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The topology and geometry of photons, gravitons, and waves in unmagnetized plasma

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The topology and geometry of photons, gravitons, and waves in unmagnetized plasma

Eric Palmerduca

A Dissertation
Presented to the Faculty
of Princeton University
in Candidacy for the Degree
of Doctor of Philosophy

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Abstract

Topology has been at the heart of many breakthroughs in modern physics including the discovery of topological insulators, the bulk-edge correspondence (BEC), and the spin Hall effects. It has been recently found that topological effects can emerge not only in lattices, but also in fluids and plasmas where topologically protected edge-modes have been identified via the BEC. While this is a major discovery on its own, it also reveals an underlying and largely unstudied topological vector bundle structure of continuum waves. We show that topologically nontrivial waves are not rare, emerging even in simple continuous media such as the vacuum—in the form of photons and gravitons—and cold unmagnetized plasmas. In this thesis we use this structure to study novel topological applications beyond the BEC. Examples include constructing nonsingular representations of massless particles, constructing globally smooth bases of polarization vectors for photons and electromagnetic plasma waves (despite claims that this would violate the hairy ball theorem), and characterizing the relationship between the geometry and topology of massless particles.

Particularly interesting applications pertain to issues surrounding the spin (SAM) and orbital angular momentum (OAM) of electromagnetic waves. OAM beams have applications in a diverse range of fields including wakefield acceleration and laser-induced magnetic field generation in plasmas. However, there has been a long controversy about whether the angular momentum (AM) of massless particles can be legitimately split into SAM and OAM. We identify geometric and topological singularities that develop in the massless limit which obstruct such an SAM-OAM decomposition for photons and gravitons; a similar obstruction occurs in plasmas. The geometric singularity is related to a nonzero curvature of the polarization space. We also show that the AM eigenstates of massless particles and unmagnetized plasma waves can be expressed as spin-weighted spherical harmonics—these harmonics concretely illustrate the lack of a genuine SAM-OAM decomposition. However, we demonstrate that the AM admits a natural decomposition, induced by rotational symmetry, into helicity quasi-AM and orbital quasi-AM. Although these operators do not generate rotations and are

therefore not true AM operators, they are gauge invariant and appear to be experimentally relevant.

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This thesis contains material that has been previously published in articles [1–5] and preprints [6, 7], and presented at conferences [8–10].

To my mom and dad.

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Chapter 1

Introduction

1.1 Background and motivation

Topology has been at the root of a number of major physical discoveries of the modern era. These include the Aharonov-Bohm effect [11], the quantum Hall effects [12–15], the intrinsic anomalous Hall effect [16], the bulk-edge correspondence (BEC) [17–20], and the existence of topological insulators [21, 22]. A common feature in most of these applications is the appearance of Chern numbers, topological invariants which are calculated for wave systems and which are often directly related to physically observable phenomena, such as resistances in Hall effects or the number of edge modes at a material interface in the BEC. The discovery of physical phenomena that depends explicitly on topology is rather surprising. In heuristic terms, topology can be thought of as the study of "pre-geometric" structures which are invariant under deformations. A famous example is that the surfaces of a coffee cup and a donut are topologically equivalent in the sense that one can be deformed into the other, even though they are geometrically quite different. Perturbing the physical parameters of a system typically causes corresponding changes to the geometry, for example, changing the Hamiltonian or Riemannian metric, which in turn changes the equations of motion. However, topological structures are typically invariant under small perturbations and so do not detect

these variations. It is very interesting to find that some physical properties depend only on the topological Chern number. For example, in the presence of a large magnetic field, materials exhibiting the integer quantum Hall effect have a transverse conductivity given by [14]

$$\sigma = C\frac{e^2}{2\pi\hbar} \tag{1.1}$$

where C is the integer Chern number. Perturbing the applied magnetic field does not result in any change in the conductivity, at least until the magnetic field is changed by such a large amount that the topology changes, at which point C will jump by an integer.

Topological effects such as these are most well-known in condensed matter systems with periodic lattice structures and corresponding periodicity in reciprocal momentum space. However, it turns out that these discrete lattices and periodic momentum spaces are not essential for the appearance of topological phenomena, and topological waves *a priori* have the possibility of appearing in almost any physical system supporting waves. In particular, topological waves in continuous media have been recently identified in neutral fluids [20, 23–28] and plasmas [19, 28–35]. The primary application in continuous media studied thus far is the BEC in which a difference in the topological Chern number between two media can be used to predict the number of edge modes at their interface. Since the properties of these edge modes are determined by topology rather than geometry, they are particularly robust, and often exhibit an immunity to backscattering even when the interface is not smooth [34].

While the physics of the aforementioned topological phenomena differ, they largely share the same mathematical structure. A common recipe which has emerged in the physics literature [13, 14, 16, 18, 22–25, 27, 28, 30–33] can be summarized as follows. One begins with a linear wave equation in a homogeneous medium, which in Fourier space can be expressed as an eigenvalue problem

$$H(\boldsymbol{k})|n(\boldsymbol{k})\rangle = \omega_n(\boldsymbol{k})|n(\boldsymbol{k})\rangle. \tag{1.2}$$

Here, H is a Hermitian operator on $\mathbb{C}^p$, the eigenmodes $|n(\boldsymbol{k})\rangle \in \mathbb{C}^p$ are discretely labeled by n, and the Hamiltonian and eigenmodes depend on the wavevector $\boldsymbol{k}$. H may also depend on some external parameters. One defines the Berry connection $\mathcal{A}_n = \langle n|\nabla_{\boldsymbol{k}}|n\rangle$ and the corresponding Berry curvature $\mathcal{B}_n = \nabla_{\boldsymbol{k}} \times \mathcal{A}_n$, and then calculates the so-called Chern number $C(n) = \frac{i}{2\pi}\int_S \mathcal{B}\, dS$ where S is some 2D surface in k-space. $C(n)$ ends up being a physically meaningful quantity in many cases, for example, it is proportional to the Hall conductivity in the anomalous Hall effect [36] and the integer quantum Hall effect [Eq. (1.1)] [14].

While this recipe is mechanically simple and quite powerful, it is also rather opaque both mathematically and physically. Indeed, it is often not immediately clear in what sense C is a topological invariant, or even of what topological object C is supposed to be an invariant of. The answer to these questions is that Chern numbers are always associated to vector bundles.[1] Vector bundles are parameterized collections of vector spaces which smoothly fit together. Most often in wave physics these vector spaces are the collections of plane waves with a given wave vector $\boldsymbol{k}$. Each value of $\boldsymbol{k}$ defines such a vector space, and when collected together, these form the vector bundle of wave solutions, which we will sometimes call the wave bundle or particle bundle, depending on context. We emphasize that the vector bundle structure naturally arises when solving linear wave equations with standard Fourier techniques, although this terminology and vector bundle methods are rarely used. The main difference in perspective that occurs when using the language of vector bundles is that waves at all $\boldsymbol{k}$ are simultaneously considered, rather than those at a fixed value of $\boldsymbol{k}$. This allows for the analysis of global, topological features of these waves.

In most physical applications of topology, the vector bundle structure is largely or entirely ignored, with the focus instead placed solely on the Chern number (although there are notable exceptions, see [19, 20]). Furthermore, the topological analysis of waves in continuous media is a new area of research, and so far work has almost exclusively focused on

[1] More generally, Chern numbers can also be defined for some types of principal G-bundles, but these are not typically used in wave applications.

the BEC. The BEC and the existence of topologically protected fluid and plasma waves is indeed a spectacular result, but one may well suspect that there is other interesting topological behavior exhibited by these systems. In this thesis we take an alternative approach to the study of continuum waves in which we make extensive use of the vector bundle structure. Vector bundles are among the most well-studied topological objects in topology and differential geometry, so this formalism allows us to apply an arsenal of powerful techniques and results from these fields to physical systems. Furthermore, it provides a clear framework which we can use to explore and uncover new topological phenomena in continuous media beyond the BEC. A key observation is that the vector bundle description of waves is particularly well-suited for simultaneously studying the topological and geometric properties of these waves. By geometry, we more specifically mean the symmetry of these waves, such as the Poincaré symmetry of photons and other massless particles, or the rotational symmetry of waves in an isotropic plasma. Symmetries are concretely described by unitary representations of symmetry groups on vector bundles, which generalize representations on vector spaces commonly used in classical and quantum mechanics. It is easy to see why vector bundles are a natural setting to discuss symmetries. Vector bundles are particularly helpful when one needs to describe waves with different $\boldsymbol{k}$ simultaneously, and symmetry actions typically change the $\boldsymbol{k}$ of a wave.

One of the surprising findings of this work is that it reveals that nontrivial topology can exist in even the simplest continuous media, including the vacuum. In particular, we will show that massless particles with nonzero helicity, such as right (R) and left (L) circularly polarized photons and (linearized) gravitons are topologically nontrivial. Likewise, the R and L electromagnetic waves in the unmagnetized cold plasma are topologically nontrivial, and in fact, are topologically equivalent to the vacuum R and L waves. Although photons, gravitons, and waves in cold unmagnetized plasma are extremely well-studied systems, using the framework of vector bundle representations we can uncover new results about these

waves, including some which contradict long-held notions in physics. Before turning to an outline of this thesis, we mention a few particular applications.

The first is that using topological techniques we can prove the existence of, and in fact explicitly construct, a global basis of polarization vectors for photons. That is, it is possible to choose globally smooth polarization vectors $\boldsymbol{\epsilon}_1(\boldsymbol{k})$ and $\boldsymbol{\epsilon}_2(\boldsymbol{k})$ which are orthonormal and transversely polarized ($\boldsymbol{\epsilon}_i \cdot \boldsymbol{k} = 0$) at each $\boldsymbol{k}$. It is claimed in some standard quantum field theory texts ([37], §6.2.1; [38], §46.4) that such a polarization basis cannot be smoothly defined and must necessarily contain singularities. The justification for this claim is that since $\boldsymbol{\epsilon}_i(\boldsymbol{k}) \cdot \boldsymbol{k} = 0$, a smooth choice of $\epsilon_i(\boldsymbol{k})$ would amount to a smoothly combing the hairs on a sphere, violating the hairy ball theorem ([39], Theorem 16.1). We show in Chapter 3 that this argument has a subtle error, namely, the hairy ball theorem does not apply since we are in Fourier space and the ϵ_i are complex, rather than real, polarization vectors. The complex extension of the hairy ball theorem does not hold, and using an adaptation of the clutching construction from algebraic topology [40, 41], we explicitly construct such a basis. The same construction can be applied to give a global basis for gravitons and electromagnetic plasma waves as well.

A second notable application pertains to the construction of smooth representations of massless particles. Wigner developed the modern framework used to classify elementary particles by their symmetry groups [42, 43]. This gives a way to construct and classify possible particle state spaces as representations of the Poincaré group (the group of spacetime symmetries). This technique produces well-behaved massive particle representations. However, Wigner's method, which is still used in modern quantum field theory textbooks [43, 44], produces singular representations of massless particles. In particular, if one begins with a smooth particle state (a wave function) and effects a change of reference frame by applying a Lorentz transformation, one typically obtains a discontinuous wave function. We show that by modifying Wigner's little group method to apply to vector bundle representations rather than vector space representations, we can produce massless particle representations

free of singularities, so that smoothness in one reference frame guarantees smoothness in all reference frames. This complements the work of Flato et al. [45] and Boya et al. [46] who also noticed singularities in Wigner's representations. Others, such as Simms [47] and Asorey et al. [48], recognized the vector bundle structure of massless particles, but explored this structure only from a mathematical perspective and did not relate it to the smoothness issues present in Wigner's representations. One of the advantages of our method is that it produces extremely simple and explicit massless representations in terms of tensor products of the photon representations. Using these, we can show the novel relationship $C = -2h$ between the geometric helicity h and topological Chern number C of massless particles.

One of the most surprising applications of the vector bundle framework pertains to the issue of defining spin (SAM) and orbital angular momentum (OAM) for photons, gravitons, and plasma waves. OAM beams have found surprisingly diverse scientific and industrial applications [49] including, but not limited to, the development of new imaging [50, 51] and optical communication [52] techniques, as well as advances in the manipulation of microstructures [53]. Given the interest in OAM beams in tandem with the groundbreaking detection of gravitational waves by LIGO [54, 55], it is unsurprising that researchers have also proposed studying the OAM of gravitational waves [56–58]. Simulations have also shown that OAM beams may be highly useful in applications involving laser-plasma interactions such as wakefield acceleration of electrons [59, 60] and laser-induced generation of extremely strong magnetic fields in plasmas [61–64]. Despite these successes, there are deep theoretical issues surrounding the concepts of "OAM" and "SAM" for electromagnetic waves. Indeed, there has been an extended debate on this issue [65–81], which began when van Enk and Nienhuis pointed out that the most obvious definitions for SAM and OAM operators are either not gauge-invariant or else do not satisfy angular momentum commutation relations [67, 68]. The angular momentum commutation relations $[J_a, J_b] = i\epsilon_{abc}J_c$ are the defining feature of angular momentum operators [67, 68, 74, 75, 78, 79, 82, 83], as they reflect that these operators are generators of rotations. Without them, there is no clear relationship

with rotational symmetry and most of the standard theoretical tools for angular momentum operators cannot be used, including the Clebsch–Gordan coefficients, the addition of angular momentum formula, and the Casimir invariance of the square angular momentum operator. Since these issues were first pointed out, there have been persistent attempts [69–72, 76, 80, 81] to find SAM and OAM operators for photons and a great deal of argument and confusion surrounding this issue. However, none of these attempts have successfully overcome the problems found by van Enk and Nienhuis, that is, none of the proposed operators are both gauge invariant and actually angular momentum operators [67, 78, 80].

There is a good reason for this failure. Using the vector bundle description of massless particles developed in this thesis, we show that there are obstructions, which can be understood both topologically and geometrically, that prevent such an SAM-OAM splitting for massless particles with nonzero helicity [1–6]. From the topological point of view, we can show that topologically nontrivial particles cannot admit an SAM-OAM decomposition. While we will explore this in much more rigorous detail, we can give a heuristic picture for why this is the case. An SAM and OAM decomposition essentially implies that one can decouple the internal polarization degrees of freedom (DOFs) and the external momentum DOFs of a system. A well-defined SAM operator generates a 3D rotational symmetry of the internal DOFs while the OAM operators generate a rotational symmetry of the momentum $\boldsymbol{k}$ without altering the polarization DOFs. These add together to give the total angular momentum which generates the simultaneous rotation of the internal and external DOFs. However, the presence of nontrivial topology indicates that the internal and external DOFs are globally twisted together and cannot be decoupled. This mixing of the internal and external DOFs obstructs an SAM-OAM decomposition. This applies to photons and gravitons as well as to plasma waves.

Nevertheless, this leaves open the question as to the theoretical origins of the commonly used "SAM" and "OAM" for massless particles. While these are not true angular momentum operators, and thus do not allow the use of any of the standard tools developed for angular

momentum operators, it appears that they are still useful experimentally in both optics and plasma physics [77, 84]. We will call these operators quasi-angular momentum operators, and show that they arise in a very subtle way as a singular limit of the relativistic massive SAM and OAM operators related to the Newton-Wigner position operator. This explanation is intimately related to the vector bundle structure of these particles. We find that the failure of these quasi-angular momentum operators to satisfy angular momentum commutation relations is related to a nonvanishing geometric curvature, and arises due to a rather strange degeneracy of transverse boosts and rotations that occurs in the massless limit.

1.2 Overview of thesis

This thesis is closely based on seven papers [1–7] studying topological and geometrical phenomena pertaining to massless particles, such as photons and gravitons, and cold unmagnetized plasma waves. Particular focus is placed on the problem of understanding the angular momentum of these waves and obstructions to its SAM-OAM decomposition.

In Chapter 2 we provide a brief mathematical introduction to vector bundles and Chern numbers. This chapter aims to give a sense of how vector bundles naturally arise in the study of waves, and why Chern numbers are the topological invariants typically discussed in wave systems. The theory of characteristic classes and numbers, which encompasses the theory of Chern classes and Chern numbers, is fairly complicated and can be discussed at varying levels of abstraction. We discuss only one of the simpler incarnations of this theory in terms of the Chern–Weil homomorphism, which is arguably the most useful for physical calculations. This chapter is included in the main text as it helps bridge and clarify the relationship between the modern mathematical formalism of vector bundles and how topological methods are typically used in wave physics. However, the reader may wish to skip this chapter and refer back to it only as needed.

Chapters 3-9 present papers [1–7] and form the core of this thesis. In chapter 3, based on Ref. [1], we describe the topology and geometry of photons, and develop the formalism which will be used to describe the topology of waves throughout this thesis. The R and L circularly polarized photon bundles are particularly important as they will be shown in Chapter 5 to be the building blocks of other massless waves as well as topologically equivalent to the electromagnetic waves in cold unmagnetized plasmas. We show that, contrary to conventional wisdom [37, 38], it is possible to construct a smooth, globally well-defined polarization basis for photons, a result which *appears* to contradict the hairy ball theorem. We also show how symmetries are naturally defined in this framework by unitary representations, and in particular we study the Poincaré symmetry of these vector bundles. We show how bundle representations of the Poincaré group naturally decompose into unitary irreducible representations (UIRs), i.e., particles, such as the R and L photons. Furthermore, these representations are smooth and free of singularities, solving the smoothness issue in the massless Wigner representations [42, 43, 46]. We show that Wigner's little group method can be recast and adapted to the vector bundle formalism, and used to classify massless particles. We will also show how in this framework it becomes clear that there is an obstruction to an SAM-OAM decomposition for photons.

In Chapter 4, which is based on Ref. [2], we extend these results to (linearized) gravitons, giving an explicit construction of the graviton bundle and a global polarization basis for it, an analysis of the global issues with linear polarizations, and the nonexistence of an SAM-OAM decomposition.

In Chapter 5, based on Ref. [3], we consider the case of massless particles of arbitrary helicity. In particular, we show that the state space of massless particles of arbitrary helicity can be constructed by taking repeated tensor products of the R and L photon bundles. This provides a simpler construction and description of massless vector bundles than that of Asorey et al. [48], who gave a more implicit construction of such vector bundles. Our explicit description is particularly useful for calculations in later chapters. In this chapter,

we also use it to describe a novel relationship between the topology and geometry of massless bosons. The relationship between the topology and geometry of *massive* particles is actually a standard result in physics (although it is not typically thought of as relating topology and geometry). In particular, for massive particles $\dim V = 2s + 1$ where $\dim V$ is the dimension of the internal polarization space (a topological quantity) while s is the particle's spin, characterizing its Poincaré geometry. This relationship completely breaks down for massless particles, which always have $\dim V = 1$. We show that instead the relationship between topology and geometry is described by $C = -2h$, where h is the helicity, which characterizes the massless Poincaré geometry, and C is the Chern number characterizing these particles' topology.

In Chapter 6, based on [4], we will take up the issue of SAM-OAM decompositions for massless particles more directly, proving a series of four no-go theorems preventing such decompositions. The first two no-go theorems show that the general argument used for photons and gravitons extends to all massless particles with nonzero helicity, and that the obstruction to an SAM-OAM decomposition can be understood both geometrically and topologically. The next two no-go theorems illustrate that this obstruction persists under certain relaxations of the requirements on SAM and OAM operators.

In Chapter 7, based on Ref. [5], we examine the origin of what we call the quasi-angular momentum operators, which have commonly been considered as an SAM-OAM-type splitting of the photon angular momentum [67, 68, 71, 72, 77]. We show how any geometric connection on a vector bundle induces a splitting of the total angular momentum into two operators which resemble in some sense SAM and OAM operators. However, these operators are only true angular momentum operators if the curvature of the connection is zero. This establishes a deep relationship between the curvature of connections—which play a very important role in general relativity and in differential geometry—and the angular momentum of particles. We show that massive particles admit two different connections, corresponding to boosts and rotations, respectively. A unique combination of these connections yields a flat connection,

which corresponds to the relativistic massive SAM and OAM operators, as well as to the underlying Newton-Wigner position operators. However, for massless particles a rather strange singularity occurs in which transverse boosts and rotations degenerate, leaving only a single curved connection for massless particles. This nonvanishing curvature implies that the resultant quasi-angular momentum operators do not constitute genuine SAM and OAM operators, giving perhaps the clearest explanation of this obstruction.

In Chapter 8, based on Ref. [6], we take up a different but related problem, pertaining to explicitly finding the definite (total) angular momentum states, or more particularly, finding the simultaneous eigenstates of energy, helicity, J^2 and J_z for massless particles. This was done recently by Dragon [85] as well, but due to his different formalism, his solutions are implicit and not related to known special functions. In contrast, we show, using our vector bundle framework and a well-chosen coordinate systems, that the eigenstates for a helicity h particle are given by the spin-weighted spherical harmonics of weight $-h$. These are special functions which generalize the ordinary spherical harmonics and have been extensively used throughout physics [86–98]. We use this description to give a concrete and intuitive illustration of the nonexistence of SAM-OAM decompositions for massless particles, showing that the angular momentum multiplet structure is far too sparse to be consistent with an SAM-OAM decomposition.

In Chapter 9, based on Ref. [7], we will apply the results developed in the previous chapters to waves in cold unmagnetized plasma. We will show that this system is very similar to the photon system, the primary difference being the introduction of longitudinal Langmuir waves. These Langmuir waves are topologically trivial, while the R and L electromagnetic modes are topologically, and to some extent geometrically, equivalent to the R and L photon modes. We can thus directly apply many of the results from the previous chapters, showing that the R and L waves are topologically nontrivial and have an explicit global polarization basis. We also show that they do not allow a genuine SAM-OAM decomposition, but that a canonical quasi-angular momentum splitting does exist which agrees with a recently proposed

splitting [84]. The R and L eigenmodes of definite angular momentum are described by the SWSHs, while the Langmuir waves are described by ordinary spherical harmonics. Finally, we discuss how the vector bundle framework we develop may be applied to more complicated plasma models which are kinetic and/or magnetized.

In Chapter 10 we give some concluding remarks and discuss possible directions for future research inspired by these results.

Chapter 2

Vector bundles, Chern classes, and Chern numbers

This chapter presents some background on one of the primary mathematical tools we will use throughout this thesis, namely vector bundles. We will introduce characteristic classes and characteristic numbers, which include the Chern classes and Chern numbers commonly used in physics. These are the most useful topological invariants used to study the vector bundles which describe waves, and we seek to give some perspective on why. We hope to convey that useful topological invariants are difficult to define, even for manifolds. Vector bundles pose additional challenges due to the fact that they are homotopy equivalent to their base manifolds, and thus homotopy invariants such as the homotopy groups and (co)homology groups are not useful, at least directly, for characterizing vector bundles. Characteristic classes, such as the Chern classes, are specifically well-suited to provide useful topological information about vector bundles, explaining their prevalence in physics. Furthermore, we will show that the vector bundles most commonly encountered in continuum wave problems are completely characterized by their Chern numbers, showing that there is often no need to search for other topological invariants.

While we believe this chapter provides useful perspective on how the topology of vector bundles arises in physical applications, the reader may choose to skip it. Theorems from this chapter, especially Theorem 2.3, will be referenced throughout this thesis and can be referred back to as needed. The theory of characteristic classes is very complicated, and we discuss only the simple version presented in Ref. [99] based on the Chern–Weil theory. This is arguably the most useful version of the theory for the purposes of physics as it is most directly related to the way Chern classes and numbers are calculated in practice. More advanced treatments of characteristic classes can be found in many standard references, including Refs. [41, 100, 101].

2.1 Vector bundles

Vector bundles are parameterized collections of vector spaces. This structure arises naturally when describing waves in homogeneous linear media. For example, linearized electromagnetic waves in a homogeneous linear medium can typically be described in the spectral representation by a wave equation of the form ([102], §1.2):

$$\boldsymbol{D}_E(\omega, \boldsymbol{k})\boldsymbol{E} = 0 \tag{2.1}$$

$$\boldsymbol{D}_E = \boldsymbol{N}\boldsymbol{N}^T - \mathbb{1}N^2 + \boldsymbol{\epsilon}(\omega, \boldsymbol{k}) \tag{2.2}$$

$$\boldsymbol{N} = \frac{c\boldsymbol{k}}{\omega} \tag{2.3}$$

where ω is the frequency, $\boldsymbol{k}$ is the wavevector, $\boldsymbol{E}$ is the electric field, $\boldsymbol{D}_E$ is the dispersion tensor, $\boldsymbol{N}$ is the refractive-index vector, and $\boldsymbol{\epsilon}$ is the dielectric tensor which depends on the plasma model. Other dynamic variables such as $\boldsymbol{B}$ and the plasma density or distribution function are determined by $\boldsymbol{E}$ via the equations of motion of the plasma model. Note that except in Chapter 9, we will set $c = 1$ and thus refer to $\boldsymbol{k}$ as the momentum. The dispersion

relation is given by

$$\det \boldsymbol{D}_E(\omega, \boldsymbol{k}) = 0. \tag{2.4}$$

Its solutions give the dispersion branches $\omega = \omega_q(\boldsymbol{k})$ for a discrete label q. For a fixed q, the wave equation

$$\boldsymbol{D}_E(\omega_q(\boldsymbol{k}), \boldsymbol{k})\boldsymbol{E} = 0 \tag{2.5}$$

defines the polarizations of the mode. At each fixed $\boldsymbol{k}$, the solution space $V_q(\boldsymbol{k})$ is a complex vector space, and in fact, is a linear subspace of $\mathbb{C}^3$. As $\boldsymbol{k}$ is varied, this vector space will vary as well, so we obtain a family of vector spaces parameterized by the momentum space. Typically $\boldsymbol{D}_E(\omega_q(\boldsymbol{k}), \boldsymbol{k})$ is a smooth function of $\boldsymbol{k}$ except at a discrete set of points; away from these points, the vector spaces $V_q(\boldsymbol{k})$ will vary smoothly with $\boldsymbol{k}$, provided we also exclude $\boldsymbol{k}$ values at which two or more branches ω_q cross. Let $M \subseteq \mathbb{R}^3$ denote the space of nondegenerate $\boldsymbol{k}$ values. As $\boldsymbol{k}$ varies on M, the dimension of the vector spaces $V_q(\boldsymbol{k})$ will be a constant value r. For each fixed $\boldsymbol{k}$ a polarization $\boldsymbol{E} \in V_q(\boldsymbol{k})$ describes a plane wave solution of the wave equation with momentum $\boldsymbol{k}$ and frequency $\omega_q(\boldsymbol{k})$. Thus, the collection of all the vector spaces $V_q(\boldsymbol{k})$ describes the collection of all plane wave solutions for the mode $\omega_q(\boldsymbol{k})$. The parameterized family of vector spaces

$$E_q = \coprod_{\boldsymbol{k}} V_q(\boldsymbol{k}) \tag{2.6}$$

(where $\coprod$ denotes the disjoint union) is an example of a mathematical structure called a vector bundle. In general, a smoothly varying collection of r-dimensional vector spaces parameterized by some manifold M is called a (smooth) rank r vector bundle. M is called the **base manifold** and the vector space $V_q(\boldsymbol{k})$ is called the **fiber** at $\boldsymbol{k} \in M$. The vector bundle is called real or complex depending on whether the fibers are real or complex vector spaces. A rank 1 vector bundle is called a **line bundle**.

More formally, vector bundles are defined as follows:

Definition 2.1 (Complex vector bundles ([99], Def. 7.1))**.** *Let M and E be smooth manifolds. A smooth surjection $\pi : E \to M$ is a complex vector bundle of rank r if*

1. *For every $k \in M$, the set $E_k \doteq E(k) \doteq \pi^{-1}(k)$ is a complex vector space of dimension $r > 0$;*

2. *every point $\boldsymbol{k} \in M$ has an open neighborhood U such that there is a fiber-preserving diffeomorphism*

$$\phi_U : \pi^{-1}(U) \to U \times \mathbb{C}^r \tag{2.7}$$

that restricts to a linear isomorphism $E_k \to \{k\} \times \mathbb{C}^r$ on each fiber.

Real vector bundles are defined by replacing complex vector spaces with real vector spaces in the above definition. Note that this is the definition for a smooth vector bundle; we will assume throughout this thesis that all vector bundles are smooth. Note also that we do not allow vector bundles to have rank 0 so as to avoid discussing trivial edge cases. Technically speaking, E is called the total space and the entire triple $\pi : E \to M$ is called the vector bundle. However, it is standard to refer to E as the vector bundle as well, especially when π and M are clear from context or irrelevant. The elements of E are called vectors and we denote them by either $v \in E_k$ or (k, v), where the latter notation indicates explicitly that the vector lies in the fiber at k.

A **section** of the vector bundle $\pi : E \to M$ is a choice of vector in each fiber. That is, a section is a map $f : M \to E$ such that $f(k) \in E_k$. In the example of waves in a linear medium, a section would be a choice of polarization vector $\boldsymbol{E}(\boldsymbol{k}) \in V(\boldsymbol{k})$ (including magnitude and phase) for each $\boldsymbol{k}$, and describes a superposition of waves. In certain contexts one may call this a (momentum space) wavefunction. The space of all sections of the bundle E is denoted by $\Gamma(E)$. $\Gamma(E)$ is itself an infinite-dimensional vector space.

Given two vector bundles E and F over the same base manifold, we say that F is a **subbundle** of E if the fibers of F are linear subspaces of the fibers of E. F is called a proper subbundle if $F \neq E$.

Definition 2.2 (Bundle map ([99], Definition 7.5)). *A bundle map between the vector bundles $\pi_E : E \to M$ and $\pi_F : F \to N$ is a pair of maps $(\phi : E \to F, \underline{\phi} : M \to N)$ such that ϕ linearly maps the fiber E_k to $F_{\underline{\phi}(k)}$ for each $k \in M$. It is typical to slightly abuse language and call ϕ a bundle map.*

Definition 2.3 (Vector bundle isomorphism). *If $\pi_E : E \to M$ and $\pi_F : F \to M$ are vector bundles and there exist bundle maps $\phi : E \to F$ and $\psi : F \to E$ such that $\psi \circ \phi : \mathbb{1}_E$ and $\phi \circ \psi = \mathbb{1}_F$ then E and F are isomorphic as vector bundles and ϕ and ψ are vector bundle isomorphisms.*

From the example of plasma waves, we see that vector bundles provide a natural way of organizing plane wave solutions. Vector bundles have emerged as one of the primary topological objects of interest in differential geometry [99] and algebraic topology [41]. By organizing our solutions into vector bundles, we can access an arsenal of powerful tools and established results to explore the properties of the solution space of a wave equation. Notice that if one is interested in studying waves with a single fixed $\boldsymbol{k}$ value, then the vector bundle description does not have any obvious benefits. The power of the vector bundle formalism lies in the fact that it encodes information about the polarization of waves for all $\boldsymbol{k}$ simultaneously. It is thus especially useful in situations when multiple $\boldsymbol{k}$ need to be considered. For example, consider the following straightforward question. Is it possible to have waves with $\omega = \omega_q(\boldsymbol{k})$ traveling in all directions simultaneously? Suppose the waves are transversely polarized, such as the electromagnetic modes with $\omega = \sqrt{\omega_p^2 + k^2c^2}$ in a cold unmagnetized plasma (with ω_p being the plasma frequency). It is not immediately clear if it is possible to smoothly choose $\boldsymbol{E}(\boldsymbol{k})$ for all directions $\boldsymbol{k}$. Indeed, one might guess that the existence of such a $\boldsymbol{E}(\boldsymbol{k})$ would violate the hairy ball theorem because the waves are transverse. In the language of vector bundles, the existence of $\boldsymbol{E}(\boldsymbol{k})$ is equivalent to asking if the vector bundle of this mode has any globally nonvanishing sections. The theory of vector bundles makes answering such questions straightforward, and in later chapters we will show that it actually is possible to choose such an $\boldsymbol{E}(\boldsymbol{k})$ (because it is complex), contrary

to what one expects from naive application of the hairy ball theorem. The vector bundle structure is also very useful for treating symmetries of waves. Indeed, one of the most useful ways of thinking about symmetries is via actions of symmetry (Lie) groups. Waves with 3D rotational symmetries, for example, have an action of the rotation group SO(3) in which a rotation $R \in \mathrm{SO}(3)$ maps waves with momentum $\boldsymbol{k}$ to waves with momentum $R\boldsymbol{k}$. Thus, in discussing rotational symmetries, is necessary to relate waves with different $\boldsymbol{k}$.

Before proceeding, we mention a couple of common types of vector bundles. Given a manifold M of dimension n, the **tangent bundle** TM is the rank n real vector bundle in which the fiber at $k \in M$ is the tangent space of the manifold at k. A vector field $s : M \to TM$ on M is a section of the tangent bundle TM. The space of all vector fields on M is denoted by $\mathfrak{X}(M) = \Gamma(TM)$.

For each rank r, the simplest vector bundles are the trivial vector bundles $M \times \mathbb{R}^r$ and $M \times \mathbb{C}^r$. These are the vector bundles formed by attaching the same vector space, $\mathbb{R}^r$ or $\mathbb{C}^r$, to every point of the manifold. A vector bundle is said to be trivial if it is isomorphic to $M \times \mathbb{R}^r$ or $M \times \mathbb{C}^r$. Vector bundles are topologically interesting because there exist nontrivial vector bundles, that is, the fibers are attached in such a way that the resulting object is not globally equivalent to a simple direct product of spaces. For example, TS^2, the tangent bundle of the 2-sphere S^2, is a real rank 2 bundle which is not equivalent to $S^2 \times \mathbb{R}^2$. Indeed, the hairy ball theorem famously says that it is not possible to continuously choose a tangent vector to the sphere at every point, that is, TS^2 has no nonvanishing sections. Meanwhile $S^2 \times \mathbb{R}^2$ always has a nonvanishing section, for example, $\psi(\hat{\boldsymbol{k}}) = (\hat{\boldsymbol{k}}, v_0)$ is such a section for any fixed nonzero $v_0 \in \mathbb{R}^2$. Thus, TS^2 and $S^2 \times \mathbb{R}^2$ cannot be equivalent.

Note that given a real vector bundle E, one can always define a corresponding complex vector bundle $E^{\mathbb{C}} \doteq E \otimes \mathbb{C}$, called the complexification of E, by defining each fiber of $E^{\mathbb{C}}$ to be the vector space complexification of the corresponding fiber of E [100]. The complexification of a tangent bundle TM is denoted $T^{\mathbb{C}}M$. We will find that $T^{\mathbb{C}}S^2$, the bundle obtained by

complexifying the tangent planes of S^2, frequently shows up in the description of transversely polarized waves.

Given a vector bundle $\pi : E \to M$ and a submanifold $N \subseteq M$, one can define the restriction $\pi : E|_N \to N$ obtained by simply restricting the base manifold.

While a generic vector bundle E is a topological object, one can endow vector bundles with a geometric metric structure. In particular, this can be done by defining a real or complex (Hermitian) inner product

$$\langle \cdot, \cdot \rangle : E_k \times E_k \to \mathbb{R} \text{ or } \mathbb{C} \tag{2.8}$$

on every fiber. A real or complex vector bundle endowed with such an inner product is called a Riemannian bundle or a Hermitian bundle, respectively. In later chapters we will consider the unitary action of symmetry groups on vector bundles, which provide further geometric structure to vector bundles.

2.1.1 (Berry) connections on vector bundles

In this subsection we will discuss connections on vector bundles. Connections, which can be thought of as generalized covariant derivatives, play a fundamental role in certain areas of physics, especially in general relativity (the Levi-Civita connection) and in various applications of topology where the Berry connection is frequently used to calculate the Chern number. It is often the case in the physics literature that a formula for the Berry connection is used without reference to the underlying vector bundle or much explanation about where the formula originates. In addition to introducing connections, one of the purposes of this subsection is to derive the commonly used formula for the Berry connection and Berry curvature. The reader may consult Ref. [103], §17.4b for a related discussion of the Berry connection for line bundles.

A section $\psi : M \to M \times \mathbb{R}^n$ of the trivial vector bundle $M \times \mathbb{R}^n$ is simply a vector-valued function on M. Given a vector field $\boldsymbol{X} \in \mathfrak{X}(M)$ on M, we can naturally define $\boldsymbol{X} \cdot \nabla\psi$, which gives the directional derivative of ψ at each point $k \in M$ in the direction $\boldsymbol{X}(k) \in T_k M$. It is often convenient to write $\boldsymbol{X} \cdot \nabla$ as $\nabla_{\boldsymbol{X}}$. Treating $\boldsymbol{X} \in \mathfrak{X}(M)$ as an argument as well, we can think of ∇ as an operator which takes both a vector field and a section, and returns another section:

$$\nabla : \mathfrak{X}(M) \times \Gamma(M \times \mathbb{R}^n) \to \Gamma(M \times \mathbb{R}^n) \tag{2.9}$$

Thus, on the trivial bundle $M \times \mathbb{R}^n$, there is a natural way to take derivatives of sections. However, on a general vector bundle $\pi : E \to M$ there is not always an unambiguous way to define such a differential operator. The essential issue is that the derivative is meant to characterize how a section $\psi(k)$ changes as $k \in M$ is varied. However, $\psi(k_1) \in E(k_1)$ and $\psi(k_2) \in E(k_2)$ live in different vector spaces (different fibers), and thus one cannot subtract these vectors from each other to measure their difference. This was not a problem for the trivial bundle $M \times \mathbb{R}^n$ because the fibers are all identified with the same vector space $\mathbb{R}^n$ and can thus be subtracted from each other. As an example, if we consider the section $\psi_0(k) = (k, v_0)$ of $M \times \mathbb{R}^n$ for some fixed $v_0 \in \mathbb{R}^n$, then $\nabla_X \psi_0 = 0$ and we naturally consider this a constant section. However, if ∇ were defined for the tangent bundle of the sphere TS^2, what would it mean for a section to be constant? There is no obvious way to say that two vectors tangent to different points on the sphere are the same.

The generalization of ∇ for a general vector bundle $\pi : E \to M$ is called a connection. As the previous discussion indicates, there is not typically a unique connection on a vector bundle.

Definition 2.4 (Connection on a vector bundle, ([99], Definition 10.1))**.** *Let $\mathfrak{X}(M)$ denote the space of vector fields $X : M \to TM$ on M. A connection $\boldsymbol{D}$ on $\pi : E \to M$ is a map*

$$\boldsymbol{D} : \mathfrak{X}(M) \times \Gamma(E) \to \Gamma(E)$$
$$(\boldsymbol{X}, \psi) \mapsto \boldsymbol{D}_{\boldsymbol{X}}\psi$$

such that for any smooth function $f : M \to \mathbb{R}$ or $\mathbb{C}$, $\boldsymbol{D}_{\boldsymbol{X}}\psi$ is f-linear in $\boldsymbol{X}$ and satisfies the Leibniz rule in ψ:

$$\boldsymbol{D}_{f\boldsymbol{X}}(\psi) = f\boldsymbol{D}_{\boldsymbol{X}}\psi \tag{2.10}$$
$$\boldsymbol{D}_{\boldsymbol{X}}(f\psi) = df(\boldsymbol{X})\psi + f\boldsymbol{D}_{\boldsymbol{X}}\psi \tag{2.11}$$

where $df : TM \to \mathbb{C}$ is the differential of f. We also write $\boldsymbol{D}_{\boldsymbol{X}}$ as $\boldsymbol{X} \cdot \boldsymbol{D}$; this should be thought of as the derivative in the direction $\boldsymbol{X}$.

Note that a connection $\boldsymbol{D}$ on $\pi : E \to M$ can be expressed locally around $k \in M$ in coordinates as follows. Let U be some open neighborhood of k and $(e_1, \ldots, e_r)$ be a set of sections which form a basis of E_k for each $k \in U$; this is always possible provided U is small enough. Then we define the **connection forms** $\omega_j^i : \mathfrak{X}(U) \to \mathbb{C}$ which are 1-forms defined by ([99], Section 11.1)

$$\boldsymbol{D}_{\boldsymbol{X}} e_j = \omega_j^i(\boldsymbol{X}) e_i \tag{2.12}$$

where summation over repeated indices is implied. $[\omega_j^i]$ is an $r \times r$ matrix of 1-forms called the **connection matrix**. It gives a local, coordinate-dependent description of the connection $\boldsymbol{D}$. One can also use (2.12) and connection forms on an open cover of M to define a connection. On a bundle with an inner product, we can choose the e_i to be orthonormal, allowing us to write

$$\omega_j^i(X) = \langle e_i, \boldsymbol{D}_{\boldsymbol{X}} e_j \rangle \tag{2.13}$$

In some cases, one has enough additional structure on a vector bundle to uniquely define a canonical connection. For example, if M is a Riemannian manifold, then there is a canonical connection on TM called the Levi-Civita connection [99]; this connection plays a fundamental role in general relativity, as its Ricci curvature appears in the Einstein field equations.

For the bundles appearing in wave physics, there is also typically a canonical connection called the Berry connection. Suppose that $\pi : E \to M$ was defined by choosing each fiber to be a particular r-dimensional subspace of $\mathbb{C}^n$ for some $n \geq r$; this was the case for the bundles E_q in Eq. (2.6) describing electromagnetic wave modes in a linear medium with $n = 3$. This makes E into a rank r subbundle of $M \times \mathbb{C}^n$. It also makes E into a Hermitian vector bundle, inheriting the inner product of $\mathbb{C}^n$. Furthermore, it allows us to consider a section $\psi \in \Gamma(E)$ as a section of the trivial bundle $M \times \mathbb{C}^n$. This allows us to take the directional derivative with the canonical connection ∇ of the trivial bundle $M \times \mathbb{C}^n$ to ψ, so that $\nabla_{\boldsymbol{X}}\psi$ is a section of $M \times \mathbb{C}^n$ (but not necessarily of E) for any vector field $\boldsymbol{X}$. We can now define the Berry connection forms. Let $U \subseteq M$ be small enough open set such that we can choose a local orthonormal basis $e_1, \ldots, e_r$ of sections of $E|_U$. Then we can define the Berry connection forms

$$\omega_j^i(\boldsymbol{X}) = \langle e_i, \nabla_{\boldsymbol{X}} e_j \rangle. \tag{2.14}$$

Note that ∇ is essentially another notation for the exterior derivative d,

$$\nabla_{\boldsymbol{X}}\psi = d\psi(\boldsymbol{X}) \tag{2.15}$$

so we can also write (2.14) as

$$\omega_j^i = \langle e_i, de_j \rangle. \tag{2.16}$$

One can show that while the Berry forms ω_j^i depend on a choice of coordinates e_i, they define a coordinate independent connection via Eq. (2.12). Eq. (2.16) is the form of the Berry connection most commonly used in physics. It is almost always the case that one can choose U to be all of M except with a finite set of points excluded, so it is not typically necessary to

consider multiple neighborhoods. For line bundles $r = 1$, there is only a single basis vector e and a single connection 1-form given by

$$\tilde{\omega} = \langle e, de \rangle \quad \text{(for line bundles)} \tag{2.17}$$

2.1.2 The curvature of a connection

The curvature of a connection is an important quantity that arises in both general relativity and in the calculation of Chern numbers via the Berry curvature. As we previously noted, if we have not chosen a connection on a vector bundle, then we cannot identify vectors in different fibers. If a connection $\boldsymbol{D}$ is chosen, one can almost do this. A connection defines a notion of parallel transport, whereby given a vector $(k_0, v_0) \in E_{k_0}$ and a path $\gamma : [0, 1] \to M$ from $k_0 = \gamma(0)$ to $k_1 = \gamma(1)$, one can transport (k_0, v_0) along γ and identify it with some uniquely defined $(k_1, v_1) \in E_{k_1}$ ([99], §29.1). However, (k_1, v_1) will typically depend on the path γ. Another way to say this is that if γ forms a loop in M so that $k_1 = k_0$, then (k_0, v_0) may return to some different (k_0, v_1) after circulating around γ. One can think of the curvature of a connection as measuring this path dependence of the induced parallel transport. A connection which has vanishing curvature will induce a path-independent parallel transport. Technically, the curvature of $\boldsymbol{D}$ is defined as the map [99]

$$F_{\boldsymbol{D}} : \mathfrak{X}(M) \times \mathfrak{X}(M) \times \Gamma(E) \to \Gamma(E) \tag{2.18}$$

$$F_{\boldsymbol{D}}(\boldsymbol{X}, \boldsymbol{Y})\psi = (\boldsymbol{D_X D_Y} - \boldsymbol{D_Y D_X} - \boldsymbol{D}_{[\boldsymbol{X},\boldsymbol{Y}]})\psi \tag{2.19}$$

where $[\boldsymbol{X}, \boldsymbol{Y}]$ is the Jacobi-Lie bracket of vector fields. One can think of $[F_{\boldsymbol{D}}(\boldsymbol{X}, \boldsymbol{Y})\psi](k_0)$ as measuring the change in a vector $\psi(k_0)$ after it is transported around an infinitesimal square loop in M with sides given by tangent vectors $\boldsymbol{X}(k_0)$ and $\boldsymbol{Y}(k_0)$.

Just as the connection is described locally as an $r \times r$ matrix of 1-forms $[\omega^i_j]$ [Eq. (2.12)], the curvature can be locally described as an $r \times r$ matrix $[\Omega^i_j]$ of 2-forms called the **curvature forms**: ([99], §11.1)

$$F(\boldsymbol{X}, \boldsymbol{Y})e_j = \Omega^i_j(\boldsymbol{X}, \boldsymbol{Y})e_i. \tag{2.20}$$

The curvature forms can be calculated directly in terms of the connection forms using the second structural equation ([99], Theorem 11.1)

$$\Omega^i_j = d\omega^i_j + \omega^i_k \wedge \omega^k_j. \tag{2.21}$$

where $\wedge$ is the wedge product of differential forms and, as always, we sum over repeated indices. Note that for line bundles ($r = 1$), which frequently appear in the study of waves as they describe nondegenerate modes, there is only a single connection 1-form ($\tilde{\omega} = \omega^1_1$) and curvature 2-form $\Omega = \Omega^1_1$ related by

$$\Omega = d\tilde{\omega} \quad \text{(for line bundles)}. \tag{2.22}$$

In such cases the Berry curvature, with $e = e_1$, is given by

$$\Omega = d\langle e, de\rangle = \langle de, de\rangle. \tag{2.23}$$

It is common in physics to express 1-forms and 2-forms as vector fields when M is $\mathbb{R}^3$ (or an open subset of $\mathbb{R}^3$. For each pair of indices $1 < i, j \leq r$, one can define the 3D vector

$$\boldsymbol{A}^i_j = -i\langle \boldsymbol{e}_i, \nabla \boldsymbol{e}_j\rangle. \tag{2.24}$$

This set of vectors is often also called the Berry connection. The factor of $-i$ is a convention commonly used in the physics literature. This convention offers one minor advantage when dealing with line bundles (nondegenerate waves). In this case $r = 1$, so the vector field $\boldsymbol{A}^1_1$

will be real. This follows from the fact that since $\boldsymbol{e}_1$ is a unit vector,

$$d\langle \boldsymbol{e}_1, \boldsymbol{e}_1\rangle = \langle \boldsymbol{e}_1, \nabla \boldsymbol{e}_1\rangle + \langle \nabla \boldsymbol{e}_1, \boldsymbol{e}_1\rangle \tag{2.25}$$

$$= \langle \boldsymbol{e}_1, \nabla \boldsymbol{e}_1\rangle + \langle \boldsymbol{e}_1, \nabla \boldsymbol{e}_1\rangle^* \tag{2.26}$$

$$= 0 \tag{2.27}$$

which shows that $\langle \boldsymbol{e}_1, \nabla \boldsymbol{e}_1\rangle$ is purely imaginary. However, the factor of $-i$ has a notable disadvantage which obscures the sense in which $\boldsymbol{A}^i_j$ is a connection. Indeed, if $\boldsymbol{D}$ is a connection and $c \neq 1$ is a complex scalar, then $c\boldsymbol{D}$ does not satisfy the Leibniz rule (2.11) and is thus not a connection. For this reason we will not use this factor of $-i$ after this section. The Berry curvature corresponding to $\boldsymbol{A}^i_j$ is defined as [104]

$$\boldsymbol{\mathcal{F}}^i_j = \nabla \times \boldsymbol{A}^i_j. \tag{2.28}$$

This is simply the vector field form of $-i\, d\omega^i_j$.[1] One will note by comparing with Eq. (2.21) that this formula is only equivalent to the actual Berry curvature if the Berry connection matrix $[\omega^i_j]$ is idempotent, that is, if $\omega^i_k \wedge \omega^k_j = 0$ for all i, j. In most physical applications this condition is satisfied and thus $\boldsymbol{\mathcal{F}}^i_j$ is indeed another representation of the Berry curvature $[\Omega^i_j]$ matrix.

2.2 Characteristic classes and Chern numbers

Topological invariants are quantities (in a general sense) that depend only on the isomorphism type of topological object being studied. For example, when dealing with manifolds a topological quantity is the same for all manifolds which are diffeomorphic to each other. When dealing with vector bundles, which are the primary topological structures in the study

[1] More precisely, first map the 2-form $-i\, d\omega^i_j$ to the 1-form $-*d\omega^i_j$ with the Hodge star, and then use the canonical map from 1-forms to vectors in Euclidean $\mathbb{R}^3$.

of waves, topological quantities are those which depend only on the isomorphism type of the vector bundle. These quantities are particularly robust because small changes to physical parameters result in small changes to the vector bundles of wave modes, that is, they may cause slight changes to the allowed polarizations for each $\boldsymbol{k}$. Such small changes change the geometry of the vector bundle, but they cannot typically change the isomorphism type, that is, the topology is unaffected.

The simplest class of complex vector bundles are the trivial bundles which are isomorphic to $M \times \mathbb{C}^n$. These vector bundles $\pi : E \to M$ are characterized by the fact that they allow a global polarization basis, that is, it is possible to define sections $\psi_1, \ldots, \psi_r$ such that at each $k \in M$, $\psi_1(k), \ldots, \psi_r(k)$ are linearly independent and span the fiber E_k. Famously, the hairy ball theorem says that the tangent bundle of the sphere TS^2 is not trivial, meaning that it is not possible to construct a global basis of tangent vectors to the sphere (in fact, it is not even possible to choose a single such ψ_1).

We will encounter many waves whose vector bundles are not trivial, and we thus wish to have a method of classifying them. This is the fundamental classification problem: find labels for the isomorphism classes of rank r real or complex vector bundles E over a given base manifold M. Such labels are called topological invariants. As will be discussed in §2.2.3, in condensed matter one is typically interested in the case that M is the 2-torus or 3-torus, and in the study of continuous media the case of $M = S^2$.

One might notice from examining the literature on topological waves that the (first) Chern number is the primary topological invariant discussed. It is natural to wonder why this is so and if there may not be other useful topological invariants used for classifying wave bundles. In this section, we seek to emphasize two points. First, Chern numbers appear because they are, in a sense, the only game in town. Topological invariants are extremely difficult to find, and it is even more difficult to find invariants which are simple enough to calculate. Finding topological invariants for vector bundles poses a particularly difficult problem due to the fact that they can be smoothly deformed into their base manifold. Characteristic classes,

which include Chern classes, are an ingeniously constructed set of topological invariants which are particularly well-suited for vector bundles. There is no obvious contender for an additional topological invariant to supplement the Chern numbers. The second point we wish to emphasize is that the vector bundles which most commonly occur in the study of continuous media are completely topologically characterized by their first Chern number. Thus, there is usually no need for other topological invariants—even if they were found, their information would be contained in the Chern numbers.

2.2.1 The difficulty of finding topological invariants for vector bundles

The most well-known classes of topological invariants of a manifold M are the dimension, homotopy groups $\pi_n(M)$, homology groups $H_n(M)$, and (de Rham) cohomology groups $H^n(M)$ [105]. The dimension offers an obvious extension to a topological invariant for a vector bundle $\pi : E \to M$. Indeed, the rank r of E is a topological invariant, as two vector bundles can only be isomorphic if they have the same rank. Vector bundles are themselves manifolds, so it is immediate that $\pi_n(E)$, $H^n(E)$, and $H_n(E)$ are all topological invariants as well. However, they are essentially useless. The reason is that the homotopy and co(homology) groups are not only diffeomorphism invariants of manifolds, but also homotopy invariants. This means that if two manifolds M and N can be continuously deformed into one another, then they have the same homotopy groups and co(homology). Notably the continuous deformations do not even need to preserve the dimension, for example, $\mathbb{R}^n$ and $\mathbb{C}^n$ can be continuously shrunk down to a single point, and have the homotopy groups and co(homology) groups of a point which are, of course, trivial. This poses a problem for vector bundles, because the fibers, that is, the vector spaces $E_k \cong \mathbb{C}^r$ attached at each $k \in M$, can all be simultaneously shrunk down to their point of attachment, completely erasing the vector bundle structure. Thus, any vector bundle E is homotopy equivalent to its base manifold M, so that $\pi_*(E) = \pi_*(M)$, $H_*(E) = H_*(M)$, and $H^*(E) = H^*(M)$. Thus, these

invariants do not directly encode any information about the vector bundle structure and cannot help distinguish two vector bundles over the same M. By this argument, we see that the homotopy invariants of manifolds are not directly useful for labeling vector bundles.

2.2.2 Characteristic classes

Characteristic classes are a cleverly constructed set of topological invariants for vector bundles that avoid the aforementioned issues. They have the critical advantage of being fairly easy to calculate; this comes at the cost of a high degree of abstraction. Indeed, the construction of characteristic classes is far from obvious. We will give an overview of only the simpler version of this theory, based on the Chern–Weil homomorphism as discussed in Ref. [99]. This is the version most closely related to how Chern numbers are typically calculated in physical applications. We refer the reader to Refs. [41, 101] for other treatments of characteristic classes.

The essential idea of characteristic classes, from the perspective of Chern–Weil theory, is to construct topological invariants from the curvature matrix $[\Omega^i_j]$. One would be right to note that this is an odd approach, as the curvature matrix depends both on the choice of connection $\boldsymbol{D}$ and on the coordinates used to express it. However, in the end it is found that the invariants calculated are independent of both these choices.

We begin by supposing we have some rank r vector bundle $\pi : E \to M$. It is always possible to choose some connection $\boldsymbol{D}$ and a local neighborhood U so that the curvature is expressed via the curvature matrix $\Omega = [\Omega^i_j]$ with respect to some local basis $e_1, \ldots, e_r$ of U. Recall that the Ω^i_j are 2-forms on $U \subseteq M$. If one changes to a new set of coordinates f_i where $f_i = a^{ij} e_j$, then the curvature matrix $[\bar{\Omega}^i_j]$ with respect to the new basis is given by

$$\bar{\Omega} = a^{-1} \Omega a. \tag{2.29}$$

We must now consider the algebra of invariant polynomials on $\mathfrak{gl}(r,\mathbb{C})$, denoted by $\mathrm{Inv}(\mathfrak{gl}(r,\mathbb{C}))$. $\mathfrak{gl}(r,\mathbb{C})$ is the space of $r\times r$ complex matrices, so a generic element $X\in\mathfrak{gl}(r,\mathbb{C})$ is given by

$$X=\begin{pmatrix} x_{11} & \dots & x_{1r} \\ \vdots & \ddots & \vdots \\ x_{r1} & \dots & x_{rr} \end{pmatrix} \tag{2.30}$$

A polynomial in $\mathfrak{gl}(r,\mathbb{C})$ is a polynomial with complex coefficients in the variables x_{ij} considered as indeterminates. For example, with $r=2$,

$$\mathrm{Tr}(X)=x_{11}+x_{22} \tag{2.31}$$

$$\det(X)=x_{11}x_{22}-x_{12}x_{21} \tag{2.32}$$

$$Q(X)=2x_{11}x_{21}+i(x_{12})^2 \tag{2.33}$$

are examples of polynomials in $\mathfrak{gl}(2,\mathbb{C})$. A polynomial P in $\mathfrak{gl}(r,\mathbb{C})$ is said to be invariant if for every invertible $r\times r$ matrix $a\in\mathrm{GL}(r,\mathbb{C})$:

$$P(a^{-1}Xa)=P(X). \tag{2.34}$$

For example, the trace $\mathrm{Tr}(X)$ and determinant $\det(X)$ are invariant polynomials but $Q(X)$ is not. Notice that we can plug Ω into an invariant polynomial, forming $P(\Omega)$ with multiplication performed via the wedge product. For example, with $r=2$,

$$\mathrm{Tr}(\Omega)=\Omega^1_1+\Omega^2_2. \tag{2.35}$$

Because the trace is $\mathfrak{gl}(r,\mathbb{C})$-invariant and from the transformation rule (2.29), $\mathrm{Tr}(\Omega)$ gives a coordinate independent differential form. We will assume that that P is a homogeneous invariant polynomial of degree k; notice then $P(\Omega)$ is a differential $2k$-form. We now list some fundamental results from Chern–Weil theory; proofs can be found in §23 of Ref. [99].

1. One can cover the base manifold M with open sets $\{U_\beta\}$ which have curvature forms Ω_β. By Eqs. (2.29) and (2.34), the $P(\Omega_\beta)$ agree whenever the U_β overlap. Thus, the $P(\Omega_\beta)$ piece together to form a well-defined global $2k$-form $P(\Omega)$ on M.

2. $P(\Omega)$ is a closed form, that is, $d\big(P(\Omega)\big) = 0$ ([99], §23.2).

3. We now consider the effect of the choice of connection $\boldsymbol{D}$. It can be shown ([99], §23.5) that if one chooses a different connection $\boldsymbol{D}'$ which has the curvature matrix Ω', then

$$P(\Omega') = P(\Omega) + d\alpha \tag{2.36}$$

   for some $(2k-1)$-form α. That is, $P(\Omega')$ and $P(\Omega)$ differ only by an exact form. Since $P(\Omega)$ and $P(\Omega')$ are closed differential forms which differ only by an exact form, they define the same de Rham cohomology class $[P(\Omega)] = [P(\Omega')] \in H^{2k}(M)$ where $H^n(M)$ is the n^{th} cohomology group.

Together, these results describe the Chern–Weil homomorphism for complex vector bundles. For any rank r complex bundle E it is the map

$$\begin{aligned} CW : \mathrm{Inv}(\mathfrak{gl}(r,\mathbb{C})) &\to H^*(M) \\ P(X) &\mapsto [P(\Omega)]. \end{aligned} \tag{2.37}$$

It is a homomorphism of algebras because $P(X)Q(X) \mapsto [P(\Omega) \wedge Q(\Omega)]$. The $CW(P) = [P(\Omega)]$ are called **characteristic classes**. One can show that if E_1 and E_2 are isomorphic vector bundles over M, then they have the same characteristic classes, and thus the characteristic classes are topological invariants ([99], Eq. 23.7). The analogous construction holds for real vector bundles as well.

Chern classes

By choosing a basis of invariant polynomials, one can obtain a basis for the characteristic classes. One such basis corresponds to the Chern classes. This basis is given by the coefficients of the characteristic polynomial ([99], Appendix B):

$$\det\left(\lambda I + \frac{i}{2\pi}X\right) = \lambda^r + c_1(X)\lambda^{r-1} + \ldots + c_{r-1}(X)\lambda + c_r(X) \tag{2.38a}$$

$$= \sum_{j=0}^{r} c_j(X)\lambda^{r-j} \tag{2.38b}$$

where the factor of $i/2\pi$ is introduced so that the Chern numbers later discussed are normalized to integers. Notice that the coefficients c_j are homogeneous polynomials of degree j. To see that they are $\mathfrak{gl}(r,\mathbb{C})$-invariant, note that for an invertible matrix a,

$$\det\left(\lambda I + \frac{i}{2\pi}X\right) = \det\left(a^{-1}(\lambda I + \frac{i}{2\pi}X)a\right) \tag{2.39}$$

$$= \det\left(\lambda I + \frac{i}{2\pi}a^{-1}Xa\right) \tag{2.40}$$

so

$$\sum_{j=0}^{r} c_j(X)\lambda^{r-j} = \sum_{j=0}^{r} c_j(a^{-1}Xa)\lambda^{r-j} \tag{2.41}$$

and thus

$$c_j(a^{-1}Xa) = c_j(X). \tag{2.42}$$

One can show that the $c_j(X)$ form a basis of the algebra of invariant polynomials $\mathrm{Inv}(\mathfrak{gl}(r,\mathbb{C})$ ([99], Theorem B.17).

Applying the Chern–Weil homomorphism to the $c_j(X)$, $1 \leq j \leq r$ gives the characteristic classes $c_j(\Omega)$, known as the **Chern classes**. The Chern classes are used extensively in the topological analysis of waves. Note that $c_j(\Omega) \in H^{2j}(M)$ and can be represented by a $2j$-form modulo an exact form.

We give a few remarks on the Chern classes. First, all of $c_1, \ldots, c_r$ are defined for a rank r vector bundle, however, they may trivially vanish after some cutoff index $k_0 < r$ for dimensional reasons. Indeed, if M has dimension n, then it possesses no differential forms of degree greater than n. Thus, $c_k(\Omega) = 0$ for any $k > n/2$. Since usually in waves applications $n \leq 3$, it is typically sufficient to consider only the first Chern class $c_1(\Omega)$. Indeed, we will only need $c_1(\Omega)$ in this thesis and often just refer to it as the Chern class. From Eq. (2.38), it is explicitly given by the trace of the curvature matrix:

$$c_1(\Omega) = \frac{i}{2\pi}\mathrm{Tr}(\Omega). \tag{2.43}$$

For line bundles, the matrix $[\Omega^i_j]$ only has a single element, so

$$c_1(\Omega) = \frac{i}{2\pi}\Omega \quad \text{(for line bundles)} \tag{2.44}$$

Since the $c_k(X)$ form a basis of the invariant polynomials, any other characteristic class $P(\Omega)$ for complex vector bundles is completely determined by the Chern classes and provides no additional information. Thus, if one seeks additional topological invariants, they would need to be constructed from a different theory. The complexity of the theory of characteristic classes should indicate that this is no easy task. Furthermore, in many physical applications, including all those we will encounter, it is not typically necessary to look for topological invariants beyond the Chern classes. While it is certainly not true that the Chern classes fully determine the topology of an arbitrary vector bundle, they sometimes do for simple enough base manifolds M and small enough rank r. In particular, we will show in Theorem 2.3 that the Chern classes fully classify the topology of the most common wave bundles arising in continuous media, namely, complex vector bundles over the two-sphere S^2. We will discuss in §3.5.1 how the "spin Chern number" defined in Ref. [104] is not actually a topological invariant.

Chern numbers

From the Chern classes, which are cohomology classes, we can often obtain simpler topological invariants which take the form of integers. Assume that M is a compact manifold of dimension n. If we can choose powers $a_r \in \mathbb{Z}$, such that $c_1^{a_1}(\Omega) \wedge \cdots \wedge c_r^{a_r}(\Omega)$ has dimension n, then we can calculate the integral

$$\int_M c_1^{a_1}(\Omega) \wedge \cdots \wedge c_r^{a_r}(\Omega). \tag{2.45}$$

These are called the Chern numbers and they are topologically invariant since the Chern classes are. Notice that by this definition only bundles over even-dimensional manifolds have Chern numbers since the integrand must have degree $2(a_1+2a_2+\cdots+ra_r) = n$. Since we will only deal with low-dimensional cases with $n \leq 3$, we will only ever encounter the simplest Chern number which is defined for bundles over two-dimensional manifolds:

$$C = \int_M c_1 = \frac{i}{2\pi} \int_M \mathrm{Tr}(\Omega). \tag{2.46}$$

This is called the **first Chern number**, but we will simply refer to this as the **Chern number**, which is common terminology used in physics. For line bundles this reduces simply to

$$C = \frac{i}{2\pi} \int \Omega \quad \text{(for line bundles)}. \tag{2.47}$$

The Chern number C is not only a topological invariant, but it is also an integer. This is not at all an obvious fact from the perspective of Chern–Weil theory. A proof that the Chern number is an integer for line bundles is sketched in [103], §17.4a. The essential idea is that the Chern number can alternatively be calculated as $\sum_j \deg(f_j)$ where $f_j : S^1 \to S^1$ is a finite set of functions between circles and $\deg(f_j)$ is the winding number of the map (which is an integer). The general result can be obtained by noting that the Chern number of a complex vector bundle can be equivalently defined as the Euler number of the real vector

bundle obtained by forgetting the complex structure ([101], §20) and using the fact that the Euler numbers are integers ([101], Theorem 11.17).

2.2.3 Wave bundles over S^2

For wave bundles, the base manifold is typically the momentum space or a subset thereof. In condensed matter applications, the base manifold M is typically the first Brillouin zone, which, due to its periodicity, is topologically equivalent to the 2-torus T^2 for 2D materials and T^3 for 3D materials. In that context, one is concerned with the classification of complex vector bundles over T^2 and T^3.

In continuous media, on which we will focus, the momentum space is continuous rather than periodic. If it consists of all of $\boldsymbol{k} \in \mathbb{R}^3$, then there can be no interesting topology, as there are no nontrivial vector bundles over $\mathbb{R}^3$, or any $\mathbb{R}^n$ for that matter. Understanding why this is the case will also help us understand how nontrivial topology does arise in continuous media. A general feature of topological structures is that they are, in some sense, invariant under smooth deformations. For vector bundles, this principle is expressed by the fact that if M and N are manifolds which are homotopy equivalent (meaning they can smoothly be deformed into each other), then the isomorphism classes of vector bundles over M bijectively map to the isomorphism classes over N ([41], Corollary 1.8). We will use a special case of this result. Suppose $N \subseteq M$ is a submanifold of M with the inclusion map $\iota : N \hookrightarrow M$, $\iota(x) = x$. A deformation retraction of M onto N is a family of continuous maps $f_t : M \to M$, $t \in [0, 1]$ such that $f_0 = \mathbb{1}_M$ is the identity map, $f_1(M) = N$ and $f_t(x) = x$ for all $x \in N$ and all t. This simply describes the space M shrinking down onto N. The inclusion ι and retraction $\rho \doteq f_1$ are inverses, $\iota \circ \rho = \mathbb{1}_M$ and $\rho \circ \iota : \mathbb{1}_N$. We have the following theorem, in which $\mathrm{Vect}^r(M)$ denotes the isomorphism classes of vector bundles (real or complex) of rank r over M:

Theorem 2.1. *If M retracts onto N, then there is a bijective correspondence between $\mathrm{Vect}^r(M)$ and $\mathrm{Vect}^r(N)$. In particular, a retraction $r : M \to N$ of M onto $N \subseteq M$ induces*

a bijection $\rho^* : \mathrm{Vect}^r(N) \to \mathrm{Vect}^r(M)$ *with inverse* ι^**, where* ρ^* *and* ι^* *are the pullbacks induced by* ρ *and* ι*.*

This is a special case of Corollary 1.8 from Ref. [41]. Thus, if M deformation retracts onto N, then we can uniquely associate vector bundles over M to those over N. This is done via ι^*, which simply maps a vector bundle $\pi : E \to M$ to its restriction $\iota^* E = E|_N$. Through this bijection, topological invariants of E and E_N can be associated with each other. There is a useful corollary that follows immediately from Theorem 2.1. A manifold M is said to be **contractible** if it deformation retracts onto a single point.

Corollary 2.2. *If* M *is contractible, then every vector bundle over* M *is trivial.*

Returning to the discussion of the momentum spaces of waves, since $\mathbb{R}^3$ can be smoothly shrunk down to a single point, it follows that there can be no nontrivial vector bundles over $\mathbb{R}^3$ and thus no nontrivial topology when the momentum space is unrestricted. However, in some systems there is a distinguished momentum $\boldsymbol{k}$ which must be removed from the base manifold in order to form a well-defined vector bundle. For example, for massless particles $\boldsymbol{k}_0 = \mathbf{0}$ must be removed since massless particles do not have a rest frame. Similarly, some systems have Weyl points, that is, points $\boldsymbol{k}_0$ at which two modes resonate and it becomes impossible to distinguish these modes. It is necessary to remove such Weyl points from the base manifold in order to form well-defined vector bundles. In many applications, including all of those examined in this thesis, there is a single $\boldsymbol{k}_0$ which is removed from the base manifold. For convenience, we will shift coordinates so that $\boldsymbol{k}_0 = \mathbf{0}$ and the momentum space is $\mathbb{R}^3 \setminus \{0\}$. Unlike $\mathbb{R}^3$, the space $\mathbb{R}^3 \setminus \{0\}$ cannot be continuously shrunk down to a point. However, it can be deformation retracted onto the unit sphere S^2 with the retraction $\rho : \mathbb{R}^3 \setminus \{0\} \to S^2$, $\rho(\boldsymbol{k}) = \hat{\boldsymbol{k}}$. Thus, the possible vector bundles arising in such problems are classified by the complex vector bundles over S^2. Note that although a vector bundle $\pi : E \to \mathbb{R}^3 \setminus \{0\}$ does not have a Chern number directly defined by Eq. (2.46) due to the odd-dimension of the base manifold, we are able to define a topologically invariant Chern

number by

$$C(E) = C(E|_{S^2}). \tag{2.48}$$

Such vector bundles are completely classified by their Chern numbers:

Theorem 2.3. *Suppose $E_1 \to M$ and $E_2 \to M$ are two vector bundles where $M \cong \mathbb{R}^3 \setminus \{0\}$ or $M = S^2$. Then $E_1 \cong E_2$ if and only if they have the same rank and the same first Chern number.*

While this result is certainly known to topologists, we were unable to find a direct proof of it in the literature. However, a proof can be pieced together from other known results as follows. First note that vector bundles can only be isomorphic if they have the same rank, so we can assume E_1 and E_2 have the same rank r. Let $\mathrm{Vect}^r_{\mathbb{C}}(S^2)$ denote the isomorphism classes of rank r complex vector bundles over S^2. Let $[S^{k-1}, \mathrm{GL}(r,\mathbb{C})]$ denote the set of homotopy classes of maps from the $(k-1)$-sphere S^{k-1} to $\mathrm{GL}(r,\mathbb{C})$. Any map $f : S^{k-1} \to \mathrm{GL}(r,\mathbb{C})$ can be used to construct a rank r complex vector bundle E_f via the clutching construction, as detailed in Ref. [41], §1.2. In this context f is called the clutching function, and the constructed bundle E_f depends only on the homotopy type of f. This allows for the following classification of complex vector bundles over the spheres:

Lemma 2.4 ([41], Proposition 1.11)**.** *The map $\Phi : [S^{k-1}, \mathrm{GL}(r,\mathbb{C})] \to \mathrm{Vect}^r_{\mathbb{C}}(S^k)$ sending a clutching function f to the vector bundle E_f is a bijection.*

Proof of Theorem 2.3. By Lemma 2.4, the complex vector bundles over S^2 are in bijection with $[S^1, \mathrm{GL}(r,\mathbb{C})]$. Notice that for any manifold N, $[S^1, N]$ is the space of loops in N, that is, it is the fundamental group $\pi_1(N)$ of N. It is a standard result that $[S^1, \mathrm{GL}(r,\mathbb{C})] = \pi_1\big(\mathrm{GL}(r,\mathbb{C})\big) \cong \mathbb{Z}$ ([106], Conclusion 13.13), showing that complex rank r vector bundles over S^2 are labeled by the integers, which in turn are determined by the clutching function. As one might expect, this integer resulting from the clutching function coincide with the first Chern number ([107], Proposition 2.8.12). This shows that the isomorphism type of a complex rank r vector bundle over S^2 is determined entirely by the first Chern number. The

bijection between vector bundles over S^2 with those over $\mathbb{R}^3 \setminus \{0\}$ from Theorem 2.1 proves the same result for complex vector bundles over $\mathbb{R}^3 \setminus \{0\}$. □

2.3 Some advantages of rigorously defining Chern classes and Chern numbers

We believe there are many advantages to the rigorous approach to Chern classes and Chern numbers. While this treatment is more involved than that often used in the physics literature, it also makes it much clearer what is actually going on at a technical level and helps one avoid subtle issues. For example, it was necessary to use the machinery of vector bundles to prove rigorous versions of the BEC for waves in continuous media [19, 20]. Many studies prior to the rigorous statements used equations for the Berry connection, Berry curvature, and Chern number borrowed from condensed matter applications [26, 28, 31–33]. Without the context of vector bundles and their topological invariants, it can be very unclear why these quantities arise. Furthermore, this led to attempts to calculate Chern numbers by integrating the Berry curvature (Chern class) over a plane P in k-space: [26, 28, 31–33]

$$\frac{i}{2\pi}\int_P \mathrm{Tr}(\Omega). \tag{2.49}$$

This formula arises from an analogy with 2D condensed matter applications, where the integral is correctly taken over the first Brillouin zone (which is topologically a 2D torus and thus compact) [12, 14, 18, 21, 22]. However, from the rigorous versions of the continuum BEC [19, 20] it is clear that the integral should be taken over a 2D manifold onto which the base manifold M retracts, typically a 2-sphere enclosing a Weyl point. Since the manifold M typically does not retract onto a plane P (if it did, the vector bundle would be trivial by Theorem 2.1 since the plane is contractible), so it is unclear how one should interpret the integral over P. Similarly, if a vector bundle $\pi : E \to M$ is restricted to a plane P, the

Chern number of $E|_P$ is zero; however, the integral in Eq. (2.49) is typically nonzero, so it is not calculating the Chern number of $E|_P$. The formal theory makes the issue obvious: the Chern number is only defined when integrating over a compact surface [Eq. (2.45)]. Since P is not compact, the integral in Eq. (2.49) is not guaranteed to be a topological invariant or an integer. Indeed, since the Chern class c_1 is only defined in terms of the cohomology class of $\mathrm{Tr}(\Omega)$, so if we choose a different connection by Eq. (2.36) we would instead obtain the integral:

$$\frac{i}{2\pi}\int_P (\mathrm{Tr}(\Omega) + d\alpha) \tag{2.50}$$

for some 1-form α. Since the the plane P is not compact, it is not guaranteed that the integral of $d\alpha$ will vanish by Stokes' theorem. This shows that this integral may fail to be topologically invariant. Furthermore, the proof that the Chern numbers are integers no longer holds. Many researchers encountered the issue of "noninteger Chern numbers" when using Eq. (2.49) [26, 28, 31–33]. The fact that these quantities are not integers is actually an indication that they are not a Chern numbers at all. Somewhat ad hoc regularization techniques have been used to modify the integrand at large $|\boldsymbol{k}|$ so as to obtain an integer [26, 28, 31–33, 108]. It is typically argued that such regularization is okay because the behavior for large $\boldsymbol{k}$ is unphysical as it corresponds to wavelengths smaller than the model's validity, and thus no physical results should change by modifying it. However, there appear to be issues with this argument. In the case that $M \cong \mathbb{R}^3 \setminus \{0\}$, the actual Chern number is given by

$$C(E) = \frac{i}{2\pi}\int_{S^2_\rho} \mathrm{Tr}(\Omega). \tag{2.51}$$

where the domain of integration can be any sphere S_ρ of radius $\rho > 0$. By taking $\rho \to \infty$, one sees that the Chern number can be calculated purely from the behavior of E at $|\boldsymbol{k}| \to \infty$. The behavior for large $\boldsymbol{k}$ can certainly change the Chern number. It should be noted that in rigorous statements of the bulk-edge correspondence [19, 20], the integral is taken over a sphere rather than a plane and there is never a need to appeal to regularization techniques.

Another advantage of the rigorous approach is that it gives a clear description of which quantities are actually topologically invariant. The fact that the Berry connection is typically used to calculate the Chern number may lead to the misconception that the Chern number fundamentally depends on the Berry connection. We emphasize that the Berry connection and curvature are geometric while the Chern number is topological. Indeed, one could have chosen any other connection (of which there are always infinitely many) and calculated the same Chern number. The Berry connection is simply the most convenient connection to use for this calculation. Furthermore, without the rigorous theory, one may get the false sense that other topological quantities can be easily defined from the Berry connection and curvature. That the Chern classes and Chern numbers are topological invariants despite being calculated via geometric means is a nonobvious result from Chern–Weil theory. As an example, Bliokh et al. [104] defined an alleged topological invariant called the spin Chern number. However, we will show in §3.5.1 that this is a geometric quantity and not in fact a topological invariant.

While the rigorous approach we use can offer improvements in the standard topological analysis of waves, our main interest in this formalism is that it provides a solid framework to explore the new physical applications of topology that form the main content of this thesis.

Chapter 3

Photon topology

This Chapter presents material published in Ref. [1] in *Physical Review D*. We will examine the topology and geometry of photons, that is, of the solutions of Maxwell's equations in vacuum. In doing so we will develop the topological formalism of the description of massless particles and other waves, as well as the description of symmetries as actions of Lie groups on vector bundle. These formalisms will be used throughout the thesis to study various aspects of other topologically nontrivial waves and their accompanying geometry. Furthermore, the topology of photons will prove extremely useful for describing the topology of other massless particles and of transversely polarized plasma waves in, as shown in Chapters 5, 8, and 9.

3.1 Introduction

Following the breakthrough discoveries of the quantum hall effect, the Berry connection, and the existence of topological insulators in condensed matters with periodic lattice structures, topology has played an increasingly important role in physics. It is now understood that there is inherent topology in wave physics, and researchers have studied the topological properties of waves in various continuous media such as fluids [20, 23–28] and plasmas [19, 28–35, 109]. Here, we examine the topology of a simpler system, that of Maxwell's equations in vacuum. That this system can exhibit topologically nontrivial behavior is due to the fact

that there are no photons with wave vector $\boldsymbol{k} = 0$, creating a hole in the momentum space M. We will show that the collection of all wave solutions in Fourier space form a vector bundle γ over M, which we call the photon bundle.

Using this framework, we give an extensive study of the topological properties of photons by examining the topological properties of γ and its subbundles. We give two proofs that the photon bundle γ is topologically trivial. The first is an abstract proof using Chern numbers and a result from low dimensional topology, namely the classification theorem for vector bundles over the two-sphere S^2 (Theorem 2.3). The second proof involves an explicit construction of two linearly-independent, globally defined polarization vector fields via the clutching construction from algebraic topology [41]. These are complex polarization vectors perpendicular to the wave vector $\boldsymbol{k}$, and are mathematically equivalent to sections of the complexification of TS^2, the tangent bundle of the 2-sphere. It has been claimed in some quantum field theory textbooks [37, 38] that such global polarization vectors do not exist as they would violate the hairy ball theorem, which states that the tangent bundle of the sphere TS^2 is nontrivial ([39], Theorem 1). However, as we show, the hairy ball theorem only applies to TS^2, not its complexification $T^{\mathbb{C}}S^2$, which resolves the apparent paradox.

While the total photon bundle is trivial, we show that the right (R) and left (L) circularly polarized photons form globally well-defined nontrivial subbundles with Chern numbers ∓ 2. These Chern numbers were also obtained by Bliokh *et al.* in their study of the quantum spin Hall effect [104]. They calculated the Chern numbers via the Berry connection, which is a geometric quantity, but the notions of vector bundles and subbundles were not defined or discussed. We emphasize that it is crucial to establish the topological structures of vector bundles before calculating their Chern numbers. Due to the Chern–Weil homomorphism [Eq. (2.37)], a deep result from differential geometry, the Berry connection can be used to calculate the topologically invariant Chern numbers via Eq. (2.46) [99]. Generally though, quantities derived from the Berry connection will depend on the geometry as well as topology of the underlying vector bundle. Indeed, while Bliokh *et al.* [104] also define the spin Chern

number of light, we show that this quantity is not a true topological invariant like the ordinary Chern numbers.

We further investigate the topology (or lack thereof) of the familiar linearly polarized photons. As circularly polarized photons have nontrivial topology characterized by nonvanishing Chern numbers, one might guess the same is true of linear polarizations. We show that this question is actually ill-defined as there are no linearly polarized subbundles of γ. This implies that linear polarization is only a locally defined concept in momentum space, and it is meaningless to discuss its topology or Chern number.

In addition to these purely topological results, we study the interplay of this topology with the geometry of light, that is, with Poincaré symmetry. Poincaré symmetry plays a fundamental role in the theory of photons since elementary particles, both massive and massless, are typically defined as unitary irreducible representations of the Poincaré group [42–44]. Conventionally, these are assumed to be vector space (more particularly, Hilbert space) representations, and are classified via Wigner's little group method [42, 43]. This theory works well for massive particles, however, a topological singularity occurs when the mass is taken to zero. In this limit, the momentum space jumps from being a topologically trivial mass hyperboloid to the nontrivial light cone. This results in nonsmooth massless representations generated by the standard little group method; this difficulty has been noted by others [45, 46, 85]. Here, we present a solution to this problem by considering vector bundle representations rather than vector space representations. Indeed, we obtain a version of the little group method for vector bundle representations that does not encounter discontinuities. We prove that massless irreducible unitary vector bundle representations of the Poincaré group induce corresponding unitary irreducible representations on the Hilbert space of bundle sections, and thus correspond to particles by the conventional definition. We apply this method in the case of photons to show that R and L polarized photons are massless unitary irreducible vector bundle representation of the Poincaré group labeled by helicity, and thus correspond to globally well-defined particles.

Lastly, there has been a long debate as to whether photon angular momentum can be split into spin and orbital parts [65–81]. It is known that the most commonly proposed "SAM" and "OAM" operators for photons satisfy peculiar commutation relations, different from the $\mathfrak{so}(3)$ commutation relations required of angular momentum operators [67, 71, 77, 78]. Our theoretical study of the massless vector bundle representations illustrate that the issues that arise in defining such operators are related a singular jump in the little group that occurs in the massless limit. We also show that a commonly proposed "SAM" is neither spin nor angular momentum, but the generator an $\mathbb{R}^3$ translational symmetry of the photon bundle characterizing the photons' internal degree of freedom. The corresponding "OAM" does not generate any symmetry at all. These issues will be explored more fully in Chapters 6, 7, and 8.

This chapter is organized as follows. In §3.2, we construct the total photon bundle γ and give two proofs that it is trivial. The second proof is used to explicitly construct two linearly independent global polarization vector fields. In §3.3, we show that R and L photons form well-defined subbundles of γ that are topologically nontrivial. We also show that no analogous linearly polarized subbundles exist. In §3.4, we define vector bundle representations and develop a modified little group construction. These are used to classify photons as unitary irreducible vector bundle representations of the Poincaré group. We prove that massless irreducible vector bundle representations of the Poincaré group naturally induce unitary irreducible Hilbert space representations, and thus correspond to particle via the conventional definition. In §3.5, we apply our theory to two problems in the theory of photons: the geometric nature of the spin Chern number and the decomposition of photon angular momentum into spin and orbital parts.

3.2 Topology of the photon bundle

Consider photons in the vacuum, defined as the eigenmodes of Maxwell equations,

$$\partial_t \boldsymbol{\mathcal{E}} = \nabla \times \boldsymbol{\mathcal{B}} \tag{3.1}$$

$$\partial_t \boldsymbol{\mathcal{B}} = -\nabla \times \boldsymbol{\mathcal{E}} \tag{3.2}$$

$$\nabla \cdot \boldsymbol{\mathcal{E}} = 0 \tag{3.3}$$

$$\nabla \cdot \boldsymbol{\mathcal{B}} = 0, \tag{3.4}$$

where t has been normalized by $1/c$ and has the unit of length. Via the Fourier transform, the fields can be expressed as

$$\boldsymbol{\mathcal{E}}(x) = \int \frac{d^4k}{(2\pi)^4} e^{ik_\mu x^\mu} \tilde{\boldsymbol{E}}(k) \tag{3.5}$$

$$\boldsymbol{\mathcal{B}}(x) = \int \frac{d^4k}{(2\pi)^4} e^{ik_\mu x^\mu} \tilde{\boldsymbol{B}}(k), \tag{3.6}$$

where we have used the four-vector notation $x^\mu = (t, \boldsymbol{x})$ and $k^\mu = (\omega, \boldsymbol{k})$, along with the flat spacetime metric $\eta_{\mu\nu} = \mathrm{diag}(-1, +1, +1, +1)$. As Maxwell's equations have a real discrete spectrum, the momentum space fields can be written as

$$\tilde{\boldsymbol{E}}(k) \doteq 2\pi \sum_j \delta\big(\omega - \omega_j(\boldsymbol{k})\big) \boldsymbol{E}_j(\boldsymbol{k}) \tag{3.7}$$

$$\tilde{\boldsymbol{B}}(k) \doteq 2\pi \sum_j \delta\big(\omega - \omega_j(\boldsymbol{k})\big) \boldsymbol{B}_j(\boldsymbol{k}) \tag{3.8}$$

where the ω_j and $\boldsymbol{E}_j$ are solutions of the eigenvalue problem

$$H \begin{pmatrix} \boldsymbol{E}_j \\ \boldsymbol{B}_j \end{pmatrix} = \omega_j \begin{pmatrix} \boldsymbol{E}_j \\ \boldsymbol{B}_j \end{pmatrix} \tag{3.9}$$

$$H = \begin{pmatrix} \mathbf{0} & -\boldsymbol{k}\times \\ \boldsymbol{k}\times & \mathbf{0} \end{pmatrix} \tag{3.10}$$

subject to the constraints

$$\boldsymbol{k} \cdot \boldsymbol{E}_j = 0 \tag{3.11}$$

$$\boldsymbol{k} \cdot \boldsymbol{B}_j = 0. \tag{3.12}$$

The spectrum of H has 6 eigenmodes (photons) divided into three two-fold degenerate groups:

1. Two transverse photons with $\omega_1 = |\boldsymbol{k}| \equiv \sqrt{k_x^2 + k_y^2 + k_z^2}$;
2. Two transverse anti-photons with $\omega_{-1} = -|\boldsymbol{k}|$;
3. Two longitudinal photons with $\omega_0 = 0$.

The longitudinal photons are forbidden by the transversality conditions (3.11) and (3.12), although they can appear as virtual photons in QED [110]. Furthermore, since the space-time fields $\boldsymbol{\mathcal{E}}$ and $\boldsymbol{\mathcal{B}}$ are real, the momentum space fields must satisfy $\boldsymbol{E}_{-1}(\boldsymbol{k}) = \boldsymbol{E}_1^*(-\boldsymbol{k})$ and $\boldsymbol{B}_{-1}(\boldsymbol{k}) = \boldsymbol{B}_1^*(-\boldsymbol{k})$, and thus the transverse anti-photons are not independent of the transverse photons. Therefore, we need only analyze the transverse photons, which will henceforth be referred to just as photons with $\boldsymbol{E} \doteq \boldsymbol{E}_1$ and $\boldsymbol{B} \doteq \boldsymbol{B}_1$. The condition $\omega = |\boldsymbol{k}|$ says that photons reside on the forward light cone $\mathcal{L}_+$ in four-momentum space. Note that the origin is not contained in $\mathcal{L}_+$. Physically, this corresponds to the fact that $k^\mu = 0$ modes are stationary and do not propagate at the speed of light, and therefore are not photons. Mathematically, the origin must be removed in order for the forward light cone to be a regular submanifold of Minkowski space. $\mathcal{L}_+$ is a three-dimensional manifold homeomorphic to

$\mathbb{R}^3 \setminus \{0\}$ with coordinates given by $\boldsymbol{k} = \{k_x, k_y, k_z\} \neq 0$. At each $\boldsymbol{k}$, the two-fold degenerate photon eigenvectors form a $\mathbb{C}^2$ vector space consisting of vectors $\boldsymbol{E}$ satisfying $\boldsymbol{E} \cdot \boldsymbol{k} = 0$. Note that $\boldsymbol{B} = \hat{\boldsymbol{k}} \times \boldsymbol{E}$, so these vector spaces can be described purely in terms of the electric field. Since H is smooth and the dimension of the eigenspaces are constant, the eigenspaces for all values of $\boldsymbol{k}$ fit together to form a rank 2 complex vector bundle $\pi : \gamma \to \mathcal{L}_+$ over the light cone, which we term the photon bundle. Here, π is the projection of the bundle γ onto the base manifold and $\gamma(\boldsymbol{k})$ will denote the fiber at $\boldsymbol{k}$. γ is a Hermitian bundle with respect to the Hermitian product inherited by embedding $\boldsymbol{E}$ in $\mathbb{C}^3$. That is, for two vectors vectors $\boldsymbol{E}_a, \boldsymbol{E}_b \in \gamma(\boldsymbol{k})$,

$$\langle \boldsymbol{E}_a, \boldsymbol{E}_b \rangle = \boldsymbol{E}_a^* \cdot \boldsymbol{E}_b. \tag{3.13}$$

We are concerned here with the topology of photons, which is equivalent to the topology of the bundle γ. If the base manifold $\mathcal{L}_+$ were contractible (meaning it can be continuously down to a point), then γ would necessarily be trivial by Corollary 2.2. Recall that a rank r vector bundle over M is trivial if it is isomorphic to the trivial bundle $M \times \mathbb{C}^r$, or equivalently, if there exist r nonvanishing sections of the bundle which are pointwise linearly independent. For example, for particles with mass $m > 0$, the momenta k^μ form a mass hyperboloid $(k^0)^2 - \boldsymbol{k} \cdot \boldsymbol{k} = m^2$ which is homeomorphic to $\mathbb{R}^3$ and thus contractible. As a result, massive particles in vacuum do not allow for nontrivial topology. However, for massless photons, the base manifold $\mathcal{L}_+ \cong \mathbb{R}^3 \setminus \{0\}$ is not topologically trivial in the sense of homotopy, so it is not obvious if γ is a trivial bundle. The following theorem is one of the fundamental result of this chapter:

Theorem 3.1. *The photon bundle $\gamma \to \mathbb{R}^3 \setminus \{0\}$ is trivial.*

We will give two proofs of this result in the next two sections. The first is purely abstract using the machinery of Chern classes and Chern numbers. The second proof utilizes clutching functions which has the advantage of giving an explicit construction of linearly independent sections of γ.

