduced by Gabai. One can define tangential and transversal

boundary of cornered manifolds similar to sutured manifolds. Again Euler class is defined as before, now an element of $H^2(M, \partial M)$, but keeping in mind that the cornered structure of a manifold contains information about $\mathcal{T}_{|\partial M}$. Again the number $\langle e(\mathcal{T}), [S] \rangle$ can be obtained by putting the surface in general position w.r.t. foliation and adding indices of tangencies with signs.

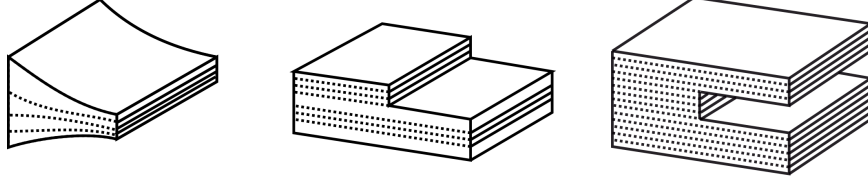


Figure 2.2: Different types of corners from left to right: convex, two-sided and concave. The dashed lines show where we have cut the manifold in order to see the cornered structure more visibly.

In order to have an analogue of inequality (1.1) for these cornered manifolds, we need to take into effect cornered structure of embedded surfaces when computing their Euler characteristic. The Poincare-Hopf formula says that for a closed surface, Euler characteristic is the obstruction for finding a section of the tangent bundle of surface; similarly for a cornered surface, it is the obstruction for finding a section of the tangent bundle of surface with prescribed boundary condition coming from the cornered structure of ∂M . The analogue of inequality (1.1) follows from existence of Thurston-Roussarie general position.

Example 2.0.9. Suppose S is a cornered surface with a combination of convex, concave and two sided corners. Let $\hat{S}$ be the surface obtained by forgetting about cornered structure of S . Let m, n be the number of convex and concave corners. Then the (sutured) Euler characteristic of S is defined as:

$$\chi(S) = \chi(\hat{S}) - \frac{1}{2}m + \frac{1}{2}n$$

Example 2.0.10. Let M be a solid torus with two sutures on its boundary. Each of the sutures goes twice in the longitudinal direction and once in the meridional direction. There is a taut foliation of M by a ‘stack of chairs’ obtained in the following way. Take an infinite ‘stack of chairs’ and glue the top to the bottom by 180° rotation (see figure 2.3). The core disk of M is a sutured disk with $\chi(D) = 1 - 4 \times (\frac{1}{2}) = -1$. The Euler class of this foliation assigns ± 1 to the core disk of M . Both ± 1 can be realized by choosing appropriate transverse orientation. One can define a similar foliation on other sutured solid tori such as the sutured solid torus N (defined in Lemma 4.3.1). For N , the foliation looks like a stack of monkey saddles. We call it the standard foliation by ‘a stack of monkey saddles’.

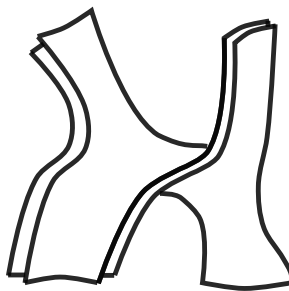


Figure 2.3: foliation by a stack of chairs

Chapter 3

Previous Work

Thurston and Roussarie realized that taut foliations and embedded incompressible surfaces have an ‘efficient intersection property’, meaning that inequality (1.1) holds for them [30] [27]. Thurston introduced a natural norm on second homology of 3-manifolds (named Thurston norm nowadays) and studied connections between taut foliations and this norm. Putting inequality (1.1) in the language of the Thurston norm, he obtained that the Euler class of any taut foliation of a 3-manifold has dual Thurston norm at most one.

Theorem 3.0.11 (Thurston). *Let M be an oriented 3-manifold and $\mathcal{F}$ a codimension one, transversely oriented foliation of M . Suppose that $\mathcal{F}$ contains no Reeb components and each component of ∂M is either a leaf of $\mathcal{F}$ or a surface T such that $\mathcal{F}$ is transverse to T and $\mathcal{F}|_{\partial M}$ has no two dimensional Reeb component. Then:*

$$x^*(e) \leq 1$$

Holds in $H^2(M, \partial M)$ (respectively $H^2(M)$ if $\partial M = \emptyset$)

Here x^ is dual Thurston norm and e is Euler class of the tangent plane bundle to foliation $\mathcal{F}$.*

This shows that not every integral second cohomology class can be realized as the Euler class of some taut foliation. In particular, the number of such classes is finite if M is an-annular and atoroidal. This is in contrast with the case of general foliations on closed 3-manifolds, where every cohomology class can be realized as the Euler class of some foliation (section 4.4.3). Conversely Thurston conjectured the following (see [30], page 129, conjecture 3).

Conjecture 3.0.12 (Thurston). *If M has no ‘essential’ singular tori, and if $a \in H^2(M; \mathbb{Z})$ is any element with $x^*(a) = 1$, then there is some (taut) foliation $\mathcal{F}$ of M such that $e(\mathcal{F}) = a$.*

Thurston showed that the unit ball for the dual Thurston norm is a convex polyhedron whose vertices are integral points [30]. Later Gabai proved the following (see [10], page 24, Remark 7.3). We present Gabai’s proof here (personal communication).

Theorem 3.0.13 (Gabai). *The conjecture holds for vertices of dual unit ball.*

Proof. Let $a \in H^2(M, \partial M)$ be a vertex of the dual unit ball. Let $\bar{a} \in H_2(M, \partial M)$ be a rational point in the face dual to the point a (note that this face is top-dimensional) and S be a norm minimizing embedded surface representing a multiple of the homology class $\bar{a}$. Let

$$\langle \cdot, \cdot \rangle : H^2(M, \partial M) \times H_2(M, \partial M) \longrightarrow \mathbb{Z}$$

be the pairing between second cohomology and homology of M . By definition we have

$$\langle a, \bar{a} \rangle = 1 = x(\bar{a}).$$

By [8], there exist a taut foliation $\mathcal{F}$ on M such that S is a leaf of $\mathcal{F}$, $\mathcal{F}$ is transverse to ∂M and $\partial \mathcal{F}$ has no Reeb component. We show that $e(\mathcal{F}) = a$.

Since $\bar{a}$ is in the interior of a top dimensional face, we can choose a basis $\bar{a}_1, \bar{a}_2, \dots, \bar{a}_n$

for the second homology of M such that

$$\bar{a} = t_1 \bar{a}_1 + \dots + t_n \bar{a}_n$$

with $0 < t_i < 1$ and $\sum_{i=1}^n t_i = 1$. For each $1 \leq i \leq n$ we have

$$\langle a, \bar{a}_i \rangle = 1 = x(\bar{a}_i).$$

Since S is a leaf of $\mathcal{F}$

$$\langle e(\mathcal{F}), [S] \rangle = \chi(S)$$

$$\implies \langle e(\mathcal{F}), \bar{a} \rangle = 1$$

Therefore we have

$$1 = \langle e(\mathcal{F}), \bar{a} \rangle = \sum_i t_i \langle e(\mathcal{F}), \bar{a}_i \rangle \underbrace{\leq}_{(1)} \sum_i t_i x(\bar{a}_i) \underbrace{=}_{(2)} x\left(\sum_i t_i \bar{a}_i\right) = x(\bar{a}) = 1$$

Where (1) is the fact that Euler class e has dual norm at most one and (2) is the linearity of Thurston norm inside a top dimensional face. So each of inequalities in (1) should be in fact equality and for each index i we have

$$\langle e(\mathcal{F}), \bar{a}_i \rangle = x(\bar{a}_i) = \langle a, \bar{a}_i \rangle$$

Since $\bar{a}_i$'s are a basis for the second homology, we have $e(\mathcal{F}) = a$. □

Chapter 4

Main Theorem

When interpreting the conjecture, we should consider a necessary ‘parity condition’. This necessary parity condition is as follows: $\chi(S)$ and $\langle e(\mathcal{F}), [S] \rangle$ have the same parity. This is a consequence of the index-sum formulas for both of these numbers [30].

Our aim here is to give a counterexample to the conjecture for the case that the cohomology class is not a vertex.

Theorem 4.0.14. *There are infinitely many closed hyperbolic 3-manifolds M for which the conjecture does not hold; i.e. there is some integral point in the unit dual ball, satisfying the parity condition, which is not realized by any taut $C^{\infty,0}$ -foliation.*

Remember that the number that a taut foliation $\mathcal{F}$ assigns to a connected and incompressible surface S is at most $|\chi(S)|$, and if S is a leaf then equality happens. This motivates the following definition.

Definition 4.0.15. Let $\mathcal{F}$ be a taut foliation on a 3-manifold M and S be a connected, embedded and incompressible surface such that $\langle e(T\mathcal{F}), [S] \rangle = \pm\chi(S)$. We call S a fully-marked surface with respect to $\mathcal{F}$. When there is no ambiguity about $\mathcal{F}$ we just say S is a fully-marked surface. Note that if $\chi(S) < 0$ then any embedded surface that satisfies the equality is automatically incompressible (in fact norm minimizing).

Compact leaves of taut foliations are fully-marked. Next theorem gives a converse to this observation up to homotopy of foliation and under some assumptions. It will be our main tool for proving theorem 4.0.14.

Theorem 4.0.16. *Let M be a closed hyperbolic 3-manifold and $\mathcal{F}$ be a transversely oriented $C^{\infty,0}$ taut foliation on M . Let S be a fully-marked surface with respect to $\mathcal{F}$. Assume that S is the unique norm-minimizing surface in its homology class. There is a new taut foliation $\mathcal{G}$ that has S as its leaf and the plane fields tangent to $\mathcal{F}$ and $\mathcal{G}$ are homotopic.*

Remark 4.0.17. Without allowing a homotopy of $\mathcal{F}$ this cannot be true. To see this, note that the Euler class is invariant of the homotopy type of foliation and starting from a foliation that has S as a leaf and perturbing the foliation by a small homotopy (and maintaining the taut property, since tautness is an open condition), the quantity $\langle e(\mathcal{F}), [S] \rangle$ does not change but the leaves change somehow radically so that S is not a leaf any more.

Remark 4.0.18. Since the foliation is taut, we can put the surface S in Thurston general position; i.e. all tangencies between surface and leaves are saddle type. So the condition $\langle e(\mathcal{F}), [S] \rangle = \pm\chi(S)$ is equivalent to requiring that all of these tangencies have the same sign (either at all points the orientation of S and the tangent leaf agree or at all points they disagree.)

Let N be a solid torus with two parallel sutures on its boundary, each of which goes three times in the longitudinal direction and once in the meridional direction (see figure 4.11). A fact about sutured solid torus N is that it violates a stronger version of Thurston's conjecture in the sense that there is some point inside (but not on the boundary of) its unit dual ball which is not realized by any taut foliation (see Lemma 4.3.1). The idea of the proof of Theorem 4.0.14 is to use surgery to plant N inside another closed manifold to make it a counterexample. Here is the rough

idea why these manifolds are counterexamples to the conjecture: starting from a taut foliation with prescribed Euler class of norm one on these manifolds, there is always a fully-marked surface S and using theorem 4.0.16, we can assume S is a leaf and so we can split the manifold along the surface and get a taut foliation on a simpler 3-manifold that we have partial information about its Euler class. Continue this by splitting along other surfaces until we get to the solid torus N with a taut foliation that has Euler class zero (our specific construction leads us to the manifold N). But such a taut foliation does not exist on solid torus N by Lemma 4.3.1 .

4.1 Construction of the counterexamples

In this section, we explain the construction of our counterexamples. The manifold is obtained by a $(\frac{3}{1})$ Dehn-surgery on a fibered 3-manifold, but we should put some constraints on the monodromy of the fibered manifold and also specify the curve on which we are doing the surgery.

Let S be a surface of genus $g \geq 3$ and M_f be a fibered 3-manifold with fiber S and monodromy $f : S \longrightarrow S$. Let γ be a non-separating simple closed curve on S . We start with the pair (M_f, γ) where the map f has the following three properties.

- 1) f is pseudo-Anosov.
- 2) $\ker(f_* - id_* : H_1(S) \longrightarrow H_1(S)) = \{0\}$. This condition is equivalent to saying that $H_2(M_f)$ has rank one (see [17], Example 2.48).
- 3) It sends a certain curve, α , in $S \times \{1\}$ to a certain curve, β , in $S \times \{0\}$, where α and β are defined in terms of γ and are non-separating. Moreover the projections of α and β to S are not homologous to each other. This condition allows us to completely control homology and Thurston norm of the manifold obtained by surgery.

Later in lemma 4.2.5 we show that f can be chosen to have these three properties. For now, let us assume that we have such a map f .

Now we specify how to do the surgery on M_f to get the desired manifold M . Let A be an annulus neighborhood of γ in S . We call its boundary components γ_+ and γ_- . Now consider the solid torus $U = A \times [\frac{1}{4}, \frac{3}{4}]$. The desired manifold M is obtained by Dehn surgery on this solid torus. $M = (M_f - U) \cup N$. One should think about the glued-in solid torus as the sutured solid torus N (figure 4.11), and $M - U$ as a sutured manifold with two sutures $\gamma_{\pm} \times [\frac{1}{4}, \frac{3}{4}]$ on its torus boundary. The gluing map sends sutures to sutures and tangential boundaries to tangential boundaries.

Now we define α and β . Pick an orientation on γ . This induces orientations on γ_+ and γ_- . Choose three points $p_i, i \in \{1, 2, 3\}$, that are cyclically ordered on γ_+ and similarly three points $q_i, i \in \{1, 2, 3\}$, on γ_- . Choose three disjoint arcs $l_i, i \in \{1, 2, 3\}$, on $A \times \{\frac{3}{4}\}$, where l_i connects $p_i \times \{\frac{3}{4}\}$ to $q_i \times \{\frac{3}{4}\}$. Likewise, choose three disjoint arcs $m_i, i \in \{1, 2, 3\}$, on $A \times \{\frac{1}{4}\}$, where m_i connects $p_i \times \{\frac{1}{4}\}$ to $q_{i+1} \times \{\frac{1}{4}\}$. (Note the shift in indices.) We choose these in such a way that m_i is obtained by adding the arc $q_i q_{i+1}$ to the projection of l_i to $A \times \{\frac{1}{4}\}$ and then perturbing them to be disjoint and properly embedded in $A \times \{\frac{1}{4}\}$ (see figure 4.1).

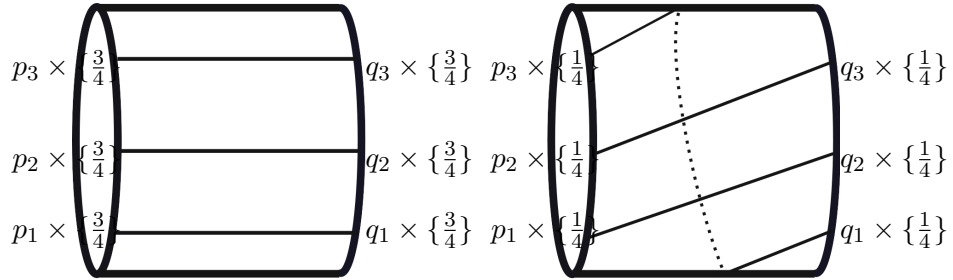


Figure 4.1: Left: Arcs l_i on $A \times \{\frac{3}{4}\}$, Right: Arcs m_i on $A \times \{\frac{1}{4}\}$

Now let $\delta_i, i \in \{1, 2, 3\}$, be disjoint arcs in $S \setminus A$ that connect p_i to q_{i-1} (again note the shift in indices). Then α_0 will be the union of the six arcs, l_i , and $\delta_i \times \{\frac{3}{4}\}$, $i \in \{1, 2, 3\}$. β_0 will be the union of m_i and $\delta_i \times \{\frac{1}{4}\}$, $i \in \{1, 2, 3\}$. $\alpha = \pi(\alpha_0)$, $\beta = \pi(\beta_0)$. By this we mean α is the parallel translation of α_0 to $S \times \{1\}$ and similarly for β . Note that the appropriate shift in indices, clockwise for l_i 's and counterclockwise for δ_i 's, implies that α and β are connected simple closed curves.

