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Doctoral Dissertations, 2011-2026

Mathematics

2023

Elliptic Curves Ordered by Faltings Height

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Princeton, NJ : Princeton University

<http://arks.princeton.edu/ark:/88435/dsp01fn1072200>

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ELLIPTIC CURVES ORDERED BY FALTINGS HEIGHT

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A DISSERTATION
PRESENTED TO THE FACULTY
OF PRINCETON UNIVERSITY
IN CANDIDACY FOR THE DEGREE
OF DOCTOR OF PHILOSOPHY

RECOMMENDED FOR ACCEPTANCE
BY THE DEPARTMENT OF
MATHEMATICS
ADVISER: MANJUL BHARGAVA

MAY 2023

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Abstract

The Faltings height — and the finiteness of semistable principally polarized abelian varieties of a given genus g over a number field K with bounded Faltings height — is a key ingredient in Faltings’s proof of the Mordell Conjecture. In the case where $g = 1$ and $K = \mathbb{Q}$, Hortsch [22] gave an expression for the number of elliptic curves with Faltings height less than X .

In this thesis, we establish asymptotics for the number of elliptic curves with bounded Faltings height over any number field. We also prove similar asymptotics for a more general class of heights. In addition, we compute an exact expression for the leading term in the asymptotic expression due to Hortsch.

One primary motivation for Hortsch’s result is to allow for the computation of arithmetic statistics of elliptic curves with respect to the ordering given by the Faltings height. Bhargava and Shankar [6, 7, 4, 5] computed the average size of the n -Selmer groups of elliptic curves over $\mathbb{Q}$ for $n \leq 5$ when ordering by naive height. We show that these averages are the same not only for the Faltings height but also for a more general class of heights that includes both the naive height and the Faltings height. These results also hold for any number field.

Acknowledgements

Many thanks to my advisor, Manjul Bhargava, who suggested the problem to me and gave me much guidance along the way. I am grateful your continued support to this point, including reading too many drafts of this thesis!

I have learned from many of the great mathematicians at Princeton University. These include Shou-wu Zhang, who graciously agreed to serve as my Reader and supported me in my preparation for my generals. Also included are my committee members, Chris Skinner and Peter Sarnak, with whom I have had the pleasure of taking multiple classes.

To my wonderful friends, roommates, and groomsmen, Wayne Zhao and Victor Wang, I am grateful for the many great times we had over the years. Thank you for being my social backbone during my time in graduate school. May we continue to have them, even as our circumstances change.

This thesis would not have been possible without the help and support of many of my friends and colleagues. To Arul Shankar, Levent Alpoge, Sameera Vemulapalli, Evan O’Dorney, Ruthi Hortsch, and others: thank you for your many helpful comments and conversations. To fellow advisee Dan Fess, thank you for stepping in to help when I broke my wrist.

I would also like to thank the staff and administrators at the Department of Mathematics at Princeton University. In particular, I would like to thank Jill LeClair for her help in the past six years.

Most of all, I would like to thank my family. This thesis is dedicated to you. To Mom and Dad, thank you for making me who I am today. To my sister Angela, thank you for being there for me, both when I wanted to brag and when I needed to vent. And to Cecilia, my lifelong partner, thank you for motivating me and giving me the energy to push this to the finish line.

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Chapter 1

Introduction

1.1 Background

Let K be a number field, i.e., a finite extension of the rational numbers $\mathbb{Q}$. Let $\mathcal{E}^K$ be the set of isomorphism classes of elliptic curves over K . The overall problem in arithmetic statistics of elliptic curves concerns the “average” values of certain invariants of an elliptic curve in $\mathcal{E}^K$, such as its arithmetic or analytic rank or the size of its Selmer groups. One common approach is to assign to each elliptic curve $E \in \mathcal{E}^K$ a certain non-negative height $H(E)$. For an invariant $f : \mathcal{E}^K \rightarrow \mathbb{R}$, define the *average* of f with respect to H to be

$$\lim_{X \rightarrow \infty} \frac{\sum_{E \in \mathcal{E}_{H < X}^K} f(E)}{\#\mathcal{E}_{H < X}^K},$$

if it exists, where $\mathcal{E}_{H < X}^K = \{E \in \mathcal{E}^K : H(E) < X\}$.

Common choices for the height function include the naive height (in the case where $K = \mathbb{Q}$, this is given by $H(E_{A,B}) = \max(4|A|^3, 27|B|^2)$), the conductor, and the (norm of) the minimal discriminant of E .

The primary focus of this thesis is the Faltings height, but many of the results we prove hold for a more general class of “singular” global height functions.

1.2 Summary of Results

1.2.1 Counting Curves of Bounded Faltings Height over a Number Field

In order to prove theorems about the average of an invariant for elliptic curves ordered by a height H , the usual first step is to determine the growth of $\#\mathcal{E}_{H < X}^K$. For Faltings height, this was done in the case of $K = \mathbb{Q}$ by Hortsch [22]. In this thesis, we generalize her result to all number fields:

Theorem 1.2.1. *Let K be a number field, and let H_F denote the Faltings height. There exists a constant c_K such that the number of elliptic curves over K with Faltings height less than X is given by*

$$\#\mathcal{E}_{H_F < X}^K = c_K X^{5/6} + O_{K,\varepsilon}(X^{5/6-1/3d+\varepsilon}).$$

We also give an exact expression for c_K which depends on (1) standard arithmetic properties of K , and (2) the volumes $\sigma_{\mathbb{R}}$ and $\sigma_{\mathbb{C}}$ of regions defined by the infinite part of the Faltings height in $\mathbb{R}^2$ and $\mathbb{C}^2$, respectively. In particular, we prove that

$$c_{\mathbb{Q}} = \frac{4209975}{128\pi^{17}} L(\Delta, 6) \approx 348716.636313766 \dots,$$

where Δ is the discriminant modular form (with suitable normalization, discussed later). Note that Hortsch [22] numerically computed $c_{\mathbb{Q}}$ to be approximately 348716.

1.2.2 Other Arithmetic Properties of Elliptic Curves Ordered by Faltings Height

We prove the following about the distribution of the sign of the discriminant of E :

Theorem 1.2.2. *Let $v_{r_1}, \dots, v_{r_1}$ be the set of real places of K , and let $t \in \{-1, +1\}^{r_1}$ be a tuple of sign signatures. When all elliptic curves E/K are ordered by Faltings height, the density of elliptic curves with $\text{sgn}(\Delta(E)_{v_j}) = t_j$ for $j = 1, \dots, r_1$ is*

$$\left(\frac{11}{15}\right)^{n_-(t)} \left(\frac{4}{15}\right)^{n_+(t)},$$

where $n_-(t)$ and $n_+(t)$ are the number of -1 's and $+1$'s, respectively, in t .

Corollary 1.2.3. *When all elliptic curves $E/\mathbb{Q}$ are ordered by Faltings height, the density of elliptic curves with positive discriminant is $\frac{4}{15}$.*

Contrast this with $\frac{1}{5}$ for the naive height and conjecturally $\frac{1}{1+\sqrt{3}}$ for the discriminant.

Associated with each elliptic curve E/K is an element of the class group of K called its Weierstrass class. One property of the Weierstrass class is that E has a global minimal model if and only if its Weierstrass class is trivial. We prove the following fact about the equidistribution of the Weierstrass class of elliptic curves (also known as the Steinitz class).

Theorem 1.2.4. *When all elliptic curves E/K are ordered by either naive height or Faltings height, the density of elliptic curves with Weierstrass class equal to a particular $c \in \text{Cl}(K)$ is $\frac{1}{h}$.*

We prove this equidistribution result for a more general class of height functions. This in contrast to the result of Bekyel [2], who proved that the Weierstrass class is not equidistributed when elliptic curves are ordered by the height resulting from the Weil height on $\mathbb{P}^2$, as follows:

$$H(E_{A,B}) = \prod_v \max\{|A|_v, |B|_v, 1\}.$$

(Note: that height is a height on pairs $(A, B) \in \mathcal{O}_K^2$ with $\Delta(A, B) \neq 0$, rather than a height on $\mathcal{E}^K$.)

1.2.3 Singular Global Heights

In Chapter 2, we review the notion of a quasi-toric height on weighted projective space due to Darda [13]. One main property of quasi-toric heights is that they are the product of a collection of local heights, one for every place of the number field K . In order to deal with the Faltings height, we allow for the height to be *singular*. That is, we allow the local heights at the infinite places to be zero on a subvariety of the weighted projective space, provided the height does not approach zero too quickly. We call such heights *subpolynomially suitable*. While the minimal discriminant is also a singular quasi-toric height, it does not satisfy the subpolynomial suitability condition that we need in order to determine the correct main term for the count of points with bounded height.

The main result of Chapter 4 is Theorem 4.1.4 (of which Theorem 1.2.1 is a special case). It shows that the main term for counting points also holds for these singular heights, an improvement on [13, Proposition 3.7.2.2]. We also obtain a power-saving error term as long as certain “tameness” conditions hold. In addition, we prove equidistribution results which are used to prove Theorem 1.2.2 and Theorem 1.2.4.

The main method of proof is to count points with bounded naive height, weighted by a mass function that gives a smaller bound on a general singular quasi-toric height H . When the height is subpolynomially suitable (or ε -suitable for sufficiently small ε), then the naive height cannot be too much larger than H , so all points with bounded H are indeed counted in this manner. Even in the case where H is not subpolynomially suitable, this gives a way to count points that are simultaneously bounded by H and the naive height.

In order to perform this count, we use the technique of Schanuel [26] to reduce the problem of counting points of bounded height in weighted projective space to the problem of counting lattice points in a homogeneously expanding compact subregion of a real vector space. In the case where our height is defined algebraically, this can be done using the classical lemma of Davenport [14]. However, as the Faltings height is not algebraic, we instead use a result of Barroero-Widmer [33], which allows us to count points in sets definable in an O-minimal

structure. The O-minimality condition is part of the “tameness” conditions and comes from a generalization of the notion of a semi-algebraic set.

In Chapter 7, we show that the Faltings Height is a singular quasi-toric height that satisfies the aforementioned “tameness” conditions. We also carry out the volume computations necessary to calculate the exact constant of proportionality, including the aforementioned values of $\sigma_{\mathbb{R}}$ and $\sigma_{\mathbb{C}}$.

1.2.4 Average Sizes of Selmer Groups over $\mathbb{Q}$ when Ordered by Faltings height

In [6], Bhargava and Shankar proved that the average size of the 2-Selmer groups of elliptic curves over $\mathbb{Q}$ when ordered by naive height is 3. Subsequently, in [4, 5, 7], they proved that the average size of the n -Selmer groups of elliptic curves under the same ordering is $\sigma(n)$ for $n = 3, 4, 5$. In Chapter 5, we prove that the analogous result holds for Faltings height.

Theorem 1.2.5. *For $n = 2, 3, 4, 5$, when all elliptic curves $E/\mathbb{Q}$ are ordered by Faltings height, the average size of the n -Selmer group of E is $\sigma(n)$.*

We prove that these results also hold when ordering elliptic curves by any sufficiently “tame” subpolynomially suitable quasi-toric height. This is done again with the method of counting points with bounded naive height, weighted by a mass function that captures the condition of also being bounded by the aforementioned subpolynomially suitable height. As such, the results and methods of [6, 7, 4, 5] are prominently featured in the proof of Theorem 1.2.5.

One corollary of Theorem 1.2.5 is that we establish for the first time that the average rank of elliptic curves over $\mathbb{Q}$ when ordered by Faltings height is bounded. We immediately get an upper bound of 1.05, which we improve upon in Chapter 8.

1.2.5 Average Sizes of Selmer Groups over Number Fields when Ordered by Faltings Height

In Chapter 6, we combine the mass function approaches of Chapter 4 and Chapter 5 and the work of Bhargava, Shankar and Wang in [8] to prove the following theorem:

Theorem 1.2.6. *For $n = 2, 3, 4, 5$, when all elliptic curves E/K are ordered by Faltings height, the average size of the n -Selmer group $S_2(E)$ is $\sigma(n)$.*

As before, the corresponding result also holds for general “tame” singular heights. Again, this implies an upper bound of 1.05 for elliptic curves when ordered by Faltings height.

We also show that Theorem 1.2.6 holds when restricting to globally minimal elliptic curves, or more generally to elliptic curves with a fixed Weierstrass class.

1.2.6 Average Ranks and Root Numbers

In Chapter 8, we prove the following bound on the average rank of elliptic curves over $\mathbb{Q}$ when ordered by Faltings height:

Proposition 1.2.7. *When all elliptic curves $E/\mathbb{Q}$ are ordered by Faltings height,*

1. *The average rank of E is at most 1.0086.*
2. *The proportion of curves that have rank 0 is at least 5.179%.*
3. *The proportion of curves that have rank 0 or 1 is at least 80.31%.*

This is done using the method of Bhargava and Shankar in [5], which constructs a positive proportion of elliptic curves E such that E and its quadratic twist E_{-1} have different root numbers. The rank bound we prove is weaker than the one proven in [5] due to the fact that Faltings height is not invariant under twists by -1 , but it is still an improvement of the bound of 1.05 purely coming from the average size of the 5-Selmer group.

Notation

For the rest of this thesis, K is a fixed number field. Let $d, r_1, r_2, h, R, w, D_K, \zeta_K$ denote the degree, number of real places, number of complex places, class number, regulator, number of roots of unity, discriminant, and Dedekind zeta function of K , respectively. $\mathcal{O}$ is the ring of integers in K and $\mathcal{O}^\times$ is the unit group inside $\mathcal{O}$. M_K is the set of places of K , M_K^∞ is the set of infinite places, and M_K^0 is the set of finite places. The notation $v \mid \infty$ and $v \nmid \infty$ denote that $v \in M_K^\infty$ and $v \in M_K^0$, respectively. Let $r = r_1 + r_2$. Let K_∞ denote the product

$$\prod_{v \mid \infty} K_v.$$

For a finite place $v \nmid \infty$,

- K_v is the completion of K at v .
- $\mathcal{O}_v$ is the ring of integers at v .
- $\mathfrak{p}_v$ is the unique maximal ideal of $\mathcal{O}_v$.
- π_v is a uniformizer for $\mathfrak{p}_v$.
- $q_v = \#(\mathcal{O}_v/\mathfrak{p}_v)$.
- $|x|_v = q_v^{-v(x)}$.

For an infinite place $v \mid \infty$,

- $\epsilon_v = [K_v : \mathbb{R}]$.
- $|x|_v = |x|_{\mathbb{R}}^{\epsilon_v}$.

The norms $|\cdot|_v$ are normalized the usual way so that the product formula $\prod_{v \in M_K} |x|_v = 1$ holds for all $x \in K^\times$.

Chapter 2

Weighted Projective Space and Global Heights

2.1 Introduction

For any field F with characteristic not equal to 2 or 3, there is a bijection via the short Weierstrass form between $\mathcal{E}^F$ and the set of equivalence classes of points $(a, b) \in F^2$ with $\Delta(a, b) := -16(4a^3 + 27b^2) \neq 0$. Two points (a_1, b_1) and (a_2, b_2) are equivalent if there exists a scalar $u \in F^\times$ such that $a_1 = u^4 a_2$ and $b_1 = u^6 b_2$. The latter set of equivalence classes can be thought of as points in the weighted projected space $\mathbb{P}^{(4,6)}$.

One simple case is that of $K = \mathbb{Q}$ and the naive height

$$H^\#(E_{A,B}) = \max(c_1 |A|^3, c_2 B^2),$$

where (A, B) is the unique pair of *weakly minimal* integers (that is, for any prime p , if $p^4 \mid A$ then $p^6 \nmid B$) such that $E_{A,B}$ can be written as $y^2 = x^3 + Ax + B$. Note that two common choices for the coefficients (c_1, c_2) are $(1, 1)$ and $(4, 27)$. This definition is not suitable for general number fields: when the class group of K is non-trivial, a single equivalence class can have zero weakly minimal points.

A different approach is to express the aforementioned naive height as the product of local heights, as follows:

$$H_{\mathbb{Q}}^{\#}(A, B) = \max(c_1 |A|^3, c_2 |B|^2) \cdot \prod_p \max(p^{-\lfloor v_p(A)/4 \rfloor}, p^{-\lfloor v_p(B)/6 \rfloor})^{12}.$$

(Here we use the convention that $v_p(0) = -\infty$.) It is clear that this expression agrees with the previous formula for weakly minimal (A, B) , and the product formula for norms of $\mathbb{Q}$ means that this does not depend on the equivalence class of (A, B) .

This leads to natural generalization of the naive height to all number fields:

$$H_K^{\#}(A, B) = \prod_{v|\infty} \max(c_1 |A|_v^3, c_2 |B|_v^2) \cdot \prod_{v \nmid \infty} \max(q_v^{-\lfloor v(A)/4 \rfloor}, q_v^{-\lfloor v(B)/6 \rfloor})^{12}.$$

When we omit the subscript, the field of definition is K . Again by the product formula, the value of $H_K^{\#}$ only depends on the equivalence class of (A, B) . This is a special case of the Size function over a general weighted projective space defined by Deng¹ [15]. One fundamental result is the following counting result of Schanuel [26]:

Theorem 2.1.1. *Let H be the classical Weil height on $\mathbb{P}_K^{m-1}$. Then*

$$\#\{x \in \mathbb{P}^n(K) : H(x) < T\} = c_K T^m + O(T^{m-1/d}),$$

where

$$c_K = \left(\frac{2^{r_1} (2\pi)^{r_2}}{\sqrt{D_{K/\mathbb{Q}}}} \right)^m m^r \frac{R}{w}.$$

Two other natural heights of (isomorphism classes of) elliptic curves are the minimal discriminant and Faltings height. The latter was defined by Faltings [18] in his proof of the Mordell and Shafarevich conjectures.

In this chapter we recap the work of Darda [13] generalizing the naive height defined

¹No relation to the author.

by Deng [15]. Then, we discuss singular heights in more detail, including those that are ε -suitable and those that are subpolynomially suitable. These notions are defined in both the local and global cases. Finally, we establish that cutting off the cusp is viable for ε -suitable global heights as long as ε is sufficiently small.

2.2 Weighted Projective Space

Here we recall the definition of weighted projective space.

Let $\mathbf{a} = (a_1, \dots, a_n)$ be a tuple of positive integers. There is an action of $\mathbb{G}_m$ on $\mathbb{A}_n$ given by

$$t \cdot (x_1, \dots, x_n) = (t^{a_1}x_1, \dots, t^{a_n}x_n). \quad (2.1)$$

Definition 2.2.1. The *weighted projective space* $\mathbb{P}^{\mathbf{a}}$ is the quotient of $\mathbb{A}^n \setminus 0$ by the action of $\mathbb{G}_m$ given by Equation (2.1).

Remark 2.2.2. As mentioned in the introduction of the chapter, for a given field F with characteristic not equal to 2 or 3, isomorphism classes of elliptic curves over F are in one-to-one correspondence with points in the space $\mathbb{P}(4, 6)(F)$ where the discriminant function Δ is nonzero.

We use $|\mathbf{a}|$ as shorthand for $a_1 + \dots + a_n$.

2.3 Local Functions — Heights and Masses

Suppose F is a local field with norm $|\cdot|_F$.

Definition 2.3.1. Let m be a positive real number. A function $f : F^n \rightarrow \mathbb{R}$ is *$\mathbf{a}$ -homogeneous of weighted degree m* if for every $\mathbf{x} \in F^n$ and every $t \in F^\times$,

$$f(t \cdot \mathbf{x}) = |t|_F^m f(\mathbf{x}).$$

(The weighted degree is assumed to be 1 unless otherwise stated.)

Definition 2.3.2. An $\mathbf{a}$ -height function is an $\mathbf{a}$ -homogeneous function $f : F^n \rightarrow \mathbb{R}$ such that $f(\mathbf{0}) = 0$, and $f(\mathbf{x}) > 0$ for $\mathbf{x} \neq \mathbf{0}$.

Given an $\mathbf{a}$ -height function f , we let $B_f(t) = \{\mathbf{x} \in F^n : f(\mathbf{x}) \leq t\}$ denote the closed ball with respect to f .

Definition 2.3.3. An $\mathbf{a}$ -mass function on F^n is a bounded $\mathbf{a}$ -homogeneous function f of weighted degree 0.

2.3.1 The Standard Height

In this section, we recall properties of the standard local height $f^\#$ defined by Darda [13]:

Definition 2.3.4. The standard local $\mathbf{a}$ -height function on F^n is

$$f^\#(\mathbf{x}) = \sup\{|t|_F : t \in F^\times, t \cdot \mathbf{x} \in \mathcal{O}_F^n\}^{-1}.$$

(Here we use $\mathcal{O}_F$ to denote the closed unit ball even in the case where F is archimedean.)

Proposition 2.3.5 ([13]). *The standard local height $f^\#$ satisfies the following:*

1. $f^\#$ is an $\mathbf{a}$ -height function, i.e., $f^\#$ is $\mathbf{a}$ -homogeneous and positive away from 0.
2. Suppose $F = K_v$. Then

$$f_v^\#(\mathbf{x}) = \begin{cases} \max_{1 \leq i \leq n} q_v^{-\lfloor v(x_i)/a_i \rfloor} & v \nmid \infty, \\ \max_{1 \leq i \leq n} |x_i|_v^{1/a_i} & v \mid \infty. \end{cases}$$

3. Let $S_{\mathbf{a}}(F) = \{\mathbf{x} \in F^n : f^\#(\mathbf{x}) = 1\}$. Then

$$F^n \setminus 0 = \bigsqcup_{t \in F^\times / \mathcal{O}_F^\times} t \cdot S_{\mathbf{a}}(F).$$

4. For every $t \in F^\times$, $f^\#$ is constant on $t \cdot S_{\mathbf{a}}$ and is equal to $|t|_F$.

Proposition 2.3.6. *Let f be another $\mathbf{a}$ -homogeneous function of weighted degree 1.*

1. *For every real number r , the indicator function $\chi_{f,r}$ for the set*

$$\{x \in F^n \setminus 0 : f(x)/f^\#(x) = r\}$$

is an $\mathbf{a}$ -mass function.

2. *Suppose $F = K_v$ where $v \nmid \infty$. If f is locally constant away from 0, then $\chi_{f,r}$ is also locally constant away from 0 and is nonzero for only finitely many r .*

Proof. Since f and $f^\#$ are $\mathbf{a}$ -homogeneous functions of weighted degree 1, their quotient is an $\mathbf{a}$ -homogeneous function of weighted degree 0. Thus, the indicator functions for the level sets of $f/f^\#$ is also an $\mathbf{a}$ -homogeneous function of weighted degree 0.

By Proposition 2.3.5, $f^\#$ is locally constant away from 0. Thus, if f is locally constant, then $f/f^\#$ is locally constant. Since $S_{\mathbf{a}}(F)$ is compact, the image of $f/f^\#$ when restricted to $S_{\mathbf{a}}(F)$ must be finite. However, by the degree 0 property of $f/f^\#$, this is also the image of $f/f^\#$ on $F^n \setminus 0$.

Finally, $\chi_{f,r}$ restricted to $S_{\mathbf{a}}(F)$ is locally constant, so it must also be locally constant away from 0. $\square$

Lemma 2.3.7. *Suppose $P \in F[x_1, \dots, x_n]$ is an $\mathbf{a}$ -homogeneous polynomial of weighted degree d . Then*

1. *There exists a constant c_P such that $|P(x)|_F \leq c_P \cdot f^\#(x)^d$ for all $\mathbf{x} \in F^n$.*

2. *If F is non-archimedean and $P \in \mathcal{O}_F[x_1, \dots, x_n]$, then we can take $c_P = 1$.*

Proof. It suffices to show that the relevant inequalities hold on $S_{\mathbf{a}}(F)$.

For 1., we may take c_P to be the sum of the F -norms of all the coefficients of $P(x)$. For 2., if F is non-archimedean, we can actually replace sum with maximum as $|x + y|_F \leq \max(|x|_F, |y|_F)$. Thus, we can take $c_P = 1$ if P has coefficients in $\mathcal{O}_F$. $\square$

We use $B^\#$ as shorthand for $B_{f^\#}$. Note that $B^\#(1) = \mathcal{O}_F^n$.

2.3.2 Singular Heights

We also need to allow for the notion of a singular height, where the height can be equal to 0 away from $\mathbf{0}$. We are mostly concerned with singular heights in the case where F is archimedean (as this is the case of concern for the Faltings height), but the following definitions also work for the non-archimedean case.

Definition 2.3.8. A *singular $\mathbf{a}$ -height function* on F^n is an $\mathbf{a}$ -homogeneous function f such that $f(\mathbf{x}) \geq 0$ for all $\mathbf{x} \in F^n$, and $f(\mathbf{x}) = 0$ for some $\mathbf{x} \neq \mathbf{0}$.

Definition 2.3.9. We say that f is *finite* if $\text{Vol } B_f(1)$ is finite (with respect to the Haar measure from $\mathcal{O}_F^n$ if F is non-archimedean or the Lebesgue measure if F is archimedean).

Certain singularities are easier to deal with than others. As referenced before, the infinite part of the Faltings height tends to 0 as the value of the discriminant polynomial Δ tends to 0, but does so in a way that is much slower than Δ itself tends to 0. To be more precise, the singularity of the Faltings height at Δ is subpolynomially suitable, which is defined as follows:

Definition 2.3.10. Let $P \in F[x_1, \dots, x_n]$ be a non-constant $\mathbf{a}$ -homogeneous polynomial of weighted degree d .

1. A function f is ε -*suitable* at P if there exists a constant C_ε such that for all $\mathbf{x} \in S_{\mathbf{a}}(F)$,

$$C_\varepsilon f(\mathbf{x}) \geq |P(\mathbf{x})|_F^{\varepsilon/d}.$$

2. A function f is *subpolynomially suitable* at P if it is ε -suitable for every $\varepsilon > 0$.

We say that f is ε -suitable (resp. subpolynomially suitable) if it is ε -suitable at P (resp. subpolynomially suitable at P) for some non-constant P .

Remark 2.3.11. By Lemma 2.3.7, if a local singular height is ε -suitable, then it is ε' -suitable for every $\varepsilon' > \varepsilon$.

We defer the proof of the subpolynomial suitability of the Faltings Height to Chapter 7. Here we prove some bounds in the case where F is archimedean:

Lemma 2.3.12. *Let F be archimedean and suppose $P \in F[x_1, \dots, x_n]$ is a fixed non-constant polynomial. Suppose f is ε -suitable at P , where $[F : \mathbb{R}] \cdot \varepsilon < a_{\min} / |\mathbf{a}|$. Then:*

1.

$$\text{Vol}(\{\mathbf{x} \in B_f(1) : \frac{f^\#(\mathbf{x})}{f(\mathbf{x})} > T\}) \ll T^{|\mathbf{a}| - a_{\min}/(\varepsilon \cdot [F:\mathbb{R}])}.$$

(Here the implied constant is allowed to depend on P but not on f .)

2. $\text{Vol}(B_f(1))$ is finite.

