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Crafting Euler Systems: Beyond the Motivic Mold

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CRAFTING EULER SYSTEMS:
BEYOND THE MOTIVIC MOLD

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Abstract

This dissertation studies Euler Systems and their arithmetic applications. Euler Systems have proven to be versatile tools for understanding Selmer groups and their connections to special values of L -functions. However, despite their importance in foundational conjectures in number theory like the Bloch–Kato conjecture, only a handful of provably non-trivial Euler systems have been constructed to date.

A significant obstacle in constructing Euler Systems lies in producing candidate Galois cohomology classes. This thesis presents a method to overcome this obstacle without relying on rare motivic classes.

In joint work with C. Skinner, I use Eisenstein classes on Siegel threefolds to construct a cyclotomic Euler System for the adjoint of an elliptic modular form. I also construct integral absolute étale classes on modular curves associated to theta series at inert primes. These theta classes should give new insight into the Iwasawa main conjecture for modular forms over quadratic imaginary fields at inert primes.

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Part I

General Introduction

Chapter 1

1.1 Literature and Philosophy

This dissertation is concerned with the mathematical objects referred to as **Euler Systems** and their arithmetic applications. Informally, given a p -adic Galois representation ρ , an Euler System for ρ is a collection of Galois cohomology classes defined over ramified abelian extensions of a fixed number field that verify certain norm compatibilities relations.

Euler Systems, when they are known to exist, have proved to be a powerful and flexible tool for establishing bounds on the ranks and orders of Selmer groups and for relating these to special values of L -functions. Inspired by known formulae for Tamagawa numbers of algebraic groups, Bloch and Kato [BK07] conjectured a precise formula for the leading term of the L -function of ρ at $s = 0$. The method of Euler Systems, through Iwasawa theory and some additional input, is able to establish such precise formulae, at least in low rank situations. Despite their relevance to foundational conjectures in Number theory like the Bloch–Kato conjecture, only a handful of Euler systems have been constructed to date by some counts – and when

they have, often one is unable to characterize their non-triviality. For arithmetic applications, relating the non-vanishing of the Euler System to the non-vanishing of the relevant p -adic L-function is crucial.

Crafting visibly non-trivial Euler Systems has historically proven to be an arduous undertaking. The first and foremost difficulty with which one is confronted amounts to actually *producing* candidate Galois cohomology classes for a given p -adic Galois representation ρ . The contribution of this thesis is centered around this obstacle: we develop a method for constructing interesting étale and Galois cohomology classes that do not involve *a priori* motivic classes.

1.1.1 The motivic mold

In order to describe the new method that I introduce in this dissertation, let me first describe what I refer to in the title of the thesis as 'the motivic mold', so that the differences and advantages with the new technique will be apparent. The 'motivic mold' represents an approach for constructing Euler Systems, reminiscent of Kummer theory. To elucidate this concept further, it is instructive to dissect its application within known instances of Euler Systems. Given the limited number of documented cases within the literature, the following enumeration of examples provides an essentially comprehensive overview of the 'motivic mold' in practice.

1.1.1.1 Units-based Euler Systems

The cyclotomic units Euler System. Although the term 'Euler System' was coined later by Kolyvagin in his celebrated article [Kol91], the 'cyclotomic units

Euler System' is arguably the first Euler System appearing in the literature with the work of Francisco Thaine [Tha88]. For any fixed rational prime p , this Euler System provides precise information about the Selmer group for the p -adic Galois representation $\rho = \mathbb{Q}_p(1)$. The Galois cohomology of ρ is related to the multiplicative group over an algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$. More precisely, if $n \geq 1$ is an integer and $F/\mathbb{Q}$ is an abelian extension of $\mathbb{Q}$, from the long exact sequence of G_F -modules

$$0 \longrightarrow \mu_{p^n} \longrightarrow \overline{\mathbb{Q}}^\times \xrightarrow{\times p^n} \overline{\mathbb{Q}}^\times \longrightarrow 0,$$

one obtains an embedding

$$F^\times \otimes_{\mathbb{Z}} \mathbb{Z}/p^n\mathbb{Z} \hookrightarrow H^1(F, \mu_{p^n}),$$

where $\mu_{p^n} \subset \overline{\mathbb{Q}}$ denotes the p^n -th roots of unity. The Kummer map is then defined by passing to the inverse limit

$$\text{Kum} : F^\times \otimes_{\mathbb{Z}} \mathbb{Z}_p \hookrightarrow H^1(F, \mathbb{Z}_p(1)).$$

Let $\mathcal{O}_F$ be the integral closure of $\mathbb{Z}$ in F . The subgroup of $H^1(F, \mathbb{Z}_p(1))$ generated by the image of $\mathcal{O}_F^\times \otimes_{\mathbb{Z}} \mathbb{Z}_p$ under the Kummer map is the Bloch-Kato Selmer group, which we denote, as is customary, by $H_f^1(F, \mathbb{Z}_p(1)) = \text{Kum}(\mathcal{O}_F^\times \otimes_{\mathbb{Z}} \mathbb{Z}_p)$.

The existence of the Kummer map implies that constructing Selmer group classes amounts to constructing units in $\mathcal{O}_F^\times$ (and vice-versa). Explicit units for abelian extensions of $\mathbb{Q}$ can be readily produced, as we shall now illustrate.

Assume that χ is a Dirichlet character such that χ (seen as a character of the Galois group of $\mathbb{Q}$ by class field theory) cuts out F . Let N be the conductor of χ , ζ be a primitive N -th root of unity, and $\mathbb{Z}_{p,\chi}$ be the valuation ring of the extension of $\mathbb{Q}_p$ generated by the values of χ . Then the element

$$u_\chi := \prod_{a=1}^{N-1} (1 - \zeta^a)^{\chi^{-1}(a)} \in \mathcal{O}_F^\times \otimes \mathbb{Z}_{p,\chi} \quad (1.1)$$

is a distinguished $\mathbb{Z}_{p,\chi}$ -linear (for the multiplicative structure) combination of units in F_χ . As is explained in [Ber+14, Section 1], these units, for varying χ , 'fit' together to form an Euler System for $\mathbb{Z}_p(1)$. Crucially, by Leopoldt's formula, one finds that the equality

$$\log_p(\text{loc}_p(u_\chi)) \doteq L_p(1, \chi) \quad (1.2)$$

holds, where $\doteq$ denotes equality up to a non-zero 'interpolation factor' (see Equation (14) of *loc cit.* for a precise formula), $L_p(1, \chi)$ denotes the Kubota Leopoldt p -adic L -function for χ , and loc_p denotes the localization-at- p map that sends the global unit to a local p -adic unit. This type of equality relating Euler System classes to special values of p -adic L -functions is often referred to as an 'explicit reciprocity law'.

This example of Euler System is the epitome of 'the motivic mold', characterized by a systematic two-step process:

1. Examination of motivic classes, exemplified by entities like the cyclotomic units u_χ .
2. Verification that their images under a Kummer map, or one of its more refined

avatars, can be arranged into an Euler System.

The elliptic units Euler System. Similarly to cyclotomic units, elliptic units are units of abelian extensions of a number field $\mathcal{K}$. However, these units are slightly less explicit than their cyclotomic counterparts: they are constructed as special values of certain modular functions defined on the upper half plane. These functions are often referred to as 'Siegel units.' Since they will play a major role in the next examples of Euler Systems, we describe them briefly. Let $N \geq 3$ be an integer, $a \in \{1, \dots, N-1\}$ an integer, and ζ be a primitive N -th root of unity. Let $Y_1(N)$ be the open modular curve that parametrizes elliptic curves with a marked point of exact order N . Then there exists an element $g_a \in \mathcal{O}_{Y_1(N)}^\times$ that has Fourier expansion given by the formula:

$$g_a(q) = q^{1/12}(1 - \zeta^a) \prod_{n>0} (1 - q^n \zeta^a)(1 - q^n \zeta^{-a}). \quad (1.3)$$

If $\tau \in Y_1(N)(\mathbb{C})$ is a CM-point for the quadratic imaginary field $\mathcal{K}$, by definition of the canonical model of $Y_1(N)$, τ is defined over an abelian extension of $\mathcal{K}$. Therefore, $g_a(\tau)$ is a unit in an abelian extension of $\mathcal{K}$.

One can then take the Kummer images of these units to obtain Galois cohomology classes for the restriction to $G_{\mathcal{K}}$ of the p -adic Galois representation $\mathbb{Q}_p(1)$. Rubin [Rub91] showed that these units 'fit' together to form an Euler System. The analog of the cyclotomic units explicit reciprocity law (see Equation 1.2) is the Coates–Wiles explicit reciprocity law [CW77].

Again, in this example, 'the motivic mold' is apparent. The motivic classes under consideration are units in abelian extensions of a quadratic imaginary field,

constructed from special values of Siegel units.

Kato's Euler System. Let f be a holomorphic modular form of weight $k \geq 2$. Let V_f be the p -adic Galois representation attached to f by Shimura, such that the L -function of f and the L -function of V_f coincide. In [Kat04], Kato constructs an Euler System for $V_f(2)$. Kato's construction relies on a more sophisticated version of a Kummer map: the p -adic Abel–Jacobi map. As this map will play a crucial role in the following example, we give a brief description.

Let X be a smooth variety over a field K . Fix a separable closure K^{sep} of K , and denote by $\overline{X}$ the base change of X to K^{sep} . Let G_K be the absolute Galois group of K . There is a Leray spectral sequence:

$$E_2^{i,j} = H^i(G_K, H_{\text{et}}^j(\overline{X}, \mathbb{Z}_p)) \Rightarrow H_{\text{et}}^{i+j}(X, \mathbb{Z}_p).$$

Denote by $E_{\infty}^{i,j}$ the stable page of the spectral sequence and by $\text{Fil}^{\bullet}$ the corresponding filtration on absolute étale cohomology. For any integer $r \geq 0$, let $H_{\text{et}}^r(X, \mathbb{Z}_p)_0$ be the kernel of the natural map

$$H_{\text{et}}^r(X, \mathbb{Z}_p) \rightarrow E_{\infty}^{0,r}.$$

Note that $E_{\infty}^{0,r} \subset H^0(G_K, H_{\text{et}}^r(\overline{X}, \mathbb{Z}_p))$. By definition if $c \in H_{\text{et}}^r(X, \mathbb{Z}_p)_0$ then $c \in \text{Fil}^1 H_{\text{et}}^r(X, \mathbb{Z}_p)$ and we can consider the natural map

$$\text{Fil}^1 H_{\text{et}}^r(X, \mathbb{Z}_p) \rightarrow E_{\infty}^{1,r-1} \hookrightarrow H^1(G_K, H_{\text{et}}^{r-1}(\overline{X}, \mathbb{Z}_p)),$$

where, for the last inclusion, we have used the fact that the Leray spectral sequence

is supported on the first quadrant. The p -adic Abel-Jacobi map is precisely this map obtained in the way just described above:

$$\mathrm{AJ}_{\mathrm{et}} : \mathrm{H}_{\mathrm{et}}^r(X, \mathbb{Z}_p)_0 \longrightarrow \mathrm{H}^1(G_K, \mathrm{H}_{\mathrm{et}}^{r-1}(\overline{X}, \mathbb{Z}_p)).$$

When the variety of interest is $X = \mathbf{G}_m$, the multiplicative group scheme over $\mathrm{Spec}(\mathbb{Q})$, the p -adic Abel-Jacobi map is essentially the Kummer map. Indeed if $x \in \mathbf{G}_m(F)$, the cycle class map defines a class $c_x \in \mathrm{H}_{\mathrm{et}}^2(\mathbf{G}_m/F, \mathbb{Z}_p(1))$, which is automatically null-homologous as $\mathbf{G}_m$ is an affine scheme. We then have the relation

$$\mathrm{AJ}_{\mathrm{et}}(c_x) = \mathrm{Kum}(x \otimes 1) \in \mathrm{H}^1(G_F, \mathbb{Z}_p(1)).$$

Hence one can view the p -adic Abel-Jacobi map as a more sophisticated version of the Kummer map.

In [Kat04], Kato considers tensor products of Siegel units (see Equation 1.3) on a product of open modular curves. By invoking an explicit description of the K_2 -group of a curve, this product of Siegel units defines absolute étale classes belonging to $\mathrm{H}_{\mathrm{et}}^2(Y_F, \mathbb{Z}_p(2))$, where $F = \mathbb{Q}(\mu_N)$ is some abelian extension of $\mathbb{Q}$ depending on the product of Siegel units under consideration. As Y is an affine curve, $\mathrm{H}^2(\overline{Y}, \mathbb{Z}_p)$ vanishes. Therefore, one can consider the images of the classes under the p -adic Abel-Jacobi map:

$$\mathrm{AJ}_{\mathrm{et}} : \mathrm{H}_{\mathrm{et}}^2(Y_F, \mathbb{Z}_p(2)) \longrightarrow \mathrm{H}^1(G_F, \mathrm{H}_{\mathrm{et}}^1(\overline{Y}, \mathbb{Z}_p(2))).$$

Since $V_f(2)$ appears as a quotient of $H_{\text{et}}^1(\overline{Y}, \mathbb{Z}_p(2))$, one obtains the desired Galois cohomology classes, which Kato arranges into an Euler System.

Again, 'the motivic mold' is apparent here. Siegel units play the role of elliptic units and the p -adic Abel–Jacobi map plays the role of the Kummer map. The explicit reciprocity law however is significantly more challenging to obtain than for the cyclotomic and elliptic units Euler Systems: it requires deep results from p -adic Hodge theory.

The Asai–Flach, Rankin–Selberg and GSp_4 -Euler Systems. For more than a decade, the examples above were the only known examples of Euler Systems. In pioneering work, Lei–Loeffler–Zerbes ([LLZ14], [LLZ18]) constructed new Euler Systems. The key insight is that Siegel units define absolute classes in the first étale cohomology group of the modular curve and that these étale classes could be *pushed forward* to a Shimura variety of dimension 2 containing the modular curve. One obtains the so-called Beilinson–Flach classes in the case of a product of modular curves, and Asai–Flach classes in the case of a Hilbert modular surface.

In [LLZ14], this strategy is carried out by considering a modular curve embedded diagonally into a product of modular curves: an Euler System for the Rankin–Selberg convolution of two modular forms is thus constructed. In [LLZ18], a modular curve is embedded diagonally into a Hilbert modular surface: an Euler System for the Asai representation of a Hilbert modular form is thus constructed. A similar strategy is employed in [LSZ21]. This time, a product of two Siegel units is pushed-forward to a Siegel three-fold: an Euler System for certain Galois representations attached to genus 2 Siegel modular forms is obtained.

'The motivic mold' is apparent in these constructions as well: Siegel units still play the crucial role in defining the Galois cohomology classes that are subsequently arranged into an Euler System. We note that explicit reciprocity laws exist for each of the above mentioned Euler Systems – in the Rankin–Selberg case they are a consequence of the work of Bertolini–Darmon–Rotger [BDR15]. In the Asai–Flach and Siegel setting, they are established in the very recent work of Grossi–Loeffler–Zerbes [GLZ23], and Loeffler–Zerbes [LZ20]. We note that obtaining the explicit reciprocity laws in these cases is increasingly difficult: both works rely on sophisticated higher Hida theory.

1.1.1.2 The cycles-based Euler System

We now turn to a seemingly different origin of interesting Galois cohomology classes: null-homologous cycles on algebraic varieties.

The Heegner point Euler System. Strictly speaking, Kolyvagin's Euler System of Heegner points is not an Euler System in the sense of Rubin [Rub00], as the norm relations are verified only along the anticyclotomic direction. However, somewhat ironically, it is the work of Kolyvagin [Kol91] that coined the term 'Euler System.' We give a modern account of the construction of Kolyvagin, which uses the p -adic Abel–Jacobi map. Let $N \geq 1$ and $\mathcal{K}$ be a quadratic imaginary field such that N satisfies the 'Heegner hypothesis' for $\mathcal{K}$: every prime dividing N splits for $\mathcal{K}$. Let $X = X_0(N)$ be the modular curve of level N . We can consider divisors $Z \subseteq X$ supported on CM points for $\mathcal{K}$, so that Z is defined over a ring class field R of $\mathcal{K}$. If $\infty \in X(\mathbb{Q})$ denotes the infinity cusp, then putting $Z_0 := Z - \deg(Z)(\infty)$,

$\text{cl}_X(Z_0) \in H_{\text{et}}^2(X_R, \mathbb{Z}_p(1))$ is null homologous, where cl_X denotes the étale cycle class map. We can therefore consider

$$\text{AJ}_{\text{et}}(Z_0) \in H_{\text{et}}^1(G_R, H_{\text{et}}^1(\overline{X}, \mathbb{Z}_p(1))) \twoheadrightarrow H^1(G_R, T_p E),$$

where $T_p E$ denotes the Tate module of some elliptic curve which has a modular parametrization by $X = X_0(N)$. Kolyvagin then showed that these classes verify certain norm compatibilities, when Z_0 and the ring class field R are varying.

The construction of the Euler System of Heegner points also fits ‘the motivic mold.’ The motivic classes defined by Siegel units are simply replaced in the construction by the images of certain null-homologous cycles under the étale cycle-class map. We note that the explicit reciprocity law verified by the Heegner point Euler System takes a more archimedean flavor: the non-triviality of the cohomology classes obtained in this way hinges on the celebrated Gross–Zagier formula [GZ86].

Other Euler Systems. The idea of leveraging null-homologous cycles defined by special subvarieties of Shimura varieties to construct Euler Systems has led to many developments in the field. As in the Heegner point case, these Euler Systems tend to be related to sign phenomena and are thus of anticyclotomic nature. We mention in particular the work of Cornut [Cor] and Graham–Shah [GS21], that construct infinite families of Euler Systems. However, there are no explicit reciprocity laws that are available in these cases. While it should be the case, at the moment these Euler Systems are not known to be non-trivial.

Let us also mention the work of Alonso–Castella–Rivero ([ACR23b] and [ACR23a])

which is based on the cycle obtained from embedding a modular curve into a triple product of modular curves. We note that, as in [LLZ15], the anticyclotomic variation of the classes comes from varying the underlying Galois representation – it does not come from the variation of the field of definition of the cycles themselves, as in the Heegner point Euler System or as in the previous examples of this paragraph.

1.1.1.3 The contribution

The literature review of the previous section leads to the following remark.

No cyclotomic Euler system, independent from modular units, has been constructed to date.

Even the cyclotomic units Euler System can be reinterpreted from the point of view of modular units, as is done in [Ber+14, Section 1]: Leopoldt’s formula is a consequence of the relation between Siegel units and Eisenstein series of weight 2. We mention however the recent work of Urban [Urb20] that reconstructs the cyclotomic units Euler System by leveraging congruences between Eisenstein series and cusp forms and the p -adic Langlands correspondence for $\mathbf{GL}_2(\mathbb{Q}_p)$. In work in progress, by generalizing the above mentioned method, Urban and collaborators are also constructing new cyclotomic Euler Systems; the construction is however conditional on some cohomological vanishing conjecture at ‘generic’ Eisenstein maximal ideals.

The main contribution of this thesis is precisely the unconditional construction of a new Euler System that is *completely independent* from modular units. In part I of the dissertation II, in joint work with C. Skinner, I construct a cyclotomic

Euler System for the adjoint of a $\mathbf{PGL}_2$ -modular form and prove the corresponding Iwasawa main conjecture.

1.1.2 The new method

How does such a construction proceed? We start from cohomology classes that are, at first, purely analytic in nature. We then show that these classes can be upgraded to absolute étale cohomology classes: these classes should thus be motivic in origin, but for the purposes of building an Euler System, we do not actually need to show that this is the case.

More precisely, the analytic cohomology classes come from Siegel Eisenstein series of weight 3. These are holomorphic functions on the genus 2 Siegel upper half plane, that can be defined by (the Hecke summation of the series):

$$E_\chi(Z) := \sum_{C,D} \chi(\det(D)) \det(CZ + D)^{-3}.$$

Here χ is an odd Dirichlet character responsible for the cyclotomic variation of the Euler System, and the sum takes place over 2×2 matrices C, D verifying some conditions modulo the conductor of χ . We upgrade the complex de Rham class to an algebraic class by a careful study of Eisenstein cohomology of the Siegel threefold: the algebraicity of the class is a consequence of the Manin–Drinfeld principle. We then analyze the resulting Galois action on the étale class by studying its image in the boundary of the Baily–Borel compactification of the Siegel threefold.

In the process of writing the construction sketched above, the authors noticed

that classical Euler Systems could be recast (namely, the cyclotomic and elliptic units Euler Systems) from this point of view as well. Let me therefore give a more precise, yet still very impressionistic, outline of the method applied to the case of the elliptic units Euler System.

Let ℓ be a prime and let $Y = Y_0(\ell)$ be the (open) modular curve and $X = X_0(\ell)$ be its compactification for the congruence subgroup $\Gamma_0(\ell)$, consisting of matrices whose reduction modulo ℓ are upper triangular. The boundary of X , C , consists of two points: the two cusps usually denoted by ∞ and 0 , both defined over $\mathbb{Q}$. Consider now the Eisenstein series E of weight 2 and level $\Gamma_0(\ell)$ with q -expansion:

$$E(q) = (1 - \ell^{-1})\zeta(-1)/2 + \sum_{n \geq 1} \sum_{\ell \nmid d|n} dq^n.$$

Let τ be the complex variable of Poincaré's upper half plane. The holomorphic differential $\omega_E := 2i\pi E(\tau)d\tau$ descends to Y , and thus defines a class $c_E \in H^1(Y, \mathbb{C})$. The residue map affords an exact sequence:

$$H^1(X, \mathbb{C}) \rightarrow H^1(Y, \mathbb{C}) \twoheadrightarrow H^0(C, \mathbb{C}).$$

The image of c_E in $H^0(C, \mathbb{C})$, is simply identified with $((1 - \ell^{-1})\zeta(-1)/2, 0)$ – here, the first coordinate corresponds to the ∞ cusp, while the second to the 0 cusp. This shows that the image of c_E actually lies in $H^0(C, \mathbb{Q})$ and since the Hecke eigenvalues of c_E violate the Deligne bounds, we deduce that c_E is rational, i.e., $c_E \in H^1(Y, \mathbb{Q})$. Then Artin's isomorphism provides a class $c_E^{\text{et}} \in H_{\text{et}}^1(Y, \mathbb{Q}_p)$. By using the étale analog of the sequence above, one can see that the Galois group of $\mathbb{Q}$ acts on c_E^{et} by

the inverse of the cyclotomic character. We are now ready to use c_E^{et} to build the relevant extension.

Let $\mathcal{K}$ be an imaginary quadratic field. Let $Z \subset Y$ be the collection of CM points attached to $\mathcal{K}$ in bijection with the class group $\text{Cl}_{\mathcal{K}}$ of $\mathcal{K}$. Then $Z \subset Y$ is defined over $\mathcal{K}$ and each point is defined over H , the Hilbert class field of $\mathcal{K}$. We view Z as a subvariety of $Y_{\mathcal{K}}$. Let ψ be a character of the class group of $\mathcal{K}$, and assume for simplicity that ψ is $\mathbb{Q}_p$ -valued. Let $\text{proj}_{\psi} : H_{\text{et}}^0(Z_{\overline{\mathbb{Q}}}, \mathbb{Q}_p) \rightarrow \mathbb{Q}_p(\psi)$ be the $G_{\mathcal{K}}$ -equivariant map given by $(x_{\mathfrak{c}})_{\mathfrak{c} \in \text{Cl}_{\mathcal{K}}} \mapsto \sum_{\mathfrak{c}} \psi(\mathfrak{c})x_{\mathfrak{c}}$. Then, relative étale cohomology for the pair $(Z, Y_{\mathcal{K}})$ affords the following diagram:

$$\begin{array}{ccccccc}
0 & \longrightarrow & \frac{H_{\text{et}}^0(Z_{\overline{\mathbb{Q}}}, \mathbb{Q}_p)}{\text{im} H_{\text{et}}^0(Y_{\overline{\mathbb{Q}}}, \mathbb{Q}_p)} & \longrightarrow & H_{\text{et}}^1(Y_{\overline{\mathbb{Q}}}, Z_{\overline{\mathbb{Q}}}, \mathbb{Q}_p) & \longrightarrow & H_{\text{et}}^1(Y_{\overline{\mathbb{Q}}}, \mathbb{Q}_p) \longrightarrow 0 \\
& & \downarrow \text{proj}_{\psi} & & \downarrow \text{dotted} & & \uparrow \\
0 & \longrightarrow & \mathbb{Q}_p(\psi) & \longrightarrow & \mathcal{E}_{\psi} & \longrightarrow & \mathbb{Q}_p c_E^{\text{et}} \longrightarrow 0
\end{array}$$

The extension class defined by $\mathcal{E}_{\psi}$ is then the base class of an Euler System for $\mathbb{Q}_p(-1)(\psi)$. The other classes are constructed in exactly the same way, replacing E with the Eisenstein series: $E_{\chi}(q) = \sum_{n \geq 1} \sum_{\ell \nmid d|n} \chi(n/d) dq^n$, for a varying Dirichlet character χ . When computing the Bloch–Kato logarithm of the above extension localized at the prime p , one ends up with Katz’s formula for the p -adic L -function for ψ ; namely, the Bloch–Kato logarithm is a multiple of a natural basis element of $D_{\text{cris}}(\mathbb{Q}_p(1)(\psi))$, with that scalar multiple being expressed as

$$\sum_{z \in Z(H)} \psi(z) E^{\text{ord}}(z),$$

where E^{ord} is a weight 0, p -ordinary Eisenstein series.

We see that central in this example is the pair of algebraic varieties (Z, Y) , the extensions (or Galois cohomology classes) being carved out of the relative cohomology of this pair. In the case of a Dirichlet character of $\mathbb{Q}$, this pair is replaced by (C, X) , where C is a collection of cusps, and in the adjoint case, by $(Y \times Y, S)$, where S is a Siegel threefold and Y is an open modular curve.

1.1.3 Families of Euler Systems

In Part II, we have decided to formulate the construction of our Euler System from the point of view of orthogonal Shimura varieties; we believe our Euler System naturally fits into a trilogy, and possibly into a much larger family of Euler Systems. I call this trilogy of Euler Systems the 'pull-back trilogy.' The overarching principle of this trifecta of Euler Systems is the following: one considers cohomology classes defined by a family of Eisenstein series on a big orthogonal group $\mathbf{G}$ and *pulls them back* to a smaller orthogonal group $\mathbf{H} \hookrightarrow \mathbf{G}$.

Is there a similar Euler System for $\mathbf{SO}(2, 3) \hookrightarrow \mathbf{SO}(2, 4)$? and more generally $\mathbf{SO}(2, r) \hookrightarrow \mathbf{SO}(2, r + 1)$? Initially, I thought that this should be the case. However, when $r > 2$ one realizes that more sophisticated Eisenstein series than the ones considered in Part II of this dissertation have to be studied. Namely, in the case of $\mathbf{SO}(2, 2) \hookrightarrow \mathbf{SO}(2, 3)$, we consider Eisenstein series induced from a parabolic subgroup that has Levi quotient $\mathbf{G}_m \times \mathbf{SO}(2, 1)$; the relevant representation on the $\mathbf{SO}(2, 1)$ -part being induced is the trivial representation. In higher rank situations $\mathbf{SO}(2, r) \hookrightarrow \mathbf{SO}(2, r + 1)$, if one is considering the analogous parabolic subgroup

Euler System	Embedding of Groups	Period Integral
Elliptic Units	$\mathbf{SO}(2) \hookrightarrow \mathbf{SO}(2, 1)$	Damerell/Waldspurger
Kato	$\mathbf{SO}(2, 1) \hookrightarrow \mathbf{SO}(2, 2)$	Rankin–Selberg
Adjoint	$\mathbf{SO}(2, 2) \hookrightarrow \mathbf{SO}(2, 3)$	Murase–Sugano/Doubling

Table 1.1: The Pull-back trilogy

Note. Here $\mathbf{SO}(2)$ is the special orthogonal algebraic group defined by the norm form of a quadratic imaginary field $\mathcal{K}$. The Damerell/Waldspurger period integral is the formula underlying Katz’s construction of the p -adic L -function for Hecke characters of CM fields. I have included descriptions of period integrals, as often both norm relations and explicit reciprocity law rely heavily on the existence of such integral representation formulae.

of $\mathbf{SO}(2, r + 1)$ with Levi quotient $\mathbf{G}_m \times \mathbf{SO}(1, r)$, one has to choose a more sophisticated representation of $\mathbf{SO}(1, r)$ to be induced up to $\mathbf{SO}(2, r + 1)$. Indeed, even though by inducing the trivial representation arithmetic Eisenstein classes can still be constructed, the Murase–Sugano period integral will unfortunately vanish unless the Eisenstein series is paired with a highly degenerate automorphic form of $\mathbf{SO}(2, r)$, i.e., whose corresponding Galois representations have very small image: the L -parameters have to factor through a copy of $\mathbf{Sp}_2$ or $\mathbf{SO}(2, 1)$.

There is a ‘dual’ picture to the pull-back trilogy described in Table 1.1. This is the ‘push-forward trilogy,’ where Eisenstein classes on Shimura varieties for an orthogonal group $\mathbf{H}$ are *pushed-forward* to cohomology classes on a Shimura variety for a larger orthogonal group $\mathbf{G}$. The pushforward trilogy should also possess a continuation into an infinite family of Euler Systems.

Although the framework described above looks very promising, it seems very difficult to extend the above trilogies to higher rank. There are two major sources of complications: on the one hand, for rank $r > 2$, there is no theory of q -expansions for

Euler System	Embedding of Groups	Period Integral
Heegner point	$\mathbf{SO}(2) \hookrightarrow \mathbf{SO}(2, 1)$	Waldspurger
Beilinson-Flach	$\mathbf{SO}(2, 1) \hookrightarrow \mathbf{SO}(2, 2)$	Rankin-Selberg
LSZ	$\mathbf{SO}(2, 2) \hookrightarrow \mathbf{SO}(2, 3)$	Piatetski-Shapiro

Table 1.2: The Push-forward trilogy

Note. The Euler System of Beilinson-Flach classes is constructed by Lei-Loeffler-Zerbes in [LLZ14]. The acronym LSZ stands for Loeffler-Skinner-Zerbes, and refers to the Euler System constructed in [LSZ21].

orthogonal Shimura varieties which leads to severe technical difficulties in showing the integrality of the Eisenstein classes. On the other hand, the theory of p -adic modular forms on orthogonal groups is still in its infancy, which makes proving any explicit reciprocity law an arduous undertaking.

However, it is possible to overcome these difficulties in the case of unitary groups. In work in progress with C. Skinner [SS], we build an infinite family of Euler Systems for certain Galois representations appearing in the middle cohomology of Shimura varieties attached to unitary groups $\mathbf{U}(n, 1)$ over $\mathbb{Q}$ with signature at infinity $(n, 1)$, for any integer $n \geq 0$. In this case, we start the construction from a Siegel Eisenstein series on $\mathbf{U}(n + 1, n + 1)$; we use Shimura’s pull-back formula to obtain cohomology classes attached to Klingen Eisenstein series on $\mathbf{U}(n + 1, 1)$ and we pull-back these classes to $\mathbf{U}(n, 1)$ – in a completely analogous fashion to the pull-back trilogy.

1.2 The theorems of the thesis

Having reviewed the literature and general philosophy underpinning the new method that is developed in this dissertation, we shall now state the main theorems.

1.2.1 The Theorems of Part I

In this part, joint with C. Skinner, we construct an Euler System for the adjoint of an elliptic modular form and prove the Iwasawa main conjecture for a large class of modular forms. The order of the Bloch–Kato Selmer group of the adjoint of a modular form is intimately related to tangent spaces of universal deformation rings of the corresponding residual Galois representations. This relationship is at the heart of Taylor–Wiles’ ([Wil95], [WT95]) proof of their celebrated modularity lifting theorem, and hence of Fermat’s Last Theorem. Three decades after this seminal work, the Iwasawa main conjecture for the adjoint of a modular form has remained open despite several attempts, most notably [LZ15] and very recently [LZ23].

The main strategy of attack has been to produce a Kolyvagin System out of classes constructed by pushing-forward modular units to a product of modular curves ([LLZ14], [LZ15]) or to a Hilbert surface ([LZ23]). Indeed Wiles’ initial attempt at proving the necessary upper bound on the order of the adjoint Selmer group was to arrange Flach’s classes [Fla92] into a Kolyvagin System. In this paper, we propose a completely different approach, where modular units – or motivic cohomology classes – make no appearance in the *construction* of the Euler System. The only place, albeit crucially, where we leverage the existence of Beilinson–Flach’s classes, is the section studying the trivial zero of the p -adic L -function: we need to invoke a Theorem of Dasgupta [Das16] that identifies the analytic L -invariant with the algebraic L -invariant as defined by Greenberg [Gre94]. Dasgupta’s theorem does leverage Beilinson–Flach’s classes.

Namely, we begin by considering de Rham cohomology classes defined by certain holomorphic Siegel Eisenstein series. These classes appear in the middle degree cohomology of the Siegel threefold and are very degenerate, in that they inherit all essential features from their image to the point boundary stratum of the Baily–Borel compactification. This degeneracy allows us to upgrade these de Rham classes to geometric étale classes on which the Galois group of $\mathbb{Q}$ acts through its abelianization. We then produce Galois extension classes by pulling-back this collection of étale classes to a product of modular curves. We then package these extensions into an Euler System. The construction of the Euler System concludes the first part of the paper. In the second part, we compute the big Perrin-Riou logarithm of the base classes and show that it coincides with Liu’s p -adic L -function [Liu20]. We deduce the Iwasawa main conjecture by showing that the upper bound that the Euler–System machinery affords is sharp, provided the L -invariant does not vanish. We then deduce the main conjecture from congruences between members of a Hida family.

Our method for constructing interesting étale classes in the cohomology of Shimura varieties is very flexible. Indeed, it can be applied to retrieve many classical Euler Systems (e.g. the cyclotomic and elliptic unit Euler Systems, Kato’s Euler System...) – this was the subject of a series of lectures of the authors gave at the MSRI during the Spring special semester on Euler Systems and special values of L -functions. In work in progress [SS], we apply these ideas to construct an Euler System for certain Galois representations realized in the étale cohomology of some unitary Shimura varieties with very simple boundary.

The reader will notice that we have written most of the paper in terms of or-

thogonal groups. As is explained in section 1.1.3, this is because the authors believe that, once the theory of p -adic modular forms on orthogonal groups is improved, this approach should generalize to pulling back certain Eisenstein series on $\mathbf{SO}(n+1, 2)$ to $\mathbf{SO}(n, 2)$. In this paper, we take $n = 2$ and use the exceptional isomorphism $\mathbf{PGSp}_4 \simeq \mathbf{SO}(3, 2)$. We note here that Kato's Euler System is the case of $n = 1$ and the Euler system of elliptic units is the case $n = 0$.

We will now introduce relevant notation and state the main Theorems that we prove in this work. Let χ be an odd Dirichlet character; denote by $F_{\chi, p}$ the finite extension of $\mathbb{Q}_p$ obtained by adjoining the values of χ . We let $\mathcal{O}_{\chi, p}$ the ring of integers of $F_{\chi, p}$. Define $\mathbf{G} := \mathbf{SO}(3, 2)$ and $\mathbf{H} := \mathbf{SO}(2, 2)$; we denote by $\mathrm{SH}_{\mathbf{G}}$ and $\mathrm{SH}_{\mathbf{H}}$ the corresponding Shimura varieties. In the introduction, we omit for simplicity level subgroups from the notations and all étale cohomology groups are geometric. In sections 3.1, 3.2 and 3.3 we construct Eisenstein classes

$$\omega_{\chi}^{\mathrm{et}} \in H_{\mathrm{et}}^3(\mathrm{SH}_{\mathbf{G}}, \mathcal{O}_{\chi, p}(3)),$$

such that if $\sigma \in \mathrm{Gal}_{\mathbb{Q}}$, then $\sigma(\omega_{\chi}^{\mathrm{et}}) = \chi^{-1}\varepsilon^{-3}(\sigma)\omega_{\chi}^{\mathrm{et}}$, where ε is the cyclotomic character and χ is seen as a character of $\mathrm{Gal}_{\mathbb{Q}}$ by class field theory (geometrically normalized).

In section 3.4, we begin the construction of the Euler System. We discuss briefly the main ingredients employed in the construction. The closed embedding of $\mathbb{Q}$ -varieties $\iota : \mathrm{SH}_{\mathbf{H}} \hookrightarrow \mathrm{SH}_{\mathbf{G}}$ yields a short exact sequence of étale sheaves on $\mathrm{SH}_{\mathbf{G}}$:

$$0 \longrightarrow j_! \underline{\mathcal{O}_{\chi, p}} \longrightarrow \underline{\mathcal{O}_{\chi, p}} \longrightarrow \iota_* \underline{\mathcal{O}_{\chi, p}} \longrightarrow 0,$$

where j is the inclusion of the complement of $\mathrm{SH}_{\mathbf{H}}$. This exact sequence induces a long exact sequence

$$\cdots \rightarrow H_{\mathrm{et}}^2(\mathrm{SH}_{\mathbf{H}}, \mathcal{O}_{\chi,p}) \rightarrow H_{\mathrm{et}}^3(\mathrm{SH}_{\mathbf{G}}, j! \mathcal{O}_{\chi,p}) \rightarrow H_{\mathrm{et}}^3(\mathrm{SH}_{\mathbf{G}}, \mathcal{O}_{\chi,p}) \rightarrow 0.$$

In the paper, p denotes an **odd** prime, fixed once and for all. Let f be a newform of weight 2 with trivial central character, and $\pi(f)$ be the automorphic representation of $\mathbf{PGL}_2(\mathbb{A})$ that is generated by f . We assume for simplicity in the introduction that $\pi(f)$ is fixed by the usual congruence subgroup $\Gamma_0(N) \subset \mathbf{PGL}_2(\mathbb{A}_f)$ of matrices that become upper triangular modulo N . Denote by ϕ_f be the Hecke-Eigensystem of f . We assume that $\mathcal{O}_{\chi,p}$ contains the Hecke eigenvalues for f . Let T_f be the free $\mathcal{O}_{\chi,p}[\mathrm{Gal}_{\mathbb{Q}}]$ -module of rank 2, defined to be the maximal quotient of $H_{\mathrm{et}}^1(\mathrm{SH}_{\mathbf{PGL}_2}, \mathcal{O}_{\chi,p}(1))$ on which the spherical Hecke algebra acts by ϕ_f . We let $\mathfrak{m}$ be the maximal ideal of the Hecke algebra defined by the reduction of $\phi_f, \overline{\phi}_f$, modulo a uniformizer of $\mathcal{O}_{\chi,p}$. Then, $\mathfrak{m}$ defines a maximal ideal $\mathfrak{M}$ of the spherical Hecke algebra for $\mathbf{H}$, defined by the algebra homomorphism $\overline{\phi}_f \boxtimes \overline{\phi}_f$. We assume for simplicity in the introduction that $\mathfrak{m}$ is not dihedral and non-Eisenstein. Under this hypothesis, localizing $H_{\mathrm{et}}^2(\mathrm{SH}_{\mathbf{H}}, \mathcal{O}_{\chi,p})$ at $\mathfrak{M}$, we obtain by push-out an extension:

$$0 \rightarrow H_{\mathrm{et}}^2(\mathrm{SH}_{\mathbf{H}}, \mathcal{O}_{\chi,p})_{\mathfrak{M}} \rightarrow \mathcal{E}_{\mathfrak{M}} \rightarrow H_{\mathrm{et}}^3(\mathrm{SH}_{\mathbf{G}}, \mathcal{O}_{\chi,p}) \rightarrow 0.$$

The classes $\omega_{\chi}^{\mathrm{et}}$ define by pull-back Galois cohomology classes

$$c_{\chi} \in H^1\left(\mathbb{Q}, H_{\mathrm{et}}^2(\mathrm{SH}_{\mathbf{H}}, \mathcal{O}_{\chi,p}(\varepsilon^3 \chi))_{\mathfrak{M}}\right).$$

We now project, after possibly taking a suitable trace, these classes down to $\mathrm{Ad}^0(T_f)(\varepsilon^2\chi)$ and obtain classes

$$c_\chi(f) \in H^1(\mathbb{Q}, \mathrm{Ad}^0(T_f)(\varepsilon^2\chi)).$$

By varying local sections away from p defining the étale classes $\omega_\chi^{\mathrm{et}}$ and by varying the character χ , we obtain the first main theorem of the paper (see Theorem 3.4.11.1).

Theorem A1: *Let χ_0 be an odd power of the Teichmüller character. Let S be the set of primes dividing the level N of the modular form f . Assume that f is ordinary at p ; that is either $p \nmid N_f$ and f admits an ordinary stabilization at p , or f is new at p and ordinary. If $p = 3$ or 5 , assume also that the level of f contains enough primes in the sense of assumption 3.3.5.2. Let $P_\ell(X) \in \mathcal{O}[X]$ be the Hecke polynomial at ℓ defined for $\ell \notin S$ as*

$$P_\ell(X) = \det(1 - X \mathrm{Frob}_\ell | \mathrm{Ad}^0(T_f)(\varepsilon^{-1}\chi_0^{-1})).$$

There exists a collection of classes (c_{Mp^a}) indexed by positive integers M and a , with M coprime to the primes of S , such that

$$c_{Mp^a} \in H^1(\mathbb{Q}(\mu_{p^aM})^+, \mathrm{Ad}^0(T_f)(\varepsilon^2\chi_0)).$$

Moreover, for any prime $\ell \notin S$

$$\mathrm{Cores}_{\mathbb{Q}(\mu_{Mp^a})^+}^{\mathbb{Q}(\mu_{M\ell p^a})^+}(c_{M\ell p^a}) = P_\ell(\mathrm{Fr}_\ell^{-1})c_{Mp^a}.$$

For all $a \geq 1$, we have also

$$\text{Cores}_{\mathbb{Q}(\boldsymbol{\mu}_{Mp^a})^+}^{\mathbb{Q}(\boldsymbol{\mu}_{Mp^{a+1}})^+}(c_{M\ell p^a}) = c_{Mp^a}.$$

Theorem A1 states that we have constructed Euler Systems in the sense of Rubin [Rub00]. Even better, we can interpolate p -adically the Euler systems constructed above by varying f in a Hida family thanks to results of Ohta [Oht95]. Let $\mathbf{f}$ be a branch of a Hida family passing through f . Let $\mathbb{Q}_\infty$ be the unique $\mathbb{Z}_p$ extension of $\mathbb{Q}$ and let $\Gamma := \text{Gal}(\mathbb{Q}_\infty/\mathbb{Q}) \simeq 1 + p\mathbb{Z}_p$ where we see $1 + p\mathbb{Z}_p$ as a quotient of $\mathbb{Z}_p^\times \simeq \text{Gal}(\mathbb{Q}(\boldsymbol{\mu}_{p^\infty})/\mathbb{Q})$. Let $\Lambda := \mathcal{O}[[\Gamma]]$ and let $T_{\mathbf{f}}$ the free rank 2 Λ -module, as constructed by Ohta [Oht99] interpolating the lattices T_f , for f a classical specialization of $\mathbf{f}$ of weight $k \geq 2$.

Theorem A2: *There exists a collection of classes $(\underline{\mathcal{C}}_M^{ord}(\mathbf{f}))$ indexed by positive integers M , coprime to the primes of S , such that*

$$\underline{\mathcal{C}}_M^{ord}(\mathbf{f}) \in H^1\left(\mathbb{Q}(\boldsymbol{\mu}_M)^+, \text{Ad}^0(T_{\mathbf{f}})(1) \otimes \Lambda\right).$$

Moreover for any prime $\ell \notin S$ these classes verify:

$$\text{Cores}_{\mathbb{Q}(\boldsymbol{\mu}_M)^+}^{\mathbb{Q}(\boldsymbol{\mu}_{M\ell})^+}(\underline{\mathcal{C}}_{M\ell}^{ord}(\mathbf{f})) = P_\ell(\text{Fr}_\ell^{-1})\underline{\mathcal{C}}_M^{ord}(\mathbf{f}).$$

Theorem A2 concludes the first part of the paper dealing with the construction of the Euler System. In Section 4, we start relating the classes of Theorem A2 to a p -adic

L -function constructed by Liu in [Liu20]– the ordinarity assumption becomes crucial in this section. The ordinary $\mathrm{Gal}_{\mathbb{Q}_p}$ -equivariant filtration $T_{\mathbf{f}}^+ \subset T_{\mathbf{f}}$ on $T_{\mathbf{f}}$ induces a three-step filtration on $\mathrm{Ad}^0(T_{\mathbf{f}})$. We let $\mathrm{Ad}^0(T_{\mathbf{f}})^+$ be the traceless endomorphisms of $T_{\mathbf{f}}$ that stabilize the line $T_{\mathbf{f}}^+$ and denote by $\mathrm{Ad}^0(T_{\mathbf{f}})^-$ the one-dimensional quotient of $\mathrm{Ad}^0(T_{\mathbf{f}})$ by $\mathrm{Ad}^0(T_{\mathbf{f}})^+$. Then, following [KLZ17], there is a morphism of $\Lambda_{\mathbf{f}} \hat{\otimes} \Lambda$ -modules that interpolates Perrin-Riou’s big logarithm in the sense of Proposition 4.2.2.1:

$$\mathcal{L} : H^1\left(\mathbb{Q}_p, \mathrm{Ad}^0(T_{\mathbf{f}})^-(1) \otimes \Lambda\right) \longrightarrow \mathbf{D}_{\mathrm{ur}}(\mathrm{Ad}^0(T_{\mathbf{f}})^-) \hat{\otimes} \Lambda$$

where $\mathbf{D}_{\mathrm{ur}}$ sends, after a suitable twist, a p -adically complete $\mathbb{Z}_p[\mathrm{Gal}_{\mathbb{Q}_p}]$ -module M to $(M \hat{\otimes}_{\mathbb{Z}_p} \widehat{\mathbb{Z}}_p^{\mathrm{ur}})^{\mathrm{Gal}_{\mathbb{Q}_p}}$. Section 4 is then dedicated to computing $\mathcal{L}(\mathcal{C}_p(\mathbf{f})^-)$, where $\mathcal{C}_p(\mathbf{f})^-$ denotes the image of $\mathcal{C}_1^{\mathrm{ord}}(\mathbf{f})$ in $H^1(\mathbb{Q}_p, \mathrm{Ad}^0(T_{\mathbf{f}})^-(1) \otimes \Lambda)$. We relate $\mathcal{L}(\mathcal{C}_p(\mathbf{f})^-)$ to Liu’s p -adic L -function [Liu20], which we denote by $\mathcal{L}_{\mathbf{f}}$. We refer the reader to Theorem 4.7.2.4 for a more precise relationship between $\mathcal{L}(\mathcal{C}_p(\mathbf{f}))$ and $c_{\mathbf{f}}\mathcal{L}_{\mathbf{f}}$. Let $C_{\mathbf{f}}$ be Hida’s congruence ideal for $\mathbf{f}$ and let $c_{\mathbf{f}} \in \Lambda_{\mathbf{f}}$ a generator.

Theorem B: *There is an equality of ideals in $\Lambda_{\mathbf{f}} \hat{\otimes} \Lambda$*

$$(\mathcal{L}(\mathcal{C}_p(\mathbf{f}))) = (c_{\mathbf{f}}\mathcal{L}_{\mathbf{f}}).$$

In Section 5, we give some arithmetic consequences of Theorem A2 and Theorem B, by invoking Rubin’s Euler System machinery [Rub00]. The major difficulty that we have to face is the trivial zero of the p -adic L -function $\mathcal{L}_{\mathbf{f}}$ at $s = 1$. In general, the presence of this trivial zero is explained by the fact that the crystalline Frobenius of

the Galois representation under consideration has 1 as an eigenvalue, as is the case for the adjoint of an elliptic modular form. Greenberg [Gre94] has investigated the algebraic implications of the presence of such a zero; he studies in particular how the algebraic L -invariant appears in the control theorem for the Λ -adic Selmer group. By a Theorem of Dasgupta [Das16] that relates the algebraic L -invariant to the analytic L -invariant, we are able to show – in the case where the L -invariant does not vanish – that the bound on the Selmer group afforded by Theorem B is tight. This uses the existence of the Galois deformation theoretic surjection

$$R^{ord} \twoheadrightarrow \mathbb{T}^{ord};$$

here $\mathbb{T}^{ord}$ is the localization at a suitable maximal ideal of Hida’s ordinary Hecke algebra, and R^{ord} is the universal p -ordinary deformation ring attached to $\rho_{\mathfrak{m}}$ (here $\rho_{\mathfrak{m}}$ is the residual representation of T_f) as defined by Hida [Hid21, Section 6.1.1].

We are then able to reprove a version of the modularity theorem of Taylor–Wiles; contrary to Taylor–Wiles we only allow ordinary deformations and do not allow flat deformations, however we allow for arbitrary level away from p . We refer to Theorem 5.3.0.1 for a more general statement.

Theorem C: (Taylor-Wiles) *The surjection*

$$R^{ord} \twoheadrightarrow \mathbb{T}^{ord}$$

is an isomorphism.

We note here that building an Euler System system for the adjoint of a modular

form of weight 2 was Wiles' initial strategy to prove his celebrated modularity theorem above and thus proving Fermat's last Theorem. We also note that Theorem C relies on the necessity of accessing modular forms of higher weight, as the L -invariant could very well vanish in weight 2. Leveraging congruences between forms with (possibly) non-vanishing L -invariant with forms with vanishing L -invariant, we obtain the Iwasawa main conjecture for adjoint Galois representations of a large class of modular forms. To state the relevant theorem we first introduce some notation.

Let S be a finite set of places containing p and $\mathbb{Q}_S$ be the maximal unramified extension of $\mathbb{Q}$ unramified outside S and ∞ . Let G_S be the Galois group of $\mathbb{Q}_S$ over $\mathbb{Q}$.

$$\mathrm{Sel}_{\mathbb{Q}_\infty}^{\mathrm{ur}}(\mathcal{M}_f) := \ker \left\{ H^1(G_S, \mathcal{M}_f) \longrightarrow H^1(I_p, \mathcal{M}_f^-) \times \prod_{\ell \in S, \ell \neq p} H^1(I_\ell, \mathcal{M}_f) \right\}.$$

We let $X_{\mathbb{Q}_\infty}^{\mathrm{ur}}(\mathcal{M}_f)$ be the Pontryagin dual of $\mathrm{Sel}_{\mathbb{Q}_\infty}^{\mathrm{ur}}(\mathcal{M}_f)$. We refer the reader to Theorem 5.4.0.2 for a more general version of the following theorem.

Theorem D: *Let f be an even weight k , trivial Nebentypus, specialization of the Hida family $\mathbf{f}$. Let c_f be a generator of the congruence ideal for f . The discrete Λ -module $X_{\mathbb{Q}_\infty}^{\mathrm{ur}}(\mathcal{M}_f)$ is torsion and we have the equality of characteristic ideals:*

$$\mathrm{Char}_\Lambda \left(X_{\mathbb{Q}_\infty}^{\mathrm{ur}}(\mathcal{M}_f) \right) = (c_f \mathcal{L}_f).$$

1.2.2 The Theorem of Part II

In Part III, I construct certain families of theta classes in the absolute étale cohomology of modular curves. This construction is the first step in building a two variable Euler System for the base change of an elliptic modular form to a quadratic imaginary field $\mathcal{K}$. We note in the case where p splits in $\mathcal{K}$, a similar Euler System can be constructed using Beilinson–Flach classes, as showed in [LLZ15]. The main innovation that I propose consists in working at an inert prime p . Of course, a similar (and easier) version of the construction works as well when p splits in $\mathcal{K}$. I omit the construction in this case, as it is more standard: general theorems of Ohta [Oht99] can be applied.

With respect to [LLZ15], there is another significant advantage in our point of view. Not only the ‘pull-back’ Euler System built with these theta classes should pass through the trivial character in the critical range, but it should naturally be related to Kato’s Euler System. Let me briefly explain why the two are related.

As is explained in section 1.1.1.1, Kato pulls back to a modular curve two absolute étale classes, defined over abelian extensions of $\mathbb{Q}$. These étale classes come from a tensor product of weight 2 Eisenstein series. The idea is simply to replace one of the weight 2 Eisenstein series with a theta series: the resulting absolute étale classes are defined over abelian extensions of $\mathcal{K}$. By the same methods of Kato, one can show that the Bloch–Kato logarithm of the resulting Galois cohomology classes is related to the integral of the cusp form of interest with a product of weight 1 Eisenstein series and a weight 1 theta series. The relation with Kato’s Euler System is then obtained

by the Siegel–Weil formula, after specializing to the cyclotomic variable. Therefore, this new Euler System should interpolate Kato’s Euler System in two variables – thus showing that the Iwasawa main conjecture (at least for ordinary forms) over $\mathbb{Q}$ implies the Iwasawa main conjecture over $\mathcal{K}$.

For applications to Iwasawa theory, the main issue in carrying out the strategy described above at inert primes is the existence of the big Perrin-Riou logarithm in the context of non-abelian extension of $\mathbb{Q}_p$. Thankfully, this big Perrin-Riou logarithm has very recently been constructed by Schneider–Venjakob [SV23]. What is still missing from the literature is the interpolation formulae at non trivial finite order characters of this big logarithm; the analogous formulae in the cyclotomic case are proven in [LZ14]. As the author hasn’t established these formulae in the Lubin–Tate setting yet, Theorem E below is subject to being slightly modified in the future. Namely, the periods $\Omega_\chi^{\text{univ}}$, which are anticyclotomic analogs of Gauss sums, are rather *ad-hoc*: it should be verified that these periods coincide with the periods appearing in the interpolation formulae of the Schneider–Venjakob big Perrin-Riou logarithm.

Let me now introduce some notation in order to state the main theorem of Part III. Let p be an odd rational prime that is inert in a quadratic imaginary field $\mathcal{K}$ with discriminant $\Delta_{\mathcal{K}}$. Let μ be a fixed Hecke character of infinity type $(1, 0)$ and conductor p . Fix embeddings $\iota_p : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}_p$ and $\iota_\infty : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$ and an isomorphism $\mathbb{C} \simeq \mathbb{C}_p$, so that ι_p and ι_∞ are compatible. Let $F/\mathbb{Q}_p$ be the finite extension of $\mathbb{Q}_p$ containing the values of the p -adic Hecke character $\hat{\mu}$ associated to μ and the image of the Hilbert class field of $\mathcal{K}$. Denote by $\mathcal{O}_F$ be the valuation ring of F . Let $\rho_{\mu,p}$ be

the p -adic Galois representation associated to $\widehat{\mu}$, by the geometric normalization of the Artin isomorphism

$$\mathrm{Art}_{\mathcal{K}} : G_{\mathcal{K}}^{\mathrm{ab}} \xrightarrow{\sim} \mathcal{K}^{\times} \backslash \mathbb{A}_{\mathcal{K}}^{(\infty), \times},$$

where $\mathbb{A}_{\mathcal{K}}^{(\infty)}$ denotes the finite adèles of $\mathcal{K}$. We view $\rho_{\mu,p}$ as a rank-one $\mathcal{O}_F$ -module, such that the action of an element $\sigma \in G_{\mathcal{K}}^{\mathrm{ab}}$ whose image in $\mathcal{K}^{\times} \backslash \mathbb{A}_{\mathcal{K}}^{(\infty), \times}$ is a , is multiplication by $a_p \mu(a)^{-1}$, where a_p is the p -component of $a = (a_v)_{v < \infty}$. Let $(A, 1 \bmod p)$ be the canonical CM -pair in the sense of section 8.2.1, so that $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathcal{K}[p])$ acts by $\rho_{\mu,p}^{-1}|_{\mathcal{K}[p]}$ on $H_{\mathrm{et}}^1(A/\overline{\mathbb{Q}}, \mathcal{O}_F(1))$. As explained in section 8.2.1, the Galois action on $H_{\mathrm{et}}^1(A/\overline{\mathbb{Q}}, \mathcal{O}_F(1))$ can be extended to $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathcal{K})$, so that it coincides with $\rho_{\mu,p}$.

Let E be the completion of $\mathcal{K}$ at p and $\Delta_E \simeq (\mathcal{O}_{\mathcal{K}}/p\mathcal{O}_{\mathcal{K}})^{\times} \subset G_E$ be the tame part of inertia. Denote by $\mathcal{R}'_{\infty}$ the compositum of all the ring-class fields of $\mathcal{K}$ of conductor co-prime to $\Delta_{\mathcal{K}}$, squarefree away from p . Put

$$\mathcal{R}_{\infty} := (\mathcal{R}'_{\infty})^{\Delta_E}.$$

For a square free integer n and $r \geq 1$, let $\mathcal{R}[np^r] \subset \mathcal{R}_{\infty}$ be the subfield of the ring class field of conductor np^r fixed by Δ_E .

Consider ξ a character of the finite Galois group $\mathrm{Gal}(\mathcal{R}[np^r]/\mathcal{R})$. Then the product $\mu\xi$ is a Hecke character of infinity type $(1, 0)$ and conductor np^r . Let $\theta_{\mu\xi}$ the new-form defined by theta lifting $\mu\xi$ to $\mathbf{GL}_2$ suitably normalized as in Definition 8.1.2.7. If F_{ξ} is the finite extension of F generated by the values of ξ , $\theta_{\mu\xi}$ defines an

algebraic de Rham class

$$\vartheta_{\mu\xi}^{\mathrm{dR}} \in H_{\mathrm{dR}}^1(X_1(\Delta_{\mathcal{K}} n^2 p^{2r})/F_{\xi}),$$

where $X_1(\Delta_{\mathcal{K}} n^2 p^{2r})$ denotes the compactified modular curve of level $\Gamma_1(\Delta_{\mathcal{K}} n^2 p^{2r})$.

Consider the geometric étale cohomology of $X_1(\Delta_{\mathcal{K}} n^2 p^{2r})$ with coefficients in $\mathcal{O}_F$, viewed as a $G_{\mathcal{K}}^{ab}$ -representation

$$\begin{aligned} H_{\mathrm{et}}(\mu, np^r) &:= H_{\mathrm{et}}^1(X_1(\Delta_{\mathcal{K}} n^2 p^{2r})/\overline{\mathbb{Q}}, \mathcal{O}_F)_{cm}^{ac} \otimes H_{\mathrm{et}}^1(A/\overline{\mathbb{Q}}, \mathcal{O}_F(1)) \\ &\simeq H_{\mathrm{et}}^1(X_1(\Delta_{\mathcal{K}} n^2 p^{2r})/\overline{\mathbb{Q}}, \mathcal{O}_F)_{cm}^{ac} \otimes \rho_{\mu,p}^{-1}. \end{aligned}$$

Here the subscript cm indicates the CM part of cohomology: that is, the subspace cut out by the kernel of the usual Hecke operators T_{ℓ} for ℓ inert in $\mathcal{K}$ and coprime to $\Delta_{\mathcal{K}} np$; and the superscript ac denotes the fixed points under the action of $\mathrm{Gal}(\mathbb{Q}^{ab}/\mathcal{K})$. Note that the $G_{\mathcal{K}}$ action on $H_{\mathrm{et}}(\mu, np^r)$ is abelian and factors through $\mathrm{Gal}(\mathcal{R}_{\infty}/\mathcal{K})$.

The combination of Theorem 8.2.3.10 and Proposition 8.2.4.2 yields Theorem E below. We note that this theorem uses crucially local input from Iwasawa theory at inert primes as developed by Burungale–Kobayashi–Ota [BKO23], [BKO].

Theorem E: Fix μ a Hecke character of infinity type $(1, 0)$ and conductor p . Let (A, α) be the canonical CM-pair in the sense of section 8.2.1.1 such that $(A, \alpha)_{/\mathbb{C}} \simeq (\mathbb{C}/p\mathcal{O}_{\mathcal{K}}, 1 \bmod p\mathcal{O}_{\mathcal{K}})$. For all square-free integers n and $r \geq 1$, there exists a unique class

$$\Theta_{\mu}(np^r) \in H_{\text{et}}(\mu, np^r)$$

such that

1. $\Theta_{\mu}(np^r)$ generates the regular representation of $\text{Gal}(\mathcal{R}[np^r]/\mathcal{K})$.
2. $\Theta_{\mu}(np^r)$ admits a unique lift

$$\Theta_{\mu}^{\text{abs}}(np^r) \in H_{\text{et}}^2\left(X_1(\Delta_{\mathcal{K}} n^2 p^{2r}) \times A/\mathcal{R}[np^r], \mathcal{O}_F(1)\right).$$

3. For all characters ξ of $\text{Gal}(\mathcal{R}[np^r]/\mathcal{K})$, we have

$$\text{comp}_{\text{dR}}^{\text{et}} \left(\sum_{\sigma \in \text{Gal}(\mathcal{R}[np^r]/\mathcal{K})} \xi^{-1}(\sigma) \sigma^* \Theta_{\mu}(np^r) \right) = \pi_{\xi}^* \vartheta_{\mu\xi}^{\text{dR}} \otimes \Omega_+(\Omega_{(\mu\xi)^{-1}}^{\text{univ}})^{-1}.$$

Here the notation is as follows:

- The map $\text{comp}_{\text{dR}}^{\text{et}}$ is the étale-de Rham p -adic comparison isomorphism for $X = X_1(\Delta_{\mathcal{K}} n^2 p^{2r})$:

$$\text{comp}_{\text{dR}}^{\text{et}} : H_{\text{dR}}^{\bullet}(X/F) \otimes_F \mathbf{B}_{\text{dR}} \xrightarrow{\sim} H_{\text{et}}^{\bullet}(X/\overline{\mathbb{Q}_p}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathbf{B}_{\text{dR}}.$$

- The map π_{ξ}^* depends only on the conductor of ξ : if the conductor of ξ is

dp^t with $d|n$ and $t \leq r$, then π_ξ is the composition of the n/d -depletion map and the pull-back under the map given by $z \mapsto p^{2(r-t)}z$ on the upper half plane.

- The period $\Omega_{\mu\xi}^{\text{univ}}$ is an 'anticyclotomic local Gauss sum' for $\mu\xi|_{G_E}$ as defined in Section 8.2.2.7, and $\Omega_+ \in \mathbf{B}_{\text{dR}}$ is Colmez's Lubin–Tate period (see section 8.2.2.3).

Part II

An Euler System for the Adjoint of a Modular Form

Marco Sangiovanni Vincentelli and Christopher Skinner.

Chapter 2

Introduction

In this paper we construct an Euler system for the adjoint of an elliptic modular newform. We also prove the corresponding Iwasawa main conjecture for a large class of such newforms.

Euler systems, when they are known to exist, have proved to be a powerful and flexible tool for establishing bounds on the ranks and orders of Selmer groups and for relating these to special values of L -functions. However, by some counts, only a handful of Euler systems have been constructed to date. The purpose of this article is two-fold: (1) to construct an Euler system for the adjoint of a newform, a case of particular interest because of the connection of the corresponding Selmer groups with deformations of Galois representations and especially ‘ $R = \mathbb{T}$ ’ theorems, and (2) provide a proof of concept for a new method of constructing Euler systems, one that applies to infinite families of Galois representations.

Throughout this paper, $p > 2$ is a fixed prime.

2.1 Results about Euler systems

Let f be a weight 2 newform (so a cuspform) of level N . Let $\mathbb{Q}(f)$ be the finite extension generated by the Hecke eigenvalues of f (equivalently, by the coefficients of the usual q -expansion $f = \sum_{n=1}^{\infty} a_n q^n$). We suppose that we have fixed an embedding $\mathbb{Q}(f) \hookrightarrow F$ with F a finite extension of $\mathbb{Q}_p$. We say that f is p -ordinary to mean that a_p is a p -adic unit with respect to this fixed embedding. Let V_f be the usual two-dimensional F -representation of $\text{Gal}_{\mathbb{Q}} = \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ associated with f , characterized by the trace of an arithmetic frobenius at all primes $\ell \nmid pN$ being equal to a_{ℓ} . Let $T_f \subset V_f$ be a $\text{Gal}_{\mathbb{Q}}$ -stable $\mathcal{O}$ -lattice, $\mathcal{O}$ being the ring of integers of F . If the residual representation of T_f is irreducible, then there is a unique homothety classes of such lattices. Let $\text{Ad}^0(V_f) = \text{Hom}_F(V_f, V_f)^0$ and $\text{Ad}^0(T_f) = \text{Hom}_{\mathcal{O}}(T_f, T_f)^0$, where the superscript ‘0’ on the right denotes the submodule of endomorphisms whose trace is 0. We refer to $\text{Ad}^0(V_f)$ as the adjoint representation of V_f (or of f); $\text{Ad}^0(T_f)$ is a $\text{Gal}_{\mathbb{Q}}$ -stable $\mathcal{O}$ -lattice in $\text{Ad}^0(V_f)$. Note that $\text{Ad}(T_f)^0$ is canonically defined up to scalar if T_f is residually irreducible.

Let $\varepsilon : \text{Gal}_{\mathbb{Q}} \rightarrow \mathbb{Z}_p^{\times}$ be the p -adic cyclotomic character. Let $\omega_p : \text{Gal}_{\mathbb{Q}} \rightarrow \mu_{p-1} \subset \mathbb{Z}_p^{\times}$ be the Teichmüller character (so $\varepsilon \equiv \omega_p \pmod{p}$). Let $\chi_0 : \text{Gal}_{\mathbb{Q}} \rightarrow \mathbb{Z}_p^{\times}$ be any odd power of ω_p . Suppose that f is p -ordinary. In Part I of this paper, comprising Sections 3.1–3.4.11, we define cohomology classes

$$c_M \in H^1\left(\mathbb{Q}(\boldsymbol{\mu}_M)^+, \text{Ad}^0(T_f)(\varepsilon^2 \chi_0)\right), \quad M = p^a m, \quad (m, N') = 1,$$

where N' is the product of p , N , and a suitable auxiliary prime. If $p = 3$ or 5 , we must assume that N contains ‘enough’ primes in the sense of Assumption [??](#). We show that the classes c_M satisfy the norm relations required of an Euler system:

Theorem A. *Suppose that f is p -ordinary. Suppose also that if $p = 3$ or 5 then N satisfies [??](#). Then the classes c_M form an Euler system for $\mathrm{Ad}^0(T_f)$. That is, letting*

$$P_\ell(X) = \det(1 - X \mathrm{Frob}_\ell | \mathrm{Ad}^0(T_f)(\varepsilon^{-1} \chi_0^{-1})) \in \mathcal{O}[X], \quad \ell \nmid Np,$$

the c_M satisfy

$$\mathrm{Cores}_{\mathbb{Q}(\mu_{p^a m})^+}^{\mathbb{Q}(\mu_{p^a \ell m})^+}(c_{p^a \ell m}) = \begin{cases} P_\ell(\mathrm{Fr}_\ell^{-1})c_{p^a m} & \ell \nmid Np \\ c_{p^a m} & \ell = p. \end{cases}$$

We refer the reader to Section 1.2.1 of the Introduction, for applications of the theorem above.

Chapter 3

The Euler systems

In this part of the paper, we construct the Euler systems. The Galois cohomology classes comprising the Euler systems are classes naturally associated with explicit Galois extensions constructed as subquotients of (relative) étale cohomology groups of Shimura varieties attached to orthogonal groups.

3.1 Orthogonal groups

In this section we introduce basic notation for the orthogonal spaces and groups that we will be working with as well as for related objects.

3.1.1 Orthogonal and symplectic groups

Let $A := \mathbb{Z}[1/2]$.

3.1.1.1 Orthogonal groups

Let (V, q) be an orthogonal space: V is a free A -module of rank n equipped with a quadratic form q . Let $\mathbf{GO}(V, q)$ be the group scheme over $\mathrm{Spec}(A)$ such that for any A -algebra R

$$\mathbf{GO}(V, q)(R) = \{g \in \mathrm{Aut}_R(V \otimes_A R) : q(g(v)) = \nu_g q(v), \nu_g \in R^\times\}.$$

Fixing a basis $\mathcal{B}$ of V , $\mathrm{Aut}_R(V \otimes_A R)$ is identified with $\mathbf{GL}_n(R)$ and, if S is the matrix of q in the basis $\mathcal{B}$, $\mathbf{GO}(V, q)$ is identified with the group scheme $\mathbf{GO}(S)$ defined by

$$\mathbf{GO}(S)(R) = \{g \in \mathbf{GL}_n(R) : gSg^\top = \nu_g S, \nu_g \in R^\times\}.$$

Let $\nu : \mathbf{GO}(V, q) \rightarrow \mathbf{G}_m$ be the character given by $\nu(g) = \nu_g$. Let $\mathbf{O}(V, q)$ be the group scheme cut out by the kernel of ν and let $\mathbf{SO}(V, q) := \mathbf{O}(V, q) \cap \mathbf{SL}(V)$. When the isomorphism class of the quadratic space is clear from the context, we will denote $\mathbf{SO}(V, q)$ by $\mathbf{SO}(V)$ or even $\mathbf{SO}_n$.

We will work with a split orthogonal group in five variables. We will sometimes need to view this group as a symplectic group in four variables – in the following, we introduce necessary notation for this purpose.

3.1.1.2 The symplectic group $\mathbf{G}'$

We will denote symplectic objects systematically with a $'$. Let $V' = A\varepsilon_1 \oplus A\varepsilon_2 \oplus A\varepsilon_3 \oplus A\varepsilon_4$ be a free A -module of rank 4. Consider the anti-symmetric matrix

$$J := \begin{pmatrix} & & & \\ & & & \\ & & 1 & \\ & & & 1 \\ -1 & & & \\ & -1 & & \end{pmatrix}.$$

Then J defines an alternating pairing (or symplectic form) on V' , which we denote by $\langle \cdot, \cdot \rangle$. Let $\mathbf{G}' = \mathbf{Sp}(V', J)$ be the group scheme over A whose R -points, for any A -algebra R , are

$$\mathbf{G}'(R) = \{g \in \mathbf{GL}(V' \otimes R) : gJg^\top = J\}.$$

Let $\mathbf{A}'$ be the diagonal maximal torus of $\mathbf{G}'$:

$$\mathbf{A}' = \left\{ a'(t_1, t_2) := \begin{pmatrix} t_1 & & & \\ & t_2 & & \\ & & t_1^{-1} & \\ & & & t_2^{-1} \end{pmatrix} : t_1, t_2 \in \mathbf{G}_m \right\}.$$

Let α', β' be the roots of $\mathbf{G}'$ such that α' is the differential of the character $a'(t_1, t_2) \mapsto t_1$ and β' is the differential of the character $a'(t_1, t_2) \mapsto t_2$. The defining symplectic form defines a $\mathbf{G}'$ -equivariant linear functional λ' on the 6-dimensional $\mathbf{G}'$ -representation $\wedge^2 V'$, namely $v_1 \wedge v_2 \mapsto \langle v_1, v_2 \rangle$. The kernel of λ' is a 5-dimensional

representation W' of $\mathbf{G}'$ and we have a decomposition

$$\wedge^2 V' = W' \oplus \mathbb{1},$$

where the trivial representation is just the image of the dual of λ' ; it is the line spanned by the vector $\varepsilon_1 \wedge \varepsilon_3 + \varepsilon_2 \wedge \varepsilon_4$. The representation W' is also the finite-dimensional irreducible representation of $\mathbf{G}'$ of highest weight $\alpha' + \beta'$. Furthermore, the symplectic form on V' defines a symmetric bilinear form on $V' \otimes V'$ given by

$$b(v_1 \otimes v_2, u_1 \otimes u_2) = 2\langle v_1, u_1 \rangle \langle v_2, u_2 \rangle.$$

Via the inclusion

$$\wedge^2 V' \hookrightarrow V' \otimes V', \quad v_1 \wedge v_2 \mapsto \frac{1}{2}(v_1 \otimes v_2 - v_2 \otimes v_1)$$

b restricts to a symmetric bilinear form on $\wedge^2 V'$. Explicitly, b is given by the formula:

$$b(v_1 \wedge v_2, u_1 \wedge u_2) = \langle v_1, u_1 \rangle \langle v_2, u_2 \rangle - \langle v_1, u_2 \rangle \langle v_2, u_1 \rangle.$$

We next study the restriction of b to W' .

3.1.1.3 The orthogonal group $\mathbf{G}$

An explicit basis of W' as a $\mathbf{G}'$ -representation is given by $\mathcal{B} := (e_i)$ for $i = 1, \dots, 5$ where $e_1 = \varepsilon_1 \wedge \varepsilon_2$, $e_2 = \varepsilon_2 \wedge \varepsilon_3$, $e_3 = \frac{1}{2}(\varepsilon_1 \wedge \varepsilon_3 - \varepsilon_2 \wedge \varepsilon_4)$, $e_4 = \varepsilon_1 \wedge \varepsilon_4$, and $e_5 = \varepsilon_3 \wedge \varepsilon_4$.

In this basis the matrix of b is

$$S_5 := \begin{pmatrix} & & & & 1 \\ & & & 1 & \\ & & 1 & & \\ & 1 & & & \\ 1 & & & & \end{pmatrix}.$$

In particular the restriction of b to W' is non-degenerate. Henceforth we write (V, q) for the orthogonal space $(W', b|_{W'})$. We let

$$\mathbf{G} := \mathbf{SO}(V, q) = \mathbf{SO}(W', b|_{W'}).$$

With respect to the basis $\mathcal{B}$ this is just $\mathbf{SO}(S_5) = \mathbf{SO}_5$.

The $\mathbf{G}'$ -representation W' affords an isogeny

$$\pi : \mathbf{G}' \rightarrow \mathbf{G},$$

the kernel of which is $\{\pm I\}$. Explicit formulas for π are given in [RS07, Appendix A.7], although with a slightly different model of $\mathbf{Sp}(V')$. Some useful formulas are recorded in Appendix (A) (see especially section (A.1)).

3.1.1.4 The doubling subgroups

Denote by $U \subseteq V'$ the plane spanned by ε_1 and ε_3 and the plane spanned by ε_2 and ε_4 by $U^\perp$. The restriction of the symplectic form to both U and $U^\perp$ is non-degenerate

and identifies the annihilator of U with $U^\perp$ (hence the notation). The decomposition

$$V' = U \oplus U^\perp$$

induces an embedding ι' of $\mathbf{H}' := \mathbf{SL}_2 \times \mathbf{SL}_2$ into $\mathbf{G}'$. Here we have identified $\mathbf{SL}(U)$ with $\mathbf{SL}_2$ (resp. $\mathbf{SL}(U^\perp)$ with $\mathbf{SL}_2$) via the basis ε_1 and ε_3 (resp. ε_2 and ε_4). Explicitly, this embedding is given by

$$\iota' : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \times \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \mapsto \begin{pmatrix} a & 0 & b & 0 \\ 0 & a' & 0 & b' \\ c & 0 & d & 0 \\ 0 & c' & 0 & d' \end{pmatrix}.$$

As an $\mathbf{H}'$ representation V decomposes as $V = \mathbf{1} \oplus V_1$, where in the notation of the preceding paragraph the trivial representation corresponds to the line spanned by e_3 and $V_1 \subset V$ is the 4-dimensional space spanned by e_1, e_2, e_4 , and e_5 . As a representation of $\mathbf{H}' \simeq \mathbf{SL}_2 \times \mathbf{SL}_2$, $V_1 \simeq V_{\text{std}} \boxtimes V_{\text{std}}^\vee$: if v_1, v_2 is the canonical basis of V_{std} and $v_1^\vee, v_2^\vee$ its dual basis, then this isomorphism identifies $v_1 \boxtimes v_1^\vee, v_2 \boxtimes v_1^\vee, v_1 \boxtimes v_2^\vee, v_2 \boxtimes v_2^\vee$ with $e_4, e_5, -e_1, e_2$, respectively.

The restriction of π to $\mathbf{H}'$ defines an isogeny, still denoted by π by abuse of notation

$$\pi : \mathbf{H}' \rightarrow \mathbf{H},$$

where $\mathbf{H} \subset \mathbf{G}$ is the isotropy subgroup of the anisotropic vector e_3 . Note that $\mathbf{H} = \mathbf{SO}(V_1, q_1) \simeq \mathbf{SO}_4$, where q_1 is the restriction of q to V_1 . The isotropy subgroup in $\mathbf{H}'$

of the identity $\text{id} \in \text{End}(V_{\text{std}}) = V_{\text{std}} \boxtimes V_{\text{std}}^\vee$ is the diagonally embedded $\mathbf{SL}_2 \hookrightarrow \mathbf{H}'$. And thus the restriction of π to $\mathbf{SL}_2$ defines an isogeny, again denoted by π by abuse of notation, $\pi : \mathbf{SL}_2 \rightarrow \mathbf{G}_0$, where $\mathbf{G}_0 \subset \mathbf{H}$ is the isometry group of $\text{id} = w_2 + w_4 \in V_1$. So $\mathbf{G}_0 = \mathbf{SO}(V_0, q_0)$ for $V'_0 \subset V_1$ the orthogonal complement of $w_2 + w_4$ and q_0 the restriction of q_1 (equiv. of q) to V_0 . In particular, $\mathbf{G}_0 \simeq \mathbf{SO}_3$.

3.1.2 Root systems and rational parabolic subgroups

Consider the maximal $\mathbb{Q}$ -split torus of $\mathbf{G}$:

$$\mathbf{A} := \left\{ \begin{pmatrix} h_1 & & & & \\ & h_2 & & & \\ & & 1 & & \\ & & & h_2^{-1} & \\ & & & & h_1^{-1} \end{pmatrix} : h_1, h_2 \in \mathbf{G}_m \right\}.$$

Here the matrices are written in the basis $\mathcal{B}$ defined in section 3.1.1. Let $\mathfrak{a}$ denote the Lie algebra of $\mathbf{A}$. The $\mathbb{Q}$ -root system Φ of $\mathbf{G}$ is of type B_2 . Let $\alpha \in \mathfrak{a}^\vee$ be the root defined by the differential of the character $a : \mathbf{A} \rightarrow \mathbf{G}_m$ given, in the coordinates above, by $\mathbf{A} \ni h \mapsto h_1 \in \mathbf{G}_m$. Similarly let β be the root defined by the character $b : \mathbf{A} \rightarrow \mathbf{G}_m$, $h \mapsto h_2$. We make the choice of positive roots given by $\Phi^+ = \{\alpha \pm \beta, \alpha, \beta\}$. Then a system of simple roots over $\mathbb{Q}$ is $\Delta := \{\alpha - \beta, \beta\}$. This data determines a set $\{\mathbf{P}_0, \mathbf{P}_1, \mathbf{P}_2\}$ of rational parabolic subgroups of $\mathbf{G}$. The parabolic subgroup $\mathbf{P}_1$ is maximal and obtained by removing β ; $\mathbf{P}_2$ is also maximal, obtained by removing $\alpha - \beta$; and $\mathbf{P}_0 = \mathbf{P}_1 \cap \mathbf{P}_2$. Note that $\mathbf{P}_1$ is the stabilizer of the

isotropic vector e_1 , and $\mathbf{P}_2$ is the stabilizer of the totally isotropic plane spanned by e_1 and $(e_2 + e_4)/2$.

The Levi subgroup of the parabolic subgroup $\mathbf{P}_i$ will be denoted by $\mathbf{M}_i$. We introduce the following notations and identifications.

- $\mathbf{G}_m \times \mathbf{SO}_3 \simeq \mathbf{M}_1$, given by

$$(t, g_0) \mapsto \begin{pmatrix} t & & \\ & g_0 & \\ & & t^{-1} \end{pmatrix}.$$

- $\mathbf{GL}_2 \simeq \mathbf{M}_2$, given by

$$a \mapsto \begin{pmatrix} a & & \\ & 1 & \\ & & \tilde{a} \end{pmatrix},$$

where

$$\tilde{a} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} (a^\top)^{-1} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

- If $t \in \mathbf{G}_m$, we denote by $[t] \in \mathbf{M}_1$ the element given by

$$[t] = \begin{pmatrix} t & & \\ & I_3 & \\ & & t^{-1} \end{pmatrix}.$$

Let $\mathbf{U}_i \subset \mathbf{P}_i$ be the unipotent radical of $\mathbf{P}_i$. Note that $\mathbf{U}_1$ is abelian and $\mathbf{U}_2$ is a

Heisenberg group.

- For $X = (x_1, x_0, x_2) \in \mathbf{G}_a^3$, we denote by $n_1(X)$ a generic element of $\mathbf{U}_1$. We have

$$n_1(X) = \begin{pmatrix} 1 & x_1 & x_0 & x_2 & Q[X]/2 \\ & 1 & 0 & 0 & * \\ & & 1 & 0 & * \\ & & & 1 & * \\ & & & & 1 \end{pmatrix}.$$

Here $Q[X]$ is a quadratic form and the $*$ are linear functions of X . We denote by $n_1^{\text{opp}}(X)$ the transpose of $n_1(X)$.

3.1.3 Weyl group combinatorics

Here we recall a few facts about the Weyl groups of the orthogonal groups introduced above and compute certain double coset representatives.

The Weyl group $\mathcal{W}$ of $\mathbf{G}$ is of order 8 and can be seen as acting by signed permutations on the simple roots α, β . We denote by $\mathcal{W}_i$ the Weyl group of the Levi $\mathbf{M}_i$. We denote by $\mathcal{W}^i$ the set of Kostant representatives of $\mathbf{P}_i$ for $i = 0, 1, 2$. These are the minimal length representatives of the cosets $\mathcal{W}_i \backslash \mathcal{W}$. Let $\Phi_i \subseteq \Phi^+$ be the roots appearing in $\mathbf{U}_i$, the unipotent radical of $\mathbf{P}_i$. Then $\Phi_0 = \Phi^+$; $\Phi_1 = \{\alpha \pm \beta, \alpha\}$; and $\Phi_2 = \{\alpha + \beta, \alpha, \beta\}$. The Kostant representatives are elements $w \in \mathcal{W}$ such that $w^{-1}\Phi^- \cap \Phi^+ \subseteq \Phi_i$. In particular, we have:

$$\mathcal{W}^1 = \{w \in \mathcal{W} : w^{-1}(\beta) > 0\} \quad \text{and} \quad \mathcal{W}^2 = \{w \in \mathcal{W} : w^{-1}(\alpha - \beta)\}.$$

We can thus give the following explicit description of the sets $\mathcal{W}^i$. Let ρ denote the half-sum of positive roots: $\rho = \frac{1}{2}(3\alpha + \beta)$. We denote by $w \bullet \lambda = w(\lambda + \rho) - \rho$ the "dot" action of the Weyl group on roots; in particular, $w \bullet 0 = w\rho - \rho$.

3.1.3.1 Lemma. *The set $\mathcal{W}^0$ is $\mathcal{W}$; the set $\mathcal{W}^1$ contains the 4 elements w given by:*

$$w : (\alpha, \beta) \mapsto \begin{cases} (\alpha, \beta) & \text{of length 0 and such that } w \bullet 0 = 0 \\ (-\alpha, \beta) & \text{of length 3 and such that } w \bullet 0 = -3\alpha \\ (\beta, \alpha) & \text{of length 1 and such that } w \bullet 0 = -\alpha + \beta \\ (\beta, -\alpha) & \text{of length 2 and such that } w \bullet 0 = -2\alpha + \beta. \end{cases}$$

The set $\mathcal{W}^2$ contains the 4 elements w given by:

$$w : (\alpha, \beta) \mapsto \begin{cases} (\alpha, \beta) & \text{of length 0 and such that } w \bullet 0 = 0 \\ (\alpha, -\beta) & \text{of length 1 and such that } w \bullet 0 = -\beta \\ (-\beta, \alpha) & \text{of length 2 and such that } w \bullet 0 = -\alpha - 2\beta \\ (-\beta, -\alpha) & \text{of length 3 and such that } w \bullet 0 = -2\alpha - 2\beta. \end{cases}$$

We next state a lemma, whose proof is straightforward, that will be used to compute the constant term of the Eisenstein series which we will later use to construct an Euler System. Let $s : (\alpha, \beta) \mapsto (\beta, \alpha)$ and $s_{-1} : (\alpha, \beta) \mapsto (\alpha, -\beta)$. Both are simple reflections in $\mathcal{W}$.

3.1.3.2 Lemma. *There are 3 double cosets in $\mathcal{W}_1 \backslash \mathcal{W} / \mathcal{W}_1$, with coset representatives $\{\text{id}, s, ss_{-1}s\}$. There are 2 double cosets in $\mathcal{W}_1 \backslash \mathcal{W}_1 / \mathcal{W}_2$, with coset representatives*

$\{\text{id}, ss_{-1}\}$.

3.1.4 The Shimura varieties

We introduce the Shimura varieties that will feature in subsequent constructions.

3.1.4.1 The Shimura datum

Let $\mathbf{S} := \text{Res}_{\mathbb{C}/\mathbb{R}} \mathbf{G}_m$ be Deligne's torus. Denote by $z, \bar{z} : \mathbf{S}(\mathbb{C}) \rightarrow \mathbb{C}^\times$ the algebraic characters of $\mathbf{S}(\mathbb{C})$ such that the restriction to $\mathbf{S}(\mathbb{R}) = \mathbb{C}^\times$ of z and $\bar{z}$ are respectively the identity and complex conjugation. Denote by X the symmetric space attached to $\mathbf{G}$. This can be identified with the $\mathbf{G}(\mathbb{R})$ -conjugacy class of homomorphisms $h : \mathbf{S} \rightarrow \mathbf{G}_{\mathbb{R}}$ containing the element:

$$h_0 : x + iy \mapsto \begin{pmatrix} \frac{x^2-y^2}{x^2+y^2} & \frac{2xy}{x^2+y^2} & & & \\ -\frac{2xy}{x^2+y^2} & \frac{x^2-y^2}{x^2+y^2} & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix}_{\mathcal{B}'}. \quad .$$

Here the subscript $\mathcal{B}'$ indicates that we are writing matrices in the basis

$$\mathcal{B}' := \left((e_1 - e_5)/2, (e_2 - e_4)/2, e_3, (e_2 + e_4)/2, (e_1 + e_5)/2 \right).$$

The pair $(\mathbf{G}, h_0)$ determines a Shimura datum, and we denote by $\text{SH}_{\mathbf{G}}$ the associated Shimura variety and by $\text{SH}_{\mathbf{G}}(K)$ the resulting Shimura variety of level K ,

for $K \subset \mathbf{G}(\mathbb{A}_f)$ a compact open subgroup. Note that the weight morphism of the Shimura variety is trivial and that the reflex field is $\mathbb{Q}$. For a more general discussion on Shimura varieties associated with special orthogonal groups, the reader can consult for example [Cor, Sections 5.1-5.9].

The stabilizer of h_0 is a maximal compact subgroup of $\mathbf{G}(\mathbb{R})$, which we denote by K_∞ . The map $\mathbf{G}(\mathbb{R}) \ni g \mapsto gh_0g^{-1} \in X$ determines a homeomorphism $\mathbf{G}(\mathbb{R})/K_\infty \simeq X$. This isomorphism provides a concrete description of the complex points of the Shimura variety of level K :

$$\mathrm{SH}_{\mathbf{G}}(K)(\mathbb{C}) = \mathbf{G}(\mathbb{Q}) \backslash [X \times \mathbf{G}(\mathbb{A}_f)/K] \simeq \mathbf{G}(\mathbb{Q}) \backslash \mathbf{G}(\mathbb{A})/K_\infty K.$$

Similarly, we let $\mathrm{SH}_{\mathbf{H}}$ be the Shimura variety resulting from the Shimura data $(\mathbf{H}, h_{0,\mathbf{H}})$, where $h_{0,\mathbf{H}} : \mathbf{S} \rightarrow \mathbf{H}$ is the group homomorphism defined by

$$h_{0,\mathbf{H}} : x + iy \mapsto \begin{pmatrix} \frac{x^2-y^2}{x^2+y^2} & \frac{2xy}{x^2+y^2} & & \\ -\frac{2xy}{x^2+y^2} & \frac{x^2-y^2}{x^2+y^2} & & \\ & & 1 & \\ & & & 1 \end{pmatrix},$$

where now the basis is $\mathcal{B}' \cap V_1 = \left((e_1 - e_5)/2, (e_2 - e_4)/2, (e_2 + e_4)/2, (e_1 + e_5)/2 \right)$. The composition of $h_{0,\mathbf{H}}$ with the inclusion $\iota : \mathbf{H} \hookrightarrow \mathbf{G}$ equals h_0 and so ι determines a morphism of Shimura varieties $i : \mathrm{SH}_{\mathbf{H}} \rightarrow \mathrm{SH}_{\mathbf{G}}$; this morphism is defined over $\mathbb{Q}$.

3.1.4.2 Models for the symmetric space X

For explicit calculations it is convenient to have explicit models of the symmetric space X .

The Projective Model. Consider the complexification $V_{\mathbb{C}} := V \otimes_A \mathbb{C}$ of V and its associated projective space $\mathbf{P}V_{\mathbb{C}}$. Consider also the quadratic hypersurface defined by

$$\mathcal{Q} := \{v \in \mathbf{P}V_{\mathbb{C}} : q(v) = 0\}.$$

This quadric can be naturally identified with the compact dual of $\mathrm{SH}_{\mathbf{G}}$. The associated cone is

$$\mathcal{Q}^- := \{v \in \mathcal{Q} : q(v, \bar{v}) < 0\},$$

which is a complex manifold of dimension 3 consisting of 2 connected components. The action of the orthogonal group $\mathbf{O}(V)(\mathbb{R})$ on $V_{\mathbb{R}} := V \otimes_A \mathbb{R}$ induces a transitive action on this cone.

The Tube Domain Model. Write the defining (complex) representation of $\mathbf{G}$ as

$$V_{\mathbb{C}} = (V_3)_{\mathbb{C}} \oplus \mathbb{C}e_1 \oplus \mathbb{C}e_5.$$

Here V_3 is the orthogonal space spanned by the vectors e_2, e_3 , and e_4 . We view V_3 as an orthogonal space endowed with the quadratic form q_3 whose matrix in the basis

(e_2, e_3, e_5) is

$$S_3 := \begin{pmatrix} & & 2 \\ & 1 & \\ 2 & & \end{pmatrix}$$

Define the tube domain:

$$\mathcal{H} := \{Z \in (V_3)_{\mathbb{C}} : q_3(\Im(Z)) < 0\}.$$

The map $\mathcal{H} \rightarrow \mathcal{Q}^-$ defined by $Z \mapsto (-S_3[Z]/2, Z, 1)^\top$ is a biholomorphism. Fix one of the two connected components $\mathcal{H}^\circ$ of $\mathcal{H}$ by setting

$$\mathcal{H}^\circ := \{Z \in \mathcal{H} : q_3(e_5, \Im(Z)) < 0\}.$$

Then $\mathcal{H}^\circ$ is the symmetric space associated with $\mathbf{G}(\mathbb{R})$.

For future reference, we give explicit formulas for the $\mathbf{G}(\mathbb{R})$ action on $\mathcal{H}^\circ$. Let

$$A := \begin{pmatrix} \alpha & a^\top & \beta \\ b & M & c \\ \gamma & d^\top & \delta \end{pmatrix} \in \mathbf{G}(\mathbb{R})$$

where $\alpha, \beta, \gamma, \delta$ are scalars $a, b, c, d \in \mathbb{R}^3$ and M is a 3×3 real matrix. The action of $g \in \mathbf{G}(\mathbb{R})$ on $\mathcal{H}^\circ$ is denoted by $Z \mapsto g\langle Z \rangle$. We have:

$$\forall A \in \mathbf{G}(\mathbb{R}), \quad A\langle Z \rangle = (-S_3[Z]b + MZ + c)J(A, Z)^{-1}$$

where

$$J(A, Z) = -\gamma S_3[Z] + d^\top Z + \delta.$$

Identification with X Let X' be the space of negative definite planes in $V_{\mathbb{R}}$. The points of X' correspond to certain polarized Hodge structures on $V_{\mathbb{Q}}$. Fix $P \in X'$, and consider a basis (e, f) of P such that $q(e) = q(f) = -1$. Define the following Hodge structure on $V_{\mathbb{C}}$:

$$V_{\mathbb{C}} \supset V_P^{r,s} := \begin{cases} \mathbb{C}(e + if) & \text{if } (r, s) = (-1, 1) \\ P_{\mathbb{C}}^\perp & \text{if } (r, s) = (0, 0) \\ \mathbb{C}(e - if) & \text{if } (r, s) = (1, -1) \\ 0 & \text{otherwise.} \end{cases}$$

Then $V_P^{\bullet, \bullet}$ is a $\mathbb{Q}$ -Hodge structure on $V_{\mathbb{Q}}$ polarized by q . In fact, each $P \in X'$ gives rise to a unique homomorphism $h_P : \mathbf{S} \rightarrow \mathbf{G}_{\mathbb{R}}$ that induces the Hodge structure $V_P^{\bullet, \bullet}$ on $V_{\mathbb{Q}}$ and whose restriction to $(\mathbf{G}_{\mathbf{m}})_{\mathbb{R}} \subset \mathbf{S}$ is the trivial map. This provides an identification of X with a connected component of X' . Moreover, the map carrying P to the line $V_P^{1, -1}$ embeds X' into $\mathcal{Q}$ and identifies X' with $\mathcal{Q}^-$. In this way, we get a biholomorphism between X and $\mathcal{H}^\circ$, and the homomorphism h_0 defines a base point $Z_0 \in \mathcal{H}$.

3.1.5 Level subgroups

Denote by $K_\ell^{\mathbf{G}}$ (resp. $K_\ell^{\mathbf{H}}$) the maximal compact subgroup $\mathbf{G}(\mathbb{Z}_\ell)$ (resp. $\mathbf{H}(\mathbb{Z}_\ell)$) of $\mathbf{G}(\mathbb{Q}_\ell)$ (resp. $\mathbf{H}(\mathbb{Q}_\ell)$). We define the following open compact subgroups: Let $r \geq 1$ be an integer.

- We let $K_1^{\mathbf{A}}(\ell^r) \subset \mathbf{G}(\mathbb{Z}_\ell)$ be the kernel of the reduction modulo ℓ^r .
- We let $K_0^{\mathbf{P}}(\ell^r) \subset \mathbf{G}(\mathbb{Z}_\ell)$ be the subgroup of elements that reduce modulo ℓ^r to elements of $\mathbf{P}_1(\mathbb{Z}/\ell^r\mathbb{Z})$.
- We let $K_1^{\mathbf{P}}(\ell^r) \subset K_0^{\mathbf{P}}(\ell^r)$ consist of the elements that reduce modulo ℓ^r to the kernel of the projection $\mathbf{P}_1(\mathbb{Z}/\ell^r\mathbb{Z}) \twoheadrightarrow \mathbf{M}_1(\mathbb{Z}/\ell^r\mathbb{Z}) \twoheadrightarrow \mathbf{G}_0(\mathbb{Z}/\ell^r\mathbb{Z})$, that is, the elements of $K_0^{\mathbf{P}}(\ell^r)$ whose images modulo ℓ^r are of the form $m(1, \bar{g})u$ for some $\bar{g} \in \mathbf{M}_1(\mathbb{Z}/\ell^r\mathbb{Z})$ and some $u \in U_1(\mathbb{Z}/\ell^r\mathbb{Z})$.
- We let

$$\tilde{K}_1^{\mathbf{P}}(\ell^r) := K_1^{\mathbf{P}}(\ell^r) \cap K_0^{\mathbf{P}}(\ell^{2r}).$$

- We let $K^{\text{vol}}(\ell^r) \subset \mathbf{G}(\mathbb{Z}_\ell)$ to be the largest open compact subgroup of $\mathbf{G}(\mathbb{Z}_\ell)$ fixing the volume section defined in (B.8) of Appendix (B).
- Similarly, we let $K^{\text{St}}(\ell)$ be the largest open compact subgroup of $\mathbf{G}(\mathbb{Z}_\ell)$ fixing the Steinberg section as defined in Proposition (B.9.2) of Appendix (B).
- We define also analogous versions of these subgroups for $\mathbf{H}$:
 - We set $K_{11}(\ell^r) := K_1^{\mathbf{A}}(\ell^r) \cap \mathbf{H}(\mathbb{Z}_\ell)$.

- We set $K_0(\ell^r) := K_0^{\mathbf{P}}(\ell^r) \cap \mathbf{H}(\mathbb{Z}_\ell)$.
 - We set $K_1(\ell^r) := K_1^{\mathbf{P}}(\ell^r) \cap \mathbf{H}(\mathbb{Z}_\ell)$.
 - We set $\tilde{K}_1(\ell^r) := \tilde{K}_1^{\mathbf{P}}(\ell^r) \cap \mathbf{H}(\mathbb{Z}_\ell)$.
 - We set $K(\ell^r) := K^{\text{vol}}(\ell^r) \cap \mathbf{H}(\mathbb{Z}_\ell)$.
 - We note that the pull-back of $K^{\text{St}}(\ell)$ to $\mathbf{H}(\mathbb{Z}_\ell)$ is $K_0(\ell)$.
- We define also the usual open compact subgroups of $\mathbf{SL}_2(\mathbb{Z}_\ell)$: We let $\Gamma_0(\ell^r)$ be the subgroup of elements that reduce modulo ℓ^r to an upper triangular matrix, $\Gamma_1(\ell^r)$ the subgroup generated by $-\text{id}$ and elements that reduce modulo ℓ^r to unipotent upper triangular matrix and $\Gamma(\ell^r)$ the subgroup of elements that reduce modulo ℓ^r to ± 1 .

Pull-backs to $\mathbf{G}'(\mathbb{Z}_\ell)$ and $\mathbf{H}'(\mathbb{Z}_\ell)$. We comment on the pull-backs (inverse images under the isogeny π) to $\mathbf{G}'(\mathbb{Z}_\ell)$ and $\mathbf{H}'(\mathbb{Z}_\ell)$ of the subgroups defined above.

- The pull-back of $K_1^{\mathbf{A}}(\ell^r)$ to $\mathbf{G}'(\mathbb{Z}_\ell)$ is the subgroup consisting of elements that reduce modulo ℓ^r to $\pm \text{id}$.
- The pull-back of $K_0^{\mathbf{P}}(\ell^r)$ to $\mathbf{G}'(\mathbb{Z}_\ell)$ is the subgroup consisting of elements that reduce modulo ℓ^r to elements of $\mathbf{P}'_1(\mathbb{Z}/\ell^r\mathbb{Z})$, that is to elements of the form
$$\begin{pmatrix} A & B \\ 0 & (A^\top)^{-1} \end{pmatrix},$$
 where A, B are 2×2 matrices with coefficients in $\mathbb{Z}/\ell^r\mathbb{Z}$, with A invertible.
- The pull-back of $K_1^{\mathbf{P}}(\ell^r)$ to $\mathbf{G}'(\mathbb{Z}_\ell)$ is the subgroup consisting of elements that

reduce modulo ℓ^r to elements of the form $\begin{pmatrix} A & B \\ 0 & (A^\top)^{-1} \end{pmatrix}$ with $\det(A) = \pm 1$.

- The pull-backs of the level subgroups $K^{\text{vol}}(\ell^r)$ and $K^{\text{St}}(\ell)$ are by definition the largest subgroups of $\mathbf{G}'(\mathbb{Z}_\ell)$ fixing the corresponding sections.
- We now identify $\mathbf{H}'(\mathbb{Z}_\ell)$ with $\mathbf{SL}_2(\mathbb{Z}_\ell) \times \mathbf{SL}_2(\mathbb{Z}_\ell)$. We then have:
 - The pull-back to $\mathbf{H}'(\mathbb{Z}_\ell)$ of $K_{11}(\ell^r)$ is identified with $\Gamma(\ell^r) \times \Gamma(\ell^r)$.
 - The pull-back to $\mathbf{H}'(\mathbb{Z}_\ell)$ of $K_0(\ell^r)$, $K'_0(\ell^r)$, is identified with $\Gamma_0(\ell^r) \times \Gamma_0(\ell^r)$.
 - The pull-back of $K_1(\ell^r)$ is identified with

$$\Gamma_1(\ell^r) \times \Gamma_0(\ell^r) \hookrightarrow \Gamma_0(\ell^r) \times \Gamma_0(\ell^r) = K'_0(\ell^r).$$

- The pull-back of $K(\ell^r)$ to $\mathbf{H}'(\mathbb{Z}_\ell)$ is identified with the image of $\mathbf{SL}_2(\mathbb{Z}_\ell) \times \Gamma(\ell^r) \hookrightarrow \mathbf{H}'(\mathbb{Z}_\ell)$ where the embedding is given, as in Lemma (B.8.2), by $(g, \gamma) \mapsto (\gamma, 1)(g, g^\vartheta)$.

3.2 The boundary cohomology

In this section we recall some facts about the boundary cohomology of the Shimura variety SH_G .

3.2.1 The Shimura data for the boundary strata

Recall that each Levi subgroup $\mathbf{M}_i$ can be decomposed into a *linear* part and a *hermitian* part:

$$\mathbf{M}_i = \mathbf{M}_{i,h} \mathbf{M}_{i,\text{lin}},$$

where $\mathbf{M}_{i,h}$ is the hermitian part of $\mathbf{M}_i$ and $\mathbf{M}_{i,\text{lin}}$ the linear part. In terms of our description of the $\mathbf{M}_i$ in 3.1.2, we have

$$\mathbf{M}_{1,\text{lin}} = \mathbf{SO}(V_3), \mathbf{M}_{1,h} = \mathbf{G}_m \quad \text{and} \quad \mathbf{M}_{2,\text{lin}} = \{1\}, \mathbf{M}_{2,h} = \mathbf{GL}_2.$$

For $i = 1, 2$ we describe the homomorphisms $h_i : \mathbf{S} \rightarrow \mathbf{M}_{i,h}$ defining the Shimura datum of the boundary stratum of the Baily–Borel compactification attached to the maximal parabolic $\mathbf{P}_i$. We denote by ω_i the weight morphism of the Shimura variety defined by the pair $(\mathbf{M}_{i,h}, h_i)$.

Case $i = 1$. The morphism h_1 is given, for all $z \in \mathbf{S}(\mathbb{R})$, by

$$h_1(z) = \begin{pmatrix} |z|^2 & & \\ & I_3 & \\ & & |z|^{-2} \end{pmatrix}, \quad \text{and so } \omega_1(t) = \begin{pmatrix} t^2 & & \\ & I_3 & \\ & & t^{-2} \end{pmatrix}.$$

Case $i = 2$ The morphism h_2 is given, for all $z = x + iy \in \mathbf{S}(\mathbb{R})$, by

$$h_2(z) = \begin{pmatrix} \begin{pmatrix} x & y \\ -y & x \end{pmatrix} \\ \begin{pmatrix} x & -y \\ y & x \end{pmatrix} \end{pmatrix} \quad \text{and so } \omega_2(t) = \begin{pmatrix} tI_2 & & \\ & 1 & \\ & & t^{-1}I_2 \end{pmatrix}.$$

In both cases the matrices are written with respect to the basis $\mathcal{B}$.

3.2.2 Cohomology of Shimura varieties

In this subsection we recall some facts about singular and analytic de Rham cohomology of Shimura varieties. We formulate the results for the Shimura variety $\mathrm{SH}_{\mathbf{G}}$, but analogous results are true for other Shimura varieties whose underlying reductive groups have $\mathbb{Q}$ -rank 2.

3.2.2.1 Generalities

Recall that for any irreducible finite-dimensional representation $\rho : \mathbf{G}(\mathbb{C}) \rightarrow \mathbf{GL}(V)$, one can define in a natural way a local system $\widetilde{V}$ on $\mathrm{SH}_{\mathbf{G}}$. If $\lambda : \mathbf{A}(\mathbb{C}) \rightarrow \mathbb{C}^\times$ is a dominant weight, then we will write V_λ for the irreducible representation of highest weight λ .

Let $\overline{\mathrm{SH}}_{\mathbf{G}}$ be the Borel–Serre compactification of $\mathrm{SH}_{\mathbf{G}}$ (see, for example, [BS73]). The natural inclusion $\mathrm{SH}_{\mathbf{G}} \hookrightarrow \overline{\mathrm{SH}}_{\mathbf{G}}$ is a homotopy equivalence. Moreover, for all dominant weights λ , the local system $\widetilde{V}_\lambda$ extends canonically to $\overline{\mathrm{SH}}_{\mathbf{G}}$. By abuse of

notation we will also denote by $\widetilde{V}_\lambda$ this canonical extension of V_λ .

Let $\partial\mathrm{SH}_{\mathbf{G}} := \overline{\mathrm{SH}}_{\mathbf{G}} \setminus \mathrm{SH}_{\mathbf{G}}$ be the boundary of the Borel–Serre compactification. By the results of [BS73] (see also [HMG94]), there is a stratification of the boundary:

$$\partial\mathrm{SH}_{\mathbf{G}} := \coprod_P \partial_P,$$

where the union ranges over all standard proper $\mathbb{Q}$ -parabolic subgroups of $\mathbf{G}$. In our case, there are three such parabolics: $\mathbf{P}_i$ for $i = 0, 1, 2$. We denote for simplicity $\partial_i := \partial_{\mathbf{P}_i}$. Note that there are closure relations between the strata: ∂_0 is a closed stratum and lies both in the closure of ∂_1 and ∂_2 . More specifically, the closure of ∂_i , $i = 1, 2$, is $\overline{\partial}_i = \partial_i \sqcup \partial_0$.

The interior cohomology of $\mathrm{SH}_{\mathbf{G}}$ is defined as

$$\mathrm{H}_!^i(\mathrm{SH}_{\mathbf{G}}, \widetilde{V}_\lambda) := \ker \left(\mathrm{H}^i(\mathrm{SH}_{\mathbf{G}}, \widetilde{V}_\lambda) \rightarrow \mathrm{H}^i(\partial\mathrm{SH}_{\mathbf{G}}, \widetilde{V}_\lambda) \right),$$

where the map $\mathrm{H}^i(\mathrm{SH}_{\mathbf{G}}, \widetilde{V}_\lambda) \rightarrow \mathrm{H}^i(\partial\mathrm{SH}_{\mathbf{G}}, \widetilde{V}_\lambda)$ is induced by the natural inclusion $\partial\mathrm{SH}_{\mathbf{G}} \hookrightarrow \overline{\mathrm{SH}}_{\mathbf{G}}$ after identifying $\mathrm{H}^i(\mathrm{SH}_{\mathbf{G}}, \widetilde{V}_\lambda)$ with $\mathrm{H}^i(\overline{\mathrm{SH}}_{\mathbf{G}}, \widetilde{V}_\lambda)$. In this simple $\mathbb{Q}$ -rank 2 case, the spectral sequence associated to the boundary stratification is just a long exact sequence:

$$\cdots \rightarrow \mathrm{H}^i(\partial\mathrm{SH}_{\mathbf{G}}, \widetilde{V}_\lambda) \rightarrow \mathrm{H}^i(\partial_1, \widetilde{V}_\lambda) \oplus \mathrm{H}^i(\partial_2, \widetilde{V}_\lambda) \rightarrow \mathrm{H}^i(\partial_0, \widetilde{V}_\lambda) \rightarrow \cdots.$$

For each $i = 0, 1, 2$, there is a decomposition, obtained using Kostant’s theorem

(see for example [HMG94, Section 11] or [HR19, Section 4]):

$$H^i(\partial_i, \widetilde{V}_\lambda) = \bigoplus_{w \in \mathcal{W}^i} \text{Ind}_{\mathbf{P}_i(\mathbb{A}_f)}^{\mathbf{G}(\mathbb{A}_f)} H^{i-\ell(w)}(S_{\mathbf{M}_i}, \widetilde{W}_{w \bullet \lambda}), \quad (3.1)$$

where $S_{\mathbf{M}_i}$ denotes the locally symmetric space attached to $\mathbf{M}_i$ - this is **not** in general a Shimura variety but only a real manifold. For $\mathbf{G}$, although the locally symmetric spaces attached to $\mathbf{M}_i$ do admit a structure of a Shimura variety, we want to stress that the above decomposition is purely in terms of the real manifold structure of $S_{\mathbf{M}_i}$.

In the above decomposition, the condition $w \in \mathcal{W}^i$ ensures that $w \bullet \lambda$ is dominant when restricted to $\mathbf{M}_i$, and we have denoted by $\widetilde{W}_{w \bullet \lambda}$ the associated local system on $S_{\mathbf{M}_i}$. We have used the letter W (and the notation $\widetilde{W}$) for local systems defined on $S_{\mathbf{M}_i}$ instead of V (and $\widetilde{V}$) so as to avoid confusion with local systems defined on $\text{SH}_{\mathbf{G}}$.

3.2.2.2 The boundary of the Borel–Serre compactification and constant terms

In this subsection, we describe the restriction maps from $\overline{\text{SH}}_{\mathbf{G}}$ to ∂_i in cohomology. References regarding the well-known results that we will outline here are [Har, Section 8], [Sch83, Section 1.5-1.10].

By the de Rham comparison theorem, the singular cohomology groups appearing in the previous subsection can be computed by the de Rham complex, which is in turn computed by $(\mathfrak{g}, K_\infty)$ -cohomology. More precisely: denote by $\mathfrak{k}$ the Lie algebra

of K_∞ ; let $\mathfrak{g}$ be the complexified Lie algebra of $\mathbf{G}$; and denote by $\Omega^\bullet(\mathrm{SH}_\mathbf{G}(K), \tilde{V}_\lambda)$ the de Rham complex computing the cohomology of the local system $\tilde{V}_\lambda$. There is a canonical isomorphism of complexes:

$$\Omega^\bullet(\mathrm{SH}_\mathbf{G}(K), \tilde{V}_\lambda) \simeq \mathrm{Hom}_{K_\infty} \left(\bigwedge^\bullet(\mathfrak{g}/\mathfrak{k}), C^\infty((\mathbf{G}(\mathbb{Q}) \backslash \mathbf{G}(\mathbb{A})/K) \otimes \tilde{V}_\lambda) \right).$$

This identification induces a Hecke-equivariant isomorphism

$$H^\bullet(\mathrm{SH}_\mathbf{G}(K), \tilde{V}_\lambda) \simeq H_{(\mathfrak{g}, \mathfrak{k})}^\bullet(C^\infty(\mathbf{G}(\mathbb{Q}) \backslash \mathbf{G}(\mathbb{A})/K) \otimes V_\lambda).$$

Moreover, by Franke's theorem [Fra98], the space of smooth functions appearing on the right-hand side can be replaced with the space of automorphic forms on $\mathbf{G}$ of level K .

There are analogous results for each boundary stratum of $\partial \mathrm{SH}_\mathbf{G}$. In particular, we have a Hecke-equivariant isomorphism (see [HR19, Section 4] or [Sch83, section 1.10])

$$H^\bullet(\partial_i(K), \tilde{V}_\lambda) \simeq H_{(\mathfrak{g}, \mathfrak{k})}^\bullet(C^\infty((\mathbf{U}_i(\mathbb{A}) \mathbf{P}_i(\mathbb{Q}) \backslash \mathbf{G}(\mathbb{A})/K) \otimes V_\lambda)$$

such that the restriction map $r_i : H^\bullet(\mathrm{SH}_\mathbf{G}(K), \tilde{V}_\lambda) \rightarrow H^\bullet(\partial_i(K), \tilde{V}_\lambda)$ is identified with the constant term map in Lie algebra (or de Rham) cohomology, i.e. integration along $\mathbf{U}_i(\mathbb{A})$. We will use this fact repeatedly in the section 3.3.

3.2.2.3 The restriction to the boundary of p -adic sheaves

In this section, we survey results of Pink [Pin92] on the restriction of p -adic sheaves to the boundary of the Baily–Borel compactification of a Shimura variety. The reader can consult [Har93, pp. 50–56] for more details.

Let j denote the inclusion of $\mathrm{SH}_{\mathbf{G}}$ into its Baily–Borel compactification, which we denote by $\mathrm{SH}_{\mathbf{G}}^{\star}$. Let $\partial^{\star} := \mathrm{SH}_{\mathbf{G}}^{\star} \setminus \mathrm{SH}_{\mathbf{G}}$ be the boundary of $\mathrm{SH}_{\mathbf{G}}^{\star}$. This is an algebraic variety defined over $\mathbb{Q}$. It has a stratification indexed by the $\mathbb{Q}$ -rational standard *maximal* parabolic subgroups of $\mathbf{G}$. We denote by $\partial_i^{\star}$ the stratum corresponding to $\mathbf{P}_i$, for $i = 1, 2$, and let ι_i be the natural inclusion of $\partial_i^{\star}$ into $\mathrm{SH}_{\mathbf{G}}^{\star}$ (see for example [Har93, Section 2.2.5]). There is a natural map $\overline{\mathrm{SH}}_{\mathbf{G}} \rightarrow \mathrm{SH}_{\mathbf{G}}^{\star}$ (of manifolds) that induces maps $\partial_i \rightarrow \partial_i^{\star}$.