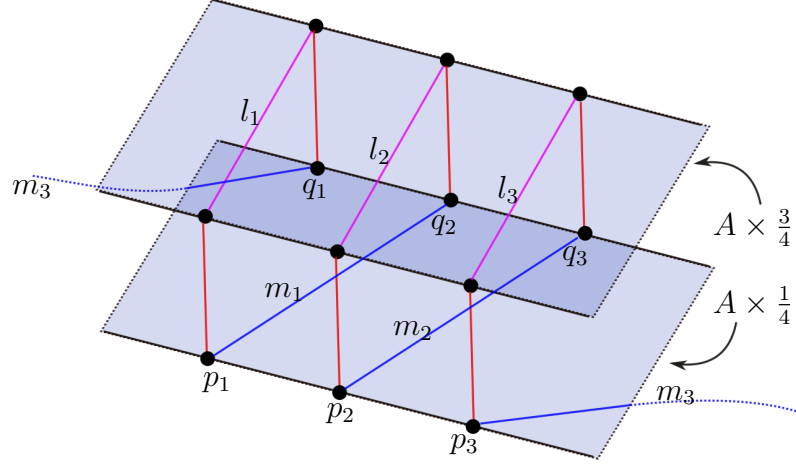


Figure 4.2: ∂D is glued to the union of blue, red and purple curves on the boundary of $A \times [\frac{1}{4}, \frac{3}{4}]$.

By choosing the arcs δ_i 's appropriately we can assume that α and β are non-separating in S . Note that $\beta = \alpha - \gamma$ as oriented sum (oriented cut and paste). Although α and β are not uniquely defined in terms of γ , any choice of them satisfying the above properties would work for the construction. Let D be the core disk of N . We choose the gluing of N to $M_f - U$ such that ∂D goes to the union of $l_i, m_i, p_i \times [\frac{1}{4}, \frac{3}{4}]$, and $q_i \times [\frac{1}{4}, \frac{3}{4}]$. Note that this is compatible with our previous assumption, namely throughout the gluing the sutures go to sutures. See Figure 4.2, where l_i 's, m_i 's and $p_i \times [\frac{1}{4}, \frac{3}{4}] \cup q_i \times [\frac{1}{4}, \frac{3}{4}]$ are shown with purple, blue and red respectively.

Lemma 4.1.1. *Let M be the manifold constructed above. $H_2(M)$ has rank two and the unit balls for Thurston norm and Thurston dual norm are as in figure 4.3.*

Proof. First we show that $H_2(M)$ has rank two. Since $H_2(M_f)$ has rank 1, by a Mayer-Vietoris argument, one can show that $H_2(M)$ has rank at most 2. More precisely, we have the following exact sequence for $M_f = (M - U^\circ) \cup U$.

$$\cdots \rightarrow H_1(\partial U) \rightarrow H_1(M - U^\circ) \oplus H_1(U) \rightarrow H_1(M_f) \rightarrow 0$$

This implies that:

$$\text{rank}H_1(M - U^\circ) + \text{rank}H_1(U) \leq \text{rank}H_1(\partial U) + \text{rank}H_1(M_f)$$

$$\Rightarrow \text{rank}H_1(M - U^\circ) \leq 2$$

We also have the exact sequence for $M = (M - U^\circ) \cup N$.

$$\cdots \rightarrow H_1(\partial U) \rightarrow H_1(M - U^\circ) \oplus H_1(N) \rightarrow H_1(M) \rightarrow 0$$

This implies that.

$$H_1(M) \cong \frac{H_1(M - U^\circ) \oplus H_1(N)}{\ker(\Phi)}$$

,where $\Phi : H_1(M - U^\circ) \oplus H_1(N) \rightarrow H_1(M)$ is defined as $\Phi(x, y) = x + y$. Therefore we have:

$$\text{rank}H_1(M) \leq \text{rank}H_1(M - U^\circ) + \text{rank}H_1(N) - \text{rank}(\ker \Phi) \leq 2$$

The last inequality is equivalent to $\text{rank}(\ker \Phi) \geq 1$. Let $y \in H_1(N)$ represent the core curve of N and $x \in H_1(M - U^\circ)$ be the element that corresponds to y via the attaching map between N and $M - U^\circ$. Then

$$(0, 0) \neq (-x, y) \in \ker(\Phi)$$

And no power of (x, y) is zero either. Hence $\text{rank}(\ker \Phi)$ is at least one.

On the other hand, the three properties assumed about f imply that $H_2(M)$ has rank at least 2. Certainly the surface $S \times \{1\}$ is a homology class. A second surface can

be obtained as follows. Let D be the core disk of N . We can assume that ∂D is the union of $l_i, m_i, p_i \times [\frac{1}{4}, \frac{3}{4}]$, and $q_i \times [\frac{1}{4}, \frac{3}{4}]$ (because this union is homologically $3m + l$, where m and l are respectively the meridian and the longitude of the removed solid torus, U , and we have done $\frac{3}{1}$ surgery). So if we attach three bands $\delta_i \times [\frac{1}{4}, \frac{3}{4}]$ to D , what we get is a surface with two boundary components, α_0 and β_0 , with Euler characteristic equal to -2 . Since f sends α to β , we can close this surface to get a closed surface of genus two, which we call F . F is the union of the core disk of N , the three mentioned bands, two vertical annuli $\alpha \times [\frac{3}{4}, 1]$ and $\beta \times [0, \frac{1}{4}]$ (identifying $\alpha \times \{1\}$ with $\beta \times \{0\}$).

Now we compute Thurston norm of $H_2(M)$. We already know that $H_2(M)$ is generated by S and F . These classes have Thurston norm at most $2g-2$ and 2 , respectively. We exhibit taut foliations on M and show that the unit balls for Thurston norm and dual Thurston norm are as in figure 4.3.

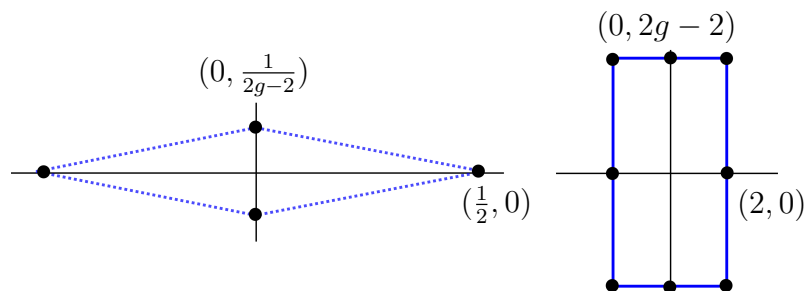


Figure 4.3: Left: Unit ball of Thurston norm, Right: Unit ball of dual Thurston norm

We will show that there is a taut foliation of M whose Euler class, e , assigns to S and F respectively the numbers $\chi(S)$ and $\chi(F)$. This implies that:

$$x([S]) = -\chi(S)$$

$$x([F]) = -\chi(F)$$

Therefore

$$x([S] + [F]) \leq x([S]) + x([F]) = -\chi(S) - \chi(F)$$

Hence

$$\langle e, [S] + [F] \rangle = \langle e, [S] \rangle + \langle e, [F] \rangle = \chi(S) + \chi(F) \leq -x([S] + [F])$$

But we know that the dual norm of e is at most one, therefore the equality should happen in the above, i.e. ,

$$x([S] + [F]) = -\chi(S) - \chi(F)$$

To construct such a foliation on M , decompose M into two pieces, $M_f - U$ and N . The first piece is equipped with a product structure coming from the fibration of M_f . On the second piece, choose a standard taut foliation of N by a ‘stack of monkey saddles’ that assigns $\chi(D) = -2$ to the core disk of N (see Example 2.0.10). Glue these two foliations together, on their common boundary, to get a taut foliation of M . The constructed foliation is taut since its compact leaves (basically the surface S) have closed transversals. The Euler class of this foliation assigns to S and F respectively the numbers $\chi(S)$ and $\chi(F)$. This is clearly true for S , since it is a leaf. To see that it also happens for F , recall that the number assigned to F can be computed from an index sum formula. F can be obtained from D by adding three bands and then gluing the two boundary curves α and β together. Since the induced foliation on each of these bands is a product by our special construction, there is no tangency on them and so they do not contribute to the index sum. Hence the number assigned to F is the same as the number assigned to D , which is equal to $\chi(D) = -2$.

Likewise, one can show the following by constructing taut foliations on M .

$$x([S] - [F]) = -\chi(S) - \chi(F)$$

Here we should construct a taut foliation on M whose Euler class assigns to S and $-F$ respectively the numbers $\chi(S)$ and $\chi(F)$. We would do the same steps, except, at the end we use a standard taut foliation of N by a ‘stack of monkey saddles’ that assigns $-\chi(D) = 2$ to the core disk of N . To sum up, we have proved the following four equalities:

$$\begin{aligned} x([S]) &= -\chi(S) \\ x([F]) &= -\chi(F) \\ x([S] + [F]) &= -\chi(S) - \chi(F) \\ x([S] - [F]) &= -\chi(S) - \chi(F) \end{aligned}$$

These show that the unit balls have the claimed shapes in figure 4.3.

□

So far, we have explained the construction of M . In lemma 4.2.2 we prove that M is atoroidal and hyperbolic. Assuming this, we can prove theorem 4.0.14.

4.2 Proof of the main Theorem

Proof of theorem 4.0.14. We show that the manifold M violates Thurston’s conjecture. Consider the point $(0, 2 - 2g)$. This point is on the boundary of the unit ball of dual Thurston norm. We want to show that it cannot be realized as Euler class of any taut foliation on M . Assume the contrary: that there is such a taut foliation $\mathcal{F}$. Note that S is a fully-marked surface, i.e., $\langle e(\mathcal{F}), [S] \rangle = \chi(S)$. Moreover, M is hyperbolic (Lemma 4.2.2) and S is the unique norm-minimizing surface in its homology class (Lemma 4.2.3). By Theorem 1.0.4, $\mathcal{F}$ can be homotoped to a new taut foliation that has S as its leaf. The new foliation has the same Euler class as $\mathcal{F}$ since they are

homotopic. By abuse of notation we still call the new foliation $\mathcal{F}$. Cut $\mathcal{F}$ along S to get a taut foliation, $\mathcal{F}'$, of $M' = M \setminus \setminus S$. $\mathcal{F}'$ assigns 0 to $F' = F \setminus \setminus \alpha$; this can be seen immediately if one thinks about the assigned numbers to F and F' as index sums. Consider the annuli $A_1 = \gamma_+ \times [0, 1]$ and $A_2 = \gamma_- \times [0, 1]$. By Lemma 4.4.24, A_1 and A_2 can be isotoped such that the induced foliations on them are suspension foliations. Now we can cut M' along A_1 and A_2 to get a taut foliation, $\mathcal{F}''$, of the sutured manifold $M'' = N \cup (A \times [\frac{3}{4}, 1]) \cup (A \times [0, \frac{1}{4}])$. M'' is homeomorphic to N . Note that Euler class of $\mathcal{F}''$ assigns 0 to the core disk of M'' . This is because the core disk can be obtained from $F' = F \setminus \setminus \alpha$ by cutting a couple of bands, and the contribution of these bands to $\langle e(\mathcal{F}''), [F'] \rangle$ is zero (the bands can be isotoped so that the induced foliations on them are products. Hence the Euler class assigns zero to them.) Thus we have a taut foliation on a sutured manifold homeomorphic to N whose Euler class assigns zero to the core disk. However we will prove in lemma 4.3.1 that such a taut foliation does not exist. Contradiction. $\square$

The next lemma is a standard position result for embedded tori in M , that is useful in proving that M is atoroidal.

Lemma 4.2.1. *Let T be an embedded essential torus in M . T can be isotoped in such a way that if we cut M along $S \times \{1\}$, it will be a collection of disjoint properly embedded essential annuli in $(S - A) \times [0, 1]$, each of which has one boundary component on each of $S \times \{0\}$ and $S \times \{1\}$.*

Proof. Firstly we isotope T such that $T \cap S$ is a collection of simple closed curves that are essential in both T and S . This can be done since both S and T are incompressible. In particular, these curves are parallel in the torus T . This set of simple closed curves cut T into a collection of annuli in $M \setminus \setminus S$. Let T_1 be one of these annuli. Now we show that T_1 can be isotoped away from N . Without loss of generality, we might throw out the top and bottom product pieces of $M \setminus \setminus S$, that is, $S \times [\frac{3}{4}, 1]$ and $S \times [0, \frac{1}{4}]$.

In other words $M \setminus S$ is homeomorphic to

$$(S - A) \times [\frac{1}{4}, \frac{3}{4}] \cup N$$

where $\partial(S - A) \times [\frac{1}{4}, \frac{3}{4}]$ is glued to $\partial_{\text{th}}N$. Let A_1 and A_2 be the annuli $\gamma^+ \times [\frac{1}{4}, \frac{3}{4}]$ and $\gamma^- \times [\frac{1}{4}, \frac{3}{4}]$, respectively. The intersection of T_1 with A_1 and A_2 is a collection of properly embedded arcs in A_1 and A_2 . Two types of arcs that can occur are shown in figure 4.4. Type one starts and ends on the same boundary component of A_i whereas, type two goes between different boundary components of A_i for $i = 1, 2$.

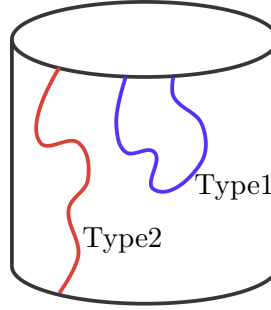


Figure 4.4: Two types of arcs

First we can get rid of type one arcs. Choose an innermost type one arc. This arc together with a portion of the boundary of A_i bounds a disk. Use this disk to push the arc out of A_i and reduce the number of type one arcs. After doing this finitely many times, we are left with a number of type two arcs. From the perspective of T_1 these arcs start from one boundary component of T_1 and end on the other boundary component of T_1 (figure 4.5). So $T_1 \cap N$ is a collection of squares. Each of these squares have two sides on $\partial_{\text{th}}N$ and two sides on $\partial_{\text{r}}N$. Therefore each of these squares are ∂ -parallel in N . Starting from the innermost square, we can isotope them out of N . We have shown that T_1 can be isotoped away from N and we can do the same for other chopped pieces of T . Now we can get rid of the pieces of T that start and end on the same boundary component of $M \setminus S$ by isotoping them (note that now pieces are totally inside the product part of $M \setminus S$.) This completes the proof of lemma.

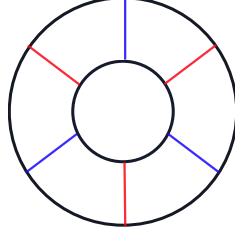


Figure 4.5: Annulus T_1 is chopped up into squares

□

Lemma 4.2.2. *The manifold M is atoroidal and hyperbolic.*

Proof. Suppose there is an essential torus inside M . Use lemma 4.2.1 to chop it into annuli pieces, each going from top boundary S to bottom boundary S of $M \setminus S$, and all disjoint from N . However this means that the collection of curves $\mathcal{C} = T \cap S$ is a reducing collection for the map f , that is, $f(\mathcal{C}) = \mathcal{C}$, contrary to f being pseudo-Anosov. So M is atoroidal. Now Thurston's hyperbolization theorem for Haken manifolds implies that M is hyperbolic, since M is Haken, atoroidal and irreducible. □

Lemma 4.2.3. *Let M and S be as defined above. Any norm-minimizing surface in the same homology class as S , is isotopic to S .*

Theorem 4.2.4. (Gabai) *Let M be a closed orientable 3-manifold. Suppose P and Q are two norm-minimizing surfaces (possibly disconnected) in M that are homologous. There exists a sequence of norm-minimizing surfaces (possibly disconnected) $P = P_0, P_1, \dots, P_n = Q$ with each term in the same homology class as $[P] = [Q]$ such that any two adjacent terms in the sequence can be isotoped to be disjoint in M .*

Proof. Directly follows from the proof of Lemma 3.6. in [8]. □

Proof of Lemma 4.2.3. Suppose F is a norm-minimizing surface that is homologous to S . By Theorem 4.2.4, we can assume that F is disjoint from S . Therefore it lies inside $M \setminus S$. $M \setminus S$ is the union of two pieces: the product part and the twisted

part. We isotope F to be in a standard form in each part and show that it is isotopic to S .

Recall that γ is a non-separating simple closed curve inside S and A is a tubular neighborhood of γ in S . We have

$$M \setminus S = (S - A) \times [0, 1] \cup N$$

where N is the twisted sutured solid torus. The two pieces are glued along the transverse boundaries i.e. $\partial A \times [0, 1] = A_1 \cup A_2$.

