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Meta-logic in Aristotle's Epistemology

Crager, Adam David

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Meta-logic in Aristotle's Epistemology

Adam David Crager

A Dissertation
Presented to the Faculty
of Princeton University
in Candidacy for the Degree
of Doctor of Philosophy

Recommended for Acceptance
by the Department of
Philosophy
Adviser: Benjamin Morison (Primary)
Hendrik Lorenz (Secondary)

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Abstract

Aristotle's *Analytics* (*Prior* and *Posterior*) is a single work. Its unifying topic is episteme. And its central aim is to elaborate and defend a view on which episteme is achieved through a certain kind of logos—a demonstrative-proof [*apodeixis*]. This dissertation studies Aristotle's use of logic in working through a debate over the nature and possibility of episteme rehearsed in *Posterior Analytics* I.3. In addition to I.3, the main texts discussed are *Posterior Analytics* I.19-23, and *Prior Analytics* II.5-7, II.14.

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My thinking about the *Posterior Analytics* has been profoundly influenced by two seminars. The first was a two-semester seminar on the *Posterior Analytics* co-taught by Hendrik Lorenz and Benjamin Morison at Princeton in 2009-2010. The second was a one-semester seminar on the project of the *Analytics* co-taught by Stephen Menn and myself at the Humboldt in 2014. I learned a great deal in both seminars, and am very grateful to everyone who participated.

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To Ronah

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Soc. To acquire an untied work of Daedalus is not worth much, like acquiring a runaway slave, for it does not remain, but it is worth much if tied down, for his works are beautiful. In what connection am I saying these things? In connection with true opinions. For true opinions as long as they remain are a fine thing and all they do is good, but they are not willing to remain long, and they escape from a man's mind so that they are not worth much until one ties them down by reasoning out the reason why. (And that, my friend Meno, is recollection as we previously agreed). After they are tied down, in the first place they become pieces of *episteme*, then they remain in place. And this is why *episteme* is prized higher than correct opinion, and *episteme* differs from correct opinion in being tied down.

Plato: *Meno* 97e-98a (trans. Grube)

After we have convinced ourselves that a boulder is immovable, by trying unsuccessfully to move it, there remains the further question, what is it that supports it so securely?

Frege: *Foundations of Arithmetic* §2 (trans. Austin)

Chapter 1

Introduction: episteme and logos

1.1 Stability

Moving through the world we are continually modifying our conception of what's the case. But there are some points on which we insist—certain views we hold fixed and are unwilling to give up without a fight. Foremost in this class is what we take ourselves to *know*. For, no one changes her mind about something so long as she takes herself to know it's the case. And rationality demands that we be well-prepared to defend our knowledge—that we try to keep our knowledge secure, that we aim to retain our knowledge in the face of whatever else we might learn. Knowledge *as such* ought to be steadfast and unyielding, resilient in the face of new information; rationality prescribes for it a stability not prescribed for beliefs *as such*, opinions *as such*, desires *as such*. If all goes well—and it often does—what we know doesn't get away from us.

All knowledge is (or should be) stable. But not to the same degree. JFK is dead. I know he's dead. Or so let's suppose. Now, I submit that there are no truths presently unknown to me that, if accepted, given everything else I believe, would make it rational to change my mind here. There just isn't any such pattern of (actual) truths that would do the trick. And I predict that so long as I remain of sound mind and body, this will always be the case for me. This gives my knowledge that JFK is dead a degree of stability unmatched by many other pieces of knowledge. But of course none of this is to say that I couldn't ever be rationally persuaded to deny that JFK is dead. We can be rationally persuaded away from the truth; we can be rationally led out of knowledge. Surely there's some pattern of *false* information I could rationally come to accept (say, as the victim of an inordinately expensive well-executed hoax) that would engender a rational change of mind on the issue. Is JFK in fact a spry 98-year-old residing in a Havana suburb with his long-time endocrinologist? No. But it's possible, at least in principle, that in a year's time I'll rationally

assent to exactly this.

There is a good deal in my conception of reality that I could, in principle, be rationally persuaded out of. Much of my knowledge is like this. But perhaps some thinkers have articles of knowledge invulnerable to all forms of rational undermining. Indeed, perhaps there are some kinds of knowledge that no rational agent, so long as she was being rational, could come to doubt or be persuaded out of. Plausibly, the kind of knowledge you and I possess of the fact that 10 is greater than 3 is like this. Somehow my grasp of this truth makes it the case that it cannot ever be rational to change my mind here, no matter what new patterns of truths or falsehoods I could ever rationally accept, no matter what arguments or ideas I might possibly be exposed to. Indeed, this would be so for anyone who grasps this truth in the way that I do.

When Plato and his followers spoke of 'episteme' they were talking about a particular type of knowledge. A very special feature of episteme was that it was supposed to possess a kind of maximal stability. If one has episteme of something, no possible argument, experience, or line of questioning should be able to rationally undermine it. The object of episteme is grasped in such a way that no one who grasps it in that same way could ever rationally call it into question. Where did this radically perfectionistic conception of episteme come from? It's possible that it was first developed by Socrates. At any rate, by the time of Plato's Academy so much was accepted philosophical commonplace. Hence the standard example of a distinguishing-property [*idion*] for episteme given in Aristotle's handbook of dialectic, his *Topics* (V.2 130b15-16, V.4 133b29ff., V.5 134a35-b1, 134b16-18, VI.8 146b1-2):

> judgment incontrovertible by reasoning [*hupolēpsis ametapeistos hupo logou*][1]

And hence the entries for episteme and opinion [*doxa*] in the Pseudo-Platonic Academic book of *Definitions*[2] (414b10-c4):

> episteme: judgment [*hupolēpsis*] of the soul which cannot be dislodged by reasoning [*ametaptōtos hupo logou*]; judgmental capacity (of something or some things) which cannot be dislodged by reason [*ametaptōtos hupo logou*]; true account in thought which cannot be dislodged [*ametaptōtos*]
>
> opinion: judgment controvertible by reasoning [*metapeistos hupo logou*]; a rational being-carried-along; thought falling into both truth and falsity by reasoning

The influence of Plato's own writings here is clear enough. There is, of course, the famous 'statues of Daedalus' passage form the *Meno* (97e-98a). And the *Timaeus* tells us that episteme is (i) 'immoveable by

[1] Unless otherwise noted, all translations are my own. As an example of a definition that is undesirably metaphorical, *Topics* VI.2 139b33-4 gives 'judgement which cannot be dislodged [*ametaptōtos*]' for episteme. Other than 'judgment incontrovertible by reasoning', Aristotle's only other example of a distinguishing-property for *episteme* is 'maximally credible [*pistotatē*] judgment' (V.3 131a23) which occurs just once.

[2] With plausibility, it is sometimes suggested—for instance by Hutchinson (in Cooper 1997)—that the 'Plato' at the head of our manuscripts of this text may simply have been meant to indicate association with Plato's school, rather than genuine Platonic authorship. The *Definitions* are standardly dated to the 3rd or 4th century BCE.

persuasion [*akinēton peithoi*]', (ii) 'accompanied by true reason', and (iii) 'possessed by gods and a small class of men', while opinion is (i) 'susceptible to persuasion' [*metapeistos*], (ii) 'unaccompanied by [true] reason', and (iii) 'shared in by all men' (51e1-6).[3] Later, Aristotle's *Posterior Analytics* will explain that while opinion is unstable [*abebaion*] (I.33 89a5-6), 'he who has episteme simpliciter is insusceptible to rational persuasion [*ametapeistos hupo logou*]' (I.2 72b2-3)—i.e. not susceptible to being persuaded out of his or her episteme. The Stoics, developing the same intuition in their own terms, will maintain that (LS 41C1-2):

> episteme is cognition which is secure and strong and unchangeable by reasoning [*ametatheton hupo logou*], but opinion is weak and false assent.[4]

1.2 Episteme and philosophy

Contemporary epistemologists tend to conceive of knowledge (in general) as a rather ordinary possession. They tend to suppose the point of investigating knowledge (in general) is to help understand ordinary thought and agency. Once we have a satisfactory general account of knowledge, we'll better understand action, belief-revision, deliberation, assertion, and so on.

Plato, Aristotle, and the Stoics give birth to western epistemology, developing sophisticated theories of episteme. But for these thinkers, episteme is no ordinary possession. It was clear that gods, should they exist, will have to possess some kind of episteme. And it was equally clear that the vast majority of human beings failed to have any episteme. In the first instance, these thinkers used 'episteme' to pick out the distinctive kind of knowledge that ordinary people lack and philosophers are trying to get—roughly: a perfected understanding of things as they really are. The conceptual role episteme plays in their philosophical investigations is thus fundamentally different than that which knowledge plays in ours. They found their radically perfectionistic conception of episteme attractive precisely because episteme was supposed to be a kind of *ideal* theoretical achievement.

[3] Strictly speaking, *Timaeus* 51e is contrasting *doxa* with *nous* rather than episteme. But like (e.g.) the *Republic* and *Philebus* (and unlike the *Seventh Letter*), the *Timaeus* sees no distinction between *episteme* and *nous*. On the only two occasions '*episteme*' occurs in the *Timaeus* (37c2, 46d7) it is linked pleonasticly with *nous*. At any rate, connection between the *Timaeus* and the *Topics* and *Definitions* texts cited above is certain. While the terms '*ametaptōtos*', '*ametapeistos*', and '*metapeistos*' each occur in epistemological discussions of the *Timaeus*, all three are exceedingly rare. Pre-Platonic uses are unattested. And while all three occur in the epistemological passages of the *Topics* and *Definitions* cited above (and no where else in those works), the only other extant 3rd or 4th century texts using any of them are the *Magna Moralia* (one use: see below), and fragments of Zeno and Chrysippus. If Aristotle and the compiler of the Academic book of *Definitions* are not directly following the *Timaeus*, all three texts would picking up on a special vocabulary for characterizing episteme/*nous* common in the 4th century Academy.

[4] Trans. Long-Sedley. Zeno (according the Cicero) already contended that episteme 'cannot be disrupted by reasoning [*convelli ratione non posset*]'. Compare also Stobaeus' enumeration of the Stoic senses of episteme (LS 41H trans. Long-Sedley):

> [The Stoics say] episteme is a cognition which is secure and unchangeable by reason. It is secondly a system of such epistemai like the rational cognition of particulars which exists in the virtuous man. It is thirdly a system of expert epistemai which has intrinsic stability, just as the virtues do. Fourthly, it is a tenor for the reception of impressions which is unchangeable by reason, and consisting, they say, in tension and power.

The focus of this dissertation is Aristotle. So I'll limit myself to making the point salient with respect to his thought. For Aristotle episteme is not a *condition* [*diathesis*] of its possessor—like a cold, a desire for coffee, or the certainty that it's Tuesday. Qualities of this kind, says Aristotle, are fleeting, easily gained and lost. Episteme is rather a *state* [*hexis*]—like literacy, courage, or an addiction—something with an intrinsic stability to it, something only alterable (if at all) with difficulty.[5] Like courage, and unlike literacy, episteme is no easy achievement. It is something that most people lack, and many do not even try to attain. But it is a profoundly excellent achievement and, like courage, a virtue [*aretē*]. This is important to see because while episteme is a type of knowledge [*gnōsis*], Aristotle does not think that all items of knowledge are achievements amounting to virtue. Aristotle is quite clear that in certain instances sense-perception [*aisthēsis*] is (or provides) *gnōsis*; but no sense-perception is itself a virtue.[6] Unlike virtues such as courage and justice, episteme is an intellectual excellence—a virtue of thought [*dianoia*] rather than a virtue of character [*ēthos*]. Episteme is specifically a theoretical form of expert knowledge, rather than an expertise in action (like *phronēsis*) or expertise in production (like *technē*). Where the exercise of *phronēsis* and *technē* paradigmatically consist in action and production respectively, the paradigm exercise of episteme is an activity of pure theoretical understanding.

When Aristotle speaks of episteme he sometimes means to indicate a body of systematic understanding. In such cases, translators will often render the term 'science'. This unproblematic so long as we keep in mind that, qua virtue, episteme is *perfected*. Episteme is not a science in the sense that we today call a discipline or branch of study (say, contemporary physics) a science. Our physicists do research and inquire into the physical world precisely *because* they do not have a perfect scientific understanding of it.[7] But for Aristotle, someone with genuine episteme of (say) physics would have no need for further investigation because her masterful understanding of the domain of physics would have reached an ideal completion. *We* say that our physicists are 'doing science'. Insofar as all outstanding problems in the area have not been correctly solved, Aristotle wouldn't say that episteme is what modern physicists are practicing, he would say that episteme is what they are trying to *achieve*.[8] For Aristotle, all parts of theoretical philosophy—metaphysics, natural philosophy, mathematics—aim at episteme.

While Aristotle sometimes uses 'episteme' to pick out a completed body of knowledge, at other times when Aristotle speaks of episteme he is naming a theorist's expert grasp of some particular truth. This is what we get, e.g., in Aristotle's celebrated 'definition' of episteme in *Posterior Analytics* I.2. I.2 tells

[5]Indeed, episteme is Aristotle's paradigm *hexis* (see *Categories* 8 8b28ff., 9a5); when present episteme should be the 'most stable [*monimōtaton*] and strongest [*biastikōtaton*] of anything in us' (*Magna Moralia* II.6 1200b35).

[6]See, e.g., *Metaphysics* A.1 981b11 which states that *aisthēseis* are 'most authoritative' with respect to *gnōsis* of particulars.

[7]Cf. *Posterior Analytics* II.1.

[8]There is, of course, also a use of the term in Aristotle on which an 'episteme' can indicate a theoretical disciple with active and not yet completed research programmes. But this is not Aristotle's main use of the term; more importantly, it is not episteme in this sense that is philosophically at issue in the *Analytics*.

us that a knower A has episteme of φ (*epistasthai*) exactly when (i) φ is a necessary-truth, (ii) A knows [*ginōskein*] φ holds because of its causal-explanation [*aitia*]. Having episteme of a truth, holds Aristotle, crucially involves recognizing *why* or *because of what* the truth holds. We have episteme of something only by understanding it in the light of the causes and principles that ground it. And like Plato, Aristotle insists that only certain kinds of objects are suitable for episteme. Aristotle somehow interprets the demand that episteme be maximally stable to rule out episteme of non-necessary truths.[9] If one has episteme of φ, one must grasp φ in a manner such that no possible agent who grasped φ in that way could ever rationally change her mind about φ.[10] And Aristotle takes this to entail that genuine episteme is limited to a certain kinds of necessary truths concerning imperishable aspects of the real.[11] The geometer shouldn't (and doesn't) aim merely to understand why transient figure X has external angles summing to 360 degrees; the *epistēmōn* understands permanent inexorable truths—e.g. that this *kind* of figure per se is such.

1.3 Logos and the possibility of episteme

We've remarked that, among Plato and his rationalist successors, the idea that episteme should possess a kind of maximal stability was deeply entrenched. If someone has episteme it should persist in the face of anything else she might rationally countenance, no matter what future arguments and ideas she might be exposed to. Her episteme must be indefeasible in the face of all possible rational challenges that could ever, in principle, arise. Clearly this is a tall order. Consider, for instance, the range of all possible disputational scenarios that could, in principle, arise. There are infinitely many; some will involve extraordinarily excellent disputational challengers with daemonic persuasive abilities. It is not surprising, then, that some late fourth-century Academics (like Arcesilaus) were accustomed to argue against the possibility of episteme.[12] Aristotle disagrees. He thinks that it is possible for us to grasp truths in such a way that it could never be rational to change our minds about them. And in his *Analytics*, Aristotle defends a bold and compelling account of how, in general, episteme can and must be achieved (*Posterior Analytics* I.2 71b17-19):

> [W]e claim that we achieve episteme through demonstrative-proof [*apodeixis*]. By a 'demonstrative-proof' I mean an epistemic syllogism [*syllogismos epistēmonikos*]; and by 'epistemic' I mean that in virtue of whose possession we have episteme.

[9] A point emphasized to me by Stephen Menn—though he may not fully endorse my development the idea.

[10] The impersonality of the criterion is important given Aristotle's views about the in principle communicability of all episteme.

[11] How exactly Aristotle understands the entailment is not perfectly clear. I suspect certain Aristotelian assumptions about content would have to play a crucial role in the story. But I will not speculate further . For texts linking the maximal stability of episteme with the need for episteme to have only necessary objects see esp. *Nicomachean Ethics* VI.3, *Metaphysics* Z.15, and *Posterior Analytics* I.33.

[12] Cf. Cooper's 'Arcesilaus: Socratic and Sceptic' (2004).

The third movement of Plato's *Theaetetus* investigates the 'great and venerable' (203e8-9) view that what converts a true opinion into episteme is the possession of a special kind of logos. But the dialogue ends in aporia. Neither Socrates nor Theaetetus are able to articulate what kind of logos could be sufficient to engender episteme. Aristotle thinks he can do better. He will insist that there is a special kind of logos whose mastery converts a grasp of a truth (judging *hoti*) into episteme of that truth. And he will express the point in terms clearly evocative of the *Theaetetus*: 'all episteme is *meta logou*' (*Posterior Analytics* II.19 100b10, *Nicomachean Ethics* VI.6 1140b33). For Aristotle, the special kind of logos which produces a perfectly inexorable grasp of a truth is a *demonstrative-proof*. This is the kind of logos mathematicians look for. And so should theoretical philosophy in general. Irrefutable episteme of something arises exactly when one has mastered how to prove it.

Generalizing Aristotle's usage at *Posterior Analytics* I.2 71b17-19, let's call any logos whose mastery is sufficient for achieving a piece of episteme an *epistemic logos*. The unifying aim of Aristotle's *Analytics* (*Prior* and *Posterior*) is to elaborate and defend an account of epistemic logoi. Aristotle contends that demonstrative-proofs [*apodeixeis*] are epistemic logoi. But every demonstrative-proof is a syllogism. So in order to account for the structure and nature of epistemic logoi in particular, Aristotle proposes to first characterize the structure and nature of syllogisms in general. That the subject of the *Analytics* is episteme is explicit from its very opening (*Prior Analytics* I.1 24a10-11):

> First, to say concerning what and of what this investigation is: that it concerns demonstrative-proof [*apodeixis*], and is of demonstrative episteme [*apodeiktikē episteme*].

And soon enough Aristotle's strategy of studying syllogistic in general for the sake of understanding demonstration in particular is made equally clear (*Prior Analytics* I.4 25b26-31)

> [L]et us now say through what [things] and when and how every syllogism comes to be. Later we must speak of demonstrative-proof. We must speak about syllogism earlier than [we speak] about demonstrative-proof since syllogism is the more universal [notion]. For every demonstrative-proof is some syllogism, but not every syllogism is a demonstrative-proof.

Aristotle invents his syllogistic theory in order to advance our understanding of logoi whose mastery produce episteme. And in so doing he is developing formal logic for dramatically different reasons than those other great logicians of classical antiquity—the Stoics. A perusal of the hundred plus titles attested for logical treatises of Chrysippus and his successors reveals deep interest in the production and 'analysis' of syllogisms, the solving of sophisms, disputational technique, etc. Yet there is no evidence of a single treatise

taking demonstration as its central subject matter.[13] Presumably this is because the Stoics did not think of demonstrations as having anything like the extraordinary epistemic powers afforded to them by Aristotle. Indeed, the Stoics will deny that the stability requisite of a piece of episteme can ever be achieved solely through the mastery of a single logos.[14] The unifying philosophical objective of Aristotle's *Analytics*—a theory of demonstrations and how they can function as epistemic logoi—was not pursued by Stoic logicians.

This dissertation is about Aristotle's use of logic in working through a certain debate concerning the existence and the nature of epistemic logoi. For Aristotle, what is at stake in the debate is nothing less than the in principle possibility of episteme—indeed, the in principle possibility of theoretical philosophy achieving its ultimate goal. Aspects of this debate over epistemic logoi are prefigured in Plato's *Theaetetus*. So let us turn briefly to the *Theaetetus* before returning to Aristotle's *Analytics*.

1.4 The *Theaetetus*' dream of foundationalism

Plato's *Theaetetus* famously records the view that what makes a true opinion episteme is the possession of a logos. Neither Socrates nor Theaetetus claim to be the view's author. Somewhat abruptly at 201c8, Theaetetus declares that he once heard someone claiming that (i) episteme *just is* true opinion accompanied by a logos, while (ii) there is no episteme of items which have no logos. Theaetetus had forgotten about this. His previous attempts to define episteme reflect no awareness of it. But having just witnessed Socrates' easy demolition of the proposal that episteme be defined as true opinion, Theaetetus is reminded of this earlier encounter (201c8-d3). Socrates too claims to have once heard a view along these lines. Maybe they've both heard some single theory, maybe they've heard two somewhat different theories—at any rate, Socrates invites Theaetetus to elaborate on the theory he's now recalled (201d4-5). But Theaetetus doesn't think he can do justice to the idea. So he asks Socrates to develop the view for him. 'Then listen,' says Socrates, 'a dream for a dream' (201d8-202c5):

> I in turn seemed to hear certain people [arguing] that **[i]** the primaries—elements [stoicheia] as it were—of which we ourselves and everything else is composed have no logos; **[ii]** (for each alone by itself can only be named, and nothing else can be said of it in addition: neither that it is nor

[13] See Book VII of Diogenes' *Lives of the Philosophers*.

[14] Aristotle and the Stoics both think that if a knower A has episteme of φ, A must grasp φ in a manner such that no possible agent who grasped φ in that way could ever rationally change her mind about φ. The Stoics think that if A grasps something in this way, then A will *in fact* never change her mind about φ because having this kind of grasp requires that A be fully rational. Aristotle denies this. Unlike the Stoics, he doesn't think that in order for one to have any episteme one has to be fully rational. Aristotle calls demonstrations *epistēmonikoi syllogismoi*. There are no such items for the Stoics; the arguments they call demonstrations [*apodeixeis*] have no remotely Aristotelian connection to causes or episteme (see Sextus: *Outlines of Pyrrhonism* 2.135-43 = LS 36B). For the Stoics, possessing episteme of anything requires having all of one's critical and argumentative dispositions perfectly honed. And mastery of no single syllogism can yield *this*. Of course, the Stoics think that the sage will have to know about syllogistic and be an expert dialectician in order to have episteme. But in the context of Stoic epistemology no special urgency attaches to the study of demonstration.

that it is not, for already that would be adding being or non-being to it, whereas it is necessary to bring nothing additional to it if we are to articulate the thing itself alone...if it were possible for [a primary item] itself to be articulated and if it had its own proper logos, it would have to be articulated independently of everything distinct from it. But in fact it is impossible for any of the primaries to be expressed in a logos: they can only be named because they only have a name); **[iii]** but with resect to the things composed of these, just as they are themselves complex, so their names are complex and form a logos: **[iv]** for to be a logos is to be a weaving together of names. **[v]** In this manner the elements have no logos [they are *aloga*] and unknowable [*agnōsta*]; yet [the elements are] are perceptible [*aisthēta*]. **[vi]** In contrast, the composites are knowable [*gnōsta*], expressible [*rhētas*] and opinable by a true opinion. **[vii]** So when someone grasps a true opinion without any logos, there is truth in his soul about it; but he has no episteme;[15] **[viii]** for one who cannot give and receive a logos of something is without episteme of it; but one who has acquired also a logos of it is capable of being all these things and perfect in episteme.

Plato's Socrates routinely assumes that if someone is unable to produce impeccable logoi in answering some fundamental questions about X, that person cannot have episteme of X. He evidently thinks that the ability to give and elaborate upon certain kinds of logoi about something is required for episteme of that thing. It is important to appreciate that the 'Dream Theory'[16] of 201d8-202c5 is a view far more demanding than this. According to the Dream Theory, for any X of which there can be episteme, there is some *particular* logos whose possession is necessary and sufficient for converting a true judgement of X into episteme. This is a bold claim. For, suppose that Xanthippe can flawlessly answer every fundamental questions about virtue that someone with episteme of virtue must to be able to answer. It is far from obvious that Xanthippe

[15]Taking *gignōskein* (following Lyons) as verbal correlate of *episteme*.

[16]With some hesitation I follow scholarly convention in naming the view adumbrated at *Theaetetus* 201d8-202c5 the 'Dream Theory'. Students of Plato—perhaps influenced by Levitt/Burnyeat's over-translation of 201d8: 'In my dream too I thought I heard...' for *'ego gar au edokoun akouein...*'—sometimes speak as if Socrates were claiming that sleeping one night he dreampt hearing the theory (cf. *Philebus* 20b6 where Socrates says he heard the 'third thing' account of the good 'either in a dream or awake'). But this is mistaken. In fact, Socrates is metaphorically characterizing both Theaetetus' partial account of the view he's heard, and Socrates' own presentation of the view, as <u>themselves</u> dreams. (At 202c5 Socrates asks Theaetetus: 'Was the dream you heard like this, or different?'). Exhibiting a metaphorical use well-attested in Plato (see *Statesman* 277d3, *Symposium* 175e3, *Charmides* 173a7, *Meno* 85c9), the point of calling the two presentations of the theory as 'dreams' is to indicate that both presentations are unclear, sketchy, and incomplete. Socrates' remark at 201d8: 'Listen, a dream for a dream [*onar anti oneiros*]' is apparently a play on 'a reality for a dream' [*hupar anti oneiros*] (attested in the *Statesman* 278e10, and also the *Third Letter* 319b7 which is probably spurious). In the *Statesman* text the waking-reality [*hupar*] is a detailed and fully worked out account, where the dream is the gappy sketch of an account they have so far (cf. *Laws* 969b6). So at 201d8, Socrates is admitting that like Theaetetus, he too is not really able to present a fully detailed and worked out presentation of the theory: instead of *hupar anti oneiros*, Theaetetus will be receiving *onar anti oneiros*. Scholars have puzzled to identify a historical person with whom we might identify the Dream Theorist of the *Theaetetus*. The name Antisthenes is sometimes raised in this connection, but the identification is doubtful (see Burnyeat 1990). As far as I'm concerned, there may not have been any single historical individual to whom Plato attributed the Dream Theory in all its particulars. Yet, I'd be very surprised to learn that Plato viewed the theory as wholly a creature of his own invention. I myself suspect that Plato saw his Dream Theorist as a sort of composite character.

possesses this ability solely on the basis of her masterful grasp of any *single* logos. Despite his perennial interest in definitions, Plato's Socrates never commits himself to anything so strong.[17] Even if Socrates does think that having episteme of any X requires mastering the definition X, it plainly does not follow that for every X this suffices for episteme of X.[18]

It shouldn't, then, surprise us that the *Theaetetus* presents a Socrates who has a hard time seeing what sort of logos could possibly have such extraordinary episteme-producing powers as posited by the Dream Theory. Ultimately, Socrates will argue against the Dream Theorist's contention that episteme *just is* true opinion accompanied by a logos. His strategy is to consider possible senses of 'logos' and argue that on whichever sense is intended, no logos can do the work demanded by the Dream Theory (206c1-210b2). By the end of the dialogue, he claims to have shown the view a 'wind-egg', unworthy of further development (210b9). But the definition of episteme as correct opinion with a logos, is only one part of the Dream Theory. And indeed, the idea that episteme *just is* true opinion with a logos is not the only aspect of the Dream Theory that Socrates finds problematic. Immediately after securing Theaetetus' endorsement of the Dream Theory, Socrates voices serious worry over what seems to be the view's 'cleverest' [*kompsotaton*] point (202d10): the claim that complexes are objects of episteme and understandable [*gnōsta*] while the primaries or 'elements' are not.

Why should Socrates think that this component of the Dream Theory is so particularly ingenious? For a moment, let's return to the opening of the dialogue. Immediately before Socrates first asks Theaetetus if he can define episteme, he secures an agreement about how to understand the definiendum. At 145d11ff., Socrates and Theaetetus agree that by 'episteme' they mean the same as 'wisdom' [*sophia*]—that is, the kind of possession one endeavors to attain in learning (say) 'geometry or other kinds of philosophy' (143d2-4, 145c7-d10). The idea that episteme consists in some kind of systematic understanding is, then, present from the start. But it is not well captured in either of Theaetetus' initial attempts at defining episteme. Prima facie, one of the most attractive features of the Dream Theory is its treatment of this point. For the Dream Theory, unlike Theaetetus' earlier proposals, successfully captures the idea that episteme, being a kind of expertise and wisdom, is systematic. On the Dream Theory, episteme is systematic in the sense

[17] In this connection, one might think of the phrase '*aitias logismōi*' (98a3) in the *Meno*'s statues of Daedalus passage. But *logismos* there is a process—it 'is' (or occurs in) recollection (98a4). And it far from obvious that Socrates conceives of this process as yielding some single unified logos as its result. Note rather the plurals in Socrates' discussion of the slave boy's recollection at 85c4-d1.

[18] Indeed, the 'gift of the gods' passage from the *Philebus* seems to positively indicate that for many values of X mastering the definition of X *cannot* suffice for episteme. Take, say, episteme of virtue. Given (e.g.) *Meno* 70a-76e, one might suppose that episteme of virtue will consist in mastering some single definitional formula. But the Socrates of the *Philebus* will insist that virtue is both one and many—many because courage, moderation, wisdom, justice, etc. are all virtues distinct from one another. From *Philebus* 16c-18d, it seems that having episteme of virtue would require not only grasping something like a definition of virtue which would explain how courage, moderation, wisdom, etc. form some single unity. It would also require the ability to flawlessly divide virtue into some determinate number of distinct further unities, and a grasp of why those exhaust the varieties of virtue. Indeed, episteme of virtue looks to require grasping the particular natures of the many kinds of virtue. On this picture, episteme of virtue is structured like literacy [*grammatikē*]. And no more will episteme of virtue be secured through mastery of a single logos, than literacy will.

that instances of episteme are systematically *structured*. Indeed, built right into the Dream Theory is an elegant and concrete account of how episteme is in fact structured. Consider the architecture of language. A paragraph consists of sentences; sentences consist of words; words consist of syllables; and syllables consist of letters. On the Dream Theory, being in general exhibits this kind of hierarchical, leveled organization. And in mirroring being epistemai too possesses such an architecture.

Now, according to the Dream Theory, to have episteme of some X is to have a logos that exactly articulates the identity of X. Such a logos must exhaustively and impeccably account for everything constitutive of the being of X. And it must not add to X anything external to its identity. Yet to exactly articulate the identity of X, a logos must systematically analyze X in terms of its lower-level constituents. These constituents of X, call them the Y's, are thereby epistemically prior to X. And if these Y's are themselves constituted in terms of even lower-level sub-constituents, having episteme of X will also require articulating the identities of the Y's in terms of constituent Z's ontologically and epistemically prior to the Y's. And so on.

But of course there's a problem looming on the horizon. In the case of language, there are primitives. Letters [*stoicheia*] are bottom-level atoms which themselves cannot be further articulated in terms of their constituents.[19] What about being and episteme? Well, either there are onto-epistemic primitives—the analog of letters—or there are not. In the latter case, an infinite regress arises and episteme looks to be impossible. Yet if there do exist onto-epistemic primitives, the Dream Theorist is committed to thinking that they cannot be objects of episteme. For such items, or 'elements', would have no constituents in terms of which they could be understood.

It is at this point that the Dream Theorist makes his apparently ingenious move. The Theorist insists that there are onto-epistemic primitives: 'elements'. He admits that the primitive elements, as such, cannot be objects of episteme. Yet he denies that episteme is thereby rendered impossible. One might have assumed that having episteme of X requires having episteme of every Y prior to X. But the Dream Theorist denies this for the special case when Y is an element. Instead he posits a special kind of immediate cognitive access to the primitive elements. While primitives as such are not possible objects of episteme, they are possible objects of *aisthēsis* (= perception or sense-perception). And insofar as primitive elements can be grasped with *aisthēsis*, they will be available for use in epistemic logoi and regress is avoided.

Theaetetus 203c-205e argues that this dream of foundationalism cannot be a reality. To avoid regress, the Dream Theorist requires a sharp distinction between (i) a knower's episteme and (ii) the correct cognitions of primitives on which the knower's episteme ultimately rests. Socrates worries that the Dream Theorist cannot reasonably maintain any such distinction. Recall that on the Dream Theory, episteme is correct

[19]Recall that in 4th century philosophical Greek: (a) '*stoicheion*' means both 'element' (e.g. Aristotle says Empedocles introduces earth, air, water, fire as *stoicheia*) as well as 'letter', and (b) *stoicheia* are no less minimal elements of speech than they are of writing.

judgment together with a logos, while the kind of logos that gives episteme is an articulation of something's identity in terms of the elements which are its component parts. The reason why primitives cannot be objects of episteme is that they have no logos. And the reason why primitives have no logos is because they are mereologically simple, having no elements and (hence) no component parts. But suppose X is an object of episteme and its elements are: Y, Z. Socrates argues (roughly) as follows. Either X is identical with Y&Z, or X is some further unity over-and-above Y&Z that Y&Z give rise to. If X is identical with Y&Z, then episteme of X requires episteme of Y&Z. For X = Y&Z, and having episteme of X depends upon understanding X *as* Y&Z. But surely, episteme of Y&Z necessitates episteme of Y and episteme of Z (how can one have episteme of both, if one neither has episteme of one nor episteme of the other?). So if X is identical with Y&Z, having episteme of a complex requires episteme of elements. On the other hand, if X is a further unity beyond Y&Z, then Y&Z cannot be parts of X. For, as Socrates argues, every whole *just is* all its parts but, ex hypothesi, it is not the case that X *just is* Y&Z. But if X does not have parts, then X is simple and cannot be an object of episteme for the very same reasons Y can't and Z can't. On neither horn, then, is the Dream Theorist's distinction between (i) and (ii) tenable.

1.5 Aristotle's dream of foundationalism

The Dream Theory described in Plato's *Theaetetus* develops a foundationalist theory of episteme and epistemic logoi in fundamentally mereological terms. When a logos yields episteme of X it is because the logos provides an inventory of X's principles or ultimate grounds, where these are understood to be the fundamental constituents of X's being. The Dream Theorist wants to say that *eptistēta* are *eptistēta* precisely because they have such principles as their parts, while principles themselves are not *eptistēta* precisely because they are simple and have no principles as parts. Socrates' argument at *Theaetetus* 203c-205a challenges any such mereological attempt to work out a foundationalist account of episteme in terms of epistemic logoi. For, claims Socrates, given his commitments the Dream Theorist will have to admit (205d-e) either (i) that both primitives and compounds of primitives are *eptistēta* and hence that both have logoi, or (ii) that neither primitives nor compounds of primitives are *eptistēta* because they both lack parts and therefore cannot have logoi. On one option, both primitives and non-primitives have logoi and are *eptistēta* in the same sense. On the other, both primitives and non-primitives lack logoi and are non-*eptistēta* in the same sense. Foundationalism survives neither option.

After the *Theaetetus*, Plato may very well have been happy to abandon the Dream Theory as a 'wind-egg', 'not worthy of rearing' (210b). But Plato is deeply attracted to, and wants to retain, the *Theaetetus'* mereological model for episteme. With clear echoes of its prequel, Plato's *Sophist* has the Eleatic Visitor

characterizing the real as woven from an alphabet of forms. On this picture, episteme of simples and complexes is interdependent and achieved only when a knower has an expert understanding of all the various combinatorial possibilities whereby elements [*stoicheia*] can come together into wholes. Similar ideas are developed by Socrates in the *Philebus*.

Aristotle moves in another direction. Aristotle wants to defend a modified version of the Dream Theory; but he will give up on a mereological model of episteme. This is not because Aristotle thinks that the mereological arguments *Theaetetus* 203c-205e deployed against the Dream Theory are insurmountable.[20] It is rather that Aristotle sees the appeal to mereology in order to make sense of epistemic logoi an unhelpful and distorting metaphor at best. To this extent, Aristotle thinks that Plato and the Dream Theorist were both caught up in a single confusion. Aristotle, like his teacher, was deeply impressed by the rapid development and epistemic successes of disciplines like mathematics. And he came to think that episteme in general can and must be achieved through demonstrative-proof. It is as a replacement for the mereological metaphor, that Aristotle turns to proof-theory.

With the shift from a mereological to a proof-theoretic model, the difficulties for a foundationalist theory of episteme and epistemic logoi articulated by *Theaetetus* are easily dealt with. The proof-theoretic picture retains the idea that episteme is achieved through epistemic logoi. Yet what makes something *eptistēton* here is not its mereological status with respect to principles, but rather the fact that it is demonstrable from principles. Having episteme of X means knowing how to prove X. But the principles of X (say: Y,Z) are not *constituents* of X in the sense that knowing how to prove X means knowing how to prove Y and knowing how to prove Z (cf. the first-horn of the *Theaetetus* 203c-205e dilemma). Moreover, both principles and objects of episteme can either have parts in the same sense (e.g. subject and predicate) or lack parts in the same sense (e.g. corporeal parts). None of this undermines the possibility of foundationalism because the cognitive asymmetry between grasp of principles and non-principles is a matter of provability and not mereological status (cf. the second-horn of the *Theaetetus* 203c-205e dilemma). As a substitute for the Dream Theorist's suggestion that *aisthēsis* functions to expertly grasp principles without an accompanying logos, Aristotle introduces '*nous*' as a state distinct from episteme that flawlessly comprehends principles without needing to demonstrate them. In insisting that *nous* is somehow 'more exact' and 'more true' than even episteme (*Posterior Analytics* II.19 100b8-14), Aristotle is conceding Socrates' final point (*Theaetetus* 206b) against the Dream Theorist that a knowing grasp of primitives ought to be 'far more clear and more authoritative' than that of *eptistēta* posterior to primitives.

So, the identification of epistemic logoi with demonstrative-proofs can help with certain philosophical

[20] See, for instance, *Metaphysics* Δ.26 in which Aristotle insists on a 'whole'/'all' distinction—a distinction on whose rejection the second horn of the *Theaetetus* 203c-205e dilemma depends.

worries about epistemic logoi. But in the proof-theoretic paradigm, new questions and new problems arise—some of them not so far from the *Theaetetus*. *Posterior Analytics* I.2 tells us that a demonstrative-proof must be (71b16-25):

> ...from [premises] which are **[i]** true, **[ii]** primary, **[iii]** immediate [*ameson*], and **[iv]** better known, **[v]** prior to, and **[vi]** explanatory of [*aitia*] their conclusion; for in this manner the first-principles [*archai*] will be appropriate to what is shown. There can be syllogisms without such [premises], but not demonstrative-proof [*apodexis*]; for premises [which do not meet these conditions] will not yield episteme.[21]

This is a picture on which a demonstration is a fully explanatory syllogistic deduction of a necessary truth. Being explanatory, a demonstration of φ must correctly explain *why* φ is the case by appealing to the features of reality which are (per se) causally responsible for things being such that φ. Being fully explanatory, a demonstration of φ must offer an appropriately *complete* explanation; in particular, beginning with the most proximate causes for φ's holding, a demonstration must construct an explanatory chain that step-by-step causally traces φ's holding all the way back to first-principles [*archai*] which are explanatorily primitive (either absolutely so, or at least relative to the domain in question). While such propositions are available for explaining other truths, they themselves are explained by nothing further.[22] This is one way to develop the idea that epistemic logoi are demonstrative-proofs. But it is not the only way. *Posterior Analytics* I.3 introduces us to two groups of rival theorists:

> Some [theorists] do not think there is episteme since [it would] require having episteme of primitives; others [think] that there is [episteme], but there is demonstration of everything [e.g. even the primitives]. Neither of them [the views] are true, nor necessary.
>
> The [first group], supposing [*hypothemenoi*] that it is not possible to have episteme in another way [than through demonstration], claim that [demonstrators] are led up [*anagesthai*] into infinity, seeing that [knowers] would not have episteme of the posteriors on account of the priors if there are no primitives. They argue rightly. For, it is impossible to traverse [*dielthein*] infinitely many [things]. But, if [chains of explanatory priority] stop and there are first-principles [= primitives],

[21] Aristotle is evidently assuming a notion of (explanatory) priority here that is transitive, but not necessarily total. Otherwise condition **[v]** would be superfluous given condition **[ii]**. (Recall that a relation R is total iff it satisfies trichotomy: for all x, y either xRy or yRx or $x = y$.) Aristotle's immediacy condition **[iii]** must be understood in the context of his syllogistic system. A syllogistic-deduction establishes a relation between two terms A and B through a series of one-step sub-deductions from two-sentence pairs. The premises of a syllogistic deduction are required to contain A and B. And the process of syllogistic-deduction works to link A with B through a series of 'middle terms' [*mesa*]. In the case of demonstration, the middle terms are required to be *explanatory*. Accordingly, in the text above, the condition **[iii]** that the premises of a demonstration be *amesa* (literally: 'lacking a middle', hence 'immediate') is supposed to amount to the requirement that the premises cannot themselves be explained by appealing to any further explanatory middle term.

[22] With similar qualifications about domain relative primacy.

these [they contend] are unknowable [*agnōstous*] since there is indeed no demonstration of them, which they say is the only [way of] having episteme. But if it is not possible to know the primitives, neither is it possible to have episteme of the things from them, without qualification [*haplōs*] and strictly speaking [*kuriōs*], rather [it is only possible to have qualified, i.e., pseudo-episteme] from a supposition [*ex hypotheseōs*]: 'If these things are so, then [such-and-such is so]'.

The [second group] agrees about having episteme, that it is through demonstration alone.[23] But [they argue that] nothing prevents there from being demonstration of all things; for it is possible for demonstration to come about in a circle and from one another [*ex allēlōn*].

We ourselves deny that all episteme is demonstrative, rather episteme of of immediates [= *nous*] is non-demonstrative. And this is clearly necessary. For if there must be episteme of priors and the items from which there is demonstration, but at some point the middles[24] come to a stop, these [immediate propositions] must be indemonstrable. And so these [claims] we thus argue. And we claim that there is not only episteme but some principle of episteme by which we come to know the limit-propositions [*horous*].[25]

Of the three views canvassed, the last (72b18-25) is Aristotle's own. It will be convenient to label Aristotle's interlocutors. Let's call the first group discussed in the text the Ad Infinitum Skeptics[26] (72b7-15). The second group we'll call the Circular Demonstrators (72b15-18).

As a historical matter, we do not have names we can confidently attach to partisans of either group. Conjectures have been made[27]; but none are founded on particularly compelling evidence. Linguistic considerations give strong support for the supposition (as far as I know, never denied in print) that Aristotle is indeed addressing two actual groups of rival theorists, and not merely speaking of positions that are the-

[23]Reading the manuscripts, with Barnes against Ross.

[24]Reading: *pote ta mesa*

[25]In the *Organon* as a whole, *horos* most commonly indicates either (i) a term or (ii) a definition/definiens. But in the *Analytics*, it is often used to mean proposition or syllogistic premise (29a21, 31b33, 24b22, etc.). And in the *Posterior Analytics* it is sometimes used to mean an explanatorily primitive-truth. For instance, 84b25-8 describes the *protaseis* articulating *horoi* as starting-points [*archai*] of demonstration. (The translation of 84b25-28 is mangled by Barnes: the *stoicheia* at 84b22 are unmiddled propositions, so if '*horoi*' at 84b27 means 'terms' as Barnes thinks, Aristotle doesn't know something he does know—namely that if a syllogism has n premises it has $n+1$ terms). In context, at I.3 72b24 '*horous*' cannot mean 'definition'. Aristotle needs to be saying something about all first-principles. But he has just explained (*Posterior Analytics* I.2 72a7ff.) that some but not all first-principles are definitions (NB that the term used in I.2 for 'definition' is *horismoi*). Cf. *Posterior Analytics* I.15, I.23 84b28-30 which explicitly say that some *archai* are negative.

[26]A more precise, though less pithy, label for the first group might be: Disbelievers in Non-hypothetical Episteme. The Ad Infinitum Skeptics are only 'skeptics' in the weak sense that they are *skeptical about* the possibility of episteme, arguing that genuine episteme is impossible—at best, they claim, there can be episteme from a hypothesis. I'll be discussing this group in detail in Section 4. And I will not be claiming that they were skeptics (or proto-skeptics) in any stronger sense than what's been indicated above. We have no evidence this group referred to themselves as skeptics. And we have no evidence that they (as self-professed ancient skeptics did) advocated any kinds of general withholdings of assent. I see no particular reason to believe that they denied the possibility of forms of knowledge [*gnōsis*] weaker than episteme.

[27]See esp. Cherniss (1944), Barnes (1976).

oretically possible.[28] This said, we do not have testimony for these groups' views independent of Aristotle. It would, of course, be interesting to know who these theorists were and what they in fact held. But we probably never shall, and I myself am far more interested in what Aristotle takes them to hold and how he proposes to engage with them. (As to whether Aristotle's criticisms are fair or not, I reserve judgment).

It is not clear to what extent the Circular Demonstrators and the Ad Infinitum Skeptics were themselves influenced by the *Theaetetus*. Aristotle himself is clearly committed to a version of the Dream Theory on which the role played by *aithēsis* is filled by *nous*. And, as we shall explain, he at any rate characterizes the two groups as defending positions effectively on opposite sides of the *Theaetetus* dilemma rehearsed above.

1.6 The road ahead

These days, we often think of Aristotle's *Prior* and *Posterior Analytics* as distinct texts. But Aristotle himself sees them as a single work. The *Analytics*' main topic is episteme. But before he gets to episteme, Aristotle thinks it important to provide a study of formal logic.

In arguing against the Ad Infinitum Skeptics, Aristotle will be defending the in principle possibility of episteme against a group who *contends* that episteme is impossible. In arguing against the Circular Demonstrators, he will be defending the in principle possibility of episteme against a group who (Aristotle argues) unwittingly expounds a view on which nearly all episteme turns out *in fact* to be impossible. On both fronts, Aristotle strategically appeals to meta-logical theses, building heavily on his work in the *Prior Analytics*. And on both fronts Aristotle aims to connect patterns of syllogistic proof structure to patterns of sub-syllogistic predicational structure. Attacking the Ad Infinitum Skeptics, *Posterior Analytics* I.19-22 works by reducing an issue about infinite proof-processes to one about infinite predicational chains. Attacking the Circular Demonstrators, *Posterior Analytics* I.3 (drawing heavily on *Prior Analytics* II.5) reduces an issue about circular proof structures to one about circular predication chains.

The present dissertation studies these texts with particular focus on Aristotle's use of meta-logic in his debates with the Ad Infinitum Skeptics and the Circular Demonstrators. I do not attempt to provide a fully complete account of all the various epistemological and metaphysical issues that come up in the debate. I discuss such issues—sometimes in detail—but I attend especially to the logical aspects of Aristotle's reply to his interlocutors.

Here is the path we follow. **Section 2** introduces Aristotle's syllogistic with a emphasis on the syllogistic

[28] In a back-reference to *Posterior Analytics* I.3, chapter I.22 refers to the Ad Infinitum Skeptics' view as 'what we were saying some people [*tinas*] claim' (84a32). We get *henois men...* for the Ad Infinitum Skeptics at 72b5 which corresponds to *tois de...* for the Circular Demonstrators at 72b6. Then *hoi men...* for the Ad Infinitum Skeptics at 72b7 matching a *hoi de...* for the Circular Demonstrators in 72b15. There's also *sumbainei tois legousi ...* referring to the Circular Demonstrators at 72b32. It is standard for Aristotle to rehearse views of other (actual) theorists when inaugurating an inquiry.

structure of demonstrative-proofs. It also presents a formal model of the (assertoric) syllogistic that we will rely on throughout. **Section 3** treats Aristotle's case against the Circular Demonstrators in *Posterior Analytics* I.3, exploring especially its relation to ideas about circular proof-patterns developed in *Prior Analytics* II.5-7. Finally, **Sections 4-5** investigate Aristotle's rather involved argument against the Ad Infinitum Skeptics in *Posterior Analytics* I.19-22. **Section 4** focuses on the more logical aspects of Aristotle's discussion, **Section 5** on the more metaphysical aspects. Throughout **Sections 2-4** we will be discussing various meta-logical results about Aristotle's syllogistic—results that are proved relative to the dissertation's formal model. Many of the results reflect claims that Aristotle himself makes about syllogistic. Some do not. In cases where Aristotle provides a proof-sketch for a result, we try to discern it and follow his proof-strategy. **Section 6** consists in an appendix of purely formal material.

It is my contention that in Aristotle's response to the Ad Infinitum Skeptics and the Circular Demonstrators, we find him competently engaged in the most sophisticated and challenging meta-logical investigations he would ever attempt. If there was anything comparable in Stoic logic, it has not survived. Among other things, in what follows I hope bring out the diversity and sophistication of Aristotle's work on meta-logic—dissipating the now seemingly common supposition that Aristotle's successes as a logician reach their zenith in the reduction-perfection proofs of *Prior Analytics* I.4-7.

Chapter 2

Syllogistic-deduction

2.1 Introduction to syllogistic

2.1.1 Conditions under which an argument is a syllogism

On the logical theory of the ancient Greeks, a *syllogism* is formally valid argument; but not every formally valid argument is a syllogism. The first half of Aristotle's *Analytics*, his so-called *Prior Analytics*, provides a systematic investigation of syllogisms. In the opening chapter of the work Aristotle offers something like a definition of 'syllogism' (*Prior Analytics* I.1 25b18-20)[1]. Drawing on this, as well as closely related Aristotelian texts, one can discern four conditions which (for Aristotle) an argument must meet in order to be a syllogism:

1. there are at least two premises
2. the information contained in the conclusion is not asserted in any premise
3. the conclusion follows of necessity from the premises
4. each of the premises is inferentially *used* in establishing the conclusion

Given condition 1., no single-premise argument can be a syllogism. For instance, though formally valid the following is not a syllogism:

$$\frac{\text{King holds of no philosopher}}{\text{Philosopher holds of no king}}$$

Condition 2. quite clearly marks circular arguments in which the conclusion is identical with a premise as non-syllogistic. It also appears to rule out formally valid arguments like:

[1] Cf. *Topics* I.1 100a25–7, *Rhetoric* I.2 1356b16–18

$$\frac{\text{King holds of no philosopher} \qquad \text{King holds of all king}}{\text{Philosopher holds of no king}}$$

Aristotle never really explains what 'follows of necessity' (*ex anagkēs sumbainei*) means in Condition 3. But he evidently has something like what we call 'formal validity' in mind. For consider this argument:

$$\frac{\text{Animal holds of all dog} \qquad \text{Dog holds of no plant}}{\text{Animal holds of no plant}}$$

Granting Aristotelian modal assumptions, it is impossible for the conclusion to be false when the premises are true (and trivially so). But Aristotle will deny that this is a syllogism because the conclusion does not follow by necessity *from* the premises in the relevant sense. And to show that the conclusion is not necessitated by the premises in the relevant sense, he'll draw our attention to the three true sentences (1) 'Animal holds of all human', (2) 'Human holds of no horse', (3) 'Animal holds of all horse'.[2] Condition 4. requires that a syllogism cannot have any unused premises. Suppose I give the following argument:

$$\frac{\text{Cat holds of no dog} \qquad \text{Dog holds of all pug} \qquad \text{Pug holds of no bat}}{\text{Cat holds of no pug}}$$

In such a case Aristotle may be willing to say that I've syllogized the conclusion. But I haven't used each of the premises I've posited. So to the extent to which I have syllogized at all, Aristotle will insist that the syllogism I've given is actually:

$$\frac{\text{Cat holds of no dog} \qquad \text{Dog holds of all pug}}{\text{Cat holds of no pug}}$$

2.1.2 Some recent history

In 1951, the great Polish logician Jan Łukasiewicz published *Aristotle's Syllogistic from the Standpoint of Modern Logic*. Despite its title, the work focused just about exclusively on *Prior Analytics* I.2, 4-7: Aristotle's account of two premise non-modalized syllogisms.[3] One of Łukasiewicz's aims was to disentangle Aristotle's *own* logical investigations from the canonical 'theory of syllogisms' handed down in textbooks of traditional logic. And challenging the long-standing orthodoxy that by 'syllogism' Aristotle meant an *argument* of a certain sort, Łukasiewicz maintained instead that an Aristotelian syllogism is a kind of true universalized conditional proposition, e.g.:

- For all A, B, C: if A holds of all B and B holds of all C, then A holds of all C.

- For all A, B, C: if B holds of no A and B holds of all C, then A holds of no C.

[2] This, of course, is Aristotle's celebrated 'method of counter-examples'.

[3] This was Łukasiewicz' first dissertation published in English. The book's second edition, which adds three chapters on Aristotle's 'modal syllogistic', was published posthumously in 1957. Like the first edition, it omits significant study of *Prior Analytics* I.23-II.

Łukasiewicz's identification of Aristotelian syllogisms with such conditionals found some early friends—most prominently, the scholar Günther Patzig. Patzig's early dissertation, the influential *Die aristotelische Syllogistik*: *Logischphilologische Untersuchungen über das Buch A der Ersten Analytiken* (1959)[4] lent some scholarly support to Łukasiewicz's contention.

It is well known that by the mid-1970's the Łukasiewicz-Patzig view of an Aristotelian syllogism as a kind of true universalized conditional had received decisive refutation. By most accounts, the death-blow came with the classic articles of John Corcoran (1972, 1973, 1974) and Timothy Smiley (1973).[5] Through the efforts of these scholars, as well as a barrage of subsequent publications,[6] the traditional view on which by 'syllogism' Aristotle means a certain kind of *valid argument* was shown to be indubitably correct. As far as I am aware, no serious scholar of Greek philosophy has defended the Łukasiewicz-Patzig heterodoxy since. Though still occasionally beaten, the horse is long dead.

Without a doubt, Corcoran and Smiley's articles of early '70s clearly constitute the most influential body of work on Aristotle's syllogistic published in the last 50 years. Although the two authors were working independently, it is with much justification that their work is often mentioned in the same breath. Both, for roughly the same reasons, were convinced that Łukasiewicz was deeply wrong about Aristotle. But both were also impressed by Łukasiewicz's method of using modern mathematic logic as a tool for investigation into the history of logic. Łukasiewicz's formal reconstruction had offered an 'axiomatic' model of Aristotle's syllogistic—a model congenial to Łukasiewicz's particular interpretive (and philosophical) views. Corcoran and Smiley each advocated on a return to the traditional consensus that syllogisms are valid arguments of a certain sort.[7] And to develop the view, both proposed formal reconstructions which invited readers to see *Prior Analytics* I.4-7 as inaugurating a 'natural deduction system'. The proof system Corcoran-Smiley proposed[8] had no axioms, but both direct and indirect (≈ reductio-style) deductions and the primitive 'rules

[4] Jonathan Barnes' translation of Patzig's book would not appear until 1968.

[5] I'm not sure how many subscribers to the Łukasiewicz' heterodoxy there actually were by the early 1970's. J.L. Austin's 1952 review of Łukasiewicz's book in *Mind* had already adduced very strong textual evidence against it. Reviewing Kurt Ebbinghaus' 1964 *Ein formales Modell der Syllogistik des Aristoteles* which, in certain respects, anticipates Corcoran and Smiley's reconstructions, D.W. Hamlyn writes (1966): '[According to Ebbinghaus, contra Łukasiewicz] Aristotle did not think of his theory as an axiomatized system, nor as one which presupposes general principles of logic. Such criticisms are not perhaps new, although the way in which they are presented here has a certain interest' (pg 35).

[6] A paper which deserves special mention here is Michael Frede's 1974 'Stoic vs. Aristotelian Syllogistic'. The essay is not, in the first instance, about Aristotle's theory of syllogism at all—'Aristotelian' in the title means 'Peripatetic'. But Frede was a student of Patzig's. And on pg 111-113 Frede presents what is clearly meant to be a sustained attack on the Łukasiewicz-Patzig view that by 'syllogism' Aristotle means a kind of sentence rather than a kind of argument. Of course, in characteristic fashion, Frede never explicitly says who his target is. Perhaps this is why Frede's paper is rarely mentioned in connection with the efforts of Corcoran and Smiley despite its date of publication. At any rate, the mountain of evidence against Łukasiewicz-Patzig that Frede adduces in these few pages is impressive (cf. Austin 1952), going well beyond what one gets in the contemporaneous Corcoran and Smiley papers.

[7] For the sake of simplicity, I'm here ignoring Smiley's rather fine-grained distinction between arguments and deductions (pg 138). Though both Corcoran and Smiley suggest that we conceive of syllogisms as 'deductions', Corcoran seems not to mark the latter distinction.

[8] For the sake of simplicity, I'm ignoring some fairly minor differences between Corcoran and Smiley's formal reconstructions. The main differences between their models can be accounted for by the authors somewhat different goals in designing their natural deduction systems. Corcoran was initially seeking just a mathematical means for accurately representing Aristotle's actual presentation of perfection/reduction proofs—something he rightly thought one couldn't do in Łukasiewicz's system.

of inference':

1. From A holds of all B, B holds of all C infer A holds of all C
2. From A holds of no B, B holds of all C infer A holds of no C
3. From A holds of all B, B holds of some C infer A holds of some C
4. From A holds of no B, B holds of some C infer A does not hold of some C
5. From A holds of all B, infer B holds of some A
6. From A holds of no B, infer B holds of no A
7. From A holds of some B, infer A holds of some B

Relative to a natural 'extensional' set-theoretic semantics, both authors obtained soundness and completeness results.

2.1.3 Aristotle's own deductive system?

Corcoran and Smiley were clearly right that by 'syllogism' Aristotle means a kind of argument—not a universalized conditional statement. The very fact that Aristotle is studying syllogisms because some of them are demonstrative-proofs makes Łukasiewicz's alternative a non-starter. (No conditional is a proof). This said, the influence of Corcoran-Smiley is also responsible for certain widespread misconceptions of how Aristotle actually goes about studying syllogistic. In particular, the Corcoran-Smiley characterization of Aristotle as the inventor of a natural deduction system has transmitted the following erroneous picture to many both inside and outside the field:

> The principal deduction system used in the *Prior Analytics*, has seven rules of inference, corresponding to the four first-figure moods and the three conversion laws, and two types of deductions, viz., 'direct' and 'indirect'.[9]

To the extent that the above statement is true at all, it is true only for *Prior Analytics* I.4-7. But these four chapters comprise just a small part of Aristotle's syllogistic investigations. Indeed, *Prior Analytics* I.4-7

Hence, Corcoran's taking of all first-figure perfect moods and all three assertoric conversion principles as primitive, and hence his added apparatus of '+', 'h', '?', etc. Smiley too thought that in principle Aristotle's perfection/reduction proofs would be better represented in a natural deduction framework than they were in Łukasiewicz's. But the specific formal model he proposed was not designed for this exact purpose. In the first instance, Smiley was trying to capture the idiosyncratic kind of relevantistic implication that he thought corresponded to Aristotle's judgments of which arguments are syllogistic. Hence his system's extreme austerity, his willingness to provide an asymmetric treatment of i- vs. a- and e-conversion, etc.

[9]From Smith's 'Completeness of an Ecthetic Syllogistic' 1982 p 225. This seems to be Smith in a moment of uncharacteristic carelessness. Later I'll be arguing against some rather more careful statements of his about the place of conversion and reductio in Aristotle's theory.

contains only a partial fragment of Aristotle's work on *two-premise* syllogisms in particular. For while these chapters deal with two-premise syllogisms without any modal premises, *Prior Analytics* I.8-22 investigates two-premise syllogisms with (at least one) modal premise. Aristotle himself sees the whole of *Prior Analytics* I.4-22 as a single unit, focused on determining exactly which two-premise categorical sentence pairs do and do not syllogistically yield conclusions.[10] For every categorical premise pair that doesn't yield a conclusion, Aristotle looks to show that it doesn't yield a conclusion—to do so, he uses his method of counterexamples. And for every categorical premise pair that does syllogistically yield a conclusion, Aristotle looks to show that it does yield a conclusion. In pursuit of the latter goal, Aristotle takes the validity of certain syllogistic forms—the 'perfect' ones—to be, in some sense, basic. Among the moods Aristotle takes to be perfect are the four assertoric moods introduced in I.4.[11] But Aristotle also considers a number of first-figure modalized moods to be perfect. Aristotle's strategy is to establish the syllogistic validity of the non-perfect moods—or as he puts it, to 'perfect' them—by 'reducing them to the first-figure', proving their validity with reference to the perfect-moods. The techniques he uses sometimes rely on the non-modal conversion principles and reductio. But Aristotle also appeals to modal conversion principles (introduced in I.3), as well as modal and non-modal ekthesis, among other techniques.[12] In *Prior Analytics* I.23 Aristotle argues that since all syllogistic arguments with more than two premises can be viewed as consisting of chains of two-premise syllogistic sub-proofs which (as he has just shown in I.4-22) can all be perfected through the first-figure, it follows that every syllogism of any arbitrary length can be so perfected as well.

This dissertation focuses on Aristotle's strategic use of meta-logic in arguments about the possibility and nature of episteme. The great majority of the texts we shall consider occur in either *Prior Analytics* I.24-II or in *Posterior Analytics* I. It is notable that after *Prior Analytics* I.22, mention of modalized syllogisms is vanishingly rare in the *Analytics*. There are but a handful of texts in *Prior Analytics* I.24-II that even indicate awareness of modalized syllogisms; and it is questionable whether modalized syllogisms make any appearance at all in the *Posterior Analytics*. At any rate, the modal syllogistic plays no role in the texts we will be discussing. Following Aristotle's lead, we strictly confine ourselves to the assertoric syllogistic.

