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Harmonic Analysis on Local Systems on Graphs -- Towards a Gross-Zagier Formula on CM-cycles over Shimura Curves

Wen, Boya

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Harmonic Analysis on Local Systems on Graphs – Towards a Gross-Zagier Formula on CM-cycles over Shimura Curves

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A Dissertation
Presented to the Faculty
of Princeton University
in Candidacy for the Degree
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Abstract

In Part I of the thesis, we study harmonic analysis on local systems over graphs, and construct Green's functions explicitly. In Part II, we apply the results in Part I towards a Gross-Zagier formula for CM cycles over Shimura curves, which connects the global height pairing of special cycles in Kuga varieties over Shimura curves with the derivatives of the L-functions associated to weight-$2k$ modular forms. More precisely, we present a suitable choice of integral model of the CM cycles, and compute local intersections at primes where the quaternion algebra associated with the Shimura curve ramifies.

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I also thank my dearest friends, with whom I could share every inch of my struggle and get all the support I need. My friends help me stay grounded and live a more authentic self. I do not like

the idea of leaving a string of names here – since the audience are not likely to know them, the names would not carry the weight I'd like them to deliver. Instead, I will make sure that each of my dear friend gets my personal thanks. There is only one exception: Xiaohong, who had left the world too soon, would not see my message. I can only leave it here my apologies for not being there in her painful but brave battle with the disease, and remember her last messages that it is always a blessing to drink water without pain.

I thank my parents for making my journey of life possible in each and every sense, for reminding me to look at the flowers and leaves from time to time, for their faith in me even when I myself lack it; in short, for their unconditional love and for giving me the privilege to know what that phrase might truly mean. They have supported me to go to wherever I want to, though it means in the time of COVID-19 I haven't seen them for more than two years. I love them and miss them. I also thank my husband for his love and support throughout the years and distances, and for his irreplaceable company and comfort during the most difficult times. Together they form my family. To my family I dedicate this thesis.

To my family

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Chapter 1

Introduction

Our work originates from the work of Gross and Zagier in [13], which connects the global height pairing of Heegner points on modular curves with the derivatives of the L-functions associated to weight-2 modular forms. In the same paper, Gross and Zagier have also pointed out two potential directions of generalization, one is to find a similar formula for weight-$2k$ modular forms when $k > 1$ (which on the geometric side means generalizing Heegner points to special cycles above them), and the other is to relax the Heegner condition (which on the geometric side means generalizing modular curves to Shimura curves). The former is addressed in [30] and the latter (keeping $k = 1$) is covered in [29], whereas our work makes progress towards the $k > 1$ case with a relaxed Heegner condition.

As a key step, we develop[1] a theory of harmonic analysis on local systems on graphs, with explicit computations of Green's functions. The theory may be of independent interest, so we first introduce it in Part I without referring to the Gross-Zagier context. In Part II, we apply the computations to give a suitable integral model of the CM cycles in question and compute certain local intersections. In this Chapter, we introduce the motivating Gross-Zagier formula and give an overview for both parts as well as their connections.

[1] While we developed the theory of harmonic analysis on local systems over graphs independently, we recently found that it had been studied in the literature (see [14] for example). The explicit construction of Green's functions in Theorem 3.2.1 and the applications in Part II are original

1.1 The Goal: a Gross-Zagier Formula for CM Cycles over Shimura Curves.

1.1.1 The Analytic Side

Let k, N be positive integers. Let D be an integer coprime to N. Let K be an imaginary quadratic field with discriminant D. Let N_- denote the product of primes dividing N that are inert in K and write $N = N_-N_+$. We shall assume the *relaxed Heegner condition* on N, which says

$$N_- \text{ is the product of an even number of distinct primes.}$$

Note that the above condition on N is indeed weaker than the *Heegner condition* assumed in both [13] and [30], which says any prime $p|N$ is split in K, or in other words $N_- = 1$.

Let $\mathscr{A}$ be an element of the class group CL_K of K. Let $f = \sum_{n=1}^{\infty} a_n e^{2\pi inz}$ be a new form of weight $2k$ on $\Gamma_0(N)$, where $\Gamma_0(N)$ denotes the subgroup of $\mathrm{SL}_2(\mathbb{Z})$ consisting of matrices that are upper-triangular modulo N. Define the L-series associated to f and $\mathscr{A}$ by

$$L_{\mathscr{A}}(f,s) = \left(\sum_{\substack{n\geq 1 \\ (n,DN)=1}} \epsilon(n) n^{2k-1-2s} \right) \cdot \left(\sum_{m\geq 1} a_m r_{\mathscr{A}}(m) m^{-s} \right), \tag{1.1}$$

where $\epsilon : (\mathbb{Z}/D\mathbb{Z})^\times \to \{\pm 1\}$ is the quadratic character associated to the extension $K/\mathbb{Q}$ and $r_{\mathscr{A}}(n)$ is the number of integral ideals $\mathfrak{a}$ in the class of $\mathscr{A}$ of norm n. The series $L_{\mathscr{A}}(f,s)$ extends analytically to an entire function of s and satisfies a functional equation about $s = k$. Moreover, when $\epsilon(N) = +1$ (which is true when N satisfies our relaxed Heegner condition), $L_{\mathscr{A}}(f,s)$ vanishes at $s = k$ and that there exists a cusp form

$$\Phi_{\mathscr{A}} = \sum_{m=1}^{\infty} a_{m,\mathscr{A}} e^{2\pi imz} \tag{1.2}$$

of weight $2k$ level N such that

$$(f, \Phi_{\mathscr{A}}) = \frac{(2k-2)!\sqrt{|D|}}{2^{4k-1}\pi^{2k}} L'_{\mathscr{A}}(f,k), \tag{1.3}$$

for any weight $2k$ new form f on $\Gamma_0(N)$, where $(f,g) = \int_{\Gamma_0(N)\backslash \mathfrak{H}} f(z)\overline{g(z)} y^{2k} \frac{dxdy}{y^2}$ is the Petersson inner product on cusp forms of weight $2k$ for $\Gamma_0(N)$. [13, Chapter 4, Theorem (5.8)]

Our goal is to realize the Fourier coefficients $a_{m,\mathscr{A}}$ as some height pairings when m is coprime

to N.

1.1.2 The Geometric Side

On the geometric side, we study cycles over *CM points* on the *Shimura curve* X_{N_+,N_-}.

We now define X_{N_+,N_-}. Let B be the (indefinite) quaternion algebra which ramifies precisely at the places $p|N_-$. Then there exists an embedding $\rho : K \to B$. Let R_{N_+,N_-} be the Eichler order in B which contains $\rho(\mathcal{O}_K)$ and has reduced discriminant N (or in other words is of level N_+ in B). Then $\Gamma = R^\times_{N_+,N_-}/\{\pm 1\}$ embeds into $(B \otimes \mathbb{R})^\times/\mathbb{R}^\times$, which is isomorphic to $\mathrm{PGL}_2(\mathbb{R})$. Identifying Γ with its image in $\mathrm{PGL}_2(\mathbb{R})$ we have $\Gamma^+ := \Gamma \cap \mathrm{PGL}_2^+(\mathbb{R})$ is a Fuchsian group acting properly discontinuously on $\mathfrak{H}$. The quotient $\Gamma^+\backslash\mathfrak{H}$ is a complete Riemann surface, which we define as $X_{N_+,N_-}(\mathbb{C})$. (See [21] for example.)

In [24], it is shown that the curve X_{N_+,N_-} has a canonical model over $\mathbb{Q}$; moreover, the curve X_{N_+,N_-} over a scheme S solves the moduli problem of quadruples $(A, \lambda, \iota, \kappa)$ over S, where A is an isogeny class of abelian surface *with quaternion multiplication (QM) by* B, the map $\lambda : A \to A^\vee$ gives a principal polarization, the embedding $\iota : B \hookrightarrow \mathrm{End}^0(A) : \mathrm{End}(A) \otimes \mathbb{Q}$ defines the B-action on A, and κ an equivalence class of left B-module isomorphisms $B \otimes \mathbb{A}_f \xrightarrow{\sim} H_1(A, \mathbb{A}_f)$, where two such isomorphisms κ_1 and κ_2 are considered equivalent if κ_1 is κ_2 composed with right multiplication by an element in $(R_{N_+,N_-} \otimes \mathbb{A}_f)^\times$. An abelian surface A is said to have QM by B if the embedding $\iota : B \hookrightarrow \mathrm{End}^0(A)$ exists.

Given an abelian surface A over $\mathbb{Q}$ with QM via a given $\iota : B \hookrightarrow \mathrm{End}^0(A)$, the subgroup $\mathrm{End}^0_B(A)$ of $\mathrm{End}^0(A)$ consisting of elements commuting with B is isomorphic to either $\mathbb{Q}$ or an imaginary quadratic field L. In the latter case, we say the abelian surface A has QM by B and CM by L, and the corresponding point in X_{N_+,N_-} is said to be a *CM point*. For the imaginary quadratic field K we fixed at the beginning, the existence of the embedding ρ implies that it gives rise to a CM point x on X_{N_+,N_-}, which is rational over the Hilbert class field H of K. Therefore the Galois group $\mathrm{Gal}(H/K)$ acts on the collection of points with CM by K, and the cycles we are interested in will lie above these points.

Over a finite cover X' of X_{N_+,N_-}, there exists a universal abelian surface $\mathcal{A}$, and we study the Kuga variety Y which is the fiber product of $(k-1)$ copies of $\mathcal{A}$ over X'. (See [17] for example.) Above a point $x \in X'$, the fiber Y_x is given by $\mathcal{A}_x^{k-1}$, where $\mathcal{A}_x$ is the abelian surface x parametrizes.

When x is a CM point by K, $\mathcal{A}_x$ is an abelian surface with QM by B and CM by K, and the algebraic cycle $S_k(x)$ we will study is a $(k-1)$-dimensional cycle lying completely inside the

$(2k-2)$-dimensional $\mathcal{A}_x^{k-1}$. Recall that for an abelian surface A with QM by B but without CM, the Néron-Severi group is 3-dimensional, whereas an abelian surface with both QM and CM has a 4-dimensional Néron-Severi group [19]. Let $Z_k(x)$ be a cycle in $\mathcal{A}_x$ whose image in the Néron-Severi group of $\mathcal{A}_x$ is perpendicular to the 3-dimensional non-CM part, then the cycle $S_k(x)$ is obtained from $Z_k(x)^{k-1}$ by averaging under permutations of $2k-2$ letters, and normalizing so that the algebraic self-intersection of $S_k(x)$ within $Y_x = \mathcal{A}_x^{k-1}$ is $(-1)^{k-1}$. (c.f. Sections 6.2 and 6.3 for more precise constructions.) In fact, an abelian surface with both QM by B and CM by K is isogenous to $E \times E$ for some elliptic curve E with CM by K (see [28] for example). So up to isogeny our CM cycle $S_k(x)$ is the same as that in [30].

The Hecke operator T_m on the Shimura curve also acts on the CM cycles, via

$$T_m S_k(x) = \sum_i m^{k-1} S_k(x_i) \text{ if } T_m(x) = \sum_i x_i.$$

The algebraic cycle $S_k(x)$ defines an arithmetic Chow cycle $\hat{S}_k(x)$ of codimension k in $\mathcal{Y}$, an integral model of the Kuga variety Y, where $\mathcal{Y}$ is a $2k$-dimensional arithmetic scheme. Therefore the intersection number $\hat{S}_k(x) \cdot \hat{S}_k(y)$ is well-defined in the sense of Gillet-Soulé [10] for two CM points x and y. We define

$$\langle S_k(x), S_k(y) \rangle := (-1)^k \hat{S}_k(x) \cdot \hat{S}_k(y).$$

In order for this to be a well-defined *global* pairing, there are conditions on the arithmetic cycles (c.f. Sec 5.4) that the Zariski closure $\overline{S_k(x)}$ of $S_k(x)$ in $\mathcal{Y}$ does not verify at places dividing N_-. Thus we need to find local adjustments at these places. This motivated Part I of the thesis, which will be applied to find the appropriate cycle adjustments in Section 6.5.

1.1.3 The Formula

The expected Gross-Zagier formula for CM cycles over Shimura curves is as follows.

Conjecture 1.1.1. *For an integer $k \geq 1$, if $\sigma \in \mathrm{Gal}(H/K)$ corresponds to $\mathscr{A} \in \mathrm{CL}_K$ via class field theory, we have*

$$\langle S_k(x), T_m S_k(x^\sigma) \rangle = u^2 a_{m,\mathscr{A}},$$

for all $m \geq 1$ and $(m, N) = 1$, where u is the order of the finite group $\mathcal{O}_K^\times / \{\pm 1\}$.

Remark 1.1.2. *While the $k = 1$ case is covered in [29], our strategy is expected to lead to a new proof for this case using explicit computations of local intersection numbers.*

Let χ be a character of $\mathrm{Gal}(H/K)$ (thus also of CL_K). Define

$$L(f,\chi,s) := \sum_{\mathscr{A}} \chi(\mathscr{A}) L_{\mathscr{A}}(f,s) \text{ and } S_\chi := \sum_{\sigma \in \mathrm{Gal}(H/K)} \chi^{-1}(\sigma) S_k(x^\sigma).$$

When f is a newform (that is a normalized Hecke eigenform) of weight-$2k$ level $\Gamma_0(N)$, we can extend it to an orthogonal basis $\{f_i\}$ of the space of cusp forms, and the cycle S_χ can be written as a sum of f_i-isotypical components S_{χ,f_i}, where $T_m S_{\chi,f} = a_m(f) S_{\chi,f}$. And assuming the conjecture, we have the following corollary as in [13, Chapter I, Theorem (6.3)] and in [30, Corollary 0.3.2].

Corollary 1.1.3. *We have*

$$L'(f,\chi,k) = \frac{2^{4k-1}\pi^{2k}(f,f)}{(2k-2)!u^2 h\sqrt{|D|}} \langle S_{\chi,f}, S_{\chi,f} \rangle,$$

where h denotes the class number of K.

1.2 The Results and the Outline

The thesis consists of two parts.

Part I focuses on harmonic analysis on local systems over graphs, with explicit computations of Green's functions over finite quotients of Bruhat-Tits tree.

In Chapter 2, we establish the theory of harmonic analysis on local systems over graphs and define vector-valued Green's functions. In Chapter 3, we compute Green's functions over the Bruhat-Tits tree and its finite quotients. Our main result in Part I is Theorem 3.2.1, Corollary 3.2.4, Corollary 3.2.5, Proposition 3.3.1, and Corollary 3.3.3, which will be used in Part II.

Part II consists of expositions to help understand Conjecture 1.1.1, as well as our progress towards the conjecture using Part I.

In Chapter 4, we give an exposition of the original Gross-Zagier formula ([13]) and the higher-weight version by Zhang in [30]. We also introduce the archimedean computations involved in both results, including the construction of Green's functions in [12]. These are the main sources of inspiration for this thesis. In Chapter 5, we give an exposition of the arithmetic intersection theory introduced by Gillet and Soulé in [10], which helps us understand the height pairing involved in Conjecture 1.1.1. We also elaborate on the conditions on the arithmetic cycles associated with the algebraic cycles so that the arithmetic intersection number defines a global height pairing, and as mentioned in Section 1.1.2, the conditions motivate the connection between Part I and Part II of the

thesis. In Chapter 6, we give a more precise definition of $S_k(x)$, building upon what we described in Section 1.1.2. We also give an appropriate integral model $\widehat{S}_k(x)$ of $S_k(x)$, applying Part I results to finding adjustments to $\overline{S_k(x)}$ in Section 6.5, which is the main contribution to Conjecture 1.1.1 of this thesis.

Finally, in Chapter 7, we give a plan of what remains to be done to prove Conjecture 1.1.1, which we are currently working on. We also give a short discussion on some further directions of future work.

Part I

Harmonic Analysis on Local Systems on Graphs

Chapter 2

Set-up and Preliminaries

2.1 Local Systems on Graphs

Let $F \subseteq \mathbb{C}$ be a field that is closed under complex conjugation. Let $\mathcal{G}$ be a connected graph whose fundamental group is finitely generated, and let E and V denote its collection of edges and vertices respectively. A rank-r local system $\mathscr{L}$ on $\mathcal{G}$ over the field F contains the following data:

- For any $v \in V$, an r-dimensional hermitian vector space $\mathscr{L}(v)$ over F;
- for any $e \in E$, an r-dimensional hermitian vector space $\mathscr{L}(e)$ over F; and
- if v is an end of e, an isometric isomorphism

$$\varphi_{ve} : \mathscr{L}(v) \to \mathscr{L}(e).$$

Let $\mathscr{L}$ and $\mathscr{L}'$ be two rank-r local systems on $\mathcal{G}$. A morphism Θ from $\mathscr{L}$ to $\mathscr{L}'$ is a collection $\{\theta_v, \theta_e\}_{v \in V, e \in E}$ of linear transformations

$$\theta_v : \mathscr{L}(v) \to \mathscr{L}'(v),$$

$$\theta_e : \mathscr{L}(e) \to \mathscr{L}'(e)$$

such that the following diagram commutes for each e and v where v is an end of e:

$$\begin{array}{ccc} \mathscr{L}(v) & \xrightarrow{\theta_v} & \mathscr{L}'(v) \\ {\scriptstyle \varphi_{ve}}\downarrow & & \downarrow{\scriptstyle \varphi'_{ve}} \\ \mathscr{L}(e) & \xrightarrow{\theta_e} & \mathscr{L}'(e) \end{array} \tag{2.1}$$

Let $\underline{\text{LocSys}}_{\mathcal{G},F,r}$ denote the category of rank-r local systems on $\mathcal{G}$ over F. Let v_0 be a vertex of $\mathcal{G}$ and let $\underline{\text{UniRep}}_{\mathcal{G},v_0,F,r}$ denote the category of r-dimensional unitary representations over F of the fundamental group $\pi_1(\mathcal{G}, v_0)$ of the graph $\mathcal{G}$. Recall that $\pi_1(\mathcal{G}, v_0)$ is the group of homotopy classes of loops in $\mathcal{G}$ with base v_0. In what follows we will not always distinguish a homotopy class with its (unique) non-backtracking representative, but whenever we define a map on $\pi_1(\mathcal{G}, v_0)$, it should be clear to see the well-definedness on homotopy classes.

Proposition 2.1.1. *The category* $\underline{\text{LocSys}}_{\mathcal{G},F,r}$ *is equivalent to the category* $\underline{\text{UniRep}}_{\mathcal{G},v_0,F,r}$.

Proof. Define the functor

$$\mathcal{F} : \underline{\text{LocSys}}_{\mathcal{G},F,r} \to \underline{\text{UniRep}}_{\mathcal{G},v_0,F,r}$$

by sending $\mathscr{L}$ to the representation $\rho : \pi_1(\mathcal{G}, v_0) \to \text{Aut}(\mathscr{L}(v_0))$, defined by sending a loop γ to the composition of φ_{ve} and φ_{ve}^{-1}'s along the loop. That is, if

$$\gamma = v_0 \xrightarrow{e_1} v_1 \xrightarrow{e_2} \cdots \xrightarrow{e_n} v_n \xrightarrow{e_{n+1}} v_0,$$

then

$$\rho(\gamma) = \varphi^{-1}_{v_0 e_{n+1}} \circ \varphi_{v_n e_{n+1}} \circ \cdots \circ \varphi^{-1}_{v_1 e_1} \circ \varphi_{v_0 e_1}. \tag{2.2}$$

Here ρ is unitary because the φ_{ve}'s are isometric isomorphisms between hermitian spaces. Given a morphism of local systems

$$\Theta : \mathscr{L} \to \mathscr{L}',$$

we define

$$\mathcal{F}(\Theta) : \mathscr{L}(v_0) \to \mathscr{L}'(v_0)$$

by

$$\mathcal{F}(\Theta)\ell(v_0) = \theta_{v_0}\ell(v_0).$$

Let $\rho = \mathcal{F}(\mathscr{L})$ and $\rho' = \mathcal{F}(\mathscr{L}')$, then from Equation (2.2) and the Commutative Diagram 2.1, we

have

$$\theta_{v_0}\rho(\gamma)\ell(v_0) = \rho'(\gamma)(\theta_{v_0}\ell(v_0)).$$

Hence the map $\mathcal{F}(\Theta)$ defines a morphism between representations $\mathcal{F}(\mathscr{L})$ and $\mathcal{F}(\mathscr{L}')$.

Define the functor

$$\mathcal{D} : \underline{\mathrm{UniRep}}_{\mathcal{G},v_0,F,r} \to \underline{\mathrm{LocSys}}_{\mathcal{G},F,r}$$

by sending a unitary representation $(W, \rho : \pi_1(\mathcal{G}, v_0) \to \mathrm{Aut}(W))$ to the following local system $\mathscr{L} = \mathcal{D}(W, \rho)$:

- Define $\mathscr{L}(v) = W$ for all $v \in V$.
- Define $\mathscr{L}(e) = W$ for all $e \in E$.
- Let $\{\gamma_1, \ldots, \gamma_g\}$ be a minimal set of generators of $\pi_1(\mathcal{G}, v_0)$. Remove an edge e_i from each γ_i, so that $\mathcal{G}\backslash\{e_1, \ldots, e_g\}$ is a tree. Define $\varphi_{ve} = \mathrm{Id}_W$ if $e \in E\backslash\{e_1, \ldots, e_g\}$ and v is an end of e.
- For each i, write e_i as $v_{i0} \xrightarrow{e_i} v_{i1}$, where the direction is inherited from γ_i. Define $\varphi_{v_{i0}e_i} = \rho(\gamma_i)$ and $\varphi_{v_{i1}e_i} = \mathrm{Id}_W$.

Note that different choices of $\{e_1, \ldots, e_g\}$ will result in isomorphic local systems.

Given a morphism f between (W, ρ) and (W', ρ'), that is, a linear transformation $f : W \to W'$ such that $f(\rho(\gamma)w) = \rho'(\gamma)f(w)$ for any $\gamma \in \pi_1(\mathcal{G}, v_0)$, we define $\Theta = \mathcal{D}(f) : \mathcal{D}(W, \rho) \to \mathcal{D}(W', \rho')$ as follows.

- define $\theta_{v_0} = f$.
- define all the θ_e's and θ_v's following Diagram 2.1. This is possible because of the connectedness of $\mathcal{G}$ (so that we have a definition of θ_e and θ_v for each $e \in E$ and $v \in V$), and the construction of $\mathcal{D}(W, \rho)$ and $\mathcal{D}(W', \rho')$ (so that θ_e's and θ_v's are well-defined.)

It is now easy to check that the functors $\mathcal{F}$ and $\mathcal{D}$ give an equivalence of categories.

□

Remark 2.1.2. *One can think of an object (W, ρ) in the category $\underline{\mathrm{UniRep}}_{\mathcal{G},v_0,F,r}$ as the vector bundle $\pi_1(\mathcal{G}, v_0)\backslash\widetilde{\mathcal{G}} \times W$ over $\pi_1(\mathcal{G}, v_0)\backslash\widetilde{\mathcal{G}} = \mathcal{G}$. Here, $\widetilde{\mathcal{G}}$ is the universal cover of $\mathcal{G}$ constructed as follows: the set of vertices $\widetilde{V}$ of $\widetilde{\mathcal{G}}$ consists of non-backtracking paths $(v_0, v_1, \ldots, v_n), n \in \mathbb{Z}_{\geq 0}$, starting from v_0 of $\mathcal{G}$, and two vertices of $\widetilde{\mathcal{G}}$ are linked by an edge if the corresponding paths differ by a one-step*

extension, i.e. if the two vertices are represented by paths $(v_0, v_1, \ldots, v_i)$ *and* $(v_0, v_1, \ldots, v_{i+1})$ *for some* i*. The group* $\pi_1(\mathcal{G}, v_0)$ *acts on* $\widetilde{\mathcal{G}}$ *by*

$$\gamma(v_0, v_1, \ldots, v_n) := (\textit{vertices } \gamma \textit{ passes in order}, v_1, \ldots, v_n),$$

where γ *is a non-backtracking path starting and ending at* v_0*. The action on the set of edges* $\widetilde{E}$ *of* $\widetilde{\mathcal{G}}$ *is induced from the action on vertices. Finally, the action of* $\pi_1(\mathcal{G}, v_0)$ *on the trivial vector bundle* $\widetilde{\mathcal{G}} \times W$ *is given by* $\gamma(\tilde{g}, w) = (\gamma\tilde{g}, \rho(\gamma)w)$*, which gives the vector bundle* $\pi_1(\mathcal{G}, v_0)\backslash\widetilde{\mathcal{G}} \times W$ *over* $\pi_1(\mathcal{G}, v_0)\backslash\widetilde{\mathcal{G}} = \mathcal{G}$*.*

2.2 Harmonic Analysis on Local Systems on Directed Graphs

From now on we assume that $\mathcal{G}$ is finite and directed, that is, both V and E are finite sets, and each edge is assigned a direction. (All the discussions in Section 2.1 still applies by forgetting the direction.) Let $\mathscr{L}$ be a local system on $\mathcal{G}$. Let $\Gamma(V, \mathscr{L})$ (resp. $\Gamma(E, \mathscr{L})$) denote the space of sections of $\mathscr{L}$ over the vertices (resp. edges) of $\mathcal{G}$. We have

$$\Gamma(V, \mathscr{L}) = \bigoplus_{v \in V} \mathscr{L}(v) \text{ and } \Gamma(E, \mathscr{L}) = \bigoplus_{e \in E} \mathscr{L}(e).$$

Both $\Gamma(V, \mathscr{L})$ and $\Gamma(E, \mathscr{L})$ are hermitian spaces over F, equipped with the hermitian inner products

$$\langle f_1, f_2 \rangle_V = \sum_{v \in V} \langle f_1(v), f_2(v) \rangle \quad \text{and} \quad \langle g_1, g_2 \rangle_E = \sum_{e \in E} \langle g_1(e), g_2(e) \rangle$$

respectively. Moreover, $\Gamma(V, \mathscr{L})$ and $\Gamma(E, \mathscr{L})$ are Hilbert spaces as they are direct sums of (finite-dimensional thus) complete hermitian spaces.