By assumption, F is disjoint from the top and bottom copies of S . Isotope F such that it is in general position with respect to A_i so that it intersects each of them in a union of simple closed curves. We can assume that there is no arc since F was disjoint from the boundary circles of A_i in the beginning. We can get rid of inessential circles by starting from an innermost circle and isotoping F out of A_i . At this point $F \cap A_i$ is a union of disjoint copies of essential circles, each of them is isotopic to the core curve of A_i . Let

$$F_1 := F \cap (S - A) \times [0, 1], F_2 := F \cap N$$

A standard argument about incompressible and ∂ -incompressible surfaces in product manifolds shows that F_1 is a union of parallel copies of $S - A$ (See Fact 1 below). But there can not be more than one copy, since otherwise

$$|\chi(F)| \geq 2|\chi(S - A)| = 2(2g - 3) > 2g - 2 = |\chi(S)|$$

And this is in contrast with our initial assumption that F is norm-minimizing and $[F] = [S]$. Hence F_1 is a single copy of $S - A$ and so F intersects each of A_i in exactly one essential simple closed curve. But the only incompressible surfaces, up to isotopy, in $(N, \partial_{\text{th}} N)$ with this oriented boundary are the two tangential boundaries

of N (See Fact 2 below). Therefore F is isotopic to either the top copy of S or the bottom copy.

Fact 1: Let K be a compact surface. The only connected, incompressible and ∂ -incompressible embedded surfaces in $K \times [0, 1]$, up to isotopy, are $K \times \{t\}$ for $t \in [0, 1]$ and $\theta \times [0, 1]$ where θ is an essential simple closed curve in K . Here ∂ -incompressible means that there is no compressing disk D with $\partial D = \alpha \cup \beta$ such that $\alpha \subset (K \times 0) \cup (K \times 1)$ and $\beta \subset \partial(\text{surface})$ and $\alpha \cap \beta = \text{two points}$.

Fact 2: Let the tangential boundary of the sutured manifold N be $N_+ \cup N_-$. Any essential annulus T in $(N, \partial_{\text{h}} N)$ with the same oriented boundary as N_+ is isotopic to either N_+ or N_- .

Proof of fact 2: An incompressible surface T in N is π_1 -injective and hence an annulus. The sutured manifold N is a disk bundle over the circle where the disk can be thought of as a hexagon and the monodromy map is the $\frac{2\pi}{3}$ rotation. Fix one of the fibers, D , and put T in general position with respect to the hexagon D . Therefore the intersection $T \cap D$ is a union of simple closed curves or arcs going between vertices of D . The simple closed curves bound disks inside D and one can isotope T to get rid of them starting from the innermost circle. Hence $T \cap D$ is a union of arcs going between the vertices of D . These arcs break T into vertical rectangles in $N \setminus D$. There are two type of arcs: the ones going between two adjacent vertices of D and the ones going between two opposite vertices of D . Note that the configuration of the arcs on D should be invariant under the monodromy map i.e. $\frac{2\pi}{3}$ rotation. This rules out the arcs that go between opposite vertices of D since such an arc gives rise to an immersed annulus in N . So the only possible arcs are the ones between adjacent vertices. Since the configurations of arcs is invariant under $\frac{2\pi}{3}$ rotation and

the annulus is connected, the only possible configurations are the ones in Figure 4.6. These give rise to the annuli N_+ and N_- . Note that this fact is a very special case of Waldhausen's theorem saying that incompressible and ∂ -incompressible surfaces in Seifert fibered manifolds are either vertical or horizontal.

□

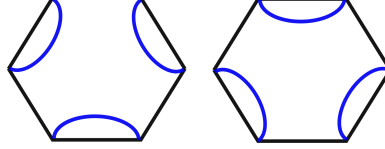


Figure 4.6: Possible configurations of arcs on D

Now we show that f can be chosen to satisfy the three conditions mentioned in the beginning.

Lemma 4.2.5. *Let S be a closed, orientable surface of genus $g \geq 6$. For suitable choice of γ and δ_i , there exists a pseudo-Anosov homeomorphism f of S with the following two properties.*

- 1) $f(\alpha) = \beta$
- 2) $\text{rank}(H_2(M_f)) = 1$

Where M_f is the mapping torus of f .

Proof. We first construct the map for a surface of genus three and from there it is clear how to generalize it for arbitrary genera. Let α , β and γ be the oriented curves shown in Figure 4.7. In this picture, δ_i 's can be thought of as the three pieces of $\alpha - \alpha \cap \beta$. We use Penner's construction of pseudo-Anosov maps to define f . For any curve η , let τ_η be the positive Dehn twist around η . Our convention is the right handed twist.

Let $a_1, \dots, a_4$ (positive twists) and $b_1, \dots, b_3$ (negative twists) be the filling system of curves shown in the left side of Figure 4.8. Define the maps f as the following:

$$f := \tau_{b_2} \circ \tau_{a_2}^{-1} \circ \tau_{a_3}^{-1} \circ \tau_{b_1} \circ \tau_{b_3} \circ \tau_{a_1}^{-1} \circ \tau_{a_4}^{-1}$$

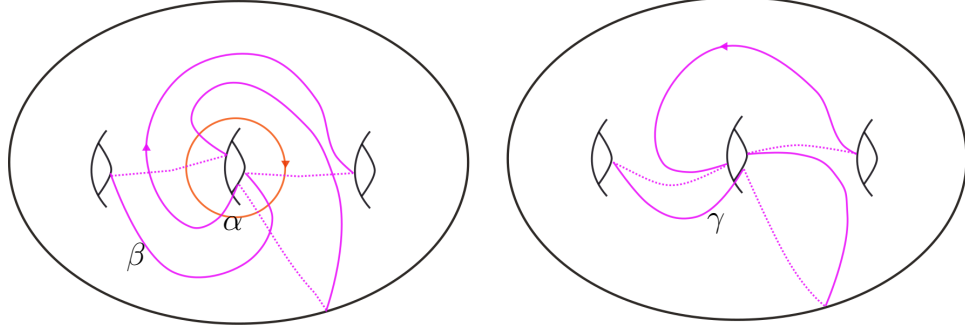


Figure 4.7: α , β and γ curves on S

By Penner's construction, the map f is pseudo-Anosov. Since the curves a_1, a_4, b_1, b_3 are disjoint from α , we have:

$$f(\alpha) = \tau_{b_2} \circ \tau_{a_2}^{-1} \circ \tau_{a_3}^{-1}(\alpha) = \beta$$

Condition (2) is equivalent to

$$\ker\{f_* - Id : H_1(S) \longrightarrow H_1(S)\} = \{0\}$$

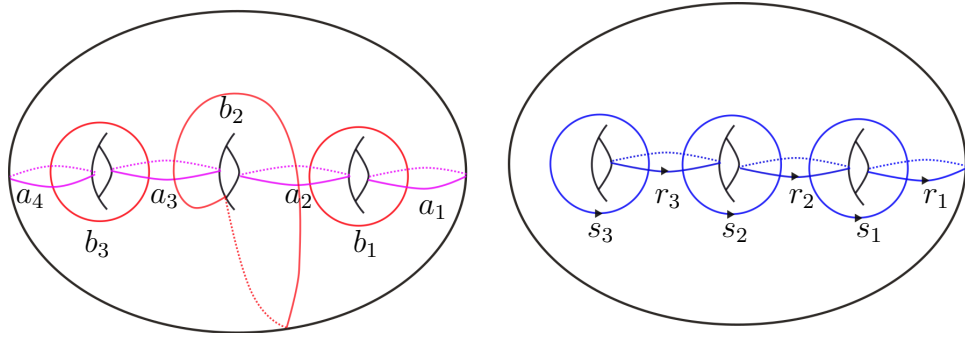


Figure 4.8: Left: Penner curves, Right: A basis for the first homology

That is $\det(f_* - Id)$ should be non-zero. Choose a basis $r_1, s_1, \dots, r_3, s_3$ for $H_1(S)$ as in the right side of Figure 4.8. One can directly see that the action of f on homology

is represented by the following matrix W .

$$W = \begin{bmatrix} 0 & 1 & 2 & -1 & -2 & 1 \\ -1 & 2 & 1 & 0 & 0 & 0 \\ -2 & 4 & 4 & -2 & -2 & 1 \\ 1 & -2 & -2 & 2 & 2 & -2 \\ 0 & 0 & 0 & 1 & 2 & -3 \\ 0 & 0 & 0 & 0 & -1 & 2 \end{bmatrix}$$

And $\det(W - Id) \neq 0$. This finishes the proof for genus three surface. For larger g , add extra handles to the left side of the picture and add suitable curves to complete the previous system of filling curves as in Figure 4.9 and Figure 4.10.

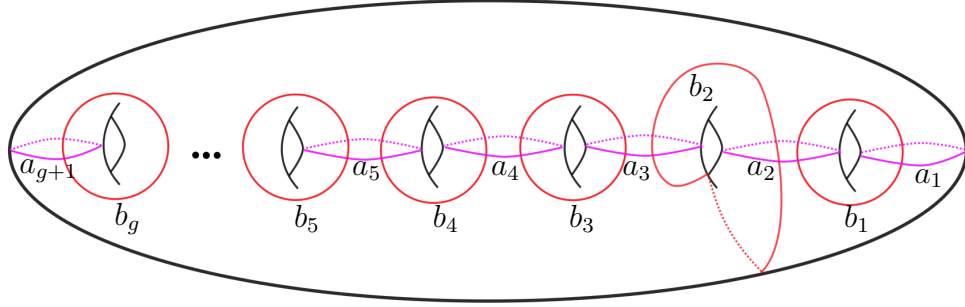


Figure 4.9: Penner curves for the general case

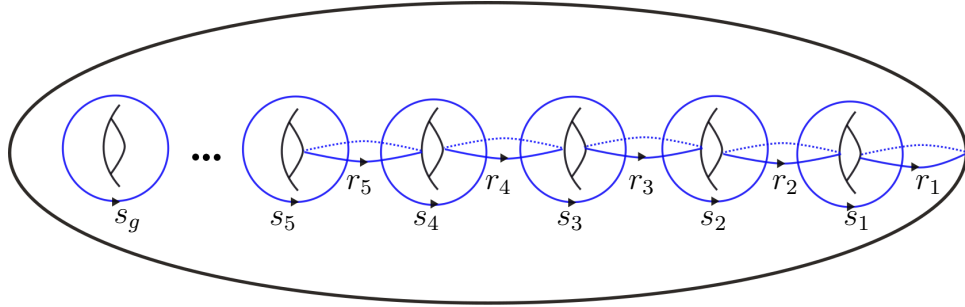


Figure 4.10: A basis for the homology

The map f is defined similarly. Firstly we do twists around the curves $\{a_1, \dots, a_{g+1}\} \setminus \{a_2, a_3\}$. Then we do twists around $\{b_1, \dots, b_g\} \setminus \{b_2\}$. And at

the end we do twists around a_2, a_3 and b_3 . Again computation shows that the action of f on homology is represented by the following matrix V (for $g \geq 6$). Here the empty entries are zero and the star shows repeating pattern. The pattern corresponds to the following for $5 \leq i \leq g-1$.

$$r_i \longmapsto s_{i-1} + r_i - s_i$$

$$s_i \longmapsto -s_{i-1} - r_i + 3s_i + r_{i+1} - s_{i+1}$$

$$V = \left[\begin{array}{cccccccc|cc|cc|cc|cc} 0 & 1 & 2 & -1 & -2 & 2 & 1 & -1 & & & & & 0 & -1 \\ -1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & & & & & 0 & 0 \\ -2 & 4 & 4 & -2 & -2 & 2 & 1 & -1 & & & & & 0 & -1 \\ 1 & -2 & -2 & 2 & 2 & -2 & -1 & 1 & & & & & 0 & 0 \\ & & & & 1 & 2 & -2 & -1 & 1 & & & & 0 & -1 \\ & & & & & -1 & 2 & 1 & -1 & & & & 0 & 0 \\ & & & & & & 1 & 1 & -1 & & & & 0 & -1 \\ & & & & & & -1 & -1 & 3 & 1 & -1 & & 0 & 0 \\ \hline & & & & & & & 1 & 1 & -1 & & & * & * \\ & & & & & & & -1 & -1 & 3 & * & * & * & * \\ \hline & & & & & & & & * & * & * & & 0 & -1 \\ & & & & & & & & * & * & * & 1 & -1 & 0 & 0 \\ \hline & & & & & & & & & 1 & 1 & -1 & 0 & -1 \\ & & & & & & & & & -1 & -1 & 3 & 1 & -1 \\ \hline & & & & & & & & & & 1 & 1 & -2 \\ & & & & & & & & & & -1 & -1 & 3 \end{array} \right]$$

By putting the matrix $V - Id$ in the row reduced form, one can see that it is invertible. We do this in detail. Put the first eight rows in the row reduced form. After this, the first eight rows of $V - Id$ changes to the following.

$$\left[\begin{array}{cccc|cc|cc|cc} 1 & & & & -1 & 1 & & & 0 & 3 \\ & 1 & & & & & & & 0 & 1 \\ & & 1 & & -1 & 1 & & & 0 & 2 \\ & & & 1 & & & & & 0 & 2 \\ & & & & 1 & & & & 0 & 4 \\ & & & & & 1 & & & 0 & 3 \\ & & & & & & 1 & & 0 & 5 \\ & & & & & & & 1 & 0 & 4 \end{array} \right]$$

Let us show the i -th row of the matrix with R_i . For $5 \leq i \leq g$, do the following.

- a) Replace R_{2i-1} with $R_{2i-1} - R_{2i-2}$.
- b) Replace R_{2i} with $R_{2i} + R_{2i-2}$.
- c) Switch R_{2i} and R_{2i-1} .
- d) Replace R_{2i} with $-R_{2i}$.

This process makes the matrix upper triangular, with all the entries on the diagonal equal to ± 1 except the last one that is equal to $-(g+1)$. Therefore the determinant of $V - Id$ is equal to $g+1$ up to sign (for $g \geq 6$). This shows that $V - Id$ has trivial kernel.

□

4.3 Taut foliations on the sutured solid torus

In this section we will prove the following lemma.

Lemma 4.3.1. *Let N be a solid torus with two parallel sutures on its boundary, where each of the sutures goes three times in longitudinal direction and once in meridional direction (see figure 4.11 where we have marked one of the sutures with diamonds in order to make it visually salient.) There is no taut foliation on N whose Euler class assigns zero to the core disk of N .*

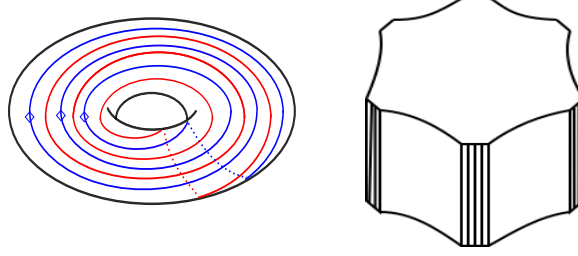


Figure 4.11: Left: Sutured solid torus N , Right: its cross section

Remark 4.3.2. This condition is equivalent to having Euler class zero since the core disk of N is the only homology class whose norm is not equal to zero.

Remark 4.3.3. The same is true for a similar sutured solid torus replacing 3 with any odd number.

Remark 4.3.4. One can think about this sutured manifold as a sutured-disk (6-gon) bundle over the circle (figure 4.11) that is twisted. Having this twist is the obstruction for the existence of such a taut foliation: otherwise, one could take two (4-gon) sutured-disk bundles over the circle with opposite orientations each of which is foliated in the standard way by a 'stack of chairs', and glue them together in the right way to get the desired foliation (see Example 2.0.10).

Definition 4.3.5. Let $\mathcal{F}$ be a transversely oriented foliation on a 3-manifold M and L be a leaf of $\mathcal{F}$. Fix a point $p \in L$. Let $\mathcal{T}(L)$ be the set of free homotopy classes of positive closed transversals for L that start and end at p . Here positive means that its orientation agrees with the transverse orientation of $\mathcal{F}$. We call this set the transversal set of L w.r.t. $\mathcal{F}$ (or just transversal set if there is no ambiguity about

L and $\mathcal{F}$). This set does not depend on the choice of p . To see this, choose another point $q \in L$ and let δ be any oriented arc from q to p . Let γ be a positive transversal based at p and denote by $\hat{\gamma}$ the concatenation of δ , γ and $-\delta$. Then a perturbation of $\hat{\gamma}$ is a positive transversal based at q that is freely homotopic to γ .

In general, this set might be empty, but when $\mathcal{F}$ is taut and M is closed, by definition $\mathcal{T}(L)$ is nonempty for each leaf L . By Novikov's theorem for a Reebless foliation $\mathcal{F}$, every element in $\mathcal{T}(L)$ is (homotopically) non-trivial [25].