The meta-logical investigations we shall be concerned with clearly presuppose knowledge of and facility with the system of assertoric figures and moods Aristotle is looking to establish in *Prior Analytics* I.4-7. The *Analytics*' extensive later researches into assertoric syllogistic thus depend on *Prior Analytics* I.4-7. However, the dependence is not what one would expect if, following Corcoran-Smiley and Smith, one sees

[10]Commentators often speak of Aristotle having two logical systems: a 'modal syllogistic' and a non-modal 'assertoric syllogistic'—and not unhelpfully. However, it's important to appreciate that the sharp distinction between a separate 'assertoric' and 'modal' syllogistic is a scholarly invention. From Aristotle's perspective, *Prior Analytics* I.4-7 and I.8-22 are not investigating two distinct logical systems. They're investigating two bordering regions of a single logical space.

[11]These correspond to the first four 'rules of inference' in the natural deduction reconstruction referred to above.

[12]See Malink and Rosen (2012) for an account of Aristotle's use of the 'possibility rule' in the modal syllogistic.

Prior Analytics I.4-7 as presenting the deduction system of Aristotle's *Analytics*. The present chapter introduces Aristotle's syllogistic with special focus on how Aristotle envisions the syllogistic structure of demonstrative-proofs. I will be arguing against two well-established interpretations of Aristotle's syllogistic in which the reduction-perfection results of *Prior Analytics* I.4-7 are centrally important to Aristotle's thinking about demonstrative-proof. I ultimately suggest that Aristotle does not see devices like conversion and reductio as having any place in demonstrative science.[13] But to help explain the relevant issues I would first like to introduce a means of representing syllogistic-deductions—a system of representation I will be employing throughout what follows.

2.1.4 General framework for reasoning about syllogistic

Following in the tradition of Łukasiewicz, Corcoran, and Smiley, this dissertation uses techniques from modern mathematical logic to explore Aristotle's thinking about syllogistic. This is not because I think that Aristotle's own approach to logic was mathematical. I am in agreement with scholars who deny that Aristotle had any conception of a formal 'logical language' in our sense—scholars who contend that the *Analytics* is not studying any such 'language' viewed (as we do these days) qua well-defined mathematical object.[14] This said, I contend that many of the texts we will encounter show Aristotle engaged in investigations reasonably described as *meta-logical*. In such texts, we find Aristotle not performing syllogistic derivations or even speaking about actual proofs, but rather using schematic letters to reason about the space of possible proof structures.

Much of the meta-logical material I will adduce from the *Analytics* is complex. I show Aristotle competently engaged in meta-logical researches reaching far beyond the relatively simple perfection proofs of *Prior Analytics* I.4-6. I have little doubt that Aristotle himself would have used sophisticated diagrams and/or other representational devices to supplement natural language explanations when presenting his more complex meta-logical studies. (Indeed, I'd be very surprised to find out Aristotle didn't use diagrams when rehearsing even the relatively straightforward perfection proofs of *Prior Analytics* I.4-6). In this context, it seems to me that to explain and think through Aristotle's work on meta-logic, a carefully developed and (hopefully) intuitive formal reconstruction can serve an invaluable interpretive-explanatory role. The model will be mathematical. But it's just a model. We allow the architect who seeks to model a bridge to use balsa wood. As a representational device, I invite the reader to view my formal reconstruction in the same spirit. Hopefully, the convenience and even necessity of proceeding as I do will gradually emerge from greater acquaintance with the textual material.

[13] This said, it's worth noting that on account of certain formal results described in the present chapter and proved in the appendix, none of the main points of chapters 3-5 depend on this suggestion.

[14] See especially Morison (2012).

2.1.4.1 Language

I begin by specifying a toy-language $\mathcal{L}$ in which we can represent Aristotelian syllogisms.

Definition 1 Language $\mathcal{L}$

1. Basic expressions
   (a) a set of terms (we'll represent them by capitalized letters)
   (b) four syllogistic copulae: a, e, i, o
2. Sentences
   (a) If X and Y are terms, then the following are sentences of $\mathcal{L}$:
      i. XaY
      ii. XeY
      iii. XiY
      iv. XoY
   (b) Nothing else is a sentence of $\mathcal{L}$

Following scholarly convention, our intended interpretation is such that the right-most term of a sentence in $\mathcal{L}$ is the grammatical subject while the left-most term is the grammatical predicate.[15] The copula a- is for universal affirmative sentences; e- is for universal negative sentences; i- is for particular affirmative sentences; and o- is for particular negative sentences. So on our intended interpretation of $\mathcal{L}$ we have:

· XaY $\Longleftrightarrow$ X is said of all Y $\Longleftrightarrow$ All Y's are X's

· XeY $\Longleftrightarrow$ X is said of no Y $\Longleftrightarrow$ No Y's are X's

· XiY $\Longleftrightarrow$ X is said of some Y $\Longleftrightarrow$ Some Y's are X's

· XoY $\Longleftrightarrow$ X is not said of some Y $\Longleftrightarrow$ Some Y's are not X's

2.1.4.2 Direct deductions

We will later discuss the issue of what kind of syllogistic inferences Aristotle envisions demonstrative-proofs as having. In order to be as ecumenical as possible I begin with a system of direct deductions in which all assertoric conversions any interpreter of Aristotle would plausibly allow are permitted, along with the 16 assertoric moods whose validity is asserted in *Prior Analytics* I.4-7. First, two definitions.

[15] The convention on which the predicate of a syllogistic sentence comes *before* the subject is preferable in contexts (like the present) where attention is paid to Aristotle's own writing on syllogistic. In describing syllogistic sentences, Aristotle himself prefers a predicate first expressions (e.g. 'X holds of all Y') to subject-first expressions ('every Y is X').

Definition 2 Basic syllogistic-deduction

For all terms X, Y, Z of $\mathcal{L}$ the following 'trees' constitute basic syllogistic-deductions (premises above the line, conclusion below the line)

First figure	Second figure	Third figure
Barbara $\frac{\text{XaY} \quad \text{YaZ}}{\text{XaZ}}$	Cesare $\frac{\text{YeX} \quad \text{YaZ}}{\text{XeZ}}$	Darapti $\frac{\text{XaY} \quad \text{ZaY}}{\text{XiZ}}$
Celarent $\frac{\text{XeY} \quad \text{YaZ}}{\text{XeZ}}$	Camestres $\frac{\text{YaX} \quad \text{YeZ}}{\text{XeZ}}$	Datisi $\frac{\text{XaY} \quad \text{ZiY}}{\text{XiZ}}$
Darii $\frac{\text{XaY} \quad \text{YiZ}}{\text{XiZ}}$	Festino $\frac{\text{YeX} \quad \text{YiZ}}{\text{XoZ}}$	Disamis $\frac{\text{XiY} \quad \text{ZaY}}{\text{XiZ}}$
Ferio $\frac{\text{XeY} \quad \text{YiZ}}{\text{XoZ}}$	Baroco $\frac{\text{YaX} \quad \text{YoZ}}{\text{XoZ}}$	Felapton $\frac{\text{XeY} \quad \text{ZaY}}{\text{XoZ}}$
Fapesmo $\frac{\text{XaY} \quad \text{YeZ}}{\text{ZoX}}$		Ferison $\frac{\text{XeY} \quad \text{ZiY}}{\text{XoZ}}$
Frisesomorum $\frac{\text{XiY} \quad \text{YeZ}}{\text{ZoX}}$		Bocardo $\frac{\text{XoY} \quad \text{ZaY}}{\text{XoZ}}$

Each basic syllogistic-deduction is a representative of one of the 16 valid syllogistic 'moods' Aristotle recognizes in *Prior Analytics* I.4-7. In first figure moods the middle term occurs as subject in one premise and predicate in the other. In second figure moods, the middle term occurs as predicate in both premises. And in third figure moods, the middle term occurs as subject in both premises. Note that the traditional names for the syllogistic moods carry useful information. The first vowel corresponds to the copula of the left premise; the second vowel corresponds to the copula of the second right premise; and the third vowel corresponds to the copula of the conclusion.[16] The first-figure moods Fapesmo and Frisesomorum introduced by Aristotle in I.7 are italicized to indicate that, unlike the other first-figure moods, Aristotle does not view them as

[16]NB that for Aristotle premise order is not logically significant.

'perfect'.[17]

Note that, at this point, we do not require that the three terms used in a basic syllogistic-deduction be distinct. For instance, we allow that the following are syllogistic-deductions even though they will not be syllogisms in Aristotle's technical sense:

$$\frac{\text{AaB} \qquad \text{BaB}}{\text{AaB}} \qquad \frac{\text{BeA} \qquad \text{BaB}}{\text{AeB}} \qquad \frac{\text{AoB} \qquad \text{BaB}}{\text{AoB}} \qquad \frac{\text{AaB} \qquad \text{BaA}}{\text{AaA}}$$

It may be noted that if we *did* require the three terms used in a basic syllogistic-deduction to be distinct, then no two-premise syllogistic-deduction would be circular—having a premise identical with its conclusion.

In **Section 4**, I will be introducing what I take to be a realistic model for (assertoric) syllogistic that respects Aristotle's idea that no syllogism has the information contained in its conclusion asserted by a premise. In **Section 3**, however, we will be investigating Aristotle's case against a group of theorists who do not accept his definition of syllogism in all its details. These are thinkers who claim that the principles of a science are demonstrated through one another. So we will begin with a representation of syllogistic which does not take account of Aristotle's insistence that the conclusion of a syllogism be different than its premises.[18]

Definition 3 Conversion-rules

For all terms X, Y, Z of $\mathcal{L}$ the following constitute conversions:

simple e-conversion $\frac{\text{XeY}}{\text{YeX}}$	accidental a-conversion $\frac{\text{XaY}}{\text{YiX}}$
simple i-conversion $\frac{\text{XiY}}{\text{YiX}}$	accidental e-conversion $\frac{\text{XeY}}{\text{YoX}}$

As is well-known, Aristotle commits himself to the validity of accidental a-conversion, as well as simple e- and i-conversion in *Prior Analytics* I.2. Somewhat less well-known is that Aristotle actually argues for these principles on the basis of some assumptions about how a-, e-, i-, and o-propositions with identical subjects and predicates are opposed to one another. I'm speaking of course of Aristotle's view that for all terms X and Y:

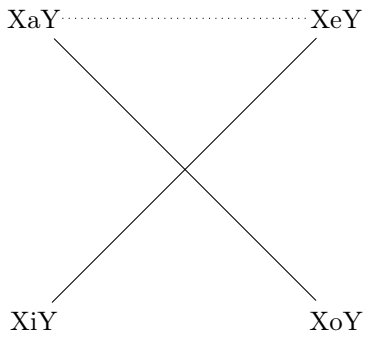
[17]Later logicians would describe these moods as belonging to a fourth-figure. Aristotle will not. On his considered view, they belong to the first-figure.

[18]NB that the central formal results proved in the remainder of this chapter will hold whether or not we require that the conclusion of a syllogistic-argument be appropriately 'different' from its premises.

1. XaY, XeY are *contraries* (they cannot be simultaneously true)

2. XeY, XiY are *contradictories* (one is true exactly if the other is false)

Hence the basis for the traditional 'square of opposition' (solid line representing a contradictory pair, a dotted-line represents a contrary pair):



It is notable that Aristotle himself does not discuss accidental e-conversion. But, as later authors have noticed, Aristotle's arguments for the other conversion principles in *Prior Analytics* I.2 clearly commit him to accidental e-conversion.[19] I highly doubt that Aristotle's failure to mention accidental e-conversion was an oversight. The fact of the matter is that given the rest of Aristotle's syllogistic, accidental e-conversion is not really useful for much of anything. And the small task it is useful for, is not needed for completing the project of *Prior Analytics* I.4-6 for which the other assertoric conversion principles were introduced.[20] This said, other authors who provide reconstructions of Aristotelian syllogistic do sometimes include accidental e-conversion. So, in an ecumenical spirit, I will as well.

Now, while Aristotle's definition of syllogism in *Prior Analytics* I.1 requires that every syllogism has at least two premises, he does not require that every syllogism has exactly two premises. Indeed, the

[19] For, assume XeY is true. Then, by simple e-conversion, YeX is true. But then it is impossible for YaX to be true. For if it were, then both YaX and YeX would be simultaneously true, violating Aristotle's contention that YaX and YeX are contraries. So, if XeY is true YaX must be false. But since YaX is the contradictory of YoX, it follows that if XeY is true then YoX is true.

[20] In *Prior Analytics* I.4-6 Aristotle is asking questions of the following form:

- Suppose we have a premises AxB and ByC (with $x, y \in$ {a,e,i,o}). Does any syllogistic conclusion with A as predicate and C as subject follow? (The question of I.4)
- Suppose we have a premises BxA and ByC (with $x, y \in$ {a,e,i,o}). Does any syllogistic conclusion with A as predicate and C as subject follow? (The question of I.5)
- Suppose we have a premises AxB and CyB (with $x, y \in$ {a,e,i,o}). Does any syllogistic conclusion with A as predicate and C as subject follow? (The question of I.6)

Note that the form of the question is existential—compare *problēmata*, i.e. 'construction problems', in Greek mathematics. Aristotle is not attempting to list *all* conclusions of the given form that follow from the premises. He just asks if there is at least one. If the answer is 'yes' Aristotle picks the strongest conclusion that follows and argues that it follows. Now, in order to validate Darapti Aristotle needs accidental a-conversion. And once accidental a-conversion is on the table, given Aristotle's assumptions about the truth-conditions for syllogistic sentences, all that the addition of accidental e-conversion would achieve is the ability to show that if some e-conclusion follows from certain premises, then an o-conclusion follows from those same premises. Given the task Aristotle introduces the conversion principles of I.2 to perform, there is thus no good reason for I.2 to add accidental e-conversion.

most interesting proof-theoretic discussions of Aristotle's *Analytics* concern syllogistic arguments with any arbitrary (but finite) number of premises. To capture this, we offer the following.

Definition 4 Syllogistic-deductions

1. If Σ is a basic syllogistic-deduction, then Σ is a syllogistic-deduction
2. If Σ_0 is a syllogistic-deduction with AxB at leaf-node n and Σ_1 is a syllogistic-deduction or a conversion with AxB at root-node m, then there is a syllogistic-deduction concatenating Σ_0 and Σ_1 by identifying node n with node m
3. If Σ_0 is a syllogistic-deduction with AxB at root-node n and Σ_1 is a conversion with AxB at leaf-node m, then there is a syllogistic-deduction concatenating Σ_0 and Σ_1 by identifying node n with node m
4. Nothing else is a syllogistic-deduction

Let's see some examples. Given **Definition 4.1**, the following are syllogistic-deductions:

$$\frac{\text{AiB} \qquad \text{CaB}}{\text{AiC}} \qquad \frac{\text{AeC} \qquad \text{CaF}}{\text{AeF}} \qquad \frac{\text{AeF} \qquad \text{DaF}}{\text{AoD}}$$

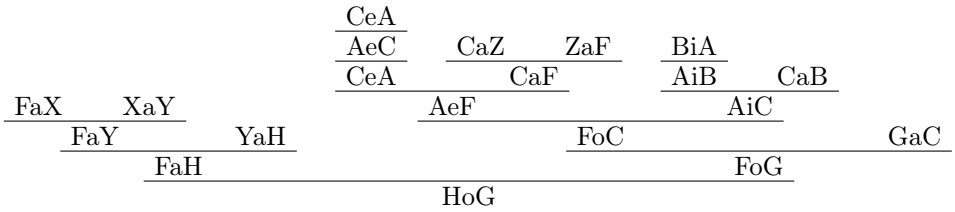
Given **Definition 4.2** the following are also syllogistic-deductions:

$$\frac{\dfrac{\text{BiA}}{\text{AiB}} \qquad \text{CaB}}{\text{AiC}} \qquad \frac{\text{DaC} \qquad \dfrac{\text{DeA}}{\text{CeA}} \qquad \text{CaF}}{\text{AeF}} \qquad \frac{\text{AaB} \qquad \dfrac{\text{AeF} \qquad \text{DaF}}{\text{AoD}}}{\text{BoD}}$$

And given **Definition 4.3** so are:

$$\frac{\dfrac{\text{AaB} \qquad \text{BaC}}{\text{AaC}} \qquad \text{CiA}}{\text{AiC}} \qquad \frac{\dfrac{\text{BiA}}{\text{AiB}} \qquad \text{CaB}}{\dfrac{\text{AiC}}{\text{CiA}}} \qquad \frac{\dfrac{\text{CeA} \qquad \text{CaF}}{\text{AeF}}}{\text{FoA}}$$

Further concatenations produce syllogistic-deductions of greater and greater length:



Given **Definition 4.4** the following trees are not syllogistic-deductions[21]:

$$\frac{\dfrac{\text{AeB}}{\text{BeA}}}{}$$

$$\frac{\text{AaB}\qquad\text{BeC}}{\text{AoC}}$$

$$\frac{\dfrac{\text{AaB}}{\text{BiA}}}{\text{AiB}}$$

$$\frac{\text{AeB}\qquad\text{BaC}}{\text{CeA}}$$

All syllogistic-deductions will result in validities Aristotle is committed to.[22] Of course, Aristotle may not be willing to call some of the resulting concatenation-patterns 'syllogisms'. No matter, we are not calling them 'syllogisms' either.

Definition 5

Where Σ is a syllogistic-deduction:

1. the conclusion of Σ, $concl(\Sigma)$, is the lowest sentence in Σ
2. the premises of Σ, $prem(\Sigma)$, is the smallest set containing only sentences occurring in Σ with no lines over them
3. if $prem(\Sigma) = \Gamma$ and $concl(\Sigma) = \varphi$, we say $\Sigma :\ \Gamma \vdash_D \varphi$
4. $\Gamma \vdash_D \varphi$ iff there's some syllogism $\Sigma*$ such that $prem(\Sigma*) = \Gamma$ and $concl(\Sigma*) = \varphi$

The subscripted 'D' on $\vdash_D$ is for syllogistic-deduction. This flags the point that some argument-forms we are counting as syllogistic-deductions (most prominently: *circular* ones) will not, strictly speaking, correspond to any Aristotelian syllogisms. In **Section 4**, we mark out the class of *syllogisms* as a proper subclass of our collection of *syllogistic-deductions*.[23]

In contemporary logic $\Gamma \vdash \varphi$ is often interpreted to mean that (relative to some specified proof-system) there exists a proof of φ that uses only premises in Γ. So interpreted, $\vdash$ trivially has the property called *strengthening*:

[21] Note also that although the tree on the far right is not a syllogistic-deduction, there is a tree with identical premises and conclusion that is:

$$\frac{\dfrac{\text{AeB}\qquad\text{BaC}}{\text{AeC}}}{\text{CeA}}$$

[22] Though clearly some validities Aristotle is committed to will correspond to no syllogistic-deductions.

[23] For instance, where $\dfrac{\text{AaB}\qquad\text{BaC}}{\text{AaC}}$ will count as a syllogism, $\dfrac{\text{AaA}\qquad\text{AaB}}{\text{AaB}}$ is a syllogistic-deduction that won't count as a syllogism.

· if Γ is a subset of Δ and $\Gamma \vdash \varphi$, then $\Delta \vdash \varphi$

Note that for $\Gamma \vdash_D \varphi$ we require that there is a syllogism proving φ whose premise-set consists of all and only sentences in Γ. This is in respect of condition 4. in Aristotle's definition of syllogism: his insistence that every premise in a syllogism be used. Even when discussing theorists who allow circular syllogisms whose conclusion is identical with one of their premises, Aristotle retains his requirement that no syllogism contains unused premises.

Note further that when $\Gamma \vdash_D \varphi$, there may be more than one way to derive φ from Γ. There may be multiple trees using exactly Γ as premises and exactly φ as conclusion. For instance, let the syllogistic-deduction on the left be called Σ_L and that on the right be called Σ_R:

$$\dfrac{\dfrac{\dfrac{\text{AeC}}{\text{CeA}} \quad \text{CaF}}{\text{AeF}} \quad \dfrac{\dfrac{\text{BiA}}{\text{AiB}} \quad \text{CaB}}{\text{AiC}}}{\text{FoC}} \qquad\qquad \dfrac{\dfrac{\text{CaB} \quad \text{BiA}}{\text{CiA}} \quad \dfrac{\text{AeC} \quad \text{CaF}}{\text{AeF}}}{\text{FoC}}$$

It will be observed that both Σ_L and Σ_R use exactly the same premises and get exactly the same conclusion. To express this we would say:

· Σ_L: {AeC, CaB, BiA, CaF} $\vdash$ FoC

· Σ_R: {AeC, CaB, BiA, CaF} $\vdash$ FoC.

If $\Gamma \vdash_D \varphi$ there must be at least one syllogistic-deduction whose premises are (exactly) Γ and whose conclusion is φ. There need not be only one.

Note finally that for all Σ , $prem(\Sigma)$ will be a set rather than multi-set or sequence. For us, the $\vdash_D$ relation is blind to how many times a premise gets used in a syllogistic-deduction. (For our purposes, there is no reason for $\vdash_D$ to track occurrences of premise-tokens rather than premise-types). And following Aristotle, our representation does not treat premise order as significant for logical consequence.

2.1.4.3 Indirect deductions

Our above system treats only 'direct' syllogistic-deductions. There is no reductio rule, no indirect syllogistic-deductions.[24] What kind of indirect syllogisms does Aristotle accept?[25]

[24]Some authors will insist on a distinction between systems with indirect proofs and systems that have a reductio rule. Nothing here turns on the issue; we harmlessly ignore the distinction. By an 'indirect' proof we effectively mean one that uses the reductio rule some number of times.

[25]NB that we are not here addressing the issue of how Aristotle understands reductio reasoning in general. For instance, to argue for assertoric conversion principles in *Prior Analytics* I.2, Aristotle will use reductio reasoning; but those arguments are not syllogisms, and we will be attempting to capture those patterns of reasoning in our formal model.

In general, one can distinguish between two kinds of reductio. Call them *strong reductio* and *weak reductio*. Suppose we have some premises $\{\alpha, \beta\}$ which are used to deduce θ and some premises $\{\varphi, \psi\}$ which are used to deduce $\neg\theta$.[26] For any χ, strong reductio allows us to deduce the contradictory of χ from $\{\alpha, \beta, \varphi, \psi\}$. In strong reductio, the 'assumption' that gets reduced and whose contradictory is concluded need not actually be *used* in the derivation of either member of the contradictory pair. In weak reductio, however, it must. Suppose again that $\{\alpha, \beta\}$ are used to deduce θ and $\{\varphi, \psi\}$ are used to deduce $\neg\theta$. By weak reductio, the only deductions we can make are:

- $\{\alpha, \beta, \varphi\}$ therefore $\neg\psi$
- $\{\alpha, \beta, \psi\}$ therefore $\neg\varphi$
- $\{\alpha, \varphi, \psi\}$ therefore $\neg\beta$
- $\{\beta, \varphi, \psi\}$ therefore $\neg\alpha$

So, an indirect deduction using weak reductio requires that the assumption that gets 'reduced' and removed from the assumptional base is actually used in the derivation of the contradiction. Unsurprisingly, Aristotle thinks that while some demonstrative-proof may use weak reductio, no truly explanatory demonstration will require the full power of strong reductio. You can't explain why something holds, he says, simply by deriving a contradiction from other propositions perfectly irrelevant to φ. The only indirect syllogistic-deductions he endorses use weak-reductio.

In a syllogistic context, it's also natural to distinguish between two different varieties of weak reductio. Suppose we use premises $\{\alpha, \beta\}$ to deduce θ and premises $\{\varphi, \psi\}$ to deduce γ. If γ is the contrary, rather than the contradictory of θ, can we still give a weak reductio? Aristotle considers this issue in *Prior Analytics* II.11 and answers yes. Of course, as Aristotle makes clear, when we do reductio we are only entitled to conclude the contradictory, and not the contrary, of the reduced proposition that gets eliminated from our assumptional base. But to apply reductio, the incompatibility we derive need not consist in a contradictory pair. It can just as well consist in a contrary pair—a pair of propositions whose simultaneous truth is impossible.

How, then, might we represent indirect deductions in our reconstruction? Suppose we have the following two direct syllogistic-deductions that derive contradictory conclusions:

[26]Note that here and in what follows I always use the '¬' symbol to indicate a function that maps a proposition to its contradictory. Aristotle himself, of course, does not have a 'sentential' conception of negation. My meta-language use of the '¬' symbol should not suggest otherwise.

$$\dfrac{\dfrac{\text{AeB}\qquad\text{BaC}}{\text{AeC}}\qquad\text{CiD}}{\text{AoD}}\qquad\qquad\dfrac{\text{AaE}\qquad\text{EaD}}{\text{AaD}}$$

We'll want to concatenate them in some way to prove the contradictory of some particular φ taken as a premise in one of the direct syllogistic-deductions. But, as Aristotle clearly explains (*Prior Analytics* II.11), in the resulting indirect deduction this φ needs to be dropped from our assumptional base. There's an easy way to do this in our tree representations. Recall our representational convention that the premises of a syllogistic-deduction are the sentences without lines over them. When concatenating the two above syllogistic-deductions to indirectly prove the contradictory of φ we should, then, simply draw a line over the original occurrence of φ to indicate that in the resulting deduction it is no longer being taken as a premise. For instance:

$$\dfrac{\dfrac{\dfrac{\text{AeB}\qquad\overline{\text{BaC}}}{\text{AeC}}\qquad\text{CiD}}{\text{AoD}}\qquad\dfrac{\text{AaE}\qquad\text{EaD}}{\text{AaD}}}{\text{BoC}}$$

So, the above would be an indirect proof whose premises are: AeB, CiD, AaE and EaD, and whose conclusion is BoC. Although we supposed BaC for reductio, and in this way used BaC, one presenting the argument represented above will not assert BaC as a premise. BaC was supposed for the sake of argument, but in the indirect deduction it is not a premise in the same sense as AeB, CiD, AaE and EaD.

In this manner, it is straightforward to extend to our system of direct syllogistic-deductions introduced in **Section 2.1.4.2** above to include indirect deductions. We would simply have to add to **Definition 4** something of the following form:

Reductio rule

Let $\neg\,\alpha$ represent the contradictory of α, and $\sim\alpha$ represent either the contrary or contradictory of α. Then if there's a syllogism:

$$\dfrac{\Gamma\quad\varphi}{\begin{array}{c}\vdots\\ \sim\ \psi\end{array}}\quad\vdots$$

and there's a syllogism (or sentence) ψ, then this is a syllogism:

$$\dfrac{\dfrac{\Gamma\quad\overline{\varphi}}{\begin{array}{c}\vdots\\ \sim\ \psi\end{array}}\qquad\begin{array}{c}\vdots\\ \psi\end{array}}{\neg\,\varphi}$$

Recall we are not interested in all Aristotelian reductio reasoning, but just *syllogistic-deductions* using reductio. Since (by **Definition 4**) $\dfrac{\text{BiA}}{\text{AiB}}$ is not a syllogistic-deduction, the above rule would not count the following as a syllogistic-deduction:

$$\dfrac{\dfrac{}{\text{BiA}}}{\dfrac{\text{AeB}\qquad\text{AiB}}{\text{BeA}}}$$

$$\vdots$$

Observe that to apply our **Reductio Rule**, ψ can be either the syllogistic sentence ψ or a syllogistic-deduction with conclusion ψ. Each of the following are generated in accord with our **Reductio Rule**:

$$\dfrac{\dfrac{\overline{\text{AaB}}\qquad\text{BaC}}{\text{AaC}}\qquad\text{AoC}}{\text{AoB}} \qquad \dfrac{\dfrac{\overline{\text{AaB}}\qquad\text{BaC}}{\text{AaC}}\qquad\dfrac{\text{AoD}\qquad\text{CaD}}{\text{AoC}}}{\text{AoB}} \qquad \dfrac{\dfrac{\text{AoD}\qquad\overline{\text{CaD}}}{\text{AoC}}\qquad\dfrac{\text{AaB}\qquad\text{BaC}}{\text{AaC}}}{\text{CoD}}$$

Note finally that we preserve the 'relevantistic' character of Aristotelian syllogistic, by reducing premise-*tokens* rather than premise-*types*.[27]

Ultimately we will want to consider what difference it makes to augment **Definition 4** with the above. So, for the time being, we leave **Definition 4** as it is. But let **Definition 4*** be identical with **Definition 4** save that the above reductio rule is added. Then we can have **Definition 5*** specify premises and conclusions in the exact same manner as **Definition 5.1-2**. Recall that **Definition 5.3** effectively stated that $\Gamma \vdash_D \varphi$ exactly if there's a direct syllogistic-deduction whose premises are Γ and conclusion is φ. We then let **Definition 5.3*** refer us to **Definition 4*** and effectively state:

· $\Gamma \vdash_R \varphi$ exactly if there's a direct or indirect syllogistic-deduction with premises Γ and conclusion φ

and mutatis mutandis for **Definition 5.4***.

2.2 Conversion, reductio, and the 16 moods

2.2.1 P-canonical form

Suppose Σ is a (direct or indirect) syllogistic-deduction of conclusion φ from premises Γ—i.e. $\Sigma : \Gamma \vdash_R \varphi$. Let's say that Σ is in *P-canonical form* ('P' for perfection) exactly if Σ uses only combinations of:

1. the four perfect first-figure assertoric moods: Barbara, Celarent, Darii, Ferio

2. the three assertoric conversion rules, and

[27] Otherwise we get non-Aristotelian results. For instance, one could indirectly syllogize AoB using only premises CeA and CaA:

$$\dfrac{\dfrac{\text{CeA}\qquad\overline{\text{AaB}}}{\text{CeB}}\qquad\dfrac{\text{CaA}\qquad\overline{\text{AaB}}}{\text{CaB}}}{\text{AoB}}$$

Cf. Smiley's (1974) treatment of reductio.

3. reductio.

For instance the deduction Σ_1 : {BeA, BaC, DiC} $\vdash_R$ AoD below to the left is not in P-canonical form (it uses the non-perfect moods Cesare and Ferison); but the deduction Σ_2 : {BeA, BaC, DiC} $\vdash_R$ AoD below to the right is in P-canonical form (using two conversion steps and the perfect moods Celarent and Ferio):

$$\text{Camestres}\ \dfrac{\dfrac{\text{BeA}\quad\text{BaC}}{\text{AeC}}\qquad\text{DiC}}{\text{AoD}}\ \text{Ferison}$$

$$\text{e-conversion}\ \dfrac{\text{BeA}}{\text{AeB}}\ \text{Celarent}\ \dfrac{\dfrac{\text{AeB}\quad\text{BaC}}{\text{AeC}}\qquad\dfrac{\text{DiC}}{\text{CiD}}\ \text{i-conversion}}{\text{AoD}}\ \text{Ferio}$$

Note that Σ_1 and Σ_2 use exactly the same premises to derive exactly the same conclusion. But they follow different argumentative paths to do so. Σ_1 goes through two inferences, Σ_2 goes through four.

2.2.2 Scholarly assessments of P-canonical form

In *Prior Analytics* I.4-7 Aristotle uses conversion and reductio to 'perfect' the 12 non-perfect assertoric moods he recognizes, 'reducing' them to perfect moods and thus (somehow) establishing (or explaining or indicating) the validity of the non-perfect moods on the basis of the validity of the perfect moods.[28] In I.23 Aristotle argues that not only two-premise syllogisms, but indeed every syllogism of any length can be perfected by these means. Corcoran-Smiley and their followers have claimed that the point of the perfection proofs of *Prior Analytics* I.4-7, 23 is to show that every syllogistic argument of any arbitrary length can be put into P-canonical form. It is far from obvious that this is the thesis Aristotle takes himself to be arguing for in these chapters. But it is fact true that if Σ : $\Gamma \vdash_R \varphi$, there's a $\Sigma*$ in P-canonical form such that $\Sigma*$: $\Gamma \vdash_R \varphi$. Given a little latitude, it is reasonable to think this result more-or-less falls out of *Prior Analytics* I.4-7, 23.

There are two views about P-canonical form, and its role in Aristotle's logic and theory of demonstration that I'd like to clear away before turning, in the next chapters, to Aristotle's arguments against the Ad Infinitum Skeptics and Circular Demonstrators. The first view has been prominently defended by Corcoran, already in his classic papers from the 1970's, and as recently as his 2009 'Aristotle's Demonstrative Logic'. On the view in question, we need to make a sharp distinction between valid syllogisms and syllogisms that establish their conclusion. All syllogisms are valid. But strictly speaking, only syllogisms which are in P-canonical form establish their conclusion. Yet, as Corcoran thinks *Prior Analytics* I.4-7, 23 tries to show, every syllogism has a P-canonical form. And every demonstrative-proof [*apodeixis*] must show how its conclusion follows from its premises. So, concludes Corcoran, strictly speaking every demonstration must be in P-canonical form.

[28] Famously, I.7 also notes that while Darii and Ferio are perfect in themselves, by using conversion and reductio one could indeed reduce them the universal moods of the first-figure: Barbara and Celarent.

The second view has been defended by Robin Smith (1986). Building on an idea in Lear (1980),[29] Smith argues that Aristotle never meant to require that demonstrations actually go through P-canonical form. This would be analogous requiring that propositional logic proofs only make use of primitive connectives $\neg$ and $\vee$, and never: $\wedge \rightarrow, \leftrightarrow$. According to Smith, what matters to Aristotle is simply the in principle possibility of doing all demonstrations in P-canonical form. Smith suggests that the point of Aristotle's perfection and reduction proofs of *Prior Analytics* I.4-7 is to simplify his meta-logical investigations. His assessment here is influenced by the practice of modern logicians. Suppose I'm studying a proof system with 6 stated rules of inference. And suppose I can show that 3 of the rules can be harmlessly eliminated because everything derivable using some pattern of the 6 rules is also derivable using some pattern of 3 of the rules. Such a result is helpful. For whatever I can show of the consequence relation determined by the 3 rule system will hold of the consequence relation determined by the full 6 rule system. For instance, to prove the 6 rule system is sound relative to some semantics, I'll just need to show that the 3 rule system is sound relative to the same semantics. This can significantly cut down on the number of cases to consider.

The sequel argues against both of the above views. Along the way, we will have opportunity to note some independently interesting facts about Aristotle's syllogistic.

2.2.3 Perfection and conversion outside of *Prior Analytics* I.4-23

The Corcoran and Smith views canvassed above are based on assumptions about what Aristotle is trying to accomplish with his perfection proofs in *Prior Analytics* I.4-7. Aristotle's manner of proceeding in the remainder of the *Analytics* does not bear these assumptions out. The distinction between perfect and imperfect syllogisms plays no role in the investigation of the *Analytics* after *Prior Analytics* I.23. Indeed, the distinction is never even mentioned again after *Prior Analytics* I.24 (41b33).[30] Pace Corcoran, Aristotle never tells us that demonstrations should (strictly speaking) only appear in P-canonical form. Indeed, when Aristotle discusses the issue of how to generate syllogisms in *Prior Analytics* I.27-30 he introduces a 'universal method' for use in both science and dialectic. The method crucially involves employing the non-perfect moods Felapton and Darapti to prove particular propositions. Use of conversion and reductio have no place in the method, and Aristotle never suggests that the syllogisms generated using the method should be somehow reformulated into P-canonical form. This is a difficulty for Corcoran's view. But it's perfectly compatible with Smith's idea that all Aristotle cares about in I.4-7, 23 is the in principle possibility of transformations into P-canonical form. On Smith's view, Aristotle will not insist that we actually build demonstrations in P-canonical form. The point of *Prior Analytics* I.4-7, 23 is rather to establish as a meta-

[29] See esp. chapter 1.

[30] 41b33 is just a back-reference to the project of I.4-23. NB that 59a41 on the figures of prosleptic syllogisms and perfection of prosleptic syllogisms is probably spurious (though see Malink 2012).

logical lemma that every syllogism has a P-canonical form. This matters to Aristotle, thinks Smith, because he ingeniously realizes that such a lemma can vastly simplify his meta-logical studies, significantly cutting down on the range of proof-forms to consider when trying to show things about syllogistic. Now, Smith is certainly right that Aristotle could have used the perfection-reduction results of *Prior Analytics* I.4-7, 23 in this fashion. Indeed, in many cases it would have been helpful for Aristotle to do so, for exactly the reasons Smith gives. But the fact of the matter is that Aristotle never actually does use his perfection-reduction results for this purpose. Many of Aristotle's investigations after *Prior Analytics* I.23 involve consideration of various forms a proof could take. We will see many examples of this in what follows. But Aristotle never appeals to perfection-results to cut down on the number of cases.

Common to Corcoran and Smith is the idea that Aristotle sees conversion principles as 'rules of inference' in his syllogistic 'deduction system'. But Aristotle's manner of proceeding after *Prior Analytics* I.4-22 doesn't really bear this out either. After *Prior Analytics* II.1, conversions of the sort discussed in I.2-3 are never again are mentioned or used in the remainder of the *Analytics*.[31] In Aristotle's presentation of his 'universal method' for the discovery of syllogisms in *Prior Analytics* I.27-30, conversion is mentioned at four points: 43b6, 44a23, 44b15, and 44a30. Conversion is not a part of the method Aristotle is presenting; it is not a rule of inference one syllogizing in accord with the method ever uses. Rather, in each of the four instances in I.27-30 that conversion principles come up, conversion is only appealed to in order to help Aristotle explain to his reader why the method works.[32] There are only two other chapters in the *Analytics* after *Prior Analytics* I.23 that even mention conversion: *Prior Analytics* I.45 and II.1. In II.1, conversion comes up at 53a7-14 where Aristotle is explaining that in some cases, when I give a syllogism more than the stated conclusion has also been syllogized. For instance, he says, when I produce this syllogism: $\frac{\text{AaB} \quad \text{BaC}}{\text{AaC}}$, I have also syllogized CiA. To explain why this is so, he tells us that AaC converts to CiA. Aristotle is not describing the person syllogizing as doing some further inference to get CiA from AaC. He's rather claiming that her syllogism establishing AaC simultaneously establishes CiA. Finally, we have I.45 which discusses the possibility of taking an existing two-premise syllogism in one figure and altering it so as to produce a syllogism in another figure. For instance, (50b10-15) if one has the first-figure syllogism: $\frac{\text{AeB} \quad \text{BaC}}{\text{AeC}}$ one can switch the order of terms in the major premise to get the second-figure syllogism: $\frac{\text{BeA} \quad \text{BaC}}{\text{AeC}}$. Aristotle is of course aware that the transformation is validity-preserving only because of entailment relations between AeB and BeA. But again, Aristotle is not appealing to conversion as 'a rule of inference' internal

[31] The word 'conversion' [*antistrephein*] is used in later parts of the *Analytics*: but with other and unrelated meanings. For instance, the *Analytics* (like the *Topics*) will often use 'X and Y convert' equivalently with 'X and Y counter-predicate'.

[32] For instance, consider 43b6. The method Aristotle describes involves collecting three lists for every term T in the relevant domain: (i) a list of terms X such that TaX, (ii) a list of terms Y such that YaT, and (iii) a list of terms Z such that ZeT. To explain why we don't need: (iv) a list of terms W such that TeW, Aristotle explains that e-propositions convert so such a list would be identical to list (iii).

to the same deduction system that includes his 16 moods in 3 figures.

As noted, after *Prior Analytics* II.1, the assertoric and modal conversion principles introduced in chapters I.2-3 are never again mentioned in the *Analytics*. Conversion's appearances between I.23 and II.1 are exhausted by the paragraph above. Pace Corcoran and Smith, not only does Aristotle fail to restrict himself to P-canonical forms in his meta-logical discussions which go through the various patterns a complex syllogism or demonstration could possibly take. Not only do such discussions always consider cases in the second and third figure that could be avoided if appeal to P-canonical forms were made. The fact is that such discussions *never* consider the possibility that a conversion inference might be used at some point within a syllogism or demonstration. Aristotle's numerous meta-logical investigations after *Prior Analytics* I.22 that proceed by the consideration-of-cases method never discuss the use of conversion as, or involved in, a possible case.

2.2.4 Reductio elimination

My own view is that Aristotle never envisioned conversion playing a role in demonstrative science. But though I believe the evidence is on my side, little in this dissertation turns on the issue. It will be harmless to retain conversion rules in our system of proof-trees.[33]

I now turn to the rather more interesting issue of indirect proofs. On Corcoran's view of the importance of P-canonical form, Aristotle ought to hold that reducio reasoning is *ineliminable* from demonstrative-proofs.[34] But, as I explain below, Aristotle himself thought otherwise. Indeed, Aristotle boldly tried to argue that in the context of his three-figured system of syllogisms, reductio reasoning can always be eliminated.

2.2.4.1 The issue

There's a common experience one has when trying to prove something. One has demonstrandum φ and some givens Γ. But where to start? There's an easy trick mathematicians throughout history have used. Try supposing that φ is false—that the contradictory of φ is true. This is helpful because it gives one a stronger base from which one can start proving. Instead of just Γ, one now has Γ and not-φ to reason from. And the

[33] When describing our tree proofs I will myself, from time to time, harmlessly refer to, say, $\dfrac{\text{AaB} \quad \text{BaC}}{\text{AaC}}$ as a 'Barbara inference' or even $\dfrac{\text{AeB}}{\text{BeA}}$ a 'conversion inference'. My occasional speaking in this way does not reflect interpretive views about Aristotle. It's just a convenience.

[34] Recall that according to Corcoran every demonstration must be in P-cannonical form. But consider Bocardo: $\dfrac{\text{AoB} \quad \text{CaB}}{\text{AoC}}$. Bocardo derives AoC from premises AoB and CaB. The only way to get this conclusion from these premises by a syllogistic-deduction in P-cannonical form is to reductio:

$$\dfrac{\dfrac{\overline{\text{AaC}} \quad \text{CaB}}{\text{AaB}} \quad \text{AoB}}{\text{AoC}}$$

For, as Corcoran himself sees (2009), the collection of 'rules': Barbara, Celarent, Darii, Ferio, e-conversion, i-conversion, a-conversion, contain nothing that takes you to any conclusion from an o-premise.

expanded assumptional base has content with direct bearing on the demonstrandum. If all goes well, the following occurs. From the combined power of Γ and not-φ, one can prove something not-ψ which is known, or legitimately assumed, to be false. One then reasons 'indirectly' to prove φ from Γ and the already known truth that ψ:

· Γ; but suppose not-φ. Then not-ψ. But ψ. Thus φ.

When one proves something 'indirectly' in this way, one is often left with a sense of dissatisfaction. Checking the proof, you can see that given Γ and ψ, on pain of contradiction φ must follow. But why really does φ hold? Despite seeing that the proof works, one cannot yet understand how Γ and ψ actually collaborate to make it the case that φ. In short, the proof doesn't seem explanatory. But often something intellectually satisfying comes next. Rewriting the proof, one sees that the initial assumption of not-φ can be dispensed with. Though helpful in proof-discovery, the reductio step was unnecessary. One can, in fact, reason directly to φ from the givens Γ and the already known fact that ψ. In the best cases, the rewritten direct proof does appear to disclose an explanation of how φ is grounded in Γ and ψ.

It's often possible to rewrite an indirect proof as a direct proof, eliminating reliance on reductio. But is it always possible? This is a question that Aristotle was deeply interested in. Aristotle invents logic in order to study demonstrative-proof. And he's interested in demonstrative-proof because he thinks that demonstrative-proof is the one and only route to genuine episteme. But Aristotle also thinks that episteme is a kind of explanatory understanding. Having episteme of something is centrally a matter of grasping why that thing must be so, on the basis of a perfectly complete and impeccably correct explanation for it. The question then arises: are any demonstrative-proofs ineluctably indirect? Or for every indirect proof with premises Γ and conclusion φ is there always a direct proof from the same premises to the same conclusion?

Aristotle answers the question immediately above in the affirmative.[35] Indeed, he provides a lengthy, albeit severely incomplete, proof-theoretic argument for an affirmative answer in *Prior Analytics* II.14. But his answer has been criticized.

I want to distinguish what I view as more and less interesting criticisms of Aristotle's thesis that reductio-reasoning is in principle eliminable from every demonstrative-proof. The least interesting kind of criticism involves pointing out that Aristotle's claim is, in fact, false. Aristotle thought that his syllogistic was *complete*. He was wrong. Contra Aristotle, there are proofs from which reductio-reasoning is not eliminable. I find this criticism uninteresting, because the resulting account of why Aristotle goes wrong is familiar and his damning oversight is well-understood. Of course, it is important to know that Aristotle makes incorrect generalizations and dramatically over-estimates the power and usefulness of his logic. But this is well known,

[35] See esp. *Prior Analytics* I.29, II.14, *Posterior Analytics* I.26.

and I don't see much new that's philosophically interesting here. For me, at any rate, a more interesting question is whether or not Aristotle is correct about the eliminability of reductio in his own system. And a more interesting line of criticism argues that even in the highly limited context of his assertoric syllogistic, Aristotle was wrong to claim that reductio is eliminable.[36]

Suppose, for the sake of argument, that Aristotle were right that (say) all mathematical proofs could be adequately expressed in assertoric syllogistic. Would he then be right that reductio would be eliminable from mathematics? Below I'll argue that there's a demonstrable sense in which the answer to this question is 'yes'. Moreover, I think a case can be made that Aristotle more or less understood why the answer is 'yes'.

2.2.4.2 The result

Suppose one wanted to show that Cesare is eliminable from a system of a system syllogistic-deductions with Celarent and simple e-conversion. For instance, we might be asked to show that if $\Gamma \vdash_R \varphi$ then there's a syllogistic-deduction with premises Γ and conclusion φ that never uses Cesare. This would be quite easy. One would simply note the obvious possibility of systematically replacing every Cesare sub-proof with an equivalent subproof in which Cesare does not appear. Whenever something with the form on the left occurs, replace it with something of the form on the right.

$$\dfrac{\dfrac{\vdots \quad \vdots}{\text{BeA} \quad \text{BaC}}}{\text{AeC}} \qquad \vdots \qquad \Longrightarrow \qquad \dfrac{\dfrac{\dfrac{\vdots}{\text{BeA}} \quad \vdots}{\text{AeB} \quad \text{BaC}}}{\text{AeC}} \qquad \vdots$$

Repeat the replacement algorithm finitely many times. Clearly, no matter what syllogistic-deduction you start with, you'll end up with a syllogistic-deduction from the same premises to the same conclusion that never uses Cesare.

As Aristotle is well aware, the eliminability of indirect syllogistic-deductions is far more difficult to prove. For every Cesare occurrence there's some particular transformation we can employ to replace Cesare. It's given above. But there's no similarly uniform way of replacing every occurrence of reductio in a syllogistic-deduction. Different cases of reductio will have to be treated differently. Proof-theoretically, reductio cannot be *defined away* in the manner than Cesare can.

In *Prior Analytics* II.14, Aristotle argues for the eliminability of indirect syllogistic-deductions. More precisely, he argues that whenever there is an indirect syllogistic-deduction of φ from some assumptions Γ, there's also a direct syllogistic-deduction of φ from Γ. It is far from obvious that this should be so. If it is so, it follows that the addition of a reductio rule to a system of direct syllogistic-deductions adds no logical

[36] As Corcoran (2009) claims.

power. Everything that can be proved with reductio can be proved without it. Or, in terms of our formal reconstruction:

- $\Gamma \vdash_D \varphi$ if and only if $\Gamma \vdash_R \varphi$

After Gödel et al., a modern logician can prove the result by appealing to a 'semantic' (i.e. *model-theoretic*) framework relative to which both deductive systems are sound and complete.[37] But we don't need such a semantics to get the result. *Prior Analytics* II.14 looks to establish it by purely proof-theoretic means. And as Aristotle correctly sees, proving it in this way involves considering a barrage of cases covering the all the different forms a reductio inference might take in syllogistic. For instance, Aristotle asks us to consider this (63a7):

$$\dfrac{\dfrac{\vdots}{\text{CeB}} \qquad \dfrac{\dfrac{\vdots}{\text{CaA}} \qquad \dfrac{}{\text{AiB}}}{\text{CiB}}}{\text{AeB}}$$

The representation above corresponds to the following state of affairs. For reductio, one supposes that AiB. Yet from CaA, AiB via Darii one concludes that CiB. But given a proof of CeB, one concludes that AiB must be false and its contradictory, AeB, true. Having described the case, Aristotle shows how the reductio step can be eliminated. Instead of arguing indirectly, one can argue directly from the same premises to the same conclusion as follows:

$$\dfrac{\dfrac{\vdots}{\text{CaA}} \qquad \dfrac{\vdots}{\text{CeB}}}{\text{AeB}}$$

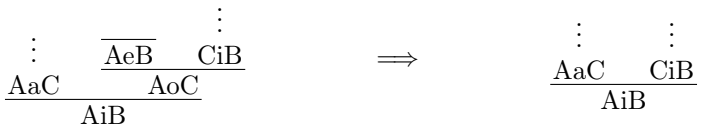
In place of an indirect proof using Darii, we can give a direct proof through Camestres.

Moving on, *Prior Analytics* II.14 considers a battery of further cases indicating transformations into and out of all the three figures. For instance, 63a25ff. considers the following elimination case:

$$\dfrac{\dfrac{\vdots}{\text{CaB}} \qquad \dfrac{\dfrac{\vdots}{\text{AaC}} \qquad \dfrac{}{\text{AoB}}}{\text{CoB}}}{\text{AaB}} \qquad \Longrightarrow \qquad \dfrac{\dfrac{\vdots}{\text{AaC}} \qquad \dfrac{\vdots}{\text{CaB}}}{\text{AaB}}$$

and 63b4ff. considers describes the elimination:

[37] Left-to-right would follow by the soundness of $\vdash_D$ and completeness of $\vdash_R$ relative to the semantics, right-to-left by the soundness of $\vdash_R$ and completeness of $\vdash_D$ relative to the semantics.



On the final analysis, Aristotle's argument for reductio elimination in *Prior Analytics* II.14 does not succeed in establishing its target result. The argument fails because it is not pursued at a sufficient level of generality to consider all the possible cases that might arise. You will note, for instance, that in all the cases covered above the assumption that is reduced is used in the inference immediately proceeding the application of reductio to a pair of incompatible propositions. But in fact, the reduced assumption might be introduced farther up in the syllogistic-deduction (there is no finite upper-bound on how far up!). Moreover, Aristotle does not explain how to deal systematically with the possibility of syllogistic-deductions in which reductio is used not once but arbitrarily many times.

But this said, Aristotle is on the right track. For, his reasoning can be generalized. And in the Appendix we prove that $\Gamma \vdash_D \varphi$ if and only if $\Gamma \vdash_R \varphi$, using a generalized version of Aristotle's own proof strategy.[38]

Consideration of reductio is in fact absent in the texts **Sections 3-5** will discuss. Following Aristotle's lead, in what follows we ignore the system with direct and indirect deduction associated with the turnstile '$\vdash_R$ ' introduced above in **Section 2.1.4.3**. We can focus exclusively on the system of direct deductions associated with our turnstile '$\vdash_D$' introduced by **Definitions 2-5** in **Section 2.1.4.2**. Whether Aristotle himself was able to prove it or not, he is right that the two are equivalent.

2.3 Epilogue

2.3.1 Conversion elimination?

The proof of reductio elimination offered in this paper's appendix uses a generalized version the proof-strategy used by Aristotle himself in his own (defective because significantly incomplete) argument for the same result in *Prior Analytics* II.14. The attentive reader may notice that the appendix's proof makes widespread use of conversion principles. Yet Aristotle himself makes no appeal to conversion anywhere in *Prior Analytics* II.14. By all appearances, Aristotle is there reasoning about a proof-system in which all direct deductions are through the three figures, and conversion principles are absent. As in Aristotle's other

[38]The key other pieces of meta-logical machinery our proof of the result uses are (i) certain observations about syllogistic Aristotle makes in *Prior Analytics* I.24, and (ii) a generalized version of a lemma that Aristotle appeals to in *Posterior Analytics* I.21.

meta-logical investigations after *Prior Analytics* I.23, he does not appear to view conversion principles as system-internal rules of inference.

In this context, it is natural to wonder if conversion, like reductio, is eliminable given the system of three figures. There is a sense in which the answer is 'yes', that's worth bringing to light.

2.3.1.1 The usefulness of conversion

The first thing to notice is that once the 16 valid assertoric moods of *Prior Analytics* I.4-7 are on the table, conversion becomes far less useful. In particular, one can show that in a system with the 16 assertoric moods, uses of conversion can be relegated to the very bottoms of syllogistic-deduction trees. For instance, the tree on the left has exactly the same premises and conclusion as the tree on the right:

$$\dfrac{\dfrac{\dfrac{\text{CeD}}{\text{DeC}}}{\text{CeD}} \qquad \dfrac{\dfrac{\text{CaA} \quad \text{AaB}}{\text{CaB}}}{\text{BiC}}}{\text{CiB}}}{\text{DoB}} \qquad\qquad \dfrac{\dfrac{\text{CeD} \quad \text{CaA}}{\text{DeA}} \qquad \text{AaB}}{\dfrac{\text{DeB}}{\dfrac{\text{BeD}}{\text{DoB}}}}$$

More generally, by a proof-theoretic argument like that for reductio elimination one can show[39]:

- If $\Sigma : \Gamma \vdash_D \varphi$ is a syllogistic-deduction, there's a syllogistic-deduction $\Sigma* : \Gamma \vdash_D \varphi$ whose only conversion inferences occur under all its non-conversion inferences.

The system of 16 assertoric moods with concatenation determines a class Π of premise-sets which from which there are valid syllogistic-deductions. Insofar as we follow Aristotle in requiring that that every member of Π contain at least two premises, adding conversion to a proof-system with only the 16 assertoric moods adds no new members to Π. Putting the point a different way, the above result establishes that for every syllogistic-deduction Σ, there is a canonical form in which use of conversion only occurs below all uses of syllogistic moods. If one thinks of constructing syllogistic-deductions as a top-down procedure[40], one can always syllogize by first applying syllogistic moods to the premise set and only appealing to conversion at the very bottom of the proof.

Now, this is not yet to say that conversion can be eliminated. Relative to the consequence relation $\vdash_D$ it cannot. For instance, consider the following syllogistic-deduction:

$$\dfrac{\dfrac{\text{AaB} \quad \text{BaC}}{\text{AaC}}}{\text{CiA}}$$

[39] See the appendix.
[40] Later, I'll encourage readers not to standardly think of it in this way.

There is no syllogistic-deduction tree which uses nothing other than combinations of the 16 assertoric moods established by *Prior Analytics* I.4-7 and has conclusion AiC whose exact premises are {AaB, BaC}. However, it is clear from the above that once the 16 assertoric moods can be taken for granted, conversion becomes far less important.

If φ converts to ψ let's say that ψ is *semantically weaker* than φ. The above result shows that once the 16 assertoric moods are available, all that conversion is good for is establishing that when some conclusion follows using the 16 moods, a semantically weaker conclusion also follows.

2.3.1.2 How to dispense with conversion

In **Definition 5**, we stipulated one natural reading of the syllogistic consequence relation. In particular, we had it that $\Gamma \vdash_D \varphi$ exactly if there is some syllogistic-deduction Σ such that (i) Γ is the set of sentences in of Σ with no lines above them, and (ii) φ is the lowest sentence in Σ. With $\vdash_D$ so interpreted, conversion is not eliminable. But this is not the only way to interpret syllogistic consequence. And it is perhaps not the most Aristotelian way to do so. It is well-known that *Prior Analytics* I.4-7 does not discuss the issue of whether valid arguments of the following forms are syllogisms:

$$\frac{\text{AaB} \quad \text{BaC}}{\text{CiA}} \qquad \frac{\text{AaB} \quad \text{BaC}}{\text{AiC}} \qquad \frac{\text{AeB} \quad \text{BaC}}{\text{CeA}} \qquad \frac{\text{BaA} \quad \text{BeC}}{\text{CoA}}$$

The closest Aristotle comes to addressing the issue is in a *Prior Analytics* II.1 text mentioned briefly above. In II.1, Aristotle points out that in many cases when one produces a syllogism further conclusions follow beyond that explicitly stated:

> Now, among syllogisms, some are universal, others are particular, the universal ones always syllogize a plurality [of conclusions], among particular syllogisms the positive ones syllogize a plurality, the privative ones syllogize only one conclusion. For the other propositions [a-, e-, and i-propositions] convert, but the privative [particular] does not convert. But the conclusion is something of something such that the rest of the syllogisms syllogize a plurality of conclusions. For instance, if A has been proved of all B or some, it is necessary that B belongs to some A, if [it has been proved] that A belongs to no B, nor does B belong to any A (this being different than the previous), but if A does not belong to some B then it is not necessary that B does not belong to some A. For it possibly belongs to all. (53a4 ff)

This text is sometimes adduced in support of the idea that Aristotle tacitly does recognize the validity of moods outside the 16 explicitly remarked on in I.4-7. But in fact, what Aristotle means to point out is simply that the premise patterns he has already mentioned often deduce more than one conclusion.

This suggests an apparently Aristotelian modification to our definition of $\vdash_D$ for our system of syllogistic-deductions. Suppose we define a function $*$ such that for all syllogistic-propositions φ, we let $\varphi*$ represent the set of propositions consisting of φ and every proposition semantically weaker than φ. So:

· (AaB)$*$ = {AaB, BiA, AiB}

· (AeB)$*$ = {AeB, BeA, AoB, BoA}

· (AiB)$*$ = {AiB, BiA}

· (AoB)$*$ = {AoB}

Then, in line with Aristotle's idea that {AaB, BaC} syllogistically-deduce not only AaC, but also further propositions like CiA, we can offer the following alternative definition of consequence:

Definition 6

$\Gamma \vdash_{D*} \varphi$ iff there is a syllogistic-deduction Σ and proposition ψ such that (i) $prem(\Sigma) = \Gamma$, (ii) $concl(\Sigma) = \psi$, and (iii) φ is a member of $\psi*$.[41]

It is easy to check that for every syllogistic proposition φ and set of syllogistic propositions Γ:

· $\Gamma \vdash_D \varphi$ iff $\Gamma \vdash_{D*} \varphi$

(Note that $\Sigma: \Gamma \vdash_{D*} \varphi$ does not entail that $\Sigma: \Gamma \vdash_D \varphi$). If the above, apparently Aristotelian, conception of consequence is introduced, conversion as well as reductio is eliminable once the 16 assertoric moods are on the table. In other words, relative to the above specification of $\vdash_{D*}$, we can show that:[42]

· $\Gamma \vdash_D \varphi$ iff there's a syllogistic-deduction Σ that never uses conversion and $\Sigma: \Gamma \vdash_{D*} \varphi$

The result is notable. But, in order not to lose the reader in a sea of definitions, the following chapters stick with $\vdash_D$ as specified in **Definition 5** .[43] Before we bring this chapter to a close, however, I would like to note one final point about the (possibly more Aristotelian) $\vdash_{D*}$.

2.3.2 Austerity

The best proof-systems spurn unnecessary complexity. The analyses above have established the following results:

[41] Equivalently: $\Gamma \vdash_{D*} \varphi$ iff for some ψ such that $\varphi \in \psi*$: $\Gamma \vdash_D \psi$

[42] See appendix

[43] Nothing substantive in **Sections 3-5** below depend on this choice.

- $\Gamma \vdash_D \varphi$ iff there is a syllogism $\Sigma : \Gamma \vdash_R \varphi$ using only (i) the 4 perfect moods of the first-figure, (ii) assertoric a-, e-, and i-conversion and (iii) reductio
- $\Gamma \vdash_D \varphi$ iff there is a syllogism $\Sigma : \Gamma \vdash_{D*} \varphi$ using only the 16 assertoric moods endorsed in *Prior Analytics* I.4-7

Given this equivalence, would Aristotle really advise demonstrators to employ a proof-system using 16 primitive patterns, rather than an equivalent one using 8?

We noted above that *Prior Analytics* I.27-30 introduces a universal method for the discovery of syllogisms. Aristotle is very enthusiastic about the method, and commends it to all would-be demonstrators. It is in this text, *Prior Analytics* I.27-30, and not *Prior Analytics* I.4-7 that Aristotle discusses the issue of actually constructing syllogisms. The method I.27-30 presents is simple. Indeed, it's so simple that commentators have been puzzled as to how Aristotle can think it universal. The method doesn't employ conversion or reductio at all. In fact, the method only involves appeal to the 7 assertoric moods with universal premises: Barbara, Celarent, Fapesmo, Camestres, Cesare, Darapti, and Felapton.

Now, a proof-system that makes use of only these 7 moods is not equivalent to one that can use all 16 moods. For, using only the 7 moods above one cannot prove anything from an i- or o-premise. Commentators have sometimes suggested that *Prior Analytics* I.27-30 might have been written before Aristotle discovered his full assertoric syllogistic. But I highly doubt this is so.

In the *Posterior Analytics*, Aristotle will claim that the ultimate premises of scientific-demonstrations are universal propositions.[44] He will deny, that is, that any particular (i.e. i- or o-) propositions are explanatorily primitive scientific-truths. Every demonstration, he maintains, proceeds from purely universal ultimate premises. It is notable that Aristotle emphasizes the usefulness of his *Prior Analytics* I.27-30 method for demonstration in particular. And in this context, Aristotle's choice to exclude from the method moods with particular premises makes a good deal of sense. Although Aristotle himself never proves anything exactly like this, the following can in fact be shown:[45]

- If $\Gamma \vdash_{D*} \varphi$ and Γ contains only universal propositions, there is a syllogism $\Sigma : \Gamma \vdash_{D*} \varphi$ that uses nothing other than the 7 assertoric moods: Barbara, Celarent, Fapesmo, Camestres, Cesare, Darapti, and Felapton.

Even without a proof, Aristotle may very well have had a justified true belief that the above result holds. Such a belief may very well have influenced the composition of *Prior Analytics* I.27-30. In this way, at least

[44] See e.g. I.4.