Let us restrict our attention to the 0-dimensional stratum $\partial_1^{\star}$. In what follows, fix $K_f \subset \mathbf{G}(\mathbb{A}_f)$ a compact-open subgroup. For an element

$$g_f \in \mathbf{P}_1(\mathbb{A}_f) \backslash \mathbf{G}(\mathbb{A}_f) / K_f,$$

we define the compact open subgroup $K_f(g_f) \subset \mathbf{M}_1(\mathbb{A}_f)$ as the projection of $\mathbf{P}_1(\mathbb{A}_f) \cap g_f K_f g_f^{-1}$ to the Levi quotient $\mathbf{M}_1(\mathbb{A}_f)$ of $\mathbf{P}_1(\mathbb{A}_f)$. Similarly let $K_{\infty}^{\mathbf{M}_1}$ be the projection of $K_{\infty} \cap \mathbf{P}_1(\mathbb{R})$ to $\mathbf{M}_1(\mathbb{R})$. Recall that

$$\mathbf{M}_1 = \mathbf{M}_{1,\mathrm{lin}} \times \mathbf{M}_{1,h},$$

where $\mathbf{M}_{1,\mathrm{lin}}$ is the linear part of $\mathbf{M}_1$ and $\mathbf{M}_{1,h}$ the hermitian part. Recall also that

$\mathbf{M}_1 = \mathbf{G}_m \times \mathbf{SO}_3$, so that $\mathbf{M}_{1,h}$ is identified with $\mathbf{G}_m$ and $\mathbf{M}_{1,\text{lin}}$ with $\mathbf{SO}_3$. Define then $K_f^{\text{lin}}(g_f) \subseteq \mathbf{M}_{1,\text{lin}}(\mathbb{A}_f)$ and $K_f^h(g_f) \subset \mathbf{M}_{1,h}(\mathbb{A}_f)$ as the projections of $K_f^{\mathbf{M}_1}(g_f)$ to $\mathbf{M}_{1,\text{lin}}(\mathbb{A}_f)$ and $\mathbf{M}_{1,h}(\mathbb{A}_f)$ respectively.

For each $g_f \in \mathbf{P}_1(\mathbb{A}_f) \backslash \mathbf{G}(\mathbb{A}_f) / K_f$, there are locally closed immersions with disjoint image $\text{SH}_{\mathbf{M}_{1,h}}(K_f^h(g_f)) \rightarrow \partial_1^*$ (see for example [Mor18, pp7-8]). The ∂_1^* stratum for $\text{SH}_{\mathbf{G}}^*(K_f)$ is then the disjoint union of the Shimura varieties $\text{SH}_{\mathbf{M}_{1,h}}(K_f^h(g_f))$ for g_f ranging over the representatives of $\mathbf{P}_1(\mathbb{A}_f) \backslash \mathbf{G}(\mathbb{A}_f) / K_f$, where the Shimura data $\text{SH}_{\mathbf{M}_{1,h}}$ is $(\mathbf{M}_{1,h}, h_1)$, with $h_1 : \mathbf{S} \rightarrow \mathbf{M}_{1,h}(\mathbb{R})$ defined in section 3.2.

To state the result of Pink [Pin92] that we will use, we need to introduce one final object. Let $\mathfrak{u}_1$ be the Lie algebra of $\mathbf{U}_1$. Then for any highest weight representation V_λ of $\mathbf{G}$, we can take the Lie algebra cohomology of V_λ : $H^\bullet(\mathfrak{u}_1, V_\lambda)$. This cohomology carries an action of $\mathbf{M}_1$, and a well-known theorem of Kostant asserts that this $\mathbf{M}_1$ -representation admits a decomposition:

$$\text{R}\Gamma(\mathfrak{u}_1, V_\lambda) = \bigoplus_{w \in \mathcal{W}^1} W_{w \bullet \lambda}[-\ell(w)]. \quad (3.2)$$

This $\mathbf{M}_1$ -module is in particular a $\mathbf{M}_{1,\text{lin}}$ -module and we denote by $\widetilde{H^\bullet(\mathfrak{u}_1, V_\lambda)}$ the local system on the locally symmetric space attached to $\mathbf{M}_{1,\text{lin}}$.

The cohomology $H^\bullet(\mathfrak{u}_1, V_\lambda)$ has also an algebraic action of the multiplicative group $\mathbf{G}_m$ given by the composition $\mathbf{G}_m \rightarrow \mathbf{S}_{\mathbb{R}} \xrightarrow{h_1} \mathbf{M}_{1,\mathbb{R}}$. Under this action of $\mathbf{G}_m$, we can decompose $H^\bullet(\mathfrak{u}_1, V_\lambda)$ in weight spaces:

$$H^\bullet(\mathfrak{u}_1, V_\lambda) = \bigoplus_{\nu \in \mathbb{Z}} H^\bullet(\mathfrak{u}_1, V_\lambda)[\nu];$$

Here $H^\bullet(\mathbf{u}_1, V_\lambda)[\nu]$ denotes the ν -eigenspace of the action of $\mathbf{G}_m$.

As is explained for example in [Har93, p. 2.2.10.1], we can state the following result:

3.2.2.4 Proposition (Pink). *The restriction of $R^\bullet j_\star(\tilde{V}_\lambda)$ to a stratum $\mathrm{SH}_{\mathbf{M}_{1,h}}(K_f^h(g_f))$ is the p -adic sheaf canonically constructed from the $\mathbf{M}_{1,h}$ -module*

$$H^\bullet\left(\mathbf{M}_{1,\mathrm{lin}}(\mathbb{Q}) \backslash \mathbf{M}_{1,\mathrm{lin}}(\mathbb{A}) / K^{\mathrm{lin}}(g_f), H^\bullet(\widetilde{\mathbf{u}_1, V_\lambda})\right).$$

Explicitly, we have

$$R^\bullet j_\star(\tilde{V}_\lambda) \Big|_{\mathrm{SH}_{\mathbf{M}_{1,h}}(K_f^h(g_f))} = \bigoplus_{\nu \in \mathbb{Z}} H^\bullet\left(\mathbf{M}_{1,\mathrm{lin}}(\mathbb{Q}) \backslash \mathbf{M}_{1,\mathrm{lin}}(\mathbb{A}) / K^{\mathrm{lin}}(g_f), H^\bullet(\widetilde{\mathbf{u}_1, V_\lambda})[\nu]\right) \otimes_{\mathbb{Q}} \mathbb{Q}_p(\nu).$$

The cohomology $H^\bullet(\partial_1^*(K_f), R^\bullet j_\star(\tilde{V}_\lambda))$ is a $\mathcal{C}_c^\infty(\mathbf{G}(\mathbb{A}_f) // K_f) \times \mathrm{Gal}_{\mathbb{Q}}$ module. Once the limit over all compact-open K_f is taken, it is identified with the $\mathbf{G}(\mathbb{A}_f) \times \mathrm{Gal}_{\mathbb{Q}}$ module:

$$\mathrm{Ind}_{\mathbf{P}_1(\mathbb{A}_f)}^{\mathbf{G}(\mathbb{A}_f)} H^\bullet(\mathrm{SH}_{\mathbf{M}_{1,h}}, R^\bullet j_\star(\tilde{V}_\lambda)|_{\mathrm{SH}_{\mathbf{M}_{1,h}}}).$$

Let us record the following lemma, which will be invoked in 3.3.8.

3.2.2.5 Lemma. *Let $w = ss_{-1}s \in \mathcal{W}^1$ be the unique Weyl element of length acting on simple roots as $(\alpha, \beta) \mapsto (-\alpha, \beta)$. Then $H^3(\mathbf{u}_1, \mathbb{C}) = W_{-3\alpha}$ and*

$$R^3 j_\star(\tilde{V}_\lambda) \Big|_{\mathrm{SH}_{\mathbf{M}_{1,h}}(K_f^h(g_f))} = H^0\left(\mathbf{M}_{1,\mathrm{lin}}(\mathbb{Q}) \backslash \mathbf{M}_{1,\mathrm{lin}}(\mathbb{A}) / K^{\mathrm{lin}}(g_f), W_{-3\alpha}\right) \otimes_{\mathbb{Q}} \mathbb{Q}_p(-3).$$

3.3 The candidate cohomology classes

In this section we begin the construction of the cohomology classes that we will arrange into an Euler System. These classes will be constructed from Eisenstein series on $\mathbf{G}$.

From here on, χ denotes an **odd** Dirichlet character with conductor **divisible by** p .

Let N_χ be the conductor of χ . The hypothesis that $p \mid N_\chi$ is imposed to simplify the cases considered in later arguments and, unlike the oddness of χ , is not essential.

We denote by F_χ (resp. $F_{\chi,p}$) the extension of $\mathbb{Q}$ (resp. $\mathbb{Q}_p$) generated by the values of χ and $\sqrt{-1}$. Let $\mathcal{O}_\chi$ (resp. $\mathcal{O}_{\chi,p}$) be the ring of integers of F_χ (resp. $F_{\chi,p}$). Let $\mathbb{F}_\chi$ (resp. $\mathbb{F}_{\chi,p}$) be the abelian extension of $\mathbb{Q}$ (resp. $\mathbb{Q}_p$) cut out by χ , that is, fixed by the kernel of χ viewed as a character of $\text{Gal}_{\mathbb{Q}}$ (resp. $\text{Gal}_{\mathbb{Q}_p}$). Let $\text{Gal}_\chi := \text{Gal}(\overline{\mathbb{Q}}/\mathbb{F}_\chi) \subset \text{Gal}_{\mathbb{Q}}$.

3.3.1 Hirai's Eisenstein series

In [Hir98] Hirai defines certain holomorphic Eisenstein series on orthogonal groups. Let $\Gamma := \mathbf{G}(\mathbb{Z})$. Recall the cocycle $J(g, Z)$ defined in 3.1.4.2 where $g \in \mathbf{G}(\mathbb{R})$ and $Z \in \mathcal{H}^\circ$. Hirai computes the Fourier expansion along the abelian unipotent radical of $\mathbf{P}_1$ of the Eisenstein series

$$E_l^{\text{Hirai}}(Z, s) := \left(\frac{q(\Im(Z))}{2} \right)^{(6s-2l+3)/4} \sum_{\gamma \in \mathbf{P}_2(\mathbb{Q}) \cap \Gamma \backslash \Gamma} J(\gamma, Z)^{-l} |J(\gamma, Z)|^{-3s+l-3/2},$$

which converges absolutely for $\Re(s) > 1/2$. As a consequence of his computation, he deduces that $E_l^{\text{Hirai}}(Z, l/3 - 1/2)$ is a holomorphic Eisenstein series of weight l , for $l > 3$.

We now rephrase Hirai's Eisenstein series from an adelic point of view. Let $K_{\mathbb{A}} = K_{\infty} \mathbf{G}(\hat{\mathbb{Z}})$, a maximal compact subgroup of $\mathbf{G}(\mathbb{A})$. By Iwasawa decomposition we can write $g \in \mathbf{G}(\mathbb{A})$ as $g = p_1(g)k(g)$, where $p_1(g) \in \mathbf{P}_1(\mathbb{A})$ and $k(g) \in K_{\mathbb{A}}$. We can define

$$f_l(g, s) := |a(p_1(g))|^{3s+3/2} J(k(g)_{\infty}, Z_0)^{-l},$$

where we recall that a is the character of $\mathbf{A}$ (which extends to a character of $\mathbf{P}_1$) whose differential is the root α of $\mathbf{G}$. Then the adelic version of Hirai's Eisenstein series is

$$E_l(g, s) := \sum_{\gamma \in \mathbf{P}_1(\mathbb{Q}) \backslash \mathbf{G}(\mathbb{Q})} f_l(\gamma g, s).$$

For $g \in \mathbf{G}(\mathbb{R})$ such that $Z = g \cdot Z_0$, we have

$$E_l(g, s) = E_l^{\text{Hirai}}(Z, s) J(g, Z_0)^{-l}.$$

3.3.2 A variation on Hirai's Eisenstein series

We now introduce a more general version of Hirai's Eisenstein series by allowing non-spherical sections at some finite places and incorporating the non-trivial odd Dirichlet character χ .

Fix $l \geq 0$ an integer. Let μ_s denote the character $|\cdot|^{3s+3/2} \chi$, where here χ is seen as an adelic character, which by abuse of notation we still denote χ . Note that

$\mu_s(a(\cdot)) = \delta_{\mathbf{P}_1}^{s+1/2} \cdot (\chi \circ a)$ for $\delta_{\mathbf{P}_1}$ the modulus character of $\mathbf{P}_1$.

3.3.2.1 Some simple sections

We now choose some simple sections of the induced representation

$$I(s, \chi) := \text{Ind}_{\mathbf{P}_1(\mathbb{A})}^{\mathbf{G}(\mathbb{A})}(\mu_s \circ a)$$

and its local counterparts. For a place v we let

$$I_v(s, \chi) := \text{Ind}_{\mathbf{P}_1(\mathbb{Q}_v)}^{\mathbf{G}(\mathbb{Q}_v)}(\mu_s \circ a).$$

Archimedean sections. At infinity, we choose the same section as Hirai. Namely, we define $f_{\infty, s}^\chi$ to be the unique function such that if $g = pk$, $g \in \mathbf{G}(\mathbb{R})$, $p \in \mathbf{P}_1(\mathbb{R})$ and $k \in K_\infty$, then

$$f_{\infty, s}^\chi(g) = (2\pi i)^{-4} |a(p)|^{3s+3/2} J(k, Z_0)^{-l}.$$

Spherical sections. At finite places ℓ where χ is unramified we define the normalized spherical sections:

$$f_{\ell, s}^\chi(g_\ell) = L_\ell(3, \chi) L_\ell(4, \chi^2) \mu_s(a(p)),$$

for $g_\ell = pk_\ell$, $p \in \mathbf{P}_1(\mathbb{Q}_\ell)$, and $k_\ell \in \mathbf{G}(\mathbb{Z}_\ell)$.

Basic ramified sections. At places ℓ where χ is ramified, we define $f_{\ell,s}^\chi$ to be A_χ times the section described in remark (B.1.3) (taking $p = \ell$, there). Here A_χ is the constant of item (3) of definition (B.1.10).

Basic adelic section. With the above choices of local sections, we let $f_s^\chi \in I(s, \chi)$ be the adelic section

$$f_s^\chi := \bigotimes f_{v,s}^\chi.$$

3.3.2.2 Remark. When $l = 3$ and $s = 1/2$, the pull-back of $f_{1/2}^\chi$ to $\mathbf{G}'(\mathbb{A})$ corresponds to the section denoted f_χ^{ord} in (B.1.10).

In classical language, the Eisenstein series defined by $f_{v,s}^\chi$ pulled-back to $\mathbf{G}'$ corresponds (up to normalization constants introduced above) to the function on the genus 2 Siegel half-space $\mathcal{H}_2$ defined by the series

$$\sum_{\gamma \in \mathbf{P}'_1(\mathbb{Q}) \cap \Gamma_0(N_\chi) \backslash \Gamma_0(N_\chi)} \chi(\gamma) j(\gamma, Z)^{-l} |j(\gamma, Z)|^{l-3s-3/2},$$

where $j(g, Z)$ is the usual cocycle on $\mathbf{G}'(\mathbb{R}) \times \mathcal{H}_2$. Note that the above series does not converge absolutely when $l = 3$, $s = 1/2$. However, arguing by Hecke summation, following for example Faltings in [Fal05, Section 9], one can show the series still defines a holomorphic function on $\mathcal{H}_2$. Consequently, the Eisenstein series $E(-; f_s^\chi)$ for $l = 3$ has a continuation to $s = 1/2$ and defines an automorphic form on $\mathbf{G}(\mathbb{A})$ such that at infinity the resulting automorphic representation is a holomorphic discrete series of "scalar weight 3".

3.3.3 The choices of sections

In this section we give the choice of sections at $l = 3$, $s = 1/2$ that we make at each place to define the Eisenstein series that will result in cohomology classes forming an Euler System. The data that goes into making these choices includes not only the character χ but also a newform f of weight 2. Let N_f be the level of f ; this is also the conductor of the irreducible automorphic representation $\pi(f)$ of $\mathbf{SL}_2(\mathbb{A})$ generated by f .

Without loss of generality, we can and do assume that N_f is minimal among all twists of f .

We may decompose N_f as follows:

$$N_f = Q_{st}Q_{sc}Q_{rps},$$

where Q_{st} , Q_{sc} and Q_{rps} are integers only divisible, respectively, by primes ℓ at which $\pi(f)$ is a special representation, supercuspidal, or a ramified principal series. The assumption on N_f ensures, among other things, that if the local constituent at ℓ is a twist of a special representation, then it is an unramified twist of the Steinberg representation. In addition to being given f we assume we are also given an auxiliary prime $\ell_{aux} \nmid pN_fN_\chi$.

The following is not needed until we claim integrality results! The prime ℓ_{aux} can just be subsumed into M in the final construction.

3.3.3.1 Assumption. If $p = 3$ or $p = 5$, we assume that

$$p^{a_p} \mid \prod_{\substack{\ell \mid N_f \\ \ell \equiv 1 \pmod{N_\chi}}} (\ell - 1),$$

where $a_3 = 3$ and $a_5 = 1$.

3.3.3.2 Remark. Fixing ρ_0 , it is always possible to choose f so that the above assumption is verified by the classical level raising result of Ribet [Rib90] and the Cebotarev density theorem.

3.3.3.3 Definition (Auxiliary prime). We say that an odd prime ℓ is *auxiliary* for ρ_0 when the image of $L_\ell(1, \text{Ad}^0(\rho_f))^{-1} \in \mathcal{O}$ in $\mathbb{F}_q$ is non-zero.

Auxiliary primes will be used to ensure that there is a closed embedding of the Shimura varieties associated to $\mathbf{H}$ and the Shimura variety associated to $\mathbf{G}$.

3.3.3.4 Lemma. *Assume that ρ_0 is irreducible and the image of ρ_0 contains an element of order p . Then, there exists auxiliary primes for ρ_0 .*

Proof. The local ℓ factor is a unit if and only if the quantity $(\ell - 1)(\text{Tr}(\rho_f(\text{Frob}_\ell)) - (\ell + 1)^2)$ is a unit. We will show that we can choose $\ell \equiv -1 \pmod{p}$ such that $\text{Tr}(\rho_0(\text{Frob}_\ell))$ is non-vanishing. For this we study the number of elements in the image of ρ_0 with vanishing trace. By Dickson's classification of subgroups of $\mathbf{PGL}_2(\mathbb{F}_q)$ (see for example the proof of Proposition 1.11 of [Wil95]), the image of ρ_0 in $\mathbf{PGL}_2(\mathbb{F}_q)$ is conjugated to some $\mathbf{PGL}_2(F)$ or $\mathbf{PSL}_2(F)$ for a subfield F of $\mathbb{F}_q$. We first assume that the image of ρ_0 is conjugated to $\mathbf{PGL}_2(F)$. Up to changing the notation for q ,

we assume that $\mathbb{F}_q = F$. The conjugacy classes of $\mathbf{PGL}_2(\mathbb{F}_q)$ that have vanishing trace are the conjugacy classes of $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 0 & u \\ 1 & 0 \end{pmatrix}$ where u is not a square in $\mathbb{F}_q$. These have centralizers of size respectively $(q-1)$ and $(q+1)$. Let Z be the kernel of the projection of the image of ρ_0 to $\mathbf{PGL}_2(\mathbb{F}_q)$ – note that necessarily $\pm \text{id} \in Z$. Then we have, in the split torus, $\#Z/2$ conjugacy classes with centralizer of size $(\#Z) \cdot (q-1)$ which have vanishing trace and $\#Z/2$ conjugacy classes in the non-split torus with centralizer of size $(\#Z) \cdot (q+1)$ that also have vanishing trace. In total, the proportion of elements in the image of ρ_0 with vanishing trace is $1/2(q-1) + 1/2(q+1)$. Since primes $\ell \equiv -1 \pmod{p}$ have Dirichlet density $1/(p-1)$, and

$$\frac{1}{p-1} - \frac{1}{2(q-1)} - \frac{1}{2(q+1)} \geq \frac{1}{p^2-1} > 0,$$

by the Chebotarev density theorem, there must exist a prime $\ell \equiv -1 \pmod{p}$ such that $\rho_0(\text{Frob}_\ell)$ has non-vanishing trace. Then, since p is odd, we deduce that the image of $(\ell-1)(\text{Tr}(\rho_f(\text{Frob}_\ell)) - (\ell+1)^2)$ in $\mathbb{F}_q$ is $-2\text{Tr}(\rho_0(\text{Frob}_\ell))^2 \neq 0$.

The case of the image of ρ_0 being conjugated to $\mathbf{PSL}_2(F)$, is treated in a very similar way and one can see that the proportion of traceless elements in $\mathbf{PSL}_2(F)$ is bounded above by the proportion of traceless elements in $\mathbf{PGL}_2(F)$; this concludes the proof. $\square$

This assumption will be used to prove integrality at small primes of some cohomology classes; see Section 3.3.10.

We fix once and for all an auxiliary prime $\ell_{aux} \nmid N_f$. Let M be a positive integer prime to $pN_f\ell_{aux}$. For a place v we define $f_{v,s;M}^\chi$ as follows:

Archimedean sections. At infinity, $f_{\infty,s;M}^\chi := f_{\infty,s}^\chi$, where the right-hand side is the section defined in 3.3.2.1.

For primes $\ell \mid Q_{st}$. If $\ell \mid Q_{st}$, we choose $f_{\ell,s;M}^\chi$ to be the Steinberg section as in Proposition (B.9.2).

For primes $\ell \mid Q_{sc}Q_{rps}M\ell_{aux}$. If $\ell \mid Q_{sc}Q_{rps}M\ell_{aux}$, we choose $f_{\ell,s;M}^\chi$ to be the volume section, as defined in Section (B.8).

For all other primes. At all other primes, namely for $\ell \nmid N_f M\ell_{aux}$, $f_{\ell,s;M}^\chi = f_{\ell,s}^\chi$ is the spherical section defined in 3.3.2.1.

We then set

$$f_{s;M}^\chi = \otimes f_{v,s;M}^\chi.$$

We will also need variations of these sections modified only at the prime p .

The depleted section. We denote by $f_{s;M}^{\chi,dep}$ the section obtained by replacing the local section at p with the section $t_\chi^{dep} f_{p,s}^\chi$, where t_χ^{dep} is the depletion operator of section (B.3). Note that when $s = 1/2$ this corresponds to the section $f_{p,\chi}^{dep}$ of Definition (B.3.1) in Appendix (B).

Liu's section. We denote by $f_{s;M}^{\chi,Liu}$ the section obtained by replacing the local section at p with Liu's section $f_{p,s}^{\chi,Liu}$. Here $f_{p,s}^{\chi,Liu}$ is defined by its pull-back to $\mathbf{G}'$:

$$f_{p,s}^{\chi,Liu} \left(\begin{bmatrix} A & B \\ C & D \end{bmatrix} \right) := \begin{cases} A_\chi \bar{\chi}(\det C) |\det(C)|^{-3s-3/2} \eta_p(C^{-1}D) & \text{if } \det C \neq 0 \\ 0 & \text{otherwise,} \end{cases}$$

where η_p is the Schwartz function on $\text{Sym}^2(\mathbb{Q}_p^{\oplus 2})$ defined by its Fourier transform

$$\hat{\eta}_p(S) = \hat{\eta}_p \left(\begin{bmatrix} s_1 & s_0 \\ s_0 & s_2 \end{bmatrix} \right) = \mathbb{1}_{p^2\mathbb{Z}_p}(s_1) \mathbb{1}_{\mathbb{Z}_p}(s_2) \chi(\det(S)) \psi(s_0) \bar{\chi}(s_0),$$

where we view χ as a function on $\mathbb{Q}_p$ by extending it by 0 away from $\mathbb{Z}_p^\times$, and ψ is the Nebentypus of the $\mathbf{SL}_2(\mathbb{A})$ modular form f .

The sections $f_{s,M}^\chi$, $f_{s,M}^{\chi,dep}$, and $f_{s,M}^{\chi,Liu}$ all give rise to Eisenstein series on $\mathbf{G}(\mathbb{A})$, denoted by $E(g; f_{s,M}^\chi)$, etc. The series defining them converges absolutely for real part of s sufficiently large and have meromorphic continuations in s to all of $\mathbb{C}$. As explained in 3.3.2, when $l = 3$ these series are holomorphic at $s = 1/2$ and correspond to holomorphic modular forms on $\mathcal{H}^0$.

3.3.3.5 Level subgroups

Let M be an integer. The section $f_{s;M}^\chi$ (and so also its associated Eisenstein series) is fixed under right translation by the open compact subgroup

$$\prod_{\ell^r || M Q_{sc} Q_{rps}} K^{\text{vol}}(\ell^r) \cdot \prod_{\ell | Q_{st}} K^{\text{St}}(\ell) \cdot \prod_{\ell \nmid M N_f, \ell | N_\chi} K_1^{\mathbf{P}}(\ell^a) \cdot K_1^{\mathbf{A}}(\ell_{aux}) \cdot \prod_{\ell \nmid \ell_{aux} M N_f N_\chi} \mathbf{G}(\mathbb{Z}_\ell) \subset \mathbf{G}(\mathbb{A}_f).$$

Similarly, $f_{s;M}^{\chi, \text{dep}}$ is fixed by the same product of local groups except with $K_1^{\mathbf{P}}(p^r)$ replaced by $\tilde{K}_1^{\mathbf{P}}(p^r)$. The local groups are as defined in 3.1.5. We call these subgroups the *level groups* of the corresponding sections.

3.3.4 The constant terms of the Eisenstein series on $\mathbf{G}$

In this subsection, we compute the constant terms of the Eisenstein series that were introduced in the previous section. Here, we take $l \geq 3$, and we will be interested in the Eisenstein series at the point $s_l = (l - 3/2)/3$.

In this section, set

$$I_0(s) := \text{Ind}_{\mathbf{P}_0(\mathbb{A})}^{\mathbf{G}(\mathbb{A})}(\mu_s)$$

and

$$I(s) := \text{Ind}_{\mathbf{P}_1(\mathbb{A})}^{\mathbf{G}(\mathbb{A})}(\mu_s),$$

where μ_s is the adelic character defined in 3.3.2. For $w \in \mathcal{W}$ and $f_s \in I_0(s)$ a flat section, let

$$M_w(s)(f_s)(g) := \int_{\mathbf{U}_0(\mathbb{A}) \cap w^{-1} \mathbf{U}_0(\mathbb{A}) w \backslash \mathbf{U}_0(\mathbb{A})} f_s(wug) du.$$

This integral converges for the real part of s sufficiently large and has a meromorphic continuation to all $s \in \mathbb{C}$. Moreover, $M_w(s)$ defines an intertwining operator:

$$M_w(s)(-) : I_0(\mu_s) \rightarrow I_0(w^{-1} \cdot \mu_s).$$

The computations for the proof of Theorem 2.3 in [GRS97] (see especially equation (2.2) of *loc cit.*) show that for $i = 1, 2$, and $g \in \mathbf{M}_i(\mathbb{A})$, we have:

$$\int_{[\mathbf{U}_i(\mathbb{A})]} E(ug, s; f_s) du = \sum_{w \in \mathcal{W}_1 \setminus \mathcal{W} / \mathcal{W}_i} E_{\mathbf{L}_i^w}(g, s'; M_w(s)(f_s)|_{\mathbf{M}_i}),$$

where:

- $\mathbf{L}_i^w := \mathbf{M}_i \cap w^{-1}\mathbf{M}_1w$ is either equal to $\mathbf{M}_i$ or the Borel subgroup $\mathbf{B}_i$ of $\mathbf{M}_i$.
Let $\mathbf{B}_0$ be the Borel subgroup of $\mathbf{G}_0$ and $\mathbf{B}$ the Borel subgroup of $\mathbf{GL}_2$. Then $\mathbf{B}_i = \mathbf{GL}_1 \times \mathbf{B}_0$ and $\mathbf{B}_2 = \mathbf{B}$.
- $M_w(f_s)|_{\mathbf{M}_i}$ is either a character of $\mathbf{M}_i$ (if $\mathbf{L}_i^w = \mathbf{M}_i$) or an element of an induced representation $\text{Ind}_{\mathbf{L}_i^w}^{\mathbf{M}_i}(\mu_s^w)$ (if $\mathbf{L}_i^w = \mathbf{B}_i$) with $\mu_s^w|_{\mathbf{A}} = \delta_{\mathbf{P}_0}^{1/2} w^{-1} \cdot \mu_s$.
- $2s' = \langle \gamma_i^\vee, \delta_{\mathbf{P}_0}^{1/2} w^{-1} \cdot \mu_s \rangle$ for $\gamma_1 = \beta$, $\gamma_2 = \alpha - \beta$.
- $E_{\mathbf{L}_i^w}$ is an Eisenstein series on $\mathbf{M}_i$ for a section of the induced representation $\text{Ind}_{\mathbf{L}_i^w}^{\mathbf{M}_i}(\mu_s^w)$.

Having already given the Weyl group representatives in Lemma 3.1.3.2, we can apply this general formula to our specific situation.

3.3.4.1 Proposition. *Let $s_l = (l - 3/2)/3$ and $f_s = f_{\infty,s}^\infty \otimes f_s^\infty$, with $f_s^\infty \in \text{Ind}_{\mathbf{P}_1(\mathbb{A}_f)}^{\mathbf{G}(\mathbb{A}_f)}(\mu_s)$ a meromorphic section that is holomorphic at s_ℓ . Then*

$$\int_{[\mathbf{U}_0(\mathbb{A})]} E(ug; f_{s_l}) du = \int_{[\mathbf{U}_1(\mathbb{A})]} E(ug; f_{s_l}) du = f_{s_l}(g).$$

Proof. Note that it suffices to show that the integral along $\mathbf{U}_1$ is equal to $f_{s_l}(g)$ as $f_s(g)$ is invariant under $\mathbf{U}_0$. The computations in [Hir98, Section 3.3] show that the three intertwining operators (corresponding to the three elements in $\mathcal{W}_1 \backslash \mathcal{W} / \mathcal{W}_1$) at ∞ do not have a pole at s_l for $l \geq 3$. From the general theory of the Langlands quotient, as our choice of section at infinity is such that f_{s_l} is in the subrepresentation of the induced representation that is a holomorphic discrete series, we deduce that the intertwining operators at infinity vanish at s_l . Furthermore, at finite places, by an application of the Gindikin–Karpelevich formula or by slightly modifying the computations of Hirai (see Theorem 2.1, (i) of *loc cit.*) to account for the presence of the Dirichlet character χ , we see that the product of the intertwining operators at finite places also do not have a pole at s_l . We then conclude the claim in the proposition from the classical computation of the constant term of [GRS97, Thm 2.3] recalled above. $\square$

When restricted to $\mathbf{M}_2 \simeq \mathbf{GL}_2$, the section f_s is a section of the induced representation $\text{Ind}_{\mathbf{B}(\mathbb{A})}^{\mathbf{GL}_2(\mathbb{A})}(|a|^{3s+3/2}(\chi \circ a))$, where we now view a as the character of the Borel of upper triangular matrices $\mathbf{B} \subset \mathbf{GL}_2$ that sends $b \in \mathbf{B}(\mathbb{A}_f)$ to its upper left entry. For $s = s_l = (l - 3/2)/3$ we let $E_{\mathbf{GL}_2,l}^\chi(g; f_s) = E_{\mathbf{GL}_2}^\chi(g; f_{s_l})$ be the resulting Eisenstein series on $\mathbf{GL}_2$. As $f_s = f_{\infty,s}^\chi \otimes f_s^\infty$, these correspond to weight l holomorphic

Eisenstein series.

3.3.4.2 Proposition. *Let f_s be as in Proposition 3.3.4.1. For g in the $\mathbf{GL}_2$ part of the Levi $\mathbf{M}_2$,*

$$\int_{[\mathbf{U}_2(\mathbb{A})]} E_l(ug; f_{s_l}) du = E_{\mathbf{GL}_2, l}^\chi(g; f_s).$$

Proof. By the classical computation of [GRS97] already cited above and Lemma 3.1.3.2, we see that the constant term along $\mathbf{U}_2$ is the sum of $E_{\mathbf{GL}_2, l}^\chi(g; f_s)$ and the $\mathbf{GL}_2$ Eisenstein series for the section $M_{ss_{-1}} f_{s; \varphi}$ restricted to $\mathbf{GL}_2$. As f_{s_ℓ} belongs at infinity to the usual holomorphic discrete series of weight l and $M_{ss_{-1}}$ restricts to the usual intertwining operator on $\mathbf{GL}_2$, we deduce that the second term vanishes, which yields the claimed result. (We could have used this to deduce the computation of the constant term along $\mathbf{U}_0$ as well). $\square$

3.3.4.3 Remark. Proposition 3.3.4.2 shows in particular that the automorphic representation generated by E_l^χ has a quotient that is not unitarizable. In particular, this representation is not unitarizable. For an alternative discussion of the non-unitarizability of E_l^χ the reader can also consult [Fal05, pp.150-151]. Note also that if χ is not quadratic, then it is obvious that this representation is not unitary as it is not even hermitian in the sense of [Ren10, §IV.2.1].

3.3.5 Some rationality and integrality properties

From here on we take $l = 3$, and $s = 1/2$. The character $\mu_{1/2}$ will be denoted by μ .

Other simplifying notation. As we have fixed the value of s , in what follows we will denote $f_{v,1/2;M}^\chi$ (resp. $f_{p,1/2;M}^{\chi,dep}, f_{p,1/2;M}^{\chi,Liu}$) by $f_{v,\chi;M}$ (resp. $f_{p,\chi;M}^{dep}, f_{p,\chi;M}^{Liu}$). If $M = 1$, then we will omit it from our notation. We adopt similar notation for the corresponding global sections (so $f_{\chi;M} = f_{1/2;M}^\chi$).

Next we highlight important rationality and integrality properties of the various choices of sections.

3.3.5.1 Proposition. *Let $f \in \{f_{\chi;M}, f_{\chi;M}^{dep}, f_{\chi;M}^{Liu}\}$. Then f is F_χ -valued on $\mathbf{G}(\mathbb{A}_f)$.*

Proof. This is Lemma (B.1.12) of Appendix (B) when $f = f_{\chi;M}$. The claim then follows for $f_{\chi;M}^{dep}$ as this section is just an $\mathcal{O}_\chi[\mathbf{G}(\mathbb{Q}_p)]$ translate of $f_{\chi;M}$. Finally, for $f_{\chi;M}^{Liu}$ the rationality can be deduced in the same way as for $f_{\chi;M}$ by noticing that η_p is F_χ -valued. $\square$

We also record an integrality property of the q -expansions (Fourier expansions) of the Eisenstein series associated with these sections. In order to deal with the small primes $p = 3$ or 5 , we make the following assumption:

3.3.5.2 Assumption. If $p = 3$ or 5 , we assume that

$$p^{a_p} \mid \prod_{\substack{\ell \mid N_f \\ \ell \equiv 1 \pmod{N_\chi}}} (\ell - 1),$$

where $a_3 = 2$ and $a_5 = 1$.

3.3.5.3 Lemma (Faltings' Lemma). *Let $K = K_1^{\mathbf{P}}(p^r)K^p$ be the level group of $E(g; f_{\chi;M})$ as defined in 3.3.3.5. The Fourier expansion of $E(Z, g_f; f_{\chi;M})$ has p -adically integral coefficients at all cusps of $\mathrm{SH}_{\mathbf{G}}(K)^\star$.*

Here by ‘cusps’ we mean the points of the 0-dimensional strata $\partial_1^\star$. These are naturally in bijection with $\mathbf{P}_1(\mathbb{Q}) \backslash \mathbf{G}(\mathbb{A}_f)/K$. The Fourier coefficients are of the associated holomorphic modular form of scalar weight 3.

Proof. As the discussion in Appendix (B) makes clear, it suffices to prove the claim in this lemma for the pullback of the Eisenstein series to $\mathbf{G}'$, where the Eisenstein series become simple variants of the classical Siegel Eisenstein series of weight 3.

We begin by showing that the coefficients of the q -expansion of $E(Z, g_f; f_{\chi; M})$ at the cusp corresponding to the identity are p -adically integral; this is the usual Fourier expansion. Indeed, by Proposition. (B.1.11) this is true for non-degenerate coefficients. Furthermore, the Fourier coefficients at degenerate T correspond to the Fourier coefficients of a usual ordinary weight 3 Eisenstein series of $\mathbf{GL}_2$ multiplied by $L^{(N_f)}(-3, \chi^2)$ (see [Tak12, Props. 5.1, 5.2]). The Fourier coefficients of this $\mathbf{GL}_2$ Eisenstein series are all p -adic integers, except possibly the constant term, which equals $L^{(N_f)}(-2, \chi)/2$. So it suffices to know that both $L^{(N_f)}(-3, \chi^2)$ and $L^{(N_f)}(-2, \chi)/2$ are p -adic integers. This is clearly true unless $p = 2, 3$, or 5 and χ has conductor a power of p . As p is odd it remains to treat the case when $p = 3$ or 5 . The desired integrality is ensured in these cases by Assumption 3.3.5.2.

The irreducibility of the Igusa tower together with the q -expansion principle for p -adic modular forms for $\mathbf{G}'$ then implies the same for all cusp represented by elements in $\mathbf{P}'_1(\mathbb{Q}_p)\mathbf{G}(\mathbb{A}_f^p)$. These include all the ‘multiplicative cusps.’ As $E(Z, g_f; f_{\chi; M})$ is ordinary at p , arguing as at the very end of Section 9 in [Fal05] yields the p -adic integrality of the q -expansions at all cusps. $\square$

3.3.6 Unique cohomological degree

Let Π_χ be the automorphic representation generated by the Eisenstein series $E(-; f)$, where $f \in I(1/2)$ is such that its archimedean constituent is just $f_{\infty, 1/2}^\chi$. Then $\Pi_\chi \simeq \Pi_\infty \otimes \Pi_\chi^\infty$ with Π_∞ the discrete series of scalar weight 3. Similarly, let $\text{Eis}_{\mathbf{GL}_2}^\infty$ be the finite part of the automorphic representation of $\mathbf{GL}_2$ generated by the Eisenstein series $E_{\mathbf{GL}_2}(-; f|_{\mathbf{M}_2})$, where f is as above (see the paragraph before Proposition 3.3.4.2).

Throughout this section, we will work with de Rham and singular cohomology. We denote $H^i(\text{SH}_{\mathbf{G}}, \mathbb{C})$ and $H_{\text{dR}}^i(\text{SH}_{\mathbf{G}}/\mathbb{C})$ the singular and de Rham cohomology of $\text{SH}_{\mathbf{G}}$, respectively. We note that we have taken implicitly direct limits over level subgroups $K \subset \mathbf{G}(\mathbb{A}_f)$ of the cohomology groups of the manifolds $\text{SH}_{\mathbf{G}}(K)(\mathbb{C})$; these spaces are admissible $\mathbf{G}(\mathbb{A}_f)$ -representations. We recall that the usual de Rham isomorphism identifies these spaces as $\mathbf{G}(\mathbb{A}_f)$ -representations.

We will show that Π_χ^∞ only occurs in de Rham cohomology in the middle cohomological degree (degree 3). We will use this to construct a rational class in the middle degree of singular cohomology that has a lift to absolute étale cohomology.

We denote by $H_{\text{dR}}^\bullet(\text{SH}_{\mathbf{G}}/\mathbb{C})[\Pi_\chi^\infty]$ the generalized Π_χ^∞ -isotypical part of the cohomology. By this we mean the maximal $\mathbf{G}(\mathbb{A}_f)$ -submodule such that every irreducible $\mathbf{G}(\mathbb{A}_f)$ -subquotient is isomorphic to the representation Π_χ^∞ . Note that Π_χ^∞ is irreducible by Proposition 3.3.6.2 below. The aim of this section is to prove the following proposition:

3.3.6.1 Proposition. *Let $q \neq 3$. Then $H_{\text{dR}}^q(\text{SH}_{\mathbf{G}}/\mathbb{C})[\Pi_\chi^\infty] = 0$.*

We start with some representation-theoretic preliminaries.

3.3.6.2 Proposition. *The $\mathbf{G}(\mathbb{A}_f)$ -representation Π_χ^∞ is irreducible.*

Proof. By remark (A.4.2) we can reason with $\mathbf{G}'$ in place of $\mathbf{G}$, and then the result is a direct consequence of [ST93, Cor. 3.10(ii)]. $\square$

The following will be useful for proving Lemma 3.3.6.5.

3.3.6.3 Lemma. *Let J_f be the Jacquet module of Π_χ^∞ along $\mathbf{P}_2$. Then $\mathrm{Eis}_{\mathbf{GL}_2}^\infty$ is an $\mathbf{M}_2(\mathbb{A}_f)$ -subquotient of J_f , and $j_f = \chi_f|3a|_f$ (restricted to $\mathbf{M}_0$) is an $\mathbf{M}_0$ -subquotient.*

Proof. This follows from the constant term computations (cf. Propositions 3.3.4.2 and 3.3.4.1). $\square$

To conclude the representation-theoretic preliminaries, we discuss the presence of Π_χ^∞ in the interior cohomology.

3.3.6.4 Proposition (Manin-Drinfeld principle). *The $\mathbf{G}(\mathbb{A}_f)$ -representation Π_χ^∞ does not appear in the interior cohomology in any degree, that is, $H_i^!(\mathrm{SH}_{\mathbf{G}}(\mathbb{C}), \mathbb{C})[\Pi_\chi^\infty] = 0$ for all i .*

Proof. Note that any $\mathbf{G}(\mathbb{A}_f)$ -subquotient of $H_i^!(\mathrm{SH}_{\mathbf{G}}(\mathbb{C}), \tilde{V}_\lambda)$ is unitarizable. Indeed, interior cohomology lies inside L^2 -cohomology, which is endowed with a natural invariant hermitian pairing as it is defined in terms of square integrable differentiable forms for a specific Riemannian metric on $\mathrm{SH}_{\mathbf{G}}(\mathbb{C})$ (see for example [Zuc89, Section 1.3] or the discussion in [Har, pp306-310, Theorem 8.1.1]). Since Π_χ^∞ is not unitarizable as is noted in Remark 3.3.4.3, the claim follows. $\square$

The next sequence of lemmas and propositions builds to a proof of Proposition 3.3.6.1.

3.3.6.5 Lemma. *We have $H_{\text{dR}}^q(\partial_0/\mathbb{C})[\Pi_\chi^\infty] = 0$ if $q \neq 3$.*

Proof. As was recalled in 3.2.2.1, we have the following description of the cohomology of the boundary:

$$H_{\text{dR}}^q(\partial_0/\mathbb{C}) = \bigoplus_{w \in \mathcal{W}^0} \text{Ind}_{\mathbf{P}_0(\mathbb{A}_f)}^{\mathbf{G}(\mathbb{A}_f)} \left(H_{\text{dR}}^{q-\ell(w)}(\text{SH}_{\mathbf{M}_0}/\mathbb{C}, \tilde{W}_{w \bullet 0}) \right).$$

As $\mathbf{M}_0$ is a split torus, $\text{SH}_{\mathbf{M}_0}$ only has cohomology in degree 0, so we can restrict to the Weyl elements of length q . Write $w \bullet 0 = a\alpha + b\beta$. From Lemma 3.3.6.3 and the Geometrical Lemma of Bernstein-Zelevinsky [BZ77] we deduce that if $H_{\text{dR}}^q(\partial_0 \text{SH}_{\mathbf{G}}/\mathbb{C})[\Pi_\chi^\infty] \neq 0$, then we must have $w \bullet 0 = \pm 3\alpha$ or $w \bullet 0 = \pm 3\beta$ (see also the discussion of [Har87, Section 2.2.5] for a general description of the cohomology of tori). The only possibility that occurs for $w \in \mathcal{W}^0$ is $w \bullet 0 = -3\alpha$, attained by the Weyl element $ss_{-1}s$ of length 3. This gives the result. $\square$

3.3.6.6 Lemma. *Let $\omega \in H_{\text{dR}}^3(\text{SH}_{\mathbf{G}}/\mathbb{C})[\Pi_\chi^\infty]$ be such that the image of ω in each of the two boundary components ∂_1 and ∂_2 vanishes. Then $\omega = 0$.*

Proof. As is recalled in subsection 3.2.2.1, we have the following exact sequence

$$H^2(\partial_0, \mathbb{C}) \longrightarrow H^3(\partial \text{SH}_{\mathbf{G}}, \mathbb{C}) \longrightarrow H^3(\partial_1, \mathbb{C}) \oplus H^3(\partial_2, \mathbb{C}).$$

Employing the de Rham version of this exact sequence, we conclude from the hypotheses of the lemma together with Lemma 3.3.6.5 that the image of ω would have

to vanish in the cohomology of the boundary. We deduce that ω lies in the cohomology of the interior. If it were non-zero, then Π_χ^∞ would appear in the cohomology of the interior. However, this is excluded by Proposition 3.3.6.4. $\square$

3.3.6.7 Lemma. *Let $q \neq 4$. If the image of $\omega \in H_{\text{dR}}^q(\text{SH}_G/\mathbb{C})[\Pi_\chi^\infty]$ in $H_{\text{dR}}^q(\partial_1/\mathbb{C})$ vanishes, then $\omega = 0$.*

Proof. If the image of ω vanishes in $H_{\text{dR}}^q(\partial_1/\mathbb{C})$, then it must also vanish in $H_{\text{dR}}^q(\partial_0/\mathbb{C})$. Therefore, the image of ω in $H_{\text{dR}}^q(\partial_2/\mathbb{C})$ must lie in the interior cohomology (of the boundary stratum ∂_2). If this image were non-zero, then it would follow that the Jacquet module $J_f \simeq \text{Eis}_f^{\text{GL}_2}$ (see Lemma 3.3.6.3) appears in this interior cohomology, which is excluded by $\text{Eis}_f^{\text{GL}_2}$ being non-unitarizable. In particular, the image of $\omega \in H_{\text{dR}}^q(\partial \text{SH}_G/\mathbb{C})$ must vanish when restricted to both ∂_1 and ∂_2 and hence must belong to the image of $H_{\text{dR}}^{q-1}(\partial_0/\mathbb{C})$. As $q-1 \neq 3$, we can invoke Lemma 3.3.6.5 to conclude that the image of ω in $H_{\text{dR}}^q(\partial_2/\mathbb{C})$ must vanish. The lemma then follows from Proposition 3.3.6.4. $\square$

We are now ready to prove the following proposition in the direction of Proposition 3.3.6.1.

3.3.6.8 Proposition. *Let $q \neq 3, 4$. Then $H_{\text{dR}}^q(\text{SH}_G/\mathbb{C})[\Pi_\chi^\infty] = 0$.*

Proof. Let $\omega \in H_{\text{dR}}^q(\text{SH}_G/\mathbb{C})[\Pi_\chi^\infty]$. We study the image of ω in ∂_1 . Let $w \in \mathcal{W}^1$ and write $w \bullet 0 = a_w \alpha + b_w \beta$. We have following description of the cohomology of the boundary:

$$H^q(\partial_1, \mathbb{C}) = \text{Ind}_{\mathbf{P}_1(\mathbb{A}_f)}^{\mathbf{G}(\mathbb{A}_f)} \bigoplus_{w \in \mathcal{W}^1} \bigoplus_{k+m=q-\ell(w)} H^k(\text{SH}_{\mathbf{G}_m}, W_{a_w \alpha}) \boxtimes H^m(\text{SH}_{\mathbf{G}_0}, W_{b_w \beta}).$$

By a similar reasoning to that in the proof of Lemma 3.3.6.5, which relied on the Geometrical Lemma of Bernstein-Zelevinsky, we deduce that if $\omega \neq 0$ then $a_w = \pm 3$ or $b_w = \pm 3$. By Lemma 3.1.3.1 there is only one possible choice for w , namely $w : (\alpha, \beta) \mapsto (-\alpha, \beta)$, which is of length 3. Therefore, if $q \neq 3$ then the restriction to ∂_1 of ω must vanish. The proposition now follows from Lemma 3.3.6.7, as $q \neq 4$. $\square$

3.3.6.9 Remark. Proposition 3.3.6.8 is enough for our purposes: it suffices to exclude the occurrence of Π_χ^∞ in degrees smaller than 3 to construct an absolute étale class. However, it may be of independent interest to show how one can use general results of Schwermer to also exclude the occurrence of Π_χ^∞ in degree 4.

To complete the proof of Proposition 3.3.6.1, we next state a result of Schwermer [Sch94] about Eisenstein cohomology, which we will need for the proof of Lemma 3.3.6.11. Note that in our case the dimension of $\mathbf{U}_0/\mathbf{U}_2$ is 1.

3.3.6.10 Proposition. *Let $\mathcal{W}_2^0 := \{id, s_{-1}\}$, then $\mathcal{W} = \mathcal{W}_2^0 \mathcal{W}^2$. Let w be an element of $\mathcal{W}$ and let ω be a cohomology class in $\text{Ind}_{\mathbf{P}_0}^{\mathbf{G}} H_{\text{dR}}^0(\text{SH}_{\mathbf{M}_0}, V_{w \bullet 0})$ represented by a cuspidal form φ . Assume $w = s_{-1}w_2$. Following the notations of [Sch94], let $E(\varphi, \Lambda)$ be the Eisenstein series that is attached to φ . Denote $\Lambda_0 = -w\rho|_{\mathfrak{a}_{\mathbf{P}_0}^{\mathbf{P}_2}}$ where $\mathfrak{a}_{\mathbf{P}_0}^{\mathbf{P}_2}$ is the Lie algebra of the maximal torus of $\mathbf{A}$ such that $\alpha(t) = \beta(t)$. Assume that $\langle \Lambda_0, \alpha^\vee \rangle > \langle \rho|_{\mathfrak{a}_{\mathbf{P}_0}^{\mathbf{P}_2}}, \alpha^\vee \rangle$. Then:*

1. *The Eisenstein series $E(\varphi, \Lambda)$ is holomorphic at Λ_0 .*
2. *Moreover, $E(\varphi, \Lambda_0)$ represents a closed form $[E(\varphi, \Lambda_0)]$ in $H_{\text{dR}}^{l(w)}(\partial_2 \text{SH}_{\mathbf{G}}/\mathbb{C})$ such that the restriction of E to the P_0 stratum is ω .*

Proof. This is a consequence of [Sch94, Section 6], in particular Theorems 6.3, 6.4. We note that the conclusion of this proposition is true although the weight of the coefficient is not regular because the hypothesis of the proposition, namely that $\Lambda_0 - \rho|_{\mathfrak{a}_{\mathbf{P}_0}^{\mathbf{P}_2}}$ is in the positive Weyl chamber of the system of simple roots of $\Delta(\mathbf{P}_0, A_{\mathbf{P}_0}^{\mathbf{P}_2})$, already implies that the Eisenstein series is holomorphic at Λ_0 and represents a closed form. Then we can use reasoning very similar to the proof of Theorem 6.4 of [Sch94] as here the condition that $\ell(s_{-1}) > d/2 = 1/2$ is trivially true. $\square$

The next lemma completes the proof of Proposition 3.3.6.1.

3.3.6.11 Lemma. *We have $H_{\text{dR}}^4(\text{SH}_{\mathbf{G}}/\mathbb{C})[\Pi_{\chi}^{\infty}] = 0$.*

Proof. Let $\omega \in H_{\text{dR}}^4(\text{SH}_{\mathbf{G}}/\mathbb{C})[\Pi_{\chi}^{\infty}]$. We already know that the image of ω in $H_{\text{dR}}^4(\partial_1/\mathbb{C})$ vanishes by the proof of Proposition 3.3.6.8. By the proof of Lemma 3.3.6.7, we also know that the image of ω in $H_{\text{dR}}^4(\partial_2/\mathbb{C})$ vanishes. It follows that the image of ω in $H_{\text{dR}}^4(\partial \text{SH}_{\mathbf{G}}/\mathbb{C})[\Pi_{\chi}^{\infty}]$ must belong to the image of $H_{\text{dR}}^3(\partial_0/\mathbb{C})[\Pi_{\chi}^{\infty}]$. However, by the proof of Lemma 3.3.6.5, Π_{χ}^{∞} can only appear in the factor $\text{Ind}_{\mathbf{P}_0(\mathbb{A}_f)}^{\mathbf{G}(\mathbb{A}_f)} H^0(\text{SH}_{\mathbf{M}_0}, V_{w \bullet 0})$ of the cohomology of ∂_0 , where w is the Weyl element $(\alpha, \beta) \mapsto (-\alpha, \beta)$. Now, this element w is not in $\mathcal{W}^2$ so we have $w = s_{-1}w_2$. The inequality in the assumptions of Proposition 3.3.6.10 is just $3 > 2$ as $w \bullet 0 = -3\alpha$. Therefore, the hypotheses of the proposition are satisfied and we can deduce from Proposition 3.3.6.10(b) that restriction from the cohomology of ∂_2 to the cohomology of ∂_0 in degree 3 is surjective onto the factor $\text{Ind}_{\mathbf{P}_0(\mathbb{A}_f)}^{\mathbf{G}(\mathbb{A}_f)} H^0(\text{SH}_{\mathbf{M}_0}, \tilde{V}_{w \bullet 0})$. We then conclude that the map

$$H^3(\partial_0, \mathbb{C})[\Pi_{\chi}^{\infty}] \rightarrow H^4(\partial \text{SH}_{\mathbf{G}}, \mathbb{C})$$

is the zero map. This implies the lemma. $\square$

3.3.7 The rational cohomology class

Recall the tube domain model $\mathcal{H}^\circ \subset (V_3)_\mathbb{C}$ of the symmetric space of $\mathbf{G}$. For $Z \in (V_3)_\mathbb{C}$ we write $Z = [z_1, z_0, z_2]^\top$ with z_1, z_2, z_0 the usual coordinates. Let $dZ := dz_1 \wedge dz_0 \wedge dz_2$; then dZ defines a trivialization of the line bundle of top degree differential forms on the tube domain.

3.3.7.1 Definition of the cohomology classes

The de Rham class Let $Z \in \mathcal{H}^\circ$ and $g_\infty \in \mathbf{G}(\mathbb{R})$ such that $g_\infty \cdot Z_0 = Z$. For $f \in \Pi_\chi^\infty$ we let $f_\chi = f_{\infty, 1/2}^\chi \otimes f$. We define

$$E^\chi(Z, g_f; f_\chi) := J(g_\infty, Z_0)^3 E(g_\infty g_f; f_\chi) \quad \text{and} \quad \omega_\chi(f) := E^\chi(Z, g_f; f_\chi) dZ.$$

Then $\omega_\chi(f)$ descends to a holomorphic differential on $\text{SH}_\mathbf{G}(\mathbb{C})$ and hence defines a class in $H_{\text{dR}}^3(\text{SH}_\mathbf{G}/\mathbb{C})$ that we continue to denote $\omega_\chi(f)$. We also set

$$\omega(f_{\chi; M}^{\text{dep}}) := E^\chi(Z, g_f; f_{\chi; M}^{\text{dep}}) dZ \quad \text{and} \quad \omega(f_{\chi; M}^{\text{Liu}}) := E^\chi(Z, g_f; f_{\chi; M}^{\text{Liu}}) dZ,$$

adopting the same notation for the corresponding de Rham classes.

The singular class By the de Rham-singular isomorphism, $\omega_\chi(f)$ defines a class

$$\omega_\chi^{\text{sing}}(f) \in H^3(\text{SH}_\mathbf{G}, \mathbb{C}).$$

In particular, the classes $\omega(f_{\chi;\varphi}^{dep})$ and $\omega(f_{\chi;\varphi}^{Liu})$ define classes

$$\omega(f_{\chi;\varphi}^{dep})^{\text{sing}}, \omega(f_{\chi;\varphi}^{Liu})^{\text{sing}} \in H^3(\text{SH}_{\mathbf{G}}, \mathbb{C}).$$

3.3.7.2 Rational structures

Recall that taking the constant term along $\mathbf{P}_1$ or $\mathbf{P}_0$ identifies Π_{χ}^{∞} with $\text{Ind}_{\mathbf{P}_1(\mathbb{A}_f)}^{\mathbf{G}(\mathbb{A}_f)}(\mu_{1/2})$.

In particular the $\mathbf{G}(\mathbb{A}_f)$ -representation Π_{χ}^{∞} has an obvious F_{χ} -structure:

$$\Pi_{\chi}^{\infty, \circ} := \{f \in \text{Ind}_{\mathbf{P}_1(\mathbb{A}_f)}^{\mathbf{G}(\mathbb{A}_f)}(\mu_{1/2}) : \forall g \in \mathbf{G}(\mathbb{A}_f), f(g) \in F_{\chi}\}.$$

To see that this is an F_{χ} -structure, we note that a function in $\text{Ind}_{\mathbf{P}_1(\mathbb{A}_f)}^{\mathbf{G}(\mathbb{A}_f)}(\mu_{1/2})^K$, $K \subset \mathbf{G}(\mathbb{A}_f)$ any open compact, is uniquely determined by its values on a set of representatives of $P_1(\mathbb{A}_f) \backslash \mathbf{G}(\mathbb{A}_f) / K$, and so $\Pi_{\chi}^{\infty, \circ, K}$ clearly has an F_{χ} -basis that is also a $\mathbb{C}$ -basis of $\Pi_{\chi}^{\infty, K}$. It follows that $H^3(\text{SH}_{\mathbf{G}}, \mathbb{C})[\Pi_{\chi}^{\infty}]$ is stable under the action of $\text{Aut}(\mathbb{C}/F_{\chi})$ on the coefficients of the cohomology group. Hence, $H^3(\text{SH}_{\mathbf{G}}, \mathbb{C})[\Pi_{\chi}^{\infty}]$ has a F_{χ} -structure, which is just $H^3(\text{SH}_{\mathbf{G}}, F_{\chi})[\Pi_{\chi}^{\infty, \circ}]$.

Let r_i be the restriction to ∂_i in singular cohomology. Then r_i is compatible with the natural F_{χ} -structures on cohomology. In particular, the restriction

$$r_1 : H^3(\text{SH}_{\mathbf{G}}, \mathbb{C})[\Pi_{\chi}^{\infty}] \rightarrow H^3(\partial_1, \mathbb{C}),$$

which is injective by Lemma 3.3.6.7, is compatible with the action of $\text{Aut}(\mathbb{C}/F_{\chi})$ on each side. Hence, to see that $\omega \in H^3(\text{SH}_{\mathbf{G}}, \mathbb{C})[\Pi_{\chi}^{\infty}]$ actually belongs to $H^3(\text{SH}_{\mathbf{G}}, F_{\chi})[\Pi_{\chi}^{\infty, \circ}]$, it suffices to check that $r_1(\omega)$ belongs to $H^3(\partial_1, F_{\chi})$. We exploit this observation to

prove the next proposition.

3.3.7.3 Proposition. *For $f \in \Pi_{\chi}^{\infty, \circ}$, $\omega_{\chi}^{\text{sing}}(f) \in H^3(\text{SH}_{\mathbf{G}}, F_{\chi})$.*

Proof. Recall from subsection 3.2.2.2 that the image $r_1(\omega(f)) \in H^3(\partial_1, \mathbb{C})$ is obtained by taking the constant term along $\mathbf{P}_1$. We make this more explicit here.

We use notation from [HMG94, Section 6, 7]. The boundary component ∂_1 of the Borel-Serre compactification would be denoted by $e_{\mathbf{P}_1} := \mathcal{H}/A_{\mathbf{P}_1}(\mathbb{R})^+$ in *loc cit.*, where $A_{\mathbf{P}_1}(\mathbb{R})^+$ is the connected component of the maximal central split $\mathbb{R}$ -torus of $\mathbf{P}_1$. In the case at hand, $A_{\mathbf{P}_1}(\mathbb{R})^+ = \{[t] : t \in \mathbb{R}_{>0}\}$, where we have used notation from Section 3.1.2. Then ∂_1 is the limit points of the orbits of $A_{\mathbf{P}_1}(\mathbb{R})^+$, seen as totally geodesic submanifolds of $\mathcal{H}$. As is explained in Section 8 and Section 11 of *loc cit.*, the decomposition 3.1 comes from the nilmanifold fibration of the Borel-Serre compactification mapping to the reductive Borel-Serre compactification. The fibre being $\Gamma_{\mathbf{u}} \backslash \mathbf{U}_1(\mathbb{R})$ where $\Gamma_{\mathbf{u}} = \Gamma \cap \mathbf{U}_1(\mathbb{R})$ and $\Gamma \subset \mathbf{G}(\mathbb{R})$ is the lattice defined by the compact open K . Therefore the fibre is diffeomorphic to a product of circles, indexed by the coordinates of the abelian group $\mathbf{U}_1$. As is well known, the integral singular cohomology of a circle has a canonical rational basis that is given by dr , if r is the coordinate on a circle $\mathbb{R}/\mathbb{Z}$, via the de Rham isomorphism. Therefore, following the notation of *loc cit.* and putting $N_{\mathbf{P}_1} = \Gamma_{\mathbf{u}} \backslash \mathbf{U}_1(\mathbb{R})$, a rational basis of the one dimensional vector space $H_{\text{dR}}^3(N_{\mathbf{P}_1}, \mathbb{C})$ is dX , if the coordinates of $\mathbf{U}_1(\mathbb{R})$ are x_1, x_0, x_2 and $dX = dx_1 \wedge dx_0 \wedge dx_2$.

By the decomposition 3.1, and the identification $H_{\text{dR}}^3(N_{\mathbf{P}_1}, \mathbb{C}) = H^3(\mathfrak{u}_1, \mathbb{C}) =$

$W_{-3\alpha}$, the rational structure on the subspace

$$\mathrm{Ind}_{\mathbf{P}_1(\mathbb{A}_f)}^{\mathbf{G}(\mathbb{A}_f)} \left(H^0(\mathrm{SH}_{\mathbf{G}_m}, W_{-3\alpha}) \boxtimes \mathbf{1} \right) \subseteq H^3(\partial_1, \mathbb{C}),$$

is therefore the space of functions on $\mathbf{G}(\mathbb{A}_f)/\mathbf{G}_0(\mathbb{A}_f)\mathbf{U}_1(\mathbb{A}_f)$ that are valued in the rational subspace $F_\chi \cdot dX$ of the $\mathbf{G}_m(\mathbb{A}_f)$ -representation $H^0(\mathrm{SH}_{\mathbf{G}_m}, W_{-3\alpha}) = \mathbb{C} \cdot dX$.

Consider now the holomorphic differential $\omega_\chi(f)$ on $\mathrm{SH}_{\mathbf{G}}$, corresponding under the trivialization $\Omega_{\mathcal{H}}^3 \simeq \mathcal{O}_{\mathcal{H}} dZ$ to $E^\chi(Z, g_f; f_\chi) dZ$. The action of $A_{\mathbf{P}_1}(\mathbb{R})^+$ on $\omega_\chi(f)$ is easily described: For all $t > 0$,

$$E^\chi([t]Z, g_f; f_\chi) dZ = E^\chi(tZ, g_f; f_\chi) dZ,$$

Hence, as $t \rightarrow \infty$ the differential $\omega_\chi(f)$ converges to the differential $C_E(g_f) dX$, where C_E is the constant term of E . That is, $r_1(\omega_\chi^{\mathrm{sing}}(f)) = r_1^{\mathrm{dR}}(\omega_\chi(f)) = C_E(g_f) dX$. From the constant term calculation of Proposition 3.3.4.1 it follows that $c_E(g_f) = f_\chi(g_f) = f(g_f)$, and so, by the above discussion, $r_1(\omega_\chi^{\mathrm{sing}}(f)) \in H^3(\partial_1, F_\chi)$. As remarked before, this then implies that $\omega_\chi^{\mathrm{sing}}(f) \in H^3(\mathrm{SH}_{\mathbf{G}}, F_\chi)[\Pi_\chi^{\infty, \circ}]$. $\square$

As an immediate consequence we also have:

3.3.7.4 Corollary. *The classes $\omega(f_{\chi; \varphi_M}^{\mathrm{dep}})$ and $\omega(f_{\chi; \varphi_M}^{\mathrm{Liu}})$ also belong to $H^3(\mathrm{SH}_{\mathbf{G}}(\mathbb{C}), F_\chi)$.*

3.3.8 The étale class and the Galois action

Recall that we have Artin's isomorphism:

$$H_{\text{ét}}^3(\text{SH}_{\mathbf{G}}/\overline{\mathbb{Q}}, F_{\chi,p}) \xrightarrow{\sim} H^3(\text{SH}_{\mathbf{G}}(\mathbb{C}), F_{\chi}) \otimes F_{\chi,p}.$$

Let $f \in \Pi_{\chi}^{\infty, \circ}$. Let $\omega_{\chi}^{\text{ét}}(f)$ be the étale class in $H_{\text{ét}}^3(\text{SH}_{\mathbf{G}}/\overline{\mathbb{Q}}, F_{\chi,p})$ corresponding to $\omega_{\chi}^{\text{sing}}(f)$ under this isomorphism.

Let $\underline{F}_{\chi,p}$ be the p -adic constant sheaf attached to $F_{\chi,p}$ on $\text{SH}_{\mathbf{G}}$. Recall that $j : \text{SH}_{\mathbf{G}} \hookrightarrow \text{SH}_{\mathbf{G}}^*$ is the inclusion of the Shimura variety $\text{SH}_{\mathbf{G}}$ into its Baily–Borel compactification. We claim that we can view $\omega_{\chi}^{\text{ét}}(f)$ as a section of $R^3j_*\underline{F}_{\chi,p}$. Let us now justify this. We can reason topologically (alternatively one can give an argument using weights and Pink's theorem 3.2.2.4). Denote by $\iota : \text{SH}_{\mathbf{G}} \hookrightarrow \overline{\text{SH}}_{\mathbf{G}}$ the natural inclusion of $\text{SH}_{\mathbf{G}}$ in its Borel–Serre compactification $\overline{\text{SH}}_{\mathbf{G}}$. We recall that ι is a homotopy equivalence. We let $p : \overline{\text{SH}}_{\mathbf{G}} \rightarrow \text{SH}_{\mathbf{G}}^*$ be the natural map - it induces natural maps $p_i : \partial_i \rightarrow \partial_i^*$ and fits into the commutative diagram:

$$\begin{array}{ccc} & & \overline{\text{SH}}_{\mathbf{G}} \\ & \nearrow \iota & \downarrow p \\ \text{SH}_{\mathbf{G}} & & \text{SH}_{\mathbf{G}}^* \\ & \searrow j & \end{array}$$

since ι is a homotopy equivalence we have $Rj_* = Rp_* \circ \iota_*$, and therefore $Rj_*\underline{F}_{\chi,p} = Rp_*\underline{F}_{\chi,p}$. It therefore suffices to show that $\omega_{\chi}^{\text{sing}}(f)$ defines a section of $R^3p_*\underline{F}_{\chi,p}$. This

is indeed the case as we have the diagram of Leray spectral sequences:

$$\begin{array}{ccc}
H^i(\mathrm{SH}_{\mathbf{G}}^{\star}, R^j p_* \underline{F}_{\chi,p}) & \Longrightarrow & H^{i+j}(\overline{\mathrm{SH}}_{\mathbf{G}}, \underline{F}_{\chi,p}) \\
\downarrow \text{res} & & \downarrow \text{res} \\
H^i(\partial_1^{\star}, R^j p_* \underline{F}_{\chi,p}) & \Longrightarrow & H^{i+j}(\partial_1, \underline{F}_{\chi,p}).
\end{array}$$

Upon taking $\Pi_{\chi}^{\infty, \circ}$ isotypical parts, the restriction $H^{i+j}(\overline{\mathrm{SH}}_{\mathbf{G}}, \underline{F}_{\chi,p})[\Pi_{\chi}^{\infty, \circ}] \rightarrow H^{i+j}(\partial_1, \underline{F}_{\chi,p})[\Pi_{\chi}^{\infty, \circ}]$ is injective (see lemma 3.3.6.7). Moreover, as $\partial_1^{\star}$ is 0-dimensional, $E_{\infty}^{i,j} = 0$ unless $i = 0$. This implies that $\omega_{\chi}^{\mathrm{sing}}(f) \in E_{\infty}^{0,3}$, and thus lifts to a global section of $R^3 p_* \underline{F}_{\chi,p}$. This concludes the proof of the claim.

By restriction, $\omega_{\chi}^{\mathrm{et}}(f)$ then defines a global section of $R^3 j_* \underline{F}_{\chi,p}|_{\partial_1^{\star}}$, which by the above diagram is the global section defined by the restriction of $\omega_{\chi}^{\mathrm{sing}}(f)$ to ∂_1 . Recall that the latter is identified with $f_{\chi}|_{\mathbf{G}(\mathbb{A}_f)} = f$. Viewing

$$f_{\chi}|_{\mathbf{M}_{1,\mathrm{lin}}} \in H^0(S_{\mathbf{M}_{1,\mathrm{lin}}}, W_{-3\alpha}) \otimes F_{\chi,p}(-3),$$

since $\mathrm{SH}_{\mathbf{M}_{1,h}} \simeq \mathbf{G}_{\mathrm{m}}$ and $\mathbf{G}_{\mathrm{m}}$ acts by $x \mapsto x^{-3}$ on $H^0(S_{\mathbf{M}_{1,\mathrm{lin}}}, W_{-3\alpha})$, we see that

$$f_{\chi}|_{\mathbf{M}_{1,h}} = \omega_{\chi}^{\mathrm{et}}(\varphi)|_{\partial_1^{\star}} \in H^0\left(\mathrm{SH}_{\mathbf{M}_{1,h}}, H^0(S_{\mathbf{M}_{1,\mathrm{lin}}}, W_{-3\alpha}) \otimes F_{\chi,p}(-3)\right).$$

We recall that the Shimura data for $\mathrm{SH}_{\mathbf{M}_{1,h}}$ is defined by the conjugacy classes of the homomorphism

$$h_1(z, \bar{z}) = \begin{bmatrix} z\bar{z} & & \\ & 1 & \\ & & (z\bar{z})^{-1} \end{bmatrix}.$$

Moreover, as $\mathbf{M}_{1,h}$ is a torus, the Galois action on $\mathrm{SH}_{\mathbf{M}_{1,h}}$ is given by the composition of $h_1(\bullet, 1)$ with the Artin map Art of class field theory sending *geometric* Frobenius elements to uniformizers. Moreover, by Lemma 3.2.2.5, the Galois action on $\mathrm{SH}_{\mathbf{M}_{1,h}}$ is twisted by ε^{-3} for sections of $R^3 j_* \underline{F}_{\chi,p}|_{\mathrm{SH}_{\mathbf{M}_{1,h}}}$.

Let $\sigma \in \mathrm{Gal}_{\mathbb{Q}}^{ab}$, such that $\mathrm{Art}(\sigma) = t^{-1}$. Let us compute explicitly $\sigma(\omega_{\chi}^{\mathrm{et}}(\varphi))$. By Lemma 3.3.6.6, the Galois action on $\omega_{\chi}(f)$ can be determined by inspecting the Galois action on the global section it defines by restriction in $R^3 j_* F_{\chi,p}|_{\partial_1^*}$. Denote by t^* the pullback by the left multiplication by t map on the cohomology $H^0(S_{\mathbf{M}_{1,\mathrm{lin}}}, W_{-3\alpha}) \otimes F_{\chi,p}(-3)$. By the above remarks, we have

$$\begin{aligned} \sigma(f_{\chi})(g_f) &= \varepsilon^{-3}(\sigma) \cdot t^* f_{\chi}(g_f) \\ &= \varepsilon^{-3}(\sigma) |t|^{-3} f_{\chi} \left(\begin{bmatrix} t & & \\ & 1 & \\ & & t^{-1} \end{bmatrix} g_f \right) \\ &= \varepsilon^{-3}(\sigma) \cdot \chi(t) f_{\chi}(g_f) \\ &= \varepsilon^{-3} \chi^{-1}(\sigma) f_{\chi}(g_f), \end{aligned}$$

where we have identified χ as a character of $\mathrm{Gal}_{\mathbb{Q}}$ under the Artin map. We therefore obtain the following proposition:

3.3.8.1 Proposition. *The absolute Galois group $\mathrm{Gal}_{\mathbb{Q}}$ acts on $\omega_{\chi}^{\mathrm{et}}(\varphi)$ by the character $\chi^{-1} \varepsilon^{-3}$. In particular, $\omega_{\chi}^{\mathrm{et}}(\varphi)$ defines a cohomology class in $H_{\mathrm{et}}^3(\mathrm{SH}_{\mathbf{G}}, F_{\chi,p}(3))$, invariant under Gal_{χ} .*

3.3.8.2 The partially compactified class

In this section, we construct an analogous class for a partially compactified Shimura variety, which will lift classes $\omega_\chi^{\text{et}}(f)$ for some choices of $f \in \Pi_\chi^{\infty, \circ}$. This subsection can be skipped at a first reading, but is used in the proof of the explicit reciprocity law.

We briefly introduce notation for toroidal compactifications of $\text{SH}_\mathbf{G}$. We refer the reader to [FC90] and [Ash+10] for additional details. Let Σ be an admissible rational polyhedral cone decomposition of the convex cone denoted by $C = C(X)$ in [FC90, Chapter IV, Section 2], so that $X = \mathbb{Z}^2$ and $C(X)$ is the cone of positive semi-definite quadratic forms on $X \otimes \mathbb{R}$ with rational radicals. Attached to Σ , there exists smooth projective varieties $\text{SH}_\mathbf{G}^\Sigma(K)$ and embeddings $\text{SH}_\mathbf{G}(K) \hookrightarrow \text{SH}_\mathbf{G}^\Sigma(K)$, such that for well-chosen Σ , $D^\Sigma = \text{SH}_\mathbf{G}^\Sigma \setminus \text{SH}_\mathbf{G}$ is a strict normal crossings divisor. We make such a choice of admissible rational polyhedral cone decomposition and we denote $\text{SH}_\mathbf{G}^\Sigma(K)$ by $\text{SH}_\mathbf{G}^{\text{tor}}(K)$. There is a natural map $\pi_{\text{tor}} : \text{SH}_\mathbf{G}^{\text{tor}}(K) \rightarrow \text{SH}_\mathbf{G}^*(K)$ extending the identity and compatible with the boundaries in the sense of Theorem 2.5 of [FC90, Chapter V].

Recall that we call a cusp $c \in \partial_1^*(K)$ multiplicative if it is indexed by an element $g \in \mathbf{P}_1(\mathbb{Q}_p)\mathbf{G}(\mathbb{A}_f^p)$. We call a connected component $Y \subseteq \partial_2^*(K)$ (Y is an open modular curve) *ordinary* when the closure of Y in $\text{SH}_\mathbf{G}^*(K)$ contains a multiplicative cusp. We now consider $D^{\star, \text{ord}}(K) \subset \partial \text{SH}_\mathbf{G}^*(K)$ to be the union of the closures of all the ordinary connected components of the boundary. We let also $D^{\text{tor}, \text{ord}} \subset D^{\text{tor}} := D^\Sigma$ the pre-image under π_{tor} of $D^{\star, \text{ord}}$, we which call the ordinary boundary in D^{tor} . We

let

$$\mathrm{SH}_{\mathbf{G}}^{\partial\text{-ord}} := \mathrm{SH}_{\mathbf{G}} \coprod D^{\mathrm{ord}}.$$

We now consider classes $\omega_{\chi}(f)$ coming from the $f \in \Pi_{\chi}^{\infty, \circ}$ such that the Eisenstein series $E^{\chi}(Z, g^{(p)}g_p; f_{\chi})$ has vanishing degenerate Fourier coefficients for all $g^{(p)} \in \mathbf{G}(\mathbb{A}^{(p)})$ and all $g_p \in \mathbf{P}_1(\mathbb{Q}_p)$. This implies, by the q -expansion principle for the modular curves in the boundary, that the $\mathbf{P}_2$ -constant term of $E^{\chi}(Z, g^{(p)}g_p; f_{\chi})$ vanishes when restricted to D^{ord} . Therefore, $\omega_{\chi}(f)$ lifts to the class in $H^0(\mathrm{SH}_{\mathbf{G}}^{\partial\text{-ord}}, \Omega^3(-D^{\mathrm{ord}}))$ defined by the holomorphic differential $E^{\chi}(Z, g^{(p)}g_p; f_{\chi})dZ$. We denote this class also by $\omega_{\chi}(f)$.

The $\mathbf{G}(\mathbb{A}_f^{(p)})$ -module generated by the image of $\omega_{\chi}(f)$ in $H_{\mathrm{dR}}^3(\mathrm{SH}_{\mathbf{G}}^{\partial\text{-ord}}/\mathbb{C})$ is not unitarizable, so that it cannot appear in the interior cohomology of $\mathrm{SH}_{\mathbf{G}}^{\partial\text{-ord}}$. We comment briefly on an often overlooked point. Strictly speaking, Hecke correspondences do not act on the (de Rham) cohomology of a fixed toroidal compactification (for a given choice of rational polyhedral cone decomposition Σ) – however, they act on the tower of toroidal compactifications (given by all refinements of Σ). The coherent cohomology of (the canonical extension of) an automorphic vector bundle (e.g. $\Omega^3(\log(D^{\Sigma}))$) does not depend on the chosen refinement: i.e. the natural proper surjection $\mathrm{SH}_{\mathbf{G}}^{\mathrm{tor}, \Sigma} \rightarrow \mathrm{SH}_{\mathbf{G}}^{\mathrm{tor}, \Sigma'}$ is an isomorphism in coherent cohomology when Σ refines Σ' . Therefore, there is a well-defined admissible $\mathbf{G}(\mathbb{A}_f^{(p)})$ -action on the subspace of $\lim_{\Sigma} H_{\mathrm{dR}}^3(\mathrm{SH}_{\mathbf{G}}^{\partial\text{-ord}, \Sigma}/\mathbb{C})$ generated by the image of $H^0(\mathrm{SH}_{\mathbf{G}}^{\partial\text{-ord}}, \Omega^3(-D^{\mathrm{ord}}))$ and we can still claim that it cannot appear in the interior cohomology of $\mathrm{SH}_{\mathbf{G}}^{\partial\text{-ord}}$ so that the restriction to boundary cohomology is injective, as before.

Therefore, also as before, $\omega_{\chi}(f)$ defines a rational class of $H^3(\mathrm{SH}_{\mathbf{G}}^{\partial\text{-ord}}, F_{\chi})$ and

thus an étale class $\omega_\chi^{\text{ét}, \partial\text{-ord}}(f) \in H_{\text{ét}}^3(\text{SH}_{\mathbf{G}}^{\partial\text{-ord}}/\overline{\mathbb{Q}}, F_{\chi,p})$. By pushing forward to the projective variety $\text{SH}_{\mathbf{G}}^*$, the same arguments from the previous section show that the absolute Galois group $\text{Gal}_{\mathbb{Q}}$ acts on $\omega_\chi^{\text{ét}, \partial\text{-ord}}(f)$ by the character $\varepsilon^{-3}\chi^{-1}$.

We have thus proved the following Proposition.

3.3.8.3 Proposition. *Let $f \in \Pi_{\chi}^{\infty, \circ}$ be such that $E^x(Z, \cdot g^{(p)} g_p; f_\chi)$ has vanishing degenerate Fourier coefficients for all $g^{(p)} \in \mathbf{G}(\mathbb{A}^{(p)})$ and all $g_p \in \mathbf{P}_1(\mathbb{Q}_p)$. Then the holomorphic differential*

$$E^x(Z, g_f; f_\chi) dZ \in H^0(\text{SH}_{\mathbf{G}}^{\partial\text{-ord}}, \Omega^3(-D^{\text{ord}}))$$

is rational and defines a class

$$\omega_\chi^{\text{ét}, \partial\text{-ord}} \in H_{\text{ét}}^3(\text{SH}_{\mathbf{G}}^{\partial\text{-ord}}/\overline{\mathbb{Q}}, F_{\chi,p}),$$

such that the absolute Galois group $\text{Gal}_{\mathbb{Q}}$ acts on the corresponding étale class $\omega_\chi^{\text{ét}, \partial\text{-ord}}(f)$ by the character $\varepsilon^{-3}\chi^{-1}$. By construction, the pull-back of $\omega_\chi^{\text{ét}, \partial\text{-ord}}(f)$ to $\text{SH}_{\mathbf{G}}$ coincides with $\omega_\chi^{\text{ét}}(f)$. In particular, f_χ^{Liu} defines such an étale class.

3.3.9 The absolute étale class

In this section we explain that the class defined in the previous sections can be lifted to an absolute cohomology class lying in $H_{\text{ét}}^3(\text{SH}_{\mathbf{G}}/\mathbb{F}_\chi, F_{\chi,p}(3))$. We note that the construction of an Euler System carried out in this paper does not make use of this absolute cohomology class: the construction is purely in terms of its geometric

realization. However, we believe to be of independent interest to prove that such a lift exists, as it suggests that $\omega_\chi^{\text{et}}(f)$ has motivic origins.

3.3.9.1 Proposition. *Let $f \in \Pi_\chi^{\infty, \circ}$. There exists a unique class $c_\chi(f) \in H_{\text{et}}^3(\text{SH}_\mathbf{G}/\mathbb{F}_\chi, F_{\chi, p}(3))$ that lifts $\omega_\chi^{\text{et}}(f)$.*

Proof. Consider the Leray spectral sequence:

$$E_2^{p, q} = H^p(\mathbb{F}_\chi, H^q(\text{SH}_\mathbf{G}/\overline{\mathbb{Q}}, F_{\chi, p}(3))) \implies H_{\text{et}}^{p+q}(\text{SH}_\mathbf{G}/\mathbb{F}_\chi, F_{\chi, p}(3)).$$

The class $\omega_\chi^{\text{et}}(f)$ defines an element of $E_2^{0, 3}$. As $\Pi_\chi^{\infty, \circ}$ does not appear in $H_{\text{et}}^2(\text{SH}_\mathbf{G}/\overline{\mathbb{Q}}, F_{\chi, p})$, ω_χ^f must lie in the kernel of the differential $d_2^{0, 3}$ and so it also defines an element of $E_3^{0, 3}$. Repeating this process, as Π_χ^∞ does not appear in degree < 3 , the class $\omega_\chi^{\text{et}}(f)$ defines an element of $E_\infty^{0, 3} = \text{gr}^0(H_{\text{et}}^3(\text{SH}_\mathbf{G}/\mathbb{F}_\chi, F_{\chi, p}(3)))$. By invoking again the fact that Π_χ^∞ does not appear in any $E_2^{0, q}$ for $q < 3$, $\omega_\chi^{\text{et}}(f)$ cannot appear in any filtered piece of $F^1 H_{\text{et}}^3(\text{SH}_\mathbf{G}/\mathbb{F}_\chi, F_{\chi, p}(3))$. Therefore we can uniquely lift $\omega_\chi^{\text{et}}(f)$ to a class $c_\chi(f) \in H_{\text{et}}^3(\text{SH}_\mathbf{G}/\mathbb{F}_\chi, F_{\chi, p}(3))$. $\square$

3.3.10 The integral class

In this subsection, following an argument in Faltings in [Fal05, Section 10], we use p -adic Hodge theory to prove that our Eisenstein classes, up to torsion, are integral. We include a few more details than in *loc cit*.

Recall that we have the exceptional isogeny $\pi : \mathbf{Sp}(4) \rightarrow \mathbf{G}$. We can pull-back our Eisenstein series and their cohomology classes via π to Eisenstein series on $\mathbf{Sp}(4)$. It suffices to prove integrality of these pull-backs, as topologically the Shimura varieties

are isomorphic. In this section only, we denote the Siegel threefold by S_K and the universal abelian scheme over S_K by $\mathcal{A}$, where the subscript K denotes a (pulled-back) level structure K . We abuse notation by using by using the same notation for the pulled-back Eisenstein series and their classes.

The level structure K will be (the pull-back of) the level $K_1^{\mathbf{P}}(p^a)K^p$ of $f_{s;M}^\chi$, where χ , of course, has log-conductor $a > 0$ at p and $M \geq 1$ is an integer. Then the Fourier expansion of $E^\chi(Z, g_f; f_{\chi;M})$ is p -adically integral at all cusps by Proposition 3.3.5.3. To show that $\omega_\chi^{\text{et}}(f_{\chi;M})$ is then an integral class we will follow quite closely arguments in Faltings ([Fal05]).

Let S be the Siegel three-fold of level $\mathbf{G}(\mathbb{Z}_p)K^p$. Let $\underline{\omega}$ be the determinant of the Hodge bundle on S , and let $\mathcal{S}$ be a smooth integral model of S over $\mathbb{Z}_\ell$. Choose a smooth toroidal compactification $\mathcal{S}^{\text{tor}}$. Note that $\underline{\omega}$ extends to $\mathcal{S}^{\text{tor}}$ as a line bundle of differentials of the corresponding universal semi-abelian scheme $\mathcal{G}$. Let $\mathcal{O}_{\mathbb{C}_p}$ denote the integral closure of $\mathbb{Z}_p$ inside the p -adic completion $\mathbb{C}_p$ of $\overline{\mathbb{Q}_p}$, and let $\rho := \zeta_p - 1$ (any other element of valuation $1/(p-1)$ will do as well). Let R be an open affine of $\mathcal{S}^{\text{tor}}$ and denote by $\overline{R}$ the integral closure of R in the maximal extension of $R[1/p]$ unramified over $S/\mathbb{Q}_p$. Denote by $\hat{R}$ the p -adic completion of $\overline{R}$.

Algebraic q expansion vs. analytic. First of all, let us recall a few facts about algebraic q -expansions before proving the next proposition. Let $\mathcal{F}$ be the Hodge bundle on S_K . Let F be a weight k holomorphic Siegel modular form: $F \in H^0(S_K, \mathcal{F}^{\otimes k})$. By pulling back F to $\mathcal{H}_2$, we identify F with a holomoprhic function on $\mathcal{H}_2$; the co-ordinate on $\mathcal{H}_2$ being Z . Over the point Z , the universal abelian variety is identified with $\mathbb{C}^2/\mathbb{Z}^2 \oplus Z\mathbb{Z}^2$. Then the pull-back of $\mathcal{F}$ is canonically isomorphic to $\mathcal{O}_{\mathcal{H}_2} \otimes_{\mathbb{C}} \wedge^2 \mathbb{C}^2$.

With this identification, let $d\tau_1, d\tau_2$ be the (fibre) coordinates on $\mathbb{C}^2$ - the identification of F with a function on the Siegel space is made by noticing that F is identified with some $F(Z)(d\tau_1 \wedge d\tau_2)^{\otimes k}$ for some holomorphic function $F(Z)$ (which is a Siegel modular form). Then the algebraic q -expansion of F corresponds to $(2\pi i)^{-2k}$ times the analytic q -expansion of F , as is remarked by Faltings-Chai [FC90, p. 141]. We use this in the next proposition.

3.3.10.1 Proposition. *Let $\mathcal{O}_{\chi,p}$ be the constant sheaf on S_K and let $\pi : S_K \rightarrow S$ be the forgetful map. Then the Eisenstein series $(2\pi i)^3 E^\chi(Z, g_f; f_{\chi;M})$ defines a compatible system of sections of $\underline{\omega}^{\otimes 3} \otimes \hat{R} \otimes \pi_* \mathcal{O}_{\chi,p}$.*

Proof. By the q -expansion principle, this is a reformulation of the fact that $E^\chi(Z, g_f; f_{\chi;M})$ has p -adically integral Fourier coefficients at all the cusps of $\mathrm{SH}_{\mathbf{G}}(K)$. $\square$

With this preliminary proposition at hand, we can deduce integrality of the class $\omega^{\mathrm{et}}(f_{\chi;M})$ as follows.

By [Fal88, p267] (see also [Fal88, Theorem 4.3]), we have a canonical extension of $\mathrm{Sym}^2(\underline{\omega}) \otimes_R \hat{R}$ by $\rho^{-1} \hat{R}(1)$:

$$0 \rightarrow \rho^{-1} \hat{R}(1) \rightarrow E \rightarrow \mathrm{Sym}^2(\underline{\omega}) \otimes_R \hat{R} \rightarrow 0.$$

This induces a three-extension of $\Omega_R^{\otimes 3} \otimes_R \hat{R}$ by $\rho^{-3} \hat{R}(3)$ represented by the exact

sequence:

$$\begin{array}{ccccccc}
0 & \longrightarrow & (\rho^{-1}\hat{R}(1))^{\otimes 3} & \longrightarrow & (\rho^{-1}\hat{R}(1))^{\otimes 2} \otimes E & \longrightarrow & \rho^{-1}\hat{R}(1) \otimes \wedge^2 E \\
& & & & & & \searrow \\
& & & & & & \nearrow \\
& & & & \wedge^3 E & \longrightarrow & \Omega_R^3 \otimes \hat{R} \longrightarrow 0
\end{array}$$

By the proposition above, $(2\pi i)^3 E(Z; g_f; f_{\chi; M})$ defines a compatible system of sections of

$$\Omega_R^{\otimes 3} \otimes_R \hat{R} \otimes \pi_* \underline{\mathcal{O}_{\chi, p}}$$

over Faltings' site – this site is defined in [Fal88], but more generally in Paragraph 1.13 of [AG15]. Varying the open affine R , these three-extensions glue and thus we obtain a cohomology class in $\rho^{-3}\mathcal{O}_{\mathbb{C}_p} \otimes R^3\pi_* \underline{\mathcal{O}_{\chi, p}}(3)$. By the main theorem of [Fal88], this cohomology is almost isomorphic to $H_{\text{et}}^3(S/\overline{\mathbb{Q}}, \pi_* \underline{\mathcal{O}_{\chi, p}}(3)) \otimes \rho^{-3}\mathcal{O}_{\mathbb{C}_p}$. Furthermore, by the functoriality of Faltings canonical extension, this construction is Hecke equivariant and compatible with residues along the point boundary stratum.

We deduce that after inverting p the cohomology class that we obtained via this construction is the Eisenstein class of the previous sections, since on $\Pi_{\chi}^{\infty, 0}$ -isotypical components, taking residues at the cusps is injective. Since this construction also induces the canonical Hodge-Tate decomposition, we can also deduce that the Eisenstein class lies in the $\text{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ invariants of $H_{\text{et}}^3(S_K/\overline{\mathbb{Q}}, \mathcal{O}_{\chi, p}(3)) \otimes_{\mathcal{O}_{\chi, p}} \rho^{-3}\mathcal{O}_{\mathbb{C}_p}$. Since for every $s > 0$ the $\text{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ invariants in $\rho^{-3}\mathcal{O}_{\mathbb{C}_p}/(p^s)$ are the sum of elements in $\mathbb{Z}_p/p^s$ and elements annihilated by ρ , we find that our integral class is uniquely the sum of an étale class with coefficients in $\mathcal{O}_{\chi, p}$ and a class annihilated by

ρ . Hence, $\omega_\chi^{\text{et}}(f_{\chi;m})$ belongs to the image of $H_{\text{et}}^3(S_K/\overline{\mathbb{Q}}, \mathcal{O}_{\chi,p}(3))$ in $H_{\text{et}}^3(S_K/\overline{\mathbb{Q}}, F_{\chi,p}(3))$.

From this we can deduce integrality of the Eisenstein classes that we will arrange into an Euler system in the next sections.

3.3.10.2 Proposition. *The classes $\omega^{\text{et}}(f_{\chi;M}^{\text{dep}})$ define an integral étale cohomology class:*

$$\omega^{\text{et}}(f_{\chi;\varphi^N}^{\text{dep}}) \in \tilde{H}_{\text{et}}^3(\text{SH}_{\mathbf{G}}/\overline{\mathbb{Q}}, \mathcal{O}_{\chi,p}) := \text{im}(H_{\text{et}}^3(\text{Sh}_{\mathbf{G}}/\overline{\mathbb{Q}}, \mathcal{O}_{\chi,p}(3)) \rightarrow H_{\text{et}}^3(\text{Sh}_{\mathbf{G}}/\overline{\mathbb{Q}}, F_{\chi,p}(3)).$$

Proof. As explained above, we know that $\omega_\chi^{\text{et}}(f_{\chi;M})$ defines an integral étale cohomology class. By definition (see Definition (B.3.1)), $f_{\chi;M}^{\text{dep}}$ is an $\mathcal{O}_\chi[\mathbf{G}(\mathbb{Q}_p)]$ -translate of $f_{\chi;M}$ and therefore also defines an integral cohomology class. $\square$

3.4 The Euler System

In this section we use the Eisenstein classes $\omega_\chi(f)$ defined in the preceding section to construct Galois cohomology classes, which we then arrange into an Euler system.

3.4.1 A dual version of the p-adic Abel-Jacobi map

Let $K \subset \mathbf{G}(\mathbb{A}_f)$ be an open compact subgroup and let $K' := K \cap \mathbf{H}(\mathbb{A}_f)$. Let $\text{SH}_{\mathbf{H}}$ be the Shimura variety attached to the Shimura datum $(\mathbf{H}, h_{0,\mathbf{H}})$ defined in 3.1.4.1. We start with the following lemma.

3.4.1.1 Lemma. *For the morphism of Shimura datum $i : (\mathbf{H}, h_{0,\mathbf{H}}) \rightarrow (\mathbf{G}, h_0)$ to*

induce a closed embedding

$$\mathrm{SH}_{\mathbf{H}}(K') \hookrightarrow \mathrm{SH}_{\mathbf{H}}(K),$$

it suffices that K contains $K_1^{\mathbf{A}}(\ell)$ for some prime $\ell > 2$.

Proof. At infinite level (that is, without quotienting on the right by an open-compact subgroup), the map is an embedding. So, let $Q_1, Q_2 \in \mathrm{SH}_{\mathbf{H}}$ be two points such that there exists $k \in K$ with $Q_2 = Q_1 k$. To show that $\mathrm{SH}_{\mathbf{H}}(K')$ injects into $\mathrm{SH}_{\mathbf{H}}(K)$, we have to show that $k \in K'$. To this end, consider

$$z := \pi \left(\begin{pmatrix} -1 & & & \\ & 1 & & \\ & & -1 & \\ & & & 1 \end{pmatrix} \right) \in \mathbf{A}(\mathbb{A}_f) \subset \mathbf{G}(\mathbb{A}_f).$$

Recall here that $\pi : \mathbf{G}' \rightarrow \mathbf{G}$ is the exceptional isogeny. Noting that z is the image of $(-\mathrm{id}, \mathrm{id})$ under the composition of $\iota' : \mathbf{H}' \simeq \mathbf{SL}_2 \times \mathbf{SL}_2 \hookrightarrow \mathbf{G}'$ with π , we see that z is central in $\mathbf{H}$. In particular, z fixes pointwise the infinite level Shimura variety.

We then have the equality of points in $\mathrm{SH}_{\mathbf{H}}$:

$$Q_1 k = Q_2 = Q_1 k z = Q_1 z k z.$$

We can therefore deduce that $z k z \cdot k^{-1}$ fixes $Q_1 \in \mathrm{SH}_{\mathbf{H}}$. Since $z \in \mathbf{A}(\hat{\mathbb{Z}})$, z normalizes $K_1^{\mathbf{A}}(\ell)$ and $K_1^{\mathbf{A}}(\ell)$ is neat¹ as ℓ is odd, we deduce that $z k z k^{-1} = 1$, that is, k

¹This is a consequence of the fact that any root of unity $\zeta \in \overline{\mathbb{Z}_p}$ that is congruent to 1 modulo $\ell > 2$ is equal to 1.

centralizes z . However the centralizer of z is easily seen to be $\mathbf{H}(\mathbb{A}_f)$. This implies that $k \in K_1^{\mathbf{A}}(\ell)\mathbf{H}(\hat{\mathbb{Z}}^{(\ell)}) \cap K \subset K \cap \mathbf{H}(\mathbb{A}_f)$. $\square$

3.4.1.2 Assumption. In what follows, we will always assume that the level K contains $K_1^{\mathbf{A}}(\ell)$ for some ℓ . Indeed, this is satisfied by the inclusion of the auxiliary prime ℓ_{aux} in 3.3.3.

Let j be the inclusion of the open complement of $\mathrm{SH}_{\mathbf{H}}(K')$ in $\mathrm{SH}_{\mathbf{G}}(K)$. We then have an exact sequence of p -adic sheaves on $\mathrm{SH}_{\mathbf{G}}(K)$:

$$0 \longrightarrow j_! F_{\chi,p} \longrightarrow F_{\chi,p} \longrightarrow i_* F_{\chi,p} \longrightarrow 0.$$

We define relative étale cohomology with coefficients in $F_{\chi,p}$ as the cohomology of the sheaf $j_! F_{\chi,p}$, that is

$$\mathrm{H}_{\mathrm{et}}^{\bullet}((\mathrm{SH}_{\mathbf{G}}(K), \mathrm{SH}_{\mathbf{H}}(K')), F_{\chi,p}) := \mathrm{H}_{\mathrm{et}}^{\bullet}(\mathrm{SH}_{\mathbf{G}}(K), j_! F_{\chi,p}).$$

We refer the reader to Section 4.1.1 and the appendix of [EG15] for more details on relative étale cohomology.

By the short exact sequence of étale sheaves described above, we have a long exact sequence in cohomology:

$$0 \longrightarrow \frac{\mathrm{H}_{\mathrm{et}}^2(\mathrm{SH}_{\mathbf{H}}(K')/\overline{\mathbb{Q}}, F_{\chi,p}(3))}{\mathrm{im} \mathrm{H}_{\mathrm{et}}^2(\mathrm{SH}_{\mathbf{G}}(K)/\overline{\mathbb{Q}}, F_{\chi,p}(3))} \longrightarrow \mathrm{H}_{\mathrm{et}}^3((\mathrm{SH}_{\mathbf{G}}(K), \mathrm{SH}_{\mathbf{H}}(K'))/\overline{\mathbb{Q}}, F_{\chi,p}(3)) \longrightarrow \mathrm{H}_{\mathrm{et}}^3(\mathrm{SH}_{\mathbf{G}}(K)/\overline{\mathbb{Q}}, F_{\chi,p}(3)) \longrightarrow \dots$$

The variety $\mathrm{SH}_{\mathbf{H}}(K')$ is an affine surface, and therefore the geometric étale cohomology of $\mathrm{SH}_{\mathbf{H}}$ vanishes in degree ≥ 3 . Consequently we obtain an extension:

$$0 \longrightarrow \frac{\mathrm{H}_{\mathrm{et}}^2(\mathrm{SH}_{\mathbf{H}}(K')/\overline{\mathbb{Q}}, F_{\chi,p}(3))}{\mathrm{im} \mathrm{H}_{\mathrm{et}}^2(\mathrm{SH}_{\mathbf{G}}(K)/\overline{\mathbb{Q}}, F_{\chi,p}(3))} \longrightarrow \mathrm{H}_{\mathrm{et}}^3((\mathrm{SH}_{\mathbf{G}}(K), \mathrm{SH}_{\mathbf{H}}(K'))/\overline{\mathbb{Q}}, F_{\chi,p}(3)) \longrightarrow \mathrm{H}_{\mathrm{et}}^3(\mathrm{SH}_{\mathbf{G}}(K)/\overline{\mathbb{Q}}, F_{\chi,p}(3)) \longrightarrow 0.$$

If $U \subseteq K$ is another compact open subgroup, then, if $U' := U \cap \mathbf{H}(\mathbb{A}_f)$, we have a natural map of pairs of varieties $(\mathrm{SH}_{\mathbf{G}}(U), \mathrm{SH}_{\mathbf{H}}(U')) \rightarrow (\mathrm{SH}_{\mathbf{G}}(K), \mathrm{SH}_{\mathbf{H}}(K'))$, induced by the natural degeneracy maps. Taking direct limits with respect to these maps, we obtain the following exact sequence:

$$0 \longrightarrow \frac{H_{\mathrm{et}}^2(\mathrm{SH}_{\mathbf{H}}/\overline{\mathbb{Q}}, F_{\chi,p}(3))}{\mathrm{im} H_{\mathrm{et}}^2(\mathrm{SH}_{\mathbf{G}}/\overline{\mathbb{Q}}, F_{\chi,p}(3))} \longrightarrow H_{\mathrm{et}}^3((\mathrm{SH}_{\mathbf{G}}, \mathrm{SH}_{\mathbf{H}})/\overline{\mathbb{Q}}, F_{\chi,p}(3)) \longrightarrow H_{\mathrm{et}}^3(\mathrm{SH}_{\mathbf{G}}/\overline{\mathbb{Q}}, F_{\chi,p}(3)) \longrightarrow 0.$$

Now consider the submodule $\Pi_{\chi}^{\infty} \boxtimes \chi^{-1} := H_{\mathrm{et}}^3(\mathrm{SH}_{\mathbf{G}}/\overline{\mathbb{Q}}, F_{\chi,p}(3))[\Pi_{\chi}^{\infty}]$. By pull-back we obtain an extension of Hecke-Galois modules $\mathcal{E}(\Pi_{\chi,f})$, and for $\omega \in \Pi_{\chi}^{\infty} \boxtimes \chi^{-1}$ an extension of Galois modules $\mathcal{E}_{\chi}(\omega)$ – these are defined by the following diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \frac{H_{\mathrm{et}}^2(\mathrm{SH}_{\mathbf{H}}/\overline{\mathbb{Q}}, F_{\chi,p}(3))}{\mathrm{im} H_{\mathrm{et}}^2(\mathrm{SH}_{\mathbf{G}}/\overline{\mathbb{Q}}, F_{\chi,p}(3))} & \longrightarrow & H_{\mathrm{et}}^3((\mathrm{SH}_{\mathbf{G}}, \mathrm{SH}_{\mathbf{H}})/\overline{\mathbb{Q}}, F_{\chi,p}(3)) & \longrightarrow & H_{\mathrm{et}}^3(\mathrm{SH}_{\mathbf{G}}/\overline{\mathbb{Q}}, F_{\chi,p}(3)) \longrightarrow 0 \\ & & \parallel & & \uparrow & & \uparrow \\ 0 & \longrightarrow & \frac{H_{\mathrm{et}}^2(\mathrm{SH}_{\mathbf{H}}/\overline{\mathbb{Q}}, F_{\chi,p}(3))}{\mathrm{im} H_{\mathrm{et}}^2(\mathrm{SH}_{\mathbf{G}}/\overline{\mathbb{Q}}, F_{\chi,p}(3))} & \longrightarrow & \mathcal{E}(\Pi_{\chi,f}) & \longrightarrow & \Pi_{\chi,f} \boxtimes \chi^{-1} \longrightarrow 0 \\ & & \parallel & & \uparrow & & \uparrow \\ 0 & \longrightarrow & \frac{H_{\mathrm{et}}^2(\mathrm{SH}_{\mathbf{H}}/\overline{\mathbb{Q}}, F_{\chi,p}(3))}{\mathrm{im} H_{\mathrm{et}}^2(\mathrm{SH}_{\mathbf{G}}/\overline{\mathbb{Q}}, F_{\chi,p}(3))} & \longrightarrow & \mathcal{E}_{\chi}(\omega) & \longrightarrow & F_{\chi,p}\omega \longrightarrow 0. \end{array}$$

This applies in particular to the classes $\omega_{\chi}(f) \in \Pi_{\chi}^{\infty} \boxtimes \chi^{-1}$ defined in 3.3.

Denote by $\widetilde{\mathrm{AJ}}^{\mathrm{et}}$ the map

$$\widetilde{\mathrm{AJ}}^{\mathrm{et}} : H_{\mathrm{et}}^3(\mathrm{SH}_{\mathbf{G}}/\overline{\mathbb{Q}}, F_{\chi,p}(3))[\Pi_{\chi}^{\infty}] \ni \omega \longmapsto \mathcal{E}_{\chi}(\omega) \in H^1\left(\mathbb{F}_{\chi}, \frac{H_{\mathrm{et}}^2(\mathrm{SH}_{\mathbf{H}}/\overline{\mathbb{Q}}, F_{\chi,p}(3))}{\mathrm{im} H_{\mathrm{et}}^2(\mathrm{SH}_{\mathbf{G}}/\overline{\mathbb{Q}}, F_{\chi,p}(3))}\right)$$

.

3.4.1.3 Remark. The usual Abel-Jacobi map (see for example [Jan88, Section 3]) is defined using a different exact sequence of p -adic sheaves where the roles of j and

i are reversed; namely, one would consider the exact sequence

$$0 \longrightarrow i_! F_{\chi,p} \longrightarrow F_{\chi,p} \longrightarrow j_* F_{\chi,p} \longrightarrow 0.$$

We view the map defined above as being "dual" to the usual étale Abel-Jacobi map, hence the notation.

Let us record the following key fact about $\widetilde{\text{AJ}}^{\text{et}}$.

3.4.1.4 Proposition. *The map $\widetilde{\text{AJ}}^{\text{et}}$ is $\mathbf{H}(\mathbb{A}_f)$ -equivariant.*

Proof. Recall that for $\omega \in \Pi_\chi^\infty \boxtimes \chi^{-1}$, $\widetilde{\text{AJ}}^{\text{et}}(\omega)$ is defined by the extension

$$0 \longrightarrow \frac{H_{\text{et}}^2(\text{SH}_{\mathbf{H}}/\overline{\mathbb{Q}}, F_{\chi,p}(3))}{\text{im} H_{\text{et}}^2(\text{SH}_{\mathbf{G}}/\overline{\mathbb{Q}}, F_{\chi,p}(3))} \longrightarrow \mathcal{E}_\chi(\omega) \longrightarrow F_{\chi,p}\omega \longrightarrow 0$$

Let $\tilde{\omega}$ be a lift of ω in $\mathcal{E}_\chi(\omega)$. Then the extension above is represented by the cocycle:

$$c_\omega = \left(\tau \mapsto \tau(\tilde{\omega}) - \tilde{\omega} \right) \in H^1 \left(\mathbb{F}_\chi, \frac{H_{\text{et}}^2(\text{SH}_{\mathbf{H}}/\overline{\mathbb{Q}}, F_{\chi,p}(3))}{\text{im} H_{\text{et}}^2(\text{SH}_{\mathbf{G}}/\overline{\mathbb{Q}}, F_{\chi,p}(3))} \right).$$

Let $h \in \mathbf{H}(\mathbb{A}_f)$. For a level $K \subset \mathbf{G}(\mathbb{A}_f)$, we denote by $[h]_K : \text{SH}_{\mathbf{G}}(K) \rightarrow \text{SH}_{\mathbf{G}}(h^{-1}Kh)$

the map induced by multiplication on the right by $\iota(h) \in \mathbf{G}(\mathbb{A}_f)$. By abuse of notation we denote also by $[h]_K$ the similarly defined map $\text{SH}_{\mathbf{H}}(K') \rightarrow \text{SH}_{\mathbf{H}}(h^{-1}K'h)$.

Note that $[h]_K$ defines a map from the pair $(\text{SH}_{\mathbf{G}}(K), \text{SH}_{\mathbf{H}}(K'))$ to the pair $(\text{SH}_{\mathbf{G}}(h^{-1}Kh), \text{SH}_{\mathbf{H}}(h^{-1}K'h))$.

By passing to the limit over small enough K (for example containing $K_1^{\mathbf{A}}(\ell)$ see Lemma 3.4.1.1), there is an action of $\mathbf{H}(\mathbb{A}_f)$ on $H_{\text{et}}^\bullet((\text{SH}_{\mathbf{G}}, \text{SH}_{\mathbf{H}})/\overline{\mathbb{Q}}, F_{\chi,p})$ such that

the natural map

$$H_{\text{et}}^{\bullet}((SH_{\mathbf{G}}, SH_{\mathbf{H}})/\overline{\mathbb{Q}}, F_{\chi,p}) \rightarrow H_{\text{et}}^{\bullet}(SH_{\mathbf{G}}/\overline{\mathbb{Q}}, F_{\chi,p})$$

is $\mathbf{H}(\mathbb{A}_f)$ -equivariant.

Putting this all together, since $h^*\tilde{\omega}$ lifts $h^*\omega$ we see that $\widetilde{AJ^{\text{et}}}(h^*\omega)$ is represented by the cocycle $\tau \mapsto h^*(\tau(\tilde{\omega}) - \tilde{\omega})$. Here we have tacitly used that the Galois action commutes with the action of $\mathbf{H}(\mathbb{A}_f)$. Therefore, we have the relation

$$\widetilde{AJ^{\text{et}}}(h^*\omega) = h^*\widetilde{AJ^{\text{et}}}(\omega),$$

as required. □

3.4.2 The Euler map and its properties

We define the Euler map

$$\mathcal{EU} : \mathcal{S}(\mathbb{A}_f^{\oplus 5}) \longrightarrow H^1\left(\mathbb{F}_{\chi}, \frac{H_{\text{et}}^2(SH_{\mathbf{H}}/\overline{\mathbb{Q}}, F_{\chi,p}(3))}{\text{im}H_{\text{et}}^2(SH_{\mathbf{G}}/\overline{\mathbb{Q}}, F_{\chi,p}(3))}\right),$$

by the following composition:

$$\begin{array}{ccccccc}
\mathcal{S}(\mathbb{A}_f^{\oplus 5}) & \xrightarrow{\varphi \mapsto f_{\chi;\varphi}^d} & \text{Ind}_{\mathbf{P}_1^d(\mathbb{A}_f)}^{\mathbf{G}(\mathbb{A}_f)}(\mu) & \xrightarrow{f \mapsto E^\chi(g;f_\chi)} & \Pi_\chi^\infty & \xrightarrow{E^\chi(g;f_\chi) \mapsto \omega_\chi(f)} & H_{\text{et}}^3(\text{SH}_{\mathbf{G}}/F_\chi, F_{\chi,p}(3))[\Pi_\chi^\infty] \\
& & & & & & \downarrow \widetilde{\text{AJ}^{\text{et}}} \\
& & & & & & H^1\left(\mathbb{F}_\chi, \frac{H_{\text{et}}^2(\text{SH}_{\mathbf{H}}/\overline{\mathbb{Q}}, F_{\chi,p}(3))}{\text{im} H_{\text{et}}^2(\text{SH}_{\mathbf{G}}/\overline{\mathbb{Q}}, F_{\chi,p}(3))}\right) \\
& \searrow \varphi \mapsto \mathcal{EU}(\varphi) & & & & &
\end{array}$$

Here $f_{\chi;\varphi}^d$ is as in remark (B.7.1) and $\mu = \mu_{1/2}$, which depends on χ , is as in 3.3.2.

We will also view $\mathcal{EU}$ as a map defined on $\text{Ind}_{\mathbf{P}_1^d(\mathbb{A}_f)}^{\mathbf{G}(\mathbb{A}_f)}(\mu)$, and for $f \in \text{Ind}_{\mathbf{P}_1(\mathbb{A}_f)}^{\mathbf{G}(\mathbb{A}_f)}(\mu) \simeq \text{Ind}_{\mathbf{P}_1^d(\mathbb{A}_f)}^{\mathbf{G}(\mathbb{A}_f)}(\mu)$, we will allow ourselves to think of $\mathcal{EU}$ as a map whose domain is $\text{Ind}_{\mathbf{P}_1(\mathbb{A}_f)}^{\mathbf{G}(\mathbb{A}_f)}(\mu)$ and write

$$\mathcal{EU}(f) \in H^1\left(\mathbb{F}_\chi, \frac{H_{\text{et}}^2(\text{SH}_{\mathbf{H}}/\overline{\mathbb{Q}}, F_{\chi,p}(3))}{\text{im} H_{\text{et}}^2(\text{SH}_{\mathbf{G}}/\overline{\mathbb{Q}}, F_{\chi,p}(3))}\right).$$

If $K \subseteq \mathbf{G}(\mathbb{A}_f)$, we let $\mathcal{EU}^K$ be the restriction of $\mathcal{EU}$ to $\text{Ind}_{\mathbf{P}_1(\mathbb{A}_f)}^{\mathbf{G}(\mathbb{A}_f)}(\mu)^K$. The image of $\mathcal{EU}^K(\varphi)$ is represented by an extension defined as above, except that one replaces $\text{SH}_{\mathbf{G}}$ with $\text{SH}_{\mathbf{G}}(K)$ and $\text{SH}_{\mathbf{H}}$ with $\text{SH}_{\mathbf{H}}(K \cap \mathbf{H})$.

3.4.2.1 Remark (Equivariance property of the Euler map). Note that by Proposition 3.4.1.4, the Euler map $\mathcal{EU}$ is $\mathbf{H}(\mathbb{A}_f)$ -equivariant. In what follows, we will use multiplicity one results to compare this map to a period integral that appears in [MS94].

Since Π_χ^∞ is $\mathbf{G}_0(\mathbb{A}_f)$ -invariant, an immediate consequence of the previous remark is the following proposition.

3.4.2.2 Proposition. *For all $f \in \text{Ind}_{\mathbf{P}_1(\mathbb{A}_f)}^{\mathbf{G}(\mathbb{A}_f)}(\mu)$, $\mathcal{EU}(f) \in H^1\left(\mathbb{F}_\chi, \frac{H_{\text{et}}^2(\text{SH}_{\mathbf{H}}/\overline{\mathbb{Q}}, F_{\chi,p}(3))}{\text{im}H_{\text{et}}^2(\text{SH}_{\mathbf{G}}/\overline{\mathbb{Q}}, F_{\chi,p}(3))}\right)$ is invariant under the $\mathbf{G}_0(\mathbb{A}_f)$ action.*

3.4.3 The Murase-Sugano period integral

We fix from here on Haar measures on $\mathbf{G}(\mathbb{Q}_\ell)$, $\mathbf{H}(\mathbb{Q}_\ell)$ and $\mathbf{G}_0(\mathbb{Q}_\ell)$, normalized so that the respective maximal compact groups are given volume 1. We also fix a choice of Haar measure at the archimedean place. Such choices determine a Haar measure on the adelic points of the group schemes under consideration. All the integrals we will consider will be understood with respect to these choices of measures.

