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# Global and Local fundamental groups in Algebraic Geometry

Figueroa Zamora, Fernando

Princeton, NJ : Princeton University

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# GLOBAL AND LOCAL FUNDAMENTAL GROUPS IN ALGEBRAIC GEOMETRY

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# Abstract

In the first part of this thesis, we study the local fundamental group of low-dimensional log canonical singularities. In dimensions 2 and 3 we establish some constraints on the possible local fundamental groups, while in dimensions 3 and 4 we construct examples of interesting groups that can appear.

In dimension 2, by classifying all the possible singularities, we can prove that the local fundamental groups are virtually solvable. Moreover, we give a bound on the number of generators and relations of the group, along with identifying the circumstances under which this maximum is achieved. In dimension 3, we show that free groups with 2 or more generators do not appear as local fundamental groups of log canonical isolated singularities.

In dimensions 3 and 4, we construct the fundamental groups of 2-dimensional closed manifolds and some special 3-dimensional manifolds, respectively, as the local fundamental groups of isolated log canonical singularities. One notable example includes the connected sum of copies of  $S^1 \times S^2$ , leading to free groups appearing in dimension 4.

In the second part, we study the orbifold fundamental groups of the smooth locus of Calabi-Yau type pairs of low coregularity. Here, we establish that the fundamental group of pairs with low coregularity exhibits similar behavior to log Calabi-Yau pairs of low dimension, specifically being virtually abelian of bounded rank. Furthermore, we prove in the case of virtually nilpotency, there are effective bounds on the index and length depending only on the dimension and coregularity.

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A mi abuela Julia.

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# Chapter 1

## Introduction

Throughout this thesis, we will work over the field of complex numbers  $\mathbb{C}$ .

The study of singularities is fundamental in algebraic geometry. An approach that dates back to the early days of algebraic geometry is to analyze the local topological structure of a singularity. It is known that the topology of a sufficiently small puncture disk stabilizes [26, 10]. Thus, one can define the Link of a point  $x \in X$  to be the intersection of  $X$  and a sphere centered at  $x$  with small enough radius. The fundamental group of the Link of  $x$  is what is called the local fundamental group of  $x \in X$ , denoted by  $\pi_1^{\text{loc}}(X; x)$ , and sometimes by abuse of notation it is also called the fundamental group of the singularity  $(X; x)$ .

In dimension one, the link of a point will be a disjoint union of circles. In higher dimensions, any smooth point will have a  $(2d - 1)$ -dimensional sphere as a link, thus a trivial local fundamental group. The converse holds for normal surfaces [31], so the fundamental group of the singularity brings a lot of geometric information. In higher dimensions this converse fails [5, 26], for example, local complete intersection singularities of dimension at least 3 are simply connected [16].

Since one can assign a CW-complex structure to the complement of a singularity, the local fundamental group of a singularity has to be finitely presented. In [19] it was proved that any finitely presented group appears as the local fundamental group of an isolated singularity. Thus, this is the only restriction that one can impose when we consider algebraic singularities as a whole.

There are two other important fundamental groups that one can assign to an isolated singularity. For them we will need a resolution of the singularity  $f : Y \rightarrow X$ , with simple normal crossing exceptional divisor  $E$ . Let the preimage by  $f$  of the intersection of a small ball around  $x$  with  $X$  be called  $Y_0$ . Then, the link of  $x \in X$  is homotopic to  $Y_0 \setminus E$ . We have the following maps:

$$\pi_1(\text{Link}(x \in X)) \simeq \pi_1(Y_0 \setminus E) \rightarrow \pi_1(Y_0) \simeq \pi_1(E) \twoheadrightarrow \pi_1(\mathcal{D}(E))$$

The kernel of the map  $\pi(Y_0 \setminus E) \rightarrow \pi_1(Y_0)$  is generated by loops around each of the irreducible components of  $E$ . The kernel of  $\pi_1(E) \twoheadrightarrow \pi_1(\mathcal{D}(E))$  is generated by loops inside each component of  $E_i$ .

Thus in general the study of the local topology of singularities is tantamount to understanding the interlocking of these different notions, i.e. one needs to study the dual complex of the exceptional divisor, the topology of each irreducible component and loops around the components of the exceptional divisor.

Previously the local fundamental groups of different classes of singularities have been studied in [19, 22, 23, 3, 4]. Fully classifying the local fundamental groups is a harder task, but there is a wider plethora of results in fully describing the possible fundamental groups of the dual complex. For example, in [19] it was proved that the groups that appear as fundamental groups of dual complexes of rational singularities are exactly the  $\mathbb{Q}$ -superperfect groups.

In dimension 2, Mumford [31] studied the case where the exceptional divisors are a tree of rational curves. In this situation the loops around the exceptional divisors give the entire picture. In the context of this thesis, we will be using his method to study the case of the local fundamental group of log canonical pairs.

In dimension 3, we will also be decomposing the local fundamental group into these three constituents. Here we will be very far from being able to give a complete list of restrictions, but in each type of loops we will be able to give restrictions that prohibit the complete group from being a free group with more than one generator.

In higher dimensions, it is extremely complicated to control effectively the possible loops around the exceptional divisors. In order to create examples, we will use the methods of [22], that amount to constructing a singularity given its resolution. Not any (reducible) variety can be the exceptional divisor of an isolated singularity. Instead of assigning the (reducible) variety as the exceptional divisor of a resolution, one can take it as the exceptional divisor of a partial resolution. Thus, one can construct this (reducible) variety as a divisor in a mildly singular variety, and contract it to a point.

Thus, the ordeal of constructing prescribed groups as the local fundamental groups of isolated singularities, is reduced to constructing some special (reducible) varieties with the prescribed fundamental group.

As any finitely presented group is the fundamental group of a closed manifold (of dimension 4), we can decompose this process into two steps. First, we will decompose a closed manifold into a polytopal complex. Secondly, to each of these smooth polytopes we will assign a smooth projective toric variety and the way that they are glued in the polytopal complex.

For our singularities to be log canonical at the end, we need the (reducible) variety to be simple normal crossing projective Calabi-Yau. Thus the polyhedral complex needs to be smooth, i.e. each  $k$ -dimensional polytope is contained in exactly  $d - k + 1$  maximal  $d$ -dimensional polytopes. This is a purely combinatorial problem, where we need to adjust some machinery about smooth polytopes.

We are able to extend this polytopal machinery to be able to decompose closed smooth 3-dimensional manifolds in  $\mathbb{R}^4$ . This allows us to create a vast variety of new examples of fundamental groups of log canonical singularities. A special case is that of free groups in any (finite) number of generators, as this is the only family of groups that we can show to require of 4-dimensions to appear as a local fundamental group of a log canonical singularity.

If we furthermore are able to decompose closed smooth 4-dimensional manifolds in  $\mathbb{R}^5$ , then this would show that any finitely presented group appears as the local fundamental group of a possibly non-isolated log canonical singularity. Thus, these methods need to be extended only in one dimension to reach our goal.

The topology of complex algebraic varieties has been the subject of research since the inception of algebraic geometry. In the classification of algebraic varieties, the study of the underlying topology as a complex manifold arises naturally. According to the positivity of their canonical bundles, there are three most distinguished classes of projective varieties. These are Fano, Calabi-Yau and canonically polarized varieties.

In the case of smooth curves, there is a correspondence between the fundamental groups and the classification by the canonical bundle. The only Fano curve is the sphere  $\mathbb{P}^1$ , which has trivial fundamental group. Calabi-Yau curves are genus 1 closed real surfaces which have fundamental group  $\mathbb{Z} \times \mathbb{Z}$ , and canonically polarized curves are closed real surfaces of genus at least 2. These have fundamental group given by the presentation  $\langle \alpha_1, \beta_1, \dots, \alpha_g, \beta_g \mid [\alpha_1 \beta_1] \dots [\alpha_g \beta_g] \rangle$ , which is infinite and not abelian.

In higher dimensions some similar results still hold. Higher-dimensional smooth Fano varieties are also simply connected, the fundamental groups of smooth Calabi-Yau varieties are virtually abelian and the fundamental groups of canonically polarized varieties can be more complicated.

In the context of birational geometry, one needs to branch out of the setting of smooth algebraic varieties. Two extensions need to be taken, First, we need to include in our study at least some mild

singularities. The two main types of singularities that we will be dealing with are kawamata log terminal and log canonical singularities, which we will abbreviate by klt and lc, respectively. These singularities can be thought of as a local analog of Fano and Calabi-Yau varieties.

The second extension is the concept of log pairs. A log pair  $(X, D)$  consists of a variety  $X$ , together with  $D$  an effective divisor on  $X$ . This notion is particularly important for induction arguments.

Just like the Fundamental groups parametrize covering spaces, the orbifold fundamental group of a pair  $(X, D)$  parametrizes the covering spaces that branch only over  $D$  with branch index bounded by the coefficients of the irreducible components of  $D$ .

In Part II we will be studying the orbifold fundamental group of the smooth locus of special klt pairs.

The possible fundamental groups of klt Fano pairs are controlled. Indeed Braun proved that they are finite [3]. Furthermore, the structure of these finite groups is better controlled, if we fix the dimension. In [4], they proved that the Jordan property holds for these groups:

**Theorem 1.0.1.** *There exists a constant  $c(d)$ , depending only on  $d$ , such that for any  $(X, B)$  klt fano pair of dimension  $d$ , there exists  $A$  an abelian group of rank at most  $n$  and a group  $N$  of order at most  $c(d)$ , fitting in the following short exact sequence:*

$$1 \rightarrow A \rightarrow \pi_1^{\text{orb}}(X, B) \rightarrow N \rightarrow 1.$$

*In other words  $\pi_1^{\text{orb}}(X, B)$  is virtually abelian of rank at most  $n$  and index at most  $c(n)$ .*

In [6], it is proven that the orbifold fundamental group of the smooth locus of Calabi-Yau pairs is virtually abelian of rank at most 4. In higher dimensions there is still plenty of work left to be done.

Our treatment of the orbifold fundamental group will finally reduce the cases that we study to that of log Calabi-Yau surfaces.

The coregularity of a Calabi-Yau pair is the dimension of a minimal log canonical center among dlt modifications. This invariant behaves well with some of the most common techniques in birational geometry. Thus, some statements that traditionally rely on having bounded dimension can be extended to bounded coregularity [12, 11].

For a Calabi-Yau type pair the absolute coregularity is the minimal coregularity amongst all of the Calabi-Yau structures on them. It is expected that this governs the geometry of Calabi-Yau type varieties.

The main result about orbifold fundamental groups in this thesis will be that the study of fundamental groups of absolute coregularity  $d$  pairs can be reduced to those of log Calabi-Yau pairs of dimension  $d$ . Hence we are able to tackle Pairs of arbitrary dimension with absolute coregularity at most 2, since the 2-dimensional case is the largest one that is fully developed.

Global and local groups relate to each other. The most simple case where this can be appreciated is in cone singularities, where the link of the singularity will be an  $S^1$ -bundle. Thus, the local fundamental group will fit in the following exact sequence.

$$\mathbb{Z} \rightarrow \pi_1^{\text{loc}}(x \in X) \rightarrow \pi_1(X) \rightarrow 1.$$

Thus, the study of  $d$ -dimensional local singularities can rely on the study of global fundamental groups of dimension  $d + 1$ .

As the links of  $d$ -dimensional singularities are contained in  $d$ -dimensional varieties, we have a natural morphism from local fundamental groups of singularities to global fundamental groups of varieties with the prescribed singularities.

These guiding techniques form part of what are called the local-to-global and global-to-local principles in birational geometry. Control of this allows to perform inductive proofs on the structure of singularities and varieties.

In our study of global fundamental groups, we will be using part of this principle, as we will be focusing on klt pairs, and the finite nature of klt local fundamental groups will be key for our arguments.

## Chapter 2

# Preliminaries

### 2.1 Virtual abelianity and solvability of groups

**Definition 2.1.1.** We say that a group  $G$  is *virtually abelian of rank  $n$*  if it has a subgroup of finite index which is abelian of rank  $n$ .

**Remark 2.1.2.** A group  $G$  being virtually abelian of rank  $n$  is equivalent to having a *normal* subgroup of finite index which is abelian of rank  $n$ , since one can take the normal core of the abelian subgroup from Definition 2.1.1, which is again abelian of the same rank and has finite index in  $G$ .

**Lemma 2.1.3.** *Let  $F, G$  and  $H$  be groups fitting in an exact sequence*

$$F \xrightarrow{\phi} G \xrightarrow{\psi} H \rightarrow 1.$$

*If  $H$  is virtually abelian of rank  $n$  and  $F$  is finite, then  $G$  is virtually abelian of rank at most  $n$ .*

*Proof.* By replacing  $F$  with its image  $\phi(F)$  we can assume that  $F$  is a subgroup of  $G$  with  $f$  elements. Since  $H$  is virtually abelian, there exists an abelian group  $A \leq H$  of rank  $n$  and of finite index in  $H$ . Let  $A$  be generated by  $\{a_i\}_{1 \leq i \leq n}$ . Let  $g_i$  be elements in  $G$  mapping to  $a_i$  via  $\psi$ . For any  $x$  in  $F$  and  $g_i$ , we have that  $g_i x g_i^{-1}$  gets mapped to the identity  $e \in H$  via  $\psi$ , hence  $g_i x g_i^{-1} \in F$ . Thus conjugation by elements  $y \in \langle g_i \rangle$  gives a group action on  $F$ . Since  $F$  is a finite set, there exists a uniform constant  $m$  such that  $m$ -iterations of all of these actions are the identity, in particular  $y^m x y^{-m} = x$  for any  $x \in F$  and  $y \in \langle g_i \rangle$ .

On the other hand, for any  $y \in \langle g_i \rangle$ , there exists  $x \in F$ , such that  $g_i^m y g_i^{-m} = xy$ , since it maps

to the same element of  $A$  as  $y$ . Therefore

$$g_i^{fm} g_j^{fm} g_i^{-fm} = g_i^{(f-1)m} x g_j^{fm} g_i^{-(f-1)m} = x g_i^{(f-1)m} g_j^{fm} g_i^{-(f-1)m} = \dots = x^f g_j^{fm} = g_j^{fm}.$$

We conclude that  $\langle g_i^{fm} \rangle_i$  is an abelian subgroup of rank  $n$  of  $G$ . This subgroup has finite index in  $G$ , as it surjects onto  $\langle a_i^{fm} \rangle_i$ , a finite index subgroup of  $H$ , via a homomorphism with finite kernel.

More explicitly, let  $h_j \langle g_i^{mf} \rangle_i$  be the left cosets in  $H$  and for every  $j$ ,  $h'_j$  a preimage of  $h_j$  in  $G$ . Then the left cosets of  $\langle g_i^{mf} \rangle_i$  in  $G$  are of the form  $x h'_j \langle g_i^{mf} \rangle_i$ , where  $x \in F$ , hence there are only finitely many cosets.

□

**Definition 2.1.4.** We say that a group  $G$  is *virtually solvable of length  $n$  and index  $i$*  if it has a subgroup of index  $i$  that is virtually solvable of length  $n$ . When we omit the index  $i$ , the solvable subgroup only needs to be of finite index.

The following two lemmas are counterparts of Lemma 2.1.3 for *virtual solvability*, which we need for effective results in Chapter 9.

**Lemma 2.1.5.** *Let  $F, G$  and  $H$  be groups fitting in an exact sequence*

$$F \xrightarrow{\phi} G \xrightarrow{\psi} H \rightarrow 1.$$

*If  $H$  is virtually solvable of length  $n$  and index  $i$  and  $F$  is finite, then  $G$  is virtually solvable of length at most  $n + 1$  and index at most  $i(|F| - 1)!$ .*

*Proof.* Let  $H'$  be the solvable subgroup of  $H$  of length  $n$  and index  $i$ . By letting  $G'$  be the preimage of  $H'$  under  $\psi$ , we obtain the exact sequence

$$F \xrightarrow{\phi} G' \xrightarrow{\psi} H' \rightarrow 1,$$

where  $[G : G'] = [H : H'] = i$ .

By taking a quotient if needed, we may assume that  $F$  is a subgroup of  $G'$ . Let  $Z$  be the centralizer of  $F$  in  $G'$ . The index of  $Z$  in  $G'$  is at most  $|\text{Aut}(F)|$ , which is at most  $(|F| - 1)!$ .

Since  $F \cap Z$  is abelian and  $\psi(Z) \leq H'$  is solvable of length  $n$ , the group  $Z$  is solvable of length at most  $n + 1$ . Hence  $G$  is virtually solvable of length at most  $n + 1$  and index at most  $i|\text{Aut}(F)| \leq i(|F| - 1)!$ .

□

**Lemma 2.1.6.** *Let  $F, G$  and  $H$  be groups fitting in an exact sequence*

$$F \xrightarrow{\phi} G \xrightarrow{\psi} H \rightarrow 1.$$

*If  $F$  is virtually solvable of length  $n_1$  and index  $r_1$ , and  $H$  is virtually solvable of length  $n_2$  and index  $r_2$ , then  $G$  is virtually solvable of length at most  $n_1 + n_2 + 1$  and index  $r_2(r_1 - 1)!$*

*Proof.* Let  $H'$  be the solvable group of length  $n_2$  and index  $r_2$  of  $H$ . By letting  $G'$  be the preimage of  $H'$  under  $\psi$ , we obtain the exact sequence:

$$F \xrightarrow{\phi} G' \xrightarrow{\psi} H' \rightarrow 1,$$

where  $[G : G'] = [H : H'] = i$ .

By taking a quotient if needed, we may assume that  $F$  is a subgroup of  $G'$ . Let  $F'$  be the solvable normal subgroup of  $F$  of smallest index.  $F'$  is a characteristic subgroup of  $F$ , otherwise there would be  $F''$ , another normal subgroup of the same index. But then the group  $F'F''$  would be a normal solvable subgroup of smaller index, a contradiction. Therefore  $F'$  is a normal subgroup of  $G'$ .

By Lemma 2.1.5 applied to the induced exact sequence

$$F/F' \xrightarrow{\phi} G'/F' \xrightarrow{\psi} H' \rightarrow 1,$$

the group  $G'/F'$  is virtually solvable of length  $n_2 + 1$  and index at most  $(r_1 - 1)!$ . Let the quotient be  $\pi : G' \rightarrow G'/F'$ . Thus there exists a subgroup  $H'' \leq G'/F'$  solvable of length  $n_2 + 1$  and index at most  $(r_1 - 1)!$ . By taking the preimage  $G'' := \pi^{-1}(H'')$ , we obtain:

$$F' \rightarrow G'' \xrightarrow{\pi} H'' \rightarrow 1.$$

Therefore  $G''$  is solvable of length  $n_1 + n_2 + 1$ , and its index in  $G$  is

$$[G : G''] = [G : G'] [G' : G''] \leq r_2(r_1 - 1)!$$

as stated. □

## 2.2 Generalized pairs and singularities of the Minimal Model Program.

In this section we recall the definitions of generalized pairs and the singularities of the Minimal Model Program. For further properties of b-divisors, we refer the reader to [8, §2.3.2].

**Definition 2.2.1.** Let  $X$  be a normal variety. A *b-divisor*  $\mathbf{M}$  on  $X$  is a function which associates any birational map  $X' \dashrightarrow X$  with an  $\mathbb{R}$ -divisor  $\mathbf{M}_{X'}$  on  $X'$ . The set of divisors  $\{\mathbf{M}_{X'} : X' \dashrightarrow X\}$  satisfies the following compatibility condition: if  $g : X_1 \rightarrow X_2$  is a birational morphism over  $X$ , then  $g_*\mathbf{M}_{X_1} = \mathbf{M}_{X_2}$ . We say that a b-divisor  $\mathbf{M}$  on  $X$  *descends* on some birational model  $X'$  of  $X$  if  $\mathbf{M}_{X'}$  is  $\mathbb{R}$ -Cartier and  $\mathbf{M}$  is equivalent to  $(X' \rightarrow X, \mathbf{M}_{X'})$ . In other words, for any birational map  $h : Y \rightarrow X'$  over  $X$ , we have  $h^*\mathbf{M}_{X'} = \mathbf{M}_Y$ . In the previous case, we say that  $\mathbf{M}$  is a *b- $\mathbb{R}$ -Cartier divisor*.

Let  $X$  be a projective variety. The b-divisor  $\mathbf{M}$  is said to be *b-Cartier* (resp. *b-nef*) if  $\mathbf{M}_{X'}$  is Cartier (resp. nef) on some birational model  $X'$  over  $X$  where  $\mathbf{M}$  descends.

The *b-Cartier closure* of an  $\mathbb{R}$ -Cartier divisor  $M$  is a b-divisor  $\mathbf{M}$  whose trace on every birational model  $f : Y \rightarrow X$  is  $f^*M$ .

**Definition 2.2.2.** A *generalized pair*  $(X, D, \mathbf{M})$  consists of a normal variety  $X$ , together with an effective  $\mathbb{Q}$ -divisor  $D$  and a nef b-divisor  $\mathbf{M}$ , such that  $K_X + D + \mathbf{M}_X$  is  $\mathbb{Q}$ -Cartier.

**Definition 2.2.3.** Let  $(X, D, \mathbf{M})$  be a generalized pair and  $Y$  a normal variety. Given a projective birational morphism  $f : Y \rightarrow X$ , we can define  $D_Y$ , by the formulas:

$$K_Y + D_Y + \mathbf{M}_Y = f^*(K_X + D + \mathbf{M}_X) \quad \text{and} \quad f_*(D_Y) = D.$$

For a prime divisor  $E \subset Y$ , we define the *generalized log discrepancy* of  $(X, D, \mathbf{M})$  at  $E$  as:

$$a_E(X, D, \mathbf{M}) = 1 - \text{coeff}_E(D_Y).$$

A generalized pair  $(X, D, \mathbf{M})$  is said to be *generalized log canonical* (respectively *generalized kawamata log terminal*), abbreviated glc (respectively gklt) if for every projective birational morphism  $Y \rightarrow X$  and every prime divisor  $E \subset Y$ , the log discrepancy  $a_E(X, D, \mathbf{M}) \geq 0$  (respectively  $a_E(X, D, \mathbf{M}) > 0$ ).

**Definition 2.2.4.** A *generalized log canonical place* (respectively *generalized non-klt place*) of  $(X, D, \mathbf{M})$  is a prime divisor  $E$  over  $X$  such that  $a_E(X, D, \mathbf{M}) = 0$  (respectively  $a_E(X, D, \mathbf{M}) \leq 0$ ).

A *generalized log canonical center* (glcc for short) of  $(X, D, \mathbf{M})$  is the image in  $X$  of a generalized log canonical place.

A generalized pair  $(X, D, \mathbf{M})$  is *generalized divisorially log terminal* (gdlt for short) if there exists an open subset  $U \subseteq X$  such that:

1.  $\text{coeff}(D) \leq 1$ ,
2.  $U$  is smooth and  $D|_U$  is simple normal crossing, and
3. all the generalized non-klt centers of  $(X, D, \mathbf{M})$  intersect  $U$  and are given by strata of  $\lfloor D \rfloor$ .

**Definition 2.2.5.** Let  $(X, D, \mathbf{M})$  be a generalized log canonical pair and  $f : Y \rightarrow X$  a birational morphism, where  $K_Y + D_Y + \mathbf{M}_Y = f^*(K_X + D + \mathbf{M}_X)$ . We say that  $(Y, D_Y, \mathbf{M})$  is a generalized dlt modification of  $(X, D, \mathbf{M})$  if  $(Y, D_Y, \mathbf{M})$  is generalized dlt and every  $f$ -exceptional divisor appears in  $D_Y$  with coefficient 1.

The following lemma is [13, Theorem 2.9].

**Lemma 2.2.6.** *Every generalized log canonical pair admits a  $\mathbb{Q}$ -factorial generalized dlt modification.*

**Definition 2.2.7.** We say that a generalized pair  $(X, D, \mathbf{M})$  is *generalized log Fano* if:

1.  $-(K_X + D + \mathbf{M}_X)$  is ample, and
2.  $(X, D, \mathbf{M})$  is generalized klt.

We say that a generalized pair  $(X, D, \mathbf{M})$  is *generalized log Calabi-Yau* if:

1.  $K_X + D + \mathbf{M}_X \sim_{\mathbb{Q}} 0$ , and
2.  $(X, D, \mathbf{M})$  is generalized log canonical.

A generalized pair  $(X, D, \mathbf{M})$  is of *generalized Fano type* (respectively *generalized Calabi-Yau type*) if for some choice of  $B \geq D$  and some nef b-divisor  $\mathbf{N}$ , the generalized pair  $(X, B, \mathbf{M} + \mathbf{N})$  is generalized log Fano (respectively generalized log Calabi-Yau).

The following is the generalized canonical bundle formula as in [14] and [17, Theorem 1.2]

**Proposition 2.2.8.** *Let  $f : X \rightarrow Z$  be a surjective projective morphism of normal varieties. Let  $(X, D, \mathbf{M})$  be a generalized log canonical pair, such that*

$$K_X + D + \mathbf{M}_X \sim_{\mathbb{Q}, f} 0.$$

Then there exists a generalized log canonical pair  $(Z, D_Z, \mathbf{N})$ , such that:

$$K_X + D + \mathbf{N}_X \sim_{\mathbb{Q}} f^*(K_Z + D_Z + \mathbf{N}_Z).$$

**Remark 2.2.9.** The pair  $(Z, D_Z, \mathbf{N})$  obtained in Proposition 2.2.8 will be called *the generalized pair induced by the canonical bundle formula*.

**Definition 2.2.10.** A *contraction* is a projective morphism of quasi-projective varieties  $f : X \rightarrow Z$ , such that  $f_*\mathcal{O}_X = \mathcal{O}_Z$ . A *fibration* is a contraction  $f : X \rightarrow Z$  such that  $\dim X > \dim Z$ .

**Lemma 2.2.11.** Let  $f : X \rightarrow Z$  be a projective fibration. Let  $(X, D, \mathbf{M})$  be a generalized projective log canonical pair, and  $f^*(K_Z + D_Z + \mathbf{N}_Z) \sim_{\mathbb{Q}} K_X + D + \mathbf{M}_X$ . Then the image of a generalized log canonical center of  $(X, D, \mathbf{M})$  contains a generalized log canonical center of  $(Z, D_Z, \mathbf{N})$ .

*Proof.* Let  $C$  be a glcc of  $(X, D, \mathbf{M})$ . Suppose that  $f(C)$  contains no glcc of  $(Z, D_Z, \mathbf{N})$ , i.e.  $(Z, D_Z, \mathbf{N})$  is gklt at  $f(C)$ . Thus we can take  $A$ , an ample effective divisor in  $Z$  containing  $f(C)$ , general enough such that  $(Z, B_Z + A, \mathbf{N})$  is generalized log canonical. But  $(X, B + f^*A, \mathbf{M})$  is not generalized log canonical at  $C$ , which contradicts [14, Theorem 1.6].  $\square$

The following lemma is taken from [28, Lemma 2.36].

**Lemma 2.2.12.** Let  $Z \rightarrow W$  be a Fano type fibration. Let  $(Z, B, \mathbf{M})$  be a generalized log Calabi-Yau pair and  $(W, B_W, \mathbf{N})$  be the generalized pair induced by the canonical bundle formula. Let  $W' \rightarrow W$  be a  $\mathbb{Q}$ -factorial gdlm modification of  $(W, B_W, \mathbf{N})$ . Then there exists a birational contraction  $\phi : Z' \dashrightarrow Z$ , only extracting generalized log canonical places of  $(Z, B, \mathbf{M})$ , such that there exists a Fano type fibration  $Z' \rightarrow W'$  making the following diagram commute:

$$\begin{array}{ccc} (Z, B, \mathbf{M}) & \xleftarrow{\phi} & (Z', B', \mathbf{M}) \\ \downarrow & & \downarrow \\ (W, B_W, \mathbf{N}) & \longleftarrow & (W', B_{W'}, \mathbf{N}) \end{array}$$

The following lemma is a slight improvement of [27, Lemma 3.7].