Observation 4.3.6. *If $\mathcal{T}(L)$ is nonempty then it is closed under multiplication by elements of $i_*(\pi_1(L))$, where $i : L \rightarrow M$ is the inclusion map.*

Proof. See figure 4.12. □

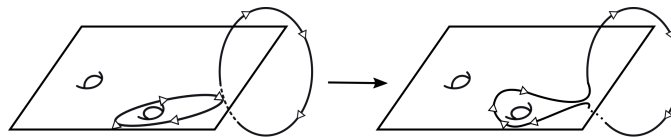


Figure 4.12: The product of a positive transversal and an element of $i_*(\pi_1(L))$ is a positive transversal.

Observation 4.3.7. *$\mathcal{T}(L)$ is closed under product.*

Proof. Let γ_1 and γ_2 be two oriented positive transversals for the leaf L . Assume that γ_1 and γ_2 intersect L at point p . Define the curve γ as the concatenation of the curves γ_1 and γ_2 . □

Corollary 4.3.8. *Let $\mathcal{F}$ be a transversely oriented taut foliation on the sutured solid torus N . Any non-compact leaf L of $\mathcal{F}$ is simply connected.*

Proof. Assume the contrary. Let L be a non-compact leaf such that $\pi_1(L) \neq \{0\}$. Let l be a generator of $\pi_1(N)$. We have $i_*(\pi_1(L)) = \langle l^k \rangle$ for some $k \in \mathbb{N}$ (L is

π_1 -injective by Novikov's theorem [25]). Since L is non-compact and $\mathcal{F}$ is taut, $\mathcal{T}(L)$ is non-empty. Thus we have $l^m \in \mathcal{T}(L)$ for some $m \in \mathbb{Z} \setminus \{0\}$. By observation 4.3.7 we have $l^{km} \in \mathcal{T}(L)$, and by observation 4.3.6 we have $0 = (l^k)^{-m} \cdot l^{km} \in \mathcal{T}(L)$. This contradicts Novikov's theorem therefore L should be simply connected. $\square$

Proof of 4.3.1. We assume there is such a foliation $\mathcal{F}$ and we will get a contradiction. Firstly, we show that *it can be assumed that $\mathcal{F}$ has no compact leaf other than its tangential boundary*. This is because every compact leaf has to be a $\partial_\tau N$ -parallel annulus (isotopic to N_+ or N_-). By Haefliger's theorem [16], the set of compact leaves inside N is closed, so one can choose the one that is farthest from the tangential boundary. Call it L . Because L is boundary parallel, the foliation between L and the tangential boundary is standard and removing it does not change the Euler class of foliation. This leaves us with a taut foliation without compact leaves (other than the tangential boundary) and with the same Euler class.

By corollary 4.3.8, *every non-compact leaf of $\mathcal{F}$ is simply connected*. This implies that *if $\mu \in \text{Homeo}^+(I)$ is the holonomy of transverse boundary of N , then μ has no fixed point except interval endpoints (it is a shift)*, since otherwise a fixed point corresponds to a nontrivial loop inside a non-compact leaf. Note that μ being a shift implies that near $\partial_\tau N$, the leaves spiral around $\partial_\tau N$. Hence if we shave a small neighborhood of the tangential boundary, we obtain a foliation, $\mathcal{G}$, on the solid torus that is transverse to the boundary and has two boundary Reeb components R_1 and R_2 . Each of R_1 and R_2 is parallel to the sutures on ∂N (see figures 4.13 and 4.14). Double this new solid torus along its boundary to obtain a foliation $\mathcal{DG}$ on the manifold $S^2 \times S^1$. The foliation $\mathcal{DG}$ is taut. To see this we should prove that every compact leaf of $\mathcal{DG}$ has a closed transversal. Any compact leaf of $\mathcal{DG}$ (if exists) is of the form DL where L is a leaf of $\mathcal{G}$ intersecting ∂N . Since L intersects ∂N , it intersects R_1 and R_2 as well (recall that μ was a shift map). Hence the core curve of

Reeb component on ∂N is a closed transversal for L . Therefore $D\mathcal{G}$ is a taut foliation on $S^2 \times S^1$. On the other hand, the Euler class of $D\mathcal{G}$ assigns 0 to the S^2 factor since the original foliation was assigning 0 to the D^2 factor (the S^2 factor is the double of the D^2 factor and $2 \times 0 = 0$). This is not possible because any taut foliation on $S^2 \times S^1$ assigns ± 2 to the S^2 factor (has S^2 factor as a leaf up to isotopy [25] [28].) Contradiction.

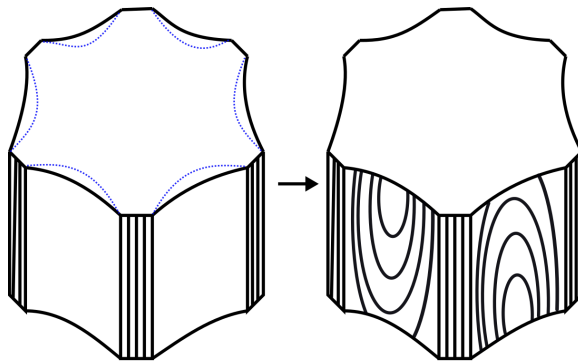


Figure 4.13: sutured manifold N before and after shaving a small neighborhood of tangential boundary. The dashed lines on the left show the neighborhood of $\partial_\tau N$ that will be shaved.

□

4.4 Proof of the fully-marked surface Theorem

Theorem 4.4.1. (*Gabai-Yazdi*) *Let M be a closed hyperbolic 3-manifold, $\mathcal{F}$ be a transversely oriented $C^{\infty,0}$ taut foliation on M , and S be a fully-marked surface in M . Assume that S is the unique (up to isotopy) norm-minimizing surface in its homology class. There exists a $C^{\infty,0}$ taut foliation $\mathcal{F}'$ that has S as a leaf and the plane fields tangent to $\mathcal{F}$ and $\mathcal{G}$ are homotopic.*

Proof of Theorem 4.4.1: Recall that by Thurston's general position after an isotopy of S , the foliation is transverse to S except at finitely many saddle singular-

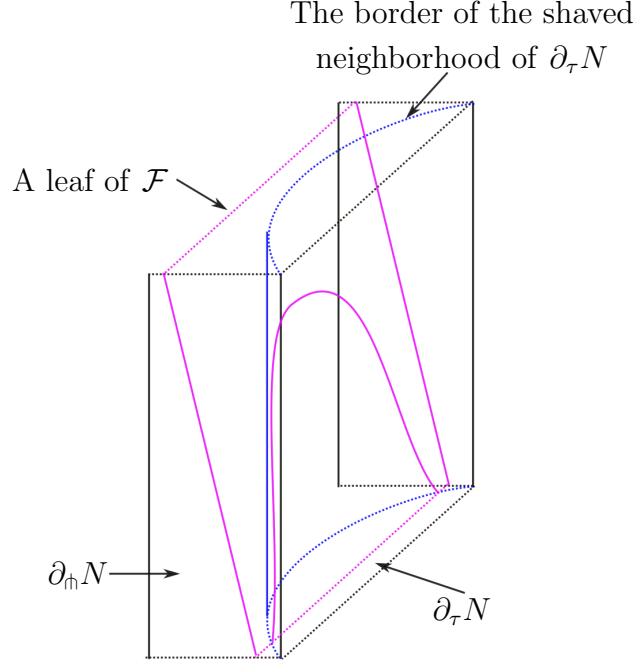


Figure 4.14: The purple surface is a leaf of foliation that spins around $\partial_\tau N$. Therefore if we shave a neighborhood of $\partial_\tau N$, up to the blue curved surface, there will be an induced Reeb component on the new boundary.

ities. Since S is the unique norm-minimizing surface in its homology class, after a possible I -bundle replacement along some of the leaves, S can be isotoped such that the induced foliation on $M \setminus S$ has no separating bad annulus and no bad solid torus (Lemma 4.4.18) without creating any center tangency on S . By Lemma 4.4.3, we can assume that the leaves of $\mathcal{F}$ make arbitrary small angle with S at each point of S . Using the operations

- 1) Smoothing Reeb components
- 2) Smoothing a saddle tangency
- 3) Smoothing along a transverse arc
- 4) Spinning the boundary leaves around
- 5) I -bundle replacement along some of the leaves of $\mathcal{F}|_{(M \setminus S)}$

one can add leaves to the foliation on $M \setminus S$ or do I -bundle replacement along some of the leaves to get a foliation that is tangential on $\partial(M \setminus S) = S_0 \cup S_1$. The term smoothing needs some explanation. Given the manifold $M \setminus S$ together with the

induced foliation on it, smoothing means gluing foliated pieces to the boundary of $M \setminus S$ so that the tangential boundary increases (see section 4.4.1). The above operations can be done in a way that the foliation remains taut after these operations (Lemma 4.4.14) and is homotopic to the original foliation (Lemma 4.4.15). Gluing back S_0 to S_1 gives the desired foliation on M .

□

Remark 4.4.2. We should explain where the assumption 'S being the unique norm-minimizing surface in its homology class' is used. Although all of the above five operations can be done regardless of the assumption on S , in order for the new foliation to be taut we need to rule out the existence of certain annuli and tori (see Lemma 4.4.14). The assumption about S is used to prove that S can be isotoped so that none of these 'problematic' annuli and tori exist (see Lemma 4.4.18).

Lemma 4.4.3. *Let $\mathcal{F}$ be a foliation on M and S be a fully-marked surface such that all tangencies on S have the same sign. For any $\epsilon > 0$, there exists a metric on M such that at every point of S , the angle between the oriented normal vectors to $\mathcal{F}$ and S is at most ϵ .*

Proof. Let g be the initial metric on M and $N(S) \cong S \times [-1, 1]$ be a regular neighborhood of S . Since M is compact and orientable, the tangent bundle, TM , is a trivial bundle (see [21], Exercise 12B). Therefore the restriction of TM to $N(S)$ is also trivial. Fix such a trivialization. Under this trivialization the tangent bundle is just the product $N(S) \times \mathbb{R}^3$ and its unit tangent bundle is $N(S) \times S^2$.

A metric is determined by a triple of orthonormal vectors. For any $s \in S = S \times 0$, let $u(s)$ be the unit normal vector to S . Extend $u(s)$ to a triple of orthonormal vectors $\{u(s), v(s), w(s)\}$, which vary smoothly by s . This is possible since the tangent bundle is trivial. We can assume that the same triple $u(s), v(s), w(s)$ is an orthonormal basis at each point $s \times t$ for $t \in [-1, 1]$. Let $n(s)$ be the unit normal to the foliation. By assumption, we know that $n(s) \neq -u(s)$ for all $s \in S$. We want to make the angle

between $u(s)$ and $n(s)$ very small by changing the orthonormal basis for the metric. Let $V(s)$ be the oriented orthogonal plane to $u(s)$, i.e., the plane spanned by $v(s)$ and $w(s)$.

The first step is to move $V(s)$ so that it has $u(s)$ and $n(s)$ on the same side, i.e. making the angle $\angle(u(s), n(s))$ less than $\frac{\pi}{2}$. Let $V'(s)$ be the oriented orthogonal plane to the segment connecting $n(s)$ and $-u(s)$ (See Figure 4.15). There is a smooth family of planes $W(s, t)$ interpolating between $V(s)$ and $V'(s)$.

$$W(s, 0) = V(s), W(s, 1) = V'(s)$$

This is because each of these oriented planes are determined by their unit normal vector, which can be identified by a point in a sphere minus a point. Since $S^2 - \{\text{point}\}$ is an open disk, one can choose such a path for each plane $V(s)$ consistently. This is schematically shown in Figure 4.15 in one dimension lower i.e. V has been drawn as a line.

The second step is to scale $u(s)$ in order to make the angle $\angle(u(s), n(s))$ arbitrary small. This is schematically shown in Figure 4.16 in one dimension lower. By projecting $v(s)$ and $w(s)$ on $W(s, t)$ and then rescaling to have norm one (with respect to the initial metric g), a basis for $W(s, t)$ can be found, which varies smoothly with respect to s and t . Call them $v(s, t)$ and $w(s, t)$. Let K be a large positive number. Define the new metric g' on M as the following: An orthonormal basis for g' coincides with one for g outside of $N(S) \cong S \times [-1, 1]$. For $s \in S$ and $\frac{1}{2} \leq |t|$, let an orthonormal basis for g at $s \times t$ consist of $u(s)$ together with an orthonormal basis for $W(s, 2 - 2|t|)$. For $|t| \leq \frac{1}{2}$ let an orthonormal basis for g consist of an orthonormal basis for $W(s, 1) = V'(s)$ together with the vector $\frac{1}{1+(1-2|t|)K}u(s)$. $\square$

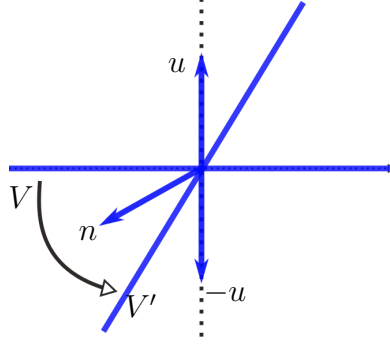


Figure 4.15: Making the angle $\angle(u, n)$ less than $\frac{\pi}{2}$ by rotating the plane V

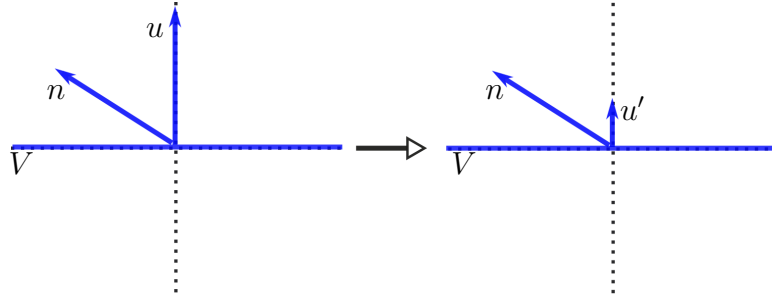


Figure 4.16: Making the angle arbitrary small by scaling the vector u

4.4.1 Smoothing the foliation along S

In order to introduce the smoothing operations, we need to define the vector field ν on the boundary of $M \setminus S$. By Lemma 4.4.3 after choosing a suitable metric we can assume that the leaves of $\mathcal{F}$ are almost tangent to S . Let n be the normal to the foliation and S_1 (respectively S_0) be the copy of S in $\partial(M \setminus S)$ that the vector n points out of (respectively into) the manifold $M \setminus S$.

Definition 4.4.4. At each point $x \in S_1$, define the vector ν as the projection of the vector $-n$ onto the tangent plane of S_1 at x .

Therefore ν is zero exactly at points where n and normal to S_1 coincide. Intuitively ν points in the direction that leaves go down (with respect to the transverse orientation of $\mathcal{F}$).

We will define certain foliated tori P_1 and P_2 as follows. Let $H \cong [0, 1] \times S^1$ be an annulus and consider $H \times I$ equipped with a foliation transverse to the I -factor

whose holonomy is a shift map. Here a shift map is a homeomorphism of the interval whose fixed points are exactly the interval endpoints. Call this foliated solid torus P . Let θ be an arc in $[0, 1] \times I$ according to the Figure 4.17. An equation for θ can be $(t, \frac{1}{2} - 2(t - \frac{1}{2})^2)$ where $0 \leq t \leq 1$. $\theta \times S^1$ is an annulus that splits P into two pieces. We call the bottom piece P_1 and the top piece P_2 . Note that since the holonomy is a shift map, there is an induced Reeb component on the common boundary of P_1 and P_2 . See figure 4.18 for a picture of P_1 and P_2 and the induces Reeb component on the common boundary

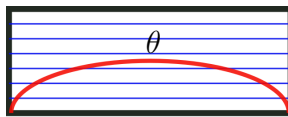


Figure 4.17: The curve θ

We show how to smooth the foliation along S_1 in $M \setminus \setminus S$ by adding leaves. The construction for smoothing along S_0 is similar (essentially by switching from ν to $-\nu$).