Let Π be an irreducible automorphic representation of $\mathbf{H}(\mathbb{A})$ and let $F \in \Pi$. Let S be the set containing all the primes at which F or χ are ramified. Define for $f_s \in \text{Ind}_{\mathbf{P}_1(\mathbb{A})}^{\mathbf{G}(\mathbb{A})}(\mu_s)$

$$\mathfrak{z}(F, f_s) := \int_{\mathbf{H}(\mathbb{Q}) \backslash \mathbf{H}(\mathbb{A})} F(h) E(h; f_s) dh.$$

This integral is a special case of the period integral considered in [MS94]; here we have specialized to the Eisenstein series of interest.

Following the notations of *loc cit.*, we define, for a pair of automorphic forms F on $\mathbf{H}(\mathbb{A})$ and σ on $\mathbf{G}_0(\mathbb{A})$:

$$\omega_{F,\sigma}(h) := \int_{\mathbf{G}_0(\mathbb{Q}) \backslash \mathbf{G}_0(\mathbb{A})} F(gh) \sigma(g) dg.$$

Then $\omega_{F,\sigma}$ is a global Shintani function associated with F and σ , and if ℓ is a prime

then we denote by $\omega_{F,\sigma}^{(\ell)}$ the restriction of $\omega_{F,f}$ to $\mathbf{H}(\mathbb{Q}_\ell)$. The restriction is simply obtained through the inclusion $\mathbf{H}(\mathbb{Q}_\ell) \hookrightarrow \mathbf{H}(\mathbb{A}_f)$, $h_\ell \mapsto (1, \dots, 1, h_\ell, 1, \dots)$. In what follows, we will be interested in the case where $\sigma = 1$ is constant.

3.4.3.1 Assumption. We assume hereafter that Π is $\mathbf{G}_0$ -distinguished, in the sense that $\omega_{F,1}(1) \neq 0$ for some $F \in \Pi$.

Because of the local Gan-Gross-Prasad conjectures for special orthogonal groups, proved by Waldspurger in [Wal12], we have that for any irreducible admissible representation π_ℓ of $\mathbf{G}_0(\mathbb{Q}_\ell)$:

$$\dim_{\mathbb{C}} \operatorname{Hom}_{\mathbf{G}_0(\mathbb{Q}_\ell)}(\Pi_\ell \otimes \pi_\ell, \mathbb{C}) \leq 1.$$

Similarly for any irreducible admissible representation Π'_ℓ of $\mathbf{G}(\mathbb{Q}_\ell)$:

$$\dim_{\mathbb{C}} \operatorname{Hom}_{\mathbf{H}(\mathbb{Q}_\ell)}(\Pi_\ell \otimes \Pi'_\ell, \mathbb{C}) \leq 1.$$

Take now π_ℓ to be the trivial representation. As $\omega_{F,1}$ provides a non-zero element of the global Hom-space, $\omega_{F,1}^{(\ell)}$ is non-zero *a fortiori* and there exists a constant c such that the global Shintani function $\omega_{F,1}$ can be written as

$$\omega_{F,1}(h) = c \prod_{\ell} \omega_{F,1}^{(\ell)}(h_p).$$

The integral $\mathfrak{z}(F, f_s)$ unfolds (see also [MS92, Proposition 4.1]) to the simple

expression:

$$\mathfrak{z}(F, f_s) = \mathfrak{z}_\infty(F, f_s) \cdot \int_{\mathbf{G}_0(\mathbb{A}_f) \backslash \mathbf{H}(\mathbb{A}_f)} f_s^d(h) \mathcal{P}_F(h) dh,$$

where $\mathfrak{z}_\infty$ is an explicit archimedean factor and we have set for simplicity

$$\mathcal{P}_F(h) := \omega_{F,1}(h) = \int_{\mathbf{G}_0(\mathbb{Q}) \backslash \mathbf{G}_0(\mathbb{A}_f)} F(gh) dg.$$

Let $\mathcal{P}_F^{(\ell)}$ be the restriction of $\mathcal{P}_F$ to $\mathbf{H}(\mathbb{Q}_\ell)$. We also define the local zeta integrals :

$$\mathfrak{z}_\ell(F, f_{\ell,s}) = \int_{\mathbf{G}_0(\mathbb{Q}_\ell) \backslash \mathbf{H}(\mathbb{Q}_\ell)} f_{\ell,s}^d(h) \mathcal{P}_F^{(\ell)}(h) dh.$$

When $f_s = f_{\ell,s;\varphi}^\chi$, we will denote $\mathfrak{z}_\ell(F, f_{\ell,s}^\chi; \varphi)$ by $\mathfrak{z}_\ell(F, \varphi_\ell)$.

3.4.3.2 Remark. The local zeta integral $\mathfrak{z}_\ell$ is an element of the Hom-space:

$$\mathrm{Hom}_{\mathbf{H}(\mathbb{Q}_\ell)}(\Pi_{\chi,\ell} \otimes \Pi_\ell, \mathbb{C}),$$

and for $\ell \notin S$, by the basic identity of [MS92], $\mathfrak{z}_\ell$ is non-zero. Therefore by the multiplicity one theorem already stated, all elements in the local Hom-space away from S are proportional to $\mathfrak{z}_\ell$.

We denote by Φ_ℓ^0 the spherical Schwartz function such that the local Siegel Gode-ment $f_{\Phi_0}(s; -)$ as defined in Section (B.8) takes the value 1 at the identity for $s = 3/2$.

3.4.3.3 Proposition. *Let $\ell \notin S$. We have*

$$\mathfrak{z}_\ell(F, f_{\Phi_\ell^0}) = \mathcal{P}_F^{(\ell)}(1) \cdot \frac{L_\ell(F \otimes \chi, 2)}{L_\ell(\chi, 2) L_\ell(\chi, 3) L_\ell(\chi^2, 4)}.$$

Proof. When χ is trivial this is a specific instance of Theorem 1.6 of [MS94]; indeed, in this case, the formula becomes:

$$\mathfrak{z}_\ell(F, \phi) = \mathcal{P}_F^{(\ell)}(1) \frac{L_\ell(F, 2)}{\zeta_\ell(2)} \cdot \frac{1}{\zeta_\ell(3)\zeta_\ell(4)}.$$

Note the factor $\zeta_\ell(2)\zeta_\ell(3)$ in the denominator comes from the factor denoted by $L_\ell(\varphi, s + 1/2)$ appearing in the formula of *loc cit.*, as φ is a constant in our case. When χ is non-trivial, one can use the explicit formula of Proposition 1.13 of *loc cit.*, noticing that one has to replace the indeterminate denoted by T in *loc cit.* with $\chi(\ell)T$. Finally, we just have to adjust the extra factor $\zeta_\ell(4)$. This proposition can also be proved by pullback to $\mathbf{Sp}(4)$, where it is an instance of the doubling zeta integral (see [Liu20, Theorem 4.1.2], for example). $\square$

Recall the definition of φ_ℓ^r from Section (B.1.1).

3.4.3.4 Proposition. *At all places ℓ , where F and χ are unramified, we have*

$$\mathfrak{z}_\ell(F, \varphi_\ell^r) = \mathcal{P}_F^{(\ell)}(1) \cdot \text{vol}(\Gamma(\ell^r)).$$

Proof. As F is unramified at ℓ , F is a pure tensor at ℓ , and we can thus apply Lemma (B.7.2) and Proposition (B.8.3). $\square$

3.4.4 A piece of the cohomology of the Shimura variety of H

3.4.4.1 Galois representations attached to modular forms

Let f be a holomorphic modular newform on $\mathbf{SL}_2$ of weight 2. By enlarging $F_{\chi,p}$ if necessary, we may assume that the Fourier coefficients of f lie in $F_{\chi,p}$. Let ρ_f be the usual p -adic Galois representation associated with f , which appears in the étale cohomology of the modular curve with $F_{\chi,p}(1)$ -coefficients. Let $V_f := \rho_f(-1)$. Then the L -function² of V_f is the just the usual L -series of f (normalized so that the critical value is at 1). We denote by $\bar{f}$ the eigenform whose Fourier coefficients are the complex conjugate of f . The coefficients of $\bar{f}$ are also valued in $F_{\chi,p}$. We define $\mathrm{Ad}(\rho_f) := \mathrm{End}(\rho_f)$ and note that

$$\mathrm{Ad}(\rho_f)(1) := \rho_f \otimes \rho_f^\vee(1) \simeq \rho_f \otimes \rho_{\bar{f}} \simeq \mathrm{Ad}^0(\rho_f)(1) \oplus \varepsilon.$$

We set also

$$\mathrm{Sym}^2(\rho_f) \subset \rho_f \otimes \rho_f$$

to be the space of symmetric 2-tensors of ρ_f . When f is an eigenform of level $\Gamma_1(Np^r)$ with nebentypus ψ_f , we have that

$$\rho_{\bar{f}} \simeq \rho_f \otimes \psi_f^{-1}.$$

²This is defined using *geometric* Frobenius elements.

So that

$$\mathrm{Sym}^2(\rho_f) \simeq \mathrm{Ad}^0(\rho_f)(\varepsilon\psi_f^{-1}).$$

Shimizu's lift. Let $\pi(f)$ be the irreducible automorphic representation of $\mathbf{SL}_2(\mathbb{A})$ generated by f . Seeing $\mathbf{SL}_2$ as a symplectic group, let Π be the theta lift of $\pi(f)$ to $\mathbf{SO}(W_4)(\mathbb{A}) \simeq (\mu_2 \backslash \mathbf{SL}_2 \times \mathbf{SL}_2)(\mathbb{A})$. It is well known (see for example [Shi72]) that this lift is the descent to $\mathbf{SO}(W_4)$ of the automorphic representation

$$\Pi := \pi(f) \boxtimes \pi(f)^\vee$$

of $\mathbf{SL}_2(\mathbb{A}) \times \mathbf{SL}_2(\mathbb{A})$.

The Galois representation attached to Π . Let $\Pi^\infty \boxtimes \sigma_\Pi$ be the Hecke-Galois summand of $H_{\mathrm{et}}^2(\mathrm{SH}_{\mathbf{H}}/\overline{\mathbb{Q}}, \mathbb{Q}_p)$ corresponding to the Π^∞ -isotypical component; here we are implicitly using that Π^∞ has a $F_{\chi,p}$ -model. We recall that a model for the quadratic space attached to $\mathbf{H}$ is the vector space of 2×2 matrices endowed with the quadratic form given by the determinant. This interpretation affords an isogeny $\mathbf{GL}_2 \times_{\mathrm{det}} \mathbf{GL}_2 \rightarrow \mathbf{H}$. Let Y be the Shimura variety associated to $\mathbf{GL}_2 \times_{\mathrm{det}} \mathbf{GL}_2$. The reflex field of Y is $\mathbb{Q}$. Let K'' be the pull-back of K to the subgroup of $\mathbf{GL}_2 \times_{\mathrm{det}} \mathbf{GL}_2(\mathbb{A})$ where the det-maps are valued in $\widehat{\mathbb{Z}}^\times$. There is then a finite Galois cover of $\mathbb{Q}$ -varieties $Y(K'') \rightarrow \mathrm{SH}_{\mathbf{H}}(K)$, and so the étale cohomology of $\mathrm{SH}_{\mathbf{H}}(K)$ is identified with the invariants of the cohomology of $Y(K'')$ under some finite quotient of $\mathbb{A}_f^\times$. From this we deduce that the only possibilities for σ_Π are tensor products $\rho_f \boxtimes \rho_g$; where f and g are two eigenforms.

By the Eichler–Shimura relations for $\mathbf{H}$, this implies that

$$\sigma_{\Pi} = V_f \otimes V_{\bar{f}} = \mathrm{Ad}^0(V_f) \oplus \varepsilon = \mathrm{Ad}^0(\rho_f) \oplus \varepsilon.$$

3.4.4.2 The cohomology of $\mathrm{SH}_{\mathbf{G}}$ in degree two

We now exclude the possibility that the summand $\mathrm{Ad}^0 \rho_f(1)$ appears in the degree 2 cohomology of $\mathbf{G}$. From here on, we make the following assumption.

The representation of ρ_f is irreducible and not dihedral.

This implies of course that f does not have complex multiplication, so that $\mathrm{Ad}^0(T_f)$ is absolutely irreducible. By the explicit description by Taylor of the interior cohomology in degree 2 for the Siegel threefold [Tay93, Section 3], we know that the only absolutely irreducible Galois representations appearing in degree 2 are one or two-dimensional. We can then deduce that the only absolutely irreducible Galois representations appearing in $H_{\mathrm{et}}^2(\mathrm{SH}_{\mathbf{G}}, F_{\chi,p}(3))$ are one or two dimensional; in particular we have:

$$\Pi_f \boxtimes \mathrm{Ad}^0(\rho_f)(2) \hookrightarrow \frac{H_{\mathrm{et}}^2(\mathrm{SH}_{\mathbf{H}}/\overline{\mathbb{Q}}, F_{\chi,p}(3))}{\mathrm{im} H_{\mathrm{et}}^2(\mathrm{SH}_{\mathbf{G}}/\overline{\mathbb{Q}}, F_{\chi,p}(3))},$$

where the inclusion above is the inclusion of a direct summand.

3.4.5 Projections in cohomology

Let Π be as in the previous section. Then Π is $\mathbf{G}_0$ -distinguished, so 3.4.3.1 is satisfied for Π . In fact, recall that Π is the descent of the automorphic representation $\pi(f) \boxtimes \pi(f)^{\vee}$ of $\mathbf{H}'(\mathbb{A})$, and Π being $\mathbf{G}_0$ -distinguished is then just a consequence of the fact

that the Petersson norm of f does not vanish: indeed, the pull-back of $\mathbf{G}_0$ to $\mathbf{H}'$ corresponds to the diagonal copy of $\mathbf{SL}_2$ inside $\mathbf{H}' = \mathbf{SL}_2 \times \mathbf{SL}_2$.

3.4.5.1 The naive projection

Let us now fix a vector $F \in \Pi_f^\vee$ of level $K \subset \mathbf{H}(\mathbb{A}_f)$. The choice of F determines an evaluation map :

$$\mathrm{ev}_F^K : \Pi_f^K \boxtimes (\sigma_\Pi(3)) \longrightarrow \sigma_\Pi(3).$$

Composing ev_F^K with the natural projection map onto the summand $\Pi_f \boxtimes (\sigma_\Pi(3))$, we obtain a projection:

$$\mathrm{proj}_F^K : H_{\mathrm{et}}^2(\mathrm{SH}_{\mathbf{H}}(K)/\overline{\mathbb{Q}}, \mathbb{Q}_p(3)) \longrightarrow \sigma_\Pi(3)$$

.

3.4.5.2 The projection in the inductive limit

Consider now a level K' such that $K' \subseteq K \subset \mathbf{H}(\mathbb{A}_f)$. Denote by

$$d_{K,K'} : \mathrm{SH}_{\mathbf{H}}(K') \rightarrow \mathrm{SH}_{\mathbf{H}}(K)$$

the natural degeneracy map. Then $d_{K,K'}$ induces a natural map in cohomology

$$d_{K,K'}^* : H_{\mathrm{et}}^2(\mathrm{SH}_{\mathbf{H}}(K)/\overline{\mathbb{Q}}, \mathbb{Q}_p(3)) \hookrightarrow H_{\mathrm{et}}^2(\mathrm{SH}_{\mathbf{H}}(K')/\overline{\mathbb{Q}}, \mathbb{Q}_p(3)).$$

Since $\text{ev}_F^K(v) = \text{ev}_F^{K'}(v)$ for all $v \in \Pi_f^K \subseteq \Pi_f^{K'}$, we deduce that we have a commutative diagram:

$$\begin{array}{ccc} H_{\text{et}}^2(\text{SH}_{\mathbf{H}}(K)/\overline{\mathbb{Q}}, \mathbb{Z}_p(3)) & \xrightarrow{d_{K,K'}^*} & H_{\text{et}}^2(\text{SH}_{\mathbf{H}}(K')/\overline{\mathbb{Q}}, \mathbb{Z}_p(3)) \\ & \searrow \text{proj}_F^K \quad \swarrow \text{proj}_F^{K'} & \\ & \sigma_{\Pi}(3) & \end{array}$$

3.4.5.3 Definition (Naive projection). We let

$$\text{proj}_F : H_{\text{et}}^2(\text{SH}_{\mathbf{H}}/\overline{\mathbb{Q}}, \mathbb{Q}_p(3)) \longrightarrow \sigma_{\Pi}(3) \xrightarrow{\sim} \text{Ad}^0(\rho_f)(2) \oplus \varepsilon^2.$$

be the projection defined at each finite level K by proj_F^K .

From now on for simplicity all étale cohomology groups will be geometric and we will omit the ‘ $/\overline{\mathbb{Q}}$ ’ from the notation.

3.4.5.4 Lattices in cohomology

Recall that we have fixed a choice of a holomorphic weight 2 modular newform f of level N for $\mathbf{SL}_2(\mathbb{A})$; the Galois representation that is attached to f is denoted by V_f and we have set $\rho_f := V_f(1)$. From hereon we assume:

The representation ρ_f is residually irreducible.

The $\mathbf{GL}_2$ -case Denote by $Y_1(N)$ the modular curve of level $\Gamma_1(N)$, and let $X_1(N)$ be its usual compactification by adding cusps. Let $\mathbb{T}_1(N)$ be the unramified $\mathcal{O}_{\chi,p}$ -

algebra of Hecke operators acting on $H^1(Y_1(N), \mathcal{O}_{\chi,p}(1))$ generated by the usual³ Hecke operators T_ℓ for primes $\ell \nmid N$. Let J_f be the ideal of $\mathbb{T}_1(N)$ defined by the kernel of the homomorphism $\phi_f : \mathbb{T}_1(N) \rightarrow \mathcal{O}_{\chi,p}$ attached to the newform f . The homomorphism ϕ_f sends T_ℓ to the ℓ -th Fourier coefficient of f . We sometimes confuse ϕ_f with its target: a one-dimensional $F_{\chi,p}$ vector space endowed with a $\mathbb{T}_1(N)$ -module structure. Let $T_f \subset V_f$ be the Galois module defined by the quotient

$$T_f := H_{\text{et}}^1(Y_1(N), \mathcal{O}_{\chi,p}(1)) / J_f \cdot H_{\text{et}}^1(Y_1(N), \mathcal{O}_{\chi,p}(1)).$$

We let similarly

$$T_{f,c} := H_{\text{et},c}^1(Y_1(N), \mathcal{O}_{\chi,p}(1)) / J_f \cdot H_{\text{et},c}^1(Y_1(N), \mathcal{O}_{\chi,p}(1)).$$

As ρ_f is residually irreducible, it follows by newform theory that the modules T_f and $T_{f,c}$ are both lattices in V_f .

Let k_χ denote the residue field of $\mathcal{O}_{\chi,p}$. To compare the lattices T_f and $T_{f,c}$, we recall that we have already assumed that the Galois module $T_f \otimes_{\mathcal{O}_{\chi,p}} k_\chi$ is simple. Since the co-kernel of the natural map $H_{\text{et},c}^1(Y_1(N), \mathcal{O}_{\chi,p}(1)) \rightarrow H_{\text{et}}^1(Y_1(N), \mathcal{O}_{\chi,p}(1))$ is Eisenstein, there is a natural isomorphism $T_{f,c} \simeq T_f$. In fact, if $\mathfrak{m}$ is the maximal ideal of $\mathbb{T}_1(N)$ corresponding to the extension of scalars along k_χ of ϕ_f , then the natural maps:

$$H_{\text{et},c}^1(Y_1(N), \mathcal{O}_{\chi,p}(1))_{\mathfrak{m}} \rightarrow H_{\text{et}}^1(X_1(N), \mathcal{O}_{\chi,p}(1))_{\mathfrak{m}} \rightarrow H_{\text{et}}^1(Y_1(N), \mathcal{O}_{\chi,p}(1))_{\mathfrak{m}} \quad (3.3)$$

³There are two choices for the action of the Hecke operators T_ℓ on cohomology and we make the choice that agrees with the subsequent discussion.

are all isomorphisms. Here the subscript $\mathfrak{m}$ denotes the localization at $\mathfrak{m}$ of a $\mathbb{T}_1(N)$ -module. Indeed, there are always primes ℓ such that $T_\ell - (\ell + 1)$ annihilates the cohomology of the boundary. The fact that ρ_f is residually irreducible, guarantees that we can find a prime ℓ such that $T_\ell - (\ell + 1)$ is not in $\mathfrak{m}$. Therefore, the localization at $\mathfrak{m}$ of the boundary cohomology vanishes and the claim follows.

There are also natural choices of lattices in ρ_f as subspaces of étale cohomology. We let $\mathcal{T}_{\bar{f}}$ be the maximal subspace of $H_{\text{et}}^1(Y_1(N), \mathcal{O}_{\chi,p}(1))$ on which $J_{\bar{f}}$ acts by 0. Then Poincaré duality affords an isomorphism

$$\text{Hom}_{\mathcal{O}_{\chi,p}}(T_{f,c}, \mathcal{O}_{\chi,p}) \simeq \mathcal{T}_{\bar{f}}(-1),$$

composing with the natural map $\mathcal{T}_{\bar{f}} \rightarrow T_{\bar{f}}$ we obtain an injection

$$\text{Hom}_{\mathcal{O}_{\chi,p}}(T_{f,c}, \mathcal{O}_{\chi,p}) \hookrightarrow T_{\bar{f}}(-1).$$

Tensoring with T_f and using that $T_{f,c} \simeq T_f$, we obtain a map:

$$\text{Ad}(T_f)(1) := \text{End}(T_f)(1) \hookrightarrow T_f \otimes T_{\bar{f}}. \quad (3.4)$$

As T_f is residually irreducible, the image of $\mathcal{T}_{\bar{f}} \rightarrow T_{\bar{f}}$ can be identified with $c_{\bar{f}}T_{\bar{f}}$, where $c_{\bar{f}} = c_f$ is a generator of the congruence ideal for f , and we can identify $\text{Ad}(T_f)(1)$ with $c_{\bar{f}}T_f \otimes T_{\bar{f}}$.

We remark that there is an isomorphism:

$$\mathcal{T}_{\bar{f}} \simeq \mathcal{T}_f(\psi_f^{-1}),$$

where ψ_f is the Nebentypus of f , so that we have, if we set

$$\mathrm{Sym}^2(T_f) \subset T_f \otimes \mathcal{T}_f,$$

that

$$\mathrm{Ad}^0(T_f)(1) \simeq \mathrm{Sym}^2(T_f)(\psi_f^{-1}).$$

The $\mathbf{H}'$ case. We now study lattices for the cohomology of $\mathrm{SH}_{\mathbf{H}'}(K_1(N))$, where we have abused notation and written $K_1(N)$ for the pull-back of $K_1(N) \subset \mathbf{H}(\mathbb{A}_f)$ to $\mathbf{H}'(\mathbb{A}_f)$. Note that $\mathrm{SH}_{\mathbf{H}'}(K_1(N)) = Y_1(N) \times Y_1(N)$. Since $Y_1(N)$ is an affine curve, from the Künneth formula we obtain a canonical isomorphism:

$$\mathrm{H}_{\mathrm{et}}^2(\mathrm{SH}_{\mathbf{H}'}(K_1(N)), \mathcal{O}_{\chi,p}) \simeq \mathrm{H}_{\mathrm{et}}^1(Y_1(N), \mathcal{O}_{\chi,p}) \otimes \mathrm{H}_{\mathrm{et}}^1(Y_1(N), \mathcal{O}_{\chi,p}).$$

There is therefore an action of $\mathbb{T}_1(N) \times \mathbb{T}_1(N)$ on $\mathrm{H}_{\mathrm{et}}^2(\mathrm{SH}_{\mathbf{H}'}(K_1(N)), \mathcal{O}_{\chi,p})$. Let $\bar{f}$ be the newform whose Hecke eigenvalues are the complex conjugates of the Hecke eigenvalues of f , and $\phi_{\bar{f}} : \mathbb{T}_1(N) \rightarrow \mathcal{O}_{\chi,p}$ the corresponding Hecke eigensystem. We denote by $\bar{\mathfrak{m}}$ the maximal ideal of $\mathbb{T}_1(N)$ corresponding to $\bar{f}$, and let $F' = f \otimes \bar{f}$ and $J_{F'}$ be the prime ideal of $\mathbb{T}_1(N)^{\otimes 2}$ corresponding to the eigensystem $\phi_{F'} := \phi_f \otimes \phi_{\bar{f}}$ and let $\mathfrak{M}$ be the maximal ideal of $\mathbb{T}_1(N)^{\otimes 2}$ corresponding to the reduction of $\phi_{F'}$ to k_{χ} . We let $T_{F'}$ be the $\mathcal{O}_{\chi,p}$ module defined as the quotient of

$H_{\text{et}}^2(\text{SH}_{\mathbf{H}'}(K_1(N)), \mathcal{O}_{\chi,p}(2))$ by the image of $J_{F'}$. By the preceding discussion of the $\mathbf{GL}_2$ case, there is a canonical isomorphism:

$$T_{F'} = T_f \otimes T_{\bar{f}}.$$

The landing lattice We will see that the Euler System classes will take values in a smaller lattice than $T_{F'}$. Let $\mathbb{T}_1(N)'$ be the subring generated by the T_ℓ for $\ell \nmid N$. Denote by $\mathbb{T}^\Delta$ the copy of $\mathbb{T}_1(N)'$ in $\mathbb{T}_1(N)^{\otimes 2}$ that is the image of the map $T_\ell \mapsto (T_\ell, T'_\ell)$ where T'_ℓ is the adjoint of T_ℓ (so $T'_\ell = \langle \ell \rangle^{-1} T_\ell$). We denote by $\mathcal{T}_{F'}$ the maximal quotient of $[H_{\text{et}}^1(Y_1(N), \mathcal{O}_{\chi,p}(1))^{\otimes 2}]^{\mathbb{T}^\Delta}$ on which $\mathbb{T}_U^{\otimes 2}$ acts by $\phi_{F'}$. Note that by Poincaré duality, we have

$$[H_{\text{et}}^1(Y_1(N), \mathcal{O}_{\chi,p}(1))^{\otimes 2}]^{\mathbb{T}^\Delta} \simeq \text{Hom}_{\mathbb{T}_1(N)'}(H_{\text{et},c}^1(Y_1(N), \mathcal{O}_{\chi,p}), H_{\text{et}}^1(Y_1(N), \mathcal{O}_{\chi,p}(1))),$$

and the maximal quotient is identified with

$$\text{Hom}_{\mathbb{T}_1(N)'}(H_{\text{et}}^1(Y_1(N), \mathcal{O}_{\chi,p})_{\mathfrak{m}}, T_f) \simeq \text{Hom}_{\mathcal{O}_{\chi,p}}(T_{f,c}(-1), T_f) \simeq \text{Ad}(T_f)(1).$$

We therefore have

$$\mathcal{T}_{F'} \simeq \text{Ad}(T_f)(1).$$

3.4.5.5 An integral projection

In this section we give an integral version of the projection map defined in the previous section, and we compare the two maps. We keep the notations from the

last section; we let F be the descent of F' to $\mathbf{H}$ and denote the level of F by K .

3.4.5.6 Remark. Let us remark now that if K' is the level of F' , the isogeny $\mathbf{H}' \rightarrow \mathbf{H}$ induces an isomorphism of complex manifolds

$$\mathrm{SH}_{\mathbf{H}'}(K')(\mathbb{C}) \xrightarrow{\sim} \mathrm{SH}_{\mathbf{H}}(K)(\mathbb{C}).$$

By Artin's étale-singular comparison theorem we then have the following commutative diagram:

$$\begin{array}{ccccc}
\mathrm{H}^2(\mathrm{SH}_{\mathbf{H}}(K)(\mathbb{C}), \mathbb{Z}_p) & \xrightarrow{\quad} & \mathrm{H}^2(\mathrm{SH}_{\mathbf{H}}(K)(\mathbb{C}), \mathbb{Q}_p) & & \\
\downarrow & \searrow \sim & \downarrow & \searrow \sim & \\
& \mathrm{H}^2(\mathrm{SH}_{\mathbf{H}'}(K')(\mathbb{C}), \mathbb{Z}_p) & \xrightarrow{\quad} & \mathrm{H}^2(\mathrm{SH}_{\mathbf{H}'}(K')(\mathbb{C}), \mathbb{Q}_p) & \\
\downarrow & \downarrow & \downarrow & \downarrow & \\
\mathrm{H}_{\mathrm{et}}^2(\mathrm{SH}_{\mathbf{H}}(K), \mathbb{Z}_p) & \xrightarrow{\quad} & \mathrm{H}_{\mathrm{et}}^2(\mathrm{SH}_{\mathbf{H}}(K), \mathbb{Q}_p) & & \\
& \searrow \sim & \downarrow & \searrow \sim & \\
& \mathrm{H}_{\mathrm{et}}^2(\mathrm{SH}_{\mathbf{H}'}(K'), \mathbb{Z}_p) & \xrightarrow{\quad} & \mathrm{H}_{\mathrm{et}}^2(\mathrm{SH}_{\mathbf{H}'}(K'), \mathbb{Q}_p) &
\end{array}$$

Here the vertical arrows are given by the functorial Artin isomorphism. Note that we do not claim the (induced) isomorphisms on the bottom square are isomorphisms of Galois modules, but just of Hecke modules.

3.4.5.7 Definition. We let T_F be the lattice corresponding to $T_{F'}$ by the diagram of Remark 3.4.5.6, and similarly we let $\mathcal{T}_F$ be the lattice corresponding to $\mathcal{T}_{F'}$.

The choice of F fixes a distinguished element in the one-dimensional (since F is

new) space of Hecke–Galois equivariant homomorphisms $\mathrm{ev}_F^K : H_{\mathrm{et}}^2(\mathrm{SH}_{\mathbf{H}}(K), F_{\chi,p}(2)) \rightarrow \phi_{F'} \boxtimes V_{F'}$; here we see here $\phi_{F'}$ as the one-dimensional $F_{\chi,p}$ -space on which the Hecke algebra acts via the homomorphism $\phi_{F'}$. The quotient map

$$P_F : H_{\mathrm{et}}^2(\mathrm{SH}_{\mathbf{H}}(K), \mathcal{O}_{\chi,p}(2)) \rightarrow \phi_{F'} \boxtimes T_{F'} \quad (3.5)$$

provides another element in the same Hom-space. Therefore, by one-dimensionality, there exists a universal constant $C_F \in F_{\chi,p}$ such that

$$\mathrm{ev}_F^K = C_F P_F.$$

By this equality, on the right hand-side we really mean the extension of scalars of P_F by $\mathcal{O}_{\chi,p} \rightarrow F_{\chi,p}$.

Let $K_2 \subseteq K_1 \subseteq \mathbf{H}(\mathbb{A}_f)$, such that K_2 is a normal subgroup of K_1 . We denote by Tr_{K_1/K_2} the element of $\mathbb{Z}[\mathbf{H}(\mathbb{A}_f)]$ given by $\sum_{k \in K_1/K_2} k$, where the sum ranges over a set of representatives of the group of cosets K_1/K_2 . Note that Tr_{K_1/K_2} induces a map

$$\mathrm{Tr}_{K_1/K_2} : H_{\mathrm{et}}^2(\mathrm{SH}_{\mathbf{H}}(K_2)/\overline{\mathbb{Q}}, \mathcal{O}_{\chi,p}(3)) \longrightarrow H_{\mathrm{et}}^2(\mathrm{SH}_{\mathbf{H}}(K_2)/\overline{\mathbb{Q}}, \mathcal{O}_{\chi,p}(3))_{\mathfrak{M}}^{K_1} \xrightarrow{\sim} H_{\mathrm{et}}^2(\mathrm{SH}_{\mathbf{H}}(K_1)/\overline{\mathbb{Q}}, \mathcal{O}_{\chi,p}(3))_{\mathfrak{M}},$$

which we still denote by Tr_{K_1/K_2} by abuse of notation. The isomorphism

$$H_{\mathrm{et}}^2(\mathrm{SH}_{\mathbf{H}}(K_2)/\overline{\mathbb{Q}}, \mathcal{O}_{\chi,p}(3))_{\mathfrak{M}}^{K_1} \simeq H_{\mathrm{et}}^2(\mathrm{SH}_{\mathbf{H}}(K_1)/\overline{\mathbb{Q}}, \mathcal{O}_{\chi,p}(3))_{\mathfrak{M}}$$

comes from the Cartan-Leray spectral sequence and the fact that localizing at $\mathfrak{M}$ kills

all other cohomology except in degree 2. In the future since the projections will all factor through localization at $\mathfrak{M}$, we will allow ourselves to identify $H_{\text{et}}^2(\text{SH}_{\mathbf{H}}(K_2)/\overline{\mathbb{Q}}, \mathcal{O}_{\chi,p}(3))^{K_1}$ and $H_{\text{et}}^2(\text{SH}_{\mathbf{H}}(K_1)/\overline{\mathbb{Q}}, \mathcal{O}_{\chi,p}(3))$. Under these identifications, Tr_{K_1/K_2} coincides with the push-forward map:

$$(d_{K_1, K_2})_* : H_{\text{et}}^2(\text{SH}_{\mathbf{H}}(K_2)/\overline{\mathbb{Q}}, \mathcal{O}_{\chi,p}(3)) \longrightarrow H_{\text{et}}^2(\text{SH}_{\mathbf{H}}(K_1)/\overline{\mathbb{Q}}, \mathcal{O}_{\chi,p}(3)).$$

3.4.5.8 Definition (Integral projection). Let $K_2 \subseteq K_1 \subseteq \mathbf{H}(\mathbb{A}_f)$ be a compact open subgroup. We define

$$p_F^{K_2} : H_{\text{et}}^2(\text{SH}_{\mathbf{H}}(K_2)/\overline{\mathbb{Q}}, \mathcal{O}_{\chi,p}(3)) \longrightarrow T_F(3)$$

as the composition of $(d_{K_1, K_2})_*$, and P_F .

3.4.5.9 Remark. Note that rationally (i.e. after base change to $F_{\chi,p}$) $p_F^{K_2}$ coincides with $C_F \text{ev}_F^K \circ \text{Tr}_{K_1/K_2}$

We now compare $p_F^{K_2}$ to $\text{proj}_F^{K_2}$. We record the comparison in the following lemma.

3.4.5.10 Lemma. *We have rationally (that is, after changing scalars from $\mathcal{O}_{\chi,p}$ to $F_{\chi,p}$)*

$$p_F^{K_2} = C_F[K_1 : K_2] \text{proj}_F^{K_2}.$$

Proof. The lemma follows from the fact that since F is of level K , we have $\text{ev}_F(k' \cdot v) = \text{ev}_F(v)$ for all vectors $v \in \Pi_f$ and $k_2 \in K_2$. $\square$

3.4.6 The projected Euler map

Let us first record the following basic lemma which provides a description of the action of $\text{Gal}_{\mathbb{Q}}$ on the classes $\mathcal{EU}(\varphi)$ as elements in a Galois cohomology group. As usual, we write χ to be the character of $\text{Gal}_{\mathbb{Q}}$ corresponding to χ .

3.4.6.1 Lemma. *The absolute Galois group $\text{Gal}_{\mathbb{Q}}$ acts by the character χ^{-1} on $\mathcal{EU}(\varphi)$.*

Proof. Recall that $\mathcal{EU}(\varphi)$ is defined by the extension:

$$0 \longrightarrow \frac{H_{\text{et}}^2(\text{SH}_{\mathbf{H}}/\overline{\mathbb{Q}}, F_{\chi,p}(3))}{\text{im}H_{\text{et}}^2(\text{SH}_{\mathbf{G}}/\overline{\mathbb{Q}}, F_{\chi,p}(3))} \longrightarrow \mathcal{E}_{\chi}(\phi) \longrightarrow F_{\chi,p}\omega_{\chi}(\varphi) \longrightarrow 0$$

Denote for simplicity in this proof $\omega := \omega_{\chi}(\varphi)$ and let $\tilde{\omega}$ be a lift of ω in $\mathcal{E}_{\chi}(\varphi)$. Then the cocycle representing $\mathcal{EU}(\phi)$ is given by

$$c_{\omega} = (\tau \mapsto \tau(\tilde{\omega}) - \tilde{\omega}).$$

The action of $\text{Gal}_{\mathbb{Q}}$ on this cocycle is simply deduced from the action of $\text{Gal}_{\mathbb{Q}}$ on $\tilde{\omega}$. Moreover, by Proposition 3.3.8.1, for any $\sigma \in \text{Gal}_{\mathbb{Q}}$, $\sigma(\tilde{\omega})$ is a lift of $\chi^{-1}(\sigma)\omega$. Then the cocycle $\sigma \cdot c_{\omega}$ defines the same cohomology class as the cocycle $c_{\chi^{-1}(\sigma)\omega}$. This last cohomology class is equal to the cohomology class of the cocycle $\chi^{-1}(\sigma)c_{\omega}$. $\square$

We can then prove:

3.4.6.2 Proposition. *The image of $\text{proj}_F \circ \mathcal{EU}$ is contained in $H^1(\mathbb{F}_{\chi}, \text{Sym}^2 \rho_f(1))$.*

Proof. Indeed, assume the summand ε^2 survives the quotient by $H_{\text{et}}^2(\text{SH}_{\mathbf{G}}/\overline{\mathbb{Q}}, F_{\chi,p}(3))$. Let us analyze the action of the exterior automorphism of $\mathbf{H}'$ corresponding to switching the two $\mathbf{SL}_2$ factors in $\mathbf{H}'$. Set $s := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$; this exterior automorphism is given by conjugation in $\mathbf{G}'$ by the element

$$\begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix} \in \mathbf{G}'.$$

This element corresponds in $\mathbf{G}$ to the element

$$- \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\text{Ad}(s) & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \mathbf{G}.$$

The Eisenstein series is invariant by right multiplication by this element, whereas the line corresponding to ε^2 is a -1 eigenspace of this action. This implies the result. $\square$

We denote by Sw the exterior automorphism of $\mathbf{H}'$ that switches the two $\mathbf{SL}_2$ factors as described in the proof of the Proposition above. By abuse of notation we still denote by Sw the corresponding automorphism of $\mathbf{H}$.

3.4.6.3 Definition (Naive projected Euler map). We denote by $\mathcal{EU}_F$ the composition

$$\mathcal{EU}_F := \text{proj}_F \circ \mathcal{EU} : \mathcal{S}(\mathbb{A}_f) \longrightarrow H^1(\mathbb{F}_\chi, \text{Ad}^0 \rho_f(2))$$

The rule $F \mapsto \mathcal{EU}_F$ defines an element of the Hom space

$$\mathrm{Hom}_{\mathbf{H}(\mathbb{A}_f)}(\Pi_f \otimes \Pi_\chi^\infty, \mathbb{C})$$

which is the same Hom space where the zeta integral $\mathfrak{z}$ of Murase-Sugano lies. Therefore by remark 3.4.3.2, $\mathcal{EU}_F(\varphi)$ is proportional to $\mathfrak{z}(F, \varphi)$ (the constant of proportionality being independent of F and φ).

3.4.6.4 The Euler System map

We now give the definition of the Euler System map; this map will preserve integrality, contrary to $\mathcal{EU}_F$, though the latter is useful in that it is related to zeta integrals by multiplicity one, as was remarked above. Furthermore, unlike $\mathcal{EU}_F$, the Euler System map may depend on the level subgroups of the Eisenstein series. Let $U \subset \mathbf{G}(\mathbb{A}_f)$ be a level subgroup whose pull-back to $\mathbf{H}(\mathbb{A}_f)$ is K . In order to define the Euler System map we will need the following lemma.

3.4.6.5 Proposition. *The image of*

$$\mathrm{H}_{\mathrm{et}}^2(\mathrm{SH}_{\mathbf{G}}(U)/\overline{\mathbb{Q}}, \mathcal{O}_{\chi,p}(2)) \rightarrow \mathrm{H}_{\mathrm{et}}^2(\mathrm{SH}_{\mathbf{H}}(K)/\overline{\mathbb{Q}}, \mathcal{O}_{\chi,p}(2))_{\mathfrak{M}}$$

is contained in the -1 eigenspace of Sw ; this eigenspace only contains one dimensional Galois representations.

Proof. Indeed, as we have localized at $\mathfrak{M}$, the residual Galois representation splits as a sum of an irreducible Galois representation of dimension 3 and a one-dimensional

representation isomorphic to the reduction of the cyclotomic character. Moreover, the exterior automorphism Sw of $\mathbf{H}'$ switching the two $\mathbf{SL}_2$ factors acts by -1 on the one-dimensional representation and by $+1$ on the three-dimensional representation. As we remarked in the proof of Proposition 3.4.6.2, there are no irreducible representations of dimension 3 in $H_{\text{et}}^2(\text{SH}_{\mathbf{G}'}(U), \mathcal{O}_{\chi,p})$, and the result follows. $\square$

Note that the action of Sw on $H_{\text{et}}^2(\text{SH}_{\mathbf{H}}(K), \mathcal{O}_{\chi,p}(2))_{\mathfrak{M}}$ descends to an action on $T_f \otimes T_{\bar{f}}$, whose -1 eigenspace is isomorphic to ε . We denote by $T_F^+ = (T_f \otimes T_{\bar{f}})^+$ the quotient of $T_F = T_f \otimes T_{\bar{f}}$ by this eigenspace. We denote by P_F^+ the composition of the map P_F defined in Equation 3.5 with the quotient map $T_F \rightarrow T_F^+$. We then have the following immediate corollary of Proposition 3.4.6.5

3.4.6.6 Corollary. *The map $P_F^+ : H_{\text{et}}^2(\text{SH}_{\mathbf{H}}(K), \mathcal{O}_{\chi,p}(2))_{\mathfrak{M}} \rightarrow T_F^+$ factors through the quotient*

$$\frac{H_{\text{et}}^2(\text{SH}_{\mathbf{H}}(K), \mathcal{O}_{\chi,p}(2))_{\mathfrak{M}}}{\text{im} H_{\text{et}}^2(\text{SH}_{\mathbf{G}}(U), \mathcal{O}_{\chi,p}(2))_{\mathfrak{M}}}.$$

Consider now the exact sequence coming from relative cohomology as in Section 3.4.1 and let φ be a choice of Schwartz function such that $\omega := \omega(\varphi) = \omega(f_{\chi;\varphi}) \in H_{\text{et}}^3(\text{SH}_{\mathbf{G}}, \mathcal{O}_{\chi,p}(3))$. Then we have:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \frac{H_{\text{et}}^2(\text{SH}_{\mathbf{H}}(K), \mathcal{O}_{\chi,p}(3))}{\text{im} H_{\text{et}}^2(\text{SH}_{\mathbf{G}}(U), \mathcal{O}_{\chi,p}(3))} & \longrightarrow & H_{\text{et}}^3((\text{SH}_{\mathbf{G}}(U), \text{SH}_{\mathbf{H}}(K), \mathcal{O}_{\chi,p}(3))) & \longrightarrow & H_{\text{et}}^3(\text{SH}_{\mathbf{G}}(U), \mathcal{O}_{\chi,p}(3)) \longrightarrow 0 \\ & & \downarrow & & \vdots & & \uparrow \\ 0 & \longrightarrow & T_F^+(1) & \longrightarrow & \mathcal{T}_{\chi}(\omega) & \longrightarrow & \mathcal{O}_{\chi,p}\omega \longrightarrow 0. \end{array}$$

Here the map $\frac{H_{\text{et}}^2(\text{SH}_{\mathbf{H}}(K), \mathcal{O}_{\chi,p}(3))}{\text{im} H_{\text{et}}^2(\text{SH}_{\mathbf{G}}(U), \mathcal{O}_{\chi,p}(3))} \rightarrow T_F^+(1)$ is given by the composition with localization at $\mathfrak{M}$, and then applying P_F^+ . The $\mathcal{O}_{\chi,p}[\text{Gal}_{\mathbb{F}_{\chi}}]$ module $\mathcal{T}_{\chi}(\omega) \subset \mathcal{E}_{\chi}(\omega)$ is a

lattice.

If $U \cap \mathbf{H}(\mathbb{A}_f) = K_2 \supset K$, then we again have a diagram:

$$\begin{array}{ccccccc}
0 & \longrightarrow & \frac{H_{\text{et}}^2(\text{SH}_{\mathbf{H}}(K), \mathcal{O}_{\chi,p}(3))}{\text{im} H_{\text{et}}^2(\text{SH}_{\mathbf{G}}(U), \mathcal{O}_{\chi,p}(3))} & \longrightarrow & H_{\text{et}}^3((\text{SH}_{\mathbf{G}}(U), \text{SH}_{\mathbf{H}}(K), \mathcal{O}_{\chi,p}(3))) & \longrightarrow & H_{\text{et}}^3(\text{SH}_{\mathbf{G}}(U), \mathcal{O}_{\chi,p}(3)) \longrightarrow 0 \\
& & \downarrow P_F^+ \circ (d_{K_2, K})^* & & \downarrow \text{dotted} & & \uparrow \\
0 & \longrightarrow & T_F^+(1) & \longrightarrow & \mathcal{T}_{\chi}^U(\omega) & \longrightarrow & \mathcal{O}_{\chi,p}\omega \longrightarrow 0.
\end{array}$$

3.4.6.7 Definition. [Euler System map] Let $U \subseteq \mathbf{G}(\mathbb{A}_f)$ be a compact open subgroup such that $U \cap \mathbf{H}(\mathbb{A}_f) = K_2 \subseteq K$, where we recall that the level of F is denoted by K . Let φ be a choice of Schwartz function such that $\omega := \omega(f_{\chi}; \varphi) \in H_{\text{et}}^3(\text{SH}_{\mathbf{G}}(U), \mathcal{O}_{\chi,p}(3))$. We define

$$\mathcal{ES}_{F,U}(\omega) = [\mathcal{T}_{\chi}^U(\omega)] \in H^1(\mathbb{F}_{\chi}, T_F^+),$$

where the brackets indicate the class of the extension.

As a consequence of Lemma 3.4.5.10, we have the following proposition.

3.4.6.8 Proposition. *We have the following relation between $\mathcal{ES}_{F,U}$ and $\mathcal{EU}_F$: if $K_2 = U \cap \mathbf{H}(\mathbb{A}_f)$ is contained in K then we have*

$$\mathcal{ES}_{F,U} = C_F[K : K_2] \mathcal{EU}_F.$$

Because of Proposition 3.4.6.10, we will denote $\mathcal{ES}_{F,U}$ also by $\mathcal{ES}_{F,K_2}$.

We now show that the Euler System map actually lands in a smaller lattice than T_F , namely $\text{Ad}^0(T_f)(2)$ where $\text{Ad}^0(T_f) \subset \text{Ad}(T_f)$ is defined as the kernel of the trace map.

3.4.6.9 Proposition. *With the notations of 3.4.2: For any Schwartz function φ , if*

$$\omega := \omega(f_\varphi) \in H_{\text{et}}^3(\text{SH}_{\mathbf{G}}(U), \mathcal{O}_{\chi,p}(3)),$$

then

$$\mathcal{ES}_{F,U}(\omega) \in H^1(\mathbb{F}_\chi, \text{Ad}^0(T_f)(2)).$$

Proof. Note that it suffices to treat the case where $U \cap \mathbf{H}(\mathbb{A}_f) = K$ is the level of F . Note also that the action of Sw commutes with the action of $\mathbf{G}_0$ (recall the pull-back of $\mathbf{G}_0$ to $\mathbf{H}'$ is the diagonal copy of $\mathbf{SL}_2$), and therefore Sw leaves stable $\mathbf{G}_0$ -invariants. With the notations of 3.4.5.4, this implies that $\mathcal{ES}_{F,U}(\omega)$ lies in the image by P_F^+ of the lattice associated to $[\text{H}_{\text{et}}^1(Y_1(N), \mathcal{O}_{\chi,p}(1))^{\otimes 2}(1)]^{\mathbb{T}^\Delta}$. And we then conclude by the last paragraph of section 3.4.5.4. $\square$

We now show that the Euler-System map does not depend too much on the level subgroup U of $\mathbf{G}(\mathbb{A}_f)$.

3.4.6.10 Proposition. *Let $U' \subset U \subset \mathbf{G}(\mathbb{A}_f)$ be two level subgroups such that*

$$U \cap \mathbf{H}(\mathbb{A}_f) = U' \cap \mathbf{H}(\mathbb{A}_f) := K' \subset K.$$

Let K_f be the level of F . Let $\omega \in H^3(\text{SH}_{\mathbf{G}}(U), \mathcal{O}_{\chi,p}(3))$ and let ω' be the pull-back of ω to $H^3(\text{SH}_{\mathbf{G}}(U'), \mathcal{O}_{\chi,p}(3))$ under the degeneracy map $d' : \text{SH}_{\mathbf{G}}(U') \rightarrow \text{SH}_{\mathbf{G}}(U)$. Then we have the equality

$$\mathcal{ES}_{F,U}(\omega) = \mathcal{ES}_{F,U'}(\omega').$$

Proof. Let $\mathcal{E}'_{\mathfrak{M}}$ be the quotient of $H_{\text{et}}^3((\text{SH}_{\mathbf{G}}(U'), \text{SH}_{\mathbf{H}}(K)), \mathcal{O}_{\chi,p}(3))$ by the image of the kernel of the localization map $H_{\text{et}}^2(\text{SH}_{\mathbf{H}}(K'), \mathcal{O}_{\chi,p}(3)) \rightarrow H_{\text{et}}^2(\text{SH}_{\mathbf{H}}(K'), \mathcal{O}_{\chi,p}(3))_{\mathfrak{M}}$. Similarly let $\mathcal{E}_{\mathfrak{M}}$ be the quotient of $H_{\text{et}}^3((\text{SH}_{\mathbf{G}}(U'), \text{SH}_{\mathbf{H}}(K')), \mathcal{O}_{\chi,p}(3))$ by the image of the kernel of the localization map $H_{\text{et}}^2(\text{SH}_{\mathbf{H}}(K'), \mathcal{O}_{\chi,p}(3)) \rightarrow H_{\text{et}}^2(\text{SH}_{\mathbf{H}}(K'), \mathcal{O}_{\chi,p}(3))_{\mathfrak{M}}$. We have the following diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & T_F^+(1) & \longrightarrow & \mathcal{T}_{\chi}^{U'}(\omega') & \longrightarrow & \mathcal{O}_{\chi,p}\omega' & \longrightarrow & 0 \\
& & \uparrow P_F^+ \circ (d_{K,K_f})_* & & \vdots & & \downarrow & & \\
0 & \longrightarrow & H_{\text{et}}^2(\text{SH}_{\mathbf{H}}(K'), \mathcal{O}_{\chi,p}(3))_{\mathfrak{M}} & \longrightarrow & \mathcal{E}'_{\mathfrak{M}} & \longrightarrow & H_{\text{et}}^3(\text{SH}_{\mathbf{G}}(U'), \mathcal{O}_{\chi,p}(3)) & \longrightarrow & 0 \\
& & \uparrow = & & \uparrow (d')^* & & \uparrow (d')^* & & \\
0 & \longrightarrow & H_{\text{et}}^2(\text{SH}_{\mathbf{H}}(K'), \mathcal{O}_{\chi,p}(3))_{\mathfrak{M}} & \longrightarrow & \mathcal{E}_{\mathfrak{M}} & \longrightarrow & H_{\text{et}}^3(\text{SH}_{\mathbf{G}}(U), \mathcal{O}_{\chi,p}(3)) & \longrightarrow & 0 \\
& & \downarrow P_F^+ \circ (d_{K,K_f})_* & & \vdots & & \uparrow & & \\
0 & \longrightarrow & T_F^+(1) & \longrightarrow & \mathcal{T}_{\chi}^U(\omega) & \longrightarrow & \mathcal{O}_{\chi,p}\omega & \longrightarrow & 0.
\end{array}$$

Note that the pull-back, which we denote by L , of the line $\mathcal{O}_{\chi,p}\omega \subset H_{\text{et}}^3(\text{SH}_{\mathbf{G}}(U), \mathcal{O}_{\chi,p}(3))$ to $\mathcal{E}_{\mathfrak{M}}$ maps under $(d')^*$ to the pull-back, L' of the line $\mathcal{O}_{\chi,p}\omega' \subset H_{\text{et}}^3(\text{SH}_{\mathbf{G}}(U'), \mathcal{O}_{\chi,p}(3))$ to $\mathcal{E}'_{\mathfrak{M}}$. Therefore $(d')^* : H_{\text{et}}^3(\text{SH}_{\mathbf{G}}(U), \mathcal{O}_{\chi,p}(3)) \rightarrow H_{\text{et}}^3(\text{SH}_{\mathbf{G}}(U'), \mathcal{O}_{\chi,p}(3))$ induces a map

$$L \rightarrow L' \twoheadrightarrow \mathcal{T}_{\chi}^{U'}(\omega').$$

The kernel M' of $L' \twoheadrightarrow \mathcal{T}_{\chi}^{U'}(\omega')$ is by definition the image in $\mathcal{E}'_{\mathfrak{M}}$ of the kernel of $P_F^+ \circ (d_{K,K_f})_*$. By the left hand-side of the diagram above, this coincides with the image under $(d')^*$ of the image, M , in $\mathcal{E}_{\mathfrak{M}}$ of the kernel of $P_F^+ \circ (d_{K,K_f})_*$. Thus we obtain an injective map

$$L/M = \mathcal{T}_{\chi}^U(\omega) \hookrightarrow \mathcal{T}_{\chi}^{U'}(\omega').$$

Since the middle arrow of the first column of the diagram above is an equality, the map $\mathcal{T}_\chi^U(\omega) \hookrightarrow \mathcal{T}_\chi^{U'}(\omega')$ induces the identity on $T_F^+(1)$. To prove that it is an isomorphism it then suffices to show that the induced map $\mathcal{O}_{\chi,p}\omega = \mathcal{T}_\chi^U(\omega)/T_F^+(1) \hookrightarrow \mathcal{T}_\chi^{U'}(\omega')/T_F^+(1) = \mathcal{O}_{\chi,p}\omega'$ is surjective. But this is obvious as $(d')^*\omega = \omega'$ by definition. $\square$

3.4.7 Equivariance at finite level

In this section, we explain how the $\mathbf{H}(\mathbb{A}_f)$ -equivariance of $\mathcal{EU}_F$ translates into an equivariance at finite level.

Let $h \in \mathbf{H}(\mathbb{A}_f)$ and $K \subset \mathbf{H}(\hat{\mathbb{Z}})$ an open compact subgroup. Multiplication on the right by h induces a map

$$[h]_K : \mathrm{SH}_{\mathbf{H}}(K) \longrightarrow \mathrm{SH}_{\mathbf{H}}(h^{-1}Kh).$$

For $U \subset \mathbf{G}(\mathbb{A}_f)$ an open compact subgroup, we denote by $[h]_U : \mathrm{SH}_{\mathbf{G}}(U) \rightarrow \mathrm{SH}_{\mathbf{G}}(h^{-1}Uh)$, the similarly defined map. Then, if U is such that $U \cap \mathbf{H}(\mathbb{A}_f) = K$, the maps $([h]_U, [h]_K)$ define a map of pairs

$$(\mathrm{SH}_{\mathbf{G}}(U), \mathrm{SH}_{\mathbf{H}}(K)) \longrightarrow (\mathrm{SH}_{\mathbf{G}}(h^{-1}Uh), \mathrm{SH}_{\mathbf{H}}(h^{-1}Kh)).$$

By the proof of Proposition 3.4.1.4 we have

$$\mathcal{ES}_{F,h^{-1}Uh}([h]_U^*\omega) = [h]_K^*\mathcal{ES}_{F,U}(\omega) = [[h]_K^*(\sigma \mapsto \tilde{\omega}^\sigma - \tilde{\omega})] \in H^1(\mathbb{F}_\chi, H_{\mathrm{et}}^2(\mathrm{SH}_{\mathbf{H}}(h^{-1}Kh), \mathcal{O}_{\chi,p}(3))). \quad (3.6)$$

Here $\tilde{\omega}$ is a lift of ω in relative cohomology and the brackets signify the cohomology class of the cocycle.

Consider now $K_2 \subseteq K_1 \subseteq \mathbf{H}(\mathbb{A}_f)$ such that K_2 is a normal subgroup of K_1 . Let $U_2 \subseteq \mathbf{G}(\mathbb{A}_f)$ such that $U_2 \cap \mathbf{H}(\mathbb{A}_f) = K_2$. Let $X = K_1/K_2$ be the quotient group. For $x \in X$, we have the map defined above $[x]_{K_2} : \mathrm{SH}_{\mathbf{H}}(K_2) \rightarrow \mathrm{SH}_{\mathbf{H}}(K_2)$, and we have the trace map in cohomology

$$\mathrm{Tr}_{K_1/K_2} : \mathrm{H}_{\mathrm{et}}^*(\mathrm{SH}_{\mathbf{H}}(K_2)/\overline{\mathbb{Q}}, \mathcal{O}_{\chi,p}(3)) \rightarrow \mathrm{H}^*(\mathrm{SH}_{\mathbf{H}}(K_1)/\overline{\mathbb{Q}}, \mathcal{O}_{\chi,p}(3)),$$

defined as in Section 3.4.5.5 by the identification provided by the Cartan-Leray spectral sequence (here again we omit the localization at $\mathfrak{M}$ to avoid cumbersome notation):

$$\mathrm{H}_{\mathrm{et}}^*(\mathrm{SH}_{\mathbf{H}}(K_2)/\overline{\mathbb{Q}}, \mathcal{O}_{\chi,p}(3))^X \simeq \mathrm{H}_{\mathrm{et}}^*(\mathrm{SH}_{\mathbf{H}}(K_1)/\overline{\mathbb{Q}}, \mathcal{O}_{\chi,p}(3)).$$

Assume that for all $x \in X$ we have $x^{-1}U_2x = U$ does not depend on x . In that case, Tr_{K_1/K_2} defines an operator $\sum_{x \in X} [x]_{U_2}^*$ still denoted as Tr_{K_1/K_2} :

$$\mathrm{Tr}_{K_1/K_2} : \mathrm{H}_{\mathrm{et}}^*(\mathrm{SH}_{\mathbf{G}}(U_2)/\overline{\mathbb{Q}}, \mathcal{O}_{\chi,p}(3)) \longrightarrow \mathrm{H}_{\mathrm{et}}^*(\mathrm{SH}_{\mathbf{G}}(U)/\overline{\mathbb{Q}}, \mathcal{O}_{\chi,p}(3)).$$

Then by the same argument as before, we have the relation:

$$\mathcal{ES}_{F,U}(\mathrm{Tr}_{K_1/K_2}\omega) = \mathrm{Tr}_{K_1/K_2}\mathcal{ES}_{F,U_2}(\omega) \in \mathrm{H}^1(\mathbb{F}_{\chi}, \mathrm{H}_{\mathrm{et}}^2(\mathrm{SH}_{\mathbf{H}}(K_1), \mathcal{O}_{\chi,p}(3))). \quad (3.7)$$

3.4.7.1 Remark. The equation above implies that $\mathcal{ES}_{F,K_1^{\mathbf{P}}(p^r)}$ is $u_p \times u_p$ (or $u'_p \times u'_p$)

equivariant in the notations of section 3.4.10: a fact that we will use often in that section.

3.4.8 The zeta norm relation for the Symmetric square

For $\Pi := \pi(f) \boxtimes \pi(f)^\vee$ as in 3.4.4, we will henceforth let

$$\mathcal{T}_\Pi := \text{Ad}^0(T_f)(1).$$

Furthermore:

We now choose once and for all a specific new vector $F \in \Pi_f = \pi(f) \boxtimes \pi(f)^\vee$ such that $\mathcal{P}_F(1) \neq 0$

Recall that the set S denotes the set of primes at which F or χ ramify, and always contains the prime p . Let us state again Proposition 3.4.3.3 in this specific case. We use that $L_\ell(F \otimes \chi, 2) = L_\ell(\text{Ad}^0(\pi(f)) \otimes \chi, 2)L_\ell(\chi, 2)$ for $\ell \notin S$.

3.4.8.1 Proposition. *For all primes $\ell \notin S$ we have:*

$$\mathfrak{z}_\ell(F, \varphi_\ell^0) = \mathcal{P}_F^{(\ell)}(1)L_\ell(\text{Ad}^0(\pi(f)) \otimes \chi, 2).$$

We can then immediately deduce the following - because of the chosen normalization - by applying proposition 3.4.3.4:

3.4.8.2 Proposition (Zeta Norm Relation). *For all primes $\ell \notin S$, we have:*

$$\mathfrak{z}_\ell(F, \varphi_\ell^r) = L_\ell(\text{Ad}^0(\pi(f)) \otimes \chi, 2)^{-1}[K_\ell^{\mathbf{H}} : K(\ell)]\mathfrak{z}_\ell(F, \varphi_\ell^0).$$

By Remark 3.4.3.2 and Lemma 3.4.5.10 we can then deduce:

3.4.8.3 Corollary. *Let $\ell \notin S$. Then we have the relation:*

$$\mathcal{ES}_{F, K^{\text{vol}}(\ell)}(\varphi^\ell) = L_\ell(\text{Ad}^0(\pi(f)) \otimes \chi, 2)^{-1} \mathcal{ES}_{F, K_\ell^\mathfrak{G}}(\varphi^0).$$

3.4.9 The horizontal Galois Norm Relations

In this section, we begin the construction of the classes that we will arrange into an Euler System.

Fix a prime $\ell \notin S$ and consider ψ an *even* Dirichlet characters modulo ℓ :

$$\psi : (\mathbb{Z}/\ell\mathbb{Z})^\times \rightarrow \mathbb{C}^\times$$

Then $\psi\chi$ is odd and we can construct the holomorphic differential $\omega_{\chi\psi}(\varphi)$, just as in section 3.3.7. Define now:

$$\varsigma_\ell^\chi := \frac{2}{\ell-1} \sum_{\psi} \omega_{\chi\psi}(\varphi^\ell),$$

where the sum is taken over all even Dirichlet characters mod ℓ . Then as in 3.3.7, we can see by looking at the constant term, as we are summing over *all* even Dirichlet character, that this class is defined over F_χ - it was only a priori defined over $\cup_\psi F_{\chi\psi}$. Indeed, the rational structure of the representation $\bigoplus_{\psi} \Pi_{\chi\psi, f}$ is determined by the

image to the boundary and the constant term of ς_ℓ^χ is

$$\sum_{\psi} f_{\chi\psi; \varphi^\ell}$$

which is an F_χ -valued section. Indeed note that $f_{\chi\psi; \varphi^\ell}(g)$ is a universal polynomial in χ (namely a special Dirichlet L -value at a negative integer) times a universal polynomial in the values of χ depending on the entries of g .

Then, denoting level structure away from ℓ by K^ℓ , ς_ℓ^χ defines an étale class

$$\varsigma_\ell^{\text{et}, \chi} \in H_{\text{et}}^3(\text{SH}_{\mathbf{G}}(K(\ell)K^\ell), F_{\chi, p}(3)).$$

It remains to justify that this class is integral, i.e. that $\varsigma_\ell^\chi \in H_{\text{et}}^3(\text{SH}_{\mathbf{G}}(K(\ell)K^\ell), \mathcal{O}_{\chi, p}(3))$.

This follows from the ordinary q -expansion principle: namely, as the Fourier coefficients of $E(-; f_{\chi\psi; \varphi^\ell})$ are universal polynomials in $\chi\psi$, we deduce that the Fourier coefficients of $\frac{2}{\ell-1} \sum_{\psi} E(-; f_{\chi\psi; \varphi^\ell})$ at the multiplicative cusps are p -adically integral. This implies by ordinarity that the Fourier coefficients at all cusps are integral by Faltings' Lemma 3.3.5.3 and therefore one obtains that

$$\varsigma_\ell^{\text{et}, \chi} \in H_{\text{et}}^3(\text{SH}_{\mathbf{G}}(K(\ell)K^\ell), \mathcal{O}_{\chi, p}(3)).$$

Note that the class $\varsigma_\ell^{\text{et}, \chi}$ is **not** invariant by $\text{Gal}(\mathbb{F}_\chi(\boldsymbol{\mu}_\ell)^+/\mathbb{F}_\chi)$. Indeed $\sigma \in \text{Gal}_{\mathbb{Q}}$ acts as :

$$\sigma(\varsigma_\ell^{\text{et}, \chi}) = \frac{2}{\ell-1} \sum_{\psi} \chi\psi(\sigma) \omega_{\chi\psi}^{\text{et}}(\varphi^\ell).$$

Here again we view, by abuse of notation, $\chi\psi$ as a character of $\text{Gal}_{\mathbb{Q}}$ by class field

theory - the Artin isomorphism being normalized by matching geometric Frobenius elements to uniformizers.

The class $\varsigma_\ell^{\text{et}, \chi}$ defines an element $\kappa_\ell^\chi = \mathcal{ES}_{F, K(\ell)}(\varsigma_\ell^{\text{et}, \chi})$ in Galois cohomology:

$$\kappa_\ell^\chi \in H^1(\mathbb{F}_\chi(\boldsymbol{\mu}_\ell)^+, \mathcal{T}_\Pi(1)),$$

and we let $\kappa_1^\chi \in H^1(\mathbb{F}_\chi, \mathcal{T}_\Pi(1))$ denote the Galois cohomology class corresponding to $\omega_\chi(\varphi^0)$.

We can then prove the following lemma.

3.4.9.1 Lemma. *We have the relation*

$$\text{Cores}_{\mathbb{F}_\chi}^{\mathbb{F}_\chi(\boldsymbol{\mu}_\ell)^+} \kappa_\ell^\chi = L_\ell(\text{Ad}^0 \pi(f) \otimes \chi, 2)^{-1} \kappa_1^\chi.$$

Proof. Note that

$$\sum_{\sigma} \psi(\sigma) = \begin{cases} \frac{\ell-1}{2} & \text{if } \psi \text{ is trivial} \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, we have

$$\text{Cores}_{\mathbb{F}_\chi}^{\mathbb{F}_\chi(\boldsymbol{\mu}_\ell)^+} \kappa_\ell^\chi = \frac{\ell-1}{2} \mathcal{ES}_{F, K(\ell)}(\varphi^\ell)$$

and we conclude by corollary 3.4.8.3, and the fact that by the inflation–restriction exact sequence the restriction map

$$H^1(\mathbb{F}_\chi, \mathcal{T}_\Pi(1)) \longrightarrow H^1(\mathbb{F}_\chi(\boldsymbol{\mu}_\ell)^+, \mathcal{T}_\Pi(1))$$

is an injection so that it suffices to check equality of two cohomology classes in the cohomology group on the right hand-side. $\square$

As Fr_ℓ , the *arithmetic* Frobenius, acts by $\chi(\ell)$ on κ_1^χ by lemma 3.4.6.1, we can deduce:

3.4.9.2 Lemma. *We have the relation :*

$$\text{Cores}_{\mathbb{F}_\chi}^{\mathbb{F}_\chi(\boldsymbol{\mu}_\ell)^+} \kappa_\ell^\chi = P_\ell(\text{Fr}_\ell^{-1}) \kappa_1^\chi,$$

where P_ℓ is the Hecke Polynomial of the Galois representation $\text{Ad}^0 \rho_f(-3) \otimes \chi^{-1}$.

Note that $\text{Ad}^0(V_f) = (\text{Ad}^0 \rho_f(1))^\vee(1)$. Moreover, by the inflation–restriction exact sequence κ_1^χ , (respectively) κ_ℓ^χ define classes in

$$H^1(\mathbb{Q}, \text{Ad}^0 \rho_f(1) \otimes \chi)$$

and in

$$H^1(\mathbb{Q}(\boldsymbol{\mu}_\ell)^+, \text{Ad}^0 \rho_f(1) \otimes \chi)$$

respectively. Here we have used that $\text{Gal}_{\mathbb{Q}}$ (resp. $\text{Gal}_{\mathbb{Q}(\mathbb{Q}(\boldsymbol{\mu}_\ell)^+)}$) acts on κ_1^χ (resp. κ_ℓ^χ) via χ^{-1} . Let us denote these classes by $\mathcal{Z}_1^\chi$ and $\mathcal{Z}_\ell^\chi$ respectively. The above norm relation can then be rephrased as:

$$\text{Cores}_{\mathbb{Q}}^{\mathbb{Q}(\boldsymbol{\mu}_\ell)^+} \mathcal{Z}_\ell^\chi = P_\ell^\chi(\text{Fr}_\ell^{-1}) \mathcal{Z}_1^\chi,$$

where P_ℓ^χ is the Hecke polynomial of the Galois representation $\text{Ad}^0 V_f(-3) \otimes \chi$.

Let M be a square-free integer. We now generalize the above arguments replacing the prime ℓ with M , in order to obtain the full horizontal Euler System norm relations. Let $M^* := \prod_{\ell|M} (\ell - 1)/2$. Define :

$$\varsigma_M^\chi := \frac{1}{M^*} \sum_{\psi} \omega_{\chi\psi}(\varphi^M),$$

where the sum is taken over all *even* Dirichlet characters of conductor M . By the same argument as before, ς_M^χ defines an integral étale class $\varsigma_M^{\text{et},\chi}$, which in turn defines a class

$$\kappa_M^\chi = \mathcal{ES}_{F,K(M)}(\varsigma_M^{\text{et},\chi}) \in H^1(\mathbb{F}_\chi(\boldsymbol{\mu}_M)^+, \mathcal{T}_\Pi(1)).$$

The following proposition records the full horizontal norm relations. The proof is very similar to 3.4.9.2.

3.4.9.3 Proposition. *Let M be a square-free integer and let ℓ be a prime that does not divide M . We have the relation :*

$$\text{Cores}_{\mathbb{F}_\chi(\boldsymbol{\mu}_M)^+}^{\mathbb{F}_\chi(\boldsymbol{\mu}_{M\ell})^+} \kappa_M^\chi = P_\ell(\text{Fr}_\ell^{-1}) \kappa_M^\chi$$

where P_ℓ is the Hecke Polynomial of the Galois representation $\text{Ad}^0 V_f(-3)$.

Proof. Just as in the proof of Lemma 3.4.9.1, we can write:

$$\text{Cores}_{\mathbb{F}_\chi(\boldsymbol{\mu}_M)^+}^{\mathbb{F}_\chi(\boldsymbol{\mu}_{M\ell})^+} \kappa_M^\chi = \frac{1}{M^*} \sum_{\psi} \mathcal{ES}_{F,K(M\ell)}(f_{\chi\psi;\varphi^{M\ell}}),$$

where the sum is indexed by all the even characters ψ of modulus M . Then, by the

zeta norm relations, we obtain:

$$\text{Cores}_{\mathbb{F}_\chi(\boldsymbol{\mu}_M)^+}^{\mathbb{F}_\chi(\boldsymbol{\mu}_{M\ell})^+} \kappa_{M\ell}^\chi = \frac{1}{M^*} \sum_{\psi} L_\ell(\text{Ad}^0 \pi(f) \otimes \chi\psi, 2)^{-1} \mathcal{ES}_{F,K(M\ell)}(f_{\chi\psi;\varphi^M}).$$

As Fr_ℓ , the *arithmetic* Frobenius, acts by $(\chi\psi)(\ell)$ on $\mathcal{ES}_{F,K(M\ell)}(f_{\chi\psi;\varphi^M})$, we deduce:

$$\text{Cores}_{\mathbb{F}_\chi(\boldsymbol{\mu}_M)^+}^{\mathbb{F}_\chi(\boldsymbol{\mu}_{M\ell})^+} \kappa_{M\ell}^\chi = \frac{1}{M^*} \sum_{\psi} P_\ell(\text{Fr}_\ell^{-1}) \mathcal{ES}_{F,K(M\ell)}(f_{\chi\psi;\varphi^M}) = P_\ell(\text{Fr}_\ell^{-1}) \kappa_M^\chi.$$

□

3.4.10 The vertical norm relations

3.4.10.1 Operators

In this subsection we introduce notation for operators that will be useful for the following sections. All bracket matrices indicate that we think of elements as elements of $\mathbb{Z}_p[\mathbf{G}(\mathbb{Q}_p)]$ acting on smooth functions on $\mathbf{G}(\mathbb{Q}_p)$ fixed on the right by $K_1^{\mathbf{P}}(p^A)$ for some integer A .

3.4.10.2 Definition. We define the following operators.

- For $a \geq 1$ an integer, we set

$$\text{tw}^a := \begin{bmatrix} 0 & 0 & p^{-a} \\ 0 & I_3 & 0 \\ p^a & 0 & 0 \end{bmatrix}.$$

We note that this elements is the image of

$$\left(\begin{bmatrix} 0 & 1 \\ p^a & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ p^a & 0 \end{bmatrix} \right) \in \mathbf{GL}_2(\mathbb{Q}_p) \times_{\det} \mathbf{GL}_2(\mathbb{Q}_p)$$

under the exceptional isogeny $\mathbf{GL}_2 \times_{\det} \mathbf{GL}_2 \rightarrow \mathbf{H}$; we view it as an element of $\mathbf{G}$ under the embedding $\iota : \mathbf{H} \hookrightarrow \mathbf{G}$. It is a version of the Atkin-Lehner involution.

- For ξ a Dirichlet character of conductor $r \geq 1$, we consider the operator:

$$t_{\xi}^{dep} := \sum_{x \bmod p^r} \xi(x) n_1(0, -x/p^r, 0);$$

and for $a \geq r$ any integer

$$t_{\xi}^{DEP,a} := \sum_{x \bmod p^a} \xi(x) n_1(0, -x/p^a, 0).$$

We recall here that $n_1(x_1, x_0, x_2)$ denotes a generic element of the unipotent radical of the parabolic $\mathbf{P}_1$ (see section 3.1.2). The element $n_1(x_1, x_0, x_2)$ is the image under the exceptional isogeny $\pi' : \mathbf{G}' \rightarrow \mathbf{G}$ of the element of the unipotent radical of the Siegel parabolic $\mathbf{P}'_1$ determined by the symmetric matrix $\begin{pmatrix} -x_1 & x_0/2 \\ x_0/2 & x_2 \end{pmatrix}$.

- For $d \in \mathbb{Z}_p^\times$, we consider the diamond operator

$$\langle d \rangle := \begin{bmatrix} d & 0 & 0 \\ 0 & I_3 & 0 \\ 0 & 0 & d^{-1} \end{bmatrix}.$$

This is the image of

$$\left(\begin{bmatrix} d & 0 \\ 0 & d^{-1} \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \in \mathbf{SL}_2 \times \mathbf{SL}_2 \simeq \mathbf{H}'$$

under $\mathbf{H}' \twoheadrightarrow \mathbf{H} \hookrightarrow \mathbf{G}$.

- We now consider the $u_p \times u_p$ operator defined as

$$u_p \times u_p := \sum_{x_1, x_2 \bmod p} n_1(x_1, 0, x_2) \begin{bmatrix} p & 0 & 0 \\ 0 & I_3 & 0 \\ 0 & 0 & p^{-1} \end{bmatrix}.$$

Each element of the sum is the image of

$$\left(\begin{pmatrix} p & x_1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} p & x_2 \\ 0 & 1 \end{pmatrix} \right) \in \mathbf{GL}_2(\mathbb{Q}_p) \times_{\det} \mathbf{GL}_2(\mathbb{Q}_p)$$

under $\mathbf{GL}_2 \times_{\det} \mathbf{GL}_2 \rightarrow \mathbf{H} \hookrightarrow \mathbf{G}$; hence the notation of $u_p \times u_p$. Consider also

its adjoint

$$u'_p \times u'_p := \sum_{x_1, x_2 \bmod p} n_1^{\text{opp}}(x_1, 0, x_2) \begin{bmatrix} p^{-1} & 0 & 0 \\ 0 & I_3 & 0 \\ 0 & 0 & p \end{bmatrix}.$$

3.4.10.3 Remark. Except for t_ξ^{dep} and $t_\xi^{\text{DEP},a}$, we will allow ourselves to view the operators defined above as acting also on smooth functions on $\mathbf{H}(\mathbb{Q}_p)$. The interaction of these operators is studied in the appendix, see in particular Section (B.4.2).

3.4.10.4 Decomposing the projection at p

In this section we spell out the decomposition of a trace map. This decomposition will be used in many instances to prove the norm relations at p . We recall the relevant notation for level subgroups from Section 3.1.5.

We consider the Euler System map $\mathcal{ES}_{F, K^{\text{vol}}(M)K_1^{\mathbf{P}}(p^a)}$. For simplicity, as F is fixed, we will omit from the notations the level structure away from p , and denote simply $\mathcal{ES}_{K_1^{\mathbf{P}}(p^a)}$, or by Proposition 3.4.6.10, $\mathcal{ES}_{K_1(p^a)}$. Define

$$K_1^{\mathbf{P}}(p^a) := K_1^{\mathbf{P}}(p^{a-1}) \cap K_0^{\mathbf{P}}(p^a).$$

Denote by $K'_1(p^a)$ its pull-back to $\mathbf{H}$ i.e.,

$$K'_1(p^a) := K_1^{\mathbf{P}}(p^a) \cap \mathbf{H}(\mathbb{Z}_p).$$

We now want to decompose in two pieces the trace map $\mathrm{Tr}_{K_1(p^{a-1})/K_1(p^a)}$. Let $D_a = (1 + p^a \mathbb{Z}_p)/(1 + p^{a-1} \mathbb{Z}_p)$ seen as a quotient the group of diamond operators $\langle d \rangle$ for $d \in \mathbb{Z}_p^\times$ as in Definition 3.4.10.2. We define

$$\mathrm{proj}_{\mathrm{Neb}}^a = \sum_{d \in D_a} \langle d \rangle.$$

Then, $\mathrm{proj}_{\mathrm{Neb}}^a$ defines a map:

$$\mathrm{proj}_{\mathrm{Neb}}^a : H_{\mathrm{et}}^2(\mathrm{SH}_{\mathbf{H}}(K_1(p^a), \mathcal{O}_{\chi,p}(3))) \rightarrow H_{\mathrm{et}}^2(\mathrm{SH}_{\mathbf{H}}(K_1'(p^a), \mathcal{O}_{\chi,p}(3))).$$

In what follows, if a is clear from the context we will denote $\mathrm{proj}_{\mathrm{Neb}}^a$ by $\mathrm{proj}_{\mathrm{Neb}}$.

We then decompose

$$\mathrm{Tr}_{K_1(p^{a-1}, K_1(p^a))} = \mathrm{proj}_{\mathrm{Neb}}^a \circ \mathrm{Tr}_{K_1(p^{a-1}), K_1'(p^a)} = \mathrm{Tr}_{K_1(p^{a-1}), K_1'(p^a)} \circ \mathrm{proj}_{\mathrm{Neb}}^a.$$

Explicitly $\mathrm{Tr}_{K_1(p^{a-1}), K_1'(p^a)}$ is defined as

$$\mathrm{Tr}_{K_1(p^{a-1}), K_1'(p^a)} = \sum_{x_1, x_2 \bmod p} n_1^{\mathrm{opp}}(p^{a-1}x_1, 0, p^{a-1}x_2).$$

3.4.10.5 The proof of the vertical norm relations

In this section, we assume that F is nearly ordinary and fixed by $K_1(p^r)$ for $r \geq 0$.

Write the level of F as $K = K_p K^p$, such that $K_p \subseteq K_1(p^r)$ for a minimal r ; and if $F = f \otimes \bar{f}$, we write $\psi_{f,p}$ for the Nebentypus at p of f . We fix once and for all an

odd branch character:

$$\chi = \chi_0 : (\mathbb{Z}/p\mathbb{Z})^\times \rightarrow \mathbb{C}^\times.$$

For example, to fix ideas the reader can assume χ_0 is the Teichmüller character. In particular $\mathbb{F}_\chi = \mathbb{Q}(\mu_p)$ and $F_{\chi,p} = \mathbb{Q}_p$. As we have possibly enlarged $F_{\chi,p}$ to contain the field of definition of the Fourier coefficients of f , we keep the notations of the previous sections where χ was arbitrary. Also, so that, χ away from p , can be chosen arbitrarily. We will denote χ by χ_0 when we want to stress that we are considering only the p -part of χ .

There are p extensions of χ_0 to a Dirichlet character of $(\mathbb{Z}/p^2\mathbb{Z})^\times$; these are naturally indexed by characters of $\mathbb{Z}/p\mathbb{Z}$. If ξ is a character of $\mathbb{Z}/p\mathbb{Z}$ denote by similarity with the horizontal norm relations $\chi_0\xi$ the extension of χ to $(\mathbb{Z}/p^2\mathbb{Z})^\times$ indexed by ξ . Similarly, for $a \geq 0$, if ξ is a character of $\mathbb{Z}/p^a\mathbb{Z}$, the extension of χ to $(\mathbb{Z}/p^{1+a}\mathbb{Z})^\times$ that is indexed by ξ will be denoted, slightly abusively, by $\chi_0\xi$.

We will often refer to lemmas in the appendix, in particular sections (B.3) and (B.4).

Since $\text{proj}_{\text{Neb}}^a$ commutes with both tw^a (for trivial central character) and $u_p \times u_p$, by proposition (B.5.3) of the Appendix (see also equation 3.7 and 3.6), we have the following Lemma.

3.4.10.6 Lemma. *We have the following relation*

$$\mathcal{ES}_{F,K_1(p^a)K^p} \circ \text{tw}^a = \mathcal{ES}_{F,K_1(p^{a-1})K^p} \circ \text{tw}^{a-1} \circ (u_p \times u_p) \circ \text{proj}_{\text{Neb}}^a.$$

We now discuss a projection to nearly ordinary cohomology. Let u'_p be the *adjoint*

of the usual u_p operator acting on the cohomology of modular curves. By the Künneth decomposition $u'_p \times u'_p$ acts on the cohomology of $\mathrm{SH}_{\mathbf{H}'}$ and thus, if we let $e' = \lim_{n \rightarrow \infty} (u'_p)^{n!}$, $e' \times e'$ acts on the cohomology of $\mathrm{SH}_{\mathbf{H}}$. We will let

$$\mathrm{H}_{ord}^*(\mathrm{SH}_{\mathbf{H}}(K), \mathcal{O}_{\chi,p}(3)) := (e' \times e') \mathrm{H}_{\mathrm{et}}^*(\mathrm{SH}_{\mathbf{H}}(K), \mathcal{O}_{\chi,p}(3)).$$

On H_{ord}^* , $u'_p \times u'_p$ acts invertibly and there is a well defined operator $(u'_p \times u'_p)^{-1}$. This can be defined as $\lim_{n \rightarrow \infty} (u'_p \times u'_p)^{n!-1}$ as $e' \times e'$ is an idempotent.

We now define the ordinary Euler–System map for level $\tilde{K}_1(p^a)$

3.4.10.7 Definition (The ordinary Euler–System map). We set

$$\mathcal{ES}_{F,K^p\tilde{K}_1(p^a)}^{ord} = (u'_p \times u'_p)^{-a} (e' \times e') \mathcal{ES}_{F,K^p\tilde{K}_1(p^a)}.$$

We will need the following Lemma in what follows: it is a modified version of Lemma 3.4.10.6.

3.4.10.8 Lemma. *We have the following relation*

$$\mathcal{ES}_{F,K^p\tilde{K}_1(p^a)}^{ord} \circ \mathrm{tw}^{2a} = \mathcal{ES}_{F,K^p\tilde{K}_1(p^{a-1})}^{ord} \circ \mathrm{tw}^{2a-2} \circ (u_p \times u_p) \circ \mathrm{proj}_{\mathrm{Neb}}^a.$$

Proof. Indeed we have

$$\mathcal{ES}_{F,K^p\tilde{K}_1(p^a)} \circ \mathrm{tw}^{2a} = \mathcal{ES}_{F,K^p\tilde{K}_1(p^{a-1})} \circ \mathrm{tw}^{2a-2} \circ (u_p \times u_p)^2 \circ \mathrm{proj}_{\mathrm{Neb}}^a.$$

Using the fact that $\mathrm{tw}^{2a-2} \circ (u_p \times u_p) = (u'_p \times u'_p) \circ \mathrm{tw}^{2a-2}$ we can conclude. $\square$

We are now ready to prove the depleted vertical norm relations. Recall the operator $t_{\chi\xi}^{DEP,a}$ defined in section (B.3). For all $b \leq a$ and all $\chi_0\xi'$ of log-conductor $\leq a$, it induces a map in cohomology still denoted by $t_{\chi_0\xi'}^{DEP,a}$:

$$t_{\chi_0\xi'}^{DEP,a} : H_{\text{et}}^* \left(\text{SH}_{\mathbf{G}}(K^p K_1^{\mathbf{P}}(p^b)), \mathcal{O}_{\chi,p}(3) \right) \longrightarrow H_{\text{et}}^* \left(\text{SH}_{\mathbf{G}}(K^p \tilde{K}_1^{\mathbf{P}}(p^a)), \mathcal{O}_{\chi,p}(3) \right).$$

We now consider the class

$$\varsigma_{p^a M}^{dep,\chi} := \frac{1}{p^{2a} M^*} \sum_{\xi', \xi} \sum_{\psi} \omega(t_{\chi_0\xi\xi'}^{DEP,a} f_{\chi_0\xi\psi;\varphi^M}) \in H_{\text{et}}^3 \left(\text{SH}_{\mathbf{G}}(K(M) \tilde{K}_1^{\mathbf{P}}(p^a)), F_{\chi,p}(3) \right).$$

Here ψ ranges over even Dirichlet characters of conductor M , ξ is such that $\chi_0\xi$ ranges over all extensions of χ_0 modulo p^a and similarly ξ' .

By interchanging the order of summation this class is equal to

$$\sum_{x \bmod p^a} \frac{1}{p^{2a} M^*} \sum_{\psi, \xi, \xi'} \chi_0\xi'\xi(x) n_1(-x/p^a)^* \omega(f_{\chi_0\xi\psi;\varphi^M}),$$

with notations as in Section (B.3). Note now that the coefficients of the q -expansions at *all cusps* of $\chi_0\xi'\xi(x) n_1(-x/p^a)^* \omega(f_{\chi_0\xi\psi;\varphi^M})$ are universal polynomials, in ξ and ξ' (depending on a only) as this is true for $\omega(f_{\chi_0\xi\psi;\varphi^M})$ by Faltings' Lemma. We deduce that the inner sum defines an integral element in cohomology and therefore $\varsigma_{p^a M}^{dep,\chi}$, by the same reasoning as in the universal and horizontal norm relations, defines an element

$$\varsigma_{p^a M}^{\text{et},dep,\chi} \in H_{\text{et}}^3(\text{SH}_{\mathbf{G}}(\tilde{K}_1^{\mathbf{P}}(p^a) K^{\text{vol}}(M)), \mathcal{O}_{\chi,p}(3)).$$

We now set

$$\kappa_{p^a M}^{dep, \chi} := \mathcal{ES}_{F, \tilde{K}_1(p^a)}^{ord}(\text{tw}^{2a} \zeta_{p^a M}^{\text{et}, dep, \chi}).$$

We now wish to compute $\text{Cores}_{\mathbb{F}_\chi(\mu_{Mp^{a-1}})^+}^{\mathbb{F}_\chi(\mu_{p^a M})^+}(\kappa_{p^a M}^{\chi, dep})$. We introduce the following notation to make the computation more legible: we write for a character $\xi \leq a$ if the log-conductor of ξ at p is $\leq a$. We have:

$$\begin{aligned} \text{Cores}_{\mathbb{F}_\chi(\mu_{Mp^{a-1}})^+}^{\mathbb{F}_\chi(\mu_{p^a M})^+}(\kappa_{p^a M}^{\chi, dep}) &= \mathcal{ES}_{F, K(M)\tilde{K}_1(p^a)}\left(\frac{1}{p^{2a-1}M^*}, \sum_{\xi' \leq a} \sum_{\xi \leq a-1} \sum_{\psi} \text{tw}^{2a} t_{\chi \xi' \xi}^{DEP, a} \omega^{\text{et}}(f_{\chi_0 \xi \psi; \varphi^M})\right) \\ &= \mathcal{ES}_{F, K(M)\tilde{K}_1(p^{a-1})}\left(\frac{1}{p^{2(a-1)}M^*}, \sum_{\xi, \xi' \leq a-1} \sum_{\psi} \text{tw}^{2a-2} (u_p \times u_p) t_{\chi \xi' \xi}^{DEP, a} \omega^{\text{et}}(f_{\chi_0 \xi \psi; \varphi^M})\right) \\ &= \mathcal{ES}_{F, K(M)\tilde{K}_1(p^{a-1})}\left(\frac{1}{p^{2(a-1)}M^*}, \sum_{\xi, \xi' \leq a-1} \sum_{\psi} \text{tw}^{2a-2} t_{\chi \xi' \xi}^{DEP, a-1} \omega^{\text{et}}(f_{\chi_0 \xi \psi; \varphi^M})\right) \\ &= \kappa_{p^{a-1}M}^{\chi, dep}. \end{aligned}$$

Here we have used Lemma 3.4.10.6 and Proposition (B.6.2) of the Appendix for the second equality. The third equality is a consequence of proposition (B.4.3) of the Appendix.

We have therefore shown the following proposition.

3.4.10.9 Proposition (Depleted vertical norm relation). *The classes $\kappa_{p^a M}^{\chi, dep} \in H^1(\mathbb{F}_\chi(\mu_{p^a M})^+, \mathcal{T}_\Pi(1))$ defined above verify the relation:*

$$\text{Cores}_{\mathbb{F}_{\chi_0}(\mu_{Mp^{a-1}})}^{\mathbb{F}_{\chi_0}(\mu_{p^a M})}(\kappa_{p^a M}^{\chi, dep}) = \kappa_{p^{a-1}M}^{\chi, dep},$$

for all $a \geq 1$.

3.4.11 The full Euler System

We now give theorems providing a summary of the constructions of the previous sections.

3.4.11.1 Theorem. *There exists a collection of classes $(c_{Mp^a}^\chi)$ indexed by positive integers M and a , with M squarefree and coprime to the primes of S , such that*

$$c_{Mp^a}^\chi \in H^1(\mathbb{F}_\chi(\boldsymbol{\mu}_{p^a M})^+, \mathcal{T}_\Pi(1)).$$

Moreover these classes verify: for any prime $\ell \notin S$

$$\text{Cores}_{\mathbb{F}_\chi(\boldsymbol{\mu}_{Mp^a})^+}^{\mathbb{F}_\chi(\boldsymbol{\mu}_{M\ell p^a})^+}(c_{M\ell p^a}^\chi) = P_\ell(\text{Fr}_\ell^{-1})c_{Mp^a}^\chi$$

and for all $a \geq 0$

$$\text{Cores}_{\mathbb{F}_\chi(\boldsymbol{\mu}_{Mp^a})^+}^{\mathbb{F}_\chi(\boldsymbol{\mu}_{Mp^{a+1}})^+}(c_{Mp^a}^\chi) = c_{Mp^a}^\chi.$$

This collection of classes can be taken to be the universal ones i.e., $c_{Mp^a}^\chi = \kappa_{Mp^a}^\chi$ constructed in section 3.4.10.5 or the depleted ones, i.e. $c_{Mp^a}^\chi = \kappa_{Mp^a}^{\chi, \text{dep}}$ constructed in section 3.4.10.5, when Π_f is assumed to be $K_1(p^r)$ fixed and ordinary at p .

And as usual by the inflation restriction-inflation exact sequence, we obtain:

3.4.11.2 Corollary. *There exists a collection of classes $(\mathcal{Z}_{Mp^a}^\chi)$ indexed by positive integers M and a with M squarefree and coprime to the primes of S , such that*

$$\mathcal{Z}_{Mp^a}^\chi \in H^1(\mathbb{Q}(\boldsymbol{\mu}_{p^a M})^+, \mathcal{T}_\Pi(1) \otimes \chi).$$

Moreover these classes verify: for any prime $\ell \notin S$

$$\text{Cores}_{\mathbb{Q}(\mu_{Mp^a})^+}^{\mathbb{Q}(\mu_{M\ell p^a})^+}(\mathcal{Z}_{M\ell p^a}^\chi) = P_\ell^\chi(\text{Fr}_\ell^{-1})\mathcal{Z}_{Mp^a}^\chi$$

and for all $a \geq 0$

$$\text{Cores}_{\mathbb{Q}(\mu_{Mp^a})^+}^{\mathbb{Q}(\mu_{Mp^{a+1}})^+}(\mathcal{Z}_{Mp^{a+1}}^\chi) = \mathcal{Z}_{Mp^a}^\chi.$$

3.4.11.3 Remark. The theorem above implies that we have constructed an Euler system for the Galois representation $V := \text{Sym}^2 \rho_f(1) \otimes \chi$. This is predicted by the Bloch-Kato conjectures for the following reason. Note that $V^\vee(1) = \text{Sym}^2 V_f \otimes \chi^{-1}$, and the L -function of $V^\vee(1)$ vanishes at order at least one at $s = 0$. Indeed, $L(V^\vee(1), s) = D(s) = D(s, f, \chi)$ in the notations of Shimura (see the introduction of [Shi75]) ; if N is the level of f , $\nu = 0, 1$ depending on χ and the nebentypus of f , and if we set

$$R(s) = N^{s/2} \pi^{-3s/2} \Gamma(s/2) \Gamma((s - \nu)/2) \Gamma((s + 1)/2) D(s),$$

then $R(s)$ verifies the functional equation $R(s) = R(3 - s)$. As Γ has a simple pole at $s = 0$, we deduce that the L -function does indeed vanish to order at least 1 at $s = 0$. More precisely, when f is a weight 2 modular form, then the nebentypus of f is even, and when χ is odd, we have $\nu = 1$: therefore the order of vanishing is exactly one. This is precisely the situation where one expects an Euler System to exist - this is the Euler System that we have constructed in this paper.

Since the classes that we constructed are universal norms at p , by Rubin-Soulé's

character twisting theory (see [Rub00, chapter VI], we can twist our Euler system by even characters of the Iwasawa algebra. Since χ is odd we can twist in particular by $\chi\varepsilon^{-1}$ and we obtain the following theorem.

3.4.11.4 Theorem. *There exists a collection of classes $(\mathcal{C}_{Mp^a})$ indexed by positive integers M and a with M squarefree and coprime to the primes of S , such that*

$$\mathcal{C}_{Mp^a} \in H^1(\mathbb{Q}(\mu_{p^a M})^+, \text{Ad}^0(T_f)(1)).$$

Here the lattice $\text{Ad}^0(T_f)$ is the trace 0 subspace of $\text{Ad}(T_f)$ as defined in Section 3.4.5.4.

Moreover these classes verify: for any prime $\ell \notin S$

$$\text{Cores}_{\mathbb{Q}(\mu_{Mp^a})^+}^{\mathbb{Q}(\mu_{M\ell p^a})^+}(\mathcal{C}_{M\ell p^a}) = P_\ell(\text{Fr}_\ell^{-1})\mathcal{C}_{Mp^a}$$

and for all $a \geq 0$

$$\text{Cores}_{\mathbb{Q}(\mu_{Mp^a})^+}^{\mathbb{Q}(\mu_{Mp^{a+1}})^+}(\mathcal{C}_{Mp^{a+1}}) = \mathcal{C}_{Mp^a}.$$

3.5 Euler System for Hida families

In this section we upgrade the construction of the previous section to Hida families.

3.5.1 Hida families: background

For a Hida family $\mathbf{f}$, we denote by $\Lambda_{\mathbf{f}}$ the localization at the prime ideal corresponding to $\mathbf{f}$ of Hida's ordinary Hecke algebra (denoted in [KLZ17] by $\mathbf{T}_{Np^\infty}$). This is a

module over the Iwasawa algebra denoted by $\Lambda_D := \mathbb{Z}_p[[\mathbb{Z}_p^\times]]$ in *loc cit.* (D for **diamond** operators). We say that the Hida family $\mathbf{f}$ is non-Eisenstein when the residual Galois representation attached to $\mathbf{f}$ valued in $\mathbf{GL}_2(\Lambda_{\mathbf{f}}/\mathfrak{m}_{\mathbf{f}})$ (here $\mathfrak{m}_{\mathbf{f}}$ is the maximal ideal of $\Lambda_{\mathbf{f}}$) is irreducible.

The algebra $\Lambda_{\mathbf{f}}$ has finitely many minimal primes and we call these the branches of the Hida family $\mathbf{f}$. These minimal primes biject with the simple direct summands of the ring $\Lambda_{\mathbf{f}} \otimes_{\Lambda_D} \text{Frac}(\Lambda_D)$. If $\mathbf{a}$ is a branch of the Hida family $\mathbf{f}$, then the quotient $\Lambda_{\mathbf{f}}/\mathbf{a}$ is an integral domain as $\mathbf{a}$ is a prime ideal and its field of fractions is a finite extensions of $\mathbb{Z}_p[[1 + p\mathbb{Z}_p]]$. We denote by $\Lambda_{\mathbf{a}}$ be the normalization of $\Lambda_{\mathbf{f}}/\mathbf{a}$ in its field of fractions – this is a normal integral domain that is finite and projective as a module over the Iwasawa algebra Λ_D . We denote by $\Lambda_{\mathbf{a}}^+$ the quotient of $\Lambda_{\mathbf{a}}$ by ± 1 . We say that the branch $\mathbf{a}$ is cuspidal if for every arithmetic prime of $\Lambda_{\mathbf{f}}$ above $\mathbf{a}$ corresponds to a cuspidal modular form and it is new if furthermore (one or all) the cuspidal specializations are new. If the Hida family $\mathbf{f}$ is non-Eisenstein then all branches are automatically cuspidal. We denote by

$$\Psi_D : \text{Gal}_{\mathbb{Q}_p} \rightarrow \mathbb{Z}_p^\times \hookrightarrow \Lambda_D^\times.$$

be the universal character; this is the character denoted by $\mathbf{k}$ in [KLZ17].

Recall that we have denoted by $T_{\mathbf{f}}$ the lattice in $\rho_{\mathbf{f}}$ that is denoted by $M(\mathbf{f})^*$ in *loc cit.* There is a two-step filtration on $T_{\mathbf{f}}$:

$$T_{\mathbf{f}}^+ \subset T_{\mathbf{f}}.$$

If $\mathbf{f}$ is non-Eisenstein and if furthermore p -distinguished then $\Lambda_{\mathbf{f}}$ is a Gorenstein ring and the Galois modules $T_{\mathbf{f}}$, $T_{\mathbf{f}}^+$ and $T_{\mathbf{f}}^-$ are all free over $\Lambda_{\mathbf{f}}$. The action of $\text{Gal}_{\mathbb{Q}_p}$ on the quotient $T_{\mathbf{f}}^- := T_{\mathbf{f}}/T_{\mathbf{f}}^+$ is by the unramified character $\delta_{\mathbf{f}}$ given associated to the unit root $\alpha_{\mathbf{f}}$ i.e., if $U_p \mathbf{f} = \alpha_{\mathbf{f}} \mathbf{f}$ then $\delta_{\mathbf{f}}(\text{Frob}_p) = \alpha_{\mathbf{f}}$. On $T_{\mathbf{f}}^+$ it is given by the character $\Xi_{\mathbf{f}} := \varepsilon \Psi_D \delta_{\mathbf{f}}^{-1} \psi_{\mathbf{f}}^{(p)}$, where $\psi_{\mathbf{f}}^{(p)}$ denotes the unramified character of $\text{Gal}_{\mathbb{Q}_p}$ giving the eigenvalue of the tame diamond operators $\langle d \rangle$ on $\mathbf{f}$. For simplicity we will assume that $\psi_{\mathbf{f}}^{(p)} = 1$ – that is, we assume that the tame level is $\Gamma_0(N)$.

The two-step filtration on $T_{\mathbf{f}}$ induces a three-step filtration on $\text{Ad}^0(T_{\mathbf{f}}) := \text{End}_{\Lambda_D}(T_{\mathbf{f}}, T_{\mathbf{f}})$:

$$\text{Fil}^1 \text{Ad}^0(T_{\mathbf{f}}) \subset \text{Fil}^0 \text{Ad}^0(T_{\mathbf{f}}) \subset \text{Ad}^0(T_{\mathbf{f}}).$$

Seeing elements of $\text{Ad}^0(T_{\mathbf{f}})$ as endomorphisms $\phi : T_{\mathbf{f}} \rightarrow T_{\mathbf{f}}$, the elements of $\text{Fil}^1 \text{Ad}^0(T_{\mathbf{f}})$ correspond to endomorphisms whose kernel contains $T_{\mathbf{f}}^+$ and elements of $\text{Fil}^0 \text{Ad}^0(T_{\mathbf{f}})$ correspond to endomorphisms that stabilize the line $T_{\mathbf{f}}^+$. We put

$$\text{Ad}^0(T_{\mathbf{f}})^+ := \text{Fil}^1 \text{Ad}^0(T_{\mathbf{f}}) \quad \text{and} \quad \text{Ad}^0(T_{\mathbf{f}})^- := \text{Ad}^0(T_{\mathbf{f}}) / \text{Fil}^0 \text{Ad}^0(T_{\mathbf{f}}).$$

Note that $\text{Ad}^0(T_{\mathbf{f}})^+$ is a one dimensional subspace on which $\text{Gal}_{\mathbb{Q}_p}$ acts by the character $\Xi_{\mathbf{f}} \delta_{\mathbf{f}}^{-1}$; $\text{Ad}^0(T_{\mathbf{f}})^-$ is a one-dimensional quotient of $\text{Ad}^0(T_{\mathbf{f}})$ on which $\text{Gal}_{\mathbb{Q}_p}$ acts by $\Xi_{\mathbf{f}}^{-1} \delta_{\mathbf{f}}$. We consider similarly:

$$\text{Sym}^2(T_{\mathbf{f}}) := \text{Ad}^0(T_{\mathbf{f}})(\varepsilon \Psi_D \psi_{\mathbf{f}}^{(p)}).$$

By the first paragraph of section 3.4.5.4, the specialization of $\text{Sym}^2(T_{\mathbf{f}})$ at a weight 2

point corresponding to a weight 2 eigenform g is $\mathrm{Sym}^2(T_g) = T_g \otimes \mathcal{T}_g$. The filtration $\mathrm{Fil}^\bullet$ on $\mathrm{Ad}^0(T_{\mathbf{f}})$ induces a filtration on $\mathrm{Sym}^2(T_{\mathbf{f}})$ such that the quotient

$$\mathrm{Sym}^2(T_{\mathbf{f}})^- = \frac{\mathrm{Sym}^2(T_{\mathbf{f}})}{\mathrm{Fil}^0 \mathrm{Sym}^2(T_{\mathbf{f}})}$$

is unramified acted on by the character $\delta_{\mathbf{f}}^2$.

For a specialization f of weight k of $\mathbf{f}$, we set

$$\alpha_f := \delta_f(p), \quad \text{and} \quad \beta_f := p^{k-1} \alpha_f^{-1} \psi_f^{(p)}(p).$$

Recall that $\psi_f^{(p)}$ is the prime-to- p part of the Nebentypus of f ; we denote the p -part of the Nebentypus of f by $\psi_{f,p}$.

3.5.2 Lattices in Hida families

Let $R \subset \overline{\mathbb{Q}_p}$ be a finite extension of $\mathbb{Z}_p$ (for example, take $R = \mathcal{O}_{\chi,p}$). Let $k \geq 2$ be an integer and $\mathbf{T}_{Np^r}^{\mathrm{ord}}(\psi; R)$ be the Hecke R -subalgebra generated by the usual operators T_ℓ for ℓ a prime coprime to Np and the diamonds operators acting on the space of nearly ordinary modular forms of weight k , tame level N , and Neben-character ψ with coefficients in R . Suppose f is a p stabilized eigenform corresponding to a algebra homomorphism $\phi_f : \mathbf{T}_{Np^r}^{\mathrm{ord}}(\psi; R) \rightarrow R$; either f is new or $r = 1$ and f is old at p but new at N . Let L be the field of fractions of R . Then the homomorphism ϕ_f affords a splitting:

$$\mathbf{T}_{Np^r}^{\mathrm{ord}}(\psi; R) \simeq \mathbf{T}' \times L.$$

We let $\mathbb{1}_f \in \mathbf{T}_{Np^r}^{ord}(\psi; R) \otimes_R L$ be the idempotent of the Hecke-algebra that projects $\mathbf{T}_{Np^r}^{ord}(\psi; R)$ onto the second factor in the above splitting. Recall that we denote by e Hida's ordinary projector associated to the u_p operator. If g is any other modular form of weight k , level Np^t for some $t \geq r$ and character ψ with coefficients in L , then

$$\mathbb{1}_f e g = c f$$

for some $c \in L$.

Let $\mathfrak{m}_f$ be the maximal ideal of $\mathbf{T}_{Np^r}^{ord}(\psi; R)$ associated with f . If R is integrally closed in L or $\mathbf{T}_{Np^r}^{ord}(\psi; R)_{\mathfrak{m}_f}$ is a Gorenstein R -algebra, then the intersection $\mathbf{T}_{Np^r}^{ord}(\psi; R) \cap (0 \times L)$ inside $\mathbf{T}_{Np^r}^{ord}(\psi; R) \otimes_R L$ is a free R -module of rank one. In this case, we let ℓ_f be an R -generator – this is well defined up to a unit of R . Then $\ell_f = c_f \mathbb{1}_f$ for some $c_f \in R$. If the residual Galois representation valued in $\mathbf{GL}_2(R/\mathfrak{m}_f)$ is irreducible then c_f is just a generator of the congruence ideal $C(f)$ of f .

We now set up the notations in Hida families. Let $\mathbb{I}$ be a domain that is a finite integral extension of $\mathbb{Z}_p[[W]] \otimes \mathbb{Z}_p[\psi]$, where ψ is a Dirichlet character modulo Np and $\mathbb{Z}_p[\psi]$ denotes the integral closure of $\mathbb{Z}_p$ in the finite extension of $\mathbb{Q}_p$ obtained by adjoining the values of ψ . We define $M(N, \psi; \mathbb{I})$ and $M^{ord}(N, \psi; \mathbb{I})$ to be the $\mathbb{I}$ -modules of $\mathbb{I}$ -adic modular forms and its ordinary quotient. We let $\mathbf{T}_{Np^\infty}^{ord}(N, \psi; \mathbb{I})$ be the $\mathbb{I}$ -subalgebra of $\mathbb{I}$ -linear endomorphisms of $M^{ord}(N, \psi; \mathbb{I})$ generated by the usual Hecke operators T_ℓ for primes ℓ away from Np .

Let $\mathbf{f} \in M^{ord}(N, \psi; \mathbb{I})$ be an ordinary cuspidal newform and denote by $L_{\mathbb{I}}$ the

fraction field of $\mathbb{I}$. Then we have a splitting

$$\mathbf{T}_{Np^\infty}^{\text{ord}}(\psi; \mathbb{I}) \otimes L_{\mathbb{I}} \simeq \mathbf{T}' \times L_{\mathbb{I}},$$

where, as before, the projection onto the second factor is the projection of a Hecke operator to its $\mathbf{f}$ -eigenvalue. We let $\mathbf{1}_{\mathbf{f}}$ be the idempotent corresponding to the second factors. Then, as before, for some $c \in L_{\mathbb{I}}$, and for any $\mathbf{g} \in M(N, \psi; \mathbb{I})$ we have

$$\mathbf{1}_{\mathbf{f}} e \mathbf{g} = c \mathbf{f}.$$

Let $\mathbf{m}_{\mathbf{f}}$ be the maximal ideal of $\mathbf{T}_{Np^\infty}^{\text{ord}}(\psi; \mathbb{I})$ associated with $\mathbf{f}$. If all specializations of the Hida family $\mathbf{f}$ is p -distinguished and residually irreducible, then $\mathbf{T}_{Np^r}^{\text{ord}}(\psi; R)_{\mathbf{m}_{\mathbf{f}}}$ is a Gorenstein $\mathbb{I}$ -algebra. And then, as before, $\mathbf{T}_{Np^r}^{\text{ord}}(\psi; R) \cap (L_{\mathbb{I}})$ is a rank one $\mathbb{I}$ -module – and we let $\ell_{\mathbf{f}}$ be a generator so that

$$\ell_{\mathbf{f}} = c_{\mathbf{f}} \mathbf{1}_{\mathbf{f}},$$

for some $c_{\mathbf{f}} \in \mathbb{I}$. Moreover for any specialization f of $\mathbf{f}$, $c_{\mathbf{f}}$ specializes to a generator of the congruence ideal of f .

For an integer N coprime to p , we introduce the notation:

$$H_{\text{ord}}^2(Np^\infty) = (e' \times e') \varprojlim_{r \geq 1} H_{\text{et}}^2(\text{SH}_{\mathbf{H}}(K_1(Np^r)), \mathcal{O}_{\chi, p}(2)) \simeq e' \varprojlim_{r \geq 1} H_{\text{et}}^1(Y_1(Np^r), \mathcal{O}_{\chi, p}(1)) \otimes e' \varprojlim_{r \geq 1} H_{\text{et}}^1(Y_1(Np^r), \mathcal{O}_{\chi, p}(1))$$

here the inverse limit is taken with respect to trace maps.

The $\mathbb{T}_{Np^\infty}^{\text{ord}}$ -module $H_{\text{ord}}^1(Np^\infty) := e' \varprojlim_{r \geq 1} H_{\text{et}}^1(Y_1(Np^r), \mathcal{O}_{\chi,p}(1))$ has been studied extensively by Ohta [Oht95]. If $\mathfrak{m}$ is a residually irreducible maximal ideal $\mathbb{T}_{Np^\infty}^{\text{ord}}$ and $\overline{\mathfrak{m}}$ its dual, by Theorem 4.2.5 and Corollary 4.2.8 of *loc cit.*, there is a perfect Λ_D -linear pairing on $H_{\text{ord}}^1(Np^\infty)_{\overline{\mathfrak{m}}}$, which interpolates the Poincaré pairing on $X_1(Np^r)$. Let $\mathfrak{M} = \mathfrak{m} \otimes \overline{\mathfrak{m}}$ be the maximal ideal of $\mathbb{T}_{Np^\infty}^{\text{ord}} \times (\mathbb{T}_{Np^\infty}^{\text{ord}})^*$ corresponding to f as defined in Paragraph ?? (the superscript $*$ indicates duals). By the existence of Ohta's perfect pairing, we have a canonical isomorphism:

$$H_{\text{ord}}^2(Np^\infty)_{\mathfrak{M}} \simeq \text{End}_{\Lambda_D^*}(H_{\text{ord}}^1(Np^\infty)_{\overline{\mathfrak{m}}}).$$

We consider the submodule:

$$\text{Ad}^0(H_{\text{ord}}^1(Np^\infty)_{\overline{\mathfrak{m}}}) := \text{End}_{(\mathbb{T}_{Np^\infty}^{\text{ord}})^*}(H_{\text{ord}}^1(Np^\infty)_{\overline{\mathfrak{m}}})^0 \subset H_{\text{ord}}^2(Np^\infty)_{\mathfrak{M}}(-1).$$

The above analysis leads to the following proposition.

3.5.2.1 Proposition. *The element $\ell_{\mathbf{f}}^* \in (\mathbb{T}_{Np^\infty}^{\text{ord}})^*$ induces a Galois-equivariant integral projection*

$$\text{pr}_{\mathbf{f}} : \text{Ad}^0(H_{\text{ord}}^1(Np^\infty)_{\overline{\mathfrak{m}}}) \twoheadrightarrow \text{Ad}^0(T_{\mathbf{f}}),$$

here $\text{Ad}^0(T_{\mathbf{f}})$ denotes the $\Lambda_{\mathbf{f}}$ -linear traceless endomorphism of $T_{\mathbf{f}}$.

3.5.3 The construction

In this section we upgrade the preceding construction to Hida families.

Recall that we have fixed an odd branch character:

$$\chi = \chi_0 : (\mathbb{Z}/p\mathbb{Z})^\times \rightarrow \mathbb{C}^\times.$$

Let $\mathbf{f}$ be a Hida family passing through f of tame level N . Again, to fix ideas and for applications χ_0 will be taken to be the inverse of the Teichmüller character.