We give some remarks about Theorem 3.1 before presenting the proofs. It is often more convenient to work with $\gamma|_{S^2}$, the bundle obtained by restricting the base manifold of γ to the unit sphere $S^2 \in \mathbb{R}^3 \setminus \{0\}$, rather than with γ itself. Since $\mathbb{R}^3 \setminus \{0\}$ deformation retracts onto S^2, the bundles γ and $\gamma|_{S^2}$ are essentially topologically equivalent. In particular, the rank r vector bundles over $\mathbb{R}^3 \setminus \{0\}$ and S^2 are in bijective correspondence by Theorem 2.1, with bundles over $\mathbb{R}^3 \setminus \{0\}$ map to bundles over S^2 by restricting the base manifold.

There are two main advantages to working with S^2, the first being the immediate observation that since the electric field is transverse to $\hat{\boldsymbol{k}}$, $\gamma|_{S^2}$ is isomorphic to $T^{\mathbb{C}}S^2$, the complexification of the tangent bundle TS^2 of the sphere. The complexification appears because $\boldsymbol{E}$ can be complex due to working in Fourier space. Thus,

Lemma 3.2. *The photon bundle γ is trivial if and only if $\gamma|_{S^2} \cong T^{\mathbb{C}}S^2$ is trivial.*

This lemma allows the use of specialized techniques that apply to bundles over spheres such as the clutching construction. Unlike $\mathbb{R}^3 \setminus \{0\}$, S^2 also has the advantage of being even-dimensional and thus possessing Chern numbers in addition to Chern classes. This is useful because Chern numbers are simply integers rather than cohomology classes.

By Lemma 3.2, the question of whether or not γ is trivial can be rephrased as asking if it is possible to smoothly choose polarization vectors $\boldsymbol{E}$ for each direction $\hat{\boldsymbol{k}}$. It can also be interpreted as asking is it possible to have photons traveling in all directions simultaneously. This issue shows up in many treatments of QED when quantizing the electromagnetic field [37, 38, 111]. Just prior to quantization, the Coulomb gauge vector potential is usually expressed in the form

$$\boldsymbol{\mathcal{A}}(x) = \int \sum_{j=1}^{2} \frac{1}{\sqrt{2|\boldsymbol{k}|}} [e^{ik_\mu x^\mu} \boldsymbol{\epsilon}_j(\boldsymbol{k}) a_j(\boldsymbol{k}) + e^{-ik_\mu x^\mu} \boldsymbol{\epsilon}_j^*(\boldsymbol{k}) a_j^*(\boldsymbol{k})] \frac{d^3\boldsymbol{k}}{(2\pi)^3}. \tag{3.14}$$

Here, $\boldsymbol{\epsilon}_j(\boldsymbol{k})$ are polarization unit vectors which are supposed to satisfy the transversality condition $\boldsymbol{\epsilon}_j(\boldsymbol{k}) \cdot \boldsymbol{k} = 0$ imposed by the Coulomb gauge condition $\nabla \cdot \boldsymbol{\mathcal{A}} = 0$. This gauge condition imposes the same transversality constraint on the $\boldsymbol{\epsilon}_j$ that Gauss's law imposes on

$\boldsymbol{E}$, and we see that $a_j(\boldsymbol{k})\boldsymbol{\epsilon}_j(\boldsymbol{k})$ and its conjugate can be considered as sections of γ. Despite using this expansion, Tong [37] and Woit [38] claim that no continuous functions $\boldsymbol{\epsilon}_j(\boldsymbol{k})$ actually exist. Indeed, this is true if one requires the $\boldsymbol{\epsilon}_j$ to be real, that is, if one considers only linear polarizations. In particular, if the $\boldsymbol{\epsilon}_j$ are real, then they are sections of TS^2 when restricted to S^2. By the hairy ball theorem ([39], Theorem 1), there are no continuous nonvanishing sections of TS^2. Despite this fact, Staruszkiewicz attempted to construct real, global polarization vectors [112]. These vector fields are not actually global though as they are only defined on stereographic coordinates and thus exclude one of the poles of the momentum sphere. Since we are working in momentum space, it is natural to work with complex polarization vectors, not just linear polarizations. R and L circular polarizations play a particularly important role due to their invariance under Lorentz transformations. The question can then be asked, is it possible to comb complex hairs on a ball flat? By explicit construction, it is indeed possible.

Theorem 3.3. *There exists at least one nonvanishing section of the bundles γ and $\gamma|_{S^2} \cong T^{\mathbb{C}}S^2$.*

Proof. Consider the sections of γ:

$$\boldsymbol{E}_1(\boldsymbol{k}) = (0, k_z, -k_y), \tag{3.15}$$

$$\boldsymbol{E}_2(\boldsymbol{k}) = (-k_z, 0, k_x). \tag{3.16}$$

Then

$$\boldsymbol{u} = \boldsymbol{E}_1 + i\boldsymbol{E}_2 \tag{3.17}$$

is a nonvanishing section of γ since

$$|\boldsymbol{u}| = \sqrt{k_x^2 + k_y^2 + 2k_z^2} > 0. \tag{3.18}$$

By restricting $\boldsymbol{u}$ to S^2 one obtains a nonvanishing section of $\gamma|_{S^2} \cong T^{\mathbb{C}}S^2$. □

Thus, it is possible to consistently choose at least one of the $\boldsymbol{\epsilon}_j$. However, to prove Theorem 3.1, one would need to construct a second linearly independent polarization, which is a much harder task. For example, while one might guess that $\boldsymbol{u}' = \boldsymbol{E}_1 - i\boldsymbol{E}_2$ may be such a vector field, $\boldsymbol{u}$ and $\boldsymbol{u}'$ are linearly dependent at any $\boldsymbol{k}$ where $\boldsymbol{E}_1$ or $\boldsymbol{E}_2$ vanishes. We note that Folland [111] states without proof that two such complex polarizations do exist. However, to our knowledge, there is neither an explicit construction of such polarizations nor a proof of their existence in the literature. We fill in these gaps in the remainder of this section.

3.2.1 Proof of Theorem 3.1 via Chern numbers

The first proof of Theorem 3.1 relies on Chern classes, the characteristic classes of complex vector bundles. An advantage of this approach is that Chern classes are commonly used in physical applications of topology, such as in the study of topological insulators and the topological properties of waves.

By Lemma 3.2, it is sufficient to show $\gamma|_{S_2}$ is trivial. We begin by proving that all of the Chern classes c_j of $\gamma|_{S_2}$ are 0.

Definition 3.1 (Conjugate vector bundle)**.** *Let $\pi\colon E \to M$ be an n-dimensional complex vector bundle. On each fiber E_p, $p \in M$, define multiplication of a complex number $a+ib \in \mathbb{C}$ with a vector $v \in E_p$ by*

$$(a + ib)v = av - ibv \tag{3.19}$$

which forms a new complex vector space $\overline{E}_p$. Then

$$\overline{E} = \bigcup \overline{E}_p \tag{3.20}$$

is defined to be the conjugate bundle of E.

In general, complex vector bundles are not isomorphic to their conjugates. However,

Lemma 3.4. *The complexification $E^{\mathbb{C}} = E^{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$ of a real vector bundle $E^{\mathbb{R}}$ is isomorphic to its own conjugate bundle $\overline{E^{\mathbb{C}}} = \overline{E^{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}}$.*

See Ref. [100], Lemma 15.1 for a proof. This helps us calculate Chern classes by the following lemma.

Lemma 3.5. *Let E be a complex vector bundle. The Chern classes c_j of E and $\overline{E}$ are related by*

$$c_j(\overline{E}) = (-1)^j c_j(E). \tag{3.21}$$

See Ref. [100], Lemma 14.9 for a proof.

Theorem 3.6. *The Chern classes and Chern numbers of $\gamma|_{S^2}$ are 0.*

Proof. $\gamma|_{S^2}$ is a vector bundle over the two-dimensional manifold S^2 and thus possesses no differential forms of degree greater than 2. Since c_j is represented by a $2j$-form on S^2, $c_j(\gamma|_{S^2}) = 0$ for all $j > 1$. By Lemmas 3.4 and 3.5,

$$\begin{aligned} c_1(\gamma|_{S^2}) &= c_1(T^{\mathbb{C}}S^2) = c_1(\overline{T^{\mathbb{C}}S^2}) \\ &= -c_1(T^{\mathbb{C}}S^2) = -c_1(\gamma|_{S^2}), \end{aligned} \tag{3.22}$$

so $c_1(\gamma|_{S^2}) = 0$. The Chern numbers are integrals of products of Chern classes [Eq. (2.45)], so it follows that the Chern numbers are also 0. □

While the Chern classes do not completely classify general complex vector bundles, this is true for certain low-dimensional and/or low-rank vector bundles. In particular, complex vector bundles over S^2 are trivial if and only if their first Chern number vanishes, as was show shown in Theorem 2.3. Thus, Theorems 2.3 and 3.6 together prove Theorem 3.1, showing that the photon bundle is trivial.

3.2.2 Proof of Theorem 3.1 via the clutching construction

Here we give a second proof that γ is trivial via the clutching construction [41]. Again, we proceed by showing $T^{\mathbb{C}}S^2$ is trivial. A major advantage of this method is that it can be adapted to explicitly furnish two independent vector fields on $T^{\mathbb{C}}S^2$.

The clutching construction is a method for determining the isomorphism type of a real or complex vector bundle over spheres S^n in arbitrary dimensions; here, we only consider the case of rank 2 complex bundles over S^2. Consider a rank 2 complex vector bundle $\pi : E \to S^2$, and decompose S^2 into its upper and lower hemispheres, $S^2 = D^2_+ \cup D^2_-$. Since each hemisphere is contractible (for example, they contract onto the poles), the restricted bundles $E|_{D^2_+}$ and $E|_{D^2_-}$ are trivial. Fix a trivialization for each of these bundles, and let $[v_{1\pm}, v_{2\pm}]$ be an orthonormal frame for $D^2_\pm$. These frames will generally not agree on the equator $D^2_+ \cap D^2_- = S^1$ where the hemispheres overlap, and there is a function $f\colon S^1 \to \mathrm{U}(2)$, where $\mathrm{U}(2)$ is the two-dimensional unitary group, which rotates one frame into the other on S^1:

$$[v_{1+}, v_{2+}](\phi) = f(\phi)[v_{1-}, v_{2-}](\phi). \tag{3.23}$$

f is referred to as a clutching function for E.

Conversely, given an arbitrary function $f\colon S^1 \to \mathrm{U}(2)$, one can construct a rank 2 complex vector bundle E_f (see Ref. [41], p. 22). For smooth manifolds M and N, let $[M, N]$ denote the set of homotopy classes of maps from M to N, and let $\mathrm{Vect}^r_{\mathbb{C}}(S^2)$ be the set of isomorphism classes of rank r complex vector bundles over S^2. The following fundamental theorem says that the homotopy type of a clutching function determines the isomorphism class of a vector bundle.

Theorem 3.7. *The map $\Phi : [S^1, \mathrm{U}(2)] \to \mathrm{Vect}^2_{\mathbb{C}}(S^2)$ which sends a clutching function f to the vector bundle E_f is a bijection.*

Proof. This is a special case of proposition 1.11 in [41]. □

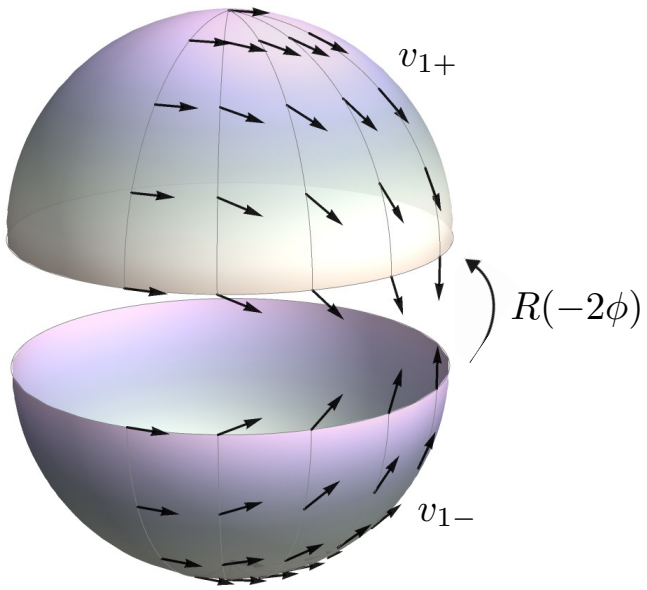


Figure 3.1: Vector fields v_{1+} on D^2_+ and v_{1-} on D^2_-. The clutching function relating them along the equator is a -2ϕ rotation.

Note that $[S^1, \mathrm{U}(2)] = \pi_1(\mathrm{U}(2))$, where π_1 denotes the fundamental group (the group of loops in U(2)), and $\pi_1(\mathrm{U}(2)) \cong \mathbb{Z}$ ([106], Proposition 13.11), so the complex vector bundles over S^2 can be labeled by the integers, with the trivial bundle corresponding to 0. It is well known that the special unitary group SU(2) is simply connected, for example, from the fact that it is diffeomorphic to S^3 ([103], §19.1c); see also Ref. [106], Proposition 13.11). Thus, we have the following corollary.

Corollary 3.8. *If a rank 2 vector bundle $\pi\colon E \to S^2$ has a clutching function f which factors through* SU(2)*, that is, if $f : S^1 \to \mathrm{SU}(2) \subseteq \mathrm{U}(2)$, then E is trivial.*

Proof. Since SU(2) is simply connected, f is homotopic to the constant map $\mathbb{1} : S^1 \to \mathrm{U}(2)$ which sends every element of S^1 to the identity of U(2). $\mathbb{1}$ is the clutching function for the trivial bundle, so the corollary follows by Theorem 3.7. □

We now use this corollary to prove $T^{\mathbb{C}}S^2$ is trivial. We construct a clutching function for $T^{\mathbb{C}}S^2$ by adapting a construction of a clutching function of the real bundle TS^2 given by Hatcher ([41], p. 22). We start by constructing orthonormal frames for the two hemispheres

$D^2_\pm$ of S^2. Choose a vector field v_{1+} by first picking some real tangent vector at the north pole and then extending it along each meridian so that it maintains a constant angle with the meridian (see Figure 3.1). Let v_{1-} be the vector field on the lower hemisphere D^2_- obtained by reflecting v_{1+} across the equatorial plane. We can produce a second real vector field $v_{2\pm}$ on each hemisphere by rotating $v_{1\pm}$ by 90 degrees counterclockwise when viewed from the exterior of the sphere. Then $[v_{1+}, v_{2+}]$ and $[v_{1-}, v_{2-}]$ are orthonormal trivializations of D^2_+ and D^2_-, respectively. Explicitly, let (k_x, k_y, k_z) and (k, θ, ϕ) be Cartesian and spherical coordinates related in the standard way, with the corresponding unit vectors $(\boldsymbol{e}_x, \boldsymbol{e}_y, \boldsymbol{e}_z)$ and $(\boldsymbol{e}_k = \hat{\boldsymbol{k}}, \boldsymbol{e}_\theta, \boldsymbol{e}_\phi)$. By choosing v_{1+} to be $\boldsymbol{e}_y$ at the north pole, we have

$$v_{1+} = \sin(\phi)\boldsymbol{e}_\theta + \cos(\phi)\boldsymbol{e}_\phi \tag{3.24}$$

$$v_{2+} = -\cos(\phi)\boldsymbol{e}_\theta + \sin(\phi)\boldsymbol{e}_\phi \tag{3.25}$$

$$v_{1-} = -\sin(\phi)\boldsymbol{e}_\theta + \cos(\phi)\boldsymbol{e}_\phi \tag{3.26}$$

$$v_{2-} = -\cos(\phi)\boldsymbol{e}_\theta - \sin(\phi)\boldsymbol{e}_\phi. \tag{3.27}$$

On the equator, the two frames are related by rotating the lower frame by -2ϕ:

$$[v_{1+}, v_{2+}] = \begin{pmatrix} \cos 2\phi & \sin 2\phi \\ -\sin 2\phi & \cos 2\phi \end{pmatrix} [v_{1-}, v_{2-}] \doteq f(\phi)[v_{1-}, v_{2-}]. \tag{3.28}$$

where $f\colon S^1 \to \mathrm{SU}(2) \subseteq \mathrm{U}(2)$ is the clutching function and the matrix of f and the column vectors are expressed in the $(\hat{\theta}, \hat{\phi})$ basis. Since f factors through $\mathrm{SU}(2)$, Corollary 3.8 proves the following.

Theorem 3.9. *$T^{\mathbb{C}}S^2$ is trivial.*

3.2.3 Explicit construction of a frame on $T^{\mathbb{C}}S^2$ and γ

Using methods related to the clutching construction, we can explicitly construct two independent vector fields on $T^{\mathbb{C}}S^2$, from which one can immediately obtain corresponding independent vector fields on γ. We do this in three steps. First, we find an explicit homotopy in SU(2) between the clutching function $f : S^1 \to \mathrm{SU}(2)$ and the identity $\mathbb{1}$. Second, we use this homotopy to extend the domain of f to all of D^2_-. From this we will obtain a continuous, piecewise smooth frame for $T^{\mathbb{C}}S^2$. The last step is to modify the construction so that the frame is everywhere smooth.

We note that the clutching construction is normally used abstractly and used to classify nontrivial vector bundles (which do not possess global bases). Our adaption of the clutching construction to produce an explicit global basis of a trivial bundle appears to be a novel application.

Step 1: Homotopy from f to the identity

From the fact that f factors through SU(2), we showed that $[f] = 0$, meaning that there exists a homotopy between

$$f(\phi) = \begin{pmatrix} \cos 2\phi & \sin 2\phi \\ -\sin 2\phi & \cos 2\phi \end{pmatrix} \quad \text{and} \quad \mathbb{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

We construct one such homotopy as follows. SU(2) is homeomorphic to the unit 3-sphere S^3 via

$$(w, x, y, z) \in S^3 \mapsto \begin{pmatrix} x + iz & y + wi \\ -y + wi & x - iz \end{pmatrix} \in \mathrm{SU}(2). \tag{3.29}$$

By setting $w = 0$, we obtain a map from S^2 into SU(2):

$$(x, y, z) \in S^2 \mapsto \begin{pmatrix} x + iz & y \\ -y & x - iz \end{pmatrix} \in \mathrm{SU}(2). \tag{3.30}$$

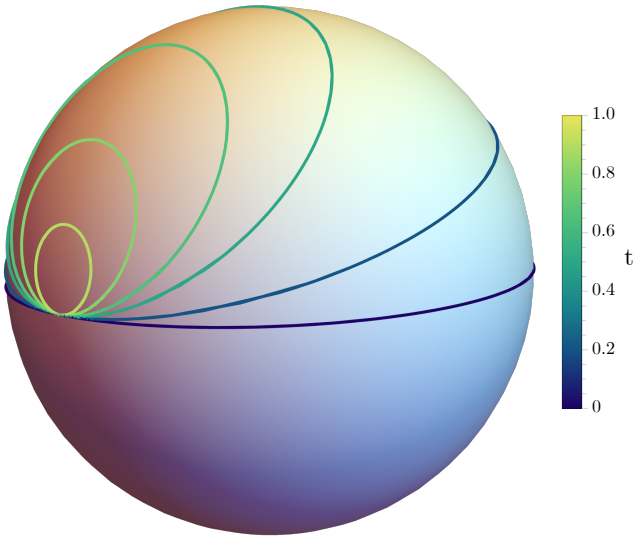


Figure 3.2: Illustration of the homotopy $F(t,\phi) : [0,1] \times S^1 \to \mathrm{SU}(2)$ from $f(\phi)$ to the identity $\mathbb{1}$. The image of F resides on an embedding of S^2 in SU(2). The equator at $t = 0$ corresponds to $f(\phi)$ and shrinks down to the identity $(1,0,0)$ at $t = 1$.

Under this map, $f(\phi)$ corresponds to a loop along the equator in the xy-plane traversed twice, while $\mathbb{1}$ corresponds to the fixed point $(x,y,z) = (1,0,0)$. By shrinking the equator down to $(1,0,0)$ while remaining on S^2, we obtain the desired homotopy as illustrated in Figure 3.2. Explicitly, choose

$$x(t,\phi) = \cos(2\phi)\cos^2\left(\frac{\pi t}{2}\right) + \sin^2\left(\frac{\pi t}{2}\right) \tag{3.31}$$

$$y(t,\phi) = \sin(2\phi)\cos\left(\frac{\pi t}{2}\right) \tag{3.32}$$

$$z(t,\phi) = \sin^2(\phi)\,\sin(\pi t). \tag{3.33}$$

Note that

$$x(t,\phi)^2 + y(t,\phi)^2 + z(t,\phi)^2 = 1, \tag{3.34}$$

so these points reside on S^2. Define the homotopy $F(t,\phi) : [0,1] \times S^1 \to \mathrm{SU}(2)$ by

$$F(t,\phi) = \begin{pmatrix} x(t,\phi) + iz(t,\phi) & y(t,\phi) \\ -y(t,\phi) & x(t,\phi) - iz(t,\phi). \end{pmatrix} \tag{3.35}$$

Since $F(t, \phi + 2\pi) = F(t, \phi)$, F is well-defined as a function on $[0, 1] \times S^1$.

Step 2: Constructing a piecewise smooth frame for $T^{\mathbb{C}}S^2$

On each hemisphere we have a frame, namely, $[v_{1+}, v_{2+}]$ and $[v_{1-}, v_{2-}]$. We can transform $[v_{1-}, v_{2-}]$ along the equator via f so that it agrees with $[v_{1+}, v_{2+}]$. If we can extend f to a function $\tilde{F}(\theta, \phi)$ on all of D^2_-, then $\tilde{F}[v_{1-}, v_{2-}]$ would extend $[v_{1+}, v_{2+}]$ to a frame on all of S^2. The primary difficulty is ensuring that $\tilde{F}$ is defined at the coordinate singularity $\theta = \pi$. Note that even if the matrix representation of $\tilde{F}$ is independent of ϕ when $\theta = \pi$, it may still be ill-defined at this point since the matrix representation is written in the $(\boldsymbol{e}_\theta, \boldsymbol{e}_\phi)$ basis and $\boldsymbol{e}_\theta$ is not defined at the singularity. One way of avoiding this issue is to require that $\tilde{F}(\pi, \phi) = \mathbb{1}$ since the matrix representation of the identity is basis independent. Thus, we can interpret the desired function $\tilde{F}(\theta, \phi)$ as a homotopy between $f(\phi)$ and $\mathbb{1}$ with $\theta \in [\pi/2, \pi]$ as the homotopy parameter. By reparameterizing from $t \in [0, 1]$ to $\theta \in [\pi/2, \pi]$ in Eq. (3.35), we obtain the desired function:

$$\tilde{F}(\theta, \phi) = \begin{pmatrix} x(\theta, \phi) + iz(\theta, \phi) & y(\theta, \phi) \\ -y(\theta, \phi) & x(\theta, \phi) - iz(\theta, \phi) \end{pmatrix}, \tag{3.36}$$

$$x(\theta, \phi) = \cos(2\phi)\sin^2(\theta) + \cos^2(\theta), \tag{3.37}$$

$$y(\theta, \phi) = \sin(2\phi)\sin(\theta), \tag{3.38}$$

$$z(\theta, \phi) = -\sin^2(\phi)\sin(2\theta). \tag{3.39}$$

Thus, the two vector fields $[v_1, v_2]$ given by $[v_{1+}, v_{2+}]$ on D^2_+ and $\tilde{F}(\theta, \phi)[v_{1-}, v_{2-}]$ on D^2_- are independent and nonvanishing on all of S^2. While $[v_1, v_2]$ are continuous everywhere, they are not smooth at $\theta = \pi/2$, and thus form a continuous, piecewise smooth frame. This is sufficient to prove that $T^{\mathbb{C}}S^2$ is trivial as a topological vector bundle, but not as a smooth vector bundle.

Step 3: Smoothing the frame

The last step is to modify the previous construction to form smooth vector fields $[\tilde{v}_1, \tilde{v}_2]$. This is done using a smooth step function as follows. Let $g : \mathbb{R} \to \mathbb{R}$ be any smooth monotonic function such that $g(t \leq \pi/2) = \pi/2$, $g(t \geq \pi) = \pi$, and $g^{(n)}(\pi/2) = g^{(n)}(\pi) = 0$ for all $n \geq 1$. For concreteness, we can define

$$h(t) = \begin{cases} e^{-1/t} & t > 0 \\ 0 & t \leq 0, \end{cases} \tag{3.40}$$

and then set

$$g(\theta) = \frac{\pi}{2}\Big(1 + \frac{h(\theta - \pi/2)}{h(\theta - \pi/2) + h(\pi - \theta)}\Big). \tag{3.41}$$

The desired smooth homotopy is obtained by replacing θ by $g(\theta)$ in Eq. (3.36):

$$\tilde{F}_s(\theta, \phi) = \begin{pmatrix} x\big(g(\theta), \phi\big) + iz\big(g(\theta), \phi\big) & y\big(g(\theta), \phi\big) \\ -y\big(g(\theta), \phi\big) & x\big(g(\theta), \phi\big) - iz\big(g(\theta), \phi\big) \end{pmatrix} \tag{3.42}$$

Since g is smooth, this replacement preserves smoothness for $\theta \neq \pi/2$. At $\theta = \pi/2$ we must show that

$$\partial_\theta^\alpha \partial_\phi^\beta [v_{1+}, v_{2+}]|_{\theta \to \pi/2^-} = \partial_\theta^\alpha \partial_\phi^\beta \tilde{F}_s[v_{1-}, v_{2-}]|_{\theta \to \pi/2^+} \tag{3.43}$$

for all nonnegative integers α, β. Since $\boldsymbol{e}_\theta$ and $\boldsymbol{e}_\phi$ are smooth at $\pi/2$, it is sufficient to show Eq. (3.43) holds when $v_{1,\pm}$ and $v_{2,\pm}$ are expressed as column vectors in the $(\boldsymbol{e}_\theta, \boldsymbol{e}_\phi)$ basis. From Eqs. (3.24-3.27), all θ derivatives of the column vectors $v_{1\pm}$ and $v_{2\pm}$ vanish, so the LHS vanishes for $\alpha \geq 1$. Since also $g^{(\alpha)}(\pi/2) = 0$ for $\alpha \geq 1$, all θ derivatives of $\tilde{F}(\theta, \phi)$ vanish as $\theta \to \pi/2^+$, so the RHS is also 0 for $\alpha \geq 1$. In the $\alpha = 0$ case, all ϕ derivatives agree since

$$[v_{1+}, v_{2+}]_{\theta = \pi/2, \phi} = \tilde{F}_s[v_{1-}, v_{2-}]\big|_{\theta = \pi/2, \phi}. \tag{3.44}$$

This completes the proof that the two vector fields $[\tilde{v}_1, \tilde{v}_2]$ given by $[v_{1+}, v_{2+}]$ on D^2_+ and $\tilde{F}_s(\theta, \phi)[v_{1-}, v_{2-}]$ form a smooth global frame on $T^{\mathbb{C}}S^2$.

We can use $\tilde{v}_1$ and $\tilde{v}_2$ to write the trivialization of γ.

$$\bar{v}_{1,2}(\boldsymbol{k}) = \tilde{v}_{1,2}\Big(\frac{\boldsymbol{k}}{|\boldsymbol{k}|}\Big) \tag{3.45}$$

are global nonvanishing sections of γ, and as such generate trivial line subbundles τ_1 and τ_2 of γ. By construction

$$\gamma = \tau_1 \oplus \tau_2. \tag{3.46}$$

3.3 Topology of circularly polarized subbundles and the nonexistence of linearly polarized subbundles

We have shown that the total photon bundle is trivial, possessing global nonvanishing sections. This decomposes γ into two trivial subbundles $\tau_{1,2}$. However, because the base manifold $\mathcal{L}_+$ is not contractible, γ can possess topologically nontrivial subbundles. Indeed, we will show that γ also decomposes into R and L circularly polarized subbundles which are nontrivial. This decomposition is important because, unlike the trivial decomposition, it is Lorentz invariant, and therefore plays a special role in particle physics.

We discuss three important types of photon polarizations— the trivial polarizations constructed in the previous sections which involve elliptical polarization as well as the most frequently used polarization, namely, the linear and circular polarizations. We will show that these three types of polarizations are significantly different in terms of their global properties. The trivial polarizations form subbundles $\tau_{1,2}$ with global nonvanishing sections. We will show that the R and L circularly polarized bundles also form well-defined subbundles of γ, however, they are topologically nontrivial, and therefore do not admit global nonvan-

ishing sections. Linear polarizations, on the other hand, do not even form vector bundles, and are thus not globally well-defined.

Definition 3.2 (Circularly polarized subbundles)**.** *A vector $v \in \gamma(\boldsymbol{k})$ is defined to be R circularly polarized if $v = \alpha(\boldsymbol{e}_1 + i\boldsymbol{e}_2)$ where $\alpha \in \mathbb{C}$ and $\boldsymbol{e}_1, \boldsymbol{e}_2$ are real unit vectors in $\gamma(\boldsymbol{k})$ such that $(\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{k})$ form a right-handed orthogonal coordinate system. The R circularly polarized vectors in $\gamma(\boldsymbol{k})$ form a vector space denoted $\gamma_+(\boldsymbol{k})$. Collected together, these vector spaces form the R circularly polarized vector bundle $\gamma_+ = \amalg_{\boldsymbol{k}} \gamma_+(\boldsymbol{k})$. The L circularly polarized bundle γ_- is defined analogously, with $(\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{k})$ left-handed. These will be called the R and L photon bundles.*

We show that γ_+ and γ_- are in fact well-defined line subbundles of γ via the subbundle criterion ([99], Theorem 20.4):

Lemma 3.10 (Subbundle criterion)**.** *Let $\pi : E \to M$ be a smooth rank r vector bundle and $F = \amalg_{p \in M} F_p$ a subset of E such that for every $p \in M$, the set F_p is a n-dimensional vector subspace of the fiber E_p. If for every $p \in M$, there exist a neighborhood U of p and $m \geq n$ smooth sections $s_1, \ldots, s_m$ of E over U that span F_q at every point $q \in U$, then F is a smooth subbundle of E.*

Theorem 3.11. *γ_+ and γ_- are complex line subbundles of γ such that $\gamma = \gamma_+ \oplus \gamma_-$.*

Proof. We first prove that γ_+ is a subbundle of γ using the subbundle criterion with $m = n = 1$. For $\boldsymbol{k}_0 \in \mathbb{R}^3 \setminus \{0\}$, let U be small ball centered around $\boldsymbol{k}_0$ which does not contain $\mathbf{0}$. U is contractible, so there exists a smooth right-handed orthonormal frame $(\boldsymbol{f}_1, \boldsymbol{f}_2, \hat{\boldsymbol{k}})$ of $\gamma|_U$. $s = \boldsymbol{f}_1 + i\boldsymbol{f}_2$ is then a smooth section of $\gamma|_U$ consisting of right-circularly polarized vectors. To show that s spans $\gamma_+(\boldsymbol{k})$ for each $\boldsymbol{k} \in U$, let $\alpha(\boldsymbol{e}_1 + i\boldsymbol{e}_2) \in \gamma(\boldsymbol{k})$ with $\alpha \in \mathbb{C}$. Since $(\boldsymbol{e}_1, \boldsymbol{e}_2, \hat{\boldsymbol{k}})$ and $(\boldsymbol{f}_1, \boldsymbol{f}_2, \hat{\boldsymbol{k}})$ are both right-handed, they are related by a rotation by some angle θ about $\boldsymbol{k}$:

$$[\boldsymbol{e}_1, \boldsymbol{e}_2] = [\boldsymbol{f}_1, \boldsymbol{f}_2] \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \tag{3.47}$$

Then at $\boldsymbol{k}$ we have

$$\begin{aligned}\alpha e^{-i\theta} s &= \alpha e^{-i\theta}(\boldsymbol{f}_1 + i\boldsymbol{f}_2) \\ &= \alpha[(\cos\theta\, \boldsymbol{f}_1 + \sin\theta\, \boldsymbol{f}_2) + i(-\sin\theta\, \boldsymbol{f}_1 + \cos\theta\, \boldsymbol{f}_2)] \\ &= \alpha(\boldsymbol{e}_1 + i\boldsymbol{e}_2),\end{aligned} \tag{3.48}$$

showing that s spans $\gamma_+(\boldsymbol{k})$. Thus, γ_+ is a rank 1 subbundle of γ by the subbundle criterion, as is γ_- by an analogous argument. For fixed $\boldsymbol{k}$, R and L polarizations form a basis for all polarization states. Allowing $\boldsymbol{k}$ to vary then shows that $\gamma = \gamma_+ \oplus \gamma_-$. □

An intuitive way to see that the complex line bundles $\gamma_\pm$ are nontrivial is to consider the underlying rank 2 real vector bundles $\gamma_\pm^{\mathbb{R}}$ obtained by forgetting the complex structure on $\gamma_\pm$. That is, $\gamma_\pm^{\mathbb{R}}$ consists of the same base space and fibers as $\gamma_\pm$, but scalar multiplication is restricted to $\mathbb{R}$.

Theorem 3.12. *The real vector bundles $\gamma_\pm^{\mathbb{R}}|_{S^2}$ and TS^2 are isomorphic, which implies that γ_+ and γ_- are nontrivial.*

Proof. Let $(\boldsymbol{k}, v) \in TS^2$ and let $R_{\boldsymbol{k}}(\frac{\pi}{2})$ denote a $\pi/2$ rotation about $\boldsymbol{k}$ in the positive sense. Define the functions $h_\pm : TS^2 \to \gamma_\pm^{\mathbb{R}}|_{S^2}$ by

$$h_\pm(\boldsymbol{k}, v) = \Big(\boldsymbol{k}, v \pm iR_{\boldsymbol{k}}\Big(\frac{\pi}{2}\Big)v\Big). \tag{3.49}$$

$h_\pm$ is a clearly a smooth, injective bundle map of real vector bundles over S^2. To show that it is also surjective, and thus an isomorphism, note that by Definition 3.2 any vector $u_\pm \in \gamma_\pm^{\mathbb{R}}|_{S^2}(\boldsymbol{k})$ can be expressed as

$$u_\pm = \alpha\Big(w \pm iR_{\boldsymbol{k}}\Big(\frac{\pi}{2}\Big)w\Big) \tag{3.50}$$

for some $w \in TS^2(\boldsymbol{k})$ and $\alpha = (a + ib) \in \mathbb{C}$. This can be rewritten as

$$u_\pm = w' \pm iR_{\boldsymbol{k}}w' = h_\pm(w') \tag{3.51}$$

$$w' \doteq aw \mp bR_{\boldsymbol{k}}\left(\frac{\pi}{2}\right)w \in TS^2(\boldsymbol{k}), \tag{3.52}$$

showing that $h_\pm$ is surjective, and thus $\gamma_\pm^{\mathbb{R}}|_{S^2} \cong TS^2$.

If $\gamma_\pm$ were trivial there would exist a nonvanishing section of s of $\gamma_\pm|_{S^2}$, which can then be considered as a nonvanishing section of $\gamma_\pm^{\mathbb{R}}|_{S^2} \cong TS^2$ which contradicts the hairy ball theorem. □

While the previous theorem shows that $\gamma_\pm$ are nontrivial, it remains to classify them as complex line bundles. As with γ, it is easier to work with the restrictions $\gamma_\pm|_{S^2}$. $\gamma_+|_{S^2}$ and $\gamma_-|_{S^2}$ are quite similar as they are conjugate bundles $\gamma_+|_{S^2} = \bar{\gamma}_-|_{S^2}$. Furthermore, they are isomorphic when considered as real vector bundles by the previous theorem. However, they are not actually isomorphic as complex vector bundles, as the complex structure detects their different orientations. To see this, note that by Lemma 3.5, the first Chern number of a line bundle and its conjugate differ by a sign:

$$C_1(\gamma_+|_{S^2}) = -C_1(\gamma_-|_{S^2}). \tag{3.53}$$

Complex line bundles over S^2 are completely classified by their first Chern number C_1 by Theorem 2.3, so $C_1(\gamma_\pm|_{S^2}) \neq 0$ as these are nontrivial bundles. Eqn. (3.53) then implies these bundles are not isomorphic. The next result fully classifies $\gamma_\pm|_{S^2}$. By the correspondence of bundles over S^2 and $\mathbb{R}^3 \setminus \{0\}$, this also classifies $\gamma_\pm$.

Theorem 3.13. $C_1(\gamma_\pm|_{S_2}) = \mp 2$.

Proof. We calculate the Chern numbers using the Berry connection. The Hermitian structure along with the identification of the electric fields of the fibers with vectors in $\mathbb{C}^3$ gives a natural Berry connection. For any local frame $e_\pm(\boldsymbol{k})$ of $\gamma_\pm|_{S^2}$ in some neighborhood U, the

Berry connection form is given by Eq. (2.17)

$$\omega_\pm = \langle e_\pm, de_\pm \rangle. \tag{3.54}$$

On the neighborhood U consisting of all unit $\hat{\boldsymbol{k}}$ not on the z-axis, we can choose the frames for $\gamma_\pm|_{S^2}$ to be

$$e_\pm = \frac{1}{\sqrt{2}}(\boldsymbol{e}_\theta \pm i\boldsymbol{e}_\phi). \tag{3.55}$$

Then,

$$\omega_\pm = \mp i \cos\theta \, d\phi. \tag{3.56}$$

and the curvature form is given on U by

$$\Omega_\pm = d\omega_\pm = \pm i \sin\theta \, d\theta \wedge d\phi = \pm i \, dA \tag{3.57}$$

where dA is the Euclidean area form on S^2. Unlike the connection forms, the curvature forms are always globally defined. Since $\Omega_\pm = \pm i dA$ on a dense neighborhood U, this relation holds on all of S^2. The first Chern number is then given by [99]

$$C_1(\gamma_\pm|_{S^2}) = \frac{i}{2\pi}\int_{S^2} \Omega_\pm = \mp 2. \tag{3.58}$$

Since C_1 is nonzero, $\gamma_\pm$ are nontrivial. □

One consequence of the nontriviality of $\gamma_\pm$ can be found in the momentum space representation of the photon wave function [72, 113]. In this representation, the wave function of a photon is written as a two component wave function, with an R and L part

$$\boldsymbol{f}(\boldsymbol{k}) = \begin{pmatrix} \boldsymbol{f}_+(\boldsymbol{k}) \\ \boldsymbol{f}_-(\boldsymbol{k}) \end{pmatrix} = \begin{pmatrix} f_+(\boldsymbol{k})\tilde{e}_+(\boldsymbol{k}) \\ f_-(\boldsymbol{k})\tilde{e}_-(\boldsymbol{k}) \end{pmatrix}. \tag{3.59}$$

where $\tilde{e}_+(\boldsymbol{k})$ is some choice of unit R polarization vector and $e_-(\boldsymbol{k}) \doteq e_+^*(\boldsymbol{k})$ is a corresponding L polarization. Frequently, the vectors $\tilde{e}_\pm$ are left implicit, and the wave function is written in terms of the two scalar functions $f_\pm$ as

$$\boldsymbol{f} = \begin{pmatrix} f_+(\boldsymbol{k}) \\ f_-(\boldsymbol{k}) \end{pmatrix}. \tag{3.60}$$

However, the $\tilde{e}_\pm$ are sections of the nontrivial bundles $\gamma_\pm$, and thus no continuous choice of $\tilde{e}_\pm$ exists! If S_a denotes the sphere of radius $a > 0$ in momentum space $\mathbb{R}^3 \setminus \{0\}$, then $\tilde{e}_+$ must be discontinuous for at least one point $\boldsymbol{k}_0(a)$ on each S_a. For $\boldsymbol{f}(\boldsymbol{k})$ to be continuous at $\boldsymbol{k}_0(a)$, the component functions must satisfy $f_\pm(\boldsymbol{k}_0(a)) = 0$. Thus, the photon wave function components $f_\pm$ are not free scalar functions. They must obey the topological constraint of having a zero on every closed surface enclosing $\boldsymbol{k} = 0$.

Our calculation of $C_1(\gamma_\pm)$ is very similar to one done by Bliokh *et al.* [104], although they use a different sign convention for the Berry connection and Chern number, resulting in an overall sign difference. Our approach differs in that it emphasizes the underlying vector bundle structures, and in particular, shows that $\gamma_\pm$ are in fact well-defined bundles. This is important, for example, because by analogy one might assume that linearly polarized photons also possess some topological structure. However, we show that unlike the circularly polarized subbundles $\gamma_\pm$ or the trivial subbundles $\tau_{1,2}$, there are no linearly polarized subbundles.

Definition 3.3 (Linearly polarized subbundle)**.** *A vector $v \in \gamma_{\boldsymbol{k}}$ is linearly polarized if it is a scalar multiple of a real polarization vector, that is, if $v = \alpha w$ for some $\alpha \in \mathbb{C}$ and $w \in TS^2$. A bundle is linearly polarized if it only contains linearly polarized vectors.*

Theorem 3.14. *There are no linearly polarized subbundles of γ.*

To prove Theorem 3.14 we use the following known result.

Lemma 3.15. *Any real line bundle over a simply connected space is trivial. In particular, all real line bundles over S^2 are trivial.*

Proof. A real line bundle is trivial if and only if it is orientable [114]. The result then follows from the fact that every real line bundle over a simply connected space is orientable ([101], Proposition 11.5). □

We can now prove Theorem 3.14 via a contradiction:

Proof of Theorem 3.14. Suppose γ_{lin} is a linearly polarized line subbundle of γ. Then $\tilde{\gamma}_{lin} \doteq \gamma_{lin}|_{S^2}$ is a subbundle of $\gamma|_{S^2} \cong T^{\mathbb{C}}S^2$. Each fiber $\tilde{\gamma}_{lin}(\hat{\boldsymbol{k}})$ contains a 1D real subspace $\gamma_{\mathbb{R}}(\boldsymbol{k})$. The collection of these subspaces $\gamma_{\mathbb{R}} = \amalg_{\hat{\boldsymbol{k}}}\gamma_{\mathbb{R}}(\hat{\boldsymbol{k}})$ forms a real line bundle over S^2 which is a subbundle of TS^2 . By Corollary 3.15, $\gamma_{\mathbb{R}}$ is trivial and must therefore have a nonvanishing section s. However, s would also then be a nonvanishing section of TS^2, violating the hairy ball theorem. □

One can also prove this with an alternative method using the Euler class, as will done for gravitons in Theorem 4.4.

We have shown that γ decomposes into both the trivial bundles $\tau_{1,2}$ and the nontrivial subbundles $\gamma_{\pm}$. These are not, however, the only possible decompositions. In fact, γ has an infinite number of topologically inequivalent decompositions by the following observation. The complex line bundles over $\mathbb{R}^3 \setminus \{0\}$ are labeled by their Chern number (Theorem 2.3); denote the line bundle with Chern number $j \in \mathbb{Z}$ by ℓ_j.

Theorem 3.16. *$\gamma \cong \ell_j \oplus \ell_{-j}$ for every $j \in \mathbb{Z}$. In particular, γ has a subbundle isomorphic to ℓ_j for every j.*

Proof. By the Whitney product formula [99], the first Chern number is additive, meaning that if E_1 and E_2 are vector bundles, then

$$C_1(E_1 \oplus E_2) = C_1(E_1) + C_1(E_2). \tag{3.61}$$

Thus,

$$C_1(\ell_j \oplus \ell_{-j}) = 0. \tag{3.62}$$

Again by Theorem 2.3, complex rank 2 vector bundles over $\mathbb{R}^3 \setminus \{0\}$ are also labeled by their first Chern number. Thus, $\ell_j \oplus \ell_{-j}$ is trivial and therefore isomorphic to γ. □

Despite the existence of these other nontrivial subbundles, only the circularly polarized subbundles $\gamma_\pm \cong \ell_{\mp 2}$ are typically used in applications. From a practical standpoint this is easy to understand, as $\gamma_\pm$ have simple descriptions, while finding explicit expressions for subbundles of γ isomorphic to ℓ_j is a nontrivial task. However, from a purely topological perspective, there is no reason to prefer the circularly polarized subbundles over the other ℓ_j. We will show in the next section that $\gamma_\pm$ are special from a geometric perspective when we consider the role of Poincaré symmetry.

3.4 Classification of photons and other massless particles by vector bundle representations of the Poincaré group

In this section we show how Poincaré symmetries of vector bundles can be used to classify massless particles such as photons. In quantum field theory, particles are unitary irreducible representations of the Poincaré group on a Hilbert space [43, 44]. We restrict our discussion to non-projective representations since this includes the photon case. The conventional method for constructing and classifying such representations is via Wigner's little group method ([42]; [43], Ch. 2). While this method produces smooth representations of massive particles, obstructions form in the massless case with the space of single-particle states becoming discontinuous [45, 85], resulting in nonsmooth representations on the Hilbert space of wave functions. By this we mean Poincaré transformations generally map smooth wave functions to nonsmooth wave functions. We will see that this issue traces back to a topological singularity that occurs in the $m \to 0$ limit as the mass hyperboloid of momentum space becomes the topologically nontrivial light cone. We will show that by considering vector

bundle representations on the single-particle states rather than vector space representations, one can obtain globally well-defined representations on the single-particle states. We prove that massless unitary irreducible vector bundle representations of the Poincaré group naturally induce unitary irreducible representations on the Hilbert space of L^2 bundle sections, and therefore such bundle representations correspond to particles by the usual definition. This result appears to be a special case of Mackey's theorem ([115], Theorem 14.2), but we give a novel proof of this which relies on much simpler methods than Mackey's general result. We show that Wigner's little group method generalizes to this bundle formalism, and can be used to decompose bundle representations into irreducible representations and classify these irreducible representations. In the case of the photon bundle, this induces the decomposition $\gamma = \gamma_+ \oplus \gamma_-$, showing that R and L photons are globally well-defined particles. This implies that any global photon wave function can be uniquely decomposed into R and L photon components, even though there exist no global bases for the R and L photons.

Our results complement the body of work which has further developed Wigner's original version of the little group method. Mackey [115–117] developed the mathematician's version of this theory which generalized the little group method from the Poincaré group to any locally compact group. Simms [47] showed that Mackey's theory had an underlying vector bundle structure, using what he called G-Hilbert bundles which are very similar to the vector bundle representations which we use. However, Simms worked with only topological vector bundles, not smooth vector bundles, and did not relate his work to the smoothness problem with Wigner's representations. Indeed, Simms's bundle formalism did not offer obvious advantages over Mackey's theory, and Mackey referred to the bundle formulation as a "digression" [117]. However, we show that our bundle formalism produces smooth representations for massless particles. Asorey *et al.* [48] connected the representations developed by Mackey and Simms [47, 115–117] to the the covariant representations more commonly used in physics. Boya *et al.* appear to be the first to identify that the massless Wigner representations had singularities, although no fix was proposed. Flato *et al.* [45] also noted

singularities in the these massless representations of the Poincaré group. Flato *et al.* remedied this issue by developing a theory of twisted delta ("twelta") functions, which they use to represent single-particle states. This produced smooth vector space representations on the functionals of sections of vector bundles. They primarily use field-theoretic techniques, with line bundles appearing implicitly via transformation properties of functions. Very recently, Dragon [85] also approached this problem, producing massless representations on sections of vector bundles using a similar implicit framework. The formalism we develop is based on the notion of symmetries on vector bundles and differs from previous methods in that it considers Poincaré actions directly on smooth vector bundles rather only on their sections. This results in particularly explicit massless representations for massless particles. We will show in this chapter and in the the following chapters that this framework is particularly useful for answer questions related to the angular momentum of massless particles.

This section is organized as follows. In §3.4.1 we define some notation related to relativistic symmetry groups. In §3.4.2, we discuss vector bundle representations of symmetry groups. In §3.4.3, we illustrate the issues with the conventional little group method for constructing irreducible representations for massless particles. In §3.4.4, we show that these issues can be overcome by developing a modified version of the little group method for bundle representations of massless particles. In particular, we will see that R and L photons are globally well-defined as irreducible bundle representations of the Poincaré group, and are thus globally well-defined as particles. In part 3.4.5 we prove that unitary irreducible massless bundle representations induce unitary irreducible Hilbert space representations, and thus correspond to particles in the usual sense.

3.4.1 Geometric notation and conventions

We use the $(-,+,+,+)$ signature for Minkowski space, with the minus sign on the time component. The spacetime metric is denoted by

$$\bar{\eta}^{\mu\nu} = \bar{\eta}_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{3.63}$$

The Poincaré group, consisting of all isometries of Minkowski space, is denoted by $\mathrm{IO}(3,1)$, where the "I" indicates the inclusion of inhomogeneous transformations. Likewise, the Euclidean group $\mathrm{ISO}(n) \cong \mathbb{R}^n \rtimes \mathrm{SO}(n)$ is the group of all isometries of n-dimensional Euclidean space and $\rtimes$ denotes the semidirect product ([38], §18.1). The improper Lorentz group $\mathrm{O}(3,1)$ is the subset of $\mathrm{IO}(3,1)$ consisting of only homogeneous isometries. A generic element of $\mathrm{IO}(3,1)$ can be expressed uniquely as $T_{\mathsf{a}} \cdot \Lambda$, where T_{a} is a spacetime translation by $\mathsf{a} \in \mathbb{R}^{3+1}$, $\Lambda \in \mathrm{O}(3,1)$, and $\cdot$ denote the group multiplication. In general the group law is given by ([38], §42.1)

$$(T_{\mathsf{a}_1} \cdot \Lambda_1) \cdot (T_{\mathsf{a}_2} \cdot \Lambda_2) = T_{\mathsf{a}_1 + \Lambda \mathsf{a}_2} \cdot \Lambda_1 \Lambda_2 \tag{3.64}$$

The orthochronous Poincaré and Lorentz groups, $\mathrm{IO}^+(3,1)$ and $\mathrm{O}^+(3,1)$, consist of transformations preserving the orientation of time:

$$\mathrm{IO}^+(3,1) = \{T_{\mathsf{a}} \cdot \Lambda \in \mathrm{IO}(3,1) | \Lambda^0{}_0 \geq +1\} \tag{3.65}$$

$$\mathrm{O}^+(3,1) = \{\Lambda \in \mathrm{O}(3,1) | \Lambda^0{}_0 \geq +1\}. \tag{3.66}$$

The proper orthochronous Poincaré and Lorentz groups, $\mathrm{ISO}^+(3,1)$ and $\mathrm{SO}^+(3,1)$, are the further restrictions of $\mathrm{IO}^+(3,1)$ and $\mathrm{O}^+(3,1)$ to transformations preserving the orientation of space, that is, with $\det \Lambda = 1$. The parity and time inversion operators are the Lorentz

transformations defined by their action on the four-vector $x = (x^0, \boldsymbol{x})$ by

$$\mathsf{P}(x^0, x^i) = (x^0, -x^i) \tag{3.67}$$

$$\mathsf{T}(x^0, x^i) = (-x^0, x^i). \tag{3.68}$$

$\mathrm{SO}^+(3,1)$ is the connected identity component of $\mathrm{O}(3,1)$ and contains neither P nor T. $\mathrm{O}^+(3,1)$ contains P but not T. As is common in physics, we will often simply refer to $\mathrm{SO}^+(3,1)$ and $\mathrm{ISO}^+(3,1)$ as the Lorentz and Poincaré groups, respectively.

3.4.2 Representations on vector bundles

Representation theory is the method of representing groups by their actions on a linear space. Typically this linear space is taken to be a vector space. However, vector bundles also possess a linear structure, albeit one that is more complicated than that of a vector space. We now define a representation of a Lie group on a vector bundle.

Definition 3.4 (Vector bundle representation)**.** *Let G be a Lie group. A vector bundle representation of G is a vector bundle $\pi : E \to M$ equipped with a smooth group action $\Sigma : G \times E \to E$ such that for each $g \in G$, $\Sigma_g = \Sigma(g, \cdot) : E \to E$ is a vector bundle isomorphism. In particular, there is a corresponding diffeomorphism $\sigma_g : M \to M$ such that*

$$\begin{array}{ccc} E & \xrightarrow{\Sigma_g} & E \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi} \\ M & \xrightarrow{\sigma_g} & M \end{array} \tag{3.69}$$

commutes. The action of g on a vector $v \in E$ or basepoint $m \in M$ is sometimes written with the shorthand $gv \doteq \Sigma_g v$ or $gm \doteq \sigma_g m$ when there is no chance of ambiguity. We also frequently use the notation $\Sigma(g) \doteq \Sigma_g$.

E is said to be a homogeneous bundle representation if the action σ on M is transitive.[1] *If E is a Hermitian bundle, then we say the representation is unitary if the restriction of Σ_g to any fiber is unitary, i.e., if $\Sigma_g|_{E_p} : E_p \to E_{\sigma_g(p)}$ is unitary. If F is a subbundle of E with nonzero rank, then we say (F, Σ) is a subrepresentation of (E, Σ) if Σ restricts to an action of G on F. E is irreducible if it has no proper subrepresentations.*

Definition 3.5 (Isomorphism of vector bundle representations)**.** *An isomorphism between vector bundle representation $\pi_1 : E_1 \to M$ and $\pi_2 : E_2 \to M$ with group actions Σ_1 and Σ_2 is a vector bundle isomorphism $f : E_1 \to E_2$ which preserves base points,*

$$f(E_1(m)) = E_2(m) \tag{3.70}$$

for $m \in M$, and which satisfies the equivariance property

$$f\big(\Sigma_1(g)v\big) = \Sigma_2(g)fv \tag{3.71}$$

for any $g \in G$ and $v \in E_1$. f is said to be a unitary isomorphism if the actions Σ_1 and Σ_2 are unitary, and if f is unitary. If such an isomorphism exists, the representations are said to be equivalent or unitarily equivalent.

These definitions are analogous to those for ordinary representations, with vector spaces and linear maps replaced by vector bundles and bundle maps. As for ordinary representations, a representation of a Lie group G on a vector bundle E describes a G-symmetry of the system represented by E. We show that the photon bundle γ is a unitary representation of the orthochronous Poincaré group, $\mathrm{IO}^+(3,1)$, describing the Poincaré symmetry of the Maxwell system. We note that it is possible to construct a bundle representation of the full Poincaré group $\mathrm{IO}(3,1)$ if one includes the backward light cone in the base manifold of γ. However, little is gained by doing so and it introduces two complications. First, the full light

[1] Meaning that for every $m_1, m_2 \in M$ there exists a $g \in G$ such that $\sigma_g m_1 = m_2$.

cone is not connected while the forward light $\mathcal{L}_+$ cone is. Second, the representation would no longer be unitary since the action of time-inversion is anti-unitary [118].

Let $(k, \boldsymbol{E}) \in \gamma$ where $k = (|\boldsymbol{k}|, \boldsymbol{k})$ is a lightlike four-vector. Both k and $\boldsymbol{E}$ have well-known transformation laws under Lorentz transformations. For $\Lambda \in \mathrm{O}(3,1)$, k transforms like a four-vector: $k \to k' = (|\boldsymbol{k}'|, \boldsymbol{k}') = \Lambda k$. As $\mathcal{L}_+$ can be parameterized by $\boldsymbol{k}$, we sometimes write $\boldsymbol{k}' = \Lambda \boldsymbol{k}$ for the spatial part of k'. When the base manifold a subset of four-momentum space, we will always assume that the action σ on the base manifold is the four-vector action unless otherwise stated (the only other action considered will be $\sigma = \mathbb{1}$). Under the subset of $\mathrm{O}(3,1)$ consisting of rotations in 3-space, $\boldsymbol{E}$ transforms like a 3-vector, since 3D rotations and the Fourier transform commute. Let $\Lambda_{\boldsymbol{v}}$ denote a boost by velocity $\boldsymbol{v}$. Under this transformation, the electric field $\boldsymbol{\mathcal{E}}(x)$ becomes [119][2]

$$\begin{aligned} \boldsymbol{\mathcal{E}}'(\Lambda_{\boldsymbol{v}} x) &= \gamma_{Lor}[\boldsymbol{\mathcal{E}}(x) - \boldsymbol{v} \times \boldsymbol{\mathcal{B}}(x)] - \frac{\gamma_{Lor}^2}{\gamma_{Lor}+1}(\boldsymbol{\mathcal{E}}(x)\cdot\boldsymbol{v})\boldsymbol{v}, \\ \gamma_{Lor} &= \frac{1}{\sqrt{1-v^2}}. \end{aligned} \tag{3.72}$$

Then,

$$\begin{aligned} \tilde{\boldsymbol{E}}'(\Lambda_{\boldsymbol{v}} k) &= \int e^{-i(\Lambda_{\boldsymbol{v}} k)^\mu x'_\mu} \boldsymbol{\mathcal{E}}'(x')\, d^4x' \\ &= \int e^{-i(\Lambda_{\boldsymbol{v}} k)^\mu (\Lambda_{\boldsymbol{v}} x)_\mu} \boldsymbol{\mathcal{E}}'(\Lambda_{\boldsymbol{v}} x) \det(\Lambda_{\boldsymbol{v}})\, d^4x \\ &= \int e^{-ik^\mu x_\mu} \boldsymbol{\mathcal{E}}'(\Lambda_{\boldsymbol{v}} x)\, d^4x \\ &= \gamma_{Lor}[\tilde{\boldsymbol{E}}(k) - \boldsymbol{v} \times (\hat{\boldsymbol{k}} \times \tilde{\boldsymbol{E}}(k))] - \frac{\gamma_{Lor}^2}{\gamma_{Lor}+1}(\tilde{\boldsymbol{E}}(k)\cdot\boldsymbol{v})\boldsymbol{v}. \end{aligned} \tag{3.73}$$

[2]We will always use active Lorentz transformations. Jackson [119] uses passive transformations so it is necessary to make the replacement $\boldsymbol{v} \mapsto -\boldsymbol{v}$.

Note that for two 4-vectors $a^\mu = (a^0, \boldsymbol{a})$ and $b^\mu = (b^0, \boldsymbol{b})$, both $\delta^4(a^\mu - b^\mu) = \delta(a^0 - b^0)\delta^3(\boldsymbol{a} - \boldsymbol{b})$ and $a^0\delta^3(\boldsymbol{a} - \boldsymbol{b})$ are Lorentz invariant [43], and therefore so is $(a^0)^{-1}\delta(a^0 - b^0)$:

$$\frac{\delta\Big((\Lambda a)^0 - (\Lambda b)^0\Big)}{(\Lambda a)^0} = \frac{\delta(a^0 - b^0)}{a^0} \tag{3.74}$$

for any Poincaré transformation Λ. It then follows from Eqs. (3.7), (3.73), and (3.74) that

$$\boldsymbol{E}'(\Lambda_{\boldsymbol{v}}\boldsymbol{k}) = \frac{|\boldsymbol{k}|}{|\Lambda_{\boldsymbol{v}}\boldsymbol{k}|}\Big[\gamma_{Lor}[\boldsymbol{E}(\boldsymbol{k}) - \boldsymbol{v} \times (\hat{\boldsymbol{k}} \times \boldsymbol{E}(\boldsymbol{k}))] - \frac{\gamma_{Lor}^2}{\gamma_{Lor} + 1}(\boldsymbol{E}(\boldsymbol{k}) \cdot \boldsymbol{v})\boldsymbol{v}\Big], \tag{3.75}$$

and thus the action of $\Lambda_{\boldsymbol{v}}$ on $(\boldsymbol{k}, \boldsymbol{E}) \in \gamma$ is given by

$$\Sigma_{\Lambda_v}(\boldsymbol{k}, \boldsymbol{E}) = (\Lambda_{\boldsymbol{v}}\boldsymbol{k}, \boldsymbol{E}') \tag{3.76}$$

$$\boldsymbol{E}' \doteq \frac{|\boldsymbol{k}|}{|\Lambda_{\boldsymbol{v}}\boldsymbol{k}|}\Big[\gamma_{Lor}[\boldsymbol{E} - \boldsymbol{v} \times (\hat{\boldsymbol{k}} \times \boldsymbol{E})] - \frac{\gamma_{Lor}^2}{\gamma_{Lor} + 1}(\boldsymbol{E} \cdot \boldsymbol{v})\boldsymbol{v}\Big]. \tag{3.77}$$

Similar calculations show that spatial inversion P and spacetime translations by $\mathsf{a} = (\mathsf{a}^0, \mathbf{a})$ act via

$$\Sigma_{\mathsf{P}}(\boldsymbol{k}, \boldsymbol{E}) = (-\boldsymbol{k}, -\boldsymbol{E}), \tag{3.78}$$

$$\Sigma_{T_{\mathsf{a}}}(\boldsymbol{k}, \boldsymbol{E}) = (\boldsymbol{k}, e^{i(-\boldsymbol{k}\cdot\mathbf{a} + |\boldsymbol{k}|\mathsf{a}^0)}\boldsymbol{E}). \tag{3.79}$$

Any element of $\mathrm{IO}^+(3,1)$ can be expressed as a combination of boosts, spatial rotations, spacetime translations, and spatial inversion, so together these relations define an action of $\mathrm{IO}^+(3,1)$ on γ. An essential property is that k transforms independently of $\boldsymbol{E}$, and thus the Poincaré action preserves fibers: $\Sigma(\Lambda)\gamma(k) \subseteq \gamma(\sigma(\Lambda)k) = \gamma(\Lambda k)$. This ensures that elements of $\mathrm{IO}^+(3,1)$ are represented by bundle maps, and therefore we have a representation on γ. We have thus proved most of the following theorem.

Theorem 3.17. *γ is a homogeneous vector bundle representation under the usual Poincaré transformations of the electric field described above. This representation is unitary with respect to the Hermitian product (3.13).*

Proof. The representation is homogeneous since $\mathrm{IO}^+(3,1)$ acts transitively on the base manifold $\mathcal{L}_+$. It remains to prove unitarity. The Hermitian product induces a norm on each fiber $\gamma(\boldsymbol{k})$ given by $|\boldsymbol{E}| = \sqrt{\boldsymbol{E}^* \cdot \boldsymbol{E}}$. The inner product of two arbitrary vectors can be expressed purely in terms of the norm via the polarization identity [120]:

$$\langle \boldsymbol{E}_1, \boldsymbol{E}_2 \rangle = \frac{1}{4}\big(|\boldsymbol{E}_1 + \boldsymbol{E}_2|^2 - |\boldsymbol{E}_1 - \boldsymbol{E}_2|^2 - i|\boldsymbol{E}_1 + i\boldsymbol{E}_2|^2 + i|\boldsymbol{E}_1 - i\boldsymbol{E}_2|^2\big) \tag{3.80}$$

It is thus sufficient to check that

$$\langle \Sigma_\Lambda \boldsymbol{E}, \Sigma_\Lambda \boldsymbol{E} \rangle = \langle \boldsymbol{E}, \boldsymbol{E} \rangle \tag{3.81}$$

for all $\Lambda \in \mathrm{IO}^+(3,1)$. Furthermore, it is sufficient to check the cases when Λ is a spatial rotation, spatial inversion, spacetime translation, or boost. The first three are trivial, and for the latter one can show directly from Eq. (3.77) that

$$\boldsymbol{E}'^* \cdot \boldsymbol{E}' = \boldsymbol{E}^* \cdot \boldsymbol{E} \tag{3.82}$$

which completes the proof. □

As in the case of vector space representations, a representation Σ of a Lie group G on a vector bundle $\pi : E \to M$ induces an action η of the Lie algebra $\mathrm{Lie}(G)$ on that bundle in the following sense. For $\mathfrak{g} \in \mathrm{Lie}(G)$, the generator $\eta_\mathfrak{g}$ associates to each vector $(m, \boldsymbol{v}) \in E$ a tangent vector in $T_{(m,\boldsymbol{v})}E$ describing the infinitesimal group action. Explicitly,

$$\eta_\mathfrak{g}(m, \boldsymbol{v}) \doteq \frac{d}{dt}\Big|_{t=0} \Sigma_{\exp(t\mathfrak{g})}(m, \boldsymbol{v}) \in T_{(m,\boldsymbol{v})}E. \tag{3.83}$$

Thus, $\eta_{\mathfrak{g}} \in \mathfrak{X}(E)$ where $\mathfrak{X}(E)$ is the set of vector fields on E. The next result says that $\eta_{\mathfrak{g}}$ respects the vector bundle structure of E in the sense that it is a lift of the corresponding vector field on the base manifold M induced by $\tilde{\Sigma}$. Define the vector field $\bar{\eta} \in \mathfrak{X}(M)$ by

$$\bar{\eta}_{\mathfrak{g}}(m) \doteq \frac{d}{dt}\Big|_{t=0} \sigma_{\exp(t\mathfrak{g})}(m) \in T_m M. \tag{3.84}$$

Proposition 3.18.

$$\bar{\eta}_{\mathfrak{g}} \circ \pi = \pi_* \circ \eta_{\mathfrak{g}} \tag{3.85}$$

where π_* is the pushforward of π. That is, the diagram

$$\begin{array}{ccc} E & \xrightarrow{\eta_{\mathfrak{g}}} & TE \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi_*} \\ M & \xrightarrow{\bar{\eta}_{\mathfrak{g}}} & TM \end{array} \tag{3.86}$$

commutes.

Proof. $\Sigma_{\exp(t\mathfrak{g})}(m, \boldsymbol{v})$ is a curve through $(m, \boldsymbol{v}) \in E$ with tangent vector $\eta_{\mathfrak{g}}(m, \boldsymbol{v})$ at $t = 0$. So

$$\begin{aligned} \pi_* \circ \eta_{\mathfrak{g}}(m, \boldsymbol{v}) &= \frac{d}{dt}\Big|_{t=0} \pi \circ \Sigma_{\exp(t\mathfrak{g})}(m, \boldsymbol{v}) \\ &= \frac{d}{dt}\Big|_{t=0} \sigma_{\exp(t\mathfrak{g})} \pi(m, \boldsymbol{v}) \\ &= \tilde{\eta}_{\mathfrak{g}} \circ \pi(m, \boldsymbol{v}). \end{aligned} \tag{3.87}$$

□

$\mathfrak{X}(E)$ is a Lie algebra with respect to the Jacobi-Lie bracket

$$[X, Y]_- = -(XY - YX). \tag{3.88}$$

The minus sign in this definition is needed for the next result and accounts for the fact that Σ and η act on the left rather than on the right. The following proposition justifies calling η a Lie algebra bundle representation.

Proposition 3.19. *$\eta : \mathrm{Lie}(G) \to \mathfrak{X}(E)$ is a Lie algebra homomorphism.*

Proof. See Ref. [121], Theorem 20.18 (a). □

Following the notation of Weinberg [43], §2.4, the infinitesimal Poincaré transformations with homogeneous and translational parts

$$\Lambda^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu} + \omega^{\mu}{}_{\nu} \tag{3.89}$$

$$\mathsf{a}^{\mu} = \epsilon^{\mu} \tag{3.90}$$

define the generators of the Poincaré group $J^{\mu\nu}$ and P^{μ} by

$$\Sigma(I + \omega, \epsilon) = 1 + \frac{1}{2} i\omega_{\sigma\eta} J^{\sigma\eta} - i\epsilon_{\rho} P^{\rho} + O(\omega^2, \epsilon^2, \omega\epsilon) \tag{3.91}$$

where $\omega_{\sigma\eta}$ is anti-symmetric. The generators are independent of ω- and ϵ-independent and go by standard names: the Hamiltonian $H = P^0$ generating time translation, momentum $\boldsymbol{P} = \{P^1, P^2, P^3\}$ generating spatial translations, angular momentum $\boldsymbol{J} = \{J^{23}, J^{31}, J^{12}\} \doteq \{J_1, J_2, J_3\}$ generating spatial rotations, and boost $\boldsymbol{K} = \{J^{10}, J^{20}, J^{30}\} \doteq \{K_1, K_2, K_3\}$ generating the boost transformations. Given $\boldsymbol{v} \in \mathbb{R}^3$, these also define the generators $J_{\boldsymbol{v}} = \boldsymbol{v} \cdot \boldsymbol{J}$ and $K_{\boldsymbol{v}} = \boldsymbol{v} \cdot \boldsymbol{K}$. The generators satisfy the commutation relations

$$[J_a, J_b] = i\epsilon_{abc} J_c\,, \quad [J_a, K_b] = i\epsilon_{abc} K_c \tag{3.92}$$

$$[K_a, K_b] = -i\epsilon_{abc} J_c\,, \quad [J_a, P_b] = i\epsilon_{abc} P_c \tag{3.93}$$

$$[K_a, P_b] = iH\delta_{ab}\,, \quad [K_a, H] = iP_a \tag{3.94}$$

$$[J_a, H] = [P_a, H] = [P_a, P_b] = [H, H] = 0. \tag{3.95}$$

The generators are related to finite rotations, boosts, and translations via

$$\Sigma\big(R(\boldsymbol{\theta})\big) = e^{-i\boldsymbol{\theta}\cdot\boldsymbol{J}} \tag{3.96}$$

$$\Sigma(\Lambda_{\boldsymbol{\zeta}}) = e^{i\boldsymbol{\zeta}\cdot\boldsymbol{K}} \tag{3.97}$$

$$\Sigma(T_{\mathsf{a}}) = e^{i(\mathsf{a}^0 H - \mathbf{a}\cdot\boldsymbol{P})}. \tag{3.98}$$

Here, $\boldsymbol{\theta} = \theta\hat{\boldsymbol{\theta}}$, $R(\boldsymbol{\theta}) \in \mathrm{SO}(3)$ is rotation by θ about $\hat{\boldsymbol{\theta}}$, $\Lambda_{\boldsymbol{\zeta}}$ is a boost with rapidity $\boldsymbol{\zeta}$, and T_{a} is a translation by $\mathsf{a} = (\mathsf{a}^0, \mathbf{a})$. These sign conventions are consistent with the textbooks of Woit [38] and Srednicki [122]. In particular, all transformations are considered from the active viewpoint. In the standard four-vector representation

$$R(\theta\boldsymbol{e}_3) = \begin{pmatrix} 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{3.99}$$

and

$$\Lambda_{\zeta\boldsymbol{e}_1} = \begin{pmatrix} \cosh\zeta & \sinh\zeta & 0 & 0 \\ \sinh\zeta & \cosh\zeta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{3.100}$$

It will also sometimes be convenient to express boosts by a velocity $v\boldsymbol{e}_3$, $0 \le v < 1$:

$$\Lambda_{v\boldsymbol{e}_1} = \begin{pmatrix} \gamma_{Lor} & \gamma_{Lor}v & 0 & 0 \\ \gamma_{Lor}v & \gamma_{Lor} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}; \tag{3.101}$$

we will always be clear if we are applying boosts in terms of velocity or rapidity. They are related by $v = \tanh\zeta$ and $\gamma_{Lor} = \cosh\zeta$.

3.4.3 Singularities in massless vector space representations

In this section we illustrate how the conventional form of the little group method [42, 43] produces vector space representations of the Poincaré group with singularities. We will discuss this method in its modern form, as presented by Weinberg [43], and in its original and slightly more rigorous form as developed by Wigner [42]. In the modern formulation, one constructs unitary representations Σ of the Poincaré group on the space of single-particle states, that is, on the (generalized) eigenvectors $\Psi_{k,a}$ of the momentum operator:

$$P^\mu \Psi_{k,a} = k^\mu \Psi_{k,a}. \tag{3.102}$$

Here, $a \in \{1, 2, \ldots, n\}$ labels the internal degrees of freedom. It is assumed that a is a discrete, finite index, reflecting the fact that only particles with a finite number of internal degrees of freedom have been experimentally observed [43]. The little group method seeks to construct all possible representations Σ as follows. One begins by fixing some momentum $\bar{k}$, and considering the little group $H_{\bar{k}}$ consisting of the elements in $\mathrm{SO}^+(3,1)$ stabilizing $\bar{k}$:

$$H_{\bar{k}} \doteq \{\Lambda \in \mathrm{SO}^+(3,1) | \Lambda(\bar{k}) = \bar{k}\}. \tag{3.103}$$

The little groups for different choices of $\bar{k}$ are conjugate to each other since if Λ is any Poincaré transformation taking k_1 to k_2, then

$$H_{k_2} = \Lambda H_{k_1} \Lambda^{-1}. \tag{3.104}$$

Thus, up to isomorphism, $H_{\bar{k}}$ is independent of $\bar{k}$. The single-particle states at $\bar{k}$ form a representation of the little group, since for $\Lambda \in H_{\bar{k}}$

$$\Sigma(\Lambda)\Psi_{\bar{k},a} = \sum_{b=1}^{n} D_{ab}(\Lambda)\Psi_{\bar{k},b} \tag{3.105}$$

for some scalars $D_{ab}(\Lambda)$. It is much easier to classify the little group representations since the little group is smaller than the Poincaré group. For massive particles, the momentum space is a mass hyperboloid, and one can choose the reference momentum $\bar{k} = (1, 0, 0, 0)$ as that of a particle at rest. The little group is then clearly SO(3). The finite-dimensional representations of SO(3) are $2s+1$-dimensional and labeled by the spin s. For non-projective representations (bosons), s is restricted to nonnegative integer values. The situation is considerably different for massless particles. In this case the momenta are lightlike and there is no preferred reference momentum as there is no rest frame. One typically chooses the reference momentum $\bar{k} = (1, 0, 0, 1)$. One can show that the little group is $\mathrm{ISO}(2) = \mathbb{R}^2 \rtimes \mathrm{SO}(2)$, the two-dimensional Euclidean group [38, 43, 44]. In contrast to SO(3), the finite-dimensional representations of ISO(2) are all one-dimensional. They are labeled not by spin but by the helicity h which, for non-projective representations, can take on all integer values. The change in the momentum space at $m = 0$ accounts for the fact that massless particles such as photons are characterized by helicity rather than by spin.

To construct representations of $\mathrm{SO}^+(3, 1)$ from representations of the little group, one must relate the states with momentum $\bar{k}$ to those with other momenta. The conventional method [42, 43] for doing so is to assign to each k in the momentum space M a Lorentz transformation $L(k)$ such that

$$L(k)\bar{k} = k. \tag{3.106}$$

Thus, $L : M \to \mathrm{SO}^+(3, 1)$. Up to normalization, one then *defines* the single-particle states of momentum k in terms of those of the reference momentum $\bar{k}$ by [43]

$$\Psi_{k,a} = \Sigma(L(k))\Psi_{\bar{k},a}. \tag{3.107}$$

Then, the action of an arbitrary $\Lambda \in \mathrm{SO}^+(3, 1)$ is given by

$$\Sigma(\Lambda)\Psi_{k,a} = \Sigma(L(\Lambda k))\Sigma\big[L(\Lambda k)^{-1}\Lambda L(k)\big]\Psi_{\bar{k},a}. \tag{3.108}$$

Since $L(\Lambda k)^{-1}\Lambda L(k) \in H_{\bar{k}}$, the action in Eq. (3.108) is completely determined by Eqs. (3.105) and (3.107):

$$\Sigma(\Lambda)\Psi_{k,a} = D_{ab}\big[L(\Lambda k)^{-1}\Lambda L(k)\big]\Psi_{\Lambda k,a} \tag{3.109}$$

Eq. (3.108) thus extends each little group action specified by D_{ab} to a Poincaré action on all single-particle states. However, the validity of this procedure relies on being able to smoothly define $L : M \to \mathrm{SO}^+(3,1)$ satisfying Eq. (3.106). If L is not smooth, then the space of single-particle states constructed in Eq. (3.107) does not have a smooth structure. It is then not possible to say that the Poincaré group acts smoothly on the single-particle states, as will be discussed in greater detail.

For massive particles, this is not an issue. For the reference momentum $\bar{k} = (1,0,0,0)$, one can smoothly choose $L(k)$ to be the unique boost taking $\bar{k}$ to k. However, for massless particles there are no smooth choices of L. This result was proved by Boya *et al.* [46] using the principal bundle structure of $\mathrm{SL}(2,\mathbb{C})$. We present a short alternative proof based on the fact that the R and L photon bundles are Poincaré symmetric:

Theorem 3.20. *Let $\bar{k} \in \mathcal{L}_+$. There exists no smooth function $L : \mathcal{L}_+ \to \mathrm{SO}^+(3,1)$ satisfying Eq. (3.106) for all k.*

Proof. Suppose there were such a function $L(k)$. We will prove in Theorem 3.27 that R and L polarization states are preserved under the action Σ of $\mathrm{SO}^+(3,1)$. Additionally, a photon will appear as a propagating wave in any reference frame, so under a Lorentz transformation a wave $(k, \boldsymbol{E}) \in \gamma_k$ with $\boldsymbol{E} \neq 0$ will transform into another wave with nonzero electric field. This also follows trivially from the fact that the Lorentz transformations are vector space isomorphisms between fibers. If one fixes a choice of $(\bar{k}, \boldsymbol{E}_+) \in \gamma_+(\bar{k})$ with $\boldsymbol{E}_+ \neq 0$ then $\Sigma_{L(k)}(\bar{k}, \boldsymbol{E}_+)$, considered as a function of k, is a continuous nonvanishing section of γ_+, contradicting the fact that γ_+ is a nontrivial bundle. Thus, no continuous function $L(k)$ exists. □

For example, Weinberg chooses the fixed reference momentum $\bar{k} = (1, 0, 0, 1)$ and chooses ([43], Eq. (2.5.44))

$$L(k) = R(\phi \boldsymbol{e}_3) R(\theta \boldsymbol{e}_2) B_z(|\boldsymbol{k}|) \tag{3.110}$$

where

$$k = |\boldsymbol{k}|(1, \sin\theta\, \cos\phi, \sin\theta\, \sin\phi, \cos\theta) \tag{3.111}$$

and $B_z(|\boldsymbol{k}|)$ is a boost in the z direction taking $(1, 0, 0, 1)$ to $(|\boldsymbol{k}|, 0, 0, |\boldsymbol{k}|)$. However, $L(k)$ is discontinuous at $\theta = 0$ since

$$\lim_{\theta \to 0} R(\phi \boldsymbol{e}_3) R(\theta \boldsymbol{e}_2) = R(\phi \boldsymbol{e}_3) \tag{3.112}$$

depends on the value of ϕ; a similar discontinuity occurs at $\theta = \pi$. This illustrates that the modern formulation of the little group method [43, 44] fails to generate smooth massless representations.

This issue presents somewhat differently in Wigner's original and slightly more rigorous formulation on the little group method [42]. The single-particle states used by Weinberg do not themselves form a Hilbert space (for example, they are delta-function normalized [43]). Wigner instead builds representations on the Hilbert space of L^2 wave functions $\psi(k, a)$ where again a is a finite index. Given a unitary n-dimensional matrix representation of the little group $H_{\bar{k}}$ in which $\Lambda \mapsto D_{ab}(\Lambda)$, Wigner defines the Hilbert space representation

$$[\Sigma(\Lambda)\psi](k, a) = \sum_b D_{ab}\Big(L(k)^{-1} \Lambda L(\Lambda^{-1} k)\Big) \psi(\Lambda^{-1} k, b). \tag{3.113}$$

In particular, this generates representations on the Hilbert space $\bigoplus_n L^2(M, d\xi)$, the n-component square-integrable complex-valued wave functions on the momentum space M with respect to the Lorentz-invariant measure $d\xi = \frac{d^3\boldsymbol{k}}{k^0}$.

This procedure encounters no difficulties for massive particles, generating spin s representations on the Hilbert space $\bigoplus_{2s+1} L^2(\mathbb{R}^3, d\xi)$. Since L can be smoothly chosen as previously

discussed, these spin representations map smooth wave functions to smooth wave functions. However, in the massless case, L is discontinuous for at least one k'. Thus, if ψ is smooth, the transformed wave function $\Sigma(\Lambda)\psi$ seen in another reference frame is discontinuous at k' and $\Lambda k'$, as seen from Eq. (3.113). The trivial exceptions to this are when Λ is the identity or when D_{ab} is the trivial representation with $D_{ab}(\Lambda) = \mathbb{1}$ for every Λ. The latter is interesting because it shows that the massless helicity 0 representation is smooth. However, all the nonzero helicity representations are not smooth. We do note that such discontinuous wave functions are still elements of the L^2 Hilbert space, and Wigner showed that these representations can be regarded as continuous in the technical sense that if $\Lambda_\alpha \to \Lambda$, then $|\Lambda_\alpha\psi| \to |\Lambda\psi|$ in the L^2 norm. Nevertheless, such nonsmooth representations are practically awkward and appear somewhat unnatural as physically one might expect that smoothness should not be dependent on the reference frame.

We will illustrate these difficulties more explicitly by examining the representations of massless helicity h particles that result from Eq. (3.113) with $L(k)$ chosen according to Eq. (3.110). In particular, we will show that the action of a rotation $\Lambda = R\big(\varphi \boldsymbol{e}_2\big)$ about $\boldsymbol{e}_2$ results in a complicated transformation which introduces singularities to a smooth wave function. As shown by Weinberg ([43], Eq. (2.5.27)), an arbitrary element W of the massless little group of $\bar{k} = (1, 0, 0, 1)$ can be expressed as

$$W(\zeta, \alpha, \beta) = S(\alpha, \beta)R(\zeta \boldsymbol{e}_3) \tag{3.114}$$

$$S(\alpha, \beta) = \begin{pmatrix} 1+\delta & \alpha & \beta & -\delta \\ \alpha & 1 & 0 & -\alpha \\ \beta & 0 & 1 & -\beta \\ \delta & \alpha & \beta & 1-\delta \end{pmatrix} \tag{3.115}$$

for some angle ζ and real parameters $\alpha, \beta, \delta = (\alpha^2+\beta^2)/2$. This gives an isomorphism of the little group at $\bar{k}$ with the two-dimensional Euclidean group ISO(2) via $(\zeta, \alpha, \beta) \mapsto W(\zeta, \alpha, \beta)$, where $(\zeta, \alpha, \beta) \in \mathrm{ISO}(2)$ denotes a rotation in the plane by ζ followed by translation by

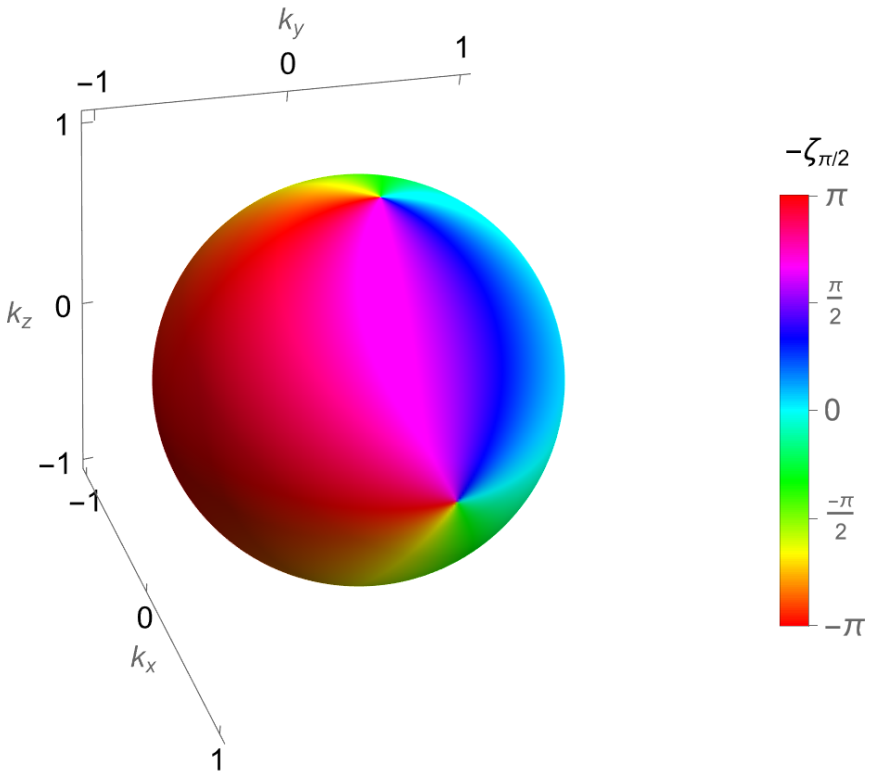


Figure 3.3: Plot of $\arg[\psi_1'(\boldsymbol{k})] = -\zeta_{\pi/2}(\theta,\phi)$ where $\psi_1 = \psi_1(|\boldsymbol{k}|)$ is a real, spherically symmetric helicity 1 wave function and $\psi' = \Sigma^1[R(\frac{\pi}{2}\boldsymbol{e}_2)]\psi = e^{-i\zeta_{\pi/2}}\psi$ is the wave function after the action of a $\pi/2$ rotation about $\boldsymbol{e}_y$. Singularities are visible at $\theta = 0$ and $(\theta,\phi) = (0,\frac{\pi}{2})$, showing that Wigner's massless helicity 1 representation Σ^1 does not preserve smoothness.

$(\alpha,\beta) \in \mathbb{R}^2$. The finite-dimensional vector space representations of the little group are the one-dimensional helicity h representations:

$$D^h(W(\zeta,\alpha,\beta)) = D^h(S(\alpha,\beta)R(\psi\boldsymbol{e}_3)) = e^{-ih\zeta}; \tag{3.116}$$

these will be discussed in more detail in the next section. Note that since these representations are all one-dimensional, D^h_{ab} has only the single component $D^h_{11} \doteq D^h$. Wigner's method furnishes representations on the Hilbert space $L^2(\mathcal{L}_+, d\xi)$ via Eq. (3.113). In particular, the

action of $R\big(\varphi \boldsymbol{e}_2\big)$ is given by

$$\Big[\Sigma^h\Big(R\big(\varphi \boldsymbol{e}_2\big)\Big)\psi\Big](\boldsymbol{k}) = D^h\big(W(\zeta_\varphi, \alpha_\varphi, \beta_\varphi)\big)\psi\Big(R\big(-\varphi \boldsymbol{e}_2\big)\boldsymbol{k}\Big) \tag{3.117}$$

$$= e^{-ih\zeta_\varphi(\boldsymbol{k})}\psi\Big(R\big(-\varphi \boldsymbol{e}_2\big)\boldsymbol{k}\Big) \tag{3.118}$$

where

$$W(\zeta_\varphi, \alpha_\varphi, \beta_\varphi) = L(\boldsymbol{k})^{-1}R\big(-\varphi \boldsymbol{e}_2\big)L\big(R\big(\varphi \boldsymbol{e}_2\big)\boldsymbol{k}\big). \tag{3.119}$$

The latter determines the angle $\zeta_\varphi(\boldsymbol{k})$. If we express $\boldsymbol{k}$ in spherical coordinates $|\boldsymbol{k}|, \theta, \phi$, then direct calculation shows that $\alpha_\varphi = \beta_\varphi = 0$ and $\zeta_\varphi = \zeta_\varphi(\theta, \phi)$ is the angle satisfying:

$$\cos\zeta_\varphi = \frac{\cos\varphi\sin\theta - \sin\varphi\cos\theta\cos\phi}{u} \tag{3.120a}$$

$$\sin\zeta_\varphi = \frac{\sin\varphi\sin\phi}{u} \tag{3.120b}$$

$$u = \sqrt{1 - (\cos\varphi\cos\theta + \sin\varphi\sin\theta\cos\phi)^2}. \tag{3.120c}$$

This can also be expressed as

$$\tan\zeta_\varphi = \frac{\sin\varphi\sin\phi}{\cos\varphi\sin\theta - \sin\varphi\cos\theta\cos\phi}. \tag{3.121}$$

The latter has two solutions, but the ambiguity is resolved by picking the solution which agrees with Eq. (3.120). Assuming $0 < \varphi < \pi$, we find that the function $\zeta_\varphi(\theta, \phi)$ is not smooth, with singularities arising in two ways. First, at the poles we have

$$\lim_{\theta\to 0}\tan\zeta_\varphi(\theta, \phi) = -\tan\phi \tag{3.122a}$$

$$\lim_{\theta\to \pi}\tan\zeta_\varphi(\theta, \phi) = \tan\phi. \tag{3.122b}$$

Thus, ζ_φ is not smooth at $\theta = 0$ or $\theta = \pi$ since the limits depend on ϕ. Second, singularities can arise when the numerator and denominator in Eq. (3.121) simultaneously go to zero.

This occurs at $(\theta, \phi) = (\varphi, 0)$ and $(\theta, \phi) = (\pi - \varphi, \pi)$, and it is straightforward to check that the limits of ζ_φ indeed do not exist at these points. Thus, if $\psi(\boldsymbol{k}) \in L^2(\mathcal{L}_+, d\xi)$ is a smooth function, then the rotated function $\Sigma^h[R(\varphi)\boldsymbol{e}_2]\psi$ will have singularities at the poles and at $(\theta, \phi) = (\varphi, 0)$ and $(\theta, \phi) = (\pi - \varphi, \pi)$. For example, under a $\varphi = \pi/2$ rotation about $\boldsymbol{e}_2$, a spherically symmetric helicity 1 wave function $\psi_1 = \psi_1(|\boldsymbol{k}|)$ transforms into the singular, non-spherically symmetric $\psi_1' = e^{-i\zeta_{\pi/2}(\theta,\phi)}\psi_1$, as depicted in Figure 3.3. Thus, smooth wave functions are not mapped to smooth wave functions under these conventional representations, as we anticipated from Eq. (3.113) and Theorem 3.20. Furthermore, we see that the action rotations is far from intuitive in these representations. It will be shown in §3.4.5 that by considering representations on different Hilbert spaces, namely on the spaces of L^2 sections of vector bundles over $\mathcal{L}_+$, we can obtain smooth massless helicity h representations. Furthermore, the action of rotations and other Lorentz transformations is easier to understand in these representations. Indeed, in the photon ($h = \pm 1$) representations, the transformations are determined by those of the electric field. The higher helicity transformation laws can be understood as iterated tensor products of the $h = \pm 1$ transformations, as will be shown in Chapter 5 [Eqs. (5.18) and (5.29)].

