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SPLINTER-TYPE SINGULARITIES VIA ULTRAPOWER

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SPLINTER-TYPE SINGULARITIES VIA ULTRAPOWER

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Abstract

We study the notions of splinter, birational derived splinter, and derived splinter for a ring A . They are defined by the splitting of the maps $A \rightarrow Rf_*\mathcal{O}_X$ for certain morphisms $X \rightarrow \mathrm{Spec}(A)$. For Noetherian A , these notions can be considered as notions of singularities of A . We establish numerous fundamental properties, including behaviours with respect to direct limit and étale extensions.

Then we study the more subtle problem of the ascent along a regular map. We make use the construction of ultrapower. A special case of ultraproduct, this construction is seen more in model theory. We give an alternative definition for local rings. Taking ultrapower does not preserve Noetherianness. We therefore need to involve non-Noetherian rings and schemes.

We also give a short discussion on the equicharacteristic zero case.

In the appendix, we perform a model-theoretic study of splinters. This is where the idea for this work originates from. For the reader's convenience, we review the basics of model theory of rings. We also discuss a variant of (birational) derived splinter that behaves better in the non-Noetherian case.

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Chapter 1

Introduction

1.1 Three notions of splinter

A Noetherian reduced ring A is called a splinter (resp. a birational derived splinter, a derived splinter) if for all finite surjective (resp. proper birational, proper surjective) maps $X \rightarrow \operatorname{Spec}(A)$, the canonical map $A \rightarrow Rf_*\mathcal{O}_X$ splits.

The direct summand theorem, that is, a regular ring is a splinter, is trivial in equal characteristic zero and known for decades in prime characteristic, see [Hoc73a]. The mixed characteristic case has been an important problem in commutative algebra and was solved only recently, see [And18]. Even better, regular rings are in fact derived splinters, see [Bha18]. Consequently, the notions of splinter, birational derived splinter, and derived splinter can be considered notions of singularities, and they are related to other interesting

notions. For a ring essentially finitely generated over a field characteristic zero, being a birational derived splinter is a characterizing property of rational singularities, see [Kov00]. Using [Mur22], the same argument works for quasi-excellent rings in equal characteristic zero, as we noted in §9.2. In prime characteristic, weakly F -regular rings are splinters, and (locally excellent) splinters are F -rational, see [Smi93].

In this thesis, we focus on the behaviour of these notions themselves. The situation is vastly different in different characteristics. In equal characteristic zero, being a splinter is the same as being normal, and being a birational derived splinter is the same as being a derived splinter. This is because you can divide the trace map by the degree. We give a short discussion in §9.1. In prime or mixed characteristic, being a splinter is the same as being a derived splinter, and thus stronger than being a birational derived splinter. This follows from [Bha12] and the forthcoming work [BL]. Consequently, in most of this thesis we could have avoided all discussions on derived splinters, and obtain the results from those of splinters and birational derived splinters. We choose not to do this, since the flexibility of our argument allows us to treat the extra case with almost no additional effort.

We establish some basic results on the notions of splinter. We consider localization (Corollaries 3.9 and 3.12), pure subrings (Lemma 3.10, Proposition 3.11, and Corollary 3.13), direct limit (Lemma 4.1 and Theorem 4.5), and étale extensions (Corollaries 5.2 and 5.6). For splinters, the results are known, see [AD21] and [DT22], but our proof for the problem of étale exten-

sions is different. For (birational) derived splinters, extra work needs to be done, especially for the problem of direct limit.

1.2 Regular extensions and openness

A regular map of rings is a flat map of rings with geometrically regular fibers. A natural question to ask is if a regular map necessarily preserves the notions of splinter. This can be seen as a relative version of the direct summand theorem. We give partial answers to this question.

Theorem 1.1 (=Theorem 7.4). *Let $(A, \mathfrak{m}) \rightarrow (B, \mathfrak{n})$ be a regular homomorphism of Noetherian local rings with $\mathfrak{n} = \mathfrak{m}B$. If A is a splinter (resp. a birational derived splinter, a derived splinter), so is B .*

This recovers the main results of [DT22], and our proof is different. One key case that our theorem covers while [DT22] does not is the case $B = A[T]_{\mathfrak{m}A[T]}$ where T is an indeterminate. This, together with the openness results we discuss below, gives the following result, which implies the direct summand theorem in prime characteristic.

Theorem 1.2 (=Theorem 8.11). *Let $A \rightarrow B$ be a regular homomorphism of Noetherian $\mathbf{F}_p$ -algebras. If A is a splinter (resp. an F -pure birational derived splinter, a derived splinter), so is B .*

Openness of the splinter locus is considered in [DT21]. Parallel results can be obtained for derived splinters (although they follow formally from

[Bha12]), and F -pure birational derived splinters. We need to involve F -purity for the argument to work, and this property is automatic for (derived) splinters. We state and prove the following result in full generality available with full details. The essential work is already done in [DT21].

Theorem 1.3 (=Theorem 8.10). *Let A be a Noetherian $\mathbf{F}_p$ -algebra. Assume one of the following holds.*

- (1) *The ring A is F -finite, that is, the Frobenius map of A is finite.*
- (2) *A is essentially of finite type over a quasi-excellent local $\mathbf{F}_p$ -algebra.*
- (3) *A admits an F -pure [Has10, (2.1)] ring map essentially of finite type from an F -pure Noetherian local $\mathbf{F}_p$ -algebra.*

Then the locus of prime ideals $\mathfrak{p}$ of A such that $A_{\mathfrak{p}}$ is a splinter (resp. an F -pure birational derived splinter, a derived splinter) is open in $\mathrm{Spec}(A)$.

Only case (3) is necessary for Theorem 1.2, and in this case, A is F -pure, so the birational derived splinter locus is open. We also have the following curious result, see Lemma 8.7 and Corollary 8.8. The corresponding result in cases (1)(2) is well-known.

Theorem 1.4. *In case (3) of Theorem 1.3, the normal locus of A is open, and the normalization of A is finite.*

Finally, we note that the equal characteristic zero case is much simpler, at least for quasi-excellent rings, due to the existence of resolution of singularities. See Chapter 9. However, even in equal characteristic zero, for

non-quasi-excellent A , Theorem 1.1 does not seem to follow easily from another argument.

1.3 Ultraproduct and ultrapower

An ultraproduct is an amalgam of a family of objects that preserves certain properties satisfied by the “majority” of the objects. There is a choice of what “majority” is, given by the choice of an ultrafilter on the index set. This construction is ubiquitous in model theory. Specifically for rings, the book [Sch10] contains all preliminaries we need here, and much more. For completeness, we discuss the very basics of model theory of rings in §§A.1–A.2. We then give a model-theoretic proof of the fact an ultraproduct of splinters is a splinter in §A.3. Some complements are given in §A.4.

In the main part, we avoid model-theoretic definitions and techniques by giving an alternative definition of ultraproduct for local rings as the localization of the Cartesian product at a maximal ideal. This probably seems much friendlier from a commutative algebraist’s perspective and is easy to work with when we consider splittings. We will also only use ultrapowers, that is, ultraproducts where the rings are the same. However, this still gets us out of the realm of Noetherian rings, so we need to prepare for it in Chapter 2. Some other preliminaries are also included in said chapter. For non-Noetherian rings, our definition for (birational) derived splinters may be too strong, although satisfied by an ultrapower of a Noetherian (birational)

derived splinter. This is discussed in Appendix B.

The use of ultrapowers is for the proof of Theorem 1.1. In that situation, after a dévissage to the case A strictly Henselian, we show that B admits a pure map to an ultrapower of A . See Chapter 7.

1.4 Summary

To summarize, this thesis is structured as follows. In Chapter 2, we give preliminaries about rings and schemes. In Chapters 3–5, we define the notions of splinter and prove fundamental properties about them. Attentions are paid since we involve non-Noetherian rings and schemes. In Chapters 6–7, we use ultrapower to prove Theorem 1.1. In Chapter 8, we borrow the arguments from [DT21] to prove Theorem 1.3. Then we deduce Theorem 1.2. A short discussion of the equal characteristic zero case is given in Chapter 9. In Appendix A we study rings and splinters from a model-theoretic perspective. In Appendix B we discuss a weaker notion of (birational) derived splinters that may be more correct for non-Noetherian rings.

Chapter 2

Preparations

2.1 Some conventions

By direct limit we mean the colimit of a system indexed by a directed set. We do not gain more generality by allowing general filtered colimits with an index category instead of a set, see for example [\[Stacks, Tag 0032\]](#).