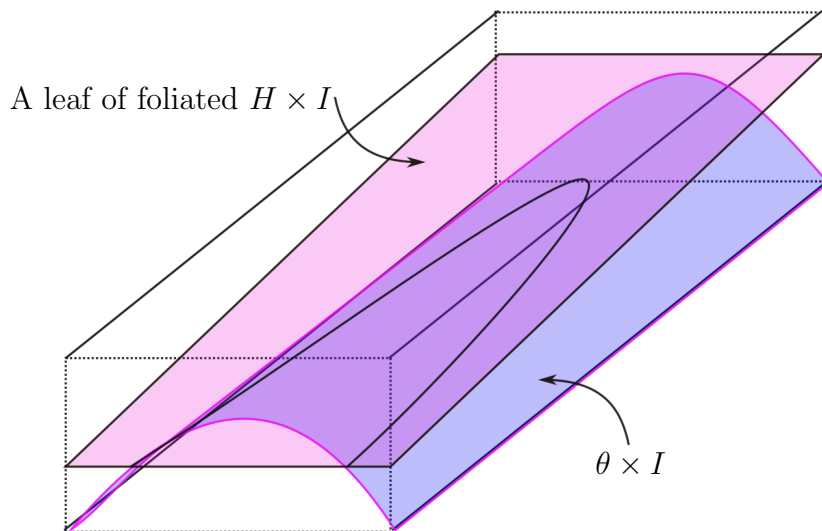


Figure 4.18: Foliated tori P_1 and P_2 . The picture shows how a leaf of the foliation on $H \times I$ intersects $\theta \times I$ in a Reeb component.

Operation 4.4.5. (*Spinning a boundary Reeb component*) Let R be a Reeb component in $S_1 \subset \partial(M \setminus \setminus S)$. Let b_1 and b_2 be boundary leaves of R and call the leaves of $\mathcal{F}$ containing them L_1 and L_2 respectively. We consider two cases:

1) If the vector ν points into R then glue P_1 to $M \setminus \setminus S$ along R . Intuitively this is the same as the following: cap off R with an annulus A that connects L_1 and L_2 . Spin the leaves that intersect the interior of R in the direction of ν such that they converge to the annulus A .

2) If the vector ν points out of R then glue P_2 to $M \setminus \setminus S$ along R . Intuitively this is the same as the following: let B be the annulus containing R . Push B slightly outside of $M \setminus \setminus S$ to get B_1 . Spin the leaves that intersect the interior of R in the direction of ν so that they converge to B_1 .

See Figure 4.19 where the top and bottom pictures correspond to the cases (1) and (2) respectively.

Remark 4.4.6. We should explain why we are smoothing the Reeb components in two different ways according to the direction of the vector field ν . Let $\mathcal{L}$ be a one dimensional foliation transverse to $\mathcal{F}$ on $M \setminus \setminus S$. After choosing a suitable metric we assume that $\mathcal{L}$ makes small angles with S_1 at each point of S_1 . One can think about the added leaves along S_1 as lying in

$$\hat{M} = M \cup (S_1 \times [0, \epsilon])$$

We want to add leaves to the foliation along S_1 such that the leaves are still transverse to the natural extension of $\mathcal{L}$ to $\hat{M}$. This will be used later in proving that the plane fields tangent to the foliations are homotopic. At this point we need to pay attention to the direction of ν in order to choose the right way of smoothing the Reeb component.

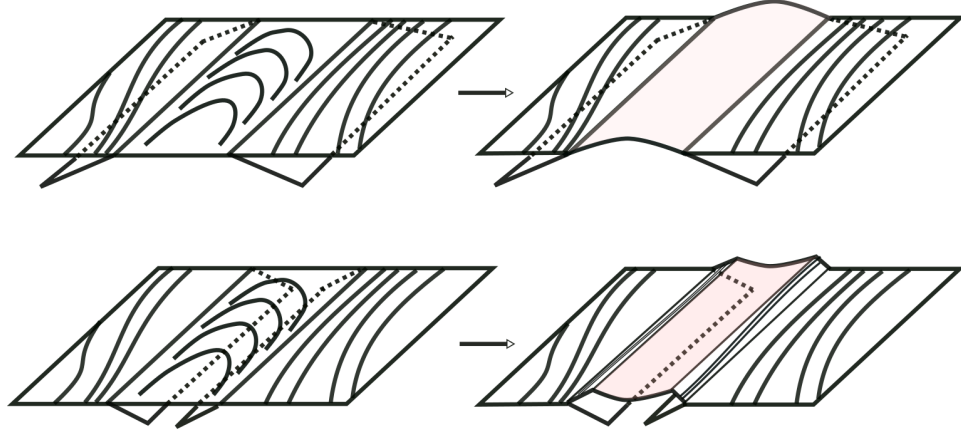


Figure 4.19: Smoothing a Reeb component

Operation 4.4.7. (*Smoothing the leaves near a saddle tangency*) For each saddle tangency p on $S_1 \subset \partial(M \setminus S)$, glue leaves according to the Figure 4.20. This replaces a neighborhood of p with a tangential rectangle. The vector field ν points out of the rectangle along two opposite sides and points in along the other two (Figure 4.21).

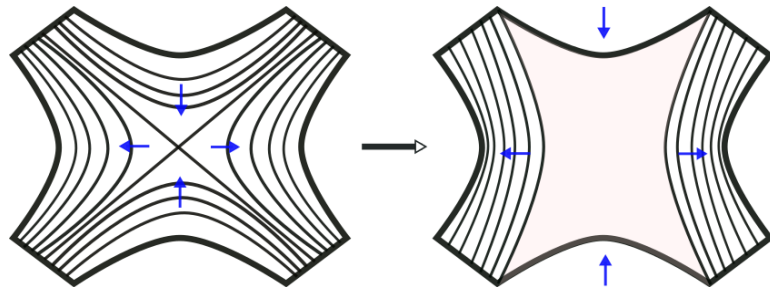


Figure 4.20: Smoothing the leaves near a saddle tangency on S_1 . The blue arrows show the direction of the vector ν .

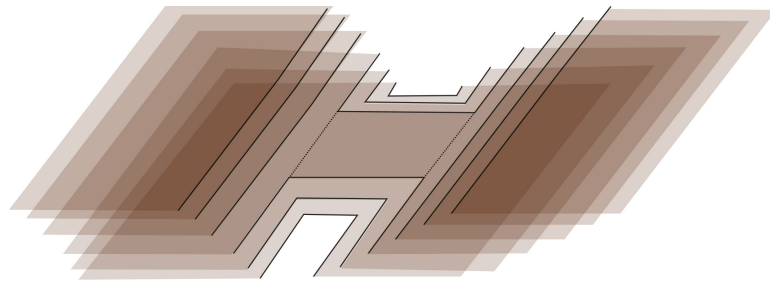


Figure 4.21: A neighborhood of a saddle point after being smoothed

If there are Reeb components on $S_1 \subset \partial(M \setminus \setminus S)$, spin these Reeb components and replace them with tangential annuli on S . For each saddle point, smooth the leaves around it. This makes the leaves tangent to S along a number of rectangles and annuli. The goal is to extend the tangential part and eventually take over the whole surface. Let T be the portion of the surface where the foliation is tangential. Each point in the boundary of T comes with the vector ν , which can point in or out of T . It is also possible that on a single boundary component of T the direction of ν changes from pointing in to pointing out or vice versa finitely many times. A **node** is a point on ∂T at which the direction of ν changes. In this case the boundary component breaks into an even number of arcs. To sum up, ∂T is a union of arc or circle pieces and each of them are equipped with a vector ν which can point in or out of T . A transverse arc is a properly embedded arc in $S - T$, transverse to $\mathcal{F}|_S$ that connects two point (which are not nodes) on ∂T together. We orient a transverse arc using the orientation of ν . Call a transverse arc γ **good** if the beginning and end point of γ lie in arc pieces of ∂T . The next lemma shows that if we find a good transverse arc, then we can extend the tangential part of S .

Operation 4.4.8. (*Smoothing the leaves along a good transverse arc*) Let T be the tangential part of S and γ be a good transverse arc. Suppose the beginning and end point of γ lie on b and c respectively, where b and c are arc pieces of ∂T . Consider a small tubular neighborhood T_1 of γ in $S - T$. The operation extends the tangential part of S to $T' := T \cup T_1$ in such a way that the nodes of T' are the nodes of T minus the endpoints of c (Figure 4.24). This can be done as follows..

By adding leaves as in Figure 4.22 we can make the leaves tangent along T_1 such that the endpoints of c are removed from the set of nodes and there is a new corner along c instead (Figure 4.23). Let L be the leaf of $\mathcal{F}$ adjacent to c and L_1 be the connected component of $L \setminus \setminus c$ on the tangential side of c . Do I -bundle replacement along L_1 with product foliation. Note that L_1 might intersect γ and as a result might

add product pieces in the foliation along T_1 . Glue the I -bundle over L_1 to the vertical boundary of the foliation along T_1 , restricted to c . The constructed foliation has the qualitative description as in Figure 4.24.

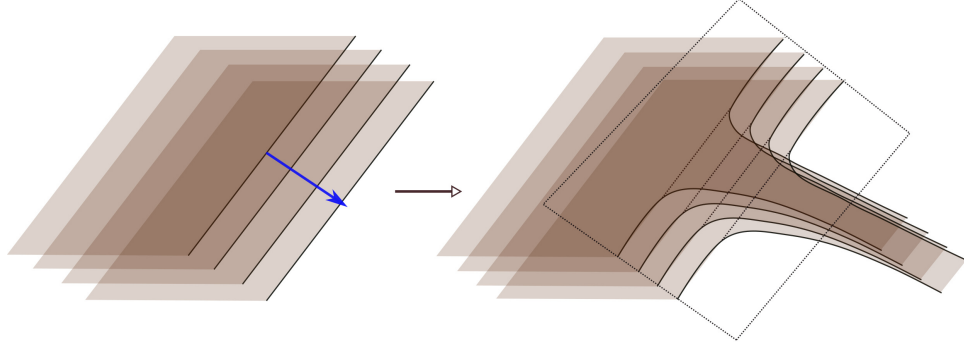


Figure 4.22: Smoothing the leaves along a transverse arc

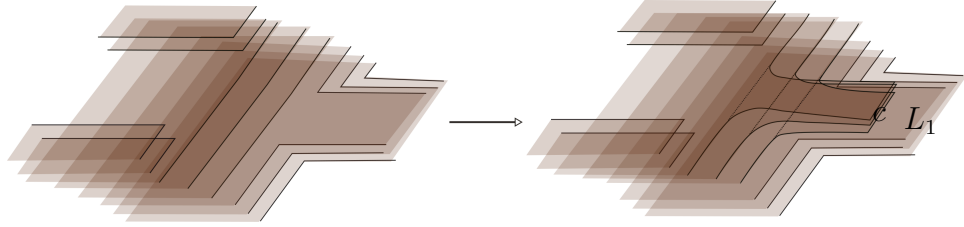


Figure 4.23: Smoothing the leaves along a good transverse arc

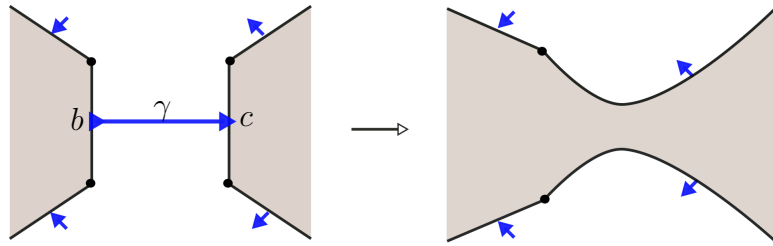


Figure 4.24: The new tangential subsurface after smoothing the leaves along a good transverse arc

Lemma 4.4.9. *One can do I -bundle replacement along some of the leaves of $\mathcal{F}_{|(M \setminus S)}$, together with an arbitrary small isotopy of S such that no two singularities of $\mathcal{F}_{|S}$ are connected by a separatrix (also no separatrix from a singularity to itself).*

Proof. We essentially repeat the proof from [5] (Theorem 7.1.10) with some adjustments. Let $p_1, \dots, p_n$ be the saddle singularities on S . Consider a small standard neighborhood U of p_1 where the surface S can be seen as the graph of function $z = x^2 - y^2$ and the foliation $\mathcal{F}$ is by horizontal planes. Choose smaller neighborhoods $W \subset V \subset U$. Let ϕ be a continuous function such that $\phi|_W = 1$ and $\phi|_{(U-V)} = 0$. Isotope S around U such that S is the graph of function $z' = z + \epsilon\phi$ where ϵ is a small number. This sends p_1 to an arbitrary close leaf of $\mathcal{F}$. Therefore by repeating this argument for other p_i 's one can make sure that $p_1, \dots, p_n$ lie on different leaves of $\mathcal{F}$. In other words no two of p_i 's are connected by a separatrix inside $\mathcal{F}|_S$. The next step is to get rid of a separatrix from a singularity to itself. Let γ be a separatrix from p_1 to itself and L be the leaf of $\mathcal{F}$ containing γ . γ is homotopically non-trivial in S since if D is a disk bounding γ in S , there should be a center tangency inside D by the Poincaré-Hopf formula. S is π_1 -injective in M therefore γ does not bound a disk in M either. In particular γ is homotopically non-trivial in L as well. Do an I -bundle replacement along L such that the germinal holonomy on either side of γ in S has no fixed points. Now if we repeat the previous argument by changing the height function z to $z' = z + \epsilon\phi$ around p_1 , there is no separatrix from p_1 to itself. Repeat this with other singularities. $\square$

By Lemma 4.4.9 we can assume that no two singularities of $\mathcal{F}|_S$ are connected by a separatrix. Now we can prove the following:

Lemma 4.4.10. *Assuming that no two singularities are connected by a separatrix, one can start from the union of rectangles and smooth the foliation along good transverse arcs or smooth Reeb components until the tangential part becomes the whole surface minus a union of annuli. Furthermore on each of these annuli the foliation is by a suspension.*

Proof. Recall the standing assumption that no two singularities of $\mathcal{F}|_S$ are connected by a separatrix. Smooth the foliation near each singularity and replace them by tangential rectangles. By choosing the rectangles small enough, we can assume that there is no trajectory connecting the nodes except those in ∂T . Similarly we can assume there is no closed trajectory near the nodes. By Haefliger's theorem, closed leaves of $\mathcal{F}|_S$ appear in finitely many packets. These packets should be disjoint from nodes by our assumption. Decompose the surface S along packets to get finitely many sub-surfaces with boundary. Each boundary of a sub-surface comes with a normal vector ν , which point either in or out of the sub-surface. Each sub-surface is fully-marked and has no simple closed trajectory in the interior. It is enough to solve the smoothing problem for each of these sub-surfaces. By this we mean that the sub-surface becomes tangential except along boundary components where the vector ν points out of the sub-surface. Along these boundary components there will be a holonomy (Figure 4.25).

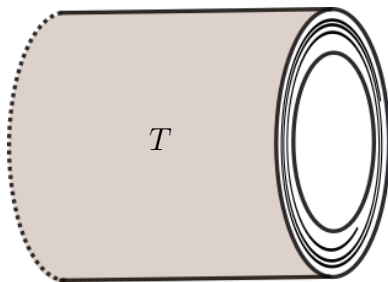


Figure 4.25: holonomy on the boundary of a tangential sub-surface

Let F_1 be one of the sub-surfaces and T_1 be its tangential part. Assume $F_1 - T_1$ is not a union of annuli. We show that there exists a good transverse arc. ∂T_1 should have at least one arc piece. This is a consequence of the Poincaré-Hopf theorem because otherwise $F_1 - T_1$ has admitted a foliation without any singularities inside or on the boundary and so $\chi(F_1 - T_1) = 0$, which implies that $F_1 - T_1$ is a union of annuli. Pick a point q on an arc piece b of ∂T_1 where ν points into T_1 . Define $\mathcal{A}$ as the set of points z where there exist a transverse arc from z to q . By Lemma 4.4.11 $\bar{\mathcal{A}}$

is a sub-surface of F_1 . Consider the connected component c of $\partial\mathcal{A}$ which contains b . If this component contains an arc piece of ∂T_1 where ν points out of T_1 then we have found the desired good transverse arc. If no such piece exists then $c - b$ contains at least one trajectory (not included in ∂T_1) connecting two nodes, which is not possible by our assumption. Let γ be a good transverse arc. Smooth the foliation along γ to extend the tangential part. If any Reeb component on F_1 is created, smooth it as in operation 4.4.5. Let $a(T)$ be the number of arcs in ∂T where ν points into T (half of the number of total arcs). Smoothing along γ reduces $a(T_1)$ by one and smoothing a Reeb component does not change it (we still call them F_1 and T_1 by abuse of notation). Also these operations do not create new trajectories connecting nodes except those in ∂T_1 . Therefore we can do this operation until $a(F_1) = 0$, which implies that $F_1 - T_1$ should be a union of annuli. $\square$

Lemma 4.4.11. *Assume we have smoothed Reeb components on S . Fix a point q in ∂T where ν points into T . Define $\mathcal{A}$ to be the set of points $z \in S - T$ such that there exists a transverse arc from z to q . The closure of $\mathcal{A}$ is a subsurface of S whose boundary is a union of simple closed curves where nodes are allowed on $\partial\mathcal{A}$.*

Proof. $\mathcal{A}$ is open and saturated. Suppose a sequence of points $x_n \in \mathcal{A}$ converges to a point $x \in \partial\mathcal{A}$. We can assume that $x \notin \mathcal{A}$ otherwise the analysis would be trivial. Let L be the leaf of $\mathcal{F}|_S$ passing through x . For each point $\bar{x} \in L$ there is a sequence of points of $\mathcal{A}$ converging to $\bar{x}$ as well. The aim is to prove that $\bar{L}$ is a simple closed curve or arc. The argument is by contradiction. If $\bar{L}$ is not a simple closed curve or arc then there is a chart $\mathcal{C}$ in the surface with the property that L passes through $\mathcal{C}$ infinitely many times. Therefore there is a point $\bar{x} \in L$ and a sequence of points $y_n \in L$ converging to $\bar{x}$ such that the points $\bar{x}, y_1, y_2, \dots$ lie on different leaves of the chart $\mathcal{C}$. Choose two points y_n and y_m with $n, m \gg 0$ and suppose the vector field ν points from y_n toward y_m . Choose a point $p \in \mathcal{A}$ very close to y_m . Let γ be a transverse arc from p to q and α be a transverse arc from y_n to p . Then $\alpha \cup \gamma$ is a

transverse arc from y_n to q implying that $y_n \in \mathcal{A}$. We came to a contradiction so $\bar{L}$ should be a simple closed curve or arc. The same argument shows that $\partial\mathcal{A}$ can have only finitely many components since otherwise, one can use a limit point to get to a similar contradiction. Hence $\partial\mathcal{A}$ is a finite union of simple closed curves where we allow nodes on these curves. This completes the proof. $\square$

Before introducing the next operation, we need a fact about foliations on surface $\times I$ and a Lemma about homeomorphisms of the interval.

