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Some results in the variational theory of the area and related functionals

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SOME RESULTS IN THE VARIATIONAL THEORY OF THE
AREA AND RELATED FUNCTIONALS

AKASHDEEP DEY

A DISSERTATION
PRESENTED TO THE FACULTY
OF PRINCETON UNIVERSITY
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Abstract

In this thesis, we prove some results in the variational theory of the minimal hypersurfaces, constant mean curvature (CMC) hypersurfaces and the Allen-Cahn equation. In Chapter 1, we summarize the main results. In Chapter 2, we show that the number of closed c -CMC hypersurfaces in a closed Riemannian manifold tends to infinity as c tends to 0^+ . In Chapter 3, we show that the space of closed singular minimal hypersurfaces (in a closed Riemannian manifold), whose areas are uniformly bounded from above and the p -th Jacobi eigenvalues are uniformly bounded from below, is sequentially compact. In Chapter 4, we prove a sub-additive inequality for the volume spectrum of a closed Riemannian manifold. In Chapter 5, we prove two results related to the question to what extent the Almgren-Pitts min-max theory and the Allen-Cahn min-max theory agree. In Chapter 6, we prove the existence of finite energy min-max solutions to the Allen-Cahn equation on a complete Riemannian manifold of finite volume.

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Chapter 1

Introduction

Minimal submanifolds are defined by the condition that they are the critical points of the area functional. In [Alm62, Alm65], Almgren studied the topology of the space of cycles and developed a min-max theory for the area functional. He proved that any closed Riemannian manifold (M^{n+1}, g) contains a minimal variety of dimension l for every $1 \leq l \leq n$. The regularity theory in the co-dimension 1 case was further developed by Pitts [Pit81] and Schoen-Simon [SS81]. They proved that in a closed Riemannian manifold (M^{n+1}, g) , $n + 1 \geq 3$, there exists a closed, minimal hypersurface which is smooth and embedded outside a singular set of Hausdorff dimension $\leq n - 7$.

In recent years, there have been a lot of research activities in the Almgren-Pitts min-max theory. By the works of Marques-Neves [MN17] and Song [Son18], every closed Riemannian manifold (M^{n+1}, g) , $3 \leq n + 1 \leq 7$, contains infinitely many closed, minimal hypersurfaces. This was conjectured by Yau [Yau82]. In [IMN18], Irie, Marques and Neves proved that for a generic metric g on M , the union of all closed, minimal hypersurfaces is dense in (M, g) . This theorem was later quantified by Marques, Neves and Song in [MNS19] where they proved that for a generic metric there exists an equidistributed sequence of closed, minimal hypersurfaces in (M, g) . In [SZ21], Song and Zhou proved the generic scarring phenomena for minimal hypersurfaces, which can be interpreted as the opposite of the equidistribution phenomena. In higher dimensions, Li [Li19] has proved that every closed Riemannian manifold, equipped with a generic metric, contains infinitely

many closed minimal hypersurfaces with optimal regularity.

In [ZZ19], Zhou and Zhu developed a min-max theory (which was further extended by Zhou in [Zho20]) to prove the existence of a closed hypersurface with constant mean curvature (CMC) c (for any $c > 0$) in an arbitrary closed Riemannian manifold (M^{n+1}, g) with $3 \leq n+1 \leq 7$. In Chapter 2 [Dey19], we show that the min-max theory for CMC hypersurfaces in [ZZ19, Zho20] can be extended in higher dimensions using the regularity theory of stable CMC hypersurfaces, developed by Bellettini-Wickramasekera [BW18, BW19] and Bellettini-Chodosh-Wickramasekera [BCW19]. Furthermore, inspired by the abundant existence of closed minimal hypersurfaces, we prove that in any closed Riemannian manifold (M^{n+1}, g) , $n+1 \geq 3$, the number of closed c -CMC hypersurfaces tends to infinity as $c \rightarrow 0^+$.

The Morse index of the minimal hypersurfaces produced by the Almgren-Pitts min-max theory has been obtained by Marques and Neves when the ambient dimension $3 \leq n+1 \leq 7$. In [MN16], Marques and Neves showed that the index of the min-max minimal hypersurface is bounded from above by the dimension of the parameter space. Zhou [Zho20] has proved that for a generic (bumpy) metric, the min-max minimal hypersurfaces have multiplicity one, which was conjectured by Marques and Neves. Using the Morse index upper bound [MN16] and multiplicity one theorem [Zho20], Marques and Neves [MN21] have proved the following theorem. For a generic (bumpy) metric there exists a sequence of closed, embedded, two-sided minimal hypersurfaces $\{\Sigma_p\}_{p=1}^\infty$ in (M^{n+1}, g) such that $\text{Ind}(\Sigma_p) = p$ and $\mathcal{H}^n(\Sigma_p) = \omega_p \sim p^{\frac{1}{n+1}}$. This theorem has been generalized by Marques, Montezuma and Neves in [MMN20] where they have proved the strong Morse inequalities for the area functional. In higher dimensions, Morse index upper bound has been proved by Li [Li20].

In [Sha17], Sharp proved certain compactness properties of the set of smooth, closed, embedded minimal hypersurfaces inside a closed Riemannian manifold (M^{n+1}, g) , $3 \leq n+1 \leq 7$, with uniformly bounded area and index. This theorem was later generalized by Ambrozio, Carlotto and Sharp in [ACS16], where instead of bounded area and index, an upper bound of the area and a lower bound of the p -th Jacobi eigenvalue λ_p (for some $p \in \mathbb{N}$) were assumed. (We note that for a smooth, closed minimal hypersurface Σ , $\text{Ind}(\Sigma) \leq I$ is equivalent to $\lambda_{I+1}(\Sigma) \geq 0$.) Sharp's compactness theorem plays an important role in many recent developments in the Almgren-Pitts min-max theory,

for instance in [MN17, IMN18, MNS19, Son18, MN16, MN21, Zho20, SZ21]. In Chapter 3 [Dey22], we show that the compactness theorems of Sharp [Sha17] and Ambrozio-Carlotto-Sharp [ACS16] can be suitably extended in the higher dimensions.

In Chapter 4 [Dey20b], we prove a sub-additive inequality satisfied by the volume spectrum of a closed Riemannian manifold. The volume spectrum is a non-linear analogue of the spectrum of the Laplacian. With respect to this analogy, the min-max minimal hypersurfaces correspond to the eigenfunctions of the Laplacian. The various properties of the volume spectrum play a major role in the above mentioned proofs of Yau's conjecture as well as in the proof of our theorem in Chapter 2.

In addition to the Almgren-Pitts min-max theory, there is an alternative PDE based approach for the min-max construction of minimal hypersurfaces. This approach was introduced by Guaraco [Gua18] and further developed by Gaspar and Guaraco [GG18]. It is based on the study of the limiting behaviour of solutions to the *Allen-Cahn equation*. The Allen-Cahn equation (with parameter $\varepsilon > 0$) is the following semi-linear, elliptic PDE

$$AC_\varepsilon(u) := -\varepsilon^2 \Delta u + W'(u) = 0 \tag{1.0.1}$$

where $W : \mathbb{R} \rightarrow [0, \infty)$ is a symmetric double well potential, e.g. $W(t) = \frac{1}{4}(1 - t^2)^2$. The solutions of this equation are precisely the critical points of the energy functional

$$E_\varepsilon(u) = \int_M \varepsilon \frac{|\nabla u|^2}{2} + \frac{W(u)}{\varepsilon}. \tag{1.0.2}$$

Informally speaking, as $\varepsilon \rightarrow 0$, the level sets of the solutions to (1.0.1), with uniformly bounded energy and Morse index, accumulate around a minimal hypersurface (called a *limit-interface*). This follows from the combined works of Hutchinson-Tonegawa [HT00], Tonegawa [Ton05], Wickramasekera [Wic14a], Tonegawa-Wickramasekera [TW12] and Guaraco [Gua18].

The overall idea of the Allen-Cahn min-max theory, as developed in [Gua18, GG18], can be summarized as follows. Let M be a closed Riemannian manifold. By the min-max theory on the

Hilbert spaces, for ε sufficiently small, one can first construct solutions to the Allen-Cahn equation (1.0.1) (on M) with uniformly bounded energy and Morse index. After that, one can pass to the limit $\varepsilon \rightarrow 0$ and obtain closed minimal hypersurfaces (which are limit-interfaces) in M .

Given that there are two independent approaches for the min-max construction of minimal hypersurfaces, i.e. the Almgren-Pitts min-max theory and the Allen-Cahn min-max theory, it is natural to ask to what extent these two theories agree. By the works of Guaraco [Gua18] and Gaspar-Guaraco [GG18], the Allen-Cahn min-max widths are greater than or equal to the Almgren-Pitts min-max widths. In Chapter 5 [Dey20a], we prove that the reverse inequalities also hold, i.e. the Allen-Cahn widths are less than or equal to the Almgren-Pitts widths. Hence, the Almgren-Pitts widths and the Allen-Cahn widths coincide. We will also show that all the closed minimal hypersurfaces, which are obtained from the Allen-Cahn min-max theory, are also produced by the Almgren-Pitts min-max theory.

In Chapter 6 [Dey21], we prove the existence of finite energy solution to the Allen-Cahn equation on certain complete, non-compact manifolds. More precisely, let M^{n+1} (with $n + 1 \geq 3$) be a complete Riemannian manifold of finite volume. Then there exists $\varepsilon_0 > 0$, depending on the ambient Riemannian metric, such that for all $0 < \varepsilon \leq \varepsilon_0$, there exists $u_\varepsilon : M \rightarrow (-1, 1)$ satisfying $AC_\varepsilon(u_\varepsilon) = 0$ with the energy $E_\varepsilon(u_\varepsilon) < \infty$ and the Morse index $\text{Ind}(u_\varepsilon) \leq 1$. Moreover, $0 < \liminf_{\varepsilon \rightarrow 0} E_\varepsilon(u_\varepsilon) \leq \limsup_{\varepsilon \rightarrow 0} E_\varepsilon(u_\varepsilon) < \infty$. This result is motivated by the theorem of Chambers-Liokumovich [CL20] and Song [Son19], which says that M contains a complete minimal hypersurface Σ with $0 < \mathcal{H}^n(\Sigma) < \infty$.

Chapter 2

Existence of multiple closed CMC hypersurfaces with small mean curvature

2.1 Introduction

Constant mean curvature (CMC) hypersurfaces are the critical points of the area functional with respect to the variations which preserve the enclosed volume. The hypersurfaces with constant mean curvature zero are called minimal hypersurfaces; they are the critical points of the area functional.

As discussed in Chapter 1, there is a well-developed existence theory for the closed minimal hypersurfaces in closed Riemannian manifolds. By the combined works of Almgren [Alm65], Pitts [Pit81] and Schoen-Simon [SS81], every closed Riemannian manifold (M^{n+1}, g) , $n + 1 \geq 3$, contains a closed minimal hypersurface, which is smooth and embedded outside a singular set of Hausdorff dimension $\leq n - 7$.

Since the cohomology ring of $\mathcal{Z}_n(M^{n+1}, \mathbb{Z}_2)$ (with coefficients in $\mathbb{Z}_2$) is isomorphic to the polynomial ring $\mathbb{Z}_2[\xi]$, from the finite dimensional Morse theory one expects that (M, g) contains infinitely

many closed minimal hypersurfaces. This was conjectured by Yau [Yau82]. Yau's conjecture has been completely resolved when the ambient dimension $3 \leq n + 1 \leq 7$. By the works of Marques-Neves [MN17] and Song [Son18], in every closed Riemannian manifold (M^{n+1}, g) , $3 \leq n + 1 \leq 7$, there exist infinitely many closed minimal hypersurfaces. For generic metrics, stronger results have been obtained in [IMN18, MNS19, Li19, MN21, Zho20, SZ21] (cf. Chapter 1).

In [ZZ19], Zhou and Zhu developed a min-max theory to construct closed CMC hypersurfaces in an arbitrary closed Riemannian manifold. More precisely, they proved that if (M^{n+1}, g) is a closed Riemannian manifold, $3 \leq n + 1 \leq 7$, given any $c \in \mathbb{R}^+$, there exists a closed, *almost embedded* hypersurface in (M, g) with constant mean curvature c . Here almost embedded means that all the self-intersections are one sided, tangential intersections. (For a precise definition, see Definition 2.3.1.) Later they generalized this theorem in [ZZ20] and proved that for a generic function $h \in C^\infty(M)$, there exists a closed, almost embedded hypersurface in (M, g) with prescribed mean curvature (PMC) h . The min-max construction of the PMC hypersurfaces was used by Zhou [Zho20] to prove the multiplicity one conjecture of Marques and Neves [MN16].

Given the previously mentioned results regarding the abundance of closed minimal hypersurfaces in closed Riemannian manifolds, it is natural to ask, for a given $c > 0$ and a closed Riemannian manifold (M, g) , whether there exist multiple closed c -CMC hypersurfaces in (M, g) . We will show that for any closed Riemannian manifold (M, g) , the number of closed c -CMC hypersurfaces in (M, g) tends to infinity as $c \rightarrow 0^+$. This will be a consequence of the following theorem. (We refer to Section 2.2 and 2.3 for the relevant notations and definitions.)

Theorem 2.1.1. *Let (M^{n+1}, g) be a closed Riemannian manifold, $n + 1 \geq 3$. Suppose $k \in \mathbb{N}$ such that $\omega_k < \omega_{k+1}$, $c \in \mathbb{R}^+$ such that $c\text{Vol}(g) < \omega_{k+1} - \omega_k$ and $\eta \in \mathbb{R}^+$ is arbitrary. Then there exists $\Omega \in \mathcal{C}(M)$ such that $\partial\Omega$ is a closed c -CMC (with respect to the inward unit normal) hypersurface with optimal regularity and $\omega_k < \mathcal{A}^c(\Omega) < \omega_k + W_0 + \eta$.*

As a corollary of Theorem 2.1.1, we obtain the following theorem.

Theorem 2.1.2. *Let (M^{n+1}, g) be a closed Riemannian manifold, $n + 1 \geq 3$. For all $m \in \mathbb{N}$, there exists a constant $c^*(m) > 0$, which depends on the metric g , such that if $0 < c < c^*(m)$, (M, g)*

contains at least m many closed c -CMC hypersurfaces with optimal regularity. More quantitatively, there exists a constant γ_0 , depending on g , such that for all $c > 0$, there exist at least $\gamma_0 c^{-\frac{1}{n+1}}$ many closed c -CMC hypersurfaces (with optimal regularity) in (M, g) .

At the end of the proof of Theorem 2.1.2, we will obtain an explicit lower bound for γ_0 .

On the other hand, by the work of Pacard and Xu [PX09], if Λ_M is the Lusternik-Shnirelman category of M (i.e. the minimum number of critical points of a smooth function $f : M \rightarrow \mathbb{R}$), for c sufficiently large, there exist at least Λ_M many closed, embedded c -CMC hypersurfaces in (M, g) .

The proof of Theorem 2.1.1 relies on the works by Zhou-Zhu [ZZ19] and Zhou [Zho20]. We will also use the regularity theory of stable CMC hypersurfaces, developed by Bellettini-Wickramasekera [BW18, BW19] and Bellettini-Chodosh-Wickramasekera [BCW19], to extend the theorem of Zhou and Zhu [ZZ19] in higher dimensions. Finally, Theorem 2.1.2 is deduced from Theorem 2.1.1 using the growth estimate of the volume spectrum, proved by Gromov [Gro03], Guth [Gut09], Marques-Neves [MN17] and Liokumovich-Marques-Neves [LMN18].

2.2 Notations and Preliminaries

2.2.1 Notations

Here we summarize the notations which will be frequently used later in this chapter.

- $\mathcal{C}(M)$: the space of Caccioppoli sets in M .
- $\mathcal{Z}_n(M; \mathcal{F}; \mathbb{Z}_2)$, $\mathcal{Z}_n(M; \mathbf{F}; \mathbb{Z}_2)$: the space of mod 2 flat hypercycles in M , which are boundaries, equipped with the $\mathcal{F}$ norm and the $\mathbf{F}$ -metric.
- $\omega_p = \omega_p(M, g)$: the p -width of (M, g) .
- $W_0 = W_0(M, g)$: the one parameter Almgren-Pitts width of (M, g) .
- $\mathcal{H}^s$: the Hausdorff measure of dimension s .
- $B(p, r)$: the open ball of radius r , centered at p .

- $A(p, r, R)$: the open annulus, centered at p , with radii $r < R$.
- $|\Sigma|$: the n -varifold associated to an n -rectifiable set Σ .
- $\|V\|$: the Radon measure associated to a varifold V .
- CA : the cone over a topological space A .
- SA : the suspension of a topological space A .
- $\Delta^k, [v_0, \dots, v_k]$: the standard k -simplex with vertices $v_0, \dots, v_k$.
- $C_i(A, \mathbb{Z}_2), C^i(A, \mathbb{Z}_2)$: the abelian group of i dimensional singular chains and cochains in A with $\mathbb{Z}_2$ coefficient.
- $H_i(A, \mathbb{Z}_2), H^i(A, \mathbb{Z}_2)$: the i -th singular homology and cohomology group of A with $\mathbb{Z}_2$ coefficient.

2.2.2 Caccioppoli sets and the $\mathcal{A}^c$ functional

In this subsection, we will briefly recall the notion of the *Caccioppoli set*; further details can be found in Simon's book [Sim83]. $E \subset (M, g)$ is called a *Caccioppoli set* if E is $\mathcal{H}^{n+1}$ -measurable and $D\chi_E$, the distributional derivative of the characteristic function χ_E , is a Radon measure. This is equivalent to

$$\sup \left\{ \int_E \operatorname{div} \omega \, d\mathcal{H}^{n+1} : \omega \in \mathfrak{X}^1(M) \text{ and } \|\omega\|_\infty \leq 1 \right\} < \infty,$$

where $\mathfrak{X}^1(M)$ is the space of C^1 vector-fields on M . The space of Caccioppoli sets in M will be denoted by $\mathcal{C}(M)$. If $E \in \mathcal{C}(M)$, there exists an n -rectifiable set, denoted by ∂E , such that the total variation measure $|D\chi_E| = \mathcal{H}^n \llcorner \partial E$. Further, there exists a $|D\chi_E|$ -measurable vector-field $\nu_{\partial E}$ such that $\|\nu_{\partial E}\| = 1$ $|D\chi_E|$ -a.e. and

$$\int_E \operatorname{div} \omega \, d\mathcal{H}^{n+1} = \int_{\partial E} g(\omega, \nu_{\partial E}),$$

for all $\omega \in \mathfrak{X}^1(M)$. $\nu_{\partial E}$ is called the *outward unit normal* to ∂E . For $c > 0$, the $\mathcal{A}^c$ functional on $\mathcal{C}(M)$ is defined by

$$\mathcal{A}^c(\Omega) = \mathcal{H}^n(\partial\Omega) - c\mathcal{H}^{n+1}(\Omega).$$

Let X be a vector-field on M and φ_t be the flow of X . The first variation of $\mathcal{A}^c$ along X is given by

$$\delta\mathcal{A}^c|_{\Omega}(X) = \frac{d}{dt}\bigg|_{t=0} \mathcal{A}^c(\varphi_t(\Omega)) = \int_{\partial\Omega} \operatorname{div}_{\partial\Omega} X d\mathcal{H}^n - c \int_{\partial\Omega} g(X, \nu_{\partial\Omega}) d\mathcal{H}^n.$$

Suppose Ω is an open set such that $\partial\Omega$ is a smooth hypersurface with mean curvature vector H . Then $\operatorname{div}_{\partial\Omega} X = -g(X, H)$. Therefore, Ω is a critical point of the $\mathcal{A}^c$ functional if and only if $H = -c\nu_{\partial\Omega}$, i.e. $\partial\Omega$ has constant mean curvature c with respect to the inward unit normal $-\nu_{\partial\Omega}$. In this case, the second variation of $\mathcal{A}^c$ along X (which is assumed to be normal along $\partial\Omega$) is given by

$$\begin{aligned} \delta^2\mathcal{A}^c|_{\Omega}(X, X) &= \frac{d^2}{dt^2}\bigg|_{t=0} \mathcal{A}^c(\varphi_t(\Omega)) \\ &= \int_{\partial\Omega} \left(|\nabla^\perp X|^2 - \operatorname{Ric}(X, X) - |A_{\partial\Omega}|^2 |X|^2 \right) d\mathcal{H}^n. \end{aligned}$$

Here $A_{\partial\Omega}$ stands for the second fundamental form of $\partial\Omega$.

2.2.3 The space of hypercycles and the volume spectrum

Let us introduce the following notation. $\mathbf{I}_l(M^{n+1}; \mathbb{Z}_2)$ is the space of l -dimensional mod 2 flat chains in M ; we only need to consider $l = n, n+1$. $\mathcal{Z}_n(M; \mathbb{Z}_2)$ denotes the space of flat chains $T \in \mathbf{I}_n(M; \mathbb{Z}_2)$ such that $T = \partial U$ for some $U \in \mathbf{I}_{n+1}(M; \mathbb{Z}_2)$. For $T \in \mathbf{I}_n(M; \mathbb{Z}_2)$, $|T|$ stands for the varifold associated to T and $\|T\|$ is the Radon measure associated to $|T|$. $\mathcal{F}$ and $\mathbf{M}$ denote the flat norm and the mass norm on $\mathbf{I}_l(M; \mathbb{Z}_2)$. When $l = n+1$, these two norms coincide. The $\mathbf{F}$ -metric

on the space of currents is defined as follows.

$$\mathbf{F}(U_1, U_2) = \mathcal{F}(U_1 - U_2) + \mathbf{F}(|\partial U_1|, |\partial U_2|) \text{ if } U_1, U_2 \in \mathbf{I}_{n+1}(M; \mathbb{Z}_2);$$

$$\mathbf{F}(T_1, T_2) = \mathcal{F}(T_1 - T_2) + \mathbf{F}(|T_1|, |T_2|) \text{ if } T_1, T_2 \in \mathcal{Z}_n(M; \mathbb{Z}_2).$$

$\mathcal{Z}_n(M; \mathcal{F}; \mathbb{Z}_2)$ and $\mathcal{Z}_n(M; \mathbf{F}; \mathbb{Z}_2)$ will stand for the space $\mathcal{Z}_n(M; \mathbb{Z}_2)$ equipped with the $\mathcal{F}$ norm and the $\mathbf{F}$ -metric respectively. We will identify $\mathcal{C}(M)$ with $\mathbf{I}_{n+1}(M; \mathbb{Z}_2)$, i.e. $E \in \mathcal{C}(M)$ will be identified with $\llbracket E \rrbracket$, the current associated with E . Similarly, ∂E (with $E \in \mathcal{C}(M)$) will be identified with $\llbracket \partial E \rrbracket = \partial \llbracket E \rrbracket$.

By the constancy theorem, if $\Omega_1, \Omega_2 \in \mathcal{C}(M)$ such that $\partial \Omega_1 = \partial \Omega_2$, then either $\Omega_1 = \Omega_2$ or $\Omega_1 = M - \Omega_2$. In [MN21], Marques and Neves proved that the space $(\mathcal{C}(M), \mathcal{F})$ is contractible and the boundary map

$$\partial : (\mathcal{C}(M), \mathcal{F}) \rightarrow \mathcal{Z}_n(M; \mathcal{F}; \mathbb{Z}_2)$$

is a 2-sheeted covering map. By the definition of the $\mathbf{F}$ -metric on $\mathcal{C}(M)$ and $\mathcal{Z}_n(M; \mathbb{Z}_2)$, this implies

$$\partial : (\mathcal{C}(M), \mathbf{F}) \rightarrow \mathcal{Z}_n(M; \mathbf{F}; \mathbb{Z}_2)$$

is also a 2-sheeted covering map. Furthermore, $\pi_1(\mathcal{Z}_n(M; \mathcal{F}; \mathbb{Z}_2)) = \mathbb{Z}_2$ and for $i \geq 2$, $\pi_i(\mathcal{Z}_n(M; \mathcal{F}; \mathbb{Z}_2)) = 0$. It was also proved in [MN21] that $\mathcal{Z}_n(M; \mathcal{F}; \mathbb{Z}_2)$ is weakly homotopy equivalent to $\mathbb{RP}^\infty$. Let $\bar{\lambda}$ denote the unique nonzero element of $H^1(\mathcal{Z}_n(M; \mathcal{F}; \mathbb{Z}_2), \mathbb{Z}_2) (= \mathbb{Z}_2)$; then the cohomology ring $H^*(\mathcal{Z}_n(M; \mathcal{F}; \mathbb{Z}_2), \mathbb{Z}_2) = \mathbb{Z}_2[\bar{\lambda}]$.

X is called a *cubical complex* if X is a subcomplex of $[0, 1]^l$ for some $l \in \mathbb{N}$. The cells of $[0, 1]^l$ are of the form $\alpha_1 \otimes \alpha_2 \otimes \cdots \otimes \alpha_l$, where each $\alpha_i \in \{[0], [1], [0, 1]\}$. By [BP02, Chapter 4], every simplicial complex is homeomorphic to a cubical complex and vice-versa.

Let X be a cubical complex. A continuous map $\Phi : X \rightarrow \mathcal{Z}_n(M; \mathcal{F}; \mathbb{Z}_2)$ is called a *k-sweepout* if

$$(\Phi^* \bar{\lambda})^k \neq 0 \in H^k(X, \mathbb{Z}_2).$$

$\Phi : X \rightarrow \mathcal{Z}_n(M; \mathbb{Z}_2)$ is said to have the property *no concentration of mass* if

$$\lim_{r \rightarrow 0} \sup \{ \|\Phi(x)\| (B(p, r)) : x \in X, p \in M \} = 0.$$

A continuous map $\Phi : X \rightarrow \mathcal{Z}_n(M; \mathbf{F}; \mathbb{Z}_2)$ has no concentration of mass [MN16, Proof of Theorem 3.8]. The set of all the k -sweepouts with no concentration of mass is denoted by $\mathcal{P}_k$. The k -width of (M, g) is defined by

$$\omega_k(M, g) = \inf_{\Phi \in \mathcal{P}_k} \sup \{ \mathbf{M}(\Phi(x)) : x \in \text{dmn}(\Phi) \},$$

where $\text{dmn}(\Phi)$ is the domain of Φ . The volume spectrum $\{\omega_k\}_{k=1}^\infty$ satisfies the following inequality, proved by Gromov [Gro03], Guth [Gut09], Marques-Neves [MN17] and Liokumovich-Marques-Neves [LMN18].

$$\omega_k \geq K_0 k^{\frac{1}{n+1}} \quad \forall k \in \mathbb{N}, \quad (2.2.1)$$

where $K_0 > 0$ is a constant, which depends on the metric g .

Let $\mathcal{S}$ be the set of all continuous maps $\Lambda : [0, 1] \rightarrow (\mathcal{C}(M), \mathcal{F})$ such that $\Lambda(0) = M$, $\Lambda(1) = \emptyset$ and $\partial \circ \Lambda : [0, 1] \rightarrow \mathcal{Z}_n(M; \mathcal{F}; \mathbb{Z}_2)$ has no concentration of mass. The *one parameter Almgren-Pitts width* of (M, g) , denoted by $W_0(M, g)$, is defined by

$$W_0(M, g) = \inf_{\Lambda \in \mathcal{S}} \sup \{ \mathbf{M}(\partial \Lambda(t)) : t \in [0, 1] \}.$$

2.2.4 Some notions from topology

Given a topological space A , let CA denote the *cone over* A , which is defined as follows.

$$CA = \frac{A \times [0, 1]}{\sim},$$

where the equivalence relation ‘ $\sim$ ’ collapses $A \times \{1\}$ to a point. The cone construction is functorial, i.e. if $f : A \rightarrow B$ is a continuous map between the topological spaces A and B , there exists a

continuous map $Cf : CA \rightarrow CB$ defined by $Cf(a, t) = (f(a), t)$. Here, by the abuse of notation, we are denoting an element of CA by a pair (a, t) with $a \in A$ and $t \in [0, 1]$. We note that if $p_0 \in CA$ is the collapsed image of $A \times \{1\}$ and $q_0 \in CB$ is the collapsed image of $B \times \{1\}$, then $Cf(p_0) = q_0$.

The *suspension* of A , denoted by SA , is defined as follows.

$$SA = \frac{A \times [-1, 1]}{\sim},$$

where the equivalence relation ' $\sim$ ' collapses $A \times \{1\}$ to a point and $A \times \{-1\}$ to another point.

$SA = C_+A \cup C_-A$, where

$$C_+A = \frac{A \times [0, 1]}{\sim}, \quad C_-A = \frac{A \times [-1, 0]}{\sim}.$$

C_+A and C_-A are cones over A . Since the cone over a topological space is contractible, using van Kampen's Theorem, one can show that if A is path-connected, SA is simply-connected. We note that the cone over a k -simplex is a $(k+1)$ -simplex. Therefore, if A is a cubical complex, CA and SA are also cubical complexes.

2.3 Regularity of the min-max CMC hypersurfaces in higher dimensions

In this section, we will modify the argument of Zhou and Zhu [ZZ19, Zho20] to extend their min-max theory (Theorem 2.3.5) in higher dimensions. We will use the compactness theory of stable CMC hypersurfaces (Theorem 2.3.3), developed by Bellettini-Wickramasekera [BW18, BW19] and Bellettini-Chodosh-Wickramasekera [BCW19]. When the ambient dimension $3 \leq n+1 \leq 7$, the compactness theorem for stable CMC hypersurfaces was also proved by Zhou and Zhu [ZZ19].

Definition 2.3.1. $\Sigma \subset (U^{n+1}, g)$ is called a *c-CMC hypersurface with optimal regularity* ($c \geq 0$) if Σ is a closed n -rectifiable subset of (U, g) and there exists a closed subset of Σ , denoted by $\text{sing}(\Sigma)$, such that the following conditions are satisfied.

- (i) $\mathcal{H}^s(\text{sing}(\Sigma)) = 0$ for all $s > n - 7$.
- (ii) $\text{reg}(\Sigma) = \Sigma \setminus \text{sing}(\Sigma)$ is *almost embedded*, i.e. if $p \in \text{reg}(\Sigma)$ such that Σ is not embedded near p , there exists an open set $B \subset U$, containing p , such that
 - $B \cap \Sigma = \Sigma_1 \cup \Sigma_2$, where Σ_1 and Σ_2 are smoothly embedded hypersurfaces in B (since Σ is not embedded at p , we necessarily have $p \in \Sigma_1 \cap \Sigma_2$);
 - $B \setminus \Sigma_1 = B_1 \cup B_2$, where B_1 and B_2 are connected and $\Sigma_2 \subset \overline{B_1}$ (hence Σ_1 and Σ_2 intersect each other tangentially).
- (iii) $|\Sigma|$ has bounded first variation and $\text{reg}(\Sigma)$ has constant mean curvature c with respect to a globally defined unit normal ν . (If $c = 0$, we do not require that $\text{reg}(\Sigma)$ is two-sided.)

Given such a Σ , following [ZZ19], let us define

$$\mathcal{R}(\Sigma) = \{p \in \text{reg}(\Sigma) : \Sigma \text{ is embedded near } p\}; \quad \mathcal{S}(\Sigma) = \text{reg}(\Sigma) \setminus \mathcal{R}(\Sigma).$$

If Σ is a minimal hypersurface (i.e. $c = 0$) with optimal regularity, then, by the maximum principle, $\mathcal{S}(\Sigma) = \emptyset$. If $c > 0$, by [ZZ19, Proposition 2.9] and [BW18, Remark 2.6], $\mathcal{S}(\Sigma)$ is $(n - 1)$ -rectifiable.

Definition 2.3.2. Let $\Sigma \subset U$ be a c -CMC hypersurface with optimal regularity ($c \geq 0$). If $c > 0$, $\text{reg}(\Sigma)$ is a two-sided, immersed hypersurface in U ; Σ is called *stable* if for all $\varphi \in C_c^\infty(\text{reg}(\Sigma))$,

$$\int_{\text{reg}(\Sigma)} \left(|\nabla_\Sigma \varphi|^2 - \text{Ric}(\nu, \nu) \varphi^2 - |A_\Sigma|^2 \varphi^2 \right) d\mathcal{H}^n \geq 0.$$

Here ν is a globally defined unit normal on $\text{reg}(\Sigma)$ and A_Σ is the second fundamental form of Σ , which is defined on $\text{reg}(\Sigma)$. On the other hand, if $c = 0$, $\text{reg}(\Sigma)$ is an embedded hypersurface in U ; Σ is called *stable* if for all compactly supported, normal vector-field $\mathcal{X}$ on $\text{reg}(\Sigma)$,

$$\int_{\text{reg}(\Sigma)} \left(|\nabla^\perp \mathcal{X}|^2 - \text{Ric}(\mathcal{X}, \mathcal{X}) - |A_\Sigma|^2 |\mathcal{X}|^2 \right) d\mathcal{H}^n \geq 0.$$

Theorem 2.3.3 ([BW18, BW19, BCW19], [ZZ19] when $3 \leq n+1 \leq 7$). *Suppose $\Sigma_k \subset U$ is a sequence of stable, c_k -CMC hypersurfaces with optimal regularity, $\sup_k \mathcal{H}^n(\Sigma_k) < \infty$ and $c_k \rightarrow c_\infty$. Then, possibly after passing to a subsequence, $|\Sigma_k|$ converges to an integral n -varifold W such that $\text{spt } \|W\| = \Sigma_\infty$ is a closed c_∞ -CMC hypersurface with optimal regularity. The convergence is smooth on compact subsets of $\text{reg}(\Sigma_\infty)$. Moreover, if each $\Sigma_k = \partial\Omega_k$ for some $\Omega_k \in \mathcal{C}(U)$ and $c_\infty > 0$, then $\Sigma_\infty = \partial\Omega_\infty$ for some $\Omega_\infty \in \mathcal{C}(U)$ and the density of W is 1 on $\mathcal{R}(\Sigma_\infty)$ and 2 on $\mathcal{S}(\Sigma_\infty)$.*

The following definition is taken from [Zho20]. We remark that in [Zho20, Definition 1.1], the homotopies are continuous in the flat norm. However, if we examine the proof of Theorem 1.7 in [Zho20], we see that the homotopies can be taken to be continuous in the $\mathbf{F}$ -metric. (In [Zho20, Proposition 1.14 and 1.15] it is proved that the Almgren extensions are homotopic to each other and to the initial map in the $\mathbf{F}$ -metric; see also [MN21, Section 3].)

Definition 2.3.4. Suppose X is a cubical complex, $Z \subset X$ is a subcomplex, $F_0 : X \rightarrow (\mathcal{C}(M), \mathbf{F})$ is a continuous map. Let Π denote the set of all sequence of continuous maps $\{F_i : X \rightarrow (\mathcal{C}(M), \mathbf{F})\}_{i=1}^\infty$ such that for every $i \in \mathbb{N}$, there exists homotopy $G_i : X \times [0, 1] \rightarrow (\mathcal{C}(M), \mathbf{F})$ with the properties $G_i(-, 0) = F_0$, $G_i(-, 1) = F_i$ and

$$\limsup_{i \rightarrow \infty} \sup \{\mathbf{F}(G_i(x, s), F_0(x)) : x \in Z, s \in [0, 1]\} = 0.$$

Π is called the (X, Z) -homotopy class of F_0 . For $c > 0$, we define

$$\mathbf{L}^c(\Pi) = \inf_{\{F_i\} \in \Pi} \limsup_{i \rightarrow \infty} \sup_{x \in X} \{\mathcal{A}^c(F_i(x))\}.$$

To prove the following theorem, we only need to slightly modify the argument given in the paper [ZZ19]. We will only mention the necessary modifications; further details can be found in [ZZ19, Section 5 and 6].

Theorem 2.3.5 ([ZZ19, Zho20]). *Let (M^{n+1}, g) be a closed Riemannian manifold, $n+1 \geq 3$, and $c > 0$. Suppose $F_0 : X \rightarrow (\mathcal{C}(M), \mathbf{F})$ is a continuous map, $Z \subset X$ and Π is the (X, Z) -homotopy*

class of F_0 such that

$$\mathbf{L}^c(\Pi) > \sup_{x \in Z} \mathcal{A}^c(F_0(x)).$$

Then there exists $\Omega \in \mathcal{C}(M)$ such that $\mathcal{A}^c(\Omega) = \mathbf{L}^c(\Pi)$ and $\partial\Omega$ is a closed c -CMC (with respect to the inward unit normal) hypersurface with optimal regularity. As a consequence, for any $c > 0$, every closed Riemannian manifold (M^{n+1}, g) , $n + 1 \geq 3$, contains a closed c -CMC hypersurface with optimal regularity.

Proof. Throughout the proof, we will use the notation used in [ZZ19]. It suffices to show that if $V \in \mathcal{V}_n(M)$ has c -bounded first variation in M and is c -almost minimizing in small annuli, then $V = |\Sigma|$, where Σ is a c -CMC hypersurface with optimal regularity. We will assume that M is isometrically embedded inside some Euclidean space $\mathbb{R}^L$. In the rest of this section, $B(p, r)$ and $A(p, r, R)$ will respectively denote the ball and the annulus in $\mathbb{R}^L$ with respect to the Euclidean metric.

For $p \in \mathbb{R}^L$ and $r > 0$, $\eta_{p,r} : \mathbb{R}^L \rightarrow \mathbb{R}^L$ is defined by $\eta_{p,r}(x) = \frac{x-p}{r}$. We start with the following proposition.

Proposition 2.3.6. *Let V be c -almost minimizing in U . Suppose $\{p_i\}$ is a sequence in U converging to $p \in U$ and $r_i > 0$ is a sequence converging to 0. If $\bar{V}$ is the varifold limit of $(\eta_{p_i, r_i})_{\#} V$, then $\bar{V}$ is induced by a minimal hypersurface with optimal regularity (possibly with integer multiplicities). In particular, if $V \in \mathcal{V}_n(M)$ has c -bounded first variation in M and is c -almost minimizing in small annuli, then V is integer rectifiable and every $C \in \text{VarTan}(V, p)$ is induced by a minimal hypersurface with optimal regularity (possibly with integer multiplicities).*

The proof of this proposition is same as the proof of Lemma 5.10 and Proposition 5.11 in [ZZ19]. One can show that $\bar{V}$ satisfies the good replacement property. This implies the proposition by the results of Schoen-Simon [SS81] and De Lellis-Tasnady [DT13].

Lemma 2.3.7. *Let $H = \{x \in \mathbb{R}^{n+1} : x_{n+1} = 0\}$ and $S \neq H$ be another hyperplane in $\mathbb{R}^{n+1}$, which is not parallel to H . Suppose W is a stationary, integral varifold in $\mathbb{R}^{n+1}$ such that $\text{spt} \|W\|$ is a minimal hypersurface with optimal regularity and $W \llcorner \{x \in \mathbb{R}^{n+1} : x_{n+1} > 0\} = m|S| \llcorner \{x \in \mathbb{R}^{n+1} :$*

$x_{n+1} > 0\}$. Then $W = m|S|$.

Proof. The proof is an adaptation of the argument of Hoffman and Meeks [HM90, Proof of Theorem 3] in the singular setting. Let $\Xi = \text{spt } \|W\|$. By our assumption,

$$\mathcal{H}^s(\text{sing}(\Xi)) = 0 \quad \text{for all } s > n - 7. \quad (2.3.1)$$

Therefore, there exist a collection $\{\Xi_j\}_{j=0}^J$ of disjoint, connected minimal hypersurfaces with optimal regularity and positive integers $\{m_j\}_{j=0}^J$ such that

$$W = \sum_{j=0}^J m_j |\Xi_j|.$$

Since $\Xi \cap \{x_{n+1} > 0\} = S \cap \{x_{n+1} > 0\}$, by (2.3.1), we have $S \subset \text{reg}(\Xi)$ (see [SS81, (7.23)]). By the theorem of Ilmanen [Ilm96, Theorem A(ii)] and Wickramasekera [Wic14b], (2.3.1) implies that the connected components of $\text{reg}(\Xi)$ are precisely $\{\text{reg}(\Xi_j)\}_{j=0}^J$. Thus $S = \text{reg}(\Xi_l)$, for some $l \in \{0, \dots, J\}$. Let us assume that $S = \text{reg}(\Xi_0)$; (2.3.1) gives $S = \Xi_0$ and hence $m_0 = m$. If $W \neq m|S|$, let us consider Ξ_1 . Since $W \llcorner \{x_{n+1} > 0\} = m|S| \llcorner \{x_{n+1} > 0\}$, $\Xi_1 \subset \{x_{n+1} \leq 0\}$. We claim that $\Xi_1 \subset \{x_{n+1} < 0\}$. Indeed, by the maximum principle proved by White [Whi10, Theorem 4] and Solomon-White [SW89], if $\Xi_1 \cap H \neq \emptyset$, then $H \subset \Xi_1$, which contradicts the fact that Ξ_1 is disjoint from $\Xi_0 = S$. As Ξ_1 is connected, Ξ_1 lies in a region $\mathcal{R}$ which is the intersection of the open half-space $\{x_{n+1} < 0\}$ and another open half-space defined by S . Using an orthogonal transformation, we can obtain a new coordinate system $\{y_1, \dots, y_{n+1}\}$ such that $\mathcal{R} \subset \{y_{n+1} > 0\}$, $\partial\mathcal{R}$ is a graph over the hyperplane $\{y_{n+1} = 0\}$ and the y_1 -axis is contained in $H \cap S$. As Ξ_1 is a closed set and is disjoint from $H \cup S$, there exists $r > 0$ such that

$$B_r = \{y \in \mathbb{R}^{n+1} : (y_1 - r)^2 + y_2^2 + \dots + y_{n+1}^2 \leq r^2\}$$

is disjoint from Ξ_1 . We note that $\partial B_r \cap \partial\mathcal{R}$ is a graph over the boundary of a convex domain in $\{y_{n+1} = 0\}$. Therefore, by [GT01, Theorem 16.8], there exists a smooth minimal hypersurface N

with boundary $\partial N = \partial B_r \cap \partial \mathcal{R}$. By the convex hull property, $N \subset B_r$; hence $N \cap \Xi_1 = \emptyset$. For $t \in [1, \infty)$, let us consider

$$N_t = \{tp : p \in N\}.$$

We claim that for all $t \in [1, \infty)$, $N_t \cap \Xi_1 = \emptyset$. Otherwise, there exists a smallest $\tau > 1$ such that $N_\tau \cap \Xi_1 \neq \emptyset$. This implies, by [SW89, Step 1, p. 687], $N_\tau \subset \Xi_1$, which contradicts the fact that $\Xi_1 \cap \partial \mathcal{R} = \emptyset$. However,

$$\mathcal{R} \cap \{y_1 > 0\} \subset B_r \cup \left(\bigcup_{t \in [1, \infty)} N_t \right).$$

Therefore, $\Xi_1 \subset \mathcal{R} \cap \{y_1 \leq 0\}$. An analogous argument gives that for any $T \in \mathbb{R}$, $\Xi_1 \subset \mathcal{R} \cap \{y_1 \leq T\}$, which forces $\Xi_1 = \emptyset$. This finishes the proof of the lemma. □

We proceed as in [ZZ19]. Let us fix $p \in \text{spt } \|V\|$. We choose $0 < r_0 < r_{am}(p)$ such that

- for all $0 < r \leq r_0$, $\partial B(p, r) \cap M$ is a smooth hypersurface in M with mean curvature greater than c ;
- $x \mapsto d_{\mathbb{R}^L}(p, x)$ is a smooth function on $(B(p, r_0) \cap M) \setminus \{p\}$.

For $r \leq r_0$ and $W \in \mathcal{V}_n(M)$, if $W \neq 0$ in $B(p, r)$ and has c -bounded first variation in $B(p, r) \cap M$, we have

$$\emptyset \neq \text{spt } \|W\| \cap \partial B(p, r) = \overline{\text{spt } \|W\| \setminus \overline{B(p, r)} \cap \partial B(p, r)}; \quad (2.3.2)$$

$$\text{spt } \|W\| \cap \overline{B(p, r)} = \overline{\bigcup_{0 < s < r} \text{spt } (\|W\| \llcorner B(p, s)) \cap \partial B(p, s)}. \quad (2.3.3)$$

Equation (2.3.3) is a consequence of [ZZ19, Lemma 6.2] (see [DT13, equation (5.6) and A.3. Proof of Lemma 5.4]). Let us fix $0 < s < t < r_0$. V^* is a c -replacement of V in $\text{Clos}(A(p, s, t) \cap M)$. By the regularity of the minimizers for the $\mathcal{A}^c$ functional [ZZ19, Theorem 2.14], [Mor03] and Theorem 2.3.3, there exists Σ_1 , a c -CMC hypersurface in $A(p, s, t) \cap M$ with optimal regularity, such that $\Sigma_1 =$

$\partial\Omega_1 \cap A(p, s, t) \cap M$ for some $\Omega_1 \in \mathcal{C}(M)$ and

$$V^* \llcorner (A(p, s, t) \cap M) = |\Sigma_1|.$$

$\mathcal{S}(\Sigma_1)$ is contained in $\cup_k \Sigma_1^{(k)}$, a countable union of $(n-1)$ -dimensional submanifolds. We choose $s_2 \in (s, t)$ such that $\partial B(p, s_2)$ intersects $\text{reg}(\Sigma_1)$ and all the $\Sigma_1^{(k)}$'s transversally. For $s_1 \in (0, s)$, let us consider c -replacement V^{**} of V^* in $\text{Clos}(A(p, s_1, s_2) \cap M)$. There exists Σ_2 , a c -CMC hypersurface in $A(p, s_1, s_2) \cap M$ with optimal regularity, such that $\Sigma_2 = \partial\Omega_2 \cap A(p, s_1, s_2) \cap M$ for some $\Omega_2 \in \mathcal{C}(M)$ and

$$V^{**} \llcorner (A(p, s_1, s_2) \cap M) = |\Sigma_2|.$$

We need to show that Σ_2 smoothly glues with Σ_1 across $\partial(B(p, s_2) \cap M)$.

Lemma 2.3.8. *For all $x \in \text{reg}(\Sigma_1) \cap \partial(B(p, s_2) \cap M)$, $\text{VarTan}(V^{**}, x) = \{\Theta^n(\|V^*\|, x)|T_x \Sigma_1|\}$, where $T_x \Sigma_1$ denotes the tangent space of Σ_1 at x with multiplicity one.*

Proof. Let $C \in \text{VarTan}(V^{**}, x)$. Denoting $\Lambda = \partial(B(p, s_2) \cap M)$, since $V^{**} = V^* = |\Sigma_1|$ in $A(p, s_2, t) \cap M$, we must have $C = \Theta^n(\|V^*\|, x)|T_x \Sigma_1|$ in an open half-space of $T_x M$, defined by $T_x \Lambda$. (By the transversality assumption, $T_x \Sigma_1 \neq T_x \Lambda$.) Using Proposition 2.3.6 and Lemma 2.3.7, we obtain $C = \Theta^n(\|V^*\|, x)|T_x \Sigma_1|$.

□

As argued in [ZZ19, Step 2], using (2.3.2), one can show that

$$\text{Clos}(\Sigma_2) \cap \partial(B(p, s_2) \cap M) \subset \Sigma_1 \cap \partial(B(p, s_2) \cap M).$$

Moreover, smallness of $\text{sing}(\Sigma_1)$ and Lemma 2.3.8 imply that

$$\Sigma_1 \cap \partial(B(p, s_2) \cap M) = \text{Clos}(\text{reg}(\Sigma_1) \cap \partial(B(p, s_2) \cap M)) \subset \text{Clos}(\Sigma_2) \cap \partial(B(p, s_2) \cap M).$$

Hence,

$$\text{Clos}(\Sigma_2) \cap \partial(B(p, s_2) \cap M) = \Sigma_1 \cap \partial(B(p, s_2) \cap M),$$

i.e. Σ_1, Σ_2 glue continuously along $\partial B(p, s_2)$. Furthermore, Lemma 2.3.8, [Sim83, Theorem 3.2(2)] and smallness of $\text{sing}(\Sigma_1)$ also give that $\|V^{**}\|(\partial B(p, s_2)) = 0$, which implies, by [ZZ19, Claim 1(c) in Section 6],

$$V^{**}\llcorner(A(p, s_1, t) \cap M) = |\partial\Omega^{**}|\llcorner(A(p, s_1, t) \cap M),$$

for some $\Omega^{**} \in \mathcal{C}(M)$.

Arguing as in [ZZ19, Step 2], we can also show that $\text{reg}(\Sigma_1)$ and $\text{reg}(\Sigma_2)$ glue smoothly across $\partial(B(p, s_2) \cap M)$; we need to use Lemma 2.3.7 in place of the half space theorem of [HM90] and Theorem 2.3.3 in place of [ZZ19, Theorem 2.11]. Let us give some details. Let ν_1, ν_2 denote the outward unit normal of $\text{reg}(\Sigma_1)$ and $\text{reg}(\Sigma_2)$ respectively;

$$\Gamma = \text{reg}(\Sigma_1) \cap \partial(B(p, s_2) \cap M), \quad \mathcal{R}(\Gamma) = \mathcal{R}(\Sigma_1) \cap \Gamma, \quad \mathcal{S}(\Gamma) = \mathcal{S}(\Sigma_1) \cap \Gamma.$$

Suppose $x \in \mathcal{R}(\Gamma)$, $y_i \in \text{reg}(\Sigma_2)$ converges to x . It is possible to choose $x_i \in \mathcal{R}(\Gamma)$ such that $r_i = |x_i - y_i|$ converges to 0 and hence $x_i \rightarrow x$. Using the fact that $V^{**}\llcorner(A(p, s_2, t) \cap M) = |\Sigma_1|\llcorner(A(p, s_2, t) \cap M)$, Proposition 2.3.6 and Lemma 2.3.7, one can prove that

$$\lim_{i \rightarrow \infty} (\eta_{x_i, r_i})_{\#} V^{**} = |T_x \Sigma_1| \quad \text{in the sense of varifolds.}$$

This, along with Theorem 2.3.3, implies that $\eta_{x_i, r_i}(\Sigma_2 \cap B(y_i, r_i/2))$ smoothly converges to a domain in $T_x \Sigma_1$, which gives

$$\lim_{\substack{y \rightarrow x \\ y \in \text{reg}(\Sigma_2)}} \nu_2(y) = \nu_1(x) \quad \text{uniformly in } x \text{ on compact subsets of } \mathcal{R}(\Gamma). \quad (2.3.4)$$

Similarly, if $x \in \mathcal{S}(\Gamma)$ and $y_i \in \text{reg}(\Sigma_2)$ converges to x , it is possible to choose $x_i \in \Gamma$ such that $r_i = |x_i - y_i|$ converges to 0 and hence $x_i \rightarrow x$. Using the fact that $V^{**}\llcorner(A(p, s_2, t) \cap M) =$

$|\Sigma_1| \mathsf{L}(A(p, s_2, t) \cap M)$, Proposition 2.3.6 and Lemma 2.3.7, one can show that

$$\lim_{i \rightarrow \infty} (\eta_{x_i, r_i})_{\#} V^{**} = \begin{cases} |T_x \Sigma_1| + |\tau_v T_x \Sigma_1| & \text{if } \liminf_{i \rightarrow \infty} d_{\mathbb{R}^L}(x_i, \mathcal{S}(\Gamma))/r_i = \infty; \\ 2 |T_x \Sigma_1| & \text{if } \liminf_{i \rightarrow \infty} d_{\mathbb{R}^L}(x_i, \mathcal{S}(\Gamma))/r_i < \infty, \end{cases}$$

where $T_x \Sigma_1$ denotes the tangent space of Σ_1 at x with multiplicity one and τ_v denotes translation by $v \in (T_x \Sigma_1)^\perp \subset T_x M$ (v might be ∞ , in which case $\tau_v T_x \Sigma_1 = \emptyset$). As before, this, together with Theorem 2.3.3, gives that

$$\lim_{\substack{y \rightarrow x \\ y \in \text{reg}(\Sigma_2)}} [T_y \Sigma_2] = [T_x \Sigma_1] \quad \text{uniformly in } x \text{ on compact subsets of } \mathcal{S}(\Gamma), \quad (2.3.5)$$

where $[T_y \Sigma_2]$ and $[T_x \Sigma_1]$ respectively denote the un-oriented, multiplicity one tangent spaces of Σ_2 and Σ_1 . (2.3.4), (2.3.5) and elliptic regularity imply that $\text{reg}(\Sigma_1)$ and $\text{reg}(\Sigma_2)$ glue smoothly across $\partial(B(p, s_2) \cap M)$. Hence, by unique continuation and by the smallness of $\text{sing}(\Sigma_1)$ and $\text{sing}(\Sigma_2)$, $\Sigma_2 = \Sigma_1$ in $A(p, s, s_2) \cap M$.

Since we will vary $s_1 \in (0, s)$, let us use the notation $V_{s_1}^{**}$ and Σ_{s_1} instead of V^{**} and Σ_2 . As argued in [ZZ19, Step 3], the above discussion implies that

$$\Sigma = \{p\} \cup \bigcup_{0 < s_1 < s} \Sigma_{s_1}$$

is a c -CMC hypersurface with optimal regularity in $B(p, s_2)$ (if necessary, p can be absorbed inside $\text{sing}(\Sigma)$). Further, for any $s_1 < s$,

$$V_{s_1}^{**} \mathsf{L}(A(p, s_1, s_2) \cap M) = |\Sigma| \mathsf{L}(A(p, s_1, s_2) \cap M). \quad (2.3.6)$$

Following the argument of Schoen-Simon [SS81] and De Lellis-Tasnady [DT13], we will show that $\text{spt} \|V\| \cap B(p, s) = \Sigma \cap B(p, s)$. By the definition of c -replacement, for all $0 < s'_1 \leq s_1$,

$$V_{s_1}^{**} \mathsf{L}(B(p, s'_1) \cap M) = V^* \mathsf{L}(B(p, s'_1) \cap M) = V \mathsf{L}(B(p, s'_1) \cap M).$$

Hence, using (2.3.3),

$$\text{spt } \|V\| \cap \partial(B(p, s_1) \cap M) = \text{spt } \|V_{s_1}^{**}\| \cap \partial(B(p, s_1) \cap M). \quad (2.3.7)$$

Moreover, by (2.3.6) and (2.3.2),

$$\text{spt } \|V_{s_1}^{**}\| \cap \partial(B(p, s_1) \cap M) = \Sigma \cap \partial(B(p, s_1) \cap M). \quad (2.3.8)$$

Therefore, combining (2.3.7) and (2.3.8),

$$\text{spt } \|V\| \cap \partial(B(p, s_1) \cap M) = \Sigma \cap \partial(B(p, s_1) \cap M),$$

for all $0 < s_1 < s$. By varying $s_1 \in (0, s)$, we obtain $\text{spt } \|V\| \cap B(p, s) = \Sigma \cap B(p, s)$. (We assumed that $p \in \text{spt } \|V\|$ and by our definition of Σ , $p \in \Sigma$.)

As $p \in \text{spt } \|V\|$ is arbitrary, we have proved that $\text{spt } \|V\|$ is a c -CMC hypersurface with optimal regularity. By a slight abuse of notation, let us denote $\text{spt } \|V\|$ by Σ . We will show that if $p \in \mathcal{R}(\Sigma)$, there exists $r > 0$ such that $\Theta^n(\|V\|, y) = 1$ for all $y \in (\mathcal{R}(\Sigma) \cap B(p, r)) \setminus \{p\}$. This will imply that $V = |\Sigma|$ as $\mathcal{H}^n(\Sigma \setminus \mathcal{R}(\Sigma)) = 0$.

The argument is similar to [ZZ19, Proof of Claim 6]. Since $p \in \mathcal{R}(\Sigma)$, one can choose $0 < r < r_0$ (where r_0 is as chosen after the proof of Lemma 2.3.7), such that $B(p, r) \cap \Sigma \subset \mathcal{R}(\Sigma)$ and for all $0 < r' \leq r$, $\partial B(p, r')$ intersects Σ transversally. Let $\rho = |p - y|$ and we consider the second replacement V_ρ^{**} . Since $V_\rho^{**} \llcorner (A(p, \rho, s) \cap M) = |\Sigma| \llcorner (A(p, \rho, s) \cap M)$ for some $s > \rho$, by the transversality assumption, Proposition 2.3.6 and Lemma 2.3.7, $\text{VarTan}(V_\rho^{**}, y) = \{|T_y \Sigma|\}$. As $V \llcorner (B(p, \rho) \cap M) = V_\rho^{**} \llcorner (B(p, \rho) \cap M)$, one can again use Proposition 2.3.6 and Lemma 2.3.7 to conclude that $\text{VarTan}(V, y) = \{|T_y \Sigma|\}$. Hence $\Theta^n(\|V\|, y) = 1$.

Finally, to show the existence of $\Omega \in \mathcal{C}(M)$ with the required properties as stated in Theorem 2.3.5, we can use the argument of Zhou and Zhu in [ZZ20, Proof of Proposition 7.3]. To adapt their argument in our situation, we need to use Theorem 2.3.3 in place of [ZZ20, Theorem 3.19] and the fact that $\text{sing}(\Sigma)$ is small. The existence of $\Omega \in \mathcal{C}(M)$ with the required properties also

follows from the notion of *h-almost minimizing in small annuli with respect to S*, used by Zhou in [Zho20, Section 1.4].

□

2.4 Proof of the main theorems

Proof of Theorem 2.1.1. The theorem will be proved in five parts.

Part 1. We choose δ such that

$$0 < \delta < \omega_{k+1} - \omega_k - c \text{Vol}(g) \quad \text{and} \quad \delta < \eta/2. \quad (2.4.1)$$

By the definition of ω_k , there exists a map $\Phi : X \rightarrow \mathcal{Z}_n(M; \mathcal{F}; \mathbb{Z}_2)$ with no concentration of mass, such that X is a connected cubical complex, Φ is a k -sweepout and

$$\sup_{x \in X} \mathbf{M}(\Phi(x)) \leq \omega_k + \delta/2. \quad (2.4.2)$$

Following the argument in [Zho20], Φ is a k -sweepout implies that

$$\Phi^* : H^1(\mathcal{Z}_n(M; \mathcal{F}; \mathbb{Z}_2), \mathbb{Z}_2) \rightarrow H^1(X, \mathbb{Z}_2)$$

is non-zero; hence,

$$\Phi_* : \pi_1(X) \rightarrow \pi_1(\mathcal{Z}_n(M; \mathcal{F}; \mathbb{Z}_2)) (= \mathbb{Z}_2)$$

is onto. Thus, $\ker(\Phi_*)$ is an index 2 subgroup of $\pi_1(X)$. From [Hat02, Proposition 1.36], it follows that there exists a two sheeted covering $\pi : \tilde{X} \rightarrow X$ such that $\tilde{X}$ is connected and if

$$\pi_* : \pi_1(\tilde{X}) \rightarrow \pi_1(X),$$

$\text{im}(\pi_*) = \ker(\Phi_*)$. By [Hat02, Proposition 1.33], Φ has a lift

$$\tilde{\Phi} : \tilde{X} \rightarrow (\mathcal{C}(M), \mathcal{F}) \quad \text{such that} \quad \partial \circ \tilde{\Phi} = \Phi \circ \pi.$$

Moreover, $\tilde{\Phi}$ is $\mathbb{Z}_2$ -equivariant, i.e. if $T : \tilde{X} \rightarrow \tilde{X}$ is the deck transformation, then for all $x \in \tilde{X}$,

$$\tilde{\Phi}(T(x)) = M - \tilde{\Phi}(x).$$

Let us recall the notation from Section 2.2.4.

$$S\tilde{X} = \underset{\sim}{\text{suspension of } \tilde{X}} = \frac{\tilde{X} \times [-1, 1]}{\sim}.$$

By the abuse of notation, an element of $S\tilde{X}$ will be denoted by a pair (x, t) with $x \in \tilde{X}, t \in [-1, 1]$.

$$S\tilde{X} = C_+ \cup C_-, \quad \text{where} \quad C_+ = \frac{\tilde{X} \times [0, 1]}{\sim}, \quad C_- = \frac{\tilde{X} \times [-1, 0]}{\sim}.$$

Since $\tilde{X}$ is connected, $S\tilde{X}$ is simply-connected. There is a free $\mathbb{Z}_2$ action on $S\tilde{X}$ given by $(x, t) \mapsto (T(x), -t)$. Let Y denote the quotient of $S\tilde{X}$ with respect to this $\mathbb{Z}_2$ action and $\rho : S\tilde{X} \rightarrow Y$ be the covering map. $\tilde{X}$ naturally sits inside $S\tilde{X}$ by the map $x \mapsto (x, 0)$; hence, X also naturally sits inside Y . We will identify $\tilde{X}$ with its image under $\tilde{X} \hookrightarrow S\tilde{X}$ and X with its image under $X \hookrightarrow Y$.

To illustrate the above constructions, let us consider the following example. If $X = \mathbb{RP}^k$, then $\tilde{X} = S^k$, $S\tilde{X} = S^{k+1}$, C_+ and C_- are respectively the upper hemisphere and the lower hemisphere of S^{k+1} , $Y = \mathbb{RP}^{k+1}$.

Part 2. We begin with the following proposition from [AFP00]. For the sake of completeness, we also include its proof (following [AFP00]).

Proposition 2.4.1 ([AFP00, Proposition 3.38]). *For an $\mathcal{H}^{n+1}$ -measurable set $E \subset M$ and an open*

set $U \subset M$, let us use the notation

$$P(E, U) = \sup \left\{ \int_{E \cap U} \operatorname{div} \omega \, d\mathcal{H}^{n+1} : \omega \in \mathfrak{X}_c^1(U) \text{ and } \|\omega\|_\infty \leq 1 \right\}.$$

Here $\mathfrak{X}_c^1(U)$ is the space of compactly supported C^1 vector-fields on U . Suppose $E, F \in \mathcal{C}(M)$. Then

$$P(E \cap F, U) \leq P(E, U) + P(F, U).$$

Hence, in particular, $E \cap F \in \mathcal{C}(M)$.

Proof. By [MPPP07, Proposition 1.4], there exist sequences $\{f_j\}_{j=1}^\infty, \{g_j\}_{j=1}^\infty \subset C^\infty(U)$ such that $0 \leq f_j, g_j \leq 1$ for all $j \in \mathbb{N}$,

$$\begin{aligned} f_j &\rightarrow \chi_E|_U \text{ and } g_j \rightarrow \chi_F|_U \text{ in } L^1(U); \\ P(E, U) &= \lim_{j \rightarrow \infty} \int_U \|Df_j\| \, d\mathcal{H}^{n+1} \text{ and } P(F, U) = \lim_{j \rightarrow \infty} \int_U \|Dg_j\| \, d\mathcal{H}^{n+1}. \end{aligned}$$

By the dominated convergence theorem,

$$f_j g_j \rightarrow \chi_E \chi_F|_U = \chi_{E \cap F}|_U \text{ in } L^1(U).$$

Therefore,

$$\begin{aligned} P(E \cap F, U) &\leq \liminf_{j \rightarrow \infty} \int_U \|D(f_j g_j)\| \, d\mathcal{H}^{n+1} \\ &\leq \liminf_{j \rightarrow \infty} \int_U \|Df_j\| \, d\mathcal{H}^{n+1} + \liminf_{j \rightarrow \infty} \int_U \|Dg_j\| \, d\mathcal{H}^{n+1} \\ &= P(E, U) + P(F, U). \end{aligned}$$

□

The map $\tilde{\Phi} : \tilde{X} \rightarrow (\mathcal{C}(M), \mathcal{F})$ (defined in Part 1) can be extended to a map $\tilde{\Psi}' : S\tilde{X} \rightarrow (\mathcal{C}(M), \mathcal{F})$

as follows. Let us define

$$\tilde{Z} = \{(x, t) \in S\tilde{X} : |t| \leq 1/2\} \subset S\tilde{X}.$$

Let $\Lambda : [0, 1] \rightarrow (\mathcal{C}(M), \mathcal{F})$ be such that $\Lambda(0) = M$, $\Lambda(1) = \emptyset$, $\partial \circ \Lambda : [0, 1] \rightarrow \mathcal{Z}_n(M; \mathcal{F}; \mathbb{Z}_2)$ has no concentration of mass and

$$\sup_{t \in [0, 1]} \mathbf{M}(\partial \Lambda(t)) \leq W_0 + \eta/2 - \delta. \quad (2.4.3)$$

We define

$$\tilde{\Psi}'(x, t) = \begin{cases} \tilde{\Phi}(x) & \text{if } (x, t) \in \tilde{Z}; \\ \tilde{\Phi}(x) \cap \Lambda(2t - 1) & \text{if } (x, t) \in C_+ \setminus \tilde{Z}; \\ M - (\tilde{\Phi}(T(x)) \cap \Lambda(-1 - 2t)) & \text{if } (x, t) \in C_- \setminus \tilde{Z}. \end{cases} \quad (2.4.4)$$

By Proposition 2.4.1, $\tilde{\Psi}'(x, t) \in \mathcal{C}(M)$ for all $(x, t) \in S\tilde{X}$.