**Lemma 2.2.13.** Let  $(X, D, \mathbf{M})$  be a generalized klt projective pair and let  $A$  be an effective ample  $\mathbb{Q}$ -divisor on  $X$ . Assume that  $\mathcal{V}$  is a finite set of divisorial valuations for which  $a_E(X, D+A, \mathbf{M}) \in (0, 1)$  for all  $E \in \mathcal{V}$ . Then, there exists an effective  $\mathbb{Q}$ -divisor  $\Delta$  on  $X$ , such that:

1.  $\Delta$  is big,
2.  $(X, \Delta)$  is a klt pair,

$$3. K_X + D + A + \mathbf{M}_X \sim_{\mathbb{Q}} K_X + \Delta,$$

$$4. \Delta \geq D, \text{ and}$$

$$5. a_E(X, \Delta) \in (0, 1) \text{ for all } E \in \mathcal{V}.$$

*Proof.* Let  $f : Y \rightarrow X$  be a log resolution of  $(X, D + A, \mathbf{M})$ , such that all the valuations in  $\mathcal{V}$  are divisorial in  $Y$ . Let the log pullback of  $(X, D + A, \mathbf{M})$  be  $(Y, D_Y + A_Y, \mathbf{M}_Y)$ . As  $\mathbf{M}_Y + A_Y$  is big and nef, for every  $m \in \mathbb{N}$  we can write  $\mathbf{M}_Y + A_Y \sim_{\mathbb{Q}} A_m + \frac{F}{m}$  [25, Example 2.2.19], where  $F$  is a fixed effective  $\mathbb{Q}$ -divisor and  $A_m$  is an ample  $\mathbb{Q}$ -divisor. By choosing  $k$  large enough and  $A_k$  general enough, we obtain a sub-klt pair  $(Y, D_Y + A_k + \frac{F}{k})$ . This pair will satisfy  $K_Y + D_Y + A_k + \frac{F}{k} \sim_{\mathbb{Q}} K_Y + D_Y + \mathbf{M}_Y + A_Y$ . Every component of  $D_Y + A_k + \frac{F}{k}$  with negative coefficients is exceptional over  $X$ . By hypothesis, for every  $E \in \mathcal{V}$ , the coefficient  $\text{coeff}_E(D_Y + A_Y) \in (0, 1)$ . Thus, setting  $A_X = f_*(A_k)$  and  $F_X = f_*(\frac{F}{k})$ , the pair  $(X, D + A_X + F_X)$  is klt with  $A_X + F_X \sim \mathbf{M} + A$ . Hence  $\Delta := D + A_X + F_X$  has the desired properties.  $\square$

The following is a version of [2, Corollary 1.4.3] for generalized pairs.

**Lemma 2.2.14.** *Let  $(X, D, \mathbf{M})$  be a generalized projective klt pair,  $(X, B, \mathbf{M})$  be a generalized lc pair, and  $\mathcal{V}$  a finite set of generalized log canonical centers of  $(X, B, \mathbf{M})$ . Then, there exists a  $\mathbb{Q}$ -factorial variety  $Y$ , and a birational morphism  $f : Y \rightarrow X$ , such that the exceptional divisors of  $f$  correspond to the elements of  $\mathcal{V}$ .*

*Proof.* Let  $A$  be an effective ample divisor general enough. As  $(X, D, \mathbf{M})$  is generalized klt, for  $\varepsilon > 0$  and  $\delta > 0$  small enough, we get

$$a_E(X, (1 - \varepsilon)B + \varepsilon D + \delta A, \mathbf{M}) \in (0, 1)$$

for every  $E \in \mathcal{V}$ . Thus by Lemma 2.2.13, there exists a klt pair  $(X, \Delta)$  such that  $a_E(X, \Delta) \in (0, 1)$  for every  $E \in \mathcal{V}$ . We can conclude by [2, Corollary 1.4.3] applied to the klt pair  $(X, \Delta)$  and valuations in  $\mathcal{V}$ .  $\square$

## 2.3 Coregularity

In this section we recall the definition of coregularity and absolute coregularity for generalized pairs, and a few lemmas regarding its behavior under the minimal model program.

**Definition 2.3.1.** Let  $(X, B, \mathbf{M})$  be a generalized log canonical pair. Let  $g : (Y, B_Y, \mathbf{M}) \rightarrow (X, B, \mathbf{M})$  be a  $\mathbb{Q}$ -factorial gdl modification, with  $\lfloor B_Y \rfloor = E_1 + \dots + E_r$  an snc divisor. We define  $\mathcal{D}(Y, B_Y, \mathbf{M})$ , the *dual complex* of  $(Y, B_Y, \mathbf{M})$ , to be:

1. For each  $E_i$ , there is a vertex  $s_{E_i}$  in  $\mathcal{D}(Y, B_Y)$ .
2. For any subset  $I \subseteq \{1, \dots, r\}$ , and any irreducible component  $Z$  of  $\cap_{i \in I} E_i$ , there is a  $(|I| - 1)$ -dimensional simplex  $s_Z$ .
3. For any
  - (a)  $I \subseteq \{1, \dots, r\}$ ,
  - (b)  $j \in I$ ,
  - (c) any irreducible component  $Z$  of  $\cap_{i \in I} E_i$ , and
  - (d)  $W$  the unique irreducible component of  $\cap_{i \in I \setminus \{j\}} E_i$  containing  $Z$ ,

there is a gluing map given by the inclusion of  $s_W$  into  $s_Z$  as the unique face that does not contain the vertex  $s_{E_j}$ .

We define the *dimension of the dual complex* to be the smallest dimension of a maximal simplex in  $\mathcal{D}(Y, B_Y, \mathbf{M})$  (with respect to inclusion). By [12, Proposition 2.21], the dimension of  $\mathcal{D}(Y, B_Y, \mathbf{M})$  does not depend on the gdl modification. Hence, we can define the *regularity* and *coregularity* of a generalized pair  $(X, B, \mathbf{M})$ , to be:

$$\text{reg}(X, B, \mathbf{M}) := \dim \mathcal{D}(Y, B_Y, \mathbf{M}).$$

$$\text{coreg}(X, B, \mathbf{M}) := \dim X - \dim \mathcal{D}(Y, B_Y, \mathbf{M}) - 1.$$

If  $Z \subset X$  is an irreducible variety with generic point  $\eta_Z$ , then we define the regularity and coregularity of  $(X, B, \mathbf{M}; Z)$  to be, respectively, the regularity and coregularity of  $(U, B|_U, \mathbf{M}|_U)$  for a sufficiently small neighborhood  $U$  of  $\eta_Z$ .

**Definition 2.3.2.** For a generalized pair  $(X, D, \mathbf{M})$  of log Calabi-Yau type, we define the *absolute coregularity*  $\text{coreg}(X, D, \mathbf{M})$  to be the smallest value of  $\text{coreg}(X, B, \mathbf{M} + \mathbf{N})$ , where  $(X, B, \mathbf{M} + \mathbf{N})$  is a generalized Calabi-Yau pair, with  $B \geq D$  and  $N$  a nef b-divisor.

The coregularity of a generalized Calabi-Yau pair is preserved after taking a gdl modification.

**Remark 2.3.3.** The absolute coregularity as it is defined here, could be lower than the one in [29], where the  $\mathbf{b}$ -divisor does not change for the generalized Calabi-Yau pair defining the coregularity.

We need to use this definition for the induction in the proof of the main theorems.

The following is [12, Lemma 2.31].

**Lemma 2.3.4.** *Let  $(X, B, \mathbf{M})$  be a generalized Calabi-Yau pair. Let  $X \dashrightarrow X'$  be a birational contraction. Denote by  $B'$  the push-forward of  $B$  on  $X'$ . Then*

$$\text{coreg}(X, B, \mathbf{M}) = \text{coreg}(X', B', \mathbf{M}).$$

The following lemma is [29, Proposition 3.27]

**Lemma 2.3.5.** *Let  $f : Y \dashrightarrow X$  be a birational contraction,  $(X, D, \mathbf{M})$  a Calabi-Yau type generalized pair, with  $f^*(K_X + D + \mathbf{M}_X) = K_Y + D_Y + \mathbf{M}_Y$ . Assume  $D_Y \geq 0$ . Then  $\text{coreg}(Y, D_Y, \mathbf{M}) \geq \text{coreg}(X, D_X, \mathbf{M})$ .*

## 2.4 Fundamental Groups

### 2.4.1 Orbifold fundamental groups

We start this subsection by defining the orbifold fundamental group of a pair and prove that they are virtually abelian in some special cases. Then, we prove some results regarding the behavior of the fundamental group after performing certain birational modifications.

**Definition 2.4.1.** A divisor  $D$  is said to have *standard* coefficients, when the coefficients are in the set  $\{1 - \frac{1}{n}\}_{n \in \mathbb{N}} \cup \{1\}$ . For an effective  $\mathbb{Q}$ -divisor  $D$ , the *standard approximation*  $D_{\text{st}}$  is the largest effective divisor with standard coefficients, such that  $D_{\text{st}} \leq D$ .

**Definition 2.4.2.** For a generalized pair  $(X, D, \mathbf{M})$ , we define the *orbifold fundamental group of the smooth locus*  $\pi_1^{\text{sm}}(X, D, \mathbf{M})$  to be

$$\pi_1(X_{\text{reg}} \setminus \text{supp}(D_{\text{st}}))/N,$$

where  $X_{\text{reg}}$  is the smooth locus of  $X$  and  $N$  is the normal subgroup of  $\pi_1(X_{\text{reg}} \setminus \text{supp}(D_{\text{st}}))$  generated by  $\gamma_i^{n_i}$ , given by  $D_{\text{st}} = \sum_i (1 - \frac{1}{n_i}) D_i$  and  $\gamma_i$  being a loop around  $D_i$ .

The following is a version of [3, Theorem 2] for generalized pairs:

**Theorem 2.4.3.** *Let  $(X, D, \mathbf{M})$  be a projective generalized klt pair, such that  $-(K_X + D + \mathbf{M}_X)$  is big and nef. Let  $\Gamma \leq D$  be an effective divisor with standard coefficients. Then, the group  $\pi_1^{\text{sm}}(X, \Gamma)$  is finite.*

*Proof.* As  $(X, D, \mathbf{M})$  is generalized klt weakly Fano, there exists  $B$  effective, such that  $(X, D+B, \mathbf{M})$  is generalized klt Calabi-Yau. Thus  $B$  is big and nef, hence it can be decomposed as  $B = A + E$ , where  $A$  is an ample  $\mathbb{Q}$ -divisor and  $E$  is an effective  $\mathbb{Q}$ -divisor.

Thus for  $0 < \varepsilon < 1$ , the divisor  $-(K_X + D + \varepsilon A + \varepsilon E + \mathbf{M}_X) \sim (1 - \varepsilon)(A + E)$  is big and nef. By Lemma 2.2.13 applied to the generalized pair  $(X, D + \varepsilon E, \mathbf{M})$  with ample divisor  $\varepsilon A$ , there exists an effective  $\mathbb{Q}$ -divisor  $\Delta$ , such that  $(X, \Delta)$  is klt and  $-(K_X + \Delta) \sim_{\mathbb{Q}} (K_X + D + \varepsilon E + \mathbf{M}_X + \varepsilon A)$  is big and nef and  $\Delta \geq D + \varepsilon E \geq \Gamma$ . Therefore by [3, Theorem 2],  $\pi_1(X, \Gamma)$  is finite.  $\square$

**Proposition 2.4.4.** *Let  $(X, D, \mathbf{M})$  be a projective generalized klt pair of dimension 1 with standard coefficients, then the following statements hold:*

1. *If  $(X, D, \mathbf{M})$  is of absolute coregularity 0, then  $\pi_1^{\text{sm}}(X, D, \mathbf{M})$  is finite.*
2. *If  $(X, D, \mathbf{M})$  is of absolute coregularity 1, then  $\pi_1^{\text{sm}}(X, D, \mathbf{M})$  is virtually abelian of rank at most 2.*

*Proof.* Since  $(X, D)$  is klt, then  $\lfloor D \rfloor = 0$ . If  $(X, D)$  is of absolute coregularity 0 and dimension 1, this means that  $(X, D + E)$  is Calabi-Yau for some effective  $\mathbb{Q}$ -divisor  $E$ , hence  $X$  is isomorphic to  $\mathbb{P}^1$ . Therefore  $(X, D)$  is Fano, and hence by [3, Theorem 2] the group  $\pi_1^{\text{sm}}(X, D_{\text{st}})$  is finite.

If  $(X, D)$  is of absolute coregularity 1 and dimension 1, then  $(X, D_{\text{st}})$  is either of absolute coregularity 0 or 1, in the first case we can reduce to the previous paragraph.

If  $(X, D_{\text{st}})$  has absolute coregularity 1, then either  $X$  is an elliptic curve with  $D = 0$ , or  $X = \mathbb{P}^1$ . In the first case, we have that  $\pi_1(X, D_{\text{st}}) = \mathbb{Z} \times \mathbb{Z}$ , which satisfies the theorem. If  $(X, D_{\text{st}})$  is Fano, then again by [3, Theorem 2], we are done. Hence we only need to consider  $(\mathbb{P}^1, D)$  klt Calabi-Yau. Therefore the only possibilities for  $D$  are  $(\frac{1}{2}P_1 + \frac{1}{2}P_2 + \frac{1}{2}P_3 + \frac{1}{2}P_4)$ ,  $(\frac{1}{2}P_1 + \frac{2}{3}P_2 + \frac{5}{6}P_3)$ ,  $(\frac{1}{2}P_1 + \frac{3}{4}P_2 + \frac{3}{4}P_3)$  and  $(\frac{2}{3}P_1 + \frac{2}{3}P_2 + \frac{2}{3}P_3)$ . We can give explicit descriptions of each of the corresponding fundamental groups:

1.  $\pi_1(\mathbb{P}^1, \frac{1}{2}P_1 + \frac{1}{2}P_2 + \frac{1}{2}P_3 + \frac{1}{2}P_4) \simeq \langle \alpha, \beta, \gamma, \delta \mid \alpha\beta\gamma\delta, \alpha^2, \beta^2, \gamma^2, \delta^2 \rangle$ , which has a normal abelian subgroup  $N = \langle \alpha\beta, \alpha\gamma \rangle \simeq \mathbb{Z}^2$  of index 2.
2.  $\pi_1(\mathbb{P}^1, \frac{1}{2}P_1 + \frac{2}{3}P_2 + \frac{5}{6}P_3) \simeq \langle \alpha, \beta, \gamma \mid \alpha\beta\gamma, \alpha^2, \beta^3, \gamma^6 \rangle$ , which has a normal abelian subgroup  $N = \langle \beta\gamma^4, \beta^2\gamma^2 \rangle \simeq \mathbb{Z}^2$  of index 6.

3.  $\pi_1(\mathbb{P}^1, \frac{1}{2}P_1 + \frac{3}{4}P_2 + \frac{3}{4}P_3) \simeq \langle \alpha, \beta, \gamma \mid \alpha\beta\gamma, \alpha^2, \beta^4, \gamma^4 \rangle$ , which has a normal abelian subgroup  $N = \langle \beta\gamma^3, \beta^3\gamma \rangle \simeq \mathbb{Z}^2$  of index 4.
4.  $\pi_1(\mathbb{P}^1, \frac{2}{3}P_1 + \frac{2}{3}P_2 + \frac{2}{3}P_3) \simeq \langle \alpha, \beta, \gamma \mid \alpha\beta\gamma, \alpha^3, \beta^3, \gamma^3 \rangle$ , which has a normal abelian subgroup  $N = \langle \alpha\beta^2, \alpha^2\beta \rangle \simeq \mathbb{Z}^2$  of index 3.

□

The following statement is [6, Theorem 2.2].

**Proposition 2.4.5.** *Let  $(X, D)$  be a projective klt Calabi-Yau pair of dimension 2, with standard coefficients, then  $\pi_1(X, D)$  is virtually abelian of rank at most 4.*

**Lemma 2.4.6.** *Let  $(X, D, \mathbf{M})$  be a generalized projective klt pair, such that there exists a generalized Calabi-Yau pair  $(X, B, \mathbf{M})$ , with  $D \leq B$ . Let  $\phi : Y \rightarrow X$  be a projective morphism, only extracting generalized log canonical places of  $(X, B, \mathbf{M})$ , with  $\phi^*(K_X + B + \mathbf{M}_X) = K_Y + B_Y + \mathbf{M}_Y$  and  $\phi^*(K_X + D + \mathbf{M}_X) = K_Y + D_Y + \mathbf{M}_Y$ .*

*Then, for  $\alpha = \frac{1}{m}$ , where  $m$  is a natural number, large and divisible enough, there exists an effective divisor  $P_Y$  on  $Y$ , such that:*

1.  $D_Y \leq P_Y \leq \alpha D_Y + (1 - \alpha)B_Y$ , and
2. *there is a surjection of orbifold fundamental groups:*

$$\pi_1^{\text{sm}}(Y, P_Y, \mathbf{M}) \twoheadrightarrow \pi_1^{\text{sm}}(X, D, \mathbf{M}).$$

*Proof.* We have that  $B_Y \geq D_Y$ . By hypothesis  $B_Y = B' + E$ , where  $B'$  is the strict transform of  $B$  and  $E$  is a reduced exceptional divisor. Let  $D'$  be the strict transform of  $D$ . Thus

$$\begin{aligned} \pi_1^{\text{sm}}(Y, D' + E, \mathbf{M}) &= \pi_1(Y_{\text{reg}} \setminus \text{supp}(D' \cup E)) / N_{D'} \\ &\simeq \pi_1(X_{\text{reg}} \setminus \text{supp}(D \cup \phi(E))) / N_D \\ &\twoheadrightarrow \pi_1(X_{\text{reg}} \setminus D) / N_D = \pi_1^{\text{sm}}(X, D, \mathbf{M}), \end{aligned}$$

where  $N_{B'}$  and  $N_B$  are the normal subgroups generated by elements of the form  $\gamma_i^{n_i}$ , where  $\gamma_i$  are loops around the divisors  $B_i$  with coefficient  $1 - \frac{1}{n_i}$  in  $B'$  and  $B$ , respectively.

Let us denote the surjection  $f : \pi_1^{\text{sm}}(Y, D' + E, \mathbf{M}) \twoheadrightarrow \pi_1^{\text{sm}}(X, D, \mathbf{M})$ , induced as displayed above by the surjection  $\hat{f} : \pi_1(Y_{\text{reg}} \setminus \text{supp} D' \cup E) \twoheadrightarrow \pi_1(X_{\text{reg}} \setminus \text{supp} D \cup \phi(E))$ .

As  $(X, D)$  is a klt pair, for any irreducible component  $C$  of  $\phi(E)$ , we have that the local fundamental group  $\pi_1^{\text{loc}}(X, D; C)$  is finite, by [3, Theorem 1]. Thus every loop  $\gamma$  around  $C$  in  $\pi_1^{\text{loc}}(X, D; C)$ , has finite order. We can choose a constant  $q$  not depending on the irreducible component  $C$ , such that  $\gamma^q$  is trivial for every loop. Therefore  $\hat{\gamma}^q$  is trivial in  $\pi_1^{\text{sm}}(X, D)$ , where  $\hat{\gamma}$  is the class of the loop  $\gamma$  in  $\pi_1^{\text{sm}}(X, D)$  induced by inclusion.

Therefore the kernel of  $\hat{f} : \pi_1(Y_{\text{reg}} \setminus \text{supp} D' \cup E) \twoheadrightarrow \pi_1(X_{\text{reg}} \setminus \text{supp} D \cup \phi(E))$  contains all elements  $\gamma^q$ , where  $\gamma$  is a loop around an irreducible component of  $E$ .

Thus  $f : \pi_1^{\text{sm}}(Y, D' + E, \mathbf{M}) \twoheadrightarrow \pi_1^{\text{sm}}(X, D, \mathbf{M})$  being a surjection, implies that we have the following surjection:

$$f : \pi_1^{\text{sm}}\left(Y, D' + \left(1 - \frac{1}{q}\right)E, \mathbf{M}\right) = \pi_1^{\text{sm}}(Y, D' + E, \mathbf{M}) / N_{(1-\frac{1}{q})E} \twoheadrightarrow \pi_1^{\text{sm}}(X, D, \mathbf{M}).$$

□

**Lemma 2.4.7.** *Let  $(X, D, \mathbf{M})$  be a projective klt pair, such that there exists a generalized Calabi-Yau pair  $(X, B, \mathbf{M})$ , with  $D \leq B$ . Let  $\phi : Y \dashrightarrow X$  be a birational contraction that only contracts generalized log canonical places from  $(X, B, \mathbf{M})$ , with  $\phi^*(K_X + B + \mathbf{M}_X) = K_Y + B_Y + \mathbf{M}_Y$  and  $\phi^*(K_X + D + \mathbf{M}_X) = K_Y + D_Y + \mathbf{M}_Y$ .*

*Then, for  $\alpha = \frac{1}{m}$ , where  $m$  is a natural number, large and divisible enough, there exists an effective divisor  $P_Y$  on  $Y$ , such that:*

1.  $D_Y \leq P_Y \leq \alpha D_Y + (1 - \alpha)B_Y$ , and
2. *there is a surjection of orbifold fundamental groups:*

$$\pi_1^{\text{sm}}(Y, P_Y, \mathbf{M}) \twoheadrightarrow \pi_1^{\text{sm}}(X, D, \mathbf{M}).$$

*Proof.* By Lemma 2.2.14 there exists a birational morphism  $f : Y' \rightarrow X$  contracting the same glc places as  $\phi : Y \dashrightarrow X$ . Let  $P_{Y'}$  be defined by the equation

$$g^*(K_Y + P_Y + \mathbf{M}_Y) = K_{Y'} + P_{Y'} + \mathbf{M}_{Y'}.$$

The map  $g : Y' \dashrightarrow Y$  is small, hence  $\pi_1^{\text{sm}}(Y', P_{Y'}, \mathbf{M}) \simeq \pi_1^{\text{sm}}(Y, P_Y, \mathbf{M})$ . By replacing  $Y$  with  $Y'$ , we can assume that we have a projective birational morphism  $f : Y \rightarrow X$  only extracting generalized log canonical places of  $(X, B, \mathbf{M})$ . Thus, we can conclude by applying Lemma 2.4.6. □

The following lemma is part of the proof of [4, Theorem 3.3] for generalized pairs.

**Lemma 2.4.8.** *Let  $(X, D, \mathbf{M})$  be a gklt pair. Let  $\phi : X \dashrightarrow X'$  be either a flip or a divisorial contraction, with  $D' := \phi_*(D)$ . Then, there is an induced surjective group homomorphism*

$$\pi_1^{\text{sm}}(X', D', \mathbf{M}) \twoheadrightarrow \pi_1^{\text{sm}}(X, D, \mathbf{M}).$$

**Lemma 2.4.9.** *Let  $(F, P_F, \mathbf{M})$  be a generalized projective klt Fano pair and  $(F, B_F, \mathbf{M})$  be a generalized Calabi-Yau pair. Let  $\phi : F_0 \rightarrow F$  be a morphism only extracting generalized log canonical places of  $(F, B_F, \mathbf{M})$ . Define  $R_{F_0} = \phi_*^{-1}(P_F) + E$ , where  $E_{\geq 0}$  is a  $\phi$ -exceptional divisor with coefficients less than 1. Then  $\pi_1^{\text{sm}}(F_0, R_{F_0}, \mathbf{M})$  is finite.*

*Proof.* Let  $B_{F_0}$  and  $P_{F_0}$  be defined by the equalities:

$$\phi^*(K_F + B_F + \mathbf{M}_F) = K_{F_0} + B_{F_0} + \mathbf{M}_{F_0} \quad \text{and} \quad \phi^*(K_F + P_F + \mathbf{M}_F) = K_{F_0} + P_{F_0} + \mathbf{M}_{F_0}.$$

For any  $0 < \alpha < 1$ , the divisor  $-K_{F_0} - (1 - \alpha)B_{F_0} - \alpha P_{F_0} - \mathbf{M}_{F_0}$  is big and nef. Indeed

$$-K_{F_0} - (1 - \alpha)B_{F_0} - \alpha P_{F_0} - \mathbf{M}_{F_0} = -(\alpha(K_{F_0} + B_{F_0} + \mathbf{M}_{F_0}) + (1 - \alpha)(K_{F_0} + P_{F_0} + \mathbf{M}_{F_0})).$$

The sub-pair  $(F_0, P_{F_0}, \mathbf{M})$  is generalized sub-klt. Since  $\phi$  only extracts glc places of  $(F, B_F, \mathbf{M})$ , the pair  $(F_0, B_{F_0}, \mathbf{M})$  is generalized log canonical with  $B_{F_0} \geq P_{F_0}$ . Hence for  $\alpha$  close enough to 0, the pair  $(F_0, (1 - \alpha)B_{F_0} + \alpha P_{F_0}, \mathbf{M})$  is a gklt weakly Fano pair with  $(1 - \alpha)B_{F_0} + \alpha P_{F_0} \geq R_{F_0}$ . Therefore by Theorem 2.4.3,  $\pi_1^{\text{sm}}(F_0, R_{F_0}, \mathbf{M})$  is finite. □

The following lemma allows us to understand the orbifold fundamental groups of log pairs admitting fibrations. It can be regarded as a version for pairs of [34, Lemma 1.5.C].

**Lemma 2.4.10.** *Let  $(Z, P)$  be a klt pair, with  $\psi : Z \rightarrow W$  a Mori fiber space. Let  $P_{st}$  be the standard approximation of  $P$ . For any prime divisor  $E$  on  $W$ , define  $n_E$  by  $\text{coeff}_{\psi^*(E)}(P_{st}) = 1 - \frac{1}{n_E}$ , let  $m_E$  be the multiplicity of the fiber at  $E$ . Let*

$$P_W := \sum_E \left(1 - \frac{1}{m_E} n_E\right) E.$$

Then we have the following exact sequence:

$$\pi_1^{\text{sm}}(F, P|_F) \rightarrow \pi_1^{\text{sm}}(Z, P) \rightarrow \pi_1^{\text{sm}}(W, P_W) \rightarrow 1,$$

where  $F$  is a general fiber of  $\psi$ .

*Proof.* By [34, Lemma 1.5.C], there is a short exact sequence:

$$\pi_1(F \setminus \text{supp}(P_{\text{st}}|_F)) \rightarrow \pi_1(Z_{\text{reg}} \setminus \text{supp}(P_{\text{st}})) \rightarrow \pi_1(W_{\text{reg}} \setminus \text{supp}(P_W)) \rightarrow 1.$$

This induces a commutative diagram:

$$\begin{array}{ccccccc} \pi_1(F \setminus P_{\text{st}}|_F) & \xrightarrow{\psi_1} & \pi_1(Z_{\text{reg}} \setminus P_{\text{st}}) & \xrightarrow{\psi_2} & \pi_1(W_{\text{reg}} \setminus \text{supp}(P_W)) & \longrightarrow & 1 \\ \downarrow \phi_F & & \downarrow \phi_X & & \downarrow \phi_C & & \\ \pi_1^{\text{sm}}(F, P|_F) & \xrightarrow{\phi_1} & \pi_1^{\text{sm}}(Z, P) & \xrightarrow{\phi_2} & \pi_1^{\text{sm}}(W, P_W) & \longrightarrow & 1 \end{array}$$

The vertical maps are surjective by the definition of the orbifold fundamental groups of pairs.

Let  $\alpha \in \pi_1^{\text{sm}}(Z, P)$  be an element such that  $\phi_2(\alpha) = 1$ . We take  $\alpha'$  a preimage of  $\alpha$  by  $\phi_X$ . Then  $\psi_2(\alpha') \in N'$ , i.e.  $\psi_2(\alpha') = \Pi_E g_E \gamma_E^{m_E n_E} g_E^{-1}$ , where  $\gamma_E$  is a loop around  $E$ .