By qcqs we mean quasi-compact and quasi-separated.

By a generic point of a scheme we mean the generic point of an irreducible component.

Let A denote a ring.

The reduction of A is denoted by A_{red} .

The Krull dimension of A is denoted by $\dim A$.

An inductively étale A -algebra is the direct limit of a system of étale

A -algebras.

We do not distinguish between A -modules and quasi-coherent $\mathcal{O}_{\mathrm{Spec}(A)}$ -modules, and similarly $D(A) = D(QCoh(\mathcal{O}_{\mathrm{Spec}(A)})) = D_{QCoh}(\mathcal{O}_{\mathrm{Spec}(A)})$, cf. [Stacks, Tag 06Z0].

Refer to [Mat80, (34.A)] and [Stacks, Tag 07QS] about excellence. A local G-ring is quasi-excellent.

2.2 Regular maps

A regular map of rings is a flat map of rings with geometrically regular fibers. See [Stacks, Tag 07BZ].

We recall the Néron-Popescu theorem of regular maps. See [Stacks, Tag 07GC].

Theorem 2.1. *A regular homomorphism of Noetherian rings $A \rightarrow B$ is a direct limit of smooth homomorphisms $A \rightarrow B_i$.*

Lemma 2.2. *Let A be a ring, B an inductively étale A -algebra, C a B -algebra. Assume that B and C are Noetherian and that $A \rightarrow C$ is regular. Then $B \rightarrow C$ is regular.*

Proof. Flatness of $B \rightarrow C$ follows from [Stacks, Tag 092C], see [Stacks, Tags 092B and 092N]. Now we need to show that for each $\mathfrak{q} \in \mathrm{Spec}(B)$ the Noetherian ring $C \otimes_B \kappa(\mathfrak{q})$ is geometrically regular over $\kappa(\mathfrak{q})$. Let $\mathfrak{p} \in \mathrm{Spec}(A)$ be the preimage of $\mathfrak{q}$ in A . The ring $C \otimes_A \kappa(\mathfrak{p})$ is geometrically regular over

$\kappa(\mathfrak{p})$ and the ring $B \otimes_A \kappa(\mathfrak{p})$ is a finite direct product of separable algebraic extensions of $\kappa(\mathfrak{p})$, see [Stacks, Tag 0AH1]. Therefore $C \otimes_B \kappa(\mathfrak{q})$ is a direct factor of $C \otimes_A \kappa(\mathfrak{p})$, and $\kappa(\mathfrak{q})$ is separable algebraic over $\kappa(\mathfrak{p})$. We conclude by [Stacks, Tag 07QH]. $\square$

2.3 Residue extensions

Definition 2.3. A *residue extension* is a flat local map $\varphi : (A, \mathfrak{m}, k) \rightarrow (B, \mathfrak{n}, l)$ of Noetherian local rings such that $\mathfrak{n} = \mathfrak{m}B$. A *separable residue extension* is a residue extension where l/k is separable [Stacks, Tag 0300]; a *regular residue extension* is a residue extension where φ is regular.

A regular residue extension is always separable (cf. [Stacks, Tag 0322]).

The converse holds in the case A is a G-ring:

Lemma 2.4 ([And74]). *Let $\varphi : A \rightarrow B$ be a separable residue extension of Noetherian local rings. If A is a G-ring then φ is a regular residue extension.*

Proof. We have a commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{\varphi} & B \\ \iota_A \downarrow & & \downarrow \iota_B \\ A^\wedge & \xrightarrow{\varphi^\wedge} & B^\wedge \end{array}$$

of flat local maps of Noetherian local rings. Since φ is a separable residue extension, so is $\varphi^\wedge$, thus $\varphi^\wedge$ is regular [Stacks, Tag 07PM], and so is ι_A by the

definition of a G-ring. Since ι_B is faithfully flat we see φ regular by [Stacks, Tags 07QI and 07NT]. $\square$

2.4 Total fraction rings

Lemma 2.5. *Let A be a ring. Let S be a multiplicative subset of A that consists of nonzerodivisors. If $\dim(S^{-1}A) = 0$, then the total fraction ring of A is $S^{-1}A$.*

Proof. It suffices to show that every nonzerodivisor $x \in S^{-1}A$ is invertible. If not, take a maximal ideal $\mathfrak{m}$ of $S^{-1}A$ that contains x . Since every prime ideal of $S^{-1}A$ is maximal, $\mathfrak{m}$ is a minimal prime and x is nilpotent in $(S^{-1}A)_{\mathfrak{m}}$, contradiction. $\square$

Lemma 2.6. *Let X be a set, A_x ($x \in X$) be a family of reduced rings with total fraction rings K_x . If $\dim K_x = 0$ for all x , then $K := \prod_x K_x$ is the total fraction ring of $A := \prod_x A_x$, and $\dim K = 0$.*

Proof. Let S_x be the set of nonzerodivisors of A_x . Then the set of nonzerodivisors of A is $S = \prod_x S_x$, and $S^{-1}A = K$, so K is the total fraction ring of A .

The fact $\dim K = 0$ can be seen as follows. A reduced local ring A is a field if and only if $a^2A = aA$ for all $a \in A$. Thus a reduced ring A has Krull dimension 0 if and only if $a^2A = aA$ for all $a \in A$. This is clearly preserved by direct product. $\square$

Remark 2.7. Without the reducedness assumption, the product may be of infinite Krull dimension. See Example [A.27](#).

2.5 Classes of proper morphisms

To define birational derived splinters, we need to make sense of what is “birational.” We use the following class of morphisms that behaves better with respect to limit and base change. The name “M-morphism” indicates its close relation to modifications [[Stacks](#), Tag [0AAZ](#)]. See Lemma [2.15](#) below.

Definition 2.8. Let A be a ring. Let $f : X \rightarrow S = \operatorname{Spec}(A)$ be a morphism of schemes. We say f is an *M-morphism* if f is proper and of finite presentation, and for the total fraction ring K of A_{red} , the base change of f to $\operatorname{Spec}(K)$ is an isomorphism.

Note that we do not assume the irreducible components of X lying above those of S .

For convenience we also give names to two other classes of morphisms.

Definition 2.9. Let $f : X \rightarrow S$ be a morphism of schemes. We say f is an *FS-morphism* if f is surjective, finite, and of finite presentation. We say f is a *PS-morphism* if f is surjective, proper, and of finite presentation.

We note that an M-morphism is always a PS-morphism, since the ring map $A_{\text{red}} \rightarrow K$ is injective and thus $\operatorname{Spec}(K) \rightarrow \operatorname{Spec}(A)$ is dominant.

Lemma 2.10. *Let $A \rightarrow B$ be a ring map and $f : X \rightarrow \operatorname{Spec}(A)$ an M -morphism.*

Assume that every nonzerodivisor of A_{red} is mapped to a nonzerodivisor of B_{red} . Then the base change of f to B is an M -morphism.

Proof. We may assume both A and B are reduced. Now every nonzerodivisor of A is mapped to a nonzerodivisor of B , so the total fraction ring L of B is naturally a K -algebra where K is the total fraction ring of A . This proves the lemma. $\square$

Lemma 2.11. *Let A be a ring, U a multiplicative subset of A , $B = U^{-1}A$.*

Let $S = \operatorname{Spec}(A)$ and let T be either a quasi-compact open subscheme of S or $\operatorname{Spec}(B)$. Let $g : Y \rightarrow T$ be a PS-morphism (resp. an FS-morphism). Then there exists a PS-morphism (resp. an FS-morphism) $f : X \rightarrow S$ such that $X \times_S T \cong Y$.