Define $\nabla : \Gamma(V, \mathscr{L}) \to \Gamma(E, \mathscr{L})$ by

$$\nabla f(e) = \varphi_{v_2 e}(f(v_2)) - \varphi_{v_1 e}(f(v_1)), \tag{2.3}$$

if e is the directed edge from v_1 to v_2.

Define $\nabla^* : \Gamma(E, \mathscr{L}) \to \Gamma(V, \mathscr{L})$ by

$$\nabla^* g(v) = \sum_{e \text{ has head } v} \varphi_{ve}^{-1}(g(e)) - \sum_{e \text{ has tail } v} \varphi_{ve}^{-1}(g(e)). \tag{2.4}$$

As the notation suggests, ∇^* is the adjoint operator of ∇:

Proposition 2.2.1. *We have $\langle \nabla f, g\rangle_E = \langle f, \nabla^* g\rangle_V$ for any $f \in \Gamma(V, \mathscr{L})$ and any $g \in \Gamma(E, \mathscr{L})$.*

Proof. We have

$$\begin{aligned} \text{LHS} &= \sum_{e\in E} \langle \nabla f(e), g(e)\rangle \\ &= \sum_{e=(v_1\to v_2)\in E} \langle \varphi_{v_2 e}(f(v_2)) - \varphi_{v_1 e}(f(v_1)), g(e)\rangle \\ &= \sum_{e=(v_1\to v_2)\in E} \langle \varphi_{v_2 e}(f(v_2)), g(e)\rangle - \langle \varphi_{v_1 e}(f(v_1)), g(e)\rangle \end{aligned} \tag{2.5}$$

and

$$\begin{aligned} \text{RHS} &= \sum_{v\in V} \langle f(v), \nabla^* g(v)\rangle \\ &= \sum_{v\in V} \langle f(v), \sum_{e \text{ has head } v} \varphi_{ve}^{-1}(g(e)) - \sum_{e \text{ has tail } v} \varphi_{ve}^{-1}(g(e))\rangle \\ &= \sum_{v\in V} \langle f(v), \sum_{e \text{ has head } v} \varphi_{ve}^{-1}(g(e))\rangle - \langle f(v), \sum_{e \text{ has tail } v} \varphi_{ve}^{-1}(g(e))\rangle \\ &= \sum_{e=(v_1\to v_2)\in E} \langle f(v_2), \varphi_{v_2 e}^{-1}(g(e))\rangle - \langle f(v_1), \varphi_{v_1 e}(g(e))\rangle. \end{aligned} \tag{2.6}$$

We see that LHS = RHS, since φ_{ve} is an isometry for each $e \in E$ and v an end of the edge e. □

Definition 2.2.2. *Define the Laplace operator $\Delta : \Gamma(V, \mathscr{L}) \to \Gamma(V, \mathscr{L})$ by $\Delta := \nabla^*\nabla$.*

We compute a point-wise formula for Δ, as follows:

$$\begin{aligned}
\Delta f(v) &= \sum_{e \text{ has head } v} \varphi_{ve}^{-1}\left(\nabla f(e)\right) - \sum_{e \text{ has tail } v} \varphi_{ve}^{-1}\left(\nabla f(e)\right) \\
&= \sum_{e=(v' \to v)} \varphi_{ve}^{-1}\left(\nabla f(e)\right) - \sum_{e=(v \to v'')} \varphi_{ve}^{-1}\left(\nabla f(e)\right) \\
&= \sum_{e=(v' \to v)} \left(\varphi_{ve}^{-1}\left(\varphi_{ve}\left(f(v)\right) - \varphi_{v'e}\left(f(v')\right)\right)\right) \\
&\quad - \sum_{e=(v \to v'')} \left(\varphi_{ve}^{-1}\left(\varphi_{v''e}\left(f(v'')\right) - \varphi_{ve}\left(f(v)\right)\right)\right) \\
&= \sum_{e=(v' \to v)} \left(f(v) - \varphi_{ve}^{-1}\varphi_{v'e}\left(f(v')\right)\right) \\
&\quad - \sum_{e=(v \to v'')} \left(\varphi_{ve}^{-1}\varphi_{v''e}\left(f(v'')\right) - f(v)\right) \\
&= \sum_{\substack{(v',e) \text{ where} \\ e \text{ connects} \\ v \text{ and } v'}} f(v) - \varphi_{ve}^{-1}\varphi_{v'e}\left(f(v')\right),
\end{aligned} \tag{2.7}$$

Remark 2.2.3. *If giving $\mathcal{G}$ a direction seems unnatural, one may also start with an undirected graph $\mathcal{G}$ and define $\nabla : \Gamma(V, \mathscr{L}) \to \Gamma(E, \mathscr{L}) \otimes \Omega^1(\mathcal{G})$ and $\nabla^* : \Gamma(E, \mathscr{L}) \otimes \Omega^1(\mathcal{G}) \to \Gamma(V, \mathscr{L})$, where $\Omega^1(\mathcal{G})$ is the free abelian group generated by directed edges of $\mathcal{G}$ and $-(v_1 \to v_2) := (v_2 \to v_1)$. The theory would be the same, but the information of direction of an edge is stored in $\Omega^1(\mathcal{G})$ rather than dictated at the beginning. While this is conceptually cleaner, many of the above equations will be very clumsy to write and may be harder to read. So we sticked to discussing directed graphs.*

Remark 2.2.4. *Note from Equation* (2.7) *that the Laplace operator is independent of the directions assigned to the edges.*

Next, we show that the Laplace operator has the properties we expect.

Proposition 2.2.5. *The Laplace operator Δ is self-adjoint; that is, $\langle \Delta f_1, f_2 \rangle_V = \langle f_1, \Delta f_2 \rangle_V$ for any $f_1, f_2 \in \Gamma(V, \mathscr{L})$. In particular, $\langle \Delta f, f \rangle_V$ is a real number for any $f \in \Gamma(V, \mathscr{L})$.*

Proof. We have for any $f_1, f_2 \in \Gamma(V, \mathscr{L})$,

$$\begin{aligned}
\langle \Delta f_1, f_2 \rangle_V = \langle \nabla^* \nabla f_1, f_2 \rangle_V = \overline{\langle f_2, \nabla^* \nabla f_1 \rangle}_V \\
= \overline{\langle \nabla f_2, \nabla f_1 \rangle}_E = \langle \nabla f_1, \nabla f_2 \rangle_E = \langle f_1, \Delta f_2 \rangle_V.
\end{aligned} \tag{2.8}$$

In particular, $\langle \Delta f, f \rangle_V = \langle f, \Delta f \rangle_V$. On the other hand, $\langle \Delta f, f \rangle_V = \overline{\langle f, \Delta f \rangle}_V$. Hence $\langle \Delta f, f \rangle_V$ is real. □

Proposition 2.2.6. *The Laplace operator Δ is semi-positive definite, that is,*

$$\langle \Delta f, f \rangle_V \geq 0$$

for any $f \in \Gamma(V, \mathscr{L})$. Equality holds if and only if $\nabla f = 0$, the zero section in $\Gamma(E, \mathscr{L})$.

Proof. We have

$$\langle \Delta f, f \rangle_V = \langle \nabla^* \nabla f, f \rangle_V = \langle \nabla f, \nabla f \rangle_E = \sum_{e \in E} \langle \nabla f(e), \nabla f(e) \rangle \geq 0, \tag{2.9}$$

since $\langle \nabla f(e), \nabla f(e) \rangle \geq 0$ for each $e \in E$. Equality holds if and only if $\langle \nabla f(e), \nabla f(e) \rangle = 0$ for each $e \in E$, i.e. $\nabla f(e) = 0$ for each $e \in E$. □

Proposition 2.2.7. *For $f \in \Gamma(V, \mathscr{L})$, the following are equivalent:*

1. *f is a constant section, i.e. $\varphi_{v_2 e}(f(v_2)) = \varphi_{v_1 e}(f(v_1))$ whenever e is an edge linking v_1 and v_2;*
2. *$\nabla f = 0$, the zero section in $\Gamma(E, \mathscr{L})$;*
3. *$\Delta f = 0$, the zero section in $\Gamma(V, \mathscr{L})$.*

In particular, $\ker(\Delta) = \ker(\nabla)$, and both spaces are finite-dimensional.

Proof. We see from Equation (2.3) that item (1) and (2) are equivalent. Next, we show (2) is equivalent to (3).

Assume that $\nabla f = 0$, then for any $g \in \Gamma(V, \mathscr{L})$, we have $\langle \Delta f, g \rangle_V = \langle \nabla^* \nabla f, g \rangle_V = \langle \nabla f, \nabla g \rangle_E = 0$. Thus $\Delta f = 0$.

Now assume that $\Delta f = 0$, then $\langle \Delta f, f \rangle = 0$. By Proposition 2.2.6, we have $\nabla f = 0$.

The space of constant sections as described in (1) clearly has the same dimension as $\mathscr{L}(v)$ for a vertex $v \in V$, and we have assumed $\mathscr{L}(v)$ to be finite dimensional. □

Proposition 2.2.8. *We have the orthogonal decomposition*

$$\Gamma(V, \mathscr{L}) = \ker(\Delta) \oplus \mathrm{Im}(\Delta).$$

Proof. Since $\ker(\Delta)$ is finite dimensional from Proposition 2.2.7, we have

$$\Gamma(V, \mathscr{L}) = \ker(\Delta) \oplus \ker(\Delta)^{\perp}.$$

It remains to show that

$$\ker(\Delta)^\perp = \mathrm{Im}(\Delta).$$

The inclusion $\mathrm{Im}(\Delta) \subseteq \ker(\Delta)^\perp$ follows from self-adjointness of Δ in Proposition 2.2.5. Next, we show $\ker(\Delta)^\perp \subseteq \mathrm{Im}(\Delta)$. Let $f \in \ker(\Delta)^\perp$. We want to show that there exists some $g \in \Gamma(V, \mathscr{L})$ such that $f = \Delta g$.

Define a linear functional ℓ on $\mathrm{Im}(\Delta)$ by $\ell(\Delta h) = \langle f, h\rangle_V$. This is well-defined, because if $\Delta h_1 = \Delta h_2$, then $h_1 - h_2 \in \ker(\Delta)$ and thus

$$\langle f, h_1\rangle_V - \langle f, h_1\rangle_V = \langle f, h_1 - h_2\rangle_V = 0.$$

By Hahn-Banach theorem, we can extend ℓ to a linear functional $\tilde{\ell}$ on the whole $\Gamma(V, \mathscr{L})$. By Riesz representation theorem, we have $\tilde{\ell}$ is represented by some $g \in \Gamma(V, \mathscr{L})$. That is, for any $u \in \Gamma(V, \mathscr{L})$, we have $\tilde{\ell}(u) = \langle g, u\rangle_V$. Then for any $h \in \Gamma(V, \mathscr{L})$, we have

$$\langle \Delta g, h\rangle_V = \langle g, \Delta h\rangle_V = \tilde{\ell}(\Delta h) = \ell(\Delta h) = \langle f, h\rangle_V.$$

Hence $f = \Delta g$ as desired. □

Let $\Gamma(V, \mathscr{L})^0$ denote the orthogonal complement of the space of constant sections i.e. $\ker(\Delta)^\perp$ above. Then clearly we have the following.

Corollary 2.2.9. *For $g \in \Gamma(V, \mathscr{L})$, the equation*

$$\Delta f = g$$

is solvable if and only if $g \in \Gamma(V, \mathscr{L})^0$. Moreover, the solution is unique if we also require $f \in \Gamma(V, \mathscr{L})^0$.

Now let $(W, \rho) = \mathcal{F}(\mathscr{L})$. Let $\widetilde{\mathscr{L}}$ denote the trivial bundle $\widetilde{\mathcal{G}} \times W$ over $\widetilde{\mathcal{G}}$. Following from the discussion in Section 2.1, we can view a section f in $\Gamma(V, \mathscr{L})$ as a section $\tilde{f}$ in $\Gamma(\widetilde{V}, \widetilde{\mathscr{L}})$ satisfying

$$\tilde{f}(\gamma\tilde{v}) = \rho(\gamma)\tilde{f}(\tilde{v}),$$

for any $\tilde{v} \in \widetilde{V}$ and any $\gamma \in \pi_1(\mathcal{G}, v_0)$. By Proposition 2.1.1 and Proposition 2.2.7, we see that

$f \in \ker(\Delta)$ corresponds to $\tilde{f} \in \Gamma(\widetilde{V}, \widetilde{\mathscr{L}})$ satisfying

$$\rho(\gamma)\tilde{f}(\tilde{v}) = \tilde{f}(\tilde{v})$$

for any $\tilde{v} \in \widetilde{V}$ and any $\gamma \in \pi_1(\mathcal{G}, v_0)$. That is, $\ker(\Delta)$ in $\Gamma(V, \mathscr{L})$ corresponds to $W^{\pi_1(\mathcal{G}, v_0)}$, the $\pi_1(\mathcal{G}, v_0)$-invariants in W. In particular, if ρ is irreducible and nontrivial, then $\ker(\Delta)$ is trivial.

Corollary 2.2.10. *Assume that ρ is irreducible and non-trivial, then for any $g \in \Gamma(V, \mathscr{L})$, $\Delta f = g$ has a unique solution f.*

2.3 Vector-valued Green's Functions

We find a formula to the unique solution in Corollary 2.2.9 via *Green's functions*, just as how we normally solve second-order differential equations.

Definition 2.3.1. *The* (vector-valued) Green's function G *associated with the local system $\mathscr{L}$ on the graph $\mathcal{G} = (V, E)$ is defined to be the unique element $G \in \Gamma(V, \mathscr{L})^0 \otimes \Gamma(V, \mathscr{L})^0$ such that*

$$\Delta_v G(v, w) = \delta_w(v) \tag{2.10}$$

as distributions. That is, if we write $G = \sum_i a_i \otimes b_i$, for some $a_i, b_i \in \Gamma(V, \mathscr{L})^0$, then for any $f \in \Gamma(V, \mathscr{L})^0$,

$$f(w) = \sum_{v \in V} \langle f(v), \Delta_v G(v, w) \rangle := \sum_{v \in V} \sum_i \langle f(v), \Delta_v a_i(v) \rangle \overline{b_i(w)}. \tag{2.11}$$

Therefore, given the Green's function $G = \sum_i a_i \otimes b_i$, for any $\varphi \in \Gamma(V, \mathscr{L})^0$, the unique solution in $\Gamma(V, \mathscr{L})^0$ to $\Delta f = \varphi$ is given by

$$f(w) = \sum_{v \in V} \langle \varphi(v), G(v, w) \rangle = \sum_{v \in V} \sum_i \langle \varphi(v), a_i(v) \rangle \overline{b_i(w)}.$$

Since $\Gamma(V, \mathscr{L})^0$ is a finite-dimensional hermitian vector space on which Δ is an invertible (by Corollary 2.2.9) linear operator, we denote by Δ^{-1} the inverse operator. Given any orthonormal basis $[\varphi_1, \ldots, \varphi_m]$ of $\Gamma(V, \mathscr{L})^0$, we have $G(v, w) = \sum_{i=1}^m \varphi_i(v) \otimes \overline{\Delta^{-1}\varphi_i(w)}$.

Since Δ is self-adjoint with respect to the hermitian inner product, we may further assume that $[\varphi_1, \ldots, \varphi_m]$ is an orthonormal eigenbasis of $\Gamma(V, \mathscr{L})^0$ with respect to Δ, with eigenvalues $\lambda_1, \ldots, \lambda_m$ respectively. Since $\Gamma(V, \mathscr{L})^0 = \ker(\Delta)^\perp$ and by Proposition 2.2.6, we have each λ_i is a positive real number. Since $\Delta\varphi_i = \lambda_i \varphi_i$, we have $\Delta^{-1}\varphi_i = \frac{\varphi_i}{\lambda_i}$. Therefore, $G(v, w) = \sum_{i=1}^m \frac{\varphi_i(v) \otimes \overline{\varphi_i(w)}}{\lambda_i}$.

Extend $[\varphi_1, \ldots, \varphi_m]$ to an orthonormal eigenbasis $[\varphi_1, \ldots, \varphi_{m+t}]$ of $\Gamma(V, \mathscr{L})$ with respect to Δ. Since $\Gamma(V, \mathscr{L}) = \Gamma(V, \mathscr{L})^0 \oplus \ker(\Delta)$, we have $\lambda_{m+1} = \cdots = \lambda_{m+t} = 0$. Let $s \in \mathbb{R}^+ \cap F$, then $\ker(\Delta + s) = 0$, and

$$G_s(v, w) = \sum_{i=1}^{m+t} \frac{\varphi_i(v) \otimes \overline{\varphi_i(w)}}{\lambda_i + s} \tag{2.12}$$

verifies the following definition:

Definition 2.3.2. *Let $s \in \mathbb{R}^+ \cap F$. The* weight-$s$ *(vector-valued) Green's function G_s associated with the local system $\mathscr{L}$ on the graph $\mathcal{G} = (V, E)$ is defined to be the unique element $G_s = \sum_i a_i \otimes b_i \in \Gamma(V, \mathscr{L}) \otimes \Gamma(V, \mathscr{L})$ such that*

$$(\Delta + s)f = \varphi \iff f(w) = \sum_{v \in V} \langle \varphi(v), G_s(v, w) \rangle = \sum_{v \in V} \sum_i \langle \varphi(v), a_i(v) \rangle \overline{b_i(w)}, \quad \forall w \in \mathcal{V},$$

for any $f, \varphi \in \Gamma(V, \mathscr{L})$.

In practice, we often find the formulas for $G_s(v, w)$ for arbitrary s first. Then, note from Equation (2.12) that G_s as a function of s has a simple pole at $s = 0$, with residue

$$\chi = \sum_{i=1}^{t} \psi_i(v) \otimes \overline{\psi_i(w)}, \tag{2.13}$$

where $[\psi_1, \ldots, \psi_t]$ $(= [\varphi_{m+1}, \ldots, \varphi_{m+t}])$ is an orthonormal basis of $\ker(\Delta)$, the space of constant sections. Then

$$G(v, w) = \lim_{s \to 0^+} (G_s(v, w) - \frac{\chi}{s}). \tag{2.14}$$

Remark 2.3.3. *Let T be the universal cover of $\mathcal{G}$ and let $(L, \rho) = \mathcal{F}(\mathscr{L})$ be the corresponding representation of $\Pi := \pi_1(\mathcal{G})$ following the notation in Proposition 2.1.1. Then up to isomorphism of local systems, we have $\mathscr{L} = \Pi \backslash (\mathcal{T} \times L)$, where the Π-action is given by $\gamma(v, \ell) = (\gamma v, \rho(v)\ell)$ and $\gamma(e, \ell) = (\gamma e, \rho(v)\ell)$, for any $\gamma \in \Pi, \ell \in L$, and for any vertex v and any edge e of T.*

Under this identification, an element $H \in \Gamma(V, \mathscr{L}) \otimes \Gamma(V, \mathscr{L})$, which we called a vector-valued function*, can also be regarded as a map from $V \times V$ to* $\mathrm{Hom}(L, L)$*. Indeed, given $H = \sum_i a_i \otimes b_i \in \Gamma(V, \mathscr{L}) \otimes \Gamma(V, \mathscr{L})$ and $(v, w) \in V \times V$, $H(v, w)$ defines an element in* $\mathrm{Hom}(L, L)$*, sending a vector $\alpha \in L$ to $\sum_i \langle \alpha, a_i(v) \rangle \overline{b_i(w)}$. In this way, $\langle *, G(v, w) \rangle$ [resp. $\langle *, G_s(v, w) \rangle$] that appeared earlier in this section can be rewritten as $G(v, w)(*)$ [resp. $G_s(v, w)(*)$].*

2.4 Green's Function on Local System of Spheres

In this section, we consider a *spherical local system* $\mathcal{S}$ on $\mathcal{G} = (V, E)$, which consists of the following data:

- for each $v \in V$, a 2-sphere $\mathcal{S}_v$ over $\mathbb{R}$;
- for each $e \in E$, a 2-sphere $\mathcal{S}_e$ over $\mathbb{R}$;
- for each pair (v, e) where v is an end of e, we have an isometry (by which we mean a bijective isometry) $\psi_{ve} : \mathcal{S}_v \to \mathcal{S}_e$.

An isometry $\psi_{ve} : \mathcal{S}_v \to \mathcal{S}_e$ induces an isometric isomorphism $\varphi_{ve} : L^2(\mathcal{S}_v) \to L^2(\mathcal{S}_e), f \mapsto f \circ \psi^{-1}$ between $\mathbb{R}$-valued square-integrable functions on the respective spheres. Therefore, $\mathcal{S}$ defines a local system $L^2(\mathcal{S})$ on $\mathcal{G}$, where the local vector spaces are given by $L^2(\mathcal{S}_v)$ for $v \in V$ and $L^2(\mathcal{S}_e)$ for $e \in E$, and the transition maps are given by φ_{ve} whenever v is an end of e.

Recall that the set $\mathrm{Aut}(S^2)$ of isometries from the standard 2-sphere S^2 to itself is given by the special orthogonal group $\mathrm{SO}(3)$. Then from the above we get a $\mathrm{SO}(3)$ representation on $L^2(S^2)$ via $g \cdot f(\alpha) = f(g^{-1}\alpha) = f(g^t\alpha)$ for any $g \in \mathrm{SO}(3)$, and any $\alpha \in S^2$. Let H_n denote the space of homogeneous harmonic polynomials of degree n in $\mathbb{R}[x, y, z]$, where by harmonic we mean vanishing under the Euclidean Laplacian operator

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}.$$

An element $f(x, y, z) \in H_n$ can be completely determined by its evaluation on S^2 and is clearly L^2 as a function on S^2, therefore H_n can be regarded as a subspace of $L^2(S^2)$, which is an invariant subspace under the $\mathrm{SO}(3)$ action. In fact, for each $n \in \mathbb{N}$, H_n is the unique (up to isomorphism) irreducible $(2n + 1)$-dimensional representation of $\mathrm{SO}(3)$. (For this and more about the *spherical harmonics* and $\mathrm{SO}(3)$ representation, see for example [16, Chapter 7].) We let $\mathcal{H}_n(\mathcal{S})$ denote the corresponding sub-local system of $L^2(\mathcal{S})$. Note that restricting a transition map φ_{ve} to $H_n(\mathcal{S}_v)$ defines an isometric isomorphism from $H_n(\mathcal{S}_v)$ to $H_n(\mathcal{S}_e)$, which we still denote by φ_{ve}.

Note however that an arbitrary element $f \in H_n$ is *not* a harmonic function on S^2, even though it is harmonic on $\mathbb{R}^3$. Recall that in terms of spherical coordinates (r, θ, ϕ), the Euclidean Laplacian Δ is given by

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\Delta_{S^2},$$

where

$$\Delta_{S^2} = -\left(\frac{\partial^2}{\partial\theta^2} + \cot\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right)$$

is the *spherical Laplacian*, which precisely equal to $-\Delta$ when r is restricted to be constantly equal to 1. For any homogeneous polynomial $f(x, y, z)$ of degree n, we have

$$f(x, y, z) = r^n Y(\theta, \phi)$$

in terms of the spherical coordinates, for some two variable function Y. Then $\Delta f = 0$ if and only if $\Delta_{S^2} f = n(n+1)f$.

Note that in most literature, one writes $\Delta = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\Delta_{S^2}$ with $\Delta_{S^2} = \frac{\partial^2}{\partial\theta^2} + \cot\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}$. We change the conventions in the above to be more consistent in Part II.

Given $p \in \mathcal{S}_v$, the stabilizer A_p of p in $\mathrm{Aut}(\mathcal{S}_v)$ consists of all the rotations fixing p. Then there exists a unique element $s_p^{(n)} \in H_n(\mathcal{S}_v)$ such that

(i) $g \cdot s_p^{(n)} = s_p^{(n)}$ for any $g \in A_p$;

(ii) $\left|\left|s_p^{(n)}\right|\right|_{L^2(\mathcal{S}_v)} = 1$; and that

(iii) $s_p^{(n)}(p) > 0$.

Indeed, recall that an orthonormal basis for H_n is given by

$$c_n^m P_n^m(\cos\theta)\cos(m\phi), \quad c_n^m P_n^m(\cos\theta)\sin(m\phi) \quad \text{for } m = 1, \ldots, n; \text{ and } c_n P_n(\cos\theta),$$

where for each m (we suppress this index when $m = 0$) and n, $P_n^m(x)$ is the asscotiated Legendre Polynomial of degree n order m, and c_n^m is a normalizing scalar. Consider the case $p = (0, 0, 1)$. Then by (i), $s_p^{(n)}$ is independent of ϕ, thus is a scalar multiple of $P_n(\cos\theta)$. Then condition (ii) and (iii) fixes the scalar to be $\frac{\sqrt{2n+1}}{4\pi}$ (which is simply $\pm c_n$ depending on your convention for c_n). More generally, for arbitrary $p \in \mathcal{S}_v$, we have

$$s_p^{(n)}(q) = \frac{\sqrt{2n+1}}{4\pi} P_n(\cos\theta(p, q)), \text{ for any } q \in \mathcal{S}_v,$$

where $\theta(p, q)$ denotes the angle between the two vectors connecting the center of the sphere with p and q respectively.

For clarity in case of different conventions in literature, recall that the Legendre polynomiral

$P_n(t)$ is given by

$$P_n(t) = \frac{1}{2^n n!}\frac{d^n}{dt^n}(t^2-1)^n,$$

which satisfies the differential equation

$$(1-t^2)P_n''(t) - 2tP_n'(t) + n(n+1)P_n(t) = 0$$

with normalization $P_n(1) = 1$.