3. For every positive real number r , $\text{Vol}(B_f(r)) = r^{|\mathbf{a}|} \text{Vol}(B_f(1))$.

Proof. Note that

$$\{\mathbf{x} \in B_f(1) : \frac{f^\#(\mathbf{x})}{f(\mathbf{x})} > T\} = \{\mathbf{x} \in B_f(1) : f^\#(\mathbf{x}) > T\} \sqcup \{\mathbf{x} \in B^\#(T) : \frac{f^\#(x)}{f(x)} > T\}. \quad (2.2)$$

If 1. is true, then we have that $\text{Vol}(\{\mathbf{x} \in B_f(1) : f^\#(\mathbf{x}) > T\}) \rightarrow 0$ as $T \rightarrow \infty$ and thus 2. follows. 3. follows from the fact that

$$B_f(r) = r^{1/[F:\mathbb{R}]} \cdot B_f(1),$$

combined with the fact that the action of t has determinant $|t|_F^{|\mathbf{a}|}$ on F^n .

Thus, it remains to show that 1. holds. Here, we make use of the multivariate Remez inequality due to Brudnyi and Ganzburg [10]:

Theorem 2.3.13. *Let V be a convex subset of $\mathbb{R}^k$ and ω is a measurable subset of V . Then for a real polynomial P on $\mathbb{R}^k$ of degree m ,*

$$\sup_V |P| \leq T_m \left(\frac{1 + \sqrt[k]{1 - \lambda}}{1 - \sqrt[k]{1 - \lambda}} \right) \sup_\omega |P|,$$

where $\lambda = \frac{\text{Vol } \omega}{\text{Vol } V}$ and $T_m(x) = \cos(m \arccos x)$ is the m th Chebyshev polynomial.

Since $T_m(x) \ll \max(1, x)^m$ and $\frac{1 + \sqrt[k]{1 - \lambda}}{1 - \sqrt[k]{1 - \lambda}} \ll \lambda^{-1}$ for $\lambda \leq 1$, the following asymptotic bound immediately follows:

Corollary 2.3.14. *Suppose V, P, m satisfy the conditions of Theorem 2.3.13. Then*

$$\frac{\text{Vol}\{x \in V : |P(x)| < \delta\}}{\text{Vol } V} \ll \left(\frac{\delta}{\sup_V |P|} \right)^{1/m}.$$

Applying this to the convex set $B^\#(t)$ and the polynomial P (of weighted degree d), we have that for any $\delta > 0$,

$$\text{Vol}\{\mathbf{x} \in B^\#(t) : |P(\mathbf{x})|_F^{1/d} < \delta\} \ll \text{Vol } B^\#(t) \cdot \left(\frac{\delta^{d/[F:\mathbb{R}]}}{t^{d/[F:\mathbb{R}]}} \right)^{a_{\min}/d} \sim t^{|\mathbf{a}|} \cdot (\delta/t)^{a_{\min}/[F:\mathbb{R}]}.$$

We first bound

$$\text{Vol}\{\mathbf{x} \in B^\#(T) : \frac{f^\#(x)}{f(x)} > T\}.$$

By the ε -suitability of f , we have

$$|P(\mathbf{x})|_F^{1/d} < f^\#(x) \cdot \left(\frac{f^\#(x)}{f(x)} \right)^{-1/\varepsilon}.$$

Thus,

$$\text{Vol}\{\mathbf{x} \in B^\#(T) : \frac{f^\#(x)}{f(x)} > T\} \leq \text{Vol}\{\mathbf{x} \in B^\#(T) : |P(\mathbf{x})|_F^{1/d} < T^{1-1/\varepsilon}\} \ll T^{|\mathbf{a}| - a_{\min}/(\varepsilon \cdot [F:\mathbb{R}])}.$$

Now we bound

$$\text{Vol}\{\mathbf{x} \in B_f(1) : f^\#(\mathbf{x}) > T\}.$$

By the ε -suitability of f , we have that

$$|P(\mathbf{x})|_F^{1/d} < f(x)^{1/\varepsilon} \cdot f^\#(x)^{1-1/\varepsilon}.$$

Thus, if $\varepsilon < 1$, then for any pair of positive real numbers t_1, t_2 with $t_1 < t_2$,

$$\begin{aligned} \text{Vol}\{\mathbf{x} \in B_f(1) : t_1 < f^\#(\mathbf{x}) \leq t_2\} &\leq \text{Vol}\{\mathbf{x} \in B^\#(t_2) : |P(\mathbf{x})|_F^{1/d} \leq t_1^{1-1/\varepsilon}\} \\ &\ll t_2^{|\mathbf{a}|} \cdot \left(t_1^{1-1/\varepsilon}/t_2\right)^{a_{\min}/[F:\mathbb{R}]}. \end{aligned}$$

It follows that if $[F:\mathbb{R}] \cdot \varepsilon < a_{\min}/|\mathbf{a}|$, then

$$\begin{aligned} \text{Vol}\{\mathbf{x} \in B_f(1) : f^\#(\mathbf{x}) > T\} &= \sum_{j=0}^{\infty} \text{Vol}\{\mathbf{x} \in B_f(1) : 2^j \cdot T < f^\#(\mathbf{x}) \leq 2^{j+1} \cdot T\} \\ &\ll \sum_{j=0}^{\infty} (2^j \cdot T)^{|\mathbf{a}| - a_{\min}/(\varepsilon \cdot [F:\mathbb{R}])} \\ &\sim T^{|\mathbf{a}| - a_{\min}/(\varepsilon \cdot [F:\mathbb{R}])}. \end{aligned}$$

The desired result now follows from Equation (2.2). □

2.4 Heights on Weighted Projective Space

Now we are ready to combine local heights to form a global height. The definitions and theorems in Section 2.4.1 and Section 2.4.2 once again come from [13].

2.4.1 The Toric Height

Given a number field K , let $f_v^\#$ be the standard local height on K_v^n .

Remark 2.4.1. For a given $\mathbf{x} \in K^n$ with $\mathbf{x} \neq \mathbf{0}$, there are only finitely many v such that $f_v^\#(\mathbf{x}) \neq 1$.

Let $H^\# : K^n \rightarrow \mathbb{R}$ be given by

$$H^\#(\mathbf{x}) = \prod_v f_v^\#(\mathbf{x}).$$

For any $t \in K^\times$, we have

$$H^\#(t \cdot \mathbf{x}) = \prod_v f_v^\#(t \cdot \mathbf{x}) = \prod_v |t|_v \cdot f_v^\#(\mathbf{x}) = \prod_v f_v^\#(\mathbf{x}).$$

The last equality follows from the product formula. Thus, the function $H^\#$ factors through $\mathbb{P}_K^{\mathbf{a}}$. This is the same height function as the Size function given by Deng [15] and the *toric height* as named by Darda [13]. We adopt the terminology of the latter.

The toric height has a finite part

$$H_{\text{fin}}^\# = \prod_{v \nmid \infty} f_v^\#$$

and an infinite part

$$H_\infty^\# = \prod_{v \mid \infty} f_v^\#.$$

2.4.2 Quasi-toric Heights

Darda considers variations on $H^\#$, as follows:

Definition 2.4.2. A collection of local $\mathbf{a}$ -height functions $(f_v : K_v^n \rightarrow \mathbb{R})_v$ is *quasi-toric* if $f_v = f_v^\#$ for all but finitely many places v . In this case, we also call the corresponding function $H : K^n \rightarrow \mathbb{R}$ given by

$$H(\mathbf{x}) = \prod_v f_v(\mathbf{x})$$

a quasi-toric height.

As with the toric height, a quasi-toric height can be decomposed into a finite part and an infinite part, as follows:

$$H_{\text{fin}} = \prod_{v \nmid \infty} f_v, \quad H_{\infty} = \prod_{v \mid \infty} f_v.$$

One basic example of a quasi-toric height is just the toric height. Using Fourier analysis of Height Zeta functions, Darda [13] deduces that for any quasi-toric height H with continuous local heights, there exists a constant c_H such that

$$\{[\mathbf{x}] \in \mathbb{P}_K^a : H([\mathbf{x}]) < T\} \sim c_H T^{|a|}.$$

2.4.3 Decomposition

In the case where $F = K_v$, we proved that $F^n \setminus 0 = \bigsqcup_{t \in F^\times / \mathcal{O}_F^\times} t \cdot S_a(F)$. A similar decomposition holds for the global field K , as follows.

By combining these local decompositions, for every $\mathbf{x} \in K^n \setminus 0$, there exists a tuple $(t_v)_{v \in M_K^0} \in \prod_{v \in M_K^0} K_v^\times$ such that for every $v \nmid \infty$, $t_v \cdot \mathbf{x} \in S_a(K_v)$. By Proposition 2.3.5, each t_v is unique up to a factor of $\mathcal{O}_v^\times$. Furthermore, $t_v \in \mathcal{O}_v^\times$ for all but finitely many v , so the tuple $(t_v)_{v \in M_K^0}$ is an element of the finite ideles $\mathbb{A}_{K, \text{fin}}^\times$. Thus,

$$K^n \setminus 0 = \bigsqcup_{\prod_{v \nmid \infty} \mathcal{O}_v^\times \setminus \mathbb{A}_{K, \text{fin}}^\times} S_\gamma, \quad (2.3)$$

where for a finite idele γ , S_γ is given by

$$S_\gamma = K^n \cap \bigcap_{v \nmid \infty} \gamma_v \cdot S_a(K_v).$$

Note that each S_γ is $\mathcal{O}^\times$ -invariant.

Proposition 2.4.3. *There exists a finite set $\Gamma \subseteq \mathbb{A}_{K, \text{fin}}^\times$ such that for every element $\mathbf{x}$ in*

$K^n \setminus 0$, there exists a unique $\gamma \in \Gamma$ such that $\mathbf{x}$ is $K^\times$ -equivalent to an element of S_γ . The set of elements in S_γ that are $K^\times$ -equivalent to $\mathbf{x}$ consists of a single $\mathcal{O}^\times$ -orbit.

Furthermore, we can choose Γ to be a subset of $\prod_{v \nmid \infty} \mathcal{O}_v$.

Proof. The main ingredient is that the class group of K , which can also be expressed as

$$\prod_{v \nmid \infty} \mathcal{O}_v^\times \backslash \mathbb{A}_{K, \text{fin}}^\times / K^\times,$$

is finite. Thus, if we let Γ be a set consisting of one element from each double coset above, it is clear from Equation (2.3) that Γ satisfies the first and second requirements of Proposition 2.4.3.

The last requirement follows from the fact that for any finite idele γ , there exists an element $t \in K^\times$ such that $t\gamma$ is $\mathcal{O}_v$ -integral for every $v \nmid \infty$. $\square$

On each $\gamma_v \cdot S_{\mathbf{a}}(K_v)$, $f_v^\#$ is equal to $|\gamma_v|_v$. Thus, for all $x \in S_\gamma$,

$$H_{\text{fin}}^\#(\mathbf{x}) = (N\gamma)^{-1}. \quad (2.4)$$

2.4.4 Singular Heights

As mentioned before, the Faltings height is a singular height: at the infinite places, there is a cusp (corresponding to a zero in the local height function) along the divisor determined by the determinant function.

The norm of the minimal discriminant is also a singular quasi-toric height with a singularity along the same locus. On the other hand, in the case where $K = \mathbb{Q}$, the conductor $N(E)$ is not even an example of a singular quasi-toric height. Though it can be written as a product of local $\mathbf{a}$ -height functions, none of the local factors are equal to the standard local height.

In order to count curves of bounded Faltings height, we need a slightly more general type of height that allows for subpolynomially suitable cusps at the infinite places (corresponding

to regions where the local height goes to 0). Note that the singularity at the minimal discriminant height is not subpolynomially suitable, so our techniques do not apply.

Instead of directly trying to count points that have height less than X , we instead count points that simultaneously have height less than X and naive height less than an auxiliary parameter T . We prove that this count of points has the correct main term that depends on X but an error term that grows with T .

In order to relax the condition on naive height, we need to pick T such that the number of points with height less than X but naive height greater than T is small. There is a balancing act here between the error term from counting points with large naive height (which decreases as T increases) and the aforementioned error term that grows with T . For the Faltings height H^F , this argument is sufficient: it turns out that for every $\varepsilon > 0$ there is a C_ε such that $H^F < C_\varepsilon \cdot (H^\#)^{1+\varepsilon}$.

However, in the case of the discriminant height function, there is no known polynomial bound for the height of a point with bounded discriminant. It is conjecturally true that for any function f such that $f(x)/x \rightarrow \infty$ as $x \rightarrow \infty$, 100% of elliptic curves with $\Delta(E) < X$ satisfy $H^\#(E) < f(X)$. (See the Brumer-McGuinness heuristics [11] and more exposition in [32].)

The main difference highlighted here is that in contrast to the discriminant height, the Faltings height is subpolynomially suitable. This means that the cusp of the Faltings height is narrow enough to extend the existing geometry of numbers techniques to show that the count of points with bounded Faltings height also has the correct main term of $c_H X^{|a|}$.

Darda's approach used the Fourier analysis of Height Zeta functions rather than the geometry of numbers techniques of Deng. While Darda was able to prove a corresponding Northcott property for logarithmically suitable heights (which are automatically subpolynomially suitable), his method is not sufficient to obtain the correct main term.

2.4.5 Subpolynomially Suitable Heights

Here we define a *subpolynomially suitable* height. It is partially motivated by Darda's notion of a logarithmically suitable height from [13].

Definition 2.4.4. Let $P \in K[x_1, \dots, x_n]$ be a non-constant $\mathbf{a}$ -homogeneous polynomial.

1. Let $\varepsilon > 0$ be a positive constant. A collection of (possibly singular) continuous $\mathbf{a}$ -homogeneous functions $(f_v : K_v^n - \{0\} \rightarrow \mathbb{R}_{\geq 0})_{v \in M_K}$ is ε -suitable at P if f_v is non-singular for $v \nmid \infty$ and f_v is ε -suitable for every $v \mid \infty$.
2. The collection (f_v) is *subpolynomially suitable* if it is ε -suitable for every $\varepsilon > 0$.

Again, we say that f is ε -suitable (resp. subpolynomially suitable) if it is ε -suitable at P (resp. subpolynomially suitable at P) for some non-constant P .

In addition, we establish some properties of subpolynomially suitable heights which are needed in Chapter 4 to count points with bounded subpolynomially suitable heights. Once again, we call the corresponding global height function, $H = \prod_v f_v$, ε -suitable or subpolynomially suitable if its underlying collection of (possibly singular) local heights is ε -suitable or subpolynomially suitable, respectively.

Remark 2.4.5. Darda [13, Proposition 3.7.2.2] originally provided an upper bound for the number of points with bounded height H when H is logarithmically suitable. However, this bound also applies to a general subpolynomially suitable height and the proof is identical.

In order to prove Theorem 1.2.1, we mostly follow the point counting technique of Deng along with cutting off the cusp as in Hortsch [22]. Just like in [22], cutting off the cusp for a subpolynomially suitable height (or a ε -singular height) is relatively straightforward. We prove one important result, as follows:

Lemma 2.4.6. *Suppose H is ε -suitable at P . Then*

$$H(\mathbf{x}) \gg_{H,\varepsilon} H^\#(\mathbf{x})^{1-\varepsilon}$$

for all $\mathbf{x} \in \mathbb{P}_K^{\mathbf{a}}$ with $P(\mathbf{x}) \neq 0$.

Proof. Let S be the set of places where $f_v \neq f_v^\#$. Then

$$\frac{H(\mathbf{x})}{H^\#(\mathbf{x})} = \prod_{v \in S} \frac{f_v(\mathbf{x})}{f_v^\#(\mathbf{x})} \gg \left(\prod_{v \in S} \frac{|P(\mathbf{x})|_v^{1/m}}{f_v^\#(\mathbf{x})} \right)^\varepsilon \gg \left(\prod_v \frac{|P(\mathbf{x})|_v^{1/m}}{f_v^\#(\mathbf{x})} \right)^\varepsilon = H^\#(\mathbf{x})^{-\varepsilon},$$

where the first inequality follows from the definition of ε -suitability and the second inequality follows from Lemma 2.4.6. The last equality follows from the product formula for norms and the fact that $P(\mathbf{x}) \in K^\times$. $\square$

Lemma 2.4.7. *Let $c_1 < c_2$ be positive real numbers. Suppose H is ε -suitable and $|\mathbf{a}| - a_{\min}/2\varepsilon < 0$. Then*

$$\text{Vol}\{\mathbf{x} \in K^\infty : H(\mathbf{x}) < 1 \wedge H^\#(\mathbf{x}) > T \wedge c_1 < \frac{f_v(\mathbf{x})^{1/\epsilon_v}}{f_v'(\mathbf{x})^{1/\epsilon_{v'}}} < c_2 \ \forall v, v' \mid \infty\} \ll T^{|\mathbf{a}| - a_{\min}/2\varepsilon},$$

where the implied constant depends on P , c_1 , c_2 , K , and $\mathbf{a}$.

This follows from Lemma 2.3.12 and either taking a routine integral or using the same method of doubling intervals that we used to prove Lemma 2.3.12.

2.5 Global Masses

From a collection of local functions $(m_v : K_v^n \setminus 0 \rightarrow \mathbb{C})$ we can associate a local mass, assuming certain conditions hold.

Definition 2.5.1. We call a collection of $\mathbf{a}$ -local masses $(m_v : K_v^n \setminus 0 \rightarrow \mathbb{C})$ *global* if for all but finitely many v , $m_v \equiv 1$. We also call the corresponding product $m : K^n \setminus 0 \rightarrow \mathbb{C}$ a global mass.

The following proposition results directly from the definitions of global masses and quasi-toric heights:

Proposition 2.5.2. *Quasi-toric heights are in one-to-one correspondence with positive global masses, i.e., masses that only take positive values, under the correspondence $H \leftrightarrow H^\# \cdot m$.*

Under this correspondence, possibly singular quasi-toric heights are in one-to-one correspondence with non-negative global masses.

Again, we denote the following two products by m_{fin} and m_∞ :

$$m_{\text{fin}} = \prod_{v \nmid \infty} m_v, \quad m_\infty = \prod_{v \mid \infty} m_v.$$

Chapter 3

O-minimal Structures and Definable Heights

3.1 O-minimal Structures

Recall the definition of an O-minimal structure:

Definition 3.1.1. A structure S is a collection of sets S_n of subsets of $\mathbb{R}^n$ for each $n = 0, 1, 2, \dots$ such that the following conditions hold:

1. Each S_n is a boolean algebra of subsets of $\mathbb{R}^n$: For each n , $\emptyset \in S_n$. Additionally, if $A, B \in S_n$, then $A \cap B \in S_n$ and $\mathbb{R}^n \setminus A \in S_n$.
2. If $A \in S_m$ and $B \in S_n$, then $A \times B$ (as a subset of $\mathbb{R}^{m+n}$) is in S_{m+n} .
3. If $A \in S_{n+1}$ and $\pi : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^n$ is any coordinate projection, then $\pi(A) \in S_n$.
4. The sets $\{x, y \in \mathbb{R}^2 : x < y\}$, $\{(x, y, z) \in \mathbb{R}^3 : x + y = z\}$, $\{(x, y, z) \in \mathbb{R}^3 : xy = z\}$ are in S_2, S_3, S_3 , respectively. In addition, for each $t \in \mathbb{R}$, $\{t\} \in S_1$.

Definition 3.1.2. For a structure S , a set $T \subseteq \mathbb{R}^n$ is *definable* (in S) if $T \in S_n$.

Definition 3.1.3. Given a set $S \subseteq \mathbb{R}^m$ and a function $f : S \rightarrow \mathbb{R}^k$, f is definable if its graph is definable. (Note that f being definable implies that S is definable.)

Definition 3.1.4. An O-minimal structure is a structure S that satisfies the following additional axiom:

5. For each $T \in S_1$, the boundary of T consists of a finite number of points.

Condition 5 can be thought of as a “tameness” condition.

Remark 3.1.5. Conditions 1, 2, 4 imply that all real semi-algebraic sets are definable in any O-minimal structure S . Conversely, the collection of semi-algebraic sets (typically denoted $\mathbb{R}^{\text{alg}}$) is an O-minimal structure: condition 5 follows from the Fundamental Theorem of Algebra and condition 3 is the Tarski-Seidenberg theorem.

3.2 Point Counting

Due to the tameness property of sets definable in O-minimal structures, we can count points in such sets with good error terms. This is made explicit by the following theorem of Barroero-Widmer [1]:

Theorem 3.2.1 ([1], Theorem 1.3). *Let m and n be positive integers, let $Z \subseteq \mathbb{R}^{m+n}$ be a definable family, and suppose the fibers Z_T are bounded. Then there exists a constant $c_Z \in \mathbb{R}$, depending only on the family Z , such that for any rank n lattice Λ ,*

$$\left| \#\{Z_x \cap \Lambda\} - \frac{\text{Vol}(Z_x)}{\det \Lambda} \right| \leq c_Z \sum_{j=0}^{n-1} \frac{V_j(Z_x)}{\lambda_1 \cdots \lambda_j},$$

where $\lambda_1, \dots, \lambda_n$ are the successive minima of Λ and $V_j(Z_x)$ is the sum of the j -dimensional volumes of the orthogonal projections of Z_x on every j -dimensional coordinate subspace of $\mathbb{R}^n$.

We bootstrap Theorem 3.2.1 into a more general version where we count points weighted by a definable function:

Theorem 3.2.2. *Let m and n be positive integers, let $Z \subseteq \mathbb{R}^m$ be a definable set and let $F : Z \times \mathbb{R}^n \rightarrow \mathbb{R}$ be a definable function such that F_x is bounded and has compact support for all $x \in Z$. Then there exists a constant $c_F \in \mathbb{R}$, depending only on F and Z , such that for any $x \in F$ and rank n lattice Λ ,*

$$\left| \sum_{v \in \Lambda} F_x(v) - \frac{1}{\det \Lambda} \int_{\mathbb{R}^n} F_x \right| \leq c_F \sum_{j=0}^{n-1} \frac{1}{\lambda_1 \cdots \lambda_j} \sum_{\substack{I \subseteq [1,n] \\ |I|=j}} \int_{\mathbb{R}^I} \sup_{\substack{v \in \mathbb{R}^n \\ \pi_I(v)=v_I}} |F_x(v)| \, dv_I.$$

Here π_I is the coordinate projection from $\mathbb{R}^n$ to $\mathbb{R}^I$. When $I = \emptyset$, the corresponding summand on the right-hand side is understood to be $\sup_{v \in \mathbb{R}^n} |F_x(v)|$.

Proof. We may assume without loss of generality that F only takes on non-negative values: given the standard decomposition $F = F^+ - F^-$, where $F^+ = \max(F, 0)$ and $F^- = \max(0, -F)$, we can compute the constant c for each of F^+ and F^- and take $c_F = c_{F^+} + c_{F^-}$.

In this case, consider the definable family $Z' \subseteq \mathbb{R}^m \times (0, \infty) \times \mathbb{R}^n$ given by

$$Z'_{x,t} = \{v \in \mathbb{R}^n : F_x(v) \geq t\}.$$

Applying Theorem 3.2.1, there exists a constant c_Z such that for any full rank lattice Λ ,

$$\left| \#(Z'_{x,t} \cap \Lambda) - \frac{\text{Vol}(Z'_{x,t})}{\det \Lambda} \right| \leq c_Z \sum_{j=0}^{n-1} \frac{V_j(Z'_{x,t})}{\lambda_1 \cdots \lambda_j} = c_Z \sum_{j=0}^{n-1} \sum_{\substack{I \subseteq [1,n] \\ |I|=j}} \int_{\mathbb{R}^I} \text{Vol}(\pi_I(Z'_{x,t})).$$

Integrating both sides with respect to t gives

$$\begin{aligned}
\left| \sum_{v \in \Lambda} F_x(v) - \frac{1}{\det \Lambda} \int_{\mathbb{R}^n} F_x \right| &= \left| \int_{t=0}^{\sup F_x} \#(Z'_{x,t} \cap \Lambda) - \frac{1}{\det \Lambda} \text{Vol}(Z'_{x,t}) \, dt \right| \\
&\leq \int_{t=0}^{\sup F_x} \left| \#(Z'_{x,t} \cap \Lambda) - \frac{1}{\det \Lambda} \text{Vol}(Z'_{x,t}) \right| \, dt \\
&\leq c_Z \int_{t=0}^{\sup F_x} \sum_{j=0}^{n-1} \frac{V_j(Z'_{x,t})}{\lambda_1 \cdots \lambda_j} \, dt \\
&= c_Z \sum_{j=0}^{n-1} \sum_{\substack{I \subseteq [1,n] \\ |I|=j}} \int_{t=0}^{\sup F_x} \text{Vol}(\pi_I(Z'_{x,t})) \, dt \\
&= c_Z \sum_{j=0}^{n-1} \sum_{\substack{I \subseteq [1,n] \\ |I|=j}} \int_{\mathbb{R}^I} \sup_{\substack{v \in \mathbb{R}^n \\ \pi_I(v)=v_I}} |F_x(v)| \, dv_I.
\end{aligned}$$

□

3.3 The O-minimal Structure $\mathbb{R}_{\text{an,exp}}$

The main O-minimal structure we work with is $\mathbb{R}_{\text{an,exp}}$, which is the smallest structure that contains the exponential function and the *restricted analytic functions*: functions f that are the restriction of an analytic function on an open neighborhood of $[-1, 1]^n$ to $[-1, 1]^n$ (and 0 otherwise). It is a deep result of van den Dries and Miller [31] that $\mathbb{R}_{\text{an,exp}}$ is in fact O-minimal.

Lemma 3.3.1. *Suppose f is a modular form (of level 1) and let Γ denote the usual fundamental domain for the action of $\text{SL}_2(\mathbb{Z})$ on the upper half-plane. Then f restricted to Γ is definable.*

Proof. This follows from [24, Theorem 4.1], but we give a quick alternative proof here. The map

$$e : \left\{ z \in \mathcal{H} : -\frac{1}{2} \leq \Re(z) < \frac{1}{2} \right\} \rightarrow \mathbb{C}$$

given by $e(z) = \exp(2\pi iz)$ is definable in $\mathbb{R}_{\text{an},\text{exp}}$ as both $\exp : \mathbb{R} \rightarrow \mathbb{R}$ and $\sin, \cos : [\pi, \pi] \rightarrow \mathbb{R}$ are definable in $\mathbb{R}_{\text{an},\text{exp}}$.

The region $e(\Gamma)$ is compact, definable in $\mathbb{R}_{\text{an},\text{exp}}$, and contained in the open unit ball. In addition, $f \circ e^{-1}$ is analytic on the unit open disc, so $f \circ e^{-1}$ is the restriction of a finite gluing of restricted analytic functions restricted to $e(\Gamma)$ and thus definable in $\mathbb{R}_{\text{an},\text{exp}}$. Thus, f restricted to the standard fundamental domain is definable in $\mathbb{R}_{\text{an},\text{exp}}$. $\square$

Corollary 3.3.2. *The function $f : \mathbb{C}^2 \rightarrow \mathbb{R}$ given by*

$$f(A, B) = \begin{cases} 0 & \Delta(A, B) = 0 \\ \frac{\Delta(A, B)}{|\Delta(\tau_{A, B})| \cdot \text{Im}(\tau_{A, B})^6} & \text{otherwise} \end{cases}$$

is definable in $\mathbb{R}_{\text{an},\text{exp}}$.