Proof. Assume first that $T = \operatorname{Spec}(B)$. We know that $B = \operatorname{colim}_{u \in U} A_u$, so there exists an element $u \in U$ and a PS-morphism (resp. an FS-morphism) $f_1 : X_1 \rightarrow \operatorname{Spec}(A_u)$ such that $X_1 \times_{\operatorname{Spec}(A_u)} T \cong Y$, see [Stacks, Tag 01ZM], and [Stacks, Tags 07RR and 081F] (resp. [Stacks, Tags 07RR and 01Z0]). So we may replace T by $\operatorname{Spec}(A_u)$, and we may always assume T a quasi-compact open subscheme of S . By Nagata compactification [Stacks, Tag 0F41] (resp. Zariski's Main Theorem [Stacks, Tag 05K0]), there exists an open immersion $j : Y \rightarrow X_2$ where X_2 is proper (resp. finite) over S . Note that j is quasi-compact as T is. We may replace X_2 by the scheme-theoretic image of j to

assume j scheme-theoretically dominant, in particular $X_2 \times_S T = Y$. Note that X_2 is not necessarily of finite presentation over S . However, by [Stacks, Tag 09ZR] (resp. [Stacks, Tag 09YY]), $X_2 = \lim X_i$ where the transition maps are closed immersions and each X_i is proper (resp. finite) and of finite presentation over S . The system $Y = X_2 \times_S T \rightarrow X_i \times_S T$ of morphisms of X_i -schemes satisfies [Stacks, Tag 081D] (as $X_2 \rightarrow X_i$ is a closed immersion), so by [Stacks, Tag 081E], we see that there exists an i_0 such that $Y = X_{i_0} \times_S T$. We take $X = X_{i_0} \sqcup Z$, where $Z \rightarrow S$ is a closed immersion of finite presentation with image $S \setminus T$. $\square$

Lemma 2.12. *Let A be a ring, U a multiplicative subset of A , $B = U^{-1}A$. Let K be the total fraction ring of A and assume $\dim K = 0$; also assume A is reduced and integrally closed in K .*

Let $S = \operatorname{Spec}(A)$, and $T = \operatorname{Spec}(B)$. Let $g : Y \rightarrow T$ be an M -morphism. Then there exists an M -morphism $f : X \rightarrow S$ such that $X \times_S T \cong Y$.

Proof. By Lemma 2.11, there exists a PS-morphism $X_0 \rightarrow S$ such that $X_0 \times_S T = Y$. In particular $h : X_0 \times_S \operatorname{Spec}(K) \rightarrow \operatorname{Spec}(K)$ is a PS-morphism. We note that h is flat since K is reduced and $\dim K = 0$, see [Stacks, Tag 092F]. By [Stacks, Tags 0D4J and 02LS], there exists a unique clopen subset E_1 of $\operatorname{Spec}(K)$ such that h is finite over E_1 and that h does not have any finite fiber outside E_1 . Then $h_* \mathcal{O}_{X_0 \times_S \operatorname{Spec}(K)}$ is finite locally free on E_1 . We now see that there is a unique clopen subset E_0 of E_1 such that h is an isomorphism over E_0 and the fibers of h outside E_0 are non-trivial. Since A is integrally

closed in K , E_0 comes from a clopen subset E of S , and it is now clear that $X := (X_0 \times_S E) \sqcup (S \setminus E) \rightarrow S$ is a well-defined M-morphism.

Finally, we show $X \times_S T \cong Y$. Since $X_0 \times_S T \cong Y$, it suffices to show that the image of the morphism $T \rightarrow S$ is contained in E . As $\dim K = 0$, $\dim(U^{-1}K) = 0$, so $U^{-1}K$ is the total fraction ring of B , see Lemma 2.5. Therefore by construction and the fact that g is an M-morphism, the image of $\text{Spec}(U^{-1}K)$ in $\text{Spec}(K)$ is contained in E_0 . Since $\text{Spec}(U^{-1}K) \rightarrow T$ is dominant (as $B \rightarrow U^{-1}K$ is injective) the image of T is therefore contained in E as desired. $\square$

Lemma 2.13. *Let A be an integral domain and $f : X \rightarrow \text{Spec}(A)$ a PS-morphism. If there exists a (nonzero) A -algebra B such that $f_B : X \times_{\text{Spec}(A)} \text{Spec}(B) \rightarrow \text{Spec}(B)$ is an M-morphism, then f is an M-morphism.*

Proof. Replace B by its reduction, and then a residue field of its total fraction ring, we may assume B a field. Let $\mathfrak{p} = \ker(A \rightarrow B)$. Since A is an integral domain it suffices to show $f_{A_{\mathfrak{p}}}$ an M-morphism. Thus we may assume $(A, \mathfrak{p}, k)$ local. B is now a faithfully flat k -algebra, so f_k is an isomorphism. By Zariski's Main Theorem (cf. [Stacks, Tag 02UP]) f is finite. By Nakayama's Lemma f is a closed immersion. Since f is surjective and A is reduced, f is an isomorphism, as desired. $\square$

Corollary 2.14. *Let $(A_i)_i$ be a direct system of integral domains and let $A = \text{colim}_i A_i$. Let $f : X \rightarrow S := \text{Spec}(A)$ be an M-morphism. Then there exists an index i and an M-morphism $f_i : X_i \rightarrow S_i := \text{Spec}(A_i)$ whose base*

change to A is f .

Proof. There exists an index 0 and a morphism $f_0 : X_0 \rightarrow S_0$ of finite presentation whose base change to A is f , see [Stacks, Tag 01ZM]. We may assume f_0 proper surjective by [Stacks, Tags 081F and 07RR]. Then f_0 is an M -morphism by Lemma 2.13. $\square$

Birational morphisms are defined as in [Stacks, Tag 01RN].

Lemma 2.15. *Let A be a Noetherian reduced ring, $f : X \rightarrow S = \operatorname{Spec}(A)$ be an M -morphism. Then there exists a reduced closed subscheme Z of X such that $Z \rightarrow S$ is birational.*

Conversely, a proper birational morphism $g : Y \rightarrow S$ is an M -morphism.

Proof. For the first part we can just take Z to be the scheme-theoretic image of $X \times_S \operatorname{Spec}(K)$ in X where K is the total fraction ring of A .

Every proper A -scheme is of finite presentation since A is Noetherian. Therefore the second part follows immediately from [Stacks, Tag 0BAB]. $\square$

Lemma 2.16. *Let A be a reduced ring and let K be the total fraction ring of A . Let $X \rightarrow S = \operatorname{Spec}(A)$ be a PS -morphism.*

If $\dim K = 0$, then there exists a closed subscheme $Z \subseteq X$ such that $Z \rightarrow S$ is a PS -morphism and that $Z \times_S \operatorname{Spec}(K) \rightarrow \operatorname{Spec}(K)$ is finite.

Proof. We first give a quick proof in the case A is Noetherian. For each generic point $\eta \in S$, take a closed point x_η of the scheme X_η . We can then take Z the closure of the set of those x_η with reduced induced structure.

The general case is similar, but we have the problem that S may have infinitely many irreducible components. We proceed as follows. We claim that for every generic point $\eta \in S$, there exists a closed immersion $Z(\eta) \subseteq X$ of finite presentation such that $Z(\eta) \times_S \operatorname{Spec}(K) \rightarrow \operatorname{Spec}(K)$ is finite, and that the image of $Z(\eta)$ in S contains η (thus $\overline{\{\eta\}}$). If the claim is true, then by Chevalley's Theorem [Stacks, Tag 054K] and compactness of the constructible topology [Stacks, Tag 0901] we see that we can find finitely many $\eta_1, \dots, \eta_m$ such that the images of $Z(\eta_i)$ in S cover S . Now we can take Z to be the closed subscheme of X whose ideal sheaf is the product of those of $Z(\eta_i)$. Finiteness of $Z \times_S \operatorname{Spec}(K) \rightarrow \operatorname{Spec}(K)$ follows from, for example, Zariski's Main Theorem [Stacks, Tag 02LS].

It remains to show the claim. Fix $\eta \in S$ a generic point and let x_η be a closed point in X_η . Let Z_0 be the closure of x_η in X with reduced induced structure, so $(Z_0)_\eta$ is a singleton. By [Stacks, Tag 01PG], applied to the ideal sheaf of Z_0 , we see $Z_0 = \lim Z_i$ where $Z_i \subseteq X$ is a closed immersion of finite presentation. There exists an i such that $(Z_i)_\eta = (Z_0)_\eta$ since X_η is Noetherian. Fix such an i and let $h : (Z_i)_K \rightarrow \operatorname{Spec}(K)$ be the base change of $Z_i \rightarrow S$ to $\operatorname{Spec}(K)$. Then h is of finite presentation, and h is flat since $\dim K = 0$, see [Stacks, Tag 092F]. By [Stacks, Tags 0D4H and 05F9] we see that there exists a finitely generated ideal J of K such that $V(J) \subseteq \operatorname{Spec}(K)$ is exactly the locus where h has finite (or empty) fibers, in particular $\eta \in V(J)$. Write $J = IK$ where I is a finitely generated ideal of A , and let $Z(\eta) = Z_i \times_S \operatorname{Spec}(A/I)$. This finishes the proof. $\square$

2.6 Splitting in the derived category: trace ideals

Let A be a ring, $f : X \rightarrow S = \operatorname{Spec}(A)$ be a qcqs morphism of schemes. Then $Rf_*\mathcal{O}_X$ lies in $D^b(A)$, see [Stacks, Tag 08D5]. The formation of $Rf_*\mathcal{O}_X$ commutes with flat base change, see [Stacks, Tag 08IB]. Therefore if $A \rightarrow B$ is a flat ring map, then the same is true after any flat base change $A \rightarrow B$.