For ξ and ψ as in the previous section consider for $b \geq a$ the class:

$$\varsigma_{p^a M}^{dep, b} := \frac{1}{p^{b+a} M^*} \sum_{\xi \leq a} \sum_{\xi' \leq b} \omega(\mathrm{tw}^{2b} t_{\chi_0 \xi \xi'}^{DEP, b} f_{\chi_0 \xi \psi; \varphi^M}) \in H_{\mathrm{et}}^3 \left(\mathrm{SH}_{\mathbf{G}}(\tilde{K}_1^{\mathbf{P}}(p^b) K^{\mathbf{M}}(M)), \mathcal{O}_{\chi, p}(3) \right)$$

Then, this class defines an extension:

$$\kappa_{p^a M}^{dep, b} = \mathcal{E} \mathcal{S}_{K(M) \tilde{K}_1(p^b)}^{ord} (\varsigma_{p^a M}^{dep, b}) \in H^1 \left(\mathbb{F}_\chi(\boldsymbol{\mu}_{p^a M})^+, H_{ord}^2(\mathrm{SH}_{\mathbf{H}}(K(N) K_1(p^b)), \mathcal{O}_{\chi, p}(3)) \right).$$

For $b \leq a$, define

$$\kappa_{p^a M}^{dep, b} := (d_{K_1(p^b), K_1(p^a)})_* \kappa_{p^a M}^{dep, a} \in H_{ord}^2(\mathrm{SH}_{\mathbf{H}}(K(N) K_1(p^b)), \mathcal{O}_{\chi, p}(3)).$$

Then an identical computation to the vertical norm relation of Section 3.4.10.5, we obtain that the classes $\kappa_{p^a M}^{dep, b}$ are compatible under pushforward along $(d_{K_1(p^{b-1}), K_1(p^b)})_*$. In particular they define a class :

$$\underline{\kappa}_{p^a M}^{ord} \in H^1\left(\mathbb{F}_{\chi_0}(\boldsymbol{\mu}_{p^a M})^+, \varprojlim_b H_{ord}^2(\mathrm{SH}_{\mathbf{H}}(K(N)K_1(p^b)), \mathcal{O}_{\chi, p}(3))\right).$$

Here the inverse limit is taken with respect to the pushforward maps $(d_{K_1(p^{b-1}), K_1(p^b)})_*$.

Now, by the same argument as in the finite level case, $\underline{\kappa}_{p^a M}^{ord}$ after localization at $\mathfrak{M}$, belongs to the sub-lattice $\mathrm{Ad}^0(H_{ord}^1(Np^\infty)_{\overline{\mathfrak{m}}})$, we can then use the projection afforded by Proposition 3.5.2.1 and set

$$\underline{\kappa}_{p^a M}^{ord}(\mathbf{f}) := \mathrm{pr}_{\mathbf{f}} \mathrm{loc}_{\mathfrak{M}} \underline{\kappa}_{p^a M}^{ord} \in H^1\left(\mathbb{F}_{\chi_0}(\boldsymbol{\mu}_{p^a M})^+, \mathrm{Ad}^0(T_{\mathbf{f}})(2)\right),$$

where $\mathrm{loc}_{\mathfrak{M}}$ denotes localization at the maximal ideal $\mathfrak{M}$.

These classes verify by the exact same reasoning as the previous sections the horizontal and vertical norm relations – by construction, they do at each specialization of $\mathbf{f}$ to a form of level $\Gamma_1(p^r)$ and nebentypus ψ as the class $\underline{\kappa}_{p^a M}^{ord}(\mathbf{f})$ specializes to the class of Section 3.4.11, as is shown in the first part of the proof of Theorem 4.2.2.8.

Let $\Lambda = \mathcal{O}_{\chi, p}[[1 + p\mathbb{Z}_p]]$. The classes $\underline{\kappa}_{p^a M}^{ord}(\mathbf{f})$ can be interpreted by Shapiro's Lemma and the inflation-restriction sequence as classes

$$\underline{\mathcal{Z}}_M^{ord}(\mathbf{f}) \in H^1\left(\mathbb{Q}(\boldsymbol{\mu}_M)^+, \mathrm{Ad}^0(T_{\mathbf{f}})(2) \otimes \Lambda(\chi_0)\right).$$

Here the Galois group acts on Λ by $\Psi : \mathrm{Gal}_{\mathbb{Q}} \rightarrow \Lambda$, the abelian character of $\mathrm{Gal}_{\mathbb{Q}}$ given by the unique $\mathbb{Z}_p$ quotient of $\mathrm{Gal}_{\mathbb{Q}}^{\mathrm{ab}}$.

3.5.3.1 Theorem (Euler System in Hida Family). *Let $\mathbf{f}$ be a Hida family passing through f and $T_{\mathbf{f}}$, $\Lambda = \mathcal{O}_{\chi, p}[[1 + p\mathbb{Z}_p]]$ be as above. Let $\chi_0 : (\mathbb{Z}/p\mathbb{Z})^\times \rightarrow \mathbb{C}^\times$ be an odd*

Dirichlet character. There exists a collection of classes $(\underline{\mathcal{Z}}_M^{ord}(\mathbf{f}))$ indexed by positive squarefree integers M , coprime to the primes of S , such that

$$\underline{\mathcal{Z}}_M^{ord}(\mathbf{f}) \in H^1\left(\mathbb{Q}(\boldsymbol{\mu}_M)^+, \text{Ad}^0(T_{\mathbf{f}})(2) \otimes \Lambda(\chi_0)\right).$$

Moreover these classes verify: for any prime $\ell \notin S$

$$\text{Cores}_{\mathbb{Q}(\boldsymbol{\mu}_M)^+}^{\mathbb{Q}(\boldsymbol{\mu}_{M\ell})^+}(\underline{\mathcal{Z}}_{M\ell}^{ord}(\mathbf{f})) = P_{\ell}(\text{Fr}_{\ell}^{-1})\underline{\mathcal{Z}}_M^{ord}(\mathbf{f}).$$

The next part of the paper will be concerned in characterizing the non-triviality of these classes. Namely, we will compute explicitly the dual Bloch-Kato exponential of the localization at p at weight two points of this family, by using Perrin-Riou's big Logarithm machinery.

Chapter 4

The Explicit Reciprocity Laws

4.1 Relative Cohomologies and comparison theorems

In this section we review basic notation and introduce relative cohomology theories.

4.1.1 Relative étale cohomology

Let X be variety over a field K (i.e. a reduced, separated, and finite type scheme over K). Let Z be a closed subscheme, and write $j : U \hookrightarrow X$ for the open immersion of the complement $U := X \setminus Z$ into X . Let F be an algebraic extension of $\mathbb{Q}_p$. We define the étale cohomology of the pair (X, Z) with coefficients in W to be the étale cohomology of the sheaf $j_! \underline{F}$ on X . We write

$$H_{\text{et}}^{\bullet}(X, Z, \underline{F}) := H_{\text{et}}^{\bullet}(X, j_! \underline{F}).$$

Let $i : Z \hookrightarrow X$ be the closed immersion of Z into X ; then we have a long exact cohomology sequence of the pair (X, Z)

$$\cdots \longrightarrow H_{\text{et}}^m(Z, \underline{F}) \longrightarrow H_{\text{et}}^{m+1}(X, Z, \underline{F}) \longrightarrow H_{\text{et}}^{m+1}(X, \underline{F}) \longrightarrow H_{\text{et}}^{m+1}(Z, \underline{F}) \longrightarrow \cdots$$

In what follows we will particularly be interested in the case of a pair $(X \setminus Y, Z \setminus Y)$, where X is a smooth proper variety over a separably closed field and Y and Z are smooth closed subschemes on X which meet transversally. In this case we have a commutative diagram where arrows are open immersions:

$$\begin{array}{ccc} X \setminus (Y \cup Z) & \xrightarrow{j} & X \setminus Y \\ \downarrow \iota' & & \downarrow \iota \\ X \setminus Z & \xrightarrow{j'} & X. \end{array}$$

The cohomology of the pair $(X \setminus Y, Z \setminus Y)$ is by definition computed as the cohomology of the sheaf $j_! \underline{F}$ on $X \setminus Y$. This cohomology is canonically isomorphic to the cohomology of the complex $R\iota_*(j_! \underline{F})$ on X . Moreover, there is a canonical isomorphism:

$$j'_!(R\iota'_* \underline{F}) \simeq R\iota_*(j_! \underline{F}).$$

For more details the reader can consult [EG15, Appendix A] or [Fal89, Section III].

4.1.2 Filtered Frobenius modules

In this subsection, we introduce basic objects of p -adic Hodge theory that we will need to state appropriate comparison theorems. Let $K/\mathbb{Q}_p$ be a finite extension. We

denote by K_0 the maximal unramified subfield of K and K_0^{ur} the maximal unramified extension of K . Let q the cardinality of the residue field.

4.1.2.1 Definition. A filtered $(\varphi, \mathcal{N}, G_K)$ -module over K . Is a finite dimensional $\mathbb{Q}_p^{\text{ur}}$ -vector space D endowed with a $\mathbb{Q}_p^{\text{ur}}$ -semilinear bijective Frobenius endomorphism φ , and a $\mathbb{Q}_p^{\text{ur}}$ -linear monodromy operator $\mathcal{N}$ satisfying $\mathcal{N}\varphi = p\varphi\mathcal{N}$. The Galois group G_K acts $\mathbb{Q}_p^{\text{ur}}$ -semilinearly on D and there is a decreasing, separated, exhaustive filtration of the K -vector space :

$$D_K := (D \otimes_{\mathbb{Q}_p^{\text{ur}}} \overline{\mathbb{Q}_p})^{G_K}$$

by K -vector spaces $\text{Fil}^i D_K$.

We will write D_{st} for the K_0 vector space D^{G_K} of Galois invariant vectors. A filtered $(\varphi, \mathcal{N}, G_K)$ -module D such that the G_K action is unramified and $N = 0$ is said to be crystalline. In this case we have

$$D = D_{\text{st}} \otimes_{K_0} \mathbb{Q}_p^{\text{ur}} \quad \text{and} \quad D_K = D_{\text{st}} \otimes_{K_0} K$$

Recall also that by results of Colmez-Fontaine, there is an equivalence of categories between potentially semi-stable p -adic Galois representations of G_K and weakly admissible filtered $(\varphi, \mathcal{N}, G_K)$ -module over K , induced by the functor $V \mapsto \mathbf{D}_{\text{pst}}(V)$. We denote by $D \mapsto \mathbf{V}(D)$ the inverse functor. On the level of objects, it is described by :

$$\mathbf{V}(D) = \text{Fil}^0(D_K \otimes B_{\text{dR}}) \cap (D \otimes B_{\text{st}})^{\mathcal{N}=0, \varphi=1}$$

4.1.3 Relative Hyodo-Kato cohomology

In this subsection we take $K/\mathbb{Q}_p$ be an algebraic extension and denote by K_0 the maximal unramified subfield of K and by q the cardinality of the residue field k .

Recall that the h -topology on $\mathcal{V}ar_K$ is the coarsest topology finer than the Zariski and proper topologies. It is generated by the pretopology whose coverings are finite families of maps $\{U_i \rightarrow X\}$ such that $\coprod U_i \rightarrow X$ is a universal topological epimorphism (i.e., a subset of X is Zariski open if and only if its preimage in $\coprod U_i$ is open). We denote by X_h the corresponding h -site. Beilinson [Bei11] shows that the h -topology on a variety over K has a basis consisting semistable models over finite extensions of $\mathcal{O}_K$.

We refer to [NW16] (see also [Bei13]) for the version of Hyodo-Kato cohomology that we will use. We denote the (arithmetic) Hyodo-Kato cohomology by $R\Gamma_{\text{HK}}(X_h)$. This is the cohomology of the h -sheafification of the preasheaf $Y \mapsto R\Gamma_{\text{HK}}(Y_0)$. Here Y_0 is the special fiber of Y a semistable scheme.

If (X, Z) is a pair as in the previous subsection, we denote by $R\Gamma_{\text{HK}}(X_h, Z_h)$ the relative Hyodo-Kato cohomology: it is the homotopy fiber defined by the following distinguished triangle:

$$R\Gamma_{\text{HK}}(X_h, Z_h) \longrightarrow R\Gamma_{\text{HK}}(X_h) \longrightarrow R\Gamma_{\text{HK}}(Z_h) \xrightarrow{+1}$$

We denote by $H_{\text{HK}}^m(X)$ the cohomology in degree m of the complex $R\Gamma_{\text{HK}}(X_h)$, and similarly $H_{\text{HK}}^m(X, Z)$ for the cohomology in degree m of the complex $R\Gamma_{\text{HK}}(X_h, Z_h)$. The Hyodo-Kato cohomology $H_{\text{HK}}^\bullet$ is a finite-dimensional K_0 vector space. Moreover,

it carries a Frobenius operator φ and a monodromy operator N such that $N\varphi = p\varphi N$.

There is also a geometric version of Hyodo-Kato cohomology, by simply base changing the varieties to $\overline{K}$. As is recalled in the introduction of [NW16], we have a natural isomorphism:

$$\mathrm{R}\Gamma_{\mathrm{HK}}(X_{\overline{K},h})^{G_K} \xrightarrow{\sim} \mathrm{R}\Gamma_{\mathrm{HK}}(X_h)$$

4.1.4 Relative de Rham cohomology

Let $\mathrm{R}\Gamma_{\mathrm{dR}}(X)$ be the extension of de Rham cohomology defined by Beilinson [Bei13]. Denote by $\mathrm{H}_{\mathrm{dR}}^\bullet(X)$ the cohomology of $\mathrm{R}\Gamma_{\mathrm{dR}}(X)$. As before, following Deligne, if (X, Z) is a pair as in subsection 4.1.1, we define relative de Rham cohomology as the homotopy fiber giving the following distinguished triangle:

$$\mathrm{R}\Gamma_{\mathrm{dR}}(X, Z) \longrightarrow \mathrm{R}\Gamma_{\mathrm{dR}}(X) \longrightarrow \mathrm{R}\Gamma_{\mathrm{dR}}(Z) \xrightarrow{+1}$$

4.1.5 Comparison theorems

In this subsection we recall comparison theorems between the different cohomology theories that were reviewed in the previous subsections. The results can be found in paragraph 1.4 and section 4 of [NW16]; the reader can also consult [BLZ16, Section 2] where some of the results are recalled.

We start by recalling the existence of period maps.

4.1.5.1 Proposition (Beilinson). *There exist canonical period maps*

$$\rho_{\mathrm{HK}} : \mathrm{R}\Gamma_{\mathrm{HK}}(X_{\overline{K},h}) \otimes_{K_0^{\mathrm{ur}}} \mathbf{B}_{\mathrm{st}} \longrightarrow \mathrm{R}\Gamma(X_{\overline{K},\mathrm{et}}, \mathbb{Q}_p) \otimes \mathbf{B}_{\mathrm{st}}$$

$$\rho_{dR} : \mathrm{R}\Gamma_{dR}(X_K) \otimes_K \mathbf{B}_{dR} \longrightarrow \mathrm{R}\Gamma(X_{\overline{K},\mathrm{et}}, \mathbb{Q}_p) \otimes \mathbf{B}_{dR}$$

$$\iota_{dR} : \mathrm{R}\Gamma_{\mathrm{HK}}(X_h) \otimes_{K_0} K \longrightarrow \mathrm{R}\Gamma_{dR}(X)$$

The following theorem is due to Beilinson and Nekovár-Niziol ([NW16], [Bei13]).

4.1.5.2 Theorem. *The period maps above induces canonical identification of $(\varphi, \mathcal{N}, G_K)$:*

$$\overline{\alpha}_{\mathrm{HK}} : \mathrm{H}_{\mathrm{HK}}^m(X_{\overline{K},h}) \xrightarrow{\sim} \mathbf{D}_{\mathrm{pst}}(\mathrm{H}_{\mathrm{et}}^m(X_{\overline{K}}, \mathbb{Q}_p)),$$

$$\alpha_{\mathrm{HK}} : \mathrm{H}_{\mathrm{HK}}^m(X_h) \xrightarrow{\sim} \mathbf{D}_{\mathrm{st}}(\mathrm{H}_{\mathrm{et}}^m(X_{\overline{K}}, \mathbb{Q}_p)).$$

When X has a semi-stable model over $\mathcal{O}_K$ we have:

$$\iota_{dR} : \mathrm{H}_{\mathrm{HK}}^m(X_h) \otimes_{K_0} K \xrightarrow{\sim} \mathrm{H}_{dR}^m(X).$$

Finally, we have the de Rham comparison theorem:

$$\alpha_{dR} : \mathrm{H}_{dR}^m(X) \xrightarrow{\sim} \mathbf{D}_{dR}(\mathrm{H}_{\mathrm{et}}^m(X_{\overline{K}}, \mathbb{Q}_p)).$$

We now state the relative version of the above theorem.

4.1.5.3 Corollary. *The period maps above induces canonical identification of $(\varphi, \mathcal{N}, G_K)$:*

$$H_{\text{HK}}^m(X_{\overline{K},h}, Z_{\overline{K},h}) \xrightarrow{\sim} \mathbf{D}_{\text{pst}}(H_{\text{et}}^m(X_{\overline{K}}, Z_{\overline{K}}, \mathbb{Q}_p)),$$

$$H_{\text{HK}}^m(X_h, Z_h) \xrightarrow{\sim} \mathbf{D}_{\text{st}}(H_{\text{et}}^m(X_{\overline{K}}, Z_{\overline{K}}, \mathbb{Q}_p)).$$

When X and Z have semi-stable models over $\mathcal{O}_K$, we have:

$$H_{\text{HK}}^m(X_h, Z_h) \otimes_{K_0} K \xrightarrow{\sim} H_{\text{dR}}^m(X, Z).$$

Finally, we have the de Rham comparison theorem:

$$H_{\text{dR}}^m(X, Z) \xrightarrow{\sim} \mathbf{D}_{\text{dR}}(H_{\text{et}}^m(X_{\overline{K}}, Z_{\overline{K}}, \mathbb{Q}_p)).$$

Proof. The long exact sequence of relative étale cohomology corresponds to the distinguished triangle

$$Rj_!j^! \longrightarrow \text{id} \longrightarrow Ri_\star i^\star \xrightarrow{+1}$$

Therefore, $Rj_!j^!$ receives a map from the (shifted) mapping cone of $\text{id} \rightarrow Ri_\star i^\star$; for constant coefficients, this is just the pullback map $\text{R}\Gamma_{\text{et}}(X_{\overline{K}}, \mathbb{Q}_p) \rightarrow \text{R}\Gamma_{\text{et}}(X_{\overline{K}}, \iota_\star \mathbb{Q}_p)$. Therefore, ρ_{HK} induces a map from $\text{R}\Gamma_{\text{HK}}(X_h, Z_h) \otimes \mathbf{B}_{\text{st}} \rightarrow \text{R}\Gamma_{\text{et}}(X_{\overline{K}}, Z_{\overline{K}}, j_! \mathbb{Q}_p) \otimes \mathbf{B}_{\text{st}}$ and thus, a natural map $H_{\text{HK}}^m(X_h, Z_h) \rightarrow \mathbf{D}_{\text{st}}(H_{\text{et}}^m(X_{\overline{K}}, Z_{\overline{K}}, \mathbb{Q}_p))$. We can thus conclude by the five Lemma that this map is an isomorphism. The proof of the other relative comparison isomorphisms is similar. $\square$

4.2 The varieties of interest

We finally fix a level structure on $\mathrm{SH}_{\mathbf{G}}$ and $\mathrm{SH}_{\mathbf{H}}$: we take $K_{\mathbf{G}} \subset \mathbf{G}(\mathbb{A}_f)$ compact open corresponding to the level of the Eisenstein series $E^{\chi}(g; f_{\phi}^{\mathrm{base}})$. Denote by $K_{\mathbf{H}}$ be the pull-back of $K_{\mathbf{G}}$ to $\mathbf{H}(\mathbb{A}_f)$. Note that $K_{\mathbf{H}} \subseteq K_1(N)$ for some N large enough so that $K_{\mathbf{H}}$ (and $K_{\mathbf{G}}$) is neat.

By [Lan22, Theorem 2.2], choose compatible collections of cone decompositions for $\mathrm{SH}_{\mathbf{G}}$ and $\mathrm{SH}_{\mathbf{H}}$ which induce closed embeddings of toroidal compactifications Shimura varieties:

$$\mathrm{SH}_{\mathbf{G}}^{\mathrm{tor}}(K_{\mathbf{G}}) \hookrightarrow \mathrm{SH}_{\mathbf{H}}^{\mathrm{tor}}(K_{\mathbf{H}})$$

For simplicity we set $Y_{\mathbf{G}} := \mathrm{SH}_{\mathbf{G}}(K_{\mathbf{G}})$ and $Y_{\mathbf{H}} := \mathrm{SH}_{\mathbf{H}}(K_{\mathbf{H}})$; denote also $\mathrm{SH}_{\mathbf{G}}^{\mathrm{tor}}(K_{\mathbf{G}})$ by $Y_{\mathbf{G}}^{\mathrm{tor}}$ and $\mathrm{SH}_{\mathbf{H}}^{\mathrm{tor}}(K_{\mathbf{H}})$ by $Y_{\mathbf{H}}^{\mathrm{tor}}$. Let $D_{\mathbf{G}}$, respectively $D_{\mathbf{H}}$, be the normal crossing divisor representing the boundary of $Y_{\mathbf{G}}^{\mathrm{tor}}$, respectively $Y_{\mathbf{H}}^{\mathrm{tor}}$. By Theorem 2.2 of *loc cit.*, we can assume that the pullback of $D_{\mathbf{G}}$ is $D_{\mathbf{H}}$. Denote by $Y_{\mathbf{G}}^{\star}$ the minimal compactification of $\mathrm{SH}_{\mathbf{G}}(K_{\mathbf{H}})$ and $D_{\mathbf{G}}^{\star}$ its boundary. There is a canonical map $Y_{\mathbf{G}}^{\mathrm{tor}} \rightarrow Y_{\mathbf{G}}^{\star}$. With obvious notations, we also introduce $Y_{\mathbf{H}}^{\star}$ and $D_{\mathbf{H}}^{\star}$. Note that contrary to $Y_{\mathbf{G}}^{\mathrm{tor}}$, $Y_{\mathbf{G}}^{\star}$ is not smooth.

We are now in the setting of subsection 4.1.1; namely with $X = Y_{\mathbf{G}}^{\mathrm{tor}}$, $Y = D_{\mathbf{G}}$ and $Z = Y_{\mathbf{H}}$.

4.2.1 p -adic Hodge theory of modular forms

Recall that we have fixed an ordinary newform f of level $\Gamma_1(N)$ and associated differential

$$\omega'_f = (2i\pi)f(\tau)d\tau \in H^0(Y_1(N), \Omega^1)$$

Denote by ψ_f be the Nebentypus of f . Let $\tau(\psi_f)$ be the Gauss-sum of ψ_f . Then the differential $\omega_f := \tau(\psi_f^{-1})\omega'_f$ is a basis of $\text{Fil}^1 H_{\text{dR}}^1(Y_1(N)/K)[f]$.

The Faltings-Tsuji de Rham comparison theorem and semistable comparison theorem give identifications

$$\mathbf{D}_{\text{pst}}(H_{\text{et}}^1(Y_1(N)/\overline{K}, K)) \simeq H_{\text{HK}}^1(Y_1(N)/\overline{K})$$

and

$$\mathbf{D}_{\text{dR}}(H_{\text{et}}^1(Y_1(N)/\overline{K}, K)) \simeq H_{\text{dR}}^1(Y_1(N)/K)$$

and thus identify $\mathbf{D}_{\text{pst}}(\rho_f)$ with $H_{\text{HK}}^1(Y_1(N)/\overline{K})[f]$ and $\mathbf{D}_{\text{dR}}(\rho_f)$ with $H_{\text{dR}}^1(Y_1(N)/K)$.

Because $\text{Fil}^1 H_{\text{HK}}^1(Y_1(N)/\overline{K})$ is one-dimensional over K^{ur} , there exists a vector $\omega_f^{\text{HK}} \in \text{Fil}^1(H_{\text{HK}}^1(Y_1(N)/\overline{K})) \simeq \text{Fil}^1 \mathbf{D}_{\text{st}}(\rho_f)$ and a scalar $\lambda \in \overline{K}$ such that

$$\alpha_{\text{HK}}(\omega_f^{\text{HK}} \otimes \lambda) = \omega_f.$$

We view ω_f (resp. ω_f^{HK}) as elements both of $\mathbf{D}_{\text{dR}}(\rho_f)$ (resp. $\mathbf{D}_{\text{st}}(\omega_f^{\text{HK}})$) and $\text{Fil}^1 H_{\text{dR}}^1(Y_1(N))$ (resp. $\text{Fil}^1 H_{\text{HK}}^1(Y_1(N))$). The semi-stable Frobenius (over K) acting on $\mathbf{D}_{\text{st}}(\rho_f)$ has distinct eigenvalues. We define similarly a vector η_f^{HK} such that

$\alpha_{\text{HK}}(\eta_f \otimes \lambda^{-1}) = \eta_f$, such that the semistable Frobenius φ_{HK} acts by the unit root α_f on η_f and $\langle \omega_f, \eta_f \rangle_{\text{dR}} = 1$

4.2.2 p -adic Hodge theory in Hida families

In this section we extend the vectors η_f and ω_f in Hida families defined in the previous section. We follow closely [KLZ17] and the terminology in *loc cit.* and we keep the notation of Section 3.5.1

Following [KLZ17, Section 8.2] we define the functor $\mathbf{D}_{\text{ur}}$ as follows. If M is an unramified, p -adically complete $\mathbb{Z}_p[\text{Gal}_{\mathbb{Q}_p}]$ -module, we let

$$\mathbf{D}_{\text{ur}}(M) := \left(M \hat{\otimes}_{\mathbb{Z}_p} \widehat{\mathbb{Z}}_p^{\text{ur}} \right)^{\text{Gal}_{\mathbb{Q}_p}}.$$

We let φ denote the Frobenius acting on $\mathbf{D}_{\text{ur}}(M)$ coming from the unramified Galois action on $\widehat{\mathbb{Z}}_p^{\text{ur}}$ (the valuation ring of the p -adic completion of the maximal unramified extension of $\mathbb{Q}_p$). As is remarked in *loc cit.* if M is free of finite rank over $\mathbb{Z}_p$ then $\mathbf{D}_{\text{ur}}(M)$ is a lattice in $\mathbf{D}_{\text{cris}}(M \otimes_{\mathbb{Z}_p} \mathbb{Q}_p)$.

Following [KLZ17, Definition 8.2.7], we write $\mathbf{D}_{\text{ur}}(\text{Ad}^0(T_{\mathbf{f}})^-)$ for the module $\mathbf{D}_{\text{ur}}(\text{Ad}^0(T_{\mathbf{f}})^-(\varepsilon^{-1}\Psi))$ equipped with the non-trivial action of Γ the Galois group of the unique $\mathbb{Z}_p$ -extension of $\mathbb{Q}$, given by the character $\varepsilon\Psi^{-1}$.

The following proposition is an immediate adaptation of [KLZ17, Theorem 8.2.8] using Theorem 4.15 and Appendix B of [LZ14] (note that our conventions for Gauss sums are different from *loc cit.*) and that the lowest Hodge Tate weight of $\text{Ad}^0(T_f)$ is $-k$ if f is of weight $k + 2$.

4.2.2.1 Proposition. *There is a morphism of $\Lambda_{\mathbf{f}} \hat{\otimes} \Lambda$ -modules:*

$$\mathcal{L} : H^1\left(\mathbb{Q}_p, \mathrm{Ad}^0(T_{\mathbf{f}})^-(1) \otimes \Lambda\right) \longrightarrow \mathbf{D}_{\mathrm{ur}}(\mathrm{Ad}^0(T_{\mathbf{f}})^-) \hat{\otimes} \Lambda$$

such that, for all specialisations of $\mathbf{f}$ to a classical eigenform f of weight $2+k \geq 2$ with Nebentypus $\psi_{f,p}$ of log-conductor r , and all characters of Γ of the form $\varepsilon^j \chi$ with χ of finite order and $j \in \mathbb{Z}$ with χ of log-conductor n , there is a commutative diagram:

$$\begin{array}{ccc} H^1\left(\mathbb{Q}_p, \mathrm{Ad}^0(T_{\mathbf{f}})^-(1) \otimes \Lambda\right) & \xrightarrow{\mathcal{L}} & \mathbf{D}_{\mathrm{ur}}(\mathrm{Ad}^0(T_{\mathbf{f}})^-) \hat{\otimes} \Lambda \\ \downarrow & & \downarrow \\ H^1\left(\mathbb{Q}_p, \mathrm{Ad}^0(T_f)^-(\varepsilon^{1-j} \chi^{-1})\right) & \longrightarrow & \mathbf{D}_{\mathrm{dR}}(\mathrm{Ad}^0(T_f)^-(1)). \end{array}$$

Here, if χ is unramified, the bottom horizontal arrow is given by

$$\left(1 - \frac{p^j}{\alpha_f \alpha_{\bar{f}}}\right) \left(1 - \frac{\alpha_f \alpha_{\bar{f}}}{p^{j+1}}\right)^{-1} \cdot \Gamma^*(1-j-k) \begin{cases} \log_{\mathrm{BK}} & \text{if } -j-k > 0 \\ \exp_{\mathrm{BK}}^* & \text{if } -j-k \leq 0. \end{cases}$$

If χ is ramified of conductor $n > 0$, then the bottom horizontal is given by

$$\left(\frac{p^{j+1}}{\alpha_f \alpha_{\bar{f}}}\right)^n \tau(\chi^{-1} \psi_{f,p}^{-1})^{-1} \cdot \Gamma^*(1-j-k) \begin{cases} \log_{\mathrm{BK}} & \text{if } -j-k > 0 \\ \exp_{\mathrm{BK}}^* & \text{if } -j-k \leq 0. \end{cases}$$

Here we have denoted by $\log_{\mathrm{BK}}$ and $\exp_{\mathrm{BK}}^*$ the Bloch-Kato logarithm and the Bloch-

Kato exponential respectively and put

$$\Gamma^*(1+l) = \begin{cases} \frac{(-1)^{l+1}}{(-l-1)!} & \text{if } l < 0, \\ l! & \text{if } l \geq 0. \end{cases}$$

4.2.2.2 Definition. With the notation of Theorem 3.5.3.1, we let $\underline{\mathcal{C}}_p(\mathbf{f}) \in H^1(\mathbb{Q}, \text{Ad}^0(T_{\mathbf{f}})(1) \otimes \Lambda)$ be the image of

$$\text{loc}_p \underline{\mathcal{Z}}_1^{\text{ord}}(\mathbf{f}) \in H^1(\mathbb{Q}_p, \text{Ad}^0(T_{\mathbf{f}})(2) \otimes \Lambda(\chi_0))$$

under the isomorphism

$$H^1(\mathbb{Q}_p, \text{Ad}^0(T_{\mathbf{f}})(1) \otimes \Lambda) \simeq H^1(\mathbb{Q}_p, \text{Ad}^0(T_{\mathbf{f}})(2) \otimes \Lambda(\chi_0)).$$

We recall here that χ_0 is taken to be the inverse of the Teichmüller character. For simplicity, write $\mathbb{Q}_{p,n}$ for the field fixed by $1 + p^n \mathbb{Z}_p \subseteq 1 + p \mathbb{Z}_p$ in $\mathbb{Q}(\mu_{p^\infty})$. Denote by

$$\underline{\mathcal{C}}_{p,n}(\mathbf{f}) \in H^1(\mathbb{Q}_{p,n}, \text{Ad}^0(T_{\mathbf{f}})(1))$$

the image of $\underline{\mathcal{C}}_p(\mathbf{f})$ in $H^1(\mathbb{Q}_{p,n}, \text{Ad}^0(T_{\mathbf{f}})(1))$ by the identification

$$\varprojlim_n H^1(\mathbb{Q}_{p,n}, M) = H^1(\mathbb{Q}_p, M \otimes \Lambda),$$

the inverse limit being taken with respect to corestriction maps. Similarly, we let $\underline{\mathcal{Z}}_{p,n}(\mathbf{f})$ be the image of $\text{loc}_p \underline{\mathcal{Z}}_1^{\text{ord}}(\mathbf{f})$ in $H^1(\mathbb{Q}_{p,n}, \text{Ad}^0(T_{\mathbf{f}})(\varepsilon^2 \chi_0))$. We let $\underline{\mathcal{C}}_p(\mathbf{f})^-$ be

the image of $\underline{\mathcal{C}}_p(\mathbf{f}) \in H^1(\mathbb{Q}, \text{Ad}^0(T_{\mathbf{f}})(1) \otimes \Lambda)$ in $H^1(\mathbb{Q}_p, \text{Ad}^0(T_{\mathbf{f}})^-(1) \otimes \Lambda)$ and we define similarly, $\underline{\mathcal{C}}_{p,n}(\mathbf{f})^-$, $\underline{\mathcal{Z}}_p(\mathbf{f})^-$, and $\underline{\mathcal{Z}}_{p,n}(\mathbf{f})^-$. Finally, for f a specialization of $\mathbf{f}$, we let $\mathcal{C}_p(f)$, $\mathcal{C}_p(f)^-$, $\mathcal{C}_{p,n}(f)$, $\mathcal{C}_{p,n}(f)^-$, $\mathcal{Z}_p(f)$, $\mathcal{Z}_p(f)^-$, $\mathcal{Z}_{p,n}(f)$, $\mathcal{Z}_{p,n}(f)^-$ be the corresponding specializations of the classes defined above.

By construction $\mathcal{L}$ is related to the Perrin-Riou big-logarithm. Let us introduce the general notation. If V is a crystalline $G_{\mathbb{Q}_p}$ representation, we denote by $\mathcal{L}^{PR}$ the Perrin-Riou big-logarithm

$$\mathcal{L}_V^{PR} : H^1(\mathbb{Q}_p, V \otimes \Lambda) \longrightarrow \mathbf{D}_{\text{cris}}(V) \otimes \mathcal{H}_{\mathbb{Q}_p}(\Gamma).$$

If a twist of $V(\sigma)$ is crystalline for some character σ of $G_{\mathbb{Q}_p}$, we define

$$\mathcal{L}_V^{PR} : H^1(\mathbb{Q}_p, V \otimes \Lambda) \longrightarrow \mathbf{D}_{\text{cris}}(V(\sigma)) \otimes \mathcal{H}_{\mathbb{Q}_p}(\Gamma)$$

by composing it with the isomorphism of Λ -modules $H^1(\mathbb{Q}_p, V \otimes \Lambda) \simeq H^1(\mathbb{Q}_p, V(\sigma) \otimes \Lambda)$ afforded by Shapiro's lemma.

Following the conventions of [LZ14, Appendix B], for χ a finite order character of $G_{\mathbb{Q}_p}$ of log conductor n , we view $\mathcal{L}^{PR}(\varepsilon^j \chi)$ as being valued in $(\mathbf{D}_{\text{cris}}(V(\varepsilon^{-j}) \otimes \mathbb{Q}_{p,n})^{\Gamma=\chi} \simeq \mathbf{D}_{\text{dR}}(V(\varepsilon^{-j} \chi^{-1})))$.

4.2.2.3 Lemma. *Let $\mathbb{M}$ be a free $\Lambda_D \simeq \Lambda$ -module of rank 1 endowed with a continuous action of $G_{\mathbb{Q}_p}$, such that a twist of $\mathbb{M}$ is unramified. Denote by $M = \mathbb{M} \otimes_{\Lambda_D, \kappa} \mathcal{O}$, where $\kappa : \Lambda_D \rightarrow \mathcal{O}$ is an arithmetic point. Let $\mathbf{Z} \in H^1(\mathbb{Q}_p, \mathbb{M} \otimes \Lambda)$ and Z be the image of $\mathbf{Z}$ in $H^1(\mathbb{Q}_p, M \otimes \Lambda)$ by extending scalars along κ .*

For all finite order characters χ of conductor n such that the conductor $n \geq r$, we have the relation

$$\mathcal{L}_M^{PR}(Z)(\chi) = \chi\left(\kappa(\mathcal{L}(\mathbf{Z}))\right).$$

We will need the following proposition extracted from [KLZ17, Proposition 10.1.1] with very slightly different notations.

4.2.2.4 Proposition. *Let $\mathbf{a}$ be a new cuspidal branch of a Hida family $\mathbf{f}$ of tame level N . Let $C(\mathbf{a})$ be the associated congruence ideal. Then there is an injective morphism of $\Lambda_{\mathbf{a}}$ -modules*

$$\eta_{\mathbf{a}} : \mathbf{D}_{\text{ur}}(T_{\mathbf{f}}^-) \otimes_{\Lambda_{\mathbf{f}}} \Lambda_{\mathbf{a}} \hookrightarrow C(\mathbf{a})$$

such that, for every arithmetic prime $\mathfrak{p}_f$ of $\Lambda_{\mathbf{f}}$ above $\mathbf{a}$ corresponding to an eigenform f of level Np^r for some $r \geq 1$, there is a commutative diagram

$$\begin{array}{ccc} \mathbf{D}_{\text{ur}}(T_{\mathbf{f}}^-) \otimes_{\Lambda_{\mathbf{f}}} \Lambda_{\mathbf{a}} & \xrightarrow{\eta_{\mathbf{a}}} & C(\mathbf{a}) \\ \downarrow & & \downarrow \\ \mathbf{D}_{\text{dR}}(\rho_f) & \longrightarrow & \text{Frac}(\Lambda_{\mathbf{f}}/\mathfrak{p}_f) \end{array}$$

where the horizontal arrows are given as in [KLZ17, Proposition 10.1.1, (2)] by pairing with a multiple of η_f (or its p -stabilization if f is old). More precisely, if f is new of level Np^r , the bottom horizontal arrow is given by pairing with

$$\frac{\alpha_f^r}{\lambda_{Np^r}(f)} \eta_f,$$

where $\lambda_{Np^r}(f)$ is the Atkin-Lehner pseudo-eigenvalue.

4.2.2.5 Remark. The Atkin-Lehner pseudo-eigenvalue $\lambda_{Np^r}(f)$ interpolates in the cuspidal branch $\mathbf{a}$ by the remark following Proposition 10.1.2 of *loc cit.* We denote the $\Lambda_{\mathbf{a}}$ -adic eigenvalue by $\lambda_N(\mathbf{a})$ following *loc cit.*, Then $\lambda_N(\mathbf{a})$ verifies

$$\lambda_N(\mathbf{a}) = \Psi_D^{-1}(-N),$$

and is in particular a p -adic unit at all specializations.

4.2.2.6 Proposition. *Let the setup be as in Proposition 4.2.2.4. There exists an injective morphism of $\Lambda_{\mathbf{a}}$ -modules*

$$\underline{\eta}_{\mathbf{a}} : \mathbf{D}_{\text{ur}}(\text{Ad}^0(T_{\mathbf{f}})^-) \otimes_{\Lambda_{\mathbf{f}}} \Lambda_{\mathbf{a}} \longrightarrow C(\mathbf{a})$$

such that, for every arithmetic prime $\mathfrak{p}_f$ of $\Lambda_{\mathbf{f}}$ above $\mathbf{a}$ corresponding to an eigenform f of level Np^r for some $r \geq 1$, there is a commutative diagram

$$\begin{array}{ccc} \mathbf{D}_{\text{ur}}(\text{Ad}^0(T_{\mathbf{f}})^-) \otimes_{\Lambda_{\mathbf{f}}} \Lambda_{\mathbf{a}} & \xrightarrow{\underline{\eta}_{\mathbf{a}}} & \Lambda_{\mathbf{a}} \\ \downarrow & & \downarrow \\ \text{Fil}^0 \mathbf{D}_{\text{dR}}(V_f) & \longrightarrow & \text{Frac}(\Lambda_{\mathbf{f}}/\mathfrak{p}_f) \end{array}$$

where the vertical arrows is given by pairing with $(\alpha_f^r \alpha_{\bar{f}}^r / \lambda_{Np^r}(f) \lambda_{Np^r}(\bar{f})) \cdot c_f \cdot (\eta_f \otimes \eta_{\bar{f}})$ when f is new of level Np^r . Here we recall that c_f is a generator of the congruence ideal of f .

Proof. Consider $\eta_{\mathbf{a}}$ and its analogue for $\bar{\mathbf{f}}$, $\eta_{\bar{\mathbf{a}}}$ that is afforded by Proposition 4.2.2.4.

Notice that all specializations at f of $\mathrm{Ad}^0(T_{\mathbf{f}})(1)$ are isomorphic to the lattice $c_{\bar{f}} \cdot T_f \otimes T_{\bar{f}} \subset \mathrm{Ad}^0(\rho_f)(1)$. In particular we can set

$$\underline{\eta}_{\mathbf{a}} := c_{\mathbf{a}} \cdot (\eta_{\mathbf{a}} \otimes \eta_{\bar{\mathbf{a}}}),$$

where $c_{\mathbf{a}}$ is the element of $\Lambda_{\mathbf{a}}$ image of $c_{\mathbf{f}} \in \Lambda_{\mathbf{f}}$, in the notations of Section 3.5.2. The result follows as $c_{\mathbf{a}}C(\mathbf{a}) = \Lambda_{\mathbf{a}}$ and as $\eta_{\mathbf{a}} \otimes \eta_{\bar{\mathbf{a}}}$ factors through $\mathbf{D}_{\mathrm{ur}}(\mathrm{Ad}(T_{\mathbf{f}})(1)) \rightarrow \mathbf{D}_{\mathrm{ur}}(\mathrm{Ad}^0(T_{\mathbf{f}})(1))$. $\square$

4.2.2.7 Definition. Consider

$$\mathcal{L} : H^1\left(\mathbb{Q}_p, \mathrm{Ad}^0(T_{\mathbf{f}})^-(1) \otimes \Lambda\right) \longrightarrow \mathbf{D}_{\mathrm{ur}}(\mathrm{Ad}^0(T_{\mathbf{f}})^-) \hat{\otimes} \Lambda$$

the morphism of $\Lambda_{\mathbf{f}} \hat{\otimes} \Lambda$ -modules defined by Proposition 4.2.2.1. We define

$$\underline{\mathcal{L}} := \underline{\eta}_{\mathbf{a}} \circ \mathcal{L}.$$

For a class $\mathcal{C} \in H^1\left(\mathbb{Q}_p, \mathrm{Ad}^0(T_{\mathbf{f}})^-(1) \otimes \Lambda\right)$, and a specialization f of $\mathbf{f}$, and all characters of the form $\varepsilon^j \chi$, where χ is finite order, we denote by $\underline{\mathcal{L}}(\mathcal{C})(f, \varepsilon^j \chi)$ the specialization of $\underline{\mathcal{L}}(\mathcal{C}) \in \Lambda_{\mathbf{a}} \hat{\otimes} \Lambda$ at f and $\varepsilon^j \chi$, so that

$$\underline{\mathcal{L}}(\mathcal{C})(f, \varepsilon^j \chi) \in H^1\left(\mathbb{Q}_p, \mathrm{Ad}^0(T_f)^-(\varepsilon^{1-j} \chi^{-1})\right).$$

Looking ahead, the aim of the next sections will be to compute the image under $\underline{\mathcal{L}}\left(\mathcal{C}_p(\mathbf{f})^-\right)$ – we will see that it will coincide with Liu’s p -adic L -function [Liu20].

The first step in this computation is computing explicitly the image of $\mathcal{C}_p(\mathbf{f})^-$ with respect to $\mathcal{C}_p(\mathbf{f})$. The following proposition completes this first step.

4.2.2.8 Theorem. *Let ξ be an even Dirichlet character trivial on $(\mathbb{Z}/p\mathbb{Z})^\times$ and let $\chi = \chi_0 \xi$ in the notation of Section 3.4.10.5 such that χ is of log-conductor $n \geq 1$. Let f be a weight 2 newform of level Np^r with non-trivial nebentypus at p , denoted by $\psi_{f,p}$, of log-conductor $r \leq n$.*

We have the equality:

$$\underline{\mathcal{L}}\left(\mathcal{C}_p(\mathbf{f})^-\right)(f, \varepsilon^{-1}\chi^{-1}) = c_f \cdot (\alpha_f \alpha_{\bar{f}})^{-n} \langle \log_{\mathbb{Q}_{p,n}}(\kappa_\chi)^-, \eta_f \otimes \eta_{\bar{f}} \rangle_{\text{dR}}.$$

Here $\kappa_\chi \in H^1(\mathbb{Q}_p, \text{Ad}^0(T_f)(\varepsilon^2\chi^{-1}))$ is the localization at p of the class

$$(e \times e) \mathcal{ES}_{f \otimes \bar{f}, K(N)K_1(p^r)} \left(\omega^{\text{et}} \left(-\tau(\chi \psi_{f,p}^{-1})^{-1} (u_p \times u_p)^n t_{\chi \psi_{f,p}^{-1}}^{\text{dep}} f_\chi^{\text{ord}} \right) \right)$$

in the notations of Section 3.4.10.5; and $\log_{\mathbb{Q}_{p,n}}(\kappa_\chi)^-$ is the image of κ_χ under the Bloch-Kato logarithm over $\mathbb{Q}_{p,n}$ composed with the natural map $\mathbf{D}_{\text{dR}}(\text{Ad}^0(T_f)(\varepsilon^2\chi^{-1})) \rightarrow \mathbf{D}_{\text{dR}}(\text{Ad}^0(T_f)^-(\varepsilon^2\chi^{-1}))$.

Proof. We use freely notation from section 3.5.3. By definition of $\mathcal{Z}_p(f)$ (see definition 4.2.2.2), we need to compute the projection of

$$\kappa_{p^a}^{\text{dep}, r} = (d_{K_1(p^r), K_1(p^n)})_* \mathcal{ES}_{f \otimes \bar{f}, K(N)\tilde{K}_1(p^n)}^{\text{ord}} \left(\varsigma_{p^n}^{\text{dep}, n} \right)$$

on $f \otimes \bar{f}$ – this is done by applying $\sum_{d \in (\mathbb{Z}/p^r\mathbb{Z})^\times} \langle d \rangle^* \psi_{f,p}^{-1}(d)$, which commutes with the adjoint ordinary projection $e' \times e'$. Noting that $\langle d \rangle^* = \langle d^{-1} \rangle$ and setting

$\kappa_{r,n}(f) := \sum_{d \in (\mathbb{Z}/p^r\mathbb{Z})^\times} \langle d \rangle^* \psi_{f,p}^{-1}(d) \kappa_{p^a}^{dep,r}$, we therefore obtain by Proposition (B.6.2):

$$\kappa_{r,n}(f) := (d_{K_1(p^r), K_1(p^r)})_* \mathcal{ES}_{f \otimes \bar{f}, K(N) \tilde{K}_1(p^n)}^{ord} \left(\frac{1}{p^n} \sum_{\xi' \equiv \psi_{f,p}^{-1}} \sum_{\xi \leq n} \omega^{\text{et}}(\text{tw}^{2n} t_{\xi \chi_0 \xi'}^{DEP,n} f_{\chi_0 \xi}^{ord}) \right).$$

Here the sum over ξ' is taken over all extensions of $\psi_{f,p}^{-1}$ modulo p^n .

By Lemma B.4 of [LZ14], and Proposition 4.2.2.6, we see that we have

$$\underline{\mathcal{L}}(\mathcal{C}_p(\mathbf{f})^-)(f, \varepsilon^{-1} \chi^{-1}) = -c_f \frac{(\alpha_f \alpha_{\bar{f}})^{r-n} \tau(\chi \psi_{f,p}^{-1})}{\lambda_{Np^r}(f) \lambda_{Np^r}(\bar{f})} \cdot \left\langle \sum_{\sigma} \chi(\sigma) \log_{\mathbb{Q}_{p,n}}(\kappa_{r,n}(f))^{\sigma}, \eta_f \otimes \eta_{\bar{f}} \right\rangle_{\text{dR}}$$

We now use the fact that σ acts by $\xi^{-1} \chi_0^{-1}(\sigma)$ on $\omega^{\text{et}}(\text{tw}^{2n} t_{\xi \chi_0 \psi_{f,p}^{-1}}^{DEP,n} f_{\chi_0 \xi}^{ord})$ and therefore obtain, by Lemma 3.4.10.8:

$$\underline{\mathcal{L}}(\mathcal{C}_p(\mathbf{f})^-)(f, \varepsilon^{-1} \chi^{-1}) = -c_f \frac{(\alpha_f \alpha_{\bar{f}})^{r-n}}{\lambda_{Np^r}(f) \lambda_{Np^r}(\bar{f})} \left\langle \log_{\mathbb{Q}_{p,n}}(\kappa_{\chi}(f)), \eta_f \otimes \eta_{\bar{f}} \right\rangle_{\text{dR}},$$

where we have set

$$\kappa_{\chi}(f) = \mathcal{ES}_{f \otimes \bar{f}, K(N) \tilde{K}_1(p^r)}^{ord} \left(\text{tw}^{2r}(u_p \times u_p)^{n-r} f_{\chi}^{dep} \right).$$

Here we have put $f_{\chi}^{dep} := \tau(\chi \psi_{f,p}^{-1}) t_{\chi \psi_{f,p}^{-1}}^{dep} f_{\chi}^{ord}$.

By using now the relation,

$$\mathcal{ES}_{f \otimes \bar{f}, K(N) \tilde{K}_1(p^n)} \circ \text{tw}^{2n} = \mathcal{ES}_{f \otimes \bar{f}, K(N) K_1(p^n)} \circ \text{tw}^n \circ (u_p \times u_p)^n,$$

we obtain:

$$\kappa_\chi(f) = (u'_p \times u'_p)^{-r} (e' \times e') \mathcal{ES}_{f \otimes \bar{f}, K(N) \tilde{K}_1(p^r)} \left(\text{tw}^r(u_p \times u_p)^n f_\chi^{\text{dep}} \right)$$

Using the fact that $u_p \times u_p$ acts by $\alpha_f \alpha_{\bar{f}}$ on $\eta_f \otimes \eta_{\bar{f}}$, we have that $\log_{\mathbb{Q}_{p,n}}(\kappa_\chi(f))$ has the same pairing with $\eta_f \otimes \eta_{\bar{f}}$ as

$$(\alpha_f \alpha_{\bar{f}})^{-r} \log_{\mathbb{Q}_{p,n}} \left[(e' \times e') \mathcal{ES}_{K(N) K_1(p^r)} (\text{tw}^r(u_p \times u_p)^n f_\chi^{\text{dep}}) \right].$$

Now by invoking the fact that at level $K_1(p^r)$, $(e' \times e') \text{tw}^r = \text{tw}^r(e \times e)$, and the fact that $\text{tw}^{-r}(\eta_f \otimes \eta_{\bar{f}}) = \lambda_{Np^r}(f) \lambda_{Np^r}(\bar{f}) \eta_f \otimes \eta_{\bar{f}}$ and rearranging the factors, we obtain that

$$\underline{\mathcal{L}}\left(\mathcal{C}_p(\mathbf{f})^-\right)(f, \varepsilon^{-1} \chi^{-1}) = -c_f \left\langle \log_{\mathbb{Q}_{p,n}}(\kappa'_\chi, \eta_f \otimes \eta_{\bar{f}}) \right\rangle_{\text{dR}},$$

where

$$\kappa_\chi = (e \times e) (\alpha_f \alpha_{\bar{f}})^{-n} \mathcal{ES}_{K(N) K_1(p^n)} ((u_p \times u_p)^n f_\chi^{\text{dep}})$$

□

4.2.2.9 Definition. Consistently with the notations of the appendix (see section (B.2)), we set

$$\omega_\chi^{\text{et}} := \omega^{\text{et}}(-\tau(\chi \psi_{f,p}^{-1}) t_{\chi \psi_{f,p}^{-1}}^{\text{dep}}(f_\chi^{\text{ord}})).$$

4.2.2.10 Remark. Note that we have

$$(e \times e)\omega_\chi^{\text{et}}|_{\mathbf{H}} = (e \times e)\omega^{\text{et}}(f_\chi^{Liu})|_{\mathbf{H}}.$$

More precisely by Proposition (B.4.2) of the appendix

$$(u_p \times u_p)^n \omega_\chi^{\text{et}} = (u_p \times u_p)^n \omega^{\text{et}}(f_\chi^{Liu}).$$

We denote the Eisenstein series defined by $\tau(\chi\psi_{f,p}^{-1})t_{\chi\psi_{f,p}^{-1}}^{dep}(f_\chi^{ord})$ simply by E_χ and the Eisenstein series defined by f_χ^{ord} by E_χ^{ord} , the corresponding étale class will be denoted by $\omega_\chi^{\text{et},ord}$.

We have the following corollary to Theorem 6.6 using the above remarks:

4.2.2.11 Corollary. *Let the notations be as in Theorem 4.2.2.8. We have*

$$\begin{aligned} \underline{\mathcal{L}}\left(\underline{\mathcal{C}}_p(\mathbf{f})^-\right)(f, \varepsilon^{-1}\chi^{-1}) &= c_f \cdot \langle \log_{\mathbb{Q}_{p,n}}(\mathcal{ES}_{K(N)\tilde{K}_1(p^n)}(\omega^{\text{et}}(f_\chi^{Liu})^-, \eta_f \otimes \eta_{\bar{f}}) \rangle_{\text{dR}} \\ &= c_f \cdot \langle \log_{\mathbb{Q}_{p,n}}(\mathcal{ES}_{K(N)\tilde{K}_1(p^n)}(\omega_\chi^{\text{et}})^-, \eta_f \otimes \eta_{\bar{f}}) \rangle_{\text{dR}}. \end{aligned}$$

From now on we consider f a weight 2 arithmetic specialization of a cuspidal new branch of $\mathbf{f}$. We recall that we denote the Nebentypus of f by ψ_f .

4.2.3 The Eisenstein class in p -adic cohomology.

For simplicity, in this section we first argue under the following assumption.

We assume *temporarily* that χ is $\mathbb{Q}_p$ -valued; that is $F_{\chi,p} = \mathbb{Q}_p$.

In this section we describe the image of the étale Eisenstein class in Hyodo-Kato cohomology.

Fix $t \in B_{\text{dR}}^+$ a uniformizer of B_{dR}^+ on which $\text{Gal}_{\mathbb{Q}_p}$ acts by the cyclotomic character. Recall that t is unique up to a unit in $\mathbb{Z}_p^\times$. Consider the element

$$\omega_\chi^{\text{et}} \otimes t^3 \in H_{\text{et}}^3(Y_{\mathbf{G}}/\overline{\mathbb{Q}_p}, \mathbb{Q}_p) \otimes B_{\text{st}}.$$

Note that $\text{Gal}_{\mathbb{Q}_p}$ acts on $\omega_\chi^{\text{et}} \otimes t^3$ by the character χ_p^{-1} , since by proposition 3.3.8.1 $\text{Gal}_{\mathbb{Q}}$ acts on ω_χ^{et} by $\varepsilon^{-3}\chi^{-1}$. Therefore, we have

$$\omega_\chi^{\text{et}} \otimes t^3 \in \mathbf{D}_{\text{st}}(H_{\text{et}}^3(Y_{\mathbf{G}}/\overline{\mathbb{Q}_p}, \mathbb{Q}_p)),$$

where Fontaine's functor $\mathbf{D}_{\text{st}}$ is taken with respect to $K := \mathbb{F}_{\chi,p}$. By Theorem 4.1.5.2, there exists a class $\omega_\chi^{\text{HK}} \in H_{\text{HK}}^3(Y_{\mathbf{G},h}/\mathbb{F}_{\chi,p})$ such that

$$\alpha_{\text{HK}}(\omega_\chi^{\text{HK}}) = \omega_\chi^{\text{et}} \otimes t^3.$$

4.3 The first Explicit Reciprocity Law

In this subsection, we give a first version of Explicit Reciprocity law: we will characterize the non-triviality of the extension represented by the Euler system base class z_1^χ .

Set

$$W := \text{Ad}^0(\rho_f)(-1)$$

By construction, the extension class c_χ^{et} is defined by the following diagram:

$$\begin{array}{ccccccc}
0 & \longrightarrow & \frac{H_{\text{et}}^2(\text{SH}_{\mathbf{H}}/\overline{\mathbb{Q}_p}, \mathbb{Q}_p)}{\text{im} H_{\text{et}}^2(\text{SH}_{\mathbf{G}}/\overline{\mathbb{Q}_p}, \mathbb{Q}_p)} & \longrightarrow & H_{\text{et}}^3((\text{SH}_{\mathbf{G}}, \text{SH}_{\mathbf{H}})/\overline{\mathbb{Q}_p}, \mathbb{Q}_p) & \longrightarrow & H_{\text{et}}^3(\text{SH}_{\mathbf{G}}/\overline{\mathbb{Q}_p}, \mathbb{Q}_p) \longrightarrow 0 \\
& & \parallel & & \vdots & & \uparrow \\
0 & \longrightarrow & \frac{H_{\text{et}}^2(\text{SH}_{\mathbf{H}}/\overline{\mathbb{Q}_p}, \mathbb{Q}_p)}{\text{im} H_{\text{et}}^2(\text{SH}_{\mathbf{G}}/\overline{\mathbb{Q}_p}, \mathbb{Q}_p)} & \longrightarrow & \mathcal{E}_\chi & \longrightarrow & \mathbb{Q}_p \omega_\chi^{\text{et}} \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \parallel \\
0 & \longrightarrow & W & \longrightarrow & \mathcal{E}_p & \longrightarrow & \mathbb{Q}_p \omega_\chi^{\text{et}} \longrightarrow 0.
\end{array}$$

From here on we denote by W_χ the one dimensional Galois representation spanned by ω_χ^{et} , i.e. $W_\chi := F_{\chi,p} \omega_\chi^{\text{et}}$. Let $W_\chi^\vee$ be the dual of W_χ . Tensoring the bottom row with the line spanned by $W_\chi^\vee$, we obtain the extension:

$$0 \longrightarrow W \otimes W_\chi^\vee \longrightarrow \mathcal{E}_p \longrightarrow W_\chi \otimes W_\chi^\vee \longrightarrow 0,$$

where $\mathcal{E}_p$ is the extension of $\text{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_{p,n})$ represented by c_χ^{et} prior to identifying $W_\chi^\vee$ (resp. W_χ) with $F_{\chi,p}(\varepsilon^3 \chi)$ (resp. $F_{\chi,p}(\varepsilon^{-3} \chi^{-1})$). Note that by the dimension computations of [BK07], we have $H_f^1(\mathbb{Q}_p, W \otimes W_\chi^\vee) = H^1(\mathbb{Q}_p, W \otimes W_\chi^\vee)$, so in particular $\mathcal{E}_p$ is semi-stable.

For simplicity we set henceforth

$$K := \mathbb{Q}_{p,n}.$$

Now apply Fontaine's $\mathbf{D}_{\text{st}}$ functor (over K) to the extension above. We obtain an extension of filtered $(\varphi, \mathcal{N}, G_K)$ -modules:

$$0 \longrightarrow \mathbf{D}_{\text{st}}(W \otimes W_\chi^\vee) \longrightarrow \mathcal{E}^{\text{st}} \longrightarrow \mathbf{D}_{\text{st}}(W_\chi \otimes W_\chi^\vee) \longrightarrow 0,$$

where we have set $\mathcal{E}^{\text{st}} := \mathbf{D}_{\text{st}}(\mathcal{E}_p)$. We are interested in the non-triviality of this extension and in particular in computing its Bloch-Kato logarithm. In the following subsections, we will use notions from p -adic Hodge theory to compute it. More precisely, we will consider the following extension deduced from $\mathcal{E}^{\text{st}}$. Projecting W onto its one dimensional quotient $W^- := \text{Ad}^0(\rho_f)^{-}(-1)$ we obtain by push-out of the extension above an extension $(\mathcal{E}^{\text{st}})^-$ given by

$$0 \longrightarrow \mathbf{D}_{\text{st}}(W^- \otimes W_{\chi}^{\vee}) \longrightarrow (\mathcal{E}^{\text{st}})^- \longrightarrow \mathbf{D}_{\text{st}}(W_{\chi} \otimes W_{\chi}^{\vee}) \longrightarrow 0.$$

4.3.1 The semistable complex

In this subsection, we follow mostly [Nek93, Section 1] and [NW16, Section 2.4].

Let D be an object in the category of weakly admissible filtered Frobenius G_K -modules. Denote by $C_{\text{st}}^{\bullet}(D)$ the following complex

$$C_{\text{st}}^{\bullet}(D) : \quad D_{\text{st}} \xrightarrow{d_0} D_{\text{st}} \oplus D_{\text{st}} \oplus D_K / \text{Fil}^0 D_K \xrightarrow{d_1} D_{\text{st}},$$

where d_0 is the map

$$d_0 : u \mapsto ((1 - \varphi)u, \mathcal{N}u, -u)$$

and d_1 is the map

$$d_1 : (w, x, y) \mapsto \mathcal{N}w - (1 - p\varphi)x.$$

We write $H_{\text{st}}^i(D)$ for the cohomology of $C_{\text{st}}^{\bullet}(D)$.

Then as is recalled in [NW16], the equivalence $D \mapsto \mathbf{V}(D)$ induces functorial maps $H_{\text{st}}^i(D) \rightarrow H^i(K, \mathbf{V}(D))$. When $i = 0$, the map is an isomorphism and when $i = 1$, it is an injection whose image is the semistable cohomology of $\mathbf{V}(D)$, which we will denote by $H_{\text{st}}^1(K, \mathbf{V}(D))$. This abelian group parametrizes Yoneda extensions of K by V in the category of semistable representations.

More precisely, following [Nek93, Section 1.20], let V be a semistable G_K module and suppose we are given an extension in the category of filtered $(\varphi, \mathcal{N})$ -modules

$$0 \longrightarrow \mathbf{D}_{\text{st}}(V) \longrightarrow D \longrightarrow K_0 \longrightarrow 0.$$

Here we recall that $K_0 = \mathbf{D}_{\text{st}}(K)$ stands for the maximal unramified subfield of K . Writing D_K for $D \otimes_{K_0} K$ we have a commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbf{D}_{\text{dR}}(V) & \longrightarrow & D_K & \longrightarrow & K & \longrightarrow & 0 \\ & & \downarrow & & \downarrow \pi_0 & & \downarrow & & \\ 0 & \longrightarrow & \mathbf{D}_{\text{dR}}(V)/\text{Fil}^0 \mathbf{D}_{\text{dR}}(V) & \xrightarrow{\sim} & D_K/\text{Fil}^0 D_K & \longrightarrow & 0 & \longrightarrow & 0 \end{array}$$

Fix a choice of a K_0 -linear section $s : K_0 \rightarrow D$. Define a triple (a, b, c) by

$$(a, b, c) := ((1 - \varphi)s(1), \mathcal{N}s(1), -\pi_0 \circ \text{can}(s(1))) \in \mathbf{D}_{\text{st}}(V) \oplus \mathbf{D}_{\text{st}}(V) \oplus \mathbf{D}_{\text{dR}}(V)/\text{Fil}^0,$$

where can denotes the canonical map $D \rightarrow D_K$. This triple is a 1-cocycle in the complex $C_{\text{st}}^\bullet(V)$. Moreover, it is readily checked that its class in $H^1(C_{\text{st}}^\bullet(V))$ does not depend on the choice of s . We recall also that by Section 1.19 and Proposition 1.24

of [Nek93], there are isomorphisms

$$H^1(C_{\text{st}}^\bullet(V)) \simeq H_{\text{st}}^1(K, V) \simeq H_g^1(\mathbb{Q}_p, V).$$

Let E be the semistable G_K -representation such that $\mathbf{D}_{\text{st}}(E) = D$ which exists by Corollary 1.22 of [Nek93].

4.3.1.1 Lemma. *Assume now that the semistable Frobenius φ does not have eigenvalue 1 on $\mathbf{D}_{\text{st}}(V)$. Then there is a well defined map*

$$H_{\text{st}}^1(K, V) \rightarrow \mathbf{D}_{\text{dR}}(V)/\text{Fil}^0 \mathbf{D}_{\text{dR}}(V)$$

induced by the map $C_{\text{st}}^1(V) \rightarrow \mathbf{D}_{\text{dR}}(V)/\text{Fil}^0 \mathbf{D}_{\text{dR}}(V)$

$$(a, b, c) \mapsto c - \text{can}((1 - \varphi)^{-1}a) \bmod \text{Fil}^0 \mathbf{D}_{\text{dR}}(V).$$

It is the Bloch–Kato logarithm of the extension.

We can sum this analysis in the following proposition

4.3.1.2 Proposition. *Let*

$$0 \longrightarrow V \longrightarrow E \longrightarrow K \longrightarrow 0$$

be an extension of semi-stable G_K representations. Let $[E] \in H_{\text{st}}^1(K, V)$ be the cohomology class of this extension. Let $D := \mathbf{D}_{\text{st}}(E)$ and $D_K := \mathbf{D}_{\text{dR}}(V) \simeq D \otimes_{K_0} K$. Consider $s : K_0 \rightarrow E$ be a K_0 -linear section which induces a sections still denoted

by $s : K_0 \rightarrow D$ and $s : K \rightarrow D_K$. Let $\pi_0 : D_K \rightarrow D_K/\text{Fil}^0 D_K$ be the natural projection and the map $V \hookrightarrow E$ induces an identification $\delta : \mathbf{D}_{\text{dR}}(V)/\text{Fil}^0 \mathbf{D}_{\text{dR}}(V) \simeq D_K/\text{Fil}^0 D_K$. With this identification, we have

$$[E] = [(1 - \varphi)s(1), \mathcal{N}s(1), -\pi_0(s(1))],$$

and

$$\log_K([E]) = -\pi_0(s(1)) - (1 - \varphi)^{-1} \delta^{-1}((1 - \varphi)s(1))$$

4.3.1.3 Remark. For our particular extension, the map δ of the above proposition will come from the connecting homomorphism coming from a long exact cohomology sequence. To compute explicitly the identification of $(1 - \varphi)s(1)$ we will see that we will have to take a p -adic "primitive" of the differential associated to the Eisenstein series.

In the case of interest to us, we recall that we have $V = W \otimes W_\chi^\vee$, $D := \mathbf{D}_{\text{st}}(V)$. By the dimension formulae of [BK07], we have $H_f^1(K, V) = H^1(K, V)$ so that $H_{\text{st}}^1(K, V) = H^1(K, V)$ and $\mathcal{E}_p$ is semistable. Moreover $\text{Fil}^0 \mathbf{D}_{\text{dR}}(V) = 0$. Moreover

To apply Proposition 4.3.1.2, we have to make some identifications. First of all, we have to identify $W_\chi \otimes W_\chi^\vee$ with the trivial representation of G_K . For this we fix the basis ω_χ^{et} of W_χ and its dual basis $\omega_\chi^{\text{et}, \vee}$. The elements $\omega_\chi^{\text{HK}} := \omega_\chi^{\text{et}} \otimes t^3$ and $\omega_\chi^{\text{HK}, \vee} := \omega_\chi^{\text{et}, \vee} \otimes t^{-3}$, provide a canonical basis of

$$\mathbf{D}_{\text{st}}(W \otimes W_\chi^\vee) \subseteq \mathbf{D}_{\text{st}}(H_{\text{et}}^3(Y_{\mathbf{G}}/\overline{K}, F_{\chi, p})) \simeq H_{\text{HK}}^3(Y_{\mathbf{G}, h}/K).$$

We denote by the same symbols the corresponding elements as cohomology classes in $H_{\text{HK}}^3(Y_{\mathbf{G},h}/K)$. We let respectively ω_χ^{dR} and $\omega_\chi^{\text{dR},\vee}$ be the images under the map induced in cohomology by ι_{dR} of Proposition 4.1.5.1 of $\omega_\chi^{\text{HK}} \otimes 1$ and $\omega_\chi^{\text{HK},\vee} \otimes 1$.

We now ought to define a specific lift of ω_χ^{HK} in relative Hyodo-Kato cohomology i.e., we need to specify a section s in the notations of Proposition 4.3.1.2. We do this as follows. Choose an affine covering $\mathcal{U} = \cup_i U_i$ of $Y_{\mathbf{G}}$. And consider $\text{Tot}^\bullet(\check{C}^\bullet(\mathcal{U}, \Omega_{Y_{\mathbf{G}}}^\bullet))$ the total complex of the Čech complex of $\Omega_{Y_{\mathbf{G}}}^\bullet$ relative to the covering $\mathcal{U}$. Consider the canonical map

$$\text{Tot}^\bullet(\check{C}^\bullet(\mathcal{U}, \Omega_{Y_{\mathbf{G}}}^\bullet)) \longrightarrow \text{R}\Gamma(Y_{\mathbf{G}}, \Omega_{Y_{\mathbf{G}}}^\bullet),$$

which is a quasi-isomorphism as the covering $\mathcal{U}$ is affine. Recall that ω_χ^{et} is associated to an algebraic differential ω_χ defined by a holomorphic Eisenstein series. As $K \supseteq F_{\chi,p}$, the differential ω_χ defines an algebraic differential over K . The restrictions to the opens forming covering define a natural map

$$H^0(X, \Omega^3 X) \in Z\left(\text{Tot}^3(\check{C}^\bullet(\mathcal{U}, \Omega_{Y_{\mathbf{G}}}^\bullet))\right)$$

where $Z(\cdot)$ indicates cocycles. Moreover, there is a natural map

$$\text{Tot}^\bullet(\check{C}^\bullet(\mathcal{U}, \Omega_{Y_{\mathbf{G}}}^\bullet)) \rightarrow \text{Tot}^\bullet(\check{C}^\bullet(\mathcal{U}_{\mathbf{H}}, \Omega_{Y_{\mathbf{H}}}^\bullet)) \quad (4.1)$$

where $\mathcal{U}_{\mathbf{H}}$ is the pullback to $Y_{\mathbf{H}}$ of the covering $\mathcal{U}$ which is still affine as $Y_{\mathbf{H}} \hookrightarrow Y_{\mathbf{G}}$ is a closed subscheme. We can compute relative de Rham cohomology by form-

ing the mapping fibre of the map of complexes 4.1. Note that the image of ω_χ in $\text{Tot}^3(\check{C}^\bullet(\mathcal{U}, \Omega_{Y_G}^\bullet))$ has an obvious lift in the mapping fibre given by $(\omega_\chi, 0)$. We denote the resulting cohomology class in relative de Rham cohomology by ω_χ^{rel} . By construction, ω_χ^{rel} defines an element in $\text{Fil}^3 \mathbf{D}_{\text{dR}}(H_{\text{et}}^3(Y_G, Y_H))$ and thus $\omega_\chi^{\text{rel}} \otimes \omega_\chi^{\text{dR}, \vee}$ defines an element of $\text{Fil}^0 \mathbf{D}_{\text{dR}}(\mathcal{E}_p)$.

Since $\mathcal{E}_p$ is semistable, there is a filtered isomorphism $\mathcal{E}^{\text{st}} \otimes_{K_0} K = \mathbf{D}_{\text{st}}(\mathcal{E}_p) \otimes_{K_0} K \simeq \mathbf{D}_{\text{dR}}(\mathcal{E}_p)$. Since the Fil^0 of the respective sides is one dimensional this implies that there is a unique element $\omega_\chi^{\text{HK}, \text{rel}}$ such that

$$\omega_\chi^{\text{HK}, \text{rel}} \otimes \omega_\chi^{\text{HK}, \vee} \in \text{Fil}^0(\mathcal{E}^{\text{st}})$$

maps to $\omega_\chi^{\text{HK}} \otimes \omega_\chi^{\text{HK}, \vee} \in \mathbf{D}_{\text{st}}(W_\chi \otimes W_\chi^\vee)$. This lift $\omega_\chi^{\text{HK}, \text{rel}}$ defines the required section s to compute the Bloch-Kato exponential.

4.3.1.4 Definition. Define the polynomial $P(T) := (1 - p^{-3}T)$.

The following Lemma follows immediately from the definition of $\omega_\chi^{\text{HK}} = \omega_\chi^{\text{et}} \otimes t^3$ as $\varphi(t) = pt$.

4.3.1.5 Lemma. *We have $P(\varphi)\omega_\chi^{\text{HK}} = 0$ (as cohomology classes) where φ is the Frobenius of Beilinson's Hyodo-Kato cohomology.*

The polynomial P will appear as we have

$$(1 - \varphi)(\omega_\chi^{\text{HK}} \otimes \omega_\chi^{\text{HK}, \vee}) = P(\varphi)\omega_\chi^{\text{HK}} \otimes \omega_\chi^{\text{HK}, \vee} = 0$$

4.3.1.6 The classes for the ordinary Eisenstein series

Note that the classes ω_χ^{HK} , ω_χ^{dR} , etc... have completely analogous classes by using, instead of ω_χ^{et} , the étale class $\omega_\chi^{\text{et}, \text{ord}}$, as defined in Definition 4.2.2.9. We will add a superscript $_{\text{ord}}$ to denote them. For example, $\omega_\chi^{\text{et}, \text{ord}} \otimes t^3 \in \mathbf{D}_{\text{st}}(H_{\text{et}}^3(Y_{\mathbf{G}}/\overline{K}, F_{\chi, p}))$ will be denoted by $\omega_\chi^{\text{HK}, \text{ord}}$.

4.3.1.7 Removing the simplifying condition that χ is $\mathbb{Q}_p$ -valued.

We now remove the simplifying assumption that χ is $\mathbb{Q}_p$ -valued; that is we no longer assume that $F_{\chi, p} = \mathbb{Q}_p$. Note that we have, as $F_{\chi, p} \subset K$

$$\mathbf{D}_{\text{st}}(H_{\text{et}}^m(X/\overline{K}, F_{\chi, p})) = \mathbf{D}_{\text{st}}(H_{\text{et}}^m(X/\overline{K}, \mathbb{Q}_p)) \otimes F_{\chi, p}^0.$$

here the superscript 0 indicates the maximal unramified sub-extension of $F_{\chi, p}$. Therefore, the comparison isomorphism between étale and Hyodo-Kato given by theorem 4.1.5.2 can be reformulated for $F_{\chi, p}$ coefficients as:

$$\alpha_{\text{HK}} : H_{\text{HK}}^m(X_h)_{F_{\chi, p}} \xrightarrow{\sim} \mathbf{D}_{\text{st}}(H_{\text{et}}^m(X/\overline{K}, F_{\chi, p})),$$

where

$$H_{\text{HK}}^m(X_h)_{F_{\chi, p}} := H_{\text{HK}}^m(X_h) \otimes_{K_0} F_{\chi, p}^0.$$

When χ is not $\mathbb{Q}_p$ valued we can still define ω_χ^{HK} as the element corresponding to $\omega_\chi^{\text{et}} \otimes t^3$ under the above comparison theorem. The above discussion then generalizes to the case where χ is not $\mathbb{Q}_p$ -valued simply by noticing that $\mathbf{D}_{\text{st}}(K) = K_0$.

For $\mathcal{F} = f \otimes \bar{f}$, let $\text{proj}_{\mathcal{F}}$ be the natural projector $H_{\text{et}}^2(\text{SH}_{\mathbf{H}}/\overline{\mathbb{Q}_p}) \rightarrow H_{\text{et}}^2(\text{SH}_{\mathbf{H}}/\overline{\mathbb{Q}_p})[\mathcal{F}]$; here $H_{\text{et}}^2(\text{SH}_{\mathbf{H}}/\overline{\mathbb{Q}_p})(3)[\mathcal{F}]$ is the maximal quotient of $H_{\text{et}}^2(\text{SH}_{\mathbf{H}}/\overline{\mathbb{Q}_p})(3)$ on which the Hecke algebra acts by $\phi_f \otimes \phi_{\bar{f}}$, with notations as in Section 3.4.5.4. It induces a map still denoted by $\text{proj}_{\mathcal{F}}$:

$$\text{proj}_{\mathcal{F}} : \mathbf{D}_{\text{st}}\left(H_{\text{et}}^2(\text{SH}_{\mathbf{H}}/\overline{\mathbb{Q}_p})(3)\right) \longrightarrow \mathbf{D}_{\text{st}}(W)$$

We can now apply Proposition 4.3.1.2 to obtain the following Proposition, noticing that $\omega_{\chi}^{\text{HK,rel}} \otimes \omega_{\chi}^{\text{HK},\vee}$ lies in $\text{Fil}^0 \mathcal{E}^{\text{st}}$.

4.3.1.8 Proposition. *We have the equality*

$$\log_K([\mathcal{E}_p]) = \delta_{\text{dR}}^{-1} \iota_{\text{dR}} \left(P(\varphi)^{-1} \text{proj}_{\mathcal{F}} P(\varphi) \omega_{\chi}^{\text{HK,rel}} \right) \otimes \omega_{\chi}^{\text{dR},\vee}$$

where δ_{dR} is the connecting homomorphism for the long exact sequence

$$\cdots \rightarrow H_{\text{dR}}^i(Y_{\mathbf{G}}, Y_{\mathbf{H}}) \rightarrow H_{\text{dR}}^i(Y_{\mathbf{G}}) \rightarrow H_{\text{dR}}^i(Y_{\mathbf{H}}) \rightarrow \cdots$$

and $\text{proj}_{\mathcal{F}}$ denotes the projection to $\mathcal{E}^{\text{st}}$ in relative Hyodo-Kato cohomology induced by $\text{proj}_{\mathcal{F}}$.

As a corollary of the above Proposition, by extending the action of the semistable Frobenius on K -linearly on $\mathbf{D}_{\text{dR}}(U) \simeq \mathbf{D}_{\text{st}}(U) \otimes_{K_0} K$ for U a semistable G_K -representation, we obtain the following Theorem.

4.3.1.9 Theorem (First Explicit Reciprocity Law). *Let $\eta_\chi^{\text{dR}} \in H_{\text{dR}}^2(Y_{\mathbf{H}})$ be such that*

$$\delta_{\text{dR}} \eta_\chi^{\text{dR}} = P(\varphi) \omega_\chi^{\text{dR,rel}}.$$

Then, we have

$$\log_K \left([\mathcal{E}_p] \right) = (P(\varphi)^{-1} \text{proj}_{\mathcal{F}} \eta_\chi^{\text{dR}}) \otimes \omega_\chi^{\text{dR},\vee}.$$

Here $\log_K$ is the Bloch-Kato logarithm over K .

We now deduce from the above Theorem the computation of the Bloch-Kato logarithm of the extension $\mathcal{E}_p^-$.

Note that the pairing $\langle \bullet, \eta_f \otimes \eta_{\bar{f}} \rangle_{\text{dR}}$ is induced by the pairing of Galois representations induced by Poincaré duality:

$$W \times W(2) \longrightarrow K$$

The Frobenius on $\mathbf{D}_{\text{st}}(W)$ is φ and the Frobenius on $W(2)$ is $p^{-2}\varphi$. Therefore, we have the relation, for all $\eta \in \mathbf{D}_{\text{st}}(W)$,

$$\langle \varphi \eta, p^{-2} \varphi \eta_f \otimes \eta_{\bar{f}} \rangle_{\text{st}} = \langle \eta, \eta_f \otimes \eta_{\bar{f}} \rangle_{\text{st}}.$$

From this we deduce that

$$\langle \varphi \eta, \eta_f \otimes \eta_{\bar{f}} \rangle_{\text{st}} = p^2 \alpha_f \alpha_{\bar{f}} \langle \eta, \eta_f \otimes \eta_{\bar{f}} \rangle_{\text{st}},$$

so

$$\langle (1 - p^{-3}\varphi)^{-1}\eta, \eta_f \otimes \eta_{\bar{f}} \rangle_{\text{st}} = (1 - p^{-1}\alpha_f\alpha_{\bar{f}})^{-1} \langle \eta, \eta_f \otimes \eta_{\bar{f}} \rangle_{\text{st}}.$$

We then obtain the following Corollary.

4.3.1.10 Corollary. *With notations as in Theorem 4.3.1.9, we have the equality:*

$$\underline{\mathcal{L}}\left(\mathcal{C}_p(\mathbf{f})^{-}\right)(f, \varepsilon^{-1}\chi^{-1}) = (1 - \alpha_f\alpha_{\bar{f}}p^{-1})^{-1} \cdot c_f \cdot \langle \eta_{\chi}^{\text{dR}}, \eta_f \otimes \eta_{\bar{f}} \rangle_{\text{dR}}.$$

This does not depend on the chosen η_{χ}^{dR} such that $\delta_{\text{dR}}\eta_{\chi}^{\text{dR}} = P(\varphi)\omega_{\chi}^{\text{dR,rel}}$ as $\langle \bullet, \eta_f \otimes \eta_{\bar{f}} \rangle$ factors through $\text{proj}_{\mathcal{F}}$.

4.3.1.11 Remark. Looking ahead, we will have to compute in terms of p -adic modular forms the de Rham pairing appearing in the formula of Corollary 4.3.1.10. A basis of $\mathbf{D}_{\text{dR}}(W)$ seen as a subspace of $\mathbf{D}_{\text{dR}}(\rho_f \otimes \rho_{\bar{f}})$ is given explicitly by $(\omega_f \otimes \omega_{\bar{f}}, \omega_f \otimes \eta_{\bar{f}} + \eta_f \otimes \omega_{\bar{f}}, \eta_f \otimes \eta_{\bar{f}})$, in the notations of section 4.2.1. In particular if the coordinate of η_{χ}^{dR} along $\omega_f \otimes \omega_{\bar{f}}$ is L_f we have that

$$\langle \eta_{\chi}^{\text{dR}}, \eta_f \otimes \eta_{\bar{f}} \rangle_{\text{dR}} = L_f.$$

Therefore it suffices to compute the coordinate of η^{dR} along the basis element $\omega_f \otimes \omega_{\bar{f}}$ of $\mathbf{D}_{\text{dR}}(W)$.

4.3.1.12 Pull-back to Sp_4

As the theory of p -adic modular forms on orthogonal groups is still in its infancy, we pull-back all the cohomology classes defined in the previous sections to the Shimura

variety attached to $\mathbf{Sp}(4)$. Therefore, from here on we will work on $\mathbf{G}'$ rather than on $\mathbf{G}$. As pulling back and then pushing forward is either the identity or multiplication by 2 depending on the level, this is innocuous. For notational simplicity, we will drop the superscript ' from the relevant data attached to $\mathbf{G}'$. This shouldn't cause any confusion as nothing the group $\mathbf{SO}_5$ will not appear anymore.

4.4 The second Explicit Reciprocity Law

We saw that in order to compute the Bloch-Kato logarithm of the relevant extension, we have to compute a primitive of ω_X^{HK} . The aim of this subsection is to relate the computation to a p -adic, rigid analytic of the Eisenstein series.

4.4.1 The Colmez-Niziol map

Let $K/\mathbb{Q}_p$ be a finite extension and let $\mathcal{O}_K$ be its ring of integers. Let K^{nr} the maximal unramified extension of K . We will eventually take $K := \mathbb{Q}_{p,n}$.

Let X be a smooth dagger variety over K . Recall that a dagger variety is a rigid analytic variety equipped with an overconvergent structure sheaf. We refer to [Gro00, Section 2] for the basic definitions and properties. Note that a dagger affinoid has de Rham cohomology that is a finite rank vector space with its natural Hausdorff topology, while de Rham cohomology of rigid analytic affinoids is, in general, infinite dimensional and not Hausdorff. For such X , Colmez-Niziol [CN19] constructed both Hyodo-Kato and rigid cohomology $\mathrm{R}\Gamma_{\mathrm{HK}}^{\mathrm{CN}}(X)$ which generalize the construction of Beilinson of Hyodo-Kato cohomology for algebraic varieties.

Let us give a very brief description of the construction. First, Colmez-Niziol show that the étale topology of X has a base consisting of semistable weak formal schemes over finite extension of $\mathcal{O}_K$ (see section 2.3 of [CN19]). The existence of such a base implies that to define étale sheaves on X it suffices to specify them on such integral models and then sheafifying for the étale topology induced from the étale topology of the generic fiber, thus mimicking the strategy of Beilinson [Bei13]. Let $\mathcal{O}_K^\times$ denote the scheme associated to $\mathcal{O}_K$ with the canonical log-structure (attached to the closed point of $\mathcal{O}_K$) and let $\mathcal{O}_K^0$ denote the scheme associated to $\mathcal{O}_K$ with the log-structure induced from the map of monoids $\mathbb{N} \rightarrow \mathcal{O}_K$, $1 \mapsto 0$. Colmez-Niziol define an étale sheaf $\mathcal{A}_{\text{HK}}$ induced from a presheaf assigning to a semistable weak formal model $\mathcal{Y}$ (i.e. an object of the category denoted by $\mathcal{M}_K^{ss,\dagger}$ in [CN19]), the complex $\text{R}\Gamma_{\text{HK}}^{\text{GK}}(\mathcal{Y}_0)$ where $\text{R}\Gamma_{\text{HK}}^{\text{GK}}(\mathcal{Y}_0)$ is as in section 5.2 of *loc cit.*. Here $\mathcal{Y}_0$ denotes the special fiber of $\mathcal{Y}$, $\text{R}\Gamma_{\text{HK}}^{\text{GK}}$ is Grosse-Klönne's overconvergent Hyodo-Kato cohomology. Let us note here that the Hyodo-Kato isomorphism of Colmez-Niziol (Theorem 1.1 of *loc cit.*) rigid cohomology is induced from Grosse-Klönne's strict quasi-isomorphism $\iota_{\text{HK}}^{\text{GK}} : \text{R}\Gamma_{\text{HK}}^{\text{GK}}(\mathcal{Y}_0) = \text{R}\Gamma_{\text{rig}}(\mathcal{Y}_0/\mathcal{O}_K^0) \simeq \text{R}\Gamma_{\text{rig}}(\mathcal{Y}_0/\mathcal{O}_K^\times)$ (see for example [CDN20, Section 3.1.3]). Here this isomorphism is *functorial* as defined by Ertl-Yamada [EY22, Proposition 2.5].

We will now freely use notations introduced at the end of section 4.2. As in [Har+16, Section 6.3] and [Gro00, Section 3.3], one can naturally attach to $Y_{\mathbf{G}}$, $Y_{\mathbf{G}}^{\text{tor}}$ and $Y_{\mathbf{G}}^\star$, dagger spaces $Y_{\mathbf{G}}^\dagger$, $Y_{\mathbf{G}}^{\text{tor},\dagger}$ and $Y_{\mathbf{G}}^{\star,\dagger}$, with $Y_{\mathbf{G}}^\dagger$, $Y_{\mathbf{G}}^{\text{tor},\dagger}$ smooth dagger spaces, i.e. objects of the category denoted by $\text{Sm}_K^\dagger$ in [CN19]. Let $Y_{\mathbf{G}}^{\text{ord},\dagger}$, $Y_{\mathbf{G}}^{\text{ord,tor},\dagger}$ and $Y_{\mathbf{G}}^{\text{ord},\star,\dagger}$ be the ordinary loci of $Y_{\mathbf{G}}^\dagger$, $Y_{\mathbf{G}}^{\text{tor},\dagger}$ and $Y_{\mathbf{G}}^{\star,\dagger}$: these spaces are admissible open sub-dagger

spaces of respectively $Y_{\mathbf{G}}^{\dagger}$, $Y_{\mathbf{G}}^{\text{tor},\dagger}$ and $Y_{\mathbf{G}}^{\star,\dagger}$ as is explained in [Har+16, p.3]. We note here that by [Lan13] these open sub-dagger spaces have models over $\mathbb{Z}_p$ that are smooth.

Take now $X \in \{Y_{\mathbf{G}}, Y_{\mathbf{G}}^{\text{tor}}\}$. Let us remark that there exists a natural map $R\Gamma_{\text{HK}}(X_h) \rightarrow R\Gamma_{\text{HK}}^{\text{CN}}(X^{\dagger})$. Indeed, Beilinson [Bei11] shows that the h -topology on a variety over K has a basis consisting semistable models over finite extensions of $\mathcal{O}_K$, therefore one has a natural map of sites from the h -site of X to the étale site of $X^{\dagger}$. Moreover, for semistable Y there is a natural map $R\Gamma_{\text{HK}}(Y_0) \rightarrow R\Gamma_{\text{HK}}^{\text{GK}}(Y_0^{\dagger})$ (see for example [EY22]). Hence we deduce the existence of a natural map

$$R\Gamma_{\text{HK}}(X_h) \longrightarrow R\Gamma_{\text{HK}}^{\text{CN}}(X^{\dagger}).$$

Note that this map is compatible with the natural Frobenius structure of both sides

We are now ready to introduce a restriction map that will allow us to compute the primitive of the Eisenstein series. We let as above $X \in \{Y_{\mathbf{G}}, Y_{\mathbf{G}}^{\text{tor}}\}$.

4.4.1.1 Definition (The Colmez-Niziol ordinary locus restriction). Let $\text{Res}_{\text{CN}}^{\text{ord}}$ be the Frobenius-compatible map defined by the following diagram:

$$\begin{array}{ccccccc} R\Gamma_{\text{HK}}(X_h) & \longrightarrow & R\Gamma_{\text{HK}}^{\text{CN}}(X^{\dagger}) & \longrightarrow & R\Gamma_{\text{HK}}^{\text{CN}}(X^{\dagger,\text{ord}}) & \longrightarrow & R\Gamma_{\text{rig}}(X_0^{\dagger,\text{ord}}) \\ & & & & \searrow & & \nearrow \\ & & & & \text{Res}_{\text{CN}}^{\text{ord}} & & \end{array}$$

Here the middle map is simply the restriction to a sub-dagger variety; $R\Gamma_{\text{rig}}(X_0^{\dagger,\text{ord}})$ denotes Berthelot's rigid cohomology for smooth rigid analytic varieties, and the

last map is a strict quasi-isomorphism by proposition 4.11 of [CN19].

We let $R\Gamma_{\text{dR}}^{\text{CN}}(X^\dagger)$ be the de Rham cohomology theory constructed by Colmez-Niziol in [CN19], by a totally analogous procedure to $R\Gamma_{\text{HK}}^{\text{CN}}(X^\dagger)$. Let $R\Gamma_{\text{dR-rig}}(X^\dagger) := R\Gamma(X^\dagger, \Omega_X^{\dagger, \bullet})$ be the rigid analytic de Rham cohomology; the cohomology of $R\Gamma_{\text{dR-rig}}(X^\dagger)$ is the hypercohomology of the overconvergent de Rham complex $\Omega_X^{\dagger, \bullet}$. We denote it by $H_{\text{dR-rig}}^i(X^\dagger)$. By Lemma 6.8 of [Har+16], there is a functorial isomorphism $H_{\text{rig}}^i(X_0^{\dagger, \text{ord}}) \otimes_{K_0} K \rightarrow H_{\text{dR-rig}}^i(X^{\dagger, \text{ord}})$ induced by a natural map $R\Gamma_{\text{rig}}(X^{\dagger, \text{ord}}) \otimes_{K_0} K \rightarrow R\Gamma_{\text{dR-rig}}(X^{\dagger, \text{ord}})$. The natural Hyodo-Kato comparison maps $R\Gamma_{\text{HK}}^{\text{GK}}(Y_0) \rightarrow R\Gamma_{\text{dR-rig}}(Y)$ induce a commutative diagram:

$$\begin{array}{ccc} R\Gamma_{\text{HK}}(X_h) & \longrightarrow & R\Gamma_{\text{HK}}^{\text{CN}}(X^\dagger) \\ \downarrow & & \downarrow \\ R\Gamma_{\text{dR}}(X) & \longrightarrow & R\Gamma_{\text{dR}}^{\text{CN}}(X^\dagger) \end{array}$$

And we still denote by $\text{Res}_{\text{CN}}^{\text{ord}}$ the map

$$\text{Res}_{\text{CN}}^{\text{ord}} : R\Gamma_{\text{dR}}(X) \rightarrow R\Gamma_{\text{dR-rig}}(X^{\dagger, \text{ord}})$$

- it is compatible with the Colmez-Niziol restriction for Hyodo-Kato cohomology under the canonical maps $R\Gamma_{\text{HK}}(X_h) \otimes_{K_0} K \rightarrow R\Gamma_{\text{dR}}(X)$ and $R\Gamma_{\text{rig}}(X^{\dagger, \text{ord}}) \otimes_{K_0} K \rightarrow R\Gamma_{\text{dR-rig}}(X^{\dagger, \text{ord}})$.

4.4.2 The computation over the ordinary locus

By the Künneth formula and the fact that the ordinary locus of modular curves is affinoid, we can deduce that $\text{Res}_{\text{CN}}^{\text{ord}} : H_{\text{dR}}^2(Y_{\mathbf{H}}/K) \rightarrow H_{\text{rig-dR}}^2(Y_{\mathbf{H}}^{\text{ord}})$ sends to zero the elements $\omega_f \otimes \eta_{\bar{f}} + \eta_f \otimes \omega_{\bar{f}}$ and $\eta_f \otimes \eta_{\bar{f}}$. Moreover it sends $\omega_f \otimes \omega_{\bar{f}}$ to the rigid de Rham cohomology class represented by the rigid analytic differential $\omega_f^{\text{rig}} \otimes \omega_{\bar{f}}^{\text{rig}}$ defined by the p -adic modular forms f and $\bar{f}$, which are overconvergent differential as f and $\bar{f}$ are nearly ordinary. Therefore we have the following proposition.

4.4.2.1 Proposition. *We have the following equality:*

$$(\mathbb{1}_f \boxtimes \mathbb{1}_{\bar{f}}) \text{Res}_{\text{CN}}^{\text{ord}}(\eta_{\chi}^{\text{dR}}) = \langle \eta_{\chi}^{\text{dR}}, \eta_f \otimes \eta_{\bar{f}} \rangle_{\text{dR}} \cdot [\omega_f^{\text{rig}} \otimes \omega_{\bar{f}}^{\text{rig}}].$$

The previous proposition shows that we can compute the Bloch-Kato logarithm of $(\mathcal{E}^{\text{st}})^-$ by restricting to the ordinary locus thanks to $\text{Res}_{\text{CN}}^{\text{ord}}$.

We now ought to express $\text{Res}_{\text{CN}}^{\text{ord}}(\eta_{\chi}^{\text{dR}})$ in terms of p -adic modular forms depending only on the p -adic modular form defined by the Eisenstein series E_{χ} . By Proposition 3.3.8.3 and Theorem [LLZ23, Theorem 3.3.10] – and the fact that the Colmez-Niziol map and the comparison Theorem of Lan et al. is compatible with Faltings' étale-de-Rham comparison, we know that $\text{Res}_{\text{CN}}^{\text{ord}}(\omega_{\chi}^{\text{dR}})$ is represented by the rigid analytic differential attached to the p -adic modular form E_{χ} , which we denote by $\omega_{\chi}^{\text{rig}} \in H_{\text{rig-dR}}^3(Y_{\mathbf{G}}^{\text{tor,ord}}, D_{\mathbf{G}}^{\text{tor,ord}})$. By the same argument as in [TX16, Section 3.4] by using Theorem 8.2.1.3 of [Lan13], we know that the cohomology $H_{\text{rig-dR}}^{\bullet}(Y_{\mathbf{G}}^{\text{tor,ord}}, D_{\mathbf{G}}^{\text{tor,ord}})$ is

computed by the complex

$$\left([H^0(Y_{\mathbf{G}}^{ord, \star, \dagger}, \Omega_{Y_{\mathbf{G}}^{ord, \star, \dagger}}^{\bullet}(-D^{\text{tor}, ord})), d] \right)$$

where d is the differential induced by the universal derivation of the de Rham complex.

Moreover by Proposition 3.3.8.3, we have a diagram (where we omit the $\mathcal{O}_{\chi, p}(3)$ coefficients):

$$\begin{array}{ccccccc} H_{\text{et}}^2(Y_{\mathbf{H}} \amalg D^{ord}) & \longrightarrow & H_{\text{et}}^3(Y_{\mathbf{G}}^{\partial-ord}, Y_{\mathbf{H}} \amalg D^{ord}) & \longrightarrow & H_{\text{et}}^3(Y_{\mathbf{G}}^{\partial-ord}) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ H_{\text{et}}^2(Y_{\mathbf{H}}) & \longrightarrow & H_{\text{et}}^3(Y_{\mathbf{G}}, Y_{\mathbf{H}}) & \longrightarrow & H_{\text{et}}^3(Y_{\mathbf{G}}) & \longrightarrow & 0 \end{array}$$

here the vertical maps are the natural pull-backs. After pulling-back by the inclusion of $\omega_{\chi}^{\text{et}} \in H_{\text{et}}^3(Y_{\mathbf{G}})$ and $\omega_{\chi}^{\text{et}, \partial-ord} \in H_{\text{et}}^3(Y_{\mathbf{G}}^{\partial-ord})$ and pushing out to the quotient $H_{\text{et}}^2(Y_{\mathbf{H}} \amalg D^{ord}) \rightarrow H_{\text{et}}^2(Y_{\mathbf{H}}) \rightarrow \text{Ad}^0(\rho_f)(2)$, we obtain isomorphic extensions.

Therefore, to compute $\text{proj}_{\mathcal{F}}(\eta_{\chi}^{\text{dR}})$, one can use the version of the extension with support condition along the ordinary boundary D^{ord} . Note that by construction the differential underlying $\omega_{\chi}^{\text{dR}, \text{rel}}$ defines actually an element of $H_{\text{dR}}^3(Y_{\mathbf{G}}^{\partial-ord}, Y_{\mathbf{H}} \amalg D^{\text{tor}, ord})$. Thus, viewing $\omega_{\chi}^{\text{dR}, \text{rel}}$ as an element of $H_{\text{dR}}^3(Y_{\mathbf{G}}^{\partial-ord}, Y_{\mathbf{H}} \amalg D^{ord})$, $\text{Res}_{\text{CN}}^{ord}(P(\varphi)\omega_{\chi}^{\text{dR}, \text{rel}})$ is identified with the cohomology class of the element $(P(\varphi)\omega_{\chi}^{\text{rig}}, 0) \in \text{MF}^3$, where MF^3 denotes the mapping fibre in degree 3 of the map of complexes:

$$\left([H^0(Y_{\mathbf{G}}^{ord, \star, \dagger}, \Omega_{Y_{\mathbf{G}}^{ord, \star, \dagger}}^{\bullet}(-D^{\text{tor}, ord})), d] \right) \rightarrow \left([H^0(Y_{\mathbf{H}}^{ord, \dagger}, \Omega_{Y_{\mathbf{H}}^{ord, \dagger}}^{\bullet}), d] \right).$$

Since $Y_{\mathbf{H}}^{ord}$ is affinoid, by the same argument as for $\mathbf{G}$, there exists an overconvergent p -adic modular form $g \in H^0(Y_{\mathbf{H}}^{ord, \dagger}, \Omega_{Y_{\mathbf{H}}^{ord, \dagger}}^2)$ such that $\text{Res}_{\text{CN}}^{ord}(\eta_{\chi}^{\text{dR}}) = [\omega_g^{\text{rig}}]$ where the bracket indicate the cohomology class of the rigid analytic differential.

Then by definition of η_{χ}^{dR} , functoriality, and Frobenius compatibility of $\text{Res}_{\text{CN}}^{ord}$, we have

$$[0, \omega_g^{\text{rig}}] = \delta_{\text{dR-rig}}([\omega_g^{\text{rig}}]) = [((1 - V)\omega_{\chi}^{\text{rig}}, 0)].$$

Here V is the usual operator acting on q -expansions as $V(\sum_T a_T q^T) = \sum_T a_T q^{pT}$ and $\delta_{\text{dR-rig}}$ is just the connecting homomorphism in the long exact sequence for the mapping fibre MF above. Therefore, by definition, there exists

$$h_{\chi} \in H^0(Y_{\mathbf{G}}^{ord, \star, \dagger}, \Omega_{Y_{\mathbf{G}}^{ord, \star, \dagger}}^2(-D_{\mathbf{G}}^{\text{tor}, ord}))$$

and

$$\alpha_{\chi} \in H^0(Y_{\mathbf{H}}^{ord, \dagger}, \Omega_{Y_{\mathbf{H}}^{ord, \dagger}}^1)$$

such that

$$dh_{\chi} = (1 - V)\omega_{\chi}^{\text{rig}} \quad \text{and} \quad h_{\chi}|_{Y_{\mathbf{H}}^{ord}} - d\alpha_{\chi} = -\omega_g^{\text{rig}}.$$

We therefore have the following Proposition:

4.4.2.2 Proposition. *With the notations of the previous paragraph, we have the equality*

$$(\mathbb{1}_f \boxtimes \mathbb{1}_{\bar{f}})(\text{Res}_{\text{CN}}^{ord}(\eta_{\chi}^{\text{dR}})) = (\mathbb{1}_f \boxtimes \mathbb{1}_{\bar{f}})[-h_{\chi}|_{Y_{\mathbf{H}}^{ord}}]$$

where $h_\chi \in H^0(Y_{\mathbf{G}}^{ord, \star, \dagger}, \Omega_{Y_{\mathbf{G}}^{ord, \star, \dagger}}^2(-D_{\mathbf{G}}^{\text{tor}, ord}))$ is a specific element such that

$$dh_\chi = (1 - V)\omega_\chi^{\text{rig}}.$$

We now ought to characterize to what extent the quantity $(\mathbb{1}_f \boxtimes \mathbb{1}_{\bar{f}})[h_\chi|_{Y_{\mathbf{H}}}]$ depends on the primitive h_χ of $(1 - V)\omega_\chi^{\text{rig}}$.

4.4.2.3 Proposition. *Assume that*

$$\langle \eta_\chi^{\text{dR}}, \eta_f \otimes \eta_{\bar{f}} \rangle_{\text{dR}} \neq 0,$$

that is assume that $\mathcal{E}_p^-$ is not split. If $h'_\chi \in H^0(Y_{\mathbf{G}}^{ord}, \Omega_{Y_{\mathbf{G}}^{ord}}^2)$ is such that $dh'_\chi = (1 - V)\omega_\chi^{\text{rig}}$, then

$$(\mathbb{1}_f \boxtimes \mathbb{1}_{\bar{f}})[h_\chi|_{Y_{\mathbf{H}}^{ord}}] = (\mathbb{1}_f \boxtimes \mathbb{1}_{\bar{f}})[h'_\chi|_{Y_{\mathbf{H}}^{ord}}].$$

Proof. Since $h_\chi - h'_\chi$ defines a cohomology class in $H_{\text{rig-dR}}^2(Y_{\mathbf{G}}^{ord})$, we deduce that $h_\chi|_{Y_{\mathbf{H}}^{ord}}$ and $h'_\chi|_{Y_{\mathbf{H}}^{ord}}$ get mapped to $[(1 - V)\omega_\chi^{\text{rig}}, 0]$. Moreover, $\mathbb{1}_f \boxtimes \mathbb{1}_{\bar{f}}$ maps $H^0(Y_{\mathbf{H}}^{ord}, \Omega_{Y_{\mathbf{H}}^{ord}}^2)$ to the line spanned by $\omega_f^{\text{rig}} \boxtimes \omega_{\bar{f}}^{\text{rig}}$, we therefore deduce that since the extension is not split, the image of $[(1 - V)\omega_\chi^{\text{rig}}, 0]$ is not zero, and therefore $\mathbb{1}_f \boxtimes \mathbb{1}_{\bar{f}}[h_\chi|_{Y_{\mathbf{H}}^{ord}}] = (\mathbb{1}_f \boxtimes \mathbb{1}_{\bar{f}})[h'_\chi|_{Y_{\mathbf{H}}^{ord}}]$ (a linear map whose source is a one dimensional vector space is either 0 or injective). $\square$

We give the summary of what we have proved in the following theorem.

4.4.2.4 Theorem (Second Explicit Reciprocity Law). *Let $\eta_\chi \in H^0(Y_{\mathbf{G}}^{ord}, \Omega_{Y_{\mathbf{G}}^{ord}}^2)$ be any element such that $d\eta_\chi = (1 - V)\omega_\chi^{\text{rig}}$. If the extension $\mathcal{E}_p^-$ is non-split, we have*

the equality:

$$\underline{\mathcal{L}}\left(\mathcal{C}_p(\mathbf{f})^-\right)(f, \varepsilon^{-1}\chi^{-1}) = -(1 - \alpha_f \alpha_{\bar{f}} p^{-1})^{-1} \cdot c_f \cdot (\mathbf{1}_f \boxtimes \mathbf{1}_{\bar{f}}) \eta_\chi|_{Y_{\mathbf{H}}^{ord}}.$$

In the following subsection we will ensure that for many members of the Hida family $\mathbf{f}$, the extension $\mathcal{E}_p^-$ must be non-split. In order to show this, we ought to consider a slightly modified extension.

4.4.3 The modified extension

Consider the elements

$$t_x = \begin{bmatrix} I & xs/p \\ 0 & I \end{bmatrix}$$

where $s = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \in \mathbf{G}(\mathbb{Q}_p)$ for $x = 0, 1, \dots, p-1$ and $s = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. These elements normalize the level group $\tilde{K}_1^{\mathbf{G}}(p^r)$ and thus act by translation on $Y_{\mathbf{G}}(\mathbb{C})$ and on the cohomology of $Y_{\mathbf{G}}$. We let

$$Y_{\mathbf{H}}(x) := Y_{\mathbf{H}} t_x \hookrightarrow Y_{\mathbf{G}},$$

be the translated Shimura varieties for $\mathbf{H}$. Note that the varieties $Y_{\mathbf{H}}(x)$ are disjoint since, as in Lemma 3.4.1.1, one can check that for all $\gamma \in \mathbf{G}(\mathbb{Q}) \cap t_x \tilde{K}_1(p^r)$, we have $\gamma(\mathcal{H} \times \mathcal{H}) \cap (\mathcal{H} \times \mathcal{H}) = \emptyset$. Here $\mathcal{H}$ denotes the Poincaré upper half-plane and $\mathcal{H} \times \mathcal{H}$ is the symmetric space for $\mathbf{H}$.

Now define

$$\tilde{Y} := \coprod_{x \bmod p} Y_{\mathbf{H}}(x).$$

We then have the extension

$$(\star) \quad 0 \rightarrow \frac{H_{\text{et}}^2(\tilde{Y}, F_{\chi,p})}{H_{\text{et}}^2(Y_{\mathbf{G}}, F_{\chi,p})} \rightarrow H_{\text{et}}^3(Y_{\mathbf{G}}, \tilde{Y}, F_{\chi,p}) \rightarrow H_{\text{et}}^3(Y_{\mathbf{G}}, F_{\chi,p}) \rightarrow 0$$

which pulls back by the inclusion $F_{\chi,p}\omega_{\chi}^{\text{et}} \hookrightarrow H_{\text{et}}^2(Y_{\mathbf{G}}, F_{\chi,p})$ to

$$(\star_{\chi}) \quad 0 \rightarrow \frac{H_{\text{et}}^2(\tilde{Y}, F_{\chi,p})}{H_{\text{et}}^2(Y_{\mathbf{G}}, F_{\chi,p})} \rightarrow \tilde{\mathcal{E}}_{\chi} \rightarrow F_{\chi,p}\omega_{\chi}^{\text{dep}} \rightarrow 0.$$

Note that $H_{\text{et}}^2(\tilde{Y}, F_{\chi,p}) = \bigoplus_x H_{\text{et}}^2(Y_{\mathbf{H}}(x), F_{\chi,p})$. Pushing-out the extension $(\star)$ under $H_{\text{et}}^2(\tilde{Y}, F_{\chi,p}) \twoheadrightarrow H_{\text{et}}^2(Y_{\mathbf{H}}(x), F_{\chi,p})$ we get the extension

$$0 \rightarrow \frac{H_{\text{et}}^2(Y_{\mathbf{H}}(x), F_{\chi,p})}{H_{\text{et}}^2(Y_{\mathbf{G}}, F_{\chi,p})} \rightarrow H_{\text{et}}^3(Y_{\mathbf{G}}, Y_{\mathbf{H}}(x); F_{\chi,p}) \rightarrow H_{\text{et}}^3(Y_{\mathbf{G}}, F_{\chi,p}) \rightarrow 0$$

and pulling-back again by the inclusion of the line spanned by $\omega_{\chi}^{\text{dep}}$, we have the extension

$$(\star_{\chi,x}) \quad 0 \rightarrow \frac{H_{\text{et}}^2(Y_{\mathbf{H}}(x), F_{\chi,p})}{H_{\text{et}}^2(Y_{\mathbf{G}}, F_{\chi,p})} \rightarrow \mathcal{E}_{\chi}(x) \rightarrow F_{\chi,p}\omega_{\chi}^{\text{dep}} \rightarrow 0.$$

Applying to the extension $(\star_{\chi,x})$ the automorphism of $Y_{\mathbf{G}}$ t_{-x} , we obtain that this extension is isomorphic to the extension constructed as $(\star_{\chi})$ with $\omega_{\chi}^{\text{et}}$ replaced by $t_x^*\omega_{\chi}^{\text{et}}$. Recall that $\omega_{\chi}^{\text{et}} = t_{\chi\psi_{f,p}^{-1}}^{\text{dep}}\omega_{\chi}^{\text{ord}}$, where $\psi_{f,p}$ is the Nebentypus of f at p . So $t_x^*\omega_{\chi}^{\text{dep}} = t_{\chi\psi_x^{-1}}^{\text{dep}}\omega_{\chi}^{\text{ord}}$ for some Dirichlet character ψ_x such that $\psi_x^p = \psi_{f,p}^p$. We denote

$t_{\chi\psi_x}^{\text{dep}}\omega_{\chi}^{\text{ord}}$ by $\omega_{\chi}^{\text{et}}(x)$. The extension $(\star_{\chi,x})$ is isomorphic to the extension

$$(\hat{\star}_{\chi,x}) \quad 0 \rightarrow \frac{H_{\text{et}}^2(Y_{\mathbf{H}}, F_{\chi,p})}{H^2(Y_{\mathbf{G}}, F_{\chi,p})} \rightarrow \hat{\mathcal{E}}_{\chi} \rightarrow F_{\chi,p}\omega_{\chi}^{\text{dep}}(x) \rightarrow 0.$$

We now henceforth identify $H_{\text{et}}^*(Y_{\mathbf{H}}(x), F_{\chi,p}) \simeq H_{\text{et}}^*(Y_{\mathbf{H}}, F_{\chi,p})$ by translation under t_x . We consider $U = \bigoplus_x U_x$ a Galois module quotient of $\bigoplus H_{\text{et}}^2(\tilde{Y}, F_{\chi,p})/H^2(Y_{\mathbf{G}}, F_{\chi,p})$; where $U_x \subset H_{\text{et}}^2(Y_{\mathbf{H}}(x), F_{\chi,p})$ is a Galois module quotient. Pushing out the extension $(\star_{\chi})$ under the projection onto U , we get the extension

$$(\star_{\chi}^U) \quad 0 \rightarrow U \rightarrow \tilde{\mathcal{E}}_{\chi}^U \rightarrow F_{\chi,p}\omega_{\chi}^{\text{dep}} \rightarrow 0.$$

Note that t_y acts on U by permuting the (U_x) as : $U_x \mapsto U_{x+y}$. We now take all the $U_x = \tilde{W} = \bigoplus_y W_y$ to be the same Galois module. Here W_y denotes the Galois module of the element f_y of the Hida family $\mathbf{f}$ with Nebentypus ψ_y . Then the enlarged extension $(\star_{\chi}^U)$ is non-trivial if and only if the push-out of $(\star_{\chi}^U)$ under $U \rightarrow U_x$ is non trivial for some x . In turn the push-out to U_x is isomorphic to the extension

$$(\star_{\chi,x}^U) \quad 0 \rightarrow W_x \rightarrow \hat{\mathcal{E}}_{\chi}^U \rightarrow F_{\chi,p}\omega_{\chi}^{\text{et}}(x) \rightarrow 0,$$

which is the extension class of $\mathcal{Z}_p(f_x)$ where f_x is as before the weight 2 member of the Hida family $\mathbf{f}$ with Nebentypus ψ_x .

To detect the non-triviality of the enlarged extension $(\star_{\chi}^U)$, we push out to a

section of the diagonal embedding $\tilde{W}^\Delta \hookrightarrow U$: this is given by the projector

$$e^{dep} = \frac{1}{p} \sum_{x \bmod p} (Id - t_x^*).$$

We denote the resulting extension by $(\star_\chi^{dep})$; noting that ω_χ^{et} is fixed by e^{dep} as the Fourier coefficients of E_χ are supported on symmetric 2×2 matrices $T = \begin{pmatrix} T_1 & T_0 \\ T_0 & T_2 \end{pmatrix}$ such that T_0 is a p -adic unit, the extension $(\star_\chi^{dep})$ is given by

$$(\star_\chi^{dep}) \quad 0 \rightarrow \tilde{W}^\Delta \rightarrow \mathcal{E}_\chi^{dep} \rightarrow F_{\chi,p} \omega_\chi^{et} \rightarrow 0$$

where $\mathcal{E}_\chi^{dep} := e^{dep} \tilde{\mathcal{E}}_\chi^U$. Whose extension class is isomorphic, by considering the action of the diamond operators to the extension still denoted by $(\star_\chi^{dep})$ given by push-out along $\tilde{W}^\Delta \rightarrow W_0^\Delta$.

We let $U^- := \bigoplus_y W_y^-$ and denote the push-out to U^- of $(\star_\chi^U)$ by $(\star_\chi^{U-})$. The pushout of this extension under e^{dep} will be denoted by $(\star_\chi^{-,dep})$.

We then have the following immediate Lemma.