Let  $\gamma_{\psi^{-1}(E)}$  be a loop around  $\psi^{-1}(E)$ , then there exists  $h'_E \in \pi_1(Z^{\text{reg}} \setminus P_{\text{st}})$ , such that  $\psi_2(h'_E \gamma_{\psi^{-1}(E)} h'^{-1}_E) = g_E \gamma_E^{m_E} g_E^{-1}$ . Indeed  $\psi_2(\gamma_{\psi^{-1}(E)}) = g \gamma_E^{m_E} g^{-1}$  for some  $g \in \pi_1(W_{\text{reg}} \setminus \text{supp}(P_W))$ , since loops around a divisor are well-defined up to conjugation. As  $\psi_2$  is surjective, there exists  $h'_E$  mapping to  $g_E g^{-1}$  in  $\pi_1(W_{\text{reg}} \setminus \text{supp}(P_W))$ .

Therefore taking  $\gamma' := \pi_E h'_E \gamma_{\psi^{-1}(E)}^{n_E} h'^{-1}_E$ , we have that  $\psi_2(\gamma') = \psi_2(\alpha')$ . Hence, by exactness of the top row, there exists  $\beta \in \psi_1(F \setminus P_{\text{st}}|_F)$ , such that  $\psi_1(\beta') = \alpha' \gamma'^{-1}$ . By definition,  $\gamma'$  is in  $\text{Ker}(\psi_X)$ , thus we have:

$$\psi_X(\psi_1(\beta')) = \psi_X(\alpha' \gamma'^{-1}) = \alpha.$$

Therefore  $\alpha = \phi_1(\psi_F(\beta'))$ , and thus  $\text{Im}(\phi_1) \supseteq \text{Ker}(\phi_2)$ .

Assume  $\alpha \in \pi_1^{\text{sm}}(Z, P)$  is of the form  $\alpha = \phi_1(\beta)$ . Since the vertical morphisms are surjective, there exists  $\beta' \in \pi_1(F \setminus P_{\text{st}}|_F)$  such that  $\psi_F(\beta') = \beta$ . Hence

$$\phi_2(\alpha) = \psi_C \circ \psi_2 \circ \psi_1(\beta') = 1.$$

Thus  $\text{Im}(\phi_1) \subseteq \text{Ker}(\phi_2)$ . Similarly, we get that  $\psi_2$  is surjective. Hence the desired sequence is exact.  $\square$

### 2.4.2 Local Fundamental Groups.

In this subsection, we give the definition of local fundamental group.

**Definition 2.4.11.** For a normal singularity  $x \in X$ , we can embed  $(X; x)$  in a smooth ambient space  $\mathbb{A}^n$ . The *link of  $x \in X$*  is the complement of  $x$  in the intersection of a small euclidean ball of  $x$  in  $\mathbb{A}^n$  with  $X$ . It is denoted by  $\text{Link}(x)$ . The topological type of this object is invariant for  $\varepsilon$  small enough. The *local fundamental group* of  $x \in X$  is defined as:

$$\pi_1^{\text{loc}}(X; x) := \pi_1(\text{Link}(x)).$$

The previous definition does not depend on the choice of the smooth ambient space. If  $(X, B; x)$  is a singularity of pairs, then we simply set  $\pi_1^{\text{loc}}(X, B; x) := \pi_1^{\text{loc}}(X; x)$ .

We define the local fundamental group of a pair analogously to the definition of orbifold fundamental groups:

**Definition 2.4.12.** Let  $(X, B)$  be a log pair and  $x \in X$  be a closed point. For each prime component  $B_i$  of  $B_s$  passing through  $x$ , we denote by  $n_i$  the positive integer for which

$$\text{coeff}_{B_i}(B_s) = 1 - \frac{1}{n_i}.$$

We denote by  $\gamma_i$  a loop in the normal circle bundle of

$$B_i \setminus \bigcup_{j \neq i} B_j \subset X \setminus \text{supp}(B_s).$$

For simplicity, this loop is called a *loop around  $B_i$* . For any open subset  $U$ , we define the group

$$\pi_1(U^{\text{sm}}, B|_U) := \pi_1(Y \setminus \text{supp}(B_s)) / \langle \gamma_i^{n_i} \rangle_n.$$

Here,  $U^{\text{sm}}$  denotes the smooth locus of  $U$  and the subscript  $n$  denotes the smallest normal subgroup generated by such elements. The *local fundamental group* of  $(X, B)$  at  $x$  is defined to be the inverse limit of  $\pi_1(U^{\text{sm}}, B|_U)$ , where  $U$  runs among all analytic neighborhoods of  $x$  in  $X$ .

The following theorem gives a characterization of the fundamental groups of klt singularities (see [28, Theorem 7]).

**Theorem 2.4.13.** *Let  $n \geq 2$  and  $r$  be two positive integers. There exists a positive integer  $c(n)$  only depending on  $n$  satisfying the following. Let  $(X; x)$  be an  $n$ -dimensional klt singularity of regularity  $r$ . Then, there exists a short exact sequence*

$$1 \rightarrow A \rightarrow \pi_1^{\text{loc}}(X; x) \rightarrow N \rightarrow 1,$$

*where  $A$  is a finite abelian group of rank at most  $r + 1$  and  $N$  is a finite group of order at most  $c(n)$ .*

## 2.5 Polyhedral complexes

In this section, we introduce the concept of smooth polyhedral complexes, blow-ups of polyhedral complexes, and prove a couple of lemmas.

**Notation 2.5.1.** Let  $M$  be a finitely generated free abelian group. We set  $N := \text{Hom}(M; \mathbb{Z})$ , the dual space of  $M$ . We denote by  $M_{\mathbb{Q}} := M \otimes_{\mathbb{Z}} \mathbb{Q}$  and  $N_{\mathbb{Q}} := N \otimes_{\mathbb{Z}} \mathbb{Q}$  the associated  $\mathbb{Q}$ -vector spaces. We denote by  $M_{\mathbb{R}} := M \otimes_{\mathbb{Z}} \mathbb{R}$  and  $N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R}$  the associated  $\mathbb{R}$ -vector spaces.

**Definition 2.5.2.** A *polyhedron* in  $M_{\mathbb{R}}$  is the convex hull of finitely many points in  $M_{\mathbb{Q}} \subset M_{\mathbb{R}}$ . Equivalently, it is the closed convex bounded subset of a real vector space  $M_{\mathbb{R}}$  that is defined by finitely many inequalities over the rational numbers.

A *lattice polyhedron* is a polyhedron with vertices in the lattice  $M \subset M_{\mathbb{Q}}$ . Let  $P$  be a lattice polyhedron with a vertex  $v$ . For each edge  $E$  of  $P$  that contains  $v$ , let  $w_E$  be the closest lattice point to  $v$  in  $E$ . Then  $P$  is said to be *smooth at the vertex  $v$* , if the vectors  $w_E - v$  form a basis of the lattice  $\mathbb{Z}^N$ . A *lattice polyhedron*  $P$  is said to be *smooth* if it is smooth at every vertex. For a polyhedron  $P$  with vertex  $v$ , the *cone in  $P$  with vertex  $v$*  is defined as the cone generated by all the rays starting at  $v$  and going through any point in  $P$ .

**Definition 2.5.3.** Let  $P$  be a polyhedron in  $\mathbb{R}^n$  and  $P'$  be the embedding of  $P$  in  $\mathbb{R}^{n+1}$ , with coordinates  $x_0, \dots, x_n$  by taking the first coordinate  $x_0$  to be zero. Let  $H_0 := \{(x_0, \dots, x_n) \mid x_0 = 0\}$ . A *pyramid* over  $P$  is any polyhedron with the same cell structure as the convex hull of  $P'$  and a point  $p$  not in  $H_0$ . A *bypyramid* over  $P$  is any polyhedron with the same cell structure as the convex hull of  $P'$  and points  $p, q$ , such that  $p$  and  $q$  are on different sides of the hyperplane  $H_0$  and have projections to  $H_0$  that lie inside of  $P'$ . Therefore, an  $n$ -dimensional simplex can be defined to be the pyramid over an  $(n - 1)$ -dimensional simplex.

**Definition 2.5.4.** Given  $M_{\mathbb{Q}}$  and  $M'_{\mathbb{Q}}$  two  $\mathbb{Q}$ -vector spaces with lattices  $M$  and  $M'$ . We say that

an affine  $\mathbb{Q}$ -linear map  $f: M'_\mathbb{Q} \rightarrow M_\mathbb{Q}$  is a *lattice embedding* if  $f$  induces an isomorphism of lattices  $M' \rightarrow f(M'_\mathbb{Q}) \cap M$ . A *polyhedral complex* is a finite category  $\mathcal{P}$  satisfying the following:

1. The elements are lattice polyhedra.
2. The morphisms are lattice embeddings.
3. The morphisms are inclusions of polyhedra to faces, where we consider the entire polyhedron to be a face.
4. For every  $P \in \mathcal{P}$  its faces are in  $\mathcal{P}$  and the inclusion of faces are morphisms in  $\mathcal{P}$ .
5. For every  $P_1, P_2 \in \mathcal{P}$ , there exists at most one morphism  $f$  in  $\mathcal{P}$  between  $P_1$  and  $P_2$ .

We denote by  $|\mathcal{P}|$  the amalgamation of the polyhedral complex  $\mathcal{P}$ . We will sometimes write  $\pi_1(\mathcal{P})$  to refer to  $\pi_1(|\mathcal{P}|)$ .

We say that  $\mathcal{P}$  is an  *$n$ -dimensional polyhedral complex* if every maximal polyhedron has the same dimension  $n$ . In this case, the number  $n$  is called the *dimension* of  $\mathcal{P}$ .

We will usually say intersections and inclusions of polyhedra in  $\mathcal{P}$  to refer to the intersections and inclusions that happen to the corresponding cells in the amalgamation  $|\mathcal{P}|$ .

**Definition 2.5.5.** Let  $\mathcal{P}$  be an  $n$ -dimensional polyhedral complex. We define the *nerve* of a cell  $P \in \mathcal{P}$  to be the dual complex of the maximal polyhedra containing  $P$ . In particular, the nerve of  $P$  will have one vertex for each maximal polyhedron in  $\mathcal{P}$  containing  $P$ . A set of vertices in the nerve of  $P$  will be in the same  $k$ -dimensional face if the corresponding maximal polyhedra in  $\mathcal{P}$  contain a common  $(n - k)$ -dimensional face.

We say a  $k$ -dimensional polyhedron  $P$  is *combinatorially smooth* in an  $n$ -dimensional complex  $\mathcal{P}$  if the nerve of  $P$  is an  $(n - k)$ -dimensional simplex. We say that  $\mathcal{P}$  is a *simple polyhedral complex* if it is  $n$ -dimensional and every polyhedron  $P \in \mathcal{P}$  is combinatorially smooth in  $\mathcal{P}$ . We say that  $\mathcal{P}$  is a *smooth polyhedral complex* if it is simple and its objects are smooth lattice polyhedra. In the case that the nerve is combinatorially equivalent to a polyhedron  $Q$ , we may simply say that the nerve is  $Q$ .

**Remark 2.5.6.** A polyhedral complex can fail to be smooth at 0-dimensional cells by being combinatorially non-smooth at the cell or non-smooth at any polyhedron containing it. While for higher dimensional cells a polyhedral complex can only fail to be smooth by combinatorial non-smoothness at the cell or non-smoothness at lower dimensional cells.

**Definition 2.5.7.** Let  $\mathcal{P}$  be a polyhedral complex. We define  $m\mathcal{P}$  to be the polyhedral complex, defined by the following:

1. For each  $Q \in \mathcal{P}$  with lattice  $M$ . The polyhedron  $mQ$ , defined by the vertices of  $Q$  inside the lattice  $\frac{1}{m}M$ , is in  $m\mathcal{P}$ .
2. For each morphism  $f : Q \rightarrow R$  in  $\mathcal{P}$ , the morphism  $f : mQ \rightarrow mR$  is in  $m\mathcal{P}$ .

**Remark 2.5.8.** The polyhedral complex  $m\mathcal{P}$  is smooth exactly when  $\mathcal{P}$  is smooth. This is because we are changing the lattice generators from  $\{e_i\}_i$  to  $\{\frac{1}{m}e_i\}_i$ .

**Definition 2.5.9.** Let  $\mathcal{P}$  be an  $n$ -dimensional polyhedral complex. Let  $v \in \mathcal{P}$  be a vertex in the complex. For each polyhedra  $P_i \in \mathcal{P}$  containing  $v$ , we can consider the cone  $C_i$  spanned by  $P_i$  with vertex  $v$ . Since the morphisms in  $\mathcal{P}$  are lattice inclusions, these cones form a complex of cones  $\mathcal{C}(\mathcal{P}, v)$  that we call the *neighborhood* of  $\mathcal{P}$  at  $v$ . We say that the neighborhood of a vertex  $v \in \mathcal{P}$  can be *embedded in*  $\mathbb{Q}^n$  if for each  $C_i \in \mathcal{C}(\mathcal{P}, v)$  there exists a linear map  $C_i \rightarrow \mathbb{Q}^n$  such that the images of two cones  $C'_i, C'_j$  in  $\mathbb{Q}^n$  are contained in each other if and only if the corresponding cones are contained in each other in  $|\mathcal{P}|$ .

The following definition is motivated by the concept of blow-up in algebraic geometry.

**Definition 2.5.10.** Let  $P$  be a polyhedron inside the  $n$ -dimensional polyhedral complex  $\mathcal{P}$ . We define a *blow-up of  $\mathcal{P}$  at  $P$*  to be a polyhedral complex  $\mathcal{P}'$  with the following objects:

1. For any polyhedron  $Q$  disjoint with  $P$ , the polyhedron  $Q$  is in  $\mathcal{P}'$ .
2. For any polyhedron  $R$  intersecting  $P$  but not contained in  $P$ , there exist polyhedra  $R'$  and  $R_P$  in  $\mathcal{P}'$ , satisfying the following:
  - (a) Let  $H_R$  be a hyperplane  $H_R$  separating  $P \cap R \subset R$  and the vertices of  $R \setminus P \cap R$ . Let  $H$  be the half-space defined by  $H_R$  that does not contain  $P$ . The polyhedron  $R'$  is the intersection of  $R$  and  $H$ .
  - (b)  $R_P$  is the intersection of  $R$  and the hyperplane  $H_R$ .
3. A polyhedron  $P'$  whose faces are the polyhedron  $R_P$  for each polyhedron  $R$  intersecting  $P$ , not contained in  $P$ .

Moreover, the morphisms are defined by the following rules:

1. For any embedding  $q : Q_1 \rightarrow Q_2$ , where  $Q_1$  is disjoint with  $P$ , the embedding  $q$  is in  $\mathcal{P}'$ .

2. For any embedding  $r : R_1 \rightarrow R_2$  between polyhedra that intersect  $P$  but are not contained in  $P$ , the restrictions of the embedding  $r : R_1 \rightarrow R_2$  to  $R_{1P} \rightarrow R_{2P}$  and  $R'_1 \rightarrow R'_2$  are in  $\mathcal{P}'$ .
3. For any  $R$  intersecting  $P$  but not contained in  $P$ , we have the morphism  $i : R_P \rightarrow P'$ .

**Remark 2.5.11.** The maximal polyhedra of a blow-up are hence:

1. The polyhedron  $Q$  for any maximal polyhedra  $Q$  not intersecting  $P$ .
2. The polyhedron  $R'$  for any maximal polyhedra  $R$  intersecting  $P$ .
3. The polyhedron  $P'$ , which is combinatorially equivalent to the dual polyhedron of the nerve of  $P$  in  $\mathcal{P}$ .

**Remark 2.5.12.** A priori, it is not clear that a blow-up always exists, as we require the existence of the rational polyhedron  $P'$ .

**Definition 2.5.13.** A lattice fan in  $\mathbb{Z}^n$  is a *strongly polytopal fan* if the lattice generators of each ray are the vertices of a convex polytope.

**Lemma 2.5.14.** *Let  $\mathcal{P}$  be a 3-dimensional polyhedral complex. Let  $v \in \mathcal{P}$  be a vertex such that all polyhedra  $P_i$  containing  $v$  are smooth at  $v$ . Assume that there is an embedding  $\phi : \mathcal{C}(\mathcal{P}, v) \rightarrow \mathbb{Q}^3$  as a strongly polytopal fan. Then, there exists a blow-up of  $2\mathcal{P}$  at  $v$ .*

*Proof.* Let  $\mathcal{Q} := \phi(\mathcal{C}(\mathcal{P}, v))$  be the embedded fan of cones with vertex  $v$  in  $\mathbb{Q}^3$ . Without loss of generality, let us assume that  $\phi(v)$  is the origin. This subcomplex gives us a 3-dimensional complete fan around the origin, generated by rays  $r_1, \dots, r_i$ , with lattice generators  $e_1, \dots, e_i$ . Since all polyhedra are smooth at  $v$ , the cones are smooth. Hence, each point in the fan can be uniquely written as a linear combination of the generators in the cone that contains it. Therefore, taking  $f_j := (e_j, 1) \in \mathbb{Q}^4$  for each generator, defines a piecewise linear map from  $\mathbb{Q}^3$  to a convex cone  $\sigma$  in  $\mathbb{Q}^4$ . Any polyhedron in  $\mathcal{P}$  containing  $p$  is a subset of one of the cones with vertex  $v$ . Hence, they also are a face of  $\sigma$ . Let  $H$  be the hyperplane  $\{2x_4 = 1\}$ . This hyperplane separates  $(0, 0, 0, 0)$  with all the other vertices of polyhedra in  $\mathcal{Q}$ . We replace the lattice  $\mathbb{Z}^4 \subset \mathbb{Q}^4$  with  $(\frac{1}{2}\mathbb{Z})^4$  and correspondingly in all the polyhedra of the complex. We perform the following replacements:

1. We replace any polyhedron containing  $v$  in  $\mathcal{Q}$  with its intersection with  $H^+$ .
2. We replace  $v$  with the intersection of  $H$  and the cone  $\sigma$  in  $\mathbb{Q}^4$ .

The polyhedral complex  $\mathcal{P}'$  obtained by performing these replacements is a blow-up of  $2\mathcal{P}$  at  $v$ .  $\square$

**Lemma 2.5.15.** *Let  $\mathcal{P}'$  be a blow-up of  $\mathcal{P}$  at  $P$ . Let  $R \in \mathcal{P}$  be a polyhedron intersecting  $P$ , but not contained in  $P$ . If the nerve at  $R \in \mathcal{P}$  is  $Q$ , then the nerve at  $R_P \in \mathcal{P}'$  is a pyramid over  $Q$ .*

*Proof.* All maximal dimensional polyhedra  $R_i$  that contain  $R$  must also intersect  $P$ . Hence, in  $\mathcal{P}'$  the maximal dimensional polyhedra that contain  $R_P$  are the polyhedra  $R'_i$  and the polyhedron  $P'$ . The polyhedra  $R'_i$  and  $R'_j$  intersect non-trivially whenever  $R_i$  and  $R_j$  do. The polyhedron  $P'$  intersects non-trivially all the polyhedra  $R_i$  by the definition of blow-up. Hence, the nerve of  $R_P$  is the pyramid over  $Q$  with vertex corresponding to the polyhedron  $P'$ .  $\square$

## 2.6 Projective toric varieties and lattice isomorphisms

In this section, we prove a result regarding embeddings of projective toric varieties and lattice embeddings.

**Lemma 2.6.1.** *Let  $P$  be a full-dimensional lattice polyhedron in  $M_{\mathbb{Q}}$ . Let  $X_P$  be the associated projective variety. Let  $k$  be a vector in  $M_{\mathbb{Q}}$ . Let  $A$  be the ample line bundle on  $X_P$  associated to  $P$  and  $A'$  be the ample line bundle on  $X_P$  associated to  $P + k$ . Then, we have that  $A \sim_{\mathbb{Q}} A'$ . Furthermore, if  $k$  is a lattice vector, then  $A \sim A'$ .*

*Proof.* Let  $t$  be such that  $tk \in M_{\mathbb{Z}}$ . The lattice polyhedron  $P$  is given by the inequalities:

$$\{m \mid \langle m, u_F \rangle \geq -a_F\}.$$

Here,  $F$  are the facets of  $P$  and  $u_F \in N$  is the inward pointing normal vector of the face  $F$ . For each face  $F$  of  $P$ , we have an associated prime torus invariant divisor  $D_F$  on  $X_P$  [9, Definition 2.3.14]. By [9, Proposition 4.2.10], we have that

$$A = \sum_F a_F D_F,$$

where the sum runs over all the faces  $F$  of  $P$ . Observe that

$$\begin{aligned} P' &= \{m + k \mid \langle m, u_F \rangle \geq -a_F\} = \{m \mid \langle m - k, u_F \rangle \geq -a_F\} \\ &= \{m \mid \langle m, u_F \rangle \geq -a_F + \langle k, u_F \rangle\}. \end{aligned}$$

Hence,  $P$  and  $P'$  have the same associated normal fan. In particular, they have the same

associated projective toric variety  $X_P$ . By [9, Proposition 4.2.10], we have that

$$A' = \sum (a_F - \langle k, u_F \rangle) D_F.$$

We conclude that

$$A - A' = \sum \langle k, u_F \rangle D_F = \sum \frac{1}{t} \langle tk, u_F \rangle D_F = \frac{1}{t} \operatorname{div}(\chi^k).$$

The last equality follows from [9, Proposition 4.1.2]. We conclude that  $A \sim_{\mathbb{Q}} A'$  holds. If  $k$  is a lattice vector, then  $t = 1$  and  $A - A'$  is principal on  $X_P$ .  $\square$

**Lemma 2.6.2.** *Let  $P$  and  $P'$  be two full-dimensional lattice polyhedra in  $M_{\mathbb{Q}}$  and  $M'_{\mathbb{Q}}$ , respectively. Let  $X_P$  (resp.  $X_{P'}$ ) be the associated projective toric variety with ample line bundle  $A_P$  (resp.  $A_{P'}$ ). Let  $H : M_{\mathbb{Q}} \rightarrow M'_{\mathbb{Q}}$  be a linear lattice isomorphism so that  $H(P) = P'$ . Then, we have an associated toric isomorphism  $\phi : X_{P'} \rightarrow X_P$ , for which  $\phi^*(A_P) = A_{P'}$ .*

*Proof.* The lattice polyhedron  $P$  is given by inequalities:

$$\{m \mid \langle m, u_F \rangle \geq -a_F\}.$$

Then, the lattice polyhedron  $P'$  is given by inequalities:

$$\{H(m) \mid \langle m, u_F \rangle \geq -a_F\} = \{m \mid \langle m, H^t(u_F) \rangle \geq -a_F\}.$$

Here,  $F'$  are the facets of  $P'$  and  $u_{F'} = H^t(u_F) \in N'$  is the inward pointing normal vector of the face  $F'$ . By [9, Proposition 4.2.10], we have that

$$A_P = \sum a_F D_F \text{ and } A_{P'} = \sum a_F D_{F'}.$$

The lattice isomorphism  $H : M_{\mathbb{Q}} \rightarrow M'_{\mathbb{Q}}$  induces a lattice isomorphism  $H^t : N'_{\mathbb{Q}} \rightarrow N_{\mathbb{Q}}$  on the dual lattices. By [9, Theorem 3.3.4], we have an associated equivariant isomorphism of projective toric varieties  $\phi : X_{P'} \rightarrow X_P$ . Using [9, Proposition 6.2.7], we can compare

$$\begin{aligned} \phi^*(A_P) &= \phi^* \left( \sum_F a_F D_F \right) = \sum_{F'} \phi_{A_{P'}}(H^t(u_F)) D_{F'} \\ &= \sum_{F'} \phi_{A_{P'}}(u_{F'}) D_{F'} = \sum_{F'} a_{F'} D_{F'} = A_{P'}. \end{aligned}$$

Here,  $\phi_D$  is the support function associated to the Cartier divisor  $D$  (see, e.g. [9, Theorem 4.2.12]). This finishes the proof of the lemma.  $\square$

**Lemma 2.6.3.** *Let  $P$  be a full-dimensional lattice polyhedron in  $M_{\mathbb{Q}}$ . Let  $X_P$  be the associated projective toric variety and  $A_P$  be the associated ample line bundle. Let  $F$  be a facet of  $P$  and  $D_F$  be the corresponding prime torus invariant divisor. Let  $M_{F,\mathbb{Q}}$  be the smallest linear subspace of  $M_{\mathbb{Q}}$  that contains  $F$  and set  $M_F := M \cap M_{F,\mathbb{Q}}$ . Let  $A_D$  be the ample line bundle in  $D$  associated to the lattice polytope  $F$  in  $M_{F,\mathbb{Q}}$ . Then, we have  $A_P|_D \sim A_D$ .*

*Proof.* Let  $\Sigma_P$  be the dual fan of  $P$ . For each cone  $\sigma \in \Sigma_P$ , we have an associated affine toric variety  $U_{\sigma}$  which is an open chart of  $X_P$ .  $D_P$  is principal on each chart of this cover. Let the local data of  $D_P$  be  $\{(U_{\sigma}, \chi^{-m_{\sigma}})\}_{\sigma \in \Sigma_P}$ . By [9, Theorem 4.2.8], we can take  $m_{\sigma}$  to be the vertex of the polytope that corresponds to  $\sigma$ . Then,  $\mathcal{O}_{X_P}(D_P)$  is the sheaf of sections of a rank 1 vector bundle  $V_P \rightarrow X_P$  with transition functions  $g_{\sigma\tau} = \chi^{m_{\tau} - m_{\sigma}}$ .

Let the local data of  $D_F$  be  $\{(U_{\sigma'}, \chi^{-m_{\sigma'}})\}_{\sigma' \in \Sigma_F}$ , where each  $\sigma' \in \Sigma_F$  is the projection to  $N_{F,\mathbb{Q}}$  of a  $\sigma \in \Sigma_P$  that corresponds to a vertex in  $F$ . Again, by [9, Theorem 4.2.8], we can take  $m_{\sigma'}$  to be the vertex in  $F$  that corresponds to  $\sigma'$ , hence also the vertex in  $P$  that corresponds to  $\sigma$ . Thus,  $\mathcal{O}_{X_F}(D_F)$  is the sheaf of sections of a rank 1 vector bundle  $V_F \rightarrow X_F$  with transition functions  $g_{\sigma'\tau'} = \chi^{m_{\tau'} - m_{\sigma'}}$ , where  $m_{\tau'}$  and  $m_{\sigma'}$  correspond to vertices of  $F$ .

Therefore, we have that  $V_F$  is the restriction of  $V_P$  to  $F$ , i.e. we have a commutative diagram:

$$\begin{array}{ccc} V_F & \hookrightarrow & V_P \\ \downarrow & & \downarrow \\ X_F & \xhookrightarrow{i} & X_P. \end{array}$$

We conclude that  $i^* \mathcal{O}_{X_P}(D_P) = \mathcal{O}_{X_F}(D_F)$ .  $\square$

**Proposition 2.6.4.** *Let  $P$  be a full-dimensional lattice polyhedron in  $M_{\mathbb{Q}}$ . Let  $X_P$  be the associated projective toric variety with induced polarization  $A_P$ . Let  $F$  be a full-dimensional lattice polyhedron in  $M'_{\mathbb{Q}}$ . Let  $X_F$  be the associated projective toric variety with induced polarization  $A_F$ . Let  $f: M'_{\mathbb{Q}} \rightarrow M_{\mathbb{Q}}$  be a lattice embedding for which  $f(F)$  is a face of  $P$ . Then, we have an associated toric embedding  $\phi: X_F \rightarrow X_P$  for which  $\phi^* A_P \sim A_F$ .*

*Proof.* The lattice embedding  $f$  can be decomposed as a translation on  $M'_{\mathbb{Q}}$ , a linear lattice isomorphism, and a sequence of linear lattice embeddings of codimension one. Then, the proposition follows from Lemma 2.6.1, Lemma 2.6.2, and Lemma 2.6.3.  $\square$

## Part I

# Fundamental groups of log canonical singularities

## Chapter 3

# Surface log canonical singularities

In this chapter, we describe the local fundamental group of a log canonical surface pair singularity  $(X, B; x)$ .

