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Asymptotically Stable Ill-Posedness of Geometric Quasilinear Wave Equations

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ASYMPTOTICALLY STABLE ILL-POSEDNESS OF
GEOMETRIC QUASILINEAR WAVE EQUATIONS

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Abstract

It has been known since the work of Smith and Tataru in [15] that quasilinear wave equations $(g^{-1})^{\alpha\beta}(\Phi)\partial_{\alpha\beta}^2\Phi = \mathcal{N}(\Phi, \partial\Phi)$ are locally well-posed in $H^{2+\epsilon} \times H^{1+\epsilon}(\mathbb{R}^3)$. The sharpness of this result was immediately known, given an older result due to Lindblad in [13] which showed that the equation $\square_m\Phi = -\Phi(\underline{L}_{(Flat)}^2\Phi)$ is illposed in $H^2 \times H^1(\mathbb{R}^3)$. We show that the recent work of Speck, Holzegel, Luk and Wong in [18] on nearly plane symmetric shock formation is intimately connected to the low-regularity illposedness of geometric quasilinear wave equations in $H^2 \times H^1(\mathbb{R}^3)$. Indeed, as with shock formation, this is actually a generic phenomenon: for almost all $(g^{-1})^{\alpha\beta}(\Phi)$ which are perturbations of the Minkowski metric we can specify initial data which is arbitrarily small in $H^2 \times H^1(\mathbb{R}^3)$ but whose $\dot{H}^1$ energy blows up arbitrarily fast in the domain of future dependence of the data. We demonstrate that illposedness is actually a corollary of the nearly planar shock formation theorem in $3 + 1$ dimensions. The nearly planar shock formation result actually gives us even more: the stability of the breakdown under asymptotically small perturbations of “Lindblad-type” initial data. In other words, Lindblad’s result is not simply an artifact of symmetry. We then show that the proof in [18] extends from $2 + 1$ to $3 + 1$ dimensions. This is largely the same argument, although elliptic estimates are now necessary to control some of the new top-order error terms which are nontrivial in the $3 + 1$ dimensional setting. In order to control these terms we follow the structure presented in [16]. Our result demonstrates that the conjectural local well-posedness of the timelike minimal surface equation in $H^3 \times H^2(\mathbb{R}^3)$ is an exceptional case: every other equation in its class of irrotational, compressible relativistic fluid equations is illposed in $H^3 \times H^2(\mathbb{R}^3)$.

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To the memory of my beloved grandfather,
Charles Ray Jaynes, Sr.
“I’ll make it, but I ain’t gonna look like much.”

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0.1 Introduction

Remark 0.1.0.1. We attempt to closely follow Jared Speck's typesetting conventions and notation in [16] and [18]. Not only are these monographs among the most readable in the field (hence our adherence to the notation), but this work is very much an elaboration of and corollary to the results in [18].

0.1.1 Well-Posedness for Nonlinear Wave Equations: A Brief Historical Overview

Ubiquitous in physics are Cauchy problems for general nonlinear wave equations of the form (following the formulation in 1.2.4 of [16]):

$$(g^{-1})^{\alpha\beta}(\Phi, \partial\Phi) \partial_\alpha \partial_\beta \Phi = \mathcal{N}(\Phi, \partial\Phi) \quad (0.1.1.1)$$

$$(\Phi|_{t=0}, \partial_t \Phi|_{t=0}) = (\mathring{\Phi}_0, \mathring{\Phi}_1) \quad (0.1.1.2)$$

where we will assume that, relative to rectangular coordinates $(x^0 = t, x^1, \dots, x^n)$ on $\mathbb{R}^{n+1}$, $g_{\alpha\beta} = g_{\beta\alpha} : \mathbb{R} \times \mathcal{T}_{0,1}(\mathbb{R}^{n+1}) \rightarrow \mathbb{R}$ is a continuous function and that

$$g_{\alpha\beta}(0, 0) = m_{\alpha\beta} \quad (0.1.1.3)$$

$$\mathcal{N}(0, 0) = \frac{\partial \mathcal{N}}{\partial \Phi}(0, 0) = \frac{\partial \mathcal{N}}{\partial(\partial\Phi)}(0, 0) = 0 \quad (0.1.1.4)$$

where $m_{\alpha\beta} = \text{diag}(-1, 1, \dots, 1)$ is the Minkowski metric on $\mathbb{R}^{n+1}$.

Hadamard defined the well-posedness of such a Cauchy problem to be the existence of Banach spaces $\mathcal{X}_{\text{data}}$ and $\mathcal{X}_{\text{evolution}}$ of sufficiently smooth functions such that for every initial data set $(\mathring{\Phi}_0, \mathring{\Phi}_1) \in \mathcal{X}_{\text{data}}$ there exists a unique solution $\mathcal{E}(\mathring{\Phi}_0, \mathring{\Phi}_1) = \Phi \in \mathcal{X}_{\text{evolution}}$ satisfying (0.1.1.1), (0.1.1.2) such that the solution map

$$\mathcal{E} : \mathcal{X}_{\text{data}} \rightarrow \mathcal{X}_{\text{evolution}} \quad (0.1.1.5)$$

is continuous.

More precisely, we will assume

$$\mathcal{X}_{\text{data}} = H^s(\mathbb{R}^n) \times H^{s-1}(\mathbb{R}^n)$$

and

$$\mathcal{X}_{\text{evolution},T} := \mathcal{C}_{loc}^0((-T, T) : H^s(\mathbb{R}^n)).$$

If, for every $(\dot{\Phi}_0, \dot{\Phi}_1) \in \mathcal{X}_{\text{data}}$, there exists an open ball

$$\mathcal{O} := \left\{ (f, g) \in \mathcal{X}_{\text{data}} : \|(f, g) - (\dot{\Phi}_0, \dot{\Phi}_1)\|_{\mathcal{X}_{\text{data}}} < \epsilon \right\} \quad (0.1.1.6)$$

and a $T > 0$ such that $\mathcal{E} : \mathcal{O} \rightarrow \mathcal{X}_{\text{evolution},T}$ is a continuous map to unique distributional solutions of (0.1.1.1), (0.1.1.2), then we say that the wave equation (0.1.1.1) is *locally well-posed* for data in $\mathcal{X}_{\text{data}}$. If T can be chosen arbitrarily large, we say that we have *global well-posedness*.

It seems clear that well-posedness is something of a minimal expectation of equations characterizing physical systems.

Wave equations (like (0.1.1.1) above) satisfy some sort of “Energy Estimate” for $\|\partial\Phi\|_{L^2(\Sigma_t^u)}$, typically of the form:

$$\|\partial\Phi\|_{L^2(\Sigma_t^u)} \lesssim \|\partial\Phi\|_{L^2(\Sigma_0)} + \int_0^t \|\mathcal{N}\|_{L^2(\Sigma_s)} ds. \quad (0.1.1.7)$$

Commuting our equation with derivative operators typically allows us to derive higher order energy estimates of the form:

$$\|\partial\Phi\|_{H^s(\Sigma_t)} \lesssim \|\partial\Phi\|_{H^s(\Sigma_0)} + \int_0^t \|\mathcal{N}\|_{H^s(\Sigma_s)} ds. \quad (0.1.1.8)$$

It is thus clear why the natural choices for $\mathcal{X}_{\text{data}}$ and $\mathcal{X}_{\text{evolution}}$ are, respectively, the Sobolev spaces $H^s(\Sigma_0)$ and the corresponding time evolution space $\mathcal{C}_{loc}^0([0, T] : H^s(\Sigma_t))$. If we assume $\mathcal{N} = \mathcal{N}(\Phi)(\partial\Phi, \partial\Phi)$ (i.e. the nonlinearity is a quadratic form in $\partial\Phi$), we can then use Gronwall (and Sobolev imbedding) or various stronger estimates such as Strichartz or bilinear estimates (if possible) to derive a priori boundedness results on the $\dot{H}^s$ energy $\|\partial\Phi\|_{\dot{H}^{s-1}(\Sigma_t)}$. This then allows us to use a fixed-point argument in the appropriate $\mathcal{X}_{\text{evolution},T}$ space which gives the desired well-posedness result.

For wave equations, we can examine the scale invariance properties of the equation, say under the transformation $\Phi_\lambda(t, x) := \lambda^\alpha \Phi\left(\frac{t}{\lambda}, \frac{x}{\lambda}\right)$ (we can think of λ as giving us the characteristic space-time scale of the solution). For instance, if $\square\Phi = m(\partial\Phi, \partial\Phi)$, then the equation is invariant under the above transformation with $\alpha = 0$. Now we can consider the homogeneous Sobolev space $\dot{H}^{sc}(\mathbb{R}^n)$ which is left invariant under the rescaling i.e. $\|\Phi_\lambda\|_{\dot{H}^{sc}(\mathbb{R}^n)} = \|\Phi\|_{\dot{H}^{sc}(\mathbb{R}^n)}$. This defines the so-called “critical regularity” of the equation.

In [8], Klainerman introduced the following “General Well Posedness Conjecture,” a guiding paradigm in the study of nonlinear wave equations:

1. (Subcritical) For all basic field theories the initial value problem is locally, strongly well posed for any data in H^s , $s > s_c$.
2. (Critical) The basic field theories are weakly, globally well posed for all initial data with small H^{s_c} norm.
3. (Supercritical) There can be no well defined solutions for $s < s_c$.

Note that for the subcritical case, $s > s_c$ we have $\|\Phi_\lambda\|_{\dot{H}^s(\mathbb{R}^n)} = \lambda^{s_c-s} \|\Phi\|_{\dot{H}^s(\mathbb{R}^n)}$ and so for $\lambda \gg 1$ (small length scales), we have that $\|\Phi_\lambda\|_{\dot{H}^s(\mathbb{R}^n)} \ll 1$. So, heuristically, controlling the subcritical (H^s , $s > s_c$) norms rules out the concentration of energy, as small length scales have will correspondingly small subcritical energies. Thus we expect subcritical problems to be well-behaved. Conversely, supercritical norms grow as we look at smaller length scales. Thus we expect supercritical problems to be very badly behaved. For critical problems, the invariance of the norm under rescaling gives us that any fine scale and/or short-time result is automatically a coarse scale, global-in-time result as well.

It was known already at the time of [8] that the story was more complicated (and the conjecture false) for certain non-geometric semilinear and quasilinear wave equations, thanks to the counterexamples of Lindblad in [12] and [13], which showed that local well-posedness fails to hold in $\dot{H}^2(\mathbb{R}^3) \times \dot{H}^1(\mathbb{R}^3)$ in the following cases:

$$\square_m \Phi = (\underline{L}_{(Flat)} \Phi)^2$$

$$\square_m \Phi = 0, \quad \square_m \Psi = (\underline{L}_{(Flat)} \Phi)^2$$

$$\square_m \Phi = -\Phi(\underline{L}_{(Flat)}^2 \Phi)$$

Where $\square_m := m^{\alpha\beta} \partial_\alpha \partial_\beta$ and where we denote our Minkowski-null vectorfields in the (t, x^1) -plane by $\underline{L}_{(Flat)} := \partial_t - \partial_{x^1}$ and $L_{(Flat)} := \partial_t + \partial_{x^1}$. Note that these equations all are invariant under the transformations $\Phi_\lambda(t, x) := \Phi\left(\frac{t}{\lambda}, \frac{x}{\lambda}\right)$ with critical exponent $s = \frac{3}{2}$.

These counterexamples all crucially exhibit a failure of what is called the “null condition.”

The null condition is satisfied for (0.1.1.1) if, when we expand the nonlinearities to second order relative to some Minkowski null frame (see below), we find that none of these “resonant” terms are

present:

$$(L_{(Flat)}\Phi)^2, (\underline{L}_{(Flat)}\Phi)^2, \quad (0.1.1.9)$$

$$\Phi(L_{(Flat)}L_{(Flat)}\Phi), \Phi(\underline{L}_{(Flat)}\underline{L}_{(Flat)}\Phi), \quad (0.1.1.10)$$

$$\partial\Phi(L_{(Flat)}L_{(Flat)}\Phi), \partial\Phi(\underline{L}_{(Flat)}\underline{L}_{(Flat)}\Phi) \quad (0.1.1.11)$$

The derivative semilinear equations which verify at least parts 1 and 2 of the WP conjecture (see for instance [19]) all verify the Null Condition, and thus have nonlinearities which exhibit an absence of the worst-case resonant terms. This seems to be a natural consequence of the fact that these equations come from Lorentz-invariant Lagrangians.

In the case of quasilinear equations, much less is known in the positive direction. Thanks to work of Klainerman and Rodnianski (for EVE) [10] and Smith and Tataru (in the general case [15] - see also a more recent geometric proof by Qian Wang [20]) it is known that quasilinear equations of the form $\square_{g(\phi)}\phi = \mathcal{N}(\phi)(\partial\phi, \partial\phi)$ are well posed in $H^{2+\varepsilon}(\mathbb{R}^3)$. Going any further for quasilinear wave equations is a tremendously challenging and interesting problem. Thanks to the results of Lindblad, it is known that more structure on the nonlinearities is clearly required. It is not entirely clear what this structure should be, although anything “from a geometric lagrangian” seems to be the first place to look. Indeed, the “Bounded L^2 Curvature Conjecture” concerning the case of the Einstein Vacuum Equations was also formulated in [8], and it has since been positively resolved by Klainerman, Rodnianski and Szeftel in [11] and the sequence of supporting papers by Szeftel (see references in [11]). It was also conjectured in the same talk that the Timelike Minimal Surface Equation would be well-posed in $H^{s_c+1/2}$, as in the case of EVE. The thesis of Ettinger [4] claims a proof of this result using the wave packet parametrix methods of Tataru.

However, Lindblad and Ettinger showed that the Bounded L^2 Conjecture is false in harmonic gauge! The blow-up is essentially a result of the illposedness of “weak null” semilinear systems which we will sketch below in the simplest case.

From this result and actually from the work in this paper it is clear that a fundamental issue with respect to low-regularity LWP is the existence/stability of a good coordinate system.

The quasilinear example of Lindblad’s that we noted above is the origin of the above mentioned and commonly quoted piece of folklore that “generic” quasilinear wave equations are illposed in H^2 . We see from all of his examples (which are all $H^{3/2}$ -critical), that the LWP conjecture clearly requires more structure. It seems that we need, at the very least, some sort of local Lorentz invariance which gives us “null structure” in the nonlinearities. Indeed, our result here makes this heuristic more

precise, at least in the negative direction.

For quasilinear equations, the issue is also seemingly tied to the issue of regularity of null hypersurfaces. Quasilinear wave equations are interesting precisely because the null/causal structure (which governs wave propagation) is a function of the solution itself. In principle, it is thus impossible to generate any a priori control on the solution without also controlling the geometry of solutions to the Eikonal equation $(g^{-1})^{\alpha\beta}\partial_\alpha u \partial_\beta u = 0$. In the setting of the Einstein Vacuum Equations it is known that without control on the curvature flux through null hyperplanes (precisely compatible with H^2 initial data), it is not possible to control the radius of injectivity and thus regularity of these surfaces. The low-regularity local well-posedness results rely on multilinear estimates for the nonlinear terms, which in turn seem to require a Fourier integral representation of the solution, with phase function u solving the Eikonal equation. Thus, as u is essentially H^2 critical - not sufficiently regular with data of any lower regularity - these tools are inapplicable for data in $H^s, s < 2$. The reader should see the discussions in [9] and in the introduction to [11] for further details.

The behavior of characteristic geometry is a crucial element of [13], where Lindblad writes that the idea is to propagate asymptotically singular data along characteristic curves. He notes there that, due to the quasilinear causal structure, the singularity focuses the solution, which thus strengthens the singularity enough to generate instantaneous blow-up in H^2 . Although not explicitly noted in [13], it turns out that Lindblad's proof follows from the collapse of characteristics of a plane symmetric solution: the solution forms a shock instantaneously, which then propagates with enough severity to make H^2 norms instantaneously infinite for his given choice of H^2 regular initial data.

In recent years, our understanding of shock formation in quasilinear wave equations has improved dramatically, thanks to the work of Christodoulou, Speck and others [1], [3], [16]. They demonstrate that the null condition is “sharp” with respect to global existence: null condition failure for sufficiently geometric quasilinear wave equations will result in shock formation for an open set of initial data. Christodoulou, in his monograph [1] on shock formation in relativistic compressible irrotational fluids, was able to provide a precise, geometric description of the solution all the way up to (and including) the boundary surface where the shock first forms. In so doing, he set the stage for his recent monograph on the continuation problem [2].

Also of interest is the work by Miao and Yu [14] which demonstrates the formation of shocks in a large data setting for a quasilinear wave equation which actually satisfies the null condition (and thus exhibits global existence for sufficiently small data).

Most importantly for the result presented here is the recent work of Speck, Holzegel, Luk and Wong in [18], which proved the stability of shock formation near planar symmetry and crucially al-

lows for certain components of the initial data to be large (unlike in the nearly spherically symmetric case).

0.1.2 Characteristic Collapse: Shock Formation and Illposedness

The fundamental mechanism underlying these shock formation results, as well as that of Lindblad's quasilinear counterexample, is the degeneration of the null/causal structure relative to the Euclidean time coordinate.

Based on this analogy, we would like to generalize Lindblad's result for the equation

$$\square_m \Phi = -\Phi(\underline{L}_{(Flat)}^2 \Phi)$$

to irrotational relativistic fluids of the type considered by Christodoulou (in [1])- roughly, equations of the form

$$(g^{-1})^{\alpha\beta}(\partial\Psi)\partial_{\alpha\beta}^2\Psi = 0$$

- which turns out to be a corollary of the result for covariant quasilinear wave equations of the form

$$\square_{g(\Phi)} \Phi = 0.$$

We ultimately want to find covariant/geometric conditions on $g(\Phi)$ which correspond to admitting a well-posed Cauchy problem in $H^2 \times H^1(\mathbb{R}^3)$. That is, we would like to better understand the quasilinear “null condition” with respect to local well-posedness. The conclusion of this work indicates a possible direction for further research regarding the exceptional case which does not exhibit shock formation and H^2 ill-posedness.

It turns out that (see [18]) the geometric quasilinear wave equations of the form

$$\square_{g(\Phi)} \Phi = 0$$

form planar shocks as long as there is some direction $\omega \in \mathbb{S}^2$ such that, if

$${}^\omega L_{(Flat)} := \partial_t + \omega \cdot \nabla_x,$$

and if

$$G_{\alpha\beta}(\Phi) := \frac{d}{d\Phi} g_{\alpha\beta}(\Phi),$$

then

$$G_{\alpha\beta}(\Phi=0)^\omega L_{(Flat)}^\alpha L_{(Flat)}^\beta \neq 0, \quad (0.1.2.1)$$

in which case we say that $\square_{g(\Phi)}\Phi=0$ *fails the null condition*.

Without loss of generality we may assume for the remainder of the paper that $L_{(Flat)} = \partial_t + \partial_{x^1}$.

Remark 0.1.2.1. One can easily check by expanding the metric to first order in the form $g_{\alpha\beta}(\Phi) = m_{\alpha\beta} + \Phi G_{\alpha\beta}(\Phi) + \mathcal{O}(\Phi^2)$ that this condition corresponds to the existence of terms $\Phi \underline{L}_{(flat)}^2 \Phi$ and $(\underline{L}_{(flat)}\Phi)^2$ in the nonlinearity.

Actually, for these geometric quasilinear wave equations which exhibit shock formation, the blowup is much stronger than for the quasilinear equation Lindblad considers, indeed is almost as strong as the semilinear blowup with RHS $(\underline{L}_{(Flat)}\Phi)^2$. This is not surprising considering that noncovariant quasilinear equations of the form $g^{\alpha\beta}(\Phi)\partial_\alpha\partial_\beta\Phi=0$ have global existence of solutions for sufficiently small data, whereas equations of the type $\square_{g(\Phi)}\Phi=0$ can form shocks under fairly general conditions and thus break down in finite time. The shock formation result crucially depends on the semilinear terms hidden in the covariant $\square_g$ operator, which drive the collapse of the optical coordinates. See [18] and [16] for more details.

Our main result is roughly the following:

Theorem 0.1.1. *Shock forming wave equations of the form $\square_{g(\Phi)}\Phi=0$ are illposed in $H^2 \times H^1(\mathbb{R}^3)$.*

We do this by producing a convergent sequence of smooth solutions with arbitrarily small data in $H^2 \times H^1(\mathbb{R}^3)$ whose $\dot{H}^1(\Sigma_t)$ -energy (i.e. $\int_{\Sigma_t} |\partial\Phi|^2(t, x)dx$) blows up arbitrarily fast as we approach the limit. Thus, in particular, no continuous dependence on data can be established with respect to the $H^2 \times H^1(\mathbb{R}^3)$ norm.

This result is actually a corollary of the following (known and stated but unproved) fact that:

Theorem 0.1.2. *For wave equations in $\mathbb{R}^{3+1}$ of the form $\square_{g(\Phi)}\Phi=0$ which satisfy (0.1.2.1), shock formation is stable under suitably small perturbations of planar symmetry.*

We will sketch a proof of this theorem, relying heavily on the $2+1$ dimensional planar case [18], utilizing some of the ideas and notation from the $3+1$ dimensional dispersive case [16]. Although the basic structure of the argument is the same as in [18], eikonal quantities such as $\hat{\chi}$ are now nontrivial and cause new top-order error terms to appear in the energy argument. The treatment of such terms, as in the dispersive setting, is via elliptic estimates on the 2-surfaces $\Sigma_t \cap \mathcal{C}_u$. We give the argument necessary to treat these additional error terms and thus show that the presence

of the additional top-order terms in our $3 + 1$ dimensional setting does not impact the structure of the energy estimates in [18]. The rest of the argument is then exactly as in [18].

This theorem actually provides us with a stability result regarding the illposedness result above. We note here that all of Lindblad's counterexamples utilized H^2 regular initial data of essentially the same form:

$$\underline{L}_{(Flat)}\Phi(t=0, x^1, x^2, x^3) = \varepsilon |\ln(x^1)|^\alpha$$

which is then localized to the ball of radius 1 centered at $(1, 0, 0) \in \mathbb{R}^3$. It turns out that the as long as our data exhibits asymptotically similar singular behavior at the edge of the ball on the initial data hypersurface then the instantaneous H^2 norm blow-up still holds:

Corollary 0.1.2.1. *Illposedness holds for all shock forming covariant wave equations away from plane symmetry as long as data is “Asymptotically Lindblad.”*

Remark 0.1.2.2. The result here is local - we only consider data in a ball on the initial hypersurface and we prove that the breakdown occurs in the domain of future dependence of this ball. Following [16] we will assume that the data is periodic in the “angular” directions i.e. we work in the space-time $\mathbb{R}^{1+1} \times \mathbb{T}^2$. This has no effect on our result, thanks to locality.

0.2 Geometric Proof of the Lindblad Counterexamples

Here we will present proofs of the three aforementioned results of Lindblad, all in the planar symmetric setting in $\mathbb{R}^{3+1}$, meaning that we only consider solutions which are a function of (t, x_1) .

0.2.1 Geometric and Comparison Notation

In the following sections we will utilize the notation

- $L_{(flat)} := \partial_t + \partial_{x_1}$,
- $\underline{L}_{(flat)} := \partial_t - \partial_{x_1}$,
- $X := -\frac{\partial}{\partial x^1}$
- $\mathcal{B}_0 := \{\vec{x} \in \mathbb{R}^3 : \|\vec{x} - (1, 0, 0)\| < 1\}$,
- $\Sigma_t := \{(t', x^1, x^2, x^3) \in \mathbb{R}^{3+1} : t' = t\}$,
- $\mathcal{D}^+(A)$ denotes the domain of future dependence of a set $A \subset \mathbb{R}^{3+1}$, relative to the causal geometry induced by a Lorentzian metric g ,

- $\mathcal{B}_t := \Sigma_t \cap \mathcal{D}^+(\mathcal{B}_0)$,
- $\mathcal{S}_{t,u'} := \mathcal{B}_t \cap \{u(t, x^1) = u'\}$.
- $\sim, \lesssim, \gtrsim$ mean that the equality (inequality) is valid up to constants which can be chosen independent of the (sufficiently small) parameters in the problem.
- Sometimes we write $\sim_C, \lesssim_C, \gtrsim_C$ to emphasize that the implicit constants in the inequality are $1 + \mathcal{O}(\mathcal{C})$.

0.2.2 Semilinear Illposedness

[12] The first equation to consider is the simplest semilinear equation exhibiting failure of the null condition:

$$\square_m \Phi = (\underline{L}_{(Flat)} \Phi)^2. \quad (0.2.2.1)$$

If we restrict our attention to planar symmetric solutions we have:

$$-L_{(Flat)} \underline{L}_{(Flat)} \Phi = (\underline{L}_{(Flat)} \Phi)^2 \quad (0.2.2.2)$$

which if we use (t, u) coordinates, where $u := t - x_1$ (corresponding to the level sets of the outgoing null characteristics), we find that $L = \frac{\partial}{\partial t} \big|_{u=const}$ and so the equation becomes

$$\partial_t \underline{L}_{(Flat)} \Phi = -(\underline{L}_{(Flat)} \Phi)^2$$

which has the solution given in terms of initial data

$$\underline{L}_{(Flat)} \Phi(t, u) = \frac{\underline{L}_{(Flat)} \Phi(0, u)}{1 + t \underline{L}_{(Flat)} \Phi(0, u)}. \quad (0.2.2.3)$$

It is easy to see that in (t, u) coordinates we have $X = \frac{\partial}{\partial u} \big|_{t=const}$.

We have that:

Proposition 0.2.2.1. *For initial data $(\mathring{\Phi}_0, \mathring{\Phi}_1)$ such that*

$$\mathring{\Phi}_0(x^1, x^2, x^3) = \int_0^{x^1} \varepsilon |\ln(x')|^\alpha dx',$$

$$\mathring{\Phi}_1 = 0,$$

and $\alpha \in (0, 1/2)$ (so, in particular, $\underline{L}_{(Flat)} \dot{\Phi}_0 = -\varepsilon |\ln(x^1)|^\alpha$), we find that

$$\left\| (\dot{\Phi}_0, \dot{\Phi}_1) \right\|_{H^2 \times H^1(\mathcal{B}_0)} \leq C\varepsilon \quad (0.2.2.4)$$

and, for any $t > 0$, we have that our solution Φ to (0.2.2.1) with this initial data satisfies

$$\|(\Phi, \partial_t \Phi)\|_{H^2 \times H^1(\mathcal{B}_t)}^2 \geq \int_{\mathcal{B}_t} |\partial \Phi(t, x)|^2 dx \geq \int_{\mathcal{B}_t} |\underline{L}_{(Flat)} \Phi|^2(t, x) dx = \infty. \quad (0.2.2.5)$$

i.e. the energy of the solution (and thus the H^2 norm) in the domain of future dependence of $\mathcal{B}_0$ is instantaneously infinite.

Proof. We may assume that $t > 0$ is small and that $0 < x^1 < 1$. It suffices to note that:

$$|\mathcal{S}_{t,u}| := \int_{\mathcal{B}_t \cap \{x^1 = t-u\}} dx^2 dx^3 = \pi(2-t-x^1)(x^1-t) \sim |u| \quad (0.2.2.6)$$

(using the fact that $\mathcal{B}_t = \{\|\vec{x} - (1, 0, 0)\| < 1-t\}$). A change of variables then shows us that:

$$\int_{\mathcal{B}_t} |\underline{L}_{(Flat)} \Phi|^2(t, x^1) dx^2 dx^3 dx^1 \quad (0.2.2.7)$$

$$= \int_u \int_{\mathcal{S}_{t,u}} |\underline{L}_{(Flat)} \Phi|^2(t, u) dx^2 dx^3 du \quad (0.2.2.8)$$

$$\gtrsim \int_{-1}^0 |u| |\underline{L}_{(Flat)} \Phi|^2(t, u) du \quad \text{Using (0.2.2.6) and planar symmetry} \quad (0.2.2.9)$$

$$= \varepsilon^2 \int_0^1 \frac{u |\ln u|^{2\alpha}}{(1-t\varepsilon |\ln u|^\alpha)^2} du \quad \text{Using (0.2.2.2) with our initial data} \quad (0.2.2.10)$$

$$= \varepsilon^2 \int_{-\infty}^0 \frac{e^{2\nu} |\nu|^{2\alpha}}{(1-t\varepsilon |\nu|^\alpha)^2} d\nu \quad (0.2.2.11)$$

which clearly blows up for $t > 0$ (t sufficiently small) due to the singularity formed at

$$\nu = -\left(\frac{1}{t\varepsilon}\right)^{1/\alpha} < 0,$$

and which is clearly finite for $t = 0$.

It is simple to check then that $\partial_{x^1}^2 \dot{\Phi}_0 \in L^2(\mathcal{B}_0)$ (and thus that $(\dot{\Phi}_0, \dot{\Phi}_1) \in H^2 \times H^1(\mathcal{B}_0)$). Note first that

$$\partial_{x^1}^2 \dot{\Phi}_0 = \alpha\varepsilon \frac{\ln x^1 |\ln x^1|^{\alpha-2}}{x^1}. \quad (0.2.2.12)$$

We merely need to check for finiteness near the singularity at the origin. A change of variables

reduces this to checking that in the region $0 \leq u \leq \delta < 1$:

$$\varepsilon^2 \alpha^2 \int_0^\delta \int_{\mathcal{S}_{t,u}} \frac{|\ln u|^{2(\alpha-1)}}{u^2} dx^2 dx^3 du \sim \varepsilon^2 \alpha^2 \int_0^\delta \frac{|\ln u|^{2(\alpha-1)}}{u} du = \varepsilon^2 \alpha^2 \int_{|\ln \delta|}^\infty \frac{1}{|\nu|^{2(1-\alpha)}} d\nu < \infty \quad (0.2.2.13)$$

as $\alpha \in (0, 1/2)$. It is also clear from the above that we have $\left\| (\dot{\Phi}_0, \dot{\Phi}_1) \right\|_{H^2 \times H^1(\mathcal{B}_0)} \leq C\varepsilon$. $\square$

0.2.3 Semilinear “Weak Null” Illposedness

[5] This next equation exhibits better behavior than the previous case, satisfying the so called “weak null” condition.

We are interested in showing that

$$\square_m \Phi = 0, \quad \square_m \Psi = -(\underline{L}_{(Flat)} \Phi)^2 \quad (0.2.3.1)$$

is illposed for data in $H^2 \times H^1(\mathbb{R}^3)$. In planar symmetry this system takes the form

$$-L_{(Flat)} \underline{L}_{(Flat)} \Phi = 0, \quad L_{(Flat)} \underline{L}_{(Flat)} \Psi = (\underline{L}_{(Flat)} \Phi)^2,$$

which, in (t, u) coordinates implies that

$$\begin{aligned} \underline{L}_{(Flat)} \Phi(t, u) &= \underline{L}_{(Flat)} \Phi(0, u), \\ \underline{L}_{(Flat)} \Psi(t, u) &= t(\underline{L}_{(Flat)} \Phi(0, u))^2. \end{aligned} \quad (0.2.3.2)$$

Note that in (t, u) coordinates, $L_{(Flat)} = \frac{\partial}{\partial t}$ and $\underline{L}_{(Flat)} = 2\frac{\partial}{\partial u} + \frac{\partial}{\partial t}$. It is then simple to check that a solution is given by:

$$\begin{aligned} \Phi(t, u) &= \Phi(0, u) \\ \Psi(t, u) &= \frac{t}{2} \int_0^u (\partial_{u'} \Phi(0, u'))^2 du' \end{aligned} \quad (0.2.3.3)$$

A similar proposition holds:

Proposition 0.2.3.1. *For initial data $(\mathring{\Phi}_0, \mathring{\Phi}_1), (\mathring{\Psi}_0, \mathring{\Psi}_1)$ such that*

$$\begin{aligned}\mathring{\Phi}_0(x^1, x^2, x^3) &= \int_0^{x^1} \varepsilon |\ln(x')|^\alpha dx', \\ \mathring{\Phi}_1 &= \mathring{\Psi}_0 = 0, \\ \mathring{\Psi}_1(x^1, x^2, x^3) &= \frac{1}{2} \int_0^{x^1} \varepsilon^2 |\ln(x')|^{2\alpha} dx'\end{aligned}\tag{0.2.3.4}$$

and $\alpha \in (1/4, 1/2)$ (so, in particular, $\underline{L}_{(Flat)}\Phi_0 = -\varepsilon |\ln(x^1)|^\alpha$), we find that

$$\left\|(\mathring{\Phi}_0, \mathring{\Phi}_1)\right\|_{H^2 \times H^1(\mathcal{B}_0)} + \left\|(\mathring{\Psi}_0, \mathring{\Psi}_1)\right\|_{H^2 \times H^1(\mathcal{B}_0)} \leq C\varepsilon\tag{0.2.3.5}$$

and, for any $t > 0$, we have that our solution Φ, Ψ to (0.2.3.1) with this initial data satisfies

$$\|\Psi\|_{H^2(\mathcal{B}_t)} = \infty.\tag{0.2.3.6}$$

i.e. the H^2 energy of the solution in the domain of future dependence of $\mathcal{B}_0$ is instantaneously infinite.

Proof. The solution in planar symmetry is given above. We can see easily from the previous section that, with our choice of data, $\|\Phi\|_{H^2(\mathcal{B}_0)} \leq \varepsilon$ and clearly $\mathring{\Psi}_1 \in L^\infty(\mathcal{B}_0) \subset L^2(\mathcal{B}_0)$. We also see that $\underline{L}_{(Flat)}\mathring{\Psi}_1(x^1, x^2, x^3) = -\frac{\varepsilon^2}{2} |\ln(x^1)|^{2\alpha} \in L^2(\mathcal{B}_0)$ (as logarithmic singularities are integrable).

Now, to see the breakdown, consider the derivative $X := \frac{\underline{L}_{(Flat)} - L_{(Flat)}}{2} = -\partial_{x^1}$ in (t, x^1) coordinates which in (t, u) coordinates is given by ∂_u .

We see from (0.2.3.3) that

$$\partial_u^2 \Psi = t(\partial_u \Phi)(\partial_u^2 \Phi)\tag{0.2.3.7}$$

and thus, again restricting our attention to the region where $0 \leq u \leq \delta < 1$ (i.e. near the singularity)

we have:

$$\|\Psi\|_{H^2(\mathcal{B}_t)}^2 \geq \int_{\mathcal{B}_t} |X^2 \Psi|^2(t, x^1) dx^2 dx^3 dx^1 \quad (0.2.3.8)$$

$$= \int_u \int_{\mathcal{S}_{t,u}} |\partial_u^2 \Psi|^2(t, u) dx^2 dx^3 du \quad (0.2.3.9)$$

$$\gtrsim \int_{-\delta}^0 |u| |\partial_u^2 \Psi|^2(t, u) dx^2 dx^3 du \quad \text{Using (0.2.2.6) and planar symmetry} \quad (0.2.3.10)$$

$$= t^2 \varepsilon^4 \alpha^2 \int_0^\delta \frac{|\ln u|^{4(\alpha - \frac{1}{2})}}{u} du \quad \text{Using (0.2.3.3), (0.2.3.7), and (0.2.2.12)} \quad (0.2.3.11)$$

$$= t^2 \varepsilon^4 \alpha^2 \int_{|\ln \delta|}^\infty \frac{1}{\nu^{4(\frac{1}{2} - \alpha)}} d\nu \quad (0.2.3.12)$$

which is clearly infinite for $t > 0$ and $\alpha \in (1/4, 1/2)$. $\square$

Remark 0.2.3.1. Note that α has a smaller range for this example. The initial data singularity needs to be stronger and only causes a blow-up at the level of the $H^2(\mathcal{B}_t)$ norm.

0.2.4 Quasilinear Illposedness

[13] This next example is a nonphysical model for quasilinear breakdown, although the symmetry assumption makes it quite straightforward to prove. The more general class of (physically relevant - see below) shock forming $\square_{g(\Phi)} \Phi = 0$ covariant equations break down with the same data and for similar reasons, although the geometry of these equations is richer.

For the toy model we have:

$$\square_m \Phi = -\Phi(\underline{L}_{(Flat)}^2 \Phi). \quad (0.2.4.1)$$

This equation is illposed for the same choice of data, although the breakdown mechanism is fundamentally different, as one sees in the proof.

Proposition 0.2.4.1. *For initial data $(\mathring{\Phi}_0, \mathring{\Phi}_1)$ such that*

$$\begin{aligned} \mathring{\Phi}_0 &= 0, \\ \mathring{\Phi}_1(x^1, x^2, x^3) &= \varepsilon |\ln(x^1)|^\alpha, \end{aligned} \quad (0.2.4.2)$$

and $\alpha \in (0, 1/2)$ (so, in particular, $\underline{L}_{(Flat)} \Phi(0, x) = \varepsilon |\ln(x^1)|^\alpha$), we find that

$$\left\| (\mathring{\Phi}_0, \mathring{\Phi}_1) \right\|_{H^2 \times H^1(\mathcal{B}_0)} \leq C\varepsilon \quad (0.2.4.3)$$

and, for any $t > 0$, we have that our solution Φ to (0.2.4.1) with this initial data satisfies

$$\|(\Phi, \partial_t \Phi)\|_{H^2 \times H^1(\mathcal{B}_t)}^2 \geq \int_{\mathcal{B}_t} |\underline{L}_{(Flat)} \Phi|^2(t, x) dx = \infty. \quad (0.2.4.4)$$

i.e. the H^2 norm of the solution in the domain of future dependence of $\mathcal{B}_0$ is instantaneously infinite.

Remark 0.2.4.1. The rough strategy for proving illposedness for all quasilinear equations in this paper is as follows

Step 1 Use a “geometric renormalization” to rewrite the wave equation and find the foliation density μ of the characteristic coordinates

Step 2.a Show that the wave equation transports $\underline{L}\Phi$ along (outgoing) u -characteristics.

Step 2.b Show that the Eikonal equation gives us a transport equation for μ .

Step 3.a Show that the smallness of Φ in L^∞ propagates.

Step 3.b Show that the characteristic/Lorentzian geometry of $\mathcal{D}^+(\mathcal{B}_0)$ is thus controlled: i.e. $u \sim t - x^1$ and $|\mathcal{S}_{t,u}| \gtrsim |u|$.

Step 4 Prove energy blow-up for the given data by changing to geometric coordinates in the appropriate energy integral.

Proof. Step 1 and 2: Geometry of equation, transporting $\underline{L}\Phi$ and μ

We again work in planar symmetry. If the solution only depends on (t, x_1) , this equation can be rewritten as

$$-L\underline{L}_{(Flat)}\Phi = 0 \quad (0.2.4.5)$$

where $L := L_{(Flat)} - \Phi \underline{L}_{(Flat)}$.

If we let u be an optical function such that

$$Lu = 0, \quad u(0, x^1) = -x^1, \quad (0.2.4.6)$$

then the above equation (0.2.4.5) merely says that

$$\underline{L}_{(Flat)}\Phi = \underline{L}_{(Flat)}\Phi(u),$$

as $\underline{L}_{(Flat)}\Phi$ is constant along the integral curves of L . In particular, on the characteristics $\{u = \text{constant}\}$ we see that $(\underline{L}_{(Flat)}\Phi)$ is determined by its value at $t = 0$:

$$\underline{L}_{(Flat)}\Phi(u(t, x^1)) = \underline{L}_{(Flat)}\Phi(u(0, x^1)) = \varepsilon |\ln(x^1)|^\alpha.$$

For sufficiently small and smooth data, level sets of the optical function foliate a region of spacetime containing the support of the data. Of interest to our problem is the behavior of this optical u foliation relative to the background coordinates (t, x^1) . We will define

$$\mu^{-1} := \underline{L}_{(Flat)}u \tag{0.2.4.7}$$

to be the foliation density¹ with $\mu^{-1}(0, x^1) = \underline{L}_{(Flat)}u(0, x^1) = \underline{L}_{(Flat)}(-x^1) = 1$.

It is straightforward to see that

$$\left[L, \underline{L}_{(Flat)} \right] = (\underline{L}_{(Flat)}\Phi)\underline{L}_{(Flat)} \tag{0.2.4.8}$$

and thus that

$$L(\mu^{-1}) = L\underline{L}_{(Flat)}u \quad \text{From (0.2.4.7)} \tag{0.2.4.9}$$

$$= \left[L, \underline{L}_{(Flat)} \right] u \quad \text{From (0.2.4.6)} \tag{0.2.4.10}$$

$$= \underline{L}_{(Flat)}\Phi \cdot \underline{L}_{(Flat)}u \quad \text{From (0.2.4.8)} \tag{0.2.4.11}$$

$$= \mu^{-1} \underline{L}_{(Flat)}\Phi \quad \text{From (0.2.4.7)} \tag{0.2.4.12}$$

In other words, we see that $L\mu = -\mu \left(\underline{L}_{(Flat)}\Phi(u) \right)$ and thus if we choose a (nonlinearly) rescaled time-coordinate $\tilde{t}$ such that

$$L\tilde{t} = 1 \quad \text{and} \quad \tilde{t}|_{t=0} = 0 \tag{0.2.4.13}$$

we find that

$$\mu = (\underline{L}_{(Flat)}u)^{-1} = e^{-\tilde{t}(\underline{L}_{(Flat)}\Phi(u))}. \tag{0.2.4.14}$$

Note that as our choice of initial data ensures $\underline{L}_{(Flat)}\Phi \geq 0$, we have $0 \leq \mu \leq 1$. Now commuting (0.2.4.5) with $\underline{L}_{(Flat)}$ and using our commutation formula (0.2.4.8) we also find that

¹i.e. measuring the local change in u relative to variations of the background coordinates in the $\partial_t - \partial_{x^1}$ direction

$$L\left(\underline{L}_{(Flat)}^2\Phi\right) = \left(\underline{L}_{(Flat)}\Phi\right)\left(\underline{L}_{(Flat)}^2\Phi\right) \quad (0.2.4.15)$$

which tells us that

$$\left(\underline{L}_{(Flat)}^2\Phi\right)(\tilde{t}, u) = e^{\tilde{t}(\underline{L}_{(Flat)}\Phi(u))}\left(\underline{L}_{(Flat)}^2\Phi\right)(0, u). \quad (0.2.4.16)$$

Step 3.a: Smallness of Φ

Now in order to show that Φ remains small given our initial conditions $(\dot{\Phi}_0, \dot{\Phi}_1)$, we need to consider the characteristics of $\underline{L}_{(Flat)}$, which we will define in (t, u) coordinates by

$$\underline{L}_{Flat}\underline{u} = 0, \quad \underline{u}(t = 0, u) = -u, \quad (0.2.4.17)$$

so that in these $(u, \underline{u})$ coordinates we have

$$\underline{L}_{(Flat)}\Phi = (\partial_u\Phi)\underline{L}_{(Flat)}u + (\partial_{\underline{u}}\Phi)\underline{L}_{(Flat)}\underline{u} = (\partial_u\Phi)\underline{L}_{(Flat)}u. \quad (0.2.4.18)$$

Integrating $\partial_u\Phi(u, \underline{u})$ in the region $0 \leq -u \leq \underline{u} \leq 1$, we find:

$$\Phi(u, \underline{u}) = \Phi(u, \underline{u}) - \Phi(-\underline{u}, \underline{u}) \quad \text{as } \Phi(-\underline{u}, \underline{u}) = \Phi(t = 0, u) = 0 \text{ by (0.2.4.17)} \quad (0.2.4.19)$$

$$= \int_{-\underline{u}}^u \partial_{u'}\Phi(u', \underline{u})du' \quad \text{by fundamental theorem of calculus} \quad (0.2.4.20)$$

$$= \int_{-\underline{u}}^u \underline{L}_{(Flat)}\Phi \cdot \boldsymbol{\mu} du' \quad \text{by (0.2.4.18) and (0.2.4.7)} \quad (0.2.4.21)$$

$$\geq 0 \quad \text{as } \underline{L}_{(Flat)}\Phi \geq 0 \text{ and } \boldsymbol{\mu} \geq 0. \quad (0.2.4.22)$$

Then note that

$$\Phi(u, \underline{u}) = \int_{-\underline{u}}^u \underline{L}_{(Flat)}\Phi \cdot \boldsymbol{\mu} du' \quad \text{by (0.2.4.18) and (0.2.4.7)} \quad (0.2.4.23)$$

$$\leq \int_{-\underline{u}}^u \underline{L}_{(Flat)}\Phi du' \quad \text{as } 0 \leq \boldsymbol{\mu} \leq 1 \text{ by (0.2.4.14)} \quad (0.2.4.24)$$

$$\leq \varepsilon \int_0^1 |\ln(u)|^\alpha du \quad \text{from the intital data, as } \underline{L}_{(Flat)}\Phi \text{ is a function of } u \quad (0.2.4.25)$$

$$\lesssim \varepsilon \quad \text{where the implicit constant only depends on } \alpha \quad (0.2.4.26)$$

and thus $|\Phi| \lesssim_\alpha \varepsilon$

Step 3.b: Controlling the characteristic geometry

Note that we have

$$u - (t - x^1) = 0 \quad \text{at} \quad t = 0$$

and that

$$L(u - (t - x^1)) = -L(t - x^1) = -(L_{(Flat)} - \Phi \underline{L}_{(Flat)})(t - x^1) = 2\Phi.$$

Thus, using that $|\Phi| \lesssim \varepsilon$ and integrating along integral curves of L , we see that in $\mathcal{D}^+(\mathcal{B}_0)$ we have

$$u \sim t - x^1. \tag{0.2.4.27}$$

It is also important to note that as $Lu = L_{(Flat)}u - \Phi \underline{L}_{(Flat)}u = 0$ we have

$$L_{(Flat)}u = \Phi \underline{L}_{(Flat)}u = \Phi \mathbf{\mu}^{-1}. \tag{0.2.4.28}$$

Putting this together with the definition $\underline{L}_{(Flat)}u = \mathbf{\mu}^{-1}$, we have that:

$$\frac{\partial}{\partial x^1}u = (L_{(Flat)}u - \underline{L}_{(Flat)}u) = (\Phi - 1)\mathbf{\mu}^{-1} \sim -\mathbf{\mu}^{-1} \tag{0.2.4.29}$$

and thus

$$\left| \frac{dx^1}{du} \right| \sim \mathbf{\mu}. \tag{0.2.4.30}$$

Finally, as $|\Phi| \lesssim \varepsilon$, our quasilinear metric $(g^{-1})^{\alpha\beta}(\Phi)$ (giving $(g^{-1})^{\alpha\beta}(\Phi)\partial_{\alpha\beta}^2\Phi = \square_m\Phi + \Phi \underline{L}^2\Phi = 0$) is uniformly close to the Minkowski metric and so we note that $\partial\mathcal{B}_t \cap \{t > 0\}$ is uniformly close to the corresponding light cone in Minkowski space (see [13] for a more detailed proof). This together with (0.2.2.6), and (0.2.4.27) tells us that for ε sufficiently small and $0 \leq |u| \leq \delta < 1$ we have

$$|\mathcal{S}_{t,u}| \sim_\varepsilon |u| \tag{0.2.4.31}$$

Step 4: Instantaneous H^2 blow-up

Putting these facts together gives us that:

$$\int_{\mathcal{B}_t \cap \{0 < x^1 < 1\}} \left| \underline{L}_{(Flat)}^2 \Phi(t, x_1) \right|^2 dx \quad (0.2.4.32)$$

$$= \int_{-1}^0 \int_{\mathcal{S}_{t,u}} \left| \underline{L}_{(Flat)}^2 \Phi(t, u) \right|^2 \left| \frac{dx^1}{du} \right| dx^2 dx^3 du \quad \text{changing to characteristic coordinates} \quad (0.2.4.33)$$

$$= \int_{-1}^0 \left| \underline{L}_{(Flat)}^2 \Phi(t, u) \right|^2 |\mathcal{S}_{t,u}| \left| \frac{dx^1}{du} \right| du \quad \text{integrating } dx^2 dx^3 \quad (0.2.4.34)$$

$$\gtrsim \int_{-\delta}^0 |u| \left| \underline{L}_{(Flat)}^2 \Phi(t, u) \right|^2 \left| \frac{dx^1}{du} \right| du \quad \text{by (0.2.4.31)}. \quad (0.2.4.35)$$

Finally, our transport equations for $\underline{L}_{(Flat)}^2 \Phi$ and μ give us that

$$\int_{-\delta}^0 |u| \left| \underline{L}_{(Flat)}^2 \Phi(t, u) \right|^2 \left| \frac{dx^1}{du} \right| du \quad (0.2.4.36)$$

$$\sim \int_0^\delta u e^{2\tilde{t}(\underline{L}_{(Flat)} \Phi(|u|))} \left| \underline{L}_{(Flat)}^2 \Phi(0, |u|) \right|^2 \mu du \quad \text{using (0.2.4.16) and (0.2.4.30)} \quad (0.2.4.37)$$

$$= \int_0^\delta u e^{\tilde{t}(\underline{L}_{(Flat)} \Phi(|u|))} \left| \underline{L}_{(Flat)}^2 \Phi(0, |u|) \right|^2 du \quad \text{using (0.2.4.14)} \quad (0.2.4.38)$$

where $0 \leq |u| \leq \delta < 1$ (here again, δ is merely restricting our attention to the region near the singularity at $u = 0$). We then simply need to check that, for our choice of “Lindblad-type” initial data, we have:

$$\int_{\mathcal{B}_0} \left| \underline{L}_{(Flat)}^2 \Phi(0, x) \right|^2 dx < \infty \quad (0.2.4.39)$$

while for $t > 0$ we have

$$\int_{\mathcal{B}_t} \left| \underline{L}_{(Flat)}^2 \Phi(t, x) \right|^2 dx = \infty. \quad (0.2.4.40)$$

It is clear from the computation in the semilinear counterexample above that our data is in $H^2(\mathcal{B}_0)$. Now we use the fact that as $|\Phi| \lesssim \varepsilon$ we have $t \sim \tilde{t}$ (note that $L(t - \tilde{t}) = (1 - \Phi) - 1 = -\Phi$ and $t - \tilde{t} = 0$ for $t = 0$, then integrate along the characteristics) and so we merely need to check that

$$\int_0^\delta u e^{t(\underline{L}_{(Flat)} \Phi(u))} \left| \underline{L}_{(Flat)}^2 \Phi(0, u) \right|^2 du = \infty \quad (0.2.4.41)$$

for all $t > 0$ with this choice of initial data. This follows from (0.2.4.38) by noting that

$$\left| \underline{L}_{(Flat)}^2 \Phi(0, u) \right|^2 = \varepsilon^2 |\partial_u (|\ln u|^\alpha)|^2 = \alpha^2 \varepsilon^2 \frac{|\ln u|^{2(\alpha-1)}}{u^2}$$

together with the following simple computation:

$$\int_0^\delta u \alpha^2 \varepsilon^2 \frac{|\ln u|^{2(\alpha-1)}}{u^2} du = \alpha^2 \varepsilon^2 \int_{|\ln \delta|}^\infty e^{\varepsilon t \nu^\alpha} \nu^{2(\alpha-1)} d\nu \quad (0.2.4.42)$$

which is clearly infinite for any $t > 0$. □

0.3 Plane Symmetric Case

This section heavily utilizes the ideas and computations in the introduction to [18] which demonstrates the formation of shocks in the following example, although we are interested here in an illposedness result along the line of Lindblad's examples presented above.