3.4 Strongly Definable Heights and Masses

Now we are ready to specify the class of “tame” global heights.

Definition 3.4.1. Suppose H is a (possibly singular) quasi-toric height with local height functions $(f_v)_{v \in M_K}$. We say that H is *strongly definable* if the following conditions hold:

1. The function $H_\infty : K_\infty^n \rightarrow \mathbb{R}$ is definable in an O-minimal structure.
2. For each v , f_v is finite.
3. For each finite v , f_v is locally constant away from $\mathbf{0}$.

It is clear that the collection $(f_v^\#)$ satisfies these conditions. In other words,

Proposition 3.4.2. *The naive height $H^\#$ is definable.*

We have a similar notion of a strongly definable global mass:

Definition 3.4.3. Suppose m is a global mass with local mass functions $(m_v)_{v \in M_K}$. We say that m is *strongly definable* if the following conditions hold:

1. There exists an O-minimal structure such that for each $v \mid \infty$, m_v is definable.
2. For each $v \nmid \infty$, m_v is locally constant away from $\mathbf{0}$.

Remark 3.4.4. Since $H^\#$ is strongly definable, the correspondence of Proposition 2.5.2 takes strongly definable heights to strongly definable masses and vice versa.

Remark 3.4.5. Condition 2 combined with the original global condition that $m_v \equiv 1$ for all but finitely many v is equivalent to the statement that the induced function $\widehat{m} : \prod_{v \nmid \infty} \mathbb{P}_{K_v}^a \rightarrow \mathbb{C}$ is locally constant. As such, its image must be a finite set.

Chapter 4

Counting Points with Bounded Singular Height

4.1 Introduction

In this section we describe how to count points in weighted projective space having bounded height for the class of singular height functions introduced at the end of the Chapter 2.

First, we provide some background for the case $\mathbf{a} = (4, 6)$ corresponding to the moduli space of elliptic curves. For the case of naive height, it is known that

$$\#\mathcal{E}_{H^\# < X}^{\mathbb{Q}} = \frac{4}{\zeta(10)} X^{5/6} + O(X^{1/2}).$$

For the case $K = \mathbb{Q}$ there are unproven heuristics for the number of elliptic curves over $\mathbb{Q}$ with bounded discriminant [11] and conductor [32]. In the case of the minimal discriminant, this heuristic can be extended to all number fields. The problem of proving these heuristics remains open even in the case where $K = \mathbb{Q}$. This is in part because it is difficult to control how far lattice points can appear in the “cusps”.

However, the situation is more tractable for the Faltings height, which we prove in Chapter 7 is subpolynomially suitable. Hortsch [22] gave the asymptotic count of the number of

elliptic curves over $\mathbb{Q}$ with Faltings height less than X :

$$\#\mathcal{E}_{H_F < X}^{\mathbb{Q}} = \frac{12\sigma}{\zeta(10)} X^{5/6} + O(X^{1/2}(\log X)^3).$$

Theorem 1.2.1 is a generalization of this result to all number fields.

For the case of a general number field, there is a one-to-one correspondence between $\mathcal{E}^K$ and the weighted projective space $\mathbb{P}_K^{(4,6)}$, along with a generalization of the naive height $H^\#$ mentioned above. Now we consider the general problem of counting points in the weighted projective space $\mathbb{P}_K^{\mathbf{a}}$. The main counting result of Deng [15] (which itself is based on the classical result of Schanuel [26]) is that

$$\#\{[\mathbf{x}] \in \mathbb{P}_K^{|\mathbf{a}|} : H^\#([\mathbf{x}]) < T\} = c_{K,\mathbf{a}}^\# 2^{nr_1} \pi^{nr_2} T^{|\mathbf{a}|} + O(T^{|\mathbf{a}| - \frac{1}{3d}}),$$

where

$$c_{K,\mathbf{a}}^\# = \frac{\gcd(w, \mathbf{a}) Rh |\mathbf{a}|^{r-1}}{w \zeta_K(|\mathbf{a}|)} \left(\frac{2^{r_2}}{\sqrt{|D_k|}} \right)^n.$$

Remark 4.1.1. The discrepancy in the exponent of the main term comes from the fact that the naive height for $\mathbb{P}^{(4,6)}$ is actually $\left(H_{\mathcal{E}}^\#\right)^{1/12}$.

The following corollary — a generalization of [13, Proposition 3.7.2.2] — follows by combining this main counting result with Lemma 2.4.6:

Corollary 4.1.2. *Suppose H is ε -suitable at P . Then*

$$\#\{[\mathbf{x}] \in \mathbb{P}_K^{|\mathbf{a}|} : P(\mathbf{x}) \neq 0, H([\mathbf{x}]) < T\} \ll X^{\frac{|\mathbf{a}|}{1-\varepsilon}}.$$

In this chapter, we prove the following:

Theorem 4.1.3. *Suppose H is a strongly definable quasi-toric height with possible singular-*

ities at the infinite places. Then

$$\begin{aligned} & \#\{[\mathbf{x}] \in \mathbb{P}_K^{\mathbf{a}} : H([\mathbf{x}]) < X \wedge H^\#([\mathbf{x}]) < T\} \\ &= c_{K,\mathbf{a}}^\# \Sigma_H X^{|\mathbf{a}|} + O(T^{|\mathbf{a}| - a_{\min}/d}) + O(X^{|\mathbf{a}|} \cdot \varepsilon_\infty(T/X)), \end{aligned}$$

where

$$\begin{aligned} \sigma_{f_v} &= \begin{cases} \text{Vol}(\{\mathbf{x} \in K_v^n : f_v(\mathbf{x}) \leq 1\}) & v \mid \infty \\ (1 - q_v^{-|\mathbf{a}|})^{-1} \int_{S_v} f_v^{-|\mathbf{a}|} & v \nmid \infty \end{cases}, \\ \Sigma_H &= \prod_{v \in M_K} \sigma_{f_v}, \\ \varepsilon_\infty(t) &= \text{Vol}\{\mathbf{x} \in R : H_\infty^\#(\mathbf{x}) > t \wedge H_\infty(\mathbf{x}) \leq 1\}. \end{aligned}$$

R is a fundamental domain for the action of $\mathcal{O}^\times$ on K_∞^n .

Note that by the strongly definable condition (in particular, the finite volume condition), we must have

$$\lim_{t \rightarrow \infty} \varepsilon_\infty(t) = 0,$$

so if $T/X \rightarrow \infty$ as $X \rightarrow \infty$, the second error term is $o(X^{|\mathbf{a}|})$.

Using Lemma 2.4.6 and Lemma 2.4.7 we can then deduce the following:

Theorem 4.1.4. *Suppose H is a (possibly singular) strongly definable quasi-toric height on $\mathbb{P}_K^{\mathbf{a}}$. There exist constants $c, \varepsilon_0 > 0$ such that if $\varepsilon < \varepsilon_0$ and H is ε -suitable, then*

$$\#\{[\mathbf{x}] \in \mathbb{P}_K^{\mathbf{a}} : P(\mathbf{x}) \neq 0, H([\mathbf{x}]) < X\} = c_{K,\mathbf{a}}^\# \Sigma_H X^{|\mathbf{a}|} + O\left(X^{\frac{|\mathbf{a}| - a_{\min}/d}{1 - c\varepsilon}}\right).$$

We later establish weighted versions of the theorems above.

Remark 4.1.5. Since $\sigma_{f_v}^\# = 1$ for all $v \nmid \infty$, the product $\prod_{v \in M_K} \sigma_{f_v}$ is well-defined.

Remark 4.1.6. In the case where $f_v(K_v^n) \subseteq |K_v|_v = \{0\} \cup q_v^{\mathbb{Z}}$, we have

$$(1 - q_v^{-|\mathbf{a}|})^{-1} \int_{E_v} f_v^{-|\mathbf{a}|} = \text{Vol}(\{\mathbf{x} \in K_v^n : f_v(\mathbf{x}) \leq 1\}).$$

Thus, it makes sense to think of σ_{f_v} as a “volume” even in the case where $v \nmid \infty$.

Corollary 4.1.7. *Suppose $\varepsilon < \varepsilon_0$ and H is an ε -suitable quasi-toric height (not necessarily strongly definable) with continuous local heights. Then*

$$\#\{[\mathbf{x}] \in \mathbb{P}_K^{\mathbf{a}} : H([\mathbf{x}]) < X\} \sim c_{K,|\mathbf{a}|}^{\#} \Sigma_H X^{|\mathbf{a}|}.$$

Remark 4.1.8. As mentioned in the introduction, this result is an improvement on [13, 3.7.2.3], in which this result is proven with the right-hand side replaced with the expression $O_{\varepsilon}(T^{|\mathbf{a}|+\varepsilon})$.

Proof of Corollary 4.1.7. We approximate H above and below by strongly definable heights. When H is non-singular, this approximation is an immediate consequence of the Stone-Weierstrass theorem.

For $v \nmid \infty$, since $S_v = \{\mathbf{x} \in K_v^n : f_v^{\#}(\mathbf{x}) = 1\}$ is compact, a continuous function can be approximated arbitrarily closely above and below on S_v by locally constant functions that are uniformly bounded away from 0.

For each infinite place v , $S_v = \{\mathbf{x} \in K^n : f_v^{\#}(\mathbf{x}) = 1\}$ is compact, so for every $s \gg 1$ there exists an $\mathbf{a}$ -homogeneous polynomial Q_s (of some degree m_Q) such that on E_v ,

$$Q_s(\mathbf{x}) - s \leq \max(f_v(\mathbf{x}), s \cdot f_v^{\#}(\mathbf{x})) \leq Q_s(\mathbf{x}).$$

We can extend Q_s to an $\mathbf{a}$ -homogeneous function on all of $K_v^n - \{0\}$ by setting $f_v^{(s)}(\mathbf{x}) = Q_s(\mathbf{x})/f_v^{\#}(\mathbf{x})^{m_Q-1}$. It is clear from the construction of Q_s that $f_v^{(s)}$ is definable.

Clearly $f_v \leq f_v^{(s)}$, so $\sigma_{f_v^{(s)}} \leq \sigma_{f_v}$ for all v . Let $H^{(s)}$ be the strongly definable height

function associated to the collection $(f_v^{(s)})_{v \in M_K}$. Then $\sigma_{H^{(s)}} \leq \sigma_H$, so

$$\begin{aligned} & \lim_{X \rightarrow \infty} \frac{\#\{[\mathbf{x}] \in \mathbb{P}_K^{\mathbf{a}} : P(\mathbf{x}) \neq 0 \wedge H([\mathbf{x}]) < X\}}{X^{|\mathbf{a}|}} \\ & \geq \lim_{X \rightarrow \infty} \frac{\#\{[\mathbf{x}] \in \mathbb{P}_K^{\mathbf{a}} : P(\mathbf{x}) \neq 0 \wedge H^{(s)}([\mathbf{x}]) < X\}}{X^{|\mathbf{a}|}} \\ & = c_{K,\mathbf{a}}^{\#} \sigma_{H^{(s)}}. \end{aligned}$$

By Lemma 2.3.12, $\sigma_{f_v^{(s)}} \rightarrow \sigma_{f_v}$ as $s \rightarrow 0$, so

$$\lim_{s \rightarrow 0} \sigma_{H^{(s)}} = \sigma_H.$$

Therefore,

$$\lim_{X \rightarrow \infty} \frac{\#\{[\mathbf{x}] \in \mathbb{P}_K^{\mathbf{a}} : P(\mathbf{x}) \neq 0 \wedge H([\mathbf{x}]) < X\}}{X^{|\mathbf{a}|}} \geq c_{K,\mathbf{a}}^{\#} \sigma_H.$$

Now we need to show the upper bound. Pick ε' to be a rational number between ε and ε_0 (defined in Theorem 4.1.4). For every $s \gg 1$ there exists an $\mathbf{a}$ -homogeneous polynomial Q_s (of some degree m_Q) such that on E_v ,

$$Q_s(\mathbf{x}) + \frac{1}{s} \geq \min(f_v(\mathbf{x}) \cdot |P(\mathbf{x})|_v^{-\varepsilon'/m}, s \cdot f_v^{\#}(\mathbf{x})) \geq Q_s(\mathbf{x}).$$

We can extend Q_s to an $\mathbf{a}$ -homogeneous function admitting an ε' -singularity at P on $K_v^n \setminus Z(P)$ by setting

$$g_v^{(s)}(\mathbf{x}) = Q_s(\mathbf{x}) / f_v^{\#}(\mathbf{x})^{m_Q-1} \cdot \left(\frac{|P(\mathbf{x})|_v^{1/m}}{f_v^{\#}(\mathbf{x})} \right)^{\varepsilon'}.$$

Again it is clear from the construction of Q_s that $g_v^{(s)}$ is definable. We have $g_v^{(s)} \leq f_v(x)$, so $\sigma_{g_v^{(s)}} \geq \sigma_{f_v}$. In addition, $\lim_{s \rightarrow \infty} g_v^{(s)}(x) = f_v(x)$. Thus, the inequality

$$\lim_{T \rightarrow \infty} \frac{\#\{[\mathbf{x}] \in \mathbb{P}_K^{\mathbf{a}} : P(\mathbf{x}) \neq 0 \wedge H([\mathbf{x}]) < X\}}{X^{|\mathbf{a}|}} \leq c_{K,\mathbf{a}}^{\#} \Sigma_H$$

follows from the sequence of functions $g_v^{(s)}$ as $s \rightarrow \infty$ much like how the $\geq$ direction followed

from the $f_v^{(s)}$'s as $s \rightarrow 0$. □

4.2 Point Counting in Affine Space

For any $\mathcal{O}^\times$ -invariant subset $\mathcal{X}$ of K^n , let

$$N(\mathcal{X}, T) = \#\{[\mathbf{x}] \in \mathcal{X}/\mathcal{O}^\times : H_\infty^\#(\mathbf{x}) < T\}.$$

More generally, given an $\mathcal{O}^\times$ -invariant function $m : \mathcal{X} \rightarrow \mathbb{C}$, we define

$$N_m(\mathcal{X}, T) = \sum_{\substack{[\mathbf{x}] \in \mathcal{X}/\mathcal{O}^\times \\ H_\infty^\#(\mathbf{x}) < T}} m(\mathbf{x}).$$

First we consider the simplest case: the mass function is equal to 1 everywhere. We recap the proof of the main counting result of [15], which itself was based on the classic results of Schanuel in [26].

Let $\mathcal{R}$ be a fundamental domain for the action of $\mathcal{O}^\times/\mu_K$ on K_∞^n , and let $\mathcal{R}(T) = \{\mathbf{x} \in \mathcal{R} : H_\infty^\#(\mathbf{x}) < T\}$. Then

$$N(\mathcal{X}, T) = \frac{\gcd(w, \mathbf{a})}{w} \cdot \#(\mathcal{X} \cap \mathcal{R}(T)).$$

This definition for $N(\mathcal{X}, T)$ makes sense even when $\mathcal{X}$ is not $\mathcal{O}^\times$ -invariant (with the caveat that this expression will then depend on the choice of $\mathcal{R}$).

Deng's construction of $\mathcal{R}$ is as follows: Consider the function $\ell : K_\infty^n \setminus 0 \rightarrow \mathbb{R}^{r_1+r_2}$ given by

$$\ell(\mathbf{x}) = (\log f_v^\#(\mathbf{x}))_{v|\infty}.$$

Let $\Gamma \subseteq \mathbb{R}^{r_1+r_2}$ be the hyperplane given by $\sum_v y_v = 0$, and let π_Γ be the projection from $\mathbb{R}^{r_1+r_2} \rightarrow \Gamma$ given by $(y_v) \mapsto (y_v - \frac{\epsilon_v}{d} \sum_v y_v)$.

Let Λ be the image of $\mathcal{O}^\times$ in $\mathbb{R}^{r_1+r_2}$ via the map $t \mapsto (\log |t|_v)_{v|\infty}$. By Dirichlet's unit theorem, Λ is a full rank sub-lattice of Γ . Pick a basis B for the lattice Λ , and let Ω_B be the usual fundamental domain for Λ in Γ . Then $\mathcal{R} = \ell^{-1}(\pi_\Gamma^{-1}(\Omega_B))$ is a fundamental domain for the action of $\mathcal{O}^\times$ on K_∞^n .

From the construction of $\mathcal{R}$ it is clear that for each $v, v' \in M_K^\infty$, $(f_v^\#)^{1/\epsilon_v} / (f_{v'}^\#)^{1/\epsilon_{v'}}$ is bounded above and below by positive real numbers on $\mathcal{R}$. Thus, for each v , $f_v^\#$ is bounded on $\mathcal{R}(1)$, so $\mathcal{R}(1)$ is bounded. Furthermore, if $\mathcal{R}(1)$ is contained in a box with side r centered at the origin, then $\mathcal{R}(T)$ is contained in the box with dimensions $(T^{1/da_1}r, T^{1/da_1}r, \dots, T^{1/da_n}r)$, where each $T^{1/da_j}r$ appears d times.

If $\mathcal{R}(1)$ is definable, then we can use Theorem 3.2.1 to deduce that

$$\#(\mathcal{O}^n \cap \mathcal{R}(T)) = \frac{1}{D_K^{n/2}} \cdot T^{|a|} \cdot \text{Vol } \mathcal{R}(1) + O(T^{|a| - a_{\min}/d}).$$

If the exponential function is definable, then $\mathcal{R}(1)$ is also definable. This is certainly the case for the O-minimal structure $\mathbb{R}_{\text{an}, \text{exp}}$, which is the relevant structure for our results on the Faltings height.

Even when $\mathcal{R}(1)$ is not definable, we use the averaging method from [6] in order to apply Theorem 3.2.1.

Let $G = \mathbb{G}_m$. We can let G_0 be a compact definable subset of $G(K_\infty) = \prod_{v|\infty} (K_v \setminus 0)$. Let L be the definable fundamental domain for the action of $G(K_\infty)$ on K_∞^n given by

$$L = \prod_{v|\infty} \{x \in K_v^n : H_v^\#(x) = 1, \text{ the leftmost nonzero coordinate of } x \text{ is in } \mathbb{R}_{>0}\},$$

and let Ω be a (not necessarily definable) fundamental domain for $G(\mathcal{O}) = \mathcal{O}^\times$ in $G(K_\infty)$. Then $\#(\mathcal{O}^n \cap \mathcal{R}(T))$ is some constant (depending on G_0) times the average of $\#(\mathcal{O}^n \cap (hG_0 \cdot L)_{H^\# < T})$, where h is averaged over Ω . Now we apply Theorem 3.2.1 and then integrate with respect to h to prove the desired result.

4.2.1 Definable Masses at ∞

Let $\mathcal{M}_\infty$ be a family of definable $\mathcal{O}^\times$ -invariant mass functions on K_∞^n . We now count

$$N_{\phi_\infty}(\mathcal{O}^n, T) = \sum_{\substack{[\mathbf{x}] \in \mathcal{O}^n / \mathcal{O}^\times \\ H_\infty^\#(\mathbf{x}) < T}} \phi_\infty(\mathbf{x})$$

in the case where $\phi_\infty \in \mathcal{M}$. Using the same logic as the previous section, we have that

$$N_{\phi_\infty}(\mathcal{O}^n, T) = \frac{\gcd(w, \mathbf{a})}{w} \sum_{\mathbf{x} \in R(T) \cap \mathcal{O}^n} \phi_\infty(\mathbf{x}).$$

Instead of Theorem 3.2.1, we can instead use Theorem 3.2.2 to deduce that

$$N_{\phi_\infty}(\mathcal{O}^n, T) = \frac{1}{D_K^{n/2}} \cdot \int_{\mathcal{R}(T)} \phi_\infty(\mathbf{x}) \, d\mathbf{x} + O(T^{|\mathbf{a}| - a_{\min}/d}),$$

where the implied constant depends only on the family $\mathcal{M}$. For a strongly definable height H , we can apply this result to the family $\mathcal{M}$ parameterized by positive real numbers, where $\mathcal{M}_t$ is the indicator function for $H_\infty < t$.

4.2.2 Volume Computations at ∞

By [15, Proposition 5.3],

$$\text{Vol } \mathcal{R}(1) = (2^{r_1} \pi^{r_2})^n R |\mathbf{a}|^{r-1}.$$

In fact, the same proof shows that

$$\text{Vol}\{\mathbf{x} \in \mathcal{R} : H_\infty(\mathbf{x}) < 1\} = R |\mathbf{a}|^{r-1} \cdot \Sigma_{H_\infty},$$

where $\Sigma_{H_\infty} = \prod_{v|\infty} \sigma_{f_v}$. More generally, if m_∞ is a definable $K_\infty^\times$ -invariant mass, then

$$\int_{\mathcal{R}_{H_\infty < 1}} m(\mathbf{x}) \, d\mathbf{x} = R |\mathbf{a}|^{r-1} \cdot \prod_v \int_{B_v(1)} m_v.$$

4.2.3 Integral Mass Functions

Now we consider mass functions defined at the finite primes. Let $\widehat{\mathcal{O}}$ denote the profinite completion of $\mathcal{O}$. Given a measurable mass function $\widehat{\phi} : \widehat{\mathcal{O}}^n \rightarrow \mathbb{C}$, we can define

$$N_{\phi_\infty \cdot \widehat{\phi}}(\mathcal{O}^n, T)$$

as the $(\phi_\infty \cdot \widehat{\phi})$ -weighted count of $\mathcal{O}^\times$ -orbits of $\mathcal{X}$ with $H^\# < T$.

In this section, we deduce the correct main term when counting in affine space weighted by any weakly good mass function. Furthermore, we obtain a power-saving error term when weighting by a locally constant function.

4.2.4 Locally Constant Masses

Suppose $\widehat{\phi} : \widehat{\mathcal{O}}^n \rightarrow \mathbb{C}$ is locally constant. Then there exist integral ideals $I_1, \dots, I_n \subseteq \mathcal{O}$ such that $\widehat{\phi}$ is defined modulo $\Lambda = I_1 \times \dots \times I_n$. Applying Theorem 3.2.2, we obtain a bound for the ϕ_∞ -weighted number of points in the intersection of $\mathcal{R}(T)$ with any translation of Λ :

$$N_{\phi_\infty}(x_0 + \Lambda, T) = \frac{1}{D_K^{n/2} \cdot [\mathcal{O}^n : \Lambda]} \cdot \int_{\mathcal{R}(T)} \phi_\infty(\mathbf{x}) \, d\mathbf{x} + O(T^{|\mathbf{a}| - a_{\min}/d}),$$

where the constant in the error term depends only on $\mathcal{M}$ and the successive minima of $I_1, \dots, I_n$. Taking the sum over all $x \in \mathcal{O}^n/\Lambda$ weighted by $\widehat{\phi}(x)$, we have

$$\begin{aligned} N_{\phi_\infty \cdot \widehat{\phi}}(\mathcal{O}^n, T) &= \frac{1}{D_K^{n/2}} \cdot \int_{\widehat{\mathcal{O}}^n} \widehat{\phi} \, d\nu \cdot \int_{\mathcal{R}(T)} \phi_\infty(\mathbf{x}) \, d\mathbf{x} + O(T^{|\mathbf{a}| - a_{\min}/d}) \\ &= N_{\phi_\infty}(\mathcal{O}^n, T) \int_{\widehat{\mathcal{O}}^n} \widehat{\phi} \, d\nu + O(T^{|\mathbf{a}| - a_{\min}/d}), \end{aligned}$$

where $d\nu$ is the Haar measure on $\widehat{\mathcal{O}}^n$. Again, the constant of proportionality in the error term only depends on $\mathcal{M}$ and $\widehat{\phi}$.

Now we calculate

$$N_{\phi_\infty \cdot \widehat{\phi}}(V_\gamma, T)$$

for a finite idele γ with $V_\gamma \subseteq \mathcal{O}^n$, where

$$\widehat{V}_\gamma := \prod_{v \nmid \infty} \gamma_v \cdot \mathcal{O}_v^n, \quad V_\gamma := V_K \cap \widehat{V}_\gamma.$$

Note that $\text{Vol}_{\mathcal{O}^n} \widehat{V}_\gamma = (N\gamma)^{-|\mathbf{a}|}$. In order to apply Theorem 3.2.1 (and Theorem 3.2.2) uniformly across all γ , we need the following lemma about the successive minima of V_γ :

Lemma 4.2.1. *There exist real numbers c_1 and c_2 such that for all finite ideles γ with $N\gamma \geq 1$ and all $1 \leq k \leq nd$,*

$$c_1 < \frac{\lambda_k(V_\gamma)}{(N\gamma)^{a'_{\lceil k/d \rceil}/d}} < c_2,$$

where $a'_1, \dots, a'_n$ is the permutation of $\mathbf{a}$ such that $a'_1 \leq a'_2 \leq \dots \leq a'_n$.

This lemma actually follows directly from the non-weighted one-dimensional version:

Lemma 4.2.2. *For all fractional ideals $I \subseteq K$,*

$$\lambda_d(I) \leq \lambda_d(\mathcal{O}) \cdot \lambda_1(I).$$

Proof. Pick linearly independent $x_1, x_2, \dots, x_d \in \mathcal{O}$ such that $x_j = \lambda_j(\mathcal{O})$.

Suppose $w \in I$ such that $|w| = \lambda_1(I)$. Then $wx_1, wx_2, \dots, wx_d \in I$ are linearly independent and $|wx_i| \leq |w| |x_i| \leq |w| \lambda_d(\mathcal{O})$. The result follows. $\square$

Note that if $\widehat{\phi}$ is defined modulo $I_1 \times \dots \times I_n$, then by the Chinese Remainder theorem, $\widehat{\phi}$ when restricted to V_γ must be defined modulo $\gamma \cdot (I_1 \times \dots \times I_n)$, and there are still at most $N(I_1) \cdot \dots \cdot N(I_n)$ congruence classes to consider.