The first main result of this section is an upper bound for the number of generators and relations in a presentation of the local fundamental group of a log canonical surface singularity:

**Theorem 3.0.1.** *Let  $(X, B; x)$  be a log canonical surface singularity. Then,  $\pi_1^{\text{loc}}(X, B; x)$  admits a presentation with at most 4 generators and at most 7 relations. Furthermore, if  $\pi_1^{\text{loc}}(X, B; x)$  admits a minimal presentation with 4 generators and 7 relations, then  $X$  is formally toric at  $x$  and  $B$  has 4 components with coefficient  $\frac{1}{2}$  through the singularity  $x \in X$ .*

**Definition 3.0.2.** Let  $x \in X$  be a closed point in an affine variety. We denote by  $\hat{X}_x$  the local completion of  $X$  at  $x$ . We say that  $(X, B)$  is *formally toric* at  $x$  if there exists an affine toric variety  $T$  so that  $\hat{X}_x \cong \hat{T}_t$  and  $B$  is identified with a toric boundary via that isomorphism.

Nevertheless, groups with few generators and relations can still be very complicated. The following result gives a structural property of the fundamental groups of log canonical surface singularities.

**Theorem 3.0.3.** *Let  $(X, B; x)$  be a log canonical surface singularity. Then, we have a short exact sequence*

$$1 \rightarrow N \rightarrow \pi_1^{\text{loc}}(X, B; x) \rightarrow G \rightarrow 1$$

*where  $N$  is a solvable group of length at most 2 and  $G$  is a finite group of order at most 6.*

In order to obtain this results, we will get a complete classification of the possible fundamental groups of log canonical surface singularities  $(X, B; x)$ . Our main tools will be the existence of dlt

modifications (see Lemma 2.2.6) and Mumford's description of the fundamental group of a normal surface singularity [31].

The following lemma follows from [31, Claim in Page 10].

**Lemma 3.0.4.** *Let  $X$  be a normal surface. Let  $E_1, \dots, E_n$  be curves on  $X$  so that  $E_1 \simeq \mathbb{P}^1$ . Assume each  $E_i$ , with  $i \geq 2$ , intersects  $E_1$  at  $p_i$ , so that  $(X, E)$  is log smooth, where  $E := E_1 + \dots + E_n$ . Define  $U$  to be a small analytic neighborhood of  $E_1$  in  $X$  intersected with the complement of the support of  $E$ . Then,  $\pi_1(U)$  is generated by a loop  $\alpha_i$  around each  $E_i$  with relations*

$$\{[\alpha_1 \alpha_i]\}_{2 \leq i \leq n} \quad \text{and} \quad \alpha_2 \alpha_3 \dots \alpha_n \alpha_1^{E_1^2}.$$

The following lemma gives an enhanced version of the main result of [31] to the case of log pairs. The idea is to treat the boundary as a codimension one orbifold singularity.

**Lemma 3.0.5.** *Let  $(X, B; x)$  be a singularity of a 2-dimensional log pair. Let  $f: (Y, B_Y) \rightarrow (X, B)$  be a log resolution whose exceptional locus is a tree of rational curves. Then, the local fundamental group  $\pi_1^{\text{loc}}(X, B; x)$  is generated by:*

- (i) a loop  $\alpha_i$  around every prime exceptional divisor  $E_i$  in  $Y$ , and
- (ii) a loop  $\gamma_i$  around every prime divisor in  $f_*^{-1}(B_s)$ ,

with the following relations:

1. Loops around intersecting divisors commute.
2. The relation  $\alpha_1 \dots \alpha_m \gamma_1 \dots \gamma_{m'} \alpha_i^{E_i^2}$ , for every divisor  $E_i$  intersecting  $E_1, \dots, E_m$  and  $B_1, \dots, B_{m'}$  in  $f_*^{-1}(B_s)$ .
3. The relation  $\gamma_i^m$ , for every prime divisor  $B_i$  of  $f_*^{-1}(B_s)$  with coefficient  $1 - \frac{1}{m}$ .

*Proof.* Note that as the singularity is isolated, the local and local fundamental groups of  $(X, B, x)$  coincide. Observe that a neighborhood of  $x \in X$  is isomorphic to a neighborhood of the exceptional locus, and the coefficients in  $B_s$  are the same as those in  $f_*^{-1}(B_s)$ . We obtain that

$$\pi_1^{\text{loc}}(X, B; x) = \pi_1^{\text{loc}}(U, f_*^{-1}(B_s)|_U) = \pi_1(U \setminus B_s) / \langle \gamma_i^{n_i} \rangle. \quad (3.1)$$

Here,  $U$  is a small analytic neighborhood of the exceptional locus in  $Y$  with the exceptional locus removed. To compute the rightmost fundamental group in (3.1), we apply the Seifert-Van Kampen Theorem to an open covering of  $U' := U \setminus B_s$ .

For each exceptional divisor  $E_i$ , we define  $U_i$  to be  $U'$  intersected with a small analytic neighborhood of  $E_i$ . Hence, we have that

$$U' = \bigcup_i U_i.$$

As the exceptional divisors form a tree, we can apply Seifert-Van Kampen Theorem for pairs to obtain the free product of all the  $\pi_1(U_i)$  modulo the amalgamation of the subgroups  $\pi_1(U_i \cap U_j)$ . By Lemma 3.0.4 the group  $\pi_1(U_i)$  is generated by:

1. a loop around  $E_i$ , and
2. loops around each prime divisor (exceptional or in  $f_*^{-1}B_s$ ) intersecting  $E_i$ ,

with relations

$$\{[\alpha_i \alpha_j]\}_j, \quad \{[\alpha_i \gamma_j]\}_j, \quad \text{and} \quad \alpha_1 \dots \alpha_m \gamma_1 \dots \gamma_{m'} \alpha_i^{E_i^2}.$$

If  $E_i$  and  $E_j$  intersect, then  $U_i \cap U_j$  is homotopic to  $(\mathbb{D}^*)^2$ . So, it has fundamental group generated by loops around  $E_i$  and  $E_j$  that commute. Therefore, the amalgamation does not introduce any further relations between the loops around the divisors. Hence,  $\pi_1(U')$  is generated by loops around the divisors with the relations

$$[\alpha_i \alpha_j]_{i,j}, \quad [\alpha_i \gamma_j]_{i,j}, \quad \text{and} \quad \alpha_1 \dots \alpha_m \gamma_1 \dots \gamma_{m'} \alpha_i^{E_i^2}.$$

Finally, we get the third relation  $\gamma_i^{m_i}$  by taking the quotient in the definition of the local fundamental group of a pair. □

**Corollary 3.0.6.** *Let  $(X, B; x)$  be a singularity of a 2-dimensional log pair, with  $B_s = 0$ . Let  $f : (Y, B_Y) \rightarrow (X, B)$  be a log resolution whose exceptional locus is a chain of  $m \geq 2$  rational curves. Then, the local fundamental group  $\pi_1^{\text{loc}}(X, B; x)$  is a finite cyclic group.*

*Proof.* Call the exceptional divisors  $E_1, \dots, E_m$ . By Lemma 3.0.5, we have that

$$\pi_1^{\text{loc}}(X, B; x) \simeq \langle x_1, \dots, x_m \mid x_2 x_1^{-r_1}, x_1 x_3 x_2^{-r_2}, \dots, x_{m-1} x_m^{-r_m} \rangle,$$

where  $x_i$  is the loop around  $E_i$  and  $-r_i = E_i^2$ . We can write

$$\frac{b_i}{b_{i-1}} = [r_{i-1}, r_{i-2}, \dots, r_1]$$

for the Hirzebruch-Jung continued fractions, we obtain  $x_i = x_1^{b_i}$  inductively. Therefore, the following

sequence of isomorphisms holds:

$$\pi_1^{\text{loc}}(X, B; x) \simeq \langle x_1 \mid x_1^{b_{m-1}}(x_1^{b_m})^{-r_m} \rangle \simeq \langle x_1 \mid x_1^{r_m b_m - b_{m-1}} \rangle \simeq \mathbb{Z}/(r_m b_m - b_{m-1})\mathbb{Z}.$$

This implies that the local fundamental group is a finite cyclic group.  $\square$

**Lemma 3.0.7.** *Consider the group*

$$G := \langle \alpha_1, \dots, \alpha_t, x_1, \dots, x_m \mid \alpha x_2 x_1^{-m_1}, \{x_{i-1} x_{i+1} x_i^{-m_i}\}_{1 < i < m}, J \rangle,$$

where  $J$  is a set of relations and  $\alpha$  is some element generated by the  $\alpha_i$ 's. Then

$$G \simeq \langle \alpha_1, \dots, \alpha_t, x_1 \mid J' \rangle,$$

where  $J'$  are the relations in  $J$  with  $x_i$  replaced by  $\alpha^{a_i} x_1^{b_i}$  for  $1 < i \leq m$ .

*Proof.* We can write

$$\begin{aligned} \frac{b_{i+1}}{b_i} &= [m_i, m_{i-1}, \dots, m_1], \text{ and} \\ \frac{a_{i+1}}{a_i} &= [m_i, m_{i-1}, \dots, m_2], \end{aligned}$$

for the Hirzebruch-Jung continued fraction. Inductively, we can obtain  $x_i = \alpha^{a_i} x_1^{b_i}$ . Therefore, we obtain the required presentation of the group.  $\square$

In the following lemma, we will classify the possible exceptional divisors of dlt modifications of log canonical surfaces singularities. We also prove a statement about the singular locus of the dlt modification.

**Lemma 3.0.8.** *Let  $(X, B; x)$  be a log canonical surface singularity. Let  $\phi : (Z, B_Z) \rightarrow (X, B)$  be a dlt modification. Let  $E_1, \dots, E_m$  be the exceptional prime divisors. Define  $E$  to be  $E_1 + \dots + E_m$ ,  $\Delta_i$  to be the different for the adjunction of  $K_Z + B_Z$  to  $E_i$ , and  $\hat{B}$  to be the strict transform of  $B$ . Then,  $E$  is one of the following:*

- (i) *a chain of rational curves,*
- (ii) *a cycle of rational curves, or*
- (iii) *an elliptic curve.*

Furthermore, the following statements hold:

1. If  $E$  is a chain of rational curves, then  $(Z, B_Z)$  is log smooth in a neighborhood of  $E_2 + \cdots + E_{m-1}$ .
2. If  $E$  is an elliptic curve or a cycle of rational curves, then  $B_Z = E$  and  $(Z, B_Z)$  is log smooth in a neighborhood of  $E$ .
3. If  $m = 1$  and  $E$  is a rational curve, then  $E_1 \cdot \hat{B} + \deg \Delta_1 = 2$ .
4. If  $m \geq 2$  and  $E$  is a chain of rational curves, then

$$E_1 \cdot \hat{B} + \deg \Delta_1 = E_m \cdot \hat{B} + \deg \Delta_m = 1.$$

*Proof.* By the definition of  $\phi$ , we have that  $K_Z + B_Z = \phi^*(K_X + B)$  and  $(Z, B_Z)$  is a dlt pair. Hence,  $K_Z + B_Z \sim_{\mathbb{Q}, X} 0$ .

By adjunction, we get that

$$0 \sim_{\mathbb{Q}} (K_Z + B_Z)|_{E_i} = K_{E_i} + \Delta_i + (B_Z - E_i)|_{E_i}.$$

Therefore,  $\deg K_{E_i} \leq 0$ , so all exceptional divisors are rational curves or elliptic curves.

If  $\deg K_{E_i} = 0$ , then  $\deg (B_Z - E_i)|_{E_i} = 0$ , so the vertex corresponding to  $E_i$  in the dual graph of  $E$  has no edges. Since  $E$  is the fiber of  $\phi$  over  $x$ ,  $E$  is connected. So, its dual graph is connected as well. Hence,  $E = E_1$  is the only exceptional divisor. In this case, we have that  $B_Z = E_1 + \hat{B}$ . Thus, by adjunction, we have that

$$0 = \deg (K_Z + B_Z)|_{E_1} = \deg \Delta_1 + \deg (B_Z - E_1)|_{E_1}.$$

Therefore,  $E_1 \cdot \hat{B} + \deg \Delta_1 = 0$ . Consequently, in a neighborhood of  $E$  the pair  $(Z, B_Z)$  is log smooth and  $B_Z = E$ .

We are only left with the case in which  $\deg K_{E_i} = -2$  for each  $E_i$ . In this case, we get that  $\deg (B_Z - E_i)|_{E_i} \leq 2$ , hence any  $E_i$  intersects at most two other divisors in  $E$ . As  $E$  is connected, it has to be a chain or a cycle of rational curves.

If  $E$  is a cycle of rational curves, then by adjunction to  $E_k$ , we have that

$$0 = \deg K_Z + B_Z|_{E_k} = -2 + \deg \Delta_k + 2 + \deg (B_Z - E_k - E_{k-1} - E_{k+1})|_{E_k} \geq 0.$$

So, there are no singularities along  $E_k$ . Furthermore,  $E_k$  only intersects  $B_Z$  at  $E_{i-1}$  and  $E_{i+1}$ , and this intersection is transversal.

If we have a chain of rational curves, then for any  $E_k$  with  $k \notin \{1, m\}$ , we have

$$0 = \deg K_Z + B_Z|_{E_k} = -2 + \deg \Delta_k + 2 + \deg (B_Z - E_k - E_{k-1} - E_{k+1})|_{E_k} \geq 0.$$

So, there are no singularities along  $E_k$ . Furthermore,  $E_k$  only intersects  $B_Z$  at  $E_{i-1}$  and  $E_{i+1}$ , and this intersection is transversal.

If  $m = 1$ , then  $B_Z = E_1 + \hat{B}$ . By adjunction to  $E_1$ , we get that

$$0 = \deg (K_Z + B_Z)|_{E_1} = -2 + \deg \Delta_1 + \deg (B_Z - E_1)|_{E_1}.$$

Hence,  $E_1 \cdot \hat{B} + \deg \Delta_1 = 2$ .

If  $m \geq 2$ , then we have the following sequence of equalities:

$$0 = \deg (K_Z + B_Z)|_{E_1} = -2 + \deg \Delta_1 + \deg (B_Z - E_1)|_{E_1} = -1 + \deg \Delta_1 + \deg (B_Z - E_1 - E_2)|_{E_1}.$$

We deduce that  $E_1 \cdot \hat{B} + \deg \Delta_1 = 1$ . We can proceed analogously for  $E_m$ . This finishes the proof.  $\square$

**Proposition 3.0.9.** *Let  $(X, B; x)$  be a log canonical surface singularity. Let  $\phi : (Z, B_Z) \rightarrow (X, B)$  be a dlt modification whose exceptional locus  $E$  is an elliptic curve. Then, the following isomorphisms hold*

$$\pi_1^{\text{loc}}(X, B; x) \simeq \pi_1^{\text{loc}}(X, B; x) \simeq \pi_1^{\text{loc}}(Z, B_Z).$$

Furthermore,  $\pi_1^{\text{loc}}(X, B; x)$  is of the form  $\mathbb{Z} \rtimes \mathbb{Z}^2$ .

*Proof.* By Lemma 3.0.8, we know that  $E$  is a smooth elliptic curve and  $B_Z$  contains no other curve. Therefore,  $\pi_1^{\text{loc}}(Z, B_Z)$  is the fundamental group of an  $S^1$ -bundle over the elliptic curve. So, it fits in the exact sequence

$$1 \rightarrow \pi_1(S^1) \simeq \mathbb{Z} \rightarrow \pi_1^{\text{loc}}(Z, B_Z) \rightarrow \pi_1(\mathbb{T}^2) \simeq \mathbb{Z}^2 \rightarrow 1.$$

Hence, it has a presentation with 3 generators.  $\square$

**Proposition 3.0.10.** *Let  $(X, B; x)$  be a log canonical surface singularity. Let  $\phi : (Z, B_Z) \rightarrow (X, B)$*

be a dlt modification whose exceptional locus  $E$  is a cycle of rational curves. Then, we have that

$$\pi_1^{\text{loc}}(X, B; x) \simeq \pi_1^{\text{loc}}(X, B; x) \simeq \pi_1^{\text{loc}}(Z, B_Z).$$

Furthermore,  $\pi_1^{\text{loc}}(X, B; x)$  is of the form  $\mathbb{Z}^2 \rtimes \mathbb{Z}$ .

*Proof.* By Lemma 3.0.8, we know that  $E$  is a chain of smooth rational curves and  $B_Z$  contains no other curve. Hence, by [32, Lemma 2.3],  $\pi_1^{\text{loc}}(X, B; x)$  is of the form  $\mathbb{Z}^2 \rtimes \mathbb{Z}$ . In particular, it fits in the short exact sequence

$$1 \rightarrow \mathbb{Z}^2 \rightarrow \pi_1^{\text{loc}}(X, B; x) \rightarrow \mathbb{Z} \rightarrow 1$$

□

**Notation 3.0.11.** The coefficients of the different in Lemma 3.0.8 have the form  $1 - \frac{1}{n}$ . This follows from the adjunction formula [24, 36]. We will write  $((n_1, n_2, \dots, n_m))$  for the un-ordered set of fractions of the form  $1 - \frac{1}{n_i}$ , allowing repetitions. For a different  $\Delta$ , we will write

$$b(\Delta) := ((n_1, \dots, n_k)),$$

for the corresponding set of non-trivial coefficients. We call  $b(\Delta)$  the *basket of singularities* of the different as it represents codimension two singularities of the ambient space. In the case that the different is trivial, we write  $b(\Delta) = ((\emptyset))$ . More generally, for any divisor  $D$  with standard coefficients, we will write:

$$b(D) := ((n_1, \dots, n_k)),$$

for the corresponding set of non-trivial coefficients. In this case, we will write  $n_i = \infty$  if the coefficient is 1.

**Proposition 3.0.12.** *Let  $(X, B; x)$  be a log canonical surface singularity. Let  $\phi : (Z, B_Z) \rightarrow (X, B)$  be a dlt modification whose exceptional locus  $E$  is a chain of rational curves and  $B_s = 0$ . Then, we have that*

$$\pi_1^{\text{loc}}(X, B; x) \simeq \pi_1^{\text{loc}}(X, B; x),$$

*has a presentation with at most 3 generators and at most 3 relations.*

*Proof.* Let  $E = E_1 + \dots + E_m$  be the chain of rational curves. By Lemma 3.0.8 (3), we may assume that

$$E_1 \cdot \hat{B} + \deg \Delta_1 = 2. \tag{3.2}$$

Here,  $\Delta_1$  is the different for the adjunction to the exceptional divisor  $E_1$ . We take a log resolution  $f : Y \rightarrow Z$  of  $(Z, B_Z)$ , where each singular point in  $E$  with  $\deg \Delta_1 = 1 - \frac{1}{n}$  has preimage a chain of rational curves with self-intersections  $-m_1, \dots, -m_l$ , and  $\frac{n}{q} = [m_1, \dots, m_l]$ . We will split the proof in two steps depending on the number of rational curves in the dlt modification.

*Step 1:* We prove the statement in the case that  $E = E_1$  is a single rational curve.

As  $E_1 \cdot \hat{B} \geq 0$ , we have that  $\deg \Delta_1 \leq 2$ . Recall that the coefficients of the different are of the form  $1 - \frac{1}{n}$  for some positive integer  $n$ . Hence, the condition  $\deg \Delta_1 \leq 2$  implies that

$$b(\Delta_1) \in \left\{ \begin{array}{l} ((2, 2, 2, 2)), ((2, 3, 6)), ((2, 4, 4)), ((3, 3, 3)), ((2, 3, 5)) \\ ((2, 3, 4)), ((2, 3, 3)), ((2, 2, n)), ((n_1, n_2)), ((n_2)), ((\emptyset)) \end{array} \right\} \quad (3.3)$$

In each of these cases, we apply Lemma 3.0.5 to  $\phi \circ f : Z \rightarrow X$ :

*Case 1.1:* We have  $b(\Delta_1) = ((2, 2, 2, 2))$ .

Then, the following isomorphisms hold:

$$\begin{aligned} \pi_1^{\text{loc}}(X, B; x) &\simeq \langle a, b, c, d, x \mid xa^{-2}, xb^{-2}, xc^{-2}, xd^{-2}, abcdx^{-m} \rangle \\ &\simeq \langle a, b, c, d \mid a^2b^{-2}, a^2c^{-2}, a^2d^{-2}, a^{1-2m}bcd \rangle \\ &\simeq \langle a, b, c \mid a^2b^{-2}, a^2c^{-2}, a^2(a^{2m-1}b^{-1}c^{-1})^{-2} \rangle. \end{aligned}$$

Then, in this case, we have at most 3 generators and 3 relations.

*Case 1.2:* We have  $b(\Delta_1) = ((n, n', n''))$ .

Then, the following isomorphisms hold:

$$\begin{aligned} \pi_1^{\text{loc}}(X, B; x) &\simeq \left\langle \begin{array}{l} a_1, \dots, a_l, b_1, \dots, b_{l'} \\ c_1, \dots, c_{l''}, x \end{array} \left| \begin{array}{l} xa_{l-1}a_l^{-n}, xb_{l'-1}b_{l'}^{-n'}, xc_{l''-1}c_{l''}^{-n''}, a_lb_{l'}c_{l''}x^{-m} \\ a_2a_1^{-q_1}, \dots, a_la_{l-2}a_{l-1}^{-q_{l-1}}, b_2b_1^{-q'_1}, \dots, b_{l'}b_{l'-2}b_{l'-1}^{-q'_{l'-1}} \\ c_2c_1^{-q''_1}, \dots, c_{l''}c_{l''-2}c_{l''-1}^{-q''_{l''-1}} \end{array} \right. \right\rangle \\ &\simeq \langle a_1, b_1, c_1, x \mid xa_1^{-n}, xb_1^{-n'}, xc_1^{-n''}, a_1^tb_1^{t'}c_1^{t''}x^{-m} \rangle \\ &\simeq \langle a_1, b_1, c_1 \mid a_1^n b_1^{-n'}, a_1^n c_1^{-n''}, a_1^{t-mn} b_1^{t'} c_1^{t''} \rangle. \end{aligned}$$

Here, the second isomorphism is a consequence of iterated applications of Lemma 3.0.7. Then, in this case, we have 3 generators and 3 relations.

*Case 1.3:* We have  $b(\Delta_1) = ((n, n'))$ ,  $b(\Delta_1) = ((n))$  or  $b(\Delta_1) = ((\emptyset))$ .

Then, the exceptional divisor of  $\phi \circ f : Z \rightarrow X$  is a chain of rational curves. Therefore, by Corollary 3.0.6, we have an isomorphism:

$$\pi_1^{\text{loc}}(X, B; x) \simeq \mathbb{Z}/n\mathbb{Z}.$$

This finishes the proof in the case that there is a unique exceptional rational curve.

*Step 2:* In this step, we prove the statement of the proposition when there are at least 2 rational curves in the exceptional divisor of the dlt modification.

In this case, the only singularities or intersections with non-exceptional divisors in  $B_Y$  happen at the two end curves. As for each of these curves, we have that  $\deg \Delta_i = 1$ , they have either

1. no singularities along the curve,
2. one orbifold singularity of index  $n$  along the curve, or
3. two orbifold singularities of index 2 along the curve.

If the exceptional divisor of  $\phi \circ f$  is a chain of rational curves, by Corollary 3.0.6, we have the following isomorphism:

$$\pi_1^{\text{loc}}(X, B; x) \simeq \mathbb{Z}/n\mathbb{Z}.$$

Depending on the singularities of  $B_Y$ , we have two remaining cases.

*Case 2.1:* Exactly one curve has two orbifold singularities of order 2. In this case, the exceptional divisor of  $\phi \circ f : Y \rightarrow X$  is the union of a chain of rational curves and two additional rational curves with self-intersection  $-2$  intersecting one end curve. By Lemma 3.0.5, we have the following isomorphism:

$$\pi_1^{\text{loc}}(X, B; x) \simeq \left\langle a, b, x_1, \dots, x_n, \begin{array}{l} x_1 a^{-2}, abx_2 x_1^{-m_1}, \{x_{i-1} x_{i+1} x_i^{-m_i}\}_{1 \leq i < n} \\ x_1 b^{-2}, x_{n-1} x_n^{-m_n}, \{[x_i, x_{i+1}]\}_{\{1 \leq i < n-1\}} \end{array} \right\rangle.$$

By Lemma 3.0.7 the group is isomorphic to:

$$\begin{aligned} \pi_1^{\text{loc}}(X, B; x) &\simeq \langle a, b, x_1 \mid x_1 a^{-2}, x_1 b^{-2}, (ab)^{a_n} x_1^{b_n} ((ab)^{a_{n+1}} x_1^{b_{n+1}})^{-m} \rangle \\ &\simeq \langle a, b \mid a^2 b^{-2}, (ab)^{a_n - m a_{n+1}} a^{2b_n - 2mb_{n+1}} \rangle. \end{aligned}$$

*Case 2.2:* Both end curves have two orbifold singularities of order 2.

In this case, the exceptional divisor of  $\phi \circ f : Y \rightarrow X$  is the union of a chain of rational curves and two additional rational curves with self-intersection  $-2$  intersecting each one a different end curve. By Lemma 3.0.5, we have the following isomorphism:

$$\pi_1^{\text{loc}}(X, B; x) \simeq \left\langle a, b, x_1, \dots, x_n, c, d \begin{array}{l} x_1 a^{-2}, x_1 b^{-2}, abx_2 x_1^{-m_1}, \{x_{i-1} x_{i+1} x_i^{-m_i}\}_{1 \leq i < n} \\ x_{n-1} c d x_n^{-m_n}, x_n c^{-2}, x_n d^{-2}, \{[x_i, x_{i+1}]\}_{\{1 \leq i \leq n-1\}} \end{array} \right\rangle$$

By Lemma 3.0.7 the group is isomorphic to:

$$\begin{aligned} \pi_1^{\text{loc}}(X, B; x) &\simeq \langle a, b, x_1, c, d \mid x_1 a^{-2}, x_1 b^{-2}, (ab)^{a_{n-1}} x_1^{b_{n-1}} ((ab)^{a_n} x_1^{b_n})^{-m_n} c d, (ab)^{a_n} x_1^{b_n} c^{-2}, (ab)^{a_n} x_1^{b_n} d^{-2} \rangle \\ &\simeq \langle a, b, c \mid a^2 b^{-2}, (ab)^{a_n} a^{2b_n} c^{-2}, c^2 ((ab)^{a_{n-1}} a^{2b_{n-1}} c^{-2m_n+1})^{-2} \rangle. \end{aligned}$$

In each of the previous cases, we have a presentation with at most 4 generators and at most 7 relations. This finishes the proof of the proposition.  $\square$

**Proposition 3.0.13.** *Let  $(X, B; x)$  be a log canonical surface singularity. Let  $\phi : (Z, B_Z) \rightarrow (X, B)$  be a dlt modification, whose exceptional locus  $E$  is a chain of rational curves, then  $\pi_1^{\text{loc}}(Z, B_Z) \simeq \pi_1^{\text{loc}}(X, B; x) \simeq \pi_1^{\text{loc}}(X, B; x)$  has a presentation with at most 4 generators and 7 relations.*

*Proof.* We take a log resolution  $f : Y \rightarrow Z$ , where each singular point in  $E$  with  $\deg \Delta = 1 - \frac{1}{n}$  has preimage a chain rational curves with self-intersections  $-m_1, \dots, -m_l$ , and  $\frac{n}{q} = [m_1, \dots, m_l]$ . We only have to deal with the case in which  $B_s \neq 0$ . Otherwise, we are in the context of Proposition 3.0.12.