Here we will consider covariant geometric wave equations of the form:

$$\square_{g(\Phi)} \Phi = \mathcal{N}(\Phi)(\partial\Phi, \partial\Phi) \quad (0.3.0.1)$$

such that

- g is a perturbation of the Minkowski metric: $g(\Phi)_{\alpha\beta} = m_{\alpha\beta} + g_{\alpha\beta}^{(Small)}(\Phi)$, $g^{(Small)}(0) = 0$,
- g fails the quasilinear null condition (for the precise algebraic condition see (0.7.1.1) below),
- $\mathcal{N}$ is null, i.e. $\mathcal{N}(\Phi)(\xi, \xi) = 0$ for all covectors ξ satisfying $g^{-1}(\xi, \xi) = 0$,
- the solution is plane symmetric (see (0.3.0.5) and (0.3.0.6)).

Remark 0.3.0.1. Note that these are precisely the shock forming plane symmetric equations discussed in the introduction to [18].

Theorem 0.3.1. *For equations (0.3.0.1) satisfying the above conditions, we can specify Lindblad type initial data with*

$$\partial_u \Phi(0, u) = \varepsilon |\ln(u)|^\alpha, \quad \alpha \in (0, 1/2)$$

such that the initial data is in $H^2 \times H^1(\mathcal{B}_0)$ and such that for any $t > 0$ we have:

$$\int_{\mathcal{B}_t} |\partial\Phi(t, x)|^2 dx = \infty \quad (0.3.0.2)$$

i.e. the energy blows up instantaneously. Thus, such equations are illposed in $H^2 \times H^1(\mathbb{R}^3)$.

Remark 0.3.0.2. The fundamental breakdown mechanism here is very much the same as in Lindblad's quasilinear counterexample, although the blowup is much stronger, as it happens at the level of the H^1 norm rather than the H^2 norm. Note that it is not quite as strong as the illposedness result for $\square_m \Phi = (\underline{L}_{(Flat)} \Phi)^2$, see (0.5.0.1) below.

Proof. Step 1, 2 and 3.a: Geometry of equation, transporting $\underline{L}\Phi$ and μ , smallness of Φ

Let $u, \underline{u}$ be eikonal functions satisfying

$$0 = (g^{-1})^{\alpha\beta}(\Phi) \partial_\alpha u \partial_\beta u \quad (0.3.0.3)$$

$$= (g^{-1})^{\alpha\beta}(\Phi) \partial_\alpha \underline{u} \partial_\beta \underline{u} \quad (0.3.0.4)$$

such that $\partial_t u, \partial_t \underline{u} > 0$ and satisfying initial conditions $u|_{t=0} = -x^1$ and $\underline{u}|_{t=0} = x^1$.

In these coordinates (assuming they are well defined and that du and $d\underline{u}$ are linearly independent), the metric takes the form

$$g = -\Omega^2(du \otimes d\underline{u} + d\underline{u} \otimes du) + \not{g}_{AB}(dx^A + \Xi^A du) \otimes (dx^B + \Xi^B du), \quad (0.3.0.5)$$

and we will make the following planar symmetry assumption:

$$\Phi = \Phi(u, \underline{u}). \quad (0.3.0.6)$$

From the fact that $\square_g \Phi = \frac{1}{|\det g|} \partial_\alpha (\sqrt{|\det g|} (g^{-1})^{\alpha\beta} \partial_\beta \Phi)$, we may derive that in these coordinates (using (0.3.0.5) and our planar symmetry assumption (0.3.0.6)) the equation $\square_{g(\Phi)} \Phi = 0$ takes the form

$$\partial_u \partial_{\underline{u}} \Phi = \mathcal{N}(\Phi) \partial_u \Phi \partial_{\underline{u}} \Phi. \quad (0.3.0.7)$$

Thus if the initial data satisfies $\partial_{\underline{u}} \Phi = 0$ then by (0.3.0.7) this condition holds in the domain of future dependence and thus our solution is simply a function of u (i.e. it is constant along the outgoing characteristics). Applying a cutoff away from the origin, we may assume $\Phi(0, u) = 0$ for $u < -1$, and it is easy to see from integrating the data with respect du that $|\Phi| \lesssim \epsilon$ if we choose Lindblad-type initial data $\partial_u \Phi(u) = \pm \epsilon |\ln(|u|)|^\alpha$. Thus our metric $g(\Phi)$ is simply a function of u , which we may denote $g(u)$.

Note that by (0.3.0.3), the vectorfield

$$L_{(Geo)} = -(g^{-1})^{\alpha\beta} \partial_\alpha u \partial_\beta \quad (0.3.0.8)$$

is a future pointing ($L_{(Geo)}^0 > 0$) vector tangent to the curves $\{u = \text{constant}\}$. Furthermore, differentiating the equation $(g^{-1})^{\alpha\beta}(u)\partial_\alpha u \partial_\beta u = 0$ with respect to ∂_ν , we find that

$$0 = 2(g^{-1})^{\alpha\beta}(u)\partial_\alpha u \partial_\beta(\partial_\nu u) + \frac{d}{du}((g^{-1})^{\alpha\beta}(u))\partial_\alpha u \partial_\beta u \partial_\nu u \quad (0.3.0.9)$$

and so rearranging and using (0.3.0.8) gives us that

$$L_{(Geo)}(\partial_\nu u) = \frac{1}{2} \left(\frac{d}{du}((g^{-1})^{\alpha\beta}(u))\partial_\alpha u \partial_\beta u \right) \partial_\nu u. \quad (0.3.0.10)$$

Thus $\partial_1 u$ and $\partial_2 u$ satisfy the same ODE along the characteristic curves $\{u = \text{constant}\}$ and so they must be constant scalar multiples of one another. We may reparameterize the characteristic such that (0.3.0.10) becomes $L_{(Geo)}(\partial_\nu u) = 0$. Thus the characteristics are clearly straight lines in (t, x^1) coordinates with possibly varying slope, determined by initial data satisfying²:

$$0 = (g^{-1})^{\alpha\beta}(u)\partial_\alpha u \partial_\beta u = (g^{-1})^{00}(\partial_0 u)^2 - 2(g^{-1})^{01}(\partial_0 u) + (g^{-1})^{11}, \quad (0.3.0.11)$$

where the $(g^{-1})^{\alpha\beta}$ are functions of u and hence of x^1 on the initial surface.

We define the lapse function (inverse foliation density) μ by:

$$\mu^{-1} := -(g^{-1})^{\alpha\beta}\partial_\alpha t \partial_\beta u \quad (0.3.0.12)$$

and we derive its associated transport equation as follows. Note that if we define the rescaled null vectorfield

$$L := -\mu \mathcal{D}u = -\mu(g^{-1})^{\alpha\beta}\partial_\beta u \partial_\alpha \quad (0.3.0.13)$$

then by the eikonal equation we have that $Lu = 0$ and so the null vectorfield L is a multiple of ∂_u .

As L is tangent to the characteristics and as $Lt = 1$ we deduce that we must have $L^\alpha = L^\alpha(u)$, where $\alpha \in \{0, 1, 2, 3\}$ denotes that we are considering the rectangular coordinates in $\mathbb{R}^{3+1}$. In (t, u) coordinates we have trivially that

$$\left. \frac{\partial}{\partial t} \right|_{u=\text{const}} = L \quad (0.3.0.14)$$

²note that a solution $\partial_0 u$ is guaranteed by hyperbolicity, which tells us that $((g^{-1})^{01})^2 - ((g^{-1})^{00}) \cdot ((g^{-1})^{11}) > 0$

and we can define

$$X := \mu^{-1} \frac{\partial}{\partial u} \Big|_{t=\text{const}}, \quad (0.3.0.15)$$

$$\check{X} := \mu X = \frac{\partial}{\partial u} \Big|_{t=\text{const}}. \quad (0.3.0.16)$$

In the rectangular coordinates we find (following [18]) that differentiating (0.3.0.12) gives us:

$$-\mu^{-2} L\mu = L\mu^{-1} \quad (0.3.0.17)$$

$$= -(L(g^{-1})^{\alpha\beta} \partial_\alpha t \partial_\beta u - (g^{-1})^{\alpha\beta} (L\partial_\alpha t) \partial_\beta u - (g^{-1})^{\alpha\beta} \partial_\alpha t (L\partial_\beta u)). \quad (0.3.0.18)$$

Now using that $Lg = g'(u) \cdot Lu = 0$ and that $L(\partial_\alpha t) = 0$ (as $\partial_\alpha t$ is constant) we see that (0.3.0.18) is equal to:

$$= -(g^{-1})^{\alpha\beta} \partial_\alpha t (L\partial_\beta u) \quad (0.3.0.19)$$

$$= \mu (g^{-1})^{\alpha\beta} \partial_\alpha t (g^{-1})^{\gamma\delta} \partial_\delta u \partial_\gamma \partial_\beta u \quad \text{from (0.3.0.13)} \quad (0.3.0.20)$$

$$= -\frac{1}{2} \mu (g^{-1})^{\alpha\beta} \partial_\alpha t \partial_\beta ((g^{-1})^{\gamma\delta} \partial_\delta u \partial_\gamma u) \quad \text{differentiating (0.3.0.3)} \quad (0.3.0.21)$$

$$= \frac{1}{2} \mu^{-1} (g^{-1})^{0\beta} \partial_\beta \Phi G_{LL} \quad \text{from (0.3.0.13) and (0.7.1.1)} \quad (0.3.0.22)$$

$$= -\frac{1}{2} \mu^{-1} (X\Phi + L\Phi) G_{LL} \quad \text{from (0.7.3.4).} \quad (0.3.0.23)$$

Recalling that $L\Phi = 0$ and using (0.3.0.16) we thus have that:

Lemma 0.3.0.1. *For plane symmetric solutions of (0.3.0.1), the inverse foliation density μ defined by (0.3.0.12) satisfies the transport equation:*

$$L\mu = \frac{1}{2} G_{LL} \check{X}\Phi. \quad (0.3.0.24)$$

Let us summarize what we understand thus far about solutions to the equation:

- The solution is constant along characteristics of constant u (the outgoing null characteristics)
- thus smallness of the initial data $\mathring{\Phi}_0$ propagates
- The collapse of the characteristics is entirely a function of initial data, and their density is given in (t, u) coordinates by $\mu(t, u) = \mu(0, u) + \frac{t}{2} G_{LL}(u) \check{X}\Phi(u)$

Step 3.b: Controlling the geometry of $\mathcal{D}^+(\mathcal{B}_0)$

Note that as $|\Phi| \lesssim \varepsilon$ we have that the domain of future dependence of $\mathcal{B}_0$ with respect to our g metric is uniformly close to the domain with respect to the Minkowski metric, so integrating in the (x^2, x^3) variables gives us that

$$\int_{\mathcal{B}_t \cap \{x^1 = \text{constant}\}} dx^2 dx^3 \sim_\varepsilon |t - x^1|. \quad (0.3.0.25)$$

Now (as Φ is small and the rectangular coordinates of L are a function only of u) we clearly have that

$$L^\alpha(u) = L_{(Flat)}^\alpha + L_{(Small)}^\alpha(u) \quad (0.3.0.26)$$

(where $L_{(Small)}^0 = 0$ as $Lt = L_{(Flat)}t = 1$) and so we may integrate the equation $L(t - x^1 - u) = L_{(Small)}^0(u) - L_{(Small)}^1(u) = -L_{(Small)}^1(u)$ along the characteristics, using that $t - x^1 = u$ at $t = 0$ and that $|L_{(Small)}^\alpha| \lesssim \varepsilon$, to get that

$$t - x^1 \sim_\varepsilon u \quad (0.3.0.27)$$

in the domain of dependence $\mathcal{D}^+(\mathcal{B}_0)$. Together, (0.3.0.25) and (0.3.0.27) imply that

$$|\mathcal{S}_{t,u}| = \int_{\mathcal{B}_t \cap \{u = \text{constant}\}} dx^2 dx^3 \sim_\varepsilon |u|. \quad (0.3.0.28)$$

Step 4: Instantaneous energy blow-up

Finally, it is not so difficult to derive that

$$\int_{\mathcal{B}_t} |X\Phi(t, x)|^2 dx$$

is instantaneously infinite for Lindblad-type data $\partial_u \Phi(t, u) \sim \varepsilon |\log(|u|)|^\alpha$ (and that this data is initially in $H^2(\mathcal{B}_0)$). Note that X is a vectorfield of g length 1 and that by smallness of Φ we have that X is approximately of Euclidean length 1 as well - thus the standard energy in rectangular coordinates $\int_{\mathcal{B}_t} |\partial \Phi|(t, x) dx$ is blowing up instantaneously.

We will denote by $\mathfrak{V}$ the coordinate transformation mapping $\mathfrak{V}(t, u) = (t, x^1)$. Note that the

Jacobian of $\mathfrak{Y}$ is

$$|\det \partial \mathfrak{Y}| = \left| \begin{pmatrix} 1 & 0 \\ L^1 & \check{X}^1 \end{pmatrix} \right| \quad (0.3.0.29)$$

$$= |\check{X}^1| \quad (0.3.0.30)$$

$$= \boldsymbol{\mu} |X^1| \quad \text{by (0.3.0.16)} \quad (0.3.0.31)$$

$$\sim_\varepsilon \boldsymbol{\mu}(t, u) \quad \text{from (0.7.3.4) and (0.3.0.26).} \quad (0.3.0.32)$$

From the above we actually also see that

$$\left| \frac{dx^1}{du} \right| = |\check{X}^1| \sim_\varepsilon \boldsymbol{\mu}(t, u) \quad (0.3.0.33)$$

The punchline is then as in the previous examples: we change variables in the energy integral and use that

$$X\Phi(t, u) = \frac{\check{X}\Phi(t, u)}{\boldsymbol{\mu}(t, u)} = \frac{\partial_u \Phi(u)}{\boldsymbol{\mu}(t, u)}. \quad (0.3.0.34)$$

Thus we can see that if $u_{(shock)}(t)$ is the value of u for which a shock forms at time t , i.e. such that $\boldsymbol{\mu}(t, u_{(shock)}(t)) = 0$ and if we set $\kappa(t) := \min \{0, u_{(shock)}(t)\}$, then we have that:

$$\int_{\mathcal{B}_t} |X\Phi(t, x)|^2 dx \geq \int_{-\delta}^{\kappa(t)} \int_{\mathcal{S}_{t,u}} \left| \frac{\check{X}\Phi(u)}{\boldsymbol{\mu}(t, u)} \right|^2 dx^2 dx^3 \left| \frac{dx^1}{du} \right| du \quad (0.3.0.35)$$

$$\sim_\varepsilon \int_{-\delta}^{\kappa(t)} \int_{\mathcal{S}_{t,u}} \left| \frac{\check{X}\Phi(u)}{\boldsymbol{\mu}(t, u)} \right|^2 \boldsymbol{\mu}(t, u) dx^2 dx^3 du \quad \text{by (0.3.0.33)} \quad (0.3.0.36)$$

$$\gtrsim \int_{-\delta}^{\kappa(t)} |u| \left| \frac{\check{X}\Phi(u)}{\boldsymbol{\mu}(t, u)} \right|^2 \boldsymbol{\mu}(t, u) du \quad \text{by (0.3.0.28)} \quad (0.3.0.37)$$

$$= \int_{-\delta}^{\kappa(t)} \frac{|u| |\partial_u \Phi(u)|^2}{\boldsymbol{\mu}(0, u) + \frac{t}{2} G_{LL} \partial_u \Phi(u)} du \quad \text{by (0.3.0.34)} \quad (0.3.0.38)$$

which will clearly blow up for any small $t > 0$ if we have $\partial_u \Phi(u) = \pm \varepsilon |\log(|u|)|^\alpha$ (depending on the sign of G_{LL}), following similiar reasoning as in the previous sections. $\square$

0.4 “Asymptotically Stable” H^2 Ill-Posedness for Generic Covariant Equations of the Form: $\square_{g(\Phi)}\Phi = 0$

Note that the above proof in planar symmetry is effectively a corollary of the proof of planar shock formation given in the introduction to [18]. It turns out that an even more general and robust illposedness statement is possible, which is likewise a corollary of the $3 + 1$ dimensional analog of the full result presented in [18].

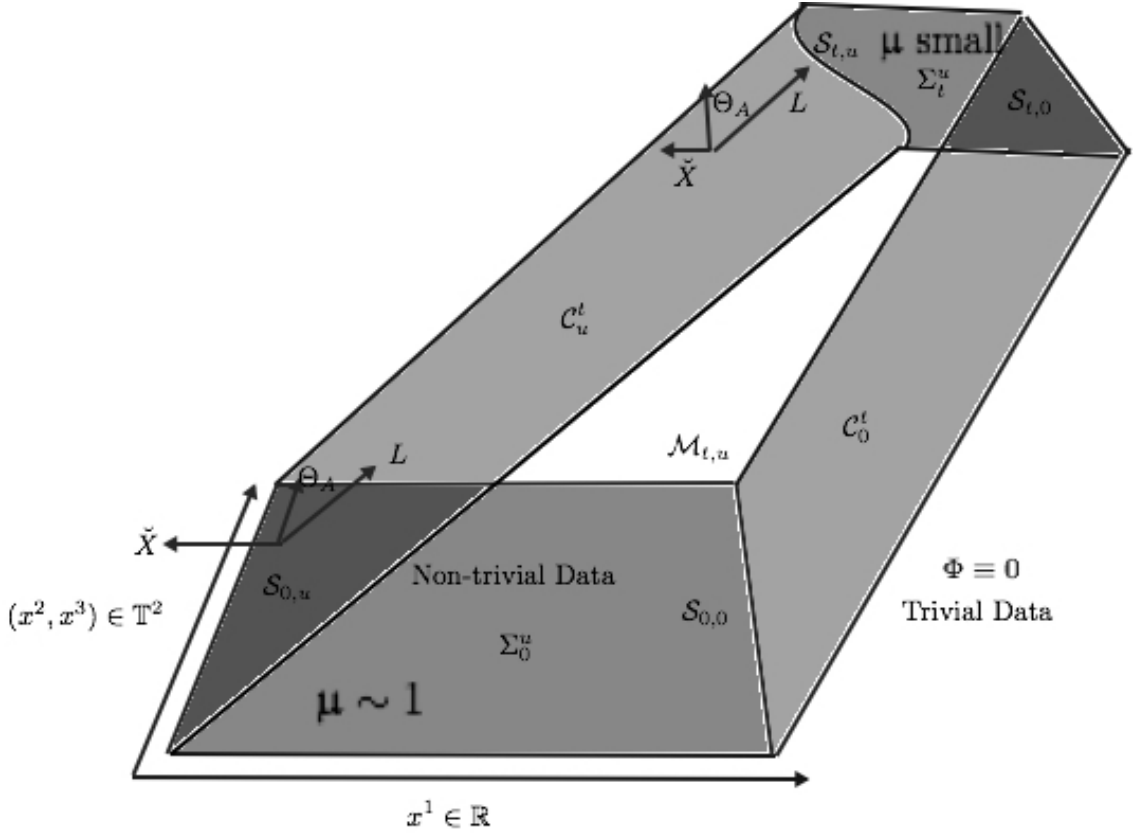


Figure 1: The spacetime region and dynamic vectorfield frame

The geometric setup and notation here is slightly different, see (0.7.2.2) and Figure 1 for reference (we note that Figure 1 parallels several of those depicted in [18]). The statement and proof of the main theorem (0.4.1) rely heavily on the following notation:

Outgoing null characteristic hypersurface C_u : We denote the level sets of $u(t, x^i) = u'$ by $C_{u'}$.

$\mathring{\delta}$ - size of “large” C_u -transverse derivatives on initial data:

$$\mathring{\delta} := \|\check{X}\Phi\|_{L^\infty(\Sigma_0^1)} + \|\check{X}^2\Phi\|_{L^\infty(\Sigma_0^1)} + \|\check{X}^3\Phi\|_{L^\infty(\Sigma_0^1)}.$$

$\mathring{\delta}_*$ - inverse time to shock formation: $\mathring{\delta}_* := \frac{1}{2} \min_{\Sigma_0^1} \left[G_{LL} \check{X} \Phi \right]_-$.

$\mathring{\epsilon}$ - size of “small” $\mathcal{C}_u$ -tangential derivatives on initial data: We consider perturbations from planar symmetry of size

$$\|\mathcal{L}_*^{\leq 18;3} \Phi\|_{L^\infty(\Sigma_0^1)}, \|\mathcal{L}_*^{\leq 20;3} \Phi\|_{L^2(\Sigma_0^1)} \leq \mathring{\epsilon},$$

where $\mathcal{L}_*^{\leq M;N}$ means that we take at least one $\mathcal{C}_u$ -tangential derivative, we take up to M total derivatives, and where at most N of them are $\mathcal{C}_u$ -transverse (i.e. $\check{X}$) derivatives. For more information on the other derivative notation $\mathcal{P}, \mathcal{P}_*$, etc. which we use below, see (0.18.2).

$\mathring{\alpha}$ - size of $\mathring{\Phi}_0$ in L^∞ : We will denote

$$\mathring{\alpha} := \|\mathring{\Phi}\|_{L^\infty(\Sigma_0)}$$

$\mathcal{O}_*$ and $\mathcal{O}$ notation: We will use $A = \mathcal{O}(B)$ to mean $|A| \leq C|B|$ where C can be made independent of $\mathring{\alpha}, \epsilon$ (a bootstrap smallness parameter, see (0.18.5.1)), and $\mathring{\epsilon}$ assuming all of these parameters are sufficiently small relative to 1 and that $\mathring{\epsilon}, \epsilon$ are sufficiently small relative to $\mathring{\delta}^{-1}$ and $\mathring{\delta}_*$. We will use $A = \mathcal{O}_*(B)$ to mean $|A| \leq C|B|$ where now C does not depend on any of the constants in the problem as long as $\mathring{\epsilon}$ and $\mathring{\alpha}$ are sufficiently small relative to 1. Essentially $\mathcal{O}$ means the implicit terms have $\mathring{\delta}, \mathring{\delta}_*$ (potentially large quantity) dependence and $\mathcal{O}_*$ means that there is no dependence of the implicit constants on these potentially large quantities.³

U_0 characteristic width of the domain of wavefront region: We are interested in the region of spacetime $\bigcup_{0 \leq u \leq U_0} \mathcal{C}_u$, for $0 \leq U_0 \leq 1$. U_0 is thus the “characteristic width” of the wavefront subregion under consideration.

μ_* - minimum of inverse foliation density over torus: $\mu_*(t, u) := \min_{\theta^A \in \mathbb{T}^2} \mu(t, u, \theta^A)$.

The statement of the theorem is nearly identical to that in [18], with minor differences in nomenclature due to the presence of our additional dimension and hence one more derivative required for Sobolev imbedding (see (0.18.3.1) and also (0.6.0.1)).

Theorem 0.4.1 (Stable Shock Formation in $\mathbb{R}^{3+1}$ Near Planar Symmetry, following Theorem 15.1 in [18]). *For geometric wave equations of the form $\square_{g(\Phi)} \Phi = 0$ which fail the null condition in the direction $L_{(Flat)} = \partial_t + \partial_{x^1}$ (see (0.1.2.1) for the exact condition), assume that the data $(\mathring{\Phi}_0, \mathring{\Phi}_1)$ satisfies the data size and regularity assumptions given above (see also (0.18.3.1) for a more extensive*

³We here are following the notation in [17] which corrects a small error in [18] involving the L^∞ size of the initial data. The correction is made by using the $\mathring{\alpha}$ and $\mathcal{O}_*$ notation which we are introducing here.

discussion). For $U_0 \in [0, 1]$, let $T_{(Lifespan); U_0}$ be the lifespan of the solution in the wavefront region $\mathcal{M}_{T_{(Lifespan); U_0}, U_0}$ (see figure 1) determined by the data in $\Sigma_0^{U_0}$ and trivial data in the region $\Sigma_0 \cap \{x_1 \geq 0\}$. For $\mathring{\epsilon}$ sufficiently small relative to $\mathring{\delta}^{-1}$ and $\mathring{\delta}_*$ and $\mathring{\alpha}$ sufficiently small relative to 1, we have that one of two mutually exclusive alternatives occurs:

- **Case 1:** $T_{(Lifespan); U_0} > 2\mathring{\delta}_*^{-1}$ and $\inf \left\{ \mu_*(s, U_0) : s \in [0, 2\mathring{\delta}_*^{-1}] \right\} > 0$
- **Case 2:** $T_{(Lifespan); U_0} \leq 2\mathring{\delta}_*^{-1}$ and

$$T_{(Lifespan); U_0} = \sup \left\{ t \in [0, 2\mathring{\delta}_*^{-1}) : \inf \{ \mu_*(s, U_0) : s \in [0, t] \} > 0 \right\}$$

where $\mu_*(t, u)$ is defined in (0.21.0.5) as the minimum of μ along Σ_t^u . For $U_0 = 1$, we have that **Case 2** occurs and $T_{(Lifespan); 1} = \{1 + \mathcal{O}(\mathring{\epsilon}) + \mathcal{O}_*(\mathring{\alpha})\} \mathring{\delta}_*^{-1}$. In **Case 2** we have for $(t, u) \in [0, T_{(Lifespan); U_0}] \times [0, U_0]$ that:

$$\left\| \mathcal{L}_*^{\leq 9} L_{(Small)}^i \right\|_{L^\infty(\Sigma_t^u)} \leq C\mathring{\epsilon}, \quad (0.4.0.1)$$

$$\left\| \mathcal{L}_*^{\leq 10} \Phi \right\|_{L^\infty(\Sigma_t^u)} \leq C\mathring{\epsilon}. \quad (0.4.0.2)$$

This is a consequence of the following more explicit result:

What happens in Case I). In case **I**), all bootstrap assumptions, the pointwise and L^∞ estimates of (0.19.6.2) and (0.20.3.1), and the energy estimates of Proposition (0.26.7.2) hold on $\text{cl}\mathcal{M}_{2\mathring{\delta}_*^{-1}, U_0}$ with the factors of ε on the RHS replaced by $C\mathring{\epsilon}$. Moreover, for $0 \leq M \leq 5$, the following estimates hold for $(t, u) \in [0, 2\mathring{\delta}_*^{-1}] \times [0, U_0]$:

$$\left\| \mathcal{P}_*^{[1, 13]} \mu \right\|_{L^2(\Sigma_t^u)}, \left\| \mathcal{P}^{\leq 13} L_{(Small)}^i \right\|_{L^2(\Sigma_t^u)}, \left\| \mathcal{L}_{\mathcal{P}}^{\leq 12} \chi \right\|_{L^2(\Sigma_t^u)} \leq C\mathring{\epsilon}, \quad (0.4.0.3)$$

$$\left\| \mathcal{P}_*^{14+M} \mu \right\|_{L^2(\Sigma_t^u)}, \left\| \mathcal{P}^{14+M} L_{(Small)}^i \right\|_{L^2(\Sigma_t^u)}, \left\| \mathcal{L}_{\mathcal{P}}^{13+M} \chi \right\|_{L^2(\Sigma_t^u)} \leq C\mathring{\epsilon} \mu_*^{-(M+4)}(t, u), \quad (0.4.0.4)$$

$$\left\| L \mathcal{P}^{19} \mu \right\|_{L^2(\Sigma_t^u)}, \left\| L \mathcal{L}^{19; 1} L_{(Small)}^i \right\|_{L^2(\Sigma_t^u)}, \left\| \mathcal{L}_L^{18; 1} \chi \right\|_{L^2(\Sigma_t^u)} \leq C\mathring{\epsilon} \mu_*^{-6.4}(t, u), \quad (0.4.0.5)$$

$$\left\| \mu Y^{19} \text{tr}_{\mathcal{G}} \chi \right\|_{L^2(\Sigma_t^u)} \leq C\mathring{\epsilon} \mu_*^{-5.9}(t, u). \quad (0.4.0.6)$$

What happens in Case II) In case **II**), all bootstrap assumptions, the pointwise and L^∞ estimates of (0.19.6.2) and (0.20.3.1), and the energy estimates of Proposition (0.26.7.2) hold on $\mathcal{M}_{T_{(Lifespan); U_0}, U_0}$ with the factors of ε on the RHS replaced by $C\mathring{\epsilon}$. Moreover, for $0 \leq M \leq 5$, the estimates (0.4.0.3), (0.4.0.4), (0.4.0.5), (0.4.0.6) hold for $(t, u) \in [0, T_{(Lifespan); U_0}] \times [0, U_0]$. In addition, the scalar functions $\mathcal{L}^{\leq 10; 1} \Phi$, $\mathcal{L}^{\leq 4; 2} \Phi$, $\check{X} \check{X} \check{X} \Phi$, $\mathcal{L}^{\leq 9; 1} L^i$, $\check{X} \check{X} L^i$, $\mathcal{P}^{\leq 9} \mu$,

$\mathcal{L}^{\leq 2;1}\mu$, and $\check{X}\check{X}\mu$ extend to $\Sigma_{T(Lifespan);U_0}^{U_0}$ as functions of the geometric coordinates (t, u, θ^A) that are uniformly bounded in L^∞ . Furthermore, the rectangular component functions $g_{\alpha\beta}(\Phi)$ verify the estimate $g_{\alpha\beta} = m_{\alpha\beta} + \mathcal{O}_*(\check{\alpha}) + \mathcal{O}(\check{\epsilon})$ (where $m_{\alpha\beta} = \text{diag}(-1, 1, 1, 1)$ is the standard Minkowski metric) and have the same extension properties as Φ and its derivatives with respect to the vectorfields mentioned above. Moreover, let $\Sigma_{T(Lifespan);U_0}^{U_0;(\text{Blowup})}$ be the (non-empty) subset of $\Sigma_{T(Lifespan);U_0}^{U_0}$ defined by

$$\Sigma_{T(Lifespan);U_0}^{U_0;(\text{Blowup})} := \{(T(Lifespan);U_0, u, \theta^A) \mid \mu(T(Lifespan);U_0, u, \theta^A) = 0\}. \quad (0.4.0.7)$$

Then for each point $(T(Lifespan);U_0, u, \theta^A) \in \Sigma_{T(Lifespan);U_0}^{U_0;(\text{Blowup})}$, there exists a past neighborhood containing it such that the following lower bound holds in the neighborhood:

$$|X\Phi(t, u, \theta^A)| \geq \frac{\check{\delta}_*}{4|G_{L(Flat)L(Flat)}(\Phi=0)|} \frac{1}{\mu(t, u, \theta^A)}. \quad (0.4.0.8)$$

In (0.4.0.8), $\frac{\check{\delta}_*}{4|G_{L(Flat)L(Flat)}(\Phi=0)|}$ is a **positive** data-dependent constant, the $\mathcal{P}_{t,u}$ -transversal vectorfield X is near-Euclidean-unit length: $\delta_{ab}X^aX^b = 1 + \mathcal{O}_*(\check{\alpha}) + \mathcal{O}(\check{\epsilon})$. In particular, $X\Phi$ blows up like $1/\mu$ at all points in $\Sigma_{T(Lifespan);U_0}^{U_0;(\text{Blowup})}$. Conversely, at all points in $(T(Lifespan);U_0, u, \theta^A) \in \Sigma_{T(Lifespan);U_0}^{U_0} \setminus \Sigma_{T(Lifespan);U_0}^{U_0;(\text{Blowup})}$, we have

$$|X\Phi(T(Lifespan);U_0, u, \theta^A)| < \infty$$

For the following illposedness corollary, note that we are working in a different domain from the examples above. Here $\mathcal{B}_0$ is the ball of radius 1 centered at the origin in $\mathbb{R}^3$.

The $\check{\epsilon} - \check{\delta}$ hierarchy in the above theorem (0.4.1) actually allows us to show that there is an asymptotic stability to the illposedness result. The following corollary provides a rough quantitative estimate on the allowable perturbations from “Lindblad-type” data.

Remark 0.4.0.1 (Importance of U_0 for the corollary). In the following corollary we will actually apply the full theorem on for every U_0 . We provide data on Σ_0^1 with a singularity at $u = 1$ and such that the transverse derivatives $\check{X}^{[1,3]}\Phi$ are strictly increasing in u . Thus for every U_0 we will have that $\check{\delta} \sim_{\check{\epsilon}} \check{\delta}(U_0)$ and $\check{\delta}_* \sim_{\check{\epsilon}} \check{\delta}_*(U_0)$ (where $\check{\delta}(U_0), \check{\delta}_*(U_0)$ denote the values in the planar symmetric case with the initial data region $\Sigma_0^{U_0}$). Thus, for our “Lindblad-type” data, we can provide a quantitative statement of “stable illposedness” for which the size of allowable perturbations essentially depends only on U_0 .

Corollary 0.4.0.1 (Asymptotically Stable Ill-Posedness). *Assume that $\square_{g(\Phi)}\Phi = 0$ is a shock forming geometric wave equation in $\mathbb{R}^{3+1}$ (the type considered in the (0.4.1) above). We may assume without loss of generality that the “null condition failure factor” satisfies $G_{L(Flat)L(Flat)}(\Phi = 0) > 0$.*

H^2 Illposedness *For all $\eta > 0, \sigma > 0$ sufficiently small and $\alpha \in (0, 1/2)$, there exists a function $U(\eta, \sigma, \alpha) < 1$ and a smooth truncation of plane-symmetric “Lindblad-type” data $(\mathring{\Phi}_0, \mathring{\Phi}_1)$ with*

$$\check{X}\mathring{\Phi}_0(0, u, \theta^A)|_{\mathcal{B}_0} = -\eta|\ln(1 - |u|)|^\alpha$$

such that

$$\left\|(\mathring{\Phi}_0, \mathring{\Phi}_1)\right\|_{H^2 \times H^1(\{u \leq U\} \cap \mathcal{B}_0)} \leq C(\alpha)\eta$$

and such that the $\dot{H}^1$ energy of the solution blows up before time σ .

Asymptotic Stability *There exists an N sufficiently large and a function $\mathfrak{S}(\alpha, \eta, U)$ such that if we have a perturbation*

$$\left\|(\mathring{\Psi}_0, \mathring{\Psi}_1)\right\|_{H^N \times H^{N-1}(\{u \leq U\} \cap \mathcal{B}_0)} \leq \mathfrak{S}(\alpha, \eta, U)$$

then the $\dot{H}^1$ energy of the solution to $\square_{g(\Phi)}\Phi = 0$ with data $(\mathring{\Phi}_0 + \mathring{\Psi}_0, \mathring{\Phi}_1 + \mathring{\Psi}_1)$ blows up before time 2σ . Furthermore, for η sufficiently small and U sufficiently close to 1, we can derive the following explicit lower bound for $\mathfrak{S}$:

$$\exp\left(\frac{-c(\alpha\eta)^k}{(1-U)^{2k}|\ln(1-U)|^{k(1-\alpha)}}\right) \leq \mathfrak{S}(\alpha, \eta, U). \quad (0.4.0.9)$$

where $c, k > 0$ are absolute constants.

Proof. The the “ H^2 Illposedness” part of the corollary was proved in the previous section. Indeed, for $\mathring{\epsilon} = 0$ we are exactly in planar symmetry. We merely need to note that for breakdown to occur at time σ , we need U to be close enough to 1 such that (see (0.3.0.38)):

$$\mu(0, U) = \frac{\sigma}{2}\eta G_{LL}(U)|\ln(1 - |U|)|^\alpha.$$

Now, by the shock formation theorem above, we can allow a small perturbation away from planar symmetry of size $\mathring{\epsilon}$ (small relative to $\mathring{\delta}^{-1}$ and $\mathring{\delta}_*$). Quantitatively we may have $\mathring{\epsilon} = \min\left\{e^{-c\mathring{\delta}^k}, e^{d\mathring{\delta}_*^{-l}}\right\}$ where $c, d, k, l > 0$ are absolute constants coming from the proof of the shock

formation theorem and are independent of any of the free parameters in the problem (see below, (0.18.6.1), also section 7.7 in [18]). It is clear from the fact that $\mathring{\delta}, \mathring{\delta}_*$ are functions of U, η, α that $\mathring{\epsilon}$ is also a function of these quantities.

Note that in the data assumptions of the shock formation proof we are allowing for perturbations which are small and smooth and which only affect the $\mathcal{C}_u$ -tangent directions (not the $\mathcal{C}_u$ -transverse/ $\check{X}$ directions). This is different from simply assuming the perturbations are small in some Sobolev space $H^N \times H^{N-1}([0, 1] \times \mathbb{T}^2)$ (N large), which would allow for small perturbations in $\mathcal{C}_u$ -transverse directions. Nonetheless, given our smallness assumptions on $\mathring{\epsilon}$ relative to $\mathring{\delta}^{-1}, \mathring{\delta}_*$, allowing sufficiently smooth size $\mathring{\epsilon}$ perturbations in the $\check{X}$ direction has a negligible impact on the proof. Thus, for the ease of exposition, we are content to state our corollary with Sobolev control on the perturbations, but may pretend here that they only affect the $\mathcal{C}_u$ tangent directions.

Also note that the proof of stable shock formation effectively addresses steps 1 through 3 in (0.2.4.1).

By (0.4.0.1) and (0.4.0.2) above, we have that $|L\Phi|, |LL^i_{(Small)}| \lesssim \mathring{\epsilon}$. Integrating along characteristics we see that:

$$\Phi \sim_{\mathring{\epsilon}} \mathring{\Phi}_0, \quad (0.4.0.10)$$

$$L^i_{(Small)} \sim_{\mathring{\epsilon}} L^i_{(Small)}|_{t=0} = \mathcal{O}(\eta). \quad (0.4.0.11)$$

As in the planar case, we have that

$$t - x^1 \sim_{\eta, \mathring{\epsilon}} u. \quad (0.4.0.12)$$

This follows from using that at time $t = 0$ we have

$$t - x^1 = u$$

and that

$$L(u - (t - x^1)) = L^1_{(Small)}.$$

We may then integrate this equation along the characteristics and use the smallness of $L^1_{(Small)}$ (from (0.4.0.11)) in $\mathcal{M}_{T(Lifespan), U}$ to arrive at (0.4.0.12).

Note that by (0.4.0.10) and our initial data assumptions we have that $|\Phi| \lesssim \eta$ in $\mathcal{M}_{T(Lifespan), U}$. This ensures that the domain of future dependence is uniformly close to the Minkowskian one. From this and (0.4.0.12) we may derive that $\int_{\mathcal{S}_{t,u}} d\theta \gtrsim_{\eta, \mathring{\epsilon}} (1 - |u|)$.

The one fact slightly different from planar symmetry is that here we have the transport equation:

$$L\mu = \frac{1}{2}G_{LL}(\Phi)\check{X}\Phi + \mathcal{O}(\dot{\epsilon}).$$

Noting that $\left|L\left(G_{LL}(\Phi)\check{X}\Phi\right)\right| \lesssim \dot{\epsilon}$ (by (0.4.0.1) and (0.4.0.2)) and integrating along characteristics gives us that

$$\left|G_{LL}(\Phi)\check{X}\Phi - G_{LL}(\check{\Phi}_0)\check{X}\check{\Phi}_0\right| \lesssim \dot{\epsilon}.$$

Thus we see that we actually have:

$$L\mu = \frac{1}{2}G_{LL}(\check{\Phi}_0)\check{X}\check{\Phi}_0(u, \theta) + \mathcal{O}(\dot{\epsilon}) \quad (0.4.0.13)$$

$$= -\frac{1}{2}G_{LL}(\check{\Phi}_0)\eta|\ln(1-|u|)|^\alpha + \mathcal{O}(\dot{\epsilon}), \quad (0.4.0.14)$$

and thus

$$\mu(t, u, \theta) = \dot{\mu}(u, \theta) - \frac{t}{2}G_{LL}(\check{\Phi}_0)\eta|\ln(1-|u|)|^\alpha + t\mathcal{O}(\dot{\epsilon}). \quad (0.4.0.15)$$

It is then easy to see from our perturbative assumptions that

$$\max_{\theta} \mu(t, u, \theta) = \dot{\mu}(u) - \frac{t}{2}G_{LL}(u)\eta|\ln(1-|u|)|^\alpha + t\mathcal{O}(\dot{\epsilon}), \quad (0.4.0.16)$$

where $\dot{\mu}(u), G_{LL}(u)$ denote $\mu|_{\Sigma_0}, G_{LL}|_{\Sigma_0}$ in the plane symmetric case.

We then have the following, just as in planar symmetry (although note that here we need to consider $\max_{\theta} \mu(t, u, \theta)$ in the denominator):

$$\int_{\mathcal{B}_t} |X\Phi|^2(t, x) dx = \int_{\mathfrak{H}^{-1}(\mathcal{B}_t)} |X\Phi|^2(t, u, \theta) \mu d\theta du \quad (0.4.0.17)$$

$$\gtrsim \int_0^U (1-u) \frac{(\eta|\ln(1-u)|^\alpha)^2}{\max_{\theta} \mu(t, u, \theta)} du \quad (0.4.0.18)$$

$$\sim \int_0^U (1-u) \frac{(\eta|\ln(1-u)|^\alpha)^2}{(\dot{\mu}(u) - (1 + \mathcal{O}(\dot{\epsilon}))\frac{1}{2}G_{LL}(u)\eta|\ln(1-u)|^\alpha t)} du \quad (0.4.0.19)$$

which blows up for $t > 0, u \leq U$ such that

$$\dot{\mu}(u) = (1 + \mathcal{O}(\dot{\epsilon}))\frac{1}{2}G_{LL}(u)\eta|\ln(1-u)|^\alpha t. \quad (0.4.0.20)$$

It is clear from our smallness assumption on $\mathring{\epsilon}$ that if

$$\mathring{\mu}(U) = \sigma \frac{1}{2} G_{LL}(U) \eta |\ln(1-U)|^\alpha, \quad (0.4.0.21)$$

then we must have

$$\mathring{\mu}(U) = (1 + \mathcal{O}(\mathring{\epsilon})) \frac{1}{2} G_{LL}(U) \eta |\ln(1-U)|^\alpha t \quad (0.4.0.22)$$

for some $t \leq 2\sigma$.

To derive a more explicit/quantitative bound on the allowable perturbations, we know that for u sufficiently close to 1 we have that

$$\eta |\ln(1-U)|^\alpha = \|\check{X}\Phi\|_{L^\infty(\{u \leq U\} \cap \mathcal{B}_0)} \leq \|\check{X}^2\Phi\|_{L^\infty(\{u \leq U\} \cap \mathcal{B}_0)} \quad (0.4.0.23)$$

$$\leq \|\check{X}^3\Phi\|_{L^\infty(\{u \leq U\} \cap \mathcal{B}_0)} \sim \frac{\alpha\eta}{(1-U)^2 |\ln(1-U)|^{1-\alpha}} \quad (0.4.0.24)$$

(where the implicit constants in $\sim$ are absolute) and thus

$$\mathring{\delta} \sim \frac{\alpha\eta}{(1-U)^2 |\ln(1-U)|^{1-\alpha}} \quad (0.4.0.25)$$

(where, again, the implied constant is absolute).

For η sufficiently small and U close to 1 we see that

$$\mathring{\delta}_* = \frac{1}{2} G_{LL}(\Phi) \cdot \eta |\ln(1-U)|^\alpha \quad (0.4.0.26)$$

and thus

$$\mathring{\epsilon} \leq \min \left\{ e^{-c\mathring{\delta}^k}, e^{d\mathring{\delta}_*^{-l}} \right\} \leq \exp \left(\frac{-c(\alpha\eta)^k}{(1-U)^{2k} |\ln(1-U)|^{k(1-\alpha)}} \right). \quad (0.4.0.27)$$

Thus we see that for sufficiently small η and U near 1 we have the bound

$$\exp \left(\frac{-c(\alpha\eta)^k}{(1-U)^{2k} |\ln(1-U)|^{k(1-\alpha)}} \right) \leq \mathfrak{S}(\alpha, \eta, U). \quad (0.4.0.28)$$

□

Remark 0.4.0.2. The proof of shock formation and hence of illposedness does not fundamentally depend on the smallness of Φ itself, but rather on the fact that $g_{\alpha\beta}(\Phi)$ is Lorentzian and the smallness of the perturbations from plane symmetry are appropriately small. Note that we needed to use this fact in the above when we approximate $L\mu$ using the initial values of $G_{LL}(\mathring{\Phi}_0)\check{X}\mathring{\Phi}_0$ rather than by

$G_{L(Flat)L(Flat)}(\Phi = 0)\check{X}\check{\Phi}_0$, which would leave us with a problematic error term $\mathcal{O}(\Phi \cdot \check{X}\check{\Phi}_0)$. See Remark 1.9 in [18].

Remark 0.4.0.3. In spite of the fact that this corollary shows that illposedness is not merely an artifact of symmetry, the very strong asymptotic requirement on the perturbations needs to be noted. Indeed, as $U \nearrow 1$ we see that we clearly have $\exp\left(\frac{-c(\alpha\eta)^k}{(1-U)^{2k}|\ln(1-U)|^{k(1-\alpha)}}\right) \searrow 0$ and so our theorem *does not* give us a small neighborhood around 0 of allowable perturbations in $H^N \times H^{N-1}(\mathcal{B}_0)$ (where N for us is large).

0.5 Illposedness for Equations of the form

$$(g^{-1})^{\alpha\beta}(\partial\Psi)\partial_{\alpha\beta}^2\Psi = 0 \text{ and for Relativistic Fluid Equations}$$

The above result gives us as a corollary the illposedness in $H^3 \times H^2(\mathbb{R}^3)$ of equations of the form $(g^{-1})^{\alpha\beta}(\partial\Psi)\partial_{\alpha\beta}^2\Psi = 0$ which fail the null condition in an appropriate sense.

See the appendices to [18] for more details, as this is only a sketch of the relevant ideas.

We are here concerned with the Cauchy problem:

$$(g^{-1})^{\alpha\beta}(\partial\Psi)\partial_{\alpha}\partial_{\beta}\Psi = 0, \tag{0.5.0.1}$$

$$(\Psi|_{\Sigma_0}, \partial_t\Psi|_{\Sigma_0}) = (\check{\Psi}_0, \check{\Psi}_1) \tag{0.5.0.2}$$

where the equation is in rectangular coordinates $\{x^\alpha\}_{\alpha=0,1,2,3}$ and we assume that $g_{\alpha\beta}(0) = m_{\alpha\beta}$ and that $(g^{-1})^{00} \equiv -1$.

In [18] it is shown that the rectangular derivatives $\Psi_\nu := \partial_\nu\Psi$ satisfy a covariant system of the form:

$$\square_{g(\vec{\Psi})}\Psi_\nu = \mathcal{Q}(\partial\vec{\Psi}, \partial\Psi_\nu) \tag{0.5.0.3}$$

where it turns out that $\mathcal{Q}(\partial\vec{\Psi}, \partial\Psi_\nu)$ is a null form. The appropriate notion of null condition failure for this system is given by

$$m_{\kappa\lambda}G_{\alpha\beta}^\kappa(\vec{\Psi} = 0)L_{(Flat)}^\alpha L_{(Flat)}^\beta L_{(Flat)}^\lambda \neq 0 \tag{0.5.0.4}$$

and shock formation is now driven by the term

$$G_{LL}^L X^a \check{X} \Psi_a \quad (0.5.0.5)$$

and thus we choose our initial data to be such that $X^a \check{X} \Psi_a \sim -\varepsilon |\ln(|u|)|^\alpha$. Thus, using that the presence of null forms on the RHS of a covariant equation is acceptable and does not affect any of our proofs (in or away from planar symmetry), we find that

Corollary 0.5.0.1. *The equation (0.5.0.1) is (asymptotically stably) illposed for data $(\check{\Psi}_0, \check{\Psi}_1) \in H^3 \times H^2(\mathbb{R}^3)$.*

Another interesting consequence is for irrotational, compressible relativistic fluid equations derived from Lagrangians of the form

$$\mathcal{L}(\sigma) \quad (0.5.0.6)$$

where

$$\sigma := -m^{\alpha\beta} \partial_\alpha \Phi \partial_\beta \Phi > 0 \quad (0.5.0.7)$$

and where $\mathcal{L}$ and its derivatives in σ satisfy certain technical conditions to ensure the resulting equation is physically meaningful.

Such lagrangians give us wave equations of the form $(g^{-1}(\partial\Phi))^{\alpha\beta} \partial_{\alpha\beta}^2 \Phi = 0$, and it turns out that the only one of these equations which verifies the null condition (and hence does not form shocks) is precisely the timelike minimal surface equation, which gives us that

Corollary 0.5.0.2. *All irrotational, compressible relativistic fluid lagrangians of the type considered in [1], except for possibly the timelike minimal surface equation are illposed in $H^3(\mathbb{R}^3)$*

Remark 0.5.0.1 (Relative Strength of Breakdown). From the above we see that we have:

$$\square_m \Phi = \left(\underline{L}_{(Flat)} \Phi \right)^2 > \square_{g(\Phi)} \Phi = 0 > \square_m \Phi = -\Phi \left(\underline{L}_{(Flat)}^2 \Phi \right) > \left\{ \begin{array}{l} \square_m \Phi = 0 \\ \square_m \Psi = \left(\underline{L}_{(Flat)} \Phi \right)^2 \end{array} \right\} \quad (0.5.0.8)$$

where the ordering is in terms of the strength of the energy blow-up. This is due to the fact that for Lindblad-type initial data (roughly of the form $\underline{L}_{(Flat)} \Phi(0, u) = \varepsilon |\ln u|^\alpha$) we have:

- Our solution to $\square_m \Phi = \left(\underline{L}_{(Flat)} \Phi \right)^2$ exhibits $H^1(\mathcal{B}_t)$ norm blow-up where the singularity comes from integrating $\frac{1}{(1-\varepsilon |\ln u|^\alpha)^2}$.