4.4.3.1 Lemma. *If $(\star_\chi^{-,dep})$ is non-trivial, then the enlarged extension $(\star_\chi^{U-})$ is non-trivial. In particular, at least one of the extension classes $\mathcal{Z}_p(f_y)^-$ is non-trivial.*

To characterize whether or not $(\star_\chi^{-,dep})$ is trivial we compute the Bloch-Kato logarithm of the push-out along $\tilde{W}^\Delta \rightarrow W_0^\Delta \simeq W_0$ where $W_0^\Delta \simeq W_0$ is the projection to the first factor, we denote this push-out by $(\star_{\chi,0}^{-,dep})$. By the exact same reasoning

of the previous section there exists an element

$$\tilde{h}_\chi \in H^0(Y_{\mathbf{G}}^{ord, \star, \dagger}, \Omega_{Y_{\mathbf{G}}^{ord, \star, \dagger}}^2(-D_{\mathbf{G}}^{\text{tor}, ord}))$$

such that

$$d\tilde{h}_\chi = (1 - V)\omega_\chi^{\text{rig}}$$

and a suitable projection of $[\tilde{h}_\chi|_{\tilde{Y}}]$ is the Bloch-Kato logarithm of $(\star_\chi^{U^-})$. Then the Bloch-Kato logarithm of $(\star_{\chi,0}^{-,dep})$ is non-zero if and only if

$$\mathbf{1}_f \boxtimes \mathbf{1}_{\tilde{f}}[(e^{dep}\tilde{h}_\chi)|_{Y_{\mathbf{H}}^{ord}}] \neq 0.$$

Note now that $d(e^{dep}\tilde{h}_\chi) = \omega_\chi^{\text{rig}}$ since $e^{dep}(1 - V)E_\chi = E_\chi$. Moreover by definition of e^{dep} the Fourier coefficients of $e^{dep}\tilde{h}_\chi$ are supported on symmetric 2×2 matrices $T = \begin{pmatrix} T_1 & T_0 \\ T_0 & T_2 \end{pmatrix}$ such that T_0 is a p -adic unit.

As a consequence of Proposition 4.6.6.3, and by the same reasoning as in Proposition 4.4.2.3 we obtain the following result.

4.4.3.2 Proposition. *The extension $(\star_{\chi,0}^{-,dep})$ defined above is non-split if and only if for any (all) $\eta_\chi \in H^0(Y_{\mathbf{G}}^{ord}, \Omega_{Y_{\mathbf{G}}^{ord}}^2)$ such that*

$$d\eta_\chi = \omega_\chi^{\text{rig}}$$

we have

$$\mathbf{1}_f \boxtimes \mathbf{1}_{\tilde{f}}[\eta_\chi|_{Y_{\mathbf{H}}^{ord}}] \neq 0.$$

In what follows, we will compute explicitly an element η_χ such that $d\eta_\chi = \omega_\chi^{\text{rig}}$.

4.5 The moduli space and the Igusa tower

In this subsection, we introduce basic notation for the Siegel moduli scheme and the Igusa tower. We will need the notations to define p -adic versions of the relevant differential operators.

Let $\mathcal{M}_{\mathbf{G}}(N)$ be the Siegel moduli scheme defined over $\mathbb{Z}_p$, parametrizing principally polarized abelian surfaces (A, λ) with a principal level N structure ψ_N over $\text{Spec}(\mathbb{Z}_p)$. Let $\mathcal{M}_{\mathbf{G}}^{\text{tor}}(N)$ be a smooth toroidal compactification of $\mathcal{M}_{\mathbf{G}}(N)$, with boundary B , over which there is the semi-abelian scheme $\mathcal{G} \rightarrow \mathcal{M}_{\mathbf{G}}^{\text{tor}}(N)$, extending the universal abelian scheme $\mathcal{A} \rightarrow \mathcal{M}_{\mathbf{G}}(N)$. Let ω be the Hodge bundle on $\mathcal{M}_{\mathbf{G}}^{\text{tor}}(N)$, $\mathcal{L}$ its determinant and let H be the Hasse invariant, which is a global section of $\mathcal{L}^{\otimes(p-1)}$ over the special fibre. A sufficiently large power of the Hasse invariant is liftable to $\mathcal{M}_{\mathbf{G}}^{\text{tor}}(N)$, as the pushforward of $\mathcal{L}$ to the minimal compactification of $\mathcal{M}_{\mathbf{G}}^{\text{tor}}(N)$ is ample.

Let F be a finite extension of $\mathbb{Q}_p$ containing all N -th roots of unity. A connected component of (the base change to $\mathcal{O}_F$) $\mathcal{M}_{\mathbf{G}}^{\text{tor}}(N)$ can be identified with a toroidal compactification of the Shimura variety attached to $\mathbf{G}$. Fix such a connected component and denote it by $S = Y_{\mathbf{G}, N}$. And let $S^{\text{ord}} := Y_{\mathbf{G}, N}^{\text{ord}}$ be the non-vanishing locus of (a lift of) the Hasse invariant. Let S_l be the base change of S to $\text{Spec}(\mathbb{Z}/p^l\mathbb{Z})$. As usual define $T_{l, m}$ as sheaf over S_l of trivializations of the étale part of the Cartier dual of the p^m -torsion of $\mathcal{G}$; namely, we set:

$$T_{l,m} := \underline{\text{Isom}}_{S_l} \left(\mathcal{G}([p^m])^{D,\text{et}}, (\mathbb{Z}/p^m\mathbb{Z})^2 \right)$$

The scheme $T_{l,m}$ is étale over S_l with Galois group $\mathbf{GL}_2(\mathbb{Z}/p^m\mathbb{Z})$.

We call the Igusa tower the diagram

$$\cdots \longrightarrow T_{l,m} \longrightarrow T_{l,m-1} \longrightarrow \cdots \longrightarrow T_{l,1} \longrightarrow S_l$$

We define the formal scheme $T_{\infty,m}$ by $T_{\infty,m} := \varprojlim_l T_{l,m}$ and we define the infinite Igusa tower as

$$\text{Ig} := \varprojlim_m T_{\infty,m}.$$

Since $T_{\infty,m}$ is a system of formal schemes, with affine transition maps, then Ig exists as a formal scheme by Proposition D.4.1 of [Far] and is a pro-étale cover of $T_{\infty,0}$, which can be identified with the formal completion along the special fibre of S^{ord} .

4.6 Differential Operators

4.6.1 Archimedean differential operators

In this section, we survey standard material on differential operators on the Siegel threefold. For more details, we invite the reader to consult [Liu20, Sections 2.1, 2.2]. Let V be a $(\mathbf{G}, \mathbf{P}_1)$ -module, then we can attach to V the automorphic vector bundle $\mathcal{E}(V)$ endowed with a Gauss-Manin connection, obtained by the C^∞ splitting of the Hodge-filtration on the relative de-Rham cohomology of the universal semi-abelian

scheme above $\overline{Y}_{\mathbf{G}}$:

$$\nabla_V : \mathcal{E}(V) \rightarrow \mathcal{E}(V) \otimes \Omega_{\overline{Y}_{\mathbf{G}}}^1(\log C);$$

this induces a Hecke-equivariant map on global sections.

Let (σ, W_σ) be any finite dimension algebraic representation of $\mathbf{GL}_2$. We view W_σ as a representation of $\mathbf{P}_1$ by identifying $\mathbf{M}_1$ with $\mathbf{GL}_2$. Let $Y = (Y_{ij})_{1 \leq i, j \leq 2}$ be a symmetric matrix whose coefficients we view as indeterminates. Attached to W_σ , there is a $(\mathrm{Lie}(\mathbf{G}), \mathbf{P}_1)$ -module V_σ . As a $\mathbb{Q}$ -vector space, V_σ is $W_\sigma[Y]$; namely, the space of polynomials in the indeterminates Y_{ij} with coefficients in W_σ . The action of $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in \mathbf{P}_1$ on any $P(Y) \in V_\sigma$ is defined to be:

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} P(Y) = aP(a^{-1}b + a^{-1}Yd).$$

We refer the reader to [Liu20, p. 9] for the $\mathrm{Lie}(\mathbf{G})$ -module structure.

As a $\mathbb{Q}$ -representation, V_σ admits an increasing filtration defined as :

$$\mathrm{Fil}^r V_\sigma := W_\sigma[Y]_{\leq r},$$

where the subscript $\leq r$ indicates we are only taking polynomials in Y_{ij} of degree less than r . Then the Gauss-Manin connection $\nabla_\sigma := \nabla_{V_\sigma}$ restricts to define an operator :

$$\nabla_\sigma : \mathcal{E}(\mathrm{Fil}^r V_\sigma) \rightarrow \mathcal{E}(\mathrm{Fil}^{r+1} V_\sigma) \otimes \Omega_{\overline{Y}_{\mathbf{G}}}^1(\log C).$$

Following Liu, denote by $\mathcal{V}_\sigma^r$ the automorphic vector bundle $\mathcal{E}(\mathrm{Fil}^r V_\sigma)$ and let τ_2 the symmetric square of the standard representation of $\mathbf{GL}_2$. Recall that the Kodaira-Spencer isomorphism identifies $\Omega_{Y/G}^1(\log C)$ with $\mathcal{V}_{\tau_2}$. Composing the connection above with the Kodaira-Spencer isomorphism we obtain a differential operator D_σ :

$$D_\sigma : \mathcal{V}_\sigma^r \xrightarrow{\nabla_\sigma} \mathcal{V}_\sigma^{r+1} \otimes \Omega_{Y/G}^1(\log C) \xrightarrow{\mathrm{KS}} \mathcal{V}_{\sigma \otimes \tau_2}^{r+1}.$$

We denote the iterated differential operator as D_σ^e :

$$D_\sigma^e : \mathcal{V}_\sigma^r \rightarrow \mathcal{V}_{\sigma \otimes \tau_2^{\otimes e}}^{r+e}.$$

4.6.2 The q -expansion map

In this section, we survey standard material for the q -expansion of a nearly holomorphic modular form and its corresponding p -adic modular form. For details, we again refer the reader to Liu's paper [Liu20, Section 2.6]. Following the notations of *loc cit.*, we denote by $\varepsilon_{q,p\text{-adic}}$ the q -expansion of a nearly holomorphic modular form f , whose polynomial q expansion is denoted by $f(q, Y)$.

We now take slightly different notations from Liu. We denote an element indexing the q -expansion by $T = (T_{ij})_{1 \leq i, j \leq 2}$, as is more usual, instead of $\beta = (\beta_{ij})_{1 \leq i, j \leq n}$. Since here $n = 2$, we set for simplicity $T_0 = T_{12} = T_{21}$, $T_1 = T_{11}$ and $T_2 = T_{22}$ and with our conventions $T_0 = 2\beta_{12}$. Let D_i for $i = 0, 1, 2$ the order one differential operator on q -expansion defined by

$$D_i(q^T) = T_i q^T.$$

Fix now a basis $X = (X_i)_{1 \leq i \leq 3}$ a basis of the $\mathbf{GL}_2$ representation τ_2 , which we recall is defined as the symmetric square of the standard representation of $\mathbf{GL}_2$. Let $f \in H^0(\overline{Y_{\mathbf{G}}}, \mathcal{V}_{\sigma}^r)$ be a nearly holomorphic form with polynomial q -expansion $f(q, Y)$. Then, by proposition 2.6.1 of *loc cit.* the polynomial q -expansion of $D_{\sigma}f$ is given by :

$$(D_{\sigma}f)(q, Y) = \sum_{1 \leq i \leq 3} (D_i f(q, Y) + \mu_i^- f(q, Y)) \otimes X_i, \quad (4.2)$$

where $\mu_k^- = \mu_{ij}^-$ are elements of $\text{Lie}(\mathbf{G})$, whose action is defined on p.9 of *loc cit.* The q -expansion map is then obtained by setting $Y = 0$ and taking the projection for each irreducible factor of $\sigma \otimes \tau_2$ denoted by ϵ_{can} in *loc cit.* – this is the projection onto the highest weight vector. The p -adic q -expansion map is denoted by $\epsilon_{q,p\text{-adic}}$ in *loc cit.*, and the inclusion of nearly holomorphic forms into p -adic modular forms by $\iota_{p\text{-adic}}$ (this inclusion is defined in more detail in the next section). Let f have polynomial q -expansion $f(q, y) = h(q) + Yg(q, Y)$. Where $h(q) = \sum_T a_T q^T$. Then we have

$$\epsilon_{q,p\text{-adic}}(\iota_{p\text{-adic}}(D_{\sigma}f)) = \sum_T \sum_{1 \leq i \leq 3} \epsilon_{can}(a_T \otimes X_i) D_{ij} q^T. \quad (4.3)$$

4.6.3 p -adic differential operators

We now define p -adic versions of the differential operators D_{σ} defined in the previous subsection.

Let $T_{\infty,m}$ be the formal scheme defined as $\varinjlim_l T_{l,m}$ over $\mathcal{O}_F$. When $m = 0$ the formal scheme $T_{\infty,0}$ is the completion of S^{ord} along its special fibre. Over $T_{\infty,0}$, we

have Dwork's unit root splitting:

$$\mathcal{H}_{\text{dR}}^1(\mathcal{A}/T_{\infty,0})^{\text{can}} = \omega(\mathcal{G}/T_{\infty,0}) \oplus \mathcal{U},$$

where the superscript *can* indicates the canonical extension of the automorphic vector bundle to the toroidal compactification. The Gauss-Manin connection induces a connection $\mathcal{U} \rightarrow \mathcal{U} \otimes \Omega_{T_{\infty,0}}^1$. If σ is an irreducible representation of $\mathbf{GL}_2$, we denote by $\mathcal{V}_{\sigma,p}$ the pull-back of the algebraic vector bundle V_σ to $T_{\infty,0}$. Recall that the Kodaira-Spencer isomorphism provides an identification

$$\text{KS} : \Omega_{Y_{\mathbf{G}}}^1 \xrightarrow{\sim} \text{Sym}^2 \omega,$$

where ω is the Hodge bundle on $Y_{\mathbf{G}}$ and $\Omega_{Y_{\mathbf{G}}}^1$ is the sheaf of 1-differentials on $Y_{\mathbf{G}}$. This induces isomorphisms, still denoted KS by abuse of notation

$$\text{KS} : \Omega_{Y_{\mathbf{G}}}^2 \xrightarrow{\sim} \text{Sym}^2 \omega \otimes \det(\omega) \quad \text{and} \quad \text{KS} : \Omega_{Y_{\mathbf{G}}}^3 \xrightarrow{\sim} \det(\omega)^3.$$

By the same procedure as in the archimedean case, using the unit root splitting instead of the C^∞ -splitting of the relative de Rham cohomology of $\mathcal{A}$, we can define over the Igusa tower, differential operators induced by the Gauss-Manin connection

$$\mathcal{D}_{\sigma,p} : \mathcal{V}_{\sigma,p} \longrightarrow \mathcal{V}_{\sigma,p} \otimes \mathcal{V}_{\tau_2,p}.$$

Let λ be a dominant weight for $\mathbf{GL}_2$ and σ the associated algebraic irreducible representation of $\mathbf{GL}_2$. By pulling back the unit root splitting to $T_{\infty,m}$, one can

define (see section 6.2.1 of [Liu20]) the Hecke-equivariant map

$$\iota_{p\text{-adic}} : H^0(Y_{\mathbf{G}, Np^m}, \mathcal{V}_{\sigma}^r) \hookrightarrow H^0(T_{\infty, m}, \mathcal{V}_{\sigma, p})^{\mathbf{U}(\mathbb{Z}/p^m\mathbb{Z})}[1/p] \hookrightarrow V[\lambda][1/p],$$

where $V[\lambda]$ denotes the λ isotypical component for the action of the diagonal maximal torus on the space of p -adic modular forms for $\mathbf{G}$. We will see $\iota_{p\text{-adic}}$ sometimes as valued in $H^0(T_{\infty, m}, \mathcal{V}_{\sigma, p})$ and sometimes in V – we will make sure to indicate it to the reader. The injectivity of $\iota_{p\text{-adic}}$ follows from the fact that the unit root splitting agrees with the C^∞ splitting of the Hodge-Filtration of the relative de Rham cohomology of the universal abelian scheme at ordinary CM points – these points are analytically dense. As a consequence of this fact, it is classical that for any nearly holomorphic section $h \in H^0(Y_{\mathbf{G}, Np^m}, \mathcal{V}_{\sigma}^r)$, we have that $\iota_{p\text{-adic}}(D_{\sigma}h) \in H^0(T_{\infty, m}, \mathcal{V}_{\sigma, p} \otimes \mathcal{V}_{\tau_2, p})$ coincides with the image of $\iota_{p\text{-adic}}(h) \in H^0(T_{\infty, m}, \mathcal{V}_{\sigma, p})$ under $\mathcal{D}_{\sigma, p}$ – that is

$$\iota_{p\text{-adic}}(D_{\sigma}h) = \mathcal{D}_{\sigma, p}(\iota_{p\text{-adic}}(h)) \in H^0(T_{\infty, m}, \mathcal{V}_{\sigma, p} \otimes \mathcal{V}_{\tau_2, p}).$$

Consistently with [Liu20] (see the discussion in section 2.6 and 6.2.1), if f is a nearly holomorphic modular form, then the q -expansion of the p -adic modular form $\iota_{p\text{-adic}}(f)$, $\varepsilon_{q, p\text{-adic}}(\iota_{p\text{-adic}}(f))$ will be simply denoted by $\varepsilon_{q, p\text{-adic}}(f)$.

4.6.4 The differential operators of interest

In this subsection we analyze the differential operators appearing in the de Rham complex. Notice that we have the following decomposition

$$\mathrm{Sym}^2\omega \otimes \mathrm{Sym}^2\omega = \mathrm{Sym}^4\omega \oplus \mathrm{Sym}^2\omega \otimes \det(\omega) \oplus \det(\omega)^2.$$

We denote by p_2 the projection of $\mathrm{Sym}^2\omega \otimes \mathrm{Sym}^2\omega$ onto to the summand $\det(\omega)^2$. This is explicitly given as follows: for $s_1 \otimes s_2 \in \mathrm{Sym}^2\omega \otimes \mathrm{Sym}^2\omega$, $p_2(s_1, s_2)$ is the bilinear symmetric pairing of s_1 with s_2 coming from the quadratic form given by the determinant— that is p_2 is bilinear symmetric and $p_2(s, s) = \det(s)$ for $s \in \mathrm{Sym}^2\omega$. Fix σ be an algebraic representation of $\mathbf{GL}_2$. Then p_2 induces a projection map

$$p_{2,\sigma} : H^0(Y_{\mathbf{G}}, \mathcal{V}_{\sigma \otimes \tau_2^{\otimes 2}}^r) \rightarrow H^0(Y_{\mathbf{G}}, \mathcal{V}_{\sigma}^r(2)).$$

Here we have put $\mathcal{V}_{\sigma}^r(i) := \mathcal{V}_{\sigma \otimes \det(\omega)^i}^r$.

We now let

$$T_{\sigma} : H^0(Y_{\mathbf{G}}, \mathcal{V}_{\sigma \otimes \tau_2}^r) \rightarrow H^0(Y_{\mathbf{G}}, \mathcal{V}_{\sigma}^{r+1}(2))$$

be the operator

$$T_{\sigma} := p_{2,\sigma} \circ D_{\sigma \otimes \tau_2}.$$

We define

$$\Theta_{\sigma} : H^0(Y_{\mathbf{G}}, \mathcal{V}_{\sigma}^r) \rightarrow H^0(Y_{\mathbf{G}}, \mathcal{V}_{\sigma}^{r+2}(2))$$

be defined by

$$\Theta_\sigma = T_\sigma \circ D_\sigma.$$

We denote the corresponding p -adic differential operators by $\Theta_{\sigma,p}$ and $\mathcal{T}_{\sigma,p}$.

We record in the following lemmas some useful facts about these differential operators.

4.6.4.1 Lemma. *Let σ be an algebraic irreducible representation of $\mathbf{GL}_2$ of highest weight λ and let $\underline{2} = (2, 2)$. There is a commutative diagram*

$$\begin{array}{ccc} H^0(Y_{\mathbf{G}, Np^m}, \mathcal{V}_\sigma^r) & \xrightarrow{\Theta_\sigma} & H^0(Y_{\mathbf{G}, Np^m}, \mathcal{V}_\sigma^{r+2}(\underline{2})) \\ \downarrow \iota_{p\text{-adic}} & & \downarrow \iota_{p\text{-adic}} \\ V[\lambda] & \xrightarrow{\Theta_p} & V[\lambda + \underline{2}] \end{array}$$

where Θ_p is the map given on p -adic q -expansions by multiplication by $\det(T)$ – that is

$$\Theta_p \sum_T a_T q^T = \sum_T \det(T) a_T q^T.$$

Proof. Let $h \in H^0(Y_{\mathbf{G}}, \mathcal{V}_\sigma^r)$ have polynomial q -expansion denoted by $h(q, Y)$. By Equation 4.2, we have

$$(D_{\sigma \otimes \tau_2} D_\sigma h)(q, 0) = \sum_T a_T \otimes (T \otimes T) q^T$$

and therefore, by the explicit description of p_2 given at the beginnig of this section we have

$$(\Theta_\sigma h)(q, 0) = \sum_T \det(T) a_T q^T,$$

which implies the result after taking $\mathfrak{e}_{can}(\det(T)a_T) = \det(T)\mathfrak{e}_{can}(a_T)$. $\square$

By density of classical modular forms, the above lemma implies that Θ_p extends to an operator

$$\Theta_p : V \rightarrow V$$

on the space of p -adic modular forms. If $h \in V$ is a p -adic modular form for $\mathbf{G}$, then by the above lemma there is an obvious congruence $\Theta_p^k h \equiv \Theta_p^\ell h \pmod{p^m}$ if $k \equiv \ell \pmod{p^m(p-1)}$. We can therefore give the following definition.

4.6.4.2 Definition. Let $\kappa \in \mathbb{Z}_p$, such that $\kappa = \lim_{n \rightarrow \infty} k_n$ where $k_n > 0$ are integers and $\kappa \equiv k_n \equiv k_0 \pmod{p-1}$. We set

$$\Theta_p^\kappa f = \lim_{n \rightarrow \infty} \Theta_p^{k_n} f.$$

This limit does not depend on the choice of sequence k_n , since for any sequence k_n and any $T \in \text{Sym}^2(\mathbb{Z}^{\oplus 2})$ the sequence $\det(T)^{k_n}$ converges to $\det(T)^\kappa$ if $\det(T) \in \mathbb{Z}_p^\times$ or 0 otherwise.

The following Lemma is proved in the exact same manner as Lemma 4.6.4.1.

4.6.4.3 Lemma. *Let σ be an algebraic irreducible representation of $\mathbf{GL}_2$ of highest weight λ and let $\underline{2} = (2, 2)$ There is a commutative diagram*

$$\begin{array}{ccc} H^0(Y_{\mathbf{G}, Np^m}, \mathcal{V}_{\sigma \otimes \tau_2}^r) & \xrightarrow{T_\sigma} & H^0(Y_{\mathbf{G}, Np^m}, \mathcal{V}_\sigma^{r+1}(2)) \\ \downarrow \scriptstyle \ell_{p\text{-adic}} & & \downarrow \scriptstyle \ell_{p\text{-adic}} \\ H^0(T_{\infty, m}, \mathcal{V}_{\sigma \otimes \tau_2, p}) & \xrightarrow{\mathcal{T}_{\sigma, p}} & H^0(T_{\infty, m}, \mathcal{V}_{\sigma, p}(2)) \end{array}$$

such that for any $h \in H^0(Y_{\mathbf{G}, Np^m}, \mathcal{V}_{\sigma \otimes \tau_2}^r)$ with polynomial q -expansion $h(q, Y) = \sum_T a_T q^T + Y g(q, Y)$ where $a_T = b_T \otimes X_T$ with $X_T = \begin{pmatrix} x_1(T) & x_0(T) \\ x_0(T) & x_2(T) \end{pmatrix}$

$$\varepsilon_{q, p\text{-adic}}(T_\sigma h) = \sum_T \mathbf{e}_{can}(b_T) p_2(X_T, T) q^T = \sum_T \mathbf{e}_{can}(b_T) \frac{x_1(T)T_1 + x_2(T)T_2 + -2x_0(T)T_0}{2} q^T.$$

4.6.5 Local Form of differential operators

In what follows we will mostly be interested in $\sigma = \det$ is the determinant character of $\mathbf{GL}_2$ so that $\mathcal{V}_\sigma = \det(\omega)$. We let

$$\mathcal{T}_p := \mathcal{T}_{\det(\omega), p}.$$

We now relate $\mathcal{T}_p$ to a local universal derivation. For this we will refer mostly to [Eis+18] and [Kat81].

Recall that by the Kodaira spencer isomorphism, we have

$$\Omega_{Y_{\mathbf{G}}}^2 \simeq \text{Sym}^2(\omega) \otimes \det(\omega) = V_{\tau_2}(1)$$

and

$$\Omega_{Y_{\mathbf{G}}}^3 \simeq \det(\omega)^3 = V_{\det^3}.$$

For any $x_0 \in S^{ord}(\mathcal{O}_{K_0})$ we write $\overline{x_0} \in S^{ord}(\overline{\mathbb{F}_p})$ for its reduction modulo p . Denote by $\widehat{S_{x_0}^{ord}}$ the formal completion of $S^{ord} \times \overline{\mathbb{F}_p}$ at $\overline{x_0}$. Let $\mathcal{R}_{x_0}$ be the complete ring over $\mathcal{O}_{K_0}$ with residue field $\overline{\mathbb{F}_p}$ such that $\widehat{S_{x_0}^{ord}} = \text{Spf}(\mathcal{R}_{x_0})$. Let A_0 denote the polarized

abelian variety over $\overline{\mathbb{F}_p}$ attached to $\bar{x}_0$ and write $A_0^\vee$ for the dual abelian varieties. We let $T_p(A_0)$ denote the physical Tate module of A_0 . By Serre–Tate theory there exists a canonical isomorphism of formal schemes

$$\widehat{S_{\bar{x}_0}^{ord}} \simeq \mathrm{Hom}_{\mathbb{Z}_p}(\mathrm{Sym}^2(T_p(A_0)), \hat{\mathbf{G}}_m),$$

which identifies $\mathcal{R}_{x_0}$ with the ring $\mathrm{Sym}^2(T_p(A_0)) \otimes \mathcal{O}_{K_0}[[t]]$. Upon choosing a basis of $\mathrm{Sym}^2(T_p(A_0))$ formal power series in three variables. By [Kat81, Corollary 4.2.2], there is a canonical Frobenius equivariant isomorphism

$$\alpha : T_p(A_0^\vee) \otimes \mathcal{R}_{x_0} \longrightarrow \omega_{x_0},$$

where ω_{x_0} is the pull-back of the Hodge bundle ω to $\mathcal{R}_{x_0}$. Combining this with the Kodaira–Spencer isomorphism, we have an identification, still denoted by α by abuse of notation

$$\alpha : \mathrm{Sym}^2(T_p(A_0)) \otimes \mathcal{R}_{x_0} \longrightarrow \Omega_{\mathcal{R}_{x_0}}^1.$$

The pull-back of $\mathcal{V}_{\sigma,p}$ to $\mathcal{R}_{x_0}$ will be denoted by $\mathcal{V}_{\sigma,p}$. As is remarked at p472 of [Eis+18], the operator $D_{\sigma,p}$ induces an operator D_{σ,x_0} on $\mathcal{V}_{\sigma,x_0}$.

Invoking the same reasoning of [Eis+18, Proposition 4.3.1] based on [Kat81, Theorem 4.3.1] we obtain the following Lemma. The precise form of the last claim of the following Lemma is a consequence of Lemma 4.6.4.3; it can also be obtained as in [Eis+18, Proposition 4.3.2, Theorem 5.1.3].

4.6.5.1 Lemma. *Let $d : \Omega_{\mathcal{R}_{x_0}}^2 \rightarrow \Omega_{\mathcal{R}_{x_0}}^2 \otimes \Omega_{\mathcal{R}_{x_0}}^1$ be the universal derivation. Let $\sigma = \text{Sym}^2 \otimes \det$ be the irreducible representation of $\mathbf{GL}_2$ of highest weight $(3, 1)$. There is a commutative diagram:*

$$\begin{array}{ccc}
\Omega_{\mathcal{R}_{x_0}}^2 & \xrightarrow{d} & \Omega_{\mathcal{R}_{x_0}}^2 \otimes \Omega_{\mathcal{R}_{x_0}}^1 \\
\downarrow \wr & & \downarrow \wr \\
\text{Sym}^2(T_p(A_0^\vee))(1) \otimes \mathcal{R}_{x_0} & \xrightarrow{\text{id} \otimes d} & \det(T_p(A_0^\vee))^3 \otimes \Omega_{\mathcal{R}_{x_0}}^1 \\
\downarrow \wr & & \downarrow \wr \\
\text{Sym}^2(\omega_{x_0}) \otimes \det(\omega_{x_0}) & \xrightarrow{D_{\sigma, x_0}} & \text{Sym}^2(\omega_{x_0})^{\otimes 2} \otimes \det(\omega_{x_0})
\end{array}$$

We then obtain, after taking universal projections $\Omega_{\mathcal{R}_{x_0}}^2 \otimes \Omega_{\mathcal{R}_{x_0}}^1 \rightarrow \Omega_{\mathcal{R}_{x_0}}^3$ and $\text{Sym}^2(\omega_{x_0})^{\otimes 2} \rightarrow \det(\omega_{x_0})^2$, a commutative diagram

$$\begin{array}{ccc}
\Omega_{\mathcal{R}_{x_0}}^2 & \xrightarrow{d} & \Omega_{\mathcal{R}_{x_0}}^3 \\
\downarrow \wr & & \downarrow \wr \\
\text{Sym}^2(\omega_{x_0}) \otimes \det(\omega_{x_0}) & \xrightarrow{\mathcal{T}_{x_0}} & \det(\omega_{x_0})^3
\end{array}$$

where $\mathcal{T}_{x_0}$ is induced by $\mathcal{T}_p$.

4.6.6 The end of the computation

We are now ready to finish the computation of the explicit reciprocity law.

4.6.6.1 Definition. Let $E_\chi^{p\text{-adic}} := \iota_{p\text{-adic}} E_\chi$ be the p -adic modular form determined by the Eisenstein series E_χ . And we let E_χ^{Liu} the p -adic modular form determined by the choice of section f_χ^{Liu} . Invoking Definition 4.6.4.2 we can form, $\Theta_p^{-1} E_\chi^{Liu} \in$

$H^0(\mathrm{Ig}, \mathcal{V}_{\det(\omega), p})$. Define

$$\eta_\chi := \mathcal{D}_{\det(\omega), p} \Theta_p^{-1} E_\chi^{Liu} \in H^0(\mathrm{Ig}, \mathcal{V}_{\tau_2(1), p}).$$

Recall the notation $\mathrm{Ig} = \varprojlim_m T_{\infty, m}$. We can then prove the following proposition.

4.6.6.2 Proposition. *The p -adic modular form η_χ verifies $\mathcal{T}_p \eta_\chi = E_\chi^{Liu}$.*

Proof. Indeed, it suffices to show that $\lim_{n \rightarrow \infty} \Theta^{p^n(p-1)} E_\chi^{p\text{-adic}}$. By looking at q -expansions, this limit is the "det"- p -depletion of $E_\chi^{p\text{-adic}}$, i.e. it has the q -expansion of $E_\chi^{p\text{-adic}}$ with the Fourier coefficients indexed by singular matrices modulo p removed. But by our choice of section at p , $E_\chi^{p\text{-adic}}$ is already det- p -depleted. The result follows. $\square$

We now give the following proposition which will show that the Bloch-Kato logarithm can be computed in terms of η .

4.6.6.3 Proposition. *Let $\eta \in H^0(\mathrm{Ig}, \mathcal{V}_{\tau_2(1), p})$. Assume that the T -th coefficient of the q -expansion for η vanishes if p^n divides every entry of T for sufficiently large n . Assume also that $\mathcal{T}_p \eta = 0$. Let $\mathrm{Ig}_{\mathbf{H}}$ be the Igusa tower for $\mathbf{H}$ and let $\eta|_{\mathbf{H}} \in H^0(\mathrm{Ig}_{\mathbf{H}}, \mathcal{V}_{\det^2 \boxtimes \det^2, p})$ be the pullback of η to $\mathrm{Ig}_{\mathbf{H}}$. Then we have*

$$(e \times e)(\eta)|_{\mathbf{H}} = 0$$

Proof. By Lemma 4.6.5.1, we see that if $X(T) = \begin{bmatrix} x_1(T) & x_0(T) \\ x_0(T) & x_2(T) \end{bmatrix}$ is the T -th coefficient of the q -expansion of η then

$$x_1(T)T_1 + x_2(T)T_2 - 2x_0(T)T_0 = 0.$$

Since $\mathcal{V}_{\det \boxtimes \det, p} \subset \mathcal{V}_{\tau_2}|_{\mathbf{H}}$ is the line spanned by the vector of weight $(1, 1)$, that is the vector $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in \tau_2$. Therefore the restriction $\eta|_{\mathbf{H}}$ has q -expansion

$$\sum_{T_1, T_2} \sum_{T_0} x_0 \left(\begin{bmatrix} T_1 & T_0 \\ T_0 & T_2 \end{bmatrix} \right) q^{T_1} q^{T_2}.$$

We then see that $(e \times e)\eta|_{\mathbf{H}}$ has q -expansion

$$\sum_{T_1, T_2} \sum_{T_0} \lim_{n \rightarrow \infty} x_0 \left(\begin{bmatrix} p^{n!} T_1 & T_0 \\ T_0 & p^{n!} T_2 \end{bmatrix} \right) q^{T_1} q^{T_2}.$$

On the other hand from the relation $x_1(T)T_1 + x_2(T)T_2 - 2x_0(T)T_0 = 0$, we deduce that for n large enough and $T_0 \neq 0$, we have $x_0 \left(\begin{bmatrix} p^{n!} T_1 & T_0 \\ T_0 & p^{n!} T_2 \end{bmatrix} \right) = 0$. And if $T_0 = 0$, for n large enough this conclusion is still valid, by hypothesis, as $p^{n!}$ divides every entry. The result follows. $\square$

We then have the following Proposition as a consequence of Proposition 4.4.3.2.

4.6.6.4 Proposition. *Let*

$$\eta_\chi = \mathcal{D}_p \Theta_p^{-1} E_\chi^{p\text{-adic}} \in V[(3, 1)],$$

with $\mathcal{T}_p \eta_\chi = E_\chi^{p\text{-adic}}$. Then the extension $(\star_{\chi, 0}^{\text{dep}, -})$ of Section 4.4.3 is non-split if and only

if

$$\mathbb{1}_f \boxtimes \mathbb{1}_{\bar{f}}(\eta_\chi) \neq 0.$$

In Section 4.7 we will show that $\mathbb{1}_f \boxtimes \mathbb{1}_{\bar{f}}(\eta_\chi)$ is up to a p -adic unit the value of Liu's two variable p -adic L-function. Therefore there exists an infinite number of members f of the Hida-family of $\mathbf{f}$ such that $\mathbb{1}_f \boxtimes \mathbb{1}_{\bar{f}}(\eta_\chi) \neq 0$. Therefore by Lemma 4.4.3.1 there exists an infinite number of members f of the Hida family $\mathbf{f}$ such that $\mathcal{Z}_p(f)^-$ is non trivial.

Then by Proposition 4.4.2.3, and noticing that $d(1 - p^{-1}V)\eta_\chi = (1 - V)d\eta_\chi = (1 - V)E_\chi$, we obtain the following Theorem.

4.6.6.5 Theorem. *Let*

$$\eta_\chi = \mathcal{D}_p \Theta_p^{-1} E_\chi^{p\text{-adic}} \in V[(3, 1)],$$

with $\mathcal{T}_p \eta_\chi = E_\chi^{p\text{-adic}}$. We have

$$\begin{aligned} \underline{\mathcal{L}}(\mathcal{C}_p(\mathbf{f}))(f, \varepsilon^{-1}\chi^{-1}) \cdot (f \otimes \bar{f}) &= -(1 - p^{-1}\alpha_f \alpha_{\bar{f}})^{-1} \cdot c_f \cdot (\mathbb{1}_f \boxtimes \mathbb{1}_{\bar{f}})(1 - p^{-1}V)\eta_\chi \\ &= -c_f \cdot (\mathbb{1}_f \boxtimes \mathbb{1}_{\bar{f}})\eta_\chi \end{aligned}$$

Note that the second equality in the Theorem above comes from the fact that V restricts to $v \times v$ on $\mathbf{H}$ and that $p^{-2}(v \times v)$ is the Frobenius on $Y_{\mathbf{H}}$.

4.7 Relation to a p -adic L -function

In this subsection, we interpret the criterion of corollary 4.6.6.5 as the non-vanishing of the symmetric square p -adic L -function outside its range of interpolation of classical L -values.

4.7.1 Liu's p -adic Eisenstein measure

In this subsection, we review Liu's construction of the p -adic Eisenstein measure, which is used in [Liu20] to construct the p -adic L -function interpolating values of the standard L -function of cuspidal automorphic representations of $\mathbf{Sp}(2n)(\mathbb{A})$ (where n is an integer as in *loc cit.*). In our case, we take $n = 1$.

For what follows, we refer to section 5.2 of *loc cit.* We review briefly the main results and specify the notations that we will use. Liu constructs a measure $\mu_{\mathcal{E}, q-\text{exp}}$ valued in formal q expansions such that

$$\int_{\mathbb{Z}_p^\times \times \mathbb{Z}_p^\times} (\kappa, \mathbf{t}) d\mu_{\mathcal{E}, q-\text{exp}}$$

provides the q -expansion of the (restricted) Eisenstein denoted in *loc cit.* by $\mathcal{E}_{\kappa, \mathbf{t}}^{\alpha_{\kappa, \mathbf{t}, p}}$.

Where $\alpha_{\kappa, \mathbf{t}, p}^1$ determines the choice of Siegel section at p defining the Eisenstein series.

For example, if $\kappa = \varepsilon^k$ and $k \geq 2$ is a positive integer, and $\mathbf{t} \geq \varepsilon^l$, $l \geq k$ is also

¹We have denoted by $\mathbf{t}$ the element denoted by $\underline{\mathbf{t}}$ in *loc cit.*, as $n = 1$ this is simply an element of $\mathbb{Z}_p^\times$.

an integer, the Eisenstein series

$$\mathcal{E}_{\kappa, \mathbf{t}} := \mathcal{E}_{\kappa, \mathbf{t}}^{\alpha_{\kappa, \mathbf{t}, p}}$$

is determined by the following choice of sections:

- At infinity the choice of section is

$$f_{\kappa, \mathbf{t}, \infty} := \left(\frac{1}{2i\pi} \hat{\mu}_0^+ \right)^{l-k} f_{\infty}^k,$$

where $\hat{\mu}_0^+$ is a differential operator defined in section 2.4 of *loc cit.*²

- Let us denote by Δ_p the p -adic analogue of $\frac{1}{2i\pi} \hat{\mu}_0^+$, as in Section 4.6.3. Then, for a p -adic modular form $g \in H^0(\mathrm{Ig}, \mathcal{V}_{\det^k, p})$, we consider the restriction $(\Delta_p g)|_{\mathbf{H}}$ of $(\Delta_p g)$ to an element of $H^0(\mathrm{Ig}_{\mathbf{H}}, \mathcal{V}_{\det^{k+1} \boxtimes \det^{k+1}, p})$. We have the equality

$$\varepsilon_{q, p\text{-adic}}((\Delta_p g)|_{\mathbf{H}}) = 2(\delta_p \varepsilon_{q, p\text{-adic}}(g))|_{\mathbf{H}}. \quad (4.4)$$

Here δ_p is the operator acting formally on q -expansions on $\mathbf{G}$ by multiplication by T_0 .

- At ramified places, Liu chooses "volume sections"; see section 4.2 of *loc cit.*. We make such choices or the choices outlined in Section (B.9) of the Appendix.
- At p , the choice is determined by $\alpha_{\kappa, \mathbf{t}, p}$ – the local section is supported on the big cell.

²Note that in our case, there is no determinant to be taken as $l = 1$ in equation 4.3.1 of *loc cit.*

4.7.1.1 Remark. In the notations introduced above, we have the equality:

$$E_{\chi}^{Liu} = \mathcal{E}_{\chi \varepsilon^3, \psi_{f,p} \varepsilon^3}.$$

The following proposition is an immediate consequence of Equation 4.4.

4.7.1.2 Proposition. *There is an equality of $\mathbb{Z}_p^{\times} \times \mathbb{Z}_p^{\times}$ -measures:*

$$\mu_{\mathcal{E}, q-exp} = \lambda_2 \star \tilde{\mu}_{\mathcal{E}, q-exp};$$

where $\tilde{\mu}_{\mathcal{E}, q-exp}$ is defined by

$$\int_{\mathbb{Z}_p^{\times} \times \mathbb{Z}_p^{\times}} (\kappa, \mathbf{t}) d\tilde{\mu}_{\mathcal{E}, q-exp} = \left(\delta_p^{\mathbf{t}-\kappa} \varepsilon_{q, p-adic} \left(\mathcal{E}_{\kappa, \kappa}^{\alpha_{\kappa, p}} \right) \right) |_{\mathbf{H}};$$

and λ_2 by

$$\int_{\mathbb{Z}_p^{\times} \times \mathbb{Z}_p^{\times}} (\kappa, \mathbf{t}) d\lambda_2 = (2)^{\mathbf{t}-\kappa}.$$

4.7.1.3 Lemma. *Let $g \in H^0(\text{Ig}, \mathcal{V}_{\det^k, p})$ be a p -adic modular form of scalar weight.*

We have the equality of q -expansions:

$$\varepsilon_{q, p-adic} \left((e \times e)(\Theta_p g) |_{\mathbf{H}} \right) = -\delta_p^2 \varepsilon_{q, p-adic} \left((e \times e)g |_{\mathbf{H}} \right).$$

Proof. Note that we have

$$\varepsilon_{q, p-adic} \left((\Theta_p g) |_{\mathbf{H}} \right) = (\theta \boxtimes \theta) \varepsilon_{q, p-adic} (g |_{\mathbf{H}}) - (\delta_p^2 \varepsilon_{q, p-adic} (g)) |_{\mathbf{H}},$$

where θ is the usual Serre-Katz differential operator of $\mathbf{SL}_2$ acting by qd/dq on q -expansions. Applying $e \times e$ to the above equality proves the claim of the Lemma. $\square$

The following Lemma is clear from the description of $\mathcal{D}_p$ from Section 4.6.3.

4.7.1.4 Lemma. *Let $g \in H^0(\mathrm{Ig}, \mathcal{V}_{\det^k, p})$ be a p -adic modular form of scalar weight. Consider $\mathcal{D}_{\det^k, p} g \in H^0(\mathrm{Ig}, \mathcal{V}_{\tau_2(k), p})$ and denote by $\cdot|_{\mathbf{H}}$ the restriction $H^0(\mathrm{Ig}, \mathcal{V}_{\tau_2(k), p}) \rightarrow H^0(\mathrm{Ig}_{\mathbf{H}}, \mathcal{V}_{\det^{k+1} \boxtimes \det^{k+1}, p})$. We have the equality of q -expansions:*

$$\varepsilon_{q, p\text{-adic}} \left((\mathcal{D}_{\det^k, p} g)|_{\mathbf{H}} \right) = (\delta_p \varepsilon_{q, p\text{-adic}}(g))|_{\mathbf{H}}.$$

4.7.2 The p -adic L-function

In this section we refer the reader to section 6.2.2 of [Liu20]. We consider the measure denoted in *loc cit.* by $\mu_{\mathcal{E}, \text{ord}}$ (we take $\underline{1}$ to be the trivial character and omit it from the notation see p 47 of *loc cit.*). Following the notations of *loc cit.* let V' be the space of p -adic forms for $\mathbf{SL}_2 = \mathbf{Sp}_2$ and V^Δ the subspace of $V \otimes V$ annihilated by $\gamma \otimes 1 - 1 \otimes \gamma$ for all γ in the $\mathbb{Z}_p$ -points of the maximal torus of $\mathbf{SL}_2$. The measure $\mu_{\mathcal{E}, \text{ord}}$ is valued in V^Δ and interpolates the ordinary projection of the q -expansion of $\mathcal{E}_{\kappa, \mathbf{t}}|_{\mathbf{H}}$.

4.7.2.1 Proposition. *Let η_χ be as in Theorem 4.6.6.5, so that $\eta_\chi|_{\mathbf{H}} \in H^0(\mathrm{Ig}_{\mathbf{H}}, \mathcal{V}_{\det^2 \boxtimes \det^2, p})$*

We have the equality

$$(e \times e) \varepsilon_{q, p\text{-adic}}(\eta_\chi|_{\mathbf{H}}) = -\frac{\chi(2) \psi_{f, p}^{-1}(2)}{2} \int_{\mathbb{Z}_p^\times \times \mathbb{Z}_p^\times} (\chi \varepsilon^3, \psi_{f, p} \varepsilon^2) d\mu_{\mathcal{E}, \text{ord}}$$

Proof. Indeed we have

$$\begin{aligned}
(e \times e)\varepsilon_{q,p\text{-adic}}(\eta_\chi|_{\mathbf{H}}) &= \varepsilon_{q,p\text{-adic}}(\mathcal{D}_p\Theta_p^{-1}\mathcal{E}_{\chi\varepsilon^3,\chi\varepsilon^3}|_{\mathbf{H}}) \\
&= (e \times e)\left(\delta_p\varepsilon_{q,p\text{-adic}}(\Theta_p^{-1}\mathcal{E}_{\chi\varepsilon^3,\psi_{f,p}\varepsilon^3})\right)|_{\mathbf{H}} \\
&= (e \times e)\left(-\delta_p^{-1}\varepsilon_{q,p\text{-adic}}(\mathcal{E}_{\chi\varepsilon^3,\psi_{f,p}\varepsilon^3})\right)|_{\mathbf{H}} \\
&= (e \times e)\left(-\delta_p^{\varepsilon^2\psi_{f,p}-\chi\varepsilon^3}\varepsilon_{q,p\text{-adic}}(\mathcal{E}_{\chi\varepsilon^3,\chi\varepsilon^3})\right)|_{\mathbf{H}} \\
&= -(e \times e) \int_{\mathbb{Z}_p^\times \times \mathbb{Z}_p^\times} (\chi\varepsilon^3, \varepsilon^2\psi_{f,p}) d\tilde{\mu}_{\mathcal{E},q\text{-exp}} \\
&= -\frac{\chi(2)\psi_{f,p}^{-1}(2)}{2} (e \times e) \int_{\mathbb{Z}_p^\times \times \mathbb{Z}_p^\times} (\chi\varepsilon^3, \varepsilon^2\psi_{f,p}) d\mu_{\mathcal{E},q\text{-exp}} \\
&= -\frac{\chi(2)\psi_{f,p}^{-1}(2)}{2} \int_{\mathbb{Z}_p^\times \times \mathbb{Z}_p^\times} (\chi\varepsilon^3, \varepsilon^2\psi_{f,p}) d\mu_{\mathcal{E},ord}
\end{aligned}$$

In the above equalities we have used, in order, Lemma 4.7.1.4, Lemma 4.7.1.3 and Proposition 4.7.1.2. $\square$

We now refer the reader to section 7 of *loc cit.* for the general construction of the p -adic L -function for $\mathcal{F}$ varying in a Hida family. We specialize to the case where $\mathcal{F} = f \otimes \bar{f} \in V^\Delta$, where f corresponds to a weight 2 ordinary Hecke Eigenform. Recall the notation $\mathbb{1}_f$ for the idempotent of the ordinary Hecke algebra corresponding to projection onto the Hecke-eigenspace generated by f (see p48 of *loc cit.* for a version of $\mathbb{1}_f$ in a Hida family.).

If $\mathbf{f}$ is a Hida family passing through f , we set

$$\mathcal{L}(\mathbf{f}, \kappa) := \mathbb{1}_{\mathbf{f}} e \int_{\mathbb{Z}_p^\times \times 1} (\kappa, 1) d\mu_{\mathcal{E},ord}.$$

And if f is a specializatioon of $\mathbf{f}$ with character $\mathbf{t}$, we set

$$\mathcal{L}(f, \kappa) := \int_{\mathbb{Z}_p^\times} \mathbf{t} d\mathcal{L}(\mathbf{f}, \kappa) = \mathbf{1}_f e \int_{\mathbb{Z}_p^\times \times \mathbb{Z}_p^\times} (\kappa, \mathbf{t}) d\mu_{\mathcal{E}, ord}.$$

We can then reformulate corollary 4.6.6.5, by the following theorem:

4.7.2.2 Lemma. *For infinitely many weight 2 specializations f of the family $\mathbf{f}$ with Nebentypus $\psi_{f,p}$ with log-conductor $\mathfrak{f}_\psi$ and for all finite order characters χ of log-conductor $\mathfrak{f}_\chi \geq \mathfrak{f}_\psi$, we have the equality:*

$$\underline{\mathcal{L}}(\mathcal{C}_p(\mathbf{f})^-)(f, \varepsilon^{-1}\chi^{-1}) = 2^{\varepsilon^{-1}\chi\psi_{f,p}^{-1}} \cdot c_f \cdot \mathcal{L}(f, \chi\varepsilon^3).$$

For f a weight k specialization of $\mathbf{f}$, let $\mathcal{L}_f \in \Lambda$ be the power series such that for all characters κ , we have $\mathcal{L}_f(\kappa) = \mathcal{L}(f, \varepsilon^k \kappa)$. Similarly denote by $\mathcal{L}_{\mathbf{f}} \in \Lambda_{\mathbf{f}} \otimes \Lambda$ the element such that for a specialization $(\kappa, \mathbf{t})$ with $\mathbf{t} = \varepsilon^k \psi$, we have $\mathcal{L}_{\mathbf{f}}(\kappa, \mathbf{t}) = \mathcal{L}_f(\varepsilon^k \kappa)$, where f is the weight $\mathbf{t}$ specialization of $\mathbf{f}$.

By the p -adic density of the points of the preceding Lemma, we have the following Proposition:

4.7.2.3 Proposition. *For all weight 2 specializations f with Nebentypus $\psi_{f,p}$ with log-conductor r and for all finite order characters χ of log-conductor $n \geq r$, we have the equality:*

$$\underline{\mathcal{L}}(\mathcal{C}_p(\mathbf{f})^-)(f, \chi^{-1}) = 2^{\varepsilon^{-2}\chi\psi_{f,p}^{-1}} \cdot c_f \cdot \mathcal{L}_f(\chi).$$

We can then deduce the following theorem.

4.7.2.4 Theorem. *There is an equality of ideals in $\Lambda_{\mathbf{f}} \otimes \Lambda$:*

$$\left(\underline{\mathcal{L}}(\mathcal{C}_p(\mathbf{f})^-) \right) = \mathcal{L}_{\mathbf{f}} \cdot C_{\mathbf{f}}.$$

Here if $a \in \Lambda_{\mathbf{f}} \otimes \Lambda$, (a) denotes the principal ideal generated by a and $C_{\mathbf{f}}$ is the congruence ideal for $\mathbf{f}$.

4.7.3 The interpolation formulae

In this section we formulate precisely the interpolation properties verified by $\mathcal{L}_{\mathbf{f}}$. Since at bad places we choose slightly different test vectors, we need to slightly modify the formulas in [Liu20].

We recall that in the notations of *loc cit.*, $\langle f, f \rangle$ is defined as

$$\langle f, f \rangle = \int_{\mathbf{SL}_2(\mathbb{Q}) \backslash \mathbf{SL}_2(\mathbb{A})} f(g) \overline{f(g)} dg.$$

If f is fixed by some compact open $K \subset \mathbf{SL}_2(\mathbb{A}_f)$, then we have

$$\langle f, f \rangle = \text{vol}(K) \langle f, f \rangle_K;$$

where $\langle \cdot, \cdot \rangle_K$ denotes the Petersson inner product at level K (given by integration on the hyperbolic Riemann surface $Y_K(\mathbb{C})$).

We recall the following:

The tame level of the Hida family $\mathbf{f}$ is K where by $\Gamma_1(N) \subseteq K \subseteq \Gamma_0(N)$.

Write $K = K^N \prod_{\ell|N} K_{\ell}$. Let us introduce some notation first:

- We let Σ_{st} be the set of primes $\ell|N$ for which $\pi(f)_\ell$ is of the ramification type of Section (B.9), that is the Steinberg representation. We let $K_{st} = \prod_{\ell \in \Sigma_{st}} K_\ell$. We notice here that if $\ell \in \Sigma_{st}$, $K_\ell = \text{Iw}_\ell = \Gamma_0(\ell)$. We set $Q_{st} := \prod_{\ell \in \Sigma_{st}} \ell$. We recall that at places Σ_{st} we choose the local section defining the Eisenstein series as given by Proposition (B.9.2).
- We let Σ_{rps} be the set of primes $\ell|N$ for which $\pi(f)_\ell \simeq \pi(\lambda, \lambda^{-1})$ is a ramified principal series (here λ is a ramified character of $\mathbb{Q}_\ell^\times$); we put $K_{rps} = \prod_{\ell \in \Sigma_{rps}} K_\ell$. We put $Q_{rps} := \prod_{\ell \in \Sigma_{rps}} \ell^{v_\ell(N)}$. We recall that at places in Σ_{st} we choose the local section defining the Eisenstein series as the trace to level $\Gamma_1(\ell^r)$ of the volume sections of Lemma (B.8.1).
- We also consider Σ_{sc} the places $\ell|N$ at which $\pi(f)_\ell$ is supercuspidal and $\Sigma_e \subseteq \Sigma_{sc}$ to be the set of primes $\ell|N$ for which $\pi(f)_\ell$ is supercuspidal and $\pi(f)_\ell \simeq \pi(f)_\ell \otimes \delta_{\mathbb{Q}_{\ell^2}}$, where $\delta_{\mathbb{Q}_{\ell^2}}$ is the unique unramified quadratic character of $\mathbb{Q}_\ell^\times$. For $\ell \in \Sigma_e$ these primes we have $L_\ell(s, \text{Ad}^0(f)) = (1 + \ell^{-s})^{-1}$ and we set

$$A_e(f) = \prod_{\ell \in \Sigma_e} L_\ell(1, \text{Ad}^0(f))^{-1} = \prod_{\ell \in \Sigma_e} (1 + \ell^{-1})$$

We let $K_e = \prod_{\ell \in \Sigma_e} K_\ell$, $Q_{sc} := \prod_{\ell \in \Sigma_{sc}} \ell^{v_\ell(N)}$, and $Q_e := \prod_{\ell \in \Sigma_e} \ell^{v_\ell(N)}$. For $\ell \in \Sigma_{sc}$, note that $K_\ell = K_0(\ell^{v_\ell(N)})$. We recall that at places in Σ_{sc} we choose the local section defining the Eisenstein series as the trace to level $\Gamma_0(\ell^r)$ of the volume sections of Lemma (B.8.1).

In the following formulas, we will ignore the issue of the auxiliary prime from assumption 3.4.1.2 (recall the definition of an auxiliary prime 3.3.3.3). This will

needlessly make the formulas look more complicated: one has to always remove the L -factor at ℓ_{aux} , which is a p -adic unit. If the reader wishes, they can insert $L_\ell(1, \text{Ad}^0(f))^{-1}$ in the unit U_f of Proposition 4.7.3.4.

4.7.3.1 Proposition (Interpolation property). *For all even weight $k \geq 2$ specializations f of $\mathbf{f}$ with trivial Nebentypus for all finite order even characters χ of log-conductor $n \geq 1$. Let $E_p(s, \text{Ad}^0(f) \otimes \chi^{-1})$ be as in equation (1.0.1) of [Liu20], so that in this case*

$$E_p(s, \text{Ad}^0(f) \otimes \chi^{-1}) = (\tau(\chi)p^{n(s-1)})^2 \cdot (p^{k-1}\alpha_f^{-2})^n.$$

We have

$$\mathcal{L}_f(\chi) = B_f \cdot \frac{\Gamma(k)}{i^{k-1}2^{2k+1}} \cdot \frac{E_p(1, \text{Ad}^0(f) \otimes \chi^{-1})L(1, \text{Ad}^0(f) \otimes \chi^{-1})}{\pi^{k+1}\langle f, f \rangle_{\Gamma_1(Np)}}.$$

Here we have set

$$B_f := \frac{\lambda_{Np}(\bar{f})\alpha_f A_e(f)\varphi(Np)}{Q_{rps}}.$$

Proof. The result follows from Liu's interpolation formula [Liu20, Theorem 7.0.2], by taking into account our slightly different choice of local sections at ramified places outlined in the Appendix (B.9). We treat each term in the interpolation formula separately. Recall that in the notations of *loc cit.*, we have $n = 1$.

- The factor denoted by $\frac{\Gamma(k-n)\Gamma_{2n}(k)}{2^{k+n-1}(i\pi)^{2nk+k-n}}$ in *loc cit.* becomes

$$\frac{\Gamma_2(2)}{2^3 i \pi^5}.$$

- As is noticed in [Liu20] at the bottom of page 23, in this case the archimedean Zeta integral is computed by Shimura [Shi95, Section 6]. Since our normalization of the Eisenstein series are different and coincides with Liu's, we rather use the formula of [Liu22, Theorem 0.0.1]. Recalling that in our notation $t_j = t_1 = k$ and the variable denoted by k in *loc cit.* is equal to 2, the contribution is exactly:

$$\frac{i^{-k+2}2^{-2k+2}}{\pi^{k-4}\Gamma_2(2)}\Gamma(k).$$

- The contribution of the above two bullet points is then

$$\frac{\Gamma(k)}{i^{k-1}2^{2k+1}\pi^{k+1}}$$

- By section ??, the contribution of the Fourier coefficients of $W(f)$ and f , is

$$\lambda_{Np}(\bar{f})\alpha_f p^{-1},$$

where $\lambda_{Np^r}(\bar{f})$ is the Atkin-Lehner pseudo-eigenvalue of $\bar{f}$.

- The volume term $\text{vol}(\hat{\Gamma}(N))^{\frac{p(p-1)}{p^2-1}}$ in *loc cit.*, becomes, with the choices of Steinberg sections and the traces of the volume sections down to level K_ℓ , with the contribution of the above bullet point, $\text{vol}(\Gamma_0(Np/Q_{rps}))\text{vol}(\Gamma_1(Q_{rps}))$. We then rewrite

$$\text{vol}(\Gamma_0(Np/Q_{rps}))\text{vol}(\Gamma_1(Q_{rps})) = \varphi(Np/Q_{rps})\text{vol}(\Gamma_1(Np)).$$

- The contribution of $E_p(1, \text{Ad}^0 f \otimes \chi^{-1})L^{(Np)}(1, \text{Ad}^0(f) \otimes \chi^{-1})$ by taking into account the choice of Steinberg sections at places in Σ_{st} , becomes

$$E_p(1, \text{Ad}^0 f \otimes \chi^{-1})L^{(N/Q_{st})}(1, \text{Ad}^0(f) \otimes \chi^{-1}).$$

We rewrite

$$L^{(N/Q_{st})}(1, \text{Ad}^0(f) \otimes \chi^{-1}) = A_e(f) \prod_{\ell \in \Sigma_{rps}} (1 - \ell^{-1}) L(1, \text{Ad}^0(f) \otimes \chi^{-1}) = \varphi(Q_{rps}) Q_{rps}^{-1} L(1, \text{Ad}^0(f) \otimes \chi^{-1})$$

- The total contribution of the previous two bullet points is then:

$$\frac{A_e(f) \varphi(Np)}{Q_{rps}} \text{vol}(\Gamma_1(Np)) \cdot E_p(1, \text{Ad}^0 f \otimes \chi^{-1}) L(1, \text{Ad}^0(f) \otimes \chi^{-1})$$

Combining these bullet points, we obtain the result. $\square$

Recall that we denoted $\mathcal{C}_p(f) \in H^1(\mathbb{Q}, \text{Ad}^0(T_f)(1) \otimes \Lambda)$ the image of $\mathcal{C}_p(\mathbf{f})$. The following proposition is an immediate consequence of Proposition 4.7.3.1 just above and Lemma 4.2.2.3.

4.7.3.2 Corollary. *Let f and χ be as in Proposition 4.7.3.1. Then we have*

$$\mathcal{L}^{PR}(\mathcal{C}_p(f)^-)(\chi) = \frac{\chi(2)B_f}{4} \cdot \frac{\Gamma(k)}{i^{k-1}2^{2k+1}} \cdot \frac{E_p(1, \text{Ad}^0(f) \otimes \chi^{-1})L(1, \text{Ad}^0(f) \otimes \chi^{-1})}{\pi^{k+1} \langle f, f \rangle_{K_1(Np)}}.$$

We now want to compare Liu's p -adic L -function $\mathcal{L}_f$ with a Symmetric square p -adic L -function appearing in the work of Schmidt and Hida, described in Theorem 3.4 of [Das16]. We will need the following well-known Lemma.

4.7.3.3 Lemma. *Let f be as in Proposition 4.7.3.1 and let f° the associated newform, so that f is an ordinary p -stabilization of f° . Assume that $f^\circ \neq f$. Then we have:*

$$\langle f, f \rangle_{\Gamma_1(Np)} = (p-1)^2 \left(1 - \frac{\beta_f}{\alpha_f} p^{-1} \right) \langle f^\circ, f^\circ \rangle_{\Gamma_1(N)}.$$

Proof. This follows from Hida's formula [Hid81, Section 5]: there exists a constant C such that, in the notation of *loc cit.*,

$$\langle f, f \rangle = C(p^2 - 1) \text{Res}_{s=k} D(f, f, s),$$

and

$$\langle f^\circ, f^\circ \rangle = C \text{Res}_{s=k} D(f^\circ, f^\circ, s).$$

We can obtain the formula of the lemma by the relation:

$$D(f_\alpha, f_\alpha, s) = \frac{(1 - p^{k-1}p^{-s})^2(1 - \beta_f^2 p^{-s})}{(1 - p^{2(k-1)}p^{-2s})} D(f, f, s).$$

□

We therefore obtain the following Proposition:

4.7.3.4 Proposition. *Let f and χ be as in Proposition 4.7.3.1 and let f° the associated newform, so that f is an ordinary p -stabilization of f° .*

$$\mathcal{L}_f(\chi) = M_f \cdot \frac{\Gamma(k)}{i^{k-1}2^{2k+1}} \cdot \frac{E_p(1, \text{Ad}^0(f) \otimes \chi^{-1}) L(1, \text{Ad}^0(f) \otimes \chi^{-1})}{\pi^{k+1} \langle f^\circ, f^\circ \rangle_{K_1(N)}};$$

where we have put

$$M_f := (p-1)^{-1} A_e(f) \varphi(N) Q_{rps}^{-1} \left(1 - \frac{\beta_f}{\alpha_f} p^{-1} \right)^{-1}.$$

4.7.3.5 Proposition. *Let f be as in the previous proposition and let $L_p(\text{Sym}^2 f, s)$ be the p -adic L -function described in Theorem 3.4 of [Das16]. There exists a constant $U_f \in \overline{\mathbb{Z}_p}^\times$ depending only on f such that, in the notations of loc cit., we have*

$$L_p(\text{Sym}^2 f, \nu_{k,\chi}) = U_f \cdot M_f \cdot \chi^2(Q_{rps}) \mathcal{L}_f(\chi).$$

Proof. This follows from comparing the interpolation properties described in Proposition 4.7.3.1 and Theorem 3.4, Equation (32) of [Das16] applied at $s = k$ and $\beta = \chi$. The constant U_f is

$$U_f = i^{-k+1} Q_{rps}$$

□

Assume henceforth that f is not multiplicative that is $f^\circ \neq f$.

Note that then $E_p(s, \text{Ad}^0(f))$ has a zero at $s = 1$. We deduce that if f is a weight k specialization of $\mathbf{f}$, we have $c_f \mathcal{L}_f \in \mathcal{O}[[\Gamma]]$, where we recall that $\Gamma = 1 + p\mathbb{Z}_p$. If γ is a topological generator of Γ , we have $\mathcal{L}_f = (\gamma - 1) \mathcal{L}'_f$ for some $\mathcal{L}'_f \in (c_f)^{-1} \mathcal{O}[[\Gamma]]$. Similarly we have

$$\mathcal{L}_{\mathbf{f}} = (\gamma - 1) \mathcal{L}'_{\mathbf{f}},$$

for some $\mathcal{L}'_{\mathbf{f}} \in (c_{\mathbf{f}})^{-1} \Lambda_{\mathbf{a}}[[\Gamma]]$. We want to study the interpolation properties of $\mathcal{L}'_{\mathbf{f}}$, in particular at the trivial character.

Note that when the Nebentypus of f is trivial at p , by Equation (1.0.1) of [Liu20], we have

$$E_p(s, \text{Ad}^0(f)) = \frac{(1 - p^{s-1})(1 - p^{k-1}\alpha_f^{-2}p^{s-1})}{(1 - p^{-s})(1 - p^{-(k-1)}\alpha_f^2p^{-s})} = \mathcal{E}_p(s, \text{Ad}^0(f)) \cdot L_p(s, \text{Ad}^0(f));$$

here we have set

$$\mathcal{E}_p(s, \text{Ad}^0(f)) := \left(1 - p^{s-1}\right) \left(1 - \frac{\beta_f}{\alpha_f}p^{s-1}\right) \left(1 - \frac{\beta_f}{\alpha_f}p^{-s}\right)$$

We set henceforth

$$\mathcal{E}'_p(s, \text{Ad}^0(f)) := \left(1 - \frac{\beta_f}{\alpha_f}p^{s-1}\right) \left(1 - \frac{\beta_f}{\alpha_f}p^{-s}\right).$$

By proposition 4.7.3.5, we obtain the following proposition.

4.7.3.6 Proposition. *Let $\mathbf{t} : \Lambda_{\mathbf{a}} \rightarrow \mathcal{O}$ be a point corresponding to a classical weight k specialization of the Hida family $\mathbf{f}$, with trivial Nebentypus, so that $\mathbf{t}(\mathcal{L}'_{\mathbf{f}}) = \mathcal{L}'_f \in (c_f)^{-1}\mathcal{O}[[\Gamma]]$. We have*

$$\mathcal{L}'_f(1) = \log_p(u)^{-1} \mathcal{L}_{an}(\text{Ad}^0(f)) \cdot L^{\text{alg}}(1, \text{Ad}^0(f));$$

Here we have put $u = 1 + p$ and $\mathcal{L}_{an}(1, \text{Ad}^0(f))$ is the L -invariant as defined by Dasgupta [Das16, Equation (11)] and

$$L^{\text{alg}}(1, \text{Ad}^0(f)) = U_f \cdot M_f \mathcal{E}'_p(1, \text{Ad}^0(f)) \cdot \frac{\Gamma(k)L(1, \text{Ad}^0(f))}{\pi^{k+1}\langle f^\circ, f^\circ \rangle_{\Gamma_1(N)}}.$$

The following Lemma is an immediate consequence of [Hid81, Theorem 5.1].

4.7.3.7 Lemma. *Let f be a twist-minimal, weight k newform fixed by $K_1(N)$. Then*

$$\left| \frac{\Gamma(k)L(1, \text{Ad}^0(f))}{\pi^{k+1}2^{2k}\langle f, f \rangle_{\Gamma_1(N)}} A_e(f)\varphi(N) \right|_p = 1$$

Proof. This follows at once from the following formula, that follows from [Hid81, Theorem 5.1]:

$$\langle f, f \rangle_{\Gamma_1(N)} = \frac{(k-1)!N\varphi(N)}{2^{2k}\pi^{k+1}} A_e(f)L(1, \text{Ad}^0(f)).$$

when f is fixed by $\Gamma_1(N)$. □

Noticing that

$$M_f \mathcal{E}'_p(1, \text{Ad}^0(f)) = (\text{unit}) \cdot \varphi(N) A_e(f),$$

we have

$$|L^{\text{alg}}(1, \text{Ad}^0(f))|_p = 1$$

Therefore we obtain the following result.

4.7.3.8 Theorem. *Let f be as in Proposition 4.7.3.1. Assume that $f \neq f^\circ$, where f° is the newform that is associated to f . We then have*

$$|\mathcal{L}'_f(1)|_p = |\log_p(u)^{-1} \mathcal{L}_{an}(\text{Ad}^0(f))|_p = |\log_p(u)^{-1} \mathcal{L}_{GS}(\text{Ad}^0(f))|_p.$$

Here $\mathcal{L}_{GS}(\text{Ad}^0(f))$ is the Greenberg-Stevens L -invariant as defined by Dasgupta [Das16, p. 7].

Chapter 5

Arithmetic applications

5.1 Adjoint Selmer groups

5.1.1 Definitions

In this section we give basic definitions for studying Selmer groups of adjoint motives. Set for simplicity $K = F_{\chi,p}$ and $\mathcal{O} = \mathcal{O}_{\chi,p}$ and let ϖ be a uniformizer for $\mathcal{O}$. We recall that f here is a weight $k \geq 2$, trivial Nebentypus, specialization of the Hida family $\mathbf{f}$, as in Proposition 4.7.3.1.

We recall that there is a filtration on T_f ,

$$T_f^+ \subset T_f,$$

such that $T_f^- := T_f/T_f^+$ is unramified and $T_f^+|_{I_p} \simeq \mathcal{O}(1)$. For simplicity, in **this**

section only, let

$$T := \text{Ad}^0(T_f) \subset \text{End}_{\mathcal{O}}(T_f).$$

The filtration $T_f^+ \subset T_f$ induces a three step filtration on T :

$$\text{Fil}^1 T \subsetneq \text{Fil}^0 T \subsetneq T,$$

where $\text{Fil}^1 T$ is the set of elements $\phi \in T$ such that $\phi(T_f^+) = 0$ and $\text{Fil}^0 T$ is the set of element $\phi \in T$ such that $\phi(T_f^+) \subset T_f^+$. Stated otherwise, if one represents the elements of $\text{Ad}^0(T)$ as matrices in a basis whose first vector spans T_f^+ , then the elements of $\text{Fil}^0 T$ are the upper triangular matrices and $\text{Fil}^1 T$ are the elements of $\text{Fil}^0 T$ that are nilpotent. Note that $\text{Gal}_{\mathbb{Q}_p}$ acts trivially on $\text{Fil}^0 T / \text{Fil}^1 T$, and inertia at p acts by the cyclotomic character on $\text{Fil}^1 T$. We let

$$T^- := T / \text{Fil}^1 T.$$

Setting $V = \text{Ad}^0(\rho_f) = T \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ and $W = V/T$, we have a completely analogous three-step filtration $\text{Fil}^\bullet V$ on V and on W .

Recall that we have set $\Gamma = 1 + p\mathbb{Z}_p$ and $\Lambda = \mathcal{O}[[\Gamma]]$. We fix γ a topological generator of Γ . Let $\mathbb{Q}_\infty \subset \mathbb{Q}(\mu_{p^\infty})$ be the unique $\mathbb{Z}_p$ -extension of $\mathbb{Q}$, and identify $\text{Gal}(\mathbb{Q}(\mu_{p^\infty})/\mathbb{Q}) \rightarrow \text{Gal}(\mathbb{Q}_\infty/\mathbb{Q})$, with the projection $\mathbb{Z}_p^\times \rightarrow 1 + p\mathbb{Z}_p = \Gamma$. Define $\Psi : G_{\mathbb{Q}} \rightarrow \Gamma \subset (\Lambda)^\times$ be the natural projection given by the inclusion of $1 + p\mathbb{Z}_p$ into $\mathcal{O}[[\Gamma]]$. This is a continuous Λ -valued character that is unramified away from p and totally ramified at p . Let Λ^* be the Pontryagin dual of Λ ; it is a discrete Λ -module.

Put now

$$\mathcal{M} := T \otimes \Lambda^*$$

with diagonal action of $G_{\mathbb{Q}}$: on Λ^* , the action is by the character Ψ^{-1} . Denote by $\mathcal{M}^+ := \text{Fil}^1 T \otimes \Lambda^*$, and $\mathcal{M}_f^- := \mathcal{M}/\mathcal{M}^+$. Similarly as before we have a three step filtration

$$\text{Fil}^1 \mathcal{M} := \mathcal{M}^+ \subsetneq \text{Fil}^0 \mathcal{M} \subsetneq \mathcal{M}.$$

We put $\mathcal{M}^- := \mathcal{M}/\mathcal{M}^+$; we also set $\mathcal{M}^{--} := \text{Fil}^0 \mathcal{M}/\text{Fil}^1 \mathcal{M} \subset \mathcal{M}^-$. We remark that $\text{Gal}_{\mathbb{Q}_p}$ acts on $\mathcal{M}^+$ by $\varepsilon^{k-1} \alpha_f^{-2} \Psi^{-1}$, by Ψ^{-1} on $\mathcal{M}^{--}$, and by $\varepsilon^{-(k-1)} \alpha_f^2 \Psi^{-1}$ on $\mathcal{M}/\text{Fil}^0 \mathcal{M}$. Here we have abused notations and denoted by α_f the unramified character of $\text{Gal}_{\mathbb{Q}_p}$ sending the Frobenius element to $\alpha_f \in \mathcal{O}$.

Let $\mathbb{Q}_S$ be the maximal unramified extension of $\mathbb{Q}$ unramified outside S and ∞ . Define $G_S := \text{Gal}(\mathbb{Q}_S/\mathbb{Q})$. Let Σ be any finite set of primes contained in S . We define a Selmer group $\text{Sel}_{\mathbb{Q}_{\infty}}^{\Sigma,1}(\mathcal{M}_f)$ by

$$\text{Sel}_{\mathbb{Q}_{\infty}}^{\Sigma,1}(\mathcal{M}) := \ker \left\{ H^1(G_S, \mathcal{M}) \longrightarrow \frac{H^1(\mathbb{Q}_p, \mathcal{M})}{\text{im} H^1(\mathbb{Q}_p, \mathcal{M}^+)} \times \prod_{\ell \in S \setminus \Sigma} H^1(I_{\ell}, \mathcal{M}) \right\}.$$

We let

$$\mathcal{F}_p^1(\mathcal{M}) := \text{im} H^1(\mathbb{Q}_p, \mathcal{M}^+) \subset H^1(\mathbb{Q}_p, \mathcal{M}).$$

We define also

$$\text{Sel}_{\mathbb{Q}_{\infty}}^{\Sigma,0}(\mathcal{M}) := \ker \left\{ H^1(G_S, \mathcal{M}) \longrightarrow \frac{H^1(\mathbb{Q}_p, \mathcal{M}^-)}{\mathcal{F}_p^0(\mathcal{M})} \times \prod_{\ell \in S \setminus \Sigma} H^1(I_{\ell}, \mathcal{M}) \right\}, \quad (5.1)$$

and

$$\mathrm{Sel}_{\mathbb{Q}_\infty}^{\Sigma, \mathrm{ur}}(\mathcal{M}) := \ker \left\{ H^1(G_S, \mathcal{M}) \longrightarrow \frac{H^1(\mathbb{Q}_p, \mathcal{M})}{\mathcal{F}_p^{\mathrm{ur}}(\mathcal{M})} \times \prod_{\ell \in S \setminus \Sigma} H^1(I_\ell, \mathcal{M}) \right\}, \quad (5.2)$$

where

$$\mathcal{F}_p^0(\mathcal{M}) = \mathrm{im} \left\{ \frac{H^1(\mathbb{Q}_p, \mathrm{Fil}^0 \mathcal{M})}{\mathrm{im} H^1(\mathbb{Q}_p, \mathrm{Fil}^1 \mathcal{M})} \hookrightarrow H^1(\mathbb{Q}_p, \mathcal{M}^-) \right\},$$

and

$$\mathcal{F}_p^{\mathrm{ur}}(\mathcal{M}) = \ker \left\{ H^1(\mathbb{Q}_p, \mathcal{M}) \longrightarrow H^1(I_p, \mathcal{M}^-) \right\}.$$

We let $X_{\mathbb{Q}_\infty}^{\Sigma, 1}(\mathcal{M})$, $X_{\mathbb{Q}_\infty}^{\Sigma, 0}(\mathcal{M})$ and $X_{\mathbb{Q}_\infty}^{\Sigma, \mathrm{ur}}$ be the Pontryagin duals, which are finite Λ -modules. We let $\mathrm{Ch}^{\mathrm{ur}, \Sigma}(\mathcal{M})$, $\mathrm{Ch}^{0, \Sigma}(\mathcal{M})$ and $\mathrm{Ch}^{1, \Sigma}(\mathcal{M})$ their respective Λ -characteristic ideal. In general, this does not depend on the choice of lattice $T \subset V$ if T is residually irreducible.

We denote similarly $\mathrm{Sel}^{\Sigma, 1}(M)$, $X^{\Sigma, 1}(M)$, $\mathrm{Sel}^{\Sigma, 0}(M)$, $X^{\Sigma, 0}(M)$, $\mathrm{Sel}^{\Sigma, \mathrm{ur}}(M)$, $X^{\Sigma, \mathrm{ur}}(M)$ for $M = T, V, W$, the Selmer groups defined by replacing $\mathcal{M}$ above by M . If Σ is empty, we simply omit it from the notation. Notice that we have the inclusions:

$$\mathrm{Sel}^{\Sigma, 1}(M) \subseteq \mathrm{Sel}^{\Sigma, \mathrm{ur}}(M) \quad \text{and} \quad \mathrm{Sel}^{\Sigma, 1}(M) \subseteq \mathrm{Sel}^{\Sigma, 0}(M).$$

Moreover, for $\Sigma_1 \subseteq \Sigma_2$ and $\bullet \in \{0, 1, \mathrm{ur}\}$, we note that we have the relation

$$\mathrm{Sel}^{\bullet, \Sigma_1}(\mathcal{M}) = \ker \left\{ \mathrm{Sel}^{\bullet, \Sigma_2}(\mathcal{M}) \longrightarrow \prod_{\ell \in \Sigma_2 \setminus \Sigma_1} H^1(I_\ell, \mathcal{M})^{\mathrm{Gal}_{\mathbb{Q}_\ell}} \right\}.$$

Each cohomology group $H^1(I_\ell, \mathcal{M})^{\mathrm{Gal}_{\mathbb{Q}_\ell}}$ is a cotorsion Λ -module; the characteristic

ideal of its Pontryagin dual is generated by $P_\ell := P_\ell(\Psi^{-1}\varepsilon^{-1}(\text{Frob}_\ell)) \in \mathcal{O}[[\Gamma]]$, where $P_\ell(X) = \det(1 - X\text{Frob}_\ell|V_{I_\ell})$, with V_{I_ℓ} being the space of I_ℓ -coinvariants of V . If S is a finite set of places, we let $P_S := \prod_{\ell \in S} P_\ell$. From this simple remark, we obtain the following Lemma.

5.1.1.1 Lemma. *For all collections $\Sigma_1 \subseteq \Sigma_2$ of finite places, the discrete Λ -module $X_{\mathbb{Q}_\infty}^{1, \Sigma_1}(\mathcal{M})$ is torsion if and only if $X_{\mathbb{Q}_\infty}^{1, \Sigma_2}(\mathcal{M})$ is torsion. And we have the inclusion of characteristic ideals:*

$$\text{Ch}^{1, \Sigma_2}(\mathcal{M}) \supseteq \text{Ch}^{1, \Sigma_1} P_{\Sigma_2 \setminus \Sigma_1}.$$

5.1.2 No-nonzero pseudo-null submodule

In this subsection we show that $X_{\mathbb{Q}_\infty}^{\Sigma, 1}(\mathcal{M})$ —if it is a torsion Λ -module—then it has no non-zero finite order submodules. We follow quite closely the arguments of [SU14]. We make the following running assumption:

The Galois representation T is residually absolutely irreducible.

This implies that the definition of the previous sections do not depend on the choice of lattice $T \subset V$. Let us introduce the following notations. For $x \in \Lambda$ and $\bullet = \emptyset, \pm$, we set

$$V_x^\bullet := \text{Hom}_{\mathcal{O}}(K, \mathcal{M}^\bullet[x]) \quad T_x^\bullet := \text{Hom}_{\mathcal{O}}(K/\mathcal{O}, \mathcal{M}^\bullet[x]).$$

Then we have isomorphisms of Galois modules $W_x^\bullet := V_x^\bullet/T_x^\bullet \simeq \mathcal{M}^\bullet[x]$ for all $x \in \Lambda$.

We also denote by Ψ_x the character

$$\mathrm{Gal}_{\mathbb{Q}_p} \xrightarrow{\Psi} \Lambda^\times \twoheadrightarrow (\Lambda/x\Lambda)^\times \simeq \mathcal{O}^\times.$$

We remark that for $x = \gamma - (1+p)^m$, for m an integer, $\Psi_x = \varepsilon^m \omega_p^{-m}$, where ω_p is the Teichmüller character. If a is a generator of $(\mathbb{Z}/p\mathbb{Z})^\times$, denote by $\zeta_{p-1} := \omega_p(a)$.

5.1.2.1 Lemma. *Let $x = \gamma - (1+p)^m \in \Lambda$ with $m \neq 0, 1, k-1$, k is an integer. Assume that $k \not\equiv 0 \pmod{p-1}$. Then*

$$H^1(\mathbb{Q}_p, \mathcal{M}^-[x]) = H^1(I_p, \mathcal{M}^-)^{\mathrm{Gal}_{\mathbb{Q}_p}}[x] = H^1(I_p, \mathcal{M}^-[x])^{\mathrm{Gal}_{\mathbb{Q}_p}} \simeq (K/\mathcal{O})^2.$$

Proof. After I_p semi-simplification, $\mathcal{M}^-$ is isomorphic to $\Lambda^*(\Psi^{-1}) \oplus \Lambda^*(\varepsilon^{-(k-1)}\Psi)$.

By definition of Ψ , we have

$$H^0(I_p, \mathcal{M}^-) \subseteq \Lambda^*[\gamma - 1] \oplus \Lambda^*[\gamma - \zeta_{p-1}^{k-1}(1+p)^{k-1}] \simeq (K/\mathcal{O})^2.$$

Therefore, since $m \neq 0, k-1$, and $K/\mathcal{O}$ is divisible, multiplication by x on $H^0(I_p, \mathcal{M}^-)$ is surjective, that is $H^0(I_p, \mathcal{M}^-)/xH^0(I_p, \mathcal{M}^-) = 0$. Therefore by the long exact sequence in cohomology associated to the short exact sequence $\mathcal{M}[x] \hookrightarrow \mathcal{M} \twoheadrightarrow \mathcal{M}$ we have:

$$H^1(I_p, \mathcal{M}^-[x]) = H^1(I_p, \mathcal{M}^-)[x].$$

By the local Euler characteristic formula we have

$$\dim_K H^1(\mathbb{Q}_p, V_x^-) = 2 + \dim_K H^0(\mathbb{Q}_p, V_x^-) + \dim_K H^0(\mathbb{Q}_p, (V_x^-)^\vee(1)) = 2$$

as $k \neq 0, 1, k-1, k$. If ϖ is a uniformizer for $\mathcal{O}$, let $\mathfrak{m} = (\gamma-1, \varpi)$ be the maximal ideal of Λ . By local duality, $H^2(\mathbb{Q}_p, \mathcal{M}^-[\mathfrak{m}])$ vanishes, as $\mathcal{M}^-[\mathfrak{m}]^*(1)^{ss}|_{I_p} \simeq \Lambda/\mathfrak{m}(\omega_p) \oplus \Lambda/\mathfrak{m}(\omega_p^k)$ is totally ramified as $k \not\equiv 0 \pmod{p-1}$. It follows that multiplication by ϖ is surjective on $H^1(\mathbb{Q}_p, \mathcal{M}^-[x])$, since $\mathfrak{m} = (\gamma-1, \varpi) = (x, \varpi)$, i.e. this last local cohomology group is divisible. By the dimension counting above, we deduce that $H^1(\mathbb{Q}_p, \mathcal{M}^-[x]) \simeq (K/\mathcal{O})^2$ and then the result, as $\mathcal{M}^-[x]^{I_p}$ is finite. $\square$

5.1.2.2 Lemma. *Suppose Σ contains the primes dividing Np and that $X_{\mathbb{Q}_\infty}^{\Sigma,1}(\mathcal{M})$ is a torsion Λ module. Then for all but finitely many $x = \gamma - (1+p)^m$*

1. *the restriction map $H^1(G_\Sigma, \mathcal{M}[x]) \rightarrow H^1(\mathbb{Q}_p, \mathcal{M}^-[x])$ is surjective;*
2. *the restriction map $H^1(G_\Sigma, T_x^\vee(1)) \rightarrow \prod_{\ell \in \Sigma} H^1(\mathbb{Q}_\ell, T_x^\vee(1))$ is injective;*
3. *the global cohomology group $H^2(G_\Sigma, \mathcal{M}[x])$ vanishes;*
4. *we have*

$$H^1(G_\Sigma, \mathcal{M})/xH^1(G_\Sigma, \mathcal{M}) = 0$$

In the proof of the Lemma above we will need the following notation. Let $\text{III}^i(\Sigma, -)$ be the kernel of the natural restriction map $H^i(G_\Sigma, -) \rightarrow \prod_{\ell \in \Sigma} H^i(\mathbb{Q}_\ell, -)$

Proof. By Poitou-Tate duality, noticing that $H^0(\mathbb{Q}_v, V_x^\vee(1)) = 0$ for $v \in \Sigma$, we have for all but finitely many x :

$$\dim_K H^1(G_\Sigma, V_x) = \dim_K \text{III}^1(\Sigma, V_x^\vee(1)) - \dim_K H^0(\mathbb{R}, V_x) + 3.$$

Since complex conjugation has only one eigenvalue equal to 1 on V_x , we have $\dim_K H^0(\mathbb{R}, V_x) = 1$, and we obtain for all but finitely many x the dimension equality:

$$\dim_K H^1(G_\Sigma, V_x) = \dim_K \text{III}^1(\Sigma, V_x^\vee(1)) + 2.$$

Since $X_{\mathbb{Q}_\infty}^{\Sigma,1}(\mathcal{M})$ is a torsion Λ module, $\text{Sel}_{\mathbb{Q}_\infty}^{\Sigma,1}(\mathcal{M})[x] = \ker\{H^1(G_\Sigma, \mathcal{M}[x]) \rightarrow H^1(\mathbb{Q}_p, \mathcal{M}^-)[x]\}$ is finite for all but finitely many x . In particular, we have an injection

$$H^1(G_\Sigma, V_x) \hookrightarrow H^1(\mathbb{Q}_p, V_x^-)$$

and thus the dimension of $H^1(G_\Sigma, V_x)$ is the dimension of its image in $H^1(I_p, V_x^-)^{\text{Gal}_{\mathbb{Q}_p}}$ for almost all x , which is a space that is at most 2-dimensional for almost all x , by Lemma 5.1.2.1.

Combining this with the dimension computation above, we see that for all but finitely many x , we have the dimension equalities

$$\dim_K H^1(G_\Sigma, V_x) = 2, \quad \text{III}^1(\Sigma, V_x^\vee(1)) = 0.$$

Part 1 of the Lemma then follows from Lemma 5.1.2.1 and the first equality just above, when $k \not\equiv 0 \pmod{p-1}$. Part 2 of the Lemma follows from the second equality just above and the fact that $\text{III}^1(\Sigma, T_x^\vee(1))$ is torsion-free (as we assumed T to be residually irreducible) of rank equal to $\dim_K \text{III}^1(\Sigma, T_x^\vee(1))$. Part 3 follows from the fact that $\text{III}^2(\Sigma, \mathcal{M}[x])$ is dual to $\text{III}^1(\Sigma, T_x^\vee(1)) = 0$ and, for almost all x , the fact that $H^2(G_\Sigma, \mathcal{M}[x]) = \text{III}^2(\Sigma, \mathcal{M}[x])$, since $H^2(\mathbb{Q}_\ell, \mathcal{M}[x]) = 0$. Part 4 follows from

Part 3 and the long exact cohomology sequence for multiplication by x on $\mathcal{M}$.

We now prove part 1 even in the case where $k \equiv 0 \pmod{p-1}$. We give a general argument using our Euler System that works in general. By Proposition 5.1.2.5 (independent from part 1 of this Lemma), and by global duality there is an exact sequence

$$0 \longrightarrow \mathrm{Sel}^{1,\Sigma}(\mathcal{M}[x]) \longrightarrow \mathrm{H}^1(G_\Sigma, \mathcal{M}[x]) \longrightarrow \mathrm{H}^1(\mathbb{Q}_p, \mathcal{M}^-[x]) \longrightarrow \mathrm{Sel}^{0,\Sigma}(\mathcal{N}_x) \longrightarrow 0.$$

Exactness on the right is a consequence of the fact that $\mathrm{III}^2(\Sigma, \mathcal{M}[x]) = 0$, as is proven above by dimension counting. Here $\mathcal{N} = \mathrm{Ad}^0(T_f)(1) \otimes \Lambda(\Psi)$, $\mathcal{N}_x = \mathcal{N}/x\mathcal{N}$, and $\mathrm{Sel}^{0,\Sigma}(\mathcal{N}_x)$ denotes global classes of $\mathrm{H}^1(G_\Sigma, \mathcal{N}_x)$ whose localization in $\mathrm{H}^1(\mathbb{Q}_p, \mathcal{N}_x)$ belong to the subspace $\mathrm{H}^1(\mathbb{Q}_p, \mathrm{Fil}^0 \mathcal{N}_x)$. Therefore one obtains an injection

$$\frac{\mathrm{H}^1(G_\Sigma, \mathcal{N}_x)}{\mathrm{Sel}^{0,\Sigma}(\mathcal{N}_x)} \hookrightarrow \frac{\mathrm{H}^1(\mathbb{Q}_p, \mathcal{N}_x)}{\mathrm{im} \mathrm{H}^1(\mathbb{Q}_p, \mathrm{Fil}^0 \mathcal{N}_x)} \simeq \mathrm{H}^1(\mathbb{Q}_p, \mathcal{N}_x / \mathrm{Fil}^0 \mathcal{N}_x),$$

where the last isomorphism holds for almost all x . By the local Euler characteristic formula (or by duality using the analysis above), we have that $\dim_K \mathrm{H}^1(G_\Sigma, V_x(1)) = 1$, as $V_x(1) = \mathcal{N}_x[1/p]$. Therefore the $\mathcal{O}$ -rank of $\mathrm{H}^1(G_\Sigma, \mathcal{N}_x)$ is at most 1. It follows that either the quotient $\mathrm{H}^1(G_\Sigma, \mathcal{N}_x) / \mathrm{Sel}^{0,\Sigma}(\mathcal{N}_x)$ is finite or $\mathrm{Sel}^{0,\Sigma}(\mathcal{N}_x)$ is finite. Notice that $\mathrm{H}^1(G_\Sigma, \mathcal{N}_x)$ is torsion-free as T is assumed to be residually irreducible. Therefore, in order to prove that $\mathrm{Sel}^{0,\Sigma}(\mathcal{N}_x) = 0$, it suffices to show that the quotient $\mathrm{H}^1(G_\Sigma, \mathcal{N}_x) / \mathrm{Sel}^{0,\Sigma}(\mathcal{N}_x)$ is not finite, which will settle the claim of point 1. Specializing the Hida variable to f , our Euler System construction produces a class

$\mathcal{C} \in H^1(G_\Sigma, \mathcal{N})$ whose image by

$$H^1(G_\Sigma, \mathcal{N}) \longrightarrow H^1(\mathbb{Q}_p, \mathcal{N}) \longrightarrow H^1(\mathbb{Q}_p, \mathcal{N}/\text{Fil}^0 \mathcal{N}) \xrightarrow{\mathcal{L}^{PR}} \Lambda$$

coincides up to a non-zero scalar with Liu's p -adic L-function. Therefore specializing at generic x (for which Liu's p -adic L-function does not vanish), there is a class $\mathcal{C}_x \in H^1(G_\Sigma, \mathcal{N}_x)$ whose image in $H^1(\mathbb{Q}_p, \mathcal{N}_x/\text{Fil}^0 \mathcal{N}_x)$ generates a free rank-1 $\mathcal{O}$ -module. We deduce that, for such x , the quotient $H^1(G_\Sigma, \mathcal{N}_x)/\text{Sel}^{0,\Sigma}(\mathcal{N}_x)$ cannot be finite and thus $\text{Sel}^{0,\Sigma}(\mathcal{N}_x) = 0$. $\square$

5.1.2.3 Proposition (No non-zero pseudonull submodule). *Assume Σ contains all primes dividing N . If $X_{\mathbb{Q}_\infty}^{\Sigma,1}(\mathcal{M})$ is a torsion Λ -module then $X_{\mathbb{Q}_\infty}^{\Sigma,1}(\mathcal{M})$ contains no non-zero pseudo-null submodule.*

Proof. It is equivalent to show that $\text{Sel}_{\mathbb{Q}_\infty}^{\Sigma,1}(\mathcal{M})$ has no non-zero finite Λ -quotient. Let $I_\Sigma \subset H^1(\mathbb{Q}_p, \mathcal{M}^-)$ be the image of $H^1(G_\Sigma, \mathcal{M}) \rightarrow H^1(\mathbb{Q}_p, \mathcal{M}^-)$. The kernel of this map is simply $\text{Sel}_{\mathbb{Q}_\infty}^{\Sigma,1}(\mathcal{M})$ since Σ contains the primes dividing Np by the second part of proposition 5.1.2.4. We thus have an exact sequence

$$0 \longrightarrow \text{Sel}_{\mathbb{Q}_\infty}^{\Sigma,1}(\mathcal{M}) \longrightarrow H^1(G_\Sigma, \mathcal{M}) \longrightarrow I_\Sigma \longrightarrow 0.$$

Let $x = \gamma - (1+p)^m \in \Lambda$ as in Lemma 5.1.2.1 such that the conclusions of Lemma 5.1.2.2 are valid. We claim that the image of $H^1(G_\Sigma, \mathcal{M}[x]) = H^1(G_\Sigma, \mathcal{M})[x]$ in $H^1(\mathbb{Q}_p, \mathcal{M}^-)[x]$ coincides with $I_\Sigma[x]$. The fact that it is contained is immediate, and equality follows from Part (i) of Lemma 5.1.2.2. By the snake Lemma applied to the above exact sequence we have an injection $\text{Sel}_{\mathbb{Q}_\infty}^{\Sigma,1}(\mathcal{M})/x\text{Sel}_{\mathbb{Q}_\infty}^{\Sigma,1}(\mathcal{M}) \hookrightarrow$

$H^1(G_\Sigma, \mathcal{M})/xH^1(G_\Sigma, \mathcal{M})$ and this quotient is 0 by the choice of x and part 4 of Lemma 5.1.2.2; so $\text{Sel}_{\mathbb{Q}_\infty}^{\Sigma,1}(\mathcal{M})$ is x -divisible. In particular, if N is a finite quotient of $\text{Sel}_{\mathbb{Q}_\infty}^{\Sigma,1}(\mathcal{M})$, then N/xN is a quotient of $\text{Sel}_{\mathbb{Q}_\infty}^{\Sigma,1}(\mathcal{M})/x\text{Sel}_{\mathbb{Q}_\infty}^{\Sigma,1}(\mathcal{M}) = 0$, so $N/xN = 0$, and therefore, by invoking Nakayama's lemma, $N = 0$. $\square$

For Σ any finite set of primes containing the divisors of Np , consider the restriction:

$$\text{Res}^\Sigma : H^1(G_\Sigma, \mathcal{M}) \longrightarrow \frac{H^1(\mathbb{Q}_p, \mathcal{M})}{\mathcal{F}_p^0(\mathcal{M})} \times \prod_{\ell \in \Sigma, \ell \neq p} H^1(I_\ell, \mathcal{M})^{\text{Gal}_{\mathbb{Q}_\ell}}.$$

The kernel of Res^Σ is independent on Σ and is of course $\text{Sel}_{\mathbb{Q}_\infty}^0(\mathcal{M})$.

We introduce some auxiliary Selmer groups

$$\mathcal{N} := \text{Ad}^0(T)^\vee(1) \otimes \Lambda = \text{Hom}_{\mathcal{O}}(\text{Ad}^0(T_f), \mathcal{O}(1)) \otimes \Lambda.$$

The $\text{Gal}_{\mathbb{Q}}$ action is given by $\text{Ad}^0(\rho_f)(1) \otimes \Psi$. Let $\mathcal{N}^+$ be the $\text{Gal}_{\mathbb{Q}_p}$ stable subspace of $\mathcal{N}$ defined by

$$\mathcal{N}^+ = \text{Hom}_{\mathcal{O}}(\text{Ad}^0(T_f)/\text{Fil}^0(T_f), \mathcal{O}(1)) \otimes \Lambda.$$

We let $\mathcal{N}^- := \mathcal{N}/\mathcal{N}^+$. For an element $x \in \mathfrak{m}$, we define the free $\mathcal{O}$ -modules $N_x := \mathcal{N}/x\mathcal{N}$, $N_x^+ := \mathcal{N}^+/x\mathcal{N}^+$, and $N_x^- := N_x/N_x^+$. We also define

$$\text{Sel}^\Sigma(\mathcal{N}) := \ker \{ H^1(G_\Sigma, \mathcal{N}) \rightarrow H^1(\mathbb{Q}_p, \mathcal{N})/\mathcal{F}_p^1(\mathcal{N}) \};$$

here $\mathcal{F}_p^1(\mathcal{N})$ denotes the image of $H^1(\mathbb{Q}_p, \mathcal{N}^+)$ in $H^1(\mathbb{Q}_p, \mathcal{N})$. We define similarly $\text{Sel}^\Sigma(N_x)$ by replacing $\mathcal{N}$ with N_x in the above definitions.

We will need the following Lemma.

5.1.2.4 Lemma. *Suppose Σ contains all primes dividing Np , and $X_{\mathbb{Q}_\infty}^{1,\Sigma}(\mathcal{M})$ is a torsion Λ -module. For almost all $x = \gamma - (1+p)^m$, $m \in \mathbb{Z}$, the natural map*

$$\text{Sel}^\Sigma(\mathcal{N})/x\text{Sel}^\Sigma(\mathcal{N}) \longrightarrow H^1(G_\Sigma, N_x)$$

factors through $\text{Sel}^\Sigma(N_x)$ and is an injection.

Proof. Let $c \in \text{Sel}^\Sigma(\mathcal{N})$ be such that its image in $H^1(G_\Sigma, N_x)$ is trivial. Then, $c = xd$ for some $d \in H^1(G_\Sigma, \mathcal{N})$, this is because $H^2(G_\Sigma, \mathcal{N})[x] = 0$. Since $c \in \text{Sel}^\Sigma(\mathcal{N})$, d is such that the image of xd in $H^1(\mathbb{Q}_p, \mathcal{N}^-)$ vanishes. We want to show that $d \in \text{Sel}^\Sigma(\mathcal{N})$.

The kernel of multiplication by x on $H^1(\mathbb{Q}_p, \mathcal{N}^-)$ is the image of $H^0(\mathbb{Q}_p, N_x^-)$, but the I_p semisimplification of N_x^- is isomorphic to $\mathcal{O}(\varepsilon^{k-1}\Psi_x) \oplus \mathcal{O}(\Psi_x)$, where we have denoted by Ψ_x the reduction of Ψ modulo x . For almost all x , this module has no $\text{Gal}_{\mathbb{Q}_p}$ -invariants. Then multiplication by x is injective in $H^1(\mathbb{Q}_p, \mathcal{N}^-)$, therefore the image of d in $H^1(\mathbb{Q}_p, \mathcal{N}^-)$ vanishes and thus $d \in \text{Sel}^\Sigma(\mathcal{N})$. $\square$

For the following proposition, we follow [Ski16, Proposition 2.3.2].

5.1.2.5 Proposition. *Suppose Σ contains all primes dividing Np , and $X_{\mathbb{Q}_\infty}^{1,\Sigma}(\mathcal{M})$ is a torsion Λ -module (equivalently: $X_{\mathbb{Q}_\infty}^1(\mathcal{M})$ is torsion by Lemma 5.1.1.1). Then, the*

restriction map Res^Σ is surjective. In particular the restriction map

$$\text{Sel}^{1,\Sigma}(\mathcal{M}) \longrightarrow \prod_{\ell \in \Sigma, \ell \neq p} \text{H}^1(I_\ell, \mathcal{M})^{\text{Gal}_{\mathbb{Q}_\ell}}$$

is surjective.

Proof. We prove first the surjectivity of Res^Σ . Since the restriction maps $\text{H}^1(\mathbb{Q}_v, \mathcal{M}) \rightarrow \text{H}^1(I_v, \mathcal{M})^{\text{Gal}_{\mathbb{Q}_v}}$ are surjective for a finite place v , it suffices to prove that :

$$\widetilde{\text{Res}}^\Sigma : \text{H}^1(G_\Sigma, \mathcal{M}) \longrightarrow \frac{\text{H}^1(\mathbb{Q}_p, \mathcal{M})}{\text{imH}^1(\mathbb{Q}_p, \text{Fil}^0 \mathcal{M})} \times \prod_{\ell \in \Sigma, \ell \neq p} \text{H}^1(\mathbb{Q}_\ell, \mathcal{M}),$$

is surjective. Denote by $\mathcal{S}$ its kernel.

Consider the natural pairing

$$(\cdot, \cdot) : \mathcal{M} \times \mathcal{N} \rightarrow \mathbb{Q}_p/\mathbb{Z}_p$$

such that on pure tensors $t \otimes \lambda^* \in \mathcal{M}$ and $\phi \otimes \lambda \in \mathcal{N}$, we have

$$(t \otimes \lambda^*, \phi \otimes \lambda) = \lambda^*(\phi(t)\lambda).$$

This pairing is $\text{Gal}_{\mathbb{Q}_p}$ -equivariant and perfect. Under this pairing, $\text{Fil}^0 \mathcal{M}$ and $\mathcal{N}^+$ are mutual annihilators. We denote by $\mathcal{F}_p^1(\mathcal{N})$ the image of $\text{H}^1(\mathbb{Q}_p, \mathcal{N}^+) \rightarrow \text{H}^1(\mathbb{Q}_p, \mathcal{N})$. Then $\mathcal{F}_p^0(\mathcal{M})$ and $\mathcal{F}_p^1(\mathcal{N})$ are mutual annihilators under local Tate duality. Let $\text{III}^1(S, \mathcal{N})$ be the set of classes of $\text{Sel}^\Sigma(\mathcal{N})$ that are locally trivial at all places of S .

Then global Tate duality identifies the dual of the cokernel of $\widetilde{\text{Res}}^\Sigma$ with $\text{Sel}^\Sigma(\mathcal{N})/\text{III}^1(\Sigma, \mathcal{N})$.

Therefore to prove the desired surjectivity, it suffices to prove that $\text{Sel}^\Sigma(\mathcal{N})$ is trivial. Since T is residually irreducible, $H^1(G_\Sigma, \mathcal{N})$ is Λ -torsion free (and hence so are $\text{Sel}^\Sigma(\mathcal{N})$ and $\text{III}^1(\Sigma, \mathcal{N})$); therefore it suffices to prove that $\text{Sel}^\Sigma(\mathcal{N})$ is torsion.

For this, we exhibit elements $x = \gamma - (1+p)^m \in \mathfrak{m}$ such that $\text{Sel}^S(\mathcal{N})/x\text{Sel}^S(\mathcal{N})$ has finite order. By Lemma 5.1.2.4, it suffices to prove that for all but finitely many x , $\text{Sel}^\Sigma(N_x)$ has finite order (and since T is absolutely irreducible, it would follow that it is trivial). For this we show that $\text{Sel}^\Sigma(N_x) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p$ is trivial for all but finitely many x – or equivalently, as this module is divisible, that it has finite order.

We put $M_x := N_x \otimes \mathbb{Q}_p/\mathbb{Z}_p$ and $M_x^- := N_x^- \otimes \mathbb{Q}_p/\mathbb{Z}_p$. Since the maps $H^1(G_\Sigma, N_x) \otimes \mathbb{Q}_p/\mathbb{Z}_p \rightarrow H^1(G_\Sigma, M_x)$ and $H^1(\mathbb{Q}_p, N_x^-) \otimes \mathbb{Q}_p/\mathbb{Z}_p \rightarrow H^1(\mathbb{Q}_p, M_x^-)$ are injections we deduce that $\text{Sel}^\Sigma(N_x) \otimes \mathbb{Q}_p/\mathbb{Z}_p$ injects into $\text{Sel}^\Sigma(M_x)$, and it therefore suffices to find an x such that $\text{Sel}^\Sigma(M_x)$ has finite order.

The collection of (M_x) for varying $x = \gamma - (1+p)^m$ is the same as the collection of $(\mathcal{M}[x])$ since $\mathcal{N} \otimes \mathbb{Q}_p/\mathbb{Z}_p \simeq \mathcal{M}(1)$. Since

$$\text{Sel}^\Sigma(\mathcal{M}[x]) \hookrightarrow \text{Sel}_{\mathbb{Q}_\infty}^{1,\Sigma}(\mathcal{M})[x].$$

Since $X^{1,\Sigma}(\mathcal{M})$ is assumed to be torsion, for all but finitely many x , $X^{1,\Sigma}(\mathcal{M})/xX^{1,\Sigma}(\mathcal{M})$ has finite order. The quotient $X^{1,\Sigma}(\mathcal{M})/xX^{1,\Sigma}(\mathcal{M})$ being dual to $\text{Sel}_{\mathbb{Q}_\infty}^{1,\Sigma}(\mathcal{M})[x]$, we deduce that for all but finitely many x , $\text{Sel}^\Sigma(\mathcal{M}[x])$ is finite, and therefore for all but finitely many x , $\text{Sel}^\Sigma(M_x)$ is finite; which suffices to prove the claim of the Proposition. $\square$

With Proposition 5.1.2.5 at hand, we can upgrade Lemma 5.1.1.1 to the following

result, by the same argument as in Lemma 2.3.4 of [Ski16].

5.1.2.6 Corollary. *For all collections $\Sigma_1 \subseteq \Sigma_2$ of finite places, the discrete Λ -module $X_{\mathbb{Q}_\infty}^{1, \Sigma_1}(\mathcal{M})$ is torsion if and only if $X_{\mathbb{Q}_\infty}^{1, \Sigma_2}(\mathcal{M})$ is torsion. And we have the equality of characteristic ideals:*

$$\mathrm{Ch}^{1, \Sigma_2}(\mathcal{M}) = \mathrm{Ch}^{1, \Sigma_1} P_{\Sigma_2 \setminus \Sigma_1}.$$

Moreover if $\mathrm{Fitt}^{1, \Sigma}$ denotes the Fitting ideal of $X_{\mathbb{Q}_\infty}^{\Sigma, 1}(\mathcal{M})$, we have the equality

$$\mathrm{Fitt}^{1, \Sigma} = \mathrm{Ch}^{1, \Sigma}.$$

The following lemma shows that if Σ is large enough then $\mathrm{Sel}_{\mathbb{Q}_\infty}^{\Sigma, 1}(\mathcal{M})[p^m]$ and $X_{\mathbb{Q}_\infty}^{\Sigma, 1}(\mathcal{M})/p^m X_{\mathbb{Q}_\infty}^{\Sigma, 1}(\mathcal{M})$ depend only on the pair of $\mathcal{O}/p^m \mathcal{O}[\mathrm{Gal}_{\mathbb{Q}}]$ -modules $(T_f/p^m T_f, T_f^+/T_f^+/p^m T_f^+)$. This Lemma will be used to deduce the Iwasawa main conjecture along the entire Hida family regardless of the vanishing of the $\mathfrak{L}(V)$ -invariant.

5.1.2.7 Lemma. *Suppose Σ contains the primes dividing N . The inclusion $\mathcal{M}[p^m] \subset \mathcal{M}$ induces an identification:*

$$\mathrm{Sel}_{\mathbb{Q}_\infty}^{\mathrm{ur}, \Sigma}(\mathcal{M})[p^m] = \ker \{ H^1(G_\Sigma, \mathcal{M}[p^m]) \rightarrow H^1(I_p, \mathcal{M}^-[p^m]). \}$$

So, $\mathrm{Sel}_{\mathbb{Q}_\infty}^{\mathrm{ur}, \Sigma}(\mathcal{M})[p^m]$ depends only on the pair of $\mathcal{O}/p^m \mathcal{O}[\mathrm{Gal}_{\mathbb{Q}}]$ -modules $(T_f/p^m T_f, T_f^+/T_f^+/p^m T_f^+)$.

Proof. Note first that $(\mathcal{M}^-)^{I_p}$ is p -divisible – this is because $\mathcal{M}^-$ has a two step filtration with a subspace acted on by Ψ^{-1} and a quotient acted on by $\varepsilon^{-(k-1)}\Psi^{-1}$; and thus $(\mathcal{M}^-)^{I_p}$ is a direct sum of $K/\mathcal{O}$. Then, since $H^1(G_\Sigma, \mathcal{M}[p^m]) = H^1(G_\Sigma, \mathcal{M})[p^m]$,

we obtain the result since the kernel of the natural $H^1(I_p, \mathcal{M}^-[p^m]) \rightarrow H^1(I_p, \mathcal{M}^-)$ is the image of $(\mathcal{M}^-)^{I_p}/p^m(\mathcal{M}^-)^{I_p} = 0$ under the boundary map.

Notice that by definition T^+ depends only on $T_f^+ \subseteq T_f$. The claim of independence now follows, as $\mathcal{M}[p^m] \simeq T/p^m T \otimes_{\mathcal{O}} \Lambda^*[p^m]$ and $\mathcal{M}^- = \mathcal{M}[p^m]/\mathcal{M}^+[p^m]$ where $\mathcal{M}^+[p^m] = T^+/p^m T^+ \otimes \Lambda^*[p^m]$. $\square$

5.1.3 Greenberg L-invariants

In this section we define Greenberg's L-invariant for adjoint Selmer and we study a few of its properties. Let $\text{Sel}^{\text{BK}, \Sigma}(V)$ be the Bloch–Kato Selmer group with relaxed conditions at Σ , a finite collection of finite places. Following Hida [Hid04], denote by $U_p(V)$ the Selmer structure at p for $\text{Sel}^{\text{BK}, \Sigma}(V)$; this coincides with $\mathcal{F}_p^1(V)$ if f is of multiplicative type and $\mathcal{F}_p^{\text{ur}}(V)$ otherwise.

Consider the defining exact sequence of $H_f^1(\mathbb{Q}, V)$, where we recall that $V = \text{Ad}^0(\rho_f)$:

$$0 \longrightarrow \text{Sel}^{\text{BK}}(V) \longrightarrow H^1(G_S, V) \longrightarrow \frac{H^1(\mathbb{Q}_p, V)}{U_p(V)} \times \prod_{\ell \in S \setminus \{p\}} H^1(I_\ell, V)^{\text{Gal}_{\mathbb{Q}_\ell}} \longrightarrow \text{Sel}^{\text{BK}}(V(1))^\vee$$

By Flach's theorem $\text{Sel}^{\text{BK}}(V) = \text{Sel}^{\text{BK}}(V(1)) = 0$. We thus obtain an isomorphism:

$$H^1(G_S, V) \xrightarrow{\sim} \frac{H^1(\mathbb{Q}_p, V)}{U_p(V)} \times \prod_{\ell \in S \setminus \{p\}} H^1(I_\ell, V)^{\text{Gal}_{\mathbb{Q}_\ell}}.$$

Consider the restriction map

$$\mathrm{Sel}^0(V) \longrightarrow H^1(\mathbb{Q}_p, \mathrm{Fil}^0 V) \longrightarrow \frac{H^1(\mathbb{Q}_p, \mathrm{Fil}^0 V)}{\mathcal{F}_p^1(V)} \hookrightarrow H^1(\mathbb{Q}_p, K). \quad (5.3)$$

The local cohomology group $H^1(\mathbb{Q}_p, K)$ is naturally identified with continuous abelian group homomorphisms

$$\mathrm{Gal}_{\mathbb{Q}_p}^{\mathrm{ab}, p} \longrightarrow K,$$

where $\mathrm{Gal}_{\mathbb{Q}_p}^{\mathrm{ab}, p}$ denotes the maximal abelian pro- p quotient of the absolute Galois group of $\mathbb{Q}_p$. Local class field theory identifies $\mathrm{Gal}_{\mathbb{Q}_p}^{\mathrm{ab}, p}$ with the p -adic completion of $\mathbb{Q}_p^\times$; that is $\varprojlim_n \mathbb{Q}_p^\times / (\mathbb{Q}_p^\times)^{p^n}$. From the decomposition $\mathbb{Q}_p^\times = p^\mathbb{Z} \times \mathbb{Z}_p^\times$ we obtain a basis $\{\lambda_{\mathrm{ur}}, \lambda_{\mathrm{cyc}}\}$ of $H^1(\mathbb{Q}_p, K)$ characterized by the properties that

$$\lambda_{\mathrm{ur}}(p) = \lambda_{\mathrm{cyc}}(u)(\log_p(u))^{-1} = 1, \quad \text{and} \quad \lambda_{\mathrm{ur}}(u) = \lambda_{\mathrm{cyc}}(p) = 0,$$

where u denotes a topological generator of $1 + p\mathbb{Z}_p$. For example, we can take $u = 1 + p$.