3.4.4 The little group method for massless vector bundle representations

We can resolve global nonsmoothness issues in the massless case by considering vector bundle representations of the Poincaré group over the light cone. We will show that a vector bundle version of the little group method can be used to canonically decompose any unitary $\mathrm{ISO}^+(3,1)$ bundle representation $\pi : E \to \mathcal{L}_+$ into irreducible bundle representations labeled by helicity.

We begin by defining bundle representations of the little group.

Definition 3.6 (Stabilizing vector bundle representation)**.** *A vector bundle representation $\pi : E \to M$ of G with group action (Σ, σ) is said to be stabilizing if $\sigma_g = \mathbb{1}$ is the identity for every $g \in G$, that is, if*

$$\Sigma_g(E_k) = E_k \tag{3.123}$$

for every $k \in M$.

Definition 3.7 (Little group of a vector bundle representation)**.** *Let $\pi : E \to M$ be a homogeneous vector bundle representation of G. The little group at $k \in M$ is defined by*

$$H_k = \{g \in G | \sigma_g(k) = k\}. \tag{3.124}$$

The little group H of the representation is defined as the isomorphism type of H_k, which is independent of k because $H_{\sigma_g k} = g H_k g^{-1}$ and E is homogeneous.

In the case that G is the Poincaré group, H and H_k are restricted to elements of $\mathrm{SO}^+(3,1)$, that is, they are defined in terms of the corresponding proper orthochronous Lorentz action.

Theorem 3.21 (Little group representation)**.** *Let $\pi : E \to M$ be a homogeneous vector bundle representation of G with action Σ and little group H. Let H_k denote the little group at $k \in M$. Suppose $f : H \times M \to G$ is a smooth map such that $f(h,k) \in H_k$ for all h, k, and for each fixed k, $f(h,k)$ is a group homomorphism from H to G. Such an f induces a stabilizing bundle representation of H on E with action Π given by*

$$\Pi_h(k, v) = \Sigma_{f(h,k)}(k, v) \tag{3.125}$$

for $(k, v) \in E$. Π is unitary if Σ is unitary. A bundle representation of H induced by such an f is called a little group representation.

Proof. Π is smooth because Σ and f are. It is a vector bundle representation because

$$\Pi_{h_1h_2}(k,v) = \Sigma_{f(h_1h_2,k)}(k,v) = \Sigma_{f(h_1,k)f(h_2,k)}(k,v) \tag{3.126}$$

$$= \Sigma_{f(h_1,k)}\Sigma_{f(h_2,k)}(k,v) = \Pi_{h_1}\Pi_{h_2}(k,v). \tag{3.127}$$

It is stabilizing since $f(h,k) \in H_k$. Since Π can be written in terms of Σ, if Σ is unitary then so is Π. □

By the definition of the little group H, for each k there exists an isomorphism $f_k : H \to H_k$. Note that f_k is not typically unique. For any choice of such f_k one could then define an f

$$f(h,k) = f_k(h). \tag{3.128}$$

However, this f is not generally smooth since there is no guarantee that the choices of f_k fit together smoothly. Our first goal is construct a canonical smooth f for an arbitrary $\mathrm{ISO}^+(3,1)$ vector bundle representation $\pi : E \to \mathcal{L}_+$, which will then furnish a canonical little group representation. Note that any such bundle E is homogeneous since $\mathrm{ISO}^+(3,1)$ acts transitively on $\mathcal{L}_+$.

For a fixed $\bar{k} \in \mathcal{L}_+$, any $\Lambda \in H_{\bar{k}}$ satisfies $\Sigma(\Lambda)E_{\bar{k}} = E_{\bar{k}}$, so Σ gives a finite-dimensional vector space representation of $H_{\bar{k}} \cong \mathrm{ISO}(2)$ on the fiber $E_{\bar{k}}$; we will use $\Sigma^{\bar{k}}$ to denote this restriction of Σ to $E_{\bar{k}}$.

Consider first the special case of $\bar{k} = (1,0,0,1)$. Weinberg ([43], §2.5) and Woit ([38], §42.3.5) showed that the little group of this $\bar{k}$ can be described by a function of three parameters $W^{\bar{k}}(\theta,\alpha,\beta) : \mathrm{ISO}(2) \to \mathrm{ISO}^+(3,1)$ and is represented by

$$\begin{aligned} \Sigma(W^{\bar{k}}(\theta,\alpha,\beta)) &= \exp[-i(\alpha A + i\beta B + i\theta J_3)], \\ A &= J_2 + K_1, \\ B &= -J_1 + K_2. \end{aligned} \tag{3.129}$$

Under the isomorphism with ISO(2), J_3 generates 2D rotations while A and B generate translations. They satisfy the commutation relations

$$[J_3, A] = +iB, \tag{3.130}$$

$$[J_3, B] = -iA, \tag{3.131}$$

$$[A, B] = 0. \tag{3.132}$$

It is known that all finite-dimensional irreducible representations of ISO(2) are one-dimensional. The general case was proved by Schwarz [123]; simpler proofs for unitary representations of ISO(2) are presented by Weinberg [43] and Maggiore [44]. Therefore, on each such irreducible representation, J_3, A, and B must all be multiplication operators and thus commute. Eqs. (3.130) and (3.131) then imply $A = B = 0$ on each irreducible representation, which in turn implies $A = B = 0$ on all of $E_{\bar{k}}$. Thus,

$$J_2 + K_1 = 0, \tag{3.133}$$

$$-J_1 + K_2 = 0 \tag{3.134}$$

when restricted to $\gamma_{\bar{k}}$. We note two subtleties here. First, the relations (3.133) and (3.134) hold only on the fiber $E_{\bar{k}}$. Second, J_2 and K_1 are not independently operators in the little group representation since neither generate transformations leaving $\bar{k}$ invariant. However, since $J_2 + K_1 = 0$ in the little group representation, this relation holds also in the bundle representation when restricted to $E_{\bar{k}}$, and in this sense the perpendicular boosts and angular momenta are related by $J_2 = -K_1$ and $J_1 = K_2$ on $E_{\bar{k}}$.

Since $A = B = 0$, the irreducible representations are completely determined by the action of the generator J_3. In fact, they are just the eigenspaces of J_3. Let $\Psi_{\bar{k},h}$ be the eigenvectors with eigenvalues h:

$$J_3 \Psi_{\bar{k},h} = h \Psi_{\bar{k},h}. \tag{3.135}$$

h defines the helicity of each irreducible representation of $H_{\bar{k}}$. Since

$$e^{i2\pi J_3}\Psi_{\bar{k},h} = e^{2\pi i h}\Psi_{\bar{k},h} = \Psi_{\bar{k},h}, \tag{3.136}$$

h must be an integer. Note that if one allows projective representations, h may also be a half-integer [43]. That also that since $A = B = 0$, the massless little group is essentially reduced from $\mathrm{ISO}(2) = \mathbb{R}^2 \rtimes \mathrm{SO}(2)$ to $\mathrm{SO}(2)$.

These results generalize easily to arbitrary $k = (|k|, \boldsymbol{k}) \in \mathcal{L}_+$ as there is nothing special about $\bar{k} = (1, 0, 0, 1)$. Choose any $\boldsymbol{f}_1, \boldsymbol{f}_2 \in \mathbb{R}^3$ such that $(\boldsymbol{f}_1, \boldsymbol{f}_2, \hat{\boldsymbol{k}})$ form a right-handed orthonormal coordinate system. Then there is an isomorphism $W^k : \mathrm{ISO}(2) \to H_k$, and the action of the little group H_k is described by

$$\Sigma(W^k(\theta, \alpha, \beta)) = \exp\left[-i(\alpha A^k + \beta B^k + \theta(\hat{\boldsymbol{k}} \cdot \boldsymbol{J}))\right], \tag{3.137}$$

$$A^k = \boldsymbol{f}_2 \cdot \boldsymbol{J} + \boldsymbol{f}_1 \cdot \boldsymbol{K} = 0, \tag{3.138}$$

$$B^k = -\boldsymbol{f}_1 \cdot \boldsymbol{J} + \boldsymbol{f}_2 \cdot \boldsymbol{K} = 0. \tag{3.139}$$

The latter two equations say that the boosts and perpendicular angular momentum are related on E_k by

$$\boldsymbol{J}_\perp = -\hat{\boldsymbol{k}} \times \boldsymbol{K}_\perp = -\hat{\boldsymbol{k}} \times \boldsymbol{K} \tag{3.140}$$

where

$$\boldsymbol{J}_\perp \doteq \boldsymbol{J} - \hat{\boldsymbol{k}}(\hat{\boldsymbol{k}} \cdot \boldsymbol{J}), \tag{3.141}$$

$$\boldsymbol{K}_\perp \doteq \boldsymbol{K} - \hat{\boldsymbol{k}}(\hat{\boldsymbol{k}} \cdot \boldsymbol{K}). \tag{3.142}$$

The only nontrivial generator of the little group is $\hat{\boldsymbol{k}} \cdot \boldsymbol{J}$, and its eigenspaces are again labeled by integer helicities.

We would like to show that the fiber-wise little group representations fit together to form a bundle representation of the little group ISO(2) via Theorem 3.21. However, $W^k(\theta, \alpha, \beta)$ considered as a function from $\mathrm{ISO}(2) \times \mathcal{L}_+$ to $\mathrm{ISO}^+(3,1)$ is not smooth because it is not possible to smoothly choose $(\boldsymbol{f}_1(k), \boldsymbol{f}_2(k))$ by the hairy ball theorem. However, the fiber-wise little group representations have the important property that the translations A^k and B^k act trivially. Thus, $f : \mathrm{ISO}(2) \times \mathcal{L}_+ \to \mathrm{SO}^+(3,1)$ given by

$$f\big((\theta, \alpha, \beta), k\big) = W^k(\theta, 0, 0) \tag{3.143}$$

is smooth and produces precisely the same action as $W^k(\theta, \alpha, \beta)$ under Σ:

$$\Sigma\big(f((\theta, \alpha, \beta), k)\big) = \Sigma(W^k(\theta, \alpha, \beta)). \tag{3.144}$$

Applying Theorem 3.21 to this f then gives a canonical little group action.

Theorem 3.22. *Every* $\mathrm{ISO}^+(3,1)$ *representation* $\pi : E \to \mathcal{L}_+$ *with group action* Σ *has an associated little group representation of* $\mathrm{ISO}(2)$ *with action* Π *given by*

$$\Pi(\theta, \alpha, \beta)(k, v) = (k, e^{-i\theta\chi} v) = (k, e^{-i\theta(\hat{\boldsymbol{k}} \cdot \boldsymbol{J})} v) \tag{3.145}$$

where

$$\chi = \hat{\boldsymbol{k}} \cdot \boldsymbol{J} \tag{3.146}$$

is the helicity operator. If Σ *is unitary, then so is* Π.

Note that the helicity operator can be written as

$$\chi = \hat{\boldsymbol{k}} \cdot \boldsymbol{J} = \frac{\boldsymbol{P}}{|\boldsymbol{k}|} \cdot \boldsymbol{J} = \frac{1}{H} \boldsymbol{P} \cdot \boldsymbol{J}. \tag{3.147}$$

This operator is smooth on the vector bundle since $H = |\boldsymbol{k}| \neq 0$ on the light cone and is thus invertible. Note that H^{-1} commutes with $\boldsymbol{P}$ and $\boldsymbol{J}$. Furthermore, for each $a \in \{1, 2, 3\}$, P_a

and J_a commute, so $\boldsymbol{P} \cdot \boldsymbol{J} = \boldsymbol{J} \cdot \boldsymbol{P}$. Thus, the terms in Eq. (3.147) can be rearranged, and there is no ordering ambiguity in the definition of χ. The following is an important property of χ.

Theorem 3.23. *The massless* $\mathrm{ISO}^+(3,1)$ *action* Σ *and its associated little group action* Π *given by Eq. (3.145) commute.*

Proof. It is sufficient to show that the generators of these actions commute, that is, we show that χ commutes with all generators of the $\mathrm{ISO}^+(3,1)$ action. In the following calculations, we implicitly sum over all repeated indices. That χ commutes with the spacetime translation generators and the Hamiltonian H is trivial. For the generators P_b, we have

$$\begin{aligned} [\chi, P_b] &= [\frac{P_a}{H} J_a, P_b] = \frac{P_a}{H}[J_a, P_b] \\ &= i\epsilon_{abc} H^{-1} P_a P_c = 0. \end{aligned} \tag{3.148}$$

For J_b,

$$\begin{aligned} [\chi, J_b] &= H^{-1}[P_a J_a, J_b] \\ &= H^{-1}(P_a[J_a, J_b] + [P_a, J_b]J_a) \\ &= iH^{-1}(\epsilon_{abc} P_a J_c + \epsilon_{abc} P_c J_a) \\ &= 0. \end{aligned} \tag{3.149}$$

To show that χ commutes with boosts, note that

$$\begin{aligned} [H^{-1}, K_b] &= H^{-1}K_b - K_b H^{-1} \\ &= H^{-1}K_b H H^{-1} - H^{-1} H K_b H^{-1} \\ &= H^{-1}[K_b, H]H^{-1} = iH^{-1}P_b H^{-1} \\ &= iH^{-2}P_b. \end{aligned} \tag{3.150}$$

Thus,

$$\begin{aligned}
[\chi, K_b] &= [H^{-1}P_aJ_a, K_b] \\
&= H^{-1}P_a[J_a, K_b] + [H^{-1}P_a, K_b]J_a \\
&= i\epsilon_{abc}H^{-1}P_aK_c + H^{-1}[P_a, K_b]J_a + [H^{-1}, K_b]P_aJ_a \\
&= -iH^{-1}(\boldsymbol{P}\times\boldsymbol{K})_b - i\delta_{ab}J_a + iH^{-2}P_bP_aJ_a \\
&= i(-J_b - (\hat{\boldsymbol{k}}\times\boldsymbol{K})_b + \hat{k}_b(\boldsymbol{k}\cdot\boldsymbol{J}))
\end{aligned} \tag{3.151}$$

By the relation (3.140) between the boost and rotation generators for massless representations, these terms cancel giving

$$[\chi, K_b] = 0. \tag{3.152}$$

□

We can now show that the massless unitary irreducible bundle representations of $\mathrm{ISO}^+(3,1)$ are the constant helicity representations.

Theorem 3.24. *Let $\pi : E \to \mathcal{L}_+$ be a unitary $\mathrm{ISO}^+(3,1)$ vector bundle representation of $\mathrm{ISO}^+(3,1)$ of rank r such that a spacetime translation by $\mathsf{a} \in \mathbb{R}^{3+1}$ acts on a vector $(k,v) \in E_k$ by $\Sigma_{T_\mathsf{a}}(k,v) = (k, e^{-ik^\mu \mathsf{a}_\mu}v)$. Then E can be decomposed as*

$$E = E_1 \oplus \cdots \oplus E_r \tag{3.153}$$

where the E_j are line subbundle of E and UIRs of $\mathrm{ISO}^+(3,1)$. Each E_j has definite helicity h_j in the sense that every element of E_j is an eigenvector of χ with helicity h_j.

Proof. Choose some $\bar{k} \in \mathcal{L}_+$. Let $(v_1, ..., v_r)$ be a basis of the fiber $E_{\bar{k}}$ consisting of eigenvectors of χ with helicities $(h_1, ..., h_r)$. Define the subset $E_j \subseteq E$ as the orbit of $(\bar{k}, v_j)$ under the group action and scalar multiplication:

$$E_j \doteq \{c\Sigma(L)(\bar{k}, v_j) | L \in \mathrm{ISO}^+(3,1), c \in \mathbb{C}\}. \tag{3.154}$$

We will show that each E_j is a rank 1 subbundle of E by the subbundle criterion. Partition E_j as

$$E_j = \amalg_{k \in \mathcal{L}_+} E_j(k) \tag{3.155}$$

where $E_j(k)$ is the subset of E_j consisting of vectors at k. Each $E_j(k)$ is a vector space and it is nonempty since if $L(k)$ is any Poincaré transformation taking $\bar{k}$ to k, then $\Sigma(L(k))(\bar{k}, v_j) \in E_j(k)$. Thus, $E_j(k)$ is at least one-dimensional. Suppose (k, w_1) and (k, w_2) are both in $E_j(k)$ and nonzero. Then

$$(k, w_1) = c_1 \Sigma(L_1)(\bar{k}, v_j) \tag{3.156}$$

$$(k, w_2) = c_2 \Sigma(L_2)(\bar{k}, v_j) \tag{3.157}$$

for some nonzero scalars (c_1, c_2) and Poincaré transformations (L_1, L_2). As $L_2^{-1} L_1$ leaves $\bar{k}$ invariant, it must be a combination of a spacetime translation by some a and an element of the little group H_k, so

$$\Sigma(L_2^{-1} L_1)(\bar{k}, v_j) = (\bar{k}, e^{-i(\theta h_j + k^\mu \mathsf{a}_\mu)} v_j) \tag{3.158}$$

for some θ. Then

$$\begin{aligned} (k, w_1) &= c_1 \Sigma(L_2) \Sigma(L_2^{-1} L_1)(\bar{k}, v_j) \\ &= c_1 c_2^{-1} e^{-i(\theta h_j + k^\mu \mathsf{a}_\mu)} (k, w_2) \end{aligned} \tag{3.159}$$

so (k, w_1) and (k, w_2) are linearly dependent. This shows that every $E_j(k)$ is one-dimensional.

Now, let $k_0 \in \mathcal{L}_+$ be arbitrary and U be a small ball about k_0. Choose the radius of U to be small enough that it does not enclose the origin. We construct a smooth function $L : U \to \mathrm{SO}^+(3,1)$ such that $L(k) k_0 = k$. Note that we showed in Theorem 3.20 that it is not possible to construct such a function if the domain is all of $\mathcal{L}_+$. However, it is possible to construct such a function locally. Indeed, we can simply choose $L(k) = R(k) B(k)$ where

$B(k)$ is the boost parallel to $\boldsymbol{k}_0$ taking $\boldsymbol{k}_0$ to $\boldsymbol{k}_0\frac{|\boldsymbol{k}|}{|\boldsymbol{k}_0|}$ and $R(k)$ is the unique rotation by angle $0 \leq \theta < \pi$ along the great circle connecting $\boldsymbol{k}_0\frac{|\boldsymbol{k}|}{|\boldsymbol{k}_0|}$ to $\boldsymbol{k}$. Let (k_0, v) be a nonzero vector in $E_j(k_0)$. Then $\Sigma(L(k))(k_0, v)$ is a smooth section of E over U that span $E_j(k)$ for each $k \in U$. Thus, each E_j is a line subbundle of E by Lemma 3.10. They are unitary subbundle representations of $\mathrm{ISO}^+(3,1)$ by their definitions in Eq. (3.154). Every vector $(k, w_1) \in E_1$ is an eigenvector of χ with helicity h_j since by Theorem 3.22,

$$\begin{aligned} \chi(k, w_1) &= \chi c_1 \Sigma(L_1)(\bar{k}, v_j) \\ &= c_1 \Sigma(L_1)\chi(\bar{k}, v_j) \\ &= h_j(k, w_1). \end{aligned} \tag{3.160}$$

□

In this bundle decomposition, it is possible for multiple line bundles E_j to have the same helicity. The next result says that such representations are equivalent, showing that the irreducible bundle representations of $\mathrm{ISO}^+(3,1)$ are completely characterized by their helicity.

Theorem 3.25. *Suppose $\pi : E_1 \to \mathcal{L}_+$ and $\pi_2 : E_2 \to \mathcal{L}_+$ are unitary line bundle representations of $\mathrm{ISO}^+(3,1)$ with actions Σ_1 and Σ_2 and helicities h_1 and h_2, and such that spacetime translations by a act via multiplication by $e^{-ik^\mu \mathsf{a}_\mu}$. They are unitarily equivalent representations if and only if $h_1 = h_2$.*

Proof. If E_1 and E_2 are equivalent representations, then there exists an isomorphism $g : E_1 \to E_2$ as in Definition 3.5. Let χ_1 and χ_2 be the helicity operators induced by Σ_1 and Σ_2. For $(k, v) \in E_1$, linearity gives

$$g(e^{i\theta\chi_1}(k, v)) = g(e^{ih_1\theta}(k, v)) = e^{ih_1\theta} g(k, v). \tag{3.161}$$

By equivariance,

$$g(e^{i\theta\chi_1}(k,v)) = e^{i\theta\chi_2}g(k,v) = e^{ih_2\theta}g(k,v). \tag{3.162}$$

Thus, $h_1 = h_2$.

Conversely, suppose $h_1 = h_2$. We will construct a unitary isomorphism of representations $g : E_1 \to E_2$. Fix some $(k_0, v_0) \in E_1$ and $(k_0, w_0) \in E_2$ such that $|v_0| = |w_0| \neq 0$. We define

$$g(k_0, v_0) = (k_0, w_0), \tag{3.163}$$

and extend this relation by equivariance and linearity. That is, for every $L \in \mathrm{ISO}^+(3,1)$ and $c \in \mathbb{C}$, define

$$g(\Sigma_1(L)(k_0, v_0)) = \Sigma_2(L)(k_0, w_0), \tag{3.164}$$

$$g(k_0, cv_0) = c(k_0, w_0). \tag{3.165}$$

Any $(k, v) \in E_1$ can be expressed as

$$(k, v) = c\Sigma_1(\Lambda)(k_0, v_0) \tag{3.166}$$

for some choice of c, Λ. Indeed, if one chooses a Lorentz transformation Λ such that $\Lambda k_0 = k$, then $\Sigma(\Lambda)(k_0, v_0) \in E_1(k)$. Since E_1 is a line bundle, any other vector in $E_1(k)$ can be obtained by scalar multiplication. We must still show that g is single-valued under this definition.

Suppose $\Sigma_1(L_1)(k_0, v_0) = \Sigma_1(L_2)(k_0, v_0)$. We need to show

$$\Sigma_2(L_1)(k_0, w_0) = \Sigma_2(L_2)(k_0, w_0) \tag{3.167}$$

or equivalently,

$$\Sigma_2(L_2^{-1}L_1)(k_0, w_0) = (k_0, w_0). \tag{3.168}$$

Since $L_2^{-1}L_1$ leaves k_0 invariant, it corresponds to $T_{\mathsf{a}} \cdot (\theta, \alpha, \beta)$ where a is a spacetime translation and (θ, α, β) is an element of $H^{k_0} \cong \mathrm{ISO}(2)$. It thus acts on the fibers E_{1,k_0} and E_{2,k_0} via multiplication by $c_1 \doteq e^{-i(\theta h + k_0^\mu \mathsf{a}_\mu)}$. By assumption

$$\Sigma_1(L_2^{-1}L_1)(k, v_0) = (k, v_0) \tag{3.169}$$

so $c_1 = 1$ and Eq. (3.168) holds. Next, suppose

$$\Sigma_1(L)(k_0, v_0) = (k_0, cv_0). \tag{3.170}$$

We need to show that

$$\Sigma_2(L)(k_0, w_0) = (k_0, cw_0). \tag{3.171}$$

Again, since L leaves k_0 invariant, it can be expressed as $T_{\mathsf{a}} \cdot (\theta, \alpha, \beta)$, and it thus acts on both E_{1,k_0} and E_{2,k_0} via multiplication by $e^{-i(h\theta + k_0^\mu \mathsf{a}_\mu)}$. Thus Eq. (3.170) implies Eq. (3.171), showing that g is well-defined.

g is a basepoint preserving unitary vector bundle isomorphism by its defining properties. All that remains to show is that g is equivariant, that is,

$$g(\Sigma_1(L)(k, v)) = \Sigma_2(L)g(k, v) \tag{3.172}$$

for arbitrary $(k, v) \in E_1$ and $L \in \mathrm{ISO}^+(3,1)$. Choose some c and Λ such that Eq. (3.166) holds. Then, using Eqs. (3.163)-(3.165) we have

$$\begin{aligned} g(\Sigma_1(L)(k,v)) &= cg(\Sigma_1(L\Lambda)(k_0,v_0)) \\ &= c\Sigma_2(L\Lambda)(k_0,w_0) \\ &= c\Sigma_2(L)\Sigma_2(\Lambda)(k_0,w_0) \\ &= \Sigma_2(L)g(c\Sigma_1(\Lambda)(k_0,v_0)) \\ &= \Sigma_2(L)g((k,v)). \end{aligned} \tag{3.173}$$

Thus, E_1 and E_2 are equivalent representations. □

We can apply this vector bundle version of the little group method to decompose the photon bundle γ into Poincaré invariant line bundles. First note that $\gamma_\pm$ have definite helicity.

Proposition 3.26. *The R and L bundles $\gamma_\pm$ have definite helicities ± 1.*

Proof. An arbitrary vector $v_\pm \in \gamma_\pm$ can be written as $v_\pm = \alpha(\boldsymbol{e}_1 \pm i\boldsymbol{e}_2)$ where $(\boldsymbol{e}_1, \boldsymbol{e}_2, \hat{\boldsymbol{k}})$ is real, right-handed, and orthonormal. $\Sigma\big(R(\theta\hat{\boldsymbol{k}})\big) = e^{-i\theta(\hat{\boldsymbol{k}}\cdot\boldsymbol{J})} = e^{-i\chi\theta}$ is the action of a rotation by θ about $\hat{\boldsymbol{k}}$. Using the $(\boldsymbol{e}_1, \boldsymbol{e}_2)$ basis,

$$R(\theta\hat{\boldsymbol{k}})v_\pm = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 1 \\ \pm i \end{pmatrix} = e^{\mp i\theta} v_\pm. \tag{3.174}$$

Differentiating with respect to θ gives

$$\chi v_\pm = (\hat{\boldsymbol{k}} \cdot \boldsymbol{J})v_\pm = \pm v_\pm. \tag{3.175}$$

□

It now follows that $\gamma_\pm$ are irreducible subrepresentations of the representation Σ on γ, provided we restrict Σ from $\mathrm{IO}^+(3,1)$ to $\mathrm{ISO}^+(3,1)$. This restriction is necessary since the parity operator P maps $\gamma_\pm$ to $\gamma_\mp$, and therefore the R and L bundles are not P-symmetric.

Theorem 3.27. *γ_+ and γ_- are unitary irreducible vector bundle representations of $\mathrm{ISO}^+(3,1)$.*

Proof. By Theorem 3.24, γ decomposes as

$$\gamma = E_1 \oplus E_2 \tag{3.176}$$

into two unitary irreducible line bundle representations of $\mathrm{ISO}^+(3,1)$ with definite helicity. By proposition 3.26, these line bundles must be $\gamma_\pm$. □

This result shows that R and L photons are globally well-defined. In particular, this construction avoids the singularities that appear in the conventional little group method for massless particles.

3.4.5 Vector bundle representations as particles

We have shown that the solutions of Maxwell's equations fit together into a smooth vector bundle γ. Furthermore, γ splits into two unitary irreducible bundle representations of the Poincaré group, γ_+ and γ_-. It is thus natural to consider these bundle representations to be particles. However, by the conventional definition, particles are unitary irreducible Hilbert space representations of the Poincaré group [42, 43]. In this section we bridge the gap between these two viewpoints, proving that massless unitary irreducible bundle representations of the Poincaré group generate corresponding unitary irreducible Hilbert space representations, and can thus be considered particles under the standard definition. These Hilbert space representations are smooth, avoiding the singularities described in §3.4.3 that occur in Wigner's little group construction. We note that a similar method of generating

Hilbert space representations from bundle representations was described by Simms [47]. Our work differs in that we use smooth bundles rather than topological bundles, which allows us to resolve the nonsmoothness issues with massless representations; it does not appear Simms was aware of the smoothness problem.

A representation of G on the vector bundle E naturally induces a vector space representation of G on the infinite-dimensional vector space $\Gamma(E)$ of smooth sections of E. We refer to this as the sectional representation.

Proposition 3.28. *A vector bundle representation (E, Σ) of G induces a vector space representation of G on $\Gamma(E)$. Given $g \in G$ and a section $\psi : M \to E$, g is represented by $\tilde{\Sigma}_g$, defined by*

$$[\tilde{\Sigma}_g \psi](m) = \Sigma_g \big[\psi(\sigma_{g^{-1}}(m))\big]. \tag{3.177}$$

Proof. The linearity of the action in $\Gamma(E)$ follows from the fact that Σ_g is a bundle map. That the identity $e \in G$ acts trivially on $\Gamma(E)$ follows from $\Sigma_e = I$. If $g_1, g_2 \in G$, then

$$\begin{aligned}
[\tilde{\Sigma}_{g_1 g_2} \psi](m) &= \Sigma_{g_1 g_2} \big[\psi(\sigma_{(g_1 g_2)^{-1} m})\big] \\
&= \Sigma_{g_1} \Sigma_{g_2} \big[\psi(\sigma_{g_2^{-1} \sigma_{g_1^{-1}}}(m))\big] \\
&= \Sigma_{g_1} \big[(\tilde{\Sigma}_{g_2} \psi)(\sigma_{g_1^{-1}}(m))\big] \\
&= \big[\tilde{\Sigma}_{g_1} \big(\tilde{\Sigma}_{g_2} \psi\big)\big](m)
\end{aligned} \tag{3.178}$$

showing that $\tilde{\Sigma}_{g_1 g_2} = \tilde{\Sigma}_{g_1} \tilde{\Sigma}_{g_2}$. Lastly, $\tilde{\Sigma}_g \psi$ is smooth since Σ_g, $\Sigma_{g^{-1}}$, and ψ are smooth. □

As an example which will be used in §3.5.2, the ISO(2) little group action on γ has a corresponding sectional representation on $\Gamma(\gamma)$, which can be simply expressed due to the splitting $\gamma = \gamma_+ \oplus \gamma_-$. A section $\boldsymbol{E}(\boldsymbol{k}) \in \Gamma(\gamma)$ can be uniquely written in terms of sections $\boldsymbol{E}_\pm$ of $\gamma_\pm$:

$$\boldsymbol{E}(\boldsymbol{k}) = \boldsymbol{E}_+(\boldsymbol{k}) + \boldsymbol{E}_-(\boldsymbol{k}). \tag{3.179}$$

That is, there are unique projections $\mathcal{P}_\pm : \Gamma(\gamma) \to \Gamma(\gamma_\pm)$. The proof of the following is immediate:

Theorem 3.29. *Let $\boldsymbol{E} \in \Gamma(\gamma)$. The sectional representation corresponding to the canonical* ISO(2)*-little group representation on γ is given by*

$$\tilde{\Pi}(\theta, \alpha, \beta)\boldsymbol{E} = e^{-i\chi\theta}\boldsymbol{E} = e^{-i\theta}\boldsymbol{E}_+ + e^{+i\theta}\boldsymbol{E}_-. \tag{3.180}$$

Since the translations α, β act trivially, the little group action can be considered as an $\mathrm{SO}(2) \cong S^1$ *action :*

$$\tilde{\Pi}_H(\theta)\boldsymbol{E} = e^{-i\chi\theta}\boldsymbol{E} = e^{-i\theta}\boldsymbol{E}_+ + e^{+i\theta}\boldsymbol{E}_- \tag{3.181}$$

Ideally, if Σ is unitary, then the sectional representation $\tilde{\Sigma}$ on $\Gamma(E)$ would also be unitary. However, this would imply that the Hermitian structure on E induces a Hermitian product on $\Gamma(E)$, which is not generally true. Suppose then that one additionally specifies a G-invariant volume form $d\xi$ on the base manifold M, meaning $g^* d\xi = d\xi$ where g^* is the pullback by the action on M, and denote by $\Gamma(E, d\xi)$ the smooth sections which are L^2-normalizable with respect to the volume form $d\xi$. Then there is an induced Hermitian structure on $\Gamma(E, d\xi)$:

$$\langle \psi_1, \psi_2 \rangle = \int_M \langle \psi_1(m), \psi_2(m) \rangle \, d\xi. \tag{3.182}$$

In the present case, the photon bundle has the light cone as its base manifold which has a canonical Lorentz invariant volume form $d\xi = \frac{d^3k}{|\boldsymbol{k}|}$. We will assume this volume form whenever the base manifold of E is $\mathcal{L}_+$.

Proposition 3.30. *Suppose $\pi : E \to M$ is a vector bundle representation of G and $d\xi$ is a G-invariant volume form on M. If the bundle representation is unitary on E, then the sectional representation on $\Gamma(E, d\xi)$ is unitary with respect to the induced Hermitian product (3.182).*

Proof. For $\psi_1, \psi_2 \in \Gamma(E)$ and $g \in G$, we have

$$\begin{aligned}
\langle \tilde{\Sigma}_g \psi_1, \tilde{\Sigma}_g \psi_2 \rangle &= \int_M \langle (\tilde{\Sigma}_g \psi_1)(\boldsymbol{k}), (\tilde{\Sigma}_g \psi_2)(\boldsymbol{k}) \rangle d\xi \\
&= \int_M \langle \Sigma_g[\psi_1(g^{-1}\boldsymbol{k})], \Sigma_g[\psi_2(g^{-1}\boldsymbol{k})] \rangle d\xi \\
&= \int_M \langle \psi_1(g^{-1}\boldsymbol{k}), \psi_2(g^{-1}\boldsymbol{k}) \rangle d\xi \\
&= \int_M \langle \psi_1(\boldsymbol{k}), \psi_2(\boldsymbol{k}) \rangle g^*(d\xi) \\
&= \langle \psi_1, \psi_2 \rangle
\end{aligned} \tag{3.183}$$

where $g^*(d\xi) = d\xi$ follows from the G-invariance of $d\xi$. □

$\Gamma(E, d\xi)$ is not a Hilbert space but rather a pre-Hilbert space since it is not complete. In particular, a sequence of smooth sections can converge in the norm to a discontinuous section. This issue is typical fo L^2 spaces, and to remedy it, one can work with the Hilbert space completion of $\Gamma(E, d\xi)$, denoted by $L^2(E, d\xi)$, which is the closure of $\Gamma(E, d\xi)$ in the norm. A unitary sectional action $\tilde{\Sigma}$ on $\Gamma(E, d\xi)$ extends to a Hilbert space action on $L^2(E, d\xi)$ in the following way. If $g \in G$ and $\psi \in L^2(E, d\xi)$, then there is a Cauchy sequence ψ_j in $\Gamma(E, d\xi)$ with $\psi_j \to \psi$. Since $\tilde{\Sigma}(g)$ acts unitarily on $\Gamma(E, d\xi)$, $\tilde{\Sigma}(g)\psi_j$ is also Cauchy in $\Gamma(E, d\xi)$, and thus converges in $L^2(E, d\xi)$. We can then define

$$\tilde{\Sigma}(g)\psi \doteq \lim_{j\to\infty} \tilde{\Sigma}(g)\psi_j. \tag{3.184}$$

This representation is still smooth in the sense that it preserves the smoothness of sections. We have thus proved the following:

Theorem 3.31. *Every $\pi : E \to M$ unitary vector bundle representation of G equipped with a G-invariant volume form $d\xi$ on M induces a smooth unitary representation on $L^2(E, d\xi)$. In particular, the sectional representations of $\mathrm{IO}^+(3,1)$ on $L^2(\gamma, d\xi)$ and $\mathrm{ISO}^+(3,1)$ on $L^2(\gamma_\pm, \xi)$ are unitary.*

The massless representations constructed in the standard little group representations are representations on the space of $\mathbb{C}$-valued L^2 functions over $\mathcal{L}_+$ which is equivalent to the space of sections of the trivial line bundle on $\mathcal{L}_+$. This leads to nonsmooth representations when $h \neq 0$. By considering representations on sections of nontrivial bundles, one can obtain smooth massless representations.

To complete this picture and prove that massless unitary irreducible bundle representation of $\mathrm{ISO}^+(3,1)$ are particles, it is necessary to show that the corresponding sectional representations are also irreducible:

Theorem 3.32. *Suppose Σ is an irreducible unitary bundle representation of $\mathrm{ISO}^+(3,1)$ on $\pi : E \to \mathcal{L}_+$ such that a spacetime translation by $\mathsf{a} \in \mathbb{R}^{3+1}$ acts on a vector $(k, v) \in E_k$ by $\Sigma_{T_\mathsf{a}}(k, v) = (k, e^{-ik_\mu \mathsf{a}^\mu} v)$. Then the induced Hilbert space representation $\tilde{\Sigma}$ on $L^2(E, d\xi)$ is irreducible.*

This is harder to prove than the previous results in this section since it depends on specific properties on the Poincaré group. Indeed, it not generally true that unitary irreducible bundle representations of an arbitrary Lie group G generate irreducible sectional representations. A counterexample can be found in elementary quantum mechanics. Consider the trivial line bundle $E = S^2 \times \mathbb{C}$ over S^2 with generic element $(\hat{\boldsymbol{k}}, c) \in E$ and with the irreducible $\mathrm{SO}(3)$ action which acts by rotations on $\boldsymbol{k}$ and trivially on the fiber:

$$R(\hat{\boldsymbol{k}}, c) = (R\hat{\boldsymbol{k}}, c). \tag{3.185}$$

The sections $\Gamma(E)$ are just $\mathbb{C}$-valued functions on S^2, and the sectional representation is

$$(Rf)(\hat{\boldsymbol{k}}) = f(R^{-1}\hat{\boldsymbol{k}}). \tag{3.186}$$

This is the standard action of SO(3) on functions on S^2, and is reducible using spherical harmonics:

$$\Gamma(S^2 \times \mathbb{C}) = V_0 \oplus V_1 \oplus V_2 \oplus \cdots \tag{3.187}$$

where V_j is the $2j+1$-dimensional vector space spanned by the spherical harmonics Y_{jm} with $|m| < j$. For example, the constant functions are a one-dimensional subrepresentation corresponding to V_0. Thus, we see that $\Gamma(E)$ is not irreducible even if E is. In fact, the sectional representation will always be reducible when G is compact. This is because $L^2(E, d\xi)$ is an infinite-dimensional vector space meanwhile any unitary irreducible representation of a compact group is finite-dimensional ([124], Theorem 3.9). However, this does not apply to noncompact groups, such as the Poincaré group.

Simms ([47], pp. 50-51) essentially claims that Theorem 3.32 is implied by a theorem of the mathematician Mackey, namely Theorem 14.2 in Ref. [115]. However, Simms does not provide any details and this implication is far from obvious as Mackey does not discuss vector bundles at all in this two part paper [115, 116]. Furthermore, Mackey's theory is based on the machinery of projection-valued measures, direct integrals, and systems of imprimitivity which are vastly more complicated than the formalism we use here. Indeed, Mackey was interested in extending Wigner's methods pertaining to the Poincaré group to representations of arbitrary locally compact groups. This is a substantially harder problem and requires correspondingly more advanced methods. However, we are interested only in the case of the Poincaré group, and we will give a novel proof of Theorem 3.32 using simpler methods from measure theory and Fourier analysis.

We first prove a simplified version of Theorem 3.32 which assumes a smoothness condition. This allows for a more transparent proof and motivates the measure theoretic techniques we will use in the general proof. Given a representation of $\mathrm{ISO}^+(3,1)$ on $L^2(E, d\xi)$, one can

decompose it into irreducible Hilbert subrepresentations $\mathcal{H}_a$:

$$L^2(E, d\xi) = \bigoplus_a \mathcal{H}_a. \tag{3.188}$$

Our goal is to ultimately show that there is only a single $\mathcal{H}_a$ in this decomposition. It is not clear *a priori* that any $\mathcal{H}_a$ contains a smooth section other than the zero section; that is, there is the possibility that every nonzero smooth section is a linear combination of nonsmooth elements from the $\mathcal{H}_a$. We call the representation normal if at least one of the $\mathcal{H}_a$ contains a smooth section which is not identically zero.

Theorem 3.33. *Theorem 3.32 holds under the additional hypothesis that the sectional representation $\tilde{\Sigma}$ is normal.*

Proof. By assumption, $L^2(E, d\xi)$ has an irreducible subrepresentation $\mathcal{H}$ which contains a smooth section ψ_0 which is not the zero section. We will prove that $L^2(E, d\xi) = \mathcal{H}$. Consider the vector space $\mathcal{V}$ generated by the orbit of ψ_0:

$$\mathcal{V} = \Big\{ \sum_{i=1}^{n} c_i [\tilde{\Sigma}(\Lambda_i)\psi_0](k) | c_i \in \mathbb{C},\ \Lambda_i \in \mathrm{ISO}^+(3,1), n \in \mathbb{N} \Big\}. \tag{3.189}$$

$\mathcal{V} \subseteq \mathcal{H}$ since $\mathcal{H}$ is invariant under the $\mathrm{ISO}^+(3,1)$ action. The closure of $\mathcal{V}$ in $L^2(E, \xi)$, denoted by $\bar{\mathcal{V}}$, is a closed subspace of $L^2(E, \xi)$ and thus a Hilbert space. Furthermore, $\bar{\mathcal{V}} \subseteq \mathcal{H}$ since $\mathcal{H}$ is complete. $\bar{\mathcal{V}}$ is also a representation of $\mathrm{ISO}^+(3,1)$ since if $\sum_{i=1}^{n_j} c_{ij}\tilde{\Sigma}(\Lambda_{ij})\psi_0 \to \bar{\psi}_0$ and $L \in \mathrm{ISO}^+(3,1)$, then $\sum_{i=1}^{n_j} c_{ij}\tilde{\Sigma}(L\Lambda_{ij})\psi_0 \to \tilde{\Sigma}(L)\bar{\psi}_0$ because the action Σ is continuous. Since $\mathcal{H}$ is irreducible by assumption, $\mathcal{H} = \bar{\mathcal{V}}$. Decompose $L^2(E, d\xi)$ as

$$L^2(E, d\xi) = \bar{\mathcal{V}} \oplus (\bar{\mathcal{V}})^{\perp} \tag{3.190}$$

where $(\bar{\mathcal{V}})^\perp$ is the orthogonal compliment of $\bar{\mathcal{V}}$. The proof is complete if we show that $(\bar{\mathcal{V}})^\perp = \{0\}$. If $\psi^\perp(k) \in (\bar{\mathcal{V}})^\perp$, then

$$\langle \psi^\perp, \tilde{\Sigma}(\Lambda)\psi_0 \rangle = \int \frac{\psi^\perp(\boldsymbol{k'})^* \cdot [(\tilde{\Sigma}(\Lambda)\psi_0](\boldsymbol{k'})}{|\boldsymbol{k'}|} d\boldsymbol{k'} = 0 \tag{3.191}$$

for every $\Lambda \in \mathrm{ISO}^+(3,1)$. As $\psi_0(\boldsymbol{k})$ is smooth and not the zero section, there is some small ball $U \in \mathcal{L}_+$ containing a point $\boldsymbol{k}_0$ on which ψ_0 is nonvanishing. For each fixed $\boldsymbol{k} \in \mathcal{L}_+$, choose a homogeneous Lorentz transformation $L_{\boldsymbol{k}} \in \mathrm{SO}^+(3,1)$ such that $L_{\boldsymbol{k}}\boldsymbol{k}_0 = \boldsymbol{k}$. Note that no continuous choice of $L_{\boldsymbol{k}}$ exists, but this is irrelevant to the current argument. Consider the action of $T_{-\mathsf{a}} \cdot L_{\boldsymbol{k}}$ where a denotes an arbitrary translation in only the spatial dimensions $\mathsf{a} = (0, \mathbf{a})$:

$$\begin{aligned} \mathcal{F}_{\boldsymbol{k}}(\mathbf{a}) &\doteq \frac{1}{(2\pi)^3} \langle \psi^\perp, \tilde{\Sigma}(T_{-\mathsf{a}} \cdot L_{\boldsymbol{k}})\psi_0 \rangle \\ &= \frac{1}{(2\pi)^3} \int e^{i\boldsymbol{k'}\cdot\mathbf{a}} \frac{\langle \psi^\perp(\boldsymbol{k'}), [\tilde{\Sigma}(L_{\boldsymbol{k}})\psi_0](\boldsymbol{k'}) \rangle}{|\boldsymbol{k'}|} d\boldsymbol{k'} \\ &= 0. \end{aligned} \tag{3.192}$$

$\mathcal{F}_{\boldsymbol{k}}(\mathbf{a})$ is the 3D Fourier transform of the function

$$F_{\boldsymbol{k}}(\boldsymbol{k'}) \doteq \frac{\langle \psi_\perp(\boldsymbol{k'}), \psi_{\boldsymbol{k}}(\boldsymbol{k'}) \rangle}{|\boldsymbol{k'}|}. \tag{3.193}$$

where

$$\psi_{\boldsymbol{k}} \doteq \tilde{\Sigma}(L_{\boldsymbol{k}})\psi_0. \tag{3.194}$$

Since $\mathcal{F}_{\boldsymbol{k}}(\boldsymbol{a}) = 0$ for all $\boldsymbol{a} \in \mathbb{R}^3$, and since the Fourier transform is an isomorphism, $F_{\boldsymbol{k}}(\boldsymbol{k'}) = 0$ almost everywhere. E is irreducible and therefore a rank 1 vector bundle by Theorem 3.24. Thus, if $F_{\boldsymbol{k}}(\boldsymbol{k'}) = 0$ for some $\boldsymbol{k'}$, either $\psi^\perp(\boldsymbol{k'}) = 0$ or $\psi_{\boldsymbol{k}}(\boldsymbol{k'}) = 0$. By construction, $\psi_{\boldsymbol{k}}(\boldsymbol{k'})$ is nonzero for $\boldsymbol{k'}$ in a neighborhood of $\boldsymbol{k}$, and so $\psi^\perp$ must vanish almost everywhere in that neighborhood. By varying $\boldsymbol{k}$, we obtain that $\psi^\perp$ vanishes almost everywhere, and in

particular is the zero section of $L^2(E, d\xi)$ since L^2 space is defined up to equivalence almost everywhere. Thus, $(\bar{\mathcal{V}})^\perp = \{0\}$ and $L^2(E, d\xi) = \bar{\mathcal{V}} = \mathcal{H}$ is irreducible. □

While the assumption that the sectional representation is normal in Theorem 3.33 appears reasonable, it is known that mathematical pathologies are common in quantum mechanics [83, 111]. Indeed, we saw earlier how subtle singularities enter into Wigner's original massless representations and produce nonsmooth representations. It is thus important to show that the theorem holds without assuming smoothness conditions from the outset.

Without assuming the representation is normal, ψ_0 cannot necessarily be chosen to be smooth. However, L^2 functions are, by definition, measurable [125], and we will use measurability as a substitute for smoothness. We will borrow standard results from measure theory and the theory of Hilbert spaces; references for these topics are [125–127]. Let ξ be the measure on the light cone $\mathcal{L} \cong \mathbb{R}^3 \setminus \{0\}$ associated to the Lorentz invariant volume form $d\xi = \frac{d\boldsymbol{k}}{|\boldsymbol{k}|}$. That is, if $A \subseteq \mathbb{R}^3 \setminus \{0\}$ is (Lebesgue) measurable, then

$$\xi(A) = \int_A d\xi. \tag{3.195}$$

ξ can be extended to a measure on all of $\mathbb{R}^3$ by assigning $\xi(\{0\}) = 0$, which is consistent with Eq. (3.195) since

$$\xi(\{0\}) = \lim_{\epsilon \to 0} \xi(B_\epsilon(\mathbf{0})) = \lim_{\epsilon \to 0} \int_{B_\epsilon(\mathbf{0})} \frac{d\boldsymbol{k}}{|\boldsymbol{k}|} = 0 \tag{3.196}$$

where $B_\epsilon(\boldsymbol{k})$ is a ball of radius ϵ about $\boldsymbol{k}$. Note that if λ denotes the standard Lebesgue measure on $\mathbb{R}^3$, then $\xi(A) = 0$ if and only if $\lambda(A) = 0$, and we can refer unambiguously to such sets as measure 0.

We will use a technical result of Buczolich [128], for which we recall a few definitions. $\boldsymbol{k} \in \mathbb{R}^3$ is a point of Lebesgue density of the measurable set $A \subseteq \mathbb{R}^3$ if

$$\lim_{\epsilon \to 0} \frac{\lambda\big(B_\epsilon(\boldsymbol{k}) \cap A\big)}{\lambda\big(B_\epsilon(\boldsymbol{k})\big)} = 1. \tag{3.197}$$

This essentially says that small enough balls around $\boldsymbol{k}$ reside mostly in A. A function $f : A \to B$ between two subsets of $\mathbb{R}^3$ is bi-Lipschitz if it is invertible and if both f and f^{-1} are Lipschitz continuous.

Lemma 3.34 (Buczolich, [128], Theorem 1)**.** *Suppose that A and B are measurable subsets of $\mathbb{R}^3$ and $f : A \to B$ is bi-Lipschitz. Then f maps points of Lebesgue density of A into points of Lebesgue density of B.*

Corollary 3.35. *Suppose A is a measurable subset of $\mathcal{L}_+ \cong \mathbb{R}^3 \setminus \{0\}$ and consider $\Lambda \in \mathrm{ISO}^+(3,1)$ as a function on $\mathcal{L}_+$. Then Λ maps points of Lebesgue density of A to points of Lebesgue density of $\Lambda(A)$.*

Proof. By the lemma, it suffices to show that the action of Λ is bi-Lipschitz. Furthermore, one can consider only the cases when Λ is a pure translation, rotation, or boost. The first two are obviously bi-Lipschitz. For the latter case, without loss of generality, suppose that Λ is a boost by velocity $v\hat{\boldsymbol{k}}_x$. Λ acts on $\boldsymbol{k} = (k_x, k_y, k_z)$ by

$$\Lambda(\boldsymbol{k}) = \Big(\gamma_{Lor}(k_x + v|\boldsymbol{k}|), k_y, k_z\Big) \tag{3.198}$$

so the Jacobian of the transformation is

$$|D\Lambda|(\boldsymbol{k}) = \gamma_{Lor}\Big(1 + \frac{vk_x}{|\boldsymbol{k}|}\Big) \leq \gamma_{Lor}(1 + |v|). \tag{3.199}$$

Since the differential has a uniform bound for all $\boldsymbol{k}$, Λ is Lipschitz. Λ^{-1} is a boost by $-v\hat{\boldsymbol{k}}_x$, and thus $|D(\Lambda^{-1})|$ is also bounded above by $\gamma_{Lor}(1 + |v|)$, so Λ^{-1} is Lipschitz. Thus, Λ is bi-Lipschitz. □

With these results in place, we can adapt the proof Theorem 3.33 to prove Theorem 3.32.

Proof of Theorem 3.32. Let $\mathcal{H}$ be an irreducible subrepresentation of $L^2(E, d\xi)$, and fix some section $\psi_0 \in \mathcal{H}$ which is not identically 0, and define $\mathcal{V}$ as the vector space generated by its orbit as in Eq. (3.189). Again let $\bar{\mathcal{V}}$ be the Hilbert space completion of $\mathcal{V}$, $(\bar{\mathcal{V}})^\perp$ be its orthogonal complement, and $\psi^\perp \in (\bar{\mathcal{V}})^\perp$. $\bar{\mathcal{V}}$ is a subrepresentation of $L^2(E, \xi)$ contained in $\mathcal{H}$, so $\bar{\mathcal{V}} = \mathcal{H}$. The strategy is again to show that $(\bar{\mathcal{V}})^\perp$ contains only the zero section. Fix some $\boldsymbol{k}_0 \in \mathcal{L}_+$ which will be specified later. For each $\boldsymbol{k} \in \mathcal{L}_+$, choose a homogeneous Lorentz transformation $L_{\boldsymbol{k}}$ such that $L_{\boldsymbol{k}}\boldsymbol{k}_0 = \boldsymbol{k}$. By precisely the same argument as in the proof of Theorem 3.33, for each fixed $\boldsymbol{k}$, at almost every $\boldsymbol{k}'$ either

$$\psi_\perp(\boldsymbol{k}') = 0 \quad \text{or} \quad \psi_{\boldsymbol{k}}(\boldsymbol{k}') = 0 \tag{3.200}$$

where

$$\psi_{\boldsymbol{k}} \doteq \tilde{\Sigma}(L_{\boldsymbol{k}})\psi_0. \tag{3.201}$$

Define the subsets

$$R_0 \doteq \{\boldsymbol{k}' \in \mathcal{L}_+,\ |\psi_0(\boldsymbol{k}')| \neq 0\} \tag{3.202}$$

$$R_{\boldsymbol{k}} \doteq \{\boldsymbol{k}' \in \mathcal{L}_+,\ |\psi_{\boldsymbol{k}}(\boldsymbol{k}')| \neq 0\} = L_{\boldsymbol{k}} R_0 \tag{3.203}$$

$$R_\perp \doteq \{\boldsymbol{k}' \in \mathcal{L}_+,\ |\psi_\perp(\boldsymbol{k}')| \neq 0\}, \tag{3.204}$$

$|\psi_0|$,$|\psi_{\boldsymbol{k}}|$ and $|\psi_\perp|$ are $L^2(d\xi)$ functions and thus measurable, so the sets R_0, $R_{\boldsymbol{k}}$ and $R_\perp$ are measurable. Eq. (3.200) says that $R_{\boldsymbol{k}}$ and $R_\perp$ are almost disjoint for every $\boldsymbol{k}$:

$$\lambda(R_\perp \cap R_{\boldsymbol{k}}) = 0. \tag{3.205}$$

Let Q_0, $Q_{\boldsymbol{k}}$, and $Q_\perp$ be the sets of points of Lebesgue density of R_0, $R_{\boldsymbol{k}}$, and $R_\perp$. By the Lebesgue density theorem ([125], Corollary 1.5), a measurable set and its set of points of

Lebesgue density differ at most by a set of measure 0, so

$$\lambda(R_0) = \lambda(Q_0) \tag{3.206}$$

$$\lambda(R_{\boldsymbol{k}}) = \lambda(Q_{\boldsymbol{k}}) \tag{3.207}$$

$$\lambda(R_\perp) = \lambda(Q_\perp). \tag{3.208}$$

By assumption, ψ_0 is not the zero section in $L^2(E, d\xi)$, so R_0 has nonzero measure, and therefore so does Q_0. We have not yet specified how $\boldsymbol{k}_0$ was chosen; choose $\boldsymbol{k}_0 \in Q_0$. Then $\boldsymbol{k} \in Q_{\boldsymbol{k}}$ by Eq. (3.203) and Corollary 3.35.

We next establish that

$$Q_\perp \cap Q_{\boldsymbol{k}} = \varnothing \tag{3.209}$$

for every $\boldsymbol{k}$. Indeed, if $\boldsymbol{k}' \in Q_\perp \cap Q_{\boldsymbol{k}}$, by the definition of Lebesgue density in Eq. (3.197), there would exist a $\delta > 0$ such that

$$\begin{aligned} \frac{1}{2}\lambda(B_\delta(\boldsymbol{k}')) &< \lambda(B_\delta(\boldsymbol{k}') \cap R_\perp), \\ \frac{1}{2}\lambda(B_\delta(\boldsymbol{k}')) &< \lambda(B_\delta(\boldsymbol{k}') \cap R_{\boldsymbol{k}}). \end{aligned} \tag{3.210}$$

By Eq. (3.205), the intersection of $B_\delta(\boldsymbol{k}') \cap R_\perp$ and $B_\delta(\boldsymbol{k}') \cap R_{\boldsymbol{k}}$ has measure 0. One would then obtain the contradiction

$$\begin{aligned} \lambda(B_\delta(\boldsymbol{k}')) &\geq \lambda\big(B_\delta(\boldsymbol{k}') \cap (R_\perp \cup R_{\boldsymbol{k}})\big) \\ &= \lambda(B_\delta(\boldsymbol{k}') \cap R_\perp) + \lambda(B_\delta(\boldsymbol{k}') \cap R_{\boldsymbol{k}}) \\ &> \lambda(B_\delta(\boldsymbol{k}')), \end{aligned} \tag{3.211}$$

establishing Eq. (3.209). Thus, $\boldsymbol{k} \notin Q_\perp$ since $\boldsymbol{k} \in Q_{\boldsymbol{k}}$. This holds for every $\boldsymbol{k} \in \mathcal{L}_+$, so $Q_\perp = \varnothing$, and therefore $\lambda(R_\perp) = 0$ by Eq. (3.208). Thus, $\psi_\perp$ is only nonzero on a set of

measure 0, and is therefore equal to the zero section in $L^2(E, d\xi)$. Then, $(\bar{\mathcal{V}})^\perp = \{0\}$, and

$$L^2(E, d\xi) = \bar{\mathcal{V}} = \mathcal{H} \tag{3.212}$$

is irreducible.

□

3.5 Applications

In this section we discuss further applications of the vector bundle description of photons. In particular, we examine the spin Chern number of light defined by Bliokh et al. [104] and obstructions to the spin-orbital decomposition of photon angular momentum.

3.5.1 Geometry of the spin Chern number

Theorem 3.27 shows that the splitting $\gamma = \gamma_+ \oplus \gamma_-$ makes sense geometrically as well as topologically. Another way to see the geometry inherent in $\gamma_\pm$ is to note that these bundles are also induced by the Berry curvature, another geometric structure. In particular, it has been shown that the R and L polarization states diagonalize the Berry curvature [104].

The geometric nature of $\gamma_\pm$ is important for understanding the so-called spin Chern number of light. The Chern classes c_j are the characteristic classes of complex vector bundles, and as such, are topological invariants. Integrals of the Chern classes define the Chern numbers C_j which are also topological invariants and commonly used in physics. Although the Chern numbers are typically calculated using a connection (often the Berry connection), which is a geometric quantity, they are true topological invariants and independent of the choice of connection [99], as discussed in §2.2.2. We have shown that the (first) Chern numbers for the photon bundles are $C(\gamma) = 0$ and $C(\gamma_\pm) = \mp 2$. In addition to the usual Chern number, a more mysterious quantity has been defined for light, the so-called spin Chern number which is physically related to the quantum spin hall effect [104]. Letting

$h_\pm = \pm 1$ denote the helicities of $\gamma_\pm$, Bliokh et al. [104] define the spin Chern number of light as

$$C_{\text{spin}} = h_+ C_1(\gamma_+) + h_- C_1(\gamma_-) = 4. \tag{3.213}$$

However, this definition clearly depends on decomposition $\gamma = \gamma_+ \oplus \gamma_-$ of the total photon bundle. From a topological standpoint, we showed in Theorem 3.16 that there are an infinite number of possible splittings $\gamma = \ell_j \oplus \ell_{-j}$. The $\gamma_\pm$ decomposition is only preferred when one considers the geometry of γ either via Poincaré symmetry or the Berry connection. Furthermore, the definition of C_{spin} explicitly involves helicity, which is a geometric quantity. Thus, the spin Chern number is not a purely topological quantity, instead reflecting both topological and geometric properties of the bundle γ. As such, the spin Chern number of light may not be as robust as the Chern number against perturbations to the underlying Maxwell system, particularly if those perturbations alter the geometric properties of the system.

3.5.2 Spin and orbital angular momenta of light

The last application of our vector bundle methods is pertains to an extended debate about the possibility of splitting photon angular momentum into spin and orbital parts [65–81]:

$$\boldsymbol{J} = \boldsymbol{J}_s + \boldsymbol{J}_o. \tag{3.214}$$

We work in the sectional representation of the Poincaré group so that the angular momentum $\boldsymbol{J}$ is represented as an operator on the vector space $\Gamma(\gamma)$. Despite many attempts, every proposed splitting of the photon angular momentum has has either failed to be gauge-invariant or else not satisfied the $\mathfrak{so}(3)$ angular momentum commutation relations [67, 78, 80], which is the defining property of angular momentum operators [1, 2, 67, 68, 74, 75, 78, 79, 82, 83]. In this section we will present a novel argument showing that the failure of the photon angular momentum to decompose into SAM and OAM is a consequence of the

singular jump of the little group from SO(3) to ISO(2) that occurs when particle mass is taken to zero. We will also show that a commonly proposed choice [71, 72] of $\boldsymbol{J}_s$ and $\boldsymbol{J}_o$ do not satisfy $\mathfrak{so}(3)$ commutation relations, and thus cannot properly be considered angular momentum operators [1, 2, 67, 68, 74, 75, 78, 79, 82, 83]. These nonstandard commutation relations have been found by others [67, 68, 71, 78], but little attempt has been made to interpret their meaning apart from the fact that they imply the operators fail to be angular momentum operators. We will use the vector bundle formalism to help explain the nature of these peculiar commutation relations.

The most straightforward attempt at an SAM-OAM decomposition would follow from treating photons like massive spin 1 particles. Consider a section $\boldsymbol{E}(\boldsymbol{k}) \in \Gamma(\gamma)$ of the photon bundle. Since each fiber $\gamma(\boldsymbol{k})$ is embedded in $\mathbb{C}^3$, we can think of the section as a map from $\mathcal{L}_+$ into $\mathbb{C}^3$. The action of a rotation $R \in \mathrm{SO}(3)$ on a section is given

$$\big[R\boldsymbol{E}\big](\boldsymbol{k}) = R\big[\boldsymbol{E}(R^{-1}\boldsymbol{k})\big] \doteq (F_R \circ \tilde{F}_{R^{-1}}\boldsymbol{E})(\boldsymbol{k}) \tag{3.215}$$

where F_R and $\tilde{F}_R$ are operators on vectors and functions, respectively:

$$F_R\boldsymbol{v} = R\boldsymbol{v} \tag{3.216}$$

$$(\tilde{F}_R f)(\boldsymbol{k}) = f(R\boldsymbol{k}). \tag{3.217}$$

The (total) angular momentum operator $\boldsymbol{J}$ is the generator of rotations. Letting $R_a(\theta)$ be a rotation by θ about a-axis, direct calculation shows

$$[J_a\boldsymbol{E}](\boldsymbol{k}) = i\frac{d}{d\theta}\Big|_{\theta=0}[R_a(\theta)\boldsymbol{E}](\boldsymbol{k}) \tag{3.218}$$

$$= i\Big(\frac{d}{d\theta}\Big|_{\theta=0}F_{R_{\hat{\boldsymbol{v}}}(\theta)}\Big)\boldsymbol{E}(\boldsymbol{k}) + i\Big(\frac{d}{d\theta}\Big|_{\theta=0}\tilde{F}_{R^{-1}_{\hat{\boldsymbol{v}}}(\theta)}\Big)\boldsymbol{E}(\boldsymbol{k}) \tag{3.219}$$

$$= S_a\boldsymbol{E}(\boldsymbol{k}) + L_a\boldsymbol{E}(\boldsymbol{k}). \tag{3.220}$$

where

$$S_a \doteq -i\epsilon_{abc} \tag{3.221}$$

$$\boldsymbol{L} \doteq -i(\boldsymbol{k} \times \partial_{\boldsymbol{k}}). \tag{3.222}$$

Here, $\boldsymbol{S} = (S_1, S_2, S_3)$ has the form of the spin 1 operator [71], and indeed, $\boldsymbol{S}$ and $\boldsymbol{L}$ resemble the SAM and OAM operators for a massive spin 1 particle. If $\boldsymbol{S}$ and $\boldsymbol{L}$ were well-defined operators on the vector space of photon states $\Gamma(\gamma)$, then Eq. (3.220) would furnish an SAM-OAM splitting since they satisfy the angular momentum commutation relations

$$[S_a, S_b] = i\epsilon_{abc}S_c \tag{3.223}$$

$$[L_a, L_b] = i\epsilon_{abc}L_c. \tag{3.224}$$

In this way they appear to generate rotations. The fundamental issue is that they are ill-defined operators for photons as they fail to map photon states to other photon states, i.e., they do not map $\Gamma(\gamma)$ into itself. Indeed, $\boldsymbol{L}$ and $\boldsymbol{S}$ generally give $\boldsymbol{E}(\boldsymbol{k})$ a nonzero component in the $\boldsymbol{k}$ direction, violating the transversality condition imposed by Gauss's law, so that $\boldsymbol{L}(\boldsymbol{E})$ and $\boldsymbol{S}(\boldsymbol{E})$ are not sections of γ. This traces back to the fact that while $F_R \circ \tilde{F}_R^{-1}$ preserves transversality, neither F_R nor $\tilde{F}_R^{-1}$ do independently.

The issues with this type of basic splitting has long been known [65], and has led many physicists to seek other potential decompositions. However, we have already encountered a hint that there may not be something fundamentally wrong with assuming that we can find SAM-OAM operators. As we have seen, elementary particles are representations of $\mathrm{ISO}^+(3,1)$. This implies the existence of one rotational symmetry which corresponds to $\mathrm{SO}(3) \subseteq \mathrm{ISO}^+(3,1)$ whose generator is $\boldsymbol{J}$. To say that there is an SAM-OAM splitting is to assume that we have another independent SO(3) symmetry $\boldsymbol{J}_s$.[3] This is a strong statement, and we should examine if there is a theoretical reason to expect that every

[3] $\boldsymbol{J}_o = \boldsymbol{J} - \boldsymbol{J}_s$, so it does not describe an independent SO(3) symmetry.

$\mathrm{ISO}^+(3,1)$ symmetry should induce an additional SO(3) symmetry. The theory is given to us by the little group method. We have seen that for massive particles the little group is $H = \mathrm{SO}(3)$ while massless particles have little group ISO(2). We showed in Theorem 3.22 that every massless particle has a canonical internal symmetry of the massless little group ISO(2). The analogous statement is true for massive particles, namely they all have a canonical internal SO(3) symmetry corresponding to their little group. This is actually much easier to show in the massive case. The massive little group at $\boldsymbol{k} = 0$ is $H_{\mathbf{0}} = \mathrm{SO}(3) \subseteq \mathrm{ISO}^+(3,1)$, the set of rotations of the spatial coordinates. Then if $\Lambda_{\boldsymbol{k}}$ is the unique boost which takes $\mathbf{0}$ to $\boldsymbol{k}$ for a mass m particle, then for $R \in \mathrm{SO}(3)$

$$\Pi^m(R) = \Sigma(\Lambda_{\boldsymbol{k}} R \Lambda_{-\boldsymbol{k}}) \tag{3.225}$$

is a stabilizing SO(3) action on the a massive particle bundle $\pi : E_m \to \mathcal{M}_m$. Thus, we see that a Poincaré representation on a particle bundle induces a representation of the little group, and furthermore, one which describes an internal symmetry (it does not change $\boldsymbol{k}$). For massive particles, we can define this action's generator to be $\boldsymbol{S}$, and then $\boldsymbol{L} = \boldsymbol{J} - \boldsymbol{S}$ gives a well-defined OAM operator. However, a geometric singularity occurs when $m = 0$ and the little group abruptly jumps from SO(3) to ISO(2), and we obtain instead an ISO(2) symmetry for massless particles. In fact, we previously saw that this is actually an SO(2) symmetry as the inhomogeneous elements act trivially. The generator of this symmetry is the scalar helicity $\chi = \hat{\boldsymbol{k}} \cdot \boldsymbol{J}$ operator; as it is not an SO(3) generator it is not an angular momentum operator (it is not even a vector operator). This provides a compelling reason to expect that there exists no SAM-OAM splitting for massless particles. Indeed, we will rigorously prove this in Chapter 6. We note that this appears to be the first work which relates the jump in the little group at $m = 0$ to the issues surrounding massless angular momentum.

While the helicity itself is not an SAM operator, one might wonder if it is valid to simply "vectorize" the helicity χ into an SAM operator by defining:

$$\boldsymbol{J}_s \doteq \boldsymbol{J}_\parallel \doteq (\boldsymbol{J}\cdot\hat{\boldsymbol{k}})\hat{\boldsymbol{k}} = \chi\hat{\boldsymbol{k}}, \tag{3.226a}$$

$$\boldsymbol{J}_o \doteq \boldsymbol{J}_\perp \doteq \boldsymbol{J} - \chi\hat{\boldsymbol{k}} = -\hat{\boldsymbol{k}}\times\boldsymbol{K}, \tag{3.226b}$$

where the last equality comes from Eq. (3.140). From Eqs. (3.92)-(3.95) it follows that

$$[J_{\parallel,a}, J_b] = i\epsilon_{abc}J_{\parallel,c} \tag{3.227}$$

$$[J_{\perp,a}, J_b] = i\epsilon_{abc}J_{\perp,c}, \tag{3.228}$$

which shows that $\boldsymbol{J}_\parallel$ and $\boldsymbol{J}_\perp$ are well defined as vector operators [83, 106], that is, they transform as three-component vectors under rotations. This splitting was proposed by similarly *ad hoc* arguments in Refs. [71, 72] and were referred to as "SAM" and "OAM". The issue with this splitting is that the $\boldsymbol{J}_\parallel$ and $\boldsymbol{J}_\perp$ satisfy the peculiar commutation relations

$$[J_{\parallel,a}, J_{\parallel,b}] = 0, \tag{3.229}$$

$$[J_{\perp,a}, J_{\parallel,b}] = i\epsilon_{abc}J_{\parallel,c}, \tag{3.230}$$

$$[J_{\perp,a}, J_{\perp,b}] = i\epsilon_{abc}(J_{\perp,c} - J_{\parallel,c}). \tag{3.231}$$

In particular, these are not the $\mathfrak{so}(3)$ commutation relations, meaning they do not generate rotations. Thus, $\boldsymbol{J}_\parallel$ and $\boldsymbol{J}_\perp$ cannot be considered angular momentum operators. Indeed, all of the standard results for angular momentum operators such as the addition of angular momentum formula, the angular momentum multiplet structure, the Clebsch–Gordan coefficients, etc. follow from the $\mathfrak{so}(3)$ commutation relations. This conclusion was also reached by van Enk and Nienhuis [67] who first obtained operators satisfying these these strange commutation relations. Since $\boldsymbol{J}_\parallel$ and $\boldsymbol{J}_\perp$ are not true angular momentum operators, we will refer to them as the helicity and orbital quasi-angular momentum, respectively. While these

strange commutation relations have been previously encountered in attempts to split the angular momentum [67, 68, 71], little attempt has been made to interpret them. However, it is natural to ask, if these operators do not generate rotations, what do they generate? One can explicitly check from the commutation relations that the $J_{\parallel,i}$ and $J_{\perp,i}$ satisfy the Jacobi identity, and therefore *together* form a well-defined Lie algebra $\mathfrak{g}$.

We examine first the fact that the $J_{s,i}$ commute. There is a notable difference between the helicity $\chi = \boldsymbol{J}\cdot\hat{\boldsymbol{k}}$ and the operator $\boldsymbol{J}_\parallel = \chi\hat{\boldsymbol{k}}$. Per Theorem 3.29, the helicity is associated with the little group action of SO(2) on $\Gamma(\gamma)$:

$$\tilde{\Pi}_H(\theta)\boldsymbol{E}(\boldsymbol{k}) = e^{-i\chi\theta}\boldsymbol{E}(\boldsymbol{k}) = e^{-i\theta}\boldsymbol{E}_+(\boldsymbol{k}) + e^{i\theta}\boldsymbol{E}_-(\boldsymbol{k}). \tag{3.232}$$

In contrast, the quasi-helicity angular momentum $\boldsymbol{J}_\parallel$ is associated with an $\mathbb{R}^3$ action. If $\boldsymbol{v}\in\mathbb{R}^3$, then

$$\tilde{\Pi}_{\mathbb{R}^3}(\boldsymbol{v})\boldsymbol{E}(\boldsymbol{k}) \doteq e^{-i\chi(\hat{\boldsymbol{k}}\cdot\boldsymbol{v})}\boldsymbol{E}(\boldsymbol{k}) = e^{-i(\hat{\boldsymbol{k}}\cdot\boldsymbol{v})}\boldsymbol{E}_+(\boldsymbol{k}) + e^{i(\hat{\boldsymbol{k}}\cdot\boldsymbol{v})}\boldsymbol{E}_-(\boldsymbol{k}). \tag{3.233}$$

Thus, the $J_{s,i}$ are associated with a translational symmetry of γ, explaining why they form a three-dimensional commuting Lie subalgebra of $\mathfrak{g}$. On the other hand, the $\boldsymbol{J}_\perp$ do not form a Lie subalgebra as seen from the $-i\epsilon_{abc}J_{\parallel,c}$ term in Eq. (3.231). Thus, $\boldsymbol{J}_\perp$ is not associated with any symmetry of the photon bundle. Since neither $\boldsymbol{J}_\perp$ nor $\boldsymbol{J}_\parallel$ are related to rotational symmetries of photons, they do not achieve a spin-orbital decomposition of the angular momentum. This traces back to the fact that photons are massless, and thus have helicity rather than spin. Indeed, it can be said that the "spin angular momentum" $\boldsymbol{J}_\parallel$ is neither spin nor angular momentum. Instead, we have seen that $\boldsymbol{J}_\parallel$ is associated with a helicity-induced translational symmetry of the photon system. We emphasize that even though $\boldsymbol{J}_\parallel$ and $\boldsymbol{J}_\perp$ are not truly angular momenta, the splitting is well-defined and has proved useful in experimental optics as argued by [77]. Thus, a proper understanding of the operators $\boldsymbol{J}_\parallel$ and $\boldsymbol{J}_\perp$ is of both theoretical and experimental important. In Chapter 7 we will show that

these operators are not actually as *ad hoc* as the discussion here may suggest, and that they arise naturally as a singular limit of the well-defined relativistic SAM and OAM operators for massive particles.

3.6 Conclusion

Despite the simplicity of the Maxwell system, it exhibits surprisingly rich and subtle topological behavior. While the total photon bundle is trivial, it has important topologically nontrivial subbundles such as the R and L bundles. This nontrivial topology traces back to the hole in momentum space at $\boldsymbol{k} = 0$, accounting for the fact that photons are massless and have no rest frame. This nontrivial topology can obstruct the smoothness of constructions that work in topologically trivial cases, such as the construction of a polarization global basis or the conventional little group construction of Hilbert space representations. We showed that vector bundle methods can be used to avoid these continuity issues in the latter. In particular, vector bundles representations have precisely the right structure to simultaneously study the topology and symmetry of waves, and allows for a modified version of the little group construction which does not encounter singularities. It also elucidated the symmetry issues that occur in attempts to separate photon angular momentum into SAM and OAM and reveals how this issue is related to the jump in the little group from SO(3) to ISO(2) that occurs in the massless limit. The formalism of vector bundle representations is very general, and can be applied to any waves with arbitrary symmetry groups. We will use it throughout the rest of this thesis to study other geometric and topological aspects of waves.

Chapter 4

Graviton topology

In this chapter we present material from Ref. [2] published in *The Journal of High Energy Physics*. We will examine the topology and geometry of linearized gravitons, solutions of the Einstein field equations linearized around the Minkowski background.

4.1 Introduction

It has been established that topologically nontrivial modes can exist in various media such as solids [129], fluids [20, 23–28] , and plasmas [19, 28–35]. The study of such modes has lead to important physical discoveries such as the quantum Hall effects and the bulk-edge correspondence. In Ch. 3 [1] it was shown that the vacuum also supports topologically nontrivial modes, in particular, left L and R circularly polarized photons form topologically nontrivial vector bundles with Chern numbers $+2$ and -2, respectively. More generally, we showed that it is possible for elementary massless particles to possess nontrivial topology due to a hole in their momentum space at $k = 0$. Here we present a thorough and rigorous examination of the topology of another massless particle, the (linearized) graviton. While gravitational waves, and their quanta, gravitons, were predicted early in the development of general relativity and quantum field theory, they are of particular interest now due to two recent experimental breakthroughs. Within the past decade, LIGO has detected grav-

itational waves from black hole and neutron star merger events, experimentally confirming the existence of gravitational waves [54, 55]. Even more recently, graviton-like quasiparticles have been observed in fractional quantum Hall liquids [130]. As topology, particularly Chern numbers, plays a fundamental role in the quantum Hall effects [12–14, 131, 132], an understanding of the topology of gravitons is needed.

Working in the transverse-traceless (TT) gauge for weak gravity [133], we show that the collection of all gravitons forms a complex rank 2 vector bundle Ξ over the the forward light cone $\mathcal{L}_+$. We prove that the graviton bundle Ξ is topologically trivial, that is,

$$\Xi \cong \tau_1 \oplus \tau_2 \tag{4.1}$$

where τ_1 and τ_2 are topologically trivial line bundles over $\mathcal{L}_+$. As such, the Chern number of Ξ is zero. This implies that it is possible to write down a smooth global basis for gravitational waves in momentum space, which we do using the modified clutching construction developed in §3.2.3 [1, 41]. While such global bases are frequently invoked in the literature [44, 134–136], until now there was no proof of their existence. In fact, the stronger assumption is typically made that there exists a smooth global basis which is linearly polarized [44, 134–136]. This stronger assumption is, however, false; we prove that Ξ has no linearly polarized subbundles, even topologically nontrivial ones. Thus, any global basis necessarily involves elliptical polarizations.

While the total graviton bundle Ξ is topologically trivial and can be broken down as in (4.1), its Poincaré symmetry gives a more natural geometric splitting into the R and L graviton bundles Ξ_+ and Ξ_-. These consist of gravitons with helicities $+2$ and -2, respectively, describing their behavior under rotations about their momentum axis. τ_1 and τ_2 are topologically trivial while linear polarizations do not even form well-defined vector bundles. In contrast, Ξ_+ and Ξ_- are well-defined vector bundles with nontrivial topology completely characterized by the Chern numbers ∓ 4. This topological nontriviality reflects the fact that

the internal (base manifold) and external (fiber) degrees of freedom (DOFs) are twisted together. This twisting acts as a topological obstruction to the splitting of graviton angular momentum into spin angular momentum (SAM) and orbital angular momentum (OAM). This is important since there have been recent efforts to study the OAM of gravitational waves [56–58].

This article is organized as follows. In Section 4.2 we show that in the limit of weak gravity, the collection of all gravitons form a smooth vector bundle Ξ over the forward light cone. In Section 4.3, we show that the Poincaré symmetry of Ξ naturally decomposes the bundle into the R and L gravitons $\Xi_\pm$, which correspond to elementary particles. In Section 4.4, we completely classify these vector bundles via their Chern numbers, showing that Ξ is trivial while $\Xi_\pm$ are nontrivial. We proceed in Section 4.5 to use the modified clutching construction described in §3.2.3 to explicitly construct a smooth global basis of Ξ. In Section 4.6, we show that unlike the trivial and circular polarizations, linear polarizations do not form well-defined vector bundles and thus have no associated topology. In Section 4.7 it is shown that the topological nontriviality of $\Xi_\pm$ obstructs the splitting of graviton angular momentum into OAM and SAM.

4.2 The graviton bundle Ξ

We work in the limit of weak gravity, in which the spacetime metric $g_{\mu\nu}$ can be expanded around the flat Minkowski metric $\eta_{\mu\nu}$ with signature $(-,+,+,+)$ as

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \tag{4.2}$$

where $h_{\mu\nu}$ is symmetric and $|h_{\mu\nu}| \ll 1$ [133]. $h_{\mu\nu}$ is the graviton field [137]. Following Schutz [133], in vacuum, far from the sources of the field, the Einstein equations in the TT gauge take the form

$$\eta^{\mu\nu} h^{\alpha\beta}{}_{,\mu\nu} = 0. \tag{4.3}$$

The solutions can be written in Fourier space as

$$h^{\alpha\beta} = A^{\alpha\beta} \exp(ik^\mu x_\mu) \tag{4.4}$$

where $k^\mu = (\omega, \boldsymbol{k})$ is the four-momentum and $A^{\alpha\beta}$ is a complex symmetric tensor. We refer to these solutions as gravitons, as they are the modes of the graviton field. k and A are both subject to constraints. The Einstein equation requires that

$$k^\alpha k_\alpha = 0, \tag{4.5}$$

so $\omega = \pm|\boldsymbol{k}|$, i.e., k resides on the light cone. As in the case of photons, it is sufficient to consider only the $\omega = |\boldsymbol{k}|$ solutions in which k is restricted to the forward light cone $\mathcal{L}_+$, as the $\omega = -|\boldsymbol{k}|$ solutions are then determined by the condition that $h^{\mu\nu}$ is real. Notably, the forward $\mathcal{L}_+$ has a hole at the origin $k^\mu = 0$, as can be seen by a number of arguments. From a physical perspective, modes with $k^\mu = 0$ are not propagating and thus do not represent gravitons. Such modes are not accessible by Lorentz transforming propagating modes since massless particles have no rest frame. Mathematically, the origin must be removed in order for $\mathcal{L}_+$ to be a regular submanifold of Minkowski space. Furthermore, as pointed out by Staruszkiewicz [138], the forward light cone is only geodesically complete if $k = 0$ is excluded. We again parametrize $\mathcal{L}_+$ by its spatial part $\boldsymbol{k}$, so $\mathcal{L}_+ \cong \mathbb{R}^3 \setminus \{0\}$.

Beyond the constraint that $A^{\alpha\beta}$ is symmetric, it must additionally satisfy the TT gauge conditions [133]:

$$A^{\alpha\beta} k_\beta = 0 \tag{4.6a}$$

$$A^\alpha{}_\alpha = 0 \tag{4.6b}$$

$$A^{\alpha\beta} U_\beta = 0, \tag{4.6c}$$

where U_β is any fixed timelike unit vector. The first two conditions reflect the transverse and traceless gauge choices, respectively. The third condition fixes a residual temporal gauge freedom. The canonical choice, which we adopt here, is to take $U_\beta = \delta^0_\beta$ since this is the distinguished timelike unit vector. The temporal gauge condition thus imposes that

$$A^{\alpha 0} = A^{0\alpha} = 0, \tag{4.7}$$

allowing us to consider just the tensors A^{ij}, where $i, j \in \{1, 2, 3\}$ span the spatial DOFs. The gauge conditions (4.6a) and (4.6b) then reduce to

$$A^{ij} k_j = 0 \tag{4.8a}$$

$$A^i{}_i = 0. \tag{4.8b}$$

Thus, the set of all gravitons is given by

$$\Xi = \{(k, A^{ij})\} \tag{4.9}$$

with $k \in \mathcal{L}_+$ and A^{ij} symmetric and subject to the constraints (4.8). Denote by $\Xi(k_0)$ the fiber at k_0 consisting of the elements of Ξ with $k = k_0$. The transverse gauge condition (4.8a) imposes three independent linear constraints on the five-dimensional complex vector space $\mathcal{V}_3$ of traceless symmetric rank 2 tensors on $\mathbb{C}^3$, and thus each fiber $\Xi(k_0)$ is a two-dimensional complex vector space. Furthermore, the transverse condition is smooth in k, and thus the fibers fit together to give Ξ the form of a rank 2 complex vector bundle over $\mathcal{L}_+ \cong \mathbb{R}^3 \setminus \{0\}$; we call Ξ the graviton bundle. This vector bundle gives a concrete way to study the topology of gravitons. The essential problem is to determine the topology of Ξ and its subbundles.

The most obvious topological question is the following: is Ξ topologically trivial, that is, does Ξ possess a global basis? It is always possible to form a local basis. For example, the

plus-cross tensors

$$B_{z,+} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{4.10a}$$

$$B_{z,\times} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{4.10b}$$

are a commonly used basis at $\boldsymbol{k}_0 = \boldsymbol{e}_z = (0, 0, 1)$, and it is straightforward to smoothly extend this locally to a small neighborhood about $\boldsymbol{e}_z$ (e.g., by parallel transport via any connection). However, it is commonly assumed that this can be smoothly extended to a global basis [44, 134–136]. One explicit globalized choice which has appeared in the literature is [44, 134]

$$B_+(\boldsymbol{k}) = \boldsymbol{u}(\boldsymbol{k}) \otimes \boldsymbol{u}(\boldsymbol{k}) - \boldsymbol{v}(\boldsymbol{k}) \otimes \boldsymbol{v}(\boldsymbol{k}) \tag{4.11a}$$

$$B_\times(\boldsymbol{k}) = \boldsymbol{u}(\boldsymbol{k}) \otimes \boldsymbol{v}(\boldsymbol{k}) + \boldsymbol{v}(\boldsymbol{k}) \otimes \boldsymbol{u}(\boldsymbol{k}), \tag{4.11b}$$

where $(\boldsymbol{u}(\boldsymbol{k}), \boldsymbol{u}(\boldsymbol{k}), \hat{\boldsymbol{k}})$ is supposed to be some orthonormal basis of $\mathbb{R}^3$ at each $\boldsymbol{k}$. The issue is that it is not actually possible to smoothly choose such an orthonormal basis, as we have seen before. Indeed, when restricted to the unit sphere S^2, $\boldsymbol{u}(\boldsymbol{k})$ and $\boldsymbol{v}(\boldsymbol{k})$ would be continuous nonvanishing tangent vector fields on S^2, violating the hairy ball theorem ([39], Theorem 1). Thus, any basis $(B_+(\boldsymbol{k}), B_\times(\boldsymbol{k}))$ of the form (4.11) has singularities.

As in the case of photons, it is not immediately obvious that *some* smooth global basis for gravitons exists because these particles are massless and thus have noncontractible light cone $\mathcal{L}_+$ as a base manifold. Indeed, as we saw in Ch. 3 ([1]), the R and L circularly polarized photon bundles $\gamma_\pm$ were topologically nontrivial [1]. Similarly, we will show that Ξ can be broken into two topologically nontrivial R and L circularly polarized subbundles $\Xi_\pm$, although they are topologically distinct from $\gamma_\pm$. Despite the nontriviality of $\Xi_\pm$, the

total graviton bundle Ξ will be trivial like the total photon bundle γ, and we will construct a globally smooth basis for it, although it is more complicated than the singular basis described by (4.11).

4.3 Decomposition of the graviton bundle into elementary particles

Although elementary particles are conventionally defined as unitary irreducible representations (UIRs) of the proper orthochronous Poincaré group $\mathrm{ISO}^+(3,1)$ on a Hilbert space [42, 43], they can be equivalently defined as UIRs of $\mathrm{ISO}^+(3,1)$ on vector bundles as shown in §3.4. Representations on vector bundles are the natural generalization of vector space representations, and describe symmetries of the underlying system. In particular, if E is a vector bundle over the base manifold M, then E is a representation of the $\mathrm{ISO}^+(3,1)$ with action Σ if for each $L \in \mathrm{ISO}^+(3,1)$, $\Sigma(L)$ is a linear isomorphism between the fibers $E(k)$ and $E(Lk)$. Furthermore, it is required Σ respect the group structure, that is, $\Sigma(L_1 L_2) = \Sigma(L_1)\Sigma(L_2)$ for any $L_1, L_2 \in \mathrm{ISO}^+(3,1)$. Massless particles correspond to representations on bundles over the light cone and massive particles to bundles over mass hyperboloids. Unlike massive particles which can correspond to higher rank (trivial) bundles, massless particles are always line bundles and can be nontrivial [1]. We will show that Ξ is a unitary vector bundle representation of $\mathrm{ISO}^+(3,1)$, and find its decomposition into UIRs, that is, into elementary particles.

Recall that $\mathrm{ISO}^+(3,1)$ consists of Lorentz transformations Λ, spacetime translations T_{a} with $a^\mu \in \mathbb{R}^{3+1}$, and compositions of the two. We define the action of T_{a} on $(k, A^{\mu\nu}) \in \Xi$ by

$$\Sigma(T_{\mathsf{a}})(k, A^{\alpha\beta}) = (k, e^{-ia^\mu k_\mu} A^{\alpha\beta}), \tag{4.12}$$

reflecting the effect of spacetime translations in momentum space. Ideally, for a general Lorentz transformation Λ, $(k, A^{\alpha\beta})$ would transform covariantly:

$$(k, A^{\alpha\beta}) \mapsto (k', A'^{\alpha\beta}) \doteq (\Lambda k, \Lambda^{\alpha}{}_{\mu}\Lambda^{\beta}{}_{\nu}A^{\mu\nu}). \tag{4.13}$$

(k', A') satisfies the transverse and traceless gauge conditions given by equations (4.6a) and (4.6b). However, when $\Lambda \notin \mathrm{SO}(3)$, A' generally has nonvanishing temporal components, violating the temporal gauge condition (4.7) and implying that $(k', A') \notin \Xi$. It is, however, always possible to find a gauge transformation which takes A' to a physically equivalent A'' such that A'' satisfies the TT gauge constraints (4.7) and (4.8) so that $(k', A'') \in \Xi$. Indeed, any C^{α} satisfying $C^{\alpha}k'_{\alpha} = 0$ induces the gauge transformation ([133], Eq. (9.17))

$$A''^{\alpha\beta} = A'^{\alpha\beta} - iC^{\alpha}k'^{\beta} - ik'^{\alpha}C^{\beta}. \tag{4.14}$$

Since $k'^{\alpha}k'_{\alpha} = C^{\alpha}k'_{\alpha} = 0$, A'' is still transverse and traceless:

$$A''^{\alpha\beta}k_{\beta} = 0 \tag{4.15a}$$

$$A''^{\alpha}{}_{\alpha} = 0. \tag{4.15b}$$

Imposing the temporal gauge condition $A''^{\alpha 0} = 0$ fixes C^{α}:

$$C^{\alpha} = \frac{i}{k'^{0}}\Big(- A'^{\alpha 0} + \frac{A'^{00}}{2k'^{0}}k'^{\alpha}\Big). \tag{4.16}$$

Then $(k', A'') \in \Xi$, so the action of Λ on Ξ is given by

$$\Sigma(\Lambda)(k, A) \to (\Lambda k, A''). \tag{4.17}$$

We note that the action of a rotation $R \in \mathrm{SO}(3) \subseteq \mathrm{ISO}^+(3,1)$ on $(k, A) \in \Xi(k)$ is simply given by

$$\Sigma(R)(k, A) = (Rk, RAR^{-1}), \tag{4.18}$$

where A and R are both considered as three-by-three matrices.