Claim 2.4.2. $\tilde{\Psi}' : S\tilde{X} \rightarrow \mathcal{C}(M)$ is continuous in the flat topology.

Proof. It is enough to prove that $(x, t) \mapsto \tilde{\Phi}(x) \cap \Lambda(2t - 1)$ is continuous in $\mathcal{F}$ for $(x, t) \in C_+ \setminus \tilde{Z}$.

We note that for arbitrary sets A_1, A_2, B_1, B_2 ,

$$(A_1 \cap A_2) \Delta (B_1 \cap B_2) \subset (A_1 \Delta B_1) \cup (A_2 \Delta B_2),$$

where $A \Delta B = (A \setminus B) \cup (B \setminus A)$ denotes the symmetric difference of A and B . Hence, if $(x_1, t_1), (x_2, t_2) \in C_+ \setminus \tilde{Z}$,

$$(\tilde{\Phi}(x_1) \cap \Lambda(2t_1 - 1)) \Delta (\tilde{\Phi}(x_2) \cap \Lambda(2t_2 - 1)) \subset (\tilde{\Phi}(x_1) \Delta \tilde{\Phi}(x_2)) \cup (\Lambda(2t_1 - 1) \Delta \Lambda(2t_2 - 1)),$$

which implies

$$\begin{aligned} & \mathcal{F}\left((\tilde{\Phi}(x_1) \cap \Lambda(2t_1 - 1)) - (\tilde{\Phi}(x_2) \cap \Lambda(2t_2 - 1))\right) \\ & \leq \mathcal{F}(\tilde{\Phi}(x_1) - \tilde{\Phi}(x_2)) + \mathcal{F}(\Lambda(2t_1 - 1) - \Lambda(2t_2 - 1)). \end{aligned} \quad (2.4.5)$$

Since $x \mapsto \tilde{\Phi}(x)$ and $t \mapsto \Lambda(t)$ are continuous in the flat topology, by (2.4.5), $(x, t) \mapsto \tilde{\Phi}(x) \cap \Lambda(2t-1)$ is also continuous in the flat topology.

□

As $\partial\tilde{\Psi}'(x, t) = \partial\tilde{\Psi}'(T(x), -t)$, $\tilde{\Psi}'$ descends to a continuous map $\Psi' : Y \rightarrow \mathcal{Z}_n(M; \mathcal{F}; \mathbb{Z}_2)$, i.e. $\Psi' \circ \rho = \partial \circ \tilde{\Psi}'$. Furthermore, $\Psi'|_X = \Phi$.

Claim 2.4.3. $\Psi' : Y \rightarrow \mathcal{Z}_n(M; \mathcal{F}; \mathbb{Z}_2)$ has no concentration of mass.

Proof. If $E \in \mathcal{C}(M)$ and $U \subset M$ is open, $P(E, U) = \|\partial E\|(U)$. By Proposition 2.4.1, for $r > 0$ and $p \in M$,

$$\sup_{y \in Y} \|\Psi'(y)\|(B(p, r)) \leq \sup_{x \in X} \|\Phi(x)\|(B(p, r)) + \sup_{t \in [0, 1]} \|\partial\Lambda(t)\|(B(p, r)).$$

This finishes the proof of the claim as the maps Φ and $\partial \circ \Lambda$ have no concentration of mass.

□

By the interpolation theorems of Marques and Neves [MN14, Theorem 13.1, 14.1], [MN17, Theorem 3.9, 3.10], the above Claim 2.4.3 implies that there exists a continuous map $\Psi : Y \rightarrow \mathcal{Z}_n(M; \mathbf{F}; \mathbb{Z}_2)$ such that Ψ is homotopic to Ψ' in the $\mathcal{F}$ -topology and for all $y \in Y$,

$$\mathbf{M}(\Psi(y)) \leq \mathbf{M}(\Psi'(y)) + \delta/2. \quad (2.4.6)$$

Part 3. We will prove that $\Psi : Y \rightarrow \mathcal{Z}_n(M; \mathbf{F}; \mathbb{Z}_2)$, constructed above, is a $(k+1)$ -sweepout.

Let $\bar{\lambda}$ be the generator of $H^*(\mathcal{Z}_n(M; \mathcal{F}; \mathbb{Z}_2), \mathbb{Z}_2)$ as in Section 2.2.3 and $\lambda = \Psi^*\bar{\lambda}$. We need to show that $\lambda^{k+1} \neq 0 \in H^{k+1}(Y, \mathbb{Z}_2)$. Since $S\tilde{X}$ is simply-connected and Y is the quotient of $S\tilde{X}$ under the $\mathbb{Z}_2$ action, $\pi_1(Y) = \mathbb{Z}_2$. Hence, $H_1(Y, \mathbb{Z}_2) = H^1(Y, \mathbb{Z}_2) = \mathbb{Z}_2$ as well.

Let $\iota : X \rightarrow Y$ be the inclusion map. Since $\Psi'|_X = \Phi$, Ψ is homotopic to Ψ' in the $\mathcal{F}$ -topology and Φ is a k -sweepout, $\iota^*\lambda = \Phi^*\bar{\lambda} \neq 0 \in H^1(X, \mathbb{Z}_2)$. Therefore, λ is the unique non-zero element of $H^1(Y, \mathbb{Z}_2)$. Hence, if $\gamma : S^1 \rightarrow Y$ is a non-contractible loop, $\lambda.[\gamma] = 1$.

To prove $\lambda^{k+1} \neq 0 \in H^{k+1}(Y, \mathbb{Z}_2)$, it is enough to find $\alpha \in H_{k+1}(Y, \mathbb{Z}_2)$ such that $\lambda^{k+1} \cdot \alpha = 1$. By [Hat02, Proof of Proposition 2B.6], there exists a map (called the *transfer homomorphism*) $\tau_* : H_k(X, \mathbb{Z}_2) \rightarrow H_k(\tilde{X}, \mathbb{Z}_2)$, which is induced by the chain map $\tau : C_k(X, \mathbb{Z}_2) \rightarrow C_k(\tilde{X}, \mathbb{Z}_2)$, defined as follows. If $u : \Delta^k \rightarrow X$, u has precisely two lifts $\tilde{u}_1, \tilde{u}_2 : \Delta^k \rightarrow \tilde{X}$. We define $\tau(u) = \tilde{u}_1 + \tilde{u}_2$.

As Φ is a k -sweepout, $(\iota^* \lambda)^k = (\Phi^* \bar{\lambda})^k \neq 0 \in H^k(X, \mathbb{Z}_2)$. Therefore, there exists $\sigma \in H_k(X, \mathbb{Z}_2)$ such that $(\iota^* \lambda)^k \cdot \sigma = 1$. Let $\sigma = [\sum_{i=1}^I \sigma_i]$, where each $\sigma_i : \Delta^k \rightarrow X$ is a singular k -chain. Let $\tau(\sigma_i) = \tilde{\sigma}_{2i-1} + \tilde{\sigma}_{2i}$, where $\tilde{\sigma}_{2i-1}, \tilde{\sigma}_{2i} : \Delta^k \rightarrow \tilde{X}$ are lifts of σ_i . Since $\partial(\sum_i \sigma_i) = 0$ and τ is a chain map, we have

$$\partial \left(\sum_i \tilde{\sigma}_{2i-1} + \tilde{\sigma}_{2i} \right) = 0 \quad (2.4.7)$$

as well. $C\Delta^k$ can be naturally identified with Δ^{k+1} such that the collapsed image of $\Delta^k \times \{1\}$ is the $(k+1)$ -st vertex v_{k+1} (see Section 2.2.4 for the notation). Let us also recall from Part 1 that C_+ is a cone over $\tilde{X}$; let $y_0 \in C_+$ be the collapsed image of $\tilde{X} \times \{1\}$. For $s = 1, \dots, 2I$, we consider

$$C\tilde{\sigma}_s : C\Delta^k (= \Delta^{k+1}) \rightarrow C\tilde{X} (= C_+); \quad C\tilde{\sigma}_s(v_{k+1}) = y_0.$$

Using equation (2.4.7), one can check that

$$\partial \left(\sum_i C\tilde{\sigma}_{2i-1} + C\tilde{\sigma}_{2i} \right) = \left(\sum_i \tilde{\sigma}_{2i-1} + \tilde{\sigma}_{2i} \right). \quad (2.4.8)$$

As $\tilde{\sigma}_{2i-1}, \tilde{\sigma}_{2i}$ are lifts of σ_i , $\rho_{\#} \tilde{\sigma}_{2i-1} = \rho_{\#} \tilde{\sigma}_{2i} = \sigma_i$. Hence,

$$\partial \left(\sum_i \rho_{\#} C\tilde{\sigma}_{2i-1} + \rho_{\#} C\tilde{\sigma}_{2i} \right) = \rho_{\#} \left(\sum_i \tilde{\sigma}_{2i-1} + \tilde{\sigma}_{2i} \right) = 0 \in C_k(X, \mathbb{Z}_2). \quad (2.4.9)$$

Let us define

$$\alpha = \left[\sum_i \rho_{\#} C\tilde{\sigma}_{2i-1} + \rho_{\#} C\tilde{\sigma}_{2i} \right] \in H_{k+1}(Y, \mathbb{Z}_2). \quad (2.4.10)$$

We claim that $\lambda^{k+1} \cdot \alpha = 1$. Let $\lambda = [\ell]$; $\ell^{k+1} \in C^{k+1}(Y, \mathbb{Z}_2)$, i.e. $\ell^{k+1} : C_{k+1}(Y, \mathbb{Z}_2) \rightarrow \mathbb{Z}_2$. By the

definition of the cup product,

$$\begin{aligned}
& \ell^{k+1} \left(\rho_{\#} C \tilde{\sigma}_{2i-1} + \rho_{\#} C \tilde{\sigma}_{2i} \right) \\
&= \ell^k \left(\rho_{\#} C \tilde{\sigma}_{2i-1} \big|_{\Delta^k} \right) \ell \left(\rho_{\#} C \tilde{\sigma}_{2i-1} \big|_{[v_k, v_{k+1}]} \right) \\
&\quad + \ell^k \left(\rho_{\#} C \tilde{\sigma}_{2i} \big|_{\Delta^k} \right) \ell \left(\rho_{\#} C \tilde{\sigma}_{2i} \big|_{[v_k, v_{k+1}]} \right) \\
&= \ell^k(\sigma_i) \ell \left(\rho_{\#} C \tilde{\sigma}_{2i-1} \big|_{[v_k, v_{k+1}]} + \rho_{\#} C \tilde{\sigma}_{2i} \big|_{[v_k, v_{k+1}]} \right).
\end{aligned} \tag{2.4.11}$$

In the second equality we have used the fact that

$$C \tilde{\sigma}_{2i-1} \big|_{\Delta^k} = \tilde{\sigma}_{2i-1} \quad \text{and} \quad C \tilde{\sigma}_{2i} \big|_{\Delta^k} = \tilde{\sigma}_{2i}.$$

We note that

$$C \tilde{\sigma}_{2i-1} \big|_{[v_k, v_{k+1}]} \text{ is a curve joining } \tilde{\sigma}_{2i-1}(v_k) \text{ and } \tilde{\sigma}_{2i-1}(v_{k+1}) = y_0,$$

and

$$C \tilde{\sigma}_{2i} \big|_{[v_k, v_{k+1}]} \text{ is a curve joining } \tilde{\sigma}_{2i}(v_k) \text{ and } \tilde{\sigma}_{2i}(v_{k+1}) = y_0.$$

As $\tilde{\sigma}_{2i-1}, \tilde{\sigma}_{2i}$ are lifts of σ_i , $T(\tilde{\sigma}_{2i-1}(v_k)) = \tilde{\sigma}_{2i}(v_k)$. Hence,

$$\left(\rho_{\#} C \tilde{\sigma}_{2i-1} \big|_{[v_k, v_{k+1}]} + \rho_{\#} C \tilde{\sigma}_{2i} \big|_{[v_k, v_{k+1}]} \right)$$

is a non-contractible loop in Y . Therefore,

$$\ell \left(\rho_{\#} C \tilde{\sigma}_{2i-1} \big|_{[v_k, v_{k+1}]} + \rho_{\#} C \tilde{\sigma}_{2i} \big|_{[v_k, v_{k+1}]} \right) = 1 \tag{2.4.12}$$

Using (2.4.11) and (2.4.12), we conclude that

$$\lambda^{k+1} . \alpha = \sum_i \ell^{k+1} \left(\rho_{\#} C \tilde{\sigma}_{2i-1} + \rho_{\#} C \tilde{\sigma}_{2i} \right) = \sum_i \ell^k(\sigma_i) = (\iota^* \lambda)^k . \sigma = 1.$$

Part 4. Since Ψ is a $(k+1)$ -sweepout, using the argument as in Part 1, Ψ has a lift

$$\tilde{\Psi} : S\tilde{X} \rightarrow (\mathcal{C}(M), \mathcal{F}) \quad \text{such that} \quad \partial \circ \tilde{\Psi} = \Psi \circ \rho.$$

$\tilde{\Psi}$ is $\mathbb{Z}_2$ -equivariant, i.e.

$$\tilde{\Psi}(T(x), -t) = M - \tilde{\Psi}(x, t),$$

for all $(x, t) \in S\tilde{X}$. Moreover, by the definition of the $\mathbf{F}$ -metric on $\mathcal{C}(M)$ and $\mathcal{Z}_n(M; \mathbb{Z}_2)$, continuity of $\Psi : Y \rightarrow \mathcal{Z}_n(M; \mathbf{F}; \mathbb{Z}_2)$ implies that $\tilde{\Psi} : S\tilde{X} \rightarrow (\mathcal{C}(M), \mathbf{F})$ is also continuous. Let $\tilde{\Pi}$ be the $(S\tilde{X}, \tilde{Z})$ -homotopy class of $\tilde{\Psi}$. We will show that

$$\mathbf{L}^c(\tilde{\Pi}) > \sup_{y \in \tilde{Z}} \mathcal{A}^c(\tilde{\Psi}(y)). \quad (2.4.13)$$

The proof closely follows [Zho20, Proof of Lemma 5.8]. By (2.4.6), (2.4.4) and (2.4.2),

$$\begin{aligned} \sup_{y \in \tilde{Z}} \mathcal{A}^c(\tilde{\Psi}(y)) &\leq \sup_{y \in \tilde{Z}} \mathbf{M}(\partial \tilde{\Psi}(y)) \leq \sup_{y \in \tilde{Z}} \mathbf{M}(\partial \tilde{\Psi}'(y)) + \delta/2 \\ &= \sup_{x \in \tilde{X}} \mathbf{M}(\partial \tilde{\Phi}(x)) + \delta/2 \leq \omega_k + \delta. \end{aligned} \quad (2.4.14)$$

Let $\{\tilde{\Psi}_i : S\tilde{X} \rightarrow (\mathcal{C}(M), \mathbf{F})\}_{i=1}^\infty$ be an arbitrary element in $\tilde{\Pi}$. There exists $H_i : S\tilde{X} \times [0, 1] \rightarrow (\mathcal{C}(M), \mathbf{F})$ such that $H_i(-, 0) = \tilde{\Psi}$, $H_i(-, 1) = \tilde{\Psi}_i$ and

$$\limsup_{i \rightarrow \infty} \sup \left\{ \mathbf{F}(H_i(y, s), \tilde{\Psi}(y)) : y \in \tilde{Z}, s \in [0, 1] \right\} = 0. \quad (2.4.15)$$

We define $\{\tilde{\Psi}_i^* : C_+ \rightarrow (\mathcal{C}(M), \mathbf{F})\}_{i=1}^\infty$ as follows.

$$\tilde{\Psi}_i^*(y) = \begin{cases} \tilde{\Psi}(y) & \text{if } y \in \tilde{X}; \\ H_i(y, 2t) & \text{if } y = (x, t) \in \tilde{Z} \cap C_+; \\ \tilde{\Psi}_i(y) & \text{if } y \in C_+ \setminus \tilde{Z}. \end{cases} \quad (2.4.16)$$

$\rho|_{C_+} : C_+ \rightarrow Y$ is a quotient map. Since $\tilde{\Psi}$ is $\mathbb{Z}_2$ -equivariant, $\tilde{\Psi}_i^*$ descends to a map $\Psi_i^* : Y \rightarrow \mathcal{Z}_n(M; \mathbf{F}; \mathbb{Z}_2)$ such that $\Psi_i^* \circ (\rho|_{C_+}) = \partial \circ \tilde{\Psi}_i^*$. Moreover, Ψ_i^* is homotopic to Ψ (in the $\mathbf{F}$ -topology) and hence, is a $(k+1)$ -sweepout. Ψ_i^* has no concentration of mass as it is continuous in the $\mathbf{F}$ -topology. Therefore,

$$\sup_{y \in C_+} \mathbf{M}(\partial \tilde{\Psi}_i^*(y)) = \sup_{y \in Y} \mathbf{M}(\Psi_i^*(y)) \geq \omega_{k+1}. \quad (2.4.17)$$

However, as noted in (2.4.14),

$$\sup_{y \in \tilde{Z}} \mathbf{M}(\partial \tilde{\Psi}(y)) \leq \omega_k + \delta < \omega_{k+1} \text{ (using (2.4.1))};$$

hence, (2.4.15) and (2.4.16) imply that if i is sufficiently large,

$$\sup_{y \in \tilde{Z} \cap C_+} \mathbf{M}(\partial \tilde{\Psi}_i^*(y)) < \omega_{k+1} \quad (2.4.18)$$

as well. Since $\tilde{\Psi}_i^*$ coincides with $\tilde{\Psi}_i$ on $C_+ \setminus \tilde{Z}$, (2.4.17) and (2.4.18) imply that for i sufficiently large,

$$\begin{aligned} \sup_{y \in C_+ \setminus \tilde{Z}} \mathbf{M}(\partial \tilde{\Psi}_i(y)) \geq \omega_{k+1} &\implies \sup_{y \in S\tilde{X}} \mathbf{M}(\partial \tilde{\Psi}_i(y)) \geq \omega_{k+1} \\ &\implies \sup_{y \in S\tilde{X}} \mathcal{A}^c(\tilde{\Psi}_i(y)) \geq \omega_{k+1} - c \text{Vol}(g). \end{aligned} \quad (2.4.19)$$

Since (2.4.19) holds for any $\{\tilde{\Psi}_i\}_{i=1}^\infty \in \tilde{\Pi}$, using (2.4.1) and (2.4.14) we conclude that

$$\mathbf{L}^c(\tilde{\Pi}) \geq \omega_{k+1} - c \text{Vol}(g) > \omega_k + \delta \geq \sup_{y \in \tilde{Z}} \mathcal{A}^c(\tilde{\Psi}(y)). \quad (2.4.20)$$

Part 5. By Theorem 2.3.5 and (2.4.13), there exists $\Omega \in \mathcal{C}(M)$ such that $\partial\Omega$ is a closed c -CMC (with respect to the inward unit normal) hypersurface with optimal regularity and $\mathcal{A}^c(\Omega) = \mathbf{L}^c(\tilde{\Pi})$. (2.4.20) implies that

$$\mathcal{A}^c(\Omega) = \mathbf{L}^c(\tilde{\Pi}) > \omega_k + \delta > \omega_k. \quad (2.4.21)$$

Moreover,

$$\begin{aligned}
\mathcal{A}^c(\Omega) = \mathbf{L}^c(\tilde{\Pi}) &\leq \sup_{(x,t) \in S\tilde{X}} \mathcal{A}^c(\tilde{\Psi}(x,t)) \\
&\leq \sup_{(x,t) \in S\tilde{X}} \mathbf{M}(\partial\tilde{\Psi}(x,t)) \\
&\leq \sup_{(x,t) \in S\tilde{X}} \mathbf{M}(\partial\tilde{\Psi}'(x,t)) + \delta/2 \quad (\text{by (2.4.6)}) \\
&\leq \sup_{x \in \tilde{X}} \mathbf{M}(\partial\tilde{\Phi}(x)) + \sup_{t \in [0,1]} \mathbf{M}(\partial\Lambda(t)) + \delta/2 \\
&\quad (\text{by (2.4.4) and Proposition 2.4.1}) \\
&\leq \omega_k + \delta/2 + W_0 + \eta/2 - \delta + \delta/2 \quad (\text{by (2.4.2) and (2.4.3)}) \\
&< \omega_k + W_0 + \eta.
\end{aligned}$$

This finishes the proof of Theorem 2.1.1. □

Proof of Theorem 2.1.2. We start with the following lemma.

Lemma 2.4.4. *Let $\Omega_1, \Omega_2 \in \mathcal{C}(M)$ and $c > 0$ such that for $i = 1, 2$, $\partial\Omega_i$ is a closed c -CMC (with respect to the inward unit normal) hypersurface with optimal regularity and $\mathcal{A}^c(\Omega_1) \neq \mathcal{A}^c(\Omega_2)$. Then $\partial\Omega_1 \neq \partial\Omega_2$.*

Proof. Suppose $\Omega_1 = \Omega_2 = \Sigma$. Since $\mathcal{A}^c(\Omega_1) \neq \mathcal{A}^c(\Omega_2)$, $\Omega_1 \neq \Omega_2$. Hence, by the constancy theorem, $\Omega_2 = M \setminus \Omega_1$; this is impossible as by our hypothesis, the mean curvature vector of Σ points inside both Ω_1 and Ω_2 . □

Since $\{\omega_p\}_{p=1}^\infty$ is a non-decreasing sequence of positive real numbers converging to $+\infty$, for every $m \in \mathbb{N}$, there exists a subsequence $\{\omega_{p_i}\}_{i=1}^m$ such that

$$\omega_{p_i} < \omega_{p_i+1} \quad \text{and} \quad \omega_{p_i} + W_0 < \omega_{p_i+1},$$

for all $i = 1, \dots, m$. Let us define

$$c^*(m) = \frac{1}{\text{Vol}(g)} \min \{\omega_{p_{i+1}} - \omega_{p_i} : i = 1, \dots, m\}. \quad (2.4.22)$$

If $0 < c < c^*(m)$, applying Theorem 2.1.1 for $k = p_i$, $i = 1, \dots, m$, we conclude that there exists $\Omega_i \in \mathcal{C}(M)$, such that $\partial\Omega_i$ is a closed c -CMC (with respect to the inward unit normal) hypersurface with optimal regularity and

$$\omega_{p_i} < \mathcal{A}^c(\Omega_i) < \omega_{p_{i+1}}.$$

Therefore, using Lemma 2.4.4, $\{\partial\Omega_i\}_{i=1}^m$ are distinct c -CMC hypersurfaces.

To prove the more quantitative statement, we will use the growth estimate of the volume spectrum (2.2.1). From the proof of Theorem 2.1.1, one can obtain the following inequality.

$$\omega_{k+1} \leq \omega_k + W_0 \quad \forall k \in \mathbb{N}. \quad (2.4.23)$$

Let us briefly repeat the argument here. We fix $\delta > 0$. There exists a k -sweepout $\Phi : X \rightarrow \mathcal{Z}_n(M; \mathcal{F}; \mathbb{Z}_2)$ with no concentration of mass such that $\sup \{\mathbf{M}(\Phi(x)) : x \in X\} \leq \omega_k + \delta$. Using the notation from the Proof of Theorem 2.1.1, Part 1, there exists a double cover $\pi : \tilde{X} \rightarrow X$ such that Φ has a lift $\tilde{\Phi} : \tilde{X} \rightarrow (\mathcal{C}(M), \mathcal{F})$. Let $\Lambda : [0, 1] \rightarrow (\mathcal{C}(M), \mathcal{F})$ be such that $\Lambda(0) = M$, $\Lambda(1) = \emptyset$, $\partial \circ \Lambda : [0, 1] \rightarrow \mathcal{Z}_n(M; \mathcal{F}; \mathbb{Z}_2)$ has no concentration of mass and $\sup \{\mathbf{M}(\partial\Lambda(t)) : t \in [0, 1]\} \leq W_0 + \delta$. We define the map $\tilde{\Psi} : S\tilde{X} \rightarrow (\mathcal{C}(M), \mathcal{F})$ as follows (cf. (2.4.4)).

$$\tilde{\Psi}(x, t) = \begin{cases} \tilde{\Phi}(x) \cap \Lambda(t) & \text{if } (x, t) \in C_+; \\ M - (\tilde{\Phi}(T(x)) \cap \Lambda(-t)) & \text{if } (x, t) \in C_-. \end{cases}$$

$\tilde{\Psi}$ descends to a map $\Psi : Y \rightarrow \mathcal{Z}_n(M; \mathcal{F}; \mathbb{Z}_2)$. As proved in Part 2 (using Proposition 2.4.1), Ψ has

no concentration of mass. Moreover, as argued in Part 3, Ψ is a $(k+1)$ -sweepout; hence,

$$\sup \{\mathbf{M}(\Psi(y)) : y \in Y\} \geq \omega_{k+1}. \quad (2.4.24)$$

Further, as estimated in Part 5 (using Proposition 2.4.1),

$$\sup \{\mathbf{M}(\Psi(y)) : y \in Y\} \leq \omega_k + W_0 + 2\delta. \quad (2.4.25)$$

Since $\delta > 0$ is arbitrary, (2.4.24) and (2.4.25) imply (2.4.23).

As a consequence of (2.4.23), for all $N \geq \omega_1$, there exist $q, r \in \mathbb{N}$ such that $\omega_q \in [N, N + W_0]$ and $\omega_r \in [N + 2W_0, N + 3W_0]$. Using (2.2.1),

$$K_0 r^{\frac{1}{n+1}} \leq \omega_r < N + 3W_0 \implies r < \left(\frac{N + 3W_0}{K_0} \right)^{n+1}.$$

Hence, there exists $s \in \mathbb{N}$, $q \leq s \leq r$, such that

$$\omega_{s+1} - \omega_s \geq \frac{\omega_r - \omega_q}{r - q} > W_0 \left(\frac{N + 3W_0}{K_0} \right)^{-(n+1)}.$$

Therefore, by Theorem 2.1.1, if

$$0 < c \operatorname{Vol}(g) \leq W_0 \left(\frac{N + 3W_0}{K_0} \right)^{-(n+1)},$$

there exists $\Omega \in \mathcal{C}(M)$, such that $\partial\Omega$ is a closed c -CMC (with respect to the inward unit normal) hypersurface with optimal regularity and

$$N \leq \omega_s < \mathcal{A}^c(\Omega) < \omega_s + W_0 + (N + 3W_0 - \omega_s) = N + 4W_0.$$

We note that $W_0 \geq \omega_1$. Setting $N = (4i - 3)W_0$, $i = 1, \dots, m$ in the above argument, we conclude that if

$$0 < c \operatorname{Vol}(g) \leq W_0 \left(\frac{4iW_0}{K_0} \right)^{-(n+1)},$$

there exists $\Omega_i \in \mathcal{C}(M)$, such that $\partial\Omega_i$ is a closed c -CMC (with respect to the inward unit normal) hypersurface with optimal regularity and

$$(4i - 3)W_0 < \mathcal{A}^c(\Omega_i) < (4i + 1)W_0.$$

Further, from the one parameter min-max construction of Zhou and Zhu [ZZ19], for any $c > 0$ there exists $\Omega_0 \in \mathcal{C}(M)$ such that $\partial\Omega_0$ is a closed c -CMC (with respect to the inward unit normal) hypersurface with optimal regularity and $\mathcal{A}^c(\Omega_0) \leq W_0$. Therefore, if

$$0 < c \operatorname{Vol}(g) \leq W_0 \left(\frac{4mW_0}{K_0} \right)^{-(n+1)}, \quad (2.4.26)$$

by Lemma 2.4.4, there exist at least $(m + 1)$ many closed c -CMC hypersurfaces in (M, g) . This implies the number of closed c -CMC hypersurfaces in (M, g) is at least

$$\left\lfloor \frac{K_0}{4W_0} \left(\frac{W_0}{c \operatorname{Vol}(g)} \right)^{\frac{1}{n+1}} \right\rfloor + 1,$$

where for $x \in \mathbb{R}^+$, $\lfloor x \rfloor$ is the largest (non-negative) integer $\leq x$ and K_0 is the constant appearing in (2.2.1).

□

Remark 2.4.5. For $m \in \mathbb{N}$, the value of $c^*(m)$, obtained from (2.4.22) for an optimal subsequence $\{\omega_{p_i}\}_{i=1}^m$, will be greater than or equal to the value, obtained from the asymptotic formula (2.4.26).

Example 2.4.6. Let us consider S^3 equipped with the round metric g_0 . By the one parameter min-max construction of Zhou and Zhu [ZZ19], for any $c \in \mathbb{R}^+$ there exists $\Omega \in \mathcal{C}(S^3)$, such that $\partial\Omega$ is a closed c -CMC (with respect to the inward unit normal) hypersurface and $\mathcal{A}^c(\Omega) \leq W_0(S^3, g_0) = 4\pi$. As proved by Nurser [Nur16], for the metric g_0 , $\omega_1 = \dots = \omega_4 = 4\pi$ and $\omega_5 = 2\pi^2$. Therefore, by Theorem 2.1.1, if

$$0 < c < \frac{\omega_5 - \omega_4}{\operatorname{Vol}(g_0)} = \frac{2\pi^2 - 4\pi}{2\pi^2} = 1 - \frac{2}{\pi} \approx \frac{1}{3},$$

there exists $\Omega' \in \mathcal{C}(S^3)$ such that $\partial\Omega'$ is a closed c -CMC (with respect to the inward unit normal)

hypersurface and $\mathcal{A}^c(\Omega') > \omega_4(S^3, g_0) = 4\pi$. Thus, for the metric g_0 (with the notation as in Theorem 2.1.2), $c^*(2)$ can be taken to be $1 - 2/\pi$. Moreover, the widths are continuous functions of the metric [IMN18, MNS19] and W_0 coincides with ω_1 when the ambient metric has positive Ricci curvature [Zho15, Zho17]. Therefore, if $\tilde{g}$ is a small perturbation of g_0 , $c^*(2)$ for $\tilde{g}$ can be taken to be close to $1 - 2/\pi$.

Chapter 3

Compactness of certain class of singular minimal hypersurfaces

3.1 Introduction

The minimal hypersurfaces, which are constructed by the Almgren-Pitts min-max theory, have bounded area and index. More precisely, suppose X is a finite simplicial complex, M is a closed Riemannian manifold and Π is a homotopy class of maps $X \rightarrow \mathcal{Z}_n(M^{n+1}, \mathbb{Z}_2)$. Let $\mathbf{M}$ denote the mass norm on $\mathcal{Z}_n(M, \mathbb{Z}_2)$. The *min-max width* of Π is defined as

$$\mathbf{L}(\Pi) = \inf_{\Phi \in \Pi} \sup\{\mathbf{M}(\Phi(x)) : x \in X\}. \quad (3.1.1)$$

Then, by the works of Almgren [Alm65], Pitts [Pit81], Schoen-Simon [SS81] and Marques-Neves [MN14, MN17], there exists a disjoint collection $\{\Sigma_i\}_{i=1}^l$ of closed minimal hypersurfaces (with optimal regularity) in M and a collection of positive integers $\{m_i\}_{i=1}^l$ such that

$$\mathbf{L}(\Pi) = \sum_{i=1}^l m_i \mathcal{H}^n(\Sigma_i). \quad (3.1.2)$$

Moreover, by the works of Marques-Neves [MN16] and Li [Li20],

$$\sum_{i=1}^l \text{Ind}(\Sigma_i) \leq \dim X. \quad (3.1.3)$$

In [Sha17], Sharp proved certain compactness properties of the set of smooth, closed, embedded minimal hypersurfaces inside a closed Riemannian manifold (M^{n+1}, g) , $3 \leq n+1 \leq 7$, with uniformly bounded area and index. This theorem was later generalized by Ambrozio, Carlotto and Sharp in [ACS16], where instead of bounded area and index, an upper bound of the area and a lower bound of the p -th Jacobi eigenvalue λ_p (for some $p \in \mathbb{N}$) were assumed. (We note that for a smooth, closed minimal hypersurface Σ , $\text{Ind}(\Sigma) \leq I$ is equivalent to $\lambda_{I+1}(\Sigma) \geq 0$.) Sharp's compactness theorem plays an important role in many recent developments in the Almgren-Pitts min-max theory, for instance in [MN17, IMN18, MNS19, Son18, MN16, MN21, Zho20, SZ21].

In this chapter we will suitably generalize the results of [Sha17] and [ACS16] in higher dimensions; for that, we need to consider the minimal hypersurfaces which may have singularities. We will state the notion of the index and the p -th Jacobi eigenvalue for a stationary n -varifold and prove the following theorem.

Theorem 3.1.1. *Let $\{M_k\}_{k=1}^\infty$ be a sequence of closed, connected, singular, minimal hypersurfaces in a closed Riemannian manifold (N^{n+1}, g) , $n+1 \geq 3$. Let $V_k = |M_k|$, the varifold associated to M_k . Suppose there exist $\Lambda > 0$, $\alpha \geq 0$, $p \in \mathbb{N}$ such that for all k ,*

- $\mathcal{H}^{n-2}(\text{sing}(M_k)) = 0$;
- $\mathcal{H}^n(M_k) = \|V_k\|(N) \leq \Lambda$;
- $\lambda_p(V_k) \geq -\alpha$.

Then there exists a stationary, integral varifold V such that, possibly after passing to a subsequence, $V_k \rightarrow V$ in the $\mathbf{F}$ metric. Moreover, denoting $M = \text{spt}(V)$, we have

- $V = m|M|$ for some $m \in \mathbb{N}$;
- $\|V\|(N) = m\mathcal{H}^n(M) \leq \Lambda$;

- $\lambda_p(V) \geq -\alpha$ or equivalently $\lambda_p(|M|) \geq -\alpha$;
- $\mathcal{H}^s(\text{sing}(M)) = 0 \quad \forall s > n - 7$;
- the convergence is smooth and graphical over the compact subsets of $\text{reg}(M) \setminus \mathcal{Y}$, where $\mathcal{Y}$ is a finite subset of $\text{reg}(M)$ with $|\mathcal{Y}| \leq p - 1$.

From the definitions of the index and the Jacobi eigenvalue, it will be clear that $\text{Ind}(V) \leq I$ if and only if $\lambda_{I+1}(V) \geq 0$. Therefore, Theorem 3.1.1 has the following theorem as a corollary, which generalizes Sharp's compactness theorem [Sha17] in higher dimensions.

Theorem 3.1.2. *Let $\{M_k\}_{k=1}^\infty$ be a sequence of closed, connected, singular, minimal hypersurfaces in a closed Riemannian manifold (N^{n+1}, g) , $n + 1 \geq 3$. Suppose for all k , $\mathcal{H}^{n-2}(\text{sing}(M_k)) = 0$, $\mathcal{H}^n(M_k) \leq \Lambda$ and $\text{Ind}(|M_k|) \leq I$. Then there is a closed, connected, singular, minimal hypersurface M in N and $m \in \mathbb{N}$ such that possibly after passing to a subsequence, $|M_k| \rightarrow m|M|$ in the $\mathbf{F}$ metric. Moreover, $m\mathcal{H}^n(M) \leq \Lambda$, $\text{Ind}(|M|) \leq I$, $\mathcal{H}^s(\text{sing}(M)) = 0 \quad \forall s > n - 7$ and the convergence is smooth and graphical over the compact subsets of $\text{reg}(M) \setminus \mathcal{Y}$, where $\mathcal{Y}$ is a finite subset of $\text{reg}(M)$ with $|\mathcal{Y}| \leq I$.*

If we take M_k to be M for all k in Theorem 3.1.1, we get the following regularity result.

Proposition 3.1.3. *Let M^n be a singular, minimal hypersurface in (N^{n+1}, g) , $n + 1 \geq 3$. Suppose, $\mathcal{H}^{n-2}(\text{sing}(M)) = 0$ and $\lambda_p(|M|) > -\infty$ for some p . Then $\mathcal{H}^s(\text{sing}(M)) = 0 \quad \forall s > n - 7$.*

The proof of Theorem 3.1.1 is similar in spirit to that of [Sha17] and [ACS16]; however there are some technical difficulties that arise in higher dimensions. For instance the definition of $\lambda_p(\cdot) \geq -\alpha$ and the regularity theory of Schoen-Simon [SS81] must be suitably extended before we may apply it to our setting. These are accomplished in Sections 3.3 and 3.4 respectively.

3.2 Notations and Preliminaries

3.2.1 Notations

Here we summarize the notations which will be frequently used later in this chapter.

$\mathcal{V}_n(U)$	the set of n -varifolds in U
$\mathcal{IV}_n(U)$	the set of integral n -varifolds in U
$\mathcal{H}^s$	the Hausdorff measure of dimension s
$\ V\ $	the Radon measure associated to the varifold V
$ \Sigma $	the varifold associated to a singular hypersurface Σ
$\delta^2 V$	2nd variation of the stationary varifold V
$\text{Ind}(\cdot)$	index (of a stationary hypersurface or varifold)
$\lambda_k(\cdot)$	k -th Jacobi eigenvalue (of a stationary hypersurface or varifold)
$\text{sing}(\Sigma)$	the singular part of Σ
$\text{reg}(\Sigma)$	the regular part of Σ
$B(p, r)$	the open ball of radius r , centered at p

3.2.2 Preliminaries from geometric measure theory

Here we will briefly discuss the notion of varifold and various related concepts; further details can be found in Simon's book [Sim83].

Given a Riemannian manifold (U^{n+1}, g) , let $G_k(U)$ denote the Grassmannian bundle of k -dimensional hyperplanes over U . A k -varifold in U is a positive Radon measure on $G_k(U)$. The topology on the space of k -varifolds $\mathcal{V}_k(U)$ is given by the weak* topology, i.e. a net $\{V_i\}_{i \in I} \subset \mathcal{V}_k(U)$ converges to V iff

$$\int_{G_k(U)} f(x, \omega) dV_i(x, \omega) \rightarrow \int_{G_k(U)} f(x, \omega) dV(x, \omega)$$

for all $f \in C_c(G_k(U))$. This topology is metrizable and the metric is denoted by $\mathbf{F}$. If $V \in \mathcal{V}_k(U)$ and $\pi : G_k(U) \rightarrow U$ denotes the canonical projection, then $\|V\| = \pi_* V$ is a Radon measure on U ; $\|V\|(A) = V(\pi^{-1}(A))$.

If $\varphi : U \rightarrow U'$ is a diffeomorphism and $V \in \mathcal{V}_k(U)$, we define $\varphi_* V \in \mathcal{V}_k(U')$ by the following formula.

$$(\varphi_* V)(g) = \int_{G_k(U)} g(\varphi(x), d_x \varphi(\omega)) J\varphi(x, \omega) dV(x, \omega),$$

where

$$J\varphi(x, \omega) = \left(\det \left((d_x \varphi|_\omega)^t \circ (d_x \varphi|_\omega) \right) \right)^{1/2}$$

is the Jacobian factor and $g \in C_c(G_k(U'))$. Given a compactly supported, smooth vector-field X on U let φ_t denote the flow of X ; the *first variation* and *second variation* of V are given by

$$\delta V(X) = \frac{d}{dt} \Big|_0 \|(\varphi_t)_* V\|(U); \quad \delta^2 V(X, X) = \frac{d^2}{dt^2} \Big|_0 \|(\varphi_t)_* V\|(U).$$

We say that V is *stationary* if $\delta V(X) = 0$ for all X and a stationary varifold V is called *stable* if $\delta^2 V(X, X) \geq 0$ for all X .

Given a k -rectifiable set $S \subset U$ and a non-negative function $\theta \in L^1_{loc}(S, \mathcal{H}^k)$, we define the k -varifold $\mathbf{v}(S, \theta)$ by

$$\mathbf{v}(S, \theta)(f) = \int_S f(x, T_x S) \theta(x) d\mathcal{H}^k(x),$$

where $T_x S$ denotes the approximate tangent space of S at x , which exists $\mathcal{H}^k$ -a.e. V is called an *integral k -varifold* if $V = \mathbf{v}(S, \theta)$ for some S and θ with θ taking non-negative integer values $\mathcal{H}^k$ -a.e.

In this thesis, we will only deal with n -varifolds and from now on we will simply write ‘varifold’ instead of ‘ n -varifold’. Given $A \subset U$, we define the regular and singular part of A

$$reg(A) = \{x \in A : \exists \text{ open } P \subset U \text{ containing } x \text{ such that}$$

$$P \cap A \text{ is a smooth, embedded hypersurface}\}$$

and

$$sing(A) = A \setminus reg(A).$$

Further, by a singular, minimal hypersurface Σ in U we will mean that $\Sigma \subset U$ is a closed, n -rectifiable set with $\mathcal{H}^{n-2}(sing(\Sigma)) = 0$ and $|\Sigma| := \mathbf{v}(\Sigma, \mathbf{1}_\Sigma)$ (where $\mathbf{1}_\Sigma$ is the constant function 1) is stationary. By [Ilm96, equation (4)], $|\Sigma|$ is stationary in U if and only if $reg(\Sigma)$ is a smooth,

minimal hypersurface and $\mathcal{H}^n(\Sigma \cap B(x, r)) \leq C(U')r^n$ for all $B(x, r) \subseteq U' \subset \subset U$.

3.3 Index and Jacobi eigenvalues of a stationary varifold

We will now state the notion of the index and the Jacobi eigenvalues of a stationary varifold, following the paper by Marques and Neves [MN16]. Suppose $\Sigma^n \subset (U^{n+1}, g)$ is a smooth, minimal hypersurface (not necessarily closed). For $\Omega \subset \subset \Sigma$, let $\lambda_k(\Omega)$ denote the k -th Jacobi eigenvalue of Ω (with Dirichlet boundary condition) and $\text{Ind}(\Omega) = \max\{k : \lambda_k(\Omega) < 0\}$. We define

$$\lambda_k(\Sigma) = \inf\{\lambda_k(\Omega) : \Omega \subset \subset \Sigma\} \quad \text{and} \quad \text{Ind}(\Sigma) = \sup\{\text{Ind}(\Omega) : \Omega \subset \subset \Sigma\}.$$

Then $\lambda_k(\Sigma)$ has the following min-max characterization.

$$\lambda_k(\Sigma) = \inf_{\dim(V)=k} \sup_{X \in V \setminus \{0\}} \left(\frac{\delta^2 \Sigma(X, X)}{\int_{\Sigma} |X|^2 d\mathcal{H}^n} \right).$$

The infimum is over all the k -dimensional linear subspaces $V \subset \Gamma_c(N\Sigma)$ = compactly supported smooth sections of the normal bundle $N\Sigma$. Therefore, $\lambda_k(\Sigma) < -a$ iff there is a k -dimensional subspace $V \subset \Gamma_c(N\Sigma)$ such that for all $X \in V \setminus \{0\}$,

$$\delta^2 \Sigma(X, X) < -a \int_{\Sigma} |X|^2 d\mathcal{H}^n.$$

Given a Riemannian manifold (U^{n+1}, g) and $k \in \mathbb{N}$ ($0 \notin \mathbb{N}$), a k -parameter family of diffeomorphisms is a smooth map $F : \overline{B}^k(0, 1) \subset \mathbb{R}^k \rightarrow \text{Diff}(U)$ such that

- $F_v (= F(v)) = (F_{-v})^{-1} \quad \forall v \in \overline{B}^k(0, 1)$ and $F_0 = \text{Id}$;
- there exists open set $U' \subset \subset U$ such that $F_v|_{U \setminus U'} = \text{Id} \quad \forall v \in \overline{B}^k(0, 1)$.

If F is a k -parameter family of diffeomorphisms, we define the vector-fields Y_i , $i = 1, \dots, k$, by

$$Y_i|_p = \frac{d}{dt} \bigg|_0 F_{te_i}(p). \tag{3.3.1}$$

Suppose further, we have a stationary varifold V in U . Then we define a smooth function A^V and a quadratic form K^V as follows.

$$A^V : \overline{B}^k(0, 1) \rightarrow [0, \infty), \quad A^V(v) = \|(F_v)_\# V\|(U); \quad (3.3.2)$$

$$K^V : \mathbb{R}^k \times \mathbb{R}^k \rightarrow \mathbb{R}, \quad K^V(u, u) = \left\| \sum_i u_i Y_i \right\|_{L^2(U, \|V\|)}^2. \quad (3.3.3)$$

Remark 3.3.1. If $V_i \rightarrow V$ in the $\mathbf{F}$ metric, then $A^{V_i} \rightarrow A^V$ ([Pit81, Section 2.3]) in the smooth topology and $K^{V_i} \rightarrow K^V$ uniformly on compact subsets.

Definition 3.3.2. Given a stationary varifold V in U , $k \in \mathbb{N}$ and $\alpha \geq 0$, we say that $\lambda_k(V) < -\alpha$ if there exists a k -parameter family of diffeomorphisms F such that

$$D^2 A^V|_0(u, u) < -\alpha K^V(u, u)$$

(where A^V and K^V are as in (3.3.2) and (3.3.3)) for all $u \in \mathbb{R}^k \setminus \{0\}$ or equivalently for all $u \in \mathbb{R}^k$ with $\|u\| = 1$. Further, by $\lambda_k(V) \geq -\alpha$ we will mean that $\lambda_k(V) < -\alpha$ does not hold. If $\lambda_k(V) < -\alpha$, then restricting F to $\overline{B}^{k-1}(0, 1) \subset \overline{B}^k(0, 1)$ we get that $\lambda_{k-1}(V) < -\alpha$ as well. Therefore, it will be natural to define

$$\text{Ind}(V) = \begin{cases} 0 & \text{if } \{I \in \mathbb{N} : \lambda_I(V) < 0\} = \emptyset, \\ \sup\{I \in \mathbb{N} : \lambda_I(V) < 0\} & \text{otherwise.} \end{cases}$$

Hence, $\text{Ind}(V) \leq I$ is equivalent to $\lambda_{I+1}(V) \geq 0$. Further, $\text{Ind}(V) = 0$ iff $\lambda_1(V) \geq 0$ iff V is stable.

Remark 3.3.3. By Remark 3.3.1 and from the above Definition 3.3.2, it follows that whenever $\lambda_k(V) < -\alpha$ and $\mathbf{F}(V, V')$ is sufficiently small, we have $\lambda_k(V') < -\alpha$ as well.

Remark 3.3.4. Suppose V is a stationary varifold and $V' = \theta V$ for some $\theta \in \mathbb{R}^+$. Then, from Definition 3.3.2 it follows that $\lambda_k(V) < -\alpha$ iff $\lambda_k(V') < -\alpha$; hence, $\text{Ind}(V) = \text{Ind}(V')$.

Proposition 3.3.5. *Given $\Lambda > 0$, $k \in \mathbb{N}$ and $\alpha \geq 0$, the following sets are compact with respect to*

the $\mathbf{F}$ metric topology.

$$\mathcal{M}_U(\Lambda, k, \alpha) = \{V \in \mathcal{V}_n(U) : V \text{ is stationary, } \|V\| \leq \Lambda \text{ and } \lambda_k(V) \geq -\alpha\};$$

$$\mathcal{M}'_U(\Lambda, k, \alpha) = \{V \in \mathcal{IV}_n(U) : V \text{ is stationary, } \|V\| \leq \Lambda \text{ and } \lambda_k(V) \geq -\alpha\}.$$

Proof. By the standard compactness theorems, if $\{V_i\}_{i=1}^\infty$ is a sequence of stationary varifolds with $\|V_i\| \leq \Lambda$ then up to a subsequence V_i converges to a stationary varifold V in the $\mathbf{F}$ metric with $\|V\| \leq \Lambda$. Further, by Allard's theorem [All72] if V_i 's are integral varifolds, V is also an integral varifold. Moreover, by Remark 3.3.3, if $\lambda_k(V) < -\alpha$, then for i large $\lambda_k(V_i) < -\alpha$ as well. Hence, $\lambda_k(V) \geq -\alpha$. $\square$

Theorem 3.3.6. *Let Σ be a singular, minimal hypersurface in U , $V = |\Sigma|$ and $\mathcal{H}^{n-2}(\text{sing}(\Sigma)) = 0$. Then, $\lambda_k(V) < -\alpha \leq 0 \iff \lambda_k(\text{reg}(\Sigma)) < -\alpha$. Hence, $\text{Ind}(V) = \text{Ind}(\text{reg}(\Sigma))$; therefore, V is stable iff $\text{reg}(\Sigma)$ is stable.*

Proof. If F is a k -parameter family of diffeomorphisms and Y_i, A^V are as in (3.3.1), (3.3.2), we note that

$$\begin{aligned} D^2 A^V|_0(u, u) &= \frac{d^2}{dt^2} \Big|_0 A^V(tu, tu) = \frac{d^2}{dt^2} \Big|_0 \|(F_{tu})^\# V\|(U) \\ &= \delta^2 V \left(\sum_i u_i Y_i, \sum_i u_i Y_i \right). \end{aligned} \quad (3.3.4)$$

Let $\lambda_k(\text{reg}(\Sigma)) < -\alpha$. Then there are k linearly independent, compactly supported normal vector-fields on $\text{reg}(\Sigma)$, say $X_1, X_2, \dots, X_k$, such that

$$\delta^2 V(X, X) < -\alpha \|X\|_{L^2(\Sigma)}^2 \quad (3.3.5)$$

for any non-zero vector-field X in the span of $\{X_i\}_{i=1}^k$. We extend each X_i to a compactly supported, globally defined vector-field on U and continue to call it by X_i ; we define $F_v = \Phi^{\sum_i v_i X_i}$, where Φ^X denotes the time 1 flow of the vector-field X . Let us check that this choice of F indeed works.

Clearly, $F_{-v} = F_v^{-1}$ and $F_0 = \text{Id}$. By (3.3.4) and (3.3.5),

$$D^2 A^V|_0(u, u) = \delta^2 V\left(\sum_i u_i X_i, \sum_i u_i X_i\right) < -\alpha \left\| \sum_i u_i X_i \right\|_{L^2(\Sigma)}^2$$

Hence, $\lambda_k(V) < -\alpha$.

For the converse, we consider the following.

$$\delta^2 V(X, X) = \int_{\Sigma} \left((\text{div}_{\Sigma} X)^2 + \sum_{i=1}^n |(\nabla_{\tau_i} X)^{\perp}|^2 - \sum_{i,j=1}^n \langle \tau_i, \nabla_{\tau_j} X \rangle \langle \tau_j, \nabla_{\tau_i} X \rangle \right) d\mathcal{H}^n,$$

where $\{\tau_i\}_{i=1}^n$ is an orthonormal basis of $T_x \Sigma$. Since $\mathcal{H}^{n-2}(\text{sing}(\Sigma)) = 0$, given any $\delta > 0$ and $0 < \kappa < 1$, we can choose balls $\{B(x_i, r_i)\}_{i=1}^K$ such that each $r_i < \kappa$, $\text{sing}(\Sigma) \subset \cup_i B(x_i, r_i)$ and $\sum_i r_i^{n-2} < \delta$. Therefore, $\sum_i r_i^{n-1} < \delta$ and $\sum_i r_i^n < \delta$ as well. We choose smooth cut-off functions $0 \leq \zeta_i \leq 1$ on U such that

$$\zeta_i = \begin{cases} 0 & \text{on } B(x_i, r_i) \\ 1 & \text{outside } B(x_i, 2r_i) \end{cases}$$

and $|\nabla \zeta_i| \leq 2/r_i$ (this can be ensured by choosing κ sufficiently small). Let $\zeta_{\delta} = \min_i \zeta_i$. From the second variation formula, we see that ¹

$$\begin{aligned} \left| \delta^2 V(X, X) - \delta^2 V(\zeta_{\delta} X, \zeta_{\delta} X) \right| &\leq \int_{\Sigma} \left((1 - \zeta_{\delta}^2) + |\nabla \zeta_{\delta}| + |\nabla \zeta_{\delta}|^2 \right) f(X, \nabla X) d\mathcal{H}^n \\ &\leq \sum_i \int_{\Sigma \cap B(x_i, 2r_i)} \left(1 + \frac{2}{r_i} + \frac{4}{r_i^2} \right) f(X, \nabla X) d\mathcal{H}^n, \end{aligned} \quad (3.3.6)$$

where f is an expression involving X and ∇X . By the monotonicity formula, the R.H.S. of this equation is bounded by $C\delta$ for some constant C , depending only on (U, g) , n , $\|V\|(U)$ and $\|X\|_{C^1}$.

We assume that $\lambda_k(V) < -\alpha$. By Definition 3.3.2 and equation (3.3.4), for all $u \in \mathbb{R}^k$ with $\|u\| = 1$,

$$\delta^2 V\left(\sum_i u_i Y_i, \sum_i u_i Y_i\right) < -\alpha \left\| \sum_i u_i Y_i \right\|_{L^2(\Sigma)}^2 \implies \sum_i u_i Y_i \neq 0 \quad (3.3.7)$$

¹Though ζ_{δ} is only a Lipschitz continuous function, its use in the subsequent calculations can be justified by an approximation argument.

Therefore,

$$\sup_{\|u\|=1} \frac{\delta^2 V(\sum_i u_i Y_i, \sum_i u_i Y_i)}{\|\sum_i u_i Y_i\|_{L^2(\Sigma)}^2} \leq -\alpha - 2\varepsilon \quad (3.3.8)$$

for some $\varepsilon > 0$.

For $p \in \text{reg}(\Sigma)$, let $Y_i^\perp(p)$ denote the normal component (with respect to $\text{reg}(\Sigma)$) of $Y_i(p)$. We note that $\zeta_\delta Y_i^\perp$ is a compactly supported normal vector field on $\text{reg}(\Sigma)$. For $u \in \mathbb{R}^k$ with $\|u\| = 1$, we have

$$\begin{aligned} \delta^2 V\left(\sum_i u_i \zeta_\delta Y_i^\perp, \sum_i u_i \zeta_\delta Y_i^\perp\right) &= \delta^2 V\left(\sum_i u_i \zeta_\delta Y_i, \sum_i u_i \zeta_\delta Y_i\right) \\ &\leq \delta^2 V\left(\sum_i u_i Y_i, \sum_i u_i Y_i\right) + C\delta. \end{aligned} \quad (3.3.9)$$

Here C depends only on $(U, g), n, \|V\|(U)$ and $\|Y_i\|_{C^1}$ and not on u .

Hence, using (3.3.9), for all $\|u\| = 1$ and δ sufficiently small,

$$\begin{aligned} &\delta^2 V\left(\sum_i u_i \zeta_\delta Y_i^\perp, \sum_i u_i \zeta_\delta Y_i^\perp\right) \\ &\leq \delta^2 V\left(\sum_i u_i Y_i, \sum_i u_i Y_i\right) + C\delta \\ &\leq (-\alpha - 2\varepsilon) \int_\Sigma \left|\sum_i u_i Y_i\right|^2 d\mathcal{H}^n + C\delta \\ &\leq (-\alpha - 2\varepsilon) \int_\Sigma \left|\sum_i u_i \zeta_\delta Y_i\right|^2 d\mathcal{H}^n + C\delta \\ &< (-\alpha - \varepsilon) \int_\Sigma \left|\sum_i u_i \zeta_\delta Y_i\right|^2 d\mathcal{H}^n \end{aligned} \quad (3.3.10)$$

$$\leq (-\alpha - \varepsilon) \left\| \sum_i u_i \zeta_\delta Y_i^\perp \right\|_{L^2(\Sigma)}^2. \quad (3.3.11)$$

To justify (3.3.10), we observe:

$$\begin{aligned}
& \int_{\Sigma} \left| \sum_i u_i \zeta_{\delta} Y_i \right|^2 d\mathcal{H}^n = \int_{\Sigma} \left| \sum_i u_i Y_i \right|^2 d\mathcal{H}^n + \int_{\Sigma} (\zeta_{\delta}^2 - 1) \left| \sum_i u_i Y_i \right|^2 d\mathcal{H}^n \\
& \geq \inf_{\|u\|=1} \int_{\Sigma} \left| \sum_i u_i Y_i \right|^2 d\mathcal{H}^n - C \sum_i r_i^n \quad (\text{using monotonicity formula}) \\
& \geq \theta - C\delta \quad (\text{for some } \theta > 0 \text{ by (3.3.7) and } C \text{ is independent of } u).
\end{aligned}$$

Since there is strict “<” in (3.3.10), (3.3.11) implies that $\{\zeta_{\delta} Y_i^{\perp}\}_{i=1}^k$ are linearly independent normal vector fields on $\text{reg}(\Sigma)$ and $\lambda_k(\text{reg}(\Sigma)) < -\alpha - \varepsilon < -\alpha$. □ □

In view of the above Theorem 3.3.6, we will use the terms $\lambda_k(V)$ and $\lambda_k(\text{reg}(\Sigma))$ interchangeably.

3.4 Modifications of the results of Schoen-Simon

Suppose the unit ball $B^{n+1}(0, 1) \subset \mathbb{R}^{n+1}$ is equipped with a Riemannian metric g ; μ_1 is a constant such that if $g = g_{ij} dx^i dx^j$,

$$\sup_{B^{n+1}(0,1)} \left| \frac{\partial g_{ij}}{\partial x_k} \right| \leq \mu_1, \quad \sup_{B^{n+1}(0,1)} \left| \frac{\partial^2 g_{ij}}{\partial x_k \partial x_l} \right| \leq \mu_1^2. \quad (3.4.1)$$

Theorem 3.4.1 (Modification of Schoen-Simon [SS81, Theorem 1, page 747]). *Suppose Σ is a singular, minimal hypersurface in $(B^{n+1}(0, 1), g)$ satisfying $\mathcal{H}^{n-2}(\text{sing}(\Sigma)) = 0$, $\mathcal{H}^n(\Sigma) \leq \mu$ and $\lambda_1(|\Sigma|) \geq -\alpha$, for some $\alpha \geq 0$. Then there exist $\delta_0 \in (0, 1)$, $r_0 \in (0, 1/4)$ and $c > 0$, depending only on n, μ, μ_1, α , such that the following holds. If $x = (x', x_{n+1}) \in \Sigma \cap B^{n+1}(0, 1/4)$, $\rho \leq r_0$, Σ' is the connected component of $\Sigma \cap C(x, \rho)$ ($C(x, \rho)$ is the cylinder on $B^n(x, \rho)$) containing x and*

$$\sup_{y=(y', y_{n+1}) \in \Sigma'} |y_{n+1} - x_{n+1}| \leq \delta_0 \rho, \quad (*)$$

then $\Sigma' \cap C(x, \rho/2)$ consists of disjoint union of graphs of functions $u_1 < u_2 < \dots < u_k$, defined on $B^n(x, \rho/2)$, satisfying the following estimate.

$$\sup_{B^n(x, \frac{\rho}{2})} (|Du_i| + \rho|DDu_i|) \leq c\delta_0$$

for $i = 1, 2, \dots, k$. In particular, Σ is smooth near x and the second fundamental form $|A_\Sigma(x)| \leq c/\rho$ (for possibly a different constant c).

Remark 3.4.2. The difference between the above Theorem 3.4.1 and Theorem 1 of Schoen-Simon [SS81] is that instead of assuming $|\Sigma|$ is stable, we have assumed that $\lambda_1(|\Sigma|) \geq -\alpha$. Indeed, under this weaker assumption, the stability inequality (1.17) of Schoen-Simon [SS81] (page 746) continues to hold with the constant c_5 replaced by $c_5 + \frac{\alpha}{\mu_1^2}$; therefore all the successive calculations in the paper [SS81] go through.

Theorem 3.4.3 (Modification of Schoen-Simon [SS81, Theorem 2, page 784]). *Let $\{\Sigma_q\}$ be a sequence of singular, minimal hypersurfaces in $(B^{n+1}(0, 1), g)$ such that $\mathcal{H}^{n-2}(\text{sing}(\Sigma_q)) = 0$, $\lambda_1(|\Sigma_q|) \geq -\alpha$ and $|\Sigma_q|$ converges to a varifold W ; $0 \in \text{spt}(W) = \Sigma$. Then $\mathcal{H}^s(\text{sing}(\Sigma) \cap B^{n+1}(0, \frac{1}{2})) = 0$ for all $s > n - 7$.*

Proof. As before, The difference between the above Theorem 3.4.3 and Theorem 2 of Schoen-Simon [SS81] is that instead of assuming $|\Sigma|$ is stable we have assumed that $\lambda_1(|\Sigma|) \geq -\alpha$.

The proof of Theorem 2 of Schoen-Simon [SS81] goes as follows. By the successive blow-up argument, one arrives at a varifold W_l , which is a stationary, integral, codimension 1 cone in $\mathbb{R}^{n-l+1}$ such that $\text{sing}(W_l) = \{0\}$ and

$$\mathbb{R}^l \times W_l = \lim_{m \rightarrow \infty} J_{\#} \circ \tau_{y_m \#} \circ \mu_{r_m \#} |\Sigma_{q_m}| \quad (3.4.2)$$

for some sequence of points $y_m \in \mathbb{R}^{n+1}$ and positive real numbers r_m and some subsequence $\{\Sigma_{q_m}\} \subset \{\Sigma_q\}$; $|y_m|, r_m \rightarrow \infty$. Here J is some orthogonal transformation of $\mathbb{R}^{n+1}$, τ_y denotes the translation of $\mathbb{R}^{n+1}$ which brings y to the origin, μ_r is the multiplication (scaling) by r . It is shown that (equation (5.22) of Schoen-Simon) $l \geq s$ for every s such that $\mathcal{H}^s(\text{sing}(\Sigma) \cap B^{n+1}(0, \frac{1}{2})) > 0$ (hence, one only needs to show that $l \leq n - 7$) and $l \leq n - 3$. Up to this point, the only facts about Σ_q which are used : Σ_q satisfies the stability inequality (1.17) and Theorem 1 of Schoen-Simon

[SS81]. After this, stability of $|\Sigma_q|$ is used to conclude that $\mathbb{R}^l \times W_l$ is stable. Hence, W_l is a smooth, stable codimension 1 cone in $\mathbb{R}^{n-l+1}$ with a singularity at origin. Therefore, $n-l \geq 7$ i.e. $l \leq n-7$.

In our context of Theorem 3.4.3, the above mentioned proof can be modified as follows. As noted above, stability inequality (1.17) and Theorem 1 of Schoen-Simon [SS81] continue to hold under the weaker assumption $\lambda_1(|\Sigma_q|) \geq -\alpha$. Moreover,

$$\lambda_1(|\Sigma_q|) \geq -\alpha \implies \lambda_1(J_{\#} \circ \tau_{y_m\#} \circ \mu_{r_m\#} |\Sigma_{q_m}|) \geq \frac{-\alpha}{r_m}. \quad (3.4.3)$$

(3.4.3) can be justified, for example, using Theorem 3.3.6. If $\lambda_1(\mathbb{R}^l \times W_l) < 0$, then $\lambda_1(\mathbb{R}^l \times W_l) < -\varepsilon$ for some $\varepsilon > 0$. (This can be seen from the proof of Theorem 3.3.6.) Since $\mathbb{R}^l \times W_l$ is the varifold limit of $J_{\#} \circ \tau_{y_m\#} \circ \mu_{r_m\#} |\Sigma_{q_m}|$, in view of Remark 3.3.3, for all large m ,

$$\lambda_1(J_{\#} \circ \tau_{y_m\#} \circ \mu_{r_m\#} |\Sigma_{q_m}|) < -\varepsilon.$$

This contradicts (3.4.3) as $\lim_{m \rightarrow \infty} r_m = \infty$. Hence, $\lambda_1(\mathbb{R}^l \times W_l) \geq 0$, i.e. $\mathbb{R}^l \times W_l$ is stable. $\square$

To prove the graphical convergence part of Theorem 3.1.1 we will need the following lemma, which is a consequence of Theorem 3.4.1.

Lemma 3.4.4. *Let (N^{n+1}, g) be a closed Riemannian manifold. Let $\{\Sigma_q\}$ be a sequence of singular, minimal hypersurfaces in N with $\mathcal{H}^{n-2}(\text{sing}(\Sigma_q)) = 0$ for all q . We also assume that $W_q = |\Sigma_q|$ varifold converges to a stationary, integral varifold W ; let $\Sigma = \text{spt}(W)$. Then, for every $x_0 \in \text{reg}(\Sigma)$, there exists $R = R(x_0) > 0$ such that if $r \leq R$ and $\lambda_1(W_q \llcorner B^N(x_0, r)) \geq -\alpha$ for all q sufficiently large, then $\Sigma_q \cap B^N(x_0, r/4)$ is smooth for large q and Σ_q converges to Σ smoothly in the ball $B^N(x_0, r/5)$.*

Proof. Pick $x_0 \in \text{reg}(\Sigma)$ and let $\Theta = \lim_{t \rightarrow 0} \frac{\|W\|(B^N(x_0, t))}{t^n}$. (The limit exists since W is stationary.)