The canonical map $A \rightarrow Rf_*\mathcal{O}_X$ splits if and only if the induced map $\operatorname{Hom}_{D(A)}(Rf_*\mathcal{O}_X, A) \rightarrow \operatorname{Hom}_{D(A)}(A, A)$ of A -modules is surjective, by general category theory. The formation of this map commutes with flat base change if $Rf_*\mathcal{O}_X$ is pseudo-coherent [Stacks, Tag 08CB], see [Stacks, Tags 0A6A and 0A64]. This holds when A is Noetherian and f is proper (cf. [Stacks, Tag 08E8]).

Using the identification $A = \operatorname{Hom}_{D(A)}(A, A)$, the morphism f determines an ideal of A , namely the image of $\operatorname{Hom}_{D(A)}(Rf_*\mathcal{O}_X, A) \rightarrow \operatorname{Hom}_{D(A)}(A, A) = A$. We denote this ideal by $\mathfrak{t}(f)$ or $\mathfrak{t}(X/A)$. As noted in the previous paragraph, $A \rightarrow Rf_*\mathcal{O}_X$ splits if and only if $\mathfrak{t}(f) = A$, and the formation of $\mathfrak{t}(f)$ commutes with flat base change if A is Noetherian and f is proper. Note that in general, for any flat ring map $A \rightarrow B$, we have at least $\mathfrak{t}(X/A)B \subseteq \mathfrak{t}(X_B/B)$.

Chapter 3

Notions of splinter

We use the classes of morphisms defined before (Definitions 2.8 and 2.9) to define three notions of splinter.

Definition 3.1. A ring A is a *splinter* (resp. an *M-derived splinter*, a *derived splinter*), if for every FS-morphism (resp. M-morphism, PS-morphism) $f : X \rightarrow \mathrm{Spec}(A)$ the map $A \rightarrow Rf_*\mathcal{O}_X$ splits in $D(A)$.

We use the shorthand “MDS” for M-derived splinter.

Definition 3.2. Let X be a scheme. The *splinter locus* $\mathrm{Spl}(X)$ (resp. *MDS locus* $\mathrm{MDS}(X)$, *derived splinter locus* $\mathrm{DS}(X)$) of X is the set of points $x \in X$ such that $\mathcal{O}_{X,x}$ is a splinter (resp. an MDS, a derived splinter). For a ring A we write $\mathrm{Spl}(A)$ for $\mathrm{Spl}(\mathrm{Spec}(A))$, similar for MDS and derived splinter.

Lemma 3.3. *A ring A is reduced and integrally closed in its total fraction ring if and only if for every finite M-morphism $X = \mathrm{Spec}(B) \rightarrow \mathrm{Spec}(A)$,*

$A \rightarrow B$ splits.

In particular, a splinter, an MDS, or a derived splinter is reduced and integrally closed in its total fraction ring.

Proof. First, assume the latter condition. Let $a \in A$ be nilpotent. Then $\mathrm{Spec}(A/aA) \rightarrow \mathrm{Spec}(A)$ is an M-morphism (Definition 2.8), so $A \rightarrow A/aA$ is a split map of A -modules. Thus $a = 0$ and A is reduced.

Let K be the total fraction ring of A and $B \subseteq K$ be a finite A -algebra. We have $B = \mathrm{colim}_i B_i$ where B_i are finite of finite presentation over A and the transition maps $B_i \rightarrow B_j$ are surjective, see [Stacks, Tag 09YY]. Using either [Stacks, Tag 081E] or [Stacks, Tag 00Q0] we see that $(B_i)_K = K$ for large i , so $\mathrm{Spec}(B_i) \rightarrow \mathrm{Spec}(A)$ is an M-morphism. Thus $A \rightarrow B_i$ is split, and we see $aB_i \cap A = aA$. Taking colimit, we see that $aB \cap A = aA$ for all $a \in A$. Since $B \subseteq K$, $A = B$, so A is integrally closed in K .

Conversely, assume that A is reduced and integrally closed in its total fraction ring. Let $X = \mathrm{Spec}(B) \rightarrow \mathrm{Spec}(A)$ be a finite M-morphism. Then by definition $B_K = K$. Let B' be the image of B in B_K . Then by assumption $B' = A$, so $A \rightarrow B$ splits, even in the category of rings. $\square$

For Noetherian rings, the following definition may be more familiar.

Definition 3.4. A Noetherian ring A is a *birational derived splinter*, if A is reduced and for every proper birational morphism $f : X \rightarrow \mathrm{Spec}(A)$ the map $A \rightarrow Rf_*\mathcal{O}_X$ splits in $D(A)$.

Here, the notion of a birational morphism to a not necessarily integral scheme is defined in [Stacks, Tag 01RN].

This is the same notion as Definition 3.1 for Noetherian rings.

Lemma 3.5. *Let A be a Noetherian ring. Then A is an MDS if and only if A is a birational derived splinter, in which case A is normal.*

Proof. A Noetherian MDS is normal by Lemma 3.3, so we always have A reduced. The equivalence of MDS and birational derived splinter follows immediately from Lemma 2.15. $\square$

In this chapter and what follows, we shall use the term “birational derived splinter” for Noetherian rings and “MDS” for general rings to state our results.

Lemma 3.6. *Let X be a set and A_x ($x \in X$) a family of splinters (resp. MDSs, derived splinters). Then $A := \prod_{x \in X} A_x$ is a splinter (resp. an MDS, a derived splinter).*

Proof. We note first that a product of reduced rings is reduced, so by Lemma 3.3 we see A and all A_x reduced.

Let $g : Y \rightarrow \operatorname{Spec}(A)$ be an FS-morphism (resp. an M-morphism, a PS-morphism) and let $g_x : Y_x \rightarrow \operatorname{Spec}(A_x)$ be the base change of g along the projection map. We see g_x is an FS-morphism (resp. an M-morphism by Lemma 2.10, a PS-morphism) and thus $A_x \rightarrow Rg_{x*}\mathcal{O}_{Y_x}$ splits in $D(A_x)$. Taking product, we see that $A \rightarrow \prod_x Rg_{x*}\mathcal{O}_{Y_x}$ splits in $D(A)$. Since $A \rightarrow A_x$ is flat, $Rg_{x*}\mathcal{O}_{Y_x} = Rg_*\mathcal{O}_Y \otimes_A^L A_x$, so $A \rightarrow \prod_x Rg_{x*}\mathcal{O}_{Y_x}$ factors through $A \rightarrow Rg_*\mathcal{O}_Y$ and thus the latter map splits. $\square$

Remark 3.7. One can do the proof in terms of explicit complexes if not comfortable with infinite products in the derived category. Cover Y by affine opens $U^{(j)}$ ($1 \leq j \leq n$). Then we have the Čech complex $\check{C}^\bullet = \check{C}^\bullet(\mathcal{U}, \mathcal{O})$ that computes $Rg_*\mathcal{O}_Y$, and we see that the complex $\check{C}_x^\bullet := \check{C}^\bullet \otimes_A A_x$ is a Čech complex that computes $Rg_{x*}\mathcal{O}_{Y_x}$. Denote by φ_x the canonical map $A_x \rightarrow \check{C}_x^\bullet$, so φ_x splits in $D(A_x)$, which means that there exists an A_x -complex $F_x^\bullet$ together with a map $\psi_x : F_x^\bullet \rightarrow A_x$ and a quasi-isomorphism $s_x : F_x^\bullet \rightarrow \check{C}_x^\bullet$ such that $H^0(\psi_x) \circ H^0(s_x)^{-1} \circ H^0(\varphi_x) = \text{id}_{A_x}$. Taking product of these data we get an A -complex $F^\bullet = \prod_x F_x^\bullet$, a map $\psi = \prod_x \psi_x$ and a quasi-isomorphism $s = \prod_x s_x$ satisfying the same conditions and we see $\prod_x A_x \rightarrow \prod_x \check{C}_x^\bullet$ splits in $D(A)$. Finally, we have the canonical factorization $A = \prod_x A_x \rightarrow \check{C}^\bullet \rightarrow \prod_x \check{C}_x^\bullet$ in the category of cochain complexes of A -modules. Thus $A \rightarrow \check{C}^\bullet$ splits in $D(A)$ as desired.