To show that this action is unitary, we must first define a Hermitian product on Ξ. Consider the vector space $\mathcal{V}_4$ consisting of symmetric, traceless rank 2 tensors on $\mathbb{C}^4$ which has the indefinite Hilbert-Schmidt product given by

$$\langle A^{\alpha\beta}, B^{\alpha\beta} \rangle = \frac{1}{2}(A^*)^{\alpha\beta} B_{\beta\alpha}. \tag{4.19}$$

When restricted to $\mathcal{V}_3$, the subspace of $\mathcal{V}_4$ with $A^{\alpha 0} = 0$, this product is positive-definite making $\mathcal{V}_3$ into a Hilbert space. On $\mathcal{V}_3$, the Hilbert-Schmidt product takes the form

$$\langle A, B \rangle = \frac{1}{2}\mathrm{Tr}(A^* B) \tag{4.20}$$

By equipping every fiber of Ξ with this inner product, Ξ is made into Hermitian vector bundle. It is easy to see that the Poincaré action Σ is unitary with respect to this product, that is, for $A, B \in \Xi(k)$:

$$\langle \Sigma(\Lambda)A, \Sigma(\Lambda)B \rangle = \langle A, B \rangle. \tag{4.21}$$

In particular, the inner product of A and B is a Lorentz scalar by (4.19), so it is preserved by the covariant transformation (4.13). The subsequent gauge transformation (4.14) preserves the product since $k'^{\alpha} k'_{\alpha} = C^{\alpha} k'_{\alpha} = 0$. We have thus proved the following result:

Theorem 4.1. *Ξ is a unitary vector bundle representation of $\mathrm{ISO}^+(3,1)$ with action Σ given by equations (4.12) and (4.17).*

This allows us to to decompose Ξ into UIRs labeled by their helicities using Thm 3.24. For $\boldsymbol{k} \in \mathcal{L}_+$, denote by $R_{\boldsymbol{k}}(\psi) \in \mathrm{SO}(3) \subseteq \mathrm{ISO}^+(3,1)$, the rotation by angle ψ about the $\boldsymbol{k}$ axis.

Theorem 4.2 (R and L circularly polarized subbundles)**.** *The graviton bundle Ξ decomposes as*

$$\Xi = \Xi_+ \oplus \Xi_- \tag{4.22}$$

where $\Xi_\pm$ are unique UIR line bundle representations of $\mathrm{ISO}^+(3,1)$ with helicities ± 2. They are thus elementary particles. Elements of $A \in \Xi_\pm(k)$ have the form

$$A = c(B_1 + iB_2) \tag{4.23}$$

where $c \in \mathbb{C}$ and $B_1, B_2 \in \Xi(k)$ are real and orthonormal. We call Ξ_+ and Ξ_- the R and L circularly polarized graviton bundles.

Proof. By Theorems 4.1 and 3.24, Ξ decomposes into irreducible bundle UIRs

$$\Xi = \Xi_+ \oplus \Xi_- \tag{4.24}$$

with helicities h_+ and h_-, respectively. In particular, this means that

$$\Sigma\big(R_{\boldsymbol{k}}(\psi)\big)(k, v_j) = e^{-ih_j\psi}(k, v_j) \tag{4.25}$$

for all $(k, v_j) \in \Xi_j$. To show that $h_\pm = \pm 2$, it suffices to consider any fixed fiber, say $\Xi(\boldsymbol{e}_z)$. The elements

$$A_{z,\pm} = \frac{1}{\sqrt{2}}(B_{z,+} \pm iB_{z,\times}) \tag{4.26a}$$

$$= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & \pm i & 0 \\ \pm i & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{4.26b}$$

of the fiber $\Xi(\boldsymbol{e}_z)$ satisfy

$$\Sigma(R_{\boldsymbol{e}_z}(\psi))A_{z,\pm} = e^{\mp 2i\psi}A_{z,\pm}, \tag{4.27}$$

for any ψ, showing that $h_\pm = \pm 2$.

Now let $L_{\boldsymbol{k}}$ be some Lorentz transformation taking $\boldsymbol{e}_z$ to $\boldsymbol{k}$. Since $\Xi_\pm$ are $\mathrm{ISO}^+(3,1)$ symmetric,

$$\Sigma(L_{\boldsymbol{k}})A_{z,\pm} = \frac{1}{\sqrt{2}}\big(\Sigma(L_{\boldsymbol{k}})B_{z,+} \pm i\Sigma(L_{\boldsymbol{k}})B_{z,\times}\big) \tag{4.28}$$

span the fibers $\Xi_\pm(\boldsymbol{k})$. The terms $\Sigma(L_{\boldsymbol{k}})B_{z,+}$ and $\Sigma(L_{\boldsymbol{k}})B_{z,\times}$ are real and orthonormal, so the elements of $\Xi_\pm$ have the form (4.23). □

4.4 The topology of the total and circularly polarized graviton bundles

In this section we characterize the topology of Ξ and $\Xi_\pm$. By Theorem 2.3, these bundles are completely classified by their Chern numbers which are defined by their Chern numbers when $\mathbb{R}^3 \setminus \{0\} \cong \mathcal{L}_+$ is restricted to the unit sphere $S^2 \subseteq \mathbb{R}^3 \setminus \{0\}$:

$$C(\Xi) \doteq C(j^*\Xi) = C(\Xi|_{S^2}). \tag{4.29}$$

Theorem 4.3 (The topology of gravitons)**.** *The total graviton bundle Ξ is topologically trivial, while the circularly polarized bundles $\Xi_\pm$ are topologically nontrivial with Chern numbers ∓ 4.*

Proof. We will determine $C_1(\Xi_\pm)$ via the Berry connection induced by the Hilbert-Schmidt product. To do so, we need smooth bases of the restricted bundles $\Xi_\pm|_{S^2}$. It is too much to ask for global bases of $\Xi_\pm|_{S^2}$ since we will find that these bundles are topologically nontrivial. However, we can find smooth bases almost everywhere and this is sufficient to calculate the Chern number. We will obtain such a basis by applying rotations to $A_{z,\pm}$. In particular, define

$$A_\pm(\theta,\phi) = \Sigma\Big(R_{\boldsymbol{e}_z}(\phi)R_{\boldsymbol{e}_x}(\theta)\Big)A_{z,\pm} \tag{4.30}$$

for $\theta \in (0, \pi)$ and $\phi \in [0, 2\pi)$. Then with

$$\hat{\boldsymbol{k}} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta),$$

we have that

$$A_\pm(\theta, \phi) \in \Xi_\pm(\hat{\boldsymbol{k}}) \tag{4.31}$$

except at the poles where $A_\pm(\theta, \phi)$ would become discontinuous. We can thus express the Berry connections on $\Xi_\pm|_{S^2}$ almost everywhere in these coordinates [103]:

$$\sigma_\pm(\theta, \phi) = \langle A_\pm(\theta, \phi), dA_\pm(\theta, \phi)\rangle \tag{4.32a}$$

$$= \mp 2i\cos\theta d\phi \tag{4.32b}$$

The Berry curvature forms are then

$$\Omega_\pm = d\sigma_\pm + \sigma_\pm \wedge \sigma_\pm \tag{4.33a}$$

$$= \pm 2i\sin\theta d\theta \wedge d\phi \tag{4.33b}$$

$$= \pm 2i\, dS. \tag{4.33c}$$

where dS is the spherical area element of S^2. Note that while our calculation excluded the poles, the Berry curvature form is a smooth globally (unlike the connection form) [99], and thus (4.33c) must hold globally. The Chern numbers can then be calculated [99]:

$$C(\Xi_\pm) = C(\Xi_\pm|_{S^2}) \tag{4.34a}$$

$$= \frac{i}{2\pi}\int_{S^2} \Omega_\pm \tag{4.34b}$$

$$= \mp 4. \tag{4.34c}$$

The Chern number is additive, so by (4.22)

$$C(\Xi) = C_1(\Xi_+) + C_1(\Xi_-) = 0. \tag{4.35}$$

Since the trivial rank 2 vector bundle over $\mathcal{L}_+$ also has vanishing Chern number, Ξ is isomorphic to the trivial bundle. □

Thus, the R and L gravitons are topologically nontrivial while the total graviton bundle is trivial. It is interesting to find that the R and L gravitons have different topology than the R and L photons, as the former have Chern number ∓ 4 while the latter have Chern numbers ∓ 2 [1]. As the Chern number can be interpreted as a measure of how nontrivial or "twisted" a line bundle is [101], this shows that R and L graviton polarization spaces are more twisted than those of R and L photons. It is clear that photons and gravitons have different geometry; this is reflected in their helicities ± 1 and ± 2, respectively. However, it is important to note that *a priori* topology and geometry are distinct and reflect different mathematical structures, in this case the vector bundle structures and particular Poincaré symmetries of those bundles. There are certainly relationships between geometry and topology, but they take the form of deep results such as the Chern–Weil homomorphism and the Gauss-Bonnet theorem [99]. We previously proved that vector bundle representations with the same helicity are topologically equivalent (Theorem 3.25). Although we did not generally establish that vector bundles with different helicities are topologically distinct, the photon and graviton cases support this conjecture. Moreover, they suggest the general relationship $C_1 = -2h$. We will prove this relationship between the topology and geometry of massless particles in Chapter 5.

We also remark that while the Berry connection was used to calculate C_1, there is no special relationship between the Berry connection and Chern numbers. Indeed, we could have used any connections on Ξ and $\Xi_\pm$ (and there are infinitely many) in this calculation and obtained the same Chern numbers. The Hilbert-Schmidt product and its induced con-

nection are geometric, whereas the Chern numbers are purely topological quantities and are independent of the various geometries one might impose on the vector bundle.

Before proceeding, we note an alternative, faster method of establishing the topological triviality of Ξ. The key observation is that Ξ is the complexification of a real vector bundle $\Xi^{\mathbb{R}}$. Indeed, the transverse gauge condition (4.8a) can be applied to the real vector space $W_3^{\mathbb{R}}$ of symmetric traceless rank 2 tensors on $\mathbb{R}^3$, generating a rank 2 real vector bundle $\Xi^{\mathbb{R}}$. It is easy to see that Ξ is the complexification of $\Xi^{\mathbb{R}}$, that is,

$$\Xi = \Xi^{\mathbb{R}} \otimes \mathbb{C}. \tag{4.36}$$

Lemma 3.5 then implies that the Chern number vanishes:

$$C(\Xi) = 0. \tag{4.37}$$

4.5 Construction of a global basis for gravitons

Since Ξ is topologically trivial there exists a smooth global basis for gravitons. However, neither of the proofs given in the previous section suggest a method to construct such a basis. In this section, we use the the modified clutching construction we developed in §3.2 [1] to form a global basis of photons.

We begin by splitting the two-sphere as $S^2 = D_U \cup D_L$ into the upper and lower closed hemispheres D_U and D_L. Since D_U and D_L are contractible, the restricted bundles $\Xi|_{D_U}$ and $\Xi|_{D_L}$ are topologically trivial and therefore admit global bases. We construct such bases as follows. For each $\boldsymbol{k} = (\theta, \phi) \in D_U$, define a rotation $R_{\boldsymbol{k}} \in \mathrm{SO}(3)$ which takes $\boldsymbol{e}_z$ to $\boldsymbol{k}$ via Rodrigues' rotation formula ([139], Eq. 15):

$$R_{\boldsymbol{k}} = I + \boldsymbol{\mathcal{K}} \sin\theta + \boldsymbol{\mathcal{K}}^2(1 - \cos\theta) \tag{4.38}$$

where

$$\boldsymbol{\mathcal{K}} = \begin{pmatrix} 0 & 0 & \cos\phi \\ 0 & 0 & \sin\phi \\ -\cos\phi & -\sin\phi & 0 \end{pmatrix}. \tag{4.39}$$

Using $R_{\boldsymbol{k}}$, we can transport $A_{z,\pm}$ from $\boldsymbol{e}_z$ to all of D_U:

$$A_{U,1}(\boldsymbol{k}) \doteq \Sigma(R_{\boldsymbol{k}})A_{z,+} \tag{4.40a}$$

$$A_{U,2}(\boldsymbol{k}) \doteq \Sigma(R_{\boldsymbol{k}})A_{z,-}. \tag{4.40b}$$

Since the action Σ is unitary and the map $\boldsymbol{k} \mapsto R_{\boldsymbol{k}}$ from D_U to SO(3) is smooth,

$$\mathcal{F}_U = (A_{U,1}, A_{U,2}) \tag{4.41}$$

forms a smooth orthonormal basis of $\Xi|_{D_U}$. The SO(3) action Σ defined by (4.18) naturally extends to an O(3) action. If we let $P = \mathrm{diag}(+1, +1, -1) \in \mathrm{O}(3)$ be the reflection across the xy plane, then

$$\mathcal{F}_L = (A_{L,1}, A_{L,2}) \tag{4.42a}$$

$$A_{L,1}(\boldsymbol{k}) \doteq \Sigma(P)[A_{U,2}(P\boldsymbol{k})] \tag{4.42b}$$

$$A_{L,2}(\boldsymbol{k}) \doteq \Sigma(P)[A_{U,1}(P\boldsymbol{k})] \tag{4.42c}$$

is a smooth orthonormal basis of $\Xi|_{D_L}$; the order of the indices has been flipped for convenience. The bases $\mathcal{F}_{U,L}$ are smooth on their respective domains $D_{U,L}$, but they do not agree on the equator $S^1 \cong D_U \cap D_L$ where they overlap. Instead, for each $\boldsymbol{k} = (\cos(\phi), \sin(\phi), 0)$ on the equator, there is a unitary transformation $T(\phi) \in \mathrm{U}(2)$ relating the two frames:

$$\mathcal{F}_U(\phi) = T(\phi)\mathcal{F}_L(\phi). \tag{4.43}$$

Direct calculation shows that

$$T(\phi) = \begin{pmatrix} e^{4i\phi} & 0 \\ 0 & e^{-4i\phi} \end{pmatrix} \in \mathrm{SU}(2). \tag{4.44}$$

$T : S^1 \to \mathrm{SU}(2) \subseteq \mathrm{U}(2)$ defines a loop in U(2). The clutching construction shows that the homotopy type of this loop fully classifies the vector bundle $\Xi|_{S^2}$ (and therefore Ξ). In particular, this bundle is trivial if and only if T it is contractible to a point i.e. if T is nullhomotopic [40, 41]. Since T is actually a loop in the simply connected subgroup SU(2) of U(2), it is necessarily nullhomotopic. We thus obtain a third proof that Ξ is topologically trivial. The advantage of this approach is that it can be adapted to give an explicit construction of a global basis for Ξ. The strategy is to extend the domain of T from the equator S^1 to all of D_L. We do this in two steps.

4.5.1 Homotopy from T to the identity

Since T is nullhomotopic, there exists a homotopy $\tilde{T} : [0,1] \times S^1 \to \mathrm{SU}(2)$ between

$$T(\phi) = \begin{pmatrix} e^{4i\phi} & 0 \\ 0 & e^{-4i\phi} \end{pmatrix} \quad \text{and} \quad \mathbb{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{4.45}$$

The map

$$(x, y, z) \in S^2 \mapsto \begin{pmatrix} x + iy & z \\ -z & x - iy \end{pmatrix} \in \mathrm{SU}(2) \tag{4.46}$$

gives an embedding of S^2 in SU(2). Under this embedding, $T(\phi)$ corresponds to the loop around the equator traversed clockwise four times, while $\mathbb{1}$ corresponds to the constant loop at the point $(1, 0, 0) \in S^2$. A homotopy can be obtained by deforming the equator into

$(1,0,0)$ as depicted in Fig. 4.1. In particular, we set

$$x(t,\phi) = \cos(4\phi)\cos^2\left(\frac{\pi t}{2}\right) + \sin^2\left(\frac{\pi t}{2}\right) \tag{4.47a}$$

$$y(t,\phi) = \sin(4\phi)\cos\left(\frac{\pi t}{2}\right) \tag{4.47b}$$

$$z(t,\phi) = \sin^2(2\phi)\,\sin(\pi t), \tag{4.47c}$$

and define the homotopy $\tilde{T} : [0,1] \times S^1 \to \mathrm{SU}(2)$ via

$$\tilde{T}(t,\phi) = \begin{pmatrix} x(t,\phi) + iy(t,\phi) & z(t,\phi) \\ -z(t,\phi) & x(t,\phi) - iy(t,\phi) \end{pmatrix}. \tag{4.48}$$

It will be more convenient to change the homotopy parameter from $t \in [0,1]$ to $\theta \in [\frac{\pi}{2}, \pi]$:

$$\bar{T}(\theta,\phi) = \begin{pmatrix} \bar{x}(\theta,\phi) + i\bar{y}(\theta,\phi) & \bar{z}(\theta,\phi) \\ -\bar{z}(\theta,\phi) & \bar{x}(\theta,\phi) - i\bar{y}(\theta,\phi) \end{pmatrix} \tag{4.49}$$

where

$$\bar{x}(\theta,\phi) = \cos(4\phi)\sin^2(\theta) + \cos^2(\theta), \tag{4.50a}$$

$$\bar{y}(\theta,\phi) = \sin(4\phi)\sin(\theta), \tag{4.50b}$$

$$\bar{z}(\theta,\phi) = -\sin^2(2\phi)\sin(2\theta). \tag{4.50c}$$

4.5.2 Construction of a smooth frame of $\Xi|_{S^2}$

Using the homotopy $\bar{T}$, we can patch together $\mathcal{F}_U$ and $\mathcal{F}_L$ into a continuous global basis $\mathcal{F}$ for Ξ:

$$\mathcal{F}(\theta,\phi) = \begin{cases} \mathcal{F}_U(\theta,\phi) & \text{if } 0 \le \theta < \frac{\pi}{2} \\ \bar{T}(\theta,\phi)\mathcal{F}_L(\theta,\phi) & \text{if } \frac{\pi}{2} \le \theta \le \pi. \end{cases} \tag{4.51}$$

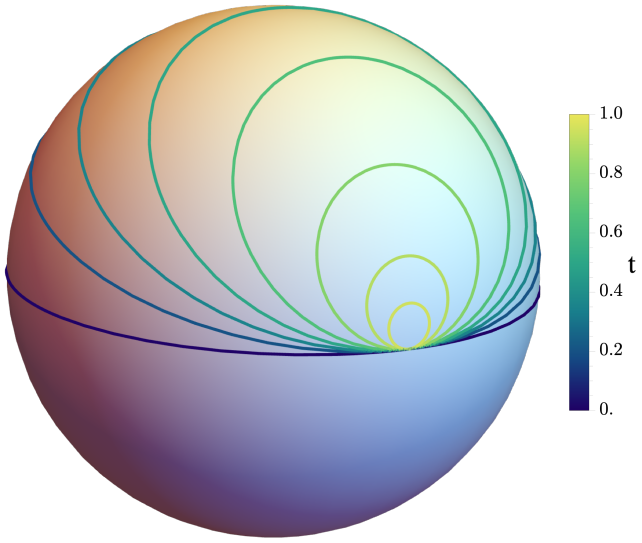


Figure 4.1: Illustration of the homotopy $\tilde{T}(t,\phi) : [0,1] \times S^1 \to \mathrm{SU}(2)$ from $T(\phi)$ to the identity $\mathbb{1}$. The image of $\tilde{T}$ resides on an embedding of S^2 in SU(2). The equator at $t = 0$ corresponds to $T(\phi)$ and shrinks down to the identity $(1,0,0)$ at $t = 1$.

The condition

$$\bar{T}(\pi/2,\phi) = T(\phi) \tag{4.52}$$

ensures that $\mathcal{F}(\theta,\phi)$ is continuous along the equator, while $\bar{T}(\pi,\phi) = \mathbb{1}$ ensures that no singularity occurs at the south pole. While $\mathcal{F}$ is continuous, it is not smooth since its θ derivatives jump abruptly at the equator. The last step is to use a smooth step function to smooth the frame at the equator.

To this end, let $g \in C^\infty(\mathbb{R})$ be any smooth monotonic function such that

$$g(\theta \leq \pi/2) = \pi/2 \tag{4.53a}$$

$$g(\theta \geq \pi) = \pi \tag{4.53b}$$

$$g^{(n)}(\pi/2) = g^{(n)}(\pi) = 0 \tag{4.53c}$$

for all $n \geq 1$. For example, we can define

$$h(t) = \begin{cases} e^{-1/t} & t > 0 \\ 0 & t \leq 0, \end{cases} \tag{4.54}$$

and then set

$$g(\theta) = \frac{\pi}{2}\Big(1 + \frac{h(\theta - \pi/2)}{h(\theta - \pi/2) + h(\pi - \theta)}\Big). \tag{4.55}$$

We then define the frame

$$\mathcal{F}_s(\theta, \phi) = \begin{cases} \mathcal{F}_U(\theta, \phi) & \text{if } 0 \leq \theta < \frac{\pi}{2} \\ \bar{T}(g(\theta), \phi)\mathcal{F}_L(\theta, \phi) & \text{if } \frac{\pi}{2} \leq \theta \leq \pi \end{cases} \tag{4.56}$$

which we claim is globally smooth. Indeed, it is obvious that $\mathcal{F}_s$ is smooth for $\theta \neq \frac{\pi}{2}$. It remains to show that

$$\partial_\phi^m \partial_\theta^n \mathcal{F}_s\big|_{\theta=\frac{\pi}{2}^-} = \partial_\phi^m \partial_\theta^n \mathcal{F}_s\big|_{\theta=\frac{\pi}{2}^+} \tag{4.57}$$

for all m, n. By (4.52), (4.53c), and (4.56), this is equivalent to the claim that

$$\partial_\phi^m \partial_\theta^n \mathcal{F}_U\big|_{\theta=\frac{\pi}{2}^-} = \partial_\phi^m \partial_\theta^n \Big[T(\phi)\mathcal{F}_L(\theta, \phi)\Big]_{\theta=\frac{\pi}{2}^+}. \tag{4.58}$$

We now note that the equations (4.40-4.42) defining $\mathcal{F}_U$ and $\mathcal{F}_L$ smoothly extend to $0 < \theta < \pi$. It follows from direct calculation that

$$\mathcal{F}_U(\theta, \phi) = T(\phi)\mathcal{F}_L(\theta, \phi) \tag{4.59}$$

whenever both sides are defined, that is, when $\theta \neq 0, \pi$. This implies that (4.58) holds, proving that $\mathcal{F}_s$ is smooth.

Although it is frequently assumed that such a global basis for gravitons exists [44, 134–136], it appears this is the first explicit construction of such a basis. It allows an explicit

decomposition of Ξ

$$\Xi = \tau_1 \oplus \tau_2 \tag{4.60}$$

where τ_1 and τ_2 are the trivial line bundles over $\mathcal{L}_+$ generated by the basis $\mathcal{F}_s$.

We note that while the trivial splitting (4.60) may be useful in practical applications such as modeling gravitational waves, the subbundles τ_1 and τ_2 are not Poincaré symmetric and thus do not represent elementary particles. In other words, the splittings $\Xi = \tau_1 \oplus \tau_2$ and $\Xi = \Xi_+ \oplus \Xi_-$ both make sense topologically, but only the latter respects the Poincaré geometry of weak gravity.

4.6 The nonexistence of linearly polarized subbundles

Unlike the basis $\mathcal{B} = (B_+, B_\times)$ from equation (4.11), the basis $\mathcal{F}_s$ is globally smooth. However, $\mathcal{F}_s$ is significantly more complicated than $\mathcal{B}$. Indeed, the simplest types of polarizations are circular polarizations and linear polarizations. Circularly polarized vectors are those of the form $c(A_1 \pm iA_2)$ where $c \in \mathbb{C}$ and A_1 and A_2 are real and orthonormal to each other. While Ξ uniquely decomposes into the sum of circularly polarized subbundles $\Xi_\pm$, these bundles are nontrivial and thus it is not possible to construct a globally smooth basis of circularly polarized gravitons. The basis $\mathcal{B}$ is an attempt to construct a linearly polarized basis, that is, a basis of real tensors; however, we showed that $\mathcal{B}$ necessarily has singularities and is not globally smooth. While this basis fails, it is natural to ask if it is possible to construct some other linearly polarized basis. $\mathcal{F}_s$ does not satisfy this criteria—while it is linearly polarized in the upper hemisphere, it is generally elliptically polarized in the southern hemisphere. In this section, we prove that it is impossible to construct a global linearly polarized basis for gravitons, but for a reason which is fundamentally different than the obstruction precluding a circularly polarized basis. While the bundles $\Xi_\pm$ are topologically nontrivial, they are still well-defined topological objects. In contrast, Ξ possesses no linearly polarized subbundles, even topologically nontrivial ones.

To prove this, we must first be more precise about the definition of linearly polarized vectors and subbundles. Indeed, if A_0 is real than it is certainly linearly polarized, however, cA_0 should also be considered linearly polarized for any $c \in \mathbb{C}$ since cA differs from A_0 only in a phase and scaling. Recall that $\Xi \cong \Xi^{\mathbb{R}} \otimes \mathbb{C}$ is the complexification of the underlying real vector bundle $\Xi^{\mathbb{R}}$. Recall also that in a tensor product $X \otimes Y$, simple tensors are those of the form $x \otimes y$ for $x \in X$ and $y \in Y$; general elements of $X \otimes Y$ are finite sums of simple tensors [140].

Definition 4.1 (Linearly polarized vectors)**.** $(k, A) \in \Xi \cong \Xi^{\mathbb{R}} \otimes \mathbb{C}$ *is linearly polarized if* A *is a simple tensor, that is, if* $A = cA_0 = A_0 \otimes c$ *for some* $c \in \mathbb{C}$ *and* $A_0 \in \Xi^{\mathbb{R}}$*. Equivalently,* (k, A) *is linearly polarized if the set* $\{Re(cA) : c \in \mathbb{C}\}$ *is a one-dimensional real vector space.*

Definition 4.2 (Linearly polarized subbundles)**.** *A line bundle* ℓ *is a linearly polarized subbundle of* Ξ *if every element of* ℓ *is linearly polarized. Equivalently,* ℓ *is linearly polarized if it is the complexification of a real line subbundle* $\ell^{\mathbb{R}}$ *of* $\Xi^{\mathbb{R}}$*.*

The main result pertaining to linear polarizations is the following theorem.

Theorem 4.4. Ξ *has no linearly polarized subbundles. As a result, there exists no smooth, linearly polarized global extension of the basis* $(B_{z,+}, B_{z,\times})$ *at* $\boldsymbol{e}_z$*.*

To prove this, we first establish the following lemma. In it, we use the Euler class e, which is a characteristic class for real, oriented vector bundles. The Euler class is less commonly used in physics than the Chern classes; expositions can be found in Refs. [41, 99, 101].

Lemma 4.5. *If* Ξ *has a linearly polarized subbundle then the Euler class of* $\Xi^{\mathbb{R}}$ *vanishes.*

Proof of Lemma 4.5. Assume Ξ has a linearly polarized subbundle, and therefore that $\Xi^{\mathbb{R}}$ possesses a real line subbundle. Since S^2 is simply connected, there exist no nontrivial real line bundles over S^2 ([1], Lemma 21). Thus, $\Xi^{\mathbb{R}}$ has a trivial line subbundle and must therefore possess a globally nonvanishing section. $\Xi^{\mathbb{R}}$ is orientable since every real vector bundle over S^2 is orientable ([101], Proposition 11.5). If a real oriented vector bundle has a

nonvanishing section, then its Euler class is zero ([41], Proposition 3.13(e)). Thus, the Euler class of $\Xi^{\mathbb{R}}$ vanishes. □

Proof of Theorem 4.4. By the lemma it suffices to show that the Euler class of $\Xi^{\mathbb{R}}$ is nonzero. The tensors $B_{z,+}$ and $B_{z_\times}$ form an orthonormal basis of $\Xi^{\mathbb{R}}(\boldsymbol{e}_z)$. Then

$$\tilde{\mathcal{B}}(\boldsymbol{k}) = (B_+(\boldsymbol{k}), B_\times(\boldsymbol{k})), \tag{4.61a}$$

$$B_+(\boldsymbol{k}) \doteq \Sigma\Big(R_{\boldsymbol{e}_z}(\phi)R_{\boldsymbol{e}_x}(\theta)\Big)B_{z,+} \tag{4.61b}$$

$$B_\times(\boldsymbol{k}) \doteq \Sigma\Big(R_{\boldsymbol{e}_z}(\phi)R_{\boldsymbol{e}_x}(\theta)\Big)B_{z,\times} \tag{4.61c}$$

forms a smooth orthonormal basis of $\Xi_{\mathbb{R}}(\theta, \phi)$ at all $\boldsymbol{k}$ except at the north and south poles, where

$$\boldsymbol{k} = |\boldsymbol{k}|(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta).$$

We can then calculate the corresponding Berry connection matrix with respect to the spherical coordinate system for $\mathcal{L}_+$:

$$\omega_{\mathbb{R}} = \langle\tilde{\mathcal{B}}, d\tilde{\mathcal{B}}\rangle \tag{4.62a}$$

$$= \begin{pmatrix} 0 & -2\cos(\theta)d\phi \\ 2\cos(\theta)d\phi & 0 \end{pmatrix}. \tag{4.62b}$$

The curvature matrix in these coordinates is

$$\Omega_{\mathbb{R}} = d\omega_{\mathbb{R}} + \omega_{\mathbb{R}} \wedge \omega_{\mathbb{R}} \tag{4.63a}$$

$$= \begin{pmatrix} 0 & 2dS \\ -2dS & 0 \end{pmatrix}. \tag{4.63b}$$

The Pfaffian of this matrix is [99]

$$\mathrm{Pf}(\Omega_{\mathbb{R}}) = 2dS. \tag{4.64}$$

The Euler class is the de Rham cohomology class of the Pfaffian [99], so

$$e(\Omega_{\mathbb{R}}) = 2[dS] \tag{4.65}$$

where the brackets denote the cohomology class. dS is not exact (since $\int_{S^2} dS = 4\pi \neq 0$), so $e(\Omega_{\mathbb{R}})$ is nonvanishing, and therefore Ξ has no linear subbundles. □

4.7 Gravitons do not have SAM or OAM

In analogy with the case of electromagnetic waves, there have been recent proposals that the angular momentum of gravitational waves can be decomposed into SAM and OAM [56–58]. While there is a long standing controversy around this issue for photons [1, 65–81], it does not appear to be known that these issues extend to gravitational waves as well. However, we will show that the gravitons indeed do not admit a genuine SAM-OAM decomposition.

The (total) angular momentum operator $\boldsymbol{J}$ of gravitons is the generator of the Poincaré action $\Sigma|_{\mathrm{SO}(3)}$ restricted to $\mathrm{SO}(3) \subseteq \mathrm{ISO}^+(3,1)$. If the angular momentum splits into SAM and OAM, then

$$\boldsymbol{J} = \boldsymbol{L} + \boldsymbol{S} \tag{4.66}$$

where $\boldsymbol{L}$ and $\boldsymbol{S}$ satisfy $\mathfrak{so}(3)$ commutation relations, and thus generate $\mathrm{SO}(3)$ actions Σ_L and Σ_S on Ξ. Furthermore, because spin symmetries are internal symmetries, $\boldsymbol{S}$ must generate an internal symmetry, that is, the action of Σ_S on (k, A) does not change k. More formally, we say that Σ_S is stabilizing.

Theorem 4.6. *There exist no nontrivial SAM-OAM splittings of the angular momentum of gravitons as in equation (4.66).*

Proof. Suppose there were such a splitting. Since Σ_S is stabilizing, each fiber $\Xi(\boldsymbol{k})$ is a two-dimensional vector space representation of $\mathrm{SO}(3)$ with the action Σ_S. However, the (nonprojective) irreducible representations of $\mathrm{SO}(3)$ are labeled by their integer spin s and

have dimensions $2s+1$ [83]. Thus, the representation Σ_S on $\Xi(\boldsymbol{k})$ must be the direct sum of two spin 0 representations. Since the spin 0 representation is simply the trivial representation which leaves the vectors unchanged, Σ_S must be the identity operator. Thus, $\boldsymbol{S} = 0$, and therefore $\boldsymbol{J} = \boldsymbol{L}$ and $\Sigma|_{\mathrm{SO}(3)} = \Sigma_L$. Thus, there is no nontrivial way to split gravitational angular momentum into SAM and OAM parts. □

This shows that if one does indeed split $\boldsymbol{J}$ into operators $\boldsymbol{S}$ and $\boldsymbol{L}$ then they cannot satisfy the angular momentum commutation relations, that is,

$$[S_a, S_b] \neq i\epsilon_{abc} S_c, \tag{4.67}$$

$$[L_a, L_b] \neq i\epsilon_{abc} L_c. \tag{4.68}$$

These commutation relations are the defining feature of angular momentum operators [1, 67, 68, 74, 75, 78, 79, 82, 83] as they ensure that $\boldsymbol{S}$ and $\boldsymbol{L}$ actually generate rotational symmetries. If these relations are not satisfied then one cannot apply the standard results for angular momentum operators such as the addition of angular momentum formula, the Clebsch–Gordan formalism, the ladder operator formalism or even guarantee that S^2, L^2, and the components of $\boldsymbol{S}$ and $\boldsymbol{L}$ will have discrete spectra. Nevertheless, it has been argued that it can still be useful to split the angular momentum of photons into operators which are not true SAM and OAM operators [77]. The same may prove true for gravitational angular momentum as is suggested by Refs. [56–58] but the one should bear in mind that $\boldsymbol{S}$ and $\boldsymbol{L}$ will not have the properties of angular momentum operators.

The argument in Theorem 4.6 against the splitting of gravitational OAM and SAM is essentially geometric, relying on the rotational symmetry of the gravitons. However, as in the case of photon angular momentum, this obstruction is also related to a topological singularity as well. We will only give a summary of this argument as it is essentially the same as that for the photon case discussed in §3.5.2. Elementary particles can be described as vector bundle representations of the Poincaré group $\mathrm{ISO}^+(3,1)$ as discussed in §3.4. The

little group of particles with four-momentum k is defined as the subset of the Lorentz group $\mathrm{SO}^+(3,1)$ which do not change k. Up to group isomorphism, the little group H is independent of k. By its definition, H depends only the momentum space M and not on the fibers (the polarizations). Furthermore, each particle has a canonical little group action which is stabilizing in the sense that the action does not change the fiber. Thus, the little group describes an internal symmetry of the particles. For massive particles, the momentum space is a mass hyperboloid and the little group is SO(3). The canonical little group action of massive particles thus describes an internal SO(3) symmetry; its generator is the spin angular momentum. However, a topological singularity occurs as the $m \to 0$ limit is realized. The momentum space jumps from the topologically trivial mass hyperboloid to the nontrivial forward light cone. This topological singularity allows the little group to jump as well from SO(3) to ISO(2), where the latter is the 2D Euclidean group consisting of translations and rotations of $\mathbb{R}^2$. However, the translations always act trivially, [1] so the little group is effectively just SO(2). The generator of this two-dimensional rotational symmetry is the helicity operator $\hat{\boldsymbol{k}} \cdot \boldsymbol{J}$. Since this is not an SO(3) generator, helicity is not an angular momentum operator. Indeed, helicity is a scalar operator, and so it is not even a vector operator. The commonly proposed SAM operator $\boldsymbol{J}_s = (\hat{\boldsymbol{k}} \cdot \boldsymbol{J})\hat{\boldsymbol{k}}$ can be seen as an ad hoc attempt to "vectorize" the helicity operator. However, this attempt fails because the components of this new operator commute, and thus describe an $\mathbb{R}^3$ symmetry rather than an SO(3) symmetry. We can summarize the situation by saying that massive particles have spin while massless particles have helicity. Something special happens for massive particles which does not happen for massless particles, namely, the little group happens to coincide with SO(3), so spin is actually an angular momentum. The same is not true of helicity.

Heuristically, it is not surprising that the topological nontriviality of R and L photons and gravitons obstructs splitting the angular momentum into SAM and OAM. An SAM-OAM de-

[1] This is violated by the hypothetical and rather exotic continuous spin particles. They have an infinite number of internal DOFs which is inconsistent with all observed particles. As such, they are not part of the Standard Model and typically ignored ([43], pp. 70-72).

composition relies on nicely splitting the internal (fiber) and external (base manifold) DOFs, and likewise splitting the rotational symmetry into parts acting internally and externally. Furthermore, the action Σ acts independently on Ξ_R and Ξ_L, so the same would be true of the spin and orbital symmetries Σ_S and Σ_L. However, the topological nontriviality of the R and L gravitons means that their internal and external DOFs are twisted together, and this precludes splitting the rotational action into internal and external parts.

4.8 Conclusion

This study of linearized gravitons gives a further illustration of the fact that there is a fundamental connection between massless particles and nontrivial topology. An immediate implication of the nontriviality of Ξ_+ and Ξ_- is that it is not possible to construct global bases of R and L gravitons. It is also not possible to construct a global basis of linearly polarized gravitons, but this is due to the more fundamental issue that such polarizations are not even globally well-defined. In light of these facts, it is rather surprising that the full set of gravitons Ξ is topologically trivial, allowing the construction of a smooth global basis. Such global bases have been previously invoked in the literature, although the explicit expressions given for them were not smooth [44, 134–136].

The Poincaré symmetry of the graviton bundle induces a geometric decomposition of Ξ into the topologically nontrivial R and L graviton bundles. These nontrivial bundles have geometry characterized by the helicities ± 2 and topology characterized by the Chern numbers ∓ 4. It is interesting to note that the -2 factor relating the helicities and Chern numbers also appears for the R and L photons which have helicities ± 1 and Chern numbers ∓ 2 as shown in Chapter 3. This naturally leads one to conjecture that the relationship $C = -2h$ between topology and geometry generally holds for massless particles. We will prove this is indeed the case in the following chapter.

An important finding is that the topological nontriviality of Ξ_+ and Ξ_- obstructs the splitting of graviton angular momentum into SAM and OAM. That is, gravitons, like photons, have well-defined angular momentum but it is not possible to decompose it into generators of internal and external rotational symmetries, respectively. Indeed, that the R and L gravitons have nontrivial topology means that they cannot simply be written as $\mathcal{L}_+ \times \mathbb{C}$, showing that the internal and external DOFs do not split nicely. They are rather twisted in a nontrivial way, ultimately preventing the splitting of angular momentum into SAM and OAM.

Chapter 5

Helicity is a topological invariant of massless particles: $C = -2h$

This Chapter is closely based on Ref. [3] published in *Physical Review Research*. Within this chapter we use the shorthand writing the action of a group element $g \in G$ on an element of a vector bundle (k, v) by $g(k, v)$ rather than as $\Sigma(g)(k, v)$.

5.1 Introduction

The presence of nontrivial topology is a unifying feature that ties together important phenomena in periodic lattice structures such as the spin Hall effects [12–15], the bulk-edge correspondence [17–20], and the existence of topological insulators [21, 22]. Over the last decade it has been found that topologically nontrivial waves exist as well in continuous media such as fluids [20, 23–28], plasmas [19, 28–35], and, as we have seen in Chapters 3 and 4, even the vacuum [1, 2, 85, 104]. The latter refers to the fact that R and L photons and gravitons are topologically nontrivial vacuum excitations. In Wigner's classification [42, 43], elementary particles are characterized by their geometry rather than their topology, in particular, they are classified as unitary irreducible representations (UIRs) of the Poincaré group $\mathrm{ISO}^+(3,1)$. In this chapter, we derive a novel relationship between the topology and

geometry of massless particles, showing that the geometric helicity h is related to the topological Chern number C via $C = -2h$. In doing so, we exhibit an Abelian group structure of massless particles with respect to the tensor product of line bundles which leads to a novel method of generating massless particle representations of all helicities from the photon representations. These representations are easier to construct and more explicit than those found by Asorey et al. [48] and Dragon [85]. They also provide a simpler alternative construction of the R and L graviton bundles than the one used in Chapter 4. These representations will be essential for the calculation of angular momentum eigenfunctions for massless particles and plasma waves presented in Chapters 8 and 9. The relationship between the geometry and topology will be used to prove general nonexistence theorems for massless SAM and OAM decompositions in Chapter 6.

5.2 The relationship between the geometry and topology of massless particles

There is an elementary but indispensable relationship between the topology and geometry of *massive* particles. Conventionally, particles are represented by UIRs of $\mathrm{ISO}^+(3,1)$ on the space of n-component square-integrable wavefunctions over the four-momentum space M, denoted by $L^2(M, \mathbb{C}^n)$ [42, 43]. Note that here we ignore quantum numbers arising from non-spacetime symmetries, such as color charge. For massive particles with $m > 0$, M is a mass hyperboloid. According to Wigner's little group method, these representations are classified by the UIRs of the little group $\mathrm{SO}(3) \subseteq \mathrm{ISO}^+(3,1)$ on the internal space $V \doteq \mathbb{C}^n$, which in turn are labeled by the nonnegative integer or half-integer spins s. The spin is related to $\dim V$ by

$$\dim V = 2s + 1. \tag{5.1}$$

Spin characterizes the geometry of the representation while the dimension of a vector space is a topological invariant. Thus, Eq. (5.1) gives the elementary but very important relationship between the geometry and topology of massive particles.

The situation is very different for massless particles. The spacetime symmetry is again characterized by a UIR of the little group, but for a massless particle the little group is ISO(2) rather than SO(3) [1, 42, 43], and as we saw in §3.21, reduces to SO(2) ([43], pp. 70-72). The UIRs of SO(2) are labeled not by their spin, but by their (possibly negative) integer helicities h (assuming all elementary massless particles are bosons, as in the Standard Model). Thus, helicity describes the geometry of massless particles. Unlike the spin representations of SO(3), the UIRs of SO(2) are all one-dimensional, so there is no relationship like Eq. (5.1) relating the helicity with the dimension of the internal space. For example, although R and L photons have helicities $h = \pm 1$ and R and L gravitons have $h = \pm 2$, each of these four particles have a single polarization degree of freedom [1, 2]. What then is the relationship between the geometry and topology of massless particles?

The topology of massless particles is far more subtle than that of massive particles. In fact, this topology is essentially obscured in Wigner's conventional representations. The momentum space of massless particles is the forward light cone

$$\mathcal{L}_+ = \{k^\mu = (\omega, \boldsymbol{k}) : k^\mu k_\mu = 0, \omega > 0\} \tag{5.2}$$

$$\cong \mathbb{R}^3 \setminus \{0\}. \tag{5.3}$$

In Wigner's representations [42], massless particles are all represented on the same Hilbert space $L^2(\mathcal{L}_+, \mathbb{C})$ because the internal spaces are all one-dimensional. However, unlike the massive representations, when $h \neq 0$ these representations are not smooth , meaning that Poincaré transformations typically map a smooth wavefunction to a nonsmooth wavefunction (cf. §3.4.3; [1, 45, 46]). This problem can be avoided by instead considering massless particles as UIRs on $L^2(E)$, the space of L^2-integrable sections of a complex Hermitian line bundle

$\pi : E \to \mathcal{L}_+$ [1, 2, 45] as we did in Chapters 3 and 4. Since $\mathcal{L}_+$ has a hole at the origin (as massless particles have no rest frame), it is noncontractible and thus E can be topologically nontrivial (unlike in the massive case when M is a mass hyperboloid). These lines bundles are topologically characterized by their Chern number C (Theorem 2.3); when $C \neq 0$, these particles are topologically nontrivial. We showed that photons ([1]) and gravitons [2] are topologically nontrivial; R and L photons have Chern numbers ∓ 2 and helicities ± 1, while R and L gravitons have Chern numbers ∓ 4 and helicities ± 2. However, the method we previously used to construct the graviton bundles becomes increasingly difficult for higher helicity particles, and does not easily generalize to arbitrary helicity. In this chapter we use an alternative construction of massless particle bundles to prove that in general $C = -2h$, giving the precise relationship between the topology and geometry of massless particles. This relationship is the massless analog of Eq. (5.1). That helicity can be viewed as a topological charge was also recently discovered by Dragon [85], but the explicit relationship $C = -2h$ was not given.

Massless particles can be considered as unitary irreducible UIRs of the Poincaré group $\mathrm{ISO}^+(3,1)$ on vector bundles over $\mathcal{L}_+$ in which translations T_{a} act via

$$T_{\mathsf{a}}(k, v) = e^{-ik^\mu \mathsf{a}_\mu}(k, v). \tag{5.4}$$

We will call such a representation a massless bundle UIR. Note that in this chapter we will use the shorthand of writing the action of $\Lambda \in \mathrm{ISO}^+(3,1)$ on a vector as $\Lambda(k, v)$ rather than $\Sigma(\Lambda)(k, v)$. Condition (5.4) ensures that (k, v) has momentum k. As in the case of Wigner's representations [42, 43], the massless bundle UIRs of $\mathrm{ISO}^+(3,1)$ are classified by their actions of the little group, which are in turn labeled by the integer helicities h [1]. In particular, by Theorem 3.24 every such bundle E is a line bundle ($r = 1$) and every vector in E is an eigenvector of the helicity operator $\hat{\boldsymbol{k}} \cdot \boldsymbol{J}$ with eigenvalue h:

$$(\hat{\boldsymbol{k}} \cdot \boldsymbol{J})v = hv \tag{5.5}$$

Here, $\boldsymbol{J}$ is the generator of the $\mathrm{SO}(3) \subseteq \mathrm{ISO}^+(3,1)$ action on E, meaning that if $R_{\hat{\boldsymbol{k}}}(\theta)$ is a rotation by θ about $\hat{\boldsymbol{k}}$ then

$$R_{\hat{\boldsymbol{k}}}(\theta)(k,v) = e^{-i\hat{\boldsymbol{k}}\cdot\boldsymbol{J}\theta}(k,v) = (k, e^{-ih\theta}v). \tag{5.6}$$

Since E is a bundle representation of $\mathrm{ISO}^+(3,1)$, it is necessary that a $\theta = 2\pi$ rotation acts trivially, so h must be an integer. Note that if one considers the generalization to projective bundle representations, then half-integer helicities can also be obtained. Since we are assuming all massless particles are bosons, we can ignore the projective case. Two massless bundle UIRs are unitarily equivalent if and only if they have the same helicity (Theorem 3.25). In this way, h fully characterizes the geometry of the these representations. However, *a priori*, it is not obvious that particles with different helicities have different topologies. We will prove that this is indeed the case.

The R and L photon bundles defined in Ch. 3 are exceptionally important, as we will show that all other massless particles can be constructed from them. Notationally, it is helpful in this chapter to denote them as $\gamma_{\pm 1} \doteq \gamma_{\pm}$. These bundles have helicity $h = \pm 1$ and Chern number $C = \mp 2$. The former comes from

$$R_{\hat{\boldsymbol{k}}}(\theta)(k, \boldsymbol{E}_{\pm}) = e^{-i(\hat{\boldsymbol{k}}\cdot\boldsymbol{J})\theta}(k, \boldsymbol{E}_{\pm}) \tag{5.7}$$

$$= (k, e^{\mp i\theta}\boldsymbol{E}_{\pm}), \tag{5.8}$$

and the latter was explicitly computed from the Berry connection in Theorem 3.13).

We present a novel construction of massless bundle UIRs of arbitrary helicity which is motivated by observations about the little group representations. In both Wigner's formalism [42, 43] and the vector bundle formalism [1, 2, 47], irreducible representations of $\mathrm{ISO}^+(3,1)$ are related to representations of the little group. The little group H_k at a fixed k in the momentum space M is defined as the subset of $\mathrm{SO}^+(3,1)$ which leaves k invariant. The little groups for different k are related by $H_{\Lambda k} = \Lambda H_k \Lambda^{-1}$ for $\Lambda \in \mathrm{SO}^+(3,1)$. Since any

two k in M are related by some Lorentz transformation, the isomorphism type of the group H_k is independent of k; the isomorphism type of H_k is called the little group H of the representation. For massive particles, M is a mass hyperboloid, and $H = \mathrm{SO}(3)$. However, for massless particles with $M = \mathcal{L}_+$, $H = \mathrm{ISO}(2) = \mathbb{R}^2 \rtimes \mathrm{SO}(2)$, the group of (inhomogeneous) isometries of $\mathbb{R}^2$ [42]. Representations in which the $\mathbb{R}^2$ translations act nontrivially on E correspond to particles with a continuum of internal degrees of freedom. As there are no known particles exhibiting this property, we make the conventional assumption that the $\mathbb{R}^2$ translations act by the identity on E [43, 44]. Thus, the little group essentially acts like SO(2). The finite-dimensional vector space UIRs of SO(3) and SO(2) are the well-known spin and helicity representations, respectively. The spin representations of SO(3) are labeled by the nonnegative integers s, and have complex dimension $2s + 1$. In contrast, the helicity representations are labeled by the complete set of integers h, but always have complex dimension 1. If $R(\theta) \in \mathrm{SO}(2)$ is a rotation by angle θ, then in the helicity h representation $R(\theta)v = e^{-ih\theta}v$ for any vector v. One can ask the following question: given two UIRs, can one produce another UIR? Suppose U_1 and U_2 are UIRs of a group G on complex vector spaces $V_1 \cong \mathbb{C}^{n_1}$ and $V_2 \cong \mathbb{C}^{n_2}$. The direct sum $U_1 \oplus U_2$ and direct product $U_1 \otimes U_2$ are well-defined representations of G on $V_1 \oplus V_2$ and $V_1 \otimes V_2$ with dimensions $n_1 + n_2$ and $n_1 n_2$, respectively [106]. For $v_i \in V_i$, they are defined by

$$g(v_1 + v_2) = gv_1 + gv_2 \tag{5.9}$$

$$g(v_1 \otimes v_2) = gv_1 \otimes gv_2. \tag{5.10}$$

These representations are unitary under the Hermitian products

$$\langle (v_1 + v_2), (w_1 + w_2) \rangle = \langle v_1, w_1 \rangle + \langle v_2, w_2 \rangle \tag{5.11}$$

$$\langle v_1 \otimes v_2, w_1 \otimes w_2 \rangle = \langle v_1, w_1 \rangle \langle v_2, w_2 \rangle \tag{5.12}$$

where the Hermitian products on the rhs are those on V_1 and V_2. $U_1 \oplus U_2$ is obviously reducible. Typically, $U_1 \otimes U_2$ is also reducible; the problem of decomposing $U_1 \otimes U_2$ into UIRs is the Clebsch–Gordon problem [106]. The most well-known cases are those of SO(3) and SU(2), which satisfy:

$$S_l \otimes S_m \cong S_{l-m} \oplus S_{l-m+1} \oplus \ldots \oplus S_{l+m}, \tag{5.13}$$

where S_k is the spin k representation, $l \geq m$, and $\cong$ denotes isomorphism of representations [106]. In particular,

$$S_1 \otimes S_1 \cong S_0 \oplus S_1 \oplus S_2. \tag{5.14}$$

However, if U_1 and U_2 both have dimension 1, then $U_1 \otimes U_2$ also has dimension 1 and is therefore irreducible. Thus if W_h is the helicity h representation, then $W_{h_1} \otimes W_{h_2}$ is irreducible. In fact, in this representation we have

$$R(\theta)(v_1 \otimes v_2) = e^{-i(h_1+h_2)\theta}(v_1 \otimes v_2), \tag{5.15}$$

so

$$W_{h_1} \otimes W_{h_2} \cong W_{h_1+h_2}. \tag{5.16}$$

Since $W_1 \otimes W_{-1} \cong W_0$, we define $(W_1)^{-1} = W_{-1}$ and $(W_{\pm 1})^0 = W_0$. Then for $h \in \mathbb{Z}$

$$W_h \cong \bigotimes_{i=1}^{|h|} W_1^{\mathrm{sgn}(h)} \doteq W_1^h. \tag{5.17}$$

Thus, the helicity representations are not just in bijective correspondence with $\mathbb{Z}$, but they are actually isomorphic to $\mathbb{Z}$ as an Abelian group, with the tensor product as the group operation. As such, they are all generated by W_1 (or W_{-1}). This is a distinctly massless phenomena, as Eq. (5.14) shows that a similar result does not apply to the massive spin

representations. Indeed, the spin representations are labeled by nonnegative integers, which do not even form a group.

We now show that we can construct all the bundle UIRs of massless particles in a similar way, by taking tensor products of $\gamma_{\pm 1}$. Suppose we have two massless bundle UIRs E_1 and E_2 with helicities h_1 and h_2. Let $E_1 \otimes E_2$ be the tensor product of the bundles, which is a line bundle over $\mathcal{L}_+$ in which the fiber at k is the vector space $E_{1,k} \otimes E_{2,k}$ [99]. It carries a Hermitian product given on each fiber by Eq. (5.12). Since every element of $\mathrm{ISO}^+(3,1)$ can be expressed as $T_{\mathsf{a}} \cdot \Lambda$ where T_{a} is a spacetime translation and Λ is a Lorentz transformation, we can define a unitary representation of $\mathrm{ISO}^+(3,1)$ on $E_1 \otimes E_2$ as follows:

$$\Lambda(k, v_1 \otimes v_2) = (\Lambda k, \Lambda v_1 \otimes \Lambda v_2) \tag{5.18}$$

$$T_{\mathsf{a}}(k, v_1 \otimes v_2) = e^{-ik^\mu \mathsf{a}_\mu}(k, v_1 \otimes v_2) \tag{5.19}$$

$$(T_{\mathsf{a}} \cdot \Lambda)(k, v_1 \otimes v_2) = T_{\mathsf{a}}[\Lambda(k, v_1 \otimes v_2)]. \tag{5.20}$$

Note that the action of translations involves a factor of $e^{ik^\mu a_\mu}$ rather than a factor of $e^{2ik^\mu \mathsf{a}_\mu}$ which one might expect from analogy with Eqs. (5.10) and (5.18). This reflects the fact that we wish to construct new single-particle representations, in accord with Eq. (5.4), rather than two-particle representations with the tensor product. To verify that (5.18)-(5.20) is actually a representation of $\mathrm{ISO}^+(3,1)$, it is necessary to check that

$$(\Lambda \cdot T_{\mathsf{a}})(k, v_1 \otimes v_2) = \Lambda[T_{\mathsf{a}}(k, v_1 \otimes v_2)]. \tag{5.21}$$

By the group law for Poincaré transformations [Eq. (3.64)], $\Lambda \cdot T_{\mathsf{a}} \cdot \Lambda^{-1} = T_{\Lambda \mathsf{a}}$ and therefore

$$\Lambda \cdot T_{\mathsf{a}} = T_{\Lambda \mathsf{a}} \cdot \Lambda. \tag{5.22}$$

Thus by Eq. (3.64)

$$(\Lambda \cdot T_{\mathsf{a}})(k, v_1 \otimes v_2) = (T_{\Lambda \mathsf{a}} \cdot \Lambda)(k, v_1 \otimes v_2) \tag{5.23}$$

$$= T_{\Lambda \mathsf{a}}(\Lambda k, \Lambda v_1 \otimes \Lambda v_2) \tag{5.24}$$

$$= e^{i(\Lambda k)^\mu (\Lambda \mathsf{a})_\mu}(\Lambda k, \Lambda v_1 \otimes \Lambda v_2) \tag{5.25}$$

$$= e^{ik^\mu a_\mu}(\Lambda k, \Lambda v_1 \otimes \Lambda v_2) \tag{5.26}$$

$$= \Lambda[T_{\mathsf{a}}(k, v_1 \otimes v_2)] \tag{5.27}$$

showing that Eq. (5.21) indeed holds and we have a valid bundle representation.

Since $E_1 \otimes E_2$ has rank 1, this representation is irreducible. Moreover, by Eq. (5.18)

$$e^{i(\hat{\boldsymbol{k}} \cdot \boldsymbol{J})\theta}(k, v_1 \otimes v_2) = e^{i(h_1 + h_2)\theta}(k, v_1 \otimes v_2), \tag{5.28}$$

showing that $E_1 \otimes E_2$ has helicity $h_1 + h_2$. Define $\gamma_1^{-1} \doteq \gamma_{-1}$ and γ_0 to be the trivial line bundle over $\mathcal{L}_+$ on which T_{a} acts by $e^{ik^\mu \mathsf{a}_\mu}$ and $\Lambda \in \mathrm{SO}^+(3,1)$ acts by the identity. Then

$$\gamma_h \doteq \gamma_1^h \doteq \bigotimes_{i=1}^{h} \gamma_1^{\mathrm{sgn}(h)} \tag{5.29}$$

is a massless bundle UIR with helicity h. In this way we obtain smooth bundle representations for massless particles with any integer helicity. By Theorem 3.21 [1], every massless particle is unitarily equivalent to one of these representations.

Due to the tensor product structure, determining the topology of γ_h is also a simple matter. According to Theorem 2.3, the topology of a line bundle over $\mathcal{L}_+$ is completely determined by its Chern number C. The Chern number can be considered as a measure of how "twisted" or nontrivial a vector bundle is; in particular, $C = 0$ corresponds to the trivial bundle. For line bundles, the Chern number is additive with respect to the tensor product (Ref. [101], Eq. 20.1):

$$C(E_1 \otimes E_2) = C(E_1) + C(E_2).$$

Note that this is a special property of line bundles which does not generally hold for higher rank bundles. As the R and L photon bundles have Chern numbers $C(\gamma_{\pm 1}) = \mp 2$, we obtain

$$C(\gamma_h) = C(\gamma_1^h) = -2h. \tag{5.30}$$

This equation describes the fundamental relationship between the geometry and topology of massless particles, giving the massless analog of (5.1). It has not previously been reported.

One consequence of (5.30) is that the helicity can be considered as a topological invariant, despite being defined purely in terms of Poincaré geometry; this result was also established by Dragon [85]. Furthermore, massless particles with different helicity are topologically distinct, with the helicity measuring how globally twisted the particle space is. This is not readily detectable using the (nonsmooth) Wigner representations of massless particles [42, 43], in which the space of wavefunctions of all massless particles are represented on the same Hilbert space $L^2(\mathcal{L}_+, \mathbb{C})$. Note in particular that Eq. (5.30) shows that massless particles with $h \neq 0$ are topologically nontrivial, and it immediately follows that it is not possible to smoothly choose polarization vectors in all directions $\boldsymbol{k}$ for such particles. Thus, it is not possible to have massless particles of a fixed nonzero helicity with flux in all directions simultaneously, generalizing results we previously obtained for photons and gravitons [1, 2].

Before discussing the implications of this relationship, we remark on the alternative vector bundle descriptions of massless particles given by Asorey et al. [48] and Dragon [85]. Asorey et al. give a method to construct massless bundle representations of helicity h in which the fibers are represented by symmetric traceless tensors of rank $|h|$ subject to a series of gauge conditions. On the other hand, in Dragon's formalism, particle states are described as sections of vector bundles, but the fibers (polarizations) of those vector bundles are never explicitly defined. Instead, the vector bundle structures are described implicitly through the transition functions of the sections. As massless particles are fully characterized by their helicity, the representations γ_h and those of Dragon and Asorey et al. are equivalent,

but differ in their mathematical presentation. We contend that both the construction and representation of γ_h via tensor products are much simpler than in those other methods. Indeed, Dragon's construction [85] requires rather messy calculations involving the action of boosts and rotations on sections and results in representations on sections in stereographic coordinates which are somewhat hard to interpret. In Asorey et al.'s method [48], the construction gets increasingly difficult as $|h|$ grows, as one must impose more and more gauge constraints on symmetric traceless rank $|h|$ tensors. The result is that the fibers are defined abstractly as quotients of symmetric tensors modded by the gauge constraints, but explicit representations of these fibers are not derived. In contrast, the γ_h representations are equally simple to explicitly construct regardless of h.

To illustrate some of the practical advantages of this construction, we will use the example of the R and L graviton bundles studied in the previous chapter. The tensor product construction immediately gives $\Xi_\pm \cong \gamma_{\pm 1} \otimes \gamma_{\pm 1}$. At $k = (|\boldsymbol{k}|, \boldsymbol{k})$, choose any right-handed basis $(\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{k})$ of $\mathbb{R}^3$. The photon fiber $\gamma_\pm(k)$ is spanned by $v_\pm = \boldsymbol{e}_1 \pm i\boldsymbol{e}_2$, so the graviton fiber $\Xi_\pm(\boldsymbol{k})$ is spanned by $v_\pm \otimes v_\pm$. By expressing the tensor product as an outer product,

$$(v_\pm \otimes v_\pm)^{ij} = v_\pm^i v_\pm^j, \tag{5.31}$$

$\Xi_\pm(k)$ is embedded in the vector space of three-by-three symmetric tensors. Furthermore, since $(v_\pm \otimes v_\pm) \cdot \boldsymbol{k} = 0$ and $\sum_{i=1}^3 v_\pm^i v_\pm^i = 0$, the vectors of $\Xi_\pm(k)$ are automatically in the typical transverse-traceless gauge commonly used to describe gravitons [2, 44, 133] as we did in the prior chapter. In this construction it was never necessary to manually impose the transverse-traceless gauge conditions as was done in 4.3 and Refs. [2, 48]—the gauge fixing one chooses for the photon bundle automatically propagates through to all higher helicity bundles. Furthermore, the technique we employed in Chapter 4 involved constructing the total graviton bundle, and then manually identifying Lorentz invariant R and L subbundles. The difficulty of that procedure increases with $|h|$ as one must embed the particle states in

spaces of increasing dimension and identify the Lorentz invariant subbundles. The tensor product construction of γ_h avoids such difficulties.

Another application of the tensor product representations will be explored in Chapters 8, which pertains to the problem of finding simultaneous eigenstates of energy, helicity, J^2, and J_z for massless particles. The corresponding massive states (with spin replacing helicity) are straight-forwardly constructed via Clebsch–Gordan sums of tensor products of spherical harmonics with definite spin states. This problem is surprisingly tricky for massless particles due to the coupling of the polarization states and the momentum (cf. [141],§16). Dragon [85] recently showed a novel method of solving this problem using vector bundle methods. However, in Dragon's formalism, the eigenstates are difficult to interpret, given as a set of two unfamiliar functions with overlapping domains in stereographic coordinates (cf. Eqns. (81) and (84), [85]). Furthermore, explicit formulas are only given for the states with $l = m$ where $l(l+1)$ and m are the eigenstates of J^2 and J_z; the states with $m < l$ must be constructed via the lowering operator $J_- = J_x - iJ_y$. However, by instead using the tensor product representations γ_h developed here, we will show that the total angular momentum eigenstates of a massless helicity h particle are simply given by [6]

$$ {}_{-h}Y_{lm}(\theta,\phi)e_h(\theta,\phi). \tag{5.32} $$

In this equation, θ and ϕ are the spherical coordinates in k-space, (e_θ, e_ϕ) are the spherical unit vectors (which exist everywhere except the poles),

$$ e_h = \bigotimes_{i=1}^{|h|}(e_\theta + \operatorname{sgn}(h) i e_\phi), \tag{5.33} $$

and the ${}_{-h}Y_{lm}$ are the spin-weighted spherical harmonics of weight $-h$ first introduced by Newman and Penrose for applications in general relativity [86]. While ${}_{-h}Y_{lm}(\theta,\phi)$ and $e_h(\theta,\phi)$ separately have singularities at the poles, the eigenstates obtained from their product in Eq. (5.32) are smooth sections of γ_h for all momenta $\boldsymbol{k}$. We will show that this also gives

a description of the angular momentum eigenfunctions for electromagnetic plasma waves in Chapter 9, and we will also show how these eigenfunctions give a clear illustration of the lack of an SAM-OAM splitting.

In the following chapter we will turn again to the issue of SAM-OAM decompositions. We will prove a series of no-go theorems showing that it is not possible to split the angular momentum operator of $h \neq 0$ massless particles into spin SAM and OAM operators. The relationship $C = -2h$ reflects the fact that this obstruction can be understood either geometrically or topologically, with $C \neq 0$ and $h \neq 0$ obstructing the existence of SAM and OAM operators, respectively.

5.3 Conclusion

It has long been known that the limit of zero particle mass is singular [42]. This is conventionally viewed as a geometric singularity, as the little group jumps abruptly from SO(3) for massive particles to SO(2) for massless particles. However, there is an accompanying topological singularity that occurs too, as the momentum space jumps from a contractible mass hyperboloid to the noncontractible light cone $\mathcal{L}_+$. This allows for massless particles to have nontrivial topology, as illustrated in the bundle representations. We have shown that this singular limit involves other nonobvious differences between massive and massless particles, such as the emergence of a completely different relationship between the topology and geometry. The degree of topological nontriviality is measured by the Chern number, encoding how globally twisted twisted the polarization space is. Meanwhile the geometric helicity is related to the particles Poincaré symmetry, and measures the amount of local twisting that occurs in a polarization vector while applying rotations about the momentum axis. The equation $C = -2h$ encodes the relationship between these local and global twists. We also see the emergence of an Abelian group structure of massless particles, by which we can higher helicity particles by simply taking tensor products of R and L photon represen-

tations. This tensor product representation will prove crucial for calculation in Chapters 8 and 9 in relating the angular momentum eigenstates of massless particles and plasmas with spin-weighted spherical harmonics.

Chapter 6

Four no-go theorems on the existence of spin and orbital angular momentum of massless bosons

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6.1 Introduction

Since it was first introduced, the concept of the orbital angular momentum (OAM) of light has been a popular area of research [49]. It has found diverse scientific and industrial applications including, but not limited to, imaging [50, 51], optical communication [52], and the manipulation of microstructures [53]. Following the detection of gravitational waves by LIGO [54, 55], researchers have also been interested in the OAM of gravitational waves [56–58]. There have also been substantial interest in potential applications of high power OAM beams in laser-plasma accelerators [59, 60, 142] and the generation of extremely large magnetic fields in plasmas [62–64]. However, there are fundamental theoretical issues with splitting the angular momentum of massless bosons, such as photons and gravitons, into spin angular momentum (SAM) and OAM [1, 2, 67, 68, 75, 78]. In general, an angular

momentum operator $\boldsymbol{\mathcal{O}} = (\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3)$, such as a spin, orbital, or total angular momentum operator, must generate some three-dimensional rotational symmetries of the particle, and must therefore satisfy $\mathfrak{so}(3)$ commutation relations [1, 2, 67, 68, 74, 75, 78, 79, 82, 83]

$$[\mathcal{O}_a, \mathcal{O}_b] = i\epsilon_{abc}\boldsymbol{\mathcal{O}}_c; \tag{6.1}$$

these are frequently referred to as the angular momentum commutation relations. Furthermore, spin is associated with the internal degrees of freedom (DOFs) [143], so SAM operators must generate an SO(3) symmetry of the internal space of a particle. For massive particles, the angular momentum naturally splits into SAM and OAM. In contrast, there is no agreed upon method to define these operators for massless bosons, leading to many proposed SAM and OAM operators [65–74, 77, 78, 80]. However, most of these operators have been shown to either not be well-defined or else not actually satisfy angular momentum commutation relations, and are therefore not genuine angular momentum operators [67, 68, 75, 78, 80]. We showed in §3.5.2 that the SAM-OAM splittings for photons described in Refs. [67, 68, 71, 72] (what we call the quasi-angular momentum splitting), the SAM operators actually generate an $\mathbb{R}^3$ symmetry rather than an SO(3) symmetry, while the OAM operators do not generate any symmetry at all. In the case of gravitons, we proved the stronger result that there is no exists no legitimate SAM-OAM splitting (Theorem 4.6). In this chapter, we generalize and extend these arguments, showing that the same is true for any massless particles and that straightforward relaxations of the definitions of SAM and OAM operators do not remedy the problem. We do so by proving four no-go theorems. The first theorem uses geometric methods to show that the angular momentum operator of massless particles cannot split into nontrivial SAM and OAM operators. The third and fourth theorems show that various generalizations of the definition of SAM and OAM operators do not remove the obstruction. More specifically, the third no-go theorem shows that it is not possible to split the angular momentum operators of massless particles into generators of non-SO(3) symmetries. Indeed,

if one of the two operators generates a non-SO(3), the other will not generate any symmetry at all. The fourth no-go theorem shows the stronger result that it is not possible to split the angular momentum operator of massless particles into two generators of any nontrivial energy-preserving symmetries, even if neither of those symmetries are internal.

While the proofs of the first, third, and fourth theorems are essentially geometric, we show in the second no-go theorem that the obstruction preventing the splitting of angular momentum can also be viewed as a consequence of the nontrivial topology of massless particles. This nontrivial topology reflects the intertwining of the internal and external DOFs which prevents an SAM-OAM decomposition. We also show how gauge redundant descriptions of massless particles can give the false impression that the angular momentum splits into SAM and OAM parts.

This chapter is organized as follows. In §6.2, we review the vector bundle formalism for elementary particles. In §6.3, §6.4, §6.5, and §6.6 we prove No-Go Theorems 6.1, 6.2, 6.3, and 6.4, respectively. In §6.7, we show how gauge redundant descriptions can lead to nonphysical SAM-OAM splittings for massless particles. In §6.8 we discuss the validity of using single-particle states rather than multiparticle Fock states to prove the no-go theorems.

6.2 Formalism

Conventionally, elementary particles are defined as unitary irreducible representations (UIRs) of the proper orthochronous Poincaré group $\mathrm{ISO}^+(3,1)$ on a Hilbert space [42, 43]. Equivalently, they can be considered as UIRs on a Hermitian vector bundle $\pi : E \to M$ where E is the total space, M is the momentum space of the particle, and π is the projection $(k, v) \mapsto k$ as we showed in Chapters 3-5 [1–3, 47]. For convenience we review some of the notation introduced in previous chapters. As usual, we often refer to the total space E as the vector bundle the vector (k, v) by just v when the base point k is irrelevant or implied. Let $\mathcal{B}$ denote Minkowski space with signature $(-,+,+,+)$. For massive particles with $m > 0$, the

momentum space M is the (positive) mass hyperboloid

$$\mathcal{M}_m = \{k^\mu = (\omega, \boldsymbol{k}) \in \mathcal{B} : k^\mu k_\mu = -m^2\} \cong \mathbb{R}^3 \tag{6.2}$$

and for massless particles M it is the forward light cone

$$\mathcal{L}_+ = \{k^\mu = (\omega, \boldsymbol{k}) \in \mathcal{B} : k^\mu k_\mu = 0, \omega > 0\} \tag{6.3}$$

$$\cong \mathbb{R}^3 \setminus \{0\}. \tag{6.4}$$

Note that massless particles have no rest frame and thus $\omega = |\boldsymbol{k}| = 0$ is excluded from the light cone. Parametrizing $\mathcal{M}_m$ and $\mathcal{L}_+$ by just their spatial parts $\boldsymbol{k}$ gives the diffeomorphisms with $\mathbb{R}^3$ and $\mathbb{R}^3 \setminus \{0\}$, respectively. The fiber at $k \in M$ is the vector space $E(k) \doteq \pi^{-1}(k)$. The momentum space M describes the external DOFs of the particle while the fiber DOFs describe the internal state (i.e., $v \in E(k)$ describes the polarization). The vector bundle E is Hermitian, meaning that for each k, there is a Hermitian inner product $\langle \cdot, \cdot \rangle$ on the fiber $E(k)$. That E is a representation of $\mathrm{ISO}^+(3,1)$ means that there are $\mathrm{ISO}^+(3,1)$-actions Σ and σ on E and M, respectively, such that for $\Lambda \in \mathrm{ISO}^+(3,1)$, Σ_Λ is a vector space isomorphism of $E(k)$ onto $E(\sigma_\Lambda k)$. Furthermore, Σ and σ respect the group structure of $\mathrm{ISO}^+(3,1)$, that is,

$$\Sigma_{\Lambda_1 \Lambda_2} = \Sigma_{\Lambda_1} \Sigma_{\Lambda_2} \tag{6.5}$$

$$\Sigma_{\Lambda^{-1}} = (\Sigma_\Lambda)^{-1}, \tag{6.6}$$

and similarly for σ. We will mostly consider bundles for which σ is the standard action of $\mathrm{ISO}^+(3,1)$ on momentum space, that is, when $\Lambda \in \mathrm{SO}^+(3,1)$ is a Lorentz transformation we simply have $\sigma_\Lambda k = \Lambda k$, while the inhomogeneous spacetime translations in $\mathrm{ISO}^+(3,1)$ do

not effect k. That the action Σ is unitary means that

$$\langle \Sigma_\Lambda v_1, \Sigma_\Lambda v_2 \rangle = \langle v_1, v_2 \rangle \tag{6.7}$$

for all $\Lambda \in \mathrm{ISO}^+(3,1)$ and all $v_1, v_2 \in E(k)$. The irreducibility of the action means that there are no proper subbundles E' of E which are preserved by Σ. These definitions generalize the notion of vector space representations to vector bundle representations. One can recover the conventional Hilbert space representations $\tilde{\Sigma}$ of the particles by considering the induced UIR on the Hilbert space $L^2(E)$ of square-integrable sections of E [1, 47]. In particular, for a wavefunction $\psi(k) \in L^2(E)$, the action is

$$[\tilde{\Sigma}_\Lambda \psi](k) \doteq \Sigma_\Lambda[\psi(\sigma_{\Lambda^{-1}} k)]. \tag{6.8}$$

The inner product of E induces an inner product on $L^2(E)$ given by

$$\langle \psi_1, \psi_2 \rangle = \int_M \langle \psi_1(k), \psi_2(k) \rangle d\xi \tag{6.9}$$

where $d\xi$ is the Lorentz invariant measure on M; if $M = \mathcal{L}_+$, then $d\xi = |\boldsymbol{k}|^{-1} d^3\boldsymbol{k}$. Under this product, $\tilde{\Sigma}$ is a UIR of $\mathrm{ISO}^+(3,1)$ on the Hilbert space $L^2(E)$. The Lie group representations Σ and $\tilde{\Sigma}$ of $\mathrm{ISO}^+(3,1)$ induce corresponding Lie algebra representations describing the infinitesimal group actions. In particular, the generators of rotations ($\mathrm{SO}(3) \subseteq \mathrm{ISO}^+(3,1)$) for Σ and $\tilde{\Sigma}$ are the (total) angular momentum operators $\boldsymbol{J} = (J_1, J_2, J_3)$ and $\tilde{\boldsymbol{J}} = (\tilde{J}_1, \tilde{J}_2, \tilde{J}_3)$ defined by Eq. (3.96), or equivalently,

$$J_a(m, v) = i \frac{d}{d\theta}\Big|_{\theta=0} \Sigma_{R_a(\theta)}(m, v) \tag{6.10}$$

$$\tilde{J}_a \psi = i \frac{d}{d\theta}\Big|_{\theta=0} \tilde{\Sigma}\big(R_a(\theta)\big)\psi \tag{6.11}$$

where $R_a(\theta)$ is a rotation about the a axis by θ. They satisfy the $\mathfrak{so}(3)$ commutation relations

$$[J_m, J_n] = i\epsilon_{mnp} J_p \tag{6.12}$$

$$[\tilde{J}_m, \tilde{J}_n] = i\epsilon_{mnp} \tilde{J}_p \tag{6.13}$$

where the bracket in the first equation is the Jacobi-Lie bracket and that in the second equation is the usual operator commutator.[1] As always, we restrict our discussion to non-projective representations of $\mathrm{ISO}^+(3,1)$, which corresponds to treating only elementary particles with integer spin or helicity, i.e., bosons. All elementary massless particles in the Standard Model are bosons, so this assumption has little effect on the generality of our results. We recall also that a three-component operator $\boldsymbol{V}$ which satisfies

$$[V_m, J_n] = i\epsilon_{mnp} V_p \tag{6.14}$$

transforms under rotations like a spatial vector, and is thus called a vector operator [83, 106].

The bundle action given by Σ and $\boldsymbol{J}$ describes the action on single-particle states while $\tilde{\Sigma}$ and $\tilde{\boldsymbol{J}}$ describe the action on wave functions; they contain equivalent information. While the action on wave functions is more commonly used, we will mostly work with the bundle action as it leads to simpler proofs of the no-go theorems. The reason is that these proofs make frequent use of sharp momentum states, that is, states which have a single momentum $p \in M$. These are simply the vectors in E in the bundle representation, while in the wave function representation these are δ-function-like states. However, Flato et al. [45] showed that for massless particles these sharp states are not standard delta functions, but rather

[1] Note that we use the physics convention for the Lie algebra, in which $\mathfrak{so}(3)$ corresponds to Hermitian transformations. Mathematicians use a different convention in which $\mathfrak{so}(3)$ corresponds to anti-Hermitian transformations; see Section 3.4, Ref. [106] for a short discussion. In the math convention, the factors of i would not appear in the $\mathfrak{so}(3)$ commutation relations. The map between the these conventions is given by $\mathfrak{g}^{\mathrm{phys}} = i\mathfrak{g}^{\mathrm{math}}$.

twisted delta ("twelta") functions, which require more care to work with. We can avoid these complications by simply working in the bundle representation.

6.3 Massless particles do not have SAM or OAM: A geometric proof

The aim of this chapter is to study possible decompositions of the total angular momentum $\boldsymbol{J}$ into nonzero vector operators $\boldsymbol{S}$ and $\boldsymbol{L}$:

$$\boldsymbol{J} = \boldsymbol{S} + \boldsymbol{L}. \tag{6.15}$$

The defining property of angular momentum operators is that they generate SO(3) symmetries [1, 2, 67, 68, 74, 75, 78, 79, 82, 83]. This ensures that standard results about angular momentum operators hold such as the addition of angular momentum equation, the Clebsch–Gordan formalism, and the Casimir invariance of the S^2 and L^2. Thus, if $\boldsymbol{S}$ and $\boldsymbol{L}$ are to be considered angular momentum operators under the standard definition, they must (up to Lie algebra isomorphism) satisfy the $\mathfrak{so}(3)$ commutation relations

$$[S_m, S_n] = i\epsilon_{mnp}S_p \tag{6.16}$$

$$[L_m, L_n] = i\epsilon_{mnp}L_p. \tag{6.17}$$

These generate corresponding bundle actions (Σ^S, σ^S) and (Σ^L, σ^L) of SO(3) on $\pi : E \to M$. Since SAM describes an SO(3) symmetry of the internal (fiber) DOFs, Σ^S must not change the momentum, i.e., $\sigma^S_R(\boldsymbol{k}) = \boldsymbol{k}$ for every $R \in \mathrm{SO}(3)$. Such actions which do not change the fiber are called stabilizing. Under such a Σ^S, each fixed fiber $E(k)$ is a finite-dimensional vector space representation of SO(3).

We now illustrate that there is a natural splitting of the form (6.15) for massive particles following from Wigner's little group method [42, 43, 82], but that this construction breaks down for massless particles. For an action Σ of $\mathrm{ISO}^+(3,1)$ on $\pi : E \to M$ with $\sigma_\Lambda = \Lambda$, the little group at $k \in M$ is the subgroup H_k of $\mathrm{SO}^+(3,1)$ consisting of elements which leave k invariant [42, 43]:

$$H_k \doteq \{\Lambda \in \mathrm{SO}^+(3,1) | \Lambda k = k\}. \tag{6.18}$$

Up to group isomorphism, H_k is independent of k, so we can just refer to the little group H [43]. There is a canonically induced unitary action of H on E which is stabilizing [1]. For a particle $\pi_m : E_m \to \mathcal{M}_m$ with mass $m > 0$, the little group is $\mathrm{SO}(3)$. Indeed, at $\boldsymbol{k} = \mathbf{0}$, we have that $H_{\mathbf{0}} = \mathrm{SO}(3) \subseteq \mathrm{ISO}^+(3,1)$ is the canonical copy of $\mathrm{SO}(3)$ in the Poincaré group consisting of pure spatial rotations. At any other $\boldsymbol{k}$, let $\Lambda_{\boldsymbol{k}}$ be the unique boost taking $(m, \mathbf{0})$ to $(\sqrt{m^2 + |\boldsymbol{k}|^2}, \boldsymbol{k})$. At $\boldsymbol{k}$, the little group is

$$H_{\boldsymbol{k}} = \Lambda_{\boldsymbol{k}} H_{\mathbf{0}} \Lambda_{-\boldsymbol{k}} = \Lambda_{\boldsymbol{k}} H_{\mathbf{0}} \Lambda_{\boldsymbol{k}}^{-1} \cong \mathrm{SO}(3). \tag{6.19}$$

The isomorphism with $\mathrm{SO}(3)$ follows from the fact that conjugate subgroups are isomorphic, and we see that each $\boldsymbol{k}$ corresponds to a different copy of $\mathrm{SO}(3)$ embedded as a subgroup of the Poincaré group. Under the action Σ, every fiber $E_m(\boldsymbol{k})$ is a vector space UIR of $H_k \cong \mathrm{SO}(3)$. Vector space UIRs of $\mathrm{SO}(3)$ are uniquely determined up to isomorphism by their dimension $2s+1$ where s is the nonnegative integer spin, so the rank of r of the vector bundle E_m determines s via $r = 2s + 1$.[2] These vector space representations on the fibers fit together to form a unitary action $\Sigma^{m,S}$ of $H = \mathrm{SO}(3)$ on E_m given by

$$\Sigma_R^{m,S}(\boldsymbol{k}, v) = \Sigma(\Lambda_{\boldsymbol{k}} R \Lambda_{-\boldsymbol{k}})(\boldsymbol{k}, v) \tag{6.20}$$

[2] s could also be a half-integer if we were considering fermions and projective representations.

where $R \in \mathrm{SO}(3)$ and $(\boldsymbol{k}, v) \in E_m$. This action describes boosting into the particle's rest frame, applying the rotation R, and then boosting back to the original momentum. The generator $\boldsymbol{S}^m$ of this action is known in the literature as the Wigner spin operator [82]. From Eq. (6.20), it is given explicitly

$$\boldsymbol{S}^m = \Sigma(\Lambda_k)\boldsymbol{J}\Sigma(\Lambda_{-k}) = \Sigma(\Lambda_k)\boldsymbol{J}_0\Sigma(\Lambda_{-k}) \tag{6.21}$$

where $\boldsymbol{J}_0 = \boldsymbol{J}|_{\boldsymbol{k}=0}$ is the restriction of $\boldsymbol{J}$ to $E_m(\boldsymbol{0})$, the fiber at $\boldsymbol{k} = 0$. $\boldsymbol{J}_0$ is the generator of the irreducible little group action at $\boldsymbol{k} = 0$, so $E_m(\boldsymbol{0})$ is spanned by the eigenvectors $(v_{-s}, \ldots, v_s)$ of $J_{0,3}$ where the integer subscripts denote the eigenvalues. If we use the eigenvectors as a basis for $E_m(\boldsymbol{0})$, then $\boldsymbol{J}_0$ is given by the spin s matrices $\boldsymbol{S}_s$ on $E_m(\boldsymbol{0})$. That is, $\boldsymbol{J}_0(\boldsymbol{0}, v) = (\boldsymbol{0}, \boldsymbol{S}_s v)$ where v is expressed in coordinates with respect to the eigenbasis. Thus, in this coordinate representation

$$\boldsymbol{S}^m = \Sigma(\Lambda_{\boldsymbol{k}})\boldsymbol{S}_s\Sigma(\Lambda_{-\boldsymbol{k}}). \tag{6.22}$$

The form of $\boldsymbol{S}^m$ can be simplified even further by labeling all polarization states by their rest frame polarizations [42, 43], that is, we define a basis of $E_m(\boldsymbol{k})$ at arbitrary $\boldsymbol{k}$ by

$$(\boldsymbol{k}, v_a) \doteq \Sigma(\Lambda_k)(\boldsymbol{0}, v_a). \tag{6.23}$$

then it follows from Eq. (6.22) that

$$\boldsymbol{S}^m(\boldsymbol{k}, v) = (\boldsymbol{k}, \boldsymbol{S}_s v). \tag{6.24}$$

With respect to this basis we can then simply write

$$\boldsymbol{S}^m = \boldsymbol{S}_s. \tag{6.25}$$

Note in particular that $(\boldsymbol{k}, v_a)$ is an eigenvector of S_3^m with eigenvalue a for any $\boldsymbol{k}$. We see, either from Eq. (6.22) or (6.25) or from the fact that $\boldsymbol{S}^m$ generates an SO(3) symmetry, that $\boldsymbol{S}^m$ satisfies the $\mathfrak{so}(3)$ commutation relations in Eq. (6.16). Furthermore, $\boldsymbol{S}^m$ generates an internal symmetry since Σ^S does not change $\boldsymbol{k}$. It also is a well-defined vector operator, satisfying Eq. (6.14) ([82], Eq. (12)). Thus, $\boldsymbol{S}^m$ is a well-defined spin angular momentum operator. Since $\boldsymbol{J}$ and $\boldsymbol{S}^m$ satisfy $\mathfrak{so}(3)$ commutation relations, so does

$$\boldsymbol{L}^m \doteq \boldsymbol{J} - \boldsymbol{S}^m, \tag{6.26}$$

completing the Wigner SAM-OAM decomposition for massive particles. Note that $\boldsymbol{S}^m$ and $\boldsymbol{L}^m$ also commute with the Hamiltonian [82], which follows from the fact that $\boldsymbol{S}^m$ and $\boldsymbol{J}$ (and therefore also $\boldsymbol{L}^m$) generate symmetries which preserve the energy $\sqrt{m^2 + \boldsymbol{k}^2}$. Thus, $\boldsymbol{S}^m$ and $\boldsymbol{L}^m$ are conserved in the free theory and, for example, one can use them to label quantum states. We note that it is sometimes stated in the literature that there are issues with relativistic spin and orbital angular momentum even for massive particles [141, 144], namely, that the spin and orbital angular momentum do not commute with the Hamiltonian and thus do not constitute good quantum numbers. This is true of the Dirac SAM and OAM operators but not of the Wigner SAM and OAM operators; see Ref. [82] for details of this distinction. The essential idea is that the Dirac spin operator results from working in the bispinor representation in which particle (positive energy) and anti-particle (negative energy) states are treated simultaneously. The total angular momentum operator independently preserves particle and anti-particle states, but the Dirac spin operator mixes the positive and negative energy states and thus does not commute with the Hamiltonian. This does not occur for the Wigner operators $\boldsymbol{S}^m$ and $\boldsymbol{L}^m$ since the bundle E_m contains only particle or anti-particle states (not both simultaneously).

The above procedure breaks down for bundle representations $\pi_0 : E_0 \to \mathcal{L}_+$ of massless particles. In this case there is no rest frame with $\boldsymbol{k} = 0$, and the little group is now

$\mathrm{ISO}(2) = \mathbb{R}^2 \rtimes \mathrm{SO}(2)$, the group of inhomogeneous isometries of $\mathbb{R}^2$ where $\rtimes$ denotes the semidirect product (cf. §3.4, [43]). It is well-known that if any of the inhomogeneous elements $\mathbb{R}^2 \subseteq \mathrm{ISO}(2)$ act nontrivially, then one obtains particles with an infinite number of internal DOFs [42, 43]. There are no known particles with such an internal space and these solutions are generally believed to be unphysical. We thus make the standard assumption that the inhomogeneous elements act trivially, effectively reducing the massless little group from ISO(2) to SO(2). Therefore, for massless particles one obtains a canonical stabilizing little group representation of SO(2) on E_0 (Theorem 3.22). Each fiber is a finite-dimensional representation of SO(2) and such representations are all one-dimensional and are uniquely labeled by a (possibly negative) integer helicity h rather than a spin [1, 42, 43]. Since this is not an SO(3) representation, the generator of this action is not an SAM operator. A clear illustration of this fact is that SO(2) is a one-dimensional Lie group, so its generator (the helicity operator) is a scalar operator rather than a vector operator, so it obviously cannot represent SAM.

Although this argument is suggestive, it does not rule out the possibility that SAM and OAM operators could be defined by some other procedure. Indeed, the many other proposed SAM and OAM operators are such attempts [65–70, 72–74, 77, 78, 80]. However, we prove in the following theorem that in general there is no SAM-OAM decomposition for massless particles. The argument uses the fundamental difference of the dimensionalities of the spin and helicity representations. Indeed, the vector space spin representations of SO(3) have dimension $2s+1$ while all of the helicity representations of SO(2) are one-dimensional. Thus the fibers of massless particles are all one-dimensional, that is, massless particles are represented by line bundles (Theorem 3.24) [1, 3]. Indeed, we previously showed that the right (R) and left (L) circularly polarized photons each form a UIR of $\mathrm{ISO}^+(3,1)$ on line bundles [1], and likewise for the R and L gravitons [2].