Here  $\hat{B}_{Y,s}$  and  $\hat{E}$  will denote the strict transforms of  $B_{Y,s}$  and  $E$  in  $Z$ .  $\hat{B}_s$  will denote the strict transform of  $B_s$  in  $Y$ . For  $x, y \in \pi_1^{\text{loc}}(X, B; x)$  we will define:

$$\delta_{t,x}(y) := \begin{cases} \{xy^{-t}\} & \text{if } y \text{ corresponds to a loop around an exceptional divisor of } f \\ \{y^t, [x, y]\} & \text{if } y \text{ corresponds to a loop around a curve in } B_{Y,s}^{\wedge} \setminus \hat{E} \\ \{[x, y]\} & \text{if } y \text{ corresponds to a loop around a curve in } B_{Y,s}^{\wedge} \setminus \hat{E} \text{ and } t = \infty \\ \emptyset & \text{if } y \text{ corresponds to a trivial loop} \end{cases} \quad (3.4)$$

*Case 1:* We prove the statement in the case that  $E = E_1$  is a single rational curve.

By Lemma 3.0.8, we have an upper bound for the degree

$$\deg(\Delta_1 + \hat{B}_s|_{E_1}) \leq 2.$$

We will prove this step in three cases depending on  $b(\Delta_1 + \hat{B}_s|_{E_1})$ . The case  $b(\Delta_1 + \hat{B}_s|_{E_1}) = ((2, 2, 2, 2))$ , gives rise to two different cases, depending on how many non-exceptional curves we have in  $B_{Y,s}$ .

*Case 1.1:* We assume that  $b(\Delta_1 + \hat{B}_s|_{E_1}) = ((2, 2, 2, 2))$  and we have one, two, or three curves in  $\hat{B}_s$ .

By Lemma 3.0.5, we have the following isomorphisms:

$$\begin{aligned} \pi_1^{\text{loc}}(X, B; x) &\simeq \langle a, b, c, d, x \mid xa^{-2}, \delta_{2,x}(b), \delta_{2,x}(c), d^2, [x, d], abcdx^{-m}, \rangle \\ &\simeq \langle a, b, c, d \mid \delta_{2,a^2}(b), \delta_{2,a^2}(c), d^2, [a^2, d], a^{1-2m}bcd \rangle \\ &\simeq \langle a, b, c \mid \delta_{2,a^2}(b), \delta_{2,a^2}(c), (a^{1-2m}bc)^2 \rangle. \end{aligned}$$

Each  $\delta_{t,x}(y)$  is at most two relations, so we have a presentation with 3 generators and at most 5 relations.

*Case 1.2:* We assume that  $b(\Delta_1 + \hat{B}_s|_{E_1}) = ((2, 2, 2, 2))$  and we have 4 curves in  $\hat{B}_s$ .

By Lemma 3.0.5, we have the following isomorphisms:

$$\begin{aligned} \pi_1^{\text{loc}}(X, B; x) &\simeq \langle a, b, c, d, x \mid a^2, b^2, c^2, d^2, abcdx^{-m}, [x, a], [x, b][x, c][x, d] \rangle \\ &\simeq \langle a, b, c, x \mid a^2, b^2, c^2, (abcx^m)^2, [a, x], [b, x], [c, x] \rangle. \end{aligned}$$

This gives a presentation with 4 generators and 7 relations.

*Case 1.3:* We assume that  $b(\Delta_1 + \hat{B}_s|_{E_1})$  is not equal to  $((2, 2, 2, 2))$ .

In these cases,  $b(\Delta_1 + \hat{B}_s|_{E_1})$  is of one of the following forms:  $((n_1, n_2, n_3)), ((n_1, n_2)), ((n)), ((\emptyset))$ . Here,  $n_i$  can be  $\infty$  only if it is a coefficient in  $B$ . By applying Lemma 3.0.5, we get the following presentation:

$$\pi_1^{\text{loc}}(X, B; x) \simeq \left\langle A, B, C, x \left| \begin{array}{l} a_{l_1} b_{l_2} c_{l_3} x^{-m}, \delta_{t_1, a_2}(a_1), \delta_{t_2, b_2}(b_1), \delta_{t_3, c_2}(c_1) \\ \{a_{i-1} a_{i+1} a_i^{-m_{1,i}}\}_{\{2 \leq i \leq l_1\}}, \{[a_i, a_{i+1}]\}_{\{1 \leq i \leq l_1\}}, \{b_{i-1} b_{i+1} b_i^{-m_{2,i}}\}_{\{2 \leq i \leq l_2\}} \\ \{[b_i, b_{i+1}]\}_{\{1 \leq i \leq l_2\}}, \{c_{i-1} c_{i+1} c_i^{-m_{3,i}}\}_{\{2 \leq i \leq l_3\}}, \{[c_i, c_{i+1}]\}_{\{1 \leq i \leq l_3\}} \end{array} \right\rangle.$$

Where  $A, B, C$  is  $\{a_1, \dots, a_{l_1}\}, \{b_1, \dots, b_{l_2}\}, \{c_1, \dots, c_{l_3}\}$ , and  $a_{l_1+1} = b_{l_2+1} = c_{l_3+1} = x$  in the previous presentation. Here, each  $l_i$  is 0 if  $n_i$  is not present in  $b(\Delta_1 + \hat{B}_s|_{E_1})$ . By applying Lemma 3.0.7 to each chain  $A, B, C$ , there exists a presentation of the local fundamental group:

$$\pi_1^{\text{loc}}(X, B; x) \simeq \langle a_1, b_1, c_1, x \mid \delta_{n_1, x}(a_1), \delta_{n_2, x}(b_1), \delta_{n_3, x}(c_1), a_1^{t_1} b_1^{t_2} c_1^{t_3} x^{-m} \rangle.$$

As each  $\delta_{t,x}(y)$  contains at most two relations, we have at most 4 generators and 7 relations. The only possible way for this to have exactly 4 generators and 7 relations is for all  $\delta_{t,x}(y)$  here to be 2 elements. Hence,  $a_1, b_1, c_1$  have to be loops around a curve in  $\hat{B}_{Y,s} \setminus \hat{E}$ . Therefore, by Lemma 3.0.5 we actually have the isomorphism:

$$\begin{aligned} \pi_1^{\text{loc}}(X, B; x) &\simeq \langle a, b, c, x \mid a^{n_1}, b^{n_2}, c^{n_3}, abc x^{-m}, [a, x], [b, x], [c, x] \rangle \\ &\simeq \langle a, b, x \mid a^{n_1}, b^{n_2}, (b^{-1} a^{-1} x^m)^{n_3}, [a, x], [b, x] \rangle. \end{aligned}$$

Therefore, in Case 1.3., there are no minimal presentations with 4 generators and 7 relations. This finishes the proof in the case that there is a unique exceptional rational curve in the dlt modification.

*Case 2:* In this case, we prove the statement of the proposition when there are at least 2 rational curves in the exceptional divisor of the dlt modification.

By Lemma 3.0.8, the only singularities in the strict transform of  $E$  happen at the end curves of the chain. Also, Lemma 3.0.8 implies that the only intersections of  $\hat{E}$  with  $B_{Y,s} \setminus \hat{E}$  happen at the end curves of the chain.

*Case 2.1:* We assume that  $b(\Delta_m + \hat{B}_s|_{E_n}) = ((2, 2))$  and  $b(\Delta_1 + \hat{B}_s|_{E_1}) \in \{((n_1)), ((\emptyset))\}$ .

In this case, by Lemma 3.0.5, we have a presentation:

$$\pi_1^{\text{loc}}(X, B; x) \simeq \left\langle \begin{array}{c} a, b, x_1, \dots, x_n \\ c_1, \dots, c_l \end{array} \middle| \begin{array}{l} \delta_{2,x_1}(a), \delta_{2,x_1}(b), abx_2x_1^{-m_1}, \{x_{i-1}x_{i+1}x_i^{-m_i}\}_{1 < i < n}, \{[x_i, x_{i+1}]\}_{1 \leq i \leq n} \\ c_l x_{n-1}x_n^{-m_n}, \delta_{m_{1,1}, c_2}(c_1), \{c_{i-1}c_{i+1}c_i^{-m_{1,i}}\}_{2 \leq i \leq l}, \{[c_i, c_{i+1}]\}_{1 \leq i \leq l} \end{array} \right\rangle.$$

Where,  $c_{l+1} = x_n$  in the previous presentation. Applying Lemma 3.0.7 to the chains  $x_1, \dots, x_n$  and  $c_1, \dots, c_l$ , we get an isomorphism:

$$\pi_1^{\text{loc}}(X, B; x) \simeq \langle a, b, x_1, c_1 \mid \delta_{2,x_1}(a), \delta_{2,x_1}(b), c_1^{t'}(ab)^{t_1}x_1^{t_2}, \delta_{r,(ab)^{t_3}x_1^{t_4}}(c_1) \rangle.$$

As each  $\delta_{t,x}(y)$  contains at most two relations, we have at most 4 generators and 7 relations. The only possible way for this group to have exactly 4 generators and 7 relations is for all  $\delta_{t,x}(y)$  in the right-hand side to contain 2 elements. Hence, in this case,  $a, b, c_1 = c_l$  are loops around a curve in  $B_{Y,s} \setminus \hat{E}$ . Therefore, by Lemma 3.0.5, we have the isomorphism:

$$\begin{aligned} \pi_1^{\text{loc}}(X, B; x) &\simeq \langle a, b, x_1, c \mid a^2, [a, x_1], b^2, [b, x_1], c^m, [c, (ab^{t_3})x_1^{t_4}], c_1(ab)^{t_1}x_1^{t_2} \rangle \\ &\simeq \langle a, b, x_1 \mid a^2, [a, x_1], b^2, [b, x_1], ((ab)^{t_1}x_1^{t_2})^m \rangle. \end{aligned}$$

Thus, under the assumptions of Case 2.1, there are no minimal presentations with exactly 4 generators and 7 relations.

*Case 2.2:* We assume that  $b(\Delta_m + \hat{B}_s|_{E_n}) = b(\Delta_1 + \hat{B}_s|_{E_1}) = ((2, 2))$ .

Here, we can apply Lemma 3.0.5 to get a presentation:

$$\pi_1^{\text{loc}}(X, B; x) \simeq \left\langle a, b, x_1, \dots, x_n, c, d \middle| \begin{array}{l} \delta_{2,x_1}(a), \delta_{2,x_1}(b), abx_2x_1^{-m_1}, \{x_{i-1}x_{i+1}x_i^{-m_i}\}_{1 < i < n} \\ \{[x_i, x_{i+1}]\}_{1 \leq i \leq n}, cdx_{n-1}x_n^{-m_n}, \delta_{2,x_n}(c), \delta_{2,x_n}(d) \end{array} \right\rangle.$$

By Lemma 3.0.7 applied to the chain  $x_1 \dots x_n$ , we get an isomorphism:

$$\pi_1^{\text{loc}}(X, B; x) \simeq \left\langle a, b, x_1, c, d \left| \begin{array}{l} (ab)^{a_{n-1}} x_1^{b_{n-1}} ((ab)^{a_n} x_1^{b_n})^{-m_n} cd, \delta_{2, x_1}(a) \\ \delta_{2, x_1}(b), \delta_{2, (ab)^{a_n} x_1^{b_n}}(c) \delta_{2, (ab)^{a_n} x_1^{b_n}}(d) \end{array} \right. \right\rangle.$$

This case turns into two different cases.

*Case 2.2.1:* We have one, two, or three non-exceptional curves in  $B_{Y,s}^\wedge$ . Then, we have the following isomorphisms due to Lemma 3.0.5:

$$\begin{aligned} \pi_1^{\text{loc}}(X, B; x) &\simeq \left\langle a, b, c, d, x_1 \left| \begin{array}{l} (ab)^{a_{n-1}} x_1^{b_{n-1}} ((ab)^{a_n} x_1^{b_n})^{-m_n} cd, x_1 a^{-2} \\ \delta_{2, x_1}(b), \delta_{2, (ab)^{a_n} x_1^{b_n}}(c), d^2, [(ab)^{a_n} x_1^{b_n}, d] \end{array} \right. \right\rangle \\ &\simeq \langle a, b, c, d \mid (ab)^{a_{n-1}} a^{2b_{n-1}} ((ab)^{a_n} a^{2b_n})^{-m_n} cd, \delta_{2, a^2}(b), \delta_{2, (ab)^{a_n} a^{2b_n}}(c), d^2, [(ab)^{a_n} a^{2b_n}, d] \rangle, \\ &\simeq \langle a, b, c, \mid \delta_{2, a^2}(b), \delta_{2, (ab)^{a_n} a^{2b_n}}(c), ((ab)^{a_{n-1}} a^{2b_{n-1}} ((ab)^{a_n} a^{2b_n})^{-m_n} c)^2 \rangle. \end{aligned}$$

As each  $\delta_{i,x}(y)$  contains at most two relations, we have a presentation with 3 generators and at most 5 relations.

*Case 2.2.2:* We have four non-exceptional curves in the strict transform  $B_{Y,s}^\wedge$ . Then, the following isomorphisms hold due to Lemma 3.0.5.

$$\begin{aligned} \pi_1^{\text{loc}}(X, B; x) &\simeq \left\langle a, b, c, d, x_1 \left| \begin{array}{l} (ab)^{a_{n-1}} x_1^{b_{n-1}} ((ab)^{a_n} x_1^{b_n})^{-m_n} cd, a^2, b^2, c^2, d^2 \\ [a, x_1], [b, x_1], [c, (ab)^{a_n} x_1^{b_n}], [d, (ab)^{a_n} x_1^{b_n}] \end{array} \right. \right\rangle \\ &\simeq \langle a, b, c, x_1 \mid a^2, b^2, [a, x_1], [b, x_1], c^2, ((ab)^{m_n a_n - a_{n-1}} x_1^{m_n b_n - b_{n-1}} c)^2, [c, (ab)^{a_n} x_1^{b_n}] \rangle. \end{aligned}$$

Thus, we get a minimal presentation with 4 generators and 7 relations.

*Case 2.3:* We assume that each  $b(\Delta_1 + \hat{B}_s|_{E_1})$  and  $b(\Delta_m + \hat{B}_s|_{E_m})$  are of the form  $((n_i))$  or  $((\emptyset))$ . By Lemma 3.0.5, we get a presentation:

$$\pi_1^{\text{loc}}(X, B; x) \simeq \left\langle \begin{array}{c|c} a_1 \dots a_{l_1}, & \delta_{m_{1,1}, a_2}(a_1), \{a_{i-1}a_{i+1}a_i^{-m_{1,i}}\}_{\{2 \leq i \leq l_1\}}, \{[a_i, a_{i+1}]\}_{\{1 \leq i \leq l_1\}} \\ x_1, \dots, x_n, & a_{l_1}x_2x_1^{-m_1}, \{x_{i-1}x_{i+1}x_i^{-m_i}\}_{1 < i < n}, \{[x_i, x_{i+1}]\}_{\{1 \leq i \leq n\}}, c_{l_2}x_{n-1}x_n^{-m_n} \\ c_1, \dots, c_{l_2} & \delta_{m_{2,1}, c_2}(c_1), \{c_{i-1}c_{i+1}c_i^{-m_{2,i}}\}_{\{2 \leq i \leq l_2\}}, \{[c_i, c_{i+1}]\}_{\{1 \leq i \leq l_2\}} \end{array} \right\rangle.$$

By applying Lemma 3.0.7 to the chains  $(a_1 \dots a_{l_1})$ ,  $(x_1, \dots, x_n)$ , and  $(c_1, \dots, c_{l_2})$ , we obtain an isomorphism:

$$\pi_1^{\text{loc}}(X, B; x) \simeq \langle a_1, x_1, c_1 \mid \delta_{n_1, x_1}(a_1), c_1^{t'}(a_1)^{t_1}x_1^{t_2}, \delta_{n_2, a_1^{t_3}x_1^{t_4}}(c_1) \rangle.$$

As each  $\delta_{t,x}(y)$  is at most two relations, we have at most 3 generators and 5 relations.

In each of the previous cases, we have a presentation with at most 4 generators and at most 7 relations. This finishes the proof of the proposition.  $\square$

*Proof of Theorem 3.0.1.* First, we prove the statement about the upper bound on the number of generators and relations. We take a dlt modification  $\phi : (Z, B_Z) \rightarrow (X, B)$ . By Lemma 3.0.8, the exceptional divisor of  $\phi$  is one of the following:

- (i) a chain of rational curves,
- (ii) a cycle of rational curves, or
- (iii) an elliptic curve.

In the first case, by Proposition 3.0.12 and Proposition 3.0.13  $\pi_1^{\text{loc}}(X, B; x)$  has a presentation with at most 4 generators and at most 7 relations. In the second case, the statement follows by Proposition 3.0.10. Finally, in the third case, the statement follows by Proposition 3.0.9.

Now, we turn to prove the second statement of the theorem. Assume that  $\pi_1^{\text{loc}}(X, B; x)$  admits a minimal presentation with 4 generators and 7 relations. By Proposition 3.0.9, Proposition 3.0.10, and Proposition 3.0.12, Proposition 3.0.13 we conclude that this can only happen in the two following cases:

1. The dlt modification  $(Y, B_Y)$  of  $(X, B)$  extracts a unique rational curve and  $B$  has 4 components with coefficient  $\frac{1}{2}$ .

2. The dlt modification  $(Y, B_Y)$  of  $(X, B)$  extracts a chain of rational curve,  $B$  has 4 components with coefficients  $\frac{1}{2}$ , and their strict transforms only intersect the first and last curve of the chain.

We denote by  $\phi: Y \rightarrow X$  the dlt modification. Let  $E$  be the reduced exceptional divisor of the dlt modification. Then, we have that

$$\phi^*(K_X + B) = K_Y + E + \hat{B},$$

where  $\hat{B}$  is the strict transform of  $B$  on  $Y$ . By construction, the variety  $Y$  is  $\mathbb{Q}$ -factorial. We run a  $(K_Y + E)$ -MMP over the base. Observe that in any of these two cases, the endpoints of the chain are  $(K_Y + E)$ -negative curves, as they intersect  $\hat{B}$  positively. Hence, this MMP will inductively contract the endpoints of the chain of curves. Thus, this minimal model program terminates on  $X$ . We conclude that  $X$  is  $\mathbb{Q}$ -factorial at  $x$ . Therefore, the local class group  $\text{Cl}(X_x)$  is torsion. Note that the sum of the coefficients of the components of  $B$  through  $x$ , denoted by  $|B|$  equals 2. Altogether, we conclude that

$$\dim X + \text{rank}_{\mathbb{Q}} \text{Cl}(X_x) - |B| = 0.$$

By [27, Theorem 2], we conclude that  $X$  must be formally toric around the point  $x$ .  $\square$

Now, we turn to prove a lemma that will be used in the proof of Theorem 3.0.3.

**Lemma 3.0.14.** *Let  $(X, B; x)$  be a log canonical surface singularity of coregularity zero. Let  $\phi: (Z, B_Z) \rightarrow (X, B)$  a dlt modification. There exists a subgroup  $H < \pi_1^{\text{loc}}(Z, B_Z)$  that is an abelian normal subgroup of  $\pi_1^{\text{loc}}(X, B; x)$ .*

*Proof.* By Lemma 3.0.8, the exceptional divisor is either a cycle or a chain of rational curves:

If it is a cycle of rational curves, then  $H \cong \mathbb{Z}^2$  is a normal subgroup of  $\pi_1^{\text{loc}}(Z, B_Z)$ , by Proposition 3.0.10. Hence, the Lemma holds.

If it is a chain of rational curves, let  $H < \pi_1^{\text{loc}}(Z, B_Z)$  be the subgroup generated by the loops around the exceptional divisor of  $\phi$ . Then we call  $x_1, \dots, x_m$  the loops around the divisors  $E_1, \dots, E_m$ . By Lemma 3.0.7 and Lemma 3.0.5,  $H = \langle x_1, x_2 \rangle = \langle x_{m-1}, x_m \rangle$ .  $H$  is abelian, since adjacent  $x_i$ 's commute. Now, we turn to prove that  $H$  is a normal subgroup of  $\pi_1^{\text{loc}}(X, B; x)$ . By the proof of Proposition 3.0.12 and Proposition 3.0.13, the remaining generators  $\gamma$  of  $\pi_1^{\text{loc}}(X, B; x)$  satisfy one of the following:

1. They commute with  $x_1$ , and  $\gamma x_2 \gamma^{-1} = x_1^2 x_2^{-1}$ , whenever  $b(\Delta_1 + \hat{B}_s|_{E_1}) = ((2, 2))$  holds.

2. They commute with  $x_m$ , and  $\gamma x_{m-1} \gamma^{-1} = x_m^2 x_{m-1}^{-1}$ , whenever  $b(\Delta_m + \hat{B}_s|_{E_m}) = ((2, 2))$  holds.
3. They are generated by  $x_1, x_2$ , whenever  $b(\Delta_1 + \hat{B}_s|_{E_1}) = ((n))$  holds.
4. They are generated by  $x_{m-1}, x_m$ , whenever  $b(\Delta_m + \hat{B}_s|_{E_m}) = ((n))$  holds.

We conclude that  $H$  is normal.  $\square$

*Proof of Theorem 3.0.3.* We start with the case where  $(X, B, x)$  is a log canonical singularity of coregularity one. We take a dlt modification  $\phi : (Z, B_Z) \rightarrow (X, B)$ . As we are in the coregularity one case, by Lemma 3.0.8 the exceptional divisor of  $\phi$  is a unique rational curve or an elliptic curve.

If the exceptional is an elliptic curve, then as in the proof of Proposition 3.0.9, we have:

$$1 \rightarrow \pi_1(S^1) \simeq \mathbb{Z} \rightarrow \pi_1^{\text{loc}}(Z, B_Z) \rightarrow \pi_1(\mathbb{T}^2) \simeq \mathbb{Z}^2 \rightarrow 1.$$

Therefore,  $\pi_1^{\text{loc}}(Z, B_Z)$  is solvable of length 2.

Now, we assume that the exceptional of the dlt modification is a rational curve. First, assume that  $B_s = 0$ . By the proof of Proposition 3.0.12, we have one of the following isomorphisms:

1.  $\pi_1^{\text{loc}}(X, B; x) \simeq \langle a, b, c, d, x \mid xa^{-2}, xb^{-2}, xc^{-2}, xd^{-2}, abcdx^{-m} \rangle$ .

In this case, we can define  $N = \langle ab, bc, x \rangle$ . Then, we have that

$$1 \rightarrow N \rightarrow \pi_1^{\text{loc}}(X, B; x) \rightarrow \mathbb{Z}/2\mathbb{Z} \rightarrow 1,$$

and  $N$  is nilpotent of length 2 as  $N/\langle x \rangle \simeq \mathbb{Z}^2$ .

2.  $\pi_1^{\text{loc}}(X, B; x) \simeq \langle a_1, b_1, c_1, x \mid xa_1^{-n}, xb_1^{-n'}, xc_1^{-n''}, a_1^t b_1^{t'} c_1^{t''} x^{-m} \rangle$ , where  $b(\Delta) = ((n, n', n''))$  is as in equation 3.3. We have that  $x$  is in the center of  $\pi_1^{\text{loc}}(X, B; x)$  and we can define

$$G' := \pi_1^{\text{loc}}(X, B; x)/\langle x \rangle \simeq \langle a_1, b_1, c_1 \mid a^n, b^{n'}, c^{n''}, abc \rangle.$$

Where  $a = a_1^t, b = b_1^{t'}, c = c_1^{t''}$ . Now, we have to check four different cases.

- (a) If  $b(\Delta) = ((3, 3, 3))$ , then  $G'$  has an abelian normal subgroup  $N' = \langle ab^{-1}, a^{-1}b \rangle$  such that the following exact sequence holds

$$1 \rightarrow N' \rightarrow \pi_1^{\text{loc}}(X, B; x)/\langle x \rangle \rightarrow \mathbb{Z}/3\mathbb{Z} \rightarrow 1.$$

Therefore, we can define  $N = \langle x, a_1^t b_1^{-t'}, a_1^{-t} b_1^{t'} \rangle$  and we have

$$1 \rightarrow N \rightarrow \pi_1^{\text{loc}}(X, B; x) \rightarrow \mathbb{Z}/3\mathbb{Z} \rightarrow 1.$$

As  $N/\langle x \rangle \simeq \mathbb{Z}^2$ , the group  $N$  is nilpotent of length 2.

- (b) If  $b(\Delta) = ((2, 4, 4))$ , then  $G'$  has an abelian normal subgroup  $N' = \langle bc^{-1}, b^{-1}c \rangle$  such that the following exact sequence holds

$$1 \rightarrow N' \rightarrow \pi_1^{\text{loc}}(X, B; x)/\langle x \rangle \rightarrow \mathbb{Z}/4\mathbb{Z} \rightarrow 1.$$

Therefore, we can define  $N = \langle x, b_1^{t'} c_1^{-t''}, b_1^{-t'} c_1^{t''} \rangle$  and we have

$$1 \rightarrow N \rightarrow \pi_1^{\text{loc}}(X, B; x) \rightarrow \mathbb{Z}/4\mathbb{Z} \rightarrow 1.$$

As  $N/\langle x \rangle \simeq \mathbb{Z}^2$ , the group  $N$  is nilpotent of length 2.

- (c) If  $b(\Delta) = ((2, 3, 6))$ , then  $G'$  has an abelian normal subgroup  $N' = \langle bc^4, c^4b \rangle$  such that the following exact sequence holds

$$1 \rightarrow N' \rightarrow \pi_1^{\text{loc}}(X, B; x)/\langle x \rangle \rightarrow \mathbb{Z}/6\mathbb{Z} \rightarrow 1.$$

Therefore, we can define  $N = \langle x, b_1^{t'} c_1^{4t''}, c_1^{4t''} b_1^{t'} \rangle$  and we have

$$1 \rightarrow N \rightarrow \pi_1^{\text{loc}}(X, B; x) \rightarrow \mathbb{Z}/6\mathbb{Z} \rightarrow 1.$$

As  $N/\langle x \rangle \simeq \mathbb{Z}^2$ , the group  $N$  is nilpotent of length 2.

- (d) If  $b(\Delta) = ((n, n')), ((n))$ , or  $((\emptyset))$ , then we have that  $G' \simeq \langle a, b \mid a^n, b^{n'}, ab \rangle \simeq \mathbb{Z}/m\mathbb{Z}$ .  
Therefore,  $\pi_1^{\text{loc}}(X, B; x)$  is nilpotent of length 2.

3.  $\pi_1^{\text{loc}}(X, B; x) \simeq \mathbb{Z}/n\mathbb{Z}$ . This is already an abelian group, hence the proposition is trivial in this case.

Now assume  $B_s \neq 0$ . By the proof of Proposition 3.0.13, we have one of the following isomorphisms

1.  $\pi_1^{\text{loc}}(X, B; x) \simeq \langle a, b, c, d, x \mid \delta_{2,x}(a), \delta_{2,x}(b), \delta_{2,x}(c), \delta_{2,x}(d), abcdx^{-m} \rangle$ .

Here we can define  $N = \langle ab, bc, x \rangle$  and we have

$$1 \rightarrow N \rightarrow \pi_1^{\text{loc}}(X, B; x) \rightarrow \mathbb{Z}/2\mathbb{Z} \rightarrow 1.$$

The group  $N$  is nilpotent of length 2, as  $N/\langle x \rangle \simeq \mathbb{Z}^2$ .

$$2. \pi_1^{\text{loc}}(X, B; x) \simeq \langle a_1, b_1, c_1, x \mid \delta_{n_1, x}(a_1), \delta_{n_2, x}(b_1), \delta_{n_3, x}(c_1), a_1^{t_1} b_1^{t_2} c_1^{t_3} x^{-m} \rangle.$$

We have that  $x$  is in the center of  $\pi_1^{\text{loc}}(X, B; x)$  and we can define

$$G' := \pi_1^{\text{loc}}(X, B; x) / \langle x \rangle \simeq \langle a, b, c \mid a^n, b^{n'}, c^{n''}, abc \rangle.$$

Where  $a = a_1^t, b = b_1^{t'}, c = c_1^{t''}$ . Hence, the subgroup  $N$  can be obtained as when  $B_s = 0$ .