The image of the map defined by the diagram 5.3 is a one-dimensional subspace of $H^1(\mathbb{Q}_p, K)$ and moreover it is not contained in the line spanned by λ_{ur} , as otherwise it would give rise to a non-trivial element of $H_f^1(\mathbb{Q}, V)$. The image of $\mathrm{Sel}^0(V)$ (equivalently $\mathrm{Sel}^{0, \Sigma}$) in $H^1(\mathbb{Q}_p, \mathrm{Fil}^0 V)$ is therefore spanned by an element of the form $x\lambda_{\mathrm{ur}} + y\lambda_{\mathrm{cyc}}$, with $y \neq 0$.

5.1.3.1 Definition. We set

$$\mathfrak{L}(V) := -y^{-1}x \in K.$$

This definition is clearly independent on the choice of vector spanning the image of $\text{Sel}^{0,\Sigma}(V)$.

We now move to an analysis of the L -invariant from an integral point of view; namely studying how the L -invariant manifests itself when considering Galois cohomology of W and T (with notations as in the first paragraph of Section 5.1.1).

5.1.3.2 Lemma. *If f is not of multiplicative type, the image of $H^1(\mathbb{Q}_p, \text{Fil}^1 W)$ in $H^1(\mathbb{Q}_p, W)$ vanishes.*

Proof. Consider the exact sequence

$$0 \longrightarrow \text{Fil}^1 W \longrightarrow W \longrightarrow W^- \longrightarrow 0.$$

Notice first that we have $(W^-)^{\text{Gal}_{\mathbb{Q}_p}} \simeq (W^{--})^{\text{Gal}_{\mathbb{Q}_p}} \simeq K/\mathcal{O}$. Therefore by the exact sequence above, we obtain the following long exact sequence

$$0 \longrightarrow (W^-)^{\text{Gal}_{\mathbb{Q}_p}} \simeq K/\mathcal{O} \longrightarrow H^1(\mathbb{Q}_p, \text{Fil}^1 W) \longrightarrow H^1(\mathbb{Q}_p, W) \longrightarrow H^1(\mathbb{Q}_p, W^-)$$

By the local Euler characteristic formula, we have $\dim_K H^1(\mathbb{Q}_p, \text{Fil}^1 V) = 1$. Moreover $H^2(\mathbb{Q}_p, \text{Fil}^1 W[\varpi]) = 0$ since f is not of multiplicative type. Therefore $H^1(\mathbb{Q}_p, \text{Fil}^1 W)$ is divisible and therefore isomorphic to $K/\mathcal{O}$. The (injective) boundary map $(W^-)^{\text{Gal}_{\mathbb{Q}_p}} \rightarrow H^1(\mathbb{Q}_p, \text{Fil}^1 W)$ must consequently be an isomorphism. This finishes the proof. $\square$

Define now $\text{Sel}^{\text{BK}}(W)$ as the image of $H_f^1(\mathbb{Q}, V)$ in $H^1(\mathbb{Q}, W)$. This coincides with the Selmer group with Selmer structure $U_p(W)^\div$ where the superscript $\div$ indicates the maximal divisible subgroup of $U_p(W)$.

5.1.3.3 Lemma. *If f is not of multiplicative type, then*

$$U_p(W) = U_p(W)^\div = \text{im}\{H^1(\mathbb{F}_p, W^{--}) \rightarrow H^1(\mathbb{Q}_p, W^-)\} \simeq K/\mathcal{O}.$$

If f is multiplicative, then

$$U_p(W) = U_p(W)^\div \simeq \text{im}H^1(\mathbb{Q}_p, \text{Fil}^1 W) \simeq K/\mathcal{O}.$$

In particular we always have $U_p(W)^\div = U_p(W) \simeq K/\mathcal{O}$. Therefore, when f is not multiplicative, $\text{Sel}^{\text{BK}}(W)$ coincides with $\text{Sel}^{\text{ur}}(W)$.

Proof. Indeed, by Lemma 5.1.3.2, $U_p(W)$ coincides with the kernel of $H^1(\mathbb{Q}_p, W^-) \rightarrow H^1(I_p, W^-)$ which is the image under inflation of $H^1(\mathbb{F}_p, (W^-)^{I_p})$, which is identified with $H^1(\mathbb{F}_p, W^{--}) \simeq K/\mathcal{O}$. $\square$

Fix a basis of $H^1(\mathbb{Q}_p, W^{--}) \simeq H^1(\mathbb{Q}_p, K/\mathcal{O})$ by taking the image of λ_{ur} and λ_{cyc} under the natural map $H^1(\mathbb{Q}_p, K) \rightarrow H^1(\mathbb{Q}_p, K/\mathcal{O})$. This fixes a decomposition

$$H^1(\mathbb{Q}_p, W^{--}) \simeq K/\mathcal{O} \times K/\mathcal{O}.$$

The first coordinate corresponding to λ_{ur} and the second to λ_{cyc} . We will denote the line spanned by λ_{ur} in $H^1(\mathbb{Q}_p, W^{--})$ by $H_{\text{ur}}^1(\mathbb{Q}_p, W^{--})$ and the line spanned by λ_{cyc}

by $H_{\text{cyc}}^1(\mathbb{Q}_p, W^{--})$ so that there is a natural decomposition:

$$H^1(\mathbb{Q}_p, W^{--}) = H_{\text{cyc}}^1(\mathbb{Q}_p, W^{--}) \times H_{\text{ur}}^1(\mathbb{Q}_p, W^{--}).$$

We denote by π_{cyc} (resp. π_{ur}) the natural projection onto the first (resp. second) factor of the above decomposition.

By the same reasoning, there is a decomposition:

$$H^1(\mathbb{Q}_p, T^{--}) = H_{\text{cyc}}^1(\mathbb{Q}_p, T^{--}) \times H_{\text{ur}}^1(\mathbb{Q}_p, T^{--}),$$

such that the choice of the basis $(\lambda_{\text{cyc}}, \lambda_{\text{ur}})$ of $H^1(\mathbb{Q}_p, V^{--})$ determines $\mathbb{Z}_p$ -module isomorphisms:

$$H_{\text{ur}}^1(\mathbb{Q}_p, T^{--}) \simeq \mathcal{O} \quad \text{and} \quad H_{\text{cyc}}^1(\mathbb{Q}_p, T^{--}) \simeq (1/\log_p(u))\mathcal{O}$$

The following proposition is essentially extracted from [Gre94, Proposition 4] applied in our particular context with slightly more general hypotheses.

5.1.3.4 Proposition. *Let $\mathcal{S}_p^0$ be the image of $\text{Sel}^0(W)$ in $H^1(\mathbb{Q}_p, W^{--})$. Assume that $\mathfrak{L}(V) \neq 0$. We then have the equality*

$$\frac{|\mathcal{S}_p^0 \cap H_{\text{cyc}}^1(\mathbb{Q}_p, W^{--})|_p}{|\mathcal{S}_p^0 \cap H_{\text{ur}}^1(\mathbb{Q}_p, W^{--})|_p} = |\log_p(u)^{-1} \mathfrak{L}(V)|_p.$$

Proof. Denote by $\mathcal{D}_p^0 \subseteq \mathcal{S}_p^0$ the image of $\text{Sel}^0(V)$ in $H^1(\mathbb{Q}_p, W^{--})$ – this is the maximal divisible subgroup of $\mathcal{S}_p^0$. Denote by d the index of $\mathcal{D}_p^0$ in $\mathcal{S}_p^0$. Since $\mathfrak{L}(V) \neq 0$, the

maps π_{ur} (and π_{cyc}) map $\mathcal{D}_p^0$ surjectively onto $K/\mathcal{O}$. Therefore we find that

$$|\mathrm{H}_{\text{ur}}^1(\mathbb{Q}_p, W^{--}) \cap \mathcal{S}_p^0| = d |\mathrm{H}_{\text{ur}}^1(\mathbb{Q}_p, W^{--}) \cap \mathcal{D}_p^0|$$

and similarly for the intersection with $\mathrm{H}_{\text{cyc}}^1(\mathbb{Q}_p, W^{--})$. It thus suffices to prove the result for $\mathcal{S}_p^0$ replaced by $\mathcal{D}_p^0$. In the basis spanned by (the image of) λ_{ur} and λ_{cyc} , $\mathcal{D}_p^0$ is the $K/\mathcal{O}$ line spanned by $(\mathfrak{L}(V)^{-1}, \log_p(u)^{-1})$. We have then that

$$\mathcal{D}_p^0 \cap \mathrm{H}_{\text{ur}}^1(\mathbb{Q}_p, W^{--}) \simeq \mathfrak{L}(V)^{-1} \mathcal{O} / \mathcal{O}$$

and

$$\mathcal{D}_p^0 \cap \mathrm{H}_{\text{cyc}}^1(\mathbb{Q}_p, W^{--}) \simeq \log_p(u)^{-1} \mathcal{O} / \mathcal{O}.$$

The result follows. □

We assume henceforth that $\mathfrak{L}(V) \neq 0$

5.1.3.5 Remark. With notations as in the proof of Proposition 5.1.3.4, we actually have $\mathcal{D}_p^0 = \mathcal{S}_p^0$ i.e., $d = 1$. Indeed, by global duality the image of $\mathrm{Sel}^0(W)$ in $\mathrm{H}^1(\mathbb{Q}_p, \mathrm{Fil}^0 W)$ is the annihilator of the image of $\mathrm{H}^1(G_S, T(1)) \rightarrow \mathrm{H}^1(\mathbb{Q}_p, T(1)) / \mathrm{H}^1(\mathbb{Q}_p, \mathrm{Fil}^1 T(1))$. By global duality the map $\mathrm{H}^1(G_S, T(1)) \rightarrow \mathrm{H}^1(G_S, T(1)) / \mathrm{H}^1(\mathbb{Q}_p, \mathrm{Fil}^0 T(1))$ is the annihilator of the image of $\mathrm{H}^1(\mathbb{Q}_p, \mathrm{Fil}^1 W)$ which is 0 if f is not multiplicative and divisible otherwise. Therefore, to show that the image of $\mathrm{Sel}^0(W)$ in $\mathrm{H}^1(\mathbb{Q}_p, \mathrm{Fil}^0 W)$ is divisible it suffices to prove that the image of $\mathrm{Sel}^0(T(1)) \rightarrow \mathrm{H}^1(\mathbb{Q}_p, T^{--}(1))$ is saturated. Indeed the elementary computation of [Har09, Chapter II] produces a $\mathbb{Z}_p$ -line with coordinates $(\mathfrak{L}(V), \log_p(u))$ in the basis determined by $\{\lambda_{\text{cyc}}, \lambda_{\text{ur}}\}$. Since we

know that the image of $\text{Sel}^0(V(1))$ is spanned by this vector by definition of the $\mathfrak{L}$ -invariant, we deduce that the image of $H^1(G_S, T(1))$ in $H^1(\mathbb{Q}_p, T^{--}(1))$ is saturated. And hence the result.

Let us record in a Lemma a fact that was used in the proof of Proposition 5.1.3.4 and the remark above and that we will use later also.

5.1.3.6 Lemma. *The image of $H^1(G_S, W)$ in $H^1(\mathbb{Q}_p, W^-)$ contains $H_{\text{ur}}^1(\mathbb{Q}_p, W^{--})$.*

We now study relationships between the Selmer groups for $\mathcal{M}$ and the Selmer groups for W .

5.1.3.7 Lemma. *There is an exact sequence:*

$$0 \longrightarrow (\mathcal{M}^{--})^{\text{Gal}_{\mathbb{Q}_p}} \simeq K/\mathcal{O} \longrightarrow H^1(\mathbb{Q}_p, W^-) \longrightarrow H^1(\mathbb{Q}_p, \mathcal{M}^-)[\gamma - 1] \longrightarrow 0.$$

Moreover, the image of $(\mathcal{M}^{--})^{\text{Gal}_{\mathbb{Q}_p}}$ in $H^1(\mathbb{Q}_p, W^-)$ is identified with the line $H_{\text{cyc}}^1(\mathbb{Q}_p, W^{--}) \subset H^1(\mathbb{Q}_p, W^{--}) \subset H^1(\mathbb{Q}_p, W^-)$.

Proof. This follows from the exact sequence

$$0 \longrightarrow W^- \longrightarrow \mathcal{M}^- \xrightarrow{\gamma-1} \mathcal{M}^- \longrightarrow 0.$$

and the fact that $(\mathcal{M}^-)^{\text{Gal}_{\mathbb{Q}_p}}/(\gamma-1)(\mathcal{M}^-)^{\text{Gal}_{\mathbb{Q}_p}} \simeq \Lambda^*[\gamma-1]$. The second claim follows easily from the fact that the image of $(\mathcal{M}^-)^{\text{Gal}_{\mathbb{Q}_p}}$ is contained in $H^1(\mathbb{Q}_p, W^{--})$ and the fact that restriction to inertia is an isomorphism $(\mathcal{M}^{--})^{\text{Gal}_{\mathbb{Q}_p}} \simeq (\mathcal{M}^{--})^{I_p} \simeq \Lambda^*[\gamma-1]$. $\square$

5.1.3.8 Lemma. *There is an exact sequence*

$$0 \longrightarrow \mathrm{Sel}_{\mathbb{Q}_\infty}^{1,\Sigma}(\mathcal{M}) \longrightarrow \mathrm{Sel}_{\mathbb{Q}_\infty}^{\mathrm{ur},\Sigma}(\mathcal{M}) \longrightarrow \mathrm{H}_{\mathrm{ur}}^1(\mathbb{Q}_p, W^{--}) \simeq K/\mathcal{O} \longrightarrow 0.$$

In particular we have the characteristic ideal equality:

$$\mathrm{Ch}^{1,\Sigma}(\mathcal{M}) = \mathrm{Ch}^{\mathrm{ur},\Sigma}(\mathcal{M})/(\gamma - 1).$$

Proof. This follows from Lemma 5.1.3.6, and the fact that $\mathrm{Sel}^{\mathrm{ur}}(\mathcal{M}) \subseteq \mathrm{Sel}^0(\mathcal{M})$ is cut out by the kernel of $\mathrm{H}^1(\mathbb{Q}_p, \mathrm{Fil}^0 \mathcal{M}) \rightarrow \mathrm{H}^1(I_p, \mathcal{M}^{--})$. $\square$

5.1.3.9 Lemma. *With the notations of Proposition 5.1.3.4, we have the equalities:*

$$|\mathrm{Sel}_{\mathbb{Q}_\infty}^{1,\Sigma}(\mathcal{M})[\gamma - 1]| = |\mathrm{Sel}^{1,\Sigma}(W)| \cdot |\mathcal{S}_p^0 \cap \mathrm{H}_{\mathrm{cyc}}^1(\mathbb{Q}_p, W^{--})|$$

and

$$|\mathrm{Sel}^{\mathrm{ur},\Sigma}(W)| = |\mathrm{Sel}^{1,\Sigma}(W)| \cdot |\mathcal{S}_p^0 \cap \mathrm{H}_{\mathrm{ur}}^1(\mathbb{Q}_p, W^{--})|$$

Proof. The second equality is immediate. The first one follows from the fact that $\mathrm{Sel}_{\mathbb{Q}_\infty}^{1,\Sigma}(\mathcal{M})[\gamma - 1]$ is the kernel of the restriction map $\mathrm{H}^1(G_S, W) \rightarrow \mathrm{H}^1(\mathbb{Q}_p, \mathcal{M})[\gamma - 1]$ (intersected with unramified classes away from $p \cup \Sigma$.) We then obtain

$$\frac{\mathrm{Sel}_{\mathbb{Q}_\infty}^{1,\Sigma}(\mathcal{M})[\gamma - 1]}{\mathrm{Sel}_{\mathbb{Q}_\infty}^{1,\Sigma}(W)} \simeq \ker\{\mathrm{H}^1(\mathbb{Q}_p, W^-) \rightarrow \mathrm{H}^1(\mathbb{Q}_p, \mathcal{M})[\gamma - 1]\} \cap \mathrm{im}\mathrm{H}^1(G_S, W).$$

We then conclude by Lemma 5.1.3.7. $\square$

We are now ready to give the following theorem, which is an immediate appli-

cation of Lemma 5.1.3.9, Proposition 5.1.3.4 and Lemma 5.1.3.8 and Proposition 5.1.2.3.

5.1.3.10 Theorem. *Assume that $\mathfrak{L}(V) \neq 0$ and that $X_{\mathbb{Q}_\infty}^{\Sigma,1}(\mathcal{M})$ is a torsion Λ module. Let h_Σ be a generator of $\mathrm{Ch}^{\mathrm{ur},\Sigma}(\mathcal{M})$ so that $\mathrm{Ch}^{1,\Sigma}(\mathcal{M})$ is generated by $h'_\Sigma := h_\Sigma/(\gamma - 1)$. Identify h_Σ and h'_Σ with power series in the variable $T := \gamma - 1$. Then, if f is not of multiplicative type, we have*

$$|\mathrm{Sel}^{\mathrm{BK},\Sigma}(W)|_p = |\log_p(u)\mathfrak{L}(V)^{-1}h'_\Sigma(0)|_p.$$

5.2 A precise bound on adjoint Selmer groups

In this section we use Rubin's Euler System machinery to deduce Iwasawa theoretic consequences for adjoint Selmer groups. We first verify Rubin's technical assumption $\mathrm{Hyp}(K_\infty, T)$. We keep the hypothesis of the previous section on the elliptic modular form f of weight 2 and further assume that f is normalized and has trivial Nebentypus; we denote by $\Gamma_0(N)$ the level of f . For simplicity assume also that the Fourier coefficients of f are rational. Let $k = \mathbb{Z}/p\mathbb{Z}$ and $\bar{k}$ an algebraic closure of k . Denote by

$$\bar{\rho}_f : \mathrm{Gal}_{\mathbb{Q}} \rightarrow T_f \otimes_{\mathbb{Z}_p} k$$

be the residual representation and let Σ the set primes at which $\bar{\rho}_f$ is ramified. If ℓ is a prime, we fix $I_\ell \subset \mathrm{Gal}_{\mathbb{Q}}$ an inertia subgroup at ℓ . We further assume the following

condition

$$\boxed{\text{There exists a prime } \ell \neq p \text{ an element of order } p \text{ in the image of } \bar{\rho}_f|_{I_\ell}.} \quad (5.4)$$

5.2.0.1 Remark. Note that this condition is verified if $\bar{\rho}_f$ is semistable but ramified at some prime ℓ . In particular, by the Remark following Lemma 1.10 in [Wil95], the projective image of $\bar{\rho}_f$ is not dihedral.

5.2.0.2 Lemma. *There exists an element $\sigma \in \text{Gal}_{\mathbb{Q}}$ such that*

1. σ acts trivially on μ_{p^∞} .
2. the $\mathbb{Z}_p$ -module $\text{Ad}^0(T_f)/(\sigma - 1)\text{Ad}^0(T_f)(1)$ is free of rank 1.

Proof. If $p > 3$ one can invoke simply [Rub00, p. 54]. So it suffices to consider the case of $p = 3$. We proceed similarly as in [Wil95]. We know that the projective image of $\bar{\rho}$ is not dihedral by Remark 5.2.0.1.

Since $p = 3$, as in [Wil95, Lemma 1.10] by Dickson's classification, we can produce an element in the derived subgroup of $\text{im} \bar{\rho}$ that is of order 4— it has minimal polynomial $X^2 + 1$. Consider now an element $\sigma \in \text{Gal}'_{\mathbb{Q}}$ mapping to this element modulo p . Let σ act on $T_f \otimes \mathbb{Z}_{p^2}$; here $\mathbb{Z}_{p^2}$ is the valuation ring of the unique unramified quadratic extension of $\mathbb{Z}_p$ obtained by adjoining a root of -1 , say w . By Hensel's lemma, σ has two distinct eigenvalues α, α^{-1} congruent to $\pm w$ modulo p . Now σ fixes the p -th roots of unity as it lies in the derived subgroup of $\text{Gal}_{\mathbb{Q}}$. Moreover, the eigenvalues of σ acting on $\text{Ad}^0(T_f)(1)$ are α^2, α^{-2} and 1. Since α^2, α^{-2} are not congruent to 1 modulo p (they are congruent to -1 and p is odd), by faithfully flat descent, this element satisfies the required conditions.

□

5.2.0.3 Lemma. *The representation $\mathrm{Ad}^0(T_f)(1) \otimes \bar{k}$ is an irreducible $\bar{k}[\mathrm{Gal}_{\mathbb{Q}(\mu_{p^\infty})}]$ -module.*

Proof. Let G' be the derived image of $\bar{\rho}_f$. Then it suffices to prove that the action by conjugation of G' on traceless 2×2 matrices is irreducible. If $p > 3$, by the proof of Lemma 1.10 (i) of [Wil95], G' surjects onto $\mathbf{PSL}_2(k)$. And the result follows. When $p = 3$, Wiles in *loc cit.* shows that $G' \subset \mathbf{SL}_2(k)$ contains an element of order 4 (fact that we already used in the proof of Lemma 5.2.0.2). Call now γ the element afforded by assumption 5.4, then γ obviously fixes the p -th roots of unity. The subgroup generated by G' and (the image of) γ has order divisible by both 4 and 3 – it must therefore contain $\mathbf{SL}_2(k)$ and we can conclude. □

To obtain the correct divisibility from the Euler System machinery, we need also the following Lemma. Recall the notation introduced before Proposition 5.1.2.5, where we have set $\mathcal{N} := \mathrm{Ad}^0(T_f)^\vee(1) \otimes \Lambda$.

5.2.0.4 Lemma. *We have an isomorphism $H^1(\mathbb{Q}_p, \mathcal{N}^-) \simeq \Lambda$, that is $H^1(\mathbb{Q}_p, \mathcal{N}^-)$ is a Λ module of rank 1. Under this identification, the image of*

$$H^1(\mathbb{Q}_p, \mathcal{N}) \rightarrow H^1(\mathbb{Q}_p, \mathcal{N}^-)$$

is $(\gamma - 1)\Lambda$.

Proof. Since $\mathcal{N}^-$ is a cyclotomic twist of an unramified character, the Iwasawa cohomology of $\mathcal{N}^-$ is a Λ -module of rank 1. To prove the second claim we consider the

exact sequence:

$$H^1(\mathbb{Q}_p, \mathcal{N}) \longrightarrow H^1(\mathbb{Q}_p, \mathcal{N}^-) \longrightarrow H^2(\mathbb{Q}_p, \mathcal{N}^+) \longrightarrow 0$$

By local duality $H^2(\mathbb{Q}_p, \mathcal{N}^+)$ is dual to $H^0(\mathbb{Q}_p, (\mathcal{N}^+)^{\vee}(1)) \simeq \Lambda^*[\gamma - 1] \simeq (\Lambda/(\gamma - 1)\Lambda)^{\vee}$, which proves the desired result. $\square$

Lemma 5.2.0.2 and Lemma 5.2.0.3 show that under assumption 5.4, Rubin's hypothesis $\text{Hyp}(K_{\infty}, T)$ for $T = \text{Ad}^0(T_f)(1)$ in [Rub00, p. 27] is verified. Using the notation of Definition 4.2.2.2, we have the Euler system classes $\mathcal{C}_M(f)$ for $\text{Ad}^0(T_f)(1)$; the localization at p of the base class $\mathcal{C}_1(f)$ is denoted by $\mathcal{C}_p(f)$. Denote by $H_{/f}^1(\mathbb{Q}_p, \mathcal{N})$ the singular quotient of $H^1(\mathbb{Q}_p, \mathcal{N})$. This quotient coincides with the quotient by the kernel of $H^1(\mathbb{Q}_p, \mathcal{N}) \rightarrow H^1(I_p, \mathcal{N}^-)$ if f is of multiplicative type and otherwise it is identified with $H^1(\mathbb{Q}_p, \mathcal{N}^-)$.

Notice now that we have the following divisibility:

$$\text{char}\left(H_{/f}^1(\mathbb{Q}_p, \mathcal{N})/\Lambda\mathcal{C}_p(f)\right) \mid (\gamma - 1) \cdot \text{char}\left((\gamma - 1)H^1(\mathbb{Q}_p, \mathcal{N}^-)/\text{im}(\mathcal{C}_p(f))\right);$$

where im denote the image of $\mathcal{C}_p(f)$ in $H^1(\mathbb{Q}_p, \mathcal{N}^-)$.

By Theorem 4.7.2.4, we have computed the big Perrin-Riou logarithm of $\text{im}(\mathcal{C}_p(f))$, which is equal to $c_f \mathcal{L}_f \neq 0$. Combining Theorem 3.8 of [Rub00], with these remarks we obtain the following Theorem.

5.2.0.5 Theorem. *The characteristic ideal of $X_{\mathbb{Q}_{\infty}}^1(\mathcal{M})$ divides $c_f \mathcal{L}'_f \in \Lambda$. Here $\mathcal{L}_f$ denotes Liu's p -adic L -function as in Theorem 4.7.2.4. In particular, $X_{\mathbb{Q}_{\infty}}^{1, \Sigma}$ is a*

torsion Λ -module and its characteristic ideal divides $c_f^\Sigma \mathcal{L}'_f$, where $c_f^\Sigma := P_\Sigma(0)c_f$.

We now combine the Theorem above with the results of Section 5.1.3 to obtain a precise on the Bloch-Kato Selmer group for W .

5.2.0.6 Corollary. *Assume that f is not multiplicative and that $\mathfrak{L}(V) \neq 0$. Let Σ be a finite collection of places containing N . Then we have the inequality:*

$$v_p(\mathrm{Sel}^{\mathrm{BK}, \Sigma}(W)) \leq v_p(c_f^\Sigma),$$

where $c_f^\Sigma := P_\Sigma(0)c_f$.

Proof. If $\mathfrak{L}(V) \neq 0$, and f is not multiplicative, then by results of Dasgupta [Das16, Theorem 3 and Theorem 4], the analytic L -invariant $\mathcal{L}_{an}(\mathrm{Ad}^0(f))$ appearing in Theorem 4.7.3.8 coincides with $\mathfrak{L}(V)$ so that

$$\mathcal{L}_{an}(\mathrm{Ad}^0(f)) = \mathfrak{L}(V) \neq 0.$$

We can then combine as before Theorem 5.1.3.10, Theorem 5.2.0.5, and Theorem 4.7.3.8 to obtain the result. $\square$

5.3 The modularity theorem

In this section we retrieve a form of Taylor–Wiles’ $R = T$ theorem [Wil95], following the original strategy of Wiles.

We now introduce relevant notations. Let us now fix once and for all a representation

$$\rho_0 : \text{Gal}_{\mathbb{Q}} \longrightarrow \mathbf{GL}_2(k),$$

such that ρ_0 is modular, ordinary at p , and verifies assumption 5.4 (which is satisfied if ρ_0 is everywhere semi-stable). Let Σ be the set of primes containing all primes $\ell \neq p$ such that ρ_0 is unramified at ℓ and containing p only when $\rho_0|_{\text{Gal}_{\mathbb{Q}_p}}$ is good and ordinary in the sense of [DDT94].

Let $N_{\Sigma_0} = p^{\delta} N(\rho_0)$ where $N(\rho_0)$ is the conductor of ρ_0 , Σ_0 are the primes dividing $N(\rho_0)$, and $\delta = 1$ if ρ_0 is good and $p \notin \Sigma_0$ and $\delta = 0$ otherwise. Let N_0 be the prime to p part of N_{Σ_0} . Denote by $\mathbb{T}_S$ the Hecke algebra defined after Lemma 3.26 in *loc cit.* Let g be the eigenform of level $\Gamma_0(N_{\Sigma_0})$ determined by Lemma 4.6 of *loc cit.* and let $\mathfrak{m}_g$ be the corresponding maximal ideal of the Hecke algebra $\mathbb{T}_{\emptyset} = \mathbb{T}_0(N_{\Sigma_0})$ acting on modular forms of level $\Gamma_0(N_{\Sigma_0})$. If Σ_0 does not contain enough primes $\ell \equiv 1 \pmod{p}$ in the sense of assumption 3.3.5.2, add to Σ_0 enough such primes to form a set of primes Σ containing p and enough primes $\ell \equiv 1 \pmod{p}$. So consider f the level raised form, ordinarily stabilized at p for this choice of $\Sigma \supseteq \Sigma_0$; denote by N the analogous quantity to N_{Σ_0} . Let $\mathfrak{m}_f \subset \mathbb{T}_0(pN)$ be the corresponding maximal ideal of f . Then there is an isomorphism

$$\mathbb{T}_0(pN)_{\mathfrak{m}_f} \simeq \mathbb{T}_{\Sigma}$$

by Lemma 3.26 of *loc cit.* Let $\phi_f : \mathbb{T}_{\Sigma} \rightarrow \mathcal{O}$ the Hecke eigensystem corresponding to

f , and denote by $\mathfrak{p}_f$ the corresponding prime ideal. We set

$$c_\Sigma := \text{Ann}_{\mathbb{T}_\Sigma}(\mathfrak{p}_f)$$

and define analogously

$$c_g := \text{Ann}_{\mathbb{T}_\emptyset}(\mathfrak{p}_g);$$

we refer to them as the congruence ideals. For any admissible set of places $S \supseteq \Sigma_0$, we can define analogous congruence ideals c_S . By the analysis in section 4.4 of *loc cit.*, they verify $c_S = T_S(\ell^{-1})c_f$; Here $T_S \in \mathbb{T}_S[X]$ is the Hecke polynomial, defined so that the pull-back of T_S along any Hecke eigensystem ϕ_f coincides with $P_\ell(\ell^{-s})$ for $\ell \neq p$ and for $\ell = p$, $T_S = 1 - T_p^2$. Finally, for S an admissible finite set of places containing we let R_S be the universal deformation rings of ρ_0 classifying lifts of ρ_0 of type S respectively in the sense of [DDT94, p. 76]. By the existence of a Galois representation that is $\mathbb{T}_{\mathfrak{m}_f}$ -valued deforming ρ_0 (as in Lemma 3.277 of *loc cit.*), we deduce the existence of a surjective homomorphism

$$R_S \twoheadrightarrow \mathbb{T}_S,$$

for any S .

We are now ready to reprove the following theorem of Taylor–Wiles.

5.3.0.1 Theorem (Taylor–Wiles). *The maps*

$$R_S \twoheadrightarrow \mathbb{T}_S,$$

are isomorphisms.

The proof of this Theorem will occupy the rest of this section.

Consider now $\wp_f$ the kernel of the homomorphism $R_\Sigma \rightarrow \mathcal{O}$ obtained by composing the surjection above with ϕ_f . We let c_f^Σ be the image of c_Σ in $\mathcal{O}$. By a very general theorem (see Theorem 3.40 of *loc cit.*), asking for the morphisms above to be isomorphisms is equivalent to asking that the order of $\wp_f/\wp_f^2$ is bounded below by the order of $\text{Sel}^{\text{BK},\Sigma}(W)$, where $W = V/T$ as in Section 5.1.1. We will prove this bound using our Euler system, in particular Corollary 5.2.0.6.

By Corollary 5.2.0.6 the map

$$R_\Sigma \longrightarrow \mathbb{T}_\Sigma.$$

is an isomorphism if $\mathfrak{L}(V) \neq 0$. Then by Corollary 3.37 of [DDT94], one deduces that also the maps $R_S \rightarrow \mathbb{T}_S$ are isomorphisms. So Theorem 5.3.0.1 follows if $\mathfrak{L}(V) \neq 0$.

We now deduce the general case. Hida [Hid04] has shown that the L-invariant can only vanish at finitely many points along a Hida family. In particular there exists a weight $k \geq 2$ and a weight k ordinary eigenform h , such that $k \equiv 2 \pmod{p-1}$ and $\mathfrak{L}(V) \neq 0$, where V is now the adjoint of ρ_h . The exact same reasoning as before then implies that the specialization of the ring denoted by R_k^{ord} in [Hid21, Section 6.1] is equal to the ring denoted by $\mathbb{T}_k$ in *loc cit.*. By Theorem 6.2.17 of *loc cit.* we can deduce that $R^{\text{ord}} = \mathbb{T}^{\text{ord}}$ and by exact control for the ordinary Hecke-algebra and R^{ord} (see Corollary 6.1.10 of *loc cit.*) we have that $R_\Sigma \simeq \mathbb{T}_\Sigma$. This concludes the proof.

5.4 The Iwasawa main conjecture

In this section we prove the Iwasawa main conjecture for adjoint Selmer groups; namely, we upgrade the divisibility of Proposition 5.2.0.5 into an equality. This is actually an immediate consequence of this Proposition and the modularity theorem. Indeed at the trivial character, by Corollary 5.2.0.6 and the modularity theorem, the bound and the size of the Selmer group coincide. The equality is then a consequence of the simple fact that two Iwasawa functions, one dividing the other, that coincide at a point differ by a unit.

We recall notations to state the next proposition, f is a weight k specialization of the Hida family $\mathbf{f}$ as in Proposition 4.7.3.1.

5.4.0.1 Proposition. *Assume that $\mathfrak{L}(V) \neq 0$ and that f is not multiplicative. We have, for any finite collection of finite places Σ , the equality of characteristic ideals:*

$$\mathrm{Char}_{\Lambda}\left(X_{\mathbb{Q}_{\infty}}^{1,\Sigma}(\mathcal{M})\right) = (P_{\Sigma}c_f\mathcal{L}'_f).$$

Equivalently

$$\mathrm{Char}_{\Lambda}\left(X_{\mathbb{Q}_{\infty}}^{\mathrm{ur},\Sigma}(\mathcal{M})\right) = (P_{\Sigma}c_f\mathcal{L}_f).$$

We now wish to remove the hypothesis of the non-vanishing of $\mathfrak{L}(V)$. We do this by using congruences, as is done in [Ski16]. We recall that P_{Σ} is defined at the end of Section 5.1.1.

5.4.0.2 Theorem. *Let f be a weight k , trivial Nebentypus, specialization of the Hida family $\mathbf{f}$. We have the equality of characteristic ideals for any finite set of finite places*

Σ :

$$\text{Char}_\Lambda \left(X_{\mathbb{Q}_\infty}^{1,\Sigma}(\mathcal{M}) \right) = (P_\Sigma c_f \mathcal{L}'_f).$$

Equivalently

$$\text{Char}_\Lambda \left(X_{\mathbb{Q}_\infty}^{\text{ur},\Sigma}(\mathcal{M}) \right) = (P_\Sigma c_f \mathcal{L}_f).$$

Proof. Let f be as in the theorem. If $\mathfrak{L}(V) \neq 0$ and f is not multiplicative, we can apply Proposition 5.4.0.1. We deduce the Theorem from this case as follows.

By Hida theory there exists newforms f_m° , $m > 0$ of weight k_m congruent to k modulo $p^m(p-1)$, such that the field generated by the Hecke eigensystem of f_m is contained in K . We denote by f_m the ordinary p -stabilization of f_m° ; it verifies the congruence $f_m \equiv f \pmod{p^m(p-1)}$, in the sense that their Hecke eigensystems are congruent modulo $p^m(p-1)$ in $\mathcal{O}$.

It suffices to prove the theorem for a given finite collection Σ of finite places. Let Σ be such a collection that contains the primes dividing the tame level N of $\mathbf{f}$. The congruence $f_m \equiv f \pmod{p^m(p-1)}$ implies the isomorphism

$$\text{Sel}_{\mathbb{Q}_\infty}^{1,\Sigma}(\mathcal{M}_f)[p^m] \simeq \text{Sel}_{\mathbb{Q}_\infty}^{1,\Sigma}(\mathcal{M}_{f_m})[p^m]$$

by Lemma 5.1.2.7. So comparing Fitting ideals gives

$$\text{Fitt}(X_{\mathbb{Q}_\infty}^{1,\Sigma}(\mathcal{M}_{f_m})) \equiv \text{Fitt}(X_{\mathbb{Q}_\infty}^{1,\Sigma}(\mathcal{M}_f)) \pmod{p^m \Lambda}.$$

Since $X_{\mathbb{Q}_\infty}^{1,\Sigma}(\mathcal{M}_{f_m})$ is torsion, by Corollary 5.1.2.6, the Fitting ideal of $X_{\mathbb{Q}_\infty}^{1,\Sigma}(\mathcal{M}_{f_m})$ is equal to its characteristic ideal. The existence of the two variable p -adic L -function

$\mathcal{L}_f$ of Liu, implies that there is a congruence

$$P_\Sigma \mathcal{L}'_{f_m} \equiv P_\Sigma \mathcal{L}'_f \bmod p^m \Lambda.$$

We therefore deduce that

$$\text{Fitt}(X_{\mathbb{Q}_\infty}^{1,\Sigma}(\mathcal{M}_f)) \equiv P_{\Sigma c_f} \mathcal{L}'_f \bmod p^m \Lambda.$$

Since this last equality is valid for all m , $X_{\mathbb{Q}_\infty}^{1,\Sigma}(\mathcal{M}_f)$ is torsion as its fitting ideal is not zero. We deduce by the usual argument of Iwasawa theory, that since $X_{\mathbb{Q}_\infty}^{1,\Sigma}(\mathcal{M}_f)[p^m]$ has no non-zero pseudo-null submodule, its Fitting ideal is its characteristic ideal, and therefore we have

$$\text{Char}_\Lambda(X_{\mathbb{Q}_\infty}^{1,\Sigma}(\mathcal{M}_f)) \equiv P_{\Sigma c_f} \mathcal{L}'_f \bmod p^m \Lambda,$$

for all m . Since the characteristic ideal is always principal, it follows that equality holds. The proof of the Theorem is now complete. $\square$

Part III

Theta Classes on Modular Curves at Inert Primes

Chapter 6

Foreword

We refer the reader to Section 1.2.2 of the General Introduction, for the statement of the main theorem of this part of the dissertation. We comment briefly on the organization of the paper. In Chapter 7, we fix general notations for modular curves. In Chapter 8, we begin the construction of the relevant Theta classes. The construction is organized in three main steps. In section 8.1, we study from the automorphic point of view theta series for $\mathbf{GL}_2$, which allows us to compute the q -expansion at all cusps of these theta series. Then, we introduce Lubin–Tate formal groups and the relevant local periods. We finish the construction by combining the preceding two steps: the computation of the q -expansion at all cusps controls the integrality properties of the theta classes by a p -adic Hodge theoretic argument and the construction of the Lubin–Tate periods allows for a correct normalization of the geometric étale classes.

6.1 Basic Notation.

Let $\overline{\mathbb{Q}}$ denote an algebraic closure of $\mathbb{Q}$ and fix an embedding $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$. Let $V = \mathbb{Q} \oplus \mathbb{Q}$ be a two dimensional $\mathbb{Q}$ -vector space; we endow V with the canonical basis. Let $\mathbf{G} = \mathbf{GL}_{2,\mathbb{Q}}$. Let $\mathbf{B} \subset \mathbf{G}$ be the Borel subgroup of $\mathbf{G}$ consisting of the upper triangular matrices (in the canonical basis of V) and let

$$\mathbf{T} := \left\{ t = \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix} : t_1, t_2 \in \mathbf{G}_{m,\mathbb{Q}} \right\}$$

be the diagonal. For $i = 1, 2$, we denote by e_i the root corresponding to the character $t \mapsto t_i$. Let

$$\mathbf{U} := \left\{ u(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} : x \in \mathbf{G}_a \right\}$$

be the unipotent radical of $\mathbf{B}$ so that $\mathbf{B} = \mathbf{T} \rtimes \mathbf{U}$. Let $\mathbf{Z}_{\mathbf{G}} \subset \mathbf{T}$ be the center of $\mathbf{G}$.

Let $\mathbf{S} := \text{Res}_{\mathbb{C}/\mathbb{R}} \mathbf{G}_m$ be Deligne's torus. Let $E/\mathbb{Q}$ be a number field and let E^{ab} denote its maximal abelian extension inside $\overline{\mathbb{Q}}$. Denote by $\mathbb{A}_E$ the ring of adèles of E and by $\mathbb{A}_E^{(\infty)}$ the ring of finite adèles of E . If S is a finite set of places of E , we denote by $\mathbb{A}^{(S)}$ the adèles away from S and $E_S := \prod_{v \in S} E_v$. The absolute Galois group of E will be denoted by G_E . Global class field theory provides a canonical surjective map, called the reciprocity map,

$$\text{rec}_E : E^\times \backslash \mathbb{A}_E^\times \longrightarrow \text{Gal}(E^{\text{ab}}/E).$$

We normalize the reciprocity map by matching uniformizers to geometric Frobenius elements. The map rec_E induces a commutative diagram:

$$\begin{array}{ccc} E^\times \backslash \mathbb{A}_E^\times & \xrightarrow{\text{rec}_E} & \text{Gal}(E^{\text{ab}}/E) \\ \downarrow & & \downarrow \\ E^\times \backslash \mathbb{A}_E^\times / N_{L/E} \mathbb{A}_L^\times & \xrightarrow[\text{rec}_{L/E}]{\sim} & \text{Gal}(L/E), \end{array}$$

for every finite abelian Galois extension L/E .

Chapter 7

Modular curves

7.1 The Shimura datum

In this section we review basic facts about modular curves from the point of view of Shimura varieties. Let $\mathcal{X}$ be the $\mathbf{G}(\mathbb{R})$ class of the homomorphism

$$h : \mathbf{S} \rightarrow \mathbf{G}_{\mathbb{R}}, \quad z = x + iy \mapsto \begin{pmatrix} x & y \\ -y & x \end{pmatrix}. \quad (7.1)$$

One can identify $\mathcal{X}$ with $\mathcal{H}^{\pm} := \mathbb{C} \setminus \mathbb{R}$, by sending h to $i \in \mathbb{C} \setminus \mathbb{R}$. The $\mathbf{G}(\mathbb{R})$ action on $\mathcal{X}$ is then identified with the usual $\mathbf{G}(\mathbb{R})$ action on $\mathcal{H}^{\pm}$ by fractional linear transformations. The datum $(\mathbf{G}, h)$ is a Shimura datum and thus determines a system of algebraic varieties $\mathrm{SH}_{\mathbf{G}}(K)$ indexed by compact open subgroup $K \subseteq \mathbf{G}(\mathbb{A}_f)$, such

that complex analytically we have

$$\mathrm{SH}_{\mathbf{G}}(K)(\mathbb{C}) = \mathbf{G}(\mathbb{Q}) \backslash (\mathcal{X} \times \mathbf{G}(\mathbb{A}_f) / K_f).$$

Note that the one dimensional complex manifold of the right hand-side is a disjoint union of usual modular curves $\Gamma \backslash \mathcal{H}$, where $\Gamma \subseteq \mathbf{SL}_2(\mathbb{Z})$ is an arithmetic subgroup. The image of $(z, g) \in \mathcal{H}^{\pm} \times \mathbf{G}(\mathbb{A}_f)$ in $\mathrm{SH}_{\mathbf{G}}(K)(\mathbb{C})$, will be denoted by $[z, g]_K$, or if z corresponds to an element $h \in \mathcal{X}$, we denote it by $[h, g]_K$.

The reflex field of $\mathrm{SH}_{\mathbf{G}}$ is $\mathbb{Q}$. Indeed, the reflex field of a Shimura datum is the field of definition of the $\mathbf{G}(\mathbb{C})$ conjugacy class of the cocharacter

$$\mu_h : \mathbf{G}_{m, \mathbb{C}} \hookrightarrow \mathbf{S}_{\mathbb{C}} \simeq \mathbf{G}_{m, \mathbb{C}} \times \mathbf{G}_{m, \mathbb{C}} \rightarrow \mathbf{G}_{\mathbb{C}}.$$

Here the decomposition $\mathbf{S}_{\mathbb{C}} \simeq \mathbf{G}_{m, \mathbb{C}} \times \mathbf{G}_{m, \mathbb{C}}$ is determined by requiring that the inclusion $\mathbb{C}^{\times} \hookrightarrow \mathbf{S}(\mathbb{R}) \hookrightarrow \mathbf{S}(\mathbb{C}) \simeq \mathbb{C}^{\times} \times \mathbb{C}^{\times}$ is the map $z \mapsto (z, \bar{z})$, where $\bar{z}$ is the complex conjugate of z . Note that $h(z)$ is the matrix of multiplication by z on $\mathbb{C} \simeq \mathbb{R}^2$. Therefore, base changing to $\mathbb{C}$, using the identification $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \simeq \mathbb{C}^2$, $z \otimes \lambda \mapsto (z\lambda, \bar{z}\lambda)$, we see that $h_{\mathbb{C}}$ is $\mathbf{G}_{\mathbb{C}}$ conjugated to

$$(z, \bar{z}) \mapsto \begin{pmatrix} z & 0 \\ 0 & \bar{z} \end{pmatrix}.$$

Explicitly $\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$ conjugates $h_{\mathbb{C}}$ to the homomorphism $\mathbf{S}_{\mathbb{C}} \rightarrow \mathbf{G}_{\mathbb{C}}$ described by the

above equation. Thus μ_h is $\mathbf{G}_{\mathbb{C}}$ conjugated to the cocharacter

$$z \mapsto \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix},$$

which is itself defined over $\mathbb{Q}$, and therefore so is the $\mathbf{G}(\mathbb{C})$ conjugacy class of μ_h .

Then $\mathrm{SH}_{\mathbf{G}}(K)$ has a *canonical model* (in the sense of Deligne [Del79]) over the reflex field $\mathbb{Q}$, which we denote by $Y(K)$. This model is characterized by the action of the Galois group on special points. Here "special points" correspond to homomorphisms $h : \mathbf{S} \rightarrow \mathbf{G}_{\mathbb{R}}$ that factor through a maximal torus $T/\mathbb{Q}$. For the Shimura datum at hand, every such homomorphism corresponds to a quadratic imaginary field $E/\mathbb{Q}$ such that $T = T_E = \mathrm{Res}_{E/\mathbb{Q}} \mathbf{G}_{m,E}$. Choosing a $\mathbb{Q}$ -basis of E , provides an embedding $\iota_E : T_E \hookrightarrow \mathbf{G}$, whose $\mathbf{G}(\mathbb{Q})$ -conjugacy class is independent of the choice of basis. The field of definition of a special point of the form $h_E : \mathbf{S} \simeq (T_E)_{\mathbb{R}} \hookrightarrow \mathbf{G}_{\mathbb{R}}$ is E . In this case, the Deligne's reciprocity morphism is the identity. Then the special point $[h_E, g]_K \in Y(K)(\mathbb{C})$ is defined over a finite abelian extension of E , such that if $t \in \mathbb{A}_E^{\times}$, and $\sigma = \mathrm{rec}_E(t) \in \mathrm{Gal}(E^{\mathrm{ab}}/E)$, then $\sigma \cdot [h_E, g]_K = [h_E, \iota_E(t_f)g]$. Here $t_f \in \mathbb{A}_E^{(\infty), \times}$ is the finite part of t .

When K is small enough, the affine curve $Y(K)$ admits an integral model over $\mathrm{Spec}(\mathbb{Z})$, smooth over $\mathrm{Spec}(\mathbb{Z}_S)$, where S is the set of primes for which $K \cap \mathbf{G}(\mathbb{Q}_p)$ is not hyperspecial, and $\mathbb{Z}_S$ denotes the localization of $\mathbb{Z}$ at S . We denote by $\mathcal{Y}(K)$ the integral model that is constructed in [KM85]. We will mainly use the moduli scheme $\mathcal{Y}_0(N)$, $\mathcal{Y}_1(N)$ and $\mathcal{Y}(N)$ defined respectively for the levels $K_0(N)$, $K_1(N)$ and $K(N)$. Here, if $\mathrm{red}_N : \mathbf{G}(\hat{\mathbb{Z}}) \rightarrow \mathbf{G}(\mathbb{Z}/N\mathbb{Z})$ denotes the reduction mod N map,

we have set:

$$K(N) := \ker(\text{red}_N), \quad K_1(N) := \text{red}_N^{-1}(\mathbf{U}(\mathbb{Z}/N\mathbb{Z})), \quad \text{and } K_0(N) = \text{red}_N^{-1}(\mathbf{B}(\mathbb{Z}/N\mathbb{Z})).$$

We will sometimes abuse notation and view these groups as subgroups of $\mathbf{GL}_2(\mathbb{Q}_S)$ where S is the set of primes dividing N .

7.1.1 Connected components

The curve $Y(K)$ is not geometrically connected in general. The geometrically connected components are described as follows. Consider the determinant map $\det : \mathbf{GL}_2 \rightarrow \mathbf{G}_m$ and the map $\mathcal{X} \rightarrow \{\pm\}$ given by sending $h \mapsto \text{sgn}(\det(h(i)))$. Then, the induced map

$$Y(K)(\mathbb{C}) \simeq \mathbf{GL}_2(\mathbb{Q}) \backslash \mathcal{X} \times \mathbf{GL}_2(\mathbb{A}^{(\infty)}) / K \rightarrow \mathbb{Q}^\times \backslash \{\pm\} \times \mathbb{A}^{(\infty), \times} / \det(K)$$

is a bijection from the geometrically connected components of $Y(K)(\mathbb{C})$ to the set on the right. Therefore, the geometric curve $Y(K)_{\overline{\mathbb{Q}}}$ decomposes as a disjoint union $Y(K)_\alpha$ indexed by elements α of $\mathbb{Q}^\times \backslash \{\pm\} \times \mathbb{A}^{(\infty), \times} / \det(K)$. Unlike $Y(K)$, the curves $Y(K)_\alpha$ are not defined over $\mathbb{Q}$ but over abelian extensions of $\mathbb{Q}$. These extensions are determined by the Galois action on the components of $Y(K)$ which we now describe. Let $(\alpha_\infty, \alpha_f) \in \{\pm\} \times \mathbb{A}^{(\infty), \times}$ be a representative of an element α of $\mathbb{Q}^\times \backslash \{\pm\} \times \mathbb{A}^{(\infty), \times} / \det(K)$. Let $\sigma \in G_{\mathbb{Q}}^{\text{ab}} := \text{Gal}(\mathbb{Q}^{\text{ab}}/\mathbb{Q})$ and let $a = (a_\infty, a_f) \in \mathbb{A}^\times$ be such that $\text{rec}_E(a) = \sigma$. Then $\sigma \cdot Y(K)_\alpha = Y(K)_{a\alpha}$.

One can obtain classical modular curves as follows: given $(\alpha_\infty, \alpha_f) \in \mathbb{Q}^\times \backslash \{\pm\} \times$

$\mathbb{A}^{(\infty), \times} / \det(K)$, let $g_f \in \mathbf{GL}_2(\mathbb{A}^{(\infty)})$ be an element such that $\det(g_f) = a_f$. Then the map

$$\mathcal{H}^+ \rightarrow Y(K)(\mathbb{C}), \quad \tau \mapsto [\alpha_\infty \tau, g_f]_K$$

induces a holomorphic covering of $Y(K)_\alpha(\mathbb{C})$ and an isomorphism $\Gamma \backslash \mathcal{H}^+ \simeq Y(K)_\alpha(\mathbb{C})$, where Γ is the lattice of $\mathbf{SL}_2(\mathbb{Z})$ defined by

$$\Gamma := \mathbf{SL}_2(\mathbb{Q}) \cap g_f K g_f^{-1}.$$

7.1.2 Compactifications of modular curves

In this section, we review basic facts about compactifications of modular curves.

7.1.2.1 The Borel-Serre compactification

We first introduce the Borel-Serre compactification of $Y(K)$, which we denote by $X^{\text{BS}}(K)$ ([BS73]). This is a real manifold such that the natural inclusion $Y(K) \hookrightarrow X^{\text{BS}}(K)$ is a homotopy equivalence. We now describe classical results in the setting of $\mathbf{GL}_2$; there are generalization that are true for all locally symmetric spaces.

As there is only one $\mathbf{G}(\mathbb{Q})$ -conjugacy class of parabolic subgroups of $\mathbf{G}$, the boundary $\partial^{\text{BS}}(K) := X^{\text{BS}}(K) - Y(K)$ of the Borel Serre compactification contains a unique stratum corresponding to the Borel subgroup $\mathbf{B}$. Topologically $\partial^{\text{BS}}(K)$ retracts onto

$$\mathbf{B}(\mathbb{Q}) \backslash \mathbf{G}(\mathbb{A}) / K_\infty^0 K.$$

Here we have set $K_\infty^0 := \mathbb{R}_{>0} \mathbf{SO}(2, \mathbb{R}) \subset \mathbf{G}(\mathbb{R})$. Let $g_1, \dots, g_r$ be a complete set of

representatives of $\mathbf{B}(\mathbb{A}^{(\infty)}) \backslash \mathbf{G}(\mathbb{A}^{(\infty)}) / K$. Then we have a decomposition:

$$\mathbf{B}(\mathbb{Q}) \backslash \mathbf{G}(\mathbb{A}) / K_{\infty}^0 K = \coprod_{1 \leq i \leq r} \mathbf{B}(\mathbb{Q}) \backslash \mathbf{B}(\mathbb{A}) / \mathbf{Z}_{\mathbf{G}}(\mathbb{R}) K^{\mathbf{B}}(g_i).$$

Here $K^P(g_i) \subseteq \mathbf{B}(\mathbb{A}^{(\infty)})$ denotes the subgroup $\mathbf{B}(\mathbb{A}^{(\infty)}) \cap g_i K g_i^{-1}$.

Denote by $\pi : \mathbf{B} \rightarrow \mathbf{T}$ the canonical projection onto the Levi quotient. Denote by $K^{\mathbf{T}}(g_i)$ the image under π of $K^{\mathbf{B}}(g_i)$. Let

$$\mathcal{S}_{\mathbf{T}}(K^{\mathbf{T}}(g_i)) := \mathbf{T}(\mathbb{Q}) \backslash \mathbf{T}(\mathbb{A}) / \mathbf{Z}_{\mathbf{G}}(\mathbb{R}) K^{\mathbf{T}}(g_i).$$

Let $K^{\mathbf{U}}(g_i) := \mathbf{U}(\mathbb{A}^{(\infty)}) \cap g_i K g_i^{-1}$. We then have the fibration:

$$\mathbf{U}(\mathbb{Q}) \backslash \mathbf{U}(\mathbb{A}) / K^{\mathbf{U}}(g_i) \hookrightarrow \mathbf{B}(\mathbb{Q}) \backslash \mathbf{B}(\mathbb{A}) / \mathbf{Z}_{\mathbf{G}}(\mathbb{R}) K^{\mathbf{B}}(g_i) \rightarrow \mathcal{S}_{\mathbf{T}}(K^{\mathbf{T}}(g_i)).$$

The corresponding Leray-Serre spectral sequence in cohomology degenerates at E_2 -level (see for example [HMG94]). The cohomology of the total space is given in terms of the cohomology of the base with coefficients in the cohomology of the fiber. To compute the cohomology of the fiber, one can invoke a classical theorem of van Est as is explained in [HMG94, (11)]. We then obtain by passing to the limit over all $K \subset \mathbf{G}(\mathbb{A}^{(\infty)})$ the following description of the cohomology of the boundary:

$$H^{\bullet}(\partial^{\text{BS}}, V_{\lambda}) = \text{Ind}_{\mathbf{B}(\mathbb{A}^{(\infty)})}^{\mathbf{G}(\mathbb{A}^{(\infty)})} \left(H^{\bullet}(\mathcal{S}_{\mathbf{T}}, H^{\bullet}(\mathfrak{u}, V_{\lambda})) \right).$$

Here $\mathfrak{u}$ denote the Lie algebra of $\mathbf{U}$ and V_{λ} is the local system on X^{BS} attached to

the irreducible representation of $\mathbf{G}(\mathbb{C})$ of highest weight λ . A classical theorem of Kostant describes $H^\bullet(\mathfrak{u}, \mathbb{C})$ as a representation of $\mathbf{T}$.

7.1.2.2 Remark. The natural restriction map

$$H^\bullet(Y, V_\lambda) \simeq H^\bullet(X^{\text{BS}}, V_\lambda) \rightarrow H^\bullet(\partial^{\text{BS}}, V_\lambda)$$

is induced by integration along $[\mathbf{U}(\mathbb{A})]$ as is explained in [HR19, Section 4.1.2], after identifying singular cohomology with $(\mathfrak{g}, K)$ -cohomology.

7.1.2.3 The Baily-Borel compactification

We now introduce the minimal or Baily-Borel compactification of $Y(K)$ from the point of view of Pink's formalism. This is a specialization in the $\mathbf{GL}_2$ -case of very general results. The reader can consult for example [Har93, pp50-56] for an concise exposition of the general theory.

We denote by $X(K)$ the minimal compactification of $Y(K)$. This is an algebraic variety defined over $\mathbb{Q}$. In the specific case of $\mathbf{GL}_2$, it is also smooth. Denote by $C(K) = X(K) \setminus Y(K)$ the boundary and $j : Y(K) \hookrightarrow X(K)$ the natural inclusion. Then $C(K)$ is a collection of points, usually referred to as "the cusps". Note that the integral model $\mathscr{Y}(K)$ of $Y(K)$ extends to an integral model $\mathscr{X}(K)$ of $X(K)$, as constructed by Deligne-Rapoport [DR73].

We now recall how $C(K)$ can be identified with a (union of) toric Shimura variety from the point of view of Pink's formalism. The unique admissible Cayley homomorphism $w_{\mathbf{T}} : \mathbf{G}_{\mathbf{m}} \rightarrow \mathbf{T}$ of the Levi $\mathbf{T} \subset \mathbf{B}$ (in the sense of [Har86, Theorem 5.1.3])

is

$$w_{\mathbf{T}}(a) = \begin{pmatrix} a^{-2} & 0 \\ 0 & 1 \end{pmatrix}.$$

Therefore by the properties listed in 5.1.9, we let $h_{\mathbf{T}} : \mathbf{S} \rightarrow \mathbf{T}_{\mathbb{R}}$ be the homomorphism given by $z = x + iy \mapsto \begin{pmatrix} x^2 + y^2 & 0 \\ 0 & 1 \end{pmatrix}$. We let $\mathbf{T}_1 \subseteq \mathbf{T}$ denote the subtorus of $\mathbf{T}$ of diagonal matrices with 1 on the bottom right coefficient. Similarly we let $\mathbf{B}_1 \subset \mathbf{B}$ the subgroup defined by the inverse image of $\mathbf{T}_1$ under the canonical projection $\mathbf{B} \rightarrow \mathbf{T}$. Denote by $\mathrm{SH}_{\mathbf{T}}$ the 0-dimensional Shimura variety attached to the Shimura datum $(\mathbf{T}_1, h_{\mathbf{T}})$. We then have an identification of $C(K)$ and a disjoint union of Shimura varieties $\mathrm{SH}_{\mathbf{T}}(K^{\mathbf{T}_1}(g_i))$ where $K^{\mathbf{T}_1}(g_i)$ is a level group depending on an element $g_i \in \mathbf{G}(\mathbb{A}^{(\infty)})$, ranging over $\mathbf{B}(\mathbb{Q})\mathbf{B}_1(\mathbb{A}^{(\infty)}) \backslash \mathbf{G}(\mathbb{A}^{(\infty)})/K$ (see for example the discussion in [Mor18, pp.8-9]). By Deligne's reciprocity law on tori, the Galois action on $C(K)$ is described as follows. If $\sigma = \mathrm{rec}_{\mathbb{Q}}(t)$, then the action of σ on $C(K)$ coincides with left translation by

$$\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix} \in \mathbf{T}(\mathbb{A}^{(\infty)}).$$

We refer the reader to [Har93, Section 2.2.7] for more details or [Pin89, Example 4.25]

Chapter 8

Theta Classes

8.1 Theta Lifting from quadratic imaginary fields

8.1.1 Local theta lifts

Let $\mathcal{K}$ be a quadratic imaginary field endowed with its norm $N_{\mathcal{K}/\mathbb{Q}}$. We will view $\mathcal{K}$ as being equipped with a privileged complex embedding $\mathcal{K} \hookrightarrow \mathbb{C}$. We denote by $\tau \in \text{Gal}(\mathcal{K}/\mathbb{Q})$ the unique non-trivial element. Let $\Delta_{\mathcal{K}}$ be the discriminant of $\mathcal{K}$. We assume that the fixed odd prime p is inert in $\mathcal{K}$ (in particular $p \nmid \Delta_{\mathcal{K}}$). We denote by E the completion of $\mathcal{K}$ at the prime p : this is the unique unramified quadratic extension of $\mathbb{Q}_p$, we denote by $\bar{\cdot}$ the unique non-trivial $\mathbb{Q}_p$ -automorphism of E and we denote by $N_{E/\mathbb{Q}_p}$ the norm of E . We let χ_E be the quadratic character of $\mathbb{Q}_p^\times$ associated to E by class field theory, so that χ_E is trivial on the image of $N_{E/\mathbb{Q}_p}$. Denote by $|\cdot|_p$ the usual p -adic absolute value of $\mathbb{Q}_p$. We let ψ be an additive unramified character of $\mathbb{Q}_p$ (for example take $\psi = \psi_p$ as in the previous section) and

denote by $\psi_E := \psi \circ \text{Tr}_{E/\mathbb{Q}_p}$ – this is an additive unramified character of E .

We recall the definition of the dihedral representations associated to the extension $E/\mathbb{Q}_p$. We follow [JL70, Section 1] Let ω_E be the Weil representation of $\mathbf{SL}_2(\mathbb{Q}_p)$ associated to $E/\mathbb{Q}_p$. The space of the representation ω_E is $\mathcal{S}(E)$ – the space of Schwartz functions on E . This representation is characterized by the following formulas.

$$\forall x \in \mathbb{Q}_p, \forall v \in E \quad \omega_E \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} f(v) := \psi(xN_E(v))f(v) \quad (8.1)$$

$$\forall a \in \mathbb{Q}_p^\times, \forall v \in E, \quad \omega_E \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} f(v) := |a|_p \chi_E(a) f(av) \quad (8.2)$$

$$\forall v \in E, \quad \omega_E \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} f(v) := \gamma \cdot \int_E f(z) \psi_E(z\bar{v}) dz. \quad (8.3)$$

The element γ is a fourth root of unity that verifies the property $\gamma^2 = \chi_E(-1)$ and the Haar measure dz on E is normalized so that the valuation ring of E , $\mathcal{O}_E$, has volume equal to 1.

For a character $\xi : E^\times \rightarrow \mathbb{C}^\times$ we let $U_\xi \subset \mathcal{S}(E)$ be the space of functions f that satisfy:

$$f(yv) = \xi(y)f(v),$$

for all $v \in E$ and $y \in \ker(N_{E/\mathbb{Q}_p})$. Then the Weil representation induces a representation ω_ξ on U_ξ . It is well-known that the isomorphism class of ω_ξ is independent of the choice of ψ .

Let $\mathbf{GL}_2(\mathbb{Q}_p)^+$ be the index 2 subgroup of $\mathbf{GL}_2(\mathbb{Q}_p)$ consisting of elements that have determinant in the image of the norm map $N_{E/\mathbb{Q}_p}$. Then the representation ω_ξ can be extended to $\mathbf{GL}_2(\mathbb{Q}_p)^+$ by requiring that

$$\forall v, b \in E, \omega_\xi \begin{pmatrix} N_{E/\mathbb{Q}_p}(b) & 0 \\ 0 & 1 \end{pmatrix} f(v) := |N_{E/\mathbb{Q}_p}(b)|_p^{1/2} \xi^{-1}(b) f(bv). \quad (8.4)$$

We note here that the extension ω_ξ of the Weil representation *does depend* on ξ more than the the space U_ξ . For example ξ and $\xi|\cdot|$ define the same space U_ξ but the extension of ω_ξ to $\mathbf{GL}_2(\mathbb{Q}_p)^+$ is clearly different. We will nevertheless allow ourselves to denote by the same symbol ω_ξ the extension of ω_ξ to $\mathbf{GL}_2(\mathbb{Q}_p)^+$. Let V_ξ be the induction of ω_ξ to $\mathbf{GL}_2(\mathbb{Q}_p)$ from $\mathbf{GL}_2(\mathbb{Q}_p)^+$. We record the following well-known facts in the following Lemma.

8.1.1.1 Lemma. *The following statments about V_ξ hold:*

- *The central character of V_ξ as a representation of $\mathbf{GL}_2(\mathbb{Q}_p)$ is $\chi_E \cdot \xi^{-1}|_{\mathbb{Q}_p^\times}$.*
- *For any character μ of $\mathbb{Q}_p^\times$, we denote by $\mu_E := \mu \circ N_{E/\mathbb{Q}_p}$. We have*

$$V_{\xi\mu_E} \simeq \mu \otimes V_\xi.$$

If V is an infinite-dimensional admissible representation of $\mathbf{GL}_2(\mathbb{Q}_p)$, we denote by $V^{K_1(p^m)}$ the space of $K_1(p^m)$ fixed vectors in V . Here $K_1(p^m)$ is the open compact

subgroup of $\mathbf{GL}_2(\mathbb{Z}_p)$ defined by

$$K_1(p^m) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{GL}_2(\mathbb{Z}_p) : c \in p^m \mathbb{Z}_p, d \in 1 + p^m \mathbb{Z}_p \right\}.$$

We define also $K^1(p^m)$ as

$$K^1(p^m) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{GL}_2(\mathbb{Z}_p) : c \in p^m \mathbb{Z}_p, a \in 1 + p^m \mathbb{Z}_p \right\}.$$

and

$$K_0(p^m) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{GL}_2(\mathbb{Z}_p) : c \in p^m \mathbb{Z}_p \right\}.$$

We call the conductor of V the smallest integer n such that $V^{K_1(p^n)} \neq 0$ (equivalently $V^{K^1(p^n)} \neq 0$). We denote this integer by $\mathfrak{f}_V$.

8.1.1.2 The Whittaker model

Having fixed the local additive character $\psi : \mathbb{Q}_p \rightarrow \mathbb{C}^\times$, we consider the space $\mathcal{W}$ of locally constant, complex-valued, functions W on $\mathbf{GL}_2(\mathbb{Q}_p)$ that verify the transformation law

$$\forall g \in \mathbf{GL}_2(\mathbb{Q}_p), \forall x \in \mathbb{Q}_p, \quad W \left[\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g \right] = \psi(x) W(g).$$

This is a representation of $\mathbf{GL}_2(\mathbb{Q}_p)$ by right multiplication on the argument of an element $W \in \mathcal{W}$. If V is an irreducible infinite dimensional admissible representation of $\mathbf{GL}_2(\mathbb{Q}_p)$, we denote by $\mathcal{W}_V$ the unique subspace of $\mathcal{W}$, that is isomorphic to V . We call $\mathcal{W}_V$ the Whittaker model of V . If n is the conductor of V , the *normalized Whittaker* newform of V is the unique $K_1(p^n)$ -fixed vector $W_V \in \mathcal{W}_V$, such that $W_V(1) = 1$. The conjugate vector W_V^* is the unique $K^1(p^n)$ -fixed vector $W_V^* \in \mathcal{W}_V$, such that $W_V^*(1) = 1$.

Let us now fix an element $\alpha := \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \in \mathbf{GL}_2(\mathbb{Q}_p)$ such that $\chi_E(a) = -1$; that is a is not in the image of the norm map $N_{E/\mathbb{Q}_p}$; for example $a = p$. Then any element $f \in V_\xi$ is clearly determined by $f(1)$ and $f(\alpha)$. Conversely, given arbitrary vectors $\phi_+, \phi_- \in U_\xi$, there exists a unique $f \in V_\xi$ such that $f(1) = \phi_+$ and $f(\alpha) = \phi_-$. Explicitly this vector is given by

$$f(g) = \begin{cases} \omega_\xi(g)\phi_+ & \text{if } g \in \mathbf{GL}_2(\mathbb{Q}_p)^+ \\ \omega_\xi(g\alpha^{-1})\phi_- & \text{if } g \notin \mathbf{GL}_2(\mathbb{Q}_p)^+. \end{cases}.$$

We will denote by f_ξ the vector of V_ξ that is determined by

$$\phi_+(v) := \xi(v)\mathbf{1}_{\mathcal{O}_E^\times}(v); \quad \phi_- := 0.$$

Then by Theorem 2.3.5 of [Sch02], or Theorem 4.2 of [JL70], when ξ does not factor through the norm map, this vector $f_\xi \in V_\xi$ generates, under the action of ω_ξ , an irreducible representation π_ξ of $\mathbf{GL}_2(\mathbb{Q}_p)$ whose newvector is f_ξ ; so that the

conductor $\mathfrak{f}_{\pi_\xi}$ of π_ξ is related to the conductor $\mathfrak{f}_\xi$ of ξ by the formula

$$\mathfrak{f}_{\pi_\xi} = 2\mathfrak{f}_\xi.$$

We let W_ξ be the normalized Whittaker newvector of π_ξ . When ξ is unramified, the spherical vector is obtained by choosing $\phi_+ = \mathbb{1}_{\mathcal{O}_E}$ and $\phi_- = 0$.

If ξ is unramified, we denote by f_ξ the " p -stabilized" vector defined by

$$f_\xi(g) := \omega_\xi(g)\mathbb{1}_{\mathcal{O}_E} + p^{-1}\omega_\xi\left(g\begin{pmatrix} p^{-1} & 0 \\ 0 & p \end{pmatrix}\right)\mathbb{1}_{\mathcal{O}_E} = \omega_\xi(g)\mathbb{1}_{\mathcal{O}_E^\times}.$$

This is a vector that is $K_0(p^2)$ -fixed. For ξ unramified, we introduce the convention that W_ξ does not denote the Whittaker newvector but rather the old vector $W_\xi(\bullet) + p^{-1}W_\xi\left(\bullet\begin{pmatrix} p^{-1} & 0 \\ 0 & p \end{pmatrix}\right)$.

8.1.1.3 Lemma. *The normalized Whittaker vector W_ξ of π_ξ is given by*

$$W_\xi(g) := f_\xi(g)(1) = \begin{cases} \omega_\xi(g)\phi_+(1) & \text{if } g \in \mathbf{GL}_2(\mathbb{Q}_p)^+ \\ 0 & \text{otherwise.} \end{cases}$$

Here we are viewing $\omega_\xi(g)\phi_+$ as a function of U_ξ , so that we can evaluate it at $1 \in E$.

Proof. Define $W_\xi(g)$ by the formula of the Lemma. We just have to show that $W_\xi(1) = 1$, and that W_ξ is $K_1(p^{2\mathfrak{f}_\xi})$ fixed. The fact that $W_\xi(1) = 1$, follows from $\phi_+(1) = 1$. The fact that it is $K_1(p^{2\mathfrak{f}_\xi})$ -fixed follows from the fact that the newvector f_ξ is $K_1(p^{2\mathfrak{f}_\xi})$ -fixed. $\square$

We give a second Lemma relating W_ξ and W_ξ^* .

8.1.1.4 Proposition. *Assume the conductor of ξ is $n \geq 1$. We have the relation*

$$(-1)^n \tau_E(\xi) W_\xi^*(g) = W_\xi \left(g \begin{pmatrix} 0 & p^{-n} \\ p^n & 0 \end{pmatrix} \right).$$

Here $\tau_E(\xi)$ is the Gauss sum defined as

$$\tau_E(\xi) = \sum_{x \in \mathcal{O}_E^\times / 1 + p^n \mathcal{O}_E} \xi(x) \psi_E(x/p^n).$$

Proof. Recall that W_ξ has conductor $2n$. The right hand-side of the formula is clearly $K^1(p^{2n})$ -invariant and verifies the transformation law of the Whittaker model. It suffices therefore to compare values at the identity. We compute $\omega_\xi \begin{pmatrix} 0 & p^{-n} \\ p^n & 0 \end{pmatrix} \phi_+(1)$ where $\phi_+ = \xi \mathbf{1}_{\mathcal{O}_E^\times}$. We now write ϕ_+ as follows

$$\phi_+ = \sum_{x \in \mathcal{O}_E^\times / 1 + p^n \mathcal{O}_E} \xi(x) \mathbf{1}_{x + p^n \mathcal{O}_E}.$$

We have

$$\omega_E \begin{pmatrix} 0 & p^{-n} \\ p^n & 0 \end{pmatrix} \mathbf{1}_{x + p^n \mathcal{O}_E} = \omega_E \begin{pmatrix} p^{-n} & 0 \\ 0 & p^n \end{pmatrix} \omega_E(w) \mathbf{1}_{x + p^n \mathcal{O}_E},$$

and for all $z \in E$

$$\omega_E(w) \mathbf{1}_{x + p^n \mathcal{O}_E}(z) = \gamma \cdot \psi_E(x\bar{z}) p^{-n} \mathbf{1}_{p^{-n} \mathcal{O}_E}(z).$$

Combining the preceding equalities, we obtain

$$\omega_\xi \begin{pmatrix} 0 & p^{-n} \\ p^n & 0 \end{pmatrix} \phi_+(1) = \sum_{x \in \mathcal{O}_E^\times / 1 + p^n \mathcal{O}_E} \chi_E(p^{-n}) \psi_E(p^{-n}x) \xi(x) = (-1)^n \tau_E(\xi).$$

□

8.1.1.5 Definition. Let $\mathcal{X}_n$ be the character group of $\mathcal{O}_E^\times / 1 + p^n \mathcal{O}_E$ – the characters of $E^\times$ of conductor n . Decompose $\mathcal{O}_E^\times = \Delta_E \times U_E$, where Δ_E is a prime-to- p finite group and U_E is a p -adically complete p -group. We will call a character $\lambda \in \mathcal{X}_1$ such that $\lambda|_{\mathcal{O}_E^\times}$ is trivial on U_E a *branch character*. For a branch character λ , we denote by $\mathcal{X}_n(\lambda)$ the characters $\xi \in \mathcal{X}_n$ such that $\lambda^{-1}\xi(p) = 1$ and $\lambda^{-1}\xi|_{\mathcal{O}_E^\times}$ is trivial on Δ_E . More generally, if $U \subset U_E$ be a subgroup containing $1 + p^N \mathcal{O}_E$ for some large integer N , we let $\mathcal{X}_U(\lambda) \subseteq \mathcal{X}_N(\lambda)$ the subset of characters $\xi \in \mathcal{X}_N(\lambda)$ such that $\lambda^{-1}\xi|_{\mathcal{O}_E^\times}$ factors through $\mathcal{O}_E^\times / U$, so that $\mathcal{X}_{1+p^n \mathcal{O}_E}(\lambda) = \mathcal{X}_n(\lambda)$

We have the following corollary to Proposition 8.1.1.4.

8.1.1.6 Corollary. *We have the equality, for all $g \in \mathbf{SL}_2(\mathbb{Q}_p)$,*

$$W_\xi \left(g \begin{pmatrix} 0 & p^{-n} \\ p^n & 0 \end{pmatrix} \right) = (-1)^n \tau_E(\xi) W_\xi(g).$$

Proof. Recall that ω_ξ is actually the extension of the Weil representation ω_E as given by Equation 8.4 – and that this representation of $\mathbf{SL}_2(\mathbb{Q}_p)$ does not depend on the character ξ . We can define a slightly different extension of ω_E that we denote

suggestively by ω_ξ^* , determined by imposing

$$\forall f \in \mathcal{S}(E), \forall v, b \in E, \omega_\xi \begin{pmatrix} 1 & 0 \\ 0 & N_{E/\mathbb{Q}_p}(b) \end{pmatrix} f(v) := |N_{E/\mathbb{Q}_p}(b)|_p^{1/2} \xi^{-1}(b) f(bv);$$

and in the same way we induce this representation ω_ξ from $\mathbf{GL}_2(\mathbb{Q}_p)^+$ to $\mathbf{GL}_2(\mathbb{Q}_p)$.

Let as usual $\phi_+ = \xi \mathbf{1}_{\mathcal{O}_E^\times}$. We then see that

$$f_\xi^*(g) := \begin{cases} \omega_\xi^*(g) \phi_+ & \text{if } g \in \mathbf{GL}_2(\mathbb{Q}_p)^+ \\ 0 & \text{otherwise} \end{cases}$$

is $K^1(p^{2f_\xi})$ -invariant and is therefore the newvector for $\pi_\xi^{K^1(p^{2f_\xi})}$. Completely analogously for W_ξ , we then see that $W_\xi^*(g) = f_\xi^*(g)(1)$. But then this expression agrees with $W_\xi(g)$ for g of determinant one, since $\omega_\xi^*(g) \phi_+ = \omega_E(g) \phi_+ = \omega_\xi(g) \phi_+$ as ω_ξ and ω_ξ^* both extend ω_E from $\mathbf{SL}_2(\mathbb{Q}_p)$. The result follows. $\square$

We now want to compute the values of W_ξ at several points explicitly and control denominators.

8.1.1.7 Computations at representatives of p -cusps.

We now introduce the following notations to denote certain distinguished elements of $\mathbf{GL}_2(\mathbb{Q}_p)$:

$$\forall x \in \mathbb{Q}_p, n(x) := \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, \quad \forall t \in \mathbb{Q}_p^\times, d(t) := \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$$

and also

$$\forall t \in \mathbb{Q}_p^\times, \quad z(t) := t \cdot \text{id}, \quad w := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

We also define for integers m, l and $v \in \mathbb{Z}_p^\times$ the elements

$$\gamma_{m,\ell,v} := d(p^m)wn(vp^{-l}) \in \mathbf{GL}_2(\mathbb{Q}_p)^+.$$

We then have, by section 3.3 of [CNS23], the decomposition for all integers $n \geq 1$,

$$\mathbf{GL}_2(\mathbb{Q}_p)^+ = \coprod_{0 \leq l \leq n} \coprod_{m \in \mathbb{Z}} \coprod_{v \in \mathbb{Z}_p^\times / 1+p^{\min(l,n-l)}\mathbb{Z}_p} \mathbf{Z}_G(\mathbb{Q}_p)\mathbf{U}(\mathbb{Q}_p)\gamma_{m,\ell,v}K_1(p^n). \quad (8.5)$$

We note here that our choice of representatives is slightly different from *loc cit.*, namely we have, in the notations of *loc cit.*,

$$g_{2m,\ell,v} = z(p^m)\gamma_{m,\ell,v}.$$

Since W_ξ vanishes outside $\mathbf{GL}_2(\mathbb{Q}_p)^+$, it suffices to study the values of W_ξ on $\gamma_{m,\ell,v}$.

8.1.1.8 Lemma. *For all integers m, ℓ and $v \in \mathbb{Z}_p^\times$, we have*

$$W_\xi(\gamma_{m,\ell,v}) = \gamma \cdot (-1)^m p^{-m} \int_{\mathcal{O}_E^\times} \xi(z) \psi(vp^{-l}N_{E/\mathbb{Q}_p}(z)) \psi_E(p^m z) dz.$$

Proof. Recall that the function $\phi_+ \in U_\xi$ defined by $\phi_+(x) = \xi(x)\mathbf{1}_{\mathcal{O}_E^\times}(x)$ for $x \in E$.

Note first that

$$\omega_\xi(n(vp^{-\ell}))\phi_+(x) = \psi(vp^{-l}N_{E/\mathbb{Q}_p}(x))\phi_+(x)$$

by Equation 8.1. Therefore

$$\omega_\xi(wn(vp^{-\ell}))\phi_+(x) = \gamma \cdot \int_E \phi_+(z)\psi(vp^{-l}N_{E/\mathbb{Q}_p}(z))\psi_E(z\bar{x})dz$$

by Equation 8.3. We then obtain, by Equation 8.2, and the simple relation $\gamma_{m,\ell,v} = d(p^m) \cdot wn(vp^{-\ell})$,

$$\omega_\xi(\gamma_{m,\ell,v})\phi_+(1) = \gamma\chi_{E/\mathbb{Q}_p}(p)^m p^{-m} \int_{\mathcal{O}_E^\times} \xi(z)\psi(vp^{-l}N_{E/\mathbb{Q}_p}(z))\psi_E(p^m z)dz.$$

We can then conclude by invoking Lemma 8.1.1.3. □

8.1.1.9 Definition. Let $\lambda \in \mathcal{X}_1$. Recall the notation $\mathcal{X}_n(\lambda)$ from definition 8.1.1.5.

Let $n \geq 1$ be an integer and define

$$\pi_n(\lambda) := \bigoplus_{\xi \in \mathcal{X}_n(\lambda)} V_\xi.$$

Then $\pi_n(\lambda)$ is an admissible representation of $\mathbf{GL}_2(\mathbb{Q}_p)$. We let also

$$W_{n,\lambda} = \bigoplus_{\xi \in \mathcal{X}_n(\lambda)} W_\xi : \pi_n(\lambda) \rightarrow \text{Ind}_{\mathbf{U}(\mathbb{Q}_p)}^{\mathbf{GL}_2(\mathbb{Q}_p)}(\psi).$$

8.1.1.10 Definition. Let $U \subseteq U_E$ be a subgroup such that $1 + p^n \mathcal{O}_E \subseteq U$ and let

λ be a branch character. We define the vector $\Theta_{U,\lambda} \in \pi_n(\lambda)$ as

$$\forall g \in \mathbf{GL}_2(\mathbb{Q}_p), \quad \Theta_{U,\lambda}(g) := \frac{1}{\#\mathcal{X}_U(\lambda)} \sum_{\xi \in \mathcal{X}_U(\lambda)} (-1)^{f_\xi} \tau_E(\xi) f_\xi(g \cdot d(p^{-(n-f_\xi)})).$$

Note that $\Theta_{U,\lambda}$ is a sum of $K_1(p^{2n})$ -fixed vectors of π_ξ for ξ ranging over $\mathcal{X}_U(\lambda)$.

8.1.1.11 Remark. We note that contrary to what the notation suggests, $\Theta_{U,\lambda}$ depends not only on U but also on a choice of an integer n such that $1 + p^n \mathcal{O}_E$. This choice shall always be made very explicit in what follows.

The following Lemma recording elementary matrix identities will be useful for the calculation of Proposition 8.1.1.14.

8.1.1.12 Lemma. *Let r, n, m, l be integers and $v \in \mathbb{Z}_p^\times$. We have the matrix identities:*

1.

$$\gamma_{m,l,v} \begin{pmatrix} 0 & p^{-n} \\ p^n & 0 \end{pmatrix} = n(-p^{2m+l}v^{-1}) \gamma_{m-(n-l), 2n-l, -v} \begin{pmatrix} v & 0 \\ 0 & v^{-1} \end{pmatrix};$$

2.

$$d(p^{-r}) \begin{pmatrix} 0 & p^{-n} \\ p^n & 0 \end{pmatrix} = \begin{pmatrix} 0 & p^{-(n-r)} \\ p^{n-r} & 0 \end{pmatrix}.$$

Proof. The second relation is obvious and the first one can be deduced from the equation in the proof of Lemma 2.18 of [Sah16] with the relation $g_{2m,l,v} = z(p^m) \gamma_{m,l,v}$, or by a simple computation. $\square$

8.1.1.13 Lemma. *For any branch character λ , for any $x = \delta \times u \in \Delta_E \times U_E \simeq \mathcal{O}_E^\times$, we have the relation*

$$\phi_U(x) := \frac{1}{\#\mathcal{X}_U(\lambda)} \sum_{\xi \in \mathcal{X}_U(\lambda)} \xi(x) = \lambda(\delta) \mathbf{1}_U(u).$$

Proof. Note first that $\xi(\delta \times u) = \lambda(\delta) \xi'(u)$ for some character ξ' of U_E/U , by definition of $\mathcal{X}_U(\lambda)$. When ξ ranges over $\mathcal{X}_U(\lambda)$, ξ' ranges over the character group of the abelian group U_E/U and so the result follows by character theory. $\square$

We are now ready to prove the following.

8.1.1.14 Proposition. *For any unit group U with $1 + p^n \mathcal{O}_E \subseteq U \subseteq U_E$, and branch character λ , we have*

$$W_{n,\lambda}(\Theta_{U,\lambda})(\gamma_{m,l,v}) = \gamma(-1)^{m-(n-l)} \psi(-p^{2m+l} v^{-1}) \cdot p^{-m+(n-l)} I_U(m, l, v),$$

where $I_U(m, l, v)$ is a p -adic integral defined as

$$I_U(m, l, v) := \int_{\mathcal{O}_E^\times} \phi_U(z) \psi(-vp^{l-2n} N_{E/\mathbb{Q}_p}(z)) \psi_E(p^{m-(n-l)} z) dz.$$

Proof. By definition of $\Theta_{U,\lambda}$ and $W_{n,\lambda}$ we have

$$W_{n,\lambda}(\Theta_{U,\lambda})(\gamma_{m,l,v}) = \frac{1}{\#\mathcal{X}_U(\lambda)} \sum_{\xi \in \mathcal{X}_U(\lambda)} (-1)^{f_\xi} \tau_E(\xi) W_\xi(\gamma_{m,l,v} d(p^{-(n-f_\xi)})). \quad (8.6)$$

Then, by Corollary 8.1.1.6, we have for all $\xi \in \mathcal{X}_U(\lambda)$:

$$(-1)^{\mathfrak{f}_\xi} \tau_E(\xi) W_\xi(\gamma_{m,l,v} d(p^{-(n-\mathfrak{f}_\xi)})) = \gamma W_\xi \left(\gamma_{m,l,v} \begin{pmatrix} 0 & p^{-n} \\ p^n & 0 \end{pmatrix} \right).$$

By (1) of Lemma 8.1.1.12, we have

$$W_\xi \left(\gamma_{m,l,v} \begin{pmatrix} 0 & p^{-n} \\ p^n & 0 \end{pmatrix} \right) = \psi(-p^{2m+l} v^{-1}) W_\xi(\gamma_{m-(n-l), 2n-l, -v}).$$

Combining these equalities we obtain:

$$W_{n,\lambda}(\Theta_{n,\lambda})(\gamma_{m,l,v}) = \gamma \psi(-p^{2m+l} v^{-1}) \cdot \frac{1}{\#\mathcal{X}_U(\lambda)} \sum_{\xi \in \mathcal{X}_U(\lambda)} W_\xi(\gamma_{m-(n-l), 2n-l, -v}).$$

By invoking Lemma 8.1.1.8 and Lemma 8.1.1.13, we obtain the formula of the proposition. $\square$

8.1.1.15 Lemma. *If $1 + p^n \mathcal{O}_E \subseteq U$, we have*

$$I_U(m, l, v) \in p^{-n} \cdot \overline{\mathbb{Z}}_p$$

Proof. Notice that for any $u \in \mathcal{O}_E^\times$, and $\in \mathcal{O}_E$,

$$N(u(1 + p^n x)) = N(u)(1 + p^n \text{Tr}_{E/\mathbb{Q}_p}(x) + p^{2n} N(x)).$$

Therefore, since $l \geq 0$, we have $\psi(-vp^{l-2n} N(u(1+xp^n))) = \psi(-vp^{l-2n} N(u)) \psi_E(-vp^{l-n} N(u)x)$.

We can therefore rewrite the integral $I_U(m, l, v)$ as

$$\begin{aligned}
I_U(m, l, v) &= p^{-n} \sum_{\delta \in \Delta_E} \lambda(\delta) \sum_{u \in U/1+p^n \mathcal{O}_E} \psi(-vp^{l-2n}N(u\delta)) \psi_E(p^{m-(n-l)}u\delta) \\
&\quad \cdot \int_{\mathcal{O}_E} \psi_E((p^{m+l}u\delta - vp^{l-n}N(u\delta))x) dx \\
&= p^{-n} \sum_{\delta \in \Delta_E} \lambda(\delta) \sum_{u \in U/1+p^n \mathcal{O}_E} \psi(-vp^{l-2n}N(u\delta)) \psi_E(p^{m-(n-l)}u\delta) \\
&\quad \cdot \mathbf{1}_{\mathcal{O}_E}(p^{m+l}u\delta - vp^{l-n}N(u\delta))
\end{aligned}$$

This last expression is clearly in $p^{-n}\overline{\mathbb{Z}}_p$. □

By Combining the Lemma just above and Proposition 8.1.1.14, we obtain the following proposition.

8.1.1.16 Proposition. *Let $U \subseteq U_E$ be a subgroup such that $1 + p^n \mathcal{O}_E \subseteq U$. We have for all integers m , and integer $l \geq 0$, $v \in \mathbb{Z}_p^\times$, and all branch characters λ :*

$$W_{n,\lambda}(\Theta_{U,\lambda})(\gamma_{m,l,v}) \in p^{-m-l} \cdot \overline{\mathbb{Z}}_p.$$

8.1.1.17 Definition (Universal Polynomial). We call a function $P : \mathcal{X}_n \rightarrow \mathbb{C}$ a *universal polynomial of index n* in ξ if there exists coefficients $(\alpha_n(x))$ indexed by $x \in \mathcal{O}_E^\times/1 + p^n \mathcal{O}_E$ such that

$$P(\xi) = \sum_{x \in \mathcal{O}_E^\times/1+p^n \mathcal{O}_E} \alpha_n(x) \xi(x).$$

We say P is an *integral universal polynomial* when the coefficients $\alpha_n(x)$ are algebraic

and belong to $\overline{\mathbb{Z}}_{(p)}$.

8.1.1.18 Proposition. *For integers $m, l \geq 0$ and $x \in \mathbb{Z}_p^\times$, the function*

$$P_{m,l,v} : \mathcal{X}_n(\lambda) \longrightarrow \mathbb{C}, \quad \xi \mapsto p^{m+l}(-1)^{\mathfrak{f}_\xi} \tau_E(\xi) W_\xi \left(\gamma_{m,l,v} d(p^{-(n-\mathfrak{f}_\xi)}) \right).$$

is an integral universal polynomial

Proof. We have

$$(-1)^{\mathfrak{f}_\xi} \tau_E(\xi) W_\xi \left(\gamma_{m,l,v} d(p^{-(n-\mathfrak{f}_\xi)}) \right) = \gamma W_\xi \left(\gamma_{m,l,v} \begin{pmatrix} 0 & p^{-n} \\ p^n & 0 \end{pmatrix} \right).$$

Therefore, by Lemma 8.1.1.8, and the same computation of 8.1.1.15, we have

$$(-1)^{\mathfrak{f}_\xi} \tau_E(\xi) W_\xi \left(\gamma_{m,l,v} d(p^{-(n-\mathfrak{f}_\xi)}) \right) = p^{-m-l} \sum_{x \in \mathcal{O}_E^\times / 1+p^n \mathcal{O}_E} \alpha_{m,l,v}(x) \xi(x);$$

where

$$\alpha_{m,l,v}(x) = \gamma (-1)^{m-(n-l)} \psi(-p^{2m+l} v^{-1}) \psi(-v p^{l-2n} N(x)) \psi_E(p^{m-(n-l)} x) \mathbf{1}_{\mathcal{O}_E}(p^{m+l} x - v p^{l-n} N(x)).$$

This implies the result. □

8.1.1.19 Classical q -expansion and global Whittaker model

In this subsection we explain the relationship between classical Fourier expansions at cusps of modular forms and the local Whittaker models of the kind that were

studied in the previous sections.

Let $\pi = \otimes'_v \pi_v$ be an irreducible automorphic representation of $\mathbf{GL}_2(\mathbb{A})$. Let $\psi = \otimes_v \psi_v$ be the additive character of $\mathbb{A}$ defined by specifying its local components

$$\psi_\infty(x_\infty) = e^{2i\pi x_\infty}, \quad \psi_\ell(x_\ell) = e^{-2i\pi x_\ell} \quad \text{for all } x = (x_v) \in \mathbb{A}.$$

Having fixed ψ , and viewing it as a character of $\mathbf{U}(\mathbb{A})$ as well, a global Whittaker function is an intertwining operator $W : \pi \rightarrow \text{Ind}_{\mathbf{U}(\mathbb{A})}^{\mathbf{GL}_2(\mathbb{A})}(\psi)$. We denote the image of an element $\Phi \in \pi$ under W by W_Φ ; then by definition W_Φ is a function $W_\Phi : \mathbf{GL}_2(\mathbb{A}) \rightarrow \mathbb{C}$, that verifies the transformation law $W_\Phi(n(x)g) = \psi(x)W_\Phi(g)$. For $\mathbf{GL}_2$, such a model always exists and is unique (up to a scalar, of course). Given $\Phi \in \pi$, we can choose the model

$$W_\Phi(g) := \int_{\mathbb{Q} \backslash \mathbb{A}} \Phi(n(x)g) \psi(-x) dx.$$

Here $dx = \prod'_p dx_p$ is the Haar measure on $\mathbb{A}$ normalized so that dx_p gives $\mathbb{Z}_p$ volume

1. One can retrieve Φ from the knowledge of W_Φ by the formula

$$\Phi(g) = \sum_{t \in \mathbb{Q}^\times} W_\Phi(a(t)g);$$

here $a(t)$ denotes the diagonal 2×2 matrix given by

$$a(t) = \begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}.$$

Let $\mathcal{H} := \mathbb{C} - \mathbb{R}$ be the hermitian symmetric domain for $\mathbf{GL}_2(\mathbb{R})$. For $z = x + iy \in \mathcal{H}$, we define $g_z := \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}$ so that $g_z \cdot i = z$.

Assume now π_∞ is a holomorphic discrete series of lowest type $k \geq 2$. For $\Phi \in \pi$ in the lowest k -type of π_∞ , we can define

$$\forall z \in \mathcal{H}, \quad h_\Phi(z) := j(g_z, i)^k \Phi(g_z).$$

Then h is a holomorphic function on $\mathcal{H}$. By admissibility of the finite part π_f of π , there exists a smallest integer $\delta \geq 0$ such that Φ is right invariant under $n(\delta x)$ for all $x \in \hat{\mathbb{Z}}$. Then h has Fourier expansion given by

$$h(z) = \sum_{\nu \geq 0} a_h(\nu) e(\nu z / \delta);$$

here $e(-)$ denotes the function $z \mapsto e^{2i\pi z}$. Additionally, for all $t \in \mathbb{Q}^\times$ we have the relation

$$W_\Phi(a(t)g_z) = \begin{cases} y^{k/2} a_h(\nu) e(\nu z / \delta) & \text{if } t \in (1/\delta)\mathbb{Z}_{\geq 0} \\ 0 & \text{otherwise.} \end{cases}$$

Assume now that Φ is new in π and that π_f is fixed by $K_1(Np^r)$ for some integers N, r , with N co-prime to p . Let h_Φ be constructed as above. Then h_Φ has Fourier expansion at infinity of the form

$$h_\Phi(z) = \sum_{\nu \geq 0} a_h(\nu) e(\nu z).$$

We then normalize the newform h_Φ so that $a_h(1) = 1$.

We now want to consider the Fourier expansion of h_Φ at other cusps. Denote by ι_f the diagonal inclusion $\mathbf{GL}_2(\mathbb{Q}) \hookrightarrow \mathbf{GL}_2(\mathbb{A}^{(\infty)})$. Let $\Gamma_1(Np^r) := K_1(Np^r)\mathbf{GL}_2(\mathbb{R})^+ \cap \mathbf{GL}_2(\mathbb{Q})$. A cusp $\mathfrak{c}$ is an element $\mathfrak{c} \in \Gamma_1(N) \backslash \mathbf{P}^1(\mathbb{Q})$. The width $\delta(\mathfrak{c}) \geq 1$ of the cusp $\mathfrak{c}$, is an integer such that, if $\gamma = \begin{pmatrix} a & b \\ L & d \end{pmatrix} \in \mathbf{SL}_2(\mathbb{Z})$, is such that $\gamma(\mathfrak{c}) = \infty$, then $\delta(\mathfrak{c})$ is defined as $\delta(\mathfrak{c}) = \frac{Np^r}{\gcd(Np^r, L^2)}$ (see for example [CNS23, Section 4.1]). Then the width of the cusp does not depend on the chosen $\gamma \in \mathbf{SL}_2(\mathbb{Z})$ mapping $\mathfrak{c}$ to infinity. However, the notion of the q -expansion at the cusp $\mathfrak{c}$ does depend on the choice of γ , but only up to a root of unity, as we shall explain.

We consider the holomorphic function $(h_\Phi|_k\gamma^{-1})(z)$. Then $h_\Phi|_k\gamma^{-1}$ is the holomorphic function associated to the vector $\Phi(\bullet \cdot \iota_f(\gamma)) \in \pi$. The width of the cusp $\mathfrak{c}$ is then the minimal integer $\delta(\mathfrak{c})$ such that $\Phi(\bullet \cdot \iota_f(\gamma))$ is invariant under $n(\delta(\mathfrak{c})x)$ for all $x \in \hat{\mathbb{Z}}$. We deduce by the above discussion that $h_\Phi|_k\gamma^{-1}$ has Fourier expansion given by

$$(h_\Phi|_k\gamma^{-1})(z) = \sum_{\nu \geq 0} a_h(\nu; \gamma) e(\nu z / \delta(\mathfrak{c})).$$

If we change the element γ such that $\gamma(\mathfrak{c}) = \infty$, in γ' then $a_h(\nu; \gamma') = e(\nu\nu' / \delta(\mathfrak{c})) a_h(\nu; \gamma)$ for some integers ν' ; in particular the p -adic valuations of the coefficients $a_h(\nu; \gamma)$ do not depend on the chosen representative γ .

We record the following Lemma combining the paragraphs above.

8.1.1.20 Lemma. *Let $\pi = \otimes'_v \pi_v$ be an irreducible automorphic representation of $\mathbf{GL}_2(\mathbb{A})$ such that π_∞ is in the holomorphic discrete series of weight k . Let Φ be the $K_1(Np^r)$ -fixed newvector of π normalized so that the holomorphic function h_Φ on $\mathcal{H}$*

associated to Φ has q -expansion at infinity $\sum_{\nu \geq 0} a_h(\nu) e(\nu z)$ with $a_h(1) = 1$. Let $\mathfrak{c}$ be a $K_1(Np^r)$ -cusp with width $\delta(\mathfrak{c})$. Let $\gamma = \begin{pmatrix} a & b \\ L & d \end{pmatrix}$ be such that $\gamma(\mathfrak{c}) = \infty$. Denote by $a_h(\nu; \gamma)$ the Fourier coefficients of h_Φ at the cusp $\mathfrak{c}$ for this choice of γ . Let W_Φ be the global Whittaker model of Φ . We then have the following:

1. The width of the cusp $\mathfrak{c}$ is given by

$$\delta(\mathfrak{c}) = \frac{Np^r}{\gcd(Np^r, L^2)}.$$

2. For all $t \in \mathbb{Q}^\times$, and $z = x + iy \in \mathcal{H}$ we have

$$W_\Phi(a(\nu/\delta(\mathfrak{c}))g_z \iota_f(\gamma)) = y^{k/2} a_h(\nu; \gamma) e(\nu z / \delta(\mathfrak{c})).$$

We now relate the values of W_Φ to local Whittaker models. Let W_v be the local Whittaker model of π_v for a place v of $\mathbb{Q}$. Since W_Φ is a global newvector, it is a pure tensor and by uniqueness of Whittaker models we must have

$$W_\Phi(g) = W_\infty(g_\infty) \prod_{\ell < \infty} W_\ell(g_\ell).$$

The local Whittaker models W_ℓ with $\ell < \infty$ are normalized so that $W_\ell(1) = 1$ and W_∞ is normalized so that the equation above holds. By Lemma 8.1.1.20, we see that we must have

$$W_\infty(a(y)) = W_\Phi(a(y)) = y^{k/2} e(iy).$$

Combining Lemma 8.1.1.20, with these remarks we obtain the following Lemma.

8.1.1.21 Lemma. *Let notations be as in Lemma 8.1.1.20. For a prime ℓ , let W_ℓ be the normalized local Whittaker model of π_ℓ . There exists a normalization of the local Whittaker model at infinity W_∞ of π_∞ such that for all $y \in \mathbb{R}^\times$, $W_\infty(a(y)) = y^{k/2}e(iy)$ and*

$$a_h(\nu; \gamma) = \left(\frac{\nu}{\delta(\mathfrak{c})} \right)^{k/2} \prod_{\ell < \infty} W_\ell \left(a(\nu/\delta(\mathfrak{c}))\gamma \right).$$

8.1.1.22 Remark. Note that $\gamma = \begin{pmatrix} a & b \\ L & d \end{pmatrix}$ has Bruhat decomposition $\gamma = n(a/L) \cdot d(1/L) \cdot wn(d/L)$. We will use this decomposition in the following section.

8.1.1.23 Definition. We say that a $K_1(Np^r)$ -cusp $\mathfrak{c}$ is a p -cusp when there is $\gamma = \begin{pmatrix} a & b \\ L & d \end{pmatrix} \in \mathbf{SL}_2(\mathbb{Z})$ such that $\gamma(\mathfrak{c}) = \infty$ and $N|L$.

Denote by $\psi_h^{(l)}$ the Neben-character of h_Φ away from p so that

$$\Phi \left(\bullet \cdot \begin{bmatrix} 1 & 0 \\ 0 & d \end{bmatrix} \right) = \psi_h^{(p)}(d)\Phi(\bullet);$$

here $\begin{bmatrix} 1 & 0 \\ 0 & d \end{bmatrix}$ is seen as an element of $\mathbf{GL}_2(\mathbb{A}_f^{(p)})$.

The following Lemma follows from the above definition and Lemma 8.1.1.21.

8.1.1.24 Lemma. *Let notations be as in Lemma 8.1.1.20. For a prime ℓ , let W_ℓ be the normalized local Whittaker model of π_ℓ . Let $\gamma = \begin{pmatrix} a & b \\ L & d \end{pmatrix} \in \mathbf{SL}_2(\mathbb{Z})$ correspond*

to a p -cusp $\mathfrak{c}$. Let $l := v_p(L)$, so that $\delta(\mathfrak{c}) = p^{r-\min(r,2l)} = p^{\max(0,r-2l)}$. Write also $\nu = p^t \nu'$ where ν' is prime to p . we have:

$$a_h(\nu; \gamma) = p^{(k/2)(t-\max(0,r-2l))} W_p\left(a(p^{t-\max(0,r-2l)})\gamma\right) \cdot \psi_h^{(p)}(d)(\nu')^{k/2} \prod_{\ell \neq p, \infty} W_\ell\left(a(\nu')\right).$$

8.1.2 Global Theta lifts of Hecke characters of quadratic imaginary fields

8.1.2.1 Hecke characters of quadratic imaginary fields

We first review some basic facts about Hecke characters of $\mathcal{K}$. We denote by $\widehat{\mathcal{O}}_{\mathcal{K}}$ the profinite completion of $\mathcal{O}_{\mathcal{K}}$ so that $\mathbb{A}_{\mathcal{K}}^{(\infty), \times} = (\widehat{\mathcal{O}}_{\mathcal{K}} \otimes \mathbb{Q})^{\times}$. For a pair of integers (j_1, j_2) a Hecke character χ of $\mathcal{K}$ of infinity type (j_1, j_2) is a continuous homomorphism

$$\chi : \mathbb{A}_{\mathcal{K}}^{\times} \rightarrow \mathbb{C}^{\times}$$

satisfying, for all $\alpha \in \mathcal{K}^{\times}$, $x \in \mathbb{A}_{\mathcal{K}}^{\times}$, $z_{\infty} \in (\mathcal{K} \otimes_{\mathbb{Q}} \mathbb{R})^{\times} \simeq \mathbb{C}^{\times}$, the equation

$$\chi(\alpha \cdot x \cdot z_{\infty}) = \chi(x) z_{\infty}^{-j_1} \bar{z}_{\infty}^{-j_2}.$$

For each prime $\mathfrak{q}$ of $\mathcal{K}$, let $\chi_{\mathfrak{q}} : \mathcal{K}_{\mathfrak{q}}^{\times} \rightarrow \mathbb{C}^{\times}$ denote the local component of χ . We denote by $\mathfrak{f}_{\chi}$ the conductor of χ . That is the restriction $\chi^{(\infty)}$ of χ to $\mathbb{A}_{\mathcal{K}}^{(\infty), \times}$ has open kernel and $\mathfrak{f}_{\chi} \subseteq \mathcal{O}_{\mathcal{K}}$ is the largest ideal such that $\chi^{(\infty)}$ factors through $(\mathcal{O}_{\mathcal{K}}/\mathfrak{f}_{\chi})^{\times}$. We will denote by the same symbol χ the character on $\mathcal{O}_{\mathcal{K}}$ ideals prime to $\mathfrak{f}_{\chi}$ that χ

defines. If $\mathfrak{a}$ is an $\mathcal{O}_K$ -ideal prime to $\mathfrak{f}_\chi$, we have

$$\chi(\mathfrak{a}) = \prod_{\mathfrak{q}|\mathfrak{a}} \chi_{\mathfrak{q}}(\varpi_{\mathfrak{q}})^{v_{\mathfrak{q}}(\mathfrak{a})}.$$

Here $\varpi_{\mathfrak{q}}$ is any uniformizer at $\mathfrak{q}$; the above equation does not depend on the choice of uniformizer. Otherwise stated, if we decompose $\mathbb{A}_K = \mathbb{C}^\times \times \mathbb{A}_K^{(\infty), \times}$, $\chi(\mathfrak{a})$ is defined as $\chi(1, a)$ where 1 is the unit element of $\mathbb{C}^\times$ and a is an element of $\mathbb{A}_K^{(\infty), \times} \cap \widehat{\mathcal{O}}_K$ such that $\widehat{\mathcal{O}}_K \mathfrak{a} = \widehat{\mathcal{O}}_K \cdot a$ and such that the image of a under $\widehat{\mathcal{O}}_K \rightarrow \widehat{\mathcal{O}}_K/\mathfrak{f}_\chi$ is 1.

For a principal ideal $\mathfrak{a} := (\alpha) \subset \mathcal{O}_K$ with $\alpha \equiv 1 \pmod{\mathfrak{f}_\chi}$, we have

$$\chi(\mathfrak{a}) = \alpha^{j_1} \overline{\alpha}^{j_2}.$$

The L -function of χ is then expressed as

$$L(\chi, s) := \sum_{\mathfrak{a} \subset \mathcal{O}_K} \chi(\mathfrak{a}) N_{K/\mathbb{Q}}(\mathfrak{a})^{-s} \quad (8.7)$$

It is known that if χ is a Hecke character of infinity type (j_1, j_2) for some integers m, n then the subfield of $\mathbb{C}$ generated by $\psi(\mathbb{A}_K^{(\infty), \times})$ is a finite extension of $\mathbb{Q}$ (see for example [Kat04, p. 255]).

8.1.2.2 The global Weil representation

Let (V, q) be an even dimensional orthogonal space over $\mathbb{Q}$. We denote the bilinear symmetric pairing attached to the quadratic form q by $\langle \cdot, \cdot \rangle$. Denote by $\mathbf{O}(V)$ the orthogonal group of V , by $\mathbf{GO}(V)$ the group of orthogonal similitudes, and by

$\mathbf{GSO}(V)$ the connected component of the identity of $\mathbf{GO}(V)$. We denote by ν the similitude character of $\mathbf{GO}(V)$.

Let $r_\psi = \otimes_v r_{\psi_v}$ be the Weil representation of $\mathbf{SL}_2(\mathbb{A}) \times \mathbf{O}(V)(\mathbb{A})$ on the space of Schwartz functions $\mathcal{S}(V(\mathbb{A}))$. Then r_ψ can be characterised as follows. For all $g \in \mathbf{O}(V)(\mathbb{Q}_v)$, $x \in \mathcal{S}(\mathbb{Q}_v)$, and $\varphi \in \mathcal{S}(V(\mathbb{Q}_v))$ we have

$$r_{\psi,v}(g)\varphi(x) = \varphi(g^{-1}x).$$

On $\mathbf{SL}_2$ the action is specified by describing the image of the generators of $\mathbf{SL}_2(\mathbb{Q}_v)$

$$U(a) := \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}, \quad D(a) := \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}, \quad \text{and } W := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

by the equations:

$$\begin{aligned} r_{\psi,v}(U(a))\varphi(x) &= \psi_v\left(\frac{1}{2}\langle ax, x \rangle\right) \varphi(x) \\ r_{\psi,v}(D(a))\varphi(x) &= \chi_{V,v}(a)|a|_v^{\dim V/2} \varphi(ax) \\ r_{\psi,v}(W)\varphi(x) &= \gamma_{V,v} \hat{\varphi}(x). \end{aligned}$$

Here $\chi_{V,v}$ is a quadratic character and $\gamma_{V,v}$ is an eighth root of unity, which are specified in the table in [Pra06, Section 3.4]. The Fourier transform $\hat{\varphi}$ is normalized as follows

$$\hat{\varphi} := \int_{V(\mathbb{Q}_v)} \varphi(y) \psi_v(\langle y, x \rangle) dy,$$

where the measure dy on $V(\mathbb{Q}_v)$ is chosen such that $\widehat{\hat{\varphi}} = \varphi(-x)$.

This is the global analog of the local Weil representation considered in section

8.1.1, for $V \simeq \mathcal{K}$, seen as a two dimensional $\mathbb{Q}$ -space with non-degenerate quadratic form $N_{\mathcal{K}/\mathbb{Q}}$, so that $\mathbf{GO}(V) \simeq \mathbb{G}_{m,\mathcal{K}}$.

8.1.2.3 A Global theta lift

Fix now a character $\varepsilon'_{\mathcal{K}} : \mathbb{A}_{\mathcal{K}}^{\times} \rightarrow \mathbb{C}^{\times}$ as in [Cha20, Definition 5.4] restricting to $\varepsilon_{\mathcal{K}/\mathbb{Q}}$. Following Section 8 of *loc cit.*, let ϕ be a Schwartz function on $\mathbb{A}_{\mathcal{K}}$. Let $\chi : \mathbb{A}_{\mathcal{K}}^{\times} \rightarrow \mathbb{C}^{\times}$ be a Hecke character of infinity type $(k, 0)$ (so $j = 0$ in the notations of *loc cit.*), where $k > 0$ is an integer. We now recall the definition of the Theta lift $\theta_{\phi}(\chi)$, as an automorphic form on $\mathbf{GL}_2(\mathbb{A})$. Recall the notation $\mathbf{GL}_2(\mathbb{A})^+$ for the subgroup of $\mathbf{GL}_2(\mathbb{A})$ defined as the elements with determinant in the image of the norm map $N_{\mathcal{K}/\mathbb{Q}} : \mathbb{A}_{\mathcal{K}}^{\times} \rightarrow \mathbb{A}^{\times}$. Then for any $h \in \mathbb{A}_{\mathcal{K}}^{\times}$ such that $\det(g) = N_{\mathcal{K}/\mathbb{Q}}(\det(g))$, one defines:

$$\theta_{\phi}(\chi) = \int_{\mathcal{K}^1 \backslash \mathbb{A}_{\mathcal{K}}^1} \Theta(r_{\psi}(hh_1, g)\phi) \cdot (\chi\varepsilon'_{\mathcal{K}})(hh_1)dh_1.$$

Here the notations are as follows:

- The group $\mathbb{A}_{\mathcal{K}}^1$ denotes the kernel of $N_{\mathcal{K}/\mathbb{Q}}$ and $\mathcal{K}^1$ denotes the intersection of $\mathbb{A}_{\mathcal{K}}^1$ with $\mathcal{K}^{\times}$.
- The symbol Θ denotes the θ -distribution:

$$\Theta : \mathcal{S}(\mathbb{A}_{\mathcal{K}}) \rightarrow \mathbb{C}, \quad \phi \mapsto \sum_{x \in \mathcal{K}} \phi(x).$$

We now extend $\theta_{\phi}(\chi)$ to the index 2-subgroup $\mathbf{GL}_2(\mathbb{Q})\mathbf{GL}_2(\mathbb{A})^+$ of $\mathbf{GL}_2(\mathbb{A})$ by

setting

$$\theta_\phi(\chi)(\gamma g) = \theta_\phi(\chi)(g), \quad \text{for } \gamma \in \mathbf{GL}_2(\mathbb{Q}), g \in \mathbf{GL}_2(\mathbb{A}).$$

We then extend by 0, $\theta_\phi(\chi)$ on $\mathbf{GL}_2(\mathbb{A})$.

We now choose specific Schwartz functions. We let $\phi_\chi^{new} = \otimes' \phi_v^{new}$ to be the vector defined (for $l = 0$) in Section 7 of [Cha20], or at the beginning of Section 8 of *loc cit.*. Then if χ does not factor through the norm map $\mathbb{A}_K^\times \rightarrow \mathbb{A}^\times$, $\theta_{\phi_\chi^{new}}(\chi)$ is the new-vector in the automorphic induction representation π_χ . At finite places v of $\mathbb{Q}$, if $\mathcal{K}_v := \mathcal{K} \otimes_{\mathbb{Q}} \mathbb{Q}_v$, ϕ_χ^{new} is given by

$$\phi_{\chi,v}^{new} := \begin{cases} \mathbf{1}_{\mathcal{O}_{\mathcal{K}_v}} & \text{if } \chi_v \text{ is unramified,} \\ \chi_v^{-1} \mathbf{1}_{\mathcal{O}_{\mathcal{K}_v}^\times} & \text{if } \chi_v \text{ is ramified.} \end{cases}$$

We then choose $\phi_{\chi,\infty}^{new}$, so that the q -expansion at infinity of $\theta_{\phi_\chi^{new}}$ is given by

$$\theta_{\phi_\chi^{new}}(q) = \sum_{\mathfrak{a} \in \mathcal{O}_K} \chi(\mathfrak{a}) q^{N_{K/\mathbb{Q}}(\mathfrak{a})}. \quad (8.8)$$

Let now χ_{Dir} be the unique Dirichlet character such that for all integers n , $\chi((n)) = n^k \chi_{\text{Dir}}(n)$ (recall that χ has infinity type $(1, 0)$). Note that we always have $\chi_{\text{Dir}}(-1) = (-1)^k$. By [Miy89, Theorem 4.8.2], $\theta_{\phi_\chi^{new}}$ is a new holomorphic cuspform of weight $k+1$, level $\Gamma_1(N_{K/\mathbb{Q}}(\mathfrak{f}_\chi) \Delta_K)$ and Nebencharakter $\varepsilon_{K/\mathbb{Q}} \chi_{\text{Dir}}$. Notice that $\varepsilon_{K/\mathbb{Q}} \chi_{\text{Dir}}(-1) = (-1)^{k+1}$.