No-Go Theorem 6.1. *Suppose $\pi_0 : E_0 \to \mathcal{L}_+$ is a massless particle, that is, it is a UIR of $\mathrm{ISO}^+(3,1)$. It is not possible to split the total angular momentum $\boldsymbol{J}$ into nonzero SAM and*

OAM operators $\boldsymbol{S}$ and $\boldsymbol{L}$, where $\boldsymbol{S}$ and $\boldsymbol{L}$ generate SO(3) *symmetries and $\boldsymbol{S}$ corresponds to an internal symmetry.*

Proof. Suppose such a decomposition exists. $\boldsymbol{S}$ generates an SO(3) action Σ^S on E_0. Since this symmetry is internal, Σ^S does not change the fiber coordinate $k \in \mathcal{L}_+$, and thus Σ^S is a vector space representation of SO(3) on each fiber $E_0(k)$. By the little group construction, we saw that since E_0 is a UIR of $\mathrm{ISO}^+(3,1)$ it must be a line bundle, so $E_0(k)$ is a one-dimensional vector space. The only representation of SO(3) on a one-dimensional vector space is the spin 0 representation, which is trivial in the sense that SO(3) always acts by the identity. Thus $\Sigma^S = I$ and $\boldsymbol{S} = 0$, and therefore $\boldsymbol{J} = \boldsymbol{L}$. There is thus no nontrivial SAM-OAM decomposition. □

Theorem 6.1 explains the issues that plague the various proposed SAM and OAM operators for photons. Van Enk and Nienhuis [67, 68] first showed that one attempt to define SAM and OAM for light fail to satisfy $\mathfrak{so}(3)$ relations and are therefore not actually SAM and OAM operators; we showed [1] that the same is true of the decompositions proposed by Bliokh [71] and Bialynicki-Birula and Bialynicka-Birula [72]. Recently, Yang et al. [80] showed through direct calculation that the SAM-OAM decompositions of Jaffe-Manohar [66], Chen et al. [69], and Wakamatsu [70] do not satisfy angular momentum commutation relations, leading again to the conclusion that none of these are legitimate SAM and OAM operators. Yang et al. propose two new sets of SAM and OAM operators $(\boldsymbol{S}_M, \boldsymbol{L}_M)$ and $(\boldsymbol{S}_M^{\mathrm{obs}}, \boldsymbol{L}_M^{\mathrm{obs}})$; see Table II in Ref. [80]. However, in accord with No-Go Theorem 6.1, neither of these sets of operators are actually well-defined SAM and OAM operators for photons by the standard definition. The first set satisfy $\mathfrak{so}(3)$ relations but, as Yang et al. themselves point out, they are not gauge invariant and are thus not measurable. As such, $\boldsymbol{S}_M$ and $\boldsymbol{L}_M$ are not well-defined vector operators on the space of physical states; this is discussed in more detail in §6.7. The other set of operators, $(\boldsymbol{S}_M^{\mathrm{obs}}, \boldsymbol{L}_M^{\mathrm{obs}})$, are gauge invariant, but the $S_{M,i}^{\mathrm{obs}}$ commute with each other rather than satisfy cyclic $\mathfrak{so}(3)$ relations (Table II, Ref. [80]), and are therefore not angular momentum operators by the standard definition.

6.4 Nontrivial topology as an obstruction to an SAM-OAM decomposition

The failure of the angular momentum of massless particles to split into SAM and OAM can be understood both in terms of the geometry and topology of such particles. The obstruction originates in the singular limit that occurs as the particle mass is taken to zero. We showed in the previous section that there is a singularity in the Poincaré geometry that occurs in this limit as the little group jumps abruptly from SO(3) to ISO(2). This leads to massive particles possessing spin while massless particles possess helicity, and the latter is not associated with an angular momentum operator.

However, the $m \to 0$ limit is also accompanied by a topological singularity as the momentum space M jumps from being a topologically trivial (contractible) mass hyperboloid $\mathcal{M}_m$ to the noncontractible light cone $\mathcal{L}_+$. Since $\mathcal{M}_m$ is contractible to a point, there are only topologically trivial vector bundles over $\mathcal{M}_m$ ([101], Corollary 6.9), and thus all massive particles $\pi_m : E_m \to \mathcal{M}_m$ are described by trivial bundles. However, $\mathcal{L}_+$ is not contractible due to the hole at $\boldsymbol{k} = 0$, and thus there are topologically nontrivial vector bundles over $\mathcal{L}_+$. Indeed, a massless particle with nonzero helicity can be described by the vector bundle $\pi_h : \gamma_h \to \mathcal{L}_+$ whose topology is fully characterized by its (first) Chern number $C(\gamma_h) = -2h$ [Eq. (5.29)]. When $h \neq 0$, γ_h is topologically nontrivial. This nontrivial topology indicates that the internal DOFs parametrizing the fibers (polarizations) and the external DOFs parametrizing $\mathcal{L}_+$ are nontrivially twisted together [101]. However, SAM and OAM are related to SO(3) symmetries of the internal and external DOFs of a particle, respectively. Thus, the nontrivial topology of massless particles with nonzero helicity can be heuristically understood as obstructing the existence of such a splitting. As for massive particles with zero spin, the $h = 0$ case is uninteresting from the perspective of SAM and OAM as the unique SAM and OAM operators are simply $\boldsymbol{S} = 0$ and $\boldsymbol{L} = \boldsymbol{J}$.

We now formalize this heuristic observation. In Theorem 6.1, we illustrated the geometric obstruction to an SAM-OAM splitting by imposing the condition that the SAM operator is internal. To illustrate the topological obstruction we will instead assume that the OAM operator is external, that is, it generates an action only changing the base manifold DOFs. In precise terms, suppose Σ^L is the action of some group G on the $\pi : E \to M$ with base manifold action σ^L. We say that Σ^L is external if the condition $\sigma^L_{g_1} k = \sigma^L_{g_2} k$ for $g_1, g_2 \in G$ and $k \in M$ implies that $\Sigma^L_{g_1}(k, v) = \Sigma^L_{g_2}(k, v)$ for every $(k, v) \in E(k)$. This is equivalent to saying that if $\sigma^L_g k = k$, then $\Sigma^L_g(k, v) = (k, v)$, i.e., the little group with respect to Σ^L acts trivially on E. We refer to a Lie algebra representation as external if it generates an external Lie group representation.

No-Go Theorem 6.2. *Suppose $\pi : E \to \mathcal{L}_+$ is a rank r bundle with an external* SO(3) *action Σ^L. Assume the corresponding action σ^L on $\mathcal{L}_+ \cong \mathbb{R}^3 \setminus \{0\}$ is the standard base manifold action $\sigma^L_R = R$ for $R \in \mathrm{SO}(3)$. Then E is a trivial bundle. Consequently, a massless particle admits an external OAM operator $\boldsymbol{L}$ if and only if the particle has zero helicity.*

Proof. Consider the restriction of E to the unit sphere $S^2 \subseteq \mathbb{R}^3 \setminus \{0\} \cong \mathcal{L}_+$, denoted by $E|_{S^2}$. $\mathcal{L}_+$ deformation retracts onto S^2, so E is trivial if and only if $E|_{S^2}$ is trivial ([41], Cor 1.8). By the assumption that $\sigma^L_R = R$, Σ^L restricts to an action on $E|_{S^2}$. Fix some $k_0 \in S^2$ and for each $k \in S^2$, define

$$G_k = \{R \in \mathrm{SO}(3) | R k_0 = k\}. \tag{6.27}$$

Let $v_1, ..., v_r$ be a basis of $E|^2_S$ at k_0, and define sections $f_1, ..., f_r$ of $E|_{S^2}$ by

$$f_j(k) \doteq \Sigma^L_{G_k}(k_0, v_j). \tag{6.28}$$

The assumption that Σ^L is external ensures that all elements of G_k have the same action on (k_0, v_j), so the f_j are well-defined. Since Σ^L is smooth and G_k varies smoothly with k, the f_j are smooth sections of $E|_{S^2}$. Furthermore, they are linearly independent at each k. To

see this, choose some $g_k \in G_k$ and note that $\Sigma^L_{g_k}$ is a linear isomorphism between the fibers at k_0 and at k. Since E possess r linearly independent sections, it is a trivial vector bundle. Since a massless particle with nonzero helicity is topologically nontrivial by the relationship $C = -2h$ between the Chern number and helicity [3], such a particle does not admit an external OAM operator. □

In practice, it is common to consider helicity h particles not in isolation, but in combination with the helicity $-h$ particles, that is, one considers the direct sum bundle $\gamma_{h,T} \doteq \gamma_h \oplus \gamma_{-h}$. For example, R and L photons are respectively described by bundles γ_1 and γ_{-1} [1], but it is necessary to consider the total photon bundle $\gamma = \gamma_{1,T} = \gamma_1 \oplus \gamma_{-1}$ if one wishes to describe linear polarizations. Since the first Chern numbers are additive with respect to the direct sum by the Whitney product formula [99] and since $C(\gamma_h) = -2h$ [3], it follows that $C(\gamma_{h,T}) = 0$. This implies that, unlike $\gamma_{\pm h}$, the bundle $\gamma_{h,T}$ is topologically trivial [1, 3]. No-Go Theorem 6.2 does not then immediately imply that $\gamma_{h,T}$ does not possess an OAM operator, although this is indeed the case.

Before proving this, we will give an informal argument showing why one would not expect an OAM operator to exist for $\gamma_{h,T}$. If we let $\Sigma^{\pm}$ denote the $\mathrm{ISO}^+(3,1)$ symmetries of $\gamma_{\pm h}$, then $\Sigma = \Sigma^+ \oplus \Sigma^-$ is an $\mathrm{ISO}^+(3,1)$ symmetry of $\gamma_{h,T} = \gamma_h \oplus \gamma_{-h}$. Suppose there were such an OAM operator $\boldsymbol{L}^T$ which generates an $\mathrm{SO}(3)$ symmetry $\Sigma^{L,T}$ on $\gamma_{h,T}$. By No-Go Theorem 6.2, $\Sigma^{L,T}$ is not a symmetry of either γ_h or γ_{-h}, but only of their direct sum. However, $\gamma_{\pm h}$ are preserved under the actions $\Sigma^{\pm}$, so $\boldsymbol{L}^T$ cannot be written purely in terms of the generators of $\Sigma^{\pm}$, that is, in terms of the Hamiltonian, momentum, angular momentum, and boost operators on $\gamma_{\pm h}$. There are no other natural operators available, so this gives a strong indication that there is no way to construct an OAM operator $\boldsymbol{L}^T$. Indeed, γ_h and γ_{-h} are related by a parity symmetry P which is part of the full Poincaré group, but not of the proper orthochronous Poincaré group $\mathrm{ISO}^+(3,1)$ [1]. Since $\boldsymbol{L}^T$ would have to generate a symmetry mixing γ_h and γ_{-h}, this suggests $\boldsymbol{L}^T$ would involve the generator of the parity symmetry P. However, parity is a discrete rather than a continuous symmetry, and thus

has no infinitesimal generator. This is essentially the reason why elementary particles are typically defined [43] as UIRs of $\mathrm{ISO}^+(3,1)$ rather than the full Poincaré group, since the Lie algebra does not detect discrete parity and time inversion symmetries.

We can give a more rigorous argument showing that no such $\boldsymbol{L}^T$ exists as follows. Note that while No-Go Theorems 6.1 and 6.2 are proved via different methods, their conclusions are related. In particular, if the OAM operator $\boldsymbol{L}^T$ is external, then $\boldsymbol{S}^T = \boldsymbol{J}^T - \boldsymbol{L}^T$ is internal. We note then that No-Go Theorem 6.1 applies equally well to the bundle $\gamma_{h,T} = \gamma_h \oplus \gamma_{-h}$. The only change in the proof of No-Go Theorem 6.1 is that the fibers now have dimension two rather than one. However, there are no two-dimensional UIRs of SO(3) and thus the representation generated by an internal $\boldsymbol{S}^T$ is reducible and must break down into two one-dimensional UIRs of SO(3) [3]. But the only such UIR is the trivial representation, so we must have $\boldsymbol{S}^T = 0$ and $\boldsymbol{L}^T = \boldsymbol{J}^T = \boldsymbol{J}^{(+h)} + \boldsymbol{J}^{(-h)}$, where $\boldsymbol{J}^{(\pm h)}$ are the total angular momentum operators of $\gamma_{\pm h}$. However, $\boldsymbol{J}^{(\pm h)}$, and therefore $\boldsymbol{J}^T$, do not generate external symmetries when $h \neq 0$ since the little group acts nontrivially, and thus, $\gamma_{h,T}$ does not admit an external OAM operator. We have thus proved the following corollary of the previous no-go theorems:

Corollary 6.1. *Let $\gamma_{\pm h}$ be line bundles describing massless particles with nonzero helicities $\pm h$ and with total angular momentum operators $\boldsymbol{J}^{\pm}$. Consider the total bundle $\gamma_{T,h} = \gamma_h \oplus \gamma_{-h}$ with the induced angular momentum operator $\boldsymbol{J}^T = \boldsymbol{J}^{(+h)} + \boldsymbol{J}^{(-h)}$. If*

$$\boldsymbol{J}^T = \boldsymbol{L}^T + \boldsymbol{S}^T, \tag{6.29}$$

where $\boldsymbol{L}^T$ and $\boldsymbol{S}^T$ generate SO(3) *symmetries, then:*

1. *If $\boldsymbol{S}^T$ is internal, then $\boldsymbol{S}^T = 0$.*

2. *$\boldsymbol{L}^T$ cannot generate an external symmetry.*

Therefore, there is no nontrivial SAM-OAM decomposition for the system of particles γ_T.

[3] There is a two-dimensional projective UIR corresponding to spin 1/2 fermions, but this is irrelevant to the case of bosons, whose representations are non-projective.

6.5 The angular momentum cannot be split into non-SO(3) symmetries

No-Go Theorems 6.1 and 6.2 establish the fundamental result that it is not possible to split the angular momentum of a massless particle into genuine SAM and OAM operators. This reflects the fact that the internal and external DOFs cannot independently carry any nontrivial SO(3) symmetries. One might still ask if $\boldsymbol{J}$ can be split into vector operators describing some non-SO(3) symmetries.

As an illustration of why this is not possible, consider the proposed splitting [71, 72]

$$\boldsymbol{J} = \boldsymbol{J}_{\parallel} + \boldsymbol{J}_{\perp} \tag{6.30}$$

$$\boldsymbol{J}_{\parallel} \doteq (\boldsymbol{J} \cdot \hat{\boldsymbol{k}})\hat{\boldsymbol{k}} \doteq \chi \hat{\boldsymbol{k}} \tag{6.31}$$

$$\boldsymbol{J}_{\perp} \doteq \boldsymbol{J} - \boldsymbol{J}_{\parallel}. \tag{6.32}$$

$\chi \doteq \boldsymbol{J} \cdot \hat{\boldsymbol{k}}$ is the helicity operator generating the internal SO(2)-action which naturally arises from the little group construction for massless particles [1]. The helicity is a scalar operator rather than a vector operator, and thus by itself cannot be a SAM operator. Eq. (6.31) can be viewed as an ad hoc attempt to "vectorize" the helicity operator into an SAM operator. However, by Theorem 6.1 this $\boldsymbol{J}_{\parallel}$ cannot actually be an angular momentum operator. For this reason we call $\boldsymbol{J}_{\parallel}$ and $\boldsymbol{J}_{\perp}$ the helicity and orbital quasi-angular momenta, respectively. Indeed, while $\boldsymbol{J}_{\parallel}$ and $\boldsymbol{J}_{\perp}$ are well-defined vector operators satisfying condition (6.14), they satisfy the nonstandard commutation relations [1, 67, 68]

$$[J_{\parallel,m}, J_{\parallel,n}] = 0, \tag{6.33}$$

$$[J_{\perp,m}, J_{\parallel,n}] = i\epsilon_{mnp} J_{\parallel,p}, \tag{6.34}$$

$$[J_{\perp,m}, J_{\perp,n}] = i\epsilon_{mnp}(J_{\perp,p} - J_{\parallel,p}). \tag{6.35}$$

Aside from not generating SO(3) symmetries, we previously noted in §3.5.2 that these suffer from an additional issue [1]. In particular, while the $J_{\parallel,m}$ operators commute with each other and thus generate an $\mathbb{R}^3$ symmetry, the $J_{\parallel,p}$ on the rhs of Eq. (6.35) shows that the components of $\boldsymbol{J}_\perp$ do not combine together to form a Lie subalgebra. Thus, $\boldsymbol{J}_\perp$ does not generate any symmetry at all. The next theorem shows that this issue is generic—it arises in any attempt to split $\boldsymbol{J}$ into generators of non-SO(3) symmetries, regardless of whether they generate internal and/or external symmetries.

No-Go Theorem 6.3. *Suppose that*

$$\boldsymbol{J} = \boldsymbol{L} + \boldsymbol{S} \tag{6.36}$$

where $\boldsymbol{J}$ is the the total angular momentum operator and $\boldsymbol{L}$ and $\boldsymbol{S}$ are nonzero vector operators, that is, they satisfy

$$[L_m, J_n] = i\epsilon_{mnp}L_p \tag{6.37a}$$

$$[S_m, J_n] = i\epsilon_{mnp}S_p. \tag{6.37b}$$

Furthermore, suppose that $\boldsymbol{L}$ and $\boldsymbol{S}$ are generators of symmetry groups G_L and G_S with associated Lie algebras $\mathfrak{g}_L$ and $\mathfrak{g}_S$. Then

$$\mathfrak{g}_L = \mathfrak{g}_S = \mathfrak{so}(3). \tag{6.38}$$

*In particular, it is not possible for either $\boldsymbol{L}$ or $\boldsymbol{S}$ to generate non-*SO(3) *symmetries.*

Proof. $\boldsymbol{L}$ and $\boldsymbol{S}$ satisfy the commutation relations of their associated Lie algebras $\mathfrak{g}_L$ and $\mathfrak{g}_S$. Therefore,

$$[L_m, L_n] = i\nu_{mnp}L_p \tag{6.39}$$

$$[S_m, S_n] = i\tau_{mnp}S_p \tag{6.40}$$

where ν_{mnp} and τ_{mnp} are the structure constants of $\mathfrak{g}_L$ and $\mathfrak{g}_S$. It then follows that

$$i\nu_{mnp}L_p = [L_m, L_n] \tag{6.41}$$

$$= [J_m - S_m, J_n - S_n] \tag{6.42}$$

$$= i\epsilon_{mnp}L_p + i(\tau_{mnp} - \epsilon_{mnp})S_p \tag{6.43}$$

Thus, either $\tau_{mnp} = \epsilon_{mnp}$ or else S_p is a multiple of L_p. If $\tau_{mnp} = \epsilon_{mnp}$, then also $\nu_{mnp} = \epsilon_{mnp}$, and therefore $\boldsymbol{L}$ and $\boldsymbol{S}$ both satisfy $\mathfrak{so}(3)$ commutation relations, proving the claim. Suppose then that $S_m = c_m L_m$ for some constants c_m. Note that $c_m \neq -1$ since $J_m \neq 0$. Then $L_m = (1 + c_m)^{-1} J_m$. From Eq. (6.37), it follows that

$$[L_m, J_n] = \frac{1 + c_p}{1 + c_m} i\epsilon_{mnp}L_p \tag{6.44}$$

$$= i\epsilon_{mnp}L_p, \tag{6.45}$$

so $c_m = c_p \doteq c$ for all m, p. Then

$$[L_m, L_n] = \frac{i}{1 + c}\epsilon_{mnp}L_p. \tag{6.46}$$

By rescaling the basis to $L'_m = (1 + c)L_m$ we obtain

$$[L'_m, L'_n] = i\epsilon_{mnp}L'_p, \tag{6.47}$$

showing that $\mathfrak{g}_L = \mathfrak{so}(3)$. An analogous argument shows that $\mathfrak{g}_S = \mathfrak{so}(3)$. □

No-Go Theorem 6.3 shows that if $\boldsymbol{S}$ or $\boldsymbol{L}$ generates a non-SO(3) symmetry, then the other operator will not generate any symmetry at all. This result thus shows a fundamental difficulty with attempts at defining generalized splittings of $\boldsymbol{J}$ such as the splitting into $\boldsymbol{J}_{\parallel}$ and $\boldsymbol{J}_{\perp}$ in Eqs. (6.30-6.32)).

Before concluding this section, we give some further remarks about the $\boldsymbol{J}_{\parallel}$, $\boldsymbol{J}_{\perp}$ splitting and its relationship to the helicity of massless particles. For massive particles, the little group is SO(3) and the generator of the corresponding internal symmetry is the spin operator $\boldsymbol{S}^m$. Massless particles instead have the little group SO(2), and the generator of this symmetry is the helicity operator $\chi = \boldsymbol{J} \cdot \hat{\boldsymbol{k}}$. In this sense, helicity is the massless analog of spin. By Wigner's little group method, massive and massless particles are classified by the representations of the little group, and therefore by their spin and helicity, respectively [1, 42, 43]. That $\boldsymbol{S}^m$ also induces a splitting of the total angular momentum $\boldsymbol{J}$ is a result of the fact that the little group of massive particles is SO(3), and thus $\boldsymbol{S}^m$ satisfies angular momentum commutation relations. In contrast, SO(2) is one-dimensional, so the helicity is a scalar operator χ and satisfies very different commutation relations than the massive spin operator. In fact, direct calculation (cf. [1], Theorem 33) shows that χ commutes with all the generators of the Poincaré group,[4], for example,

$$[J_a, \chi] = 0, \tag{6.48}$$

$$[\chi, \chi] = 0. \tag{6.49}$$

These commutation relations are important for the classification of the massless particle representations, as they show that the helicity can be used to classify massless particles in the same way $(S^m)^2$ classifies massive representations [1]. However, the helicity does not

[4] χ is a central element (*i.e.*, commutes with all other elements) of the finite helicity massless representations of the Poincaré algebra. However, χ is not central for massive representations or continuous spin representations, and is therefore not a Casimir invariant of the Poincaré algebra.

induce an SAM-OAM splitting the way spin does for massive particles. One might attempt to "vectorize" the scalar helicity operator from χ to $\chi\hat{\boldsymbol{k}}$ as in Eqs. (6.30-6.32)) to obtain a splitting. On the upside, the vector operators $\boldsymbol{J}_{\parallel}$ and $\boldsymbol{J}_{\perp}$ are gauge-invariant and thus observable. There are, however, two notable drawbacks to this splitting. The first is that, as previously discussed, neither $\boldsymbol{J}_{\perp}$ nor $\boldsymbol{J}_{\parallel}$ generate SO(3) symmetries, and the vector operator $\boldsymbol{J}_{\perp}$ does not generate any symmetry at all. Thus, they are not angular momentum operators in the typical sense and tools such as the Clebsch–Gordan theory do not apply to $\boldsymbol{J}_{\parallel}$ and $\boldsymbol{J}_{\perp}$. The second drawback is that this splitting appears to be *ad hoc*, as χ is the generator of the little group representation, not the vectorized operator $\boldsymbol{J}_{\parallel} = \chi\hat{\boldsymbol{k}}$. This is to say, there is no known theory which justifies the $\boldsymbol{J}_{\parallel}, \boldsymbol{J}_{\perp}$. In summary, while $\boldsymbol{J}_{\parallel}$ and $\boldsymbol{J}_{\perp}$ are measurable, they are not SAM and OAM operators in the standard sense, and their general theoretical significance is not well-understood. However, the fact that these operators are expressed entirely in terms of Poincaré generators is an indication that there may be some natural construction for representations of the Poincaré group which produces this splitting. Ideally, such a construction would produce the standard SAM-OAM splitting when applied to massive representations. We will present such a construction in Chapter 7.

6.6 Generalization to non-internal SAM operators

In this section we prove a stronger version of No-Go Theorem 6.1 by relaxing the constraint that the SAM operator is stabilizing. Assume again that $\pi_0 : E_0 \to \mathcal{L}_+$ and Σ are a representation of a massless particle, and that

$$\boldsymbol{J} = \boldsymbol{L} + \boldsymbol{S} \tag{6.50}$$

where $\boldsymbol{L}$ and $\boldsymbol{S}$ are vector operators generating symmetries. By No-Go Theorem 6.3, these must generate SO(3) symmetries Σ^L and Σ^S on E. Let σ, σ^L, and σ^S be the actions on the base manifold $\mathcal{L}_+$ corresponding to Σ, Σ^L, and Σ^S. For massless particles, the energy

is given by $\omega = |\boldsymbol{k}|$. If one enforces the constraint that Σ^S is internal, then $\sigma^S_R = I$ and $\sigma^L_R = \sigma_R = R$ for any $R \in \mathrm{SO}(3)$. In this case,

$$|\sigma^L_R \boldsymbol{k}| = |\sigma^S_R \boldsymbol{k}| = |\boldsymbol{k}|, \tag{6.51}$$

so $\boldsymbol{L}$ and $\boldsymbol{S}$ both generate symmetries which preserve the energy. The first no-go theorem shows that there are no nontrivial splittings such that $\boldsymbol{S}$ is internal. We now prove this conclusion is still true if we replace the assumption that $\boldsymbol{S}$ is internal with the weaker assumption that $\boldsymbol{S}$ and $\boldsymbol{L}$ conserve energy in the sense of (6.51). To this end, we first establish the following lemmas.

Lemma 6.2. *Suppose $\boldsymbol{L}$ and $\boldsymbol{S}$ are vector operators generating symmetries and satisfy equation (6.50). Then $\boldsymbol{L}$ and $\boldsymbol{S}$ commute, and therefore so do σ^L and σ^S.*

Proof. From equations (6.50) and (6.17) we have that

$$[L_m, J_n] = i\epsilon_{mnp} L_p + [L_m, S_n] \tag{6.52}$$

$$= i\epsilon_{mnp} L_p. \tag{6.53}$$

Thus

$$[L_m, S_n] = 0, \tag{6.54}$$

and since the generators commute, so do the generated actions Σ^L and Σ^S. This in turn implies that σ^L and σ^S commute. □

Let $\mathrm{Diff}(M)$ be the group of diffeomorphisms on a manifold M. A G-action

$$\zeta : G \to \mathrm{Diff}(M) \tag{6.55}$$

$$g \mapsto \zeta_g \tag{6.56}$$

on a manifold M is a continuous group homomorphism from the Lie group G to $\mathrm{Diff}(M)$. ζ is transitive if for any $m_1, m_2 \in M$, there is a $g \in G$ such that $\zeta_g m_1 = m_2$. The orbit of $x \in M$ is the subset $\zeta_G x \subseteq S^2$. Under the energy conserving assumption (6.51), σ, σ^L, and σ^S are all SO(3)-actions on the unit sphere $S^2 \subseteq \mathcal{L}_+$.

Lemma 6.3. *Suppose that $\zeta : \mathrm{SO}(3) \to \mathrm{Diff}(S^2)$ is an SO(3)-action on S^2. Then*

(i) Either ζ is the identity operator or else it is transitive.

(ii) For every $R \in \mathrm{SO}(3)$, ζ_R has a fixed point, that is, there exists an $x \in S^2$ such that $\zeta_R x = x$.

Proof. (i) [145] For $x \in S^2$, let G_x be the isotropy group of x with respect to ζ:

$$G_x = \{g \in G | \zeta_g(x) = x\}. \tag{6.57}$$

Note first that G_x is a closed subgroup of SO(3). Indeed, if we consider the map $\zeta(x) : G \to M$, $g \mapsto \zeta_g(x)$, then $G_x = (\zeta(x))^{-1}(x)$ which is closed since it is the preimage of the closed set $\{x\}$. The only closed subgroups of SO(3) are isomorphic to either a finite group, SO(2), O(2), or SO(3) [146]. Consider the map $F : \mathrm{SO}(3)/G_x \to S^2$ given by $F(R) = \zeta_R x$. This gives a homeomorphism of $\mathrm{SO}(3)/G_x$ with the orbit of x in S^2 [103]. If G_x is a finite group, then $\mathrm{SO}(3)/G_x$ would be three-dimensional but homeomorphic to a subset of S^2, which is an obvious contradiction. Similarly, $\mathrm{SO}(3)/\mathrm{O}(2) \cong \mathbb{RP}^2$, but the latter cannot be embedded in S^2 [147], so $G_x \ncong \mathrm{O}(2)$. If $G_x \cong \mathrm{SO}(2)$, the orbit of x is homeomorphic to $\mathrm{SO}(3)/\mathrm{SO}(2) \cong S^2$ [103]. There are no proper subspaces of S^2 homeomorphic to S^2, thus the orbit of x must be all of S^2. Thus, if $G_x \cong \mathrm{SO}(2)$ for some x, then ζ is transitive.

Lastly, if $G_x = \mathrm{SO}(3)$, then $\zeta_R x = x$ for all $R \in \mathrm{SO}(3)$. Furthermore, this implies that $\zeta_R y = y$ for all $R \in \mathrm{SO}(3)$ and $y \in S^2$. For suppose there were a y for which this were not true. Then $G_y \cong \mathrm{SO}(2)$ by the previous argument and the action is transitive and thus x would be in the orbit of y, which is a contradiction. Thus, if there exists any x for which $G_x = \mathrm{SO}(3)$, then ζ is just the identity action on S^2.

(ii) Let I be the identity in SO(3), and note that $\zeta_I = \mathbb{1}$ is the identity operator on S^2. Since SO(3) is path-connected, $\zeta_R : S^2 \to S^2$ is homotopic to the identity for any $R \in \mathrm{SO}(3)$. The degree of a map $f : S^2 \to S^2$ is homotopy invariant [105], so $\deg(\zeta_R) = \deg(\mathbb{1}) = 1$. If a map $f : S^2 \to S^2$ has no fixed points, then $\deg(f) = -1$ ([105], p. 134). Thus ζ_R must have a fixed point. □

No-Go Theorem 6.4. *Assume that the* SO(3) *action* Σ *of a massless particle splits into UIRs of* SO(3)*,* Σ^L *and* Σ^S*, which are generated by the vector operators* $\boldsymbol{L}$ *and* $\boldsymbol{S}$ *as in Eq. (6.50). Assume further that these actions preserve energy in the sense of Eq. (6.51). Then either* Σ^L *or* Σ^S *acts trivially. Thus, either* $\boldsymbol{L} = 0$ *or* $\boldsymbol{S} = 0$*, and consequently the splitting is trivial.*

Proof. If either of Σ^L or Σ^S is an internal action then the result follows from No-Go Theorem 6.1, so consider the case when neither Σ^L or Σ^S are internal. Let $E_{S_r^2}$ be the bundle obtained by restricting the base manifold of E_0 to S_r^2 where $S_r^2 \subseteq \mathbb{R}^3 \setminus \{0\} \cong \mathcal{L}_+$ is the sphere of radius $r > 0$. By the energy preserving condition, Σ^L and Σ^S restrict to SO(3)-actions on $E_{S_r^2}$. Consider the corresponding actions $\sigma^{L,r}$ and $\sigma^{S,r}$ on the base manifold $S_r^2 \cong S^2$. By Lemma 2 (i), for each r, $\sigma^{L,r}$ is either transitive or the identity. Since Σ^L is not internal, there must be some r_0 such that σ^{L,r_0} is transitive. Furthermore, because the action σ^L is continuous, by varying r_0 to any other $r > 0$, we obtain that $\sigma^{L,r}$ is also transitive. The same argument shows that $\sigma^{S,r}$ is transitive for every r.

Fix some $r > 0$. By Lemma 6.2,

$$\sigma^{S,r}_{R_1}\sigma^{L,r}_{R_2} = \sigma^{L,r}_{R_2}\sigma^{S,r}_{R_1} \tag{6.58}$$

for every $R_1, R_2 \in \mathrm{SO}(3)$. Consider an arbitrary but fixed R_1. By Lemma 6.3 (ii), $\sigma^{S,r}_{R_1}$ has some fixed point $y \in S_r^2$. By (6.58),

$$\sigma^{S,r}_{R_1}(\sigma^{L,r}_{R_2} y) = \sigma^{L,r}_{R_2} y, \tag{6.59}$$

showing that $\sigma^{L,r}_{R_2} y$ is a fixed point of $\sigma^{S,r}_{R_1}$ for every R_2. Since $\sigma^{L,r}$ is transitive, $\sigma^{L,r}_{R_2} y$ spans all of S^2_r as R_2 is varied over SO(3). Therefore, every point is a fixed point of $\sigma^{S,r}_{R_1}$, and thus $\sigma^{S,r}_{R_1}$ is the identity. Since this argument can be applied to every R_1, it follows that $\sigma^{S,r}$ is the trivial action. However, $\sigma^{S,r}$ was nontrivial by assumption, so a contradiction has been reached, showing that either Σ^L or Σ^S must be trivial. □

6.7 Non-physical SAM operators arising from gauge redundancy

In this section we give an example of how gauge redundancies in the description of massless particles can lead to the identification of nonphysical SAM operators for massless particles. For concreteness, we consider the case of photons. It is common to describe photons in momentum space using the Lorentz gauge vector potential $A^\mu(k) = (A^0(k), \boldsymbol{A}(k))$, which satisfies the Lorentz gauge condition

$$A^\mu k_\mu = 0. \tag{6.60}$$

Note that while the Lorentz gauge has the advantage of being Lorentz covariant, it is not complete, i.e., there is still residual redundancy in this description [37]. Condition (6.60) determines A^0 in terms of k and $\boldsymbol{A}$, so the potential can be parameterized just by $\boldsymbol{A}(k)$. One can then form the vector potential description of the photon bundle $\pi_\gamma : \gamma \to \mathcal{L}_+$ as the collection of all $(k, \boldsymbol{A})$ where $k \in \mathcal{L}_+$ and $\boldsymbol{A} \in \mathbb{C}^3$. Notice that this bundle is topologically trivial $\gamma \cong \mathcal{L}_+ \times \mathbb{C}^3$ since the fiber does not depend on k. As such, it is similar to a mass $m > 0$ particle with spin 1, which is described by the vector bundle $M_m \times \mathbb{C}^3$. In particular, one can define an SAM operator $\boldsymbol{S}^\gamma$ for γ which acts on the fibers by the spin 1 matrices

$$(S^\gamma_n)_{pq} = -i\epsilon_{npq}. \tag{6.61}$$

By defining $\boldsymbol{L}^\gamma = \boldsymbol{J} - \boldsymbol{S}^\gamma$, which generates rotations in M_m without affecting $\mathbb{C}^3$, it might then seem that we have obtained an SAM-OAM decomposition of the photon angular momentum. Indeed, this was one of the early attempts at such a splitting [65, 67, 68]. However, it is known that this splitting is nonphysical and is only possible due to residual redundancy in the Lorentz gauge. To see this, let $\Sigma^{\gamma,S}$ denote the SO(3) symmetry on γ generated by $\boldsymbol{S}^\gamma$. γ is a rank-3 bundle, but there are only two photon polarizations. A typical way of removing the remaining gauge freedom is to impose the Coulomb gauge condition $\boldsymbol{k} \cdot \boldsymbol{A} = 0$. Applying this condition in each fiber gives the rank-2 Coulomb gauge photon bundle in which all vectors represent physically different states [2], it is equivalent to the description of γ using the electric field $\boldsymbol{E}$. More generally, the remaining gauge freedom can be removed without choosing a specific gauge via the procedure of Asorey et al. [48]. At each fixed $k \in \mathcal{L}_+$, $\boldsymbol{A}$ and $\boldsymbol{A} + \alpha \boldsymbol{k}$ are physically equivalent in the Lorentz gauge for any $\alpha \in \mathbb{C}$, and thus we can use this to define an equivalence relation $\sim$ on each fiber. We then remove all gauge redundancy by forming the rank-2 vector bundle $\gamma/\sim$ by taking the quotient in each fiber. Regardless of how the gauge redundancy is removed, one obtains a rank-2 vector bundle $\tilde{\gamma}$ of physical states. If $\boldsymbol{S}^\gamma$ is a true SAM operator, it must descend to an operator on $\tilde{\gamma}$, generating an SO(3) action $\Sigma^{\tilde{\gamma},S}$ on $\tilde{\gamma}$. This is equivalent to saying that if $(k, \boldsymbol{A})$ and $(k, \boldsymbol{A}')$ are physically equivalent, then $\Sigma_R^{\gamma,S}(k, \boldsymbol{A})$ and $\Sigma_R^{\gamma,S}(k, \boldsymbol{A}')$ must be physically equivalent for any $R \in \mathrm{SO}(3)$. However, this is not possible for the same dimensional reasons that lead to No-Go Theorem 6.1. In particular, $\Sigma^{\tilde{\gamma},S}$ would be an SO(3) representation on the two-dimensional fibers of $\tilde{\gamma}$, but the only such representation is the trivial representation. Thus $\boldsymbol{S}^\gamma$ would also have to be trivial, contradicting Eq. (6.61). Therefore $\boldsymbol{S}^\gamma$ cannot descend to a well-defined operator on the space of physical photon states; it is an artifact of gauge redundancy in the Lorentz gauge. Similarly, massless particles characterized by helicity $\pm h$ with $h > 0$ can together be described by a spin h representation [48]. However, the spin h representation has dimension $2h + 1$, while the the $\pm h$ massless particles only have two independent internal degrees of freedom, and thus there is a dimension $2h - 1$ gauge redundancy in this description. While

this gauge redundant description might appear to give rise to an SAM-OAM decomposition, this decomposition cannot be gauge independent and is thus not physical.

6.8 Single-particle states vs. Fock space

We note that we have obtained the No-Go Theorems by examining the symmetries of the single-particle states rather than the multiparticle Fock space. Under very mild physical assumptions, these treatments are equivalent for the SAM-OAM decomposition problem. Let $\mathcal{H} = L^2(E)$ be the single-particle Hilbert space. The boson Fock space $\mathcal{F}$ is constructed from symmetrized sums of copies of $\mathcal{H}$ as [111]

$$\mathcal{F} = \bigoplus_{n=0}^{\infty} \mathrm{Sym}^n(\mathcal{H}). \tag{6.62}$$

Thus, any symmetry of the single-particle space $\mathcal{H}$ induces a corresponding symmetry of $\mathcal{F}$ given by applying the symmetry to each copy of $\mathcal{H}$. *A priori*, there is the possibility that there exist SO(3) symmetries of the Fock space that do not arise in this manner, that is, actions of SO(3) which mix states of different particle number. Our analysis has not accounted for such symmetries. However, it is reasonable to make the assumption that such exotic symmetries, if they exist, do not describe SAM or OAM operators. Indeed, the total angular momentum operator does not mix states of different particle number, nor do any of the previously proposed SAM or OAM operators for massless particles. For example, see Yang et al.'s [80] review of the various proposed SAM and OAM operators, and note that in every term creation and annihilation operators appear in equal numbers so that particle number is conserved. Indeed, it seems to be implicitly assumed in discussions of the SAM or OAM of elementary particles that these properties are defined at the single-particle level and are not emergent multiparticle phenomena. We also make this assumption.

6.9 Conclusion

The fundamental results of this chapter are No-Go Theorems 6.1 and 6.2, showing that the total angular momentum operator of massless bosons cannot be split into SAM and OAM operators. This result may be surprising given that the angular momentum of massive particles readily decomposes into SAM and OAM. On the other hand, this result could have been anticipated. Despite the relevance of the SAM and OAM of light to optics and photonics and despite sustained interest in this problem, there is no agreement in the literature as to the correct definition of such SAM and OAM operators. This suggests the existence of a fundamental obstruction to such a splitting, one which we showed traces back to the singular limit that occurs as the mass of a particle goes to zero. This limit entails a topological singularity, as the momentum space jumps abruptly from a contractible mass hyperboloid to the noncontractible light cone and the topologically trivial rank-$(2s+1)$ massive vector bundles are replaced by nontrivial massless line bundles. This topological singularity is accompanied by a geometric singularity as the little group jumps from SO(3) for massive particles to SO(2) for massless particles. Particles carry a canonical internal little group symmetry. As SAM is associated with an internal SO(3) symmetry rather than an SO(2) symmetry, this singularity can be viewed as the essential obstruction to an SAM-OAM decomposition.

Furthermore, we have shown that this obstruction is not easily circumvented by generalizing the definition of angular momentum operators. Perhaps the simplest generalization is to allow spin to describe non-internal symmetries, however, No-Go Theorem 6.4 shows that this does not yield nontrivial operators. Another option is to attempt to define SAM and OAM operators which generate non-rotational symmetries; while this is less natural, it has been attempted. Indeed, as Yang et al. [80] showed, most of the proposed SAM and OAM operators do not actually generate 3D rotations (a conclusion also reached by Leader and Lorcé after an extensive review [78]). No-Go Theorem 6.3 shows that in these cases it

is not possible for both the generalized SAM and OAM to generate *any* symmetries, even non-SO(3) symmetries.

These no-go theorems raise the following question. Although photon angular momentum does not split into SAM and OAM, why is it that OAM beams have found such remarkable success in experimental optics? The answer to this question certainly requires additional theoretical work and examination of specific experimental results. However, we suspect that an important point in addressing this question is that while photon angular momentum cannot be split into SAM and OAM, photons can still carry an arbitrarily large amount of angular momentum. We thus hypothesize that many of the useful features of so-called OAM beams result because they carry large amounts of angular momentum, not necessarily because that angular momentum is orbital.

We also note that as we have considered only non-projective representations, this analysis does not directly apply to Weyl fermions. Nevertheless, we suspect that the no-go theorems can be extended to projective representations, and thus to Weyl fermions; future research will explore the possibility of such extensions.

Chapter 7

Zero curvature is a necessary and sufficient condition for a spin-orbital decomposition

This chapter is based on an article [5] published in *Physical Review D*.

7.1 Introduction

There has been an extended controversy [1, 2, 4, 65–74, 77, 78, 80, 81] surrounding a basic question: can the angular momentum operator of massless particles, such as photons and gravitons, be split into well-defined spin and orbital components? This can of course be done for massive particles, but issues immediately arise in the massless case as we have seen in Chapters 3-5. For massive particles, the internal (polarization) and external (momentum) degrees of freedom (DOFs) are independently rotationally symmetric, and the generators of these SO(3) symmetries are the spin (SAM) and orbital angular momentum (OAM) operators, respectively [82]. For massless particles, the situation is different, as the internal and external DOFs are coupled [4]. For example, a photon's momentum $\boldsymbol{k}$ and polarization $\boldsymbol{E}$ must satisfy the transversality constraint $\boldsymbol{k} \cdot \boldsymbol{E} = 0$. $\boldsymbol{k}$ and $\boldsymbol{E}$ can be simultaneously rotated,

and the generator of this symmetry is the total angular momentum operator $\boldsymbol{J} = (J_1, J_2, J_3)$. However, $\boldsymbol{k}$ and $\boldsymbol{E}$ cannot be independently rotated without violating the transversality constraint, that is, without creating unphysical states. This shows that the most straightforward attempt at an SAM-OAM decomposition fails. It has been suggested that instead the decomposition for massless particles should be [71, 72]

$$\boldsymbol{J} = \boldsymbol{J}_\parallel + \boldsymbol{J}_\perp \tag{7.1a}$$

$$\boldsymbol{J}_\parallel = (\boldsymbol{J} \cdot \hat{\boldsymbol{P}})\hat{\boldsymbol{P}} = \chi \hat{\boldsymbol{P}} \tag{7.1b}$$

$$\boldsymbol{J}_\perp = -\frac{1}{H}\boldsymbol{P} \times \boldsymbol{K} \tag{7.1c}$$

with $\boldsymbol{J}_\parallel$ and $\boldsymbol{J}_\perp$ being the SAM and OAM operators, respectively. Here $\boldsymbol{P}$ is the momentum operator, $\boldsymbol{K}$ is the generator of boosts, H is the Hamiltonian, $\hat{\boldsymbol{P}} \doteq \boldsymbol{P}/|\boldsymbol{P}|$, and $\chi \doteq \boldsymbol{J} \cdot \hat{\boldsymbol{P}}$ is the helicity operator. $\boldsymbol{J}_\parallel$ and $\boldsymbol{J}_\perp$ are well-defined vector operators, that is,

$$[J_{\parallel,a}, J_b] = i\epsilon_{abc} J_{\parallel,c} \tag{7.2}$$

$$[J_{\perp,a}, J_b] = i\epsilon_{abc} J_{\perp,c}. \tag{7.3}$$

Furthermore, these operators are defined purely in terms of Poincaré generators, so the expression is coordinate-independent. However, there is the fundamental issue that $\boldsymbol{J}_\parallel$ and $\boldsymbol{J}_\perp$ are not actually angular momentum operators. Indeed, they do not generate rotational SO(3) symmetries, or equivalently, they do not satisfy the $\mathfrak{so}(3)$ angular momentum commutation relations:

$$[J_{\parallel,a}, J_{\parallel_b}] = 0 \neq i\epsilon_{abc} J_{\parallel,c} \tag{7.4}$$

$$[J_{\perp,a}, J_{\perp,b}] = i\epsilon_{abc}(J_{\perp,c} - J_{\parallel,c}) \neq i\epsilon_{abc} J_{\perp,c}. \tag{7.5}$$

This issue was first observed by van Enk and Nienhuis [67]. We call $\boldsymbol{J}_\parallel$ and $\boldsymbol{J}_\perp$ the helicity and orbital quasi-angular momentum, respectively. We showed in the previous chapter [4] that it

is in fact impossible to produce a genuine SAM-OAM splitting for massless particles, as the internal polarization space does not permit any SO(3) symmetry. Any attempted splitting will result in operators which are either not gauge invariant or else fail to satisfy angular momentum commutation relations. Leader [77] pointed out that even though $\boldsymbol{J}_\parallel$ and $\boldsymbol{J}_\perp$ do not satisfy SAM-OAM commutation relations, they appear to be relevant for understanding laser-probe interactions. Yet given that $\boldsymbol{J}_\parallel$ and $\boldsymbol{J}_\perp$ fail to be angular momentum operators, this splitting appears *ad hoc* from a theoretical point of view. We note that the helicity operator $\chi = \boldsymbol{J} \cdot \hat{\boldsymbol{P}}$ has a clear theoretical significance. Indeed, χ is the massless analog of the massive spin operator $\boldsymbol{S}$ in Wigner's classification of particles, generating the massless little group[1] SO(2) in a similar way that $\boldsymbol{S}$ generates the massive little group SO(3) [1, 4, 42, 43]. However, it is the helicity χ, not its "vectorization" $\boldsymbol{J}_\parallel = \chi\hat{\boldsymbol{P}}$, which shows up in Wigner's method. One of the main goals of this chapter is to show that the massless $\boldsymbol{J}_\parallel$, $\boldsymbol{J}_\perp$ splitting does in fact result from a general geometric construction for particles, albeit one that is different from Wigner's little group construction.

To understand this, we recall some of the subtleties of the SAM-OAM splitting for relativistic massive particles. Wigner showed that massive particles are classified by their spin s and that their internal polarization space has dimension $2s+1$ (neglecting DOFs describing nonspacetime symmetries, such as color charge) [42, 43]. It is then possible to choose coordinates such that [82]

$$\boldsymbol{J} = -\boldsymbol{P} \times \boldsymbol{Q}^{NW} + \boldsymbol{S}. \tag{7.6}$$

In this coordinate representation, $\boldsymbol{J}$ acts on $2s+1$ component wave functions in $\boldsymbol{k}$ space, $\boldsymbol{S}$ acts as the $(2s+1) \times (2s+1)$ spin matrices, and $\boldsymbol{Q}^{NW}$ acts as $i\nabla$, the component-wise gradient in $\boldsymbol{k}$ space. It is important to note that this simple description of $\boldsymbol{S}$ and $\boldsymbol{Q}^{NW}$ is only valid in a very specific set of coordinates [148] and it hides considerable complexity.

[1]Technically, the massless little group is the two-dimensional Euclidean group ISO(2). However, if one considers only particles with a finite number of internal DOFs (which includes all particles in the Standard Model), then the inhomogeneous elements of ISO(2) act by the identity [43], so that the effective action is that of SO(2).

Indeed, the coordinate independent descriptions of $\boldsymbol{S}$ and $\boldsymbol{Q}^{NW}$ are much more complicated [148]:

$$\boldsymbol{S} = \frac{1}{m}(H\boldsymbol{J} + \boldsymbol{P} \times \boldsymbol{K}) - \frac{1}{m(H+m)}(\boldsymbol{P} \cdot \boldsymbol{J})\boldsymbol{P} \tag{7.7}$$

$$\boldsymbol{Q}^{NW} = \frac{1}{H}\Big(\boldsymbol{K} - \frac{i\boldsymbol{P}}{2H}\Big) - \frac{1}{mH(H+m)}\boldsymbol{P} \times (H\boldsymbol{J} + \boldsymbol{P} \times \boldsymbol{K}). \tag{7.8}$$

$\boldsymbol{Q}^{NW}$ is known as the Newton–Wigner position operator. By choosing an internal basis of eigenvectors of S_3, one obtains the simpler coordinate representation in terms of spin matrices and gradients [148]. The operator $\boldsymbol{Q}^{NW}$ was defined by Newton and Wigner in their search for a relativistic position operator for massless particles [149] and by Pryce in his investigations of relativistic commuting coordinates [150] and center of mass coordinates [151]. The coordinate independent form (7.8) was derived by Jordan [148] using the Pauli–Lubański vector; a modern derivation is given in Lorcé [152]. In this article we will present a novel and physically intuitive construction of $\boldsymbol{Q}^{NW}$ and the relativistic SAM-OAM splitting (7.6) based on the theory of connections on vector bundles. What is particularly important about this construction is that it also produces the quasi-angular momentum splitting in Eq. (7.1) when applied to massless particles. This shows that a single geometric procedure results in both the known massive and massless splittings, and particularly that the latter is not *ad hoc*. Furthermore, we will see the precise singularity that causes the massless operators to fail to be true angular momentum operators. In contrast, existing derivations of the Newton–Wigner position operator [148, 151, 152] do not generalize in a straightforward way to massless particles.

This chapter is organized as follows. In Sec. 7.2 we will review the vector bundle description of particles. In §7.3 we show how connections on vector bundles induce splittings of $\boldsymbol{J}$, but that only flat connections induce SAM-OAM decompositions. In §7.4 we explore specific examples of such splittings for elementary particles induced by Poincaré symmetry, showing that massive particles admit a unique SAM-OAM splitting which is obstructed when

the mass is taken to zero. We discuss the similarities and differences of this approach with existing methods [151, 152] in §7.4.5.

7.2 The vector bundle description of particles

We will use the vector bundle description of particles developed in previous chapters. The notation is the same with the following exception. We will denote the generators of the Poincare action on the space of wave functions $L^2(E)$ as $(H, \boldsymbol{P}, \boldsymbol{J}, \boldsymbol{K})$, dropping the tilde that was used elsewhere to distinguish these from the action directly on the vector bundle. This is because we will be have little use for the latter. We provide here a review of the vector bundle formalism and notation so that this chapter can be read independently of the rest of the thesis. A basis for the states of an elementary particle are given by the momentum eigenstates (k, v) where $k^\mu = (\omega, \boldsymbol{k})$ is the 4-momentum and v is the internal polarization vector [43]. $\omega = \sqrt{m^2 + |\boldsymbol{k}|^2}$ for a mass m particle, so k^μ is restricted to the mass m hyperboloid $\mathcal{M}_m$ for massive particles or to the (forward) light cone $\mathcal{L}_+$ for massless particles. In either case, the momentum can be labeled just by $\boldsymbol{k}$, so states can also be written as $(\boldsymbol{k}, v)$. Note that with this parametrization $\mathcal{M}_m \cong \mathbb{R}^3$ and $\mathcal{L}_+ \cong \mathbb{R}^3 \setminus \{0\}$; the origin is removed in the latter case because massless particles have no rest frame. The collection of all states $(\boldsymbol{k}_0, v)$ with fixed momentum $\boldsymbol{k}_0$ forms a vector spaces denoted $E(\boldsymbol{k}_0)$. Thus the collection of all states, denoted E, is a family of vector spaces parametrized by $M = \mathcal{M}_m$ or $M = \mathcal{L}_+$. Such a parametrized collection of vector spaces $\pi : E \to M$ is called a vector bundle with base manifold M. π is the canonical projection $\pi(\boldsymbol{k}, v) = \boldsymbol{k}$ and the rank r of the vector bundle is the dimension of the vector spaces $E(\boldsymbol{k})$. The vector space $E(\boldsymbol{k}_0)$ is called the fiber at $\boldsymbol{k}_0$. A section $\psi : M \to E$ is a choice of vector in each fiber, i.e., $\psi(\boldsymbol{k}) \in E(\boldsymbol{k})$. A section describes a superposition of momentum eigenfunctions and can be considered as a wave function. The space of wave functions is denoted $L^2(E)$. Each fiber $E(\boldsymbol{k})$ is actually an inner product space, that is, $\langle(\boldsymbol{k}, v_1), (\boldsymbol{k}, v_2)\rangle$ is well defined. This makes

E into a Hermitian vector bundle. This induces an inner product on $L^2(E)$:

$$\langle \psi_1, \psi_2 \rangle_s = \int \frac{d^3k}{\omega} \langle \psi_1(\boldsymbol{k}), \psi_2(\boldsymbol{k}) \rangle \tag{7.9}$$

where $\omega^{-1} d^3k$ is the Lorentz invariant measure. As suggested by the notation $L^2(E)$, we consider only square-integrable sections with respect to this measure.

Elementary particles correspond to unitary irreducible representations (UIRs) of the (proper orthochronous) Poincaré group $\mathrm{ISO}^+(3,1)$, which consists of all combinations of boosts, rotations, and spacetime translations [42, 43]. Thus, there is a unitary irreducible action Σ on the states in E. For simplicity, we will restrict our discussion to bosons since our focus is on massless particles and there are no known massless fermions. Thus, we will not consider projective representations. For any $\Lambda \in \mathrm{ISO}^+(3,1)$, Σ_Λ is a vector bundle isomorphism on E, linearly mapping a fiber $E(k)$ to $E(\Lambda k)$, where Λk denotes the standard Poincaré action on momentum four-vectors. That is,

$$\Sigma_\Lambda(k, v) = (\Lambda k, \Sigma_\Lambda v). \tag{7.10}$$

Σ is unitary in that it preserves the Hermitian product between vectors. It is also irreducible, meaning there are no proper subbundles of E (with nonzero rank) which are also unitary representations. This makes E into a vector bundle UIR of $\mathrm{ISO}^+(3,1)$. The bundle representation induces a corresponding vector space UIR $\tilde{\Sigma}$ on the space of wave function $L^2(E)$, namely, [1, 47]

$$(\tilde{\Sigma}_\Lambda \psi)(\boldsymbol{k}) = \Sigma_\Lambda[\psi(\Sigma_{\Lambda^{-1}} \boldsymbol{k})]. \tag{7.11}$$

The generators of $\tilde{\Sigma}$ are the Hamiltonian H, momentum $\boldsymbol{P} = (P_1, P_2, P_3)$, angular momentum $\boldsymbol{J}$, and boost $\boldsymbol{K}$ operators. In particular,

$$J_a\psi = i\frac{d}{dt}\Big|_{t=0}\tilde{\Sigma}_{R_a(t)}\psi \tag{7.12}$$

$$K_a\psi = -i\frac{d}{dt}\Big|_{t=0}\tilde{\Sigma}_{\Lambda_a(t)}\psi \tag{7.13}$$

$$\tag{7.14}$$

where $R_a(t) \in \mathrm{SO}(3) \subseteq \mathrm{SO}^+(3,1)$ is a rotation by t about the a axis and $\Lambda_a(t)$ is a boost with rapidity $t\boldsymbol{e}_a$. Similarly, H and $\boldsymbol{P}$ are the generators of time and space translations, given by

$$\boldsymbol{P}\psi(\boldsymbol{k}) = \boldsymbol{k}\psi(\boldsymbol{k}) \tag{7.15}$$

$$H\psi(\boldsymbol{k}) = \sqrt{m^2 + |\boldsymbol{k}|^2}\psi(\boldsymbol{k}). \tag{7.16}$$

The generators satisfy the Poincaré algebra commutation relations given in Eq. (A.1) in Appendix A, and in particular, the angular momentum satisfies the $\mathfrak{so}(3)$ relations

$$[J_a, J_b] = i\epsilon_{abc}J_c. \tag{7.17}$$

Using Wigner's little group method [42, 43], the elementary particle bundles can be classified [1, 3]. The massive bundle representation are labeled by their spin s and are isomorphic to the rank $2s+1$ trivial bundles $\pi : \mathcal{M}_m \times \mathbb{C}^{2s+1} \to \mathcal{M}_m$. Massless bosons, on the other hand, are all line bundles (rank 1 bundles) over the light cone and are distinguished by a difference in topology. They are labeled by their integer helicity h and are isomorphic to line bundles $\pi : \gamma_h \to \mathcal{L}_+ \cong \mathbb{R}^3 \setminus \{0\}$. Line bundles over $\mathcal{L}_+$ are completely characterized by their (first) Chern number C, and the γ_h can be abstractly described as the line bundles with $C(\gamma_h) = -2h$. C can be considered as a measure of how topologically nontrivial or globally "twisted" the state space is [101].

7.3 Connection induced angular momentum decompositions

Our goal is to decompose the angular momentum operator into two vector operators

$$\boldsymbol{J} = \boldsymbol{L} + \boldsymbol{S}. \tag{7.18}$$

That $\boldsymbol{L}$ and $\boldsymbol{S}$ are vector operators just means that they rotate as vectors under $\mathrm{SO}(3) \subseteq \mathrm{ISO}^+(3,1)$, or equivalently, that they satisfy [83]

$$[L_a, J_b] = i\epsilon_{abc} L_C \tag{7.19}$$

$$[S_a, J_b] = i\epsilon_{abc} S_C. \tag{7.20}$$

Ideally, we would like $\boldsymbol{L}$ and $\boldsymbol{S}$ to be angular momentum operators, that is, to satisfy the $\mathfrak{so}(3)$ angular momentum commutation relations

$$[L_a, L_b] = i\epsilon_{abc} L_c, \tag{7.21}$$

$$[S_a, S_b] = i\epsilon_{abc} S_c. \tag{7.22}$$

In nonrelativistic quantum mechanics, the orbital angular momentum operator is given in the momentum basis by $-i\boldsymbol{k} \times \nabla$ where ∇ is the momentum space gradient on scalar functions. To generalize this to the current setting of vector bundle representations of particles, we must find a suitable replacement for the operator ∇. However, ∇ is not uniquely defined on sections of vector bundles even when the base manifold M is identified with an open subset of $\mathbb{R}^3$ as in $\mathcal{M}_m \cong \mathbb{R}^3$ or $\mathcal{L}_+ \cong \mathbb{R}^3 \setminus \{0\}$ [99]. The core issue is that there is not a unique way to identify the vectors at different base points in M (even if one specifies a path on M between the points). As such, there is ambiguity in how one measures the change in a wave function $\psi(\boldsymbol{k}) \in L^2(E)$ as $\boldsymbol{k}$ is varied. It turns out that there are many inequivalent

ways to remove this ambiguity, and each describes a connection $\boldsymbol{D}$ on the vector bundle E. Such connections can be thought of as covariant derivatives, and indeed, they generalize the affine connections that occur in general relativity; a detailed treatment of connections on vector bundles can be found in Ref. [99]. The essential program will be to replace ∇ with connections on vector bundles and study the resultant splittings of the angular momentum. Technically, a connection is defined as follows:

Definition 7.1 ([99])**.** *Let $\mathfrak{X}(M)$ denote the space of vector fields $X : M \to TM$ on M. A connection $\boldsymbol{D}$ on $\pi : E \to M$ is a map*

$$\boldsymbol{D} : \mathfrak{X}(M) \times L^2(E) \to L^2(E)$$
$$(X, \psi) \mapsto \boldsymbol{D}_X \psi$$

such that for any smooth function $f : M \to \mathbb{C}$, $\boldsymbol{D}_X\psi$ is f-linear in X and satisfies the Leibniz rule in ψ:

$$\boldsymbol{D}_{fX}(\psi) = f\boldsymbol{D}_X\psi \tag{7.23}$$
$$\boldsymbol{D}_X(f\psi) = df(X)\psi + f\boldsymbol{D}_X\psi \tag{7.24}$$

where $df : TM \to \mathbb{C}$ is the differential of f. We also write $\boldsymbol{D}_X$ as $\boldsymbol{X} \cdot \boldsymbol{D}$; this can be thought of as the derivative in direction $\boldsymbol{X}$.

In the case when M is a an open subset of $\mathbb{R}^3$ and $\boldsymbol{Y} \in \mathfrak{X}(M)$ is a vector field, we define the operator

$$\boldsymbol{Y} \times \boldsymbol{D} : \mathfrak{X}(M) \times L^2(E) \to L^2(E) \tag{7.25}$$

such that for $X \in \mathfrak{X}(M)$,

$$\boldsymbol{X} \cdot (\boldsymbol{Y} \times \boldsymbol{D})\psi = [(\boldsymbol{X} \times \boldsymbol{Y}) \cdot \boldsymbol{D}]\psi = \boldsymbol{D}_{\boldsymbol{X}\times\boldsymbol{Y}}\psi. \tag{7.26}$$

Given a vector field $\boldsymbol{X}(\boldsymbol{k}) = v^a(\boldsymbol{k})\boldsymbol{e}_a$ on an open subset of $\mathbb{R}^3$, where $\boldsymbol{e}_a$ are an orthogonal basis of $\mathbb{R}^3$, we can define the operator on $L^2(E)$

$$\boldsymbol{X} \cdot \boldsymbol{K} = v^a K_a \tag{7.27}$$

and similarly for $\boldsymbol{J}$ and $\boldsymbol{P}$. In this sense, we can think of the generators $\boldsymbol{K}$, $\boldsymbol{J}$, and $\boldsymbol{P}$ as having the same domain and range as a connection:

$$\boldsymbol{K} : \mathfrak{X}(M) \times L^2(E) \to L^2(E), \tag{7.28}$$

with the Cartesian components K_a corresponding to the constant vector fields $\boldsymbol{e}_a$, $K_a = \boldsymbol{e}_a \cdot \boldsymbol{K}$. Note also that

$$[(\boldsymbol{P} \times \boldsymbol{D})\psi] = (\boldsymbol{k} \times \boldsymbol{D})\psi \tag{7.29}$$

where the $\boldsymbol{k}$ on the rhs is considered as a vector field.

Ideally, we would like for $\boldsymbol{Q} \doteq i\boldsymbol{D}$ to be resemble a position operator. One of the necessary requirements for this correspondence is that $\boldsymbol{Q}$ be Hermitian. In particular, we will ensure that $\boldsymbol{Q}$ is constructed out a Hermitian combination of the Poincaré generators $(H, \boldsymbol{P}, \boldsymbol{K}, \boldsymbol{J})$ (which are themselves Hermitian). We will call $\boldsymbol{D}$ an anti-Hermitian connection if this holds. We will also be interested in the cases when $\boldsymbol{D}$ possesses rotational symmetry:

Definition 7.2. *The connection $\boldsymbol{D}$ is rotationally symmetric if*

$$\boldsymbol{D}_{R\boldsymbol{X}} = \Sigma_R \boldsymbol{D}_{\boldsymbol{X}} \Sigma_{R^{-1}} \tag{7.30}$$

for any $R \in \mathrm{SO}(3)$, or equivalently ([106], Proposition C.7), if $\boldsymbol{D}$ is a vector operator:

$$[\boldsymbol{D}_a, \boldsymbol{J}_b] = i\epsilon_{abc}\boldsymbol{D}_c. \tag{7.31}$$

We will later discuss particular choices of a connection on E. For now, assume we have chosen a connection $\boldsymbol{D}$. We then define the splitting of $\boldsymbol{J}$ induced by $\boldsymbol{D}$:

$$\boldsymbol{L}^{\boldsymbol{D}} \doteq -i\boldsymbol{P} \times \boldsymbol{D} = -i\boldsymbol{k} \times \boldsymbol{D} \tag{7.32}$$

$$\boldsymbol{S}^{\boldsymbol{D}} \doteq \boldsymbol{J} - \boldsymbol{L}^{\boldsymbol{D}} \tag{7.33}$$

where $\boldsymbol{k}$ is considered as a vector field on M, that is, $\boldsymbol{k} = |\boldsymbol{k}|\boldsymbol{e}_k$ where $\boldsymbol{e}_k$ is the radially directed unit vector field on M. We will explore the extent to which $\boldsymbol{S}^{\boldsymbol{D}}$ and $\boldsymbol{L}^{\boldsymbol{D}}$ can be considered SAM and OAM operators. In particular, we will show how the properties of $\boldsymbol{D}$ determine if they are vector operators and satisfy angular momentum commutation relations.

Theorem 7.1. *Suppose $\boldsymbol{D}$ is rotationally symmetric. Then $\boldsymbol{L}^{\boldsymbol{D}}$ and $\boldsymbol{S}^{\boldsymbol{D}}$ are vector operators:*

$$[L^{\boldsymbol{D}}_m, J_n] = i\epsilon_{mnp} L^{\boldsymbol{D}}_p \tag{7.34}$$

$$[S^{\boldsymbol{D}}_m, J_n] = i\epsilon_{mnp} S^{\boldsymbol{D}}_p \tag{7.35}$$

where the components are defined with respect to some right-handed orthonormal basis of $(\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$ of $\mathbb{R}^3$, that is, $L^{\boldsymbol{D}}_m \doteq \boldsymbol{e}_m \cdot \boldsymbol{L}^{\boldsymbol{D}}$ and $S^{\boldsymbol{D}}_m \doteq \boldsymbol{e}_m \cdot \boldsymbol{S}^{\boldsymbol{D}}$.

Proof. Let $\boldsymbol{v}$ be a constant vector field on M. Then by rotational symmetry

$$\boldsymbol{D}_{R\boldsymbol{v}\times\boldsymbol{k}} = \tilde{\Sigma}_R \boldsymbol{D}_{\boldsymbol{v}\times R^{-1}\boldsymbol{k}} \tilde{\Sigma}_{R^{-1}} = \tilde{\Sigma}_R \boldsymbol{D}_{\boldsymbol{v}\times\boldsymbol{k}} \tilde{\Sigma}_{R^{-1}} \tag{7.36}$$

where we have used the fact that the vector field $\boldsymbol{k} = |\boldsymbol{k}|\boldsymbol{e}_k$ is rotationally invariant. Thus by Eq. (7.26):

$$R\boldsymbol{v} \cdot (\boldsymbol{k} \times \boldsymbol{D}) = \tilde{\Sigma}_R \boldsymbol{v} \cdot (\boldsymbol{k} \times \boldsymbol{D}) \tilde{\Sigma}_{R^{-1}} \tag{7.37}$$

$$(R\boldsymbol{v}) \cdot \boldsymbol{L}^{\boldsymbol{D}} = \tilde{\Sigma}_R (\boldsymbol{v} \cdot \boldsymbol{L}^{\boldsymbol{D}}) \tilde{\Sigma}_{R^{-1}} \tag{7.38}$$

Let $R_n(t)$ denote the rotation by angle t about $\boldsymbol{e}_n$. Then

$$\frac{d}{dt}\Big|_{t=0}(R_n(t)\boldsymbol{e}_m)\cdot\boldsymbol{L}^{\boldsymbol{D}} = \frac{d}{dt}\Big|_{t=0}\tilde{\Sigma}_{R_n(t)}(\boldsymbol{e}_m\cdot\boldsymbol{L}^{\boldsymbol{D}})\tilde{\Sigma}_{R_n(-t)} \tag{7.39}$$

$$(\boldsymbol{e}_n\times\boldsymbol{e}_m)\cdot\boldsymbol{L}^{\boldsymbol{D}} = \frac{d}{dt}\Big|_{t=0}e^{-i(\boldsymbol{e}_n\cdot\boldsymbol{J})t}L_m^{\boldsymbol{D}}e^{i(\boldsymbol{e}_n\cdot\boldsymbol{J})t} \tag{7.40}$$

$$-\epsilon_{mnp}L_p^{\boldsymbol{D}} = -i(J_nL_m^{\boldsymbol{D}} - L_m^{\boldsymbol{D}}J_n) \tag{7.41}$$

$$i\epsilon_{mnp}L_p^{\boldsymbol{D}} = [L_m^{\boldsymbol{D}}, J_n] \tag{7.42}$$

proving that $\boldsymbol{L}^{\boldsymbol{D}}$ is a vector operator. Since $\boldsymbol{J}$ and $\boldsymbol{L}^{\boldsymbol{D}}$ are vector operators, $\boldsymbol{S}^{\boldsymbol{D}}$ is also a vector operator. □

We note that the rotational invariance of $\boldsymbol{D}$ is a very mild condition and is expected of any useful connection; it essentially imposes the condition that $\boldsymbol{D}$ be constructed from vector operators. We now turn to the question of whether $\boldsymbol{S}^{\boldsymbol{D}}$ is an internal operator as one would expect for an SAM operator. An operator $\bar{\boldsymbol{S}}$ on a vector bundle $\pi : E \to M$ is internal if it preserves fibers, that is, if it only changes the polarization but not the momentum. The corresponding operator $\boldsymbol{S}$ on $L^2(E)$ is thus internal if for any ψ and $\boldsymbol{k}_0$, $[\boldsymbol{S}\psi](\boldsymbol{k}_0)$ depends only on $\psi(\boldsymbol{k}_0)$ and not on ψ at any other $\boldsymbol{k}$. This is equivalent to the condition that $\boldsymbol{S}$ is a point operator on $L^2(E)$ [99].

Theorem 7.2. *$\boldsymbol{S}^{\boldsymbol{D}}$ is a an internal operator for any connection $\boldsymbol{D}$.*

Proof. By Lemma 7.23 in Ref. [99], to show $\boldsymbol{S}^{\boldsymbol{D}}$ is internal it is sufficient to show that for any $\psi \in L^2(E)$ and any smooth function $f : M \to \mathbb{C}$,

$$\boldsymbol{S}^{\boldsymbol{D}}(f\psi) = f\boldsymbol{S}^{\boldsymbol{D}}(\psi), \tag{7.43}$$

and for this it is sufficient to prove each component of $\boldsymbol{S^D}$ is f-linear. We have

$$[J_m(f\psi)](\boldsymbol{k}) = i\frac{d}{dt}\Big|_{t=0}[\tilde{\Sigma}_{R_m(t)}(f\psi)](\boldsymbol{k}) \tag{7.44}$$

$$= i\frac{d}{dt}\Big|_{t=0}\Sigma_{R_m(t)}\big[f\big(R_m(-t)\boldsymbol{k}\big)\psi\big(R_m(-t)\boldsymbol{k}\big)\big] \tag{7.45}$$

$$= if(\boldsymbol{k})[\tilde{\Sigma}_{R_m(t)}\psi](\boldsymbol{k}) + i\psi(\boldsymbol{k})\frac{d}{dt}\Big|_{t=0}f\big(R_m(-t)\boldsymbol{k}\big) \tag{7.46}$$

$$= f(\boldsymbol{k})[J_m\psi](\boldsymbol{k}) + i\psi(\boldsymbol{k})\frac{d}{dt}\Big|_{t=0}f\big(\boldsymbol{k} - \boldsymbol{e}_m \times \boldsymbol{k}t + O(t^2)\big) \tag{7.47}$$

so that

$$J_m(f\psi) = fJ_m\psi - i\psi(\boldsymbol{k})df(\boldsymbol{e}_m \times \boldsymbol{k}). \tag{7.48}$$

By the Leibniz rule for $\boldsymbol{D}$,

$$L_m^{\boldsymbol{D}}(f\psi) = -i(\boldsymbol{k} \times \boldsymbol{D})_m(f\psi) \tag{7.49}$$

$$= -i\boldsymbol{D}_{\boldsymbol{e}_m \times \boldsymbol{k}}(f\psi) \tag{7.50}$$

$$= fL_m^{\boldsymbol{D}}\psi - i\psi df(\boldsymbol{e}_m \times \boldsymbol{k}). \tag{7.51}$$

Thus,

$$S_m^{\boldsymbol{D}}(f\psi) = J_m(f\psi) - L_m^{\boldsymbol{D}}(f\psi) \tag{7.52}$$

$$= f(J_m\psi - L_m^{\boldsymbol{D}}\psi) \tag{7.53}$$

$$= fS_m^{\boldsymbol{D}}(\psi), \tag{7.54}$$

proving the result. □

We now address the fundamental question of whether or not $\boldsymbol{S^D}$ and $\boldsymbol{L^D}$ satisfy $\mathfrak{so}(3)$ commutation relations so that they are valid angular momentum operators. In contrast to the previous two results, this imposes a very stringent condition on $\boldsymbol{D}$—one which we will show cannot be satisfied for massless particles. As for the affine connections encountered in

general relativity, the curvature of a connection $\boldsymbol{D}$ is defined as the map [99]

$$F_{\boldsymbol{D}} : \mathfrak{X}(M) \times \mathfrak{X}(M) \times L^2(E) \to L^2(E)$$
$$F_{\boldsymbol{D}}(\boldsymbol{X}, \boldsymbol{Y})\psi = (\boldsymbol{D}_{\boldsymbol{X}}\boldsymbol{D}_{\boldsymbol{Y}} - \boldsymbol{D}_{\boldsymbol{Y}}\boldsymbol{D}_{\boldsymbol{X}} - \boldsymbol{D}_{[\boldsymbol{X},\boldsymbol{Y}]})\psi \tag{7.55}$$

where $[\boldsymbol{X}, \boldsymbol{Y}]$ is the Jacobi-Lie bracket of vector fields. The curvature is most intuitively understood in terms of holonomy. That is, one can think of $F_{\boldsymbol{D}}(\boldsymbol{X}, \boldsymbol{Y})$ as measuring the change in a vector which is parallel transported via $\boldsymbol{D}$ along an infinitesimal loop in M in the plane specified by $\boldsymbol{X}$ and $\boldsymbol{Y}$. We note here the important fact that $F_{\boldsymbol{D}}$ is defined pointwise in all three arguments ([99], Sec. 10.3), so $F_{\boldsymbol{D}}(\boldsymbol{u_1}, \boldsymbol{u_2})v$ is well defined for $\boldsymbol{u}_1, \boldsymbol{u}_2 \in T_{\boldsymbol{k}}M$ and $v \in E(\boldsymbol{k})$. The same is not true of $\boldsymbol{D}$ which is defined pointwise in $\mathfrak{X}(M)$ but only locally in $L^2(E)$, that is, to define $[D_{\boldsymbol{u}}\psi](\boldsymbol{k})$, the section ψ must be specified on a neighborhood of $\boldsymbol{k}$. The connection $\boldsymbol{D}$ is called flat if $F_{\boldsymbol{D}}$ vanishes identically. Let $S^2_r \subseteq \mathbb{R}^3$ denote the sphere of radius r in $\boldsymbol{k}$ space, and let $E|_{S^2_r}$ be the bundle obtained by restricting the base manifold of E to S^2_r. Physically, this corresponds to considering monochromatic waves with energy $\sqrt{m^2 + r^2}$. We then define a slightly weaker notion of flatness of a connection:

Definition 7.3 (S^2-flat connections)**.** *A connection $\boldsymbol{D}$ on E is S^2-flat if it is flat on $E|_{S^2_r}$ for every $r > 0$. Equivalently, $\boldsymbol{D}$ is S^2-flat if $F_{\boldsymbol{D}}(\boldsymbol{X}, \boldsymbol{Y}) = 0$ for all vector fields transverse to $\boldsymbol{k}$:*

$$\boldsymbol{X}(\boldsymbol{k}) \cdot \hat{\boldsymbol{k}} = 0 \tag{7.56a}$$
$$\boldsymbol{Y}(\boldsymbol{k}) \cdot \hat{\boldsymbol{k}} = 0. \tag{7.56b}$$

Remarkably, the condition that $\boldsymbol{S^D}$ and $\boldsymbol{L^D}$ satisfy angular momentum relations is equivalent to the condition that the connection $\boldsymbol{D}$ is S^2-flat.

Theorem 7.3. *$\boldsymbol{S^D}$ and $\boldsymbol{L^D}$ satisfy the angular momentum commutation relations*

$$[S_a^{\boldsymbol{D}}, S_b^{\boldsymbol{D}}] = i\epsilon_{abc} S_c^{\boldsymbol{D}} \tag{7.57}$$

$$[L_a^{\boldsymbol{D}}, L_b^{\boldsymbol{D}}] = i\epsilon_{abc} L_c^{\boldsymbol{D}} \tag{7.58}$$

if and only if the connection $\boldsymbol{D}$ is S^2-flat.

Proof. We first note that since $[J_a, J_b] = i\epsilon_{abc} J_c$, Eqs. (7.57) and (7.58) are equivalent, so it suffices to consider only the commutation relations for $\boldsymbol{L^D}$. In general, we have

$$[L_a^{\boldsymbol{D}}, L_b^{\boldsymbol{D}}] = -(\boldsymbol{D}_{\boldsymbol{e}_a \times \boldsymbol{k}} \boldsymbol{D}_{\boldsymbol{e}_b \times \boldsymbol{k}} - \boldsymbol{D}_{\boldsymbol{e}_b \times \boldsymbol{k}} \boldsymbol{D}_{\boldsymbol{e}_a \times \boldsymbol{k}}) \tag{7.59}$$

$$= -\boldsymbol{D}_{[\boldsymbol{e}_a \times \boldsymbol{k}, \boldsymbol{e}_b \times \boldsymbol{k}]} - F_{\boldsymbol{D}}(\boldsymbol{e}_a \times \boldsymbol{k}, \boldsymbol{e}_b \times \boldsymbol{k}). \tag{7.60}$$

By direct computation $[\boldsymbol{e}_a \times \boldsymbol{k}, \boldsymbol{e}_b \times \boldsymbol{k}] = -\epsilon_{abc}(\boldsymbol{e}_c \times \boldsymbol{k})$, so

$$[L_a^{\boldsymbol{D}}, L_b^{\boldsymbol{D}}] = \epsilon_{abc} \boldsymbol{D}_{\boldsymbol{e}_c \times \boldsymbol{k}} - F_{\boldsymbol{D}}(\boldsymbol{e}_a \times \boldsymbol{k}, \boldsymbol{e}_b \times \boldsymbol{k}) \tag{7.61}$$

$$= i\epsilon_{abc} L_c^{\boldsymbol{D}} - F_{\boldsymbol{D}}(\boldsymbol{e}_a \times \boldsymbol{k}, \boldsymbol{e}_b \times \boldsymbol{k}). \tag{7.62}$$

We thus see that $\boldsymbol{L^D}$ satisfies angular momentum commutation relations if and only if

$$F_{\boldsymbol{D}}(\boldsymbol{e}_a \times \boldsymbol{k}, \boldsymbol{e}_b \times \boldsymbol{k}) = 0 \tag{7.63}$$

for each a, b. Since $F_{\boldsymbol{D}}$ is defined pointwise in all arguments, this condition is equivalent to

$$F_{\boldsymbol{D}}(\boldsymbol{X} \times \boldsymbol{k}, \boldsymbol{Y} \times \boldsymbol{k}) = 0 \tag{7.64}$$

for any vector fields $\boldsymbol{X}, \boldsymbol{Y}$, and this in turn is equivalent to

$$F_{\boldsymbol{D}}(\boldsymbol{X}^\perp, \boldsymbol{Y}^\perp) = 0 \tag{7.65}$$

for all vector fields $\boldsymbol{X}^\perp$ and $\boldsymbol{Y}^\perp$ satisfying the S^2-transversality condition Eq. (7.56). Thus, the angular momentum commutation relations Eqs. (7.57) and (7.58) are satisfied if and only if $\boldsymbol{D}$ is S^2-flat. □

7.4 Applications

7.4.1 Massless particles do not admit an SAM-OAM decomposition

By Theorem 7.3 a valid SAM-OAM decomposition of the form (7.32)-(7.33) requires the existence of an S^2-flat connection. No such connection exists for massless particles with helicity $h \neq 0$.

Theorem 7.4 (Massless SAM-OAM no-go theorem)**.** *The massless particle bundle γ_h with $h \neq 0$ admits no SAM-OAM decomposition of the form $\boldsymbol{J} = \boldsymbol{L}^{\boldsymbol{D}} + \boldsymbol{S}^{\boldsymbol{D}}$ where $\boldsymbol{L}^{\boldsymbol{D}} = -i\boldsymbol{k} \times \boldsymbol{D}$ and $\boldsymbol{D}$ is a connection.*

Proof. The massless particle bundle of a helicity h particle is isomorphic to $\pi : \gamma_h \to \mathbb{R}^3 \setminus \{0\}$ and has (first) Chern number $C(\gamma_h) = -2h$ [3]. When calculating the Chern number, we can restrict the base manifold to the unit sphere S^2, that is, $C(\gamma_h) = C(\gamma_h|_{S^2})$ [Eq. (2.48)]. The Chern number can be calculated in terms of any connection $\boldsymbol{D}$ on γ_h, even though the result is independent of the choice of connection (cf. §2.2.2 or [99]). In particular, smoothly choose unit vectors $\boldsymbol{e}_h(\hat{\boldsymbol{k}}) \in \gamma_h|_{S^2}$ for $\hat{\boldsymbol{k}} \in S^2 \setminus p$. We have removed a single point p so that a smooth choice is possible. Then, we can express the curvature in this basis [Eq. (2.20)]:

$$F_{\boldsymbol{D}}(\boldsymbol{X}^\perp, \boldsymbol{Y}^\perp)\boldsymbol{e}_h = \Omega(\boldsymbol{X}^\perp, \boldsymbol{Y}^\perp)\boldsymbol{e}_h \tag{7.66}$$

where Ω is a differential 2-form on $S^2 \setminus p$, called the curvature form of $\boldsymbol{D}$ with respect to $\boldsymbol{e}_h$ [99]. Then by Eq. (2.47)

$$C(\gamma_h) = C(\gamma_h|_{S^2}) = \int_{S^2} \Omega = -2h \neq 0. \tag{7.67}$$

Since the result is nonzero, Ω must be nonzero for some points on S^2. Therefore, there exist $\boldsymbol{X}^\perp$ and $\boldsymbol{Y}^\perp$ such that $F_{\boldsymbol{D}}(\boldsymbol{X}^\perp, \boldsymbol{Y}^\perp) \neq 0$, and thus $\boldsymbol{D}$ is not S^2-flat. By Theorem 7.3, $\boldsymbol{D}$ does not produce an SAM-OAM decomposition. $\boldsymbol{D}$ is a completely arbitrary connection on γ_h, so this proves the theorem. □

This result follows the series of no-go theorems (6.1-6.4) [4] we proved in the previous chapter pertaining to massless SAM-OAM splittings. Strictly speaking, Theorem 7.3 is a special case of no-go Theorem 6.1, which showed that no massless SAM-OAM decomposition is possible, without constraints on the form of SAM and OAM operators. However, the proofs proceed by very different arguments, with that in Ref. [4] based on the classification of vector space UIRs of SO(3), whereas here the argument is based on the nonvanishing curvature of connections. As we will see in the following sections, an advantage of this proof is that it will result in a very clear geometric description of how this obstruction occurs in the massless limit.

7.4.2 The boost connection and the $\boldsymbol{J}_\parallel$, $\boldsymbol{J}_\perp$ splitting

We now show that the quasi-angular momentum splitting [71, 72] discussed in the introduction

$$\boldsymbol{J} = \boldsymbol{J}_\perp + \boldsymbol{J}_\parallel \tag{7.68}$$

$$\boldsymbol{J}_\perp = -\frac{1}{H}\boldsymbol{P} \times \boldsymbol{K} = -\frac{1}{|\boldsymbol{k}|}\boldsymbol{k} \times \boldsymbol{K} \tag{7.69}$$

$$\boldsymbol{J}_\parallel = (\hat{\boldsymbol{P}} \cdot \boldsymbol{J})\hat{\boldsymbol{P}} \tag{7.70}$$

is induced by a connection, but that the connection has nonvanishing curvature, explaining why $\boldsymbol{J}_\parallel$ and $\boldsymbol{J}_\perp$ do not satisfy angular momentum commutation relations.

Theorem 7.5 (Boost connection)**.** *Suppose E is a particle bundle with mass $m \geq 0$. Then*

$$\boldsymbol{D}^K = -\frac{i}{2}\Big(\frac{1}{H}\boldsymbol{K} + \boldsymbol{K}\frac{1}{H}\Big) \tag{7.71}$$

$$= -\frac{i}{H}\Big(\boldsymbol{K} - \frac{i\boldsymbol{P}}{2H}\Big) \tag{7.72}$$

is an anti-Hermitian, rotationally symmetric connection on E which we call the boost connection. For massless particles, this connection induces the separation of $\boldsymbol{J}$ into $\boldsymbol{L}^K = \boldsymbol{J}_\perp$ and $\boldsymbol{S}^K = \boldsymbol{J}_\parallel$ (which are vector operators but not angular momentum operators).

Proof. We will first show that

$$\bar{\boldsymbol{D}}^K = -\frac{i}{H}\boldsymbol{K} \tag{7.73}$$

is a connection. For $\boldsymbol{X} \in \mathfrak{X}(M)$, $\psi \in L^2(E)$, and smooth $f : M \to \mathbb{C}$, we have

$$\bar{\boldsymbol{D}}^K_{f\boldsymbol{X}}\psi = -i\frac{f}{k^0}(\boldsymbol{X}\cdot\boldsymbol{K})\psi \tag{7.74}$$

so $\boldsymbol{D}^K$ is f-linear in X. We now check that the Leibniz rule is satisfied in ψ. Let $\Lambda_{\boldsymbol{v}}$ denote a boost by rapidity $\boldsymbol{v} \in \mathbb{R}^3$. Then

$$[\bar{\boldsymbol{D}}^K_{\boldsymbol{X}}(f\psi)](\boldsymbol{k}) = -\frac{i}{k^0}(\boldsymbol{X}\cdot\boldsymbol{K})[(f\psi)](\boldsymbol{k}) \tag{7.75}$$

$$= \frac{1}{k^0}\frac{d}{dt}\Big|_{t=0}\Big[\tilde{\Sigma}(\Lambda_{-\boldsymbol{X}(\boldsymbol{k})t})(f\psi)\Big](\boldsymbol{k}) \tag{7.76}$$

$$= \frac{1}{k^0}\frac{d}{dt}\Big|_{t=0}\Sigma(\Lambda_{-\boldsymbol{X}(\boldsymbol{k})t})f\big(\Lambda_{\boldsymbol{X}(\boldsymbol{k})t}\boldsymbol{k}\big)\psi(\Lambda_{\boldsymbol{X}(\boldsymbol{k})t}) \tag{7.77}$$

$$= f(\boldsymbol{k})\boldsymbol{D}^K_{\boldsymbol{X}}\psi(\boldsymbol{k}) + \frac{\psi(\boldsymbol{k})}{k^0}\frac{d}{dt}\Big|_{t=0}f\big(\Lambda_{\boldsymbol{X}(\boldsymbol{k})t}\boldsymbol{k}\big). \tag{7.78}$$

Note that

$$(\Lambda_{\boldsymbol{X}(\boldsymbol{k})t}\boldsymbol{k})^m = (\Lambda_{\boldsymbol{X}(\boldsymbol{k})t})^m{}_\mu k^\mu = k^m + k^0 X^m(\boldsymbol{k})t + O(t^2) \tag{7.79}$$

so

$$\bar{\boldsymbol{D}}^K_{\boldsymbol{X}}(f\psi) = f\bar{\boldsymbol{D}}^K_{\boldsymbol{X}}\psi + \psi(\boldsymbol{k})df(\boldsymbol{X}) \tag{7.80}$$

proving that $\bar{\boldsymbol{D}}^K$ satisfies Eq. (7.24) and is thus a connection. Now note that $\frac{\boldsymbol{P}}{H^2}$ is a point operator, i.e., for any functions f, g,

$$\Big[(f\boldsymbol{X})\cdot\frac{\boldsymbol{P}}{H^2}\Big](g\psi) = fg(\boldsymbol{X}\cdot\frac{\boldsymbol{P}}{H^2})\psi. \tag{7.81}$$

It thus follows that

$$\boldsymbol{D}^K = \bar{\boldsymbol{D}}^K - i\frac{\boldsymbol{P}}{H^2} \tag{7.82}$$

also satisfies f-linearity in $\boldsymbol{X}$ and the Leibniz rule in ψ, and is thus a connection. Using Eq. (A.15a), $\boldsymbol{D}^K$ can be expressed in the manifestly anti-Hermitian form

$$\boldsymbol{D}^K = -\frac{i}{2}\Big(\frac{1}{H}\boldsymbol{K} + \boldsymbol{K}\frac{1}{H}\Big). \tag{7.83}$$

$\boldsymbol{D}^K$ is rotationally invariant since $H^{-1}\boldsymbol{K}$ and $\boldsymbol{K}H^{-1}$ are vector operators. In the massless case, this connection induces the splitting of $\boldsymbol{J}$ into $\boldsymbol{L}^K$ and $\boldsymbol{S}^K$ according to Eqs. (7.32)-(7.33), where

$$\boldsymbol{L}^K \doteq -i\boldsymbol{P}\times\boldsymbol{D}^K \tag{7.84}$$

$$= -\frac{1}{H}\boldsymbol{P}\times\boldsymbol{K} \tag{7.85}$$

$$= \boldsymbol{J}_\perp. \tag{7.86}$$

This implies too that $\boldsymbol{S}^K \doteq \boldsymbol{J} - \boldsymbol{L}^K = \boldsymbol{J}_\parallel$. By Theorem 7.4, $\boldsymbol{D}^K$ cannot be S^2-flat when $m = 0$ and thus $\boldsymbol{L}^K$ and $\boldsymbol{S}^K$ are not angular momentum operators. It will be shown in Theorem 7.6 that this is actually true for all $m \geq 0$. □

Intuitively, $\boldsymbol{D}^K$ can be understood in terms of the corresponding parallel transport. If $s(t)$ is a path between $\boldsymbol{k}_0$ and $\boldsymbol{k}_1$, $\boldsymbol{D}^K$ identifies the vectors in $E(\boldsymbol{k}_0)$ with those in $E(\boldsymbol{k}_1)$ by boosting along the path $s(t)$. This connection is defined for both massive and massless particles. Regardless of the particle's mass, this connection is not S^2-flat and thus the operators $\boldsymbol{L}^K$ and $\boldsymbol{S}^K$ induced via Eqs. (7.32) and (7.33) are not angular momentum operators. This is ultimately due to Wigner rotation [42]. Boosts in different directions do not commute ($[K_a, K_b] = -i\epsilon_{abc}J_c$) so boosting around a loop results in a rotation rather than an identity transformation, reflecting the nonzero curvature of this connection.