- Our solution to $\square_{g(\Phi)}\Phi = 0$ exhibits $H^1(\mathcal{B}_t)$ norm blow-up where the singularity comes from integrating $\frac{1}{(1-\varepsilon)|\ln u|^\alpha}$.
- Our solution to $\square_m\Phi = -\Phi\left(\underline{L}_{(Flat)}^2\Phi\right)$ exhibits $H^2(\mathcal{B}_t)$ norm blow-up for $\alpha \in (0, 1/2)$.
- Our solution to $\left\{ \begin{array}{l} \square_m\Phi = 0 \\ \square_m\Psi = \left(\underline{L}_{(Flat)}\Phi\right)^2 \end{array} \right\}$ exhibits $H^2(\mathcal{B}_t)$ norm blow-up for $\alpha \in (1/4, 1/2)$.

Remark 0.5.0.2. This effectively concludes the central part of the thesis. The rest of the thesis is simply to sketch the additional details needed to extend the result from [18] to 3+1 dimensions. The existing literature on stable planar shock formation is set in 2+1 dimensions. It was known that this result is also true in higher dimensions, but the proof requires an additional argument using elliptic estimates to control top-order terms containing $\hat{\chi}$. This elliptic argument is in the spirit of what was done in [16], although some of the details are different and the notation is somewhat simplified. We provide a sketch of the proof of the 3 + 1 result, including a more explicit description of these additional details, and note where our lemmas, notation, or proofs differ from that in [18]. However, we refer to [18] wherever possible. The fundamental novelty in this thesis is the connection between shock formation and illposedness.

0.6 Sketch of stable shock formation result near planar symmetry

The stable shock formation result is essentially the proof of steps 1 through 3 in (0.2.4.1). Due to the quasilinear nature of the equation, we are essentially solving the system

$$\square_{g(\Phi)}\Phi = 0, \quad (g^{-1})^{\alpha\beta}(\Phi)\partial_\alpha u \partial_\beta u = 0.$$

Solving the eikonal equation gives us a natural coordinate system (a foliation of spacetime by null hyperplanes of constant u) which is geometrically adapted to the wave equation. As in the plane symmetric example above, we define the inverse foliation density μ , which tells us the scale of the null normal vectorfield $-\mathcal{D}u$ with respect to the rectangular background coordinate system on $\mathbb{R}^{3+1}$. We can think about μ as a measure of how “pinched” the characteristic surfaces appear with respect to the background coordinates, where $\mu = 0$ means that the characteristics have collapsed and hence we expect a shock has formed.

We can then use μ to define a geometric frame $\{L, \check{X}, \Theta_A\}$, where L is tangent to the $u = \text{constant}$ null hyperplanes (denoted by $\mathcal{C}_u$), $\check{X} \sim \partial_u \sim -\mu\partial_{x^1}$ denotes the “bad” derivative which is transverse to the null hyperplanes and tangent to the $t = \text{constant}$ surfaces Σ_t , and Θ_A denotes the derivatives on the 2-torus $\mathcal{P}_{t,u} := \Sigma_t \cap \mathcal{C}_u$.

As in the planar symmetric case, we then find that μ satisfies an equation of the form

$$L\mu = \frac{1}{2}G_{LL}\check{X}\Phi + \mathcal{O}(\mu L\Phi). \quad (0.6.0.1)$$

Unlike [16], which was essentially a small data result, we are here interested in demonstrating that shocks form for data which is large in the $\mathcal{C}_u$ -transverse direction, assuming that it is suitably small in the $\mathcal{C}_u$ -tangent directions. The authors in [18] denote the size of the transverse derivatives $\check{X}\Phi, \check{X}^2\Phi$, etc. by $\mathring{\delta}$ and the size of any terms with tangential derivatives (basically measuring the perturbations from planar symmetry) by $\mathring{\epsilon}$. We will require $\mathring{\delta} \gg \mathring{\epsilon}$. The goal will be to prove that this so-called “ $\mathring{\epsilon} - \mathring{\delta}$ hierarchy” propagates in the wave front region in a precise way, which allows us to demonstrate that $\mu \searrow 0$ linearly in time and gives us precise asymptotics for our solution all the way up to the shock.

In [18] it was shown (similar to [16]) that rescaling the equation with μ , we find that we can use our geometric frame to rewrite $\mu\Box_g\Phi = 0$ as

$$-L(\mu L\Phi + 2\check{X}\Phi) + \mu\Delta\Phi = \mathcal{N} \quad (0.6.0.2)$$

where the terms in $\mathcal{N}$ depend on ≤ 1 derivative of Φ and ≤ 2 derivatives of u . The remarkable “null structure” which allows us to propagate the $\mathring{\epsilon} - \mathring{\delta}$ hierarchy is that the size $\mathring{\delta}$ terms in $\mathcal{N}$ are always paired with a term which is size $\mathring{\epsilon}$! This enables us to treat the above equation (0.6.0.2) as a transport equation for $\check{X}\Phi$ which in turn allows us to track the behavior of μ using (0.6.0.1).

Now, being away from planar symmetry, in order to prove that Φ remains small all the way to the shock we can no longer use the method of characteristics. This means that we must rely on energy inequalities in our geometric coordinates. It turns out that we may commute the equation with the geometric $\mathcal{C}_u$ -tangent vectorfields $\{L, Y_{(1)}, Y_{(2)}\}$,⁴ while still retaining this remarkable null structure in the equation. We may then use the Sobolev inequality on the 2-surfaces $\mathcal{P}_{t,u}$ to recover the smallness.

However several interesting problems arise in the course of the energy argument (all of which

⁴The $Y_{(l)}$ are $\mathcal{P}_{t,u}$ -tangent vectorfields, chosen for technical reasons to be different from the Lie-propagated coordinate vectorfields $\{\Theta_A\}$. See (0.11) for details.

were essentially dealt with in [1]):

- **Derivative loss at top order.** A problem arises from the fact that commuting the equation to top order with our geometric $\mathcal{C}_u$ tangent commutators P will give us error terms of the form $P^{\leq TOP} \partial(\mu \partial u) \cdot \partial \Phi$. One example is the term $Y^{TOP} \text{tr}_{\mathcal{G}} \chi \cdot \check{X} \Phi$. In order to control $Y^{TOP} \text{tr}_{\mathcal{G}} \chi$ we can integrate a transport equation schematically of the form

$$L(Y^{TOP} \text{tr}_{\mathcal{G}} \chi) \sim (\text{tr}_{\mathcal{G}} \chi) \cdot Y^{TOP} \text{tr}_{\mathcal{G}} \chi + Y^{TOP} \text{Ric}_{LL}.$$

Unfortunately the term $Y^{TOP} \text{Ric}_{LL}$ contains $\partial^{TOP+2} \Phi$ terms, while the top-order energy only controls $\partial^{TOP+1} \Phi$ terms. It turns out, however, that $Y^{TOP} \text{Ric}_{LL} = L(Y^{TOP+1} \Phi) + l.o.t.$ and we can move the $L(Y^{TOP+1} \Phi)$ terms to the LHS of the transport equation. This allows us to avoid losing derivatives in the top-order energies.

In $n \geq 3$ spatial dimensions, there are other top-order eikonal quantities which appear in the commutation error terms, roughly of the form $\hat{\chi} \cdot \mathcal{L}_{\bar{P}}^{\leq TOP} \hat{\chi}$ which we cannot control with a transport equation. We require elliptic estimates on the $\mathcal{P}_{t,u}$ surfaces in order to bound them in $L^2(\Sigma_t)$ by the top-order energies for Φ . The need for this additional elliptic argument is essentially the main difference between the rest of the thesis and [18]. The form of the argument is not new - we rely heavily on the presentation in [16] - although the details are somewhat different in a few places. The treatment is thus somewhat fast and proofs given are fairly minimal. We present the argument only for the sake of completeness. The interested reader is referred to [16] for a more thorough and involved treatment of the same ideas.

- **Degenerate μ weights in energies.** The energies and fluxes that we use have the form

$$\int_{\Sigma_t} \mu |L\Phi|^2 + |\check{X}\Phi|^2 + \mu |\nabla \Phi|^2$$

and

$$\int_{\mathcal{C}_u} |L\Phi|^2 + \mu |\nabla \Phi|^2,$$

where there are no μ weights associated to the implied volume form on the hypersurface. Thus our coercive control over angular derivatives degenerates as we approach the shock $\mu = 0$. It turns out that in the bulk error integral

$$\int_{\mathcal{M}} \mu Q_{\alpha\beta}^{(T)} \pi^{\alpha\beta}$$

coming from the T multiplier vectorfield there is a term with a good sign and no problematic μ weights, of the form:

$$\int_{\mathcal{M}} [L\mu]_- |\Phi|^2,$$

which gives us coercive control over the angular derivatives near $\mu = 0$.

- **Blow-up of energies at top order.** It turns out that the only control we can obtain for the top order energies is of the form

$$\mathbb{E}_{TOP}[\Phi] \lesssim \frac{1}{\mu^M},$$

i.e. they are blowing up as we approach the shock. Christodoulou devised a so-called “descent scheme” which uses our estimates on μ to show that by passing to lower order energies we can actually gain powers of μ i.e. we have estimates of the form:

$$\mathbb{E}_{TOP-k}[\Phi] \lesssim \frac{1}{\mu^{M-k}}.$$

By descending far enough, we actually find that the lower order energies (at a certain level and below) are actually bounded. This allows us to demonstrate smallness of Φ all the way to the shock, and hence closes the argument. ⁵

Remark 0.6.0.1 (Regularity requirements on the data). The need to keep track of the size of $\|\check{X}^{[1,3]}\Phi\|_{L^\infty(\Sigma_0^+)}$ stems from the need to control a dangerous bulk error term which appears in the multiplier argument. The need for such high regularity of the perturbation terms (i.e. we require $\|\mathcal{P}^{[1,N]}\Phi\|_{L^2(\Sigma_0^+)} \leq \epsilon$ for a large N) basically stems from the descent scheme and the product terms appearing in our commutator estimates. In order to eliminate the $\mu_\star$ degeneracy at top-order we need to descend a certain number of derivative levels. This, together with Sobolev embedding, gives us something like

$$\mathbb{E}^{1/2}[\mathcal{P}^{\leq TOP}\Phi] \lesssim \epsilon \mu_\star^{-M} \quad (0.6.0.3)$$

$$\|\mathcal{P}^{\leq TOP-M-2}\Phi\|_{L^\infty} \lesssim \mathbb{E}^{1/2}[\mathcal{P}^{\leq TOP-M}\Phi] \lesssim \epsilon. \quad (0.6.0.4)$$

Now in order to prove these estimates we need to have assumed that we could bound the lower order part of product terms appearing in the commutator estimates in L^∞ , i.e. closure requires something like $\|\mathcal{P}^{\leq \frac{TOP}{2}+1}\Phi\|_{L^\infty} \leq \|\mathcal{P}^{\leq TOP-M-2}\Phi\|_{L^\infty}$ which necessitates $TOP - M - 2 \geq \frac{TOP}{2} + 1$ and thus

⁵We will later discuss the descent scheme in slightly more detail but, as the proof is actually identical to that in [18], we refer the interested reader to that paper for the precise details.

$TOP \geq 2(M + 3)$, where M is a structural constant.

0.7 Geometric Preliminaries

0.7.1 The Equation in Rectangular Coordinates, Null Condition Failure (Corresponding to Section 2.2 in [18])

We denote the time coordinate by $t = x^0 \in \mathbb{R}$ and $(x_1, x_2, x_3) \in \mathbb{R} \times \mathbb{T}^2$ to denote the standard coordinates on Σ_t^u . These are the so-called rectangular coordinates on our spacetime, with respect to which we have $m_{\alpha\beta} = \text{diag}(-1, 1, 1, 1)$. We assume that in rectangular coordinates our metric is of the form

$$g_{\alpha\beta}(\Phi) = m_{\alpha\beta} + g_{\alpha\beta}^{(small)}(\Phi) \quad (0.7.1.1)$$

where $g_{\alpha\beta}^{(small)}(\Phi)$ is a smooth function of Φ satisfying $g_{\alpha\beta}^{(small)}(0) = 0$. In rectangular coordinates we find that

$$\square_{g(\Phi)}\Phi = (g^{-1})^{\alpha\beta}\partial_\alpha\partial_\beta\Phi - (g^{-1})^{\alpha\beta}\Gamma_{\alpha\beta}^\gamma\partial_\gamma\Phi = 0 \quad (0.7.1.2)$$

where

$$\begin{aligned} \Gamma_{\alpha\beta}^\gamma(\Phi, \partial\Phi) &:= \frac{1}{2}(g^{-1})^{\gamma\kappa}(\partial_\alpha g_{\kappa\beta} + \partial_\beta g_{\kappa\alpha} - \partial_\kappa g_{\alpha\beta}) \\ &= \frac{1}{2}(g^{-1})^{\gamma\kappa} \{G_{\kappa\beta}\partial_\alpha\Phi + G_{\alpha\kappa}\partial_\beta\Phi - G_{\alpha\beta}\partial_\kappa\Phi\} \end{aligned} \quad (0.7.1.3)$$

where we define

$$G_{\alpha\beta} = G_{\alpha\beta}(\Phi) := \frac{d}{d\Phi}g_{\alpha\beta}(\Phi).$$

We will also assume that we are in the setting $(g^{-1})^{00}(\Phi) \equiv -1$. If this condition is not satisfied, note that we may rescale the metric by a conformal factor $\Omega^2(\Phi)$ to ensure that $(g^{-1})^{00}(\Phi) \equiv -1$ and that the resulting rescaled equation has the form

$$\square_{g(\Phi)}\Phi = \mathcal{N}(\Phi)g^{\alpha\beta}(\Phi)\partial_\alpha\Phi\partial_\beta\Phi,$$

i.e. the now nontrivial RHS satisfies the strong null condition which does not at all affect the proof of the shock formation theorem (as “bad”/large size $\mathring{\delta}$ derivatives of Φ are always multiplied by “good”/small $\mathring{\epsilon}$ derivatives of Φ).

Definition 0.7.1.1 (Null Condition Failure). We say that $\square_{g(\Phi)}\Phi = 0$ *fails the null condition* if

$$G_{\alpha\beta}(\Phi = 0)^\omega L_{(flat)}^\alpha{}^\omega L_{(flat)}^\beta \neq 0.$$

0.7.2 Eikonal Functions (Corresponding to Section 2.3 in [18])

Definition 0.7.2.1 (Eikonal Function). The eikonal function u is defined by the solutions of

$$(g^{-1})^{\alpha\beta}(\Phi)\partial_\alpha u \partial_\beta u = 0 \tag{0.7.2.1}$$

such that $\partial_t u > 0$ and $u|_{\Sigma_0} = -x_1$

Definition 0.7.2.2. [Spacetime regions] We define the following subsets of $\mathbb{R}^{1+1} \times \mathbb{T}^2$

$$\Sigma_{t'} := \{(t, x^1, x^2, x^3) \mid t = t'\} \tag{0.7.2.2}$$

$$\Sigma_{t'}^{u'} := \{(t, x^1, x^2, x^3) \mid t = t', 0 \geq u \geq u'\} \tag{0.7.2.3}$$

$$\mathcal{C}_{u'}^{t'} := \{(t, x^1, x^2, x^3 \mid u = u', 0 \leq t \leq t')\} \tag{0.7.2.4}$$

$$\mathcal{P}_{t', u'} := \mathcal{C}_{u'}^{t'} \cap \Sigma_{t'}^{u'} \tag{0.7.2.5}$$

$$\mathcal{M}_{t', u'} := \{(t, x^1, x^2, x^3) \mid 0 \geq u \geq u', 0 \leq t \leq t'\} \tag{0.7.2.6}$$

$$\mathcal{B}_t := \Sigma_t^u \cap \mathcal{D}^+(\mathcal{B}_0) \tag{0.7.2.7}$$

Where by $\mathcal{D}^+(A)$ we mean the domain of future dependence of the set A .

0.7.3 Frame Vectorfields, Projections, Fundamental Forms, Metric and Determinant Expressions (Corresponding to Sections 2.4-2.7 in [18])

Definition 0.7.3.1 (outgoing null geodesic vectorfield). We set

$$L_{(geo)} := -(g^{-1})^{\alpha\beta}(\Phi)\partial_\alpha u \partial_\beta u = -\mathcal{D}u \tag{0.7.3.1}$$

It is a standard fact that $L_{(geo)}$ is g -null, future-directed, and g -orthogonal to $\mathcal{C}_u$, which are thus null hypersurfaces.

Definition 0.7.3.2 (inverse foliation density). We define the inverse foliation density μ by

$$\mu^{-1} = L_{(geo)}^0 = -(g^{-1})^{\alpha\beta}(\Phi)\partial_\alpha u \partial_b t \tag{0.7.3.2}$$

Definition 0.7.3.3 (rescaled outgoing null vectorfield). We define the rescaled vectorfield by

$$L := \mu L_{(geo)}$$

We use geometric coordinates given by $(t, u, \theta^1, \theta^2)$ where $\theta^A|_{\Sigma_0} = x^{A+1}$ for $A = 1, 2$ and where we extend θ^A by $L\theta^A = 0$.

Definition 0.7.3.4 (additional vectorfields). We define the spacelike vectorfield X as the unique Σ_t^u -tangent vectorfield which is g -orthogonal to the $\mathcal{P}_{t,u}$ surfaces and normalized by $g(L, X) = -1$. We define the rescaled spacelike vectorfield $\check{X}$ by

$$\check{X} := \mu X$$

and the timelike unit normal N by

$$N := L + X$$

Lemma 0.7.3.1 (some useful facts about our vectorfields). *We easily find that*

$$\begin{array}{ll} Lu = 0 & Lt = 1 \\ \check{X}u = 1 & \check{X}t = 0 \\ Nu = 1 & Nt = 1 \end{array}$$

and that

$$\begin{aligned} g(L, L) &= 0 \\ g(\check{X}, \check{X}) &= \mu^2 \\ g(X, X) &= 1 \\ g(L, \check{X}) &= -\mu \\ g(N, N) &= -1 \end{aligned} \tag{0.7.3.3}$$

we also see that $N^\alpha = -(g^{-1})^{0\alpha}$, that $[L, \check{X}] \in T\mathcal{P}_{t,u}$, and that in rectangular coordinates we have

$$X_\alpha = -L_\alpha - \delta_\alpha^0, \quad X^\alpha = -L^\alpha - (g^{-1})^{0\alpha}. \tag{0.7.3.4}$$

Definition 0.7.3.5. Let $\Theta_A := \frac{\partial}{\partial \theta^A} |_{t,u}$ be the geometric angular vectorfield. We will denote the *rescaled frame* by:

$$\{L, \check{X}, \Theta_1, \Theta_2\}$$

and the *nonrescaled frame* by:

$$\{L, X, \Theta_1, \Theta_2\}$$

It is not difficult to see from the above that in our geometric (/rescaled) frame we have that $L = \partial_t$ and $\check{X} = \partial_u - \Xi$ where $\Xi \in T\mathcal{P}_{t,u}$.

Lemma 0.7.3.2 (transport equation for Ξ). *We find that*

$$L\Xi^A = [L, \Xi]^A = -[L, \check{X}]^A \quad (0.7.3.5)$$

Definition 0.7.3.6 (Projection operators). We let $\underline{\Pi}$ and $\mathbb{I}$ denote the projections onto Σ_t^u and $\mathcal{P}_{t,u}$, respectively.

Lemma 0.7.3.3. *It is elementary to derive that*

$$\underline{\Pi}_\alpha^\beta = \delta_\alpha^\beta - N_\alpha N^\beta = \delta_\alpha^\beta + \delta_\alpha^0 L^\beta + \delta_\alpha^0 X^\beta \quad (0.7.3.6)$$

$$\mathbb{I}_\alpha^\beta = \delta_\alpha^\beta + X_\alpha L^\beta + L_\alpha (L^\beta + X^\beta) = \delta_\alpha^\beta - \delta_\alpha^0 L^\beta + L_\alpha X^\beta \quad (0.7.3.7)$$

Definition 0.7.3.7 (Projected derivative operators). For tensorfields ξ , vectorfields V , and scalar functions f , we define the following projected derivative operators:

$$\underline{\mathcal{L}}_V \xi := \underline{\Pi} \mathcal{L}_V \xi \quad (0.7.3.8)$$

$$\mathbb{L}_V \xi := \mathbb{I} \mathcal{L}_V \xi \quad (0.7.3.9)$$

$$\mathbb{d}f := \mathbb{V}f = \mathbb{I} \mathcal{D}f \quad (0.7.3.10)$$

Definition 0.7.3.8 (First fundamental form). We denote:

$$\underline{g}_{ij} := \underline{\Pi}_i^\alpha \underline{\Pi}_j^\beta g_{\alpha\beta}$$

and

$$\mathbb{g}_{AB} := \mathbb{I}_A^\alpha \mathbb{I}_B^\beta g_{\alpha\beta}$$

The inverse fundamental forms are defined by raising indices with (g^{-1}) , as is standard.

Definition 0.7.3.9 (Second fundamental form). We define the second fundamental form k of Σ_t^u by

$$k := \frac{1}{2} \underline{\mathcal{L}}_N \underline{g}$$

and the null second fundamental form χ of $\mathcal{P}_{t,u}$ by

$$\chi := \frac{1}{2} \underline{\mathcal{L}}_L \not{g}$$

Lemma 0.7.3.4. *Our metric inverse $(g^{-1})^{\alpha\beta}$ can be written as:*

$$\begin{aligned} (g^{-1})^{\alpha\beta} &= -L^\alpha L^\beta - \frac{1}{\mu} \left(L^\alpha \check{X}^\beta + \check{X}^\alpha L^\beta \right) + \not{g}^{\alpha\beta} \\ &= -L^\alpha L^\beta - (L^\alpha X^\beta + X^\alpha L^\beta) + \not{g}^{\alpha\beta} \end{aligned} \tag{0.7.3.11}$$

Lemma 0.7.3.5 (Geometric coordinate decomposition of the metric). *We find that in the $(t, u, \theta^1, \theta^2)$ coordinates we have:*

$$g = -\mu dt \otimes du - \mu du \otimes dt + \mu^2 du \otimes du + \not{g}_{AB} (d\theta^A + \Xi^A du) \otimes (d\theta^B + \Xi^B du) \tag{0.7.3.12}$$

$$\underline{g} = \mu^2 du \otimes du + \not{g}_{AB} (d\theta^A + \Xi^A du) \otimes (d\theta^B + \Xi^B du) \tag{0.7.3.13}$$

From this it easily follows that

Corollary 0.7.3.6 (Geometric volume forms).

$$|\det g| = \mu^2 \det \not{g} \tag{0.7.3.14}$$

$$\det \underline{g} = \mu^2 \det \not{g} \tag{0.7.3.15}$$

0.7.4 Change of Variables

Lemma 0.7.4.1.

$$\det \frac{\partial(x^\alpha)}{\partial(t, u, \theta^A)} = \mu (\det \underline{g})^{-1/2} \sqrt{\det \not{g}} \tag{0.7.4.1}$$

0.8 Commutation Identities and Deformation Tensors

Definition 0.8.0.1 (Deformation tensor of a vectorfield). For any vectorfield V we define its *deformation tensor* ${}^{(V)}\pi$ to be the following tensorfield:

$${}^{(V)}\pi_{\alpha\beta} := \mathcal{L}_V g_{\alpha\beta} = \mathcal{D}_\alpha V_\beta + \mathcal{D}_\beta V_\alpha$$

The following lemmas are straightforward.

Lemma 0.8.0.1. *For any $\mathcal{P}_{t,u}$ -tangent vectorfield V we have $[L, V], [\check{X}, V] \in T\mathcal{P}_{t,u}$. We also have that $[L, \check{X}]$ is a $\mathcal{P}_{t,u}$ -tangent vectorfield and that:*

$$[L, \check{X}] = {}^{(\check{X})}\sharp_L^\sharp,$$

$$[L, V] = {}^{(V)}\sharp_L^\sharp,$$

$$[\check{X}, V] = {}^{(V)}\sharp_{\check{X}}^\sharp.$$

For $Z \in \mathcal{L}$ we have

$$\mathcal{L}_Z \mathcal{G} = {}^{(Z)}\sharp, \quad \mathcal{L}_Z \mathcal{G}^{-1} = -{}^{(Z)}\sharp^{\sharp\sharp}.$$

Lemma 0.8.0.2. *For any type $\binom{0}{n}$ spacetime tensorfield ξ and $V \in \mathcal{L}$ or V tangent to $\mathcal{P}_{t,u}$ we have that*

$$\underline{\mathcal{L}}_N \xi = \underline{\mathcal{L}}_N (\underline{\Pi} \xi) \tag{0.8.0.1}$$

$$\mathcal{L}_V \xi = \mathcal{L}_V \xi \tag{0.8.0.2}$$

Corollary 0.8.0.3. *For any type $\binom{0}{n}$ spacetime tensorfield ξ and $V, W \in \mathcal{L}$ or tangent to $\mathcal{P}_{t,u}$ we have that*

$$[\mathcal{L}_V, \mathcal{L}_W] \xi = \mathcal{L}_{[V, W]} \xi \tag{0.8.0.3}$$

Lemma 0.8.0.4. *For any scalar function f and $V \in \{L, \check{X}, Y_A\}$ we have that*

$$\mathcal{L}_V \mathcal{D}f = \mathcal{D}Vf.$$

0.9 Transport Equations for Eikonal Quantities (Corresponding to Section 2.11 in [18])

Proofs here are identical to those in [18]. The following lemma gives us the fundamental transport equation governing the behavior of μ :

Lemma 0.9.0.1 (Transport equation for μ). *We have that the inverse foliation density μ satisfies*

$$L\mu = \frac{1}{2}G_{LL}\check{X}\Phi - \frac{1}{2}\mu G_{LL}L\Phi - \mu G_{LX}L\Phi \quad (0.9.0.1)$$

and that the rectangular components $L_{(small)}^i$ satisfy:

$$LL_{(small)}^i = -\frac{1}{2}G_{LL}(L\Phi)L^i - \frac{1}{2}G_{LL}(L\Phi)(g^{-1})^{0i} - G_{LA}(\check{g}^{-1})^{AB}\not{d}_Bx^i(L\Phi) + \frac{1}{2}G_{LL}(\check{g}^{-1})^{AB}\not{d}_Ax^i(\not{d}_B\Phi)$$

Lemma 0.9.0.2. *For $i = 1, 2, 3$ we have that*

$$\check{X}L_{(small)}^i = \left\{ -\frac{1}{2}G_{LL}\check{X}\Phi + \frac{1}{2}\mu G_{LL}L\Phi + \mu G_{LX}L\Phi + \frac{1}{2}\mu G_{XX}L\Phi \right\} L^i \quad (0.9.0.2)$$

$$+ \left\{ -\frac{1}{2}G_{LL}\check{X}\Phi + \frac{1}{2}\mu G_{LL}L\Phi + \mu G_{LX}L\Phi + \frac{1}{2}G_{XX}L\Phi \right\} (g^{-1})^{0i} \quad (0.9.0.3)$$

$$- \left\{ G_L^A \check{X}\Phi + \frac{1}{2}\mu G_{XX}\not{d}^A\Phi \right\} \cdot \not{d}_Ax^i + (\not{d}^A\mu) \cdot \not{d}_Ax^i. \quad (0.9.0.4)$$

0.10 Geometry of the Covariant Wave Operator

0.10.1 Ricci Coefficients (Corresponding to 2.12 in [18])

Definition 0.10.1.1 (Ricci Coefficients). For our rescaled geometric frame we define the following quantities:

$$\chi_{AB} := g(\mathcal{D}_{\Theta_A} L, \Theta_B) \quad (0.10.1.1)$$

$$k_{AB} := g(\mathcal{D}_{\Theta_A} N, \Theta_B) \quad (0.10.1.2)$$

$$\zeta_A := g(\mathcal{D}_{\Theta_A} L, X) = \frac{1}{\mu} g(\mathcal{D}_{\Theta_A} L, \check{X}) \quad (0.10.1.3)$$

$$\mathrm{tr}_{\mathcal{G}} \chi := \mathcal{G}^{AB} \chi_{AB} \quad (0.10.1.4)$$

$$\mathrm{tr}_{\mathcal{G}} k := \mathcal{G}^{AB} k_{AB} \quad (0.10.1.5)$$

$$\widehat{\chi}_{AB} := \chi_{AB} - \frac{1}{2} \mathcal{G}_{AB} \mathrm{tr}_{\mathcal{G}} \chi \quad (0.10.1.6)$$

Remark 0.10.1.1. One can easily check that these definitions agree with the definitions of χ and k in terms of Lie derivatives in a previous section.

We can express the covariant derivatives of our frame vectorfield in terms of the Ricci coefficients as follows.

Lemma 0.10.1.1. *We use the notation $\omega := L\mu$*

$$\mathcal{D}_L L = \mu^{-1} \omega L \quad (0.10.1.7)$$

$$\mathcal{D}_{\check{X}} L = -\omega L + \mu \zeta^A \Theta_A + \mathcal{L}^A \mu \Theta_A \quad (0.10.1.8)$$

$$\mathcal{D}_A L = -\zeta_A L + \chi_A{}^B \Theta_B \quad (0.10.1.9)$$

$$\mathcal{D}_L \check{X} = -\omega L - \mu \zeta^A \Theta_A \quad (0.10.1.10)$$

$$\mathcal{D}_{\check{X}} \check{X} = \mu \omega L + \left\{ \mu^{-1} \check{X} \mu + \omega \right\} \check{X} - \mu \mathcal{L}^A \mu \Theta_A \quad (0.10.1.11)$$

$$\mathcal{D}_A \check{X} = \mu \zeta_A L + \zeta_A \check{X} + \mu^{-1} (\mathcal{L}_A \mu) \check{X} + \mu k_A{}^B \Theta_B - \mu \chi_A{}^B \Theta_B \quad (0.10.1.12)$$

$$\mathcal{D}_L \Theta_A = \mathcal{D}_A L \quad (0.10.1.13)$$

$$\mathcal{D}_A \Theta_B = \nabla_A \Theta_B + k_{AB} L + \mu^{-1} \chi_{AB} \check{X} \quad (0.10.1.14)$$

and we can write

$$\zeta_A = \boldsymbol{\mu}^{-1} \zeta_A^{(Trans-\Phi)} + \zeta_A^{(Tan-\Phi)} \quad (0.10.1.15)$$

$$k_{AB} = \boldsymbol{\mu}^{-1} k_{AB}^{(Trans-\Phi)} + k_{AB}^{(Tan-\Phi)} \quad (0.10.1.16)$$

where we define the transverse components by

$$\zeta_A^{(Trans-\Phi)} := -\frac{1}{2} \mathcal{G}_{LA} \check{X} \Phi \quad (0.10.1.17)$$

$$k_{AB}^{(Trans-\Phi)} := \frac{1}{2} \mathcal{G}_{AB} \check{X} \Phi \quad (0.10.1.18)$$

and the tangent components by

$$\zeta_A^{(Tan-\Phi)} := \frac{1}{2} \mathcal{G}_{XA} L \Phi - \frac{1}{2} G_{LX} \not{d}_A \Phi - \frac{1}{2} G_{XX} \not{d}_A \Phi \quad (0.10.1.19)$$

$$k_{AB}^{(Tan-\Phi)} := \frac{1}{2} \mathcal{G}_{AB} L \Phi - \frac{1}{2} \mathcal{G}_{LB} \not{d}_A \Phi - \frac{1}{2} \mathcal{G}_{LA} \not{d}_B \Phi - \frac{1}{2} \mathcal{G}_{XB} \not{d}_A \Phi - \frac{1}{2} \mathcal{G}_{XA} \not{d}_B \Phi \quad (0.10.1.20)$$

Lemma 0.10.1.2. *We have that*

$$\not{d}_A L^i = (\mathcal{G}^{-1})^{BC} \chi_{AB} \not{d}_C x^i + (\mathcal{G}^{-1})^{BC} \Lambda_{AB}^{Tan-\Phi} \not{d}_C x^i + \lambda_A^{Tan-\Phi} X^i \quad (0.10.1.21)$$

$$\not{d}_A X^i = -(\mathcal{G}^{-1})^{BC} \chi_{AB} \not{d}_C x^i + (\mathcal{G}^{-1})^{BC} E_{AB}^{(Tan-\Phi)} \not{d}_C x^i + \eta_A^{(Tan-\Phi)} X^i \quad (0.10.1.22)$$

where we denote

$$\lambda_A^{(Tan-\Phi)} := -G_{LX} \not{d}_A \Phi - \frac{1}{2} G_{XX} \not{d}_A \Phi \quad (0.10.1.23)$$

$$\eta_A^{(Tan-\Phi)} := -\frac{1}{2} G_{XX} \not{d}_A \Phi \quad (0.10.1.24)$$

$$\Lambda_{AB}^{(Tan-\Phi)} := -\frac{1}{2} \mathcal{G}_{AB} L \Phi + \frac{1}{2} \mathcal{G}_{LA} \not{d}_B \Phi - \frac{1}{2} \mathcal{G}_{LB} \not{d}_A \Phi \quad (0.10.1.25)$$

$$E_{AB}^{(Tan-\Phi)} := \frac{1}{2} \mathcal{G}_{AB} L \Phi - \frac{1}{2} \mathcal{G}_{LA} \not{d}_B \Phi - \frac{1}{2} \mathcal{G}_{LB} \not{d}_A \Phi - \mathcal{G}_{XB} \not{d}_A \Phi \quad (0.10.1.26)$$

Lemma 0.10.1.3. *We have the following identities involving χ :*

$$\chi_{AB} = g_{\alpha\beta} \not{d}_A L_{(small)}^\alpha \not{d}_B x^b + \frac{1}{2} \mathcal{G}_{AB} L \Phi + \{ \mathcal{G}_L \wedge \not{d} \Phi \}_{AB} \quad (0.10.1.27)$$

$$= g_{\alpha\beta} \not{d}_A L_{(small)}^\alpha \not{d}_B x^b - \Lambda_{AB}^{(Tan-\Phi)} \quad (0.10.1.28)$$

$$tr_{\mathcal{G}} \chi = g_{\alpha\beta} (\mathcal{G}^{-1})^{AB} (\not{d}_A L_{(small)}^\alpha \not{d}_B x^b + \frac{1}{2} (tr_{\mathcal{G}} \mathcal{G}) L \Phi) \quad (0.10.1.29)$$

Proof. Write $\mathcal{D}_A L^\beta = \partial_A L^\beta + \Gamma_{AL}^\beta$ and use (0.7.1.3), then substitute this into the definition of χ_{AB} . $\square$

0.10.2 Frame Decomposition of the Wave Operator

Lemma 0.10.2.1. *Relative to our frame $\{L, \check{X}, \Theta_1, \Theta_2\}$ we find that*

$$\mu \square_{g(\Phi)} f = -L(\mu L f + 2\check{X} f) + \mu \Delta f - (tr_{\check{g}} \chi) \check{X} f - \mu (tr_{\check{g}} k) L f - 2\mu \zeta^A \not{d}_A f \quad (0.10.2.1)$$

$$\mu \square_{g(\Phi)} f = -(\mu L + 2\check{X})(L f) + \mu \Delta f - (tr_{\check{g}} \chi) \check{X} f - \omega L f + 2\mu \zeta^A \not{d}_A f + 2(\not{d}^A \mu) \cdot \not{d}_A f. \quad (0.10.2.2)$$

0.11 Commutation Vectorfields (Corresponding to Chapter 6 in [16])

Remark 0.11.0.1. These commutators seem unnatural. However, choosing the rotations to be the solutions to $\mathcal{L}_L X_A = 0$ creates the problem that L is one degree more regular than the X_A s, which this construction circumvents. See explanation in section 2.8 of [18].

Definition 0.11.0.1 (Angular Commutators). We will define the angular commutation vectorfields $Y_{(l)}$ ($l=1,2$) by

$$Y_{(l)}^i := \mathbb{M}_{l+1}^i \quad (0.11.0.1)$$

I.e. we simply project the rectangular vectorfields δ_2^i, δ_3^i onto $T\mathcal{P}_{t,u}$

Definition 0.11.0.2. We define the set of commutation vectorfields to be

$$\mathcal{L} := \{L, \check{X}, Y_{(1)}, Y_{(2)}\}$$

and the set of $\mathcal{C}_u$ tangent commutation vectorfields to be

$$\mathcal{P} := \{L, Y_{(1)}, Y_{(2)}\}$$

Definition 0.11.0.3. For $i = 1, 2, 3$ we will define:

$$\begin{aligned} L_{(small)}^i &:= L^i - \delta_1^i \\ X_{(small)}^i &:= X^i + \delta_1^i \\ Y_{(l),(small)}^i &:= Y_{(l)}^i - \delta_{l+1}^i \end{aligned} \tag{0.11.0.2}$$

Definition 0.11.0.4 (X component of flat “angular vectorfield”). We define

$$\rho_{(l)} := g(\delta_{l+1}, X).$$

Lemma 0.11.0.1. For $l = 1, 2$ and $i = 1, 2, 3$ we have

$$\delta_{l+1}^i = Y_{(l)}^i + \rho_{(l)} X^i \tag{0.11.0.3}$$

$$\rho_{(l)} = g_{(l+1)b} X^b = g_{(l+1)b} X_{(small)}^b - g_{(l+1)1} \tag{0.11.0.4}$$

Lemma 0.11.0.2.

$$Y_{(l)}^i = \delta_{l+1}^i - \rho_{(l)} X^i = \delta_{l+1}^i - \rho_{(l)} X_{(small)}^i + \rho_{(l)} \delta_1^i \tag{0.11.0.5}$$

Lemma 0.11.0.3.

$$\not{g}(Y_{(l)}, \Theta_B) = g_{(l+1)b} \not{d}_B x^b \tag{0.11.0.6}$$

The following lemma appears in [16] (but not in [18] for obvious reasons).

Lemma 0.11.0.4. We have that

$$\mathcal{L}_{Y_{(1)}} Y_{(2)} = [Y_{(1)}, Y_{(2)}] = W^\sharp \tag{0.11.0.7}$$

Where $W^\sharp$ is the vectorfield in $T\mathcal{P}_{t,u}$ which is $\not{g}$ -dual to

$$W_A = \rho_{(2)} Y_{(1)}^B \chi_{AB} - \rho_{(1)} Y_{(2)}^B \chi_{AB} \tag{0.11.0.8}$$

$$+ \rho_{(1)} Y_{(2)}^B E_{AB}^{(Tan-\Phi)} - \rho_{(2)} Y_{(1)}^B E_{AB}^{(Tan-\Phi)} \tag{0.11.0.9}$$

0.12 Deformation Tensors of Commutator Vectorfields

The deformation tensors here are at the origin of the most complicated error terms in the proof of shock formation. They appear whenever we commute an $\mathcal{L}$ vectorfield with a $\mathcal{P}$ vectorfield (see

(0.8.0.1)) and whenever we commute an $\mathcal{L}$ vectorfield with $\square_g$ (see (0.15.0.2) below).

Lemma 0.12.0.1. *The covariant derivatives of our projection operator $\mathbb{I}$ expressed in our frame are given by*

$$(\mathcal{D}_L \mathbb{I}) \cdot X = \zeta^A \Theta_A \quad (0.12.0.1)$$

$$(\mathcal{D}_L \mathbb{I}) \cdot \Theta_A = -\zeta_A L \quad (0.12.0.2)$$

$$(\mathcal{D}_{\check{X}} \mathbb{I}) \cdot X = \not{d}^A \boldsymbol{\mu} \cdot \Theta_A \quad (0.12.0.3)$$

$$(\mathcal{D}_{\check{X}} \mathbb{I}) \cdot \Theta_A = \boldsymbol{\mu} \zeta_A L + \boldsymbol{\mu} \zeta_A X + (\not{d}_A \boldsymbol{\mu}) X \quad (0.12.0.4)$$

$$(\mathcal{D}_A \mathbb{I}) \cdot X = \chi_A{}^B \Theta_B - \not{k}_A{}^B \Theta_B \quad (0.12.0.5)$$

$$(\mathcal{D}_A \mathbb{I}) \cdot \Theta_B = \not{k}_{AB} L + \chi_{AB} X. \quad (0.12.0.6)$$

Lemma 0.12.0.2. *The following frame decompositions hold for the deformation tensors of our commutation vectorfields*

$$(\check{X}) \pi_{LL} = 0 \quad (0.12.0.7)$$

$$(\check{X}) \pi_{\check{X}X} = 2\check{X} \boldsymbol{\mu} \quad (0.12.0.8)$$

$$(\check{X}) \pi_{L\check{X}} = -\check{X} \boldsymbol{\mu} \quad (0.12.0.9)$$

$$(\check{X}) \pi_{LA} = -\not{d}_A \boldsymbol{\mu} - 2\zeta_A^{(Trans-\Phi)} - 2\boldsymbol{\mu} \zeta_A^{(Tan-\Phi)} \quad (0.12.0.10)$$

$$(\check{X}) \pi_{\check{X}A} = 0 \quad (0.12.0.11)$$

$$(\check{X}) \pi_{AB} = -2\boldsymbol{\mu} \chi_{AB} + 2\not{k}_{AB}^{(Trans-\Phi)} + 2\boldsymbol{\mu} \not{k}^{(Tan\Phi)} \quad (0.12.0.12)$$

$$^{(L)} \pi_{LL} = 0 \quad (0.12.0.13)$$

$$^{(L)} \pi_{\check{X}X} = 2L \boldsymbol{\mu} \quad (0.12.0.14)$$

$$^{(L)} \pi_{L\check{X}} = -L \boldsymbol{\mu} \quad (0.12.0.15)$$

$$^{(L)} \pi_{LA} = 0 \quad (0.12.0.16)$$

$$^{(L)} \pi_{\check{X}A} = \not{d}_A \boldsymbol{\mu} + 2\zeta_A^{(Trans-\Phi)} + 2\boldsymbol{\mu} \zeta_A^{(Tan-\Phi)} \quad (0.12.0.17)$$

$$^{(L)} \pi_{AB} = 2\chi_{AB} \quad (0.12.0.18)$$

$$^{(Y_{(l)})}\pi_{LL} = 0 \quad (0.12.0.19)$$

$$^{(Y_{(l)})}\pi_{\check{X}X} = 2Y_{(l)}\mathbf{\mu} \quad (0.12.0.20)$$

$$^{(Y_{(l)})}\pi_{L\check{X}} = -Y_{(l)}\mathbf{\mu} \quad (0.12.0.21)$$

$$^{(Y_{(l)})}\pi_{LA} = -\chi_{AB}Y_{(l)}^B + \mathbf{Error}[\pi, L, Y_{(l)}]_A \quad (0.12.0.22)$$

$$^{(Y_{(l)})}\pi_{\check{X}A} = \mathbf{\mu}\chi_{AB}Y_{(l)}^B + \rho_{(l)}\not{d}_A\mathbf{\mu} + \mathbf{Error}[\pi, \check{X}, Y_{(l)}]_A \quad (0.12.0.23)$$

$$^{(Y_{(l)})}\pi_{AB} = 2\rho_{(l)}\chi_{AB} + \mathbf{Error}[\pi, \Theta, Y_{(l)}]_{AB} \quad (0.12.0.24)$$

where

$$\begin{aligned} \mathbf{Error}[\pi, L, Y_{(l)}]_A &= \frac{1}{2}(\not{G}_A \cdot Y_{(l)})L\Phi + \rho_{(l)}\not{G}_{XA}L\Phi \\ &\quad - \frac{1}{2}\rho_{(l)}G_{XX}\not{d}_A\Phi - \rho_{(l)}G_{LX}\not{d}_A\Phi + (\not{G}_L \wedge \not{d}\Phi)_{AB}Y_{(l)}^B \\ \mathbf{Error}[\pi, \check{X}, Y_{(l)}]_A &= -\mathbf{\mu}k_{AB}^{(Tan-\Phi)}Y_{(l)}^B + \rho_{(l)}\not{G}_{XA}\check{X}\Phi - \frac{1}{2}\mathbf{\mu}\rho_{(l)}G_{XX}\not{d}_A\Phi + \mathbf{\mu}(\not{G}_X \wedge \not{d}\Phi)_{AB}Y_{(l)}^B \\ \mathbf{Error}[\pi, \Theta, Y_{(l)}]_{AB} &= -2\rho_{(l)}k_{AB}^{(Tan-\Phi)} + 2(\not{G} \cdot Y_{(l)} \wedge \not{d}\Phi)_{AB} + 2\rho_{(l)}(\not{G} \cdot X \wedge \not{d}\Phi)_{AB} + \not{G}_{AB}Y_{(l)}\Phi \end{aligned} \quad (0.12.0.25)$$

and where we have defined the wedge product of ξ, η cotangent vectors on $\mathcal{P}_{t,u}$ by $(\xi \wedge \eta)_{AB} := \frac{1}{2}(\xi_A\eta_B - \eta_A\xi_B)$

Proof. We only prove the last few results for the deformation tensor of $Y_{(l)}$ as the others are straightforward and do not rely on any special cancellations. We first have that

$$\begin{aligned} ^{(Y_{(l)})}\pi_{LA} &= g(\mathcal{D}_LY_{(l)}, \Theta_A) + g(\mathcal{D}_AY_{(l)}, L) \\ &= g(\mathcal{D}_LY_{(l)}, \Theta_A) - g(\mathcal{D}_AL, Y_{(l)}) \quad \text{from } g(L, Y_{(l)}) = 0 \\ &= g((\mathcal{D}_L\mathbb{H}) \cdot \delta_{l+1}, \Theta_A) + g(\mathcal{D}_L\delta_{l+1}, \Theta_A) - \chi_{AB}Y_{(l)}^B \quad \text{using } Y_{(l)}^j = \mathbb{H}^j_i\delta_{l+1}^i \\ &= \mathbf{\mu}^{-1}\rho_{(l)}\zeta_A^{(Trans-\Phi)} + \rho_{(l)}\zeta_A^{(Tan-\Phi)} \\ &\quad + \Gamma_{LAB}Y_{(l)}^B + \rho_{(l)}\Gamma_{LAX} - \chi_{AB}Y_{(l)}^B \\ &= \rho_{(l)}\zeta_A^{(Tan-\Phi)} + \frac{1}{2}\not{G}_{LA}Y_{(l)}\Phi + \frac{1}{2}\not{G}_{AB}Y_{(l)}^BL\Phi - \frac{1}{2}\not{G}_{LB}Y_{(l)}^B\not{d}_A\Phi \\ &\quad + \frac{1}{2}\rho_{(l)}\not{G}_{XA}L\Phi - \frac{1}{2}\rho_{(l)}G_{LX}\not{d}_A\Phi - \chi_{AB}Y_{(l)}^B \end{aligned}$$

where in the last line we have used the cancellation of the bad term in $\rho_{(l)}\Gamma_{LAX}$, namely

$$\frac{1}{2}\mathbf{\mu}^{-1}\rho_{(l)}\mathcal{G}_{LA}\check{X}\Phi$$

with

$$\mathbf{\mu}^{-1}\rho_{(l)}\zeta_A^{(Trans-\Phi)} = -\frac{1}{2}\mathbf{\mu}^{-1}\rho_{(l)}\mathcal{G}_{LA}\check{X}\Phi.$$

□

Corollary 0.12.0.3. *We have that*

$$\nabla_A Y_{(l)}^B = \rho_{(l)}\chi_A^B + \Gamma_{AD}^B Y_{(l)}^D + \rho_{(l)} (\mathcal{G}_X \wedge \mathcal{H}\Phi)_A^B \quad (0.12.0.26)$$

Proof. Similar to the proof above, we have:

$$\nabla_A Y_{(l)}^B = (\mathcal{G}^{-1})^{BC} \mathcal{G}(\mathcal{D}_A Y_{(l)}, \Theta_C) \quad (0.12.0.27)$$

$$= (\mathcal{G}^{-1})^{BC} \mathcal{G}(\mathcal{D}_A (\mathbb{M}\delta_{l+1}), \Theta_C) \quad (0.12.0.28)$$

$$= (\mathcal{G}^{-1})^{BC} \{ \mathcal{G}((\mathcal{D}_A \mathbb{M})\delta_{l+1}, \Theta_C) + \mathcal{G}(\mathcal{D}_A \delta_{l+1}, \Theta_C) \} \quad (0.12.0.29)$$

$$= (\mathcal{G}^{-1})^{BC} \{ \mathcal{G}((\mathcal{D}_A \mathbb{M})Y_{(l)}, \Theta_C) \quad (0.12.0.30)$$

$$+ \rho_{(l)} \mathcal{G}((\mathcal{D}_A \mathbb{M})X, \Theta_C) + \mathcal{G}(\mathcal{D}_A \delta_{l+1}, \Theta_C) \} \quad (0.11.0.3) \quad (0.12.0.31)$$

$$= (\mathcal{G}^{-1})^{BC} \left\{ \rho_{(l)}(\chi_{AC} - \mathcal{K}_{AC}) + \Gamma_{ACD} Y_{(l)}^D + \rho_{(l)} \Gamma_{ACX} \right\} \quad \text{using (0.12.0.6), (0.12.0.5), (0.11.0.3)} \quad (0.12.0.32)$$

Now note that the bad term

$$-\rho_{(l)}\mathbf{\mu}^{-1}\mathcal{K}_{AC}^{(Trans-\Phi)} = -\rho_{(l)}\mathbf{\mu}^{-1}\frac{1}{2}\mathcal{G}_{AC}\check{X}\Phi$$

from $\rho_{(l)}\mathcal{K}$ and the bad term

$$\rho_{(l)}\mathbf{\mu}^{-1}\frac{1}{2}\mathcal{G}_{AC}\check{X}\Phi$$

from $\rho_{(l)}\Gamma_{ACX}$ cancel out exactly. □

Remark 0.12.0.1 (Compare Remark 7.6 in [16]). There are several miraculous properties to the above commutators and their deformation tensors. Note the absence of bad $X\Phi$ (transverse derivative) terms, which would carry dangerous $\mathbf{\mu}^{-1}$ weights. We also have that $^{(Z)}\pi_{LL}$ for $Z \in \mathcal{L}$, which means that the uncontrollable terms $\mathbf{\mu}^{-1(Z)}\pi_{LL}\check{X}\Phi$ and $\mathbf{\mu}^{-1}\{^{(Z)}\pi_{L\check{X}} + Z\mathbf{\mu}\}$ do not appear on the

RHS after commutation in (0.15.0.2) or in the divergence of the commutation current (0.15.0.4).