Lemma 3.8. *Let A be a ring and let K be the total fraction ring of A . Assume that A is a splinter (resp. an MDS with $\dim K = 0$, a derived splinter) Then any localization $S^{-1}A$ is a splinter (resp. an MDS, a derived splinter).*

Proof. We need to find a splitting $S^{-1}A \rightarrow Rg_*\mathcal{O}_Y$ for every FS-morphism (resp. M-morphism, PS-morphism) $g : Y \rightarrow \text{Spec}(S^{-1}A)$. By Lemma 3.3 A is reduced and integrally closed in K , so by Lemma 2.11 or Lemma 2.12, g extends to an FS-morphism (resp. an M-morphism, a PS-morphism) $f : X \rightarrow \text{Spec}(A)$, and $A \rightarrow Rf_*\mathcal{O}_X$ splits. By flat base change (§2.6), $S^{-1}A \rightarrow Rg_*\mathcal{O}_Y$ splits, as desired. $\square$

A ring map $A \rightarrow B$ is *pure* if it is universally injective as a map of A -modules. See [Stacks, Tag 058H]. A faithfully flat ring map is pure.

Corollary 3.9 (cf. [DT22, Lemma 2.1.3]). *The followings are equivalent for an arbitrary ring A .*

- (1) *A is a splinter.*
- (2) *For every prime ideal $\mathfrak{p}$ of A , $A_{\mathfrak{p}}$ is a splinter.*
- (3) *For every maximal ideal $\mathfrak{m}$ of A , $A_{\mathfrak{m}}$ is a splinter.*

Proof. Lemma 3.8 gives that (1) implies (2); (2) trivially implies (3); and (3) implies (1) because for a finite and finitely presented ring map splitting is equivalent to purity, see [Stacks, Tag 058L], and purity can be checked locally, see [Stacks, Tag 05CL]. $\square$

Lemma 3.10 ([AD21, Proposition 5.2.5]). *Let $A \rightarrow B$ be a pure ring map. If B is a splinter, so is A .*

For the notions of MDS and derived splinter we introduce a Noetherian assumption to make our lives easier. However, note that B is not necessarily Noetherian, which is essential later on.

Proposition 3.11. *Let $A \rightarrow B$ be a pure ring map (resp. a pure map that sends nonzerodivisors to nonzerodivisors). Assume that A is Noetherian and that B is a derived splinter (resp. an MDS). Then A is a derived splinter (resp. a birational derived splinter).*

Proof. We note that B is reduced by Lemma 3.3, so A is reduced as a pure ring map is injective.

Let $f : X \rightarrow S = \operatorname{Spec}(A)$ be a PS-morphism (resp. an M-morphism), and $T = \operatorname{Spec}(B)$. The base change $f_T : X_T \rightarrow T$ is also a PS-morphism (resp. an M-morphism by Lemma 2.10). Since B is a derived splinter (resp. an MDS), $B \rightarrow Rf_{T*}\mathcal{O}_{X_T}$ splits in $D(B)$.

Let $\mathfrak{m}$ be a maximal ideal of A and let A' be the completion of A at $\mathfrak{m}$. By flat base change (§2.6), $B' \rightarrow Rf'_{T*}\mathcal{O}_{X'_T}$ splits in $D(B')$, where $(-)'$ is the base change along $A \rightarrow A'$. We also note that $A' \rightarrow B'$ is pure, and that A' is a Noetherian complete local ring, so $A' \rightarrow B'$ splits as a map of A' -modules by [Fed83, Lemma 1.2]. Therefore $A' \rightarrow Rf'_{T*}\mathcal{O}_{X'_T}$ splits in $D(A')$, and since this map factors through $A' \rightarrow Rf'_*\mathcal{O}_{X'}$, we see that $A' \rightarrow Rf'_*\mathcal{O}_{X'}$ splits in $D(A')$ as well.

Using the observations made in §2.6, we see that the ideal $\mathfrak{t}(f)$ satisfies $\mathfrak{t}(f)A' = A'$. Since $\mathfrak{m}$ was arbitrary, we see $\mathfrak{t}(f) = A$, as desired. $\square$

Corollary 3.12. *Let A be a Noetherian ring. The followings are equivalent.*

- (1) *A is a derived splinter (resp. a birational derived splinter).*
- (2) *$A_{\mathfrak{p}}$ is a derived splinter (resp. a birational derived splinter) for all prime ideals $\mathfrak{p}$ of A .*
- (3) *$A_{\mathfrak{m}}$ is a derived splinter (resp. a birational derived splinter) for all maximal ideals $\mathfrak{m}$ of A .*

Proof. We have (1) $\implies$ (2) by Lemma 3.8, and (2) $\implies$ (3) is trivial.

The implication (3) $\implies$ (1) follows from a similar but much simpler argument as in the proof of Proposition 3.11, but we can also obtain the result directly as follows. For a Noetherian ring A , the map $A \rightarrow \prod_{\mathfrak{m}} A_{\mathfrak{m}}$, where $\mathfrak{m}$ runs over all maximal ideals of A , is faithfully flat, cf. [Stacks, Tags 05CZ and 05CY]. Our result now follows from Lemma 3.6 and Proposition 3.11. $\square$

In the Noetherian case, we also have the following strengthening of Lemma 3.10 and Proposition 3.11.

Corollary 3.13. *Let $A \rightarrow B$ be a cyclically pure ring map, that is, for all ideals $I \subseteq A$ we have $IB \cap A = I$. If B is a Noetherian splinter (resp. birational derived splinter, derived splinter), so is A , and $A \rightarrow B$ is pure.*

Proof. From Lemma 3.3 or Lemma 3.5 we see B is normal. By [Has10, Corollary 3.12] (which ultimately comes from [Hoc77]), A is a Noetherian normal ring and $A \rightarrow B$ is pure.

The splinter (resp. derived splinter) case follows from Lemma 3.10 (resp. Proposition 3.11), so we assume B a birational derived splinter and we want to show so is A . For every $\mathfrak{p} \in \operatorname{Spec}(A)$, $A_{\mathfrak{p}} \rightarrow B_{\mathfrak{p}}$ is pure, so [HH95, Lemma 2.2] shows that there exists $\mathfrak{q} \in \operatorname{Spec}(B)$ such that $\mathfrak{q} \cap A \subseteq \mathfrak{p}$ and $A_{\mathfrak{p}} \rightarrow B_{\mathfrak{q}}$ is pure. Now $A_{\mathfrak{p}}$ and $B_{\mathfrak{q}}$ are integral domains, so $A_{\mathfrak{p}} \rightarrow B_{\mathfrak{q}}$ sends nonzerodivisors to nonzerodivisors. By Proposition 3.11 (and Lemma 3.8), $A_{\mathfrak{p}}$ is a birational derived splinter. Since this holds for all $\mathfrak{p} \in \operatorname{Spec}(A)$ we see A is a birational derived splinter from Corollary 3.12. $\square$

Chapter 4

Direct limit

The following result is not particularly difficult, see [AD21, Proposition 5.2.5].

Lemma 4.1. *A direct limit of splinters is a splinter.*

However, the corresponding result for derived splinters (resp. birational derived splinters) is not as easy. We need to do some work on derived categories and direct limit.

CONVENTION: by a complex of modules over a ring A we mean a cochain complex in the abelian category of A -modules. We usually denote it by $\square^\bullet$ where $\square$ is a capital Latin letter, in which case $\square$ denotes the corresponding object in the derived category $D(A)$.

4.1 Splitting in the derived category: from perfect to pseudo-coherent

In this section we establish the following result. This is inspired by a notion of purity in general categories, cf. [AR94, §2.D].

Lemma 4.2. *Let A be a ring, E a perfect object in $D(A)$ [Stacks, Tag 0657], F a pseudo-coherent object in $D(A)$ [Stacks, Tag 064Q].*

Let $\varphi : E \rightarrow F$ be a map in $D(A)$. If for all perfect objects $K \in D(A)$ and all factorizations $E \rightarrow K \rightarrow F$ of φ the map $E \rightarrow K$ is a split mono in $D(A)$, then φ is a split mono in $D(A)$.

Proof. By definition, we may represent the object F by a complex $F^\bullet$ of finite free A -modules such that $F^i = 0$ for i outside an interval $(-\infty, b]$ ($b \in \mathbf{Z}$); and we may represent the object E by a complex $E^\bullet$ of finite projective modules such that E^i are zero for i outside an interval $[a, b']$ ($a, b' \in \mathbf{Z}, a \leq b'$). Then φ is represented by a map of complexes $\varphi^\bullet : E^\bullet \rightarrow F^\bullet$, cf. [Stacks, Tag 064B].