We now show this more rigorously. The calculation of curvatures is rather delicate. We will calculate the curvature of the boost connection in two ways; the first method is simpler but does not easily generalize. The second method requires more work but allows for more complicated calculations which we will need later in this chapter.

Let $(\boldsymbol{e}_k, \boldsymbol{e}_\theta, \boldsymbol{e}_\phi)$ be the standard spherical unit vectors

$$\boldsymbol{e}_k = \sin(\theta)\cos(\phi)\boldsymbol{e}_1 + \sin(\theta)\sin(\phi)\boldsymbol{e}_2 + \cos(\theta)\boldsymbol{e}_3 \tag{7.87}$$

$$\boldsymbol{e}_\theta = \cos(\theta)\cos(\phi)\boldsymbol{e}_1 + \cos(\theta)\sin(\phi)\boldsymbol{e}_2 - \sin(\theta)\boldsymbol{e}_3 \tag{7.88}$$

$$\boldsymbol{e}_\phi = -\sin(\phi)\boldsymbol{e}_1 + \cos(\phi)\boldsymbol{e}_2. \tag{7.89}$$

It will be useful to use the radial cylindrical unit vector $\boldsymbol{e}_R$ and its relation to the spherical unit vectors:

$$\boldsymbol{e}_R = \cos(\phi)\boldsymbol{e}_1 + \sin(\phi)\boldsymbol{e}_2 = \sin(\theta)\hat{\boldsymbol{k}} + \cos(\theta)\boldsymbol{e}_\theta. \tag{7.90}$$

At $\boldsymbol{k}_0$, let $(\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3 = \hat{\boldsymbol{k}}_0)$ be an orthonormal basis. We will extend $\boldsymbol{u}_a$ as constant vector fields on all of M, specifically, $\boldsymbol{u}_3(\boldsymbol{k}) \neq \boldsymbol{e}_k$ for general $\boldsymbol{k} \neq \boldsymbol{k}_0$.

Theorem 7.6. *For particles of any mass, the boost connection $\boldsymbol{D}^K$ is not S^2-flat, in particular, the S^2-curvature at $\boldsymbol{k}_0$ is given by*

$$F_K(\boldsymbol{u}_1, \boldsymbol{u}_2) = \frac{i}{H^2}J_3. \tag{7.91}$$

This can also be expressed as

$$F_K(\boldsymbol{e}_\theta, \boldsymbol{e}_\phi) = \frac{i}{H^2} J_k. \tag{7.92}$$

Since the connection is not S^2-flat, the operators

$$\boldsymbol{L}^K \doteq -i\boldsymbol{P} \times \boldsymbol{D}^K \tag{7.93}$$

$$\boldsymbol{S}^K \doteq \boldsymbol{J} - \boldsymbol{L}^K \tag{7.94}$$

are not angular momentum operators.

Proof 1. This first proof uses Cartesian components of the generators. We will calculate the curvature at some arbitrary but fixed $\boldsymbol{k}_0$. We orient our axes so that $K_a = \boldsymbol{u}_a \cdot \boldsymbol{K}$ and similarly for other generators. The Cartesian components of $\boldsymbol{D}^K$ are

$$\boldsymbol{D}^K_a \doteq \boldsymbol{D}^K_{\boldsymbol{u}_a} = -i\Big(\frac{1}{H} K_a - i\frac{P_a}{H^2}\Big). \tag{7.95}$$

Since the $\boldsymbol{u}_a$ are constant vector fields, their Jacobi-Lie bracket vanishes: $[\boldsymbol{u}_a, \boldsymbol{u}_b] = 0$. Thus

$$F_K(\boldsymbol{u}_1, \boldsymbol{u}_2) = [\boldsymbol{D}^K_1, \boldsymbol{D}^K_2] \tag{7.96}$$

$$= -\Big[\frac{1}{H} K_1 - i\frac{1}{H^2} P_1, \frac{1}{H} K_2 - i\frac{1}{H^2} P_2\Big]. \tag{7.97}$$

We are interested only in calculating this quantity at $\boldsymbol{k}_0$. We note that while $F_K(\boldsymbol{u}_1, \boldsymbol{u}_2)$ and P_a are defined pointwise, this is not true of the operators $\boldsymbol{D}^K_a$, K_a, and J_a which are locally defined. Thus, even though $P_1 = 0$ at $\boldsymbol{k}_0$, $[P_1, J_2] = iP_3 \neq 0$, illustrating that we must evaluate the commutators before evaluating pointwise. By Eq. (A.15a),

$$[H^{-1}K_1, H^{-1}K_2] = H^{-2}[K_1, K_2] + H^{-1}[K_1, H^{-1}]K_2 + H^{-1}[H^{-1}, K_2]K_1 \tag{7.98}$$

$$= -iH^{-2}J_3 + iH^{-3}(-P_1K_2 + P_2K_1). \tag{7.99}$$

By Eq. (A.15e),

$$[H^{-1}K_1, H^{-2}P_2] = -2iH^{-4}P_1P_2 \tag{7.100}$$

$$[H^{-2}P_1, H^{-1}K_2] = iH^{-4}P_1P_2 \tag{7.101}$$

Thus, evaluating $F_K(\boldsymbol{u}_1, \boldsymbol{u}_2)$ at $\boldsymbol{k}_0$ where $P_1 = P_2 = 0$, we have

$$F_K(\boldsymbol{u}_1, \boldsymbol{u}_2) = -[H^{-1}K_1, H^{-1}K_2] \tag{7.102}$$

$$+ i[H^{-1}K_1, H^{-2}P_2] + i[H^{-2}P_1, H^{-1}K_2] \tag{7.103}$$

$$= iH^{-2}J_3. \tag{7.104}$$

This result holds for any orthonormal basis with $\boldsymbol{u}_3 = \boldsymbol{k}_0$, so we can choose $\boldsymbol{u}_1 = \boldsymbol{e}_\theta(\boldsymbol{k}_0)$ and $\boldsymbol{u}_2 = \boldsymbol{e}_\phi(\boldsymbol{k}_0)$. By varying $\boldsymbol{k}_0$ we obtain

$$F_K(\boldsymbol{e}_\theta, \boldsymbol{e}_\phi) = iH^{-2}J_k. \tag{7.105}$$

□

Proof 2. We can instead directly do the global calculation

$$F_K(\boldsymbol{e}_\theta, \boldsymbol{e}_\phi) = [\boldsymbol{D}^K_{\boldsymbol{e}_\theta}, \boldsymbol{D}^K_{\boldsymbol{e}_\phi}] - \boldsymbol{D}^K_{[\boldsymbol{e}_\theta, \boldsymbol{e}_\phi]}. \tag{7.106}$$

This requires more work but there are critical advantages of this method which we will later exploit. One computation advantage is immediately apparent. If we let $P_\theta = \boldsymbol{e}_\theta \cdot \boldsymbol{P}$, $P_\phi = \boldsymbol{e}_\phi \cdot \boldsymbol{P}$, and similarly for the other generators, then we have

$$[P_\theta \psi](\boldsymbol{k}) = (\boldsymbol{e}_\theta \cdot \boldsymbol{k})\psi(\boldsymbol{k}) = 0. \tag{7.107}$$

So globally $P_\theta = 0$ and likewise $P_\phi = 0$. Therefore,

$$\boldsymbol{D}^K_{\boldsymbol{e}_\theta} = -iH^{-1}K_\theta = -\frac{i}{H}(\cos\theta\cos\phi K_1 + \cos\theta\sin\phi\, K_2 - \sin\theta\, K_3) \tag{7.108a}$$

$$\boldsymbol{D}^K_{\boldsymbol{e}_\phi} = -iH^{-1}K_\phi = -\frac{i}{H}(-\sin\phi\, K_1 + \cos\phi\, K_2). \tag{7.108b}$$

The most important advantage is that while the Cartesian boost generators K_a do not commute with $|\boldsymbol{P}|$ and H, the spherical boosts $K_{\theta,\phi}$ do. Indeed, using Eq. (A.15b) we have

$$[K_{\theta,\phi}, |\boldsymbol{P}|] = \boldsymbol{e}_{\theta,\phi}\cdot[\boldsymbol{K}, |\boldsymbol{P}|] = i\frac{H}{|\boldsymbol{P}|}P_{\theta,\phi} = 0 \tag{7.109}$$

$$[K_{\theta,\phi}, H] = \boldsymbol{e}_{\theta,\phi}\cdot[\boldsymbol{K}, H] = iP_{\theta,\phi} = 0. \tag{7.110}$$

The Jacobi-Lie bracket of $\boldsymbol{e}_\theta$ and $\boldsymbol{e}_\phi$ is

$$[\boldsymbol{e}_\theta, \boldsymbol{e}_\phi] = -\frac{\cot\theta}{|\boldsymbol{k}|}\boldsymbol{e}_\phi. \tag{7.111}$$

Thus,

$$\boldsymbol{D}^K_{[\boldsymbol{e}_\theta,\boldsymbol{e}_\phi]} = -\frac{\cot\theta}{|\boldsymbol{k}|}\boldsymbol{D}^K_{\boldsymbol{e}_\phi} = \frac{i\cot\theta}{|\boldsymbol{k}|H}K_\phi. \tag{7.112}$$

We now turn to the remaining term in Eq. (7.106). Using Eqs. (7.108) and the Leibniz rule, we obtain

$$\boldsymbol{D}^K_\phi \boldsymbol{D}^K_\theta \psi = -\frac{i}{H}\boldsymbol{D}^K_\phi(\cos\theta\cos\phi\, K_1\psi + \cos\theta\sin\phi\, K_2\psi - \sin\theta\, K_3\psi) \tag{7.113}$$

$$\begin{aligned} = -\frac{i}{h}\Big\{ &\cos\theta\cos\phi\, \boldsymbol{D}^K_\phi(K\psi) + \frac{\cos\theta}{\sin\theta\,|\boldsymbol{k}|}\frac{\partial}{\partial\phi}(\cos\phi)(K_1\psi) \\ &+ \cos\theta\sin\phi\, \boldsymbol{D}^K_\phi(K_2\psi) + \frac{\cos\theta}{\sin\theta\,|\boldsymbol{k}|}\frac{\partial}{\partial\phi}(\sin\phi)(K_2\psi) \\ &- \sin\theta \boldsymbol{D}^K_\phi(K_3\psi)\Big\} \end{aligned} \tag{7.114}$$

$$\begin{aligned} = \frac{1}{H^2}\Big(&\cos\theta\cos\phi\sin\phi\, K_1^2 - \cos\theta\cos^2\phi\, K_2K_1 \\ &+ \cos\theta\sin^2\phi\, K_1K_2 - \cos\theta\cos\phi\sin\phi\, K_2^2 \\ &- \sin\theta\sin\phi\, K_1K_3 + \sin\theta\cos\phi\, K_2K_3\Big)\psi + \frac{\cot\theta}{|\boldsymbol{k}|}\boldsymbol{D}^K_{\boldsymbol{e}_\phi}\psi \end{aligned} \tag{7.115}$$

By the same method we find

$$\begin{aligned} \boldsymbol{D}^K_\theta \boldsymbol{D}^K_\phi = \frac{1}{H^2}\Big(&\cos\theta\cos\phi\sin\phi\, K_1^2 + \cos\theta\sin^2\phi K_2K_1 - \sin\theta\sin\phi\, K_3K_1 \\ &- \cos\theta\cos^2\phi\, K_1K_2 - \cos\theta\cos\phi\sin\phi K_2^2 + \sin\theta\cos\phi K_3K_2\Big). \end{aligned} \tag{7.116}$$

From Eqs. (7.115) and (7.116) we obtain

$$\begin{aligned} [\boldsymbol{D}^K_\theta, \boldsymbol{D}^K_\phi] &= \frac{1}{H^2}\Big(- \cos\theta\sin^2\phi\,[K_1,K_2] - \cos\theta\cos^2\phi\,[K_1,K_2] \\ &\qquad - \sin\theta\sin\phi\,[K_3,K_1] + \sin\theta\cos\phi[K_3,K_2]\Big) - \frac{\cot\theta}{|\boldsymbol{k}|}\boldsymbol{D}^K_\phi \\ &= \frac{i}{H^2}(\ \cos\theta\, J_3 + \sin\theta\sin\phi\, J_2 + \sin\theta\cos\phi J_1) - \frac{\cot\theta}{|\boldsymbol{k}|}\boldsymbol{D}^K_\phi \\ &= \frac{i}{H^2}\ \ J_k - \frac{\cot\theta}{|\boldsymbol{k}|}\boldsymbol{D}^K_\phi. \end{aligned} \tag{7.117}$$

Then from Eqs. (7.106), (7.112),and (7.117) we obtain

$$F_K(\boldsymbol{e}_\theta, \boldsymbol{e}_\phi) = \frac{i}{H^2} J_k. \tag{7.118}$$

□

7.4.3 The rotation connection for elementary particles

We have seen that any elementary particle bundle is endowed with a natural anti-Hermitian connection $\boldsymbol{D}^K$ resulting from the Poincaré action, particularly from the boost transformations. However, there is another such connection induced by the rotations. Indeed, for a path along a spherical (fixed energy) surface in $\boldsymbol{k}$ space, one can transport vectors along the path by applying infinitesimal rotations rather than infinitesimal boosts. If the path involves changes in energy, i.e., radial changes in k-space, these are still accomplished by boosts. This gives the intuitive description of the following connection.

Theorem 7.7 (Rotation connection)**.** *Suppose E is a particle bundle with mass $m \geq 0$. Let $\tilde{E}$ be the restriction of this bundle to the base manifold $\mathbb{R}^3 \setminus \{0\}$; note that $\tilde{E} = E$ for massless particles. Then*

$$\boldsymbol{D}^R = -i\left(\frac{\hat{\boldsymbol{P}} \times \boldsymbol{J}}{|\boldsymbol{P}|} + \frac{1}{H}\hat{\boldsymbol{P}}(\hat{\boldsymbol{P}} \cdot \boldsymbol{K}) - \frac{i\boldsymbol{P}}{2H^2}\right) \tag{7.119}$$

$$= -i\left(\frac{\hat{\boldsymbol{P}} \times \boldsymbol{J}}{|\boldsymbol{P}|} - \frac{i\hat{\boldsymbol{P}}}{|\boldsymbol{P}|}\right) - \frac{i}{2}\left(\frac{1}{H}\hat{\boldsymbol{P}}(\hat{\boldsymbol{P}} \cdot \boldsymbol{K}) + (\boldsymbol{K} \cdot \hat{\boldsymbol{P}})\hat{\boldsymbol{P}}\frac{1}{H}\right) \tag{7.120}$$

is an anti-Hermitian connection on $\tilde{E}$, which we call the rotation connection.

Proof. We first show that

$$\bar{\boldsymbol{D}}^R \doteq -i\left(\frac{\hat{\boldsymbol{P}} \times \boldsymbol{J}}{|\boldsymbol{P}|} + \frac{1}{H}\hat{\boldsymbol{P}}(\hat{\boldsymbol{P}} \cdot \boldsymbol{K})\right) \tag{7.121}$$

is a connection. It is trivial that $\bar{\boldsymbol{D}}^R_{f\boldsymbol{X}}\psi = f\bar{\boldsymbol{D}}^R_{\boldsymbol{X}}\psi$. It remains to check that the Leibniz rule (7.24) holds . It suffices to check this when $\boldsymbol{X}$ is everywhere orthogonal or parallel to $\boldsymbol{e}_k$. In the former case, we take $\boldsymbol{X}^\perp$ to an arbitrary vector field such that $\boldsymbol{X}^\perp(\boldsymbol{k})\cdot\boldsymbol{e}_k = 0$ for all $\boldsymbol{k}$. Then

$$\Big[\boldsymbol{X}^\perp\cdot\big[\frac{1}{H}\hat{\boldsymbol{P}}(\hat{\boldsymbol{P}}\cdot\boldsymbol{K})\big]\psi\Big](\boldsymbol{k}) = [H^{-1}(\boldsymbol{X}^\perp\cdot\boldsymbol{e}_k)(\boldsymbol{e}_k\cdot\boldsymbol{K})]\psi = 0 \tag{7.122}$$

so

$$\begin{aligned}
[\bar{\boldsymbol{D}}^R_{\boldsymbol{X}^\perp}(f\psi)](\boldsymbol{k}) &= -\frac{i}{|\boldsymbol{k}|}\big(\boldsymbol{X}^\perp\cdot(\boldsymbol{e}_k\times\boldsymbol{J})\big)[f\psi](\boldsymbol{k}) && (7.123)\\
&= -\frac{i}{|\boldsymbol{k}|}\big((\boldsymbol{X}^\perp\times\boldsymbol{e}_k)\cdot\boldsymbol{J}\big)[f\psi](\boldsymbol{k}) && (7.124)\\
&= \frac{1}{|\boldsymbol{k}|}\left[\frac{d}{dt}\Big|_{t=0} e^{-i(\boldsymbol{X}^\perp\times\boldsymbol{e}_k)\cdot\boldsymbol{J}t}(f\psi)\right](\boldsymbol{k}) && (7.125)\\
&= \frac{1}{|\boldsymbol{k}|}\frac{d}{dt}\Big|_{t=0}\tilde{\Sigma}\big(R_{\boldsymbol{X}^\perp\times\boldsymbol{e}_k}(t)\big)[f\psi](\boldsymbol{k}) && (7.126)\\
&= \frac{1}{|\boldsymbol{k}|}\frac{d}{dt}\Big|_{t=0} f\big(R_{\boldsymbol{X}^\perp\times\boldsymbol{e}_k}(-t)\boldsymbol{k}\big)\Sigma\big(R_{\boldsymbol{X}^\perp\times\boldsymbol{e}_k}(t)\big)\psi\big(R_{\boldsymbol{X}^\perp\times\boldsymbol{e}_k}(-t)\boldsymbol{k}\big) && (7.127)\\
&= f(\boldsymbol{k})\boldsymbol{D}^R_{\boldsymbol{X}^\perp}\psi(\boldsymbol{k}) + \frac{\psi(\boldsymbol{k})}{|\boldsymbol{k}|}\frac{d}{dt}\Big|_{t=0} f\big(\boldsymbol{k} - t(\boldsymbol{X}^\perp\times\boldsymbol{e}_k)\times\boldsymbol{k} + O(t^2)\big) && (7.128)\\
&= f(\boldsymbol{k})\boldsymbol{D}^R_{\boldsymbol{X}^\perp}\psi(\boldsymbol{k}) + \frac{\psi(\boldsymbol{k})}{|\boldsymbol{k}|}\frac{d}{dt}\Big|_{t=0} f\big(\boldsymbol{k} + t|\boldsymbol{k}|\boldsymbol{X}^\perp\big) && (7.129)\\
&= f(\boldsymbol{k})\boldsymbol{D}^R_{\boldsymbol{X}^\perp}\psi(\boldsymbol{k}) + \psi(\boldsymbol{k})df(\boldsymbol{X}^\perp)(\boldsymbol{k}). && (7.130)
\end{aligned}$$

This shows that the Leibniz rule holds when $\boldsymbol{X} = \boldsymbol{X}^\perp$. We now consider the parallel case, where $\boldsymbol{X} = \boldsymbol{e}_k$. In this case, dotting $\boldsymbol{e}_k$ into Eq. (7.121) kills the first term. Then using the fact that $\bar{\boldsymbol{D}}^K$ defined in Eq. (7.73) is a connection, we have

$$\begin{aligned}
[\bar{\boldsymbol{D}}^R_{\boldsymbol{e}_k}(f\psi)](\boldsymbol{k}) &= -\frac{i}{H}(\boldsymbol{e}_k\cdot\boldsymbol{K})[f\psi] = \bar{\boldsymbol{D}}^K_{\boldsymbol{e}_k}(f\psi) && (7.131)\\
&= f\bar{\boldsymbol{D}}^K_{\boldsymbol{e}_k}\psi(\boldsymbol{k}) + \psi(\boldsymbol{k})df(\boldsymbol{e}_k) && (7.132)\\
&= f\bar{\boldsymbol{D}}^R_{\boldsymbol{e}_k}\psi(\boldsymbol{k}) + \psi(\boldsymbol{k})df(\boldsymbol{e}_k), && (7.133)
\end{aligned}$$

showing that $\bar{\boldsymbol{D}}^R$ is a connection. Since

$$-\frac{i\boldsymbol{P}}{2H^2}(f\psi) = -f\frac{i\boldsymbol{P}}{2H^2}(\psi) \tag{7.134}$$

it follows that

$$\boldsymbol{D}^K = \bar{\boldsymbol{D}}^K - \frac{i\boldsymbol{P}}{2H^2} \tag{7.135}$$

also satisfies the Leibniz rule and is thus a connection. By application of the Poincaré commutation relations, we have

$$\frac{1}{H}\hat{\boldsymbol{P}}(\hat{\boldsymbol{P}}\cdot\boldsymbol{K}) = (\boldsymbol{K}\cdot\hat{\boldsymbol{P}})\hat{\boldsymbol{P}}\frac{1}{H} + \frac{i\boldsymbol{P}}{H^2} - \frac{2i\hat{\boldsymbol{P}}}{|\boldsymbol{P}|} \tag{7.136}$$

from which the equivalence of Eqs. (7.119) and (7.120) follows.

$\boldsymbol{D}^R$ is an anti-Hermitian connection if

$$\boldsymbol{Q}^R \doteq i\boldsymbol{D}^R = \frac{1}{|\boldsymbol{P}|^2}\left(\boldsymbol{P}\times\boldsymbol{J} - i\boldsymbol{P}\right) + \frac{1}{2}\left(\frac{1}{H}\hat{\boldsymbol{P}}(\hat{\boldsymbol{P}}\cdot\boldsymbol{K}) + (\boldsymbol{K}\cdot\hat{\boldsymbol{P}})\hat{\boldsymbol{P}}\frac{1}{H}\right) \tag{7.137}$$

is Hermitian. The second term in parenthesis is clearly Hermitian. We have that

$$(\boldsymbol{P}\times\boldsymbol{J})^\dagger_a = \epsilon_{abc}(P_bJ_c)^\dagger = \epsilon_{abc}J_cP_b \tag{7.138}$$

$$= \epsilon_{abc}(P_bJ_c - [P_b, J_c]) \tag{7.139}$$

$$= \epsilon_{abc}(P_bJ_c - i\epsilon_{bcd}P_d) \tag{7.140}$$

$$= \epsilon_{abc}P_aJ_b - 2iP_a \tag{7.141}$$

so

$$(\hat{\boldsymbol{P}}\times\boldsymbol{J})^\dagger = (\hat{\boldsymbol{P}}\times\boldsymbol{J}) - 2i\hat{\boldsymbol{P}} \tag{7.142}$$

so the first term in Eq. (7.137) is also Hermitian. Therefore $\boldsymbol{Q}^K$ is Hermitian. □

The rotation connection induces another splitting of the angular momentum via Eqs. (7.32) and (7.33) with

$$\boldsymbol{L}^R = -i\hat{\boldsymbol{P}} \times (\hat{\boldsymbol{P}} \times \boldsymbol{J}). \tag{7.143}$$

Note that while $\boldsymbol{D}^R$ has a $|\boldsymbol{P}|^{-1}$ singularity at the origin (for massive particles), $\boldsymbol{L}^R$ does not suffer from a diverging singularity. However, $\hat{\boldsymbol{P}}$ is still discontinuous at $\boldsymbol{k} = 0$, so even if ψ is smooth, $\boldsymbol{L}^R\psi$ will typically be discontinuous at the origin. This does not cause fundamental mathematical difficulties since ψ is only defined almost-everywhere in the L^2 theory, but it is still unusual from a physical perspective and one does not expect such singularities in physically meaningful operators. It is thus unsurprising that we will find $\boldsymbol{D}^R$ is also not S^2-flat and thus $\boldsymbol{L}^R$ and $\boldsymbol{S}^R$ are not angular momentum operators.

Theorem 7.8. *The rotation connection is not S^2-flat, in particular,*

$$F^R(\boldsymbol{e}_\theta, \boldsymbol{e}_\phi) = \frac{iJ_k}{|\boldsymbol{P}|^2}. \tag{7.144}$$

Thus, this connection does not induce an SAM-OAM splitting of the total angular momentum by Theorem 7.3.

Proof. Using the second proof technique used for Theorem 7.6, we find

$$[\boldsymbol{D}^R_\theta, \boldsymbol{D}^R_\phi] = \frac{i}{|\boldsymbol{P}|^2}(J_k + \cot\theta\, J_\theta). \tag{7.145}$$

Also,

$$(\boldsymbol{D}^R_{[\boldsymbol{e}_\theta, \boldsymbol{e}_\phi]}\psi)(\boldsymbol{k}) = -\left[\frac{\cot\theta}{|\boldsymbol{P}|}\boldsymbol{D}^R_{\boldsymbol{e}_\phi}\right]\psi(\boldsymbol{k}) = \left[\frac{i\cot\theta}{|\boldsymbol{P}|^2}\boldsymbol{e}_\phi \cdot (\boldsymbol{e}_k \times \boldsymbol{J})\psi\right](\boldsymbol{k}) \tag{7.146}$$

$$= \left[\frac{i\cot\theta}{|\boldsymbol{P}|^2}J_\theta\psi\right](\boldsymbol{k}) \tag{7.147}$$

so

$$\boldsymbol{D}^R_{[\boldsymbol{e}_\theta,\boldsymbol{e}_\phi]} = \frac{i\cot\theta}{|\boldsymbol{P}|^2} J_\theta. \tag{7.148}$$

We thus obtain the nonzero S^2-curvature

$$F^R(\boldsymbol{e}_\theta, \boldsymbol{e}_\phi) = [\boldsymbol{D}^R_\theta, \boldsymbol{D}^R_\phi] - \boldsymbol{D}^R_{[\boldsymbol{e}_\theta,\boldsymbol{e}_\phi]} = \frac{iJ_k}{|\boldsymbol{P}|^2}. \tag{7.149}$$

□

7.4.4 The spin-orbital decomposition for massive particles and its singularity in the massless limit

We have seen that boosts and rotations each induce connections on particle bundles of arbitrary mass. However, neither of these connections are S^2-flat and thus neither gives an SAM-OAM decomposition or a path independent way to identify the internal states at different momenta. We observe though that the S^2-curvatures of these two connections, given in Eqs. (7.92) and (7.144), are very similar, being proportional to J_k and differing only by a factor of

$$\frac{|\boldsymbol{P}|^2}{H^2} = \frac{|\boldsymbol{k}|^2}{|\boldsymbol{k}|^2 + m^2}. \tag{7.150}$$

It is thus natural to ask if these connections can be combined in such a way that the curvatures cancel, producing an S^2-flat connection. We will see that, for massive particles, the answer is yes. A general result from the theory of connections indicates how to proceed. While the sum of two connections is never a connection, any *affine* sum of connections is. That is,

$$\boldsymbol{D}^f \doteq f\boldsymbol{D}^K + (1-f)\boldsymbol{D}^R \tag{7.151}$$

is a connection for any smooth $f \in C^\infty(M)$ ([99], Proposition 10.5). As we wish to construct a rotationally symmetric connection, we will assume that $f = f(|\boldsymbol{k}|)$ depends only on $|\boldsymbol{k}|$.

The S^2-curvature of $\boldsymbol{D}^f$ is given by

$$\begin{aligned} F^f(\boldsymbol{e}_\theta, \boldsymbol{e}_\theta) &= [f\boldsymbol{D}^K_\theta + (1-f)\boldsymbol{D}^R_\theta, f\boldsymbol{D}^K_\phi + (1-f)\boldsymbol{D}^R_\phi] \\ &\quad - f\boldsymbol{D}^K_{[\boldsymbol{e}_\theta,\boldsymbol{e}_\phi]} - (1-f)\boldsymbol{D}^R_{[\boldsymbol{e}_\theta,\boldsymbol{e}_\phi]} \qquad (7.152) \\ &= f^2[\boldsymbol{D}^K_\theta, \boldsymbol{D}^K_\phi] + (1-f)^2[\boldsymbol{D}^R_\theta, \boldsymbol{D}^R_\phi] \\ &\quad + f(1-f)\Big([\boldsymbol{D}^K_\theta, \boldsymbol{D}^R_\phi] + [\boldsymbol{D}^R_\theta, \boldsymbol{D}^K_\phi]\Big) \\ &\quad - f\boldsymbol{D}^K_{[\boldsymbol{e}_\theta,\boldsymbol{e}_\phi]} - (1-f)\boldsymbol{D}^R_{[\boldsymbol{e}_\theta,\boldsymbol{e}_\phi]}. \qquad (7.153) \end{aligned}$$

We note that we have made use of the fact that the connection terms $\boldsymbol{D}^{K,R}_{\theta,\phi}$ commute with $|\boldsymbol{k}|$ and thus with f. This is only true because we are working with the spherical components of the connection; had we worked with Cartesian components this would not be the case and this calculation would be substantially more difficult, if not intractable. The commutators $[\boldsymbol{D}^K_\theta, \boldsymbol{D}^K_\phi]$ and $[\boldsymbol{D}^R_\theta, \boldsymbol{D}^R_\phi]$ were calculated in Eqs. (7.117) and (7.145) and the Jacobi-Lie terms $\boldsymbol{D}^K_{[\boldsymbol{e}_\theta,\boldsymbol{e}_\phi]}$ and $\boldsymbol{D}^K_{[\boldsymbol{e}_\theta,\boldsymbol{e}_\phi]}$ in Eqs. (7.112) and (7.148). We can calculate the remaining commutators using the same method:

$$[\boldsymbol{D}^K_\theta, \boldsymbol{D}^R_\phi] = \frac{iJ_k}{|\boldsymbol{P}|^2} + \frac{i\cot\theta}{H|\boldsymbol{P}|}K_\phi \qquad (7.154)$$

$$[\boldsymbol{D}^R_\theta, \boldsymbol{D}^K_\phi] = \frac{iJ_k}{|\boldsymbol{P}|^2} + \frac{i\cot\theta}{|\boldsymbol{P}|^2}K_\phi. \qquad (7.155)$$

Plugging all of these expressions into Eq. (7.153) gives

$$F^f(\boldsymbol{e}_\theta, \boldsymbol{e}_\phi) = \Big(\frac{f^2}{H^2} + \frac{1-f^2}{|\boldsymbol{P}^2|}\Big)J_k. \qquad (7.156)$$

Thus, if we require $\boldsymbol{D}^f$ to be S^2-flat so as to produce an SAM-OAM decomposition of $\boldsymbol{J}$, then we find two solutions

$$f_\pm(|\boldsymbol{k}|) = \pm\frac{H}{m} = \pm\frac{\sqrt{|\boldsymbol{k}|^2 + m^2}}{m} \qquad (7.157)$$

and we denote the corresponding connections by $\boldsymbol{D}^{\pm}$. Only f_+ produces a nonsingular connection at the origin. Indeed, *a priori* $\boldsymbol{D}^f$ is only a legitimate connection if we remove the origin $\boldsymbol{k} = 0$ due to the $|\boldsymbol{k}|^{-1}$ singularity in $\boldsymbol{D}^R$. However, for $|\boldsymbol{k}| \ll 1$

$$(1 - f_+) \sim -\frac{|\boldsymbol{k}|^2}{2m^2}, \tag{7.158}$$

so that for small $|\boldsymbol{k}|$

$$(1 - f_+)\boldsymbol{D}^R \sim -\frac{i}{2m^2}\Big(\boldsymbol{P} \times \boldsymbol{J} + \frac{1}{H}\boldsymbol{P}(\boldsymbol{P} \cdot \boldsymbol{K}) - \frac{i|\boldsymbol{P}|^2}{2H^2}\boldsymbol{P}\Big) \tag{7.159}$$

which is nonsingular at $\boldsymbol{k} = 0$. On the other hand

$$\lim_{|\boldsymbol{k}|\to 0}(1 - f_-) = 2, \tag{7.160}$$

showing that $\boldsymbol{D}^-$ is singular at the origin. Thus, only $\boldsymbol{D}^+$ gives a globally well-defined flat connection for massive particle bundles.

Plugging in f_+ we find that

$$\boldsymbol{D}^+ = \frac{H}{m}\boldsymbol{D}^K + \Big(1 - \frac{H}{m}\Big)\boldsymbol{D}^R \tag{7.161}$$

$$= -\frac{i}{H}(\boldsymbol{K} - i\boldsymbol{P}/2H) + \frac{i}{Hm(H+m)}\boldsymbol{P} \times (H\boldsymbol{J} + \boldsymbol{P} \times \boldsymbol{J}) \tag{7.162}$$

In this latter form, we find that $\boldsymbol{Q}^{NW} = i\boldsymbol{D}^+$ is precisely the Newton–Wigner position operator in Eq. (7.8). We see that we can recover this operator via a novel method by studying connections on massive particle bundles which produce well-defined SAM-OAM splittings. Our method helps explain the rather complicated expression for $\boldsymbol{Q}^{NW}$: it corresponds to an affine sum of the two natural connections induced by Poincaré symmetry, namely the rotation and boost connections. The affine parameter H/m is the unique choice which pro-

duces a flat connection and, in turn, SAM and OAM operators which satisfy the angular momentum commutation relations.

Importantly, this derivation of the massive SAM-OAM operators gives a very clear picture of the singularity that occurs in the massless limit that prevents such an SAM-OAM decomposition for massless particles. We found that the boost and rotation connections $\boldsymbol{D}^K$ and $\boldsymbol{D}^R$ are anti-Hermitian connections which are naturally defined on particle bundles of any mass. Our method of deriving the flat connection for massive particles relied on the fact that these are two different connections with different curvatures, and thus by taking affine sums we could produce a family of connections with different curvatures. Among this family was a unique globally well-defined flat connection. However, it is a strange fact of relativity that rotations and boosts orthogonal to the momentum degenerate when $m = 0$. In particular, we have (Eq. (3.140); [43], Eq. (2.5.38))

$$\boldsymbol{J}_\perp = -\hat{\boldsymbol{P}} \times \boldsymbol{K} \tag{7.163}$$

$$\boldsymbol{K}_\perp = \hat{\boldsymbol{P}} \times \boldsymbol{J} \tag{7.164}$$

where

$$\boldsymbol{J}_\perp \doteq \boldsymbol{J} - \hat{\boldsymbol{P}}(\hat{\boldsymbol{P}} \cdot \boldsymbol{J}) \tag{7.165}$$

$$\boldsymbol{K}_\perp \doteq \boldsymbol{K} - \hat{\boldsymbol{P}}(\hat{\boldsymbol{P}} \cdot \boldsymbol{K}). \tag{7.166}$$

This is a nonobvious relationship which originates from imposing the condition that massless particles have a finite number of internal degrees of freedom. It would not hold for massless particles with continuous spin, but such particles have never been observed and are not part of the Standard Model ([43], pp. 70-72). This condition says that, infinitesimally, boosting in some direction $\hat{\boldsymbol{v}}$ perpendicular to $\hat{\boldsymbol{k}}$ is equivalent to rotating in the plane containing $\hat{\boldsymbol{v}}$ and $\hat{\boldsymbol{k}}$. Thus, parallel transporting a vector via boosts or rotations will give the same result.

Indeed, for massless particles we have $H = |\boldsymbol{P}|$ and

$$\boldsymbol{K} = \hat{\boldsymbol{P}}(\hat{\boldsymbol{P}} \cdot \boldsymbol{K}) + \hat{\boldsymbol{P}} \times \boldsymbol{J} \tag{7.167}$$

as shown in Eq. (A.12). Plugging these into Eq. (7.72) and comparing with Eq. (7.119) shows that the boost and rotation connections degenerate for massless particles:

$$\boldsymbol{D}^K = \boldsymbol{D}^R. \tag{7.168}$$

There is now a single non-flat connection induced by Poincaré symmetry and one cannot produce a non-flat connection via an affine combination of $\boldsymbol{D}^K$ and $\boldsymbol{D}^R$. Thus, the lack of an SAM-OAM decomposition for massless particles can be traced back to the degeneracy of transverse boosts and rotations.

However, this also shows that there is a theoretical origin of the massless splitting in Eq. (7.1). The most naive splitting of the massless angular momentum leads to operators which are not well-defined as they violate the transversality of EM waves [65]. The $\boldsymbol{J}_\parallel$, $\boldsymbol{J}_\perp$ split was an *ad hoc* fix, solving the transversality issue, but leading to operators which are not legitimate SAM-OAM operators. However, here we show see that these operators naturally arise as the unique splitting induced by Poincaré symmetry, and belongs to the same class of splittings as the well-defined SAM-OAM splitting for massive particles.

7.4.5 Comparison with center of energy and center of mass position operators

Our approach of taking a weighted average of the connections $\boldsymbol{D}^K$ and $\boldsymbol{D}^R$ to obtain a massive SAM-OAM decomposition, and which also produced the Newton–Wigner position operator, is similar in spirit to the approach used by Pryce [151] and Lorcé [152] of averaging different types of generalized position operators (whose components do not commute) to

obtain the Newton–Wigner operator (whose components do commute). Indeed, for mass $m > 0$ particles there is some resemblance in form between $i\boldsymbol{D}^K$ [Eq. (7.72)] and $i\boldsymbol{D}^R$ [Eq. (7.120)] and the centers of energy $\boldsymbol{Q}^E$ ([152], Eq. (8)) and mass $\boldsymbol{Q}^m$([152], Eq. (17)), respectively:

$$\boldsymbol{Q}^E = \mathrm{sym}\Big[t\frac{\boldsymbol{P}}{H} - \frac{\boldsymbol{K}}{H}\Big] \tag{7.169}$$

$$Q^{m,a} = \mathrm{sym}\Big[\Big(t + \frac{\boldsymbol{P}\cdot\boldsymbol{K}}{m^2}\Big)\frac{P^a}{H} - \frac{P_\mu M^{\mu a}}{m^2}\Big]. \tag{7.170}$$

See also Ref. [153] for additional information on the relativistic center of mass and its relationship to massive spin operators. Here, t is a time coordinate, $M^{\mu\nu}$ is the Lorentz tensor with $J^a = \frac{1}{2}\epsilon_{abc}M^{bc}$ and $K^a = M^{0a}$ for $a \in \{1,2,3\}$, and sym denotes the symmetrization of products, e.g.,

$$\mathrm{sym}(\boldsymbol{K}H^{-1}) = \frac{1}{2}(\boldsymbol{K}H^{-1} + H^{-1}\boldsymbol{K}).$$

The Newton–Wigner position operator can be obtained by an appropriate average of these operators ([152], Eq. (22))

$$\boldsymbol{Q}^{NW} = \mathrm{sym}\Big[\frac{H\boldsymbol{Q}^E + m\boldsymbol{Q}^m}{H+m}\Big]. \tag{7.171}$$

While this approach is similar to ours, the centers of energy and mass are not equivalent to $i\boldsymbol{D}^K$ and $i\boldsymbol{D}^R$. A simple way to confirm this is to note that the affine coefficients of $\boldsymbol{Q}^E$ and $\boldsymbol{Q}^M$ used to form the Newton–Wigner position operators in Eq. (7.171) differ from those of $\boldsymbol{D}^K$ and $\boldsymbol{D}^R$ in Eq. (7.161). Another important difference to note is that $\boldsymbol{D}^K$ and $\boldsymbol{D}^R$ are well-defined, and in fact degenerate, for massless particles with $m = 0$. The center of mass, however, is ill defined when $m = 0$ (as one would expect). This difficulty was noted by Pryce [151], who had to adopt an alternative approach to treat massless particles. A critical advantage of the geometric connection-based formalism presented in this chapter is that the same formalism applies to both massive and massless particles, showing how the massive spin

operator and the massless $\boldsymbol{J}_\parallel$ operator can be derived from the same theory. In furthermore allows us to carefully trace the singularity that develops in the massless limit, showing that the degeneration of transverse rotations and boosts degenerate causes an obstruction to a massless SAM-OAM decomposition.

7.5 Conclusion

We have shown that any rotationally symmetric connection $\boldsymbol{D}$ on a particle bundle induces a splitting of the angular momentum operator into two parts $\boldsymbol{S}^{\boldsymbol{D}}$ and $\boldsymbol{L}^{\boldsymbol{D}}$. $\boldsymbol{S}^{\boldsymbol{D}}$ is always internal and $\boldsymbol{L}^{\boldsymbol{D}}$ has the form $-i\boldsymbol{k}\times\boldsymbol{D}$, and in this sense they *resemble* spin and orbital angular momentum operators. However, the defining feature of angular momentum operators is that they satisfy angular momentum commutation relations, and we find that $\boldsymbol{S}^{\boldsymbol{D}}$ and $\boldsymbol{L}^{\boldsymbol{D}}$ do so if and only if the S^2-curvature of $\boldsymbol{D}$ vanishes. Poincaré symmetry induces two connection, corresponding to parallel transport via boosts or rotations. For massive particles these are different connections and there is a unique affine combination of them which produces a well-defined SAM-OAM splitting. The result agrees with the SAM-OAM operators for relativistic massive particles resulting from the Newton–Wigner position operator. The method presented gives a geometric derivation of these rather complicated operators. For massless particles, however, we find that the connections $\boldsymbol{D}^R$ and $\boldsymbol{D}^K$ degenerate, and it is not possible to combine them to produce a flat connection. This gives a geometric picture of why the massless limit obstructs an SAM-OAM splitting for massless particles such as photons and gravitons. Nevertheless, the degeneracy of $\boldsymbol{D}^R$ and $\boldsymbol{D}^K$ means that for massless particles there is a unique splitting of $\boldsymbol{J}$ induced by Poincaré symmetry, namely the $\boldsymbol{J}_\parallel$, $\boldsymbol{J}_\perp$ quasi-angular momentum splitting. This splitting has been previously suggested as an *ad hoc* fix for the SAM-OAM decomposition problem for photons, but it was difficult to interpret or justify since neither $\boldsymbol{J}_\parallel$ nor $\boldsymbol{J}_\perp$ are actually angular momentum operators. Here we show the theoretical origins of these operators.

The methods in this chapter can be extended to the analysis of electromagnetic waves in cold unmagnetized plasma as we will heuristically describe. Introducing a homogeneous isotropic medium introduces a preferred frame, namely the rest frame of the medium. One can always solve for the wave solutions in this rest frame, and doing so removes the boost symmetry of the solution space. The solutions will then only have a rotational SO(3) and translational $\mathbb{R}^{3+1}$ spacetime symmetry, regardless of whether one chooses to work relativistically or nonrelativistically. Thus, the only symmetry-induced connection is the rotation connection which, for transverse waves, is curved. As such, the rotation connection will induce a splitting of the total angular momentum into operators which are not true angular momentum operators. By this reasoning, one can conjecture that most homogeneous isotropic media will not allow an SAM-OAM decomposition for EM waves. We will show this in the case of cold unmagnetized plasma in Chapter 9.

Chapter 8

Spin-weighted spherical harmonics as total angular momentum eigenstates of massless particles

This chapter presents material from Ref. [6] in which we illustrate the relationship between the spin-weighted spherical harmonics and the eigenstates of the (total) angular momentum of massless particles.

8.1 Introduction

We have shown in previous chapters that the state space of an elementary massless particle with helicity h can be described by $L^2(\gamma_h)$, the space of sections (wave functions) over the topologically nontrivial particle bundle $\pi : \gamma_h \to \mathcal{L}_+$. Furthermore, as an elementary particle, it carries a unitary irreducible representation (UIR) of the Poincaré group $\mathrm{ISO}^+(3,1)$ of spacetime symmetries, and in particular, it also a representation of the rotation group $\mathrm{SO}(3) \subseteq \mathrm{ISO}^+(3,1)$ whose generators are the angular momentum operators $\boldsymbol{J}$. One can ask the basic and natural question: what are the angular momentum eigenstates, or more specifically, what are the eigenstates of J^2 and J_z? One can phrase this question slightly

differently. One might immediately note that $L^2(\gamma_h)$ is an infinite-dimensional vector space. However, there are no irreducible infinite-dimensional representations of the rotation group SO(3) since it is compact ([124], Theorem 3.9). Thus, $L^2(\gamma_h)$ is not an irreducible representation of SO(3) and must decompose into an infinite number of rotationally invariant subspaces. The problem of finding these rotationally invariant subspaces is equivalent to the problem of finding eigenstates of J^2 and J_z. Note that the analogous states for massive particles, with helicity replaced by spin, are simply given as Clebsch–Gordan sums of tensor products of the spherical harmonics with definite spin states. The problem is much more difficult for massless particles where the internal (polarization) and external (momentum) DOFs are coupled.

This problem was recently approached by Dragon [85], who used vector bundle techniques to construct simultaneous eigenstates of energy (H), helicity (χ), J^2 and J_z for massless particles. This gives a decomposition of the state space into finite-dimensional rotationally invariant subspaces and gives a countable orthonormal basis for monochromatic waves (as opposed to the uncountable basis of momentum eigenstates). However, the form of these solutions have certain undesirable properties. One can describe vector bundles with a number of different formalisms. In that used by Dragon, the fibers are never explicitly defined; instead the vector bundle structure is probed solely through transition functions on overlapping domains of the north and south stereographic wavefunctions over momentum space. While this is valid mathematically, it leads to eigenstates which are represented as pairs of functions in stereographic coordinates which are not easily interpreted. Furthermore, Dragon only gives the explicit formula for a single eigenstate within each angular momentum multiplet (cf. [85], Eqs. (81) and (84)); to obtain the others one would need to repeatedly apply the angular momentum lowering operator, which itself has a somewhat complicated form (cf. [85], Eq. (69)).

In contrast to Dragon's implicit framework, we showed in Eq. (5.29) [3] that massless particles of arbitrary helicity can be explicitly represented as iterated tensor products of the

R and left L photon bundles $\gamma_{\pm}$, with these photon bundles taking the form of line subbundles of the complexified tangent bundle of the sphere (when restricted to S^2). In this chapter, we use this alternative framework to derive more explicit forms of the simultaneous eigenstates of H, χ, J^2, and J_z for massless particles. We show that in appropriate coordinates, these eigenstates actually take the form spin-weighted spherical harmonics (SWSHs), generalizing the ordinary spherical harmonics that appear in the study of massive particles. SWSHs have shown up in many physical applications, including the study of asymptotically flat spacetimes [86], the dynamics of charged particles in the presence of a magnetic monopole [87–90], the study of complex spacetimes and twistor theory [91], in geophysical [92] and computer graphics applications [154], and in studies of the cosmic microwave background (CMB) [93–95]. They have been used in treatments of electromagnetic and gravitational waves in position space where the theory is more complicated and the topological nature of the SWSHs is not as apparent [96–98]. Here, we show that when working in momentum space, the SWSHs give a simple description of the angular momentum eigenfunctions for massless particles of arbitrary helicity. In Chapter 9 we will show how this description applies to plasma waves as well.

We showed in Chapter 6 that there are topological and geometric obstructions which prevent the angular momentum $\boldsymbol{J}$ of massless particles from splitting into SAM and OAM [1–3, 6]. The second goal of this chapter is to show that the description of the rotationally invariant subspaces of massless particles in terms of SWSHs gives a concrete illustration of this no-go result. In particular, we show that the angular momentum multiplet structure of the SWSHs is too sparse to be consistent with any reasonable spin-orbital decomposition. Furthermore, we also find that the helicity acts as lower bound on the total angular momentum of a massless particles. In contrast, massive particles possess states of total angular momentum 0, regardless of the spin, and such states are unavoidable if the the total OAM l can assume all nonnegative integer values.

We will restrict our attention to massless bosons since this leads to simplifications and there are no known elementary massless fermions. We also note that an alternative form of the simultaneous eigenfunctions of energy, helicity, and angular momentum in terms of Wigner D-functions and $3j$ symbols can be found in Ref. [141]. However, the derivation is more complicated and the topological nature of these eigenstates is not clear in this form of the solution. Furthermore, the relationship between these eigenstates and the problem of SAM-OAM decomposition has not been previously considered.

This chapter is organized as follows. In §8.2 we will discuss the description of massless particles in terms of tensor products of the R and L photon bundles. We then show how massless particles states can be described by spin-weighted functions in Sec. 8.3. In Sec. 8.4 we construct the total angular momentum operators for massless particles and show that they decompose the state space into angular momentum multiplets corresponding to the SWSHs. In Sec. 8.5 we show this decomposition illustrates the topological obstruction to a spin-orbital splitting of the massless angular momentum operator.

8.2 Normal form of massless particle bundles

Recall from Chapter 3 that the photon bundle : $\pi : \gamma \to \mathcal{L}_+$ consists of all plane wave solutions of Maxwell's equations in vacuum, labeled by $(\boldsymbol{k}, \boldsymbol{E})$ where the Fourier transformed electric field satisfies $\boldsymbol{E} \cdot \boldsymbol{k} = 0$. The photon bundle is a representation of the Poincaré group $\mathrm{ISO}^+(3,1)$: for a Lorentz transformation $\Lambda \in \mathrm{SO}^+(3,1)$ we have

$$\Lambda(k^\mu, \boldsymbol{E}) = (\Lambda k^\mu, \boldsymbol{E}') \tag{8.1}$$

where k^μ transforms like a four-vector and $\boldsymbol{E}'$ transforms under rotations like a three-vector and under boosts according to Eq. (3.77). Under a spacetime translation T_{a} by $\mathsf{a}^\mu \in \mathbb{R}^4$,

$$T_a(k^\mu, \boldsymbol{E}) = e^{-ik^\mu a_\mu}(k^\mu, \boldsymbol{E}). \tag{8.2}$$

Note that in this chapter we write the actions of Λ and T_{a} multiplicatively rather than writing $\Sigma(\Lambda)$ and $\Sigma(T_{\mathsf{a}})$. Elementary particles correspond to UIRs of the Poincaré group; γ decomposes into the R and L photon bundle UIRs $\gamma = \gamma_+ \oplus \gamma_-$ where $\gamma_\pm(\boldsymbol{k})$ consists of vectors of the form

$$c(\boldsymbol{E}_1 \pm i\boldsymbol{E}_2) \tag{8.3}$$

where $c \in \mathbb{C}$ and $(\boldsymbol{E}_1, \boldsymbol{E}_2, \hat{\boldsymbol{k}})$ forms a right-handed orthonormal basis of $\mathbb{R}^3$. In Chapter 5 we showed that a massless particle bundle of helicity h can be put into the normal form

$$\gamma_h = \underbrace{\gamma_\pm \otimes \cdots \otimes \gamma_\pm}_{|h| \text{ times}} \tag{8.4}$$

where the $\pm$ corresponds to the sign of h; note that this is a distinctly massless result that relies on the fact that massless particles have a single internal DOF. We can write a vector in γ_h as $v = v_1 \otimes \cdots \otimes v_h$ with $v_n \in \gamma_\pm$, and the Poincaré action on γ_h is given by

$$\Lambda(\boldsymbol{k}, v_1 \otimes \cdots \otimes v_n) = (\Lambda\boldsymbol{k}, \Lambda v_1 \otimes \cdots \otimes \Lambda v_n), \tag{8.5}$$

$$T_a(\boldsymbol{k}, v_1 \otimes \cdots \otimes v_n) = e^{-ik^\mu a_\mu}(\boldsymbol{k}, v_1 \otimes \cdots \otimes v_n). \tag{8.6}$$

Eqs. (8.1) and (8.5) describe the Lorentz transformation on single-particle states with definite momentum $\boldsymbol{k}$. The Hilbert space of particle states is given by $L^2(\gamma_h)$, the collection of square-integrable sections of γ_h with respect to the Lorentz invariant measure $d^3\boldsymbol{k}/|\boldsymbol{k}|$. An element $\boldsymbol{\alpha} \in L^2(\gamma_h)$ can be considered as a wave function, where $\boldsymbol{\alpha}(\boldsymbol{k}) \in \gamma_h(\boldsymbol{k})$. The inner product is given by

$$\langle \boldsymbol{\alpha}, \boldsymbol{\beta} \rangle = \int_{\mathcal{L}_+} \frac{d^3\boldsymbol{k}}{|\boldsymbol{k}|} \boldsymbol{\alpha}^*(\boldsymbol{k}) \cdot \boldsymbol{\beta}(\boldsymbol{k}) \tag{8.7}$$

$$= \prod_{n=1}^{|h|} \int_{\mathcal{L}_+} \frac{d^3\boldsymbol{k}}{|\boldsymbol{k}|} \boldsymbol{\alpha}_n^*(\boldsymbol{k}) \cdot \boldsymbol{\beta}_n(\boldsymbol{k}). \tag{8.8}$$

The Poincaré action on γ_h induces a representation of the Poincaré group on $L^2(\gamma_h)$ given by

$$(\Lambda\boldsymbol{\alpha})(\boldsymbol{k}) = \Lambda\Big(\boldsymbol{\alpha}(\Lambda^{-1}\boldsymbol{k})\Big) \tag{8.9}$$

for $\Lambda \in \mathrm{ISO}^+(3,1)$. We are particularly concerned with the subgroup $\mathrm{SO}(3) \subseteq \mathrm{ISO}^+(3,1)$ corresponding to rotations of the spacelike coordinates. The infinitesimal generators of this SO(3) action are the angular momentum operators $\boldsymbol{J}$. That is, for some direction $\hat{\boldsymbol{v}} \in \mathbb{R}^3$, the scalar operator $\hat{\boldsymbol{v}} \cdot \boldsymbol{J}$ is defined by

$$[(\hat{\boldsymbol{v}} \cdot \boldsymbol{J})\boldsymbol{\alpha}](\boldsymbol{k}) = i\frac{d}{d\psi}\Big|_{\psi=0} \boldsymbol{R}^{\psi}_{\hat{\boldsymbol{v}}}\boldsymbol{\alpha}\big(\boldsymbol{R}^{-\psi}_{\hat{\boldsymbol{v}}}\boldsymbol{k}\big) \tag{8.10}$$

where $\boldsymbol{R}^{\psi}_{\hat{\boldsymbol{v}}}$ is a rotation by ψ about $\hat{\boldsymbol{v}}$. Similarly, the momentum operator $\boldsymbol{P}$ and Hamiltonian H are the generators of spacetime translations and act simply by

$$\boldsymbol{P}\boldsymbol{\alpha}(\boldsymbol{k}) = \boldsymbol{k}\boldsymbol{\alpha}(\boldsymbol{k}), \tag{8.11}$$

$$H\boldsymbol{\alpha}(\boldsymbol{k}) = |\boldsymbol{k}|\boldsymbol{\alpha}(\boldsymbol{k}). \tag{8.12}$$

8.3 Massless particle states as spin-weighted functions

In this section we will review the formalism of spin-weighted functions and show that massless particle states of helicity h can be described using these functions if appropriate coordinates are chosen. One of the standard definitions of spin-weighted functions is as follows [89, 155, 156]. Denote the points on the 2-sphere S^2 by the normal unit vector $\hat{\boldsymbol{k}}$. Suppose that $(\boldsymbol{a}, \boldsymbol{b}, \hat{\boldsymbol{k}})$ is a right-handed orthonormal basis at each $\hat{\boldsymbol{k}}$. Note that by the hairy ball theorem $\boldsymbol{a}$ and $\boldsymbol{b}$ will have singularities at at least one $\hat{\boldsymbol{k}}$; we will consider $\boldsymbol{a}, \boldsymbol{b}$ such that they vary smoothly with $\hat{\boldsymbol{k}}$ except at a finite number of points. We can encode the choice of $\boldsymbol{a}$ and $\boldsymbol{b}$

in the complex vector

$$\boldsymbol{m}_+ = \frac{\boldsymbol{a} + i\boldsymbol{b}}{\sqrt{2}}. \tag{8.13}$$

A different choice of orthonormal frame $(\boldsymbol{a}', \boldsymbol{b}', \hat{\boldsymbol{k}})$ corresponds to a multiplication of $\boldsymbol{m}_+$:

$$\boldsymbol{m}'_+ = e^{i\xi(\hat{\boldsymbol{k}})}\boldsymbol{m}_+ \tag{8.14}$$

$$\boldsymbol{a}' = \cos(\xi)\boldsymbol{a} - \sin(\xi)\boldsymbol{b} \tag{8.15}$$

$$\boldsymbol{b}' = \sin(\xi)\boldsymbol{a} + \cos(\xi)\boldsymbol{b} \tag{8.16}$$

for some $\xi : S^2 \to \mathbb{R}$. We consider $\mathbb{C}$-valued functions $f(\boldsymbol{m}_+, \hat{\boldsymbol{k}})$ on the sphere that depend not only on $\hat{\boldsymbol{k}} \in S^2$, but on the choice of right-handed orthonormal basis defined by $\boldsymbol{m}_+$. Such a function is said to have spin-weight s and is written as ${}_sf$ if

$${}_sf(e^{i\xi}\boldsymbol{m}_+, \hat{\boldsymbol{k}}) = e^{is\xi}{}_sf(\boldsymbol{m}_+, \hat{\boldsymbol{k}}) \tag{8.17}$$

for any ξ. We note that it is straightforward to see that spin-weighted functions can equivalently be defined in terms of

$$\boldsymbol{m}_- = \frac{\boldsymbol{a} - i\boldsymbol{b}}{\sqrt{2}}. \tag{8.18}$$

The transformation $\boldsymbol{m}_+ \to e^{i\xi}\boldsymbol{m}_+ \doteq \boldsymbol{m}'_+$ corresponds to $\boldsymbol{m}_- \to e^{-i\xi}\boldsymbol{m}_- \doteq \boldsymbol{m}'_-$, that is, these both describe $(\boldsymbol{a}, \boldsymbol{b}, \hat{\boldsymbol{k}}) \to (\boldsymbol{a}', \boldsymbol{b}', \hat{\boldsymbol{k}})$. Then a function ${}_sf$ which depends on $\hat{\boldsymbol{k}}$ and a choice of orthonormal basis has spin-weight s if

$${}_sf(e^{-i\xi}\boldsymbol{m}_-, \hat{\boldsymbol{k}}) = e^{is\xi}{}_sf(\boldsymbol{m}_-, \hat{\boldsymbol{k}}). \tag{8.19}$$

We will now show how the state space $L^2(\gamma_h)$ of a massless helicity h particles corresponds to the space of spin-weight $-h$ functions. $\boldsymbol{m}_+(\hat{\boldsymbol{k}}) \in \gamma_+(\boldsymbol{k})$ by (8.3) and (8.13), and since γ_+ is a line bundle, $\boldsymbol{m}_+$ forms a basis of the fiber $\gamma_+(\hat{\boldsymbol{k}})$. Therefore, we can express any section

$\boldsymbol{\alpha} \in L^2(\gamma_+)$ by

$$\boldsymbol{\alpha}(\boldsymbol{k}) = a(\boldsymbol{k})\boldsymbol{m}_+(\hat{\boldsymbol{k}}) = \big(e^{-i\xi}a(\boldsymbol{k})\big)\big(e^{i\xi}\boldsymbol{m}_+(\hat{\boldsymbol{k}})\big) \tag{8.20}$$

$$= \big(e^{-i\xi}a(\boldsymbol{k})\big)\boldsymbol{m}'_+(\hat{\boldsymbol{k}}) \tag{8.21}$$

We can thus define a basis dependent function ${}_{-1}a(\boldsymbol{m}_+, \boldsymbol{k})$ such that

$$\boldsymbol{\alpha}(\boldsymbol{k}) = {}_{-1}a(\boldsymbol{m}_+, \boldsymbol{k})\boldsymbol{m}_+ \tag{8.22}$$

for any choice of $\boldsymbol{m}_+$. We can express $\boldsymbol{k} = |\boldsymbol{k}|\hat{\boldsymbol{k}} \doteq \omega\hat{\boldsymbol{k}}$, and write ${}_{-1}a$ as ${}_{-1}a(\boldsymbol{m}_+, \hat{\boldsymbol{k}}, \omega)$. Then for any fixed ω_0, ${}_{-1}\alpha(\boldsymbol{m}_+, \hat{\boldsymbol{k}}, \omega_0)$ is a function of spin-weight -1. Therefore, we see that R photon states $L^2(\gamma_+)$ are described by spin-weight -1 functions. The spin-weighting encodes the local basis dependence of any description of $\boldsymbol{\alpha}$. By an analogous argument with $\boldsymbol{m}_+$ replaced by $\boldsymbol{m}_-$, we find that the L-photon states, $L^2(\gamma_-)$, are described by spin-weight 1 functions. We can extend these results to bundles of higher helicity using Eq. (8.4). A section $\boldsymbol{\beta} \in L^2(\gamma_h)$ can be written as

$$\boldsymbol{\beta}(\boldsymbol{k}) = b(\boldsymbol{k}) \bigotimes_{n=1}^{|h|} \boldsymbol{m}_\pm \tag{8.23}$$

$$= e^{\mp i|h|\xi} b(\boldsymbol{k}) \bigotimes_{n=1}^{|h|} e^{\pm i\xi}\boldsymbol{m}_\pm \tag{8.24}$$

$$= e^{-ih\xi} b(\boldsymbol{k}) \bigotimes_{n=1}^{|h|} \boldsymbol{m}'_\pm \tag{8.25}$$

where the $\pm$ sign is corresponds to the sign of h. Thus, massless particle states of helicity h are described by function of spin-weight $-h$, and we see that this is essentially a different (and arguably bulkier) way to describe the fact that $\boldsymbol{\beta}$ is a section of the vector bundle γ_h. The relationship between massless particle states and spin-weighted functions helps explain

why SWSHs appear in the study of the angular momentum eigenstates which we examine in the next section.

8.4 Expressing angular momentum eigenstates as spin-weighted spherical harmonics

In this section we will show that the angular momentum eigenstates of massless particles are described by the SWSHs, provided we choose our coordinates appropriately. We can peel off the radial behavior in $\boldsymbol{k}$ space since it is unaffected by the angular momentum $\boldsymbol{J}$. Denote by γ_{h,S^2} the bundle γ_h with base manifold restricted to the unit sphere S^2 in $\boldsymbol{k}$ space. Then ([157], p. 160)

$$L^2(\gamma_h) = L^2(\mathbb{R}^+) \otimes L^2(\gamma_{h,S^2}) \tag{8.26}$$

where $\mathbb{R}^+$ denotes the positive real numbers, $L^2(\mathbb{R}^+)$ denotes the space of functions of the radial variable $f : \mathbb{R}^+ \to \mathbb{C}$ which are L^2 with respect to the measure $|\boldsymbol{k}|^2\, d|\boldsymbol{k}|$, and $L^2(\gamma_{h,S^2})$ represents the space of monochromatic waves (by which we mean waves with fixed $|\boldsymbol{k}|$). Note that this decomposition can be considered as induced by the operators H and $\boldsymbol{J}$; H acts trivially on $L^2(\gamma_{h,S^2})$ while $\boldsymbol{J}$ acts trivially on $L^2(\mathbb{R}^+)$. Our goal then is to find the simultaneous eigenstates of J^2 and J_z in $L^2(\gamma_{h,S^2})$, which is equivalent to decomposing $L^2(\gamma_{h,S^2})$ into UIRs of SO(3).

8.4.1 Decomposing the angular momentum

It will be useful to decompose $\boldsymbol{J}$ as

$$\boldsymbol{J} = \boldsymbol{J}_\parallel + \boldsymbol{J}_\perp \tag{8.27a}$$

$$\boldsymbol{J}_\parallel \doteq (\hat{\boldsymbol{P}} \cdot \boldsymbol{J})\hat{\boldsymbol{P}} \tag{8.27b}$$

$$\boldsymbol{J}_\perp \doteq \boldsymbol{J} - (\hat{\boldsymbol{P}} \cdot \boldsymbol{J})\hat{\boldsymbol{P}}. \tag{8.27c}$$

into the helicity and orbital quasi-angular momentum operators $\boldsymbol{J}_\parallel$ and $\boldsymbol{J}_\perp$ discussed in §3.5.2, §6.5, and Chapter 7. Recall that these do not satisfy the angular momentum commutation relations,

$$[J_{\parallel,a}, J_{\parallel,b}] \neq i\epsilon_{abc} J_{\parallel,c} \tag{8.28a}$$

$$[J_{\perp,a}, J_{\perp,b}] \neq i\epsilon_{abc} J_{\perp,c}, \tag{8.28b}$$

and are thus not true angular momentum operators. Nevertheless, as we showed in Ch. 7, they arise as the unique massless splitting induced by Poincaré symmetry and give essentially the closest approach to an SAM-OAM decomposition. It is noteworthy that $\boldsymbol{J}_\parallel$ is a point operator (Theorem 7.2) [5, 99], meaning that $(\boldsymbol{J}_\parallel \boldsymbol{\alpha})(\boldsymbol{k}_0)$ depends only on $\boldsymbol{\alpha}(\boldsymbol{k}_0)$ and not on the value of $\boldsymbol{\alpha}$ at any other $\boldsymbol{k}$. In physical terms, this means that $\boldsymbol{J}_\parallel$ only generates transformations of the internal polarization DOFs. In contrast, $\boldsymbol{J}_\perp$ is a local differential operator, with $(\boldsymbol{J}_\perp \boldsymbol{\alpha})(\boldsymbol{k}_0)$ depending on the value of $\boldsymbol{\alpha}(\boldsymbol{k})$ in a neighborhood of $\boldsymbol{k}_0$.

It is of both theoretical and practical interest to relate $\boldsymbol{J}_\parallel$ and $\boldsymbol{J}_\perp$ to the more familiar SAM and OAM $\boldsymbol{S}$ and $\boldsymbol{L}$ of massive particles. The fibers of $\gamma_\pm(\boldsymbol{k})$ are embedded in the Hermitian vector space $\mathbb{C}^3$ via Eq. (8.3); let $\mathcal{P} : \mathcal{L}_+ \times \mathbb{C}^3 \to \gamma_\pm$ be the projection onto the photon states. Then for a section $\boldsymbol{\alpha} \in L^2(\gamma_\pm)$,

$$[(\hat{\boldsymbol{v}} \cdot \boldsymbol{J})\boldsymbol{\alpha}](\boldsymbol{k}) = i\frac{d}{d\psi}\Big|_{\psi=0} \boldsymbol{R}^\psi_{\hat{\boldsymbol{v}}} \boldsymbol{\alpha}\big(\boldsymbol{R}^{-\psi}_{\hat{\boldsymbol{v}}} \boldsymbol{k}\big) \tag{8.29}$$

$$= i\mathcal{P}\frac{d}{d\psi}\Big|_{\psi=0} \boldsymbol{R}^\psi_{\hat{\boldsymbol{v}}} \boldsymbol{\alpha}\big(\boldsymbol{R}^{-\psi}_{\hat{\boldsymbol{v}}} \boldsymbol{k}\big) \tag{8.30}$$

$$= i\mathcal{P}\Big\{\frac{d}{d\psi}\Big|_{\psi=0} \boldsymbol{R}^\psi_{\hat{\boldsymbol{v}}} \boldsymbol{\alpha}(\boldsymbol{k}) + \frac{d}{d\psi}\Big|_{\psi=0} \boldsymbol{\alpha}\big(\boldsymbol{R}^{-\psi}_{\hat{\boldsymbol{v}}} \boldsymbol{k}\big)\Big\} \tag{8.31}$$

$$= \hat{\boldsymbol{v}} \cdot (\mathcal{P} \circ \boldsymbol{S})\boldsymbol{\alpha}(\boldsymbol{k}) + \hat{\boldsymbol{v}} \cdot (\mathcal{P} \circ \boldsymbol{L})\boldsymbol{\alpha}(\boldsymbol{k}). \tag{8.32}$$

Here,

$$(S_a)_{bc} = -i\epsilon_{abc} \tag{8.33}$$

and in spherical coordinates [158]

$$\boldsymbol{L} = -i\boldsymbol{e}_\phi \partial_\theta - \boldsymbol{e}_\theta \frac{1}{\sin\theta}\partial_\phi. \tag{8.34}$$

These extend to operators on the tensor product bundles γ_h via the product rule, that is,

$$(\mathcal{P}\circ\boldsymbol{S})(\boldsymbol{\alpha}_1\otimes\cdots\otimes\boldsymbol{\alpha}_{|h|}) \doteq \sum_{m=1}^{|h|} \boldsymbol{\alpha}_1\otimes\cdots\otimes(\mathcal{P}\circ\boldsymbol{S})\boldsymbol{\alpha}_m\otimes\cdots\otimes\boldsymbol{\alpha}_{|h|} \tag{8.35}$$

and similarly for $\mathcal{P}\circ\boldsymbol{L}$. The following lemma states that the projected operators $\mathcal{P}\circ\boldsymbol{S}$ and $\mathcal{P}\circ\boldsymbol{L}$ are in fact the same as $\boldsymbol{J}_\parallel$ and $\boldsymbol{J}_\perp$.

Lemma 8.1. *$\boldsymbol{J}_\parallel = \mathcal{P}\circ\boldsymbol{S}$ and $\boldsymbol{J}_\perp = \mathcal{P}\circ\boldsymbol{L}$. In particular,*

$$\boldsymbol{J} = (\hat{\boldsymbol{P}}\cdot\boldsymbol{J})\hat{\boldsymbol{P}} + \mathcal{P}\circ\boldsymbol{L} \tag{8.36}$$

Proof. $\mathcal{P}\circ\boldsymbol{S}$ restricts to an operator on each fiber of γ_h. Suppose first that $h=\pm1$ so that $\boldsymbol{\alpha}(\boldsymbol{k})$ is embedded in $\mathbb{C}^3$. Let $\hat{\boldsymbol{v}}\in\mathbb{R}^3$ be a unit vector and consider the fiber at $\boldsymbol{k}$, with $\hat{\boldsymbol{k}}\cdot\hat{\boldsymbol{v}} = \cos(\theta)$. Decompose $\hat{\boldsymbol{v}} = \cos(\theta)\hat{\boldsymbol{k}} + \hat{\boldsymbol{v}}_\perp$ into parallel and perpendicular components with respect to $\hat{\boldsymbol{k}}$. Then

$$[\hat{\boldsymbol{v}}\cdot(\mathcal{P}\circ\boldsymbol{S})]\boldsymbol{\alpha}(\boldsymbol{k}) = \Big(\cos(\theta)\hat{\boldsymbol{k}}\cdot(\mathcal{P}\circ\boldsymbol{S}) + \hat{\boldsymbol{v}}_\perp\cdot(\mathcal{P}\circ\boldsymbol{S})\Big)\boldsymbol{\alpha}(\boldsymbol{k}). \tag{8.37}$$

Using the identity

$$\frac{d}{d\psi}\Big|_{\psi=0} R^\psi_{\hat{\boldsymbol{v}}_1}\boldsymbol{v}_2 = \hat{\boldsymbol{v}}_1\times\boldsymbol{v}_2 \tag{8.38}$$

we find that

$$[\hat{\boldsymbol{v}}_\perp \cdot (\mathcal{P} \circ \boldsymbol{S})]\boldsymbol{\alpha}(\hat{\boldsymbol{k}}) = i\mathcal{P}\frac{d}{d\psi}\Big|_{\psi=0} R^{\psi}_{\hat{\boldsymbol{v}}_\perp}\boldsymbol{\alpha}(\boldsymbol{k}) \tag{8.39}$$

$$= i\mathcal{P}(\boldsymbol{v}_\perp \times \boldsymbol{\alpha}) \tag{8.40}$$

$$= 0. \tag{8.41}$$

The last equality follows since $\hat{\boldsymbol{v}}_\perp \times \boldsymbol{\alpha}(\boldsymbol{k})$ is proportional to $\hat{\boldsymbol{k}}$ and is thus annihilated by $\mathcal{P}$. $\mathcal{P}(\hat{\boldsymbol{k}} \cdot \boldsymbol{S}) = \hat{\boldsymbol{k}} \cdot \boldsymbol{S}$ since $(R^{\psi}_{\hat{\boldsymbol{k}}}\boldsymbol{\alpha}(\boldsymbol{k})) \cdot \hat{\boldsymbol{k}} = 0$, so

$$[\hat{\boldsymbol{v}} \cdot (\mathcal{P} \circ \boldsymbol{S})]\boldsymbol{\alpha}(\boldsymbol{k}) = \cos(\theta)\mathcal{P}(\hat{\boldsymbol{k}} \cdot \boldsymbol{S})\boldsymbol{\alpha}(\boldsymbol{k}) \tag{8.42}$$

$$= (\hat{\boldsymbol{v}} \cdot \hat{\boldsymbol{k}})(\hat{\boldsymbol{k}} \cdot \boldsymbol{S})\boldsymbol{\alpha}(\boldsymbol{k}) \tag{8.43}$$

$$= i(\hat{\boldsymbol{v}} \cdot \hat{\boldsymbol{k}})\frac{d}{d\psi}\Big|_{\psi=0} R^{\psi}_{\hat{\boldsymbol{k}}}\boldsymbol{\alpha}(\boldsymbol{k}) \tag{8.44}$$

$$= i(\hat{\boldsymbol{v}} \cdot \hat{\boldsymbol{k}})\frac{d}{d\psi}\Big|_{\psi=0} R^{\psi}_{\hat{\boldsymbol{k}}}\boldsymbol{\alpha}(R^{-\psi}_{\hat{\boldsymbol{k}}}\boldsymbol{k}) \tag{8.45}$$

$$= (\hat{\boldsymbol{v}} \cdot \hat{\boldsymbol{k}})(\hat{\boldsymbol{k}} \cdot \boldsymbol{J})\boldsymbol{\alpha}(\boldsymbol{k}) \tag{8.46}$$

$$= (\hat{\boldsymbol{v}} \cdot \boldsymbol{J}_\parallel)\boldsymbol{\alpha}(\boldsymbol{k}), \tag{8.47}$$

proving that $\boldsymbol{J}_\parallel = (\mathcal{P} \circ \boldsymbol{S})$. It then follows from Eqs. (8.27) and (8.32) that $\boldsymbol{J}_\perp = (\mathcal{P} \circ \boldsymbol{L})$, proving the lemma for $h = \pm 1$. The extension to arbitrary h follows from Eq. (8.35). □

$\boldsymbol{J}_\parallel = \mathcal{P} \circ \boldsymbol{S}$ and $\boldsymbol{J}_\perp = \mathcal{P} \circ \boldsymbol{L}$ are well-defined operators on the Hilbert space of sections $L^2(\gamma_h)$, however, the same is not true of $\boldsymbol{S}$ and $\boldsymbol{L}$. Indeed, $(\boldsymbol{S}\boldsymbol{\alpha})(\boldsymbol{k})$ and $(\boldsymbol{L}\boldsymbol{\alpha})(\boldsymbol{k})$ are not generally orthogonal to $\boldsymbol{k}$ and thus are not sections of γ_h. While $\boldsymbol{S}$ and $\boldsymbol{L}$ do each satisfy $\mathfrak{so}(3)$ commutation relations,

$$[S_a, S_b] = i\epsilon_{abc}S_c, \tag{8.48}$$

$$[L_a, L_b] = i\epsilon_{abc}L_b, \tag{8.49}$$

composition with the projection operator destroys this rotational symmetry. The projected operators instead satisfy nonstandard commutation relations

$$[J_{\parallel,a}, J_{\parallel,b}] = 0, \tag{8.50}$$

$$[J_{\perp,a}, J_{\parallel,b}] = i\epsilon_{abc} J_{\parallel,c}, \tag{8.51}$$

$$[J_{\perp,a}, J_{\perp,b}] = i\epsilon_{abc}(J_{\perp,c} - J_{\parallel,c}), \tag{8.52}$$

with $\boldsymbol{J}_\parallel$ generating a 3D translational symmetry while the vector operator $\boldsymbol{J}_\perp$ is not the generator of any symmetry group at all (since it is not closed under commutations).

8.4.2 Solving for angular momentum eigenstates

We now decompose the angular Hilbert space $L^2(\gamma_{h,S^2})$ into UIRs of SO(3), which is, by the standard ladder operator formalism, equivalent to solving for the angular momentum multiplets of sections $({}_h\boldsymbol{\alpha}_{jm})$ with $-j \leq m \leq j$ which satisfy

$$J_z \, {}_h\boldsymbol{\alpha}_{jm} = m \, {}_h\boldsymbol{\alpha}_{jm}, \tag{8.53a}$$

$$J_\pm \, {}_h\boldsymbol{\alpha}_{jm} = [(j \mp m)(j + 1 \pm m)]^{1/2} \, {}_h\boldsymbol{\alpha}_{jm\pm1}, \tag{8.53b}$$

$$J^2 \, {}_h\boldsymbol{\alpha}_{jm} = j(j+1) \, {}_h\boldsymbol{\alpha}_{jm}, \tag{8.53c}$$

where $J_\pm = J_x \pm iJ_y$. To proceed, we will express the $\boldsymbol{J}$ operators in a particular choice of basis for γ_{h,S^2} which will elucidate the relationship between the total angular momentum states and the spin-weighted spherical harmonics. If we choose the standard spherical unit vectors

$$\boldsymbol{e}_\theta = \cos(\theta)\cos(\phi)\boldsymbol{e}_x + \cos(\theta)\sin(\phi)\boldsymbol{e}_y - \sin(\theta)\boldsymbol{e}_z \tag{8.54a}$$

$$\boldsymbol{e}_\phi = -\sin(\phi)\boldsymbol{e}_x + \cos(\phi)\boldsymbol{e}_y. \tag{8.54b}$$

then $\boldsymbol{e}_\pm \doteq \frac{1}{\sqrt{2}}(\boldsymbol{e}_\theta \pm i\boldsymbol{e}_\phi)$ forms a smooth basis of $\gamma_\pm$ at all $\hat{\boldsymbol{k}} \in S^2$ except the poles; similarly,

$$\boldsymbol{e}_h \doteq \boldsymbol{e}_\pm^{|h|} \doteq 2^{-\frac{|h|}{2}} \underbrace{(\boldsymbol{e}_\theta \pm i\boldsymbol{e}_\phi) \otimes \cdots \otimes (\boldsymbol{e}_\theta \pm i\boldsymbol{e}_\phi)}_{|h| \text{ times}} \tag{8.55}$$

forms such a basis of γ_h, where the $\pm$ signs correspond to the sign of h. A globally smooth section $\boldsymbol{\alpha}$ of γ_{h,S^2} is expressed as

$$\boldsymbol{\alpha}(\theta, \phi) = f(\theta, \phi)\boldsymbol{e}_h. \tag{8.56}$$

$\boldsymbol{\alpha}$ and $\boldsymbol{e}_h$ are smooth for $\theta \neq 0, \pi$, so the same is true of $f(\theta, \phi)$. Since $\boldsymbol{\alpha}$ is also smooth at the poles, the limits

$$\lim_{\theta \to 0} f(\theta, \phi)\boldsymbol{e}_h(\theta, \phi) = \lim_{\theta \to 0} f(\theta, \phi)e^{-ih\phi}(\boldsymbol{e}_x \pm i\boldsymbol{e}_y)^{|h|} \tag{8.57a}$$

$$\lim_{\theta \to \pi} f(\theta, \phi)\boldsymbol{e}_h(\theta, \phi) = \lim_{\theta \to \pi} f(\theta, \phi)e^{ih\phi}(\boldsymbol{e}_x \pm i\boldsymbol{e}_y)^{|h|} \tag{8.57b}$$

must exist and agree for all ϕ. This implies that asymptotically

$$f(\theta, \phi) \sim c_N e^{ih\phi}, \quad (\theta \to 0^+) \tag{8.58a}$$

$$f(\theta, \phi) \sim c_S e^{-ih\phi}, \quad (\theta \to \pi^-) \tag{8.58b}$$

for constants c_N and c_S. If c_N or c_S are nonzero, then $f(\theta, \phi)$ is not smooth at $\theta = 0$ or $\theta = \pi$, respectively. However, we see that such cases correspond only to coordinate singularities, and not actual singularities as the state $\boldsymbol{\alpha}$ is still globally smooth.

To solve for the angular momentum multiplets, we express J_z, $J_\pm$, and J^2 in the $\boldsymbol{e}_h$ frame so that

$$J_z(f\boldsymbol{e}_h) = J'_z(f)\boldsymbol{e}_h, \tag{8.59a}$$

$$J_\pm(f\boldsymbol{e}_h) = J'_\pm(f)\boldsymbol{e}_h, \tag{8.59b}$$

$$J^2(f\boldsymbol{e}_h) = J'^2(f)\boldsymbol{e}_h, \tag{8.59c}$$

for operators J'_z, $J'_\pm$, J'^2 which will be determined. By the definition of $\boldsymbol{J}$ [Eq. (8.10)],

$$\boldsymbol{J}(f\boldsymbol{e}_h) = \boldsymbol{L}(f)\boldsymbol{e}_h + f\boldsymbol{J}(\boldsymbol{e}_h). \tag{8.60}$$

Since $\boldsymbol{e}_h$ is rotationally invariant about the z-axis, it follows from Eq. (8.10) that

$$J'_z = -i\partial_\phi. \tag{8.61}$$

To find $J'_\pm$ and J'^2 we will use the decomposition in Eq. (8.36). We have

$$\boldsymbol{J}_\perp = -i\Big(\boldsymbol{e}_\phi \mathcal{P}\circ\partial_\theta - \frac{\boldsymbol{e}_\theta}{\sin\theta}\mathcal{P}\circ\partial_\phi\Big) \tag{8.62}$$

where $\mathcal{P}$ annihilates any $\hat{\boldsymbol{k}}$ components produced by the angular derivatives. Direct calculation gives

$$J_{\perp,x}\boldsymbol{e}_h = h\frac{\cos^2\theta}{\sin\theta}\cos\phi\,\boldsymbol{e}_h \tag{8.63}$$

$$J_{\perp,y}\boldsymbol{e}_h = h\frac{\cos^2\theta}{\sin\theta}\sin\phi\,\boldsymbol{e}_h \tag{8.64}$$

$$J_{\perp,z}\boldsymbol{e}_h = -h\cos\theta\,\boldsymbol{e}_h \tag{8.65}$$

$$(\boldsymbol{v}\cdot\boldsymbol{J}_\parallel)\boldsymbol{e}_h = (\boldsymbol{v}\cdot\hat{\boldsymbol{k}})h\boldsymbol{e}_h. \tag{8.66}$$

Defining $J_{\perp,\pm} = J_{\perp,x} + iJ_{\perp,y}$ and similarly for $J_{\parallel,\pm}$ and $L_\pm$, we have

$$J_\pm(f\boldsymbol{e}_h) = \big(L_\pm(f) + fJ_{\parallel,\pm} + fJ_{\perp,\pm}\big)\boldsymbol{e}_h \tag{8.67}$$

$$= e^{\pm i\phi}\Big[\Big(\pm i\partial_\theta + i\frac{\cos\theta}{\sin\theta}\,\partial_\phi + \frac{h}{\sin\theta}\Big)f\Big]\boldsymbol{e}_\pm, \tag{8.68}$$

so that

$$J'_\pm = e^{\pm i\phi}\Big(\pm\partial_\theta + i\frac{\cos\theta}{\sin\theta}\partial_\phi + \frac{h}{\sin\theta}\Big). \tag{8.69}$$

To calculate J^2 we note that by Eq. (8.62), $\boldsymbol{J}_\perp \cdot \boldsymbol{J}_\parallel = \boldsymbol{J}_\parallel \cdot \boldsymbol{J}_\perp = 0$ so

$$J^2 = (\boldsymbol{J}_\parallel + \boldsymbol{J}_\perp)^2 = J_\parallel^2 + J_\perp^2 \tag{8.70}$$

and thus

$$J^2(f\boldsymbol{e}_h) = h^2 f\boldsymbol{e}_h + J_\perp^2(f\boldsymbol{e}_h). \tag{8.71}$$

Application of the product rule gives

$$J^2(f\boldsymbol{e}_h) = h^2 f\boldsymbol{e}_h + (L^2 f)\boldsymbol{e}_h + |h| f(J_\perp^2\boldsymbol{e}_\pm)\otimes\boldsymbol{e}_h^{(1)} \tag{8.72}$$

$$+ \sum_{p=1}^{3} 2|h|(L_p f)(J_{\perp,p}\boldsymbol{e}_\pm)\otimes\boldsymbol{e}_h^{(1)} + (h^2 - |h|)f(J_{\perp,p}\boldsymbol{e}_\pm \otimes J_{\perp,p}\boldsymbol{e}_\pm \otimes \boldsymbol{e}_\pm^{(2)}) \tag{8.73}$$

where

$$\boldsymbol{e}_h^{(q)} \doteq \underbrace{\boldsymbol{e}_\pm \otimes \cdots \otimes \boldsymbol{e}_\pm}_{|h|-q \text{ times}}. \tag{8.74}$$

From Eqs. (8.63)-(8.65),

$$J_\perp^2 \boldsymbol{e}_\pm = \cot^2\theta \boldsymbol{e}_\pm \tag{8.75}$$

$$\sum_{p=1}^{3} (J_{\perp,p}\boldsymbol{e}_\pm \otimes J_{\perp,p}\boldsymbol{e}_\pm \otimes \boldsymbol{e}_\pm^{(2)}) = \cot^2\theta \boldsymbol{e}_h \tag{8.76}$$

$$\sum_{p=1}^{3} |h|(L_p f)(J_{\perp,p}\boldsymbol{e}_\pm) \otimes \boldsymbol{e}_h^{(1)} = -h\frac{\cos\theta}{\sin^2\theta}(L_z f)\boldsymbol{e}_h. \tag{8.77}$$

Substituting these into Eq. (8.72) gives

$$J^2(f\boldsymbol{e}_h) = \left[\nabla^2 f + \frac{h^2}{\sin^2\theta} f - 2h\frac{\cos\theta}{\sin^2\theta} L_z f\right]\boldsymbol{e}_h. \tag{8.78}$$

Thus we find that the helicity h angular momentum operators in the $\boldsymbol{e}_h$ basis are given by

$$J'_z = -i\partial_\phi, \tag{8.79}$$

$$J'_\pm = e^{\pm i\phi}\Big(\pm\partial_\theta + i\frac{\cos\theta}{\sin\theta}\partial_\phi + \frac{h}{\sin\theta}\Big), \tag{8.80}$$

$$J'^2 = -\nabla^2 - \frac{2h\cos\theta}{\sin^2\theta} L_z + \frac{h^2}{\sin^2\theta}, \tag{8.81}$$

so that Eq. (8.53) reduces to solving

$$J'_z\, {}_h f_{jm} = m\, {}_h f_{jm}, \tag{8.82a}$$

$$J'_\pm\, {}_h f_{lm} = [(j\mp m)(j+1\pm m)]^{1/2}\, {}_h f_{jm\pm1}, \tag{8.82b}$$

$$J'^2\, {}_h f_{jm} = j(j+1)\, {}_h f_{lm}. \tag{8.82c}$$

These equations are solved by the SWSHs of spin-weight $-h$ [89]

$${}_h f_{jm}(\theta,\phi) = {}_{-h}Y_{jm}(\theta,\phi), \tag{8.83}$$

so that the simultaneous eigenstates of helicity, J^2, J_z, and energy are given by

$$ {}_h\boldsymbol{\alpha}_{jm,\boldsymbol{k}_0} = {}_{-h}Y_{jm}(\theta,\phi)\delta(|\boldsymbol{k}|-|\boldsymbol{k}_0|)\boldsymbol{E}_h. \tag{8.84} $$

The SWSHs are given explicitly by [155]

$$ {}_hY_{jm}(\theta,\phi) = \Big[\frac{(j+m)!(j-m)!(2j+1)}{4\pi(j+h)!(j-h)!}\Big]^{\frac{1}{2}}(\sin\theta/2)^{2j} \tag{8.85} $$

$$ \times\sum_q \binom{j-h}{q}\binom{j+h}{q+h-m}(-1)^{j-q-h-m}e^{im\phi}(\cot\theta/2)^{2q+h-m} \tag{8.86} $$

where $j \geq |h|$, $-j \leq m \leq j$ and the sum ranges from $q = \max(0, m-h)$ to $q = \min(j-h, j+m)$. Interestingly, Dray [89] discovered this in the inverse context, reverse engineering angular momentum operators $J'_z, J'_\pm, J'^2$ for the SWSHs. Here, we instead naturally encounter these angular momentum operators in the study of massless particles, demonstrating one of their physical origins. An arbitrary section $\alpha(\boldsymbol{k})$ of γ_h can be expressed by

$$ \boldsymbol{\alpha}(\boldsymbol{k}) = \sum_{j=|h|}^{\infty}\sum_{m=-j}^{j} a_{jm}(|\boldsymbol{k}|)\,{}_{-h}Y_{jm}(\theta,\phi). \tag{8.87} $$

Technically, this follows from the equivalence of sections of γ_h with spin-weight $-h$ functions established in Sec. 8.3 and the completeness of the SWSHs with respect to the spin-weighted functions which was established by Newman and Penrose [86]. The coefficients a_{jm} can be calculated from the orthogonality of the SWSHs [155]:

$$ \int_0^{2\pi} d\phi \int_{-1}^{1} d(\cos(\theta))\,{}_hY_{jm}(\theta,\phi)\,{}_{h'}Y_{j'm'} = \delta_{jj'}\delta_{mm'}. \tag{8.88} $$

We also see how the SWSHs ${}_{-h}Y_{jm}$ are better understood as sections of the bundle γ_h rather than as scalar functions. For example, the ${}_{-h}Y_{jm}(\theta,\phi)$ have seemingly anomalous

singularities at ${}_{-h}Y_{jh}(0,\phi)$ and ${}_{-h}Y_{j,-h}(\pi,\phi)$. However, a closer examination shows that

$$ {}_{-h}Y_{jh}(\theta,\phi) \sim (-1)^h \sqrt{\frac{2j+1}{4\pi}} e^{ih\phi}, \quad (\theta \to 0^+) \tag{8.89} $$

$$ {}_{-h}Y_{j,-h}(\theta,\phi) \sim (-1)^j \sqrt{\frac{2j+1}{4\pi}} e^{-ih\phi}, \quad (\theta \to \pi^-) \tag{8.90} $$

so that by Eq. (8.58), the sections ${}_{-h}Y_{jm}\boldsymbol{E}_h$ are indeed smooth at the poles; the apparent singularities of the SWSHs vanish when they are considered as sections rather than functions. We note that the relationship between SWSHs and sections of vector bundles was rigorously studied in the mathematics literature by Eastwood and Tod [159], who showed that the SWSHs can be considered as sections of line bundles over the complex projective line $\mathbb{P}^1(\mathbb{C})$. This is essentially the same structure we encounter since $\mathbb{P}^1(\mathbb{C})$ is homeomorphic to the Riemann sphere S^2, and thus the line bundles over $\mathbb{P}^1(\mathbb{C})$ are in bijective correspondence with the γ_h. Our findings show a physical interpretation of the sectional nature of the SWSHs, showing that they are angular momentum eigenstates of massless particle bundles.