Let ρ_0 be such that

$$\frac{\|W\|(B^N(x_0, t))}{t^n} \leq \Theta + 1/2 \quad \forall t \leq \rho_0. \quad (3.4.4)$$

Let δ_0 and r_0 be the constants which are provided by Theorem 3.4.1, when we set $\mu = \Theta + 1$ and $\mu_1 = 1$. Let $s_0 = \frac{\delta_0 r_0}{2}$. For $0 < a < \frac{1}{2} \text{inj}_N$, identifying $\mathbb{R}^{n+1}$ with $T_{x_0}N$ (and $\mathbb{R}^n \times \{0\} \subset \mathbb{R}^{n+1}$ is identified with $T_{x_0}\Sigma$), we define

$$\Phi_a : B^{n+1}(0, 2) \rightarrow B^N(x_0, 2a), \quad \Phi_a(v) = \exp_{x_0}(av), \quad g_a = a^{-2} \Phi_a^* g$$

We can choose $R > 0$ so that

- $R < \min \left\{ \frac{1}{2} \text{inj}_N, \frac{1}{2} d^N(x_0, \text{sing}(\Sigma)), 1, \rho_0 \right\}$

and whenever $r \leq R$,

- $(B^{n+1}(0, 1), g_r)$ satisfies (3.4.1) with $\mu_1 = 1$;
- $\tilde{\Sigma} = \Phi_r^{-1}(\Sigma \cap B^N(x_0, 2r)) \subset B^{n+1}(0, 2) \cap \{x : |x_{n+1}| < s_0\}$.²

We will show that the above choice of $R = R(x_0)$ works. Let us fix an $r \leq R$. We define

$$C = B^n(0, 1) \times [-1, 1] \subset B^{n+1}(0, 2), \quad C' = C \setminus \{x : |x_{n+1}| < s_0\}.$$

Then

$$C' \cap \left(\tilde{\Sigma} \cup \partial B^{n+1}(0, 2) \right) = \emptyset; \quad d := \text{dist}_{g_r} \left(C', \left(\tilde{\Sigma} \cup \partial B^{n+1}(0, 2) \right) \right).$$

Since W_q varifold converges to W and W_q 's are stationary varifolds, Σ_q converges to Σ in the Hausdorff distance. We choose q_0 (depending on r) so that for all $q \geq q_0$,

- $0 < \frac{\|W_q\|(B^N(x_0, r))}{r^n} \leq \Theta + 1$;³
- $\lambda_1(W_q \llcorner B^N(x_0, r)) \geq -\alpha$;
- the Hausdorff distance $d_H(\Sigma_q, \Sigma) < dr$.

²This is possible since Σ is smooth near x_0 .

³This is possible because of (3.4.4) and the varifold convergence of W_q to W .

Hence, denoting $\tilde{\Sigma}_q = \Phi_r^{-1}(\Sigma_q \cap B^N(x_0, 2r))$, for all $q \geq q_0$, we have

$$0 < \mathcal{H}^n(\tilde{\Sigma}_q \cap B^{n+1}(0, 1)) = \frac{\mathcal{H}^n(\Sigma_q \cap B^N(x_0, r))}{r^n} \leq \Theta + 1,$$

$$\lambda_1(|\tilde{\Sigma}_q \cap B^{n+1}(0, 1)|) \geq -\alpha r \geq -\alpha$$

and

$$\tilde{\Sigma}_q \cap C' = \emptyset \quad \text{i.e.} \quad (\tilde{\Sigma}_q \cap C) \subset C \cap \{x : |x_{n+1}| < s_0\}^4. \quad (3.4.5)$$

We can now apply Theorem 3.4.1 to the singular, minimal hypersurface

$$(\tilde{\Sigma}_q \cap B^{n+1}(0, 1)) \subset (B^{n+1}(0, 1), g_r)$$

for $q \geq q_0$. Since $B^{n+1}(0, 1) \subset C$, equation (3.4.5) implies that for all $q \geq q_0$, $x \in \tilde{\Sigma}_q \cap B^{n+1}(0, 1/4)$ and for $\rho = r_0$, the oscillation bound (*) of Theorem 3.4.1 is satisfied in the cylinder $C(x, r_0)$. Hence, $\tilde{\Sigma}_q \cap B^{n+1}(0, 1/4)$ is smooth with uniform bound on $A_{\tilde{\Sigma}_q \cap B^{n+1}(0, 1/4)}$ and we have already noted above a uniform upper bound on $\mathcal{H}^n(\tilde{\Sigma}_q \cap B^{n+1}(0, 1))$. The counterpart of the Arzela-Ascoli theorem for the smooth, minimal hypersurfaces gives that in a smaller ball $B^{n+1}(0, 1/5)$, $\tilde{\Sigma}_q$ converges to $\tilde{\Sigma}$ smoothly and graphically. (Here we do not have to pass to a further subsequence since we already know that $\tilde{\Sigma}_q$ Hausdorff converges to $\tilde{\Sigma}$). When we rescale back to go back to N , we get that in the ball $B^N(x_0, r/5)$, Σ_q converges to Σ smoothly and graphically.

□

3.5 Proof of the main theorem

In this section, we will give a proof of Theorem 3.1.1.

By Allard's compactness theorem [All72], possibly after passing to a subsequence, $V_k \rightarrow V$ in the $\mathbf{F}$ metric with V is a stationary, integral varifold and $\|V\|(N) \leq \Lambda$. We also know that $\text{spt}(V_k) = M_k$ converges to $\text{spt}(V) = M$ in the Hausdorff topology. In Remark 3.6.1, we will show

⁴This follows from $d_H(\Sigma_q, \Sigma) < dr$.

that $V = m|M|$. If $\lambda_p(V) < -\alpha$, then $\lambda_p(V_k) < -\alpha$, for k sufficiently large, by Remark 3.3.3. Hence $\lambda_p(V) \geq -\alpha$, which is equivalent to $\lambda_p(|M|) \geq -\alpha$, by Remark 3.3.4.

Lemma 3.5.1. *Let W be a stationary varifold in U^{n+1} with $\lambda_p(W) \geq -\alpha$, $\alpha \geq 0$. Let $U_1, \dots, U_p$ be mutually disjoint open subsets of U such that $\|W\|(U_j) \neq 0$ for each $j = 1, \dots, p$. Then there exists $i \in \{1, \dots, p\}$ such that $\lambda_1(W \llcorner U_i) \geq -\alpha$.*

Proof. Let us assume that for all $i = 1, \dots, p$, we have $\lambda_1(W \llcorner U_i) < -\alpha$. Hence, from Definition 3.3.2 and equation (3.3.4), we have the maps

$$F^i : \bar{B}^1 \rightarrow \text{Diff}(U_i)$$

and the vector-fields Y^i , compactly supported in U_i ,

$$Y^i|_x = \frac{d}{dt} \Big|_0 F_t^i(x)$$

such that

$$\delta^2 W(Y^i, Y^i) < -\alpha \|Y^i\|_{L^2(U, \|W\|)}^2.$$

We define

$$F : \bar{B}^p \rightarrow \text{Diff}(U), \quad F_v = \Phi^{\sum_i v_i Y^i},$$

where Φ^X denotes the time 1 flow of the vector field X . For this choice of F , we have

$$\begin{aligned} D^2 A^W|_0(u, u) &= \delta^2 W \left(\sum_i u_i Y^i, \sum_i u_i Y^i \right) \\ &= \sum_i u_i^2 \delta^2 W(Y^i, Y^i) \quad (\text{since } \text{spt}(Y^i)\text{'s are mutually disjoint}) \\ &< -\alpha \sum_i u_i^2 \|Y^i\|_{L^2(U, \|W\|)}^2 \\ &= -\alpha \left\| \sum_i u_i Y^i \right\|_{L^2(U, \|W\|)}^2 \quad (\text{since } \text{spt}(Y^i)\text{'s are mutually disjoint}) \\ &= -\alpha K^W(u, u). \end{aligned}$$

This contradicts $\lambda_p(W) \geq -\alpha$. $\square$

Returning back to the proof of the main Theorem 3.1.1, let $G \subset M$ be the set of points $x \in M$ for which there exists $r = r(x) > 0$ and some subsequence $\{V_{k'}\} \subset \{V_k\}$ such that for each k' , $\lambda_1(V_{k'} \llcorner B^N(x, 2r)) \geq -\alpha$. Therefore, using Theorem 3.4.3, $\mathcal{H}^s(\text{sing}(M) \cap B^N(x, r(x))) = 0$ for all $x \in G$ and $s > n - 7$.

Lemma 3.5.2. *The set $M \setminus G$ has at most $p - 1$ points.*

Proof. Suppose there exist p points $\{x_i\}_{i=1}^p \subset M \setminus G$. Let t be small enough so that the normal geodesic balls $\{B^N(x_i, t)\}_{i=1}^p$ are mutually disjoint. By the definition of G , there exists k_0 such that for all $k \geq k_0$, $\lambda_1(V_k \llcorner B^N(x_i, t)) < -\alpha$ for each i . By Lemma 3.5.1, this implies $\lambda_p(V_k) < -\alpha$ for all $k \geq k_0$, a contradiction. $\square$

We note that

$$\text{sing}(M) \cap G = \bigcup_{x \in \text{sing}(M) \cap G} (\text{sing}(M) \cap B^N(x, r(x))). \quad (3.5.1)$$

We can extract a countable subcover of the R.H.S. of (3.5.1) and write⁵

$$\text{sing}(M) \cap G = \bigcup_{i=1}^{\infty} (\text{sing}(M) \cap B^N(x_i, r(x_i))), \quad (3.5.2)$$

where each x_i is in $\text{sing}(M) \cap G$. By the definition of G , for $s > n - 7$, $\mathcal{H}^s(\text{sing}(M) \cap B^N(x_i, r(x_i))) = 0$ for each i . Therefore, (3.5.2) and Lemma 3.5.2 imply that $\mathcal{H}^s(\text{sing}(M)) = 0$. (Here we are implicitly assuming that $n \geq 7$ so that for $s > n - 7 \geq 0$, $\mathcal{H}^s(M \setminus G) = 0$; when $n < 7$ further arguments are required to show that M is smooth at the points of $M \setminus G$, as explained in [ACS16].)

We can now complete the proof of the graphical convergence. We will produce a set $\mathcal{Y} \subset \text{reg}(M)$ and a subsequence $\{M_{k'}\} \subset \{M_k\}$ such that $M_{k'}$ converges smoothly and graphically on compact

⁵This is possible because N , being a manifold, is second-countable and a subspace of a second-countable space is second-countable.

subsets of $\text{reg}(M) \setminus \mathcal{Y}$. Let X be a countable, dense subset of $\text{reg}(M)$ and

$$\mathcal{B} = \{B^N(x, r) : x \in X, r \in \mathbb{Q}^+, r < d^N(x, \text{sing}(M)), r < \text{inj}_N\}.$$

Then $\mathcal{B}$ is a countable collection of balls, say $\mathcal{B} = \{B_i\}_{i=1}^\infty$. We will mark each B_i as good or bad and to each B_i we will assign an infinite index set $I_i \subset \mathbb{N}$ as follows. At the first step, we examine whether there exists an infinite set $J \subset \mathbb{N}$ such that $\{M_j\}_{j \in J}$ converges to M smoothly in B_1 . If it exists, we mark B_1 as good and define I_1 to be that J . Otherwise we mark B_1 as bad and define I_1 to be $\mathbb{N}$. Suppose we have marked B_{i-1} as good or bad and defined I_{i-1} . Then we examine whether there exists an infinite set $J \subset I_{i-1}$ such that $\{M_j\}_{j \in J}$ converges to M smoothly in B_i . If it exists we mark B_i as good and define I_i to be that J . Otherwise we mark B_i as bad and define I_i to be I_{i-1} . Let $\mathcal{G}$ be the union of the good balls. Denoting $\mathcal{Y} = \text{reg}(M) \setminus \mathcal{G}$, we claim that $|\mathcal{Y}| \leq p - 1$. Otherwise, there exists p distinct points $x_1, \dots, x_p$ in $\mathcal{Y}$. To arrive at a contradiction, we will apply Lemma 3.4.4 to the sequence of singular, minimal hypersurfaces $\{M_k\}$; $|M_k|$ converges in the varifold sense to V , which is supported on M ; Lemma 3.4.4 provides a function $R : \text{reg}(M) \rightarrow \mathbb{R}^+$. Let τ be a positive number such that $5\tau \leq R(x_l)$ for each l and the balls $\{B^N(x_l, 5\tau)\}_{l=1}^p$ are mutually disjoint. There exists $B_{i_l} \in \mathcal{B}$ such that $x_l \in B_{i_l} \subset B^N(x_l, \tau)$. As $x_l \in \mathcal{Y}$, B_{i_l} is a bad ball. Hence $\{M_j\}_{j \in I_{i_l}}$ does not have a subsequence which smoothly converges to M in $B^N(x_l, \tau)$. Therefore, by Lemma 3.4.4, $\lambda_1(V_j \llcorner B^N(x_l, 5\tau)) < -\alpha$ for all large $j \in I_{i_l}$. Without loss of generality, we can assume that $i_1 < \dots < i_p$ so that $I_{i_1} \supset \dots \supset I_{i_p}$. Hence $\lambda_1(V_j \llcorner B^N(x_l, 5\tau)) < -\alpha$ for each $l = 1, \dots, p$ and for all large $j \in I_{i_p}$. By Lemma 3.5.1, this gives $\lambda_p(V_j) < -\alpha$ for all large $j \in I_{i_p}$, a contradiction. Hence, $|\mathcal{Y}| \leq p - 1$.

By a diagonal argument, we can choose an infinite set $I \subset \mathbb{N}$ such that $|I \setminus I_i|$ is finite for all i . Then, by the definition of good ball, $\{M_i\}_{i \in I}$ is a sequence which converges smoothly and graphically on the compact subsets of $\text{reg}(M) \cap \mathcal{G} = \text{reg}(M) \setminus \mathcal{Y}$.

3.6 Some further remarks

Besides the main compactness theorems of [Sha17] and [ACS16], some additional results proved in these two papers can be suitably generalized in higher dimensions. In this last section, we will state them as a sequence of remarks. Below we will assume that M_k 's and M are as in Theorem 3.1.1 (and Theorem 3.1.2).

Remark 3.6.1. We have assumed that each M_k is connected. Since $\{M_k\}$ converges to M in the Hausdorff topology, this implies that M is connected as well. From [Ilm96, Theorem A (ii)] it follows that $\text{reg}(M)$ and hence $\text{reg}(M) \setminus \mathcal{Y}$ is also connected. Therefore, the number of sheets in the graphical convergence is constant over $\text{reg}(M) \setminus \mathcal{Y}$. In particular, $V = m|M|$, where m is the number of sheets in the graphical convergence.

Remark 3.6.2. If the number of sheets in the graphical convergence is 1, then $\mathcal{Y} = \emptyset$. This is Claim 4 in [Sha17] and the proof presented there works in our case as well.

Remark 3.6.3. Suppose $\text{reg}(M)$ is two sided. If the number of sheets in the graphical convergence is at least 2 or if the number of sheets is 1 and $M_k \cap M = \emptyset$ for large k , we can construct a positive Jacobi field on $\text{reg}(M)$. In this case, $\text{reg}(M)$ and hence M is stable. The proof is same as presented in [Sha17].

Remark 3.6.4. Continuing with Remark 3.6.3, suppose $\text{Ric}(N, g) > 0$. Then the convergence of M_k to M is always single sheeted. This can be thought of as a higher dimensional analogue of the theorem of Choi-Schoen [CS85], which says that in a three manifold with positive Ricci curvature, the space of closed, embedded minimal surfaces with bounded genus is compact in the smooth topology. Indeed, in our case, if $H_n(N, \mathbb{Z}_2) = 0$ then $\text{reg}(M)$ is two sided; hence, if the number of sheets is ≥ 2 , M is stable by the above Remark 3.6.3. However, as proved in [Zho17, Lemma 2.8], positive Ricci curvature of (N, g) implies that M can not be stable. The general case can be obtained by lifting M_k 's and M to the universal cover $\tilde{N}$ of N (by [Fra66] and [Zho15, Lemma 2.10], the lifts $\tilde{M}_k, \tilde{M}$ are connected).

Remark 3.6.5. Theorems 3.1.1 and 3.1.2 hold in the varying metric set-up. More precisely, instead of assuming $|M_k|$ is stationary with respect to the fixed metric g if we assume that $|M_k|$ is stationary

with respect to the metric g_k and g_k converges to g in C^3 , Theorems 3.1.1 and 3.1.2 continue to hold.

Chapter 4

A sub-additive inequality for the volume spectrum

4.1 Introduction

For a closed Riemannian manifold (M^{n+1}, g) , the spectrum of the Laplacian on M , denoted by $\{\lambda_p\}_{p=1}^\infty$, $0 < \lambda_1 \leq \lambda_2 \leq \dots$, has the following min-max characterization. Let $H^1(M)$ be the Sobolev space of real valued functions $\varphi \in L^2(M)$ such that $\nabla\varphi$, the distributional derivative of φ , is also in $L^2(M)$. Suppose $\mathcal{V}_p$ denotes the set of all p -dimensional vector subspaces of $H^1(M)$. Then

$$\lambda_p = \inf_{V \in \mathcal{V}_p} \sup_{f \in V \setminus \{0\}} R(f) \quad \text{where} \quad R(f) = \frac{\int_M |\nabla f|^2}{\int_M f^2}. \quad (4.1.1)$$

The Rayleigh quotient $R(f)$ is invariant under the scaling, i.e. for all $c \in \mathbb{R} \setminus \{0\}$, $R(f) = R(cf)$. Hence, it descends to a well-defined functional on $\mathbb{P}(H^1(M))$, the projective space associated to the real vector space $H^1(M)$.

In [Gro88, Gro03, Gro09], Gromov introduced various non-linear analogues of the spectrum of the Laplacian. In the context of the area functional and the minimal hypersurfaces, the relevant spectrum is the volume spectrum $\{\omega_p\}_{p=1}^\infty$. ω_p is defined by a min-max quantity, similar to

(4.1.1). One replaces the vector space $H^1(M)$ by $\mathcal{Z}_n(M; \mathbb{Z}_2)$, the space of mod 2 flat hypercycles which bound a region in M and the Rayleigh quotient by the area functional (see Section 4.2.2 for the precise definition). By the works of Almgren [Alm62] and Marques-Neves [MN21], the space $\mathcal{Z}_n(M; \mathbb{Z}_2)$ is weakly homotopy equivalent to $\mathbb{RP}^\infty$. The cohomology ring $H^*(\mathcal{Z}_n(M; \mathbb{Z}_2), \mathbb{Z}_2)$ is the polynomial ring $\mathbb{Z}_2[\bar{\lambda}]$ where $\bar{\lambda} \in H^1(\mathcal{Z}_n(M; \mathbb{Z}_2), \mathbb{Z}_2)$. In the definition of ω_p , instead of considering p -dimensional vector subspaces of $H^1(M)$, one considers all the $S \subset \mathcal{Z}_n(M; \mathbb{Z}_2)$ such that $\bar{\lambda}^p|_S \neq 0$ in $H^p(S, \mathbb{Z}_2)$.

The connection between the volume spectrum and the minimal hypersurfaces comes from the Almgren-Pitts min-max theory, developed by Almgren [Alm65], Pitts [Pit81], Schoen-Simon [SS81], Marques-Neves [MN14, MN17]. If Π is a homotopy class of maps $X \rightarrow \mathcal{Z}_n(M; \mathbb{Z}_2)$, by the Almgren-Pitts min-max theory, the width of Π is achieved by the area of a closed, minimal hypersurface (which can have a singular set of Hausdorff dimension $\leq n-7$ if the ambient dimension $n+1 \geq 8$), possibly with multiplicities. The index upper bound of the min-max minimal hypersurfaces was proved by Marques-Neves [MN16] and Li [Li20]. Sharp [Sha17] proved the compactness of the space of minimal hypersurfaces with bounded area and index. As a consequence, one can conclude that [IMN18, Proposition 2.2], [Li20, Theorem 3] for each $p \in \mathbb{N}$, ω_p is achieved by the area of a closed, minimal hypersurface with optimal regularity (possibly with multiplicities), whose index is bounded above by p (see also [Li19, Corollary 3.2]). Moreover, if the ambient dimension $3 \leq n+1 \leq 7$, by the works of Marques-Neves [MN21] and Zhou [Zho20], for a generic (bumpy) metric, ω_p is realized by the area of a closed, two-sided, minimal hypersurface with multiplicity one, whose index $= p$. When the ambient dimension is 3, this was also proved by Chodosh and Mantoulidis [CM20], in the Allen-Cahn setting.

By the works of Gromov [Gro88, Gro03], Guth [Gut09], Marques and Neves [MN17], there exist positive constants C_1 and C_2 , depending on the ambient metric g , such that

$$C_1 p^{\frac{1}{n+1}} \leq \omega_p \leq C_2 p^{\frac{1}{n+1}} \quad \forall p \in \mathbb{N}. \quad (4.1.2)$$

In [LMN18], Liokumovich, Marques and Neves proved that $\{\omega_p\}_{p=1}^\infty$ satisfies the following Weyl

law, which was conjectured by Gromov [Gro03].

$$\lim_{p \rightarrow \infty} \omega_p(M) p^{-\frac{1}{n+1}} = a(n) \text{Vol}(M)^{\frac{n}{n+1}}, \quad (4.1.3)$$

where $a(n) > 0$ is a constant, which depends only on the ambient dimension.

The above mentioned theorems regarding the asymptotic behaviour of $\{\omega_p\}_{p=1}^{\infty}$, play important roles in the proofs of Yau's conjecture in [MN17, IMN18, MNS19, Son18, Li19, SZ21]. (The arguments in the papers [Son18] and [SZ21] use the Weyl law for the volume spectrum on certain *non-compact* manifolds with cylindrical ends, proved by Song [Son18].)

The properties of the volume spectrum are also important in the context of constant mean curvature (CMC) hypersurfaces. In Chapter 2, we obtained a quantitative estimate for the number of closed c -CMC hypersurfaces in an arbitrary closed Riemannian manifold. To prove this estimate, we used the lower bound of ω_p , stated in (4.1.2) and the following inequality.

$$\omega_{p+1} \leq \omega_p + W_0 \quad \forall p \in \mathbb{N}, \quad (4.1.4)$$

where W_0 is the one parameter Almgren-Pitts width of M .

In this chapter, we will prove a more general sub-additive inequality for the volume spectrum, as stated below.

Theorem 4.1.1. *Let (M, g) be a closed Riemannian manifold. Let $\{\omega_p\}_{p=1}^{\infty}$ be the volume spectrum and W_0 be the one parameter Almgren-Pitts width of M (see Section 4.2.2). Then, for all $k, m \geq 0$,*

$$\omega_{k+m+1} \leq \omega_k + \omega_m + W_0, \quad (4.1.5)$$

where we set $\omega_0 = 0$.

A similar inequality also holds for the phase-transition spectrum (as stated below in Theorem 4.1.2), which is the Allen-Cahn analogue of the volume spectrum. The Allen-Cahn min-max theory is a PDE based approach to the min-max construction of minimal hypersurfaces, which

was introduced by Guaraco [Gua18] and further extended by Gaspar and Guaraco [GG18]. The regularity of the minimal hypersurfaces, obtained from the Allen-Cahn theory, depends on the previous works by Hutchinson-Tonegawa [HT00], Tonegawa [Ton05], Wickramasekera [Wic14a] and Tonegawa-Wickramasekera [TW12]. If the ambient dimension $n + 1 = 3$, the regularity of the Allen-Cahn minimal surfaces can also be obtained from the curvature estimates of Chodosh and Mantoulidis [CM20]. In [GG18], Gaspar and Guaraco defined the phase-transition spectrum and proved a sub-linear growth estimate for them (which is similar to (4.1.2)). In [GG19], they proved a Weyl law for the phase-transition spectrum (which is similar to the Weyl law for the volume spectrum (4.1.3)).

Theorem 4.1.2. *Let (M, g) be a closed Riemannian manifold. For $\varepsilon > 0$, let $\{c_\varepsilon(p)\}_{p=1}^\infty$ be the ε -phase-transition spectrum and γ_ε be the one parameter ε -Allen-Cahn width of M (see Section 4.2.3). Then, for all $k, m \geq 0$,*

$$c_\varepsilon(k + m + 1) \leq c_\varepsilon(k) + c_\varepsilon(m) + \gamma_\varepsilon, \quad (4.1.6)$$

where $c_\varepsilon(0) = 0$. Hence, letting $\varepsilon \rightarrow 0^+$, we obtain the following inequality for the phase-transition spectrum $\{\ell_p\}_{p=1}^\infty$.

$$\ell_{k+m+1} \leq \ell_k + \ell_m + \gamma,$$

where $\ell_0 = 0$ and γ is the one parameter Allen-Cahn width of M .

4.2 Notation and Preliminaries

4.2.1 Caccioppoli sets

In this subsection, we will briefly recall the notion of the Caccioppoli set; further details can be found in [Sim83] and [AFP00]. An $\mathcal{H}^{n+1}$ -measurable set $E \subset M$ is called a *Caccioppoli set* if $D\chi_E$, the distributional derivative of the characteristic function χ_E , is a finite Radon measure on M .

This is equivalent to

$$\sup \left\{ \int_E \operatorname{div} \omega \, d\mathcal{H}^{n+1} : \omega \in \mathfrak{X}^1(M), \|\omega\|_\infty \leq 1 \right\} < \infty;$$

where $\mathfrak{X}^1(M)$ denotes the space of C^1 vector-fields on M . Let us use the notation $\mathcal{C}(M)$ to denote the space of all Caccioppoli sets in M . If $E \in \mathcal{C}(M)$, there exists an n -rectifiable set ∂E such that the total variation measure $|D\chi_E| = \mathcal{H}^n \llcorner \partial E$. Hence, for all $\omega \in \mathfrak{X}^1(M)$,

$$\int_E \operatorname{div} \omega \, d\mathcal{H}^{n+1} = \int_{\partial E} \langle \omega, \nu_E \rangle \, d\mathcal{H}^n,$$

where ν_E is a $|D\chi_E|$ -measurable vector-field; $|\nu_E| = 1$ $|D\chi_E|$ -a.e. The following proposition can be found in [AFP00]. For the sake of completeness, we also include its proof (following [AFP00]).

Proposition 4.2.1 ([AFP00, Proposition 3.38]). *Let $E \subset M$ be $\mathcal{H}^{n+1}$ -measurable and $U \subset M$ be an open set. Let*

$$P(E, U) = \sup \left\{ \int_E \operatorname{div} \omega \, d\mathcal{H}^{n+1} : \omega \in \mathfrak{X}_c^1(U), \|\omega\|_\infty \leq 1 \right\};$$

where $\mathfrak{X}_c^1(U)$ denotes the space of compactly supported C^1 vector-fields on U . If $E, F \in \mathcal{C}(M)$,

$$P(E \cap F, U) + P(E \cup F, U) \leq P(E, U) + P(F, U). \quad (4.2.1)$$

Hence, if $E, F \in \mathcal{C}(M)$, $E \cap F$ and $E \cup F$ also belong to $\mathcal{C}(M)$.

Proof. Since $E, F \in \mathcal{C}(M)$, by [MPPP07, Proposition 1.4], there exist $\{f_i\}_{i=1}^\infty, \{g_i\}_{i=1}^\infty \subset C^\infty(U)$ such that $0 \leq f_i, g_i \leq 1$ for all i ;

$$\begin{aligned} f_i &\rightarrow \chi_E|_U \text{ in } L^1(U) \text{ and pointwise a.e.;} & P(E, U) &= \lim_{i \rightarrow \infty} \int_U |\nabla f_i|; \\ g_i &\rightarrow \chi_F|_U \text{ in } L^1(U) \text{ and pointwise a.e.;} & P(F, U) &= \lim_{i \rightarrow \infty} \int_U |\nabla g_i|. \end{aligned}$$

By the dominated convergence theorem,

$$f_i g_i \rightarrow \chi_{E \cap F}|_U \text{ in } L^1(U) \text{ and } f_i + g_i - f_i g_i \rightarrow \chi_{E \cup F}|_U \text{ in } L^1(U).$$

Therefore,

$$\begin{aligned} & P(E \cap F, U) + P(E \cup F, U) \\ & \leq \liminf_{i \rightarrow \infty} \int_U |\nabla(f_i g_i)| + |\nabla f_i + \nabla g_i - \nabla(f_i g_i)| \\ & \leq \liminf_{i \rightarrow \infty} \int_U g_i |\nabla f_i| + f_i |\nabla g_i| + (1 - g_i) |\nabla f_i| + (1 - f_i) |\nabla g_i| \\ & = P(E, U) + P(F, U). \end{aligned}$$

□

4.2.2 The space of hypercycles and the volume spectrum

For $l \in \mathbb{N}$, let $\mathbf{I}_l(M^{n+1}; \mathbb{Z}_2)$ be the space of l -dimensional flat chains in M with coefficients in $\mathbb{Z}_2$. We will only need to consider $l = n, n + 1$. $\mathcal{Z}_n(M^{n+1}; \mathbb{Z}_2)$ denotes the space of flat chains $T \in \mathbf{I}_n(M; \mathbb{Z}_2)$ such that $T = \partial\Omega$ for some $\Omega \in \mathbf{I}_{n+1}(M; \mathbb{Z}_2)$. If $T \in \mathbf{I}_n(M; \mathbb{Z}_2)$, $|T|$ stands for the varifold associated to T and $\|T\|$ is the Radon measure associated to $|T|$. $\mathcal{F}$ and $\mathbf{M}$ denote the flat norm and the mass norm on $\mathbf{I}_l(M; \mathbb{Z}_2)$. When $l = n + 1$, these two norms coincide. We will always assume that the spaces $\mathbf{I}_{n+1}(M; \mathbb{Z}_2)$ and $\mathcal{Z}_n(M; \mathbb{Z}_2)$ are equipped with the $\mathcal{F}$ norm. We will also identify $\mathcal{C}(M)$ with $\mathbf{I}_{n+1}(M; \mathbb{Z}_2)$, i.e. $E \in \mathcal{C}(M)$ will be identified with $\llbracket E \rrbracket$, the current associated with E . Similarly, ∂E (with $E \in \mathcal{C}(M)$) will be identified with $\llbracket \partial E \rrbracket = \partial \llbracket E \rrbracket$.

In [MN21], Marques and Neves proved that the space $\mathcal{C}(M)$ is contractible and the boundary map $\partial : \mathcal{C}(M) \rightarrow \mathcal{Z}_n(M; \mathbb{Z}_2)$ is a double cover. Indeed, by the constancy theorem, for $\Omega_1, \Omega_2 \in \mathcal{C}(M)$, $\partial\Omega_1 = \partial\Omega_2$ if and only if either $\Omega_1 = \Omega_2$ or $\Omega_1 = M - \Omega_2$. Thus $\pi_1(\mathcal{Z}_n(M; \mathbb{Z}_2)) = \mathbb{Z}_2$. It was also proved in [MN21] that $\mathcal{Z}_n(M; \mathbb{Z}_2)$ is weakly homotopy equivalent to $\mathbb{RP}^\infty$. The cohomology ring $H^*(\mathcal{Z}_n(M; \mathbb{Z}_2), \mathbb{Z}_2)$ is the polynomial ring $\mathbb{Z}_2[\bar{\lambda}]$ where $\bar{\lambda}$ is the unique non-zero cohomology class

in $H^1(\mathcal{Z}_n(M; \mathbb{Z}_2), \mathbb{Z}_2)$.

X is called a cubical complex if X is a subcomplex of $[0, 1]^N$ for some $N \in \mathbb{N}$. By [BP02, Chapter 4], every cubical complex is homeomorphic to a finite simplicial complex and vice-versa. In this chapter, we choose to work with simplicial complexes.

Let X be a finite simplicial complex. Suppose $\Phi : X \rightarrow \mathcal{Z}_n(M; \mathbb{Z}_2)$ is a continuous map. Φ is called a p -sweepout if $\Phi^*(\bar{\lambda}^p) \neq 0$ in $H^p(X, \mathbb{Z}_2)$. Φ is said to have *no concentration of mass* if

$$\lim_{r \rightarrow 0^+} \sup \{ \|\Phi(x)\| (B(p, r)) : x \in X, p \in M \} = 0.$$

Let $\mathcal{P}_p$ denote the set of all p -sweepouts with no concentration of mass. The *volume spectrum* $\{\omega_p\}_{p=1}^\infty$ is defined by

$$\omega_p = \inf_{\Phi \in \mathcal{P}_p} \sup \{ \mathbf{M}(\Phi(x)) : x \in \text{domain of } \Phi \}.$$

Let us also recall the definition of the one parameter Almgren-Pitts width, denoted by W . Suppose $\mathcal{S}$ is the set of all continuous maps $\Lambda : [0, 1] \rightarrow \mathcal{C}(M)$ such that $\Lambda(0) = M$, $\Lambda(1) = \emptyset$ and $\partial \circ \Lambda : [0, 1] \rightarrow \mathcal{Z}_n(M; \mathbb{Z}_2)$ has no concentration of mass. We define

$$W_0 = \inf_{\Lambda \in \mathcal{S}} \sup \{ \mathbf{M}(\partial \Lambda(t)) : t \in [0, 1] \}.$$

4.2.3 The phase-transition spectrum

Here we briefly recall the definition of the phase-transition spectrum, which was originally defined by Gaspar and Guaraco in [GG18]. Let $\mathfrak{p} : E \rightarrow B$ be a universal $\mathbb{Z}_2$ -principal bundle. Then B is homotopy equivalent to $\mathbb{RP}^\infty$; hence $H^*(B, \mathbb{Z}_2)$ is isomorphic to the polynomial ring $\mathbb{Z}_2[\xi]$, where $\xi \in H^1(B, \mathbb{Z}_2)$. Suppose $\tilde{X}$ is a finite simplicial complex with a free, simplicial $\mathbb{Z}_2$ action (i.e. $\mathbb{Z}_2$ acts on $\tilde{X}$ by simplicial homeomorphism) and X is the quotient of $\tilde{X}$ by $\mathbb{Z}_2$. There exists a continuous $\mathbb{Z}_2$ -equivariant map $\tilde{f} : \tilde{X} \rightarrow E$, which is unique up to $\mathbb{Z}_2$ -homotopy (see [Die08, Chapter 14.4]). $\tilde{f}$ descends to a map $f : X \rightarrow B$, i.e. if $\pi : \tilde{X} \rightarrow X$ is the projection map, $f \circ \pi = \mathfrak{p} \circ \tilde{f}$. Let

$f^* : H^*(B, \mathbb{Z}_2) \rightarrow H^*(X, \mathbb{Z}_2)$ be the map induced by f . One defines

$$\text{Ind}_{\mathbb{Z}_2}(\tilde{X}) = \sup \{p \in \mathbb{N} : f^*(\xi^{p-1}) \neq 0 \in H^{p-1}(X, \mathbb{Z}_2)\}.$$

$\mathcal{C}_p$ denotes the set of all finite simplicial complex $\tilde{X}$, with a free, simplicial $\mathbb{Z}_2$ action, such that $\text{Ind}_{\mathbb{Z}_2}(\tilde{X}) \geq p + 1$.

Let us recall that

$$H^1(M) = \{\varphi \in L^2(M) : \text{the distributional derivative } \nabla \varphi \in L^2(M)\}.$$

$H^1(M)$ is an infinite dimensional, separable Hilbert space; hence $H^1(M) \setminus \{0\}$ is contractible. There is a free $\mathbb{Z}_2$ action on $H^1(M) \setminus \{0\}$ given by $u \mapsto -u$. Thus $H^1(M) \setminus \{0\}$, equipped with this $\mathbb{Z}_2$ action, is the total space of a universal $\mathbb{Z}_2$ -principal bundle (see [Die08, 14.4.12]). If $\tilde{X}$ is a finite simplicial complex with a free, simplicial $\mathbb{Z}_2$ action, the set of all continuous $\mathbb{Z}_2$ -equivariant maps $\tilde{X} \rightarrow H^1(M) \setminus \{0\}$ is denoted by $\Gamma(\tilde{X})$.

For $\varepsilon > 0$, the ε -Allen-Cahn functional $E_\varepsilon : H^1(M) \rightarrow [0, \infty)$ is defined by

$$E_\varepsilon(u) = \int_M \varepsilon \frac{|\nabla u|^2}{2} + \frac{W(u)}{\varepsilon},$$

where $W : \mathbb{R} \rightarrow [0, \infty)$ is a smooth, symmetric double-well potential. More precisely, W is bounded; $W(t) = W(-t)$ for all $t \in \mathbb{R}$; W has precisely three critical points $0, \pm 1$; $W(\pm 1) = 0$ and $W''(\pm 1) > 0$; 0 is a local maximum of W . We note that $E_\varepsilon(u) = E_\varepsilon(-u)$. The ε -phase-transition spectrum $\{c_\varepsilon(p)\}_{p=1}^\infty$ can be defined as follows (see [GG18, Lemma 6.2]).

$$c_\varepsilon(p) = \inf_{\tilde{X} \in \mathcal{C}_p} \left(\inf_{f \in \Gamma(\tilde{X})} \sup_{x \in \tilde{X}} E_\varepsilon(f(x)) \right).$$

The phase-transition spectrum $\{\ell_p\}_{p=1}^\infty$ is defined by

$$\ell_p = \frac{1}{2\sigma} \lim_{\varepsilon \rightarrow 0^+} c_\varepsilon(p),$$

where $\sigma = \int_{-1}^1 \sqrt{W(t)/2} dt$. By the combined works of Hutchinson-Tonegawa [HT00], Tonegawa [Ton05], Wickramasekera [Wic14a], Tonegawa-Wickramasekera [TW12], Guaraco [Gua18], Gaspar-Guaraco [GG18] and Gaspar [Gas19], ℓ_p is achieved by the area of a closed, minimal hypersurface with optimal regularity (possibly with multiplicities), whose index is bounded above by p [GG19, Corollary 2.3]. From the results in [Gua18], [GG18] and Chapter 5, it follows that $\ell_p = \omega_p$ for all $p \in \mathbb{N}$. Let us also recall the definition of the *one parameter Allen-Cahn width* γ , which was defined by Guaraco [Gua18]. Suppose

$$\mathcal{S} = \{f : [0, 1] \rightarrow H^1(M) : f \text{ is continuous, } f(0) \equiv 1, f(1) \equiv -1\}$$

Then

$$\gamma_\varepsilon = \inf_{f \in \mathcal{S}} \sup_{t \in [0, 1]} E_\varepsilon(f(t)); \quad \gamma = \frac{1}{2\sigma} \lim_{\varepsilon \rightarrow 0^+} \gamma_\varepsilon.$$

Again, by the results in [Gua18] and Chapter 5, $\gamma = W_0$.

4.2.4 The join of two topological spaces

Let A and B be two topological spaces. The *join of A and B* , denoted by $A * B$, is defined as follows.

$$A * B = \frac{A \times B \times [0, 1]}{\sim},$$

where the equivalence relation ' $\sim$ ' is such that

$$(a, b_1, 0) \sim (a, b_2, 0) \quad \forall b_1, b_2 \in B \quad \text{and} \quad (a_1, b, 1) \sim (a_2, b, 1) \quad \forall a_1, a_2 \in A,$$

i.e. $A \times B \times \{0\}$ is collapsed to A and $A \times B \times \{1\}$ is collapsed to B . If either A or B is path-connected, using van Kampen's Theorem, one can show that $A * B$ is simply connected.

By the abuse of notation, an element of $A * B$ will be denoted by a triple (a, b, t) with $a \in A$, $b \in B$, $t \in [0, 1]$. If $f : A \rightarrow C$ and $g : B \rightarrow D$ are continuous maps, one can define a continuous map

$f * g : A * B \rightarrow C * D$ as follows.

$$(f * g)(a, b, t) = (f(a), g(b), t); \quad a \in A, b \in B, t \in [0, 1].$$

Let us recall that the standard d -simplex Δ^d is defined as follows.

$$\Delta^d = \{x = (x_1, x_2, \dots, x_{d+1}) \in \mathbb{R}^{d+1} : x_1 + x_2 + \dots + x_{d+1} = 1 \text{ and } x_i \geq 0 \ \forall i\}.$$

$\Delta^p * \Delta^q$ can be canonically identified with the simplex Δ^{p+q+1} . Indeed, $F_{pq} : \Delta^p * \Delta^q \rightarrow \Delta^{p+q+1}$, defined by

$$F_{pq}(u, v, t) = ((1-t)u, tv) \ (\in \mathbb{R}^{p+q+2}),$$

is a homeomorphism. Hence, the join of two simplicial complexes is again a simplicial complex.

If $\mathcal{X}$ is a topological space, let $C_d(\mathcal{X}, \mathbb{Z}_2)$ denote the abelian group of singular d -chains in $\mathcal{X}$ with coefficients in $\mathbb{Z}_2$. For $p, q \in \mathbb{N}_0$, we define a bilinear map $\mathcal{F}_{pq} : C_p(A, \mathbb{Z}_2) \times C_q(B, \mathbb{Z}_2) \rightarrow C_{p+q+1}(A * B, \mathbb{Z}_2)$ as follows. If $\varphi_1 : \Delta^p \rightarrow A$ and $\varphi_2 : \Delta^q \rightarrow B$ are continuous maps,

$$\mathcal{F}_{pq}(\varphi_1, \varphi_2) = (\varphi_1 * \varphi_2) \circ (F_{pq}^{-1});$$

for a general pair $(\varphi_1, \varphi_2) \in C_p(A, \mathbb{Z}_2) \times C_q(B, \mathbb{Z}_2)$, $\mathcal{F}_{pq}(\varphi_1, \varphi_2)$ is defined by bilinearity. For convenience, let us denote $\mathcal{F}_{pq}(\varphi_1, \varphi_2)$ simply by $\varphi_1 * \varphi_2$. The following identity can be verified by explicit computation.

$$\partial(\varphi_1 * \varphi_2) = (\partial\varphi_1) * \varphi_2 + \varphi_1 * (\partial\varphi_2). \tag{4.2.2}$$

4.3 Proof of Theorem 4.1.1

If $k = m = 0$, (4.1.5) reduces to $\omega_1 \leq W$, which holds by the definitions of ω_1 and W . So let us assume that $k \geq 1$. The proof of Theorem 4.1.1 is divided into four parts.

Part 1. Let us fix $\delta > 0$. We choose a k -sweepout $\Phi_1 : X \rightarrow \mathcal{Z}_n(M; \mathbb{Z}_2)$, with no concentration of

mass, such that X is connected and

$$\sup_{x \in X} \{\mathbf{M}(\Phi_1(x))\} \leq \omega_k + \delta. \quad (4.3.1)$$

Following the argument of Zhou [Zho20], Φ_1 is a k -sweepout implies that

$$\Phi_1^* : H^1(\mathcal{Z}_n(M; \mathbb{Z}_2), \mathbb{Z}_2) \rightarrow H^1(X, \mathbb{Z}_2)$$

is non-zero. Hence,

$$(\Phi_1)_* : \pi_1(X) \rightarrow \pi_1(\mathcal{Z}_n(M; \mathbb{Z}_2)) (= \mathbb{Z}_2)$$

is onto. Thus $\ker(\Phi_1)_*$ is an index 2 subgroup of $\pi_1(X)$. By [Hat02, Proposition 1.36], there exists a double cover $\rho_1 : \tilde{X} \rightarrow X$ such that $\tilde{X}$ is connected and if

$$(\rho_1)_* : \pi_1(\tilde{X}) \rightarrow \pi_1(X),$$

then $\text{im}(\rho_1)_* = \ker(\Phi_1)_*$. Using [Hat02, Proposition 1.33], Φ_1 has a lift $\tilde{\Phi}_1 : \tilde{X} \rightarrow \mathcal{C}(M)$ such that $\partial \circ \tilde{\Phi}_1 = \Phi_1 \circ \rho_1$. $\tilde{\Phi}_1$ is $\mathbb{Z}_2$ -equivariant, i.e. if $T_1 : \tilde{X} \rightarrow \tilde{X}$ is the deck transformation,

$$\tilde{\Phi}_1(T_1(x)) = M - \tilde{\Phi}_1(x) \quad \forall x \in \tilde{X}.$$

Similarly, for $m \geq 1$, we can choose an m -sweepout $\Phi_2 : Y \rightarrow \mathcal{Z}_n(M; \mathbb{Z}_2)$ with no concentration of mass such that Y is connected and

$$\sup_{y \in Y} \{\mathbf{M}(\Phi_2(y))\} \leq \omega_m + \delta. \quad (4.3.2)$$

There exists a double cover $\rho_2 : \tilde{Y} \rightarrow Y$ such that $\tilde{Y}$ is connected and Φ_2 has a lift $\tilde{\Phi}_2 : \tilde{Y} \rightarrow \mathcal{C}(M)$ such that $\partial \circ \tilde{\Phi}_2 = \Phi_2 \circ \rho_2$. If $T_2 : \tilde{Y} \rightarrow \tilde{Y}$ is the deck transformation, then

$$\tilde{\Phi}_2(T_2(y)) = M - \tilde{\Phi}_2(y) \quad \forall y \in \tilde{Y}.$$

On the other hand, if $m = 0$, we simply choose Y to be a singleton set, say $Y = \{0\}$ and $\Phi_2(0)$ to be the zero cycle. Moreover, in this case, $\tilde{Y} = \{-1, 1\}$ with $\tilde{\Phi}_2(-1) = M$ and $\tilde{\Phi}_1(1) = \emptyset$. We also choose $\Lambda : [0, 1] \rightarrow \mathcal{C}(M)$ such that $\Lambda(0) = M$, $\Lambda(1) = \emptyset$, $\partial \circ \Lambda : [0, 1] \rightarrow \mathcal{Z}_n(M; \mathbb{Z}_2)$ has no concentration of mass and

$$\sup_{t \in [0, 1]} \{\mathbf{M}(\partial \Lambda(t))\} \leq W_0 + \delta. \quad (4.3.3)$$

Part 2. Let us consider $\tilde{X} * \tilde{Y}$. As remarked in Section 4.2.4, $\tilde{X} * \tilde{Y}$ is simply-connected. There exists a free $\mathbb{Z}_2$ action on $\tilde{X} * \tilde{Y}$ given by

$$(x, y, t) \mapsto (T_1(x), T_2(y), t). \quad (4.3.4)$$

$\tilde{X}$ and $\tilde{Y}$ can be equipped with simplicial complex structures such that the maps $T_1 : \tilde{X} \rightarrow \tilde{X}$ and $T_2 : \tilde{Y} \rightarrow \tilde{Y}$ are simplicial homeomorphisms. Let $\{\Delta_{1,i} : i = 1, \dots, 2I\}$ and $\{\Delta_{2,j} : j = 1, \dots, 2J\}$ be the set of simplices in $\tilde{X}$ and $\tilde{Y}$. As discussed in Section 4.2.4, we can put a canonical simplicial complex structure on $\tilde{X} * \tilde{Y}$, whose simplices are $\{\Delta_{1,i}, \Delta_{2,j}, \Delta_{1,i} * \Delta_{2,j} : i = 1, \dots, 2I; j = 1, \dots, 2J\}$, such that the above action of $\mathbb{Z}_2$ on $\tilde{X} * \tilde{Y}$ ((4.3.4)) is a simplicial action.

One can define a $\mathbb{Z}_2$ -equivariant map $\tilde{\Psi} : \tilde{X} * \tilde{Y} \rightarrow \mathcal{C}(M)$ as follows.

$$\tilde{\Psi}(x, y, t) = \left(\tilde{\Phi}_1(x) \cap \Lambda(t) \right) \bigcup \left(\tilde{\Phi}_2(y) \cap (M - \Lambda(t)) \right). \quad (4.3.5)$$

This definition is motivated by the work of Marques and Neves [MN21, Proof of Claim 5.3], where they proved that $\mathcal{C}(M)$ is contractible. By Proposition 4.2.1, R.H.S. of (4.3.5) indeed belongs to $\mathcal{C}(M)$. Moreover, $\tilde{\Psi}$ is well-defined on $\tilde{X} * \tilde{Y}$ as $\Lambda(0) = M$ and $\Lambda(1) = \emptyset$.

Claim 4.3.1. *The map $\tilde{\Psi} : \tilde{X} * \tilde{Y} \rightarrow \mathcal{C}(M)$, defined above, is continuous in the flat topology.*

Proof. Since $\tilde{\Phi}_1$, $\tilde{\Phi}_2$ and Λ are continuous in the flat topology, it is enough to prove that the maps $\mu_1, \mu_2 : \mathcal{C}(M) \times \mathcal{C}(M) \rightarrow \mathcal{C}(M)$ defined by

$$\mu_1(E, F) = E \cap F \quad \text{and} \quad \mu_2(E, F) = E \cup F$$

are continuous. For arbitrary sets A_1, A_2, B_1, B_2 ,

$$(A_1 \cap B_1) \Delta (A_2 \cap B_2) \subset (A_1 \Delta A_2) \cup (B_1 \Delta B_2),$$

and

$$(A_1 \cup B_1) \Delta (A_2 \cup B_2) \subset (A_1 \Delta A_2) \cup (B_1 \Delta B_2).$$

Here $A \Delta B = (A \setminus B) \cup (B \setminus A)$ is the symmetric difference of A and B . Therefore,

$$\mathcal{F}(\mu_1(E, F) - \mu_1(E', F')) \leq \mathcal{F}(E - E') + \mathcal{F}(F - F'),$$

$$\mathcal{F}(\mu_2(E, F) - \mu_2(E', F')) \leq \mathcal{F}(E - E') + \mathcal{F}(F - F'),$$

which proves the continuity of μ_1 and μ_2 . $\square$

Let Z denote the quotient of $\tilde{X} * \tilde{Y}$ under the above mentioned $\mathbb{Z}_2$ action (4.3.4) (Z is a simplicial complex as the $\mathbb{Z}_2$ action on $\tilde{X} * \tilde{Y}$ is simplicial) and $\rho : \tilde{X} * \tilde{Y} \rightarrow Z$ be the covering map. Since $\tilde{\Psi} : \tilde{X} * \tilde{Y} \rightarrow \mathcal{C}(M)$ is $\mathbb{Z}_2$ -equivariant, $\tilde{\Psi}$ descends to a continuous map $\Psi : Z \rightarrow \mathcal{Z}_n(M; \mathbb{Z}_2)$, i.e. $\partial \circ \tilde{\Psi} = \Psi \circ \rho$. If $E \in \mathcal{C}(M)$, $P(E, U) = \|\partial E\|(U)$. Since $\Phi_1, \Phi_2, \partial \circ \Lambda$ have no concentration of mass, from Proposition 4.2.1, it follows that Ψ also has no concentration of mass.

Part 3. We will show that $\Psi : Z \rightarrow \mathcal{Z}_n(M; \mathbb{Z}_2)$ is a $(k + m + 1)$ -sweepout. Since $\tilde{X} * \tilde{Y}$ is simply-connected and Z is the quotient of $\tilde{X} * \tilde{Y}$ by $\mathbb{Z}_2$, $\pi_1(Z) = \mathbb{Z}_2$; hence $H^1(Z, \mathbb{Z}_2) = \mathbb{Z}_2$. Let λ be the unique non-zero element of $H^1(Z, \mathbb{Z}_2)$ so that

$$\lambda.[\kappa] = 1 \tag{4.3.6}$$

for all non-contractible loop $\kappa : S^1 \rightarrow Z$.

As discussed in Section 4.2.4, $\tilde{X}$ naturally sits inside $\tilde{X} * \tilde{Y}$, via the map $x \mapsto (x, \bar{y}, 0)$ and $\tilde{Y}$ naturally sits inside $\tilde{X} * \tilde{Y}$, via the map $y \mapsto (\bar{x}, y, 1)$. Hence, X and Y can be naturally identified as subspaces of Z ; let $\iota_1 : X \hookrightarrow Z$ and $\iota_2 : Y \hookrightarrow Z$ be the inclusion maps. By the definition of $\tilde{\Psi}$ in

(4.3.5),

$$\Psi|_X = \Phi_1 \quad \text{and} \quad \Psi|_Y = \Phi_2. \quad (4.3.7)$$

Thus, recalling the notation from Section 4.2.2 that $H^*(\mathcal{Z}_n(M; \mathbb{Z}_2), \mathbb{Z}_2) = \mathbb{Z}_2[\bar{\lambda}]$,

$$\Psi^* \bar{\lambda}|_X = \Phi_1^* \bar{\lambda} \quad \text{and} \quad \Psi^* \bar{\lambda}|_Y = \Phi_2^* \bar{\lambda}. \quad (4.3.8)$$

In particular, this implies $\Psi^* \bar{\lambda}$ is non-zero and hence $\Psi^* \bar{\lambda} = \lambda$. Therefore, to prove that $\Psi : Z \rightarrow \mathcal{Z}_n(M; \mathbb{Z}_2)$ is a $(k+m+1)$ -sweepout, it is enough to find $c \in H_{k+m+1}(Z, \mathbb{Z}_2)$ such that $\lambda^{k+m+1}.c = 1$.

Since Φ_1 is a k -sweepout, there exists $\alpha \in H_k(X, \mathbb{Z}_2)$ such that

$$(\Phi_1^* \bar{\lambda})^k . \alpha = 1. \quad (4.3.9)$$

Let $\alpha = \left[\sum_{i=1}^I \alpha_i \right]$, where each $\alpha_i : \Delta^k \rightarrow X$ is a singular k -simplex and

$$\partial \left(\sum_i \alpha_i \right) = 0. \quad (4.3.10)$$

Similarly, as Φ_2 is an m -sweepout, there exists $\beta \in H_m(Y, \mathbb{Z}_2)$ such that

$$(\Phi_2^* \bar{\lambda})^m . \beta = 1. \quad (4.3.11)$$

Let $\beta = \left[\sum_{j=1}^J \beta_j \right]$, where each $\beta_j : \Delta^m \rightarrow Y$ is a singular m -simplex and

$$\partial \left(\sum_j \beta_j \right) = 0. \quad (4.3.12)$$

Let $\tau : C_k(X, \mathbb{Z}_2) \rightarrow C_k(\tilde{X}, \mathbb{Z}_2)$ be the transfer homomorphism defined as follows (see the discussion after the statement of Proposition 2B.6 in [Hat02]). If $\eta : \Delta^k \rightarrow X$ is a continuous map, η has exactly two lifts $\tilde{\eta}_1, \tilde{\eta}_2 : \Delta^k \rightarrow \tilde{X}$. One defines $\tau(\eta) = \tilde{\eta}_1 + \tilde{\eta}_2$. For a general $\eta \in C_k(X, \mathbb{Z}_2)$, $\tau(\eta)$ is defined by linearity. Let $\tilde{\alpha}_{i,1}, \tilde{\alpha}_{i,2} : \Delta^k \rightarrow \tilde{X}$ be the lifts of α_i so that $\tau(\alpha_i) = \tilde{\alpha}_{i,1} + \tilde{\alpha}_{i,2}$. We also choose $\tilde{\beta}_j : \Delta^m \rightarrow \tilde{Y}$, which is a lift of β_j .

Claim 4.3.2.

$$\partial \rho_{\#} \left(\sum_{i,j} \tau(\alpha_i) * \tilde{\beta}_j \right) = 0 \quad \text{in } C_{k+m}(Z, \mathbb{Z}_2). \quad (4.3.13)$$

Proof. By (4.2.2),

$$\partial \left(\sum_{i,j} \tau(\alpha_i) * \tilde{\beta}_j \right) = \sum_{i,j} \partial \tau(\alpha_i) * \tilde{\beta}_j + \sum_{i,j} \tau(\alpha_i) * \partial \tilde{\beta}_j. \quad (4.3.14)$$

Since τ is a chain map (i.e. τ commutes with ∂), (4.3.10) implies

$$\sum_i \partial \tau(\alpha_i) = 0. \quad (4.3.15)$$

Further, by (4.3.12),

$$\rho_{\#} \left(\sum_j \partial \tilde{\beta}_j \right) = 0.$$

Hence, there exist singular $(m-1)$ -simplices $\theta_{r,1}, \theta_{r,2} : \Delta^{m-1} \rightarrow \tilde{Y}$ such that (see the short exact sequence in the discussion after the statement of Proposition 2B.6 in [Hat02]) $\theta_{r,2} = T_2 \circ \theta_{r,1}$ and

$$\sum_j \partial \tilde{\beta}_j = \sum_{r=1}^R \theta_{r,1} + \theta_{r,2}.$$

Thus

$$\sum_{i,j} \tau(\alpha_i) * \partial \tilde{\beta}_j = \sum_{i,r} \tilde{\alpha}_{i,1} * \theta_{r,1} + \tilde{\alpha}_{i,2} * \theta_{r,1} + \tilde{\alpha}_{i,1} * \theta_{r,2} + \tilde{\alpha}_{i,2} * \theta_{r,2}. \quad (4.3.16)$$

However,

$$\rho \circ (\tilde{\alpha}_{i,1} * \theta_{r,1}) = \rho \circ (\tilde{\alpha}_{i,2} * \theta_{r,2}) \quad \text{and} \quad \rho \circ (\tilde{\alpha}_{i,2} * \theta_{r,1}) = \rho \circ (\tilde{\alpha}_{i,1} * \theta_{r,2}). \quad (4.3.17)$$

Therefore, combining (4.3.14), (4.3.15), (4.3.16) and (4.3.17), we obtain (4.3.13). $\square$

Claim 4.3.2 implies that

$$c = \left[\rho_{\#} \left(\sum_{i,j} \tau(\alpha_i) * \tilde{\beta}_j \right) \right] \quad (4.3.18)$$

is a well-defined homology class in $H_{k+m+1}(Z, \mathbb{Z}_2)$.

Claim 4.3.3. $\lambda^{k+m+1}.c = 1$, where c is as defined in (4.3.18).

Proof. Let us choose $h : C_1(Z, \mathbb{Z}_2) \rightarrow \mathbb{Z}_2$ such that $\lambda = [h]$. We recall from Section 4.2.4 that Δ^{k+m+1} can be canonically identified with $\Delta^k * \Delta^m$ and there are canonical inclusions $\Delta^k \hookrightarrow \Delta^k * \Delta^m$ and $\Delta^m \hookrightarrow \Delta^k * \Delta^m$. Suppose $\{\nu_1, \dots, \nu_{k+m+2}\}$ are the vertices of Δ^{k+m+1} so that $\{\nu_1, \dots, \nu_{k+1}\}$ (resp. $\{\nu_{k+2}, \dots, \nu_{k+m+2}\}$) are the vertices of Δ^k (resp. Δ^m). We note that

$$\rho_{\#}(\tilde{\alpha}_{i,1} * \tilde{\beta}_j)|_{[\nu_1, \dots, \nu_{k+1}]} = \iota_1 \circ \alpha_i = \rho_{\#}(\tilde{\alpha}_{i,2} * \tilde{\beta}_j)|_{[\nu_1, \dots, \nu_{k+1}]}. \quad (4.3.19)$$

Similarly,

$$\rho_{\#}(\tilde{\alpha}_{i,1} * \tilde{\beta}_j)|_{[\nu_{k+2}, \dots, \nu_{k+m+2}]} = \iota_2 \circ \beta_j = \rho_{\#}(\tilde{\alpha}_{i,2} * \tilde{\beta}_j)|_{[\nu_{k+2}, \dots, \nu_{k+m+2}]}. \quad (4.3.20)$$

Moreover,

$$\begin{aligned} (\tilde{\alpha}_{i,1} * \tilde{\beta}_j)|_{[\nu_{k+1}, \nu_{k+2}]} &\text{ is a curve joining } \tilde{\alpha}_{i,1}(\nu_{k+1}) \text{ and } \tilde{\beta}_j(\nu_{k+2}); \\ (\tilde{\alpha}_{i,2} * \tilde{\beta}_j)|_{[\nu_{k+1}, \nu_{k+2}]} &\text{ is a curve joining } \tilde{\alpha}_{i,2}(\nu_{k+1}) \text{ and } \tilde{\beta}_j(\nu_{k+2}). \end{aligned}$$

Since $\tilde{\alpha}_{i,2} = T_1 \circ \tilde{\alpha}_{i,1}$,

$$\rho_{\#}(\tilde{\alpha}_{i,1} * \tilde{\beta}_j)|_{[\nu_{k+1}, \nu_{k+2}]} + \rho_{\#}(\tilde{\alpha}_{i,2} * \tilde{\beta}_j)|_{[\nu_{k+1}, \nu_{k+2}]}$$

is a non-contractible loop in Z . Hence, using (4.3.19), (4.3.20) and (4.3.6), we obtain

$$\begin{aligned} &h^{k+m+1} \left(\rho_{\#}(\tilde{\alpha}_{i,1} * \tilde{\beta}_j) + \rho_{\#}(\tilde{\alpha}_{i,2} * \tilde{\beta}_j) \right) \\ &= h^k (\iota_1 \circ \alpha_i) h \left(\rho_{\#}(\tilde{\alpha}_{i,1} * \tilde{\beta}_j)|_{[\nu_{k+1}, \nu_{k+2}]} + \rho_{\#}(\tilde{\alpha}_{i,2} * \tilde{\beta}_j)|_{[\nu_{k+1}, \nu_{k+2}]} \right) h^m (\iota_2 \circ \beta_j) \\ &= h^k (\iota_1 \circ \alpha_i) h^m (\iota_2 \circ \beta_j). \end{aligned} \quad (4.3.21)$$

(4.3.8), (4.3.9), (4.3.11) and (4.3.21) imply that $\lambda^{k+m+1}.c = 1$. $\square$

Part 4. Since $\Psi : Z \rightarrow \mathcal{Z}_n(M; \mathbb{Z}_2)$ is a $(k + m + 1)$ -sweepout with no concentration of mass,

$$\sup_{z \in Z} \{\mathbf{M}(\Psi(z))\} \geq \omega_{k+m+1}. \quad (4.3.22)$$

The following proposition essentially follows from the proof of Proposition 4.2.1.

Proposition 4.3.4. *For all $(x, y, t) \in \tilde{X} * \tilde{Y}$,*

$$\mathbf{M}(\partial\tilde{\Psi}(x, y, t)) \leq \mathbf{M}(\Phi_1(x)) + \mathbf{M}(\Phi_2(y)) + \mathbf{M}(\partial\Lambda(t)).$$

Proof. By [MPPP07, Proposition 1.4], there exist $\{u_i\}_{i=1}^\infty$, $\{v_i\}_{i=1}^\infty$, $\{w_i\}_{i=1}^\infty \subset C^\infty(M)$ such that $0 \leq u_i, v_i, w_i \leq 1$ for all i ;

$$\begin{aligned} u_i &\rightarrow \chi_{\tilde{\Phi}_1(x)} \text{ in } L^1(M) \text{ and pointwise a.e.;} & \mathbf{M}(\Phi_1(x)) &= \lim_{i \rightarrow \infty} \int_M |\nabla u_i|; \\ v_i &\rightarrow \chi_{\tilde{\Phi}_2(y)} \text{ in } L^1(M) \text{ and pointwise a.e.;} & \mathbf{M}(\Phi_2(y)) &= \lim_{i \rightarrow \infty} \int_M |\nabla v_i|; \\ w_i &\rightarrow \chi_{\Lambda(t)} \text{ in } L^1(M) \text{ and pointwise a.e.;} & \mathbf{M}(\partial\Lambda(t)) &= \lim_{i \rightarrow \infty} \int_M |\nabla w_i|. \end{aligned}$$

By the dominated convergence theorem,

$$u_i w_i + v_i (1 - w_i) \rightarrow \chi_{\tilde{\Psi}(x, y, t)} \text{ in } L^1(M).$$

Therefore,

$$\begin{aligned} \mathbf{M}(\partial\tilde{\Psi}(x, y, t)) &\leq \liminf_{i \rightarrow \infty} \int_M |\nabla(u_i w_i) + \nabla(v_i (1 - w_i))| \\ &\leq \liminf_{i \rightarrow \infty} \int_M |\nabla u_i| + |\nabla v_i| + |\nabla w_i| \\ &= \mathbf{M}(\Phi_1(x)) + \mathbf{M}(\Phi_2(y)) + \mathbf{M}(\partial\Lambda(t)). \end{aligned}$$

$\square$

Combining Proposition 4.3.4, (4.3.1), (4.3.2), (4.3.3) and (4.3.22), we obtain

$$\omega_{k+m+1} \leq \omega_k + \omega_m + W_0 + 3\delta.$$

Since $\delta > 0$ is arbitrary, this finishes the proof of Theorem 4.1.1.