Repeating the same logic as the previous section, we have that as long as $N\gamma \ll T$,

$$N_{\phi_\infty \cdot \hat{\phi}}(V_\gamma, T) = N_{\phi_\infty}(\mathcal{O}^n, T) \int_{\hat{V}_\gamma} \hat{\phi} \, d\nu + O((T/N\gamma)^{|\mathbf{a}| - a_{\min}/d}),$$

where the constant of proportionality does not depend on γ . Now we calculate

$$N_{m_\infty \cdot \hat{\phi}}(S_{\gamma_0}, T)$$

for a *fixed* finite idele γ_0 using the inclusion-exclusion sieve. We need the following lemma:

Lemma 4.2.3. *Suppose $\mathbf{x} \in V_\gamma \setminus 0$. Then $H_\infty^\#(\mathbf{x}) \geq N\gamma$.*

Proof. Note that $x_i \in \gamma^{a_i} \mathcal{O}$. Suppose without loss of generality that $x_1 \neq 0$. Then

$$N\gamma \leq N(x_i)^{1/a_i} = \prod_{v|\infty} |x_i|_v^{1/a_i} \leq H_\infty^\#(\mathbf{x}).$$

□

Let $\mu : \mathbb{A}_{K, \text{fin}} \rightarrow \{\pm 1\}$ denote the (ideal) Möbius function. By Möbius inversion,

$$\begin{aligned} N_{m_\infty \hat{\phi}}(S_{\gamma_0}, T) &= \sum_{\gamma} \mu(\gamma) N_{m_\infty \hat{\phi}}(V_{\gamma\gamma_0}, T) \\ &= \sum_{N\gamma \leq T/N\gamma_0} \left[\mu(\gamma) N_{\phi_\infty}(\mathcal{O}^n, T) \int_{\hat{V}_{\gamma\gamma_0}} \hat{\phi} \, d\nu + O((T/N\gamma)^{|\mathbf{a}| - a_{\min}/d}) \right] \\ &= N_{\phi_\infty}(\mathcal{O}^n, T) \int_{\hat{S}_{\gamma_0}} \hat{\phi} \, d\nu + O \left(N(\mathcal{O}^n, T) \sum_{N\gamma > T/N\gamma_0} \text{Vol}_{\hat{\mathcal{O}}^n} \hat{V}_{\gamma\gamma_0} \right) + O(T^{|\mathbf{a}| - a_{\min}/d}) \\ &= N_{\phi_\infty}(\mathcal{O}^n, T) \int_{\hat{S}_{\gamma_0}} \hat{\phi} \, d\nu + O(T^{|\mathbf{a}| - a_{\min}/d}). \end{aligned} \tag{4.1}$$

Remark 4.2.4. Equation (4.1) also holds if $\hat{\phi}$ is equal to $\prod_{v \in S} m_v$, where S is finite and each m_v is $K_v^\times$ -invariant and locally constant away from 0. We will need this in the reduction from counting points in $\mathbb{P}_K^{\mathbf{a}}$ to counting $\mathcal{O}^\times$ -equivalence classes in $\mathcal{O}^n$.

4.2.5 Some Generalities about Point Counting

We have shown that for every locally constant function $\widehat{\phi} : \widehat{\mathcal{O}}_K^n \rightarrow \mathbb{C}$,

$$N_{\phi_\infty \widehat{\phi}}(\mathcal{O}^n, T) = N_{\phi_\infty}(\mathcal{O}^n, T) \int_{\widehat{\mathcal{O}}_K^n} \widehat{\phi} d\nu + O_{\widehat{\phi}}(T^{|\mathbf{a}| - a_{\min}/d}),$$

where $d\nu$ is the Haar measure on $\widehat{\mathcal{O}}_K^n$. For the rest of this subsection, assume that $N_{\phi_\infty}(\mathcal{O}^n, T)$ grows faster than $T^{|\mathbf{a}| - a_{\min}/d}$. Then for every locally constant $\widehat{\phi}$,

$$\lim_{T \rightarrow \infty} \frac{N_{\phi_\infty \widehat{\phi}}(\mathcal{O}^n, T)}{N_{\phi_\infty}(\mathcal{O}^n, T)} = \int_{\widehat{\mathcal{O}}_K^n} \widehat{\phi} d\nu. \quad (4.2)$$

Since $\widehat{\mathcal{O}}$ is compact, any continuous $\widehat{\mathcal{O}}_K^n \rightarrow \mathbb{C}$ can be approximated uniformly arbitrarily closely by locally constant function. Thus, Equation (4.2) holds if $\widehat{\phi}$ is continuous.

More generally, if $\widehat{\phi}$ is continuous on an open set of full measure, then for every $\epsilon > 0$ there exists a locally constant function $\widehat{\phi}'$ and a compact open set $U \subseteq \phi$ such that

1. The measure of U is greater than $1 - \epsilon$.
2. On U , $|\widehat{\phi}' - \widehat{\phi}| < \epsilon$.
3. Away from U , $\widehat{\phi} = 0$.

Thus, Equation (4.2) again holds if ϕ is bounded everywhere and continuous on an open set of full measure. One example of such a function is the indicator function of a set whose boundary has measure 0.

Acceptable functions

Suppose we have collection of local functions $\phi_v : \mathcal{O}_v^n \rightarrow \mathbb{C}$, one for each $v \mid \infty$, such that the following conditions hold:

1. For each $\mathbf{x} \in \mathcal{O}^n$, $\prod_{v \mid \infty} \phi_v(\mathbf{x})$ converges to $\widehat{\phi}(\mathbf{x})$.

2. For each finite v , ϕ_v is bounded everywhere and continuous outside a set of measure 0.

3. For all but finitely many v , $|\phi_v| \leq 1$.

4. The limit

$$\lim_{M \rightarrow \infty} \limsup_{T \rightarrow \infty} \frac{N_{|\phi_\infty|}(\bigcup_{Nv > M} W_v, T)}{|N_{\phi_\infty}(\mathcal{O}^n, T)|},$$

where $W_v = \{\mathbf{x} \in \mathcal{O}^n : \phi_v(\mathbf{x}) \neq 1\}$, exists and is equal to 0.

Let $\phi_M = \prod_{Nv \leq M} \phi_v$. For each M , ϕ_M is bounded and continuous outside a set of measure 0, so Equation (4.2) holds for ϕ_M . In addition, we have

$$\begin{aligned} \lim_{M \rightarrow \infty} \left| \lim_{T \rightarrow \infty} \frac{N_{\phi_\infty \cdot \hat{\phi}}(\mathcal{O}^n, T)}{N_{\phi_\infty}(\mathcal{O}^n, T)} - \int \phi_M \, d\nu \right| &= \lim_{M \rightarrow \infty} \left| \lim_{T \rightarrow \infty} \frac{N_{\phi_\infty \cdot (\hat{\phi} - \phi_M)}(\mathcal{O}^n, T)}{N_{\phi_\infty}(\mathcal{O}^n, T)} \right| \\ &\leq \lim_{M \rightarrow \infty} \lim_{T \rightarrow \infty} \left| \frac{N_{\phi_\infty \cdot (\hat{\phi} - \phi_M)}(\mathcal{O}^n, T)}{N_{\phi_\infty}(\mathcal{O}^n, T)} \right| \\ &\leq \lim_{M \rightarrow \infty} \lim_{T \rightarrow \infty} \frac{N_{|\phi_\infty \cdot (\hat{\phi} - \phi_M)|}(\mathcal{O}^n, T)}{|N_{\phi_\infty}(\mathcal{O}^n, T)|} = 0. \end{aligned}$$

The last equality follows from conditions 2 and 4. Thus, $\lim_{M \rightarrow \infty} \int \phi_M \, d\nu$ exists and is equal to $\hat{\phi}$, so $\hat{\phi}$ again satisfies Equation (4.2).

Remark 4.2.5. Condition 4 is called a *uniformity estimate* and is often difficult to verify directly. One way is to use the geometric sieve in combination with the reduction sieve. This is described in more detail in [8].

Remark 4.2.6. Without Condition 4, if we assume that ϕ_∞ and ϕ_v are always non-negative, then

$$\limsup_{T \rightarrow \infty} \frac{N_{\phi_\infty \cdot \hat{\phi}}(\mathcal{O}^n, T)}{N_{\phi_\infty}(\mathcal{O}^n, T)} \leq \int_{\hat{\mathcal{O}}^n} \hat{\phi} \, d\nu,$$

provided the product on the right exists. This follows from the proof of [6, Theorem 2.21].

4.3 Reduction from Global Masses to Integral Masses

4.3.1 Naive Height, Single Mass

Let m be a strongly definable global mass function.

Recall that for each element $\mathbf{x}$ of $K^n \setminus 0$ we can associate an element γ of the finite ideles, unique up to an element of $\prod_{v \nmid \infty} \mathcal{O}_v^\times$. Thus, for an element $[\mathbf{x}] \in \mathbb{P}_K^a$, there is a canonical element $[\gamma]$ of the class group of K . This is the generalized Weierstrass class of $[\mathbf{x}]$. Let $\mathbb{P}^\gamma$ denote the subset of $\mathbb{P}_K^a$ with generalized Weierstrass class equal to γ .

By Proposition 2.4.3, elements of $\mathbb{P}^\gamma$ are in one-to-one correspondence with $\mathcal{O}^\times$ -orbits of elements of S_γ , on which $H^{\text{fin}} = (N\gamma)^{-1}$. Thus,

$$\sum_{[\mathbf{x}] \in \mathbb{P}^\gamma, H^\#([\mathbf{x}]) < T} m([\mathbf{x}]) = N_m(S_\gamma, T \cdot N\gamma).$$

Now, m_{fin} is locally constant away from 0. In particular, m_{fin} is locally constant on S_γ . Thus,

$$\begin{aligned} N_m(S_\gamma, T \cdot N\gamma) &= N_{m_\infty}(\mathcal{O}^n, T \cdot N\gamma) \int_{\widehat{S}_\gamma} m_{\text{fin}} \, d\nu + O(T^{|\mathbf{a}| - a_{\min}/d}) \\ &= (T \cdot N\gamma)^{|\mathbf{a}|} \cdot \frac{1}{D_K^{n/2}} \int_{\mathcal{R}(1)} m_\infty(\mathbf{x}) \, d\mathbf{x} \cdot \frac{1}{(N\gamma)^{|\mathbf{a}|}} \int_{\widehat{S}} m_{\text{fin}} \, d\nu + O(T^{|\mathbf{a}| - a_{\min}/d}) \\ &= T^{|\mathbf{a}|} \zeta_K(|\mathbf{a}|)^{-1} \frac{1}{D_K^{n/2}} \int_{\mathcal{R}(1)} m_\infty(\mathbf{x}) \, d\mathbf{x} \int_{\widehat{\mathcal{O}}^n} m_{\text{fin}} \, d\nu + O(T^{|\mathbf{a}| - a_{\min}/d}). \end{aligned}$$

The last equality follows from the fact that m_{fin} is $\widehat{\mathcal{O}}^\times$ -invariant.

4.3.2 Singular Quasi-toric Heights, No Mass

First we treat the case where $H_{\text{fin}} = H_{\text{fin}}^\#$. Let $\phi_X : K_\infty^n \rightarrow \{0, 1\}$ be the indicator function for the set $\{\mathbf{x} \in K_\infty^n : H_\infty(\mathbf{x}) < X\}$. Then for any fixed finite idele γ ,

$$\#\{[\mathbf{x}] \in \mathbb{P}^\gamma, H^\#([\mathbf{x}]) < T, H([\mathbf{x}]) < X\} = N_{\phi_X \cdot N\gamma}(S_\gamma, T \cdot N\gamma).$$

Applying Equation (4.1) with $\widehat{\phi} = 1$,

$$N_{\phi_X \cdot N\gamma}(S_\gamma, T \cdot N\gamma) = \frac{1}{D_K^{n/2}} \int_{\mathcal{R}(T \cdot N\gamma)} \phi_{X \cdot N\gamma}(\mathbf{x}) \, d\mathbf{x} \frac{1}{(N\gamma)^{|\mathbf{a}|} \zeta_K(|\mathbf{a}|)} + O(T^{|\mathbf{a}| - a_{\min}/d}).$$

The integral is equal to

$$\begin{aligned} & \text{Vol}\{\mathbf{x} \in R : H_\infty^\#(\mathbf{x}) N\gamma \cdot T \wedge H_\infty(\mathbf{x}) < N\gamma \cdot X\} \\ &= (N\gamma \cdot X)^{|\mathbf{a}|} \text{Vol}\{\mathbf{x} \in R : H_\infty^\#(\mathbf{x}) < T/X \wedge H_\infty(\mathbf{x}) < 1\} \\ &= (N\gamma \cdot X)^{|\mathbf{a}|} R |\mathbf{a}|^{r-1} (\Sigma_{H_\infty} - \varepsilon_\infty(T/X)). \end{aligned}$$

Thus,

$$\begin{aligned} N_{\phi_X \cdot N\gamma}(S_\gamma, T \cdot N\gamma) &= \frac{1}{h} c_{K,\mathbf{a}}^\# \sigma_H X^{|\mathbf{a}|} + O(T^{|\mathbf{a}| - a_{\min}/d} + X^{|\mathbf{a}|} \varepsilon_\infty(T/X)), \\ \#\{[\mathbf{x}] \in \mathbb{P}^\gamma, H^\#([\mathbf{x}]) < T, H([\mathbf{x}]) < X\} &= c_{K,\mathbf{a}}^\# \sigma_H X^{|\mathbf{a}|} + O(T^{|\mathbf{a}| - a_{\min}/d} + X^{|\mathbf{a}|} \varepsilon_\infty(T/X)). \end{aligned}$$

The analogous equation for m_{fin} -weighted orbits follows similarly; we obtain that for a locally constant mass m_{fin} ,

$$\sum_{\substack{[\mathbf{x}] \in \mathbb{P}_{\mathbf{a}}^K \\ H^\#([\mathbf{x}]) < T, H([\mathbf{x}]) < X}} m_{\text{fin}}([\mathbf{x}]) = c_{K,\mathbf{a}}^\# \sigma_H X^{|\mathbf{a}|} \int_{\widehat{\mathcal{O}}^n} m_{\text{fin}} \, d\nu + O(T^{|\mathbf{a}| - a_{\min}/d} + X^{|\mathbf{a}|} \varepsilon_\infty(T/X)).$$

Now we consider a general strongly definable quasi-toric height. By Remark 3.4.4 and Remark 3.4.5, the function $H_{\text{fin}}/H_{\text{fin}}^\#$ takes on finitely many values $\lambda_1, \dots, \lambda_j$, and for each j , the indicator function χ_{λ_j} for $H_{\text{fin}} = \lambda_j H_{\text{fin}}^\#$ (in $\widehat{\mathcal{O}}^n$) is a strongly definable global mass.

Let

$$\mathbb{P}_\lambda^\gamma = \{[\mathbf{x}] \in \mathbb{P}^\gamma : H_{\text{fin}} = \lambda H_{\text{fin}}^\#\}.$$

Then

$$\begin{aligned}
& \#\{[\mathbf{x}] \in \mathbb{P}_\lambda^\gamma, H^\#([\mathbf{x}]) < T, H([\mathbf{x}]) < X\} \\
&= N_{\phi_{X \cdot N\gamma/\lambda} \cdot \chi_\lambda}(S_\gamma, T \cdot N\gamma) \\
&= \frac{1}{h} c_{K, \mathbf{a}}^\# \sigma_{H^\infty} \left(\frac{X}{\lambda} \right)^{|\mathbf{a}|} \int_{\hat{\mathcal{O}}^n} \chi_\lambda \, d\nu + O(T^{|\mathbf{a}| - a_{\min}/d} + \left(\frac{X}{\lambda} \right)^{|\mathbf{a}|} \varepsilon_\infty(T\lambda/X)).
\end{aligned}$$

Summing over all possible values of λ , we have

$$\begin{aligned}
& \#\{[\mathbf{x}] \in \mathbb{P}^\gamma, H^\#([\mathbf{x}]) < T, H([\mathbf{x}]) < X\} \\
&= \frac{1}{h} c_{K, \mathbf{a}}^\# \sigma_{H^\infty} X^{|\mathbf{a}|} \int_{\hat{\mathcal{O}}^n} \left(\frac{H_{\text{fin}}}{H_{\text{fin}}^\#} \right)^{-|\mathbf{a}|} d\nu + O(T^{|\mathbf{a}| - a_{\min}/d} + X^{|\mathbf{a}|} \varepsilon_\infty(T\lambda_{\min}/X)).
\end{aligned}$$

Summing over the general Weierstrass class $\gamma \in \text{Cl}(K)$ gives Theorem 4.1.3.

Now we add in a strongly definable mass m . Using the same logic,

$$\begin{aligned}
\sum_{\substack{[\mathbf{x}] \in \mathbb{P}^\gamma \\ H^\#([\mathbf{x}]) < T \\ H([\mathbf{x}]) < X}} m([\mathbf{x}]) &= \frac{1}{h} c_{K, \mathbf{a}}^\# \left(\prod_{v|\infty} \int_{B_v(1)} m_v \right) X^{|\mathbf{a}|} \int_{\hat{\mathcal{O}}^n} \left(m_{\text{fin}} \frac{H_{\text{fin}}}{H_{\text{fin}}^\#} \right)^{-|\mathbf{a}|} d\nu \\
&\quad + O(T^{|\mathbf{a}| - a_{\min}/d} + X^{|\mathbf{a}|} \varepsilon_\infty(T\lambda_{\min}/X)).
\end{aligned}$$

Remark 4.3.1. None of the main terms in this section depend on γ .

4.4 Equidistribution of (Quasi-)Scaling Ideals

Assume that X grows faster than $T^{1-a_{\min}/d|\mathbf{a}|}$ but slower than T . One immediate consequence of the above proof is that the generalized Weierstrass class γ is equidistributed.

In fact, we can say something more general. There is an analog of the relationship between the toric height and quasi-toric heights to scaling ideals and quasi-scaling ideals.

Proposition 4.4.1. *Suppose $I_v : K_v^n \setminus 0 \rightarrow K_v^\times / \mathcal{O}_v^\times$ satisfies*

1. For all $v \nmid \infty$, I_v is locally constant, and

$$I_v(\alpha \cdot \mathbf{x}) = (\alpha) \cdot I_v(\mathbf{x})$$

for all $\alpha \in K_v^\times$, $\mathbf{x} \in K_v^n \setminus 0$.

2. For all but finitely many $v \nmid \infty$, $I_v(\mathbf{x}) \cdot \mathbf{x} \in S_v$.

One can associate to $(I_v)_{v \in M_K^0}$ the map $I : K^n \setminus 0 \rightarrow \mathbb{A}_{K, \text{fin}}^\times / \prod_{v \nmid \infty} \mathcal{O}_v^\times$ given by

$$I(\mathbf{x}) = \prod_v I_v(\mathbf{x}).$$

For a class $c \in \text{Cl}(K)$, define

$$\mathcal{N}_{\mathcal{I}}(c, X, T) = \{[\mathbf{x}] \in \mathbb{P}_K^{\mathbf{a}} : \mathcal{I}(\mathbf{x}) \in c, H^\#(\mathbf{x}) < T, H(\mathbf{x}) < X\}.$$

Then

$$\mathcal{N}_{\mathcal{I}}(c, X, T) = \frac{1}{h} \#\{[\mathbf{x}] \in \mathbb{P}_K^{\mathbf{a}} : H^\#([\mathbf{x}]) < T, H([\mathbf{x}]) < X\} + O(T^{|\mathbf{a}| - a_{\min}/d} + X^{|\mathbf{a}|} \varepsilon_\infty(T \lambda_{\min}/X)).$$

Definition 4.4.2. A function $\mathcal{I}$ satisfying the conditions of Proposition 4.4.1 is called a *quasi-scaling ideal* (for K and $\mathbf{a}$).

Remark 4.4.3. For every $[\mathbf{x}] \in \mathbb{P}(\mathbf{a})(K)$, the class of $\mathcal{I}(\mathbf{x})$ in $\text{Cl}(K)$ does not depend on the choice of $\mathbf{x} \in [\mathbf{x}]$. This proposition states that the class of $\mathcal{I}(\mathbf{x})$ is equidistributed in the class group of K .

Proof. Note that for each $c \in \text{Cl}(K)$, the indicator function for $\gamma^{-1}\mathcal{I} = c$ is a locally constant global mass. Thus, by the previous section, if we restrict to the points where $\gamma^{-1}\mathcal{I} = c$, then γ is equidistributed in $\text{Cl}(K)$. Summing over all c , we deduce that $\mathcal{I}$ must also be equidistributed in $\mathcal{I}$. $\square$

Chapter 5

Average Sizes of Selmer Groups of Elliptic Curves over $\mathbb{Q}$ Ordered by a Singular Height

5.1 Introduction

In this chapter we will prove the following theorem:

Theorem 5.1.1. *Let $H : \mathbb{P}(4, 6)(\mathbb{Q}) - Z(\Delta) \rightarrow \mathbb{R}_{>0}$ be any subpolynomially suitable strongly definable quasi-toric height. For $n = 2, 3, 4, 5$, when all elliptic curves $E/\mathbb{Q}$ are ordered by H , the average size of the n -Selmer group $S_2(E)$ is $\sigma(n)$.*

We prove in Chapter 7 that the Faltings height is a strongly definable quasi-toric height. Just like in the previous cases $([6, 7, 4, 5])$, we prove that the same average holds if we restrict to a *large* family defined by congruence conditions.

Note that by Lemma 2.4.6, it suffices to show the following generalization:

Theorem 5.1.2. *Let $H^\#(E) = \max(|I|^3, |J|^2/4)$ denote the naive height. For $n = 2, 3, 4, 5$, there exists an absolute constant $\delta > 0$ such that the following statement holds:*

Let $H : \mathbb{P}(4, 6) \rightarrow \mathbb{R}_{>0}$ be any (possibly singular) strongly definable quasi-toric height. When all elliptic curves $E/\mathbb{Q}$ are ordered by $\max(H(E), H^\#(E)^{1+\delta})$, the average size of the n -Selmer group $S_2(E)$ is $\sigma(n)$.

We also prove that the analogous generalization to large subfamilies holds.

5.2 Outline of Proof

Most of the proofs in [6, 7, 4, 5] continue to hold for any height function. We provide a summary of the methods used there and indicate the sections that depend on the height function.

We first prove Theorem 5.1.2 in the case where $H = H^\infty \cdot H_{\text{fin}}^\#$.

For each $n = 2, 3, 4, 5$, there exists a representation (G, V) such that n -Selmer elements of the curve $E : y^2 = x^3 - \frac{I}{3}x - \frac{J}{27}$ are in bijection with locally soluble $G_{\mathbb{Q}}$ -orbits of $V_{\mathbb{Q}}$.

This is then translated to a problem of counting $G_{\mathbb{Z}}$ -orbits of points in $V_{\mathbb{Z}}$, weighted by a $G_{\mathbb{Z}}$ -invariant mass function m on $V_{\mathbb{Z}}$. Here the mass m is nonzero if and only if its $G_{\mathbb{Q}}$ orbit is locally soluble.

To do this, we first count all *irreducible* $G_{\mathbb{Z}}$ -orbits of $V_{\mathbb{Z}}$ as follows:

We give a fundamental domain R for the action of $G_{\mathbb{R}}$ on $V_{\mathbb{R}}$, where every pair of invariants (I, J) is represented a finite number of times in R .

We construct a fundamental domain $\mathcal{F}$ for the action of $G_{\mathbb{Z}}$ in $G_{\mathbb{R}}$ and divide it into a main body $\mathcal{F}^{\text{main}}$ and a cusp $\mathcal{F}^{\text{cusp}}$ depending *only* on T such that the following statements hold:

1. The number of points in $\mathcal{F}^{\text{main}} \cdot R(X, T)$ is equal to the volume of the multiset $\mathcal{F}^{\text{main}} \cdot R(X, T)$ plus a small error term. This is accomplished using an averaging argument over a compact ball inside $G_{\mathbb{R}}$, and then using the generalized principle of Lipschitz due to Barroero-Widmer.

2. The number of irreducible points in $\mathcal{F}^{\text{cusp}} \cdot R(X, T)$ is small. This follows directly from the existing bounds from counting Selmer elements with bounded naive height.
3. The volume of the multiset $\mathcal{F} \cdot R(X, T)$ is equal to a constant times $\text{inv}(R(X, T))$.
4. The number of reducible points and points with large stabilizer in $\mathcal{F}^{\text{main}} \cdot R(X, T)$ is small.

It then follows that the number of irreducible $G_{\mathbb{Z}}$ -orbits of $V_{\mathbb{Z}}$ with height less than X and naive height less than T has the desired main term, as long as T is asymptotically not too much larger than X .

Now the problem of counting $G_{\mathbb{Z}}$ -orbits in $V_{\mathbb{Z}}$ weighted by m is accomplished using a squarefree sieve. Central to this sieve is a uniformity estimate using the geometric sieve [3].

In order to generalize the count to those of any arbitrary *large* family, a corresponding uniformity estimate is needed in the space of elliptic curves.

The subsequent mass calculations then follow exactly as in [6, 7, 4, 5].

Finally, we deduce the general case of a strongly definable height H from the case where $H_{\text{fin}} = H_{\text{fin}}^{\#}$. This follows from the logic used in Section 4.3.

5.3 Parameterization of Selmer Elements

n	V	G	$d = \dim(V)$
2	$\wedge^4(2)$	PGL_2	5
3	$\wedge^3(3)$	PGL_3	10
4	$\text{Sym}^2(4) \otimes 2$	$\{(g_1, g_2) \in \text{GL}_4 \times \text{GL}_2 : \det(g_1)^2 \det(g_2) = 1\} / \{(\lambda, \lambda^{-2})\}$	20
5	$\wedge^2(5) \otimes 5$	$\{(g_1, g_2) \in \text{GL}_5 \times \text{GL}_5 : \det(g_1)^2 \det(g_2) = 1\} / \{(\lambda, \lambda^{-2})\}$	50

Table 5.1: Representations parameterizing n -Selmer elements

Let G, V be one of the four representations in Table 5.1. The ring of invariants for the action of G on V is generated freely by two polynomials I and J , where $\deg(I) = \frac{2}{5}d$ and $\deg(J) = \frac{3}{5}d$. For each of the representations (G, V) , there is a choice of I, J such that the following parameterization of n -Selmer elements holds:

Theorem 5.3.1. *Let $E : y^3 = x^3 - \frac{I}{3}x - \frac{J}{27}$ be an elliptic curve over K . Then the elements in the n -Selmer group of E are in bijective correspondence with G_K -equivalence classes on the set of locally soluble elements in V_K having invariants equal to $I(E)$ and $J(E)$.*

Let V^{stab} be the locus of V where the discriminant is nonzero.

We say that an element of $V_{\mathbb{Z}}$ is *irreducible* if it has nonzero discriminant and the corresponding element of the Selmer group of E has order exactly equal to n . Let $V_{\mathbb{Z}}^{\text{irr}}$ be the set of irreducible elements in $V_{\mathbb{Z}}^{\text{stab}}$.

The problem of counting (locally sifted) $G_{\mathbb{Q}}$ -orbits in $V_{\mathbb{Q}}$ can be translated to the problem of counting weighted $G_{\mathbb{Z}}$ -orbits in $V_{\mathbb{Z}}$ as follows: There exists a positive integer M such that for each $v \in V_{\mathbb{Q}}$ having integer invariants $I(v)$ and $J(v)$, there exists an integral representative $v' \in V_{\mathbb{Z}}$ such that

$$I(v') = M^4 I(v) \text{ and } J(v') = M^6 J(v).$$

To count irreducible Selmer elements, it remains to count $G_{\mathbb{Z}}$ -orbits of elements in $V_{\mathbb{Z}}^{\text{irr}}$ with bounded invariants I and J , weighting each $G_{\mathbb{Z}}$ -orbit by the reciprocal of the number of $G_{\mathbb{Z}}$ -orbits contained in the single $G_{\mathbb{Q}}$ orbit.

5.4 Reduction Theory

In order to count $G_{\mathbb{Z}}$ -orbits in $V_{\mathbb{Z}}$, one constructs a finite cover of a fundamental domain for the action of $G_{\mathbb{Z}}$ in $V_{\mathbb{R}}$ and then counts lattice points in this fundamental domain using an averaging argument and a principle of Lipschitz.