Eq. (8.87) shows that the SWSHs give a countable basis for monochromatic massless waves. The vector bundle description also furnishes a basis in terms of the momentum eigenstates, $(\hat{\boldsymbol{k}}, \boldsymbol{e}_h(\hat{\boldsymbol{k}}))$, however, this basis has certain disadvantages. First, the basis vectors are labeled by $\hat{\boldsymbol{k}} \in S^2$ and thus uncountable in number, making this a much larger basis than that given by the SWSHs. This corresponds to the spectrum of $\boldsymbol{k}$ being continuous whereas that of $\boldsymbol{J}^2$ and J_z is discrete. Second, the momentum basis has discontinuities at the poles due to the singularities in $\boldsymbol{e}_h$; such singularities cannot be avoided by replacing $\boldsymbol{e}_h$ with some other choice of unit vectors due to the topological nontriviality of γ_h (except when $h \neq 0$). In contrast, the basis given by ${}_{-h}Y_{jm}\boldsymbol{e}_h$ is countable and composed of globally smooth states. We see that the SWSH furnish a spectral decomposition of monochromatic waves in which a state is simply described by a countable set of of numbers α_{jm} which label the components with total angular momentum $\sqrt{j(j+1)}$ and z angular momentum m.

In the case of photons, this results in an expansion of polarization states in terms of the angular momentum eigenstates ${}_{\mp 1}Y_{jm}\boldsymbol{e}_{\pm 1}$. It is notable that this expansion is quite different form that used in Refs. [93–95] for analyzing the CMB in terms of SWSHs. Indeed, these studies, as well as a related application in computer graphics [154], actually use the spin-weight ± 2 harmonics ${}_{\pm 2}Y_{jm}$. This is due to the fact that they are analyzing Stokes vectors, which encode time averaged polarization intensities, and thus do not have any phase information. Furthermore, they are only analyzing linear polarizations since the CMB does not contain circularly polarized light.[1] It turns out that linear polarizations without phase information can essentially be described by helicity ± 2 representations and can thus be expressed in terms of the spin-weight ± 2 spherical harmonics [93]. In contrast, we are analyzing the eigenstates of the full polarization space, allowing for circular polarizations and retaining phase information. In this context, photons are described by the spin-weight ± 1 harmonics.

8.5 The nonexistence of spin and orbital angular momentum operators of massless particles

As discussed in Chapters 3-7, there has been a long controversy [1, 2, 4, 65–74, 77, 78, 80, 81] surrounding the question of whether or not massless particles admit an SAM-OAM decomposition. We showed in Chapter 6 that it is impossible to decompose the total angular momentum operator of massless particles into SAM and OAM operators, generalizing the findings of van Enk and Nienhuis who first found that a particular attempted SAM-OAM decomposition did not result in legitimate angular momentum operators [67, 68]. This result holds even if one significantly relaxes axiomatic constraints on the SAM and OAM operators, as shown in the no-go Theorems 6.3 and 6.4. However, the proofs of these theorems

[1] Unpolarized light cannot become circularly polarized via Thomson scattering and is thus absent from the CMB [93].

are somewhat abstract. Here, we can use the properties of the total angular momentum eigenstates to give a concrete illustration of the no-go results, showing that the structure of the total angular momentum multiplets given by the SWSHs rules out any reasonable spin-orbital decomposition of $\boldsymbol{J}$.

The (nonprojective) irreducible representations of SO(3) are the complex vector spaces V_a of dimension $2a+1$ where a is a nonnegative integer. The tensor product of two such representations with $a \geq b$ is given by the addition of angular momentum formula:

$$V_a \otimes V_b \cong V_{a-b} \oplus V_{a-b+1} \oplus \cdots \oplus V_{a+b}. \tag{8.91}$$

From Eq. (8.26), we have

$$L^2(\gamma_h) = L^2(\mathbb{R}^+) \otimes \left(V_{|h|} \oplus V_{|h|+1} \oplus V_{|h|+2} \oplus \cdots\right) \tag{8.92}$$

$$= L^2(\mathbb{R}^+) \otimes \Big(\bigoplus_{n=0}^{\infty} V_{|h|+n}\Big) \tag{8.93}$$

where the copy of $V_{|h|+n}$ is spanned by ${}_{-h}Y_{|h|+n,m}\,\boldsymbol{e}_h$ for $-(|h|+n) \leq m \leq |h|+n$ and $L^2(\mathbb{R}^+)$ describes the radial dependence, *i.e.*, the energy of the states (which is unimportant for the analysis of angular momentum). There is precisely one total angular momentum j multiplet if $j \geq |h|$ and zero multiplets for $j < |h|$. This multiplet structure is distinctly different from what one typical obtains for a particle whose total angular momentum splits into well-defined spin and orbital parts as $\boldsymbol{J} = \boldsymbol{L} + \boldsymbol{S}$. Consider, for example, a massive spin s particle. In nonrelativistic quantum mechanics, the Hilbert space is given by

$$\mathcal{H} = L^2(\mathbb{R}^3) \otimes \mathbb{C}^{2s+1} = \underbrace{L^2(\mathbb{R}^+)}_{\mathcal{H}_r} \otimes \underbrace{L^2(S^2)}_{\mathcal{H}_o} \otimes \underbrace{\mathbb{C}^{2s+1}}_{\mathcal{H}_s}. \tag{8.94}$$

$\mathcal{H}$, $L^2(S^2)$ and $\mathbb{C}^{2s+1}$ are each representations SO(3) whose generators are the total ($\boldsymbol{J}$), orbital ($\boldsymbol{L}$), and spin ($\boldsymbol{S}$) angular momentum operators and are related by $\boldsymbol{J} = \boldsymbol{L} + \boldsymbol{S}$. $\mathcal{H}_r$

is a trivial representation of SO(3) and has generator 0. We write

$$\mathcal{H}_J = \mathcal{H}_o \otimes \mathcal{H}_s \tag{8.95}$$

for the part of the Hilbert space containing nontrivial rotational symmetry. $\mathcal{H}_s = \mathbb{C}^{2s+1}$ is the irreducible representation V_s while $L^2(S^2)$, the space of square-integrable functions on the sphere, decomposes as

$$L^2(S^2) \cong V_0 \oplus V_1 \oplus V_2 \oplus \cdots \tag{8.96}$$

where V_l corresponds to the subspace spanned by the ordinary spherical harmonics Y_{lm} with $|m| \leq l$. By the addition of angular momentum, for massive particles with integer spin we have

$$\mathcal{H}_J \cong V_s \otimes (V_0 \oplus V_1 \oplus V_2 \oplus \cdots) \tag{8.97}$$

$$\cong \Big(\bigoplus_{m=0}^{s-1}(2m+1)V_m\Big) \oplus \Big(\bigoplus_{n=s}^{\infty}(2s+1)V_n\Big). \tag{8.98}$$

$$aV_m \doteq \underbrace{V_m \oplus \cdots \oplus V_m}_{a \text{ times}} \tag{8.99}$$

Eq. (8.98) follows since for $0 \leq m < s$, one copy of V_m will result from $V_s \otimes V_{s+q}$ precisely when $-m \leq q \leq m$, and thus there are $2m+1$ copies of V_m in $\mathcal{H}$. Meanwhile, for $n \geq s$, one copy of V_n results from $V_s \otimes V_{n+q}$ precisely when $-s \leq q \leq s$, giving $2s+1$ copies of V_n in $\mathcal{H}$.

Comparing Eqs. (8.92) and (8.98), we see that massive particles (which possess spin) and massless particles (which possess helicity) have very different angular momentum structures. For example, a massive spin 1 particle has the multiplet structure

$$\mathcal{H}_J^{s=1} \cong V_0 \oplus 3V_1 \oplus 3V_2 \oplus 3V_3 \oplus \ldots \tag{8.100}$$

while a massless helicity 1 particle has

$$\mathcal{H}_J^{h=1} \cong V_1 \oplus V_2 \oplus V_3 \oplus \cdots . \tag{8.101}$$

Notably, the helicity of a massless particle enforces a lower bound on the total angular momentum which does not exist for massive particles. This is reflected in the properties of the SWSHs ${}_{-h}Y_{jm}$, which only exist if $j \geq |h|$. Massless particles also only have at most a single angular momentum multiplet of each total angular momentum j, while massive particles have up to $2s+1$ degenerate multiplets for each j. This is related to the fact that for massive particles, the spherical harmonics describe the orbital angular momentum which are then tensor multiplied by the internal spin space $\mathbb{C}^{2s+1}$, producing additional copies of the multiplets. In contrast, the SWSHs ${}_{-h}Y_{jm}$ describe the *total* angular momentum states for massless particles.

From examining the addition of angular momentum formula in Eq. (8.91), it becomes apparent that the multiplet structure of massless particles [Eq. (8.92)] would not result from any reasonable spin-orbital splitting of the angular momentum. In particular, it is difficult to avoid generating repeated multiplets and or low angular momentum multiplets. Thus, from examining the difference between massive and massless angular momentum states, and the differences between ordinary and spin-weighted spherical harmonics, we come to the conclusion that massless particles should not admit an SAM-OAM decomposition.

We note that it is technically possible to contrive spin and orbital multiplet structures which would tensor product to give Eq. (8.91), but they are highly unnatural. For example, the simplest such splitting for a helicity 1 massless particle is

$$V_1 \otimes (V_0 \oplus V_3 \oplus V_6 \oplus \cdots) \cong V_1 \oplus V_2 \oplus V_3 \oplus \cdots \tag{8.102}$$

in which the candidate orbital angular momentum l could only take on integer multiples of 3, a strange hypothetical which has never been suggested. One can rigorously prove that

the rotationally symmetry of the massless Hilbert spaces $L^2(\gamma_{h,S^2})$ cannot be described by a tensor product of two nontrivial SO(3) representations (no-go Theorem 6.1; [4]), so these exotic splittings are not realized in massless particles.

We note that the massless SAM-OAM obstruction can also be understood by examining the complete set of commuting observables (CSCO). As we saw in the previous section, (H, χ, J^2, J_z) form a CSCO for massless particles, that is, a basis of states can be labeled by $|\boldsymbol{k}_0|, h, j, j_z$. Compare this with massive particles, for which we know the angular momentum splits into well-defined SAM and OAM parts:

$$\boldsymbol{J} = \boldsymbol{S} + \boldsymbol{L}. \tag{8.103}$$

Since $\boldsymbol{S}$ and $\boldsymbol{L}$ are SO(3) generators, it follows from the representation theory of SO(3) that S^2 and L^2 commute with each other and with J^2, and J_z. It is also true that $\boldsymbol{S}$ and $\boldsymbol{L}$ commute with H, and therefore so do S^2 and L^2 [82]. Indeed, a CSCO for massive particles with spin, assuming no additional internal structure (e.g., no color charge), is given by (H, J^2, J_z, S^2, L^2), and a basis is labeled by $(|\boldsymbol{k}_0|, j, j_z, s, l)$ [160]. s is determined by the identity of the particles, as is h in the massless case. We see though a discrepancy between massive and massless particles: one must specify both the orbital and total angular momentum eigenvalues l and j in the massive case, whereas in the massless case only j is needed.

Suppose for a massless particle we assume that Eq. (8.103) is true for some angular momentum operators $\boldsymbol{S}$ and $\boldsymbol{L}$ which commute with H. It then it follows that L^2 commutes with (H, χ, J^2, J_z). However, the latter is already a complete set of observables, so it must be that L^2 gives redundant information such that the value of l can be inferred from $|\boldsymbol{k}_0|, h, j, j_z$. This corresponds precisely to the unnatural splittings of the form Eq. (8.102), since each V_j on the rhs (labeled by j) is produced by a single tensor product $V_1 \otimes V_l$ on the lhs. As

previously mentioned, these unnatural splittings can be rigorously ruled out by the no-go theorems of Chapter 6 [4].

8.6 Conclusion

Massive and massless particles have very different Hilbert spaces. Massive particles are topologically trivial, allowing the internal and external DOFs to be separated. The external Hilbert space decomposes into rotationally invariant subspaces spanned by the ordinary spherical harmonics. Massless particles, on the other hand, are topologically nontrivial (for $h \neq 0$). Even though they only have a single internal DOF, this DOF is twisted together with the external momentum DOFs. The result is that states are described not by ordinary functions, but by sections of topologically nontrivial line bundles (or equivalently by spin-weighted functions). The rotationally invariant subspaces are spanned by the total angular momentum multiplets, which in this case are SWSHs (${}_{-h}Y_{jm}$) rather than ordinary spherical harmonics (Y_{jm}). This harmonic decomposition gives a smooth countable basis for monochromatic massless waves, something which is not provided by the momentum eigenstates. The angular momentum multiplet structure of the SWSHs is also quite different than that which occurs for massive particles. Indeed, in the massless case we obtain a very sparse multiplet structure, with a single multiplet for each $j \geq |h|$ and no multiplet if $j < |h|$. In contrast, there is no lower bound on the angular momentum of massive particles with integer spin and typical multiplets are degenerate (except in the case of spin 0 particles). From taking into account the way angular momenta add, one sees that the massless multiplet structure cannot result from any reasonable SAM-OAM decomposition of the angular momentum $\boldsymbol{J}$. Indeed, in Chapters 6 and 7 we showed via more abstract geometric and topological arguments that such a massless SAM-OAM decomposition is impossible [6], [5]; here we see that this conclusion is concretely illustrated by carrying out the explicit angular

momentum decomposition of massless particles. We will see in the following chapter that this analysis can be applied to plasma waves as well.

Chapter 9

The topology, geometry, and angular momentum of waves in cold unmagnetized plasma

This chapter is based on Ref. [7]. In this chapter we will use cgs units rather than natural units, in particular, we will not set $c = 1$. We will also denote the angular momentum operator on sections of vector bundles by $\boldsymbol{J}$ rather than $\tilde{\boldsymbol{J}}$, and similarly for the other generators. This is because we will not need to distinguish between generators on sections and those directly on the vector bundle, as we will only use the former.

9.1 Introduction

Topological methods have helped uncover and describe novel phenomena in wave systems, first in topological insulators with periodic lattices [21, 22], and later in continuous media such as fluids [20, 23–27] and plasmas [19, 29–31, 33–35, 161]. In continuous media, the primary topological application studied thus far is use of the bulk-edge correspondence (BEC) to predict the existence of topologically protected edge modes. In this method, one typically follows a prescription of calculating the Berry connection and Berry curvature, and the

latter is then integrated over some 2D surface in k space to obtain a topological Chern number. The difference in Chern numbers across a boundary then determines the number of edge modes. As we discussed in Chapter 2, Chern numbers are associated with vector bundles. The success of the BEC in plasma physics indicates that plasma waves have such a topological structure, in particular, the collection of all plane wave solutions forms a vector bundle parameterized by the momentum space. In previous chapters we have shown in the case of massless particle waves, such as photons and gravitons, that the vector bundle structure provides a flexible framework to explore topological properties of waves beyond the BEC. In this chapter, we show that most of the methods and results from previous chapters [1–6] can be applied to the study of waves in a cold isotropic unmagnetized plasma, and in fact, electromagnetic (EM) plasma waves have essentially the same topological structure of photons. We will show that there exists a global polarization basis for EM plasma waves, which is surprising as one might guess that this would violate the hairy ball theorem. We further show that any such basis must involve elliptical polarizations as it is not possible to form a global basis using only circularly or linearly polarized waves.

Our main use of the vector bundle structure pertains to the angular momentum of plasma waves and the question of whether or not it permits a spin-orbital decomposition. In laser-plasma accelerators, the angular momentum of the electron beam is one of the two key parameters (along with the emittance) for characterizing the beam profile [162, 163]. Recent simulations have suggested that the use of so-called orbital angular momentum (OAM) beams in such accelerators may lead to enhanced tuning of the electron beam and the resultant X-ray production [59, 60]. It has also been suggested that OAM beams may be used in the production of extremely large magnetic fields in plasmas, with simulations suggesting that fields in the kilo-Tesla range may be accessible for short pulses [62, 63] and ~ 40 T in the steady state [64]. Despite the great success of so-called OAM beams in applications [49], it has long been noted that there are deep theoretical issues with the concept of splitting the angular momentum of light waves into spin angular momentum (SAM) and OAM. The

fundamental issue, as we have seen in Chapters 6, 7, and 8.4 is that the alleged SAM and OAM operators for light do not satisfy angular momentum commutation relations and are thus not true angular momentum operators [1, 4, 67, 68, 78]. One might at first guess that this issue does not apply to waves propagating in plasmas. Indeed, we have shown the obstruction to an SAM-OAM splitting for photons and gravitons is closely related to the fact that these waves are massless. However, in EM plasma waves, the plasma frequency ω_p shows up as a mass-like term in the dispersion relation, so one may wonder if this restores the SAM-OAM decomposition. We will show that this is not the case and that the obstructions to an SAM-OAM survive the introduction of a background plasma. Nevertheless, we show that the rotational symmetry of plasma waves induces a unique splitting of the angular momentum operator $\boldsymbol{J} = \boldsymbol{J}_{\parallel} + \boldsymbol{J}_{\perp}$ into the well-defined helicity and orbital quasi-angular momentum. They are not true angular momentum operators since they do not satisfy angular momentum commutation relations and are thus not generators of 3D rotations. We show, for example, that the components of $\boldsymbol{J}_{\parallel}$ and $\boldsymbol{J}_{\perp}$ have continuous spectra, something which is impossible for legitimate angular momentum operators. Despite this deficiency, these operators appear to be of both theoretical and experimental interest.

We also use the angular momentum operator $\boldsymbol{J}$, which is the generator of the SO(3) symmetry of the plasma waves, to give a novel harmonic decomposition of these waves, mirroring that found in Ch. 8. Indeed, plasma waves can be decomposed into rotationally invariant finite-dimensional multiplets of wave solutions. More particularly, these are simultaneous eigenstates of energy, helicity, J^2, and J_z. This gives a basis for plasma waves in terms of spin-weighted spherical harmonics (SWSHs), which are generalizations of the more common ordinary spherical harmonics. The resultant angular momentum multiplet structure gives a concrete illustration of the issues that arise when attempting a spin-orbital decomposition for plasma waves as well as a discrete basis which can be used to express monochromatic waves.

We also remark on our use of the cold fluid model to study the topology of electromagnetic (EM) and electrostatic (ES) waves. This simple plasma model allows us to clearly illustrate the topology in these waves and has the particular advantage of making it easier to relate the expectation values of the Hamiltonian, momentum, and angular momentum operators with the conserved fluid quantities. However, we point out that these methods appear to apply to kinetic descriptions of plasmas as well.

This chapter is organized as follows. In §9.2, we show how sets of plane waves in plasmas are described by vector bundles. In §9.3, we characterize the topology of the EM and electrostatic (ES) Langmuir waves, showing that Langmuir waves are topologically trivial, as is the collection of all EM plasma waves. This allows for an explicit construction of global polarization bases for EM and ES waves which vary smoothly with $\boldsymbol{k}$. However, if we decompose the EM waves into rotationally invariant parts, namely R and L waves, then we find that these subspaces are topologically nontrivial. Although the plasma introduces an effective mass, which typically indicates topological triviality, we show that topologically nontrivial waves are able to exist due to the resonance of EM and ES waves at the plasma frequency. In §9.4, we show that we can define a global basis for R, L, and Langmuir waves in terms of SWSHs, and how this harmonic decomposition illustrates the obstruction to an SAM-OAM splitting. In §9.5, we show that there is a symmetry-induced decomposition of the plasma angular momentum, but that it is not a legitimate SAM-OAM splitting. In §9.6, we will discuss how our results generalize to kinetic plasmas.

9.2 Vector bundle structure of cold plasma waves

We will consider linear waves in a cold homogeneous unmagnetized plasma. Assume that electrons have density $n = n_0 + n_1$ where n_0 is constant and $n_1 \ll n_0$. We will work in the background plasma's rest frame, and let $\boldsymbol{v}$ be the perturbed electron velocity. With ions

treated as a neutralizing background, the linearized fluid equations are

$$m\frac{\partial \boldsymbol{v}}{\partial t} = -e\boldsymbol{\mathcal{E}} \tag{9.1}$$

$$\frac{\partial n_1}{\partial t} = -n_0 \nabla \cdot \boldsymbol{v} \tag{9.2}$$

$$\frac{\partial \boldsymbol{\mathcal{E}}}{\partial t} = c\nabla \times \boldsymbol{\mathcal{B}} + 4\pi e n_0 \boldsymbol{v} \tag{9.3}$$

$$\frac{\partial \boldsymbol{\mathcal{B}}}{\partial t} = -c\nabla \times \boldsymbol{\mathcal{E}} \tag{9.4}$$

$$\nabla \cdot \boldsymbol{\mathcal{E}} = -4\pi e n_1 \tag{9.5}$$

$$\nabla \cdot \boldsymbol{\mathcal{B}} = 0 \tag{9.6}$$

where $e > 0$ and m is the electron mass. We consider plane wave solutions with $(t, \boldsymbol{x})$ dependence of the form $e^{i(\boldsymbol{k}\cdot\boldsymbol{x}-\omega t)}$. The system simplifies to an eigenvalue problem

$$\boldsymbol{D}_E(\omega, \boldsymbol{k}) \cdot \boldsymbol{E}(\boldsymbol{k}) = 0 \tag{9.7a}$$

$$D_{E,ab}(\omega, \boldsymbol{k}) \doteq \frac{1}{\omega^2}\left[k_a k_b + \delta_{ab}(\omega^2 - \omega_p^2 - k^2)\right] \tag{9.7b}$$

where $D_{E,ab}$ is the dispersion tensor and $\omega_p = (4\pi n_0 e^2/m)^{1/2}$ is the background plasma frequency. $\boldsymbol{\mathcal{E}}(\boldsymbol{x}, t)$ and $\boldsymbol{E}(\boldsymbol{k})$ are related by

$$\boldsymbol{\mathcal{E}}(\boldsymbol{x}, t) = \frac{1}{2(2\pi)^{3/2}} \int d^3k\, \boldsymbol{E}(\boldsymbol{k}) e^{i(\boldsymbol{k}\cdot\boldsymbol{x}-\omega(\boldsymbol{k})t)} + c.c.. \tag{9.8}$$

The other dynamical variables are determined by $\boldsymbol{E}(\boldsymbol{k})$:

$$n_1(\boldsymbol{k}) = -\frac{i}{4\pi e}\boldsymbol{k} \cdot \boldsymbol{E}(\boldsymbol{k}) \tag{9.9}$$

$$\boldsymbol{v}(\boldsymbol{k}) = -\frac{ie}{m\omega}\boldsymbol{E}(\boldsymbol{k}) \tag{9.10}$$

$$\boldsymbol{B}(\boldsymbol{k}) = \frac{c}{\omega}\boldsymbol{k} \times \boldsymbol{E}(\boldsymbol{k}). \tag{9.11}$$

Note that we use the same symbols n_1 and $\boldsymbol{v}$ for the position space variables and their Fourier transform; it will always be clear from context which is being used. The branches of the solution are determined by the dispersion relation $\det \boldsymbol{D} = 0$. The positive frequency solutions correspond to the nondegenerate Langmuir mode with $\omega = \omega_p$ and the two degenerate EM modes with $\omega = \sqrt{\omega_p^2 + |\boldsymbol{k}|^2 c^2}$.[1] The ES modes can be labeled by $(k, E\hat{\boldsymbol{k}})$ where $E \in \mathbb{C}$ and $k^\mu = (\omega_p/c, \boldsymbol{k})$ is the wave four-vector; note that we will also refer to this as the momentum four-vector. We denote the collection of all such modes by ζ_0. For each fixed four-vector k_0, the subset $\zeta_0(k_0)$ of all modes with k_0 comprise a one-dimensional vector space. Similarly, the EM modes can be labeled by $(k, \boldsymbol{E})$ for $k = (\sqrt{\omega_p^2/c^2 + |\boldsymbol{k}|^2}, \boldsymbol{k})$ and any $\boldsymbol{E} \in \mathbb{C}^3$ satisfying the transversality condition $\boldsymbol{E} \cdot \boldsymbol{k} = 0$. Denote the collection of all such modes by ζ_{EM} and those with momentum k_0 by $\zeta_{EM}(k_0)$. Each $\zeta_{EM}(k_0)$ is a two-dimensional vector space. Thus, ζ_0 and ζ_{EM} form collections of vector spaces smoothly parameterized by subsets of the four-momentum space; that is, ζ_0 and ζ_{EM} are vector bundles. The parameterizing space is known as the base manifold. Note that at $(\omega/c, \boldsymbol{k}) = (\omega_p/c, 0)$, all modes degenerate and it is not possible to identify subspaces corresponding to ES and EM modes. In other words, the limits $\lim_{\boldsymbol{k}\to 0} \zeta_0(k)$ and $\lim_{\boldsymbol{k}\to 0} \zeta_{EM}(k)$ do not exist as vector spaces. It is thus necessary to remove the degeneracy point $(\omega/c, \boldsymbol{k}) = (\omega_p/c, \boldsymbol{0})$ from the base manifold. For EM modes, the base manifold is the punctured mass hyperboloid

$$\mathcal{M}_{\omega_p} = \left\{(\omega/c, \boldsymbol{k}) \in \mathbb{R}^{3+1} \setminus \{(\omega_p/c, \boldsymbol{0})\} | \omega = \sqrt{\omega_p^2 + |\boldsymbol{k}|^2 c^2}\right\}. \tag{9.12}$$

For ES modes, the base manifold is the plane $\omega = \omega_p$ with the point $(\omega_p/c, \boldsymbol{0})$ removed; we denote this plane by $\mathcal{P}_{\omega_p}$. This is depicted in Fig. 9.1. Both $\mathcal{M}_{\omega_p}$ and $\mathcal{P}_{\omega_p}$ can be parameterized by just the three vector $\boldsymbol{k}$, and so the base manifolds for ES and EM waves are homeomorphic to the punctured space $\mathbb{R}^3 \setminus \{0\}$. We write the vector bundles as $\pi_{EM} : \zeta_{EM} \to \mathbb{R}^3 \setminus \{0\}$ and $\pi_0 : \zeta_0 \to \mathbb{R}^3 \setminus \{0\}$. Here π_{EM} is the projection onto the base manifold

[1] Modes can be unambiguously labeled as ES or EM when working in the background plasma rest frame. We note, however, that this distinction is not valid in reference frames where the background plasma is flowing [161].

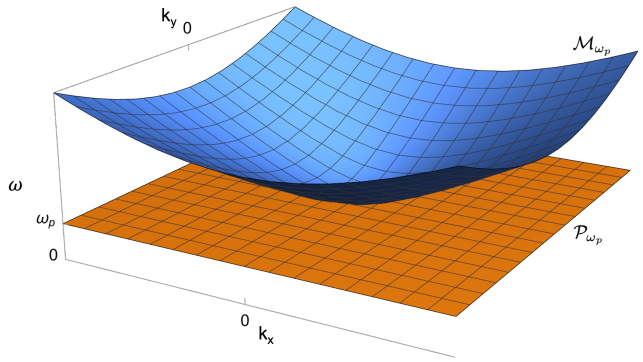


Figure 9.1: Electrostatic (orange) and electromagnetic (blue) dispersion relations for fixed $k_z = 0$. The ES waves reside on the plane $\mathcal{P}_{\omega_p}$ in four-momentum space, while EM waves reside on a mass hyperboloid $\mathcal{M}_{\omega_p}$. The waves degenerate at the Weyl point $\boldsymbol{k} = 0$, $\omega = \omega_p$.

$(\boldsymbol{k}, \boldsymbol{E}) \mapsto \boldsymbol{k}$, and similarly for π_0. The vector spaces $\zeta_{EM}(\boldsymbol{k})$ and $\zeta_0(\boldsymbol{k})$ are called the fibers at $\boldsymbol{k}$. Vector bundles are inherently topological objects, and this description allows a direct study of the topology of plasma waves.

As in the case of the photon and graviton bundles, the vector bundle structure of plasma waves also allows for precise treatment of their geometry, in particular, their metric and symmetry properties. We will do so by using the formalism of vector bundle representations of symmetry groups developed in Ch. 3. Each fiber is a Hermitian vector space with inner product defined by

$$\langle (\boldsymbol{k}, \boldsymbol{E}), (\boldsymbol{k}, \boldsymbol{E}') \rangle_{\text{pw}} = \boldsymbol{E}^* \cdot \boldsymbol{E}', \tag{9.13}$$

making ζ_{EM} and ζ_0 into Hermitian vector bundles; the subscript emphasizes that this is a pointwise inner product at fixed $\boldsymbol{k}$.

We now consider the spacetime symmetry of these waves. In studying the geometry of EM vacuum waves, the fully relativistic spacetime symmetry group is relevant, namely the (proper orthochronous) Poincaré group $\text{ISO}^+(3,1)$. This group consists of all combinations of spatial rotations, relativistic boosts, and spacetime translations. There are two primary differences in treating plasma waves. The most important is that the rest frame of the background plasma introduces a preferred reference frame. We have written the cold fluid equations assuming the background plasma is stationary, thus the solution space is not

symmetric under boosts.[2] The second difference is that we will work nonrelativistically, which is not possible in the case of photons. However, this does not introduce substantial modifications since this primarily affects boost transformations, which the solution space not possess anyway. Thus, the symmetry group of interest is $G = \mathbb{R}^{3+1} \rtimes \mathrm{SO}(3)$, where $\mathbb{R}^{3+1}$ corresponds to spacetime translations and SO(3) to rotations. The semidirect product $\rtimes$ reflects the fact that spatial rotations and translations do not commute, and in particular that, generically, $T_{\mathsf{a}} R T_{\mathsf{a}^{-1}} \notin \mathrm{SO}(3)$ where $R \in \mathrm{SO}(3)$ is a rotation and T_{a} is a translation by $\mathsf{a} \in \mathbb{R}^{3+1}$. G is the subgroup of the Galilean group not containing boosts; SO(3) is the homogeneous part of G. The actions of $\mathsf{a} \in \mathbb{R}^{3+1}$ and $R \in \mathrm{SO}(3)$ on a mode $(\boldsymbol{k}, \boldsymbol{E})$ are described by

$$\Sigma_{T_{\mathsf{a}}}(\boldsymbol{k}, \boldsymbol{E}) = e^{-ik^\mu \mathsf{a}_\mu}(\boldsymbol{k}, \boldsymbol{E}) \tag{9.14a}$$

$$\Sigma_R(\boldsymbol{k}, \boldsymbol{E}) = (R\boldsymbol{k}, R\boldsymbol{E}) \tag{9.14b}$$

where we use summation notation with the $(-+++)$ signature and $R\boldsymbol{k}$ and $R\boldsymbol{E}$ denote the standard action of SO(3) on 3-vectors. For any $g_1, g_2 \in G$,

$$\Sigma_{g_1 g_2}(\boldsymbol{k}, \boldsymbol{E}) = \Sigma_{g_1} \Sigma_{g_2}(\boldsymbol{k}, \boldsymbol{E}) \tag{9.15}$$

so we have a vector bundle representation of G. The representation is unitary with respect to the Hermitian product (9.13):

$$\langle \Sigma_g(\boldsymbol{k}, \boldsymbol{E}), \Sigma_g(\boldsymbol{k}, \boldsymbol{E}') \rangle_{\mathrm{pw}} = \langle (\boldsymbol{k}, \boldsymbol{E}), (\boldsymbol{k}, \boldsymbol{E}') \rangle_{\mathrm{pw}} \tag{9.16}$$

for any g.

[2] Of course, we could produce a solution space that is boost symmetric by allowing the background flow velocity to be arbitrary, but this would lead to many solutions which are physically equivalent and complicate our analysis.

The representation Σ on the vector bundles also induces a representation on the Hilbert spaces of (square-integrable) sections of ζ_0 and ζ_{EM}, denoted by $L^2(\zeta_0)$ and $L^2(\zeta_{EM})$. A section, written as $\boldsymbol{E}(\boldsymbol{k})$ or $|\boldsymbol{E}(\boldsymbol{k})\rangle$, is a choice of vector in each fiber and describes a superposition of plane waves. The inner product of sections is given by

$$\langle \boldsymbol{E}_1(\boldsymbol{k}), \boldsymbol{E}_2(\boldsymbol{k})\rangle = \int \frac{d^3k'}{\omega(\boldsymbol{k}')} \boldsymbol{E}_1^*(\boldsymbol{k}') \cdot \boldsymbol{E}_2(\boldsymbol{k}'). \tag{9.17}$$

The action of $g \in G$ on a section $\boldsymbol{E}(\boldsymbol{k})$ induced by (9.14) is given by

$$(\tilde{\Sigma}_g \boldsymbol{E})(\boldsymbol{k}) = \Sigma_g \boldsymbol{E}\big(\Sigma_{g^{-1}}\boldsymbol{k}\big), \tag{9.18}$$

that is, the external degree of freedom (DOF) $\boldsymbol{k}$ and the internal DOF (the polarization) are simultaneously transformed. The generators of the SO(3) symmetries are the angular momentum operators $\boldsymbol{J} = (J_1, J_2, J_3)$, the generators of the $\mathbb{R}^{3+1}$ translational symmetry are the momentum operators $\boldsymbol{P} = (P_1, P_2, P_3)$ and the Hamiltonian H. Explicitly,

$$J_n \boldsymbol{E} = i \frac{d}{d\varphi}\Big|_{\varphi=0} \tilde{\Sigma}(R_n(\varphi))\boldsymbol{E}, \tag{9.19}$$

$$[P_n \boldsymbol{E}](\boldsymbol{k}) = k_n \boldsymbol{E}(\boldsymbol{k}), \tag{9.20}$$

$$[H\boldsymbol{E}](\boldsymbol{k}) = \omega(\boldsymbol{k})\boldsymbol{E}(\boldsymbol{k}), \tag{9.21}$$

where $R_a(\varphi)$ is a rotation by angle φ about the n axis. The expectation values of these operators coincide with the total fluid energy, momentum, and angular momentum, as shown in §9.5, Eqs. (9.68)-(9.70) and Appendix B.

9.3 Topology of plasma waves

By Theorem 2.3, we can classify the topology of EM and ES waves by their Chern numbers, which can be thought of as a measure of how globally nontrivial or "twisted" the vector bundle is [101].

9.3.1 Topology of electrostatic waves

It is straightforward to see that ES waves are topologically trivial and thus have Chern number 0. A vector bundle is topologically trivial if and only if it has a smoothly varying basis over the whole bundle. That is to say, a rank r vector bundle is trivial if it has r nowhere-vanishing sections $(\psi_1, \dots \psi_r)$ such that $(\psi_1(\boldsymbol{k}), \dots, \psi_r(\boldsymbol{k}))$ are linearly independent every $\boldsymbol{k}$. Since ζ_0 is a line bundle ($r = 1$), it is trivial because $\boldsymbol{E}(\boldsymbol{k}) = E_0\hat{\boldsymbol{k}}$ is such a smooth basis for any nonzero $E_0 \in \mathbb{C}$. Thus, ζ_0 is isomorphic to the bundle $\mathbb{R}^3 \setminus \{0\} \times \mathbb{C}$.

9.3.2 Topology of ζ_{EM}

It is much less obvious that ζ_{EM} is topologically nontrivial. Because any $(\boldsymbol{k}, \boldsymbol{E}) \in \zeta_{EM}$ satisfies the transversality condition $\boldsymbol{k} \cdot \boldsymbol{E} = 0$, a nonvanishing section $\boldsymbol{E}(\boldsymbol{k})$ would imply a choice of transverse vector at every point on the unit $\boldsymbol{k}$ sphere. Naively, this would appear to violate the hairy ball theorem ([39], Theorem 1). However, as Theorem 3.3 shows, the hairy ball theorem only applies to real vector fields [1]; it does not apply in the present setting as $\boldsymbol{E}(\boldsymbol{k})$ is generally complex.

In fact, we can show that ζ_{EM} is topologically equivalent (vector bundle isomorphic) to the photon bundle γ constructed in Chapter 3 [1], and therefore, ζ_{EM} is trivial by the same arguments applied to γ. Recall that the photon bundle consists of all of the vacuum solutions of Maxwell's equations in $\boldsymbol{k}$ space. It is constructed in exactly the same way as ζ_{EM} except with $\omega_p = 0$. For all values of ω_p, the EM modes are described by $(\boldsymbol{k}, \boldsymbol{E})$ with $\boldsymbol{E}$ transverse to $\boldsymbol{k}$, thus, the bundles ζ_{EM} and γ are isomorphic for all values of ω_p, including for $\omega_p = 0$.

Thus, we can apply all of the results from Ch. 3 on the topology of the photon bundle to the EM plasma bundle ζ_{EM}. In particular, one can show that the Chern number of ζ_{EM} is 0 by observing that ζ_{EM} is the complexification of a real vector bundle and that such bundles always have vanishing first Chern number (§3.2.1). This is sufficient to prove that ζ_{EM} is topologically trivial, and thus there exists a globally smooth orthonormal basis for these EM waves. It takes substantially more work to explicitly find such a basis, but as we saw it can be accomplished with the use of the modified clutching construction developed in §3.2.3 [1, 41]. The explicit basis found for the photon bundle in §3.2.3 also gives a basis for ζ_{EM}. In particular, define

$$\boldsymbol{\epsilon}_{1+}(\boldsymbol{k}) = \quad \sin(\phi)\boldsymbol{e}_\theta + \cos(\phi)\boldsymbol{e}_\phi \tag{9.22a}$$

$$\boldsymbol{\epsilon}_{2+}(\boldsymbol{k}) = -\cos(\phi)\boldsymbol{e}_\theta + \sin(\phi)\boldsymbol{e}_\phi \tag{9.22b}$$

$$\boldsymbol{\epsilon}_{1-}(\boldsymbol{k}) = -\sin(\phi)\boldsymbol{e}_\theta + \cos(\phi)\boldsymbol{e}_\phi \tag{9.22c}$$

$$\boldsymbol{\epsilon}_{2-}(\boldsymbol{k}) = -\cos(\phi)\boldsymbol{e}_\theta - \sin(\phi)\boldsymbol{e}_\phi. \tag{9.22d}$$

where (θ, ϕ) are spherical coordinates in $\boldsymbol{k}$ space and $(\hat{\boldsymbol{k}}, \boldsymbol{e}_\theta, \boldsymbol{e}_\phi)$ are the standard spherical unit vectors and $\boldsymbol{k} \neq 0$. The orthonormal basis for ζ_{EM} is given by $[\boldsymbol{\epsilon}_1(\boldsymbol{k}), \boldsymbol{\epsilon}_2(\boldsymbol{k})]$ where

$$[\boldsymbol{\epsilon}_1(\boldsymbol{k}), \boldsymbol{\epsilon}_2(\boldsymbol{k})] = \begin{cases} [\boldsymbol{\epsilon}_{1+}, \boldsymbol{\epsilon}_{2+}] & \text{for } 0 \leq \theta \leq \frac{\pi}{2} \\ [\mathcal{F}(\theta,\phi)\boldsymbol{\epsilon}_{1-}, \mathcal{F}(\theta,\phi)\boldsymbol{\epsilon}_{2-}] & \text{for } \frac{\pi}{2} < \theta \leq \pi \end{cases} \tag{9.23}$$

and $\mathcal{F}(\theta,\phi)$ is a 2D linear transformation which in the $(\boldsymbol{e}_\theta, \boldsymbol{e}_\phi)$ basis is given by

$$F(\theta,\phi) \doteq \begin{pmatrix} x\big(g(\theta),\phi\big) + iz\big(g(\theta),\phi\big) & y\big(g(\theta),\phi\big) \\ -y\big(g(\theta),\phi\big) & x\big(g(\theta),\phi\big) - iz\big(g(\theta),\phi\big) \end{pmatrix} \tag{9.24a}$$

$$x(\theta,\phi) \doteq \cos(2\phi)\sin^2(\theta) + \cos^2(\theta), \tag{9.24b}$$

$$y(\theta,\phi) \doteq \sin(2\phi)\sin(\theta), \tag{9.24c}$$

$$z(\theta,\phi) \doteq -\sin^2(\phi)\sin(2\theta) \tag{9.24d}$$

$$h(t) \doteq \begin{cases} e^{-1/t} & t > 0 \\ 0 & t \leq 0, \end{cases} \tag{9.24e}$$

$$g(\theta) \doteq \frac{\pi}{2}\Big(1 + \frac{h(\theta - \pi/2)}{h(\theta-\pi/2) + h(\pi - \theta)}\Big). \tag{9.24f}$$

It is carefully constructed to be smooth at the poles and along the equator. This basis is fairly complicated, but it demonstrates that it is possible to construct an orthonormal basis for EM waves. Thus in theory it may be possible to find a simpler global bases via other methods. This may be useful, for example, in computational plasma physics by providing a smoothly varying spectral basis for the fields without restricting the domain of $\boldsymbol{k}$.

Since $\zeta_{EM} \cong \gamma$, the other topological results obtained for γ in Ch. 3 apply directly to ζ_{EM} as well. In particular, it is not possible to construct a global basis consisting of only linear polarizations.Intuitively, this is because linear polarizations are scalar multiples of real polarization vectors, and thus, in this case, the hairy ball theorem does apply to preclude any globally nonvanishing linearly polarized vector fields. A global linear polarization basis would imply that $\zeta_{EM} \cong \gamma \cong \ell_1 \oplus \ell_2$ where ℓ_1 and ℓ_2 are trivial linearly polarized line bundles. However, there do not exist *any* linearly polarized subbundles of $\zeta_{EM} \cong \gamma$, let alone topologically trivial ones (Theorem 3.14).

9.3.3 R and L waves are topologically nontrivial

While the complete set of EM waves is topologically trivial, it is not possible to trivialize it using only linear polarizations as we just argued. One may ask an alternative question: is it possible to write a global basis with one basis vector R circularly polarized and the other L circularly polarized? The answer is no, but the obstruction is quite different from that which prevents a linear basis. To begin with, the circularly polarized waves become relevant once one considers the geometry of the EM waves, that is, once we consider the action of the spacetime symmetry group G. While ζ_{EM} is a unitary representation of G, it is not an irreducible representation, meaning that there are rotationally and translationally symmetric subbundles of ζ_{EM}. These subbundles are the R and L subbundles ζ_+ and ζ_-. $\zeta_\pm$ consists of all $(\boldsymbol{k}, \boldsymbol{E}) \in \zeta_{EM}$ such that $\boldsymbol{E} = c(\boldsymbol{e}_1 \pm i\boldsymbol{e}_2)$ for some $c \in \mathbb{C}$ and some right-handed orthonormal coordinate system $(\boldsymbol{e}_1, \boldsymbol{e}_2, \hat{\boldsymbol{k}})$. ζ_{EM} is the direct sum of these bundles

$$\zeta_{EM} = \zeta_+ \oplus \zeta_-. \tag{9.25}$$

That $\zeta_\pm$ are irreducible representations of G means that an R or L polarized wave will always remain R or L polarized after rotation or translation. It is important to note that the R and L waves behave differently from the spin up and spin down states of an electron or a massive spin 1 particle. The spin up and spin down states can be transformed into each other via a change in reference frame, while no change of reference frame (no element of G) can transform an R wave into an L wave. We note that decomposing a representation into its irreducible subrepresentations is a central problem in representation theory and particle physics. This is true in the latter because particles are typically defined as irreducible representations of the Poincaré group [43, 44]; in the present case we are essentially identifying the quasiparticles of our plasma system. The Langmuir waves are also quasiparticles in this sense, as ζ_0 has rank 1 and is thus irreducible. Unlike the total EM bundle ζ_{EM}, the circularly polarized bundles $\zeta_\pm$ are not topologically trivial. This is most easily seen by calculating the Chern

numbers via the Berry connection; the calculation is the same as that for vacuum photons given in Theorem (3.13) which shows that the Chern numbers are $C(\zeta_\pm) = \mp 2$. The nonzero Chern numbers indicate that the bundles $\zeta_\pm$ are nontrivial. It is therefore not possible to form a global nonvanishing basis of either ζ_+ or ζ_-; put differently, if $\boldsymbol{E}_R(\boldsymbol{k})$ denotes a superposition of R polarized modes which varies continuously with $\boldsymbol{k}$, then there must be at least one $\boldsymbol{k}_0$ such that $\boldsymbol{E}_R(\boldsymbol{k}_0) = 0$, and similarly for L polarized waves. This means that it is not possible to have monochromatic R (or L) polarized waves traveling in all directions simultaneously; there must be at least one dark spot. Notice that this obstruction to a global circularly polarized basis is a result of the topological nontriviality of the well-defined vector bundles $\zeta_\pm$. In contrast, the obstruction to a global linearly polarized basis was due to the nonexistence of linearly polarized subbundles.

The geometric separation into R and L waves can be carried out more systematically by using Wigner's little group method [42, 43], and this shows how these waves are characterized by their helicity. This method was originally used to characterize the UIRs of the Poincaré group, resulting in Wigner's classification of particles. We can apply the same idea here to classify the quasiparticles in the cold plasma equations. The vector bundle version of the little group method was developed in detail in §3.4 [1, 47]. Here we will use a simplified version of this technique which suffices because the symmetry group $G = \mathbb{R}^{3+1} \rtimes \mathrm{SO}(3)$ does not contain boosts. Heuristically, the idea of the little group method is to find an action of a group H which is smaller than G, but which still contains enough information to classify the action of G.

Consider $\pi : \zeta \to \mathbb{R}^3 \setminus \{0\}$ where $\zeta = \zeta_0$ or ζ_{EM}. The little group at $\boldsymbol{k}$ is defined as the subgroup of $H_{\boldsymbol{k}}$ of homogeneous transformations which leave $\boldsymbol{k}$ invariant, that is,

$$H_{\boldsymbol{k}} \doteq \{R \in \mathrm{SO}(3) | R\boldsymbol{k} = \boldsymbol{k}\}. \tag{9.26}$$

It is easy to see that $H_{\boldsymbol{k}}$ consists of all rotations $R_{\boldsymbol{k}}(\varphi)$ about $\boldsymbol{k}$ for $\varphi \in [0, 2\pi)$, and thus $H_{\boldsymbol{k}}$ is isomorphic to SO(2) via $R_{\boldsymbol{k}}(\varphi) \mapsto \varphi$. Since $H_{\boldsymbol{k}}$ does not change $\boldsymbol{k}$, the fiber $\zeta(\boldsymbol{k})$ is a unitary representation of SO(2). The generator of this action is the helicity operator $\chi = \hat{\boldsymbol{P}} \cdot \boldsymbol{J} = \hat{\boldsymbol{k}} \cdot \boldsymbol{J}$. The representations of SO(2) are particularly simple; they are all 1D and are spanned by the eigenvectors of the generator $\hat{\boldsymbol{k}} \cdot \boldsymbol{J}$. Denote the eigenvectors by $(\boldsymbol{k}, \boldsymbol{E}_n)$ and eigenvalues as h_n so that

$$(\hat{\boldsymbol{k}} \cdot \boldsymbol{J})(\boldsymbol{k}, \boldsymbol{E}_n) = h_n(\boldsymbol{k}, \boldsymbol{E}_n). \tag{9.27}$$

Exponentiating gives

$$\Sigma\big(R_{\boldsymbol{k}}(\varphi)\big)(\boldsymbol{k}, \boldsymbol{E}) = e^{-ih_n\varphi}(\boldsymbol{k}, \boldsymbol{E}). \tag{9.28}$$

The condition that $\Sigma\big(R_{\boldsymbol{k}}(2\pi)\big) = 1$ implies that h_n must be an integer. Since the integers are discrete, they cannot vary with $\boldsymbol{k}$. Thus, the vector space spanned by $(\boldsymbol{k}, \boldsymbol{E}_n)$ smoothly varies with $\boldsymbol{k}$ and these vector spaces fit together to form a line bundle ζ_{h_n} (assuming no two h_n are degenerate). Thus, we can decompose ζ as

$$\zeta = \zeta_{h_1} \oplus \cdots \oplus \zeta_{h_r}. \tag{9.29}$$

Furthermore, it is straightforward to check explicitly that the helicity operator $\hat{\boldsymbol{k}} \cdot \boldsymbol{J}$ commutes with all the generators of G, and thus the ζ_{h_n} are irreducible bundle representations of G. Applying this to the case of ζ_{EM}, we note that if $(\boldsymbol{e}_1, \boldsymbol{e}_2, \hat{\boldsymbol{k}})$ is a right-handed orthonormal basis of $\mathbb{R}^3$, then

$$\Sigma\big(R_{\boldsymbol{k}}(\varphi)\big)(\boldsymbol{k}, \boldsymbol{e}_1 \pm i\boldsymbol{e}_2) = e^{\mp i\varphi}(\boldsymbol{k}, \boldsymbol{e}_1 \pm i\boldsymbol{e}_2), \tag{9.30}$$

so the helicities are ± 1. This shows that the R and L bundles, $\zeta_\pm$, defined earlier result from the little group method as the helicity ± 1 bundles. As the notation ζ_0 for the ES bundle suggests, it is the helicity zero bundle since the modes $(\boldsymbol{k}, E\hat{\boldsymbol{k}})$ are invariant under the action

of $R_{\boldsymbol{k}}(\varphi)$. The upshot is that the little group method furnishes the helicity operator $\boldsymbol{J}\cdot\hat{\boldsymbol{k}}$ which lifts the internal two-fold degeneracy of the EM waves.

9.3.4 The electromagnetic-electrostatic resonance as a precondition for topological nontriviality

The EM dispersion relation $\omega_\pm = \sqrt{\omega_p^2 + k^2c^2}$ is the same as that of a relativistic particle with mass $m_0 = \hbar\omega_p/c^2$. This can be understood as the inertia of the plasma giving the EM waves an effective mass. Massive particles are typically associated with trivial topology, and for this reason it is surprising that we find $\zeta_\pm$ are topologically nontrivial. Indeed, for elementary massive particles the momentum space is the *non-punctured* mass hyperboloid [1]

$$\bar{\mathcal{M}}_{m_0} = \left\{(\omega/c, \boldsymbol{k}) \in \mathbb{R}^{3+1} \middle| \omega = \sqrt{|\boldsymbol{k}|^2c^2 + \frac{m^2c^4}{\hbar^2}}\right\}. \tag{9.31}$$

This is a contractible manifold, meaning that it can be smoothly deformed onto a single point. As every vector bundle over a contractible base manifold is topologically trivial (Corollary 2.2), the vector bundles describing the states of elementary massive particles are trivial. Massless particles, such as vacuum photons, are not subject to this constraint since their momentum space is given by the forward light cone

$$\mathcal{L}_+ = \left\{(\omega/c, \boldsymbol{k}) \in \mathbb{R}^{3+1} \setminus \{(0, \boldsymbol{0})\} | \omega = |\boldsymbol{k}|c\right\} \tag{9.32}$$

with the vertex at $\omega = 0$, $\boldsymbol{k} = 0$ removed since massless particles have no rest frame. Note that $\mathcal{L}_+ \cong \mathbb{R}^3 \setminus \{0\}$, since it can be parameterized by just $\boldsymbol{k}$. $\mathcal{L}_+$ is not contractible due to this hole at the origin, and as such this momentum space can support topologically nontrivial particle bundles such as the R and L photon bundles $\gamma_\pm$.

The reason the EM plasma bundles $\zeta_\pm$ can be topologically nontrivial is due to the resonance with the ES bundle ζ_0 at $(\omega/c, \boldsymbol{k}) = (\omega_p/c, \boldsymbol{0})$. The ES modes can be thought of as longitudinal photons in plasma [161]. In the vacuum, such longitudinal modes are

nonphysical as they violate Gauss's law, but these modes reappear once an effective mass is introduced. As discussed in §9.2, the point of resonance must be removed from the base manifolds of $\zeta_\pm$ and ζ_0 to form well-defined vector bundles. This creates a hole in momentum space which allows for the possibility of topologically nontrivial behavior of the plasma waves.

9.4 Angular momentum eigenmodes, spin-weighted spherical harmonics, and the nonexistence of SAM and OAM operators

We now consider the following question: what are the eigenmodes of the angular momentum operator for plasma waves? The angular momentum operator $\boldsymbol{J}$ is the generator of the rotational symmetry, so we will restrict our focus to the symmetry group SO(3). It is helpful to first separate the radial and angular variations in momentum space since only the latter is important for studying the angular momentum: ([157], p. 160)

$$L^2(\zeta_h) = L^2(\mathbb{R}^+) \otimes L^2(\zeta_h|_{S^2}). \tag{9.33}$$

Here, $L^2(\mathbb{R}^+)$ is the space of square-integrable functions of the radial coordinate $|\boldsymbol{k}|$ with respect to the measure $|\boldsymbol{k}|^2 d|\boldsymbol{k}|$ and $\gamma_h|_{S^2}$ is the bundle of monochromatic waves obtained from ζ by restricting the base manifold $\mathbb{R}^3 \setminus \{0\}$ to the unit sphere S^2 in momentum space. This expresses the fact that any field configuration $\boldsymbol{E}_h(\boldsymbol{k}) \in \zeta_h$ can be written as a convergent sum of terms of the form $E(|\boldsymbol{k}|)\boldsymbol{\alpha}_h(\hat{\boldsymbol{k}})$ where $\boldsymbol{\alpha}_h \in L^2(\zeta_h|_{S^2})$. The angular momentum operator $\boldsymbol{J}$ only acts nontrivially on $L^2(\zeta|_{S^2})$ since $|\boldsymbol{k}|$ is invariant under rotations.

The Hilbert space $\Gamma(\zeta_h|_{S^2})$ is an infinite-dimensional unitary representation of SO(3) for $h = 0, \pm 1$. Since the UIRs of SO(3) are all finite-dimensional, as is true of any compact group (cf. [124], Theorem 3.9), $\Gamma(\zeta_h|_{S^2})$ must decompose into an infinite number of UIRs. The (non-projective) vector space UIRs of SO(3) are classified by the nonnegative integers

n and have dimension $2n+1$; this follows from the usual ladder operator argument [83]. This means that there must exist multiplets of monochromatic field configurations $\boldsymbol{E}_{hjm}(\hat{\boldsymbol{k}})$ which are closed under rotations, where h and j label a multiplet and m spans all integers between $-j$ and j. That is, if R is a rotation, then $\tilde{\Sigma}_R \boldsymbol{E}_{hjm}$ is a linear combination of the $\boldsymbol{E}_{hjm'}$ for $|m'| \leq j$. Indeed, by the representation theory of SO(3), these multiplets solve the equations

$$J_3 \boldsymbol{E}_{hjm} = m \boldsymbol{E}_{hjm}, \tag{9.34a}$$

$$J_\pm \boldsymbol{E}_{hjm} = [(j \mp m)(j+1 \pm m)]^{1/2} \boldsymbol{E}_{hjm}, \tag{9.34b}$$

$$J^2 \boldsymbol{E}_{hjm} = j(j+1) \boldsymbol{E}_{hjm} \tag{9.34c}$$

where $J_\pm = J_1 \pm i J_2$ and $J^2 = J_1^2 + J_2^2 + J_3^2$. Note that if $\boldsymbol{E}_{hjm}$ were a scalar function rather than a section of a vector bundle, then these equations would be equivalent to the angular part of the time-independent Schrödinger equation in a spherically symmetric potential, which is solved by the ordinary spherical harmonics. In fact, this is precisely the solution for Langmuir waves since such waves are effectively scalar. Indeed, if we fix $h = 0$, then we can write $\boldsymbol{E}_{0jm}(\hat{\boldsymbol{k}}) = E_{0jm}(\hat{\boldsymbol{k}})\hat{\boldsymbol{k}}$. Since the vector field $\hat{\boldsymbol{k}}$ is invariant under all rotations, it follows that $J_a \hat{\boldsymbol{k}} = 0$ for any a. Thus,

$$(J_a \boldsymbol{E}_{0jm})(\hat{\boldsymbol{k}}) = \frac{d}{d\varphi}\Big|_{\varphi=0} R_a(\varphi) \boldsymbol{E}_{0jm}\big(R_a(-\varphi)\hat{\boldsymbol{k}}\big) \tag{9.35a}$$

$$= \hat{\boldsymbol{k}} \frac{d}{d\varphi}\Big|_{\varphi=0} E_{0jm}\big(R_a(-\varphi)\hat{\boldsymbol{k}}\big) \tag{9.35b}$$

$$= (L_a E_{0jm})\hat{\boldsymbol{k}} \tag{9.35c}$$

where

$$\boldsymbol{L} = -i\Big(\boldsymbol{e}_\phi \partial_\theta - \boldsymbol{e}_\theta \frac{1}{\sin\theta} \partial_\phi\Big) \tag{9.36}$$

is the angular momentum operator on functions ([158], Eq. B.24) and $L_a = \boldsymbol{e}_a \cdot \boldsymbol{L}$. For $h = 0$, Eq. (9.34) then reduces to

$$L_3 E_{0jm} = m E_{0jm}, \tag{9.37a}$$

$$L_\pm E_{0jm} = [(j \mp m)(j + 1 \pm m)]^{1/2} E_{0jm}, \tag{9.37b}$$

$$L^2 E_{0jm} = j(j+1) E_{0jm}. \tag{9.37c}$$

These are solved by the ordinary spherical harmonics

$$E_{0jm}(\hat{\boldsymbol{k}}) = Y_{jm}(\theta, \phi). \tag{9.38}$$

Langmuir waves can thus be expanded in terms of the spherical harmonics, and the labels j and m describe the total and z angular momentum, respectively.

The situation is more interesting for EM waves with $h = \pm 1$ due to the nontrivial vector polarizations. We will work in a particular basis for R and L waves, namely,

$$\boldsymbol{e}_h = \frac{1}{\sqrt{2}}(\boldsymbol{e}_\theta + i \operatorname{sgn}(h) \boldsymbol{e}_\phi). \tag{9.39}$$

This basis will allow us to express the angular momentum basis in terms of a familiar set of special functions. Note that this basis has discontinuities at the poles, thus, if we write

$$\boldsymbol{E}_{hjm}(\hat{\boldsymbol{k}}) = E_{hjm}(\hat{\boldsymbol{k}}) \boldsymbol{e}_h \tag{9.40}$$

where $\boldsymbol{E}_{hjm}(\hat{\boldsymbol{k}})$ is everywhere smooth, then $E_{hjm}(\hat{\boldsymbol{k}})$ will be discontinuous at the poles (unless $\boldsymbol{E}_{hjm}$ vanishes at the poles); importantly, these are coordinate singularities in the representation, not true singularities of $\boldsymbol{E}_{hjm}$. We also define

$$\boldsymbol{e}_0 = \hat{\boldsymbol{k}} \tag{9.41}$$

so that the following analysis also applies to the $h = 0$ case. In this basis, we can write

$$J_3 \boldsymbol{E}_{hjm} = (J'_{h3} E_{hjm}) \boldsymbol{e}_h \tag{9.42a}$$

$$J_\pm \boldsymbol{E}_{hjm} = (J'_{h\pm} E_{hjm}) \boldsymbol{e}_h \tag{9.42b}$$

$$J^2 \boldsymbol{E}_{hjm} = (J'^2_h E_{hjm}) \boldsymbol{e}_h \tag{9.42c}$$

for some operators $\boldsymbol{J}'_h$ on scalar functions. The same set of equations (for arbitrary integer h) appeared in our analysis of the angular momentum eigenstates of massless particles §8.4 [6], where it was shown that

$$J'_{hz} = -i\partial_\phi, \tag{9.43a}$$

$$J'_{h\pm} = e^{\pm i\phi}\Big(\pm \partial_\theta + i\frac{\cos\theta}{\sin\theta}\partial_\phi + \frac{h}{\sin\theta}\Big), \tag{9.43b}$$

$$J'^2_h = -\nabla^2 - \frac{2h\cos\theta}{\sin^2\theta} L_z + \frac{h^2}{\sin^2\theta}. \tag{9.43c}$$

The E_{hjm} thus solve

$$J'_{hz} E_{hjm} = m E_{hjm}, \tag{9.44a}$$

$$J'_{h\pm} E_{hjm} = [(j \mp m)(j + 1 \pm m)]^{1/2} E_{hjm}, \tag{9.44b}$$

$$J'^2_h E_{hjm} = j(j+1) E_{hjm}. \tag{9.44c}$$

Dray [89] showed that these equations are solved by the SWSHs ${}_{-h}Y_{jm}$ rather than the ordinary spherical harmonics. The SWSHs are given explicitly by [155]

$$\begin{aligned} {}_hY_{jm}(\theta,\phi) &= \Big[\frac{(j+m)!(j-m)!(2j+1)}{4\pi(j+h)!(j-h)!}\Big]^{\frac{1}{2}} (\sin\theta/2)^{2j} \\ &\times \sum_q \binom{j-h}{q}\binom{j+h}{q+h-m} (-1)^{j-q-h-m} e^{im\phi} (\cot\theta/2)^{2q+h-m} \end{aligned} \tag{9.45}$$

where $j \geq |h|$, $-j \leq m \leq j$ and the sum ranges from $q = \max(0, m-h)$ to $q = \min(j-h, j+m)$. These are special functions which were first defined by Newman and Penrose in their study of asymptotically flat spacetimes [86], but have since found applications in many other areas including the interaction of charged particles with magnetic monopoles [87–90], the study of complex spacetimes [91], analysis of the cosmic microwave background radiation [93–95], in geophysical [92] and computer graphics applications [154], and in the analysis of the angular momentum of massless particles [6]. This appears to be the first time the SWSHs have shown up in the context of plasma physics.

The simultaneous eigenstates of J^2 and J_z in $L^2(\zeta_h|_{S^2})$ are thus given by

$$\boldsymbol{E}_{hjm}(\hat{\boldsymbol{k}}) = {}_{-h}Y_{jm}(\theta,\phi)\boldsymbol{e}_h \tag{9.46}$$

Note that when $h = 0$, the SWSHs reduce to the ordinary spherical harmonics, ${}_0Y_{jm} = Y_{jm}$, so we recover the ES result in Eq. (9.38). For fixed h, the SWSHs are orthogonal [89],

$$\int_{S^2} {}_hY^*_{jm}\,{}_hY_{j'm'}\sin\theta\,d\theta\,d\phi = \delta_{jj'}\delta_{mm'}, \tag{9.47}$$

so that any $\boldsymbol{E}(\boldsymbol{k}) \in L^2(\zeta_+) \oplus L^2(\zeta_0) \oplus L^2(\zeta_-)$ is easily expanded in terms of SWSHs:

$$\boldsymbol{E}(\boldsymbol{k}) = \sum_{h=-1}^{1}\sum_{j=|h|}^{\infty}\sum_{|m|\leq j}\alpha_{hjm}(|\boldsymbol{k}|)\,{}_{-h}Y_{jm}(\theta,\phi)\boldsymbol{e}_h, \tag{9.48a}$$

$$\alpha_{hjm}(|\boldsymbol{k}|) = \int {}_{-h}Y^*_{jm}(\theta,\phi)\boldsymbol{e}^*_h\cdot\boldsymbol{E}(\boldsymbol{k})\sin\theta\,d\theta\,d\phi. \tag{9.48b}$$

Note that this gives a countable basis for monochromatic waves, as an arbitrary monochromatic wave $\boldsymbol{E}(\hat{\boldsymbol{k}})$ is specified by a countable set of complex numbers α_{hjm}. Compare this, for example, to the expansion $\boldsymbol{E}(\hat{\boldsymbol{k}}) = \sum_h E_h(\hat{\boldsymbol{k}})\boldsymbol{e}_h$ in terms of the momentum eigenstates $\boldsymbol{e}_h$—in this case the state is specified by an uncountable set of numbers $E_h(\hat{\boldsymbol{k}})$ as $\hat{\boldsymbol{k}}$ spans over the unit sphere. We have essentially followed the technique frequently employed in

quantum mechanics of lifting degeneracies using commuting operators. In this case, the ${}_{-h}Y_{jm}\boldsymbol{e}_h\delta(|\boldsymbol{k}|-|\boldsymbol{k}_0|)$ are simultaneous eigenvectors of the energy H, helicity χ, J^2, and J_3. Indeed, since we can describe any wave using these eigenstates, these four operators give a complete set of commuting operators (CSCO) for plasma waves, that is, a basis for the states is determined by the eigenvalues $(\omega_h(|\boldsymbol{k}|), h, j, m)$. Compare this to a massive particle with spin and orbital DOFs, for which a CSCO is given by the *five* operators H, J^2, J_3, S^2, L^2 [160]. S^2 and h are determined by the identity of the particle (i.e., by the UIR of the symmetry group), but in the massive case it is necessary to specify both J^2 and L^2 (assuming nonzero spin), while in the massless case J^2 suffices. This observation suggests that there may be issues with defining SAM and OAM operators for EM plasma waves, a topic that we will examine in greater detail.

The form of the SWSHs reveals some interesting features of the rotational symmetry and angular momentum of plasma waves. We note first that there are no SWSHs with $j < |h|$, showing that $|h|$ is a lower bound on the total angular momentum, in particular,

$$\frac{\langle \boldsymbol{E}_h | J^2 | \boldsymbol{E}_h \rangle}{|\boldsymbol{E}_h|^2} \geq |h|(|h|+1). \tag{9.49}$$

Thus, R and L waves have no $j = 0$ state, reflecting the fact that there is no wave $\boldsymbol{E}_\pm(\hat{\boldsymbol{k}})$ that is rotationally invariant. This agrees with our topological analysis of R and L waves, for such a rotationally invariant $\boldsymbol{E}_\pm(\hat{\boldsymbol{k}})$ would have to be nonvanishing for all $\hat{\boldsymbol{k}}$, which is prohibited by the topological nontriviality of $\zeta_\pm$. Notice that this lack of $j = 0$ states is not consistent with the existence of a standard SAM-OAM decomposition of the angular momentum, as we argued in §8.5. For example, consider a massive spin 1 particle, with the SO(3) symmetry of its spin space described by the representation V_1 and its orbital space by

$$\mathcal{H}_o = V_0 \oplus V_1 \oplus V_2 \oplus \cdots . \tag{9.50}$$

Here, V_l denotes the canonical SO(3) representations with dimension $2l+1$, and in Eq. (9.50) they are spanned by the ordinary spherical harmonic multiplets Y_{lm} with $|m| \leq l$. The total angular momentum multiplet structure is determined by the addition of angular momentum formula, where for $a \geq b$,

$$V_a \otimes V_b \cong V_{a-b} \oplus V_{a-b+1} \oplus \cdots \oplus V_{a+b}. \tag{9.51}$$

The complete angular Hilbert space for a spin 1 particle is then

$$V_1 \otimes (V_0 \oplus V_1 \oplus \cdots) = V_0 \oplus 3V_1 \oplus 3V_2 \oplus \cdots . \tag{9.52}$$

In particular, the copy of V_0 shows that there is a total angular momentum 0 state. In general, if the total orbital angular momentum can assume the same value s as the total spin angular momentum, as is true for all massive particles particles, then there must exist a total angular momentum 0 multiplet resulting from $V_s \otimes V_s$. R and L plasma waves possess no such multiplet, further suggesting that plasma waves do not admit an SAM-OAM splitting. Additional support for this conclusion comes from analyzing the full multiplet structure of R and L plasma waves. Indeed, we see that for $h = \pm 1$,

$$\Gamma(\zeta_\pm|_{S^2}) = V_1 \oplus V_2 \oplus V_3 \oplus \cdots \tag{9.53}$$

where each V_j corresponds to the multiplet ${}_{\mp 1}Y_{jm}\boldsymbol{e}_\pm$. There is only one multiplet for each $j \geq 1$. This is inconsistent with the typical multiplet structure of particles with an SAM-OAM decomposition, which typically produce higher multiplicities as in Eq. (9.52). This argument is highly suggestive, but not fully rigorous since it is possible to contrive exotic OAM multiplet structures with missing OAM states which are consistent with Eq. (9.53) [6], for example,

$$V_1 \otimes (V_0 \otimes V_3 \otimes V_6 \otimes \cdots) = V_1 \oplus V_2 \oplus V_3 \oplus \cdots . \tag{9.54}$$

However, as discussed in §8.5, such an OAM multiplet structure is very unnatural, with the OAM label l restricted to integer multiples of 3, a situation that does not show up in any standard physics applications.

We can rigorously show that no SAM-OAM decomposition is possible, ruling out such exotic OAM structures, by essentially repeating an argument that applies to massless particles (cf. no-go Theorem 6.1; [4]). Assume $\boldsymbol{J} = \boldsymbol{S} + \boldsymbol{L}$ where $\boldsymbol{S}$ and $\boldsymbol{L}$ commute with the Hamiltonian and generate SO(3) symmetries. Assume further that $\boldsymbol{S}$ generates an internal symmetry, meaning that it does not alter $\boldsymbol{k}$. Since they commute with the Hamiltonian, $\boldsymbol{S}$ and $\boldsymbol{L}$ do not mix $L^2(\zeta_{EM})$ and $L^2(\zeta_0)$, and thus can be considered to act on EM and Langmuir waves separately. Since $\boldsymbol{S}$ is internal, it is an operator on each fiber $\zeta_0(\boldsymbol{k})$ and $\zeta_{EM}(\boldsymbol{k})$. Since $\boldsymbol{S}$ generates an SO(3) symmetry, the fibers $\zeta_0(\boldsymbol{k})$ and $\zeta_{EM}(\boldsymbol{k})$ are vector space representations of SO(3) of dimension 1 and 2, respectively. We can decompose these into UIRs of SO(3), which are isomorphic to the complex vector spaces V_n with dimension $2n+1$. ζ_0 is one-dimensional, so it must be equivalent to the trivial representation V_0 whose generator is 0. Similarly, there are no non-projective representations of dimension 2, so $\zeta_{EM}(\boldsymbol{k}) \cong V_0 \oplus V_0$ and therefore also corresponds to a trivial representation with generator 0. Thus, we find that $\boldsymbol{S} = 0$ and therefore there is no nontrivial SAM-OAM decomposition.

9.5 Helicity and orbital quasi-angular momenta

We showed in the previous section that there is no true SAM-OAM decomposition for cold plasma waves. By this we mean that if one splits $\boldsymbol{J}$ into two operators $\boldsymbol{S}$ and $\boldsymbol{L}$, then these operators do not generate 3D rotations, or equivalently, they will not satisfy the angular momentum commutation relations. This has significant implications, because the majority of the theoretical tools pertaining to SAM and OAM rely on their SO(3) structure. For example, the addition of angular momentum formula, the Clebsch–Gordan basis, and the Casimir invariance of S^2 and L^2 do not generally hold in the absence of SO(3) symmetry.

Furthermore, we are accustomed to S_z, S^2, L_z, and L^2 having discrete spectra; this too may fail if we consider $\boldsymbol{S}$ and $\boldsymbol{L}$ which do not generate rotational symmetries.

However, we now show that, as in the case of photons [1, 5, 71], it is possible to decompose $\boldsymbol{J}$ into two well-defined vector operators defined purely in terms of symmetry generators:

$$\boldsymbol{J} = \boldsymbol{J}_\parallel + \boldsymbol{J}_\perp \tag{9.55a}$$

$$\boldsymbol{J}_\parallel = (\boldsymbol{J}\cdot\hat{\boldsymbol{P}})\hat{\boldsymbol{P}} = \chi\hat{\boldsymbol{P}} \tag{9.55b}$$

$$\boldsymbol{J}_\perp = -\hat{\boldsymbol{P}}\times(\hat{\boldsymbol{P}}\times\boldsymbol{J}) \tag{9.55c}$$

where $\hat{\boldsymbol{P}} = \boldsymbol{P}/|\boldsymbol{P}|$. That they are vector operators means that they satisfy [83]

$$[J_{\parallel,a}, J_b] = i\epsilon_{abc}J_{\parallel,c} \tag{9.56}$$

$$[J_{\perp,a}, J_b] = i\epsilon_{abc}J_{\perp,c}. \tag{9.57}$$

$\boldsymbol{J}_\parallel$ and $\boldsymbol{J}_\perp$ are the helicity quasi-angular momentum and orbital quasi-angular momentum encountered in §3.5.2 and Chapter 7. The term quasi-angular momentum reflects the fact that these operators cannot be generators of rotation. While the analogous splitting has been previously applied to photons in vacuum [1, 5, 67, 71], this is the first time this operator splitting has been given for EM waves in plasmas, although we will later discuss its similarity with a recently proposed splitting in Ref. [84]. The validity of this decomposition can be established as follows. If $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$ are vector operators such that $\boldsymbol{A}$ and $\boldsymbol{B}$ commute, then by Proposition A.1

$$\boldsymbol{A}\times(\boldsymbol{B}\times\boldsymbol{C}) = \boldsymbol{B}(\boldsymbol{A}\cdot\boldsymbol{C}) - (\boldsymbol{A}\cdot\boldsymbol{B})\boldsymbol{C}. \tag{9.58}$$

Setting $\boldsymbol{A} = \boldsymbol{B} = \hat{\boldsymbol{P}}$ and $\boldsymbol{C} = \boldsymbol{J}$ and using the fact that $(\boldsymbol{J}\cdot\hat{\boldsymbol{P}})$ commutes with $\hat{\boldsymbol{P}}$ gives Eq. (9.55).

While it is not difficult to check that this splitting is technically valid, it can in fact be derived from a more fundamental theoretical consideration—rotational symmetry. We

showed in Chapter 7, in the context of elementary particles, that this splitting arises from considering connections on vector bundles [5, 99], the same type of mathematical objects which are ubiquitous in general relativity. We showed that every connection $\boldsymbol{D}$ (which can be thought of as a covariant derivative) on a vector bundle over momentum space induces a splitting of the angular momentum of the form

$$\boldsymbol{J} = \boldsymbol{L}^{\boldsymbol{D}} + \boldsymbol{S}^{\boldsymbol{D}} \tag{9.59a}$$

$$\boldsymbol{L}^{\boldsymbol{D}} \doteq -i\boldsymbol{P} \times \boldsymbol{D} \tag{9.59b}$$

$$\boldsymbol{S}^{\boldsymbol{D}} \doteq \boldsymbol{J} - \boldsymbol{L}^{\boldsymbol{D}}. \tag{9.59c}$$

Notice that this is what one would expect from the semiclassical heuristic of taking $\boldsymbol{L} = \boldsymbol{x} \times \boldsymbol{P}$ and making the replacement $\boldsymbol{x} \to i\boldsymbol{D}$. However, $\boldsymbol{L}^{\boldsymbol{D}}$ and $\boldsymbol{S}^{\boldsymbol{D}}$ satisfy SO(3) commutation relations if and only if $\boldsymbol{D}$ has vanishing S^2-curvature (Theorem 7.3; [5], Theorem 3), which is not possible for the EM bundles $\zeta_\pm$ since they are topologically nontrivial. Nevertheless, there is a unique connection $\boldsymbol{D}$ on $\zeta_\pm|_{S^2}$ induced by their rotational symmetry, namely,

$$\boldsymbol{D} = -i\frac{\hat{\boldsymbol{P}} \times \boldsymbol{J}}{|\boldsymbol{P}|}, \tag{9.60}$$

and this induces the splitting into the orbital and helicity quasi-angular momenta:

$$\boldsymbol{L}^{\boldsymbol{D}} = -\hat{\boldsymbol{P}} \times (\hat{\boldsymbol{P}} \times \boldsymbol{J}) = \boldsymbol{J}_\perp \tag{9.61}$$

$$\boldsymbol{S}^{\boldsymbol{D}} = \boldsymbol{J} - \boldsymbol{L}^{\boldsymbol{D}} = \boldsymbol{J}_\parallel. \tag{9.62}$$

Note that the Langmuir bundle is topologically trivial and $\boldsymbol{D}$ is flat on this bundle. However, the induced splitting is trivial in this case since it has helicity $\chi = 0$ and thus $\boldsymbol{J}_\parallel = 0$ and $\boldsymbol{J}_\perp = \boldsymbol{J}$. We have thus shown that the quasi-angular momentum decomposition arises from the same general theory which describes the corresponding decomposition for massless elementary particles as well as the standard SAM-OAM splitting for massive particles [5].

The quasi-angular momentum operators have a number of features which distinguish them from true angular momentum operators. Most importantly, they do not satisfy angular momentum commutation relations, but rather the peculiar relations

$$[J_{\parallel,a}, J_{\parallel,b}] = 0 \neq i\epsilon_{abc} J_{\parallel,c} \tag{9.63a}$$

$$[J_{\perp,a}, J_{\perp,b}] = i\epsilon_{abc}(J_{\perp,c} - J_{\parallel,c}) \neq i\epsilon_{abc} J_{\perp,c}. \tag{9.63b}$$

The commutation relations for the components of $J_\parallel$ show that they actually generate a 3D translational $\mathbb{R}^3$ symmetry rather than a 3D rotation symmetry. That the $\boldsymbol{J}_\perp$ components are not closed under the commutator (due to the $\boldsymbol{J}_{\parallel,c}$ term) shows that the vector operator $\boldsymbol{J}_\perp$ does not generate any symmetry at all.

Spectral analysis reveals another strange feature of these operators. In terms of spherical coordinates $(|\boldsymbol{k}|, \theta, \phi)$ for momentum space,

$$J_{\parallel,z}[\delta(\theta - \theta_0)e^{im\phi}\boldsymbol{e}_\pm] = \frac{P_z}{|\boldsymbol{P}|}\chi[\delta(\theta - \theta_0)e^{im\phi}\boldsymbol{e}_\pm] \tag{9.64a}$$

$$= \pm\frac{k_z}{|\boldsymbol{k}|}\delta(\theta - \theta_0)e^{im\phi}\boldsymbol{e}_\pm \tag{9.64b}$$

$$= \pm\cos(\theta_0)\delta(\theta - \theta_0)e^{im\phi}\boldsymbol{e}_\pm \tag{9.64c}$$

for any integer m, showing that

$$\boldsymbol{\alpha}^\pm_{\theta_0}(\boldsymbol{k}) = \delta(\theta - \theta_0)e^{im\phi}\boldsymbol{e}_\pm \tag{9.65}$$

is a (generalized [83]) eigenvector of $J_{\parallel,z}$ with eigenvalue $\pm\cos(\theta_0)$ on $L^2(\zeta_\pm)$. From the fact that χ and $\hat{\boldsymbol{P}}$ are both bounded operators with norms $||\chi|| = ||\hat{\boldsymbol{P}}|| = 1$, we see that $\boldsymbol{J}_\parallel$ does not have any eigenvalues of magnitude greater than 1. $J_{\parallel,z}$ thus has a continuous spectrum given by the interval $[-1, 1]$. This is in stark contrast to the components of a true angular momentum operator, which always have discrete integer or half-integer eigenvalues.

Furthermore, $J_z\delta(\theta-\theta_0)\boldsymbol{e}_\pm = 0$ since $\delta(\theta-\theta_0)\boldsymbol{e}_\pm$ is invariant about the k_z axis, so it follows that

$$J_z\boldsymbol{\alpha}^\pm_{\theta_0} = \delta(\theta-\theta_0)\boldsymbol{e}_\pm J_z e^{im\phi} \tag{9.66}$$

$$= m\boldsymbol{\alpha}^\pm_{\theta_0}, \tag{9.67}$$

showing that $\boldsymbol{\alpha}^\pm_{\theta_0}$ is also an eigenvector of J_z with eigenvalue m. Thus, the spectrum of $J_{\perp,z} = J_z - J_{\parallel,z}$ contains the interval $[m-1, m+1]$ for each m. Since m can be any integer, the spectrum of $J_{\perp,z}$ is all of $\mathbb{R}$. This again contrasts with the discrete spectral behavior that occurs for true OAM operators.

Nevertheless, operators satisfying the non-SO(3) commutation relations in Eq. (9.63) have long been used in the optics literature and are often called SAM and OAM operators [67, 71, 74, 77, 78]. They are gauge-invariant vector operators and appear to be important in experimental optics, despite not being true SAM and OAM operators [77]. We show in Appendix B that the expectation values of $\boldsymbol{J}_\parallel$ and $\boldsymbol{J}_\perp$, as well as those of $H, \boldsymbol{P}$, and $\boldsymbol{J}$, can be calculated by integrals over position or momentum space:

$$\begin{aligned}
\frac{1}{8\pi}\langle \boldsymbol{E}|H|\boldsymbol{E}\rangle &= \sum_h \frac{1}{8\pi}\int \frac{d^3k}{\omega_h}\boldsymbol{E}_h^*(\boldsymbol{k})\cdot[H\boldsymbol{E}_h](\boldsymbol{k}) \\
&= \sum_h \frac{1}{8\pi}\int d^3k\,|\boldsymbol{E}_h(\boldsymbol{k})|^2 \\
&= \int d^3x\left[\frac{1}{8\pi}\big(\mathcal{E}^2(\boldsymbol{x})+\mathcal{B}^2(\boldsymbol{x})\big)+\frac{1}{2}mn_0v^2(\boldsymbol{x})\right] \qquad (9.68)\\
\frac{1}{8\pi}\langle \boldsymbol{E}|\boldsymbol{P}|\boldsymbol{E}\rangle &= \sum_h \frac{1}{8\pi}\int d^3k\frac{\boldsymbol{k}}{\omega_h}|\boldsymbol{E}_h(\boldsymbol{k})|^2 \\
&= \int d^3x\left[\frac{1}{4\pi c}\boldsymbol{\mathcal{E}}(\boldsymbol{x})\times\boldsymbol{\mathcal{B}}(\boldsymbol{x})+mn_{1,h}(\boldsymbol{x})\boldsymbol{v}(\boldsymbol{x})\right] \qquad (9.69)
\end{aligned}$$

$$\frac{1}{8\pi}\langle \boldsymbol{E}|\boldsymbol{J}|\boldsymbol{E}\rangle = -i\sum_h \frac{1}{8\pi}\int \frac{d^3k}{\omega_h}\Big(\sum_n E^*_{h,n}\cdot(\boldsymbol{k}\times\nabla_{\boldsymbol{k}})E_{h,n} + \boldsymbol{E}^*_h\times\boldsymbol{E}_h\Big)$$
$$= \int d^3x\left[\frac{1}{4\pi c}\boldsymbol{x}\times\big(\boldsymbol{\mathcal{E}}(\boldsymbol{x})\times\boldsymbol{\mathcal{B}}(\boldsymbol{x})\big) + mn_1(\boldsymbol{x})\boldsymbol{x}\times\boldsymbol{v}\right] \tag{9.70}$$
$$\frac{1}{8\pi}\langle \boldsymbol{E}|\boldsymbol{J}_\parallel|\boldsymbol{E}\rangle = -i\sum_h \frac{1}{8\pi}\int \frac{d^3k}{\omega_h}\boldsymbol{E}^*_h\times\boldsymbol{E}_h$$
$$= \frac{1}{4\pi c}\int d^3x\,\boldsymbol{\mathcal{E}}^\perp(\boldsymbol{x})\times\boldsymbol{\mathcal{A}}^\perp(\boldsymbol{x}) \tag{9.71}$$
$$\frac{1}{8\pi}\langle \boldsymbol{E}|\boldsymbol{J}_\perp|\boldsymbol{E}\rangle = -i\sum_{h,n} \frac{1}{8\pi}\int \frac{d^3k}{\omega_h}E^*_{h,n}(\boldsymbol{k}\times\nabla_{\boldsymbol{k}})E_{h,n}$$
$$= \int d^3x\Big[\frac{1}{4\pi c}\sum_n \mathcal{E}^\perp_n(\boldsymbol{x}\times\nabla)\mathcal{A}^\perp_n + mn_1\boldsymbol{x}\times\boldsymbol{v}$$
$$-\frac{e}{c}n_1\boldsymbol{x}\times\boldsymbol{\mathcal{A}}^\perp\Big] \tag{9.72}$$

where $\omega_0 = \omega_p$, $\omega_\pm = \sqrt{\omega_p^2 + |\boldsymbol{k}|^2}$, and $\boldsymbol{E}_h = \boldsymbol{e}^*_h\cdot\boldsymbol{E}$. $\boldsymbol{\mathcal{A}}^\perp$ is the transverse vector potential in position space, the divergence-free part of the Helmholtz decomposition of the vector potential. It is the component of the vector potential generating the EM waves $\boldsymbol{E}_\pm$, but not the Langmuir waves $\boldsymbol{E}_0$. The transverse electric field is given by $\boldsymbol{\mathcal{E}}^\perp = c^{-1}\partial_t\boldsymbol{\mathcal{A}}^\perp$. Unlike the full vector potential, $\boldsymbol{\mathcal{A}}^\perp$ is gauge-invariant, so the expectation values of $\boldsymbol{J}_\parallel$ and $\boldsymbol{J}_\perp$ are also gauge-invariant and thus physically measurable [67]. These expectation values of $\boldsymbol{J}_\parallel$ and $\boldsymbol{J}_\perp$ agree with expressions given recently by Bliokh and Bliokh ([84], cf. Table 1). We note that Ref. [84] referred to these as the spin and orbital components of the canonical angular momentum, but as we have shown, these do not correspond to generators of rotational symmetries, and are thus not true angular momenta. We also note that, unlike the expressions in Ref. [84], Eqs. (9.68)-(9.72) do not assume that the waves are monochromatic or time-averaged. Bliokh and Bliokh show that $\boldsymbol{J}_\parallel$ is related to the induced plasma magnetization in the inverse Faraday effect, which illustrates that the quasi-angular momentum decomposition is experimentally relevant in plasma physics, despite it not being a true SAM-OAM splitting. The $\boldsymbol{J}_\parallel$, $\boldsymbol{J}_\perp$ decomposition is also useful for theoretical calculations, as we illustrated in §8.4.2 when we used these operators to calculate the expressions for J'_z, J'^2,

and $J'_{\pm}$ in Eq. (9.43). In summary, the quasi-angular momentum operators are naturally induced by the rotational connection of cold plasma waves, and can be relevant in both experimental and theoretical contexts. It is important, however, to bear in mind that these are not true SAM and OAM operators, and thus most of the theoretical tools used for angular momentum operators do not apply to them.

9.6 Extension to kinetic plasma models

We indicate in this section how one might generalize our methods to the treatment of kinetic plasmas. The true (generalized) eigenmodes of a kinetic plasma are the Case–van Kampen modes which have a continuous spectrum [102, 164]. The continuous spectrum makes these modes difficult to treat using the vector bundle formalism since it would produce infinite-dimensional fibers. Since vector bundles are typically defined to have finite-dimensional fibers, and since many vector bundle methods require this constraint, treating the Case–van Kampen modes would require a modified formalism. While this may be possible, it is not clear that it would be particularly helpful. Case–van Kampen modes are non-normalizable and correspond to nonphysical initial conditions ([164], §8-10; [102], §9.1), and thus are not particularly convenient for treating many physical problems. The standard and often more useful formalism is the Landau initial value approach [102, 164, 165], which allows one to focus on plasma configurations corresponding to smooth initial conditions. Following the treatment in Ref. [102], by linearizing around a spatially homogeneous distribution function $f_0(\boldsymbol{v})$, one can express the wave equation in terms of the plasma dispersion tensor

$$\boldsymbol{D}_E(\omega, \boldsymbol{k}) = \boldsymbol{N}\boldsymbol{N}^T - \mathbb{1}N^2 + \boldsymbol{\epsilon}(\omega, \boldsymbol{k}) \tag{9.73}$$

where $\boldsymbol{\epsilon}$ is the dielectric tensor of the plasma model and $\boldsymbol{N} = c\boldsymbol{k}/\omega$ is the index of refraction tensor. The dispersion relation is given by

$$\det \boldsymbol{D}_E(\omega, \boldsymbol{k}) = 0. \tag{9.74}$$

Denoting the solution branches by $\omega_n(\boldsymbol{k})$, the quasimodes $\boldsymbol{E}_n(\boldsymbol{k})$ are determined by

$$\det \boldsymbol{D}_E(\omega_n(\boldsymbol{k}), \boldsymbol{k})\boldsymbol{E}_n(\boldsymbol{k}) = 0. \tag{9.75}$$

The term quasimode is used to distinguish these from the Case–van Kampen modes. The problem thus takes the same form as Eq. (9.7) and we see that the quasimodes will globalize to form vector bundles as $\boldsymbol{k}$ is varied. Thus we can study the topology of the quasimodes by the same formalism as we have applied to the cold fluid model. One minor difference is that the ω_n can generally be complex, reflecting the fact that quasimodes can be damped. This does not cause obvious issues since the base manifold can always be parameterized by $\boldsymbol{k}$.