In the next section, we modify slightly the choice of Schwartz function ϕ_χ^{new} .

Normalization when p is inert in $\mathcal{K}$. Assume that p is inert in $\mathcal{K}$. Let $E := \mathcal{K}_p$ the unique quadratic unramified extension of $\mathbb{Q}_p$. Let χ_p be the local component at p of χ . We then define, at all places $v \neq p$ of $\mathbb{Q}$

$$\phi_{\chi,v} := \phi_{\chi}^{new},$$

and at p :

$$\phi_{\chi,p} := (-1)^{f_{\chi p}} \tau_E(\chi_p^{-1}) \chi_p^{-1} \mathbf{1}_{\mathcal{O}_E^{\times}},$$

with notations as in section 8.1.1.2. We also define for any integer $n \geq 0$,

$$\phi_{\chi,p}^{(n)} := \omega_{\chi_p^{-1}}(d(p^{-n})) \phi_{\chi,p},$$

so that $\theta_{\phi_{\chi,p}^{(n)}} = \iota_p(d(p^{-n})) \cdot \theta_{\phi_{\chi,p}}$; here $\iota_p : \mathbf{GL}_2(\mathbb{Q}_p) \hookrightarrow \mathbf{GL}_2(\mathbb{A}_f)$ is the natural inclusion and $\cdot$ is the right translation action of $\mathbf{GL}_2(\mathbb{A}_f)$ on automorphic forms for $\mathbf{GL}_2(\mathbb{A})$.

8.1.2.4 The q -expansion at all cusps of certain theta series

In this section, we assume that p is inert in $\mathcal{K}$.

We analyze the q -expansion of $\theta_{\phi_{\chi}}$ at all p -cusps (see Definition 8.1.1.23). We also give an analogous definition for global Hecke characters of definition 8.1.1.17. We denote by $\widehat{\mathcal{O}}_{\mathcal{K}}$ the profinite completion of $\mathcal{O}_{\mathcal{K}}$ and view $\widehat{\mathcal{O}}_{\mathcal{K}}^{\times}$ as a topological subgroup of $\mathbb{A}_{\mathcal{K}}^{\times}$. Let $\mathfrak{a} \subset \mathcal{O}_{\mathcal{K}}$ be an ideal; we denote by $\widehat{U}_{\mathfrak{a}} \subset \widehat{\mathcal{O}}_{\mathcal{K}}^{\times}$ the subgroups of units $(u_v)_v$ such that $u_v \in 1 + \mathfrak{p}_v^{v(\mathfrak{a})} \mathcal{O}_{\mathcal{K}_v}$; here $\mathfrak{p}_v$ is the prime ideal corresponding to the place

v of $\mathcal{K}$, and $\mathfrak{a}$ has prime decomposition $\mathfrak{a} = \prod_v \mathfrak{p}_v^{v(\mathfrak{a})}$. Let $\mathcal{X}_{\mathfrak{a}}$ be the set of Hecke characters χ such that χ is of conductor dividing $\mathfrak{a}$; i.e the characters of $\mathcal{K}^{\times} \backslash \mathbb{A}_{\mathcal{K}}^{\times}$ that factor through $\mathcal{K}^{\times} \backslash \mathbb{A}_{\mathcal{K}}^{\times} / U_{\mathfrak{a}}$. We let $\mathcal{X}_{\mathfrak{a}}^k$ be the subset of Hecke characters that have infinity type $(k, 0)$.

8.1.2.5 Definition. A function

$$P : \mathcal{X}_{\mathfrak{a}}^k \rightarrow \mathbb{C}$$

is a universal polynomial when there exists coefficients $\alpha_{\mathfrak{a},k}(x) \in \mathbb{C}$, for $x \in (\hat{\mathcal{O}}_{\mathcal{K}}^{\times} / \hat{U}_{\mathfrak{a}})$, such that

$$P(\chi) = \sum_{x \in \hat{\mathcal{O}}_{\mathcal{K}}^{\times} / \hat{U}_{\mathfrak{a}}} \alpha_{\mathfrak{a},k}(x) \chi(x).$$

Moreover we say that P is a universal p -integral polynomial when the $\alpha_{\mathfrak{a},k}(x)$ are algebraic and in $\overline{\mathbb{Z}}_{(p)}$.

We have the following proposition.

8.1.2.6 Proposition. Let $\gamma = \begin{pmatrix} a & b \\ L & d \end{pmatrix}$ be such that $\mathfrak{c} := \gamma^{-1}(\infty)$ is a p -cusp.

Denote by $(a_{\phi}(\nu; \gamma))$ the coefficients of the q -expansion of θ_{ϕ} at the cusp $\mathfrak{c}$. Let $\mathfrak{a} \subset \mathcal{O}_K$ be a prime to p ideal. Then for any integers ν and n ,

$$\chi \in \mathcal{X}_{p^n \mathfrak{a}}(\lambda) \mapsto a_{\phi_{\chi}^{(n-f_{\chi})}}(\nu; \gamma),$$

is a p -integral universal polynomial in the sense of definition 8.1.2.5.

Proof. We apply Lemma 8.1.1.24, for $h = \theta_{\phi_\chi^{(n-\mathfrak{f}_{\chi_p})}}$. Then h is of level $\Gamma_1(Np^r)$ where $r = 2n$ and $N := N_{\mathcal{K}/\mathbb{Q}}(\mathfrak{f}_\chi^{(p)})\Delta_{\mathcal{K}}$; here $\mathfrak{f}_\chi$ denotes the conductor of χ away from p . Writing as in Lemma 8.1.1.24, $l = v_p(L)$, $\nu = p^t\nu'$, with ν' prime to p , we have

$$a_h(\nu; \gamma) = p^{((k+1)/2)(t-2\max(0,n-l))} W_p\left(a(p^{t-2\max(0,n-l)})\gamma\right) \cdot (\varepsilon_{\mathcal{K}/\mathbb{Q}}\chi_{\text{Dir}}^{(p)}(d) \cdot (\nu')^{k/2} \prod_{\ell \neq p, \infty} W_\ell(a(\nu'))).$$

By Equation 8.8 or the general formulas for the Kirillov new-vector, the term $(\nu')^{k/2} \prod_{\ell \neq p, \infty} W_\ell(a(\nu'))$, is a universal p -integral polynomial in χ . By definition of χ_{Dir} the same is true for the term $\varepsilon_{\mathcal{K}/\mathbb{Q}}(d)\chi_{\text{Dir}}^{(p)}(d)$. Remains to treat the term

$$P_{\gamma, \nu}(\chi) := p^{((k+1)/2)(t-2\max(0,n-l))} W_p\left(a(p^{t-2\max(0,n-l)})\gamma\right).$$

Since p is not in the image of the norm, we must have $t = 2x$ is an even integer. Therefore, since by Lemma 8.1.1.1 the central character of $\pi_{\chi_p} = V_{\chi_p^{-1}}$ is $\varepsilon_{\mathcal{K}/\mathbb{Q}}\chi_p|_{\mathbb{Q}_p^\times}$, we have

$$W_p\left(a(p^{t-2\max(0,n-l)})\gamma\right) = \varepsilon_{\mathcal{K}/\mathbb{Q}}(p^{x-\max(0,n-l)})\chi_p(p^{x-\max(0,n-l)})W_p\left(d(p^{x-\max(0,n-l)})\gamma\right).$$

Now notice that $\chi_p|_{\mathbb{Q}_p^\times} = |\cdot|^{-k}\xi$ where ξ is a finite order character of $E^\times$ as χ is of infinity type $(k, 0)$, and so it suffices to prove that

$$p^{x-\max(0,n-l)} W_p\left(d(p^{x-\max(0,n-l)})\gamma\right)$$

is an integral universal polynomial in ξ .

We have

$$\begin{aligned} d(p^{x-\max(0,n-l)})\gamma &= n(ap^{2(x-\max(0,n-l))}/L) \cdot d(p^{x-\max(0,n-l)-l})wn(dp^{-l}) \\ &= n(ap^{2(x-\max(0,n-l))}/L)\gamma_{x-l-\max(0,n-l),l,d}. \end{aligned}$$

Now by Proposition 8.1.1.18, we know that $p^{x-\max(0,n-l)} \cdot W_p(\gamma_{x-l-\max(0,n-l),l,d})$ is an integral universal polynomial in ξ^{-1} , and thus in ξ . This remark concludes the proof. $\square$

We now give the global formula for the q -expansion at a specific cusp. Given the computation of Proposition 8.1.2.8 it is convenient to renormalize the section $\phi_\chi^{(n-\mathfrak{f}_{\chi p})}$. We give the renormalization in the following definition.

8.1.2.7 Definition. We define

$$\tilde{\phi}_{\chi,n} := \zeta_n \cdot \langle p^n \rangle^p \phi_\chi^{(n-\mathfrak{f}_{\chi p})}.$$

Here ζ_n is $2n$ -th root of unity, specified by

$$\zeta_n^{-1} = \gamma(-1)^n \psi_p(p^{-2n}),$$

and $\langle \cdot \rangle^{(p)}$ denotes the local diamond operator away from p .

The following proposition shows that with the normalization above, the q -expansion at infinity of $\theta_{\phi_\chi^{new}}$ is the q -expansion of $\theta_{\tilde{\phi}_{\chi,n}}$ at the 0 cusp.

8.1.2.8 Proposition. Let $\gamma_0 := \begin{pmatrix} 1 & 0 \\ M & 1 \end{pmatrix} \in \mathbf{SL}_2(\mathbb{Z})$ where M is an integer divisible by N such that $M \equiv 1 \pmod{p^r}$. Then the q -expansion of $\theta_{\phi_\chi^{(n-\mathfrak{f}_{\chi p})}}$ at the $\Gamma_1(Np^{2n})$ -cusp labelled by γ_0 is given by

$$\theta_{\tilde{\phi}_{\chi,n}}(q) = \sum_{\mathfrak{a} \subseteq Np\mathcal{O}_K} \chi(\mathfrak{a}) q^{N_{K/\mathbb{Q}}(\mathfrak{a})}.$$

Here the sum is taken with respect to all ideals $\mathfrak{a} \subseteq \mathcal{O}_K$ coprime to pN , and we recall that $(k, 0)$ is the infinity type of χ .

Proof. Let $\mathfrak{c}$ be the cusp labelled by γ_0 , that is $\mathfrak{c} = \gamma^{-1}(\infty)$. Then by (1) of Lemma 8.1.1.20, we have $\delta(\mathfrak{c}) = p^r = p^{2n}$. With the notations of Proposition 8.1.2.6, we have $l = 0$. Writing again $\nu = p^{2x}\nu'$, and specialising the computation of Proposition 8.1.2.6, we have

$$a_{\phi_\chi^{(n-\mathfrak{f}_\chi)}}(\nu, \gamma_0) = (-1)^{x-n} \mu_p(p^{x-n}) p^{x-n} \psi_p(p^{2(x-n)}) W_\xi(\gamma_{x-n,0,1}) \cdot (\nu')^{k/2} \prod_{\ell \neq p, \infty} W_\ell(a(\nu')).$$

Invoking the formulas in the proof of Proposition 8.1.1.18, we have, recalling that $E = K_p$, and writing $\chi_p^{-1}|_{\mathbb{Q}_p^\times} = |\cdot|_p^k \mu_p$

$$a_{\phi_\chi^{(n-\mathfrak{f}_\chi)}}(\nu, \gamma_0) = \gamma (-1)^n \mu_p(p^{x-n}) \cdot \left(\sum_{z \in \mathcal{O}_E^\times / 1+p^n \mathcal{O}_E} \beta_0(z) \xi(z) \right) \cdot (\nu')^{k/2} \prod_{\ell \neq p, \infty} W_\ell(a(\nu')),$$

where for all $z \in \mathcal{O}_E^\times$

$$\beta_0(z) := \psi(-p^{-2n}N_{E/\mathbb{Q}_p}(z))\psi_E(p^{-2n}z)\mathbf{1}_{p^n\mathcal{O}_E}(p^x z - N_{E/\mathbb{Q}_p}(z)).$$

Now note that if $p^x z - N_{E/\mathbb{Q}_p}(z) \in p^n\mathcal{O}_E$ then necessarily as $z \in \mathcal{O}_E^\times$, we must have $x = 0$, and then $z(1 - \bar{z}) \in p^n\mathcal{O}_E$, which implies that $z \in 1 + p^n\mathcal{O}_E$. Therefore $\beta_0(z) = 0$ unless $z = 1$, and in this case $\beta_0(1) = \psi(p^{-2n})$. Taking into consideration the renormalization of Definition 8.1.2.7, and Equation 8.8, we obtain the result. $\square$

8.1.2.9 The almost maximal abelian extension of $\mathcal{K}$

Denote by $\mathbb{Q}^{cyc} = \mathbb{Q}^{ab}$, where $\mathbb{Q}^{ab}$ is the maximal abelian extension of $\mathbb{Q}$. Let $\mathcal{K}^{ac}$ be the compositum of all the ring class fields of $\mathcal{K}$ and let $\mathcal{K}_\infty := \mathbb{Q}^{cyc}\mathcal{K}^{ac}$ – then $\mathcal{K}^{ac}$ is generated by the j invariants of Elliptic curves with complex multiplication by an order of $\mathcal{O}_\mathcal{K}$. We record in the following Lemma the basic properties of $\mathcal{K}_\infty$.

8.1.2.10 Lemma. *The extension $\mathcal{K}_\infty/\mathcal{K}$ verifies the following properties.*

1. *The Galois group $\text{Gal}(\mathcal{K}^{ab}/\mathcal{K}_\infty)$ is an $\mathbb{F}_2$ -vector space.*
2. *Let $\mathfrak{q}$ be a prime of $\mathcal{K}$ lying above an odd prime of $\mathbb{Q}$ and denote by $\mathcal{K}(\mathfrak{q})$ the maximal p -extension inside the ray class field of $\mathcal{K}$ modulo $\mathfrak{q}$. Then $\mathcal{K}_\infty$ contains $\mathcal{K}(\mathfrak{q})$. In particular, for any odd prime q , $\mathcal{K}_\infty$ contains the unique maximal abelian $\mathbb{Z}_q^2$ extension of $\mathcal{K}$.*
3. *No finite prime of $\mathcal{K}$ splits completely in $\mathcal{K}_\infty$.*

4. The conjugation action of $\text{Gal}(\mathcal{K}/\mathbb{Q})$ on $\text{Gal}(\mathcal{K}_\infty/\mathcal{K})$ induces on $\text{Gal}(\mathcal{K}^{ac}/\mathcal{K})$ the action given by $g \mapsto g^{-1}$ and induces the trivial action on $\text{Gal}(\mathbb{Q}^{cyc}/\mathcal{K})$.
5. If G is a topological group we denote by $X(G)$ the set of complex-valued continuous characters of G . The natural map:

$$X(\text{Gal}(\mathcal{K}^{ac}/\mathcal{K})) \times X(\text{Gal}(\mathbb{Q}^{cyc}/\mathcal{K})) \longrightarrow X(\text{Gal}(\mathcal{K}_\infty/\mathcal{K}))$$

is surjective and the kernel is identified with the maximal $\mathbb{F}_2$ -vector space contained in $\text{Gal}(\mathcal{K}_\infty/\mathbb{Q})$.

Proof. The first fact is the Theorem in Section 3 of [Ser67]. The second fact follows immediately from the first and the third from the second. The second to last fact comes from Section 2 of [Ser67]. For the last fact: the surjectivity of the map is by definition of $\mathcal{K}_\infty$ and the second claim comes from the fact that if χ is in the image of $X(\text{Gal}(\mathcal{K}^{ac}/\mathcal{K}))$ then τ acts on χ as $\chi(\tau\sigma\tau) = \chi^{-1}(\sigma)$, and τ acts trivially on the image of $X(\text{Gal}(\mathbb{Q}^{cyc}/\mathcal{K}))$. $\square$

8.1.2.11 Remark. For a character $\chi \in X(\text{Gal}(\mathcal{K}_\infty/\mathcal{K}))$, writing $\chi = \chi_{cyc}\chi_{ac}$ where χ_{cyc} is the restriction of to $\text{Gal}(\mathbb{Q}^{cyc}/\mathcal{K})$ of an element in $X(\text{Gal}(\mathbb{Q}^{cyc}/\mathbb{Q}))$, we note that χ_{cyc}^2 is defined without ambiguity because of point 5 of Lemma 8.1.2.10.

8.2 Theta Classes in étale cohomology

In this section, we construct certain étale cohomology classes on the modular curve associated to the Theta series whose Fourier expansion was analyzed in the previous

section.

8.2.1 CM theory

8.2.1.1 p -adic Hecke characters

We first review basic facts about the p -adic Hecke character attached to a Hecke character $\chi : \mathcal{K}^\times \backslash \mathbb{A}_{\mathcal{K}}^\times \rightarrow \mathbb{C}^\times$ and its associated Galois representation. In order to do so, one has to make a choice of embedding $\iota_p : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}_p$ and $\iota_\infty : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$ and an isomorphism $\mathbb{C} \simeq \mathbb{C}_p$ of fields such that the diagram

$$\begin{array}{ccc} & & \mathbb{C} \\ & \nearrow \iota_\infty & \downarrow \sim \\ \overline{\mathbb{Q}} & & \mathbb{C}_p \\ & \searrow \iota_p & \end{array}$$

We also assume that the embedding $\mathcal{K} \hookrightarrow \mathbb{C}$ induced by ι_∞ coincides with our fixed embedding as before.

Let now $\chi : \mathcal{K}^\times \backslash \mathbb{A}_{\mathcal{K}}^\times \rightarrow \mathbb{C}^\times$ be a Hecke character of infinity type $(k, 0)$ as in section 8.1.2.4. We recall that p is assumed to be inert in $\mathcal{K}$. If $x \in \mathbb{A}_{\mathcal{K}}$ we denote by x_v the local component of x if v is a place of $\mathcal{K}$ and $x^{(\infty)}$ for the finite component of x . The p -adic Hecke character $\widehat{\chi}$ attached to χ is defined as

$$\forall x \in \mathbb{A}_{\mathcal{K}}^\times, \quad \widehat{\chi}(x) = x_p^{-k} \iota_p(\chi(x^{(\infty)})). \quad (8.9)$$

Note that as defined $\widehat{\chi}$ is invariant under translation by global elements of $\mathcal{K}^\times$. Indeed, if $t \in \mathcal{K}^\times$, we have

$$\widehat{\chi}(tx) = \iota_p(t^{-k})\iota_p(\chi(t^{(\infty)}))\widehat{\chi}(x) = \widehat{\chi}(x),$$

where the last equality is a consequence of the commutativity of the diagram above relating ι_p to ι_∞ under the chosen isomorphism $\mathbb{C} \simeq \mathbb{C}_p$. We still denote by $\widehat{\chi} : \mathcal{K}^\times \backslash \mathbb{A}_\mathcal{K}^\times \rightarrow \mathbb{C}_p^\times$ the corresponding descended character. Note that since the values of χ on $\mathbb{A}_\mathcal{K}^{(\infty), \times}$ generate a finite extension of $\mathbb{Q}$, we can view $\widehat{\chi}$, as being valued in a finite extension of $\mathbb{Q}_p$. We denote by $\mathcal{K}_{\chi, p}$ the minimal such extension so that $\widehat{\chi}$ defines a character $\mathcal{K}^\times \backslash \mathbb{A}_\mathcal{K}^\times \rightarrow \mathcal{K}_{\chi, p}^\times$.

Let C^0 be the connected component of the identity of the idèle class group $\mathcal{K}^\times \backslash \mathbb{A}_\mathcal{K}^\times$. Then since $\mathcal{K}$ is complex imaginary, C^0 is the image of $(\mathcal{K} \otimes \mathbb{R})^\times$. By Equation 8.9 above, $\widehat{\chi}$ is clearly invariant under C^0 and therefore defines a character still denoted by $\widehat{\chi} : \mathcal{K}^\times C^0 \backslash \mathbb{A}_\mathcal{K}^\times \rightarrow \mathcal{K}_{\chi, p}^\times$. The class field theory reciprocity map $\text{rec}_\mathcal{K} : \mathcal{K}^\times \backslash \mathbb{A}_\mathcal{K}^\times \rightarrow G_\mathcal{K}^{\text{ab}}$ is surjective and the kernel is identified with C^0 . We denote by $\text{Art}_\mathcal{K}$ the inverse of $\text{rec}_\mathcal{K}$:

$$\text{Art}_\mathcal{K} : G_\mathcal{K}^{\text{ab}} \xrightarrow{\sim} \mathcal{K}^\times C^0 \backslash \mathbb{A}_\mathcal{K}^\times$$

To $\widehat{\chi}$, there is therefore a p -adic one-dimensional Galois representation attached to χ defined as, defined as

$$\rho_\chi : G_\mathcal{K}^{\text{ab}} \rightarrow \mathcal{K}_{\chi, p}^\times, \quad \rho_\chi := \widehat{\chi}^{-1} \circ \text{Art}_\mathcal{K}.$$

For the description of the CM part of cohomology of modular curves it will be actually convenient to think of (p -adic) Hecke characters as characters on fractional ideals of $\mathcal{K}$. We now introduce the relevant notation. Let I be the group of fractional ideals of $\mathcal{K}$. Let $P \subset I$ be the subgroup of principal ideals. We denote by $\text{Cl} := I/P$ the class group of $\mathcal{K}$. For a modulus $\mathfrak{m} = \prod_{\mathfrak{p}} \mathfrak{p}^{e_{\mathfrak{p}}}$, we let $I_{\mathfrak{m}}$ the subgroup of fractional ideals that coprime to $\mathfrak{m}$. Recall the notation $\hat{U}_{\mathfrak{m}}$ for the subgroup of $\hat{\mathcal{O}}_{\mathcal{K}}^{\times}$ for the elements locally in $1 + \mathfrak{p}^{e_{\mathfrak{p}}}$ at all primes for which the integer $e_{\mathfrak{p}}$ is strictly positive. Let $P_{\mathfrak{m}} \subseteq I_{\mathfrak{m}}$ be the subgroup of principal ideals generated by elements of whose image in $\hat{\mathcal{O}}_{\mathcal{K}}^{\times} \otimes \mathbb{Q}$ lies in $U_{\mathfrak{m}} \cap \mathcal{O}_{\mathfrak{p}}$ at all places $\mathfrak{p}$ for which $e_{\mathfrak{p}} > 0$. The ray class group $\text{Cl}_{\mathfrak{m}}$ is then defined as $\text{Cl}_{\mathfrak{m}} := I_{\mathfrak{m}}/P_{\mathfrak{m}}$. We denote its cardinality by $h_{\mathfrak{m}}$. Let $\mathbb{A}_{\mathcal{K}}^{\mathfrak{m}} \subseteq \mathbb{A}_{\mathcal{K}}^{(\infty), \times}$ be the subgroup consisting of the finite idèles (x_v) with $x_{\mathfrak{p}} \in 1 + \mathfrak{p}^{e_{\mathfrak{p}}}$ for every $\mathfrak{p}$ with $e_{\mathfrak{p}} > 0$. Let $\mathcal{K}_1^{\mathfrak{m}} := \mathbb{A}_{\mathcal{K}}^{\mathfrak{m}} \cap \mathcal{K}^{\times}$ and $\mathcal{K}^{\mathfrak{m}}$ be the elements α of $\mathcal{K}^{\times}$ such that (α) is coprime to $\mathfrak{m}$. There is a surjective map

$$\mathbb{A}_{\mathcal{K}}^{\mathfrak{m}} \twoheadrightarrow I_{\mathfrak{m}}$$

that discards the $\mathfrak{m}$ -components of its input and matches local uniformizers and prime ideals. The kernel of this map is $\hat{U}_{\mathfrak{m}}$.

As in the discussion of Paragraph 8.1.2.1, a Hecke character χ of conductor $\mathfrak{f}_{\chi}$, defines a character

$$\chi : I_{\mathfrak{f}_{\chi}} \rightarrow \mathcal{K}_{\chi},$$

by viewing the values of $\chi^{(\infty)}$ as being elements of $\mathcal{K}_{\chi}$. The character on ideals above uniquely determines χ and $\hat{\chi}$. This is because the character on fractional ideals is

simply the restriction $\chi|_{\mathbb{A}_{\mathcal{K}}^{\mathfrak{f}_X}}$, which descends to a character $\hat{U}_{\mathfrak{f}_X} \backslash \mathbb{A}_{\mathcal{K}}^{\mathfrak{f}_X} \simeq I_{\mathfrak{f}_X}$ and since $\mathbb{A}_{\mathcal{K}}^{\times} = \mathbb{A}_{\mathcal{K}}^{\mathfrak{m}} \mathcal{K}^{\times}$, the claim is clear. Conversely, if one is given a character $\chi : I_{\mathfrak{f}} \rightarrow \mathcal{K}_{\chi,p}$, such that $\chi((\alpha)) = \alpha^{-j_1} \bar{\alpha}^{-j_2}$ for all $\alpha \in \mathcal{K}^{\mathfrak{f}}$, one views it as a character of $\mathbb{A}_{\mathcal{K}}^{\mathfrak{m}}$, and then defines it in the unique manner on the infinity component of $\mathbb{A}_{\mathcal{K}}^{\times}$ so that the resulting character be invariant under $\mathcal{K}^{\times}$. We allow ourselves to still write ρ_{χ} for the p -adic Galois representation attached to the Hecke character underlying the ideal character $\chi : I_{\mathfrak{f}_X} \rightarrow \mathcal{K}_{\chi}$.

For a modulus $\mathfrak{m}$, denote by $\mathcal{K}[\mathfrak{m}] \subset \mathcal{K}^{ab}$ the ray class field of $\mathcal{K}$ of modulus $\mathfrak{m}$. For a field $K'/\mathcal{K}$ CM-pair (A, α) with modulus $\mathfrak{m}$ over K' is a pair of an elliptic curve $A_{/K'}$ defined over K' endowed with an isomorphism $\mathcal{O}_{\mathcal{K}} \simeq \text{End}(A)$ such that the composite map

$$\mathcal{O}_{\mathcal{K}} \rightarrow \text{End}(A) \rightarrow \text{End}_{K'}(\text{Lie}(A)) = K'$$

coincides with the inclusion of $\mathcal{K}$ into K' ; $\alpha \in E(K')$ is a torsion point such that the annihilator of α in $\mathcal{O}_{\mathcal{K}}$ is the ideal $\mathfrak{m}$. A morphism of CM-pairs $(A, \alpha) \rightarrow (B, \beta)$ is simply a map of group schemes $A \rightarrow B$ such that $f(\alpha) = \beta$. If one assumes that $\mathcal{O}_{\mathcal{K}}^{\times} \rightarrow (\mathcal{O}_{\mathcal{K}}/\mathfrak{m})^{\times}$ is assumed to be injective (automatic if $\mathcal{O}_{\mathcal{K}}^{\times} = \pm 1$), and two CM pairs (A, α) and (B, β) are isomorphic, the isomorphism $(A, \alpha) \rightarrow (B, \beta)$ is unique.

By the theory of complex multiplication, for any modulus $\mathfrak{m}$ there exists a CM pair of modulus $\mathfrak{m}$ over $\mathcal{K}[\mathfrak{m}]$ which is isomorphic to the CM pair $(\mathbb{C}/\mathfrak{m}, 1 \bmod \mathfrak{m})$ over $\mathbb{C}$. This CM pair of modulus $\mathfrak{m}$ over $\mathcal{K}[\mathfrak{m}]$ is unique up to unique isomorphism by the remark above if $\mathcal{O}_{\mathcal{K}}^{\times} \hookrightarrow (\mathcal{O}_{\mathcal{K}}/\mathfrak{m})^{\times}$, which we always assume to be the case. We call this CM pair the *canonical CM pair of modulus $\mathfrak{m}$* .

Moreover, if $K'/\mathcal{K}$ is an extension of $\mathcal{K}$ and (A, α) is a CM-pair of modulus $\mathfrak{m}$ defined over K' , then there exists a unique embedding $\mathcal{K}[\mathfrak{m}] \hookrightarrow K'$ such that (A, α) is the base change of the canonical CM pair over $\mathcal{K}[\mathfrak{m}]$ to K' .

Let $K' \subset \mathcal{K}^{ab}$ be a finite abelian extension of K . Let $\mathfrak{a}$ be a non-zero ideal of $\mathcal{O}_{\mathcal{K}}$, that is divisible only by primes of $\mathcal{K}$ that are unramified in K' . Let $\sigma = (\mathfrak{a}, K'/K) \in \text{Gal}(K'/K)$ be the Artin symbol. Let (A, α) be the canonical CM-pair of modulus $\mathfrak{m}$ over K' ; and let $(A^\sigma, \alpha^\sigma)$ be the base change of (A, α) along $\sigma : K' \rightarrow K'$. Then there is a unique isomorphism of CM pairs

$$\Phi_{\mathfrak{a}} : (A/A[\mathfrak{a}], \alpha \bmod A[\mathfrak{a}]) \xrightarrow{\sim} (A^\sigma, \alpha^\sigma);$$

Here $A[\mathfrak{a}]$ is the $\mathfrak{a}$ -torsion part of A .

Let χ be a Hecke character of infinity type $(1, 0)$. We review here the motive that is attached to χ . We assume that the conductor $\mathfrak{f}$ of χ is small enough so that $\mathcal{O}_{\mathcal{K}}^\times \rightarrow (\mathcal{O}_{\mathcal{K}}/\mathfrak{f})^\times$ is injective. We recall that we denote by $\mathcal{K}_\chi$ the subfield of $\mathbb{C}$ generated by the values of χ . Let (A, α) be the canonical CM-pair over $\mathcal{K}[\mathfrak{f}]$ defined in the previous paragraph. We define

$$\rho_\chi := H_{\text{sing}}^1(A(\mathbb{C}), \mathbb{Q}) \otimes_{\mathcal{K}} \mathcal{K}_\chi;$$

here the the singular cohomology $H^1(A(\mathbb{C}), \mathbb{Q})$ is viewed as a module over $\text{End}(A) \simeq \mathcal{K}$. Denote by $\mathcal{K}_{\chi, p}$ the completion of $\mathcal{K}_\chi$ at the place above p determined by ι_p . Let $\mathcal{O}_{\chi, p}$ be the valuation ring of $\mathcal{K}_{\chi, p}$. By Artin's étale-singular comparison isomorphism,

we have a canonical identification

$$\rho \otimes_{\mathcal{K}_\chi} \mathcal{K}_{\chi,p} = H_{\text{et}}^1(A \otimes_{\mathcal{K}[\mathfrak{f}]} \overline{\mathbb{Q}}, \mathbb{Q}_p) \otimes_{\mathcal{K} \otimes \mathbb{Q}_p} \mathcal{K}_{\chi,p}.$$

This identification endows $\rho_\chi \otimes \mathcal{K}_{\chi,p}$ with a $\mathcal{K}_{\chi,p}$ -linear action of $\text{Gal}(\overline{\mathbb{Q}}/\mathcal{K}[\mathfrak{f}])$. We will denote for simplicity $\rho_{\chi,p} := \rho_\chi \otimes_{\mathcal{K}_\chi} \mathcal{K}_{\chi,p}$. We extend this action to an action of $G_\mathcal{K}$ as follows. Let $\sigma \in \text{Gal}(\overline{\mathbb{Q}}/\mathcal{K})$, we then define the action of σ as the composite:

$$\rho_{\chi,p} \xrightarrow{\sigma} H_{\text{et}}^1(A^\sigma \otimes_{\mathcal{K}[\mathfrak{f}]} \overline{\mathbb{Q}}, \mathbb{Q}_p) \otimes_{\mathcal{K} \otimes \mathbb{Q}_p} \mathcal{K}_{\chi,p} \xrightarrow{\sim^{\Phi_\chi}} \rho_{\chi,p}.$$

The isomorphism Φ_χ is defined as follows. For an ideal $\mathfrak{a}$ of $\mathcal{O}_\mathcal{K}$ coprime to $\mathfrak{f}$ such that the Artin-symbol $(\mathfrak{a}, K[\mathfrak{f}]/\mathcal{K})$ is the image of σ in $\text{Gal}(K[\mathfrak{f}]/\mathcal{K})$. We define $\Phi_\chi := \chi(\mathfrak{a})^{-1} \Phi_\mathfrak{a}^*$, where $\Phi_\mathfrak{a}$ is the CM pair isomorphism of the previous paragraph. This definition of Φ_χ does not depend on the choice of ideal $\mathfrak{a}$ by uniqueness of $\Phi_\mathfrak{a}$.

The action of $G_\mathcal{K}$ on $\rho_{\chi,p}$ is abelian. Since $(\mathcal{K} \otimes \mathbb{R})^\times$ is the connected component of the identity of $\mathbb{A}_\mathcal{K}^\times$, we recall the Artin isomorphism

$$\text{Art}_\mathcal{K} : G_\mathcal{K}^{ab} \simeq \mathcal{K}^\times \backslash \mathbb{A}_\mathcal{K}^{(\infty), \times}.$$

Then an element $\sigma \in G_\mathcal{K}^{ab}$ whose image in $\mathcal{K}^\times \backslash \mathbb{A}_\mathcal{K}^{(\infty), \times}$ is a acts on $\rho_{\chi,p}$ as multiplication by $a_p \chi(a)^{-1}$.

For a Hecke character of infinity type $(1, 0)$ and conductor $\mathfrak{N}p^r$ coprime to $\Delta_\mathcal{K}$

with $\mathfrak{N} \subseteq \mathcal{O}_K$ an ideal with $N_{K/\mathbb{Q}}(\mathfrak{N}) = N$, we let

$$\vartheta_\chi := \theta_{\tilde{\phi}_{\chi,n}}$$

be the weight 2 newform defined by the theta series of Paragraph 8.1.2.3. Let

$$T(\chi) := H^1(X_1(\Delta_K N p^{2r}), \mathcal{O}_{\chi,p})[\vartheta_\chi] \subseteq H_{\text{et}}^1(X_1(\Delta_K N p^{2r}), \mathcal{O}_{\chi,p}) \quad (8.10)$$

be the subspace of the étale cohomology of the modular curve cut out by the Hecke eigenvalues of ϑ_χ . By newform theory, this is a free rank 2 $\mathcal{O}_{\chi,p}$ -module with a continuous action of $G_{\mathbb{Q}}$. We define $V(\chi) := T(\chi) \otimes_{\mathcal{O}_{\chi,p}} \mathcal{K}_{\chi,p}$. Then, having geometrically normalized Art_K , we have

$$V(\chi) \simeq \text{Ind}_{G_K}^{G_{\mathbb{Q}}}(\rho_{\chi,p}).$$

Induction from a principal ideal character We now construct certain Hecke character of K of infinity type $(1, 0)$; so $k = 1$ in all of what will follow.

Let $\mathfrak{N} \subseteq \mathcal{O}_K$ be an integral ideal and $r \geq 1$ be an integer. We denote by $\mathcal{F}(\mathfrak{N}p^r)$ be the representation of $(\mathcal{O}_K/\mathfrak{N}p^r)^\times$ defined by the set of K -valued functions on $(\mathcal{O}_K/\mathfrak{N}p^r)^\times$, i.e. the regular representation of $(\mathcal{O}_K/\mathfrak{N}p^r)^\times$. We view also $\mathcal{F}(\mathfrak{N}p^r)$ as a $K^{\mathfrak{N}p^r}$ representation by inflation $K^{\mathfrak{N}p^r} \rightarrow (\mathcal{O}_K/\mathfrak{N}p^r)^\times$. Consider also κ the tautological representation of $K^{\mathfrak{N}p^r}$ acting by translation on K . Denote by $\mathcal{F}(\mathfrak{N}p^r)^-$ the subrepresentation of $\mathcal{F}(\mathfrak{N}p^r)$ consisting of odd functions – if $\mathcal{O}_K^\times \neq \{\pm 1\}$ we are calling here a K -valued function f odd if for all $u \in \mathcal{O}_K^\times$, $f(ug) = u^{-1}f(g)$. Then $\mathcal{F}(\mathfrak{N}p^r)^- \otimes_K \kappa$ descends to a representation of the group $P_{\mathfrak{N}p^r}$ of principal ideals

prime to $\mathfrak{N}p^r$. We now define

$$\mathcal{I}(\mathfrak{N}p^r) := \text{Ind}_{P_{p^r\mathfrak{N}}}^{I_{\mathfrak{N}p^r}} \left(\mathcal{F}(\mathfrak{N}p^r)^- \otimes \kappa \right). \quad (8.11)$$

Then $\mathcal{I}(\mathfrak{N}p^r) \otimes_{\mathcal{K}} F$ decomposes, for a sufficiently large abelian extension F of $\mathcal{K}$, as the sum of all Hecke characters of infinity type $(1, 0)$ and conductor dividing $\mathfrak{N}p^r$.

We also define some subrepresentations of $\widehat{\mathcal{I}}(\mathfrak{N}p^r)$. Let $\mathfrak{n}|\mathfrak{N}p^r$ and let $\mu^{(\infty)} : (\mathcal{O}_{\mathcal{K}}/\mathfrak{n})^\times \rightarrow \overline{\mathbb{Q}}^\times$ denote an odd character, i.e. $\mu \in \mathcal{F}(\mathfrak{N}p^r)^- \otimes \overline{\mathbb{Q}}$. Let $\mathcal{K}_\mu$ be the field generated by $\mathcal{K}$ and the values of $\mu^{(\infty)}$. Then $\mu^{(\infty)}$ defines a character

$$\mu : P_{\mathfrak{n}} \rightarrow \mathcal{K}_\mu^\times, \quad (\alpha) \mapsto \alpha \mu^{(\infty)}(\alpha)$$

of the principal ideals of $\mathcal{O}_{\mathcal{K}}$ coprime to $\mathfrak{n}$. We consider the representation $I(\mu)$ of $I_{\mathfrak{n}}$ defined by

$$I(\mu) := \text{Ind}_{P_{\mathfrak{n}}}^{I_{\mathfrak{n}}}(\mu);$$

as a model for the induced representation we view $I(\mu)$ as the set of functions $I_{\mathfrak{n}} \rightarrow \mathcal{K}_\mu$ that transform under translation by $P_{\mathfrak{n}}$ by μ . Then $I(\mu)$ is a $\mathcal{K}_\mu$ vector space of dimension $h_{\mathcal{K}} = \#\text{Cl}_{\mathcal{K}}$. If $F/\mathcal{K}_\mu$ is an extension containing enough roots of unity, then $I(\mu) \otimes_{\mathcal{K}_\mu} F$ splits as the sum of one-dimensional representations defined by characters $\lambda : I_{\mathfrak{n}} \rightarrow F^\times$. Having fixed λ_0 one such character, all the other characters are given by $\lambda_0 \cdot \xi$ where ξ is a character of the ideal class group $\text{Cl}_{\mathcal{K}} = I_1/P_1 \simeq I_{\mathfrak{n}}/P_{\mathfrak{n}}$. If χ is a Hecke character of infinity type $(1, 0)$ we write $\chi \leq \mathcal{I}(\mathfrak{N}p^r)$ if χ appears in $\mathcal{I}(\mathfrak{N}p^r)$, that is if the conductor of χ divides $\mathfrak{N}p^r$. Let $\mathcal{K}_{\chi,p}$ be the field of values

of the p -adic character $\widehat{\chi}$ that is attached to χ as in section 8.2.1.1. We recall the notation of $\rho_\chi : G_{\mathcal{K}}^{\text{ab}} \rightarrow \mathcal{K}_{\chi,p}$ of the p -adic Galois representation attached to $\widehat{\chi}$, i.e. $\rho_\chi = \widehat{\chi} \circ \text{Art}_{\mathcal{K}}$.

Let $S_{\mathfrak{N}p^r}$ be the set of primes dividing $\mathfrak{N}p^r$ and let $\mathcal{K}[S_{\mathfrak{N}p^r}]$ be the maximal extension of $\mathcal{K}$ unramified away from $S_{\mathfrak{N}p^r}$. Then by the classical global Artin homomorphism

$$I_{\mathfrak{N}p^r} \longrightarrow \text{Gal}(\mathcal{K}[S_{\mathfrak{N}p^r}]/\mathcal{K})^{ab}$$

matching prime ideals to geometric Frobenius elements, we can view through ι_p the $I_{\mathfrak{N}p^r}$ -representation $\mathcal{I}(\mathfrak{N}p^r)$ as a continuous $\text{Gal}(\mathcal{K}[S_{\mathfrak{N}p^r}]/\mathcal{K})^{ab}$ representation with coefficients in $E = \mathcal{K}_p$ which we denote by $\widehat{\mathcal{I}}(\mathfrak{N}p^r)$. Concretely,

$$\widehat{\mathcal{I}}(\mathfrak{N}p^r) := \text{Ind}_{P_{p^r\mathfrak{N}}}^{I_{p^r\mathfrak{N}}} \left(\mathcal{F}(\mathfrak{N}p^r)^- \otimes \iota_p \circ \kappa \right) \otimes_{\mathcal{K}_p} \kappa_p^{-1},$$

where here κ_p is the character of $G_{\mathcal{K}}^{ab}$ defined by class field theory that sends a finite idèle (a_v) to its p -th component $a_p \in \mathcal{K}_p$. We note that after extending scalar to an extension $F_p/\mathcal{K}_p$ $\widehat{\mathcal{I}}(\mathfrak{N}p^r)$ splits as the direct sum of one dimensional representations ρ_χ where $\rho_\chi = \widehat{\chi} \circ \text{Art}_{\mathcal{K}}$ is as in section 8.2.1 and $\chi \leq \mathcal{I}(\mathfrak{N}p^r)$ is as above a Hecke character of infinity type $(1, 0)$

The induction in the cohomology of modular curves In this paragraph we study a certain part of the cohomology of the modular curves in terms of the representation $\widehat{\mathcal{I}}(\mathfrak{N}p^r)$ defined in the previous paragraph. Recall that $E = \mathcal{K}_p$ denotes the completion of $\mathcal{K}$ at the inert prime p — so that $E/\mathbb{Q}_p$ is the unique quadratic

unramified extension of $\mathbb{Q}_p$.

Fix an ideal $\mathfrak{N} \subseteq \mathcal{O}_K$ coprime to p and define $N := N_{K/\mathbb{Q}}(\mathfrak{N})$.

We now consider the étale cohomology groups $H_{\text{et}}^1(X_1(Np^{2r})/\overline{\mathbb{Q}}, \mathbb{Z}_p)$ of the modular curves $X_1(Np^{2r})$. We denote by π_n the degeneracy map

$$\pi_{r,2} : X_1(Np^{2(r+1)}) \rightarrow X_1(Np^{2r})$$

induced by $z \mapsto p^2 z$ on the upper half-plane. We denote by π_r^* the induced pull-back map in cohomology:

$$\pi_r^* : H_{\text{et}}^1(X_1(Np^{2r})/\overline{\mathbb{Q}}, \mathbb{Z}_p) \longrightarrow H_{\text{et}}^1(X_1(Np^{2(r+1)})/\overline{\mathbb{Q}}, \mathbb{Z}_p).$$

Note that there are two other degeneracy maps $\pi_{r,0}$ and $\pi_{r,1}$ given respectively by $z \mapsto z$ and $z \mapsto pz$ on the upper half plane. For a locally constant étale sheaf Λ . We define $H_{\text{et}}^1(X_1(Np^{2(r+1)})/\overline{\mathbb{Q}}, \Lambda)^{\text{old}}$ to be the subspace spanned by the image of the degeneracy maps $\pi_{r,0}$, $\pi_{r,1}$ and π_r , that is the subspace spanned by $\pi_{r,\bullet}^* H_{\text{et}}^1(X_1(Np^{2r})/\overline{\mathbb{Q}}, \Lambda)$ for $\bullet \in \{0, 1, 2\}$. We define $H_{\text{et}}^1(X_1(Np^{2(r+1)})/\overline{\mathbb{Q}}, \Lambda)^{\text{new}}$ to be the subspace of $H_{\text{et}}^1(X_1(Np^{2(r+1)})/\overline{\mathbb{Q}}, \Lambda)$ that is the annihilator under Poincaré duality of $H_{\text{et}}^1(X_1(Np^{2(r+1)})/\overline{\mathbb{Q}}, \Lambda)^{\text{old}}$.

8.2.1.2 Definition. We define

$$H_{\text{et}}^1(X_1(Np^{2r})/\overline{\mathbb{Q}}, \Lambda)_{CM}^{\text{new}} \subset H_{\text{et}}^1(X_1(Np^{2r})/\overline{\mathbb{Q}}, \Lambda)^{\text{new}},$$

to be the subspace of $H_{\text{et}}^1(X_1(Np^{2r})/\overline{\mathbb{Q}}, \Lambda)^{\text{new}}$ cut out by the kernel of the usual

Hecke operators T_ℓ of the spherical Hecke algebra, for ℓ a rational prime inert in $\mathcal{K}$.

8.2.1.3 Definition. We define the *p-admissible new CM space*

$$H_{ncm}(N, r) \subset H_{\text{et}}^1\left(X_1(\Delta_{\mathcal{K}} N p^{2r})/\overline{\mathbb{Q}}, \mathbb{Z}_p(1)\right)$$

inductively to be the subspace generated by $\pi_{r-1,2}^* H_{ncm}(N; r-1)$ (if $r > 1$) and by $H_{\text{et}}^1\left(X_1(\Delta_{\mathcal{K}} N p^{2r})/\overline{\mathbb{Q}}, \mathbb{Z}_p\right)_{CM}^{new}$.

8.2.1.4 Definition. We define the *total p-admissible CM space*

$$H_{tcm}(N, r) \subset H_{\text{et}}^1\left(X_1(\Delta_{\mathcal{K}} N p^{2r})/\overline{\mathbb{Q}}, \mathbb{Z}_p(1)\right)$$

defined as the space spanned by the image of $H_{ncm}(N/d, r)$, for $d|N$ under all degeneracy maps $z \mapsto d'z$ on the upper half plane for $d'|d$.

We let $J_1(Np^{2r}) = \text{Pic}_{X_1(Np^{2r})/\mathbb{Q}}^0$ denote the Jacobian of $X_1(Np^{2r})$. The choice of the (in our model) rational cusp ∞ embeds $X_1(Np^{2r}) \hookrightarrow J_1(Np^{2r})$ a point P being sent to the line bundle defined by the divisor $(P) - (\infty)$. This embedding defines isomorphisms:

$$H_{\text{et}}^1\left(J_1(Np^{2r})/\overline{\mathbb{Q}}, \mathbb{Q}_p(1)\right) \simeq H_{\text{et}}^1\left(X_1(Np^{2r})/\overline{\mathbb{Q}}, \mathbb{Q}_p(1)\right)$$

and on cotangent spaces

$$H^0\left(J_1(Np^{2r}), \Omega_{J_1(Np^{2r})/\mathbb{Q}}^1\right) \simeq H^0\left(X_1(Np^{2r}), \Omega_{X_1(Np^{2r})/\mathbb{Q}}^1\right).$$

We define also $J_{ncm}(N, r)$ and $J_{tcm}(N, r)$ the sub-abelian varieties of $J_1(\Delta_{\mathcal{K}} N p^{2r})$ cut out by as before by the kernel of the Hecke operators T_ℓ for ℓ inert in $\mathcal{K}$ and the images of degeneracy maps as in Definition 8.2.1.3 and 8.2.1.4, so that we have isomorphism

$$H_{\text{et}}^1(J_{ncm}(N, r)/\overline{\mathbb{Q}}, \mathbb{Q}_p) \simeq H_{ncm}(N, r) \otimes \mathbb{Q}_p$$

and

$$H_{\text{et}}^1(J_{tcm}(N; r)/\overline{\mathbb{Q}}, \mathbb{Q}_p) \simeq H_{tcm}(N; r) \otimes \mathbb{Q}_p$$

8.2.1.5 Lemma. *There is a $G_{\mathcal{K}}$ -equivariant decomposition:*

$$H_{\text{et}}^1(X_1(\Delta_{\mathcal{K}} p^{2r})/\overline{\mathbb{Q}}, E) \supseteq H_{tcm}(1; r) \otimes E \simeq \widehat{\mathcal{I}}(p^r) \oplus \widehat{\mathcal{I}}(p^r)^\tau.$$

We recall that $\mathcal{I}(p^r) = \text{Ind}_{P_{p^r}}^{I_{p^r}}(\mathcal{F}(p^r)^- \otimes \kappa)$ is viewed as a Galois representation as explained in section 8.2.1.1 and $\widehat{\mathcal{I}}(p^r)^\tau$ is the twist of $\widehat{\mathcal{I}}(p^r)$ by the outer automorphism of $G_{\mathcal{K}}$ defined by the unique non trivial element $\tau \in \text{Gal}(\mathcal{K}/\mathbb{Q})$.

Proof. First remark that if F is an extension of $E = \mathcal{K}_p$ containing enough roots of unity, $H_{\text{et}}^1(X_1(\Delta_{\mathcal{K}} p^{2r})/\overline{\mathbb{Q}}, E)_{CM}^{new} \otimes_E F$ is spanned exactly the Galois representations $\text{Ind}_{G_{\mathcal{K}}}^{G_{\mathbb{Q}}}(\rho_\lambda)$ where λ is a Hecke character of conductor p^r – this is the p -adic Galois representation attached by Shimura to the theta series coming from λ , whose q -expansion at the identity is $\sum_{\mathfrak{a} \subseteq \mathcal{O}_{\mathcal{K}}} \lambda(\mathfrak{a}) q^{N_{\mathcal{K}/\mathbb{Q}}(\mathfrak{a})}$. Note that $\text{Ind}_{G_{\mathcal{K}}}^{G_{\mathbb{Q}}}(\rho_\lambda)|_{G_{\mathcal{K}}} = \rho_\lambda \oplus \rho_\lambda^\tau$ by Mackey theory. Then since degeneracy maps are injective and Galois equivariant, we obtain the claimed decomposition after tensoring with F as $\widehat{\mathcal{I}}(p^r)$ decomposes in the same manner (we are using tacitly multiplicity one for $\mathbf{GL}_2$). Therefore by

faithfully flat descent the same is true over E . □

We now want to decompose rationally the CM part of cohomology with non-trivial tame part. In order to do so we need to cut down the space $H_{tcm}(N; r)$.

8.2.1.6 Definition (Admissible tame level). We say the tame level N of $H_{tcm}(N; r)$ is admissible when: the integer N is prime to Δ_K and divisible at most by the square of a prime and additionally we require that if $\ell|N$ and ℓ is inert in K then $\ell^2|N$.

8.2.1.7 Definition (ℓ -depletion). For any integer N , We denote by $c \mapsto c^{[\ell]}$ let ℓ be a prime not dividing N . The map

$$H_{\text{et}}^1(X_1(N)/\overline{\mathbb{Q}}, \mathbb{Z}_p) \rightarrow H_{\text{et}}^1(X_1(N\ell^2)/\overline{\mathbb{Q}}, \mathbb{Z}_p)$$

induced by the ℓ -depletion map

$$\pi_{\ell^2,0}^* - \frac{1}{\ell}\pi_{\ell^2,1}^* \circ T_\ell + \frac{1}{\ell}\pi_{\ell^2,2}^* \circ \langle \ell \rangle$$

Here the notation is as follows:

- The pullback maps

$$\pi_{\ell^2,i}^* : H_{\text{et}}^1(X_1(N), \mathbb{Z}_p) \rightarrow H_{\text{et}}^1(X_1(N\ell^2), \mathbb{Z}_p)$$

are induced by $z \mapsto \ell^i z$ on the upper half plane.

- The operators T_ℓ and $\langle \ell \rangle$ are respectively the usual Hecke operator of the spherical Hecke algebra acting on $H_{\text{et}}^1(X_1(N), \mathbb{Z}_p)$ and the usual diamond operators

indexed by elements $d \in (\mathbb{Z}/N\mathbb{Z})^\times$.

Take F a large enough extension of E so that one has the decomposition

$$H_{\text{et}}^1\left(X_1(\Delta_{\mathcal{K}}Np^{2r})/\overline{\mathbb{Q}}, \mathcal{K}_p\right)_{CM}^{new} \otimes_{\mathcal{K}_p} F \simeq \bigoplus_{N_{\mathcal{K}/\mathbb{Q}}(\mathfrak{N})=N} \bigoplus_{\lambda: \mathfrak{f}_{\lambda}=\mathfrak{N}p^r} \rho_{\lambda} \oplus \rho_{\lambda}^{\tau}. \quad (8.12)$$

Here the representation ρ_{λ} are viewed as one dimensional over F . For simplicity we let

$$V_{\lambda} := \rho_{\lambda} \oplus \rho_{\lambda}^{\tau},$$

for λ a Hecke character appearing in the decomposition above.

8.2.1.8 Definition ($\mathfrak{l}$ -stabilization). Let ℓ be a prime that splits in $\mathcal{K}$. And choose a prime $\mathfrak{l}$ of $\mathcal{O}_{\mathcal{K}}$ above ℓ . Let F be a field such that the decomposition of Equation 8.12 holds and let $\mathcal{O}_F$ be its valuation ring. We define the subspace $H_{tcm}(N\ell; r)^{\mathfrak{l}} \otimes_{\mathbb{Z}_p} \mathcal{O}_F \subseteq H_{tcm}(N\ell; r) \otimes_{\mathbb{Z}_p} \mathcal{O}_F$ (the notation is justified by Lemma 8.2.1.11) as the image under the composition

$$H_{tcm}(N; r) \otimes F \oplus H_{tcm}(N; r) \otimes F \xrightarrow{S_{\mathfrak{l}}} H_{tcm}(N\ell; r) \otimes F.$$

Here $S_{\mathfrak{l}}$ is the operator defined as follows. On V_{λ} , where λ ranges over all Hecke characters of infinity type $(1, 0)$ appearing in the decomposition of Equation 8.12, we have

$$S_{\mathfrak{l}} = \bigoplus_{\lambda} S_{\mathfrak{l}}^{\lambda}, \quad S_{\mathfrak{l}}^{\lambda} : V_{\lambda} \rightarrow H_{tcm}(N\ell; r) \otimes \mathcal{O}_F, \quad v \mapsto \pi_{\ell, 0}^* - \frac{1}{\ell} \lambda(\bar{\mathfrak{l}}) \pi_{\ell, 1}^* v.$$

8.2.1.9 Remark. With notation just as in the definition above, note that $S_{\mathfrak{l}}^{\lambda} S_{\mathfrak{l}}^{\lambda} v = S_{\mathfrak{l}}^{\lambda} S_{\mathfrak{l}}^{\lambda} v = v^{[l]}$. So $S_{\mathfrak{l}}^{\lambda}$ can be viewed as a partial version of ℓ -depletion. This remark motivates the following definition.

8.2.1.10 Definition. Let $\mathfrak{N}$ be an ideal prime to p such that $N = N_{\mathcal{K}/\mathbb{Q}}(\mathfrak{N})$ is an admissible tame level. Let $M \geq 1$ be an integer. We define

$$S_{\mathfrak{N}} : H_{tcm}(M, r) \longrightarrow H_{tcm}(NM; r)$$

as the composition $S_{\mathfrak{N}} = \prod_{\mathfrak{q}|\mathfrak{N}} S_{\mathfrak{q}}$, where, by convention, if $\mathfrak{q}$ lies above an inert rational prime q , $S_{\mathfrak{q}}$ denotes the q -depletion map (see Definition 8.2.1.7).

8.2.1.11 Lemma. *There exists a sub- $\mathbb{Z}_p$ -module $H_{tcm}(Nl; r)^{\mathfrak{l}} \subseteq H_{tcm}(Nl; r)$, of rank $\dim_{\mathbb{Q}_p}(H_{ncm}(N; r) \otimes \mathbb{Q}_p)$, such that image of $S_{\mathfrak{l}}$ in $H_{tcm}(Nl; r) \otimes F$ appearing in Definition 8.2.1.8 is $H_{tcm}(Nl; r)^{\mathfrak{l}} \otimes F$.*

Proof. We can assume that F is Galois over $\mathbb{Q}_p$, up to enlarging F . We let the Galois group of F act on the coefficients of $H_{\text{et}}^1(X_1(N)/\overline{\mathbb{Q}}, F)$, and $H_{\text{et}}^1(X_1(N\ell)/\overline{\mathbb{Q}}, F)$. Denote for simplicity $H = H_{tcm}(N\ell; r) \otimes \mathbb{Q}_p$ and the image of $S_{\mathfrak{l}}$ in $H \otimes F$ by W_F . Note first that $\sigma(S_{\mathfrak{l}}^{\lambda}) = S_{\mathfrak{l}}^{\lambda^{\sigma}}$, where λ^{σ} is the Galois character of $G_{\mathcal{K}}^{ab}$ given by the composition

$$\lambda^{\sigma} : G_{\mathcal{K}}^{ab} \xrightarrow{\lambda} F^{\times} \xrightarrow{\sigma} F^{\times}.$$

Noting that λ^{σ} also appears in the decomposition of Equation 8.12, and that $\sigma(V_{\lambda}) = V_{\lambda^{\sigma}}$, we see that σ preserves the space W_F . By Galois descent, there exists a vector space W over $\mathbb{Q}_p$ such that $W \otimes_{\mathbb{Q}_p} F = W_F$. We then define $H_{tcm}(N\ell; r)^{\mathfrak{l}}$ to be the intersection of W with the image of $H_{ncm}(N; r)^{\oplus 2} \otimes \mathcal{O}_F$ under $S_{\mathfrak{l}}$. $\square$

Let N be an integer and ℓ be a prime that splits completely in $\mathcal{K} : (\ell) = \mathfrak{l}\bar{\mathfrak{l}}$. Then the new p -admissible subspace $H_{ncm}(N\ell, r)$ decomposes $G_{\mathcal{K}}$ -equivariantly as a direct sum

$$H_{ncm}(N\ell, r) = H_{ncm}(N\ell, r)[\mathfrak{l}] \oplus H_{ncm}(N\ell, r)[\bar{\mathfrak{l}}].$$

Here $H_{ncm}(N\ell, r)[\mathfrak{l}]$ denotes the space where the action of inertia at $\mathfrak{l}$ acts trivially. Indeed, any character appearing in $H_{\text{et}}^1(X_1(N\ell), \mathbb{Z}_p(1))_{CM}^{new} \otimes F$ is of conductor $\mathfrak{f}$ such that $N_{\mathcal{K}/\mathbb{Q}}(\mathfrak{f}) = N\ell$, and so it is ramified either at $\mathfrak{l}$ or at $\bar{\mathfrak{l}}$.

We are now ready to define the space where the theta classes that we will arrange in an Euler System live.

8.2.1.12 Definition (Admissible CM space). Let N be an admissible tame level (see Definition 8.2.1.6). Let $S_{spl} = \{\mathfrak{l} \subset \mathcal{O}_{\mathcal{K}}\}$ be a choice of a prime above a split rational prime ℓ . Denote by $\mathfrak{N}$ the unique ideal of $\mathcal{O}_{\mathcal{K}}$ divisible only by primes in S_{spl} such that $N_{\mathcal{K}/\mathbb{Q}}(\mathfrak{N}) = N$. We define a subspace $H_{acm}(N; r)$

$$H_{ncm}(N, r) \subseteq H_{acm}(N; r) \subseteq H_{tcm}(N; r)$$

as follows.

- If $N = 1$, $H_{acm}(1; r) = H_{ncm}(1; r)$.
- For each admissible divisor $d|N$ (that is, d is an admissible tame level), $H_{acm}(N; r)$ contains exactly one copy of $H_{ncm}(N/d; r)$, as the image of the map $S_{\mathfrak{D}}$, where $\mathfrak{D}$ is the unique $\mathcal{O}_{\mathcal{K}}$ -ideal divisible only by primes in S_{spl} such that $N_{\mathcal{K}/\mathbb{Q}}(\mathfrak{D}) = d$.

That is we have

$$H_{acm}(N, r) = \bigoplus_{d|N} S_{\mathfrak{D}} H_{ncm}(N/d, r) \subseteq H_{tcm}(N, r),$$

where the direct sum ranges only over admissible divisors of N .

8.2.1.13 Remark. Stated imprecisely, $H_{acm}(N; r)$ is the smallest space containing the p -admissible new-forms $H_{ncm}(N; r)$, the depletions and fixed ℓ -stabilizations of old forms. Note that the space $H_{acm}(N, r)$ depends on the choice of S_{spl} but we omit this choice for convenience.

We now want to give a similar decomposition of $H_{acm}(N, r)$ to the one that appears in Lemma 8.2.1.5. Note that the choice of S_{spl} determines uniquely an ideal $\mathfrak{N}$ such that $N_{\mathcal{K}/\mathbb{Q}}(\mathfrak{N}) = N$ for each admissible tame level N as S_{spl} defines a bijection between the primes of $\mathbb{Q}$ and the primes of $\mathcal{K}$. By construction and by the same reasoning as Proposition 8.2.1.5, we obtain the following proposition.

8.2.1.14 Proposition. *There is a $G_{\mathcal{K}}$ -equivariant decomposition*

$$H_{acm}(N; r) \simeq \widehat{\mathcal{I}}(\mathfrak{N}p^r) \oplus \widehat{\mathcal{I}}(\mathfrak{N}p^r)^{\tau},$$

where $\mathfrak{N} \subseteq \mathcal{O}_{\mathcal{K}}$ is the unique ideal divisible only by primes in S_{spl} such that $N_{\mathcal{K}/\mathbb{Q}}(\mathfrak{N}) = N$.

8.2.1.15 A rational basis of the CM part of cohomology

In this section, we fix a natural basis of the representation $\mathcal{I}(\mathfrak{N}p^r)$ and its p -adic analog $\widehat{\mathcal{I}}(\mathfrak{N}p^r)$ and corresponding étale classes in $H_{acm}(N, r)$. We assume that $r \geq 1$ and that $\mathfrak{N}p^r$ is large enough so that $\mathcal{O}_K^\times$ injects into $(\mathcal{O}_K/\mathfrak{N}p^r)^\times$. For $\alpha \in (\mathcal{O}_K/\mathfrak{N}p^r)^\times$ denote by $f_\alpha \in \mathcal{F}(\mathfrak{N}p^r)^-$ the function $f_\alpha : (\mathcal{O}_K/\mathfrak{N}p^r)^\times \rightarrow \mathcal{K}$ defined as

$$f_\alpha(x) = \begin{cases} u & \text{if } x = u\alpha \text{ for } u \in \mathcal{O}_K^\times \\ 0 & \text{otherwise.} \end{cases}$$

When $\mathcal{O}_K^\times = \{\pm 1\}$, we remark that $f_\alpha = \delta_\alpha - \delta_{-\alpha}$, where $\delta_\bullet$ denotes the indicator function of $\{\bullet\}$. Note that if α and α' are in the same $\mathcal{O}_K^\times$ -orbit then f_α and $f_{\alpha'}$ are proportional by a unit. More precisely we have $f_{u\alpha} = u^{-1}f_\alpha$. Choose once and for all a set of representatives $\{\alpha_i\}$ such that $(\mathcal{O}_K/\mathfrak{N}p^r)^\times = \coprod_i \mathcal{O}_K^\times \alpha_i$, where $i = 1, \dots, M$ where M is the order of the quotient.

Recall that there is an exact sequence:

$$1 \rightarrow (\mathcal{O}_K/\mathfrak{N}p^r)^\times / \mathcal{O}_K^\times \rightarrow \text{Cl}_{\mathfrak{N}p^r} \rightarrow \text{Cl}_K \rightarrow 1$$

If $\mathfrak{a}$ is an ideal coprime to $\mathfrak{N}p^r$, i.e. an element of $I_{\mathfrak{N}p^r}$, we denote by $[\mathfrak{a}]$ its image in Cl_K and $[\mathfrak{a}]_{\mathfrak{N}p^r}$ its image in $\text{Cl}_{\mathfrak{N}p^r}$.

Choose representatives $\mathfrak{a}_i \in I_{\mathfrak{N}p^r}$ of the Class group of $\mathcal{K}$, $i = 1, \dots, h_K$: these can be seen as coset representatives of $P_{\mathfrak{N}p^r} \subseteq I_{\mathfrak{N}p^r}$. We always take $\mathfrak{a}_1 = \mathcal{O}_K$. Having made such choices, given a class $\mathfrak{c} \in \text{Cl}_{\mathfrak{N}p^r}$, we write $\omega_{\mathfrak{c}} \in \mathcal{I}(\mathfrak{N}p^r)$ for the element

$\alpha_j(\mathfrak{a}_i \otimes f_{\alpha_j})$. Here the notation is as follows:

1. The ideal $\mathfrak{a}_i$ is the fixed representative such that $[\mathfrak{a}_i]_1 = [\mathfrak{c}]_1$;
2. The element $\alpha_j \in \mathcal{K}$ is such that $\mathfrak{a}_i^{-1}\mathfrak{a} = (\alpha)$ and $\alpha\mathcal{O}_{\mathcal{K}}^\times = \alpha_j\mathcal{O}_{\mathcal{K}}^\times$ in $(\mathcal{O}_{\mathcal{K}}/\mathfrak{N}p^r)^\times$, where $\mathfrak{a}$ is any lift of $\mathfrak{c}$ to $I_{\mathfrak{N}p^r}$
3. The tensor product $\mathfrak{a}_i \otimes f_{\alpha_j}$ is of course taken with respect to the $P_{\mathfrak{N}p^r}$ module structure.

By definition of induction we have the following Lemma.

8.2.1.16 Lemma. *Given the choices of representatives (α_j) of $(\mathcal{O}_{\mathcal{K}}/\mathfrak{N}p^r)^\times/\mathcal{O}_{\mathcal{K}}^\times$ and $(\mathfrak{a}_i)$ of $\text{Cl}_{\mathcal{K}}$, the elements $\{\omega_{\mathfrak{c}} : \mathfrak{c} \in \text{Cl}_{\mathfrak{N}p^r}\}$ is a basis of $\mathcal{I}(\mathfrak{N}p^r)$.*

8.2.1.17 Definition. We define $\omega_{\mathfrak{c}}^{\text{et}}$ and $\bar{\omega}_{\mathfrak{c}}^{\text{et}}$ to be the corresponding basis of $H_{acm}(N, r)$ under the decomposition of Proposition 8.2.1.14, where $\bar{\omega}_{\mathfrak{c}}^{\text{et}}$ is the image of $\omega_{\mathfrak{c}}^{\text{et}}$ by the unique non-trivial automorphism of $\mathcal{K}$.

We now define, in terms of the $\omega_{\mathfrak{c}}$, the classes in $\mathcal{I}(\mathfrak{N}p^r)$ that are acted on by Hecke characters of infinity type $(1, 0)$ after a suitably large extension of scalars. Given an ideal $\mathfrak{a} \in I_{\mathfrak{N}p^r}$, we define $\omega_{\mathfrak{a}} \in \mathcal{I}(\mathfrak{N}p^r)$ to be the element $\alpha(\mathfrak{a}_i \otimes f_{\alpha})$, where α is such that $\mathfrak{a}_i^{-1}\mathfrak{a} = (\alpha)$. Note that $\omega_{\mathfrak{a}}$ is well defined because of the relation $f_{u\alpha} = u^{-1}f_{\alpha}$ for all units $u \in \mathcal{O}_{\mathcal{K}}^\times$. Now let χ be a Hecke character appearing in $\mathcal{I}(\mathfrak{N}p^r)$. The element $\chi(\mathfrak{a})^{-1}\omega_{\mathfrak{a}} \in \mathcal{I}(\mathfrak{N}p^r)$ depends only on the class $[\mathfrak{a}]_{\mathfrak{N}p^r}$. Indeed, let $\mathfrak{a} = \mathfrak{b}(\beta)$ where $\beta \equiv 1 \pmod{p^r\mathfrak{N}}$. and consider $\mathfrak{a}_i$ such that $[\mathfrak{a}_i]_1 = [\mathfrak{b}]_1 = [\mathfrak{a}]_1$. then

$$\omega_{\mathfrak{a}} = \alpha(\mathfrak{a}_i \otimes f_{\alpha}) = \beta^{-1}\omega_{\mathfrak{b}}.$$

On the other hand

$$\chi(\mathfrak{a}) = \chi(\mathfrak{b})\beta,$$

as $\beta \equiv 1 \pmod{\mathfrak{N}p^r}$ and χ is of conductor dividing $\mathfrak{N}p^r$. This finishes proving the claim. Moreover, by construction we have for any ideals $\mathfrak{a}, \mathfrak{b} \in I_{\mathfrak{N}p^r}$,

$$\mathfrak{b} \cdot \omega_{\mathfrak{a}} = \omega_{\mathfrak{ab}}.$$

Therefore, the element

$$\omega_{\chi} := \sum_{\mathfrak{c} \in \text{Cl}_{\mathfrak{N}p^r}} \chi(\mathfrak{c})^{-1} \omega_{\mathfrak{c}} \in \mathcal{I}(\mathfrak{N}p^r),$$

where the sum runs over any set of representatives in $I_{\mathfrak{N}p^r}$ of $\text{Cl}_{\mathfrak{N}p^r}$ (for example $\mathfrak{a}_i(\alpha_j)$ to have compatible notations) is acted on by $I_{\mathfrak{N}p^r}$ by the character χ . As before we denote by $\omega_{\chi}^{\text{et}} \in H_{\text{acm}}(N, r)$ the corresponding class and by $\bar{\omega}_{\chi}^{\text{et}}$ the image of $\omega_{\chi}^{\text{et}}$ under $\tau \in \text{Gal}(\mathcal{K}/\mathbb{Q})$. By class field theory, this class is acted on by $G_{\mathcal{K}}$ by the character $\hat{\chi} \circ \text{Art}_{\mathcal{K}}$. We summarize the discussion above in the following Lemma.

8.2.1.18 Lemma. *Let F/E be an extension that contains all the values of the p -adic Hecke characters $\hat{\chi}$ of conductor dividing $\mathfrak{N}p^r$. Then we have the decomposition*

$$H_{\text{acm}}(N, r) \otimes_E F = \bigoplus_{\chi \leq \mathcal{I}(\mathfrak{N}p^r)} F\omega_{\chi}^{\text{et}} \oplus F\bar{\omega}_{\chi}^{\text{et}},$$

where

$$\omega_{\chi}^{\text{et}} = \sum_{\mathfrak{c} \in \text{Cl}_{\mathfrak{N}p^r}} \chi(\mathfrak{c})^{-1} \omega_{\mathfrak{c}}^{\text{et}}.$$

The Galois group of $\mathcal{K}$ acts on ω_χ^{et} by $\rho_\chi = \widehat{\chi} \circ \text{Art}_\mathcal{K}$.

We now also want to retrieve $\omega_\mathfrak{c}^{\text{et}}$ given the ω_χ^{et} . Fix a Hecke character λ_0 appearing in $\mathcal{I}(\mathfrak{N}p^r)$. Then $\lambda_0\chi^{-1}$ is a finite order Hecke character of conductor dividing $\mathfrak{N}p^r$ for all $\chi \leq \mathcal{I}(\mathfrak{N}p^r)$. Therefore all Hecke character appearing in $\mathcal{I}(\mathfrak{N}p^r)$ are of the form $\lambda_0\xi$ where ξ is a finite order character of $\text{Cl}_{\mathfrak{N}p^r}$. By counting, all such finite order characters appear exactly once. Therefore, having fixed λ_0 , there is a bijection $\xi \mapsto \lambda_0\xi$ between finite order Hecke characters ξ of $\text{Cl}_{\mathfrak{N}p^r}$ and Hecke characters appearing in $\mathcal{I}(\mathfrak{N}p^r)$. Invoking the Fourier inversion relation:

$$\frac{1}{h_{\mathfrak{N}p^r}} \sum_{\xi} \xi(\mathfrak{c}) \omega_{\lambda_0\xi}^{\text{et}} = \lambda_0(\mathfrak{c})^{-1} \omega_{\mathfrak{c}},$$

one obtains

$$\frac{1}{h_{\mathfrak{N}p^r}} \sum_{\chi \leq \mathcal{I}(\mathfrak{N}p^r)} \chi(\mathfrak{c}) \omega_\chi^{\text{et}} = \omega_{\mathfrak{c}}^{\text{et}}. \quad (8.13)$$

Let us now study the action of the decomposition group at p , $G_E \subset G_\mathcal{K}$ determined by the embedding ι_p , on the class ω_χ^{et} . We know that $G_\mathcal{K}$ acts on $\widehat{\chi}$ by $\rho_\chi = \widehat{\chi} \circ \text{Art}_\mathcal{K}$. Let $\sigma \in G_E$ corresponding under $\text{Art}_\mathcal{K}$ to the idèle with $a = (1, \dots, 1, a_p, 1, \dots)$, where $a_p = p^{n(\sigma)} u_p$, where $n(\sigma)$ is an integer and $u_p \in \mathcal{O}_E^\times$ is a local unit. Let κ_{LT} be the Lubin–Tate character as defined in section 8.2.2. This is the $\mathcal{O}_E^\times$ -valued character of G_E that cuts out the maximal abelian totally ramified extension of E whose sign is fixed by the requirement that $\kappa_{LT} \bar{\kappa}_{LT} = \varepsilon_{\text{cyc}}$, so that, under class field theory, $\kappa_{LT}(a_p) = u_p$. Writing χ_p for the local component at p of the Hecke character χ , and $\chi_p = |\cdot|^{-1} \chi_{\text{fin},p}$ for some finite order character $\chi_{\text{fin},p}$ of

$E^\times$, one finds that

$$\sigma \cdot \omega_\chi^{\text{et}} = \kappa_{LT}^{-1}(\sigma) \chi_{\text{fin},p}(\sigma) \omega_\chi^{\text{et}}.$$

It will be convenient for us to reindex the definition of $\omega_\mathfrak{c}^{\text{et}}$ on the Galois group of the ring class field $\mathcal{K}[\mathfrak{N}p^r]$ and to twist by κ_{LT} .

8.2.1.19 Definition. Let $\widehat{\chi} \leq \widehat{\mathcal{I}}(\mathfrak{N}p^r)$ be a p -adic Hecke character of conductor dividing $\mathfrak{N}p^r$. For all $\sigma \in \text{Gal}(\mathcal{K}[S_{\mathfrak{N}p^r}]/\mathcal{K})^{\text{ab}}$ define

$$\omega_\sigma^{\text{et}} := \sum_{\chi \leq \mathcal{I}(\mathfrak{N}p^r)} \widehat{\chi}(\sigma) \omega_\chi^{\text{et}}.$$

Fixing a set of representatives $\{\sigma_i\}$ in $\text{Gal}(\mathcal{K}[S_{\mathfrak{N}p^r}]/\mathcal{K})^{\text{ab}}$ of $\text{Gal}(\mathcal{K}[\mathfrak{N}p^r]/\mathcal{K})$, so that σ_i corresponds to the representatives $\mathfrak{c}_i \in I_{\mathfrak{N}p^r}$ chosen above for $\text{Cl}_{\mathfrak{N}p^r}$. We have the relation

$$\omega_\chi^{\text{et}} = \frac{1}{h_{\mathfrak{N}p^r}} \sum_i \widehat{\chi}(\sigma_i)^{-1} \omega_{\sigma_i}^{\text{et}}.$$

In the following proposition, we reformulate the discussion in terms of the quantities defined above.

8.2.1.20 Proposition. *Let F/E be an extension that contains all the values of the p -adic Hecke characters $\widehat{\chi}$ of conductor dividing $\mathfrak{N}p^r$. Let $\{\sigma_i\}$ be representatives in $\text{Gal}(\mathcal{K}[S_{\mathfrak{N}p^r}]/\mathcal{K})^{\text{ab}}$ of $\text{Gal}(\mathcal{K}[\mathfrak{N}p^r]/\mathcal{K})$. Then the elements $\omega_{\sigma_i}^{\text{et}}$ and $\overline{\omega}_{\sigma_i}^{\text{et}}$ defined in Definition 8.2.1.19 form a basis of $H_{\text{acm}}(N, r) \otimes E$. Moreover if F/E is a sufficiently large extension of E containing the values of all the p -adic Hecke characters appearing*

in $\widehat{\mathcal{I}}(\mathfrak{N}p^r)$, we have the decomposition

$$H_{acm}(N, r) \otimes F \bigoplus_{\chi \leq \mathcal{I}(\mathfrak{N}p^r)} F\omega_{\chi}^{\text{et}} \oplus F\overline{\omega}_{\chi}^{\text{et}},$$

where

$$\omega_{\chi}^{\text{et}} = \frac{1}{h} \sum_{i=1}^h \widehat{\chi}(\sigma_i)^{-1} \omega_{\sigma_i}^{\text{et}}$$

does not depend on the chosen set of representatives σ_i . If $\sigma \in G_{\mathcal{K}}^{\text{ab}}$ put

$$\omega_{\sigma}^{\text{et}} = \sum_{\chi \leq \mathcal{I}(\mathfrak{N}p^r)} \widehat{\chi}(\sigma) \omega_{\chi}^{\text{et}}.$$

Then, for all $\sigma, \sigma' \in G_{\mathcal{K}}$, we have

$$\sigma' \cdot \omega_{\sigma}^{\text{et}} = \omega_{\sigma'\sigma}^{\text{et}}.$$

8.2.2 Lubin–Tate formal groups, and periods

As in section 8.1.1, we denote by E the unique unramified quadratic extension of $\mathbb{Q}$, with valuation ring $\mathcal{O}_E$. We first review some facts about Fontaine’s period rings and general material relating to formal groups over local valuation rings. We specialize the discussion to Galois extensions of $K/\mathbb{Q}_p$.

8.2.2.1 Period rings

Let W_p be the Weil group of $\mathbb{Q}_p$ that is the subgroup of elements $g \in G_{\mathbb{Q}_p}$ acting on the residue field of $\overline{\mathbb{Z}_p}$ by an integer power of the Frobenius endomorphism

φ_p . Let W_K be the quotient of W_p by the inertia subgroup of K . We now review some material found in the first two sections of [OK94]. Let $\mathbf{B}_{\text{dR}}^+$ be the universal pro-infinitesimal thickening ([OK94, Section 1.1]) of $\mathbb{C}_p^\times$ and $\theta : \mathbf{B}_{\text{dR}}^+ \rightarrow \mathbb{C}_p$ be the canonical homomorphism afforded by the universal property of $\mathbf{B}_{\text{dR}}^+$. Then $\mathbf{B}_{\text{dR}}^+$ is a discrete valuation ring whose residue field is $\mathbb{C}_p$ and maximal ideal is $\ker(\theta)$. We let $\mathbf{B}_{\text{dR}} = \text{Frac}(\mathbf{B}_{\text{dR}}^+)$ be the field of fractions of $\mathbf{B}_{\text{dR}}^+$. One can define a decreasing filtration of $\mathbf{B}_{\text{dR}}^+$, by setting $\text{Fil}^i \mathbf{B}_{\text{dR}}^+ := \ker(\theta)^i$. We can define $i_{\text{dR}}(x) = i$ if $x \in \text{Fil}^i \mathbf{B}_{\text{dR}}^+$ but $x \notin \text{Fil}^{i+1} \mathbf{B}_{\text{dR}}^+$. We denote by $\mathbf{B}_{HT}$ the graded ring with respect to this filtration. We let $\mathbf{A}_{\text{inf}} \subset \mathbf{B}_{\text{dR}}^+$ be the universal infinitesimal thickening of the completion of $\overline{\mathbb{Z}_p}$, that we denote by $\mathcal{O}_{\mathbb{C}_p}$. Then $\mathbf{A}_{\text{inf}}$ contains the ring of Witt vectors of $\mathcal{O}_{\mathbb{C}_p}/p\mathcal{O}_{\mathbb{C}_p}$, which can be identified with the completion of the ring of integers of the maximal unramified extension of $\mathbb{Q}_p$. Equivalently, if we let $\mathcal{O}_{\mathbb{C}_p^\flat} := \varprojlim_{x \mapsto x^p} \mathcal{O}_{\mathbb{C}_p}/p\mathcal{O}_{\mathbb{C}_p}$ then $\mathcal{O}_{\mathbb{C}_p^\flat}$ is a ring of characteristic p , with a discrete valuation $v^\flat$ for which it is complete. Then $\mathbf{A}_{\text{inf}}$ is defined as the ring of Witt vectors of $\mathcal{O}_{\mathbb{C}_p^\flat}$. Then $\mathbf{A}_{\text{inf}} \cap \ker(\theta)$ is a principal ideal of $\mathbf{A}_{\text{inf}}$ and we fix a generator $\hat{p}^1$ of this ideal. We let $\mathbf{A}_{\text{cris}}$ be the completion, inside $\mathbf{B}_{\text{dR}}^+$, of the sub- $\mathbf{A}_{\text{inf}}$ -algebra of $\mathbf{B}_{\text{dR}}^+$ that is generated by $\hat{p}^n/n!$ for $n \geq 1$. If $L/\mathbb{Q}_p$ is a finite Galois extension of $\mathbb{Q}_p$ and $L_0 \subseteq L$ be the maximal unramified subextension of $L/\mathbb{Q}_p$. We let $\mathbf{B}_{\text{cris},L}^+ := L \otimes_{\mathcal{O}_{L_0}} \mathbf{A}_{\text{cris}}$ be the sub-algebra of $\mathbf{B}_{\text{dR}}^+$ generated by L and $\mathbf{A}_{\text{cris}}$. We still denote by φ_p the absolute Frobenius acting on $\mathbf{A}_{\text{inf}}, \mathbf{A}_{\text{cris}}, \mathbf{B}_{\text{cris},L}^+$. Then the absolute Galois group of $\mathbb{Q}_p$ acts continuously on $\mathbf{B}_{\text{dR}}^+, \mathbb{C}_p$, and $\mathbf{B}_{\text{cris},L}^+$ (endowed with the topology induced by the $\mathbf{A}_{\text{cris}}$ which is finer than that induced from $\mathbf{B}_{\text{dR}}^+$) for any L . This action commutes with θ .

¹Sometimes this generator is denoted by ξ , but ξ for us will be a letter reserved to characters.

Having already fixed embeddings of $\overline{\mathbb{Q}}$ into $\overline{\mathbb{Q}_p}$ ι_p , and fixing a choice (ζ_n) of compatible primitive p^n -th roots of unity, we denote by $\varepsilon_n := \iota_p(\zeta_n) \in \mathcal{O}_{\mathbb{C}_p}$. Then the image of the system (ε_n) in $\mathcal{O}_{\mathbb{C}_p}/p\mathcal{O}_{\mathbb{C}_p}$ is an element of $[\varepsilon] \in \mathcal{O}_{\mathbb{C}_p^\flat}$. We then put

$$t := \log([\varepsilon]) = - \sum_{k \geq 1} \frac{(1 - [\varepsilon])^k}{k},$$

then $t \in \mathbf{A}_{\text{cris}}$ is a uniformizer of $\mathbf{B}_{\text{dR}}^+$ and $G_{\mathbb{Q}_p}$ acts on t by the cyclotomic character ε_{cyc} . Then t is a uniformizer for $\mathbf{B}_{\text{dR}}^+$ and $\mathbf{B}_{\text{dR}} = \mathbf{B}_{\text{dR}}^+[1/t]$. We let $\mathbf{B}_{HT}$ be the graded ring associated to the obvious filtration on $\mathbf{B}_{\text{dR}}$ induced by t , so that as a $G_{\mathbb{Q}_p}$ -module $\mathbf{B}_{HT} \simeq \bigoplus_{i \in \mathbb{Z}} \mathbb{C}_p(i)$.

8.2.2.2 Commutative formal groups

We now review some facts about formal commutative groups. We follow closely the exposition of [Col93]. Let K be a finite Galois extension of $\mathbb{Q}_p$ and K_0 be its maximal unramified sub-extension. A triple (V, ϕ, Fil) is a Dieudonné module when V is a finite dimensional K_0 -vector space of dimension h , ϕ is an endomorphism of V that is φ_p -semilinear, injective, and topologically nilpotent of V , which moreover satisfies the condition that if $T \subset V$ is a lattice in V stable by ϕ , then $pT \subseteq \phi T$. Finally, Fil is a decreasing filtration on $V_K := V \otimes_{K_0} K$, such that $\text{Fil}^0 V_K = V_K$, $\text{Fil}^2(V_K) = 0$ and $\dim \text{Fil}^1 V_K = \dim_k T/\phi T$ where k here is the residue field of K . Let $\mathfrak{G}$ be a commutative formal group defined over $\mathcal{O}_K$, of dimension d and height h . To such a formal group one can associate a Dieudonné module in the previous sense in the following way. Let $A_{\mathfrak{G}} := \mathcal{O}_K[[X]] = \mathcal{O}_K[[X_1, \dots, X_d]]$ be the affine algebra of $\mathfrak{G}$. We

let $K(\mathfrak{G}) := K[[X]]$ the field of fraction of $A_{\mathfrak{G}}$. Denote by $\oplus$ the formal law of $\mathfrak{G}$. If $\omega = \sum_i \alpha_i dX_i$ where $\alpha_i \in K(\mathfrak{G})$ is a closed differential form, following [Col93, Section 1], we denote by $F_{\omega} \in \text{Frac}(A_{\mathfrak{G}})$ such that $dF_{\omega} = \omega$, and $F_{\omega}(0) = 0$. We let F_{ω}^2 denote the formal power series in $2d$ variables defined by $F_{\omega}^2 := F_{\omega}(X \oplus Y) - F_{\omega}(X) - F_{\omega}(Y)$; here $X = (X_1, \dots, X_d)$ and $Y = (Y_1, \dots, Y_d)$. A closed differential form ω will be called exact if F_{ω} has bounded coefficients, of the second kind is F_{ω}^2 has also bounded coefficients, and invariant if F_{ω}^2 . Denote by $\Omega_{\mathfrak{G}}$ the K -vector space of invariant differential forms. The dimension of $\Omega_{\mathfrak{G}}$ is d . If $F \in K(\mathfrak{G})$ is an element such that $F(0) = 0$ and $dF \in \Omega_{\mathfrak{G}}$, we call F a logarithm of $\mathfrak{G}^2$. We let $H_{\text{dR}}^1(\mathfrak{G})$ be the quotient of the K -vector space of differential forms of the second kind by the K -space of exact forms. Then $H_{\text{dR}}^1(\mathfrak{G})$ is a K -vector space of dimension h , it is endowed with the filtration $\text{Fil}^0 H_{\text{dR}}^1(\mathfrak{G}) = H_{\text{dR}}^1(\mathfrak{G})$, $\text{Fil}^1 H_{\text{dR}}^1(\mathfrak{G}) = \Omega_{\mathfrak{G}}$, $\text{Fil}^2 H_{\text{dR}}^1(\mathfrak{G}) = 0$. Let $V_{\mathfrak{G}}$ be the sub- K_0 -vector-space of $H_{\text{dR}}^1(\mathfrak{G})$ generated by the image differential forms ω of the second kind with coefficients in K_0 . We endow $V_{\mathfrak{G}}$ with the endomorphism ϕ obtained by applying φ_p to the coefficients of ω and by raising the variable X_i to the p -th power. The triple $D(\mathfrak{G}) = (V_{\mathfrak{G}}, \phi, \text{Fil})$ is a Dieudonné module in the sense above; we will call it the Dieudonné module of $\mathfrak{G}$. Let us note that $H_{\text{dR}}^1(\mathfrak{G})$ is endowed with an action of W_K , given by $g(x \otimes \omega) = g(x) \otimes \phi^{n(g)}(\omega)$, where $n(g)$ is determined by the condition that g act by $\varphi_p^{n(g)}$ on the residue field of K . Here we recall that W_K is defined at the beginning of the previous paragraph. Two formal groups are isogenous if and only if their Dieudonné modules are isomorphic.

²Because for the formal multiplicative group $d \log = dt/t$

8.2.2.3 Periods of Lubin–Tate groups

We define

$$T_p\mathfrak{G} := \varprojlim_n \mathfrak{G}[p^n],$$

where $\mathfrak{G}[p^n]$ is the subgroup of p^n -torsion points of $\mathfrak{G}(\mathfrak{m}_{\overline{\mathbb{Q}_p}})$ —here $\mathfrak{m}_{\overline{\mathbb{Q}_p}}$ is the maximal ideal of the valuation ring of $\overline{\mathbb{Q}_p}$. This $\mathbb{Z}_p$ -module is of rank h and endowed with an obvious continuous action of G_K . We let $V_p\mathfrak{G} := T_p\mathfrak{G}[1/p]$. It is well-known (for example see Proposition I.1.3 of *loc cit.*) that two formal groups are isomorphic over $\mathcal{O}_K$ if and only if the associated Tate modules are isomorphic as $\mathbb{Z}_p[G_K]$ -modules. The following proposition, extracted from [Col93, p. I.1.4] will be very useful for us.

8.2.2.4 Proposition. *Let ω be a differential of the second kind on $\mathfrak{G}$ and let $u = \varprojlim u_n \in T_p\mathfrak{G}$. Let $\hat{u}_n \in \mathbf{A}_{\text{inf},K}^d$ be such that $\theta(\hat{u}_n) = u_n$. Let F_ω be as above the unique element such that $F_\omega(0) = 0$ and $dF_\omega = \omega$. Then the sequence $-p^n F_\omega(\hat{u}_n)$ converges in $\mathbf{B}_{\text{cris},K}^+$ to a limit denoted by $\int_u \omega$ that only depends on u and the image of ω in $H_{\text{dR}}^1(\mathfrak{G})$. The pairing*

$$H_{\text{dR}}^1(\mathfrak{G}) \times T_p\mathfrak{G} \rightarrow \mathbf{B}_{\text{cris},K}^+, \quad (\omega, u) \mapsto \int_u \omega$$

is bilinear, respects filtration and commutes with the action of G_K . (The action of G_K on $H_{\text{dR}}^1(\mathfrak{G})$ factoring through W_K).

We now specialize construct certain periods for Lubin–Tate formal groups over $\mathcal{O}_E$; here $E = \mathcal{K}_p$ is the unique unramified quadratic extension of $\mathbb{Q}_p$. So in the set-up of Section 2 of [Col93], $E = K \subseteq \overline{\mathbb{Q}_p}$ is a finite Galois extension of $\mathbb{Q}_p$. Let κ_{LT} be the

universal character of the inertia subgroup of G_E , I_E , afforded by class field theory and let $\bar{\kappa}_{LT}$ be the composition of κ_{LT} with the unique non-trivial automorphism $\bar{\cdot} \in \text{Gal}(E/\mathbb{Q}_p)$. Then $\kappa_{LT}\bar{\kappa}_{LT} = \varepsilon_{\text{cyc}}$ where ε_{cyc} is the (restriction to G_E of the) cyclotomic character. We fix the uniformizer p of E . In the notations of *loc cit.*, for $\sigma = \text{id}$, and $\tau \in \{\text{id}, \bar{\cdot}\}$, we define

$$l_\tau(X) := \begin{cases} \sum_{n=0}^{+\infty} \frac{X^{p^{2n}}}{p^n} & \text{if } \tau = \text{id} \\ \sum_{n=0}^{+\infty} \frac{X^{p^{2n+1}}}{p^{n+1}} & \text{if } \tau = \bar{\cdot} \end{cases}$$

Then $l_\tau^{-1}(l_\tau(X) + l_\tau(Y))$ is a formal group law of a formal group $\mathfrak{G}_\tau$ defined over $\mathcal{O}_E$. The formal group obtained in this way is isomorphic over $\mathcal{O}_E$ to the Lubin–Tate formal group for the uniformizer $\tau(p) = p$ (see p. 642 of *loc cit.*), we denote it by $\mathfrak{G} \simeq \mathfrak{G}_\tau$. In particular, it is of height h and $\text{End}_{\mathcal{O}_E}(\mathfrak{G}_\tau) \simeq \mathcal{O}_E$ – we denote by $x \mapsto [\alpha]x$ the endomorphism of $\mathfrak{G}_\tau$ associated to an $\alpha \in \mathcal{O}_E$. By construction one has that $\Omega_{\mathfrak{G}_\tau} = E\omega_\tau$ where $\omega_\tau = dl_\tau$ and $[\alpha]_\tau^* \omega_\tau = \tau(\alpha)\omega_\tau$ for all $\alpha \in \mathcal{O}_E$. Then we have the decomposition

$$H_{\text{dR}}^1(\mathfrak{G}) = E\omega_+ \oplus E\omega_-,$$

where we have denoted for simplicity $\omega_- = \omega_{\bar{\cdot}}$ and $\omega_+ = \omega_{\text{id}}$. Moreover the lines $E\omega_+$ and $E\omega_-$ are eigenspaces for the action of $\text{End}_{\mathcal{O}_E}(\mathfrak{G}) \simeq \mathcal{O}_E$: we have

$$[\alpha]^* \omega_+ = \alpha \omega_+, \quad [\alpha] \omega_- = \bar{\alpha} \omega_-.$$

Fixing a generator u of $T_p\mathfrak{G}$, we define

$$\widehat{\Omega}_-(u) := \int_u \omega_- \in \mathbf{B}_{\text{cris}}^+ \quad (8.14)$$

Then $\widehat{\Omega}_-(u)$ differs from $\widehat{\Omega}_-(u')$ by a unit of $\mathcal{O}_E$, if u' is a different choice of basis of the Tate-module of $\mathfrak{G}$. We allow ourselves to write $\widehat{\Omega}_-$ when the choice of u is implicit. If $g \in I_E$ is an inertial element we have, by Théorème I.2.1 of *loc cit.*

$$g(\widehat{\Omega}_-) = \overline{\kappa_{LT}}(g)\widehat{\Omega}_- ;$$

and moreover $v_p(\Omega_-) = \frac{p}{p^2-1}$. By (iii) of the theorem cited above, $\widehat{\Omega}_-$ does not belong to $\ker(\theta)$ (contrary to $\widehat{\Omega}_+$ who does belong to $\ker(\theta)$). We then define

$$\Omega_- := \theta(\widehat{\Omega}_-) \in \mathbb{C}_p.$$

Then by Galois equivariance of θ , we know that G_K^{ab} acts by the character $\overline{\kappa_{LT}}$ on Ω_- . The reader can consult [Fou09] For the same construction of Ω_- (denoted by ξ_ρ for $\rho = \bar{\cdot}$ in *loc cit.*), but in a more elementary fashion. Recalling the element $t \in \mathbf{A}_{\text{cris}}$ defined at the end of Paragraph 8.2.2.1, we now define

$$\Omega_+ := \frac{t}{\Omega_-} \in \mathbf{B}_{\text{cris}}.$$

Since $\kappa_{LT}\overline{\kappa_{LT}} = \varepsilon_{\text{cyc}}|_{G_E}$ and $G_{\mathbb{Q}_p}$ acts on t by ε_{cyc} , we deduce that G_E acts by κ_{LT} on Ω_+ .

We sum-up the properties of Ω_- and Ω_+ in the following Lemma.

8.2.2.5 Lemma. *There exists elements $\Omega_- \in \mathcal{O}_{\mathbb{C}_p}$ and $\Omega_+ = t\Omega_-^{-1} \in \text{Fil}^1 \mathbf{B}_{\text{dR}}^+$ uniquely determined up to an element of $\mathcal{O}_E^\times$ such that*

- *We have $v_p(\Omega_-) = \frac{p}{p^2-1}$,*

- *For all $g \in G_E$,*

$$g(\Omega_-) = \overline{\kappa_{LT}}(g)\Omega_-,$$

- *For all $g \in G_E$,*

$$g(\Omega_+) = \kappa_{LT}(g)\Omega_+.$$

8.2.2.6 The cyclotomic and anticyclotomic towers

We fix as in paragraph 8.2.2.3 Lubin–Tate formal group $\mathfrak{G}$ over $\mathcal{O}_E$ with uniformizing parameter $\varpi = -p$. For $n \geq 0$, write $E_n := E(\mathfrak{G}[\varpi^{n+1}])$, the extension of E in $\mathbb{C}_p$ generated by the ϖ^{n+1} torsion points of $\mathfrak{G}$ and put $E_\infty := \bigcup_n E_n$. The Galois action on the ϖ -adic Tate module $T_\varpi \mathfrak{G}$ defines a natural isomorphism

$$\kappa_{LT} : \text{Gal}(E_\infty/E) \xrightarrow{\sim} \text{Aut}(T_\varpi \mathfrak{G}) \simeq \mathcal{O}_E^\times.$$

We put $\Delta_E := \text{Gal}(E_0/E) \simeq (\mathcal{O}_E/\varpi\mathcal{O}_E)^\times$ and note $\text{Gal}(E_\infty/E) \simeq \Delta_E \times \text{Gal}(E_\infty/E_0)$ canonically. The action of $\text{Gal}(E/\mathbb{Q}_p)$, decomposes $G_\infty := \text{Gal}(E_\infty/E_0)$ as follows:

$$G_\infty \simeq G_\infty^+ \oplus G_\infty^-,$$

where, as a topological group, we have $G_\infty^\pm \simeq \mathbb{Z}_p$ and G_∞^+ (resp. G_∞^-) is the maximal subgroup of G_∞ on which $\text{Gal}(E/\mathbb{Q}_p)$ acts via the trivial (resp. non-trivial) character.

The group G_∞^+ is naturally isomorphic to the Galois group of the cyclotomic $\mathbb{Z}_p$ extension of E , E_∞^{cyc} , and G_∞^- is the Galois group of the anticyclotomic $\mathbb{Z}_p$ -extension E_∞^{ac} . Let $U_E := 1 + p\mathcal{O}_E$. Then, under local class field theory, $G_\infty \simeq U_E$, and the decomposition $G_\infty = G_\infty^+ \oplus G_\infty^-$, induces a decomposition

$$U_E \simeq U_E^+ \times U_E^-,$$

where U_E^+ is naturally identified with $\mathbb{Z}_p \simeq 1 + p\mathbb{Z}_p \hookrightarrow 1 + p\mathcal{O}_E = U_E$. We denote by γ_+ (resp. γ_-) the basis of G_∞ (as a free $\mathbb{Z}_p$ -modules of rank 2) corresponding to the decomposition $G_\infty = G_\infty^+ \oplus G_\infty^-$. We view $\gamma_\pm$ also as elements of U_E (and thus of $\mathcal{O}_E^\times$).

We call a finite order character of $\text{Gal}(E_\infty/E)$ that factors through G_∞^- (resp. G_∞^+) an *anticyclotomic* (resp. *cyclotomic*) character. If a finite order character ξ of $\text{Gal}(E_\infty/E)$ factors through $\text{Gal}(E_n/E)$ but not through $\text{Gal}(E_{n-1}/E)$, then ξ has log-conductor n .

We define the full Iwasawa algebra

$$\Lambda_E := \mathcal{O}_E[[G_\infty]] \tag{8.15}$$

and their partial versions

$$\Lambda_E^{cyc} := \mathcal{O}_E[[G_\infty^+]], \quad \Lambda_E^{ac} := \mathcal{O}_E[[G_\infty^-]]. \tag{8.16}$$

If one puts $T_\pm := \gamma_\pm - 1$, then we can identify $\Lambda_E^{cyc} \simeq \mathbb{Z}_p[[T_+]]$ as a topological

$\mathbb{Z}_p$ -algebra and similarly $\Lambda_E^{ac} \simeq \mathbb{Z}_p[[T_-]]$.

8.2.2.7 Finite order periods

We end this section by giving analogues of the period Ω_+ and Ω_- for finite order characters of inertia $I_E \simeq \mathcal{O}_E^\times$. We keep the notation $\mathfrak{G}$ for the Lubin–Tate formal group with parameter $\varpi = -p$, of the previous paragraph. Let $\lambda : \mathfrak{G} \rightarrow \hat{\mathbf{G}}_a$ be the formal group logarithm. If one endows $E[[X]]$ with the product topology, we have $\lambda(X) = \lim_{n \rightarrow \infty} \frac{[\varpi^n](X)}{\varpi^n}$. Following the notations of [BKO23, Section 2], we let Ξ be the set of finite characters of G_∞^- . Let

$$\Xi^+ = \{\chi \in \Xi : \text{the conductor of } \chi \text{ is an even power of } p\},$$

and

$$\Xi^- = \{\chi \in \Xi : \text{the conductor of } \chi \text{ is an odd power of } p\}.$$

Let E_n^{cyc} (resp. E_n^{ac}) be the n -th anticyclotomic (resp. cyclotomic layer); that is $E_n^{cyc} = (E_\infty^{cyc})^{p^n \mathbb{Z}_p}$ and $E_n^{ac} = (E_\infty^{ac})^{p^n \mathbb{Z}_p}$. For $n \geq 0$, we let $\Xi_n^\pm$ denote the set of $\chi \in \Xi^\pm$ factoring through $\text{Gal}(E_n^{ac}/E)$. For $\chi \in \Xi_n^\pm$, we define for any point x of $\mathfrak{G}(\mathfrak{m}_n)$,

$$\lambda_\chi(x) = \frac{1}{p^n} \sum_{\sigma \in \text{Gal}(E_n^{ac}/E)} \chi(\sigma) \lambda(x)^\sigma. \quad (8.17)$$

Let $\mathfrak{m}_n$ be the maximal ideal of the valuation ring of E_n^{ac} . Define

$$\mathfrak{G}(\mathfrak{m}_n)^\pm = \{x \in \mathfrak{G}(\mathfrak{m}_n) : \forall \chi \in \Xi_n^\pm, \lambda_\chi(x) = 0\}.$$

Note that as is remarked in [BKO, Proposition 4.3], if $\epsilon = (-1)^n$, $\mathfrak{G}(\mathfrak{m}_n)^{-\epsilon} \subseteq \mathfrak{G}(\mathfrak{m}_{n-1})$.

The following proposition is extracted from Theorem 2.7 and Lemma 2.6 of [BKO23].

8.2.2.8 Proposition. *Let $\Lambda_n^{ac} = \mathcal{O}_E[\text{Gal}(E_n^{ac}/E)]$. The Λ_n^{ac} -module $\mathfrak{G}(\mathfrak{m}_n)$ decomposes as*

$$\mathfrak{G}(\mathfrak{m}_n) = \mathfrak{G}(\mathfrak{m}_n)^+ \oplus \mathfrak{G}(\mathfrak{m}_n)^-.$$

Moreover, there exists a system of points $c_n^\pm \in \mathfrak{G}(\mathfrak{m}_n)$ that are generators of $\mathfrak{G}(\mathfrak{m}_n)^\pm$ as a Λ_n^{ac} -module such that if $(-1)^{n+1} := \pm$, $c_{n+1}^\pm$ is a uniformizer and

$$\sum_{\sigma \in \text{Gal}(E_{n+1}^{ac}/E_n^{ac})} \sigma(c_{n+1}^\pm) = c_{n-1}^\pm.$$

8.2.2.9 Lemma. *Let $c_n^\pm \in \mathfrak{G}(\mathfrak{m}_n)$ be the elements defined in Proposition 8.2.2.8.*

Let $\epsilon = (-1)^n$. Define

$$t_n := p^{-(n+1)/2}(c_n^\epsilon + c_{n-1}^{-\epsilon}).$$

Then, for all $n \geq 3$

$$\sum_{\sigma \in \text{Gal}(E_n^{ac}/E_{n-2}^{ac})} \sigma(t_n) = t_{n-2}.$$

Proof. Indeed since $c_{n-1}^{-\epsilon} \in \mathfrak{G}(\mathfrak{m}_{n-1})$ and E_n^{ac}/E_{n-1}^{ac} is a degree p extension, we have

$$\sum_{\sigma \in \text{Gal}(E_n^{ac}/E_{n-1}^{ac})} \sigma(t_n) = p^{-(n+1)/2}(c_{n-2}^\epsilon + pc_{n-1}^{-\epsilon}).$$

Therefore, applying the same reasoning again,

$$\sum_{\sigma \in \text{Gal}(E_n^{ac}/E_{n-2}^{ac})} \sigma(t_n) = p^{-(n+1)/2+1} (c_{n-2}^\epsilon + c_{n-3}^{-\epsilon}) = t_{n-2}.$$

□

8.2.2.10 Definition. Let χ be any finite order character and n be an integer such that χ factors through $\text{Gal}(E_{2n+1}^{ac}/E)$. Define

$$\Omega_\chi^{\text{univ}} := \sum_{\sigma \in \text{Gal}(E_{2n+1}^{ac}/E)} \chi(\sigma) t_{2n+1}^\sigma.$$

Note that because of Lemma 8.2.2.9, $\Omega_\chi^{\text{univ}}$ does not depend on the chosen integer n .

8.2.2.11 Remark. Note that if χ factors also through E_{2n}^{ac} then we have

$$\Omega_\chi^{\text{univ}} = \sum_{\sigma \in \text{Gal}(E_{2n}^{ac}/E)} \chi(\sigma) p^{-n} (c_{2n-1}^- + c_{2n}^+)^\sigma.$$

Putting $t'_n = p^{-n/2} (c_n^\epsilon + c_{n-1}^{-\epsilon})$, we could have defined $\Omega_\chi^{\text{univ}}$ for even conductors as well and the definitions coincide.

8.2.2.12 Definition. If χ is a character factoring through $\text{Gal}(E_n^{ac}/E)$, we also define the period

$$\Omega_{\chi,n} := \sum_{\sigma \in \text{Gal}(E_n^{ac}/E)} \chi(\sigma) (c_n^\epsilon + c_{n-1}^{-\epsilon})^\sigma,$$

for $\epsilon = (-1)^n$.

8.2.2.13 Remark. Unlike $\Omega_\chi^{\text{univ}}$, as the subscript n indicates, $\Omega_{\chi,n}$ depends on the

chosen integer n such that χ factors through $\text{Gal}(E_n^{ac}/E)$. In [BKO], this Gauss-sum-like period is called a "local-resolvent". Note also that if r is the log-conductor of χ , then we have

$$\Omega_\chi^{\text{univ}} = \begin{cases} p^{-(n+1)} \sum_{\sigma \in \text{Gal}(E_{2n+1}^{ac})} \chi(\sigma) c_{2n+1}^- & \text{if } r \text{ is odd,} \\ p^{-n} \sum_{\sigma \in \text{Gal}(E_{2n}^{ac})} \chi(\sigma) c_{2n}^+ & \text{if } r \text{ is even.} \end{cases}$$

By the main p -adic valuation calculation of [BKO], $\Omega_\chi^{\text{univ}}$ has p -adic valuation $1/2$ if χ has odd conductor, otherwise it is a p -adic unit.

We are now ready to give the salient properties of the periods for finite order anticyclotomic characters defined above. Point number 2 is a consequence of Section 2 of [BKO].

8.2.2.14 Proposition. *Let χ be any character of $\text{Gal}(E_\infty/E)$ factoring through $\text{Gal}(E_n^{ac}/E)$. following hold.*

1. *For all $\sigma \in G_E$,*

$$\sigma(\Omega_\chi^{\text{univ}}) = (\chi^\sigma)^{-1}(\sigma) \cdot \Omega_{\chi^\sigma}^{\text{univ}},$$

and similarly

$$\sigma(\Omega_{\chi,n}) = (\chi^\sigma)^{-1}(\sigma) \cdot \Omega_{\chi^\sigma,n}.$$

2. *The period $\Omega_\chi^{\text{univ}}$ is non-zero – it is either p -adic unit or it has valuation $1/2$ according to whether the conductor of χ is odd or even.*

We finish this section by constructing periods for tame characters $\chi : \text{Gal}(E_0/E) \rightarrow \overline{\mathbb{Q}}_p^\times$. Fix ϖ_0 be a non-trivial solution to the equation $[-p](X) = 0$, then ϖ_0 is uniformizer of the valuation ring of E_0 — we recall that E_0 is obtained by adjoining the $[-p]$ torsion points of $\mathfrak{G}$; if $\mathfrak{G}$ is the standard Lubin–Tate formal group with Frobenius power series $-pX + X^{p^2}$, we are simply considering solutions of the equation $X^{p^2-1} = p$. Let Teich_E be the Teichmüller character of G_E defined as

$$G_E \rightarrow \text{Gal}(E_0/E) \simeq (\mathcal{O}_E/p\mathcal{O}_E)^\times \hookrightarrow \mathcal{O}_E^\times.$$

Then by definition we have $\sigma(\varpi_0) = \text{Teich}_E(\sigma)\varpi_0$, for all $\sigma \in G_E$. Since $(\mathcal{O}_E/p\mathcal{O}_E)^\times \simeq \mathbb{F}_{p^2}$ is cyclic, all characters of G_E factoring through $\text{Gal}(E_0/E)$ are a power of Teich_E — if μ is such a character we denote by $t(\mu) \in \{0, 1, \dots, p^2 - 2\}$ this integer.

8.2.2.15 Definition. This definition extends Definition 8.2.2.10. Let $\chi : G_E \rightarrow \overline{\mathbb{Q}}_p^\times$ be a finite order character. Assume that $\chi(\text{Frob}_p) = 1$ and that $\chi|_{I_E}$ factors through $\text{Gal}(E_n^{ac}/E)$. Let $t(\chi)$ be the integer such that $\chi|_{\text{Gal}(E_0/E)} = (\text{Teich}_E)^{t(\chi)}$. We define

$$\Omega_\chi^{\text{univ}} := \varpi_0^{t(\chi)} \Omega_\chi^{\text{univ}}.$$

Then for all $\sigma \in G_E$, we have

$$\sigma(\Omega_\chi^{\text{univ}}) = (\chi^\sigma)^{-1}(\sigma) \cdot \Omega_\chi^{\text{univ}}.$$

8.2.2.16 p -adic Hodge Theory à la Faltings

In this section we review a construction due to Faltings [Fal87], and [Fal88]. Let $N \geq 1$ be a fixed integer prime to p and let $X_r := X_1(Np^r)$ for $r \geq 0$ the Baily-Borel compactification of the modular curve $Y_r := Y(K_1(Np^r))$. We assume N large enough so that X_1 has genus ≥ 2 . The universal elliptic curve $\mathcal{E} \rightarrow Y_r$ extends to a semi-abelian variety over X_r , still denoted by $\mathcal{E}$ by abuse of notation. Let ω denote the Hodge bundle over X_r , so that $\omega_r^{\otimes 2} \simeq \Omega_{X_r}^1(\log(C))$; here $\Omega_{X_r}^1(\log(C))$ denotes the sheaf of 1-forms on X_r with log poles at the cusps C . Let $\mathcal{X}_0$ and $\mathcal{Y}_0$ the smooth models over any spectrum of a ring where N is invertible. Let V be the $\mathfrak{p}$ -adic completion of the ring of integers of a number field (containing the N -th roots of unity) for a prime $\mathfrak{p}$ dividing p . As in [Fal87, Section 4], we assume that V is big enough so that X_r has a (unique) semi-stable model $\mathcal{X}_r$ over V . Let $\pi_r : \mathcal{X}_r \rightarrow \mathcal{X}_0$ be the projection map. As remarked in p145 of *loc cit.*, the Kodaira-Spencer class of $\mathcal{E}$, induces an isomorphism

$$(\pi_r)^* \Omega_{\mathcal{X}_0}^1(\log(C)) \simeq (\pi_r)^* \omega^{\otimes 2} \simeq \Omega_{\mathcal{X}_r}^1(\log(C_r)).$$

Let $\widehat{V}$ be the completion of the integral closure of V in an algebraic closure of $\text{Frac}(V)$. We choose as in *loc cit.*, an affine coverint $\mathcal{X}_r = \bigcup_j U_j$, such that $U_j = \text{Spec}(R_j)$ with R_j a ring of the type considered in Section 2b) of *loc cit.*. We denote by $\widehat{R}_j$ the base change of R_j to $\widehat{V}$, and define $\widehat{U}_j := \text{Spec}(\widehat{R}_j)$.

With this setup, using the Kodaira-Spencer class coming from $\mathcal{E}$ in the case of a modular curve, and for any variety with good reduction in [Fal88], Faltings constructs

a functorial complex

$$0 \rightarrow \widehat{R}_j(1) \rightarrow E \rightarrow \Omega_{R_j/V}^1 \otimes_R \widehat{R}_j \rightarrow 0.$$

We note here that we have twisted the extension appearing on p145 by the Hodge bundle; Moreover, by Theorem 4 and 3e) of *loc cit.*, the cohomology of this complex is killed by ρ — here $\rho \in \widehat{V}$ is an element of valuation $1/(p-1)$.