0.13 A Brief But Important Interlude On Shorthand Notation (Corresponds to section 2.16 in [18])

Following Speck et al in [18], we will define the following arrays of fundamental unknowns:

Definition 0.13.0.1.

$$\gamma := \left(\Phi, L^i_{(Small)} \right) \quad (0.13.0.1)$$

$$\underline{\gamma} := \left(\Phi, \mu - 1, L^i_{(Small)} \right) \quad (0.13.0.2)$$

Note that all $\mathcal{P}$ -derivatives of γ quantities are small but when L hits an $\underline{\gamma}$ quantity it can be large (although taking additional $\mathcal{P}$ -derivatives regains the smallness).

The following lemma is a major conceptual simplification in [18] relative to [16]. By clarifying the dependence of our eikonal and metric quantities on the fundamental unknowns, our bookkeeping becomes considerably easier.

Lemma 0.13.0.1.

$$g_{\alpha\beta}, (g^{-1})^{\alpha\beta}, (\underline{g}^{-1})^{\alpha\beta}, \not{g}_{\alpha\beta}, (\not{g}^{-1})^{\alpha\beta}, G_{\alpha\beta}, G'_{\alpha\beta}, \mathbb{W}^\beta_\alpha, L^\alpha, X^\alpha, Y^\alpha_{(l)} = f(\gamma), \quad (0.13.0.3)$$

$$G_{LL}, G_{LX}, G_{XX}, G'_{LL}, G'_{LX}, G'_{XX} = f(\gamma) \quad (0.13.0.4)$$

$$g^{(Small)}_{\alpha\beta}, Y^\alpha_{(l), (Small)}, X^\alpha_{(Small)}, \rho_{(l)} = f(\gamma)\gamma \quad (0.13.0.5)$$

$$\check{X}^\alpha = f(\underline{\gamma}) \quad (0.13.0.6)$$

$$\not{g}, \not{G}_{LA}, \not{G}_{XA}, \not{G}_{AB}, \not{G}'_{LA}, \not{G}'_{XA}, \not{G}'_{AB} = f(\gamma, \not{d}x^i) \quad (0.13.0.7)$$

$$Y_{(l)} = f(\gamma, \not{g}^{-1}, \not{d}x^i) \quad (0.13.0.8)$$

$$\nabla Y_{(l)} = f(\gamma, \not{g}^{-1}, \not{d}x^i) P\gamma \quad (0.13.0.9)$$

$$\zeta^{(Tan-\Phi)}, \not{k}^{(Tan-\Phi)}, \lambda^{(Tan-\Phi)}, \eta^{(Tan-\Phi)}, \Lambda^{(Tan-\Phi)}, E^{(Tan-\Phi)} = f(\gamma, \not{d}x^i) P\Phi \quad (0.13.0.10)$$

$$\zeta^{(Trans-\Phi)}, \not{k}^{(Trans-\Phi)} = f(\gamma, \not{d}x^i) \check{X}\Phi \quad (0.13.0.11)$$

$$\chi = f(\gamma, \not{d}x^i) P\gamma \quad (0.13.0.12)$$

$$W^\# = f(\gamma, \not{g}^{-1}, \not{d}x^i) \gamma \cdot P\gamma \quad (0.13.0.13)$$

$$Error[\pi, Y] = f(\gamma, \not{g}^{-1}, \not{d}x^i) P\Phi \quad (0.13.0.14)$$

0.14 Energies and Fluxes

In this section we assume we are dealing with solutions to a μ -weighted geometric wave equation $\mu \square_g \Psi = \mathfrak{F}$

Definition 0.14.0.1 (Energy-Momentum Tensor). We define $Q_{\alpha\beta}[\Psi]$ by

$$Q_{\alpha\beta} := \mathcal{D}_\alpha \Psi \mathcal{D}_\beta \Psi - \frac{1}{2} g_{\alpha\beta} g(\mathcal{D}\Psi, \mathcal{D}\Psi)$$

Lemma 0.14.0.1 (Divergence Identity). *For solutions to $\mu \square_g \Psi = \mathfrak{F}$ and vectorfields X we define the multiplier current ${}^{(X)}\mathcal{P}^\alpha[\Psi] := Q^\alpha{}_\beta[\Psi] X^\beta$ which satisfies*

$$\mu \mathcal{D}_\alpha {}^{(X)}\mathcal{P}^\alpha[\Psi] = \mathfrak{F} X^\alpha + \frac{1}{2} \mu Q_{\alpha\beta} {}^{(X)}\pi^{\alpha\beta}$$

Lemma 0.14.0.2 (Null decomposition of $Q_{\alpha\beta}$). *If we let $Q_{\alpha\beta}[\Psi]$ be the energy-momentum tensor defined above, then relative to the rescaled null frame $\{L, \check{X}, Y_{(1)}, Y_{(2)}\}$ we find that*

$$\begin{aligned}
Q_{LL} &= (L\Psi)^2 \\
Q_{\check{X}\check{X}} &= \frac{1}{2}\mu^2(L\Psi)^2 + (\check{X}\Psi)^2 + \mu(L\Psi)\check{X}\Psi - \frac{1}{2}\mu^2|\not{d}\Psi|^2 \\
Q_{L\check{X}} &= -\frac{1}{2}\mu(L\Psi)^2 + \frac{1}{2}\mu|\not{d}\Psi|^2 \\
Q_{LA} &= (L\Psi)\not{d}_A\Psi \\
Q_{\check{X}A} &= (\check{X}\Psi)\not{d}_A\Psi \\
Q_{AB} &= (\not{d}_A\Psi)(\not{d}_B\Psi) - \frac{1}{2}\not{g}_{AB} \left\{ -(L\Psi)^2 - 2\mu^{-1}(L\Psi)(\check{X}\Psi) + |\not{d}\Psi|^2 \right\}
\end{aligned} \tag{0.14.0.1}$$

Definition 0.14.0.2 (Multiplier Vectorfield). We define the timelike multiplier field by

$$T := (1 + 2\mu)L + 2\check{X}$$

Lemma 0.14.0.3 (Null components of $^{(T)}\pi$). *For T the above defined timelike multiplier we find that its deformation tensor admits the following decomposition relative to the rescaled null frame $\{L, \check{X}, Y_1, Y_2\}$*

$$\begin{aligned}
^{(T)}\pi_{LL} &= 0 \\
^{(T)}\pi_{\check{X}\check{X}} &= 2(1 + 2\mu)L\mu \\
^{(T)}\pi_{L\check{X}} &= -\left\{ L\mu + 4\mu L\mu + 2\check{X}\mu \right\} \\
^{(T)}\not{\pi}_{LA} &= -2\not{d}_A\mu - 4\mu\zeta_A \\
^{(T)}\not{\pi}_{\check{X}A} &= \not{d}_A\mu + 2(1 + 2\mu)\mu\zeta_A \\
^{(T)}\hat{\not{\pi}}_{AB} &= 2\hat{\chi}_{AB} + 4\mu\hat{k}_{AB} \\
tr_{\not{g}}^{(T)}\not{\pi} &= 2tr_{\not{g}}\chi + 4\mu tr_{\not{g}}k
\end{aligned} \tag{0.14.0.2}$$

Remark 0.14.0.1 (nondegenerate volume forms). Note that in this section all of the volume forms in our integrals are geometric and nondegenerate i.e. they are not written with μ weights. Rather, as the $L\Phi$, $\check{X}\Phi$, and $\not{d}\Phi$ terms exhibit different behavior with respect to μ we write all integrals such that the μ weights are explicitly paired with the respective derivative terms.

Definition 0.14.0.3 (Multiplier Current). We will use the following timelike multiplier current:

$${}^{(T)}\mathcal{P}^\alpha[\Psi] := Q^\alpha{}_\beta[\Psi]T^\beta$$

Definition 0.14.0.4 (Energy and Null Flux). We will define the Energy and Null Flux of Ψ to be

$$\begin{aligned}\mathbb{E}[\Psi](t, u) &:= \int_{\Sigma_t^u} \mu^{(T)} \mathcal{P}_N[\Psi] dv_g du \\ \mathbb{F}[\Psi](t, u) &:= \int_{C_u^t} {}^{(T)}\mathcal{P}_L[\Psi] dv_g dt\end{aligned}\tag{0.14.0.3}$$

Remark 0.14.0.2. A simple computation reveals that we have

$$\begin{aligned}\mathbb{E}[\Psi](t, u) &= \int_{\Sigma_t^u} \left\{ \frac{\mu}{2} (1 + 2\mu) (L\Psi)^2 + \frac{1}{2} (1 + 2\mu) \mu |\not{d}\Psi|^2 + 2(\check{X}\Psi)^2 + 2\mu(L\Psi)\check{X}\Psi \right\} dv_g du \\ \mathbb{F}[\Psi](t, u) &= \int_{C_u^t} \left\{ (1 + \mu) (L\Psi)^2 + \mu |\not{d}\Psi|^2 \right\} dv_g dt\end{aligned}\tag{0.14.0.4}$$

Lemma 0.14.0.4 (Divergence Theorem on $\mathcal{M}_{t,u}$). *For any vectorfield $\mathcal{P}$ have that*

$$\int_{\Sigma_t^u} \mu \mathcal{P}_N dv_g du + \int_{C_u^t} \mathcal{P}_L dv_g dt - \int_{\Sigma_0^u} \mu \mathcal{P}_N dv_g du - \int_{C_0^t} \mathcal{P}_L dv_g dt = - \int_{\mathcal{M}_{t,u}} \mu \mathcal{D}_\alpha \mathcal{P}^\alpha dv_g dudt \tag{0.14.0.5}$$

From the above we easily derive that

Lemma 0.14.0.5 (Energy-Flux Identity). *For solutions to $\mu \square_g \Psi = \mathfrak{F}$ which vanish along the outer null hypersurface C_0 we have*

$$\begin{aligned}\mathbb{E}[\Psi](t, u) + \mathbb{F}[\Psi](t, u) &= \mathbb{E}[\Psi](0, u) \\ &\quad - \int_{\mathcal{M}_{t,u}} \left\{ (1 + 2\mu) L\Psi + 2\check{X}\Psi \right\} \mathfrak{F} dv_g dudt \\ &\quad - \frac{1}{2} \int_{\mathcal{M}_{t,u}} \mu Q_{\alpha\beta}[\Psi] {}^{(T)}\pi^{\alpha\beta} dv_g dudt\end{aligned}\tag{0.14.0.6}$$

Lemma 0.14.0.6 (Bulk Error Term). *We find that the bulk error term in the above energy-flux identity is given by*

$$\mathfrak{E}^{(T)}[\Psi] := -\frac{1}{2} \mu Q_{\alpha\beta}[\Psi] {}^{(T)}\pi^{\alpha\beta} = -\frac{1}{2} \mu |\not{d}\Psi|^2 \frac{[L\mu]_-}{\mu} + \sum_{i=1}^6 \mathfrak{E}_{(i)}^{(T)}[\Psi] \tag{0.14.0.7}$$

where $f_+ := \max\{f, 0\}$ and $f_- := \max\{-f, 0\}$ and

$$\mathfrak{E}_{(1)}^{(T)}[\Psi] := (L\Psi)^2 \left\{ -\frac{1}{2}L\mu + \check{X}\mu - \frac{1}{2}\mu \text{tr}_{\mathfrak{g}}\chi - \mu^2 \text{tr}_{\mathfrak{g}}\mathring{k} \right\} \quad (0.14.0.8)$$

$$\mathfrak{E}_{(2)}^{(T)}[\Psi] := -(L\Psi)(\check{X}\Psi) \left\{ \text{tr}_{\mathfrak{g}}\chi + 2\mu \text{tr}_{\mathfrak{g}}\mathring{k} \right\} \quad (0.14.0.9)$$

$$\mathfrak{E}_{(3)}^{(T)}[\Psi] := \mu |\not{d}\Psi|^2 \left\{ \frac{1}{2} \frac{[L\mu]_+}{\mu} + \frac{\check{X}\mu}{\mu} + 2L\mu \right\} \quad (0.14.0.10)$$

$$\mathfrak{E}_{(4)}^{(T)}[\Psi] := (L\Psi)(\not{d}_A\Psi) \left\{ (1-\mu)\not{d}^A\mu + 2\zeta^A + 2\mu(2-\mu)\zeta^A \right\} \quad (0.14.0.11)$$

$$\mathfrak{E}_{(5)}^{(T)}[\Psi] := -2(\check{X}\Psi)\not{d}_A\Psi \left\{ \not{d}^A\mu + 2\mu\zeta^A \right\} \quad (0.14.0.12)$$

$$\mathfrak{E}_{(6)}^{(T)}[\Psi] := -\mu \left\{ \hat{\chi}(\not{d}\Psi, \not{d}\Psi) + 2\mu \hat{k}(\not{d}\Psi, \not{d}\Psi) \right\} \quad (0.14.0.13)$$

Lemma 0.14.0.7 (Derivatives of Surface Integrals). *For scalar functions ϕ we find that*

$$\frac{\partial}{\partial t} \int_{\mathcal{P}_{t,u}} \phi dv_{\mathfrak{g}} = \int_{\mathcal{P}_{t,u}} \left\{ L\phi + (\text{tr}_{\mathfrak{g}}\chi)\phi \right\} dv_{\mathfrak{g}}, \quad (0.14.0.14)$$

$$\frac{\partial}{\partial u} \int_{\mathcal{P}_{t,u}} \phi dv_{\mathfrak{g}} = \int_{\mathcal{P}_{t,u}} \left\{ \check{X}\phi + \frac{1}{2}(\text{tr}_{\mathfrak{g}}^{(\check{X})}\mathring{\pi})\phi \right\} dv_{\mathfrak{g}}, \quad (0.14.0.15)$$

$$\frac{\partial}{\partial t} \int_{\Sigma_t^u} \phi dv_{\mathfrak{g}} du = \int_{\Sigma_t^u} \left\{ L\phi + (\text{tr}_{\mathfrak{g}}\chi)\phi \right\} dv_{\mathfrak{g}} du \quad (0.14.0.16)$$

Lemma 0.14.0.8 (Integration by parts on $\mathcal{M}_{t,u}$). *Let ϕ and ψ be scalar functions on $\mathcal{M}_{t,u}$. We find that*

$$\begin{aligned} \int_{\mathcal{M}_{t,u}} \phi(L\psi) dv_{\mathfrak{g}} du dt &= - \int_{\mathcal{M}_{t,u}} (L\phi)\psi dv_{\mathfrak{g}} du dt - \int_{\mathcal{M}_{t,u}} (\text{tr}_{\mathfrak{g}}\chi)\phi\psi dv_{\mathfrak{g}} du dt \\ &\quad + \int_{\Sigma_t^u} \phi\psi dv_{\mathfrak{g}} du - \int_{\Sigma_0^u} \phi\psi dv_{\mathfrak{g}} du. \end{aligned} \quad (0.14.0.17)$$

Lemma 0.14.0.9 (Integration by parts on $\mathcal{P}_{t,u}$). *Let ϕ and ψ be scalar functions on $\mathcal{M}_{t,u}$ and let $Z \in T\mathcal{P}_{t,u}$ be a vectorfield.*

$$\int_{\mathcal{P}_{t,u}} (Z\phi)\psi dv_{\mathfrak{g}} = - \int_{\mathcal{P}_{t,u}} \phi(Z\psi) dv_{\mathfrak{g}} - \frac{1}{2} \int_{\mathcal{P}_{t,u}} \left(\text{tr}_{\mathfrak{g}}^{(Z)}\mathring{\pi} \right) \phi\psi dv_{\mathfrak{g}}, \quad (0.14.0.18)$$

where $\frac{1}{2} \text{tr}_{\mathfrak{g}}^{(Z)}\mathring{\pi} = d\mathring{k}vZ$.

Lemma 0.14.0.10 (Weighted integration by parts for top-order error terms). *For scalar functions*

Ψ we have that

$$\begin{aligned}
& \int_{\mathcal{M}_{t,u}} (1+2\mu) \left(\check{X}\Phi \right) (L\mathcal{P}^N\Phi) Y_{(l)} \Psi dv_g dudt = \int_{\mathcal{M}_{t,u}} (1+2\mu) \left(\check{X}\Phi \right) (Y_{(l)} \mathcal{P}^N\Phi) L\Psi dv_g dudt \\
& - \int_{\Sigma_t^u} (1+2\mu) \left(\check{X}\Phi \right) (Y_{(l)} \mathcal{P}^N\Phi) \Psi dv_g du + \int_{\Sigma_0^u} (1+2\mu) \left(\check{X}\Phi \right) (Y_{(l)} \mathcal{P}^N\Phi) \Psi dv_g du \\
& + \int_{\mathcal{M}_{t,u}} Error_{1,(l)}[\mathcal{P}^N\Phi; \Psi] dv_g dudt + \int_{\Sigma_t^u} Error_{2,(l)}[\mathcal{P}^N\Phi; \Psi] dv_g du - \int_{\Sigma_0^u} Error_{2,(l)}[\mathcal{P}^N\Phi; \Psi] dv_g du
\end{aligned} \tag{0.14.0.19}$$

where

$$\begin{aligned}
Error_{1,(l)}[\mathcal{P}^N\Phi; \Psi] &:= 2(L\mu) \left(\check{X}\Phi \right) (Y_{(l)} \mathcal{P}^N\Phi) \Psi + (1+2\mu) \left(L\check{X}\Phi \right) (Y_{(l)} \mathcal{P}^N\Phi) \Psi \\
&+ (1+2\mu) \left(\check{X}\Phi \right) \left((Y_{(l)})^\sharp \not{X}_L \cdot \not{d} \mathcal{P}^N\Phi \right) \Psi + (1+2\mu) \left(\check{X}\Phi \right) tr_g \chi (Y_{(l)} \mathcal{P}^N\Phi) \Psi \\
&+ 2(Y_{(l)}\mu) \left(\check{X}\Phi \right) (\mathcal{P}^N\Phi) L\Psi + (1+2\mu)(Y\check{X}\Phi)(\mathcal{P}^N\Phi) L\Psi \\
&+ \frac{1}{2}(1+2\mu)(\check{X}\Phi) tr_g^{Y_{(l)}} \not{X}(\mathcal{P}^N\Phi) L\Psi \\
&+ 2(LY_{(l)}\mu)(\check{X}\Phi)(\mathcal{P}^N\Phi)\Psi + 2(Y_{(l)}\mu)(L\check{X}\Phi)(\mathcal{P}^N\Phi)\Psi \\
&+ 2(Y_{(l)}\mu)(\check{X}\Phi) tr_g \chi(\mathcal{P}^N\Phi)\Psi + (L\mu)(\check{X}\Phi) tr_g^{(Y_{(l)})} \not{X}(\mathcal{P}^N\Phi)\Psi \\
&+ (1+2\mu)(LY_{(l)}\check{X}\Phi)(\mathcal{P}^N\Phi)\Psi + (1+2\mu)(\check{X}\Phi)(Y_{(l)} tr_g \chi)(\mathcal{P}^N\Phi)\Psi \\
&+ (1+2\mu)(Y_{(l)}\check{X}\Phi) tr_g \chi(\mathcal{P}^N\Phi)\Psi + \frac{1}{2}(1+2\mu)(L\check{X}\Phi) tr_g^{(Y_{(l)})} \not{X}(\mathcal{P}^N\Phi)\Psi \\
&+ (1+2\mu)(\check{X}\Phi)(d\sharp v^{(Y_{(l)})} \not{X}_L^\sharp)(\mathcal{P}^N\Phi)\Psi + \frac{1}{2}(1+2\mu)(\check{X}\Phi)(tr_g \chi)(tr_g^{(Y_{(l)})} \not{X})(\mathcal{P}^N\Phi)\Psi,
\end{aligned} \tag{0.14.0.20}$$

$$\begin{aligned}
Error_{2,(l)}[\mathcal{P}^N\Phi; \Psi] &:= -2(Y_{(l)}\mu)(\check{X}\Phi)(\mathcal{P}^N\Phi)\Psi - (1+2\mu)(Y_{(l)}\check{X}\Phi)(\mathcal{P}^N\Phi)\Psi \\
&- \frac{1}{2}(1+2\mu)(\check{X}\Phi) tr_g^{(Y_{(l)})} \not{X}(\mathcal{P}^N\Phi)\Psi
\end{aligned} \tag{0.14.0.21}$$

Proof. The tedious computation is exactly as in [18], with the one difference that Y is now $Y_{(l)}$. $\square$

0.15 Analysis of Once Commuted Equation

Lemma 0.15.0.1. *If $\xi_{\alpha_1 \dots \alpha_n}$ is a $\binom{0}{n}$ spacetime tensorfield, V is a vectorfield with deformation tensor ${}^{(V)}\pi$ then we have that*

$$\mathcal{D}_\beta \mathcal{L}_V \xi_{\alpha_1 \dots \alpha_n} - \mathcal{L}_V \mathcal{D}_\beta \xi_{\alpha_1 \dots \alpha_n} = \frac{1}{2} \sum_{i=1}^n \left\{ \mathcal{D}_{\alpha_i} {}^{(V)}\pi_\beta{}^\kappa + \mathcal{D}_\beta {}^{(V)}\pi_{\alpha_i}{}^\kappa - \mathcal{D}^\kappa {}^{(V)}\pi_{\alpha_i \beta} \right\} \xi_{\alpha_1 \dots \alpha_{i-1} \kappa \alpha_{i+1} \dots \alpha_n}. \quad (0.15.0.1)$$

Lemma 0.15.0.2. *If we assume that ${}^{(Z)}\pi_{LL} = 0$ then we have that*

$$\begin{aligned} \mu \square_{g(\Phi)}(Z\Phi) = & \mu \mathcal{D}_\alpha \left({}^{(Z)}\pi^{\alpha\beta} \mathcal{D}_\beta \Phi - \frac{1}{2} tr_g {}^{(Z)}\pi \mathcal{D}^\alpha \Phi \right) \\ & + Z(\mu \square_{g(\Phi)} \Phi) - \mu^{-1} \left\{ {}^{(Z)}\pi_{L\check{X}} + Z\mu \right\} (\mu \square_{g(\Phi)} \Phi) + \frac{1}{2} tr_g {}^{(Z)}\pi (\mu \square_{g(\Phi)} \Phi). \end{aligned} \quad (0.15.0.2)$$

Lemma 0.15.0.3. *Let $\mathcal{J}$ be a vectorfield with decomposition*

$$\mathcal{J} = -\mathcal{J}_L L - \mu^{-1} \mathcal{J}_{\check{X}} L - \mu^{-1} \mathcal{J}_L \check{X} + \mathcal{J}.$$

Then we find that $\mathcal{J}$ satisfies the following weighted divergence identity

$$\begin{aligned} \mu \mathcal{D}_\alpha \mathcal{J}^\alpha = & -L(\mu \mathcal{J}_L) - L(\mathcal{J}_{\check{X}}) - \check{X}(\mathcal{J}_L) + \text{div}(\mu \mathcal{J}) \\ & - \mu tr_g \mathcal{J} \mathcal{J}_L - tr_g \chi \mathcal{J}_{\check{X}} \end{aligned} \quad (0.15.0.3)$$

The following proposition is crucial. It differs somewhat from what is found in both [16] and [18].

Proposition 0.15.0.4 (Frame decomposition of divergence term). *For $Z \in \mathcal{L}$ satisfying ${}^{(Z)}\pi_{LL} = 0$ and ${}^{(Z)}\pi_{L\check{X}} = -Z\mu$, we have that*

$$\begin{aligned} \mu \mathcal{D}_\alpha \left\{ {}^{(Z)}\pi^{\alpha\beta} \mathcal{D}_\beta \Psi - \frac{1}{2} tr_g {}^{(Z)}\pi \mathcal{D}^\alpha \Psi \right\} = & \mathfrak{K}_{(\pi-Dangerous)}^{(Z)}[\Psi] \\ & + \mathfrak{K}_{(\pi-Cancel-1)}^{(Z)}[\Psi] + \mathfrak{K}_{(\pi-Cancel-2)}^{(Z)}[\Psi] \\ & + \mathfrak{K}_{(\pi-Elliptic)}^{(Z)}[\Psi] + \mathfrak{K}_{(\pi-Good)}^{(Z)}[\Psi] \\ & + \mathfrak{K}_{(\Psi)}^{(Z)}[\Psi] + \mathfrak{K}_{(Low)}^{(Z)}[\Psi] \end{aligned} \quad (0.15.0.4)$$

where we denote

$$\begin{aligned}
\mathfrak{K}_{(\pi-Danger)}^{(Z)}[\Psi] &:= -(d\dot{k}v^{(Z)}\not{\pi}_L^\#)\check{X}\Psi, \\
\mathfrak{K}_{(\pi-Cancel-1)}^{(Z)}[\Psi] &:= \left\{ \frac{1}{2}\check{X}tr_{\not{g}}^{(Z)}\not{\pi} - d\dot{k}v\dot{\pi}_{\check{X}}^\# - \mu d\dot{k}v\dot{\pi}_L^\# \right\} L\Psi \\
\mathfrak{K}_{(\pi-Cancel-2)}^{(Z)}[\Psi] &:= \left\{ -\not{\mathcal{L}}_{\check{X}}^{(Z)}\not{\pi}_L^\# + \dot{\pi}^{(Z)}\not{\pi}_{L\check{X}} \right\} \cdot \not{\pi}\Psi \\
\mathfrak{K}_{(\pi-Elliptic)}^{(Z)}[\Psi] &:= \mu(d\dot{k}v^{(Z)}\hat{\pi})^\# \cdot \not{\pi}\Psi \\
\mathfrak{K}_{(\pi-Good)}^{(Z)}[\Psi] &:= \frac{1}{2}\mu(Ltr_{\not{g}}^{(Z)}\not{\pi})L\Psi + (L\pi_{L\check{X}})L\Psi + (L\pi_{X\check{X}})L\Psi \\
&\quad + \frac{1}{2}(Ltr_{\not{g}}^{(Z)}\not{\pi})\check{X}\Psi - \left(\not{\mathcal{L}}_L^{(Z)}\not{\pi}_{\check{X}}^\# \right) \cdot \not{\pi}\Psi - \mu \left(\not{\mathcal{L}}_L^{(Z)}\not{\pi}_L^\# \right) \cdot \not{\pi}\Psi \\
\mathfrak{K}_{(\Psi)}^{(Z)}[\Psi] &:= \left\{ \frac{1}{2}\mu tr_{\not{g}}^{(Z)}\not{\pi} + {}^{(Z)}\pi_{L\check{X}} + {}^{(Z)}\pi_{\check{X}X} \right\} L^2\Psi \\
&\quad + tr_{\not{g}}^{(Z)}\not{\pi}L\check{X}\Psi \\
&\quad - 2\mu^{(Z)}\not{\pi}_L^\# \cdot \not{\pi}L\Psi - 2\dot{\pi}_{\check{X}}^\# \not{\pi}L\Psi - 2{}^{(Z)}\not{\pi}_L^\# \cdot \not{\pi}\check{X}\Psi \\
&\quad + \pi_{L\check{X}}\not{\Delta}\Psi + \mu^{(Z)}\hat{\pi} \cdot \not{\nabla}^2\Psi
\end{aligned} \tag{0.15.0.5}$$

and

$$\begin{aligned}
\mathfrak{K}_{(Low)}^{(Z)}[\Psi] &:= \left\{ \frac{1}{2}(L\mu)tr_{\not{g}}^{(Z)}\not{\pi} + \frac{1}{2}\mu tr_{\not{g}}\not{k}tr_{\not{g}}^{(Z)}\not{\pi} + tr_{\not{g}}\chi^{(Z)}\pi_{L\check{X}} + tr_{\not{g}}\chi^{(Z)}\pi_{X\check{X}} - {}^{(Z)}\not{\pi}_L^\# \cdot \not{\pi}\mu \right\} L\Psi \\
&\quad + \frac{1}{2}tr_{\not{g}}\chi tr_{\not{g}}^{(Z)}\not{\pi}\check{X}\Psi \\
&\quad + \left\{ -(L\mu)^{(Z)}\not{\pi}_L^\# - \mu tr_{\not{g}}\not{k}^{(Z)}\not{\pi}_L^\# - tr_{\not{g}}\chi^{(Z)}\not{\pi}_{\check{X}}^\# + \hat{\pi} \cdot \not{\pi}^\# \mu + \frac{1}{2}(tr_{\not{g}}\not{\pi})\not{\pi}^\# \mu + \mu(tr_{\not{g}}\not{\pi})\zeta^\# \right\} \cdot \not{\pi}\Psi
\end{aligned} \tag{0.15.0.6}$$

Proof. Tedious computation using the previous lemma. We set $\mathcal{J}^\alpha := {}^{(Z)}\pi^{\alpha\beta}\mathcal{D}_\beta\Psi - \frac{1}{2}tr_{\not{g}}^{(Z)}\pi\mathcal{D}^\alpha\Psi$ and compute that

$$\mathcal{J}_L = -\frac{1}{2}(tr_{\not{g}}\not{\pi})L\Psi + {}^{(Z)}\not{\pi}_L^\# \cdot \not{\pi}\Psi \tag{0.15.0.7}$$

$$\mathcal{J}_{\check{X}} = -{}^{(Z)}\pi_{L\check{X}}L\Psi - {}^{(Z)}\pi_{\check{X}X}L\Psi + {}^{(Z)}\not{\pi}_{\check{X}}^\# \cdot \not{\pi}\Psi - \frac{1}{2}tr_{\not{g}}^{(Z)}\not{\pi}\check{X}\Psi \tag{0.15.0.8}$$

$$\begin{aligned}
\mu\mathcal{J} &= -\mu^{(Z)}\not{\pi}_L^\#L\Psi - {}^{(Z)}\not{\pi}_{\check{X}}^\#L\Psi - {}^{(Z)}\not{\pi}_L^\#\check{X}\Psi + {}^{(Z)}\pi_{L\check{X}}\not{\pi}^\#\Psi + \mu^{(Z)}\not{\pi}^\# \cdot \not{\pi}\Psi - \frac{1}{2}\mu(tr_{\not{g}}^{(Z)}\not{\pi})\not{\pi}^\#\Psi.
\end{aligned} \tag{0.15.0.9}$$

Now use the divergence formula and make use of the above commutation lemma and the identity $[L, \check{X}]^A = -\not{\pi}^A\mu - 2\mu\zeta^A$. $\square$

0.16 Commutation Identities (Corresponding to Section 5 in [18])

Definition 0.16.0.1. We say that $Z_{(1)} = L, Z_{(2)} = Y_{(1)}, Z_{(3)} = Y_{(2)}, Z_{(4)} = \check{X}$ and that $\vec{I} = (\iota_1, \dots, \iota_N)$ is a multiindex of order $|\vec{I}| = N$ with $\iota_i \in \{1, 2, 3, 4\}$, and we will define

$$\mathcal{L}^{\vec{I}} := Z_{(\iota_1)} \cdots Z_{(\iota_N)} \quad (0.16.0.1)$$

and

$$\mathcal{L}_{\mathcal{L}}^{\vec{I}} := \mathcal{L}_{Z_{(\iota_1)}} \cdots \mathcal{L}_{Z_{(\iota_N)}} \quad (0.16.0.2)$$

We will denote a permutation and separation of the indices into subcollections by $\vec{I}_1 + \cdots + \vec{I}_M = \vec{I}$.

The following lemmas are proved as in [18], with one minor difference: they require Lemmas 8.9 and 8.11 in [16] to handle commutation with ∇ and ∇^2 , as $\mathcal{P}_{t,u}$ is two dimensional in our setting.

Lemma 0.16.0.1. For $\mathcal{P}_{t,u}$ tangent $\binom{0}{n}$ tensorfields ξ we have that

$$\left[\nabla, \mathcal{L}_{\mathcal{L}}^{\vec{I}} \right] \xi = \sum_{M=1}^{|\vec{I}|} \sum_{\substack{\vec{I}_1 + \cdots + \vec{I}_{M+1} = \vec{I} \\ |\vec{I}_a| \geq 1 \text{ for } 1 \leq a \leq M}} C_{\vec{I}_1, \dots, \vec{I}_{M+1}, n} (\mathcal{L}^{\vec{I}_1} \mathcal{L}^{\vec{I}_2} \cdots \mathcal{L}^{\vec{I}_{M+1}} \xi) \underbrace{(\nabla \mathcal{L}^{\vec{I}_M} \mathcal{L}^{\vec{I}_{M+1}} \xi)}_{\text{absent when } M=1} \quad (0.16.0.3)$$

and for symmetric $\mathcal{P}_{t,u}$ tangent $\binom{0}{2}$ tensorfields ξ we have

$$\left[d\mathbb{I}v, \mathcal{L}_{\mathcal{L}}^{\vec{I}} \right] \xi = \sum_{i_1 + i_2 = 1} \sum_{M=1}^{|\vec{I}|} \sum_{\substack{\vec{I}_1 + \cdots + \vec{I}_{M+1} = \vec{I} \\ |\vec{I}_a| \geq 1 \text{ for } 1 \leq a \leq M}} C_{\vec{I}_1, \dots, \vec{I}_{M+1}, n} (\mathcal{L}^{\vec{I}_1} \mathcal{L}^{\vec{I}_2} \cdots \mathcal{L}^{\vec{I}_{M+1}} \xi) \underbrace{(\nabla^{i_1} \mathcal{L}^{\vec{I}_M} \mathcal{L}^{\vec{I}_{M+1}} \xi)}_{\text{absent when } i_1 = M=1} \quad (0.16.0.4)$$

For functions f we have

$$\left[\nabla^2, \mathcal{L}_{\mathcal{L}}^{\vec{I}} \right] f = \sum_{M=1}^{|\vec{I}|} \sum_{\substack{\vec{I}_1 + \cdots + \vec{I}_{M+1} = \vec{I} \\ |\vec{I}_a| \geq 1 \text{ for } 1 \leq a \leq M}} C_{\vec{I}_1, \dots, \vec{I}_{M+1}, n} (\mathcal{L}^{\vec{I}_1} \mathcal{L}^{\vec{I}_2} \cdots \mathcal{L}^{\vec{I}_{M+1}} f) \underbrace{(\nabla \mathcal{L}^{\vec{I}_M} \mathcal{L}^{\vec{I}_{M+1}} f)}_{\text{absent when } M=1} \quad (0.16.0.5)$$

$$\begin{aligned}
\left[\mathbb{A}, \mathcal{L}^{\vec{I}} \right] f &= \sum_{i_1+i_2=1} \sum_{M=1}^{|\vec{I}|} \sum_{\substack{\vec{I}_1+\dots+\vec{I}_{M+1}=\vec{I} \\ |\vec{I}_a| \geq 1 \text{ for } 1 \leq a \leq M}} \\
C_{\vec{I}_1, \dots, \vec{I}_{M+1}, n} (\mathcal{G}^{-1})^{M+1} &\underbrace{(\mathcal{L}^{\vec{I}_1} \mathcal{G}) \dots (\mathcal{L}^{\vec{I}_{M-1}} \mathcal{G})}_{\text{absent when } i_1=M=1} (\nabla^{i_1} \mathcal{L}^{\vec{I}_M} \mathcal{G}) (\nabla^{i_2+1} \mathcal{L}^{\vec{I}_{M+1}} f)
\end{aligned} \tag{0.16.0.6}$$

The following lemma is different from the one in [18] due to the presence of an extra angular commutation vectorfield, although the proof is nearly identical.

Lemma 0.16.0.2. *Let f be a function and let ξ be a $\binom{m}{n} \mathcal{P}_{t,u}$ tangent tensorfield with $m+n \geq 1$. Then we have*

$$\begin{aligned}
\left\{ \mathcal{L}^{\vec{I}} - \mathcal{L}^{\vec{I}'} \right\} f &= \sum_{\substack{\vec{I}_1+\vec{I}_2+\iota_{k_1}+\iota_{k_2}=\vec{I} \\ Z_{(\iota_{k_1})} \in \{L, \check{X}, Y_{(l)}\}, Z_{(\iota_{k_2})} \in \{\check{X}, Y_{(l)}\}, Z_{(\iota_{k_1})} \neq Z_{(\iota_{k_2})}}} C_{\vec{I}_1, \vec{I}_2, \iota_{k_1}, \iota_{k_2}} \mathcal{L}^{\vec{I}_1} \mathfrak{H}^\sharp(Z_{(\iota_{k_1})}, Z_{(\iota_{k_2})}) \cdot \mathcal{L}^{\vec{I}_2} f,
\end{aligned} \tag{0.16.0.7}$$

$$\begin{aligned}
\left\{ \mathcal{L}^{\vec{I}} - \mathcal{L}^{\vec{I}'} \right\} \xi &= \sum_{\substack{\vec{I}_1+\vec{I}_2+\iota_{k_1}+\iota_{k_2}=\vec{I} \\ Z_{(\iota_{k_1})} \in \{L, \check{X}, Y_{(l)}\}, Z_{(\iota_{k_2})} \in \{\check{X}, Y_{(l)}\}, Z_{(\iota_{k_1})} \neq Z_{(\iota_{k_2})}}} C_{\vec{I}_1, \vec{I}_2, \iota_{k_1}, \iota_{k_2}} \mathcal{L}^{\vec{I}_1} \mathfrak{H}^\sharp(Z_{(\iota_{k_1})}, Z_{(\iota_{k_2})}) \mathcal{L}^{\vec{I}_2} \xi
\end{aligned} \tag{0.16.0.8}$$

Where $\mathfrak{H}^\sharp(Z_{(\iota_{k_1})}, Z_{(\iota_{k_2})}) = {}^{(Z_{(\iota_{k_2})})} \mathfrak{H}^\sharp_{Z_{(\iota_{k_1})}}$ if at least one $Z_{(\iota_{k_i})} \notin \{Y_{(1)}, Y_{(2)}\}$ and $= W^\sharp$ otherwise

The following is lemma 8.8 in [16].

Lemma 0.16.0.3. *For $Z \in \mathcal{L}$ and ξ be a $\mathcal{P}_{t,u}$ tangent $\binom{0}{2}$ tensor field,*

$$\mathcal{L}_Z \widehat{\xi}_{AB} - \widehat{\mathcal{L}} \xi_{AB} = \frac{1}{2} {}^{(Z)} \widehat{\mathfrak{F}}^{CD} \widehat{\xi}_{CD} \mathcal{G}_{AB} - \frac{1}{2} (tr_{\mathcal{G}} \xi)^{(Z)} \widehat{\mathfrak{F}}_{AB} \tag{0.16.0.9}$$

0.17 Geometric Lemmas for Top Order Estimates

The following section establishes the geometric machinery which we must use to avoid derivative loss in our top-order error terms. This section is similar to Section 6 in [18] and Chapter 11 in [16], but is a bit different from both. Nonetheless the computations are basically the same as those two works so they are omitted here. We will use the following convention:

Definition 0.17.0.1 (Riemann and Ricci Curvature Tensors). The *Riemann curvature* of the space-

time metric g is the spacetime tensorfield defined by

$$R(U, V, W, Z) = R_{\alpha\beta\gamma\delta} U^\alpha V^\beta W^\gamma Z^\delta := g(\mathcal{D}_{VU}^2 W - \mathcal{D}_{UV}^2 W, Z), \quad (0.17.0.1)$$

for spacetime vectors U, V, W, Z , where $\mathcal{D}_{VU}^2 W := V^\alpha U^\beta \mathcal{D}_\alpha \mathcal{D}_\beta W$ is the Hessian of a locally-defined vectorfield W . Similarly, we define for A, B, C, D tangent vectors to $\mathcal{P}_{t,u}$ the Riemann curvature of $\mathcal{g}$ to be

$$\mathfrak{R}(A, B, C, D) := \mathcal{g}(\nabla_{BA}^2 C - \nabla_{AB}^2 C, D). \quad (0.17.0.2)$$

We will also denote the *Ricci curvature* of g by the tensorfield

$$\text{Ric}_{\alpha\beta} := (g^{-1})^{\gamma\delta} R_{\alpha\gamma\beta\delta}. \quad (0.17.0.3)$$

Lemma 0.17.0.1 (Rectangular Coordinate decomposition of $R_{\mu\nu\alpha\beta}$).

$$\begin{aligned} R_{\mu\nu\alpha\beta} = & \frac{1}{2} \{ G_{\beta\mu} D_{\alpha\nu}^2 \Phi + G_{\alpha\nu} D_{\beta\mu}^2 \Phi - G_{\beta\nu} D_{\alpha\mu}^2 \Phi - G_{\alpha\mu} D_{\beta\nu}^2 \Phi \} \\ & + \frac{1}{4} G_{\mu\alpha} G_{\nu\beta} (g^{-1})^{\kappa\lambda} (\partial_\kappa \Phi) (\partial_\lambda \Phi) - \frac{1}{4} G_{\mu\beta} G_{\nu\alpha} (g^{-1})^{\kappa\lambda} (\partial_\kappa \Phi) (\partial_\lambda \Phi) \\ & + \frac{1}{4} (g^{-1})^{\kappa\lambda} [G_{\lambda(\nu} \partial_{\alpha)} \Phi] [G_{\kappa(\mu} \partial_{\beta)} \Phi] - \frac{1}{4} (g^{-1})^{\kappa\lambda} [G_{\lambda(\mu} \partial_{\alpha)} \Phi] [G_{\kappa(\nu} \partial_{\beta)} \Phi] \\ & + \frac{1}{2} \{ G'_{\beta\mu} (\partial_\alpha \Phi) (\partial_\nu \Phi) + G'_{\alpha\nu} (\partial_\beta \Phi) (\partial_\mu \Phi) - G'_{\beta\nu} (\partial_\alpha \Phi) (\partial_\mu \Phi) - G'_{\alpha\mu} (\partial_\beta \Phi) (\partial_\nu \Phi) \} \end{aligned} \quad (0.17.0.4)$$

The following idea goes back to Klainerman and Rodnianski (see [7])

Lemma 0.17.0.2 (Regularized expressions for Ric_{LL}). *If $\square_{g(\Phi)} \Phi = 0$ we have that*

$$\mu \text{Ric}_{LL} = L \left\{ -G_{LL} \check{X} \Phi - \frac{1}{2} \mu \text{tr}_{\mathcal{g}} \mathcal{G} L \Phi - \frac{1}{2} \mu G_{LL} L \Phi + \mu \mathcal{G}_L^\# \cdot \not{d} \Phi \right\} + \mathfrak{A} \quad (0.17.0.5)$$

where $\mathfrak{A}$ has the form

$$\mathfrak{A} = \sum_{l=0}^1 \begin{pmatrix} G_{(frame)}^2 \not{g}^{-1} \\ G'_{(frame)} \end{pmatrix} (\not{g}^{-1})^l \begin{pmatrix} \mu L \Phi \\ \check{X} \Phi \\ \mu \not{d} \Phi \end{pmatrix} \begin{pmatrix} L \Phi \\ \not{d} \Phi \end{pmatrix} \quad (0.17.0.6)$$

$$= f(\underline{\gamma}, \not{g}^{-1}, \not{d} x^i, Z \Phi) P \Phi \quad (0.17.0.7)$$

Where $Z \in \mathcal{L}$ and $P \in TC_u$. If $\square_{g(\Phi)}\Phi \neq 0$ we have that

$$Ric_{LL} = \frac{L\mu}{\mu} tr_{\mathcal{G}}\chi + L \left\{ -\frac{1}{2} tr_{\mathcal{G}}\mathcal{G}L\Phi - \frac{1}{2} G_{LL}L\Phi + \mathcal{G}_L^\# \cdot \mathcal{G}\Phi \right\} - \frac{1}{2} G_{LL}\mathcal{A}\Phi + \mathfrak{B} \quad (0.17.0.8)$$

where we have

$$\mathfrak{B} = G_{(frame)} tr_{\mathcal{G}}\chi L\Phi + \sum_{l=0}^1 \begin{pmatrix} G_{(frame)}^2 \mathcal{G}^{-1} \\ G'_{(frame)} \end{pmatrix} (\mathcal{G}^{-1})^l \begin{pmatrix} L\Phi \\ \mathcal{G}\Phi \end{pmatrix}^2 \quad (0.17.0.9)$$

$$= f(\gamma, \mathcal{G}^{-1}, \mathcal{G}x^i)(P\Phi)P\gamma \quad (0.17.0.10)$$

Definition 0.17.0.2. Set

$$\mathcal{X} := \mu tr_{\mathcal{G}}\chi + \mathfrak{X},$$

$$\mathfrak{X} := -G_{LL}\check{X}\Phi - \frac{1}{2}\mu tr_{\mathcal{G}}\mathcal{G}L\Phi - \frac{1}{2}\mu G_{LL}L\Phi + \mu \mathcal{G}_L^\# \cdot \mathcal{G}\Phi$$

Lemma 0.17.0.3. If $\square_{g(\Psi)}\Psi = 0$ we find that $\mathcal{X}$ satisfies the following transport equation

$$L\mathcal{X} = 2(L\mu)tr_{\mathcal{G}}\chi - \frac{1}{2}\mu(tr_{\mathcal{G}}\chi)^2 - \mu|\hat{\chi}|^2 - \mathfrak{A} \quad (0.17.0.11)$$

Lemma 0.17.0.4. If $\square_{g(\Psi)}\Psi = 0$ and $P \in \mathcal{P}$, we define

$$^{(P)}\mathcal{X} := \mu P tr_{\mathcal{G}}\chi + P\mathfrak{X} \quad (0.17.0.12)$$

then we have that

$$\begin{aligned} L^{(P)}\mathcal{X} &= -(tr_{\mathcal{G}}\chi)^{(P)}\mathcal{X} + \mu[L, P]tr_{\mathcal{G}}\chi + 2(L\mu)P tr_{\mathcal{G}}\chi + \frac{1}{2}tr_{\mathcal{G}}\chi P\mathfrak{X} \\ &\quad - P(\mu|\hat{\chi}|^2) + ^{(P)}\mathfrak{J} \end{aligned} \quad (0.17.0.13)$$

where we define

$$\begin{aligned} ^{(P)}\mathfrak{J} &:= -P\mathfrak{A} + \frac{1}{2}tr_{\mathcal{G}}\chi P(2L\mu + \mathfrak{X}) \\ &\quad - (P\mu) \left\{ Ltr_{\mathcal{G}}\chi + \frac{1}{2}(tr_{\mathcal{G}}\chi)^2 \right\} + [L, P]\mathfrak{X} \end{aligned} \quad (0.17.0.14)$$

and note that $2L\mu + \mathfrak{X}$ contains no $G_{LL}\check{X}\Psi$ terms.