Consider the example of an unmagnetized isotropic plasma, so that $f_0(\boldsymbol{v}) = f_0(v)$ where $v = |\boldsymbol{v}|$. As in the case of our treatment of the cold fluid plasma, we will ignore the ion contribution to the dispersion tensor. If we pick an orthonormal basis $(\boldsymbol{e}_x, \boldsymbol{e}_y, \hat{\boldsymbol{k}})$ at a fixed $\boldsymbol{k}$, then ([102], Eq. (10.14))

$$\boldsymbol{D}_E = \begin{pmatrix} \epsilon_\perp - N^2 & 0 & 0 \\ 0 & \epsilon_\perp - N^2 & 0 \\ 0 & 0 & \epsilon_\parallel \end{pmatrix} \tag{9.76}$$

where $\boldsymbol{N} = c\boldsymbol{k}/\omega$ and

$$\epsilon_\parallel(\omega, \boldsymbol{k}) = 1 - \frac{\omega_p^2}{k^2} \int_L dv \frac{f_0'(v)}{v - \omega/k} \tag{9.77}$$

$$\epsilon_\perp(\omega, \boldsymbol{k}) = 1 - \frac{\omega^2}{\omega_p^2}\Big(1 + \int_L d\boldsymbol{v} \frac{kv_y^2}{\omega - kv_x} \frac{\partial f_0}{\partial v_x}\Big). \tag{9.78}$$

The integral is taken over the the Landau contour L, which bends under the pole. Thus, there is still a clear distinction between the ES modes $\epsilon_{\parallel}(\omega, \boldsymbol{k}) = 0$, which are longitudinally polarized with $\boldsymbol{E} \times \boldsymbol{k} = 0$, and EM modes $\epsilon_{\perp} - N^2 = 0$, which are transversely polarized with $\boldsymbol{E} \cdot \boldsymbol{k} = 0$. We also need to consider the set of Weyl points where ES and EM modes degenerate. For concreteness, let us take the background to be Maxwellian with electron thermal velocity v_T:

$$f_0(v) = \frac{1}{\sqrt{2\pi} v_T} e^{-\frac{v^2}{2v_T^2}}. \tag{9.79}$$

Then, if we neglect relativistic corrections ($v_T \ll c$), the EM dispersion relation is still given by ([102], Eq. (10.20))

$$\omega_{EM}^2 \approx \omega_p^2 + k^2 c^2. \tag{9.80}$$

For small k ($k \ll \omega / v_T$) the real part of the ES frequency is given by ([102], Eq. (11.35))

$$\omega_{ES,r}^2 \approx \omega_p^2 + 3k^2 v_T^2. \tag{9.81}$$

Thus, the modes will still only degenerate at $\boldsymbol{k} = 0$. Thus, we see that in these limits, ζ_{ES}^K and ζ_{EM}^K are vector bundles over the momentum space $\mathbb{R}^3 \setminus \{0\}$ which are topologically equivalent to the cold plasma bundles ζ_{ES} and ζ_{EM}^K. This suggest that moving to a kinetic model does not necessarily modify the topology. Furthermore, these waves will still be rotationally symmetric. Thus, it appears that the description of the angular momentum eigenmodes in terms of SWSHs, the nonexistence of an SAM-OAM decomposition, and the quasi-angular momentum splitting should still apply in a kinetic unmagnetized plasma. However, we have made a number of assumptions, such as neglecting the effect of ions and assuming only a Maxwellian background. We emphasize that our treatment has been somewhat informal, and is meant to indicate a potential route for future research on the topology and geometry of kinetic plasma waves. The formalism described should also allow for the topological analysis of magnetized kinetic plasmas by appropriately modifying $\boldsymbol{D}_E$.

9.7 Conclusion

We have shown that the rotational symmetry of waves in a cold isotropic plasma induces a natural decomposition into R, L, and Langmuir waves. The state space for each type of wave globally forms a topological object known as a vector bundle. Langmuir waves have Chern number $C(\zeta_0) = 0$ and are topologically trivial, while R and L waves are topologically nontrivial with $C(\zeta_\pm) = \mp 2$. This topological nontriviality has surprising implications, particularly pertaining to the angular momentum. Instead of the ordinary spherical harmonics which describe the orbital angular momentum eigenstates of massive particles, we find that the total angular momentum eigenstates of R and L waves are described by SWSHs. This gives a harmonic decomposition of plasma waves, and in particular, it gives a countable basis for monochromatic waves. SWSHs have been utilized in many branches of physics, but this appears to be their first application in plasma physics. The resulting angular momentum multiplets have a very different structure than that which results for massive particles. In particular, the multiplet structure for R and L waves is far sparser than that of massive particles with spin and also has a lower bound of $j = 1$. This angular momentum multiplet structure reflects the fact that plasma waves do not admit an SAM-OAM decomposition, a somewhat surprising result in light of recent work which has implied such a decomposition exists [84]. We show that the total angular momentum can be decomposed into the helicity and orbital quasi-angular momenta, $\boldsymbol{J}_\parallel$ and $\boldsymbol{J}_\perp$, and that this splitting is induced by the rotational symmetry of R and L waves. It has previously been found that $\boldsymbol{J}_\parallel$ is related to the inverse Faraday effect in plasmas [84]. While this splitting is naturally induced by the symmetry of the waves, $\boldsymbol{J}_\parallel$ and $\boldsymbol{J}_\perp$ do not themselves generate rotational symmetries and are thus not legitimate angular momentum operators. Indeed, these operators do not satisfy angular momentum commutation relations and the components have continuous spectra (in contrast to the discrete spectrum of the components of true angular momentum operators). As a result, we find that most of the theoretical tools pertaining to SAM and OAM operators, such as the Clebsch–Gordan formalism, do not apply to $\boldsymbol{J}_\parallel$ and $\boldsymbol{J}_\perp$, or to any

other attempted splitting of $\boldsymbol{J}$. These are important limitations to account for as further applications of this splitting are explored.

This chapter demonstrates how the formalism of vector bundles, and representations of symmetry groups on these bundles, can be used to rigorously study the topology and geometry of plasma waves. We have only applied these methods to a very simple plasma model, but this formalism is very flexible and can be applied to more complicated and accurate plasma descriptions such as MHD and kinetic models. Future research will explore such applications.

Chapter 10

Conclusion

We review the main results of this thesis and discuss some potential directions for future research.

10.1 Thesis summary and highlights

Throughout this thesis we have developed a flexible and rigorous framework to study the topology and geometry of waves, and applied it to simple plasma waves as well as to vacuum EM waves (photons), linearized gravitons, and more generally to massless helicity h particles. We showed that even the simplest continuous media such as the vacuum or a cold unmagnetized plasma support topologically nontrivial waves. Using this framework, we were able to explore a variety of physical manifestations of the topology and geometry of these systems. Many applications pertain to a surprising connection we discovered between the topology and angular momentum of waves, particularly the emergence of topological obstructions to SAM-OAM decompositions. Below we highlight the major contributions of this thesis.

In Chapters 3 and 4 we showed that photons and linearized gravitons have a similar structure. The solutions of the vacuum Maxwell equations fit together to form a vector bundle γ over the momentum space which takes the form of the light cone $\mathcal{L}_+$; the solutions of the linearized Einstein field equations similarly fit together to form the graviton bundle

Ξ. Somewhat surprisingly we show that γ and Ξ are topologically trivial, and this allows us to construct the first globally smooth bases of photon and graviton polarizations. This finding contradicts claims in some standard QFT references, such as Tong [37] and Woit [38]. Indeed, in the second quantization of the EM field in the Coulomb gauge, the EM field is typically expanded in a transversely polarized basis, but standard texts mention that this is technically impossible due to the hairy ball theorem. It is claimed that this does not ultimately result in issues because the resultant QED scattering amplitudes are not singular ([37], p. 129), but it is still theoretically disquieting. Our findings show, however, that it actually is possible to choose these polarization vectors as the polarizations are complex and the hairy ball theorem does not apply. Essentially we show that it actually is possible to comb the hairs on a sphere flat, provided the hairs are complex.

By considering Poincaré symmetry, we find that the photon and graviton bundles naturally decompose into R and L polarized bundles $\gamma_\pm$ and $\Xi_\pm$ which form unitary irreducible representations (UIRs) of $\mathrm{ISO}^+(3,1)$. Unlike γ and Ξ, these bundles are topologically nontrivial with $C(\gamma_\pm) = \mp 2$ and $C(\Xi_\pm) = \mp 4$. The Chern numbers for gravitons have not been previously reported. The Chern numbers for photons were also obtained by Bliokh et al. [104], albeit through less rigorous means without discussing vector bundles. This less rigorous approach led them to attempt to define a new topological quantity, the so-called spin-Chern number; we show that this quantity is inherently geometric rather than topological. The finding that photons and gravitons are topologically nontrivial shows that even the simplest continuous medium, namely the vacuum, can support topologically nontrivial waves. One immediate consequence of this nontriviality is that there exist no globally smooth bases for R and L gravitons. We show that the same is true of linear polarization but for a very different reason—there exist no vector bundles consisting exclusively of linearly polarized waves.

An important observation is that the circularly polarized photon and graviton bundles $\gamma_\pm$ and $\Xi_\pm$ are smooth UIRs of the Poincaré group, meaning that the polarizations globally fit together and that they vary smoothly under Poincaré transformations (changes of reference

frame). In contrast, the conventional Wigner representations for photons and other massless particles with $h \neq 0$ are singular. We prove in Theorem 3.32 that vector bundle UIRs of massless particles induce corresponding smooth UIRs on the Hilbert space of sections of particles, showing that one recovers elementary particles according to the standard definition, and moreover, one obtains smooth representations of these particles. It has been suggested [47, 48] that this result can be derived from a theorem of Mackey's ([115], Theorem 14.2), however, Mackey's theorem is not phrased in the language of vector bundles and the details of this correspondence are not provided. Instead of appealing to this theorem, we offer a novel proof which uses far simpler methods than Mackey's theorem; this is possible because our result only applies to representations of the Poincaré group while Mackey treated the far broader class of locally compact symmetry groups.

In Chapter 5, we obtained the novel result that massless particle bundles of arbitrary helicity can be constructed from iterated tensor products of the photon bundles $\gamma_{\pm}$,

$$\gamma_h = \underbrace{\gamma_{\pm} \otimes \cdots \otimes \gamma_{\pm}}_{|h| \text{ times}}. \tag{10.1}$$

This is a distinctly massless phenomenon and results in an exceptionally simple description of massless particle bundles which is useful for explicit calculations, such as those in Chapter 8 and Chapter 9 pertaining to the spin-weighted spherical harmonics (SWSHs). We see that the photon bundles $\gamma_{\pm}$ form the core building blocks of topologically nontrivial waves over the sphere, and indeed, we show in Chapter 9 that these describe R and L plasma waves as well and that Langmuir waves can be identified with γ_0.

We used this tensor product description of γ_h in Chapter 5 to uncover a previously unreported relationship between the topology and geometry of massless particles. According to Wigner's classification, which we show in Chapter 3 applies to vector bundle representations, the Poincaré geometry of elementary particles is characterized by the representation of the little group H on the internal polarization space. For massive particles, $H = \mathrm{SO}(3)$ and

the geometry is characterized by the nonnegative spin s which is famously related to the topological dimension of the internal vector space V by $\dim V = 2s+1$. A singularity occurs when $m = 0$ and the little group jumps to $H = \mathrm{SO}(2)$.[1] This results in all massless particles being characterized by an integer helicity h, but they all have a single internal DOF; it is clear then that the dimension of the internal space does not encode the topology. Instead, we have shown that massless particles form line bundles which are topologically characterized by their Chern number C. We show that this is related to the geometry of massless particles via $C = -2h$, giving the nontrivial massless analogy of $\dim V = 2s + 1$. Heuristically, this relationship shows that the local behavior of polarizations under rotations is directly related to the degree to which the polarization states are globally twisted together.

Much of this thesis focused on showing connections between the issues surrounding SAM-OAM decompositions of massless particles and plasma waves and the topological nontriviality of these waves. In the formal framework we have employed, it becomes immediately apparent that one will encounter issues when attempting to decompose the angular momentum into SAM and OAM. The (total) angular momentum $\boldsymbol{J}$ is the generator of the global SO(3) rotational symmetry of these systems. The existence of an SAM $\boldsymbol{S}$ implies that there exists an internal SO(3) symmetry of the polarization spaces. Particles have a canonical internal symmetry of the little group H; for massive particles $H = \mathrm{SO}(3)$ and the generator of this symmetry furnishes just such an SAM. However, $H = \mathrm{SO}(2)$ for massless particles, producing a scalar helicity operator, but not an SAM operator. This relationship between the jump in the little group from SO(3) to SO(2) and the issues encountered with defining massless SAM and OAM does not appear to have been previously reported.

We extensively studied this question of the existence of an SAM-OAM decomposition for massless particles in Chapter 6. We proved a series of four no-go theorems, the first two showing that a genuine SAM-OAM decomposition for massless particles is impossible. They provide two different perspectives on the obstruction, no-go Theorem 6.1 showing that

[1] Technically the little group is ISO(2) but the inhomogeneous elements act trivially for Standard Model particles, so the little group effectively acts like SO(2).

it is related to the geometric singularity of the little group jump just described, and no-go Theorem 6.2 showing that it can be related to the topologically singular jump of the momentum space from the contractible mass hyperboloid to the noncontractible light cone. No-go theorem 6.3 illustrates that one cannot split $\boldsymbol{J}$ into $\boldsymbol{S} + \boldsymbol{L}$ where both $\boldsymbol{S}$ and $\boldsymbol{L}$ generate non-SO(3) symmetries. No-go theorem 6.4 shows that relaxing the constraint that the SAM is internal does not remove the obstruction. Thus, our results show why a genuine SAM-OAM splitting for massless particles have not been found, and puts constraints on what one can hope to obtain by generalizing the definitions of SAM and OAM.

Nevertheless, a commonly proposed splitting $\boldsymbol{J} = \boldsymbol{J}_{\parallel} + \boldsymbol{J}_{\perp}$ has appeared repeatedly in the literature [68, 71, 72, 77, 84]. These are not genuine angular momentum operators, and in line with no-go Theorem 6.3, $\boldsymbol{J}_{\perp}$ is not the generator of a symmetry. Nevertheless, this splitting is well-defined and expressed entirely in terms of natural operators, and furthermore, it has been argued that it is physically relevant for both photons and plasma waves [77, 84]. Thus in Chapter 7 we examined the question of whether there is a compelling theoretical origin of this splitting. We showed that every geometric connection $\boldsymbol{D}$ on a vector bundle naturally induces a splitting of $\boldsymbol{J}$ but that this splitting is only an SAM-OAM decomposition if the connection is (S^2)-flat. Both rotational symmetry and boost symmetry define curved connections, $\boldsymbol{D}^R$ and $\boldsymbol{D}^K$, and thus induce non-SAM-OAM splittings of $\boldsymbol{J}$. However, for massive particles one can form a uniquely combine $\boldsymbol{D}^R$ and $\boldsymbol{D}^K$ to form a flat connection, and this recovers the Newton-Wigner position operator and the corresponding SAM-OAM decomposition for massive particles. However, a strange singularity occurs when $m = 0$, in which transverse boosts and rotations degenerate, causing $\boldsymbol{D}^K = \boldsymbol{D}^R$. There is only a single connection in this case, and it produces the $\boldsymbol{J}_{\parallel}, \boldsymbol{J}_{\perp}$ splitting. This connection is, however, curved, explaining why these fail to be SAM and OAM. In the case of plasma waves, the background plasma breaks boost symmetry, leaving only the curved connection $\boldsymbol{D}^R$ which again induces the $\boldsymbol{J}_{\parallel}$, $\boldsymbol{J}_{\perp}$ splitting.

An interesting question arises when one considers the analog of the spherical harmonics for plasma waves or massless particles, which we explored in Chapters 8 and 9. Ordinary spherical harmonics decompose the orbital Hilbert space for massive particles. However, the Hilbert spaces of plasma waves and massless particle waves do not decompose into spin and orbital parts. Using the tensor product representation in Eq. (10.1) we show that the monochromatic Hilbert space decomposes into the spin-weighted spherical harmonics ${}_{-h}Y_{jm}\boldsymbol{e}_{\pm}^{|h|}$ of weight $-h$. This breaks the Hilbert space into rotationally invariant subspaces spanned by simultaneous eigenstates of helicity, energy, J^2, and J_z, giving a rather elegant decomposition of the fields. They give a much simpler and more explicit description of these eigenstates than that recently obtained by Dragon [85]. We used the properties of the SWSHs to give a concrete illustration that there is no SAM-OAM decomposition for massless particles or plasma waves. Indeed, the angular momentum multiplet structure of the SWSHs is extremely sparse, consisting of at most a single multiplet for each j. Furthermore, there are no multiplets with $j < |h|$. Neither of these properties are consistent with any reasonable SAM-OAM decomposition, as the addition of angular momentum formula typically creates degenerate multiplets and multiplets with j extending all the way down to $j = 0$.

10.2 Suggested future research

10.2.1 Extension to kinetic and dissipative fluid models

The topological analysis of plasma waves is a fairly new area of research, and so far only models without dissipation have been analyzed such as cold fluids [19, 28–31, 33, 34], ideal MHD [32], and Hall MHD [35, 109]. A natural direction for future research is to apply our methods to the analysis of waves in a kinetic or dissipative fluid plasma model. The vector bundle methods developed are well-suited to apply in such theories, provided in the kinetic case one treats quasimodes rather than Case–van Kampen modes as discussed in §9.6. Changing the plasma model will change the dispersion tensor, and then one can follow the procedure

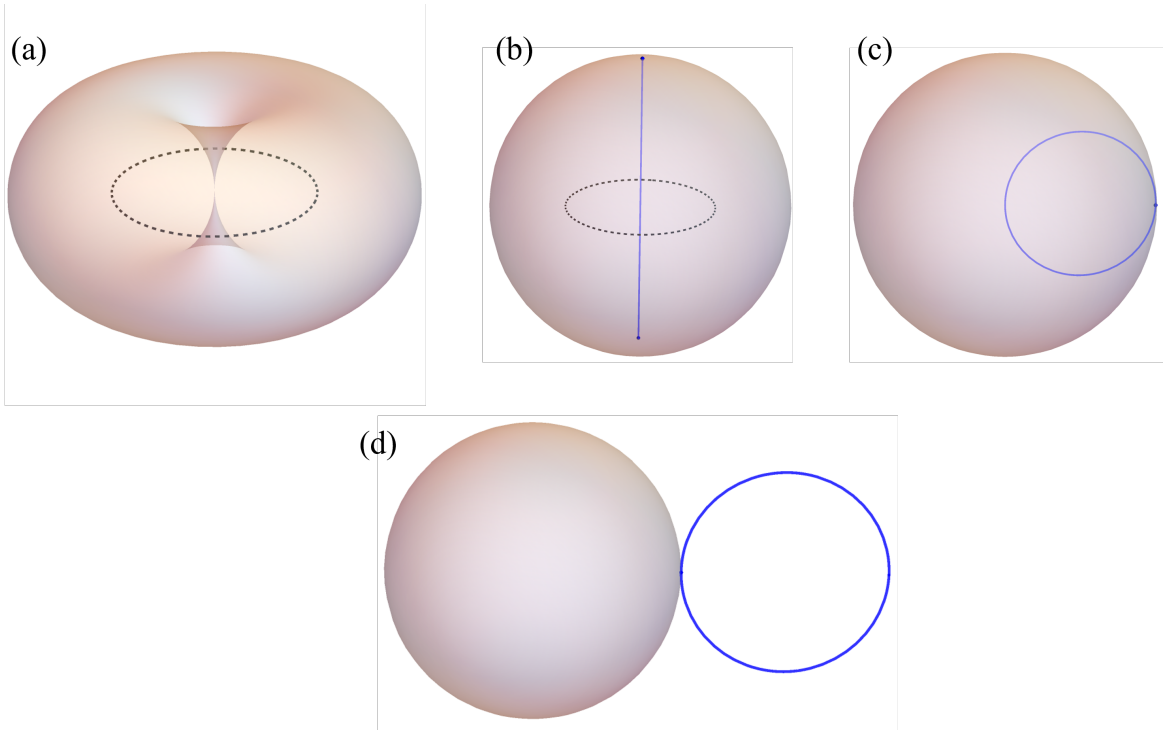


Figure 10.1: Illustration of the homotopy equivalence of $\mathbb{R}^3 \setminus S^1$ with the horn torus and the wedge sum of a 2-sphere and circle, $S^2 \vee S^1$. (a) $\mathbb{R}^3 \setminus S^1$, with the excluded circle shown in dashed black, deformation retracts onto the horn torus (the torus with equal major and minor radii). (b) The cusp at the center of (a) can be stretched into a line, so that the horn torus is deformed into a 2-sphere along with a diameter (blue) connecting the poles. (c) The space is deformed into $S^2 \vee S^1$ by moving the poles in (b) along S^2 until they meet. (d) Same as (c) with the copy of S^1 depicted outside the sphere for clarity.

of identifying the different modes $\omega_n(\boldsymbol{k})$ from $\det \boldsymbol{D}_E = 0$ and forming vector bundles by varying $\boldsymbol{k}$, excluding the points at which modes degenerate (Weyl points). One difference is that for dissipative waves the ω_n will be complex, but this should not cause any difficulties for topological analysis. It will still be the case that the set of points $M_n = \{(\omega_n(\boldsymbol{k}), \boldsymbol{k})\}_{\boldsymbol{k}}$ will form a manifold which can be parameterized by $\boldsymbol{k}$. The more interesting question is what the set of excluded Weyl points looks like. Introducing dissipation corresponds to adding a non-Hermitian term to the Hamiltonian. In condensed matter applications this has led to the phenomenon of an isolated Weyl point splitting into a ring of Weyl points [166, 167], and it has been argued that this is actually a common effect of non-Hermitian perturbations [167, 168]. This does not occur in the case of a nonrelativistic unmagnetized kinetic plasma

discussed in §9.6; indeed, this phenomenon cannot occur in a spherically symmetric system because a ring of Weyl points would break that symmetry. It is possible, however, that such rings may occur in systems with a preferred direction, such as magnetized plasmas. If such a ring of singularities occurs for waves in a plasma, the base manifold would correspond to momentum space with that ring removed, that is, $M \cong \mathbb{R}^3 \setminus S^1$. Unlike $\mathbb{R}^3 \setminus \{0\}$, M does not deformation retract onto a sphere S^2. As depicted in Fig. 10.1(a), M instead retracts onto the horn torus, that is, the torus with major and minor radii equal. Note that due to the singular point at the center, the horn torus is not actually a manifold, but rather a cell complex (a space formed by gluing together spheres of various dimensions) [105]. As depicted in Fig. 10.1, this space can be deformed into a 2-sphere tangent to a circle, that is, it is homotopy equivalent to the wedge sum $S^2 \vee S^1$. This description helps illustrate a rather peculiar feature of the momentum space $\mathbb{R}^3 \setminus S^1$, namely, that it does not deformation retract onto *any* closed 2D surface. Indeed, it is straightforward to calculate the first homology group from the cell complex structure (cf. §2.2 [105])

$$H_1(\mathbb{R}^3 \setminus S^1) \cong H_1(S^2 \vee S^1) \cong \mathbb{Z} \tag{10.2}$$

which reflects the single one-dimensional hole corresponding to S^1. However, an orientable genus g surface M_g has $H_1(M_g) \cong \mathbb{Z}^{2g}$ while a nonorientable genus g surface N_g has $H_1(N_g) \cong \mathbb{Z}^{g-1} \oplus \mathbb{Z}_2$ where $\mathbb{Z}_2$ is the group with two elements (cf. [105], Examples 2.36 and 2.37). Since no closed 2D surface has first homology $\mathbb{Z}$, $\mathbb{R}^3 \setminus S^1$ cannot retract onto any such surface. Recall that we used the fact that $\mathbb{R}^3 \setminus \{0\}$ deformation retracted onto the 2D surface S^2 to define the Chern numbers of vector bundles over $\mathbb{R}^3 \setminus \{0\}$ in §2.2.3. It is thus unclear if one can define a meaningful analogy of the Chern number for vector bundles that arise when $M \cong \mathbb{R}^3 \setminus S^1$. We note that the Chern *class* is still a well-defined topological invariant for these bundles, so it may be the case that one must work instead with this object rather than a Chern number. This discussion suggests that damped waves in magnetized plasmas may

have a very different topological characterization compared to their non-damped Hermitian counterparts due to the presence of ring degeneracies. Thus, it may be the case that these waves exhibit novel topological properties; additional research is needed to examine this possibility.

It particular, it may be interesting to study the degree to which the bulk-edge correspondence (BEC) is valid for kinetic plasmas, for example, to study the kinetic analog of the topological Langmuir-cyclotron wave [19, 33] and determine if it exhibits topologically protected behavior. Known rigorous statements of the BEC for continuum waves [19, 20] assume Hermitian bulk Hamiltonians. It was recently shown by Fu and Qin [35] that the non-Hermiticity in the transition region between Hermitian bulk media does not modify the BEC. However, it has been found that the BEC is sometimes violated in lattice systems if the bulk Hamiltonians themselves are non-Hermitian [169, 170]. It may be that a shift in the momentum spaces from $\mathbb{R}^3 \setminus \{0\}$ to $\mathbb{R}^3 \setminus S^1$ could cause similar issues for the plasma BEC. More work is needed to understand the nature of the BEC for dissipative plasma waves.

It is also interesting to note that the quasi-angular momentum decomposition $\boldsymbol{J} = \boldsymbol{J}_{\parallel} + \boldsymbol{J}_{\perp}$ in Eq. (9.55) can be applied to unmagnetized kinetic plasma waves, and one could obtain the kinetic forms of Eqs. (9.71) and (9.72) for their expectation values. Indeed, this splitting is naturally induced for any wave system that is invariant under 3D rotations. As it has been argued that the fluid version of this splitting is useful, for example, in describing wave-induced magnetization [84], it may be the case that this provides a useful splitting of the kinetic plasma angular momentum as well.

10.2.2 Magnetized plasmas and odd Chern numbers

One might notice that all of the Chern numbers encountered in this thesis are even: $C = 0$ for trivial bundles, $C = \mp 2$ for R and L photons and EM plasma waves, $C = \mp 4$ for gravitons, and more generally, $C = -2h$ for massless bosons with helicity h. It is thus interesting to note that bundles with Chern number $C = \pm 1$ are encountered in the study of Hall MHD

waves [35, 109]. Similarly, some magnetized fluid waves have odd Chern numbers [33]. One can ask if there is a meaningful reason why odd Chern numbers show up in these contexts. It may be that this is related to the fact that in both magnetized fluid plasmas and Hall MHD, the background magnetic field breaks the spherical symmetry of the system. That is, the wave bundles will not be SO(3) symmetric and thus do not possess a helicity so the equation $C = -2h$ is not applicable, allowing for odd Chern numbers. This raises a number follow-up questions. Are waves with odd Chern number common in non-spherically symmetric configurations? In magnetized plasmas, the waves will still possess SO(2) symmetry about the background magnetic field; is there any interesting relationship between the topology and geometry (symmetry) of such waves? If the conjecture is true that breaking spherical symmetry often leads to waves with odd Chern number, this would imply via the BEC that one may typically expect the appearance of edge modes at the boundary of spherically symmetric (which have even C) and non-spherically symmetric bulk media. This gives one reason why it is interesting to explore potential relationships between the parity of Chern numbers and symmetries of wave equations.

10.2.3 Continuous spin particles

Throughout this thesis we have excluded what are known as the continuous spin particles. These are the massless representations where the full massless little group $\mathrm{ISO}(2) = \mathbb{R}^2 \rtimes \mathrm{SO}(2)$ acts nontrivially, and does not reduce to to an SO(2) symmetry [42, 43]. In particular, if $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3 = \boldsymbol{k}$ is an orthonormal coordinate system, then the generators of the little group $H_{\boldsymbol{k}}$ action at $\boldsymbol{k}$ are α, β, and J_3 where

$$\alpha = J_2 + K_1 \tag{10.3}$$

$$\beta = -J_1 + K_2 \tag{10.4}$$

and $\boldsymbol{J}$ and $\boldsymbol{K}$ are the angular momentum and boost operators. J_3 generates SO(2) part of the H_k action while α and β generate the translational $\mathbb{R}^2$ part. Continuous spin representations are those in which α and β are not zero. Wigner showed that such particles have an infinite number of internal degrees ([42]; [43], pp. 70-72). As this finding is inconsistent with all experimentally observed particles, they are not part of the Standard Model [43]. Nevertheless, there has been renewed interest in these hypothetical particles since 2015 after action principals for them were discovered [171]. Recent work suggests that these particles can consistently interact with matter, even in flat spacetime [172–178]. There are an infinite number of continuous spin particles and all can be obtained from mass m, spin s particle representations in the limit $m \to 0$ and $s \to \infty$, so in a sense they correspond to "infinite spin" [175]. It is interesting to consider the question of whether or not such particles admit an SAM-OAM decomposition. One will immediately notice that most of our results do not apply to continuous spin particles. For example, no-go Theorem 6.1, which proved that massless particles do not have an SAM-OAM decomposition, relies on the fact that massless UIRs have only one internal DOF. This is true for massless particles in the Standard Model which have $\alpha = \beta = 0$ but not for the infinite-dimensional continuous spin particles. It is more interesting to consider how the argument in §7.4.4 fails. We saw that for typical massless particles, as $m \to 0$, the actions of transverse boosts and transverse rotations degenerate [Eq. (7.163)], corresponding to $\alpha \to 0$ and $\beta \to 0$. This caused the degeneracy of the boost and rotation connections $\boldsymbol{D}^K = \boldsymbol{D}^R$, so there was no way to affinely combine $\boldsymbol{D}^K$ and $\boldsymbol{D}^R$ to produce an S^2-flat connection, and thus no corresponding SAM-OAM decomposition. However, continuous spin particles are precisely those for which $\alpha \neq 0$ and $\beta \neq 0$, so $\boldsymbol{D}^K \neq \boldsymbol{D}^R$ and this argument fails. Thus it may be possible to construct an SAM-OAM splitting for continuous spin particles by balancing $\boldsymbol{D}^K$ and $\boldsymbol{D}^R$ as in the case of massive particles. More careful analysis is needed to explore this possibility, as $\boldsymbol{D}^K \neq \boldsymbol{D}^R$ alone does not guarantee that such a balancing is possible.

There is also the issue of actually defining $\boldsymbol{D}^K$ and $\boldsymbol{D}^R$ when there are infinitely many DOFs; after all, the typical definition of vector bundles requires that the fibers are finite-dimensional vector spaces. We believe this difficulty can be overcome by using the more generalized notion of Hilbert bundles which allow fibers to be infinite-dimensional Hilbert spaces [179]. In fact, if it is indeed the case that continuous spin particles form Hilbert bundles one obtains an immediate and rather surprising result: continuous spin particles are topologically trivial. This is because all Hilbert bundles with infinite-dimensional fibers are actually topologically trivial ([180], p. 67). Thus the topological obstruction to an SAM-OAM operator discussed in no-go Theorem 6.2 does not appear to apply to continuous spin particles, leading further credence to the notion that an SAM-OAM decomposition may exist. On the other hand, the little group is still ISO(2) rather than SO(3) so heuristically it seems that there should not be an internal SO(3) representation for these particles. In summary, there are qualitative arguments that both support and oppose the contention that continuous spin particles admit an SAM-OAM decomposition. Given the recent interest in these particles by the high energy physics community, it may be worthwhile to examine this issue more closely.

Appendix A

Noncommutativity of the Poincaré generators

The Poincaré generators satisfy

$$[J_a, J_b] = i\epsilon_{abc}J_c\,, \quad [J_a, K_b] = i\epsilon_{abc}K_c \tag{A.1a}$$

$$[K_a, K_b] = -i\epsilon_{abc}J_c\,, \quad [J_a, P_b] = i\epsilon_{abc}P_c \tag{A.1b}$$

$$[K_a, P_b] = iH\delta_{ab}\,, \quad [K_a, H] = iP_a \tag{A.1c}$$

$$[J_a, H] = [P_a, H] = [P_a, P_b] = [H, H] = 0. \tag{A.1d}$$

In this appendix we give some useful results pertaining to the noncommutativity of the generators.

A.1 Noncommutative BAC-ABC rule

Ordinary 3D vectors $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ satisfy the well-known BAC-CAB rule

$$\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = \boldsymbol{b}(\boldsymbol{a} \cdot \boldsymbol{c}) - \boldsymbol{c}(\boldsymbol{a} \cdot \boldsymbol{b}). \tag{A.2}$$

However, this relationship does not hold for general vector operators $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$ which fail to commute. If $\boldsymbol{A}$ and $\boldsymbol{B}$ commute, then we have the following useful relationship

Proposition A.1 (Noncommutative BAC-ABC relation)**.** *Let $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$ be vector operators such that $\boldsymbol{A}$ and $\boldsymbol{B}$ commute: $[A_m, B_n] = 0$. Then*

$$\boldsymbol{A} \times (\boldsymbol{B} \times \boldsymbol{C}) = \boldsymbol{B}(\boldsymbol{A} \cdot \boldsymbol{C}) - (\boldsymbol{A} \cdot \boldsymbol{B})\boldsymbol{C} \tag{A.3}$$

Proof. From the relation

$$\epsilon_{ijk}\epsilon_{mnk} = \delta_{im}\delta_{jn} - \delta_{in}\delta_{jm} \tag{A.4}$$

we obtain

$$\begin{aligned}
[\boldsymbol{A} \times (\boldsymbol{B} \times \boldsymbol{C})]_i &= \epsilon_{ijk} A_j \epsilon_{kmn} B_m C_n && \text{(A.5)}\\
&= (\delta_{im}\delta_{jn} - \delta_{in}\delta_{jm}) A_j B_m C_n && \text{(A.6)}\\
&= A_j B_i C_j - A_j B_j C_i && \text{(A.7)}\\
&= \boldsymbol{B}(\boldsymbol{A} \cdot \boldsymbol{C}) - (\boldsymbol{A} \cdot \boldsymbol{B})\boldsymbol{C} && \text{(A.8)}
\end{aligned}$$

□

This is useful for decomposing a vector operator, such as $\boldsymbol{J}$, into parallel and perpendicular parts with respect to another such as $\hat{\boldsymbol{P}}$. In particular, since $\hat{\boldsymbol{P}} \cdot \hat{\boldsymbol{P}} = 1$, applying the above result with $\boldsymbol{A} = \boldsymbol{B} = \hat{\boldsymbol{P}}$ and $\boldsymbol{C} = \boldsymbol{J}$ gives

$$\begin{aligned}
\boldsymbol{J} &= \hat{\boldsymbol{P}}(\hat{\boldsymbol{P}} \cdot \boldsymbol{J}) - \hat{\boldsymbol{P}} \times (\hat{\boldsymbol{P}} \times \boldsymbol{J}) && \text{(A.9)}\\
&\doteq \boldsymbol{J}_{\parallel} + \boldsymbol{J}_{\perp} && \text{(A.10)}
\end{aligned}$$

where

$$\boldsymbol{J}_{\parallel} \doteq \hat{\boldsymbol{P}}(\hat{\boldsymbol{P}} \cdot \boldsymbol{J}) \tag{A.11a}$$

$$\boldsymbol{J}_{\perp} \doteq -\hat{\boldsymbol{P}} \times (\hat{\boldsymbol{P}} \times \boldsymbol{J}). \tag{A.11b}$$

Similarly,

$$\boldsymbol{K} = \boldsymbol{K}_{\parallel} + \boldsymbol{K}_{\perp} \tag{A.12a}$$

$$\boldsymbol{K}_{\parallel} \doteq \hat{\boldsymbol{P}}(\hat{\boldsymbol{P}} \cdot \boldsymbol{K}) \tag{A.12b}$$

$$\boldsymbol{K}_{\perp} \doteq -\hat{\boldsymbol{P}} \times (\hat{\boldsymbol{P}} \times \boldsymbol{K}). \tag{A.12c}$$

A.2 Useful commutation relations

The boost generators $\boldsymbol{K}$ satisfy more complicated commutation relations than the other Poincaré generators. Indeed, unlike the momentum operators, the boosts do not commute with H, nor do different components of the boost commute with each other. This gives rise to interesting physics, such as the Wigner rotation, but it also greatly complicates calculations. In this section we collect a number of useful commutation relations involving the Poincaré generators, mostly involving $\boldsymbol{K}$.

If the operator B^{-1} is defined, then

$$\begin{aligned}
[A, B^{-1}] &= AB^{-1} - B^{-1}A \\
&= B^{-1}BAB^{-1} - B^{-1}ABB^{-1} \\
&= -B^{-1}[A, B]B^{-1}.
\end{aligned} \tag{A.13}$$

If $\big[[A, B], B\big] = 0$, then for nonnegative n: ([158], Eq. (2.90))

$$[A, B^n] = n[A, B]B^{n-1}. \tag{A.14}$$

If B^{-1} exists, then Eq. (A.14) holds for all integers n by Eq. (A.13). Using the Poincaré commutation relations and Eq. (A.14) we obtain the following:

$$[K_a, H^n] = inP_a H^{n-1} \tag{A.15a}$$

$$[K_a, |\boldsymbol{P}|^n] = inHP_a|\boldsymbol{P}|^{n-2} \tag{A.15b}$$

$$[(\boldsymbol{P}\cdot\boldsymbol{K}), P_a] = iHP_a \tag{A.15c}$$

$$[\hat{P}_a, K_b] = -\frac{i\delta_{ab}H}{|\boldsymbol{P}|} + \frac{iHP_aP_b}{|\boldsymbol{P}|^3} \tag{A.15d}$$

$$[H^m K_a, H^n P_b] = iH^{m+n+1}\delta_{ab} + inP_aP_bH^{m+n-1} \tag{A.15e}$$

$$\hat{\boldsymbol{P}}\cdot\boldsymbol{K} - \boldsymbol{K}\cdot\hat{\boldsymbol{P}} = -\frac{2iH}{|\boldsymbol{P}|} \tag{A.15f}$$

$$\boldsymbol{P}\cdot\boldsymbol{K} - \boldsymbol{K}\cdot\boldsymbol{P} = -3iH. \tag{A.15g}$$

Appendix B

Expectation values of plasma operators

In this appendix we derive Eqs. (9.68)-(9.72), showing that the expectation values of the cold plasma operators $H, \boldsymbol{P}, \boldsymbol{J}, \boldsymbol{J}_{\parallel}$, and $\boldsymbol{J}_{\perp}$ can be calculated in terms of integrals over momentum space or position space. Because each of these operators commutes with the Hamiltonian, the expectation values are all conserved quantities. We use the notation $\boldsymbol{E} \doteq \boldsymbol{E}(\boldsymbol{k})$, $\boldsymbol{E}' \doteq \boldsymbol{E}(\boldsymbol{k}')$, $\bar{\boldsymbol{E}} \doteq \boldsymbol{E}(-\boldsymbol{k})$, and similarly for other variables. We also write $\omega_\gamma(\boldsymbol{k}) \doteq \omega_+(\boldsymbol{k}) = \omega_-(\boldsymbol{k})$.

Proposition B.1 (Expectation value of H)**.**

$$\frac{1}{8\pi}\langle \boldsymbol{E}|H|\boldsymbol{E}\rangle = \sum_h \frac{1}{8\pi}\int \frac{d^3k}{\omega_h}\boldsymbol{E}_h^*(\boldsymbol{k})\cdot[H\boldsymbol{E}_h](\boldsymbol{k}) \tag{B.1a}$$

$$= \sum_h \frac{1}{8\pi}\int d^3k\,|\boldsymbol{E}_h|^2 \tag{B.1b}$$

$$= \sum_h \int d^3x\left[\frac{1}{8\pi}\left(\mathcal{E}_h^2+\mathcal{B}_h^2\right)+\frac{1}{2}mn_0v_h^2\right] \tag{B.1c}$$

$$= \int d^3x\left[\frac{1}{8\pi}\left(\mathcal{E}^2+\mathcal{B}^2\right)+\frac{1}{2}mn_0v^2\right)\Big] \tag{B.1d}$$

Proof. We can decompose any state into eigenvectors of the helicity χ:

$$\boldsymbol{E} = \sum_{h=0,\pm1}\boldsymbol{E}_h \tag{B.2}$$

Since χ commutes with H and since states with different helicity are orthogonal, it follows that

$$\langle \boldsymbol{E}|H|\boldsymbol{E}\rangle = \sum_h \langle \boldsymbol{E}_h|H|\boldsymbol{E}_h\rangle = \sum_h \int \frac{d^3k}{\omega_h} \boldsymbol{E}_h^*(\boldsymbol{k}) \cdot [H\boldsymbol{E}_h](\boldsymbol{k}), \tag{B.3}$$

establishing Eq. (B.1a), and $[H\boldsymbol{E}](\boldsymbol{k}) = \omega_h \boldsymbol{E}(\boldsymbol{k})$ gives Eq. (B.1b). Analogous statements for the operators $\boldsymbol{J}, \boldsymbol{P}, \boldsymbol{J}_\parallel$ and $\boldsymbol{J}_\perp$ also hold because these commute with χ. We establish the equivalence of (B.1b) and (B.1c) by showing that the summands are equal for each h. We have that

$$\boldsymbol{\mathcal{E}}_h(\boldsymbol{x}) = \frac{1}{2(2\pi)^{3/2}} \int d^3k \left(\boldsymbol{E}_h(\boldsymbol{k}) e^{i(\boldsymbol{k}\cdot\boldsymbol{x}-\omega t)} + c.c. \right) \tag{B.4a}$$

$$\boldsymbol{\mathcal{B}}_h(\boldsymbol{x}) = \frac{c}{2(2\pi)^{3/2}} \int \frac{d^3k}{\omega_h} \left(\boldsymbol{k} \times \boldsymbol{E}_h(\boldsymbol{k}) e^{i(\boldsymbol{k}\cdot\boldsymbol{x}-\omega t)} + c.c. \right) \tag{B.4b}$$

so

$$\int d^3x \, |\boldsymbol{\mathcal{E}}_h|^2 = \frac{1}{4(2\pi)^3} \int d^3x \, d^3k \, d^3k' \left(\boldsymbol{E}_h e^{i(\boldsymbol{k}\cdot\boldsymbol{x}-\omega t)} + \boldsymbol{E}_h^* e^{-i(\boldsymbol{k}\cdot\boldsymbol{x}-\omega t)} \right) \cdot \left(\boldsymbol{E}'_h e^{i(\boldsymbol{k}'\cdot\boldsymbol{x}-\omega' t)} + \boldsymbol{E}_h'^* e^{-i(\boldsymbol{k}'\cdot\boldsymbol{x}-\omega' t)} \right) \tag{B.5}$$

Using

$$\int d^3x \, e^{i\boldsymbol{k}\cdot\boldsymbol{x}} = (2\pi)^3 \delta(\boldsymbol{k}) \tag{B.6}$$

we obtain

$$\frac{1}{8\pi} \int d^3x \, |\boldsymbol{\mathcal{E}}_h|^2 = \frac{1}{32\pi} \int d^3k \left(2|\boldsymbol{E}_h|^2 + \boldsymbol{E}_h \cdot \bar{\boldsymbol{E}}_h e^{-2i\omega t} + \boldsymbol{E}_h^* \cdot \bar{\boldsymbol{E}}_h^* e^{2i\omega t} \right) \tag{B.7}$$

Using

$$\boldsymbol{v}_h(\boldsymbol{k}) = -i \frac{e}{m\omega_h} \boldsymbol{E}_h(\boldsymbol{k}) \tag{B.8}$$

we similarly find

$$\frac{mn_0}{2}\int d^3x\, v_h^2 = \frac{mn_0}{8}\int d^3k \left(2|\boldsymbol{v}_h|^2 + \boldsymbol{v}_h\cdot\bar{\boldsymbol{v}}_h e^{-2i\omega t} + \boldsymbol{v}_h^*\cdot\bar{\boldsymbol{v}}_h^* e^{2i\omega t}\right) \tag{B.9}$$

$$= \frac{1}{32\pi}\int d^3k\,\frac{\omega_p^2}{\omega_h^2}\Big(2|\boldsymbol{E}_h|^2 - \boldsymbol{E}_h\cdot\bar{\boldsymbol{E}}_h e^{-2i\omega t} - \boldsymbol{E}_h^*\cdot\bar{\boldsymbol{E}}_h^* e^{2i\omega t}\Big) \tag{B.10}$$

($h=0$ case): In the $h=0$ case, $\omega_0=\omega_p$ and $\boldsymbol{\mathcal{B}}=0$, so from Eqs. (B.7) and (B.10) we have

$$\int d^3x\left[\frac{1}{8\pi}(\mathcal{E}_0^2+\mathcal{B}_0^2)+\frac{1}{2}mn_0v^2\right] = \frac{1}{8\pi}\int d^3k\,|\boldsymbol{E}_0|^2 \tag{B.11}$$

giving the desired result.

($h=\pm1$ case): In the EM case, $\omega_\pm^2=\omega_p^2+|\boldsymbol{k}|^2c^2$ and $\boldsymbol{k}\cdot\boldsymbol{E}_\pm=0$. By replacing $\boldsymbol{\mathcal{E}}$ with $\boldsymbol{\mathcal{B}}$ in Eq. (B.7), and using the relationship

$$\boldsymbol{B}_\pm = \frac{c|\boldsymbol{k}|}{\omega_\gamma}\hat{\boldsymbol{k}}\times\boldsymbol{E}_\pm \tag{B.12}$$

we obtain

$$\frac{1}{8\pi}\int d^3x\,\mathcal{B}_\pm^2 = \frac{1}{32\pi}\int d^3k\frac{c^2|\boldsymbol{k}|^2}{\omega_\gamma^2}\left(2|\boldsymbol{E}_\pm|^2 - \boldsymbol{E}_\pm\cdot\bar{\boldsymbol{E}}_\pm e^{-2i\omega t} + \boldsymbol{E}_\pm^*\cdot\bar{\boldsymbol{E}}_\pm^* e^{2i\omega t}\right) \tag{B.13}$$

This equation combines with Eqs. (B.7) and (B.10) to give

$$\int d^3x\left[\frac{1}{8\pi}(\mathcal{E}_\pm^2+\mathcal{B}_\pm^2)+\frac{1}{2}mn_0v_\pm^2\right] = \frac{1}{8\pi}\int d^3k\,|\boldsymbol{E}_\pm|^2, \tag{B.14}$$

completing the EM case and establishing the equivalence of (B.1b) and (B.1c). Lastly, we note that equation (B.1d) is established if we show that the cross terms in the inner products

vanish, namely, if

$$\int d^3x \left[\frac{1}{8\pi}\left(\boldsymbol{\mathcal{E}}_h \cdot \boldsymbol{\mathcal{E}}_{h'} + \boldsymbol{\mathcal{B}}_h \cdot \boldsymbol{\mathcal{B}}_{h'}\right) + \frac{1}{2} m n_0 \boldsymbol{v}_h \cdot \boldsymbol{v}_{h'}\right] = 0 \tag{B.15}$$

for $h \neq h'$. In general,

$$\int d^3x\, \boldsymbol{\mathcal{E}}_h \cdot \boldsymbol{\mathcal{E}}_{h'} = \frac{1}{4}\int d^3k \left[\boldsymbol{E}_h \cdot \boldsymbol{E}^*_{h'} e^{-i(\omega_h - \omega_{h'})} + \boldsymbol{E}_h \cdot \bar{\boldsymbol{E}}_{h'} e^{-i(\omega_h + \omega_{h'})} + c.c.\right]. \tag{B.16}$$

The first term vanishes since $\boldsymbol{e}_h \cdot \boldsymbol{e}^*_{h'} = 0$ for $h \neq h'$. If exactly one of h or h' is 0, then the second dot product clearly vanishes. If $h = \pm 1$ and $h' = \mp 1$, then since $\boldsymbol{E}_\mp(-\boldsymbol{k})$ is proportional to $\boldsymbol{E}_\pm(\boldsymbol{k})$,

$$\boldsymbol{E}_\pm(\boldsymbol{k}) \cdot \boldsymbol{E}_\mp(-\boldsymbol{k}) = c\boldsymbol{E}_\pm(\boldsymbol{k}) \cdot \boldsymbol{E}_\pm(\boldsymbol{k}) = 0. \tag{B.17}$$

Thus

$$\int d^3x\, \boldsymbol{\mathcal{E}}_h \cdot \boldsymbol{\mathcal{E}}_{h'} = 0. \tag{B.18}$$

Similar calculations show that the other two terms on the lhs of (B.15) vanish, establishing (B.1d). □

Proposition B.2 (Expectation value of $\boldsymbol{P}$)**.**

$$\frac{1}{8\pi}\langle \boldsymbol{E}|\boldsymbol{P}|\boldsymbol{E}\rangle = \sum_h \frac{1}{8\pi}\int \frac{d^3k}{\omega_h} \boldsymbol{E}^*_h(\boldsymbol{k}) \cdot [\boldsymbol{P}\boldsymbol{E}_h](\boldsymbol{k}) \tag{B.19a}$$

$$= \sum_h \frac{1}{8\pi}\int d^3k \frac{\boldsymbol{k}}{\omega_h}|\boldsymbol{E}_h|^2 \tag{B.19b}$$

$$= \sum_h \int d^3x \left[\frac{1}{4\pi c}\boldsymbol{\mathcal{E}}_h \times \boldsymbol{\mathcal{B}}_h + m n_{1,h} \boldsymbol{v}_h\right] \tag{B.19c}$$

$$= \int d^3x \left[\frac{1}{4\pi c}\boldsymbol{\mathcal{E}} \times \boldsymbol{\mathcal{B}} + m n_{1,h} \boldsymbol{v}\right] \tag{B.19d}$$

Proof. From $\boldsymbol{P}\boldsymbol{E}(\boldsymbol{k}) = \boldsymbol{k}\boldsymbol{E}(\boldsymbol{k})$ we obtain

$$\langle \boldsymbol{E}|\boldsymbol{P}|\boldsymbol{E}\rangle = \sum_h \int d^3k\, \frac{\boldsymbol{k}_h}{\omega_h}|\boldsymbol{E}_h|^2. \tag{B.20}$$

We now show that the summands in Eqs. (B.19b) and (B.19c) are equal for each h.

($h = \pm 1$ case): If $h = \pm 1$, then $n_{1,h} = 0$, so the only nonvanishing term on the rhs of Eq. (B.19c) is the EM term. We have

$$\begin{aligned}\int d^3x\, \boldsymbol{\mathcal{E}}_h \times \boldsymbol{\mathcal{B}}_h = \frac{1}{4(2\pi)^3}\int d^3x\, d^3k\, d^3k'\, \frac{c|\boldsymbol{k}'|}{\omega'_h}\Big(\boldsymbol{E}_h e^{i(\boldsymbol{k}\cdot\boldsymbol{x}-\omega_h t)} + \boldsymbol{E}_h^* e^{-i(\boldsymbol{k}\cdot\boldsymbol{x}-\omega_h t)}\Big) \\ \times\Big(\hat{\boldsymbol{k}}'\times\boldsymbol{E}'_h e^{i(\boldsymbol{k}'\cdot\boldsymbol{x}-\omega'_h t)} + \hat{\boldsymbol{k}}'\times\boldsymbol{E}'^*_h e^{-i(\boldsymbol{k}'\cdot\boldsymbol{x}-\omega'_h t)}\Big)\end{aligned} \tag{B.21}$$

Note that

$$\begin{aligned}I &= \frac{1}{(2\pi)^3}\int d^3x\, d^3k\, d^3k'\, \frac{c|\boldsymbol{k}'|}{\omega'_h}\boldsymbol{E}_h\times\big(\hat{\boldsymbol{k}}'\times\boldsymbol{E}'_h\big)e^{i\boldsymbol{x}\cdot(\boldsymbol{k}+\boldsymbol{k}')-it(\omega+\omega')} \\ &= -\int d^3k\, \frac{c|\boldsymbol{k}|}{\omega'_h}\boldsymbol{E}_h\times\big(\hat{\boldsymbol{k}}\times\boldsymbol{E}'_h\big)e^{-2i\omega_h t}\end{aligned} \tag{B.22}$$

By substituting $\boldsymbol{k}\to-\boldsymbol{k}$ and noting that

$$\boldsymbol{E}_\pm(-\boldsymbol{k})\times\big(\hat{\boldsymbol{k}}\times\boldsymbol{E}_\pm(\boldsymbol{k})\big) = \boldsymbol{E}_\pm(\boldsymbol{k})\times\big(\hat{\boldsymbol{k}}\times\boldsymbol{E}_\pm(-\boldsymbol{k})\big), \tag{B.23}$$

we see that $I = -I$ so that

$$I = 0. \tag{B.24}$$

Similarly, the cross term in Eq. (B.21) involving $\boldsymbol{E}^*(\boldsymbol{k})$ and $\boldsymbol{E}^*(\boldsymbol{k}')$ also vanishes. The remaining two terms give

$$\frac{1}{4\pi c}\int d^3x\, \boldsymbol{\mathcal{E}}_h\times\boldsymbol{\mathcal{B}}_h = \frac{1}{8\pi}\int d^3k\frac{\boldsymbol{k}}{\omega_h}|\boldsymbol{E}_h|^2 \tag{B.25}$$

for $h = \pm 1$.

(h = 0 case): For Langmuir waves we have $\boldsymbol{\mathcal{B}}_0 = 0$ and in Fourier space

$$n_{1,0} = -\frac{i}{4\pi e}\boldsymbol{k}\cdot\boldsymbol{E}_0 \tag{B.26a}$$

$$\boldsymbol{v}_h = -\frac{ie}{m\omega_h}\boldsymbol{E}_h \tag{B.26b}$$

$$(\boldsymbol{E}_0\cdot\boldsymbol{k})\boldsymbol{E}_0^* = \boldsymbol{k}|\boldsymbol{E}_0|^2. \tag{B.26c}$$

The only nonvanishing term on the rhs of Eq. (B.19c) is

$$\int d^3x\, mn_{1,0}\boldsymbol{v}_0 = -\frac{1}{16\pi(2\pi)^3}\int d^3x\, d^3k\, d^3k'\frac{1}{\omega_p}\left[\boldsymbol{k}\cdot(\boldsymbol{E}_0 e^{i(\boldsymbol{k}\cdot\boldsymbol{x}-\omega_p t)} - \boldsymbol{E}_0^* e^{-i(\boldsymbol{k}\cdot\boldsymbol{x}-\omega_p t)})\right]$$
$$\times\left(\boldsymbol{E}_0' e^{i(\boldsymbol{k}'\cdot\boldsymbol{x}-\omega_p t)} - \boldsymbol{E}_0'^* e^{-i(\boldsymbol{k}'\cdot\boldsymbol{x}-\omega_p t)}\right) \tag{B.27a}$$

$$= \frac{1}{8\pi}\int d^3k\,\frac{\boldsymbol{k}}{\omega_p}|\boldsymbol{E}_0|^2 + \left(e^{-2i\omega_p t}\int\frac{d^3k}{\omega_p}(\boldsymbol{k}\cdot\boldsymbol{E}_0)\bar{\boldsymbol{E}}_0 + c.c.\right). \tag{B.27b}$$

The last vanishes is zero because by writing $\boldsymbol{E}_0(\boldsymbol{k}) = \alpha(\boldsymbol{k})\hat{\boldsymbol{k}}$ and taking $\boldsymbol{k}\to-\boldsymbol{k}$:

$$\int d^3k(\boldsymbol{k}\cdot\boldsymbol{E}_0)\bar{\boldsymbol{E}}_0 = \int d^3k\,\alpha(\boldsymbol{k})\alpha(-\boldsymbol{k})\hat{\boldsymbol{k}}$$
$$= -\int d^3k\alpha(-\boldsymbol{k})\alpha(\boldsymbol{k})\hat{\boldsymbol{k}}. \tag{B.28a}$$

Therefore

$$\int d^3x\, mn_{1,0}\boldsymbol{v}_0 = \frac{1}{8\pi}\int d^3k\,\frac{\boldsymbol{k}}{\omega_p}|\boldsymbol{E}_0|^2, \tag{B.29}$$

establishing the equality of Eqs. (B.19b) and (B.19c).

To show the equivalence of Eqs. (B.19c) and (B.19d) it is sufficient to show that the helicity cross terms in (B.19d) vanish, namely that,

$$\int d^3x\left[\frac{1}{4\pi c}\boldsymbol{\mathcal{E}}_h\times\boldsymbol{\mathcal{B}}_{h'} + mn_{1,h}\boldsymbol{v}_{h'}\right] = 0 \tag{B.30}$$

for $h \neq h'$. For $h = \pm 1$ and $h' = \mp 1$:

$$\int d^3x\, \boldsymbol{\mathcal{E}}_h \times \boldsymbol{\mathcal{B}}_{h'} = \frac{1}{4(2\pi)^3} \int d^3x\, d^3k\, d^3k'\, \frac{c|\boldsymbol{k}'|}{\omega'_\gamma} \Big(\boldsymbol{E}_\pm e^{i(\boldsymbol{k}\cdot\boldsymbol{x} - \omega_\gamma t)} + \boldsymbol{E}^*_\pm e^{-i(\boldsymbol{k}\cdot\boldsymbol{x} - \omega_\gamma t)} \Big) \times \Big(\hat{\boldsymbol{k}}' \times \boldsymbol{E}'_\mp e^{i(\boldsymbol{k}'\cdot\boldsymbol{x} - \omega'_\gamma t)} + \hat{\boldsymbol{k}}' \times \boldsymbol{E}'^*_\mp e^{-i(\boldsymbol{k}'\cdot\boldsymbol{x} - \omega'_\gamma t)} \Big). \tag{B.31}$$

Then

$$\frac{1}{(2\pi)^3} \int d^3x\, d^3k\, d^3k'\, \frac{c|\boldsymbol{k}'|}{\omega'_\gamma} \boldsymbol{E}_\pm \times \big(\hat{\boldsymbol{k}}' \times \boldsymbol{E}'_\mp \big) e^{i\boldsymbol{x}\cdot(\boldsymbol{k}+\boldsymbol{k}') - it(\omega_\gamma + \omega'_\gamma)} \tag{B.32a}$$

$$= -\int d^3k\, \frac{c|\boldsymbol{k}|}{\omega_\gamma} \boldsymbol{E}_\pm \times \big(\hat{\boldsymbol{k}} \times \bar{\boldsymbol{E}}_\mp \big) e^{-2i\omega_\gamma t} \tag{B.32b}$$

$$= \mp i \int d^3k\, \frac{c\boldsymbol{k}}{\omega_\gamma} [\boldsymbol{E}_\pm(\boldsymbol{k}) \cdot \bar{\boldsymbol{E}}_\mp] e^{-2i\omega_\gamma t} \tag{B.32c}$$

$$= 0. \tag{B.32d}$$

The last line follows because $\boldsymbol{E}_\mp(-\boldsymbol{k}) = C(\boldsymbol{k})\boldsymbol{E}_\pm(\boldsymbol{k})$ for some scalar function C and $\boldsymbol{E}_\pm(\boldsymbol{k}) \cdot \boldsymbol{E}_\pm(\boldsymbol{k}) = 0$. The term involving $\boldsymbol{E}^*_\pm$ and $\boldsymbol{E}'^*_\mp$ in Eq. (B.31) similarly vanishes. The term involving $\boldsymbol{E}_\pm$ and $\boldsymbol{E}^*_\mp$ (and the term involving their conjugates) also vanishes:

$$\frac{1}{(2\pi)^3} \int d^3x\, d^3k\, d^3k'\, \frac{c|\boldsymbol{k}'|}{\omega'} \boldsymbol{E}_\pm \times \big(\hat{\boldsymbol{k}}' \times \boldsymbol{E}'^*_\mp \big) e^{i\boldsymbol{x}\cdot(\boldsymbol{k}+\boldsymbol{k}') - it(\omega_\gamma + \omega'_\gamma)} \tag{B.33a}$$

$$= -\int d^3k\, \frac{c|\boldsymbol{k}|}{\omega_\gamma} \boldsymbol{E}_\pm \times \big(\hat{\boldsymbol{k}} \times \boldsymbol{E}^*_\mp \big) e^{-2i\omega_\gamma t} \tag{B.33b}$$

$$= -\int d^3k\, \frac{c\boldsymbol{k}}{\omega_\gamma} (\boldsymbol{E}_\pm \cdot \boldsymbol{E}^*_\mp) e^{-2i\omega_\gamma t} \tag{B.33c}$$

$$= 0. \tag{B.33d}$$

Thus Eq. (B.30) holds for $h = \pm 1$ and $h' = \mp 1$. The $h = \pm 1$, $h' = 0$ case is trivial since $n_{1,\pm} = 0$ and $\mathcal{B}_0 = 0$. The final case, $h = 0$ and $h' = \pm 1$, follows because

$$\int d^3x\, m n_{1,0} v_\pm = -\frac{1}{16\pi} \int \frac{d^3k}{\omega_\gamma} \Big[(\boldsymbol{k} \cdot \boldsymbol{E}_0) \bar{\boldsymbol{E}}_\pm e^{-it(\omega_\gamma + \omega_p)} - (\boldsymbol{k} \cdot \boldsymbol{E}_0) \bar{\boldsymbol{E}}^*_\pm e^{it(\omega_\gamma - \omega_p)} + c.c. \Big] \tag{B.34a}$$

$$= -\int d^3x\, \frac{1}{4\pi c} \boldsymbol{\mathcal{E}}_0 \times \boldsymbol{\mathcal{B}}_\pm. \tag{B.34b}$$

□

Proposition B.3 (Expectation value of $\boldsymbol{J}$)**.**

$$\frac{1}{8\pi} \langle \boldsymbol{E} | \boldsymbol{J} | \boldsymbol{E} \rangle = \sum_h \frac{1}{8\pi} \int \frac{d^3k}{\omega_h} \boldsymbol{E}^*_h(\boldsymbol{k}) \cdot [\boldsymbol{J} \boldsymbol{E}_h](\boldsymbol{k}) \tag{B.35a}$$

$$= -i \sum_h \frac{1}{8\pi} \int \frac{d^3k}{\omega_h} \Big(\sum_n E^*_{h,n} \cdot (\boldsymbol{k} \times \nabla_{\boldsymbol{k}}) E_{h,n} + \boldsymbol{E}^*_h \times \boldsymbol{E}_h \Big) \tag{B.35b}$$

$$= \sum_h \int d^3x \left[\frac{1}{4\pi c} \boldsymbol{x} \times \big(\boldsymbol{\mathcal{E}}_h \times \boldsymbol{\mathcal{B}}_h \big) + m n_{1,h} \boldsymbol{x} \times \boldsymbol{v}_h \right] \tag{B.35c}$$

$$= \int d^3x \left[\frac{1}{4\pi c} \boldsymbol{x} \times \big(\boldsymbol{\mathcal{E}} \times \boldsymbol{\mathcal{B}} \big) + m n_1 \boldsymbol{x} \times \boldsymbol{v} \right] \tag{B.35d}$$

Proof. It was shown in Ref. [5], Lemma 1 that $\boldsymbol{J}_\parallel$ and $\boldsymbol{J}_\perp$ can be expressed as

$$\boldsymbol{J}_\parallel = \mathcal{P}_h \circ \boldsymbol{S} \tag{B.36a}$$

$$\boldsymbol{J}_\perp = \mathcal{P}_h \circ \boldsymbol{L} \tag{B.36b}$$

where $\boldsymbol{S}$ and $\boldsymbol{L}$ are the usual spin and orbital angular momentum operators

$$(S_a)_{bc} = -i\epsilon_{abc} \tag{B.37a}$$

$$\boldsymbol{L} = -i\hat{\boldsymbol{k}} \times \nabla_{\boldsymbol{k}} \tag{B.37b}$$

and $\mathcal{P} : \mathbb{R}^3 \setminus \{0\} \times \mathbb{C}^3 \to \gamma_h$ is the projection onto the helicity h states. For a component J_a of the angular momentum, we generally have

$$\langle \boldsymbol{E}|J_a|\boldsymbol{E}\rangle = \sum_h \int \frac{d^3k}{\omega} \boldsymbol{E}_h^*(\boldsymbol{k}) \cdot [J_a \boldsymbol{E}_h](\boldsymbol{k}) \tag{B.38a}$$

$$= \sum_h \int \frac{d^3k}{\omega} \Big(\boldsymbol{E}_h^* \cdot (\mathcal{P}_h \circ L_a) \boldsymbol{E}_h + \boldsymbol{E}_h^* \cdot (\mathcal{P}_h \circ S_a) \boldsymbol{E}_h \Big) \tag{B.38b}$$

$$= \sum_h \int \frac{d^3k}{\omega} \Big(\boldsymbol{E}_h^* \cdot L_a \boldsymbol{E}_h + \boldsymbol{E}_h^* \cdot S_a \boldsymbol{E}_h \Big) \tag{B.38c}$$

$$= -i \int \frac{d^3k}{\omega} \Big(\boldsymbol{E}_h^* \cdot (\boldsymbol{k} \times \nabla_{\boldsymbol{k}})_a \boldsymbol{E}_h + E_{h,b}^* \epsilon_{abc} E_{h,c} \Big) \tag{B.38d}$$

$$= -i \int \frac{d^3k}{\omega} \Big(\sum_n E_{h,n}^* (\boldsymbol{k} \times \nabla_{\boldsymbol{k}})_a E_{h,n} + (\boldsymbol{E}_h^* \times \boldsymbol{E}_h)_a \Big), \tag{B.38e}$$

establishing Eqs. (B.35a) and (B.35b). We show that the equivalence of (B.35b) and (B.35c) by showing the integrands are equal for each h.

($h = \pm 1$ case): In the EM case, $n_{1,\pm} = 0$, so the claim is that

$$\frac{1}{4\pi c} \int d^3x \, \boldsymbol{x} \times \big(\boldsymbol{\mathcal{E}}_\pm \times \boldsymbol{\mathcal{B}}_\pm \big) = -i \int \frac{d^3k}{\omega} \Big(\sum_n E_{\pm,n}^* \cdot (\boldsymbol{k} \times \nabla_{\boldsymbol{k}}) E_{\pm,n} + \boldsymbol{E}_\pm^* \times \boldsymbol{E}_\pm \Big) \tag{B.39}$$

One can prove this by similar techniques used for Propositions B.1 and B.2. However, we omit the proof since a detailed derivation of this result is given by Li in the Appendix of Ref. [181]. While Li was concerned with the case of EM waves in the vacuum, the derivation only uses the relationship

$$\boldsymbol{\mathcal{B}}(\boldsymbol{x}) = \frac{1}{2(2\pi)^{3/2}} \int \frac{d^3k}{\omega} \left(\boldsymbol{k} \times \boldsymbol{E} e^{i(\boldsymbol{k}\cdot\boldsymbol{x} - \omega t)} + c.c. \right) \tag{B.40}$$

and is equally valid in the vacuum ($\omega = |\boldsymbol{k}|$) and plasma ($\omega = \sqrt{\omega_p^2 + c^2|\boldsymbol{k}|^2}$) cases.

(h=0 case): For Langmuir waves, $\boldsymbol{E}_0^* \times \boldsymbol{E}_0 = 0$, $\boldsymbol{\mathcal{B}}_0 = 0$, and we can write $\boldsymbol{E}(\boldsymbol{k}) = \alpha(\boldsymbol{k})\hat{\boldsymbol{k}}$. We thus wish to show that

$$\int d^3x m_e n_{1,0} \boldsymbol{x} \times \boldsymbol{v}_0 = -\frac{i}{8\pi} \int \frac{d^3k}{\omega_p} \boldsymbol{E}_0^* \cdot (\boldsymbol{k} \times \nabla_{\boldsymbol{k}}) \boldsymbol{E}_0 \tag{B.41a}$$

$$= -\frac{i}{8\pi} \int \frac{d^3k}{\omega_p} \alpha^* (\boldsymbol{k} \times \nabla_{\boldsymbol{k}}) \alpha. \tag{B.41b}$$

Starting with the lhs, we have

$$\int d^3x m_e n_{1,0} \boldsymbol{x} \times \boldsymbol{v}_0 = \frac{m_e}{4(2\pi)^3} \int d^3x\, d^3k\, d^3k'\, \boldsymbol{x} \times \big[(n_{1,0}(\boldsymbol{k}') e^{i(\boldsymbol{k}' \cdot \boldsymbol{x} - \omega_p t)} + c.c.) \tag{B.42a}$$

$$\cdot (\boldsymbol{v}_0(\boldsymbol{k}) e^{i(\boldsymbol{k} \cdot \boldsymbol{x} - \omega_p t)} + c.c.) \big]. \tag{B.42b}$$

Using Eqs. (B.26a) and (B.26b) we get

$$\int d^3x m_e n_{1,0} \boldsymbol{x} \times \boldsymbol{v}_0 = \frac{1}{8\pi} \big(\mathrm{Re}(I_1) + \mathrm{Re}(I_2) \big) \tag{B.43a}$$

$$I_1 = \frac{1}{(2\pi)^3} \int d^3x\, d^3k\, d^3k' \frac{d^3k}{\omega_p} (\boldsymbol{k}' \cdot \boldsymbol{E}_0') \boldsymbol{E}_0^* \times \boldsymbol{x} e^{i\boldsymbol{x} \cdot (\boldsymbol{k} - \boldsymbol{k}')} \tag{B.43b}$$

$$I_2 = -\frac{1}{(2\pi)^3} \int d^3x\, d^3k\, d^3k' \frac{d^3k}{\omega_p} (\boldsymbol{k}' \cdot \boldsymbol{E}_0') e^{-2i\omega_p t} \boldsymbol{E}_0 \times \boldsymbol{x} e^{i\boldsymbol{x} \cdot (\boldsymbol{k} + \boldsymbol{k}')}. \tag{B.43c}$$

Using

$$\int d^3x\, \boldsymbol{x} e^{i\boldsymbol{x} \cdot (\boldsymbol{k} - \boldsymbol{k}')} = -(2\pi)^3 i \nabla_{\boldsymbol{k}'} \delta(\boldsymbol{k} - \boldsymbol{k}') \tag{B.44}$$

we have

$$I_1 = -i \int \frac{d^3k}{\omega_p} d^3k' \, (\boldsymbol{k}' \cdot \boldsymbol{E}_0') \nabla_{\boldsymbol{k}'} \times (\boldsymbol{E}_0^* \delta(\boldsymbol{k} - \boldsymbol{k}')) \tag{B.45a}$$

$$= i \int \frac{d^3k}{\omega_p} \nabla_{\boldsymbol{k}} (\boldsymbol{k} \cdot \boldsymbol{E}_0) \times \boldsymbol{E}_0^* \tag{B.45b}$$

$$= -i \int \frac{d^3k}{\omega_p} \alpha^* \boldsymbol{k} \times \nabla_{\boldsymbol{k}} \alpha \tag{B.45c}$$

$$= \langle \boldsymbol{E}_0 | \boldsymbol{J} | \boldsymbol{E}_0 \rangle \in \mathbb{R} \tag{B.45d}$$

Similar algebra shows that

$$I_2 = -ie^{-2i\omega_p t}\int \frac{d^3k}{\omega_p}\alpha(\boldsymbol{k})\boldsymbol{k}\times(\nabla_{\boldsymbol{k}}\alpha)(-\boldsymbol{k}) \tag{B.46a}$$

$$= ie^{-2i\omega_p t}\int \frac{d^3k}{\omega_p}\alpha(-\boldsymbol{k})\boldsymbol{k}\times(\nabla_{\boldsymbol{k}}\alpha)(\boldsymbol{k}) \tag{B.46b}$$

$$= ie^{-2i\omega_p t}\Big(\int \frac{d^3k}{\omega_p}\alpha(\boldsymbol{k})\boldsymbol{k}\times(\nabla_{\boldsymbol{k}}\alpha)(-\boldsymbol{k}) \tag{B.46c}$$

$$-\int d^3k\,\nabla_{\boldsymbol{k}}\times\Big[\frac{\alpha(\boldsymbol{k})\alpha(-\boldsymbol{k})\boldsymbol{k}}{\omega_p}\Big]\Big) \tag{B.46d}$$

$$= -I_2 \tag{B.46e}$$

so that

$$I_2 = 0. \tag{B.47}$$

Combining Eqs. (B.43a), (B.45c), and (B.47) shows that

$$\int d^3x m_e n_{1,0}\boldsymbol{x}\times\boldsymbol{v}_0 = -\frac{i}{8\pi}\int \frac{d^3k}{\omega_p}\alpha^*(\boldsymbol{k}\times\nabla_{\boldsymbol{k}})\alpha, \tag{B.48}$$

establishing the equivalence of (B.35b) and (B.35c).

We verify that (B.35c) and (B.35d) are equivalent by checking that the helicity cross terms ($h \neq h'$) vanish:

$$\int d^3x\left[\frac{1}{4\pi c}\boldsymbol{x}\times\left(\boldsymbol{\mathcal{E}}_h\times\boldsymbol{\mathcal{B}}_{h'}\right) + mn_{1,h}\boldsymbol{x}\times\boldsymbol{v}_{h'}\right] = 0. \tag{B.49}$$

This is trivial for $h = \pm 1$ and $h' = 0$ since $\boldsymbol{\mathcal{B}}_0 = 0$ and $n_{1,\pm} = 0$. For the $h = 0$ and $h' = \pm 1$, it is straight-forward to show that

$$\int d^3x\, m n_{1,0} \boldsymbol{x} \times \boldsymbol{v}_\pm \tag{B.50a}$$

$$= \int d^3k \frac{i}{4\pi\omega_\gamma} \Big[- e^{it(\omega_\pm - \omega_p)} \nabla_{\boldsymbol{k}} [\boldsymbol{k} \cdot \boldsymbol{E}_0] \times \boldsymbol{E}^*_\pm \tag{B.50b}$$

$$+ e^{-it(\bar{\omega}_\pm + \omega_p)} \nabla_{\boldsymbol{k}} [\boldsymbol{k} \cdot \boldsymbol{E}_0] \times \bar{\boldsymbol{E}}_\pm \Big] + c.c.$$

$$= -\frac{1}{4\pi c} \int d^3x \boldsymbol{x} \times (\boldsymbol{\mathcal{E}}_0 \times \boldsymbol{\mathcal{B}}_\pm) \tag{B.50c}$$

so that Eq. (B.49) holds. For $h = \pm 1$ and $h = \mp 1$ the second term in Eq. (B.49) clearly vanishes. The remaining EM term can be split into two parts:

$$\int d^3x\, \boldsymbol{x} \times (\boldsymbol{\mathcal{E}}_\pm \times \boldsymbol{\mathcal{B}}_\mp) = I_3 + I_4 \tag{B.51a}$$

$$I_3 = \int d^3x\, d^3k\, d^3k'\, \boldsymbol{x} \times (\boldsymbol{E}_\pm e^{i(\boldsymbol{k}\cdot\boldsymbol{x} - \omega_\gamma t)} \times \boldsymbol{B}'^*_\mp e^{-i(\boldsymbol{k}'\cdot\boldsymbol{x} - \omega'_\gamma t)}) + c.c. \tag{B.51b}$$

$$I_4 = \int d^3x\, d^3k\, d^3k'\, \boldsymbol{x} \times (\boldsymbol{E}_\pm e^{i(\boldsymbol{k}\cdot\boldsymbol{x} - \omega_\gamma t)} \times \boldsymbol{B}'_\mp e^{i(\boldsymbol{k}'\cdot\boldsymbol{x} - \omega'_\gamma t)}) + c.c.. \tag{B.51c}$$

Standard manipulations, along with the fact that $\boldsymbol{E}^*_\mp \times \boldsymbol{E}_\pm = 0$, give

$$I_3 = \pm \int d^3k \frac{c|\boldsymbol{k}|}{\omega_\gamma} \nabla \times (\boldsymbol{E}^*_\mp \times \boldsymbol{E}_\pm) = 0. \tag{B.52}$$

Similarly, since $\bar{\boldsymbol{E}}_\mp \times \boldsymbol{E}_\pm = 0$, we have

$$I_4 = \pm \int d^3k\, \frac{c|\boldsymbol{k}|}{2\omega_\gamma} \nabla \times (\bar{\boldsymbol{E}}_\mp \times \boldsymbol{E}_\pm) + c.c. = 0. \tag{B.53}$$

Thus

$$\int d^3x\, \boldsymbol{x} \times (\boldsymbol{\mathcal{E}}_\pm \times \boldsymbol{\mathcal{B}}_\mp) = 0, \tag{B.54}$$

completing the verification. □

Proposition B.4 (Expectation value of $\boldsymbol{J}_\parallel$)**.**

$$\frac{1}{8\pi}\langle \boldsymbol{E}|\boldsymbol{J}_\parallel|\boldsymbol{E}\rangle = \sum_h \frac{1}{8\pi}\int \frac{d^3k}{\omega_h}\boldsymbol{E}_h^*(\boldsymbol{k})\cdot[\boldsymbol{J}_\parallel \boldsymbol{E}_h](\boldsymbol{k}) \tag{B.55a}$$

$$= -i\sum_h \frac{1}{8\pi}\int \frac{d^3k}{\omega_h}\boldsymbol{E}_h^*\times \boldsymbol{E}_h \tag{B.55b}$$

$$= \sum_h \frac{1}{4\pi c}\int d^3x\, \boldsymbol{\mathcal{E}}_h^\perp \times \boldsymbol{\mathcal{A}}_h^\perp \tag{B.55c}$$

$$= \frac{1}{4\pi c}\int d^3x\, \boldsymbol{\mathcal{E}}^\perp \times \boldsymbol{\mathcal{A}}^\perp \tag{B.55d}$$

Proof. Eqs. (B.55a) and (B.55b) follow from considering only the $\boldsymbol{J}_\parallel = \mathcal{P}_h \circ \boldsymbol{S}$ term in Proposition B.3. Since $\boldsymbol{E} = \boldsymbol{E}_0 + \boldsymbol{E}^\perp$ and since $\boldsymbol{E}_0^\perp \times \boldsymbol{E}_0 = \boldsymbol{E}_0^* \times \boldsymbol{E}_0 = 0$, the sum can be taken only over $h = \pm 1$, or equivalently, $\boldsymbol{E}$ can be replaced by $\boldsymbol{E}^\perp$. Eq. (B.55c) follow from $\boldsymbol{\mathcal{A}}^\perp = c^{-1}\partial_t \boldsymbol{\mathcal{E}}^\perp$. Lastly, we prove the equivalence of (B.55c) and (B.55d). Since $\boldsymbol{\mathcal{E}}_0^\perp = \boldsymbol{\mathcal{A}}_0^\perp = 0$, it is sufficient to show that

$$\int d^3x \boldsymbol{\mathcal{E}}_\pm^\perp \times \boldsymbol{\mathcal{A}}_\mp^\perp = 0. \tag{B.56}$$

This follows because

$$\int d^3x\, \boldsymbol{\mathcal{E}}_\pm^\perp \times \boldsymbol{\mathcal{A}}_\mp^\perp = \int d^3k\, \frac{1}{\omega_\gamma}\big[\boldsymbol{E}_\pm \times \boldsymbol{E}_\mp(-\boldsymbol{k})e^{-2it\omega_\pm(\boldsymbol{k})} + \boldsymbol{E}_\pm(\boldsymbol{k})\times \boldsymbol{E}_\mp^*(\boldsymbol{k}) + c.c.\big]. \tag{B.57}$$

The first term on the rhs vanishes since $\boldsymbol{E}_\mp(-\boldsymbol{k}) = C_1(\boldsymbol{k})\boldsymbol{E}_\pm(\boldsymbol{k})$ and the second vanishes since $\boldsymbol{E}_\mp^*(\boldsymbol{k}) = C_2(\boldsymbol{k})\boldsymbol{E}_\pm(\boldsymbol{k})$ for some scalar functions C_1 and C_2. □

Proposition B.5 (Expectation value of $\boldsymbol{J}_\perp$)**.**

$$\frac{1}{8\pi}\langle \boldsymbol{E}|\boldsymbol{J}_\perp|\boldsymbol{E}\rangle = \sum_h \frac{1}{8\pi}\int \frac{d^3k}{\omega_h}\boldsymbol{E}_h^*(\boldsymbol{k})\cdot[\boldsymbol{J}_\perp \boldsymbol{E}_h](\boldsymbol{k}) \tag{B.58a}$$

$$= -i\sum_{h,n}\frac{1}{8\pi}\int \frac{d^3k}{\omega_h}E_{h,n}^*(\boldsymbol{k}\times\nabla_{\boldsymbol{k}})E_{h,n} \tag{B.58b}$$

$$= \sum_h \int d^3x\Big[\frac{1}{4\pi c}\sum_n \mathcal{E}_{h,n}^\perp(\boldsymbol{x}\times\nabla)\mathcal{A}_{h,n}^\perp + mn_{1,h}\boldsymbol{x}\times\boldsymbol{v}_h\Big] \tag{B.58c}$$

$$= \int d^3x\Big[\frac{1}{4\pi c}\sum_n \mathcal{E}_n^\perp(\boldsymbol{x}\times\nabla)\mathcal{A}_n^\perp + mn_1\boldsymbol{x}\times\boldsymbol{v} - \frac{e}{c}n_1\boldsymbol{x}\times\boldsymbol{\mathcal{A}}^\perp\Big] \tag{B.58d}$$

Proof. Ref. [182] (pp. 45-47) gives the decomposition

$$\frac{1}{4\pi c}\int d^3x\,\boldsymbol{x}\times(\boldsymbol{\mathcal{E}}\times\boldsymbol{\mathcal{B}}) = \int d^3x\Big\{\sum_n \mathcal{E}_n^\perp(\boldsymbol{x}\times\nabla)\mathcal{A}_n^\perp + \boldsymbol{\mathcal{E}}^\perp\times\boldsymbol{\mathcal{A}}^\perp - en_1\boldsymbol{x}\times\boldsymbol{\mathcal{A}}^\perp\Big\}. \tag{B.59}$$

The equivalence of (B.58a), (B.58b), and (B.58d) then follow directly the corresponding equations for $\boldsymbol{J}$ (Proposition B.3) and $\boldsymbol{J}_\parallel$ (Proposition B.4) by using $\boldsymbol{J}_\perp = \boldsymbol{J} - \boldsymbol{J}_\parallel$. Those Propositions also directly imply that

$$\frac{1}{8\pi}\langle \boldsymbol{E}|\boldsymbol{J}_\perp|\boldsymbol{E}\rangle = = \sum_h \int d^3x\Big[\frac{1}{4\pi c}\sum_n \mathcal{E}_{h,n}^\perp(\boldsymbol{x}\times\nabla)\mathcal{A}_{h,n}^\perp + mn_{1,h}\boldsymbol{x}\times\boldsymbol{v}_h - en_{1,h}\boldsymbol{x}\times\boldsymbol{\mathcal{A}}_h^\perp]. \tag{B.60}$$

Eq. (B.58c) then follows because $n_{1,h}\boldsymbol{\mathcal{A}}_h^\perp = 0$ for all h. □

Appendix C

List of symbols

Below is a list of symbols used frequently throughout this thesis and their primary usage. Some symbols are occasional used for other purposes.

Symbol	**Primary usage**
$\mathrm{O}(n)$	n-dimensional orthogonal group
$\mathrm{SO}(n)$	n-dimensional rotation group (special orthogonal group)
$\mathfrak{so}(n)$	Lie algebra of $\mathrm{SO}(n)$ and $\mathrm{O}(n)$
$\mathrm{U}(n)$	n-dimensional unitary group
$\mathrm{SU}(n)$	n-dimensional special unitary group
$\mathfrak{su}(n)$	Lie algebra of $\mathrm{SU}(n)$ and $\mathrm{U}(n)$
$\mathrm{IO}(3,1)$	improper Poincaré group, including parity and time inversion
$\mathrm{IO}^+(3,1)$	orthochronous Poincaré group, includes parity inversion
$\mathrm{ISO}^+(3,1)$	(proper orthochronous) Poincaré group
$\mathfrak{iso}(3,1)$	Lie algebra of $\mathrm{ISO}^+(3,1)$, $\mathrm{IO}(3,1)$, and $\mathrm{IO}^+(3,1)$
$\mathrm{O}(3,1)$	improper Lorentz group, includes parity and time inversion
$\mathrm{O}^+(3,1)$	orthochronous Lorentz group, includes parity inversion
$\mathrm{SO}^+(3,1)$	(proper orthochronous) Lorentz group

(list of symbols)

$\mathfrak{so}(3,1)$	Lie algebra of $\mathrm{SO}^+(3,1)$, $\mathrm{O}(3,1)$, and $\mathrm{O}^+(3,1)$
$\Gamma(E)$	space of smooth sections of vector bundle E
γ	total photon bundle
$\gamma_\pm$	right (+) and left (-) circularly polarized photon bundles
γ_h	helicity h bundle
γ_{Lor}	relativistic Lorentz factor
ζ_{EM}	electromagnetic plasma wave bundle
$\zeta_\pm$	right (+) and left (-) circularly polarized electromagnetic plasma wave bundles
ζ_0	electrostatic plasma wave bundle
$\eta^{\mu\nu}$	Minkowski metric in the mostly plus $(-+++)$ signature
Λ	Lorentz transformation
Ξ	total graviton bundle
$\Xi_\pm$	right (+) and left (-) circularly polarized graviton bundles
$\pi : E \to M$	vector bundle with total space E, base manifold M, and projection π. It is standard shorthand to refer to the vector bundle as E.
$\pi_1(M)$	fundamental group (first homotopy group) of a manifold M
Σ	action of symmetry (Lie) group on vector bundle
Σ_g	linear action of a symmetry transformation $g \in G$ on a vector bundle
$\tilde{\Sigma}$	induced Lie group action on vector bundle of sections
χ	helicity operator
ω^i_j	(basis-dependent) connection 1-forms for some connection $\boldsymbol{D}$
Ω^i_j	(basis-dependent) curvature 2-forms for some connection $\boldsymbol{D}$
A^*	Hermitian conjugate of an operator A
$\boldsymbol{\mathcal{B}}(\boldsymbol{x}, t)$	position-space magnetic field
$\boldsymbol{B}(\boldsymbol{k})$	momentum-space (Fourier-transformed) magnetic field

$\mathbb{C}^n$	standard n-dimensional complex vector space
$\boldsymbol{D}$	connection on a vector bundle
$\boldsymbol{D}^K$	boost connection
$\boldsymbol{D}^R$	rotation connection
$E\|_N$	vector bundle obtained by restricting base manifold to N
$\boldsymbol{\mathcal{E}}(\boldsymbol{x}, t)$	position-space electric field
$\boldsymbol{E}(\boldsymbol{k})$	momentum-space (Fourier-transformed) electric field
$F_{\boldsymbol{D}}$	curvature of the connection $\boldsymbol{D}$
f^*	complex conjugate of a constant or function f
H	Hamiltonian
$H_*(M)$	(singular) homology complex of a manifold M
$H^*(M)$	de Rham cohomology complex of a manifold M
$H^n(M)$	n^{th} de Rham cohomology group of a manifold M
Lie(G)	Lie algebra of the Lie group G
$\mathcal{L}_+$	forward light cone
$L^2(E)$	space of square integrable sections of vector bundle E
$L^2(E, d\mu)$	space of square integrable sections with respect to measure $d\mu$
$\mathcal{M}_m$	mass m hyperboloid
$\boldsymbol{J}$	angular-momentum operator
$\boldsymbol{K}$	boost operator
P	parity operator
$\boldsymbol{P}$	momentum operator
$\mathbb{R}^n$	n-dimensional Euclidean space
$\mathbb{R}^+$	the set of positive real numbers
S^n	n-dimensional sphere

sgn	sign function on real numbers
sym	symmetrized product of operators or Hilbert spaces
T^n	n-dimensional torus
TM	tangent bundle of manifold M
T_pM	tangent space at p to M
$T^{\mathbb{C}}M$	complexified tangent bundle of M
$T_p^{\mathbb{C}}\mathrm{M}$	complexified tangent space at p
T_{a}	translation by $\mathsf{a}^\mu \in \mathbb{R}^{3+1}$
$\mathfrak{X}(M)$	space of vector fields on manifold M
T	time-inversion operator
Y_{jm}	(ordinary) spherical harmonics
${}_sY_{jm}$	spin-weighted spherical harmonics of weight s
$\coprod_I$	disjoint union over the index set I
$G_1 \rtimes G_2$	semidirect product of groups, with G_1 as a normal subgroup
$\otimes$	tensor product of vector spaces, vector bundles, or representations
$\varnothing$	the empty set
$\wedge$	wedge product of differential forms
$\vee$	wedge sum of topological spaces

Appendix D

List of abbreviations

BEC	bulk-edge correspondence
c.c.	complex conjugate
CMB	cosmic microwave background
CSCO	complete set of commuting observables/operators
DOF	degree of freedom
DOFs	degrees of freedom
EM	electromagnetic
Eq.	Equation
ES	electrostatic
lhs	left-hand side
L	left
OAM	orbital angular momentum
R	right
Ref.	Reference
rhs	right-hand side
SAM	spin angular momentum
TT	transverse-traceless

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