[45] Proved in the appendix. The possibility of proving that moods which argue from i- and o-propositions are eliminable from complex syllogisms whose premises are all universal was originally suggested to me in conversation with Stephen Menn. The particular proof given in the appendix for that result is my own; but I believe Stephen also proved it using a more-or-less similar strategy.

in scientific contexts, the austerity of P-canonical form can in fact be bettered by a system without reductio or conversion.[46]

[46]It is worth noting that despite Aristotle's great enthusiasm for the method of *Prior Analytics* I.27-30, the meta-logical discussions in the remainder of the *Analytics* never restrict their focus to the class of proofs generated by following it.

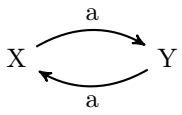
Chapter 3

Demonstrating in a circle

3.1 Circular syllogistic and circular predication

Aristotle builds a non-circularity condition into his definition of *syllogism*. 'A syllogism', he says, 'is a logos in which certain things having been posited something different from what was laid down follows by necessity because those things are so' (*Prior Analytics* I.1 24b18-20)[1]. At minimum, the 'something different' requirement means that no syllogism has its conclusion identical with one of its premises. It is then perhaps surprising that one sees in Aristotle's *Analytics* sustained attention to arguments conforming to his three figures that exhibit circularity. Aristotle turns out to be quite interested in what syllogistic and demonstrative-proof *would* look like if his presumptions against circularity were dropped, and we find Aristotle diligently investigating exactly how various circular proof-phenomena might arise in a syllogistic framework.

A striking feature of Aristotelian texts on circularities in syllogistic is their regular mention of counter-predication.[2] For Aristotle, syllogistic terms X and Y counter-predicate exactly if both XaY and YaX are true. It is natural to think of counter-predication as a circularity of a-predications:

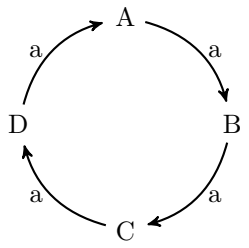


Aristotle sometimes speaks of two terms counter-predicating, but other times he will describe situations in which several more terms all counter-predicate with one another. For instance, suppose that AaB, BaC, CaD,

[1] Cf. *Topics* I.1 100a25–7, *Rhetoric* I.2 1356b16–18

[2] Esp. *Prior Analytics* II.5-7, II.16, *Posterior Analytics* I.3, II.4, II.12, cf. *Generation and Corruption* II.10.

DaA are all true. Given the transitivity of the 'said of all' a-relation we will then have each of {A,B,C,D} counter-predicating with one another:



By following transitive a-arrows, one reaches any point in the circle from any other point.

In discussing circular phenomena in syllogistic, Aristotle is constantly bringing up counter-predication. He evidently intuits some kind of important connection between circularity on the level of predication, and circularity on the level of syllogistic-argument. Aristotle's intuition is a good one. For, it turns out that there is a whole family of circular proof-phenomena that, in the context of Aristotelian syllogistic, admit of precise characterization in terms of counter-predication. For instance, consider the following result. Suppose we call a syllogistic-deduction *circular* exactly if its conclusion is identical with one of its premises. Then one can prove:

Fact 1

Let Σ be a circular syllogistic-deduction whose conclusion is AxB. If Σ's premises are true, every other term used in Σ counter-predicates with A or with B

The result above is not one that Aristotle himself proves. It can, however, be proved, and proved using generalized versions of lemmas that Aristotle appeals to in the *Analytics*.[3] Let's consider for a moment the implications of **Fact 1**. In effect, **Fact 1** illustrates a severe expressive limitation of valid circular argument in Aristotelian syllogistic. Suppose we call terms *extensionally distinct* if they do not counter-predicate with one another. **Fact 1** shows that if the premises of a circular syllogistic-deduction are true, the deduction uses no more than two extensionally distinct terms. For, all other terms used in the syllogistic-deduction will have to counter-predicate with either the subject or the predicate of the conclusion. In a sense, a circular syllogism can talk about no more than two parts of reality, and can indicate no extensional distinctions within them. This will occur no matter how long the circular syllogism is, no matter how many billions of premises it may have. Nothing like this will hold in a modern first-order logic—be it 'classical', intuitionistic, or relevantistic. But it does hold in syllogistic. In fact, this is just one in a whole family of syllogistic phenomena of a circular

[3]Proof given in the appendix. The crucial lemmas are the 'a-space Lemmas' which (on my reconstruction) are just generalizations of results that Aristotle presents in *Posterior Analytics* I.21, in the course of demolishing the view of the Ad Infinitum Skeptics. We discuss I.21 in **Section 4** below.

character whose behavior can be captured in terms of counter-predication and whose expressive power is thereby limited. In the *Analytics*, we find Aristotle himself trying to characterize various other circular proof-phenomena in terms of counter-predication with an eye to these kinds of issues.

Connections between circularity on the level of syllogistic-deductions and circularity on the level of predication are interesting in their own right. But Aristotle was a busy man. And he is not studying circularity in syllogistic solely out of an interest in pure logic. It's on account of his desire to make progress in epistemology that Aristotle thinks it's urgent to investigate how circular phenomena can arise in a syllogistic framework. It is certainly possible that the idea of identifying epistemic logoi with demonstrative-proofs in particular originated with Aristotle. But by the time he wrote the *Posterior Analytics*, there were evidently several versions of the view around. Aristotle's *Analytics* witnesses the existence of lively debates on the structure and nature of epistemic logoi. In the first instance, Aristotle wants to understand how circularity can arise in syllogistic precisely in order to make progress in such debates.

3.2 Three views on the structure and possibility of demonstrative episteme

Posterior Analytics I.2 (71b17ff.) boldly announces that episteme is achieved through mastery of a certain kind of logos. There are epistemic logoi: such logoi are demonstrative-proofs. I.2 proceeds to adumbrate criteria which, according to Aristotle, a syllogism must meet in order to be a demonstrative-proof. But exposition of Aristotle's own theory of demonstration is interrupted by I.3. This is reasonable. For there are different ways to develop the idea that episteme requires mastery of a demonstrative-proof. *Posterior Analytics* I.3 introduces a dispute between three parties—all of whom identify epistemic logoi with demonstrations in (more or less) Aristotle's sense. Recalling worries about episteme of 'primaries' [*prōta*] raised in connection with the Dream Theory at *Theaetetus* 202e-205e, the debate described in *Posterior Analytics* I.3 centers on questions about the principles of demonstrative-proofs (72b5-25):

> Some [theorists] do not think there is episteme since [it would] require having episteme of primitives; others [think] that there is [episteme], but there is demonstration of everything [e.g. even the primitives]. Neither of them [the views] are true, nor necessary.
>
> The [first group], supposing [*hypothemenoi*] that it is not possible to have episteme in another way [than through demonstration], claim that [demonstrators] are led up [*anagesthai*] into infinity, seeing that [knowers] would not have episteme of the posteriors on account of the priors if there are no primitives. They argue rightly. For, it is impossible to traverse [*dielthein*] infinitely many

[things]. But, if [chains of explanatory priority] stop and there are first-principles [= primitives], these [they contend] are unknowable [*agnōstous*] since there is indeed no demonstration of them, which they say is the only [way of] having episteme. But if it is not possible to know the primitives, neither is it possible to have episteme of the things from them, without qualification [*haplōs*] and strictly speaking [*kuriōs*], rather [it is only possible to have qualified, i.e., pseudo-episteme] from a supposition [*ex hypotheseōs*]: 'If these things are so, then [such-and-such is so]'.

The [second group] agrees about having episteme, that it is through demonstration alone.[4] But [they argue that] nothing prevents there from being demonstration of all things; for it is possible for demonstration to come about in a circle and from one another [*ex allēlōn*].

We ourselves deny that all episteme is demonstrative, rather episteme of of immediates [= *nous*] is non-demonstrative. And this is clearly necessary. For if there must be episteme of priors and the items from which there is demonstration, but at some point the middles[5] come to a stop, these [immediate propositions] must be indemonstrable. And so these [claims] we thus argue. And we claim that there is not only episteme but some principle of episteme by which we come to know the limit-propositions [*horous*].[6]

Of the three views canvassed, the last (72b18-25) is Aristotle's own. Recall that we've dubbed the first group discussed in the text the Ad Infinitum Skeptics[7] (72b7-15); the second group we call the Circular Demonstrators (72b15-18). The text indicates that Aristotle, Ad Infinitum Skeptics, and the Circular Demonstrators all accept that epistemic logoi would have to be demonstrative-proofs. But they evidently disagree over three foundational issues about the relationship between episteme and demonstration:

Question (1) Is all scientific understanding grounded in demonstrative-proof?

Question (2) Are there knowable [*gnōsta*] first-principles of episteme?

Question (3) Are first-principles, as such, indemonstrable?

One attracted to the view that episteme is achieved through demonstration might be tempted to answer all three questions in the affirmative. But if, as both Aristotle and his interlocutors assume, episteme requires a scientific understanding of principles, a position answering 'yes' to all three will not be tenable. As we shall see, Aristotle, the Ad Infinitum Skeptics, and the Circular Demonstrators each answer two of the above questions in the affirmative and one in the negative, differing among each other by their pattern of responses.

[4]Reading the manuscripts, with Barnes against Ross.
[5]Reading: *pote ta mesa*
[6]For my rendering of *horous* see footnote 25 in **Section 1.5**.
[7]For the sense in which we mean to suggest that the Ad Infinitum Skeptics were *skeptics* see footnote 26 in **Section 1.5**.

	Ad Infinitum Skeptics	Circular Demonstrators	Aristotle
Q1	YES	YES	*NO*
Q2	*NO*	YES	YES
Q3	YES	*NO*	YES

Let's begin with the first issue. Both the Ad Infinitum Skeptics and the Circular Demonstrators contend that all scientific understanding is purely demonstrative. In particular, they hold that every piece of episteme will have to be demonstrated on the basis of theses that are themselves demonstrated. Aristotle denies this (72b18-25). For he accepts the Dream Theorist's idea that pieces of episteme can rest on elemental cognitions that are not secured through logoi. In the text above, Aristotle speaks of 'non-demonstrative episteme' of primitives. More often, however, he marks a distinction between non-demonstrative *nous* which grasps first-principles in such a way as to make them available as premises for demonstrations, and episteme which is (strictly speaking) always and only achieved through demonstrating from first-principles grasped by *nous*.[8] Note that *nous* is introduced as an analog of *aisthēsis* in the Dream Theory. Like the Dream Theorist's *aisthēsis* that grasps elements [*stoicheia*], Aristotle's *nous* is a kind of cognition which is not *meta logou*.[9] Indeed, Aristotle himself will sometimes describe the proper objects *nous* as *stoicheia*.[10] By identifying the state that grasps principles with *nous* in particular (opposed to *aisthēsis* or any other cognitive state), Aristotle is granting Socrates' point against the Dream Theorist (*Theatetus* 206a-b) that the knower's grasp of primitives must be 'much clearer [*enargesteran*]...and more authoritative [*kuriōteran*]'[11] than her grasp of non-primitives.

While Aristotle splits against Circular Demonstrators and the Ad Infinitum Skeptics on issue (1), it is the Ad Infinitum Skeptics who split against Aristotle and the Circular Demonstrators on issue (2). Aristotle and the Circular Demonstrators both insist on the possibility of episteme. And both think that episteme can

[8] See e.g. *Posterior Analytics* I.23, I.33, II.19, *Nicomachean Ethics* VI.6.
[9] Again see esp. *Posterior Analytics* II.19, *Nicomachean Ethics* VI.6
[10] See e.g. *Posterior Analytics* I.23 84b19-30ff.
[11] Cf. Plato's *Seventh Letter* 342d (probably authentic).

be achieved because there are scientifically intelligible first-principles. The Ad Infinitum Skeptics dissent, suggesting that there just aren't any first-principles. Suppose every scientifically explicable truth has distinct further truths explanatorily prior to it. Since episteme of X can only be achieved by a demonstrative logos that provides a fully complete explanation of X, it follows that a genuine demonstration can never be constructed. As all episteme is purely demonstrative, they conclude that in principle no episteme is possible. Indeed, they add, even if there were scientific first-principles, episteme would still be impossible. For all episteme is purely demonstrative, but genuine first-principles would be indemonstrable as such. There can be no episteme, for either there are no first-principles, or there are first-principles and they are not *epistēta*.[12]

On issue (3) Aristotle and the Ad Infinitum Skeptics concur: first-principles, as such, are indemonstrable. But here the Circular Demonstrators protest, maintaining that first-principles not only would have to, but actually *can* be known through demonstrative-proof. Like Aristotle, the Circular Demonstrators contend that episteme is possible. Dissenting from Aristotle they hold that what makes episteme possible is the fact that scientific first-principles can be demonstrated 'in a circle and from one another' (72b17-18).

The remainder of this dissertation works through Aristotle's case against the Ad Infinitum Skeptics and Circular Demonstrators. We focus in particular on Aristotle's use of meta-logic in responding to these alternative accounts of episteme. We will find Aristotle strategically deploying meta-logical results to help establish, against his interlocutors, theses concerning the nature and possibility of episteme. The present chapter discusses Aristotle's arguments against Circular Demonstrators. The bulk of Aristotle's case is presented in the second half *Posterior Analytics* I.3—the immediate sequel to 72b5-25, quoted above; but the argument depends crucially on material from the *Prior Analytics* which we shall also have to understand. The dissertation's next two chapters discuss Aristotle's attack on the Ad Infinitum Skeptics in *Posterior Analytics* I.19-22.

3.3 Proving claims 'in a circle and from one another'

The Circular Demonstrators claim that epistemai are founded on the basis of propositions demonstrated 'in a circle and from one another'. To understand the view's motivation, let's begin by considering how the Circular Demonstrators respond to the Ad Infinitum Skeptics.

Like the Ad Infinitum Skeptics, the Circular Demonstrators think that all scientific understanding is through demonstrative-proof. The Ad Infinitum Skeptics charge that this rules out the possibility of unqualified episteme. For, they claim, the process of constructing a demonstrative-proof can never be terminated in a way that yields episteme. Suppose I want to construct a demonstration of χ, so I provide a sound

[12] Collectively, the six aporiai in opening of the *Parmenides* have a similar force: either there are no forms, or if there are forms they cannot be objects of episteme.

scientific argument for χ on the basis of some explanatory truths Γ_0. A demonstration must give a fully complete explanation of the demonstrandum. So, in order to build a demonstration of χ the non-primitive truths in Γ_0 will themselves have to be demonstrated through further truths Γ_1, whose non-primitive truths will also have to be demonstrated through even further truths Γ_2, and so on. Now, either (i) this goes on ad infinitum, or (ii) at some point I reach principles. In case (i) I might halt the demonstrative-process at some point, and have a fine partial explanation of χ on the basis of an incomplete and partial demonstration of χ. But this is not sufficient for genuine episteme; genuine episteme would require a complete demonstration. In case (ii), episteme is again impossible. For even if I reach primitives and stop there, no episteme will follow. If χ has principles, having episteme of χ requires episteme of its principles. But there cannot in fact be any episteme of χ's principles since they cannot be demonstrated.

Against the Ad Infinitum Skeptics, the Circular Demonstrators insist that every scientific truth can be demonstrated through a finitely terminating proof-process. And this can be done, they add, in a way that produces episteme. For, in a rightly guided process of demonstrating χ, at some point first-principles Γ_n will be reached. But principles are demonstrated 'in a circle and from one another'. Suppose one such first-principle in Γ_n is φ. In order to have episteme of χ, this φ will also have to be demonstrated. But suppose I demonstrate φ from further principles θ and ψ which are each proved in a circle with φ. Then regress is averted and episteme can be achieved:

$$\dfrac{\dfrac{\theta \quad \varphi}{\psi} \quad \dfrac{\psi \quad \varphi}{\theta}}{\varphi}$$
$$\vdots$$
$$\chi$$

If $\dfrac{\theta \quad \varphi}{\psi}$, $\dfrac{\psi \quad \varphi}{\theta}$, and $\dfrac{\psi \quad \theta}{\varphi}$ are all demonstrations, no more demonstrative work needs to be done on the branch of the proof that terminates in the principle φ. For in the proof above, each of the principles φ, ψ, θ has been proved in a sub-proof. There can, thus, be episteme of φ, ψ, θ without any further demonstrating. If all the other branches could be closed off in this manner, with principles demonstrated in a circle and from one another, episteme of χ would follow in finitely many steps.

So far, this is all very abstract. For instance, the toy example above posited the existence of two demonstrations $\dfrac{\theta \quad \varphi}{\psi}$, and $\dfrac{\psi \quad \varphi}{\theta}$. But why think there could even be valid *syllogisms* of such forms? In physics, Aristotle often reminds us that what comes to be does not come to be from just anything. And the same is true in logic. For, as the *Analytics* emphasizes, conclusions do not come to be [*ginesthai*] from just any premises; only certain kinds of premise-sets can syllogistically produce a given conclusion, and only certain kinds of conclusion can come to be from a given premise-set. *Prior Analytics* I claims to investigate 'how <u>all</u> syllogisms come to be [*ginetai*]' (I.4 25b26-28, I.26 43a16-19, cf. I.32 46b38-40, II.1 52b38-39).

Ultimately, Aristotle is prepared to see his theory of syllogisms as complete, thinking he's shown that every possible syllogism is exhausted by two-premise argument patterns that conform to the cannons of his three figures. So to determine whether and in what cases the proofs Circular Demonstrators envision are even logically possible, Aristotle proposes to study how proving 'in a circle and from one another' would work in the framework of his three-figured theory of syllogistic. Aristotle executes this task in *Prior Analytics* II.5-7—a text which his discussion of the Circular Demonstrators in *Posterior Analytics* I.3 explicitly refers back to (73a14). The Circular Demonstrators' 'in a circle and from one another' locution is ubiquitous in *Prior Analytics* II.5-7. And in line with its epistemological context, *Prior Analytics* II.5-7 almost always speaks of 'proving [*deiknusthai*] in a circle and from one another' rather than merely 'syllogizing' or 'arguing'.[13]

To better appreciate how Aristotle understands the Circular Demonstrators, let's start with a look at the account of 'proving in a circle and from one another' given in *Prior Analytics* II.5-7. II.5 opens with a kind of definition for what it is to prove propositions 'in a circle and from one another' (57b18-30):

> *To prove in a circle and from one another* is to conclude something which one posited [as a premise] in another syllogism, through the conclusion [of that syllogism] and the other proposition posited as reversed in predication.[14] For instance, [suppose] one has to prove that A belongs to all of C, and one proved it through B. If, in turn, one would show that A belongs to B, by positing that A belongs to [all] C and C to [all] B, [one would prove circularly and from one other]; (previously one posited the reverse: that B belongs to all C). Or suppose one must prove that B belongs to C. [He would prove in a circle and from one another] if he should posit that A [holds of all] C—that which was previously a conclusion—and that B holds of A; (earlier it was posited in reverse that A holds of [all] B). Proving from one another is not possible otherwise. For if a different middle is to be posited, it won't be [proving] in a circle; for none of the same things is taken.

Aristotle is describing the following circumstance. Suppose I'm convinced that A holds of all C, and I aim to prove it. I discover that I can establish AaC through a syllogism:

$$\frac{\text{AaB} \quad \text{BaC}}{\text{AaC}}$$

But I also want to prove that A holds of all B. So I proceed as follows:

$$\frac{\text{AaC} \quad \text{CaB}}{\text{AaB}}$$

[13] A point also emphasized by Barnes 1976.

[14] More literally: '...is to conclude through a conclusion and another premise taken as reversed in predication, the remaining premise which one took in the other syllogism'.

Having given two such 'proofs' , Aristotle will say that I've proved AaC and AaB 'in a circle and from one another'. For, I've proved AaC by taking AaB as an assumed premise, and I've proved AaB by taking AaC as an assumed premise. Return for a moment to our original syllogism of AaC. There AaB was the major premise, and BaC was the minor. Just as the major and the conclusion could (in principle) be used to syllogistically deduce one another, so could the minor and the conclusion. In the simplest case, I might prove BaC with:

$$\frac{\text{BaA} \qquad \text{AaC}}{\text{BaC}}$$

If so, given my first syllogism I will have proved AaC and BaC 'in a circle and from one another'.

From *Prior Analytics* II.5 57b18-30, it is clear that Aristotle understands 'proving in a circle and from one another' as a process whereby one assumes some φ in order to prove some ψ, and then assumes ψ in order to prove φ. One proves φ and ψ in a circle and from one another by constructing two syllogisms:

- $\Sigma_1 : \Gamma, \varphi \vdash_D \psi$
- $\Sigma_2 : \Delta, \psi \vdash_D \varphi$

It is not difficult to see why Aristotle, or anyone else, would characterize using φ to prove ψ and using ψ to prove φ as an episode of proving φ and ψ 'in a circle and from one another'.[15] Yet commentators have tended to see Aristotle's *Prior Analytics* II.5 definition of 'proving in a circle and from one another' as rather mysterious. The bump in the rug of course is the bit about 'reversing the other premise in predication' (57b19-20). Given the broader context of *Prior Analytics* II.5-7, the intended meaning of this phrase is clear. Aristotle is talking about using something of the form XaY in one syllogistic-deduction and using something of the form YaX in the other.[16] Now, in *Prior Analytics* II.5-7 and *Posterior Analytics* I.3 Aristotle will often describe propositions XaY and YaX as expressing counter-predications—irrespective of whether both are actually true. But we ourselves (tracing another of Aristotle's uses) earlier stipulated that syllogistic terms X and Y counter-predicate exactly if both XaY and YaX are in fact true. So taking a cue from Aristotle, let's also introduce a notion of relative counter-predication:

Definition 1 Syllogistic closure and relative counter-predication

Where Γ is a set of syllogistic propositions:

1. $\Gamma^{\vdash} = \{\varphi : \Delta \vdash_D \varphi \text{ for some } \Delta \subseteq \Gamma\}$

[15] On my usage, one 'uses α to prove β' exactly when one produces a syllogism $\Sigma : \Theta, \alpha \vdash_D \beta$. In saying that one uses α to prove β I do not mean that no propositions beyond α are appealed to in proving β. Likewise, if one produces a syllogism $\Sigma : \Theta \cup \Lambda \vdash_D \beta$ and Λ is a set of propositions, I will say (i) she used Λ to prove β and (ii) she used exactly $\Theta \cup \Lambda$ to prove β.

[16] On prosletic conversion see below.

2. X and Y counter-predicate$_\Gamma$ iff {XaY,YaX}$\subseteq \Gamma \cup \Gamma^{\vdash}$

In other words, X and Y counter-predicate$_\Gamma$ (counter-predicate relative to Γ) exactly if one of the following situations obtain—either: (i) both XaY and YaX are in Γ, or (ii) both XaY and YaX can be syllogized using only propositions in Γ, or (iii) one of {XaY,YaX} is in Γ and the other can be syllogized using only propositions in Γ.

Scholars have been puzzled by the reference to counter-predication in *Prior Analytics* II.5's definition of proving in a circle and from one another. Indeed, they have tended to see the definition as highly artificial, describing a phenomenon of questionable logical or philosophical interest. Ross, for instance, has called the whole of *Prior Analytics* II.5-7 a kind of 'mental gymnastic'[17]; Smith[18] too characterizes II.5-7 as the product of logical doodling.[19] Such scholars are, of course, aware of the back-reference to *Prior Analytics* II.5-7 in *Posterior Analytics* I.3. But they don't see how Aristotle's *Prior Analytics* II.5 definition of proving in a circle and from one another could possibly be motivated. Indeed, they express suspicion as to whether the conception of proving in a circle investigated in *Prior Analytics* II.5-7 can really be seen as the same conception at issue in *Posterior Analytics* I.3.[20]

Commentators have tended to read the *Prior Analytics* II.5 definition of 'proving in a circle and from one another' as stipulative. But I contend that Aristotle means to offer a substantive characterization of when—at least as an abstract logical possibility—there can be two demonstrations $\Sigma_1 : \Gamma, \varphi \vdash_D \psi$ and $\Sigma_2 : \Delta, \psi \vdash_D \varphi$. Moreover, I propose to defend the adequacy of Aristotle's definition.

Recall that Aristotle's definition says that 'to prove in a circle and from one another is to conclude something which one posited [as a premise] in another syllogism, *through the conclusion* [of that syllogism] *and the other proposition posited as reversed in predication*'. We've already seen examples of Barbara described in *Prior Analytics* II.5 (57b18-30) that fit this pattern:

$$\Sigma_L \ \frac{\text{AaB} \quad \text{BaC}}{\text{AaC}} \qquad\qquad \Sigma_R \ \frac{\text{AaC} \quad \text{CaB}}{\text{AaB}}$$

Observe that Σ_L uses premises {AaB, BaC} to prove AaC, where Σ_R uses premises {AaC, CaB} to prove AaB. The conclusion of Σ_L is a premise in Σ_R, and the conclusion of Σ_R is a premise in Σ_L. Σ_R proves one of Σ_L's premises (i.e. AaB) by taking the other of Σ_L's premises 'reversed in predication' (Σ_L has premise

[17] See Ross (1944) p 441.

[18] Smith (1989) suggests following genealogy to explain Aristotle's definition. Aristotle began by wondering if both premises of a two-premise syllogism could ever be deduced from the conclusion. He noticed that this was impossible. He next wondered if it might ever be possible to syllogize both (i) χ from ϕ, ψ, and (ii) φ from χ, ψ. But he concluded that this too is impossible. Then, finally, Aristotle noticed that in some cases something similar can happen—sometimes we can syllogize: (i) χ from ϕ, ψ, and (ii) φ from χ and ψ in a counter-predicated form. As to the logical interest of the exercise, Smith says nothing.

[19] I owe this memorable phrase to Alan Code.

[20] For instance Barnes: '[h]ere as elsewhere, syllogistic does not readily provide a formalization of the argument structures considered in *APst.*' (p 106) 1994. Cf. Smith 1989 p192-3.

BaC, Σ_R has premise CaB). Σ_L proves one of Σ_R's premises (i.e. AaC) by taking the other of Σ_R's premises 'reversed in predication' (Σ_R has premise CaB, Σ_L has premise BaC). Here's a further example from *Prior Analytics* II.6:

$$\Sigma_L \ \frac{\text{BaA} \quad \text{BeC}}{\text{AeC}} \qquad\qquad \Sigma_R \ \frac{\text{AaB} \quad \text{AeC}}{\text{BeC}}$$

Here AeC and BeC are proved in a circle and from one another. Note that BeC combines with AaB and AeC combines with BaA, conforming again to Aristotle's *Prior Analytics* II.5 characterization.

Now, the discussion of *Prior Analytics* II.5-7 focuses on instances of 'proving in a circle and from one another' involving just two-premise syllogisms. In other words, Aristotle is asking when it can occur that both (i) $\varphi, \alpha \vdash_D \psi$, and (ii) $\psi, \beta \vdash_D \varphi$. By consideration of cases, Aristotle effectively proves that unless extra-syllogistic devices like prosleptic propositions are introduced, the following holds:[21]

Fact 2

If $\varphi, \alpha \vdash_D \psi$ and $\psi, \beta \vdash_D \varphi$, then α is of the form XaY and β of the form YaX

It is presumably after noticing the above fact that Aristotle first hit upon his *Prior Analytics* II.5 definition of proving in a circle and from one another. But scholars have also noted that more complex cases of using φ to prove ψ and using ψ to prove φ do not fit this pattern. For instance, consider the following pair:

$$\Sigma_L \ \frac{\text{AaD} \quad \dfrac{\text{AeB} \quad \text{BaF}}{\text{AeF}}}{\dfrac{\text{DeF} \qquad\qquad \text{FaC}}{\text{DeC}}} \qquad\qquad \Sigma_R \ \frac{\text{GaA} \quad \dfrac{\text{DaG} \quad \text{DeC}}{\text{GeC}}}{\dfrac{\text{AeC} \qquad\qquad \text{CaB}}{\text{AeB}}}$$

Note that Σ_L uses AeB as a premise to derive DeC, while Σ_R uses DeC as a premise to derive AeB. What AeB combines with in Σ_L, and what DeC combines with in Σ_R are not a pair of the form XaY, YaX. Indeed, there is no pair of propositions of the form XaY, YaX such that XaY occurs in Σ_L, and YaX occurs in Σ_R. Nonetheless, I submit that there is a sense in which Aristotle might still maintain that Σ_L and Σ_R do fall under his definition of proving in a circle and from one another. For consider the following pair:

$$\Sigma_L* \ \frac{\text{AaD} \quad \dfrac{\text{AeB} \quad \dfrac{\text{BaF} \quad \text{FaC}}{\text{BaC}}}{\text{AeC}}}{\text{DeC}} \qquad\qquad \Sigma_R* \ \frac{\dfrac{\text{DaG} \quad \text{GaA}}{\text{DaA}} \quad \dfrac{\text{DeC} \quad \text{CaB}}{\text{DeB}}}{\text{AeB}}$$

[21] This was pointed out by Lear in his 1980. Somewhat surprisingly Smith's (1989, 1986) and Barnes (1994) discussions of II.5-7/I.3 in make no reference to this point.

Compare Σ_L to Σ_L^* and Σ_R and Σ_R^*. They derive the same conclusions from the same premises. But now compare Σ_L^* and Σ_R^*. In particular, focus on the inferences in Σ_L^* that depend on AeB and those in Σ_R^* that depend on DeC. The only propositions that AeB, DeC, and their consequences combine with in Σ_L^* and Σ_R^* are a-propositions. And something of the form XaY combines with AeB or a consequence of AeB in Σ_L^* exactly if YaX combines with DeC or a consequence of DeC in Σ_R^*.

If we take Aristotle's *Prior Analytics* II.5 definition as describing the form that any syllogistic-deductions Σ_1 and Σ_2 actually have if $\Sigma_1 : \Gamma, \varphi \vdash_D \psi$ and $\Sigma_2 : \Delta, \psi \vdash_D \varphi$, the definition is a failure. But one can also take the definition as indicating a *canonical form* for pairs of demonstrations (or 'demonstrations' in the Circular Demonstrators' sense) witnessing: $\Gamma, \varphi \vdash_D \psi$ and $\Delta, \psi \vdash_D \varphi$. Both Aristotle and the Circular Demonstrators will claim that demonstrations must have true premises. So if there are to be demonstrations $\Sigma_1 : \Gamma, \varphi \vdash_D \psi$ and $\Sigma_2 : \Delta, \psi \vdash_D \varphi$, the collection of premises $\Gamma \cup \Delta \cup \{\varphi, \psi\}$ must be consistent. Moreover, since we're talking about science Aristotle will be focused on cases where Σ_1, Σ_2 are maximally interesting in the sense that neither proves a particular conclusion (e.g. XiY) when $\Gamma \cup \Delta \cup \{\varphi, \psi)$ in fact contains material to prove the corresponding universal (e.g. XaY). It can in fact be shown that when there are two maximally interesting syllogisms of the form $\Sigma_1 : \Gamma, \varphi \vdash_D \psi$ and $\Sigma_2 : \Delta, \psi \vdash_D \varphi$ whose premise-sets are consistent, there must exist syllogisms $\Sigma_1^* : \Gamma, \varphi \vdash_D \psi$ and $\Sigma_2^* : \Delta, \psi \vdash_D \varphi$ that conform to the pattern indicated by *Prior Analytics* II.5 (with $\Gamma_0 \cup \Gamma_1 = \Gamma$ and $\Delta_0 \cup \Delta_1 = \Delta$):

$$\Sigma_1* \quad \dfrac{\dfrac{\Gamma_0 \quad \varphi}{\vdots} \quad \Gamma_1}{\dfrac{\text{AaB} \quad \alpha}{\psi} \quad \dfrac{\vdots}{\text{CaD}}} \qquad\qquad \Sigma_2* \quad \dfrac{\dfrac{\Delta_0 \quad \psi}{\vdots} \quad \Delta_1}{\dfrac{\text{BaA} \quad \beta}{\varphi} \quad \dfrac{\vdots}{\text{DaC}}}$$

According to *Prior Analytics* II.5, when I prove both $\Gamma, \varphi \vdash_D \psi$ and $\Delta, \psi \vdash_D \varphi$, I'll go through propositions 'reversed in predication'. More specifically, Aristotle says that the proposition I combine with φ in syllogizing $\Gamma, \varphi \vdash_D \psi$ must be 'reversed in predication' compared to the proposition I combine with ψ in syllogizing $\Delta, \psi \vdash_D \varphi$. The above shows that provided we're dealing with consistent premise sets and proving maximally interesting conclusions, whenever $\Gamma, \varphi \vdash_D \psi$ and $\Delta, \psi \vdash_D \varphi$ there will be syllogistic-deductions witnessing $\Gamma, \varphi \vdash_D \psi$ and $\Delta, \psi \vdash_D \varphi$ that proceed exactly in fashion Aristotle describes.[22] Given the existence of such canonical forms, it would very reasonable for Aristotle characterize the phenomenon of proving in a circle just as he does. Aristotle would be indicating that even if demonstrations (as described by the Circular Demonstrators) witnessing $\Gamma, \varphi \vdash_D \psi$ and $\Delta, \psi \vdash_D \varphi$ do not actually go through propositions 'reversed in predication', given the very same premise-sets it is always possible to do so. Indeed, by

[22] If requirements consistency and the maximally interesting conditions are dropped something very similar, but slightly less strong holds. See appendix.

mentioning counter-predication in his definition of proving in a circle, Aristotle would be offering a far more interesting and informative characterization of proving in a circle than if he had simply said 'φ, ψ are proved in a circle exactly if both: $\Gamma, \varphi \vdash_D \psi$ and $\Delta, \psi \vdash_D \varphi$'.

It is certainly possible that when Aristotle was composing *Prior Analytics* II.5 he was not himself able to prove the result mentioned above. No proof or proof-sketch for it appears in the text. But it is clear that Aristotle thought that going through premises related in counter-predication was somehow definitionally fundamental to situations in which both $\Sigma_1 : \Gamma, \varphi \vdash_D \psi$ and $\Sigma_2 : \Delta, \psi \vdash_D \varphi$. Whether or not Aristotle was ever able to work out all the details, he was certainly on the right track of something correct. His definition of 'proving in a circle and from one another' can in fact be seen as a well-motivated and successful attempt to substantively characterize a logically interesting phenomenon connected to a live philosophical issue. Later we will be looking at a further text from *Prior Analytics* II.5-7 in which Aristotle links a different (albeit related) circular proof-phenomenon to counter-predication. But first I'd like to return to Aristotle's discussion of the Circular Demonstrators in *Posterior Analytics* I.3.

3.4 The first two arguments of *Posterior Analytics* I.3: proving in a circle cannot produce episteme

Posterior Analytics I.3 presents three arguments against the Circular Demonstrators' thesis that 'all the postulates [i.e. principles] can be demonstrated from one another' (73a12-13). The chapter's final argument draws heavily on the analyses of *Prior Analytics* II.5-7. But *Posterior Analytics* I.3's first two arguments proceed otherwise. They depend on no specific technical results about syllogistic. But they do presuppose the same conception of proving in a circle and from one another that Aristotle studies in *Prior Analytics* II.5-7. As we have seen, to prove φ and ψ in a circle and from one another is to produce two syllogistic-deductions—one that uses φ to show ψ, and one that uses ψ to show φ:

- $\Sigma_1 : \Gamma, \varphi \vdash_D \psi$
- $\Sigma_2 : \Delta, \psi \vdash_D \varphi$

The first two arguments of *Posterior Analytics* I.3 look to establish that 'proving' two propositions φ, ψ in a circle and from another cannot produce demonstrative-episteme of both φ and ψ. In other words, if there are two syllogisms: $\Sigma_1 : \Gamma, \varphi \vdash_D \psi$ and $\Sigma_2 : \Delta, \psi \vdash_D \varphi$, no more than *one* of syllogisms Σ_1, Σ_2 can be demonstrative-proof—no more than one of the syllogistic-deductions can yield episteme of its conclusion. Aristotle's first argument runs as follows (72b25-32):

> It is clear that strictly speaking it's impossible to demonstrate in a circle, if indeed demonstration is from prior and more intelligible propositions. For it is impossible for the same things to be simultaneously prior and posterior to one another, unless [they are prior and posterior to one another] in different respects (for instance, certain things [prior and more intelligible] to us, others [prior and more intelligible] simpliciter). But if this is so, understanding simpliciter [*to haplōs eidenai*] has not been well defined, rather it is double. Or one demonstration is not [a demonstration] simpliciter coming to be from [premises] more intelligible to us.

We've seen that on Aristotle's view, the premises of a demonstrative-proof must be explanatorily *prior* to the conclusions they prove. But the relations of prior-to and posterior-to are *converses* of one another: x is prior to y iff y is posterior to x. And considered individually, both relations constitute *strict partial order*s.[23] For nothing is prior to itself, and if x is prior to y while y is prior to z then x is prior to z (mutatis mutandis for posterior-to). It follows that if x is prior to y, then it is not the case that x posterior to y (and vice versa). But Aristotle charges that the Circular Demonstrators are committed to exactly this. For, suppose there are two syllogistic-deductions: $\Sigma_1 : \Gamma, \varphi \vdash_D \psi$ and $\Sigma_2 : \Delta, \psi \vdash_D \varphi$. If Σ_1 is a demonstration, φ must be prior to ψ. But if Σ_2 is a demonstration ψ must be posterior to φ. So it results that Σ_1 is a demonstrative-proof of φ only if Σ_2 is not a demonstrative-proof of ψ (and vice versa). It is, thus, impossible for both Σ_1 and Σ_2 to be demonstrative-proofs. But this is precisely what the Circular Demonstrators need if proving 'in a circle and from one another' is to provide episteme of first-principles.[24] One might, Aristotle suggests (I.3 72b28-32), attempt to resolve the problem by introducing different senses of priority/posteriority. There is no difficulty in φ being <u>both</u> prior to ψ <u>and</u> posterior to ψ provided that it is in different respects that φ is prior to/posterior to ψ. For instance, maybe in the order of nature φ is prior to and more intelligible than ψ, but relative to our idiosyncratic cognitive faculties φ is less intelligible than and posterior to ψ. But this unhelpful. For the Circular Demonstrators will be left with two unattractive options. They might insist that both Σ_1 and Σ_2 are productive of episteme despite the fact that only one of the syllogisms offers a genuinely causal explanation of its conclusion. Yet in this case they will have to admit that there is more than one way of having episteme (72b30-31: 'understanding simpliciter...is double')—something they claim to avoid, but will presumably charge Aristotle of doing when he introduces *nous*. Alternatively, the Circular Demonstrators might admit that only one of Σ_1, Σ_2 is a genuine demonstration (72b31: 'the other demonstration is not [a demonstration] simpliciter'). But the Circular Demonstrators claim that all episteme is through demonstration. So again, by their own lights, they will then have failed to show how episteme of

[23] Recall that a strict partial order is transitive and irreflexive (irreflexive: i.e. $(\forall x)\neg xRx$).
[24] Cf. our discussion above of how the Circular Demonstrators respond to the Ad Infinitum Skeptics.

principles is possible through demonstrating them 'in a circle and from one another'.

Modern commentators don't much like this argument. They point out that Aristotle is quite flagrantly begging the question against his opponents.[25] The argument essentially depends on the assumption that the the premises of a demonstration must be prior to and more intelligible than its conclusion. This certainly is Aristotle's *own* view. But why should anyone grant it unless she already thinks that propositions cannot be demonstrated in a circle and from one another?

The argument is indeed swift. Aristotle himself is free to view it as providing a good explanation of why *in fact* no two propositions can (strictly speaking) be demonstrated in a circle and from one another. But the commentators are certainly right that the argument ought not to move any Circular Demonstrators. Nonetheless, these commentators seem to miss something that Aristotle himself presumably sees as a delicious irony. Aristotle is, after all, presenting an *obviously* question-begging argument to opponents who think that all scientific knowledge is grounded in (what sure look like) question-begging patterns of argumentation. If the Circular Demonstrators complain that this argument is defective because it illegitimately begs the question, they will have to explain why—on their view—all scientific demonstrations are not just as illegitimately question-begging and (hence) defective. Indeed, the central point of *Posterior Analytics* I.3's next argument is that for the Circular Demonstrators, illegitimate question-begging argumentation must be ubiquitous in science (72b35-73a6):

> For those who claim that demonstration is in a circle, there results not only what's just been said, but that they say nothing else than that 'if this is the case, this is the case'. But in this way it is easy to prove everything. It is clear that this results with three terms having been posited. [...] Whenever it occurs that A holding it is necessitated that B, and B holding it is necessitated that C, then A holding it is necessitated that C. So, let A be C (for this would be to prove in a circle: when A holds it is necessitated that B and when B holds it is necessitated that A. Then to say when B holds to C holds, is to say that when B holds A holds. And [recall] this, that when A holds C holds. But A is the same as C. So it results that those who claim demonstration is in a circle say nothing other than that when A is the case A is the case. But it is easy to show everything in this way.

It is possible to prove propositions in a circle and from one another without offering any single question begging argument. For instance, consider the two syllogisms:

[25] Barnes (1976, 1994), Malink (2013); Smith (1986).

$$\dfrac{\dfrac{\text{XaY}\quad\text{YiZ}}{\text{XiZ}}\quad\text{WaX}}{\text{WiZ}} \qquad\qquad \dfrac{\dfrac{\text{XaW}\quad\text{WiZ}}{\text{XiZ}}\quad\text{YaX}}{\text{YiZ}}$$

Aristotle's point in his second argument against the Circular Demonstrators is that by 'demonstrating' in this manner one is effectively arguing 'if A, then A'. And this (cf. 64b38-65a8) is one of Aristotle's standard ways of describing the most egregious kind of question-begging argument where the reasoner takes a premise the very conclusion she intends to show . Suppose one uses A to prove B, and uses B to prove C. As Aristotle explains above (72b37-39), deducibility is transitive.[26] So in such a case one can use A to prove C. But now let A = C. Using our example above, let A = YiZ, B = WiZ, and C = YiZ. One uses YiZ to prove WiZ. And one uses WiZ to prove YiZ. So one uses YiZ to prove YiZ. And indeed, says Aristotle, insofar as the Circular Demonstrators will use the syllogism on the left to establish one of the premises for the syllogism on the right, this is exactly what they are doing. So 'it results that those who claim demonstration is in a circle say nothing other than that when A is the case A is the case' (73a5). But arguments of this kind are not truly probative. It is easy to 'prove' anything in this way:[27]

$$\dfrac{\text{AaB}\quad\text{BaB}}{\text{AaB}} \qquad \dfrac{\text{AeB}\quad\text{BaB}}{\text{AeB}} \qquad \dfrac{\text{AiB}\quad\text{BaB}}{\text{AiB}} \qquad \dfrac{\text{AoB}\quad\text{BaB}}{\text{AoB}}$$

[26] Cf. Malink (2013)

[27] Malink (2013) has suggested that since Aristotle denies that single premise inferences are syllogistic, the letters A, B, Γ in the argument of 72b35-73a6 should be read as placeholders for collections of syllogistic propositions. In developing the view, Malink claims that the circular proof structure Aristotle is describing in the argument is a sequence of sets of syllogistic propositions: (A, B, Γ), such that (i) every proposition in Γ can be proved by some propositions in B, (ii) every proposition in B can be proved by some propositions in A, and (iii) every proposition in A can be proved by some propositions in Γ. The proposal is ingenious, but with little textual support. I myself think the proposal is neither correct nor necessary. It is not necessary because when Aristotle says (e.g.) 'when A holds, B holds' he can easily be taken to mean that B is syllogized using A (along with other propositions). Aristotle need not mean, as Malink assumes he does, that B is syllogized using A alone. Reading Aristotle's prose here as telegraphic, we are free (as I do above) to take A, B, Γ as placeholders for propositions rather than placeholders for pluralities of propositions. On this reading, Aristotle will just be discussing proving in a circle and from one another in the sense of his *Prior Analytics* II.5 definition. He will be describing a pretty standard case where we have: $\Gamma, \varphi \vdash_D \psi$, $\Delta, \psi \vdash_D \chi$, and $\Theta, \chi \vdash_D \varphi$. Now a reason for thinking Malink's proposal false—one local to the text above. At 73a6 Aristotle says that it is 'easy' to prove 'everything' from itself. But if we follow Malink's interpretation of A, B, Γ as placeholders for pluralities of propositions this is quite false. Firstly, it is not true that *every* plurality of syllogistic propositions can deduced from itself in Malink's sense. (For instance, let A = {XaY, YaZ}); you can't syllogistically prove every proposition in A using only propositions in A). Secondly, even when we have collections for which this is true, it is not necessarily a simple matter to check whether this is so, or to in fact do so. In contrast, taken as I propose, Aristotle's claim comes out true on both counts. Aristotle would just be thinking about the 'easy' universally applicable proof patterns with an BaB premise in the main text. Or alternatively he would be thinking about these 'easy' patterns available whenever at least one term of our proposition counter-predicates with some other term. For the subject term:

$$\dfrac{\dfrac{\text{AaB}\quad\text{BaC}}{\text{AaC}}\quad\text{CaB}}{\text{AaB}} \qquad \dfrac{\dfrac{\text{AeB}\quad\text{BaC}}{\text{AeC}}\quad\text{CaB}}{\text{AeB}} \qquad \dfrac{\dfrac{\text{AiB}\quad\text{CaB}}{\text{AiC}}\quad\text{BaC}}{\text{AiB}} \qquad \dfrac{\dfrac{\text{AoB}\quad\text{CaB}}{\text{AoC}}\quad\text{BaC}}{\text{AaB}}$$

and for the predicate term:

$$\dfrac{\text{AaC}\quad\dfrac{\text{CaA}\quad\text{AaB}}{\text{CaB}}}{\text{AaB}} \qquad \dfrac{\text{CaA}\quad\dfrac{\text{AaC}\quad\text{AeB}}{\text{CeB}}}{\text{AeB}} \qquad \dfrac{\text{AaC}\quad\dfrac{\text{CaA}\quad\text{AiB}}{\text{CiB}}}{\text{AiB}} \qquad \dfrac{\text{CaA}\quad\dfrac{\text{AaC}\quad\text{AoB}}{\text{CoB}}}{\text{AoB}}$$

3.5 The third argument of *Posterior Analytics* I.3: expressive limitations of proving in a circle

The Circular Demonstrators contend that a certain style of demonstrating plays a crucial role in science. In particular, they maintain that scientific principles can be demonstrated 'in a circle and from one another' and they attempt to ground the whole of episteme on this basis. The first two arguments of *Posterior Analytics* I.3 undermine the idea that demonstrating propositions in a circle and from one another produces episteme in the way the Circular Demonstrators maintain. Suppose I offer two syllogistic-deductions: $\Sigma_1 : \Gamma, \psi \vdash_D \varphi$ and $\Sigma_2 : \Delta, \varphi \vdash_D \psi$. From the first argument it follows that at most one of $\{\Sigma_1, \Sigma_2\}$ is a demonstration. But given the second argument, we should doubt whether someone who propounds both Σ_1 and Σ_2 and claims that both are demonstrations could ever thereby have episteme of either φ or ψ. For, such an individual would have to think that her episteme of φ rests on a proof that assumes φ while her episteme of ψ rests on a proof that assumes ψ. But how can one have episteme of something on the basis of assuming it in its own demonstration? The argument form is not probative.

The third argument of *Posterior Analytics* I.3 is like the first two in aiming to show that the technique of proving in a circle cannot do the work that the Circular Demonstrators demand of it. The first two arguments establish that proving items in a circle and from one another can never produce episteme of all of them. And since the technique could never produce the amount of episteme the Circular Demonstrators want to get out of it, it will not help them establish the possibility of episteme in the face of the challenge brought by the Ad Infinitum Skeptics. The third argument will not develop the idea that proving in a circle doesn't produce episteme of what our proofs talk about. Its insight is rather that in syllogistic, employing the technique of proving in a circle dramatically constrains the range of structures that can be talked about. Given the extremely limited range of cases in which proving in a circle is possible, nothing which actually exemplified what the Circular Demonstrators say about science could be a science. And this is why what the Circular Demonstrators say is impossible—nothing which conformed to their model of 'science' would actually be a science. This, in any case, is the reading of Aristotle's third argument against the Circular Demonstrators I will ultimately defend.

We begin with the text presenting the argument, the final lines of *Posterior Analytics* I.3. For ease of discussion, I've appended numbers to Aristotle's text:

> **[i]** Nor in fact is this [what the Circular Demonstrators claim] possible, except in the case of as many things as follow one another, as propria. **[ii]** It has been proved that when one thing is assumed, never is it necessary for something different to be the case (by one thing I mean

> neither one term nor one posit), two posits are the first and fewest though which it is possible, if at all, to syllogistically deduce [a conclusion]. **[iii]** But if A should follow B and C, and these things [i.e. B and C] should follow one another and A, in this way it is possible to prove all the postulates from one another in the first-figure, as was shown in our [writings] on syllogism [i.e. *Posterior Analytics* II.5]. And it was also shown that in the case of the other figures, either there is no syllogism or none concerning the assumptions. **[iv]** But in the case of things which are not counter-predicated, in no way is it possible to prove [everything] in a circle. **[v]** So because there are few [counter-predicated items] in demonstrative-proofs, **[vi]** it is clear that it is empty and impossible to claim that demonstration [of principles] is from one another, and that because of this there can be demonstration of everything.

The Circular Demonstrators have endorsed a picture on which 'all the postulates [i.e. first-principles of a science] are proved from one another', on which 'everything can be proved in a circle'. Aristotle contends that in fact this is only possible in a very limited range of cases. In **[ii]**, Aristotle points us back to his investigations of the *Prior Analytics*. And **[iii]-[iv]** point us specifically towards a result concerning demonstrating items 'in a circle and from one another' Aristotle claims to have established in *Prior Analytics* II.5-7. The result to which Aristotle refers amounts to a bi-conditional—roughly: one can prove some propositions in a circle from one another iff their terms all counter-predicate. With back reference to *Prior Analytics* II.5 (73a14), **[iii]** states the right-to-left direction of this bi-conditional. And **[iv]**, recapitulating **[i]**, asserts the left-to-right side of the biconditional. Given the left-to-right side, Aristotle thinks he's shown that demonstrating items in a circle and from one another demands some kind of ubiquitous counter-predication. But in **[v]** Aristotle asserts that in demonstrations there isn't this kind of ubiquitous counter-predication. He concludes in **[vi]** that not only are the Circular Demonstrators incorrect—in fact what they are saying is 'empty and impossible'.

To adequately approach this argument we will need to get clearer on several points. First we will have to give a more precise characterization of the view Aristotle takes himself to be arguing against. And relatedly, we will have to determine what result Aristotle thinks he's proved in *Prior Analytics* II.5-7, and how exactly he thinks the result helps him in his debate with the Circular Demonstrators. I would like to begin, however, by introducing and arguing against an competing approach to these questions recently articulated by Marko Malink in his 2013 'Aristotle on Circular Proof'.

3.5.1 Weak-circularity

Concerning *Prior Analytics* II.5 and *Posterior Analytics* I.3, Malink and I differ on numerous points of interpretive detail. But our most fundamental disagreements concern (i) the view we take Aristotle to be arguing against ('all the postulates are proved [in a circle and] from one another' (73a12-13)), and (ii) the question of what technical result Aristotle means to use in his I.3 attack on the view. Malink has assumed that (according to Aristotle) the Circular Demonstrators claim that every scientific principle can be proved using some scientific principles. Expressing the point our framework, the Circular Demonstrators would then be claiming that for every principle φ of a science, there is a collection of principles Γ such that $\Gamma \vdash_D \varphi$. Now, in proof-systems for contemporary logics, there is often an 'identity rule' licensing inferences: $\frac{\varphi}{\varphi}$. Given the presence of such a rule, it will trivially hold of any collection of sentences Π that for every φ in Π: there's a subset Γ of Π such that $\Gamma \vdash \varphi$. But as Malink rightly points out, in Aristotelian syllogistic—which licenses no such identity inference—the condition is not always trivially satisfied. Consider, for instance the following collection of syllogistic propositions:

· Π = {AaB, BaC, AaC}

It is not the case that for every φ in Π there is a syllogistic-deduction of φ that uses only propositions in Π. It will be helpful here to introduce a definition.

Definition 3 Weakly-circular set

Let Π be a collection of syllogistic propositions. Then:

· Π is *weakly-circular* iff for all φ in Π there's a subset Γ of Π such that: $\Gamma \vdash_D \varphi$

Using **Definition 3**, we can state Malink's view as follows: according to Aristotle, the Circular Demonstrators hold that the principles of a science form a weakly-circular set. I call the property above 'weak-circularity' in anticipation of what will follow. Ultimately, I will be arguing that it is in some stronger sense that Aristotle takes the Circular Demonstrators to hold 'all the postulates are demonstrated from one another' (73a12-13). But first, I'd like explain my dissatisfaction with Malink's proposal. I'll begin with some tractable worries, and work up to objections I take to be decisive.

Recall the strategy of the Circular Demonstrators in responding to the Ad Infinitum Skeptics. The Ad Infinitum Skeptics claim that when I attempt to demonstrate some φ, I'll need to posit some further claims Γ_0 that will themselves have to be demonstrated from some further claims Γ_1. But all the claims in Γ_1 will also have to be demonstrated from even further claims Γ_2, which will have to be proved from further

claims Γ_3, and so on ad infinitum. On this view, there can be no episteme of φ because there can be no complete demonstration of φ. And there can be no complete demonstration of φ because there are infinitely many further truths that would have to be demonstrated to produce a complete demonstration of φ. So demonstrating φ would require performing an infinite task as impossible as counting to infinity by beginning with a finite number and successively adding one. The Circular Demonstrators agree with the Ad Infinitum Skeptics that every claim used in a demonstration must itself be demonstrated. But they claim that items can be demonstrated through one another, with the result that the process of demonstration construction can reach a stopping point in which everything has been demonstrated and there is no need to introduce further truths. Let's look at a syllogistic example coming from *Prior Analytics* II.5. Suppose I'm trying to prove some φ and I've led my proof up to a principle AaC. Suppose in fact that {AaB, BaA, BaC, CaB, AaC, CaA} are principles. I may proceed as follows. I begin with:

$$\dfrac{\text{AaB} \quad \text{BaC}}{\text{AaC}}$$
$$\vdots$$
$$\varphi$$

But in order to have episteme of the conclusion I'll need demonstrations of AaB and BaC. So I can continue:

$$\dfrac{\dfrac{\text{AaC} \quad \text{CaB}}{\text{AaB}} \quad \dfrac{\text{BaA} \quad \text{AaC}}{\text{BaC}}}{\text{AaC}}$$
$$\vdots$$
$$\varphi$$

now demonstrating AaC and AaB through one another and AaC and BaC through one another. And eventually I build:

$$\dfrac{\text{AaC} \quad \dfrac{\dfrac{\text{CaB} \quad \text{BaA}}{\text{CaA}}}{\text{AaB}} \quad \dfrac{\dfrac{\text{BaC} \quad \text{CaA}}{\text{BaA}} \quad \text{AaC}}{\text{BaC}}}{\text{AaC}}$$
$$\vdots$$
$$\varphi$$

But now I can stop. For, this is a syllogistic-deduction which concludes AaC from certain premises. But, as the Circular Demonstrators will point out, none of the premises used is unproved. Every premise that gets used is itself also established in the syllogistic-deduction. This is how the possibility of demonstrating 'in a circle and from one another' is supposed to help with the possibility of regress.

Say we grant Malink that (according to Aristotle) the Circular Demonstrators claim that the principles of every science constitute

a weakly-circular set. Weak-circularity *per se* will not give the Circular Demonstrators the epistemological result they want. For consider an a-dense collection

· $\Theta = \{X_i a X_k : i < k \text{ and } i, k \in \mathbb{Q}\}$

Since between every two rational numbers i, k there is a further rational number j, it is clear that Θ is weakly-circular. For, whenever we have rational numbers i, j, k such that $i < j < k$, there will be a syllogism:

$$\frac{X_i a X_j \qquad X_j a X_k}{X_i a X_k}$$

Yet while Θ is weakly circular, no two sentences in Θ can be proved in a circle and from one another. If one tries to improve upon the demonstration of $X_i a X_k$, there will be no stopping point.

So relative to a weakly-circular set, it may not be possible to prove any propositions in a circle and from one another. As such, weak-circularity per se doesn't offer the Circular Demonstrators a solution to the problem with which they began. This is a difficulty for Malink's interpretation, but not a fatal one. For he may, and effectively does, claim that the Circular Demonstrators in fact hold two theses about the principles of a science: (i) that they constitute a weakly-circular set, and (ii) that they contain no infinite chains of a-propositions that use infinitely many distinct terms:

· $\{Y a X_0, X_0 a X_1, X_1 a X_2, ..., X_n a X_{n+1}, ...\}$

· $\{X_0 a Y, X_1 a X_0, X_2 a X_1, ..., X_{n+1} a X_n, ...\}$

This in fact does secure the epistemic result the Circular Demonstrators are looking for. One might worry that the reinterpreting move here is ad hoc. In responding to the Ad Infinitum Skeptics in *Posterior Analytics* I.19-22, Aristotle does argue against scientific principles forming infinite chains of the above sort. But nothing in either *Posterior Analytics* I.3 or *Prior Analytics* II.5-7 really indicates that Aristotle's response to the Circular Demonstrators is supposed to depend on this rejection—that Aristotle thinks his third argument requires a premise only stated and defended in *Posterior Analytics* I.22.[28] Be this as it may, let's consider the result of attributing to the Circular Demonstrators both (i) and (ii).

So, suppose we follow Malink in thinking that (by Aristotle's lights) the Circular Demonstrators' position consists of two theses: (i) that the principles of every science constitute a weakly-circular set, and (ii) that no science has infinite chains of a-propositions among its principles. How, then, will Aristotle's third argument in *Posterior Analytics* I.3 against the Circular Demonstrators go?

It's clear enough that Aristotle is arguing on the basis of something from *Prior Analytics* II.5 that the Circular Demonstrators view is an 'impossible' model for science. He thinks he's shown as a matter of pure *logic*—apart from any considerations about epistemic priority—that what the Circular Demonstrators are describing could only work in a very limited range of cases. In particular, he claims to have established that

[28]Indeed, Aristotle back reference to I.3 at 90b25 is reasonably read as indicating that Aristotle's case against the Circular Demonstrators is not supposed to depend on *Posterior Analytics* I.19-22.

Circular Demonstrators' view would require some kind of ubiquitous counter-predication. But in fact, says Aristotle, there is no such ubiquitous counter-predication of the relevant sort in scientific demonstrations. In interpreting *Posterior Analytics* I.3, Malink suggests that the logical result Aristotle has in mind is the following:

Fact 3

Suppose that no single-inference circular arguments are syllogistically valid. If Π is weakly-circular and contains no infinite chains of a-propositions, then at least distinct 2 terms used in Π counter-predicate$_\Pi$ with one another.

In order to prove any connection between weakly-circular premise sets and counter-predication, Malink (2013) is forced to consider a proof system weaker than those we've been discussing here. In particular, he reads Aristotle's argument as resting on the assumption that no single-inference circular arguments, for instance $\frac{\text{AaB} \quad \text{BaB}}{\text{AaB}}$ or $\frac{\text{AaA} \quad \text{AiB}}{\text{AiB}}$, are syllogistically valid. For suppose we allow single-inference arguments with their conclusion identical with a premise. Then Π = {AeB, BaB} is weakly circular. For each proposition in Π can be syllogistically derived using other propositions in Π as witnessed by the following:

$$\frac{\text{AeB} \quad \text{BaB}}{\text{AeB}} \qquad\qquad \frac{\text{BaB} \quad \text{BaB}}{\text{BaB}}$$

Note that no distinct terms in Π counter-predicate relative to Π.[29] Malink's 2013 article gives a successful reductio argument for **Fact 3**. In the appendix below, I offer a full characterization of weak-circularity in a proof system without single-inference circular arguments.[30] And I prove a slightly stronger result on this basis:

Fact 4

Suppose that no single-inference circular arguments are syllogistically valid. If Π is weakly-circular and contains no infinite chains of a-propositions, then (relative to Π) at least 3 terms used in Π counter-predicate with one another.

Let us, then, consider Malink's proposal that Aristotle is arguing against an opponent who holds that the principles of a science form a weakly-circular set, and that Aristotle is appealing to something like **Facts 3-4** in *Posterior Analytics* I.3 to attack the view. I will raise three objections to the proposal that I take to be decisive.

[29] I.e. if every proposition in Π is true, it does not logically follow that two distinct terms counter-predicate (recall **Definition 1** above).

[30] In particular, I show how weak circularity can be captured in terms of a counter-predication condition.

My first worry is more philosophical than textual. If something like the above were what Aristotle had in mind, his final argument against the Circular Demonstrators looks like a washout. Note firstly that for an appeal to **Fact 3** or **4** to get any traction, the Circular Demonstrators will have to accept the illegitimacy (or scientific illegitimacy) of single-inference circular arguments. But why should the Circular Demonstrators grant this? Perhaps a case could be made. But Aristotle isn't making it. And I see nothing in either *Posterior Analytics* I.3 or *Prior Analytics* II.5-7 that suggests some presupposition against single-inference circular arguments is in force. In any case, even if the Circular Demonstrators are willing to grant so much, Aristotle's objection would still be very weak. For what great problem is there in having 2 or 3 terms in a science counter-predicate with one another? Aristotle himself thinks that a science will have to contain certain a-propositions that express definitions. But on Aristotle's understanding of such definitions, definiens and definiendum will be distinct and they will counter-predicate. Aristotle also grants that most or even all sciences will contain some non-definitional propria terms that will counter-predicate with their subjects. Any science with names for two propria of a single named subject will have three counter-predicating terms; any science that contains a definable entity with one proprium will have three counter-predicating terms. If Aristotle's final objection to the Circular Demonstrators is simply that on their view every science will have to contain 2 or 3 mutually counter-predicating terms, he hasn't come close to showing their view 'impossible' and 'empty'. Given the same assumptional base as **Facts 3-4** it can be proved that weak-circularity does not require any more than three mutually counter-predicating terms. Indeed, in our appendix we show:

Fact 5

For every $n \geq 0$ there's a weakly-circular Π with 3 terms that counter-predicate$_\Pi$ with each other, and n terms that do not counter-predicate$_\Pi$ with any other terms in Π

If it were weak-circularity along with **Fact 3** or **4** that were on Aristotle's mind at the end of *Posterior Analytics* I.3, he'd be complaining that the Circular Demonstrators are committed to something not obviously false. For the argument to show their view 'impossible' and 'empty' Aristotle would need to make a case that at least some sciences cannot have three mutually counter-predicating terms. Aristotle never does argue anything like this. And I have a hard time seeing how he would.