4.4 Proof of Theorem 4.1.2

If $k = m = 0$, (4.1.6) reduces to $c_\varepsilon(1) \leq \gamma_\varepsilon$, which holds by the definitions of $c_\varepsilon(1)$ and γ_ε (see [GG18, Remark 3.6]). So let us assume that $k \geq 1$. We fix $\delta > 0$. There exists a connected simplicial complex $\tilde{X} \in \mathcal{C}_k$ and $\tilde{h}_1 \in \Gamma(\tilde{X})$ such that

$$\sup_{x \in \tilde{X}} E_\varepsilon(\tilde{h}_1(x)) \leq c_\varepsilon(k) + \delta. \quad (4.4.1)$$

Similarly, for $m \geq 1$, there exists a connected simplicial complex $\tilde{Y} \in \mathcal{C}_m$ and $\tilde{h}_2 \in \Gamma(\tilde{Y})$ such that

$$\sup_{y \in \tilde{Y}} E_\varepsilon(\tilde{h}_2(y)) \leq c_\varepsilon(m) + \delta. \quad (4.4.2)$$

If $m = 0$, we choose $\tilde{Y} = \{-1, 1\}$ and $\tilde{h}_2(\pm 1) \equiv \pm 1$. Let $w : [0, 1] \rightarrow H^1(M)$ be such that $w(0) \equiv 1$, $w(1) \equiv -1$ and

$$\sup_{t \in [0, 1]} E_\varepsilon(w(t)) \leq \gamma_\varepsilon + \delta. \quad (4.4.3)$$

Suppose $\mathcal{T} : H^1(M) \rightarrow H^1(M)$ is the following truncation map (see [Gua18, Section 4]).

$$\mathcal{T}(u) = \min\{\max\{u, -1\}, 1\}.$$

Then $\mathcal{T}$ is continuous, $-1 \leq \mathcal{T}(u) \leq 1$, $\mathcal{T}(-u) = -\mathcal{T}(u)$ and $E_\varepsilon(\mathcal{T}(u)) \leq E_\varepsilon(u)$. Therefore, without loss of generality, we can assume that

$$-1 \leq \tilde{h}_1(x), \tilde{h}_2(y), w(t) \leq 1 \quad (4.4.4)$$

for all $x \in \tilde{X}$, $y \in \tilde{Y}$ and $t \in [0, 1]$.

For $a \in \mathbb{R}$, let $a^+ = \max\{a, 0\}$; $a^- = \min\{a, 0\}$. The following maps $\phi, \psi, \theta : H^1(M) \times H^1(M) \times H^1(M) \rightarrow H^1(M)$ are defined in Chapter 5.

$$\begin{aligned}\phi(u_1, u_2, v) &= \min\{\max\{u_1, -v\}, \max\{u_2, v\}\}; \\ \psi(u_1, u_2, v) &= \max\{\min\{u_1, v\}, \min\{u_2, -v\}\}; \\ \theta(u_1, u_2, v) &= \phi(u_1, u_2, v)^+ + \psi(u_1, u_2, v)^-.\end{aligned}$$

Proposition 4.4.1 (Proposition 5.3.10). *The map θ , defined above, has the following properties.*

- (i) θ is continuous.
- (ii) $\theta(-u_1, -u_2, v) = -\theta(u_1, u_2, v)$.
- (iii) If $-1 \leq u_1, u_2 \leq 1$, $\theta(u_1, u_2, 1) = u_1$ and $\theta(u_1, u_2, -1) = u_2$.
- (iv) $E_\varepsilon(\theta(u_1, u_2, v)) \leq E_\varepsilon(u_1) + E_\varepsilon(u_2) + E_\varepsilon(v)$.

We define $\tilde{q} : \tilde{X} * \tilde{Y} \rightarrow H^1(M)$ as follows.

$$\tilde{q}(x, y, t) = \theta(\tilde{h}_1(x), \tilde{h}_2(y), w(t)). \quad (4.4.5)$$

By (4.4.4) and Proposition 4.4.1, items (i), (iii), $\tilde{q}$ is a well-defined, continuous map and

$$\tilde{q}|_{\tilde{X}} = \tilde{h}_1; \quad \tilde{q}|_{\tilde{Y}} = \tilde{h}_2. \quad (4.4.6)$$

(We recall from Part 3 of the proof of Theorem 4.1.1 that $\tilde{X}$ and $\tilde{Y}$ can be naturally identified as subspaces of $\tilde{X} * \tilde{Y}$.)

Claim 4.4.2. $\tilde{Z} = \tilde{X} * \tilde{Y} \in \mathcal{C}_{k+m+1}$ and there exists $\tilde{h} \in \Gamma(\tilde{Z})$ such that

$$E_\varepsilon(\tilde{h}(z)) \leq E_\varepsilon(\tilde{q}(z)) + \delta \quad \forall z \in \tilde{Z}. \quad (4.4.7)$$

Proof. Let $T_1 : \tilde{X} \rightarrow \tilde{X}$ be such that the $\mathbb{Z}_2$ action on $\tilde{X}$ is given by $x \mapsto T_1(x)$ and $T_2 : \tilde{Y} \rightarrow \tilde{Y}$ be such that the $\mathbb{Z}_2$ action on $\tilde{Y}$ is given by $y \mapsto T_2(y)$. Let $\{\Delta_{1,i} : i = 1, \dots, 2I\}$ and $\{\Delta_{2,j} : j = 1, \dots, 2J\}$ be the set of simplices in $\tilde{X}$ and $\tilde{Y}$. As discussed in Section 4.2.4, one can put a canonical simplicial complex structure on $\tilde{Z} = \tilde{X} * \tilde{Y}$, whose simplices are $\{\Delta_{1,i}, \Delta_{2,j}, \Delta_{1,i} * \Delta_{2,j} : i = 1, \dots, 2I; j = 1, \dots, 2J\}$. There is a free, simplicial $\mathbb{Z}_2$ action on $\tilde{Z}$ given by

$$z = (x, y, t) \mapsto T(z) = (T_1(x), T_2(y), t).$$

By Proposition 4.4.1(ii), $\tilde{q}(T(z)) = -\tilde{q}(z)$ for all $z \in \tilde{Z}$. There might exist $z \in \tilde{Z}$ such that $\tilde{q}(z) = 0 \in H^1(M)$. However, as we discuss below, one can perturb $\tilde{q}$ and obtain a new map $\tilde{h} : \tilde{Z} \rightarrow H^1(M) \setminus \{0\}$, which is $\mathbb{Z}_2$ -equivariant and satisfies

$$E_\varepsilon(\tilde{h}(z)) \leq E_\varepsilon(\tilde{q}(z)) + \delta \quad \forall z \in \tilde{Z}; \quad (4.4.8)$$

$$\tilde{h}|_{\tilde{X}} = \tilde{q}|_{\tilde{X}} = \tilde{h}_1; \quad \tilde{h}|_{\tilde{Y}} = \tilde{q}|_{\tilde{Y}} = \tilde{h}_2. \quad (4.4.9)$$

For $R > 0$, let

$$B_R = \{u \in H^1(M) : \|u\|_{H^1(M)} < R\}.$$

$B_R \setminus \{0\}$, being homeomorphic to $H^1(M) \setminus \{0\}$, is contractible. Let $r > 0$ be such that

$$|E_\varepsilon(u) - E_\varepsilon(0)| < \delta/2 \quad \forall u \in B_r. \quad (4.4.10)$$

Let $\{e_s, f_s : s = 1, \dots, S\}$ be the set of all simplices in $\tilde{Z}$, indexed in such a way that $T(e_s) = f_s$ and $s_1 \leq s_2$ implies $\dim(e_{s_1}) \leq \dim(e_{s_2})$. Suppose

$$\tilde{Z}_0 = \{z \in \tilde{Z} : \tilde{q}(z) = 0 \in H^1(M)\}.$$

Using barycentric subdivision, without loss of generality one can assume that

$$\text{if } \alpha \text{ is a simplex in } \tilde{Z} \text{ and } \alpha \cap \tilde{Z}_0 \neq \emptyset, \text{ then } \tilde{q}(\alpha) \subset B_r. \quad (4.4.11)$$

We will define $\tilde{h}$ inductively on the simplices of $\tilde{Z}$ so that

(i) $\tilde{h} : \tilde{Z} \rightarrow H^1(M) \setminus \{0\}$ is $\mathbb{Z}_2$ -equivariant;

(ii) if α is a simplex in $\tilde{Z}$ with $\alpha \cap \tilde{Z}_0 = \emptyset$ then

$$\tilde{h}|_{\alpha} = \tilde{q}|_{\alpha}; \quad (4.4.12)$$

(iii) if α is a simplex in $\tilde{Z}$ with $\alpha \cap \tilde{Z}_0 \neq \emptyset$ then

$$\tilde{h}(\alpha) \subset B_r \setminus \{0\}. \quad (4.4.13)$$

By abuse of notation, let us identify a simplex of $\tilde{Z}$ with its support. If s is such that $\dim(e_s) = 0$, we set

$$\tilde{h}(e_s) = \begin{cases} \tilde{q}(e_s) & \text{if } \tilde{q}(e_s) \neq 0 \\ \text{an arbitrary point in } B_r \setminus \{0\} & \text{if } \tilde{q}(e_s) = 0 \end{cases} \quad (4.4.14)$$

and $\tilde{h}(f_s) = -\tilde{h}(e_s)$. Let us assume that we have defined

$$\tilde{h} : \bigcup_{s \leq l-1} (e_s \cup f_s) \rightarrow H^1(M) \setminus \{0\}$$

in such a way that

(i) $\tilde{h}(T(z)) = -\tilde{h}(z)$ for all $z \in \bigcup_{s \leq l-1} (e_s \cup f_s)$;

(ii) $\tilde{h}|_{e_s \cup f_s} = \tilde{q}|_{e_s \cup f_s}$ if $s \leq l-1$ and $e_s \cap \tilde{Z}_0 = \emptyset$ (which is equivalent to $f_s \cap \tilde{Z}_0 = \emptyset$);

(iii) $(\tilde{h}(e_s) \cup \tilde{h}(f_s)) \subset B_r \setminus \{0\}$ if $s \leq l-1$ and $e_s \cap \tilde{Z}_0 \neq \emptyset$ (which is equivalent to $f_s \cap \tilde{Z}_0 \neq \emptyset$).

We define $\tilde{h} : (e_l \cup f_l) \rightarrow H^1(M) \setminus \{0\}$ as follows. By the induction hypothesis, $\tilde{h}$ has already been defined on $(\partial e_l \cup \partial f_l)$. If $e_l \cap \tilde{Z}_0 = \emptyset = f_l \cap \tilde{Z}_0$, we set

$$\tilde{h}|_{e_l \cup f_l} = \tilde{q}|_{e_l \cup f_l}.$$

Otherwise $e_l \cap \tilde{Z}_0, f_l \cap \tilde{Z}_0 \neq \emptyset$, which, together with (4.4.11) and the induction hypothesis, implies that $\tilde{h}(\partial e_l \cup \partial f_l) \subset B_r \setminus \{0\}$. $\tilde{h}$ is defined on e_l in such a way that the definition of $\tilde{h}$ on ∂e_l matches with its previous definition and $\tilde{h}(e_l) \subset B_r \setminus \{0\}$. Such an extension is possible as $B_r \setminus \{0\}$ is contractible. For $z \in f_l$, $\tilde{h}(z)$ is defined to be equal to $-\tilde{h}(T(z))$.

Thus we have defined $\tilde{h} : \tilde{Z} \rightarrow H^1(M) \setminus \{0\}$, which is $\mathbb{Z}_2$ -equivariant. (4.4.8) and (4.4.9) can be proved using (4.4.10) – (4.4.13)

Suppose Z and $\mathcal{H}(M)$ denote the quotients of $\tilde{Z}$ and $H^1(M) \setminus \{0\}$, respectively, by $\mathbb{Z}_2$. Then $\tilde{h}$ descends to a map $h : Z \rightarrow \mathcal{H}(M)$. As discussed in Section 4.2.3, $H^*(\mathcal{H}(M), \mathbb{Z}_2) = \mathbb{Z}_2[\xi]$. Furthermore, $\tilde{Z} \in \mathcal{C}_{k+m+1}$ if and only if

$$h^*(\xi^{k+m+1}) \neq 0 \in H^{k+m+1}(Z, \mathbb{Z}_2). \quad (4.4.15)$$

However, since $\tilde{X} \in \mathcal{C}_k$, if X denotes the quotient of $\tilde{X}$ by $\mathbb{Z}_2$, $\tilde{h}_1$ descends to a map $h_1 : X \rightarrow \mathcal{H}(M)$ such that

$$h_1^*(\xi^k) \neq 0 \in H^k(X, \mathbb{Z}_2). \quad (4.4.16)$$

Similarly, as $\tilde{Y} \in \mathcal{C}_m$, if Y denotes the quotient of $\tilde{Y}$ by $\mathbb{Z}_2$, $\tilde{h}_2$ descends to a map $h_2 : Y \rightarrow \mathcal{H}(M)$ such that

$$h_2^*(\xi^m) \neq 0 \in H^m(Y, \mathbb{Z}_2). \quad (4.4.17)$$

One can prove (4.4.15) by using (4.4.16), (4.4.17), (4.4.9) and an argument similar to Part 3 of the proof of Theorem 4.1.1. $\square$

By the above Claim 4.4.2,

$$\sup_{z \in \tilde{X} * \tilde{Y}} E_\varepsilon(\tilde{h}(z)) \geq c_\varepsilon(k + m + 1). \quad (4.4.18)$$

On the other hand, by (4.4.7) and Proposition 4.4.1(iv), for all $(x, y, t) \in \tilde{X} * \tilde{Y}$,

$$E_\varepsilon(\tilde{h}(x, y, t)) \leq E_\varepsilon(\tilde{q}(x, y, t)) + \delta \leq E_\varepsilon(\tilde{h}_1(x)) + E_\varepsilon(\tilde{h}_2(y)) + E_\varepsilon(w(t)) + \delta. \quad (4.4.19)$$

Combining (4.4.18), (4.4.19), (4.4.1), (4.4.2) and (4.4.3), we obtain

$$c_\varepsilon(k+m+1) \leq c_\varepsilon(k) + c_\varepsilon(m) + \gamma_\varepsilon + 4\delta.$$

Since $\delta > 0$ is arbitrary, this finishes the proof of Theorem 4.1.2.

Chapter 5

A comparison of the Almgren-Pitts and the Allen-Cahn min-max theory

5.1 Introduction

In [Gua18], Guaraco introduced a new approach for the min-max construction of minimal hypersurfaces which was further developed by Gaspar and Guaraco in [GG18]. This approach is based on the study of the limiting behaviour of solutions to the Allen-Cahn equation. The Allen-Cahn equation (with parameter $\varepsilon > 0$) is the following semi-linear, elliptic PDE

$$AC_\varepsilon(u) := \varepsilon \Delta u - \varepsilon^{-1} W'(u) = 0 \tag{5.1.1}$$

where $W : \mathbb{R} \rightarrow [0, \infty)$ is a double well potential, e.g. $W(t) = \frac{1}{4}(1 - t^2)^2$. The solutions of this equation are precisely the critical points of the energy functional

$$E_\varepsilon(u) = \int_M \varepsilon \frac{|\nabla u|^2}{2} + \frac{W(u)}{\varepsilon}.$$

Informally speaking, as $\varepsilon \rightarrow 0$, the level sets of the solutions to (5.1.1) (with uniformly bounded energy) accumulate around a generalized minimal hypersurface (called a *limit-interface*). In particular, Modica [Mod87] and Sternberg [Ste88] proved that as $\varepsilon \rightarrow 0$, the energy minimizing solutions to (5.1.1) converge to a area minimizing hypersurface. For general solutions to (5.1.1), Hutchinson and Tonegawa [HT00] proved that the limit-interface is a stationary, integral varifold. Moreover, if the solutions are stable, by the works of Tonegawa [Ton05], Wickramasekera [Wic14a] and Tonegawa-Wickramasekera [TW12], the limit-interface is a stable minimal hypersurface with optimal regularity. In [Gua18], Guaraco proved that the limit-interface has optimal regularity if the solutions have uniformly bounded Morse index. Furthermore, by a mountain-pass argument he proved the existence of critical points of E_ε with uniformly bounded energy and Morse index. In this way he obtained a new proof of the previously mentioned theorem of Almgren-Pitts-Schoen-Simon. The index of the limit-interface is bounded by the index of the solutions. This was proved by Hiesmayr [Hie18] assuming the limit-interface is two-sided and by Gaspar [Gas19] in the general case.

In [GG18, GG19], Gaspar and Guaraco studied the phase transition spectrum which is the Allen-Cahn analogue of the volume spectrum. They proved that the phase transition spectrum satisfies a Weyl law similar to the volume spectrum and gave alternative proofs of the density [IMN18] and the equidistribution [MNS19] theorems. In [CM20], Chodosh and Mantoulidis proved the multiplicity one conjecture in the Allen-Cahn setting in dimension 3 and the upper semi-continuity of the Morse index when the limit-interface has multiplicity one. As a consequence, they proved that for a generic (bumpy) metric g on a closed manifold M^3 , there exists a sequence of closed, embedded, two-sided minimal surfaces $\{\Sigma_p\}_{p=1}^\infty$ in (M^3, g) such that $\text{Ind}(\Sigma_p) = p$ and $\text{area}(\Sigma_p) \sim p^{1/3}$. In higher dimensions, the multiplicity one conjecture for the one parameter Allen-Cahn min-max has

been proved by Bellettini [Bel20a, Bel20b]. In [GMN19], Guaraco, Marques and Neves proved that a strictly stable limit-interface must have multiplicity one.

The Allen-Cahn equation on the surfaces has been studied by Mantoulidis [Man21] and Chodosh-Mantoulidis [CM21]. In particular, they have proved that the geodesic limit-interface, obtained from the Allen-Cahn min-max theory, consists of immersed, closed geodesics. On the other hand, the Almgren-Pitts min-max theory on the surfaces produce geodesic networks, which may not be smooth [MN16, Remark 1.1].

In [BW20], Bellettini and Wickramasekera proved the existence of closed prescribed mean curvature (PMC) hypersurfaces in arbitrary closed Riemannian manifolds using the min-max solutions of the inhomogeneous Allen-Cahn equations. To prove the regularity of the Allen-Cahn PMC hypersurfaces, they used their earlier works [BW18, BW19] on the regularity and compactness theory of stable PMC hypersurfaces. Previously, Zhou and Zhu [ZZ19, ZZ20] developed a min-max theory for the construction of closed PMC hypersurfaces which is parallel to the Almgren-Pitts min-max theory. The estimates for the index and nullity of the Allen-Cahn PMC hypersurfaces have been proved by Mantoulidis [Man20].

The asymptotic behaviour of the critical points of the Ginzburg-Landau functional (which approximates the codimension-2 area functional) has been studied by Stern [Ste21], Cheng [Che17], Pigati-Stern [PS21] and Parise-Pigati-Stern [PPS21]. In particular, in [PS21], Pigati and Stern proved the existence of a codimension-2 stationary, integral varifold in an arbitrary closed Riemannian manifold. This theorem was previously proved by Almgren [Alm65] using more complicated geometric measure theory approach. The Γ -convergence of the $U(1)$ -Yang-Mills-Higgs functional to the codimension-2 area functional has been proved by Parise, Pigati and Stern [PPS21].

If Σ is a non-degenerate, separating, closed, embedded minimal hypersurface in a closed Riemannian manifold, Pacard and Ritoré [PR03] constructed solutions of the Allen-Cahn equation $AC_\varepsilon(u) = 0$ for sufficiently small $\varepsilon > 0$, whose level sets converge to Σ . The uniqueness of these solutions has been proved by Guaraco, Marques and Neves [GMN19]. The construction of Pacard and Ritoré has been extended by Caju and Gaspar [CG19] in the case when all the Jacobi fields of Σ are induced by the ambient isometries. Assuming a positivity condition on the Ricci curvature

of the ambient manifold, del Pino-Kowalczyk-Wei-Yang [dKWY10] constructed solutions of the Allen-Cahn equation whose energies concentrate on a non-degenerate, closed minimal hypersurface with multiplicity > 1 .

In this chapter we will be interested in the question to what extent the Almgren-Pitts min-max theory and the Allen-Cahn min-max theory agree. Part of this question has been answered by Guaraco [Gua18] and Gaspar-Guaraco [GG18]; they proved that the Almgren-Pitts widths are less than or equal to the Allen-Cahn widths. (The analogous inequality in the codimension-2 case has been proved by Parise-Pigati-Stern [PPS21].) In this chapter, we will prove the reverse inequality, i.e the Allen-Cahn widths are less than or equal to the Almgren-Pitts widths.

To precisely state our main result, we need some facts about the universal G -principal bundle. We will follow the book by Dieck [Die08, Chapter 14.4] and the paper by Gaspar and Guaraco [GG18, Appendix B], where further details can be found. Let G be a topological group and $p_G : EG \rightarrow BG$ be a universal G -principal bundle (which is unique up to isomorphism). Given a topological space B , there exists a one-to-one correspondence between the set of homotopy classes of maps $B \rightarrow BG$ and the set of isomorphism classes of numerable principal G -bundles over B . If $f_1, f_2 : B \rightarrow BG$ are homotopic, $f_1^* EG$ and $f_2^* EG$ are isomorphic numerable principal G -bundles over B . Conversely, if E is a numerable free G -space, there exists a G -map from E to EG which is unique up to G -homotopy. Denoting $B = E/G$, if $F_1, F_2 : E \rightarrow EG$ are G -maps, they descend to homotopic maps $f_1, f_2 : B \rightarrow BG$. We also note the following facts: a numerable G -principal bundle $\mathfrak{p} : \mathcal{E} \rightarrow \mathcal{B}$ is universal if $\mathcal{E}$ is a contractible topological space [Die08, 14.4.12]; each open covering of a paracompact space is numerable [Die08, 13.1.3].

We refer to Section 5.2 for the definitions and notations used in the rest of this section. Let M^{n+1} be a closed Riemannian manifold. Suppose X is a cubical complex which has a non-trivial double cover $\pi : \tilde{X} \rightarrow X$. Since the space $\mathbf{I}_{n+1}(M; \mathcal{F}; \mathbb{Z}_2)$ is contractible [MN21] and every metric space is paracompact, $\partial : \mathbf{I}_{n+1}(M; \mathcal{F}; \mathbb{Z}_2) \rightarrow \mathcal{Z}_n(M; \mathcal{F}; \mathbb{Z}_2)$ is a universal $\mathbb{Z}_2$ -principal bundle. As a consequence, there exists a homotopy class Π' of maps $X \rightarrow \mathcal{Z}_n(M; \mathcal{F}; \mathbb{Z}_2)$ corresponding to the double cover $\pi : \tilde{X} \rightarrow X$. More concretely, by [Hat02, Propositions 1.33, 1.36], Π' is the set of all

maps $\Phi : X \rightarrow \mathcal{Z}_n(M; \mathcal{F}; \mathbb{Z}_2)$ such that $\ker(\Phi_*) = \text{im}(\pi_*)$, where

$$\Phi_* : \pi_1(X) \rightarrow \pi_1(\mathcal{Z}_n(M; \mathcal{F}; \mathbb{Z}_2)) (= \mathbb{Z}_2) \quad \text{and} \quad \pi_* : \pi_1(\tilde{X}) \rightarrow \pi_1(X)$$

are the maps induced by Φ and π . However, in the Almgren-Pitts min-max theory, one needs to work with the $\mathbf{F}$ metric instead of the flat topology. The space $\mathbf{I}_{n+1}(M; \mathbf{F}; \mathbb{Z}_2)$ is simply-connected and $\partial : \mathbf{I}_{n+1}(M; \mathbf{F}; \mathbb{Z}_2) \rightarrow \mathcal{Z}_n(M; \mathbf{F}; \mathbb{Z}_2)$ is a double cover. Let Π denote the set of all maps $\Phi : X \rightarrow \mathcal{Z}_n(M; \mathbf{F}; \mathbb{Z}_2)$ such that $\ker(\Phi_*) = \text{im}(\pi_*)$ (where $\Phi_* : \pi_1(X) \rightarrow \pi_1(\mathcal{Z}_n(M; \mathbf{F}; \mathbb{Z}_2)) (= \mathbb{Z}_2)$). By the above discussion, any two maps $\Phi_1, \Phi_2 \in \Pi$ are homotopic in the flat topology. To show that Π is a well-defined homotopy class with respect to the $\mathbf{F}$ metric, we consider $\tilde{\Phi}_i : \tilde{X} \rightarrow \mathbf{I}_{n+1}(M; \mathbf{F}; \mathbb{Z}_2)$ ($i = 1, 2$), which is a lift of Φ_i , i.e. $\partial \circ \tilde{\Phi}_i = \Phi_i \circ \pi$. One can explicitly define a $\mathbb{Z}_2$ -homotopy $\tilde{H} : [0, 1] \times \tilde{X} \rightarrow \mathbf{I}_{n+1}(M; \mathcal{F}; \mathbb{Z}_2)$ between $\tilde{\Phi}_1$ and $\tilde{\Phi}_2$ by

$$\tilde{H}(t, x) = \tilde{\Phi}_1(x) \mathsf{L} \{f > t\} + \tilde{\Phi}_2(x) \mathsf{L} \{f < t\},$$

where $f : M \rightarrow [0, 1]$ is a Morse function. Then $\tilde{H}$ descends to a homotopy $H : [0, 1] \times X \rightarrow \mathcal{Z}_n(M; \mathcal{F}; \mathbb{Z}_2)$ between Φ_1 and Φ_2 , which has no concentration of mass. Therefore, by the interpolation theorems of Marques and Neves [MN21, Section 3], Φ_1 and Φ_2 are homotopic in the $\mathbf{F}$ metric.

On the other hand, $H^1(M) \setminus \{0\}$ is also contractible and there is a free $\mathbb{Z}_2$ action on this space given by $u \mapsto -u$. Therefore, $H^1(M) \setminus \{0\}$ (equipped with the $\mathbb{Z}_2$ action) is the total space of a universal $\mathbb{Z}_2$ -principal bundle. Let $\tilde{\Pi}$ denote the set of all $\mathbb{Z}_2$ -equivariant maps $h : \tilde{X} \rightarrow H^1(M) \setminus \{0\}$, i.e. if $T : \tilde{X} \rightarrow \tilde{X}$ is the deck transformation, $h(T(x)) = -h(x)$ for all $x \in \tilde{X}$.

The following theorem follows from the works of Guaraco [Gua18] and Gaspar-Guaraco [GG18]. We refer to Chapter 4 for the definitions of the volume spectrum $\{\omega_p\}_{p=1}^\infty$ and the ε -phase-transition spectrum $\{c_\varepsilon(p)\}_{p=1}^\infty$.

Theorem 5.1.1 ([Gua18, GG18]). *Let $\mathbf{L}_{AP}(\Pi)$ be the Almgren-Pitts width of Π ((5.2.1)) and $\mathbf{L}_\varepsilon(\tilde{\Pi})$*

be the ε -Allen-Cahn width of $\tilde{\Pi}$ ((5.2.2)). Then there holds

$$\mathbf{L}_{AP}(\Pi) \leq \frac{1}{2\sigma} \liminf_{\varepsilon \rightarrow 0^+} \mathbf{L}_\varepsilon(\tilde{\Pi}). \quad (5.1.2)$$

As a consequence, the following inequality holds between the volume spectrum and the phase transition spectrum.

$$\omega_p \leq \frac{1}{2\sigma} \liminf_{\varepsilon \rightarrow 0^+} c_\varepsilon(p) \quad \forall p \in \mathbb{N}. \quad (5.1.3)$$

In this chapter we will show that the reverse inequality also holds. More precisely, we will prove the following theorem.

Theorem 5.1.2. *The following inequality holds between the Almgren-Pitts width and the Allen-Cahn width.*

$$\frac{1}{2\sigma} \limsup_{\varepsilon \rightarrow 0^+} \mathbf{L}_\varepsilon(\tilde{\Pi}) \leq \mathbf{L}_{AP}(\Pi). \quad (5.1.4)$$

As a consequence,

$$\frac{1}{2\sigma} \limsup_{\varepsilon \rightarrow 0^+} c_\varepsilon(p) \leq \omega_p \quad \forall p \in \mathbb{N}. \quad (5.1.5)$$

Hence, combining (5.1.2) and (5.1.4), we conclude that $\frac{1}{2\sigma} \lim_{\varepsilon \rightarrow 0^+} \mathbf{L}_\varepsilon(\tilde{\Pi})$ exists and is equal to $\mathbf{L}_{AP}(\Pi)$. Similarly, (5.1.3) and (5.1.5) together imply that $\frac{1}{2\sigma} \lim_{\varepsilon \rightarrow 0^+} c_\varepsilon(p)$ exists and is equal to ω_p for all $p \in \mathbb{N}$. When the ambient dimension $3 \leq n+1 \leq 7$, it was proved by Gaspar and Guaraco [GG19] that $\lim_{\varepsilon \rightarrow 0^+} c_\varepsilon(p)$ exists.

The next theorem essentially follows from the works of Hutchinson-Tonegawa [HT00], Guaraco [Gua18] and Gaspar-Guaraco [GG18]. Informally speaking, it says that all the minimal hypersurfaces, obtained from the Allen-Cahn min-max theory, are also produced by the Almgren-Pitts min-max theory. This can be interpreted as the min-max analogue of the above mentioned theorem of Modica [Mod87] and Sternberg [Ste88], which says that as $\varepsilon \rightarrow 0$, the minimizers of E_ε converge to a minimizer of the area functional.

Theorem 5.1.3. *Let $\mathbf{C}_{AC}(\tilde{\Pi})$ be as defined at the end of Section 5.2.5 and $\mathbf{C}_{AP}(\Pi)$ be as defined at the end of Section 5.2.3. If $V \in \mathbf{C}_{AC}(\tilde{\Pi})$, then $V \in \mathbf{C}_{AP}(\Pi)$ as well.*

Combining the index estimate of Gaspar [Gas19] (Theorem 5.2.2) and the above Theorem 5.1.3, one can obtain an alternative proof of the following Morse index upper bound in the Almgren-Pitts min-max theory, proved by Marques-Neves [MN16] and Li [Li20].

Theorem 5.1.4 ([MN16, Li20]). *Let $\dim(X) = \dim(\tilde{X}) = k$. There exists $V \in \mathbf{C}_{AP}(\Pi)$ such that $\text{Ind}(\text{spt}(V)) \leq k$.*

Indeed, by the min-max theory for the Allen-Cahn functional (see Section 5.2.4), for all sufficiently small $\varepsilon > 0$ there exists a min-max critical point u_ε of E_ε (corresponding to the homotopy class $\tilde{\Pi}$) such that $\text{Ind}(u_\varepsilon) \leq k$. Hence, by Theorem 6.2.1, 5.2.2 and 5.1.3, there exists

$$V \in \mathbf{C}_{AC}(\tilde{\Pi}) \subset \mathbf{C}_{AP}(\Pi)$$

such that $\text{Ind}(\text{spt}(V)) \leq k$.

5.2 Notations and Preliminaries

5.2.1 Notations

Here we summarize the notations which will be frequently used later in this chapter.

$[m]$	the set $\{1, 2, \dots, m\}$
$\mathcal{H}^s$	the Hausdorff measure of dimension s
$A \dot{\cup} B$	the disjoint union of A and B
$\text{int}(A), \overline{A}$	the interior of A , the closure of A (in a topological space)
∂A	the topological boundary of $A = \overline{A} \setminus \text{int}(A)$; ∂ will also denote the boundary of a current or the boundary of a cell in a cell-complex.
$\mathcal{C}(M)$	the space of Caccioppoli sets in M
$\partial^* E$	the reduced boundary of a Caccioppoli set E
$\llbracket S \rrbracket$	the current associated to the rectifiable set S

$ \Sigma $	the varifold associated to the rectifiable set Σ
$\ V\ $	the Radon measure associated to the varifold V
B^c	the complement of B in M , i.e. $M \setminus B$
$B(p, R)$	the geodesic ball centered at p with radius R
$A(p, r, R)$	the annulus centered at p with radii $r < R$
$d(-, S)$	distance from a set $S \subset (M, g)$
$\mathcal{N}_\rho(S)$	the set of points $y \in M$ such that $d(y, S) \leq \rho$
$\mathcal{T}_\rho(S)$	the set of points $y \in M$ such that $d(y, S) = \rho$
$H^1(M)$	the Sobolev space $\{f \in L^2(M) : \text{the distributional derivative } \nabla f \in L^2(M)\}$

5.2.2 Varifolds

Here we will briefly discuss the notion of the varifold; further details can be found in Simon's book [Sim83]. Given a Riemannian manifold M^{n+1} , let $G_n M$ denote the Grassmanian bundle of unoriented n -dimensional hyperplanes over M . An n -varifold in M is a positive Radon measure on $G_n M$. If V is an n -varifold and $\mathbf{p} : G_n M \rightarrow M$ is the canonical projection map, then $\|V\| := \mathbf{p}_* V$ is a Radon measure on M ; $\|V\|(A) = V(\mathbf{p}^{-1}(A))$. The topology on the space of n -varifolds is given by the weak* topology, i.e. a net $\{V_i\}_{i \in \mathcal{I}}$ converges to V if and only if

$$\int_{G_n M} h(y, \omega) dV_i(y, \omega) \rightarrow \int_{G_n M} h(y, \omega) dV(y, \omega)$$

for all $h \in C_c(G_n M)$. If $\Sigma \subset M$ is n -rectifiable and $\theta : \Sigma \rightarrow [0, \infty)$ is in $L^1_{\text{loc}}(\Sigma, \mathcal{H}^n)$, the n -varifold $\mathbf{v}(\Sigma, \theta)$ is defined by

$$\mathbf{v}(\Sigma, \theta)(h) = \int_{\Sigma} h(y, T_y \Sigma) \theta(y) d\mathcal{H}^n(y)$$

for all $h \in C_c(G_n M)$. Here $T_y \Sigma$ is the approximate tangent space of Σ at y which exists for $\mathcal{H}^n$ -a.e. $y \in \Sigma$. Such a varifold is called a *rectifiable n -varifold*. When θ is the constant function 1, $\mathbf{v}(\Sigma, \theta)$ is denoted by $|\Sigma|$; it is called the *varifold associated to Σ* .

If $\varphi : M \rightarrow M$ is a C^1 map and V is an n -varifold in M , the push-forward varifold $\varphi_{\#} V$ is

defined as follows.

$$(\varphi_{\#}V)(h) = \int_{G_n^+M} h(\varphi(y), D\varphi|_y(\omega)) J\varphi(y, \omega) dV(y, \omega),$$

where

$$J\varphi(y, \omega) = \left(\det \left((D\varphi(y)|_{\omega})^t \circ (D\varphi(y)|_{\omega}) \right) \right)^{1/2}$$

is the Jacobian factor and

$$G_n^+M = \{(y, \omega) \in G_nM : J\varphi(y, \omega) \neq 0\}.$$

If $V = \mathbf{v}(\Sigma, \theta)$ is a rectifiable n -varifold and $\varphi : M \rightarrow M$ is a C^1 map, then $\varphi_{\#}V = \mathbf{v}(\varphi(\Sigma), \tilde{\theta})$, where $\tilde{\theta} : \varphi(\Sigma) \rightarrow [0, \infty)$ is defined by

$$\tilde{\theta}(z) = \sum_{y \in \varphi^{-1}(z) \cap \Sigma} \theta(y).$$

We denote by $\mathcal{V}_n(M)$ the closure of the space of rectifiable n -varifolds in M with respect to the above mentioned varifold weak topology. The $\mathbf{F}$ metric on $\mathcal{V}_n(M)$ is defined as follows [Pit81, page 66].

$$\mathbf{F}(V, W) = \sup\{V(h) - W(h) : h \in C_c^{0,1}(G_nM), |h| \leq 1, \text{Lip}(h) \leq 1\}.$$

For every $a > 0$, the $\mathbf{F}$ -metric topology and the varifold weak topology coincide on the set $\{V \in \mathcal{V}_n(M) : \|V\|(M) \leq a\}$.

5.2.3 Almgren-Pitts min-max theory

In this subsection, we will recall some of the definitions in the Almgren-Pitts min-max theory; we refer to the papers by Marques-Neves [MN14, MN17, MN16, MN21], Schoen-Simon [SS81] and the book by Pitts [Pit81] for more details. To discuss the Almgren-Pitts min-max theory, we need to introduce the following spaces of currents. Let (M^{n+1}, g) be a closed Riemannian manifold.

$\mathbf{I}_l(M^{n+1}; \mathbb{Z}_2)$ is the space of l -dimensional mod 2 flat chains in M ; we only need to consider $l = n, n+1$. The space $\mathbf{I}_{n+1}(M; \mathbb{Z}_2)$ can be canonically identified with $\mathcal{C}(M)$, the space of Caccioppoli sets in M . $\mathcal{Z}_n(M; \mathbb{Z}_2)$ denotes the space of flat chains $T \in \mathbf{I}_n(M; \mathbb{Z}_2)$ such that $T = \partial U$ for some $U \in \mathbf{I}_{n+1}(M; \mathbb{Z}_2)$. For $T \in \mathcal{Z}_n(M; \mathbb{Z}_2)$, $|T|$ stands for the varifold associated to T and $\|T\|$ stands for the Radon measure associated to $|T|$. $\mathcal{F}$ and $\mathbf{M}$ denote the flat and mass norm on $\mathbf{I}_l(M; \mathbb{Z}_2)$. When $l = n+1$, these two norms coincide. The $\mathbf{F}$ metric on the space of currents is defined as follows.

$$\begin{aligned} \mathbf{F}(U_1, U_2) &= \mathcal{F}(U_1, U_2) + \mathbf{F}(|\partial U_1|, |\partial U_2|) \text{ if } U_1, U_2 \in \mathbf{I}_{n+1}(M; \mathbb{Z}_2); \\ \mathbf{F}(T_1, T_2) &= \mathcal{F}(T_1, T_2) + \mathbf{F}(|T_1|, |T_2|) \text{ if } T_1, T_2 \in \mathbf{I}_n(M; \mathbb{Z}_2). \end{aligned}$$

It was proved in [MN21] that the space $\mathbf{I}_{n+1}(M; \mathcal{F}; \mathbb{Z}_2)$ is contractible and the boundary map $\partial : \mathbf{I}_{n+1}(M; \mathcal{F}; \mathbb{Z}_2) \rightarrow \mathcal{Z}_n(M; \mathcal{F}; \mathbb{Z}_2)$ is a two sheeted covering map. Moreover, from the interpolation theorems of Marques and Neves [MN21, Section 3], it follows that $\mathbf{I}_{n+1}(M; \mathbf{F}; \mathbb{Z}_2)$ is simply-connected (see [MMN20, Proposition 5.3]) and $\partial : \mathbf{I}_{n+1}(M; \mathbf{F}; \mathbb{Z}_2) \rightarrow \mathcal{Z}_n(M; \mathbf{F}; \mathbb{Z}_2)$ is a double cover. By the constancy theorem, if $U_1, U_2 \in \mathbf{I}_{n+1}(M; \mathbb{Z}_2)$ such that $\partial U_1 = \partial U_2$, either $U_1 = U_2$ or $U_1 + U_2 = \llbracket M \rrbracket$.

Let X and Π be as in Section 5.1. The *Almgren-Pitts width* of the homotopy class Π is defined by

$$\mathbf{L}_{AP}(\Pi) = \inf_{\Phi \in \Pi} \sup_{x \in X} \{\mathbf{M}(\Phi(x))\}. \quad (5.2.1)$$

A sequence of maps $\Phi_i : X \rightarrow \mathcal{Z}_n(M; \mathbf{F}; \mathbb{Z}_2)$ in Π is called a *minimizing sequence* if

$$\lim_{i \rightarrow \infty} \sup_{x \in X} \{\mathbf{M}(\Phi_i(x))\} = \mathbf{L}_{AP}(\Pi).$$

The *critical set* of a minimizing sequence $\{\Phi_i\}$, denoted by $\mathbf{C}(\{\Phi_i\})$, is the set of all varifolds $V \in \mathcal{V}_n(M)$ such that $\|V\|(M) = \mathbf{L}_{AP}(\Pi)$ and there exist sequences $\{i_j\} \subset \{i\}$ and $\{x_j\} \subset X$ such that

$$\lim_{j \rightarrow \infty} \mathbf{F}(V, |\Phi_{i_j}(x_j)|) = 0.$$

We define $\mathbf{C}_{AP}(\Pi)$ to be the set of all varifolds $V \in \mathcal{V}_n(M)$ such that $V \in \mathbf{C}(\{\Phi_i\})$ for some minimizing sequence $\{\Phi_i\} \subset \Pi$, V is a stationary, integral varifold and $\text{spt}(V)$ is a closed, minimal hypersurface with optimal regularity (i.e. smooth and embedded outside a singular set of Hausdorff dimension $\leq n - 7$). By the works of Almgren [Alm62], Pitts [Pit81], Schoen-Simon [SS81] and Marques-Neves [MN14, MN17], $\mathbf{C}_{AP}(\Pi)$ is non-empty.

5.2.4 Allen-Cahn min-max theory

In this subsection, we briefly discuss the min-max theory for the Allen-Cahn functional following the papers by Guaraco [Gua18] and Gaspar-Guaraco [GG18], where further details can be found. Let $W : \mathbb{R} \rightarrow [0, \infty)$ be a smooth, symmetric, double well potential. More precisely, W has the following properties. W is bounded; $W(-t) = W(t)$ for all $t \in \mathbb{R}$; W has exactly three critical points $0, \pm 1$; $W(\pm 1) = 0$ and $W''(\pm 1) > 0$, i.e. ± 1 are non-degenerate minima; 0 is a local maximum. The Allen-Cahn energy (with parameter $\varepsilon > 0$) is given by

$$E_\varepsilon(u) = \int_M \varepsilon \frac{|\nabla u|^2}{2} + \frac{W(u)}{\varepsilon}.$$

As mentioned earlier,

$$AC_\varepsilon(u) := -\varepsilon^2 \Delta u + W'(u) = 0$$

if and only if u is a critical point of E_ε .

Let $\tilde{X}$ and $\tilde{\Pi}$ be as in Section 5.1. The ε -Allen-Cahn width of the homotopy class $\tilde{\Pi}$ is defined by

$$\mathbf{L}_\varepsilon(\tilde{\Pi}) = \inf_{h \in \tilde{\Pi}} \sup_{x \in \tilde{X}} E_\varepsilon(h(x)). \quad (5.2.2)$$

A sequence of maps $h_i : \tilde{X} \rightarrow H^1(M) \setminus \{0\}$ in $\tilde{\Pi}$ is called a *minimizing sequence* for E_ε if

$$\lim_{i \rightarrow \infty} \sup_{x \in \tilde{X}} E_\varepsilon(h_i(x)) = \mathbf{L}_\varepsilon(\tilde{\Pi}).$$

u is called a *min-max critical point* of E_ε (corresponding to the homotopy class $\tilde{\Pi}$) if u is a critical

point of E_ε with $E_\varepsilon(u) = \mathbf{L}_\varepsilon(\tilde{\Pi})$ and there exists a minimizing sequence $\{h_i\}$ for E_ε in $\tilde{\Pi}$ so that

$$\lim_{i \rightarrow \infty} d_{H^1(M)}(u, h_i(\tilde{X})) = 0.$$

As W is an even function, E_ε is invariant under the $\mathbb{Z}_2$ action on $H^1(M)$ given by $u \mapsto -u$, i.e. $E_\varepsilon(u) = E_\varepsilon(-u)$. Moreover, as proved in [Gua18, Proposition 4.4], E_ε satisfies the Palais-Smale condition for bounded sequences. Hence, as explained in [GG18, Section 3.3], if $\varepsilon > 0$ satisfies

$$\mathbf{L}_\varepsilon(\tilde{\Pi}) < E_\varepsilon(0) = \frac{W(0)}{\varepsilon} \text{Vol}(M) \quad (5.2.3)$$

(which holds if ε is sufficiently small by (5.1.4)), one can apply Corollary 10.5 of [Gho93] to the $\mathbb{Z}_2$ -homotopic family $\tilde{\Pi}$ to conclude that there exists a min-max critical point u_ε of E_ε (corresponding to the homotopy class $\tilde{\Pi}$) such that $\text{Ind}(u_\varepsilon) \leq k$. (Here k is the dimension of the parameter space $\tilde{X}$.) The restriction on ε given by (5.2.3) is due to the fact that the space $H^1(M) \setminus \{0\}$ is not complete; (5.2.3) ensures that a minimizing sequence for E_ε is bounded away from 0.

5.2.5 Convergence of the phase interfaces

Let $F : \mathbb{R} \rightarrow \mathbb{R}$ and the energy constant σ be defined as follows.

$$F(t) = \int_0^t \sqrt{W(s)/2} \, ds; \quad \sigma = \int_{-1}^1 \sqrt{W(s)/2} \, ds \quad \text{so that} \quad F(\pm 1) = \pm \frac{\sigma}{2}. \quad (5.2.4)$$

For $u \in C^1(M)$, let $\tilde{u} = F \circ u$. The n -varifold $V[u]$ associated to u is defined by

$$V[u](A) = \frac{1}{\sigma} \int_{-\infty}^{\infty} |\{\tilde{u} = s\}|(A) \, ds,$$

for every Borel set $A \subset G_n M$. On a closed manifold, if $AC_\varepsilon(u) = 0$ and u is not identically equal to ± 1 , $|u| < 1$ [GG18, Lemma 2.2]. Hence, in that case, in the definition of $V[u]$, the integral can be taken over the interval $(-\sigma/2, \sigma/2)$.

Building on the works of Hutchinson-Tonegawa [HT00], Tonegawa [Ton05] and Tonegawa-

Wickramasekera [TW12], Guaraco [Gua18] has proved the following theorem.

Theorem 5.2.1 ([HT00, Ton05, TW12, Gua18]). *Let $\{u_i : M \rightarrow (-1, 1)\}_{i=1}^\infty$ be a sequence such that*

(i) $AC_{\varepsilon_i}(u_i) = 0$ with $\varepsilon_i \rightarrow 0$ as $i \rightarrow \infty$;

(ii) *there exist $E_0 > 0$ and $I_0 \in \mathbb{N}_0$ such that $E_{\varepsilon_i}(u_i) \leq E_0$ and $\text{Ind}(u_i) \leq I_0$ for all $i \in \mathbb{N}$.*

Then there exists a stationary, integral varifold V such that possibly after passing to a subsequence, $V[u_i] \rightarrow V$ in the sense of varifolds. Moreover,

$$\|V\|(M) = \frac{1}{2\sigma} \lim_{i \rightarrow \infty} E_{\varepsilon_i}(u_i)$$

and $\text{spt}(V)$ is a closed, minimal hypersurface with optimal regularity.

The proof of the regularity of the limit-interface depends on the regularity theory of stable minimal hypersurfaces, developed by Wickramasekera [Wic14a]. In the ambient dimension $n+1 = 3$, the regularity of the limit-interface can also be obtained from the curvature estimates of Chodosh and Mantoulidis [CM20].

The upper bound for the Morse index of the limit-interface in the above theorem was proved by Gaspar [Gas19] and Hiesmayr [Hie18] (when the limit-interface is two sided).

Theorem 5.2.2 ([Gas19, Hie18]). *The Morse index of the limit-interface $\text{spt}(V)$ in the above Theorem 6.2.1 is less than or equal to I_0 .*

Lastly, let us introduce the following definition. $\mathbf{C}_{AC}(\tilde{\Pi})$ is the set of all stationary, integral n -varifolds V such that $\text{spt}(V)$ is a closed, minimal hypersurface with optimal regularity and V is the varifold limit of $V[u_i]$ for some sequence $\{u_i\}_{i=1}^\infty$ such that u_i is a min-max critical point of E_{ε_i} (with $\varepsilon_i \rightarrow 0$) corresponding to the homotopy class $\tilde{\Pi}$. By the discussion of Section 5.2.4 and Theorem 6.2.1, $\mathbf{C}_{AC}(\tilde{\Pi})$ is non-empty.

5.2.6 Approximation of a Caccioppoli set by open sets with smooth boundary

In this subsection, following the book by Giusti [Giu84] and the paper by Miranda-Pallara-Paronetto-Preunkert [MPPP07], we briefly discuss the fact that a Caccioppoli set can be approximated by open sets with smooth boundary. Let us begin with the following theorem.

Theorem 5.2.3 ([Giu84, Theorem 1.17], [MPPP07, Proposition 1.4]). *Suppose $E \in \mathcal{C}(M)$. Then there exists a sequence of smooth functions $\{u_j : M \rightarrow [0, 1]\}_{j=1}^\infty$ such that*

$$\lim_{j \rightarrow \infty} \int_M |u_j - \chi_E| d\mathcal{H}^{n+1} = 0 \quad \text{and} \quad \int_M |D\chi_E| = \lim_{j \rightarrow \infty} \int_M |Du_j|. \quad (5.2.5)$$

Following [Giu84, Proof of Theorem 1.24], for $t \in (0, 1)$, let us define $E_{j,t} = \{u_j > t\}$ (where u_j is as in the above Theorem 5.2.3). Then

$$|u_j - \chi_E| > t \text{ on } E_{j,t} \setminus E \quad \text{and} \quad |u_j - \chi_E| \geq 1 - t \text{ on } E \setminus E_{j,t}, \quad (5.2.6)$$

which imply

$$\int_M |u_j - \chi_E| d\mathcal{H}^{n+1} \geq \min\{t, 1 - t\} \int_M |\chi_{E_{j,t}} - \chi_E| d\mathcal{H}^{n+1}. \quad (5.2.7)$$

Hence, by (5.2.5), for all $t \in (0, 1)$,

$$\lim_{j \rightarrow \infty} \int_M |\chi_{E_{j,t}} - \chi_E| d\mathcal{H}^{n+1} = 0 \quad (5.2.8)$$

$$\implies \int_M |D\chi_E| \leq \liminf_{j \rightarrow \infty} \int_M |D\chi_{E_{j,t}}|. \quad (5.2.9)$$

Therefore, using (5.2.5), the co-area formula for the BV functions and (5.2.9), we obtain the following inequalities.

$$\int_M |D\chi_E| = \lim_{j \rightarrow \infty} \int_M |Du_j| \geq \int_0^1 \left(\liminf_{j \rightarrow \infty} \int_M |D\chi_{E_{j,t}}| \right) dt \geq \int_M |D\chi_E|.$$

This implies

$$\liminf_{j \rightarrow \infty} \int_M |D\chi_{E_j,t}| = \int_M |D\chi_E|, \quad \text{for a.e. } t \in (0, 1). \quad (5.2.10)$$

Let us choose $t_0 \in (0, 1)$ such that t_0 is a regular value of u_j for all j and (5.2.10) holds for $t = t_0$.

Further, possibly after passing to a subsequence, one can assume that

$$\lim_{j \rightarrow \infty} \int_M |D\chi_{E_j,t_0}| = \int_M |D\chi_E|. \quad (5.2.11)$$

Let us define $E_j = \overline{E_{j,t_0}}$. Since

$$E_{j,t_0} \subset \overline{E_{j,t_0}} \subset E_{j,t_0} \cup \{u_j = t_0\},$$

we have $\mathcal{H}^{n+1}(\overline{E_{j,t_0}} \setminus E_{j,t_0}) = 0$. From (5.2.8) and (5.2.11) one obtains

$$\chi_{E_j} \rightarrow \chi_E \text{ in } L^1(M) \quad \text{and} \quad \lim_{j \rightarrow \infty} \int_M |D\chi_{E_j}| = \int_M |D\chi_E|. \quad (5.2.12)$$

By [Pit81, 2.1(18)(f), page-63], (5.2.12) implies that $\llbracket \partial^* E_j \rrbracket$ converges to $\llbracket \partial^* E \rrbracket$ in $\mathbf{F}$.

Let us fix $y \in M$ and $R > 0$. Using (5.2.12),

$$\lim_{j \rightarrow \infty} \int_0^R \left(\int_{\partial B(y,t)} |\chi_{E_j} - \chi_E| d\mathcal{H}^n \right) dt = \lim_{j \rightarrow \infty} \int_{B(y,R)} |\chi_{E_j} - \chi_E| d\mathcal{H}^{n+1} = 0.$$

Hence, there exists a subsequence $\{\chi_{E_{j_s}}\} \subset \{\chi_{E_j}\}$ such that

$$\lim_{s \rightarrow \infty} \int_{\partial B(y,t)} |\chi_{E_{j_s}} - \chi_E| d\mathcal{H}^n = 0, \quad \text{for a.e. } t \in (0, R).$$

Next, let us define $F_j = \overline{\{u_j < t_0\}}$ and $F = M \setminus E$. Then

$$E_j \cap F_j \subset \{u_j = t_0\} \quad \text{and} \quad E_j \cup F_j = M \quad (\text{as } \{u_j = t_0\} \text{ is a smooth hypersurface}).$$

Therefore,

$$\llbracket E_j \rrbracket + \llbracket F_j \rrbracket = \llbracket M \rrbracket, \quad \partial^* E_j = \partial^* F_j \quad \text{and} \quad \chi_{F_j} \rightarrow \chi_F \text{ in } L^1(M). \quad (5.2.13)$$

We note that $\partial E_j, \partial F_j \subset \{u_j = t_0\}$. As the reduced boundary is a subset of the topological boundary, we also have $\partial^* E_j = \partial^* F_j \subset \{u_j = t_0\}$. Since $\{u_j = t_0\}$ is a smooth hypersurface in M , for each $a \in \{u_j = t_0\}$ there exist $\rho > 0$ and co-ordinates $\{x_1, x_2, \dots, x_{n+1}\}$ on $B(a, \rho)$ such that $x_i(a) = 0$ for all i and

$$(B(a, \rho) \cap \{u_j = t_0\}) = \{x \in B(a, \rho) : x_{n+1} = 0\}.$$

Let $G_1 = \{x \in B(a, \rho) : x_{n+1} < 0\}$ and $G_2 = \{x \in B(a, \rho) : x_{n+1} > 0\}$ so that

$$(B(a, \rho) \setminus \{u_j = t_0\}) = G_1 \dot{\cup} G_2.$$

G_1 and G_2 are connected open sets. We have the following three mutually exclusive cases.

- (1) $u_j > t_0$ on both G_1 and G_2 ; this implies $a \in \text{int}(E_j) \setminus F_j$;
- (2) $u_j < t_0$ on both G_1 and G_2 ; this implies $a \in \text{int}(F_j) \setminus E_j$;
- (3) $u_j > t_0$ on one of G_1 and G_2 , and $u_j < t_0$ on the other; this implies $a \in E_j \cap F_j$.

From the above three cases, one can conclude that $\partial^* E_j = \partial^* F_j = E_j \cap F_j = \partial E_j = \partial F_j$, which is a smooth, closed, embedded hypersurface in M . Indeed, the set of points $a \in \{u_j = t_0\}$ for which the above item (3) holds is both open and closed in $\{u_j = t_0\}$; hence it is the union of certain connected components of $\{u_j = t_0\}$.

From the above discussion, we arrive at the following proposition.

Proposition 5.2.4. *Let $E \in \mathcal{C}(M)$ and $F = M \setminus E$. Then there exist two sequences $\{E_j\}_{j=1}^\infty$ and $\{F_j\}_{j=1}^\infty$ of closed subsets of M with the following properties.*

- (i) $E_j, F_j \in \mathcal{C}(M)$, $\llbracket E_j \rrbracket + \llbracket F_j \rrbracket = \llbracket M \rrbracket$ and $M = E_j \cup F_j$ (for every $j \in \mathbb{N}$).
- (ii) $\chi_{E_j} \rightarrow \chi_E$ and $\chi_{F_j} \rightarrow \chi_F$ in $L^1(M)$.

(iii) $\partial^* E_j = \partial^* F_j = E_j \cap F_j = \partial E_j = \partial F_j$ is a smooth, closed, embedded hypersurface in M (for every $j \in \mathbb{N}$). Moreover, for all $p \in E_j \cap F_j$, there exists a normal geodesic ball $\mathbf{U}$, centered at p , such that

$$\mathbf{U} \setminus (E_j \cap F_j) = G_1 \dot{\cup} G_2,$$

where G_1, G_2 are connected open sets, $G_1 \subset E_j$ and $G_2 \subset F_j$.

(iv) $\llbracket \partial^* E_j \rrbracket = \llbracket \partial^* F_j \rrbracket$ converges to $\llbracket \partial^* E \rrbracket = \llbracket \partial^* F \rrbracket$ in $\mathbf{F}$.

(v) For every $y \in M$ and $R > 0$, there exists a subsequence $\{j_s\} \subset \{j\}$ such that

$$\lim_{s \rightarrow \infty} \int_{\partial B(y,t)} |\chi_{E_{j_s}} - \chi_E| d\mathcal{H}^n = \lim_{s \rightarrow \infty} \int_{\partial B(y,t)} |\chi_{F_{j_s}} - \chi_F| d\mathcal{H}^n = 0,$$

for a.e. $t \in (0, R)$.

5.3 Proof of the width inequality

In this section we will prove our main Theorem 5.1.2. Let us fix $\eta > 0$. Let $L = \mathbf{L}_{AP}(\Pi)$. By the interpolation theorems of Pitts and Marques-Neves [MN17, MN21], there exists $\Phi : X \rightarrow \mathcal{Z}_n(M^{n+1}; \mathbf{M}; \mathbb{Z}_2)$ with $\Phi \in \Pi$ such that

$$\sup_{x \in X} \{\mathbf{M}(\Phi(x))\} < L + \eta. \quad (5.3.1)$$

Let us choose $\tilde{\Phi} : \tilde{X} \rightarrow \mathcal{C}(M)$ which is a lift of Φ , i.e. for all $x \in \tilde{X}$,

$$\llbracket \partial^* \tilde{\Phi}(x) \rrbracket = \partial \llbracket \tilde{\Phi}(x) \rrbracket = \Phi(\pi(x)).$$

$\tilde{\Phi}$ is $\mathbb{Z}_2$ -equivariant, i.e. if $T : \tilde{X} \rightarrow \tilde{X}$ is the deck transformation, then $\llbracket \tilde{\Phi}(x) \rrbracket + \llbracket \tilde{\Phi}(T(x)) \rrbracket = \llbracket M \rrbracket$ for all $x \in \tilde{X}$. The proof of Theorem 5.1.2 is divided into five subsections.

5.3.1 Preliminary constructions

Let D be a countable, dense subset of M and $\text{inj}(M)$ be the injectivity radius of M . We consider

$$\mathcal{B} = \{B(p, t) : p \in D, t \in (0, \text{inj}(M)) \cap \mathbb{Q}\},$$

which is also a countable set. Let us assume that M is isometrically embedded in some Euclidean space $\mathbb{R}^m$. The following theorem is a consequence of Sard's theorem.

Theorem 5.3.1. ([Nic11, Corollary 1.25]) *Let Σ be a closed submanifold of $\mathbb{R}^m$. For $v \in \mathbb{R}^m$, $f_v : \mathbb{R}^m \rightarrow \mathbb{R}$ is defined by $f_v(x) = \langle x, v \rangle$. Then, for almost every $v \in \mathbb{R}^m$, $f_v|_\Sigma$ is a Morse function on Σ .*

By the above theorem, there exists $\omega \in \mathbb{R}^m$ such that $f_\omega|_M$ and $f_\omega|_{\partial B}$ are Morse functions for all $B \in \mathcal{B}$. By composing scaling and translation with f_ω , one can assume that $f_\omega(M) = [1/3, 2/3]$. From now on, whenever we will consider f_ω , it will be assumed that $f_\omega : M \rightarrow [1/3, 2/3]$. Let us choose $r_0 \in (0, \text{inj}(M))$ such that

- $\mathcal{H}^n(f_\omega^{-1}(t) \cap \overline{B}(p, r_0)) < \eta/2$ for all $t \in [1/3, 2/3]$, $p \in M$;
- $\|\Phi(x)\|(\overline{B}(p, r_0)) < \eta$ for all $x \in X$, $p \in M$.

Such a choice of r_0 is possible because of the ‘no concentration of mass property’ [MN17, Lemma 5.2, Lemma 3.5]. Let us choose $\{p_i\}_{i=1}^I \subset D$ so that

$$M = \bigcup_{i=1}^I B(p_i, r_0/4); \quad \mathbf{B}_i^0 := B(p_i, r_0). \quad (5.3.2)$$

Hence, in particular,

$$\begin{aligned} \mathcal{H}^n(f_\omega^{-1}(t) \cap \overline{\mathbf{B}_i^0}) &< \eta/2 \quad \forall t \in [1/3, 2/3], i \in [I]; \\ \|\Phi(x)\|(\overline{\mathbf{B}_i^0}) &< \eta \quad \forall x \in X, i \in [I]. \end{aligned} \quad (5.3.3)$$

Lemma 5.3.2. *There exists $r_1 \in (r_0/2, 3r_0/4) \cap \mathbb{Q}$, $\delta \in (0, r_0/8)$ such that*

$$\|\Phi(x)\| (\overline{A}(p_i, r_1 - 2\delta, r_1 + \delta)) < \frac{\eta}{I}$$

for all $x \in X$ and $i \in [I]$.

Proof. By the compactness of X , there exist $\{x_j\}_{j=1}^J \subset X$ such that

$$\forall x \in X, \exists j \in [J] \text{ such that } \mathbf{M}(\Phi(x) - \Phi(x_j)) < \frac{\eta}{2I}. \quad (5.3.4)$$

For $i \in [I], j \in [J]$ let us define

$$\mathcal{R}_{ij} = \left\{ r \in (r_0/2, 3r_0/4) : \|\Phi(x_j)\| (\partial B(p_i, r)) < \frac{\eta}{2I} \right\}.$$

We note that $(r_0/2, 3r_0/4) \setminus \mathcal{R}_{ij}$ is finite for all i, j . Hence, one can choose

$$r_1 \in \left(\bigcap_{i \in [I], j \in [J]} \mathcal{R}_{ij} \right) \cap (r_0/2, 3r_0/4) \cap \mathbb{Q}. \quad (5.3.5)$$

Therefore,

$$\|\Phi(x_j)\| (\partial B(p_i, r_1)) < \frac{\eta}{2I} \quad \forall i, j.$$

Hence, there exists $\delta \in (0, r_0/8)$ such that

$$\|\Phi(x_j)\| (\overline{A}(p_i, r_1 - 2\delta, r_1 + \delta)) < \frac{\eta}{2I} \quad \forall i, j.$$

Thus, by (5.3.4),

$$\|\Phi(x)\| (\overline{A}(p_i, r_1 - 2\delta, r_1 + \delta)) < \frac{\eta}{I} \quad \forall x \in X, i \in [I].$$

□

Lemma 5.3.3. *Let r_1 be as in Lemma 5.3.2. There exists $\delta \in (0, r_0/8)$ such that*

$$\mathcal{H}^n(f_\omega^{-1}(t) \cap \overline{A}(p_i, r_1 - 2\delta, r_1 + \delta)) < \frac{\eta}{2I}$$

for all $t \in [1/3, 2/3]$, $i \in [I]$.