Fact: If F is any surface with boundary which is not compact planar and b is a boundary component of F , then there are foliations of $F \times I$ (I is a closed interval), transverse to I factor that have a given holonomy on b and trivial holonomy on all other boundary components. In the remaining case that F is compact planar (not a disk), if b and b' are two boundary components with the induced orientations from F , then there exists a foliation transverse to I factor that has a given holonomy μ on b and μ^{-1} on b' and trivial holonomy on all other boundary components [9].

The next lemma is a modification of Lemma 2.1 in [9].

Lemma 4.4.12. *Suppose u, v are given homeomorphisms of the interval. There exist a homemorphism τ such that τ is conjugate to:*

a) $u\tau^{-1}v$

b) $u\tau v$

c) $u\tau$

d) τv

e) $u\tau^{-1}$

f) $\tau^{-1}v$

Here $u\tau v$ shows concatenation, likewise for $u\tau^{-1}v$.

Proof. a) Identify the interval with $[-1, 1]$. Break this interval into symmetric pieces as:

$$[-1, -\frac{1}{2}], [-\frac{1}{2}, -\frac{1}{3}], \dots, [\frac{1}{3}, \frac{1}{2}], [\frac{1}{2}, 1]$$

define τ to be conjugate to u and v respectively on $[-1, -\frac{1}{2}]$ and $[\frac{1}{2}, 1]$, then define it to be conjugate with v^{-1} and u^{-1} respectively on $[-\frac{1}{2}, -\frac{1}{3}]$ and $[\frac{1}{3}, \frac{1}{2}]$, and continue so on. Finally set $\tau(0) = 0$. $\diamond$

b) Similar to previous part.

c) The constructed homeomorphism in part (b) for $v = \text{Id}$ works for this case.

d,e,f) Similar to part (c). $\square$

Operation 4.4.13. (*Smoothing two-sided corners*) Let $\mathcal{F}_1$ be a foliation on $M \setminus \setminus S$ which only has two-sided corners on $\partial(M \setminus \setminus S) = S_0 \cup S_1$ (Figure 4.26). One can do I-bundle replacement and spin the boundary leaves of $\mathcal{F}_1$ to obtain a foliation on $M \setminus \setminus S$ that is tangential to the boundary.

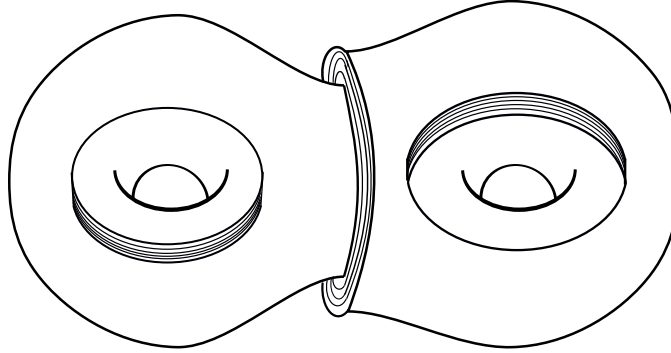


Figure 4.26:

Proof. Let A be an annulus corresponding to a two-sided corner with holonomy μ , and let b be its bottom boundary curve. We show how to resolve this two-sided corner. Let M' be $M \setminus \setminus S$ cut along the curve b and L be the leaf of M' adjacent to b which is on tangent boundary side of b (see Figure 4.27). We do an I-bundle replacement

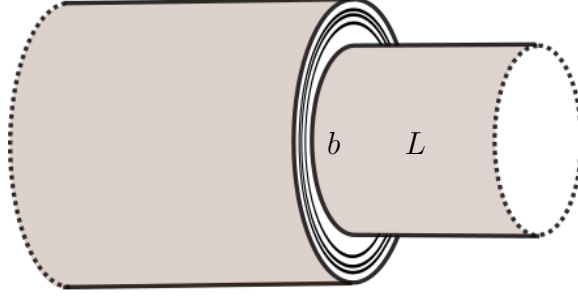


Figure 4.27:

for L and foliate it to have certain holonomies on different boundary components, in such a way that the holonomy of thickened b matches with the new holonomy of A . Then glue them together to resolve this two-sided corner. Consider three different cases:

First) L is not compact planar. Let the product foliation have holonomy τ on b and identity on all other boundary components, where $\tau = \mu'$ and μ' can be obtained from μ in the following way: for each boundary component of L in A , replace that circle by a thickened circle/line with trivial foliation. In other words we are doing the Denjoy blow-up. Write μ' for the new holonomy of A .

Second) L is compact planar but has some boundary component c disjoint from A . Let the product foliation have holonomy τ on b and τ^{-1} on c and identity on all other ones. Since c is disjoint from A , this is similar to the previous case.

Third) L is compact planar and all of its boundary components intersect A . Let c be one such boundary component. Choose the product foliation so that it has holonomy τ on b and τ^{-1} on c and the identity on all other ones. Choose τ such that τ is conjugate to $f'\tau g'$ or $\tau\mu'$ depending on whether $c = b$ or not. Here f, g are holonomies of the two parts of A separated by c . Also f', g' can be obtained from f, g similarly to the way μ' was obtained from μ in the first part.

By repeating the same procedure for other two-sided corners, we can smooth the foliation along $\partial(M \setminus S) = S_0 \cup S_1$. □

Lemma 4.4.14. *Assume that there are no separating bad annuli and no bad solid tori. The foliation remains taut after smoothing along S .*

Proof. Throughout the smoothing we have used the following operations:

- 1) Smoothing Reeb components
- 2) Smoothing a saddle tangency
- 3) Smoothing along a good transverse arc
- 4) Spinning the leaves around
- 5) I-bundle replacement along some of the leaves of $\mathcal{F}|_{(M \setminus S)}$.

Let L be a leaf of the induced foliation on $M \setminus S$. In the absence of separating bad annuli and bad solid tori, L has a transversal γ going from S_0 to S_1 (Lemma 4.4.23). Let $\hat{M}$ be the manifold obtained by I -bundle replacement along some of the leaves and adding product pieces to the boundary components of $M \setminus S$:

$$\hat{M} = (M \setminus S) \cup (S_0 \times [0, \epsilon]) \cup (S_1 \times [0, \epsilon])$$

Obviously $\hat{M}$ is homeomorphic to $M \setminus S$. Since in each of the first four operations we are adding leaves, we can think about the added leaves as lying in $(S_0 \times [0, \epsilon]) \cup (S_1 \times [0, \epsilon])$ and the smoothed surfaces at the end will be $S_0 \times \epsilon$ and $S_1 \times \epsilon$. If $\hat{\gamma}$ is the natural extension of γ to $\hat{M}$ then $\hat{\gamma}$ has the same property as γ did i.e. it hits L transversely (here we use the fact that S and $\mathcal{F}$ make small angles at each point of S and all of above operations can be done in a way that preserves this property). Therefore after gluing back $S_0 \times \epsilon$ to $S_1 \times \epsilon$, $\hat{\gamma}$ gives rise to a closed transversal for L . We have shown that each leaf of $\mathcal{F}'$ that is an extension of a leaf of $\mathcal{F}$ has a closed transversal. Therefore each leaf of $\mathcal{F}'$ has a closed transversal. Hence $\mathcal{F}'$ is taut. $\square$

Lemma 4.4.15. *The new foliation is homotopic to the original one (as plane fields).*

Proof. Let $\mathcal{L}$ be a one-dimensional foliation transverse to $\mathcal{F}$. By adjusting the metric on M we can assume that along S , the direction of $\mathcal{L}$ and the normal vector to S differ by an arbitrary small amount (less than ϵ). Similar to the previous Lemma let $\hat{M}$ be the manifold obtained by I -bundle replacement along some of the leaves and adding product pieces to the boundary components of $M \setminus \setminus S$. $\hat{M}$ is homeomorphic to M . Let $\hat{\mathcal{L}}$ be the extension of $\mathcal{L}$ in $\hat{M}$. The new foliation is transverse to $\hat{\mathcal{L}}$ as well. Since the tangent bundle of M is trivial, one can think about $\mathcal{F}$ and $\mathcal{F}'$ (more precisely their normal bundles) as sections of a D^2 bundle over M . Here D^2 is identified with the set of directions that make angle less than $\frac{\pi}{2}$ with the direction of $\mathcal{L}$. Since D^2 is contractible, there is a homotopy between $\mathcal{F}$ and $\mathcal{F}'$ (not necessarily through integrable plane fields). $\square$

As a final remark, if the constructed foliation is not $C^{\infty,0}$, isotope it to a $C^{\infty,0}$ foliation using Calegari's theorem [1].

4.4.2 Isotoping out certain annuli

Definition 4.4.16. A **bad annulus** is a properly embedded leaf $L \subset M \setminus \setminus S$ that is homeomorphic to an annulus with both boundary components lying on the same copy of S in $\partial(M \setminus \setminus S) = S_0 \cup S_1$.

By a separating bad annulus, we mean one that is separating in $M \setminus \setminus S$.

Definition 4.4.17. A **bad solid torus** is a solid torus B in $M \setminus \setminus S$ which is bounded by non-separating bad annuli U_i 's together with annuli subsurfaces A_j 's of $S_1 \cup S_0$ (See Figure 4.31). Furthermore, the normal to the foliation $\mathcal{F}$ points out of B along all U_i 's.

Lemma 4.4.18. *Let M be a closed hyperbolic 3-manifold, $\mathcal{F}$ be a taut foliation on M and S be a fully-marked surface. Assume that any norm-minimizing surface that is homologous to S , is isotopic to S . One can do I -bundle replacement along some of*

the leaves and isotope S such that the induced foliation on $M \setminus S$ has no separating bad annulus and no bad solid torus and there is no center tangency on S .

Lemma 4.4.19. *Let M be a closed hyperbolic 3-manifold, $\mathcal{F}$ be a taut foliation on M and S be a fully-marked surface such that there is no center tangency on it. A bad annulus is separating in $M \setminus S$ if and only if its boundary curves are parallel in $\partial(M \setminus S) = S_0 \cup S_1$.*

Proof. Let U be a bad annulus and $\partial U = b_1 \cup b_2$. If b_1 and b_2 are parallel then they bound an annulus A in $\partial(M \setminus S)$. $A \cup U$ is a torus that has to bound a solid torus since M is hyperbolic. Therefore U is separating. To prove the other direction, assume U is separating a submanifold Q . Let E be the part of $S_0 \cup S_1$ that is cut by Q . Since all singularities on S have the same sign, Lemma 4.4.3 implies that there is a metric on M such that S and $\mathcal{F}$ make small angles at every point of S . An application of the Poincaré-Hopf formula for the normal vector to $\mathcal{F}$ on the manifold Q shows that:

$$\chi(E) \pm \chi(U) = 0$$

Hence E is an annulus and b_1 is parallel to b_2 . The fact that E is an annulus will be used in the proof of Lemma 4.4.18. $\square$

Definition 4.4.20. A **string** is an annulus $S^1 \times [0, 1]$, lying inside a leaf of $\mathcal{F}$ such that the interval direction can be decomposed as $0 = t_0 < t_1 < \dots < t_k = 1$ and each of $S^1 \times [t_i, t_{i+1}]$ is an annulus with boundary components on S . If the t_i 's are chosen such that that none of $S^1 \times (t_i, t_{i+1})$ intersect S then the **height** of string is defined to be equal to k and $S^1 \times [t_i, t_{i+1}]$ are called the pieces of the string. A **maximal** string is one that can not be extended to a string of larger height. A **packet of strings** is defined similarly. A packet of strings intersects S in annuli that are foliated as suspensions. A single string is also considered as a special case of a packet. Since

packets of strings are special classes of sutured manifolds, it makes sense to talk about their tangential/transversal boundary.

Lemma 4.4.21. *Given $\mathcal{F}$ and S , there exists a number K such that the height of any string is at most K .*

Proof. By Lemma 4.4.9, after an I -bundle replacement and a small isotopy of S we can assume that in the induced foliation on S no two singularities are connected by a separatrix. By Haefliger's theorem, there are finitely many packets of leaves in $M \setminus \setminus S$ which contain all annuli leaves of $M \setminus \setminus S$. Let n be the number of these packets and set $K = n$. If a string $S^1 \times [0, 1]$ has height $k > n$ then at least two of the pieces of the string lie in the same packet. Following the portion of string between them we obtain an annulus A lying in a leaf such that $\partial A = b_1 \cup b_2$ lies on S and the induced foliation on S between b_1 and b_2 is a suspension foliation of an annulus B . Therefore $A \cup B$ is an embedded annulus which has to bound a solid torus since M is hyperbolic. Hence the leaf containing A can not have a closed transversal. Contradicting the assumption that $\mathcal{F}$ is taut. $\square$

Lemma 4.4.22. *There exist finitely many packets of strings that*

a) include all maximal strings.

b) only intersect on their tangential boundary, i.e., for any two string packets P_1 and P_2 we have $P_1 \cap P_2 \subset \partial_\tau P_1 \cap \partial_\tau P_2$.

*We call such a family of string packets a **string covering**.*

Proof. Again possibly after an I -bundle replacement and a small isotopy of S , there are finitely many packets of leaves of $M \setminus \setminus S$ that cover all annuli leaves of $M \setminus \setminus S$. Call this family $\mathcal{M}$. We construct the string covering inductively. Let Q be one of the packets of $\mathcal{M}$ and consider $\partial_{\mathfrak{n}} Q$. Looking at other packets of $\mathcal{M}$, we can decompose $\partial_{\mathfrak{n}} Q$ into finitely many pieces such that each piece lies in the transverse boundary of one element of $\mathcal{M}$. Decompose Q compatible with this decomposition of $\partial_{\mathfrak{n}} Q$ and

enlarge each piece of Q using other packets of $\mathcal{M}$. This gives a number of string packets of length at most three which satisfy condition (b). We repeat this process with each of these packets. This process terminates since the length of strings are uniformly bounded. If any packet of $\mathcal{M}$ is left, we do the same process with that as well until all packets of $\mathcal{M}$ are used. It is easy to see that conditions (a) and (b) are satisfied. $\square$

Proof of Lemma 4.4.18: For a string covering $\mathcal{C}$, let $\text{Height}(\mathcal{C})$ be sum of the heights of string packets in $\mathcal{C}$. Define the following complexity function for the pair $(\mathcal{F}, S)$:

$$C = (c_1, c_2)$$

where c_1 is the number of Reeb components on S and

$$c_2 = \min\{\text{Height}(\mathcal{C}) \mid \mathcal{C} \text{ is a string covering}\}$$

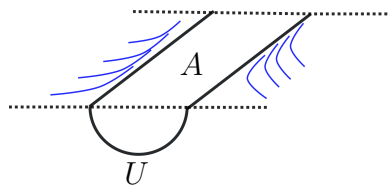


Figure 4.28: Separating annulus U

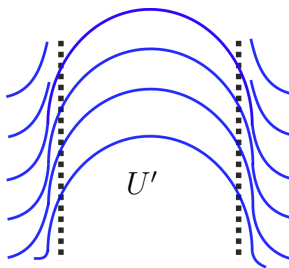


Figure 4.29: Induced foliation on U' (the area between dashed lines)

Equip C with the lexicographic order, i.e.,

$$(c_1, c_2) \leq (d_1, d_2) \iff c_1 < d_1 \quad \text{or} \quad c_1 = d_1, c_2 < d_2$$

We show that if there exists a separating bad annulus or a bad solid torus then one can replace S with a new surface S' , isotopic to S , such that the complexity function C for the pair $(\mathcal{F}, S')$ is less than the corresponding one for $(\mathcal{F}, S)$. Therefore after repeating this finitely many times there should be no separating bad annulus and bad solid torus. Firstly we outline the argument for a separating bad annulus.