My final two worries about Malink's proposal are more textual in character. The first concerns *Posterior Analytics* I.3. Aristotle says that the Circular Demonstrators' thesis could only work in the case of items that counter-predicate (73a7), that 'in the case of things which are not counter-predicated, in no way is it possible to prove [everything] in a circle' (73a16-18). Aristotle certainly doesn't seem to be saying that the Circular Demonstrators are committed to just 2 or 3 mutually counter-predicating terms. In fact, it would be very

strange if Aristotle did mean this. For he goes on to say that in fact 'in demonstrative-proofs such things [= counter-predicating terms] are few' (73a17-18), and that because there are but few counter-predicating terms in demonstrative-proofs that the Circular Demonstrator's thesis is impossible. But surely two or three terms is a few, and on Malink's reading this is all the Circular Demonstrators are committed to.[31]

Finally, there is the issue of *Prior Analytics* II.5. When Aristotle says 'in the case of things which are not counter-predicated, in no way is it possible to prove [everything] in a circle' (73a12-13), this is supposed to recapitulate or develop something from *Prior Analytics* II.5. Malink and I agree about which piece of text in II.5 that Aristotle is referring his readers back to in his final argument against the Circular Demonstrators: 57b32-58a15.[32] But as Malink admits, Aristotle neither proves nor even states **Fact 3** or **4** at II.5 57b32-58a15. Malink remarks that Aristotle could have been aware of his **Fact 3**, and indeed have seen how to prove it. I agree. But on Malink's own reading II.5 57b32-58a15 doesn't even contain the rudiments of a proof for **Fact 3**. More generally, I see no evidence that weak-circularity is at issue anywhere in II.5-7.

In what follows, I develop an alternative to Malink's account of both (i) the thesis Aristotle takes the Circular Demonstrators to hold, and (ii) the meta-logical result Aristotle is bringing to bear in the final argument of *Posterior Analytics* I.3. My reading better matches the text of I.3 than Malink's. And it has Aristotle providing a far stronger case against the Circular Demonstrators than on Malink's alternative. Finally, the meta-logical result I take Aristotle to be appealing to in I.3 is one that falls directly out of a result Aristotle himself argues for in the *Prior Analytics* II.5 passage referred to by *Posterior Analytics* I.3.

3.5.2 Stronger forms of circularity

According to Aristotle, the Circular Demonstrators maintain that 'all the postulates can be proved from one another' (I.3 73a12-13). In context, this 'from one another' is elliptical for 'in a circle and from one another'. And as Aristotle explains in the opening to *Prior Analytics* II.5, to prove φ and ψ in a circle and from one another is to produce syllogisms $\Sigma_1 : \Gamma, \varphi \vdash_D \psi$ and $\Sigma_2 : \Delta, \psi \vdash_D \varphi$. It will be convenient to introduce some short hand for 'proving in a circle and from one another'.

Definition 4 Inter-derivable propositions

Let Θ be a collection of syllogistic propositions with subsets Γ, Δ. Then:

- φ, ψ are *inter-derivable* relative to Θ iff $\Gamma, \varphi \vdash_D \psi$ and $\Delta, \psi \vdash_D \varphi$

As an example let $\Theta = \{$AaD, BaF, FaC, GaA, DaG, CaB, JeK, LiM$\}$. Then relative to Θ, AeB and DeC are inter-derivable. This may be shown by adducing the following:

[31] *Metaphysics* Iota 6 says that strictly speaking we shouldn't call one 'few', that strictly speaking two is the first quantity we should call 'few' because a few must be a plurality and one is not a plurality.

[32] The textual parallels are very close; there are really no other plausible candidates in II.5-7.

$$\dfrac{\dfrac{\text{AaD}\quad\dfrac{\text{AeB}\quad\text{BaF}}{\text{AeF}}}{\text{DeF}}\qquad\text{FaC}}{\text{DeC}} \qquad\qquad \dfrac{\dfrac{\text{GaA}\quad\dfrac{\text{DaG}\quad\text{DeC}}{\text{GeC}}}{\text{AeC}}\qquad\text{CaB}}{\text{AeB}}$$

Call the syllogistic-deduction of the left Σ_L and that on the right Σ_R. Note that Σ_L uses AeB as a premise to derive DeC, while Σ_R uses DeC as a premise to derive AeB. Setting Γ = {AaD, BaF, FaC}, Δ = {GaA, DaG, CaB}, φ = AeB, ψ = DeC we have the pattern of **Definition 4**.

For ease of expression, let's introduce one further stipulation. In the context of this chapter's discussion of the Circular Demonstrators, when we speak of propositions being inter-derivable we will always ignore the distinction between propositions that are inter-derivable through genuine demonstrations, and those that are inter-derivable only in sound syllogistic-deductions. This will allow for far less awkward prose, and in the present context will prove harmless.

An upshot of our discussion in **Section 3.5.1** above is that weak-circularity does a poor job of capturing the reciprocal ('from one another') aspect of the Circular Demonstrators' thesis. Recall that a non-empty set of propositions Π is weakly-circular exactly if it is possible to prove every proposition in Π using only propositions in Π. But, as we've seen, this might well hold without there being any two propositions in Π that are inter-derivable.

I submit that when Aristotle attributes to the Circular Demonstrators the view that 'all the postulates can be proved from one another' (I.3 73a12-13) he has something stronger than weak-circularity in mind. It seems to me that 'all the postulates can be proved from one another' is most plausibly read in one of two ways. Firstly it might be taken to mean that all the principles of a science are inter-derivable with one another. Secondly it might be taken to mean that every principle in a demonstration can be demonstrated using only principles with which it is inter-derivable. This gives rise to two further kinds of circularity to compare with weak-circularity:[33]

Definition 5

Where Π is a non-empty collection of syllogistic propositions:

1. Π is *strongly-circular* iff every two propositions in Π are inter-derivable relative to Π
2. Π is *mildly-circular* iff for every φ in Π there's subset Γ of Π such that (i) $\Gamma \vdash_D \varphi$, and (ii) every ψ in Γ is inter-derivable with φ relative to Π
3. Π is *weakly-circular* iff for every φ in Π there's a subset Γ of Π such that: $\Gamma \vdash_D \varphi$

[33]NB that the definition of weak-circularity given here is identical with that provided above. I repeat it for ease of reference.

Note that every strongly-circular set is a mildly-circular set (but not vice versa). And every mildly-circular set is a weakly-circular set (but not vice versa). The claim that the principles of a science form a strongly-circular set corresponds to the first interpretation of 'all the postulates...' canvassed above—that on which all of a science's principles are inter-derivable with one another. The claim that the principles of a science form a mildly-circular set corresponds to the second interpretation of 'all the postulates...' canvassed above—that on which every principle φ in a given demonstration is demonstrated from other principles which which φ is inter-derivable.

Let's see some examples illustrating **Definition 5**. Consider the set Θ = {AaB, BaC, CaA}. This is a strongly-circular set. For consider the following syllogistic-deductions showing that each proposition in Θ is inter-derivable with every other proposition in Θ:

$$\dfrac{\dfrac{\dfrac{\text{AaB}\quad\text{BaC}}{\text{AaC}}\quad\dfrac{\text{CaA}\quad\text{AaB}}{\text{CaB}}}{\text{AaB}}}{}$$

$$\dfrac{\dfrac{\dfrac{\text{BaC}\quad\text{CaA}}{\text{BaA}}\quad\dfrac{\text{AaB}\quad\text{BaC}}{\text{AaC}}}{\text{BaC}}}{}\qquad\qquad\dfrac{\dfrac{\dfrac{\text{CaA}\quad\text{AaB}}{\text{CaB}}\quad\dfrac{\text{BaC}\quad\text{CaA}}{\text{BaA}}}{\text{CaA}}}{}$$

Since Θ is strongly-circular, it is mildly- and weakly-circular as well. But consider Θ' = {AaB, BaC, CaA, XaX}. Θ' is not strongly-circular. For, not every proposition in Θ' is inter-derivable relative to Θ'. For instance, XaX and AaB are not inter-derivable relative to Θ'. Pace Malink I do not think that the argument of *Posterior Analytics* I.3 requires the Circular Demonstrators to reject single-inference circular arguments. Indeed, our turnstile '$\vdash_D$' as defined above allows for them. We accept $\dfrac{\text{XaX}\quad\text{XaX}}{\text{XaX}}$ as syllogistic-deduction. As such, given the three syllogistic-deductions above it is clear that Θ' = {AaB, BaC, CaA, XaX} is mildly-circular. For every proposition in Θ' can be proved, and every proposition can be proved through propositions with which it is inter-derivable. Since Θ' is mildly-circular it is weakly-circular as well. But now consider Θ'' = {AaB, BaC, CaA, DeA}. Θ'' is weakly-circular. For in addition to the syllogisms adduced above, we have:

$$\dfrac{\dfrac{\dfrac{\text{DeA}\quad\text{AaB}}{\text{DeB}}\quad\text{BaC}}{\text{DeC}}\quad\text{CaA}}{\text{DeA}}$$

Yet Θ'' is not mildly-circular. For it is impossible to use DeA to prove any of the other propositions in Θ''. But these propositions must be used to prove DeA. Relative to Θ'', DeA cannot be proved using only propositions with which it is inter-derivable.

3.5.3 That the Circular Demonstrators' thesis is 'impossible'

In addition to Malink's proposal, we now have two further ways to interpret what Aristotle takes the Circular Demonstrators to hold:

Option 1 For the Circular Demonstrators, the principles of a science form a strongly-circular set

Option 2 For the Circular Demonstrators, the principles of a science form a mildly-circular set

How would Aristotle's third argument against the Circular Demonstrators go if he were understanding their position in one of these ways? On Malink's reading, Aristotle's argument depends on two additional assumptions independent of their view that the principles of a science form a weakly-circular set: (i) that no science contains infinite a-chains among its principles, and (ii) that no single-inference circular argument is syllogistically valid. But on **Options 1** and **2**, some rather severe absurdities follow without any such additional assumptions. For instance, consider the following:

Fact 6

1. If Π is mildly-circular, then every proposition in Π is an a-proposition.
2. If Π is strongly-circular, then every proposition in Π is an a-proposition.

It would be easy for Aristotle to see either **Fact 6.1** or **Fact 6.2**. For they follow immediately from a handful of correct meta-logical observations about his syllogistic that he himself emphasizes in *Prior Analytics* I.24, and uses throughout the *Analytics*—and in *Prior Analytics* II.5-7 in particular. The observations in question concern patterns of proposition quantity and quality that can arise in syllogistic. And since we ourselves will make frequent use of this material it is worth making it explicit.

Laws of Quantity and Quality

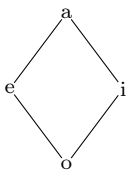
1. A syllogism proving a universal conclusion asserts only universal premises
2. A syllogism proving a positive conclusion asserts only positive premises
3. A syllogism proving a negative conclusion asserts exactly one negative premise
4. A syllogism proving a particular conclusion asserts no more than one particular premise

Although formulated in terms of syllogisms, the above laws hold for syllogistic-deductions (in our sense) as well.[34] In our formal framework this is readily shown by induction on proof length.

To exercise our grasp of the above four laws, let's see how they can be used to quickly prove **Facts 6.1-2**. Note first that every strongly-circular set is mildly-circular. So we can consider an arbitrary mildly-circular set to prove both **6.1** and **6.2**.[35] So let Π be mildly-circular, and let ψ be any proposition in Π. We need to show that ψ is an a-proposition. As Π is mildly circular, ψ must be the conclusion of some syllogistic-deduction Σ all of whose premises are inter-derivable with ψ. We claim that ψ cannot be particular. For, every syllogistic-deduction uses at least two premises, so by Law 4 it follows that Σ uses at least one universal premise. Call this universal premise φ. By Law 1, one cannot prove a universal conclusion using any particular premises. But we've assumed that ψ is inter-derivable with every premise of Σ, where φ is a premise of Σ that is universal. Hence, ψ must be universal. Moreover, ψ must be positive. For suppose that ψ is negative. Since every syllogistic-deduction uses at least two premises, by Law 3 Σ must have a positive premise, call it χ. Given Law 2 one cannot prove a positive conclusion from a negative premise, but we're assuming that χ can be proved with ψ. So ψ must be positive in addition to being universal. So it follows that ψ is an a-proposition.

For someone who has spent as much time with syllogistic as Aristotle, the above the proof is quick and easy. And given **Fact 6**, some substantive limitations of the Circular Demonstrators' view already follow. For note Law 3 above. To prove any negative conclusion, a syllogistic-deduction must assert a negative premise. But **Fact 6** says if the principles of a science are strongly- or mildly-circular, no principle will be negative. So if the principles constitute a strongly- or mildly-circular set, no negative propositions can be demonstrated by the science. Suppose that someone conjectures or defends some positive proposition φ relevant to science X that turns out to be false. If the principles of X are mildly- or strongly-circular, it will be logically impossible to *demonstrate* that φ is false. And this looks like a problem. For surely, in certain

[34] In learning Aristotelian syllogistic, many of us found the traditional square of opposition helpful. When trying to recall Aristotle's Laws of Quality and Quantity, one can think of a lattice:



For nodes x, y in the lattice let $x \geq y$ iff either (i) $x = y$, or (ii) x is above y. Then Laws of Quality and Quantity can be remembered in terms of:

Rule 1 An x-proposition can be used to prove an y-proposition exactly if $x \geq y$

Rule 2 A syllogism can have more than one x-premise exactly if $x \geq$a

[35] Note that an independent argument for **Fact 6.2** would be nearly identical with that given here for **Fact 6.1**; it's just as easy to prove **Fact 6.2** without showing **Fact 6.1**.

cases someone with episteme should be able to demonstrate something false is false.

So in syllogistic it follows that if the principles of a science are strongly- or mildly-circular, the science can never (as Aristotle would put it) prove of what's not, that it is not. If Aristotle interprets the Circular Demonstrators in line with **Option 1** or **2** above, he'll see this as an embarrassing consequence of their view. But this is not yet the most damning absurdity. And it's not the exact absurdity Aristotle has in mind in the final argument of *Posterior Analytics* I.3. For recall that the objection there concerns counter-predication. Aristotle thinks he's proved in *Prior Analytics* II.5 that the thesis of the Circular Demonstrators requires some kind of ubiquitous counter-predication. But in fact, he says, demonstrative-proofs will only contain a 'few' mutually counter-predicating terms. In **Section 3.5.5** below, I'll argue that if we read *Prior Analytics* II.5 57b32-58a15—the passage that Aristotle's final argument in *Posterior Analytics* I.3 refers us back to—in the context of Aristotle's **Laws of Quality and Quantity**, we can discern in that text a proof-sketch targeting a result from which the following two facts immediately follow:

Fact 7

If Π is strongly-circular and every member of Π is true, then all distinct terms used in Π counter-predicate with one another.

Fact 8

If Π is mildly-circular and every member of Π is true, then the two terms of every proposition in Π counter-predicate.

I submit that Aristotle is interpreting the Circular Demonstrators in line with either **Option 1** or **2** described above. And I contend that the consequence of *Prior Analytics* II.5 that Aristotle is appealing to in his final argument against the Circular Demonstrators is either **Fact 7** or **Fact 8**. The more natural reading of *Posterior Analytics* I.3 seems to be the one on which Aristotle is attacking an opponent who claims the principles of a science are strongly-circular (**Option 1**), rather than an opponent who claims that the principles of a science are mildly-circular (**Option 2**).[36] But I also think that both interpretive hypotheses give a philosophically and textually stronger reading of *Posterior Analytics* I.3 than Malink's. And moreover, I'll argue that both **Fact 7** and **Fact 8** easily fall out of *Prior Analytics* II.5.

So let's have a look at the absurdities involving counter-predication that the Circular Demonstrators would be committed to on both **Option 1** and on **Option 2**. Since every strongly-circular set is mildly-circular, let's begin with examining **Option 2**. For whatever absurdities follow from thinking the principles of a science mildly-circular,will also follow from thinking the principles of a science strongly-circular.

[36]This seems to be the easier reading of the Greek at 73a12-13: 'all the postulates can be proved from one another'.

Suppose that the principles of a science X are mildly-circular. The Circular Demonstrators will surely agree with Aristotle that every scientific principle must be a truth. So combining **Fact 6** and **Fact 8** it follows that every principle of science X is an a-proposition that links two terms that counter-predicate. We've seen that **Fact 6** puts a constraint on what demonstrations in science X can say about the beings that they talk about. No principle nor theorem of science X can be negative. More damningly, **Fact 8** imposes limitations on the range of beings that any demonstration in science X could talk about at all. For by **Fact 8**, the two terms of every principle in science X will counter-predicate with one another. But suppose AaB is a principle in science X. In Aristotelian logic, AaB can only syllogistically combine with another principle φ if φ has one of {A,B} as its subject or predicate. But since science X is mildly-circular and φ is itself a principle, if φ uses a term distinct from A and B, that term must also counter-predicate with A and with B. Generalizing the point, it follows that every demonstration in science X can only use terms that all counter-predicate with one another.

We call terms X and Y *extensionally distinct* when they do not counter-predicate with one other—when, that is, it is false that both XaY and YaX. What the above remarks show is that a demonstration in a science whose principles are mildly-circular can never use two extensionally distinct terms. Demonstrations in such a science can never state relations between terms of different levels of generality (for instance, genera and species). And they can never establish connections between non-counter-predicating terms. This is absurd, thinks Aristotle, because 'in demonstrations such things [counter-predications] are few'. It obviously should not hold of demonstrations in general that all their terms counter-predicate.

All of this follows if the principles of a science are mildly-circular. Now, although no *single* proof of a science whose premises are mildly-circular can use two extensionally distinct terms, it is possible for such a science to contain separate proofs that collectively refer to extensionally distinct entities. Consider for instance Θ = {AaB, BaC, CaA, XaY, YaZ, ZaX} or, more simply, Θ' = {AaA, XaX}. Both Θ and Θ' are mildly-circular. And in each case, all their constituent propositions might be true without all the terms in the science counter-predicating. If every proposition in Θ is true, {A,B,C} counter-predicate with one another, and {X,Y,Z} counter-predicate with one another. But there is no necessity for any of {A,B,C} to counter-predicate with any of {X,Y,Z}. Of course, even if the terms in {A,B,C} were extensionally distinct from terms in {X,Y,Z}, no such distinctness claim could be proved in a science whose principles were a mildly-circular set. For this would require a demonstration of a negative, for instance AoX. But as we've seen, no mildly-circular science can prove any negatives.

Now, it is arguable that the most plausible reading of 'all the postulates can be proved from one another' is that on which the Circular Demonstrators are committed to the strong-circularity of a science's principles—the view that all of a science's principles are inter-derivable with one another. In this case the absurdities

above will also hold. For suppose that science X has principles that form a strongly-circular set. Every strongly-circular set is mildly-circular. So as before every principle, proof, and theorem in science X can only use terms that counter-predicate with one another. Indeed, as before, no negative propositions can be proved in science X. But if Aristotle in fact holds that the Circular Demonstrators think the principles of a science must form a strongly-circular set, an even greater absurdity results. For, given **Fact 7** it follows that every term in the science counter-predicates with every other. The science cannot contain two extensionally distinct terms. It can only talk about one extensional region of reality. And it can make no extensional distinctions within that region. If the principles of a science are strongly-circular, from an extensional perspective the science speaks only of one thing. Since negatives are inexpressible, in Parmenidean silence the science can never say of this one that it is not something else. Indeed, the episteme can only know of the one is that it is it itself.

3.5.4 That the Circular Demonstrators' thesis is 'empty'

Aristotle's third argument against the Circular Demonstrators concludes that 'it is empty and impossible to say that demonstration is from one another and on account of this there can be demonstrations of all things' (73a18-19). We have seen why Aristotle thinks that what the Circular Demonstrators say is impossible. Their model for science turns out to be hopeless. On account of deep connections between circular syllogistic-phenomena and circular (i.e counter-) predication, scientific proofs conforming to the Circular Demonstrator's claims can only talk about an incredibly limited range of structures. If every science has all its principles demonstrated from one another, no scientific proofs can provide explanations that appeal to extensionally distinct items; indeed no sciences can even represent what's extensionally distinct as distinct. And if, as I suspect, Aristotle ultimately interprets the Circular Demonstrators in line with strong-circularity, he will claim that on their view no science can even refer to two extensionally distinct portions of the real. In bringing our discussion of *Posterior Analytics* I.3 to a close, I'd like to make a suggestion about why Aristotle characterizes their claims as 'empty'.

In his recent 'Empty Words',[37] Sean Kelsey provides thematic study of Aristotle's accusation that a particular theory is 'empty' [*kenon*], or that certain theorists engage in 'empty talk' [*kenologein*, *legein kenōs*, etc.]. Drawing on texts from Aristotle's ethical and physical writings, Kelsey reveals a kind of systematic deployment of the accusation. For Aristotle, to say a theory is 'empty' is to charge its proponents with propounding a wildly erroneous view, arrived at without serious investigation of phenomena at issue. Those who engage in empty talk are often intellectually undisciplined persons who manifest a kind of lack of diligence in theorizing; they are pretenders to wisdom, talkers who lack sufficient experience [*emperia*] to

[37] Forthcoming in *Theory and Practice in Aristotle's Natural Science*, ed. Ebrey

speak authoritatively about what they claim to be authoritatively speaking about.

I'd like to suggest that when Aristotle charges that Circular Demonstrators' thesis is empty [*kenon*] (I.3 73a18), it is this kind of sentiment he is expressing. In the *Prior Analytics*, Aristotle claims to have a systematic account of 'how *all* syllogisms are generated' (I.4 25b25-6, I.26 43a15, cf. 47a2-3). Arguing for the universality of the system of figures and going through all the various premise pairs in each figure is hard painstaking work. On the basis of such work, Aristotle has a method for concretely investigating the possibility of various logical phenomena and the conditions under which various logical phenomena can arise. The Circular Demonstrators, of course, haven't done any of this. Before one has rigorously worked through logical theory and learned things like the **Laws of Quality and Quantity**, it may seem clever and plausible to argue that there is episteme of principles because principles are proved in a circle and from one another. But without a theory of syllogistic, such thinkers are in no position to even investigate what such a view of logico-epistemic structure would really amount to. The Circular Demonstrators set out trying to defend the in principle possibility of episteme. But they've unwittingly presented a picture on which most (or maybe even all) episteme is impossible. In propounding their view, the Circular Demonstrators are engaging in empty talk. If they really understood what they were saying, they wouldn't be saying it. To a serious logician who's investigated the matter, their account of episteme looks like joke.

3.5.5 The counter-predication result of *Prior Analytics* II.5

I've argued that Aristotle's attack on the Circular Demonstrators depends on a result like **Fact 7** or **8**, and not **Fact 3** as Malink has suggested. We've noted that Malink does not find an argument for **Fact 3** in the *Prior Analytics* II.5 57b32-58a15 text that Aristotle's final argument against the Circular Demonstrators refers back to. I have claimed that this passage does contain a proof-sketch, in fact a good proof-sketch, for a result that immediately will get you either **Fact 7** or **8**. Here, I explain my reading of *Prior Analytics* II.5 57b32-58a15.

In **Section 3.3** above, we quoted *Prior Analytics* 57b18-30, the opening of II.5 in which proving 'in a circle and from one another' is defined. We saw Aristotle to be discussing cases in which the conclusion of a syllogistic-deduction Σ can be used to establish one of the premises of Σ. For instance, suppose we have $\frac{\text{AaB} \quad \text{BaC}}{\text{AaC}}$. Then one might also use AaC to derive AaB: $\frac{\text{AaC} \quad \text{CaB}}{\text{AaB}}$. In the text that follows (57b32-58a15), Aristotle asks in what cases the conclusion of a syllogistic-deduction Σ can be used to derive *every* premise asserted in Σ. When, that is, is it possible to demonstrate a conclusion using only premises that are inter-derivable with it? Indeed, *can* it ever occur that $\Gamma \vdash_D \varphi$ such that for every ψ in Γ: $\Delta \vdash_D \psi$ for some Δ? Aristotle answers in the affirmative. And he gives an accurate account of precisely when this

is and isn't possible (57b32-58a15):

> **[i]** Now, in the case of [terms] which do not convert, the syllogism must come to be from one undemonstrated premise. **[ii]** For it is not possible to demonstrate through these premises that the third [term] belongs to the middle [term] or the middle [term] belongs to the first [term]. **[iii]** But in the case of terms which do convert, they can all be proved through each other, as for example if A, B, and C convert with each other. **[iv]** For let conclusion AC have been proved by middle B, and again AB by means of the conclusion together with premise BC converted, and likewise also BC by means of the conclusion and premise AB converted. Then both premises CB and BA need to be demonstrated, for these are the only [premises] we have used without demonstrating. Accordingly, if B is taken to belong to all C and C to all A, then there will be a syllogism of B in relation to A. Again, if C is taken to belong to all A and A to all B, then it is necessary for C to belong to all B. Now in both of these syllogisms, premise CA was taken without being demonstrated; for the others had been shown. Consequently, if we can demonstrate this premise, then all of the premises will have been shown through one another. Accordingly, if C is taken to belong to all B and B to all A, then both the premises taken have been demonstrated, and also it is necessary for C to belong to A. **[v]** It is clear, then, that in the case where there are only converting terms,[38] it is possible for demonstrations to come about in a circle and from one another; **[vi]** but in other cases [i.e. if we do not have only converting terms], it is as we said earlier [i.e. in **[i]**].

The first point to note is that context makes clear that by 'converting terms' Aristotle means 'counter-predicating terms'. The second point to note in this text is the '*en men...tois*'/'*en de tois ...*' contrast between **[i]** and **[iii]**. Effectively, **[i]** and **[iii]** articulate a bi-conditional:

- Where Σ is a syllogistic-deduction: every premise Σ is inter-derivable with Σ's conclusion iff every term used in Σ counter-predicates with every other relative to the premises of Σ

Text **[i]** states the left-to-right direction of the bi-conditional (in contrapositive form), and text **[iii]** states the right-to-left direction of the bi-conditional. Note further that **[ii]** offers telegraphic argument for **[i]**, and **[iv]** offers argument for **[iii]**. Finally, **[v]** gives a restatement of **[iii]** as proved, and **[vi]** gives a restatement of **[i]** as proved. The bi-conditional Aristotle states can in fact be proved. And I contend that Aristotle offers here a proof sketch for the bi-conditional which, although telegraphic and incomplete, shows him more or

[38] I'm translating reading *en monois tois antistrephousi* at 58a13 differently than Smith (1989) and Malink (2013). Strangely they take *monois* as a modifying *endechetai* at the end of the sentence. I don't see how this is possible.

less in possession of a successful proof. Note that the left-to-right direction of the bi-conditional immediately entails **Fact 8**.[39] And just slightly less immediately, left-to-right also entails **Fact 7**.[40]

Let's try to follow Aristotle's argument for the bi-conditional, beginning with his highly telegraphic case for **[i]**, the left-to-right direction. I want to suggest that Aristotle is relying on an unstated background assumption. In particular, I suggest that he takes for granted that every syllogism such that all of its premises are inter-derivable with its conclusion only asserts a-propositions. Aristotle would be entitled to assume so much. For it is in fact true, and follows easily from his Laws of Quantity and Quality articulated in *Prior Analytics* I.24.[41] Of course, for a proper proof of the bi-conditional Aristotle would have to make this assumption explicit, and even perhaps argue for it. But what we get in the text of II.5 isn't an attempt at a full proof of his target conclusion. It's just a proof-*sketch*. And when confronted with a proof-sketch we the readers are supposed to supply the missing steps.

So then, suppose we grant Aristotle the unstated assumption that every syllogism such that all of its premises are inter-derivable from its conclusion asserts only a-propositions. What argument for **[i]** will he then be providing in the text above? Recall that **[i]** is a conditional stating that if a syllogistic-deduction is such that its conclusion can be used to syllogistically-prove all its premises, then all the terms it uses must counter-predicate (relative to its premises). Let P represent the antecedent of this conditional, and let Q represent the consequent. I submit that Aristotle means to establish $P \to Q$ through something R which implies Q and is implied by P. Though Aristotle does not make the connection entirely explicit, I'll argue that text **[ii]** in fact correctly states such an R. So, while **[i]** states $P \to Q$, and **[ii]** just says 'for, [if $\neg Q$] then $\neg R$', what Aristotle means is that $P \to Q$ since $P \to R$ and $R \to Q$.

Tracing Aristotle's reasoning, let's begin by assuming P: i.e. that we have a syllogism Σ such that all of its premises are inter-derivable with its conclusion. Without loss of generality (more on this below), let's assume that our syllogism has just two premises. So we have three terms A, B, C. And granting Aristotle his observations in *Prior Analytics* I.24 (and reductio elimination), we assume syllogism of the form:

$$\Sigma \qquad \frac{\text{AaB} \qquad \text{BaC}}{\text{AaC}}$$

Suppose that $\neg Q$: i.e. that some of the terms {A,B,C} don't convert with one another. From the assumption

[39] For let $\varphi \in \Pi$. If Π is mildly-circular, there's a Σ to which we can apply the left-to-right direction of the bi-conditional and conclude that the terms used in Σ counter-predicate.

[40] To get **Fact 7** from the bi-conditional we just need to unpack the definition of strong-circularity and apply the bi-conditional left-to-right. Let Π be strongly-circular, and let X and Y be any terms used in Π. We show that X and Y counter-predicate$_\Pi$. Let $\varphi \in \Pi$ be a proposition in which X occurs, and ψ be one in which Y occurs. Since Π is strongly-circular, there's a $\Gamma \subseteq \Pi$ such that $\Gamma, \psi \vdash_D \varphi$ where for all $\chi \in \Gamma$: there's a $\Delta \subseteq \Pi$ such that $\Delta, \varphi \vdash_D \chi$. So we can apply left-to-right on any syllogistic-deduction witnessing $\Gamma, \psi \vdash_D \varphi$, and get that X and Y counter-predicate$_\Pi$ as $\Gamma \cup \{\psi\} \subseteq \Pi$.

[41] The argument would be nearly identical to that given for **Fact 6** above. Let Σ be such that all of its premises are inter-derivable with its conclusion ψ. It is impossible that ψ is particular since (by the Laws of Quantity and Quality) Σ must have a universal premise but (again by the Laws of Quantity and Quality) a particular cannot be asserted in the proof of a universal. It is also impossible that ψ is negative since (by the Laws of Quantity and Quality) Σ must have a positive premise but (again by the Laws of Quantity and Quality) a negative cannot be asserted in the proof of a positive. So ψ must be an a-proposition.

that $\neg Q$, what Aristotle says in **[ii]** follows: 'it is not possible to demonstrate through these terms (i) that the third belongs to the middle or (ii) that the middle belongs to the first'. This gives us our $\neg Q \rightarrow \neg R$. Why is $\neg Q \rightarrow \neg R$ the case? Well, the 'first' is A, the 'middle' is B, the 'third' is C. If we could demonstrate both that the third belongs to the middle and that the middle belongs to the first, we could demonstrate CaB and BaA. But in this case we could demonstrate CaA in addition to AaC so all of A,B,C would counter-predicate. So Aristotle correctly observes that $\neg Q \rightarrow \neg R$, i.e. $R \rightarrow Q$ (where R states that both CaB and BaA can be proved). And although Aristotle does not make the point explicit, he has stated an R such that $P \rightarrow R$ is also the case. For if all of Σ's premises can be demonstrated through the conclusion AaC, there will have to be demonstrations of CaB and of BaA. For, since we're assuming that AaC can be used to prove AaB and BaC, we know there will be:

- Σ_1 : Γ, AaC $\vdash_D$ AaB
- Σ_2 : Δ, AaC $\vdash_D$ BaC

But every syllogistic-deduction whose conclusion is an a-proposition has premises of the form: {XaY_0, $\mathrm{Y}_0\mathrm{aY}_1$, ... $\mathrm{Y}_{n-1}\mathrm{aY}_n$, $\mathrm{Y}_n\mathrm{aZ}$}.[42] So since AaC can be used to prove both AaB and BaC, these proofs must go through premises:

1. {AaC, CaY_0, $\mathrm{Y}_0\mathrm{aY}_1$, ..., $\mathrm{Y}_{n-1}\mathrm{aY}_n$, $\mathrm{Y}_n\mathrm{aB}$}
2. {BaY_0, $\mathrm{Y}_0\mathrm{aY}_1$, ... $\mathrm{Y}_{n-1}\mathrm{aY}_n$, $\mathrm{Y}_n\mathrm{aA}$, AaC}

If all the premises in 1. are true, it must be the case CaB ('the third belongs to the middle'). And if all the premises in 2. are true, it must be the case that the BaA ('the middle belongs to the first'). Given both CaB and BaA, we have that A, B, and C all counter-predicate with one another. In short, Aristotle sees that if AaC can be used to prove AaB, then the same premises entail CaB. Moreover, he sees that if AaC can be used to prove BaC, then the same premises entail BaA. He says that if any two of {A,B,C} do not convert, then we can't demonstrate both CaB and BaA. But if every premise in Σ can be proved using AaC, we'll have to be able to demonstrate both CaB, BaA. It follows that all of {A,B,C} will convert.

The argument above is successful. But one point needs flagging. At the start we assumed that the syllogism Σ, all of whose premises were inter-derivable through its conclusion, had just two premises. This seems to be what Aristotle himself does. In order to prove his result for all syllogistic-deductions whose premises are inter-derivable with their conclusions, needn't Aristotle account for instances of Σ with more than two premises? Strictly speaking, the answer is yes. But there's a sense in which Aristotle is licensed to proceed as he does—assuming that Σ has exactly two premises. Hence our rider above 'without loss of

[42] Cf. Lemma 1.1-2 in the appendix

generality'. The fact of the matter is that if we were to drop the assumption that Σ has exactly two premises, the proof of the result would go in exactly the same manner. For Σ will have at least one pair of premises of the form {JaK, KaL}. And by applying reasoning above just once to the case of {JaK, KaL} we will get: CaK and KaA which show CaA—the conversion of our original conclusion AaC. Now, Aristotle explicitly argues in *Prior Analytics* II.22 (67b27ff.), that if A and C convert, all middles to AaC will also convert both with each other and with A and C. So it will follow that if all the premises of Σ are true, then all the terms in Σ will convert with one another. What Aristotle establishes for the limited case where Σ has two premises, thus holds for the general case by just about the exact same argument. Hence, it is reasonable for him to consider only the two premise case without loss of generality. Though telegraphic and just a proof-sketch, what is apparently the intended argument is successful.

What about the other direction of the bi-conditional? What is Aristotle's argument that if every term used in Σ counter-predicates with every other relative to the premises of Σ, every premise in Σ is inter-derivable with its conclusion? To really prove this point, Aristotle would have to use something like mathematical induction. He's in no position to do so (though it can in fact be done without difficulty[43]). Instead he simply constructs a case where what he says does hold. Presumably we are supposed to observe (what is in fact true) that the result will hold in other cases as well.

It is **[iv]** that looks to show that if one has a collection of converting terms all premises can all be proved through each other. What Aristotle goes on to describe is either probably either a series of six syllogisms (in the following order):

(1) $\dfrac{\text{AaB} \quad \text{BaC}}{\text{AaC}}$ (2) $\dfrac{\text{AaC} \quad \text{CaB}}{\text{AaB}}$ (3) $\dfrac{\text{BaA} \quad \text{AaC}}{\text{BaC}}$

(4) $\dfrac{\text{BaC} \quad \text{CaA}}{\text{BaA}}$ (5) $\dfrac{\text{CaA} \quad \text{AaB}}{\text{CaB}}$ (6) $\dfrac{\text{CaB} \quad \text{BaA}}{\text{CaA}}$

or the complex syllogism:

$$\dfrac{\text{(2) AaC} \quad \dfrac{\text{(6) CaB} \quad \text{BaA}}{\text{(5) CaA} \quad \text{AaB}}}{\text{(1) AaB}} \quad \dfrac{\dfrac{\text{BaC} \quad \text{CaA}}{\text{BaA}}\text{ (4)} \quad \text{AaC}}{\text{BaC}}\text{ (3)}}{\text{AaC}}$$

one builds in steps (1)-(6). (NB that every premise used in the above complex syllogism is also established by a subproof in the same syllogism). From the above it is easy to see that every premise is inter-derivable with AaC. AaB and BaC are directly proved through AaC via syllogisms (2) and (3). But since CaB and BaA are proved through AaB and BaC respectively (syllogisms (4) and (5)), the former are also proved from

[43]Left as exercise to the reader.

AaC. Finally, since CaA is proved through CaB and BaA (syllogism (6)) which are proved through AaC, we have that CaA is as well.

Chapter 4

Demonstrating ad infinitum I: logical issues

4.1 Introduction

4.1.1 The possibility of episteme

Philosophical debates take place in contexts of shared assumption. The central commitment shared by Aristotle, the Circular Demonstrators, and the Ad Infinitum Skeptics is that having episteme of X is always matter of mastering some particular logos in connection with X. As noted in **Section 1**, a view of this kind is discussed in detail in the third movement of Plato's *Theaetetus*. On the 'Dream Theory' there, one converts true judgment of X into episteme of X by obtaining a logos that (somehow) articulates X in terms of its primitive 'elements' [*stoicheia*]. But for episteme of X, these primitives must, of course, themselves be grasped. And to avoid regress, the Dream Theorist 'most cleverly' (202d10) posits a sharp distinction between episteme and elemental cognition of primitives in themselves. According to the Dream Theorist, primitives as such do not have logoi and thus cannot be objects of episteme. When primitives are correctly cognized so as to be available for the understanding of complexes, primitives are grasped through some kind of perception [*aithēsis*] without an accompanying logos. Having rehearsed the Dream Theory, Socrates proceeds to develop a series of problems for the view. The first such problem he presents is a dilemma challenging the Dream Theorist's foundationalism about episteme and epistemic logoi by attacking his appeal to a cognitive asymmetry between the knower's (= *episthmōn*'s) grasp of primitives and her grasp of complexes (205d-e). Unless the Dream Theorist can defend some kind of sophisticated mereology—or, perhaps, present some

alternative model for understanding the relation between epistemic logoi and their primitives—he will have to concede that either (i) both primitives and complexes are objects of episteme and both knowable in the same sense, or (ii) *neither* primitives nor complexes are objects of episteme and both are unknowable in the same sense.[1] Let's call this *Dream Theorist's Dilemma*.

We've remarked that Aristotle defends a version of the Dream Theory on which the role played by *aithēsis* is filled by *nous*. He clearly thinks his suitably reformulated foundationalism about epistemic logoi survives the aporiai targeting the Dream Theory developed by the *Theaetetus*. Against the Dream Theory, the Ad Infinitum Skeptics and Circular Demonstrators insist that every piece of scientific understanding requires a logos and reject any distinction between the understanding of primitives and that of non-primitives (cf. 72b15-16). From Aristotle's perspective, the two groups are defending views about epistemic logoi that put them on opposite sides of Dream Theorist's Dilemma. For, the Circular Demonstrators have it that both primitives and non-primitives are possible objects of episteme, and both knowable in the same sense (as in (i) above). The Ad Infinitum Skeptics, in contrast, claim that neither non-primitives nor primitives are possible objects of episteme with the result that both are *un*knowable in the same sense (as in (ii) above). Like the Circular Demonstrators, the Ad Infinitum Skeptics contend that episteme must be through a logos (a demonstrative-proof). But dissenting from the Circular Demonstrators, they argue that in fact demonstrative-proof is impossible, that nothing is possible objects of episteme.[2]

Now, against the Ad Infinitum Skeptics, Aristotle agrees with the Circular Demonstrators that episteme is possible. But, as we've seen, he thinks their account of how episteme is in fact possible has disastrous consequences. The logical operation of 'proving in a circle and from one another' is completely incapable of doing the work that the Circular Demonstrators demand of it. In a science conforming to the view of the Circular Demonstrators, no demonstration could use more than one extensionally distinct term. Purely as a matter of logic, the Circular Demonstrators will have to deny that there can be episteme that (say) all triangles have external angles summing to 360 degrees, or that (say) moving does not hold of some mover. Unwittingly, then, the Circular Demonstrators are defending a view on which an enormous amount of content that they themselves presumably take to be *epistēton* is not *epistēton*. When this is combined with their mistaken ideas about causal priority, it sure seems that 'science' as described by the Circular Demonstrators cannot be science at all. Or so concludes Aristotle.

An important aim of the *Posterior Analytics* is to explain on what basis episteme in fact is possible. But there is no sense in seeking *how* something is possible before one is confident *that* it is possible at

[1] Socrates' development of the dilemma was discussed above in **Section 1.4-5**.

[2] The fact that I.3 72b5 introduces the debate as one about *prōta* rather than *archai* is, perhaps, a nod to the *Theaetetus*. It is not clear to what extent (if any) the Circular Demonstrators and the Ad Infinitum Skeptics were themselves influenced by the arguments against the Dream Theory in the *Theaetetus*.

all.[3] The Circular Demonstrators' ideas about demonstrations through demonstrable principles fails to substantiate the possibility of episteme. But Aristotle thinks he can do better. In particular, he proposes to account for the possibility of episteme by working out a suitably modified version of the Dream Theorist's foundationalism about episteme and epistemic logoi. Yet to truly establish the in principle possibility of episteme, Aristotle must confront the challenge of the Ad Infinitum Skeptics. For, these thinkers present a sophisticated argument against the in principle possibility of episteme. Here, again, is Aristotle's testimony on the Ad Infinitum Skeptics from *Posterior Analytics* I.3 (72b5-15):

> Some [= the Ad Infinitum Skeptics] do not think there is episteme since [it would] require having episteme of primitives; others [=the Circular Demonstrators] think that there is [episteme], but there is demonstration of everything [e.g. even the primitives]. Neither of them [the views] are true, nor necessary.
>
> The [Ad Infinitum Skeptics], supposing [*hypothemenoi*] that it is not possible to have episteme in another way [than through demonstration], claim that [demonstrators] are led up [*anagesthai*] into infinity, seeing that [knowers] would not have episteme of the posteriors on account of the priors if there are no primitives. They argue rightly. For, it is impossible to traverse [*dielthein*] infinitely many [things]. But, if [chains of explanatory priority] stop and there are first-principles [= primitives], these [they contend] are unknowable [*agnōstous*] since there is indeed no demonstration of them, which they say is the only [way of] having episteme. But if it is not possible to know the primitives, neither is it possible to have episteme of the things from them, without qualification [*haplōs*] and in the authoritative sense [*kuriōs*], rather [it is only possible to have qualified, i.e., pseudo-episteme] from a supposition [*ex hypotheseōs*]: 'If these things are so, then [such-and-such else is so]'.

The Ad Infinitum Skeptics deny that reality is scientifically intelligible—they deny, that is, that episteme is possible for any intellects. If episteme of any φ were possible, it would require a demonstrative-proof that correctly explains *why* φ is the case by appealing to the features of reality which are (per se) causally responsible for things being such that φ. In particular, the demonstrative-proof of φ would have to offer an appropriately *complete* explanation on the basis of primitive first-principles. But why think there are any such principles in the first place? The world might be otherwise: for instance, it could be such that every truth is partially explained by some distinct further truths. This is exactly what the Ad Infinitum Skeptics suggest. They propose that for every truth φ_n, there are some distinct truth(s) Γ_n explanatorily prior to

[3]Cf. *Posterior Analytics* II.1-2.

it. Accordingly, they suggest that the process of constructing a demonstration for a scientifically explicable truth can never successfully terminate. There can, then, no episteme because in principle epistemic logoi are impossible to produce.[4]

What argument do the Ad Infinitum Skeptics have for this conception of reality? How can they be sure that world's causal-explanatory chains have no foundations? Aristotle doesn't tell us that Ad Infinitum Skeptics do present arguments for this conclusion; he says, rather, that they 'hypothesize' that it is so.[5] Indeed, the Ad Infinitum Skeptics maintain that even if there turn out to be explanatorily primitive first-principles, episteme will still prove impossible. For suppose there are first-principles such that all scientifically explicable truths could be syllogized from them in finitely many steps. One does not get episteme of φ simply by memorizing a logos which happens to be a demonstration of φ. For a demonstration to give a thinker episteme of its conclusion, the thinker must to have episteme of the demonstration's premises. But this, the Ad Infinitum Skeptics claim, would require that the first-principles which are the demonstration's premises themselves be demonstrated. And insofar as the items in question are first-principles they cannot be demonstrable, and cannot be objects of episteme.

The Ad Infinitum Skeptics conclude that whether there are first-principles or not, episteme in the 'unqualified' and 'authoritative' sense is impossible. At best, they claim, thinkers can possess a kind of suppositional quasi-episteme of certain truths: episteme 'from a hypothesis'. Suppose I construct a syllogistic-proof of φ from truths Γ that are indeed explanatorily prior to φ. Provided all is well, it follows that if I *had* episteme of the truths in Γ, I *would* have episteme of φ. But whether or not the propositions in Γ are in fact first-principles, the Ad Infinitum Skeptics will argue that I cannot have episteme of them. In either case, the result of my constructing a proof of φ from Γ will not yield any substantive understanding of reality.

The basic form of the Ad Infinitum Skeptics' argument is simple. Either there are first-principles or not; but in either case episteme is not possible. To defend his foundationalism against the challenge of the Ad Infinitum, Aristotle will have to accomplish two tasks. First, he will have to establish that there are first-principles. More precisely, he will need to show that every demonstrable truth is fully grounded in a finite collection of explanatorily primitive truths (its principles), and can indeed be demonstrated from its principles in finitely many steps. Next he will have to show that although first-principles cannot be objects of episteme strictly speaking, first-principles are not unknowable [*agnōsta*] in every sense. In particular, Aristotle will have to argue that there is a knowing state even more perfect than episteme; that this state—

[4] A likely influence on the Ad Infinitum Skeptics is the notion of an unhypothetical principle from *Republic* VI. The Ad Infinitum Skeptics would be charging that there cannot be episteme since in demonstrating one never reaches unhypothetical principles, because there aren't any.

[5] There is undoubtably some connection between *hypothemenoi* in 72b8 and *ex hypotheseōs* in 72b15. What the connection is, is not clear. It is conceivable that 'hypothesizing' is part of the Ad Infinitum Skeptics' self-presentation—that they themselves describe the two horns of the 72b7-15 dilemma as proceeding from hypotheses. Alternatively, *hypothemenoi* in 72b8 might be Aristotle punning on their idea of episteme *ex hypotheseōs* .

which Aristotle will call *nous*—is not *meta logou*; and that a noetic understanding of first-principles is sufficient to ground demonstrative episteme which proceeds from first-principles.

The *Posterior Analytics* records a sophisticated and ingenious attempt to accomplish the first task in I.19-22. Aristotle characterizes *Posterior Analytics* I.19-22 negatively as showing that it is not possible for demonstration [*apodeixis*] to continue ad infinitum.[6] He characterizes I.19-22 positively as showing that every demonstration (*apodeixis* in the sense of demonstrative-process) 'terminates' or 'comes to a stop' [*histanai*], thus reaching first-principles [*archai*] or primitives [*prōta*].[7] Aristotle adds that this conclusion goes against 'what we said in the beginning that some people [*tinas*] claim' (84a32), referring us back to his presentation of the Ad Infinitum Skeptics in I.3.[8] Whether or not the Ad Infinitum Skeptics themselves had arguments against the existence of first-principles, Aristotle thinks he can prove them wrong by showing that there are first-principles on which episteme can be founded. And to accomplish his end he offers a truly remarkable analysis of the logical structure presupposed by non-terminating demonstrative processes. These final two chapters of the dissertation present a novel reading of I.19-22, focusing especially on the argument's logical aspects.

Concerning the second task described above—that of establishing the existence and sufficiency for episteme of non-demonstrative *nous*—Aristotle's *Analytics* gives us far less. It is arguable that *Posterior Analytics* I.33 provides some help here. *Posterior Analytics* II.19 explicitly asks 'what is the state that knows [first-principles]?' (99b18). The concluding paragraph of the chapter, and the *Analytics* as a whole, 100a5-16 asserts that the state in question is *nous* which—unlike episteme—is not *meta logou* (100a5-16). But it is not entirely clear what Aristotle's case for the possibility of *nous* as a kind of knowledge [*gnōsis*] fit to ground episteme is supposed to be. This is a vexed and important question, and not one usually addressed in the voluminous body of literature on *Posterior Analytics* II.19. But it is outside the scope of the present undertaking. Our aim here is simply to understand I.19-22.

4.1.2 Demonstrating ad infinitum in a syllogistic context

Commentators will sometimes say that the aim of *Posterior Analytics* I.19-22 is to show that every demonstrative-proof is finite, that no demonstrative-proof uses infinitely many premises.[9] This is incorrect. A demonstrative-proof is a logos. And both Aristotle and the Ad Infinitum Skeptics are in agreement that every logos is finitely long. What the Ad Infinitum Skeptics are suggesting is that the process of constructing a demonstrative-proof can never successfully terminate—that the *process* of demonstrating does not 'come to a stop' but

[6] 82a5-6, 84a32-33, 84a37-38.
[7] 84a30-32, 82a21-2, a36-37, b34-35.
[8] Cf. 82a6-8
[9] For instance Ross in his commentary on the *Analytics* (pg. 567ff.), McKirihan in his *Principles and Proofs: Aristotle's Demonstrative Science* (p 159, 150), Corcoran in 'Aristotle's Natural Deduction System' (1974).

rather 'continues ad infinitum'.[10] There cannot be epistemic logoi, since epistemic logoi in principle cannot be formulated. Aristotle disagrees. And he wants to show that every scientifically explicable truth is a possible object of episteme, because for every scientifically explicable truth a genuine demonstrative-proof from first-principles can be constructed in finitely many steps.

Our discussion of the Ad Infinitum Skeptics' reasoning in 4.1.1 above is abstract: it does not assume that demonstrative-proofs take any particular form. Such too is Aristotle's own presentation of Ad Infinitum Skeptics in *Posterior Analytics* I.3. Aristotle himself, of course, does envision demonstrative-proofs as possessing a particular logical structure. Indeed, Aristotle thinks that every possible proof will have to conform to his theory of syllogistic. Insofar as the Ad Infinitum Skeptics are talking about proofs at all, they must be inchoately talking about syllogisms that conform to the three figures.

We've already remarked on the well-known fact that Aristotle is wrong to think that his syllogistic is complete in the way he claims. But from the logician's perspective, what makes *Posterior Analytics* I.19-22 seriously interesting is that it presents a detailed *concrete* analysis of the possibility of demonstration-explanation in a syllogistic system continuing ad infinitum.

When the question of non-terminating proof-processes is studied in the context of Aristotelian syllogistic it acquires a peculiar character. In Aristotle's system, every proposition is composed of two terms and a copula (copulae: e.g. 'holds of all', 'holds of no', etc.). Syllogisms work to establish a copulative connection between the two 'extreme' terms A, B of the conclusion, by connecting them through a 'middle term' C. For instance:

$$\frac{\text{AaC} \quad \text{CaB}}{\text{AaB}}$$

Now, if an argument is to constitute not merely a syllogisism, but indeed a demonstrative-proof, it is required that its premises be fully explanatory of its conclusion. Every 'middle' term used to connect the two 'extreme' terms of the conclusion must be an *explanatory* middle. And in some cases more than one middle term will be required. For instance, it might occur that the premise 'A holds of all C' in the above is not explanatorily primitive. If this is so, the above cannot constitute a demonstration, and a new explanatory middle (one between A and C, explaining their connection) will be required in order to derive and more fully explain connection between A and B:

$$\frac{\dfrac{\text{AaD} \quad \text{DaC}}{\text{AaC}} \qquad \text{CaB}}{\text{AaB}}$$

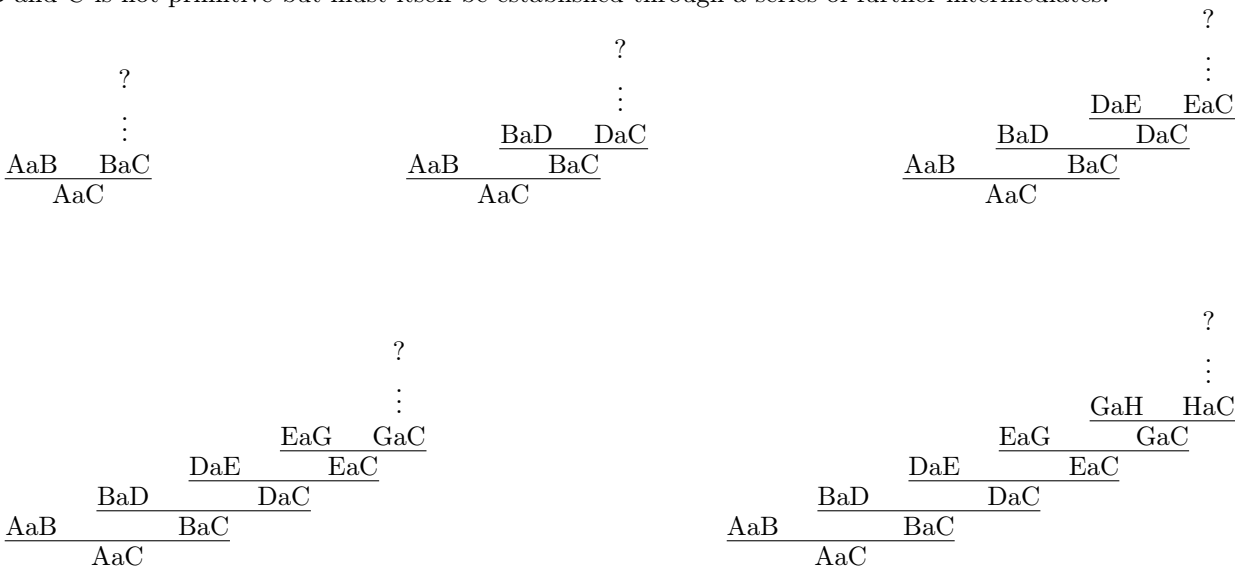
For Aristotle, notions like 'extremes', 'middle', and 'intermediate' are topological in origin. They have an important role to play in his kinematics,[11] and enter logic by metaphorical extension. *Posterior Analytics*

[10] *Posterior Analytics* 82a5-6, 84a32-33, 84a37-38, 84a30-32, 82a21-2, a36-37, b34-35.
[11] See esp. *Physics* V-VI.

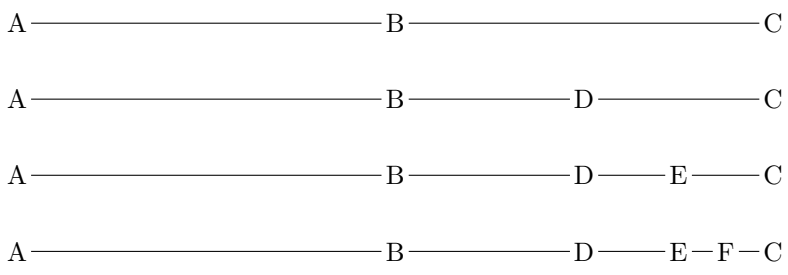
I.19-22 will often describe the possibility of a demonstrative-process continuing ad infinitum in basically topological terms. Note, for instance, Aristotle's recapitulation of I.3 72b-11 in *Posterior Analytics* I.19 (82a2-8):

> When the extremes are fixed is it possible for the intermediates to be infinite? I mean, for instance, if A belongs to C, and B is a middle [between] them, but of A and B there are other middles, and different middles of these, is it indeed possible for these to continue ad infinitum, or is it impossible? And this is the same as to investigate whether demonstrations can continue ad infinitum and if there is demonstration of all things, or some are limited by one another. (Likewise I also mean in the case of negative syllogisms and propositions...).

Consider a demonstrative process whereby AaC is shown through a middle B, but the connection between B and C is not primitive but must itself be established through a series of further intermediates:



To bring out the topological metaphor of finding successive 'middles' between two 'extremes' we can picture the iterated process above with a series of lines. The issue is to connect A and C. But to do so we must go through middle B. Yet to connect B and C we must go through a further middle, D, and so on:



From the peculiar perspective of Aristotle's system, the question of demonstration ad infinitum is the question of whether establishing a connection between two 'extremes' might require first going through infinitely many 'middles'. The Ad Infinitum Skeptics suggest that establishing such a connection always requires going through infinitely many middles. But as 'it is impossible to go through [*dielthein*] infinitely many things' (72b10-11), they contend that demonstration and episteme are in principle impossible.

When developed in the context of Aristotelian syllogistic, the difficulty brought by the Ad Infinitum Skeptics bears strong structural parallels to Zenonian arguments against the possibility of change [*kinēsis*]. Recall, in particular, Zeno's so-called 'dichotomy' argument.[12] Zeno's dichotomy argues that change is impossible on the grounds that a changing object which changes from A to B (e.g., a runner running from point A to point B) would have to traverse infinitely many half distances. However, claims Zeno, it is impossible to 'traverse' or 'go through' [*dielthein*, *diexelthein*] infinitely many things. Hence, change too must be impossible. The Ad Infinitum Skeptics argue not that change [*kinēsis*] is impossible but that demonstration [*apodeixis*] is impossible. Yet these thinkers argue from analogous considerations: the impossibility of traversing [*dielthein*] the infinite due to the requirement of traversing infinitely many middles. Whereas Zeno's challenge is derived from the alleged infinite divisibility of physical space, the Ad Infinitum Skeptics' challenge is derived from the alleged infinite divisibility of explanatory space.

In the context of such structural similarities between the Zeno's argument and Ad Infinitum Skeptics' argument, it is notable how structurally dissimilar Aristotle's solutions to the two puzzles are. In response to Zeno, Aristotle will reject the atomistic alternative and uphold the possibility of change while preserving the infinite divisibility of the continuum. In contrast, *Posterior Analytics* I.19-22 aims to argue that explanatory space has an atomic architecture. If the connection between two universals A and B is demonstrable, that connection is grounded in a finite number of explanatorily indivisible, atomic intervals. Indeed, Aristotle will regularly call explanatorily primitive propositions *atoma* or indivisibles [*adiareta*], or say that their predicates hold of their subjects 'atomically'.[13] For Aristotle, a successful demonstration traverses explanatory space in the way the atomists say a mobile traverses a distance.[14]

[12]For Aristotle's discussion of Zeno's dichotomy see esp. *Physics* VI.2, VIII.8.

[13]See e.g. *Posterior Analytics* I.15 79a33-36, texts cited below.

[14]Compare: *Posterior Analytics* I.22 84a32-37):

> If there are principles, it is not the case that every [scientific proposition] is demonstrable, nor is it possible to continue [demonstrating] ad infinitum: for either of these things to be the case is simply for there to be no immediate and indivisible [*ameson kai adiaireton*] intervals but for all of them to be divisible.

and *Posterior Analytics* I.23 84b34-85a2:

> The middle terms will always be thickened until they become indivisible and unitary [*hen*]. They are unitary when they have become immediate, and a unitary proposition simpliciter is an immediate proposition. And just as in other things the principle [*archē*] is unitary, but not the same everywhere: a *mna* in weight, a semi-tone in a musical tune, and other things in other cases, similarly in deduction the unit [to hen] is the immediate proposition, and in demonstration and *episteme* it is *nous*.

4.1.3 *Posterior Analytics* I.19-22 and Compactness?

Scholarly interest in *Posterior Analytics* I.19-22 has lain dormant over the last few decades. I suspect that the cause for scholarly neglect of these chapters during a period of otherwise great interest in the *Posterior Analytics* is due more to the perceived difficulty of the material, than a sense that existing work on I.19-22 is satisfactory. For, consider where our most recent scholarship on *Posterior Analytics* I.19-22 leaves off. On the view of Barnes (1975)[15], the argument of chapters I.19-22 looks like a disaster. To make any sense of Aristotle's attempted defense of the crucial lemma of I.21, Barnes maintains that we must delete seven lines from the transmitted text.[16] But even with the deletion, says Barnes, Aristotle's argument for the lemma remains obviously invalid. Aristotle's arguments in the important *Posterior Analytics* I.22, Barnes adds, are both invalid and unsound by Aristotle's own lights. So Barnes, summing up his pessimistic verdict on I.22: 'None of the arguments [of I.22] is successful; and they cannot be reformulated in such a way as to furnish proof of Aristotle's contention'.[17] In contrast to Barnes, Lear (1979/1980) sees *Posterior Analytics* I.19-22 as less hopeless. A persistent goal in Lear's work is to emphasize Aristotle's positive achievements as a meta-logician. Nonetheless, Lear agrees that Aristotle's proof of the I.21 lemma is deeply flawed. As to whether the arguments of I.22 can possibly (contra Barnes) 'be reformulated in such a way as to furnish proof of Aristotle's contention', Lear omits comment. Next there is the note of Scanlan (1982). Scanlan's piece is mostly just an attack on Lear's reconstruction: the aim is to show that Lear's reading is thoroughly marred by a flawed understanding of the mathematics of infinite sequences. Most memorably, Scanlan objects that (what he takes to be) Lear's central interpretive claim about I.19-22—that Aristotle is trying to establish a 'proof-theoretic analogue of compactness'—is both obscure and unacceptably anachronistic. Scanlan does not develop any alternative in detail. Smith (1986) mostly concerns himself with the place of I.19-22 in the project of the *Analytics* and the relation of the *Prior* to the *Posterior Analytics*. He largely passes over the main philosophical and logical difficulties raised by Barnes-Lear-Scanlan.[18]

A superficial glance at the most recent literature on *Posterior Analytics* I.19-22 would suggest that the question of whether Aristotle is trying to prove a compactness result in I.19-22 is controversial and important.[19] I don't think that this is so; but the issue merits comment. Once the red herring has been dealt with, we will be free to pursue our own study of the chapters.