Proof. Let us assume by contradiction that for some $i \in [I]$, there exist $\{t_j\}_{j=1}^\infty \subset [1/3, 2/3]$ and $\{d_j\}_{j=1}^\infty \subset (0, r_0/8)$ such that $d_j \rightarrow 0$ and

$$\mathcal{H}^n(f_\omega^{-1}(t_j) \cap \overline{A}(p_i, r_1 - 2d_j, r_1 + d_j)) \geq \frac{\eta}{2I} \quad \forall j \in \mathbb{N}.$$

Without loss of generality one can assume that $t_j \rightarrow t$. Denoting

$$\mu_j = \mathcal{H}^n \llcorner f_\omega^{-1}(t_j) \quad \text{and} \quad \mu = \mathcal{H}^n \llcorner f_\omega^{-1}(t),$$

we have that μ_j weakly converges to μ (in the sense of Radon measures). Let us fix $l \in \mathbb{N}$. There exists $j_0 \in \mathbb{N}$ such that $d_j \leq l^{-1}$ for all $j \geq j_0$. Hence,

$$\mu_j(\overline{A}(p_i, r_1 - 2l^{-1}, r_1 + l^{-1})) \geq \frac{\eta}{2I}$$

for all $j \geq j_0$. Therefore,

$$\mu(\overline{A}(p_i, r_1 - 2l^{-1}, r_1 + l^{-1})) \geq \limsup_{j \rightarrow \infty} \mu_j(\overline{A}(p_i, r_1 - 2l^{-1}, r_1 + l^{-1})) \geq \frac{\eta}{2I}.$$

This holds for all $l \in \mathbb{N}$. Denoting $A_l = \overline{A}(p_i, r_1 - 2l^{-1}, r_1 + l^{-1})$ we have

$$A_{l+1} \subset A_l \quad \text{and} \quad \bigcap_{l=1}^\infty A_l = \partial B(p_i, r_1).$$

This implies

$$\mathcal{H}^n(f_\omega^{-1}(t) \cap \partial B(p_i, r_1)) = \mu(\partial B(p_i, r_1)) = \lim_{l \rightarrow \infty} \mu(A_l) \geq \frac{\eta}{2I}.$$

However, this is not possible. Indeed, $r_1 \in \mathbb{Q}$ (Lemma 5.3.2) and by the choice of ω , $f_\omega|_{\partial B(p_i, r_1)}$ is a Morse function on $\partial B(p_i, r_1)$; hence $\mathcal{H}^n(f_\omega^{-1}(t) \cap \partial B(p_i, r_1))$ must be 0. $\square$

One can assume that the two δ 's appearing in Lemma 5.3.2 and Lemma 5.3.3 are the same. Next, we modify f_ω near the points where it achieves local maxima or minima (which are not global maxima or minima) to get another Morse function $f : M \rightarrow [1/3, 2/3]$ such that f has no non-global local maxima or minima and

$$\mathcal{H}^n(f^{-1}(t) \cap \overline{\mathbf{B}_i^0}) < \eta \quad \text{and} \quad \mathcal{H}^n(f^{-1}(t) \cap \overline{A}(p_i, r_1 - 2\delta, r_1 + \delta)) < \frac{\eta}{I} \quad (5.3.6)$$

for all $t \in [1/3, 2/3]$, $i \in [I]$. Let us introduce the following notation which will be used later.

$$\begin{aligned} \mathbf{B}_i^1 &= B(p_i, r_1); & \mathcal{A}_0 &= \bigcup_{i=1}^I A(p_i, r_1 - 2\delta, r_1 + \delta); \\ \mathcal{A}_1 &= \bigcup_{i=1}^I A(p_i, r_1 - 3\delta/2, r_1 + \delta/2); & \mathcal{A} &= \bigcup_{i=1}^I A(p_i, r_1 - \delta, r_1). \end{aligned} \quad (5.3.7)$$

5.3.2 Cell complex structure on the parameter space

For $l \in \mathbb{N}$, $\mathcal{J}[l]$ is the cell complex on $\mathcal{J} = [0, 1]$ whose 0-cells are

$$[0], [l^{-1}], \dots, [1 - l^{-1}], [1]$$

and 1-cells are

$$[0, l^{-1}], [l^{-1}, 2l^{-1}], \dots, [1 - l^{-1}, 1].$$

$\mathcal{J}^m[l]$ denotes the cell complex on $\mathcal{J}^m = [0, 1]^m$ whose cells are of the form $\alpha_1 \otimes \alpha_2 \otimes \dots \otimes \alpha_m$, where each $\alpha_j \in \mathcal{J}[l]$. We note that $\mathcal{J}^m[1]$ is the standard cell complex on $\mathcal{J}^m$. By abuse of notation, a cell $\alpha_1 \otimes \alpha_2 \otimes \dots \otimes \alpha_m$ will be identified with its support $\alpha_1 \times \alpha_2 \times \dots \times \alpha_m \subset \mathcal{J}^m$. Similarly, if Y is a subcomplex of $\mathcal{J}^m[1]$, $Y[l]$ is the union of all the cells in $\mathcal{J}^m[l]$ whose support is contained in some cell of Y . If $\mathcal{Y}$ is a cell complex, $\mathcal{Y}_p$ will denote the set of all p -cells in $\mathcal{Y}$; if $\alpha, \beta \in \mathcal{Y}$ such that β is a face of α (in the definition of the face, we do not insist that $\dim(\beta) < \dim(\alpha)$ or

$\dim(\beta) = \dim(\alpha) - 1$, we use the notation $\beta \prec \alpha$.

If $\lambda = [il^{-1}, (i+1)l^{-1}] \in \mathcal{J}[l]_1$, there exists a canonical map

$$\Delta_\lambda : \mathcal{J} \rightarrow \mathcal{J}; \quad \Delta_\lambda(t) = (i+t)l^{-1} \quad (5.3.8)$$

such that $\Delta_\lambda : \mathcal{J} \rightarrow \lambda$ is a homeomorphism. Similarly, if $\alpha = \alpha_1 \otimes \alpha_2 \otimes \cdots \otimes \alpha_m \in \mathcal{J}^m[l]_p$, there exists a canonical map $\Delta_\alpha : \mathcal{J}^p \rightarrow \mathcal{J}^m$ defined as follows. There exist precisely p indices

$$j_1 < j_2 < \cdots < j_p \text{ such that } \dim(\alpha_{j_s}) = 1 \ \forall s = 1, 2, \dots, p.$$

We define

$$(\Delta_\alpha(t_1, t_2, \dots, t_p))_j = \begin{cases} \Delta_{\alpha_{j_s}}(t_s) & \text{if } j = j_s, \\ \alpha_j & \text{if } j \notin \{j_1, j_2, \dots, j_p\} \Leftrightarrow \dim(\alpha_j) = 0. \end{cases} \quad (5.3.9)$$

$\Delta_\alpha : \mathcal{J}^p \rightarrow \alpha$ is a homeomorphism. Let $D_\alpha : \alpha \rightarrow \mathcal{J}^p$ be the inverse of Δ_α .

If $\alpha \in \mathcal{J}^m[l]_p$ (for some $m, l, p \in \mathbb{N}$), $\beta \prec \alpha$ and $D_\alpha(\beta) = \varrho$, then the following compatibility relation holds. (If $\dim(\beta) = q$, we think of ϱ as a q -cell of $\mathcal{J}^p[1]$ and $\Delta_\varrho : \mathcal{J}^q \rightarrow \mathcal{J}^p$.)

$$\Delta_\alpha \circ \Delta_\varrho = \Delta_\beta. \quad (5.3.10)$$

Let us recall from Section 5.1 that X is a cubical complex (i.e. X is a subcomplex of $\mathcal{J}^K[1]$, for some $K \in \mathbb{N}$) and $\pi : \tilde{X} \rightarrow X$ is a double cover. For $l \in \mathbb{N}$, $X[l]$ is the cell complex on X as defined above. $\tilde{X}[l]$ denotes the cell complex on $\tilde{X}$ whose cells are the pre-images of the cells of $X[l]$ via the map π . More precisely, if c is a cell of $X[l]$, then

$$\pi^{-1}(c) = \tilde{c}_1 \dot{\cup} \tilde{c}_2;$$

since c is contractible, $\pi|_{\tilde{c}_1} : \tilde{c}_1 \rightarrow c$ and $\pi|_{\tilde{c}_2} : \tilde{c}_2 \rightarrow c$ are homeomorphisms. The cells of $\tilde{X}[l]$ are

precisely

$$\{\tilde{c}_1, \tilde{c}_2 : c \text{ is a cell of } X[l]\}.$$

We choose $N \in \mathbb{N}$ such that the following two conditions are satisfied. (Here Φ , $\tilde{\Phi}$, η , δ , I are as defined previously in this section.)

- If $x, x' \in X[N]_0$ belong to a common cell in $X[N]$, then

$$\mathbf{M}(\Phi(x) - \Phi(x')) < \eta. \quad (5.3.11)$$

- If $x_1, x_2 \in \tilde{X}[N]_0$ belong to a common cell in $\tilde{X}[N]$, then

$$\mathcal{H}^{n+1}(\tilde{\Phi}(x_1) \Delta \tilde{\Phi}(x_2)) < \frac{\eta\delta}{I}. \quad (5.3.12)$$

Let $\{c_q : q \in [Q]\}$ be all the cells of $X[N]$, indexed in such a way that $\dim(c_{q_1}) \leq \dim(c_{q_2})$ if $q_1 \leq q_2$. Let $\{e_q, f_q : q \in [Q]\}$ be the cells of $\tilde{X}[N]$ so that

$$\pi^{-1}(c_q) = e_q \dot{\cup} f_q.$$

As mentioned above, $\pi|_{e_q} : e_q \rightarrow c_q$ and $\pi|_{f_q} : f_q \rightarrow c_q$ are homeomorphisms. We introduce the following notations ($m = \dim(c_q)$).

- $\Delta_{c_q} : \mathcal{J}^m \rightarrow c_q$ is denoted by Δ_q ; $D_{c_q} : c_q \rightarrow \mathcal{J}^m$ is denoted by D_q .
- $(\pi|_{e_q})^{-1} \circ \Delta_q : \mathcal{J}^m \rightarrow e_q$ is denoted by Δ_q^1 ; $D_q \circ (\pi|_{e_q}) : e_q \rightarrow \mathcal{J}^m$ is denoted by D_q^1 .
- $(\pi|_{f_q})^{-1} \circ \Delta_q : \mathcal{J}^m \rightarrow f_q$ is denoted by Δ_q^2 ; $D_q \circ (\pi|_{f_q}) : f_q \rightarrow \mathcal{J}^m$ is denoted by D_q^2 .

5.3.3 Construction of an almost smooth sweepout

The next proposition follows from Proposition 5.2.4.

Proposition 5.3.4. *There exists a sequence $\{\tilde{\Phi}_j : \tilde{X}[N]_0 \rightarrow \mathcal{C}(M)\}_{j=1}^\infty$ such that the following conditions are satisfied for all $x \in \tilde{X}[N]_0$.*

- (i) For all $j \in \mathbb{N}$, $\tilde{\Phi}_j(x)$ is a closed subset of M .
- (ii) For all $j \in \mathbb{N}$, $\llbracket \tilde{\Phi}_j(x) \rrbracket + \llbracket \tilde{\Phi}_j(T(x)) \rrbracket = \llbracket M \rrbracket$ and $M = \tilde{\Phi}_j(x) \cup \tilde{\Phi}_j(T(x))$.
- (iii) As $j \rightarrow \infty$, $\chi_{\tilde{\Phi}_j(x)} \rightarrow \chi_{\tilde{\Phi}(x)}$ in $L^1(M)$.
- (iv) For all $j \in \mathbb{N}$, $\partial \tilde{\Phi}_j(x) = \partial \tilde{\Phi}_j(T(x)) = \tilde{\Phi}_j(x) \cap \tilde{\Phi}_j(T(x)) = \partial^* \tilde{\Phi}_j(x) = \partial^* \tilde{\Phi}_j(T(x))$ is a smooth, closed, embedded hypersurface in M , which we denote by $\Phi_j(x)$ ($= \Phi_j(T(x))$). Moreover, for every $p \in \Phi_j(x)$, there exists a normal geodesic ball $\mathbf{U}$, centered at p , such that

$$\mathbf{U} \setminus \Phi_j(x) = G_1 \dot{\cup} G_2,$$

where G_1, G_2 are connected open sets, $G_1 \subset \tilde{\Phi}_j(x)$ and $G_2 \subset \tilde{\Phi}_j(T(x))$.

- (v) As $j \rightarrow \infty$, $\llbracket \Phi_j(x) \rrbracket \rightarrow \Phi(\pi(x))$ in $\mathbf{F}$.
- (vi) For all $i \in [I]$,

$$\lim_{j \rightarrow \infty} \int_{\partial B(p_i, t)} |\chi_{\tilde{\Phi}_j(x)} - \chi_{\tilde{\Phi}(x)}| d\mathcal{H}^n = 0$$

for a.e. $t \in (0, r_1)$.

We will approximate $\tilde{\Phi}$ by a discrete and ‘almost smooth’ sweepout $\tilde{\Psi} : \tilde{X}[NI]_0 \rightarrow \mathcal{C}(M)$, which will be constructed using the map $\tilde{\Phi}_\gamma$ (for some sufficiently large $\gamma \in \mathbb{N}$) and certain collections of balls. The construction is motivated by the interpolation theorems of Almgren [Alm62], Pitts [Pit81, Theorem 4.5], Marques-Neves [MN14, Theorem 14.1] and Chambers-Liokumovich [CL20, Lemma 6.2]. The construction is divided into three parts.

Part 1. Let us recall that $\{e_q, f_q : q \in [Q]\}$ are the cells of $\tilde{X}[N]$. For every $q \in [Q]$ we define the following collection of balls $(\{p_i\}_{i=1}^I, r_1, \delta)$ are as in Subsection 5.3.1)

$$\mathcal{B}(q) = \{B(p_i, r_i(q)) : i \in [I]\}; \tag{5.3.13}$$

$r_i(q) \in (r_1 - \delta, r_1)$ is chosen inductively so that the following conditions are satisfied.

- (i) $\|\Phi(\pi(x))\|(\partial B(p_i, r_i(q))) = 0$ for all $x \in (e_q)_0$.

- (ii) $\partial B(p_i, r_i(q))$ is transverse to $\Phi_j(x)$ for all $x \in (e_q)_0$ and $j \in \mathbb{N}$.
- (iii) $\partial B(p_i, r_i(q))$ is transverse to $\partial B(p_s, r_s(q))$ for all $s < i$.
- (iv) $\partial B(p_i, r_i(q))$ is transverse to $\partial B(p_j, r_j(q'))$ for all $q' < q$ and $j \in [I]$.
- (v) For all $x \in (e_q)_0 \cup (f_q)_0$,

$$\lim_{j \rightarrow \infty} \int_{\partial B(p_i, r_i(q))} |\chi_{\tilde{\Phi}_j(x)} - \chi_{\tilde{\Phi}(x)}| d\mathcal{H}^n = 0.$$

(For this item one needs to use Proposition 5.3.4 (vi).)

- (vi) If $m = \dim(e_q) = \dim(f_q)$,

$$\int_{\partial B(p_i, r_i(q))} |\chi_{\tilde{\Phi}(x)} - \chi_{\tilde{\Phi}(x')}| d\mathcal{H}^n < \frac{2^{2m}\eta}{I}$$

for all $x, x' \in (e_q)_0$ and for all $x, x' \in (f_q)_0$. (For this item one needs to use (5.3.12).)

Next let us choose

$$r_1 > r^* > \max\{r_i(q) : i \in [I], q \in [Q]\} \quad (5.3.14)$$

so that the followings hold.

$$\|\Phi(\pi(x))\|(\partial B(p_i, r^*)) = 0 \quad \forall x \in \tilde{X}[N]_0, i \in [I]; \quad (5.3.15)$$

$$\|\Phi_j(x)\|(\partial B(p_i, r^*)) = 0 \quad \forall j \in \mathbb{N}, x \in \tilde{X}[N]_0, i \in [I]. \quad (5.3.16)$$

We introduce the notation

$$B_i(q) = B(p_i, r_i(q)); \quad \mathbf{B}_i = B(p_i, r^*). \quad (5.3.17)$$

(5.3.2), the definition of r_1 (in Lemma 5.3.2) and (5.3.14) imply that for all $q \in [Q]$,

$$\bigcup_{i \in [I]} B_i(q) = M = \bigcup_{i \in [I]} \mathbf{B}_i. \quad (5.3.18)$$

Part 2. Let us introduce the following definitions.

$$\mathcal{R}_1 = \{\mathbf{B}_i : i \in [I]\} \cup \{M \setminus \overline{\mathbf{B}}_i : i \in [I]\}, \quad \mathcal{R}_2 = \{U_1 \cap U_2 \cap \cdots \cap U_s : s \in \mathbb{N} \text{ and each } U_j \in \mathcal{R}_1\},$$

$$\mathcal{R} = \{V_1 \cup V_2 \cup \cdots \cup V_t : t \in \mathbb{N} \text{ and each } V_j \in \mathcal{R}_2\}$$

$\mathcal{R}_1, \mathcal{R}_2$ and $\mathcal{R}$ are finite sets. In a topological space, for any two sets A and B ,

$$\partial(A \cap B), \partial(A \cup B) \subset \partial A \cup \partial B. \quad (5.3.19)$$

Hence, for any $R \in \mathcal{R}$,

$$\partial R \subset \bigcup_{i \in [I]} \partial \mathbf{B}_i \text{ which implies } \|\Phi(\pi(x))\|(\partial R) = 0 \quad \forall x \in \tilde{X}[N]_0 \quad (\text{by (5.3.15)}).$$

Therefore, by Proposition 5.3.4 (v), as $j \rightarrow \infty$,

$$\|\Phi_j(x)\|(R) \rightarrow \|\Phi(\pi(x))\|(R) \quad \forall x \in \tilde{X}[N]_0, R \in \mathcal{R}. \quad (5.3.20)$$

We also note that by (5.3.18), $M = \cup_{i \in [I]} \mathbf{B}_i \in \mathcal{R}$.

Proposition 5.3.5. *There exists $\gamma \in \mathbb{N}$ such that the following conditions are satisfied. ($\mathbf{B}_i^0, \mathcal{A}_0$ are as in Subsection 5.3.1.)*

$$(i) \quad \left| \|\Phi_\gamma(x)\|(R) - \|\Phi(\pi(x))\|(R) \right| < \eta \text{ for all } x \in \tilde{X}[N]_0 \text{ and } R \in \mathcal{R}.$$

$$(ii) \quad \mathcal{H}^n(\Phi_\gamma(x)) < L + 2\eta \text{ for all } x \in \tilde{X}[N]_0.$$

$$(iii) \quad \mathcal{H}^n(\Phi_\gamma(x) \cap \overline{\mathbf{B}}_i^0) < \eta \text{ for all } x \in \tilde{X}[N]_0 \text{ and } i \in [I].$$

$$(iv) \quad \mathcal{H}^n(\Phi_\gamma(x) \cap \overline{\mathcal{A}}_0) < \eta \text{ for all } x \in \tilde{X}[N]_0.$$

$$(v) \quad \text{For } q \in [Q] \text{ and } i \in [I],$$

$$\int_{\partial B_{i(q)}} |\chi_{\tilde{\Phi}_\gamma(x)} - \chi_{\tilde{\Phi}_\gamma(x')}| d\mathcal{H}^n < \frac{2^{2m}\eta}{I} \quad (\text{where } m = \dim(e_q) = \dim(f_q)), \quad (5.3.21)$$

if $x, x' \in (e_q)_0$ or if $x, x' \in (f_q)_0$.

Proof. (i) follows from (5.3.20). Since, $M \in \mathcal{R}$, (ii) follows from (i) and (5.3.1). To obtain (iii), we note that Proposition 5.3.4 (v) and (5.3.3) imply for each $x \in \tilde{X}[N]_0$ and $i \in [I]$,

$$\limsup_{j \rightarrow \infty} \|\Phi_j(x)\|(\overline{\mathbf{B}_i^0}) \leq \|\Phi(\pi(x))\|(\overline{\mathbf{B}_i^0}) < \eta.$$

Similarly, (iv) follows from Proposition 5.3.4 (v) and Lemma 5.3.2. Finally, item (v) of the proposition follows from items (v) and (vi) of the definition of $r_i(q)$. $\square$

Next we define

$$\mathcal{S} = \left\{ \Phi_\gamma(x) \cap \partial B_i(q) : x \in \tilde{X}[N]_0, i \in [I], q \in [Q] \right\} \cup \left\{ \partial B_i(q) \cap \partial B_j(q') : i, j \in [I]; q, q' \in [Q] \right\},$$

where γ is as in Proposition 5.3.5. Let $\mathcal{S}$ be the closed subset of M which is the union of all the elements of $\mathcal{S}$. By the transversality assumptions in the definition of $r_i(q)$, each non-empty element of $\mathcal{S}$ is a smooth, closed, embedded co-dimension 2 submanifold of M . Hence, by the results of [HK78, Section 3], for all $\Sigma \in \mathcal{S}$,

$$\mathcal{H}^n(\mathcal{T}_\rho(\Sigma)) = O(\rho),$$

where the constants in $O(\rho)$ depend only on the second fundamental tensor of Σ and the ambient Riemannian metric on M . Therefore, there exist constants C and ρ_0 , depending on the submanifolds contained in the set $\mathcal{S}$, such that

$$\mathcal{H}^n(\mathcal{T}_\rho(\mathcal{S})) \leq C\rho \quad \forall \rho \leq \rho_0. \quad (5.3.22)$$

Part 3. Let

$$\mathcal{K}(M) = \{S \in \mathcal{C}(M) : S \text{ is closed subsets of } M\}.$$

We are now going to prove that there exists a discrete, ‘almost smooth’ sweepout $\tilde{\Psi} : \tilde{X}[NI]_0 \rightarrow \mathcal{K}(M)$ which ‘approximates’ $\tilde{\Phi} : \tilde{X} \rightarrow \mathcal{C}(M)$.

Let $K, K' \in \mathcal{K}(M)$, B be a normal geodesic ball and $A = M \setminus \overline{B}$. We define

$$\Omega(K, K'; B) = (K \cap \overline{A}) \cup (K' \cap \overline{B}), \quad (5.3.23)$$

which is also an element of $\mathcal{K}(M)$. This definition is motivated by the construction of “nested sweepouts” by Chambers and Liokumovich [CL20, Section 6]. In the following proposition and in its proof, we will use various notations which were introduced previously.

Proposition 5.3.6. *There exists a map $\tilde{\Psi} : \tilde{X}[NI]_0 \rightarrow \mathcal{K}(M)$ such that $\tilde{\Psi}$ has the property*

$$(P)_0 \quad \tilde{\Psi}(x) = \tilde{\Phi}_\gamma(x) \text{ for all } x \in \tilde{X}[N]_0.$$

Moreover, using the notation

$$\Psi(v) = \Psi(T(v)) = \tilde{\Psi}(v) \cap \tilde{\Psi}(T(v)) \text{ for } v \in \tilde{X}[NI]_0, \quad (5.3.24)$$

$$\tilde{\Psi} \circ \Delta_q^1 = \tilde{\Xi}_q^1, \quad \tilde{\Psi} \circ \Delta_q^2 = \tilde{\Xi}_q^2,$$

for $q \in [Q]$, $\tilde{\Psi}|_{e_q[I]_0}$ and $\tilde{\Psi}|_{f_q[I]_0}$ have the following properties. (Here $m = \dim(e_q) = \dim(f_q)$.)

(P0)_{m,q} For $\xi \in \mathcal{J}^m$, let us write $\xi = (\xi', \xi_m)$ where $\xi' \in \mathcal{J}^{m-1}$. Then, for $s = 1, 2$,

$$\tilde{\Xi}_q^s(\xi', iI^{-1}) = \Omega\left(\tilde{\Xi}_q^s(\xi', (i-1)I^{-1}), \tilde{\Xi}_q^s(\xi', 1); B_i(q)\right)$$

for all $\xi' \in \mathcal{J}^{m-1}[I]_0 \setminus (\partial \mathcal{J}^{m-1})[I]_0$ and $i \in [I]$.

(P1)_{m,q} For all $v \in e_q[I]_0$, $M = \tilde{\Psi}(v) \cup \tilde{\Psi}(T(v))$.

(P2)_{m,q} For all $v \in e_q[I]_0$,

$$\tilde{\Psi}(v) \subset \bigcup_{x \in e_q[1]_0} \tilde{\Phi}_\gamma(x).$$

Similarly, for all $v' \in f_q[I]_0$,

$$\tilde{\Psi}(v') \subset \bigcup_{x' \in f_q[1]_0} \tilde{\Phi}_\gamma(x').$$

(P3)_{m,q} Let

$$\mathcal{F}_q = \{s \in [Q] : e_s \prec e_q \text{ or } f_s \prec e_q\}.$$

Then, for all $v \in e_q[I]_0$,

$$\partial\tilde{\Psi}(v), \partial\tilde{\Psi}(T(v)) \subset \left(\bigcup_{x \in e_q[1]_0} \Phi_\gamma(x) \right) \bigcup \left(\bigcup_{j \in [I], s \in \mathcal{F}_q} \partial B_j(s) \right).$$

(P4)_{m,q} For all $v \in e_q[I]_0$,

$$\Psi(v) \setminus \mathcal{S} = \mathbb{S}_1 \dot{\cup} \mathbb{S}_2 \dot{\cup} \dots \dot{\cup} \mathbb{S}_l \quad (\text{for some } l \in \mathbb{N}),$$

where each $\mathbb{S}_j$ is an open subset of a hypersurface belonging the set

$$\left\{ \Phi_\gamma(x) : x \in e_q[1]_0 \right\} \bigcup \left\{ \partial B_j(s) : j \in [I], s \in \mathcal{F}_q \right\}.$$

Here $\mathcal{F}_q$ is as in (P3)_{m,q}. Moreover, for every $p \in \Psi(v) \setminus \mathcal{S}$, there exists a normal geodesic ball $\mathbf{U}$, centered at p , such that $\mathbf{U} \cap \Psi(v)$ is a smooth hypersurface and

$$\mathbf{U} \setminus \Psi(v) = G_1 \dot{\cup} G_2,$$

where G_1, G_2 are connected open sets, $G_1 \subset \tilde{\Psi}(v)$ and $G_2 \subset \tilde{\Psi}(T(v))$.

(P5)_{m,q} Suppose $v, v' \in e_q[I]_0 \cup f_q[I]_0$, $e \in e_q[I]_1 \cup f_q[I]_1$ such that $v, v' \prec e$. Then there exists $\mathcal{B} = \mathbf{B}_\ell$, for some $\ell = \ell(e) \in [I]$, such that

$$\begin{aligned} \tilde{\Psi}(v) \cap (M \setminus (\mathcal{A} \cup \mathcal{B})) &= \tilde{\Psi}(v') \cap (M \setminus (\mathcal{A} \cup \mathcal{B})); \\ \tilde{\Psi}(T(v)) \cap (M \setminus (\mathcal{A} \cup \mathcal{B})) &= \tilde{\Psi}(T(v')) \cap (M \setminus (\mathcal{A} \cup \mathcal{B})); \\ \Psi(v) \cap (M \setminus (\mathcal{A} \cup \mathcal{B})) &= \Psi(v') \cap (M \setminus (\mathcal{A} \cup \mathcal{B})). \end{aligned}$$

Moreover, $\mathcal{B}$ can be characterized as follows. Without loss of generality, let $v, v' \in e_q[I]_0$. If

$D_q^1(v) = (\xi_1, \xi_2, \dots, \xi_m)$ and $D_q^1(v') = (\xi'_1, \xi'_2, \dots, \xi'_m)$, there exists a unique index $p \in [m]$ such that $\xi_p \neq \xi'_p$. Let $\ell \in [I]$ be such that $\{\xi_p, \xi'_p\} = \{(\ell - 1)I^{-1}, \ell I^{-1}\}$. Then $\mathcal{B} = \mathbf{B}_\ell$.

(P6)_{m,q} For all $l \in [I]$ and $v \in e_q[I]_0$,

$$\mathcal{H}^n(\Psi(v) \cap \overline{\mathbf{B}_l^0}) < 2^{4m+2}\eta.$$

(P7)_{m,q} For all $v \in e_q[I]_0$,

$$\mathcal{H}^n(\Psi(v) \cap \overline{\mathcal{A}_0}) < 2^{4m+2}\eta.$$

(P8)_{m,q} For all $v \in e_q[I]_0$, $x \in e_q[1]_0$ and $R \in \mathcal{R}$,

$$\left| \mathcal{H}^n(\Psi(v) \cap R) - \mathcal{H}^n(\Phi_\gamma(x) \cap R) \right| < 2^{4m+2}\eta.$$

Hence, in particular,

$$\mathcal{H}^n(\Psi(v)) < L + (2^{4m+2} + 2)\eta. \quad (5.3.25)$$

Proof. We will inductively construct $\tilde{\Psi} : e_q[I]_0 \cup f_q[I]_0 \rightarrow \mathcal{K}(M)$. If $\dim(e_q) = \dim(f_q) = 0$, then $e_q[I] = e_q$ and $f_q[I] = f_q$. In that case we define

$$\tilde{\Psi}(e_q) = \tilde{\Phi}_\gamma(e_q) \quad \text{and} \quad \tilde{\Psi}(f_q) = \tilde{\Phi}_\gamma(f_q),$$

which is precisely the property $(P)_0$. Moreover, for this definition of $\tilde{\Psi}$ on the 0-cells e_q and f_q , properties $(P0)_{0,q} - (P8)_{0,q}$ are also satisfied. (One needs to use Propositions 5.3.4 and 5.3.5.)

Let us assume that for some $d \geq 1$, $\tilde{\Psi}$ has been defined on

$$\bigcup_{\dim(e_q) \leq d-1} (e_q[I]_0 \cup f_q[I]_0)$$

and if $\dim(e_s) = \dim(f_s) = (d-1)$, $\tilde{\Psi}|_{e_s[I]_0}$ and $\tilde{\Psi}|_{f_s[I]_0}$ have the properties $(P0)_{(d-1),s} - (P8)_{(d-1),s}$.

Let $\nu \in [Q]$ be such that $\dim(e_\nu) = \dim(f_\nu) = d$. Let us define $\tilde{\Xi}_\nu^1, \tilde{\Xi}_\nu^2$ on $\mathcal{J}^d[I]_0$ and $\tilde{\Psi}$ on

$e_\nu[I]_0, f_\nu[I]_0$ as follows. By our assumption, $\tilde{\Psi}$ is already defined on $(\partial e_\nu)[I]_0$ and $(\partial f_\nu)[I]_0$. If $z \in (\partial \mathcal{J}^d)[I]_0$, then $\Delta_\nu^1(z) \in (\partial e_\nu)[I]_0$ and $\Delta_\nu^2(z) \in (\partial f_\nu)[I]_0$; for $s = 1, 2$, we define

$$\tilde{\Xi}_\nu^s(z) = \left(\tilde{\Psi} \circ \Delta_\nu^s \right) (z). \quad (5.3.26)$$

If $z \in \mathcal{J}^d[I]_0 \setminus (\partial \mathcal{J}^d)[I]_0$, we write $z = (z', iI^{-1})$ with $z' \in \mathcal{J}^{d-1}[I]_0$ and $1 \leq i \leq I-1$; for $s = 1, 2$, we define

$$\tilde{\Xi}_\nu^s(z', iI^{-1}) = \Omega \left(\tilde{\Xi}_\nu^s(z', (i-1)I^{-1}), \tilde{\Xi}_\nu^s(z', 1); B_i(\nu) \right). \quad (5.3.27)$$

For $v \in e_\nu[I]_0 \cup f_\nu[I]_0$, we define

$$\tilde{\Psi}(v) = \begin{cases} \left(\tilde{\Xi}_\nu^1 \circ D_\nu^1 \right) (v) & \text{if } v \in e_\nu[I]_0; \\ \left(\tilde{\Xi}_\nu^2 \circ D_\nu^2 \right) (v) & \text{if } v \in f_\nu[I]_0. \end{cases} \quad (5.3.28)$$

By (5.3.26), $\tilde{\Psi}$ is well-defined on $(\partial e_\nu)[I]_0 \cup (\partial f_\nu)[I]_0$. Let us also set

$$\Xi_\nu(z) = \tilde{\Xi}_\nu^1(z) \cap \tilde{\Xi}_\nu^2(z), \quad z \in \mathcal{J}^d[I]_0, \quad (5.3.29)$$

so that

$$\Psi(v) = \begin{cases} \left(\Xi_\nu \circ D_\nu^1 \right) (v) & \text{if } v \in e_\nu[I]_0; \\ \left(\Xi_\nu \circ D_\nu^2 \right) (v) & \text{if } v \in f_\nu[I]_0. \end{cases} \quad (5.3.30)$$

We need to show that $\tilde{\Psi}$, defined in this way on $e_\nu[I]_0$ and $f_\nu[I]_0$, have the properties $(P0)_{d,\nu} - (P8)_{d,\nu}$.

Before we start, for convenience let us introduce the following notation.

$$B_i = B_i(\nu), \quad A_i = M \setminus \overline{B}_i, \quad S_i = \partial B_i = \partial A_i = \overline{A}_i \cap \overline{B}_i.$$

We fix $z' \in \mathcal{J}^{d-1}[I]_0 \setminus (\partial \mathcal{J}^{d-1})[I]_0$ and define (for $i = 0, 1, \dots, I$)

$$K_i = \tilde{\Xi}_\nu^1(z', iI^{-1}), \quad L_i = \tilde{\Xi}_\nu^2(z', iI^{-1}), \quad \Sigma_i = \Xi_\nu(z', iI^{-1}) = K_i \cap L_i.$$

From (5.3.27), for $i = 1, 2, \dots, (I-1)$, we obtain

$$\begin{aligned} K_i &= (K_{i-1} \cap \overline{A}_i) \cup (K_I \cap \overline{B}_i); \\ L_i &= (L_{i-1} \cap \overline{A}_i) \cup (L_I \cap \overline{B}_i); \\ \Sigma_i &= (\Sigma_{i-1} \cap \overline{A}_i) \cup (\Sigma_I \cap \overline{B}_i) \cup (K_{i-1} \cap L_I \cap S_i) \cup (K_I \cap L_{i-1} \cap S_i). \end{aligned} \tag{5.3.31}$$

Using induction one can prove the following. For $i = 1, 2, \dots, (I-1)$,

$$\begin{aligned} K_i &= \left[K_0 \cap (\cup_{j=1}^i B_j)^c \right] \cup \left[K_I \cap (\cup_{j=1}^i \overline{B}_j) \right]; \\ L_i &= \left[L_0 \cap (\cup_{j=1}^i B_j)^c \right] \cup \left[L_I \cap (\cup_{j=1}^i \overline{B}_j) \right]; \\ \Sigma_i &= \left[\Sigma_0 \cap (\cup_{j=1}^i B_j)^c \right] \cup \left[\Sigma_I \cap (\cup_{j=1}^i \overline{B}_j) \right] \cup \Lambda_i, \quad \text{where } \Lambda_i \subset \bigcup_{t=1}^i S_t; \end{aligned} \tag{5.3.32}$$

setting $\Lambda_0 = \emptyset$,

$$\Lambda_i = (\Lambda_{i-1} \cap \overline{A}_i) \cup (K_{i-1} \cap L_I \cap S_i) \cup (K_I \cap L_{i-1} \cap S_i). \tag{5.3.33}$$

From the equations in (5.3.32) (and using (5.3.18)), we conclude that the equations in (5.3.31) (and hence (5.3.27) also) hold for $i = I$ as well. Let us denote the top and bottom faces of e_ν and f_ν by the indices τ and β , i.e.

$$\begin{aligned} \left\{ \Delta_\nu^1(\mathcal{J}^{d-1} \times \{1\}), \Delta_\nu^2(\mathcal{J}^{d-1} \times \{1\}) \right\} &= \{e_\tau, f_\tau\}; \\ \left\{ \Delta_\nu^1(\mathcal{J}^{d-1} \times \{0\}), \Delta_\nu^2(\mathcal{J}^{d-1} \times \{0\}) \right\} &= \{e_\beta, f_\beta\}. \end{aligned}$$

We note that the top face of e_ν could be either e_τ (in that case the top face of f_ν is f_τ) or f_τ (in that case the top face of f_ν is e_τ); similar remark applies for the bottom face. For the ease of the

presentation, we will assume that e_τ is the top face of e_ν and e_β is the bottom face of e_ν . The other cases will be entirely analogous.

We now proceed in steps to check that $\tilde{\Psi}|_{e_\nu[I]_0}$ and $\tilde{\Psi}|_{f_\nu[I]_0}$ have the properties $(P0)_{d,\nu} - (P8)_{d,\nu}$.

Step 0. $(P0)_{d,\nu}$ is satisfied because of our definition of $\tilde{\Xi}_\nu^1$ and $\tilde{\Xi}_\nu^2$, i.e. equation (5.3.27). As noted above, (5.3.27) holds for $i = I$ as well.

Step 1. By the induction hypothesis, $\tilde{\Psi}$ restricted to $e_\beta[I]_0$, $f_\beta[I]_0$ (and $e_\tau[I]_0$, $f_\tau[I]_0$) satisfy $(P1)_{(d-1),\beta}$ (and $(P1)_{(d-1),\tau}$). Hence

$$K_0 \cup L_0 = M = K_I \cup L_I,$$

which together with (5.3.32) imply

$$M = K_i \cup L_i \quad \forall i \in [I-1].$$

Step 2. By our assumption, $\tilde{\Psi}$ restricted to $e_\beta[I]_0$, $f_\beta[I]_0$ (and $e_\tau[I]_0$, $f_\tau[I]_0$) satisfy $(P2)_{(d-1),\beta}$ (and $(P2)_{(d-1),\tau}$). Hence,

$$K_0 \subset \bigcup_{x \in e_\beta[1]_0} \tilde{\Phi}_\gamma(x); \quad K_I \subset \bigcup_{x \in e_\tau[1]_0} \tilde{\Phi}_\gamma(x).$$

Therefore, using (5.3.32),

$$K_i \subset \bigcup_{x \in e_\nu[1]_0} \tilde{\Phi}_\gamma(x) \quad \forall i \in [I-1].$$

Similarly, we can show that

$$L_i \subset \bigcup_{x \in f_\nu[1]_0} \tilde{\Phi}_\gamma(x) \quad \forall i \in [I-1].$$

Step 3. By the induction hypothesis $((P3)_{(d-1),\beta})$ and $(P3)_{(d-1),\tau})$,

$$\partial K_0 \subset \left(\bigcup_{x \in e_\beta[1]_0} \Phi_\gamma(x) \right) \bigcup \left(\bigcup_{j \in [I], s \in \mathcal{F}_\beta} \partial B_j(s) \right); \quad (5.3.34)$$

$$\partial K_I \subset \left(\bigcup_{x \in e_\tau[1]_0} \Phi_\gamma(x) \right) \bigcup \left(\bigcup_{j \in [I], s \in \mathcal{F}_\tau} \partial B_j(s) \right). \quad (5.3.35)$$

Moreover, by (5.3.19) and (5.3.32),

$$\partial K_i \subset \partial K_0 \cup \partial K_I \cup \left(\bigcup_{t=1}^i S_t \right) \quad \forall i \in [I-1]. \quad (5.3.36)$$

Combining (5.3.34), (5.3.35) and (5.3.36), we obtain

$$\partial K_i \subset \left(\bigcup_{x \in e_\nu[1]_0} \Phi_\gamma(x) \right) \bigcup \left(\bigcup_{j \in [I], s \in \mathcal{F}_\beta \cup \mathcal{F}_\tau} \partial B_j(s) \right) \bigcup \left(\bigcup_{t=1}^i \partial B_t \right) \quad \forall i \in [I-1]. \quad (5.3.37)$$

One can arrive at a similar conclusion for ∂L_i as well.

Step 4. For $j = 0, 1, \dots, I$, let $\hat{\Sigma}_j := \Sigma_j \setminus \mathcal{S}$. By the induction hypothesis $((P4)_{(d-1),\beta})$ and $(P4)_{(d-1),\tau})$

$$\hat{\Sigma}_0 = \mathbb{S}_1^0 \dot{\cup} \mathbb{S}_2^0 \dot{\cup} \dots \dot{\cup} \mathbb{S}_{l_0}^0,$$

where each $\mathbb{S}_j^0$ is an open subset of a hypersurface belonging to the set

$$\left\{ \Phi_\gamma(x) : x \in e_\beta[1]_0 \right\} \bigcup \left\{ \partial B_j(s) : j \in [I], s \in \mathcal{F}_\beta \right\};$$

and

$$\hat{\Sigma}_I = \mathbb{S}_1^I \dot{\cup} \mathbb{S}_2^I \dot{\cup} \dots \dot{\cup} \mathbb{S}_{l_I}^I,$$

where each $\mathbb{S}_j^I$ is an open subset of a hypersurface belonging to the set

$$\left\{ \Phi_\gamma(x) : x \in e_\tau[1]_0 \right\} \bigcup \left\{ \partial B_j(s) : j \in [I], s \in \mathcal{F}_\tau \right\}. \quad (5.3.38)$$

Let us assume that

$$\hat{\Sigma}_{i-1} = \mathbb{S}_1^{i-1} \dot{\cup} \mathbb{S}_2^{i-1} \dot{\cup} \dots \dot{\cup} \mathbb{S}_{l_{i-1}}^{i-1},$$

where each $\mathbb{S}_j^{i-1}$ is an open subset of a hypersurface belonging to the set

$$\left\{ \Phi_\gamma(x) : x \in e_\nu[1]_0 \right\} \bigcup \left\{ \partial B_j(s) : j \in [I], s \in \mathcal{F}_\beta \cup \mathcal{F}_\tau \right\} \bigcup \left\{ \partial B_t : t \in [i-1] \right\}. \quad (5.3.39)$$

Then

$$(\Sigma_{i-1} \cap \bar{A}_i) \setminus \mathcal{S} = (\hat{\Sigma}_{i-1} \cap \bar{A}_i) \setminus \mathcal{S} = \hat{\Sigma}_{i-1} \cap A_i \quad (5.3.40)$$

as (5.3.39) implies $\hat{\Sigma}_{i-1} \cap \partial B_i \subset \mathcal{S}$. Similarly, using (5.3.38),

$$(\Sigma_I \cap \bar{B}_i) \setminus \mathcal{S} = \hat{\Sigma}_I \cap B_i. \quad (5.3.41)$$

Furthermore,

$$(K_{i-1} \cap L_I \cap S_i) \setminus \mathcal{S} = \text{int}(K_{i-1} \cap L_I) \cap S_i \cap \mathcal{S}^c. \quad (5.3.42)$$

Indeed, by (5.3.19), $\partial(K_{i-1} \cap L_I) \subset \partial K_{i-1} \cup \partial L_I$; hence (5.3.37) and $(P3)_{(d-1),\tau}$ imply

$$(\partial K_{i-1} \cup \partial L_I) \cap S_i \subset \mathcal{S}.$$

Similarly,

$$(K_I \cap L_{i-1} \cap S_i) \setminus \mathcal{S} = \text{int}(K_I \cap L_{i-1}) \cap S_i \cap \mathcal{S}^c. \quad (5.3.43)$$

Moreover,

$$M = A_i \dot{\cup} B_i \dot{\cup} S_i.$$

Therefore, using (5.3.31) and equations (5.3.38) – (5.3.43), we obtain

$$\begin{aligned} \hat{\Sigma}_i &= (\hat{\Sigma}_{i-1} \cap A_i) \cup (\hat{\Sigma}_I \cap B_i) \cup (\text{int}(K_{i-1} \cap L_I) \cap S_i \cap \mathcal{S}^c) \cup (\text{int}(K_I \cap L_{i-1}) \cap S_i \cap \mathcal{S}^c) \\ &= \mathbb{S}_1^i \dot{\cup} \mathbb{S}_2^i \dot{\cup} \dots \dot{\cup} \mathbb{S}_{l_i}^i, \end{aligned} \quad (5.3.44)$$

where each $\mathbb{S}_j^i$ is an open subset of a hypersurface belonging to the set

$$\left\{ \Phi_\gamma(x) : x \in e_\nu[1]_0 \right\} \bigcup \left\{ \partial B_j(s) : j \in [I], s \in \mathcal{F}_\beta \cup \mathcal{F}_\tau \right\} \bigcup \left\{ \partial B_t : t \in [i] \right\}. \quad (5.3.45)$$

(5.3.31) and (5.3.44) also imply the existence of the geodesic ball $\mathbf{U}$ around $p \in \hat{\Sigma}_i$, as stated in (P4). Indeed, for $p \in \hat{\Sigma}_{i-1} \cap A_i$ or $p \in \hat{\Sigma}_I \cap B_i$, one can argue by induction. If $p \in \text{int}(K_{i-1} \cap L_I) \cap S_i \cap \mathcal{S}^c$, it is possible to choose a ball $\mathbf{U}$ around p so that $\mathbf{U} \subset \text{int}(K_{i-1} \cap L_I) \cap \mathcal{S}^c$, $\mathbf{U} \cap \Sigma_i = \mathbf{U} \cap S_i$ and $\mathbf{U} \setminus S_i = G_1 \dot{\cup} G_2$, where $G_1 = \mathbf{U} \cap A_i$ and $G_2 = \mathbf{U} \cap B_i$. Therefore, using (5.3.31),

$$G_1 \subset \text{int}(K_{i-1} \cap L_I) \cap A_i \cap \mathcal{S}^c \subset K_{i-1} \cap A_i = K_i \cap A_i;$$

$$G_2 \subset \text{int}(K_{i-1} \cap L_I) \cap B_i \cap \mathcal{S}^c \subset L_I \cap B_i = L_i \cap B_i.$$

The argument for the case when $p \in \text{int}(K_I \cap L_{i-1}) \cap S_i \cap \mathcal{S}^c$ is similar.

Step 5. Suppose $v, v' \in (\partial e_\nu)[I]_0$. Let $\rho \in [Q]$ such that $\lambda_\rho \prec e_\nu$ (λ_ρ is either e_ρ or f_ρ) with $\dim(\lambda_\rho) = d - 1$ and $v, v' \in \lambda_\rho[I]_0$. Then, by the induction hypothesis $(P5)_{(d-1),\rho}$ and by the compatibility relation (5.3.10), we get $(P5)_{d,\nu}$ for this case.

Now, let us assume that $v \notin (\partial e_\nu)[I]_0$. Suppose

$$D_q^1(v) = (z_1, z_2, \dots, z_d) = (z', z_d), \quad D_q^1(v') = (\mathfrak{z}_1, \mathfrak{z}_2, \dots, \mathfrak{z}_d) = (\mathfrak{z}', \mathfrak{z}_d).$$

If $z_d \neq \mathfrak{z}_d$, using (5.3.31) we get $(P5)_{d,\nu}$. Otherwise, $z_d = \mathfrak{z}_d$ and there are two possibilities: either $v' \notin (\partial e_\nu)[I]_0$ or $v' \in (\partial e_\nu)[I]_0$. Let us assume the second possibility (the other case is similar), i.e. there exists $\rho \in [Q]$ such that $\lambda_\rho \prec e_\nu$ ($\lambda_\rho = e_\rho$ or f_ρ) with $\dim(\lambda_\rho) = d - 1$ and $v' \in \lambda_\rho[I]_0$. Let us define,

$$B'_i = B_i(\rho), \quad A'_i = M \setminus \overline{B'_i}, \quad S'_i = \partial B'_i = \partial A'_i;$$

$$K'_i = \tilde{\Xi}_\nu^1(\mathfrak{z}', iI^{-1}), \quad L'_i = \tilde{\Xi}_\nu^2(\mathfrak{z}', iI^{-1}), \quad \Sigma'_i = \Xi_\nu(\mathfrak{z}', iI^{-1}) = K'_i \cap L'_i.$$

Using $(P0)_{(d-1),\rho}$ and the compatibility relation (5.3.10), one can deduce the following relations,

similar to those in (5.3.32).

$$\begin{aligned}
K'_i &= \left[K'_0 \cap (\cup_{j=1}^i B'_j)^c \right] \cup \left[K'_I \cap (\cup_{j=1}^i \overline{B'_j}) \right]; \\
L'_i &= \left[L'_0 \cap (\cup_{j=1}^i B'_j)^c \right] \cup \left[L'_I \cap (\cup_{j=1}^i \overline{B'_j}) \right]; \\
\Sigma'_i &= \left[\Sigma'_0 \cap (\cup_{j=1}^i B'_j)^c \right] \cup \left[\Sigma'_I \cap (\cup_{j=1}^i \overline{B'_j}) \right] \cup \Lambda'_i \quad \text{where} \quad \Lambda'_i \subset \bigcup_{t=1}^i S'_t.
\end{aligned} \tag{5.3.46}$$

$(P5)_{(d-1),\beta}$ and $(P5)_{(d-1),\tau}$ imply, for $s \in \{0, I\}$,

$$\begin{aligned}
K_s \cap (M \setminus (\mathcal{A} \cup \mathcal{B})) &= K'_s \cap (M \setminus (\mathcal{A} \cup \mathcal{B})); \\
L_s \cap (M \setminus (\mathcal{A} \cup \mathcal{B})) &= L'_s \cap (M \setminus (\mathcal{A} \cup \mathcal{B})); \\
\Sigma_s \cap (M \setminus (\mathcal{A} \cup \mathcal{B})) &= \Sigma'_s \cap (M \setminus (\mathcal{A} \cup \mathcal{B})),
\end{aligned} \tag{5.3.47}$$

where $\mathcal{B} = \mathbf{B}_\ell$ and ℓ is such that if $p \in [d]$ is the unique index for which $z_p \neq \mathfrak{z}_p$, then $\{z_p, \mathfrak{z}_p\} = \{(\ell-1)I^{-1}, \ell I^{-1}\}$. We note that for all $j \in [I]$,

$$B_j \Delta B'_j, \partial B_j, \partial B'_j \subset \mathcal{A}.$$

Hence,

$$\begin{aligned}
(\cup_{j=1}^i B_j)^c \Delta (\cup_{j=1}^i B'_j)^c &\subset \mathcal{A}, \quad \text{which implies} \quad (\cup_{j=1}^i B_j)^c \cap (\mathcal{A} \cup \mathcal{B})^c = (\cup_{j=1}^i B'_j)^c \cap (\mathcal{A} \cup \mathcal{B})^c; \\
(\cup_{j=1}^i \overline{B_j}) \Delta (\cup_{j=1}^i \overline{B'_j}) &\subset \mathcal{A}, \quad \text{which implies} \quad (\cup_{j=1}^i \overline{B_j}) \cap (\mathcal{A} \cup \mathcal{B})^c = (\cup_{j=1}^i \overline{B'_j}) \cap (\mathcal{A} \cup \mathcal{B})^c; \\
\Lambda_i \cap (\mathcal{A} \cup \mathcal{B})^c &= \emptyset = \Lambda'_i \cap (\mathcal{A} \cup \mathcal{B})^c.
\end{aligned} \tag{5.3.48}$$

Using the equations in (5.3.32), (5.3.46), (5.3.47) and (5.3.48), we obtain, for all $i \in [I]$,

$$K_i \cap (M \setminus (\mathcal{A} \cup \mathcal{B})) = K'_i \cap (M \setminus (\mathcal{A} \cup \mathcal{B}));$$

$$L_i \cap (M \setminus (\mathcal{A} \cup \mathcal{B})) = L'_i \cap (M \setminus (\mathcal{A} \cup \mathcal{B}));$$

$$\Sigma_i \cap (M \setminus (\mathcal{A} \cup \mathcal{B})) = \Sigma'_i \cap (M \setminus (\mathcal{A} \cup \mathcal{B})).$$

Step 6. Using (5.3.33), for $i \in [I - 1]$,

$$\mathcal{H}^n(\Lambda_i) \leq \mathcal{H}^n(\Lambda_{i-1}) + \mathcal{H}^n(K_{i-1} \cap L_I \cap S_i) + \mathcal{H}^n(K_I \cap L_{i-1} \cap S_i). \quad (5.3.49)$$

By $(P2)_{d,\nu}$,

$$K_{i-1} \subset \bigcup_{x \in e_\nu[1]_0} \tilde{\Phi}_\gamma(x), \quad L_I \subset \bigcup_{x \in f_\nu[1]_0} \tilde{\Phi}_\gamma(x).$$

Hence,

$$\mathcal{H}^n(S_i \cap K_{i-1} \cap L_I) \leq \sum_{\substack{x_1 \in e_\nu[1]_0, \\ x_2 \in f_\nu[1]_0}} \mathcal{H}^n(S_i \cap \tilde{\Phi}_\gamma(x_1) \cap \tilde{\Phi}_\gamma(x_2)). \quad (5.3.50)$$

For $x_1 \in e_\nu[1]_0$, $x_2 \in f_\nu[1]_0$,

$$\begin{aligned} \mathcal{H}^n(S_i \cap \tilde{\Phi}_\gamma(x_1) \cap \tilde{\Phi}_\gamma(x_2)) &= \mathcal{H}^n(S_i \cap \tilde{\Phi}_\gamma(x_1) \cap \tilde{\Phi}_\gamma(x_2) \cap \tilde{\Phi}_\gamma(T(x_2))) \\ &\quad + \mathcal{H}^n(S_i \cap \tilde{\Phi}_\gamma(x_1) \cap \tilde{\Phi}_\gamma(x_2) \cap (M - \tilde{\Phi}_\gamma(T(x_2)))) \\ &= 0 + \mathcal{H}^n(S_i \cap (\tilde{\Phi}_\gamma(x_1) - \tilde{\Phi}_\gamma(T(x_2)))) \\ &< \frac{2^{2d}\eta}{I}. \end{aligned} \quad (5.3.51)$$

In the second equality, we have used the fact that S_i is transverse to $\Phi_\gamma(x_2) = \tilde{\Phi}_\gamma(x_2) \cap \tilde{\Phi}_\gamma(T(x_2))$ (which was assumed in the definition of $r_i(q)$) and $(P1)_{d,\nu}$. In the last inequality, we have used Proposition 5.3.5 (v). Therefore, combining (5.3.49) – (5.3.51), we obtain

$$\mathcal{H}^n(\Lambda_i) < \mathcal{H}^n(\Lambda_{i-1}) + \frac{2^{4d+1}\eta}{I}.$$

Since $\Lambda_0 = \emptyset$, for all $i \in [I - 1]$,

$$\mathcal{H}^n(\Lambda_i) < \frac{2^{4d+1}\eta^i}{I} < 2^{4d+1}\eta. \quad (5.3.52)$$

Using this inequality along with (5.3.32) and $(P6)_{(d-1),\beta}$, $(P6)_{(d-1),\tau}$, one can conclude that for all $l \in [I]$,

$$\begin{aligned} \mathcal{H}^n(\Sigma_i \cap \overline{\mathbf{B}}_l^0) &\leq \mathcal{H}^n(\Sigma_0 \cap \overline{\mathbf{B}}_l^0) + \mathcal{H}^n(\Sigma_I \cap \overline{\mathbf{B}}_l^0) + \mathcal{H}^n(\Lambda_i) \\ &< (2^{4d-2} + 2^{4d-2} + 2^{4d+1})\eta \\ &< 2^{4d+2}\eta. \end{aligned} \quad (5.3.53)$$

Step 7. Using (5.3.52), (5.3.32) and $(P7)_{(d-1),\beta}$, $(P7)_{(d-1),\tau}$, we obtain as in Step 6,

$$\begin{aligned} \mathcal{H}^n(\Sigma_i \cap \overline{\mathcal{A}}_0) &\leq \mathcal{H}^n(\Sigma_0 \cap \overline{\mathcal{A}}_0) + \mathcal{H}^n(\Sigma_I \cap \overline{\mathcal{A}}_0) + \mathcal{H}^n(\Lambda_i) \\ &< (2^{4d-2} + 2^{4d-2} + 2^{4d+1})\eta \\ &< 2^{4d+2}\eta. \end{aligned} \quad (5.3.54)$$

Step 8. We note that for all $j \in [I]$,

$$\overline{\mathbf{B}}_j^c \subset B_j^c \subset \overline{\mathbf{B}}_j^c \cup \overline{\mathcal{A}}; \quad \overline{B}_j \subset \mathbf{B}_j \subset \overline{B}_j \cup \overline{\mathcal{A}}.$$

Therefore, for all $i \in [I]$,

$$\begin{aligned} \left(\bigcup_{j=1}^i \overline{\mathbf{B}}_j \right)^c &\subset \left(\bigcup_{j=1}^i B_j \right)^c \subset \left(\bigcup_{j=1}^i \overline{\mathbf{B}}_j \right)^c \cup \overline{\mathcal{A}}; \\ \bigcup_{j=1}^i \overline{B}_j &\subset \bigcup_{j=1}^i \mathbf{B}_j \subset \left(\bigcup_{j=1}^i \overline{B}_j \right) \cup \overline{\mathcal{A}}. \end{aligned}$$

Hence, using (5.3.32), one can deduce the following inequalities. For $R \in \mathcal{R}$,

$$\mathcal{H}^n(\Sigma_i \cap R) \leq \mathcal{H}^n\left(\Sigma_0 \cap (\cup_{j=1}^i \overline{\mathbf{B}}_j)^c \cap R\right) + \mathcal{H}^n(\Sigma_0 \cap \overline{\mathcal{A}}) + \mathcal{H}^n(\Sigma_I \cap (\cup_{j=1}^i \mathbf{B}_j) \cap R) + \mathcal{H}^n(\Lambda_i); \quad (5.3.55)$$

$$\begin{aligned} \mathcal{H}^n(\Sigma_i \cap R) &\geq \mathcal{H}^n\left(\left(\Sigma_0 \cap (\cup_{j=1}^i B_j)^c \cap R\right) \cup \left(\Sigma_I \cap (\cup_{j=1}^i \overline{B}_j) \cap R\right)\right) \\ &\geq \mathcal{H}^n\left(\left(\Sigma_0 \cap (\cup_{j=1}^i B_j)^c \cap R\right) \cup \left(\Sigma_I \cap R \cap ((\cup_{j=1}^i \overline{B}_j) \cup \overline{\mathcal{A}})\right)\right) - \mathcal{H}^n(\Sigma_I \cap R \cap \overline{\mathcal{A}}) \\ &\geq \mathcal{H}^n\left(\Sigma_0 \cap (\cup_{j=1}^i \overline{\mathbf{B}}_j)^c \cap R\right) + \mathcal{H}^n(\Sigma_I \cap (\cup_{j=1}^i \mathbf{B}_j) \cap R) - \mathcal{H}^n(\Sigma_I \cap \overline{\mathcal{A}}). \end{aligned} \quad (5.3.56)$$

Denoting

$$R_1 = (\cup_{j=1}^i \overline{\mathbf{B}}_j)^c \cap R, \quad R_2 = (\cup_{j=1}^i \mathbf{B}_j) \cap R, \quad (5.3.57)$$

using (5.3.55), (5.3.56), $(P7)_{(d-1),\beta}$, $(P7)_{(d-1),\tau}$ and (5.3.52), we obtain

$$\begin{aligned} &|\mathcal{H}^n(\Sigma_i \cap R) - \mathcal{H}^n(\Sigma_0 \cap R_1) - \mathcal{H}^n(\Sigma_I \cap R_2)| \\ &\leq \mathcal{H}^n(\Sigma_0 \cap \overline{\mathcal{A}}) + \mathcal{H}^n(\Sigma_I \cap \overline{\mathcal{A}}) + \mathcal{H}^n(\Lambda_i) \\ &\leq (2 \cdot 2^{4d-2} + 2^{4d+1})\eta. \end{aligned} \quad (5.3.58)$$

Let $x \in e_\nu[1]_0$; without loss of generality we can assume that $x \in e_\beta[1]_0$, i.e. $D_\nu^1(x) = (\xi, 0)$ for some $\xi \in \mathcal{J}^{d-1}$. Let $x' \in e_\tau[1]_0$ such that $x' = \Delta_\nu^1(\xi, 1)$. We note that $R \in \mathcal{R}$ implies $R_1, R_2 \in \mathcal{R}$ as well. Further, by the definition of $\mathbf{B}_j$ ((5.3.16)), $\|\Phi_\gamma(x)\|(\partial \mathbf{B}_j) = 0$; hence

$$\mathcal{H}^n(\Phi_\gamma(x) \cap R) = \mathcal{H}^n(\Phi_\gamma(x) \cap R_1) + \mathcal{H}^n(\Phi_\gamma(x) \cap R_2).$$

Therefore, using (5.3.58), $(P8)_{(d-1),\beta}$, $(P8)_{(d-1),\tau}$, Proposition 5.3.5 (i) and (5.3.11) we get the

following estimate.

$$\begin{aligned}
& |\mathcal{H}^n(\Sigma_i \cap R) - \mathcal{H}^n(\Phi_\gamma(x) \cap R)| \\
& \leq |\mathcal{H}^n(\Sigma_i \cap R) - \mathcal{H}^n(\Sigma_0 \cap R_1) - \mathcal{H}^n(\Sigma_I \cap R_2)| + |\mathcal{H}^n(\Sigma_0 \cap R_1) - \mathcal{H}^n(\Phi_\gamma(x) \cap R_1)| \\
& \quad + |\mathcal{H}^n(\Sigma_I \cap R_2) - \mathcal{H}^n(\Phi_\gamma(x') \cap R_2)| + |\mathcal{H}^n(\Phi_\gamma(x') \cap R_2) - \mathcal{H}^n(\Phi_\gamma(x) \cap R_2)| \\
& < (2^{4d-1} + 2^{4d+1} + 2^{4d-2} + 2^{4d-2} + 3)\eta \\
& < 2^{4d+2}\eta.
\end{aligned} \tag{5.3.59}$$

Since $M \in \mathcal{R}$, (5.3.59) and Proposition 5.3.5 (ii) imply

$$\mathcal{H}^n(\Sigma_i) \leq L + (2^{4d+2} + 2)\eta.$$

□

5.3.4 An approximate solution of the Allen-Cahn equation

Here we will briefly discuss an approximate solution of the Allen-Cahn equation whose energy is concentrated in a tubular neighbourhood of a closed, two-sided hypersurface (with mild singularities).

We will follow the paper by Guaraco [Gua18]; further details can be found there.

Let $\mathbf{g} : \mathbb{R} \rightarrow \mathbb{R}$ be the unique solution to the following ODE.

$$\varphi'(t) = \sqrt{2W(\varphi(t))}; \quad \varphi(0) = 0. \tag{5.3.60}$$

For all $t \in \mathbb{R}$, $-1 < \mathbf{g}(t) < 1$ and

$$\text{as } t \rightarrow \pm\infty, (\mathbf{g}(t) \mp 1) \text{ converges to 0 exponentially fast.} \tag{5.3.61}$$

$\mathbf{g}_\varepsilon(t) = \mathbf{g}(t/\varepsilon)$ is a solution to the one dimensional Allen-Cahn equation

$$\varepsilon^2 \varphi''(t) = W'(\varphi(t))$$

with finite total energy:

$$\int_{-\infty}^{\infty} \left(\frac{\varepsilon}{2} (\mathfrak{g}'_{\varepsilon}(t))^2 + \frac{1}{\varepsilon} W(\mathfrak{g}_{\varepsilon}(t)) \right) dt = 2\sigma. \quad (5.3.62)$$

For $\varepsilon > 0$, we define the Lipschitz continuous function

$$g_{\varepsilon}(t) = \begin{cases} \mathfrak{g}_{\varepsilon}(t) & \text{if } |t| \leq \sqrt{\varepsilon}; \\ \mathfrak{g}_{\varepsilon}(\sqrt{\varepsilon}) + \left(\frac{t}{\sqrt{\varepsilon}} - 1 \right) (1 - \mathfrak{g}_{\varepsilon}(\sqrt{\varepsilon})) & \text{if } \sqrt{\varepsilon} \leq t \leq 2\sqrt{\varepsilon}; \\ 1 & \text{if } t \geq 2\sqrt{\varepsilon}; \\ \mathfrak{g}_{\varepsilon}(-\sqrt{\varepsilon}) + \left(\frac{t}{\sqrt{\varepsilon}} + 1 \right) (1 + \mathfrak{g}_{\varepsilon}(-\sqrt{\varepsilon})) & \text{if } -2\sqrt{\varepsilon} \leq t \leq -\sqrt{\varepsilon}; \\ -1 & \text{if } t \leq -2\sqrt{\varepsilon}. \end{cases}$$

Suppose $d : M \rightarrow \mathbb{R}$ is a Lipschitz continuous function such that $|\nabla d| = 1$, $\mathcal{H}^{n+1}$ -a.e. Let $u_{\varepsilon} = g_{\varepsilon} \circ d$ and $U \subset M$ be $\mathcal{H}^{n+1}$ -measurable. Using the notation

$$E_{\varepsilon}(u, U) = \int_U \varepsilon \frac{|\nabla u|^2}{2} + \frac{W(u)}{\varepsilon},$$

we compute

$$\begin{aligned} E_{\varepsilon}(u_{\varepsilon}, U) &= \int_{U \cap \{|d| \leq 2\sqrt{\varepsilon}\}} \left[\frac{\varepsilon}{2} (g'_{\varepsilon}(d(y)))^2 + \frac{1}{\varepsilon} W(g_{\varepsilon}(d(y))) \right] d\mathcal{H}^{n+1}(y) \\ &= \int_{|\tau| \leq 2\sqrt{\varepsilon}} \left[\frac{\varepsilon}{2} (g'_{\varepsilon}(\tau))^2 + \frac{1}{\varepsilon} W(g_{\varepsilon}(\tau)) \right] \mathcal{H}^n(U \cap \{d = \tau\}) d\tau \\ &\leq \int_{|\tau| \leq \sqrt{\varepsilon}} \left[\frac{\varepsilon}{2} (\mathfrak{g}'_{\varepsilon}(\tau))^2 + \frac{1}{\varepsilon} W(\mathfrak{g}_{\varepsilon}(\tau)) \right] \mathcal{H}^n(U \cap \{d = \tau\}) d\tau \end{aligned} \quad (5.3.63)$$

$$+ \left(\frac{1}{2} (1 - \mathfrak{g}_{\varepsilon}(\sqrt{\varepsilon}))^2 + \frac{1}{\varepsilon} W(\mathfrak{g}_{\varepsilon}(\sqrt{\varepsilon})) \right) \text{Vol}(M). \quad (5.3.64)$$

Let

$$\Lambda_{\varepsilon} = \sup_{|\tau| \leq \sqrt{\varepsilon}} \mathcal{H}^n(U \cap \{d = \tau\}).$$

Then, by (5.3.62), the integral in (5.3.63) is bounded by $2\sigma\Lambda_{\varepsilon}$. Further, (6.4.2) implies that there exists $\varepsilon_0 = \varepsilon_0(W, \text{Vol}(M), \eta)$ such that for all $0 < \varepsilon \leq \varepsilon_0$, the expression in (5.3.64) is bounded by

$2\sigma\eta$. Hence,

$$E_\varepsilon(u_\varepsilon, U) \leq 2\sigma(\Lambda_\varepsilon + \eta) \quad \forall 0 < \varepsilon \leq \varepsilon_0. \quad (5.3.65)$$

For $v \in \tilde{X}[NI]_0$, we define $d_v : M \rightarrow \mathbb{R}$ as follows.

$$d_v(y) = \begin{cases} -d(y, \Psi(v)) & \text{if } y \in \tilde{\Psi}(v); \\ d(y, \Psi(v)) & \text{if } y \in \tilde{\Psi}(T(v)). \end{cases} \quad (5.3.66)$$

By the definition of $\Psi(v)$ (in (5.3.24)) and (P1) of Proposition 5.3.6, d_v is a well-defined continuous function on M . Moreover, by [Gua18, Proposition 9.1], d_v is Lipschitz continuous and $|\nabla d_v| = 1$, $\mathcal{H}^{n+1}$ -a.e.