Definition 2.4.1. *Given a local system $\mathcal{S}$ of spheres over the graph $\mathcal{G} = (V, E)$. Assume that the Green's function G [resp. weight-s Green's function G_s] associated with the local system $\mathcal{H}_n(\mathcal{S})$ are given by $G = \sum_i a_i \otimes b_i$ [resp. $G_s = \sum_j c_j \otimes d_j$]. Define the scalar-valued Green's function $G^{(n)}$ [resp. weight-s scalar-valued Green's function $G_s^{(n)}$] associated with $\mathcal{H}_n(\mathcal{S})$ on $\coprod_{v\in V}\mathcal{S}_v \times \coprod_{v\in V}\mathcal{S}_v$ by*

$$G^{(n)}(p,q) = \sum_i \langle a_i(v), s_p^{(n)}\rangle\langle\overline{b_i(w)}, s_q^{(n)}\rangle$$

and

$$G_s^{(n)}(p,q) = \sum_j \langle c_j(v), s_p^{(n)}\rangle\langle\overline{d_j(w)}, s_q^{(n)}\rangle$$

respectively, for any $p \in \mathcal{S}_v$ and $q \in \mathcal{S}_w$.

Chapter 3

Applications to Quotients of the Bruhat-Tits Tree

In what follows we apply the theory above to quotients of the Bruhat-Tits Tree. We first recall the construction of the Bruhat-Tits tree following [6].

3.1 The Bruhat-Tits Tree

Let p be a prime. Consider $V = \mathbb{Q}_p^2$, and V^* its dual space. A free rank-2 $\mathbb{Z}_p$ module L in V^* is called a lattice in V^*. We say two lattices L_1 and L_2 are equivalent if there is a scalar $a \in \mathbb{Q}_p$ such that $L_1 = aL_2$.

The vertices of the Bruhat-Tits tree $\mathcal{T}$ are equivalence classes $[L]$ of lattices $L \subseteq V^*$. Two vertices $[L_1]$ and $[L_2]$ are joined by an edge if

$$pL_1 \subsetneq L_2 \subsetneq L_1 \tag{3.1}$$

for some representatives L_1 and L_2. Note that this is symmetric on $[L_1]$ and $[L_2]$ as Equation (3.1) implies

$$p\left(\frac{1}{p}L_2\right) \subsetneq L_1 \subsetneq \frac{1}{p}L_2. \tag{3.2}$$

One can also think of Equation (3.1) as the following – there exist $e_1, e_2 \in V^*$ such that $[L_1]$ can be represented by $\langle e_1, e_2\rangle$, the lattice generated by e_1 and e_2, while $[L_2]$ can be represented by $\langle e_1, pe_2\rangle$.

Proposition 3.1.1. *The Bruhat-Tits tree $\mathcal{T}$ is a homogeneous tree of degree $p+1$.*

Proof. First we show the combinatorial object described is indeed a tree. Assume to the contrary

that there is a cycle, then there is a chain of $\mathbb{Z}_p$-modules

$$p^r L_0 \subsetneq L_d \subsetneq L_{d-1} \subsetneq \cdots \subsetneq L_1 \subsetneq L_0,$$

where $d \geq 2$ and $L_0, L_1, L_2, \ldots, L_d$ are pairwisely non-equivalent and for each i, $[L_i]$ is joined with $[L_{i+1}]$ by an edge. Note that L_0/L_1 is cyclic but $L_0/p^r L_0$ is not. So there is a smallest i such that L_0/L_i is cyclic but L_0/L_{i+1} is not. So L_{i-1}/L_{i+1} is a non-cyclic, length-2 $\mathbb{Z}_p$-module, so $L_{i+1} = pL_{i-1}$, a contradiction.

Next, we show that the Bruhat-Tits tree has degree $p+1$. This is because

$$\{\text{the edges coming out of } [L]\} \xrightarrow{1:1} \mathbb{P}\left(L/pL\right) \xrightarrow{1:1} \mathbb{P}^1(\mathbb{F}_p), \tag{3.3}$$

which has cardinality $p+1$. □

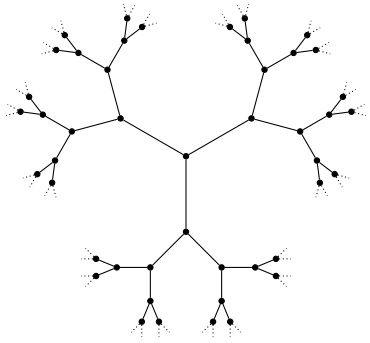


Figure 3.1: Bruhat-Tits tree when $p = 2$

From the construction, $\mathrm{PGL}_2(\mathbb{Q}_p)$ acts transitively on the vertices (resp. edges) of $\mathcal{T}$. The stabilizer of a vertex is isomorphic to $\mathrm{PGL}_2(\mathbb{Z}_p)$ and the stabilizer of an edge is isomorphic to

$$K_0(p) = \left\{ g \in PGL_2(\mathbb{Z}_p) : g \equiv \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \mod p. \right\}$$

Thus the set $\mathcal{V}$ of vertices of $\mathcal{T}$ is in bijection with $\mathrm{PGL}_2(\mathbb{Q}_p)/\mathrm{PGL}_2(\mathbb{Z}_p)$ and the set $\mathcal{E}$ of edges of $\mathcal{T}$ is in bijection with $\mathrm{PGL}_2(\mathbb{Q}_p)/K_0(p)$. Recall that the Hecke operator T_p acts on $\mathrm{PGL}_2(\mathbb{Q}_p)/\mathrm{PGL}_2(\mathbb{Z}_p)$ by taking the formal sum of index-p sub-lattices. This translates to the Hecke action on $\mathcal{V}$ as the formal sum over the neighboring $p+1$ vertices of a vertex. That is,

$$T_p(v) = \sum_{\substack{v \text{ is connected to} \\ v' \text{ by an edge}}} v', \tag{3.4}$$

for each $v \in \mathcal{V}$. This formal sum means that whenever we have a function f on $\mathcal{V}$,

$$T_p f(v) = \sum_{\substack{v \text{ is connected to} \\ v' \text{ by an edge}}} f(v'), \tag{3.5}$$

for each $v \in \mathcal{V}$.

Given a local system on $\mathcal{T}$ such that all the transition maps φ_{ve} are constant, then by Equation (2.7) we see that

$$\Delta f(v) = (p + 1 - T_p) f(v). \tag{3.6}$$

3.2 Green's Functions on the Bruhat-Tits Tree

In this section we first construct $\mathbb{C}$-valued weight-s Green's functions on the Bruhat-Tits tree $\mathcal{T} = \mathrm{PGL}_2(\mathbb{Q}_p)/\mathrm{PGL}_2(\mathbb{Z}_p)$. More precisely, we will show the following theorem.

Theorem 3.2.1. *Let $\mathcal{V}$ be the set of vertices of $\mathcal{T}$. Let $s \in \mathbb{R}$. There exists a function $g_s(v, w)$ from $\mathcal{V} \times \mathcal{V}$ to $\mathbb{C}$ with the following three properties if and only if $s < -(2p+2)$ or $s > 0$. When this is the case, the function $g_s(v, w)$ is uniquely determined by the three properties and the range of g_s is in $\mathbb{R}$.*

1. *It is point-pair invariant under the automorphism group $\mathrm{PGL}_2(\mathbb{Q}_p)$ of $\mathcal{T}$. That is,*

$$g_s(\gamma v, \gamma w) = g_s(v, w) \quad \text{for all } \gamma \in \mathrm{PGL}_2(\mathbb{Q}_p) \tag{3.7}$$

2. *There exists a constant N such that*

$$\sum_{w \in \mathcal{V}} |g_s(v, w)| \leq N \tag{3.8}$$

   for all $v \in \mathcal{V}$.

3. *For every bounded function f on $\mathcal{V}$,*

$$(\Delta + s) f = \phi \iff f(w) = \sum_{v \in \mathcal{V}} \phi(v) \overline{g_s(v, w)}, \quad \forall w \in \mathcal{V}. \tag{3.9}$$

Remark 3.2.2. *Since a tree has trivial fundamental group, up to isomorphism of local systems (c.f. Section 2.1), any local system $\mathscr{L}$ over $\mathcal{T}$ can be regarded as a constant local system; that is $\mathscr{L}(v) = \mathscr{L}(e) = L$ for some fixed hermitian vector space L, for any vertex v and any edge e, and the transition maps are identity maps. Let $[\ell_1, \ldots, \ell_r]$ be an orthonormal basis of L, then*

$$\vec{g}_s(v, w) = g_s(v, w) \sum_i \ell_i \otimes \overline{\ell_i}$$

verifies definition 2.3.2 for bounded f. We need boundedness here and in Theorem 3.2.1 to compensate for the fact that the graph is now infinite.

We will spend the rest of this section proving Theorem 3.2.1.

From Equation (3.7), the function $g_s(v, w)$ only depends on the distance $d(v, w)$ of the two vertices v and w, that is, the number of edges in the unique non-backtracking path connecting v and w. Indeed, if $d(v, w) = d(v', w') = n$, then for some basis e_1, e_2 of the vector space V^*, the vertex v can be represented by the lattice $\langle e_1, e_2 \rangle$ while w by $\langle e_1, p^n e_2 \rangle$; similarly for some basis e_1', e_2' of the vector space V^*, the vertex v' can be represented by the lattice $\langle e_1', e_2' \rangle$ while w' by $\langle e_1', p^n e_2' \rangle$. Let γ be the change of basis matrix from $[e_1, e_2]$ to $[e_1', e_2']$, then $v' = \gamma v$ and $w' = \gamma w$ thus $g_s(v', w') = g_s(\gamma v, \gamma w) = g_s(v, w)$.

Corollary 3.2.3. *The function $g_s(v, w)$ satisfying the properties in Theorem 3.2.1 is symmetric in the two variables. That is,*

$$g_s(v, w) = g_s(w, v) \tag{3.10}$$

for any $v, w \in \mathcal{V}$.

Fix a vertex v_0 on the Bruhat-Tits tree, and view it as the center of the tree, then $g_s(v_0, v)$ depends only on the distance $d(v_0, v)$, so each "layer" of vertices should give the same function value. We look for a sequence $\{a_n\}_{n \geq 1}$ so that $\overline{g_s(v_0, v)} = a_n$ if v is on the n-th layer outside v_0, i.e. if $d(v_0, v) = n$.

Then Property 2 can be rewritten as

$$|a_0| + \sum_{n=1}^{\infty} |a_n| \, p^{n-1}(p+1) \quad \text{is finite,} \tag{3.11}$$

since for each $n \geq 1$, there are $p^{n-1}(p+1)$ vertices on the n-th layer. We now look at Property 3 in

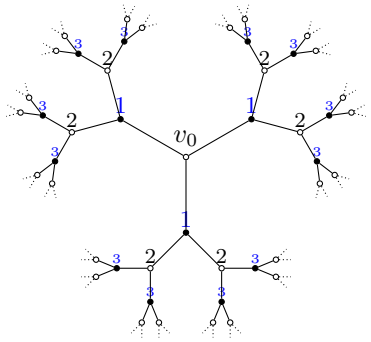


Figure 3.2: Bruhat-Tits tree when $p = 2$, with marked layer-count

Theorem 3.2.1. By definition, $(\Delta + s)\, f = \phi$ means for any $w \in \mathcal{V}$, we have

$$\phi(w) = (p+1+s)f(w) - \sum_{d(w,w')=1} f(w') = \lambda f(w) - \sum_{d(w,w')=1} f(w'), \tag{3.12}$$

if we let

$$\lambda = s + p + 1. \tag{3.13}$$

Since f is bounded, so is ϕ. Indeed, if there exists some $M > 0$ such that $|f(v)| \leq M$ for each $v \in V$, then for each $w \in V$,

$$|\phi(w)| = \left| \lambda f(w) - \sum_{d(w,w')=1} f(w') \right| \leq |\lambda|\, |f(w)| + \sum_{d(w,w')=1} |f(w')| \leq |\lambda|\, M + (p+1)M, \tag{3.14}$$

which is finite. We are looking to satisfy

$$\begin{aligned} f(v) = \sum_{w \in \mathcal{V}} \phi(w) \overline{g_s(v,w)} &= \sum_{n=0}^{\infty} \sum_{d(w,v)=n} \phi(w) a_n \\ &= \sum_{n=0}^{\infty} \sum_{d(w,v)=n} \left(\lambda f(w) - \sum_{d(w',w)=1} f(w') \right) a_n \\ &= (\lambda a_0 - (p+1)a_1) f(v) + \sum_{n=1}^{\infty} \sum_{d(w,v)=n} \left(-a_{n-1} + \lambda a_n - p a_{n+1} \right) f(w). \end{aligned} \tag{3.15}$$

We briefly explain the equation above. The series $\sum_{w \in \mathcal{V}} \phi(w) \overline{g_s(v,w)}$ is absolutely convergent since ϕ is bounded and $\sum_{w \in \mathcal{V}} |g_s(v,w)|$ is finite. Therefore we may rearrange the summands as we like. The first two lines should be clear and we now explain how to get the last line from the second. The term $f(v)$ appears in the $n = 0$ summand in the second line as $\lambda f(v) a_0$, and in the $n = 1$ summand as $-f(v)a_1$ for each of the $p + 1$ neighbors of v, and in none of the $n \geq 2$ summands,

hence the overall coefficient of $f(v)$ is $\lambda a_0 - (p+1)a_1$. If $d(w,v) = n \geq 1$, then $f(w)$ only appears in the $n-1$, n, and $n+1$ summands of the second line, in the $n-1$ summand as $-f(w)a_{n-1}$, in the n summand as $\lambda f(w)a_n$, and in the $n+1$ summand as $-f(w)a_{n+1}$ for each of the p neighbors of w that is on the $(n+1)$-th layer outside v. Therefore when $d(w,v) = n \geq 1$, the coefficient for $f(w)$ is $(-a_{n-1} + \lambda a_n - pa_{n+1})$.

Since Equation (3.15) should hold for every bounded f, we must have

$$\begin{cases} \lambda a_0 - (p+1)a_1 = 1, & \text{and} \\ -a_{n-1} + \lambda a_n - pa_{n+1} = 0, & \text{for each } n \geq 1. \end{cases} \tag{3.16}$$

We solve the difference equation in the second line first. We have for all $n \geq 1$,

$$pa_{n+1} = \lambda a_n - a_{n-1}$$

so for any α,

$$pa_{n+1} - \alpha a_n = (\lambda - \alpha)a_n - a_{n-1}.$$

In particular, when

$$\frac{\lambda - \alpha}{p} = \frac{1}{\alpha} \tag{3.17}$$

i.e.

$$\alpha = \frac{\lambda \pm \sqrt{\lambda^2 - 4p}}{2} \tag{3.18}$$

we have

$$p\left(a_{n+1} - \frac{\alpha}{p}a_n\right) = (\lambda - \alpha)\left(a_n - \frac{1}{\lambda - \alpha}a_{n-1}\right) = (\lambda - \alpha)\left(a_n - \frac{\alpha}{p}a_{n-1}\right)$$

so

$$a_{n+1} - \frac{\alpha}{p}a_n = \frac{\lambda - \alpha}{p}\left(a_n - \frac{\alpha}{p}a_{n-1}\right) = \frac{1}{\alpha}\left(a_n - \frac{\alpha}{p}a_{n-1}\right). \tag{3.19}$$

That is, if for $n \geq 1$ we let

$$b_n = a_n - \frac{\alpha}{p}a_{n-1}, \tag{3.20}$$

then

$$b_{n+1} = \frac{1}{\alpha}b_n \tag{3.21}$$

From Equation (3.21) we get

$$b_n = \frac{1}{\alpha^{n-1}}b_1, \tag{3.22}$$

and hence

$$a_n - \frac{\alpha}{p}a_{n-1} = \frac{1}{\alpha^{n-1}}\left(a_1 - \frac{\alpha}{p}a_0\right). \tag{3.23}$$

Iterating Equation (3.23), we get

$$\begin{aligned}
a_n &= \frac{\alpha}{p}a_{n-1} + \frac{1}{\alpha^{n-1}}\left(a_1 - \frac{\alpha}{p}a_0\right) \\
&= \frac{\alpha}{p}\left(\frac{\alpha}{p}a_{n-2} + \frac{1}{\alpha^{n-2}}\left(a_1 - \frac{\alpha}{p}a_0\right)\right) + \frac{1}{\alpha^{n-1}}\left(a_1 - \frac{\alpha}{p}a_0\right) \\
&= \left(\frac{\alpha}{p}\right)^2 a_{n-2} + \frac{1}{\alpha^{n-1}}\left(a_1 - \frac{\alpha}{p}a_0\right)\left(1 + \frac{\alpha^2}{p}\right) \\
&= \cdots \\
&= \left(\frac{\alpha}{p}\right)^n a_0 + \frac{1}{\alpha^{n-1}}\left(a_1 - \frac{\alpha}{p}a_0\right)\left(1 + \frac{\alpha^2}{p} + \left(\frac{\alpha^2}{p}\right)^2 + \cdots + \left(\frac{\alpha^2}{p}\right)^{n-1}\right).
\end{aligned} \tag{3.24}$$

Note that when $\lambda^2 - 4p = 0$, $\lambda = \pm 2\sqrt{p}$ and $\alpha = \frac{\lambda}{2} = \pm\sqrt{p}$, and we have

$$a_n = \frac{1}{\alpha^n}(n(\alpha a_1 - a_0) + a_0). \tag{3.25}$$

When $\lambda^2 - 4p \neq 0$ ($\iff \alpha^2 \neq p \iff \frac{\alpha^2}{p} \neq 1$), we further have

$$\begin{aligned}
a_n &= \left(\frac{\alpha}{p}\right)^n a_0 + \frac{1}{\alpha^{n-1}}\left(a_1 - \frac{\alpha}{p}a_0\right)\left(\frac{1 - \left(\frac{\alpha^2}{p}\right)^n}{1 - \frac{\alpha^2}{p}}\right) \\
&= \left(\frac{\alpha}{p}\right)^n a_0 + \frac{pa_1 - \alpha a_0}{\alpha^{n-1}(p - \alpha^2)}\left(1 - \left(\frac{\alpha^2}{p}\right)^n\right) \\
&= \left(\frac{\alpha}{p}\right)^n a_0 + \frac{pa_1 - \alpha a_0}{p - \alpha^2} \cdot \frac{1}{\alpha^{n-1}} - \frac{pa_1\alpha - a_0\alpha^2}{p - \alpha^2}\left(\frac{\alpha}{p}\right)^n \\
&= \left(\frac{p}{p - \alpha^2}a_0 - \frac{p\alpha}{p - \alpha^2}a_1\right)\left(\frac{\alpha}{p}\right)^n + \left(\frac{p\alpha}{p - \alpha^2}a_1 - \frac{\alpha^2}{p - \alpha^2}a_0\right)\left(\frac{1}{\alpha}\right)^n \\
&= \left(\frac{\frac{1}{\alpha}}{\frac{1}{\alpha} - \frac{\alpha}{p}}a_0 - \frac{1}{\frac{1}{\alpha} - \frac{\alpha}{p}}a_1\right)\left(\frac{\alpha}{p}\right)^n + \left(\frac{\frac{\alpha}{p}}{\frac{\alpha}{p} - \frac{1}{\alpha}}a_0 - \frac{1}{\frac{\alpha}{p} - \frac{1}{\alpha}}a_1\right)\left(\frac{1}{\alpha}\right)^n.
\end{aligned} \tag{3.26}$$

For the ease of writing we will use the shorthanded notation:

$$c_1 = \frac{\frac{1}{\alpha}}{\frac{1}{\alpha} - \frac{\alpha}{p}}a_0 - \frac{1}{\frac{1}{\alpha} - \frac{\alpha}{p}}a_1, \quad c_2 = \frac{\frac{\alpha}{p}}{\frac{\alpha}{p} - \frac{1}{\alpha}}a_0 - \frac{1}{\frac{\alpha}{p} - \frac{1}{\alpha}}a_1 \tag{3.27}$$

Then

$$a_n = c_1\left(\frac{\alpha}{p}\right)^n + c_2\left(\frac{1}{\alpha}\right)^n. \tag{3.28}$$

From Equation (3.13) and Equation (3.18), when $\alpha = \frac{\lambda+\sqrt{\lambda^2-4p}}{2}$, we have

$$\frac{\alpha}{p} = \frac{\lambda+\sqrt{\lambda^2-4p}}{2p}, \quad \frac{1}{\alpha} = \frac{\lambda-\sqrt{\lambda^2-4p}}{2p}. \tag{3.29}$$

Note that when we take $\alpha = \frac{\lambda-\sqrt{\lambda^2-4p}}{2}$, the values of $\frac{\alpha}{p}$ and $\frac{1}{\alpha}$ are simply swapped compared to those in Equation (3.29). And when $\frac{\alpha}{p}$ and $\frac{1}{\alpha}$ are swapped, so are c_1 and c_2. Thus the formula for a_n is invariant under the interchange of $\frac{\alpha}{p}$ and $\frac{1}{\alpha}$. Therefore, without any loss of generality, in the formula (3.26) we may simply take α to be

$$\alpha = \frac{\lambda+\sqrt{\lambda^2-4p}}{2} \tag{3.30}$$

and the values of $\frac{1}{\alpha}$ and $\frac{\alpha}{p}$ as in Equation (3.29). The formula for a_n is now unique up to the choice of a_0 and a_1.

We will now work towards Property 2 (or (3.11)). We will discuss the following cases

1. $\lambda^2 - 4p > 0$ and $\lambda > 0$, which includes $s > 0$ since when $s > 0$ we must have

$$\lambda = s + p + 1 > p + 1 > 0 \text{ and } \lambda^2 - 4p > (p+1)^2 - 4p = (p-1)^2 \geq 0;$$

2. $\lambda^2 - 4p > 0$ and $\lambda < 0$, which includes $s < -(2p+2)$ since when $s < -(2p+2)$ we must have

$$\lambda = s + p + 1 < -(p+1) < 0 \text{ and } \lambda^2 - 4p > (p+1)^2 - 4p = (p-1)^2 \geq 0;$$

3. $\lambda^2 - 4p \leq 0$.

We first assume $\lambda^2 - 4p > 0$. Then both $\frac{\alpha}{p}$ and $\frac{1}{\alpha}$ are real numbers. Hence

$$\begin{aligned}
&\sum_{n=1}^{\infty} |a_n| \, p^{n-1}(p+1) = \sum_{n=1}^{\infty} \sqrt{a_n \bar{a}_n} p^{n-1}(p+1) \\
=&\sum_{n=1}^{\infty} \sqrt{\left(c_1 \left(\frac{\alpha}{p}\right)^n + c_2 \left(\frac{1}{\alpha}\right)^n\right)\left(\bar{c}_1 \left(\frac{\alpha}{p}\right)^n + \bar{c}_2 \left(\frac{1}{\alpha}\right)^n\right)} p^{n-1}(p+1) \\
=&(p+1)\sum_{n=1}^{\infty} \sqrt{\frac{|c_1|^2}{p^2}\alpha^{2n} + \frac{|c_2|^2}{p^2}\left(\frac{p}{\alpha}\right)^{2n} + (c_1\bar{c}_2 + \bar{c}_1 c_2)p^{n-2}}.
\end{aligned} \tag{3.31}$$

From Equation (3.29), when $\lambda > 0$ we have

$$\frac{\alpha}{p} > \frac{1}{\alpha} > 0 \tag{3.32}$$

so

$$\frac{\alpha^2}{p} > 1 \text{ and } \alpha > \sqrt{p}, \tag{3.33}$$

and the series in Equation (3.31) converges only if $c_1 = 0$, or equivalently

$$a_1 = \frac{a_0}{\alpha}. \tag{3.34}$$

However, $c_1 = 0$ does not guarantee the convergence: we need to compare

$$\left|\frac{p}{\alpha}\right| = \frac{p}{\alpha} = \frac{2p}{\lambda + \sqrt{\lambda^2 - 4p}}$$

with 1. Note that under the condition $\lambda > 0$ and $\lambda^2 - 4p > 0$, we have

$$\begin{aligned}
&\frac{2p}{\lambda + \sqrt{\lambda^2 - 4p}} < 1 \iff \sqrt{\lambda^2 - 4p} > 2p - \lambda = -s + p - 1 \\
\iff & -s + p - 1 < 0 \text{ or } \lambda^2 - 4p > 4p^2 + \lambda^2 - 4p\lambda \\
\iff & -s + p - 1 < 0 \text{ or } 4p(-1 - p + \lambda) > 0 \\
\iff & s > p - 1 \text{ or } s > 0 \\
\iff & s > 0.
\end{aligned} \tag{3.35}$$

Thus when $s \leq 0$, in order for Equation (3.31) to converge, we must have $c_1 = c_2 = 0$ and cannot satisfy the first line of Equation (3.16), a contradiction. When $s > 0$, Equation (3.31) does converge when c_1=0 and together with the first line of Equation (3.16) and Equaton (3.27), we get

$$a_0 = \alpha a_1 = \frac{\alpha}{\lambda\alpha - (p+1)}, \quad a_1 = \frac{1}{\lambda\alpha - (p+1)}, \quad c_2 = \frac{\frac{\alpha}{p}}{\frac{\alpha}{p} - \frac{1}{\alpha}} a_0 - \frac{\frac{1}{\alpha}}{\frac{\alpha}{p} - \frac{1}{\alpha}} a_0 = a_0 \tag{3.36}$$

so we have

$$a_n = \frac{a_0}{\alpha^n} = \frac{1}{(\lambda\alpha - (p+1))\alpha^{n-1}}, \tag{3.37}$$

for all $n \geq 0$. This completes Case 1.