Definition 0.17.0.3. We define higher-order fully modified quantities by

$$({}^{\mathcal{P}^N})\mathcal{X} := \mu \mathcal{P}^N \text{tr}_{\mathcal{G}} \chi + \mathcal{P}^N \mathfrak{X} \quad (0.17.0.15)$$

And the so-called *partially modified quantities*:

$$({}^{\mathcal{P}^N})\widetilde{\mathcal{X}} := \mathcal{P}^N \text{tr}_{\mathcal{G}} \chi + ({}^{\mathcal{P}^N})\widetilde{\mathfrak{X}}, \quad (0.17.0.16)$$

$$({}^{\mathcal{P}^N})\widetilde{\mathfrak{X}} := -\frac{1}{2} \text{tr}_{\mathcal{G}} \mathcal{G} L \mathcal{P}^N \Phi - \frac{1}{2} G_{LL} L \mathcal{P}^N \Phi + \mathcal{G}_L^{\sharp} \cdot \mathcal{P}^N \Phi. \quad (0.17.0.17)$$

We will also need the following partially modified quantity:

$$\widetilde{\mathfrak{X}} := -\frac{1}{2} \text{tr}_{\mathcal{G}} \mathcal{G} L \Phi - \frac{1}{2} G_{LL} L + \mathcal{G}_L^{\sharp} \cdot \Phi. \quad (0.17.0.18)$$

We find that the following transport equation is satisfied by the higher-order fully modified quantities

Proposition 0.17.0.5.

$$\begin{aligned} L({}^{\mathcal{P}^N})\mathcal{X} - \left\{ 2 \frac{L\mu}{\mu} - \text{tr}_{\mathcal{G}} \chi \right\} ({}^{\mathcal{P}^N})\mathcal{X} &= \left\{ \frac{1}{2} \text{tr}_{\mathcal{G}} \chi - 2 \frac{L\mu}{\mu} \right\} \mathcal{P}^N \mathfrak{X} + \mu [L, \mathcal{P}^N] \text{tr}_{\mathcal{G}} \chi \\ &\quad - 2\mu \hat{\chi} \cdot \mathcal{L}_{\mathcal{P}^N}^N \hat{\chi} + ({}^{\mathcal{P}^N})\mathfrak{J} \end{aligned} \quad (0.17.0.19)$$

where

$$\begin{aligned} ({}^{\mathcal{P}^N})\mathfrak{J} &:= \mathcal{P}^{N-1(P)}\mathfrak{J} + [L, \mathcal{P}^{N-1}]P\mathfrak{X} \\ &\quad - [\mathcal{P}^{N-1}, \mu \text{tr}_{\mathcal{G}} \chi] P \text{tr}_{\mathcal{G}} \chi - \frac{1}{2} [\mathcal{P}^{N-1}, \text{tr}_{\mathcal{G}} \chi] P\mathfrak{X} \\ &\quad + [\mathcal{P}^{N-1}, L\mu] P \text{tr}_{\mathcal{G}} \chi - [\mathcal{P}^{N-1}, \mu] PL \text{tr}_{\mathcal{G}} \chi \\ &\quad + \left\{ 2\mu \hat{\chi} \cdot \mathcal{L}_{\mathcal{P}^N}^N \hat{\chi} - \mathcal{P}^N (\mu |\hat{\chi}|^2) \right\} \end{aligned} \quad (0.17.0.20)$$

and for the partially modified quantities we find that

Proposition 0.17.0.6.

$$L({}^{\mathcal{P}^{N-1}})\widetilde{\mathcal{X}} = \frac{1}{2} G_{LL} \mathcal{A} \mathcal{P}^{N-1} \Phi + ({}^{\mathcal{P}^{N-1}})\mathfrak{B} \quad (0.17.0.21)$$

where

$$\begin{aligned}
({\mathcal{P}}^{N-1})\mathfrak{B} &:= -{\mathcal{P}}^{N-1}\mathfrak{B} - {\mathcal{P}}^{N-1} \left(|\hat{\chi}|^2 + \frac{1}{2}(tr_{\mathcal{G}}\chi)^2 \right) \\
&+ \frac{1}{2} [{\mathcal{P}}^{N-1}, G_{LL}] \Delta\Phi + \frac{1}{2} G_{LL} [{\mathcal{P}}^{N-1}, \Delta] \Phi + [L, {\mathcal{P}}^{N-1}] tr_{\mathcal{G}}\chi \\
&+ [L, {\mathcal{P}}^{N-1}] \tilde{\mathfrak{X}} + L \left\{ ({\mathcal{P}}^{N-1})\tilde{\mathfrak{X}} - {\mathcal{P}}^{N-1}\tilde{\mathfrak{X}} \right\}
\end{aligned} \tag{0.17.0.22}$$

0.18 Norms, Bootstrap and Smallness Assumptions (Corresponding to Section 7 in [18])

0.18.1 Norms

Our main quantities of interest are scalar functions and $\mathcal{P}_{t,u}$ -tangent tensorfields, to which we associate the following geometrically natural norms:

Definition 0.18.1.1 (Norms for $\mathcal{P}_{t,u}$ -tangent tensorfields). For ξ is a type $\binom{m}{n}$ $\mathcal{P}_{t,u}$ -tangent tensorfield, we define:

$$|\xi|^2 := \mathcal{G}_{\mu_1\mu'_1} \cdots \mathcal{G}_{\mu_m\mu'_m} (\mathcal{G}^{-1})^{\nu_1\nu'_1} \cdots (\mathcal{G}^{-1})^{\nu_n\nu'_n} \xi^{\mu_1\cdots\mu_m}_{\nu_1\cdots\nu_n} \xi^{\mu'_1\cdots\mu'_m}_{\nu'_1\cdots\nu'_n}. \tag{0.18.1.1}$$

Definition 0.18.1.2 (L^2 and L^∞ norms for $\mathcal{P}_{t,u}$ -tangent tensorfields). For ξ is a type $\binom{m}{n}$ $\mathcal{P}_{t,u}$ -tangent tensorfield, we define:

$$\|\xi\|_{L^2(\mathcal{P}_{t,u})}^2 := \int_{\mathcal{P}_{t,u}} |\xi|^2 dv_{\mathcal{G}}, \tag{0.18.1.2}$$

$$\|\xi\|_{L^2(\Sigma_t^u)}^2 := \int_{\Sigma_t^u} |\xi|^2 dv_{\mathcal{G}} du, \tag{0.18.1.3}$$

$$\|\xi\|_{L^2(\mathcal{C}_u^t)}^2 := \int_{\mathcal{C}_u^t} |\xi|^2 dv_{\mathcal{G}} dt, \tag{0.18.1.4}$$

$$\|\xi\|_{L^\infty(\mathcal{P}_{t,u})}^2 := \operatorname{ess\,sup}_{\theta \in \mathbb{T}^2} |\xi|(t, u, \theta), \tag{0.18.1.5}$$

$$\|\xi\|_{L^\infty(\Sigma_t^u)}^2 := \operatorname{ess\,sup}_{(u', \theta) \in [0, u] \times \mathbb{T}^2} |\xi|(t, u', \theta), \tag{0.18.1.6}$$

$$\|\xi\|_{L^\infty(\mathcal{C}_u^t)}^2 := \operatorname{ess\,sup}_{(t', \theta) \in [0, t] \times \mathbb{T}^2} |\xi|(t', u, \theta). \tag{0.18.1.7}$$

0.18.2 Differentiation Notation

Here our notation follows that in [18].

$\mathcal{L}_*^{N;M}$ is a string of N vectorfields from $\mathcal{L}$, $*$ means there is at least one vectorfield from $\mathcal{P}$,

and $\cdot^M$ means there are at most M factors of $\check{X}$. These lead to a gain in smallness except for maybe when they hit μ .

$\mathcal{P}_*^N$ is a string of N vectorfields from $\mathcal{P}$, $*$ means there is at least one angular factor $Y_{(l)}$ or at least two factors of L . These always lead to a gain in smallness.

$|\mathcal{L}^{[K,N];M}f|$ is defined to be the sum of $|\mathcal{L}^{N';M}f|$ for $K \leq N' \leq N$, and $|\mathcal{L}^{\leq N;M}f|$ is defined to be the sum of $|\mathcal{L}^{N';M}f|$ for $0 \leq N' \leq N$. Other terms are defined analogously.

0.18.3 Assumptions on initial data

In order to demonstrate H^2 -illposedness, we need to assume that our initial data is a perturbation of the Lindblad-type data $\check{X}\Phi_0(u) = -\eta|\log(u)|^\alpha$, but will not worry about this here.

For the purpose of proving the shock formation result, we will assume that our initial data satisfies

$$\|\mathcal{L}_*^{\leq 18;3}\Phi\|_{L^\infty(\Sigma_0^1)}, \|\mathcal{L}_*^{\leq 20;3}\Phi\|_{L^2(\Sigma_0^1)} \leq \mathring{\epsilon}, \quad (0.18.3.1)$$

$$\|\check{X}^{[1,3]}\Phi\|_{L^\infty(\Sigma_0^1)} =: \mathring{\delta} > 0 \quad (0.18.3.2)$$

$$\|\Phi\|_{L^\infty(\Sigma_0^1)} =: \mathring{\alpha} \quad (0.18.3.3)$$

Remark 0.18.3.1. The number of derivatives above is one more than that planar shock formation paper [18]. This is due to the fact that $\mathcal{P}_{t,u}$ is 2-dimensional and so the $L^2 - L^\infty$ Sobolev imbedding requires 1 more derivative and so, in order to close our L^∞ bootstrap estimates (see below) we need control on one more derivative in L^2 . We must also note that this does not cause an issue when we apply our commutator estimates (see below) where we frequently will want to put, say, $\lceil \frac{N}{2} \rceil$ derivatives in L^∞ and N derivatives in L^2 .

Definition 0.18.3.1 (Quantity Controlling Time of Shock Formation). We define

$$\mathring{\delta}_* := \frac{1}{2} \sup_{\Sigma_0} \left[G_{LL} \check{X}\Phi \right]_- \quad (0.18.3.4)$$

The following three lemmas are exactly as in [18], with the same proofs.

Lemma 0.18.3.1 (Identities for $\partial_i u$). *We have that*

$$(\underline{g}^{-1})^{ab} \partial_a u \partial_b u = \mu^{-1} \quad (0.18.3.5)$$

Where $\underline{g}^{-1}$ is the inverse of the metric $\underline{g}$ on $\Sigma_t^{U_0}$. We also have that

$$\mu \partial_i u = X^i \quad (0.18.3.6)$$

Lemma 0.18.3.2 (Identities on Σ_0). *The following identities are true on the initial hypersurface Σ_0 , for $i = 1, 2, 3$:*

$$\mu = \frac{1}{\sqrt{(\underline{g}^{-1})^{11}}}, \quad (0.18.3.7)$$

$$L_{(small)}^i = \frac{(\underline{g}^{-1})^{i1}}{\sqrt{(\underline{g}^{-1})^{11}}} - \delta^{i1} - (\underline{g}^{-1})^{0i}, \quad (0.18.3.8)$$

$$\Xi^i = \frac{(\underline{g}^{-1})^{i1}}{(\underline{g}^{-1})^{11}} - \delta^{i1}. \quad (0.18.3.9)$$

Using the above lemmas, commutator estimates, and Sobolev imbedding (keeping track of $\mathring{\epsilon} - \mathring{\delta}$ pairings) allows us to derive that

Lemma 0.18.3.3. *Under our initial data assumptions, for $\mathring{\epsilon}$ sufficiently small we have the following estimates along Σ_0^1 , where the implicit constant depends on $\mathring{\delta}$.*

$$\|\mathcal{L}_*^{\leq 20;3} L_{(small)}^i\|_{L^2(\Sigma_0^1)} \lesssim \mathring{\epsilon}, \quad (0.18.3.10)$$

$$\|\check{X}^{[1,3]} L_{(small)}^i\|_{L^2(\Sigma_0^1)} \lesssim 1, \quad (0.18.3.11)$$

$$\|\mu - 1\|_{L^2(\Sigma_0^1)} \lesssim \mathring{\alpha}, \quad (0.18.3.12)$$

$$\|\mathcal{P}_*^{[1,20]} \mu\|_{L^2(\Sigma_0^1)} \lesssim \mathring{\epsilon}, \quad (0.18.3.13)$$

$$\|L \check{X}^{[0,2]} \mu\|_{L^2(\Sigma_0^1)}, \|\check{X}^{[0,2]} L \mu\|_{L^2(\Sigma_0^1)}, \|\check{X} L \check{X} \mu\|_{L^2(\Sigma_0^1)}, \|\check{X}^{[0,2]} \mu\|_{L^2(\Sigma_0^1)} \lesssim 1, \quad (0.18.3.14)$$

$$\|\mathcal{L}_*^{\leq 18;2} L_{(small)}^i\|_{L^\infty(\Sigma_0^1)}, \|\check{X}^{[0,2]} L_{(small)}^i\|_{L^\infty(\Sigma_0^1)} \lesssim 1, \quad (0.18.3.15)$$

$$\|\mu - 1\|_{L^\infty(\Sigma_0^1)} \lesssim \mathring{\alpha}, \quad (0.18.3.16)$$

$$\|\mathcal{P}_*^{[1,18]} \mu\|_{L^\infty(\Sigma_0^1)} \lesssim \mathring{\epsilon}, \quad (0.18.3.17)$$

$$\|L \check{X}^{[0,2]} \mu\|_{L^\infty(\Sigma_0^1)}, \|\check{X}^{[0,2]} L \mu\|_{L^\infty(\Sigma_0^1)}, \|\check{X} L \check{X} \mu\|_{L^\infty(\Sigma_0^1)}, \|\check{X}^{[0,2]} \mu\|_{L^\infty(\Sigma_0^1)} \lesssim 1. \quad (0.18.3.18)$$

0.18.4 $T_{(Boot)}$, Positivity of μ , Diffeomorphism of Geometric and Rectangular Coordinates in Bootstrap Region

We will fix

$$0 < T_{(Boot)} \leq 2\mathring{\delta}_*^{-1} \quad (0.18.4.1)$$

and we will assume that $\mu > 0$ on $\mathcal{M}_{T_{(Boot)}, U_0}$.

We will also assume that the map $\mathfrak{Y} : [0, T_{(Boot)}] \times [0, U_0] \times \mathbb{T}^2 \rightarrow \mathbb{R}^{3+1}$ is a $\mathcal{C}^1$ diffeomorphism onto its image.

0.18.5 L^∞ bootstrap assumptions

Our main bootstrap assumption is that on $\mathcal{M}_{T_{(Boot)}, U_0}$ we have

$$\|\mathcal{P}^{[1,11]}\Phi\|_{L^\infty(\Sigma_t^u)} \leq \varepsilon \quad (0.18.5.1)$$

0.18.6 Auxiliary L^∞ bootstrap assumptions

We assume that we have the following on our spacetime region $\mathcal{M}_{T_{(Boot)}, U_0}$.

$$\|\mathcal{L}_*^{\leq 10;1}\Phi\|_{L^\infty(\Sigma_t^u)} \leq \varepsilon^{1/2} \quad (0.18.6.1)$$

$$\|L\mathcal{P}^{[1,9]}\mu\|_{L^\infty(\Sigma_t^u)} \leq \varepsilon^{1/2} \quad (0.18.6.2)$$

$$\|\mathcal{P}_*^{[1,9]}\mu\|_{L^\infty(\Sigma_t^u)} \leq \varepsilon^{1/2} \quad (0.18.6.3)$$

$$\|\mathcal{L}_*^{\leq 9;1}L^i_{(Small)}\|_{L^\infty(\Sigma_t^u)} \leq \varepsilon^{1/2} \quad (0.18.6.4)$$

$$\|\mathcal{L}_*^{\leq 8;1}\chi\|_{L^\infty(\Sigma_t^u)} \leq \varepsilon^{1/2} \quad (0.18.6.5)$$

$$\|\check{X}\Phi\|_{L^\infty(\Sigma_t^u)} \leq \|\check{X}\Phi\|_{L^\infty(\Sigma_0^u)} + \varepsilon^{1/2} \quad (0.18.6.6)$$

$$\|L\mu\|_{L^\infty(\Sigma_t^u)} \leq \frac{1}{2} \|G_{LL}\check{X}\Phi\|_{L^\infty(\Sigma_0^u)} + \varepsilon^{1/2} \quad (0.18.6.7)$$

$$\|\mu\|_{L^\infty(\Sigma_t^u)} \leq 1 + 2\delta_*^{-1} \|G_{LL}\check{X}\Phi\|_{L^\infty(\Sigma_0^u)} + \mathring{\alpha}^{1/2} + \varepsilon^{1/2} \quad (0.18.6.8)$$

$$\|\check{X}L^i_{(Small)}\|_{L^\infty(\Sigma_t^u)} \leq \|\check{X}L^i_{(Small)}\|_{L^\infty(\Sigma_0^u)} + \varepsilon^{1/2} \quad (0.18.6.9)$$

Remark 0.18.6.1 (Smallness/Largeness Assumptions, see 7.7 in [18]). We will need to assume that ε is small with respect to $\mathring{\delta}^{-1}$ and $\mathring{\delta}_*$, that $\mathring{\alpha}$ is small with respect to 1, and that $\mathring{\epsilon} \leq \varepsilon < \mathring{\alpha}$. Following [18], when we say “A is small with respect to B” we mean that $B > 0$ and that there is some positive increasing continuous function f such that $A \leq f(B)$. Smallness of ε with respect to $\mathring{\delta}^{-1}$ will be used to control products of size ε terms with size $\mathring{\delta}$ terms. Smallness of ε with respect to $\mathring{\delta}_*$ will be used to show that size ε terms remain small up to time $t < 2\mathring{\delta}_*^{-1}$. As the number of “bad derivatives” taken is finite we find that before the final Gronwall argument (see Section 14.9 in [18]) we have constants on the order of $c\mathring{\delta}^k$. Due to Gronwall this becomes an exponential, and thus we

must have $\varepsilon \leq e^{-c\tilde{\delta}^k}$ to satisfy the first condition. As we only take a finite number of integrals in time (and then Gronwall) it turns out that $\varepsilon \leq e^{-d\tilde{\delta}_*^{-l}}$ will suffice for the second condition, where $c, d, k, l > 0$ are absolute structural constants.

0.19 First Pointwise Estimates (Corresponding to Section 8 in [18])

This section consists of the most immediate estimates which can be derived from our bootstrap assumptions. Proofs are straightforward but somewhat tedious. The interested reader should refer to [18] and [16] for details.

0.19.1 Differential Operator Comparison Lemmas

Lemma 0.19.1.1. *For any $\mathcal{P}_{t,u}$ tangent tensors we have, under our above bootstrap, smallness, and initial data assumptions, that*

$$|\xi| = \left\{1 + \mathcal{O}(\varepsilon^{1/2})\right\} \left(\sum_{A,B,\dots,Z} |\xi_{Y_{(A)}Y_{(B)}\dots Y_{(Z)}}| \right) \sim \sum |\xi_{(Rectangular)}| \quad (0.19.1.1)$$

Lemma 0.19.1.2. *For any scalar function f on $\mathcal{P}_{t,u}$ under our bootstrap/smallness/data assumptions satisfies*

$$|\not{d}f| \leq (1 + C\varepsilon^{1/2}) \sum_{i=1}^2 |Y_{(i)}f| \quad (0.19.1.2)$$

and

$$|\not{\nabla}^2 f| \leq C\varepsilon^{1/2} |\not{d}f| + (1 + C\varepsilon^{1/2}) \sum_{i=1}^2 |\not{d}(Y_{(i)}f)| \quad (0.19.1.3)$$

Lemma 0.19.1.3. *For ξ any $\mathcal{P}_{t,u}$ tangent $\binom{0}{n}$ tensorfield and V a $\mathcal{P}_{t,u}$ tangent vectorfield we have (again assuming we are in the bootstrap region under our bootstrap, smallness, and initial data assumptions) that*

$$|\not{\mathcal{L}}_V \xi| \lesssim |V| \left(\sum_{i=1}^2 |\not{\mathcal{L}}_{Y_{(i)}} \xi| \right) + |\xi| \left(\sum_{i=1}^2 |\not{\mathcal{L}}_{Y_{(i)}} V| \right) + \varepsilon^{1/2} |\xi| |V| \quad (0.19.1.4)$$

$$|\not{\nabla} \xi| \lesssim \left(\sum_{i=1}^2 |\not{\mathcal{L}}_{Y_{(i)}} \xi| \right) + \varepsilon^{1/2} |\xi| \quad (0.19.1.5)$$

Lemma 0.19.1.4. *Under our bootstrap/smallness/data assumptions we have for $\mathcal{P}_{t,u}$ tangent ten-*

sorfields that

$$|\mathcal{L}_Y \xi| \lesssim |\nabla \xi| + \varepsilon^{1/2} |\xi| \quad (0.19.1.6)$$

Proof. Use that

$$\mathcal{L}_{Y_{(l)}} \xi_{A_1 \dots A_k} = \nabla_{Y_{(l)}} \xi_{A_1 \dots A_k} + \sum_{i=1}^k \xi_{A_1 \dots B \dots A_k} \nabla_{A_i} Y_{(l)}^B$$

together with (0.13.0.9), (0.19.1.1), and our bootstrap assumptions. $\square$

0.19.2 Pointwise Estimates for Rectangular Components

Lemma 0.19.2.1. *Let $x^i = x^i(t, u, \theta^A)$ denote the rectangular coordinate function. Given our bootstrap/smallness/data assumptions we have that for $N \leq 19$ and $V \in \{L, X, Y_{(l)}\}$:*

$$|V^i| \lesssim 1 + |\gamma|, \quad (0.19.2.1)$$

$$|\mathcal{P}^{[1,N]} V^i| \lesssim |\mathcal{P}^{\leq N} \gamma|, \quad (0.19.2.2)$$

$$|\mathcal{L}_*^{[1,N];1} V^i| \lesssim |\mathcal{L}_*^{\leq N;1} \gamma|, \quad (0.19.2.3)$$

$$|\mathcal{L}^{[1,N];1} V^i| \lesssim |\mathcal{L}^{\leq N;1} \gamma|, \quad (0.19.2.4)$$

$$|\check{X}^i| \lesssim 1 + |\underline{\gamma}|, \quad (0.19.2.5)$$

$$|\mathcal{P}^{[1,N]} \check{X}^i| \lesssim |\mathcal{P}^{\leq N} \underline{\gamma}|, \quad (0.19.2.6)$$

$$|\mathcal{L}_*^{[1,N];1} \check{X}^i| \lesssim |\mathcal{L}_*^{\leq N;1} \underline{\gamma}|, \quad (0.19.2.7)$$

$$|\mathcal{L}^{[1,N];1} \check{X}^i| \lesssim |\mathcal{L}^{\leq N;1} \underline{\gamma}|, \quad (0.19.2.8)$$

$$|x^i - \hat{x}^i| \lesssim 1, \quad (0.19.2.9)$$

$$|\mathcal{d}x^i| \lesssim 1 + |\gamma|, \quad (0.19.2.10)$$

$$|\mathcal{d}\mathcal{P}^{[1,N]} x^i| \lesssim |\mathcal{P}^{\leq N} \gamma|, \quad (0.19.2.11)$$

$$|\mathcal{d}\mathcal{L}^{[1,N];1} x^i| \lesssim |\mathcal{L}_*^{\leq N;1} \gamma| + |\mathcal{P}_*^{[1,N]} \underline{\gamma}|, \quad (0.19.2.12)$$

$$|\mathcal{P}^{[1,N]} Y_{(l),(small)}^i| \lesssim |\mathcal{P}^{\leq N} \gamma|, \quad (0.19.2.13)$$

$$|\mathcal{L}_*^{[1,N];1} Y_{(l),(small)}^i| \lesssim |\mathcal{L}_*^{\leq N;1} \gamma|, \quad (0.19.2.14)$$

$$|\mathcal{L}^{[1,N];1} Y_{(l),(small)}^i| \lesssim |\mathcal{L}^{\leq N;1} \gamma|. \quad (0.19.2.15)$$

0.19.3 Pointwise estimates for $\mathcal{P}_{t,u}$ -tangent tensor fields and their Lie derivatives

The following lemma is a straightforward corollary of the above lemma (0.19.2.1).

Lemma 0.19.3.1. *Under our bootstrap, smallness, and data assumptions we have for $N \leq 19$ that*

$$|\mathcal{L}_{\mathcal{P}}^{N+1} \mathcal{G}|, |\mathcal{L}_{\mathcal{P}}^{N+1} \mathcal{G}^{-1}|, |\mathcal{L}_{\mathcal{P}}^N \chi|, |\mathcal{P}^N \text{tr}_{\mathcal{G}} \chi| \lesssim |\mathcal{P}^{\leq N+1} \gamma| \quad (0.19.3.1)$$

$$|\mathcal{L}_{\mathcal{L}_*}^{N+1;1} \mathcal{G}|, |\mathcal{L}_{\mathcal{L}_*}^{N+1;1} \mathcal{G}^{-1}|, |\mathcal{L}_{\mathcal{L}}^{N;1} \chi|, |\mathcal{L}^{N;1} \text{tr}_{\mathcal{G}} \chi| \lesssim |\mathcal{L}_*^{\leq N+1;1} \gamma| + |\mathcal{P}_*^{[1,N+1]} \underline{\gamma}| \quad (0.19.3.2)$$

$$|\mathcal{L}_{\mathcal{L}}^{N+1;1} \mathcal{G}|, |\mathcal{L}_{\mathcal{L}}^{N+1;1} \mathcal{G}^{-1}| \lesssim |\mathcal{L}^{\leq N+1;1} \gamma| + |\mathcal{P}_*^{[1,N+1]} \underline{\gamma}| \quad (0.19.3.3)$$

Lemma 0.19.3.2. *As usual, we are working with our bootstrap, smallness, and data assumptions.*

Here Y denotes any $Y_{(l)}$ and $N \leq 19$

$$|Y| \leq 1 + C|\gamma|, \quad (0.19.3.4)$$

$$|\mathcal{L}_{\mathcal{P}}^{[1,N]} Y| \leq C|\mathcal{P}^{\leq N} \gamma|, \quad (0.19.3.5)$$

$$|\mathcal{L}_{\mathcal{L}_*}^{[1,N];1} Y| \leq C|\mathcal{L}_*^{\leq N;1} \gamma| + C|\mathcal{P}_*^{[1,N]} \underline{\gamma}|, \quad (0.19.3.6)$$

$$|\mathcal{L}_{\mathcal{L}}^{[1,N];1} Y| \leq C|\mathcal{L}_*^{\leq N;1} \gamma| + C|\mathcal{P}_*^{[1,N]} \underline{\gamma}|, \quad (0.19.3.7)$$

$$|\mathcal{L}_{\mathcal{P}}^{N(Y)} \mathcal{F}_L|, |\mathcal{L}_{\mathcal{P}}^{N(Y)} \mathcal{F}_L^{\#}| \lesssim |\mathcal{P}^{\leq N+1} \gamma|, \quad (0.19.3.8)$$

$$|\mathcal{L}_{\mathcal{L}}^{N;1(Y)} \mathcal{F}_L|, |\mathcal{L}_{\mathcal{L}}^{N;1(Y)} \mathcal{F}_L^{\#}| \lesssim |\mathcal{L}_*^{\leq N+1;1} \gamma| + \varepsilon^{1/2} |\mathcal{L}^{\leq N;1} \gamma| + \varepsilon^{1/2} |\mathcal{P}_*^{[1,N]} \underline{\gamma}|, \quad (0.19.3.9)$$

$$|\mathcal{L}_{\mathcal{P}}^{N(Y)} \mathcal{F}_{\check{X}}|, |\mathcal{L}_{\mathcal{P}}^{N(Y)} \mathcal{F}_{\check{X}}^{\#}| \lesssim |\mathcal{L}_*^{\leq N+1;1} \Phi| + |\mathcal{P}^{\leq N+1} \gamma| + |\mathcal{P}_*^{[1,N]} \underline{\gamma}|, \quad (0.19.3.10)$$

$$|(\check{X}) \mathcal{F}_L|, |(\check{X}) \mathcal{F}_L^{\#}| \lesssim |\mathcal{L}^{\leq 1} \Phi| + |\mathcal{P}_* \mu|, \quad (0.19.3.11)$$

$$|\mathcal{L}_{\mathcal{P}}^{[1,N](\check{X})} \mathcal{F}_L|, |\mathcal{L}_{\mathcal{P}}^{[1,N](\check{X})} \mathcal{F}_L^{\#}| \lesssim |\mathcal{L}_*^{\leq N+1;1} \Phi| + |\mathcal{P}^{\leq N+1} \gamma| + |\mathcal{P}_*^{[1,N+1]} \underline{\gamma}|, \quad (0.19.3.12)$$

$$|\mathcal{L}_{\mathcal{P}}^{N(L)} \mathcal{F}|, |\mathcal{L}_{\mathcal{P}}^{N(Y)} \mathcal{F}| \lesssim |\mathcal{P}^{\leq N+1} \gamma|, \quad (0.19.3.13)$$

$$|\mathcal{L}_{\mathcal{L}}^{N;1(L)} \mathcal{F}|, |\mathcal{L}_{\mathcal{L}}^{N;1(Y)} \mathcal{F}| \lesssim |\mathcal{L}_*^{\leq N+1;1} \gamma| + \varepsilon^{1/2} |\mathcal{L}^{\leq N;1} \gamma| + \varepsilon^{1/2} |\mathcal{P}_*^{[1,N]} \underline{\gamma}|, \quad (0.19.3.14)$$

$$|\mathcal{L}_{\mathcal{P}}^N W^{\#}| \lesssim |\mathcal{P}^{\leq N+1} \gamma| \quad (0.19.3.15)$$

0.19.4 Commutator Estimates

The following central estimates are slightly different for us than for [18] (for instance we need $|[\nabla^2, \mathcal{L}_{\mathcal{P}}^N]f|$ rather than $|[\nabla^2, \mathcal{P}^N]f|$). The proofs, however, are nearly identical. The idea is to use

the lemmas in (0.16) together with our bootstrap/smallness/data

Lemma 0.19.4.1. *For multi-index $|I| = N + 1$ we have under our bootstrap/smallness/data assumptions that under any permutation I' of I*

$$|\mathcal{P}^I f - \mathcal{P}^{I'} f| \lesssim \varepsilon^{1/2} |\mathcal{P}_*^{[1,N]} f| + |\mathcal{P}^{[1, \lfloor N/2 \rfloor]} f| |\mathcal{P}^{\leq N} \gamma| \quad (0.19.4.1)$$

If $1 \leq N \leq 19$ and $|I| = N + 1$ then we have

$$|[\nabla^2, \mathcal{L}_{\mathcal{P}}^N] f| \lesssim \varepsilon^{1/2} |\mathcal{P}_*^{[1,N]} f| + |\mathcal{P}^{[1, \lfloor N/2 \rfloor]} f| |\mathcal{P}^{\leq N+1} \gamma|, \quad (0.19.4.2)$$

$$|[\Delta, \mathcal{P}^N] f| \lesssim \varepsilon^{1/2} |\mathcal{P}_*^{[1, N+1]} f| + |\mathcal{P}^{[1, \lfloor N/2 \rfloor]} f| |\mathcal{P}^{\leq N+1} \gamma|, \quad (0.19.4.3)$$

$$|\mathcal{L}_{\mathcal{P}}^I \xi - \mathcal{L}_{\mathcal{P}}^{I'} \xi| \lesssim \varepsilon^{1/2} |\mathcal{L}_{\mathcal{P}}^{[1,N]} \xi| + |\mathcal{L}_{\mathcal{P}}^{\leq \lfloor N/2 \rfloor} \xi| |\mathcal{P}^{\leq N+1} \gamma|, \quad (0.19.4.4)$$

$$|[\nabla, \mathcal{L}_{\mathcal{P}}^N] \xi| \lesssim \underbrace{\varepsilon^{1/2} |\mathcal{L}_{\mathcal{P}}^{[1, N-1]} \xi|}_{\text{Absent when } N=1} + |\mathcal{L}_{\mathcal{P}}^{\leq \lfloor N/2 \rfloor} \xi| |\mathcal{P}^{\leq N+1} \gamma|, \quad (0.19.4.5)$$

$$|[\text{div}, \mathcal{L}_{\mathcal{P}}^N] \xi| \lesssim \varepsilon^{1/2} |\mathcal{L}_{\mathcal{P}}^{[1,N]} \xi| + |\mathcal{L}_{\mathcal{P}}^{\leq \lfloor N/2 \rfloor} \xi| |\mathcal{P}^{\leq N+1} \gamma|, \quad (0.19.4.6)$$

we also have

$$|[\nabla^2, \mathcal{L}_{\mathcal{P}}^N] f| \lesssim |\mathcal{P}^{\leq \lfloor N/2 \rfloor + 1} \gamma| |\mathcal{P}_*^{[1,N]} f| + |\mathcal{P}^{[1, \lfloor N/2 \rfloor]} f| |\mathcal{P}^{\leq N+1} \gamma|, \quad (0.19.4.7)$$

Lemma 0.19.4.2. *Here we assume $\mathcal{L}^I$ contains exactly one $\check{X}$ and assume that $|I| = N + 1$. Under our bootstrap and smallness assumptions we have that for $1 \leq N \leq 19$:*

$$|\mathcal{L}^I f - \mathcal{L}^{I'} f| \lesssim |\mathcal{P}_*^{[1,N]} f| + \underbrace{\varepsilon^{1/2} |Y \mathcal{L}^{\leq N-1; 1} f|}_{\text{Absent if } N=1} + |\mathcal{P}_*^{[1, \lfloor N/2 \rfloor]} f| \left| \begin{pmatrix} \mathcal{P}_*^{[1,N]} \gamma \\ \mathcal{L}_*^{\leq N; 1} \gamma \end{pmatrix} \right| + \underbrace{|Y \mathcal{L}^{\leq \lfloor N/2 \rfloor - 1; 1} f| |\mathcal{P}^{\leq N} \gamma|}_{\text{Absent if } N \leq 3} \quad (0.19.4.8)$$

For $1 \leq N \leq 18$ we have that:

$$|[\nabla^2, \mathcal{L}_{\mathcal{L}}^{N;1}]f| \lesssim |\mathcal{L}_*^{\leq N;1}f| + |\mathcal{P}^{\leq \lceil N/2 \rceil}f| \left| \begin{pmatrix} \mathcal{P}_*^{[1,N+1]} \underline{\gamma} \\ \mathcal{L}_*^{\leq N+1;1} \gamma \end{pmatrix} \right| + |\mathcal{L}_*^{\leq \lceil N/2 \rceil}f| |\mathcal{P}^{\leq N+1}\gamma|, \quad (0.19.4.9)$$

$$|[\Delta, \mathcal{L}^{N;1}]f| \lesssim |\mathcal{L}_*^{\leq N+1;1}f| + |\mathcal{P}^{\leq \lceil N/2 \rceil}f| \left| \begin{pmatrix} \mathcal{P}_*^{[1,N+1]} \underline{\gamma} \\ \mathcal{L}_*^{\leq N+1;1} \gamma \end{pmatrix} \right| + |\mathcal{L}_*^{\leq \lceil N/2 \rceil}f| |\mathcal{P}^{\leq N+1}\gamma| \quad (0.19.4.10)$$

Corollary 0.19.4.3. *We have under our smallness/bootstrap assumptions that for $1 \leq N \leq 19$:*

$$|\mathcal{P}^{N-1}\Delta\Phi| \lesssim |\mathcal{P}^{\leq N+1}\Phi| + |\mathcal{P}^{\leq N}\gamma|. \quad (0.19.4.11)$$

0.19.5 Commutator Estimates Involving Trace and Trace-Free Parts

This section does not appear in [18] as it is irrelevant in the $\mathbb{R}^{2+1}$ dimensional case. The proofs are essentially those in [16] Section 12.25 although here we make use of the notational simplifications given above.

Lemma 0.19.5.1. *Let ξ be a symmetric $\mathcal{P}_{t,u}$ tangent $\binom{0}{2}$ tensorfield. Under our bootstrap and smallness assumptions we have that for $0 \leq N \leq 19$:*

$$|\mathcal{L}_{\mathcal{P}}^N \widehat{\xi}| \lesssim |\mathcal{L}_{\mathcal{P}}^N \chi| + \|\mathcal{L}^{\lfloor N/2 \rfloor} \xi\|_{L^\infty(\mathcal{P}_{t,u})} |\mathcal{P}^{N-1}\gamma| \quad (0.19.5.1)$$

while for $1 \leq N \leq 19$ we have

$$|\mathcal{L}_{\mathcal{P}}^N \widehat{\xi} - \widehat{\mathcal{L}_{\mathcal{P}}^N \xi}| \lesssim |\mathcal{L}_{\mathcal{P}}^{\leq N-1} \xi| + \|\mathcal{L}^{\lfloor N/2 \rfloor} \xi\|_{L^\infty(\mathcal{P}_{t,u})} |\mathcal{P}^{N-1}\gamma|, \quad (0.19.5.2)$$

$$|tr_g \mathcal{L}_{\mathcal{P}}^N \xi - \mathcal{P}^N tr_g \xi| \lesssim |\mathcal{L}_{\mathcal{P}}^{\leq N-1} \xi| + \|\mathcal{L}^{\lfloor N/2 \rfloor} \xi\|_{L^\infty(\mathcal{P}_{t,u})} |\mathcal{P}^{N-1}\gamma|, \quad (0.19.5.3)$$

$$|\mathcal{L}_{\mathcal{P}}^N \widehat{\xi} - \widehat{(\mathcal{L}_{\mathcal{P}}^N \xi)}| \lesssim |\mathcal{L}_{\mathcal{P}}^{\leq N-1} \xi| + \|\mathcal{L}^{\lfloor N/2 \rfloor} \xi\|_{L^\infty(\mathcal{P}_{t,u})} |\mathcal{P}^{N-1}\gamma|, \quad (0.19.5.4)$$

$$\left| \mathcal{L}_{\mathcal{P}}^N \xi - \left\{ \widehat{\mathcal{L}_{\mathcal{P}}^N \xi} + \frac{1}{2} (\mathcal{P}^N tr_g \xi) g \right\} \right| \lesssim |\mathcal{L}_{\mathcal{P}}^{\leq N-1} \xi| + \|\mathcal{L}^{\lfloor N/2 \rfloor} \xi\|_{L^\infty(\mathcal{P}_{t,u})} |\mathcal{P}^{N-1}\gamma|, \quad (0.19.5.5)$$

and if $0 \leq N \leq 18$ we have:

$$\left| \nabla \mathcal{L}_{\mathcal{P}}^N \xi - \left\{ \nabla \widehat{\mathcal{L}}_{\mathcal{P}}^N \widehat{\xi} + \frac{1}{2} (\mathcal{L}_{\mathcal{P}}^N \text{tr}_{\mathcal{G}} \xi) \mathcal{G} \right\} \right| \lesssim |\mathcal{L}_{\mathcal{P}}^{\leq N} \xi| + \|\mathcal{L}^{\lfloor N/2 \rfloor} \xi\|_{L^\infty(\mathcal{P}_{t,u})} |\mathcal{P}^N \gamma|, \quad (0.19.5.6)$$

$$|\nabla \mathcal{L}_{\mathcal{P}}^N \widehat{\xi} - \nabla \widehat{\mathcal{L}}_{\mathcal{P}}^N \widehat{\xi}| \lesssim |\mathcal{L}_{\mathcal{P}}^{\leq N} \xi| + \|\mathcal{L}^{\lfloor N/2 \rfloor} \xi\|_{L^\infty(\mathcal{P}_{t,u})} |\mathcal{P}^N \gamma|, \quad (0.19.5.7)$$

$$|\mathcal{L}_{\mathcal{P}}^N d\dot{v} \xi - d\dot{v} \mathcal{L}_{\mathcal{P}}^N \xi| \lesssim |\mathcal{L}_{\mathcal{P}}^{\leq N} \xi| + \|\mathcal{L}^{\lfloor N/2 \rfloor} \xi\|_{L^\infty(\mathcal{P}_{t,u})} |\mathcal{P}^N \gamma|, \quad (0.19.5.8)$$

$$|\mathcal{L}_{\mathcal{P}}^N d\dot{v} \widehat{\xi} - d\dot{v} \widehat{\mathcal{L}}_{\mathcal{P}}^N \widehat{\xi}| \lesssim |\mathcal{L}_{\mathcal{P}}^{\leq N} \xi| + \|\mathcal{L}^{\lfloor N/2 \rfloor} \xi\|_{L^\infty(\mathcal{P}_{t,u})} |\mathcal{P}^N \gamma|. \quad (0.19.5.9)$$

0.19.6 Transportation Inequalities and Improving Auxiliary Bootstrap Assumptions

In this section we rely heavily on the null structure of the renormalized equations which always pair dangerous $\mathcal{O}(\mathring{\delta})$ quantities with $\mathcal{O}(\varepsilon)$ quantities. We may then use this structure to show that the smallness (largeness) of ε ($\mathring{\delta}$) quantities propagates throughout our bootstrap wavefront region. For instance, using (0.10.2.1) and our equation for $L\mu$, we find that:

$$L\check{X}\Phi = f(\underline{\gamma})\mathring{\Delta}\Phi + f(\gamma, \mathring{g}^{-1}, \mathring{\Delta}x^i, P\Phi, \check{X}\Phi)P^2\Phi + f(\underline{\gamma}, \mathring{g}^{-1}, \mathring{\Delta}x^i, P\Phi, \check{X}\Phi)P\gamma, \quad (0.19.6.1)$$

which clearly exhibits this pairing. The results in this section are a bit different from [18], as we have to worry about $\chi, \chi^\#, \hat{\chi}$. Nonetheless, the proofs are virtually the same- see Proposition 8.10 in [18] for details.

To handle these new nontrivial terms we may use the following, together with the above commutator estimates:

Lemma 0.19.6.1 (Decomposition of $\mathring{\mathcal{L}}_L \chi$). *We have*

$$\mathring{\mathcal{L}}_L \chi = \left(G_{ab} \mathring{\Delta}_A x^a \mathring{\Delta}_B L_{(Small)}^b \right) \cdot L\Phi + g_{ab} \mathring{\Delta}_A L_{(Small)}^a \mathring{\Delta}_B L_{(Small)}^b + g_{ab} \mathring{\Delta}_A x^a \mathring{\Delta}_B L \left(L_{(Small)}^b \right) - \mathring{\mathcal{L}}_L \Lambda_{AB} \quad (0.19.6.2)$$

Proof. Use (0.10.1.27) together with the Leibniz rule and the fact that

$$\mathring{\mathcal{L}}_L \mathring{\Delta}_A x^i = \mathring{\Delta}_A L x^i = \mathring{\Delta}_A L_{(Small)}^i,$$

which follows from (0.8.0.4). □

However, using the schematic relation $\chi = f(\gamma, \mathring{g}^{-1}, \mathring{\Delta}x^i)P\gamma$, applying Lie derivatives, the Leibniz

rule, and our commutator estimates, taking care to note that $\mathcal{L}_L \mathcal{G}^{-1} = (\mathcal{G}^{-1})^2 \chi$ is perhaps an easier (and equally correct) way to handle these terms.

Proposition 0.19.6.2. *For all of the following we are making our usual size and bootstrap assumptions.*

For μ we have that

$$|L\mu| \lesssim |\mathcal{L}^{\leq 1}\Phi| \quad (0.19.6.3)$$

and for $1 \leq N \leq 19$

$$|L\mathcal{P}^N \mu|, |\mathcal{P}^N L\mu| \lesssim |\mathcal{L}_*^{\leq N+1;1}\Phi| + |\mathcal{P}^{\leq N}\gamma| + \varepsilon |\mathcal{P}_*^{[1,N]}\underline{\gamma}| \quad (0.19.6.4)$$

For $L_{(small)}^i$ and χ

$$\left| \begin{pmatrix} L\mathcal{P}^N L_{(small)}^i \\ L\mathcal{P}^{N-1} tr_{\mathcal{G}}\chi \\ \mathcal{L}_L \mathcal{L}^{N-1} \chi \\ \mathcal{L}_L \mathcal{L}^{N-1} \chi^\# \\ \mathcal{L}_L \mathcal{L}^{N-1} \widehat{\chi} \end{pmatrix} \right|, \left| \begin{pmatrix} \mathcal{P}^N L L_{(small)}^i \\ \mathcal{P}^{N-1} L tr_{\mathcal{G}}\chi \\ \mathcal{L}^N \mathcal{L}_L \chi \\ \mathcal{L}^N \mathcal{L}_L \chi^\# \\ \mathcal{L}^N \mathcal{L}_L \widehat{\chi} \end{pmatrix} \right| \lesssim |\mathcal{P}^{\leq N+1}\Phi| + \varepsilon |\mathcal{P}^N \gamma|, \quad (0.19.6.5)$$

$$\left| \begin{pmatrix} L\mathcal{L}^{N;1} L_{(small)}^i \\ L\mathcal{L}^{N-1;1} tr_{\mathcal{G}}\chi \\ \mathcal{L}_L \mathcal{L}^{N-1;1} \chi \\ \mathcal{L}_L \mathcal{L}^{N-1;1} \chi^\# \\ \mathcal{L}_L \mathcal{L}^{N-1;1} \widehat{\chi} \end{pmatrix} \right|, \left| \begin{pmatrix} \mathcal{L}^{N;1} L L_{(small)}^i \\ \mathcal{L}^{N-1;1} L tr_{\mathcal{G}}\chi \\ \mathcal{L}^{N-1;1} \mathcal{L}_L \chi \\ \mathcal{L}^{N-1;1} \mathcal{L}_L \chi^\# \\ \mathcal{L}^{N-1;1} \mathcal{L}_L \widehat{\chi} \end{pmatrix} \right| \lesssim |\mathcal{L}_*^{\leq N+1;1}\Phi| + \left| \begin{pmatrix} \varepsilon \mathcal{P}_*^{[1,N]}\underline{\gamma} \\ \mathcal{L}_*^{\leq N;1}\gamma \end{pmatrix} \right|. \quad (0.19.6.6)$$

For at most one transversal derivative of Φ we have that

$$\|\check{X}\Phi\|_{L^\infty(\Sigma_t^u)} \leq \|\check{X}\Phi\|_{L^\infty(\Sigma_0^u)} + C\varepsilon \quad (0.19.6.7)$$

$$\|\mathcal{L}_*^{\leq 10;1}\Phi\|_{L^\infty(\Sigma_t^u)} \leq C\varepsilon. \quad (0.19.6.8)$$

For μ we have:

$$\|L\mu\|_{L^\infty(\Sigma_t^u)} = \frac{1}{2}\|G_{LL}\check{X}\Phi\|_{L^\infty(\Sigma_0^u)} + \mathcal{O}_*(\check{\alpha}) + \mathcal{O}(\varepsilon), \quad (0.19.6.9)$$

$$\|L\mathcal{P}^{[1,9]}\mu\|_{L^\infty(\Sigma_t^u)}, \|\mathcal{P}_*^{[1,9]}\mu\|_{L^\infty(\Sigma_t^u)} \leq C\varepsilon, \quad (0.19.6.10)$$

$$\|\mu - 1\|_{L^\infty(\Sigma_t^u)} \leq 2\check{\delta}_*^{-1}\|G_{LL}\check{X}\Phi\|_{L^\infty(\Sigma_0^u)} + C\varepsilon. \quad (0.19.6.11)$$

For $L_{(small)}^i$ and χ we have:

$$\|L\mathcal{P}^{\leq 10}L_{(small)}^i\|_{L^\infty(\Sigma_t^u)}, \|\mathcal{P}^{\leq 10}L_{(small)}^i\|_{L^\infty(\Sigma_t^u)} \leq C\varepsilon, \quad (0.19.6.12)$$

$$\|L\mathcal{L}^{\leq 9;1}L_{(small)}^i\|_{L^\infty(\Sigma_t^u)}, \|\mathcal{L}_*^{\leq 9;1}L_{(small)}^i\|_{L^\infty(\Sigma_t^u)} \leq C\varepsilon, \quad (0.19.6.13)$$

$$\|\check{X}L_{(small)}^i\|_{L^\infty(\Sigma_t^u)} \leq \|\check{X}L_{(small)}^i\|_{L^\infty(\Sigma_0^u)} + C\varepsilon, \quad (0.19.6.14)$$

$$\|\check{\mathcal{L}}_{\mathcal{P}}^{\leq 9}\chi\|_{L^\infty(\Sigma_t^u)}, \|\check{\mathcal{L}}_{\mathcal{P}}^{\leq 9}\chi^\#\|_{L^\infty(\Sigma_t^u)}, \|\mathcal{P}^{\leq 9}tr_{\check{g}}\chi\|_{L^\infty(\Sigma_t^u)} \leq C\varepsilon, \quad (0.19.6.15)$$

$$\|\check{\mathcal{L}}_{\mathcal{L}}^{\leq 8;1}\chi\|_{L^\infty(\Sigma_t^u)}, \|\check{\mathcal{L}}_{\mathcal{L}}^{\leq 8;1}\chi^\#\|_{L^\infty(\Sigma_t^u)}, \|\mathcal{L}^{\leq 8;1}tr_{\check{g}}\chi\|_{L^\infty(\Sigma_t^u)} \leq C\varepsilon. \quad (0.19.6.16)$$

Proof. (Idea) Use our commutator estimates, combined with the fundamental theorem of calculus and our auxiliary bootstrap assumptions $\square$

Corollary 0.19.6.3. *All of the estimates in previous sections which were stated with $\varepsilon^{1/2}$ hold instead with $C\varepsilon$.*

0.20 L^∞ Estimates With Transversal Derivatives

Remark 0.20.0.1. The following two sections, 19 and 20 are straight from [16]. They apply directly to our setting without modification (although we note that in our case $Y \in \{Y_{(1)}, Y_{(2)}\}$). We have simply copied the definitions and results without proof for convenience of the reader. For the proofs, see [16]. The proofs are in the spirit of the previous sections: the wave equation $\square_{g(\Phi)}\Phi = 0$ (viewed as a transport equation for $\check{X}\Phi$) and the associated transport equations for the Eikonal quantities all exhibit nice null structure pairing $\check{\delta}$ quantities with $\check{\epsilon}$ quantities. In the previous section we saw that commuting with the “good” $\mathcal{C}_u$ -tangent derivatives does not disturb this structure and thus we can propagate smallness at appropriately low order. It turns out that even commuting with a few $\check{X}$ derivatives does not affect this fundamental structure.

The purpose of this section is obtaining the estimate $\|\check{X}^2\mu\|_{L^\infty(\Sigma_t^u)} \lesssim 1$. This is done by commuting the equation $L\mu = \frac{1}{2}G_{LL}\check{X}\Phi + \dots$ with $\check{X}^2$ and using the fundamental theorem of calculus.

Hence our need for control of $\|\check{X}^3\Phi\|_{L^\infty(\Sigma_t^u)}$. We will need to use this estimate on $\check{X}^2\mu$ to derive sharp estimates for μ in the next section.

0.20.1 (More) Auxiliary Bootstrap Assumptions

$$\|\mathcal{L}^{\leq 4;2}\Phi\|_{L^\infty(\Sigma_t^u)} \leq \varepsilon^{1/2} \quad (0.20.1.1)$$

Here Y denotes any $Y_{(l)}$

$$\|\check{X}Y\mu\|_{L^\infty(\Sigma_t^u)}, \|\check{X}LL\mu\|_{L^\infty(\Sigma_t^u)}, \|\check{X}YY\mu\|_{L^\infty(\Sigma_t^u)}, \|\check{X}LY\mu\|_{L^\infty(\Sigma_t^u)} \leq \varepsilon^{1/2}, \quad (0.20.1.2)$$

$$\|\mathcal{L}_*^{\leq 3;2}L_{(small)}^i\|_{L^\infty(\Sigma_t^u)} \leq \varepsilon^{1/2} \quad (0.20.1.3)$$

$$\|\check{X}^M\Phi\|_{L^\infty(\Sigma_t^u)} \leq \|\check{X}^M\Phi\|_{L^\infty(\Sigma_0^u)} + \varepsilon^{1/2}, \quad (2 \leq M \leq 3) \quad (0.20.1.4)$$

$$\|L\check{X}^M\Phi\|_{L^\infty(\Sigma_t^u)} \leq \frac{1}{2} \left\| \check{X}^M \left\{ G_{LL}\check{X}\Phi \right\} \right\|_{L^\infty(\Sigma_0^u)} + \varepsilon^{1/2}, \quad (1 \leq M \leq 2) \quad (0.20.1.5)$$

$$\|\check{X}^M\mu\|_{L^\infty(\Sigma_t^u)} \leq \|\check{X}^M\mu\|_{L^\infty(\Sigma_0^u)} + 2\delta_*^{-1} \left\| \check{X}^M \left\{ G_{LL}\check{X}\Phi \right\} \right\|_{L^\infty(\Sigma_0^u)} + \varepsilon^{1/2}, \quad (1 \leq M \leq 2) \quad (0.20.1.6)$$

$$\|\check{X}\check{X}L_{(small)}^i\|_{L^\infty(\Sigma_t^u)} \leq \|\check{X}\check{X}L_{(small)}^i\|_{L^\infty(\Sigma_0^u)} + \varepsilon^{1/2}. \quad (0.20.1.7)$$

0.20.2 Commutator Estimates with two transversal derivatives

As usual, Y denotes any $Y_{(l)}$.