Recall that elements of $\mathcal{M}$ were packets of leaves of $M \setminus S$ and their union include all annuli leaves. Consider a maximal packet P in $\mathcal{M}$ such that components of $\partial_{\text{in}} P$ lie on the same copy of S and are parallel (Figure 4.28). Let A be the annulus portion of the surface which is cut by P (A includes the transverse boundary of P). $P \cup A$ is a torus that should bound a solid torus since M is hyperbolic. Let U be the outer leaf of P and replace S with $S - A + U$ and push it out of P slightly to make it disjoint from U . Call this new surface S' . S' is homologous to S since $U - A$ bounds a solid torus. Hence S' is isotopic to S . We show that the complexity function for the pair $(\mathcal{F}, S')$ is less than the one for $(\mathcal{F}, S)$. First we examine what happens to the number of Reeb components. Note that there is at least one Reeb component on A since otherwise the transverse orientation of U would be inconsistent. After replacing S with S' all the Reeb components on A disappear. We show that at most one new Reeb component can be created (therefore c_1 is non-increasing). Let $\partial U = b_1 \cup b_2$. Since P was a maximal packet, the germinal holonomy of b_i on the side not contained in A can not have fixed points except for the origin. Since b_1 and b_2 are freely homotopic in U they have the same germinal holonomy. Let U' be the portion of S' obtained from U after pushing out. The induced foliation on U' is as in Figure 4.29. Let R be a Reeb component of S' . If $R \cap U' = \emptyset$ then R is a Reeb component of S as well. If $R \cap U' \neq \emptyset$ then the leaves in U' can not be the boundary leaves of R since they have a closed transversal. Therefore all of them should be part of interior leaves of R . The only ways that U' can be completed to a Reeb component are as in Figure 4.30. In the first scenario, there are closed trajectories $b'_1, b'_2 \subset S - A$ parallel

to b_1 and b_2 respectively such that the induced foliation on the annulus connecting b_i and b'_i is a suspension of a shift homeomorphism for $i = 1, 2$. In the second scenario, there are closed trajectories $b'_1, b'_2 \subset S - A$ parallel to b_1 and b_2 respectively such that the induced foliation on the annulus connecting b_i and b'_i is a Reeb component for $i = 1, 2$. Note that in the second scenario, the number of Reeb components still decreases since two Reeb components are replaced with one new Reeb component. So the only way that c_1 does not decrease is that there should be exactly one Reeb component on A and the induced foliation in a tubular neighborhood of A is as described in the first scenario. We show that assuming the first scenario, c_2 decreases.

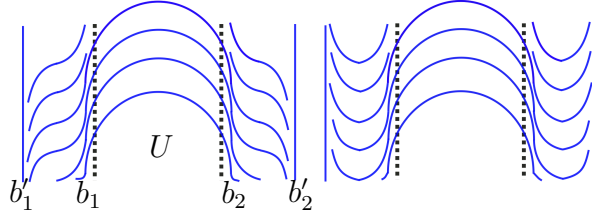


Figure 4.30: Two scenarios for the induced foliation on a neighborhood of U'

Let $\mathcal{C}$ be a string covering for $(\mathcal{F}, S)$ where $\text{Height}(\mathcal{C})$ is minimized. We define a string covering $\mathcal{C}'$ for $(\mathcal{F}, S')$ with smaller Height. Let $J \in \mathcal{C}$ be a string packet. If $J \cap A = \emptyset$ then let $J \in \mathcal{C}'$. Note that there is at least one string packet $J \in \mathcal{C}'$ such that $J \cap A \neq \emptyset$ (there should be some maximal string including U for example). If $J \cap A \neq \emptyset$ then define $J' \in \mathcal{C}'$ as follows. Assume $\text{Height}(J) = k$ and $0 = t_0 < t_1 < \dots < t_k = 1$ be the interval decomposition for J ; therefore J has k pieces corresponding to sub-intervals $[t_i, t_{i+1}]$ for $0 \leq i \leq k - 1$. If J hits the annulus A in the first (last) consecutive r moments $t_0, \dots, t_{r-1}$ then throw out the pieces of J corresponding to the first (last) r sub-intervals. Note that if a connected component of $\partial_{\text{th}} J$ has non-empty intersection with A , then it lies entirely in A by maximality of U and conditions (a) and (b) of string packets. After doing this, the beginning and the end point of packet lie on $S - A$. Next, if the packet hits A at moments t_j and t_{j+1} , remove t_j and t_{j+1}

from the list. This has the effect of joining some pieces of packet together. Call the new string packet J' . Note that $\text{Height}(J') < \text{Height}(J)$ is immediate, however we need to prove $\mathcal{C}'$ is a string covering. We should prove that $\mathcal{C}'$ satisfies conditions (a) and (b). Condition (b) is immediate since any packet $J \in \mathcal{C}$ was satisfying (b) as well. To prove the condition (a), let s be a maximal string for S' . Since none of the leaves inside U' are part of a closed trajectory, the transverse boundary of pieces of s should lie on $S' - U'$, which is the same as $S - A$. Extend s to a maximal string for S by adding annuli pieces to the beginning and end of it and also subdividing the pieces if it intersects A . Call the new string $\hat{s}$. $\hat{s}$ is a maximal string for S so it is included in a packet $J \in \mathcal{C}$. By construction, s is included in the packet $J' \in \mathcal{C}'$. This completes the proof that c_2 decreases.

Now consider the case that there exists a bad solid torus B . We can assume that B is maximal (can not be extended in the obvious way). Let

$$\partial B = \bigcup_{i=1}^m A_i \cup \bigcup_{i=1}^m U_i$$

where U_i is an annulus leaf of $\mathcal{F}_{|M \setminus S}$ (a non-separating bad annulus) and A_i is an annulus sub-surface of S_1 . Then each A_i contains at least one Reeb component since the normal vector to $\mathcal{F}$ points out of B along all U_i 's. Let $\hat{S}$ be the following surface:

$$\hat{S} := S - \bigcup_{i=1}^m A_i + \bigcup_{i=1}^m U_i$$

and S' is obtained from $\hat{S}$ by pushing it slightly out of B and removing any possible torus component. Throwing out torus components does not change the homology class since any torus bounds. Therefore S' is homologous to S and so isotopic to S . Each A_i contains at least one Reeb component. Hence if the number of Reeb components does not decrease after the isotopy then the induced foliation on a neighborhood of U_i in S has to be one of the two possible scenarios explained before. Thus repeating

the previous argument, we see that c_2 decreases.

Note that during this procedure no center tangency on S is created. $\square$

Lemma 4.4.23. *Suppose that there is no separating bad annulus and no bad solid torus. For every leaf L of $\mathcal{F}$, there is a transverse arc hitting L and going from S_0 to S_1 .*

Proof. Define $\mathcal{A}$ as follows:

$$\mathcal{A} = \{q \in M \setminus \setminus S \mid \text{there exists an arc, transverse to } \mathcal{F} \text{ and going from } S_0 \text{ to } q\}$$

We repeat an argument of Goodman [14] to show that $\mathcal{A} = M \setminus \setminus S$. Assuming this for the moment, one can get the desired transverse arc by putting together a transverse arc from S_0 to q and another one from q to S_1 (existence of the second one can be proved similarly).

We show that the assumption $\mathcal{A} \neq M \setminus \setminus S$ leads to a contradiction. Note that $\mathcal{A}$ is open and saturated. Moreover $\partial\mathcal{A}$ lies on the right side of $\mathcal{A}$ (with respect to the transverse orientation of $\mathcal{F}$) since otherwise one could extend a transversal further. Each leaf in $\partial\mathcal{A}$ is compact because otherwise a limit point provides points of $\partial\mathcal{A}$ that lie on the left side of $\mathcal{A}$. Also there are finitely many leaves in $\partial\mathcal{A}$ for the same reason. Therefore $\partial\mathcal{A}$ is a finite union of compact surfaces. Suppose $L_1, \dots, L_k$ are the pieces of $\partial\mathcal{A}$ where the normal vector to the foliation points out of $\mathcal{A}$ (the pieces lying on S_1) and $L_{k+1}, \dots, L_n$ are the ones that the normal points into $\mathcal{A}$ (the surface S_0 together with the pieces lying inside $M \setminus \setminus S$). The Poincaré-Hopf index formula implies that

$$\sum_{i=1}^k \chi(L_i) = \sum_{i=k+1}^n \chi(L_i)$$

This shows that if $\mathcal{A} \neq M \setminus \setminus S$, then $\{L_{k+1}, \dots, L_n\} - \{S_0\}$ should be annuli with both boundary curves on S_1 . Therefore $L_{k+1}, \dots, L_n$ are bad annuli. Since we assumed there is no separating bad annulus $L_{k+1}, \dots, L_n$ should be non-separating.

Let B be a connected component of the complement of $\mathcal{A}$ in $M \setminus \setminus S$. Consider the normal vector to the foliation $\mathcal{F}$. Since this vector points out of B at each point of ∂B , an application of the Poincare-Hopf formula shows that ∂B is a union of tori. However, M is hyperbolic so every torus in M bounds a solid torus. Hence

$$\partial B = \bigcup_{i=1}^m A_i \cup \bigcup_{i=1}^m U_i$$

where U_i is an annulus leaf of $\mathcal{F}$ and A_i is an annulus sub-surface of S_1 . Figure 4.31 shows B schematically in one dimension lower. Note that $U_i \in \{L_{k+1}, \dots, L_n\}$ is non-separating. Hence B is a bad solid torus which contradicts our initial assumption.

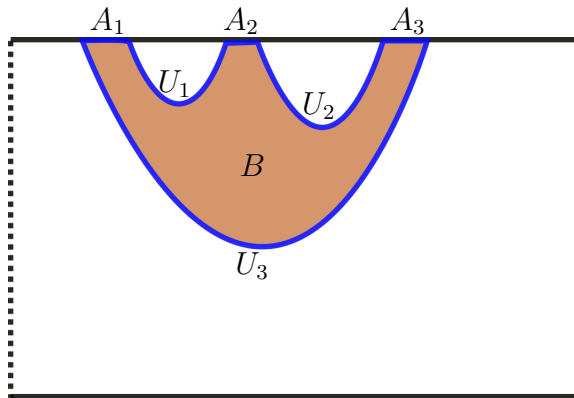


Figure 4.31: Schematic picture for a bad solid torus

□

4.4.3 After smoothing

Lemma 4.4.24. *(Gabai) Let $\mathcal{F}$ be a taut foliation on $M \setminus \setminus S$ with S_0 and S_1 as leaves. Recall that*

$$M \setminus \setminus S = (S - A) \times [0, 1] \cup N$$

and $\partial A \times [0, 1] = A_1 \cup A_2$. One can isotope A_1 and A_2 such that the induced foliations on them are by suspensions (has no Reeb component).

Proof. Let $F := S - A$ and $F_i := (S - A) \times \{i\}$ for $i = 0, 1$.

Step 1: *Isotope A_i 's such that $\mathcal{F} \pitchfork A_i$ and there is no annulus leaf in $(F \times [0, 1], A_1 \cup A_2)$:* Since A_i 's are incompressible, by Thurston's general position they can be isotoped so that they are transverse to the foliation (there might be Reeb components on them though). Consider the induced foliation on $F \times [0, 1]$. By abuse of notation, we still call it $\mathcal{F}$. Leaves of $\mathcal{F}$ are incompressible by Novikov's theorem. An annulus leaf in $(F \times [0, 1], A_1 \cup A_2)$ is ∂ -incompressible as well since its boundary consist of two essential circles on $A_1 \cup A_2$ and there can not be any ∂ -compressing disk. Here a ∂ -compressing disk D should have $\partial D = \alpha \cup \beta$ such that α, β are arcs with $\alpha \subset \partial(\text{annulus})$ and $\beta \subset F_0 \cup F_1$ and $\alpha \cap \beta = \{\text{endpoints}\}$. Incompressible and ∂ -incompressible surfaces in $F \times [0, 1]$ are either horizontal or vertical. Therefore any such annulus leaf has to be vertical, i.e. isotopic to A_1 or A_2 . Hence we can consider the outermost annuli on A_1, A_2 and isotope A_1, A_2 to push all annuli leaves out of $F \times [0, 1]$. After this isotopy, the foliation is still transverse to A_i 's and there is no annulus leaf in $(F \times [0, 1], A_1 \cup A_2)$.

Step 2: Let $Q := F \times [0, 1]$, therefore $\partial Q = \partial_v \cup \partial_h$ where $\partial_v = A_1 \cup A_2$ is the vertical boundary and $\partial_h = F_0 \cup F_1$ is the horizontal boundary. Let $D_{\partial_v} Q$ be the double of Q along its vertical boundary and $D\mathcal{F}$ be the induced transversely oriented foliation on it. *Then $D\mathcal{F}$ is taut. In fact, each leaf has a transverse arc going from DF_0 to DF_1 .* The proof is essentially the same as in Lemma 4.4.23.

To see this, define $\mathcal{A}$ as follows:

$$\mathcal{A} = \{q \in D_{\partial_v} Q \mid \text{there exists an arc, transverse to } D\mathcal{F} \text{ and going from } DF_0 \text{ to } q\}$$

We want to show that $\mathcal{A} = D_{\partial_v} Q$. Assuming this for the moment, it follows that $D\mathcal{F}$ is taut. This is because for each leaf L and $q \in L$ there is a transverse arc

from DF_0 to q and similarly there is another transverse arc from q to DF_1 . Putting them together one obtains an arc transverse to L and going from DF_0 to DF_1 . In particular $D\mathcal{F}$ is taut.

Note that $\mathcal{A}$ is open and saturated. Moreover $\partial\mathcal{A}$ lies on the right side of $\mathcal{A}$ (with respect to the transverse orientation of $D\mathcal{F}$) and $\partial\mathcal{A}$ is a finite union of compact surfaces. An application of the Poincaré-Hopf index formula shows that if $\partial\mathcal{A} \neq DF_0 \cup DF_1$ then $\partial\mathcal{A}$ should include tori. But there can not be any such torus since there was no torus leaf in Q and we had isotoped out the annuli leaves from Q in the previous step. So $\partial\mathcal{A} = DF_0 \cup DF_1$ and $\mathcal{A} = D_{\partial_v}Q$.

Step 3: *For every leaf L of $\mathcal{F}$ on $Q = F \times [0, 1]$, there is a transverse arc intersecting L and going from F_0 to F_1 :* By the previous step, there is a transverse arc γ in $D_{\partial_v}Q$ hitting DL and going from DF_0 to DF_1 . Isotope γ to make it transverse to $A_1 \cup A_2$. This breaks γ into arcs that lie entirely in one of the two copies of Q in $D_{\partial_v}Q$. Flip the arcs that are in one copy to the other one and perturb it slightly to lie entirely inside Q . This gives us the desired transverse arc.

Step 4: Let $\tilde{Q}$ be the universal cover of Q with the induced foliation $\tilde{\mathcal{F}}$. $\tilde{Q}$ is homeomorphic to $\tilde{F} \times [0, 1]$ where $\tilde{F}$ is the universal cover of F . *Fix a transversal J for $\tilde{\mathcal{F}}$ going from $\tilde{F}_0$ to $\tilde{F}_1$. Then every leaf of $\tilde{\mathcal{F}}$ hits J exactly once.*

Firstly we show that every leaf of $\tilde{\mathcal{F}}$ hits J at most once. If not, we get a closed transversal γ for $\tilde{\mathcal{F}}$ that is homotopically trivial since it lies in the simply connected manifold $\tilde{Q}$. Projecting γ to Q gives an immersed closed transversal for $\mathcal{F}$ which has to be homotopically trivial (since γ was). It can be perturbed to an embedded closed transversal for $\mathcal{F}$ that is homotopically trivial. This contradicts Novikov's theorem. Secondly every leaf of $\tilde{\mathcal{F}}$ hits J at least once. Consider two transversals J_1 and J_2 for $\tilde{\mathcal{F}}$ going from $\tilde{F}_0$ to $\tilde{F}_1$. Since each of them are isotopic to a vertical fiber in

$\tilde{F} \times [0, 1]$, we can consider a singular disk $D \cong [0, 1] \times [0, 1]$ whose vertical boundary is identified with $J_1 \cup J_2$ and whose horizontal boundary lies on $\tilde{F}_0 \cup \tilde{F}_1$. It is enough to show that every leaf intersecting J_1 intersects J_2 as well. Isotope D such that it is transverse to $\tilde{F}$ and consider the induced transversely oriented singular foliation on it. The singularities can be of either center or saddle.