Let $V^{(1)}, \dots, V^{(k)}$ be the connected components of $V_{\mathbb{R}}^{\text{stab}}$. For each i , the function Δ is either always positive or always negative; let $\text{sign}(V^{(i)})$ denote this sign.

For each of the representations in Table 5.1, a fundamental domain $\mathcal{F}$ for the left action of $G_{\mathbb{Z}}$ on $G_{\mathbb{R}}$ can be given as follows:

$$\mathcal{F} = \{nak : n \in N'(a), a \in A, k \in K\},$$

where K is a maximal compact subgroup of $G_{\mathbb{R}}$, A is parameterized by coordinates $s_1, \dots, s_m$ where all the s_j are bounded below by an absolute constant c , and $N'(a)$ is a subset of a fixed compact subset of the unipotent elements of $G_{\mathbb{R}}$.

For each i , there exists an algebraically defined fundamental region $R^{(i)}$ for the action of $G_{\mathbb{R}}$ on $V^{(i)}$. This fundamental region has the property such that for all I, J with $\text{sign}(\Delta(I, J)) = \text{sign}(V^{(i)})$, there exists a unique $f \in R^{(i)}$ such that $I(f) = I$ and $J(f) = J$. Furthermore, for each i , the quantity $\frac{\#\text{Stab}_{G_{\mathbb{R}}}(f)}{\#\text{Stab}_{G_{\mathbb{Z}}}(f)}$ is equal to a constant n_i for almost all $f \in V^{(i)}$.

We direct the reader to each of the sources [6, 7, 4, 5] for more details.

5.5 Counting Irreducible Elements with Bounded Height(s)

Let $H_{\infty} : \mathbb{R}^2 \rightarrow \mathbb{R}_{>0}$ be the infinite component of H . By assumption, H_{∞} is a definable $(4, 6)$ -homogeneous function on $\mathbb{R}^2$ and

$$R(1) = \{(I, J) \in \mathbb{R}^2 : H_{\infty}(I, J) < 1\}$$

is definable and has finite volume. Again, we do not assume for the time being that H_{∞} has a subpolynomial singularity at Δ .

Precomposing the invariant polynomials $I, J : V_{\mathbb{R}} \rightarrow \mathbb{R}$ gives a continuous height $H_{\infty} : V_{\mathbb{R}} \rightarrow \mathbb{R}_{>0}$.

Let $H^{\#}$ denote the naive height, that is the one coming from $H^{\#} : \mathbb{R}^2 \rightarrow \mathbb{R}$ given by $H^{\#}(I, J) = \max(|I|^3, J^2/4)$. In this section we describe the changes that need to be made from the case of the naive height to the more general case of a truncated singular height.

Definition 5.5.1. For a $G_{\mathbb{Z}}$ invariant set $S \subset V_{\mathbb{Z}}$, let

1. $N(S; X)$ denote the number of $G_{\mathbb{Z}}$ -equivalence classes of irreducible elements $f \in S$ satisfying $H_{\infty}(f) < X$,

2. $N^\bullet(S; X)$ denote the number of $G_{\mathbb{Z}}$ -equivalence classes of all elements $f \in S$ satisfying $H_\infty(f) < X$.

It is not a priori true that $N(V_{\mathbb{Z}}; X)$ is finite for all X . However, if we also impose a constraint on the naive height $H^\#$, then the results of [6, 7, 4, 5] guarantee finiteness (among other things)!

Definition 5.5.2. For a $G_{\mathbb{Z}}$ invariant set $S \subset V_{\mathbb{Z}}$, let

1. $N(S; X, T)$ denote the number of $G_{\mathbb{Z}}$ -equivalence classes of irreducible elements $f \in S$ satisfying $H_\infty(f) < X$ and $H_\infty^\#(f) < T$,
2. $N^\bullet(S; X, T)$ denote the number of $G_{\mathbb{Z}}$ -equivalence classes of all elements $f \in S$ satisfying $H_\infty(f) < X$ and $H_\infty^\#(f) < T$.

Let G_0 be a fixed compact K -invariant subgroup of $G_{\mathbb{R}}$. Just as in [6], for any $G_{\mathbb{Z}}$ -invariant set $S \subset V_{\mathbb{Z}}$, we have

$$N(S; X, T) = \frac{\int_{na \in \mathcal{F}} \#\{f \in naG_0R \cap S^{\text{irr}} : H_\infty(f) < X, H_\infty^\#(f) < T\} \, dnd^\times a}{n_i \cdot C_{G_0}} \quad (5.1)$$

where $C_{G_0} = \int_{h \in G_0} dh$. Thus we can extend our definitions of N and $N^\bullet$ to sets that are not $G_{\mathbb{Z}}$ -invariant. In each of the representations in Table 5.1, there is a minimal coordinate a ; points in $V_{\mathbb{Z}}$ with $a = 0$ are in the *cusp*; points with $a \neq 0$ are in the *main body*.

Proposition 5.5.3. *There exists a real number $\delta > 0$ such that*

$$N(V_{\mathbb{Z}} \cap \{a = 0\}; X, T) = O(T^{5/6-\delta'}).$$

Proposition 5.5.3 is proven in each of the representations in Table 5.1 without the dependence on H or X . (In fact, for the 2-Selmer representation, the left-hand side is zero.)

Let

$$B(n, a; X)^{(i)} = \{f \in naG_0R \cap V_{\mathbb{R}}^{(i)} : H_{\infty}(f) < X\}, \quad (5.2)$$

$$B^{\#}(n, a; T)^{(i)} = \{f \in naG_0R \cap V_{\mathbb{R}}^{(i)} : H^{\#}(f) < T\}, \quad (5.3)$$

$$B(n, a; X, T)^{(i)} = B(n, a; X)^{(i)} \cap B^{\#}(n, a; T)^{(i)}. \quad (5.4)$$

Proposition 5.5.4. *There exists a $\delta' > 0$ such that*

$$N^{\bullet}(V_{\mathbb{Z}}^{(i)} \cap \{a \neq 0\}; X, T) = \frac{1}{n_i} \text{Vol}(\mathcal{F} \cdot R^{(i)}(X, T)) + O(T^{5/6-\delta'}). \quad (5.5)$$

By Proposition 5.5.3 and the point counting principle of Barroero-Widmer (Theorem 3.2.1), we have

$$\int_{na \in \mathcal{F}} \#(B(n, a; X, T) \cap V_{\mathbb{Z}}^{\text{irr}}) \, dnd^{\times}a \quad (5.6)$$

$$= \int_{\substack{na \in \mathcal{F} \\ T^{1/d}w(a_{\min}) \gg 1}} \#(B(n, a; X, T) \cap V_{\mathbb{Z}}^{\text{irr}}) \, dnd^{\times}a + O(T^{5/6-\delta}) \quad (5.7)$$

$$= \int_{\substack{na \in \mathcal{F} \\ T^{1/d}w(a_{\min}) \gg 1}} [\text{Vol}(B(n, a; X, T)) + O(T^{5/6-1/d}/w(a_{\min}))] \, dnd^{\times}a + O(T^{5/6-\delta}) \quad (5.8)$$

$$\begin{aligned} &= \int_{na \in \mathcal{F}} \text{Vol}(B(n, a; X, T)) + \int_{\substack{na \in \mathcal{F} \\ T^{1/d}w(a_{\min}) \gg 1}} O(T^{5/6-1/d}/w(a_{\min})) \, dnd^{\times}a \\ &\quad - \int_{\substack{na \in \mathcal{F} \\ T^{1/d}w(a_{\min}) \ll 1}} \text{Vol}(B(n, a; X, T)) \, dnd^{\times}a + O(T^{5/6-\delta}). \end{aligned} \quad (5.9)$$

Now $\text{Vol}(B(n, a; X, T)) \leq \text{Vol}(B^{\#}(n, a; T)) = O(T^{5/6})$. In addition, there exist $\delta_2, \delta_3 > 0$ such that

$$\int_{\substack{na \in \mathcal{F} \\ T^{1/d}w(a) \gg 1}} O(T^{5/6-1/d}/w(a_{\min})) \, dnd^{\times}a = O(T^{5/6-\delta_2}), \quad (5.10)$$

$$\int_{\substack{na \in \mathcal{F} \\ T^{1/d}w(a) \ll 1}} dnd^{\times}a = O(T^{-\delta_3}). \quad (5.11)$$

Finally,

$$\int_{na \in \mathcal{F}} \text{Vol}(B(n, a; X, T)) \, dnd^\times a = \int_{h \in G_0} \text{Vol}(\mathcal{F}h \cdot R^{(i)}(X, T)) \, dnd^\times a, \quad (5.12)$$

and the integrand does not depend on h . Thus, there exists a δ' such that

$$N^\bullet(V_{\mathbb{Z}}^{(i)} \cap \{a \neq 0\}; X, T) = \frac{1}{n_i} \text{Vol}(\mathcal{F} \cdot R^{(i)}(X, T)) + O(T^{5/6-\delta'}). \quad (5.13)$$

For the rest of the chapter, we assume that the parameter T is chosen so that $T^{5/6-\delta} = o(X^{5/6})$ and $X = o(T)$. Then $\text{Vol } R(X, T) = \text{Vol } R(X) + o(X^{5/6})$, and so we have the following:

Corollary 5.5.5. *There exists $\delta_0 > 0$ not depending on the height H such that*

$$N^\bullet(V_{\mathbb{Z}}^{(i)} \cap \{a \neq 0\}; X, X^{1+\delta_0}) = \frac{1}{n_i} \text{Vol}(\mathcal{F} \cdot R^{(i)}(1)) \cdot X^{5/6} + o(X^{5/6}).$$

5.6 Congruence Conditions and Main Body Errors

Note that if we replace $V_{\mathbb{Z}}$ in Proposition 5.5.4 with any $V_{\mathbb{Z}}$ -translate of $v_0 + mV_{\mathbb{Z}}$ for some fixed integer m , we get

$$N^\bullet(V_{\mathbb{Z}}^{(i)} \cap v_0 + mV_{\mathbb{Z}} \cap \{a \neq 0\}; X, T) = \frac{1}{m^d} \frac{1}{n_i} \text{Vol}(\mathcal{F} \cdot R^{(i)}(X, T)) + o(X^{5/6}), \quad (5.14)$$

where the constant of proportionality in the error term depends only on m . Thus, we have the following proposition:

Proposition 5.6.1. *Suppose S is a subset of $V_{\mathbb{Z}}$ defined by finitely many congruence conditions. Then we have*

$$N^\bullet(S \cap V_{\mathbb{Z}} \cap \{a \neq 0\}; X, T) = N^\bullet(V_{\mathbb{Z}} \cap \{a \neq 0\}; X, T) \prod_p \mu_p(S) + o(X^{5/6}),$$

where $\mu_p(S)$ denotes the p -adic density of S in $V_{\mathbb{Z}}$.

In a manner identical to the proofs of [5, Proposition 22, Lemma 30], we have the following propositions:

Proposition 5.6.2.

$$N^{\bullet}(V_{\mathbb{Z}}^{\text{red}} \cap \{a \neq 0\}; X, T) = o(X^{5/6}).$$

Proposition 5.6.3.

$$N(V_{\mathbb{Z}}^{\text{bigstab}} \cap \{a \neq 0\}; X, T) = o(X^{5/6}).$$

Central to the proofs is the fact that the p -adic density of points that are irreducible is $\Omega(1/p)$. Likewise, the p -adic density of points that have nontrivial $G_{\mathbb{Z}}$ -stabilizer is also $\Omega(1/p)$. Thus, we have

Theorem 5.6.4. *There exists $\delta_0 > 0$ (not depending on the height H) such that if T grows faster than X but slower than $X^{1+\delta_0}$, then*

$$N(V_{\mathbb{Z}}^{(i)}; X, T) = \frac{1}{n_i} \text{Vol}(\mathcal{F} \cdot R^{(i)}(1)) \cdot X^{5/6} + o(X^{5/6}).$$

Finally, we have the following weighted version of Proposition 5.6.1:

Proposition 5.6.5. *Let P be a finite set of primes. For $p \in P$, let $\phi_p : V_{\mathbb{Z}_p} \rightarrow \mathbb{C}$ be a $G_{\mathbb{Z}_p}$ -invariant function on $V_{\mathbb{Z}_p}$ such that $\phi_p(f)$ depends only on the congruence class of v modulo some bounded power of p .*

For a $G_{\mathbb{Z}}$ -invariant set S , let $N_{\phi}(S; X, T)$ denote the number of irreducible $G_{\mathbb{Z}}$ -orbits in S satisfying $H_{\infty}(f) < X$ and $H^{\#}(f) < T$ where each orbit $G_{\mathbb{Z}} \cdot f$ is counted with weight $\phi(f) = \prod_{p \in P} \phi_p(f)$. Then

$$N_{\phi}(V_{\mathbb{Z}}^{(i)}; X, T) = N(V_{\mathbb{Z}}^{(i)}; X, T) \prod_{p \in P} \int_{f \in V_{\mathbb{Z}_p}} \phi_p(f) df + o(X^{5/6}).$$

Here the measure df on $V_{\mathbb{Z}_p}$ is normalized so that $\int_{v \in V_{\mathbb{Z}_p}} df = 1$.

As with [6], we need to allow for congruence conditions defined modulo infinitely many primes, which means that we need a squarefree sieve with a uniformity estimate.

5.7 A Squarefree Sieve

Theorem 5.7.1. *For a prime p , let $\mathcal{W}_p$ denote the set of elements in $V_{\mathbb{Z}}$ whose discriminants are divisible by p^2 . Let $\varepsilon > 0$ be fixed. Then we have:*

$$N(\cup_{p>M} \mathcal{W}_p; X, X^{1+\delta_0}) = O_{\varepsilon}(X^{5/6}/(M \log M) + X^{5/6-1/d}) + O(\varepsilon X^{5/6}),$$

where the implied constant is independent of M and X .

We show a modified version of the Ekedahl sieve (derived in a quantitative form in [3, Theorem 3.3]):

Theorem 5.7.2. *Suppose B_1, B_2 are definable regions in $\mathbb{R}^n$ such that the volume of B_1 is finite and B_2 is precompact. Let Z be any closed subscheme of $\mathbb{A}_{\mathbb{Z}}^n$ of codimension $k \geq 2$. Let r_1, r_2, M be positive real numbers. Then*

$$\begin{aligned} & \#\{a \in r_1 B_1 \cap r_2 B_2 \cap \mathbb{Z}^n : a \pmod{p} \in Z(\mathbb{F}_p) \text{ for some prime } p > M\} \\ &= O\left(\frac{r_1^n}{M^{k-1} \log M} + \frac{r_2^{n-1}}{M^{k-2} \log M} + r_2^{n-k+1}\right). \end{aligned} \quad (5.15)$$

Proof of 5.7.2. As in the proof of [3, Theorem 3.3], we split the primes p into two cases: $p < r_2$ and $p \geq r_2$. Note it is shown there that

$$\#\{a \in r_2 B_2 \cap \mathbb{Z}^n : a \pmod{p} \in Z(\mathbb{F}_p) \text{ for some prime } p > r_2\} = O(r_2^{n-k+1}). \quad (5.16)$$

Now suppose $p \leq r_2$. We once again use the point counting principle of Barroero–Widmer

(Theorem 3.2.1). For each $v \in \mathbb{Z}^n$, we have

$$\begin{aligned} \#(r_1 B_1 \cap r_2 B_2 \cap v + (p\mathbb{Z})^n) &= \frac{1}{p^n} \text{Vol}(r_1 B_1 \cap r_2 B_2) + \sum_{k=0}^n O((r_2/p)^k) \\ &= O(r_1^n/p^n + r_2^{n-1}/p^{n-1}). \end{aligned}$$

Since $\#Z(\mathbb{F}_p) = O(p^{n-k})$, we have

$$\#\{a \in r_1 B_1 \cap r_2 B_2 \cap \mathbb{Z}^n : a \pmod{p} \in Z(\mathbb{F}_p)\} = O(r_1^n/p^k + r_2^{n-1}/p^{k-1}).$$

Summing over $p \in [M, r_2)$ gives the desired result. $\square$

Theorem 5.7.1 now follows for each of the representations in Table 5.1 in the same way that [5, Theorem 27] follows from [3, Theorem 3.3]. Note that the proof of the uniformity estimate is more complicated for the 2-Selmer case, but this proof still holds. We thus obtain the following:

Theorem 5.7.3. *Let $\phi : V_{\mathbb{Z}} \rightarrow [0, 1]$ be an acceptable function that is defined by congruence conditions via the local functions $\phi_p : V_{\mathbb{Z}_p} \rightarrow [0, 1]$. Then with notation as in Proposition 5.6.1, we have:*

$$N_{\phi}(V_{\mathbb{Z}}^{(i)}; X, T) = N(V_{\mathbb{Z}}^{(i)}; X, T) \prod_p \int_{f \in V_{\mathbb{Z}_p}} \phi_p(f) df + o(X^{5/6}).$$

Using the same volume calculations, we get Theorem 5.1.2. Like in the naive case, Theorem 5.1.2 also holds if we restrict to a large subfamily (in the sense of [6, pg. 22]) of the family of all elliptic curves. We need a slightly stronger version of this generalization for root number calculations in Chapter 8:

For F a subset of $\mathcal{O}_K^2$, let

$$\mu(F) := \lim_{T \rightarrow \infty} \frac{\#\{I, J \in F_T, H(I, J) < T\}}{\#\{I, J \in \mathcal{O}_K^2 : H(I, J) < T\}},$$

assuming the limit exists. Note that if F is large, then this limit must exist.

Theorem 5.7.4. *Suppose F is a large family and $\{F_s\}_{s \in S}$ is a countable partition of F into large families such that $\mu(F) = \sum_{s \in S} \mu(F_s)$. Then for any nonempty $T \subset S$, the set $F_T = \bigcup_{t \in T} F_t$ satisfies $\mu(F_T) = \sum_{t \in T} \mu(F_t)$ and that the average size of the n -Selmer group for elements of F_T is $\sigma(n)$.*

Proof. Note that for any finite $T_0 \subseteq S$, $\mu(F_{T_0}) = \sum_{t \in T_0} \mu(F_t)$. Thus,

$$\mu(F_T) \geq \sum_{t \in T} \mu(F_t).$$

Combining this with the analogous equation for $S \setminus T$, we get that equality must hold. The same argument holds if we replace the function μ with the function μ_{Sel} , where

$$\mu_{\text{Sel}}(F) := \lim_{T \rightarrow \infty} \frac{\sum_{I, J \in F_T, H(I, J) < T} \# \text{Sel}(E_{I, J})}{\sum_{I, J \in \mathcal{O}_K^2, H(I, J) < T} \# \text{Sel}(E_{I, J})}$$

Thus, the average size of the Selmer groups of the elliptic curves in F_T is $\sigma(n)$ for any $T \subseteq S$. □

Theorem 5.7.4 immediately implies the following:

Theorem 5.7.5. *Suppose G is a finite abelian group. Let $\tilde{H} = \max(H, (H^\#)^{1-\delta})$, and suppose $\phi : \mathcal{E} \rightarrow G$ is an acceptable function, that is, $\phi_p : \mathbb{Z}_p^2 \rightarrow G$ is continuous for all p and for all but finitely many p , $\phi_p(f) = 1$ if $p^2 \nmid \Delta(f)$, and $\phi = \prod_p \phi_p$. Suppose F is any large family, and let $F_{g_0} = \{E \in F : \phi(E) = g_0\}$. Then*

1. *The density of F_{g_0} in F is equal to*

$$\frac{\mu(F_{g_0})}{\mu(F)} = \frac{1}{|G|} \sum_{\chi \in G^*} \chi(g_0) \prod_p \frac{\int_{f \in \text{Inv}(F)} \chi(\phi(f)) df}{\int_{f \in \text{Inv}(F)} df}.$$

2. *If $\mu(F_{g_0}) \neq 0$, then when elliptic curves in F_{g_0} are ordered by $\tilde{H}$, the average size of the n -Selmer group is $\sigma(n)$.*

Indeed, this is seen by applying Theorem 5.7.4 with S being the set of finite sets of prime numbers and for such a set $s \in S$, F_s is the set of (I, J) with $\phi_p(I, J) = 1$ if and only if p is not in s .

5.8 Reduction to Quasi-Toric Heights

For a general subpolynomially suitable strongly definable height H , there exist a finite number of large families $F_1, \dots, F_k$ and real numbers $r_1, \dots, r_k$ such that for each k and each $x \in F_k$, $H_{\text{fin}}(x) = r_k \cdot H_{\text{fin}}^{\#}(x)$. Thus, if the average size of the n -Selmer group of elliptic curves E in each F_k ordered by $H_{\infty} \cdot H_{\text{fin}}^{\#}$ is $\sigma(n)$, then the average size of the n -Selmer group when ordered by H must also be $\sigma(n)$.

5.9 The Average Rank is Bounded

Note that

$$\#\text{Sel}_5(E) \geq \#(E/5E) \geq 5^{\text{rk}(E)} \geq 20 \text{rk}(E) - 15.$$

Thus, we have the following corollary:

Corollary 5.9.1. *When elliptic curves $E/\mathbb{Q}$ are ordered by a subpolynomially suitable strongly definable height H , the average rank of E is at most 1.05.*

We improve this upper bound in Chapter 8.

Chapter 6

Average Sizes of Selmer Groups of Elliptic Curves over Number Fields Ordered by a Singular Height

6.1 Introduction

In this chapter, we show the analog of Theorem 5.1.1 for number fields, much like how [8, Theorem 4] is a generalization of the results of [6, 7, 4, 5]:

Theorem 6.1.1. *Let $H : \mathbb{P}(4, 6)(K) \setminus V(\Delta) \rightarrow \mathbb{R}_{>0}$ be any strongly definable quasi-toric height subpolynomially suitable at Δ . For $n = 2, 3, 4, 5$, when all elliptic curves E/K are ordered by H , the average size of the n -Selmer group $S_2(E)$ is $\sigma(n)$.*

Theorem 6.1.1 has a more general form:

Theorem 6.1.2. *Let $H : \mathbb{P}(4, 6)(K) \rightarrow \mathbb{R}$ be a singular strongly definable quasi-toric height. For $n = 2, 3, 4, 5$, there exists a constant $\delta_{n,K} > 0$ such that when all elliptic curves E/K are ordered by $\max(H, c \cdot (H^\#)^{1-\delta_{n,K}})$, the average size of the n -Selmer group $S_2(E)$ is $\sigma(n)$.*

Indeed, Lemma 2.4.6 implies that Theorem 6.1.1 follows from Theorem 6.1.2.

6.2 Outline of Proof

We mostly follow the arguments of [27] and [8] and highlight with (*) the paragraphs that contain changes to account for general quasi-toric heights. Note that in Chapter 7 we prove that the Faltings height is a subpolynomially suitable strongly definable quasi-toric height.

Recall Theorem 5.3.1, which states that for $n = 2, 3, 4, 5$, there exists a representation (G, V) such that n -Selmer elements of an elliptic curve $E_{I,J}$ defined over K correspond to *locally soluble* G_K -orbits of V_K with invariants equal to I, J .

(*) By Chapter 4, there exist families $\mathcal{A}_1, \dots, \mathcal{A}_t \subseteq \mathcal{O}_K^2$ such that

1. For every $E \in \mathcal{E}_K$, there exists a unique choice of k and $(I, J) \in \mathcal{A}_k$ such that $E \simeq E_{I,J}$.
2. For each $\mathcal{A}_k$, H_{fin} and $H_{\text{fin}}^\#$ are constant.
3. $\mathcal{A}_k$ is defined by congruence conditions.

It suffices to show that for a set $\mathcal{A} \subseteq \mathcal{O}_K^2$ defined by local conditions, when $(I, J) \in \mathcal{A}$ are ordered by H_∞ , the average number of locally soluble G_K -orbits of V_K with invariants equal to I and J is equal to $\sigma(n)$.

Unlike the case of $K = \mathbb{Q}$, it is not generally true that every G_K -orbit of V_K with integral invariants contains an element of $V_{\mathcal{O}_K}$, but due to the finiteness of the class group of G over K , there is a result that is almost as good.

For a fixed $\beta \in G_{\mathbb{A}_{K,\text{fin}}}$, there exists a lattice $V_\beta \subseteq V_K$ commensurable with $V_{\mathcal{O}_K}$ and a corresponding subgroup $G_\beta \subseteq G_K$ commensurable with $G_{\mathcal{O}_K}$ acting on V_β . The result is that there exists a finite set $\mathcal{B} \subseteq G_{\mathbb{A}_{K,\text{fin}}}$ such that every G_K -orbit of V_K with integral invariants contains an element of V_β for some $\beta \in \mathcal{B}$.

Thus, we reduce the problem of counting G_K -orbits of V_K with invariants in $\mathcal{A}$ to counting weighted G_β -orbits of V_β with invariants in $\mathcal{A}$.

We first count unweighted G_β -orbits of V_β with invariants in $\mathcal{A}_\infty$. It suffices to construct a fundamental domain for the action of G_β on V_{K_∞} and count the number of points

of V_β in this fundamental domain with invariants in $\mathcal{A}_\infty$. In general, the Principle of Lipschitz states that the number of points of a lattice Λ inside a region $\mathcal{R}$ should be close to $\text{Vol}(\mathcal{R})/\det \Lambda$. When $\mathcal{R}$ is a precompact semi-algebraic region, this is made precise using Davenport's lemma. However, this Principle is not true in the case of the groups parameterizing Selmer elements. It is only true after filtering to orbits corresponding to the irreducible elements of the corresponding Selmer groups, as shown in [8] when $\mathcal{A}_\infty$ is the entire fundamental region for $K_\infty^2/\mathcal{O}^\times$.

(*) In this chapter, we demonstrate that this principle also holds when $\mathcal{A}_\infty$ is a sufficiently “full” subregion of $K_\infty^2/\mathcal{O}^\times$. In order to accomplish this, we need to keep better track of the error terms from the difference between the lattice point count and the region volume. Since the subregions are no longer guaranteed to be semi-algebraic, we instead use Theorem 3.2.2 to count points in regions parameterized by definable families.

The task of counting weighted orbits with invariants in $\mathcal{A}$ is accomplished with a square-free sieve.

(*) This requires more general versions of the uniformity estimates used in [8] to accommodate more general $\mathcal{A}_\infty$, which we deduce by bootstrapping the uniformity estimates in [8].

6.3 Minimization

As in the previous chapter, we reduce the weighted counts of G_K -orbits of V_K into m' -weighted counts of G' -orbits of V' , where $G' \subseteq G_{K_\infty}$ is *commensurable* with $G_{\mathcal{O}_K}$ (that is, both groups are finite index in the same group) and $V' \subseteq V_{K_\infty}$ is a G' -invariant lattice commensurable with V_{K_∞} .

For $\beta \in G_{\mathbb{A}_{K,\text{fin}}}$, let

$$V_\beta = V_K \cap \bigcap_{v \nmid \infty} \beta_v^{-1} V_{\mathcal{O}_v},$$

$$G_\beta = G_K \cap \bigcap_{v \nmid \infty} \beta_v^{-1} G_{\mathcal{O}_v} \beta_v.$$

Since $\beta_v \in G_{\mathcal{O}_v}$ for all but finitely many $v \nmid \infty$, V_β is a lattice in V_{K_∞} commensurable with $V_{\mathcal{O}_K}$ and G_β is a subgroup of G_{K_∞} commensurable with $G_{\mathcal{O}_K}$.