In contemporary logic, compactness is standardly thought of as a model-theoretic property. In a very

[15]Revised 1994, though Barnes' reading of I.19-22 are not significantly altered in the revision.

[16]The paragraph that Barnes proposes to delete occurs in all manuscripts of the *Analytics* in our possession, as well as those known to John Philoponus (6th century CE).

[17]Barnes (1994) 181

[18]In terms of understanding the thesis of I.19-22 and why Aristotle is interested in the issue, I think Smith is closer to the truth than either Lear or Scanlan. This said, as will come out in what follows, I also think Smith misunderstands how the argument of I.19-22 is supposed to work.

[19]Lear's (1979) was titled 'Aristotle's Compactness in Proof'; Scanlan's (1982) was titled 'Compactness in Aristotle?'.

(very) general way, the model-theorist will define a *logic* as a pair $L = (\mathscr{L}, \models_{\mathscr{L}})$, such that

1. $\mathscr{L}$ is a map from vocabularies τ (i.e. non-empty collections of symbols) to classes of sentences composed from from vocabularies: $\mathscr{L}[\tau]$ ($\mathscr{L}$-sentences)

2. $\models_{\mathscr{L}}$ ($\mathscr{L}$-satisfaction) is a binary-relation between (some specified class of extra-linguistic) τ-structures and particular $\mathscr{L}$-sentences[20]

As usual, given some sentences $\Gamma \subseteq \mathscr{L}[\tau]$ and a τ-structure $\mathfrak{A}$, we say that $\mathfrak{A} \models_{\mathscr{L}} \Gamma$ ($\mathfrak{A}$ is a model of Γ) iff for all $\varphi \in \Gamma$: $\mathfrak{A} \models_{\mathscr{L}} \varphi$. Moreover, for any $\Gamma \cup \{\varphi\} \subseteq \mathscr{L}[\tau]$, we say $\Gamma \models_L \varphi$ (φ is a logical consequence of Γ relative to L) iff for every $\mathfrak{A}$ such that $\mathfrak{A} \models_{\mathscr{L}} \Gamma$, it holds that $\mathfrak{A} \models_{\mathscr{L}} \varphi$. Finally, the model theorist defines (strong) compactness in one of the following two equivalent ways:

Compactness A $L = (\mathscr{L}, \models_{\mathscr{L}})$ is compact iff for all τ and all $\Gamma \subseteq \mathscr{L}[\tau]$: Γ has a model whenever every finite subset $\Gamma_0 \subseteq \Gamma$ has a model.

Compactness B $L = (\mathscr{L}, \models_{\mathscr{L}})$ is compact iff for all τ and all $\Gamma \subseteq \mathscr{L}[\tau]$: whenever $\Gamma \models_{\mathscr{L}} \varphi$, there is a finite subset $\Gamma_0 \subseteq \Gamma$ such that $\Gamma_0 \models_{\mathscr{L}} \varphi$

It is **Compactness B** that Lear especially has in mind in calling the thesis of *Posterior Analytics* I.19-22 a 'proof-theoretic analogue of compactness'. Effectively, a logic L is compact iff whenever a conclusion semantically L-entailed by infinitely many sentences Γ, is L-entailed by some finite subset of that Γ.[21] The I.19-22 thesis is supposed to be like compactness in the sense that (according to Lear) Aristotle wants to prove in I.19-22 that every demonstrable conclusion can be demonstrated from finitely many premises.

Scanlan (1982) complains of anachronism. If one does not (as Aristotle surely didn't) countenance models or the possibility of something being logically entailed by infinitely many sentences, the question of compactness cannot arise. This is, of course, correct. But in focusing on this point, Scanlan is attacking a straw-man. For, Lear was never claiming that Aristotle *really* confronted the problem of compactness.[22] The idea was simply that Aristotle confronted a problem *analogous* to compactness. Read more charitably, Lear's train of thought looks to be the following. For Aristotle, syllogistic deductions are finitary objects. Every basic syllogistic inference involves exactly two premises, and no chain of syllogistic inferences can be infinitely long. In *Posterior Analytics* I.19-22, Aristotle is addressing the question whether the finitary nature of syllogistic deduction imposes any limitation on the amount of episteme that can be gained by

[20]NB The above definition captures everything the model-theorist will normally call a 'logic'; but to ensure that it does not capture more, additional conditions are normally imposed (an isomorphism condition requiring uniformity of $\models_{\mathscr{L}}$ across isomorphic structures, etc.: see Barwise and Feferman 28). For our purposes, such conditions may be ignored.

[21]First-order logic and S5 modal logic, for instance, are compact; second-order logic, infinitary logics, and the modal logic of actions/programs called PDL (propositional dynamic logic with the Kleene-star) are not.

[22]The title of Lear's original article is misleading in this regard.

syllogistic deduction. He argues that it does not. For, everything demonstrable is demonstrable from finitely many premises. And this result is suggestively analogous to compactness.

Certainly, one might question how *helpful* Lear's comparison with compactness ultimately is. But not much of what Lear actually says about *Posterior Analytics* I.19-22 turns on the issue. The real problem with Lear (and Scanlan) is that neither interpret *Posterior Analytics* I.19-22 as addressing the problem that Aristotle himself tells us he is addressing.[23] Contra Lear, I.19-22 is not responding to an objector who thinks that syllogistic deductions—because they are finitary—are inadequate for purposes of science, and worries that a different logic may be needed. Rather, Aristotle is engaging with a skepticism about the possibility of episteme. He is responding to an objector who denies that there could be genuine science as Aristotle understands it. Moreover, in arguing against to this objector, Aristotle's aim is not (pace Lear) to establish that some special property holds of syllogistic consequence in general; he is not (pace Lear) trying to show his syllogistic's proof-theoretic strength. Aristotle's aim is rather to establish something that holds of demonstrative-proof [*apodeixis*] in particular—that a genuine process of demonstration can never continue ad infinitum. Indeed, in *Posterior Analytics* I.19 81b18-23 Aristotle allows that a processes of mere syllogistic-deduction can (potentially at least) continue ad infinitum in a way that a demonstrative process cannot.

4.2 Syllogisms, demonstrations, demonstrative processes

4.2.1 Syllogisms

Posterior Analytics I.19-22 investigates the logical possibility of demonstrative processes which are not finitely terminating. If no (rightly-guided) process for constructing a demonstrative-proof of φ terminates after finitely many steps, there will be no demonstrative-proofs for φ. The Ad Infinitum Skeptics suggest that this holds for all values of φ. And I.19-22 argues that if φ is a scientifically explicable truth[24], it holds for no values of φ. Now, the argument of I.19-22 depends on a good deal of technical analysis of syllogistic. To track Aristotle's line of thought, formal representation will again prove enormously helpful. We will need a way of marking out demonstrations from mere syllogisms; but we must be careful to characterize the notion of demonstration in such a way that we do not prejudge the question of whether there actually are any demonstrations. For, this is exactly what's at stake in I.19-22. Indeed, we will need a way of characterizing what it would be for a demonstrative-process to have no (successful) termination. We must begin, however, by introducing a way to represent syllogisms—opposed to mere syllogistic-deductions.

[23] This is something Smith gets right—both Scanlan and Lear manage to ignore connections between I.19-22 and I.3.
[24] For more on this notion, see below.

The notion of a syllogistic-deduction was introduced in **Section 2.1.4**. With an eye towards our discussion of the Circular Demonstrators in **Section 3**, we allowed for the possibility of syllogistic arguments whose conclusions were identical with their premises. In moving forward, however, we can narrow our focus to capturing the (proper) subclass of syllogistic-deductions that Aristotle himself would consider syllogisms. Both Aristotle and the Ad Infinitum Skeptics reject circular syllogisms. Indeed, the arguments of *Posterior Analytics* I.20-21 will fail if circular syllogisms are allowed in science.

It is straightforward to produce an adequate representation for the class of Aristotelian syllogisms. Recall that in **Section 3**, we said that a syllogistic-deduction is circular exactly if its conclusion is identical with one of its premises. We can then provide the following:

Definition 1 Syllogism

A syllogistic-deduction Σ is a *syllogism* iff

1. neither Σ nor any of its subtrees is circular, and

2. every single-inference subtree of Σ uses three distinct terms

For instance, by **Definition 1.1** the following syllogistic-deductions are not syllogisms:

$$\dfrac{\text{AaB}\qquad\dfrac{\text{BaA}\qquad\text{AeC}}{\text{BeC}}}{\text{AeC}} \qquad\qquad \dfrac{\dfrac{\text{AaB}\qquad\text{BaC}}{\text{AaC}}\qquad\dfrac{\text{CaA}\qquad\dfrac{\text{AaE}\qquad\text{EaD}}{\text{AaD}}}{\text{CaD}}}{\text{AaD}}$$

By **Definition 1.2** this instance of Cesare is not a syllogism: $\dfrac{\text{AeB}\qquad\text{AaA}}{\text{BeA}}$. And on account of **Definition 1.2**, no syllogism can have a conclusion of the form XaX.

4.2.2 Demonstration as the termination of a processes

Not every syllogism is a demonstration. Indeed, not every syllogism with true premises is a demonstration. Like Plato before him, Aristotle will deny that every truth can be an object of episteme (or *nous*). For, not all truths admit the type of maximally stable grasp distinctive of scientific understanding—episteme and Aristotelian *nous*. Roughly, Aristotle will identify the class of truths which are potential objects of scientific understanding with general and necessary truths about reality. Bracketing the somewhat complex issue of how exactly Aristotle proposes to specify the class of truths apt for scientific understanding, let's just call any such truth a *scientific-truth*.

So, every demonstration is a sound syllogism having a scientific-truth as its conclusion. But some sound syllogisms whose conclusions are scientific-truths will not be demonstrations. For, as we noted above, Aristotle requires that the premises on which a demonstration rests be explanatorily prior to their conclusion. To adapt a (somewhat perfunctory) example of Aristotle's let A = *twinkling stars*, B = *star a certain distance from earth,* and C = *these stars.*[25] Plausibly there are two syllogisms with true premises:

$$\frac{\text{AaB} \qquad \text{BaC}}{\text{AaC}} \qquad\qquad \frac{\text{BaA} \qquad \text{AaC}}{\text{BaC}}$$

And, plausibly, the syllogism on the left uses premises explanatorily prior to their conclusion, while that on the right does not. (These stars are not such-and-such distance from earth because they twinkle; it's rather that they twinkle because the are stars such-and-such distance from earth).

According to *Posterior Analytics* I.2, a demonstration has to proceed from truths prior to, more intelligible than, and causal of their conclusion. For Aristotle, 'prior to', 'more intelligible than', and 'causal of' will be different notions. But such distinctions play no role in the argument of I.19-22. So let's harmlessly use 'φ is prior to ψ' as a stand in for 'φ is prior to, more intelligible than, and causal of ψ'. To represent demonstrative processes we must make some use of this priority notion. Following Aristotle's lead, let's view priority as a binary-relation, one that's transitive and irreflexive (a strict partial order). In the context of I.19-22, it will be harmless to make the simplifying assumption that the only propositions that are related by the priority relation are scientific-truths. This will allow for a straightforward way of representing Aristotelian demonstration and the possibility of demonstration ad infinitum. The key will be an intermediary notion of a *partial demonstration.*

Definition 2 Partial demonstration

1. Where Σ is a syllogism whose conclusion is a scientific-truth φ:

   · Σ is a partial demonstration iff every single-inference sub-syllogism $\dfrac{\alpha \qquad \beta}{\gamma}$ in Σ is such that (i) α is prior to γ, and (ii) β is prior to γ

2. A sentence $\psi = \text{A}x\text{B}$, with $x \in \{\text{a,e,i,o}\}$, is a *demonstrable x-proposition* iff there is a partial demonstration of ψ

Intuitively, think of a partial demonstration as an explanatory, but possibly incomplete, syllogistic proof of a scientific-truth. We will want a partial demonstration to represent a sub-proof constructed at some stage in a (rightly-guided) demonstrative process. We will ultimately set things up so that a partial demonstration

[25] Cf. *Posterior Analytics* I.13.

fails to be a demonstration if it is not fully explanatory because it does not proceed from premises that have nothing explanatorily prior to them.

In the *Analytics,* Aristotle is accustomed to envisioning proof-processes whereby one begins with the desired conclusion, and works one's way back to prior premises on which the conclusion can be shown. It will be helpful to have a name for this kind of process of proof-construction. Let's call it *improvement.*

Definition 3 Improvement

1. Let Σ and $\Sigma*$ be distinct partial demonstrations. Then $\Sigma*$ *improves* Σ iff they have either the forms:

   · $\Sigma =$ $\dfrac{\begin{array}{cc}\vdots & \\ \varphi \quad \psi & \\ \theta & \\ \vdots & \\ \chi & \end{array}}{}$ and $\Sigma* =$ $\dfrac{\begin{array}{cc}\alpha \quad \beta & \vdots \\ \varphi & \psi \\ \theta & \\ \vdots & \\ \chi & \end{array}}{}$, or:

   · $\Sigma =$ $\dfrac{\begin{array}{cc}\vdots & \\ \varphi \quad \psi & \\ \theta & \\ \vdots & \\ \chi & \end{array}}{}$ and $\Sigma* =$ $\dfrac{\begin{array}{cc}\vdots & \alpha \quad \beta \\ \varphi & \psi \\ \theta & \\ \vdots & \\ \chi & \end{array}}{}$

2. A sequence of partial demonstrations $\overline{D} = (D_0, D_1, ..., D_n, D_{n+1}, ...)$ is an *improvement-path* originating in $D*$ iff

   (a) $D_0 = D*$ and

   (b) for all D_i, D_{i+1} in $\overline{D}$: D_{i+1} improves D_i

3. D is an *improvement* of $D*$ iff there is an improvement-path $(D_0, D_1, ..., D_k)$ such that $D = D_k$ and $D* = D_0$

Let's see an example. Suppose the following are all partial demonstrations:

D_1:

$$\dfrac{\text{AaB} \quad \text{BaC}}{\text{AaC}}$$

D_2:

$$\dfrac{\text{AaB} \quad \dfrac{\text{BaD} \quad \text{DaC}}{\text{BaC}}}{\text{AaC}}$$

D_3:

$$\dfrac{\text{AaB} \quad \dfrac{\dfrac{\text{BaE} \quad \text{EaD}}{\text{BaD}} \quad \text{DaC}}{\text{BaC}}}{\text{AaC}}$$

D_4:

$$\dfrac{\dfrac{\text{AaF} \quad \text{FaB}}{\text{AaB}} \quad \dfrac{\dfrac{\text{BaE} \quad \text{EaD}}{\text{BaD}} \quad \text{DaC}}{\text{BaC}}}{\text{AaC}}$$

By **Definition 3.1** we have that (i) D_2 improves D_1, (ii) D_3 improves D_2, and (iii) D_4 improves D_3. By **Definition 3.2**, the following (among others) are improvements paths: (i) (D_1, D_2, D_3, D_4), (ii) (D_1, D_2, D_3), (iii) (D_2, D_3). But the following (among others) is not an improvement-path: (D_1, D_3). Finally, given **Definition 3.3**, we have that D_4 is an improvement of D_1, and that D_4 is an improvement of D_3 and D_2. Note that if D improves D', the pair are required to have the same ultimate conclusion. By definition, all the partial demonstrations in any improvement-path have the same ultimate conclusion. Note finally that while we allow for the possibility of infinitely long improvement-paths, our definitions ensure that every improvement-path is of length $\leq \omega$; no infinitely long improvement-path can have a final member.

The notion of a demonstrative improvement-path will prove crucial in what follows. For one, we use the notion to capture the idea of a demonstrative-proof:

Definition 4 Demonstration

D is a demonstration iff it is a partial demonstration and there is no distinct partial demonstration D' that improves D.

Suppose there were an infinite demonstrative improvement-path $\overline{D}$. No partial demonstration in $\overline{D}$ could be a demonstration. A demonstration, having reached explanatory primitive first-principles, cannot be improved, and will always be the *final* member of any improvement-path of which it is a member. Let D be a demonstration that uses a premise φ. One might, of course, find some syllogism Σ which provides a sound argument for φ and concatenate Σ with the demonstration at φ. But insofar as D is a demonstration, φ must be a first-principle with nothing explanatorily prior to it. So by assumption the premises used in Σ

will not be explanatorily prior to φ. And concatenating D and Σ at φ will not produce an improvement of D.

Finally we introduce the following in terms of which the thesis of *Posterior Analytics* I.19-22 can be neatly stated:

Definition 5 Demonstration ad infinitum

We say that *the process of demonstrating φ continues ad infinitum* iff

1. there is a partial demonstration D of φ, and

2. there is an infinite demonstrative improvement-path originating in D

4.3 Argument structure: *Posterior Analytics* I.19-22

4.3.1 Global argument structure

In the opening of *Posterior Analytics* I.2, Aristotle asserts that A has episteme of X when (i) X is a necessary-truth, (ii) A knows [*ginōskein*] X holds because of its causal-explanation [*aitia*]. Aristotle goes on to explain that this occurs exactly when one has mastered an epistemic logos—a demonstration of X from the first-principles prior to and causally-explanatory of X [*aitia tou*].

Above, we introduced a priority relation in order to track such causal-explanatory chains. Following Aristotle, we let *prior-to* be a binary relation on the collection of scientific-truths, and we required that the prior-to relation be a strict partial order.[26] Aristotle himself will sometimes refer to the complete causal-explanation of an item as simply 'the cause' of that item. The reader is invited to understand the following definition in the same spirit:

Definition 6

For every demonstrable truth φ, we let $\textsc{cause}(\varphi)$ name the smallest collection containing every truth ψ prior to φ.[27]

Aristotle and the Ad Infinitum Skeptics will both say that having episteme of a scientific-truth φ requires demonstrating φ. A demonstrative-proof of φ will have to reveal φ's dependence on further truths in

[26] I.e. irreflexive and transitive.

[27] In order-theoretic terms, $\textsc{cause}(\varphi)$ will be the up-set for φ relative to the priority relation. Representing 'ψ is prior to φ' as $\psi >_p \varphi$:

· $\textsc{cause}(\varphi) = \{\psi : \psi >_p \varphi\}$.

CAUSE(φ). In particular, the demonstrative-proof must terminate in truths from CAUSE(φ) that have no further truths explanatorily prior to them. The Ad Infinitum Skeptics charge that no partial demonstration of φ will contain only premises that are truly primary. Whenever one tries to demonstrate φ one will run into an infinite explanatory regress.

It is well-known that Aristotle has principled reasons for denying the existence of 'actual' infinities. So, one might have expected him to bring such considerations to bear in his case against the Ad Infinitum Skeptics, arguing perhaps that every CAUSE(φ) will have to be finitely large. But this is not how Aristotle proceeds. Indeed, Aristotle does not argue directly against the possibility of there being an infinite series of propositions each explanatorily prior to the last. *Posterior Analytics* I.19-22 rather argues through the following two steps:

1. For every scientific-truth φ: the process of demonstrating φ can continue ad infinitum[28] only if CAUSE(φ) contains an infinite chain of a-propositions

2. For no scientific-truth φ does CAUSE(φ) contain an infinite chain of a-propositions

After the introductory *Posterior Analytics* I.19, the task of establishing (1) falls to I.20-1; that of establishing (2) falls to I.22. It will be convenient to have a label for (1), the result Aristotle aims to prove in I.20-1. We call it the *Main Lemma*.

When Aristotle introduces the idea of infinite chains in *Posterior Analytics* I.19 he neatly distinguishes two cases: chains that infinite in the upwards direction (towards the more general), and chains that are infinite in the downwards direction (towards the more particular). There is a sense in which a chain of infinitely iterated counter-predicables: (AaB, BaC, CaA, AaB, BaC, CaA, AaB ...) might be described as infinite. But, as is clear from the argument for (2) in I.22, chains of this sort are not infinite in Aristotle's intended sense. Is the chain: (AaB, BaC, CaA, AaB, ...) infinite in the direction of the more general or in the direction of the more particular? Properly viewed, thinks Aristotle, the answer is neither. The infinite a-chains at issue in I.19-22, however, must proceed ad infinitum either in the direction of the general, or in the direction of the particular.

Definition 7

An infinite chain of a-propositions (or an *infinite a-chain*) is an infinite sequence using infinitely many terms of one of the forms:

1. (YaX_0, X_0aX_1, X_1aX_2, ..., X_naX_{n+1}, ...) infinite down-chain

2. (X_0aY, X_1aX_0, X_2aX_1, ..., $X_{n+1}aX_n$, ...) infinite up-chain

[28] 'Can continue...' i.e. there's an infinite demonstrative improvement chain with conclusion φ.

It is important to appreciate that *Posterior Analytics* I.19-22 does not try to establish that there can be no infinite a-chains composed of *truths*. Aristotle means only to establish that for no scientific-truth φ does CAUSE(φ) contain an infinite a-chain. Indeed, Aristotle will accept that infinite chains of true a-statements. A toy example would be:[29]

Less than $2\frac{1}{2}$ long holds of everything less than $2\frac{1}{4}$ long

Less than $2\frac{1}{4}$ long holds of everything less than $2\frac{1}{8}$ long

Less than $2\frac{1}{8}$ long holds of everything less than $2\frac{1}{16}$ long

$\vdots$

∞

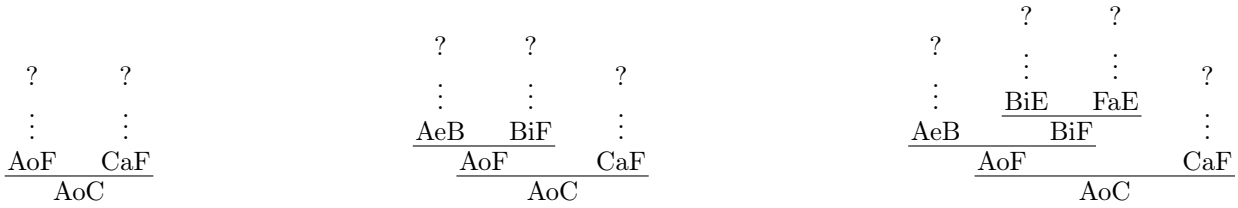
Since Aristotle holds that space is dense, he will admit that every such proposition in the down-chain above is true. What he will deny is that going through the sequence above is necessary for constructing a complete and perfectly correct account of why less than $2\frac{1}{2}$ holds of everything less than 2. For, he will contend that the 'middles' in the above sequence are not explanatory in the relevant sense.

This present chapter focuses on Aristotle's case for the Main Lemma in I.20-21—Aristotle's claim that for every scientific-truth φ: the process of demonstrating φ can continue ad infinitum only if CAUSE(φ) contains an infinite chain of a-propositions. **Section 5** below will discuss turn to Aristotle's argument agains the relevant kind of infinite a-chain in I.22.

4.3.2 Local argument structure: *Posterior Analytics* I.20-21

The Main Lemma states that for every scientific-truth φ: the process of demonstrating φ can continue ad infinitum only if CAUSE(φ) contains an infinite chain of a-propositions. Aristotle offers a sophisticated and remarkable argument for this claim in *Posterior Analytics* I.20-21. The Main Lemma can indeed be proved relative to the formal reconstruction of the present dissertation. But it is far from obvious that the Main Lemma should be true. Consider, for instance, the following improvement-path:

[29] Whether or not any entity could actually cognitively formulate such a sequence, every member of the infinite sequence would be true.



The syllogism on the right uses a-premises, e-premises, and i-premises. Why think that in order to keep improving it ad infinitum there would have to be an infinite *chain* of a-propositions? Why aren't other infinitely large collections of scientific-truths sufficient—for instance, collections that contain infinitely many a-propositions but no infinite chains of a-propositions?

Below, I shall argue that *Posterior Analytics* I.20-21 sketches the basics of a successful proof for the Main Lemma. But first I will need to introduce the proof-strategy I take I.20-21 to employ. My approach to I.20-21 differs rather significantly from the alternatives one finds in the literature. So let me begin by indicating some of the central interpretive issues.

4.3.2.1 Previous work on I.20-21

One sometimes encounters scholars of Aristotle who think that the conclusion of an Aristotelian demonstration must always be a-proposition. Indeed, the view is sometimes expressed in print.[30] But it is quite definitely wrong. Among other texts, there is *Posterior Analytics* I.15 which remarks on negative propositions in demonstrative science, and explicitly says that there should be immediate or 'atomic' e-propositions among scientific first-principles. A more plausible view holds that for Aristotle only a- and e-propositions can be conclusions of demonstration. Notable adherents of the latter are Barnes (1994) and Smith (1986). I myself think that this too is wrong, and on both textual and philosophical ground would defend the traditional view[31] on which Aristotle allows for demonstrations with a-, e-, i, or o-conclusions.[32] But whether or not I'm correct in this regard, we must ask whether the argument of *Posterior Analytics* I.20-21 depends on the assumption that every demonstrable truth is an a- or an e-proposition. We must ask this because the most recent literature on the argument of I.20-21 reads Aristotle's argument as resting on this assumption. Barnes (1975/1994), Lear (1980), and Smith (1986) all see I.20-21 as trying establish a weak version of what I've been calling the the Main Lemma that is restricted to the fragment of assertoric syllogistic consisting of just the moods: Barbara, Celarent, Cesare, Camestres. On this reading, in I.20-21 Aristotle would only be

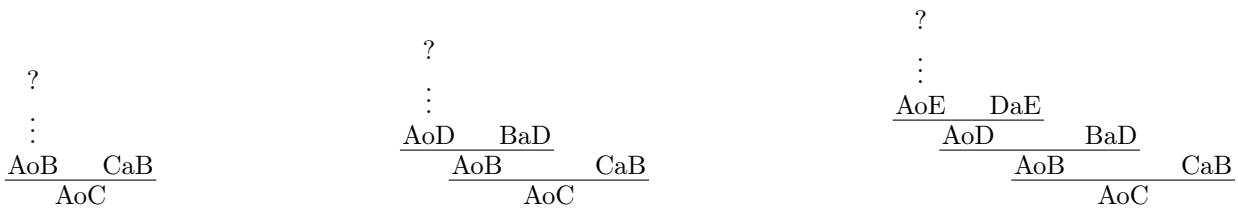
[30] Owen Goldin says this in his *Explaining an Eclipse* book on the Post An says this. Mariska says this in a recent article

[31] Accepted, for instance, by Philoponus and Ross.

[32] *Posterior Analytics* I.4-6 just shows Aristotle requiring that scientific *first-principles* be a- or e-propositions. This would only rule out demonstrations with particular conclusions if one couldn't prove a particular conclusion from a set of wholly universal premises. But in syllogistic, of course, one can do this using the moods validated by a-to-i conversion: Darapti, Felapton, or Fapesmo.

trying to show that an unending process of demonstrating an a- or e-conclusion requires a certain kind of infinite chain of a-propositions; he would not concerned with whether this also holds for i- and o-conclusions.

Against Barnes-Lear-Smith, I contend that *Posterior Analytics* I.20-21 is in fact an attempt to establish the unrestricted version of the Main Lemma stated above—i.e. that a non-terminating demonstrative process targeting an a-, e-, i-, or o-conclusion can only continue ad infinitum if there are infinite a-chains of the relevant type. The establishment of this thesis would give Aristotle a far stronger case against the Ad Infinitum Skeptics than the Barnes-Lear-Smith alternative. And given the transmitted text of I.20-21, the Barnes-Lear-Smith alternative faces a serious problem. *Posterior Analytics* I.21 considers the possibility of a demonstrative-process that targets a negative conclusion continuing ad infinitum. The text shows Aristotle concerned with the architecture of various improvement-paths (in the sense of Section **4.2.2** above). And I.21 describe the effects of demonstrative processes iterating: (i) Celarent-improvement, (ii) Camestres-improvement, and (iii) Bocardo-improvement. Indeed, I.21 explicitly discuss a demonstrative improvement-paths of the form (82b21-28):



The tenability of the Barnes-Lear-Smith reading of I.20-21 crucially depends on deleting I.21's seven line discussion of Bocardo from the text. Indeed, it would also require further deletions—for instance, Aristotle's mention of the third figure at I.2182b31. But there are no good philological grounds to suspect any of the material Barnes et al. want to delete. The material appears in all our manuscripts, and was clearly known to the ancient commentators.

So why do Barnes-Lear-Smith think I.21's discussion of Bocardo is spurious? Their main reason is failure to see why Aristotle would think that I.20-21 needs to discuss exactly (i) Barbara, (ii) Celarent, (iii) Camestres, and (iv) Bocardo. *Posterior Analytics* I.21 treats demonstrative improvement in the case of negative demonstrations. Although Aristotle never says he wants to focus on negative-universal demonstrations, I.21 does say that three cases need to be considered. But why would he think this? And why do the three cases turn out to be improvement by Celarent, improvement by Camestres, and improvement by Bocardo? What about the 8 other negative moods? Barnes notes that if the *Posterior Analytics* I.21 that Aristotle wrote *were* restricting its focus to demonstrations of e-conclusions, it would make sense for Aristotle to say that three cases need to be considered. The cases in question would be: Celarent, Camestres, and Cesare.

Unable to make sense of the text as transmitted, Barnes suggests that Aristotle himself composed I.21's discussions of Celarent and Camestres, adding something like 'and similarly in the remaining case' which was later crossed out and bizarrely replaced by a discussion of Bocardo penned by a 'wrong-headed glossator' (1994 pg 173). With discussion of Bocardo expunged, Barnes-Lear-Smith find the argument strategy of I.20-21 intelligible. But they do not think the proof sketch Aristotle provides is successful. Even with I.21 read as restrictedly treating e-conclusions (and not pertaining to o-conclusions), Barnes et al. take Aristotle's argument to be fallacious, and indeed quite obviously so.[33]

An alternative to the Barnes-Smith-Lear approach to I.21 is found in Philoponus and Ross. Reading the text as transmitted, Philoponus and Ross think that Aristotle is trying to show something concerning both e- and o-propositions. Why else would Aristotle bother discussing Bocardo? And why else would he mention three syllogistic figures in 82b30-1?[34]As to why Aristotle considers exactly Celarent, Camestres, and Bocardo, Philoponus-Ross claim that Aristotle is simply picking a negative mood from each of the three figures. In explaining how the argument of I.21 is actually supposed to work we find Ross uncritically repeating the analysis of Philoponus. Yet the Philoponus reconstruction has a vanishingly small basis in the text. One gets the impression that Philoponus didn't see how Aristotle's own argument was supposed to work, so is instead presenting an alternative and (by Philoponus' lights) simpler argument for the same conclusion.[35] Neither Ross nor Philoponus do much to ground their interpretation in I.21 itself. Most damningly, infinite a-chains play no role in the Ross-Philoponus analysis. But as clear from the opening of the chapter, Aristotle himself clearly wants the argument to turn on the issue of infinite a-chains.

For the moment, let's leave aside the issue of whether or not *Posterior Analytics* I.21 presents a valid argument. I hope it is clear from the preceding that both Philoponus-Ross and Barnes-Lear-Smith do a poor job of explaining the transmitted text. The Barnes-Lear reading is a non-starter unless we accept Barnes' conjectural excisions. And in the Philoponus-Ross interpretation, what Aristotle takes to be a central assumption of the argument plays no role.

It is not difficult to see why commentators have had such a hard time with *Posterior Analytics* I.21. The

[33]For Barnes-Lear (Smith is less clear) Aristotle's thought is supposed to be the following. For all C∈{Celarent, Camestres, Cesare}, if there are no infinite a-chains every token syllogistic string containing only C-proofs must be finite. Moreover, the relevant collection of argument-types: {Celarent, Camestres, Cesare}, is finite. So, since no other moods show a universal negative conclusion, every process of demonstrating a universal negative must be finite. The argument is obviously invalid because it does not rule out infinite alternating improvement patterns like (e.g.) the following (with C_a = Camestres and C_e = Celarent):

$$[\text{e-conclusion}]\longleftarrow[C_e\text{-string}]\longleftarrow[C_a\text{-string}]\longleftarrow[C_e\text{-string}]\longleftarrow[C_a\text{-string}]\longleftarrow\cdots$$

Noting the argument's invalidity, Barnes and Lear locate the fault in Aristotle. Really, though, given what Aristotle actually says in I.21 (see e.g. 82b16-17) he ought to be aware of such alternating improvement patterns.

[34]Recall that no universal conclusions (positive or negative) are proved in the third figure.

[35]Not only is the alternative most definitely not the argument Aristotle gives; it is also an argument Aristotle should not accept (endangering, as it does, the status of arithmetic as a science).

version of the argument of I.21 we have presents a bare-bones proof sketch. In order to (re)construct a proof from a proof sketch, one must have in view the overall proof strategy that the sketch's author intends his or her proof to follow. Aristotle might very well have carefully explained his overall I.21 proof strategy in oral presentation of the chapter's material. But no such detailed explanation appears in the transmitted text.

I present here a novel account of a proof strategy Aristotle might have intended the proof sketched in *Posterior Analytics* I.21 to follow. The hypothesis that Aristotle *did* intend the proof sketched in I.21 to follow the proof strategy I describe (or something very close to it), is eminently plausible. The hypothesis does a good job of explaining why the sketch given in I.21 takes the exact form it does. In any case, the hypothesis does considerably better in explaining the transmitted text of I.21 than any of the alternatives available in the commentary tradition; it explains in particular why Aristotle should consider exactly Celarent, Camestres, and Bocardo in I.21. Finally, it is notable that my reading extracts from I.20-21 a valid argument for a conclusion targeting *all* negative syllogistic moods—a result considerably stronger than that countenanced by Barnes-Lear-Smith.

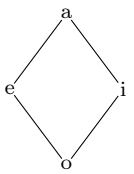
4.3.2.2 Argument structure of I.20-21

I contend that Aristotle's proof of the Main Lemma crucially relies on a handful of correct observations about syllogistic that *Posterior Analytics* I.20-21 does not make explicit. Against the background of these observations, one can discern in I.20-21 the basics of a successful proof of the Main Lemma in its full generality. I maintain that once the proof strategy of I.21 is understood, it becomes clear why Aristotle should consider exactly improvement by Celarent, improvement by Camestres, and improvement by Bocardo. However, we shall also need to understand the strategy of I.20-21 as a whole—how it is Aristotle thinks he can establish the Main Lemma, and how he thinks the proof sketched in I.20-21 yields the Main Lemma.

The first such observation I.20-21 takes for granted concerns what kinds of premises can be used to syllogistically establish what kinds of conclusions. We noted above that *Prior Analytics* I.24 describes some syllogistic laws of quantity and quality:

Laws of Quality and Quantity [36]

[36]Here again is our mnemonically helpful lattice:



For nodes x, y in the lattice let $x \geq y$ iff either (i) $x = y$, or (ii) x is above y. Laws of Quality and Quantity can be remembered in terms of:

Rule 1 An x-proposition can be used to prove an y-proposition exactly if $x \geq y$

1. A syllogism with a universal conclusion asserts only universal premises
2. A syllogism with a positive conclusion asserts only positive premises
3. A syllogism with a negative conclusion asserts exactly one negative premise
4. A syllogism that a particular conclusion asserts no more than one particular premise

The above Laws of Quality and Quantity allow *Posterior Analytics* I.20-21 to employ an elegant proof-strategy to establish the Main Lemma.

Recall that the Main Lemma states for any scientific a-, e-, i-, or o-proposition φ, the process of demonstrating φ can continue ad infinitum only if CAUSE(φ) contains an infinite chain a-chain. To get the Main Lemma, Aristotle sets out to establish the following three theorems:

Theorem 1

If for no φ does CAUSE(φ) contain an infinite a-chain, the process of demonstrating an a-proposition is always finitely terminating

Theorem 2

If for no φ does CAUSE(φ) contain an infinite a-chain, the process of demonstrating an i-proposition is always finitely terminating

Theorem 3

If for no φ does CAUSE(φ) contain an infinite a-chain, the process of demonstrating an e- or o-proposition is always finitely terminating

It is easily seen that the conjunction of **Theorems 1-3** is equivalent to the Main Lemma.

Now, it is clear from the opening of *Posterior Analytics* I.21 that Aristotle correctly sees that the argumentative burden for proving **Theorem 3** will be far lessened if he can assume that **Theorems 1** and **2** are true.[37] For, in I.21 Aristotle will argue for **Theorem 3** by assuming the truth of **Theorems 1-2**. It is equally clear that Aristotle's aim in I.20 is to argue for **Theorem 1**. But what of **Theorem 2**? The transmitted text provides no explicit argument for **Theorem 2**. Yet I will argue below that once the proof of I.21 is understood, we will immediately have the resources to show that in syllogistic **Theorem 2** is true if **Theorem 1** is. In sum, I submit that Aristotle wants to establish **Theorems 1-3** and, hence, the Main Lemma in the following steps:

Rule 2 A syllogism can have more than one x-premise exactly if $x \geq$a

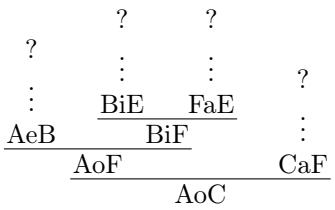
[37]Better: it is clear that this is so provided that we don't follow Barnes and delete I.21's mention of Bocardo and the third figure!

Step 1: Prove that **Theorem 1** is true (goal of *Post. An.* I.20)

Step 2: Prove that if **Theorem 1** and **2** are true, one can prove **Theorem 3** (*Post. An.* I.21)

Step 3: Prove that if **Theorem 1** is true, one can prove **Theorem 2** (by argument parallel to I.21)

The appeal of the above strategy is salient in the context of Aristotle's Laws of Quality and Quantity. For instance, suppose that we wanted to show that a non-terminating improvement-path establishing an o-proposition requires an infinite a-chain. We would have to consider the possibility of an infinite demonstrative improvement-path which at some stage reaches:



Given the Laws of Quality and Quantity we know that no e- or o-proposition can ever occur above the BiE, FaE, or CaF branches. For, a negative premise can never establish a positive conclusion. Suppose we follow Aristotle in assuming that **Theorems 1** and **2** are true in order to prove **Theorem 3**. Then we can be certain that improvements targeting the BiE, FaE, and CaF will all be finitely terminating provided that CAUSE(AoC) contains no infinite a-chains. We can, then, restrict our attention to the AeB branch. Given the Laws of Quantity and Quality, we know that no o-proposition can ever occur above AeB. For a particular premise can never establish a universal conclusion. Indeed, we also know that every improvement of the syllogism above will only contain exactly one e-premise. So, without loss of generality, all we shall have to worry about is further improvement using combinations of Celarent, Camestres, and Cesare.

Below, I present what I take to be Aristotle's strategy for proving (i) that **Theorem 1** is true, (ii) that if **Theorem 1** and **Theorem 2**, then **Theorem 3**, and (iii) if **Theorem 1**, then **Theorem 2**. The proof I describe is successful. But it is rather is complex. I will be attributing to Aristotle argumentative moves that the text does not make explicit. However, the hypothesis that Aristotle himself intended to follow something very close to this proof-strategy I describe does do better in explaining Aristotle's text than any alternative readings yet proposed. Even if Aristotle never fully worked out all the details, there is good reason to think that what follows represents the proof he in fact (perhaps somewhat inchoately) had in mind.

For the purposes of defending my account of *Posterior Analytics* I.20-21, I propose to invert Aristotle's own order. I begin, that is, with Aristotle's argument in I.21 for the claim that in syllogistic **Theorem 3** follows if **Theorems 1** and **2** are true. Then I will work back to discuss Aristotle case for **Theorem 1**, and his claim that in syllogistic **Theorem 2** is true if **Theorem 1** is.

4.4 The Main Lemma proved: *Posterior Analytics* I.20-21

4.4.1 The case of negative demonstrative improvement

Let's see what sense we can make of *Posterior Analytics* I.21 if we reject Barnes' conjectural excisions and read the text as transmitted. There are, after all, no good philological reasons to accept them. Given the analysis of Bocardo (82b21-28) and the mention of three figures (82b30-31), Aristotle evidently wants to prove a result in I.21 about all negative moods—not just the universal ones. Now, Aristotle recognizes 16 valid (assertoric) syllogistic moods: eleven negative (concluding an e- or o-proposition) and five positive (concluding an a- or i- proposition). All 16 moods can occur in improvement-paths for partial demonstrations that prove negative conclusions.

As noted above, Aristotle sees correctly that his argumentative burden for **Theorem 3** is lessened if we assume **Theorems 1** and **2**: i.e. that all positive demonstrations are finitely terminating if there are no infinite a-chains. Indeed, Aristotle begins the chapter by assuming **Theorems 1** and **2**, for the sake of proving **Theorem 3**. *Posterior Analytics* I.21 82a36-37 states negative demonstration terminates (i.e. **Theorem 3**) if positive demonstration does (i.e. if **Theorem 1** and **2**). No restriction to universal positive moods is given here. He just says that negative demonstrations terminate if positive ones do.

Note that **Theorem 3** is itself a conditional. If one wants to prove a conditional using the assumption of two further propositions, one begins by assuming the two further propositions as well as the antecedent of the conditional one is trying to prove. Indeed, after Aristotle assumes the truth of **Theorems 1** and **2** at 82a36-37 his next move is to assume the antecedent of **Theorem 3** at 82a38-82b4. But since the antecedent of **Theorem 3** is identical with the antecedents of **Theorems 1-2**, Aristotle is entitled to ignore improvement using positive moods in what follows. Indeed, Aristotle explicitly notes (82b6-8) that in considering improvement of negative partial demonstrations we can ignore possible improvement targeting positive premises. For, consider, e.g. a negative demonstrative process that reaches the following stage:

$$\dfrac{\dfrac{\text{DaA} \quad \text{DeB}}{\text{AeB}} \qquad \text{BaC}}{\text{AeC}}$$

Without loss of generality, we are allowed to suppose that DaA and BaC are indemonstrable principles. For, since we're assuming that **Theorem 1** is true, even if they are not principles, demonstration of them is finitely terminating. Similarly in other cases.

So given Aristotle's assumptional base, to establish the consequent of **Theorem 3** which will get Aristotle the target result of I.21, he need only consider improvement-paths that make use of the the eleven negative syllogistic moods he recognizes: Celarent, Ferio, Fapesmo, Frisero, Camestres, Cesare, Festino, Baroco, Felapton, Bocardo, and Ferison. Aristotle's task is to show something concerning all possible processes of

improvement using only combinations of these eleven. Let's call any process of improvement using only a combination of the eleven negative moods a process of *pure negative improvement*. Aristotle, then, needs to show that if there is to be a non-terminating purely negative improvement-path targeting a negative proposition φ then CAUSE(φ) contains an infinite a-chain.

The task is apt to seem daunting. For, in principle there are infinitely many possible construction processes for pure negative improvement to follow. So, an argument covering infinitely many distinct possibilities is needed. Here one might recall that Aristotle confronts a similarly large task in *Prior Analytics* I.23. There the issue was to show that every possible syllogism (of any arbitrary length) can be 'perfected' through single-inference first-figure moods. But there are infinitely many possible syllogisms. And to argue for this conclusion, Aristotle uses an elegant strategy. First, Aristotle gathers a small finite collection of patterns and proves that anything conforming to one of the patterns has the relevant property. Then he argues that each of the (infinitely many) syllogisms to be considered are wholly constituted out of argument tokens instantiating patterns in the collection. But since all deductions are so constituted, each must itself have the relevant property. So by explicitly treating only finitely many patterns, Aristotle thus aims to establish a conclusion covering a unlimited collection of cases.

Aristotle, I hypothesize, is using an analogous strategy in *Posterior Analytics* I.21. He wants to prove a conclusion covering a potentially infinite class of cases, by establishing something about a well-chosen finite collection of patterns. He sees that pure negative improvement is governed by certain determinate laws. And given these laws, he argues that non-terminating pure negative improvement can only occur through infinite improvement-paths that use only a small number of moods. So Aristotle looks to show that all the infinite improvement-paths of this kind require infinite a-chains.

Let's begin by considering which mood combinations occur in which patterns in purely negative improvement. Given Aristotle's work in the *Prior Analytics* on the 'reduction' of the second and third figures to the first, the following grouping of syllogistic moods would already be salient to him[38]:

C-group: {Celarent, Cesare, Camestres}

F-group: {Ferio, Festino, Felapton, Ferison, Fapesmo, Frisesomorum}

B-group: {Baroco, Bocardo}

[38]The Aristotelian salience of the above grouping, is conveniently reflected in the traditional names for the negative syllogistic moods given to us by the medievals. For, it was with an eye to the workings of Aristotle's 'reduction' of the other figures to the first that the initial letters of the traditional names were assigned. Hence, the important grouping for us conveniently turns out to be a partition of the 11 mood-names under the equivalence relation 'having a name with the same first letter'. The **C-Group** consists of Celarent (first figure) and the moods in other figures whose validity Aristotle establishes through the evident validity of **C**elarent and conversion; the **F-group** consists of Ferio (first figure) and the moods in other figures whose validity Aristotle establishes through the validity of **F**erio and conversion; the **B-group** consists of the moods whose validity Aristotle notes that we can only be establish through the evident validity of **B**arbara via reductio (or *ekthesis*).

As Aristotle says in *Prior Analytics* I.24, every syllogism proving a negative conclusion has exactly one negative premise. And the above grouping reflects correspondence between quantity of negative conclusion and quantity of the premise used to prove it:

	Conclusion type	Negative premise type
C-group	e	e
F-group	o	e
B-group	o	o

The non-existence of negative moods in which an o-premise is used to show an e-conclusion, witnesses Aristotle's oft-repeated claim that one cannot prove a universal through a particular premise. Note that syllogisms in the F-group always prove o-conclusions, but do so without using o-premises. As a result, syllogisms in the F-group cannot be concatenated with one another. For instance, suppose we have a partial demonstration:

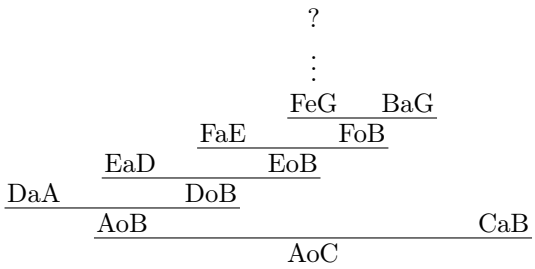
$$\frac{\text{AeB} \qquad \text{BiC}}{\text{AoC}}$$

One cannot improve on the above using any other syllogism from the F-group because its premises include no o-propositions. In fact, since neither an e-conclusion nor an i-conclusion can be established from an o-premise, no improvement-path originating with the above syllogism can ever employ a second mood from the F-group. The B-group and C-group differ. Their negative premise type matches their conclusion type. So improvement-paths can exhibit C-group iteration or B-group iteration:

$$\cfrac{\text{DaA} \qquad \cfrac{\cfrac{?}{\vdots}\ \text{FeD} \qquad \text{FaE}}{\text{DeE}} \qquad \text{EaB}}{\text{AeB}} \qquad \text{BaC}}{\text{AeC}}$$

$$\cfrac{\text{DaA} \qquad \cfrac{\cfrac{?}{\vdots}\ \text{FaE} \qquad \text{FoB}}{\text{EaD}} \qquad \text{EoB}}{\text{AoB}} \qquad \text{CaB}}{\text{AoC}}$$

Note however, that no mood from the B-group can ever appear above a C-group mood. For, again, it is impossible to prove a universal conclusion (here: an e-proposition) from a particular premise (here: an o-proposition). For the same reason, no mood from the B-group can appear above a mood from the F-group. Though, of course, an F-group mood can appear above a B-group mood. For instance, improvement of the syllogism above on the right might continue with a Felapton:



If improvement did so continue, no B- or F-group mood could later occur. If pure negative improvement were to continue, it would have to involve only further use of C-group moods. In line with the above, let's introduce the following definitions.

Definition

1. Where Σ is a syllogism:

   (a) Σ is a *C-string* iff it uses only C-group moods

   (b) Σ is a *F-string* iff it uses only F-group moods

   (c) Σ is a *B-string* iff it uses only B-group moods

2. Where $\overline{D}$ is an improvement-path:

   (a) $\overline{D}$ is a *C-improvement-path* iff $\overline{D}$ contains only C-strings

   (b) $\overline{D}$ is a *F-improvement-path* iff $\overline{D}$ contains only F-strings

   (c) $\overline{D}$ is a *B-improvement-path* iff $\overline{D}$ contains only B-strings

From Aristotle's Laws of Quantity and Quality the following observation about how C-, F-, and B-strings interact is immediate:

Observation 1

A syllogism which uses only negative inferences must instantiate exactly one of the following forms:

1. [Conclusion]⟵[C-string]⟵[Single negative premise]

2. [Conclusion]⟵[B-string]⟵[Single negative premise]

3. [Conclusion]⟵[F-string]⟵[Single negative premise]

4. [Conclusion]⟵[B-string]⟵[F-string]⟵[Single negative premise]

5. [Conclusion]⟵[B-string]⟵[F-string]⟵[C-string]⟵[Single negative premise]

6. [Conclusion]⟵[F-string]⟵[C-string]⟵[Single negative premise]

There are examples illustrating **Observation 1** above. But let's see some more. if a process of pure negative improvement reaches the following stage, no C-string can be appended before an F-string is appended:

$$\frac{\text{BaA} \quad \text{BoC}}{\text{AoC}}$$

Another example—if a process of pure negative improvement reaches the following stage, every further negative improvement must use one of Celarent, Cesare, Camestres:

$$\frac{\dfrac{\text{AeD} \quad \text{DiB}}{\text{AoB}} \quad \text{CaB}}{\text{AoC}}$$

For anyone who has spent a good deal of time with syllogistic, **Observation 1** is easy to grasp. It falls right out of the Laws of Quality and Quantity. And Aristotle's I.21 idea of treating negative improvement processes by assuming a prior fact about positive improvement processes, shows that in inventing the proof of *Posterior Analytics* I.19-22 Aristotle paid very much attention to these kinds of facts. I submit that the argument of *Posterior Analytics* I.21 is crucially guided by **Observation 1**. The important upshot of **Observation 1** is that the consequent of **Theorem 3**—the thesis argued for in *Posterior Analytics* I.21—follows immediately from Aristotle's assumptional base if he can show that given this assumptional base: demonstrative C-string, F-string and B-string construction are each finitely terminating. For, what **Observation 1** makes clear is that if (i) all demonstrative C-improvement-paths are finite, (ii) all demonstrative F-improvement-paths are finite, and (iii) all demonstrative B-improvement-paths are finite, then purely negative demonstrative improvement must be finitely terminating. Finitely many finite things added together are finite.[39] I propose that Aristotle's strategy in the proof sketched by I.21 is to show that (i)-(iii) hold if there are no infinite a-chains.

Note that the argument from the finitude of all demonstrative C-, F-, and B-improvement-paths to the finitude of all negative demonstrative processes is valid. I hypothesize that this is what Aristotle is trying to show. Let us, then, test the hypothesis. If Aristotle was pursuing this line of proof in I.21—trying to show that given his assumptions: (i) all demonstrative C-improvement-paths are finite, (ii) all demonstrative F-improvement-paths are finite, and (iii) all demonstrative B-improvement-paths are finite—then how would he proceed? What cases, and which of the 11 negative moods should he consider? The case of F-improvement-paths can be immediately set aside. We have already noted that one cannot improve on an F-subsyllogism by appending another F-syllogism. It is clear, then, that every F-string is of length one. Indeed by **Observation 1** it follows that every negative syllogism can have at most one inference in a mood from the **F-group**. We've noted that case is otherwise with syllogisms in the **C-group** (e-premise,

[39] Contra Barnes-Lear, I hypothesize that this is the idea behind the text expressing the 'generalizing step' of 82b29-33.

e-conclusion) and **B-group** (o-premise, o-conclusion). Aristotle himself notes (82b14-16, 82b23-25), that one can (say) improve a syllogism in Camestres using Celarent or Cesare; and one can improve a syllogism in Bocardo using Bacoco. Accordingly, while all F-strings are of length one, C-strings and B-strings can, in principle, be of any arbitrary length. So, given the interpretive hypotheses that I.21 is trying to show something concerning all negative demonstrations (universal and particular) using the proof strategy set up by **Observation 1**, we would expect Aristotle to focus on arguing that every C-improvement-path and every B-improvement-path must be finite if there are no infinite chains of a-propositions.

This, I maintain, is precisely what the sketch of *Posterior Analytics* I.21 is doing. And once we see this it makes perfect sense that Aristotle discusses exactly: improvement by Celarent, improvement by Camestres, and improvement by Bocardo. There are three syllogistic moods in in **C-group** and two syllogistic moods in the **B-group**. We want to show that both C-improvement and B-improvement are finitely terminating if no infinite a-chains in any CAUSE(φ). Aristotle correctly observes that this can be shown once we see that the behavior of C-improvment and B-improvement is determined by three well-defined patterns. One pattern is exemplified by Celarent; a second is exemplified by Camestres; and the only mood that exemplifies the third pattern is Bocardo. In I.21 Aristotle is trying to get his auditors to understand these three patterns. For once the three patterns are understood, and once it is seen that Cesare and Baroco also conform to one of these patterns, Aristotle will have what he needs to prove the necessary thesis about B- and C-improvment paths. Here, then, is the relevant observation about patterns.

Observation 2

With respect to improvement, each of the five moods in **C-group** $\cup$ **B-group** instantiate one of three patterns:

1. In applying Celarent or Cesare in an improvement targeting a premise φ, we (i) assume a fresh positive premise stating that our new middle term Y holds of all of the subject of φ, and (ii) assume a fresh negative premise that relates Y negatively to the predicate of φ.

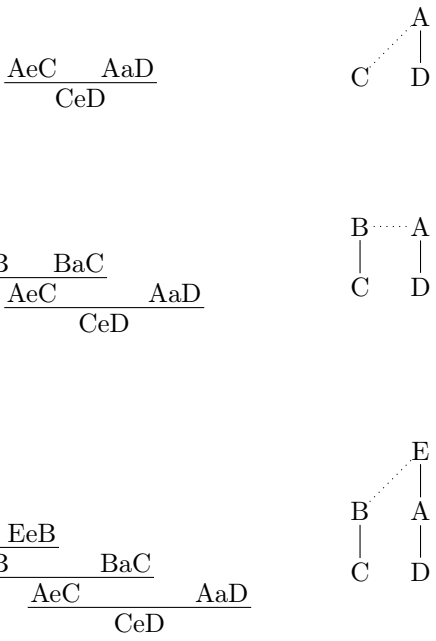
2. In applying Camestes and Baroco in an improvement targeting a premise φ, we (i) assume a fresh positive premise stating that our new middle term Y holds of all of the predicate of φ, and (ii) assume a fresh negative premise that relates Y negatively to the subject of φ.

3. In applying Bocardo in an improvement targeting a premise φ, we (i) assume a fresh positive premise stating that the subject of φ holds of all of our new middle term Y, and (ii) relate Y negatively to the predicate of φ.

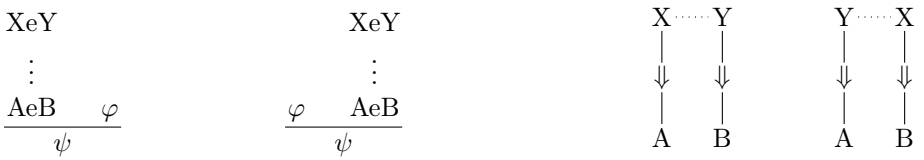
In **Observation 2.1-3**, both (i) and (ii) are significant. The relevant facts are perhaps clearer in diagrammatic presentation. Let us adopt a pictorial convention on which we represent a premise GaH by placing G above H and connecting them with a single line, and represent a negative premise GxH (with $x \in$ {e,o}) by connecting G and H with a dotted line. Then **Observation 2** may be helpfully displayed:

Pattern 1	Pattern 2	Pattern 3
Celarent $\dfrac{\text{XeY} \quad \text{YaZ}}{\text{XeZ}}$	Camestres $\dfrac{\text{YaX} \quad \text{YeZ}}{\text{XeZ}}$	Bocardo $\dfrac{\text{XoY} \quad \text{ZaY}}{\text{XoZ}}$
Cesare $\dfrac{\text{YeX} \quad \text{YaZ}}{\text{XeZ}}$	Baroco $\dfrac{\text{YaX} \quad \text{YoZ}}{\text{XoZ}}$	
		

We begin with how the above observation can be used to show that C-improvment must be finitely terminating given our assumptional base. As shall be clear soon, the case for B-improvement-paths is similar. Suppose we have a C-improvement process with initial segment displayed below:



Different C-moods are used at different improving steps. But the upshot of **Observation 2** is that in each improving step our new middle term appears at the 'top' of a chain of a-linked terms—a chain which corresponds to a chain of a-propositions. More precisely, the new middle term always appears at the 'top' of a term-chain whose 'bottom' is either the subject or predicate of the original conclusion CeD. Such is the insight **Observation 2** expresses about C-improvement: it always conforms to Patterns 1 and 2 (see diagram for **Observation 2** above). Accordingly, iterative C-improvement works by extending *pre-existing* premise chains of a-connected terms: chains which originate in the major and minor terms of the conclusion. Parts (i) and (ii) of **Observation 2.1-3** jointly explain why this is so. The negative premise that an instance of C-improvement targets always contains exactly the two terms X,Y that 'top' the two pre-existing chains of a-connected terms:

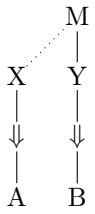


Then what C-improvement does is assume a fresh middle M that it puts above one of {X,Y} and relates negatively to the other, thus yielding a new negative premise containing the exact two terms that 'top' the new term up-chains. This happens with Celarent and Cesare:

$$\frac{\text{XeM} \quad \text{MaY}}{\text{XeY}} \qquad \frac{\text{MeX} \quad \text{MaY}}{\text{XeY}}$$

$$\vdots \qquad\qquad \vdots$$

$$\text{AeB} \qquad\qquad \text{AeB}$$

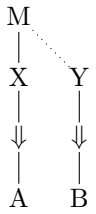


and with the remaining C-mood, Camestres:

$$\frac{\text{MaX} \quad \text{MeY}}{\text{XeY}}$$

$$\vdots$$

$$\text{AeB}$$



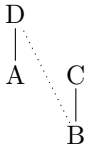
On the basis of the above, Aristotle correctly sees that a C-improvement process AeB only continue ad infinitum if there were infinite upwards extending chains originating in A or B.

Observation 2 points to a parallel insight about B-improvement. But here it is not only up-chains but also down-chains that are involved. It was, perhaps, by noting such proof-theoretic patterns that Aristotle's originally discovered the importance of both predicational up- and down-chains for the argument of *Posterior Analytics* I.21. Iterative B-improvement always works by extending either a *pre-existing* a-proposition up-chain originating from the predicate of the conclusion (Baroco) or a *pre-existing* a-proposition down-chain originating from the subject of the conclusion (Bocardo). To see this, the reader may want to consider the following:

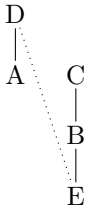
$$\frac{\text{AoB} \quad \text{CaB}}{\text{AoC}}$$



$$\frac{\dfrac{\text{DaA} \quad \text{DoB}}{\text{AoB}} \quad \text{CaB}}{\text{AoC}}$$

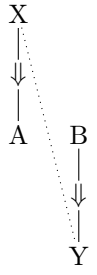


$$\frac{\dfrac{\text{DaA} \quad \dfrac{\text{DoE} \quad \text{BaE}}{\text{DoB}}}{\text{AoB}} \quad \text{CaB}}{\text{AoC}}$$



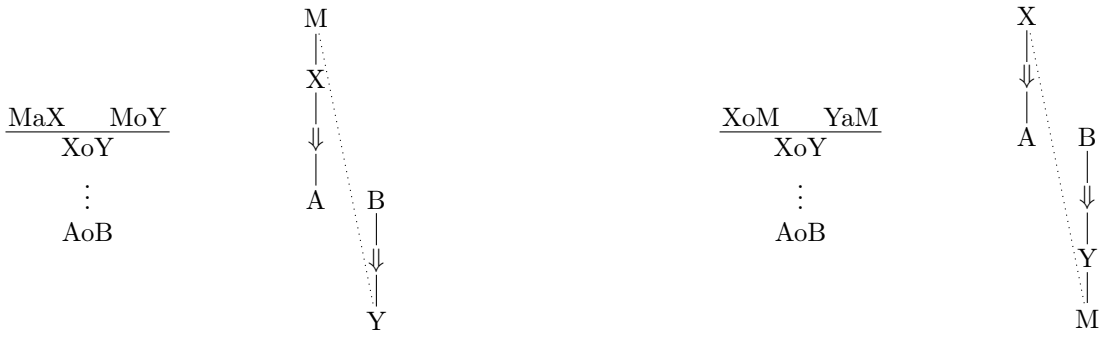
Again, Aristotle's claim will be that the above improvement-path could only continue ad infinitum if there were an infinite predicational down-chain originating in C or an infinite predicational up-chain originating in A. For iterative B-improvement works by extending pre-existing chains of a-connected terms: chains which originate in the major and minor terms of the conclusion. Again, **Observation 2** explains why this is so. The negative premise XoY that an instance of B-improvement targets always contains exactly the two terms X,Y that 'top' two pre-existing chains of a-connected terms:

$$\frac{\begin{array}{c}\text{XoY}\\ \vdots\\ \text{AoB} \quad \varphi\end{array}}{\psi} \qquad \frac{\begin{array}{c}\text{XoY}\\ \vdots\\ \varphi \quad \text{AoB}\end{array}}{\psi}$$



B-improvement takes a fresh middle M that it either puts above X or below Y, and relates M negatively to the other thus yielding a new o-premise. Whether using Baroco or Bocardo: one term used in the new

o-premise will always be at the top of the chain originating in the major of the conclusion, while the other will be on the bottom of the chain originating in the minor of the conclusion. Thus Baroco (on the left) and thus Bocardo (on the right):



Admittedly, *Posterior Analytics* I.21 does not provide a complete analysis of C-improvement-path or B-improvement-path construction.[40] Nonetheless, we have very good reasons to believe that the train of thought outlined above lies behind the proof sketch of I.21. For not only does the interpretive hypothesis have Aristotle sketching a valid argument for the desired conclusion.[41] The interpretation provides an excellent explanation for why I.21 considers exactly Celarent, Camestres, and Bocardo. Once syllogistic Laws of Quality and Quantity are in the background, Aristotle only needs to show that C-improvement and B-improvement require infinite a-chains to proceed ad infinitum. To see that this is so, one must understand the specifics of how C- and B-improvement work by building extended chains of a-propositions. But this can be understood once one understands exactly the three patterns indicated in **Observation 2**. Celarent illustrates the first pattern. Camestres illustrates the second. The *only* mood that illustrates the third pattern is Bocardo. Consideration of no other mood would be helpful for the proof. To consider further moods would be redundant. And to consider (say) Baroco or Cesare rather than Bocardo would leave the third pattern unaccounted for. Once the three patterns are grasped and it is seen that all various moods conform, one can see that ad infinitum C-improvement and B-improvement require infinite a-chains. Contra Barnes-Lear, it is in no way surprising that I.21 discusses Bocardo. And contra Philoponus-Ross, Aristotle has no need for parallel discussions of all the other 8 negative moods (the **F-group**, for instance can be

[40]Concerning the latter, Aristotle may be assuming his audience's familiarity with the account of C-string construction provided a few pages earlier in *Posterior Analytics* I.15 (79b14-18):

> If there is a middle [between A and B, for a proof of AeB], one of them [i.e. either A or B] must be in some whole. For, the syllogism [showing AeB] will be in either the first or the middle figure. So, [i] if [he syllogism] is in the first [figure], then B will be in some whole (for the premise concerning this [i.e., the premise concerning B, the minor premise] must be made affirmative) and [ii] if [the syllogism] is in the middle [figure], then one or the other of them, [either the major term A or the minor term B], will be in some whole.