Similarly, when $\lambda < 0$ (still assuming $\lambda^2 - 4p > 0$) we have

$$0 > \frac{\alpha}{p} > \frac{1}{\alpha} \tag{3.38}$$

so

$$0 < \frac{\alpha^2}{p} < 1, \quad \frac{p}{\alpha^2} > 1, \text{ and } \frac{p^2}{\alpha^2} > p. \tag{3.39}$$

and the series in Equation (3.31) converges only if $c_2 = 0$, or equivalently

$$a_1 = \frac{\alpha a_0}{p}. \tag{3.40}$$

This time we need to compare

$$|\alpha| = -\alpha = \frac{-\lambda - \sqrt{\lambda^2 - 4p}}{2}$$

with 1, and under the condition $\lambda < 0$ and $\lambda^2 - 4p > 0$ we have

$$\frac{-\lambda - \sqrt{\lambda^2 - 4p}}{2} < 1 \iff \sqrt{\lambda^2 - 4p} > -\lambda - 2 \iff s < -(2p + 2) \tag{3.41}$$

Thus when $s \geq -(2p + 2)$, in order for Equation (3.31) to converge, we must have $c_1 = c_2 = 0$ and cannot satisfy the first line of Equation (3.16), a contradiction. When $s < -(2p+2)$, Equation (3.31) does converge when c_2=0 and together with the first line of Equation (3.16) and Equaton (3.27), we get

$$a_0 = \frac{p}{\lambda p - (p+1)\alpha}, \quad a_1 = \frac{\alpha}{\lambda p - (p+1)\alpha}, \quad c_1 = \frac{\frac{1}{\alpha}}{\frac{1}{\alpha} - \frac{\alpha}{p}} a_0 - \frac{\frac{\alpha}{p}}{\frac{1}{\alpha} - \frac{\alpha}{p}} a_0 = a_0 \tag{3.42}$$

so we have

$$a_n = a_0 \left(\frac{\alpha}{p}\right)^n = \frac{p}{\lambda p - (p+1)\alpha} \left(\frac{\alpha}{p}\right)^n, \tag{3.43}$$

for all $n \geq 0$. This completes Case 2.

Now assume $\lambda^2 - 4p < 0$. Then $\frac{\alpha}{p}$ and $\frac{1}{\alpha}$ are non-real and are complex conjugates of each other.

Hence

$$\begin{aligned}
&\sum_{n=1}^{\infty} |a_n| \, p^{n-1}(p+1) = \sum_{n=1}^{\infty} \sqrt{a_n \bar{a}_n} p^{n-1}(p+1) \\
&= \sum_{n=1}^{\infty} \sqrt{\left(c_1 \left(\frac{\alpha}{p}\right)^n + c_2 \left(\frac{1}{\alpha}\right)^n \right) \left(\bar{c}_1 \left(\frac{1}{\alpha}\right)^n + \bar{c}_2 \left(\frac{\alpha}{p}\right)^n \right)} p^{n-1}(p+1) \\
&= \sum_{n=1}^{\infty} \sqrt{\left(\left(|c_1|^2 + |c_2|^2 \right) \frac{1}{p^n} + c_2 \bar{c}_1 \left(\frac{1}{\alpha^2}\right)^n + c_1 \bar{c}_2 \left(\frac{\alpha^2}{p^2}\right)^n \right)} p^{n-1}(p+1) \\
&= \sum_{n=1}^{\infty} (p+1) \sqrt{(|c_1|^2 + |c_2|^2) p^{n-2} + \frac{c_2 \bar{c}_1}{p^2} \left(\frac{p}{\alpha}\right)^{2n} + \frac{c_1 \bar{c}_2}{p^2} \alpha^{2n}}
\end{aligned} \tag{3.44}$$

converges if and only if

$$c_1 = c_2 = 0 \iff a_n = 0 \text{ for every } n \geq 0, \tag{3.45}$$

contradicting the first line of Equation (3.16).

Finally when $\lambda^2 - 4p = 0$, $\lambda = \pm 2\sqrt{p}$ and $\alpha = \frac{\lambda}{2} = \pm\sqrt{p}$. From Equation (3.25) we have

$$\sum_{n=1}^{\infty} |a_n| \, p^{n-1}(p+1) = \sum_{n=1}^{\infty} p^{n/2-1}(p+1) \, |n(\alpha a_1 - a_0) + a_0| \, ,$$

which converges if and only if $a_0 = \alpha a_1 - a_0 = 0$, i.e. $a_0 = a_1 = 0$, contradicting the first line of Equation (3.16). This completes Case 3 and we have proved the Theorem.

We summarize the useful formulas from our computation in the following Corollary.

Corollary 3.2.4. *Let $\lambda = s + p + 1$ and $\alpha = \frac{\lambda + \sqrt{\lambda^2 - 4p}}{2}$. The (weight-s) Green's function $g_s(v, w)$ as defined in Theorem 3.2.1 on the Bruhat-Tits tree $\mathcal{T}$ for $\mathbb{Q}_p$ is given by*

$$g_s(v, w) = \frac{1}{(\lambda\alpha - (p+1))\alpha^{d(v,w)-1}}, \tag{3.46}$$

when $s > 0$; by

$$g_s(v, w) = \frac{p}{\lambda p - (p+1)\alpha} \left(\frac{\alpha}{p}\right)^{d(v,w)} \tag{3.47}$$

when $s < -(2p+2)$; and does not exist when $-(2p+2) \leq s \leq 0$.

We may also explain the necessity of the condition on s from a spectral perspective.

Corollary 3.2.5. *When $s \in [-(2p+2), 0]$, the operator $\Delta + s$ has a nontrivial kernel on the space of bounded functions over $\mathcal{V}$ of the Bruhat-Tits tree $\mathcal{T}$ for $\mathbb{Q}_p$. That is, any $\mu \in [0, 2p+2]$ is an eigenvalue of Δ.*

Proof. The corollary follows directly from our proof of Theorem 3.2.1, but here we give another proof by exhibiting some bounded functions in $\ker(\Delta + s)$.

Given a vertex $v_0 \in \mathcal{V}$, a subspace of $\ker(\Delta + s)$ is the collection of functions of the form $f(v) = a'_{d(v,v_0)}$ where the sequence $\{a'_n\}_{n\geq 0}$ satisfies (c.f. (3.12))

$$\begin{cases} \lambda a'_0 - (p+1)a'_1 = 0, & \text{and} \\ -a'_{n-1} + \lambda a'_n - pa'_{n+1} = 0, & \text{for each } n \geq 1. \end{cases} \tag{3.48}$$

Note that this is the same as Equation (3.16) except for the first line, so the computations and reasoning from Equation (3.17) to Equation (3.30) still applies (replacing a_n by a'_n for each n) and we in addition have $a'_1 = \frac{\lambda a'_0}{p+1}$. Hence if $\lambda^2 - 4p = 0$, i.e. $s = -(\sqrt{p} \pm 1)^2 \in [-(2p+2), 0]$, then

$$a'_n = \frac{1}{\alpha^n}(n(\alpha a'_1 - a'_0) + a'_0) = \frac{1}{\alpha^n}(n(\alpha \frac{\lambda a'_0}{p+1} - a'_0) + a'_0) \tag{3.49}$$

verifies Equation (3.48) and is bounded regardless of the choice of a'_0, giving a one-dimensional subspace of $\ker(\Delta + s)$.

When $\lambda^2 - 4p \neq 0$,

$$a'_n = \frac{\frac{1}{\alpha} - \frac{\lambda}{p+1}}{\frac{1}{\alpha} - \frac{\alpha}{p}} a'_0 \left(\frac{\alpha}{p}\right)^n + \frac{\frac{\alpha}{p} - \frac{\lambda}{p+1}}{\frac{\alpha}{p} - \frac{1}{\alpha}} a'_0 \left(\frac{1}{\alpha}\right)^n, \tag{3.50}$$

verifies Equation (3.48) and it remains to check boundedness. When $\lambda^2 < 4p$ (then necessarily $-(2p+2) \leq s \leq 0$), $\sqrt{\lambda^2 - 4p}$ is purely imaginary, thus we have

$$\left|\frac{\alpha}{p}\right| = \left|\frac{1}{\alpha}\right| = \sqrt{\frac{\lambda + \sqrt{\lambda^2 - 4p}}{2p} \cdot \frac{\lambda - \sqrt{\lambda^2 - 4p}}{2p}} = \frac{1}{\sqrt{p}} < 1. \tag{3.51}$$

When $\lambda^2 > 4p$ but $s \in [-(2p+2), 0]$, we've also seen from the analysis above that $\left|\frac{\alpha}{p}\right|, \left|\frac{1}{\alpha}\right| \leq 1$. Therefore whenever $s \in [-(2p+2), 0]$,

$$|a'_n| \leq \left|\frac{\frac{1}{\alpha} - \frac{\lambda}{p+1}}{\frac{1}{\alpha} - \frac{\alpha}{p}} a'_0\right| \left|\frac{\alpha}{p}\right|^n + \left|\frac{\frac{\alpha}{p} - \frac{\lambda}{p+1}}{\frac{\alpha}{p} - \frac{1}{\alpha}} a'_0\right| \left|\frac{1}{\alpha}\right|^n \leq \left|\frac{\frac{1}{\alpha} - \frac{\lambda}{p+1}}{\frac{1}{\alpha} - \frac{\alpha}{p}} a'_0\right| + \left|\frac{\frac{\alpha}{p} - \frac{\lambda}{p+1}}{\frac{\alpha}{p} - \frac{1}{\alpha}} a'_0\right|, \tag{3.52}$$

so a'_n is bounded, and with the freedome to choose a'_0, functions of the form $f(v) = a'_{d(v,v_0)}$ provides a one-dimensional subspace of $\ker(\Delta + s)$. □

3.3 Green's Functions on Finite Quotients of the Bruhat-Tits Tree

Let Γ be a discrete subgroup of $\mathrm{PGL}_2(\mathbb{Q}_p)$ such that $\mathcal{G} = \Gamma\backslash\mathcal{T}$ is a finite graph. Let $\mathscr{L}$ be a local system on $\mathcal{G}$, and let $(L, \rho) = \mathcal{F}(\mathscr{L})$ be the corresponding representation of Γ following the notation in Proposition 2.1.1. Then up to isomorphism of local systems, we have $\mathscr{L} = \Gamma\backslash(\mathcal{T} \times L)$, where the Γ-action is given by $\gamma(v, \ell) = (\gamma v, \rho(\gamma)\ell)$ and $\gamma(e, \ell) = (\gamma e, \rho(\gamma)\ell)$, for any $\ell \in L$, and for any vertex v and any edge e of $\mathcal{T}$. We shorthand $\rho(\gamma)\ell$ as $\gamma\ell$. Let $[\ell_1, \ldots, \ell_r]$ be an orthonormal basis of L.

Proposition 3.3.1. *Let $s > 0$, and let V denote the set of vertices of $\mathcal{G} = \Gamma\backslash\mathcal{T}$. For any $v, w \in V$, let $\tilde{v}, \tilde{w}$ be any liftings of v, w in the set of vertices $\mathcal{V}$ of $\mathcal{T}$. Let*

$$G_s(v, w) = \sum_{\gamma\in\Gamma} g_s(\gamma\tilde{v}, \tilde{w}) \sum_{i=1}^{r} \gamma^{-1}\ell_i \otimes \overline{\ell_i}. \tag{3.53}$$

Then G_s is absolutely convergent and is the weight-s vector-valued Green's function associated with the local system $\mathscr{L}$ on the graph $\mathcal{G}$. (c.f. Definition 2.3.2.)

Proof. First notice that Inequality (3.8) guarantees absolute convergence of (3.53). Moreover, for any $\alpha, \beta \in \Gamma$, we have

$$\begin{aligned}
&\sum_{\gamma\in\Gamma} g_s(\gamma\alpha\tilde{v}, \beta\tilde{w}) \sum_{i=1}^{r} \gamma^{-1}\ell_i \otimes \overline{\ell_i} = \sum_{\gamma\in\Gamma} g_s(\beta^{-1}\gamma\alpha\tilde{v}, \tilde{w}) \sum_{i=1}^{r} \gamma^{-1}\ell_i \otimes \overline{\ell_i} \\
=&\sum_{\delta\in\Gamma} g_s(\delta\tilde{v}, \tilde{w}) \sum_{i=1}^{r} \alpha\delta^{-1}\beta^{-1}\ell_i \otimes \overline{\ell_i} = \sum_{\delta\in\Gamma} g_s(\delta\tilde{v}, \tilde{w}) \sum_{i=1}^{r} \alpha\left(\delta^{-1}\ell_i\right) \otimes \overline{\beta\ell_i},
\end{aligned}$$

which shows that the right-hand-side of (3.53) is a well-defined element of $\Gamma(V, \mathscr{L}) \otimes \Gamma(V, \mathscr{L})$.

Now let $f, \varphi \in \Gamma(V, \mathscr{L})$ such that $(\Delta + s)f = \varphi$. Then $f = \sum_{i=1}^{r} f_i\ell_i$ and $\varphi = \sum_{i=1}^{r} \varphi_i\ell_i$ for some scalar-valued functions $f_1, \ldots, f_r$ and $\varphi_1, \ldots, \varphi_r$, and $(\Delta + s)f_i = \varphi_i$ for each i. Let $\tilde{\varphi}$ (resp. $\tilde{f}$) be a lifting of φ (resp. f) to $\mathcal{V}$ with the constant local system L on $\mathcal{T}$ such that $\tilde{\varphi}(\tilde{v}) = \varphi(v)$ (resp. $\tilde{f}(\tilde{v}) = f(v)$). Write $\tilde{\varphi} = \sum_{i=1}^{r} \tilde{\varphi}_i\ell_i$ and $\tilde{f} = \sum_{i=1}^{r} \tilde{f}_i\ell_i$ for some scalar functions $\tilde{\varphi}_i$ and $\tilde{f}_i$ on $\mathcal{V}$. Then for any $x \in \mathcal{V}$ and $\gamma \in \Gamma$, we have

$$\tilde{\varphi}(\gamma x) = \gamma\tilde{\varphi}(x), \quad \text{and } \tilde{f}(\gamma x) = \gamma\tilde{f}(x),$$

or equivalently,

$$\sum_{i=1}^{r} \tilde{\varphi}_i(\gamma x)\ell_i = \sum_{i=1}^{r} \tilde{\varphi}_i(x)\gamma\ell_i, \quad \text{and} \quad \sum_{i=1}^{r} \tilde{f}_i(\gamma x)\ell_i = \sum_{i=1}^{r} \tilde{f}_i(x)\gamma\ell_i.$$

Also, $(\Delta + s)\tilde{f}_i = \tilde{\varphi}_i$ on $\mathcal{T}$, and $\tilde{f}_i, \tilde{\varphi}_i$ are bounded for each i. Therefore,

$$\begin{aligned}
&\sum_{v\in V} \langle \varphi(v), G(v,w)\rangle = \sum_{v\in V}\sum_{i=1}^{r} \langle \tilde{\varphi}(\tilde{v}), \sum_{\gamma\in\Gamma} g_s(\gamma\tilde{v},\tilde{w})\gamma^{-1}\ell_i\rangle \ell_i \\
=&\sum_{v\in V}\sum_{i=1}^{r}\langle \sum_{j=1}^{r} \tilde{\varphi}_j(\tilde{v})\ell_j, \sum_{\gamma\in\Gamma} g_s(\gamma\tilde{v},\tilde{w})\gamma^{-1}\ell_i\rangle\ell_i \\
=&\sum_{v\in V}\sum_{\gamma\in\Gamma}\sum_{i=1}^{r}\langle \sum_{j=1}^{r} \tilde{\varphi}_j(\gamma\tilde{v})\gamma^{-1}\ell_j, g_s(\gamma\tilde{v},\tilde{w})\gamma^{-1}\ell_i\rangle\ell_i \\
=&\sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{v\in V}\sum_{\gamma\in\Gamma}\left(\tilde{\varphi}_j(\gamma\tilde{v})\overline{g_s(\gamma\tilde{v},\tilde{w})}\right)\langle\gamma^{-1}\ell_j,\gamma^{-1}\ell_i\rangle\ell_i \\
=&\sum_{i=1}^{r}\sum_{v\in V}\sum_{\gamma\in\Gamma}\left(\tilde{\varphi}_i(\gamma\tilde{v})\overline{g_s(\gamma\tilde{v},\tilde{w})}\right)\ell_i \\
=&\sum_{i=1}^{r}\tilde{f}_i(\tilde{w})\ell_i = f(w),
\end{aligned}$$

as desired. □

Remark 3.3.2. *Following Remark 2.3.3, we can also understand the construction* (3.53) *in a* coordinate-free *way. Indeed, for* $H_\gamma := \sum_{i=1}^{r}\gamma^{-1}\ell_i \otimes \overline{\ell_i} \in \Gamma(\mathcal{V},\mathscr{L}) \times \Gamma(\mathcal{V},\mathscr{L})$, $\tilde{v},\tilde{w} \in \mathcal{V}$, *and* $a \in L$, *we have*

$$H_\gamma(\tilde{v},\tilde{w})(a) = \sum_i \langle a, \gamma^{-1}\ell_i\rangle\ell_i = \sum_i \langle \gamma a, \ell_i\rangle\ell_i = \gamma a.$$

From the discussion in Section 2.3, we have the vector-valued Green's function on $\mathcal{G}$ with local system $\mathscr{L}$ as follows.

Corollary 3.3.3. *Following the notations in the previous proposition, and let* $\psi_1,\dots,\psi_t$ *be an orthonormal basis for* $\ker(\Delta)$ *on* $\mathscr{L}$ *over* $\mathcal{G}$, *which is the space of constant sections with values in the subspace of* L *invariant under* Γ*-action. Then*

$$G(v,w) = \lim_{s\to 0^+}\left(G_s(v,w) - \left(\sum_{i=1}^{t}\psi_i\otimes\overline{\psi_i}\right)s^{-1}\right). \tag{3.54}$$

is the vector-valued Green's function on $\mathcal{G}$ *with respect to the local system* $\mathscr{L}$.

Now let $\mathcal{S}$ be a local system of spheres over $\mathcal{G} = \Gamma\backslash\mathcal{T}$. Fix a vertex $v_0 \in V$ and fix an isomorphism

$\mathcal{S}_{v_0} \cong S^2$, then there exists a representation $\pi : \Gamma \to \mathrm{SO}(3)$ such that $\mathcal{S} = \Gamma\backslash\mathcal{T} \times S^2$, where Γ acts on $\mathcal{T} \times S^2$ by $\gamma(v, P) = (\gamma v, \pi(\gamma)P)$ and $\gamma(e, P) = (\gamma e, \pi(\gamma)P)$, for any $P \in S^2$, any vertex v and any edge e of $\mathcal{T}$. Consider the local system $\mathscr{L} = \mathcal{H}_n(\mathcal{S})$. Then $\mathscr{L} = \Gamma\backslash\mathcal{T} \times H_n$, where the Γ-action on $\mathcal{T} \times H_n$ is induced from that on $\mathcal{T} \times S^2$. (c.f. Section 2.4.) Then the weight-s scalar-valued Green's function (c.f. Definition 2.4.1) is given by

$$\begin{aligned} G_s^{(n)}(p,q) &= \sum_{\gamma\in\Gamma} g_s(\gamma\tilde{v},\tilde{w}) \sum_{i=1}^{r} \langle \gamma^{-1}\ell_i, s_P^{(n)}\rangle \langle \overline{\ell_i}, s_Q^{(n)}\rangle \\ &= \sum_{\gamma\in\Gamma} g_s(\gamma\tilde{v},\tilde{w}) \sum_{i=1}^{r} \langle \ell_i, \gamma s_P^{(n)}\rangle \langle \overline{\ell_i}, s_Q^{(n)}\rangle \\ &= \sum_{\gamma\in\Gamma} g_s(\gamma\tilde{v},\tilde{w}) \langle \gamma s_P^{(n)}, s_Q^{(n)}\rangle \\ &= \sum_{\gamma\in\Gamma} g_s(\gamma\tilde{v},\tilde{w}) \langle s_{\gamma P}^{(n)}, s_Q^{(n)}\rangle, \\ &= \sum_{\gamma\in\Gamma} g_s(\gamma\tilde{v},\tilde{w}) P_n(\theta(\gamma P, Q)), \end{aligned} \tag{3.55}$$

for any $p \in \mathcal{S}_v$, $q \in \mathcal{S}_w$, and any lifting P, Q of p, q respectively in S^2, where $\theta(\gamma P, Q)$ denotes the angle between the two vectors connecting the origin with γP and Q respectively. The last equality above comes from the following observation: let $\Phi_n(P, Q) := \langle s_P^{(n)}, s_Q^{(n)}\rangle$ for any $P, Q \in S^2$, then Φ_n is characterized by

1. $\Phi_n(\alpha P, \alpha Q) = \Phi_n(P, Q)$ for all $\alpha \in \mathrm{SO}(3)$ and any $P, Q \in S^2$,
2. $\Phi_n(P, Q) = \Phi_n(Q, P)$, for any $P, Q \in S^2$,
3. $\Delta_{S^2,P}\Phi_n(P, Q) = n(n + 1)\Phi_n(P, Q)$, for any $P, Q \in S^2$,
4. $\Phi_n(P, P) = 1$.

The first three conditions yields that $\Phi_n(P, Q)$ is a constant multiple of $s_P^{(n)}(Q)$, whereas the condition in the last normalization determines $\Phi_n(P, Q) = P_n(\cos\theta(P, Q))$.

Example 3.3.4. *Consider a quaternion algebra D over $\mathbb{Q}$ that splits at p and is non-split at ∞. Take an order O in D. Then $D_\infty = \mathbb{H} = \mathbb{R} + \mathbb{R}i + \mathbb{R}j + \mathbb{R}k$, the Hamiltonian quaternions. Let $\mathbb{H}_0 = \mathbb{R}i + \mathbb{R}j + \mathbb{R}k$ denote the traceless elements. Let $\mathbb{H}_0^\times$ be norm-1 elements in $\mathbb{H}_0$, which has a conjugation action by $\mathbb{H}^\times$. That is, $\mathbb{H}_0^\times = \{ai+bj+ck : a^2+b^2+c^2 = 1\}$, which is isomorphic to S^2 as a metric space. Then $\Gamma := O[1/p]^\times/\{\pm p^{\mathbb{Z}}\}$ can be embedded as a discrete subgroup in $\mathrm{PGL}_2(\mathbb{Q}_p)$ and also as a subgroup of $\mathbb{H}^\times/\mathbb{R}^\times$. Then Γ acts on both $\mathcal{T}$ and $\mathbb{H}_0^\times$. This demonstrates that local systems of spheres over quotients of Bruhat-Tits tree do occur.*

Let K be an imaginary quadratic field inert at p embedded in D. Then the image A of $(K \cap O[1/p])^\times$ in Γ is a subgroup of Γ that has a unique fixed vertex $\tilde{v}$ of $\mathcal{T}$ and a unique fixed point $P \in \mathbb{H}_0^\times$. We call the image (v, x) of the pair $(\tilde{v}, P)$ in the local system of spheres $\mathcal{S} := \Gamma\backslash\mathcal{T} \times S^2$ a CM point *of $\mathcal{S}$. And we call the corresponding vector $s_x^{(n)}$ a* CM cycle. *Given two embeddings of K into D, we get two CM cycles $s_x^{(n)}$ and $s_y^{(n)}$, and our constructions provide a scalar-valued function $G(s_x^{(n)}, s_y^{(n)})$. This is the major original ingredient of Part II.*

Part II

Towards a Gross-Zagier Formula on CM-cycles over Shimura Curves

Chapter 4

Background: A Review of Some Gross-Zagier Type Results

4.1 Gross-Zagier's Work

In this section we summarize the work of Gross and Zagier in [13], one of the main inspirations of this thesis. Following notations from the introduction, the assumption in their work is that $k = 1$ and $N_- = 1$.

Consider the modular curve $X = X_0(N)$, which is the completion of the coarse moduli space of isomorphism classes of pairs (E, C), where E is a generalized elliptic curve and C is a cyclic subgroup of order N. A generalized elliptic curve is an elliptic curve or a polygon. It is known that X is a complete, non-singular and geometrically connected curve over $\mathbb{Q}$.

Over a field k of characteristic zero, the above description is the same as saying that X is the moduli space of isogenies

$$\phi : E \to E' \tag{4.1}$$

of generalized elliptic curves over k such that $\ker(\phi) \cong \mathbb{Z}/N\mathbb{Z}$ over the algebraic closure $\bar{k}$. When the isogeny ϕ is between polygons, we further require $\ker(\phi)$ to meet every irreducible component of E, and such an isogeny corresponds to a cusp on X. The cusp corresponding to when E is the 1-gon and $\ker(\phi) = \mu_N$ is denoted by 0, while the cusp corresponding to when E is the N-gon and $\ker(\phi) = \mathbb{Z}/N\mathbb{Z}$ is denoted by ∞. Over the complex numbers $\mathbb{C}$, X is given by the Riemann surface $\Gamma_0(N)\backslash(\mathfrak{H} \sqcup \mathbb{P}^1(\mathbb{Q}))$, where $\mathfrak{H} := \{\tau \in \mathbb{C} : \mathrm{Im}(\tau) > 0\}$ is the upper half plane, and $\Gamma_0(N)$ is the

subgroup of $\mathrm{PSL}_2(\mathbb{Z})$ consisting of matrices that are upper-triangular modulo N.

Let m be a positive integer. The Hecke action T_m on X is given by

$$T_m(x) = \sum_S (x_S), \tag{4.2}$$

where the sum is taken over all subgroups S of index m in E_x such that $\ker(\phi_x) \cap S = \{O_x\}$, if x is representing the isogeny $\phi_x : E_x \to E'_x$ and O_x is the identity element of the group E_x, and x_S representing the induced isogeny

$$\phi_{x_S} : E_x/S \to E'_x/\phi_x(S).$$

Let D be an integer coprime to N. Let K be an imaginary quadratic field with discriminant D. Let $\mathcal{O}_K$ be the ring of integers in K, h the class number of K, and u the order of the finite group $\mathcal{O}_K^\times/\{\pm 1\}$. A point $x = \{\phi_x : E_x \to E'_x\}$ on X is called a *Heegner point* of discriminant D if both E_x and E'_x has complex multiplication by $\mathcal{O}_K$. Such points exist under the Heegner condition and are all rational over the Hilbert class field $H = K(j(E))$ of K.