The main result of this section is as follows:

Proposition 6.3.1. *There exists a finite set $\mathcal{B} \subseteq G_{\mathbb{A}_{K,\text{fin}}}$ such that for any $w \in V_K$ with $\text{inv}(w) \in \kappa \cdot \mathcal{O}_K^2$, $G_K w \cap V_\beta$ is nonempty for some $\beta \in \mathcal{B}$.*

The crucial ingredient is the minimization result of Cremona, Fisher, Stoll [12, Theorem 3.4] (for the cases where $n = 2, 3, 4$) and Fisher [19, Theorem 2.1] (when $n = 5$):

Theorem 6.3.2. *Let $v \nmid \infty$ be a finite place. Suppose $w \in V_{K_v}, w_{\mathcal{O}_v} \in V_{\mathcal{O}_v}$ are K_v -soluble and $\text{inv}(w) \in \kappa \cdot \mathcal{O}_v^2$. Then there exists $g \in G_{K_v}$ such that $g \cdot w = w_{\mathcal{O}_v}$.*

It is a result of Borel [9] that

$$\text{Cl}(G, K) := \left(\prod_{v \nmid \infty} G_{\mathcal{O}_v} \right) \backslash G_{\mathbb{A}_{K,\text{fin}}} / G_K$$

is finite. Pick $\mathcal{B}$ to be any set of representatives in $G_{\mathbb{A}_{K,\text{fin}}}$ for $\text{Cl}(G, K)$.

If $w \in V_K$ is locally soluble and $\text{inv}(w) \in \text{inv}(V_{\mathcal{O}_v})$, then there exists a $\beta \in \mathcal{B}$ and a $\gamma(w) \in G_K$ such that

$$\beta_{j,v} \cdot \gamma(w) \cdot w \in V_{\mathcal{O}_v}$$

for all $v \nmid \infty$. Equivalently, w is G_K -equivalent to an element of V_β for some $\beta \in \mathcal{B}$.

6.4 Stabilizer Mass Functions

Let m be a G_K -invariant mass function on V_K defined by congruence conditions. Our goal is to count

$$N_m(V_K, G_K, T) := \sum_{\substack{G_K \cdot w \in V_K/G_K \\ H(w) < T}} \frac{m(w)}{\# \text{Stab}_{G_K}(w)}.$$

Remark 6.4.1. Note that $\text{Stab}_{G_K}(w)$ is equal to the K -rational n -torsion of the elliptic curve with invariants $I(w), J(w)$, so the denominator of the summand is an absolutely bounded positive integer.

Define the mass function m_β on V_β by

$$m_\beta(w) = \frac{m(w)}{\sum_{\beta \in \mathcal{B}} \sum_{w_\beta \in G_K v \cap V_\beta/G_\beta} \frac{\# \text{Stab}_{G_K}(w)}{\# \text{Stab}_{G_\beta}(w_\beta)}}.$$

Then

$$N_m(V_K, G_K, T) = \sum_{\beta \in \mathcal{B}} N_{m_\beta}(V_\beta, G_\beta, T).$$

For $v \nmid \infty$, let $m_v : V_{K_v} \rightarrow \mathbb{R}$ be defined as follows:

$$m_v(w) = \begin{cases} \left(\sum_{w_v \in G_{K_v} w \cap V_{\mathcal{O}_v}} \frac{\# \text{Stab}_{G_{K_v}}(w)}{\# \text{Stab}_{G_{\mathcal{O}_v}}(w)} \right)^{-1} & \text{if } w \text{ is } K_v\text{-soluble and } G_{K_v} w \cap V_{\mathcal{O}_v} \neq \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

Following Shankar [27, §4], if w is locally soluble, then

$$\sum_{\beta \in \mathcal{B}} \sum_{w_\beta \in G_K v \cap V_\beta/G_\beta} \frac{\# \text{Stab}_{G_K}(w)}{\# \text{Stab}_{G_\beta}(w_\beta)} = \prod_{v \nmid \infty} m_v(w)^{-1}.$$

In addition, it is proven in [27, §4] that if $v(\Delta(w)) < 2$, then w is K_v -soluble and the local mass is equal to 1.

6.5 Point Counting in Affine Space

Let ϕ be a G_β -invariant function on V_β . The goal of this section is to evaluate

$$N_\phi(V_\beta, G_\beta, T) = \sum_{\substack{G_\beta \cdot \mathbf{x} \in V_\beta^{\text{irr}}/G_\beta \\ H_\infty^\#(\mathbf{x}) < T}} \frac{\phi(\mathbf{x})}{\# \text{Stab}_{G_\beta}(\mathbf{x})}.$$

Let R denote a fundamental domain for the action of G_{K_∞} on V_{K_∞} , and let $R(T) = \{w \in R : H^\#(w) < T\}$.

Let $\mathcal{F}$ denote a fundamental domain for the action of G_β on G_{K_∞} . The precise constructions of R and $\mathcal{F}$ can be found in [8]. One important property of $\mathcal{F}$ is as follows:

There exists a basis of V_{K_∞} and a diagonal matrix coefficient a of G_{K_∞} such that if $h \in \mathcal{F}$ and Z is a fixed compact subset of V_{K_∞} , then there exists a real number t such that $h \cdot Z$ is contained in a coordinate-aligned box with bounded volume such that every dimension of the box is $\gg a(h)$. Further, the volume (under the Haar measure of G_{K_∞}) of $\mathcal{F}$ is finite, and the portion of $\mathcal{F}$ where $a(h) < \epsilon$ has volume at most ϵ^{t_2} .

Let G_0 be a definable pre-compact subset of G_{K_∞} of finite nonzero measure invariant under the action of a maximal compact subgroup of G_{K_∞} . As before, we have the following chain of equalities:

$$\begin{aligned} N_\phi(V_\beta, G_\beta, T) &= \sum_{\substack{G_\beta \cdot \mathbf{x} \in V_\beta^{\text{irr}}/G_\beta \\ H_\infty^\#(\mathbf{x}) < T}} \frac{\phi(\mathbf{x})}{\# \text{Stab}_{G_\beta}(\mathbf{x})} \\ &= \sum_{\mathbf{x} \in \mathcal{F} \cdot R(T) \cap V_\beta^{\text{irr}}} \frac{\phi(\mathbf{x})}{\# \text{Stab}_{G_{K_\infty}}(\mathbf{x})} \\ &= \frac{1}{\tau_\infty(G_0)} \int_{G_0} \sum_{\mathbf{x} \in \mathcal{F}g \cdot R(T) \cap V_\beta^{\text{irr}}} \frac{\phi(\mathbf{x})}{\# \text{Stab}_{G_{K_\infty}}(\mathbf{x})} dg \\ &= \frac{1}{\tau_\infty(G_0)} \int_{\mathcal{F}} \sum_{\mathbf{x} \in hG_0 \cdot R(T) \cap V_\beta^{\text{irr}}} \frac{\phi(\mathbf{x})}{\# \text{Stab}_{G_{K_\infty}}(\mathbf{x})} dh. \end{aligned}$$

First we assume that ϕ is the restriction of a bounded definable function $\phi_\infty : V_{K_\infty} \rightarrow \mathbb{R}$.

We now use the generalized Principle of Lipschitz from Theorem 3.2.2. Before we can do that, we first partition $\mathcal{F}$ into $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3$ in a way that only depends on T . Fix $\delta_0 > 0$. Let

$$\begin{aligned}\mathcal{F}_3 &= \{h \in \mathcal{F} : a(h) \cdot \left(\sup_{x \in G_0 \cdot R(T)} |x| \right) \leq 1\}, \\ \mathcal{F}_2 &= \{h \in \mathcal{F} : a(h) \cdot \left(\sup_{x \in G_0 \cdot R(T)} |x| \right) \leq X^{\delta_0}\} \setminus \mathcal{F}_1, \\ \mathcal{F}_1 &= \mathcal{F} \setminus (\mathcal{F}_2 \cup \mathcal{F}_3).\end{aligned}$$

Here $\mathcal{F}_1$ is the main body, $\mathcal{F}_2$ is the medium cusp, and $\mathcal{F}_3$ is the thin cusp. Let V^{irr} and V^{red} be the subsets of V corresponding to irreducible and reducible points in Selmer groups, respectively. We first prove that there exist positive constants $\delta_1, \delta_2, \delta_3$ such that

$$\frac{1}{\tau_\infty(G_0)} \int_{\mathcal{F}_1} \sum_{\mathbf{x} \in \langle G_0 \cdot R(T) \cap V_\beta^{\text{irr}} \rangle} \frac{\phi_\infty(\mathbf{x})}{\# \text{Stab}_{G_\beta}(\mathbf{x})} d\tau_\infty(h) = \frac{1}{\det V_\beta} \int_{\mathcal{F} \cdot R(T)} \frac{\phi_\infty(\mathbf{x})}{\# \text{Stab}_{G_\beta}(\mathbf{x})} d\mathbf{x} + O(T^{5/6-\delta_1}), \quad (6.1)$$

$$\int_{\mathcal{F}_1} \#\{hG_0 \cdot R(T) \cap V_\beta^{\text{red}}\} d\tau_\infty(h) = O(T^{5/6-\delta_2}), \quad (6.2)$$

$$\int_{\mathcal{F}_2} \#\{hG_0 \cdot R(T) \cap V_\beta\} d\tau_\infty(h) = O(T^{5/6-\delta_3}), \quad (6.3)$$

$$\int_{\mathcal{F}_3} \#\{hG_0 \cdot R(T) \cap V_\beta^{\text{irr}}\} d\tau_\infty(h) = O(T^{5/6-\delta_4}). \quad (6.4)$$

Equation (6.1) and Equation (6.3) follow from Theorem 3.2.2. Equation (6.2) follows from the Selberg Sieve and [23, Proposition 1.14]. Equation (6.4) is proven in [6, 7, 4, 5] in the case where $K = \mathbb{Q}$ and more generally in [8] using combinatorial arguments about the roots of G . Thus we deduce the following main result:

Theorem 6.5.1. *Suppose $\phi_\infty : V_{K_\infty} \rightarrow \mathbb{C}$ is bounded and definable. Then*

$$N_{\phi_\infty}(V_\beta, G_\beta, T) = \frac{1}{\det V_\beta} \int_{\mathcal{F} \cdot R(T)} \frac{m_{\beta, \infty}(\mathbf{x})}{\# \text{Stab}_{G_{K_\infty}}(\mathbf{x})} d\mathbf{x} + O(T^{5/6-\delta}).$$

6.6 Congruence Conditions

Note that the above theorem holds if we replace V_β with any translation of a fixed sublattice $\mathbf{x} + V'$.

Now we incorporate the mass function at the finite places. Let $\widehat{V}_\beta = V_\beta \otimes_{\mathcal{O}_K} \widehat{\mathcal{O}}_K$ and suppose $\widehat{\phi} : \widehat{V}_\beta \rightarrow \mathbb{C}$ is locally constant. Then there exists a sublattice Λ of V_β such that $\widehat{\phi}$ restricted to V_β is defined modulo V' . Now suppose m_{fin} is any function on V_β defined modulo a finite sublattice Λ . Then by summing over the various translations of Λ , we deduce that for any locally constant $\widehat{\phi}$,

$$\begin{aligned} N_{\phi_\infty \widehat{\phi}}(G_\beta, V_\beta, T) &= \frac{1}{\det V_\beta} \int_{\widehat{V}_\beta} \widehat{\phi} d\nu_\beta \cdot \int_{\mathcal{F} \cdot R(T)} \frac{\phi_\infty(\mathbf{x})}{\# \text{Stab}_{G_{K_\infty}}(\mathbf{x})} d\mathbf{x} + O_{\widehat{\phi}}(T^{5/6-\delta}) \\ &= N_{\phi_\infty}(G_\beta, V_\beta, T) \int_{\widehat{V}_\beta} \widehat{\phi} d\nu_\beta + O_{\widehat{\phi}}(T^{5/6-\delta}) \end{aligned}$$

where $d\nu_\beta$ is chosen so that $\nu_\beta(\widehat{V}_\beta) = 1$. Using the same logic as in Section 4.2.5, if $N_{\phi_\infty}(G_\beta, V_\beta, T)$ grows faster than $T^{5/6-\delta}$ and $\widehat{\phi}$ is continuous outside a set of measure 0, then

$$\lim_{T \rightarrow \infty} \frac{N_{\phi_\infty \widehat{\phi}}(G_\beta, V_\beta, T)}{N_{\phi_\infty}(G_\beta, V_\beta, T)} = \int_{\widehat{V}_\beta} \widehat{\phi} d\nu_\beta.$$

Again following the logic of Section 4.2.5, suppose $\widehat{\phi}$ is given as a product of local functions $\phi_v : V_{\beta,v} \rightarrow \mathbb{C}$. Suppose each ϕ_v is bounded everywhere and continuous outside a set of measure 0. Suppose we also have the following uniformity estimate:

$$\lim_{M \rightarrow \infty} \limsup_{T \rightarrow \infty} \frac{N_{|\phi_\infty|}(\bigcup_{Nv > M} W_v, G_\beta, T)}{|N_{\phi_\infty}(V_\beta, G_\beta, T)|} = 0,$$

where $W_v = \{\mathbf{x} \in V_\beta : \phi_v(\mathbf{x}) \neq 1\}$. Then

$$\lim_{T \rightarrow \infty} \frac{N_{\phi_\infty \widehat{\phi}}(V_\beta, G_\beta, T)}{N_{\phi_\infty}(V_\beta, G_\beta, T)} = \prod_v \int_{V_{\beta,v}} \phi_v d\nu_v. \quad (6.5)$$

Here we prove the following two uniformity estimates, one for the space of elliptic curves over

K and one for the space of Selmer elements over K . We first prove the following uniformity estimate for the space of Selmer elements over K :

Theorem 6.6.1. *There exists $\delta > 0$ such that the following statement holds: Let $W_{\beta,v}$ denote the set of elements w of V_β such that $v(\Delta(w)) \geq 2$. Then for any positive real number M ,*

$$N_{\phi_\infty}(\bigcup_{Nv > M} W_{\beta,v}, G_\beta, T) = O\left(T^{5/6-\delta} + \frac{1}{M \log M} N_{\phi_\infty}(G_\beta, V_\beta, T)\right). \quad (6.6)$$

Proof. Note that the condition $v(\Delta(w)) \geq 2$ can be checked modulo π_v^2 and has density $O(Nv^{-1})$. Thus, there exists some δ_1 such that

$$N_{\phi_\infty}(W_{\beta,v}, G_\beta, T) = O((Nv)^{-1} N_{\phi_\infty}(G_\beta, V_\beta, T) + T^{5/6-\delta_1}).$$

The unweighted version of Equation (6.6) is proven in [8]:

Proposition 6.6.2. *There exists some $\delta_2 > 0$ such that*

$$N(\bigcup_{Nv > M} W_{\beta,v}, G_\beta, T) = O\left(T^{5/6-\delta_2} + \frac{T^{5/6}}{M \log M}\right). \quad (6.7)$$

Take $\delta = \min(\delta_1/2, \delta_2)$. If $M > T^{\delta_1/2}$, then we are done. Otherwise,

$$\begin{aligned} N(\bigcup_{Nv > M} W_{\beta,v}, G_\beta, T) &\leq \sum_{M < Nv \leq T^{\delta_1/2}} [O((Nv)^{-1} N_{\phi_\infty}(G_\beta, V_\beta, T) + T^{5/6-\delta_1})] \\ &\quad + N(\bigcup_{|v| > T^{\delta_1/2}} W_{\beta,v}, G_\beta, T) \\ &= O(T^{5/6-\delta_1/2} + \frac{1}{M \log M} N_{\phi_\infty}(G_\beta, V_\beta, T)). \end{aligned}$$

□

As long as $N_{\phi_\infty}(V_\beta, G_\beta, T)$ grows faster than $T^{5/6-\delta}$, then any mass function $\hat{\phi}$ satisfying $\phi_v(x) = 1$ when $v(\Delta(x)) < 2$ is acceptable, and thus Equation (6.5) holds.

The proof of the uniformity estimate for elliptic curves follows similarly. Suppose $\varphi_\infty : K_\infty^2 \rightarrow \mathbb{C}$ is definable.

Theorem 6.6.3. *There exists $\delta > 0$ such that the following statement holds: Let W_v denote the set of elements f of $\mathcal{O}_K^2$ such that $v(\Delta(f)) \geq 2$. Then for any positive real number M ,*

$$N_{\varphi_\infty}(\bigcup_{Nv > M} W_v, T) = O\left(T^{5/6-\delta} + \frac{1}{M \log M} N_{\varphi_\infty}(\mathcal{O}_K^2, T)\right).$$

Proof. Theorem 6.6.3 follows from the following result of [8] in the same way that Theorem 6.6.1 follows from Proposition 6.6.2:

There exists some δ_0 such that

$$N(\bigcup_{Nv > M} W_v, T) = O(T^{5/6-\delta_0} + \frac{T^{5/6}}{M \log M}).$$

Indeed, this is proven in [8, §3-§4]. □

6.7 Mass Formulas

This section is identical to the corresponding section in [8]. These results do not depend on the mass functions, but we include them for reference.

Proposition 6.7.1. *There exists a nonzero constant $\mathcal{J} \in K^\times$ such that for any place v , any open subset R of K_v^2 , any continuous function $s : R \rightarrow V_{K_v}$ such that $\text{inv} \circ s = \text{id}_R$, and any measurable function ϕ on V_{K_v} ,*

$$\int_{G_{K_v} \cdot s(R)} \phi(\mathbf{x}) \, d\nu_v(\mathbf{x}) = |\mathcal{J}|_v \cdot \int_R \int_{G_{K_v}} \phi(g \cdot s(f)) \, d\tau_v(g) \, d\mu_v(f).$$

Now, a global section $s : \mathbb{A}^2 \rightarrow V$ over $\mathbb{Z}[1/d_n]$ for some positive integer d_n is constructed for each of the cases where $n = 2, 3, 4, 5$ in [6, 7, 4, 5]. We apply this theorem to the region $\mathcal{R}(T)$ from Chapter 4.

Suppose we have a function $\varphi = \varphi_\infty \cdot \widehat{\varphi}$, where $\varphi : K_\infty^2 \rightarrow \mathbb{C}$ and $\widehat{\varphi} : \widehat{\mathcal{O}}_K^2 \rightarrow \mathbb{C}$ are masses. Pulling back by inv , we get corresponding masses on V_{K_∞} and $V_{\widehat{\mathcal{O}}_v}$. Suppose $\phi = m_\beta \cdot (\varphi \circ \text{inv})$. Then

$$\begin{aligned} \int_{\mathcal{F} \cdot R(T)} \frac{m_{\beta, \infty}(\mathbf{x}) \varphi_\infty(\text{inv}(\mathbf{x}))}{\# \text{Stab}_{G_{K_\infty}}(\mathbf{x})} d\nu_\infty \mathbf{x} &= |\mathcal{J}|_\infty \tau_\infty(\mathcal{F}_\beta) \int_{\mathcal{R}(T)} \sum_{\mathbf{x} \in G_{K_\infty} \setminus V_{K_\infty}^f} \frac{\varphi_\infty(f)}{\# \text{Stab}_{G_{K_\infty}}(\mathbf{x})} d\mu_\infty(f) \\ &= |\mathcal{J}|_\infty \tau_\infty(\mathcal{F}_\beta) \int_{\mathcal{R}(T)} \varphi_\infty(f) \cdot \frac{\#E_f(K_\infty)/nE_f(K_\infty)}{\#E_f[n](K_\infty)} d\mu_\infty(f) \\ &= |\mathcal{J}|_\infty \tau_\infty(\mathcal{F}_\beta) \frac{1}{n^d} \int_{\mathcal{R}(T)} \varphi_\infty(f) d\mu_\infty(f), \end{aligned}$$

$$\begin{aligned} \int_{\widehat{V}_\beta} \widehat{m}_\beta \cdot (\widehat{\varphi} \circ \text{inv}) d\nu_\beta &= \widehat{\tau}(G_{\widehat{\mathcal{O}}_K}) \prod_v |\mathcal{J}|_v \int_{\widehat{\mathcal{O}}_K^2} \widehat{\varphi}(f) \left(\prod_v \sum_{x \in G_{K_v} \setminus V_{K_v}^f} \frac{1}{\# \text{Stab}_{G_{K_v}}(x)} \right) d\widehat{\mu}(f) \\ &= \widehat{\tau}(G_{\widehat{\mathcal{O}}_K}) \prod_v |\mathcal{J}|_v \int_{\widehat{\mathcal{O}}_K^2} \widehat{\varphi}(f) \prod_v \frac{\#E_f(K_v)/nE_f(K_v)}{\#E_f[n](K_v)} d\widehat{\mu}(f) \\ &= \widehat{\tau}(G_{\widehat{\mathcal{O}}_K}) \prod_v |\mathcal{J}|_v n^d \int_{\widehat{\mathcal{O}}_K^2} \widehat{\varphi} d\widehat{\mu}. \end{aligned}$$

6.8 Average Sizes of Selmer Groups

Let δ be small enough so that the uniformity estimates hold. Suppose $\varphi_\infty, \widehat{\varphi}$ satisfy the following conditions:

1. $\varphi_\infty : K_\infty^2 \rightarrow \mathbb{C}$ is definable.

2.

$$\liminf_{T \rightarrow \infty} \left| \frac{N_{\varphi_\infty}(\mathcal{O}_K^2, T)}{N_{|\varphi_\infty|}(\mathcal{O}_K^2, T)} \right| > 0, \quad \limsup_{T \rightarrow \infty} \left| \frac{T^{5/6-\delta}}{N_{\varphi_\infty}(\mathcal{O}_K^2, T)} \right| = 0.$$

3. $\widehat{\varphi}$ is acceptable (in the sense of Section 4.2.5).

4. The restriction of $\widehat{\varphi}$ to $\mathcal{O}_K^2$ is supported on the set $\kappa \cdot \mathcal{O}_K^2$.

Then we have

$$\begin{aligned}
\lim_{T \rightarrow \infty} \frac{\sum_{f \in \mathcal{O}_K^2 / \mathcal{O}_K^\times} \varphi(f) \# \text{Sel}_n^{\text{irr}}(E_f)}{\sum_{f \in \mathcal{O}_K^2 / \mathcal{O}_K^\times} \varphi(f)} &= \lim_{T \rightarrow \infty} \frac{N_\varphi(V_K, G_K, T)}{N_\varphi(\mathcal{O}_K^2, T)} \\
&= \lim_{T \rightarrow \infty} \frac{\sum_{\beta} N_{m_{\beta} \cdot (\varphi \circ \text{inv})}(V_{\beta}, G_{\beta}, T)}{N_\varphi(\mathcal{O}_K^2, T)} \\
&= \lim_{T \rightarrow \infty} \frac{\widehat{\tau}(G_{\widehat{\mathcal{O}}_K}) \sum_{\beta} \tau_{\infty}(\mathcal{F}_{\beta}) \int_{\mathcal{R}(T)} \varphi_{\infty} d\mu_{\infty} \int_{\widehat{\mathcal{O}}_K^2} \varphi d\widehat{\mu}}{\int_{\mathcal{R}(T)} \varphi_{\infty} d\mu_{\infty} \int_{\widehat{\mathcal{O}}_K^2} \varphi d\widehat{\mu}} \\
&= \text{The Tamagawa number of } G/K = n.
\end{aligned}$$

If we specialize to the case where φ_{∞} is the indicator for $H_{\infty} < X$, then as long as we pick $T \leq X^{1+\delta}$, we get the following result: When elliptic curves are ordered by $\max(H, (H^{\#})^{1+\delta})$ and weighted by φ , the average number of irreducible n -Selmer elements is n .

Using the same reduction from global masses to integral masses as Section 4.3, we get that there are finitely many subfamilies of $\mathcal{O}_K^2$ such that each isomorphism class of elliptic curves over K is represented in exactly one such subfamily. In addition, within a subfamily, the ordering of elliptic curves by height is the same under both H_{∞} and H . Since the average size of the n -Selmer group in each family is $\sigma(n)$, we now deduce Theorem 6.1.2.

Remark 6.8.1. With Remark 4.3.1 and Proposition 4.4.1 in mind, the same logic as above shows that if we restrict to elliptic curves with quasi-scaling ideal $\mathcal{I}$ in a fixed class of the class group of K , the average size of the n -Selmer group is still $\sigma(n)$.

6.9 The Average Rank is Bounded

Much like the proof at the end of Chapter 5, we get the following corollary of Theorem 6.1.1:

Corollary 6.9.1. *Suppose elliptic curves E/K are ordered by a subpolynomially suitable*

strongly definable height H . Then the average rank of E is at most 1.05.

This result again applies for any large subfamily of $\mathcal{E}^K$.

Chapter 7

Faltings Height

7.1 Introduction

In his proof of the Mordell and Shafarevich Conjectures, Faltings introduced a height on abelian varieties, now known as the Faltings Height. It is defined as follows:

Given an abelian variety A of dimension g over K , let $\mathcal{A}$ be the Néron minimal model of A over $\text{Spec } \mathcal{O}_K$. Let $\Omega_{\mathcal{A}/R}$ be the sheaf of relative differentials of $\mathcal{A}$. Then the pullback of $\Omega_{\mathcal{A}/R}$ with respect to the zero section $\text{Spec } R \rightarrow \mathcal{A}$ gives rise to a metrized line bundle on $\text{Spec } R$, where the norm at an infinite place v is given by

$$|\omega|_v = \int_{\mathcal{A}(\overline{K_v})} |\alpha \wedge \bar{\alpha}|.$$

The Faltings height $H_F(A)$ is the height of this metrized line bundle.

For a given positive integer g and positive real number X , Faltings showed that there are finitely many principally polarized abelian varieties with dimension g and Faltings Height less than X .

In the case of elliptic curves, Silverman computes a simple formula for the Faltings height in [28, p. 254]:

Theorem 7.1.1. *The Faltings height of an elliptic curve E/K is given by*

$$H_F(E) = \frac{N(\Delta^{\min}(E))}{\prod_{v|\infty} |\Delta(\tau_v) \operatorname{Im}(\tau_v)^6|_v}.$$

Here τ_v is chosen in the upper half-plane such that $E(\overline{K_v}) \simeq \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau_v)$, and

$$\Delta(\tau) = (2\pi)^{12} q \prod_{n=1}^{\infty} (1 - q^n)^{24}$$

is the modular discriminant function.

Remark 7.1.2. Different sources use different normalizations of Δ (and as a result, different normalizations of the Faltings height). Silverman uses the normalization factor $\frac{1}{(2\pi)^{12}}$. We instead use the normalization where the leading coefficient in the q -expansion of Δ is $(2\pi)^{12}$, which matches [22, 18]. As a result, Theorem 1.2.1 agrees with [22, Theorem 2.1] in the case where $K = \mathbb{Q}$.