Next, let $K^\bullet$ be the complex defined by $K^i = F^i$ ($i \geq a - 1$) and $K^i = 0$ ($i < a - 1$). (This is the stupid truncation $\sigma_{\geq a-1} F^\bullet$, see [Stacks, Tag 0118].) Then $K^\bullet$ is a perfect complex of Tor-amplitude $[a-1, b]$ that canonically maps to $F^\bullet$, and the map $\varphi^\bullet$ factors as $E^\bullet \xrightarrow{\psi^\bullet} K^\bullet \rightarrow F^\bullet$.

Now, by assumption, the map $\psi : E \rightarrow K$ is a split mono in $D(A)$. As $K^\bullet$ is a bounded above complex of projectives, there exists a map of complexes $\psi'^\bullet : K^\bullet \rightarrow E^\bullet$ such that $\psi' \circ \psi = \text{id}_E \in D(A)$, cf. [Stacks, Tag 064B].

Since $E^i = 0$ for $i < a$ and $F^i = K^i$ for $i \geq a - 1$, $\psi'^\bullet : K^\bullet \rightarrow E^\bullet$ extends canonically to a map $\varphi'^\bullet : F^\bullet \rightarrow E^\bullet$ of complexes by putting $\varphi'^i = 0$ when $i < a - 1$. Again, since F and K only differ in degrees $< a - 1$, $\varphi'^\bullet \circ \varphi^\bullet = \psi'^\bullet \circ \psi^\bullet$. Thus $\varphi' \circ \varphi = \text{id}_E \in D(A)$, as desired. $\square$

4.2 Splitting in the derived category: direct limit

In this section fix a direct system $(A_i)_{i \in \Lambda}$ of rings with colimit A .

Let $C_i^\bullet$ be complexes of A_i -modules equipped with compatible maps of A_i -complexes $C_i^\bullet \rightarrow C_j^\bullet$ for $i \leq j$. Then the termwise colimit $C^\bullet = \text{colim}_i C_i^\bullet$ is canonically a complex of A -modules and $H^n(C^\bullet) = \text{colim}_i H^n(C_i^\bullet)$ for all $n \in \mathbf{Z}$, since filtered colimit is exact.

Let $K^\bullet$ be a complex of finitely presented R -modules with finitely many nonzero terms. Then there exists an index i_0 and a complex $K_{i_0}^\bullet$ of finitely presented A_{i_0} -modules with finitely many nonzero terms such that $K_{i_0}^\bullet \otimes_{A_{i_0}} A = K^\bullet$ (cf. [Stacks, Tag 05N7]). Note that in general $K_{i_0} \otimes_{A_{i_0}}^L A \neq K \in D(A)$. Fix such a choice of i_0 and $K_{i_0}^\bullet$ and denote by $K_i^\bullet$ the tensor product $K_{i_0}^\bullet \otimes_{A_{i_0}} A_i$ for $i \geq i_0$, so $K^\bullet = \text{colim}_{i \geq i_0} K_i^\bullet$. Then we have (see [Stacks, Tags

[05DQ](#) and [0G8P](#)])

$$\begin{aligned}
\mathrm{Hom}_A(K^\bullet, C^\bullet) &= \mathrm{Hom}_{A_{i_0}}(K_{i_0}^\bullet, C^\bullet) \\
&= \mathrm{Hom}_{A_{i_0}}(K_{i_0}^\bullet, \mathrm{colim}_{i \geq i_0} C_i^\bullet) \\
&= \mathrm{colim}_{i \geq i_0} \mathrm{Hom}_{A_{i_0}}(K_{i_0}^\bullet, C_i^\bullet) \\
&= \mathrm{colim}_{i \geq i_0} \mathrm{Hom}_{A_i}(K_i^\bullet, C_i^\bullet).
\end{aligned}$$

Now, let $K \in D(A)$ be perfect [[Stacks](#), Tag [0657](#)] and let $\psi : K \rightarrow C$ be a map in $D(A)$. Then there exists a complex $K^\bullet$ of finite projective A -modules with finitely many nonzero terms and a map of complexes $\psi^\bullet : K^\bullet \rightarrow C^\bullet$ (see the proof of [[Stacks](#), Tag [0658](#)]) that represents ψ . Fix a choice of i_0 and $K_{i_0}^\bullet$ as above, and enlarge i_0 if necessary, we may assume that there exists a map of A_{i_0} -complexes $\psi_{i_0} : K_{i_0}^\bullet \rightarrow C_{i_0}^\bullet$ compatible with ψ . Since being finite projective is the same as being a direct summand of a finite free module we see from [[Stacks](#), Tag [05N7](#)] that enlarging i_0 if necessary we may assume each term of $K_{i_0}^\bullet$ finite projective, in particular flat. Then $K_{i_0} \otimes_{A_{i_0}}^L A = K \in D(A)$, and we have the following commutative diagram in $D(A)$:

$$\begin{array}{ccc}
K_{i_0} \otimes_{R_{i_0}}^L A & \xrightarrow{\psi_{i_0} \otimes 1} & C_{i_0} \otimes_{A_{i_0}}^L A \\
\downarrow \wr & & \downarrow \\
K & \xrightarrow{\psi} & C
\end{array}$$

If, furthermore, we are given an integer n and a cohomology class $\alpha_{i_0} \in H^n(C_{i_0}^\bullet)$ such that the image $\alpha \in H^n(C^\bullet)$ comes from a $\beta \in H^n(K^\bullet)$, then β

comes from some $\beta_{i_0} \in H^n(K_{i_0}^\bullet)$ after possibly enlarging i_0 . The class α_{i_0} does not necessarily agree with $\psi_{i_0}(\beta_{i_0})$. However, since $H^n(C^\bullet) = \operatorname{colim}_i H^n(C_i^\bullet)$, after possibly enlarging i_0 we will have $\alpha_{i_0} = \psi_{i_0}(\beta_{i_0})$. The discussions above lead us to

Lemma 4.3. *Let $A = \operatorname{colim}_i A_i$ be a direct limit of rings. Let i_0 be an index and $f_{i_0} : X_{i_0} \rightarrow \operatorname{Spec}(A_{i_0})$ be a quasi-compact and separated morphism of schemes. Let $f_i : X_i \rightarrow \operatorname{Spec}(A_i)$ and $f : X \rightarrow \operatorname{Spec}(A)$ be the base change of f_{i_0} to A_i and A respectively.*

Let $K \in D(A)$ be a perfect object and $\psi : K \rightarrow Rf_*\mathcal{O}_X \in D(A)$ a map in $D(A)$. Assume that there exists a class $\beta \in H^0(K)$ that maps to $1 \in f_*\mathcal{O}_X$.

Then for large enough i , ψ factors through the canonical map $Rf_{i*}\mathcal{O}_{X_i} \otimes_{A_i}^L A \rightarrow Rf_*\mathcal{O}_X$, in a way that $\beta \in H^0(K)$ is mapped to $1 \in H^0(Rf_{i*}\mathcal{O}_{X_i} \otimes_{A_i}^L A)$.

Proof. Fix a finite affine cover $X_{i_0} = \bigcup_{j=1}^m U_{i_0}^{(j)}$. Then the Čech complex $\check{C}_{i_0}^\bullet$ associated with this open cover and the structure sheaf $\mathcal{O}_{X_{i_0}}$ computes $Rf_{i_0*}\mathcal{O}_{X_{i_0}}$, see [Stacks, Tag 01XD]. We can use the base change of this open cover to compute $Rf_{i*}\mathcal{O}_{X_i}$ ($i \geq i_0$) and $Rf_*\mathcal{O}_X$ with corresponding Čech complexes $\check{C}_i^\bullet = \check{C}_{i_0}^\bullet \otimes_{A_{i_0}} A_i$ and $\check{C}^\bullet = \check{C}_{i_0}^\bullet \otimes_{A_{i_0}} A = \operatorname{colim}_{i \geq i_0} \check{C}_i^\bullet$. The canonical class $1 \in H^0(\check{C}_i^\bullet)$ is defined by the unit element of the ring $\check{C}_i^0 = \prod_{j=1}^m \mathcal{O}_{X_i}(U_i^{(j)})$. Therefore we can apply the discussions above to the system