Let $y \in U \cap \{d_v = \tau\}$. Then either $d(y, \mathcal{S}) = |\tau|$ or there exists $z \in \Psi(v) \setminus \mathcal{S}$ with $d(y, z) = |\tau|$ such that $y = \exp_z(\tau \mathbf{n}(z))$, where $\mathbf{n}(z)$ is the unit normal to $\Psi(v) \setminus \mathcal{S}$ at z , pointing towards $\text{int}(\tilde{\Psi}(T(v)))$. (By Proposition 5.3.6 (P4), $z' \mapsto \mathbf{n}(z')$ is a well-defined normal vector-field on $\Psi(v) \setminus \mathcal{S}$.) We must have

$$z \in \mathcal{N}_{|\tau|}(U) \cap (\Psi(v) \setminus \mathcal{S}).$$

Hence

$$\mathcal{H}^n(U \cap \{d_v = \tau\}) \leq \mathcal{H}^n(\mathcal{T}_{|\tau|}(\mathcal{S})) + \int_{\mathcal{N}_{|\tau|}(U) \cap (\Psi(v) \setminus \mathcal{S})} |J \exp_z(\tau \mathbf{n}(z))| d\mathcal{H}^n(z), \quad (5.3.67)$$

where $J \exp_z(\tau \mathbf{n}(z))$ is the Jacobian factor of the map $z \mapsto \exp_z(\tau \mathbf{n}(z))$, which can be estimated as follows.

Let

$$\mathcal{S}' = \{\Phi_\gamma(x) : x \in \tilde{X}[NI]_0\} \bigcup \{\partial B_i(q) : i \in [I], q \in [Q]\}.$$

Using (P4) of Proposition 5.3.6 and [War66, Corollary 4.2, Theorem 4.3], [Gua18, Proposition 9.4], one can deduce the following estimate. There exist τ_0 and C_1 , depending only on the hypersurfaces

contained in $\mathcal{S}'$ and the ambient Riemannian metric on M , such that

$$|J \exp_z(\tau \mathbf{n}(z))| \leq (1 + C_1 |\tau|) \quad \forall z \in \Psi(v) \setminus \mathcal{S}, \quad |\tau| \leq \tau_0.$$

Hence, using (5.3.22) and (5.3.67), there exists $\tau_1 = \tau_1(\mathcal{S}, \mathcal{S}', M)$ such that for all $|\tau| \leq \tau_1$,

$$\mathcal{H}^n(U \cap \{d_v = \tau\}) \leq C|\tau| + (1 + C_1 |\tau|) \mathcal{H}^n(\mathcal{N}_{|\tau|}(U) \cap \Psi(v)).$$

This estimate along with (P6) and (P7) of Proposition 5.3.6, (5.3.25) and (5.3.65) gives the following proposition. We recall that k is the dimension of the parameter spaces X and $\tilde{X}$.

Proposition 5.3.7. *There exists $\varepsilon_1 = \varepsilon_1(C, C_1, \eta, \tau_1, \varepsilon_0, \delta, r_0 - r_1)$ such that for all $v \in \tilde{X}[NI]_0$, $0 < \varepsilon \leq \varepsilon_1$ and $i \in [I]$, denoting $\vartheta_\varepsilon^v = g_\varepsilon \circ d_v$, we have*

$$E_\varepsilon(\vartheta_\varepsilon^v, \mathbf{B}_i^1) \leq 2\sigma(2^{4k+2} + 2)\eta; \quad E_\varepsilon(\vartheta_\varepsilon^v, \mathcal{A}_1) \leq 2\sigma(2^{4k+2} + 2)\eta; \quad E_\varepsilon(\vartheta_\varepsilon^v, M) \leq 2\sigma(L + (2^{4k+2} + 4)\eta).$$

Let us recall from Section 5.3.1 that $f : M \rightarrow [1/3, 2/3]$ is a Morse function with no non-global local maxima or minima; hence the map $t \mapsto f^{-1}(t)$ is continuous in the Hausdorff topology. Following [Gua18, Section 7], let us define the following functions. For $t \in [1/3, 2/3]$,

$$d^t(y) = \begin{cases} -d(y, f^{-1}(t)) & \text{if } f(y) \leq t; \\ d(y, f^{-1}(t)) & \text{if } f(y) \geq t. \end{cases}$$

We define $w_\varepsilon : [0, 1] \rightarrow H^1(M)$ as follows.

$$w_\varepsilon(t) = \begin{cases} g_\varepsilon \circ d^t & \text{if } \frac{1}{3} \leq t \leq \frac{2}{3}; \\ 1 - 3t(1 - w_\varepsilon(1/3)) & \text{if } 0 \leq t \leq \frac{1}{3}; \\ -1 + 3(1 - t)(1 + w_\varepsilon(2/3)) & \text{if } \frac{2}{3} \leq t \leq 1. \end{cases} \quad (5.3.68)$$

Since $f^{-1}(t)$ varies continuously in the Hausdorff topology, $w_\varepsilon : [0, 1] \rightarrow H^1(M)$ is continuous

[Gua18, Proposition 9.2]. The following proposition follows from [Gua18, Section 9.6] and the estimates (5.3.6), (5.3.65).

Proposition 5.3.8. *There exists $\varepsilon_2 = \varepsilon_2(\eta, f, \varepsilon_0, \delta, r_0 - r_1)$ such that for all $t \in [0, 1]$, $0 < \varepsilon \leq \varepsilon_2$ and $i \in [I]$, we have*

$$E_\varepsilon(w_\varepsilon(t), \mathbf{B}_i^1) \leq 6\sigma\eta; \quad E_\varepsilon(w_\varepsilon(t), \mathcal{A}_1) \leq 6\sigma\eta.$$

5.3.5 Construction of sweepout in $H^1(M) \setminus \{0\}$ and proof of Theorem 5.1.2

The next proposition essentially follows from the argument in [GT01, Proof of Lemma 7.5].

Proposition 5.3.9. *$u \mapsto |u|$ is a continuous map from $H^1(M)$ to itself.*

Proof. Let $\kappa : \mathbb{R} \rightarrow \mathbb{R}$ be defined as follows

$$\kappa(t) = \begin{cases} 1 & \text{if } t > 0; \\ 0 & \text{if } t = 0; \\ -1 & \text{if } t < 0. \end{cases}$$

By [GT01, Lemma 7.6] $v \in H^1(M)$ implies $|v| \in H^1(M)$ with $D|v| = \kappa(v)Dv$.

We need to show that if $u_i \rightarrow u$ in $H^1(M)$, then $|u_i| \rightarrow |u|$ in $H^1(M)$. We will use the following fact. Let $\{x_i\}$ be a sequence in a topological space $\mathcal{X}$ and $x \in \mathcal{X}$ such that every subsequence $\{x_{i_j}\}$ of $\{x_i\}$ has a further subsequence $\{x_{i_{j_s}}\}$ which converges to x . Then $\{x_i\}$ converges to x . Therefore, without loss of generality one can assume that u_i converges to u pointwise a.e.

Since $||u_i| - |u|| \leq |u_i - u|$, $|u_i| \rightarrow |u|$ in $L^2(M)$. Using the fact that $Du = 0$ a.e. on $\{u = 0\}$, we compute

$$\begin{aligned} \int_M |\kappa(u_i)Du_i - \kappa(u)Du|^2 &\leq \int_M 2|\kappa(u_i)|^2|Du_i - Du|^2 + 2|\kappa(u_i) - \kappa(u)|^2|Du|^2 \\ &\leq 2 \int_M |Du_i - Du|^2 + 2 \int_{\{u \neq 0\}} |\kappa(u_i) - \kappa(u)|^2|Du|^2. \end{aligned}$$

The first integral converges to 0. If $u(y) \neq 0$ and $u_i(y)$ converges to $u(y)$, then $\kappa(u_i(y))$ converges to

$\kappa(u(y))$ as well. Hence, by the dominated convergence theorem, the second integral also converges to 0. $\square$

For $t \in \mathbb{R}$, let $t^+ = \max\{t, 0\}$; $t^- = \min\{t, 0\}$. We define the maps $\phi, \psi, \theta : H^1(M) \times H^1(M) \times H^1(M) \rightarrow H^1(M)$ as follows.

$$\begin{aligned}\phi(u_0, u_1, w) &= \min\{\max\{u_0, -w\}, \max\{u_1, w\}\}; \\ \psi(u_0, u_1, w) &= \max\{\min\{u_0, w\}, \min\{u_1, -w\}\}; \\ \theta(u_0, u_1, w) &= \phi(u_0, u_1, w)^+ + \psi(u_0, u_1, w)^-.\end{aligned}\tag{5.3.69}$$

Proposition 5.3.10. *The map θ , defined above, has the following properties.*

- (i) θ is a continuous map.
- (ii) $\theta(-u_0, -u_1, w) = -\theta(u_0, u_1, w)$.
- (iii) If $|u_0|, |u_1| \leq 1$, then $\theta(u_0, u_1, \mathbf{1}) = u_0$, $\theta(u_0, u_1, -\mathbf{1}) = u_1$, where $\mathbf{1}$ denotes the constant function 1.
- (iv) If $u_0(y) = u_1(y)$, then $\theta(u_0, u_1, w)(y) = u_0(y) = u_1(y)$.
- (v) For every $\mathcal{H}^{n+1}$ -measurable set $U \subset M$,

$$E_\varepsilon(\theta(u_0, u_1, w), U) \leq E_\varepsilon(u_0, U) + E_\varepsilon(u_1, U) + E_\varepsilon(w, U).$$

- (vi) $\theta(u_0, u_1, w)(y) = 0$ implies $y \in \{u_0 = 0\} \cup \{u_1 = 0\} \cup \{w = 0\}$.

Proof. We have the following identities.

$$\max\{a, b\} = \frac{a + b + |a - b|}{2}; \quad \min\{a, b\} = \frac{a + b - |a - b|}{2}.$$

Hence, the continuity of ϕ , ψ and θ follows from Proposition 5.3.9. We note that $\phi(-u_0, -u_1, w) = -\psi(u_0, u_1, w)$, which implies (ii). (iii) also follows from direct computation. Indeed, if $|u_0|, |u_1| \leq 1$,

then $\phi(u_0, u_1, \mathbf{1}) = u_0 = \psi(u_0, u_1, \mathbf{1})$ and $\phi(u_0, u_1, -\mathbf{1}) = u_1 = \psi(u_0, u_1, -\mathbf{1})$.

To prove (iv), we need the following identity. If $a \geq 0$,

$$\min\{\max\{a, -b\}, \max\{a, b\}\} = a. \quad (5.3.70)$$

Indeed, the identity holds if $\max\{a, -b\} = a = \max\{a, b\}$. It also holds if $\max\{a, -b\} = a$ and $\max\{a, b\} = b > a$. It is not possible for both $\max\{a, -b\}$ and $\max\{a, b\}$ to be different from a as this will imply $-b > a, b > a \implies 0 > a$. Moreover, if $a \geq 0$, either $\min\{a, b\} \geq 0$ (if $b \geq 0$) or $\min\{a, -b\} \geq 0$ (if $b \leq 0$); hence

$$\max\{\min\{a, b\}, \min\{a, -b\}\} \geq 0. \quad (5.3.71)$$

In the case when $u_0(y) = u_1(y) \geq 0$, (5.3.70) and (5.3.71) give (iv). The other case $u_0(y) = u_1(y) \leq 0$ is similar.

For $i = 0, 1$, let $U_i = \{u_i < 0\}$ and $V_i = \{u_i > 0\}$. We also set $G = \{w < 0\}$, $H = \{w > 0\}$, $Z = \{w = 0\}$. Then we have (suppressing the dependence of ϕ, ψ, θ on u_0, u_1, w)

$$\begin{aligned} \phi(y) < 0 &\Leftrightarrow y \in (U_0 \cap H) \dot{\cup} (U_1 \cap G); \\ \phi(y) > 0 &\Leftrightarrow y \in (V_0 \cap H) \dot{\cup} (V_1 \cap G) \dot{\cup} (V_0 \cap V_1 \cap Z); \\ \psi(y) < 0 &\Leftrightarrow y \in (U_0 \cap H) \dot{\cup} (U_1 \cap G) \dot{\cup} (U_0 \cap U_1 \cap Z); \\ \psi(y) > 0 &\Leftrightarrow y \in (V_0 \cap H) \dot{\cup} (V_1 \cap G). \end{aligned} \quad (5.3.72)$$

Further,

$$\theta(y) = \begin{cases} \phi(y) & \text{if } y \in \{\phi \geq 0\} \cap \{\psi \geq 0\}; \\ \psi(y) & \text{if } y \in \{\phi \leq 0\} \cap \{\psi \leq 0\}. \end{cases} \quad (5.3.73)$$

It follows from the equations in (5.3.72) that

$$M = (\{\phi \geq 0\} \cap \{\psi \geq 0\}) \cup (\{\phi \leq 0\} \cap \{\psi \leq 0\}).$$

Hence, using the definition of ϕ and ψ , (5.3.73) implies

$$\theta(y) \in \{u_0(y), u_1(y), w(y), -w(y)\}; \quad \nabla\theta(y) \in \{\nabla u_0(y), \nabla u_1(y), \nabla w(y), -\nabla w(y)\}. \quad (5.3.74)$$

Therefore, the following point-wise estimates hold.

$$|\nabla\theta(y)|^2 \leq |\nabla u_0(y)|^2 + |\nabla u_1(y)|^2 + |\nabla w(y)|^2; \quad W(\theta(y)) \leq W(u_0(y)) + W(u_1(y)) + W(w(y)),$$

which give item (v) of the proposition. (5.3.74) also implies item (vi). $\square$

Let $\alpha \in \tilde{X}[NI]_j$ so that $\pi(\alpha) = \mu \in X[NI]_j$. We introduce the following notation.

$$(\pi|_\alpha)^{-1} \circ \Delta_\mu : \mathcal{J}^j \rightarrow \alpha \text{ is denoted by } \Delta_\alpha^\bullet; \quad D_\mu \circ (\pi|_\alpha) : \alpha \rightarrow \mathcal{J}^j \text{ is denoted by } D_\alpha^\bullet.$$

Proposition 5.3.11. *Let $\varepsilon^* = \min\{\varepsilon_1, \varepsilon_2, (\delta/4)^2, 4^{-1}(r_1 - r^*)^2\}$. For $0 < \varepsilon \leq \varepsilon^*$ there exists a continuous, $\mathbb{Z}_2$ -equivariant map $\zeta_\varepsilon : \tilde{X} \rightarrow H^1(M) \setminus \{0\}$ with the property*

$$(\mathcal{P})_0 \quad \zeta_\varepsilon(v) = \vartheta_\varepsilon^v \text{ for all } v \in \tilde{X}[NI]_0,$$

where ϑ_ε^v is as in Proposition 5.3.7. Moreover, if $\alpha \in \tilde{X}[NI]_j$, $\zeta_\varepsilon|_\alpha$ has the following properties.

$$(\mathcal{P}0)_{j,\alpha} \quad \text{Let } \hat{\zeta}_\varepsilon^\alpha = \zeta_\varepsilon \circ \Delta_\alpha^\bullet. \text{ Then, for all } z = (z', z_j) \in \mathcal{J}^j \text{ (with } z' \in \mathcal{J}^{j-1}),$$

$$\hat{\zeta}_\varepsilon^\alpha(z', z_j) = \theta \left(\hat{\zeta}_\varepsilon^\alpha(z', 0), \hat{\zeta}_\varepsilon^\alpha(z', 1), w_\varepsilon(z_j) \right),$$

where w_ε is as defined in (5.3.68).

$$(\mathcal{P}1)_{j,\alpha} \quad \text{We define } \mathcal{I}_\alpha = \{\ell(e) : e \prec \alpha, \dim(e) = 1\}, \text{ where } \ell(e) \text{ is as in Proposition 5.3.6 (P5). For all } 0 < \varepsilon \leq \varepsilon^*, t \in \alpha \text{ and } v \prec \alpha \text{ with } \dim(v) = 0,$$

$$y \notin \mathcal{A}_1 \bigcup \left(\bigcup_{i \in \mathcal{I}_\alpha} \mathbf{B}_i^1 \right) \implies \zeta_\varepsilon(t)(y) = \zeta_\varepsilon(v)(y).$$

$(\mathcal{P}2)_{j,\alpha}$ For all $0 < \varepsilon \leq \varepsilon^*$, $i \in [I]$ and $t \in \alpha$,

$$\frac{1}{2\sigma} E_\varepsilon (\zeta_\varepsilon(t), \mathbf{B}_i^1) \leq 3^j (2^{4k+2} + 2)\eta; \quad \frac{1}{2\sigma} E_\varepsilon (\zeta_\varepsilon(t), \mathcal{A}_1) \leq 3^j (2^{4k+2} + 2)\eta.$$

As a consequence, for all $0 < \varepsilon \leq \varepsilon^*$ and $t \in \alpha$,

$$\frac{1}{2\sigma} E_\varepsilon (\zeta_\varepsilon(t), M) \leq L + (2^{4k+2} + 4)\eta + (j+1)3^j (2^{4k+2} + 2)\eta. \quad (5.3.75)$$

Proof. The map ζ_ε will be defined inductively on the cells of $\tilde{X}[NI]$. The equivariance of ζ_ε and the fact that $\zeta_\varepsilon(x)$ is not identically 0 for all $x \in \tilde{X}$ will be proved at the end (after proving the other properties of ζ_ε). If $v \in \tilde{X}[NI]_0$, we define $\zeta_\varepsilon(v) = \vartheta_\varepsilon^v$, which is precisely the property $(\mathcal{P})_0$. The properties $(\mathcal{P}0)_{0,v}$ and $(\mathcal{P}1)_{0,v}$ are vacuous; $(\mathcal{P}2)_{0,v}$ is satisfied because of Proposition 5.3.7. Let us assume that for some $p \geq 1$, ζ_ε has been defined on

$$\bigcup_{s \leq p-1} \tilde{X}[NI]_s$$

and if $\varrho \in \tilde{X}[NI]_{p-1}$, $\zeta_\varepsilon|_{\varrho}$ has the properties $(\mathcal{P}0)_{(p-1),\varrho} - (\mathcal{P}2)_{(p-1),\varrho}$.

Let $\lambda \in \tilde{X}[NI]_p$. By our assumption, ζ_ε is already defined on $\partial\lambda$. For $z \in \partial\mathcal{J}^p$, we define

$$\hat{\zeta}_\varepsilon^\lambda(z) = (\zeta_\varepsilon \circ \Delta_\lambda^\bullet)(z).$$

For arbitrary $z = (z', z_p) \in \mathcal{J}^p$ with $z' \in \mathcal{J}^{p-1}$, we define

$$\hat{\zeta}_\varepsilon^\lambda(z', z_p) = \theta \left(\hat{\zeta}_\varepsilon^\lambda(z', 0), \hat{\zeta}_\varepsilon^\lambda(z', 1), w_\varepsilon(z_p) \right), \quad (5.3.76)$$

where w_ε is as defined in (5.3.68). By Proposition 5.3.10 (iii), this equation indeed holds for $z_p = 0, 1$. Further, (5.3.76) is also valid if $z \in \partial\mathcal{J}^p$ because of our induction hypothesis that ζ_ε restricted $\partial\lambda$ satisfies $(\mathcal{P}0)$ and the compatibility relation (5.3.10). For $t \in \lambda$, $\zeta_\varepsilon(t)$ is defined by $\zeta_\varepsilon(t) = \left(\hat{\zeta}_\varepsilon^\lambda \circ D_\lambda^\bullet \right)(t)$. ζ_ε is continuous on λ as $\zeta_\varepsilon|_{\partial\lambda}$ is continuous and the maps θ , w_ε are

continuous.

It remains to verify that $\zeta_\varepsilon|_\lambda$ satisfies the properties $(\mathcal{P}0)_{p,\lambda} - (\mathcal{P}2)_{p,\lambda}$. $(\mathcal{P}0)_{p,\lambda}$ holds by the definition (5.3.76).

To prove $(\mathcal{P}1)_{p,\lambda}$, we first show that if $v, v' \in \tilde{X}[NI]_0$ and $e \in \tilde{X}[NI]_1$ such that $v, v' \prec e$, then

$$y \notin \mathcal{A}_1 \cup \mathbf{B}_{\ell(e)}^1 \implies \zeta_\varepsilon(v)(y) = \zeta_\varepsilon(v')(y). \quad (5.3.77)$$

We set $\mathcal{B}^1 = \mathbf{B}_{\ell(e)}^1$ and $\mathcal{B} = \mathbf{B}_{\ell(e)}$. Either $y \in \tilde{\Psi}(v)$ or $y \in \tilde{\Psi}(T(v))$. Let us assume that $y \in \tilde{\Psi}(T(v))$ (the other case is analogous); hence (using Proposition 5.3.6 (P5))

$$y \in \tilde{\Psi}(T(v)) \cap (\mathcal{A}_1 \cup \mathcal{B}^1)^c = \tilde{\Psi}(T(v')) \cap (\mathcal{A}_1 \cup \mathcal{B}^1)^c.$$

So $d_v(y) = d(y, \Psi(v))$ and $d_{v'}(y) = d(y, \Psi(v'))$. If both $d_v(y)$ and $d_{v'}(y)$ are $\geq 2\sqrt{\varepsilon}$, then $\zeta_\varepsilon(v)(y) = 1 = \zeta_\varepsilon(v')(y)$. Otherwise, let us say $d_v(y) = d(y, \Psi(v)) < 2\sqrt{\varepsilon}$. We will show that $d_v(y) = d_{v'}(y)$. Let $y' \in \Psi(v)$ such that $d(y, y') = d(y, \Psi(v))$. Therefore,

$$d(y, y') = d(y, \Psi(v)) < 2\sqrt{\varepsilon} \leq \min\{\delta/2, r_1 - r^*\}. \quad (5.3.78)$$

Hence, $y \notin \mathcal{A}_1 \cup \mathcal{B}^1$ implies $y' \notin \mathcal{A} \cup \mathcal{B}$. Thus, (again using Proposition 5.3.6 (P5))

$$y' \in \Psi(v) \cap (\mathcal{A} \cup \mathcal{B})^c = \Psi(v') \cap (\mathcal{A} \cup \mathcal{B})^c.$$

This gives

$$d(y, \Psi(v)) = d(y, \Psi(v) \cap (\mathcal{A} \cup \mathcal{B})^c) \quad (5.3.79)$$

and $d(y, \Psi(v')) \leq d(y, y') < 2\sqrt{\varepsilon}$, which also implies (by the above reasoning)

$$d(y, \Psi(v')) = d(y, \Psi(v') \cap (\mathcal{A} \cup \mathcal{B})^c). \quad (5.3.80)$$

Since $\Psi(v) \cap (\mathcal{A} \cup \mathcal{B})^c = \Psi(v') \cap (\mathcal{A} \cup \mathcal{B})^c$, from (6.1.1) and (6.5.103), we get $d_v(y) = d_{v'}(y)$.

Thus we have proved (5.3.77), which, together with (5.3.76), induction hypothesis $(\mathcal{P}1)_{(p-1),\partial\lambda}$ and Proposition 5.3.10 (iv), gives $(\mathcal{P}1)_{p,\lambda}$.

Finally we prove $(\mathcal{P}2)_{p,\lambda}$. Using (5.3.76) together with Proposition 5.3.10 (v), the induction hypothesis $(\mathcal{P}2)_{(p-1),\partial\lambda}$ and Proposition 5.3.8, one obtains the following estimate ($\mathcal{U}$ stands for $\mathbf{B}_i^1$ for some $i \in [I]$ or $\mathcal{A}_1$.)

$$\begin{aligned} \frac{1}{2\sigma} E_\varepsilon \left(\hat{\zeta}_\varepsilon^\lambda(z', z_p), \mathcal{U} \right) &\leq \frac{1}{2\sigma} E_\varepsilon \left(\hat{\zeta}_\varepsilon^\lambda(z', 0), \mathcal{U} \right) + \frac{1}{2\sigma} E_\varepsilon \left(\hat{\zeta}_\varepsilon^\lambda(z', 1), \mathcal{U} \right) + \frac{1}{2\sigma} E_\varepsilon (w_\varepsilon(z_p), \mathcal{U}) \\ &\leq 2 \cdot 3^{p-1} (2^{4k+2} + 2) \eta + 3\eta \\ &\leq 3^p (2^{4k+2} + 2) \eta. \end{aligned}$$

This estimate along with $(\mathcal{P}1)_{p,\lambda}$, $(\mathcal{P})_0$ and Proposition 5.3.7 gives for all $0 < \varepsilon \leq \varepsilon^*$, $t \in \alpha$ and $v \prec \alpha$ with $\dim(v) = 0$,

$$\begin{aligned} \frac{1}{2\sigma} E_\varepsilon (\zeta_\varepsilon(t), M) &\leq \frac{1}{2\sigma} E_\varepsilon (\zeta_\varepsilon(v), M) + \sum_{i \in \mathcal{I}_\lambda} \frac{1}{2\sigma} E_\varepsilon (\zeta_\varepsilon(t), \mathbf{B}_i^1) + \frac{1}{2\sigma} E_\varepsilon (\zeta_\varepsilon(t), \mathcal{A}_1) \\ &\leq L + (2^{4k+2} + 4) \eta + (p+1) 3^p (2^{4k+2} + 2) \eta. \end{aligned}$$

In the last line one needs to use the fact that $|\mathcal{I}_\lambda| \leq p$, which follows from the characterization of $\ell(e)$ in Proposition 5.3.6 (P5).

Let us show that the map ζ_ε , constructed above, is $\mathbb{Z}_2$ -equivariant. From the definition of d_v ((5.3.66)), it follows that $d_{T(v)} = -d_v$ for all $v \in \tilde{X}[NI]_0$. Hence, by $(\mathcal{P})_0$, $\zeta_\varepsilon(T(v)) = -\zeta_\varepsilon(v)$ for all $v \in \tilde{X}[NI]_0$. Now we can use induction along with $(\mathcal{P}0)$ and Proposition 5.3.10 (ii) to conclude that $\zeta_\varepsilon(T(x)) = -\zeta_\varepsilon(x)$ for all $x \in \tilde{X}$.

Lastly, we prove that for all $x \in \tilde{X}$, $\mathcal{H}^{n+1}(\{\zeta_\varepsilon(x) = 0\}) = 0$. This will in particular imply that for all $x \in \tilde{X}$, $\zeta_\varepsilon(x)$ is not identically equal to 0. If $v \in \tilde{X}[NI]_0$, by (5.3.66) and $(\mathcal{P})_0$, $\{\zeta_\varepsilon(v) = 0\} = \Psi(v)$ and $\mathcal{H}^{n+1}(\Psi(v)) = 0$ by Proposition 5.3.6 (P4). Hence, $\mathcal{H}^{n+1}(\{\zeta_\varepsilon(v) = 0\}) = 0$ for all $v \in \tilde{X}[NI]_0$. Now, as before we can use induction along with $(\mathcal{P}0)$ and Proposition 5.3.10 (vi) to conclude that $\mathcal{H}^{n+1}(\{\zeta_\varepsilon(x) = 0\}) = 0$ for all $x \in \tilde{X}$. $\square$

Proof of Theorem 5.1.2. By Proposition 5.3.11, for every $\eta > 0$ there exists $\varepsilon^* > 0$ such that for all $0 < \varepsilon \leq \varepsilon^*$ there exists $\zeta_\varepsilon \in \tilde{\Pi}$ such that

$$\frac{1}{2\sigma} \sup_{x \in \tilde{X}} E_\varepsilon(\zeta_\varepsilon(x)) \leq L + \alpha(k)\eta \quad (\text{by (5.3.75)}),$$

where $\alpha(k)$ is a constant which depends only on $k = \dim(\tilde{X})$. Hence,

$$\frac{1}{2\sigma} \mathbf{L}_\varepsilon(\tilde{\Pi}) \leq L + \alpha(k)\eta \quad \forall 0 < \varepsilon \leq \varepsilon^* \implies \frac{1}{2\sigma} \limsup_{\varepsilon \rightarrow 0^+} \mathbf{L}_\varepsilon(\tilde{\Pi}) \leq L + \alpha(k)\eta.$$

Since $\eta > 0$ is arbitrary, this implies (5.1.4).

The following facts follow from [GG18, Section 6]. $\text{Ind}_{\mathbb{Z}_2}(\tilde{X}) \geq p + 1$ if and only if each map $\Phi \in \Pi$ is a p -sweepout. Moreover,

$$c_\varepsilon(p) = \inf_{\tilde{\Pi}} \mathbf{L}_\varepsilon(\tilde{\Pi}),$$

where the infimum is taken over all $\tilde{\Pi}$ such that $\text{Ind}_{\mathbb{Z}_2}(\tilde{X}) \geq p + 1$. Similarly,

$$\omega_p = \inf_{\Pi} \mathbf{L}_{AP}(\Pi),$$

where the infimum is taken over all Π such that each map in the homotopy class Π is a p -sweepout.

We fix $p \in \mathbb{N}$. For every $j \in \mathbb{N}$, there exists double cover $\tilde{X}_j \rightarrow X_j$ and the corresponding homotopy classes $\Pi_j, \tilde{\Pi}_j$ (as discussed in Section 1) such that $\text{Ind}_{\mathbb{Z}_2}(\tilde{X}_j) \geq p + 1$ and

$$\mathbf{L}_{AP}(\Pi_j) < \omega_p + \frac{1}{j}.$$

Therefore, using (5.1.4),

$$\frac{1}{2\sigma} \limsup_{\varepsilon \rightarrow 0^+} c_\varepsilon(p) \leq \frac{1}{2\sigma} \limsup_{\varepsilon \rightarrow 0^+} \mathbf{L}_\varepsilon(\tilde{\Pi}_j) \leq \mathbf{L}_{AP}(\Pi_j) < \omega_p + \frac{1}{j}.$$

Since this holds for all $j \in \mathbb{N}$, we obtain (5.1.5). □

5.4 Proof of Theorem 5.1.3

In this section, we will discuss how Theorem 5.1.3 can be proved using the results contained in the papers [HT00, Gua18, GG18]. Let us recall the function F , defined in (5.2.4). For $\beta \in (0, 1)$, we set $F(1 - \beta) = \sigma_\beta/2$.

Proposition 5.4.1. *Let $\{u_i : M \rightarrow (-1, 1)\}_{i=1}^\infty$ be a sequence such that the following conditions are satisfied.*

- (i) $AC_{\varepsilon_i}(u_i) = 0$ with $\varepsilon_i \rightarrow 0$ as $i \rightarrow \infty$.
- (ii) There exists E_0 such that $E_{\varepsilon_i}(u_i) \leq E_0$ for all $i \in \mathbb{N}$.
- (iii) V is a stationary, integral varifold such that $V_i = V[u_i] \rightarrow V$ and $\text{spt}(V)$ has optimal regularity.

Then for all $s > 0$ there exists $b_0 \in (0, 1/2]$ such that the following holds. Denoting $w_i = F \circ u_i$, for all $b \in (0, b_0]$ there exists $i_0 \in \mathbb{N}$, depending on s and b , such that for all $i \geq i_0$ there exists $t_i \in [-\sigma_b/2, \sigma_b/2]$ for which

- $\{w_i > t_i\}, \{w_i < t_i\} \in \mathcal{C}(M)$ and $[\{w_i > t_i\}] + [\{w_i < t_i\}] = [M]$;
- $\{w_i = t_i\}$ is a smooth hypersurface, $\partial[\{w_i > t_i\}] = [\{w_i = t_i\}]$ and $\mathbf{F}(V, |\{w_i = t_i\}|) \leq s$.

Proof. The proof is divided into three parts.

Part 1. For simplicity, let us assume that $V = N|\Sigma|$, where $N \in \mathbb{N}$ and Σ is a closed, connected minimal hypersurface with optimal regularity. The general case (i.e. when $\text{spt}(V)$ is not connected) will be similar. Let

$$s_0 = \frac{2\sigma}{N + 3 + \omega(n)^{-1} + 2\sigma N}, \quad (5.4.1)$$

where $\omega(n)$ is the n -volume of the n -dimensional unit Euclidean ball in $\mathbb{R}^n$ and we fix $0 < s < s_0$. (The assumption $s < s_0$ will be useful later.) By [HT00, Proposition 5.1] there exists $b_0 \in (0, 1/2]$ such that

$$\limsup_{i \rightarrow \infty} \int_{\{|u_i| \geq 1-b_0\}} \frac{W(u_i)}{\varepsilon_i} \leq s.$$

Let us fix $b \in (0, b_0]$. Using the above equation and [HT00, Proposition 4.3],

$$\limsup_{i \rightarrow \infty} \int_{\{|u_i| \geq 1-b\}} |\nabla w_i| \leq s. \quad (5.4.2)$$

We choose $\mathcal{U}_1 \subset \mathcal{U}_2 \subset \mathcal{U}_3 \subset \text{reg}(\Sigma)$ such that

$$N\mathcal{H}^n(\Sigma \setminus \mathcal{U}_1) < s. \quad (5.4.3)$$

For $\mathcal{U} \subset \text{reg}(\Sigma)$, let

$$\mathfrak{N}_r \mathcal{U} = \{v \in T_p M : p \in \mathcal{U}, v \in T_p^\perp \Sigma \text{ with } |v| < r\}.$$

There exists $\rho > 0$ such that $\exp : \mathfrak{N}_{3\rho} \mathcal{U}_3 \rightarrow M$ is a diffeomorphism onto its image. Let

$$U_3 = \exp(\mathfrak{N}_{3\rho} \mathcal{U}_3), \quad U_2 = \exp(\mathfrak{N}_{2\rho} \mathcal{U}_2), \quad U_1 = \exp(\mathfrak{N}_\rho \mathcal{U}_1).$$

One can assume that (choosing a smaller ρ if necessary) $d_\Sigma = d(-, \Sigma)$ is smooth on $U_3 \setminus \Sigma$ with $d_\Sigma(\exp_p(v)) = |v|$, the nearest point projection map $P : U_3 \rightarrow \mathcal{U}_3$, $P(\exp_p(v)) = p$, is smooth and the Jacobian factor

$$JP(y, S) \leq 2 \quad \forall (y, S) \in G_n \overline{U}_2. \quad (5.4.4)$$

We note that

$$P(U_j) = \mathcal{U}_j = \Sigma \cap U_j \quad (j = 1, 2, 3) \quad \text{and} \quad U_1 \subset \subset U_2 \subset \subset U_3.$$

Part 2. We are now going to discuss some results from [HT00]. We note that in [HT00], the ambient dimension is assumed to be n . Let g denote the the ambient Riemannian metric on M^{n+1} . For $p \in M$, we define

$$f_{p,r} : T_p M \rightarrow M \quad \text{by} \quad f_{p,r}(v) = \exp_p(rv) \quad \text{and} \quad g_{p,r} = r^{-2} f_{p,r}^* g.$$

As explained in [Gua18, Section 10], all the results of [HT00] continue to hold on a closed Riemannian manifold. In particular, if $p \in M$ and $r > 0$ is sufficiently small, all the estimates of [HT00], which are proved under the hypothesis that $AC_\varepsilon(u^\varepsilon) = 0$ with respect to the Euclidean metric, continue to hold if one instead assumes that $AC_\varepsilon(u^\varepsilon) = 0$ with respect to $g_{p,r}$. For the sake of completeness, we restate below Propositions 5.5 and 5.6 of [HT00], adapted to the Riemannian manifold setup. Let us introduce the following notations.

- For $p \in M$, $B_p(0, r) = \{v \in T_p M : g(v, v) < r^2\}$.
- Given a hyperplane $\Gamma \subset T_p M$, which contains 0, $P_\Gamma : T_p M \rightarrow \Gamma$ is the orthogonal projection onto Γ with respect to g_p .
- $e_\varepsilon(u) = \varepsilon \frac{|\nabla u|^2}{2} + \frac{W(u)}{\varepsilon}$, $\xi_\varepsilon(u) = \varepsilon \frac{|\nabla u|^2}{2} - \frac{W(u)}{\varepsilon}$, where $|\cdot|$ is with respect to the Riemannian metric as specified in the following propositions.

Proposition 5.4.2 ([HT00, Propositions 5.5]). *There exists $0 < r_0 < \text{inj}(M)$ such that the following holds. Given $E > 0$, $\hat{s} \in (0, 1)$ and $N_0 \in \mathbb{N}$, there exists $\eta > 0$ satisfying the following property. Let $B_p(0, 2)$ be equipped with the Riemannian metric $g_{p,\hat{r}}$ for some $p \in M$ and $\hat{r} \in (0, r_0]$. (Subsequently, all the metric dependent objects e.g. ∇ , AC_ε , the distance function, the geodesic balls etc. are assumed to be with respect to $g_{p,\hat{r}}$.) Suppose*

(i) $\Gamma \subset T_p M$ is a hyperplane containing 0, $Y \subset B_p(0, 1)$ with the cardinality $|Y| \leq N_0 + 1$, $P_\Gamma(x) = 0$ for all $x \in Y$, $a > 0$, $d(x, x') > 3a$ for all $x, x' \in Y$ and $\text{diameter } Y \leq \eta$.

(ii) $\varepsilon > 0$ and $u : B_p(0, 2) \rightarrow (-1, 1)$ satisfies $AC_\varepsilon(u) = 0$ with $\sup_{B_p(0, 2)} \xi_\varepsilon(u) \leq \eta$.

(iii)

$$\tau_\varepsilon(u) = \varepsilon \left(|\nabla u|^2 - \langle \nabla u, \nabla d \rangle^2 \right),$$

where $d : B_p(0, 2) \rightarrow \mathbb{R}$ denotes the signed distance from $\Gamma \cap B_p(0, 2)$. For all $x \in Y$ and $a \leq r \leq 1$,

$$\int_{B(x, r)} |\xi_\varepsilon(u)| + \tau_\varepsilon(u) \leq \eta r^n, \quad \int_{B(x, r)} \varepsilon |\nabla u|^2 \leq E r^n.$$

Then

$$\sum_{x \in Y} a^{-n} \int_{B(x,a)} e_\varepsilon(u) \leq \hat{s} + (1 + \hat{s}) \int_{\{x': d(x', Y) < 1\}} e_\varepsilon(u).$$

The next Proposition 5.4.3 follows from the proof of Propositions 5.6 in [HT00]. Before we state it, we need to introduce some notation. Let us recall that $\mathbf{g}$ is the canonical 1-D solution to the Allen-Cahn equation (see (5.3.60)). For $\hat{s}, \hat{b} \in (0, 1)$, $L = L(\hat{s}, \hat{b}) \in (1, \infty)$ is defined as follows. Whenever $t \in \mathbb{R}$ satisfies $|\mathbf{g}(t)| \leq 1 - \hat{b}$, there holds (see [HT00, Equation (5.4)])

$$\left| \frac{1}{\omega(n)L^n} \int_{B_p(0,L)} \left[\frac{1}{2} \mathbf{g}'(x_{n+1} + t)^2 + W(\mathbf{g}(x_{n+1} + t)) \right] d\mathcal{L}^{n+1}(x) - 2\sigma \right| \leq \frac{\hat{s}}{2}, \quad (5.4.5)$$

where $\mathcal{L}^{n+1}$ denotes the Lebesgue measure on $\mathbb{R}^{n+1}$ and $\omega(n)$ is the n -volume of the n -dimensional unit Euclidean ball in $\mathbb{R}^n$.

Proposition 5.4.3 ([HT00, Proposition 5.6]). *Given $\hat{s}, \hat{b} \in (0, 1)$, there exist $r_0 \in (0, \text{inj}(M))$ and $\eta \in (0, 1)$ satisfying the following property. Let $L = L(\hat{s}, \hat{b})$, $B_p(0, 4L)$ be equipped with the Riemannian metric $g_{p, \hat{r}}$ for some $p \in M$ and $\hat{r} \in (0, r_0]$. (Subsequently, all the metric dependent objects are assumed to be with respect to $g_{p, \hat{r}}$.) Suppose*

(i) $u : B_p(0, 4L) \rightarrow (-1, 1)$ satisfies $AC_1(u) = -\Delta u + W'(u) = 0$ with $\xi_1(u) = \frac{1}{2} |\nabla u|^2 - W(u) \leq \eta$ on $B_p(0, 4L)$, $|u(0)| \leq 1 - \hat{b}$.

(ii) $\Gamma \subset T_p M$ is a hyperplane containing 0; $0 < \hat{\eta} \leq \eta$;

$$\tau_1(u) = |\nabla u|^2 - \langle \nabla u, \nabla d \rangle^2,$$

where $d : B_p(0, 4L) \rightarrow \mathbb{R}$ denotes the signed distance from $\Gamma \cap B_p(0, 4L)$;

$$\int_{B_p(0, 4L)} |\xi_1(u)| + \tau_1(u) \leq \hat{\eta} (4L)^n.$$

Then, there holds (with either + or -)

$$\|d(x) + \mathfrak{g}^{-1}(u(0)) \pm \mathfrak{g}^{-1}(u(x))\|_{C^1(B_p(0,3L))} \leq c(\hat{s}, \hat{b}) \hat{\eta}^{\frac{1}{n+2}},$$

where $c(\hat{s}, \hat{b}) > 0$ is a constant depending on $\hat{s}, \hat{b}$.

Proposition 5.4.4 ([HT00, Proposition 5.6]). *Given $\hat{s}, \hat{b} \in (0, 1)$, there exist $r_0 \in (0, \text{inj}(M))$ and $\eta \in (0, 1)$ satisfying the following property. Let $L = L(\hat{s}, \hat{b})$, $B_p(0, 4L)$ be equipped with the Riemannian metric $g_{p, \hat{r}}$ for some $p \in M$ and $\hat{r} \in (0, r_0]$. (Subsequently, all the metric dependent objects are assumed to be with respect to $g_{p, \hat{r}}$.) Suppose*

(i) $u : B_p(0, 4L) \rightarrow (-1, 1)$ satisfies $AC_1(u) = 0$ with $\xi_1(u) \leq \eta$ on $B_p(0, 4L)$, $|u(0)| \leq 1 - \hat{b}$.

(ii) $\Gamma \subset T_p M$ is a hyperplane containing 0;

$$\tau_1(u) = |\nabla u|^2 - \langle \nabla u, \nabla d \rangle^2,$$

where $d : B_p(0, 4L) \rightarrow \mathbb{R}$ denotes the signed distance from $\Gamma \cap B_p(0, 4L)$;

$$\int_{B_p(0, 4L)} |\xi_1(u)| + \tau_1(u) \leq \eta(4L)^n.$$

Then, there holds

$$\{x \in B_p(0, 3L) : P_\Gamma(x) = 0, u(x) = u(0)\} = \{0\}$$

and

$$\left| \frac{1}{\omega(n)L^n} \int_{B_p(0, L)} e_1(u) - 2\sigma \right| \leq \hat{s}.$$

Part 3. Coming back to the proof of Proposition 5.4.1, on U_3 , following [HT00, Section 5], we define

$$v_i = \begin{cases} \frac{|\langle \nabla u_i, \mathbf{n} \rangle|}{|\nabla u_i|} & \text{on } \{|\nabla u_i| \neq 0\}, \\ 0 & \text{on } \{|\nabla u_i| = 0\}, \end{cases}$$

where $\mathbf{n}(y)$ is a unit normal at y to the hypersurface $\{d_\Sigma = d_\Sigma(y)\}$. Let

$$\tau_i = (1 - v_i^2)\varepsilon_i|\nabla u_i|^2.$$

We choose a cut-off function $0 \leq \chi_1 \leq 1$ such that $\text{spt}(\chi_1) \subset U_3$ and $\chi_1 \equiv 1$ on $\overline{U}_2$. Since, $V_i \rightarrow V$ is the varifold sense, using [HT00, Proposition 4.3] we obtain ([HT00, (5.7)])

$$\lim_{i \rightarrow \infty} \int_{U_3} \chi_1 \tau_i = \lim_{i \rightarrow \infty} 2 \int_{U_3} \chi_1 (1 - v_i^2) |\nabla w_i| = 0 \implies \lim_{i \rightarrow \infty} \int_{\overline{U}_2} \tau_i = 0. \quad (5.4.6)$$

Let us use the notation

$$\xi_i = \frac{\varepsilon_i |\nabla u_i|^2}{2} - \frac{W(u_i)}{\varepsilon_i}.$$

By [HT00, Proposition 4.3] and (5.4.6),

$$\lim_{i \rightarrow \infty} \int_{\overline{U}_2} |\xi_i| + \tau_i = 0.$$

Let us choose a sequence $\eta_i \rightarrow 0$ such that

$$\lim_{i \rightarrow \infty} \eta_i^{-1} \int_{\overline{U}_2} |\xi_i| + \tau_i = 0. \quad (5.4.7)$$

For our fixed choice of s and b , we choose $L = L(s, b)$ via (6.5.86). Let

$$G_i = \left\{ y \in \overline{U}_1 : \int_{B(y, r)} |\xi_i| + \tau_i \leq \eta_i r^n \text{ if } 4\varepsilon_i L \leq r \leq \tilde{\rho} \right\},$$

where $\tilde{\rho} \leq d(\overline{U}_1, \partial U_2)$ is such that the monotonicity formula for the scaled energy holds on balls (in M) with radii $\leq \tilde{\rho}$ (see [Gua18, Section 10.2]). For our later purpose, we choose a compact set $H_i \subset G_i$ as follows. By the Besicovitch covering theorem, there exist $\{\mathcal{B}_j\}_{j=1}^J$ such that each $\mathcal{B}_j$

is a collection of at-most countably many mutually disjoint open balls and

$$\overline{U}_1 \setminus G_i \subset \bigcup_{j=1}^J \bigcup_{B \in \mathcal{B}_j} B.$$

Then

$$H_i = \left(\bigcup_{j=1}^J \bigcup_{B \in \mathcal{B}_j} B \right)^c \cap \overline{U}_1.$$

Using the monotonicity formula for the scaled energy, proved in [HT00, Proposition 3.4] and [Gua18, Section 10.2], one can show the following ([HT00, (5.9)]).

$$\|V_i\|(\overline{U}_1 \setminus H_i) + \mathcal{H}^n(P(\overline{U}_1 \setminus H_i)) \leq c(s, E_0, g) \eta_i^{-1} \int_{\overline{U}_2} |\xi_i| + \tau_i, \quad (5.4.8)$$

which converges to 0 as $i \rightarrow \infty$ by our choice of the sequence $\{\eta_i\}$.

Let us define

$$S_i^t = \{w_i = t\}.$$

For $t \in [-\sigma_b/2, \sigma_b/2]$ and $\varphi \in C^{0,1}(G_n M^{n+1})$ with $|\varphi| \leq 1$ and $\text{Lip}(\varphi) \leq 1$, we obtain the following estimates.

$$\left| |S_i^t|(\varphi) - N |\Sigma|(\varphi) \right| \leq \left| |S_i^t \cap \overline{U}_1|(\varphi) - N |\Sigma \cap \overline{U}_1|(\varphi) \right| + \|S_i^t\|(U_1^c) + N \|\Sigma\|(U_1^c). \quad (5.4.9)$$

$$\begin{aligned} \left| |S_i^t \cap \overline{U}_1|(\varphi) - N |\Sigma \cap \overline{U}_1|(\varphi) \right| &\leq \left| |S_i^t \cap H_i|(\varphi) - N |P(H_i)|(\varphi) \right| \\ &\quad + \|S_i^t\|(\overline{U}_1 \setminus H_i) + N \mathcal{H}^n(P(\overline{U}_1 \setminus H_i)). \end{aligned} \quad (5.4.10)$$

$$\begin{aligned} \left| |S_i^t \cap H_i|(\varphi) - N |P(H_i)|(\varphi) \right| &\leq \left| |S_i^t \cap H_i|(\varphi) - P_\# |S_i^t \cap H_i|(\varphi) \right| \\ &\quad + \left| P_\# |S_i^t \cap H_i|(\varphi) - N |P(H_i)|(\varphi) \right|. \end{aligned} \quad (5.4.11)$$

$$\begin{aligned}
& \left| |S_i^t \cap H_i| (\varphi) - P_{\#} |S_i^t \cap H_i| (\varphi) \right| \\
& \leq \int_{S_i^t \cap H_i} \left| \varphi(y, T_y S_i^t) - \varphi(P(y), DP|_y(T_y S_i^t)) JP(y, T_y S_i^t) \right| d\mathcal{H}^n(y). \quad (5.4.12)
\end{aligned}$$

Claim 5.4.5. *As $i \rightarrow \infty$,*

$$d_{G_n M}((y, T_y S_i^t), (P(y), DP|_y(T_y S_i^t))) \rightarrow 0 \quad \text{and} \quad JP(y, T_y S_i^t) \rightarrow 1,$$

uniformly for $t \in [-\sigma_b/2, \sigma_b/2]$ and $y \in S_i^t \cap H_i$. (Here $d_{G_n M}$ denotes the distance in $G_n M$.)

Proof. To prove this claim, we use some arguments from [HT00, Proof of Theorem 1]. For a fixed $t \in [-\sigma_b/2, \sigma_b/2]$ and $y \in S_i^t \cap H_i$, $|u_i(y)| \leq 1 - b$ and

$$\int_{B(y, 4\varepsilon_i L)} |\xi_i| + \tau_i \leq \eta_i(4\varepsilon_i L)^n.$$

Using the notations from Part 2, let $\mathbf{u}_i = u_i \circ f_{y, \varepsilon_i}$. Then $\mathbf{u}_i : (B_y(0, 4L), g_{y, \varepsilon_i}) \rightarrow (-1, 1)$ satisfies $AC_1(\mathbf{u}_i) = 0$,

$$\sup_{B_y(0, 4L)} \xi_1(\mathbf{u}_i) \leq O(\varepsilon_i) \quad (\text{by [HT00, Proposition 3.3]})$$

and $|\mathbf{u}_i(0)| \leq 1 - b$. Thus, if i is sufficiently large, the hypothesis (i) of Proposition 5.4.3 is satisfied for $\hat{s} = s$, $\hat{b} = b$, $p = y$, and $u = \mathbf{u}_i$. Let

$$\Gamma_y = \{d_{\Sigma} = d_{\Sigma}(y)\}.$$

Since $f_{y, \varepsilon}^{-1}(\Gamma_y)$ smoothly converges to $T_y \Gamma_y$ as $\varepsilon \rightarrow 0$, for sufficiently large i , the hypothesis (ii) of Proposition 5.4.3 is also satisfied for $\Gamma = T_y \Gamma_y$, $u = \mathbf{u}_i$ and $\hat{\eta} = \tilde{\eta}_i$ for some sequence $\{\tilde{\eta}_i\} \rightarrow 0$. Therefore, Proposition 5.4.3 implies that inside the ball $B(y, 3L\varepsilon_i)$, $S_i^t = \{u_i = u_i(y)\}$ is a normal graph of a function h_i on $B(y, 3L\varepsilon_i) \cap \Gamma_y$ such that

$$|\nabla h_i| \leq c(s, b) \tilde{\eta}_i^{\frac{1}{n+2}}.$$

This, along with the fact that $\{|u_i| \leq 1 - b\}$ converges to Σ in the Hausdorff sense, implies our claim. $\square$

Hence (5.4.12) and Claim 5.4.5 imply that there exists a sequence of positive real numbers $\{\theta_i\}_{i=1}^\infty$ (which does not depend on t) such that

$$\lim_{i \rightarrow \infty} \theta_i = 0 \quad \text{and} \quad \sup_{\varphi} \left| |S_i^t \cap H_i|(\varphi) - P_{\#} |S_i^t \cap H_i|(\varphi) \right| \leq \theta_i \mathcal{H}^n(S_i^t) \quad (5.4.13)$$

for all $t \in [-\sigma_b/2, \sigma_b/2]$.

For $y \in P(H_i)$, let

$$m_i^t(y) = \text{the cardinality of } P^{-1}(y) \cap S_i^t \cap H_i.$$

The proof of the following claim is essentially contained in [HT00, Proof of Theorem 1].

Claim 5.4.6. *There exists $i_1 \in \mathbb{N}$ such that for all $i \geq i_1$, $y \in P(H_i)$ and $|t| \leq \sigma_b/2$, $m_i^t(y) \leq N$.*

Proof. Let $y \in P(H_i)$, $z \in P^{-1}(y) \cap S_i^t \cap H_i$, $\mathbf{u}_i = u_i \circ f_{z, \varepsilon_i}$. Then $\mathbf{u}_i : (B_z(0, 4L), g_{z, \varepsilon_i}) \rightarrow (-1, 1)$ satisfies $AC_1(\mathbf{u}_i) = 0$,

$$\sup_{B_z(0, 4L)} \xi_1(\mathbf{u}_i) \leq O(\varepsilon_i) \quad (\text{by [HT00, Proposition 3.3]})$$

and $|\mathbf{u}_i(0)| \leq 1 - b$. Therefore, if i is sufficiently large, one can set $\hat{s} = s$, $\hat{b} = b$, $p = z$, $u = \mathbf{u}_i$ and $\Gamma = T_z \Gamma_z$ in Proposition 5.4.4 and conclude the followings.

$$\{z' \in B(z, 3L\varepsilon_i) : P(z') = P(z), u_i(z') = u_i(z)\} = \{z\}; \quad (5.4.14)$$

$$\left| \frac{1}{\omega(n)(L\varepsilon_i)^n} \int_{B(z, L\varepsilon_i)} e_{\varepsilon_i}(u) - 2\sigma \right| \leq s. \quad (5.4.15)$$

Since $\{|u_i| \leq 1 - b\}$ converges to Σ in the Hausdorff sense, it is possible choose a sequence $\{r_i\}$ such that

$$r_i \rightarrow 0; \quad \{|u_i| \leq 1 - b\} \subset \mathcal{N}_{r_i^2}(\Sigma); \quad \tilde{\varepsilon}_i = \varepsilon_i/r_i \rightarrow 0.$$

Then, for $y \in P(H_i)$ and $|t| \leq \sigma_b/2$,

$$\tilde{Y} = f_{y,r_i}^{-1} (P^{-1}(y) \cap S_i^t \cap H_i) \subset B_y(0, r_i).$$

Moreover, if $\tilde{u}_i = u_i \circ f_{y,r_i}$, then $\tilde{u}_i : (B_y(0, 2), g_{y,r_i}) \rightarrow (-1, 1)$ satisfies $AC_{\tilde{\varepsilon}_i}(\tilde{u}_i) = 0$ with

$$\sup_{B_y(0,2)} \xi_{\tilde{\varepsilon}_i}(\tilde{u}_i) \leq O(r_i) \quad (\text{by [HT00, Proposition 3.3]}).$$

(5.4.14) implies that $d(x, x') > 3L\tilde{\varepsilon}_i$ for all $x, x' \in \tilde{Y}$.

If there exists $Y' \subset \tilde{Y}$ with the cardinality $|Y'| = N + 1$, then, for sufficiently large i , one can set $\hat{s} = s$, $p = y$, $\hat{r} = r_i$, $\Gamma = T_y \Sigma$, $Y = Y'$, $a = L\tilde{\varepsilon}_i$ and $\varepsilon = \tilde{\varepsilon}_i$ in Proposition 5.4.2 and conclude that

$$\sum_{x \in Y'} \frac{1}{(L\tilde{\varepsilon}_i)^n} \int_{B(x, L\tilde{\varepsilon}_i)} e_{\tilde{\varepsilon}_i}(\tilde{u}_i) \leq s + (1 + s) \int_{B(0, 1+r_i)} e_{\tilde{\varepsilon}_i}(\tilde{u}_i). \quad (5.4.16)$$

Since $V[u_i] \rightarrow N|\Sigma|$, $V[\tilde{u}_i] \rightarrow N|T_y \Sigma \cap B_y(0, 2)|$. Hence, if i is sufficiently large, (6.3.2) and (6.3.3) imply

$$(N + 1)(2\sigma - s) \leq \omega(n)^{-1}s + (1 + s)(2\sigma N + s),$$

which contradicts the assumption that $s < s_0$ ((5.4.1)). $\square$

We choose a cut-off function $0 \leq \chi_2 \leq 1$ with $\text{spt}(\chi_2) \subset U_2$ and $\chi_2 \equiv 1$ on $\overline{U}_1$. If $\mathbf{p} : G_n M \rightarrow M$ is the canonical projection map, let $\tilde{\chi}_2 = \chi_2 \circ \mathbf{p}$. Since $P_{\#} |S_i^t \cap H_i| = \mathbf{v}(P(H_i), m_i^t)$, for all $i \geq i_1$

$$\begin{aligned} & \left| P_{\#} |S_i^t \cap H_i| (\varphi) - N |P(H_i)| (\varphi) \right| \\ & \leq \int_{P(H_i)} (N - m_i^t(y)) d\mathcal{H}^n(y) \\ & = N |P(H_i)| (\tilde{\chi}_2) - P_{\#} |S_i^t \cap H_i| (\tilde{\chi}_2) \\ & \leq N |\Sigma| (\tilde{\chi}_2) - P_{\#} |S_i^t| (\tilde{\chi}_2) + 2 \|S_i^t\| (\overline{U}_1 \setminus H_i) + 2 \|S_i^t\| (U_1^c) \quad (\text{by (5.4.4)}). \end{aligned} \quad (5.4.17)$$

In (5.4.17), $P_{\#} |S_i^t| (\tilde{\chi}_2)$ is well-defined as $\text{spt}(\chi_2) \subset U_2$.

We will now integrate the various error terms, obtained above, with respect to t in the interval $[-\sigma_b/2, \sigma_b/2]$ and let $i \rightarrow \infty$.

$$\limsup_{i \rightarrow \infty} \int_{-\sigma_b/2}^{\sigma_b/2} \|S_i^t\| (U_1^c) dt \leq \limsup_{i \rightarrow \infty} \sigma \|V_i\| (U_1^c) \leq \sigma s \quad (\text{by (5.4.3)}); \quad (5.4.18)$$

$$\int_{-\sigma_b/2}^{\sigma_b/2} N \|\Sigma\| (U_1^c) dt \leq \sigma s \quad (\text{by (5.4.3)}); \quad (5.4.19)$$

$$\lim_{i \rightarrow \infty} \int_{-\sigma_b/2}^{\sigma_b/2} \|S_i^t\| (\bar{U}_1 \setminus H_i) + N \mathcal{H}^n(P(\bar{U}_1 \setminus H_i)) dt = 0 \quad (\text{by (5.4.8)}); \quad (5.4.20)$$

$$\limsup_{i \rightarrow \infty} \int_{-\sigma_b/2}^{\sigma_b/2} \mathcal{H}^n(S_i^t) dt \leq \limsup_{i \rightarrow \infty} \sigma \|V_i\| (M) = \sigma N \mathcal{H}^n(\Sigma). \quad (5.4.21)$$

We note that $V_i \rightarrow N|\Sigma|$ in the varifold sense implies $P_{\#}V_i(\tilde{\chi}_2) \rightarrow N|\Sigma|(\tilde{\chi}_2)$. ($P_{\#}V_i(\tilde{\chi}_2)$ is well-defined as $\text{spt}(\chi_2) \subset U_2$.) Hence, using (5.4.4) and (5.4.2),

$$\begin{aligned} & \limsup_{i \rightarrow \infty} \int_{-\sigma_b/2}^{\sigma_b/2} (N|\Sigma|(\tilde{\chi}_2) - P_{\#}|S_i^t|(\tilde{\chi}_2)) dt \\ & \leq \lim_{i \rightarrow \infty} \sigma (N|\Sigma|(\tilde{\chi}_2) - P_{\#}V_i(\tilde{\chi}_2)) + 2 \limsup_{i \rightarrow \infty} \int_{\{|u_i| \geq 1-b\}} |\nabla w_i| \\ & \leq 2s. \end{aligned} \quad (5.4.22)$$

Therefore, using the equations (5.4.9) – (5.4.13) and (5.4.17) – (5.4.22), we conclude that for all $i \geq i_1$ there exists a measurable function $\Theta_i : [-\sigma_b/2, \sigma_b/2] \rightarrow \mathbb{R}$ such that

$$\mathbf{F}(N|\Sigma|, |S_i^t|) \leq \Theta_i(t) \quad \forall t \in [-\sigma_b/2, \sigma_b/2] \quad \text{and} \quad \limsup_{i \rightarrow \infty} \int_{-\sigma_b/2}^{\sigma_b/2} \Theta_i(t) dt \leq (2 + 4\sigma)s.$$

Hence, there exists $i_2 \in \mathbb{N}$ such that for all $i \geq i_2$,

$$\int_{-\sigma_b/2}^{\sigma_b/2} \Theta_i(t) dt \leq (3 + 4\sigma)s.$$

Let $\mathcal{T}_i \subset [-\sigma_b/2, \sigma_b/2]$ be the set of all $t \in [-\sigma_b/2, \sigma_b/2]$ such that t is a regular value of w_i ,

$$\{w_i > t\}, \{w_i < t\} \in \mathcal{C}(M), \quad \llbracket \{w_i > t\} \rrbracket + \llbracket \{w_i < t\} \rrbracket = \llbracket M \rrbracket, \quad \partial \llbracket \{w_i > t\} \rrbracket = \llbracket \{w_i = t\} \rrbracket.$$

Then $[-\sigma_b/2, \sigma_b/2] \setminus \mathcal{T}_i$ has Lebesgue measure 0. Therefore, for $i \geq i_2$, there exists $t_i \in \mathcal{T}_i$ such that

$$\mathbf{F}(N|\Sigma|, |S_i^{t_i}|) \leq \Theta_i(t_i) \leq (3 + 4\sigma)\sigma_b^{-1}s.$$

Since $s \in (0, s_0)$ is arbitrary and $b \leq b_0 \leq 1/2$, this finishes the proof of the proposition. $\square$

Proposition 5.4.7. *Let $\{u_i : M \rightarrow (-1, 1)\}_{i=1}^\infty$ be a sequence such that items (i) – (iii) of Proposition 5.4.1 are satisfied. Additionally, we assume that u_i is a min-max critical point of E_{ε_i} corresponding to the homotopy class $\tilde{\Pi}$. Let us set $L_{\varepsilon_i} = \mathbf{L}_{\varepsilon_i}(\tilde{\Pi})$ so that*

$$L = \mathbf{L}_{AP}(\Pi) = \frac{1}{2\sigma} \lim_{i \rightarrow \infty} L_{\varepsilon_i} = \|V\|(M).$$

Then, (using the notation from Proposition 5.4.1) for every $s > 0$ and $b \in (0, b_0(s)]$, there exists $i_0^* \geq i_0(s, b)$ such that the following holds. For all $i \geq i_0^*$, there exists $\Phi_i : X \rightarrow \mathcal{Z}_n(M; \mathbf{M}; \mathbb{Z}_2)$, $x_i^* \in X$ and $\delta_i > 0$ such that $\Phi_i \in \Pi$, $\delta_i \rightarrow 0$,

$$\sup_{x \in X} \mathbf{M}(\Phi_i(x)) \leq \max \left\{ \frac{1}{2\sigma_b}(L_{\varepsilon_i} + \varepsilon_i), L + s \right\} + \delta_i \quad \text{and} \quad \mathbf{F}(V, |\Phi_i(x_i^*)|) \leq s. \quad (5.4.23)$$

Proof. Since u_i is a min-max critical point of E_{ε_i} corresponding to the homotopy class $\tilde{\Pi}$, for each i , there exists a sequence of continuous, $\mathbb{Z}_2$ -equivariant maps $\{h_j^i : \tilde{X} \rightarrow H^1(M) \setminus \{0\}\}_{j=1}^\infty$ such that

$$\sup_{x \in \tilde{X}, j \in \mathbb{N}} E_{\varepsilon_i}(h_j^i(x)) \leq L_{\varepsilon_i} + \varepsilon_i \quad \text{and} \quad \lim_{j \rightarrow \infty} d_{H^1(M)}(u_i, h_j^i(\tilde{X})) = 0. \quad (5.4.24)$$

The next lemma is a restatement of Lemmas 8.10 and 8.11 of [Gua18].