Claim: Let δ be a trajectory of the induced foliation on D . The omega limit set of δ is a union of finitely many isolated points which are either fixed points or lie on $J_1 \cup J_2$.

This is because otherwise one can use a limit point to construct a closed transversal for $\tilde{F}$ in $\tilde{Q}$ which contradicts Novikov's theorem as seen before.

Take any trajectory θ with one endpoint p on J_1 . Intuitively if we follow J_1 for an infinite time (i.e. passing through singularities), it should land on J_2 for at least one such path. To make the intuition precise, define the sets U_i for $i \in \mathbb{N}$ as follows:

$$Y = \{\text{singularities inside } D\}$$

$$U_1 = \{p\}$$

$$U_i = \{q \in J_1 \cup J_2 \cup Y \mid \exists \text{ a maximal trajectory joining } q \text{ to a point in } U_{i-1}\} \cup U_{i-1}$$

Where a maximal trajectory is one that can not be extended any more. Informally speaking, U_i is just the set of points we can get to them by at most $i - 1$ maximal trajectories starting from p . Note that there exist a k such that for $i \geq k$, $U_{i+1} = U_i$ since Y is a finite set. Consider U_k together with all maximal trajectories connecting points of U_k to obtain a graph G . A parity argument counting the number of edges shows that G has an even number of edges connecting $J_1 \cup J_2$ to the inside vertices. To see this, let V be the number of saddle points inside D that are a vertex of G , e_1 be the number of edges between elements of V and e_2 be the number of edges joining

$J_1 \cup J_2$ to V . Since each saddle point has degree 4 we have the following:

$$4V + e_2 = 2(e_1 + e_2)$$

Which shows that e_2 is even as promised before. Therefore there exists an edge $\theta' \neq \theta$ that joins $J_1 \cup J_2$ to V . We want to say that θ' joins J_2 to V . It is enough to prove that θ is the only edge of G that connects J_1 to V . To prove this, assume the contrary to get a contradiction. If there are two different edges from J_1 to V , there is a path from J_1 to itself where a path is a sequence of consecutive maximal trajectories. Consider an innermost path P from J_1 to itself. Let $x_0, s_1, \dots, s_j, x_1$ be the sequence of vertices and $m_1, \dots, m_{j+1}$ be the sequence of edges of this path. m_i and m_{i+1} are adjacent trajectories for the saddle point s_i for $1 \leq i \leq j$ since otherwise the innermost hypothesis would be contradicted. Therefore following the path P we see that the transverse orientations of points x_0 and x_1 disagree on J_1 . Contradiction. Therefore, there is an edge from J_2 to V . By connectivity of G , there is a path from J_1 to J_2 .

Corollary 4.4.25. *If $\tilde{\mathcal{F}}$ is the induced foliation on $\tilde{Q}$ then the leaves of $\tilde{\mathcal{F}}$ are parametrized by the interval $[0, 1]$.*

Step 5: If $B \subset \partial_v$ is a Reeb component and $\tilde{B}$ is a lift of B to $\tilde{Q}$ then:

- a) Components of $\partial\tilde{B}$ lie in distinct leaves of $\tilde{\mathcal{F}}$.
- b) If L_1 is a component of ∂B , $L_2 \neq L_1$ is a leaf of $\mathcal{F}|_B$ and $\tilde{L}_1, \tilde{L}_2$ are lifts of them to $\tilde{B}$ then $\tilde{L}_1$ and $\tilde{L}_2$ lie in distinct leaves of $\tilde{\mathcal{F}}$.

The proof uses a doubling argument together with Novikov's theorem. Suppose $B \subset A_1$ and let $\tilde{A}_1$ be the lift of A_1 to $\tilde{Q}$ that contains $\tilde{B}$. Let $\tilde{b}_1$ and $\tilde{b}_2$ be components of $\partial\tilde{B}$. For part (a), assume the contrary and let $\tilde{L}$ be the leaf containing $\tilde{b}_1$ and $\tilde{b}_2$. Consider an arc $\tilde{\alpha}$ in $\tilde{L}$ joining a point in $\tilde{b}_1$ to a point in $\tilde{b}_2$. Let $D\tilde{Q}$ be the double of $\tilde{Q}$ along $\tilde{A}_1$. Let $D\tilde{L}$ (respectively $D\tilde{\alpha}$) be the double of $\tilde{L}$ (respectively

$\tilde{\alpha}$) along $\tilde{A}_1$. $D\tilde{\alpha}$ is homotopically nontrivial in $D\tilde{L}$ since it intersects the proper line $\tilde{b}_1 \subset D\tilde{L}$ exactly at one point and transversely. But this contradicts Novikov's theorem on π_1 -injectivity of $D\tilde{L}$. The proof of part (b) is similar.

Step 6: Let $B \subset \partial_v$ be a Reeb component. There exist distinct leaves L_1 and L_2 in $\mathcal{F}|_B$ such that

- 1) At least one of them is a component of ∂B .
- 2) If $\tilde{L}_1$ and $\tilde{L}_2$ are lifts of them to the same boundary component of $\tilde{Q}$ then they lie on the same leaf of $\tilde{\mathcal{F}}$.

We use the same notation as in step 5. By corollary 4.4.25 the leaves of $\tilde{\mathcal{F}}$ are parametrized by $[0, 1]$. Assume that the parameter of leaves increases in the positive transverse direction. Let $x, y \in [0, 1]$ be the parameters associated to the leaves of $\tilde{\mathcal{F}}$ containing $\tilde{b}_1$ and $\tilde{b}_2$ respectively. By the previous step $x \neq y$. Assume that $x > y$. We consider two cases.

First case: The transverse orientation to the foliation points out of B . Let L_2 be the leaf of $\tilde{\mathcal{F}}$ containing b_2 and consider a small transverse arc γ_2 for $\tilde{b}_2$. Let K_1 be a leaf of $\tilde{\mathcal{F}}$ intersecting γ_2 at a point inside $\tilde{B}$ and very close to $\tilde{b}_2$. This guaranties that the parameter corresponding to K_1 , z , is smaller than y . Following the intersection $K_1 \cap \tilde{B}$ we see that K_1 also intersects a small transverse arc γ_1 for $\tilde{b}_1$ inside $\tilde{B}$. By mean value theorem, at least one of the leaves intersecting γ_1 should have parameter equal to y since its endpoints have parameters x and z and we have $x > y > z$. This shows the existence of such L_2 .

The second case where the transverse orientation points into B is similar.

Note that steps 5 and 6 show that the existence of a Reeb component on ∂_v leads to a contradiction. This completes the proof of lemma. $\square$

Chapter 5

Euler classes of general foliations

Let M be a closed orientable 3-manifold. In this section we show that every second cohomology of M , satisfying the parity condition, can be realized as Euler class of a foliation (not necessarily taut) of M . This follows from some well known theorems and we bring it here for the sake of completeness.

Observation 5.0.26. *Let M be a closed orientable 3-manifold with $H_1(M)$ infinite and $a \in H^2(M; \mathbb{Z})$ be any element that satisfies the parity condition. There is a foliation $\mathcal{F}$ of M (possibly with Reeb components) such that $e(\mathcal{F}) = a$.*

Remark 5.0.27. Here the parity condition means that a has even coefficients. This is because every closed orientable surface has even Euler characteristic.

Proof. By a theorem of Wood [32] every plane field on M is homotopic to an integrable one (coming from a foliation). In particular they have the same Euler class. Therefore it is sufficient to find a plane bundle over M with the Euler class equal to a . The tangent bundle of a closed orientable 3-manifold is trivial. Fix a trivialization of TM (tangent bundle of M). Then the set of plane bundles over M can be parametrized by maps $f : M \rightarrow S^2$. One can show that twice the pullback of the fundamental cohomology class of S^2 under the map f does not depend on the choice of trivialization and is equal to the Euler class of the plane bundle [13]. So it is enough to show that

every element in $H^2(M)$ can be obtained by pulling back the fundamental cohomology class of S^2 under some map $f : M \longrightarrow S^2$. Note that this would be true if we had CP^∞ instead of S^2 , since it is a $K(\mathbb{Z}, 2)$. We can replace CP^∞ by S^2 here, since every map $f : S^2 \longrightarrow CP^\infty$ can be homotoped so that its image lies in the three-skeleton of CP^∞ that is a copy of S^2 . □

Chapter 6

Open Questions

Although Thurston's conjecture is for the unit sphere of dual norm, a priori we only know that the Euler class of a taut foliation is inside the unit ball, so a natural question is the following.

Question 6.0.28. *Which points inside the unit dual ball can be realized as the Euler class of some taut foliation on M ?*

A point strictly inside the unit dual ball cannot correspond to a taut foliation with a compact leaf. This makes it difficult to construct taut foliations with Euler class strictly inside the unit dual ball. An interesting case is taut foliations with trivial Euler class. Anosov flows provide one way of constructing taut foliations of Euler class zero, since the stable (unstable) foliation of an Anosov flow has trivial Euler class (the flow direction is a section.)

Question 6.0.29. *Which 3-manifolds with infinite first homology have a taut foliation with trivial Euler class?*

As a concrete example we do not even know if the Whitehead link complement has a taut foliation with Euler class zero. Such a taut foliation, if it exists, should have boundary Reeb components because of the parity condition. More precisely, if

there is no Reeb component on ∂M , then $\chi(S)$ and $\langle e(\mathcal{F}), [S] \rangle$ have the same parity for each properly embedded surface S . Therefore choosing S to be a twice-punctured disk bounding one of the link components implies that $e(\mathcal{F})$ is not identically zero ($\langle e(\mathcal{F}), [S] \rangle$ is an odd integer).

Since [30], other classes of geometric structures such as pseudo-Anosov flows on closed 3-manifolds and quasi-geodesic flows on closed hyperbolic 3-manifolds are known to have a similar ‘efficient intersection property’, i.e., inequality (1.1) holds for their Euler class (here the Euler class is associated to the orthogonal plane bundle to the flow) [2].

Question 6.0.30. *Let Φ be the set of pseudo-Anosov flows on a closed 3-manifold (respectively quasi-geodesic flows on a closed hyperbolic 3-manifold). What points inside the dual unit ball can be realized as Euler class of elements of Φ ?*

Gabai and Mosher [24] proved that for any finite depth taut foliation on a closed, hyperbolic 3-manifold there is an almost transverse pseudo-Anosov flow. Fenley and Mosher [7] proved that these flows are quasi-geodesic as well. Moreover it can be easily seen that they have the same Euler class (See section 6.6 of [3] for a good exposition). Hence Gabai’s theorem about vertices of unit dual ball (theorem 3.0.13), implies similar results for these classes of flows as well, that is, the vertices of dual unit ball can be realized as Euler classes of pseudo-Anosov flows (respectively quasi-geodesic flows) on closed hyperbolic 3-manifolds.

Another class of geometric structures that satisfy a similar ‘efficient intersection property’ is tight contact structures. Every $C^{1,0}$ taut foliation can be perturbed to a tight contact structure [6] [19] so Euler classes of tight contact structures include Euler classes of taut foliations.

Question 6.0.31. *Which points inside the unit dual ball can be realized as Euler class of some tight contact structure on M ?*

Associated to any taut foliation $\mathcal{F}$ is a faithful representation:

$$\rho : \pi_1(M) \longrightarrow \text{Homeo}^+(S^1).$$

This has been done using the universal circle construction [4] [31]. There is a canonical way to assign an Euler class $e(\rho) \in H^2(M)$ to each such representation. Firstly construct a circle bundle associated to this representation using Borel's construction:

$$E = (\tilde{M} \times S^1)/(m, \theta) \sim (\alpha(m), \rho(\alpha)(\theta)), \forall \alpha \in \pi_1(M)$$

Define the Euler class of the representation to be the Euler class of the associated circle bundle. When the representation comes from a taut foliation, these two Euler classes coincide [2], i.e.,

$$e(\rho_{\mathcal{F}}) = e(\mathcal{F})$$

Therefore, if Thurston's Euler class one conjecture about taut foliations was true, then for every integral second cohomology class a of dual norm one and satisfying the parity condition, there would be a representation with Euler class equal to a . On the other hand, there is a similar efficient intersection property for Euler classes of representation into $\text{Homeo}^+(S^1)$, called Milnor-Wood inequality [20] [33]. It says that for any embedded surface S in M we have:

$$\langle e(\rho), [S] \rangle \leq |\chi(S)|$$

So it makes sense to conjecture the following that is weaker than Thurston's original conjecture.

Conjecture 6.0.32. *Let M be a closed hyperbolic 3-manifold with infinite first homology. Then any integral class $a \in H^2(M; \mathbb{R})$ of dual norm exactly one and satisfying the parity condition, can be realized as the Euler class of some faithful representation $\rho : \pi_1(M) \longrightarrow \text{Homeo}^+(S^1)$.*

Miyoshi [22] proved that the above conjecture is true for closed orientable Seifert fibered manifolds under the condition that $H_1(M; \mathbb{Z})$ is torsion free if the Euler number of Seifert fibration is zero (he worked with cohomology with integral coefficients and so he considered torsions as well.) One cannot in general replace $\text{Homeo}^+(S^1)$ with $\text{Diff}^+(S^1)$ instead since Miyoshi [23] gave examples of cohomology classes that are representable as continuous Euler classes but not as smooth ones. He showed, using a rigidity theorem of Ghys [12], that in fact the Euler class of any fibration of a closed hyperbolic 3-manifold is not smoothly representable.

One can ask if a virtual version of Thurston's conjecture is true or not. Given any taut foliation $\mathcal{F}$ on M with Euler class a and a finite covering map $p : \hat{M} \longrightarrow M$, one can pull back the foliation to a foliation $\hat{\mathcal{F}}$ on $\hat{M}$ with Euler class $p^*(a)$. If $a \in H^2(M)$ is any integral class of norm 1 (no matter if it is realized as the Euler class of a taut foliation on M or not) and $p : \hat{M} \longrightarrow M$ is any finite covering then the pullback of a under the covering map has norm 1 again (proposition 2.0.8). Therefore it might be possible that every class $a \in H^2(M)$ of norm 1 could be realized as Euler class of some taut foliation after passing to a suitable finite cover.

Question 6.0.33. *Let $a \in H^2(M)$ be an integral point with dual norm 1 that satisfies the parity condition. Is there a finite covering map $p : \hat{M} \longrightarrow M$ and a taut foliation, $\hat{\mathcal{F}}$, of $\hat{M}$ such that $e(\hat{\mathcal{F}}) = p^*(a)$?*

In fact for our counterexamples, the above question has a positive answer. Here is the sketch of a proof. There is a sutured manifold hierarchy that almost gives us the desired foliation with prescribed Euler class a . The only problem is that at the

last step there is no foliation on N with Euler class 0. N does not admit such a foliation because of the $\frac{2\pi}{3}$ twist. Now by the LERF property of the fundamental group, one can untwist N at the expense of going to a finite cover $p : \tilde{M} \rightarrow M$. We construct a foliation $\tilde{\mathcal{F}}$ of $\tilde{M}$ with $e(\tilde{\mathcal{F}}) = p^*(a)$. Decompose $\tilde{M}$ as the union of $p^{-1}(N)$ and $\tilde{M} - p^{-1}(N)$. On $\tilde{M} - p^{-1}(N)$ define $\tilde{\mathcal{F}}$ to be the lift of the foliation of $M - N$ obtained by the sutured manifold hierarchy. On $p^{-1}(N)$, define $\tilde{\mathcal{F}}$ to be a foliation of $p^{-1}(N)$ which has Euler class 0. While gluing the pieces, one should be careful about holonomies. To overcome this issue, one can change the holonomies by attaching product pieces (similar to section 5).

Question 6.0.34. *Is Theorem 1.0.4 true without assuming that S is the unique norm-minimizing surface in its homology class?*

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