Therefore this extension induces a canonical map

$$H^0(\mathcal{X}_r, (\pi_r)^* \omega^2) \otimes \rho \widehat{V} \rightarrow H_{\text{et}}^1(X_r, \mathbb{Z}_p(1)) \otimes \rho^{-1} \widehat{V}.$$

We denote by Falt the resulting map

$$\text{Falt} : H^0(\mathcal{X}_r, (\pi_r)^* \omega^{\otimes 2}) \otimes \rho^2 \widehat{V} \longrightarrow H_{\text{et}}^1(X_r, \mathbb{Z}_p(1)) \otimes \widehat{V}. \quad (8.18)$$

We will also denote by Falt the map obtained after tensoring Falt with $\mathbb{C}_p$ (that is, after inverting p .)

8.2.2.17 Remark. If one uses the formalism of prismatic cohomology as in [SS23] one can improve the ρ^2 in the above map to ρ .

8.2.3 The geometric étale classes

8.2.3.1 The de Rham comparison theorem

Let $K/\mathbb{Q}_p$ be a finite extension of $\mathbb{Q}_p$ and let X/K be a smooth algebraic variety over K . We let

$$\mathrm{comp}_{\mathrm{dR}}^{\mathrm{et}} : \quad \mathrm{H}_{\mathrm{dR}}^{\bullet}(X/K) \otimes_K \mathbf{B}_{\mathrm{dR}} \xrightarrow{\sim} \mathrm{H}_{\mathrm{et}}^{\bullet}(X/\overline{K}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathbf{B}_{\mathrm{dR}}$$

be the Faltings comparison theorem. We omit X and the cohomological degree from the notation as it will be clear from the context which algebraic variety and cohomological degree one is considering. We recall that $\mathrm{comp}_{\mathrm{dR}}^{\mathrm{et}}$ is $G_K = \mathrm{Gal}(\overline{K}/K)$ -equivariant, where G_K acts on the de Rham side only on the period ring $\mathbf{B}_{\mathrm{dR}}$ and acts diagonally on the étale side. Moreover, $\mathrm{comp}_{\mathrm{dR}}^{\mathrm{et}}$ is functorial in X and compatible with the natural filtrations on both sides: the natural filtration on $\mathbf{B}_{\mathrm{dR}}$ on the étale side and the convolution of the Hodge filtration and the natural $\mathbf{B}_{\mathrm{dR}}$ filtration on the de Rham side. In particular, $\mathrm{comp}_{\mathrm{dR}}^{\mathrm{et}}$ is compatible with the Hodge-Tate comparison map Falt of the previous section when X is a modular curve.

8.2.3.2 The construction of the first layer

Let χ be a Hecke character of conductor $\mathfrak{N}p^r$ and infinity type $(1, 0)$. We denote by $N = N_{\mathcal{K}/\mathbb{Q}}(\mathfrak{N})$.

We fix once and for all a Hecke character μ of infinity type $(1, 0)$ and conductor p .

Given μ , all the other Hecke characters of conductor p and infinity type μ are of the form $\mu\xi$ where ξ is a finite order character of $\text{Cl}_p \simeq \text{Gal}(\mathcal{K}[p]/\mathcal{K})$. It will be convenient for us to assume that the local component of μ at p verifies $\mu_p = \text{Teich}_E^{-1} \cdot |\cdot|_p^{-1}$.

Let F be an extension of $\mathbb{Q}$ containing the values of all the characters χ as above. With notation as at the end of section 8.2.1.1, we view ϑ_χ as an element of $H^0(X_1(\Delta_{\mathcal{K}}p^2), \Omega^1) \otimes F$ (it lies in this space by the q -expansion principle and Proposition 8.1.2.8), where $\Omega^1 \simeq \omega^{\otimes 2}$ denotes the sheaf of 1-forms on the modular curve $X_1(\Delta_{\mathcal{K}}p^2)$. By Proposition 8.1.2.6 ϑ_χ lies in the lattice

$$H^0(\mathcal{X}_1, (\pi_1)^*\omega^{\otimes 2}) \otimes \mathcal{O}_{\mathbb{C}_p} \subset H^0(X_1(\Delta_{\mathcal{K}}p^2), \Omega^1) \otimes \mathbb{C}_p.$$

Therefore, in the notations of Section 8.2.2.16, we can consider

$$\text{Falt}(\vartheta_\chi \otimes \rho) \in H_{\text{et}}^1(X_1(\Delta_{\mathcal{K}}p^2), \mathbb{Z}_p(1)) \otimes \mathcal{O}_{\mathbb{C}_p}.$$

Recall the notation $V(\chi)$ and $T(\chi)$ at the end of section 8.2.1.1. Since Falt is Hecke equivariant we know that

$$\text{Falt}(\vartheta_\chi \otimes \rho) \in T(\chi)(1) \otimes \mathcal{O}_{\mathbb{C}_p} \subset H_{\text{et}}^1(X_1(\Delta_{\mathcal{K}}p^2), \mathbb{Z}_p(1))_{CM}^{\text{new}} \otimes \mathcal{O}_{\mathbb{C}_p} \simeq \widehat{\mathcal{I}}(p)(1) \oplus \widehat{\mathcal{I}}(p)^\tau(1).$$

Better yet, since the Hodge-filtration on the CM part of cohomology splits rationally (by the action of the extra endomorphisms of J_{ncm} ; for notation, see section 8.2.1.1),

we have that

$$\text{Falt}(\vartheta_\chi \otimes \rho) \in \widehat{\mathcal{I}}(p)(1),$$

so that after fixing a basis of $\widehat{\mathcal{I}}(p)(1)$ as in section 8.2.1.15, we know that there exists a scalar $\lambda_\chi \in \mathbb{C}_p$ such that

$$\text{Falt}(\vartheta_\chi \otimes 1) = \omega_\chi^{\text{et}} \otimes \lambda_\chi.$$

We stress that for now ω_χ^{et} (and thus λ_χ) depends on the choice of the basis. Our aim will be to find a convenient choice of basis and pinn down the scalar λ_χ for this choice.

To this end, we invoke the de Rham–étale comparison theorem. Let z be the variable on the symmetric space $\mathcal{H}^\pm$ for $\mathbf{GL}_2$. We consider the holomorphic differential on $\mathcal{H}^\pm$ defined by $\vartheta_\chi(z)dz$, then this holomorphic differential descends to $X_1(\Delta_\kappa p^2)$ and defines a de Rham cohomology class

$$\vartheta_\chi^{\text{dR}} \in H_{\text{dR}}^1(X_1(\Delta_\kappa p^2)/\mathbb{C})_{CM}^{\text{new}}.$$

Again, interpreting this de Rham cohomology as the de Rham cohomology of an abelian variety with CM, the Hodge filtration splits rationally and therefore we have

$$\vartheta_\chi^{\text{dR}} \in H_{\text{dR}}^1(X_1(\Delta_\kappa p^2)/F).$$

By the compatibility of the Hodge–Tate comparison theorem with the de Rham

comparison theorem we have

$$\mathrm{comp}_{\mathrm{dR}}^{\mathrm{et}}(\vartheta_{\chi}^{\mathrm{dR}} \otimes t^{-1}) = \mathrm{Falt}(\vartheta_{\chi} \otimes 1) = \omega_{\chi}^{\mathrm{et}} \otimes \lambda_{\chi}.$$

By the Galois equivariance of $\mathrm{comp}_{\mathrm{dR}}^{\mathrm{et}}$ the relation above allows one to study how the Galois group of F acts on λ_{χ} . However, we would like to have information on the full action of the Galois group of E . We argue as follows.

Let $h = h_p = \#\mathrm{Cl}_p$. Consider

$$\vartheta_1 = \frac{1}{h} \sum_{\xi} \vartheta_{\mu\xi} \in \mathrm{H}^0(X_1(\Delta_{\mathcal{K}}p^2), \Omega^1) \otimes E,$$

where the sum ranges over all characters ξ of $\mathrm{Gal}(\mathcal{K}[p]/\mathcal{K})$. We note here that the Fourier coefficients of ϑ_1 at the 0-cusp indeed lie in E .

As before ϑ_1 defines a de Rham class

$$\vartheta_1^{\mathrm{dR}} = \frac{1}{h} \sum_{\xi} \vartheta_{\mu\xi}^{\mathrm{dR}} \in \mathrm{H}_{\mathrm{dR}}^1(X_1(\Delta_{\mathcal{K}}p^2)/E),$$

and we have

$$\mathrm{comp}_{\mathrm{dR}}^{\mathrm{et}}(\vartheta_1^{\mathrm{dR}} \otimes 1) = \frac{1}{h} \sum_{\xi} \omega_{\mu\xi}^{\mathrm{et}} \otimes \lambda_{\mu\xi},$$

for some scalars $\lambda_{\mu\xi} \in \mathbf{B}_{\mathrm{dR}}$.

Since the left hand-side of the above equation is fixed by G_E , the right hand-side is also fixed by G_E . Recall the notation Ω_+ for the Lubin–Tate period, as defined in

section 8.2.2.3. Consider now

$$\text{comp}_{\text{dR}}^{\text{et}}(\vartheta_1 \otimes \Omega_+) \in \left(H_{\text{et}}^1(X_1(\Delta_{\mathcal{K}} p^2), E)(\kappa_{LT}^{-1}) \otimes \mathbf{B}_{\text{dR}} \right)^{G_E}.$$

(This amounts to dividing all the λ_χ 's by Ω_+ but by abuse of notation, we continue denoting them by λ_χ .) Notice that on $H_{\text{et}}^1(X_1(\Delta_{\mathcal{K}} p^2), E)(\kappa_{LT}^{-1})$ the G_E action factors through a finite extension L/E . So that by the basic properties of $\mathbf{B}_{\text{dR}}$, we can view

$$\text{comp}_{\text{dR}}^{\text{et}}(\vartheta_1 \otimes \Omega_+) \in H_{\text{et}}^1(X_1(\Delta_{\mathcal{K}} p^2), L)(\kappa_{LT}^{-1})^{G_E},$$

where G_E acts diagonally. If c is an étale class, for $\sigma \in G_E$, we denote by $\sigma \cdot c$ the diagonal action, by c^σ the action on the coefficients and $\sigma^* c$ the usual Galois action coming from the action on the base change of the modular curve to $\overline{\mathbb{Q}}$.

We now make a few remarks, we state them in the form of a Lemma.

8.2.3.3 Lemma. *1. By the analysis of section 8.2.1.15, we know that*

$$(\omega_\chi^{\text{et}})^\sigma = \omega_{\chi^\sigma}^{\text{et}}.$$

2. Choose representatives σ_i of $\text{Gal}(\mathcal{K}[p]/\mathcal{K})$ in $\text{Gal}(\mathcal{K}[\{p\}]/\mathcal{K})^{\text{ab}}$ such that

$$\text{Art}_{\mathcal{K}}(\sigma_i) = x(\sigma_i) := a_{\mathfrak{a}} \cdot y_p,$$

where the collection $\{\mathfrak{a}\}$ are ideals generating $\text{Cl}_{\mathcal{K}}$ coprime to p and $a_{\mathfrak{a}}$ are the corresponding product of local uniformizers, and $y_p \in \mathcal{O}_E^\times$ are representatives

of $(\mathcal{O}_E/p\mathcal{O}_E)^\times$. We remark that the elements

$$\omega_{\sigma_i}^{\text{et}} := \frac{1}{h} \sum_{\xi} \sigma_i^* \omega_{\mu\xi}^{\text{et}} \in \widehat{\mathcal{I}}(p)(\kappa_{LT}^{-1}) \subset H_{\text{et}}^1(X_1(\Delta\kappa p^2), E)(\kappa_{LT}^{-1})$$

depend only on the class of y_p modulo p such that $\text{Art}_{\mathcal{K}}(\sigma_i) = a_{\mathfrak{a}} \cdot y_p$. It is then easily checked that the action of G_E permutes this E -basis of $\widehat{\mathcal{I}}(p)(\kappa_{LT}^{-1})$.

3. We have the relation

$$\omega_{\mu\xi}^{\text{et}} = \sum_{i=1}^h \mu\xi(\sigma_i) \omega_{\sigma_i}^{\text{et}},$$

since $\sigma_i^* \omega_{\mu\xi}^{\text{et}} = \mu\xi(a_{\mathfrak{a}} y_p)^{-1} \omega_{\mu\xi}^{\text{et}}$, if $\text{Art}_{\mathcal{K}}(\sigma_i) = a_{\mathfrak{a}} y_p$.

4. If the F -basis consisting of the classes $\omega_{\chi}^{\text{et}}$ is replaced by the F -basis $t_{\chi} \omega_{\chi}^{\text{et}}$ for some $t_{\chi} \in F^\times$ such that $t_{\chi}^{\sigma} = t_{\chi^{\sigma}}$, then

$$\omega'_{\sigma_i} := \frac{1}{h} \sum_{\xi} t_{\mu\xi} \sigma_i^* \omega_{\mu\xi}^{\text{et}} \in H^1(X_1(\Delta\kappa p^2), E)(\kappa_{LT}^{-1}),$$

is an E -basis of $\widehat{\mathcal{I}}(p)(\kappa_{LT}^{-1})$.

By invoking point number 3³ of the Lemma just above, we have:

$$\text{comp}_{\text{dR}}^{\text{et}}(\vartheta_1 \otimes \Omega_+) = \sum_{i=1}^h \left(\frac{1}{h} \sum_{\xi} \mu\xi(\sigma_i) \lambda_{\mu\xi} \right) \omega_{\sigma_i}^{\text{et}} \in \left(H_{\text{et}}^1(X_1(\Delta\kappa p^2), L)(\kappa_{LT}^{-1}) \right)^{G_E}$$

Therefore, since the basis $\omega_{\sigma_i}^{\text{et}}$ is permuted by G_E by point 2 of Lemma 8.2.3.3,

³We note the difference between $\mu\xi(\sigma_i)$ and $\widehat{\mu\xi}(\sigma_i)$ the two differ only at p by the Lubin-Tate character.

we obtain the relation:

$$\forall i = 1, \dots, h, \forall \sigma \in G_E, \quad \frac{1}{h} \sum_{\xi} \mu\xi(\sigma_i) \lambda_{\mu\xi} = \frac{1}{h} \sum_{\xi} (\mu\xi)^{\sigma} (\sigma^{-1} \sigma_i) \lambda_{\mu\xi}^{\sigma}$$

Therefore, by Fourier inversion on $\text{Gal}(\mathcal{K}[p]/\mathcal{K})$, we obtain

$$\forall \sigma \in G_E, \lambda_{\chi}^{\sigma} = \chi^{\sigma}(\sigma) \lambda_{\chi^{\sigma}}.$$

This is exactly the type of relation verified by the universal period $\Omega_{\chi^{-1}}^{\text{univ}}$. This suggests setting:

$$t_{\chi} := \lambda_{\chi}(\Omega_{\chi^{-1}}^{\text{univ}})^{-1};$$

here by abuse of notation we denote $\Omega_{\chi}^{\text{univ}} = \Omega_{\chi|G_E}$. Then t_{χ} verifies the relation

$$t_{\chi}^{\sigma} = t_{\chi^{\sigma}}.$$

By point number 4 of Lemma 8.2.3.3, we can re-normalize the basis $\omega_{\chi}^{\text{et}}$ (and thus $\omega_{\sigma_i}^{\text{et}}$), so that $t_{\chi} = 1$. Therefore under this renormalization, we obtain:

$$\text{comp}_{\text{dR}}^{\text{et}}(\vartheta_1 \otimes \Omega_+) = \sum_{i=1}^h \left(\frac{1}{h} \sum_{\xi} \mu\xi(\sigma_i) \Omega_{(\mu\xi)^{-1}}^{\text{univ}} \right) \omega_{\sigma_i}^{\text{et}}. \quad (8.19)$$

Note that when ξ ranges over characters of $\text{Cl}_p \simeq \text{Gal}(\mathcal{K}[p]/\mathcal{K})$, the local component at p of $\mu\xi$ ranges over all *odd* characters of $(\mathcal{O}_{\mathcal{K}}/p\mathcal{O}_{\mathcal{K}})^{\times}$ in the sense of Section 8.2.1.1.

We assume for simplicity hereafter that $\mathcal{O}_{\mathcal{K}}^{\times} = \{\pm 1\}$

The hypothesis above implies that "odd" characters χ of $(\mathcal{O}_K/p\mathcal{O}_K)^\times$ has the usual meaning, i.e. $\chi(-1) = -1$.

We now need to understand the averaged universal periods that appear in the above formula. Recall the notation ϖ_0 for the uniformizer of E_0 , where E_0 is the maximal totally ramified extension of E of prime to p degree. The following Lemma is an immediate consequence of the definition and the summation formula for geometric series.

8.2.3.4 Lemma. *Let $\chi : G_E \rightarrow \overline{\mathbb{Q}}_p^\times$ be a finite order character. Assume that $\chi(\text{Frob}_p) = 1$ and that $\chi|_{I_E}$ factors through $\text{Gal}(E_n^{ac}/E)$, $n \geq 0$. We say that χ is odd when $t(\chi)$ (in the notations of Definition 8.2.2.15) is odd. Let r be such that such that $n = 2r + 1$ if n is odd or $n = 2r$ if n is even, and $r = 0$ if χ factors through $\text{Gal}(E_0/E)$. Let $h = \#\text{Cl}_{p^n}$. We have*

$$\forall \sigma \in G_E, \quad \frac{1}{h} \sum_{\chi \text{ odd}} \chi(\sigma) \Omega_{\chi^{-1}}^{\text{univ}} = \left(2\text{Teich}_E(\sigma) \varpi_0 \frac{\varpi_0^{p^2-1} - 1}{(p^2 - 1)(\varpi_0 - 1)} \right) \cdot \begin{cases} \frac{(c_{2r+1}^- + c_{2r}^+)^\sigma}{p^{r+1}} & \text{if } n \text{ is odd} \\ \frac{(c_{2r}^+ + c_{2r-1}^-)^\sigma}{p^r} & \text{if } n \text{ is even,} \end{cases}$$

where the sum over χ ranges over all characters χ factoring through $\text{Gal}(E_n^{ac}/E)$. In particular if $n \geq 1$, we have :

$$\forall \sigma \in G_E, \quad v_p \left(\frac{1}{h} \sum_{\chi \text{ odd}} \chi(\sigma) \Omega_{\chi^{-1}}^{\text{univ}} \right) = \frac{1}{p^2 - 1} + \epsilon_n - \left\lceil \frac{n}{2} \right\rceil,$$

where $\epsilon_n \in \mathbb{Q}$ verifies $\epsilon_n \leq \frac{1}{p^{2(n-1)}(p^2-1)}$. And if $n = 0$, the average has p -adic valuation $\frac{1}{p^2-1}$.

Combining this Lemma with Equation 8.19, we have the following proposition.

8.2.3.5 Proposition. *Having fixed the basis $\{\omega_{\sigma_i}\}$, there exists a finite extension $L/\mathbb{Q}_p$ and elements $\alpha(\sigma_i) \in L$ such that*

$$\text{comp}_{\text{dR}}^{\text{et}}(\vartheta_1 \otimes 1) = \sum_{i=1}^h \alpha(\sigma_i) \omega_{\sigma_i}^{\text{et}} \otimes t^{-1} \Omega_-.$$

Moreover the $\alpha(\sigma_i)$ have p -adic valuation $\frac{1}{p^2-1}$.

Since the ω_{σ_i} are all Galois images of one another, there exists a minimal integer n_0 such that, for all $i = 1, \dots, h$,

$$p^{n_0} \omega_{\sigma_i} \in H_{\text{et}}^1(X_1(\Delta_{\mathcal{K}} p^2), \mathcal{O}_E(\kappa_{LT}^{-1}));$$

that is, such that the $p^{n_0} \omega_{\sigma_i}$ is a basis of $\widehat{\mathcal{I}}(p)(\kappa_{LT}^{-1}) \otimes (\mathcal{O}_E/p\mathcal{O}_E)$. But since the image of $p\text{Falt}(\vartheta_1 \otimes \rho)$ is 0 modulo p , and since $\Omega_- \alpha(\sigma_i)$ has p -adic valuation $\frac{1}{p-1}$, it follows that $n_0 = 0$ and that the ω_{σ_i} are integral ($p \geq 3$).

We have therefore proven the following proposition.

8.2.3.6 Proposition. *Let $h = \#\text{Cl}_p$. There exists a basis $\{\omega_{\sigma_i}^{\text{et}}\}_{1 \leq i \leq h}$,*

$$\omega_{\sigma_i}^{\text{et}} \in \widehat{\mathcal{I}}(p) \subseteq H_{\text{et}}^1(X_1(\Delta_{\mathcal{K}} p^2), \mathcal{O}_E),$$

such that, setting

$$\omega_{\chi}^{\text{et}} := \sum_{i=1}^h \rho_{\chi,p}(\sigma_i)^{-1} \omega_{\sigma_i}^{\text{et}} \in H_{\text{et}}^1(X_1(\Delta_{\mathcal{K}} p^2), \mathcal{O}_F),$$

we have

$$\mathrm{comp}_{\mathrm{dR}}^{\mathrm{et}}(\vartheta_{\chi}^{\mathrm{dR}} \otimes 1) = \omega_{\chi}^{\mathrm{et}} \otimes t^{-1} \Omega_{-} \Omega_{\chi^{-1}}^{\mathrm{univ}}.$$

8.2.3.7 The construction of the anticyclotomic tower

We now generalize the construction of the previous section to allow for arbitrary level in the anticyclotomic tower. Let μ be a Hecke character of infinity type $(1, 0)$ and level p . Let n be a squarefree integer coprime to p and let $N = n^2$. Let $R[np^r]$ be the ring class field of $\mathcal{K}$ of conductor np^r . We now consider characters

$$\xi : \mathrm{Gal}(\mathcal{K}[np^r]/\mathcal{K}) \rightarrow \mathbb{C}^{\times},$$

such that there exists a character μ' of $\mathrm{Gal}(\mathcal{K}[p]/\mathcal{K})$ such that $\xi(\mu')^{-1}$ factors through $\mathrm{Gal}(R[np^r]/\mathcal{K})$. If ξ is of this form, we say that ξ is almost anticyclotomic. Stated differently, these are all the characters of $\mathrm{Gal}(R[np^r]\mathcal{K}[p]/\mathcal{K})$. If χ is a Hecke character of infinity type $(1, 0)$ and conductor np^r such that $\chi = \mu\xi$ with ξ almost anticyclotomic, we say that χ is almost anticyclotomic.

For simplicity, we set

$$h = h(np^r) = \#\mathrm{Gal}(R[np^r]\mathcal{K}[p]/\mathcal{K}).$$

We note that when $n = 1$ and $r = 1$, $h = \#\mathrm{Cl}_p$, so notation is compatible with the previous section. We let $\mathcal{X}^{aac}(np^r)$ be the set of almost anticyclotomic characters of conductor np^r . Let $\widehat{\mathcal{I}}^{aac}(np^r) \subseteq \widehat{\mathcal{I}}(np^r)$ be the subrepresentation of I_{np^r} defined by the kernel of the operators $(\alpha) - \alpha \cdot \mathrm{id}$, for $\alpha \in \mathcal{K}$ ranging over elements that reduce

modulo n to a (rational) integer and modulo p^r to an integer times an element of prime-to- p order. Denote for simplicity

$$\mathcal{R} := R[np^r]\mathcal{K}[p].$$

We now consider

$$\vartheta_{r,n} := \frac{1}{h} \sum_{\chi \in \mathcal{X}^{aac}(np^r)} \pi_\chi^* \vartheta_\chi \in H^0(X_1(\Delta_{\mathcal{K}} N p^{2r}), \Omega^1) \otimes E.$$

Here π_χ is defined as follows. If the conductor of χ is dp^t for some $d|n$ and $t \leq r$, then $\pi_\chi = S_{n/d} \pi_{r-t}$, where $S_{n/d}$ is the depletion operator as defined in Section 8.2.1.1, and π_{r-t} is the pull-back under $z \mapsto p^{2(n-t)}z$. The modular form $\vartheta_{r,n}$ is the analogue of the average of theta series ϑ_1 defined in the previous section: more precisely, we have $\vartheta_{1,1} = \vartheta_1$. Let $\vartheta_{r,n}^{\text{dR}}$ be the de Rham class defined by $\vartheta_{r,n}$.

Consider lifts $\sigma_i \in \text{Gal}(\mathcal{K}[\{np^r\}]/\mathcal{K})^{ab}$ of $\text{Gal}(\mathcal{R}/\mathcal{K})$, such that $\text{Art}_{\mathcal{K}}(\sigma_i) = a_{\mathfrak{a}} \cdot y_n \cdot y_p$, where $a_{\mathfrak{a}}$ is exactly as in Lemma 8.2.3.3, $y_p \in \mathcal{O}_E$ ranges over representatives of $(\mathcal{O}_E/p^r \mathcal{O}_E)^\times / (1 + p\mathbb{Z}_p)$ and $y_n \in \widehat{\mathcal{O}}_{\mathcal{K},n}$ over representatives of $(\mathcal{O}_{\mathcal{K}}/n\mathcal{O}_{\mathcal{K}})/(\mathbb{Z}/n\mathbb{Z})^\times$. Then setting

$$\omega_{\sigma_i}^{\text{et}} := \frac{1}{h} \sum_{\chi \in \mathcal{X}^{aac}(np^r)} \sigma_i^* \pi_\chi^* \omega_\chi^{\text{et}}$$

defines a rational basis of $\widehat{\mathcal{I}}^{aac}(np^r) \subseteq H_{\text{et}}^1(X_1(\Delta_{\mathcal{K}} N p^{2r}), E)$. As before, we can

renormalize the vectors ω_χ^{et} for characters χ of conductor exactly np^r such that

$$\text{comp}_{\text{dR}}^{\text{et}}(\vartheta_{r,n}^{\text{dR}} \otimes \Omega_+) = \sum_{i=1}^h \left(\frac{1}{h} \sum_{\chi \in \mathcal{X}^{\text{aac}}(np^r)} \chi(\sigma_i) \Omega_{\chi^{-1}}^{\text{univ}} \right) \omega_{\sigma_i}^{\text{et}}.$$

When ξ ranges over characters of $\text{Gal}(\mathcal{R}/\mathcal{K})$, the local component at p of $\mu\xi$ ranges over characters totally ramified characters of G_E that factor through $\Delta_E \times \text{Gal}(E_r^{\text{ac}}/E)$ and that are odd on Δ_E (in the same sense as in the previous paragraph). By Lemma 8.2.3.4, and exactly the same reasoning of the previous paragraph, we obtain that ω_{σ_i}/p^a is integral, where $a = \lceil n/2 \rceil$ and *a fortiori* ω_{σ_i} .⁴

We therefore obtain the following Proposition, generalizing proposition 8.2.3.6.

8.2.3.8 Proposition. *Let $\mathcal{R} := R[np^r]\mathcal{K}[p]$, where $R[np^r]$ is the ring class field of $\mathcal{K}$ of conductor np^r . Put $h := \#\text{Gal}(\mathcal{R}/\mathcal{K})$ and $N = n^2$. There exists lifts $\{\sigma_i \in G_{\mathcal{K}}^{\text{ab}}\}_{1 \leq i \leq h}$ of $\text{Gal}(\mathcal{R}/\mathcal{K})$ and a basis $\{\omega_{\sigma_i}\}_{1 \leq i \leq h}$ of $\widehat{\mathcal{I}}^{\text{aac}}(np^r) \subset H_{\text{et}}^1(X_1(\Delta_{\mathcal{K}}Np^{2r}), E)$, such that, for any almost anticyclotomic character χ of conductor exactly np^r , setting*

$$\omega_\chi^{\text{et}} = \sum_{i=1}^h \rho_{\chi,p}(\sigma_i)^{-1} \omega_{\sigma_i}^{\text{et}},$$

we have

$$\text{comp}_{\text{dR}}^{\text{et}}(\vartheta_\chi^{\text{dR}}) = \omega_\chi^{\text{et}} \otimes t^{-1} \Omega_- \Omega_{\chi^{-1}}^{\text{univ}}.$$

Moreover, $\omega_{\sigma_i} \in H_{\text{et}}^1(X_1(\Delta_{\mathcal{K}}Np^{2r}), \mathcal{O}_E)$ and if χ is of conductor $\mathfrak{f} = dp^t$ dividing np^r ,

⁴The author does not know why integrality improves as one climbs the anticyclotomic tower.

we have

$$\sum_{i=1}^h \rho_{\chi,p}(\sigma_i)^{-1} \omega_{\sigma_i}^{\text{et}} = \pi_{\mathfrak{f}}^* \omega_{\chi}^{\text{et}},$$

where $\pi_{\mathfrak{f}} = S_{N/d^2} \pi_{n-t}$ and S_{N/d^2} is the depletion operator defined in Section 8.2.1.1, while π_{n-t} is the map defined by $z \in \mathcal{H}^{\pm} \mapsto p^{2(n-t)} z \in \mathcal{H}^{\pm}$.

We now combine the above proposition for varying level n and r into the following theorem.

8.2.3.9 The construction for varying levels

Let μ be a fixed Hecke character of infinity type $(1, 0)$ and conductor p . Let $F/\mathbb{Q}_p$ be the field of values of the p -adic Hecke character $\hat{\mu}$ and denote by $\mathcal{O}_F$ be the valuation ring of F . We define

$$\mathrm{H}_{\text{et}} := \varinjlim_{n,r} \mathrm{H}_{\text{et}}^1(X_1(\Delta_{\mathcal{K}} n^2 p^{2(r+1)}), \mathcal{O}_F)_{cm},$$

where:

- the subscript cm indicates the kernel of the spherical Hecke operators T_{ℓ} for ℓ inert in $\mathcal{K}$,
- the direct limit is taken with respect to integers $r \geq 0$ and squarefree n with n coprime to p over the depletion maps $S_d : \mathrm{H}_{\text{et}}^1(X_1(\Delta_{\mathcal{K}} n^2 p^{2(r+1)}), \mathcal{O}_F) \rightarrow \mathrm{H}_{\text{et}}^1(X_1(\Delta_{\mathcal{K}} n^2 d^2 p^{2(r+1)}), \mathcal{O}_F)$ away from p and $z \mapsto p^2 z$ at p .

Similarly, let

$$\mathrm{H}_{\text{dR}} = \varinjlim_{n,r} \mathrm{H}_{\text{dR}}^1(X_1(\Delta_{\mathcal{K}} n^2 p^{2(r+1)})/F)_{cm}.$$

Define also $H_{\text{et}}(\mu) \subset H_{\text{et}}$ to be the sub-representation defined by

$$H_{\text{et}}(\mu) := (H_{\text{et}} \otimes \rho_{\mu,p}^{-1})^{\Delta_E \text{Gal}(\mathbb{Q}^{ab}/\mathcal{K})}.$$

Here recall that $\Delta_E \simeq \text{Gal}(E_0/E) \hookrightarrow G_E^{ab}$, with notations as in subsection 8.2.2.6.

Denote by $\mathcal{R}'_{\infty}$ the compositum of all the ring-class fields of $\mathcal{K}$ of conductor coprime to $\Delta_{\mathcal{K}}$ and squarefree away from p . Put

$$\mathcal{R}_{\infty} := (\mathcal{R}'_{\infty})^{\Delta_E}.$$

The $G_{\mathcal{K}}$ action on $H_{\text{et}}(\mu)$ factors through $\text{Gal}(\mathcal{R}_{\infty}/\mathcal{K})$. Let $\mathcal{C}_{lc}(\text{Gal}(\mathcal{R}_{\infty}/\mathcal{K}), \mathcal{O}_F)$ be the space of $\mathcal{O}_F$ -valued locally constant function defined on $\text{Gal}(\mathcal{R}_{\infty}/\mathcal{K})$. We endow this space with a $\text{Gal}(\mathcal{R}_{\infty}/\mathcal{K})$ -action: $\sigma \cdot \phi = \phi(\sigma^{-1} \cdot)$ for $\phi \in \mathcal{C}_{lc}(\text{Gal}(\mathcal{R}_{\infty}/\mathcal{K}), \mathcal{O}_F)$ and $\sigma \in \text{Gal}(\mathcal{R}_{\infty}/\mathcal{K})$

Proposition 8.2.3.8 can be reformulated as follows.

8.2.3.10 Theorem. *Let notations be as above. There exists a unique Galois equivariant embedding:*

$$\Theta_{\mu} : \mathcal{C}_{lc}(\text{Gal}(\mathcal{R}_{\infty}/\mathcal{K}), \mathcal{O}_F) \hookrightarrow H_{\text{et}}(\mu)$$

such that if $\xi \in \mathcal{C}_{lc}(\text{Gal}(\mathcal{R}_{\infty}/\mathcal{K}), \mathcal{O}_F) \otimes \mathcal{K}_{\chi,p}$ is a finite order character of $\text{Gal}(\mathcal{R}_{\infty}/\mathcal{K})$, then

$$\text{comp}_{\text{dR}}^{\text{et}}(\Theta_{\mu}(\xi)) = \text{im}(\vartheta_{\mu\xi}^{\text{dR}}) \otimes \Omega_+(\Omega_{(\mu\xi)^{-1}}^{\text{univ}})^{-1}.$$

Here $\text{im}(\vartheta_{\mu\xi}^{\text{dR}})$ denotes the image in H_{dR} of the de Rham class $\vartheta_{\mu\xi}^{\text{dR}}$ as defined in section 8.2.3.7.

8.2.4 The absolute étale classes

We now comment on liftings of the classes $\Theta_\mu(\phi)$ for $\phi \in \mathcal{C}_{lc}(\mathrm{Gal}(\mathcal{R}_\infty/\mathcal{K}), \mathcal{O}_F)$ to absolute étale cohomology classes.

8.2.4.1 Lemma. *There exists infinitely many rational primes ℓ such that ℓ is inert in $\mathcal{K}$ and $\ell \not\equiv -1 \pmod{p}$.*

Proof. Indeed, if $p > 3$, one can reason with Dirichlet densities. If $p = 3$, the claim of the Lemma is a consequence of quadratic reciprocity and the fact that we assumed $p \nmid 4\Delta_{\mathcal{K}}$. $\square$

Consider now $\phi \in \mathcal{C}_{lc}(\mathrm{Gal}(\mathcal{R}_\infty/\mathcal{K}), \mathcal{O}_F)$. Let ℓ be a prime satisfying the conditions of the previous Lemma and at which ϕ is unramified (ϕ is invariant under the action of inertia at ℓ .)

Let N, r be integers such that if we set $X := X_1(\Delta_{\mathcal{K}} N p^{2r})$, we have

$$\Theta_\mu(\phi) \in H_{\mathrm{et}}^1(X/\overline{\mathbb{Q}}, \mathcal{O}_F) \otimes H_{\mathrm{et}}^1(A/\overline{\mathbb{Q}}, \mathcal{O}_F(1)) \hookrightarrow H_{\mathrm{et}}^2(X \times A/\overline{\mathbb{Q}}, \mathcal{O}_F(1)).$$

Here $(A, 1)$ is the canonical CM-pair of modulus p , and we have extended the action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathcal{K}[p])$ on $H_{\mathrm{et}}^1(A/\overline{\mathbb{Q}}, \mathcal{O}_F(1))$ so that it is given by $\rho_{\mu, p}^{-1}$ in a similar fashion as in section 8.2.1.

Then, for the rational primes ℓ considered above, we have

$$T_\ell \Theta_\mu(\phi) = 0.$$

Let be $\mathcal{R}(\phi)$ the finite extension of $\mathcal{K}$, such that the translation action of $\text{Gal}(\mathcal{R}_\infty/\mathcal{K})$ factors through $\text{Gal}(\mathcal{R}_\phi/\mathcal{K})$. There is a Leray Serre spectral sequence

$$E_2^{i,j} = H^i\left(\mathcal{R}(\phi), H_{\text{et}}^j(X \times A/\overline{\mathbb{Q}}, \mathcal{O}_F(1))\right) \Rightarrow H_{\text{et}}^{i+j}(X \times A/\mathcal{R}(\phi), \mathcal{O}_F(1)).$$

We denote by $E_\infty^{i,j}$ the limit of the spectral sequence.

By Galois-equivariance of Θ_μ , we notice that

$$\Theta_\mu(\phi) \in H^0(\mathcal{R}(\phi), H_{\text{et}}^2(X \times A/\overline{\mathbb{Q}}, \mathcal{O}_F(1))).$$

Moreover, $\Theta_\mu(\phi) \in E_\infty^{0,1}$ because T_ℓ acts by multiplication by $\ell + 1$ on $H_{\text{et}}^0(X/\overline{\mathbb{Q}})$. Therefore, since $\ell + 1 \not\equiv 0 \pmod{p}$ by our choice of ℓ , and by considering the action of the extra endomorphisms of A , $\Theta_\mu(\phi) \in H_{\text{et}}^2(X \times A/\mathcal{R}(\phi), \mathcal{O}_F(1))/\text{Fil}^1 H_{\text{et}}^2(X \times A/\mathcal{R}(\phi), \mathcal{O}_F(1))$. But again T_ℓ acts by multiplication by $\ell + 1$ or the extra endomorphisms of A act trivially on $\text{Fil}^i H_{\text{et}}^2(X \times A/\mathcal{R}(\phi), \mathcal{O}_F(1))$ for $i \geq 1$, so that $\Theta_\mu(\phi)$ must lift uniquely to an absolute étale class in $H_{\text{et}}^2(X \times A/\mathcal{R}(\phi), \mathcal{O}_F(1))$.

We sum up this discussion in the following proposition.

8.2.4.2 Proposition. *Let notations be as in Proposition 8.2.3.8. Then the class*

$$\omega_1 \in H_{\text{et}}^1(X_1(\Delta_{\mathcal{K}} N p^{2r})/\overline{\mathbb{Q}}, \mathcal{O}_F) \otimes \rho_{\mu,p}^{-1} \subseteq H_{\text{et}}^2(X_1(\Delta_{\mathcal{K}} N p^{2r}) \times A/\overline{\mathbb{Q}}, \mathcal{O}_F(1))$$

has a unique lift in absolute étale cohomology

$$\omega_1^{\text{abs}} \in H_{\text{et}}^2(X_1(\Delta_{\mathcal{K}} N p^{2r}) \times A/\mathcal{R}[np^r], \mathcal{O}_F(1)).$$

Part IV

Appendix

Chapter 9

Appendix to "An Euler System for the Adjoint of a Modular Form"

(A) Eisenstein series on $\mathbf{SO}(3, 2)$ vs. $\mathbf{Sp}(4)$

In this section we elaborate on the relationship between the Eisenstein series on the orthogonal group $\mathbf{G} = \mathbf{SO}(W', q) \simeq \mathbf{SO}(3, 2)$ under consideration in this paper and the analogous Eisenstein series on $\mathbf{G}' = \mathbf{Sp}(V', J) \simeq \mathbf{Sp}(4)$. Here $\mathbf{G}$ and $\mathbf{G}'$ are as in Section 3.1.1.

As Hirai remarks, just after Theorem 7.4 in [Hir98], the orthogonal Eisenstein series lifts to an Eisenstein series on $\mathbf{Sp}(4)$: Hirai remarks that the computation he just carried out in *loc cit.* retrieves the formula he finds in [Hir99], taking the quaternion algebra in the latter article to be everywhere split. We instead elaborate on the relationship between Hirai's Eisenstein series and the Siegel Eisenstein series

appearing in [Liu20].

As in 3.1.1.2, we will systematically decorate with a ' all data pertaining to the symplectic group, so as to avoid any confusion in the notations.

(A.1) The Siegel parabolic

We give explicit formulas for the 2-to-1 isogeny $\pi : \mathbf{G}' \rightarrow \mathbf{G}$ on the Siegel parabolic.

Let $\mathbf{P}'_1$ be the subgroup of $\mathbf{G}'$ defined as the stabilizer of the totally isotropic plane spanned by ε_1 and ε_2 . Let $\mathbf{U}'_1$ be its unipotent radical, and let $\mathbf{M}'_1$ be its usual Levi subgroup. All elements of $\mathbf{P}'_1$ leave invariant the line spanned by $e_1 = \varepsilon_1 \wedge \varepsilon_2 \in W'$, so $\mathbf{P}'_1$ is mapped to $\mathbf{P}_1$ under π . Here $\mathbf{P}_1$ is the parabolic subgroup of $\mathbf{G}$ stabilizing the isotropic line spanned by e_1 (see 3.1.2).

Let $\text{Ad} : \mathbf{SL}_2 \rightarrow \mathbf{SO}(V_3)$ be the adjoint representation of $\mathbf{SL}_2$. We can write elements $u \in \mathbf{U}'_1(R)$ and $m \in \mathbf{M}'_1(R)$ as

$$u = u'(T) := \begin{pmatrix} I_2 & T \\ 0 & I_2 \end{pmatrix} \quad \text{and} \quad m = m'(A) := \begin{pmatrix} A & \\ & (A^\top)^{-1} \end{pmatrix},$$

where $T \in \text{Sym}^2(R^2)$ and $A \in \mathbf{GL}_2(R)$. Then π restricted to the Siegel parabolic $\mathbf{P}'_1$ is described by the formulas (here matrices are written in the basis $\mathcal{B}$):

$$\pi(m'(A)) = m(\det(A), \text{Ad}(A)) := \begin{pmatrix} \det(A) & & \\ & \text{Ad}(A) & \\ & & \det(A)^{-1} \end{pmatrix}$$

and

$$\pi(u'(T)) = n_1(X_T) := \begin{pmatrix} 1 & X_T & -S_3[X_T]/2 \\ & 1 & S_3X_T^\top \\ & & 1 \end{pmatrix}$$

where if $T = \begin{pmatrix} t_1 & t_0 \\ t_0 & t_2 \end{pmatrix}$ then $X_T = [-t_1, 2t_0, t_2]$. In particular, as 2 is invertible in R , π defines an isomorphism $\mathbf{U}'_1(R) \simeq \mathbf{U}_1(R)$.

(A.2) Similitude groups

Here we briefly explain how the isogeny π extends to similitude groups.

We define $\mathbf{GG}'$ to be the algebraic group whose R -points for any A -algebra R are given by $\mathbf{GG}'(R) := \mathbf{GSp}(V', J)(R)$. We denote by $\nu' : \mathbf{GG}' \rightarrow \mathbf{G}_m$ the similitude factor. Then $g \in \mathbf{GG}'$ leaves invariant b up to a factor of $\nu(g)^2$. Thus, π extends to an isogeny $\pi : \mathbf{GG}' \rightarrow \mathbf{GSO}(W', q) \subset \mathbf{GO}(W', q)$, such that on R -points the image of $\mathbf{GG}'(R)$ in $\mathbf{GSO}(W', q)(R)$ is exactly the image of the natural map $R^\times \times \mathbf{SO}(W_R) \rightarrow \mathbf{GSO}(W', q)(R)$. We denote the algebraic group whose R -points are given by this image by $\mathbf{GG}$.

(A.3) Symmetric spaces

Let $\mathcal{H}_2$ be the space of complex 2×2 symmetric matrices with positive definite imaginary part. Then $\mathcal{H}_2$ can be identified with the symmetric space associated to $\mathbf{G}'$. Recall from section 3.1.4, that the symmetric space attached to $\mathbf{G}$ is $\mathcal{H}^\circ$.

Consider the map

$$\beta : \mathcal{H}_2 \longrightarrow \mathcal{H}^\circ$$

$$Z \longmapsto X_Z$$

where if $Z = \begin{pmatrix} z_1 & z_0 \\ z_0 & z_2 \end{pmatrix}$, $X_Z = [-z_1, 2z_0, z_2]$. Note that $X_Z \in \mathcal{H}^\circ$ as $q_3(\Im(X_Z)) = -4\det(\Im(Z)) < 0$ and $q_3(e_5, \Im(X_Z)) = -2\Im(z_1)$ which is negative by Sylvester's criterion.

We refer to the main body of the text (see Section 3.1.4.2) for explicit formulas for the $\mathbf{G}(\mathbb{R})$ action on $\mathcal{H}^\circ$. One can then see explicitly that β intertwines the group action of $\mathbf{G}(\mathbb{R})$ with the action of $\mathbf{G}'(\mathbb{R})$ on $\mathcal{H}_2$, so that β defines a biholomorphism between $\mathcal{H}_2$ and $\mathcal{H}^\circ$ that intertwines the respective actions of $\mathbf{G}'(\mathbb{R})$ and $\mathbf{G}(\mathbb{R})$; that is, for all $g' \in \mathbf{G}'(\mathbb{R})$, $\beta(g' \cdot Z) = \pi(g') \cdot \beta(Z)$. This can be checked, by the Iwasawa decomposition, only using the formulas for $\mathbf{P}_1$, which is easy to verify.

Comparison of q -expansions. Let f be a holomorphic automorphic form on $\mathcal{H}^\circ$; let Γ the periodic lattice of f . Then β^*f is a holomorphic automorphic form on $\mathcal{H}_2$ with periodic lattice $\pi^{-1}(\Gamma)$. As π identifies $\mathbf{U}_1(\mathbb{R})$ and $\mathbf{U}'_1(\mathbb{R})$, the pull-back identifies the Fourier expansions of f and of β^*f . From the Adelic perspective, for any additive character $\psi : \mathbf{U}(\mathbb{A}) \rightarrow \mathbb{C}$ we have the equality, for all $g' \in \mathbf{G}(\mathbb{A})$:

$$\int_{\mathbf{U}_1(\mathbb{Q}) \backslash \mathbf{U}_1(\mathbb{A})} f(u\pi(g'))\psi(u)du = \int_{\mathbf{U}'_1(\mathbb{Q}) \backslash \mathbf{U}'_1(\mathbb{A})} \beta^*f(u'g')\psi(u')du'.$$

Here ψ' denotes the character that ψ defines on $\mathbf{U}'_1(\mathbb{A})$ under the isomorphism induced by π on unipotent radicals.

Let $e_{\mathbb{A}} = \otimes e_v$ be the usual additive character of $\mathbb{A}$, where e_v is defined by

$$e_v = \begin{cases} e^{-2\pi i \{x\}_v} & \text{if } v \neq \infty \\ e^{2\pi i x} & \text{if } v = \infty. \end{cases}$$

(A.4) The pullback Eisenstein series

In this subsection, we give some consequences of the previous discussion on the Eisenstein series we study in this paper. In particular, we write down the q expansions of the pull-backs of Hirai's Eisenstein series to $\mathbf{G}'$. We assume that the character χ is only ramified at p , for simplicity.

Let $\delta_{\mathbf{P}_1}$ be the modulus character of $\mathbf{P}_1$ and $\delta_{\mathbf{P}'_1}$ be the modulus character for the Siegel parabolic $\mathbf{P}'_1$. Recall that, $\delta_{\mathbf{P}_1} = |a^3|$ and $\delta_{\mathbf{P}'_1} = \det^3$.

For a prime ℓ , denote by $I_{\ell}(s, \chi) := \text{Ind}_{\mathbf{P}_1(\mathbb{Q}_{\ell})}^{\mathbf{G}(\mathbb{Q}_{\ell})}(\mu_s)$ and $I'_{\ell}(s, \chi) = \text{Ind}_{\mathbf{P}'_1(\mathbb{Q}_{\ell})}^{\mathbf{G}'(\mathbb{Q}_{\ell})}(\mu'_s)$, where $\mu_s = \delta_{\mathbf{P}_1}^{s+1/2}(\chi \circ a)$ and $\mu'_s = \delta_{\mathbf{P}'_1}^{s+1/2}(\chi \circ \det)$ are as in section 3.3.2.

(A.4.1) Lemma. *The pullback by π , where π is the morphism defined in section 3.1.1, defines an embedding*

$$\pi^* : I_{\ell}(s, \chi) \hookrightarrow I'_{\ell}(s, \chi)$$

that sends the spherical section of $I_{\ell}(s, \chi)$ to the spherical section of $I'_{\ell}(s, \chi)$ if χ is unramified at ℓ . Moreover, when $\ell = \infty$, the section $f_{\infty}(s) = J(\bullet, Y_0)^{-k} |J(\bullet, Z_0)|^{k-3s-3/2}$

maps to the section $f'_\infty(s) = j(\bullet, i)^{-k} |j(\bullet, i)|^{k-3s-3/2}$.

Proof. This is immediate, as pullback by π preserves the value at the identity. $\square$

(A.4.2) Remark. Actually, π induces an isomorphism of the induced representations. The inverse of the pull-back by π is $f' \mapsto f(g) = f'(\pi(g'))$ for any lift g' of g , this is well defined as the determinant of $-\text{id}$ is 1 on a 4 dimensional space.

(A.4.3) Remark. Denote by $\mathbf{GP}'_1$ the Siegel parabolic subgroup of $\mathbf{GG}'$. Note that there exists a $\mathbf{G}'$ -equivariant inclusion:

$$\text{Ind}_{\mathbf{P}'_1(\mathbb{A}_f)}^{\mathbf{G}'(\mathbb{A}_f)}(\mu_s) \hookrightarrow \text{Ind}_{\mathbf{GP}'_1(\mathbb{A}_f)}^{\mathbf{GG}'(\mathbb{A}_f)}(\mu_s),$$

defined by $f \mapsto \tilde{f}$, where for all $g \in \mathbf{GG}'$,

$$\tilde{f}(g) := f \left(\begin{bmatrix} I_2 & \\ & \nu'(g)^{-1} I_2 \end{bmatrix} g \right).$$

Note that this is well defined since f is left invariant under the unipotent radical of $\mathbf{P}'_1$. Moreover the restriction of functions in $\text{Ind}_{\mathbf{GP}'_1(\mathbb{A}_f)}^{\mathbf{GG}'(\mathbb{A}_f)}(\mu_s)$ provides a $\mathbf{G}'$ -equivariant inverse to this map. Therefore by transport of structure $\text{Ind}_{\mathbf{P}'_1(\mathbb{A}_f)}^{\mathbf{G}'(\mathbb{A}_f)}(\mu_s)$ acquires a right action of $\mathbf{GG}'(\mathbb{A}_f)$. We will use this fact in section (B.4).

(B) On q -expansions of Eisenstein series

If $S \in \text{Sym}^2(\mathbb{Q}^{\oplus 2})$ is a symmetric matrix and $\mathcal{F}'$ is a modular form on $\mathbf{G}'$, we denote by $\mathcal{F}'_T$ the T -th Fourier coefficient of $\mathcal{F}'$:

$$\mathcal{F}'_T(g') := \int_{\text{Sym}^2(\mathbb{Q}^{\oplus 2}) \backslash \text{Sym}^2(\mathbb{A}^{\oplus 2})} \mathcal{F}'(u'(S)g) \psi(-\text{Tr}(TS)) dS$$

Similarly, if $X \in (V_3)_{\mathbb{Q}}$, we denote by $\mathcal{F}_X$ the X -th Fourier coefficient of an automorphic form $\mathcal{F}$ on $\mathbf{G}$.

Let $E(g, s; f)$ be the Eisenstein series on $\mathbf{G}(\mathbb{A})$ resulting from the choice of a section $f = \otimes_v f_v$ of $\text{Ind}_{\mathbf{P}_1(\mathbb{A})}^{\mathbf{G}(\mathbb{A})}(\mu_s)$. And let $f' = \pi^* f$ be the section defined by the previous Lemma. Then by the previous section, if $T \leftrightarrow X_T$ and $g' \in \mathbf{G}'(\mathbb{A})$, we have an equality:

$$E_{X_T}(\pi(g'), s; f) = E'_T(g', s; f')$$

Moreover if $f = \otimes f_v$ and $f'_v = \pi^* f_v$, and $\det(T) \leftrightarrow S_3[T] \neq 0$, then we have the factorizations:

$$E_{X_T}(\pi(g'), s; f) = \prod_v W_{X_T, v}(\pi(g'), s; f) \quad \text{and} \quad E'_T(g', s; f') = \prod_v W'_{T, v}(g', s; f')$$

where for T non-degenerate we have:

$$W'_{T, v}(g', s; f') = \int_{\text{Sym}^2(\mathbb{Q}_v)} f'_v(s)(wu'(S)g') e_v(-\text{Tr}(TS)) dS.$$

We have equality of local Fourier coefficients:

$$W_{X_{T,v}}(\pi(g'), s; f) = W'_{T,v}(g', s; f').$$

(B.1) The ordinary Eisenstein series

By the discussion above, it suffices to compute the Fourier expansion of the Eisenstein series on $\mathbf{G}' = \mathbf{Sp}_4$. In this section, we define the ordinary Eisenstein series that will be used to prove integrality of the cohomology classes that form the Euler system we construct in this paper.

If χ is ramified at p we denote the log-conductor of χ at o by $\mathfrak{f}_\chi$, so that for example if χ is only ramified at p , we have $N_\chi := p^{\mathfrak{f}_\chi}$. We denote by χ_v the local component of χ at v . Recall that we denote by $\tau(\chi)$ the Gauss sum:

$$\tau(\chi) := \sum_{x \bmod \mathbb{Z}/p^{\mathfrak{f}_\chi}\mathbb{Z}} \chi(x) e(x/p^{\mathfrak{f}_\chi}).$$

Let $\alpha : \mathbb{Q}^\times \backslash \mathbb{A}^\times \rightarrow \mathbb{C}^\times$ be the adelic character attached to χ . The local component of α at a place v of $\mathbb{Q}$ is described by the following:

- If $v = \infty$, $\alpha_v(x) = \text{sgn}(x)$.
- If $v = q$ is finite and does not divide N_χ , $\alpha_v(q) = \overline{\chi}(q)$ and $\alpha_v(u) = 1$ for $u \in \mathbb{Z}_q^\times$.
- If $v = p$ divides N_χ , $\alpha_v(p) = \prod_{q|N_\chi/p^{\mathfrak{f}_\chi}} \overline{\chi}_q(p)$ and $\alpha_v(u) = \chi(u)$ for $u \in \mathbb{Z}_p^\times$.

(B.1.1) Choices of sections

In this section we follow the notations of [Tak12, p. 2.2] and denote by $C_0(\ell^r) \subset \mathbf{G}'(\mathbb{Z}_\ell)$ the subgroup of elements that reduce to elements of $\mathbf{P}'_1(\mathbb{Z}/\ell^r\mathbb{Z})$. This is the pull-back of $K_0^{\mathbf{P}}(\ell^r)$ to $\mathbf{G}'(\mathbb{Z}_\ell)$ (see section ?? for notations.). We will view α as also a character of $C_0(N_\chi)$ by composing α with the determinant of the 2×2 lower right block of a matrix in $C_0(N_\chi)$.

We now define the sections that we will use to define ordinary Eisenstein series. We rescale the s variable slightly from section 3.3.2, so as to be closer to the notations of [Tak12]. We caution the reader that the character ψ of *loc cit.* in our notations is $\bar{\chi}$.

(B.1.2) Definition. For v a place of $\mathbb{Q}$ define the function $f_{v,s}^\chi \in I'_v(s, \chi)$ of $\mathbf{G}'(\mathbb{Q}_v)$ as:

1. If $v = \infty$,

$$f_{\infty,s}^\chi(g_\infty) := J(g_\infty, Z_0)^{-3} |J(g_\infty, Z_0)|^{-s+1/2}.$$

2. If v is finite, and χ unramified at v , then for $g_v = u\xi$ where $u \in \mathbf{P}'_1(\mathbb{Q}_v)$, $\xi \in \mathbf{G}'(\mathbb{Z}_v)$, with

$$u = \begin{pmatrix} a_u & b_u \\ 0 & d_u \end{pmatrix}$$

we set:

$$f_{p,s}^\chi(g_p) = |\det(a_u)|_p^{s-1/2+3} \bar{\alpha}_p(\det(d_u)).$$

That is $f_{v,s}^\chi$ is the spherical vector of $I'_v(s, \chi)$ with value 1 at the identity.

3. For $v = p$, and χ is ramified at p , $f_{p,s}$ is supported on the small cell $\mathbf{P}'_1(\mathbb{Q}_p)C_0(p^{\mathbf{f}_\chi})$,
and for $g_p = u\xi$, $u \in \mathbf{P}'_1(\mathbb{Q}_p)$, $\xi \in C_0(p^{\mathbf{f}_\chi})$, we set

$$f_{p,s}^\chi(g_p) = |\det(a_u)|_p^{s-1/2+3} \overline{\alpha}_p(\det(d_u)) \alpha_p(\xi).$$

For $g \in \mathbf{G}(\mathbb{A})$, $g = (g_v)_v$, we set

$$f_s^\chi(g) := \prod_v f_{v,s}(g_v)$$

(B.1.3) Remark. For the sake of clarity, we discuss here the sections of $I_v(s, \chi)$, whose pull-back to $\mathbf{G}'(\mathbb{Q}_v)$ by π are the section $f_{v,s}^\chi$. At infinity and at places $v \neq p$ it is given by Lemma (A.4.1). At a place p where χ is ramified, we define $f_{p,s}^\chi \in I_p(s, \chi)$ by

$$f_{p,s}^\chi(g_p) := \begin{cases} 0 & \text{if } g_p \notin \mathbf{P}_1(\mathbb{Q}_p)K_0(p^{\mathbf{f}_\chi}) \\ |t_u|^{3+s-1/2} \alpha_p(t_u) \alpha_p^{-1}(\xi) & \text{if } g_p = u\xi, \end{cases}$$

where

$$u = \begin{pmatrix} t_u & * & * \\ 0 & A_u & * \\ 0 & 0 & t_u^{-1} \end{pmatrix} \in \mathbf{P}_1(\mathbb{Q}_p) \quad \text{and } \xi \in K_0(p^{\mathbf{f}_\chi})$$

and we view α_p as a character of $K_0(p^{\mathbf{f}_\chi})$ by composing with the matrix coefficient of the bottom right entry of an element ξ of $K_0(p^{\mathbf{f}_\chi})$.

(B.1.4) Remark. Note that for all $g \in \mathbf{G}'(\mathbb{A})$ and $\gamma \in C_0(N_\chi)$

$$f_s^\chi(g\gamma) = \left(\prod_{p|N_\chi} \alpha_p(\gamma_p) \right) f_s^\chi(g).$$

In particular f_s^χ is of level $K_1^{\mathbf{P}}(N_\chi)$ and Nebentypus χ .

From f_s^χ defined above, we can define the Eisenstein series:

$$E(g; f_s^\chi) := \sum_{\gamma \in \mathbf{P}'_1(\mathbb{Q}) \backslash \mathbf{G}'(\mathbb{Q})} f_s^\chi(\gamma g).$$

The right hand-side is absolutely convergent when $\Re(s) > 1/2$.

Classically, choosing $Z_0 = iI_2 \in \mathcal{H}_2$. Then a point $Z \in \mathcal{H}_2$ can be written as $Z = X + iY$, with $\text{Im}(Y) > 0$. We notice that $g_\infty I = Z$ where $g_\infty = \begin{pmatrix} y^{1/2} & xy^{-1/2} \\ 0 & y^{-1/2} \end{pmatrix}$.

We can then write:

$$E_\chi(g_\infty; f_s^\chi) J(g_\infty, Z_0)^3 = |\det(y)|^{(s-1)/2} \sum_{\gamma \in \mathbf{P}_1(\mathbb{Q}) \cap \Gamma_0(N_\chi) \backslash \Gamma_0(N_\chi)} \chi(\gamma) J(\gamma, Z)^{-3} |J(\gamma, Z)|^{-s+1/2}.$$

Here $\Gamma_0(N_\chi)$ denotes the subgroup of $\mathbf{G}'(\mathbb{Z})$ such that the lower left 2×2 block is divisible by N modulo N_χ and we regard χ as a character of $\Gamma_0(N)$ by composing with the determinant of the lower right 2×2 block.

(B.1.5) The Fourier expansion of the ordinary Eisenstein series

In this subsection, we give the Fourier expansion of the Eisenstein series defined by the choice of local sections of Definition (B.1.2). The letter w will always denote the element $w := \begin{pmatrix} 0 & -I_2 \\ I_2 & 0 \end{pmatrix} \in \mathbf{G}'$. Let $T \in \text{Sym}^2(\mathbb{Z}_p^{\oplus 2})$ be non-degenerate half-integral and ψ be a Dirichlet character modulo p^n . Following [Tak12], we set

$$S_p(T, \psi, s) := \sum_{X \in \text{Sym}^2(\mathbb{Q}_p^{\oplus 2}) / \text{Sym}^2(\mathbb{Z}_p^{\oplus 2})} \nu(X)^{-s} e_p(-\text{Tr}(TX)) \quad \text{if } \psi \text{ is trivial.}$$

$$S_p(T, \psi, s) := \sum_{X \in \text{Sym}^2(\mathbb{Q}_p^{\oplus 2}) / \text{Sym}^2(\mathbb{Z}_p^{\oplus 2})} \psi(\nu(X) \det(X)) \nu(X)^{-s} e_p(-\text{Tr}(TX)) \quad \text{if } n \geq 1$$

where ν is the function on $\text{Sym}^2(\mathbb{Q}_p^{\oplus 2})$ defined in *loc cit.* right below equation (2.26).

We also set

$$A_p(T, \psi; p^{-s}) := S_p(T, \psi, s),$$

this is a formal power series in p^{-s} . When ψ is trivial we omit ψ from the notation:

we denote $A_p(T, \psi; p^{-s})$ by $A_p(T, p^{-s})$.

We have the relation:

$$\int_{\text{Sym}^2(\mathbb{Q}_p^{\oplus 2})} f_{p,s}^\chi(wu(X)) e_p(-\text{Tr}(TX)) = A_p(T, \bar{\chi}; p^{-k-s+1/2})$$

We denote by $F_p(T; p^{-s})$ the polynomial in p^{-s} defined in *loc cit.* in equation (2.32) (see also [Tak12, Proposition 2.6]). This is a polynomial with integer coefficients and constant term 1. Let ξ the function denoted by $\xi(y, h; \alpha, \beta)$ in theorem 2.1 of *loc*

cit.. Finally let λ_T be the quadratic character attached to the abelian extension $\mathbb{Q}(\sqrt{-\det(2T)})$ of $\mathbb{Q}$. We then have introduced all the relevant information to state the following result, which is a reformulation theorem 2.2 of *loc cit.*.

(B.1.6) Proposition. *Suppose $\Re(s) > 0$. For $T \in \text{Sym}^2(\mathbb{Q}_p^{\oplus 2})$ we denote by $a_T(y, s, \chi)$ be the T -th Fourier coefficient at infinity of $E(- + iy; f_s^\chi)$. We have the formula:*

$$a_T(y, s, \chi) = \det(y)^{s/2-1/4} \xi(y, T; 3 + s/2 - 1/4, s/2 - 1/4) \frac{L^{(N_\chi)}(2, \lambda_T \chi)}{L(3, \chi) L^{(N_\chi)}(4, \chi^2)} \\ \times \prod_{q \nmid N_\chi} F_v(T, \chi(q) q^{-k}) \cdot \prod_{p \mid N_\chi} A_p(T, \bar{\chi}; p^{-k-s+1/2})$$

We have shown in section 3.3.4 that the Eisenstein $E(-; f_s^\chi)$ can be analytically continued at $s = 1/2$ (the point $s = 0$ for Takemori [Tak12]). Therefore the Fourier coefficient also can be analytically continued. In particular the result stated in *loc cit.* for $k > 3$ are also valid at $k = 3$ when χ is odd. We now state the final result on the Fourier expansion of the ordinary Eisenstein series: this is theorem 4.1 of *loc cit.*. We only state the result when χ_p is not quadratic for any $p \mid N_\chi$. When χ_p is quadratic one has to take a slight modification of the section at p (which is explained in *loc cit.*) and the Fourier expansion looks slightly different. Thus, when χ_p is quadratic for some p , one should take the Eisenstein series defined in proposition 5.1 of *loc cit.*, whose Fourier expansion is of the same form of the non-quadratic case. As we will be only interested in the final form of the Fourier expansion, the final results are still true when χ_p is quadratic for some $p \mid N_\chi$.

We record the following proposition which is a direct consequence of theorem 4.1

of *loc cit.*, combined with the initial remark of the above paragraph.

(B.1.7) Proposition. *At the point $s = 1/2$, $k = 3$ We denote $a_T(y, s, \chi)$ by $a_T(\chi)$.*

Write $\det(T) = d_T g_T^2$, where d_T is square free, and set $\beta_p = v_p(g_T)$. We have

$$a_T(\chi) = 2 \frac{L^{(N_\chi)}(-1, \overline{\lambda_T \chi})}{L(-2, \overline{\chi}) L^{(N_\chi)}(-3, \overline{\chi^2})} \prod_{q \nmid N_\chi} F_q(T, \overline{\chi}(q)).$$

Using the functional equation of F_p (see proposition 2.6 of *loc cit.*), one obtains the following corollary (this can also be obtained directly from the expression in the middle of p1256 of *loc cit.*).

(B.1.8) Corollary. *With the notations of the above proposition, we have for T positive definite :*

$$a_T(\chi) = -2i \det(2T)^{3/2} p^{-3\beta_p} \left(\prod_{q \nmid N_\chi} \overline{\alpha}_q(\det(2T)) \overline{\alpha}_q(N_{\lambda_T}) \right) \cdot N_{\lambda_T}^{-3/2} \frac{L^{(N_\chi)}(-1, \overline{\lambda_T \chi})}{L(-2, \overline{\chi}) L^{(N_\chi)}(-3, \overline{\chi^2})} \prod_{q \nmid N_\chi} F_p(T, \chi(p)p)$$

(B.1.9) Normalization of sections

In order to get rational and integral cohomology classes, we ought to normalize the sections above so as to have both integral q -expansions (up to a power of $(2\pi i)$) and rational valued sections.

(B.1.10) Definition. Let $f_\chi^{ord} = \otimes_v f_{v,\chi}^{ord}$ be the section defined by:

1. For $v = \infty$, we set

$$f_{\infty,\chi}^{ord}(g_\infty) := (2\pi i)^{-7} f_{\infty,1/2}^\chi.$$

2. For $v \nmid N_\chi$ finite, we set

$$f_{v,\chi}^{ord} = L_v(3, \chi) L_v(4, \chi^2) f_{v,1/2}^\chi(g_v),$$

where $f_{v,1/2}^\chi(g_v)$ is the section of item 2 in definition (B.1.2).

3. For $v | N_\chi$, we set

$$f_{p,\chi}^{ord}(g_p) = A_\chi f_{p,1/2}^\chi(g_p),$$

where $f_{p,1/2}^\chi(g_p)$ is the section of item 3 in definition (B.1.2) and A_χ is a constant given by:

$$A_\chi := 2^4 \cdot 3 \cdot N_\chi N_{\chi^2} \tau(\overline{\chi}) \tau(\overline{\chi}^2)$$

Denote by $a_T^{ord}(\chi)$ the T -th Fourier coefficient of $E(-; f_\chi^{ord})$. By Corollary (B.1.8) and the functional equation of Dirichlet L-functions, we obtain the following proposition.

(B.1.11) Proposition. *We have for T positive definite:*

$$a_T^{ord}(\chi) = -2i \det(2T)^{3/2} p^{-3\beta_p} \det(2h)^{3/2} \prod_{p|N_\chi} \overline{\alpha}_q(\det(2T)) \overline{\alpha}_q(N_{\lambda_T}) N_{\lambda_T}^{-3/2} L^{(N_\chi)}(-1, \overline{\lambda_T \chi}) \prod_{q \nmid N_\chi} F_q(T, \chi(q) q^{-3})$$

Equivalently,

$$a_T^{ord}(\chi) = -2i L^{(N_\chi)}(-1, \overline{\lambda_T \chi}) \prod_{q \nmid N_\chi} F_q(T, \overline{\chi}(q)).$$

In particular, the Fourier coefficients (at infinity) $a_T^{ord}(\chi)$ are universal integral polynomials in χ .

To prove rationality of the cohomology classes defined by the ordinary Eisenstein series, we need the following result.

(B.1.12) Lemma. *The section $f_{p,\chi}^{ord}$ is F_χ -valued on $\mathbf{G}'(\mathbb{A}_f)$.*

Proof. As the local sections of definition (B.1.2) are clearly F_χ -valued on $\mathbf{G}'(\mathbb{A}_f)$, it suffices to prove that

$$(2\pi i)^{-7} A_\chi L(3, \chi) L(4, \chi^2)$$

belongs to F_χ . This is the case by the functional equation of Dirichlet L-functions and well-known special value formulas in terms of generalized Bernoulli numbers: the above constant is $L(-2, \bar{\chi}) L(-3, \bar{\chi}^2)$. $\square$

We now explain why we refer to this choice of section as "ordinary"

(B.1.13) Remark. Takemori in [Tak12] shows that f_χ^{ord} yields an ordinary Eisenstein series (ordinary for the operator denoted by $U(p)$ in *loc cit.*, or $U_{p,2}$ in [Liu20, Section 2.5]).

(B.2) Comparison with Liu's section

In this section we compare the ordinary Eisenstein series resulting from f_χ^{ord} and Liu's Eisenstein series resulting from a different choice of section at p .

(B.2.1) The q -expansion of Liu's Eisenstein series

Let f_p^{Liu} be the section at p that Liu defines in [Liu20, Section 5.1]. In the notations of *loc cit.*, this corresponds to $f_{\kappa,p}$, where $\kappa = 3$. Explicitly, f_p^{Liu} is a big cell section defined by:

$$f_p^{Liu} \left(\begin{bmatrix} A & B \\ C & D \end{bmatrix} \right) = \begin{cases} \overline{\chi}(\det C) |\det(C)|^{-3} \eta_p(C^{-1}D) & \text{if } \det C \neq 0 \\ 0 & \text{otherwise,} \end{cases}$$

where η_p is a Schwartz function on $\text{Sym}^2(\mathbb{Q}_p^{\oplus 2})$ (denoted by $\alpha_{\kappa,p}$ in *loc cit.*) defined by its Fourier transform

$$\hat{\eta}_p(S) = \hat{\eta}_p \left(\begin{bmatrix} s_1 & s_0 \\ s_0 & s_2 \end{bmatrix} \right) = \mathbb{1}_{p^2\mathbb{Z}_p}(s_1) \mathbb{1}_{\mathbb{Z}_p}(s_2) \chi(\det(2S)) \overline{\chi}(s_0),$$

where we view χ as a function on $\mathbb{Q}_p$ by extending it by 0 away from $\mathbb{Z}_p^\times$.

Denote by f^{Liu} the section defined by the same local sections at $v \neq p$ as in Definition (B.1.10), and at p we choose respectively f_p^{Liu} . We now record the Fourier expansion of f^{Liu} .

(B.2.2) Proposition. *The Fourier expansion of $E(-; f^{Liu})$ (resp $E(-; f^{Liu})$) is supported on the integral positive definite T . Denote by $a_T^{Liu}(\chi)$ (resp. $a_T^{pre}(\chi)$) the T -th Fourier coefficient. We have:*

$$a_T^{Liu}(\chi) = i \det(2T)^{3/2} N_{\lambda_T}^{-3/2} \overline{\chi}(N_{\lambda_T}) L^{(N_{\lambda_T})}(-1, \overline{\chi \lambda_T}) \hat{\eta}_p(T) \prod_{q \nmid p} F_q(T; \chi(q)q^{-3}).$$

Proof. This is [Liu20, Proposition 3.5.1], rephrased in our specific setting. Note that we have used the relation $\tau(\lambda_T) = -iN_{\lambda_T}^{1/2}$. □

(B.3) Depleting the ordinary Eisenstein series.

In this section we move from the ordinary Eisenstein series to an Eisenstein series defined by a section that is more similar to Liu's choice ([Liu20]) of sections supported on the big cell. We write with bracket matrices elements of $\mathbf{G}'(\mathbb{Q}_p)$ seen as acting on spaces of functions on $\mathbf{G}'(\mathbb{A}_f)$ spherical away from p or of level $K_0^{\mathbf{P}'}(N)$.

Define first the operator $t_\chi^{dep} \in \mathcal{O}_\chi[\mathbf{G}'(\mathbb{Q}_p)]$ defined by the formula:

$$t_\chi^{dep} := \sum_{x \bmod p^{\mathfrak{f}_X}} \chi(x) U(-xs/2p^{\mathfrak{f}_X}),$$

where we recall that s denotes the 2×2 matrix

$$s := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

and that $U(S)$ for a symmetric 2×2 matrix S denotes the element of $\mathbf{G}'(\mathbb{Q}_p)$ given by

$$U(S) = \begin{bmatrix} I_2 & S \\ & I_2 \end{bmatrix}.$$

The corresponding element of $\mathbb{Z}[\mathbf{G}(\mathbb{Q}_p)]$ still denoted by t_χ^{dep} is given by

$$t_\chi^{dep} := \sum_{x \bmod p^{\mathfrak{f}_X}} \chi(x) n_1(0, -x/p^{\mathfrak{f}_X}, 0),$$

where $n_1(X)$ is the unipotent element defined in Section (A.1).

We now define the "depleted" section.

(B.3.1) Definition. Fix a Dirichlet character $\psi : (\mathbb{Z}/p^r\mathbb{Z})^\times \rightarrow \mathbb{C}$ (in the body of the paper ψ is the Nebentypus of the modular form f under consideration). We denote by $f_{p,\chi}^{dep}$ the section defined by

$$f_{p,\chi}^{dep} := t_{\chi\psi^{-1}}^{dep} f_{p,\chi}^{ord}.$$

And denote f_χ^{dep} the resulting global section such that at a place $v \neq p$, $f_{v,\chi}^{dep}$ coincides with the ordinary choice from definition (B.1.10).

We now relate the Fourier coefficients of $E(-; f_\chi^{dep})$ to the Fourier coefficients of $E(-; f_\chi^{ord})$. Denote the T -th Fourier coefficient of the former Eisenstein series by $a_T^{dep}(\chi)$.

(B.3.2) Proposition. *If $T = \begin{pmatrix} T_1 & T_0 \\ T_0 & T_2 \end{pmatrix}$ we have:*

$$a_T^{dep}(\chi) = \tau(\chi\psi^{-1})\psi\overline{\chi}(T_0)a_T^{ord}(\chi).$$

In particular $a_T^{dep}(\chi) = 0$ if T_0 is not a p -adic unit.

To prove integrality of the Euler system classes, we will need a "universal" version of the above depletion. For $a \geq f_\chi$, we denote by $t_\chi^{DEP,a}$ the following operator:

$$t_\chi^{DEP,a} := \sum_{x \bmod p^a} \chi(x)U(-xs/2p^a)$$

We then define

$$f_{p,\chi}^{DEP,a} := t_{\chi\psi^{-1}}^{DEP,a} f_p^{ord},$$

and denote $f_{\chi}^{DEP,a}$ the resulting global section such that at a place $v \neq p$, $f_{v,\chi}^{DEP,a}$ coincides with the ordinary choice from definition (B.1.10). We then have the following proposition, that is proven by a very similar computation to proposition (B.3.2).

(B.3.3) Proposition. *If $T = \begin{pmatrix} T_1 & T_0 \\ T_0 & T_2 \end{pmatrix}$, denote by $a_T^{DEP,a}(\chi)$ the T -th Fourier coefficient of $E(-, f_{\chi}^{DEP,a})$. We then have:*

$$a_T^{DEP,a}(\chi) = p^{a-\mathfrak{f}_X} \tau(\chi \psi^{-1}) \mathbb{1}\left(p^{a-\mathfrak{f}_X} | T_0\right) \psi \bar{\chi}(T_0/p^{a-\mathfrak{f}_X}) a_T^{ord}(\chi).$$

(B.4) U_p operators

We now study the effect of the ordinary projector of $\mathbf{H}'$ on the sections defined in the previous subsection. This operator is denoted by $e \times e$ in [Liu20]; it is defined with respect to the operator denoted by $U_p \times U_p$ in *loc cit.*. To distinguish the U_p operator on $\mathbf{SL}_2$ from the U_p operator of $\mathbf{Sp}_4$, we will denote by u'_p the U_p operator of $\mathbf{SL}_2$, and by U'_p the U_p operator of $\mathbf{G}' = \mathbf{Sp}_4$; U'_p is the operator denoted by $U_{p,2}$ in [Liu20, Section 2.5]. As an element of $\mathbb{Z}[\mathbf{G}'(\mathbb{Q}_p)]$ it is defined by:

$$U'_p := \sum_{S \bmod p^2} \begin{bmatrix} I_2 & S \\ 0 & I_2 \end{bmatrix} \begin{bmatrix} pI_2 & 0 \\ 0 & p^{-1}I_2 \end{bmatrix} = \sum_{S \bmod p^2} \begin{bmatrix} pI_2 & p^{-1}S \\ 0 & p^{-1}I_2 \end{bmatrix}.$$

We can view $u'_p \times u'_p$ as an element of $\mathbb{Z}[\mathbf{G}'(\mathbb{Q}_p)]$ through the usual embedding $\mathbf{H}' \hookrightarrow \mathbf{G}'$. Explicitly, we have:

$$u'_p \times u'_p = \sum_{x_1, x_2 \bmod p^2} \begin{bmatrix} & x_1 & \\ I_2 & & \\ & x_2 & \\ 0 & & I_2 \end{bmatrix} \begin{bmatrix} pI_2 & 0 \\ 0 & p^{-1}I_2 \end{bmatrix}$$

(B.4.1) Remark. Note that the U'_p operator defined above, up to multiplication by p^{-1} is the *square* of the usual U_p operator of $\mathbf{GSp}_4$; the latter is defined by:

$$U_p = \sum_{S \bmod p} \begin{bmatrix} I_2 & S \\ 0 & I_2 \end{bmatrix} \begin{bmatrix} pI_2 & 0 \\ 0 & I_2 \end{bmatrix}.$$

Similarly $u'_p \times u'_p$ is the square of the operator

$$u_p \times u_p := \sum_{x_1, x_2 \bmod p} \begin{bmatrix} & x_1 & \\ I_2 & & \\ & x_2 & \\ 0 & & I_2 \end{bmatrix} \begin{bmatrix} pI_2 & 0 \\ 0 & I_2 \end{bmatrix}.$$

On q -expansions at infinity, U_p acts by $U_p \sum_T a_T q^T = \sum_T a_{pT} q^T$. Similarly, $u_p \times u_p$ acts on q -expansions by

$$(u_p \times u_p) \sum_T a_T q^T = \frac{1}{p} \sum_T \mathbb{1}(p|T_1, T_2) a_T q^{T/p}.$$

In what follows we will view the operators U_p and $u_p \times u_p$ acting on sections of $\text{Ind}_{\mathbf{P}'_1}^{\mathbf{G}'}(\mu)$ by remark (A.4.3).

We now compare the ordinary projection for $\mathbf{H}'$ of the Eisenstein series defined by f_χ^{dep} and f_χ^{Liu} .

(B.4.2) Proposition. *For $a \geq \mathfrak{f}_\chi$, we have*

$$(u_p \times u_p)^a E(-; f_\chi^{dep})|_{\mathbf{H}} = (u_p \times u_p)^a E(-; -2\tau(\chi\psi^{-1})f_\chi^{Liu})|_{\mathbf{H}}$$

If the p -adic valuation of T_1 and T_2 is large enough $E(-; f_\chi^{dep})$ and $E(-; -2\tau(\chi)f_\chi^{Liu})$ have the same T -th Fourier coefficient.

Proof. The operator $u_p \times u_p$ acts on a form with q -expansion at infinity $\sum_T a_T q^T$ by $\frac{1}{p} \sum_T a_T \mathbb{1}_{p|T_1, T_2} q^{T/p}$, as we mentioned above. Then the result follows by combining proposition (B.1.11), (B.2.2) and (B.3.2). $\square$

We record here another proposition that compares the action of $t_\chi^{DEP, a}$ and t_χ^{dep} , using the $u_p \times u_p$ operator.

(B.4.3) Proposition. *Let $a \geq \mathfrak{f}_\chi + 1$. We have*

$$(u_p \times u_p)t_\chi^{DEP, a} E(-; f_\chi^{ord}) = t_\chi^{DEP, a-1} E(-; f_\chi^{ord})$$

Proof. Using proposition (B.3.3), and the description of the action of $u_p \times u_p$ on q -expansion already given in the proof of the previous proposition, we have that the q -expansion of $(u_p \times u_p)t_\chi^{DEP, a} E(-; f_\chi^{ord})$ is:

$$\sum_T p^{a-1-\mathfrak{f}_\chi} \tau(\chi\psi^{-1}) \mathbb{1}(p|T_1, T_2) \mathbb{1}(p^{a-\mathfrak{f}_\chi}|T_0) \psi \bar{\chi}(T_0/p^{a-\mathfrak{f}_\chi}) a_T^{ord}(\chi) q^{T/p}.$$

Now using the fact that $a_{pT}^{ord}(\chi) = a_T^{ord}(\chi)$, we obtain the result. $\square$

We thus obtain the following immediate corollary.

(B.4.4) Corollary. *We have $(u_p \times u_p)^{(a-1)\chi} t_\chi^{DEP,a} E(-; f_\chi^{ord}) = t_\chi^{dep} E(-; f_\chi^{ord}) = E(-; f_\chi^{dep})$*

(B.5) Twisting the ordinary and depleted sections

Here we describe a modification of $f_{p,\chi}^{ord}$ that we use to prove the "universal" vertical norm relations.

Let $a \geq 0$ be an integer and consider the operator $\text{Tw}^a \in \mathbb{Z}[\mathbf{G}'(\mathbb{Q}_p)]$ defined by

$$\text{Tw}^a := \sum_{x \bmod p} \begin{bmatrix} 0 & I_2 \\ p^a I_2 & 0 \end{bmatrix} \begin{bmatrix} I_2 & \begin{pmatrix} x/p^a \\ x/p^a \end{pmatrix} \\ 0 & I_2 \end{bmatrix},$$

where the sum is taken with respect to representatives x of $\mathbb{Z}/p^a\mathbb{Z}$.

Let us record first the following lemma whose proof is an elementary computation.

Let $K'_1(p^a) := K_1(p^{a-1}) \cap K_0(p^a)$

(B.5.1) Lemma. *For all $X = \begin{pmatrix} x_1 & x_0 \\ x_0 & x_2 \end{pmatrix}$ the following relation holds.*

$$\begin{bmatrix} I_2 & 0 \\ \begin{pmatrix} p^{a-1}x_1 \\ p^{a-1}x_2 \end{pmatrix} & I_2 \end{bmatrix} \begin{bmatrix} 0 & I_2 \\ p^a I_2 & 0 \end{bmatrix} \begin{bmatrix} I_2 & \begin{pmatrix} x_0/p \\ x_0/p \end{pmatrix} \\ 0 & I_2 \end{bmatrix} = \begin{bmatrix} 0 & I_2 \\ p^{a-1} I_2 & 0 \end{bmatrix} \begin{bmatrix} I_2 & X \\ 0 & I_2 \end{bmatrix} \begin{bmatrix} pI_2 & 0 \\ 0 & I_2 \end{bmatrix}$$

We now use notations from ??, and consider the operator:

$$\mathrm{Tr}_{K'_1(p^a)/K_1(p^{a-1})} = \sum_{x_1, x_2 \bmod p} \left[\begin{array}{c|c} I_2 & 0 \\ \hline \left(\begin{array}{c} p^{a-1}x_1 \\ p^{a-1}x_2 \end{array} \right) & I_2 \end{array} \right],$$

where the sum is taken over representatives x_1, x_2 of $\mathbb{Z}/p\mathbb{Z}$. From Lemma (B.5.1), one deduces immediately the following relationship between Tw^a and U_p .

(B.5.2) Proposition. *We have the relation:*

$$\mathrm{Tr}_{K'_1(p^a)/K_1(p^{a-1})} \mathrm{Tw}^a = \mathrm{Tw}^{a-1} U_p$$

Similarly let

$$\mathrm{tw}^a := \left[\begin{array}{cc} 0 & I_2 \\ p^a I_2 & 0 \end{array} \right] \in \mathbb{Z}[\mathbf{GG}'(\mathbb{Q}_p)].$$

The corresponding element in $\mathbb{Z}[\mathbf{GG}(\mathbb{Q}_p)]$ is, still denoted by tw^a ,

$$\mathrm{tw}^a := \left[\begin{array}{c} 1 \\ I_3 \\ p^a \end{array} \right] \in \mathbb{Z}[\mathbf{GG}(\mathbb{Q}_p)].$$

By a completely analogous computation we have the following proposition

(B.5.3) Proposition. *We have the relation:*

$$\mathrm{Tr}_{K'_1(p^a)/K_1(p^{a-1})} \mathrm{tw}^a = \mathrm{tw}^{a-1}(u_p \times u_p)$$

(B.6) Nebentypus questions

Here we study the relationship of the operators tw^a and $t_\chi^{DEP,a}$ with "diamond" operators. Fix $a \geq 1$ an integer and $d \in (\mathbb{Z}/p^a\mathbb{Z})^\times$.

We let $\langle d \rangle \in \mathbb{Z}[\mathbf{G}(\mathbb{Q}_p)]$ be the operator defined by $\langle d \rangle := \begin{bmatrix} d & & \\ & I_3 & \\ & & d^{-1} \end{bmatrix}$. We have

the following immediate lemma:

(B.6.1) Lemma. *There are the following relations:*

1.

$$\langle d \rangle \mathrm{tw}^a = \mathrm{tw}^a \langle d^{-1} \rangle.$$

2.

$$\langle d \rangle t_\chi^{DEP,a} = \chi(d^{-1}) t_\chi^{DEP,a} \langle d \rangle$$

Note that $\langle d \rangle$ acts by multiplication by $\chi^{-1}(d)$ on $f_{p,\chi}^{ord}$. As a consequence of the previous lemma and this remark, we have the following proposition:

(B.6.2) Proposition. *We have the relation, for all characters ξ of log-conductor at p verifying : $\mathfrak{f}_\xi \leq \mathfrak{f}_\chi \leq a$:*

$$\langle d \rangle (\mathrm{tw}_{\chi\xi}^a t_{\chi\xi}^{DEP,a} f_{p,\chi}^{ord}) = \xi(d) (\mathrm{tw}_{\chi\xi}^a t_{\chi\xi}^{DEP,a} f_{p,\chi}^{ord}).$$

(B.7) The Doubling zeta integral

Here we introduce the doubling Zeta integral on $\mathbf{G}'$. We refer the reader to section 4 of [Liu20] for more details. Recall that the embedding $\iota' : \mathbf{H}' \hookrightarrow \mathbf{G}'$ of section 3.1.1.4 is defined in the basis $(\varepsilon_i)_{i=1,\dots,4}$ by

$$\iota' : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \times \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \mapsto \begin{pmatrix} a & 0 & b & 0 \\ 0 & a' & 0 & b' \\ c & 0 & d & 0 \\ 0 & c' & 0 & d' \end{pmatrix}.$$

Recall the notation the element denoted by $\vartheta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ in [Liu20]. Define now P' as the maximal isotropic subspace of V' spanned by ε_1 and ε_2 - this corresponds to the isotropic line spanned by $\varepsilon_1 \wedge \varepsilon_2$ in W . We can write abstractly $V' \simeq P' \oplus P'$ by identifying ε_3 (resp. ε_4) and ε_1 (resp. ε_2). We denote by $P^d := \{(v, \vartheta(v)) : v \in P\} \subset V'$: P^d is a maximal isotropic subspace of V' . The doubling Siegel parabolic $\mathbf{P}_1^{d'}$ is defined as the stabilizer of P^d . It is conjugated to the standard Siegel parabolic $\mathbf{P}_1'$:

$$\mathbf{P}_1^{d'} = \gamma \mathbf{P}_1' \gamma^{-1},$$

where

$$\gamma := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}.$$

If f is a section of an induced representation from $\mathbf{P}'_1(\mathbb{A})$ we set $f^d := f(\gamma^{-1}-)$. Then f^d is a section of an induced representation from $\mathbf{P}_1^{d'}$. When $f \in \text{Ind}_{\mathbf{P}'_1(\mathbb{A})}^{\mathbf{G}'(\mathbb{A})}(\mu_s)$, we have $E(-; f) = E(-; f^d)$.

(B.7.1) Remark. The element γ correspond to the element $n_1^{\text{opp}}([0, 2, 0]) \in \mathbf{G}(\mathbb{Q})$ under π . By abuse of notation, we denote $n_1^{\text{opp}}([0, 2, 0])$ by γ ; it will be clear from the context whether $\Gamma \in \mathbf{G}'(\mathbb{Q})$ or $\gamma \in \mathbf{G}(\mathbb{Q})$. We also set by $\mathbf{P}_1^d := \gamma \mathbf{P}_1 \gamma^{-1} \subseteq \mathbf{G}$.

We denote by $g \mapsto g^\vartheta = \vartheta g \vartheta$ the MVW involution of an irreducible representation of $\mathbf{SL}_2(\mathbb{Q}_v)$. Given an irreducible cuspidal automorphic representation π of $\mathbf{SL}_2(\mathbb{A})$ and its complex conjugate $\bar{\pi}$, we fix isomorphis $\pi \simeq \bigotimes'_v \pi_v$, $\bar{\pi} \simeq \bigotimes'_v \bar{\pi}_v$ such that for factorizable $\varphi_1 = \bigotimes \varphi_{1,v} \in \bigotimes'_v \pi_v$ and $\varphi_2 = \bigotimes'_v \bar{\varphi}_{2,v}$, we have

$$\langle \varphi_1, \bar{\varphi}_2 \rangle_{\text{Pet}} = \prod_v \langle \varphi_{1,v}, \bar{\varphi}_{2,v} \rangle_v.$$

Here the pairing on the left hand-side is the Petersson inner product and on the right hand-side the pairing is the duality pairing between π and its contragredient naturally isomorphic to $\bar{\pi}$. If $\varphi \in \pi$, we denote by φ^ϑ the function of $\bar{\pi}$ defined by

$$\varphi^\vartheta(g) = \varphi^\vartheta(g) = \varphi(\vartheta g \vartheta).$$

For a local section $f_{v,s}^\chi \in I'_v(s, \chi)$, we follow Liu and define the operator $T(f_{v,s}^\chi) : \pi \rightarrow \pi$ by the formula:

$$T(f_{v,s}^\chi)\varphi(g) := \int_{\mathbf{SL}_2(\mathbb{Q}_v)} f_{v,s}^{\chi,d}(\iota'(g_v, 1))\varphi(gg_v)dg_v. \quad (\text{B.7.1.a})$$

We refer to [Liu20, p.21] regarding the well-definedness of $T(f_{v,s}^\chi)$.

Again following the conventions of Liu, the local doubling zeta integral is defined as a bilinear pairing between $I'_v(s, \chi)$ and $\pi \boxtimes \bar{\pi} \rightarrow \mathbb{C}$. For $f_{v,s}^\chi \in I'_v(s, \chi)$ and $v_1 \otimes \bar{v}_2 \in \pi \boxtimes \bar{\pi}$, we set:

$$Z_v(f_{v,s}^\chi, v_1, \bar{v}_2) := \int_{\mathbf{SL}_2(\mathbb{Q}_v)} f_{v,s}^{\chi,d}(\iota'(g_v, 1))\langle \pi_v(g_v)v_1, \bar{v}_2 \rangle_v dg_v.$$

Then Z_v defines an element of the Hom space:

$$\text{Hom}_{\mathbf{H}'(\mathbb{Q}_v)}(I'_v(s, \chi) \otimes (\pi \boxtimes \bar{\pi}), \mathbb{C}).$$

We now compare the doubling Zeta integral to the Zeta integral of Murase and Sugano defined in section 3.4.2. We freely use notations of that section. Let ℓ be a finite place. We further assume that the vector that F defines in $\Pi_\ell = \pi_\ell \boxtimes \bar{\pi}_\ell$ is a pure tensor, i.e. $F_\ell = v \otimes \bar{v}$.

(B.7.2) Lemma. *For a finite place ℓ , verifying the hypothesis above, we have*

$$\mathfrak{z}_\ell(F, f_{v,s}^\chi) = Z_\ell(f_{v,s}^\chi, v, \bar{v})$$

Proof. Indeed, the integral Z_ℓ descends an integral on $\mathbf{SL}_2/\mu_2$. Then the matrix coefficient appearing in Z_ℓ is exactly the period integral $\mathcal{P}_F(h_v)$, as $\mathbf{G}_0$ corresponds to the diagonal copy of $\mathbf{SL}_2$ in $\mathbf{H}'$. We thus obtain the result. $\square$

(B.8) The volume sections and their zeta integral

Here we define local sections analogous to the ones that Liu introduces in [Liu20, Section 4.2]. We start by introducing the relevant Siegel-Godement sections.

Let ℓ be a place at which χ is unramified. Consider the space $M_\ell := M(2, 4, \mathbb{Q}_\ell)$ of 2×4 matrices with coefficients in $\mathbb{Q}_\ell$. We denote by $M_n(A)$ the space of $n \times n$ matrices with coefficients in the ring A . We decompose M_ℓ as a direct sum in two ways

$$M_\ell = M_2(\mathbb{Q}_\ell) \oplus M_2(\mathbb{Q}_\ell) = M'_2(\mathbb{Q}_\ell) \oplus M'_2(\mathbb{Q}_\ell).$$

In the first decomposition, the first factor $M_2(\mathbb{Q}_\ell)$ corresponds to the first two columns of the element $M \in M_\ell$ and the second factor corresponds to the last two. In the second decomposition, the first factor $M'_2(\mathbb{Q}_\ell)$ corresponds to the first and third columns, while the second to the second and fourth columns of an element $M \in M_\ell$.

Let $\Phi \in \mathcal{S}(M_\ell)$ be a Schwartz-Bruhat function on M_ℓ . We will always choose Φ of the form

$$\Phi = \phi_1 \otimes \phi_2 \otimes \phi_3 \otimes \phi_4$$

where $\phi_i \in \mathcal{S}(\mathbb{Q}_\ell^2)$ is a Schwartz-Bruhat function defined on the i -th column of an

element $M \in M_\ell$. We will also write

$$\Phi = \Phi_1 \otimes \Phi_2 = \Phi'_1 \otimes \Phi'_2,$$

where Φ_i and Φ'_i are Schwartz Bruhat functions on $M_2(\mathbb{Q}_\ell)$ and $M'_2(\mathbb{Q}_\ell)$ respectively, defined as $\Phi_1 = \phi_1 \otimes \phi_2$, $\Phi_2 = \phi_3 \otimes \phi_4$ and $\Phi'_1 = \phi_1 \otimes \phi_3$, $\Phi'_2 = \phi_2 \otimes \phi_4$.

Endow M_ℓ with the right action of $\mathbf{G}'$ by right multiplication. Consider now for $g \in \mathbf{G}'(\mathbb{Q}_\ell)$:

$$f_\Phi(s; g) := \int_{\mathbf{GL}_2(\mathbb{Q}_\ell)} \Phi((0, Z)g) |\det(Z)|^{s+3/2} d^\times Z.$$

Here $d^\times Z$ is the Haar measure of $\mathbf{GL}_2(\mathbb{Q}_\ell)$ normalized so that $\mathbf{GL}_2(\mathbb{Z}_\ell)$ has volume equal to 1 and $(0, Z) \in M_2(\mathbb{Q}_\ell) \oplus M_2(\mathbb{Q}_\ell)$, so that

$$f_\Phi(s; g) := \int_{\mathbf{GL}_2(\mathbb{Q}_\ell)} \Phi_1(ZC) \Phi_2(ZD) |\det(Z)|^{s+3/2} d^\times Z,$$

for $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathbf{G}'(\mathbb{Q}_\ell)$, $A, B, C, D \in M_2(\mathbb{Q}_\ell)$.

Then $f_\Phi(s; -)$ defines a section of $I_\ell(s) := \text{nInd}_{\mathbf{P}'_1(\mathbb{Q}_\ell)}^{\mathbf{G}'(\mathbb{Q}_\ell)}(|\det|^s)$. Recall the notation of Liu $\vartheta := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in M_2(\mathbb{Z})$. We now take Φ of the form: for all $M = (Z_1, Z_2) \in M_2(\mathbb{Q}_\ell) \oplus M_2(\mathbb{Q}_\ell)$,

$$\Phi(Z_1, Z_2) = \Phi_1(Z_1) \cdot \Phi_2(Z_2),$$

for some $\Phi_1, \Phi_2 \in \mathcal{S}(M_2(\mathbb{Q}_\ell))$. There is an action of $\mathbf{GL}_2(\mathbb{Q}_\ell)$ on $\mathcal{S}(M_2(\mathbb{Q}_\ell))$; if

$h \in \mathbf{GL}_2(\mathbb{Q}_\ell)$ we set $(h \cdot \Phi)(M) = \Phi(Mg)$. We define the subgroup $\Gamma(\ell^r) \subseteq \mathbf{GL}_2(\mathbb{Z}_\ell)$ as the elements that reduce modulo ℓ^r to the identity. Let ϕ^r be the indicator function of $\Gamma(\ell^r)$. We then take $\Phi_1 = \vartheta \cdot \phi^r$ and $\Phi_2 = \phi^r$. Notice that $\Phi'_1 = \Phi_1$ and $\Phi'_2 = \Phi_2$.

For $\Phi = \text{vol}(\Gamma(\ell^r))^{-1} \Phi_1 \otimes \Phi_2$, we want to compute $f_\Phi^d(s, \iota'(g, 1))$ for $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{SL}_2(\mathbb{Q}_\ell)$. Note first that

$$\gamma^{-1} \iota'(g, 1) = \begin{pmatrix} a & 0 & b & 0 \\ 0 & 1 & 0 & 0 \\ c & 1 & d & 0 \\ a & 0 & b & 1 \end{pmatrix}.$$

We then have, for $Z \in \mathbf{GL}_2(\mathbb{Q}_\ell)$,

$$\Phi((0, Z) \gamma^{-1} \iota'(g, 1)) = \Phi'_1(Z \vartheta g \vartheta) \Phi'_2(Z).$$

Since if $Z \in \Gamma(\ell^r)$, $\Phi'_1(ZM) = \Phi'_1(M)$ for all $M \in M'_2(\mathbb{Q}_\ell)$, we then obtain

$$f_\Phi^d(s; \iota'(g, 1)) = \text{vol}(\Gamma(\ell^r))^{-1} \int_{\mathbf{GL}_2(\mathbb{Q}_\ell)} |\det(Z)|^{s+3/2} \phi^r(Z) \phi^r(g) = \phi^r(g).$$

Therefore we obtain the following Lemma.

(B.8.1) Lemma. *Let $\Gamma(\ell^r) \subset \mathbf{GL}_2(\mathbb{Z}_\ell)$ be the elements that reduce modulo ℓ^r to diagonal matrices. Let ϕ^r be the indicator function of $\Gamma(\ell^r)$ in $M_2(\mathbb{Z}_\ell)$. We set*

$$\Phi := \text{vol}(\Gamma(\ell^r))^{-1} \cdot (\vartheta \cdot \phi^r) \otimes \phi^r \in \mathcal{S}(M(2, 4, \mathbb{Q}_\ell)).$$

Then if $v \in \pi_\ell$ is $\Gamma(\ell^r)$ -fixed we have

$$Z_\ell(f_\Phi(s; -), v, \bar{v}) = \text{vol}(\Gamma(\ell^r)) \langle v, \bar{v} \rangle_\ell$$

We now study the level of the restriction of $f_\Phi^d(s; -)$ to $\mathbf{H}'$ for Φ as in the Lemma just above. Recall the formula $f_\Phi^d(s; \iota'(g, 1)) = \phi^r(g)$. Following the notations of Liu, denote by $g^\vartheta = \vartheta g \vartheta$ for $g \in \mathbf{SL}_2(\mathbb{Q}_\ell)$. We first compute

$$\gamma^{-1} \iota'(g, g^\vartheta) = \gamma^{-1} \iota'(g, g^\vartheta) = \begin{pmatrix} a & 0 & b & 0 \\ 0 & d & 0 & c \\ c & d & d & c \\ a & b & b & a \end{pmatrix}.$$

Since $g^\vartheta := \begin{pmatrix} d & c \\ b & a \end{pmatrix} = \begin{pmatrix} c & d \\ a & b \end{pmatrix} \vartheta$, we see that by changing variable $Z' = Zg^\vartheta$, we have for all $h \in \mathbf{G}'(\mathbb{Q}_\ell)$

$$f_\Phi^d(s; \iota(g, g^\vartheta)h) = f_\Phi^d(s; h).$$

From this we deduce that if $h = \iota'(h_1, h_2) \in \mathbf{G}'(\mathbb{Q}_\ell)$, we have actually

$$f_\Phi^d(s; h) = f_\Phi^d(s; \iota'((h_2^\vartheta)^{-1}h_1, 1)) = \phi^r((h_2^\vartheta)^{-1}h_1).$$

Since $\Gamma(\ell^r)$ is a normal subgroup of $\mathbf{SL}_2(\mathbb{Z}_\ell)$, we deduce that f_Φ^d is right-invariant under $\mathbf{SL}_2(\mathbb{Z}_\ell) \hookrightarrow \mathbf{SL}_2(\mathbb{Z}_\ell)^2$, where the embedding is given by $g \mapsto (g, g^\vartheta)$. We deduce the following Lemma.

(B.8.2) Lemma. *The section f_{Φ}^d as defined in Lemma (B.8.1) is invariant under the the image of*

$$\mathbf{SL}_2(\mathbb{Z}_\ell) \times \Gamma(\ell^r) \hookrightarrow \mathbf{H}'(\mathbb{Z}_\ell) = \mathbf{SL}_2(\mathbb{Z}_\ell) \times \mathbf{SL}_2(\mathbb{Z}_\ell), \quad (g, h) \mapsto (h, 1)(g, g^\vartheta).$$

We now recall a few facts about the "volume" sections that Liu introduces in [Liu20, Section 4.2]. We define $f_{\phi^r}(s; -) \in I'_v(s, \chi)$ as

$$f_{\phi^r}(s; -) \left(\begin{bmatrix} A & B \\ C & D \end{bmatrix} \right) := \begin{cases} \mu_{s,v}(\det(C))^{-1} \phi^r(C^{-1}D\vartheta) & \text{if } \det(C) \neq 0 \\ 0 & \text{otherwise.} \end{cases}$$

This section coincides with the section $f_{\Phi}(s; -)$ defined in just above for $\Phi_1 = \vartheta \cdot \phi^r$ and $\Phi_2 = \phi^r$.

We record now the local Fourier coefficient attached to $f_{\phi^r}(s; -)$. We have, as is mentioned at the bottom of p. 22 in *loc cit.*,

$$W_{T,v}(1, f_{\phi^r}(s; -)) = \ell^{-3r} e_v(2T_0) \mathbb{1}(T \in \ell^{-r} \text{Sym}^2(\mathbb{Z}_\ell^2)).$$

We then have the following proposition, which is an immediate consequence of proposition 4.2.1 of *loc cit.*

(B.8.3) Proposition. *Let $\varphi \in \pi_\ell$ be invariant under $\Gamma(\ell^r)$. Then*

$$T(f_{\phi^r}(s; -))\varphi = -\text{vol}(\Gamma(\ell^r))\varphi.$$

Therefore:

$$Z_\ell(f_{\phi^r}(s; -), \varphi, \bar{\varphi}) = \text{vol}(\Gamma(\ell^r)) \langle \varphi_\ell, \bar{\varphi}_\ell \rangle_\ell.$$

(B.9) Sections at bad places

(B.9.1) Special representations

In this section we choose Siegel-Godement sections $f_{\Phi_{ST}}$ at places where π_ℓ is the Steinberg representation $\text{St}_\ell \subseteq \text{nInd}_{\mathbf{B}(\mathbb{Q}_\ell)}^{\mathbf{SL}_2(\mathbb{Q}_\ell)}(\delta_{\mathbf{B}}^{1/2})$, $\pi_\ell = \text{St}_\ell$. Let $\Gamma_0(\ell)$ be the subgroup of elements of $\mathbf{SL}_2(\mathbb{Z}_\ell)$ that reduce to upper triangular elements modulo ℓ . Then St_ℓ has unique $\Gamma_0(\ell)$ -fixed vector φ determined by the condition that φ lie in the kernel of the intertwining operator, that is $\varphi(1) + \ell\varphi(w) = 0$; where $w := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Consider the dual induced representation $\text{nInd}_{\mathbf{B}(\mathbb{Q}_\ell)}^{\mathbf{SL}_2(\mathbb{Q}_\ell)}(\delta_{\mathbf{B}}^{-1/2})$. Then St_ℓ appears as the quotient of $\text{nInd}_{\mathbf{B}(\mathbb{Q}_\ell)}^{\mathbf{SL}_2(\mathbb{Q}_\ell)}(\delta_{\mathbf{B}}^{-1/2})$ by the trivial representation $\mathbf{1} \hookrightarrow \text{nInd}_{\mathbf{B}(\mathbb{Q}_\ell)}^{\mathbf{SL}_2(\mathbb{Q}_\ell)}(\delta_{\mathbf{B}}^{-1/2})$. The vector space $\text{nInd}_{\mathbf{B}(\mathbb{Q}_\ell)}^{\mathbf{SL}_2(\mathbb{Q}_\ell)}(\delta_{\mathbf{B}}^{-1/2})^{\Gamma_0(\ell)}$ is 2-dimensional— and we choose a basis given by $(1, \bar{v})$, where 1 here denotes the constant function and $\bar{v}$ is a small cell section, i.e. the unique function on $\mathbf{SL}_2(\mathbb{Q}_\ell)$ supported on $\mathbf{B}(\mathbb{Q}_\ell)\Gamma_0(\ell)$, such that $\bar{v}(1) = 1$. We will define a Schwartz-function Φ and we will compute $T_{f_\Phi(s; -)}\bar{v}$, as defined in Equation B.7.1.a.

Define $\Phi_1 := \mathbf{1}_{\Gamma_0(\ell)}/\text{vol}(\Gamma_0(\ell))$ and take

$$\Phi_2 = \text{the indicator function of } M_2(\mathbb{Z}_\ell)$$

and $\Phi := \Phi_1 \otimes \Phi_2$.

Since Φ_2 is left $\Gamma_0(\ell)$ -invariant we have by a change of variables:

$$f_\Phi \left(s; \begin{bmatrix} A & B \\ C & D \end{bmatrix} \right) = \begin{cases} |\det(C)|^{-s-3/2} \Phi_2(C^{-1}D) & \text{if } \det(C) \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Note also that we have

$$\Phi'_1 = \text{the indicator function of } \begin{pmatrix} \mathbb{Z}_\ell^\times & \mathbb{Z}_\ell \\ \ell\mathbb{Z}_\ell & \mathbb{Z}_\ell \end{pmatrix}$$

and

$$\Phi'_2 = \text{the indicator function of } \begin{pmatrix} \mathbb{Z}_\ell & \mathbb{Z}_\ell \\ \mathbb{Z}_\ell^\times & \mathbb{Z}_\ell \end{pmatrix}.$$

We then have

$$f_\Phi^d(s; \iota'(g, 1)) = \int_{\mathbf{GL}_2(\mathbb{Q}_\ell)} \Phi'_1(Z\vartheta g) \Phi'_2(Z) |\det(Z)|^{s+3/2} d^\times Z.$$

From this expression it is clear that $f_\Phi^d(s; \iota(-; -))$ is right $\Gamma_0(\ell) \times \Gamma_0(\ell)$ invariant.

We now compute the integral

$$T_{f_\Phi(s; -)} \bar{v}(1) = \int_{\mathbf{SL}_2(\mathbb{Q}_\ell)} f_\Phi^d(s; \iota(g, 1)) \bar{v}(g) dg = I_{sm} + I_b.$$

Here I_{sm} is the integral over the small cell and I_b over the big cell. Note first that $I_b = 0$, since by definition $\bar{v}(g) = 0$ for g in the big cell. We now compute I_{sm} . Since $f_\Phi^d(s; \iota(g, 1))$ is $\Gamma_0(\ell)$ invariant it suffices to compute this function on elements g of

the form $g = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$, $x \in \mathbb{Q}_\ell$, $\alpha \in \mathbb{Q}_\ell^\times$.

We find that

$$\gamma^{-1}\iota(g, 1) = \begin{pmatrix} * & * \\ C & D \end{pmatrix}$$

where $C = \begin{pmatrix} 0 & 1 \\ \alpha & 0 \end{pmatrix}$ and $D = \begin{pmatrix} \alpha^{-1} & 1 \\ x\alpha & 0 \end{pmatrix}$ so that we have $C^{-1}D = -\alpha^{-1} \begin{pmatrix} -x\alpha & -1 \\ 1 & 0 \end{pmatrix}$.

Therefore, for $g = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$,

$$f_\Phi^d(s; g) = |\alpha|^{-s-3/2} \mathbf{1}_{\mathbb{Z}_\ell}(x) \mathbf{1}_{\mathbb{Z}_\ell}(\alpha^{-1}).$$

We thus find

$$T_{f_\Phi(s; -)} \bar{v}(1) = \text{vol}(\Gamma_0(\ell)) \zeta_\ell(s + 3/2) \bar{v}(1).$$

We now compute $T_{f_\Phi(s; -)} \bar{v}(w)$. We decompose again this quantity as $T_{f_\Phi(s; -)} \bar{v}(w) = I'_b + I'_{sm}$. Note here that $I'_{sm} = 0$ since for all $x \in \mathbb{Z}_\ell$ and $\alpha \in \mathbb{Q}_\ell^\times$ we have

$$\bar{v} \left(w \begin{pmatrix} \alpha & x\alpha \\ 0 & \alpha^{-1} \end{pmatrix} \right) = \bar{v} \left(w \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \right) = 0$$

; and since $f_\Phi^d(s; -)$ is supported on such elements on the small cell as was seen before. We now compute the integral on the big cell – we show that it vanishes.

We write $g = \begin{pmatrix} \alpha & \alpha x \\ 0 & \alpha^{-1} \end{pmatrix} w = \begin{pmatrix} x & -\alpha \\ \alpha^{-1} & 0 \end{pmatrix}$. With notations as before, we have $C^{-1}D = -(x\alpha)^{-1} \begin{pmatrix} \alpha & -1 \\ 1 & \alpha^{-1} \end{pmatrix}$. Thus if $f_{\Phi}^d(s; g) \neq 0$ we must have $x^{-1} \in \mathbb{Z}_{\ell}$. But then

$$\bar{v} \left(w \begin{pmatrix} \alpha & \alpha x \\ 0 & \alpha^{-1} \end{pmatrix} w \right) = \bar{v} \left(\begin{pmatrix} 1 & 0 \\ -x & 1 \end{pmatrix} \right) = 0$$

by the decomposition

$$\begin{pmatrix} 1 & 0 \\ -x & 1 \end{pmatrix} = \begin{pmatrix} -x^{-1} & 1 \\ 0 & -x \end{pmatrix} w \begin{pmatrix} 1 & -x^{-1} \\ 0 & 1 \end{pmatrix}.$$

Since

$$\Phi_2(C^{-1}D) = \mathbb{1}_{\mathbb{Z}_{\ell}}(x^{-1}) \mathbb{1}_{\mathbb{Z}_{\ell}}(\alpha) \mathbb{1}_{\mathbb{Z}_{\ell}^{\times}}(\alpha^{-1}x^{-1}),$$

we must have $I'_b = 0$. Therefore, we obtain

$$T_{f_{\Phi}(s; -)} \bar{v} = \text{vol}(\Gamma_0(\ell)) \bar{v}.$$

Passing to the quotient by the trivial representation in $\text{nInd}_{\mathbf{B}(\mathbb{Q}_{\ell})}^{\mathbf{SL}_2(\mathbb{Q}_{\ell})}(\delta_{\mathbf{B}}^{-1/2})$ the operator

$$T_{f_{\Phi}(s; -)}$$

$$T_{f_{\Phi}(s; -)} : \text{St}_{\ell}^{\Gamma_0(\ell)} \rightarrow \text{St}_{\ell}^{\Gamma_0(\ell)}$$

is multiplication by the quantity $\text{vol}(\Gamma_0(\ell)) \zeta_{\ell}(s + 3/2)$. Noticing that $\zeta_{\ell}(s + 3/2) = L(s + 1/2, \text{Ad}^0(\text{St}_{\ell}))$ We can therefore deduce the following proposition.

(B.9.2) Proposition. *Let $\Phi = \Phi_1 \otimes \Phi_2$, where $\Phi_1 = \mathbb{1}_{\Gamma_0(\ell)}/\text{vol}(\Gamma_0(\ell))$ and $\Phi_2 = \mathbb{1}_{M_2(\mathbb{Z}_\ell)}$. Consider the corresponding Siegel-Godement section $f_\Phi(s; -)$. Then the following statements hold.*

- *The section $f_\Phi(s; -)$ is $\Gamma_0(\ell) \times \Gamma_0(\ell)$ invariant.*
- *We have, for all $T \in \text{Sym}^2(\mathbb{Q}_\ell^2)$,*

$$W_{T,\ell}(f_\Phi(s; -), 1) = \Phi_2(T) \in \mathbb{Z}_p.$$

- *For all $\varphi \in \text{St}_\ell^{\Gamma_0(\ell)}$, we have the local Zêta integral*

$$Z(f_\Phi(s; -), \varphi, \overline{\varphi}) = \text{vol}(\Gamma_0(\ell)) L(s + 1/2, \text{Ad}^0(\text{St}_\ell)) \langle \varphi, \overline{\varphi} \rangle.$$

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