Now, we prove the case where  $(X, B, x)$  is a log canonical singularity of coregularity zero. We take a dlt modification  $\phi : (Z, B_Z) \rightarrow (X, B)$ . As we are in the coregularity zero case, by Lemma 3.0.8 the exceptional divisor is a chain of rational curves or a cycle of rational curves.

We use the notation of Lemma 3.0.14 for the normal subgroup  $H$ . If it is a cycle of rational curves, then by Proposition 3.0.10, we have the exact sequence  $1 \rightarrow H \cong \mathbb{Z}^2 \rightarrow \pi_1^{\text{loc}}(X, B; x) \rightarrow \mathbb{Z} \rightarrow 1$ , hence  $\pi_1^{\text{loc}}(X, B; x)$  is solvable of length 2.

If it is a chain of rational curves. First, assume  $B_s = 0$ . By the proof Proposition 3.0.12, we have one of the following isomorphisms:

$$1. \pi_1^{\text{loc}}(X, B; x) \simeq \langle a, b \mid a^2 b^{-2}, (ab)^{a_n - ma_{n+1}} a^{2b_n - 2mb_{n+1}} \rangle.$$

Here we can define  $N = \langle ab, a^2 \rangle = H$ , and we get:

$$1 \rightarrow N \rightarrow \pi_1^{\text{loc}}(X, B; x) \rightarrow \mathbb{Z}/2\mathbb{Z} \rightarrow 1.$$

The group  $N$  is abelian.

$$2. \pi_1^{\text{loc}}(X, B; x) \simeq \langle a, b, c \mid a^2 b^{-2}, (ab)^{a_n} a^{2b_n} c^{-2}, c^2 ((ab)^{a_{n-1}} a^{2b_{n-1}} ((ab)^{a_n} a^{2b_n})^{-m_n} c)^{-2} \rangle.$$

Here, we can define  $N = \langle ac, ab, a^2 \rangle = \langle ac, H \rangle$  and we get:

$$1 \rightarrow N \rightarrow \pi_1^{\text{loc}}(X, B; x) \rightarrow \mathbb{Z}/2\mathbb{Z} \rightarrow 1.$$

The group  $N$  is solvable of length 2 as  $N/H \simeq \mathbb{Z}$

Now assume that  $B_s \neq 0$ . By the proof of Proposition 3.0.13, we have one of the following isomorphisms:

$$1. \pi_1^{\text{loc}}(X, B; x) \simeq \langle a, b, x_1, c_1 \mid \delta_{2, x_1}(a), \delta_{2, x_1}(b), c_1^{t'}(ab)^{t_1} x_1^{t_2}, \delta_{r, (ab)^{t_3} x_1^{t_4}}(c_1) \rangle.$$

We can define  $N = \langle ab, ac, x_1 \rangle = \langle ac, H \rangle$  and we get:

$$1 \rightarrow N \rightarrow \pi_1^{\text{loc}}(X, B; x) \rightarrow \mathbb{Z}/2\mathbb{Z} \rightarrow 1.$$

The group  $N$  is solvable of length 2, as  $N/H \simeq \mathbb{Z}$

$$2. \pi_1^{\text{loc}}(X, B; x) \simeq \langle a, b, c, \mid \delta_{2, a^2}(b), \delta_{2, (ab)^{a_n} a^{2b_n}}(c), ((ab)^{a_{n-1}} a^{2b_{n-1}} ((ab)^{a_n} a^{2b_n})^{-m_n} c)^2 \rangle.$$

We can define  $N = \langle ab, ac, a^2 \rangle = \langle ac, H \rangle$  and we get:

$$1 \rightarrow N \rightarrow \pi_1^{\text{loc}}(X, B; x) \rightarrow \mathbb{Z}/2\mathbb{Z} \rightarrow 1.$$

The group  $N$  is solvable of length 2, as  $N/H \simeq \mathbb{Z}$

$$3. \pi_1^{\text{loc}}(X, B; x) \simeq \langle a, b, c, x_1 \mid a^2, b^2, [a, x_1], [b, x_1], c^2, ((ab)^{m_n a_n - a_{n-1}} x_1^{m_n b_n - b_{n-1}} c)^2, [c, (ab)^{a_n} x_1^{b_n}] \rangle.$$

We can define  $N = \langle ab, ac, x_1 \rangle = \langle ac, H \rangle$  and we get:

$$1 \rightarrow N \rightarrow \pi_1^{\text{loc}}(X, B; x) \rightarrow \mathbb{Z}/2\mathbb{Z} \rightarrow 1.$$

$N$  is solvable of length 2, as  $N/H \simeq \mathbb{Z}$

$$4. \pi_1^{\text{loc}}(X, B; x) \simeq \langle a_1, x_1, c_1 \mid \delta_{n_1, x_1}(a_1), c_1^{t'}(a_1)^{t_1} x_1^{t_2}, \delta_{n_2, a_1^{t_3} x_1^{t_4}}(c_1) \rangle.$$

Here  $H = \pi_1^{\text{loc}}(X, B; x)$ , hence the group is abelian.

□

## Chapter 4

# Smooth polyhedral complexes

In this chapter we will describe a process for constructing log canonical singularities with prescribed local fundamental groups. We will begin this process with smooth polyhedral complexes having the desired fundamental group for our singularities.

The main result of this chapter is the following theorem:

**Theorem 4.0.1.** *For any  $\mathcal{P}$ , a smooth polyhedral complex of dimension  $n$ , there exists an  $(n+1)$ -dimensional log canonical singularity  $(X; x)$  for which  $\pi_1^{\text{loc}}(X; x) \simeq \pi_1(|\mathcal{P}|)$ . Furthermore, for  $n \leq 4$ , the singularity  $(X; x)$  is isolated.*

### 4.1 From polyhedral complexes to simple normal crossing toric varieties

In this section, we construct projective snc toric Calabi-Yau varieties from smooth polyhedral complexes.

**Proposition 4.1.1.** *Let  $\mathcal{P}$  be a smooth polyhedral complex of dimension  $n$ . Then, there exists an  $n$ -dimensional simple normal crossing projective toric Calabi-Yau variety  $T$ , with  $\pi_1(T) \cong \pi_1(|\mathcal{P}|)$*

*Proof.* Each polyhedra  $F$  in  $\mathcal{P}$  is contained in an affine space  $M_{F, \mathbb{Q}} \simeq \mathbb{Q}^{N_F}$  so that the vertices of  $F$  are contained in  $\mathbb{Z}^{N_F}$ . By replacing  $F$  with its affine span, we may assume  $N_F = \dim F$ , i.e.,  $F$  is a full-dimensional polyhedron in  $M_{F, \mathbb{Q}}$ . Hence, for each polyhedron  $P \in \mathcal{P}$ , we have an associated toric projective variety  $T_P$  and an ample Cartier divisor  $A_P$ . Assume that  $F$  is a face of  $P$ , then we have an induced lattice embedding  $i_{F, P}: M_{F, \mathbb{Q}} \rightarrow M_{P, \mathbb{Q}}$ . By Proposition 2.6.4, this lattice embedding

induces a toric embedding  $i_{F,P}: T_F \hookrightarrow T_P$  for which

$$i_{F,P}^* A_P \sim A_F. \quad (4.1)$$

Furthermore, if  $G$  is a face of  $F$  and  $F$  is a face of  $P$ , then we have the following equality:

$$i_{F,P} \circ i_{G,F} = i_{G,P}. \quad (4.2)$$

Let  $P_1, \dots, P_k$  be the  $n$ -dimensional faces of  $\mathcal{P}$ . We denote by  $T_1, \dots, T_k$  the associated projective  $n$ -dimensional toric varieties. Note that all the polyhedra  $P_i$  are smooth, so each  $T_i$  is a smooth projective toric variety. Then, we can glue the varieties  $T_i$  whenever the polyhedra  $P_i$  have a common facet. To obtain a normal crossing scheme, we define the gluing by taking affine covers and glue affine locally by taking the fiber product of rings. This gluing is well-defined due to the compatibility condition (4.2). We obtain a scheme  $T$ . Since  $\mathcal{P}$  is a smooth polyhedral complex and the gluing of irreducible components is normal crossing, the scheme  $T$  has snc singularities. Note that each irreducible component  $T_i$  comes with a line bundle  $L_i \rightarrow T_i$ . Let  $T_{i,j} := T_i \cap T_j$  and  $L_{i,j} \rightarrow T_{i,j}$  be the induced line bundle. Due to (4.1), we have closed embeddings  $L_{i,j} \hookrightarrow L_i$  and  $L_{i,j} \hookrightarrow L_j$  for each  $i$  and  $j$ . The push-out of closed embeddings exists in the category of schemes [35, Corollary 3.9]. Hence, we obtain a scheme  $L_T$  by gluing the  $L_i$ 's. By the universal property of push-outs,  $L_T$  admits a morphism  $L_T \rightarrow T$  that restricts to  $L_i \rightarrow T_i$  on each component  $T_i$ . In particular,  $L_T$  is a line bundle over the snc variety  $T$ . By the Nakai-Moshezon criterion, this line bundle is ample. Hence,  $T$  is a projective snc variety.

We claim that  $T$  is a Calabi-Yau variety, i.e., for each component  $T_i$  of  $T$ , we have that  $K_T|_{T_i} \sim 0$ . Indeed, since  $\mathcal{P}$  is simple, every face of dimension  $n-1$  in  $\mathcal{P}$  is contained in exactly two maximal polyhedra. Hence, by adjunction formula, for each  $T_i$ , we have that

$$K_T|_{T_i} \sim K_{T_i} + B_{T_i} \sim 0.$$

Here,  $B_{T_i}$  is the reduced toric boundary of  $T_i$ . This proves that  $T$  is a Calabi-Yau variety.

The moment maps  $\phi_P: T_P \rightarrow P$  glue together to a map  $\phi_T: T \rightarrow |\mathcal{P}|$ . We claim that  $\phi_T$  induces an isomorphism

$$\phi_{T*}: \pi_1(T) \rightarrow \pi_1(|\mathcal{P}|). \quad (4.3)$$

Indeed, the restriction of  $\phi_T$  to

$$\bigcup_{\dim F=1} T_F \rightarrow |\mathcal{P}|_1$$

induces an isomorphism between fundamental groups. Here,  $|\mathcal{P}|_1$  is the 1-skeleton of  $|\mathcal{P}|$ . Indeed, both spaces have the same graph structure, when considering each  $\mathbb{P}^1$  in  $\bigcup_{\dim F=1} T_F$  as a topological  $S^2$ . Moreover, we have a commutative diagram

$$\begin{array}{ccc} \pi_1 \left( \bigcup_{\dim F=1} T_F \right) & \xrightarrow{\phi_{T*}} & \pi_1(|\mathcal{P}|_1) \\ \downarrow & & \downarrow \\ \pi_1 \left( \bigcup_{\dim F \leq 2} T_F \right) & \xrightarrow{\phi_{T*}} & \pi_1(|\mathcal{P}|_2). \end{array}$$

The kernel of the right vertical map is the smallest normal subgroup of  $\pi_1(|\mathcal{P}|_1)$  containing all the loops around 2-skeleta. On the other hand, the kernel of the left vertical arrow is the smallest normal subgroup generated by the torus invariant boundary of the toric surfaces  $T_F$  with  $\dim F = 2$ . Indeed, the fundamental group of a projective toric surface is trivial [9, Theorem 10.4.3]. We conclude that the restriction of  $\phi_T$  to

$$\bigcup_{\dim F \leq 2} T_F \rightarrow |\mathcal{P}|_2$$

induces an isomorphism between fundamental groups. Observe that gluing cells of dimension at least 3 does not change the fundamental group of a CW-complex. Analogously, gluing a toric manifold of dimension at least 3 along the torus invariant boundary does not change the fundamental group of an snc variety. Hence, the isomorphism 4.3 holds. We deduce that  $T$  is an snc CY variety with fundamental group  $\pi_1(|\mathcal{P}|)$ .  $\square$

## 4.2 From simple normal crossing toric varieties to lc singularities

In this section, we construct log canonical singularities of dimension  $n + 1$  from reducible simple normal crossing toric varieties of dimension  $n$ . Putting together this construction with the one in the previous section, we will give a proof for Theorem 4.0.1

In order to construct log canonical singularities, we will use a result due to Kollár [22], (see also [19, Theorem 35]). This proposition allows us to construct singularities admitting a prescribed partial resolution. The following theorem is proved in [22, Theorem 8, Propositions 9 and 10].

**Proposition 4.2.1.** *Let  $T$  be an  $n$ -dimensional projective variety with simple normal crossing singularities and  $n \geq 2$ . Let  $L$  be an ample line bundle on  $T$ . Then, for  $m \gg 1$  there is a germ of a normal singularity  $(X; x)$  with a partial resolution:*

$$\begin{array}{ccc} T & \hookrightarrow & Y \\ \downarrow & & \downarrow \phi \\ x & \hookrightarrow & (X; x) \end{array}$$

*satisfying the following conditions:*

1.  $T$  is a Cartier divisor on  $Y$ .
2. The normal bundle of  $T$  in  $Y$  is  $K_T \otimes L^{-m}$ .
3. We have an isomorphism  $\pi_1(Y) \simeq \pi_1(T)$ .
4. The kernel of  $\pi_1^{\text{loc}}(X; x) \rightarrow \pi_1(Y)$  is cyclic, central, and generated by any loop around an irreducible component of  $T$ .
5. If  $\dim T \leq 4$ , then  $(x \in X)$  is an isolated singular point.
6. If  $K_T \sim 0$ , then  $K_X$  is Cartier and  $(x \in X)$  is lc.

**Remark 4.2.2.** In Proposition 4.2.1 we only need  $L^m$  to be very ample. In our application of this proposition, we will have two ample line bundles  $L_1$  and  $L_2$ . By [18, Chapter II. Exercise 7.5], for  $k \gg 1$ , we have that  $L_1 \otimes L_2^k$  is very ample. Thus, in the previous proposition, we may replace  $L^m$  with  $L_1 \otimes L_2^k$  and yield the same conclusion, up to replacing  $L^{-m}$  with  $L_1 \otimes L_2^k$  in Proposition 4.2.1.(2).

**Definition 4.2.3.** Let  $T$  be an  $n$ -dimensional simple normal crossing variety. Let  $F$  be an  $(n-1)$ -dimensional stratum. Let  $Z$  be a divisor in  $F$  that intersects transversally all the strata of  $T$ . We will describe blow-ups in each irreducible component of  $T$  and then glue them together. In this definition, when we say the blow-up of a subvariety  $V \subset W$ , we mean the blow-up defined by the reduced scheme  $V$  in  $W$ . Let  $E$  be an irreducible component, we will say  $F_E$  and  $Z_E$  for the intersections of  $F$  and  $Z$  with  $E$ , respectively.

We first perform the blow-up of  $Z_E$  in  $E$ , call it  $p : E' \rightarrow E$ . As  $Z_E$  in  $F_E$  has codimension at most 1, we can identify  $F_E$  with its strict transform in  $E'$ . Let  $D_E$  be the exceptional divisor of  $p : E' \rightarrow E$ . Then, we perform the blow-up of  $D_E \cap F_E$  in  $E'$  and call it  $p' : E'' \rightarrow E'$ . Similarly, the locus of the blow-up has codimension at most 1 in  $F$ . Thus, we can identify again  $F_E$  with its

strict transform in  $E''$ . Therefore, we still have the datum for gluing the blow-ups of the irreducible components in  $T$ . So, we can glue the irreducible components  $E''$ , to obtain a simple normal crossing variety  $T''$ . This will be called the *iterated blow-up of  $Z$  in  $T$* . Here we can also identify  $Z_F$  in  $T''$  as the intersection of the exceptional divisor with  $F_E$  in  $E''$ . We call  $D_1$  the union of all the strict transforms of  $D_E$  for every irreducible component  $E''$ , and we call  $D_2$  the union of the exceptional divisors of  $p' : E'' \rightarrow E'$  for every irreducible component  $E''$ . Then,  $D_1$  and  $D_2$  are Cartier Divisors on  $T''$ .

**Remark 4.2.4.** Let  $T$  be an  $n$ -dimensional simple normal crossing Calabi-Yau variety, such that all the codimension  $c$  strata are contained in exactly  $c + 1$  irreducible components. Then,  $T''$  the iterated blow-up defined in Definition 4.2.3 is also Calabi-Yau. Indeed, we only need to check the statement after the first blow-up, our snc variety  $T'$  is also Calabi-Yau. This can be checked at each irreducible component  $E_i$ . Let us call  $Z_i := Z \cap E_i$  and  $F_i := F \cap E_i$ . Let  $p : E'_i \rightarrow E_i$  be the blow-up of  $Z_i$  on  $E_i$ . Let  $D$  be the exceptional divisor of  $p : E' \rightarrow E$ .

Let  $c$  be the codimension of  $Z_i$  in  $E_i$ . Then,  $p^*(K_{E_i}) = K_{E_i} - (c - 1)D$ . And for any other component  $E_j$  that contains  $Z_i$ , we have that  $p^*(E_j)|_{E_i} = E_j|_{E_i} + D$ . By our hypothesis,  $F'_i$  is contained in  $c$  irreducible components of  $T'$ . Thus,  $Z'_i$  is contained in  $c - 1$  irreducible components of  $T'$ , other than  $E_i$ . Hence, adding our equalities for every irreducible component containing  $Z_i$ , we obtain that  $K_{T'}|_{E_i}$  is trivial for each  $E_i$ .

**Lemma 4.2.5.** *Let  $T$  be an snc variety with an irreducible component  $E$ . Let  $F$  be a codimension one stratum contained in  $E$ . Moreover, let  $Z$  be a smooth divisor in  $F$  that intersects transversally all the strata of  $T$ . Let  $T'' \rightarrow T$  be the iterated blow-up of  $Z$  in  $T$ , as in Definition 4.2.3. Let  $D_1$  and  $D_2$  be the exceptional divisors over  $T$ , as in Definition 4.2.3. Then, the divisors  $-3D_2 - 2D_1$  and  $-4D_2 - 3D_1$  are relatively ample over  $T$ .*

*Proof.* It suffices to check that the restrictions of these divisors to every irreducible component are ample over  $T$ . Let  $E_i$  be an irreducible component of  $T$ . Let us call  $Z_i := Z \cap E_i$  and  $F_i := F \cap E_i$ . We will call  $F''_i$  the strict transform of  $F_i$  in  $E''_i$ . So, we have the blow-ups  $p_2 : E''_i \rightarrow E'_i$  and  $p_1 : E'_i \rightarrow E_i$ . By abuse of notation, the intersections of  $D_1$  and  $D_2$  with  $E''_i$  are also called  $D_1$  and  $D_2$ . We will define  $Z''_i$  to be the intersection of the exceptional divisor with  $F''_i$ .

As  $Z_i$  in  $E_i$  has codimension  $c \geq 2$ . The fiber of the exceptional divisor of  $p_1 : E'_i \rightarrow E_i$  over  $Z'_i$  are  $\mathbb{P}^{c-1}$ . When we perform the second blow-up  $p_2 : E''_i \rightarrow E'_i$ , in the fibers of  $D_1$  over  $Z'_i$  we are performing the blow-up at one smooth point. Hence, the fibers of  $D_1$  over  $Z''_i$  are the blow-ups of  $\mathbb{P}^{c-1}$  at one point, and the fibers of  $D_2$  over  $Z''_i$  are  $\mathbb{P}^{c-1}$ .

Let  $C_1$  be a curve in  $D_1$  that intersects  $D_2$  transversally in one point and  $C_2$  a curve in  $D_2$  that intersects  $D_1$  transversally in one point.

We first prove that  $C_1$  and  $C_2$  generate the relative cone of curves of  $E'' \rightarrow E$ . Indeed, let  $N$  be a relatively nef divisor that is positive on both curves. There must be a curve  $C'_2 \sim C_2$ , such that  $C'_2$  lies in  $D_1 \cap D_2$  and is contracted on  $E$ . We have that  $C_1$  and  $C'_2$  generate the relative cone of curves of  $E'$ . Hence,  $N$  is relatively ample on  $D_1$ . Furthermore,  $N$  is relatively ample on  $D_2$ , since it intersects a curve positively and  $D_2$  has relative Picard rank 1. Therefore,  $N$  intersects positively any exceptional curve, hence it is relatively ample. Thus, there cannot be any other generator of the relative cone of curves.

As  $C_1$  and  $C_2$  generate the relative cone of curves, we only need to compute the intersection products with these curves. We have that  $p_1^*(K_{E'}) = K_{E''} - (c-1)D_2$ . Hence,  $K_{E''} \sim_{E'_i} (c-1)D_2$ . By adjunction, we obtain the value of  $cD_2 \cdot C_2 = (K_{E''} + D_2) \cdot C_2 = K_{D_2} \cdot C_2 = -c$ . Therefore,  $D_2 \cdot C_2 = -1$ . Similarly, we have that  $(p_2 \circ p_1)^*(K_E) = K_{E''} - (c-1)D_1 - (2c-2)D_2$ , hence  $K_{E''} \sim_E (c-1)D_1 + (2c-2)D_2$ . By adjunction,  $(cD_1 + (2c-2)D_2) \cdot D_1 = (K_{E''} + D_1) \cdot C_1 = K_{D_1} \cdot C_1 = -2$ , we conclude that  $D_1 \cdot C_1 = -2$ . This implies that  $-3D_2 - 2D_1$  and  $-4D_2 - 3D_1$  are relatively ample over  $T$ .  $\square$

*Proof of Theorem 4.0.1.* By Proposition 4.1.1, we have a toric Calabi-Yau snc variety  $T$ , with an ample line bundle  $H$ . We will perform iterated blow-ups with loci contained in disjoint irreducible components. Let  $E$  be an irreducible component of  $T$ . Pick  $F$  a codimension 1 stratum of  $T$  contained in  $E$ . Let  $Z$  be a smooth divisor of  $F$  that intersects transversally all the strata.

We perform the iterated blow-up of  $Z$  in  $T$ , as in Definition 4.2.3. Let us call these maps  $p_1 : T'' \rightarrow T'$  and  $p_2 : T' \rightarrow T$ . Let  $D_1$  be the exceptional divisor of the first blow-up and  $D_2$  be the exceptional divisor of the second blow-up in  $E'$ . By Lemma 4.2.5, the divisors  $p^*(mH) - 3D_2 - 2D_1$  and  $p^*(mH) - 4D_2 - 3D_1$  are ample. We can perform this same procedure for  $G$  an irreducible component disjoint from  $E$ , with exceptional divisors  $G_1$  and  $G_2$ . Thus,  $p^*(mH) - 3G_2 - 2G_1$  and  $p^*(mH) - 4G_2 - 3G_1$  are ample.

Now, consider the following line bundles  $L_1 := p^*(mH) - 3D_2 - 2D_1 - 4G_2 - 3G_1$  and  $L_2 := p^*(mH) - 3D_2 - 2D_1 - 3G_2 - 2G_1$ . By our construction, we have smooth rational curves  $P_1$  and  $P_2$  that do not intersect any irreducible component other than  $E$  and  $G$ , respectively. By Remark 4.2.4, we have that  $T''$  is a simple normal crossing Calabi-Yau variety. For  $k \gg 1$ , the line bundle  $L_1 \otimes L_2^k$  is very ample. Hence, as mentioned in Remark 4.2.2, we can use Proposition 4.2.1. Then, there

exists an  $(n + 1)$ -dimensional singularity  $(X; x)$  and a partial resolution:

$$\begin{array}{ccc} T & \hookrightarrow & Y \\ \downarrow & & \downarrow \phi \\ x & \hookrightarrow & X \end{array}$$

satisfying the following statements:

1.  $T$  is a Cartier divisor on  $Y$ .
2. The normal bundle of  $T$  in  $Y$  is  $L_1^{-1} \otimes L_2^{-k}$ .
3. We have an isomorphism  $\pi_1(Y) \simeq \pi_1(|\mathcal{P}|)$ .
4. The kernel of  $\pi_1^{\text{loc}}(X; x) \twoheadrightarrow \pi_1(Y)$  is cyclic, central, and generated by any loop around an irreducible component of  $T$ .
5. The singularity  $(X; x)$  is log canonical.

Since  $T$  and  $Y$  are smooth along  $P_i$ , the restriction of the normal bundle to  $P_i$  are lens spaces  $S_i$ , with  $|\pi_1(S_i)| = c_1(N) \cap P_i$ . The order of any loop around  $T$  divides  $|\pi_1(S_i)|$ .

The restriction of  $L_1$  to  $P_1$  is  $\mathcal{O}(2)$  and its restriction to  $P_2$  is  $\mathcal{O}(1)$ . Furthermore, the restriction of  $L_2$  to  $P_1$  is  $\mathcal{O}(1)$  and its restriction to  $P_2$  is  $\mathcal{O}(1)$ . Hence, the restrictions of the normal bundle  $L_1 \otimes L_2^k$  to these curves are  $\mathcal{O}(k)$  and  $\mathcal{O}(k + 1)$ . Thus, the Lens spaces  $S_1$  and  $S_2$  have coprime orders, therefore all the loops around  $T$  are trivial in the local fundamental group of  $(X; x)$ . Therefore, the kernel of  $\pi_1^{\text{loc}}(X; x) \twoheadrightarrow \pi_1(Y)$  is trivial, so we have the following isomorphisms:

$$\pi_1^{\text{loc}}(X; x) \cong \pi_1(Y) \cong \pi_1(|\mathcal{P}|)$$

□

## Chapter 5

# Threefold log canonical singularities

In this chapter we will study the local fundamental groups of isolated threefold log canonical singularities. The objective will be twofold, for the fundamental groups of threefold isolated log canonical singularities we will show a family of groups that can be obtained and a family that cannot occur.

### 5.1 Surface groups in 3-dimensional singularities

In this section we will be using the main result of Chapter 4 to construct isolated threefold log canonical singularities with surface fundamental groups:

**Theorem 5.1.1.** *Let  $S$  be a compact 2-dimensional manifold without boundary. Then, there exists an isolated 3-fold log canonical singularity  $(X; x)$  for which  $\pi_1^{\text{loc}}(X; x) \simeq \pi_1(S)$ .*

*Proof of Theorem 5.1.1.* Let  $S$  be a 2-dimensional manifold and  $G_S$  be its fundamental group. Let  $\mathcal{T}$  be a triangulation of  $S$ . We can consider  $\mathcal{T}$  as a polyhedral complex where each triangle is given by

$$T = \text{conv}\{(0, 0), (0, 3), (3, 0)\} \subset \mathbb{Q}^2.$$

We may assume that each vertex of  $\mathcal{T}$  is contained in at most 7 triangles. Let  $v$  be a vertex of degree  $d$  at least 4. Let  $P_d$  be a smooth lattice polygon with no lattice points in the relative interior of its edges. Let  $e_1, \dots, e_d$  be its edges. Let  $T_1, \dots, T_d$  be the triangles containing  $v$ .

We replace  $T_i$  with  $T'_i$ , where:

1. If  $v = (0, 0) \in T_i$ , then we replace  $T_i$  with  $T_i \cap H_1^+$ , where  $H_1 = \{(x, y) \mid x + y \geq 1\}$ . By doing so, we delete the vertex  $(0, 0)$  and add the vertices  $(1, 0)$  and  $(0, 1)$ .
2. If  $v = (0, 3) \in T_i$ , then we replace  $T_i$  with  $T_i \cap H_2^-$ , where  $H_2 = \{(x, y) \mid y = 2\}$ . By doing so, we delete the vertex  $(0, 3)$  and add the vertices  $(1, 2)$  and  $(0, 2)$ .
3. If  $v = (3, 0) \in T_i$ , then we replace  $T_i$  with  $T_i \cap H_3^-$ , where  $H_3 = \{(x, y) \mid x = 2\}$ . By doing so, we delete the vertex  $(3, 0)$  and add the vertices  $(2, 0)$  and  $(2, 1)$ .