Lemma 0.20.2.1. *Working under our bootstrap/smallness/data assumptions, we have that $\mathcal{L}$ contains exactly two factors of $\check{X}$ and that $3 \leq |I| := N+1 \leq 4$*

$$|\mathcal{L}^I f - \mathcal{L}^{I'} f| \lesssim |Y\mathcal{L}^{\leq N-1;1} f| + \underbrace{\varepsilon |Y\mathcal{L}^{N-1;2} f|}_{\text{Absent if } N=2} \quad (0.20.2.1)$$

$$|[\Delta, \check{X}\check{X}]f| \lesssim |Y\mathcal{L}^{\leq 2;1} f|. \quad (0.20.2.2)$$

0.20.3 Estimates for Higher Order Transversal Derivatives

Proposition 0.20.3.1. *Under our bootstrap/smallness/data assumptions we find that for derivatives of Φ we have:*

$$\|L\mathcal{L}_*^{\leq 4;2}\Phi\|_{L^\infty(\Sigma_t^u)} \leq C\varepsilon, \quad (0.20.3.1)$$

$$\|\mathcal{L}_*^{\leq 4;2}\Phi\|_{L^\infty(\Sigma_t^u)} \leq C\varepsilon, \quad (0.20.3.2)$$

$$\|\check{X}\check{X}\Phi\|_{L^\infty(\Sigma_t^u)} \leq \|\check{X}\check{X}\Phi\|_{L^\infty(\Sigma_0^u)} + C\varepsilon, \quad (0.20.3.3)$$

$$\|L\check{X}\check{X}\Phi\|_{L^\infty(\Sigma_t^u)} \leq C\varepsilon, \quad (0.20.3.4)$$

$$\|\check{X}\check{X}\check{X}\Phi\|_{L^\infty(\Sigma_t^u)} \leq \|\check{X}\check{X}\check{X}\Phi\|_{L^\infty(\Sigma_0^u)} + C\varepsilon \quad (0.20.3.5)$$

For derivatives of μ we have:

$$\|L\check{X}\mu\|_{L^\infty(\Sigma_t^u)} \leq \frac{1}{2} \left\| \check{X} \left\{ G_{LL}\check{X}\Phi \right\} \right\|_{L^\infty(\Sigma_0^u)} + C\varepsilon, \quad (0.20.3.6)$$

$$\|\check{X}\mu\|_{L^\infty(\Sigma_t^u)} \leq \|\check{X}\mu\|_{L^\infty(\Sigma_0^u)} + \check{\delta}_*^{-1} \left\| \check{X} \left\{ G_{LL}\check{X}\Phi \right\} \right\|_{L^\infty(\Sigma_0^u)} + C\varepsilon, \quad (0.20.3.7)$$

$$\|L\check{X}Y\mu\|_{L^\infty(\Sigma_t^u)}, \|L\check{X}LL\mu\|_{L^\infty(\Sigma_t^u)}, \|L\check{X}YY\mu\|_{L^\infty(\Sigma_t^u)}, \|L\check{X}LY\mu\|_{L^\infty(\Sigma_t^u)} \leq C\varepsilon, \quad (0.20.3.8)$$

$$\|\check{X}Y\mu\|_{L^\infty(\Sigma_t^u)}, \|\check{X}LL\mu\|_{L^\infty(\Sigma_t^u)}, \|\check{X}YY\mu\|_{L^\infty(\Sigma_t^u)}, \|\check{X}LY\mu\|_{L^\infty(\Sigma_t^u)} \leq C\varepsilon, \quad (0.20.3.9)$$

$$\|L\check{X}\check{X}\mu\|_{L^\infty(\Sigma_t^u)} \leq \frac{1}{2} \left\| \check{X}\check{X} \left\{ G_{LL}\check{X}\Phi \right\} \right\|_{L^\infty(\Sigma_0^u)} + C\varepsilon, \quad (0.20.3.10)$$

$$\|\check{X}\check{X}\mu\|_{L^\infty(\Sigma_t^u)} \leq \|\check{X}\check{X}\mu\|_{L^\infty(\Sigma_0^u)} + \check{\delta}_*^{-1} \left\| \check{X}\check{X} \left\{ G_{LL}\check{X}\Phi \right\} \right\|_{L^\infty(\Sigma_0^u)} + C\varepsilon. \quad (0.20.3.11)$$

For derivatives of $L_{(small)}^i$ we have:

$$\|\mathcal{L}_*^{\leq 3;2}L_{(small)}^i\|_{L^\infty(\Sigma_t^u)} \leq C\varepsilon, \quad (0.20.3.12)$$

$$\|\check{X}\check{X}L_{(small)}^i\|_{L^\infty(\Sigma_t^u)} \leq \|\check{X}\check{X}L_{(small)}^i\|_{L^\infty(\Sigma_0^u)} + C\varepsilon. \quad (0.20.3.13)$$

And the following estimates involving G_{LL} hold for $0 \leq M \leq 2$ and $0 \leq s \leq t < T_{(Boot)}$:

$$|\check{X}^M G_{LL}(t, u, \theta) - \check{X}^M G_{LL}(s, u, \theta)| \leq C\varepsilon(t - s), \quad (0.20.3.14)$$

$$\left| \check{X}^M \left\{ G_{LL} \check{X} \Phi \right\} (t, u, \theta) - \check{X}^M \left\{ G_{LL} \check{X} \Phi \right\} (s, u, \theta) \right| \leq C\varepsilon(t - s), \quad (0.20.3.15)$$

$$L\mu(t, u, \theta) = \frac{1}{2} G_{L(flat)L(flat)}(\Phi = 0) \check{X} \Phi(t, u, \theta) + \mathcal{O}(\varepsilon). \quad (0.20.3.16)$$

Proof. (Idea) Use the commutator estimates in conjunction with the auxiliary bootstrap assumptions and the fundamental theorem of calculus. See [18] for details. $\square$

0.21 Sharp Estimates for μ

This section is of crucial importance due to proposition (0.21.0.3), which allows one to implement the “descent scheme” of Christodoulou - deriving nondegenerate lower order energies from higher order energies which exhibit blow-up near the shock as a function of $\mu_\star$. Another crucial result in this section is the bound (0.21.0.19) for $\left\| \frac{[\check{X}\mu]}{\mu} \right\|_{L^\infty(\Sigma_u^y)}$, which in turn makes use of the bound on $\|\check{X}^2 \mu\|_{L^\infty(\Sigma_u^y)}$ of the previous section. We need this bound to control the term (0.14.0.10) which appears in the bulk error from our multiplier field T .

Definition 0.21.0.1 (Auxiliary Quantities for Controlling μ).

$$M(s, u, \theta; t) := \int_{s'=s}^{s'=t} \{L\mu(t, u, \theta) - L\mu(s', u, \theta)\} ds', \quad (0.21.0.1)$$

$$\mathring{\mu}(u, \theta) := \mu(0, u, \theta), \quad (0.21.0.2)$$

$$\widetilde{M}(s, u, \theta; t) := \frac{M(s, u, \theta; t)}{\mathring{\mu}(u, \theta) - M(0, u, \theta; t)} \quad (0.21.0.3)$$

$$\mu_{(Approx)}(s, u, \theta; t) := 1 + \frac{L\mu(t, u, \theta)}{\mathring{\mu}(u, \theta) - M(0, u, \theta; t)} s + \widetilde{M}(s, u, \theta; t). \quad (0.21.0.4)$$

Definition 0.21.0.2.

$$\mu_\star(t, u) := \min \left\{ 1, \min_{\Sigma_t^u} \mu \right\} \quad (0.21.0.5)$$

The following lemma is a straightforward corollary of the above definitions.

Lemma 0.21.0.1. *Under our bootstrap/smallness/data assumptions we find that:*

$$\mathring{\mu}(u, \theta) = 1 + \mathcal{O}_\star(\mathring{\alpha}) + \mathcal{O}(\varepsilon), \quad (0.21.0.6)$$

$$\mathring{\mu}(u, \theta) = 1 + M(0, u, \theta; t) + \mathcal{O}_\star(\mathring{\alpha}) + \mathcal{O}(\varepsilon). \quad (0.21.0.7)$$

$$|L\boldsymbol{\mu}(t, u, \theta) - L\boldsymbol{\mu}(s, u, \theta)| \lesssim \varepsilon(t - s), \quad (0.21.0.8)$$

$$|M(s, u, \theta; t)|, |\widetilde{M}(s, u, \theta; t)| \lesssim \varepsilon(t - s)^2 \quad (0.21.0.9)$$

$$\boldsymbol{\mu}(s, u, \theta) = (1 + \mathcal{O}(\varepsilon)) \boldsymbol{\mu}_{(Approx)}(s, u, \theta; t). \quad (0.21.0.10)$$

Definition 0.21.0.3. We say that

$$[0, u] \times \mathbb{T}^2 = {}^{(+)}\mathcal{V}_t^u \cup {}^{(-)}\mathcal{V}_t^u, \quad (0.21.0.11)$$

$$\Sigma_s^u = {}^{(+)}\Sigma_{s;t}^u \cup {}^{(-)}\Sigma_{s;t}^u, \quad (0.21.0.12)$$

where we define

$${}^{(+)}\mathcal{V}_t^u := \left\{ (u', \theta) \in [0, u] \times \mathbb{T}^2 \left| \frac{L\boldsymbol{\mu}(t, u', \theta)}{\mathring{\boldsymbol{\mu}}(u', \theta) - M(0, u', \theta; t)} \geq 0 \right. \right\}, \quad (0.21.0.13)$$

$${}^{(-)}\mathcal{V}_t^u := \left\{ (u', \theta) \in [0, u] \times \mathbb{T}^2 \left| \frac{L\boldsymbol{\mu}(t, u', \theta)}{\mathring{\boldsymbol{\mu}}(u', \theta) - M(0, u', \theta; t)} < 0 \right. \right\}, \quad (0.21.0.14)$$

$${}^{(+)}\Sigma_{s;t}^u := \left\{ (s, u', \theta) \in \Sigma_s^u \left| (u', \theta) \in {}^{(+)}\mathcal{V}_t^u \right. \right\}, \quad (0.21.0.15)$$

$${}^{(-)}\Sigma_{s;t}^u := \left\{ (s, u', \theta) \in \Sigma_s^u \left| (u', \theta) \in {}^{(-)}\mathcal{V}_t^u \right. \right\}. \quad (0.21.0.16)$$

Proposition 0.21.0.2. Under our smallness/bootstrap/data assumptions we have that:

$$\left\| \frac{[L\boldsymbol{\mu}]_+}{\boldsymbol{\mu}} \right\|_{L^\infty(\Sigma_s)} \leq C. \quad (0.21.0.17)$$

$$\boldsymbol{\mu}(s, u, \theta) \leq \frac{1}{4} \Rightarrow L\boldsymbol{\mu}(s, u, \theta) \leq -\frac{1}{4} \mathring{\boldsymbol{\delta}}_* \quad (0.21.0.18)$$

$$\left\| \frac{[\check{X}\boldsymbol{\mu}]_+}{\boldsymbol{\mu}} \right\|_{L^\infty(\Sigma_s)} \leq \frac{C}{\sqrt{T_{(Boot)} - s}}. \quad (0.21.0.19)$$

Define the constant $\boldsymbol{\kappa}$ by

$$\boldsymbol{\kappa} := - \min_{(u', \theta) \in [0, u] \times \mathbb{T}^2} \frac{L\boldsymbol{\mu}(t, u', \theta)}{\mathring{\boldsymbol{\mu}}(u', \theta) - M(0, u', \theta; t)}. \quad (0.21.0.20)$$

Then we have that:

$$\boldsymbol{\mu}_*(s, u) = \{1 + \mathcal{O}_*(\mathring{\alpha}) + \mathcal{O}(\varepsilon)\} \{1 - \boldsymbol{\kappa}s\}, \quad (0.21.0.21)$$

$$\| [L\boldsymbol{\mu}]_- \|_{L^\infty(\Sigma_s)} \leq \begin{cases} \{1 + \mathcal{O}(\varepsilon^{1/2})\} \boldsymbol{\kappa}, & \text{if } \boldsymbol{\kappa} \geq \sqrt{\varepsilon} \\ \mathcal{O}(\varepsilon^{1/2}), & \text{if } \boldsymbol{\kappa} \leq \sqrt{\varepsilon}. \end{cases} \quad (0.21.0.22)$$

$$\boldsymbol{\kappa} \leq \{1 + \mathcal{O}(\varepsilon)\} \mathring{\boldsymbol{\delta}}_* \quad (0.21.0.23)$$

and when $u = 1$ we have

$$\boldsymbol{\kappa} = \{1 + \mathcal{O}(\varepsilon)\} \mathring{\boldsymbol{\delta}}_* \quad (0.21.0.24)$$

We also have that

$$\sup_{(u', \theta) \in (+)\mathcal{V}_t^u} \frac{\boldsymbol{\mu}(s_2, u', \theta)}{\boldsymbol{\mu}(s_1, u', \theta)} \leq C \quad (0.21.0.25)$$

and that

$$\inf_{(+)\Sigma_{s;t}^u} \boldsymbol{\mu} \geq 1 - C\varepsilon \quad (0.21.0.26)$$

$$\left\| \frac{[L\boldsymbol{\mu}]_-}{\boldsymbol{\mu}} \right\|_{L^\infty((+)\Sigma_{s;t}^u)} \leq C\varepsilon. \quad (0.21.0.27)$$

For $(-)\mathcal{V}_t^u$ nonempty we have

$$\sup_{\substack{0 \leq s_1 \leq s_2 \leq t \\ (u', \theta) \in (-)\mathcal{V}_t^u}} \frac{\boldsymbol{\mu}(s_2, u', \theta)}{\boldsymbol{\mu}(s_1, u', \theta)} \leq 1 + C\varepsilon \quad (0.21.0.28)$$

$$\| [L\boldsymbol{\mu}]_+ \|_{L^\infty((-)\Sigma_{s;t}^u)} \leq C\varepsilon. \quad (0.21.0.29)$$

$$\| [L\boldsymbol{\mu}]_- \|_{L^\infty((-)\Sigma_{s;t}^u)} \leq \begin{cases} \{1 + \mathcal{O}(\varepsilon^{1/2})\} \boldsymbol{\kappa}, & \text{if } \boldsymbol{\kappa} \geq \sqrt{\varepsilon} \\ \mathcal{O}(\varepsilon^{1/2}), & \text{if } \boldsymbol{\kappa} \leq \sqrt{\varepsilon}. \end{cases} \quad (0.21.0.30)$$

$$\boldsymbol{\mu}_*^{-1}(s_1, u) \leq (1 + C\varepsilon) \boldsymbol{\mu}_*^{-1}(s_2, u) \quad (0.21.0.31)$$

The following proposition is fundamental for the Gronwall estimates which are used in the descent scheme. It allows us to control the integrating factors which we expect from the estimates in (0.26.7).

It is essentially a consequence of (0.21.0.21) and (0.21.0.22):

Proposition 0.21.0.3. *Let $B > 1$. Then under our smallness and bootstrap assumptions we have that if $B\sqrt{\varepsilon} \leq 1$ there exists a $C > 0$ such that*

$$\int_0^t \frac{\| [L\mu]_- \|_{L^\infty(\Sigma_s^u)}}{\mu_*^B(s, u)} ds \leq \frac{1 + C\sqrt{\varepsilon}}{B-1} \mu_*^{1-B}(t, u) \quad (0.21.0.32)$$

$$\|L\mu\|_{L^\infty((-)\Sigma_{t,t}^u)} \int_0^t \frac{1}{\mu_*^B(s, u)} ds \leq \frac{1 + C\sqrt{\varepsilon}}{B-1} \mu_*^{1-B}(t, u) \quad (0.21.0.33)$$

If $B\sqrt{\varepsilon} \leq 1$ then

$$\int_0^t \frac{1}{\mu_*^B(s, u)} ds \leq C \left\{ 2^B + \frac{1}{B-1} \right\} \mu_*^{1-B}(t, u) \quad (0.21.0.34)$$

$$\int_0^t \frac{\| [L\mu]_- \|_{L^\infty(\Sigma_s^u)}}{\mu_*(s, u)} ds \leq (1 + C\sqrt{\varepsilon}) \ln \mu_*^{-1}(t, u) + C\sqrt{\varepsilon} \quad (0.21.0.35)$$

$$\int_0^t \frac{1}{\mu_*(s, u)} ds \leq C \{ \ln \mu_*^{-1}(t, u) + 1 \} \quad (0.21.0.36)$$

$$\int_0^t \frac{1}{\mu_*^{9/10}(s, u)} ds \leq C \quad (0.21.0.37)$$

0.22 Top Order Pointwise Commutator Estimates Involving the Eikonal Function

The goal here is to isolate the worst error terms which arise from commuting the wave equation with vectorfields from $\mathcal{P}$. We need to show that $\nabla^2 \mu$ and $\Delta \mu$ can be controlled in terms of $\mathcal{L}_{\check{X}} \chi$ (up to error terms).

We first need the following, which is derived from (0.17.0.1)

Lemma 0.22.0.1.

$$\begin{aligned} R_{\check{X}ALB} &= \frac{1}{2} \left\{ \mu \mathcal{G}_{XA} \mathcal{G}_B L\Phi + \mathcal{G}_{LA} \mathcal{G}_B \check{X}\Phi - \mathcal{G}_{AB} L\check{X}\Phi - \mu G_{LX} \nabla_{AB}^2 \Phi \right\} \\ &\quad + \frac{1}{4} \mu^{-1} \mathcal{G}_{LA} \mathcal{G}_{LB} (\check{X}\Phi)^2 - \frac{1}{2} \mu^{-1} \mathcal{G}_{LA} (\mathcal{G}_B \mu) \check{X}\Phi \\ &\quad + \frac{1}{2} \left\{ G_{LX} \chi_{AB} \check{X}\Phi + \mu \mathcal{G}_{LAX}{}^C \mathcal{G}_C \Phi \right\} \\ &\quad + \mathfrak{A} \end{aligned} \quad (0.22.0.1)$$

where $\mathfrak{A} = f(\underline{\gamma}, \mathcal{G}^{-1}, \mathcal{G} x^i, Z\Phi) P\Phi$.

from which we can derive:

Lemma 0.22.0.2.

$$\begin{aligned}\mathfrak{L}_{\check{X}}\chi_{AB} &= \nabla_{AB}^2 \mathfrak{u} + f(\gamma, \mathfrak{g}^{-1}, \mathfrak{d}x^i) P \check{X} \Phi + f(\underline{\gamma}, \mathfrak{g}^{-1}, \mathfrak{d}x^i) P P \Phi \\ &\quad + f(\underline{\gamma}, \check{X} \Phi, P \underline{\gamma}, \mathfrak{d}x^i) \cdot (P \gamma, \nabla^2 x^i)\end{aligned}\tag{0.22.0.2}$$

The following lemma does not appear in [18]. It is similar to what is used in [16], although we use the simplified schematic notation from [18].

Lemma 0.22.0.3.

$$\begin{aligned}d\mathfrak{h}v\chi, 2d\mathfrak{h}v\hat{\chi} &= \mathfrak{d}tr_{\mathfrak{g}}\chi + f(\gamma, \mathfrak{g}^{-1}, \mathfrak{d}x^i) P P \Phi \\ &\quad + f(\mathfrak{g}^{-1}, \gamma, P \gamma, \mathfrak{d}x^i) \cdot (P \gamma, \nabla^2 x^i)\end{aligned}\tag{0.22.0.3}$$

0.22.1 Commutator Estimates

Definition 0.22.1.1. Under our smallness/bootstrap assumptions and for $1 \leq N \leq 19$ we denote:

$$|\text{Harmless}^{\leq N}| \lesssim |\mathcal{L}_*^{\leq N+1;1} \Phi| + |\mathcal{L}_*^{\leq N;1} \gamma| + |\mathcal{P}_*^{[1,N]} \underline{\gamma}| \tag{0.22.1.1}$$

The following lemma is obvious:

Lemma 0.22.1.1. *Under our bootstrap and smallness assumptions it holds that:*

$$|\text{Harmless}^{\leq 9}| \lesssim \varepsilon \tag{0.22.1.2}$$

Lemma 0.22.1.2. *Under our bootstrap and smallness assumptions we have, for $1 \leq N \leq 19$*

$$\begin{aligned}&|\nabla^2 \mathcal{P}^{N-1} \mathfrak{u} - \mathfrak{L}_{\mathcal{P}}^{N-1} \mathfrak{L}_{\check{X}} \chi|, \\ &|\hat{\nabla}^2 \mathcal{P}^{N-1} \mathfrak{u} - \mathfrak{L}_{\check{X}} \mathfrak{L}_{\mathcal{P}}^{N-1} \hat{\chi}|, \\ &|\Delta \mathcal{P}^{N-1} \mathfrak{u} - \mathcal{P}^{N-1} \check{X} tr_{\mathfrak{g}} \chi|, \\ &|\Delta \mathcal{P}^{N-1} \mathfrak{u} - \mathcal{P}^{N-1} \check{X} tr_{\mathfrak{g}} \chi| \lesssim |\mathcal{L}_*^{\leq N+1;1} \Phi| + |\mathcal{P}^{\leq N} \gamma| + \varepsilon |\mathcal{P}_*^{[1,N]} \underline{\gamma}|\end{aligned}\tag{0.22.1.3}$$

Proof. write

$$\mathfrak{L}_{\check{X}}\chi_{AB} = \nabla_{AB}^2 \mathfrak{u} + \mathfrak{J}_{AB} \tag{0.22.1.4}$$

so we have that

$$\mathfrak{L}_{\mathcal{P}}^{N-1} \mathfrak{L}_{\check{X}} \chi_{AB} - \nabla_{AB}^2 \mathcal{P}^{N-1} \mu = - \left(\left[\nabla^2, \mathfrak{L}_{\mathcal{P}}^{N-1} \right] \mu \right)_{AB} + \mathfrak{L}_{\mathcal{P}}^{N-1} \mathfrak{J}_{AB} \quad (0.22.1.5)$$

use commutator estimates along with the fact that $|\mathfrak{L}_{\mathcal{P}}^M \nabla^2 x^i| \lesssim |\mathcal{P}^{M+1} \gamma|$. There is one commutator estimate which we need to check, which is that

$$\left| [\mathfrak{L}_{\check{X}}, \mathfrak{L}_{\mathcal{P}}^{N-1}] \chi \right| \lesssim |\mathcal{L}_*^{\leq N+1;1} \Phi| + |\mathcal{P}^{\leq N} \gamma| + \varepsilon |\mathcal{P}_*^{[1,N]} \underline{\gamma}| \quad (0.22.1.6)$$

The key is to use (0.16.0.2) together with (0.19.1.4) and our above pointwise estimates $\square$

Lemma 0.22.1.3. *Under our bootstrap and smallness assumptions and for $1 \leq N \leq 19$, we have:*

$$\begin{aligned} & |\mathfrak{L}_{\mathcal{P}}^{N-1} d\dot{v}\chi - \not{d}\mathcal{P}^{N-1} tr_{\not{g}}\chi|, \\ & \left| \mathfrak{L}_{\mathcal{P}}^{N-1} d\dot{v}\hat{\chi} - \frac{1}{2} \not{d}\mathcal{P}^{N-1} tr_{\not{g}}\chi \right|, \\ & |d\dot{v}\mathfrak{L}_{\mathcal{P}}^{N-1} \chi - \not{d}\mathcal{P}^{N-1} tr_{\not{g}}\chi|, \\ & \left| d\dot{v}\hat{\mathfrak{L}}_{\mathcal{P}}^{N-1} \hat{\chi} - \frac{1}{2} \not{d}\mathcal{P}^{N-1} tr_{\not{g}}\chi \right| \lesssim |\mathcal{L}_*^{\leq N+1;1} \Phi| + |\mathcal{P}^{\leq N} \gamma| + \varepsilon |\mathcal{P}_*^{[1,N-1]} \underline{\gamma}| \end{aligned} \quad (0.22.1.7)$$

Proof. Similar to the proof of the previous lemma, but simpler. We use (0.22.0.3) instead of (0.22.0.2) together with our previous pointwise estimates $\square$

0.22.2 Worst Case Error Terms

Proposition 0.22.2.1. *Under our bootstrap and smallness assumptions we have for $1 \leq N \leq 19$ that:*

$$\mu \square_g (Y^{N-1} L \Phi) = (\not{d}^A \Phi) (\mu \not{d}_A Y^{N-1} tr_{\not{g}} \chi) + Harmless^{\leq N} \quad (0.22.2.1)$$

$$\mu \square_g (Y^{N-1} Y_{(l)} \Phi) = (\check{X} \Phi) Y^N tr_{\not{g}} \chi + \rho_{(l)} (\not{d}^A \Phi) (\mu \not{d}_A Y^{N-1} tr_{\not{g}} \chi) + Harmless^{\leq N} \quad (0.22.2.2)$$

while for all other $\mathcal{P}^N$ commutators we have that

$$\mu \square_g (\mathcal{P}^N \Phi) = Harmless^{\leq N} \quad (0.22.2.3)$$

Lemma 0.22.2.2. For $\mathcal{P}^N = \mathcal{P}^{N-1}Z$ we have

$$\mu \square_g(\mathcal{P}^N \Phi) = {}^{(\mathcal{P}^N)}\mathfrak{F} \quad (0.22.2.4)$$

$${}^{(\mathcal{P}^N)}\mathfrak{F} = \mathcal{P}^{P-1}(\mu \mathcal{D}_\alpha^{(Z)} \mathcal{J}^\alpha[\Phi]) + {}^{(\mathcal{P}^N)}\mathfrak{F}_{(Eikonal-Low)} \quad (0.22.2.5)$$

where for $N = 1$ we have ${}^{(\mathcal{P}^N)}\mathfrak{F}_{(Eikonal-Low)} = 0$ and otherwise

$$|{}^{(\mathcal{P}^N)}\mathfrak{F}_{(Eikonal-Low)}| \lesssim \sum_{\substack{N_1+N_2+N_3 \leq N-1 \\ N_1, N_2 \leq N-2}} \sum_{Z_1, Z_2 \in \mathcal{P}} \left(|\mathcal{P}^{N_1} tr_{\mathfrak{g}}^{(Z_1)} \not{A}| + 1 \right) |\mathcal{P}^{N_2}(\mu \mathcal{D}_\alpha^{(Z_2)} \mathcal{J}^\alpha[\mathcal{P}^{N_3} \Phi])| \quad (0.22.2.6)$$

0.22.3 Derivatives of ${}^{(L)}\pi$ and ${}^{(Y_{(l)})}\pi$

The following lemmas are different from those in [18], although the proofs are rather straightforward applications of the machinery developed above.

Lemma 0.22.3.1 (Isolating the top-order terms in ${}^{(L)}\pi$ and ${}^{(Y_{(l)})}\pi$). *Under our smallness and bootstrap assumptions we have that for ${}^{(L)}\pi$:*

$$\left| \mathcal{P}^{N-1} \check{X} tr_{\mathfrak{g}}^{(L)} \not{A} - 2\Delta \mathcal{P}^{N-1} \mu \right| \lesssim |\mathcal{L}_*^{\leq N+1;1} \Phi| + |\mathcal{L}_*^{\leq N;1} \gamma| + \varepsilon |\mathcal{P}^{[1,N]} \underline{\gamma}|, \quad (0.22.3.1)$$

$$\begin{aligned} & \left| \mathfrak{L}_{\mathcal{P}}^{N-1} \not{A}^{\#(L)} \pi_{L\check{X}} \right|, \left| \mathcal{P}^{N-1} d\dot{v}^{(L)} \not{A}_{\check{X}}^{\#} - \Delta \mathcal{P}^{N-1} \mu \right|, \left| \mathfrak{L}_{\mathcal{P}}^{N-1} d\dot{v}^{(L)} \hat{\not{A}}^{\#\#} - \not{A}^{\#} \mathcal{P}^{N-1} tr_{\mathfrak{g}} \chi \right| \\ & \lesssim |\mathcal{L}_*^{\leq N+1;1} \Phi| + |\mathcal{L}_*^{\leq N;1} \gamma| + \varepsilon |\mathcal{P}^{[1,N]} \underline{\gamma}| \end{aligned} \quad (0.22.3.2)$$

For ${}^{(Y_{(l)})}\pi$ we find that

$$\begin{aligned} & \left| \mathfrak{L}_{\mathcal{P}}^{N-1} \not{A}_{\check{X}}^{(Y)} \not{A}_L^{\#} + \left(\nabla^{2\#} \mathcal{P}^{N-1} \mu \right) \cdot Y_{(l)} \right|, \left| \mathcal{P}^{N-1} \check{X} tr_{\mathfrak{g}}^{(Y_{(l)})} \not{A} - \rho_{(l)} \Delta \mathcal{P}^{N-1} \mu \right| \\ & \lesssim |\mathcal{L}_*^{\leq N+1;1} \Phi| + |\mathcal{L}_*^{\leq N;1} \gamma| + \varepsilon |\mathcal{P}^{[1,N]} \underline{\gamma}| \end{aligned} \quad (0.22.3.3)$$

$$\begin{aligned}
& \left| \mathcal{P}^{N-1} d\dot{v}^{(Y_{(l)})} \sharp_L^\# + Y_{(l)} \mathcal{P} tr_{\mathfrak{g}} \chi \right|, \left| \mathcal{P}^{N-1} d\dot{v}^{(Y_{(l)})} \sharp_{\check{X}}^\# - \left\{ \mathfrak{u} Y_{(l)} \mathcal{P}^{N-1} tr_{\mathfrak{g}} \chi + \rho_{(l)} \Delta \mathcal{P}^{N-1} \mathfrak{u} \right\} \right|, \\
& \left| \mathfrak{L}_{\mathcal{P}}^{N-1} \mathfrak{d}^\#(Y_{(l)}) \pi_{L\check{X}} + \left(\nabla^{2\#} \mathcal{P}^{N-1} \mathfrak{u} \right) \cdot Y_{(l)} \right|, \left| \mathfrak{L}_{\mathcal{P}}^{N-1} d\dot{v}^{(Y_{(l)})} \widehat{\sharp}^{\#\#} - 2\rho_{(l)} \mathfrak{d}^\# \mathcal{P}^{N-1} tr_{\mathfrak{g}} \chi \right| \\
& \lesssim |\mathcal{L}_*^{\leq N+1;1} \Phi| + |\mathcal{L}_*^{\leq N;1} \gamma| + \varepsilon |\mathcal{P}^{[1,N]} \underline{\gamma}|
\end{aligned} \tag{0.22.3.4}$$

Lemma 0.22.3.2 (Estimates for the Easy to Bound Derivatives of $^{(L)}\pi$ and $^{(Y)}\pi$). *Let $P \in \mathcal{P}$.*

Under our bootstrap and smallness assumptions we find for the following top-order terms that

$$\begin{aligned}
& |\mathcal{P}^{\leq N-1} L tr_{\mathfrak{g}}^{(P)} \sharp|, |\mathcal{P}^{\leq N-1} L^{(P)} \pi_{L\check{X}}|, |\mathcal{P}^{\leq N-1} L^{(P)} \pi_{\check{X}X}|, |\mathfrak{L}_{\mathcal{P}}^{\leq N-1} \mathfrak{L}_L^{(P)} \sharp_L^\#|, |\mathfrak{L}_{\mathcal{P}}^{\leq N-1} \mathfrak{L}_L^{(P)} \sharp_{\check{X}}^\#| \\
& \lesssim |\mathcal{L}_*^{\leq N+1;1} \Phi| + |\mathcal{P}^{\leq N} \gamma| + |\mathcal{P}^{[1,N]} \underline{\gamma}|
\end{aligned} \tag{0.22.3.5}$$

while for the below top order terms we have

$$\begin{aligned}
& |\mathcal{P}^{\leq N-1(P)} \sharp|, |\mathcal{P}^{\leq N-1(P)} \widehat{\sharp}|, |\mathcal{P}^{\leq N-1} tr_{\mathfrak{g}}^{(P)} \sharp|, |\mathcal{P}^{\leq N-1(P)} \pi_{L\check{X}}|, \\
& |\mathcal{P}^{\leq N-1(P)} \pi_{\check{X}X}|, |\mathfrak{L}_{\mathcal{P}}^{\leq N-1(P)} \sharp_L^\#|, |\mathfrak{L}_{\mathcal{P}}^{\leq N-1(P)} \sharp_{\check{X}}^\#| \\
& \lesssim |\mathcal{L}_*^{\leq N;1} \Phi| + |\mathcal{P}^{\leq N} \gamma| + |\mathcal{P}^{[1,N]} \underline{\gamma}| + 1
\end{aligned} \tag{0.22.3.6}$$

0.22.4 Proof of (0.22.2.1)

Proof. The proof is essentially the same as that in [18], with a few minor modifications accounted for by using our above lemmas (0.22.3.1) and (0.22.3.2) on pointwise estimates (which account for the fact that in our case $\mathcal{P}_{t,u}$ is 2-dimensional).

We find that:

$$Y^{N-1} \mathfrak{K}_{(\pi\text{-Danger})}^{(Z)}[\Phi] = \begin{cases} (\check{X}\Phi)(Y_{(l)} Y^{N-1} tr_{\mathfrak{g}} \chi) + \text{Harmless}^{\leq N} & Z = Y_{(l)} \\ \text{Harmless}^{\leq N} & Z = L \end{cases} \tag{0.22.4.1}$$

$$Y^{N-1} \mathfrak{K}_{(\pi\text{-Cancel-1})}^{(Z)}[\Phi] = \text{Harmless}^{\leq N} \tag{0.22.4.2}$$

$$Y^{N-1} \mathfrak{K}_{(\pi\text{-Cancel-2})}^{(Z)}[\Phi] = \text{Harmless}^{\leq N} \tag{0.22.4.3}$$

$$Y^{N-1}\mathfrak{K}_{(\pi-\text{Elliptic})}^{(Z)}[\Phi] = \begin{cases} \mu\rho_{(l)}(\not{d}Y^{N-1}\text{tr}_{\not{g}}\chi) \cdot \not{d}^{\#}\Phi + \text{Harmless}^{\leq N} & Z = Y_{(l)} \\ \mu(\not{d}Y^{N-1}\text{tr}_{\not{g}}\chi) \cdot \not{d}^{\#}\Phi + \text{Harmless}^{\leq N} & Z = L \end{cases} \quad (0.22.4.4)$$

$$Y^{N-1}\mathfrak{K}_{(\pi-\text{Good})}^{(Z)}[\Phi] = \text{Harmless}^{\leq N} \quad (0.22.4.5)$$

$$Y^{N-1}\mathfrak{K}_{(\Phi)}^{(Z)}[\Phi] = \text{Harmless}^{\leq N} \quad (0.22.4.6)$$

$$Y^{N-1}\mathfrak{K}_{(\text{Low})}^{(Z)}[\Phi] = \text{Harmless}^{\leq N} \quad (0.22.4.7)$$

The proof that the other $\mathscr{P}^N$ commutators are harmless is as in [18]. □

0.22.5 Pointwise Estimates For Error Integrands Arising From $^{(T)}\pi$

Remark 0.22.5.1. Note that the following pointwise bound utilizes the estimate (0.21.0.19) to control the dangerous bulk term

$$\mu|\not{d}\Phi|^2 \frac{\check{X}\mu}{\mu}$$

with

$$\frac{1}{\sqrt{T_{(Boot)} - t}} \mu|\not{d}\Phi|^2.$$

This bound is the only reason we require control on the initial quantity $\|\check{X}^3\Phi\|_{L^\infty(\Sigma_0^1)}$!

Lemma 0.22.5.1. *Under our bootstrap and smallness assumptions and with $\varsigma > 0$ we have (where the implicit constant in $\lesssim$ does not depend on ς), without taking the absolute value of the LHS, that:*

$$\sum_{i=1}^6 \mathfrak{E}_{(i)}^{(T)}[\Phi] \lesssim (1 + \varsigma^{-1})(L\Phi)^2 + (1 + \varsigma^{-1})(\check{X}\Phi)^2 + \mu|\not{d}\Phi|^2 + \varsigma\mathring{\mathfrak{d}}_*|\not{d}\Phi|^2 \quad (0.22.5.1)$$

$$+ \frac{1}{\sqrt{T_{(Boot)} - t}} \mu|\not{d}\Phi|^2 \quad (0.22.5.2)$$

Proof. Essentially the same as in [18]. □

0.22.6 Pointwise Estimates Needed To Close Elliptic Estimates

Lemma 0.22.6.1. *Under our bootstrap and smallness assumptions we have that for $1 \leq N \leq 19$:*

$$\left| \nabla \mathcal{L}_{\mathcal{P}}^{N-1} \chi - \left\{ \nabla \mathcal{L}_{\mathcal{P}}^{N-1} \hat{\chi} + \frac{1}{2} (\not{d} \mathcal{P}^{N-1} \text{tr}_{\not{g}} \chi) \not{g} \right\} \right| \lesssim |\mathcal{P}^N \gamma|, \quad (0.22.6.1)$$

$$\left| \nabla \mathcal{L}_{\mathcal{P}}^{N-1} \hat{\chi} - \nabla \hat{\mathcal{L}}_{\mathcal{P}}^{N-1} \hat{\chi} \right| \lesssim \varepsilon |\mathcal{P}^{\leq N} \gamma|, \quad (0.22.6.2)$$

$$|Y_{(l)} \mathcal{P}^{N-1} \text{tr}_{\not{g}} \chi| \lesssim |\nabla \mathcal{L}_{\mathcal{P}}^{N-1} \chi| + \varepsilon, \quad (0.22.6.3)$$

$$\left| \hat{\mathcal{L}}_{\mathcal{P}}^{N-1} \hat{\chi} \right| \lesssim |\mathcal{P}^{\leq N} \gamma|. \quad (0.22.6.4)$$

Proof. For the first two estimates, we merely need to check that the top order terms cancel. The second two estimates are straightforward. $\square$

0.23 Pointwise estimates for Worst Error Integrands

Lemma 0.23.0.1. *For $N \leq 19$ and under our smallness and bootstrap assumptions we find that:*

$$|\mathcal{P}^{\leq N} \mathfrak{X}| \lesssim |\mathcal{L}_*^{\leq N+1;1} \Phi| + |\mathcal{P}^{\leq N} \gamma| + |\mathcal{P}_*^{[1,N]} \underline{\gamma}| \quad (0.23.0.1)$$

$$|\mathcal{P}^N \mathfrak{X} + G_{LL} \check{X} \mathcal{P}^N \Phi| \lesssim \mu |\mathcal{P}^{\leq N+1} \Phi| + |\mathcal{L}_*^{\leq N;1} \Phi| + |\mathcal{P}^{\leq N} \gamma| + |\mathcal{P}_*^{[1,N]} \underline{\gamma}| \quad (0.23.0.2)$$

$$|\mathcal{P}^{N-1(P)} \mathfrak{J}| \lesssim \varepsilon \left(|\mathcal{L}_*^{\leq N+1;1} \Phi| + |\mathcal{P}^{\leq N} \gamma| + |\mathcal{P}_*^{[1,N]} \underline{\gamma}| \right) \quad (0.23.0.3)$$

$$|^{(\mathcal{P}^N)} \mathfrak{J}| \lesssim \varepsilon \left(|\mathcal{L}_*^{\leq N+1;1} \Phi| + |\mathcal{P}^{\leq N} \gamma| + |\mathcal{P}_*^{[1,N]} \underline{\gamma}| \right) \quad (0.23.0.4)$$

$$|\mathcal{P}^N \tilde{\mathfrak{X}}| \lesssim |\mathcal{P}^{\leq N+1} \Phi| + |\mathcal{P}^{\leq N} \gamma| \quad (0.23.0.5)$$

$$|P^{(\mathcal{P}^{N-1})} \tilde{\mathfrak{X}}| \lesssim |\mathcal{P}_*^{\leq N+1} \Phi| + \varepsilon |\mathcal{P}^{\leq N} \Phi| \quad (0.23.0.6)$$

$$|^{(\mathcal{P}^{N-1})} \mathfrak{B}| \lesssim \varepsilon |\mathcal{P}^{\leq N} \gamma| \quad (0.23.0.7)$$

Proof. The only estimates not included in [18] are those for $|\mathcal{P}^{N-1(P)} \mathfrak{J}|$ and $|^{(\mathcal{P}^N)} \mathfrak{J}|$.

The estimate for $|\mathcal{P}^{N-1(P)} \mathfrak{J}|$ is rather straightforward if we write

$$\begin{aligned} ^{(P)} \mathfrak{J} &= -P(f(\underline{\gamma}, \not{g}^{-1}, \not{d} x^i, \check{X} \Phi, P \Phi) P \Phi) + \frac{1}{2} (\text{tr}_{\not{g}} \chi) P(2L\mu + \mathfrak{X}) \\ &\quad - (P\mu) \left\{ L \text{tr}_{\not{g}} \chi + \frac{1}{2} (\text{tr}_{\not{g}} \chi)^2 \right\} + [L, P] \mathfrak{X}, \end{aligned} \quad (0.23.0.8)$$

then make use of the Leibniz rule, the estimate for $\mathcal{P}^{\leq N} \mathfrak{X}$ and our previous pointwise estimates.

The estimate for $|^{(\mathcal{P}^N)} \mathfrak{J}|$ is a bit more tedious to check. We have that

1. (using our commutator estimates)

$$|[L, \mathcal{P}^{N-1}]P\mathfrak{X}| \lesssim \varepsilon |\mathcal{P}_*^{[1, N-1]}P\mathfrak{X}| + |\mathcal{P}^{[1, \lfloor (N-1)/2 \rfloor]}| |\mathcal{P}^{\leq N}\gamma| \quad (0.23.0.9)$$

2.

$$\begin{aligned} & \left| \mathcal{P}^{N-1}(\mu \text{tr}_{\mathcal{G}} \chi P \text{tr}_{\mathcal{G}} \chi) - \mu \text{tr}_{\mathcal{G}} \chi \mathcal{P}^N \text{tr}_{\mathcal{G}} \chi \right| \\ & \lesssim \sum_{\substack{N_1+N_2+N_3 \leq N-1 \\ N_3 \leq N-2}} |\mathcal{P}^{N_1} \mu| |\mathcal{P}^{N_2} \text{tr}_{\mathcal{G}} \chi| |\mathcal{P}^{N_3} P \text{tr}_{\mathcal{G}} \chi| \end{aligned} \quad (0.23.0.10)$$

3.

$$\begin{aligned} & \left| \mathcal{P}^{N-1}(\text{tr}_{\mathcal{G}} \chi P \mathfrak{X}) - \text{tr}_{\mathcal{G}} \chi \mathcal{P}^N \mathfrak{X} \right| \\ & \lesssim \sum_{\substack{N_1+N_2 \leq N-1 \\ N_2 \leq N-2}} |\mathcal{P}^{N_1} \text{tr}_{\mathcal{G}} \chi| |\mathcal{P}^{N_2} P \mathfrak{X}| \end{aligned} \quad (0.23.0.11)$$

4.

$$\begin{aligned} & \left| \mathcal{P}^{N-1}(L \mu P \text{tr}_{\mathcal{G}} \chi) - L \mu \mathcal{P}^N \text{tr}_{\mathcal{G}} \chi \right| \\ & \lesssim \sum_{\substack{N_1+N_2 \leq N-1 \\ N_2 \leq N-2}} |\mathcal{P}^{N_1} L \mu| |\mathcal{P}^{N_2} P \text{tr}_{\mathcal{G}} \chi| \end{aligned} \quad (0.23.0.12)$$

5.

$$\begin{aligned} & \left| \mathcal{P}^{N-1}(\mu P L \text{tr}_{\mathcal{G}} \chi) - \mu \mathcal{P}^N L \text{tr}_{\mathcal{G}} \chi \right| \\ & \lesssim \sum_{\substack{N_1+N_2 \leq N-1 \\ N_2 \leq N-2}} |\mathcal{P}^{N_1} \mu| |\mathcal{P}^{N_2} P L \text{tr}_{\mathcal{G}} \chi| \end{aligned} \quad (0.23.0.13)$$

6.

$$\begin{aligned} & \left| 2\mu \widehat{\chi} \cdot \mathcal{L}_{\mathcal{P}}^N \widehat{\chi} - \mathcal{P}^N(\mu |\widehat{\chi}|^2) \right| \\ & \lesssim \sum_{\substack{N_1+N_2+N_3+N_4+N_5 \leq N \\ N_4+N_5 \leq N-1}} |\mathcal{P}^{N_1} \mu| |\mathcal{L}_{\mathcal{P}}^{N_2} \mathcal{G}^{-1}| |\mathcal{L}_{\mathcal{P}}^{N_3} \mathcal{G}^{-1}| |\mathcal{L}_{\mathcal{P}}^{N_4} \widehat{\chi}| |\mathcal{L}_{\mathcal{P}}^{N_5} \widehat{\chi}| \end{aligned} \quad (0.23.0.14)$$

Now apply (0.19.3.1) and (0.23.0.1) and our bootstrap assumptions to conclude. $\square$

Remark 0.23.0.1. The following pointwise estimates for the modified top-order terms have basi-

cally the same as the proof as in [18], except we now have to worry about the top order eikonal terms $\widehat{\chi} \cdot \mathcal{L}^N \widehat{\chi}$.

Lemma 0.23.0.2. *For $1 \leq N \leq 19$ and with $\mathcal{P}^N = Y^N$, under our smallness and bootstrap assumptions we find that:*

$$\begin{aligned}
\left| (Y^N) \mathcal{X} \right| (t, u, \theta) &\leq C \left| \mathcal{X} \right| (0, u, \theta) \\
&+ 2(1 + C\varepsilon) \int_{s=0}^t \frac{[L\mu]_-(s, u, \theta)}{\mu(s, u, \theta)} |Y^N \mathcal{X}|(s, u, \theta) ds \\
&+ 2(1 + C\varepsilon) \int_{s=0}^t \left(\frac{\mu(t, u, \theta)}{\mu(s, u, \theta)} \right)^2 |\widehat{\chi}| \cdot |\mu \mathcal{L}^{\leq N} \widehat{\chi}|(s, u, \theta) ds \\
&+ C \int_{s=0}^t \left\{ |\mathcal{L}_*^{\leq N+1;1} \Phi| + |\mathcal{P}^{\leq N} \gamma| + \left| \mathcal{P}_*^{[1,N]} \underline{\gamma} \right| \right\} (s, u, \theta) ds
\end{aligned} \tag{0.23.0.15}$$

Lemma 0.23.0.3. *For $1 \leq N \leq 19$ and with $\mathcal{P}^N = Y^N$, under our smallness and bootstrap assumptions we find that:*

$$\begin{aligned}
\left| (\check{X}\Phi) Y^N \text{tr}_{\sharp} \chi \right| (t, u, \theta) &\leq 2 \left\| \frac{[L\mu]_-}{\mu} \right\|_{L^\infty(\Sigma_t^u)} |\check{X} Y^N \Phi|(t, u, \theta) \\
&+ 4(1 + C\varepsilon) \frac{\left\| [L\mu]_- \right\|_{L^\infty(\Sigma_t^u)}}{\mu_*(t, u)} \int_{s=0}^t \frac{\left\| [L\mu]_- \right\|_{L^\infty(\Sigma_s^u)}}{\mu_*(s, u)} |\check{X} Y^N \Phi|(s, u, \theta) ds \\
&+ \frac{C\varepsilon}{\mu_*(t, u)} \int_{s=0}^t \left| \mu \nabla \mathcal{L}^{\leq N-1} \widehat{\chi} \right| (s, u, \theta) ds \\
&+ \text{Error}
\end{aligned} \tag{0.23.0.16}$$

where

$$\begin{aligned}
|\text{Error}|(t, u, \theta) &\lesssim \frac{1}{\mu_*(t, u)} \left| (Y^N) \mathcal{X} \right| (0, u, \theta) + |\mathcal{L}_*^{\leq N+1;1} \Phi|(t, u, \theta) \\
&+ \frac{1}{\mu_*(t, u)} |\mathcal{L}_*^{\leq N;1} \Phi|(t, u, \theta) \\
&+ \frac{1}{\mu_*(t, u)} \left\{ |\mathcal{P}^{\leq N} \gamma| + \left| \mathcal{P}_*^{[1,N]} \underline{\gamma} \right| \right\} (t, u, \theta) \\
&+ \varepsilon \frac{1}{\mu_*(t, u)} \int_{s=0}^t \left(\frac{1}{\mu_*(s, u)} |\mathcal{L}_*^{\leq N+1;1} \Phi|(s, u, \theta) \right) + |\mu \mathcal{L}^{\leq N-1} \widehat{\chi}|(s, u, \theta) ds \\
&+ \frac{1}{\mu_*(t, u)} \int_{s=0}^t |\mathcal{L}_*^{\leq N+1;1} \Phi|(s, u, \theta) ds \\
&+ \frac{1}{\mu_*(t, u)} \int_{s=0}^t \frac{1}{\mu_*(s, u)} \left\{ |\mathcal{L}_*^{\leq N;1} \Phi| + |\mathcal{P}^{\leq N} \gamma| + \left| \mathcal{P}_*^{[1,N]} \underline{\gamma} \right| \right\} (s, u, \theta) ds
\end{aligned} \tag{0.23.0.17}$$

Lemma 0.23.0.4 (Nonsharp Pointwise Estimate for $\mu Y^N \text{tr}_{\mathcal{G}} \chi$). *Under our smallness/bootstrap/data assumptions we find that:*

$$\begin{aligned}
|\mu Y^N \text{tr}_{\mathcal{G}} \chi| &\lesssim \left| {}^{(Y^N)} \mathcal{X} \right| (0, u, \theta) + \mu \left| \mathcal{P}^{N+1} \Phi \right| (t, u, \theta) \\
&+ \left| \check{X} \mathcal{P}^N \Phi \right| (t, u, \theta) + \left| \mathcal{L}_*^{\leq N;1} \right| (t, u, \theta) \\
&+ \left\{ \left| \mathcal{P}^{\leq N} \gamma \right| + \left| \mathcal{P}_*^{[1,N]} \underline{\gamma} \right| \right\} (t, u, \theta) \\
&+ \int_{s=0}^t \frac{1}{\mu_*(s, u)} \left| \check{X} \mathcal{P}^N \Phi \right| (s, u, \theta) ds \\
&+ \int_{s=0}^t \left| \mathcal{L}_*^{\leq N+1;1} \Phi \right| (s, u, \theta) ds + \varepsilon \int_{s=0}^t \left| \mu \mathcal{L}_{\mathcal{P}}^{\leq N} \hat{\chi} \right| (s, u, \theta) ds \\
&+ \int_{s=0}^t \frac{1}{\mu_*(s, u)} \left\{ \left| \mathcal{L}_*^{\leq N;1} \Phi \right| + \left| \mathcal{P}^{\leq N} \gamma \right| + \left| \mathcal{P}_*^{[1,N]} \underline{\gamma} \right| \right\} (s, u, \theta) ds
\end{aligned} \tag{0.23.0.18}$$

Lemma 0.23.0.5 (Pointwise estimates for $\widetilde{\mathcal{X}}$). *With our smallness/bootstrap/data assumptions we have that:*

$$\left| L^{(Y^{N-1})} \widetilde{\mathcal{X}} \right| \leq \frac{1}{2} |G_{LL}| |\Delta Y^{N-1} \Phi| + C\varepsilon |\mathcal{P}^{\leq N} \gamma| \tag{0.23.0.19}$$

$$\begin{aligned}
\left| {}^{(Y^{N-1})} \widetilde{\mathcal{X}} \right| (t, u, \theta) &\leq \left| {}^{(Y^{N-1})} \widetilde{\mathcal{X}} \right| (0, u, \theta) + \frac{1}{2} |G_{LL}| (t, u, \theta) \int_{s=0}^t |\Delta Y^{N-1} \Phi| (s, u, \theta) ds \\
&+ C\varepsilon \int_{s=0}^t \left\{ \left| \mathcal{P}^{\leq N+1} \Phi \right| + \left| \mathcal{P}^{\leq N} \gamma \right| \right\} (s, u, \theta) ds.
\end{aligned} \tag{0.23.0.20}$$

0.24 Elliptic Estimates and Sobolev Embedding on $\mathcal{P}_{t,u}$ surfaces (Corresponding to Section 18 in [16])

In order to bound the top-order derivatives of $\nabla \chi$ in our top-order a priori L^2 estimates, we will need to use the following elliptic estimates.