[41]The details are worked out in the appendix.

ignored entirely). The problem is that both Philoponus-Ross and Barnes-Lear fail to discern the overall proof strategy in the context of which Aristotle's choice of considering exactly Celarent, Camestres and Bocardo—and no others—would make perfect sense.

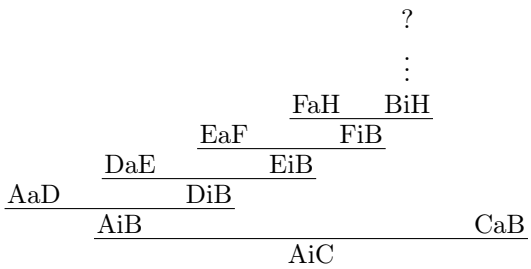
This is not yet a fully formed proof. It is still a proof sketch. The appendix to this paper fills in the technical gaps, and brings it to a successful conclusion. It is arguable that because Aristotle lacked technique of mathematical induction, he was in no position to make any of this rigorous and fully general. In oral presentation he presumably would have given a fuller explanation than what we get in the bare bones sketch preserved in *Posterior Analytics* I.21. Maybe he would have used diagrams like the ones described in I.19. Nonetheless, I do believe that there is good reason to hold this to be more-or-less the proof Aristotle is aiming at.

4.4.2 Positive particular improvement: the Bridge Lemma

I've been developing a reading of *Posterior Analytics* I.20-21 on which Aristotle is (correctly) assuming that in syllogistic **Theorem 2** if **Theorem 1**. Let's call this the *Bridge Lemma*. The text at I.21 82a36-39, in the context of I.20 which deals only with Barbara, certainly seems to show Aristotle relying on the Bridge Lemma. But no proof of the Bridge Lemma is given in the extant text. Yet Aristotle needs it to complete his proof of the Main Lemma. I contend that Aristotle omits a proof of the Bridge Lemma while including a proof-sketch for the claim that **Theorem 1** and **Theorem 2** collectively entail **Theorem 3**, simply because the Bridge Lemma can be proved by an argument exactly parallel to the sub-argument for the finitude of C-improvement-paths witnessed by the text of I.21. Once one sees how to prove the more difficult result that gets you **Theorem 3** from an argument assuming **Theorems 1-2**, it is straightforward to prove the easier result of establishing that if **Theorem 1**, **Theorem 2** follows. It is not simply that there are strong similarities between what it takes to prove Bridge Lemma and the case of C-improvement considered in I.21. The Bridge Lemma can be proved through argument following what are essentially exactly the same steps, as those considered in proving that infinite C-improvement requires infinite a-chains in I.21. A modern logician could straightforwardly produce a single generalized proof that covers both cases simultaneously. The absence of a discussion of the Bridge Lemma in the transmitted text may very well mean that Aristotle was aware of something like this.

Proof of the Bridge Lemma will begin in the same way Aristotle proceeds in I.21. First one will assume **Theorem 1** and the antecedent of **Theorem 2**—namely that for no φ does CAUSE(φ) contain an infinite a-chain. Since the antecedent of **Theorem 2** is the same as the antecedent of **Theorem 1**, we can again restrict attention to improvement-paths that do not use Barbara. Consider, for instance, a partial demonstration of

the form:



Given our assumptional base, without loss of generality we can assume that every a-proposition in the partial demonstration above is a principle. For (again) if any such a-proposition isn't a principle, given our assumptional base we know that it can be proved from first-principles in just finitely many steps. So like before, and on account of the same reasons, we can exclusively focus on improvement-paths which at every step use moods in {Darii, Disamis, Datisi, Darapti}: the four positive particular moods Aristotle countenances. Again, the proof will be guided by an elementary observation which falls directly out of Prior Analytics I.24-5. Note the following:

	Conclusion type	Premise types
Darii, Datisi, Disamis	i	a, i
Darapti	i	a, a

Definition

1. A *D-string* is a syllogism using only inferences in {Disamis, Datisi, Darii}

2. A *D*-string* is a syllogism using only inferences in Darapti

Observe that every D*-string is of length 1. For Darapti has a particular conclusion, and a particular premise cannot be used to prove a universal conclusion.

$$\frac{\text{AaB} \quad \text{CaB}}{\text{AiC}}$$

One cannot improve on the tree above by applying Darapti in a subproof of either premise. Indeed, one also cannot improve on the tree above by putting a D-string over any premise. If an infinite demonstrative improvement-path were to begin with the above, every subsequent improvement would go through Barbara. The case is different with D-strings. Improvements using {Disamis, Datisi, Darii} can be iterated as in the example partial demonstration above our last definition. And given **Laws of Quality and Quantity**, we have the following observation:

Observation 3

A syllogism which uses only inferences in {Darii, Disamis, Datisi, Darapti} must instantiate exactly one of the following forms:

1. [Conclusion]⟵[D*-string]
2. [i-conclusion]⟵[D-string]⟵[Single i-premise]
3. [i-conclusion]⟵[D-string]⟵[D*-string]⟵[Single i-premise]

From the above, it is clear that to prove the Bridge Lemma, one only needs show that given the relevant assumptions every D-improvement-path must be finite. But the argument for this is again perfectly parallel to the case for C-improvement-paths above. For , proof-theoretically the D-string trio of Darii, Disamis, Datisi is in a very close correspondence to the C-string trio Celarent, Cesare, Camestres. Indeed, C-improvement is more structurally similar to D-improvement than it is to B-improvement. For our purposes, one need only note the following, analogous to **Observation 2**.

Observation 4

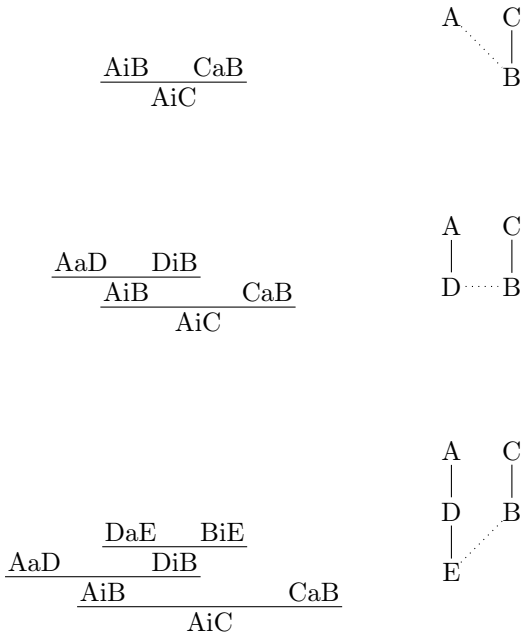
With respect to improvement, each of the three moods in {Darii, Disamis, Datisi} instantiate one of two patterns:

1. In applying Darii or Datisi in an improvement targeting a premise φ, we (i) assume a fresh positive premise stating that the predicate of φ holds of all of our new middle term Y, and (ii) relate Y in an i-proposition to the subject of φ.
2. In applying Disamis in an improvement targeting a premise φ, we (i) assume a fresh positive premise stating that the subject of φ holds of all of our new middle term Y, and (ii) relate Y in an i-proposition to the predicate of φ.

Again, pictorial representation may be helpful. As before we represent a premise GaH by placing G above H and connecting them with a solid line. But now we represent a particular premise GiH with a dotted line between G and H. **Observation 4** is then displayed:

Pattern 4	Pattern 5
Disamis $\dfrac{\text{XiY} \quad \text{ZaY}}{\text{XiZ}}$	Darii $\dfrac{\text{XaY} \quad \text{YiZ}}{\text{XiZ}}$ Datisi $\dfrac{\text{XaY} \quad \text{ZiY}}{\text{XiZ}}$
	

In a certain sense, D-string construction is just C-string construction in a mirror. To see an aspect of this, the reader is invited to compare Patterns 1 and 2 in our pictorial rendering of **Observation 4** with Patterns 4 and 5 above. For the perspective of later logical traditions, this congruence should not be surprising. A modern logician will explain it by saying that while e-propositions are left-monotone decreasing and right-monotone decreasing, i-propositions are left-monotone increasing and right-monotone increasing. A medieval logician would explain it by pointing out that the subject and predicate of e-propositions both have distributed supposition, while those of i-propositions both have determinate supposition, and then appealing to rules of ascent and descent. Aristotle doesn't have a technical terminology for any of this. But he is capable of noticing that in the C-case we always build up, while in the D-case we always build down. Suppose we have a D-improvement process with initial segment displayed below:



In building a D-improvement-path we always continue downwards, extending a pre-existing chain of a-propositions, just as in case of C-improvement we went upwards. The above observation—essentially the mirror image of the observation concerning C-improvement-paths he makes explicit in I.21—yields the needed result for D-improvement-paths. And from this, the Bridge Lemma follows.[42]

4.4.3 The case of Barbara (I.20)

The above arguments establish the following two results for demonstrative-proofs:

- If **Theorem 1** and **Theorem 2**, then **Theorem 3**
- If **Theorem 1** then **Theorem 2**

Posterior Analytics I.20 provides argument for **Theorem 1**: that if there are no infinite predicational up-chains and down-chains, then the process of demonstrating a (demonstrable) a-proposition is finitely terminating. Given the results established above, the Main Lemma in its full generality will follow if **Theorem 1** is secured.

Prima facie, the task of proving **Theorem 1** might appear simpler than proving the results above. After all, since exactly one syllogistic mood (Barbara) can prove an a-proposition, we can ignore all 15 other

[42] Our appendix supplies the remaining technical details.

moods. But appearances can be deceiving. In every syllogism proving an e-conclusion, there is exactly one e-premise; in every syllogism proving an o-conclusion there is at most one o-premise; and in every syllogism proving an i-conclusion there is at most one i-premise. But for all n (with $2 \leq n < \omega$), there is a syllogism establishing an a-proposition with n-many a-premises. As a result, iterated improvement through Barbara will be fundamentally different than iterated C-improvement, D-improvement, or B-improvement. So the structure of the problem confronting Aristotle when he attempts an argument for **Theorem 1**, is of a rather different character than the others.

The brief and highly condensed argument of *Posterior Analytics* I.20 runs as follows (82a21-30):

> So then, that the intermediates [mediating AaZ] cannot be infinite, if the predications come to a stop upwards and downwards is clear. By upwards I mean towards the more universal, by downwards I mean towards the more particular. For if when A is predicated of Z the intermediates (the B's) are infinite, it is clear that it would be possible [for predications to continue ad infinitum]; such that [for instance] both from A downwards one thing is predicated of another ad infinitum (for before one has reached Z there are infinitely many intermediates), and from Z upwards there are infinite [intermediates] before one has reached A; with the result that if these things are impossible, also it is impossible for there to be infinitely many intermediates.

As in I.19, Aristotle here represents the possibility that the process of demonstrating AaZ continues ad infinitum as a possibility on which there are infinitely many intermediates between two extremes. It is reasonable of Aristotle to do so. For it is indeed the case that infinitely iterated Barbara demonstrative improvement would require a structure with the characteristics Aristotle here describes.[43]

In I.20 82a24-30, Aristotle reasons by contra-position. He supposes there are infinitely many intermediates between A and Z. And states that in this case there must be an infinite a-chain down from A or an infinite up-chain up from Z. Aristotle finds the truth of this conditional 'clear' [*dēlon*], and provides no substantial defense of it in the extant text of the *Posterior Analytics*. Viewing his intuition as sufficient, he moves on to negative improvement in *Posterior Analytics* I.21. Given the truth of the conditional, **Theorem 1**—that if demonstration of a demonstrable a-proposition continues ad infinitum, then reality contains either infinite predicational up-chains or infinite predicational down-chains—does follow. The conditional is indeed true, and can be viewed as a special case of a well-known theorem in Order Theory.[44] But Aristotle was in no position to prove this fact. Doing so requires something like our mathematics of the actually infinite, and Aristotle knew nothing of the sort. He attempts no such proof. As noted above, when Aristotle comes in

[43] See appendix.

[44] A partially-ordered set contains no infinite totally-ordered sets iff it has no infinite up-chains or down-chain. The theorem essentially depends on Zorn's lemma.

sight of it he just says that it is 'clear'. Even if Aristotle is unable to explain why the conditional holds, he is correct that it does. For someone who has principled reasons to reject actually infinite structures, Aristotle has remarkably good counter-factual intuitions about what such structures would be like, were they to exist.

Chapter 5

Demonstrating ad infinitum II: metaphysical issues

5.1 Introduction

The goal of *Posterior Analytics* I.19-22 is to prove a claim about demonstrative processes. To establish the possibility of episteme, Aristotle wants to show that every scientific truth is grounded in first-principles from which it can be demonstrated in just finitely many steps. It is not obvious that this should be so, and indeed 'some people' [*tines*] (I.22 84a30-33), the Ad Infinitum Skeptics of I.3, suggest that there are no first-principles with the result that demonstrative processes never successfully terminate.

Demonstrative processes can make use of all 16 syllogistic moods.[1] But the line of argument witnessed by *Posterior Analytics* I.20-21 shows that the only way for a demonstrative-process targeting φ to continue ad infinitum is if there is an infinite a-chain in CAUSE(φ). In particular, the argument shows that CAUSE(φ) would have to contain an infinitely long chain of one of the following forms:

- $(YaX_0, X_0aX_1, X_1aX_2, ..., X_naX_{n+1}, ...)$
- $(X_0aY, X_1aX_0, X_2aX_1, ..., X_{n+1}aX_n, ...)$

Aristotle is indeed right, that in his syllogistic a non-terminating improvement-path would require an infinitely long up-chain or down-chain. Insofar as both he and the Ad Infinitum Skeptics rule out circular demonstrations, it will follow that the infinitely long chains in question will have to use infinitely many

[1] Or so, at any rate, Aristotle is prepared to concede in I.19-22.

distinct terms.[2]

We noted above that the argument of *Posterior Analytics* I.20-22 proceeds in two steps. Aristotle first argues that the process of demonstrating a scientific truth φ can continue ad infinitum only if CAUSE(φ) contains an infinite chain of a-propositions. He then argues that in fact, for no scientific-truth φ does CAUSE(φ) contain an infinite chain of a-propositions. It is this latter claim that Aristotle defends in *Posterior Analytics* I.22.

The discussion of *Posterior Analytics* I.22 is infamously hard to follow. The chapter seems to offer three arguments against the possibility of infinite a-chains of the relevant type. But it is far from clear how they are supposed to establish the target conclusion. The three arguments are not entirely independent—the first (82b37-83b31) and the third (84a7-28) clearly use several of the same premises. And the second (83b32-84a6) looks like a recapitulation of what's at stake in I.19-22, rather than any kind of argument at all. Providing a fully detailed treatment of I.22 is beyond the scope of the present undertaking. Instead, I aim to present only an account of the chapter's first argument. This first argument is, by far, the chapter's longest and most developed. Moreover, it introduces several ideas on which the third apparently also depends. We designate it the *Main Argument* of I.22.

Aristotle's basic strategy the Main Argument is simple enough. He wants to show that if φ is a scientific truth, then CAUSE(φ) contains no infinite a-chains. So he presents a criterion that a-propositions have to meet if they enter into any such CAUSE(φ). And he goes on to argue that no chain of a-propositions which met the criterion in question could be infinitely long in either the upwards or downward direction. The criterion is easy to state: the a-propositions in any CAUSE(φ) must express *genuine predications*. But what exactly the criterion amounts to, and why it rules out infinite a-chains of the relevant sort are vexed questions. Barnes has written of I.22: 'None of the arguments is successful; and they cannot be reformulated in such a way as to furnish proof of Aristotle's contention'.[3] In what follows, I present a reading of I.22 on which the Main Argument comes out valid, and plausibly sound by Aristotelian lights.

We begin with the topic of predication and Aristotle's notion of 'genuine predication' [*katēgoriein haplōs*]. On the Main Argument, for an a-proposition to enter into demonstrative science not only must it be true; it must express a genuine predication. Getting clear on the nature of genuine predication will involve us in a discussion of how Aristotle conceives of truth conditions for a-propositions. And in order to develop my view I will be introducing a novel, though by no means unprecedented, semantics for assertoric syllogistic.

[2]For details on how the prohibition of circular demonstrations forces an infinity of terms in such a chain, see esp. **Lemma 11** in the appendix.

[3]Barnes 181.

5.2 Predication and truth

5.2.1 Genuine predication

In the *Analytics*, Aristotle conceives of predicating [*katēgoriein*] as a linguistic action. It is something one does with words, and it can be done truly or falsely. However, things that get predicated, one of another, are not mere representations of beings. It is beings themselves that get predicated.[4] Described from an operational perspective, predicating is well characterized as a function that takes exactly two (usually extra-linguistic) entities as input, and yields an affirmative statement as output. So, while the result of predicating, say, human being of Socrates is a simple sentence ('Socrates is a human being'), both what gets predicated (here, the species: *human being*), and what gets something predicated of it (here, the individual: *Socrates*) are beings.

Following Aristotle, let us call the result of an act of predicating: a predication [*katēgoria*]. What all genuine predications have in common is affirming that a being X underlies [*hupokeitai*] a being Y as its metaphysical subject—affirming that a being Y holds of [*huparchei*] or belongs to a being X. Predication (qua linguistic action) is a form of affirmation, not denial.

Plato did not systematically distinguish between beings that are universals and beings that are individuals. Aristotle thinks this gets Plato into philosophical problems. And Aristotle emphasizes the distinction between between universal and individual beings in his own writings on logic and metaphysics. A universal (being) is such as to hold of a plurality of other beings; an individual (being) is not.[5] However, Aristotle does not systematically distinguish singular from general *terms*. And he does not see *holding-of* as a relation that in every case an individual bears to a universal. By Aristotle's lights, the *belonging* or *holding-of* relation *animal* bears to *whale*, is not categorically different from that which *animal* bears to *Moby Dick*. An individual can be truly predicated of an individual, and a universal can be truly predicated of both individuals and universals. Accordingly, in contrast to Frege-Russell, Aristotle does not think that the predications:

- Moby Dick is a whale.
- Whales are mammals.
- Moby Dick is Moby Dick.

possess distinct 'logical forms'.

[4] A nice text bringing out this dual (linguistic/extra-linguistic) aspect in predication is *Prior Analytics* I.27 (43a25-6): 'Among all beings [*hapantōn ontōn*], some are such that they are predicated truly and universally [*katēgorieisthai alēthōs katholou*] of nothing else...'. *Prior Analytics* I.27 contains a forward reference to *Posterior Analytics* I.22; we discuss the ontology of I.27 43a25-6 below.

[5] See, e.g. *De Interpretatione* 7; 'such as to hold': if every particular human is destroyed but one, the species human being is not thereby an individual.

So, Aristotle evidently ignores a distinction on whose importance Frege-Russell insist. Of course, Frege-Russell are themselves ignoring a distinction on whose importance Aristotle insists. For, Aristotle famously contends that there are irreducibly different kinds of predication [*genē tōn katēgoriōn*] corresponding to irreducibly different ways for one being to belong to another. Hence Aristotle's division of 'categories' [*katēgoriai*]: some predications [*katēgoriai*] indicate what a subject is ('Rocinante is a horse', 'Triangle is a figure'), others indicate how a subject is quantified ('Rocinante is 450kg'), or qualified ('Rocinante is trustworthy), or related to something else ('Rocinante is a companion'), etc. These predications differ because the *way* that (say) being-a-horse belongs to Rocinante, is irreducibly different from the way that (say) being-450kg belongs to Rocinante. And mutandis mutatis, for the other 'categories'. So, as Aristotle ignores what Frege-Russell see as a 'categorical difference' between (e.g.): (i) 'Moby Dick is a whale' (expressing $\in$), and (ii) 'Whales are animals' (expressing $\subseteq$), Frege-Russell ignore Aristotle's 'categorical difference' between (e.g.): (iii) 'Moby Dick is a whale' (expressing what Moby Dick is), and (iv) 'Moby Dick is 60,000kg' (expressing how much Moby Dick is).[6]

Now, Aristotle takes it that Greek speakers typically make predications by uttering simple sentences of the form: 'X is Y'. And Aristotle further assumes that a Greek speaker who says 'X is Y'—i.e. utters a simple affirmative statement with grammatical subject 'X' and grammatical predicate 'Y'—standardly makes a predication indicating that the being referred to by 'X' underlies the being referred to by 'Y' as its metaphysical subject. But Aristotle also recognizes that there are exceptions. A sentence of the form 'X is Y' does not always predicate (the referent of) 'Y' of (the referent of) 'X'. Of special importance for us is his treatment of the issue in *Posterior Analytics* I.22 (83a1-17)[7]:

> It is possible to speak truly: [**case (a)**] 'the white is walking' and 'this large thing is a log'; again [it is possible to speak truly]: [**case (b)**] 'the log is large' and 'the man is walking'. But to speak in the latter way is different than [to speak] in the former. For, when I say [*phō*] 'the white is log', then I mean [*legō*]: that to which being-white accidentally belongs is a log; I do not mean that the white is the subject for log... In contrast, when I say 'the log is white' I do not [mean]: some different thing is white and being-a-log belongs accidentally to that [i.e. the different thing]... Rather, [I mean that] the log is the subject, the very thing that became white, not in virtue of being something different than [a thing] which just is a log or a certain log. If, then, we must legislate, let it be that saying [*legein*] in the latter way [**case (b)**: e.g. 'the log is white'] is predicating [*katēgorein*]; but saying in the former way [**case (a)**: e.g. 'the white is a log'] is either predicating not at all [*mēdamōs*], or predicating not genuinely [*mē haplōs*] but

[6]The basic historical narrative of this paragraph (in one instance, where Frege-Russell see many Aristotle sees one; in another, where Frege-Russell see one Aristotle sees many) is also developed in Code (1983).

[7]Cf. I.19 (81b24-29) which already hints at the importance of genuine predication for the discussion to follow

predicating incidentally [*kata sumbebēkos*]...

This difficult text is frequently called upon to witness various interpretations of Aristotle's view that substances are ultimate metaphysical subjects. But, the central contrast Aristotle wants to draw here tends to be obscured in such discussions. For, the passage's point is not to defend, or even to state, any specific thesis about substance or *ultimate* metaphysical subjects. Its point is to make a distinction between two ways of 'speaking' or 'saying' [*legein*]—in particular, two ways of speaking *truly*.[8] And to make this distinction Aristotle is just assuming a simplified version of his usual conception of metaphysical subject-hood on which, paradigmatically, perceptible substances are metaphysical subjects for qualities and quantities.[9]

Let us consider, then, the two ways in which Aristotle says 'it is possible to speak [*eipein*] truly'. One kind of speaking Aristotle calls 'predicating' [*katēgorein*]. The other he alternatively describes as 'predicating not at all', or 'predicating not genuinely' but 'predicating incidentally' (a14ff.). Aristotle explains the difference between genuinely predicating ('The log is white') and this predicating 'not at all' or 'incidentally' ('The white is a log') as follows. Typically when a speaker says [*phēsi*] 'The white is log', she will not mean [*legei*] that this white is the metaphysical subject in which being-a-log inheres. Insofar as she speaks truly, the speaker presumably means that this log is a metaphysical subject in which white inheres. Indeed, the statement 'The white is a log' is true precisely because this log is a metaphysical subject of white, and therefore (per accidens) what 'the white' picks out is numerically the same as this log. (That these are the correct truth-conditions is something Aristotle simply takes for granted). So, while a speaker may utter 'The white is a log' and do so truly, there is a certain disconnect between what she means (the truth she communicates) and the grammatical structure of what she says. What is metaphysically a subject is referred to by an expression in grammatical predicate position; and what is metaphysically a predicate is referred to by an expression in grammatical subject position. Such is incidental, non-genuine predication. In contrast, for a speaker who genuinely predicates (saying, e.g. 'The log is white') there is no such disconnect between what is said and what is meant. In a genuine predication, the entity that the grammatical subject picks out really is the metaphysical subject for what the grammatical predicate picks out. Grammatical and metaphysical structure are well-aligned so that the truth-conditions for what the speaker who predicates says, are perfectly mirrored in the grammatical structure of her utterance. Such is genuine predication.

It is the custom of modern commentators to follow the usage of Philoponus, in referring to predications of the form 'The log is white' as *natural predications* and predications of the form 'The white is log' or 'The white is musical' as *unnatural predications*. But this is not entirely felicitous. For one, it is clear from the

[8]That this passage's point is to contrast two ways of speaking is quite clear from its structure. 83a1-4 states the thesis; the *gar* in a5 indicates that the aim of the *hotan men* (a5)/*hotan de* (a9) contrast is to explain the thesis. Finally the distinction between the two ways of speaking is labeled at a14ff.

[9]Simplified: on Aristotle's considered view, the per se subject of a color like white is not a substantial body (like log); it is the surface of a body. See, e.g., *De Sensu* 3.

relevant passages of Philoponus' commentary that the distinction between natural and unnatural predication was a technical development of later Peripatetic logicians involving sub-distinctions beyond any Aristotle makes in *Posterior Analytics* I.22. Accordingly, the Peripatetics' natural/unnatural distinction ought to be studied by historians of logic in its own right, and not terminologically conflated with the distinction of I.22 83a1-17 (whatever it amounts to).

So, I prefer 'genuine predication'—more often, I just say 'predication' (it's *katēgoriein haplōs*)—for what other commentators on *Posterior Analytics* I.22 often call 'natural predication'. Of course, to insist on this would be pedantry. What actually matters here, is to see that Aristotle does not think of (genuine) predication as some specialized case of predication. For Aristotle, predications should not be divided into the 'natural' and 'unnatural' any more than trees should be divided into genuine and plastic. Predication in the style of 'The log is white' or 'The human being is an animal', is predication full-stop: predication in the basic and standard sense. Usually, Aristotle's only reason for even mentioning non-genuine predication is to set it aside as irrelevant for whatever claim about predication he does want to make. And in I.22, the immediate sequel to the 83a1-17 text quoted above, we get the claim that the predicational chains which reflect the real structures that demonstrative sciences make their demonstrations about—that is, the predicational chains at issue in I.22—are chains of (genuine) predications:

> So, let it be assumed that [in the chains we are considering] the predicate is always predicated of what it is predicated of genuinely, not incidentally. For this is how demonstrations demonstrate. Therefore, whenever one thing is predicated of one thing [in a demonstration], either it is in the essence [*ti esti*] or [it indicates] that [the subject is] qualified or quantified or related to something or doing something or undergoing something or somewhere or some-when.

Note that Aristotle sees it as a consequence of the fact that propositions in demonstrative science are predicational (= express genuine predications), that his division of categories will be applicable to them. Indeed, I think Aristotle's considered view is that the division of categories applies to genuine predications and genuine predications alone.[10]

5.2.2 Genuine predication and metaphysical subject-hood

Posterior Analytics I.22 83a1-17 is difficult to understand partly because Aristotle's presentation is excessively digressive. To bring out the passage's central point, I translated it above with several elipses. Now I remove the elipses (83a1-17):

[10]What category could the predication: 'The white is a log', be in? In substance/essence because log is what the thing which underlies the white is? In quality, because the sentence just means the log is white? Cf. esp. *Topics* I.9.

> It is possible to speak truly: **[case (a)]** 'the white is walking' and 'this large thing is a log' ; again [it is possible to speak truly]: **[case (b)]** 'the log is large' and 'the man is walking'. But to speak in the latter way is different than [to speak] in the former. For, when I say [*phō*] 'the white is log', then I mean [*legō*]: that to which being-white accidentally belongs is a log; I do not mean that the white is the subject for log; for neither by being white nor [a thing] which just is white did it become a log. In contrast, when I say 'the log is white' I do not [mean]: some different thing is white and being-a-log belongs accidentally to that [i.e. the different thing]. (For instance, when I utter 'the musical is white', then I mean that the man is white to whom being-musical happens to belong). Rather, [I mean that] the log is the subject, the very thing that became white, not in virtue of being something different than [a thing] which just is a log or a certain log.

As noted above, interpreters have often seen this text as articulating Aristotle's view that substances are ultimate metaphysical subjects: items that underlie everything else while nothing distinct underlies them. Unfortunately, the mistaken impression that Aristotle is making a claim about substance in 83a1-17, is often developed into an analysis of genuine predication in terms of substance. So, for instance, Hamlyn reads I.22 as arguing: 'X is Y' is genuine iff X is a primary substance (in the sense of the *Categories*). And Barnes reads I.22 as arguing: 'X is Y' is genuine iff there is no Z such that Z is Y and Z is accidentally X—making clear that he intends this to amount to: 'X is Y' is genuine iff X is a primary or secondary substance (in the sense of the *Categories*). But both readings are hopeless. *Posterior Analytics* I.22 requires that all propositions which enter into demonstrative sciences are/express genuine predications. But Aristotle most definitely holds that mathematics is a demonstrative science, and most definitely denies that the subject-terms of mathematical propositions are always substances. Aristotle's favorite example of a demonstrable truth is, after all, 'Every triangle has internal angles summing to 180 degrees'. And triangle, for Aristotle, is certainly no substance.

Here is the corrective. *Posterior Analytics* I.22 83a1-17 does use a notion of metaphysical subject-hood to articulate an account of genuine predication. But *ultimate* metaphysical subject-hood is not at issue. For Aristotle, 'X is Y' is a genuine predication iff X *in fact* underlies Y as its metaphysical subject. Construed rightly, this does not entail that X is a substance. Let it be that X and Y both hold of Z. On Aristotle's view, X is a metaphysical subject for Y iff it is in virtue of being X (or in virtue of being a certain X) that Z is available to receive Y. So (83a12-14): 'The log is the subject, the very thing that became white, not in virtue of being something different than [a thing] which just is a log or a certain log'. Likewise, triangle has internal angles summing to 180 degrees not in virtue of being something other than a certain triangle (e.g. bronze, wooden).

In the *Analytics*, for X to be a metaphysical subject it is a sufficient that X is a what and not a such. Here, whats will include beings like Rocinante, log, redness, triangle, walking (the type of action), night; suches will include beings like red, triangular, walks, nocturnal.[11] For Aristotle, this is a metaphysical and not a linguistic distinction. We shall meet it again in our analysis of the Main Argument in **Section 5.3** below.

5.2.3 Genuine predication in a syllogistic context

Up to this point we have discussed Aristotelian syllogistic in purely proof theoretic terms with little reference to how Aristotle conceives of truth-conditions for syllogistic propositions. Aristotle himself says remarkably little about truth-conditions for syllogistic propositions—the great majority of his logical reflections are purely proof-theoretic. But the argument of *Posterior Analytics* I.19-22 turns not only on issues about the internal structure of syllogistic proofs, but also on metaphysical considerations about extra-linguistic reality. The argument of I.22 in particular depends on a view of how demonstrations would have to represent extra-linguistic reality. For *Posterior Analytics* I.22 Aristotle requires that the predications involved in demonstrations be genuine predications. However, it is not perfectly obvious what this requirement amounts to. Aristotle's examples of genuine predications are of the form 'X is Y'. But syllogistic propositions are of the form AxB (with $x \in$ {a,e,i,o}). Here I present a formal and philosophical interpretation of this requirement. The proposal is closely related to my preferred semantics (i.e. truth-conditions) for syllogistic propositions, which I should first like to rehearse.

5.2.3.1 Predicable semantics for syllogistic

I begin by specifying the models in terms of which I'll be describing truth conditions for syllogistic propositions.

Definition 1 Predication structures

A *predication structure* is a triple $K = (\mathcal{D}, P, [\![\cdot]\!])$ such that:

1. $\mathcal{D}$ (domain) is a non-empty set of beings

2. P (predication) is a serial and transitive binary relation on $\mathcal{D}$, i.e.:

[11] In this context the reader is invited to recall *Topics* I.9 (103b27-35):

> It is clear... that someone indicating the what-is-it sometimes indicates a substance , sometimes a quantity, sometimes a quality, and sometimes one of the other categories. For, whenever one says of a given human that the given [thing] is a human or an animal, one says what it is and indicates a substance. But whenever one says of a given white color that it is white or is a color he says what it is and indicates quality. Likewise too if one says of a given foot-long length that it is a foot-long length he says what it is and indicates quantity. Likewise too for the other [categories].

(a) $(\forall x \in \mathcal{D})(\exists y \in \mathcal{D})\ xPy$

(b) $(\forall x, y, z \in \mathcal{D})\ xPy \wedge yPz \rightarrow xPz$

3. $[\![\cdot]\!]$ (semantic value) is an unary function such that for every term T in our language $\mathcal{L}$: $[\![\mathrm{T}]\!] \in \mathcal{D}$

On the intended interpretation, xPy represents the relation 'x is (truly) predicated of y'. And predication here is exactly genuine predication in the sense of I.22. The transitivity of Aristotelian predication, is witnessed by numerous texts.[12] And I submit that the seriality of the predication relation P falls immediately out of Aristotle's basic conception of what a being is. On Aristotle's view, a being is an item which some subject *is*: Socrates is a being just in case some subject (this human being) is Socrates; likewise, the walking is a being just in case some subject is walking. So x is a being iff x is predicated of some y. But since Aristotle assumes that everything is a being, it follows straightaway that everything is predicated of something: that predication is serial.[13]

Note that for each term T in our language $[\![\mathrm{T}]\!]$ assigns T an element of $\mathcal{D}$—not a set of elements of $\mathcal{D}$—as its semantic value. This differs from the standard 'extensional' or 'set-theoretic' semantics one gets for Aristotelian syllogistic in (e.g.) the influential papers of Corcoran (1972) and Smiley (1973). These authors treat syllogistic terms as contemporary logicians treat predicate expressions in first-order logic. In contrast, the predication structures adduced here treat syllogistic terms in the manner that first-order logic handles *names*. And this, I submit, is more in line with how Aristotle himself conceives of syllogistic terms. These days, we build a distinction between general and singular terms into the very design of our logical languages. In standard first-order logic, we reserve lower-case letters for name-constants and variables which pick out 0-order individuals[14]. And we use upper-case letters for predicates and relational expressions which pick out a *sets* of 0-order individuals (or sets of ordered tuples of 0-order individuals). In contrast, Aristotle will treat both terms like 'Socrates' and terms like 'human being' on a par. One picks out a species (a universal entity) the other picks out an individual (a particular entity). But they are both names in the same sense.

Definition 2 Truth conditions

For any predication structure $K = (\mathcal{D}, P, [\![\cdot]\!])$

- $K \models \mathrm{AaB}$ iff $(\forall x \in \mathcal{D})([\![\mathrm{B}]\!]Px \rightarrow [\![\mathrm{A}]\!]Px)$
- $K \models \mathrm{AeB}$ iff $(\forall x \in \mathcal{D})([\![\mathrm{B}]\!]Px \rightarrow \neg[\![\mathrm{A}]\!]Px)$
- $K \models \mathrm{AiB}$ iff $(\exists x \in \mathcal{D})([\![\mathrm{B}]\!]Px \wedge [\![\mathrm{A}]\!]Px)$

[12] See e.g. *Categories* 3*; Prior Analytics* 25b32-35, 44b4ff.; *Posterior Analytics* 96a12-15, *Topics* 122a35-37, etc.
[13] Here see, e.g. Metaphysics Z.1, Γ.2, Δ.6.
[14] The latter relative to an assignment function.

· $K \models \text{AoB}$ iff $(\exists x \in \mathcal{D})([\![\text{B}]\!]Px \wedge \neg[\![\text{A}]\!]Px)$

As usual, we say that a syllogistic proposition φ is *true in* K iff $K \models \varphi$; also that a set of syllogistic propositions Γ are true in K (i.e. $K \models \Gamma$) iff for every ψ in Γ: $K \models \psi$. Finally, we say $\Gamma \models \varphi$ iff for every K such that $K \models \Gamma$ it holds that $K \models \varphi$. With 'semantic validity' so defined, Aristotle's assertoric syllogistic—represented here by the proof system introduced in **Section 2** above—is sound on the class of predication structures. This can easily be checked by appeal to P-canonical form (or P-canonical form minus Ferio and Darii).[15] Given soundness, predication structures can be used for representing Aristotle's method in the *Analytics* of showing invalidity and inconclusiveness via counterexamples. It will be noted, of course, that assertoric syllogistic is far from being complete on the class of predication structures. For instance, given that every syllogistic-deduction requires at least two premises, the semantic validity YaZ$\vDash$XaX corresponds to no syllogistic-deduction. But insofar as our purpose is the formal reconstruction of something in Aristotle, this is how things should be. For, Aristotle's notion of a syllogism isn't designed to capture anything like what we'd call 'semantic validity'. In formulating his syllogistic Aristotle is just not after a proof system that is complete in our sense. [16]

We note, finally, a feature of the semantics important for capturing Aristotle's requirement that the only a-propositions used in demonstrations be genuine predications.

Observation

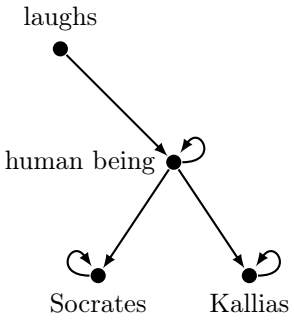
1. In any predication structure K, if $[\![\text{A}]\!]P[\![\text{B}]\!]$, then $K \vDash \text{AaB}$
2. However, there are predication structures $K = (\mathcal{D}, P, [\![\cdot]\!])$ according to which $K \vDash \text{AaB}$ but not $[\![\text{A}]\!]P[\![\text{B}]\!]$.

It may be useful to consider a predication structure K witnessing our **Observation** (xPy indicated by an arrow from x pointing to y; transitive arrows not pictured)[17]:

[15] Note in particular, that transitivity and seriality together with the assumption that for every term T in our language $\mathcal{L}$: $[\![\text{T}]\!] \in \mathcal{D}$, ensure a-conversion: XaY$\models$YiX, giving the 'existential import' of a-propositions.

[16] NB The completeness results of (e.g.) Corcoran and Smith come at the cost of disallowing syllogistic propositions of the form XaX. There is little basis for this in the text, and I reject it here (see my specification of our language $\mathcal{L}$: **Definition 1** of **Section 2**). Of course, it is true that Aristotle rarely talks about syllogistic propositions of the form XaX the argument of Corcoran for disallowing them is entirely based on this omission]. But the case holds no water. The reason why Aristotle rarely talks about such syllogistic propositions is that given his definition of syllogism no proposition of the form XaX can ever occur in a syllogism. For, if XaX occurs in a premise, the conclusion must be identical to the other premise (hence, no syllogism). And no syllogism can conclude XaX because nothing of this form will hold *due to* the premises (hence, no syllogism). Since nothing of the form XaX can occur in as either a premise or a conclusion of a syllogism it is not at all surprising that he rarely discusses them in the *Analytics*. The omissions don't show what Corcoran and Smith want them to.

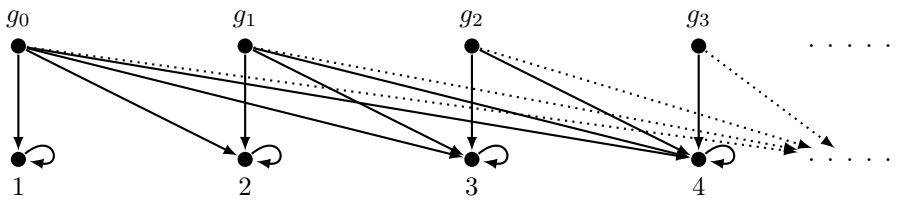
[17] Transitive arrows not pictured: i.e. there is an arrow from laughs to Socrates that is not pictured and an arrow from laughs to Kallias that is not pictured. Recall that the P relation in every predication structure is transitive.



For the structure K pictured above let $[\![\text{H}]\!]$ = human being and $[\![\text{L}]\!]$ = laughs. Adding all transitive arrows, we have both that $K \models$LaH and $K \models$HaL: all human beings laugh and everything that laughs is a human being. But while $K \models$HaL is true, human being is not predicated of laughs: on the structure above it is not the case that $[\![\text{H}]\!]P[\![\text{L}]\!]$. Taking a cue from Aristotle, we may describe this state of affairs as one in which (evaluating at K) a speaker can say HaL truly, but in so saying not make a genuine predication.

The machinery of predication structures, thus allows us to capture a rather fine-grained distinction Aristotle insists on in *Posterior Analytics* I.22—the distinction between a-propositions that are *true*, and a-propositions which *express a genuine predication*. In a predication structure K, for an a-proposition AaB to be true is just for $K \models$ AaB. For an a-proposition to express a genuine predication, it must also hold that relative to the structure K described above: $[\![\text{A}]\!]P[\![\text{B}]\!]$. But as we've just seen, while $[\![\text{A}]\!]P[\![\text{B}]\!]$entails $K \models$ AaB, $K \models$ AaB does not entail $[\![\text{A}]\!]P[\![\text{B}]\!]$. One upshot of this is that we can make precise sense of a possibility on which there is an infinite chain of true a-propositions, but not an infinite chain of genuine predications. Let's see an example.

Suppose that for every natural number n there's a distinct entity g_n: the property *being-numerically-greater-than-n*. Now, no such g_n is itself a natural number. And let's suppose also, reasonably enough, that for all n: the only entities of which the *being-numerically-greater-than-n* property belongs to are natural numbers. For instance, the property *being-numerically-greater-than-5* (i.e. g_5) does not itself have the property *being-numerically-greater-than-4* (i.e. g_4); it's rather the number 5 that has the property *being-numerically-greater-than-4*. In Aristotelian terms, the being g_5 does not have g_4 genuinely predicated of it; but the number 5—a being distinct from the property g_5—does have g_4 genuinely predicated of it. Indeed everything of which g_5 is predicated has g_4 predicated of it. Finally, let's suppose that every natural number is predicated of itself. If we indicate xPy by drawing an arrow from x to y, the structure we've described looks like this:



Note that the P relation pictured is serial (everything bears P to something) and (trivially) transitive. So imagine a predication structure $K = (\mathcal{D}, P, [\![\cdot]\!])$ whose domain is the natural numbers along with all the g_n properties, with P exactly as described above. Suppose that our language has a term for each *being-numerically-greater-than-n* property g_n: terms {G_1, G_2, ...} such that such that relative to K: G_0 names the property g_0 (i.e. $[\![G_0]\!] = g_0$), G_1 names the property g_1 (i.e. $[\![G_1]\!] = g_1$), and so on. Then relative to K there is an infinite down-chain of true a-propositions:

$$G_0aG_1$$

$$G_1aG_2$$

$$G_2aG_3$$

$$G_3aG_4$$

$$\vdots$$

$$\infty$$

Yet there are no infinite P-chains and no infinite chains of a-propositions that express *genuine predications*. For note that for no g_n, g_m is it the case that g_nPg_n. For instance, the properties of *being-numerically-greater-than-4* and *being-numerically-greater-than-5* are not themselves related by P because (as Aristotle might put it) strictly speaking neither property underlies the other as its subject. So, no G_naG_{n+1} is genuine. Although the above structure manifests an infinite chain of true a-propositions, it doesn't manifest an infinite a-chain of true *genuine predications*.

5.2.3.2 Interlude: comparison with related semantic approaches

The semantics of **Definitions 1-2**, constitute a version of the so-called 'non-extensional' or 'predicable semantics' for Aristotelian syllogistic.[18] Support for such an approach emerges from recent work of Mario

[18] I myself find 'non-extensional' a bit misleading. I would say that given **Definitions 1-2** the truth-conditions for a-, e-, i-, and o-propositions *are* functions of terms' extensions (where the extension of a term T is just all the beings of which the

Mignucci (2000), Marko Malink (2006, 2009, 2013), and Benjamin Morison (2008).[19] As mentioned, the chief alternative to predicable semantics is the natural 'set-theoretic' or 'extensional' approach to semantics for Aristotelian syllogistic one gets (e.g.) in Smiley and Corcoran's classic papers and modern introductory logic texts that bother to mention Aristotelian syllogistic.[20]

One can use a semantic interpretation of syllogistic propositions for a variety of purposes. In different contexts, different alternatives are to be preferred. But in the context of representing Aristotle's own conception of truth conditions for syllogistic propositions, predicable semantics strikes me as vastly superior. As mentioned above Aristotle will view terms like 'whale' and 'Socrates' on all fours, and the predicable semantics treats them accordingly. Malink has persuasively argued that there are aspects of Aristotle's logical theory in the *Prior Analytics* for which the familiar extensional semantics can give no plausible reconstruction. And my own formulation of the predicable semantics yields a natural representation of the distinction between a-propositions that do and do not express genuine predications unavailable in the familiar extensional semantics.[21] I will offer no further defense of employing a predicable rather than 'extensional' approach to semantics of syllogistic here. Instead, I propose to argue for the superiority of my particular version of predicable semantics over the alternative and closely related formal rendering of predicable semantics found in the pioneering work of Malink (2006, 2009, 2013).

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entity named by T, i.e. $[\![\mathrm{T}]\!]$, is predicated: $extention_K(\mathrm{T}) = \{x : [\![\mathrm{T}]\!]Px\}$). But though the object language is extensional, the framework allows us systematically track certain non-extensional distinctions that don't show up in our truth-conditions. For instance, if H is human being and L is having the disposition to laugh, the extensions of H and L will be the same—recall that $[\![\mathrm{L}]\!]$ is not predicated of itself. Yet only one of {LaH,HaL} is a genuine predication even though the extensions of subject and predicate in each proposition are identical. For these and other reasons, I myself prefer 'predicable' semantics to 'non-extensional' semantics.

[19] In the context of Barnes (2007), the latter two authors trace their proposals to the late Michael Frede.

[20] Insofar as there's an *interpretive* dispute here, the debate is centered around a disagreement over the correct interpretation of Aristotle's *dictum de omni et nullo*. The *dictum* is, essentially, Aristotle's only explicit statement in the *Analytics* as to how we should understand the meaning of a-propositions and e-propositions. And is apparently supposed to underwrite the 'perfection' (e.g.) of Barbara and Celarent. It is notable that no *dictum* for i- and o-propositions are given (a *dictum de aliquo et aliquo non*, as it were). And concerning e-propositions, the text in fact says next to nothing. So *Prior Analytics* I.1 24b28-30 reads (translating *tou hupokeimenou* with the manuscripts at b29, against Alexander—but as Mignucci (2000) argues nothing turns on this):

> We say that '[A] is predicated of all [B]' when nothing of the subject [i.e., B] can be taken of which the other [i.e., A] will not be said. And [for A is predicated] of no [B], similarly.

Let's use $[\![\ ]\!]$ as a function symbol to represent semantic value. Speaking generally (and ignoring a few small details), proponents of both the 'set-theoretic' and 'predicable' approaches understand the *dictum de omni* as stating:

$$\mathrm{AaB} \text{ iff } \neg(\exists x)([\![\mathrm{B}]\!]Rx \wedge \neg[\![\mathrm{A}]\!]Rx).$$

The disagreement is over (1) whether the variable x (in our metalinguistic representation) should range over items of the same semantic type as $[\![\mathrm{A}]\!]$ and $[\![\mathrm{B}]\!]$, and (2) what precisely R is. Proponents of the set-theoretic semantics assume that x ranges over individuals while the semantic values of syllogistic terms are sets of individuals—items of of a different (semantic) type than individuals. The relation R is then given a set-theoretic interpretation. So AaB will be true just in case $[\![\mathrm{A}]\!] \supseteq [\![\mathrm{B}]\!]$. And more generally, any syllogistic proposition AyB (with $y \in$ {a,e,i,o}), is true just in case the sets $[\![\mathrm{A}]\!]$ and $[\![\mathrm{B}]\!]$ stand in the appropriate set-theoretic relation indicated by y. In contrast, proponents of the predicable semantics insist that the kind of distinction between different semantic types which the set-theoretic semantics presupposes is post-Aristotelian. We argue that one should instead take x to range over items of the same semantic type as $[\![\mathrm{A}]\!]$ and $[\![\mathrm{B}]\!]$. Accordingly, we interpret R as a binary predication relation. On these issues see esp Barnes (2007) Malink (2009, 2013), Morison (2008).

[21] I see no way to track such a distinction in the set-theoretic semantics for syllogistic that (i) isn't purely stipulative (i.e. we could just primitively stipulate that among the a-propositions true in this model given our set-theoretic semantics *these ones* are genuine), and (ii) doesn't render the semantic machinery that gets extended basically a third-wheel.

As I see it, my predicable semantics differs from Malink's centrally on two issues. Firstly, the predication structures introduced here include a semantic value function $[\![\cdot]\!]$; in contrast, Malink uses no semantic value function but (effectively) treats each term as its own semantic value. Secondly, while Malink's semantics is formulated in terms of a primitive predication relation that he requires to be a preorder (reflexive and transitive), my own approach makes the weaker assumption that P is serial and transitive.[22]

First the question of term reference. Malink, of course, does not think that Aristotle failed to distinguish terms and (non-linguitic) beings. Malink's choice to treat the semantic value of a term as the term itself is supposed to be a harmless formal simplification. Now, with respect to the parts of Aristotle's logic that Malink's system was developed to explore—the analysis of modalized syllogisms in *Prior Analytics* I.3, 8-22—I am inclined to agree that the distinction between a term and its referent plays no crucial role. This said, I worry that that the identification of a term with its semantic value can give a misleading picture of how predicable semantics actually works.[23] But at any rate, maintaining a term/being distinction is important for capturing important aspects of Aristotle's reasoning in the *Posterior Analytics* in general and I.19-22 in particular. Regarding the former, a specific disadvantage of Malink's approach is its inability to represent definitional a-propositions—a class of a-propositions crucial in the *Posterior Analytics*. For, on Aristotle's view, the subject and the predicate of a definitional a-proposition XaY pick out the same entity. However, the subject term (say, 'human being') and the predicate term (say, 'rational animal') of a definitional a-proposition are distinct. The present framework has no difficulty in representing definitional a-propositions as Aristotle conceives of them. For there may be instances where for distinct terms A,B: $[\![A]\!]=[\![B]\!]$. In all such instances it will hold that AaB and BaA are both true. Yet there there will also be cases where both AaB and BaA are true but $[\![A]\!]\neq[\![B]\!]$: for instance, if A = laughs and B = human being. Nothing like this is possible in Malink's framework in which distinctness of terms entails distinctness of refence. As to *Posterior Analytics* I.19-22 in particular it strikes me as important that Aristotle's reasoning in I.19-22 involves both considerations about the internal structure of syllogistic proofs (esp. I.20-21) and metaphysical considerations about extra-linguistic reality (esp. I.22). Failing to formally represent a distinction between language and being would obscure this double aspect of I.19-22's argument.

Now to the second main difference between Malink and I: reflexivity versus (mere) seriality of the predi-

[22]Malink and I also importantly differ on how we read the *dictum de omni et nullo*, and how it relates to predicable semantics. But that dispute has little bearing on present issues.

[23]In both his 2006 and 2009, Malink claims that unlike the set-theotetic alternative, his predicable semantics demands no special assumption of existential import. For, given that Malink requires the predication relation to be reflexive, if AaB is true then B is predicated of B. So there is always at least one thing to which A belongs, namely B itself. In fact, however, it is hard to see how Malink has *dispensed* with the usual assumption of existential import opposed to simply moving it elsewhere in the model. For, Malink's assumption that the semantic value of a term is the term itself looks like no less an added assumption of existential import, than the 'set-theoretic' semanticist's assumption that no term picks out the empty set, or my own requirement that in a predication structure for all terms T: $[\![T]\!]\in D$. In fact, I think there is something unusual and interesting about Malink's approach to existential import; but I don't think it's connected to predicable semantics as his writings seem to suggest.

cation relation. I argued above that the seriality of (genuine) predication falls immediately out of Aristotle's basic understanding of what a being is. The claim that predication is a serial relation, I submit, has much independent motivation. The same cannot be said for reflexivity. It is important for both Malink and I that the predication relation in our models be construed as genuine predication in the sense of *Posterior Analytics* I.19 and I.22 (texts that Malink himself cites). But notably, there is also a good deal of negative evidence against taking genuine predication as reflexive. For, Aristotle explicitly tells us that the 'The walking is walking' and 'The white is white' are not genuine.[24] But even without such passages, I think there are decisive reasons for preferring seriality to reflexivity. Recall the predication structure K adduced as a witness to **Observation 1.2**. Since according to K, laughs is not predicated of laughs Malink will reject K as an admissible model for syllogistic. And indeed, in Malink's system AaB does entail APB (where I have an 'if' Malink has an 'iff'). Now, both Malink and I agree that for Aristotle 'laughing is a human' is not a genuine predication. But on Malink's semantics this entails that 'Everything that laughs is a human being' is false.[25] This is dubious. And it also goes against what Aristotle actually about genuine predication in *Posterior Analytics* I.22. For recall that Aristotle's point there was to contrast genuine predicating ('the log is white') with incidentally predicating ('the white is log') as two ways of *speaking truly*. But on Malink's semantics non-genuine predications can never be true.

There are, then, two important distinctions that Malink's approach is not fine-grained enough to capture. The first is the distinction between definitional a-propositions and a-propositions between counter-predicating terms. The second is the distinction between merely true a-propositions an a-propositions that express genuine predications. I believe this gives strong support for the utility of my own framework over his for certain purposes—and the task at hand in particular. But I have given no reasons for thinking that Malink's approach to syllogistic semantics are insufficient for the interpretive work that he wants them to do. Though we differ on the points rehearsed above, my approach to the semantics of syllogistic is considerably closer to Malink's than any other writer.

5.3 The Main Argument of I.22: interpretive problems

We've noted that *Posterior Analytics* I.20-21 work to show that for any syllogistic proposition φ, if there is an infinite demonstrative improvement-path of partial demonstrations concluding that φ, there must be an infinitely long chain of one of the following forms in $\textsc{cause}(\varphi)$:

[24] See, e.g. *Posterior Analytics* I.4, *Metaphysics* N.1.

[25] It does, at any rate, if Malink's '$\models$' indicates satisfaction in the sense of truth. Of course, Malink's system can, and perhaps *should*, be interpreted otherwise. But the point would stand that his framework still won't be able to represent a distinction between true a-propositions and a-propositions that express genuine predications.

· $\{YaX_0, X_0aX_1, X_1aX_2, ..., X_naX_{n+1}, ...\}$

· $\{X_0aY, X_1aX_0, X_2aX_1, ..., X_{n+1}aX_n, ...\}$

Given that both Aristotle and the Ad Infinitum Skeptics deny circular demonstration, it follows that the chains in question will have to use infinitely many distinct terms.[26] We've noted as well that the Main Argument works by imposing a criterion on the type of a-propositions that can enter into any CAUSE(φ). In particular, Aristotle contends that for an a-proposition to enter into CAUSE(φ) it must express a genuine predication. Suppose that 'A' names the being a while 'B' names the being b. In the framework of our model, we would make the point by saying that if AaB is in CAUSE(φ), not only must it be the case that AaB is true; it also has to be the case that aPb. So, in *Posterior Analytics* I.22 Aristotle will effectively argue that there are no infinite predicational chains containing infinitely many entities[27]:

· $(x_0Px_1, x_1Px_2, x_2Px_3, \ldots x_nPx_{n+1}, \ldots)$

· $(x_1Px_0, x_2Px_1, x_3Px_2, \ldots x_{n-1}Px_n, \ldots)$

If Aristotle can show this, it will follow given his other assumptions that for no φ does CAUSE(φ) contain an infinite a-chain. But in this case, by the argument of I.20-21 it follows that at least in principle, every demonstrable scientific truth φ can be demonstrated from first-principles in just finitely many steps.

Now, *Posterior Analytics* I.19-21 discuss predication largely in the abstract;[28] effectively, these chapters appeal to the notion an unanalyzed primitive. I.19-21 do not hint at any distinction between different kinds of (genuine) predication. Chapter I.22, however, does introduce an anatomy of genuine predication, dividing it into two (exclusive and exhaustive) kinds: essential and accidental. The strategy of the Main Argument is to use this division to establish that all predicational chains are finite.[29] Much of our work in interpreting the Main Argument will involve getting clearer on Aristotle's I.22 anatomy of predication and its implications for the predicational structure of reality. Only then will we be in a position to explicitly formulate the Main Argument's case that reality contains no infinite predicational up- or down-chains.

Commentators from Philoponus, to Barnes and Lear, have found it difficult to see the Main Argument as sound by Aristotelian lights. One difficulty concerns the interpretation of genuine predication. The Main

[26] On this issue see esp. **Lemma 11** in the appendix.

[27] The only way for there to be an infinite down-chain $\{X_0aY, X_1aX_0, X_2aX_1, ..., X_{n+1}aX_n, ...\} \subseteq$ CAUSE(φ) of genuine predications that uses infinitely many terms without there being an infinite down-chain $(x_0Px_1, x_1Px_2, x_2Px_3, \ldots x_nPx_{n+1}, \ldots)$ in which infinitely many entities appear, would be if infinitely many terms in $\{X_0aY, X_1aX_0, X_2aX_1, ..., X_{n+1}aX_n, ...\}$ name the same entity (and mutatis mutandis for up-chains). As we shall see, the argument of I.22 clearly assumes that for no φ is this so. Indeed, it is very hard to see what argument anyone could have for the claim that in some cases the possibility of scientific explanation requires infinitely many names for a single entity. I doubt any such hypothesis would seem plausible to even the Ad Infinitum Skeptics.

[28] The only actual *example* predications we get in I.19-21, occur in I.19's brief remarks on genuine predication. Those remarks anticipate Aristotle's much fuller discussion of genuine predication in I.22.

[29] Unless otherwise noted, by 'predicational chain' I always mean a chain of *genuine* predications.

Argument evidently requires that all propositions involved in a demonstration express genuine predications. But on the reading of some interpreters (e.g. Barnes, Hamlyn) it is required for a genuine predication to have a subject-term which picks out a (primary or secondary) substance. Since Aristotle most definitely does think that some demonstrative sciences (e.g. mathematics) use propositions whose subject-terms refer to non-substances, Barnes concludes that Aristotle cannot in the end endorse this premise. Another difficulty concerns Aristotle's case for the finitude of essentialist predicational chains. The Main Argument argues from the possibility of real definition to the thesis that every essence has finitely many parts. Barnes and Lear contend that Aristotle's argument for the finitude of essence relies on an implausibly strong premise. Aristotle's thought is supposed to be that since we are finite creatures and we can define and know every essence, surely every essence must be finite. But what grounds could Aristotle have for such extreme epistemological optimism? A final difficulty developed at length by Barnes but found already in Philoponus, questions not the acceptability of Aristotle's premises, but rather the validity of the Main Argument. Aristotle claims to be arguing against infinite chains of predication in both the upwards and the downwards directions. Indeed, as is clear from **Section 4** above Aristotle needs to argue against chains in both directions. But, it is charged, Main Argument only bears on predicational up-chains. Could Aristotle be inferring that there are no infinite predicational down-chains from the conclusion that there are no predicational up-chains? This is a bad inference. And there is good evidence both in the *Posterior Analytics* itself (chapter I.19-20) and elsewhere in the corpus (e.g. *De Caelo* I.5-7), that Aristotle himself knows this is a bad inference.