Let us call  $f_i$  the edge of  $T'_i$  joining the two new vertices. For each  $i \in \{1, \dots, d\}$  we can consider a lattice isomorphism that glues  $e_i$  with  $f_i$ . By doing so, we obtained a 2-dimensional polyhedral complex  $\mathcal{T}'$  homotopic to  $T$ , that does not contain the singular point  $v$  of  $\mathcal{T}$  and has no other singular point introduced. Since our construction works for any singular point of  $T$ , even after doing it for any other vertex. Performing this construction for all the singular vertices, yields a 2-dimensional smooth polyhedral complex  $\mathcal{P}_S$  that is homotopic to  $S$ . Then, by Theorem 4.0.1, we obtain an isolated 3-dimensional lc singularity  $(X_S; x)$  for which the isomorphism  $\pi_1^{\text{loc}}(X_S; x) \simeq G_S$  holds.  $\square$

## 5.2 Free groups in 3-dimensional singularities

In this section we prove that no free group with 2 or more generators appears as the local fundamental group of an isolated log canonical 3-fold singularity.

**Theorem 5.2.1.** *No isolated 3-fold log canonical singularity  $(X; x)$  satisfy that  $\pi^{\text{loc}}(X; x) \simeq F_r$  with  $r \geq 2$ .*

In order to prove this theorem, we will need the following lemma:

**Lemma 5.2.2.** *Let  $(X; x)$  be an isolated lc 3-fold singularity. There exists a  $\mathbb{Q}$ -factorial dlt modification  $\phi: Y \rightarrow X$  and a divisor  $E_Y$  fully supported on  $\text{Ex}(\phi)$  that is effective and satisfy: for every movable curve  $C_i$  on an irreducible component  $E_i$  of  $E$ , we have that  $-E_Y \cdot C_i > 0$ .*

*Proof.* Let  $\psi: X' \rightarrow X$  be a log resolution obtained by blowing-up centers of codimension at least 2. Then, there exists an effective divisor  $E_{X'}$  supported on  $E_0 := \text{Ex}(\psi)$  such that  $-E_{X'}$  is ample over the base. We run a  $(K_{X'} + E_0)$ -MMP over  $X$  that terminates with a dlt modification  $(Y, E)$ . Let  $X'' \rightarrow X'$  be a resolution for which there exists a morphism  $p: X'' \rightarrow Y$ . Therefore, there exists an effective divisor  $E_{X''}$  supported on  $\text{Ex}(X'' \rightarrow X)$  for which  $-E_{X''}$  is ample over  $X$ . We set  $E_Y$  to

be the push-forward of  $E_{X''}$  to  $Y$ . Then, the statement follows from the negativity lemma. Indeed, we can write

$$p^*(-E_Y) = -E_{X''} + F,$$

where  $F$  is an effective divisor exceptional over  $Y$ . Let  $C_i$  be a movable curve on  $E_i$ . Then,  $C_i$  is not contained in  $p(F)$  as  $p(F)$  has codimension at least two in  $Y$ . Let  $C_i''$  be the strict transform of  $C_i$  on  $X''$ . The following equalities hold by the projection formula and the fact that  $C_i''$  is not contained in the support of  $F$

$$-E_Y \cdot C_i = p^*(-E_Y) \cdot C_i = (-E_{X''} + F) \cdot C_i \geq -E_{X''} \cdot C_i > 0.$$

This finishes the proof of the lemma.  $\square$

*Proof of Theorem 5.2.1.* Note that any  $F_r$  has  $F_{2(r-1)+1}$  as a normal subgroup of index 2. Thus, we may assume that  $r$  is an odd number greater than 3.

Let  $(X; x)$  be a 3-dimensional isolated lc singularity. We assume that  $\pi_1^{\text{loc}}(X; x) \simeq F_r$  for some odd number  $r \geq 3$ . Let  $(X'; x') \rightarrow (X; x)$  be the index one cover of  $K_X$ . Then  $(X'; x')$  is again an isolated log canonical singularity with  $K_X'$  Cartier. Note that we have an exact sequence

$$1 \rightarrow \pi_1^{\text{loc}}(X'; x') \rightarrow \pi_1^{\text{loc}}(X; x) \rightarrow \mathbb{Z}/k\mathbb{Z} \rightarrow 1,$$

so the local fundamental group of  $(X'; x')$  is free with at least 3 generators. We may replace  $(X; x)$  with  $(X'; x')$  so we can assume that  $K_X$  is Cartier.

If  $(X; x)$  is klt, then its fundamental group is finite due to Theorem 2.4.13. Hence, we may assume that  $(X; x)$  is log canonical,  $K_X$  is Cartier, and  $\{x\}$  is a log canonical center. In this case, the dual complex  $\mathcal{D}(X; x)$  is a manifold of dimension at most 2.

Let  $(Y, E)$  be a  $\mathbb{Q}$ -factorial dlt modification of  $(X; x)$ . We let  $E_1, \dots, E_r$  be the irreducible components of  $E$  and  $\gamma_i$  be a loop around the divisor  $E_i$ . We have a sequence of surjective homomorphisms and isomorphisms:

$$\pi_1^{\text{loc}}(X; x) \simeq \pi_1(Y \setminus E) \xrightarrow{\psi_1} \pi_1(Y \setminus Y^{\text{sing}}) \xrightarrow{\psi_2} \pi_1(Y) \simeq \pi_1(E) \xrightarrow{\psi_3} \pi_1(\mathcal{D}(E)) \simeq \pi_1(\mathcal{D}(X; x)).$$

The kernel of  $\psi_1$  is the smallest normal group containing  $\gamma_i$  for  $i \in \{1, \dots, r\}$ . The kernel of  $\psi_2$  is generated by torsion elements as  $Y$  has isolated klt singularities and klt singularities have finite local fundamental groups [3, Theorem 1]. The kernel of  $\psi_3$  is the smallest normal group containing

the image of  $\pi_1(E_i) \rightarrow \pi_1(E)$  for each irreducible component  $E_i$ . We will proceed in three different cases depending on the coregularity of  $(X; x)$ .

*Case 1:* In this case, we assume that  $\text{coreg}(X; x) = 2$ .

In this case,  $E$  only has one component. Thus, the kernel of  $\psi_1$  is normally generated by a single loop  $\gamma_1$ . We conclude that the group  $\pi_1(Y \setminus Y^{\text{sing}})$  is a one-relator group. So, there is an isomorphism

$$\pi_1(Y \setminus Y^{\text{sing}}) \simeq \langle x_1, \dots, x_r \mid s \rangle.$$

By the Magnus-Karras-Solitar theorem [20, Theorem 1], every torsion element of  $\pi_1(Y \setminus Y^{\text{sing}})$  has the form  $nrn^{-1}$  where  $n \in \pi_1(Y \setminus Y^{\text{sing}})$  and  $r^k = s$  for some integer  $k$ . Since the kernel of  $\psi_2$  is torsion, we conclude that  $\pi_1(E)$  is a one-relator group as well. Observe that in this case,  $E$  is a Calabi-Yau surface with canonical singularities. Hence, its fundamental group is isomorphic to the fundamental group of a smooth Calabi-Yau surface. We use the classification of such surfaces, to show that their fundamental groups cannot be a one-relator group with at least 3 generators. Indeed, the fundamental group of a smooth Calabi-Yau surface is either:

1. trivial,
2. a finite cyclic group,
3. the free abelian group  $\mathbb{Z}^4$ , or
4. an extension of  $\mathbb{Z}^4$  by a finite cyclic or a finite bi-cyclic group.

The first two cases have rank at most 2 and are not free, so they are not one-relators with at least 3 generators. By [33], we know that an abelian finitely generated subgroup of a one-relator group has rank at most 2. We conclude that the third and fourth cases cannot be one-relator groups. This leads to a contradiction.

*Case 2:* In this case, we assume that  $\text{coreg}(X; x) = 1$ .

We show that the loops  $\gamma_i$  and  $\gamma_j$  commute. We run a  $(K_Y + E - E_i)$ -MMP over  $X$ . This minimal model program terminates with a good minimal model  $\phi': Y' \rightarrow X$ . Let  $E'_i$  be the strict transform of  $E_i$  on  $Y'$ . In the minimal model  $Y'$  the divisor  $-E'_i$  is nef over  $X$ . We claim that  $\phi'^{-1}(x) = E'_i$

holds set-theoretically. Indeed, if there is another divisor  $E'_j$  in this fiber, then by connectedness we may assume  $E'_i \cap E'_j \neq \emptyset$ . Thus, a general ample curve<sup>1</sup> in  $E'_j$  will intersect  $E'_i$  positively and hence will intersect  $-E'_i$  negatively. We conclude that, at some step of this minimal model program, we lead to a model  $Y''$  on which the strict transform of  $E_i$  and  $E_j$  intersect along a codimension 2 subvariety. We write  $\phi'': Y'' \rightarrow X$  for the associated projective morphism. We let  $E''_i$  and  $E''_j$  the strict transform of  $E_i$  and  $E_j$  on  $Y''$ , respectively. We let  $\gamma''_i$  and  $\gamma''_j$  be the loops around  $E''_i$  and  $E''_j$ . Note that at the generic point of  $Z$ , the variety  $Y''$  has a toric surface singularity. In particular, the local fundamental group  $\pi_1^{\text{loc}}(Y''; \eta_Z)$  is abelian. We can move homotopically the loops  $\gamma''_i$  and  $\gamma''_j$  so that the corresponding circles around  $E''_i$  and  $E''_j$  are in a small neighborhood of  $Z$ . Hence, the loops  $\gamma''_i$  and  $\gamma''_j$  commute as loops in  $\pi_1(Y'' \setminus \phi''^{-1}(x))$ . There is a natural isomorphism

$$\pi_1(Y \setminus E) \rightarrow \pi_1(Y'' \setminus \phi''^{-1}(x)),$$

that sends  $\gamma_i$  (resp.  $\gamma_j$ ) to  $\gamma''_i$  (resp.  $\gamma''_j$ ). We conclude that  $\gamma_i$  and  $\gamma_j$  commute.

Now, we turn to prove that every  $\gamma_i$  is a torsion element of  $\pi_1(Y \setminus E)$ . Observe that each surface  $E_i$  admits a morphism  $E_i \rightarrow C$  to an elliptic curve  $C$  with general fiber a rational curve. Let  $C_i$  be a general fiber of  $E_i \rightarrow C$ . Let  $-m_i := C_i \cdot E_i$ . Then, we have a relation of the form  $\gamma_i^{m_i} = \gamma_{i-1} \gamma_{i+1}$  in the fundamental group  $\pi_1(Y \setminus E)$ . Indeed, this relation holds in the restriction of the normal bundle to  $C_i$ . The  $r \times r$  matrix  $M$  with entries  $M_{i,j} := E_i \cdot C_j$  is invertible. This fact and the commutativity of the loops  $\gamma_i$  implies that there exists  $k \gg 0$  such that  $\gamma_i^k = 1$  for each  $i \in \{1, \dots, r\}$ . We are assuming that  $\pi_1(Y \setminus E)$  is free. Hence, it has no non-trivial torsion elements. Since the kernel of  $\psi_1$  is normally generated by torsion elements, we conclude that  $\psi_1$  is an isomorphism. Since the kernel of  $\psi_2$  is torsion, again we conclude that  $\psi_2$  is an isomorphism. Hence, we get that  $\pi_1(E) \simeq F_r$  for some  $r \geq 3$ .

Finally, we turn to analyze the kernel of  $\psi_3$  in this case. Note that  $\pi_1(\mathcal{D}(E)) \simeq \mathbb{Z}$ . By Lemma 2.4.10, we conclude that  $\pi_1(E_i) \simeq \pi_1(C) \simeq \mathbb{Z}^2$  for each  $i$ . Since  $C$  has a lifting to  $E_i$  via the isomorphism  $C \rightarrow E_i \cap E_{i-1}$ , we conclude that the image of every homomorphism  $\pi_1(E_i) \rightarrow \pi_1(E)$  consists of the same two commuting loops in  $\pi_1(E)$ . By the previous paragraph, we have that  $\pi_1(E) \simeq F_r$  so these two commuting elements have the form  $g^s$  and  $g^k$  for some  $g \in F_r$ . In particular, we obtain a presentation  $\mathbb{Z} \simeq \langle x_1, x_2, \dots, x_r \mid s \rangle$  with a single relation  $s$  and  $r \geq 3$ . This leads to a contradiction.

---

<sup>1</sup>We say that  $C$  is a *general ample curve* in an  $n$ -dimensional variety if it is the intersection of  $(n-1)$  general effective ample divisors.

*Case 3:* In this case, we assume that  $\text{coreg}(X; x) = 0$ . In particular,  $\mathcal{D}(X; x)$  is a closed 2-manifold.

As in the previous case, the loops  $\gamma_i$  around the components  $E_i$  commute. Hence the subgroup generated by this loops has to be trivial or  $\mathbb{Z}$ . Therefore the kernel of  $\psi_1$  is normally generated by a single element. We conclude that the group  $\pi_1(Y \setminus Y^{\text{sing}})$  is a one-relator group with  $r$  generators. As in the first case, by the Magnus-Karras-Solitar Theorem, the group  $\pi_1(E)$  must also be a one-relator group with rank  $r$  or a free group of rank  $r - 1$ .

Each component  $E_i$  of  $E$  is simply connected. Indeed the minimal resolution of  $E_i$  is a smooth rationally connected projective surface. This implies that  $\psi_3$  is an isomorphism. Hence  $\pi_1(D(E))$  is a free group or a one-relator group with odd rank. Either case is impossible for the fundamental groups of closed 2-manifolds, hence we have a contradiction.  $\square$

## Chapter 6

# Fourfold log canonical singularities

In this chapter we will construct isolated fourfold log canonical singularities with a wide variety of local fundamental groups

The main result of this chapter will be the following:

**Theorem 6.0.1.** *Let  $M$  be a connected 3-dimensional manifold without boundary, smoothly embedded in  $\mathbb{R}^4$ . There exists a 4-dimensional isolated lc singularity  $(X; x)$  for which  $\pi_1^{\text{loc}}(X; x) \simeq \pi_1(M)$ . In particular, every free group appears as the fundamental group of an isolated 4-dimensional lc singularity.*

In order to prove this we will use the main result of Chapter 4 to reduce this to the construction of a special smooth polyhedral complex:

**Theorem 6.0.2.** *Let  $M$  be a connected 3-dimensional manifold without boundary, smoothly embedded in  $\mathbb{R}^4$ . There exists a smooth 3-dimensional polyhedral complex  $\mathcal{P}_r$  for which  $\pi_1(|\mathcal{P}|_r) = \pi_1(M)$ .*

### 6.1 The Freudenthal decomposition

In this section, we introduce the Freudenthal decomposition of  $\mathbb{R}^4$  and prove a property of its dual. This decomposition will be our starting point to go from 3-dimensional real manifolds to smooth 3-dimensional polyhedral complexes.

**Definition 6.1.1.** For a cube  $C$  with coordinates  $x_j \in [i_j, i_j + 1]$  for  $j \in \{1, 2, 3, 4\}$ , the *Freudenthal decomposition* of  $C$  is defined in the following way. For each path  $P$  in the edges of  $C$  that:

1. starts in  $(i_1, i_2, i_3, i_4)$ ,

2. ends in  $(i_1 + 1, i_2 + 1, i_3 + 1, i_4 + 1)$ , and
3. is non-decreasing in each coordinate

we take the simplex with the vertices that are in the path  $P$  and the cube  $C$ . The decomposition of  $\mathbb{R}^4$  obtained by taking the Freudenthal decomposition of each unit cube is called the *Freudenthal decomposition of  $\mathbb{R}^4$*  or the *F-decomposition of  $\mathbb{R}^4$* .

**Definition 6.1.2.** For each 4-dimensional simplex  $\Delta$  in the  $F$ -decomposition of  $\mathbb{R}^4$  define  $B_\Delta$  to be its barycenter. For each simplex  $\Delta$  in the  $F$ -decomposition, we define its *dual polyhedron*  $\Delta^\vee$  to be the convex hull of the vertices  $B_{\Delta_i}$ , for each  $\Delta \subseteq \Delta_i$ .

**Lemma 6.1.3.** *Let  $\Delta$  be a  $k$ -dimensional simplex of the  $F$ -decomposition of  $\mathbb{R}^4$ . The dual  $\Delta^\vee$  is a  $(4 - k)$ -dimensional polyhedron whose vertices are exactly the  $B_{\Delta_i}$  for each  $\Delta \subset \Delta_i$ . Moreover, the dual polyhedra tile  $\mathbb{R}^4$ .*

*Proof.* The dual of the  $F$ -decomposition of  $\mathbb{R}^4$  is the projection of the Voronoi diagram of the  $A_4$  lattice in  $\mathbb{R}^5$ , obtained by deleting the last coordinate. See [7, Chapter 21, Theorem 7] for more details.

□

**Definition 6.1.4.** Let  $\mathcal{F}$  be the  $F$ -decomposition of  $\mathbb{R}^4$ . By Lemma 6.1.3, the set  $\{\Delta^\vee \mid \Delta \in \mathcal{F}\}$  gives a convex polyhedral decomposition of  $\mathbb{R}^4$ . This decomposition will be called the *Freudenthal dual decomposition of  $\mathbb{R}^4$*  or simply the  *$F$ -dual decomposition*. Note that both the  $F$ -decomposition and the  $F$ -dual decomposition can be regarded as complexes. For the  $F$ -decomposition, we can use the lattice  $\mathbb{Z}^4$ , while for the  $F$ -dual decomposition, we can use the lattice  $(\frac{1}{5}\mathbb{Z})^4$ .

**Definition 6.1.5.** Let  $M$  be a subcomplex of the Freudenthal decomposition in  $\mathbb{R}^4$ . We define the *dual  $M^\vee$*  of  $M$  in the following way: For each vertex  $v \in M$  the dual polyhedron  $v^\vee$  is in  $M^\vee$ . We write  $\partial M^\vee$  for the boundary of the dual.

## 6.2 Subcomplexes of the Freudenthal decomposition

In this section, we prove the following proposition regarding subcomplexes of the  $F$ -decomposition.

**Proposition 6.2.1.** *Let  $M$  be a subcomplex of the Freudenthal decomposition in  $\mathbb{R}^4$ . Let  $M_0$  be the boundary of the dual of  $M$ . Then, there exists a 3-dimensional smooth polyhedral complex  $M'$  that is homotopy equivalent to  $M_0$ .*

*Proof.* We will prove the statement in several steps. First, we will study the polyhedral decomposition of  $M_0$ . Then, we will proceed by performing blow-ups on  $M_0$  in order to obtain a smooth polyhedral complex. Recall from Remark 2.5.11 that blow-ups may not exist in the general setting. Hence, we will need to carefully analyze the nerves at each simplex to argue that the blow-up does exist in this setting.

*Step 1:* In this step, we study the polyhedra in  $M_0$  and its nerves. We show the following statements:

- (i) The maximal polyhedra of  $M_0$  are truncated octahedrons and hexagonal prisms.
- (ii) The nerves of the vertices of  $M_0$  are 3-dimensional simplices or triangular prisms.
- (iii) The nerves of the edges of  $M_0$  are triangles or quadrilaterals.

The maximal polyhedra in  $M_0$  are 3-dimensional faces of the dual Freudenthal decomposition of  $\mathbb{R}^4$ . Thus, they are three-dimensional faces of the 4-dimensional permutahedron, i.e. truncated octahedra or hexagonal prisms. This shows (i).

A 3-dimensional polyhedron of the dual Freudenthal decomposition is in  $M_0$  if and only if it is a face of a 4-dimensional polyhedron not in  $M^\vee$  and a face of a 4-dimensional polyhedron in  $M^\vee$ . Therefore, a 3-dimensional polyhedron is in  $M_0$  if its dual edge has one vertex in  $M$  and one vertex not in  $M$ . For a vertex  $v$  in  $M_0$ , its dual in the Freudenthal decomposition is a 4-dimensional simplex  $\Delta$ . The maximal polyhedra of  $M_0$  containing  $v$  correspond to the edges of  $\Delta$  that have one vertex in  $M$  and the other vertex not in  $M$ . Two of these maximal polyhedra intersect in a 1 or 2-dimensional cell if the corresponding edges share a 3 or 2-dimensional cell in  $\Delta$ , respectively. The vertices in the nerve of  $v$  are connected by an edge if the corresponding maximal polyhedra of  $M_0$  share a 2-dimensional cell. This is the same as asking for the corresponding edges in the Freudenthal decomposition to be in the same 2-dimensional cell. Since  $\Delta$  has 5 vertices there are combinatorially 3 possibilities:

1. The simplex  $\Delta$  contains 0 or 5 vertices of  $M$ . Then no maximal polyhedron of  $M_0$  contains  $p$ .  
So, the point  $v$  is not in  $M_0$ .
2. The simplex  $\Delta$  contains 1 or 4 vertices of  $M$ . Then exactly 4 edges of  $\Delta$  have one vertex in  $M$  and one vertex outside of  $M$ . Since all these edges in  $\Delta$  share a common vertex, each pair

is contained in a common triangle. Therefore, in the nerve of  $v$  in  $M_0$  all the points are joined by an edge, hence it is a 3-dimensional simplex.

3. The simplex  $\Delta$  contains 2 or 3 points of  $M$ . Then exactly 6 edges of  $\Delta$  have one vertex in  $M$  and one vertex outside of  $M$ . Call  $A, B, 1, 2, 3$  the points of  $\Delta$ . Where  $A$  and  $B$  are both inside or outside of  $M$  and the same holds for  $1, 2$ , and  $3$ . The edge formed by points  $\{1, A\}$  shares a triangle only with  $\{1, B\}$ ,  $\{2, A\}$  and  $\{3, A\}$ . We can proceed similarly for each other of the 6 edges. Hence, the nerve of  $v$  in  $M_0$  is a triangular prism.

This finishes the proof of (ii).

For an edge  $E$  in  $M_0$  its dual in the Freudenthal decomposition is a 3-dimensional simplex  $\Delta$ . The maximal polyhedra of  $M_0$  containing  $E$  correspond to the edges of  $\Delta$  that have one vertex in  $M$  and the other vertex not in  $M$ . Two of these maximal polyhedra intersect in a 2-dimensional cell if the corresponding edges share a 2-dimensional cell in  $\Delta$ . Hence, the vertices in the nerve of  $E$  are connected by an edge if the corresponding edges in the Freudenthal decomposition are in the same 2-dimensional cell.

Since  $\Delta$  has 4 vertices there are combinatorially 3 possibilities:

1. The simplex  $\Delta$  contains 0 or 4 vertices of  $M$ . Then, no maximal polyhedron of  $M_0$  contains  $v$ . So, the vertex  $v$  is not in  $M_0$ .
2. The simplex  $\Delta$  contains 1 or 3 vertices of  $M$ . Hence, exactly 3 edges of  $\Delta$  have one vertex in  $M$  and one vertex outside of  $M$ . Therefore, the nerve of  $E$  in  $M_0$  is a 2-dimensional simplex.
3. The simplex  $\Delta$  contains 2 vertices of  $M$ . Hence, exactly 4 edges of  $\Delta$  have one vertex in  $M$  and one vertex outside of  $M$ . Therefore, the nerve of  $E$  in  $M_0$  is a quadrilateral.

This finishes the proof of (iii).

*Step 2:* In this step, we show that the neighborhood of any non-smooth vertices of  $M_0$  admits an embedding in  $\mathbb{Q}^3$  as a strongly polytopal fan.

A vertex  $v$  in  $M_0$  corresponds to a simplex  $\Delta$  in the Freudenthal decomposition that has 1, 2, 3 or 4 vertices in  $M$ . If it corresponds to a non-smooth vertex, then it must have 2 or 3 vertices in  $M$ , without loss of generality, we will assume that two vertices are in  $M$ . Up to a permutation of

coordinates, we can assume that the simplex has vertices  $(0, 0, 0, 0)$ ,  $(1, 0, 0, 0)$ ,  $(1, 1, 0, 0)$ ,  $(1, 1, 1, 0)$ , and  $(1, 1, 1, 1)$ .

The vertex  $v$  in  $M_0$ , then has coordinates  $\frac{1}{5}(4, 3, 2, 1)$ . The vertices of  $M_0$  that have edges connecting to  $v$  will correspond to the barycenters of 4-dimensional simplices sharing a 3-dimensional simplex with  $\Delta$ . Hence, they will be one of the following:

$$\frac{1}{5}(3, 2, 1, -1), \frac{1}{5}(4, 3, 1, 2), \frac{1}{5}(4, 2, 3, 1), \frac{1}{5}(3, 4, 2, 1), \text{ and } \frac{1}{5}(6, 4, 3, 2).$$

Therefore, after translating to the origin the lattice generators of the edges in  $M_0$  with vertex  $v$  will be:

$$A := \frac{1}{5}(-1, -1, -1, -2), B := \frac{1}{5}(0, 0, -1, 1), C := \frac{1}{5}(0, -1, 1, 0), \\ D := \frac{1}{5}(-1, 1, 0, 0), \text{ and } E := \frac{1}{5}(2, 1, 1, 1).$$

As all the cones are smooth, defining linear maps for each of the cones to a common  $\mathbb{Z}^4$  is the same as defining an element in  $\mathbb{Q}^3$  for each lattice ray generator in  $\mathbb{Z}^4$ . The edges of  $\Delta$  that have one vertex in  $M$  and one vertex not in  $M$  correspond to the 3-dimensional polytopes in  $M_0$ . Up to symmetry with respect to the origin, there are 6 possibilities for the 2 vertices in  $M$ .

1. The vertices are  $(0, 0, 0, 0)$  and  $(1, 0, 0, 0)$ . Hence, the cones in  $M_0$ , will have generators:  $BDE$ ,  $BCE$ ,  $BCD$ ,  $ADE$ ,  $ACE$ , and  $ACD$ . In this case, we define the map as mapping the vertices  $A$ ,  $B$ ,  $C$ ,  $D$ , and  $E$  to  $(0, 0, 1)$ ,  $(0, 0, -1)$ ,  $(1, 0, 0)$ ,  $(0, 1, 0)$ , and  $(-1, -1, 0)$ , respectively.
2. The vertices are  $(0, 0, 0, 0)$  and  $(1, 1, 0, 0)$ . So, the cones in  $M_0$  have generators:  $CDE$ ,  $BCE$ ,  $BCD$ ,  $ADE$ ,  $ABE$ , and  $ABD$ . We can define the map as sending the vertices  $A$ ,  $B$ ,  $C$ ,  $D$ , and  $E$  to  $(0, 0, 1)$ ,  $(1, 0, 0)$ ,  $(0, 0, -1)$ ,  $(0, 1, 0)$ , and  $(-1, -1, 0)$ , respectively.
3. The vertices are  $(0, 0, 0, 0)$  and  $(1, 1, 1, 0)$ . The cones in  $M_0$  have generators:  $CDE$ ,  $BDE$ ,  $BCD$ ,  $ACE$ ,  $ABE$  and  $ABC$ . We consider the map sending the vertices  $A$ ,  $B$ ,  $C$ ,  $D$ , and  $E$  to  $(0, 0, 1)$ ,  $(1, 0, 0)$ ,  $(0, 1, 0)$ ,  $(0, 0, -1)$ , and  $(-1, -1, 0)$ , respectively.
4. The vertices are  $(0, 0, 0, 0)$  and  $(1, 1, 1, 1)$ . Hence, the cones in  $M_0$ , will have generators:  $CDE$ ,  $BDE$ ,  $BCE$ ,  $ACD$ ,  $ABD$ , and  $ABC$ . We define the map as sending the vertices  $A$ ,  $B$ ,  $C$ ,  $D$  and  $E$  to  $(0, 0, 1)$ ,  $(1, 0, 0)$ ,  $(0, 1, 0)$ ,  $(-1, -1, 0)$ , and  $(0, 0, -1)$ , respectively.
5. The vertices are  $(1, 0, 0, 0)$  and  $(1, 1, 0, 0)$ . The cones in  $M_0$  have generators:  $CDE$ ,  $ACE$ ,

$ACD$ ,  $BDE$ ,  $ABE$ , and  $ABD$ . We consider the morphism that maps the vertices  $A$ ,  $B$ ,  $C$ ,  $D$ , and  $E$  to  $(1, 0, 0)$ ,  $(0, 0, 1)$ ,  $(0, 0, -1)$ ,  $(0, 1, 0)$ , and  $(-1, -1, 0)$ , respectively.