The following lemma is standard:

Lemma 0.24.0.1. *We have that*

$$\mathfrak{R}_{ABCD} = \mathfrak{K} \left\{ \mathcal{G}_{AC} \mathcal{G}_{BD} - \mathcal{G}_{AD} \mathcal{G}_{BC} \right\} \tag{0.24.0.1}$$

where $\mathfrak{R}_{ABCD}$ is the Riemann curvature of $\mathcal{G}$ on $\mathcal{P}_{t,u}$ and $\mathfrak{K}$ is its Gauss curvature

The following is identical to lemma 18.4 in [16]

Lemma 0.24.0.2. *Let*

$$\mathfrak{R}_{AB} := (\mathfrak{g}^{-1})^{CD} \mathfrak{R}_{ACBD} \quad \text{and} \quad \mathfrak{R} := (\mathfrak{g}^{-1})^{AB} \mathfrak{R}_{AB}$$

denote the Ricci curvature and the scalar curvature of $\mathfrak{g}$. Let $(\lambda)\mathfrak{g}$ be a one parameter family of Riemannian metrics on $\mathcal{P}_{t,u}$ with $(0)\mathfrak{g} = \mathfrak{g}$ and denote $\dot{\mathfrak{g}} := \frac{d}{d\lambda}|_{\lambda=0} (\lambda)\mathfrak{g}$. We let $(\lambda)\mathfrak{R}$ denote the scalar curvature of $(\lambda)\mathfrak{g}$ and let $\dot{\mathfrak{R}} := \frac{d}{d\lambda}|_{\lambda=0}$. Then for the first variation of the scalar curvature we have that:

$$\dot{\mathfrak{R}} = -\mathfrak{R}^{AB} \dot{\mathfrak{g}}_{AB} - \Delta \text{tr}_{\mathfrak{g}} \dot{\mathfrak{g}} + (\mathfrak{g}^{-1})^{AB} (\mathfrak{g}^{-1})^{CD} \nabla_{AC}^2 \dot{\mathfrak{g}}_{BD} \quad (0.24.0.2)$$

The following corollary is immediate:

Corollary 0.24.0.3.

$$\check{X}\mathfrak{R} = -\mathfrak{R}^{AB(\check{X})} \check{\nabla}_{AB} - \Delta \text{tr}_{\mathfrak{g}}^{(\check{X})} \check{\nabla} + \nabla_{AB}^2 (\check{X}) \check{\nabla}^{AB}. \quad (0.24.0.3)$$

From this we can derive that

Lemma 0.24.0.4. *Under our small data and bootstrap assumptions we find that in the bootstrap region we have:*

$$|\mathfrak{K}| \lesssim \varepsilon \quad (0.24.0.4)$$

Proof. Note that as the above inequality is invariant under coordinate changes, we may choose coordinates on $\Sigma_t^{U_0}$ such that $\check{X} = \frac{\partial}{\partial u}$. Now we may use the fact that $\mathfrak{R}_{AB} = \mathfrak{K} \mathfrak{g}_{AB}$ and that $\mathfrak{R} = 2\mathfrak{K}$ in conjunction with the above lemma and Gronwall to derive that

$$|\mathfrak{K}| \lesssim \left(\int \left| \nabla^{2(\check{X})} \check{\nabla} \right| du \right) e^{c \int |\text{tr}_{\mathfrak{g}}^{(\check{X})} \check{\nabla}| du} \lesssim \varepsilon \quad (0.24.0.5)$$

□

The proofs of the following elliptic estimates are exactly as in [16]:

Lemma 0.24.0.5 (Elliptic Estimate for Solutions of the Poisson Equation on $\mathcal{P}_{t,u}$).

$$\int_{\mathcal{P}_{t,u}} \mu^2 \left\{ \frac{1}{2} |\nabla^2 f|^2 + \mathfrak{K} |\mathfrak{d}f|^2 \right\} dv_{\mathfrak{g}} \leq 2 \int_{\mathcal{P}_{t,u}} \mu^2 |\Delta f|^2 dv_{\mathfrak{g}} + 3 \int_{\mathcal{P}_{t,u}} |\mathfrak{d}\mu|^2 |\mathfrak{d}f|^2 dv_{\mathfrak{g}} \quad (0.24.0.6)$$

Lemma 0.24.0.6 (Elliptic Estimate for Symmetric Traceless $(0, 2)$ Tensorfields on $\mathcal{P}_{t,u}$).

$$\int_{\mathcal{P}_{t,u}} \mu^2 \left\{ \frac{1}{2} |\nabla \xi|^2 + 2\mathfrak{K} |\xi|^2 \right\} dv_{\mathcal{G}} \leq 3 \int_{\mathcal{P}_{t,u}} \mu^2 |\mathring{d}\nabla \xi|^2 dv_{\mathcal{G}} + 3 \int_{\mathcal{P}_{t,u}} |\mathring{d}\mu|^2 |\xi|^2 dv_{\mathcal{G}} \quad (0.24.0.7)$$

Remark 0.24.0.1. Unlike Speck/Christodoulou in the dispersive regime, we do not have a sign on the curvature $\mathfrak{K}$, we only have smallness of $\mathfrak{K}$. This is only enough to give us

$$\|\mu |\nabla \xi|\|_{L^2(\mathcal{P}_{t,u})} \lesssim \|\mu |\mathring{d}\nabla \xi|\|_{L^2(\mathcal{P}_{t,u})} + \varepsilon^{1/2} \|\xi\|_{L^2(\mathcal{P}_{t,u})},$$

whereas they have a $\varepsilon \|\xi\|_{L^2(\mathcal{P}_{t,u})}$ term on the RHS. It turns out that this loss of some smallness is unimportant for our purposes.

As in [18], the following estimate is essentially a consequence of the standard Sobolev embedding estimate on $\mathbb{T}^2$ together with the smallness of Φ , which gives us that $\mathcal{G}$ is approximately flat (with volume form approximately equal to the flat torus volume form, up to an $\mathcal{O}(\varepsilon)$ correction) and also that $|\Theta^{\leq 2} f| \sim_{\varepsilon} |Y^{\leq 2} f|$ for any scalar function f on $\mathcal{P}_{t,u}$.

Lemma 0.24.0.7 (Sobolev Embedding on $\mathcal{P}_{t,u}$).

$$\|f\|_{L^\infty(\mathcal{P}_{t,u})} \leq C \|Y^{\leq 2} f\|_{L^2(\mathcal{P}_{t,u})} \quad (0.24.0.8)$$

0.25 Coerciveness and L^2 -Controlling Quantities (Corresponding to Section 13 in [18])

This section we present the higher order energy-flux quantities which we utilize in the energy argument. They give us coercive control on certain homogeneous Sobolev norms of the solution which will ultimately allow us to improve our L^∞ bootstrap assumptions via Sobolev imbedding.

Definition 0.25.0.1.

$$\mathbb{Q}_N(t, u) := \max_{|\bar{I}|=N} \sup_{(t', u') \in [0, t] \times [0, u]} \{ \mathbb{E}[\mathcal{P}^{\bar{I}}\Phi](t', u') + \mathbb{F}[\mathcal{P}^{\bar{I}}\Phi](t', u') \} \quad (0.25.0.1)$$

$$\mathbb{Q}_{[1, N]}(t, u) := \max_{1 \leq M \leq N} \mathbb{Q}_M(t, u) \quad (0.25.0.2)$$

$$\mathbb{K}[\Phi](t, u) := \frac{1}{2} \int_{\mathcal{M}_{t, u}} [L\bar{\mu}]_- |\not{d}\Phi|^2 d\omega \quad (0.25.0.3)$$

$$\mathbb{K}_N(t, u) := \max_{|I|=N} \mathbb{K}[\mathcal{P}^{\bar{I}}\Phi](t, u) \quad (0.25.0.4)$$

$$\mathbb{K}_{[1, N]}(t, u) := \max_{1 \leq M \leq N} \mathbb{K}_M(t, u) \quad (0.25.0.5)$$

The following bound is crucial for controlling the energies of angular derivatives in the regions where $\bar{\mu}$ is small.

Lemma 0.25.0.1 (Coercive Spacetime Integral Bound).

$$\mathbb{K}[\Phi](t, u) \geq \frac{1}{8} \mathring{\delta}_* \int_{\mathcal{M}_{t, u}} \mathbf{1}_{\{\bar{\mu} \leq 1/4\}} |\not{d}\Phi|^2 d\omega \quad (0.25.0.6)$$

The following lemma is Lemma 13.2 in [18], with effectively the same proof:

Lemma 0.25.0.2. *Let f be a scalar function on $\mathcal{M}_{T_{(Boot)}, U_0}$ and let*

$$F(t, u, \theta) := \int_{t'=0}^t f(t', u, \theta) dt'. \quad (0.25.0.7)$$

Then under our data/smallness/bootstrap assumptions we have the following estimate for $(t, u) \in [0, T_{(Boot)}] \times [0, U_0]$:

$$\|F\|_{L^2(\Sigma_t^u)} \leq (1 + C\varepsilon) \int_{t'=0}^t \|f\|_{L^2(\Sigma_{t'}^u)} dt'. \quad (0.25.0.8)$$

Lemma 0.25.0.3 (Coercivity of $\mathbb{Q}_{[1, N]}$). *Under our bootstrap/smallness assumptions we have for $1 \leq M \leq N \leq 19$ that on $(t, u) \in [0, T_{(Boot)}] \times [0, U_0]$ the following coercive bounds hold:*

$$\begin{aligned} \mathbb{Q}_{[1, N]}(t, u) \geq \max \left\{ \frac{1}{2} \|\sqrt{\bar{\mu}} L \mathcal{P}^M \Phi\|_{L^2(\Sigma_t^u)}^2, \|\check{X} \mathcal{P}^M \Phi\|_{L^2(\Sigma_t^u)}^2, \frac{1}{2} \|\sqrt{\bar{\mu}} \not{d} \mathcal{P}^M \Phi\|_{L^2(\Sigma_t^u)}^2, \right. \\ \left. C^{-1} \|\mathcal{P}^M \Phi\|_{L^2(\Sigma_t^u)}^2, \|L \mathcal{P}^M \Phi\|_{L^2(C_t^u)}^2, \|\sqrt{\bar{\mu}} \not{d} \mathcal{P}^M \Phi\|_{L^2(C_t^u)}^2, \right. \\ \left. C^{-1} \|\mathcal{P}^M \Phi\|_{L^2(\mathcal{P}_{t, u})}^2 \right\} \quad (0.25.0.9) \end{aligned}$$

$$\|\Phi\|_{L^2(\Sigma_t^u)} \leq C\check{\alpha} + C\mathbb{Q}_1^{1/2}(t, u),$$

$$\|\check{X}\Phi\|_{L^2(\Sigma_t^u)} \leq \|\check{X}\Phi\|_{L^2(\Sigma_0^u)} + C\check{\epsilon} + C\mathbb{Q}_1^{1/2}(t, u).$$

Lemma 0.25.0.4 (L^∞ bounds for Φ). *Under our smallness and bootstrap assumptions we have that*

$$\|\mathcal{P}^{[1,M]}\Phi\|_{L^\infty(\Sigma_t^u)} \lesssim \mathbb{Q}_{[1,M+2]}^{1/2}(t, u) + \epsilon \quad (0.25.0.10)$$

0.26 Energy Estimates

Remark 0.26.0.1. This section is really the crux of the argument. We derive a very complicated (and coupled) set of energy inequalities (0.26.7) for top-order terms (which do not lose derivatives but exhibit blowup near the shock $\mu = 0$) together with below-top-order estimates which lose derivatives. The below-top-order estimate is straightforward and follows from applying our L^∞ estimates to the error terms which arise in the bulk integral for the commuted wave equation. The top-order estimate is rather delicate and requires the use of the modified quantities. The full argument is treated in detail in [18]. Here we merely present a sketch of the argument which demonstrates how to apply the elliptic estimates which do not appear in the $2+1$ dimensional setting but are required in $3+1$ dimensions due to the nontriviality of $\hat{\chi}$. Basically the modified quantities allow us to estimate derivatives $P^N \text{tr}_g \chi$ without losing derivatives in Φ , but leave us with uncontrolled top-order error terms $\mathcal{L}_{\mathcal{P}}^N \hat{\chi}$. The above elliptic estimates on $\mathcal{P}_{t,u}$ allow us to bound those terms again by the estimate on $P^N \text{tr}_g \chi$, which, after applying Gronwall to eliminate the RHS $\mathcal{L}_{\mathcal{P}}^N \hat{\chi}$ terms gives us control on all of the top-order eikonal quantities in terms of the top-order energies.

0.26.1 Preliminary L^2 -Estimates for Below Top-Order Quantities

The quantities here are allowed to lose derivatives, and the following estimates are straightforward applications of our above L^∞ estimates in (0.19.6.2).

Lemma 0.26.1.1. *Under our smallness/bootstrap/data assumptions, for $1 \leq N \leq 19$ we have that:*

$$\begin{pmatrix} \|L\mathcal{P}_*^{[1,N]}\boldsymbol{\mu}\|_{L^2(\Sigma_t^u)} \\ \|L\mathcal{P}^{\leq N}L_{(Small)}^i\|_{L^2(\Sigma_t^u)} \\ \|L\mathcal{P}^{\leq N-1}tr_{\mathcal{G}}\chi\|_{L^2(\Sigma_t^u)} \\ \|\mathcal{L}_L\mathcal{L}^{\leq N-1}\chi\|_{L^2(\Sigma_t^u)} \\ \|\mathcal{L}_L\mathcal{L}^{\leq N-1}\widehat{\chi}\|_{L^2(\Sigma_t^u)} \end{pmatrix} \lesssim \varepsilon + \frac{\mathbb{Q}_{[1,N]}^{1/2}(t,u)}{\boldsymbol{\mu}_*^{1/2}(t,u)}, \quad (0.26.1.1)$$

$$\begin{pmatrix} \|L\mathcal{L}^{\leq N;1}L_{(Small)}^i\|_{L^2(\Sigma_t^u)} \\ \|L\mathcal{L}^{\leq N-1}tr_{\mathcal{G}}\chi\|_{L^2(\Sigma_t^u)} \\ \|\mathcal{L}_L\mathcal{L}^{\leq N-1;1}\chi\|_{L^2(\Sigma_t^u)} \\ \|\mathcal{L}_L\mathcal{L}^{\leq N-1;1}\widehat{\chi}\|_{L^2(\Sigma_t^u)} \end{pmatrix} \lesssim \varepsilon + \frac{\mathbb{Q}_{[1,N]}^{1/2}(t,u)}{\boldsymbol{\mu}_*^{1/2}(t,u)} \quad (0.26.1.2)$$

$$\begin{pmatrix} \|\mathcal{P}_*^{[1,N]}\boldsymbol{\mu}\|_{L^2(\Sigma_t^u)} \\ \|\mathcal{P}^{\leq N}L_{(Small)}^i\|_{L^2(\Sigma_t^u)} \\ \|\mathcal{P}^{\leq N-1}tr_{\mathcal{G}}\chi\|_{L^2(\Sigma_t^u)} \\ \|\mathcal{L}^{\leq N-1}\chi\|_{L^2(\Sigma_t^u)} \\ \|\mathcal{L}^{\leq N-1}\widehat{\chi}\|_{L^2(\Sigma_t^u)} \end{pmatrix} \lesssim \varepsilon + \int_0^t \frac{\mathbb{Q}_{[1,N]}^{1/2}(s,u)}{\boldsymbol{\mu}_*^{1/2}(s,u)} ds, \quad (0.26.1.3)$$

$$\begin{pmatrix} \|\mathcal{L}^{\leq N;1}L_{(Small)}^i\|_{L^2(\Sigma_t^u)} \\ \|\mathcal{L}^{\leq N-1}tr_{\mathcal{G}}\chi\|_{L^2(\Sigma_t^u)} \\ \|\mathcal{L}^{\leq N-1;1}\chi\|_{L^2(\Sigma_t^u)} \\ \|\mathcal{L}^{\leq N-1;1}\widehat{\chi}\|_{L^2(\Sigma_t^u)} \end{pmatrix} \lesssim \varepsilon + \int_0^t \frac{\mathbb{Q}_{[1,N]}^{1/2}(s,u)}{\boldsymbol{\mu}_*^{1/2}(s,u)} ds. \quad (0.26.1.4)$$

Corollary 0.26.1.2. *For $1 \leq N \leq 19$, under our data/smallness/bootstrap assumptions, we have that*

$$\|\mathcal{L}_*^{N;1}\Phi\|_{L^2(\Sigma_t^u)} \lesssim \mathbb{Q}_{[1,N]}^{1/2}(t,u) + \varepsilon \quad (0.26.1.5)$$

0.26.2 Easiest Error Integrals

The following are stated and proven exactly as in [18]:

Lemma 0.26.2.1. *For $1 \leq N \leq 19$ we have under our smallness/bootstrap/data assumptions that*

$$\mathbb{Q}_N(0,u) \lesssim \varepsilon^2 \quad (0.26.2.1)$$

Lemma 0.26.2.2. For $1 \leq N \leq 19$ we have under our smallness/bootstrap/data assumptions that

$$\begin{aligned} \int_{\mathcal{M}_{t,u}} \sum_{i=1}^6 \mathfrak{E}_i^{(T)}[\mathcal{P}^N \Phi] d\omega &\lesssim \int_0^t \frac{1}{\sqrt{T_{(Boot)} - s}} \mathbb{Q}_{[1,N]}(s, u) ds \\ &\quad + (1 + \varsigma^{-1}) \int_0^t \mathbb{Q}_{[1,N]}(s, u) ds \\ &\quad + (1 + \varsigma^{-1}) \int_0^u \mathbb{Q}_{[1,N]}(s, u') du' + \varsigma \mathbb{K}_{[1,N]}(t, u). \end{aligned} \quad (0.26.2.2)$$

Lemma 0.26.2.3. For $1 \leq N \leq 19$ we have under our smallness/bootstrap/data assumptions that

$$\begin{aligned} \int_{\mathcal{M}_{t,u}} \left| \begin{pmatrix} (1 + \mu)L\mathcal{P}^N \Phi \\ \check{X}\mathcal{P}^N \Phi \end{pmatrix} \right|_{|Harmless^{\leq N}|} d\omega &\lesssim (1 + \varsigma^{-1}) \int_0^t \mathbb{Q}_{[1,N]}(s, u) ds + (1 + \varsigma^{-1}) \int_0^u \mathbb{Q}_{[1,N]}(t, u) du \\ &\quad + \varsigma \mathbb{K}_{[1,N]}(t, u) + \varepsilon^2 \end{aligned} \quad (0.26.2.3)$$

$$\begin{aligned} \int_{\mathcal{M}_{t,u}} |\mathfrak{d}\mathcal{P}^N \Phi|_{|Harmless^{\leq N}|} d\omega &\lesssim \int_0^t \mathbb{Q}_{[1,N]}(s, u) ds + \int_0^u \mathbb{Q}_{[1,N]}(t, u) du + \mathbb{K}_{[1,N]}(t, u) + \varepsilon^2 \end{aligned} \quad (0.26.2.4)$$

0.26.3 Controlling Top-Order Derivatives of χ

Remark 0.26.3.1. This section is the main, crucial difference from [18], although the line of the argument basically follows [16]. See the above comments in (0.26.0.1) regarding the use of elliptic estimates in our $3 + 1$ dimensional setting.

Lemma 0.26.3.1 (Preliminary L^2 Bounds for Some Top-order Derivatives of χ). For $1 \leq N \leq 19$, under our smallness/bootstrap/data assumptions, we have that:

$$\begin{aligned} \left\| \mu \nabla \mathfrak{L}_{\mathcal{P}}^{N-1} \chi \right\|_{L^2(\Sigma_t^u)} &\lesssim \sum_{l=1}^2 \left\| \mu Y_{(l)} \mathcal{P}^{N-1} tr_{\sharp} \chi \right\|_{L^2(\Sigma_t^u)} \\ &\quad + \left\| \mu \nabla \widehat{\mathfrak{L}}_{\mathcal{P}}^{N-1} \widehat{\chi} \right\|_{L^2(\Sigma_t^u)} \\ &\quad + \mathbb{Q}_{[1,N]}^{1/2}(t, u) + \int_0^t \frac{1}{\mu_{\star}^{1/2}(s, u)} \mathbb{Q}_{[1,N]}^{1/2}(s, u) ds + \varepsilon \end{aligned} \quad (0.26.3.1)$$

$$\begin{aligned}
\left\| \boldsymbol{\mu} \not{d} \mathcal{P}^{N-1} \text{tr}_{\not{g}} \chi \right\|_{L^2(\Sigma_t^u)}, \left\| \boldsymbol{\mu} Y_{(l)} \mathcal{P}^{N-1} \text{tr}_{\not{g}} \chi \right\|_{L^2(\Sigma_t^u)} &\lesssim \left\| \boldsymbol{\mu} \nabla \not{L}^{\mathcal{P}^{N-1}} \chi \right\|_{L^2(\Sigma_t^u)} \\
&+ \mathbb{Q}_{[1,N]}^{1/2}(t, u) + \int_0^t \frac{1}{\boldsymbol{\mu}_\star^{1/2}(s, u)} \mathbb{Q}_{[1,N]}^{1/2}(s, u) ds + \varepsilon
\end{aligned} \tag{0.26.3.2}$$

$$\begin{aligned}
\left\| \boldsymbol{\mu} \nabla \not{L}^{\mathcal{P}^{N-1}} \hat{\chi} \right\|_{L^2(\Sigma_t^u)} &\lesssim \left\| \boldsymbol{\mu} \nabla \not{L}^{\mathcal{P}^{N-1}} \chi \right\|_{L^2(\Sigma_t^u)} \\
&+ \mathbb{Q}_{[1,N]}^{1/2}(t, u) + \int_0^t \frac{1}{\boldsymbol{\mu}_\star^{1/2}(s, u)} \mathbb{Q}_{[1,N]}^{1/2}(s, u) ds + \varepsilon
\end{aligned} \tag{0.26.3.3}$$

$$\begin{aligned}
\left\| \boldsymbol{\mu} d\not{v} \not{L}^{\mathcal{P}^{N-1}} \hat{\chi} \right\|_{L^2(\Sigma_t^u)} &\lesssim \sum_{l=1}^2 \left\| \boldsymbol{\mu} Y_{(l)} \mathcal{P}^{N-1} \text{tr}_{\not{g}} \chi \right\|_{L^2(\Sigma_t^u)} \\
&+ \mathbb{Q}_{[1,N]}^{1/2}(t, u) + \int_0^t \frac{1}{\boldsymbol{\mu}_\star^{1/2}(s, u)} \mathbb{Q}_{[1,N]}^{1/2}(s, u) ds + \varepsilon
\end{aligned} \tag{0.26.3.4}$$

$$\left\| \not{L}^{\mathcal{P}^{N-1}} \hat{\chi} \right\|_{L^2(\Sigma_t^u)} \lesssim \mathbb{Q}_{[1,N]}^{1/2}(t, u) + \int_0^t \frac{1}{\boldsymbol{\mu}_\star^{1/2}(s, u)} \mathbb{Q}_{[1,N]}^{1/2}(s, u) ds + \varepsilon \tag{0.26.3.5}$$

Proof. (idea) The proof, much as in Lemma 20.27 in [16], is a straightforward application of (0.22.6) and (0.22.1.3). $\square$

Lemma 0.26.3.2. [Preliminary Non-Sharp L^2 -Bounds for $\boldsymbol{\mu} \mathcal{P}^N \text{tr}_{\not{g}} \chi$] For $1 \leq N \leq 19$, under our smallness and bootstrap assumptions, we have that:

$$\begin{aligned}
\left\| \boldsymbol{\mu} \not{d} \mathcal{P}^{N-1} \text{tr}_{\not{g}} \chi \right\|_{L^2(\Sigma_t^u)} &\lesssim \mathbb{Q}_{[1,N]}^{1/2}(t, u) \\
&+ \int_0^t \frac{1}{\boldsymbol{\mu}_\star(s, u)} \mathbb{Q}_{[1,N]}^{1/2}(s, u) ds \\
&+ \varepsilon \{ \ln \boldsymbol{\mu}_\star^{-1}(t, u) + 1 \} \\
&+ \varepsilon \int_0^t \left\| \boldsymbol{\mu} \nabla \not{L}^{\leq N-1} \hat{\chi} \right\|_{L^2(\Sigma_s^u)} ds
\end{aligned} \tag{0.26.3.6}$$

Proof. (idea) This follows from taking $L^2(\Sigma_t^u)$ norms of our nonsharp pointwise bound (0.23.0.4). $\square$

Now note that from applying (0.26.3.1) to the bound (0.26.3.3) we have

$$\begin{aligned}
\left(\begin{array}{l} \left\| \boldsymbol{\mu} \nabla \mathcal{L}_{\mathcal{P}}^{\leq N-1} \chi \right\|_{L^2(\Sigma_t^u)} \\ \left\| \boldsymbol{\mu} \nabla \mathcal{L}_{\mathcal{P}}^{\leq N-1} \widehat{\chi} \right\|_{L^2(\Sigma_t^u)} \end{array} \right) &\lesssim \sum_{l=1}^2 \left\| \boldsymbol{\mu} Y_{(l)} \mathcal{P}^{N-1} \text{tr}_{\mathcal{P}} \chi \right\|_{L^2(\Sigma_t^u)} \\
&+ \left\| \boldsymbol{\mu} \nabla \widehat{\mathcal{L}}_{\mathcal{P}}^{N-1} \widehat{\chi} \right\|_{L^2(\Sigma_t^u)} \\
&+ \mathbb{Q}_{[1,N]}^{1/2}(t, u) + \int_0^t \frac{1}{\boldsymbol{\mu}_{\star}^{1/2}(s, u)} \mathbb{Q}_{[1,N]}^{1/2}(s, u) ds + \varepsilon
\end{aligned} \tag{0.26.3.7}$$

and that from our elliptic estimate (0.24.0.7) together with (0.26.3.4) and (0.26.3.5) we have that:

$$\begin{aligned}
\left\| \boldsymbol{\mu} \nabla \widehat{\mathcal{L}}_{\mathcal{P}}^{N-1} \widehat{\chi} \right\|_{L^2(\Sigma_t^u)} &\lesssim \left\| \boldsymbol{\mu} \text{div} \widehat{\mathcal{L}}_{\mathcal{P}}^{N-1} \widehat{\chi} \right\|_{L^2(\Sigma_t^u)} + \varepsilon^{1/2} \left\| \widehat{\mathcal{L}}_{\mathcal{P}}^{N-1} \widehat{\chi} \right\|_{L^2(\Sigma_t^u)} \\
&\lesssim \sum_{l=1}^2 \left\| \boldsymbol{\mu} Y_{(l)} \mathcal{P}^{N-1} \text{tr}_{\mathcal{P}} \chi \right\|_{L^2(\Sigma_t^u)} \\
&+ \mathbb{Q}_{[1,N]}^{1/2}(t, u) + \int_0^t \frac{1}{\boldsymbol{\mu}_{\star}^{1/2}(s, u)} \mathbb{Q}_{[1,N]}^{1/2}(s, u) ds + \varepsilon.
\end{aligned} \tag{0.26.3.8}$$

Now from (0.26.3.2), (0.26.3.7), and (0.26.3.8) we see that

$$\begin{aligned}
\left(\begin{array}{l} \left\| \boldsymbol{\mu} \mathcal{L}_{\mathcal{P}}^{\leq N-1} \text{tr}_{\mathcal{P}} \chi \right\|_{L^2(\Sigma_t^u)} \\ \left\| \boldsymbol{\mu} \nabla \mathcal{L}_{\mathcal{P}}^{\leq N-1} \chi \right\|_{L^2(\Sigma_t^u)} \\ \left\| \boldsymbol{\mu} \nabla \mathcal{L}_{\mathcal{P}}^{\leq N-1} \widehat{\chi} \right\|_{L^2(\Sigma_t^u)} \end{array} \right) &\lesssim \mathbb{Q}_{[1,N]}^{1/2}(t, u) \\
&+ \int_0^t \frac{1}{\boldsymbol{\mu}_{\star}(s, u)} \mathbb{Q}_{[1,N]}^{1/2}(s, u) ds \\
&+ \varepsilon \{ \ln \boldsymbol{\mu}_{\star}^{-1}(t, u) + 1 \} \\
&+ \varepsilon \int_0^t \left\| \boldsymbol{\mu} \nabla \mathcal{L}_{\mathcal{P}}^{\leq N-1} \chi \right\|_{L^2(\Sigma_s^u)} ds
\end{aligned} \tag{0.26.3.9}$$

Lemma 0.26.3.3 (Top Order Bounds on Derivatives of χ). *For $1 \leq N \leq 19$ we have, under our*

smallness and bootstrap assumptions, that:

$$\begin{aligned}
& \left(\begin{array}{l} \|\mu d\mathcal{P}^{\leq N-1} tr_{\mathcal{G}} \chi\|_{L^2(\Sigma_t^u)} \\ \|\mu \nabla \mathcal{L}^{\leq N-1} \chi\|_{L^2(\Sigma_t^u)} \\ \|\mu \nabla \mathcal{L}^{\leq N-1} \hat{\chi}\|_{L^2(\Sigma_t^u)} \end{array} \right) \lesssim \mathbb{Q}_{[1,N]}^{1/2}(t, u) \\
& + \int_0^t \frac{1}{\mu_{\star}(s, u)} \mathbb{Q}_{[1,N]}^{1/2}(s, u) ds \\
& + \varepsilon \{ \ln \mu_{\star}^{-1}(t, u) + 1 \}
\end{aligned} \tag{0.26.3.10}$$

$$\left\| \int_0^s \left(\begin{array}{l} |\mu d\mathcal{P}^{\leq N-1} tr_{\mathcal{G}} \chi|(s, \cdot) \\ |\mu \nabla \mathcal{L}^{\leq N-1} \chi|(s, \cdot) \\ |\mu \nabla \mathcal{L}^{\leq N-1} \hat{\chi}|(s, \cdot) \end{array} \right) ds \right\|_{L^2(\Sigma_t^u)} \lesssim \mathbb{Q}_{[1,N]}^{1/2}(t, u) + \varepsilon \tag{0.26.3.11}$$

Proof. Use (0.26.3.9) and Gronwall to derive (0.26.3.10). To derive (0.26.3.11) we use the monotonicity of $\mathbb{Q}$ and (0.21.0.36) to bound (0.26.3.10) by $\lesssim (1 + \ln \mu_{\star}^{-1}) \mathbb{Q}_{[1,N]}^{1/2}(t, u) + \varepsilon \{ \ln \mu_{\star}^{-1}(t, u) + 1 \}$. Now apply (0.25.0.8) and the bound $\int \ln \mu_{\star}^{-1} \leq B$ to conclude. $\square$

We are finally able to then prove the following central lemma which bounds the most dangerous error term without derivative loss:

Lemma 0.26.3.4 (L^2 bound for the most difficult top order term). *Under our smallness and bootstrap assumptions, for $1 \leq N \leq 19$, that*

$$\begin{aligned}
\|(\check{X}\Phi)Y^N tr_{\mathcal{G}} \chi\|_{L^2(\Sigma_t^u)} & \leq 2 \frac{\|[L\mu]_{-}\|_{L^{\infty}(\Sigma_t^u)}}{\mu_{\star}(t, u)} \mathbb{Q}_{[1,N]}^{1/2}(t, u) \\
& + 4.05 \frac{\|[L\mu]_{-}\|_{L^{\infty}(\Sigma_t^u)}}{\mu_{\star}(t, u)} \int_0^t \frac{\|[L\mu]_{-}\|_{L^{\infty}(\Sigma_s^u)}}{\mu_{\star}(s, u)} \mathbb{Q}_{[1,N]}^{1/2}(s, u) ds \\
& + C\varepsilon \frac{1}{\mu_{\star}(t, u)} \int_0^t \frac{1}{\mu_{\star}(s, u)} \mathbb{Q}_{[1,N]}^{1/2}(s, u) ds \\
& + C \frac{1}{\mu_{\star}(t, u)} \int_0^t \frac{1}{\mu_{\star}(s', u)} \int_0^{s'} \frac{1}{\mu_{\star}^{1/2}(s, u)} \mathbb{Q}_{[1,N]}^{1/2}(s, u) ds ds' \\
& + C \frac{1}{\mu_{\star}(t, u)} \int_0^t \frac{1}{\mu_{\star}^{1/2}(s, u)} \mathbb{Q}_{[1,N]}^{1/2}(s, u) ds \\
& + C\varepsilon \frac{1}{\mu_{\star}(t, u)} \mathbb{Q}_{[1,N]}^{1/2}(t, u) \\
& + C \frac{1}{\mu_{\star}^{1/2}(t, u)} \mathbb{Q}_{[1,N]}^{1/2}(t, u) + C \frac{1}{\mu_{\star}^{3/2}(t, u)} \mathbb{Q}_{[1,N-1]}^{1/2}(t, u) \\
& + C \frac{1}{\mu_{\star}^{3/2}(t, u)} \varepsilon
\end{aligned} \tag{0.26.3.12}$$

Proof. The proof is essentially as in [18] and derives from our sharp pointwise bound (0.23.0.3) on $(\check{X}\Phi)Y^N \text{tr}_{\not{g}}\chi$. There is, however, one crucial difference in our setting: the nontrivial top-order $\not{L}^N \hat{\chi}$ terms which appear on the RHS of (0.23.0.3). These can nonetheless be bounded using (0.26.3.11) from the previous lemma. Thus, due to our use of the above elliptic estimates, our lemma here differs from that in [18] only by the term $C\varepsilon \frac{1}{\mu_\star(t,u)} \mathbb{Q}_{[1,N]}^{1/2}(t,u)$. $\square$

0.26.4 Less Degenerate Top-Order Error Terms

Due to the fact that our estimate for $\|\mu Y^N \text{tr}_{\not{g}}\chi\|_{L^2(\Sigma_t^u)}$ (after the elliptic estimates, (0.26.3.10)) has the same bound as that in [18] (see equation (14.15) there), the following estimate is the same

Lemma 0.26.4.1. *Under our smallness and bootstrap assumptions, for $1 \leq N \leq 19$, that:*

$$\begin{aligned} & \left| \int_{\mathcal{M}_{t,u}} \rho_{(l)}(\check{X}Y^N\Phi)(\check{X}\Phi)(\not{d}^\# \Phi) \cdot (\mu \not{d}Y^{N-1} \text{tr}_{\not{g}}\chi) d\omega \right| \\ & \lesssim \int_0^t \{\ln \mu_\star^{-1}(s,u) + 1\}^2 \mathbb{Q}_{[1,N]}(s,u) du + \varepsilon^2, \end{aligned} \quad (0.26.4.1)$$

$$\begin{aligned} & \left| \int_{\mathcal{M}_{t,u}} (1 + 2\mu) \rho_{(l)}(LY^N\Phi)(\check{X}\Phi)(\not{d}^\# \Phi) \cdot (\mu \not{d}Y^{N-1} \text{tr}_{\not{g}}\chi) d\omega \right| \\ & \lesssim \int_0^t \{\ln \mu_\star^{-1}(s,u) + 1\}^2 \mathbb{Q}_{[1,N]}(s,u) ds + \int_0^u \mathbb{Q}_{[1,N]}(t,u') du' + \varepsilon^2. \end{aligned} \quad (0.26.4.2)$$

0.26.5 Error Integrals Requiring Integration By Parts In L

As our pointwise estimates (0.23.0.5) for the modified quantities $(Y^{N-1}) \widetilde{\mathcal{X}}$ are identical to the corresponding estimates in 2 + 1 dimensions (see Lemma 11.2 in [18]), the $\not{L}_{\mathcal{P}}^{TOP} \hat{\chi}$ quantities are not a problem in these terms. Thus the following estimates (and their proofs) are identical to those in [18].

Lemma 0.26.5.1. *Under our smallness and bootstrap assumptions, for $1 \leq N \leq 19$, that*

$$\begin{aligned} \left\| \frac{1}{\sqrt{\mu}} (\check{X}\Phi) L^{(Y^{N-1})} \widetilde{\mathcal{X}} \right\|_{L^2(\Sigma_t^u)} & \leq \sqrt{2} \frac{\| [L\mu]_- \|_{C^0(\Sigma_t^u)}}{\mu_\star(t,u)} \mathbb{Q}_{[1,N]}^{1/2}(t,u) \\ & + C \frac{1}{\mu_\star^{1/2}(t,u)} \mathbb{Q}_{[1,N]}^{1/2}(t,u) + C\varepsilon \frac{1}{\mu_\star(t,u)} \mathbb{Q}_{[1,N]}^{1/2}(t,u) \\ & + C\varepsilon \frac{1}{\mu_\star^{1/2}(t,u)} \end{aligned} \quad (0.26.5.1)$$

$$\begin{aligned}
\left\| \frac{1}{\sqrt{\mu}} (\check{X}\Phi)^{(Y^{N-1})} \widetilde{\mathcal{X}} \right\|_{L^2(\Sigma_t^u)} &\leq \sqrt{2} \| [L\mu]_- \|_{C^0(\Sigma_t^u)} \frac{1}{\mu_\star^{1/2}(t, u)} \int_0^t \frac{1}{\mu_\star^{1/2}(s, u)} \mathbb{Q}_{[1, N]}^{1/2}(s, u) ds \\
&+ C \int_0^t \frac{1}{\mu_\star^{1/2}(t, u)} \mathbb{Q}_{[1, N]}^{1/2}(t, u) \\
&+ C\varepsilon \frac{1}{\mu_\star^{1/2}(t, u)} \int_0^t \frac{1}{\mu_\star^{1/2}(s, u)} \mathbb{Q}_{[1, N]}^{1/2}(s, u) ds \\
&+ C\varepsilon \frac{1}{\mu_\star^{1/2}(t, u)}
\end{aligned} \tag{0.26.5.2}$$

$$\left\| L^{(Y^{N-1})} \widetilde{\mathcal{X}} \right\|_{L^2(\Sigma_t^u)} \lesssim \frac{1}{\mu_\star^{1/2}(t, u)} \mathbb{Q}_{[1, N]}^{1/2}(t, u) + \varepsilon \tag{0.26.5.3}$$

$$\left\| (Y^{N-1}) \widetilde{\mathcal{X}} \right\|_{L^2(\Sigma_t^u)} \lesssim \int_0^t \frac{1}{\mu_\star^{1/2}(s, u)} \mathbb{Q}_{[1, N]}^{1/2}(s, u) ds + \varepsilon \tag{0.26.5.4}$$

Lemma 0.26.5.2. *Under our smallness and bootstrap assumptions, for $1 \leq N \leq 19$, that*

$$\begin{aligned}
&\int_{\mathcal{M}_{t, u}} \left| \text{Error}_{1, (l)} [Y^N \Phi; (Y^{N-1}) \check{\mathcal{X}}] \right| d\omega \\
&(1 + \varsigma^{-1}) \int_0^t \frac{1}{\mu_\star^{1/2}(s, u)} \mathbb{Q}_{[1, N]}(s, u) ds + \int_0^t \frac{1}{\mu_\star^{3/2}(s, u)} \mathbb{Q}_{[1, N-1]}(s, u) ds \\
&+ \varsigma \mathbb{K}_{[1, N]}(t, u) + (1 + \varsigma^{-1}) \varepsilon^2
\end{aligned} \tag{0.26.5.5}$$

$$\int_{\Sigma_t^u} \left| \text{Error}_{2, (l)} [Y^N \Phi; (Y^{N-1}) \check{\mathcal{X}}] \right| d\omega \lesssim \varepsilon^2 + \varepsilon \mathbb{Q}_{[1, N]}(t, u) \tag{0.26.5.6}$$

$$\int_{\Sigma_0^u} \left| \text{Error}_2 [Y^N \Phi; (Y^{N-1}) \check{\mathcal{X}}] \right| d\omega \lesssim \varepsilon^2 \tag{0.26.5.7}$$

$$\int_{\Sigma_0^u} \left| (1 + 2\mu) (\check{X}\Phi) (Y \mathcal{P}^N \Phi)^{(Y^{N-1})} \check{\mathcal{X}} \right| d\omega \lesssim \varepsilon^2 \tag{0.26.5.8}$$

Lemma 0.26.5.3. *Under our smallness and bootstrap assumptions, for $1 \leq N \leq 19$, that*

$$\begin{aligned}
&\left| \int_{\mathcal{M}_{t, u}} (1 + 2\mu) (\check{X}\Phi) (Y^{N+1} \Phi) L^{(Y^{N-1})} \widetilde{\mathcal{X}} d\omega \right| \\
&\leq 2 \int_0^t \frac{\| [L\mu]_- \|_{L^\infty(\Sigma_s^u)}}{\mu_\star(s, u)} \mathbb{Q}_{[1, N]}^{1/2}(s, u) ds \\
&+ C\varepsilon \int_0^t \frac{1}{\mu_\star(s, u)} \mathbb{Q}_{[1, N]}^{1/2}(s, u) ds + C \int_0^t \frac{1}{\mu_\star^{1/2}(s, u)} \mathbb{Q}_{[1, N]}^{1/2}(s, u) ds \\
&+ \varepsilon^2
\end{aligned} \tag{0.26.5.9}$$

$$\begin{aligned}
& \left| \int_{\mathcal{M}_{t,u}} (1 + 2\mu)(\check{X}\Phi)(Y^{N+1}\Phi)^{(Y^{N-1})} \widetilde{\mathcal{X}} d\omega \right| \\
& \leq 2 \frac{\|L\mu\|_{L^\infty((-)\Sigma_t^u)}}{\mu_\star^{1/2}(s,u)} \mathbb{Q}_{[1,N]}^{1/2}(t,u) \int_0^t \frac{1}{\mu_\star^{1/2}(s,u)} \mathbb{Q}_{[1,N]}^{1/2}(s,u) ds \\
& + C\varepsilon \frac{1}{\mu_\star^{1/2}(s,u)} \mathbb{Q}_{[1,N]}^{1/2}(t,u) \int_0^t \frac{1}{\mu_\star^{1/2}(s,u)} \mathbb{Q}_{[1,N]}^{1/2}(s,u) ds \\
& + C \mathbb{Q}_{[1,N]}^{1/2}(t,u) \int_0^t \frac{1}{\mu_\star^{1/2}(s,u)} \mathbb{Q}_{[1,N]}^{1/2}(s,u) ds \\
& + C\varsigma \mathbb{Q}_{[1,N]}(t,u) + C\varsigma^{-1}\varepsilon^2 \frac{1}{\mu_\star(t,u)}.
\end{aligned} \tag{0.26.5.10}$$

0.26.6 Error Integrals that lose a derivative

This lemma is identical to Lemma 14.13 in [18], with the same proof.

Lemma 0.26.6.1. *Under our smallness and bootstrap assumptions, for $1 \leq N \leq 19$, that*

$$\begin{aligned}
& \left| \int_{\mathcal{M}_{t,u}} (\check{X} \mathcal{P}^{N-1}\Phi)(\check{X}\Phi) Y^{N-1} \text{tr}_{\mathcal{G}} \chi d\omega \right| \\
& \lesssim \int_0^t \mathbb{Q}_{[1,N-1]}^{1/2}(t',u) \left\{ \int_0^{t'} \frac{1}{\mu_\star^{1/2}(s,u)} \mathbb{Q}_{[1,N]}^{1/2}(s,u) ds \right\} dt' + \varepsilon \int_0^t \frac{1}{\mu_\star^{1/2}(s,u)} \mathbb{Q}_{[1,N-1]}^{1/2}(s,u) ds
\end{aligned} \tag{0.26.6.1}$$

$$\begin{aligned}
& \left| \int_{\mathcal{M}_{t,u}} (1 + 2\mu)(L\mathcal{P}^{N-1}\Phi)(\check{X}\Phi) Y^{N-1} \text{tr}_{\mathcal{G}} \chi d\omega \right| \\
& \lesssim \int_0^t \mathbb{Q}_{[1,N-1]}^{1/2}(t',u) \left\{ \int_0^{t'} \frac{1}{\mu_\star^{1/2}(s,u)} \mathbb{Q}_{[1,N]}^{1/2}(s,u) ds \right\} dt' + \varepsilon \int_0^t \frac{1}{\mu_\star^{1/2}(s,u)} \mathbb{Q}_{[1,N-1]}^{1/2}(s,u) ds
\end{aligned} \tag{0.26.6.2}$$

0.26.7 Final Top-order and Below Top-order Energy Estimates

In deriving the top-order energy inequalities for the $\mathcal{P}^{TOP}$ commuted wave equation, we find that the one difference in our $3+1$ dimensional setting with respect to the $2+1$ dimensional setting of [18] occurs when we bound the worst top-order product term:

$$- \int_{\mathcal{M}_{t,u}} (\check{X} Y^{TOP} \Phi)(\check{X}\Phi) Y^{TOP} \text{tr}_{\mathcal{G}} \chi dv_{\mathcal{G}} du dt. \tag{0.26.7.1}$$

As in [18] we apply Cauchy-Schwarz and (0.25.0.3) to bound this term by

$$\leq 2 \int_{s=0}^t \mathbb{Q}_{[1, TOP]}^{1/2}(s, u) \left\| (\check{X}\Phi)Y^{TOP} \text{tr}_{\mathcal{G}} \chi \right\|_{L^2(\Sigma_s^u)} ds \quad (0.26.7.2)$$

and now we apply our estimate (0.26.3.4) for $\left\| (\check{X}\Phi)Y^{TOP} \text{tr}_{\mathcal{G}} \chi \right\|_{L^2(\Sigma_s^u)}$. The only term in (0.26.3.4) which is different in our 3 + 1 dimensional setting is $C\varepsilon \frac{1}{\mu_*(t, u)} \mathbb{Q}_{[1, TOP]}^{1/2}(t, u)$, which generates a RHS term of the form

$$C\varepsilon \int_0^t \frac{1}{\mu_*(t', u)} \mathbb{Q}_{[1, TOP]}(t', u) dt'. \quad (0.26.7.3)$$

This RHS term also appears in the 2 + 1 dimensional case when we bound the error integral

$$- \int_{\mathcal{M}_{t, u}} (1 + 2\mu)(LY^{TOP}\Phi)(\check{X}\Phi)Y^{TOP} \text{tr}_{\mathcal{G}} \chi dv_{\mathcal{G}} dudt \quad (0.26.7.4)$$

which reduces to bounding

$$- \int_{\mathcal{M}_{t, u}} (1 + 2\mu)(LY^{TOP}\Phi)(\check{X}\Phi)Y^{(Y^{TOP}-1)} \widetilde{\mathcal{X}} dv_{\mathcal{G}} dudt \quad (0.26.7.5)$$

by integrating by parts using (0.14.0.10), using Cauchy-Schwarz and then applying (0.26.5.1) (see [18] for details). Thus, at the end of the day, our elliptic estimates allow us to control the top-order Eikonal terms $\mu \hat{\chi} \cdot \mathcal{L}^{\leq TOP} \hat{\chi}$ by the term $C\varepsilon \frac{1}{\mu_*(t, u)} \mathbb{Q}_{[1, TOP]}^{1/2}(t, u)$ which appears already in the RHS of (0.26.5.1). Thus the energy estimates for top-order and below top-order are identical to that in [18]. We copy them here for convenience of the reader.