Let x be a Heegner point of discriminant D, then the class of the divisor

$$c = (x) - (\infty) \tag{4.3}$$

is an element of $J(H)$, where J is the Jacobian of X.

Next, we define the global height pairing $\langle\ ,\ \rangle : J(H) \times J(H) \to \mathbb{R}$.

Recall that for a global field k, at each place v we can define the valuation homomorphism $|\ |_v : k_v^\times \to \mathbb{R}_+^\times$ on the completion k_v of k at v by

$$|\alpha|_v = \begin{cases} |\alpha| \text{ if } k_v = \mathbb{R}, \\ |\alpha|^2 \text{ if } k_v = \mathbb{C}, \\ q_v^{-\mathrm{val}(\alpha)} \text{ if } k_v \text{ is non-archimedean}, \end{cases} \tag{4.4}$$

where $\mathrm{val}(\cdot)$ is the additive valuation normalized so that $\mathrm{val}(\varpi) = 1$ for the uniformizer ϖ of k_v, and q_v is the number of elements in the residue field of k_v. And Néron's theory gives rise to local height pairing of relatively prime degree-zero divisors over a local field k_v.

Proposition 4.1.1. *[12, Proposition 2.3] Let X be a complete, non-singular, geometrically irre-*

ducible curve defined over a local field k_v. Let $\mathrm{Div}^0(X/k_v)$ denote the group of degree-0 divisors of X rational over k_v and let $Z^0(X/k_v)$ denote the group of degree-0 elements in the free abelian group generated by $X(k_v)$, which is a subgroup of $\mathrm{Div}^0(X/k_v)$. Assume that the k_v-rational points $X(k_v)$ is non-empty. Then there is a unique function $\langle a, b\rangle_v$ on relatively prime divisors $a \in Z^0(X/k_v)$, $b \in \mathrm{Div}^0(X/k_v)$ with values in $\mathbb{R}$ which satisfies the following four properties

1. $\langle a, b\rangle_v + \langle a, c\rangle_v = \langle a, b + c\rangle_v$.
2. $\langle a, b\rangle_v = \langle b, a\rangle_v$ *if* $b \in Z^0(X/k_v)$.
3. $\langle a, \mathrm{Div}(f)\rangle_v = \log |f(a)|_v$, *where* $f(\sum m_x(x)) := \prod f(x)^{m_x}$. *That is,*

$$\langle a, \mathrm{Div}(f)\rangle_v = \sum m_x \log |f(x)|_v \text{ , if } a = \sum m_x(x). \tag{4.5}$$

4. *Fix the divisor b and a point $x_0 \in X(k_v) - |b|$. Then the function $X(k_v) - |b| \to \mathbb{R}$, defined by $x \mapsto \langle (x) - (x_0), b\rangle_v$ is continuous.*

Therefore, for any degree-0 relatively prime divisors a, b on X, we can define the global height pairing over H by

$$\langle a, b\rangle = \sum_v \langle a, b\rangle_v, \tag{4.6}$$

where the sum is taken over all places of H. Note that by the product formula, $\langle a, b\rangle$ only depends on the images of a, b in $J(H)$. When a, b are not relatively prime, the definition of local heights would depend on the choice of a tangent vector at each point in the common support, but the sum over all places does not depend on this choice and the global height pairing is still well defined. Let σ be an element in $\mathrm{Gal}(H/K)$. The geometric side of the equation involves the global height pairing

$$\langle c, T_m c^\sigma\rangle, \tag{4.7}$$

while the analytic side is covered in the introduction. One of the main theorem of Gross-Zagier's work can be written as follows.

Theorem 4.1.2. *[13, Chapter I, (9.3)] When $k = 1$,*

$$\langle c, T_m c^\sigma\rangle = u^2 a_{m,\mathscr{A}}, \tag{4.8}$$

for all $m \geq 1$ and $(m, N) = 1$.

4.2 Zhang's Work

In [30], Zhang generalized Theorem 4.1.2 to higher weight cases while the other assumptions remain. While the right-hand-side of Equation (4.8) is defined when $k > 1$ and will be kept, the left-hand-side of the equation will be replaced by arithmetic intersection (see [10], also introduced in Chapter 5) of *Heegner cycles* $S_k(x)$ and $T_m S_k(x^\sigma)$ in the *Kuga-Sato variety* Y over the modular curve X

The Kuga-Sato variety Y is the canonical resolution of the $(2k-2)$-th fiber product ($k \geq 2$) of the universal elliptic curve $\mathcal{E}$ over a finite cover X' of X. We denote this covering map by $p : X' \to X$. For a Heegner point $x \in X$, we shall define

$$S_k(x) = \frac{1}{\sqrt{\deg p}} \sum_{x' \in X' : p(x') = x} S_k(x'), \tag{4.9}$$

where $S_k(x')$ will be defined as follows. The cycle $S_k(x')$ lies in the fiber $Y_{x'} = \mathcal{E}_{x'}^{2k-2}$. We define

$$S_k(x') = c_k \sum_{g \in G_{2k-2}} (\operatorname{sgn} g) \cdot g(Z(\mathcal{E}_{x'})^{k-1}), \tag{4.10}$$

where $Z(\mathcal{E}_{x'})$ is a cycle whose image in $\mathrm{NS}(\mathcal{E}_{x'} \times \mathcal{E}_{x'})$ is perpendicular to that of $\{0\} \times \mathcal{E}_{x'}$, $\mathcal{E}_{x'} \times \{0\}$, and the diagonal $\Delta_{\mathcal{E}_{x'} \times \mathcal{E}_{x'}}$; G_{2k-2} is the permutation group of $2k-2$ letters, which acts on $Z(\mathcal{E}_{x'})^{k-1}$ by permuting the $2k-2$ copies of $\mathcal{E}_{x'}$; $\operatorname{sgn} g$ is the sign of the permutation g; and c_k is a constant such that the self-intersection[1] $S_k(x') \cdot S_k(x')$ is precisely $(-1)^{k-1}$. The Hecke operators T_m act on $S_k(x)$ by

$$T_m[S_k(x)] = \sum_i m^{k-1}[S_k(x_i)] \text{ when } T_m(x) = \sum_i x_i. \tag{4.11}$$

In an integral model $\mathcal{Y}$ of the Kuga-Sato variety Y, define the arithmetic Chow cycle $\hat{S}_k(x)$ by $(\overline{S_k(x)}, g_k(x))$, where $\overline{S_k(x)}$ is the Zariski closure of $S_k(x)$ in $\mathcal{Y}$ and $g_k(x)$ is a Green's current associated to $S_k(x)$. The cycle $\hat{S}_k(x)$ is of codimension k in $\mathcal{Y}$, and the intersection $\hat{S}_k(x) \cdot \hat{S}_k(y)$ is well-defined in the sense of Gillet-Soulé [10] for two Heegner points x and y. Moreover, $\hat{S}_k(x)$ satisfies certain conditions (c.f. Section 5.4) so that the pairing defined by

$$\langle S_k(x), S_k(y) \rangle := (-1)^k \hat{S}_k(x) \cdot \hat{S}_k(y), \tag{4.12}$$

is well-defined as a *global* height pairing. We have the following theorem.

Theorem 4.2.1. *[30, Theorem 0.2.1] Following notations above, for an integer $k \geq 2$, we have for*

[1] as self-intersection of a $k-1$ dimensional algebraic variety in the $2k-2$ dimensional variety $Y_{x'}$

all $m \geq 1$ *and* $(m, N) = 1$,

$$\langle S_k(x), T_m S_k(x^\sigma) \rangle = u^2 a_{m,\mathscr{A}}. \tag{4.13}$$

4.3 Archimedean Computations

In this section we recall the archimedean computations from [12], [13], and [30], the analogy with which provided most inspriations for Part I of this thesis.

4.3.1 Green's Function on the Shimura Curve $\Gamma\backslash\mathfrak{H}$

Recall from [12] the following proposition:

Proposition 4.3.1. *[12, Proposition 6.1] Let* X *be a complete, non-singular, geometrically irreducible curve defined over* $\mathbb{C}$*. Let* $M = X(\mathbb{C})$*. Let* $d\mu$ *be a positive real analytic 2-form on* M *with* $\int_M d\mu = 1$*. For each divisor* a *on* X *there is a unique real analytic function* $G(a, \cdot) : M - |a| \to \mathbb{R}$ *such that:*

1. $G(a, \cdot) - m_x \log |\varphi|^2$ *is real analytic near* x*, where* φ *is any uniformizing parameter at* x *and* m_x *is the multiplicity of* x *in* a*,*

2. $G(a, \cdot)$ *satisfies the differential equation:*

$$\partial\bar{\partial} G(a, \cdot) = 2\pi i(\deg(a)\dot{d\mu} - \delta_a), \tag{4.14}$$

   where δ_a *is the* $(1, 1)$*-current which represents the evaluation of* $(0, 0)$*-forms at* a*.*

3. $\int_M G(a, \cdot) d\mu = 0$*.*

The system of functions $\{g_a\}_{a \in \mathrm{Div}(X)}$ *satisfy the further identities:*

4. $G(a + b, \cdot) = G(a, \cdot) + G(b, \cdot)$*.*

5. $G(a, b) = G(b, a)$ *whenever* a, b *are disjoint.*

In particular, for $x, y \in X(\mathbb{C})$, with $x \neq y$, we have $G(x, y)$, which we call *the Green's function on* $X(\mathbb{C})$ *with respect to* $d\mu$. So the Green's function enables us to extend Néron's pairing (in Proposition 4.1.1) to points (thus divisors of arbitrary degree by bi-additivity) by $\langle x, y \rangle_v = G(x, y)$ when the local field $k_v = \mathbb{C}$.

Next we recall the construction of Green's function on curves uniformized by $\mathfrak{H}$ as constructed in [12].

From now on we assume that $X = \mathfrak{H}/\Gamma$, where Γ is a discrete subgroup of $\mathrm{SL}(2,\mathbb{R})$, and assume that X has finite volume (hyperbolic area). Let $z, w \in \mathfrak{H}$ and let x, y be the corresponding points in $X(\mathbb{C})$.

We first find the *weight k-Green's function* $G_k(x,y)$, which satisfies a slightly different differential equation as $G(x,y)$.

Definition 4.3.2. *The weight k-Green's function $G_k(x,y)$ is the unique function on $X \times X$ minus the diagonal that satisfies the following three properties. In what follows we will often write $G_k(x,y)$ as $G_k(z,w)$, a function on $\mathfrak{H} \times \mathfrak{H}$ minus the preimage of the diagonal of $X \times X$, invariant under Γ-action on either variable.*

1. *$\Delta_z G_k(z,w) = k(k-1)G_k(z,w)$, where $\Delta_z = b^2\left(\frac{\partial^2}{\partial a^2} + \frac{\partial^2}{\partial b^2}\right)$, if we use the coordinate $z = a+ib$.*
2. *$G_k(z,w) = \log|z-w|^2 + O(1)$ as $z \to z'$.*
3. *In a neighborhood of a cusp $\gamma^{-1}\infty$, the function $\mathrm{Im}(\gamma z)^{k-1}G_k(z,w)$ extends to a continuous function.*

Remark 4.3.3. *Note that our definition of* weight *of a Green's function in Part I is not exactly analogous to the definition of weight here. The function G_k defined above is more analogous to the (scalar-valued) $G_{k(k-1)}$ in Part I.*

Remark 4.3.4. *The existence of $G_k(x,y)$ is shown by the construction below from [12], and the uniqueness is proved in [30, Page 132].*

For $x \neq y$ (i.e. $z \notin \Gamma w$), we let

$$G_k(z,w) = \sum_{\gamma\in\Gamma} g_k(z,\gamma w), \tag{4.15}$$

where

$$g_k(z,w) = -2Q_{k-1}(\cosh d(z,w)) = -2Q_{k-1}\left(\frac{|z-w|^2}{2\,\mathrm{Im}\,z\,\mathrm{Im}\,w} + 1\right). \tag{4.16}$$

Here $Q_{k-1}(u) = \int_0^\infty (u + \sqrt{u^2-1}\cosh t)^{-k}dt$ for $u > 1$ is the Legendre function of the second kind designed to satisfy the Legendre differential equation

$$(1-u^2)\frac{d^2f}{du^2} - 2u\frac{df}{du} + k(k-1)f = 0. \tag{4.17}$$

Thus $g_k(z,w)$ on $\mathfrak{H}\times\mathfrak{H}$ minus the diagonal is constructed to satisfy

$$\begin{cases} g_k(\gamma z,\gamma w)=g_k(z,w) & \text{for all } \gamma\in \mathrm{PSL}_2(\mathbb{R}),\\ g_k(z,w)\sim \log|z-w|_v & \text{as } z\to w,\\ \Delta_z g_k(z,w)=k(k-1)g_k(z,w). \end{cases} \tag{4.18}$$

Then, $G_k(z,w)$ as defined in Equation (4.15) is invariant under Γ action on either variable and verifies the properties in Definition 4.3.2. Note that in Equation (4.18), the second and third conditions together are equivalent to $(\Delta_z - k(k-1))g_k(z,w)=\delta_w$, hence the analogy with our definition of Green's functions in Part I.

Note also that $G_k(z,w)$, when considered as a function of k, has a pole at $k=1$. We will denote the residue at $k=1$ by $\mathrm{Res}_{k=1}G_k$. Then we can construct the Green's function $G(x,y)$ by

$$G(x,y)=G(z,w)=\lim_{k\to 1}\left\{G_k(z,w)-\frac{\mathrm{Res}_{k=1}G_k}{k(k-1)}\right\}, \tag{4.19}$$

One can check that this definition of Green's function, when extended to relatively prime divisors of arbitrary degree by bi-linearity, indeed verifies the properties in Theorem 4.3.1 if we choose $d\mu$ to be the normalized hyperbolic area distribution so that $\int_X d\mu = 1$. Interested readers can consult [12] for more details.

To allow common support, we may also define

$$G(x,x):=\lim_{y\to x}(G(x,y)-\log|\varphi(y)|^2), \tag{4.20}$$

where φ is a chosen local uniformizing parameter at x.

4.3.2 Weight-k Green's Function on Modular Curves and the Archimedean Intersection of Heegner Cycles

While the Green's function gives the archimedean intersection on the Shimura curve, we are interested in the archimedean intersection of CM cycles in the Kuga variety of relative dimension $2k-2$. In [30], Zhang showed that the intersection of the Heegner cycles in the Kuga-Sato variety (of relative dimension $2k-2$) over a modular curve is half of $G_k(x,y)$, whose formula is given by Equation (4.15), and whose defining properties are listed in Definition 4.3.2. Below we explain his work. That is, consider $X=X_0(N)$, x,y Heegner points on X, Y the Kuga-Sato variety of relative

dimension $2k-2$, and $S_k(x), S_k(y)$ Heegner cycles in Y as defined in Section 4.2. We'll explain the relation between the weight-k Green's function $G_k(x,y)$ and the archimedean intersection of Heegner cycles $S_k(x)$ and $S_k(y)$.

Let $g_k(x)$ be the green's current corresponding to $S_k(x)$, that is, a $(k-1,k-1)$-current such that

1. $\frac{\partial\bar{\partial}}{\pi i}g_k(x) = \delta_{S_k(x)}$, and
2. the integration $\int_{Y(\mathbb{C})} g_k(x)\eta = 0$ for any $\frac{\partial\bar{\partial}}{\pi i}$-closed form η on $Y(\mathbb{C})$.

Let η_y be the closed form representing the homology class of $S_k(y)$ in $H^{2k-2}(Y_y^{2k-2},\mathbb{C})$, where Y_y is the elliptic curve corresponding to the CM point y.

We define

$$H_k(x,y) = (-1)^k \int_{Y_y} \eta_y g_k(x). \tag{4.21}$$

When y is a CM point, we have

$$\int_{Y_y} \eta_y g_k(x) = \int_{S_k(y)} g_k(x), \tag{4.22}$$

thus from Chapter 5 we know that $H_k(x,y)$ gives the archimedean intersection of $S_k(x)$ and $S_k(y)$. Note however that the cycle $S_k(y)$ is defined only when y is a CM point, so if we want to consider $H_k(x,y)$ as a function of $y \in X - \{x\}$, we would need a definition of η_y that does not require y being a CM point. We will accomplish this by showing that η_y can be considered as the restriction of a form η on $Y(\mathbb{C})$ to the fiber Y_y. We will write out η explicitly in terms of local coordinates. Then, in order to show the claimed

$$H_k(x,y) = \frac{1}{2}G_k(x,y), \tag{4.23}$$

we only need to show that $2H_k(x,y)$ satisfies the properties listed in Definition 4.3.2.

For the ease of exposition, we will focus on the $k=2$ case, that is, the Kuga-Sato variety of relative dimension $2k-2=2$. The general relative dimension as dealt with in [30] is similar but requires a very careful bookkeeping of indices. We will also omit the discussion of Property 3 about the growth behavior near the cusp as it is less related to this thesis work – in our setting introduced in Section 1.1.2, the Shimura curve is free of cusps.

Over a neighborhood V of $y \in X_0(N)$, the universal elliptic curve $\mathscr{E}$ can be trivialized as $\mathbb{C}\times V/$

$\mathbb{Z}^2$, where $\mathbb{Z}^2$ acts on $\mathbb{C} \times V$ by

$$(m, n) \cdot (u, z) = (u + m + nz, z). \tag{4.24}$$

Indeed, the fiber above z is the elliptic curve $\mathbb{C}/\langle 1, z\rangle$. If we write $u = s + it$ and $z = a + ib$, then the form $du - \frac{t}{b}dz$ is invariant under the $\mathbb{Z}^2$ action described, and thus descends to a form α on $\mathscr{E}$. Indeed,

$$d(u + m + nz) - \frac{t + nb}{b}dz = du + ndz - \frac{t}{b}dz - ndz = du - \frac{t}{b}dz. \tag{4.25}$$

Moreover, when restricted to the fiber over y (or similarly to any other fiber for this step), we have $\alpha_y = du$ on $E := \mathscr{E}_y = \mathbb{C}/\langle 1, y\rangle$, and

$$\int_E du \wedge d\bar{u} = \int_E (ds + idt) \wedge (ds - idt) = \int_E -2ids \wedge dt = -2ib_y, \tag{4.26}$$

where b_y denotes the imaginary part of y and is also the area of the parallelogram spanned by 1 and y.

As we are focusing on the case where $k = 2$, we are looking for a form on $\mathscr{E} \times_{X_0(N)} \mathscr{E}$, which locally is $\mathbb{C} \times \mathbb{C} \times V/\mathbb{Z}^4$, where $\mathbb{Z}^4$ acts by

$$(m_1, n_1, m_2, n_2) \cdot (u_1, u_2, z) = (u_1 + m_1 + n_1 z, u_2 + m_2 + n_2 z, z) \tag{4.27}$$

We write $u_1 = s_1 + it_1$, $u_2 = s_2 + it_2$. Then from Equation (4.25) we know that $du_1 - \frac{t_1}{b}dz$ and $du_2 - \frac{t_2}{b}dz$ are invariant under the $\mathbb{Z}^4$ action and descends from forms on $\mathscr{E} \times_{X_0(N)} \mathscr{E}$, which we call α_1 and α_2 respectively.

Define

$$\eta = -2i\sqrt{D}c_2(\alpha_1\bar{\alpha}_2 + \bar{\alpha}_1\alpha_2), \tag{4.28}$$

where c_2 is as defined in Equation (6.21). Then η is a form on the Kuga-Sato variety. Moreover, when restricted to the fiber over y, we have

$$\eta_y = -2i\sqrt{D}c_2(du_1 d\bar{u}_2 + d\bar{u}_1 du_2), \tag{4.29}$$

which we claim to represent $S_2(y)$ on $E_y \times E_y$. From Equation (4.10), it suffices to show that

$-i\sqrt{D}\,(du_1 d\bar{u}_2 + d\bar{u}_1 du_2)$ represents $Z(E_y)$. Then we have

$$c_2\left(-i\sqrt{D}\,(du_1 d\bar{u}_2 + d\bar{u}_1 du_2) - \left(-i\sqrt{D}\,(du_2 d\bar{u}_1 + d\bar{u}_2 du_1)\right)\right) = -2i\sqrt{D}c_2(du_1 d\bar{u}_2 + d\bar{u}_1 du_2)$$

represents $c_2\,(Z(E_y) - \sigma Z(E_y)) = S_2(y)$.

Indeed, du_1 and du_2 form a basis for $(1,0)$-forms on $E_y \times E_y$, so every cohomological class in $H^{1,1}(E_y \times E_y)$ can be represented by a form

$$\xi = a_1 du_1 d\bar{u}_1 + a_2 du_1 d\bar{u}_2 + a_3 du_2 d\bar{u}_1 + a_4 du_2 d\bar{u}_2,$$

where $a_1, \ldots, a_4$ are constants. We have

$$\begin{aligned}
&\int_{Z(E_y)} \xi = \int_{\Gamma - E\times\{0\} - D\{0\}\times E} a_1 du_1 d\bar{u}_1 + a_2 du_1 d\bar{u}_2 + a_3 du_2 d\bar{u}_1 + a_4 du_2 d\bar{u}_2 \\
=&\int_{\Gamma - E\times\{0\}} a_1 du_1 d\bar{u}_1 + \int_{\Gamma} a_2 du_1 d\bar{u}_2 + \int_{\Gamma} a_3 du_2 d\bar{u}_1 + \int_{\Gamma - D\{0\}\times E} a_4 du_2 d\bar{u}_2 \\
=&0 - \sqrt{-D}(-2ib_y)a_2 + \sqrt{-D}(-2ib_y)a_3 + 0 \\
=&2\sqrt{D}b_y(a_3 - a_2)
\end{aligned} \tag{4.30}$$

and

$$\begin{aligned}
&\int_{E_y\times E_y} \xi \wedge (du_1 d\bar{u}_2 + d\bar{u}_1 du_2) = \int_{E_y\times E_y} a_3 du_2 d\bar{u}_1 du_1 d\bar{u}_2 + \int_{E_y\times E_y} a_2 du_1 d\bar{u}_2 d\bar{u}_1 du_2 \\
=&\int_{E_y\times E_y} (a_2 - a_3) du_1 d\bar{u}_1 du_2 d\bar{u}_2 = 2ib_y(a_3 - a_2).
\end{aligned} \tag{4.31}$$

So

$$\int_{Z(E_y)} \xi = -i\sqrt{D} \int_{E_y\times E_y} \xi \wedge (du_1 d\bar{u}_2 + d\bar{u}_1 du_2)\,, \tag{4.32}$$

as desired.

We now show that $H_k(x,y) = H_k(z,w)$ (recall that z and w are representatives of x and y in $\mathfrak{H}$ respectively) satisfies the differential equation in Definition 4.3.2. That is, in the example of $k = 2$

$$\Delta_z H_2(z,w) = 2H_2(z,w), \tag{4.33}$$

i.e.

$$\Delta_z \int_{Y_y} \eta_y g_2(z) = 2 \int_{Y_y} \eta_y g_2(z), \tag{4.34}$$

where $g_2(z) = g_2(x)$ is the $(1,1)$-Green's current corresponding to $S_2(x)$. As before, on a neighborhood V of x we can write Y_V as $\mathbb{C} \times \mathbb{C} \times V/\mathbb{Z}^4$, where $\mathbb{Z}^4$ acts as in Equation (4.27). There is a slight abuse of notation here: we used the same notation on a neighborhood of x as that of y. We will also use the coordinates $u_1 = s_1 + it_1$, $u_2 = s_2 + it_2$ and z, and the forms α_1, α_2, now all on the neighborhood of x.

Since α_1, α_2 and dz form a basis of $(1,0)$-forms on Y_V, we may write

$$g_2(z) = \sum_{j,k=1,2} f_{j,k}\alpha_j\bar{\alpha}_k + \sum_{k=1,2} f_{0,k}dz\bar{\alpha}_k + \sum_{j=1,2} f_{j,0}\alpha_j d\bar{z} + f_{0,0}dzd\bar{z}. \tag{4.35}$$

By the properties of $g_k(x)$, we may assume that

$$g_2(z) = f(z)(\alpha_1\bar{\alpha}_2 + \bar{\alpha}_1\alpha_2) + \sum_{k=1,2} f_{0,k}dz\bar{\alpha}_k + \sum_{j=1,2} f_{j,0}\alpha_j d\bar{z} + f_{0,0}dzd\bar{z}. \tag{4.36}$$

Note that only the first term of Equation (4.36) contributes to $\int_{Y_y} \eta_y \partial\bar{\partial} g_2(z)$ or to $\int_{Y_y} \eta_y g_2(z)$, thus only the first term of Equation (4.36) matters on either side of Equation (4.34).

Then from the properties of $g_2(x)$ again, in particular, since $\partial\bar{\partial} g_2(x) = 0$ in a neighborhood of $y \neq x$, we can deduce by computation that

$$b^2 \frac{\partial^2}{\partial z \partial \bar{z}} f(z) = \frac{1}{2} f(z), \tag{4.37}$$

where $z = a + ib$. Then

$$\Delta_z f(z) = 4b^2 \frac{\partial^2}{\partial z \partial \bar{z}} f(z) = 2f(z), \tag{4.38}$$

and hence Equation (4.34).

Chapter 5

Arithmetic Intersection Theory

In [10], Gillet and Soulé introduced *arithmetic Chow groups* for *arithmetic varieties* over *arithmetic rings* and their *intersections*, which is essential for the geometric side of higher-weight Gross-Zagier formulas, so we will recall the theory in this chapter.