$\check{C}_i^\bullet$ and for some $i \geq i_0$ find a commutative diagram

$$\begin{array}{ccc} K_i \otimes_{A_i}^L A & \xrightarrow{\psi_i \otimes 1} & \check{C}_i \otimes_{A_i}^L A \\ \wr \downarrow & & \downarrow \\ K & \xrightarrow{\psi} & \check{C} \end{array}$$

in $D(A)$ and a class $\beta_i \in H^0(K_i^\bullet)$ that maps to $\beta \in H^0(K)$ and to $1 \in H^0(\check{C}_i)$ by ψ_i . We win. $\square$

Remark 4.4. Using the language of derived ∞ -categories [HA, §1.3.2], we can prove the lemma above as follows. We may assume that i_0 is the smallest element of the index set. The Čech complex consideration tells us $Rf_*\mathcal{O}_X = \operatorname{colim}_i Rf_{i*}\mathcal{O}_{X_i}$ in the (bounded-above) derived ∞ -category of abelian groups, and this identification preserves A_i -structure. (This colimit is the “homotopy colimit” (cf. [Stacks, Tag 090Z]).) Therefore

$$Rf_*\mathcal{O}_X = \operatorname{colim}_{(i,j): i \leq j} (Rf_{i*}\mathcal{O}_{X_i} \otimes_{A_i}^L A_j)$$

since $\{(i, i) \mid i \in \Lambda\}$ is a cofinal subset of $\{(i, j) \mid i \leq j\}$. Taking colimit with respect to j we get

$$Rf_*\mathcal{O}_X = \operatorname{colim}_i (Rf_{i*}\mathcal{O}_{X_i} \otimes_{A_i}^L A)$$

in the (bounded-above) derived ∞ -category of A -modules. The perfect object K is a compact object in this ∞ -category (cf. [HA, Propositions 7.2.4.2 and

7.1.1.15]), so we obtain a factorization $K \rightarrow Rf_{i*}\mathcal{O}_{X_i} \otimes_{A_i}^L A$ for some i .

Enlarging i we may assume β is mapped to 1 and we win.

4.3 Direct limit of Noetherian (birational) derived splinters

We turn to the main result of this chapter.

Theorem 4.5. *Let $(A_i)_i$ be a direct system of Noetherian rings with colimit A . Assume that A is Noetherian, and that each A_i is a derived splinter (resp. a birational derived splinter). Then A is a derived splinter (resp. a birational derived splinter).*

Proof. Since A and all A_i are Noetherian, the problem is local by Corollary 3.12. We may therefore assume that A is local, that each A_i is local, and that the transition maps are local. By Lemma 3.5 each A_i is normal, hence an integral domain. (For the derived splinter case this reduction is not necessary.)

Let $f : X \rightarrow \operatorname{Spec}(A)$ be a PS-morphism (resp. an M-morphism). By [Stacks, Tags 01ZM, 081F, and 07RR] (resp. Corollary 2.14), there exists an index i_0 and a PS-morphism (resp. an M-morphism) $f_{i_0} : X_{i_0} \rightarrow \operatorname{Spec}(A_{i_0})$ whose base change to A gives f . As usual, denote by $f_i : X_i \rightarrow \operatorname{Spec}(A_i)$ the base change of f_{i_0} to $\operatorname{Spec}(A_i)$ where $i \geq i_0$. Each f_i is a PS-morphism (resp. an M-morphism by Lemma 2.13) and thus $A_i \rightarrow Rf_{i*}\mathcal{O}_{X_i}$ splits in $D(A_i)$.

Therefore we obtain a map $\epsilon_i : Rf_{i*}\mathcal{O}_{X_i} \otimes_{A_i}^L S \rightarrow S$ in $D(A)$ that sends 1 to 1.

Now, let $K \in D(A)$ be perfect and let $A \xrightarrow{\varphi} K \xrightarrow{\psi} Rf_*\mathcal{O}_X$ be a factorization of the canonical map $A \rightarrow Rf_*\mathcal{O}_X$. Since A is Noetherian, $Rf_*\mathcal{O}_X \in D(A)$ is pseudo-coherent (cf. [Stacks, Tag 08E8]). Thus by Lemma 4.2 it suffices to show that φ is a split mono in $D(A)$.

Denote by β the image of $1 \in A$ in $H^0(K)$, so ψ maps β to 1. By Lemma 4.3, for large i there exists a map $\delta_i : K \rightarrow Rf_{i*}\mathcal{O}_{X_i} \otimes_{A_i}^L A$ in $D(A)$ that sends β to 1. Composing with ϵ_i , we get a map $K \rightarrow A$ in $D(A)$ that sends β to 1, in other words, a splitting of φ , as desired. $\square$

Chapter 5

Étale ascent

Lemma 5.1. *Let $T \rightarrow S$ be an étale morphism of schemes. Let $Y \rightarrow T$ be a surjective integral morphism of schemes. Then the followings hold.*

- (1) *Assume that S has finitely many irreducible components. Then there exists a surjective integral morphism $\tilde{S} \rightarrow S$ that admits a T -morphism $T \times_S \tilde{S} \rightarrow Y$.*
- (2) *Assume that S and T are qcqs and $Y \rightarrow T$ is of finite presentation (i.e. an FS-morphism). Then there exists an FS-morphism $\tilde{S} \rightarrow S$ that admits a T -morphism $T \times_S \tilde{S} \rightarrow Y$.*

Proof. For (1), fix an algebraic closure Ω_η of $\kappa(\eta)$ for each generic point $\eta \in S$, and let $\Omega = \prod_\eta \Omega_\eta$. Note that this is a finite product. We get a dominant morphism $\mathrm{Spec} \Omega \rightarrow S$, which is affine because every open subscheme of $\mathrm{Spec} \Omega$ is affine. Let $\tilde{S}$ be the normalization of S in $\mathrm{Spec} \Omega$ [Stacks, Tag

[035H](#)]. Then $\tilde{S}$ is normal by [\[Stacks, Tag 035L\]](#), and it is clear that $\tilde{S} \rightarrow S$ is surjective and integral.

Consider the base change $T \times_S \text{Spec } \Omega \rightarrow T \times_S \tilde{S} \rightarrow T$. By [\[Stacks, Tag 033C\]](#) $T \times_S \tilde{S}$ is normal, so it is the normalization of T in $T \times_S \text{Spec } \Omega$. On the other hand, $T \times_S \text{Spec } \Omega$ is a disjoint union of spectra of algebraically closed fields, see [\[Stacks, Tag 02GL\]](#). Since $Y \rightarrow T$ is integral and surjective, there exists a T -morphism $T \times_S \text{Spec } \Omega \rightarrow Y$. Thus there exists a T -morphism $T \times_S \tilde{S} \rightarrow Y$ by the universal property [\[Stacks, Tag 035I\]](#).

To show [\(2\)](#), note that $T \rightarrow S$ is of finite presentation, so by the general theory of limits [\[Stacks, Tags 01ZA, 01ZM, 01ZO, 07RP, and 07RR\]](#), we may further assume S of finite type over $\mathbf{Z}$, in particular having finitely many irreducible components. Now the result follows from [\(1\)](#) and a further application of the general theory of limits [\[Stacks, Tags 09YZ and 01ZC\]](#). $\square$

We can now remove the Noetherian assumption in [\[DT22, Theorem A\]](#). Our proof is different from theirs.

Corollary 5.2. *Let A be a splinter and let B be an inductively étale A -algebra. Then B is a splinter.*

Proof. By [Lemma 4.1](#) we may assume $A \rightarrow B$ étale. The result follows immediately from [Lemma 5.1\(2\)](#). $\square$

Remark 5.3. For the sake of completeness, we note that [Lemma 5.1\(1\)](#) holds as well if we replace the assumption S having finitely many irreducible components by that S and T are qcqs. To see this, let Ω_η and Ω be as in the proof

of Lemma 5.1, so $\dim \Omega = 0$, see Lemma 2.6. From [Bha16, Theorem 1.3] we get a dominant morphism $\mathrm{Spec} \Omega \rightarrow S^1$. This morphism is affine because every quasi-compact open of $\mathrm{Spec} \Omega$ is clopen, see for example [Stacks, Tag 0903]. We still have $\tilde{S}$ normal, in the sense that $\mathcal{O}_{\tilde{S},s}$ is a normal domain for all $s \in \tilde{S}$, cf. Discussion B.8. Therefore $T \times_S \tilde{S}$ is the normalization of T in $T \times_S \mathrm{Spec} \Omega$. Now, it is easy to see that for each η , there exists a T -morphism $T \times_S \mathrm{Spec} \Omega_\eta \rightarrow Y$. Thus there exists a T -morphism $T \times_S \mathrm{Spec} \Omega \rightarrow Y$ by [Bha16, Theorem 1.3] again.