Remark 7.1.3. Since $\Delta(\tau)$ is a modular form of weight 12, the value of the expression $|\Delta(\tau_v) \operatorname{Im}(\tau_v)^6|$ (and thus $H_F(E)$) does not depend on the choices of τ_v .

The main goal of this chapter is to show that the Faltings height satisfies the conditions of Theorem 4.1.4. Theorem 1.2.1 then follows after an explicit computation of the volume factors.

7.2 The Faltings Height is Quasi-toric

Let H_F denote the Faltings Height. The goal of this section is to prove the following theorem:

Theorem 7.2.1. $H_F^{1/12}$ is a singular quasi-toric height that is subpolynomially suitable at Δ .

We first consider the local heights at finite places. Let $f_v(A, B)$ be the extent to which the model $E_{A,B} : y^2 = x^3 + Ax + B$ is non-minimal at v . More precisely, we have the following:

Proposition 7.2.2. *For $v \nmid \infty$, define $f_v : K_v^2 \setminus Z(\Delta) \rightarrow \mathbb{R}_{>0}$ as follows:*

$$\lambda_v(A, B) = \frac{1}{12} [v(\Delta(A, B)) - v(\Delta^{\min}(E_{A,B}))], \quad f_v(A, B) = q_v^{-\lambda_v(A, B)}.$$

Then

1. f_v is a locally constant $(4, 6)$ -homogeneous function.
2. f_v is bounded below away from 0.
3. f_v can be extended to a locally constant function on $E_v = \{\mathbf{x} \in K_v^2 \setminus 0 : f_v^\#(\mathbf{x}) = 1\}$.
4. If $v \nmid 6$, then $f_v = f_v^\#$.

Proof.

1. Homogeneity follows from the fact that $E_{A,B}$ is isomorphic to $E_{\pi_v^4 A, \pi_v^6 B}$. The fact that f_v is locally constant follows from Tate's algorithm.
2. Suppose $\lambda_v(A, B) = k$. Then there exist $u, r, s, t \in \mathcal{O}_v$ with $v(u) = k$ such that the change of variables

$$x = u^2 x' + r, \quad y = u^3 y' + s u x' + t,$$

leads to a minimal equation $(y')^2 + a_1 x' y' + a_3 y' = (x')^3 + a_2 (x')^2 + a_4 x' + a_6$. Then using the well-known invariants c_4 and c_6 , we have

$$u^4 c'_4 = c_4 = -48A, \quad u^6 c'_6 = c_6 = -864B.$$

Thus, we have

$$\begin{aligned} 1 > \max \left(\frac{v(A)}{4}, \frac{v(B)}{6} \right) &\geq \max \left(\frac{v(u^4) - v(48)}{4}, \frac{v(u^6) - v(864)}{6} \right) \\ &= k - \min(v(48)/4, v(864)/6). \end{aligned} \tag{7.1}$$

So k is bounded above.

3. Now we show that for each k , $\lambda(A, B) \geq k$ is a condition that can be checked for $(A, B) \bmod \pi^k$. In fact, $\lambda(A, B) \geq k$ is equivalent to the statement that there exist $r, s, t \in \mathcal{O}_v$ such that after the change of coordinates

$$x = x' + r, \quad y = y' + sx' + t,$$

the resulting equation

$$(y')^2 + a_1 x' y' + a_3 y' = (x')^3 + a_2 (x')^2 + a_4 x' + a_6$$

satisfies $\pi^{ik} \mid a_i$ for $i = 1, 2, 3, 4, 6$. It is now clear that the existence of such r, s, t only depends on the class of $A, B \bmod \pi^{6k}$.

Thus, if we let $N = \sup_{A, B \in E_v} \lambda_v(A, B)$, then λ_v is determined by the class of $(A, B) \bmod \pi_v^{6N}$, which means that λ_v (and thus f_v) is locally constant on E_v .

4. This follows from standard facts about the reduction of elliptic curves over local fields, but can also be read off of (7.1).

□

Now we consider the infinite places:

Proposition 7.2.3. *For $v \mid \infty$ and for $A, B \in K_v^2 - Z(\Delta)$, define*

$$f_v(A, B) = \left| \frac{\Delta(A, B)}{\Delta(\tau_{A, B}) \operatorname{Im}(\tau_{A, B})^6} \right|_v^{1/12}.$$

Then

1. f_v is a $(4, 6)$ -homogeneous function.
2. f_v is ε -suitable at Δ for every $\varepsilon > 0$.

3. f_v is definable in the O -minimal structure $\mathbb{R}_{\text{an}, \text{exp}}$.

Proof.

1. This is clear from the definition of f_v , as $\tau_{A,B} = \tau_{\lambda^4 A, \lambda^6 B}$ for any nonzero complex number λ .
2. Following [28, p. 257], we have that for any $A, B \in \mathbb{C}$ with $\Delta(A, B) \neq 0$,

$$\Delta(\tau) \asymp \max(|j(\tau)|, 1)^{-1},$$

$$\text{Im}(\tau) \asymp \max(\log |j(\tau)|, \log |j(\tau) - 1728|).$$

However, $j(\tau) = j(A, B) = 1728 \frac{-64A^3}{\Delta(A, B)}$, so

$$\Delta(\tau) \asymp \max(|A|^3, |B|^2),$$

$$\text{Im}(\tau) \asymp \max \left[1, \log \left(\frac{\max |A|^3, |B|^2}{\Delta(A, B)} \right) \right].$$

Thus,

$$\begin{aligned} \frac{f^\#(A, B)}{f_v(A, B)} &= \left(\frac{\max(|A|^3, |B|^2)}{\frac{\Delta(A, B)}{\Delta(\tau) \text{Im}(\tau)^6}} \right)^{\epsilon_v/12} \ll \max \left[1, \log^{\epsilon_v/2} \left(\frac{\max |A|^3, |B|^2}{\Delta(A, B)} \right) \right] \\ &\ll \left(\frac{f_v^\#(A, B)}{|\Delta(A, B)|_v^{1/12}} \right)^\epsilon. \end{aligned}$$

3. This follows directly from Lemma 3.3.1.

□

It is clear that the product formula

$$H_F^{1/12} = \prod_v f_v$$

holds. Thus, we have proven Theorem 7.2.1.

7.3 Equidistribution of Weierstrass Class

Recall the definition of the Weierstrass class of an elliptic curve $E_{A,B}/K$ from [29, p. 243-244]. By construction, there exists a fractional ideal $\alpha = \alpha(A, B)$ such that $\Delta^{\min}(E_{A,B}) = (\Delta(A, B))\alpha^{12}$, and two different α s are related by an element of $K^\times$. Thus, α defines a unique element in the class group of K ; this element $\bar{\alpha}_E$ is called the *Weierstrass class* of E . In fact, we have

$$\alpha(A, B) = \prod_v \mathfrak{p}_v^{\lambda_v(A, B)},$$

and by the proof of Proposition 7.2.2, α is a quasi-scaling ideal. Thus, the following is a direct consequence of Proposition 4.4.1:

Proposition 7.3.1. *Let H be any subpolynomially suitable strongly definable quasi-toric height on $\mathcal{E}_K \hookrightarrow \mathbb{P}(4, 6)(K)$. Then for any $c \in \text{Cl}(K)$, the density of elliptic curves (when ordered by H) that have Weierstrass class equal to c is $\frac{1}{h}$.*

Since both the Faltings height and the naive height are quasi-toric heights, this proves Theorem 1.2.4. It is well-known [29, p. 244] that an elliptic curve E/K has a global minimal Weierstrass equation if and only if its Weierstrass class is trivial. Thus, if we order by either the naive height or by Faltings height, the density of elliptic curves having a global minimal Weierstrass equation is $\frac{1}{h}$. This is in contrast to the results of Bekyel [2], which uses a different ordering on elliptic curves.

7.4 Volume Computations at Finite Places

Proposition 7.4.1. *Suppose $v \nmid \infty$. Then*

$$(1 - q_v^{-10})^{-1} \int_{E_v} f_v^{-10} = \left| \frac{1}{12} \right|_v. \quad (7.2)$$

Remark 7.4.2. We already know that both sides of Equation (7.2) are equal to 1 when

$v \nmid 6$, as $f_v = f_v \#$.

Proof. By Proposition 2.3.5, λ_v is bounded above by a constant N_0 . Thus,

$$\int_{E_v} f_v^{-10} = \sum_{k=0}^{N_0} m_k q_v^{10k},$$

where m_k is the density of $(A, B) \in \mathcal{O}_v \times \mathcal{O}_v$ such that $\lambda_v = k$ and either $v(A) < 4$ or $v(B) < 6$.

We have that

1. The condition $\lambda_v \geq k$ can be checked modulo π_v^{6k} . This follows from Tate's algorithm and was already shown in the proof of Proposition 7.2.2.
2. $\lambda_v(\pi_v^4 A, \pi_v^6 B) = \lambda_v(A, B) + 1$. This is clear from the definition of λ_v .

Let s_k, t_k be the density of (A, B) such that $\lambda_v(A, B) = k$ and $\lambda_v(A, B) \geq k$, respectively.

Then

$$m_0 = s_0, \quad m_k = s_k - q_v^{-10} s_{k-1} \text{ for } k \geq 1.$$

Since $m_k = 0$ for $k > N_0$, $s_k = q_v^{-10} s_{k-1}$ for $k > N_0$,

$$t_k = \sum_{j=k}^{\infty} s_j = \frac{s_k}{1 - q_v^{-10}}$$

for $k > N_0$. Thus, for any $N > N_0$,

$$\sum_{k=0}^N m_k q_v^{10k} = s_N q_v^{10N} = (1 - q_v^{-10}) t_N q_v^{10N}.$$

Let W_n be the space of integral Weierstrass equations modulo π_v^{6n} :

$$W_n = \{(y^2 + a_1 xy + a_3 y = x^3 + a_2 x + a_4 x^2 + a_6) : a_1, a_2, a_3, a_4, a_6 \in \mathcal{O}_v / \pi_v^{6n}\}.$$

Let W_n^{red} be the subset of W_n where $v(a_i) \geq ni$ for $i \in \{1, 2, 3, 4, 6\}$, and let W_n^0 be the

subset of W_n where $a_1 \equiv a_2 \equiv a_3 \equiv 0 \pmod{\pi_v^{6n}}$.

The group of unipotent upper triangular matrices

$$U_n = \left\{ \begin{pmatrix} 1 & s & t \\ 0 & 1 & r \\ 0 & 0 & 1 \end{pmatrix} : r, s, t \in O_v/\pi_v^{6n} \right\}$$

acts on W_n by coordinate transformation. Let U_n^{red} be the subgroup of U where $v(r) \geq 2n, v(s) \geq n, v(t) \geq 3n$.

Lemma 7.4.3. *For any $w \in W_n^{\text{red}}$, U_n^{red} is the set of elements $u \in U_n$ that $uw \in W_n^{\text{red}}$.*

Proof. Identical to the proof of [29, VII.1.3(b)]. □

Lemma 7.4.4. *If $n > 2v(2) + v(3)$, then for any $w \in W_n^{\text{red}}$, there are $q^{2v(2)+v(3)}$ elements $u \in U_n$ such that $uw \in W_n^0$.*

Proof. Given a Weierstrass equation $w : y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$ such that $v(a_i) \geq ni$ for $i = 1, 2, 3, 4, 6$, an element $u = u(r, s, t)$ satisfies $uw \in W_n^0$ if and only if each of the elements of the set

$$\{a_1 + 2s, a_2 - sa_1 + 3r - s^2, a_3 + ra_1 + 2t\}$$

are divisible by π_v^6 . There are $q^{v(2)}$ choices for s , $q^{v(3)}$ choices for r given s , and $q^{v(2)}$ choices of t given r , for a total of $q^{2v(2)+v(3)}$ choices of u . □

Now, $t_N = \frac{1}{q^{12N}} \#(U_N W_N^{\text{red}} \cap W_N^0)$, but by the previous two lemmas:

$$\#(U_N W_N^{\text{red}} \cap W_N^0) = \sum_{w \in W_N^{\text{red}}} \frac{\#\{u \in U_N : uw \in W_N^0\}}{\#\{u \in U_N : uw \in W_N^{\text{red}}\}} = \sum_{w \in W_N^{\text{red}}} q^{2v(2)+v(3)-12N} = q^{2v(2)+v(3)+2N}.$$

Thus,

$$\int_{E_v} f_v^{10} = (1 - q_v^{-10}) t_N q_v^{10N} = (1 - q_v^{-10}) q_v^{2v(2)+v(3)},$$

which completes the proof. □

7.5 Volume Computations at Infinite Places

Finally, we compute the volumes at the infinite places in terms of special values of the L -functions of certain modular forms:

Proposition 7.5.1. *Let $R_{\mathbb{R}}$ and $R_{\mathbb{C}}$ denote the regions*

$$\{(A, B) \in \mathbb{R}^2 : H_F^\infty(A, B) < 1\}$$

and

$$\{(A, B) \in \mathbb{C}^2 : H_F^\infty(A, B) < 1\},$$

respectively. Then

$$\begin{aligned} \text{Vol}(R_{\mathbb{R}}) &= \frac{15}{512\pi^7} L(\Delta, 6) \approx 29088.62176724 \dots \\ \text{Vol}(R_{\mathbb{C}}) &= \frac{1}{11520\pi} \langle \Delta, \Delta \rangle_{\text{Petersson}} = \frac{\Gamma(12)}{45 \cdot 2^{31} \cdot \pi^{14}} L(\text{Sym}^2 \Delta, 12) \approx 410021598.13169 \dots \end{aligned}$$

Proof. Consider the transformation $\phi : (A, B) \rightarrow (D, \tau)$ given by

$$A^3 = \frac{-27j(\tau)D}{1728^2}, \quad B^2 = \frac{4(j(\tau) - 1728)D}{1728^2}.$$

The magnitude of the determinant of the Jacobian of ϕ is

$$\left| \frac{\partial(A, B)}{\partial(D, \tau)} \right| = \left| \frac{18D \cdot j'}{1728^3 A^2 B} \right| = \frac{1}{144} \left| \frac{(j')^6}{D j^4 (j - 1728)^3} \right|^{1/6} = \frac{1}{288\pi} \left| \frac{\Delta(\tau)}{D} \right|^{1/6}.$$

Real ($\mathbb{R}$) case: The map ϕ is (generically) 2-to-1 onto the region

$$\{(D, \tau) \in (-\infty, 0] \times \gamma^- \cup [0, \infty) \times \gamma^+ : |D| < |\Delta(\tau) \text{Im}(\tau)^6|\},$$

where γ^-, γ^+ are paths in the upper half plane that map diffeomorphically via j to $(-\infty, 1728]$ and $[1728, \infty)$, respectively. Thus,

$$\sigma_{\mathbb{R}} = \sigma_{\mathbb{R}}^- + \sigma_{\mathbb{R}}^+,$$

where

$$\sigma_{\mathbb{R}}^- = 2 \cdot \frac{1}{288\pi} \int_{\gamma^-} \int_{-|\Delta(\tau)| \operatorname{Im}(\tau)^6}^0 \left| \frac{\Delta(\tau)}{D} \right|^{1/6} dD |d\tau| = \frac{1}{120\pi} \int_{\gamma^-} |\Delta(\tau)| \operatorname{Im}(\tau)^5 |d\tau|.$$

Likewise,

$$\sigma_{\mathbb{R}}^+ = \frac{1}{120\pi} \int_{\gamma^+} |\Delta(\tau)| \operatorname{Im}(\tau)^5 |d\tau|.$$

We can select the following paths:

$$\begin{aligned} \gamma^- &= \frac{1}{2} + i\left[\frac{1}{2}, \infty\right) \text{ or } \frac{1}{2} + i(0, \frac{1}{2}], \\ \gamma^+ &= i(1, \infty) \text{ or } i(0, 1). \end{aligned}$$

From the formula $\Delta(\tau) = (2\pi)^{12} q \prod_{k=1}^{\infty} (1 - q^k)^{24}$, it is clear that $\Delta(\tau)$ is a positive real number whenever $\Re(\tau) \in \mathbb{Z}$ and a negative real number when $\Re(\tau) \in \frac{1}{2}(2\mathbb{Z} + 1)$. Thus,

$$\sigma_{\mathbb{R}}^- = -\frac{1}{240\pi} \int_0^\infty \Delta\left(\frac{1}{2} + it\right) t^6 \frac{dt}{t} = \frac{1}{240\pi} \cdot \frac{\Gamma(6)}{(2\pi)^6} \cdot \frac{2 - (1 - \frac{\tau(2)}{2^6} + \frac{2^{11}}{2^{12}})^{-1}}{(1 - \frac{\tau(2)}{2^6} + \frac{2^{11}}{2^{12}})^{-1}} \cdot L(\Delta, 6) = \frac{11}{512\pi^7} L(\Delta, 6)$$

and

$$\sigma_{\mathbb{R}}^+ = \frac{1}{240\pi} \int_0^\infty \Delta(it) t^6 \frac{dt}{t} = \frac{1}{240\pi} \cdot \frac{\Gamma(6)}{(2\pi)^6} \cdot L(\Delta, 6) = \frac{1}{128\pi^7} L(\Delta, 6).$$

Complex ($\mathbb{C}$) case: The map ϕ is (generically) 6-to-1 on the region

$$\{(\Delta, \tau) \in \mathbb{C} \times \mathfrak{h} : |D| < |\Delta(\tau) \operatorname{Im}(\tau)^6|\},$$

where $\mathfrak{h}$ is the usual fundamental domain for the action of $\Gamma = \mathrm{SL}(2, \mathbb{Z})$ on the upper half-plane H . Thus,

$$\begin{aligned}\sigma_{\mathbb{C}} &= 6 \iint_{\mathfrak{h}} \iint_{|D| < |\Delta(\tau) \mathrm{Im}(\tau)^6|} \left(\frac{1}{288\pi} \right)^2 \left| \frac{\Delta(\tau)}{D} \right|^{1/3} dA_D dA_\tau \\ &= \frac{1}{11520\pi} \iint_{\mathfrak{h}} |\Delta(\tau)|^2 \mathrm{Im}(\tau)^{10} dA_\tau \\ &= \frac{1}{11520\pi} \langle \Delta, \Delta \rangle_{\mathrm{Petersson}} = \frac{\Gamma(12)}{45 \cdot 2^{31} \cdot \pi^{14}} L(\mathrm{Sym}^2 \Delta, 12).\end{aligned}$$

□

Theorem 1.2.2 now follows from the fact that $\frac{\sigma_{\mathbb{R}}^+}{\sigma_{\mathbb{R}}} = \frac{4}{15}$, combined with the equidistribution discussed in Chapter 3.

7.6 Final Computations

By Theorem 4.1.4 and Theorem 7.2.1,

$$\begin{aligned}\#\{E \in \mathcal{E}_K : H_F(\mathcal{E}) < X\} &= \#\{[\mathbf{x}] \in \mathbb{P}(4, 6)(K) - Z(\Delta) : H_F^{1/12}(\mathbf{x}) < X^{1/12}\} \\ &= c_{H_F} 2^{2r_2} \frac{2hR}{w |D_K| \zeta_K(10)} 10^{r_1+r_2-1} X^{5/6} + O_\varepsilon(X^{5/6-1/3d+\varepsilon}).\end{aligned}$$

By Proposition 7.5.1 and Proposition 7.4.1,

$$c_{H_F} = \prod_{v \nmid \infty} \left| \frac{1}{12} \right|_v \prod_{v \mid \infty} \sigma_{K_v} = 12^d \sigma_{\mathbb{R}}^{r_1} \sigma_{\mathbb{C}}^{r_2}.$$

Thus, Theorem 1.2.1 is proven.

Chapter 8

Root Numbers and Average Rank

8.1 Introduction

It is shown in [5] that the average rank of elliptic curves when ordered by naive height is less than 0.885. Key to this result is the following result about the equidistribution of root numbers. Bhargava and Shankar showed that there exists a family F of elliptic curves, having density greater than 55.01% when ordered by naive height and defined by congruence conditions, such that the root number of elliptic curves in F is equidistributed. Using this equidistribution result, they also show that at least 20.62% of curves have rank 0 and at least 83.75% have rank 0 or 1.

We do not obtain as strong a set of results, but we do obtain the bounds of Proposition 1.2.7, which is an improvement of the bound of 1.05 purely from considering the average size of the 5-Selmer group.

Central to the construction of Bhargava and Shankar is the fact that the naive height is invariant under twisting by -1 .

However, this true less than 25% of the time for Faltings height (and conjecturally for the discriminant). This is the main reason Proposition 1.2.7 is weaker than the aforementioned density results of [5].

For elliptic curves $E/\mathbb{Q}$, the root number $W(E)$ is the sign of the functional equation for the L function associated to E , and is conjectured to equal the parity of the algebraic rank of E (the “parity conjecture”). For more exposition on this topic, see [16]. However, we make use of the following unconditional result of Dokchitser–Dokchitser:

Theorem 8.1.1 (Dokchitser–Dokchitser [17]). *Let E be an elliptic curve over $\mathbb{Q}$ and let p be any prime. Let $s_p(E)$ and $t_p(E)$ denote the rank of the p -Selmer group of E and the rank of $E(\mathbb{Q})[p]$, respectively. Then the quantity $s_p(E) - t_p(E)$ is even if and only if the root number of E is $+1$.*

Since $E(\mathbb{Q})[5]$ is trivial for 100% of curves in $\mathcal{E}$, we have that the parity of the 5-Selmer rank of E equals $W_p(E)$ for 100% of elliptic curves E .

Recall that the root number $W(E)$ of an elliptic curve $E/\mathbb{Q}$ can be written as a product of local root numbers, $-\prod_p W_p(E)$. One might hope that a local-to-global principle holds:

Conjecture 8.1.2.

$$\text{avg}_{E \in \mathcal{E}}(W(E)) = - \prod_p \text{avg}_{E \in \mathcal{E}}(W_p(E)).$$

One can show that $1 - \text{avg}_{E \in \mathcal{E}} W_p(E) \gg O(1/p)$. Thus, this conjecture would imply the equidistribution of root numbers. (For the treatment of this problem in certain families of elliptic curves; see [21].)

We follow the approach taken in [5] to get a nonzero amount of equidistribution for (global) root numbers, namely to show that the product

$$\alpha(E) := W(E)W(E_{-1})$$

is equal to -1 for a positive proportion of elliptic curves. Surprisingly, this proportion is not 50%! In addition, we provide marginal improvements for equidistribution by also including twists by 2 and -2 .

8.2 Relation Between Equidistribution and Rank

Using the Theorem 8.1.1 and basic convexity arguments, the following corollary is established in [5]:

Corollary 8.2.1. *Let F be any set of elliptic curves such that the average size of the 5-Selmer group is 6 and the density of curves with nontrivial 5-torsion is 0. Then:*

1. *The average 5-Selmer rank of elliptic curves in F is bounded above by 1.05.*
2. *The set of elliptic curves in F with 5-Selmer rank ≤ 1 has density at least $\frac{19}{24}$.*

Suppose further that the average root number of the curves in F is 0. Then

1. *The average 5-Selmer rank of elliptic curves in F is bounded above by $\frac{3}{4}$.*
2. *The set of elliptic curves in F with 5-Selmer rank ≤ 1 has density at least $\frac{7}{8}$.*
3. *The set of elliptic curves in F with 5-Selmer rank 0 has density at least $\frac{3}{8}$.*

Note that the algebraic rank of E does not exceed the 5-Selmer rank of E , so the results in the previous proposition also hold if we replace the 5-Selmer rank of E with the algebraic rank of E .

It is in fact conjectured that when the set of all elliptic curves is ordered by a “reasonable” height, then the average root number is 0.

8.3 Equidistribution Construction

For D a fundamental discriminant, let E^D denote the quadratic twist of E by D , and let

$$\alpha^{(D)}(E) = W(E)W(E^D). \tag{8.1}$$

Let $\chi_D : (\mathbb{Z}/|D|\mathbb{Z})^\times \rightarrow \{\pm 1\}$ be the quadratic character with conductor D induced by the Jacobi symbol. Note for any nonzero integer n and any positive integer M , we have

$$\frac{n}{\gcd(n, M)} = \text{sign}(n) \prod_{p \nmid M} p^{v_p(n)}.$$

For $D \mid M$, define

$$\begin{aligned} \alpha_p^{(D)}(E) &= W_p(E)W_p(E^D)\chi_D(\Delta_p(E)), \\ \alpha_\infty^{(D)}(E) &= \chi_D(\text{sign}(\Delta(E))), \\ \beta_M^{(D)}(E) &= \chi_D(\Delta_{M^{-1}}(E)) \cdot \prod_{p \mid M} W_p(E)W_p(E^D). \end{aligned}$$

Then

$$\alpha^{(D)}(E) = \alpha_\infty^{(D)}(E) \beta_M^{(D)}(E) \prod_{p \nmid M} \alpha_p^{(D)}(E). \quad (8.2)$$

By [25] and [20], $W_p(E)$ is determined by congruence conditions at p , so for $p \nmid M$, $\alpha_p^{(D)}(E)$ is also determined by congruence conditions at p . In addition, for $p \nmid D$, $\alpha_p^{(D)}(E) = 1$ as long as E is semistable at p , which means $\alpha_p(E) = 1$ whenever $p^2 \nmid \Delta(E)$.

In addition, for each d with $\gcd(d, D) = 1$, the set $\{E \in F : \Delta_{D^{-1}}(E) \equiv d \pmod{D}\}$ is a finite union of large subfamilies of F . Thus, Theorem 5.7.5 implies the following:

Theorem 8.3.1. *Let F be a large family, let $\mathbf{D} = (D_1, \dots, D_k)$ be a sequence of fundamental discriminants, and let $M = \gcd(D_1, \dots, D_k)$. Fix $\mathbf{t} = (t_1, \dots, t_k) \in \{\pm 1\}^k$. Consider*

$$F_{\mathbf{D}, \mathbf{t}} = \{E \in \mathcal{F} : \alpha^{(D_i)}(E) = t_i\}.$$

Then

1. The density of $F_{\mathbf{D},\mathbf{t}}$ in F is

$$\frac{\mu(F_{\mathbf{D},\mathbf{t}})}{\mu(F)} = \frac{1}{2^k} \sum_{e_1, \dots, e_k \in \{0,1\}^k} \left(\prod_{i=1}^k t_i^{e_i} \right) \cdot \nu \left(\prod_{i=1}^k (\alpha^{(D_i)})^{e_i} \right). \quad (8.3)$$

Here $\nu(f) = \xi_M(f) \cdot \nu_\infty(f) \cdot \prod_{p|M} \nu_p(f)$, where

$$\xi_M(f) = \text{avg}_{E \in \mathcal{F}} \left(\prod_{p|M} f_p \right),$$

$$\nu_\infty(f) = \text{avg}_{E \in \mathcal{F}} (f_\infty),$$

$$\nu_p(f) = \text{avg}_{E \in \mathcal{F}} (f_p).$$

2. If $F_{\mathbf{D},\mathbf{t}}$ is nonempty, the average size of the 5-Selmer group of elliptic curves in $F_{\mathbf{D},\mathbf{t}}$ when ordered by H is 6.