6. The vertices are  $(1, 0, 0, 0)$  and  $(1, 1, 1, 0)$ . Hence, the cones in  $M_0$  will have generators:  $DCE$ ,  $ADE$ ,  $ACD$ ,  $BCE$ ,  $ABE$  and  $ABC$ . We define the map as sending the vertices  $A$ ,  $B$ ,  $C$ ,  $D$ , and  $E$  to  $(1, 0, 0)$ ,  $(0, 0, 1)$ ,  $(0, 1, 0)$ ,  $(0, 0, -1)$ , and  $(-1, -1, 0)$ , respectively.

In any case, we have lattice maps that send each of the cones in  $\mathcal{P}$  with vertex  $v$  to a common fan in  $\mathbb{Z}^3$ . As the vertices  $(0, 0, 1)$ ,  $(0, 0, -1)$ ,  $(1, 0, 0)$ ,  $(0, 1, 0)$ , and  $(-1, -1, 0)$  form a convex bipyramid, the fan in  $\mathbb{Z}^3$  is strongly polytopal.

*Step 3:* In this step, we produce a polyhedral complex  $M_1$  obtained from  $M_0$  by a sequence of blow-ups at points.

By Lemma 2.5.14 and Step 2, we can perform blow-ups at each non-smooth point of  $2M$ . The construction of the blow-up in the proof of Lemma 2.5.14 depends on the choice of: an embedding into  $\mathbb{Q}^3$ , a piecewise linear function  $\mathbb{Q}^3 \rightarrow \mathbb{Q}^4$ , and a hyperplane  $H$  in  $\mathbb{Q}^4$ . By the second step, we can choose the same embedding, piecewise linear function, and hyperplane for every two vertices in  $M_0$  with isomorphic neighborhoods. By doing so, we produce a polyhedral complex  $M_1$  where all the non-smooth edges are disjoint.

*Step 4:* In this step, we blow-up at the edges of  $M_1$  to obtain a smooth complex.

To do so, we show that the blow-up of  $M_1$  at any edge with nerve a quadrilateral exists.

In  $M_0$  the only nerves of edges that are not 2-dimensional simplices are quadrilaterals. The complex  $M_1$  is obtained by performing blow-ups at vertices of  $M_0$  and all the nerves at 2-dimensional cells are simplices. Therefore, by Lemma 2.5.15, all the new edges in  $M_1$  have simplices as nerves. Hence, by Step 1, the only edges in  $M_1$ , without simplices as nerves were edges in  $M_0$  without simplices as nerves. Such edge had non-simplicial nerves in both vertices, hence there were blow-ups performed in  $M_0$  at both of these vertices.

Let  $L$  be one such edge, with vertices  $A$  and  $B$ . In  $M_1$  this edge is contained in four 3-dimensional simplices. The points  $A$  and  $B$  are contained in these four 3-dimensional simplices and also in polyhedrons  $P_A$  and  $P_B$ , respectively. The polyhedrons  $P_A$  and  $P_B$  are the result of the blow-ups at points  $A'$  and  $B'$  in  $M_0$ .

The cone at  $P_A$  with vertex  $A$  and the cone at  $P_B$  with vertex  $B$  are isomorphic by the choice of blow-ups in Step 2. We take a hyperplane  $H_A$  in  $P_A$  separating  $A$  and the other vertices in  $P_A$ . We define a hyperplane  $H_B$  in  $P_B$  by the same equations that define  $H_A$  in  $P_A$ . The intersection points of  $H_A \cap P_A$  and  $H_B \cap P_B$  define four points in each of the four 3-dimensional polyhedra that contain  $L$ . In each of these 3-dimensional polyhedra, the 2 points coming from  $R_A$  are joined to the 2 points coming from  $R_B$  with lines parallel to  $L$ . Therefore, these four points define a parallelogram. Therefore, we define the blow-up at  $L$  by cutting with the hyperplanes defined by the parallelograms. The edge  $L$  is replaced with the quadrilateral prism with faces given by the parallelograms in the 3-dimensional polyhedra containing  $L$  and the quadrilaterals in  $H_A$  and  $H_B$ . This is the prism over the quadrilateral  $H_A \cap P_A \cong H_B \cap P_B$ .

In  $M_1$  the only strata with non-simplicial nerves are disjoint edges and the points in these edges. Hence, after blowing-up these edges, by Lemma 2.5.15, all the nerves are simplicial. Therefore, after performing the aforementioned blow-up at each non-smooth edge, we end up with  $M_2$  a smooth polyhedral complex.

*Step 5:* In this step, we define  $M' := M_2$  and finish the proof.

Observe that  $M'$  is a smooth polyhedral complex of dimension 3 by construction. Furthermore, since blow-ups are simply homotopy equivalences, we conclude that  $M'$  is homotopic to  $M_0$ . This finishes the proof.  $\square$

### 6.3 Free fundamental groups of 4-dimensional singularities

In this section, we show that every free group appears as the fundamental group of a log canonical 4-dimensional singularity.

*Proof of Theorem 6.0.2.* As  $M$  is compact and smooth in  $\mathbb{R}^4$ , there exists large enough  $N$ , such that the union of the lattice cubes of size  $\frac{1}{N}$  that intersect  $M$  gives a set homotopic to a tubular neighborhood of  $M$ . Equivalently dilating  $M$  by a factor of  $N$  and taking all the unit cubes intersecting  $M_N$  gives us a subcomplex of the  $F$ -decomposition of  $\mathbb{R}^4$  homotopic to a tubular neighborhood of  $M$ . Call  $M'$  this subcomplex of the  $F$ -decomposition. The boundary of the dual complex of  $M'$  in the  $F$ -dual decomposition of  $\mathbb{R}^4$  is two disjoint complexes, homotopic to  $M$ . Let  $M_0$  be one of these subcomplexes.

By Proposition 6.2.1 there exists a smooth 3-dimensional polyhedral complex  $\mathcal{P}_r$  for which  $\pi_1(|\mathcal{P}|_r) = \pi_1(M)$ . This finishes the proof.  $\square$

*Proof of Theorem 6.0.1.* This follows from Theorem 6.0.2 and Theorem 4.0.1.  $\square$

## Part II

**Fundamental Groups of low  
coregularity Calabi-Yau type pairs.**

## Chapter 7

# Fundamental Groups and coregularity

The main result of this chapter is the following Theorem:

**Theorem 7.0.1.** *Let  $(X, D)$  be a projective klt pair with absolute coregularity  $c \leq 2$ . Then  $\pi_1^{\text{sm}}(X, D)$  is virtually abelian of rank at most  $2c$ .*

An important particular case of this theorem is the following:

**Corollary 7.0.2.** *Let  $(X, D)$  be a projective klt pair of absolute coregularity 0. Then  $\pi_1^{\text{sm}}(X, D)$  is finite.*

The proof of the main theorem will rely on the following reduction from pairs of coregularity  $c$  to pairs of dimension  $c$ .

**Theorem 7.0.3.** *Assume that the orbifold fundamental group of the smooth locus of any klt Calabi-Yau pair with standard coefficients of dimension at most  $c$  is virtually abelian of rank at most  $2c$ . Then for any  $(X, D)$  projective klt pair with absolute coregularity at most  $c$ , the fundamental group  $\pi_1^{\text{sm}}(X, D)$  is virtually abelian of rank at most  $2c$ .*

*Proof.* We proceed by induction on the dimension, assume that  $(X, D)$  is a projective klt pair of absolute coregularity  $c$ , with  $\dim X = d$ , and that the statement of the theorem holds for pairs with  $\dim X < d$  and coregularity at most  $c$ .

As  $\pi_1^{\text{sm}}(X, D) = \pi_1^{\text{sm}}(X, D_{\text{st}})$  and  $\text{coreg}(X, D_{\text{st}}) \leq \text{coreg}(X, D)$  it is enough to prove the statement for pairs with standard coefficients. Thus, we may assume that  $D = D_{\text{st}}$

*Case 1:* In this case, we prove the statement when  $(X, D)$  is a Calabi-Yau pair.

If  $(X, D)$  is a Calabi-Yau pair, then the dimension of  $X$  is  $c$  as  $(X, D)$  is a klt pair. Since  $D$  has standard coefficients, the group  $\pi_1^{\text{sm}}(X, D)$  is virtually abelian by the statement of the theorem.

*Case 2:* In this case, we prove the statement when  $(X, D)$  is not Calabi-Yau.

If  $(X, D)$  is not Calabi-Yau, then there exists a generalized Calabi-Yau pair  $(X, B, \mathbf{M})$  of coregularity  $c$  with  $B \geq D$  and  $B - D + \mathbf{M}_X \neq 0$ .

Let  $\phi : (Y, B_Y, \mathbf{M}) \rightarrow (X, B, \mathbf{M})$  be a  $\mathbb{Q}$ -factorial gdl modification. Let  $\phi^*(K_X + D + \mathbf{M}_X) = K_Y + D_Y + \mathbf{M}_Y$ . Then by Lemma 2.4.6, we can choose  $P_Y$ , such that  $\pi_1^{\text{sm}}(Y, P_Y, \mathbf{M}) \twoheadrightarrow \pi_1^{\text{sm}}(X, D, \mathbf{M})$ .

We have that  $(Y, B_Y, \mathbf{M})$  is generalized Calabi-Yau of coregularity  $c$  and that  $(Y, D_Y, \mathbf{M})$  is generalized sub-klt. As by Lemma 2.4.6,  $D_Y \leq P_Y \leq \alpha D_Y + (1 - \alpha)(B_Y)$  for appropriate  $\alpha$ . For  $\alpha$  small enough,  $(Y, P_Y, \mathbf{M})$  is generalized klt of Calabi-Yau type with absolute coregularity at most  $c$ . In fact it has absolute coregularity exactly  $c$ , as  $P_Y \geq D_Y$  and Lemma 2.3.5 imply that

$$\text{coreg}(Y, P_Y, \mathbf{M}) \geq \text{coreg}(Y, D_Y, \mathbf{M}) \geq \text{coreg}(X, D, \mathbf{M}) = c.$$

Since  $K_Y + P_Y \sim -\alpha B_Y + \alpha D_Y - \mathbf{M}_Y$  is not pseudo-effective, we can run a  $(K_Y + P_Y)$ -MMP with scaling of an ample divisor that ends on a Mori fiber space  $f : Z \rightarrow W$ .

By monotonicity of log discrepancies and Lemma 2.3.4, the generalized pair  $(Z, P_Z, \mathbf{M})$  is again gklt of absolute coregularity at most  $c$ .

By Lemma 2.4.8, we have  $\pi_1^{\text{sm}}(Z, P_Z, \mathbf{M}) \twoheadrightarrow \pi_1^{\text{sm}}(Y, P_Y, \mathbf{M})$ , where  $\varphi : Y \dashrightarrow Z$  induces  $\varphi_*(P_Y) = P_Z$  and  $\varphi_*(B_Y) = B_Z$ .

Let  $(W, B_W, \mathbf{N})$  be the generalized pair induced by the canonical bundle formula (Lemma 2.2.8), i.e.

$$f^*(K_W + B_W + \mathbf{N}_W) \sim_{\mathbb{Q}} K_Z + B_Z + \mathbf{M}_Z.$$

Let  $(W', B_{W'}, \mathbf{N})$  be a  $\mathbb{Q}$ -factorial gdl modification of  $(W, B_W, \mathbf{N})$ . By Lemma 2.2.12, we have the following commutative diagram:

$$\begin{array}{ccc} (Z, B_Z, \mathbf{M}) & \xleftarrow{\quad g \quad} & (Z', B_{Z'}, \mathbf{M}) \\ \downarrow f & & \downarrow f' \\ (W, B_W, \mathbf{N}) & \longleftarrow & (W', B_{W'}, \mathbf{N}), \end{array}$$

where  $g$  is a birational contraction only extracting generalized log canonical places and  $f'$  is of Fano type.

By construction,  $(Z', B_{Z'}, \mathbf{M})$  has a generalized log canonical center of dimension at most  $c$ . By Lemma 2.2.11, its image contains a glcc of  $(W', B_{W'}, \mathbf{N})$  of dimension at most  $c$ . Since  $(W', B_{W'}, \mathbf{N})$  is gdlt, we have  $c \geq \text{coreg}(W', B_{W'}, \mathbf{N}) \geq \text{coreg}(W', B_{W'})$ .

By Lemma 2.4.7, there exists an effective divisor  $R_{Z'}$ , satisfying that  $P_{Z'} \leq R_{Z'} \leq B_{Z'}$ , where  $K_{Z'} + P_{Z'} + \mathbf{M}_{Z'} = g^*(K_Z + P_Z + \mathbf{M}_Z)$ , such that:

$$\pi_1^{\text{sm}}(Z', R_{Z'}, \mathbf{M}) \twoheadrightarrow \pi_1^{\text{sm}}(Z, P_Z, \mathbf{M}).$$

Let  $R_{W'}$  be as in Lemma 2.4.10, applied to  $(Z', R_{Z'}, \mathbf{M})$ , and  $F'$  ( $F$ , respectively) the general fiber of  $f'$  (of  $f$ , respectively). Consequently, we obtain the exact sequence

$$\pi_1^{\text{sm}}(F', R_{F'}, \mathbf{S}) \rightarrow \pi_1^{\text{sm}}(Z', R_{Z'}, \mathbf{M}) \rightarrow \pi_1^{\text{sm}}(W', R_{W'}, \mathbf{N}) \rightarrow 1,$$

where we define  $R_{F'} := R_{Z'}|_{F'}$  and  $\mathbf{S} = \mathbf{M}|_{F'}$ . Similarly we define  $B_F := B_Z|_F$ .

The divisor  $R_{W'}$  satisfies  $R_{W'} \leq B_{W'}$  and  $\text{coeff}(R_{W'}) < 1$ . Since  $(W', B_{W'}, \mathbf{N})$  is gdlt of coregularity at most  $c$ , the pair  $(W', R_{W'}, \mathbf{N})$  is gklt of absolute coregularity at most  $c$ . By the induction hypothesis,  $\pi_1^{\text{sm}}(W', R_{W'}, \mathbf{N}) = \pi_1^{\text{sm}}(W', R_{W'})$  is virtually abelian of rank at most  $2c$ .

By Lemma 2.2.14 applied to the generalized pair  $(F, B_F, \mathbf{S})$  and the generalized log canonical places extracted by  $g|_{F'} : F' \dashrightarrow F$ , there exists a birational morphism  $p : F_0 \rightarrow F$  extracting the same generalized lc places. Therefore the induced birational map  $q : F_0 \dashrightarrow F'$  is small. Hence

$$\pi_1^{\text{sm}}(F', R_{F'}, \mathbf{S}) \simeq \pi_1^{\text{sm}}(F_0, R_{F_0}, \mathbf{M}|_{F_0}),$$

where  $R_{F_0} = q_*^{-1}(R_{F'}) = p_*^{-1}(R_F) = p_*^{-1}(P_F) + E$  satisfies the hypotheses of Lemma 2.4.9. Therefore  $\pi_1^{\text{sm}}(F', R_{F'}, \mathbf{S})$  is finite.

Hence, by Lemma 2.1.3, the group  $\pi_1^{\text{sm}}(Z', R_{Z'}, \mathbf{M})$  is also virtually abelian of rank at most  $2c$ . By the surjections

$$\pi_1^{\text{sm}}(Z', R_{Z'}, \mathbf{M}) \twoheadrightarrow \pi_1^{\text{sm}}(Z, P_Z, \mathbf{M}) \twoheadrightarrow \pi_1^{\text{sm}}(Y, P_Y, \mathbf{M}) \twoheadrightarrow \pi_1^{\text{sm}}(X, D, \mathbf{M}) \simeq \pi_1^{\text{sm}}(X, D),$$

a virtually abelian group of rank at most  $2c$  surjects onto  $\pi_1^{\text{sm}}(X, D)$ . Therefore  $\pi_1^{\text{sm}}(X, D)$  is also virtually abelian of rank at most  $2c$ .

□

*Proof of Theorem 7.0.1.* Due to Proposition 2.4.4 and Proposition 2.4.5, the hypothesis of Theorem 7.0.3 holds for dimension at most 2, hence the statement holds for pairs of absolute coregularity at most 2. □

## Chapter 8

# log Calabi-Yau surfaces

The main result of this chapter is the following constraint on the fundamental groups of log Calabi-Yau pairs:

**Theorem 8.0.1.** *Let  $(X, B)$  be a projective log Calabi-Yau pair of dimension 2. Then, we have that*

$$\text{rank}(\pi_1^{\text{sm}}(X, B)_{\mathbb{Q}}^{\text{ab}}) \leq 4.$$

The following lemma allows us to understand the fundamental group under contractions of curves.

**Lemma 8.0.2.** *Let  $(X, B)$  be a 2-dimensional log Calabi-Yau pair. Let  $X \rightarrow Y$  be a divisorial contraction. Let  $B_Y$  be the push-forward of  $B$  to  $Y$ . Then, there is a surjective homomorphism*

$$\pi_1^{\text{sm}}(Y, B_Y) \rightarrow \pi_1^{\text{sm}}(X, B).$$

*Proof.* Let  $C \subset X$  be the exceptional curve. If the image of  $C$  on  $Y$  is a smooth point  $y$ , then we have

$$\pi_1^{\text{sm}}(Y, B_Y) \simeq \pi_1^{\text{sm}}(Y \setminus \{y\}, B_Y) \simeq \pi_1^{\text{sm}}(X \setminus C, B|_{X \setminus C}) \rightarrow \pi_1^{\text{sm}}(X, B),$$

where the last homomorphism is surjective. On the other hand, if  $y$  lies in the smooth locus, then we have that

$$\pi_1^{\text{sm}}(Y, B_Y) \simeq \pi_1^{\text{sm}}(X \setminus C, B|_{X \setminus C}) \rightarrow \pi_1^{\text{sm}}(X, B),$$

where the last homomorphism is surjective. This finishes the proof.  $\square$

*Proof.* First, note that if  $(C, B_C)$  is of log Calabi-Yau type, then  $\text{rank}(\pi_1^{\text{sm}}(C, B_C)_{\mathbb{Q}}^{\text{ab}}) \leq 2$ .

Let  $(X, B)$  be a log Calabi–Yau surface. We run a  $K_X$ -MMP that we denote by  $X \rightarrow X'$ . Let  $B'$  be the push-forward of  $B$  to  $X'$ . By Lemma 8.0.2, we may replace  $(X, B)$  with  $(X', B')$  in order to prove the statement. Hence, we may assume that one of the following statements holds:

- (i) the variety  $X$  is Calabi–Yau and  $B = 0$ ,
- (ii) the variety  $X$  admits a Mori fiber space to a curve  $X \rightarrow C$ , or
- (iii) the variety  $X$  is Fano of Picard rank one.

Assume that  $X$  is Calabi–Yau. Let  $Y \rightarrow X$  be the index one cover of  $K_X$ , so  $Y$  is Calabi–Yau with canonical singularities and  $K_Y \sim 0$ . It suffices to prove the statement for  $Y$ . The minimal resolution of  $Y$  is either an abelian surface or a K3 surface. If the minimal resolution of  $Y$  is an abelian surface, then  $Y$  is itself an abelian surface and the statement follows. We may replace  $X$  with  $Y$  and assume that its minimal resolution is a K3 surface. Assume that

$$\pi_1^{\text{sm}}(X) \rightarrow \pi_1^{\text{sm}}(X)^{\text{ab}} \simeq \mathbb{Z} \oplus A,$$

where  $A$  is a finitely generated abelian group. We let  $N_k := k\mathbb{Z} \oplus A \leq \pi_1^{\text{sm}}(X)$  a normal subgroup of index  $k$ . Then, we have a short exact sequence

$$1 \rightarrow N'_k \rightarrow \pi_1^{\text{sm}}(X) \rightarrow \mathbb{Z}/k\mathbb{Z} \rightarrow 1.$$

We denote  $G_k := \mathbb{Z}/k\mathbb{Z}$ . Let  $Z_k \rightarrow X$  be the branched cover associated to  $N'_k$ . Then,  $Z_k$  admits the action of  $G_k$ . Let  $Z'_k$  be the  $G_k$ -equivariant minimal resolution of  $Z_k$ . Let  $X'_k$  be the quotient of  $Z'_k$  by  $G_k$ . We have a commutative diagram

$$\begin{array}{ccc} Z'_k & \longrightarrow & X' \\ \downarrow & & \downarrow \\ Z & \longrightarrow & X. \end{array}$$

We may assume that  $Z'_k$  is a smooth K3 surface. Otherwise,  $Z$  is an abelian surface and the statement follows. Note that  $X' \rightarrow X$  only extracts divisors with log discrepancy at most 1. Hence, the projective birational morphism  $X' \rightarrow X$  is crepant. This implies that  $X'$  carries a nowhere vanishing 2-form. Hence, the automorphism group  $G_k$  must act symplectically on the smooth K3 surface  $Z'_k$ . This leads to a contradiction, as a finite group acting symplectically on a smooth K3 surface has order bounded above [30].

It suffices to prove the statement for  $\pi_1(Z)$ . Let  $Z' \rightarrow Z$  be a minimal resolution. Then  $Z'$  is a smooth Calabi–Yau variety and  $\pi_1(Z') \simeq \pi_1(Z)$ . Thus, the statement follows from the classification of smooth Calabi–Yau surfaces.

Assume that  $X$  admits a Mori fiber space  $X \rightarrow C$  to a curve. Then, the statement follows from Lemma 2.4.10.

Assume that  $X$  is a Fano variety of Picard rank one. By means of contradiction, assume that

$$\rho: \pi_1^{\text{sm}}(X, B) \rightarrow \pi_1^{\text{sm}}(X, B)^{\text{ab}} \simeq \mathbb{Z}^m \oplus F,$$

where  $F$  is a finite group and  $m \geq 5$ . Let  $N_k = (k\mathbb{Z})^m \oplus F \leq \pi_1^{\text{sm}}(X, B)$  a normal subgroup of index  $k^m$ . Let  $N'_k := \psi^{-1}(N_k)$ . Then, we have a short exact sequence

$$1 \rightarrow N'_k \rightarrow \pi_1^{\text{sm}}(X, B) \rightarrow (\mathbb{Z}/k\mathbb{Z})^m \rightarrow 1.$$

We denote  $G_k := (\mathbb{Z}/k\mathbb{Z})^m$ . Let  $Y \rightarrow X$  be the branched cover associated to  $N'_k$ . Then  $Y$  admits the action of  $G_k$  and the log pull-back of  $(X, B)$  to  $Y$  is a  $G_k$ -equivariant log Calabi–Yau pair  $(Y, B_Y)$ . By assumption, we have that

$$\text{rank}(\pi_1^{\text{sm}}(Y, B_Y)_{\mathbb{Q}}^{\text{ab}}) \geq m \geq 5.$$

Then, by Lemma 8.0.2, case (i), and case (ii) above, we conclude that the  $G_k$ -equivariant MMP  $Y \rightarrow Y'$  for  $K_Y$  must terminate in a Fano surface of Picard rank one. We obtain a Fano type surface  $Y'$  that admits the action of  $(\mathbb{Z}/k\mathbb{Z})^m$  for  $k$  arbitrarily large and  $m \geq 5$ . By [28, Lemma 2.24] there exists a  $G_k$ -equivariant boundary  $B_{Y'}$  for which  $(Y', B_{Y'})$  is log Calabi–Yau and  $N(K_{Y'} + B_{Y'}) \sim 0$ , where  $N$  is independent of  $k$ . This contradicts [28, Theorem 6]. We conclude that  $m \leq 2$  in case (iii). This finishes the proof.  $\square$

## Chapter 9

# Effective Bounds

The main result of this chapter is that we can obtain effective bounds on the index and length of normal solvable subgroups of the orbifold fundamental groups of pairs of small coregularity:

**Theorem 9.0.1.** *There exists a constant  $i(d, c)$  depending only on  $d$  and  $c$ , such that any klt Calabi-Yau type pair  $(X, D)$  of dimension  $d$  and absolute coregularity  $c$  has fundamental group  $\pi_1^{\text{sm}}(X, D)$  with a normal solvable subgroup of length at most  $2d - 1$  and index at most  $i(d, c)$ .*

Similarly to the proof of these fundamental groups being virtually solvable, we will rely on the reduction of the statement to pairs of low dimensions:

**Theorem 9.0.2.** *Assume that there exists a constant  $j(c)$ , such that the orbifold fundamental group of the smooth locus of any klt Calabi-Yau pair with standard coefficients of dimension at most  $c$  has a solvable subgroup of index at most  $j(c)$  and length at most  $2c - 1$ . Then, there exists constants  $i(d, c)$  depending only on  $d$  and  $c$ , such that any klt Calabi-Yau type pair  $(X, D)$  of dimension  $d$  and absolute coregularity  $c$  has  $\pi_1^{\text{sm}}(X, D)$  containing a normal solvable subgroup of length at most  $2d - 1$  and index at most  $i(d, c)$ .*

*Proof.* We proceed by induction on the dimension, assume that  $(X, D)$  is a projective klt pair of absolute coregularity  $c$ , with  $\dim X = d$ , and that the statement of the theorem holds for pairs with  $\dim X < d$  and coregularity at most  $c$ .

As  $\pi_1^{\text{sm}}(X, D) = \pi_1^{\text{sm}}(X, D_{\text{st}})$  and  $\text{coreg}(X, D_{\text{st}}) \leq \text{coreg}(X, D)$  it is enough to prove the statement for pairs with standard coefficients. Thus, we may assume that  $D = D_{\text{st}}$

*Case 1:* In this case, we prove the statement when  $(X, D)$  is a Calabi-Yau pair.

If  $(X, D)$  is a Calabi-Yau pair, then the dimension of  $X$  is  $c$  as  $(X, D)$  is a klt pair. Then the statement holds by the hypothesis of the theorem

*Case 2:* In this case, we prove the statement when  $(X, D)$  is not Calabi-Yau.

By the proof of Theorem 7.0.3, we have a surjection

$$\pi_1^{\text{sm}}(Z', R_{Z'}, \mathbf{M}) \twoheadrightarrow \pi_1^{\text{sm}}(X, D),$$

and an exact sequence

$$\pi_1^{\text{sm}}(F', R_{F'}, \mathbf{S}) \rightarrow \pi_1^{\text{sm}}(Z', R_{Z'}, \mathbf{M}) \rightarrow \pi_1^{\text{sm}}(W', R_{W'}, \mathbf{N}) \rightarrow 1,$$

where  $(F', R_{F'}, \mathbf{S})$  is Fano of dimension at most  $d$ . Hence by [4, Theorem 3] there exists a constant  $c(d)$  depending only on  $d$ , such that  $\pi_1^{\text{sm}}(F', R_{F'}, \mathbf{S})$  has an abelian subgroup of index  $c(d)$ .  $(W', R_{W'}, \mathbf{N})$  is klt of Calabi-Yau type of coregularity at most  $c$  and dimension less than  $d$ , therefore by induction  $\pi_1^{\text{sm}}(W', R_{W'}, \mathbf{N})$  has a solvable subgroup of length  $2d - 3$  and index  $i(d - 1, c)$ .

Therefore by Lemma 2.1.6, the group  $\pi_1^{\text{sm}}(Z', R_{Z'}, \mathbf{M})$  is solvable of length at most  $2d - 3 + 2$  and index at most  $c(d)!i(d - 1, c)$ .

As  $\pi_1^{\text{sm}}(Z', R_{Z'}, \mathbf{M})$  surjects onto  $\pi_1^{\text{sm}}(X, D)$ , the desired bounds hold.

□

*Proof of Theorem 9.0.1.* By [15, Theorem 3] there is an effective bound on the index of an abelian subgroup of  $\pi_1^{\text{sm}}(X, D)$  of klt Calabi-Yau surfaces. Therefore, the result follows in coregularity 2 by Proposition 9.0.2.

□

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