5.1 Arithmetic Chow Groups

While the set-up is more general in [10], for our considerations, an arithmetic ring is a triple $(\mathcal{O}_F, \Sigma, f_\infty)$, where $\mathcal{O}_F$ is the ring of integers of a number field F, $\Sigma = \mathrm{Hom}(\mathcal{O}_F, \mathbb{C})$ is the set of complex embeddings, and $f_\infty : \mathbb{C}^\Sigma \to \mathbb{C}^\Sigma$ the conjugate-linear involution of $\mathbb{C}$-algebras such that the diagram

$$\begin{array}{ccc} \mathcal{O}_F & \xrightarrow{\delta} & \mathbb{C}^\Sigma \\ {\scriptstyle \mathrm{Id}}\downarrow & & \downarrow{\scriptstyle f_\infty} \\ \mathcal{O}_F & \xrightarrow{\delta} & \mathbb{C}^\Sigma \end{array} \tag{5.1}$$

commutes, where $\delta(\alpha) = (\sigma(\alpha))_{\sigma\in\Sigma}$. Under the identification $\mathbb{C}^\Sigma \cong \mathbb{C} \otimes_{\mathbb{Q}} F, (\alpha\sigma(\beta))_{\sigma\in\Sigma} \leftarrow\!\shortmid \alpha \otimes \beta$, f_∞ on $\mathbb{C}^\Sigma$ is induced by $\alpha \otimes \beta \mapsto \bar{\alpha} \otimes \beta$ on $\mathbb{C} \otimes_{\mathbb{Q}} F$. For example, when F is imaginary quadratic, $f_\infty : \mathbb{C}^2 \to \mathbb{C}^2$ is given by $f_\infty(a, b) = (\bar{b}, \bar{a})$.

An arithmetic variety over $(\mathcal{O}_F, \Sigma, f_\infty)$ is a scheme $\pi : X \to \mathrm{Spec}(\mathcal{O}_F)$ that is flat and of finite type over $S = \mathrm{Spec}(\mathcal{O}_F)$, where the generic fiber X_F is smooth. For each $s \in S$, $X(s) = \pi^{-1}(s)$ denotes the fiber over s; for each $\sigma \in \Sigma$, we define $X_\sigma := X \otimes_\sigma \mathbb{C}$. We write

$$X_\Sigma = \coprod_{\sigma\in\Sigma} X_\sigma = X \otimes_{\mathcal{O}_F} \mathbb{C}^\Sigma, \tag{5.2}$$

and $X_\infty = X_\Sigma(\mathbb{C})$ the associated (disjoint union of) complex manifold(s), on which f_∞ is a continuous involution.

Let $A^n(X)$ denote the space of C^∞ complex-valued n-forms on X_∞, and $A^{i,j}(X)$ the subspace of $(i,j)-$forms (in terms of local coordinates, these are the forms of the shape $\sum f_{IJ} dz_I \wedge d\bar{z}_J$ where I is an i-tuple and J is an j-tuple). Let $A^n_c(X)$ and $A^{i,j}_c(X)$ be the subspace consisting of compactly supported forms of $A^n(X)$ and $A^{i,j}(X)$ respectively. Let $\mathscr{D}_n(X)$ [resp. $\mathscr{D}_{i,j}(X)$] denote the space of n-currents [resp. (i,j)-currents], which is the dual space of $A^n_c(X)$ [resp. $A^{i,j}_c(X)$]; let $\mathscr{D}^c_n(X)$ [resp. $\mathscr{D}^c_{i,j}(X)$] denote the space of compactly supported n-currents [resp. (i,j)-currents], which is the dual space of $A^n(X)$ [resp. $A^{i,j}(X)$]. For our considerations X_∞ will be compact so forms and currents will be compactly supported.

Suppose X_∞ is of (complex) dimension d, then there is a map

$$A^{i,j}(X) \to \mathscr{D}_{d-i,d-j}(X), \quad \alpha \mapsto [\alpha], \tag{5.3}$$

where for any compactly-supported $(d-i,d-j)$-form β, $[\alpha]\beta := \int_X \alpha \wedge \beta$. The space $\mathscr{D}_{d-i,d-j}(X)$ has a natural topology for which the above map is continuous with dense image, and we can think of $\mathscr{D}_{d-i,d-j}(X)$ as the space of (i,j)-forms with distribution coefficients. ([7], as cited in [10].) This motivates the notation

$$\mathscr{D}^{i,j}(X) := \mathscr{D}_{d-i,d-j}(X) \quad \text{and} \quad \mathscr{D}^n := \mathscr{D}_{2d-n}. \tag{5.4}$$

The operators

$$\partial : A^{i,j} \to A^{i+1,j}, \quad \sum_{I,J} f_{IJ} dz_I \wedge d\bar{z}_J \mapsto \sum_{I,J} \sum_{k=1}^{d} \frac{\partial f_{IJ}}{\partial z_k} dz_k \wedge dz_I \wedge d\bar{z}_J,$$

$$\bar{\partial} : A^{i,j} \to A^{i,j+1}, \quad \sum_{I,J} f_{IJ} dz_I \wedge d\bar{z}_J \mapsto \sum_{I,J} \sum_{k=1}^{d} \frac{\partial f_{IJ}}{\partial \bar{z}_k} d\bar{z}_k \wedge dz_I \wedge d\bar{z}_J,$$

and

$$d : A^n \to A^{n+1}, \quad d = \partial + \bar{\partial}$$

induce dual operators $\partial^*, \bar{\partial}^*, d^*$ on $\mathscr{D}^{i,j}$ and $\mathscr{D}^n$. However we do not necessarily have $[d\alpha] = d^*[\alpha]$.

Rather, for forms $\alpha \in A^n(X)$, and $\beta \in A_c^{2d-n-1}(X)$,

$$\begin{aligned}[d\alpha](\beta) &= \int_X d\alpha \wedge \beta = \int_X d(\alpha \wedge \beta) - \int_X (-1)^n \alpha \wedge d\beta \\ &= 0 - \int_X (-1)^n \alpha \wedge d\beta = (-1)^{n+1}(d^*[\alpha])\beta,\end{aligned} \tag{5.5}$$

where $\int_X d(\alpha \wedge \beta) = 0$ by Stokes Theorem. Therefore, we shall write the operator $(-1)^{n+1}d^*$ on $\mathscr{D}^n$ (or on $\mathscr{D}^{i,j}$ where $i+j=n$) simply as d, and under this notation, d commutes with the map in Equation (5.3). We shall likewise write $\partial = (-1)^{n+1}\partial^* : \mathscr{D}^{i,j} \to \mathscr{D}^{i+1,j}$ and $\bar{\partial} = (-1)^{n+1}\bar{\partial}^* : \mathscr{D}^{i,j} \to \mathscr{D}^{i,j+1}$ if $i+j=n$, so that on $\mathscr{D}^n$ or $\mathscr{D}^{i,j}$ we still have $d = \partial + \bar{\partial}$. We write

$$\begin{aligned}\widetilde{A}^{i,j}(X) &:= A^{i,j}(X)/(\partial A^{i-1,j}(X) + \bar{\partial} A^{i,j-1}(X)), \text{ and} \\ \widetilde{\mathscr{D}}^{i,j}(X) &:= \mathscr{D}^{i,j}(X)/(\partial \mathscr{D}^{i-1,j}(X) + \bar{\partial} \mathscr{D}^{i,j-1}(X)).\end{aligned}$$

Then the map $\widetilde{A}^{i,j}(X) \to \widetilde{\mathscr{D}}^{i,j}(X)$ induced from ((5.3)) is injective.

Note that f_∞ acts on $A^{i,i}(X)$ and $\mathscr{D}^{i,i}(X)$, and we define $A^{i,i}(X_\mathbb{R})$ [resp. $\mathscr{D}^{i,i}(X_\mathbb{R})$] to be the subspace of $A^{i,i}(X)$ [resp. $\mathscr{D}^{i,i}(X)$] consisting of real forms [resp. currents] satisfying $f_\infty^*\alpha = (-1)^i\alpha$. We write $\widetilde{A}^{i,i}(X_\mathbb{R})$ [resp. $\widetilde{\mathscr{D}}^{i,i}(X_\mathbb{R})$] to be the image of $A^{i,i}(X_\mathbb{R})$ [resp. $\mathscr{D}^{i,i}(X_\mathbb{R})$] in $\widetilde{A}^{i,i}(X)$ [resp. $\widetilde{\mathscr{D}}^{i,i}(X)$]. We also define $d^c := \frac{i}{4\pi}(\bar{\partial} - \partial)$. Then both d and d^c maps real forms [resp. currents] to real forms [resp. currents] and $dd^c = -d^c d = \frac{i}{2\pi}\partial\bar{\partial}$.

The group $Z^i(X)$ of cycles of codimension i on X is the free abelian group generated by the set of codimension-i integral subschemes of X. That is,

$$Z^i(X) = \bigoplus_{x \in X^{(i)}} \mathbb{Z}, \tag{5.6}$$

where $X^{(i)} = \{x \in X : \mathcal{O}_{X,x} \text{ has Krull dimension } i\}$.

If Y is a codimension-i integral subscheme of X, then Y_∞ is an analytic subspace of X_∞, invariant under f_∞. Therefore, integration over Y_∞, which we denote by δ_Y, defines a current in $\mathscr{D}^{i,i}(X_\mathbb{R})$. A *Green's current* for Y_∞ (or for Y) is an element $g \in \widetilde{\mathscr{D}}^{i-1,i-1}(X_\mathbb{R})$ such that $dd^c g + \delta_Y$ is represented by a smooth (i,i)-form $\omega(Y,g)$. It is shown in [10, Section 1.2] that Green's current exists when X_∞ is compact and Kähler (which is true when X_F is smooth and projective), and that two Green's currents for Y differ by an element of $\widetilde{A}^{i-1,i-1}(X) \hookrightarrow \widetilde{\mathscr{D}}^{i-1,i-1}(X)$.

We could extend the above definitions to cycles in $Z^i(X)$ linearly. We have the map

$$Z^i(X) \to \mathscr{D}^{i,i}(X_{\mathbb{R}}), \quad Z := \sum n_i Z_i \mapsto \delta_Z := \sum n_i \delta_{Z_i}, \tag{5.7}$$

and we can associate with $Z \in Z^i(X)$ a Green's current g satisfying

$$dd^c g + \delta_Z = \omega(Z, g) \tag{5.8}$$

for some smooth (i,i)-form $\omega(Z,g)$, whose cohomology class is the Poincaré dual of Z.

The group $\widehat{Z}^i(X)$ of codimension-i arithmetic cycles on X is the subgroup of $Z^i(X) \oplus \widetilde{\mathscr{D}}^{i-1,i-1}(X_{\mathbb{R}})$ consisting of pairs (Z,g) such that g is a Green's current for Z.

Let W be an $(i-1)$-codimensional reduced irreducible subscheme, with generic point $w \in X^{(i-1)}$. Let f be a nonzero element in the function field of w, i.e. $f \in k(w)^*$. Then f gives rise to a codimension-i cycle

$$\mathrm{Div}(f) = \sum_V \mathrm{ord}_V(f)[V], \tag{5.9}$$

where the sum is taken over all integral subschemes V of W that is of codimension i in X. Let $\pi : \widetilde{Y}_\infty \to Y_\infty$ be a proper resolution of singularities of Y_∞. Then f induces a rational function $\widetilde{f}$ on $\widetilde{Y}_\infty$ and $\log|\widetilde{f}|^2$ defines a current in $\mathscr{D}^{0,0}(\widetilde{Y})$. Let ι denote the inclusion $Y \hookrightarrow X$ and $\widetilde{\iota} : \widetilde{Y}_\infty \to X_\infty$ the induced map on $\widetilde{Y}_\infty$. Then $\widetilde{\iota}_* \left(\log|\widetilde{f}|^2\right)$ defines an element in $\mathscr{D}^{i-1,i-1}(X_{\mathbb{R}})$ and thus a class in $\widetilde{D}^{i-1,i-1}(X_{\mathbb{R}})$. Moreover, $\widetilde{\iota}_* \left(\log|\widetilde{f}|^2\right)$ does not depend on the choice of Y_∞ so we shall write $\iota_* \log|f|^2$ instead. By the Poincaré-Lelong Lemma (as seen in [10, Page 127]) $dd^c \iota_* \log|f|^2 = \delta_{\mathrm{Div}(f)}$, so $-\iota_* \log|f|^2$ is a Green's current for $\mathrm{Div}(f)$.

The chow group of codimension i is defined to be the quotient

$$\mathrm{CH}^i(X) := Z^i(X)/\mathrm{Rat}^i(x), \tag{5.10}$$

where $\mathrm{Rat}^i(x) \subset Z^i(X)$ is the subgroup generated by rational cycles, i.e. cycles of the form $\mathrm{Div}(f)$. The arithmetic Chow group $\widehat{\mathrm{CH}}^i(X)$ is defined to be the quotient

$$\widehat{\mathrm{CH}}^i(X) := \hat{Z}^i(X)/\hat{R}^i(X), \tag{5.11}$$

where $\hat{R}^i(X)$ is the subgroup generated by all pairs of the form $(\mathrm{Div}(f), -\iota_* \log|f|^2)$ for some $f \in k(W)^*$. The group $\widehat{\mathrm{CH}}^i(X)$ is an additive group with addition defined by $(Y, g_Y) + (Z, g_Z) =$

$(Y + Z, g_Y + g_Z)$.

Example 5.1.1. *[10, Section 3.4.3] Consider* $S = \mathrm{Spec}(\mathcal{O}_F)$ *as an arithmetic variety. Then* $\mathrm{CH}^1(S) = \mathrm{CL}(\mathcal{O}_F) \cong F^\times \backslash \mathbb{A}_F^\times / U_F$*, where*

$$\mathbb{A}_F^\times = \prod_v{}' (F_v^\times, \mathcal{O}_v^\times) = \{(a_v) \in \prod_v F_v^\times : a_v \in \mathcal{O}_v^\times \text{ for all but finitely many } v\}$$

is the idèle group of F*, and* $U_F = \prod_v \mathcal{O}_v^\times$ *the maximal compact subgroup of* $\mathbb{A}_F^\times$*. Here,* $\mathcal{O}_v$ *is the valuation subring of* F_v *if* v *is non-archimedean, and* $\mathcal{O}_v = F_v$ *if* v *is archimedean. The map from* $F^\times \backslash \mathbb{A}_F^\times / U_F$ *to* $\mathrm{CL}(\mathcal{O}_F)$ *is given by mapping the class of* (a_v) *to the class of* $Z = \sum_{v \text{ finite}} v(a_v)[v]$*. In fact, we also have* $\widehat{\mathrm{CH}}^1(S) \cong F^\times \backslash \mathbb{A}_F^\times / U_F$*, where the class of* (a_v) *to the class of* (Z, g)*,* Z *as above, and* $g = g_v := -\log |a_v|^2$ *when* v *is real and* $g = g_v := -2 \log |a_v|^2$ *when* v *is complex.*

Furthermore, there is a homomorphism from $\widehat{\mathrm{CH}}^1(S)$ *to* $\mathbb{R}$*, called the degree map, given by*

$$\deg : \widehat{\mathrm{CH}}^1(S) \to \mathbb{R}, \quad (Z, g) \mapsto \sum n_i \log \#(\mathcal{O}_F/\wp_i) + \frac{1}{2} \sum_{\sigma \in \Sigma} g_\sigma, \tag{5.12}$$

if $Z = \sum n_i[\wp_i]$*. The degree map is an isomorphism when* $F = \mathbb{Q}$*.*

Remark 5.1.2. *Note that there is a typo in [10, Section 3.4.3] when defining the degree formula. They wrote* $\log n_i \#(\mathcal{O}_F/\wp_i)$ *instead of* $n_i \log \#(\mathcal{O}_F/\wp_i)$*, but that was not compatible with the product formula.*

5.2 Intersection Pairing of Arithmetic Chow Cycles

While [10] studied arithmetic intersection in a more general setting, for our purposes we focus on the following case.

Theorem 5.2.1. *[10, Theorem 4.3.2] Let* $\pi : Y \to S$ *be a proper map between equidimensional nonsingular arithmetic varieties over an arithmetic ring such that the map is smooth on generic fibers* $Y_F \to S_F$*. If* $d = \dim Y - \dim S$ *and* $i + j = d + 1$*, there is a bi-additive pairing*

$$\widehat{\mathrm{CH}}^i(Y) \otimes \widehat{\mathrm{CH}}^j(Y) \to \widehat{\mathrm{CH}}^1(S). \tag{5.13}$$

When $S = \mathrm{Spec}\, \mathcal{O}_F$, we have $\dim S = 1$ so $\dim Y = d+1$; composing with the degree map (5.12), we get a pairing on arithmetic Chow cycles of complimentary co-dimension in the arithmetic vari-

ety Y:

$$\widehat{\mathrm{CH}}^i(Y) \times \widehat{\mathrm{CH}}^j(Y) \to \mathbb{R}, \qquad (\widehat{Z}_1, \widehat{Z}_2) \mapsto \widehat{Z}_1 \cdot \widehat{Z}_2 \tag{5.14}$$

where $i + j = d + 1 = \dim Y = \dim Y_F + 1$. We call $\widehat{Z}_1 \cdot \widehat{Z}_2$ the *arithmetic intersection number* of the Chow cycles $\widehat{Z}_1 = (Z_1, g_{Z_1})$ and $\widehat{Z}_2 = (Z_2, g_{Z_2})$.

We recall the construction of the pairings from [10]. Given two Chow cycles (with representatives) $\widehat{Z}_1 = (Z_1, g_{Z_1}) \in \widehat{\mathrm{CH}}^i(Y)$ and $\widehat{Z}_2 = (Z_2, g_{Z_2}) \in \widehat{\mathrm{CH}}^j(Y)$, by Chow's moving lemma on Y_F [22, Section 3], there exists a cycle Z_2' rationally equivalent to Z_2 such that Z_2' and Z_1 intersect properly (when $i+j = \dim Y_F + 1$, intersecting properly means the set theoretic intersection is empty) on the generic fiber Y_F (though not necessarily on the entire arithmetic variety Y). So we may assume Z_1 and Z_2 intersects properly on the generic fiber to start with. (For details of the well-definedness of the pairing (5.13), see the proof of [10, Theorem 4.3.2].) By bi-additivity, we may also assume that Z_1 and Z_2 are integral subschemes of Y. Under these assumptions, we have the following decomposition into local intersections

$$\widehat{Z}_1 \cdot \widehat{Z}_2 = \sum_{v \text{ finite}} \left(\widehat{Z}_1 \cdot \widehat{Z}_2\right)_v \cdot \log \#(\mathcal{O}_F/v) + \sum_{v|\infty} \left(\widehat{Z}_1 \cdot \widehat{Z}_2\right)_v, \tag{5.15}$$

where if v is a finite place $\left(\widehat{Z}_1 \cdot \widehat{Z}_2\right)_v = (Z_1 \cdot Z_2)_v$ is the intersection multiplicity of Z_1 and Z_1 at v, which is zero for all but the finitely many places where $Z_1 \cap Z_2$ is supported; and when $v|\infty$

$$\left(\widehat{Z}_1 \cdot \widehat{Z}_2\right)_v = \frac{1}{2} \int_{Y_v(\mathbb{C})} g_{Z_1} * g_{Z_2}, \tag{5.16}$$

where $Y_v = Y \otimes_{\mathcal{O}_{F,\sigma}} \mathbb{C}$ if v is the place corresponding to the embedding $\sigma : F \to \mathbb{C}$. (See [10, Sections 2.2.1 and 4.3.2 - 4.3.8], and [30, Section 1.2].)

5.3 Computation of the Local Intersection Pairing in a Special Case

The local intersection numbers at finite places to be computed in this thesis fall into the following special case. Suppose $\pi : Y \to S$ factor through X, where $S = \operatorname{Spec} \mathcal{O}_F$ and X is an arithmetic variety of relative dimension 1 over S. Let x and y be two divisors of X, and Y_x, Y_y the respective fibers of Y above x, y. Suppose Z_1 and Z_2 are (formal linear combinations of) integral subschemes of Y_x and Y_y respectively and are of complimentary arithmetic dimensions in Y (in the sense of the

previous section), and that $Z_1 \cdot Z_2$ has a flat presentation over $x \cap y$, then

$$(Z_1 \cdot Z_2)_v = (x \cdot y)_v (Z_1^0 \cdot Z_2^0), \tag{5.17}$$

where $Z_1^0 \cdot Z_2^0$ is the geometric intersection taken within $Y_{x \cap y}$ over the geometric fiber at v. [30, Identity 3.3.1]

5.4 Height Pairing

In this section we recall the height pairing discussed in [30, Section 1.3].

Let Y be an algebraic variety of dimension $2n$ over a number field F, Z_{1F} and Z_{2F} be algebraic cycles of co-dimension n in Y_F. Let $\mathcal{Y}$ be a regular flat projective integral model of Y. Assume that

(a) The cycle Z_{1F} has an integral model $\mathcal{Z}_1$ in $\mathcal{Y}$ such that for any finite place v and any cycle $\mathcal{Z}$ of dimension n supported on the special fiber of $\mathcal{Y}$ above v, $(\mathcal{Z}_1 \cdot \mathcal{Z})_v = 0$; and that

(b) for any infinite place v, the class of $Z_{1v}(\mathbb{C})$ vanishes in $H^{2n}(Y_v(\mathbb{C}), \mathbb{C})$ and thus there is a green's current g_1 such that $\omega(Z, g_1) = 0$, i.e. $dd^c g_1 + \delta_{Z_1} = 0$.

Let $\widehat{Z}_1 := (\mathcal{Z}_1, g_1)$ where $\mathcal{Z}_1$ and g_1 are as chosen above. Let $\widehat{Z}_2 = (\mathcal{Z}_2, g_2)$ where $\mathcal{Z}_2$ is an integral model of Z_2 and g_2 a green's current of Z_2. By the construction of $\widehat{Z}_1$, the intersection number $\widehat{Z}_1 \cdot \widehat{Z}_2$ does not depend on the choice of $\mathcal{Z}_2$ and g_2.

We define the global height pairing

$$\langle Z_{1F}, Z_{2F} \rangle := (-1)^n \widehat{Z}_1 \cdot \widehat{Z}_2. \tag{5.18}$$

Since $\widehat{Z}_1 \cdot \widehat{Z}_2$ only depends on the class of $\widehat{Z}_1$ and $\widehat{Z}_2$ in $\widehat{\mathrm{CH}}^n(\mathcal{Y})$ and because of the assumptions on the choice of $\widehat{Z}_1$, the global height pairing $\langle Z_{1F}, Z_{2F} \rangle$ only depends on the rational equivalence class of Z_{1F} and Z_{2F} in Y_F.

The height pairings we discuss in this thesis will fall into the following special case. The map $Y \to F$ factors through a curve $X \to F$ and that there is a divisor D of X such that Z_1 and Z_2 are contained in the divisor Y_D. Moreover, we may choose the integral models $\mathcal{Y}$ of Y and $\mathcal{X}$ of X such that $\mathcal{Y} \to \operatorname{Spec} \mathcal{O}_F$ factors through $\mathcal{X} \to \operatorname{Spec} \mathcal{O}_F$ and there exists a divisor $\mathcal{D}$ of $\mathcal{X}$ such that both $\mathcal{Z}_1$ and $\mathcal{Z}_2$ are contained in the fiber $\mathcal{Y}_{\mathcal{D}}$. In this case, $\langle Z_{1F}, Z_{2F} \rangle$ only depends on the rational equivalence classes of Z_{1F} and Z_{2F} in Y_D.

When local intersection numbers are well-defined, we also write

$$\langle Z_{1F}, Z_{2F} \rangle_v := (-1)^n \left(\widehat{Z}_1 \cdot \widehat{Z}_2 \right)_v . \tag{5.19}$$

The height pairing (5.18) above coincides with the Beilinson-Bloch height pairing. ([2], [4], and [23].) Regarding the positivity of the pairing, we have the following conjecture:

Conjecture 5.4.1. *([2] as cited in [30, Conjecture 1.3.1]) Let $\mathscr{L}$ be an ample line bundle on Y_F and let $\mathrm{CH}^n(Y)^{00}$ be the subgroup of $\mathrm{CH}^n(Y)$ containing Chow cycles satisfying conditions (a) and (b), and having 0 intersections with the first Chern class $c_1(\mathscr{L})$. Then the height pairing* (5.18) *is positive definite on $\mathrm{CH}^n(Y)^{00}$.*

The algebraic cycles studied in this thesis is inside $\mathrm{CH}^n(Y)^{00}$. We shall prove that they satisfy conditions (a) and (b) in Sections 6.4 and 6.5. The zero intersection with $c_1(\mathscr{L})$ follows from the choice of $S_k(x)$ described in the next Chapter.

Chapter 6

Kuga Varieties and CM Cycles

In this Chapter, we go into more details about the geometric side introduced in Section 1.1.2.

6.1 Universal Abelian Surfaces over Shimura Curves

Recall that we let $N = N_- N_+$ where $N_- > 1$ is a product of an even number of distinct primes such that every prime $p|N_-$ is inert in the quadratic imaginary field K while every $p|N_+$ is split in K. We let

$$\Sigma = \{p : p \text{ is a finite prime dividing } N_-\},$$

a non-empty set with an even number of elements. Let B be a quaternion algebra whose set of places of ramification is Σ and fix an embedding $\rho : K \to B$. The arithmetic of quaternion algebra used below follows from [25] as cited in [3]. The readers can also consult [1] and [26] for more details about the claims used.