For ease, we define

$$\nu^{(D)} = \text{avg}_{E \in F} (\alpha^{(D)}(E)),$$

$$\nu_p^{(D)} = \text{avg}_{E \in F} (\alpha_p^{(D)}(E)).$$

(Note that the values $\nu^{(D)}$ and $\nu_p^{(D)}$ depend on the height H and the family F .) Then (8.2) becomes

$$\nu^{(D)} = \prod_{v \in M_{\mathbb{Q}}} \nu_v^{(D)}. \quad (8.4)$$

8.4 Symmetric Heights

In the case where H is *symmetric* on the large family F , that is, $H(E) = H(E_{-1})$ for all $E \in F$, we can now apply Theorem 8.3.1 and Corollary 8.2.1:

Proposition 8.4.1. *Suppose $H(E) = H(E_{-1})$ for all $E \in F$, and let $\mu = \mu(F_{(-4),(-1)})$. Then when $\mathcal{E}$ is ordered by H ,*

1. Their average rank is at most $1.05 - 0.3 \cdot \mu$.
2. The density of curves with rank at most 1 is at least $\frac{19}{24} + \frac{1}{12} \cdot \mu$.
3. The density of curves with rank equal to 0 is at least $\frac{3}{8}\mu$.

8.5 Density Computations

For the rest of this chapter, we consider only the twists by -4 , 8 , and -8 . In these cases, $\beta^{(D)}$ is defined by congruence conditions at only the prime $p = 2$.

8.5.1 Naive Height

Suppose we have the naive height $H^N(E_{A,B}) = \max(4|A|^3, 27|B|^2)$ for all A, B such that $p^4 \mid A$ implies $p^6 \nmid B$. Then for all p , H_p (the local factor of H at p) is equal to $H_p^\#$.

Proposition 8.5.1. *Suppose $p > 3$, $p \nmid D$, and $H_p = H_p^\#$. If $\chi_D(p) = 1$, then $\text{avg}_{E \in \mathcal{E}}(\alpha_p^{(D)}(E)) = 1$. Otherwise,*

$$\nu_p^{(D)} = \nu_p := 1 - 2(1 - p^{-10})^{-1}(1 - p^{-1}) [p^{-3} + p^{-8} + (p^{-2} + p^{-6})(1 + p^{-1})^{-1}].$$

Proof. By [25], $W_p(E)W_p(E_D) = 1$ if either E has good or additive reduction at p or D is a quadratic residue modulo p (equivalently, $\chi_D(p) = 1$). Thus, if $\chi_D(p) = 1$, then $\alpha_p^{(D)}(E) = 1$.

Now suppose $\chi_D(p) \neq 1$. Then $\alpha_p^{(D)}(E) = -1$ if and only if one of the following two conditions holds:

1. E has multiplicative reduction at p and $v_p(\Delta(E))$ is even.
2. E has additive reduction at p and $v_p(\Delta(E))$ is odd.

The former case occurs when the Kodaira symbol of E at p is I_{2k} for a positive integer k ; the latter occurs when the Kodaira symbol of E at p is one of III , III^* , or I_{2k-1}^* for a positive integer k . The density of such curves now follows from Table 8.3. $\square$

For $p \leq 3$, the situation is more complicated. At the end of this chapter are Table 8.1 and Table 8.2, which are completed versions of the partial tables computed in [5] using the results of [25] and [20]. We compute the following:

$$\beta_2^{(-4)} = \frac{23}{93}, \quad \nu_3^{(-4)} = \nu_3 = \frac{6821}{8052}.$$

Finally, it is a straightforward calculation that $\nu_\infty^{(-4)}$ is equal to $-\frac{3}{5}$. (Note that this constant depends on the exact choice of coefficients for $|A|^3$ and B^2 in the naive height.) Plugging this and the values from Table 8.5 into Equation (8.3), we get

$$\mu = \frac{1}{2} \left(1 + \frac{23}{93} \cdot -\frac{3}{5} \cdot \prod_{p \equiv 3 \pmod{4}} \nu_3 \right) \approx 0.5587030988 \dots$$

Thus, by Corollary 8.2.1, the average rank of elliptic curves when ordered by naive height is at most 0.8824.

Compare this to the upper bound of 0.8826 from [5] — the method is essentially the same, but we are a bit more precise in the calculation of μ .

8.5.2 Symmetric Faltings Height

Let H_F^{sym} be the symmetric Faltings height, that is,

$$H_F^{\text{sym}}(E) = \max(H(E), H(E_{-1})).$$

This is again a quasi-toric height, where only the factor at 2 is different from the (non-symmetric) Faltings Height. In this case, we have $H_p = H_p^\#$ for $p \neq 3$, and for $p = 3$, we can use Table 8.2 to get $\nu_3^{(-4)} = \frac{2249}{2684}$. Finally, by Theorem 1.2.2, $\nu_\infty^{(-4)} = -\frac{11}{15}$. Plugging these values in to Equation (8.3), we get

$$\mu_{\text{Sym}} = 0.5451626 \dots \tag{8.5}$$

This corresponds to an upper bound of 0.8865 for the average rank when elliptic curves are ordered by symmetric Faltings height.

8.5.3 A Further Improvement for $f_2^\#$

Suppose further that the local height at $p = 2$ is equal to $f_2^\#$. This is clearly the case for the naive height, and one can show that this is also the case for the symmetric Faltings height. We can improve the average rank bound by additionally considering the twists E_2 and E_{-2} . Note that for every E ,

$$\frac{H(E_2)}{H(E)} \in \{2^{-6}, 2^6\}.$$

If we let $H'(E) = \max(H(E), H(E_2))$, then the density of curves with $H' < X$ in the set of curves with $H < X$ is $\frac{2}{33}$. The height H' now has the property that

$$H'(E) = H'(E_{-1}) = H'(E_2) = H'(E_{-2})$$

for all elliptic curves E . Consider the family of elliptic curves E where

$$W(E) = W(E_{-1}) = -W(E_2) = -W(E_{-2}). \quad (8.6)$$

This family is disjoint from $F_{(-4),(-1)}$. The average root number in this family (when ordered by H') is 0 and the average size of the 5-Selmer group is 6. With the local height $f_2'(A, B) = \max(f_2^\#(A, B), f_2^\#(4A, 8B))$, the density of the possible values of $(\beta^{(-4)}, \beta^{(8)}, \beta^{(-8)})$ is computed in Table 8.6 from Table 8.1. Table 8.7 then shows the averages of various products of β s. The density of such curves is

$$\begin{aligned} \mu' &:= \mu(\mathcal{E}_{(-4,8,-8),(1,-1,-1)}) \\ &= \frac{1}{8} \text{Avg}(\alpha^{(-4)}\alpha^{(8)}\alpha^{(-8)} - \alpha^{(-4)}\alpha^{(8)} - \alpha^{(-4)}\alpha^{(-8)} + \alpha^{(8)}\alpha^{(-8)} + \alpha^{(-4)} - \alpha^{(8)} - \alpha^{(-8)} + 1). \end{aligned}$$

Taking into account the fact that $\frac{2}{33}$ of curves that satisfy $H < T$ satisfy $H' < T$, we have

$$\lim_{T \rightarrow \infty} \frac{\sum_{E \in \mathcal{E}_{H < T}} \text{rk}(E)}{\sum_{E \in \mathcal{E}_{H < T}} 1} \leq 1.05 - 0.3 \cdot \left(\mu + \frac{2}{33}\mu'\right).$$

In the case of naive height, we have

$$\mu' = 0.127212829 \dots$$

The corresponding rank upper bound is 0.8801.

In the case of symmetric Faltings height, we have

$$\mu'_{\text{Sym}} = 0.120808165 \dots$$

The corresponding rank upper bound is 0.8843.

It is worth noting that the latter two results of Proposition 8.4.1 hold with μ replaced by $\mu + \frac{2}{33}\mu'$ for essentially the same reason as the above.

It is worth noting that we could further improve these bounds by incorporating twists by $3, -3, 5, -5$, and so on. However, the marginal gains in the quantity μ are quickly diminishing: the improvement by incorporating a twist by N is bounded above by $N^{-5+o(1)}$ and involves more ad hoc local root number calculations. We have included this section to both demonstrate that an improvement is possible by considering more twists but also to show that the potential gains are small and bounded.

8.5.4 Faltings Height

As alluded to before, the Faltings height H_F is not symmetric. Indeed, if (A, B) is in one of the following 12 congruence classes modulo 2^6 , then $H_F(E_{A,B}) = 2^{-12}H_F(E_{A,-B})$.

(5, 22)	(13, 14)	(21, 38)	(29, 30)
(37, 54)	(45, 46)	(53, 6)	(61, 12)
(0, 16)	(16, 16)	(32, 16)	(48, 16)

Let Ω be the union of these 12 congruence classes. Note that

$$\mathcal{E}_{H_F < X} = \mathcal{E}_{H_F^{\text{Sym}} < X} \cup \Omega_{H_F \in [2^{-12}X, X]}.$$

In the set $\mathcal{E}_{H_F < X}$, $\mathcal{E}_{H_F^{\text{Sym}} < X}$ has density $1/4$. By the previous section, the average rank of curves in $\mathcal{E}_{H_F^{\text{Sym}} < X}$ is at most $1.05 - 0.3 \cdot (\mu_{\text{Sym}} + \frac{2}{33}\mu'_{\text{Sym}})$, and the proportion of curves that have rank ≤ 1 and rank 0 are at least $\frac{19}{24} + \frac{1}{12} \cdot (\mu_{\text{Sym}} + \frac{2}{33}\mu'_{\text{Sym}})$ and $\frac{3}{8} \cdot (\mu_{\text{Sym}} + \frac{2}{33}\mu'_{\text{Sym}})$, respectively.

On the other hand, Ω is a large family, so by Corollary 8.2.1, its average rank when ordered by Faltings height is at most 1.05 and at least $\frac{19}{24}$ of curves in Ω have rank 0 or 1. This is still true in the limit for the subset where $H_F \in [2^{-12}X, X]$. Thus, we get the following propositions:

Proposition 8.5.2. *When elliptic curves over $\mathbb{Q}$ are ordered by Faltings height,*

1. *Their average rank is at most $1.05 - (0.075 \cdot \mu_{\text{Sym}} + \frac{2}{33}\mu'_{\text{Sym}})$.*
2. *The density of curves with rank at most 1 is at least $\frac{19}{24} + \frac{1}{48} \cdot (\mu_{\text{Sym}} + \frac{2}{33}\mu'_{\text{Sym}})$.*
3. *The density of curves with rank 0 is at least $\frac{3}{32} \cdot (\mu_{\text{Sym}} + \frac{2}{33}\mu'_{\text{Sym}})$.*

Proposition 1.2.7 follows.

8.5.5 Tables

Included in this section are several tables containing calculations made in this chapter. We used Sage [30] to calculate the local root numbers at 2 and 3.

In Table 8.1, each row corresponds to a tuple $v_2(A), v_2(B), v_2(\Delta_{A,B})$. Note that if $3v(A) \neq 2v(B) - 2$, $v_2(\Delta_{A,B})$ depends only on $v(A)$ and $v(B)$.

For example, consider the family of elliptic curves where $v_2(A) = v_2(B) = 2$. This family has density $2^{-6}\zeta_2(10)$ in the set of all elliptic curves when ordering by naive height. Within this family, the tuple $(\beta^{-4}, \beta^{(8)}, \beta^{(-8)})$ is equal to $(-1, 1, -1)$ 50% of the time and equal to each of the tuples $(1, 1, -1)$ and $(1, -1, 1)$ 25% of the time.

The row labelled “ < 2 or < 3 ” gives the distribution of the possible values of $(\beta^{-4}, \beta^{(8)}, \beta^{(-8)})$ within the family of elliptic curves where $v_2(A) < 2$ or $v_2(B) < 3$. This family has density $\frac{32}{33}$ in the set of all elliptic curves when ordering by naive height.

For space reasons we have only included rows corresponding to (A, B) where either $4 \nmid A$ or $8 \nmid B$. The cases where $4 \mid A$ and $8 \mid B$ can be deduced from the existing rows using the following formulas:

$$\begin{aligned}\beta^{(-4)}(E) &= \beta^{(-8)}(E_2)\beta^{(8)}(E_2), \\ \beta^{(-8)}(E) &= \beta^{(-4)}(E_2)\beta^{(8)}(E_2), \\ \beta^{(8)}(E) &= \beta^{(-4)}(E_2)\beta^{(-8)}(E_2).\end{aligned}$$

For example, within the family $v_2(A) = 4$ and $v_2(B) = 5$, the tuple $(\beta^{-4}, \beta^{(8)}, \beta^{(-8)})$ is equal to $(-1, 1, -1)$ 50% of the time and equal to each of the tuples $(-1, -1, 1)$ and $(-1, 1, -1)$ 25% of the time.

The values in Table 8.6 are the *unweighted* averages of the two rows labeled “ < 2 or < 3 ” and “ ≥ 2 and ≥ 3 ”. The reason for taking the unweighted average is that these two families have density 50% in the set of all elliptic curves when the local height at 2 is equal to f'_2 .

				Relative Density of $(\beta^{(-4)}, \beta^{(8)}, \beta^{(-8)})$							
$v_2(A)$	$v_2(B)$	$v_2(\Delta_{A,B})$	(Naive) Density								
0	0	4	$2^{-2} \cdot \zeta_2(10)$	-	$\frac{3}{8}$	$\frac{3}{8}$	-	-	-	$\frac{1}{4}$	-
0	2	6	$2^{-4} \cdot \zeta_2(10)$	-	-	-	-	$\frac{1}{2}$	-	-	$\frac{1}{2}$
0	≥ 3	6	$2^{-4} \cdot \zeta_2(10)$	$\frac{1}{4}$	-	-	$\frac{1}{4}$	-	$\frac{1}{4}$	$\frac{1}{4}$	-
1	0	4	$2^{-3} \cdot \zeta_2(10)$	-	$\frac{1}{4}$	$\frac{1}{4}$	-	-	$\frac{1}{2}$	-	-
1	1	6	$2^{-4} \cdot \zeta_2(10)$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	-	-	-	-
1	2	8	$2^{-5} \cdot \zeta_2(10)$	-	$\frac{1}{4}$	$\frac{1}{4}$	-	$\frac{1}{4}$	-	-	$\frac{1}{4}$
1	3	9	$2^{-6} \cdot \zeta_2(10)$	$\frac{1}{4}$	-	-	$\frac{1}{4}$	-	$\frac{1}{4}$	$\frac{1}{4}$	-
1	≥ 4	9	$2^{-6} \cdot \zeta_2(10)$	$\frac{1}{4}$	-	-	$\frac{1}{4}$	-	$\frac{1}{4}$	$\frac{1}{4}$	-
≥ 2	0	4	$2^{-3} \cdot \zeta_2(10)$	-	$\frac{1}{2}$	$\frac{1}{2}$	-	-	-	-	-
≥ 2	1	6	$2^{-4} \cdot \zeta_2(10)$	-	-	-	-	-	-	1	-
2	2	8	$2^{-6} \cdot \zeta_2(10)$	-	$\frac{1}{4}$	$\frac{1}{4}$	-	-	$\frac{1}{2}$	-	-
≥ 3	2	10	$2^{-6} \cdot \zeta_2(10)$	-	$\frac{1}{2}$	$\frac{1}{2}$	-	-	-	-	-
		7	$2^{-4} \cdot \zeta_2(10)$	-	$\frac{1}{2}$	$\frac{1}{2}$	-	-	-	-	-
		8	$2^{-5} \cdot \zeta_2(10)$	-	$\frac{1}{8}$	$\frac{1}{8}$	-	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{4}$
		9	$2^{-6} \cdot \zeta_2(10)$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	-	-	-	-
0	1	10	$2^{-7} \cdot \zeta_2(10)$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$
		11	$2^{-8} \cdot \zeta_2(10)$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$
		12	$2^{-9} \cdot \zeta_2(10)$	$\frac{1}{4}$	-	-	$\frac{1}{4}$	-	$\frac{1}{4}$	$\frac{1}{4}$	-
		≥ 13	$2^{-9} \cdot \zeta_2(10)$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$
< 2 or < 3			$\frac{32}{33}$	$\frac{185}{3968}$	$\frac{1079}{3968}$	$\frac{1079}{3968}$	$\frac{185}{3968}$	$\frac{199}{3968}$	$\frac{409}{3968}$	$\frac{633}{3968}$	$\frac{199}{3968}$
≥ 2 and ≥ 3			$\frac{1}{33}$	$\frac{185}{3968}$	$\frac{199}{3968}$	$\frac{199}{3968}$	$\frac{185}{3968}$	$\frac{1079}{3968}$	$\frac{409}{3968}$	$\frac{633}{3968}$	$\frac{1079}{3968}$
All			1	$\frac{185}{3968}$	$\frac{3157}{11904}$	$\frac{3157}{11904}$	$\frac{185}{3968}$	$\frac{677}{11904}$	$\frac{409}{3968}$	$\frac{633}{3968}$	$\frac{677}{11904}$
Density of $\beta_2^{(-4)}$					$\frac{58}{93}$				$\frac{35}{93}$		

Table 8.1: Density of elliptic curves having prescribed values of $(\beta^{(-4)}, \beta^{(8)}, \beta^{(-8)})$, ordered by naive height.

$v_3(A)$	$v_3(B)$	Density		Relative density of $(\chi_4(\Delta_3), \alpha_3^{(-4)})$			
		Naive	Faltings	(+1, +1)	(+1, -1)	(-1, +1)	(-1, -1)
0	≥ 0	$\frac{2}{3} \cdot \zeta_3(10)$	$\frac{2}{3^2} \cdot \zeta_3(10)$	1	-	-	-
1	1	$\frac{4}{3^4} \cdot \zeta_3(10)$	$\frac{4}{3^5} \cdot \zeta_3(10)$	-	-	-	1
1	≥ 2	$\frac{2}{3^4} \cdot \zeta_3(10)$	$\frac{2}{3^5} \cdot \zeta_3(10)$	-	-	1	-
≥ 2	0	$\frac{2}{3^3} \cdot \zeta_3(10)$	$\frac{2}{3^4} \cdot \zeta_3(10)$	-	-	$\frac{1}{3}$	$\frac{2}{3}$
≥ 2	1	$\frac{2}{3^4} \cdot \zeta_3(10)$	$\frac{2}{3^5} \cdot \zeta_3(10)$	-	-	-	1
2	2	$\frac{4}{3^6} \cdot \zeta_3(10)$	$\frac{4}{3^7} \cdot \zeta_3(10)$	1	-	-	-
2	≥ 3	$\frac{2}{3^6} \cdot \zeta_3(10)$	$\frac{2}{3^7} \cdot \zeta_3(10)$	1	-	-	-
≥ 3	2	$\frac{2}{3^6} \cdot \zeta_3(10)$	$\frac{2}{3^7} \cdot \zeta_3(10)$	-	-	-	1
3	4	$\frac{4}{3^9} \cdot \zeta_3(10)$	$\frac{4}{3^{10}} \cdot \zeta_3(10)$	-	-	-	1
3	≥ 5	$\frac{2}{3^9} \cdot \zeta_3(10)$	$\frac{2}{3^{10}} \cdot \zeta_3(10)$	-	-	1	-
≥ 4	3	$\frac{2}{3^8} \cdot \zeta_3(10)$	$\frac{2}{3^9} \cdot \zeta_3(10)$	-	-	$\frac{1}{3}$	$\frac{2}{3}$
≥ 4	4	$\frac{2}{3^9} \cdot \zeta_3(10)$	$\frac{2}{3^{10}} \cdot \zeta_3(10)$	-	-	-	1
4	5	$\frac{4}{3^{11}} \cdot \zeta_3(10)$	$\frac{4}{3^{12}} \cdot \zeta_3(10)$	1	-	-	-
≥ 5	5	$\frac{2}{3^{11}} \cdot \zeta_3(10)$	$\frac{2}{3^{12}} \cdot \zeta_3(10)$	-	-	-	1
1	0						
$v_3(\Delta_{A,B}) = 3$		$\frac{2}{3^3} \cdot \zeta_3(10)$	$\frac{2}{3^4} \cdot \zeta_3(10)$	-	-	$\frac{1}{3}$	$\frac{2}{3}$
4		$\frac{4}{3^4} \cdot \zeta_3(10)$	$\frac{4}{3^5} \cdot \zeta_3(10)$	1	-	-	-
5		$\frac{4}{3^5} \cdot \zeta_3(10)$	$\frac{4}{3^6} \cdot \zeta_3(10)$	-	-	-	1
$2 \cdot \mathbb{Z}_{\geq 3}$		$\frac{1}{2 \cdot 3^4} \cdot \zeta_3(10)$	$\frac{1}{2 \cdot 3^5} \cdot \zeta_3(10)$	1	-	-	-
$2 \cdot \mathbb{Z}_{\geq 3} + 1$		$\frac{1}{2 \cdot 3^5} \cdot \zeta_3(10)$	$\frac{1}{2 \cdot 3^6} \cdot \zeta_3(10)$	-	-	1	-
3	3						
$v_3(\Delta_{A,B}) = 9$		$\frac{2}{3^8} \cdot \zeta_3(10)$	$\frac{2}{3^9} \cdot \zeta_3(10)$	-	-	$\frac{1}{3}$	$\frac{2}{3}$
10		$\frac{4}{3^9} \cdot \zeta_3(10)$	$\frac{4}{3^{10}} \cdot \zeta_3(10)$	1	-	-	-
11		$\frac{4}{3^{10}} \cdot \zeta_3(10)$	$\frac{4}{3^{11}} \cdot \zeta_3(10)$	-	-	-	1
12		$\frac{4}{3^{11}} \cdot \zeta_3(10)$	$\frac{4}{3^{12}} \cdot \zeta_3(10)$	1	-	-	-
$2 \cdot \mathbb{Z}_{\geq 6} + 1$		$\frac{1}{2 \cdot 3^{10}} \cdot \zeta_3(10)$	$\frac{1}{2 \cdot 3} \cdot \zeta_3(10)$	-	-	-	1
$2 \cdot \mathbb{Z}_{\geq 7}$		$\frac{1}{2 \cdot 3^{11}} \cdot \zeta_3(10)$	$\frac{1}{2 \cdot 3^2} \cdot \zeta_3(10)$	-	1	-	-
Total (Naive)		1	-	$\frac{258883}{6 \cdot (3^{10}-1)}$	$\frac{1}{6 \cdot (3^{10}-1)}$	$\frac{9027}{2 \cdot (3^{10}-1)}$	$\frac{68323}{6 \cdot (3^{10}-1)}$
Total (Faltings)		-	1	$\frac{731267}{18 \cdot (3^{10}-1)}$	$\frac{6561}{2 \cdot (3^{10}-1)}$	$\frac{3009}{2 \cdot (3^{10}-1)}$	$\frac{245467}{18 \cdot (3^{10}-1)}$

Table 8.2: Densities of elliptic curves having prescribed values of $\chi_4(\Delta_3)$ and $\alpha_3^{(-4)}$.

Kodaira Symbol	Reduction Type	$v_p(\Delta)$	Density	ν_p weight
I_0	Good	0	$\frac{p-1}{p} \cdot \zeta_p(10)$	+1
I_n ($n \geq 1$)	Multiplicative	n	$\frac{(p-1)^2}{p^{n+2}} \cdot \zeta_p(10)$	$(-1)^{n+1}$
II	Additive	2	$\frac{(p-1)^2}{p^3} \cdot \zeta_p(10)$	+1
III	Additive	3	$\frac{(p-1)^2}{p^4} \cdot \zeta_p(10)$	-1
IV	Additive	4	$\frac{(p-1)^2}{p^5} \cdot \zeta_p(10)$	+1
I_0^*	Additive	6	$\frac{(p-1)}{p^6} \cdot \zeta_p(10)$	+1
I_n^* ($n \geq 1$)	Additive	$n + 6$	$\frac{(p-1)^2}{p^{n+7}} \cdot \zeta_p(10)$	$(-1)^n$
II*	Additive	10	$\frac{(p-1)^2}{p^{10}} \cdot \zeta_p(10)$	+1
III*	Additive	9	$\frac{(p-1)^2}{p^9} \cdot \zeta_p(10)$	-1
IV*	Additive	8	$\frac{(p-1)^2}{p^8} \cdot \zeta_p(10)$	+1

Table 8.3: Density of curves by Kodaira symbol at p for $p > 3$ (ordered by Naive height).

p	ν_p	
	Naive	Faltings
3	$\frac{6821}{8052}$	$\frac{2249}{2684}$
$p > 3$	$1 - 2\zeta_p(10)(1 - p^{-1})(p^{-3} + p^{-8} + (p^{-2} + p^{-6})(1 + p^{-1})^{-1})$	
∞	$-\frac{3}{5}$	$-\frac{7}{15}$

Table 8.4: Values of $\nu_p = \text{avg } W_p$ for $p \in M_{\mathbb{Q}}$ for both Naive and Faltings height.

Product	$p \pmod{8}$				∞
	1	3	5	7	
$\alpha_p^{(-4)}$	1	ν_p	1	ν_p	ν_{∞}
$\alpha_p^{(8)}$	1	ν_p	ν_p	1	1
$\alpha_p^{(-8)}$	1	1	ν_p	ν_p	ν_{∞}
$\alpha_p^{(-4)} \alpha_p^{(8)}$	1	1	ν_p	ν_p	ν_{∞}
$\alpha_p^{(-4)} \alpha_p^{(-8)}$	1	ν_p	ν_p	1	1
$\alpha_p^{(8)} \alpha_p^{(-8)}$	1	ν_p	1	ν_p	ν_{∞}
$\alpha_p^{(-4)} \alpha_p^{(8)} \alpha_p^{(-8)}$	1	1	1	1	1

Table 8.5: Averages of combinations of products of $\alpha_p^{(-4)}, \alpha_p^{(8)}, \alpha_p^{(-8)}$.

$\beta^{(-4)}$	$\beta^{(8)}$	$\beta^{(-8)}$	Density
1	1	1	$\frac{185}{3968}$
1	1	-1	$\frac{639}{3968}$
1	-1	1	$\frac{639}{3968}$
1	-1	-1	$\frac{185}{3968}$
-1	1	1	$\frac{639}{3968}$
-1	1	-1	$\frac{409}{3968}$
-1	-1	1	$\frac{633}{3968}$
-1	-1	-1	$\frac{639}{3968}$

Table 8.6: Density of elliptic curves having prescribed values of $(\beta^{(-4)}, \beta^{(8)}, \beta^{(-8)})$, where the local height at 2 is f'_2 .

Product	Average
$\beta^{(-4)}$	$-\frac{21}{124}$
$\beta^{(8)}$	$-\frac{7}{124}$
$\beta^{(-8)}$	$\frac{7}{124}$
$\beta^{(-4)} \beta^{(8)}$	$\frac{7}{124}$
$\beta^{(-4)} \beta^{(-8)}$	$-\frac{7}{124}$
$\beta^{(8)} \beta^{(-8)}$	$-\frac{21}{124}$
$\beta^{(-4)} \beta^{(8)} \beta^{(-8)}$	$-\frac{143}{496}$

Table 8.7: Averages of products of β s where the local height at 2 is f'_2 .

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