Lemma 5.4.8 ([Gua18, Lemmas 8.10, 8.11]). *Let $h_1, h_2 \in H^1(M)$. For $\delta \in (0, 1)$, we set $C_\delta = W(1 - \delta) > 0$. Then, for all $\varepsilon > 0$,*

$$\mathcal{H}^{n+1}(\{|h_1| \leq 1 - \delta\}) \leq \varepsilon C_\delta^{-1} E_\varepsilon(h_1).$$

Let $\alpha_0 \in (-1 + \delta, 1 - \delta)$ be such that for $j = 1, 2$, $\Omega_j = \{h_j > \alpha_0\} \in \mathcal{C}(M)$. Then, for all $\varepsilon > 0$,

$$\mathcal{H}^{n+1}(\Omega_1 \setminus \Omega_2) \leq \varepsilon C_\delta^{-1} E_\varepsilon(h_2) + (\alpha_0 + 1 - \delta)^{-2} \|h_1 - h_2\|_{H^1(M)}^2.$$

As a consequence, for $j = 1, 2$, if $\lambda_j \in (-1 + \delta, 1 - \delta)$ such that $T_j = \partial[\{h_j > \lambda_j\}] \in \mathcal{Z}_n(M; \mathbb{Z}_2)$, then

$$\mathcal{F}(T_1, T_2) \leq 2\varepsilon C_\delta^{-1} (E_\varepsilon(h_1) + E_\varepsilon(h_2)) + 2(\alpha_0 + 1 - \delta)^{-2} \|h_1 - h_2\|_{H^1(M)}^2 \quad \forall \varepsilon > 0.$$

Let us recall that X is a subcomplex of $\mathcal{S}^K[1]$. There exists $\alpha \in (-1 + b, 1 - b)$ such that for all $i, j \in \mathbb{N}$ and $x \in \pi^{-1}(X \cap \mathbb{Q}^K)$, $\{h_j^i(x) > \alpha\} \in \mathcal{C}(M)$. Let us fix $i \geq i_0$ ($i_0 = i_0(s, b)$ is as in Proposition 5.4.1); let $j_0 \in \mathbb{N}$ such that

$$d_{H^1(M)}(u_i, h_{j_0}^i(\tilde{X})) \leq \varepsilon_i/2.$$

For simplicity, let us denote the map $h_{j_0}^i$ by h . We choose $l_i \in \mathbb{N}$ such that if x, x' belong to a common cell in $\tilde{X}[3^{-l_i}]$, then $\|h(x) - h(x')\|_{H^1(M)} \leq \varepsilon_i/4$. Let $x_i \in \tilde{X}[3^{-l_i}]_0$ such that

$$d_{H^1(M)}(u_i, h(x_i)) = d_{H^1(M)}(u_i, h(\tilde{X}[3^{-l_i}]_0)),$$

which is bounded by $3\varepsilon_i/4$ by our choices of j_0 and l_i . It follows from the arguments in [Gua18] and [GG18, Section 6.7] that there exists a function $\lambda : \tilde{X} \rightarrow (-1 + b, 1 - b)$ such that for all $x \in \tilde{X}$ the following conditions are satisfied.

- $\lambda(T(x)) = -\lambda(x)$;
- $\{h(x) > \lambda(x)\}, \{h(x) < \lambda(x)\} \in \mathcal{C}(M)$ with $[\{h(x) > \lambda(x)\}] + [\{h(x) < \lambda(x)\}] = [M]$;
- Denoting $\tilde{h} = F \circ h$ and $\tilde{\lambda} = F \circ \lambda$, $2\sigma_b \mathbf{M}(\partial[\{\tilde{h}(x) > \tilde{\lambda}(x)\}]) \leq E_{\varepsilon_i}(h(x))$.

One can define a discrete, $\mathbb{Z}_2$ -equivariant map $\tilde{\varphi}_i : \tilde{X}[3^{-l_i}]_0 \rightarrow \mathbf{I}_{n+1}(M; \mathbb{Z}_2)$ as follows (w_i, t_i are

as in Proposition 5.4.1).

$$\tilde{\varphi}_i(x) = \begin{cases} \llbracket \{\tilde{h}(x) > \tilde{\lambda}(x)\} \rrbracket & \text{if } x \notin \{x_i, T(x_i)\}; \\ \llbracket \{w_i > t_i\} \rrbracket & \text{if } x = x_i; \\ \llbracket \{w_i < t_i\} \rrbracket & \text{if } x = T(x_i). \end{cases}$$

$\tilde{\varphi}_i$ descends to a map $\varphi_i : X[3^{-l_i}]_0 \rightarrow \mathcal{Z}_n(M, \mathbb{Z}_2)$ such that $\varphi_i \circ \pi = \partial \circ \tilde{\varphi}_i$. By Lemma 5.4.8 and (5.4.24), the fineness of φ_i with respect to the flat norm

$$\mathbf{f}^{\mathcal{F}}(\varphi_i) \leq 4\varepsilon_i C_b^{-1}(L_{\varepsilon_i} + \varepsilon_i) + 2(1 - b + \alpha)^{-2} \varepsilon_i^2.$$

Thus we have a sequence of maps $\{\varphi_i : X[3^{-l_i}]_0 \rightarrow \mathcal{Z}_n(M, \mathbb{Z}_2)\}_{i \geq i_0}$ such that $\mathbf{f}^{\mathcal{F}}(\varphi_i) \rightarrow 0$. Moreover, $\mathbf{F}(V, |\{w_i = t_i\}|) \leq s$ implies that $\mathbf{M}(\llbracket \{w_i = t_i\} \rrbracket) \leq L + s$. Hence,

$$\sup_{x \in X[3^{-l_i}]_0} \mathbf{M}(\varphi_i(x)) \leq \max \left\{ \frac{1}{2\sigma_b}(L_{\varepsilon_i} + \varepsilon_i), L + s \right\}.$$

As argued in [Gua18] and [GG18, Sections 6.8, 6.9], one can apply the interpolation theorem of Zhou [Zho17, Proposition 5.8] to produce a sequence of discrete maps whose fineness with respect to the mass norm converges to 0 and then, using the interpolation theorem of Marques and Neves [MN14, Theorem 14.1], one can find a sequence of maps in Π , which are continuous in the mass norm. More precisely, there exists $i_0^* \geq i_0$ such that for all $i \geq i_0^*$ there exist $\Phi_i : X \rightarrow \mathcal{Z}_n(M; \mathbf{M}; \mathbb{Z}_2)$ and $\delta_i > 0$ such that $\Phi_i \in \Pi$, $\delta_i \rightarrow 0$ and

$$\sup_{x \in X} \mathbf{M}(\Phi_i(x)) \leq \max \left\{ \frac{1}{2\sigma_b}(L_{\varepsilon_i} + \varepsilon_i), L + s \right\} + \delta_i.$$

Moreover, $\Phi_i(x) = \varphi_i(x)$ for all $x \in X[3^{-l_i}]_0$. In particular, denoting $x_i^* = \pi(x_i)$, $\Phi_i(x_i^*) = \varphi_i(x_i^*) = \llbracket \{w_i = t_i\} \rrbracket$; hence,

$$\mathbf{F}(V, |\Phi_i(x_i^*)|) \leq s.$$

□

Proof of Theorem 5.1.3. By letting $s \rightarrow 0$, $b \rightarrow 0$ and $i \rightarrow \infty$ in the above Proposition 5.4.7, we obtain Theorem 5.1.3. More precisely, let $\{s_m\}_{m=1}^\infty$ be a sequence such that $s_m \rightarrow 0$. We choose $b_m \in (0, b_0(s_m)]$ such that $b_m \rightarrow 0$. By Proposition 5.4.7, for every $m \in \mathbb{N}$, there exist $\Psi_m : X \rightarrow \mathcal{Z}_n(M; \mathbf{M}; \mathbb{Z}_2)$, $i(m) \in \mathbb{N}$ and $x_m^\bullet \in X$ such that $\Psi_m \in \Pi$, $i(m) > i(m-1)$,

$$\sup_{x \in X} \mathbf{M}(\Psi_m(x)) \leq \max \left\{ \frac{1}{2\sigma_{b_m}} (L_{\varepsilon_{i(m)}} + \varepsilon_{i(m)}), L + s_m \right\} + s_m \quad \text{and} \quad \mathbf{F}(V, |\Psi_m(x_m^\bullet)|) \leq s_m.$$

This implies $\{\Psi_m\}_{m=1}^\infty$ is a minimizing sequence in Π and $V \in \mathbf{C}(\{\Psi_m\})$. □

Chapter 6

The Allen-Cahn equation on the complete Riemannian manifolds of finite volume

6.1 Introduction

In all the theorems, mentioned in the previous chapters, the ambient manifolds are assumed to be closed. If M is a complete non-compact manifold, Gromov [Gro14] proved that either M contains a complete minimal hypersurface with finite area or every compact domain of M admits a (possibly singular) strictly mean convex foliation. In [Mon16], Montezuma proved that a complete Riemannian manifold with a bounded, strictly mean concave domain contains a complete minimal hypersurface with finite area. The existence of minimal surfaces in hyperbolic 3-manifolds has been proved by Collin-Hauswirth-Mazet-Rosenberg [CHMR17], Huang-Wang [HW17] and Coskunuzer [Cos21]. In [CK18], Chodosh and Ketover proved the existence of minimal planes in asymptotically flat 3-manifolds. In [CL20], Chambers and Liokumovich proved that every complete Riemannian manifold with finite volume contains a complete minimal hypersurface with finite area. In

[Son19], Song proved Yau's conjecture on certain complete non-compact manifolds. Moreover, he also proved the local version of the above mentioned theorem of Gromov [Gro14], using which he gave alternative proofs of the above mentioned theorems of Montezuma [Mon16] and Chambers-Liokumovich [CL20].

In this chapter we will show the existence of finite energy min-max solution to the Allen-Cahn equation (for ε sufficiently small) on complete Riemannian manifolds of finite volume. More precisely, we will prove the following theorem, which is motivated by the previously mentioned theorem of Chambers-Liokumovich [CL20] and Song [Son19].

Theorem 6.1.1. *Let M^{n+1} be a complete, Riemannian manifold, $n + 1 \geq 3$, such that $\text{Vol}(M)$ is finite. Then there exists $\varepsilon_0 > 0$, depending on the ambient Riemannian metric, such that for all $0 < \varepsilon \leq \varepsilon_0$, there exists $\mathbf{u}_\varepsilon : M \rightarrow (-1, 1)$ satisfying $AC_\varepsilon(\mathbf{u}_\varepsilon) = 0$, $\text{Ind}(\mathbf{u}_\varepsilon) \leq 1$ and $E_\varepsilon(\mathbf{u}_\varepsilon) < \infty$. Moreover, there exists a good set $U \subset M$ (see Section 6.2.4) such that*

$$0 < \liminf_{\varepsilon \rightarrow 0^+} E_\varepsilon(\mathbf{u}_\varepsilon, U) \leq \limsup_{\varepsilon \rightarrow 0^+} E_\varepsilon(\mathbf{u}_\varepsilon) < \infty. \quad (6.1.1)$$

We will prove Theorem 6.1.1 by adapting the argument of Chambers-Liokumovich [CL20] in the Allen-Cahn setting. From Theorem 6.1.1, one can recover the above mentioned theorem of Chambers-Liokumovich [CL20] and Song [Son19].

Theorem 6.1.2. [CL20, Son19] *Let M^{n+1} be a complete, Riemannian manifold, $n + 1 \geq 3$, such that $\text{Vol}(M)$ is finite. Then there exists a complete minimal hypersurface $\Sigma \subset M$ such that $0 < \mathcal{H}^n(\Sigma) < \infty$ and Σ has optimal regularity, i.e. Σ is smooth and embedded outside a singular set of Hausdorff dimension $\leq n - 7$.*

As in [CL20] and [Son19], Theorem 6.1.1 and Theorem 6.1.2 continue to hold if the assumption $\text{Vol}(M) < \infty$ is replaced by the weaker assumption that there exists a sequence $\{U_i\}_{i=1}^\infty$, where each $U_i \subset M$ is a bounded open set with smooth boundary, such that $U_i \subset U_{i+1}$ for all $i \in \mathbb{N}$ and $\lim_{i \rightarrow \infty} \mathcal{H}^n(\partial U_i) = 0$.

6.2 Notations and Preliminaries

6.2.1 Notations

Here we summarize the notations which will be frequently used later in this chapter.

- $\mathcal{H}^k$: the Hausdorff measure of dimension k .
- $B(p, r)$: the geodesic ball centered at p with radius r .
- $d(-, S)$: distance from a set S .
- $H^1(N)$: the Sobolev space $\{u \in L^2(N) : \text{the distributional derivative } \nabla u \in L^2(N, TN)\}$.
- $e_\varepsilon(u) = \varepsilon \frac{|\nabla u|^2}{2} + \frac{W(u)}{\varepsilon}$.
- $E_\varepsilon(u) = \int_N e_\varepsilon(u)$, where N is the ambient manifold.
- $E_\varepsilon(u, S) = \int_S e_\varepsilon(u)$, where S is a measurable set.
- $AC_\varepsilon(u) = -\varepsilon^2 \Delta u + W'(u)$.
- 2σ = the energy of the 1-D solution to the Allen-Cahn equation (see (6.2.1), (6.4.3)).
- For two measurable functions u and v , we say that $u \leq v$ (resp. $u \geq v$) if $u(x) \leq v(x)$ (resp. $u(x) \geq v(x)$) for a.e. x .

6.2.2 The Allen-Cahn equation and convergence of the phase interfaces

In this subsection we will briefly discuss about the Allen-Cahn equation and its connection with the minimal hypersurfaces. Suppose Ω^{n+1} is the interior of a compact Riemannian manifold. Let $W : \mathbb{R} \rightarrow [0, \infty)$ be a smooth, symmetric, double well potential. More precisely, W has the following properties. W is bounded; $W(-t) = W(t)$ for all $t \in \mathbb{R}$; W has exactly three critical points $0, \pm 1$; $W(\pm 1) = 0$ and $W''(\pm 1) > 0$ i.e. ± 1 are non-degenerate minima; 0 is a local maximum. For $u \in H^1(\Omega)$, the ε -Allen-Cahn energy of u is given by

$$E_\varepsilon(u) = \int_\Omega \varepsilon \frac{|\nabla u|^2}{2} + \frac{W(u)}{\varepsilon}.$$

As mentioned earlier,

$$AC_\varepsilon(u) := -\varepsilon^2 \Delta u + W'(u) = 0$$

if and only if u is a critical point of E_ε .

Let $F : \mathbb{R} \rightarrow \mathbb{R}$ and the energy constant σ be defined as follows.

$$F(t) = \int_0^t \sqrt{W(s)/2} \, ds; \quad \sigma = \int_{-1}^1 \sqrt{W(s)/2} \, ds \quad \text{so that} \quad F(\pm 1) = \pm \frac{\sigma}{2}. \quad (6.2.1)$$

For an n -rectifiable set $S \subset \Omega$, let $|S|$ denote the n -varifold defined by S . Given $u \in C^1(\Omega)$, we set $\tilde{u} = F \circ u$. The n -varifold associated to u is defined by

$$V[u](A) = \frac{1}{\sigma} \int_{-\infty}^{\infty} |\{\tilde{u} = s\}|(A) \, ds,$$

for every Borel set $A \subset G_n \Omega$ (where $G_n \Omega$ denotes the Grassmannian bundle of unoriented n -dimensional hyperplanes on Ω).

Building on the works of Hutchinson-Tonegawa [HT00], Tonegawa [Ton05] and Tonegawa-Wickramasekera [TW12], Guaraco [Gua18] has proved the following theorem.

Theorem 6.2.1 ([HT00, Ton05, TW12, Gua18]). *Suppose Ω^{n+1} , $n+1 \geq 3$, is the interior of a compact Riemannian manifold. Let $\{u_i : \Omega \rightarrow (-1, 1)\}_{i=1}^\infty$ be a sequence of smooth functions such that*

$$(i) \quad AC_{\epsilon_i}(u_i) = 0 \text{ with } \epsilon_i \rightarrow 0 \text{ as } i \rightarrow \infty;$$

(ii)

$$\sup_{i \in \mathbb{N}} E_{\epsilon_i}(u_i) < \infty \quad \text{and} \quad \sup_{i \in \mathbb{N}} \text{Ind}(u_i) < \infty.$$

Then there exists a stationary, integral varifold V in Ω such that possibly after passing to a subsequence, $V[u_i] \rightarrow V$ in the sense of varifolds. Moreover, $\text{spt}(V)$ is a minimal hypersurface with

optimal regularity in Ω . Furthermore, if $\|V\|$ denotes the Radon measure associated to V , then

$$\frac{1}{2\sigma} \left(\epsilon_i \frac{|\nabla u_i|^2}{2} + \frac{W(u_i)}{\epsilon_i} \right) d\text{Vol}_\Omega \rightarrow \|V\|, \quad (6.2.2)$$

in the sense of Radon measures.

The proof of the regularity of the limit-interface depends on the regularity theory of stable, minimal hypersurfaces, developed by Wickramasekera [Wic14a]. In the ambient dimension $n+1 = 3$, the regularity of the limit-interface can also be obtained from the curvature estimates of Chodosh and Mantoulidis [CM20].

We also state here the theorem proved by Smith [Smi16] about the generic finiteness of the number of solutions to the Allen-Cahn equation on a closed manifold. This theorem will be used to prove the Morse index upper bound in Theorem 6.1.1.

Let N^{n+1} , $n+1 \geq 3$, be a closed manifold and $\mathcal{M}$ be the space of all smooth Riemannian metrics on N , endowed with the C^∞ topology. For $\varepsilon > 0$ and $\gamma \in \mathcal{M}$, we define

$$\mathcal{Z}_{\varepsilon, \gamma} = \{u \in C^\infty(N) : -\varepsilon^2 \Delta_\gamma u + W'(u) = 0\}. \quad (6.2.3)$$

Theorem 6.2.2. [Smi16, Theorem 1.1 (2)] *There exists a generic set $\widetilde{\mathcal{M}} \subset \mathcal{M}$ such that if $\gamma \in \widetilde{\mathcal{M}}$ and $\varepsilon^{-1} \notin \text{Spec}(-\Delta_\gamma)$ (here we are using the convention that $\text{Spec}(-\Delta_\gamma) \subset [0, \infty)$), then $\mathcal{Z}_{\varepsilon, \gamma}$ is finite.*

6.2.3 Min-max theorem on the Hilbert space

Let $\mathcal{H}$ be a separable Hilbert space and $\mathcal{E} : \mathcal{H} \rightarrow \mathbb{R}$ be a C^2 functional. Suppose B_0, B_1 are closed subsets of $\mathcal{H}$. We define

$$\mathcal{F} = \{\zeta : [0, 1] \rightarrow \mathcal{H} : \zeta \text{ is continuous, } \zeta(0) \in B_0, \zeta(1) \in B_1\} \quad (6.2.4)$$

and

$$c = \inf_{\zeta \in \mathcal{F}} \sup_{t \in [0,1]} \mathcal{E}(\zeta(t)). \quad (6.2.5)$$

A sequence $\{\zeta_i\}_{i=1}^\infty \subset \mathcal{F}$ is called a *minimizing sequence* if

$$\lim_{i \rightarrow \infty} \sup_{t \in [0,1]} \mathcal{E}(\zeta_i(t)) = c.$$

For a minimizing sequence $\{\zeta_i\} \subset \mathcal{F}$, let $\mathcal{K}(\{\zeta_i\})$ denote the set of all $v \in \mathcal{H}$ for which there exist sequences $\{i_j\} \subset \{i\}$ and $\{t_j\} \subset [0,1]$ such that

$$v = \lim_{j \rightarrow \infty} \zeta_{i_j}(t_j).$$

Definition 6.2.3. Given a minimizing sequence $\{\zeta_i\}$ in $\mathcal{F}$, we say that $\mathcal{E}$ satisfies the *Palais-Smale condition* along $\{\zeta_i\}$ if every sequence $\{v_i\}$, satisfying the conditions

$$\lim_{i \rightarrow \infty} \mathcal{E}'(v_i) = 0 \quad \text{and} \quad \lim_{i \rightarrow \infty} d(v_i, \zeta_i([0,1])) = 0,$$

has a convergent subsequence.

Definition 6.2.4. [Gho91, Section 3, page 53] Let

$$K_c = \{v \in \mathcal{H} : \mathcal{E}'(v) = 0, \mathcal{E}(v) = c\}.$$

A compact subset $\mathcal{C}$ of K_c is called an *isolated critical set* for $\mathcal{E}$ in K_c if there exists an open set $\mathcal{U} \subset \mathcal{H}$ such that $\mathcal{C} \subset \mathcal{U}$ and

$$K_c \cap \mathcal{U} = \mathcal{C}.$$

The following min-max theorem, which was proved by Ghoussoub [Gho91] in a much more general setting (see [Gho91, Theorem (1.bis) in page 30 and Theorem (4) in page 53]), will be used to prove Theorem 6.1.1.

Theorem 6.2.5. [Gho91] (a) Let $L \subset \mathcal{H}$ be a closed set such that the following conditions are

satisfied:

$$(a1) \quad L \cap (B_0 \cup B_1) = \emptyset;$$

$$(a2) \quad \text{for all } \zeta \in \mathcal{F}, \quad L \cap \zeta([0, 1]) \neq \emptyset;$$

$$(a3) \quad \inf_{v \in L} \mathcal{E}(v) \geq c.$$

Suppose $\mathcal{E}$ satisfies the Palais-Smale condition along a minimizing sequence $\{\zeta_i\}_{i=1}^\infty$. Then

$$K_c \cap L \cap \mathcal{K}(\{\zeta_i\}) \neq \emptyset. \quad (6.2.6)$$

(b) In addition to the assumptions stated in part (a), let us also assume that:

$$(b1) \quad K_c \cap L \text{ is an isolated critical set for } \mathcal{E} \text{ in } K_c;$$

$$(b2) \quad \mathcal{E}'' \text{ is Fredholm on } K_c.$$

Then there exists

$$v \in K_c \cap L \cap \mathcal{K}(\{\zeta_i\}) \text{ such that } m(v) \leq 1, \quad (6.2.7)$$

where $m(v)$ is the Morse index of the critical point v , i.e. $m(v)$ is equal to the index of the bilinear form $\mathcal{E}''|_v$.

Remark 6.2.6. In the definition of $\mathcal{F}$ in (6.2.4) and in Theorem 6.2.5, we have assumed that B_0 and B_1 are closed subsets of $\mathcal{H}$. This is slightly different from the hypothesis made in [Gho91, Theorem (1.bis) in page 30 and Theorem (4) in page 53], where B_0 and B_1 are assumed to be singleton sets. However this does not affect the proof of the above Theorem 6.2.5 in [Gho91] for the following reason (see [Gho91, Remark (3) in page 32 and Remark (11) in page 60]). If $\zeta \in \mathcal{F}$ (as defined in (6.2.4)) and $\zeta' : [0, 1] \rightarrow \mathcal{H}$ is another map satisfying $\zeta'(0) = \zeta(0)$ and $\zeta'(1) = \zeta(1)$, then $\zeta' \in \mathcal{F}$ as well.

6.2.4 The notion of the good set

In this subsection we will recall the definition of the *good set* from [CL20, Section 2.2]. Let N be a complete Riemannian manifold and $\Omega \subset N$ be a bounded open set with smooth boundary $\partial\Omega$.

$\mathbf{I}_{n+1}(\Omega; \mathbb{Z}_2)$ denotes the space of $(n+1)$ -dimensional mod 2 flat chains in Ω ; $\mathcal{Z}_{n,\text{rel}}(\Omega, \partial\Omega; \mathbb{Z}_2)$ denotes the space of n -dimensional mod 2 relative flat cycles in Ω and $\partial : \mathbf{I}_{n+1}(\Omega; \mathbb{Z}_2) \rightarrow \mathcal{Z}_{n,\text{rel}}(\Omega, \partial\Omega; \mathbb{Z}_2)$ is the boundary map. Both the spaces $\mathbf{I}_{n+1}(\Omega; \mathbb{Z}_2)$ and $\mathcal{Z}_{n,\text{rel}}(\Omega, \partial\Omega; \mathbb{Z}_2)$ are assumed to be equipped with the flat topology. (We refer to [LMN18, Section 2] and [LZ21, Section 3] for more details about these spaces.) Let $\mathcal{S}$ be the set of all continuous maps $\Gamma : [0, 1] \rightarrow \mathbf{I}_{n+1}(\Omega; \mathbb{Z}_2)$ such that $\Gamma(0) = \emptyset$ and $\Gamma(1) = \Omega$. The *(relative) width of Ω* , denoted by $\mathbb{W}(\Omega)$, is defined as follows [Gro03, Gut09, LMN18, CL20, LZ21].

$$\mathbb{W}(\Omega) = \inf_{\Gamma \in \mathcal{S}} \sup_{t \in [0, 1]} \mathbf{M}(\partial\Gamma(t)). \quad (6.2.8)$$

Ω is called a *good set* if

$$\mathbb{W}(\Omega) > 4\mathcal{H}^n(\partial\Omega). \quad (6.2.9)$$

6.3 Nested families in the Sobolev space $H^1(N)$

The notion of the nested family of open sets played an important role in the proof of the main theorem of Chambers-Liokumovich in [CL20]. In this section, we will deal with the notion of the nested family in the function space $H^1(N)$. Throughout this section, N will be assumed to be a closed Riemannian manifold (of dimension $n+1$) and $\varepsilon > 0$. We begin with the following lemma.

Lemma 6.3.1. *Let $u, v \in H^1(N)$ such that $v \geq u$ and $|u|, |v| \leq 1$. Suppose*

$$\mathcal{S} = \{w \in H^1(N) : v \geq w \geq u\}.$$

Then there exists $w^* \in \mathcal{S}$ such that

$$E_\varepsilon(w^*) = \inf \{E_\varepsilon(w) : w \in \mathcal{S}\}.$$

Proof. Let

$$\alpha = \inf \{E_\varepsilon(w) : w \in \mathcal{S}\}$$

and $\{w_i\}_{i=1}^\infty \subset \mathcal{S}$ be such that

$$\int_N \varepsilon \frac{|\nabla w_i|^2}{2} + \frac{W(w_i)}{\varepsilon} \leq \alpha + \frac{1}{i}. \quad (6.3.1)$$

Since $|u|, |v| \leq 1$,

$$|w_i| \leq 1 \quad \forall i \in \mathbb{N}. \quad (6.3.2)$$

(6.3.1) and (6.3.2) imply that $\{w_i\}_{i=1}^\infty$ is a bounded sequence in $H^1(N)$. Therefore, by Rellich's compactness theorem, there exist $w^* \in H^1(N)$ and a subsequence $\{w_{i_k}\}_{k=1}^\infty$ such that

$$w_{i_k} \rightarrow w^* \text{ strongly in } L^2(N) \text{ and pointwise a.e.} \quad (6.3.3)$$

and

$$\nabla w_{i_k} \rightarrow \nabla w^* \text{ weakly in } L^2(N). \quad (6.3.4)$$

(6.3.3) implies that $w^* \in \mathcal{S}$. (6.3.3), (6.3.4) and (6.3.1) together imply that $E_\varepsilon(w^*) \leq \alpha$ and hence $E_\varepsilon(w^*) = \alpha$. $\square$

Definition 6.3.2. A continuous map $u : [a, b] \rightarrow H^1(N)$ is called *nested* if $u(t) \geq u(s)$ whenever $t \leq s$.

6.3.1 Truncation and concatenation of the nested maps

The following Lemmas 6.3.3 and 6.3.4 are the Allen-Cahn counterparts of [CL20, Lemma 5.1] and [CL20, Proposition 6.3], respectively.

Lemma 6.3.3. (a) Let $u : [0, 1] \rightarrow H^1(N)$ be nested. Suppose $v \in H^1(N)$ has the following properties:

- $v \geq u(1)$;
- for any $v' \in H^1(N)$ with $v \geq v' \geq u(1)$, we have $E_\varepsilon(v) \leq E_\varepsilon(v')$.

Then there exists $\tilde{u} : [0, 1] \rightarrow H^1(N)$ such that

(i) $\tilde{u}$ is nested;

(ii) $\tilde{u}(0) \geq u(0)$ and $\tilde{u}(1) = v$; moreover if $u(0) \geq v$, one can choose $\tilde{u}(0) = u(0)$;

(iii) $E_\varepsilon(\tilde{u}(t)) \leq E_\varepsilon(u(t))$ for all $t \in [0, 1]$;

(iv) if $\|v\|_{L^\infty(N)} \leq 1$ and $\sup_{t \in [0, 1]} \|u(t)\|_{L^\infty(N)} \leq 1$, then $\sup_{t \in [0, 1]} \|\tilde{u}(t)\|_{L^\infty(N)} \leq 1$ as well.

(b) Let $u : [0, 1] \rightarrow H^1(N)$ be nested. Suppose $v \in H^1(N)$ has the following properties:

- $u(0) \geq v$;
- for any $v' \in H^1(N)$ with $u(0) \geq v' \geq v$, we have $E_\varepsilon(v) \leq E_\varepsilon(v')$.

Then there exists $\tilde{u} : [0, 1] \rightarrow H^1(N)$ such that

(i) $\tilde{u}$ is nested;

(ii) $\tilde{u}(0) = v$ and $\tilde{u}(1) \leq u(1)$; moreover if $u(1) \leq v$, one can choose $\tilde{u}(1) = u(1)$;

(iii) $E_\varepsilon(\tilde{u}(t)) \leq E_\varepsilon(u(t))$ for all $t \in [0, 1]$;

(iv) if $\|v\|_{L^\infty(N)} \leq 1$ and $\sup_{t \in [0, 1]} \|u(t)\|_{L^\infty(N)} \leq 1$, then $\sup_{t \in [0, 1]} \|\tilde{u}(t)\|_{L^\infty(N)} \leq 1$ as well.

Proof. To prove part (a), we define

$$\tilde{u}(t) = \max\{u(t), v\}.$$

Items (i) and (ii) follow from the assumptions that u is nested and $v \geq u(1)$, respectively. Item (iv) follows from the definition of $\tilde{u}$. To prove (iii), we consider

$$u'(t) = \min\{u(t), v\}.$$

Then $v \geq u'(t) \geq u(1)$ for all $t \in [0, 1]$. By our hypothesis,

$$E_\varepsilon(v) \leq E_\varepsilon(u'(t)) = E_\varepsilon(u(t), \{u(t) < v\}) + E_\varepsilon(v, \{u(t) \geq v\}).$$

Hence

$$E_\varepsilon(v, \{u(t) < v\}) \leq E_\varepsilon(u(t), \{u(t) < v\}). \quad (6.3.5)$$

Therefore, using (6.3.5),

$$E_\varepsilon(\tilde{u}(t)) = E_\varepsilon(u(t), \{u(t) \geq v\}) + E_\varepsilon(v, \{u(t) < v\}) \leq E_\varepsilon(u(t)).$$

Part (b) can also be proved in a similar way by defining

$$\tilde{u}(t) = \min\{u(t), v\}$$

and using the fact that

$$u(0) \geq \max\{u(t), v\} \geq v.$$

□

Lemma 6.3.4. *Let $u_1, u_2 : [0, 1] \rightarrow H^1(N)$ be such that*

- u_1, u_2 are nested;
- $\sup_{t \in [0, 1]} \|u_i(t)\|_{L^\infty(N)} \leq 1$ for $i = 1, 2$;
- $u_2(0) \geq u_1(1)$;
- $\sup_{t \in [0, 1]} E_\varepsilon(u_i(t)) \leq A$ for $i = 1, 2$.

Then there exists $\tilde{u} : [0, 1] \rightarrow H^1(N)$ such that

- (i) $\tilde{u}$ is nested;
- (ii) $\sup_{t \in [0, 1]} \|\tilde{u}(t)\|_{L^\infty(N)} \leq 1$;
- (iii) $\tilde{u}(0) \geq u_1(0)$ and $\tilde{u}(1) \leq u_2(1)$;
- (iv) $\sup_{t \in [0, 1]} E_\varepsilon(\tilde{u}(t)) \leq A$.

Proof. Let

$$\mathcal{S} = \{v \in H^1(N) : u_2(0) \geq v \geq u_1(1)\}.$$

By Lemma 6.3.1, there exists $v^* \in \mathcal{S}$ such that

$$E_\varepsilon(v^*) = \inf\{E_\varepsilon(v) : v \in \mathcal{S}\}.$$

We note that $v^* \geq v' \geq u_1(1)$ implies that $v' \in \mathcal{S}$ and hence $E_\varepsilon(v^*) \leq E_\varepsilon(v')$. Therefore, by Lemma 6.3.3, part (a), there exists a nested map $\tilde{u}_1 : [0, 1] \rightarrow H^1(N)$ such that

$$\tilde{u}_1(0) \geq u_1(0), \quad \tilde{u}_1(1) = v^*, \quad \sup_{t \in [0, 1]} \|\tilde{u}_1(t)\|_{L^\infty(N)} \leq 1 \quad \text{and} \quad \sup_{t \in [0, 1]} E_\varepsilon(\tilde{u}_1(t)) \leq A.$$

Similarly, $u_2(0) \geq v' \geq v^*$ implies that $v' \in \mathcal{S}$ and hence $E_\varepsilon(v^*) \leq E_\varepsilon(v')$. Therefore, by Lemma 6.3.3, part (b), there exists a nested map $\tilde{u}_2 : [0, 1] \rightarrow H^1(N)$ such that

$$\tilde{u}_2(0) = v^*, \quad \tilde{u}_2(1) \leq u_2(1), \quad \sup_{t \in [0, 1]} \|\tilde{u}_2(t)\|_{L^\infty(N)} \leq 1 \quad \text{and} \quad \sup_{t \in [0, 1]} E_\varepsilon(\tilde{u}_2(t)) \leq A.$$

Finally, we define $\tilde{u} : [0, 1] \rightarrow H^1(N)$ by

$$\tilde{u}(t) = \begin{cases} \tilde{u}_1(2t) & \text{if } t \in [0, 1/2]; \\ \tilde{u}_2(2t - 1) & \text{if } t \in [1/2, 1]. \end{cases}$$

□

6.3.2 Approximation by nested maps

In [CL20, Proposition 6.1], Chambers and Liokumovich proved that if $\{\Omega_t\}_{t \in [0, 1]}$ is a family of open sets and $\kappa > 0$, there exists a nested family of open sets $\{\tilde{\Omega}_t\}_{t \in [0, 1]}$ such that $\tilde{\Omega}_0 \subset \Omega_0$, $\tilde{\Omega}_1 \supset \Omega_1$ and

$$\sup_{t \in [0, 1]} \mathcal{H}^n(\partial \tilde{\Omega}_t) \leq \sup_{t \in [0, 1]} \mathcal{H}^n(\partial \Omega_t) + \kappa.$$

The following Proposition 6.3.5 is the Allen-Cahn analogue of this theorem.

Proposition 6.3.5. *Let $\phi : [0, 1] \rightarrow H^1(N)$ be a continuous map such that $|\phi(t)| \leq 1$ for all $t \in [0, 1]$ and $\kappa > 0$. Then there exists a nested map $\psi : [0, 1] \rightarrow H^1(N)$ such that $\psi(0) \geq \phi(0)$, $\psi(1) \leq \phi(1)$, $|\psi(t)| \leq 1$ for all $t \in [0, 1]$ and*

$$\sup_{t \in [0, 1]} E_\varepsilon(\psi(t)) \leq \sup_{t \in [0, 1]} E_\varepsilon(\phi(t)) + \kappa.$$

Before we prove Proposition 6.3.5, we need to prove few lemmas.

Lemma 6.3.6. *Let $\varepsilon > 0$ and $w : [0, 1] \rightarrow H^1(N)$ be a continuous map. Then, for all $\delta > 0$, there exists $r > 0$ such that*

$$E_\varepsilon(w(t), B(p, r)) \leq \delta,$$

for all $t \in [0, 1]$ and $p \in N$.

Proof. We assume by contradiction that there exist $\delta > 0$ and sequences $\{t_i\}_{i=1}^\infty \subset [0, 1]$ and $\{p_i\}_{i=1}^\infty \subset N$ such that

$$E_\varepsilon(w(t_i), B(p_i, i^{-1})) > \delta. \tag{6.3.6}$$

Without loss of generality, we can assume that $t_i \rightarrow t_0$ and $p_i \rightarrow p_0$. Then, for all $m \in \mathbb{N}$, if i is sufficiently large, $B(p_i, i^{-1}) \subset B(p_0, m^{-1})$. Therefore, by (6.3.6), for all $m \in \mathbb{N}$,

$$E_\varepsilon(w(t_0), B(p_0, m^{-1})) = \lim_{i \rightarrow \infty} E_\varepsilon(w(t_i), B(p_0, m^{-1})) \geq \delta.$$

This contradicts the fact that

$$\lim_{m \rightarrow \infty} E_\varepsilon(w(t_0), B(p_0, m^{-1})) = 0.$$

□

Lemma 6.3.7. *Let $u_0, u_1 \in L^\infty(N)$ with $u_0 \geq u_1$ and $|u_0|, |u_1| \leq 1$. Suppose $h : \mathbb{R} \rightarrow [-1, 1]$ is a piecewise C^1 function so that $h' \in L^\infty(\mathbb{R})$ and $e_\varepsilon(h)$ is compactly supported inside the compact*

interval $[-a, a]$. For $p \in N$ and $r > 0$, let $b^r : N \rightarrow \mathbb{R}$ be defined by $b^r(x) = h(d_p(x) - r)$, where $d_p(x) = d(x, p)$. Then, for all $0 < s_1 < s_2$ and $\varepsilon > 0$,

$$\int_{s_1}^{s_2} E_\varepsilon(b^r, \{u_0 > b^r > u_1\}) \, dr \leq \int_{B(p, s_2+a)} \int_{\{u_0(x) > h > u_1(x)\}} e_\varepsilon(h)(t) \, dt \, d\mathcal{H}^{n+1}(x).$$

Proof. $d_p : N \rightarrow \mathbb{R}$ is a Lipschitz continuous function with $|\nabla d_p| = 1$, $\mathcal{H}^{n+1}$ -a.e. As a consequence,

$$\mathcal{H}^{n+1}(\{d_p = s\}) = 0 \quad \forall s \geq 0, \quad (6.3.7)$$

since otherwise $\nabla d_p = 0$ on a set of positive $\mathcal{H}^{n+1}$ -measure. For all $r > 0$, b^r is Lipschitz continuous; by [GT01, Theorem 7.8] and (6.3.7),

$$\nabla b^r(x) = h'(d_p(x) - r) \nabla d_p(x) \quad \text{for } \mathcal{H}^{n+1}\text{-a.e. } x \in N. \quad (6.3.8)$$

Therefore,

$$\begin{aligned} & \int_{s_1}^{s_2} E_\varepsilon(b^r, \{u_0 > b^r > u_1\}) \, dr \\ &= \int_{s_1}^{s_2} \int_{\{u_0 > b^r > u_1\}} \left[\frac{\varepsilon}{2} h'(d_p(x) - r)^2 + \frac{1}{\varepsilon} W(h(d_p(x) - r)) \right] d\mathcal{H}^{n+1}(x) \, dr \\ &= \int_{s_1}^{s_2} \int_{-a}^a \left[\frac{\varepsilon}{2} h'(t)^2 + \frac{1}{\varepsilon} W(h(t)) \right] \mathcal{H}^n(\{d_p - r = t\} \cap \{u_0 > b^r > u_1\}) \, dt \, dr. \end{aligned} \quad (6.3.9)$$

In the last step we have used the co-area formula. It follows from the definition of b^r that for fixed r and t ,

$$\{d_p - r = t\} \cap \{u_0 > b^r > u_1\} = \{d_p = r + t\} \cap \{u_0 > h(t) > u_1\}.$$

Hence, by Fubini's theorem and the co-area formula, (6.3.9) implies that

$$\begin{aligned}
& \int_{s_1}^{s_2} E_\varepsilon(b^r, \{u_0 > b^r > u_1\}) \, dr \\
&= \int_{s_1}^{s_2} \int_{-a}^a \left[\frac{\varepsilon}{2} h'(t)^2 + \frac{1}{\varepsilon} W(h(t)) \right] \mathcal{H}^n(\{d_p = r + t\} \cap \{u_0 > h(t) > u_1\}) \, dt \, dr \\
&\leq \int_{-a}^a e_\varepsilon(h)(t) \mathcal{H}^{n+1}(B(p, s_2 + a) \cap \{u_0 > h(t) > u_1\}) \, dt \\
&= \int_{-a}^a e_\varepsilon(h)(t) \int_{B(p, s_2 + a)} \chi_{\{u_0 > h(t) > u_1\}}(x) \, d\mathcal{H}^{n+1}(x) \, dt \\
&= \int_{B(p, s_2 + a)} \int_{\{u_0(x) > h > u_1(x)\}} e_\varepsilon(h)(t) \, dt \, d\mathcal{H}^{n+1}(x).
\end{aligned}$$

□

For $\rho > 0$, let $h^\rho : \mathbb{R} \rightarrow [-1, 1]$ be defined by

$$h^\rho(t) = \begin{cases} \frac{t}{\rho} & \text{if } |t| \leq \rho; \\ 1 & \text{if } t \geq \rho; \\ -1 & \text{if } t \leq -\rho. \end{cases} \quad (6.3.10)$$

Setting $h = h^\rho$ in Lemma 6.3.7 and using the notation $b^{\rho, r}(x) = h^\rho(d_p(x) - r)$, one obtains

$$\int_{s_1}^{s_2} E_\varepsilon(b^{\rho, r}, \{u_0 > b^{\rho, r} > u_1\}) \, dr \leq C(\varepsilon, \rho) \|u_0 - u_1\|_{L^1(N)}, \quad (6.3.11)$$

where u_0, u_1 are as in Lemma 6.3.7 and

$$C(\varepsilon, \rho) = \frac{\varepsilon}{2\rho^2} + \frac{1}{\varepsilon} \|W\|_{L^\infty([-1, 1])}. \quad (6.3.12)$$

Lemma 6.3.8. *Let $\varepsilon, \delta > 0$. Suppose $u_0, u_1 \in H^1(N) \cap L^\infty(N)$ such that $u_0 \geq u_1$; $|u_0|, |u_1| \leq 1$ and*

$$\int_N |e_\varepsilon(u_0) - e_\varepsilon(u_1)| \leq \delta. \quad (6.3.13)$$

We fix a nested map $w : [0, 1] \rightarrow H^1(N)$ such that $w(0) \equiv 1$, $w(1) \equiv -1$ and $|w(t)| \leq 1$ for all $t \in [0, 1]$. Let $R > 0$ be such that for all $p \in N$,

$$E_\varepsilon(u_0, B(p, 4R)) \leq \delta, \quad E_\varepsilon(u_1, B(p, 4R)) \leq \delta \quad \text{and} \quad \sup_{t \in [0, 1]} E_\varepsilon(w(t), B(p, 4R)) \leq \delta. \quad (6.3.14)$$

Suppose N can be covered by I balls of radius R . Then, using the notation of (6.3.12),

$$\|u_0 - u_1\|_{L^1(N)} \leq \frac{\delta R}{C(\varepsilon, R)I} \quad (6.3.15)$$

implies that there exists a nested map $u : [0, 1] \rightarrow H^1(N)$ such that $u(0) = u_0$, $u(1) = u_1$, $|u(t)| \leq 1$ for all $t \in [0, 1]$ and

$$\sup_{t \in [0, 1]} E_\varepsilon(u(t)) \leq \min\{E_\varepsilon(u_0), E_\varepsilon(u_1)\} + 9\delta.$$

Proof. Let

$$N = \bigcup_{i=1}^I B(p_i, R). \quad (6.3.16)$$

For $r > 0$ and $i = 1, 2, \dots, I$, let $b_i^r : N \rightarrow [-1, 1]$ be defined by (using the notation as in (6.3.10))

$$b_i^r(x) = h^R(d_{p_i}(x) - r).$$

We inductively define a sequence $\{v_i\}_{i=0}^I$ with

$$v_0 \geq v_1 \geq \dots \geq v_I$$

as follows. Set $v_0 = u_0$. Let us assume that v_k has been defined for $1 \leq k \leq i-1$ and

$$u_0 = v_0 \geq v_1 \geq \dots \geq v_{i-1} \geq u_1,$$

which implies that

$$\|v_{i-1} - u_1\|_{L^1(N)} \leq \|u_0 - u_1\|_{L^1(N)}. \quad (6.3.17)$$

Therefore, using (6.3.11), (6.3.17) and (6.3.15),

$$\int_{2R}^{3R} E_\varepsilon(b_i^r, \{v_{i-1} > b_i^r > u_1\}) \, dr \leq \frac{\delta R}{I}.$$

So there exists $r_i \in (2R, 3R)$ such that

$$E_\varepsilon(b_i^{r_i}, \{v_{i-1} > b_i^{r_i} > u_1\}) \leq \frac{\delta}{I}. \quad (6.3.18)$$

We define

$$v_i = \min\{v_{i-1}, \max\{u_1, b_i^{r_i}\}\} = \begin{cases} v_{i-1} & \text{on } \{b_i^{r_i} \geq v_{i-1}\}; \\ b_i^{r_i} & \text{on } \{v_{i-1} > b_i^{r_i} > u_1\}; \\ u_1 & \text{on } \{u_1 \geq b_i^{r_i}\}. \end{cases} \quad (6.3.19)$$

Then $v_{i-1} \geq v_i \geq u_1$.

Using the definition of v_i in (6.3.19), one can prove by induction that for each $1 \leq i \leq I$, there exist pairwise disjoint, $\mathcal{H}^{n+1}$ -measurable sets $G_i^0, G_i^1, \{G_{k,i}\}_{k=1}^i$ with

$$N = G_i^0 \cup G_i^1 \cup \left(\bigcup_{k=1}^i G_{k,i} \right) \quad (6.3.20)$$

such that the following conditions are satisfied.

(i)

$$v_i = \begin{cases} u_0 & \text{on } G_i^0; \\ u_1 & \text{on } G_i^1; \\ b_k^{r_k} & \text{on } G_{k,i}, \, 1 \leq k \leq i. \end{cases} \quad (6.3.21)$$

(ii)

$$G_{i,i} = \{v_{i-1} > b_i^{r_i} > u_1\}. \quad (6.3.22)$$

(iii) For $1 < i \leq I$ and $1 \leq k \leq i-1$,

$$G_{i-1}^0 \supset G_i^0; \quad G_{i-1}^1 \subset G_i^1; \quad G_{k,i-1} \supset G_{k,i}. \quad (6.3.23)$$

(iv)

$$\bigcup_{k=1}^i B(p_k, R) \subset G_i^1 \quad (6.3.24)$$

(For this item one needs to use the fact that $b_k^{r_k} \equiv -1$ on $B(p_k, R)$.)

(6.3.24) implies that (by (6.3.16)) $v_I = u_1$. Moreover,

$$\begin{aligned} E_\varepsilon(v_i) &= E_\varepsilon(u_0, G_i^0) + E_\varepsilon(u_1, G_i^1) + \sum_{k=1}^i E_\varepsilon(b_k^{r_k}, G_{k,i}) \text{ (by (6.3.20), (6.3.21))} \\ &\leq \min\{E_\varepsilon(u_0, N), E_\varepsilon(u_1, N)\} + \int_N |e_\varepsilon(u_0) - e_\varepsilon(u_1)| + \sum_{k=1}^i E_\varepsilon(b_k^{r_k}, G_{k,k}) \text{ (by (6.3.23))} \\ &\leq \min\{E_\varepsilon(u_0), E_\varepsilon(u_1)\} + 2\delta \text{ (by (6.3.13), (6.3.22), (6.3.18)).} \end{aligned} \quad (6.3.25)$$

Similarly, using (6.3.20) and (6.3.21), for any $p \in N$,

$$\begin{aligned} &E_\varepsilon(v_i, B(p, 4R)) \\ &= E_\varepsilon(u_0, G_i^0 \cap B(p, 4R)) + E_\varepsilon(u_1, G_i^1 \cap B(p, 4R)) + \sum_{k=1}^i E_\varepsilon(b_k^{r_k}, G_{k,i} \cap B(p, 4R)) \\ &\leq 3\delta \text{ (by (6.3.14), (6.3.23), (6.3.22), (6.3.18)).} \end{aligned} \quad (6.3.26)$$

For each $1 \leq i \leq I$, we define $\beta_i : [0, 1] \rightarrow H^1(N)$ by

$$\beta_i(t) = \min\{v_{i-1}, \max\{v_i, w(t)\}\} = \begin{cases} v_{i-1} & \text{on } \{w(t) \geq v_{i-1}\}; \\ w(t) & \text{on } \{v_{i-1} > w(t) > v_i\}; \\ v_i & \text{on } \{v_i \geq w(t)\}. \end{cases}$$

Here $w : [0, 1] \rightarrow H^1(N)$ is as stated in the Lemma 6.3.8. Since w is nested, β_i is also nested. It

follows from (6.3.19) that $|v_k| \leq 1$ for all $0 \leq k \leq I$. Hence $|\beta_i(t)| \leq 1$ for all $t \in [0, 1]$. Moreover, $w(0) \equiv 1$ (resp. $w(1) \equiv -1$) implies that $\beta_i(0) = v_{i-1}$ (resp. $\beta_i(1) = v_i$). Using the fact that $b_i^{r_i} \equiv 1$ on $N \setminus B(p_i, 4R)$, it also follows from (6.3.19) that $v_{i-1} = v_i$ on $N \setminus B(p_i, 4R)$; hence for all $t \in [0, 1]$,

$$\beta_i(t) = v_{i-1} = v_i \text{ on } N \setminus B(p_i, 4R).$$

Therefore, for all $t \in [0, 1]$,

$$\begin{aligned} E_\varepsilon(\beta_i(t)) &\leq E_\varepsilon(v_i, N) + E_\varepsilon(v_{i-1}, B(p_i, 4R)) + E_\varepsilon(v_i, B(p_i, 4R)) + E_\varepsilon(w(t), B(p_i, 4R)) \\ &\leq \min\{E_\varepsilon(u_0), E_\varepsilon(u_1)\} + 9\delta \text{ (by (6.3.25), (6.3.26), (6.3.14)).} \end{aligned}$$

Finally we obtain the required map $u : [0, 1] \rightarrow H^1(N)$ by concatenating all the maps β_i , $i = 1, 2, \dots, I$. $\square$

Proof of Proposition 6.3.5. We fix a nested map $w_0 : [0, 1] \rightarrow H^1(N)$ such that $w_0(0) \equiv 1$, $w_0(1) \equiv -1$ and $|w_0(t)| \leq 1$ for all $t \in [0, 1]$. (For instance, one can define $w_0(t)$ to be equal to the constant function $1 - 2t$.) Let $\delta_0 = \kappa/9$ and

$$A_0 = \sup_{t \in [0, 1]} E_\varepsilon(\phi(t)).$$

By Lemma 6.3.6, there exists $R_0 > 0$ such that for all $p \in N$ and $t \in [0, 1]$,

$$E_\varepsilon(\phi(t), B(p, 4R_0)) \leq \frac{\delta_0}{2} \quad \text{and} \quad E_\varepsilon(w_0(t), B(p, 4R_0)) \leq \delta_0. \quad (6.3.27)$$

Suppose N can be covered by I_0 balls of radius R_0 . One can choose $m \in \mathbb{N}$ such that $|t_1 - t_2| \leq 1/m$ implies

$$\int_N |e_\varepsilon(\phi(t_1)) - e_\varepsilon(\phi(t_2))| \leq \frac{\delta_0}{2} \quad (6.3.28)$$

and

$$\|\phi(t_1) - \phi(t_2)\|_{L^1(N)} \leq \frac{\delta_0 R_0}{C(\varepsilon, R_0) I_0}. \quad (6.3.29)$$

We define the sequence $\{\hat{\phi}_i\}_{i=0}^{2m}$ by setting $\hat{\phi}_{2k} = \phi(k/m)$ and

$$\hat{\phi}_{2k+1} = \min\{\hat{\phi}_{2k}, \hat{\phi}_{2k+2}\} = \begin{cases} \hat{\phi}_{2k} & \text{on } \{\hat{\phi}_{2k} \leq \hat{\phi}_{2k+2}\}; \\ \hat{\phi}_{2k+2} & \text{on } \{\hat{\phi}_{2k} > \hat{\phi}_{2k+2}\}. \end{cases}$$

Hence, (6.3.28) implies that for $0 \leq k \leq m-1$,

$$\int_N |e_\varepsilon(\hat{\phi}_{2k}) - e_\varepsilon(\hat{\phi}_{2k+1})| = \int_{\{\hat{\phi}_{2k} > \hat{\phi}_{2k+2}\}} |e_\varepsilon(\hat{\phi}_{2k}) - e_\varepsilon(\hat{\phi}_{2k+2})| \leq \frac{\delta_0}{2}. \quad (6.3.30)$$

Similarly, (6.3.29) implies that for $0 \leq k \leq m-1$,

$$\|\hat{\phi}_{2k} - \hat{\phi}_{2k+1}\|_{L^1(N)} \leq \frac{\delta_0 R_0}{C(\varepsilon, R_0) I_0}.$$

By (6.3.27) and (6.3.30), for all $0 \leq i \leq 2m$ and $p \in N$,

$$E_\varepsilon(\hat{\phi}_i, B(p, 4R_0)) \leq \delta_0.$$

By Lemma 6.3.8, for $0 \leq i \leq m-1$, there exists a nested map $\gamma_i : [0, 1] \rightarrow H^1(N)$ such that $\gamma_i(0) = \hat{\phi}_{2i}$, $\gamma_i(1) = \hat{\phi}_{2i+1}$, $|\gamma_i(t)| \leq 1$ for all $t \in [0, 1]$ and

$$\sup_{t \in [0, 1]} E_\varepsilon(\gamma_i(t)) \leq E_\varepsilon(\hat{\phi}_{2i}) + \kappa \leq A_0 + \kappa.$$

Therefore,

$$\gamma_i(1) = \hat{\phi}_{2i+1} \leq \hat{\phi}_{2i+2} = \gamma_{i+1}(0).$$

One obtains the map ψ in Proposition 6.3.5 from the maps $\{\gamma_i\}_{i=0}^{m-1}$ by repeatedly applying Lemma 6.3.4. More precisely, setting $u_1 = \gamma_0$ and $u_2 = \gamma_1$ in Lemma 6.3.4, we get a nested map $\bar{\gamma}_1 : [0, 1] \rightarrow H^1(N)$ such that

$$\bar{\gamma}_1(0) \geq \hat{\phi}_0, \bar{\gamma}_1(1) \leq \hat{\phi}_3, \sup_{t \in [0, 1]} \|\bar{\gamma}_1(t)\|_{L^\infty(N)} \leq 1 \text{ and } \sup_{t \in [0, 1]} E_\varepsilon(\bar{\gamma}_1(t)) \leq A_0 + \kappa.$$

Let us assume that there exists a nested map $\bar{\gamma}_i : [0, 1] \rightarrow H^1(N)$, $1 \leq i < m - 1$, such that

$$\bar{\gamma}_i(0) \geq \hat{\phi}_0, \quad \bar{\gamma}_i(1) \leq \hat{\phi}_{2i+1} \leq \hat{\phi}_{2i+2} = \gamma_{i+1}(0), \quad (6.3.31)$$

$$\sup_{t \in [0, 1]} \|\bar{\gamma}_i(t)\|_{L^\infty(N)} \leq 1 \quad \text{and} \quad \sup_{t \in [0, 1]} E_\varepsilon(\bar{\gamma}_i(t)) \leq A_0 + \kappa. \quad (6.3.32)$$

Then choosing $u_1 = \bar{\gamma}_i$ and $u_2 = \gamma_{i+1}$ in Lemma 6.3.4, one gets a nested map $\bar{\gamma}_{i+1} : [0, 1] \rightarrow H^1(N)$ such that

$$\bar{\gamma}_{i+1}(0) \geq \hat{\phi}_0, \quad \bar{\gamma}_{i+1}(1) \leq \hat{\phi}_{2i+3}, \quad \sup_{t \in [0, 1]} \|\bar{\gamma}_{i+1}(t)\|_{L^\infty(N)} \leq 1 \quad \text{and} \quad \sup_{t \in [0, 1]} E_\varepsilon(\bar{\gamma}_{i+1}(t)) \leq A_0 + \kappa.$$

The map ψ in Proposition 6.3.5 is obtained by setting $\psi = \bar{\gamma}_{m-1}$. □

6.4 A deformation lemma

The following Lemma 6.4.1 is motivated by [CL20, Lemma 7.1 (3)]. To prove this lemma, we adapt the argument of Chambers and Liokumovich [CL20, Proof of Lemma 7.1] in the Allen-Cahn setting.

Lemma 6.4.1. *Let N be a closed Riemannian manifold and $\Omega \subset N$ be an open set with smooth boundary $\partial\Omega$. Suppose $f : N \rightarrow [1/3, \infty)$ is a Morse function so that in the interval $[1/3, 2/3]$, f has no critical value which is a non-global local maxima or minima;*

$$\min_N f = 1/3; \quad \max_N f > 1; \quad \Omega \subset\subset f^{-1}([1/3, 2/3]).$$

We set

$$\tilde{\Omega} = f^{-1}([1/3, 1]).$$

Then, for all $\eta > 0$, there exist $\varepsilon_1, \tilde{\eta} > 0$, depending on $\eta, \Omega, \tilde{\Omega}, f|_{\tilde{\Omega}}$, such that the following two conditions are satisfied.

(i) If $0 < \varepsilon \leq \varepsilon_1$ and $u_0 \in H^1(N)$ satisfies $|u_0| \leq 1$, $\|1 - u_0\|_{L^1(\Omega)} \leq \tilde{\eta}$, then there exists $u :$

$[0, 1] \rightarrow H^1(N)$ such that $u(0) = u_0$, $u(1)|_{\Omega} \equiv 1$ and

$$\sup_{t \in [0, 1]} E_{\varepsilon}(u(t)) \leq E_{\varepsilon}(u_0) + 2\sigma \mathcal{H}^n(\partial\Omega) + \eta.$$

(ii) If $0 < \varepsilon \leq \varepsilon_1$ and $u_0 \in H^1(N)$ satisfies $|u_0| \leq 1$, $\|1 + u_0\|_{L^1(\Omega)} \leq \tilde{\eta}$, then there exists $u : [0, 1] \rightarrow H^1(N)$ such that $u(0) = u_0$, $u(1)|_{\Omega} \equiv -1$ and

$$\sup_{t \in [0, 1]} E_{\varepsilon}(u(t)) \leq E_{\varepsilon}(u_0) + 2\sigma \mathcal{H}^n(\partial\Omega) + \eta.$$

Proof. Let $q : \mathbb{R} \rightarrow \mathbb{R}$ be the unique solution of the following ODE.

$$\varphi'(t) = \sqrt{2W(\varphi(t))}; \quad \varphi(0) = 0. \quad (6.4.1)$$

For all $t \in \mathbb{R}$, $-1 < q(t) < 1$ and

$$\text{as } t \rightarrow \pm\infty, (q(t) \mp 1) \text{ converges to zero exponentially fast.} \quad (6.4.2)$$

$q_{\varepsilon}(t) = q(t/\varepsilon)$ is a solution of the one dimensional Allen-Cahn equation

$$\varepsilon^2 \varphi''(t) = W'(\varphi(t))$$

with finite total energy:

$$\int_{-\infty}^{\infty} \left[\frac{\varepsilon}{2} (q'_{\varepsilon}(t))^2 + \frac{1}{\varepsilon} W(q_{\varepsilon}(t)) \right] dt = 2\sigma. \quad (6.4.3)$$

For $\varepsilon > 0$, we define Lipschitz continuous function

$$\hat{q}_\varepsilon(t) = \begin{cases} q_\varepsilon(t) & \text{if } |t| \leq \sqrt{\varepsilon}; \\ q_\varepsilon(\sqrt{\varepsilon}) + \left(\frac{t}{\sqrt{\varepsilon}} - 1\right)(1 - q_\varepsilon(\sqrt{\varepsilon})) & \text{if } \sqrt{\varepsilon} \leq t \leq 2\sqrt{\varepsilon}; \\ 1 & \text{if } t \geq 2\sqrt{\varepsilon}; \\ q_\varepsilon(-\sqrt{\varepsilon}) + \left(\frac{t}{\sqrt{\varepsilon}} + 1\right)(1 + q_\varepsilon(-\sqrt{\varepsilon})) & \text{if } -2\sqrt{\varepsilon} \leq t \leq -\sqrt{\varepsilon}; \\ -1 & \text{if } t \leq -2\sqrt{\varepsilon}. \end{cases} \quad (6.4.4)$$

For $t \in \mathbb{R}$ and $x \in N$, we set

$$d_1^t(x) = d_{\partial\Omega}(x) - t, \quad \text{where} \quad d_{\partial\Omega}(x) = \begin{cases} -d(x, \partial\Omega) & \text{if } x \in \Omega; \\ d(x, \partial\Omega) & \text{if } x \notin \Omega. \end{cases} \quad (6.4.5)$$

For $t \in [1/3, 1]$ and $x \in N$, we set

$$d_2^t(x) = \begin{cases} -d(x, f^{-1}(t)) & \text{if } f(x) \leq t; \\ d(x, f^{-1}(t)) & \text{if } f(x) \geq t. \end{cases} \quad (6.4.6)$$

Following [Gua18, Section 7 and Section 9], we define the continuous maps $w_{1,\varepsilon} : \mathbb{R} \rightarrow H^1(N)$ and $w_{2,\varepsilon} : [0, 1] \rightarrow H^1(N)$ by

$$w_{1,\varepsilon}(t) = \hat{q}_\varepsilon \circ d_1^t; \quad (6.4.7)$$

$$w_{2,\varepsilon}(t) = \begin{cases} \hat{q}_\varepsilon \circ d_2^t & \text{if } \frac{1}{3} \leq t \leq \frac{2}{3}; \\ 1 - 3t(1 - w_{2,\varepsilon}(1/3)) & \text{if } 0 \leq t \leq \frac{1}{3}; \\ -1 + 3(1 - t)(1 + w_{2,\varepsilon}(2/3)) & \text{if } \frac{2}{3} \leq t \leq 1. \end{cases} \quad (6.4.8)$$

Since in the interval $[1/3, 2/3]$, f has no critical value which is a non-global local maxima or minima, $t \mapsto f^{-1}(t)$ is continuous on $[1/3, 2/3]$ in the Hausdorff topology. This implies that $w_{2,\varepsilon}$ is continuous (see [Gua18, Proposition 9.2]).

Let us fix $\eta > 0$. From the argument in [Gua18, Section 9], it follows that there exist $\varepsilon', t_0 > 0$, depending on η , $\partial\Omega$ and $\tilde{\Omega}$, such that if $0 < \varepsilon \leq \varepsilon'$ and $|t| \leq 2t_0$ then

$$E_\varepsilon(w_{1,\varepsilon}(t)) \leq 2\sigma\mathcal{H}^n(\partial\Omega) + \frac{\eta}{2}. \quad (6.4.9)$$

By the “no concentration of mass” property ([MN17, Lemma 5.2]), there exists $0 < R < t_0/5$, depending on η , Ω , $\tilde{\Omega}$ and $f|_{\tilde{\Omega}}$, such that

$$\mathcal{H}^n(f^{-1}(t) \cap B(p, 5R)) \leq \frac{1}{2\sigma} \frac{\eta}{3}, \quad (6.4.10)$$

for all $t \in [1/3, 2/3]$ and $B(p, 5R) \subset \Omega$. Moreover, $w_{2,\varepsilon}(1/3) > 0$ on N and $w_{2,\varepsilon}(2/3) < 0$ on Ω . As a consequence, by the results in [Gua18, Section 9], there exists $0 < \varepsilon'' \leq R^2/4$, depending on Ω , $\tilde{\Omega}$ and $f|_{\tilde{\Omega}}$ such that

$$E_\varepsilon(w_{2,\varepsilon}(t), B(p, 4R)) \leq \frac{\eta}{2}, \quad (6.4.11)$$

for all $t \in [0, 1]$ and $B(p, 5R) \subset \Omega$. We define $\varepsilon_1 = \min\{\varepsilon', \varepsilon''\}$. By our definitions of R and ε'' ,

$$2\sqrt{\varepsilon_1} \leq R < \frac{t_0}{5}. \quad (6.4.12)$$

Let us fix $\varepsilon \in (0, \varepsilon_1]$. Using the notation of (6.4.5), let

$$\Omega_r = \{x \in N : d_{\partial\Omega}(x) \leq r\}. \quad (6.4.13)$$

For $r > 0$ and for a fixed $p \in \Omega_{-t_0}$, we define $\omega_\varepsilon^r : N \rightarrow \mathbb{R}$ by

$$\omega_\varepsilon^r(x) = \hat{q}_\varepsilon(r - d_p(x)),$$

where $d_p(x) = d(x, p)$.