Then for each $\ell \in \Sigma$, $B_\ell := B \otimes \mathbb{Q}_\ell$ is a division algebra, and thus has a unique maximal order, which we denote by R_ℓ. For each place $\ell \notin \Sigma$ of $\mathbb{Q}$, including the place at infinity, we have $B_\ell := B \otimes \mathbb{Q}_\ell \cong M_{2\times 2}(\mathbb{Q}_\ell)$; for such an ℓ and for any isomorphism $\phi_\ell : B_\ell \to M_2(\mathbb{Q}_\ell)$, the order $R_\ell := \phi_\ell^{-1}(M_2(\mathbb{Z}_\ell))$ is a maximal order of B_ℓ. Let $\hat{\mathbb{Z}} := \prod_{\ell \neq \infty} \mathbb{Z}_\ell$ be the profinite completion of $\mathbb{Z}$ and let $\hat{\mathbb{Q}} := \hat{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q} = \prod'_{\ell \neq \infty} \mathbb{Q}_\ell$ be the ring of finite adèles over $\mathbb{Q}$. Let $\hat{B} = B \otimes \hat{\mathbb{Q}}$. We choose the ϕ_ℓ's so that

$$\mathcal{O}_B := \mathrm{Diag}^{-1}\left(\mathrm{Diag}(B) \cap \prod_{\ell \text{ a place of } \mathbb{Q}} R_\ell\right) \tag{6.1}$$

is nonempty and contains $\rho(\mathcal{O}_K)$, where Diag denotes the diagonal embedding of B in $\hat{B}$. Then $\mathcal{O}_B$

is a maximal order of B.

An involution $* : B \to B$ is said to be *positive* if $\mathrm{tr}(\alpha^*\alpha) > 0$ for all $\alpha \in B\backslash\{0\}$. It is known that every involution on B is either the canonical involution $b \mapsto \bar{b} = \mathrm{tr}\, b - b$, or of the form

$$b \mapsto \gamma_\xi b := \xi \bar{b} \xi^{-1}, \qquad \xi \in B^\times, \bar{\xi} = -\xi, \tag{6.2}$$

and that an involution on B is positive if and only if $* = \gamma_\xi$ with $\xi^2 < 0$ (i.e. $\mathrm{nm}(\xi) > 0$). [19, Chapter IV] Fix such a ξ and therefore a corresponding positive involution $* : B \to B$.

For any open compact subgroup U of $\widehat{B}^\times$, there is a Shimura curve X_U, which can be canonically defined over $\mathbb{Q}$ as the moduli space of equivalence classes of (A, ι, P, κ), where

- A is an abelian surface over $\mathbb{Q}$,
- $\iota : B \hookrightarrow \mathrm{End}(A) \otimes \mathbb{Q} =: \mathrm{End}^0(A)$,
- P a principal polarization of A such that the induced rosati involution restricts to $*$ on B via the map ι,
- κ an U-class of isomorphisms $\kappa : B \otimes \hat{\mathbb{Q}} \xrightarrow{\sim} H_1(A, \hat{\mathbb{Q}})$.

Two tuples $(A_1, \iota_1, P_1, \kappa_1)$ and $(A_2, \iota_2, P_2, \kappa_2)$ are equivalent if there exists an isogeny $\phi : A_1 \to A_2$ that induces identification of ι_1 with ι_2 and P_1 with P_2 and such that $H_1(\phi) \circ \kappa_1 = \kappa_2$. Given A and ι, the compatible polarization P is uniquely determined and thus can be omitted from the tuple.

The analytic points $X_U(\mathbb{C})$ are given by the double coset space

$$X_U(\mathbb{C}) = B^\times \backslash (\mathfrak{H}^\pm \times \hat{B}^\times)/U, \tag{6.3}$$

where $\mathfrak{H}^\pm = \mathbb{P}^1(\mathbb{C}) - \mathbb{P}^1(\mathbb{R}) = \mathbb{C} - \mathbb{R}$. The group $B^\times$ acts on $\mathfrak{H}^\pm$ via the fractional linear transformation of $\phi_\infty(B_\infty^\times) = \mathrm{GL}_2(\mathbb{R})$ on $\mathfrak{H}^\pm$, and on $\hat{B}^\times$ by left multiplication (after the diagonal map). The group U acts on $\mathfrak{H}^\pm$ trivially, and on $\hat{B}^\times$ by right multiplication.

We consider the Shimura curve X_U where U is given by $U = \hat{R}^\times_{N_+,N_-}$, which we denote by X_{N_+,N_-}. Here the Eichler order R_{N_+,N_-} is given by

$$R_{N_+,N_-} = \left\{ x \in \mathcal{O}_B : \phi_\ell(x) \equiv \begin{bmatrix} * & * \\ 0 & * \end{bmatrix} \mod \ell^r, \text{ if } r \geq 1 \text{ and } \ell^r | N_+ \right\}. \tag{6.4}$$

In other words,

$$R_{N_+,N_-} = \mathrm{Diag}^{-1}\left(\mathrm{Diag}(B) \cap \prod_{\ell \text{ a place of } \mathbb{Q}} R'_\ell\right), \tag{6.5}$$

where $R'_\ell = R_\ell$ if $\ell \nmid N_+$ (including $\ell = \infty$), and $R'_\ell = \phi_\ell^{-1}\left(\begin{bmatrix} \mathbb{Z}_\ell & \mathbb{Z}_\ell \\ N_+\mathbb{Z}_\ell & \mathbb{Z}_\ell \end{bmatrix}\right)$ if $\ell | N_+$. We write $\hat{R}_{N_+,N_-}$ for $R_{N_+,N_-} \otimes_{\mathbb{Z}} \hat{\mathbb{Z}} = \prod_{\ell \neq \infty}(R_{N_+,N_-} \otimes \mathbb{Z}_\ell)$.

Let M be an integer that is a product of two relatively prime integers no less than 3, coprime to N_-, and such that $N_+|M$. Let R'_{M,N_-} be the suborder of R defined by

$$R'_{M,N_-} = \left\{ x \in \mathcal{O}_B : \phi_\ell(x) \equiv \pm \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \mod \ell^r, \text{ if } r \geq 1 \text{ and } \ell^r | M \right\}. \tag{6.6}$$

Consider Shimura curve $X'_{M,N_-} := X_{\hat{R}'^{\times}_{M,N_-}}$

Proposition 6.1.1. *Assuming that $M = M_1M_2$ where $M_1, M_2 \in \mathbb{Z}_{\geq 3}$ and $(M_1, M_2) = 1$, the Shimura curve X'_{M,N_-} has a model over $\mathbb{Q}[\zeta_M]$ and admits a universal abelian surface (with level-M structure) $\pi_{\mathbb{Q}[\zeta_M]} : \mathcal{A}(M) \to X'_{M,N_-}$, and $\pi_{\mathbb{Q}[\zeta_M]}$ can be uniquely extended to a morphism $\pi_{\mathbb{Z}[\zeta_M]}$ of regular, flat, and projective $\mathbb{Z}[\zeta_M]$-schemes.*

Proof. We have integral models

$$\mathcal{A}(M_i)_{\mathbb{Z}[\zeta_{M_i}, \frac{1}{M_i}]} \to (X'_{M_i,N_-})_{\mathbb{Z}[\zeta_{M_i}, \frac{1}{M_i}]}, \tag{6.7}$$

for each $i = 1, 2$.

Get integral models

$$\mathcal{A}(M)_{\mathbb{Z}[\zeta_M, \frac{1}{M_i}]} \to (X'_{M,N_-})_{\mathbb{Z}[\zeta_M, \frac{1}{M_i}]}, \tag{6.8}$$

for each $i = 1, 2$, and glue along

$$\mathcal{A}(M)_{\mathbb{Z}[\zeta_M, \frac{1}{M}]} \to (X'_{M,N_-})_{\mathbb{Z}[\zeta_M, \frac{1}{M}]} \tag{6.9}$$

to get $\pi_{\mathbb{Z}[\zeta_M]}$. (c.f. [18, Section 7.2]) □

In what follows, we base change X'_{M,N_-} to $H[\zeta_M]$, where H is the Hilbert class field of K, so that the CM point x is a closed algebraic point, and we denote this curve by $X_{H[\zeta_M]}$ or simply X. We consider the fiber product Y of $(2k-2)$ copies of $\mathcal{A}(M)_{H[\zeta_M]}$ over X. Then there exist regular, flat

and projective integral models $\mathcal{Y}$ of Y and $\mathcal{X}$ of X over $\mathcal{O}_H[\zeta_M]$ together with a morphism $\mathcal{Y} \to \mathcal{X}$.

6.2 CM Cycles in an Abelian Surface with QM and CM

Let A an abelian surface with quaterion multiplication by B, i.e. $B \hookrightarrow \mathrm{End}^0(A)$. Given the positive involution $* : B \to B$ fixed in the previous section, there is a unique ample divisor D of A such that $\gamma_D = *$ when restricted on the image of B in $\mathrm{End}^0(A)$, where

$$\gamma_D(\alpha) := \lambda_D^{-1} \circ \alpha^\vee \circ \lambda_D, \qquad \alpha \in \mathrm{End}^0(A) \tag{6.10}$$

is the Rosati involution and

$$\lambda_D : A \to A^\vee := \mathrm{Pic}^0(A), \qquad x \mapsto \mathcal{O}([D+x]-D) \tag{6.11}$$

is the polarization defined by D, where $D + x$ denotes the translation of D by x with the group operation "+" on A. Note that $\lambda_D^\vee = \lambda_D$.

The Neron-Severi group $\mathrm{NS}(A)$ is the group of divisors of A modulo algebraic equivalence. We define the subspace $\mathrm{NS}(A)_{\mathbb{Q}}^{ord}$ of $\mathrm{NS}(A)_{\mathbb{Q}}$ to be generated by an ample divisor D described above under the action by B. Note that the subspace $\mathrm{NS}(A)_{\mathbb{Q}}^{ord}$ does not depend on the choice of D and is three dimensional. If A is non-CM, i.e. $\mathrm{End}_B^0(A) = \mathbb{Q}$ or equivalently $B = \mathrm{End}^0(A)$, we then have that $\mathrm{NS}(A)_{\mathbb{Q}}^{ord} = \mathrm{NS}(A)_{\mathbb{Q}}$.

We can take the example $B = M_{2\times 2}(\mathbb{Q})$ to illustrate the above. In this case we have $A = E \times E$, on which B acts by matrix multiplication. The 3-dimensional $\mathbb{Q}$-vector space $\mathrm{NS}(A)_{\mathbb{Q}}^{ord}$ are generated by $E \times \{0\}, \{0\} \times E, \Delta_E$. This is in fact generated by ample divisor $D := E \times \{0\} + \{0\} \times E$ corresponding to the positive involution

$$\gamma_D : \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto \begin{bmatrix} a & b \\ c & d \end{bmatrix}^t = \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}^{-1}.$$

To see that γ_D is indeed given by taking the transpose, we note that for $(x, y) \in E \times E$,

$$\lambda_D(x,y) = \mathcal{O}([E \times \{y\}] + [\{x\} \times E] - ([E \times \{0\}] + [\{0\} \times E])) = p_1^* \mathcal{O}([x] - [0]) + p_2^* \mathcal{O}([y] - [0])$$

Where $p_i : E \times E \to E$ denotes the projection onto the i-th component. Then for $\alpha = [\alpha_{ij}] \in$

$M_{2\times 2}(\mathbb{Z})$ we have

$$\alpha^\vee\lambda_D(x,y) = \alpha^*(p_1^*\mathcal{O}([x]-[0]) + p_2^*\mathcal{O}([y]-[0])) = (p_1\circ\alpha)^*\mathcal{O}([x]-[0]) + (p_2\circ\alpha)^*\mathcal{O}([y]-[0]).$$

Since $\mathrm{Pic}^0(X\times Y)\cong \mathrm{Pic}^0(X)\times\mathrm{Pic}^0(Y)$ for any algebraic varieties X and Y, we have

$$\alpha^\vee\lambda_D(x,y)\cong p_1^*\mathcal{L}_1 + p_2^*\mathcal{L}_2,$$

for some line bundles $\mathcal{L}_1$ and $\mathcal{L}_2$ on E. Let $\iota_1 : E\to E\times E, x\mapsto (x,0)$, and $\iota_2 : E\to E\times E, x\mapsto (0,x)$, then

$$\begin{aligned}\mathcal{L}_i &\cong \iota_i^*(\alpha^\vee\lambda_D(x,y)) = \iota_i^*\left((p_1\circ\alpha)^*\mathcal{O}([x]-[0]) + (p_2\circ\alpha)^*\mathcal{O}([y]-[0])\right)\\ &= (p_1\circ\alpha\circ\iota_i)^*\mathcal{O}([x]-[0]) + (p_2\circ\alpha\circ\iota_i)^*\mathcal{O}([y]-[0])\\ &= \alpha_{1i}^*\mathcal{O}([x]-[0]) + \alpha_{2i}^*\mathcal{O}([y]-[0]) \cong \mathcal{O}([\alpha_{1i}x+\alpha_{2i}y]-[0]).\end{aligned}$$

Hence

$$\gamma_D(x,y) = \lambda_D^{-1}\circ\alpha^\vee\circ\lambda_D(x,y) = (\alpha_{11}x+\alpha_{21}y, \alpha_{12}x+\alpha_{22}y) = \alpha^t\begin{bmatrix}x\\ y\end{bmatrix}.$$

We get the same result for $\alpha\in M_{2\times 2}(\mathbb{Q})$ after base changing to $\mathbb{Q}$.

We are interested in the cohomology groups $H^2(A,L) = \wedge^2 H^1(A,L)$ where L is the coefficient field of character 0. When we take $L=\mathbb{Q}$, $H^i(A,L)$ would mean Betti cohomology, and when we take $L=\mathbb{Q}_\ell$ where ℓ is prime to the characteristic of the field over which A is defined, $H^i(A,L)$ would mean étale cohomology. The polarization λ_D induces a homomorphism

$$H_1(A,L)\to H_1(A^\vee,L)(1) = H_1(A,L)^\vee(1) \tag{6.12}$$

and thus a form

$$\psi_D : H_1(A,L)\otimes H_1(A,L)\to L(1), \tag{6.13}$$

where $L(1)$ denotes the Tate twist: when $L=\mathbb{Q}_\ell$, $\mathbb{Q}_\ell(1) = (\varprojlim \mu_{\ell^n})\otimes\mathbb{Q}_\ell$; when $L=\mathbb{Q}$, $\mathbb{Q}(1) = 2\pi i\mathbb{Q}$.

The form ψ_D is symplectic following from the self-duality of λ_D, and thus defines an element in

$H^2(A,L) = \wedge^2 H^1(A,L)$. The Rosati involution γ_D can be characterized by

$$\psi_D(\alpha x, y) = \psi_D(x, \gamma_D \alpha y), \qquad x, y \in H_1(A,L), \alpha \in \mathrm{End}^0(A). \tag{6.14}$$

Fix an isomorphism $H_1(A,L) \simeq B_L := B \otimes L$ as a left B_L-module. We define the subspace $H^2(A,L)^{ord}$ consisting of elements of the form

$$\psi_\xi^{ord}(x,y) = \mathrm{tr}_{B/\mathbb{Q}}(\xi x \bar{y}), \qquad \xi \in B_L^{\mathrm{tr}=0}, x, y \in H_1(A,L). \tag{6.15}$$

Had we chosen a different isomorphism $H_1(A,L) \simeq B_L$ by a right multiplication by $t \in B_L^\times$, then ξ would be replaced by $\mathrm{nm}(t)\xi$, since $\mathrm{tr}_{B/\mathbb{Q}}(\xi x t \overline{yt}) = \mathrm{tr}_{B/\mathbb{Q}}(\xi x t \bar{t} \bar{y}) = \mathrm{tr}_{B/\mathbb{Q}}(\mathrm{nm}(t)\xi x \bar{y})$. Thus the space $H^2(A,L)^{ord}$ does not depend on the choice of the isomorphism.

We have

$$\psi : \mathrm{NS}(A)_L^{ord} \xrightarrow{\sim} H^2(A,L)^{ord}, \qquad D \mapsto \psi_D. \tag{6.16}$$

Therefore $\dim_L(H^2(A,L)^{ord}) = \dim_L(\mathrm{NS}(A)_L^{ord}) = 3$. Since $\dim_L(H^1(A,L)) = \dim_L B_L = 4$, we have $H^2(A,L) = \wedge^2 H^1(A,L)$ is six dimensional.

Fix an isomorphism $H_1(A,L) \simeq B_L$ as a left B_L-module and define the subspace *of CM cycles* $H^2(A,L)^{cm}$ consisting of elements of the form

$$\psi_\eta^{cm}(x,y) = \mathrm{tr}_{B/\mathbb{Q}}(\eta \bar{y} x), \qquad \eta \in B_L^{\mathrm{tr}=0}, x, y \in H_1(A,L). \tag{6.17}$$

As with $H^2(A,L)^{ord}$, the space $H^2(A,L)^{cm}$ is independent of the choice of the isomorphism $H_1(A,L) \simeq B_L$ – had we chosen a different isomorphism $H_1(A,L) \simeq B_L$ by a right multiplication by $t \in B_L^\times$, then η would be replaced by $t\eta\bar{t}$, since $\mathrm{tr}_{B/\mathbb{Q}}(\eta \overline{yt} x t) = \mathrm{tr}_{B/\mathbb{Q}}(\eta \bar{t} \bar{y} x t) = \mathrm{tr}_{B/\mathbb{Q}}(t\eta\bar{t}\bar{y}x)$.

We then have a decomposition

$$H^2(A,L) = H^2(A,L)^{ord} \oplus H^2(A,L)^{cm}, \tag{6.18}$$

which is in fact an orthogonal decomposition with respect to the symmetric quadratic form:

$$H^2(A,L) \otimes H^2(A,L) \to H^4(A,L)(2) \simeq L(2). \tag{6.19}$$

Assume that A has CM by K, that is, $\mathrm{End}_B^0(A) = K$. Then we can embed K into B. Let $i \in K^\times$ with trace 0. Then we have a CM cycle given by ψ_i^{cm}. We shall show that this cycle comes

from $\mathrm{NS}(A)_{\mathbb{Q}}$.

Note that A has two actions by K, via $\iota : K \hookrightarrow \mathrm{End}_B^0(A) \subset \mathrm{End}^0(A)$ and $\rho : K \hookrightarrow B \to \mathrm{End}^0(A)$. Then there exists an isogeny:

$$\Phi : E_1 \times E_2 \to A, \tag{6.20}$$

where E_1 and E_2 are elliptic curves with CM by K, on $\Phi(E_1)$ the two actions of K (via ι and ρ) agrees and on $\Phi(E_2)$ the two actions are conjugate to each other.

Since K is an imaginary quadratic field which splits B, we may write as $B = K + Kj$ where the (traceless) element $j \in B$ satisfies $ji = -ij$. We may identify $H_1(E_1, \mathbb{Q})$ with K and $H_1(E_2, \mathbb{Q})$ with Kj. Then the action via ι and ρ are multiplication by K on the left and right respectively. For any $a + bi \in H_1(E_1, \mathbb{Q})$ and $cj + dij \in H_1(E_2, \mathbb{Q})$, we have

$$\psi_i^{cm}(a + bi, cj + dij) = \mathrm{tr}_{B/\mathbb{Q}}(i(-cj - dij)(a + bi)) = \mathrm{tr}_{B/\mathbb{Q}}((bc - ad)i^2 j + (bdi^2 - ac)ij) = 0,$$

therefore

$$\psi_i^{cm} = \alpha\psi_{\Phi(E_1)} + \beta\psi_{\Phi(E_2)},$$

for some constants α and β, and therefore ψ_i^{cm} is represented by $Z(A) := \alpha[\Phi(E_1)] + \beta[\Phi(E_2)]$.

Continuing the example $A = E \times E$ and now assuming E has CM by K, then $\mathrm{End}^0(A) \cong M_2(K)$. We may choose an embedding $\rho : K \hookrightarrow B \hookrightarrow M_2(K)$ by $\sqrt{D} \mapsto \begin{bmatrix} 0 & 1 \\ D & 0 \end{bmatrix}$, and $\iota : K \hookrightarrow \mathrm{End}_B^0(A) \hookrightarrow M_2(K)$ by $\sqrt{D} \mapsto \begin{bmatrix} \sqrt{D} & 0 \\ 0 & \sqrt{D} \end{bmatrix}$. In this case we have

$$E_1 = \{(x, \sqrt{D}x) : \quad x \in E\}, \qquad E_2 = \{(x, -\sqrt{D}x) : \quad x \in E\},$$

since

$$\begin{bmatrix} 0 & 1 \\ D & 0 \end{bmatrix} \begin{bmatrix} x \\ \sqrt{D}x \end{bmatrix} = \begin{bmatrix} \sqrt{D}x \\ Dx \end{bmatrix} = \begin{bmatrix} \sqrt{D} & 0 \\ 0 & \sqrt{D} \end{bmatrix} \begin{bmatrix} x \\ \sqrt{D}x \end{bmatrix}$$

and

$$\begin{bmatrix} 0 & 1 \\ D & 0 \end{bmatrix} \begin{bmatrix} x \\ -\sqrt{D}x \end{bmatrix} = \begin{bmatrix} -\sqrt{D}x \\ Dx \end{bmatrix} = \overline{\begin{bmatrix} \sqrt{D} & 0 \\ 0 & \sqrt{D} \end{bmatrix} \begin{bmatrix} x \\ -\sqrt{D}x \end{bmatrix}}.$$

6.3 CM Cycles in the Kuga Variety

Let A be an abelian surface with QM by B. Let F be a quadratic extension [1] of $\mathbb{Q}$ that splits B, in other words, $B \otimes F \cong M_{2\times 2}(F)$. Then $M_{2\times 2}(F) \hookrightarrow \mathrm{End}(A) \otimes F$ and $M_{(2k-2)\times(2k-2)}(F) \hookrightarrow \mathrm{End}(A^{k-1}) \otimes F$. The permutation group $\mathfrak{S}_{2k-2}$ naturally embeds into $M_{(2k-2)\times(2k-2)}(F)$ as the permutation matrices, and therefore has an action on A^{k-1} that is algebraic over F.

When A also has CM by K, we define

$$Z_k(A) := \sum_{g\in\mathfrak{S}_{2k-2}} \mathrm{sgn}(g) g^*(Z(A)^{k-1}),$$

and

$$S_k(A) := c_k Z_k(A), \tag{6.21}$$

where c_k is a positive constant such that the self intersection of $S_k(A)$ in A^{2k-2} is $(-1)^{k-1}$. Let x' be a CM point on the Shimura curve X'_{M,N_-} with CM by K, we define:

$$S_k(x') := S_k(A_{x'}), \tag{6.22}$$

which defines an algebraic cycle in the Kuga variety Y.

Let x be a CM point with CM by K on the Shimura curve X_{N_+,N_-}, we define

$$S_k(x) := \frac{1}{\sqrt{\deg(P)}} \sum_i S_k(x_i), \tag{6.23}$$

where $P : X'_{M,N_-} \to X_{N_+,N_-}$ is the natural projection and $P^*(x) = \sum_i x_i$.

6.3.1 The Local System $H^{2k-2}(A^{k-1})$

Let ℓ be a prime not dividing MN_-, such that F can be embedded in to $\mathbb{Q}_\ell$.

We associate with a Shimura curve X_U the local system $V(x) = V(A_x) := H^1(A_x, \mathbb{Q}_\ell)$, dual to $H_1(A_x, \mathbb{Q}_\ell)$. Then V is a free B_ℓ module of rank-1. Let $G := \mathrm{Aut}_{B_\ell}(V)$ then $G \cong B_\ell^\times \cong \mathrm{GL}_2(\mathbb{Q}_\ell)$.

Now for each integer $k \geq 2$, consider the local system $W_k(x) = W_k(A_x) := H^{2k-2}(A_x^{k-1}, \mathbb{Q}_\ell) \cong \wedge^{2k-2} H^1(A_x^{k-1}, \mathbb{Q}_\ell) \cong \wedge^{2k-2}(V(x)^{k-1})$. Both V and W_k are G representations. By normalizing the G action on W_k by $\mathrm{nm}^{-(k-1)}$, W_k is also a $G/Z(G)$ representation. Via the identification $G \cong B_\ell^\times \cong \mathrm{GL}_2(\mathbb{Q}_\ell)$, W_k is also $\mathrm{GL}_2(\mathbb{Q}_\ell)$ and $\mathrm{PGL}_2(\mathbb{Q}_\ell)$ representations, where the $\mathrm{PGL}_2(\mathbb{Q}_\ell)$

[1] the quadratic imaginary extension K that we fixed earlier could work, but it is not the unique choice and may cause unnecessary confusion, so we use F here

action is obtained by the $\mathrm{GL}_2(\mathbb{Q}_\ell)$ action normalized by $\det^{-(k-1)}$.

Each representation of $\mathrm{PGL}_2(\mathbb{Q}_\ell)$ is semi-simple and each irreducible component of W_k is isomorphic to $\mathrm{Sym}^{2i}\mathbb{Q}_\ell^2$ for some integer i, which we call a *weight*-$2i$ representation of $G/Z(G)$. There is a unique weight-$(2k-2)$ component L_k of W_k. To see this, we take a decomposition (of G modules) $V = W \oplus W$ where $W \cong \mathbb{Q}_\ell^2$. Then

$$W_k = \wedge^{2k-2}(W^{2k-2}) \cong \bigoplus_{\substack{I=(i_1,\ldots,i_{2k-2}) \\ \text{a partition of } 2k-2}} \wedge^{i_1} W \otimes \cdots \otimes \wedge^{i_r} W.$$

Since $\wedge^i W$ is 0 for $i \neq 0, 1, 2$ and is the trivial $\mathbb{Q}_\ell$ representation if $i = 0, 2$, we have

$$W_k \cong \bigoplus_{t=0}^{k-1} \left(\bigoplus_{i=1}^{m_t} W^{\otimes 2t} \right),$$

where the multiplicity m_t is given by $\begin{pmatrix} 2k-2 \\ 2t \end{pmatrix} \begin{pmatrix} 2k-2-2t \\ k-1-t \end{pmatrix}$. In particular, $m_{k-1} = 1$. That is, the component $W^{\otimes(2k-2)}$ has multiplicity one. Moreover, the subspace $\mathrm{Sym}^{2k-2}W$ of $W^{\otimes(2k-2)}$ generated by the subset $\{\sum_{\sigma \in \mathfrak{S}_{2k-2}} \mathrm{sgn}(\sigma) a_{\sigma(1)} \otimes \cdots \otimes a_{\sigma(2k-2)} \mid a_1, \ldots, a_{2k-2} \in W\}$, is invariant and irreducible under $G/Z(G)$ action, and the component L_k we mentioned earlier is the unique subspace identified with $\mathrm{Sym}^{2k-2}W$ under the above isomorphism, on which the intersection pairing (induced by Poincare duality) is non-degenerate.