Proposition 0.26.7.1. *Under our bootstrap and smallness assumptions we find that for $1 \leq N \leq 19$*

we have

$$\begin{aligned}
& \max \{ \mathbb{Q}_{[1,N]}(t, u), \mathbb{K}_{[1,N]}(t, u) \} \\
& \leq C(1 + \varsigma^{-1}) \mathring{\mathbf{e}}^2 \boldsymbol{\mu}_\star^{-3/2} \\
& + 6 \int_0^t \frac{\| [L\mu] - \|_{L^\infty(\Sigma_{t'}^u)} }{\boldsymbol{\mu}_\star(t', u)} \mathbb{Q}_{[1,N]}(t', u) dt' \\
& + 8.1 \int_0^t \frac{\| [L\mu] - \|_{L^\infty(\Sigma_{t'}^u)} }{\boldsymbol{\mu}_\star(t', u)} \mathbb{Q}_{[1,N]}^{1/2}(t', u) \int_0^{t'} \frac{\| [L\mu] - \|_{L^\infty(\Sigma_s^u)} }{\boldsymbol{\mu}_\star(s, u)} \mathbb{Q}_{[1,N]}^{1/2}(s, u) ds dt' \\
& + 2 \frac{1}{\boldsymbol{\mu}_\star^{1/2}(t, u)} \mathbb{Q}_{[1,N]}^{1/2}(t, u) \| [L\mu] \|_{L^\infty(\Sigma_{t,t}^u)} \int_0^t \frac{1}{\boldsymbol{\mu}_\star^{1/2}(t', u)} \mathbb{Q}_{[1,N]}^{1/2}(t', u) dt' \\
& + C\varepsilon \int_0^t \frac{1}{\boldsymbol{\mu}_\star(t', u)} \mathbb{Q}_{[1,N]}^{1/2}(t', u) \int_0^{t'} \frac{1}{\boldsymbol{\mu}_\star(s, u)} \mathbb{Q}_{[1,N]}^{1/2}(s, u) ds dt' \\
& + C\varepsilon \int_0^t \frac{1}{\boldsymbol{\mu}_\star(t', u)} \mathbb{Q}_{[1,N]}(t', u) dt' \\
& + C\varepsilon \frac{1}{\boldsymbol{\mu}_\star^{1/2}(t, u)} \mathbb{Q}_{[1,N]}^{1/2}(t, u) \int_0^t \frac{1}{\boldsymbol{\mu}_\star^{1/2}(t', u)} \mathbb{Q}_{[1,N]}^{1/2}(t', u) dt' \\
& + C \mathbb{Q}_{[1,N]}^{1/2}(t, u) \int_0^t \frac{1}{\boldsymbol{\mu}_\star^{1/2}(t', u)} \mathbb{Q}_{[1,N]}^{1/2}(t', u) dt' \tag{0.26.7.6} \\
& + C \int_0^t \frac{1}{\sqrt{T_{(Boot)} - t'}} \mathbb{Q}_{[1,N]}(t', u) dt' \\
& + C(1 + \varsigma^{-1}) \int_0^t \frac{1}{\boldsymbol{\mu}_\star^{1/2}(t', u)} \mathbb{Q}_{[1,N]}(t', u) dt' \\
& + C \int_0^t \frac{1}{\boldsymbol{\mu}_\star(t', u)} \mathbb{Q}_{[1,N]}^{1/2}(t', u) \int_0^{t'} \frac{1}{\boldsymbol{\mu}_\star^{1/2}(s, u)} \mathbb{Q}_{[1,N]}^{1/2}(s, u) ds dt' \\
& + C \int_0^t \frac{1}{\boldsymbol{\mu}_\star(t', u)} \mathbb{Q}_{[1,N]}^{1/2}(t', u) \int_0^{t'} \frac{1}{\boldsymbol{\mu}_\star(s, u)} \int_0^s \frac{1}{\boldsymbol{\mu}_\star^{1/2}(s', u)} \mathbb{Q}_{[1,N]}^{1/2}(s', u) ds' ds dt' \\
& + C(1 + \varsigma^{-1}) \int_0^u \mathbb{Q}_{[1,N]}(t, u') du' \\
& + C\varepsilon \mathbb{Q}_{[1,N]}(t, u) + C\varsigma \mathbb{Q}_{[1,N]}(t, u) + C\varsigma \mathbb{K}_{[1,N]}(t, u) \\
& + C \int_0^t \frac{1}{\boldsymbol{\mu}_\star^{5/2}(t', u)} \mathbb{Q}_{[1,N-1]}(t', u) dt'
\end{aligned}$$

and for $2 \leq N \leq 19$ we have:

$$\begin{aligned}
& \max \{ \mathbb{Q}_{[1,N-1]}(t, u), \mathbb{K}_{[1,N-1]}(t, u) \} \\
& \leq C\mathring{\epsilon}^2 \\
& + C \int_0^t \frac{1}{\mu_\star^{1/2}(t', u)} \mathbb{Q}_{[1,N-1]}^{1/2}(t', u) \int_0^{t'} \frac{1}{\mu_\star^{1/2}(s, u)} \mathbb{Q}_{[1,N]}^{1/2}(s, u) ds dt' \\
& + C \int_0^t \frac{1}{\sqrt{T_{(Boot)} - t'}} \mathbb{Q}_{[1,N-1]}(t', u) dt' \\
& + C(1 + \varsigma^{-1}) \int_0^t \frac{1}{\mu_\star^{1/2}(t', u)} \mathbb{Q}_{[1,N-1]}(t', u) dt' \\
& + C\mathring{\epsilon} \int_0^t \frac{1}{\mu_\star^{1/2}(t', u)} \mathbb{Q}_{[1,N-1]}(t', u) dt' \\
& + C(1 + \varsigma^{-1}) \int_0^u \mathbb{Q}_{[1,N-1]}(t, u') du' \\
& + C\varsigma \mathbb{K}_{[1,N-1]}(t, u)
\end{aligned} \tag{0.26.7.7}$$

Remark 0.26.7.1. Thus we have demonstrated that the $3 + 1$ dimensional case actually gives us the same energy inequality as in the $2 + 1$ dimensional case. The subsequent Gronwall argument (the aforementioned “descent scheme”) used to derive the final a priori energy estimates and the proof of stable shock formation are then almost exactly as in [18], with very minor modifications needed in our case. Nonetheless we will sketch the rest of the proof of the stable shock formation theorem.

Proposition 0.26.7.2 (The Final A Priori Energy Estimates). *Under our data size and bootstrap assumptions we have that there exists a $C > 0$ such that in $\mathcal{M}_{T_{(Boot)}, U_0}$ we have*

$$\mathbb{Q}_{[1,14+M]}^{1/2}(t, u) + \mathbb{K}_{[1,14+M]}^{1/2}(t, u) \leq C\mathring{\epsilon}\mu_\star^{-(M+9)}(t, u), \quad (0 \leq M \leq 5), \tag{0.26.7.8}$$

$$\mathbb{Q}_{[1,1+M]}^{1/2}(t, u) + \mathbb{K}_{[1,1+M]}^{1/2}(t, u) \leq C\mathring{\epsilon}, \quad (0 \leq M \leq 12). \tag{0.26.7.9}$$

Remark 0.26.7.2. Note that the only difference from Proposition 14.1 in [18] is that we need to go up one order higher, due to using the Sobolev embedding on our 2 dimensional rather than 1 dimensional surfaces $\mathcal{P}_{t,u}$.

Proof. (Rough idea of proof) The top-order and below-top-order estimates (0.26.7) are actually highly coupled and the Gronwall argument giving (0.26.7.2) is rather delicate, the interested reader should consult [18] for details. Nonetheless the basic idea is fairly easy to see.

Assume that we have the μ -degenerate top-order bound

$$\mathbb{E}^{1/2}[Y^{TOP}\Phi](t, u) \lesssim \mathring{\epsilon}\mu_\star^{-M}(t, u). \quad (0.26.7.10)$$

Note that our commuted (partially modified) equation for $Y^{TOP-1}\text{tr}_g\chi$ is of the form

$$LY^{TOP-1}\text{tr}_g\chi = \frac{1}{2}G_{LL}\Delta Y^{TOP-1}\Phi + \dots \quad (0.26.7.11)$$

Taking $L^2(\Sigma_t^u)$ norms of both sides and using our coercive bound (0.25.0.3) gives us that

$$\left\| LY^{TOP-1}\text{tr}_g\chi \right\|_{L^2(\Sigma_t^u)} \lesssim \mu_\star^{-1/2}(t, u) \mathbb{E}^{1/2}[Y^{TOP}\Phi](t, u) \quad (0.26.7.12)$$

$$\lesssim \mathring{\epsilon}\mu_\star^{-M-1/2}(t, u). \quad (0.26.7.13)$$

We may integrate $LY^{TOP-1}\text{tr}_g\chi$ along the characteristics and apply (0.25.0.8) together with (0.21.0.34) to get

$$\left\| Y^{TOP-1}\text{tr}_g\chi \right\|_{L^2(\Sigma_t^u)} \lesssim \mathring{\epsilon}\mu_\star^{-M+1/2}(t, u). \quad (0.26.7.14)$$

Now we consider only the error term $(\check{X}\Phi)(Y^{TOP-1}\text{tr}_g\chi)(\check{X}Y^{TOP-1}\Phi)$ in the energy estimate for one level below top-order:

$$\mathbb{E}[Y^{TOP-1}\Phi](t, u) + \mathbb{F}[Y^{TOP-1}\Phi](t, u) \quad (0.26.7.15)$$

$$\lesssim \mathring{\epsilon}^2 + \int_{\mathcal{M}_{t,u}} (\check{X}\Phi)(Y^{TOP-1}\text{tr}_g\chi)(\check{X}Y^{TOP-1}\Phi) \quad (0.26.7.16)$$

$$\lesssim \mathring{\epsilon}^2 + \int_0^t \left\| Y^{TOP-1}\text{tr}_g\chi \right\|_{L^2(\Sigma_s^u)} \mathbb{E}^{1/2}[Y^{TOP-1}\Phi](s, u) ds \quad \text{using } |\check{X}\Phi| \lesssim 1, \text{ Cauchy-Schwarz, and (0.25.0.3)} \quad (0.26.7.17)$$

$$\lesssim \mathring{\epsilon}^2 + \mathring{\epsilon} \left(\sup_{0 \leq s' \leq t} \mathbb{E}^{1/2}[Y^{TOP-1}\Phi](s', u) \right) \int_0^t \mu_\star^{-M+1/2}(s, u) ds \quad \text{by (0.26.7.14)} \quad (0.26.7.18)$$

$$\lesssim \mathring{\epsilon}^2 + \mathring{\epsilon} \left(\sup_{0 \leq s' \leq t} \mathbb{E}^{1/2}[Y^{TOP-1}\Phi](s', u) \right) \mu_\star^{-M+3/2}(t, u) \quad \text{by (0.21.0.34)}. \quad (0.26.7.19)$$

Thus, for $\mathring{\epsilon}$ sufficiently small we easily see that

$$\mathbb{E}[Y^{TOP-1}\Phi](t, u) + \mathbb{F}[Y^{TOP-1}\Phi](t, u) \lesssim \mathring{\epsilon}\mu_\star^{-M+3/2}(t, u). \quad (0.26.7.20)$$

Thus our below-top-order estimate gains back powers of $\mu_\star$ and hence by descending far enough

we may eliminate the $\mu_\star$ degeneracy.⁶ We thus show that the lower order energies are actually bounded. $\square$

0.27 Stable Shock Formation

Now with (0.26.7) we are effectively done. Using (0.25.0.4) together with the nondegenerate lower-order energy bound, we are able to improve our bootstrap assumptions if $\mathring{\epsilon}$ is sufficiently small. We then see that all of the above proven estimates hold with ε replaced by $C\mathring{\epsilon}$.

See the theorem (0.4.1) in (0.4) for a complete statement of the stable shock formation theorem in our $3 + 1$ dimensional setting. See [18] for the proof.

0.28 Conclusion

We have here demonstrated that H^2 illposedness is to be expected of a generic covariant quasilinear wave equation of the form $\square_{g(\Phi)}\Phi = 0$ in $\mathbb{R}^{3+1}$. The condition for illposedness (i.e. the planar shock formation condition) that

$$G_{\alpha\beta}(\Phi = 0)^\omega L_{(Flat)}^\alpha{}^\omega L_{(Flat)}^\beta \neq 0$$

for some $\omega \in \mathbb{S}^2$ is fairly robust. Indeed, if

$$G_{\alpha\beta}(\Phi = 0)^\omega L_{(Flat)}^\alpha{}^\omega L_{(Flat)}^\beta = 0$$

for all $\omega \in \mathbb{S}^2$, then $g(\Phi)$ is necessarily conformally Minkowski up to quadratic order terms i.e. $g(\Phi)_{\alpha\beta} = (1 + f(\Phi))m_{\alpha\beta} + \mathcal{O}(\Phi^2)$. We also showed that the instantaneous energy blow-up observed for “Lindblad-type” initial data $\underline{L}\Phi|_{t=0} = \eta|\ln(x^1)|^\alpha$ is actually stable under perturbations, albeit under a fairly strong asymptotic condition which ensures that the singularity of the data is essentially the same as the singularity at 0 of $\eta|\ln(x^1)|^\alpha$. Thus the breakdown behavior is not merely a manifestation of exceptional symmetry.

The exceptional geometric character of the equations for which we cannot demonstrate illposedness (or planar shock formation) highlights the following open questions regarding the positive question of when H^2 wellposedness (or better) is possible for quasilinear wave equations.

⁶In actuality there are worse behaved error terms and we are only able to gain 1 power of $\mu_\star$ per derivative level.

0.28.1 Quasilinear Low-Regularity Local Well-Posedness in Spherical Symmetry

In [6], it was proved that quasilinear wave equations of the form

$$\partial_t^2 \Phi - \Delta \Phi + h(\Phi) \Delta \Phi = F(\partial \Phi, \partial \Phi)$$

are well-posed for spherically symmetric initial data in $H^2 \times H^1(\mathbb{R}^3)$. The proof is fairly robust, essentially relying on a Morawetz-type space-time estimate and the Sobolev inequality on spheres. It is of immediate interest to reproduce the proof in our more geometric setting and to see if we can make improvements in the case of quasilinear null structure. The hope would be to show $H^{3/2}$ local (and thus global) well-posedness if the equation is not shock forming.

0.28.2 Timelike Minimal Surface Equation

The most interesting (and challenging) problem related to this paper is the question of H^2 local well posedness in the case where the null condition is satisfied. One wonders whether it is possible to give a much simplified geometric proof of “Bounded L^2 ” for the timelike minimal surface equation, which, as we noted above, is the only relativistic lagrangian fluid equation satisfying the null condition (i.e. the only one which does not form shocks or exhibit H^3 illposedness). Note that this is an equation of the form $\partial_\alpha((g^{-1})^{\alpha\beta}(\partial\Psi)\partial_\beta\Psi) = 0$, although this is essentially the equation in “wave gauge” or harmonic coordinates. Thus, perhaps similar to the Einstein vacuum equations in harmonic coordinates, it is possible that there is a more subtle counterexample to H^3 local well posedness (although the null structure is seemingly much stronger for the timelike minimal surface). Nonetheless, the equation also admits a more geometric formulation in the form of the Gauss-Codazzi equations together with constraints on initial data (see the recent paper by Willie Wong: [21]):

Definition 0.28.2.1 (Timelike Minimal Surface). Assume $\mathfrak{h}$ is a symmetric 2-tensor on the Lorentzian spacetime manifold $(\mathcal{M}^{3+1}, g)$. We say that $\mathfrak{h}$ is the second fundamental form of a *timelike minimal surface* if

$$\mathrm{tr}_g \mathfrak{h} = 0 \tag{0.28.2.1}$$

and if the following Gauss-Codazzi equations are satisfied:

$$R_{\alpha\beta\gamma\delta} = \mathfrak{h}_{\alpha\gamma}\mathfrak{h}_{\beta\delta} - \mathfrak{h}_{\alpha\delta}\mathfrak{h}_{\beta\gamma} \quad (0.28.2.2)$$

$$\mathcal{D}_\alpha \mathfrak{h}_{\beta\gamma} = \mathcal{D}_\beta \mathfrak{h}_{\alpha\gamma} \quad (0.28.2.3)$$

It is well-known that:

Theorem 0.28.1 (Fundamental Theorem of (Lorentzian) Submanifolds). *For $(\mathcal{M}^{3+1}, g, \mathfrak{h})$ satisfying (0.28.2.1), (0.28.2.2), and (0.28.2.3), there exists an isometric immersion $\iota : (\mathcal{M}^{3+1}, g, \mathfrak{h}) \rightarrow (\mathbb{R}^{4+1}, m)$ such that $\mathfrak{h}$ is the second fundamental form of the immersion*

From (0.28.2.1) and (0.28.2.3) it is easy to see that for such $\mathfrak{h}$ we have:

$$(\operatorname{div} \mathfrak{h})_\alpha = \mathcal{D}^\beta \mathfrak{h}_{\beta\alpha} = 0 \quad (0.28.2.4)$$

From the above it follows that $\mathfrak{h}$ satisfies the tensorial wave equation:

$$(\square_g \mathfrak{h})_{\alpha\beta} = g^{\gamma\delta} \mathcal{D}_\gamma \mathcal{D}_\delta \mathfrak{h}_{\alpha\beta} = -\mathfrak{h}_{\alpha\beta} |\mathfrak{h}|_g^2 \quad (0.28.2.5)$$

where $|\mathfrak{h}|_g^2 := g^{\alpha\beta} g^{\gamma\delta} \mathfrak{h}_{\alpha\gamma} \mathfrak{h}_{\beta\delta} = \mathfrak{h}_{\alpha\beta} \mathfrak{h}^{\alpha\beta} = \frac{1}{2} \mathfrak{h}_{LL} \mathfrak{h}_{\underline{L}\underline{L}} + \frac{1}{2} (\mathfrak{h}_{L\underline{L}})^2 - 2\mathcal{D}^\#(\mathfrak{h}_L^\#, \mathfrak{h}_{\underline{L}}^\#) + |\mathfrak{h}|_{\mathcal{D}}^2$. Also note that $\mathfrak{h}_{L\underline{L}} = \operatorname{tr}_{\mathcal{D}} \mathfrak{h}$ for the timelike minimal surface. It is easy to see that if $\operatorname{tr}_g \mathfrak{h}|_{\Sigma_0} = 0$, $\partial_t \operatorname{tr}_g \mathfrak{h}|_{\Sigma_0} = 0$ for an initial hypersurface Σ_0 then this condition propagates to the domain of future dependence of Σ_0 .

This equation should be well-posed for initial data in: $\|(g_{ij}, k_{ij})\|_{H^2 \times H^1(\Sigma_0)} + \|\mathfrak{h}_{\alpha\beta}\|_{H^1(\Sigma_0)}$.

0.29 Appendix: Notation and Geometry

This section largely parallels Appendix C in [18], although we reproduce it here for the convenience of the reader. Also there are some places where our notation follows that in [16].

0.29.1 Coordinates

- (x^0, x^1, x^2, x^3) are the rectangular spacetime coordinates on $\mathbb{R}^{1+1} \times \mathbb{T}^2$, relative to which the Minkowski metric has the form $m_{\alpha\beta} = \operatorname{diag}(-1, 1, 1, 1)$.
- $t = x^0$, and we use these interchangeably.
- u denotes the eikonal function

- $(t, u, \theta^1, \theta^2)$ denote the geometric coordinates on $\mathbb{R}^{1+1} \times \mathbb{T}^2$.

0.29.2 Indices

- Lowercase Greek indices μ, ν , etc. will be used to denote rectangular spacetime coordinate indices.
- Lowercase Latin indices $a, b, \dots, i, j, k$, etc. will be used to denote rectangular spatial coordinate indices.
- All lowercase greek indices are raised and lowered with g^{-1} and g , respectively.
- Capital Latin indices A, B , etc. denote the components relative to the geometric coordinate frame $\{\Theta_1 = \frac{\partial}{\partial \theta^1}, \Theta_2 = \frac{\partial}{\partial \theta^2}\}$.
- We use Einstein summation convention: repeated indices are summed over their ranges.

0.29.3 Constants and Data Size Parameters

- $\mathring{\delta}$ is the parameter corresponding to the initial size of the $\mathcal{C}_u$ - transversal derivatives of the solution.
- $\mathring{\epsilon}$ is a *relatively small* parameter corresponding to the initial size of Φ and its derivatives involving at least one $\mathcal{C}_u$ - tangential differentiation. We explain the kind of smallness that we impose on $\mathring{\epsilon}$ in (0.18.6.1).
- $\mathring{\delta}_* = \frac{1}{2} \sup_{\Sigma_0^1} \left[G_{LL} \check{X} \Phi \right]_-$ is the key quantity that controls the blowup time.
- $\mathring{\alpha}$ denotes the size of Φ in L^∞ on the initial hypersurface Σ_0^1 .
- $C, C_\star$ denote uniform constants that are free to vary from line to line.
- The constants C are allowed to depend on the data-size parameters $\mathring{\delta}$ and $\mathring{\delta}_*^{-1}$.
- The constants $C_\star$ are independent of the data-size parameters $\mathring{\delta}$ and $\mathring{\delta}_*^{-1}$.
- If we want to emphasize that the constant C depends on a quantity $\mathcal{Q}$, then we use notation such as " $C_{\mathcal{Q}}$ ".
- We use the notation

$$f_1 \lesssim f_2$$

To indicate that there exists a uniform constant $C > 0$ such that $f_1 \leq Cf_2$. We sometimes use the alternate notation $\mathcal{O}(f_2)$ to denote a quantity f_1 that verifies $|f_1| \lesssim |f_2|$.

- If we want to emphasize that the implicit constant C depends on a quantity $\mathcal{Q}$, then we use the alternate notation:

$$f_1 \lesssim_{\mathcal{Q}} f_2$$

0.29.4 Spacetime Subsets

- $\mathcal{B}_0 := \{\mathbf{x} \in \mathbb{R}^3 : \|\mathbf{x} - (1, 0, 0)\| \leq \varepsilon\}$
- $\mathcal{B}_t := \mathcal{D}^+(\mathcal{B}_0) \cap \Sigma_t^u$
- $\mathcal{C}_u := \{(t, x) \in \mathbb{R}^{3+1} : u(t, x) = u\}$
- $\mathcal{S}_{t,u} := \mathcal{C}_u \cap \mathcal{B}_t$

0.29.5 Metrics

- $g = g(\Phi)$ denotes the spacetime metric.
- Relative to rectangular coordinates, $g_{\mu\nu}(\Phi) = m_{\mu\nu} + g_{\mu\nu}^{(Small)}(\Phi)$, where $m_{\mu\nu} = \text{diag}(-1, 1, 1)$.
- Relative to rectangular coordinates, $G_{\mu\nu}(\Phi) = \frac{d}{d\Phi} g_{\mu\nu}(\Phi)$ and $G'_{\mu\nu}(\Phi) = \frac{d}{d\Phi} G_{\mu\nu}(\Phi)$.
- $\underline{g}$ denotes the first fundamental form of Σ_t , that is $\underline{g}_{ij} = g_{ij}$.
- $\underline{g}^{-1}$ denotes the inverse first fundamental form of Σ_t .
- $\mathcal{g}$ denotes the first fundamental form of $\mathcal{P}_{t,u}$.

0.29.6 Musical Notation, Contractions, and Inner Products.

- We denote the $\mathcal{g}$ dual of a $\mathcal{P}_{t,u}$ -tangent one-form ξ by $\xi^\#$. Similarly, if Y is an $\mathcal{P}_{t,u}$ -tangent vector, then Y_b denotes the $\mathcal{g}$ -dual of Y , which is an $\mathcal{P}_{t,u}$ -tangent covector. Similarly, if ξ is a symmetric type $\binom{0}{2}$ $\mathcal{P}_{t,u}$ -tangent tensor, then $\xi^\#$ denotes the type $\binom{1}{1}$ tensor that is $\mathcal{g}$ -dual to ξ , and $\xi^{\#\#}$ denotes the type $\binom{2}{0}$ tensor that is $\mathcal{g}$ -dual to ξ . We use similar notation to denote the $\mathcal{g}$ -duals of general type $\binom{m}{0}$ and type $\binom{0}{n}$ $\mathcal{P}_{t,u}$ -tangent tensors.
- $g(X, Y) = g_{\alpha\beta} X^\alpha Y^\beta$ denotes the inner product of the vector X and Y with respect to the metric g . Similarly, if X and Y are $\mathcal{P}_{t,u}$ -tangent, then $\mathcal{g}(X, Y) = \mathcal{g}_{ab} X^a Y^b$.

- $\cdot$ denotes the natural contraction between two tensors. For example, if ξ is a spacetime one-form and V is a spacetime vectorfield, then $\xi \cdot V := \xi_\alpha V^\alpha$. As a second example, if T is a symmetric type $\binom{2}{0}$ $\mathcal{P}_{t,u}$ -tangent tensorfield and ξ is a $\mathcal{P}_{t,u}$ -tangent one-form, then $(\text{div } T) \cdot \xi = (\nabla_a T^{ab}) \xi_b$.
- If ξ is a one-form and V is a vectorfield, then $\xi_V := \xi_\alpha V^\alpha$. Similarly, if W is a vectorfield, then $W_V := W_\alpha V^\alpha = g(W, V)$. We use similar notation when contracting higher-order tensorfields against vectorfields. Similarly, if $\Gamma_{\alpha\kappa\beta}$ are the rectangular Christoffel symbols, then $\Gamma_{UVW} := U^\alpha V^\kappa W^\beta \Gamma_{\alpha\kappa\beta}$.
- If ξ is a symmetric type $\binom{0}{2}$ spacetime tensor and V is a vector, then ξ_V is the one-form with rectangular components $(\xi_V)_\nu = \xi_{\alpha\nu} V^\alpha$.

0.29.7 Tensor Products and the Trace of Tensors

- $\text{tr}_g \pi = (g^{-1})^{\alpha\beta} \pi_{\alpha\beta}$ denotes the g -trace of the type $\binom{0}{2}$ spacetime tensor $\pi_{\mu\nu}$.
- $\text{tr}_g \xi = (g^{-1})^{\alpha\beta} \xi_{\alpha\beta}$ denotes the g -trace of the type $\binom{0}{2}$ $\mathcal{P}_{t,u}$ -tangent tensor ξ .
- $\hat{\xi}_{AB} = \xi_{AB} - \frac{1}{2} g_{AB} \text{tr}_g \xi$ denotes the trace-free part of the type $\binom{0}{2}$ $\mathcal{P}_{t,u}$ -tangent tensor ξ_{AB} .
- $(\xi \otimes \omega)_{AB} = \xi_A \omega_B$ denotes the tensor product of the $\mathcal{P}_{t,u}$ -tangent one-forms ξ_A and ω_A .
- $(\xi \wedge \omega)_{AB} = \frac{1}{2} (\xi_A \omega_B - \xi_B \omega_A)$ denotes the wedge product of the $\mathcal{P}_{t,u}$ -tangent one-forms ξ_A and ω_A .

0.29.8 Eikonal Function and Related Quantities

- The eikonal function u verifies the eikonal equation $(g^{-1})^{\alpha\beta} \partial_\alpha u \partial_\beta u = 0$ and has the initial condition $u|_{t=0} = 1 - x^1$.
- $\mu = - \{ (g^{-1})^{\alpha\beta} \partial_\alpha u \partial_\beta t \}^{-1}$ denotes the inverse foliation density.
- $L_{(Geo)}^\nu = -(g^{-1})^{\nu\alpha} \partial_\alpha u$ denotes the $\mathcal{C}_u$ -tangent outgoing null geodesic vectorfield.
- $L^\nu = \mu L_{(Geo)}^\nu$ denotes a rescaled outgoing null vectorfield.
- $L_{(Small)}^i = L^i - \delta_1^i$ is the perturbed part of L^i .
- $X_{(Small)}^i = X^i + \delta_1^i$ is the perturbed part of X^i .
- $\chi = \frac{1}{2} \mathcal{L}_L g$ is the null second fundamental form of $\mathcal{C}_u$ relative to g .

0.29.9 Additional $\mathcal{P}_{t,u}$ Tensorfields related to the Frame Connection Coefficients.

- $\omega = L\mu$
- $k = \frac{1}{2}\mathcal{L}_N g$ is the second fundamental form of Σ_t relative to g .
- $\zeta_A = \mathbb{k}_{XA} = g(\mathcal{D}_A L, X)$
- $\zeta_A = \mu^{-1}\zeta_A^{(Trans-\Psi)} + \zeta_A^{(Tan-\Psi)}$ is a splitting of ζ_A .
- $\mathbb{k}_{AB} = \mu^{-1}\mathbb{k}_{AB}^{(Trans-\Psi)} + \mathbb{k}_{AB}^{(Tan-\Psi)}$ is a splitting of $\mathbb{k}_{AB}$.

0.29.10 Vectorfields

- L is $\mathcal{C}_u$ -tangent, outward pointing, and verifies $g(L, L) = 0$ and $Lt = 1$.
- $L = \frac{\partial}{\partial t}$ relative to the geometric coordinates.
- $\check{X}$ is Σ_t -tangent, $\mathcal{P}_{t,u}$ -orthogonal, and verifies $g(\check{X}, \check{X}) = \mu^2$, $\check{X}u = 1$.
- $\check{X} = \frac{\partial}{\partial u} - \Xi$ relative to the geometric coordinates, where Ξ is $\mathcal{P}_{t,u}$ -tangent.
- $X = \mu^{-1}\check{X}$.
- $\Theta_A = \frac{\partial}{\partial \theta^A}$, $A = 1, 2$ is the geometric torus coordinate partial derivative vectorfield.
- $N = L + X$ is the future-directed unit normal to Σ_t .
- $\{L, \check{X}, \Theta_1, \Theta_2\} = \{L, \check{X}, \Theta_A\}$ denotes the rescaled frame.
- $\{L, X, \Theta_1, \Theta_2\} = \{L, X, \Theta_A\}$ denotes the non-rescaled frame.

0.29.11 Projection Operators and Frame Components

- Π denotes the type $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ tensorfield that projects onto Σ_t .
- $\mathbb{I}$ denotes the type $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ tensorfield that projects onto $\mathcal{P}_{t,u}$.
- If ξ is a spacetime tensor, then $\mathbb{g} = \mathbb{I}\xi$ is the projection of ξ onto $\mathcal{P}_{t,u}$.
- If ξ is a type $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$ spacetime tensor, then $\mathbb{g}_V = \mathbb{I}(\xi_V)$.
- $G_{(Frame)} = (G_{LL}, G_{LX}, G_{XX}, \mathbb{G}_L, \mathbb{G}_X, \mathbb{G})$ is the array of components of $G_{\mu\nu}$ relative to the non-rescaled frame $\{L, X, \Theta_A\}$.

- $G'_{(Frame)} = (G'_{LL}, G'_{LX}, G'_{XX}, \mathcal{G}'_L, \mathcal{G}'_X, \mathcal{G}')$ is the array of components of $G'_{\mu\nu}$ relative to the non-rescaled frame $\{L, X, \Theta_A\}$.

0.29.12 Arrays of Solution Variables and Schematic Functional Dependence

- $\gamma = \left(\Phi, L^1_{(Small)}, L^2_{(Small)}, L^3_{(Small)} \right)$
- $\underline{\gamma} = \left(\Phi, \mu - 1, L^1_{(Small)}, L^2_{(Small)}, L^3_{(Small)} \right)$
- $f(\xi_{(1)}, \xi_{(2)}, \dots, \xi_{(m)})$ schematically denotes an expression depending smoothly on the $\mathcal{P}_{t,u}$ -tangent tensorfields $\xi_{(1)}, \xi_{(2)}, \dots, \xi_{(m)}$.

0.29.13 Rescaled Frame Components of a Vector

- If $\mathcal{J}$ is a spacetime vector, then $\mu \mathcal{J} = -\mu \mathcal{J}_L L - \mathcal{J}_{\check{X}} L - \mathcal{J}_L \check{X} + \mu \mathcal{J}$ denotes its decomposition relative to the rescaled frame $\{L, \check{X}, \Theta_A\}$, where $\mathcal{J} = \mathcal{J}^\alpha L_\alpha$, $\mathcal{J}_{\check{X}} = \mathcal{J}^\alpha \check{X}_\alpha$, and $\mathcal{J} = \mathbb{I} \mathcal{J}$.

0.29.14 Energy-Momentum Tensorfield and Multiplier Vectorfields

- $\mathcal{Q}_{\mu\nu}[\Phi] = \mathcal{D}_\mu \Phi \mathcal{D}_\nu \Phi - \frac{1}{2} g_{\mu\nu} (g^{-1})^{\alpha\beta} \mathcal{D}_\alpha \Phi \mathcal{D}_\beta \Phi$ denotes the energy-momentum tensorfield associated to Φ .
- $T = (1 + 2\mu)L + 2\check{X}$ denotes the timelike multiplier vectorfield.

0.29.15 Commutation Vectorfields

In this section we have $l \in \{1, 2\}$.

- $Y_{(l),(Flat)}$ denotes the Σ_t -tangent vectorfield with rectangular spatial components $Y^i_{(l),(Flat)} = \delta^i_{l+1}$.
- $Y_{(l)} = \mathbb{I} Y_{(l),(Flat)}$ denotes the $\mathcal{P}_{t,u}$ -tangent commutation vectorfields.
- $Y^i_{(l)} = \delta^i_{l+1} + Y^i_{(l),(Small)}$ is a splitting of $Y_{(l)}$ into $Y_{(l),(Flat)}$ and a perturbation.
- $\mathcal{L} = \{L, \check{X}, Y_{(1)}, Y_{(2)}\} = \{L, \check{X}, Y_{(l)}\}$ denotes the full set of commutation vectorfields.
- $\mathcal{P} = \{L, Y_{(1)}, Y_{(2)}\}$ are the $\mathcal{C}_u$ -tangent commutation vectorfields.

0.29.16 Differential Operators and Commutator Notation

- ∂_μ denotes the rectangular coordinate partial derivative vectorfield $\frac{\partial}{\partial x^\mu}$.
- $\frac{\partial}{\partial t}, \frac{\partial}{\partial u}, \Theta_A = \frac{\partial}{\partial \theta^A}$ denote the geometric coordinate partial derivative vectorfields.
- $Vf = V^\alpha \partial_\alpha f$ denote the V -directional derivative of a function f .
- df denotes the standard differential of a function f on a spacetime.
- $\mathring{d}f = \mathring{\Pi}df$, where f is a function of spacetime and $\mathring{\Pi}$ denotes projection onto $\mathcal{P}_{t,u}$. Alternatively, $\mathring{d}f$ can be viewed as the inherent differential of a function f defined on $\mathcal{P}_{t,u}$.
- $\mathcal{D}$ = Levi-Civita connection of g .
- ∇ = Levi-Civita connection of $\mathring{g}$.
- $\nabla =$ Levi-Civita connection of the Minkowski metric m .
- $\mathcal{D}_{X,Y}^2 = X^\alpha Y^\beta \mathcal{D}_\alpha \mathcal{D}_\beta$ and similarly for the other connections (contractions against X and Y are taken after the two covariant differentiations).
- ∇^2 denotes the second $\mathcal{P}_{t,u}$ covariant derivative corresponding to $\mathring{g}$.
- $\mathring{\Delta} = \text{tr}_{\mathring{g}} \nabla^2 f$ denotes the covariant Laplacian on $\mathcal{P}_{t,u}$ corresponding to $\mathring{g}$.
- If ξ is a $\mathcal{P}_{t,u}$ -tangent one-form, then $\text{div} \xi$ is the scalar-valued function $\text{div} \xi := \mathring{g}^{-1} \cdot \nabla \xi$. Similarly, if V is a $\mathcal{P}_{t,u}$ -tangent vectorfield, then $\text{div} V := \mathring{g}^{-1} \cdot \nabla V_b$, where V_b is the one-form $\mathring{g}$ -dual to V . If ξ is a symmetric type $\binom{0}{2}$ $\mathcal{P}_{t,u}$ -tangent tensorfield, then $\text{div} \xi$ is the $\mathcal{P}_{t,u}$ -tangent one-form $\text{div} \xi := \mathring{g}^{-1} \cdot \nabla \xi$, where the two contraction indices is $\nabla \xi$ correspond to the operator ∇ and the first index of ξ .
- $\mathcal{L}_V \xi$ denotes the Lie derivative of ξ with respect to V .
- $[V, W] = \mathcal{L}_V W$ when V and W are vectorfields.
- More generally, if P and Q are two operators, then $[P, Q] = PQ - QP$ denotes their commutator.
- $\underline{\mathcal{L}}_V \xi = \mathring{\Pi} \mathcal{L}_V \xi$ is the Σ_t -projected Lie derivative of ξ with respect to V .
- $\mathring{\mathcal{L}}_V \xi = \mathring{\Pi} \mathcal{L}_V \xi$ is the $\mathcal{P}_{t,u}$ -projected Lie derivative of ξ with respect to V .

0.29.17 Floor and Ceiling Functions and Repeated Differentiation

- If M is a non-negative integer, then $\lfloor M/2 \rfloor = M/2$ for M even and $\lfloor M/2 \rfloor = (M-1)/2$ for M odd, while $\lceil M/2 \rceil = M/2$ for M even and $\lceil M/2 \rceil = (M+1)/2$ for M odd.
- We label the four vectorfields in $\mathcal{L}$ as follows: $Z_{(1)} = L, Z_{(2)} = Y_{(1)}, Z_{(3)} = Y_{(2)}, Z_{(4)} = \check{X}$. Note that $\mathcal{P} = \{Z_{(1)}, Z_{(2)}, Z_{(3)}\}$.
- If $\vec{I} = (\iota_1, \iota_2, \dots, \iota_N)$ is a multi-index of order $|\vec{I}| := N$ with $\vec{I} = (\iota_1, \iota_2, \dots, \iota_N) \in \{1, 2, 3, 4\}$, then $\mathcal{L}^{\vec{I}} := Z_{(\iota_1)} Z_{(\iota_2)} \cdots Z_{(\iota_N)}$ denotes the corresponding N^{th} order differential operator. We write $\mathcal{L}^N$ rather than $\mathcal{L}^{\vec{I}}$ when we are not concerned with the structure of $\vec{I}$.
- Similarly, $\mathcal{L}^{\vec{I}} := \mathcal{L}_{Z_{(\iota_1)}} \mathcal{L}_{Z_{(\iota_2)}} \cdots \mathcal{L}_{Z_{(\iota_N)}}$ denotes an N^{th} order $\mathcal{P}_{t,u}$ -projected Lie derivative operator, and we write $\mathcal{L}_{\mathcal{L}}^N$ when we are not concerned with the structure of $\vec{I}$.
- If $\vec{I} = (\iota_1, \iota_2, \dots, \iota_N)$, then $\vec{I}_1 + \vec{I}_2 = \vec{I}$ means that $\vec{I}_1 = (\iota_{k_1}, \iota_{k_2}, \dots, \iota_{k_m})$ and $\vec{I}_2 = (\iota_{k_{m+1}}, \iota_{k_{m+2}}, \dots, \iota_{k_N})$, where $1 \leq m \leq N$ and $k_1, k_2, \dots, k_N$ is a permutation of $1, 2, \dots, N$.
- Sums such as $\vec{I}_1 + \vec{I}_2 + \cdots + \vec{I}_M = \vec{I}$ have an analogous meaning.
- $\mathcal{C}_u$ -tangent operators such as $\mathcal{P}^{\vec{I}}$ are defined analogously, except in this case we clearly have $\iota_1, \iota_2, \dots, \iota_N \in \{1, 2, 3\}$.
- $\mathcal{L}^{N;M} f$ denotes an arbitrary string of N commutation vectorfields in $\mathcal{L}$ applied to f , where the string contains *at most* M factors of the $\mathcal{C}_u$ -transversal vectorfield $\check{X}$.
- $\mathcal{P}^N f$ denotes an arbitrary string of N commutation vectorfields in $\mathcal{P}$ applied to f .
- For $N \geq 1$, $\mathcal{L}_*^{N;M} f$ denotes an arbitrary string of N commutation vectorfields in $\mathcal{L}$ applied to f , where the string contains *at least* one $\mathcal{C}_u$ -tangent factor and *at most* M factors of $\check{X}$. We also set $\mathcal{L}_*^{0;0} f := f$.
- For $N \geq 1$, $\mathcal{P}_*^N f$ denotes an arbitrary string of N commutation vectorfields in $\mathcal{P}$ applied to f , where the string contains *at least one factor* of Y or *at least two factors* of L .
- For $\mathcal{P}_{t,u}$ -tangent tensorfields ξ , we similarly define strings of $\mathcal{P}_{t,u}$ -projected Lie derivatives such as $\mathcal{L}_{\mathcal{L}}^{N;M} \xi$.
- $|\mathcal{L}^{\leq N;M} f|$ is the sum over all terms of the form $|\mathcal{L}^{N';M} f|$ with $N' \leq N$ and $\mathcal{L}^{N';M} f$ as defined above. When $N = M = 1$, we sometimes write $|\mathcal{L}^{\leq 1} f|$ with $1 \leq N' \leq N$ and $\mathcal{L}^{N';M} f$ as defined above.

- Sums such as $|\mathcal{P}_*^{[1,N]}f|$, $|\mathcal{L}_{\mathcal{L}}^{\leq N;M}\xi|$, etc. are defined analogously. We write $|\mathcal{P}_*f|$ instead of $|\mathcal{P}_*^{[1,1]}f|$. We also use the notation $|\check{X}^{[1,N]}f| = |\check{X}f| + |\check{X}\check{X}f| + \cdots + \overbrace{|\check{X}\check{X}\cdots\check{X}f|}^{N \text{ copies}}$.

0.29.18 Norms

- $|\xi|^2 = \not{g}_{\mu_1\tilde{\mu}_1} \cdots \not{g}_{\mu_m\tilde{\mu}_m} (\not{g}^{-1})^{\nu_1\tilde{\nu}_1} \cdots (\not{g}^{-1})^{\nu_n\tilde{\nu}_n} \xi_{\nu_1\cdots\nu_n}^{\mu_1\cdots\mu_m} \xi_{\tilde{\nu}_1\cdots\tilde{\nu}_n}^{\tilde{\mu}_1\cdots\tilde{\mu}_m}$ denotes the square of the norm of the type $\binom{m}{n}$ - $\mathcal{P}_{t,u}$ tensor ξ .
- $\|\xi\|_{L^\infty(\mathcal{P}_{t,u})} = \text{ess sup}_{\theta^A \in \mathbb{T}^2} |\xi|(t, u, \theta^A)$.
- $\|\xi\|_{L^\infty(\Sigma_t^u)} = \text{ess sup}_{(u', \theta^A) \in [0, u] \times \mathbb{T}^2} |\xi|(t, u', \theta^A)$.
- $\|\xi\|_{L^\infty(\mathcal{C}_u^t)} = \text{ess sup}_{(t', \theta^A) \in [0, t] \times \mathbb{T}^2} |\xi|(t', u, \theta^A)$.
- $\|f\|_{H_e^n(\Sigma_0)}^2 = \sum_{|\vec{I}| \leq n} \int_{\Sigma_0} (\partial_{\vec{I}} f)^2 d^3x$, where $\partial_{\vec{I}}$ is a multi-indexed differential operator representing repeated differentiation with respect to the spatial coordinate partial derivatives and d^3x is the area form corresponding to the standard Euclidean metric e on Σ_0 .
- $\|\xi\|_{L^2(\mathcal{P}_{t,u})}^2 = \int_{\theta^A \in \mathbb{T}^2} |\xi|^2(t, u, \theta^A) dv_{\not{g}} = \int_{\mathcal{P}_{t,u}} |\xi|^2 dv_{\not{g}}$.
- $\|\xi\|_{L^2(\Sigma_t^u)}^2 = \int_{u'=0}^u \int_{\theta^A \in \mathbb{T}^2} |\xi|^2(t, u', \theta^A) dv_{\not{g}} du'$.
- $\|\xi\|_{L^2(\mathcal{C}_u^t)}^2 = \int_{t'=0}^t \int_{\theta^A \in \mathbb{T}^2} |\xi|^2(t', u, \theta^A) dv_{\not{g}} dt'$.
- We use similar notation for the norms $\|\cdot\|_{L^2(\Omega)}$ and $\|\cdot\|_{L^\infty(\Omega)}$ of functions defined on subsets Ω of $\mathcal{P}_{t,u}$, Σ_t^u , or $\mathcal{C}_u^t$.

0.29.19 L^2 -controlling quantities.

- $\mathbb{E}[\Phi](t, u)$ denotes the energy of Φ along Σ_t^u corresponding to the multiplier T .
- $\mathbb{F}[\Phi](t, u)$ denotes the null flux of Φ along $\mathcal{C}_u^t$ corresponding to the multiplier T .
- $\mathbb{Q}_N(t, u) = \max_{|\vec{I}|=N} \sup_{(t', u') \in [0, t] \times [0, u]} \left\{ \mathbb{E}[\mathcal{P}^{\vec{I}}\Phi](t', u') + \mathbb{F}[\mathcal{P}^{\vec{I}}\Phi](t', u') \right\}$.
- $\mathbb{Q}_{[1,N]}(t, u) = \max_{1 \leq M \leq N} \mathbb{Q}_M(t, u)$.
- $\mathbb{K}[\Phi](t, u) = \frac{1}{2} \int_{\mathcal{M}_{t,u}} [L\mu]_- |\not{d}\Phi|^2 dv_{\not{g}} du dt$ denotes the coercive spacetime integral associated to Φ .
- $\mathbb{K}_N(t, u) = \max_{|\vec{I}|=N} \mathbb{K}[\mathcal{P}^{\vec{I}}\Phi](t, u)$.
- $\mathbb{K}_{[1,N]}(t, u) = \max_{1 \leq M \leq N} \mathbb{K}_M(t, u)$.

0.29.20 Modified Quantities

- $(\mathcal{P}^N)\mathcal{X} = \mu\mathcal{P}^N\mathrm{tr}_{\mathcal{g}}\chi + \mathcal{P}^N\mathfrak{X}$ is the fully modified version of $\mathcal{P}^N\mathrm{tr}_{\mathcal{g}}\chi$.
- $(\mathcal{P}^N)\widetilde{\mathcal{X}} = \mathcal{P}^N\mathrm{tr}_{\mathcal{g}}\chi + (\mathcal{P}^N)\widetilde{\mathfrak{X}}$ is the partially modified version of $\mathcal{P}^N\mathrm{tr}_{\mathcal{g}}\chi$.

0.29.21 Curvature Tensors

- $R_{\mu\nu\alpha\beta}$ is the Riemann curvature tensor of g .
- $R(W, X, Y, Z) = g(\mathcal{D}_X^2 Y - \mathcal{D}_W^2 Y, Z)$ indicates our curvature sign convention. The same sign convention is used for $\mathfrak{R}_{ABCD}$.
- $\mathrm{Ric}_{\alpha\beta} = (g^{-1})^{\kappa\lambda}\mathcal{R}_{\alpha\kappa\beta\lambda}$ is the Ricci curvature tensor of g .
- $\mathfrak{R}_{ABCD}$ is the Riemann curvature tensor of $\mathcal{g}$.
- $\mathfrak{R}_{AB}$ is the Ricci curvature tensor of $\mathcal{g}$.
- $\mathfrak{R}$ is the scalar curvature of $\mathcal{g}$.
- $\mathfrak{K}$ is the Gaussian curvature of $\mathcal{g}$.

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Bibliography

- [1] Demetrios Christodoulou. *The formation of shocks in 3-dimensional fluids*. EMS Monographs in Mathematics. European Mathematical Society (EMS), Zürich, 2007.
- [2] Demetrios Christodoulou. *The Shock Development Problem*. pre-print, 2017. arXiv:1705.00828 [math.AP].
- [3] Demetrios Christodoulou and Shuang Miao. *Compressible flow and Euler’s equations*, volume 9 of *Surveys of Modern Mathematics*. International Press, Somerville, MA; Higher Education Press, Beijing, 2014.
- [4] Boris Ettinger. *Well-posedness of the three-form field equation and the minimal surface equation in Minkowski space*. ProQuest LLC, Ann Arbor, MI, 2013. Thesis (Ph.D.)—University of California, Berkeley.
- [5] Boris Ettinger and Hans Lindblad. A sharp counterexample to local existence of low regularity solutions to Einstein equations in wave coordinates. *Ann. of Math. (2)*, 185(1):311–330, 2017.
- [6] Kunio Hidano, Chengbo Wang, and Kazuyoshi Yokoyama. On almost global existence and local well posedness for some 3-D quasi-linear wave equations. *Adv. Differential Equations*, 17(3-4):267–306, 2012.
- [7] S. Klainerman and I. Rodnianski. Improved local well-posedness for quasilinear wave equations in dimension three. *Duke Math. J.*, 117(1):1–124, 2003.
- [8] Sergiu Klainerman. PDE as a unified subject. *Geom. Funct. Anal.*, special(Special Volume, Part I):279–315, 2000. GAFA 2000 (Tel Aviv, 1999).
- [9] Sergiu Klainerman and Igor Rodnianski. Causal geometry of Einstein-vacuum spacetimes with finite curvature flux. *Invent. Math.*, 159(3):437–529, 2005.

- [10] Sergiu Klainerman and Igor Rodnianski. Rough solutions of the Einstein-vacuum equations. *Ann. of Math. (2)*, 161(3):1143–1193, 2005.
- [11] Sergiu Klainerman, Igor Rodnianski, and Jeremie Szeftel. The bounded L^2 curvature conjecture. *Invent. Math.*, 202(1):91–216, 2015.
- [12] Hans Lindblad. Counterexamples to local existence for semi-linear wave equations. *Amer. J. Math.*, 118(1):1–16, 1996.
- [13] Hans Lindblad. Counterexamples to local existence for quasilinear wave equations. *Math. Res. Lett.*, 5(5):605–622, 1998.
- [14] Shuang Miao and Pin Yu. On the formation of shocks for quasilinear wave equations. *Invent. Math.*, 207(2):697–831, 2017.
- [15] Hart F. Smith and Daniel Tataru. Sharp local well-posedness results for the nonlinear wave equation. *Ann. of Math. (2)*, 162(1):291–366, 2005.
- [16] Jared Speck. *Shock formation in small-data solutions to 3D quasilinear wave equations*, volume 214 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2016.
- [17] Jared Speck. Multidimensional nonlinear geometric optics for transport operators with applications to stable shock formation. *pre-print*, 2017. arXiv:1709.04509 [math.AP].
- [18] Jared Speck, Gustav Holzegel, Jonathan Luk, and Willie Wong. Stable shock formation for nearly simple outgoing plane symmetric waves. *Ann. PDE*, 2(2):Art. 10, 198, 2016.
- [19] Daniel Tataru. Rough solutions for the wave maps equation. *Amer. J. Math.*, 127(2):293–377, 2005.
- [20] Qian Wang. A Geometric Approach for Sharp Local Well-Posedness of Quasilinear Wave Equations. *Ann. PDE*, 3(1):3:12, 2017.
- [21] Willie Wai Yeung Wong. Global existence for the minimal surface equation on $\mathbb{R}^{1,1}$. *Proc. Amer. Math. Soc. Ser. B*, 4:47–52, 2017.