By (6.4.12), $B(p, 3R + 2\sqrt{\varepsilon_1}) \subset \Omega$. Therefore, it follows from Lemma 6.3.7 that for $u \in L^\infty(N)$

with $|u| \leq 1$,

$$\int_{2R}^{3R} E_\varepsilon(\omega_\varepsilon^r, \{1 > \omega_\varepsilon^r > u\}) dr \quad (6.4.14)$$

$$\leq \int_\Omega \int_{\{1 > \hat{q}_\varepsilon > u(x)\}} e_\varepsilon(\hat{q}_\varepsilon)(t) dt d\mathcal{H}^{n+1}(x) \quad (\text{since } \hat{q}_\varepsilon \text{ is an odd function}) \quad (6.4.15)$$

$$= \int_\Omega \int_{u(x)}^1 \left[\frac{\varepsilon}{2} \hat{q}'_\varepsilon(\hat{q}_\varepsilon^{-1}(s)) + \frac{1}{\varepsilon} \frac{W(s)}{\hat{q}'_\varepsilon(\hat{q}_\varepsilon^{-1}(s))} \right] ds d\mathcal{H}^{n+1}(x). \quad (6.4.16)$$

In (6.4.16), $\hat{q}_\varepsilon$ is thought of as a bijective map from $[-2\sqrt{\varepsilon}, 2\sqrt{\varepsilon}]$ to $[-1, 1]$. We claim that there exists $C_0 = C_0(W, \varepsilon_1) > 0$ such that the $L^\infty([-1, 1])$ norm of the integrand in (6.4.16) is bounded by C_0 . (In particular, C_0 does not depend on ε .) Indeed, the integrand is a non-negative, even function. If $0 \leq t < \sqrt{\varepsilon}$, then

$$\frac{\varepsilon}{2} \hat{q}'_\varepsilon(t) + \frac{1}{\varepsilon} \frac{W(\hat{q}_\varepsilon(t))}{\hat{q}'_\varepsilon(t)} \quad (6.4.17)$$

$$= \frac{1}{2} q'(t/\varepsilon) + \frac{W(q(t/\varepsilon))}{q'(t/\varepsilon)} \quad (6.4.18)$$

$$= \sqrt{2W(q(t/\varepsilon))} \quad (\text{by (6.4.1)}) \quad (6.4.19)$$

$$\leq \sqrt{2} \|W\|_{L^\infty([-1, 1])}. \quad (6.4.20)$$

If $\sqrt{\varepsilon} < t < 2\sqrt{\varepsilon}$, using the fact that

$$\sup_{t \in [-1, 1]} \frac{W(t)}{(1-t)^2} = C_1 < \infty,$$

we obtain

$$\frac{\varepsilon}{2} \hat{q}'_\varepsilon(t) + \frac{1}{\varepsilon} \frac{W(\hat{q}_\varepsilon(t))}{\hat{q}'_\varepsilon(t)} \quad (6.4.21)$$

$$\leq \frac{\sqrt{\varepsilon}}{2} \left(1 - q(\varepsilon^{-1/2}) \right) + \frac{1}{\sqrt{\varepsilon}} \frac{W(q(\varepsilon^{-1/2}))}{(1 - q(\varepsilon^{-1/2}))} \quad (6.4.22)$$

$$\leq \left(\frac{\sqrt{\varepsilon}}{2} + \frac{C_1}{\sqrt{\varepsilon}} \right) \left(1 - q(\varepsilon^{-1/2}) \right). \quad (6.4.23)$$

By (6.4.2), the expression in (6.4.23) is bounded by some constant $C_2 = C_2(\varepsilon_1, C_1)$. Thus our claim follows from (6.4.20) and (6.4.23). (6.4.16), together with the claim, implies that

$$\int_{2R}^{3R} E_\varepsilon(\omega_\varepsilon^r, \{1 > \omega_\varepsilon^r > u\}) dr \leq C_0 \|1 - u\|_{L^1(\Omega)}. \quad (6.4.24)$$

We choose a covering

$$\Omega_{-t_0} = \bigcup_{i=1}^I B(p_i, R); \quad (6.4.25)$$

each $p_i \in \Omega_{-t_0}$ so that (by (6.4.12)) $B(p_i, 5R) \subset \Omega$. To prove part (i) of Lemma 6.4.1, we set

$$\tilde{\eta} = \frac{\eta R}{2C_0 I}, \quad (6.4.26)$$

where C_0 is as in the above claim. Let u_0 be as in the statement of Lemma 6.4.1, part (i). We inductively define a sequence $\{v_i\}_{i=0}^I$,

$$-1 \leq v_0 \leq v_1 \leq \cdots \leq v_I \leq 1,$$

as follows. Set $v_0 = u_0$. Suppose v_k has been defined for $0 \leq k \leq i-1$ so that

$$u_0 = v_0 \leq v_1 \leq \cdots \leq v_{i-1} \leq 1;$$

hence

$$\|1 - v_{i-1}\|_{L^1(\Omega)} \leq \|1 - u_0\|_{L^1(\Omega)} \leq \tilde{\eta}. \quad (6.4.27)$$

Let $\omega_{i,\varepsilon}^r(x) = \hat{q}_\varepsilon(r - d_{p_i}(x))$. By (6.4.24), (6.4.27) and (6.4.26), there exists $r_i \in (2R, 3R)$ such that

$$E_\varepsilon(\omega_{i,\varepsilon}^{r_i}, \{v_{i-1} < \omega_{i,\varepsilon}^{r_i} < 1\}) \leq \frac{\eta}{2I}. \quad (6.4.28)$$

We define

$$v_i = \max\{v_{i-1}, \omega_{i,\varepsilon}^{r_i}\} = \begin{cases} v_{i-1} & \text{on } \{v_{i-1} \geq \omega_{i,\varepsilon}^{r_i}\}; \\ \omega_{i,\varepsilon}^{r_i} & \text{on } \{v_{i-1} < \omega_{i,\varepsilon}^{r_i}\}. \end{cases} \quad (6.4.29)$$

It follows from (6.4.29) that $v_{i-1} \leq v_i \leq 1$. Moreover, since $\omega_{i,\varepsilon}^{r_i} \equiv -1$ on $N \setminus B(p_i, 4R)$,

$$v_i = v_{i-1} \text{ on } N \setminus B(p_i, 4R). \quad (6.4.30)$$

Thus we have obtained the sequence $\{v_i\}_{i=0}^I$ with $v_0 = u_0$ and we set $v_I = \bar{u}$. As $\omega_{i,\varepsilon}^{r_i} \equiv 1$ on $B(p_i, R)$, using (6.4.29) and (6.4.25), one can prove by induction that

$$\bar{u}|_{\Omega_{-t_0}} \equiv 1. \quad (6.4.31)$$

By (6.4.29) and (6.4.28),

$$E_\varepsilon(v_i, \{v_i \neq v_{i-1}\}) = E_\varepsilon(\omega_{i,\varepsilon}^{r_i}, \{v_{i-1} < \omega_{i,\varepsilon}^{r_i} < 1\}) \leq \frac{\eta}{2I}. \quad (6.4.32)$$

Thus

$$E_\varepsilon(v_i) = E_\varepsilon(v_{i-1}) + E_\varepsilon(v_i, \{v_i \neq v_{i-1}\}) \leq E_\varepsilon(v_{i-1}) + \frac{\eta}{2I} \quad (6.4.33)$$

$$\implies E_\varepsilon(v_i) \leq E_\varepsilon(u_0) + \frac{\eta i}{2I} \quad \forall 0 \leq i \leq I. \quad (6.4.34)$$

For $1 \leq i \leq I$, let $\beta_i : [0, 1] \rightarrow H^1(N)$ be defined by

$$\beta_i(t) = \max\{v_{i-1}, \min\{v_i, -w_{2,\varepsilon}(t)\}\} = \begin{cases} v_{i-1} & \text{on } \{-w_{2,\varepsilon}(t) \leq v_{i-1}\}; \\ -w_{2,\varepsilon}(t) & \text{on } \{v_{i-1} < -w_{2,\varepsilon}(t) < v_i\}; \\ v_i & \text{on } \{v_i \leq -w_{2,\varepsilon}(t)\}. \end{cases} \quad (6.4.35)$$

Since $w_{2,\varepsilon}(0) \equiv 1$ and $w_{2,\varepsilon}(1) \equiv -1$, $\beta_i(0) = v_{i-1}$ and $\beta_i(1) = v_i$. Moreover, by (6.4.30),

$$\beta_i(t) = v_{i-1} = v_i \text{ on } N \setminus B(p_i, 4R). \quad (6.4.36)$$

Therefore, using (6.4.35), (6.4.34), (6.4.32) and (6.4.11), one obtains

$$E_\varepsilon(\beta_i(t)) \leq E_\varepsilon(v_{i-1}) + E_\varepsilon(v_i, \{v_i \neq v_{i-1}\}) + E_\varepsilon(-w_{2,\varepsilon}(t), B(p_i, 4R)) \quad (6.4.37)$$

$$\leq E_\varepsilon(u_0) + \frac{\eta(i-1)}{2I} + \frac{\eta}{2I} + \frac{\eta}{2} \quad (6.4.38)$$

$$\leq E_\varepsilon(u_0) + \eta. \quad (6.4.39)$$

Concatenating all the β_i 's we get a map $\beta' : [0, 1] \rightarrow H^1(N)$ such that $\beta'(0) = u_0$, $\beta'(1) = \bar{u} = v_I$ and

$$\sup_{t \in [0, 1]} E_\varepsilon(\beta'(t)) \leq E_\varepsilon(u_0) + \eta. \quad (6.4.40)$$

Let $\beta'' : [-2t_0, t_0] \rightarrow H^1(N)$ be defined by

$$\beta''(t) = \max\{\bar{u}, -w_{1,\varepsilon}(t)\}. \quad (6.4.41)$$

Then $\beta''(-2t_0) = \bar{u}$ as $\bar{u} \equiv 1$ on Ω_{-t_0} ((6.4.31)) and $w_{1,\varepsilon}(-2t_0) \equiv 1$ on $N \setminus \Omega_{-t_0}$ (by (6.4.12)). Moreover, $\beta''(t_0)|_\Omega \equiv 1$ as by (6.4.12), $w_{1,\varepsilon}(t_0)|_\Omega \equiv -1$. Using (6.4.9) and (6.4.34), we conclude that for all $t \in [-2t_0, t_0]$,

$$E_\varepsilon(\beta''(t)) \leq E_\varepsilon(\bar{u}) + E_\varepsilon(-w_{1,\varepsilon}(t)) \leq E_\varepsilon(u_0) + 2\sigma\mathcal{H}^n(\partial\Omega) + \eta.$$

Finally, the map u in Lemma 6.4.1 part (i) is obtained by concatenating β' and β'' . This finishes the proof of part (i) of Lemma 6.4.1; part (ii) of the lemma can be deduced from part (i) by replacing u_0 by $-u_0$. $\square$

The next lemma is motivated by the properties of the isoperimetric profile of a compact Riemannian manifold (see [CL20, Lemma 7.1 (2)]).

Lemma 6.4.2. *Let Ω be a compact Riemannian manifold (not necessarily closed). For all $\eta_1 > 0$, there exist $\varepsilon_2, \eta_2 > 0$, depending on Ω and η_1 , such that the following holds. If $0 < \varepsilon \leq \varepsilon_2$ and $u \in H^1(\Omega)$ satisfies $\|u\|_{L^\infty(\Omega)} \leq 1$ and $\min\{\|1 - u\|_{L^1(\Omega)}, \|1 + u\|_{L^1(\Omega)}\} > \eta_1$, then $E_\varepsilon(u, \Omega) > \eta_2$.*

Proof. We assume by contradiction that there exist sequences $\{u_i\}_{i=1}^\infty \subset H^1(\Omega)$ with $|u_i| \leq 1$ for all i and $\{\alpha_i\}_{i=1}^\infty \subset (0, \infty)$ with $\alpha_i \rightarrow 0$ such that

$$\min\{\|1 - u_i\|_{L^1(\Omega)}, \|1 + u_i\|_{L^1(\Omega)}\} > \eta_1 \quad \forall i \in \mathbb{N}, \quad (6.4.42)$$

and

$$E_{\alpha_i}(u_i, \Omega) \rightarrow 0. \quad (6.4.43)$$

Let $F : [-1, 1] \rightarrow [-\sigma/2, \sigma/2]$ be as defined in (6.2.1) and $v_i = F \circ u_i$. As argued in [HT00, Section 2.1], for all i , $|v_i| \leq \sigma/2$ and

$$\int_{\Omega} |\nabla v_i| \leq \frac{1}{2} E_{\alpha_i}(u_i, \Omega). \quad (6.4.44)$$

Therefore, there exists a subsequence $\{v_{i_k}\} \subset \{v_i\}$ and $v_\infty \in BV(\Omega)$ such that

$$v_{i_k} \rightarrow v_\infty \text{ in } L^1(\Omega) \text{ and pointwise a.e.} \quad (6.4.45)$$

and (using (6.4.43) and (6.4.44))

$$\int_{\Omega} |Dv_\infty| \leq \liminf_{k \rightarrow \infty} \int_{\Omega} |\nabla v_{i_k}| = 0. \quad (6.4.46)$$

Thus v_∞ is a constant function. Denoting $u_\infty = F^{-1}(v_\infty)$, by the dominated convergence theorem,

$$u_{i_k} \rightarrow u_\infty \text{ pointwise a.e. and in } L^1(\Omega). \quad (6.4.47)$$

Moreover, by (6.4.43),

$$\int_{\Omega} W(u_\infty) = \lim_{k \rightarrow \infty} \int_{\Omega} W(u_{i_k}) = 0.$$

Since u_∞ is a constant function, either $u_\infty \equiv 1$ or $u_\infty \equiv -1$. However, this contradicts the assumption (6.4.42). $\square$

6.5 Proof of Theorem 6.1.1

Let N be a closed Riemannian manifold and $\Omega \subset N$ be an open set with smooth boundary $\partial\Omega$. For $\varepsilon > 0$, the ε -Allen-Cahn width of Ω , which we denote by $\lambda_\varepsilon(\Omega)$, is defined as follows [Gua18]. Let $\mathcal{A}$ be the set of all continuous maps $\zeta : [0, 1] \rightarrow H^1(\Omega)$ such that $\zeta(0) \equiv 1$ and $\zeta(1) \equiv -1$. Then

$$\lambda_\varepsilon(\Omega) = \inf_{\zeta \in \mathcal{A}} \sup_{t \in [0, 1]} E_\varepsilon(\zeta(t), \Omega). \quad (6.5.1)$$

It follows from [Gua18, Section 8] that

$$\mathbb{W}(\Omega) \leq \frac{1}{2\sigma} \liminf_{\varepsilon \rightarrow 0^+} \lambda_\varepsilon(\Omega), \quad (6.5.2)$$

where $\mathbb{W}(\Omega)$ is as defined in (6.2.8). Motivated by [CL20, Section 2.2], we also make the following definition.

Definition 6.5.1. Let $\mathcal{B}$ be the set of all continuous maps $\zeta : [0, 1] \rightarrow H^1(N)$ such that $\zeta(0)|_\Omega \equiv 1$ and $\zeta(1)|_\Omega \equiv -1$. For $\varepsilon > 0$, we define

$$\tilde{\lambda}_\varepsilon(\Omega) = \inf_{\zeta \in \mathcal{B}} \sup_{t \in [0, 1]} E_\varepsilon(\zeta(t), N). \quad (6.5.3)$$

Given $\tilde{\zeta} \in \mathcal{B}$ one can define $\zeta \in \mathcal{A}$ by $\zeta(t) = \tilde{\zeta}(t)|_\Omega$; hence, for all $\varepsilon > 0$,

$$\lambda_\varepsilon(\Omega) \leq \tilde{\lambda}_\varepsilon(\Omega). \quad (6.5.4)$$

The following Proposition 6.5.2 is the Allen-Cahn analogue of [CL20, Proposition 2.1].

Proposition 6.5.2. *Let N be a closed Riemannian manifold and $\Omega \subset N$ be a good set (as defined in Section 6.2.4). Suppose $f : N \rightarrow [1/3, \infty)$ is a Morse function so that in the interval $[1/3, 2/3]$,*

f has no critical value which is a non-global local maxima or minima;

$$\min_N f = 1/3; \quad \max_N f > 1; \quad \Omega \subset\subset f^{-1}([1/3, 2/3]); \quad 1 \text{ is a regular value of } f.$$

We set

$$\tilde{\Omega} = f^{-1}([1/3, 1]).$$

Then there exist $\varepsilon^*, \eta^* > 0$, depending on Ω , $\tilde{\Omega}$ and $f|_{\tilde{\Omega}}$, such that for all $0 < \varepsilon \leq \varepsilon^*$ the following condition is satisfied. For every $\zeta \in \mathcal{B}$, there exists $t^0 \in [0, 1]$ such that

$$E_\varepsilon(\zeta(t^0)) \geq \tilde{\lambda}_\varepsilon(\Omega) \quad \text{and} \quad E_\varepsilon(\zeta(t^0), \Omega) \geq \eta^*. \quad (6.5.5)$$

Remark 6.5.3. The constants ε^* and η^* in the above Proposition 6.5.2 depend on the ambient Riemannian metric restricted to $\tilde{\Omega}$. (By our hypothesis, $\partial\tilde{\Omega}$ is smooth.) Let us fix a Riemannian metric g_0 on $\tilde{\Omega}$. If g' is an arbitrary Riemannian metric on N , from the proofs of Proposition 6.5.2, Lemmas 6.4.1 and 6.4.2, it follows that there exists $\varrho > 0$, depending on g_0 and $g'|_{\tilde{\Omega}}$, such that the following holds. One can choose ε^* and η^* in Proposition 6.5.2 in such a way that the proposition holds for all Riemannian metrics g'' on N satisfying

$$\|g'|_{\tilde{\Omega}} - g''|_{\tilde{\Omega}}\|_{C^2(\tilde{\Omega}, g_0)} < \varrho. \quad (6.5.6)$$

Proof of Proposition 6.5.2. The proof will be presented in four parts.

Part 1. Let

$$\tau = \frac{\sigma}{2} \mathcal{H}^n(\partial\Omega). \quad (6.5.7)$$

We set $\eta = \tau$ in Lemma 6.4.1 and choose $\varepsilon_1^* > 0$ and

$$0 < \tau_1 < \mathcal{H}^{n+1}(\Omega) \quad (6.5.8)$$

so that Lemma 6.4.1 holds for $\eta = \tau$, $\varepsilon_1 = \varepsilon_1^*$ and $\tilde{\eta} = 3\tau_1$. Next, we set $\eta_1 = \tau_1$ in Lemma 6.4.2

and choose $\varepsilon_2^* > 0$ and

$$0 < \tau_2 \leq \frac{\sigma}{12} \mathcal{H}^n(\partial\Omega) \quad (6.5.9)$$

so that Lemma 6.4.2 holds for $\eta_1 = \tau_1$, $\varepsilon_2 = \varepsilon_2^*$ and $\eta_2 = \tau_2$. Let us define $\eta^* = \tau_2$. We also define ε^* to be a positive real number so that the following conditions are satisfied.

- $\varepsilon^* \leq \min\{\varepsilon_1^*, \varepsilon_2^*\}$.

- Let

$$w_\varepsilon = w_{1,\varepsilon}(-2\sqrt{\varepsilon}), \quad (6.5.10)$$

where $w_{1,\varepsilon}$ is as defined in (6.4.7). Then, for all $0 < \varepsilon \leq \varepsilon^*$,

$$E_\varepsilon(w_\varepsilon) \leq 2\sigma \mathcal{H}^n(\partial\Omega) + \tau_2. \quad (6.5.11)$$

(For this item one needs to use (6.4.9).)

- For all $0 < \varepsilon \leq \varepsilon^*$, using the notation of (6.4.13),

$$\mathcal{H}^{n+1}(\Omega \setminus \Omega_{-4\sqrt{\varepsilon}}) \leq \frac{\tau_1}{2}. \quad (6.5.12)$$

- For all $0 < \varepsilon \leq \varepsilon^*$,

$$\frac{1}{2\sigma} \lambda_\varepsilon(\Omega) > \frac{7}{2} \mathcal{H}^n(\partial\Omega). \quad (6.5.13)$$

(For this item one needs (6.5.2) and the hypothesis (6.2.9) that Ω is a good set.)

We will show that Proposition 6.5.2 holds for the above choices of ε^* and η^* . Let us assume by contradiction that there exist $\alpha \in (0, \varepsilon^*]$ and $h \in \mathcal{B}$ such that

$$\text{for } t \in [0, 1], \text{ if } E_\alpha(h(t)) \geq \tilde{\lambda}_\alpha(\Omega) \text{ then } E_\alpha(h(t), \Omega) < \eta^* = \tau_2. \quad (6.5.14)$$

Without loss of generality, we can assume that

$$|h(t)| \leq 1 \quad \forall t \in [0, 1]. \quad (6.5.15)$$

Indeed, if

$$\hat{h}(t) = \min \{1, \max\{-1, h(t)\}\},$$

then for all $t \in [0, 1]$, $|\hat{h}(t)| \leq 1$ and $e_\varepsilon(\hat{h}(t)) \leq e_\varepsilon(h(t))$. Therefore,

$$E_\alpha(\hat{h}(t)) \geq \tilde{\lambda}_\alpha(\Omega) \implies E_\alpha(h(t)) \geq \tilde{\lambda}_\alpha(\Omega) \implies \eta^* > E_\alpha(h(t), \Omega) \geq E_\alpha(\hat{h}(t), \Omega).$$

To prove Proposition 6.5.2, we will show that the existence of such $h \in \mathcal{B}$ and $\alpha \in (0, \varepsilon^*]$ imply there exists $\gamma \in \mathcal{B}$ satisfying $\sup_{t \in [0, 1]} E_\alpha(\gamma(t)) < \tilde{\lambda}_\alpha(\Omega)$.

Part 2. Let h and α be as defined above in (6.5.14) and (6.5.15).

Lemma 6.5.4. *There exist $0 < a < b < 1$ such that the following conditions are satisfied.*

- $E_\alpha(h(a), \Omega) = \tau_2 = E_\alpha(h(b), \Omega)$.
- $\|1 - h(a)\|_{L^1(\Omega)} \leq \tau_1$ and $\|1 + h(b)\|_{L^1(\Omega)} \leq \tau_1$.
- $E_\alpha(h(t), \Omega) \geq \tau_2$ for all $t \in [a, b]$.

Proof. Let

$$S_1 = \left\{ t \in [0, 1] : E_\alpha(h(t), \Omega) \leq \tau_2 \text{ and } \|1 - h(t)\|_{L^1(\Omega)} \leq \tau_1 \right\}; \quad (6.5.16)$$

$$S_2 = \left\{ t \in [0, 1] : E_\alpha(h(t), \Omega) \leq \tau_2 \text{ and } \|1 + h(t)\|_{L^1(\Omega)} \leq \tau_1 \right\}. \quad (6.5.17)$$

By (6.5.8),

$$\nexists t \in [0, 1] \text{ such that } \max \left\{ \|1 - h(t)\|_{L^1(\Omega)}, \|1 + h(t)\|_{L^1(\Omega)} \right\} \leq \tau_1. \quad (6.5.18)$$

(6.5.18), together with the choices of τ_1 , τ_2 and Lemma 6.4.2, implies that

$$S_1 \cap S_2 = \emptyset; \quad S_1 \cup S_2 = \{t \in [0, 1] : E_\alpha(h(t), \Omega) \leq \tau_2\}. \quad (6.5.19)$$

S_1 and S_2 are closed subsets of $[0, 1]$. Since $h \in \mathcal{B}$, $0 \in S_1$ and $1 \in S_2$. Let

$$a = \max S_1, \quad b = \min (S_2 \cap [a, 1]). \quad (6.5.20)$$

(6.5.19) implies that $\|1 + h(a)\|_{L^1(\Omega)} > \tau_1$. Suppose $E_\alpha(h(a), \Omega) < \tau_2$. By continuity, there exists $a' > a$ such that

$$E_\alpha(h(a'), \Omega) < \tau_2 \quad \text{and} \quad \|1 + h(a')\|_{L^1(\Omega)} > \tau_1,$$

which implies (by (6.5.19)) $a' \in S_1$. This contradicts the definition of a ; hence $E_\alpha(h(a), \Omega) = \tau_2$.

A similar argument shows that $E_\alpha(h(b), \Omega) = \tau_2$ as well. Suppose there exists $t' \in (a, b)$ such that $E_\alpha(h(t'), \Omega) < \tau_2$. Then by (6.5.19), $t' \in S_1 \cup S_2$, which contradicts the definitions of a and b .

This finishes the proof of the lemma. $\square$

Part 3. By (6.5.14) and Lemma 6.5.4, there exists $\delta > 0$ such that

$$\sup_{t \in [a, b]} E_\alpha(h(t)) \leq \tilde{\lambda}_\alpha(\Omega) - \delta. \quad (6.5.21)$$

By Proposition 6.3.5, there exists a nested map $\tilde{h} : [0, 1] \rightarrow H^1(N)$ such that $\tilde{h}(0) \geq h(a)$, $\tilde{h}(1) \leq h(b)$, $|\tilde{h}(t)| \leq 1$ for all $t \in [0, 1]$ and

$$\sup_{t \in [0, 1]} E_\alpha(\tilde{h}(t)) \leq \tilde{\lambda}_\alpha(\Omega) - \frac{\delta}{2}. \quad (6.5.22)$$

We recall from (6.5.10) that $w_\varepsilon = w_{1,\varepsilon}(-2\sqrt{\varepsilon})$; hence using the notation of (6.4.13),

$$w_\varepsilon \equiv \begin{cases} -1 & \text{on } \Omega_{-4\sqrt{\varepsilon}}; \\ 1 & \text{on } (N \setminus \Omega). \end{cases} \quad (6.5.23)$$

Lemma 6.5.5. *Let $T : H^1(N) \rightarrow H^1(N)$ be defined by*

$$T(u) = \min\{-w_\alpha, \max\{w_\alpha, u\}\}. \quad (6.5.24)$$

If $|u| \leq 1$, then denoting $\hat{u} = T(u)$, we have $|\hat{u}| \leq 1$;

$$E_\alpha(\hat{u}) \leq E_\alpha(w_\alpha) + E_\alpha(u, \{-w_\alpha \geq u \geq w_\alpha\}) \leq E_\alpha(w_\alpha) + E_\alpha(u, \Omega); \quad (6.5.25)$$

$$\|1 - \hat{u}\|_{L^1(\Omega)} \leq \|1 - u\|_{L^1(\Omega)} + 2\mathcal{H}^{n+1}(\Omega \setminus \Omega_{-4\sqrt{\alpha}}); \quad (6.5.26)$$

$$\|1 + \hat{u}\|_{L^1(\Omega)} \leq \|1 + u\|_{L^1(\Omega)} + 2\mathcal{H}^{n+1}(\Omega \setminus \Omega_{-4\sqrt{\alpha}}). \quad (6.5.27)$$

Proof.

$$\hat{u} = \begin{cases} u & \text{on } \{-w_\alpha \geq u \geq w_\alpha\}; \\ \pm w_\alpha & \text{otherwise.} \end{cases} \quad (6.5.28)$$

Therefore, $|\hat{u}| \leq 1$. Moreover,

$$\{-w_\alpha \geq u \geq w_\alpha\} \subset \{w_\alpha \leq 0\} \subset \Omega \quad (\text{by (6.5.23)}). \quad (6.5.29)$$

Combining (6.5.28) and (6.5.29), one gets (6.5.25). It follows from (6.5.23) and (6.5.28) that

$$\hat{u} = u \text{ on } \Omega_{-4\sqrt{\alpha}}. \quad (6.5.30)$$

Moreover,

$$0 \leq 1 \pm w_\alpha \leq 2. \quad (6.5.31)$$

Equations (6.5.26) and (6.5.27) both follow from (6.5.28), (6.5.30) and (6.5.31). $\square$

Let us define

$$\ell = \min\{h(a), -h(b)\}; \quad (6.5.32)$$

so

$$-\ell = \max\{-h(a), h(b)\}. \quad (6.5.33)$$

Moreover,

$$0 \leq 1 - \ell \leq (1 - h(a)) + (1 + h(b)). \quad (6.5.34)$$

Lemma 6.5.6. *Let $\tilde{h}$ be as in (6.5.22). There exists $t^* \in [0, 1]$ such that*

$$E_\alpha \left(\tilde{h}(t^*), \{w_\alpha \leq \tilde{h}(t^*) \leq -w_\alpha\}^c \cup \{-\ell \leq \tilde{h}(t^*) \leq \ell\}^c \right) \leq E_\alpha(w_\alpha) + 2\tau_2. \quad (6.5.35)$$

Here, for $S \subset N$, $S^c = (N \setminus S)$.

Proof. Let $h' : [0, 1] \rightarrow H^1(N)$ be defined by

$$h'(t) = \min\{\ell, \max\{-\ell, \tilde{h}(t)\}\} = \begin{cases} \tilde{h}(t) & \text{on } \{\ell \geq \tilde{h}(t) \geq -\ell\}; \\ \pm\ell & \text{otherwise.} \end{cases} \quad (6.5.36)$$

Since

$$\ell \leq h(a) \leq \tilde{h}(0) \leq \max\{-\ell, \tilde{h}(0)\} \quad \text{and} \quad -\ell \geq h(b) \geq \tilde{h}(1),$$

we have

$$h'(0) = \ell \quad \text{and} \quad h'(1) = \min\{\ell, -\ell\}. \quad (6.5.37)$$

Let $h'' : [0, 1] \rightarrow H^1(N)$ be defined by $h''(t) = T(h'(t))$, where T is as in Lemma 6.5.5. By

Lemma 6.5.5, (6.5.37), (6.5.11) and Lemma 6.5.4,

$$E_\alpha(h''(0)) \leq E_\alpha(w_\alpha) + E_\alpha(\ell, \Omega) \leq 2\sigma\mathcal{H}^n(\partial\Omega) + 3\tau_2; \quad (6.5.38)$$

$$E_\alpha(h''(1)) \leq E_\alpha(w_\alpha) + E_\alpha(\ell, \Omega) \leq 2\sigma\mathcal{H}^n(\partial\Omega) + 3\tau_2. \quad (6.5.39)$$

Moreover, by Lemma 6.5.5, (6.5.34), Lemma 6.5.4 and (6.5.12),

$$\|1 - h''(0)\|_{L^1(\Omega)} \leq \|1 - \ell\|_{L^1(\Omega)} + 2\mathcal{H}^{n+1}(\Omega \setminus \Omega_{-4\sqrt{\alpha}}) \leq 3\tau_1; \quad (6.5.40)$$

$$\|1 + h''(1)\|_{L^1(\Omega)} \leq \|1 - \ell\|_{L^1(\Omega)} + 2\mathcal{H}^{n+1}(\Omega \setminus \Omega_{-4\sqrt{\alpha}}) \leq 3\tau_1. \quad (6.5.41)$$

Therefore, by our choices of τ_1 , τ_2 , τ and Lemma 6.4.1, there exists a continuous map $\beta_0 : [0, 1] \rightarrow H^1(N)$ such that $\beta_0(0) = h''(0)$, $\beta_0(1)|_\Omega \equiv 1$ and

$$\sup_{t \in [0, 1]} E_\alpha(\beta_0(t)) \leq 4\sigma\mathcal{H}^n(\partial\Omega) + 3\tau_2 + \tau \leq 5\sigma\mathcal{H}^n(\partial\Omega) \text{ (by (6.5.7) and (6.5.9))}. \quad (6.5.42)$$

Similarly, there exists a continuous map $\beta_1 : [0, 1] \rightarrow H^1(N)$ such that $\beta_1(0) = h''(1)$, $\beta_1(1)|_\Omega \equiv -1$ and

$$\sup_{t \in [0, 1]} E_\alpha(\beta_1(t)) \leq 4\sigma\mathcal{H}^n(\partial\Omega) + 3\tau_2 + \tau \leq 5\sigma\mathcal{H}^n(\partial\Omega). \quad (6.5.43)$$

Let us define $\beta : [0, 1] \rightarrow H^1(N)$ by

$$\beta(t) = \begin{cases} \beta_0(1 - 3t) & \text{if } 0 \leq t \leq 1/3; \\ h''(3t - 1) & \text{if } 1/3 \leq t \leq 2/3; \\ \beta_1(3t - 2) & \text{if } 2/3 \leq t \leq 1. \end{cases} \quad (6.5.44)$$

Since $\beta \in \mathcal{B}$, there exists $t^\bullet \in [0, 1]$ such that

$$E_\alpha(\beta(t^\bullet)) \geq \tilde{\lambda}_\alpha(\Omega) \geq \lambda_\alpha(\Omega) > 7\sigma\mathcal{H}^n(\partial\Omega) \text{ (using (6.5.3), (6.5.4) and (6.5.13))}.$$

Therefore, by (6.5.42) and (6.5.43), there exists $t^* \in [0, 1]$ such that

$$E_\alpha(h''(t^*)) \geq \tilde{\lambda}_\alpha(\Omega). \quad (6.5.45)$$

However, by Lemma 6.5.5, (6.5.36) and Lemma 6.5.4, for all $t \in [0, 1]$,

$$E_\alpha(h''(t)) \leq E_\alpha(w_\alpha) + E_\alpha(h'(t), \{w_\alpha \leq h'(t) \leq -w_\alpha\}) \quad (6.5.46)$$

$$\leq E_\alpha(w_\alpha) + E_\alpha(\tilde{h}(t), \{w_\alpha \leq \tilde{h}(t) \leq -w_\alpha\} \cap \{-\ell \leq \tilde{h}(t) \leq \ell\}) + E_\alpha(\ell, \Omega) \quad (6.5.47)$$

$$\leq E_\alpha(w_\alpha) + E_\alpha(\tilde{h}(t), \{w_\alpha \leq \tilde{h}(t) \leq -w_\alpha\} \cap \{-\ell \leq \tilde{h}(t) \leq \ell\}) + 2\tau_2. \quad (6.5.48)$$

Further, by (6.5.22),

$$E_\alpha(\tilde{h}(t^*)) < \tilde{\lambda}_\alpha(\Omega). \quad (6.5.49)$$

Combining (6.5.45), (6.5.48) and (6.5.49), one obtains (6.5.35). $\square$

Part 4. For $r \in \mathbb{R}$, let $r^+ = \max\{r, 0\}$; $r^- = \min\{r, 0\}$. The maps $\Phi, \Psi, \Theta : H^1(N) \times H^1(N) \times H^1(N) \rightarrow H^1(N)$ are defined as follows (Chapter 5, Equation (5.3.69)).

$$\begin{aligned} \Phi(u_0, u_1, w) &= \min\{\max\{u_0, -w\}, \max\{u_1, w\}\}; \\ \Psi(u_0, u_1, w) &= \max\{\min\{u_0, w\}, \min\{u_1, -w\}\}; \\ \Theta(u_0, u_1, w) &= \Phi(u_0, u_1, w)^+ + \Psi(u_0, u_1, w)^-. \end{aligned} \quad (6.5.50)$$

Lemma 6.5.7. *Let $u_0, u_1, w \in H^1(N)$ such that $|u_0|, |u_1|, |w| \leq 1$; $\phi = \Phi(u_0, u_1, w)$, $\psi = \Psi(u_0, u_1, w)$, $\theta = \Theta(u_0, u_1, w)$.*

(i) *If $w(x) = 1$, then $\theta(x) = u_0(x)$; if $w(x) = -1$, then $\theta(x) = u_1(x)$.*

(ii) *If $u_0(x) = u_1(x)$, then $\theta(x) = u_0(x) = u_1(x)$.*

(iii) *For all $x \in N$, either $\theta(x) = \phi(x)$ or $\theta(x) = \psi(x)$; hence $\theta(x) \in \{u_0(x), u_1(x), w(x), -w(x)\}$.*

(iv) For $\varepsilon > 0$ and $S \subset N$,

$$E_\varepsilon(\theta, S) \leq E_\varepsilon(w, S) + E_\varepsilon(u_0, S \cap (\{u_0 > -w\} \cup \{u_0 < w\})) \quad (6.5.51)$$

$$+ E_\varepsilon(u_1, S \cap (\{u_1 > w\} \cup \{u_1 < -w\})). \quad (6.5.52)$$

(v) If $v_0, v_1 \in H^1(N)$ such that $v_0 \geq u_0, u_1 \geq v_1$, then $v_0 \geq \theta \geq v_1$.

Proof. For the proofs of items (i) – (iii), we refer to the proof of Proposition 5.3.10 in Chapter 5. (iv) follows from (iii) and the definitions of Φ and Ψ . To prove item (v), we note the following. If $r_1, r'_1, r_2, r'_2 \in \mathbb{R}$ such that $r_1 \geq r_2$ and $r'_1 \geq r'_2$, then $\max\{r_1, r'_1\} \geq \max\{r_2, r'_2\}$ and $\min\{r_1, r'_1\} \geq \min\{r_2, r'_2\}$. In particular, if we set $r'_1 = r'_2 = s$, then $\max\{r_1, s\} \geq \max\{r_2, s\}$ and $\min\{r_1, s\} \geq \min\{r_2, s\}$. Hence $v_0 \geq u_0, u_1 \geq v_1$ implies

$$\Phi(v_0, v_0, w) \geq \Phi(u_0, u_1, w) \geq \Phi(v_1, v_1, w), \quad (6.5.53)$$

$$\Psi(v_0, v_0, w) \geq \Psi(u_0, u_1, w) \geq \Psi(v_1, v_1, w). \quad (6.5.54)$$

Therefore, using item (ii), we obtain

$$v_0 \geq \Theta(u_0, u_1, w) \geq v_1. \quad (6.5.55)$$

□

Let t^* be as in Lemma 6.5.6 and ℓ be as defined in (6.5.32). We define

$$\ell_0 = \max\{\tilde{h}(t^*), \ell\}; \quad \ell_1 = \min\{\tilde{h}(t^*), -\ell\}; \quad (6.5.56)$$

$$h_0^* = \Theta(\tilde{h}(t^*), \ell_0, w_\alpha); \quad h_1^* = \Theta(\tilde{h}(t^*), \ell_1, w_\alpha). \quad (6.5.57)$$

Using the fact that

$$\ell_0 = \begin{cases} \ell & \text{on } \{\tilde{h}(t^*) \leq \ell\} \\ \tilde{h}(t^*) & \text{on } \{\tilde{h}(t^*) > \ell\}, \end{cases} \quad (6.5.58)$$

and Lemma 6.5.7 (ii), (iv), we obtain

$$E_\alpha(h_0^*) = E_\alpha\left(\Theta(\tilde{h}(t^*), \ell, w_\alpha), \{\tilde{h}(t^*) \leq \ell\}\right) + E_\alpha\left(\tilde{h}(t^*), \{\tilde{h}(t^*) > \ell\}\right) \quad (6.5.59)$$

$$\leq E_\alpha(w_\alpha) + E_\alpha(\ell, \{\ell > w_\alpha\} \cup \{\ell < -w_\alpha\}) \quad (6.5.60)$$

$$+ E_\alpha\left(\tilde{h}(t^*), \{\tilde{h}(t^*) > \ell\} \cup \{\tilde{h}(t^*) > -w_\alpha\} \cup \{\tilde{h}(t^*) < w_\alpha\}\right) \quad (6.5.61)$$

$$\leq 2E_\alpha(w_\alpha) + 4\tau_2. \quad (6.5.62)$$

In the last step we have used Lemma 6.5.4, Lemma 6.5.6 and the fact that

$$\{\ell > w_\alpha\} \cup \{\ell < -w_\alpha\} \subset \{w_\alpha < 1\} \subset \Omega \text{ (by (6.5.23)).}$$

By a similar argument,

$$E_\alpha(h_1^*) \leq 2E_\alpha(w_\alpha) + 4\tau_2. \quad (6.5.63)$$

By Lemma 6.5.7 (i) and (6.5.23), $h_0^* = \ell_0$ on $\Omega_{-4\sqrt{\alpha}}$. Further, $1 \geq \ell_0 \geq \ell$ and by Lemma 6.5.7 (v), $|h_0^*| \leq 1$. Hence,

$$\|1 - h_0^*\|_{L^1(\Omega)} \leq \|1 - \ell_0\|_{L^1(\Omega_{-4\sqrt{\alpha}})} + 2\mathcal{H}^{n+1}(\Omega \setminus \Omega_{-4\sqrt{\alpha}}) \quad (6.5.64)$$

$$\leq \|1 - \ell\|_{L^1(\Omega)} + \tau_1 \text{ (by (6.5.12))} \quad (6.5.65)$$

$$\leq 3\tau_1 \text{ (by Lemma 6.5.4).} \quad (6.5.66)$$

Similarly, one can show that

$$\|1 + h_1^*\|_{L^1(\Omega)} \leq 3\tau_1. \quad (6.5.67)$$

Using Lemma 6.5.7 (v) and the definitions of $\tilde{h}$ and ℓ , we obtain

$$\tilde{h}(0) \geq h_0^* \geq \tilde{h}(t^*); \quad \tilde{h}(t^*) \geq h_1^* \geq \tilde{h}(1). \quad (6.5.68)$$

By Lemma 6.3.1, there exists $\tilde{h}(0) \geq h_0^\bullet \geq h_0^*$ such that

$$E_\alpha(h_0^\bullet) = \inf\{E_\alpha(u) : \tilde{h}(0) \geq u \geq h_0^*\} \quad (6.5.69)$$

$$\leq E_\alpha(h_0^*) \quad (6.5.70)$$

$$\leq 2E_\alpha(w_\alpha) + 4\tau_2 \text{ (by (6.5.62))} \quad (6.5.71)$$

$$\leq 4\sigma\mathcal{H}^n(\partial\Omega) + 6\tau_2 \text{ (by (6.5.11))}. \quad (6.5.72)$$

Moreover, (6.5.66) and $1 \geq h_0^\bullet \geq h_0^*$ imply that

$$\|1 - h_0^\bullet\|_{L^1(\Omega)} \leq \|1 - h_0^*\|_{L^1(\Omega)} \leq 3\tau_1. \quad (6.5.73)$$

Similarly, by Lemma 6.3.1, there exists $h_1^* \geq h_1^\bullet \geq \tilde{h}(1)$ such that

$$E_\alpha(h_1^\bullet) = \inf\{E_\alpha(u) : h_1^* \geq u \geq \tilde{h}(1)\} \quad (6.5.74)$$

$$\leq E_\alpha(h_1^*) \quad (6.5.75)$$

$$\leq 2E_\alpha(w_\alpha) + 4\tau_2 \text{ (by (6.5.63))} \quad (6.5.76)$$

$$\leq 4\sigma\mathcal{H}^n(\partial\Omega) + 6\tau_2 \text{ (by (6.5.11))}. \quad (6.5.77)$$

Moreover, (6.5.67) and $h_1^* \geq h_1^\bullet \geq -1$ imply that

$$\|1 + h_1^\bullet\|_{L^1(\Omega)} \leq \|1 + h_1^*\|_{L^1(\Omega)} \leq 3\tau_1. \quad (6.5.78)$$

Setting $u = \tilde{h}$ and $v = h_1^\bullet$ in Lemma 6.3.3 (a), we conclude that there exists a nested map $\gamma_1 : [0, 1] \rightarrow H^1(N)$ such that $\gamma_1(0) = \tilde{h}(0)$, $\gamma_1(1) = h_1^\bullet$ ((6.5.68) implies that $\tilde{h}(0) \geq h_1^\bullet$) and

$$\sup_{t \in [0, 1]} E_\alpha(\gamma_1(t)) \leq \sup_{t \in [0, 1]} E_\alpha(\tilde{h}(t)) \leq \tilde{\lambda}_\alpha(\Omega) - \frac{\delta}{2} \text{ (by (6.5.22))}. \quad (6.5.79)$$

Next, setting $u = \gamma_1$ and $v = h_0^\bullet$ in Lemma 6.3.3 (b), we obtain another nested map $\gamma_2 : [0, 1] \rightarrow$

$H^1(N)$ such that $\gamma_2(0) = h_0^\bullet$, $\gamma_2(1) = h_1^\bullet$ ((6.5.68) implies that $h_0^\bullet \geq h_1^\bullet$) and

$$\sup_{t \in [0,1]} E_\alpha(\gamma_2(t)) \leq \sup_{t \in [0,1]} E_\alpha(\gamma_1(t)) \leq \tilde{\lambda}_\alpha(\Omega) - \frac{\delta}{2}. \quad (6.5.80)$$

By the definitions of τ_1 , τ and Lemma 6.4.1, (6.5.72) and (6.5.73) imply that there exists $\tilde{\gamma}_0 : [0, 1] \rightarrow H^1(N)$ such that $\tilde{\gamma}_0(0) = h_0^\bullet$, $\tilde{\gamma}_0(1)|_\Omega \equiv 1$ and

$$\sup_{t \in [0,1]} E_\alpha(\tilde{\gamma}_0(t)) \leq 6\sigma\mathcal{H}^n(\partial\Omega) + 6\tau_2 + \tau \leq 7\sigma\mathcal{H}^n(\partial\Omega) \text{ (by (6.5.7) and (6.5.9))}. \quad (6.5.81)$$

Similarly (6.5.77) and (6.5.78) imply that there exists $\tilde{\gamma}_1 : [0, 1] \rightarrow H^1(N)$ such that $\tilde{\gamma}_1(0) = h_1^\bullet$, $\tilde{\gamma}_1(1)|_\Omega \equiv -1$ and

$$\sup_{t \in [0,1]} E_\alpha(\tilde{\gamma}_1(t)) \leq 6\sigma\mathcal{H}^n(\partial\Omega) + 6\tau_2 + \tau \leq 7\sigma\mathcal{H}^n(\partial\Omega) \text{ (by (6.5.7) and (6.5.9))}. \quad (6.5.82)$$

Let $\gamma : [0, 1] \rightarrow H^1(N)$ be defined by

$$\gamma(t) = \begin{cases} \tilde{\gamma}_0(1-3t) & \text{if } 0 \leq t \leq 1/3; \\ \gamma_2(3t-1) & \text{if } 1/3 \leq t \leq 2/3; \\ \tilde{\gamma}_1(3t-2) & \text{if } 2/3 \leq t \leq 1. \end{cases} \quad (6.5.83)$$

Then $\gamma \in \mathcal{B}$; (6.5.80), (6.5.81), (6.5.82), (6.5.13) and (6.5.4) imply that

$$\sup_{t \in [0,1]} E_\alpha(\gamma(t)) < \tilde{\lambda}_\alpha(\Omega), \quad (6.5.84)$$

which contradicts the definition of $\tilde{\lambda}_\alpha(\Omega)$ ((6.5.3)). This finishes the proof of Proposition 6.5.2. $\square$

Theorem 6.5.8. *Let (N^{n+1}, g) , $n+1 \geq 3$, be a closed Riemannian manifold and $\Omega \subset N$ be a good set. Suppose ε^* and η^* are as in Proposition 6.5.2 and Remark 6.5.3 (where we set $g' = g$ in Remark 6.5.3). Then for all $0 < \varepsilon \leq \varepsilon^*$, there exists $\vartheta_\varepsilon : N \rightarrow (-1, 1)$ satisfying $AC_\varepsilon(\vartheta_\varepsilon) = 0$, $\text{Ind}(\vartheta_\varepsilon) \leq 1$, $E_\varepsilon(\vartheta_\varepsilon) = \tilde{\lambda}_\varepsilon(\Omega)$ and $E_\varepsilon(\vartheta_\varepsilon, \Omega) \geq \eta^*$.*

Proof. To prove this theorem, we apply Theorem 6.2.5 to the functional $E_\varepsilon : H^1(N) \rightarrow \mathbb{R}$, $0 < \varepsilon \leq \varepsilon^*$. In Theorem 6.2.5, we set

$$B_0 = \{u \in H^1(N) : u|_\Omega \equiv 1\}; \quad B_1 = \{u \in H^1(N) : u|_\Omega \equiv -1\}; \quad (6.5.85)$$

$\mathcal{F} = \mathcal{B}$ so that $c = \tilde{\lambda}_\varepsilon(\Omega)$ and

$$L = \{u \in H^1(N) : E_\varepsilon(u) \geq \tilde{\lambda}_\varepsilon(\Omega)\} \cap \{u \in H^1(N) : E_\varepsilon(u, \Omega) \geq \eta^*\}. \quad (6.5.86)$$

Since $u|_\Omega \equiv 1$ or $u|_\Omega \equiv -1$ imply that $E_\varepsilon(u, \Omega) = 0$, the condition (a1) of Theorem 6.2.5 is satisfied. By Proposition 6.5.2, the condition (a2) is also satisfied. It follows from (6.5.86) that (a3) is satisfied as well. Following [Gua18, Section 4], if $\{\tilde{h}_i\}_{i=1}^\infty$ is an arbitrary minimizing sequence for E_ε in $\mathcal{B}$, we define $\{h_i\}_{i=1}^\infty \subset \mathcal{B}$ by

$$h_i(t) = \min\{1, \max\{-1, \tilde{h}_i(t)\}\}.$$

Then

$$-1 \leq h_i(t) \leq 1 \quad \forall i \in \mathbb{N}, t \in [0, 1] \quad (6.5.87)$$

and $E_\varepsilon(h_i) \leq E_\varepsilon(\tilde{h}_i)$. Hence $\{h_i\}$ is again a minimizing sequence. By [Gua18, Proposition 4.4], E_ε satisfies the Palais-Smale condition along $\{h_i\}$. Therefore, by Theorem 6.2.5, part (a), there exists $\vartheta_\varepsilon \in \mathcal{K}(\{h_i\})$ such that

$$AC_\varepsilon(\vartheta_\varepsilon) = 0, \quad E_\varepsilon(\vartheta_\varepsilon) = \tilde{\lambda}_\varepsilon(\Omega), \quad E_\varepsilon(\vartheta_\varepsilon, \Omega) \geq \eta^*.$$

Moreover, by (6.5.87), $|\vartheta_\varepsilon| \leq 1$; hence by the strong maximum principle $|\vartheta_\varepsilon| < 1$. In addition, if the ambient metric $g \in \widetilde{\mathcal{M}}$ (where $\widetilde{\mathcal{M}}$ is as defined in Theorem 6.2.2) and $\varepsilon^{-1} \notin \text{Spec}(-\Delta_g)$, then $\mathcal{Z}_{\varepsilon, g}$ is finite. In that case, the condition (b1) of Theorem 6.2.5, part (b) is satisfied and one can ensure that ϑ_ε satisfies $\text{Ind}(\vartheta_\varepsilon) \leq 1$.

To get the Morse index upper bound for arbitrary metric g and $\varepsilon \in (0, \varepsilon^*]$, we use an approximation argument. Since, by Theorem 6.2.2, $\widetilde{\mathcal{M}}$ is a generic subset of $\mathcal{M}$, it is possible to choose

$\{g_i\}_{i=1}^\infty \subset \widetilde{\mathcal{M}}$ such that g_i converges to g smoothly. Let $\{\epsilon_i\}_{i=1}^\infty$ be a sequence in $(0, \epsilon^*]$ such that $\epsilon_i^{-1} \notin \text{Spec}(-\Delta_{g_i})$ and $\epsilon_i \rightarrow \epsilon$. Since the width $\mathbb{W}(\Omega)$ depends continuously on the ambient metric [IMN18, Lemma 2.1], Ω is a good set with respect to g_i if i is sufficiently large. Therefore, by the above discussion, Theorem 6.2.2, Theorem 6.2.5, Proposition 6.5.2 and Remark 6.5.3 imply that for all i sufficiently large, there exists $\vartheta^i : N \rightarrow (-1, 1)$ such that

$$AC_{\epsilon_i, g_i}(\vartheta^i) = 0; \quad \text{Ind}_{g_i}(\vartheta^i) \leq 1; \quad E_{\epsilon_i, g_i}(\vartheta^i) = \tilde{\lambda}_{\epsilon_i, g_i}(\Omega); \quad E_{\epsilon_i, g_i}(\vartheta^i, \Omega) \geq \eta^*. \quad (6.5.88)$$

In this equation, the subscript g_i indicates that these quantities are computed with respect to the metric g_i . Since $|\vartheta^i| < 1$, by the elliptic regularity and the Arzela-Ascoli theorem, there exists $\vartheta^\infty : N \rightarrow [-1, 1]$ such that up to a subsequence ϑ^i converges to ϑ^∞ in $C^2(N)$. Using (6.5.88) and the fact that the min-max quantity $\tilde{\lambda}_\epsilon(\Omega)$ depends continuously on the ambient metric [GG18, Lemma 5.4], we obtain

$$AC_{\epsilon, g}(\vartheta^\infty) = 0; \quad \text{Ind}_g(\vartheta^\infty) \leq 1; \quad E_{\epsilon, g}(\vartheta^\infty) = \tilde{\lambda}_{\epsilon, g}(\Omega); \quad E_{\epsilon, g}(\vartheta^\infty, \Omega) \geq \eta^*. \quad (6.5.89)$$

Furthermore, by the strong maximum principle, $|\vartheta^\infty| < 1$. This finishes the proof of Theorem 6.5.8. $\square$

Proof of Theorem 6.1.1. As proved in [CL20, Section 8.1], $\text{Vol}(M) < \infty$ implies that there exists a sequence $\{U_i\}_{i=1}^\infty$, where each $U_i \subset M$ is a bounded open set with smooth boundary, such that $U_i \subset U_{i+1}$ for all $i \in \mathbb{N}$ and $\lim_{i \rightarrow \infty} \mathcal{H}^n(\partial U_i) = 0$. As a consequence, there exists $i_0 \in \mathbb{N}$ such that U_{i_0} is a good set. For simplicity, let us denote U_{i_0} by U .

The following proposition was proved in [Mon16, Section 12.2].

Proposition 6.5.9. [Mon16, Section 12.2] *Let M' be a complete Riemannian manifold and $f' : M' \rightarrow [a_0, \infty)$ be a proper Morse function. Suppose a_1 is a regular value of f' and define*

$$\mathcal{R} = \{x \in M' : f'(x) \leq a_1\}.$$

Then there exists a closed Riemannian manifold N' and a Morse function $f'' : N' \rightarrow [a_0, \infty)$ such that N' contains an isometric copy of $\mathcal{R}$, f'' coincides with f' on $\mathcal{R}$ and $f'' > a_1$ on $N' \setminus \mathcal{R}$.

Coming back to the proof of Theorem 6.1.1, we choose a proper Morse function $f_1 : M \rightarrow [0, \infty)$ with $\min_M f_1 = 0$. Let t_1 be a regular value of f_1 so that $U \subset \subset f_1^{-1}([0, t_1/2])$. Furthermore, by suitably modifying f_1 , we can assume that in the interval $[0, t_1/2]$, f_1 has no critical value which is a non-global local maxima or minima. Let

$$f_2 = \frac{1}{3} + \frac{2f_1}{3t_1}.$$

Then $f_2 : M \rightarrow [1/3, \infty)$, $U \subset \subset f_2^{-1}([1/3, 2/3])$ and we set $\tilde{U} = f_2^{-1}([1/3, 1])$.

We choose an increasing sequence $\{s_i\}_{i=1}^\infty$ such that $s_1 \geq 1$, each s_i is a regular value of f_2 and $s_i \rightarrow \infty$ as $i \rightarrow \infty$. Let

$$Q_i = \{x \in M : f_2(x) < s_i\}, \quad \bar{Q}_i = \{x \in M : f_2(x) \leq s_i\}.$$

By Proposition 6.5.9, there exists a sequence $\{N_i\}_{i=1}^\infty$ of closed Riemannian manifolds such that N_i contains an isometric copy of $\bar{Q}_i$. Moreover, for every i , there exists a Morse function $\tilde{f}_i : N_i \rightarrow [1/3, \infty)$ such that $\tilde{f}_i$ coincides with f_2 on $\bar{Q}_i$ and $\tilde{f}_i > s_i$ on $N_i \setminus \bar{Q}_i$. In particular, N_i contains isometric copies of U and $\tilde{U}$; suppose $\mathcal{U}_i$ denotes the isometric copy of U in N_i . Setting $N = N_i$ and $f = \tilde{f}_i$ in Proposition 6.5.2 and Remark 6.5.3 and using Theorem 6.5.8, we obtain $\varepsilon_0, \eta_0 > 0$, which depend only on $U, \tilde{U}$, the ambient metric on M restricted to $\tilde{U}$ and $f_2|_{\tilde{U}}$ such that the following holds.¹ For all $0 < \varepsilon \leq \varepsilon_0$, there exists $\vartheta_{\varepsilon,i} : N_i \rightarrow (-1, 1)$ satisfying

$$AC_\varepsilon(\vartheta_{\varepsilon,i}) = 0; \quad \text{Ind}(\vartheta_{\varepsilon,i}) \leq 1; \quad E_\varepsilon(\vartheta_{\varepsilon,i}) = \tilde{\lambda}_\varepsilon(\mathcal{U}_i); \quad E_\varepsilon(\vartheta_{\varepsilon,i}, \mathcal{U}_i) \geq \eta_0. \quad (6.5.90)$$

Let $b \in [1/3, \infty)$ such that

$$\{x \in M : d(x, U) \leq 2\sqrt{\varepsilon_0}\} \subset f_2^{-1}([1/3, b]). \quad (6.5.91)$$

¹In particular, ε_0 and η_0 do not depend on i .

One can modify f_2 on $f_2^{-1}([1/3, b])$ and define another proper Morse function $f_3 : M \rightarrow [1/3, \infty)$ so that

$$\{x \in M : d(x, U) \leq 2\sqrt{\varepsilon_0}\} \subset f_3^{-1}([1/3, b]) \quad (6.5.92)$$

and in the interval $[1/3, b]$, f_3 has no critical value which is a non-global local maxima or minima. For $1/3 \leq t \leq b$, let $d^t : M \rightarrow \mathbb{R}$ be defined by

$$d^t(x) = \begin{cases} -d(x, f_3^{-1}(t)) & \text{if } f_3(x) \leq t; \\ d(x, f_3^{-1}(t)) & \text{if } f_3(x) \geq t. \end{cases} \quad (6.5.93)$$

(6.5.92) and the fact that b is not a local maximum value of f_3 imply

$$U \subset \{d^b \leq -2\sqrt{\varepsilon_0}\}. \quad (6.5.94)$$

We choose $i_1 \in \mathbb{N}$ so that

$$\{d^b \leq 2\sqrt{\varepsilon_0}\} \subset Q_{i_1}. \quad (6.5.95)$$

For $0 < \varepsilon \leq \varepsilon_0$, let $\zeta_\varepsilon : [0, b] \rightarrow H^1(Q_{i_1})$ be defined by

$$\zeta_\varepsilon(t) = \begin{cases} \hat{q}_\varepsilon \circ d^t & \text{if } \frac{1}{3} \leq t \leq b \\ 1 - 3t(1 - \zeta_\varepsilon(1/3)) & \text{if } 0 \leq t \leq \frac{1}{3}, \end{cases} \quad (6.5.96)$$

where $\hat{q}_\varepsilon$ is as defined in (6.4.4). ζ_ε is continuous (see the discussion after equation (6.4.8)), $\zeta_\varepsilon(0) \equiv 1$ and (6.5.94) implies that $\zeta_\varepsilon(b)|_U \equiv -1$. Further, by [Gua18, Section 9],

$$\Lambda_\varepsilon := \sup_{0 \leq t \leq b} E_\varepsilon(\zeta_\varepsilon(t)) \quad (6.5.97)$$

satisfies

$$\limsup_{\varepsilon \rightarrow 0^+} \Lambda_\varepsilon \leq 2\sigma \sup_{1/3 \leq t \leq b} \mathcal{H}^n(f_3^{-1}(t)). \quad (6.5.98)$$

For each $i \geq i_1$, N_i contains an isometric copy of Q_{i_1} . Hence (6.5.95) implies that ζ_ε canonically

defines a continuous map $\zeta_{\varepsilon,i} : [0, b] \rightarrow H^1(N_i)$ ($\zeta_{\varepsilon,i}(t) \equiv 1$ on $N_i \setminus Q_{i_1}$ for all $t \in [0, b]$) so that $\zeta_\varepsilon(0) \equiv 1$ and $\zeta_\varepsilon(b)|_{\mathcal{U}_i} \equiv -1$. Thus

$$\tilde{\lambda}_\varepsilon(\mathcal{U}_i) \leq \Lambda_\varepsilon \quad \forall i \geq i_1. \quad (6.5.99)$$

Restricting $\vartheta_{\varepsilon,i}$ to the isometric copy of Q_i contained in N_i , one gets $\mathbf{u}_{\varepsilon,i} : Q_i \rightarrow (-1, 1)$ such that

$$AC_\varepsilon(\mathbf{u}_{\varepsilon,i}) = 0; \quad \text{Ind}(\mathbf{u}_{\varepsilon,i}) \leq 1; \quad E_\varepsilon(\mathbf{u}_{\varepsilon,i}, Q_i) \leq \tilde{\lambda}_\varepsilon(\mathcal{U}_i); \quad E_\varepsilon(\mathbf{u}_{\varepsilon,i}, U) \geq \eta_0. \quad (6.5.100)$$

For each fixed $j \in \mathbb{N}$, by the elliptic estimates, $\|\mathbf{u}_{\varepsilon,i}\|_{C^{2,\alpha}(Q_j)}$ is uniformly bounded for all $i > j$ (since $|\mathbf{u}_{\varepsilon,i}| < 1$ for all $i \in \mathbb{N}$). Using a diagonal argument and the Arzela-Ascoli theorem, we conclude that there exists $\mathbf{u}_\varepsilon : M \rightarrow [-1, 1]$ such that a subsequence $\{\mathbf{u}_{\varepsilon,i_k}\}$ converges to $\mathbf{u}_\varepsilon$ in $C_{\text{loc}}^2(M)$. Hence, using (6.5.100) and (6.5.99),

$$AC_\varepsilon(\mathbf{u}_\varepsilon) = 0; \quad \text{Ind}(\mathbf{u}_\varepsilon) \leq 1; \quad E_\varepsilon(\mathbf{u}_\varepsilon) \leq \Lambda_\varepsilon; \quad E_\varepsilon(\mathbf{u}_\varepsilon, U) \geq \eta_0. \quad (6.5.101)$$

By the strong maximum principle, $|\mathbf{u}_\varepsilon| < 1$. Further, as mentioned in (6.5.98), $\limsup_{\varepsilon \rightarrow 0^+} \Lambda_\varepsilon < \infty$. This completes the proof of Theorem 6.1.1. $\square$

Proof of Theorem 6.1.2 using Theorem 6.1.1. Let

$$\Omega_1 \subset \cdots \subset \Omega_i \subset \Omega_{i+1} \subset \cdots$$

be an exhaustion of M by bounded open subsets with smooth boundaries and $U \subset \subset \Omega_1$. Using Theorem 6.2.1 and a diagonal argument, there exist a sequence $\{\epsilon_i\}_{i=1}^\infty$ converging to 0 and a stationary, integral varifold V_k in Ω_k (for each $k \in \mathbb{N}$) such that the following conditions are satisfied.

•

$$V \left[\mathbf{u}_{\epsilon_i} \Big|_{\Omega_k} \right] \rightarrow V_k \quad (6.5.102)$$

in the sense of varifolds.

- $\text{spt}(V_k)$ is a minimal hypersurface with optimal regularity in Ω_k .

•

$$\|V_k\|(\text{Clos}(U)) \geq \frac{1}{2\sigma} \liminf_{i \rightarrow \infty} E_{\epsilon_i}(\mathbf{u}_{\epsilon_i}, U). \quad (6.5.103)$$

•

$$\|V_k\|(\Omega_k) \leq \frac{1}{2\sigma} \limsup_{i \rightarrow \infty} E_{\epsilon_i}(\mathbf{u}_{\epsilon_i}, \Omega_k) \leq \frac{1}{2\sigma} \limsup_{i \rightarrow \infty} E_{\epsilon_i}(\mathbf{u}_{\epsilon_i}). \quad (6.5.104)$$

By (6.5.102), $V_i \llcorner \Omega_j = V_j$ if $i > j$. Therefore, there exists a stationary, integral varifold V in M such that $V \llcorner \Omega_i = V_i$ and $\text{spt}(V)$ is a minimal hypersurface with optimal regularity. Further, (6.1.1), (6.5.103) and (6.5.104) imply

$$0 < \|V\|(\text{Clos}(U)) \leq \|V\|(M) < \infty.$$

□

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