For abelian varieties A with CM by K, the cohomological cycle corresponding to the algebraic cycle $S_k(A)$ defined in the previous section is in $L_k(A)$.

6.4 The Archimedean Picture

In this section we show that the cycle $S_k(x)(\mathbb{C})$ is homologically trivial in the complex manifold $Y(\mathbb{C})$, thus $S_k(x)(\mathbb{C})$ satisfies condition (b) in Section 5.4. It suffices to show that for a CM point $x' \in X$, $W_k(A_{x'})(\mathbb{C})$ is homologically trivial, or in other words,

$$\int_{W_k(A_{x'})(\mathbb{C})} \alpha = 0, \tag{6.24}$$

for any closed $(k-1, k-1)$-form α.

In [17], Kuga studied *Kuga varieties* over Shimura curves whose fibers are powers of abelian surfaces with QM, following which we give an archimedean description of the varieties in question

and then prove identity (6.24).

Fix an isomorphism $\phi : B \otimes \mathbb{R} \to M_2(\mathbb{R})$ such that $\Gamma^{\pm} := \phi_\infty(R'^{\times}_{M,N_-})$ is contained in $\mathrm{GL}_2(\mathbb{R})$. Then $\Gamma^{\pm}$ is a discrete subgroup of $\mathrm{GL}_2(\mathbb{R})$ containing elements with determinant ± 1, and let Γ be the subgroup of $\Gamma^{\pm}$ with determinant 1.

Let $G = \mathrm{SL}(2,\mathbb{R})$, $K = \mathrm{SO}(2,\mathbb{R})$, then $G/K \cong \mathfrak{H} := \{x + iy \colon x, y \in \mathbb{R}, y > 0\}$, where the isomorphism is given by

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto \frac{ai+b}{ci+d}.$$

We have $X(\mathbb{C}) = X'_{M,N_-}(\mathbb{C}) = \Gamma\backslash\mathfrak{H}$, which is a compact Riemann surface. The universal cover of the universal abelian surface $\mathcal{A}(\mathbb{C})$, considered as a real manifold, is the Cartesian product $\mathfrak{H} \times \mathbb{R}^4$.

Consider the vector bundle $V \xrightarrow{\pi} X = \Gamma\backslash\mathfrak{H} = \Gamma\backslash G/K$ where the fiber above $x \in X$ is given by $H_1(A_x, \mathbb{R})$. Note that $H_1(A_x,\mathbb{R}) \cong M_{2\times 2}(\mathbb{R})$ and under this isomorphism $H_1(A_x,\mathbb{Z})^{k-1}$ is mapped to a maximal order Λ in $M_{2\times 2}(\mathbb{R})$. Up to isomorphism of vector bundles, V is determined by the trivial bundle $M_{2\times 2}(\mathbb{R})$ on G together with the left regular representation ρ of G on $M_{2\times 2}(\mathbb{R})$, which has the integrality property $\rho(\gamma)\Lambda = \Lambda$ for any $\gamma \in \Gamma$. Let $\tilde{F}$ denote this representation space, that is, $\tilde{F} = M_{2\times 2}(\mathbb{R})$ together with the G-action via ρ.

Then as real manifolds, $Y(\mathbb{C}) \cong \Gamma \times \Lambda^{k-1}\backslash\mathfrak{H} \times \tilde{F}^{k-1}$, where the $\Gamma \times \Lambda^{k-1}$ action on $\mathfrak{H} \times \tilde{F}^{k-1}$ is given by $(\gamma, \ell_1, \ldots, \ell_{k-1}) \cdot (x, w_1 \ldots, w_{k-1}) = (\gamma x, \rho(\gamma)w_1 + \ell_1, \ldots, \rho(\gamma)w_{k-1} + \ell_{k-1})$, and each fiber of $Y(\mathbb{C})$ over $X(\mathbb{C})$ is isomorphic to the torus $(\Lambda\backslash\tilde{F})^{k-1}$ in the real-analytic sense.

It is also shown in [17, Section IV.1] that one can choose

(i) a simplectic form β on $M_{2\times 2}(\mathbb{R})$ that is integral on Λ and such that $\beta(\rho(g)u, \rho(g)v) = \beta(u,v)$ for each $g \in G$; and

(ii) a complex structure $\tilde{J}$ on $\mathfrak{H} \times \tilde{F}^{k-1}$

such that each fiber F_x of $\mathfrak{H} \times \tilde{F}^{k-1}$ above $\mathfrak{H}$ together with the complex structure $J(x)$ induced by $\tilde{J}$ and the Riemann form $-\beta(u, J(x)v) + i\beta(u,v)$ is a polarized abelian variety that is the $(k-1)$-th power of an abelian surface, and is invariant under Γ actions, that is,

$$(F_x, J(x), -\beta(u, J(x)v) + i\beta(u,v)) \cong (F_{\gamma(x)}, J(\gamma(x)), -\beta(u, J(x)v) + i\beta(u,v)).$$

The complex structure $\tilde{J}$ on $\mathfrak{H} \times \tilde{F}^{(k-1)}$ can be described as follows: it agrees with the complex

structure on $\mathfrak{H} \times \mathbb{C}^{2k-2}$ under the bijection $\mathfrak{H} \times \tilde{F}^{(k-1)} \to \mathfrak{H} \times \mathbb{C}^{2k-2}$,

$$\left(\tau, \begin{pmatrix} \xi_1 & \xi_2 \\ \eta_1 & \eta_2 \end{pmatrix}, \ldots, \begin{pmatrix} \xi_{2k-3} & \xi_{2k-2} \\ \eta_{2k-3} & \eta_{2k-2} \end{pmatrix}\right) \mapsto (\tau, z_1, \ldots, z_{2k-2}),$$

where $z_i = \tau\eta_i - \xi_i$. The inverse map of this map is given by

$$(\tau, z_1, \ldots, z_{2k-2}) \mapsto \left(\tau, \begin{pmatrix} \xi_1 & \xi_2 \\ \eta_1 & \eta_2 \end{pmatrix}, \ldots, \begin{pmatrix} \xi_{2k-3} & \xi_{2k-2} \\ \eta_{2k-3} & \eta_{2k-2} \end{pmatrix}\right),$$

where

$$\eta_i = \frac{z_i - \bar{z}_i}{\tau - \bar{\tau}}, \quad \text{and } \xi_i = \frac{\bar{\tau} z_i - \tau \bar{z}_i}{\tau - \bar{\tau}}.$$

The action of $\Gamma \times \Lambda^{k-1}$ then translates to

$$\begin{aligned} &\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} s_1 & s_2 \\ t_1 & t_2 \end{pmatrix} \ldots, \begin{pmatrix} s_{2k-3} & s_{sk-2} \\ t_{2k-3} & t_{2k-2} \end{pmatrix}\right) \cdot (\tau, z_1, \ldots, z_{2k-2}) \\ &= \left(\frac{a\tau + b}{c\tau + d}, \frac{ad - bc}{c\tau + d} z_1 + \frac{(a\tau + b)}{c\tau + d} t_1 - s_1, \ldots, \frac{ad - bc}{c\tau + d} z_{2k-2} + \frac{(a\tau + b)}{c\tau + d} t_{2k-2} - s_{2k-2}\right) \end{aligned}$$

We shall use the coordinate system $(\tau, z_1, \ldots, z_{2k-2})$ and denote $z_0 := \tau$ for notational simplicity. Then a $(k-1, k-1)$ form α on $Y(\mathbb{C})$ descends from a $\Gamma \times \Lambda^{k-1}$-invariant form on $\mathfrak{H} \times \mathbb{C}^{2k-2}$:

$$\sum_{I,J} \alpha_{IJ} dz_I \wedge d\bar{z}_J,$$

where $I = \{i_1, \ldots, i_{k-1}\}, J = \{j_1, \ldots, j_{k-1}\}$ run through length-$(k-1)$ subsets of $\{0, 1, \ldots, 2k-2\}$; assuming $i_1 < i_2 < \cdots < i_{k-1}$ and $j_1 < j_2 < \cdots < j_{k-1}$, we denote $dz_I := dz_{i_1} \wedge \cdots \wedge dz_{i_{k-1}}$ and $d\bar{z}_J := d\bar{z}_{j_1} \wedge \cdots \wedge d\bar{z}_{j_{k-1}}$, and α_{IJ} is a function of $\tau = z_0, z_1, \ldots, z_{2k-2}$. We will not spell out the $\Gamma \times \Lambda^{k-1}$-invariance condition explicitly, but keep in mind that the form descends to a form on a compact manifold.

Assume α is closed, and we would like to show Equality (6.24). By complex hodge theory [11, Chapter 0, Section 6-7], a closed form on a complex torus is equal to the sum of a harmonic form with an exact form. Therefore, by Stoke's theorem, we may assume without lose of generality that $\alpha(\tau)$ is harmonic (thus constant by compactness) for each fixed $\tau \in \mathfrak{H}$. That is, each α_{IJ} is a function of τ only. Moreover, if $0 \in I$ or $0 \in J$, since the cycle $W_k(A_{x'})$ concentrates in a fiber above a fixed $z_0 = \tau$, the integration $\int_{W_k(A_{x'})(\mathbb{C})} \alpha_{IJ} dz_I \wedge d\bar{z}_J$ is clearly zero.

It now suffice to show

$$\sum_{I,J\subseteq\{1,\ldots,2k-2\}} \int_{W_k(A_{x'})(\mathbb{C})} \alpha_{IJ}(\tau) dz_I \wedge d\bar{z}_J = 0,$$

for each fixed τ, which follows from the construction of $W_k(A_{x'})$.

6.5 The Arithmetic Cycle

In this section we find a suitable integral model for $S_k(x)$ satisfying condition (a) in Section 5.4. Let $\overline{S_k(x)}$ denote the Zariski closure of $S_k(x)$ in $\mathcal{Y}$.

What we have proven in the previous section shows that $S_k(x)$ is a homologically trivial algebraic cycle in the Kuga variety Y, so it satisfies condition (a) at most places. More precisely, for any place $v \nmid MN_-$, $\mathcal{X}$ has good reduction over v, and thus for any cycle $\mathcal{Z}$ of dimension k supported on the special fiber of $\mathcal{Y}$ above v, we have $\left(\overline{S_k(x)} \cdot \mathcal{Z}\right)_v = 0$. If $v|M$, then the special fiber of $\mathcal{Y}$ above v is copies of $X'_{\frac{M}{p},N_-,v}$ meeting transversally and $\bar{x}$ does not support on the intersection points of the copies. ([15] as cited in [21, Section 4.3].) So for such places, we have $\left(\overline{S_k(x)} \cdot \mathcal{Z}\right)_v = 0$ for any cycle $\mathcal{Z}$ of dimension k supported on the special fiber of $\mathcal{Y}_v$ as well.

However, at places v above a prime $p|N_-$, $\overline{S_k(x)}$ may not satisfy condition (a) in Section 5.4. For such a place v, the curve $\mathcal{X}_v$ is a Mumford curve; that is, the special fiber mod v is a finite union of copies of $\mathbb{P}^1$ meeting transversally. We label them by $\mathbb{P}^1_1, \mathbb{P}^1_2, \ldots, \mathbb{P}^1_n$ respectively. Let G be the dual graph of $\mathcal{X}_v$, that is, $\mathbb{P}^1_1, \ldots, \mathbb{P}^1_n$ correspond to the vertices $v_1, \ldots, v_n$ of G, and each intersection point $a \in \mathbb{P}^1_i \cap \mathbb{P}^1_j$ corresponds to an edge e connecting v_i and v_j. Denote the set of edges of G by $\{e_1, \ldots, e_m\}$.

By Mumford's theory [20] the curve $\mathcal{X}_v$ has a p-adic uniformization, which is made explicit by Cerednik-Drinfeld (see [8], [5], and [21, Section 4.4]). In particular, the dual graph G is a quotient of the Bruhat-Tits tree of degree $p+1$ (see Section 3.1). Since vector bundles above $\mathbb{P}^1$ are all trivial, $\mathcal{Y}_v$ is quasi-isogeneous to $\mathbb{P}^1_1 \times A_1^{k-1}, \ldots, \mathbb{P}^1_n \times A_n^{k-1}$, intersecting transversally, where each A_i is a supersingular abelian surface over k_v, which implies that each element of $H^2(A_i, \mathbb{Q}_\ell)$ is algebraic.

Assume that e is an edge connecting v_i and v_j in G, which corresponds to an intersection point a of $\mathbb{P}^1_i$ and $\mathbb{P}^1_j$. Let A_e denote the intersection of $\mathbb{P}^1_i \times A_i$ and $\mathbb{P}^1_j \times A_j$ above the point a, and let

$$\psi_{v_i e} : A_i^{k-1} \to A_e^{k-1}$$

denote the transition quasi-isogeny and ψ_{ev_i} its inverse. The cycle $\overline{S_k(x)} \cap \mathcal{Y}_v$ is within one piece $\mathbb{P}^1_i \times A_i^{k-1}$ for some i, and we assume without loss of generality that $i = 1$. Moreover, $\overline{S_k(x)} \cap \mathcal{Y}_v$ is of the form $\{x\} \times S$, where $x \in \mathbb{P}^1_1$ and $x \notin \mathbb{P}^1_j$ for $j \neq 1$ (i.e. x is not at an intersection) and S is a $(k-1)$-dimensional cycle in A_1^{k-1}. Now let $\mathcal{Z}$ be a k-dimensional cycle supported on $\mathcal{Y}_v$, then $\mathcal{Z}$ is a linear combination of cycles of the following types:

1. Cycles of the form $\{y\} \times T$ for some $y \in \mathbb{P}^1_j$ and T a k-dimensional cycle in A_j^{k-1}, for some j; and

2. Cycles of the form $\mathbb{P}^1_j \times R$ for some $(k-1)$-dimensional cycle R in A_j^{k-1}, for some j.

For Type 1 cycles, if $j \neq 1$, then clearly $\left(\overline{S_k(x)} \cdot (\{y\} \times T)\right)_v = 0$; if $j = 1$, via linear equivalence, we may assume without loss of generality that $y \neq x$, thus $\left(\overline{S_k(x)} \cdot (\{y\} \times T)\right)_v = 0$ as well. For Type 2 cycles, if $j \neq 1$, then $\left(\overline{S_k(x)} \cdot (\mathbb{P}^1_j \times R)\right)_v = 0$; if $j = 1$ however, $\left(\overline{S_k(x)} \cdot (\mathbb{P}^1_1 \times R)\right)_v = \{x\} \times (S \cdot R)$, where $(S \cdot R)$ denotes the intersection of S and R within the $(2k-2)$-dimensional A_1^{k-1}, i.e., within the fiber of $\mathcal{Y}_v$ above $x \in \mathcal{X}_v$. This is in general nonzero. For example, when $R = S$, the intersection multiplicity of $(S \cdot S)$ is $(-1)^{k-1}$ (c.f. Section 6.3).

Assume that $\mathcal{S}_v(x)$ is a k-dimensional cycle supported on the special fiber $\mathcal{Y}_v$ such that $\left(\overline{S_k(x)} \cdot \mathcal{Z}\right)_v = \left(\mathcal{S}_v(x) \cdot \mathcal{Z}\right)_v$ for any k-dimensional cycle $\mathcal{Z}$ supported on $\mathcal{Y}_v$, then the cycle

$$\widehat{S}_k(x) := \overline{S_k(x)} - \sum_{v|N_-} \mathcal{S}_v(x) \tag{6.25}$$

would satisfy condition (a) in Section 5.4. That is, $\left(\widehat{S}_k(x) \cdot \mathcal{Z}\right)_v = 0$ for any finite place v and any k-dimensional cycle $\mathcal{Z}$ supported on $\mathcal{Y}_v$.

We construct $\mathcal{S}_v(x)$ as follows. Consider

$$\mathcal{S}_v(x) = \sum_{i=1}^{n} \mathbb{P}^1_i \times S_i, \tag{6.26}$$

where each S_i is a $(k-1)$-cycle in A_i^{k-1} to be solved below. Since Type 1 cycles can be moved to a different component via algebraic equivalence, their intersections with each $\mathbb{P}^1_i \times S_i$ are always 0, regardless of the choice of S_i. It remains to guarantee that $\left(\overline{S_k(x)} \cdot \mathcal{Z}\right)_v = \left(\mathcal{S}_v(x) \cdot \mathcal{Z}\right)_v$ for any Type 2 cycle $\mathcal{Z}$. That is,

$$\begin{aligned}
&\left(\mathcal{S}_v(x)\cdot(\mathbb{P}^1_1\times R)\right)_v = \{x\}\times(S\cdot R) \text{ for any } (k-1)-\text{cycle } R \text{ in } A_1^{k-1};\text{ and}\\
&\left(\mathcal{S}_v(x)\cdot(\mathbb{P}^1_j\times R)\right)_v = 0 \text{ for any } (k-1)-\text{cycle } R \text{ in } A_j^{k-1} \text{ and any } j\neq 1,
\end{aligned} \tag{6.27}$$

where the equality is in the sense of numerical equivalence. To solve for each S_i from (6.27), we first note that

$$\left((\mathbb{P}^1_i\times S_i)\cdot(\mathbb{P}^1_j\times R)\right)_v = \begin{cases} 0, & \text{if } \mathbb{P}^1_i \text{ and } \mathbb{P}^1_j \text{ do not intersect;}\\ \sum\limits_{a\in\mathbb{P}^1_i\cap\mathbb{P}^1_j}\{a\}\times\left(\psi^*_{e_a v_i}(S_i)\cdot\psi^*_{e_a v_j}(R)\right), & \text{if } i\neq j \text{ and } \mathbb{P}^1_i \text{ intersects } \mathbb{P}^1_j;\\ -\sum\limits_{\ell\neq i}\sum\limits_{a\in\mathbb{P}^1_i\cap\mathbb{P}^1_\ell}\{a\}\times\left(\psi^*_{e_a v_i}(S_i)\cdot\psi^*_{e_a v_i}(R)\right), & \text{if } i=j, \end{cases}$$

where e_a denotes the edge in G corresponding to the intersection point a. Then the system of equations (6.27) is equivalent to

$$\sum_{i\neq 1}\sum_{a\in\mathbb{P}^1_i\cap\mathbb{P}^1_1}\{a\}\times\left((\psi^*_{e_a v_i}(S_i)-\psi^*_{e_a v_1}(S_1))\cdot\psi^*_{e_a v_1}(R)\right) = \{x\}\times(S\cdot R), \tag{6.28}$$

for any $(k-1)$-cycle R in A_1^{k-1}; and

$$\sum_{i\neq j}\sum_{a\in\mathbb{P}^1_i\cap\mathbb{P}^1_j}\{a\}\times\left((\psi^*_{e_a v_i}(S_i)-\psi^*_{e_a v_j}(S_j))\cdot\psi^*_{e_a v_j}(R)\right) = 0 \tag{6.29}$$

for any $(k-1)$-cycle R in A_j^{k-1}, for any $j\neq 1$.

Let C denote the cohomology class of S in $H^{2k-2}(A_1^{k-1},\mathbb{Q}_\ell)$, which lies in the subspace $L_k(A_1)$ (c.f. Subsection 6.3.1). Since we only need to find S_i up to numerical equivalence, it suffices to find a cohomology class C_i in $L_k(A_i)$ for each i such that

$$\sum_{\substack{(v_i,e) \text{ where}\\ e \text{ connects}\\ v_1 \text{ and } v_i}} \varphi^{-1}_{v_1 e}\varphi_{v_i e}(C_i) - C_1 = C \tag{6.30}$$

and that

$$\sum_{\substack{(v_i,e) \text{ where}\\ e \text{ connects}\\ v_j \text{ and } v_i}} \varphi^{-1}_{v_j e}\varphi_{v_i e}(C_i) - C_j = 0 \tag{6.31}$$

for each $j\neq 1$, where $\varphi_{v_i e}$ is the isomorphism of vector spaces $H^{2k-2}(A_i^{k-1},\mathbb{Q}_\ell)\to H^{2k-2}(A_e^{k-1},\mathbb{Q}_\ell)$

induced by $\psi_{v_i e}$.

Fix an embedding $\mathbb{Q}_\ell \hookrightarrow \mathbb{C}$, the above problem translates to solving

$$-\Delta f = C\delta_{v_1} \tag{6.32}$$

on the local system L_k over a finite quotient of the Bruhat-Tits tree, which we solved in Part I. Then let S_i be an algebraic cycle in A_i^{k-1} whose cohomological class is C_i, which is always possible since A_i is supersingular. Then (6.26) defines the appropriate adjustments so that $\widehat{S}_k(x)$ defined in (6.25) is an integral model of $S_k(x)$ making the global height pairing well-defined.

Then at a place $v|N_-$, we have

$$\begin{aligned}
&\left(\widehat{S}_k(x)\cdot\widehat{S}_k(y)\right)_v = \left(\widehat{S}_k(x)\cdot(\overline{S_k(y)}-\mathcal{S}_v(y))\right)_v \\
=&\left(\widehat{S}_k(x)\cdot\overline{S_k(y)}\right)_v = \left(\overline{S_k(x)}\cdot\overline{S_k(y)}\right)_v - \left(\mathcal{S}_v(x)\cdot\overline{S_k(y)}\right)_v \\
=&\left(\overline{S_k(x)}\cdot\overline{S_k(y)}\right)_v - \left(\mathcal{S}_v(x)\cdot\mathcal{S}_v(y)\right)_v,
\end{aligned} \tag{6.33}$$

since $\left(\widehat{S}_k(x)\cdot\mathcal{S}_v(y)\right)_v = \left(\mathcal{S}_v(x)\cdot\widehat{S}_k(y)\right)_v = 0$.

Let $\bar{x},\bar{y}$ denote the Zariski closure of x,y in $\mathcal{X}$. Then the first term $\left(\overline{S_k(x)}\cdot\overline{S_k(y)}\right)_v$ is nonzero if and only if $\bar{x}$ and $\bar{y}$ intersect at a point z in the special fiber $\mathcal{X}_v$, in which case

$$\left(\overline{S_k(x)}\cdot\overline{S_k(y)}\right)_v = (\bar{x}\cdot\bar{y})_v\left(\left(\overline{S_k(x)}\cap\mathcal{Y}_{v,z}\right)\cdot\left(\overline{S_k(y)}\cap\mathcal{Y}_{v,z}\right)\right)_{\mathcal{Y}_{v,z}},$$

where the second multiplicand is a algebraic intersection number taken within the $(2k-2)$-dimensional fiber $\mathcal{Y}_{v,z}$ (c.f. (5.17)).

Chapter 7

Future Work

7.1 To Get the Gross-Zagier Formula

In order to prove Conjecture 1.1.1, we will need to match the analytic formula for $a_{m,\mathscr{A}}$ in [13, Chapter IV Theorem 5.8] with the height pairing computations.

We expect the local height pairings to match the analytic formula in a way similar to Proposition 4.2.2 and 4.2.3 in [30]. We expect the computations at archimedean places and finite places not dividing N_- to be similar to [30]. In this thesis we constructed the arithmetic cycles involved and made progress towards the local height pairings at places dividing N_-, though further computation is needed to precisely match the analytic formula. We hope to complete the proof Conjecture 1.1.1 in an upcoming work.

7.2 Higher Dimensional Generalizations

We are interested in generalizing the theory of local systems over trees and graphs and the analysis on them in Part I to complexes and buildings. This will be useful in the study of Shimura varieties of unitary type that assumes a p-adic uniformization by the Drinfeld upper half space. In particular, it could be a step towards certain cases of the arithmetic Gan-Gross-Prasad conjecture ([9]), which is a higher-dimensional generalization of the Gross-Zagier formula.

7.3 Potential Applications

The original Gross-Zagier formula was a step towards the BSD conjecture, which says that the order of vanishing $\mathrm{ord}_{s=1} L(E,s)$ (smallest non-negative integer r so that $L^{(r)}(E,1) \neq 0$) at the center $s = 1$ of the L-function associated with an elliptic curve E is equal to the rank of the Mordell-Weil group $E(\mathbb{Q})$, and predicts a formula for $L^{(r)}(E,1)$. Via the correspondence between weight-2 modular forms and elliptic curves (which was an assumption at the time of [13] but no longer so because of the modularity theorem [27]), the Gross-Zagier formula contributed to the $r = 1$ case of the BSD conjecture. For a weight-$2k$ ($k > 1$) modular form f, the corresponding geometric object is the f-isotypical component of the Chow group of $\mathcal{Y}$, whose rank is predicted to be equal to the order of vanishing of $L(f, \mathbf{1}, s)$ (where $\mathbf{1}$ denotes the trivial character) at $s = k$ by the Beilinson-Bloch conjecture. There were applications to the Beilinson-Bloch conjecture in [30], and if we get a Gross-Zagier formula for CM cycles over Shimura curves, it could contribute to the Beilinson-Bloch conjecture under the relaxed Heegner condition.

Gross and Zagier also provided an example where the L-function is of order 3 [13, Chapter I Proposition 7.4]. It is also compelling to ask if we can find a higher weight modular form whose L-function is of order 3 (or even larger). Since under our relaxed Heegner condition $L(f, \chi, s)$ still has an odd order as in the original Gross-Zagier case, it suffices to find an example where $L'(f, \chi, s) = 0$. Because of Corollary 1.1.3 this boils down to find Chow cycles of height 0, and it would be nice to either have a theoretical proof of certain height pairing being zero, or a gap principal on the height pairing so that we can search for examples computationally.

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