I maintain that the Main Argument can plausibly be read as valid. Moreover, its premises, when properly understood, are indeed acceptable by Aristotelian lights. The first difficulty adduced above has already dealt with. Aristotle's I.22 account of predication need not and should not be read as committing him to the view that all (genuine) predications have substances as subjects. I tackle the remaining two difficulties in what follows, offering a reconstruction of how I take it the Main Argument works. For purposes of this dissertation, I do not offer a complete textual defense of my reconstruction. Numerous delicate interpretive details will have to be passed over. I leave a more thorough textual analysis of I.22 for a latter occasion.

5.3.1 Finitude, essence, intelligibility

Roughly, the text presenting I.22's Main Argument is structured as follows. First Aristotle treats essential predication, arguing that no essence is infinitely complex. Then Aristotle turns to predication 'in general' (83a1), arguing that if xPy is accidental then x can be a subject for no further predications (essential or accidental). Following Aristotle's own order we begin with essential predication. Aristotle's case for the finitude of essence is highly compressed. But it repays close scrutiny. *Posterior Analytics* I.22 opens as

follows (82b27-83a1):

> So then, [**(a)**] as for [items] predicated in the what-is-it [in the *ti esti*, i.e. essentially], it is clear [that they are finite]. For, [**(b)**] if it is possible to define [*horisasthai*], or (**[c]**) if the what-it-is-to-be [*to ti ēn einai*: i.e. essence] is knowable but it is not possible to traverse [*dielthein*] infinitely many [items], then [**(d)**] it is necessary that the [items] predicated [of a subject] in its what-is-it be finite.

For ease of discussion, I've broken the text into four. Aristotle is arguing that for every being x which has an essence, there are only finitely many items predicated of x in its what-it-is [*ti esti*]: i.e., essentially. For Aristotle, this is basically[30] equivalent to saying that every essence has finitely many parts, that no essence is infinitely complex. (I will often formulate the thesis of **(d)**/**(a)** accordingly). Clearly, **(b)** and **(c)** are the argumentative support for this thesis. Note that the text is presuming familiarity with the usual Platonist/Aristotelian conception of definition. In particular, the definability at issue in **(b)** is what a contemporary metaphysician would call *real definability*, where the *definiendum* of a real definition is not a word or a concept, but a being x, and what the real definition of x states is what it is for x to be. With his colleagues in the Academy, Aristotle is more specifically assuming that the real definition [*horismos*] of a being x captures the essence of x, and that episteme of a being x is paradigmatically a matter of grasping the (real) definition of x . Moreover, as is clear from his subsequent recapitulation of the argument (83a39-b8), Aristotle is thinking of real definitions as taking the form: genus + differentia. Note finally that Aristotle is assuming that that everything predicated in the what-is-it of x (i.e. essentially) is part of the essence of x. Accordingly, since essential predication is transitive, essential part-hood turns out to be transitive as well : i.e. if x is in the what-it-is of y (e.g. animal is in the what-is-it of human being) and y is in the what-is-it of z (e.g. human being is in the what-is-it of Socrates), then x is in the what-is-it of z (animal is in the what-is-it of Socrates). All of this is standard in the *Organon*.

Barnes and Lear (perhaps Ross as well), read **(b)**-**(c)** as arguing that every essence is finite because every essence can be comprehended by a finite human intellect. According to these interpreters, the definability assumed in **(b)** is definability *by us*; and the knowability assumed in **(c)** is knowability *by us*. Aristotle's guiding assumption is, then, supposed to be that although we human beings are finite we can define and know every essence. But defining an essence and knowing an essence requires surveying all the essence's parts. Since finite beings like us cannot go through infinitely many items, but can know every essence, every essence must be finite.

[30] I'm bracketing here some issues about unity of definition, and essence qua form functioning as principle of unity, which arise outside the *Organon*. This is reasonable. When Aristotle himself turns to such issues in the *Metaphysics* (see e.g. Z.12), he speaks of them as unresolved in and problematized by the framework of the *Analytics*.

Strikingly, this reading credits Aristotle with a dubious and extraordinarily strong assumption about the epistemic capacities of human beings. For a modern, I suppose, it is natural to expect the definability/knowability at issue in **(b)** and **(c)** to be definability/knowability by us. After all, our own epistemological inquiries are usually cast from a first-personal perspective (What should *I* believe?, What do *we* know?), and often reflect anthropocentric preoccupation. Moreover, Aristotle is a theist. And we are quite used to theists who affirm that God is omniscient and omnipotent, developing this philosophically by attributing to God an 'infinite intellect' outstripping our 'finite intellects'. But all of this is very much post-Aristotelian. In *Metaphysics* Λ, Aristotle identifies the gods with perfect intellects which exist for all eternity. But Aristotle nowhere characterizes these gods as being infinite intellects, nor as having infinite intellectual powers. Indeed, the distinction between finite and infinite intellects (or minds, or whatever) is entirely absent from pre-Christian Greek philosophy. Equally post-Aristotelian is the first-personal orientation of epistemology. For Aristotle, the investigation of episteme is not conceived as an investigation into a distinctively human cognitive state.

It should give us pause, then, that neither in I.22 82b27-83a1 itself nor in its recapitulations elsewhere (e.g. 83b5-7), does Aristotle actually mention human beings. Nor do any such texts use first- or second-person pronouns/verbs (Barnes' translation is misleading on this point), or give contextual indications that they specifically concern the capacities of creatures like us. Dispensing with the philosophically unattractive assumption the Barnes-Lear reading of 82b27-83a1 attributes to Aristotle, I submit that the passage is better read in the context of Aristotle's particular conception of the infinite. Characterizing what the term 'infinite' means, *Physics* III.4 tells us: x is infinite iff (i) x is the sort of item that can be gone through one thing after another, but (ii) it is impossible to 'traverse' [*dielthein*] all of x.[31] It is important to recall that the impossibility here is impossibility full-stop. For Aristotle, the nature of the infinite is to be endless, and to admit no complete traversal. The infinite as such cannot be fully gone through: neither by us nor by anything else. Of course, since Aristotle does not himself think of gods as omnipotent, and indeed argues that it would be irrational for gods to pay attention to anything beyond themselves, he never really raises the explicit question whether the unmoved movers of *Metaphysics* Λ and *Physics* VIII in principle *could* perform an infinite task of this sort. But in *Physics* I.4 (188a5-11) he argues that Anaxagoras' god (which, like Aristotle's, is an eternally existing perfect intellect) cannot separate every bit of an infinitely divisible quantity since there will be no minimum magnitude, and that Anaxagoras thus absurdly has his God attempting an impossible (because infinite) task. In any case, *Posterior Analytics* I.22 82b27-83a1 really should be read alongside Aristotle's claims in both in *Physics* I.4 (187b7-13) and III.6 (207a25-32) that 'the

[31]The verbs *dielthein* and *diexelthein* appear frequently in Aristotle's discussions of the infinite. As mentioned above, they are used in articulating Zeno's paradoxes of the infinite in particular. So it is notable that *dielthein* appears in connection with the infinite in 82b27-83a1.

infinite qua infinite is unknowable [*agnōston*]'. Linguistically, the two *Physics* texts are remarkably close to *Posterior Analytics* I.22. And in both of those contexts, the idea is clearly that the infinite is unknowable per se, by its own nature, and not just unknowable by us. I see no reason to attribute to Aristotle something different in *Posterior Analytics* I.22 82b27-83a1 .

In this light, **(b)** and **(c)** look to present two distinct but closely related arguments for the finitude of essence. Both are from the in principle intelligibility of reality: one from the possibility of (real) definition, another from the possibility of knowledge. With respect to knowledge, Aristotle is registering his commitment to the view that every essence is in principle comprehensible—that no essence is intrinsically impenetrable by thought, and that all essences are (in principle) objects of knowledge. This is a deep Parmenidean strand in Aristotle's philosophy.[32] And Aristotle believes that his interlocutors, Platonists and Academics foremost among them, share in this commitment. There can be no infinitely complex essence because such an essence would be in principle unknowable. In this context, Aristotle is evidently supposing (with, e.g., the Dream Theorist) that knowing an essence requires comprehending all of its parts. But since the parts of an essence are here taken to be actual beings prior to and principles of the whole that they compose, it follows that traversing an infinite essence would require going through each part *individually* one by one. In particular, one cannot traverse an infinite essence in the manner a body traverses an infinitely divisible (but finitely long) magnitude—a point Aristotle makes in a passage quoted below. (Nor, in this framework, could one 'comprehend' an infinite essence by means of a technique like recursive definition). From Aristotle's own perspective, the main controversial claim **(c)** relies on, is an element of his positive conception of the infinite: his claim that the infinite, by its very nature, is untraverseable and that due to the inexhaustability of its parts, must thus be incomprehensible. No strong assumption about our epistemic capacities is required.

So much for **(c)**. Now, interpreters usually to read **(b)** and **(c)** together as making a single point. This is not unreasonable, because definition and knowledge of essence are closely related for Aristotle. But the grammar of the sentence (no epexegetic *kai*) may indicate that **(b)** is referring to an independent or semi-independent argument for the finitude of essence from considerations of definability in particular. In any case, Aristotle elsewhere thus breaks his case for the finitude of essence into two. Consider *Metaphysics* α.2 994b16-25:

> And look, neither can an essence always be brought back to another definition [*horismos*] lengthening [the last definition] in formula . For it is always the case that that the prior [i.e. the further expanded definition] is more [a definition], and the later not [more a definition]; and [a sequence] which does not have a first does not have a second. Further [*eti*], those who argue in this way

[32] Cf. Parmenides Fragment B3.

> destroy understanding [*epistasthai*]; for it is not possible to know until one reaches atoms; and [if definitions are infinite in the way sketched above] there is no knowing, for how is it possible to think things infinite in this way? For, it is not like the case of a line which does not stop in division [i.e. is infinitely divisible], but cannot be conceived without stopping [the division], (thus one going through the infinite[ly divisible line] will not count the cuts).

In *Metaphysics* α.2, Aristotle is arguing against the possibility of infinite causal chains. In the text just quoted he is focusing on chains of formal causes. And since Aristotle is assuming both that the formal cause of x is its essence, and the transitivity of essential part-hood, Aristotle's target is the claim that no essence can have infinitely many parts. Note that the argument appealing to definability is kept distinct (see the *eti*) from that concerning knowability. Regarding the former, Aristotle looks to be considering this sort of case. Say we are asked what x is and reply correctly:

$$x{=}_{def}\ \mathrm{G}_0\mathrm{D}_0$$

It may occur that G_0 is itself definable:

$$\mathrm{G}_0 =_{def}\ \mathrm{G}_1\mathrm{D}_1$$

so that (substituting equivalents) a fuller definition of x would be:

$$x{=}_{def}\ \mathrm{G}_1\mathrm{D}_1\mathrm{D}_0$$

Suppose it occurred that for all G_n ($n \geq 1$), there's a fresh G_{n+1} and D_{n+1} such that:

$$\mathrm{G}_n =_{def}\ \mathrm{G}_{n+1}\mathrm{D}_{n+1}$$

This would be a case in which 'an essence [can] always be brought back to another definition lengthening [the last definition] in formula'. We would then have an infinite sequence of successively fuller definitions of x:

def 1: $x =_{def}\ \mathrm{G}_0\mathrm{D}_0$

def 2: $x =_{def}\ \mathrm{G}_1\mathrm{D}_1\mathrm{D}_0$

def 3: $x =_{def}\ \mathrm{G}_2\mathrm{D}_2\mathrm{D}_1\mathrm{D}_0$

def 4: $x =_{def}\ \mathrm{G}_3\mathrm{D}_3\mathrm{D}_2\mathrm{D}_1\mathrm{D}_0$

$\vdots$

∞

Metaphysics α.2 argues that this cannot occur and (hence) no essence is infinitely complex. More precisely, Aristotle contends that the envisaged situation is incoherently described. It was stipulated that x was definable. But it in fact represents a possibility on which is x is not definable. For, *the* definition of x (strictly speaking, every being has exactly one full definition) would have to capture every aspect of the essential being of x. It would have to be complete, and given in a formula which admits of no expansion. Since the sequence above has no final element, it contains no element that may rightly be called *the* definition of x. However, for every n, n+1: **def n** is less of a definition of x than **def n+1**. So, since nothing in the sequence is *the* definition of x, it turns out that no element in the sequence is even a partial definition of x.

Given the grammar of *Posterior Analytics* I.22 82b27-83a1 and the testimony of *Metaphysics* α.2 994b16-25, it is perhaps attractive to see the latter as telegraphically pointing to two semi-distinct arguments for the finitude of essence: one from the in principle definability of essence, another from in principle knowability of essence.

5.3.2 Essence and accident in *Posterior Analytics* I.22

Having thus argued that every essence has finitely many parts, the Main Argument turns to predication in general (83a1). Aristotle's first task (83a1-23) is to clarify what predication is. This yields the discussion of genuine predication analyzed above. It is here Aristotle insists that all the predications which appear in demonstration are genuine predications (83a17-21). And an upshot of this is supposed to be that all predications in demonstration have a predicate which falls into one of the categories (83a21-23): 'Therefore, whenever one thing is predicated of one thing [in a demonstration], either it is in the essence [*ti esti*] or [it indicates] that [the subject is] qualified or quantified or related to something or doing something or undergoing something or somewhere or some-when.'

Aristotle's next task is to sketch a distinction between essence and accident. As noted above, the Main Argument's case for the finitude of predicational chains crucially relies on this distinction. 83a24-32 reads as follows:

> Further, [a predicate y] which indicates essence [of that x] of which it is predicated, indicates that [x] *just is* that [y], or that [x] *just is* a certain [y]. In contrast, [a predicate y] which does not indicate essence but is said of a different subject [x] which neither *just is* that [y] nor *just is* a certain [y], is an accident [*sumbebēkos*]: for instance, pale [predicated] of human being. For human being neither *just is* pale nor *just is* a certain [shade of] pale; rather [human being] *just is* (perhaps) [a certain] animal: for, human being *just is* animal. But it is necessary that [items]

> which do not indicate essence are predicated of some [distinct] subject; and [it is necessary] that there not be some pale which, not in virtue of being something else, is pale.

For Aristotle, there is a fundamental difference between predicates which indicate a mere characteristic of their subject and predicates which indicate something of their subject's *identity*. What is picked out by 'human being' in 'Socrates is human being' is an element of Socrates' identity. For, part of what it is to be Socrates is to be a human being: being human is part of the nature of Socrates, part of the essence of Socrates. In contrast, when one says of the deathly-white hemlock-drunk man 'Socrates is pale' one indicates a mere characteristic of Socrates, not part of what-Socrates-is.

In general, one should note that Aristotle's essence/accident distinction is not a distinction between an object's necessary and contingent properties. For Aristotle, as for us, having internal angles summing to two right angles (call it having-2R) belongs necessary to triangle. However, Aristotle quite clearly insists that having-2R is not essential to triangle. For even while no triangle can fail to have 2R, having-2R is not part of what a triangle is. Accordingly, Aristotle will say that having-2R property is an accident of triangle. Notably, since the fact that every triangle has 2R can be proved directly from the essence of triangle (at least so Aristotle assumes), having-2R turns out to be a special kind of accident. In the terminology of the *Analytics* and the *Metaphysics*: having-2R is a 'per se accident' of triangle. Nevertheless, qua accident, having-2R attaches to triangle in a similar way that being-pale attaches to Socrates: as a characteristic external to the identity of its subject. Making the contrast in English we might affirm 'Hillary Clinton is something which *just is* a certain human being' (i.e. that is her nature), and in the same sense deny 'Hillary Clinton is something which *just is* smart'. In Aristotle's Greek, the *hoper* idiom plays this very role. At any rate, in accord with his use elsewhere in the *Organon* and the *Metaphysics*, the truth condition for Aristotle's *hoper* idiom at 83a24-32 amounts to:

x is *hoper* y or *hoper* a certain y iff (i) $x = y$, or (ii) y is part of the real definition of x

The text above (esp 83a24-26) makes quite clear that if x is an accident of y then x is not part of the essence of y. But it also indicates a view about the nature of accidents—about beings x such that for some y: x is an accident of y. The view is further developed later in the chapter, and yields an important premise for the Main Argument. So I will draw both on 83a24-32 itself and as well as subsequent material in explaining Aristotle's view, and formulating the relevant premise of the Main Argument.

In the *Organon*, Aristotle conceives of the essence of an object as its identity. Accordingly, in some sense an essential predicate indicates nothing outside its subject's identity. The predicate of 'human being is rational animal' indicates something wholly identical to its subject; the predicate of 'human being is animal' indicates something partially identical to its subject: that is, something identical to an (essential)

part of its subject. A predicate indicating accident, in contrast, indicates something wholly external to the subject's identity. And Aristotle takes this to mean that an accident is wholly distinct from (≈ identical with no essential part of) its subject. *Posterior Analytics* I.22 develops this by claiming that an accident qua accident is always and only predicated of something different from it (82a26, cf. 83b23-24, *Met.* Γ.4 1007a33-b1).

But what of this different thing, an accident's x's distinct subject y? Fundamental to his metaphysics is Aristotle's multifaceted insistence that whatness (essence/essential identity) is both ontologically primary and ontologically prior to suchness (accident). And on Aristotle's conception of metaphysical subject-hood, for every such (= accident), there is a what (= essence) that underlies it. After all, for an accident to have being is simply for it to characterize some what: for it to be a such of that what. The very existence of an accident, then, depends on its belonging to a what. Speaking from of the reverse side of the same coin, Aristotle contends that a what can receive a such as an accident only in virtue of its being a what. In a certain metaphysical sense, a what x characterized by an accident y, *is* x prior to *being* y. And Aristotle is putting both sides of the coin together when he says in I.22 83a32 that there is nothing pale which is not pale in virtue of its (essentially) being something else.

The above line of thought gives Aristotle the following claim about the role accidents play in predicational structures. While a what can be a subject for both essential and accidental predications, a such can be subject for no predication: neither essential nor accidental. As Aristotle puts it towards the end of the Main Argument (83b18-24):

> As many [things] as do not [indicate] essence are not predicated, these things of themselves. For they are accidents, all of them... And we assert that all these things [accidents] are predicated of a subject. But an accident is no subject. For, we posit that none of these are such that not in virtue of being something else they are called what they are called, rather [each is] itself [predicated] of another and that of something different.

The above states a necessary condition for metaphysical subject-hood. If this x thing (e.g. pale Socrates) is x (e.g. pale) in virtue of being essentially identical to something else (i.e. a certain human) modified by x, then that x (e.g. pale, the accident) is no metaphysical subject. The philosophical grounds for this view are, of course, in need of further clarification. But for the argument of I.22 the upshot is clear enough. No accident can receive anything else (accident or essence) as subject. Hence no term which picks out an accident can be a subject term of a genuine predication. Making a related point about accidental predication in *Metaphysics* Γ.4 1007a33-b3, Aristotle writes:

> If everything were predicated accidentally, there would be no first universal [i.e. no universal

> predicated of only non-universals][33], (supposing that accident always indicates predication of some [other] subject). So it would necessary for [accidental predication] to continue [downwards] ad infinitum. But this is impossible; for [accidental predication in fact] does not even connect more than two things [i.e. a what and an accident]: for, there is no accident of an accident...

5.3.3 Interlude: accidents and ontology

Accidents in the above sense are not themselves subjects for any predications: essential or accidental. Hence, denizens of the Aristotelian ontology like triangles, redness, and surfaces cannot be counted as accidents in the sense of *Posterior Analytics* I.22 (and *Metaphysics* Γ.4). For, according to Aristotle, triangle is subject to the per se accident having-2R; color is an essential predicate of redness; and surfaces are the immediate metaphysical subjects to color accidents (see *De Sensu* 3).

Posterior Analytics I.22 commits Aristotle to a distinction between, for instance, *redness* (a kind of color: a what) and *red* (a way of being colored: a such and an accident). One finds this distinction elsewhere in the corpus. See for instance *Metaphysics* Z.1 1028a20-27:

> ...[O]ne might puzzle about walking [*to badizein*], being healthy [*to hugiainein*], and sitting [*to kathēsthai*], whether each of them is a being or a non-being...and rather, if indeed [it is] the walking [*to badizon*], the sitting [*to kathēmenon*], and the healthy [*to hugiainon*] that are among beings. For these things are more clearly beings, since there is for each a certain definite subject [*hupokeimenon*]...

Note that the distinction in the passage above is not a distinction between (e.g.) walking in general and a particular trope-like individual walking property (say, Socrates' walking). Unfortunately, this is clearer in the Greek than in the English (cf. the infinitive *badizein* vs. the participle *badizon*). Expressing the relevant contrast more clearly in English, the distinction at issue is between (i) both walking (a what, a type of action) and Socrates' walking (a what, an instance of the aforementioned type of action) on one side, and (ii) walks (a such, something which a subject does to exemplify the type of action: walking). NB walks does not itself exemplify walking: it is the thing that walks which does.

Recent commentators on *Posterior Analytics* I.22 (Barnes, Hamlyn) have not endorsed this interpretation of the chapter's view of accidents. And this seems to be because they read into the chapter the ontological framework of *Categories* 2. Now in *Categories* 2 Aristotle does offer a picture of what there is. The picture, however, is not put forward as presenting a complete ontology. And indeed, the ontology of Aristotle's

[33]Reading the manuscript and rejecting the conjecture of Alexander (accepted by Ross, Jaeger).

Metaphysics is considerably richer.[34] In particular, the fourfold division of beings articulated in *Categories* 2 comprehends only the sort of beings that have essences. Both the individual substances (e.g. Socrates) and individual non-substances (e.g. an individual whiteness) of *Categories* 2, are clearly conceived of as subjects of essential predications: e.g. 'Socrates is a human being', '(this individual) whiteness is color'. And mutatis mutandis, for their species/genera ('color is a quality', 'human being is an animal'). In contrast, the items which *Posterior Analytics* I.22 and several texts in the *Metaphysics* call 'accidents' [*sumbebēkota*] are subjects of no predications: neither essential nor accidental. Hence, they simply cannot fall under the fourfold division of *Categories* 2. One might think the issue here that the *Categories* calls one sort of entity (color, triangle) 'accident', while the *Analytics* and *Metaphysics* call another sort of entity (colored, triangular) 'accident'. But this is wrong. Commentators, of course, often speak of Aristotle's *Categories* as giving a 'theory of accidents'. And from antiquity to the present, one finds interpreters who refer to the non-substances of the *Categories* which are 'in a subject' (e.g. color, triangle) as accidents.[35] But in the *Categories*, Aristotle himself never calls these entities *sumbebēkota* (= 'accidents').[36] The fact that Aristotle's draws a contrast between *ousia* and *sumbebēkos* in *Posterior Analytics* I.22 does not put us in the framework of the *Categories.*

In fact, the ontological framework on which *Posterior Analytics* I.22 depends, is articulated not in *Categories* 2 but in *Prior Analytics* I.27. That interpreters from Philoponus, to Ross and Barnes, have failed to recognize the importance of *Prior Analytics* I.27 for understanding *Posterior Analytics* I.22 is remarkable. For, *Prior Analytics* I.27 clearly refers forward to *Posterior Analytics* I.22. The key portion of *Prior Analytics* I.27 reads as follows (43a25-42):

> Now, among all beings: [**Group I**] some (e.g., Kleon and Kallias, the particular and perceptible) are such that truly and universally they are predicated [*katēgoreisthai*] of nothing else, but different [things] are predicated of them (for each of these is human being an animal); [**Group II**] others are themselves predicated of different [things], while different [things] are not predicated of them in a prior way; [**Group III**] others are both themselves [predicated] of different [things] and have further different [things] predicated of them (e.g., human being [is predicated] of Kallias, and animal of human being). So then, that [*hoti men oun*] some beings are by nature not predicated of anything [else] is clear (for among perceptibles just about each is such that it is not predicated of anything [else], except in an incidental way: for we sometimes say: 'This pale is Socrates' and 'The approaching is Kallias'). But that [*hoti de*] also in the case of [chains] going upwards,

[34]This should not be surprising, if indeed (as several recent scholars have concluded) the *Categories* is not supposed to be a metaphysical work but was rather written as an introduction to the study of dialectic.

[35]For instance, Philoponus does this in commenting on *Posterior Analytics* I.22.

[36]The term and its cognates rarely occur at all in the *Categories* and never in chapters 1-5. There are three times *sumbebēkos* occurs as a substantive in the *Categories* (7a27, 7a33, 7a36). In no instance is it used to pick out the class of items in a subject.

> at some point they stop we will argue later [i.e. in *Posterior Analytics* I.22]; for now let it be assumed. So, of these [items in **Group II**] it is not possible to demonstrate [*apodeixai*] another predicate [*katēgoroumenon*] unless it is [a demonstration] by opinion [*kata doxan*: i.e. a merely dialectical and not properly scientific demonstration], rather these things [can be demonstrated] of other things. Nor [is it possible to demonstrate] the particulars of other things, rather other things [can be demonstrated] of them. But it is [with respect to] the intermediates that it is possible in both ways (for they themselves are said of other things and other things of them)...

In language and structure the opening of 43a25-42 ('among all beings... some... others...', *hapantōn tōn ontōn...ta men...ta de...*) is strongly reminiscent of *Categories* 2 (1a16ff.: 'among beings...some...others' *tōn ontōn...ta men...ta de....*). The absence of 'all' in *Categories* 2, given its presence in *Prior Analytics* I.27, is perhaps significant—suggesting a generality for the latter division not reached by the former. In any case, it is clear that the threefold division of *Prior Analytics* I.27 is supposed to be applicable to everything there is.

To explore the division of beings outlined by *Prior Analytics* I.27 43a25-42, let us consider some examples. The following items are mentioned in the passage:

- Kallias, Socrates
- human being, animal
- (the) pale, (the) approaching

Kallias and Socrates are adduced as items which fall into **Group I**. For, other beings are 'predicated truly and universally' of them (e.g. 'Kallias is human being'); but they themselves are predicated 'of nothing else'. Except, Aristotle later adds, incidentally. Relying on a version of the *Posterior Analytics* I.22 contrast between genuine and non-genuine predication, Aristotle remarks that we sometimes say 'The pale is Socrates'. And the implication is clearly supposed to be that such cases are not real counter-examples to his claim, because in some sense they aren't really predications. Human being, in contrast, is given as an example of something in **Group III**. For, it is both the case that human being is predicated of other things (e.g. Socrates, Kallias), and the case that other things (e.g. animal) are predicated of human being. Though the above text does not exactly say so, we are presumably supposed to put animal in **Group III** as well. Aristotle's usual view would be that body is predicated (truly and universally) of animal; and certainly animal is predicated of human being. What about **Group II**? Beings in **Group II** are supposed to be those which are themselves predicated of other things, but do not have others predicated of them. Commentators traditionally conjecture that the denizens of **Group II** are the categories themselves, thinking here of the

categories along Porphyrian lines as maxima genera. Now, I would agree that Aristotelian maxima genera (whatever they are) must go in **Group II**. However, I think the commentators are wrong to think of maxima genera as the only denizens of **Group II**. The text announces a distinction dividing **all** beings. And Aristotle's examples '(the) pale' and '(the) approaching' fit in no other group. These are beings which are subject for no (genuine) predication: they are accidents in the terminology of *Posterior Analytics* I.22. Since these items are predicated of other beings while nothing else is predicated of them they must belong in **Group II**. Indeed, Aristotle seems to have accidents in particular on his mind in his remarks on **Group II** at 43a37ff. Issues of demonstration *kata doxan* vs. demonstration *pros alētheian* (cf. *Posterior Analytics* I.19 81b18-29) concern especially whether once can demonstrate a conclusion with an accident as its subject.

5.3.4 The Main Argument reconstructed

Prior Analytics I.27 43a25-42 refers forward to *Posterior Analytics* I.22 as a text which will establish that reality has a metaphysical top: that every predicational chain has an upper bound. In contrast, *Prior Analytics* I.27 43a25-42 positively endorses the view that reality has a metaphysical bottom: that every predicational chain has a lower bound that is not predicated of anything further. The claim that every predicational chain has a lower bound is not made explicit in *Posterior Analytics* I.22. However, I believe a case can be made that it is a piece of standard Aristotelian doctrine—hinted at already in texts like *Categories* 5.[37]

The Main Argument is not usually read in the context of the Aristotelian assumption that reality has a bottom: that every predicational chain has a lower bound. However, if we add it to three claims that are explicitly argued for in the text, we get a valid argument that there are no infinite predicational up-chains and no infinite predicational down-chains. Let me begin, then, by stating what I take the premises of the Main Argument to be:

1. Every predication is either essential or accidental

2. Nothing has infinitely many items predicated of it essentially

3. The predicate of an accidental predication is subject in no further predication

4. Every predicational chain has a lower bound

[37]The assumption that every predicational chain has a lower bound does not entail that there are no infinite predicational down-chains. So it is notable that despite the background in *Prior Analytics* I.27, Aristotle thinks in *Posterior Analytics* I.19-22 that he still needs to establish that reality has no infinite predicational down-chains. (If an example explaining the non-entailment is wanted let Θ be the collection consisting of exactly (a) the number 0, and (b) every rational number q such that $0 < q < 1$. Relative to the strictly-less-than relation $<$, Θ contains lower-bound (for there's no q in Θ such that $q < 0$). But relative to $<$, Θ also contains infinitely many down-chains—for instance, the one whose first member is $\frac{1}{2}$).

We've seen above that 1. - 3. defended in the text of I.22. Premise 4. is not.

Here, then is how the Main Argument would work. First the case of up-chains. For reductio suppose reality contains an infinite predicational up-chain $\overline{P}$ which has element x. By premise 1. , there are two options. Either (a) every link in $\overline{P}$ is a link of essential predication, or (b) at least one link in $\overline{P}$ is a link of accidental predication. Option (a) is impossible. For given premise 2., x cannot have infinitely many things predicated of it essentially. Yet option (b) is also impossible. For, given premise 3. there must be exactly one accidental predication in $\overline{P}$; it will have to occur just finitely far above x; and the chain $\overline{P}$ then could not infinite but will have to terminate with this accidental predication. So, on both option (a) and on option (b) there cannot be an infinite predicational up-chains. So there cannot be infinite predicational up-chains.

Now the case of down-chains. For reductio suppose reality contains an infinite predicational down-chain $\overline{P}$. Again, by premise 1., there are two options. Either (a) every link in $\overline{P}$ is a link of essential predication, or (b) at least one link in $\overline{P}$ is a link of accidental predication. Given premise 4. it follows that $\overline{P}$ has a lower bound: let's call it r. It follows that option (a) is impossible. For, by premise 2., r cannot have infinitely many things predicated essentially of it. So consider option (b). By premise 3. it follows that $\overline{P}$ contains exactly one accidental link—namely it's top link—and that all the infinitely many links below it are links of essential predication. In this case, it would again happen that r has infinitely many items predicated of it essentially. So both option (a) and option (b) are impossible. So there cannot be any predicational down-chains.

The Main Argument leaves us, finally, with the following view of reality predicational architecture (*Posterior Analytics* I.22 83b24-28):

> So, it has been argued that neither upwards nor downwards does one thing belong to another [ad infinitum]. For the accidents are predicated of these, as many as are in the essence of each thing, but those [items in the essence] are not infinite. And upwards there are both these things [items in the essence] and accidents, neither infinite[ly].

5.4 Afterword

Aristotle, the Circular Demonstrators, and the Ad Infinitum Skeptics all agree that genuine episteme can only be achieved through demonstrative-proof. At the surface, the debate between the three parties is logical—it is about logoi—but it's ultimately a debate about the nature and in principle possibility of episteme.

The central logical theses of both the Circular Demonstrators and the Ad Infinitum concern specific proof-processes. The Circular Demonstrators claim to establish the possibility of episteme by appealing to the possibility of a process whereby principles are proved in a circle and from one another. The Ad

Infinitum Skeptics suggest there can be no episteme given that the process of proving a scientific truth can never terminate. Aristotle approaches the two groups in similar ways. In both cases he will argue that the proof-process under discussion is only realizable relative to a particular type of predicational structure. The possibility of demonstrating ad infinitum rests on the availability of a certain type of infinite predicational chain; the possibility of demonstrating everything in a circle rests on certain types of ubiquitous circular predicational chains. And having reduced an issue about proof-structure to a (not obviously) related issue about predicational structure, Aristotle looks to refute his opponents by adducing absurdities. The Ad Infinitum Skeptics turn out to be committed to the impossibility of genuine essences and real definition. The Circular Demonstrators turn out to be committed to the impossibility of episteme relating two extensionally distinct terms.

The Circular Demonstrators insist on a criterion for what kinds of propositions can appear in a demonstration. And as we've seen, Aristotle argues that the criterion in question is far too restrictive to be plausible. Yet we've also seen that in responding to the Ad Infinitum Skeptics, Aristotle himself will impose a criterion on the kinds of propositions that can appear in demonstration. In no uncertain terms, the Main Argument of *Posterior Analytics* I.22 requires that an a-proposition may appear in a demonstration only if it expresses a *genuine predication.* On the final analysis, for the Main Argument to succeed Aristotle will have to defend this criterion against the worry that it too is overly restrictive, problematically excluding some genuine demonstrative-proofs from the sphere of science. For instance, suppose there were (say) a mathematical proof that concluded:

$$\frac{\text{AaB} \qquad \text{BaC}}{\text{AaC}}$$

Suppose also that B is a per se accident of C. Do the commitments of the I.22's Main Argument rule out the possibility of this being a demonstration? Maybe, maybe not. They do not if the per se accident named by 'A' turns out to be the very same per se accident as that named by 'B'. But given the Main Argument of I.22, the above cannot be a demonstration if 'A' and 'B' turn out to name two genuinely different per se accidents.

My suspicion is that at the time Aristotle wrote *Posterior Analytics* I.22, he was genuinely unsure about whether this is problematic or not. If Aristotle thought he actually *knew* some epistemic logoi and had correct judgments about the individuation of per se accidents therein, he might have looked for counter-examples. But I'm skeptical that Aristotle did think he had that kind of knowledge.[38] Perhaps, it is on account of this sort of uncertainty that *Posterior Analytics* I.22 does not end with—what I've been calling—the Main Argument. Aristotle in fact provides a second argument against infinite predicational chains, that works by

[38]The toy examples of demonstration one gets in the *Analytics* seem to me highly perfunctory in character.

assuming a somewhat different criterion for propositions occurring in a demonstration to meet. I leave a reading of that text for another occasion.

Chapter 6

Appendix of proofs

6.1 Basic definitions and commonly used lemmas

6.1.1 Basic definitions

Definition 1 Language $\mathcal{L}$

1. Basic expressions
   (a) a set of terms (we'll represent them by capitalized letters)
   (b) four syllogistic copulae: a, e, i, o
2. Sentences
   (a) If X and Y are terms, then the following are sentences of $\mathcal{L}$:
      i. XaY
      ii. XeY
      iii. XiY
      iv. XoY
   (b) Nothing else is a sentence of $\mathcal{L}$

Definition 2 Conversion-rules

For all terms X, Y, Z of $\mathcal{L}$ the following constitute conversions:

simple e-conversion $\frac{\text{XeY}}{\text{YeX}}$	accidental a-conversion $\frac{\text{XaY}}{\text{YiX}}$
simple i-conversion $\frac{\text{XiY}}{\text{YiX}}$	accidental e-conversion $\frac{\text{XeY}}{\text{YoX}}$

Definition 3 Basic syllogistic-deduction

For all terms X, Y, Z of $\mathcal{L}$ the following 'trees' constitute basic syllogistic-deductions

First figure	Second figure	Third figure
Barbara $\frac{\text{XaY} \quad \text{YaZ}}{\text{XaZ}}$	Cesare $\frac{\text{YeX} \quad \text{YaZ}}{\text{XeZ}}$	Darapti $\frac{\text{XaY} \quad \text{ZaY}}{\text{XiZ}}$
Celarent $\frac{\text{XeY} \quad \text{YaZ}}{\text{XeZ}}$	Camestres $\frac{\text{YaX} \quad \text{YeZ}}{\text{XeZ}}$	Datisi $\frac{\text{XaY} \quad \text{ZiY}}{\text{XiZ}}$
Darii $\frac{\text{XaY} \quad \text{YiZ}}{\text{XiZ}}$	Festino $\frac{\text{YeX} \quad \text{YiZ}}{\text{XoZ}}$	Disamis $\frac{\text{XiY} \quad \text{ZaY}}{\text{XiZ}}$
Ferio $\frac{\text{XeY} \quad \text{YiZ}}{\text{XoZ}}$	Baroco $\frac{\text{YaX} \quad \text{YoZ}}{\text{XoZ}}$	Felapton $\frac{\text{XeY} \quad \text{ZaY}}{\text{XoZ}}$
Fapesmo $\frac{\text{XaY} \quad \text{YeZ}}{\text{ZoX}}$		Ferison $\frac{\text{XeY} \quad \text{ZiY}}{\text{XoZ}}$
Frisesomorum $\frac{\text{XiY} \quad \text{YeZ}}{\text{ZoX}}$		Bocardo $\frac{\text{XoY} \quad \text{ZaY}}{\text{XoZ}}$

Definition 4 Syllogistic-deductions

1. If Σ is a basic syllogistic-deduction, then Σ is a syllogistic-deduction

2. If Σ_0 is a syllogistic-deduction with AxB at leaf-node n and Σ_1 is a syllogistic-deduction or a conversion with AxB at root-node m, then there is a syllogistic-deduction concatenating Σ_0 and Σ_1 by identifying node n with node m

3. If Σ_0 is a syllogistic-deduction with AxB at root-node n and Σ_1 is a conversion with AxB at leaf-node m, then there is a syllogistic-deduction concatenating Σ_0 and Σ_1 by identifying node n with node m

4. Nothing else is a syllogistic-deduction

Definition 5 R-syllogistic-deductions

Identical with **Definition 4** but with the following added:

· **Reductio** Let $\neg\,\alpha$ represent the contradictory of α, and $\sim\,\alpha$ represent either the contrary or contradictory of α. Then if there's a syllogism:

$$\dfrac{\Gamma \quad \varphi}{\sim\ \psi} \quad \text{and there's a syllogism (or sentence)} \quad \psi \quad \text{, then this is a syllogism:} \quad \dfrac{\Gamma \quad \overline{\varphi} \qquad \vdots}{\dfrac{\sim\ \psi \qquad \psi}{\neg\,\varphi}}$$

Defintion 7 Conclusions, Term-sets, Premise-sets

Where Σ is syllogistic-deduction or R-syllogistic-deduction:

1. $concl(\Sigma)$ is the lowest sentence in Σ

2. $term(\Sigma) = \{\text{T} : \text{T is a term occurring in } \Sigma\}$

3. $prem(\Sigma) = \{\varphi : \varphi \text{ is a sentence occurring in } \Sigma \text{ without a line over it}\}$

Definition 7 M-syllogistic-deductions

A syllogistic-deduction Σ is an M-syllogistic-deduction iff every single-inference subtree of Σ uses three distinct terms

Definition 8 Circular-syllogisms

A syllogistic-deduction Σ is circular iff its conclusion is identical with one of its premises

Definition 9 Syllogisms

A syllogistic-deduction Σ is a syllogism iff

1. neither Σ nor any of its subtrees is circular, and

2. every single-inference subtree of Σ uses three distinct terms

Definition 10 Consequence and consequence-closure

1. $\Gamma \vdash_D \varphi$ iff for some syllogistic-deduction Σ: $prem(\Sigma) = \Gamma$ and $concl(\Sigma) = \varphi$

2. $\Gamma \vdash_R \varphi$ iff for some R-syllogistic-deduction Σ: $prem(\Sigma) = \Gamma$ and $concl(\Sigma) = \varphi$

3. $\Gamma \vdash_M \varphi$ iff for some M-syllogistic-deduction Σ: $prem(\Sigma) = \Gamma$ and $concl(\Sigma) = \varphi$

4. $\Gamma \vdash \varphi$ iff for some syllogism Σ: $prem(\Sigma) = \Gamma$ and $concl(\Sigma) = \varphi$

5. Where $\vdash_x$ is one of our consequence relations:

   · $\Sigma : \Gamma \vdash_x \varphi$ iff Σ witnesses the fact that $\Gamma \vdash_x \varphi$

   · $\Gamma^{\vdash}_x = \{\varphi : \Delta \vdash_x \varphi \text{ for some } \Delta \subseteq \Gamma\}$

Definition 10 Strings

Let Σ be a direct syllogistic-deduction

1. Σ is an a-string iff Σ has an a-conclusion

2. Σ is a e-string iff Σ has an e-conclusion

3. Σ is an i-string iff Σ has an i-premise and i-conclusion

4. Σ is an o-string iff Σ has an o-premise and o-conclusion

6.1.2 Commonly used lemmas: the a-space lemmas

Definition Syllogistic a-Space

Let there be a syllogistic-deduction $\Sigma : \Gamma \vdash_D \varphi$ with $\tau \subseteq term(\Sigma)$. Define a relational structure $A^{\Sigma}_{\tau} = (\tau, \mathrm{a}_{\tau})$ in terms of the binary relation:

· $\mathrm{a}_{\tau} = \{(\mathrm{X},\mathrm{Y}) : \mathrm{X},\mathrm{Y} \in \tau \text{ and } \mathrm{XaY} \in \Gamma \cup \Gamma^{\vdash}\}$

Lemma 1.1

$\Gamma \vdash_D \mathrm{AaC}$ iff $|\Gamma| \geq 2$ and Γ may be put in the form $\{\mathrm{AaB}_0, \mathrm{B}_0\mathrm{aB}_1, \ldots \mathrm{B}_{n-1}\mathrm{aB}_n, \mathrm{B}_n\mathrm{aC}\}$

Lemma 1.2

Let Σ be an a-string with premises ordered left-to-right $p_1, p_2, ..., p_{k-1}, p_k$. For all n, m such that $1 \leq n < m \leq k$, there exists an a-string $\Gamma \vdash$XaY such that (i) $\Gamma = \{p_x \ : \ n \leq x \leq m\}$, and (ii) X is predicate in p_n while Y is subject in p_m.

Lemma 2

Let $\Sigma:\ \Gamma \vdash_D$ XaY. If $\tau = term(\Sigma)$, A^{Σ}_{τ} is a total order in which X is maximal and Y is minimal.[1]

Lemma 3

Let $\Sigma:\ \Gamma$, XeY $\vdash_D$ BeC. Then there are sets of terms $\tau \cup \tau* = term(\Sigma)$ such that:

1. $A^{\Sigma}_{\tau} = (\tau, \mathrm{a}_{\tau})$ and $A^{\Sigma}_{\tau*} = (\tau*, \mathrm{a}_{\tau*})$ are total orders
2. A^{Σ}_{τ} has minimal element B, and $A^{\Sigma}_{\tau*}$ has minimal element C
3. A^{Σ}_{τ} has maximal element one of {X,Y} and $A^{\Sigma}_{\tau*}$ has as maximal element the other

Lemma 4

Let $\Sigma:\ \Gamma$, XiY $\vdash_D$ BiC be an i-string. Then there are sets of terms $\tau \cup \tau* = term(\Sigma)$ such that:t

1. $A^{\Sigma}_{\tau} = (\tau, \mathrm{a}_{\tau})$ and $A^{\Sigma}_{\tau*} = (\tau*, \mathrm{a}_{\tau*})$ are total orders
2. A^{Σ}_{τ} has maximal element B, and $A^{\Sigma}_{\tau*}$ has maximal element C
3. A^{Σ}_{τ} has minimal element one of {X,Y} and $A^{\Sigma}_{\tau*}$ has as minimal element the other

Lemma 5

Let $\Sigma:\ \Gamma$, XoY $\vdash_D$ BoC be an a o-string. Then there are sets of terms $\tau \cup \tau* = term(\Sigma)$ such that:

1. $A^{\Sigma}_{\tau} = (\tau, \mathrm{a}_{\tau})$ and $A^{\Sigma}_{\tau*} = (\tau*, \mathrm{a}_{\tau*})$ are total orders
2. A^{Σ}_{τ} has minimal element B, and $A^{\Sigma}_{\tau*}$ has maximal element C
3. A^{Σ}_{τ} has maximal element X and $A^{\Sigma}_{\tau*}$ has as minimal element Y

[1] NB for the lemmas in this section we don't require a total order to be either irreflexive or reflexive—just that it's transitive and meets the totality condition: $(\forall$A,B$\in term(\Sigma))$(Aa$_{\tau}$B $\vee$ Ba$_{\tau}$A $\vee$ A=B). To be clear: we say an element A$\in \tau$ is maximal in A^{Σ}_{τ} iff for all other B$\in \tau$: Aa$_{\tau}$B$\in A^{\Sigma}_{\tau}$. We say an element A$\in \tau$ is minimal in A^{Σ}_{τ} iff for all other B$\in \tau$: Ba$_{\tau}$A$\in A^{\Sigma}_{\tau}$.

Proofs. For **Lemma 1.1**: left-to-right is shown by induction on the length of the proof; right-to-left by induction on the size of Γ. **Lemma 1.2** follows right away from **Lemma 1.1**. We get **Lemmas 2-4** by easy induction on the length of Σ and appealing **Lemmas 1.1-2**.

Corollary

Let. $z \in \{\text{e, i, o}\}$. If Γ, XzY $\vdash_D$ AzB, and QaR$\in$ Γ there is a Σ$*$: Γ, XzY $\vdash_D$ AzB in which QaR occurs in a binary inference with a z-proposition.

Proof. Consideration of cases using **Lemmas 3-5**. (Details left as exercise for reader).

6.2 Proofs of results mentioned in Section 2

6.2.1 Reductio elimination

Lemma 6

Let Σ be a single-inference syllogistic-deduction.

1. If Σ is an instance of Barbara, Celarent, Darii, Baroco, or Bocardo and has a premise of the form XaX, the conclusion of Σ is identical with its other premise
2. If Σ is an instance of Ferio or Darii and has a premise of the form XiX, the conclusion of Σ is the subaltern of the major premise.

Proof. Easy check of cases.

Theorem

Γ $\vdash_D \varphi$ iff Γ $\vdash_R \varphi$

Proof. Left to right is trivial; right to left demonstrandum. Let there be a syllogistic-deduction Σ: Γ $\vdash_R \varphi$ that uses reductio and let Σ$*$: Δ $\vdash_R$ ψ be a sub-tree of Σ such that (i) the final inference in Σ$*$ is a reductio, and (ii) none of Σ$*$'s inferences above its lowest inference is a reductio. Without loss of generality we may assume that the reductio step in Σ$*$ goes through a contradictory pair. Now, consider Σ$*$: Δ $\vdash_R$ ψ in isolation from the remainder of Σ. (So, if reductio is applied to any line in Σ that depends on Σ$*$ with the result that some other leaves of Σ$*$ distinct from that reduced in its final inference have reductio lines over them, erase those reductio lines). By multiple (but finite) repetition of argument, we have Γ $\vdash_D \varphi$ if we show Δ $\vdash_D \psi$ for all possible such Σ$*$: Δ $\vdash_R$ ψ considered in isolation. To control the number of cases

that need be considered, we take advantage of a well-known corollary of *Prior Analytics* I.4-6. Without loss of generality, we assume that $\Sigma*$ uses exactly some combination of: Barbara, Celarent, Darii, Ferio, Baroco, Bocardo, simple (= e-, i-) conversion, accidental (a-, e-) conversion, and—of course—reductio. We proceed by considering all forms a highest reductio proof $\Sigma*$ might have depending on whether its conclusion is an a-, e-, i-, or o-proposition. Given **Lemma 6**, the following abstract schemata will suffice for encoding transformation procedures for all possible cases.[2] Note that since the only reductio step in $\Sigma*$ is the last inference, we can be certain that **Lemmas 2-5** hold for the two subtrees of $\Sigma*$ that derive the contrary or contradictory pair of propositions on which reductio is performed (recall that these lemmas were proved for direct syllogistic-deductions). In every case that follows we apply some combination of **Lemmas 2-5** to the form on the left (which lemmas are employed will obvious from the copulae at issue in the case), and then use the result to establish the existence of a syllogism of the form on the right.

1. Suppose that $concl(\Sigma*)$ = XaY. Then the reduced assumption must be XoY, and the reductio step's contradiction must bring together the conclusion of an a-string with the conclusion of an o-string. The derivation must be of the form on the left, which can be replaced by the reductio free form on the right:

$$\text{(a)}\quad \dfrac{\begin{array}{cc}\vdots & \overline{\text{XoY}}\\ \text{AaB} & \text{AoB}\end{array}}{\text{XaY}} \qquad\Longrightarrow\qquad \dfrac{\dfrac{\begin{array}{cc}\vdots & \vdots\\ \text{XaA} & \text{AaB}\end{array}}{\text{XaB}} \quad \begin{array}{c}\vdots\\ \text{BaY}\end{array}}{\text{XaY}}$$

   Making our use of **Lemma 6** explicit, note that if the syllogistic-deduction above on the left never uses Baroco then X = A and our abstract representation on the right represents Σ_1 below; if the syllogistic-deduction above on the left never uses Bocardo then Y = B and our abstract representation above on the right represents Σ_2 below:

$$\Sigma_1\quad \dfrac{\begin{array}{cc}\vdots & \vdots\\ \text{AaB} & \text{BaY}\end{array}}{\text{AaY}} \qquad\qquad \Sigma_2\quad \dfrac{\begin{array}{cc}\vdots & \vdots\\ \text{XaA} & \text{AaB}\end{array}}{\text{XaB}}$$

   (Cf. **Observation 2** in our discussion of improvement by Bocardo and Baroco in **Section 4.4.1**). **Lemma 6** plays a parallel role ensuring generality throughout the considered below. In what follows we leave our reliance on it implicit.

2. Suppose that $concl(\Sigma*)$ = XeY. The reduced assumption must be XiY. And the largest reductio-free subtree of $\Sigma*$ with XiY as premise must have either (a) an i-conclusion or (b) an o-conclusion. In both cases there are two transformational possibilities depending on the form of $\Sigma*$:

[2] Though not strictly necessary, to help the reader, I always provide transformations into a canonical form using nothing other than Barbara, Celarent, Darii, Ferio, Baroco, Bocardo, and conversions.

(a) $\dfrac{\dfrac{\vdots}{\text{AeB}} \quad \dfrac{\vdots}{\text{AiB}}}{\text{XeY}}$ $\overline{\text{XiY}}$ $\Longrightarrow$ $\dfrac{\dfrac{\dfrac{\vdots}{\text{AeB}}}{\text{BeA}} \quad \dfrac{\vdots}{\text{AaX}}}{\dfrac{\text{BeX}}{\text{XeB}} \quad \dfrac{\vdots}{\text{BaY}}}$ XeY or $\dfrac{\dfrac{\dfrac{\vdots}{\text{AeB}}}{\text{BeA}} \quad \dfrac{\vdots}{\text{AaY}}}{\dfrac{\text{BeY}}{\text{YeB}} \quad \dfrac{\vdots}{\text{BaX}}}$ $\dfrac{\text{YeX}}{\text{XeY}}$

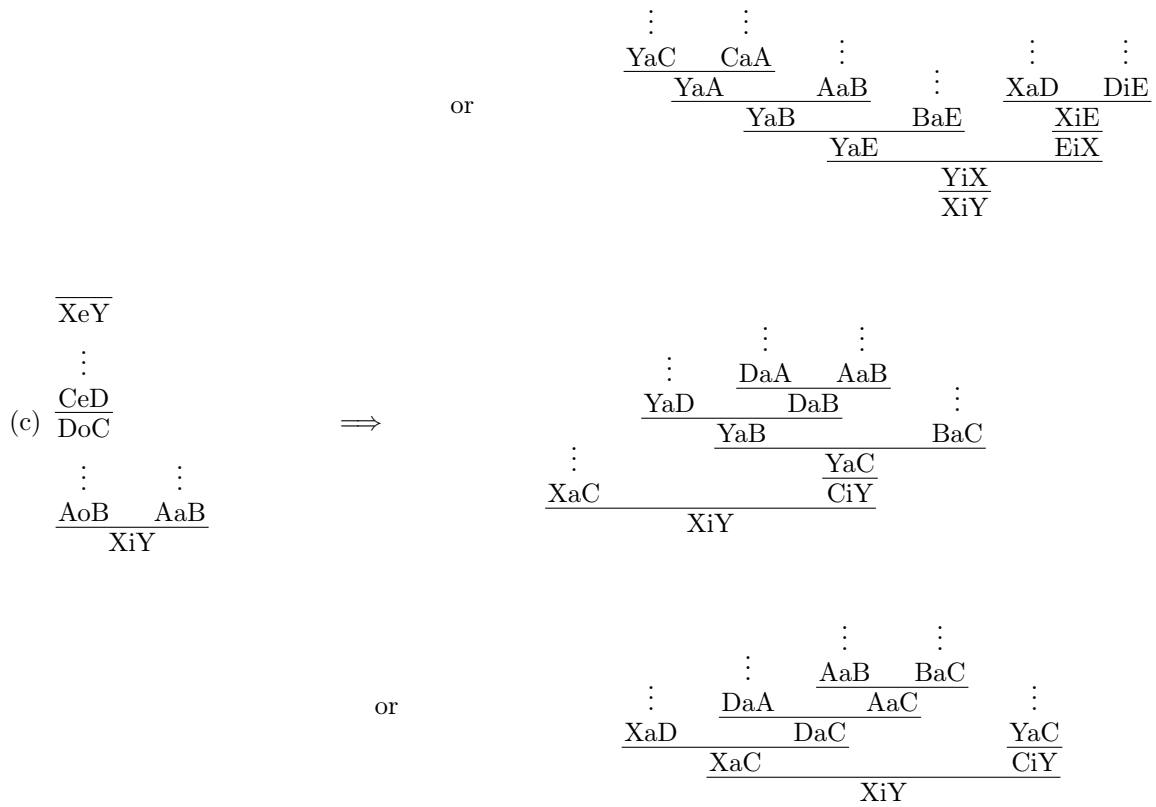
(b) $\dfrac{\dfrac{\vdots}{\text{CeD}} \quad \dfrac{\vdots}{\text{DiE}}}{\text{CoE}} \quad \dfrac{\vdots}{\text{AoB}} \quad \dfrac{\vdots}{\text{AaB}}$ $\overline{\text{XiY}}$ XeY $\Longrightarrow$ $\dfrac{\dfrac{\vdots}{\text{CeD}}}{\text{DeC}} \quad \dfrac{\dfrac{\vdots}{\text{CaA}} \quad \dfrac{\dfrac{\vdots}{\text{AaB}} \quad \dfrac{\dfrac{\vdots}{\text{BaE}} \quad \dfrac{\vdots}{\text{EaX}}}{\text{BaX}}}{\text{AaX}}}{\text{CaX}}$ $\dfrac{\text{DeX}}{\text{XeD}} \quad \dfrac{\vdots}{\text{DaY}}$ XeY

or $\dfrac{\dfrac{\vdots}{\text{CeD}}}{\text{DeC}} \quad \dfrac{\dfrac{\vdots}{\text{CaA}} \quad \dfrac{\dfrac{\vdots}{\text{AaB}} \quad \dfrac{\dfrac{\vdots}{\text{BaE}} \quad \dfrac{\vdots}{\text{EaY}}}{\text{BaY}}}{\text{AaY}}}{\text{CaY}}$ $\dfrac{\text{DeY}}{\text{YeD}} \quad \dfrac{\vdots}{\text{DaX}}$ $\dfrac{\text{YeX}}{\text{XeY}}$

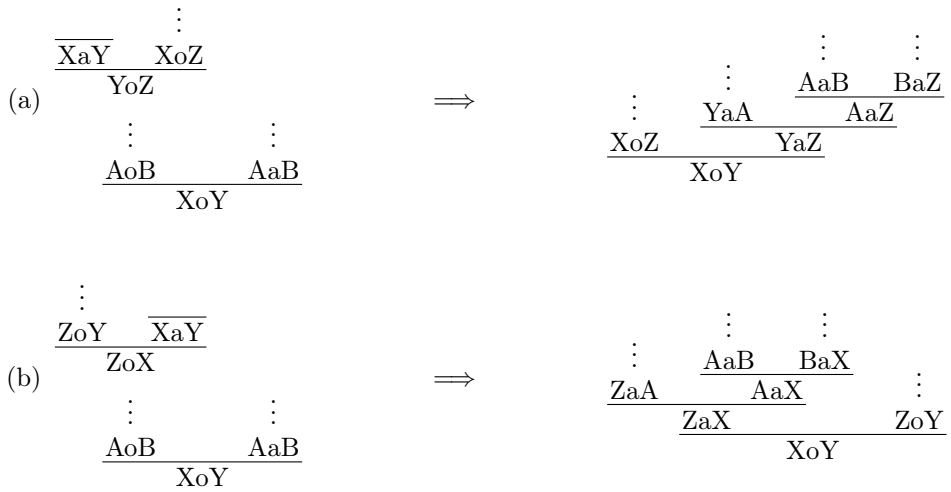
3. Suppose that $\mathit{concl}(\Sigma*) = \text{XiY}$. The reduced assumption must be XeY. And the largest reductio-free subtree of $\Sigma*$ with XeY as premise must have either an e-conclusion or an o-conclusion. The former case is dealt with in (a) below; the latter case in (b)-(c).

(a) $\dfrac{\dfrac{\vdots}{\text{AiB}} \quad \dfrac{\vdots}{\text{AeB}}}{\text{XiY}}$ $\overline{\text{XeY}}$ $\Longrightarrow$ $\dfrac{\vdots}{\text{XaB}} \quad \dfrac{\dfrac{\vdots}{\text{YaA}} \quad \dfrac{\vdots}{\text{AiB}}}{\dfrac{\text{YiB}}{\text{BiY}}}$ XiY or $\dfrac{\vdots}{\text{YaB}} \quad \dfrac{\dfrac{\vdots}{\text{XaA}} \quad \dfrac{\vdots}{\text{AiB}}}{\dfrac{\text{XiB}}{\text{BiX}}}$ $\dfrac{\text{YiX}}{\text{XiY}}$

(b) $\dfrac{\dfrac{\vdots}{\text{CeD}} \quad \dfrac{\vdots}{\text{DiE}}}{\text{CoE}} \quad \dfrac{\vdots}{\text{AoB}} \quad \dfrac{\vdots}{\text{AaB}}$ $\overline{\text{XeY}}$ XiY $\Longrightarrow$ $\dfrac{\dfrac{\vdots}{\text{XaC}} \quad \dfrac{\vdots}{\text{CaA}}}{\text{XaA}} \quad \dfrac{\vdots}{\text{AaB}} \quad \dfrac{\vdots}{\text{BaE}} \quad \dfrac{\dfrac{\vdots}{\text{YaD}} \quad \dfrac{\vdots}{\text{DiE}}}{\text{YiE}}$ $\dfrac{\text{XaB}}{\text{XaE}} \quad \dfrac{\text{EiY}}{\text{XiY}}$



4. Suppose that $concl(\Sigma*)$ = XoY. The reduced assumption must be XaY. And the largest reductio-free subtree of $\Sigma*$ with XaY as premise can have either an o-conclusion (cases (a)-(f)), an a-conclusion (case (g)), an e-conclusion (case (h)), or an i-conclusion (case (i)-(j)). Possible transformations are presented as above. (NB that the **Corollary** to our a-space **Lemmas 3-5** ensures the generality of cases (a)-(e), (h)-(i)).



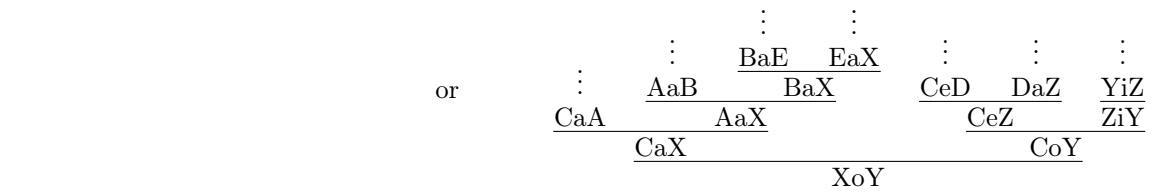
(c)
$$\dfrac{\dfrac{\vdots}{\dfrac{\mathrm{ZeX}\quad\overline{\mathrm{XaY}}}{\mathrm{ZeY}}}\quad\dfrac{\vdots\qquad\vdots}{\dfrac{\mathrm{CeD}\qquad\mathrm{DiE}}{\mathrm{CoE}}}\quad\dfrac{\vdots\qquad\vdots}{\dfrac{\mathrm{AoB}\qquad\mathrm{AaB}}{\mathrm{XoY}}}}{\mathrm{XoY}}$$
$\Longrightarrow$
$$\dfrac{\dfrac{\vdots}{\mathrm{ZeX}}\quad\dfrac{\dfrac{\dfrac{\vdots\qquad\vdots}{\mathrm{ZaC}\quad\mathrm{CaA}}\quad\vdots}{\mathrm{ZaA}\qquad\mathrm{AaB}}\quad\dfrac{\vdots}{\mathrm{ZaB}\qquad\mathrm{BaE}}\quad\dfrac{\vdots\qquad\vdots}{\mathrm{YaD}\quad\mathrm{DiE}}}{\mathrm{XeZ}\qquad\mathrm{ZaE}\qquad\mathrm{YiE}}}{\mathrm{XeE}\qquad\mathrm{EiY}}$$
$$\mathrm{XoY}$$

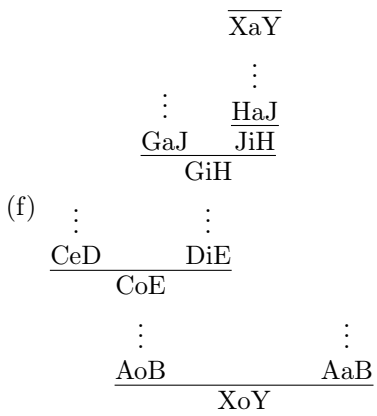
or
$$\dfrac{\dfrac{\vdots}{\mathrm{ZeX}}\quad\dfrac{\vdots}{\mathrm{XeZ}\quad\mathrm{ZaD}}\quad\dfrac{\vdots}{\mathrm{XeD}\qquad\mathrm{DiE}}\quad\dfrac{\vdots}{\mathrm{YaC}}\quad\dfrac{\vdots\qquad\vdots}{\mathrm{CaA}\qquad\mathrm{AaB}\quad\mathrm{BaE}}}{\mathrm{XoE}\qquad\mathrm{CaE}\qquad\mathrm{AaE}}$$
$$\dfrac{\mathrm{YaE}}{\mathrm{XoY}}$$

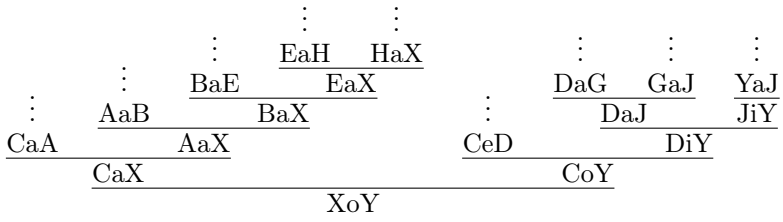
(d)
$$\dfrac{\dfrac{\vdots}{\dfrac{\mathrm{ZeX}\quad\overline{\mathrm{XaY}}}{\mathrm{ZeY}}}\quad\dfrac{\vdots}{\dfrac{\mathrm{CeD}}{\mathrm{DoC}}}\quad\dfrac{\vdots\qquad\vdots}{\dfrac{\mathrm{AoB}\qquad\mathrm{AaB}}{\mathrm{XoY}}}}{\mathrm{XoY}}$$
$\Longrightarrow$
$$\dfrac{\dfrac{\vdots}{\mathrm{ZeX}}\quad\dfrac{\vdots}{\mathrm{XeZ}\quad\mathrm{ZaC}}\quad\dfrac{\vdots}{\mathrm{YaD}}\quad\dfrac{\dfrac{\vdots\qquad\vdots}{\mathrm{DaA}\quad\mathrm{AaB}}\quad\vdots}{\mathrm{DaB}\qquad\mathrm{BaC}}}{\mathrm{XeC}\qquad\mathrm{YaC}\qquad\mathrm{DaC}}$$
$$\dfrac{\mathrm{CiY}}{\mathrm{XoY}}$$

or
$$\dfrac{\dfrac{\vdots}{\mathrm{ZeX}}\quad\dfrac{\vdots}{\mathrm{XeZ}\quad\mathrm{ZaD}}\quad\dfrac{\vdots}{\mathrm{DaA}}\quad\dfrac{\vdots\qquad\vdots}{\mathrm{AaB}\quad\mathrm{BaC}}}{\mathrm{XeZ}\qquad\mathrm{ZaC}\qquad\mathrm{DaC}\qquad\mathrm{AaC}}\quad\dfrac{\vdots}{\mathrm{YaC}}$$
$$\dfrac{\mathrm{XeC}\qquad\mathrm{CiY}}{\mathrm{XoY}}$$

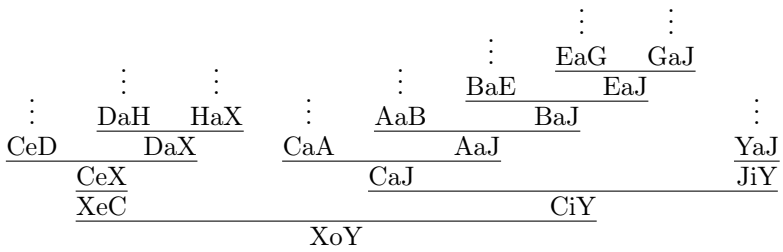
(e)
$$\dfrac{\dfrac{\dfrac{\overline{\mathrm{XaY}}\quad\mathrm{YiZ}}{\mathrm{XiZ}}\quad\vdots}{\dfrac{\mathrm{CeD}\qquad\mathrm{DiE}}{\mathrm{CoE}}}\quad\dfrac{\vdots\qquad\vdots}{\dfrac{\mathrm{AoB}\qquad\mathrm{AaB}}{\mathrm{XoY}}}}{\mathrm{XoY}}$$
$\Longrightarrow$
$$\dfrac{\dfrac{\vdots}{\mathrm{CeD}}\quad\dfrac{\vdots}{\mathrm{DeC}\quad\mathrm{CaA}}\quad\dfrac{\dfrac{\vdots}{\mathrm{AaB}}\quad\dfrac{\dfrac{\vdots}{\mathrm{BaE}}\quad\dfrac{\vdots\qquad\vdots}{\mathrm{EaY}\quad\mathrm{YaZ}}}{\mathrm{EaZ}}}{\mathrm{BaZ}}}{\mathrm{AaZ}\qquad\mathrm{CaZ}}$$
$$\dfrac{\dfrac{\mathrm{DeZ}}{\mathrm{ZeD}}\quad\dfrac{\vdots}{\mathrm{DaX}}}{\mathrm{ZeX}}$$
$$\dfrac{\mathrm{XeZ}\qquad\vdots}{\mathrm{XoY}}\quad\mathrm{ZiY}$$

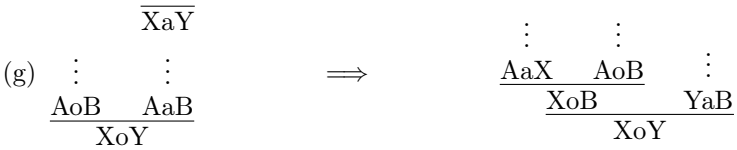

or
BaE EaX
AaB BaX
CeD DaZ YiZ
CaA AaX
CeZ ZiY
CaX CoY
XoY

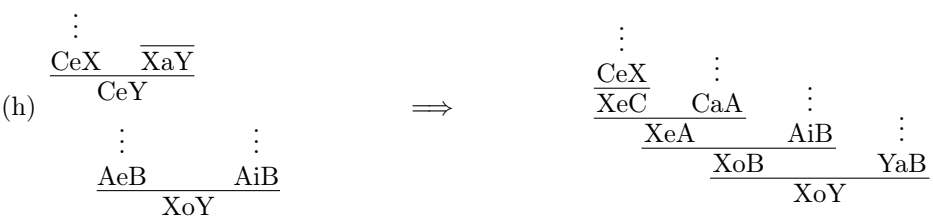

XaY
HaJ
GaJ JiH
GiH
(f)
CeD DiE
CoE
AoB AaB
XoY

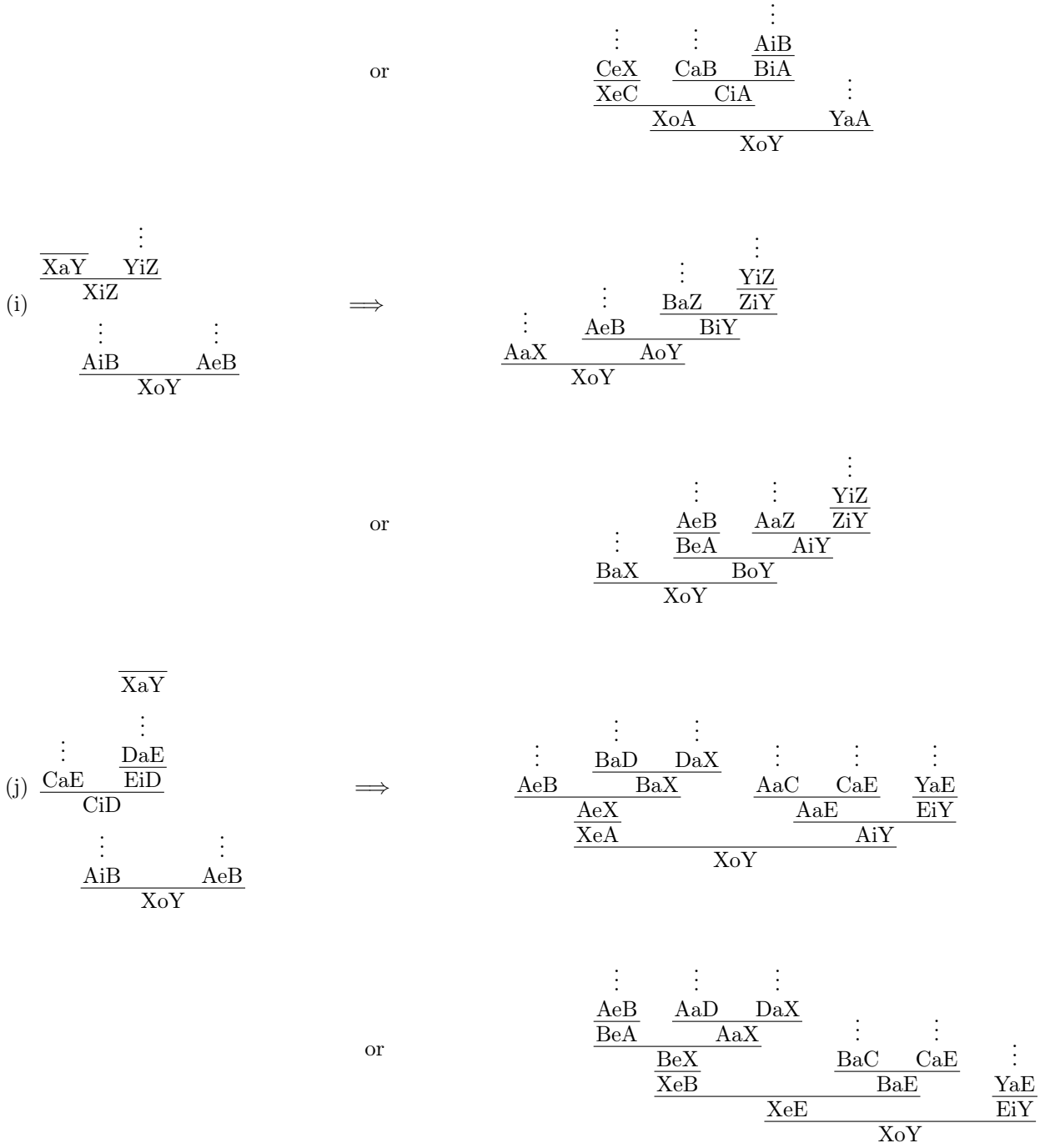

EaH HaX
BaE EaX
DaG GaJ YaJ
AaB BaX
DaJ JiY
CaA AaX
CeD DiY
CaX CoY
XoY

or


EaG GaJ
BaE EaJ
DaH HaX
AaB BaJ
CeD DaX
CaA AaJ
YaJ
CeX
CaJ JiY
XeC CiY
XoY


XaY
(g)
AoB AaB
XoY
AaX AoB
XoB YaB
XoY


CeX XaY
CeY
(h)
AeB AiB
XoY
CeX
XeC CaA
XeA AiB
XoB YaB
XoY



6.2.2 Conversion elimination

Definition Define a $*$-function on syllogistic propositions:

- $(\mathrm{AaB})* = \{\mathrm{AaB}, \mathrm{BiA}, \mathrm{AiB}\}$

- (AeB)$*$ = {AeB, BeA, AoB, BoA}
- (AiB)$*$ = {AiB, BiA}
- (AoB)$*$ = {AoB}

Definition $*$-consequence

$\Gamma \vdash_{D*} \varphi$ iff for some ψ such that $\varphi \in \psi*$: $\Gamma \vdash_D \psi$

Observation

It is easy to check that $\Gamma \vdash_D \varphi$ iff $\Gamma \vdash_{D*} \varphi$.

Theorem

$\Gamma \vdash_D \varphi$ iff there's a syllogistic-deduction Σ that never uses conversion and $\Sigma : \Gamma \vdash_{D*} \varphi$

Proof. Right-to-left is trivial. Left-to-right follows immediately given **Lemma 7** proved below.

Lemma 7

Let $\Sigma : \Gamma \vdash_D \varphi$. There's a $\Sigma* : \Gamma \vdash_D \varphi$ such that the only conversions in $\Sigma*$ are below all basic syllogisms in $\Sigma*$.

Proof. Let h be some standard measure of inference height (distance away from the conclusion) on syllogistic-deductions. Note that in any syllogistic-deduction, each of the four sets below are totally ordered by h:

1. the set of simple e-conversions
2. the set of simple i-conversions
3. the set of accidental e-conversions
4. the set of accidental a-conversions

Let X be one of the four conversion types, and let $\Sigma : \Gamma \vdash_D \varphi$ be a syllogistic-deduction whose highest X-inference has $n > 0$ non-conversion inferences in the branch below it. The result follows if we prove that for every such X there's a syllogistic-deduction $\Sigma* : \Gamma \vdash_D \varphi$ whose highest X-inference has $m < n$ non-conversion inferences in the branch below it. We treat the four conversion types one by one. Without loss of generality, we assume that Σ contains no strings of consecutive simple conversions.

1. Simple e-conversion.

   (a) Suppose the highest simple e-conversion has an instance of Camestres directly below it. Our $\Sigma*$ is obtained as follows:

$$\dfrac{\dfrac{\text{BaA}\quad\dfrac{\text{CeB}}{\text{BeC}}}{\text{AeC}}}{} \quad\Longrightarrow\quad \dfrac{\dfrac{\text{CeB}\quad\text{BaA}}{\text{CeA}}}{\text{AeC}}$$

   (b) If the highest simple e-conversion has an instance of Celarent, Cesare, Ferio, Festino, Ferison, Fiseso, Fampesmo, Felapton directly below it, the conversion can be eliminated in one of the following manners:

      i. replace [e-conversion + Celarent] with Cesare

      ii. replace [e-conversion + Cesare] with Celarent

      iii. replace [e-conversion + Ferio] with Festino

      iv. replace [e-conversion + Festino] with Ferio

      v. replace [e-conversion + Ferison] with Frisesomorum

      vi. replace [e-conversion + Frisesomorum] with Ferison

      vii. replace [e-conversion + Fapesmo] with Felapton

      viii. replace [e-conversion + Felapton] with Fapesmo

   (c) If the highest simple e-conversion has an instance of accidental e-conversion directly below it, our $\Sigma*$ is obtained in one of the following manners:

      i. $\dfrac{\dfrac{\dfrac{\text{AeC}}{\text{CeA}}}{\text{AoC}}\quad\text{BaC}}{\text{AoB}} \quad\Longrightarrow\quad \dfrac{\text{AeC}\quad\text{BaC}}{\text{AoB}}$

      ii. $\dfrac{\text{AaB}\quad\dfrac{\dfrac{\text{AeC}}{\text{CeA}}}{\text{AoC}}}{\text{BoC}} \quad\Longrightarrow\quad \dfrac{\text{AeC}\quad\text{AaB}}{\dfrac{\text{CeB}}{\text{BoC}}}$

2. i-conversion

   (a) Suppose the highest i-conversion has an instance of Disamis directly below it. Our $\Sigma*$ is obtained as follows:

$$\dfrac{\dfrac{\dfrac{\text{BiA}}{\text{AiB}}\quad\text{CaB}}{\text{AiC}}}{} \quad\Longrightarrow\quad \dfrac{\dfrac{\text{CaB}\quad\text{BiA}}{\text{CiA}}}{\text{AiC}}$$

(b) If the highest i-conversion has an instance of Darii, Datisi, Ferio, Ferison, Festino, or Frisesomorum directly below it, the conversion can be eliminated in one of the following manners:

i. replace [i-conversion + Darii] with Datisi

ii. replace [i-conversion + Datisi] with Darii

iii. replace [i-conversion + Ferio] with Ferison

iv. replace [i-conversion + Ferison] with Ferio

v. replace [i-conversion + Festino] with Frisesomorum

vi. replace [i-conversion + Frisesomorum] with Festino

3. accidental e-conversion

(a) Suppose the highest accidental e-conversion has an instance of Baroco directly below it. Our $\Sigma*$ is obtained as follows:

$$\dfrac{\text{BaA} \quad \dfrac{\text{CeB}}{\text{BoC}}}{\text{AoC}} \qquad \Longrightarrow \qquad \dfrac{\dfrac{\text{CeB} \quad \text{BaA}}{\text{CeA}}}{\text{CoA}}$$

(b) If the highest accidental e-conversion has an instance of Bocardo directly below it, the conversion can be eliminated through replacement with Fapesmo

4. a-conversion

(a) Suppose the highest a-conversion has an instance of Disamis directly below it. Our $\Sigma*$ is obtained as follows:

$$\dfrac{\dfrac{\text{BaA}}{\text{AiB}} \quad \text{CaB}}{\text{AiC}} \qquad \Longrightarrow \qquad \dfrac{\dfrac{\text{CaB} \quad \text{BaA}}{\text{CaA}}}{\text{AiC}}$$

(b) Suppose the highest a-conversion has an instance of Datisi directly below it. Our $\Sigma*$ is obtained as follows:

$$\dfrac{\text{AaB} \quad \dfrac{\text{BaC}}{\text{CiB}}}{\text{AiC}} \qquad \Longrightarrow \qquad \dfrac{\dfrac{\dfrac{\text{AaB} \quad \text{BaC}}{\text{AaC}}}{\text{CiA}}}{\text{AiC}}$$

(c) Suppose the highest a-conversion has an instance of Ferison directly below it. Our $\Sigma*$ is obtained as follows:

$$\dfrac{\text{AeB} \quad \dfrac{\text{BaC}}{\text{CiB}}}{\text{AoC}} \qquad \Longrightarrow \qquad \dfrac{\dfrac{\dfrac{\text{AeB} \quad \text{BaC}}{\text{AeC}}}{\text{CeA}}}{\text{AoC}}$$

(d) Suppose the highest a-conversion has an instance of Frisesomorum directly below it. Our $\Sigma*$ is obtained as follows:

$$\dfrac{\dfrac{\text{BaA}}{\text{AiB}} \quad \text{BeC}}{\text{CoA}} \qquad \Longrightarrow \qquad \dfrac{\text{BaA} \quad \text{BeC}}{\dfrac{\text{AeC}}{\text{CoA}}}$$

(e) If the highest a-conversion has an instance of Darii, Ferio, or Festino directly below it, the conversion can be eliminated in one of the following manners:

i. replace [a-conversion + Darii] with Darapti

ii. replace [a-conversion + Ferio] with Felapton

iii. replace [a-conversion + Festino] with Fapesmo

(f) Suppose the highest a-conversion has an instance of i-conversion directly below it. Our $\Sigma*$ is obtained in one of the following manners:

i. $$\dfrac{\text{CaA} \quad \dfrac{\dfrac{\text{AaB}}{\text{BiA}}}{\text{AiB}}}{\text{CiB}} \qquad \Longrightarrow \qquad \dfrac{\text{CaA} \quad \text{AaB}}{\dfrac{\dfrac{\text{CaB}}{\text{BiC}}}{\text{CiB}}}$$

ii. $$\dfrac{\text{CaB} \quad \dfrac{\dfrac{\text{AaB}}{\text{BiA}}}{\text{AiB}}}{\text{CiA}} \qquad \Longrightarrow \qquad \dfrac{\text{CaB} \quad \text{AaB}}{\text{CiA}}$$

iii. $$\dfrac{\dfrac{\dfrac{\text{AaB}}{\text{BiA}}}{\text{AiB}} \quad \text{CaB}}{\text{AiC}} \qquad \Longrightarrow \qquad \dfrac{\text{AaB} \quad \text{CaB}}{\text{AiC}}$$

iv. $$\dfrac{\text{CeA} \quad \dfrac{\dfrac{\text{AaB}}{\text{BiA}}}{\text{AiB}}}{\text{CoB}} \qquad \Longrightarrow \qquad \dfrac{\text{CeA} \quad \text{AaB}}{\dfrac{\dfrac{\text{CeB}}{\text{BeC}}}{\text{CoB}}}$$

v. $$\dfrac{\text{AeC} \quad \dfrac{\dfrac{\text{AaB}}{\text{BiA}}}{\text{AiB}}}{\text{CoB}} \qquad \Longrightarrow \qquad \dfrac{\text{AaB} \quad \text{AeC}}{\dfrac{\text{BeC}}{\text{CoB}}}$$

vi. $$\dfrac{\text{CeB} \quad \dfrac{\dfrac{\text{AaB}}{\text{BiA}}}{\text{AiB}}}{\text{CoA}} \qquad \Longrightarrow \qquad \dfrac{\text{CeB} \quad \text{AaB}}{\text{CoA}}$$

vii. $$\dfrac{\dfrac{\dfrac{\text{AaB}}{\text{BiA}}}{\text{AiB}} \quad \text{BeC}}{\text{CoA}} \qquad \Longrightarrow \qquad \dfrac{\text{AaB} \quad \text{BeC}}{\text{CoA}}$$

6.2.3 Science and mood elimination

Definition

A syllogistic-deduction Σ is io-universal iff every premise of Σ is universal and the conclusion of Σ is an i-proposition or an o-proposition

Theorem

If $\Sigma: \ \Gamma \vdash_D \varphi$ is an io-universal syllogistic-deduction there is a syllogistic-deduction $\Sigma*: \ \Gamma \vdash_D \varphi$ such that every inference in Σ^* is one of: Barbara, Celarent, Cesare, Camestres, Darapti, Felapton, Fapesmo.

Proof. Assume some standard measure of inference height. Call any syllogistic inference with a particular premise (Darii, Ferio, etc.) a *p-inference*. Since every syllogistic tree has at most one particular premise leaf, it is clear that the p-inferences in Σ are totally ordered by inference height. Since Σ is finite, repeated stage-by-stage application of the following transformation procedures (whichever is appropriate at each stage) on the highest p-inference generates Σ^* in finitely many steps.

1. Highest p-inference is Darii:

$$\dfrac{\text{AaB} \quad \dfrac{\text{BaE} \quad \text{CaE}}{\text{BiC}}}{\text{AiC}} \quad \Longrightarrow \quad \dfrac{\dfrac{\text{AaB} \quad \text{BaE}}{\text{AaE}} \quad \text{CaE}}{\text{AiC}}$$

2. Highest p-inference is Disamis:

$$\dfrac{\dfrac{\text{AaD} \quad \text{BaD}}{\text{AiB}} \quad \text{CaB}}{\text{AiC}} \quad \Longrightarrow \quad \dfrac{\text{AaD} \quad \dfrac{\text{CaB} \quad \text{BaD}}{\text{CaD}}}{\text{AiC}}$$

3. Highest p-inference is Datisi:

$$\dfrac{\text{AaB} \quad \dfrac{\text{CaD} \quad \text{BaD}}{\text{CiB}}}{\text{AiC}} \quad \Longrightarrow \quad \dfrac{\dfrac{\text{AaB} \quad \text{BaD}}{\text{AaD}} \quad \text{CaD}}{\text{AiC}}$$

4. Highest p-inference is Ferio:

$$\dfrac{\text{AeB} \quad \dfrac{\text{BaD} \quad \text{CaD}}{\text{BiC}}}{\text{AoC}} \quad \Longrightarrow \quad \dfrac{\dfrac{\text{AeB} \quad \text{BaD}}{\text{AeD}} \quad \text{CaD}}{\text{AoC}}$$

5. Highest p-inference is Festino:

$$\dfrac{\text{BeA} \quad \dfrac{\text{BaD} \quad \text{CaD}}{\text{BiC}}}{\text{AoC}} \quad \Longrightarrow \quad \dfrac{\dfrac{\text{BeA} \quad \text{BaD}}{\text{AeD}} \quad \text{CaD}}{\text{AoC}}$$

6. Highest p-inference is Ferison:

$$\dfrac{\mathrm{AeB} \quad \dfrac{\mathrm{CaD} \quad \mathrm{BaD}}{\mathrm{CiB}}}{\mathrm{AoC}} \quad \Longrightarrow \quad \dfrac{\dfrac{\mathrm{AeB} \quad \mathrm{BaD}}{\mathrm{AeD}} \quad \mathrm{CaD}}{\mathrm{AoC}}$$

7. Highest p-inference is Frisesomorum:

$$\dfrac{\dfrac{\mathrm{AaD} \quad \mathrm{BaD}}{\mathrm{AiB}} \quad \mathrm{BeC}}{\mathrm{CoA}} \quad \Longrightarrow \quad \dfrac{\dfrac{\mathrm{BeC} \quad \mathrm{BaD}}{\mathrm{CeD}} \quad \mathrm{AaD}}{\mathrm{CoA}}$$

8. If the highest p-inference is Baroco, there are two cases to consider (depending whether the o-premise of Baroco is introduced by Felapton or Fapesmo):

(a) $$\dfrac{\mathrm{BaA} \quad \dfrac{\mathrm{BeD} \quad \mathrm{CaD}}{\mathrm{BoC}}}{\mathrm{AoC}} \quad \Longrightarrow \quad \dfrac{\dfrac{\mathrm{BaA} \quad \mathrm{BeD}}{\mathrm{AeD}} \quad \mathrm{CaD}}{\mathrm{AoC}}$$

(b) $$\dfrac{\mathrm{BaA} \quad \dfrac{\mathrm{CaD} \quad \mathrm{DeB}}{\mathrm{BoC}}}{\mathrm{AoC}} \quad \Longrightarrow \quad \dfrac{\mathrm{CaD} \quad \dfrac{\mathrm{DeB} \quad \mathrm{BaA}}{\mathrm{DeA}}}{\mathrm{AoC}}$$

9. If the highest p-inference is Baroco, there are two cases to consider (depending whether the o-premise of Baroco is introduced by Felapton or Fapesmo):

(a) $$\dfrac{\dfrac{\mathrm{AeD} \quad \mathrm{BaD}}{\mathrm{AoB}} \quad \mathrm{CaB}}{\mathrm{AoC}} \quad \Longrightarrow \quad \dfrac{\mathrm{AeD} \quad \dfrac{\mathrm{CaB} \quad \mathrm{BaD}}{\mathrm{CaD}}}{\mathrm{AoC}}$$

(b) $$\dfrac{\dfrac{\mathrm{BaD} \quad \mathrm{DeA}}{\mathrm{AoB}} \quad \mathrm{CaB}}{\mathrm{AoC}} \quad \Longrightarrow \quad \dfrac{\dfrac{\mathrm{CaB} \quad \mathrm{BaD}}{\mathrm{CaD}} \quad \mathrm{DeA}}{\mathrm{AoC}}$$

6.3 Proofs of results mentioned in Section 3

6.3.1 Canonical form for 'proving in a circle and from one another'

Definition

Two syllogistic propositions φ, ψ are inter-derivable exactly if for some Γ, Δ:

1. $\Gamma, \varphi \vdash_D \psi$, and
2. $\Delta, \psi \vdash_D \varphi$

Theorem

1. Both $\Gamma, \varphi \vdash_D \psi$ and $\Delta, \psi \vdash_D \varphi$ iff for some $\Gamma_0 \cup \Gamma_1 = \Gamma$ and $\Delta_0 \cup \Delta_1 \supseteq \Delta$ such that $\Delta_0 \cup \Delta_1 \subseteq \Delta \cup \Gamma$, there are syllogistic-deductions:

$$\dfrac{\dfrac{\Gamma_0 \quad \vdots}{\mathrm{AaB} \quad \varphi}\ \alpha \qquad \dfrac{\Gamma_1 \quad \vdots}{\mathrm{CaD}}}{\psi} \qquad\qquad \dfrac{\dfrac{\Delta_0 \quad \vdots}{\mathrm{BaA} \quad \psi}\ \beta \qquad \dfrac{\Delta_1 \quad \vdots}{\mathrm{DaC}}}{\varphi}$$

2. Let $\Sigma_1 : \Gamma, \varphi \vdash_D \psi$ and $\Sigma_2 : \Delta, \psi \vdash_D \varphi$ be such that (i) $\Gamma \cup \Delta \cup \{\varphi, \psi\}$ are consistent and (ii) Σ_1 and Σ_2 are maximally interesting[3]. Then for some $\Gamma_0 \cup \Gamma_1 = \Gamma$ and $\Delta_0 \cup \Delta_1 = \Delta$ there are syllogistic-deductions:

$$\dfrac{\dfrac{\Gamma_0 \quad \vdots}{\mathrm{AaB} \quad \varphi}\ \alpha \qquad \dfrac{\Gamma_1 \quad \vdots}{\mathrm{CaD}}}{\psi} \qquad\qquad \dfrac{\dfrac{\Delta_0 \quad \vdots}{\mathrm{BaA} \quad \psi}\ \beta \qquad \dfrac{\Delta_1 \quad \vdots}{\mathrm{DaC}}}{\varphi}$$

Proof of 1. Right-to-left is trivial. Left-to-right demonstrandum. It is easy to check that:

1. φ is an a-proposition iff ψ is an a-proposition
2. φ is an e-proposition iff ψ is an e-proposition
3. φ is an i-proposition iff ψ is an i-proposition
4. φ is an o-proposition iff ψ is an o-proposition

In case 1, our a-space lemmas tell us there must be syllogistic-deductions of the following forms (with $\Delta_0 \cup \Delta_1 = \Delta$):

$$\dfrac{\dfrac{\Gamma_0 \quad \vdots}{\mathrm{AaB} \quad \mathrm{BaC}}\ \mathrm{AaC} \qquad \dfrac{\Gamma_1 \quad \vdots}{\mathrm{CaD}}}{\mathrm{AaD}} \qquad\qquad \dfrac{\dfrac{\Delta_0 \quad \vdots}{\mathrm{BaA} \quad \mathrm{AaD}}\ \mathrm{BaD} \qquad \dfrac{\Delta_1 \quad \vdots}{\mathrm{DaC}}}{\mathrm{BaC}}$$

In case 4, our a-space lemmas tell us there must be syllogistic-deductions of the following forms (with $\Delta_0 \cup \Delta_1 = \Delta$):

[3] I.e. neither proves something of the form XiY if something of the form XaY can be proved using only sentences from $\Gamma \cup \Delta \cup \{\varphi, \psi\}$. This is just the 'maximally interesting' condition of **Section 3.3** above.

$$\dfrac{\dfrac{\dfrac{\Gamma_0}{\vdots}\quad\dfrac{\Gamma_1}{\vdots}}{\dfrac{\mathrm{AaB}\quad\mathrm{AoD}}{\mathrm{BoD}}}\quad\dfrac{\Gamma_1}{\vdots}}{\dfrac{\mathrm{BoD}\qquad\mathrm{CaD}}{\mathrm{BoC}}} \qquad\qquad \dfrac{\dfrac{\dfrac{\Delta_0}{\vdots}\quad\dfrac{\Delta_1}{\vdots}}{\dfrac{\mathrm{BaA}\quad\mathrm{BoC}}{\mathrm{AoC}}}\quad\dfrac{\Delta_1}{\vdots}}{\dfrac{\mathrm{AoC}\qquad\mathrm{DaC}}{\mathrm{AoD}}}$$

In case 2, our a-space lemmas tell us there must be a syllogistic-deduction of the following forms (with $\Delta_0 \cup \Delta_1 = \Delta$ or $\Delta_0 \cup \Delta_1 = \Gamma \cup \Delta$):

$$\dfrac{\dfrac{\dfrac{\Gamma_0}{\vdots}\quad\dfrac{\Gamma_1}{\vdots}}{\dfrac{\mathrm{AaB}\quad\mathrm{AeD}}{\mathrm{BeD}}}\quad\dfrac{\Gamma_1}{\vdots}}{\dfrac{\mathrm{BeD}\qquad\mathrm{DaC}}{\mathrm{BeC}}} \qquad\qquad \dfrac{\dfrac{\dfrac{\Delta_0}{\vdots}\quad\dfrac{\Delta_1}{\vdots}}{\dfrac{\mathrm{BaA}\quad\mathrm{BeC}}{\mathrm{AeC}}}\quad\dfrac{\Delta_1}{\vdots}}{\dfrac{\mathrm{AeC}\qquad\mathrm{CaD}}{\mathrm{AeD}}}$$

In case 3, our a-space lemmas tell us there must be a syllogistic-deduction of the following forms (with $\Delta_0 \cup \Delta_1 = \Delta$ or $\Delta_0 \cup \Delta_1 = \Gamma \cup \Delta$):

$$\dfrac{\dfrac{\dfrac{\Gamma_0}{\vdots}\quad\dfrac{\Gamma_1}{\vdots}}{\dfrac{\mathrm{AaB}\quad\mathrm{BiC}}{\mathrm{AiC}}}\quad\dfrac{\Gamma_1}{\vdots}}{\dfrac{\mathrm{AiC}\qquad\mathrm{DaC}}{\mathrm{AiD}}} \qquad\qquad \dfrac{\dfrac{\dfrac{\Delta_0}{\vdots}\quad\dfrac{\Delta_1}{\vdots}}{\dfrac{\mathrm{BaA}\quad\mathrm{AiD}}{\mathrm{BiD}}}\quad\dfrac{\Delta_1}{\vdots}}{\dfrac{\mathrm{BiD}\qquad\mathrm{CaD}}{\mathrm{BiC}}}$$

Proof of 2. From **Proof of 1** . It is straightforward to check that conditions (i) and (ii) respectively ensure that when cases 3. and 4. in our **Proof of 1** of are encountered: $\Delta_0 \cup \Delta_1 = \Delta$.

6.3.2 Proof of Circularity Facts 1-2

Fact 1

Let Σ be a circular syllogistic-deduction whose conclusion is AxB. If Σ's premises are true, every other term used in Σ counter-predicates with A or with B

Proof. Let Σ be a circular syllogistic-deduction witnessing: $\Gamma \vdash$ AxB, with $x \in$ {a,e,i,o}. We say A $\sim_\Sigma$ B iff (i) A = B, or (ii) both AaB$\in \Gamma \cup \Gamma^{\vdash}$ and BaA$\in \Gamma \cup \Gamma^{\vdash}$. We need to show that for every term T$\in$ *term*(Σ): A $\sim_\Sigma$ T or B $\sim_\Sigma$ T. Now, it is easy to check that every circular syllogism is either

1. composed entirely of a-strings, or

2. composed entirely of a- and e-strings, or

3. composed entirely of a- and i-strings, or

4. composed entirely of a- and o-strings

Beginning with the first case, let Σ be a circular Barbara-string concluding AaB. And let's stipulate that the premise occurrences of any syllogism Σ are numbered $p_1, p_2, ..., p_{n-1}, p_n$ tracing its tree left-to-right. In Σ's tree pick the left-most occurrence of AaB: premise p_k. WLOG assume that $k > 1$. By **Lemma 1.1** we know there are Barbara-strings Σ' witnessing $\{p_1, ..., p_{k-2}, p_{k-1}\} \vdash$AaA and Σ'' witnessing $\{p_{k+1}, p_{k+2}, ..., p_n\} \vdash$BaB, with $\{p_1, ..., p_{k-2}, p_{k-1}\} \cup \{p_{k+1}, p_{k+2}, ..., p_n\} \subseteq premises(\Sigma)$. By **Lemma 1.2** together with the fact that $term(\Sigma') \cup term(\Sigma'') = term(\Sigma)$ the required result holds. Next we consider cases of 2-4 above. The required result for case 2 follows immediately given **Lemma 2** in conjunction with **Lemma 1.2** (consider how can Barbara-strings can combine with C-strings). Mutatis mutandis, case 3 is handed by **Lemma 3** in conjunction with **Lemma 1.2** and case 4 by **Lemma 4** in conjunction with **Lemma 1.2**.

Fact 2

If $\varphi, \alpha \vdash_D \psi$ and $\psi, \beta \vdash_D \varphi$, then α is of the form XaY and β of the form YaX

Proof. Consideration of cases.

6.3.3 Proof of various results concerning weak-circularity including Facts 3-5

We study weak-circularity relative to Malink's consequence relation $\vdash_M$ (defined in **6.1.1** above).

Definition Weak-circularity

Where Π is a non-empty collection of syllogistic sentences and $x \in$ {a, e, i, o}:

1. Π is weakly-circular iff $\Pi \subseteq \Pi^{\vdash}_M$

2. Π is x-weakly-circular iff $\Pi_x \subseteq \Pi^{\vdash}_M$ (where Π_x consists of all and only the x-propositions in Π)

Definition Relative counter-predication for $\vdash_M$

Where Γ is a collection of syllogistic sentences:

· A $\sim_\Gamma$ B (A and B counter-predicate$_\Gamma$) iff (i) A = B, or (ii) both AaB$\in \Gamma \cup \Gamma^{\vdash}_M$ and BaA$\in \Gamma \cup \Gamma^{\vdash}_M$

Definition

Where Γ is a collection of syllogistic sentences:

1. for T$\in$ $terms(\Gamma)$: $[\mathrm{T}]^{\sim}_{\Gamma} = \{\mathrm{T}' \in terms(\Gamma) : \mathrm{T} \sim_{\Gamma} \mathrm{T}'\}$

2. $[\mathrm{A}]^{\sim}_{\Gamma} > [\mathrm{B}]^{\sim}_{\Gamma}$ iff (i) AaB$\in \Gamma \cup \Gamma^{\vdash}_{M}$, and (ii) $[\mathrm{A}]^{\sim}_{\Gamma} \neq [\mathrm{B}]^{\sim}_{\Gamma}$

3. $[\mathrm{A}]^{\sim}_{\Gamma} \succ [\mathrm{B}]^{\sim}_{\Gamma}$ iff (i) $[\mathrm{A}]^{\sim}_{\Gamma} > [\mathrm{B}]^{\sim}_{\Gamma}$ and (ii) there is no C $\in terms(\Gamma)$ such that $[\mathrm{A}]^{\sim}_{\Gamma} > [\mathrm{C}]^{\sim}_{\Gamma} > [\mathrm{B}]^{\sim}_{\Gamma}$

4. Γ is $>$-dense iff for all A,B$\in$ $terms(\Gamma)$: if $[\mathrm{A}]^{\sim}_{\Gamma} > [\mathrm{B}]^{\sim}_{\Gamma}$, then $[\mathrm{A}]^{\sim}_{\Gamma} \not\succ [\mathrm{B}]^{\sim}_{\Gamma}$

Observations

For all Γ, $\sim_{\Gamma}$ will be an equivalence relation. So, for all T, $[\mathrm{T}]^{\sim}_{\Gamma}$ will be an equivalence class. So $\{[\mathrm{T}]^{\sim}_{\Gamma} : \mathrm{T} \in terms(\Gamma)\}$ partitions $terms(\Gamma)$. Note too that so defined, $>$ is a strict partial-order on $\{[\mathrm{T}]^{\sim}_{\Gamma} : \mathrm{T} \in terms(\Gamma)\}$. Intuitively, $>$ indicates level of extensional generality: $[\mathrm{A}]^{\sim}_{\Gamma} > [\mathrm{B}]^{\sim}_{\Gamma}$ iff A is (strictly) more general than B. It is easy to check that, $>$ can be equivalently defined:

- $[\mathrm{A}]^{\sim}_{\Gamma} > [\mathrm{B}]^{\sim}_{\Gamma}$ iff for some X$\in[\mathrm{A}]^{\sim}_{\Gamma}$, Y$\in$ $[\mathrm{B}]^{\sim}_{\Gamma}$: XaY$\in \Gamma \cup \Gamma^{\vdash}_{M}$ & YaX$\notin \Gamma \cup \Gamma^{\vdash}_{M}$

Intuitively, $[\mathrm{A}]^{\sim}_{\Gamma} \succ [\mathrm{B}]^{\sim}_{\Gamma}$ indicates that A and B are $>$-consecutive: that $[\mathrm{A}]^{\sim}_{\Gamma}$ is the immediate $>$-successor of $[\mathrm{B}]^{\sim}_{\Gamma}$.

Theorem 1

Let Π be a non-empty collection of syllogistic sentences. Then Π is weakly-circular iff

1. $|\Pi| \geq 3$

2. nothing of the form XaX occurs in Π

3. if X$\in$ $terms(\Pi)$, then $|[\mathrm{X}]^{\sim}_{\Pi}| \neq 2$

4. if $[\mathrm{X}]^{\sim}_{\Pi} \succ [\mathrm{Y}]^{\sim}_{\Pi}$, either $|[\mathrm{X}]^{\sim}_{\Pi}| \neq 1$ or $|[\mathrm{Y}]^{\sim}_{\Pi}| \neq 1$, and

5. for all $z \in \{\mathrm{e}, \mathrm{i}, \mathrm{o}\}$: Π is z-weakly-circular

Proof. Left-to-right. By assumption, there is some $\varphi \in \Pi$. As Π is weakly-circular, given our definition of $\vdash_{M}$, conjuncts 1-2 are immediate. For conjunct 3, let there be distinct X,Y$\in$ $terms(\Pi)$ such that $[\mathrm{X}]^{\sim}_{\Pi} = \{\mathrm{X},\mathrm{Y}\}$. Since Π is weakly-circular and given our definition of $\vdash_{M}$, there must be a Z distinct from X and Y such that: XaZ, ZaY$\in \Pi \cup \Pi^{\vdash}_{M}$. But then, as YaX$\in \Pi \cup \Pi^{\vdash}_{M}$, $|[\mathrm{X}]^{\sim}_{\Pi}| \neq 2$. For conjunct 4, let there be X,Y$\in$ $terms(\Pi)$ such that $[\mathrm{X}]^{\sim}_{\Pi} \succ [\mathrm{Y}]^{\sim}_{\Pi}$, but both $|[\mathrm{X}]^{\sim}_{\Pi}| = 1$ and $|[\mathrm{Y}]^{\sim}_{\Pi}| = 1$. As $[\mathrm{X}]^{\sim}_{\Pi} \succ [\mathrm{Y}]^{\sim}_{\Pi}$, it follows that XaY$\in \Pi \cup \Pi^{\vdash}_{M}$. Given our assumptions there must be a Z distinct from X and Y such that: XaZ, ZaY$\in \Pi \cup \Pi^{\vdash}_{S}$. But $|[\mathrm{X}]^{\sim}_{\Pi}| = 1$ and $|[\mathrm{Y}]^{\sim}_{\Pi}| = 1$. So $[\mathrm{X}]^{\sim}_{\Pi} \not\succ [\mathrm{Y}]^{\sim}_{\Pi}$. Conjunct 5 requires no argument. Right-to-left. Given conjunct 5, it suffices to show that Π is a-weakly-circular (see definition above) if conjuncts

1-4 hold. So let XaY$\in \Pi$. Given conjunct 2, X and Y are distinct. Now, either $[X]^{\sim}_{\Pi} = [Y]^{\sim}_{\Pi}$ or $[X]^{\sim}_{\Pi} \neq [Y]^{\sim}_{\Pi}$. In the former case, conjunct 3 guarantees an M-syllogistic-deduction of XaY:

$$\dfrac{\begin{array}{cc}\Phi & \Psi \\ \vdots & \vdots \\ \text{XaZ} & \text{ZaY}\end{array}}{\text{XaY}}$$

with $\Phi \cup \Psi \subseteq \Pi$. So let $[X]^{\sim}_{\Pi} \neq [Y]^{\sim}_{\Pi}$. Then since XaY$\in \Pi$, we have that: $[X]^{\sim}_{\Pi} > [Y]^{\sim}_{\Pi}$. Well, either $[X]^{\sim}_{\Pi} \succ [Y]^{\sim}_{\Pi}$ or not. In the latter case, we will have $[X]^{\sim}_{\Pi} > [Z]^{\sim}_{\Pi} > [Y]^{\sim}_{\Pi}$ for some Z distinct from X and Y, and there will be an M-syllogistic-deduction of XaY from some $\Phi \cup \Psi \subseteq \Pi$ as above. So assume that $[X]^{\sim}_{\Pi} \succ [Y]^{\sim}_{\Pi}$. Conjunct 3 and 4 ensure that XaY has an M-syllogistic-deduction of one of the following forms using only premises from Π:

$$\dfrac{\begin{array}{ccc}[\Phi] & [\Theta] & [\Lambda] \\ \vdots & \vdots & \vdots \\ \text{XaZ} & \dfrac{\text{ZaW} \quad \text{WaY}}{\text{ZaY}} & \end{array}}{\text{XaY}} \qquad \dfrac{\begin{array}{ccc}[\Gamma] & [\Delta] & \\ \vdots & \vdots & [\Psi] \\ \dfrac{\text{XaW} \quad \text{WaZ}}{\text{XaZ}} & & \vdots \\ & & \text{ZaY}\end{array}}{\text{XaY}}$$

Theorem 2

Let Π be weakly-circular. Then:

1. If A$\sim_{\Pi}$B for distinct terms A,B, there's a third term C such that B$\sim_{\Pi}$C.
2. If Π is not $>$-dense, then A$\sim_{\Pi}$B for some distinct A, B.
3. A$\sim_{\Pi}$B for some distinct A, B iff some finite $\Delta \subseteq \Pi^{\vdash}_{M}$ is weakly-circular.

Proof of Theorem 2.1 From conjunct 3 of **Theorem 1**, we get **Theorem 2.1**. From conjunct 4 of **Theorem 1**, we get **Theorem 2.2**. We show **Theorem 2.3**. For right-to-left, let $\Delta \subseteq \Pi^{\vdash}_{M}$ be as described. Since $\Delta \cup \Delta^{\vdash}_{M} \subseteq \Pi \cup \Pi^{\vdash}_{M}$, it suffices to show A$\sim_{\Delta}$B for A$\neq$B. Now, by **Theorem 2.2**: A$\sim_{\Delta}$B for A$\neq$B if Δ is not $>$-dense. So suppose Δ is $>$-dense. Since $|\Delta| < \aleph_0$, it must be trivially $>$-dense: i.e. for all X,Y$\in$ *terms*(Δ): $[X]^{\sim}_{\Delta} \not\succ [Y]^{\sim}_{\Delta}$. But since Δ is weakly-circular: $\Delta \neq \emptyset$ and AaB$\in \Delta$, for some distinct A,B. As AaB and $[A]^{\sim}_{\Delta} \not\succ [B]^{\sim}_{\Delta}$, it follows that $[A]^{\sim}_{\Delta} = [B]^{\sim}_{\Delta}$ which is all we need. For left-to-right let Π be weakly-circular and let A$\sim_{\Pi}$B for A$\neq$B. By **Theorem 2.1**, there is a C distinct from A and B such that $\{A,B,C\} \subseteq [A]^{\sim}_{\Pi}$. Set $\Delta = \{$AaB, BaA, BaC, CaB, AaC, CaA$\} \subseteq \Pi \cup \Pi^{\vdash}_{M}$. It is easy to check that Δ is weakly circular.

Fact 3

Suppose that no single-inference circular arguments are syllogistically valid. If Π is weakly-circular and contains no infinite chains of a-propositions, then at least distinct 2 terms used Π counter-predicate$_\Pi$ with one another.

Fact 4

Suppose that no single-inference circular arguments are syllogistically valid. If Π is weakly-circular and contains no infinite chains of a-propositions, then (relative to Π) at least 3 terms used Π counter-predicate with one another.

Proofs. We get of **Facts 3-4** immediately from **Theorem 2.1**.

Fact 5

For every $n \geq 0$ there's a weakly-circular Π with 3 terms that counter-predicate$_\Pi$ with each other, and n terms that do not counter-predicate$_\Pi$ with any other terms in Π

Proof. Let $\Phi = \{A,B,C\}$. Pick a $\Psi = \{Lx M : x \in \{a,e,i,o\}$ and $L \in \Phi$ iff $M \notin \Phi\}$ such that $|\Psi| = n$. It is straightforward to check that $\Pi = \{AaB, BaC, CaA\} \cup \Psi$ is weakly-circular containing n terms that do not counter-predicate$_\Pi$.

6.3.4 Mild- and Strong-circularity: proofs of Facts 6-8

Definition

Where Π is a non-empty collection of syllogistic propositions:

1. Π is *strongly-circular* iff every $\varphi, \psi \in \Pi$: there are $\Gamma, \Delta \subseteq \Pi$ such that $\Gamma, \varphi \vdash_D \psi$ and $\Delta, \psi \vdash_D \varphi$
2. Π is *mildly-circular* iff for every φ in Π: there's a $\Gamma \subseteq \Pi$ such that (i) $\Gamma \vdash_D \varphi$, and (ii) for all $\psi \in \Gamma$ there's a $\Delta \subseteq \Pi$ such that $\Delta, \varphi \vdash_D \psi$

Definition

Where Γ is a collection of syllogistic sentences:[4]

1. $term(\Gamma) = \{T : T$ is a term used in $\Gamma\}$
2. $A \sim_\Gamma B$ (A and B counter-predicate$_\Gamma$) iff (i) $A = B$, or (ii) both $AaB \in \Gamma \cup \Gamma^{\vdash}_D$ and $BaA \in \Gamma \cup \Gamma^{\vdash}_D$
3. for $T \in term(\Gamma)$: $[T]^{\sim}_\Gamma = \{T' \in terms(\Gamma) : T \sim_\Gamma T'\}$

[4]NB that we are redefining $\sim$ as it was before defined for the relation $\vdash_M$.

Theorem 3

Π is strongly-circular iff

1. $\Pi \neq \emptyset$ and Π contains only a-propositions
2. $term(\Pi) = \bigcap\{[\mathrm{T}]^{\sim}_{\Pi} : \mathrm{T} \in term(\Pi)\}$

Proof. Right-to-left. Let {XaY, AaB}$\subseteq \Pi$. Without loss of generality, we let $|\{\mathrm{A,B,X,Y}\}| = 4$. With $\Pi_0 \cup \Pi_1 \cup \Pi_2 \cup \Pi_3 \subseteq \Pi$ we have the following syllogistic-deductions:

$$\dfrac{\dfrac{\dfrac{\Pi_0}{\vdots}\ \mathrm{XaA} \quad \mathrm{AaB}}{\mathrm{XaB}}\quad \dfrac{\dfrac{\Pi_1}{\vdots}}{\mathrm{BaY}}}{\mathrm{XaY}} \qquad\qquad \dfrac{\dfrac{\dfrac{\Pi_2}{\vdots}\ \mathrm{AaX} \quad \mathrm{XaY}}{\mathrm{AaY}}\quad \dfrac{\dfrac{\Pi_3}{\vdots}}{\mathrm{YaB}}}{\mathrm{AaB}}$$

Left-to-right. By definition $\Pi \neq \emptyset$. Since Π is strongly-circular, if Π contains an e-, i-, or o-proposition α it must contain an a-proposition β. But then α cannot be used in a syllogistic-deduction of β. So every $\varphi \in \Pi$ must be an a-proposition. Pick an arbitrary AaB, XaY$\in \Pi$. By our a-space lemmas, we'll have {AaC_0, $\mathrm{C}_0\mathrm{aC}_1$, ..., $\mathrm{C}_{n-1}\mathrm{aC}_n$, $\mathrm{C}_n\mathrm{aX}$, XaY, YaD_0, $\mathrm{D}_0\mathrm{aD}_1$, ..., $\mathrm{D}_{m-1}\mathrm{aD}_m$, $\mathrm{D}_m\mathrm{aB}$} $\subseteq \Pi$. But as Π is strongly-circular there will be $\Theta, \Lambda \subseteq \Pi$ such that: Θ, AaB $\vdash_D$ AaC_0 and Λ, AaB $\vdash_D$ $\mathrm{D}_m\mathrm{aB}$. So we'll get: {AaB, BaX, XaY, YaA}$\subseteq \Pi \cup \Pi^{\vdash}_D$.

Theorem 4

Π is mildly-circular iff

1. $\Pi \neq \emptyset$ and Π contains only a-propositions
2. if XaY$\in \Pi$ then $[\mathrm{X}]^{\sim}_{\Pi} = [\mathrm{Y}]^{\sim}_{\Pi}$

Proof. Right-to-left. Let AaB$\in \Pi$. The only non-obvious case is with A $\neq$ B and BaA$\in \Pi^{\vdash}_D$. By condition 2., with $\Pi_0 \subseteq \Pi$ we have the following syllogistic-deduction:

$$\dfrac{\dfrac{\dfrac{\Pi_0}{\vdots}\ \mathrm{AaB} \quad \mathrm{BaA}}{\mathrm{AaA}}\quad \mathrm{AaB}}{\mathrm{AaB}}$$

But for every $\varphi \in \Pi_0$, φ is of the form XaY and {X,Y} $\subseteq [\mathrm{A}]^{\sim}_{\Pi}$. So define $\Pi* = \Pi_0 \cup \{\mathrm{AaB}\}$. It is easy to check that by **Theorem 3**, $\Pi*$ is strongly-circular. And as $\Pi* \subseteq \Pi$, it follows that for every $\varphi \in \Pi_0$ there's a $\Gamma \subseteq \Pi$ such that Γ, AaB $\vdash_D$ φ. Left-to-right. The case for condition 1. follows our proof of **Theorem 3**. For 2., let AaB$\in \Pi$. There's a $\Gamma \subseteq \Pi$ such that $\Gamma \vdash_D$ AaB. Let XaY $\in \Gamma$. Then Γ = {AaC_0, $\mathrm{C}_0\mathrm{aC}_1$, ...,

$C_{n-1}aC_n$, C_naX, XaY, YaD_0, D_0aD_1, ..., $D_{m-1}aD_m$, $D_maB\}$. But Π is mildly-circular. So (as above) there are $\Theta, \Lambda \subseteq \Pi$ such that: Θ, $AaB \vdash AaC_0$ and Λ, $AaB \vdash D_maB$. Hence we have: AaB, BaX, XaY, YaA. So $[A]^{\sim}_{\Pi} = [B]^{\sim}_{\Pi}$.

Fact 6

If Π is strong- or mildly-circular, then every proposition in Π is an a-proposition.

Fact 7

If Π is strongly-circular and every member of Π is true, then all distinct terms used in Π counter-predicate with one another

Fact 8

If Π is mildly-circular and every member of Π is true, then the two terms of every proposition in Π counter-predicate

Proofs of Facts 6-8. Immediate from **Theorems 3-4**.

6.4 Proofs of results mentioned in Section 4

6.4.1 Definitions

Definition Priority and CAUSE(φ)

1. For any collection of scientific-truths $\mathcal{T}$, let $>_p$ (priority) be a strict partial order on $\mathcal{T}$
2. For any scientific-truth φ let CAUSE(φ) = $\{\psi : \psi >_p \varphi\}$

Definition Partial demonstration

1. Where Σ be a syllogism whose conclusion is a scientific-truth φ:
   - Σ is a partial demonstration iff every single-inference sub-syllogism $\dfrac{\alpha \quad \beta}{\gamma}$ in Σ is such that (i) $\alpha >_p \gamma$, and (ii) $\beta >_p \gamma$
2. A sentence $\psi = AxB$, with $x \in \{a,e,i,o\}$, is a *demonstrable x-proposition* iff there is a partial demonstration of ψ

Definition Improvement

1. Let Σ and $\Sigma*$ be distinct partial demonstrations. Then Σ^* *improves* Σ iff they have either the forms:

$$\cdot\ \Sigma = \dfrac{\dfrac{\vdots}{\varphi \quad \psi}}{\begin{array}{c}\theta\\ \vdots\\ \chi\end{array}} \quad \text{and} \quad \Sigma* = \dfrac{\dfrac{\alpha \quad \beta}{\varphi} \quad \dfrac{\vdots}{\psi}}{\begin{array}{c}\theta\\ \vdots\\ \chi\end{array}} \quad \text{, or:}$$

$$\cdot\ \Sigma = \dfrac{\begin{array}{c}\vdots\\ \varphi \quad \psi\end{array}}{\begin{array}{c}\theta\\ \vdots\\ \chi\end{array}} \quad \text{and} \quad \Sigma* = \dfrac{\begin{array}{c}\vdots \quad \dfrac{\alpha \quad \beta}{\psi}\\ \varphi\end{array}}{\begin{array}{c}\theta\\ \vdots\\ \chi\end{array}}$$

2. A sequence of partial demonstrations $\overline{D} = (D_0, D_1, ..., D_n, D_{n+1}, ...)$ is an *improvement-path* originating in $D*$ iff

   (a) $D_0 = D*$ and

   (b) for all $D_i, D_{i+1} \in \overline{D}$: D_{i+1} improves D_i[5]

3. D is an *improvement* of $D*$ iff there is an improvement-path $(D_0, D_1, ..., D_k)$ such that $D = D_k$ and $D* = D_0$

Definition Demonstration

D is a demonstration iff there is no distinct partial demonstration D' that improves D.

Definition Demonstration ad infinitum

We say that *the process of demonstrating φ continues ad infinitum* iff

1. there is a partial demonstration D of φ, and

2. there is an infinite demonstrative improvement-path originating in D

Definition

An *infinite a-chain* is an infinite sequence using infinitely many distinct terms of one of the forms:

1. ($\mathrm{YaX_0}$, $\mathrm{X_0aX_1}$, $\mathrm{X_1aX_2}$, ..., $\mathrm{X_naX_{n+1}}$, ...) infinite up-chain

2. ($\mathrm{X_0aY}$, $\mathrm{X_1aX_0}$, $\mathrm{X_2aX_1}$, ..., $\mathrm{X_{n+1}aX_n}$, ...) infinite down-chain

[5] In describing sequences, as before, we harmlessly abuse notation.

6.4.2 Proof of Theorems 1-3

Recall **Theorems 1-3**:

Theorem 1

If for no φ does CAUSE(φ) contain an infinite a-chain, the process of demonstrating an a-proposition cannot continue ad infinitum

Theorem 2

If for no φ does CAUSE(φ) contain an infinite a-chain, the process of demonstrating an i-proposition cannot continue ad infinitum

Theorem 3

If for no φ does CAUSE(φ) contain an infinite a-chain, the process of demonstrating an e- or o-proposition cannot continue ad infinitum

Proof. Following Aristotle's lead, we prove the above by **Theorem 1** and the following two lemmas:

Lemma 8

If **Theorem 1** and **Theorem 2**, then **Theorem 3**

Lemma 9

If **Theorem 1** then **Theorem 2**

Proof of Lemma 8. Assume the antecedent and let $\overline{D}$ be an infinite improvement-path an e- or o-conclusion φ.Without loss of generality we can assume that $\overline{D}$ uses only negative moods. Let a B-improvement-path be a demonstrative improvement-path in which only Bocardo and Baroco are used; let a C-improvement-path be a demonstrative improvement-path in which only Celarent, Camestres, and Cesare are used. By the **Laws of Quality and Quantity** (cf. **Observation 1** in **Section 4.4.1**) $\overline{D}$ must contain an infinite B- or C-improvement-path $\overline{D}_X \subseteq \overline{D}$. Let $\Theta = \{\text{XaY} : \text{XaY} \in prem(D_n)$ for some $D_n \in \overline{D}_X\}$. By **Lemma 11** we know that $|\Theta| \not< \aleph_0$ (one can check that $|\Theta| = \aleph_0$). So given **Lemmas 3** and **5**, it follows that for some Φ, Ψ : (i) $\Phi \cup \Psi = \Theta \subseteq$ CAUSE(φ), and (ii) at least one of Φ, Ψ is an infinite a-chain.

Proof of Lemma 9. Assume the antecedent and let $\overline{D}$ be an infinite improvement-path an i-conclusion φ. Without loss of generality we can assume that every $D_n \in \overline{D}$ is an i-string. Let a D-improvement-path be a demonstrative improvement-path in which only Darii, Datisi, and Disamis are used. Then $\overline{D}$ is an infinite D-improvement-path. Let $\Theta = \{\text{XaY} : \text{XaY} \in prem(D_n) \text{ for some } D_n \in \overline{D}\}$. By **Lemma 11** we know that $|\Theta| \not< \aleph_0$. So given **Lemma 4**, it follows that for some Φ, Ψ : (i) $\Phi \cup \Psi = \Theta \subseteq$ CAUSE(φ), and (ii) at least one of Φ, Ψ is an infinite a-chain.

Lemma 10

Let $\overline{D} = (D_0, D_1, D_2, ...)$ be a demonstrative-improvement-path such that (i) $D_0 : \Gamma \vdash \varphi$, and (ii) for all $D_i, D_j \in \overline{D}$: $term(D_i) = term(D_j)$. Then $\overline{D}$ is finitely long.

Proof. For our language $\mathcal{L}$, let $\Delta = \{\text{A}x\text{B} \in \mathcal{L} : \text{A,B} \in term(D_0)\}$, and $\Phi = \Delta \cap$ CAUSE(φ). Since $|\Delta| < \aleph_0$, $|\Phi| < \aleph_0$. Now, consider D_0. Since Φ is finite, on pain of circularity only finitely many demonstrative improvements can be done on any branch of D_0. But D_0 contains just finitely many branches. So for some $n \in \mathbb{N}$, $\overline{D} = (D_0, D_1, ...D_{n-1}, D_n)$.

Lemma 11

If a demonstrative improvement-path $\overline{D} = (D_0, D_1, D_2, ...)$ uses just finitely many terms, $\overline{D}$ is finitely long.

Proof. Let $\Psi = \bigcup\{term(D_n) : D_n \in \overline{D}\}$, where $|\Psi| = k < \aleph_0$. There must be a D_i such that : (i) $|term(D_i)| = k$, and (ii) for all $j < i$ and $h > i$: $term(D_j) \subsetneq term(D_i) = term(D_h)$. By **Lemma 10**, there's a finite $m \geq i$ such that $\overline{D} = (D_0, D_1, ..., D_{m-1}, D_m)$.

Proof of Theorem 1 Let $\overline{D}$ be an infinite improvement-path whose conclusion is AaB. Let $\Theta = \bigcup\{prem(D_n) : D_n \in \overline{D}\}$. By **Lemma 2** with **Lemma 11** we get that Θ is infinite, and totally ordered by the a-relation with maximal element A and minimal element B. By standard order-theoretic reasoning, we know there's a $\Psi \subseteq \Theta \subseteq$ CAUSE(AaB) such that Ψ is an infinite a-chain.

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