Note that you cannot take, for example, the disjoint union of all irreducible components of S , since it is not quasi-compact and hence the map to S is not integral.

For (birational) derived splinters, we need the following result. It can be understood as a version of Chow's lemma for algebraic spaces [RG71, Corollaire 5.7.13]. Nevertheless, for the reader's convenience, we (re-)do the proof using just schemes. The arguments here are similar, but not identical, to those in [RG71, §5.7].

Theorem 5.4. *Let A be a Noetherian reduced ring, $A \rightarrow B$ an étale ring map. Let $g : Y \rightarrow T := \mathrm{Spec}(B)$ be a PS-morphism (resp. an M -morphism). Then there exists a PS-morphism (resp. an M -morphism) $f : X \rightarrow S := \mathrm{Spec}(A)$ that admits a morphism $X \times_S T \rightarrow Y$ of T -schemes.*

¹Since each Ω_η is a field, we can use an elementary argument as follows. $\mathrm{Spec} \Omega$ is the Stone-Čech compactification of $\sqcup_\eta \mathrm{Spec} \Omega_\eta$, so we get a map $\mathrm{Spec} \Omega \rightarrow S$ of sets continuous with respect to the constructible topologies. If U is an affine open of S , then both U and $S \setminus U$ are constructible. Thus one sees that the preimage of U is exactly $\mathrm{Spec} \left(\prod_{\eta \in U} \Omega_\eta \right)$, and we get a map of sheaves. Details omitted.

In particular, if A is a derived splinter (resp. birational derived splinter), so is B .

Remark 5.5. We can weaken the assumption A Noetherian to that the total fraction ring of A has Krull dimension zero. The proof is almost the same, with some technical caution regarding finite presentation.

Proof. The “in particular” statement follows from flat base change, see §2.6. By Lemma 2.16 we may assume g generically finite.

By Zariski’s Main Theorem [Stacks, Tag 00QB] there exists a factorization $T \xrightarrow{j} \bar{T} \xrightarrow{\pi} S$ where π is finite and j is a scheme-theoretically dominant open immersion. Write $\bar{T} = \operatorname{Spec}(\bar{B})$. For the total fraction ring K of A , we have $B \otimes_A K = \bar{B} \otimes_A K$, because a dense subset of the discrete space $\operatorname{Spec}(\bar{B} \otimes_A K)$ must be the whole space.

By Lemma 2.11 (which has a much shorter proof in the Noetherian case), there exists a PS-morphism $\bar{g} : \bar{Y} \rightarrow \bar{T}$ such that $\bar{Y} \times_{\bar{T}} T = Y$. We know $B \otimes_A K$ is the total fraction ring of $B = B_{\text{red}}$ (cf. Lemma 2.5). Since g is generically finite (resp. an M-morphism), g is finite (resp. an isomorphism) after the base change $A \rightarrow K$. Since $\bar{B} \otimes_A K = B \otimes_A K$ the same is true for $\bar{g}$. Picture:

$$\begin{array}{ccc}
 \bar{Y} \times_S \operatorname{Spec}(K) & \longrightarrow & \bar{Y} \\
 \wr \downarrow & & \downarrow \bar{g} \\
 \bar{T} \times_S \operatorname{Spec}(K) & \longrightarrow & \bar{T} \\
 \text{finite flat} \downarrow & & \downarrow \\
 \operatorname{Spec}(K) & \longrightarrow & S
 \end{array}$$

By [Stacks, Tag 02KB] we know that a finite flat morphism of finite presentation is finite locally free (see [Stacks, Tag 02KA] for definition). By [Stacks, Tag 06AC], the composition $\bar{Y} \rightarrow \bar{T} \rightarrow S$ is finite locally free after the base change $\text{Spec}(A_a) \rightarrow S$ for some nonzerodivisor $a \in A$. By a consequence [Stacks, Tag 0B49] of “flattening” [RG71, Théorème 5.2.2], there exists a blow-up $f : X_0 \rightarrow S$ of an ideal supported on $\text{Spec}(A/aA)$ set-theoretically and a closed subscheme $\bar{Y}' \subseteq \bar{Y} \times_S X_0$ finite locally free over X_0 that agrees with $\bar{Y} \times_S X_0$ over $\text{Spec}(A_a) \subseteq S$. The last bit follows from the definition of strict transform [Stacks, Tag 080D]. We see that $f : X_0 \rightarrow S$ is proper and that the base change of f to $\text{Spec}(A_a)$ is an isomorphism, so f is an M-morphism. Picture:

$$\begin{array}{ccccc}
\bar{Y}' & \longrightarrow & \bar{Y} \times_S X_0 & \longrightarrow & \bar{Y} \\
& & \downarrow & & \downarrow \bar{g} \\
& & \bar{T} \times_S X_0 & \longrightarrow & \bar{T} \\
& & \downarrow & & \downarrow \bar{\pi} \\
& & X_0 & \xrightarrow{f} & S
\end{array}$$

Let $Y' = \bar{Y}' \times_{\bar{T}} T$ and $h : Y' \rightarrow T \times_S X_0$ be the morphism induced by $\bar{g}$.

Then the following diagram of schemes is commutative:

$$\begin{array}{ccc}
Y' & \longrightarrow & Y \\
h \downarrow & & \downarrow g \\
T \times_S X_0 & \longrightarrow & T \\
\downarrow & & \downarrow \\
X_0 & \xrightarrow{f} & S
\end{array}$$

By our choice $Y' \rightarrow X_0$ is quasi-finite and flat. Since $T \times_S X \rightarrow X$ is étale, h is quasi-finite and flat (cf. [Stacks, Tag 04R3]). However, by construction, h is proper, so h is finite flat. Note that $X_0 \rightarrow S$ is a blowup, so a nonzerodivisor in A is always a nonzerodivisor on X_0 . Thus $X_0 \times_S \text{Spec}(K) \rightarrow X_0$ is (scheme-theoretically) dominant. Then the same is true after the flat base change $T \rightarrow S$, *i.e.* the canonical map $b : T \times_S X_0 \times_S \text{Spec}(K) \rightarrow T \times_S X_0$ is dominant.

In the case g is an M-morphism, we have $\overline{Y}' = \overline{Y} \times_S X_0$ over $\text{Spec}(A_a)$, so $Y' = Y \times_S X_0$ over $\text{Spec}(A_a)$; also $Y = T$ after the base change $A \rightarrow K$ as noted before. Thus h is an isomorphism after the base change $A \rightarrow K$. Therefore the locus where the degree of h is 1 contains the image of b , hence equals to $T \times_S X_0$, so we can take $X = X_0$.

In the case g is a (generically finite) PS-morphism, we still have that h agrees with g after the base change $A \rightarrow K$, so h is surjective. Applying Lemma 5.1(2), we see that there exists an FS-morphism $X \rightarrow X_0$ that admits a morphism $T \times_S X \rightarrow Y'$ of $(T \times_S X_0)$ -schemes. This X is what we want. $\square$

Corollary 5.6. *Let A be a Noetherian derived splinter (resp. birational derived splinter) and let B be an inductively étale A -algebra. If B is Noetherian, then B is a derived splinter (resp. a birational derived splinter).*

Proof. Immediate from Theorems 4.5 and 5.4. Alternatively, one can apply Corollaries 3.12 and 2.14 and flat base change (§2.6) to avoid the materials in Chapter 4; details of this route are left to the reader. $\square$

Remark 5.7. We can also deduce from Theorem 5.4 that if $A \rightarrow B$ is a local map of Noetherian local rings that is inductively étale, and A is pseudo-rational in the sense of [LT81], then so is B .

Remark 5.8. By [BS15, Theorem 2.3.4] and flat descent, we see that if a Noetherian ring A is a splinter (resp. a birational derived splinter, a derived splinter) and $A \rightarrow B$ is a weakly étale ring map [Stacks, Tag 092B], then B is a splinter (resp. a birational derived splinter, a derived splinter).

Chapter 6

Ultraproduct of some local rings

We study ultraproducts in the appendix, §A.2. The following alternative definition for local rings is convenient for us. It is equivalent with the usual Definition A.10, see Theorem A.14, so we may apply results from the appendix, and from [Sch10].

Definition 6.1. Let X be an index set and A_x ($x \in X$) be a family of local rings. An *ultraproduct* of the rings A_x ($x \in X$) is the localization of the ring $\prod_x A_x$ at a maximal ideal.

An *ultrapower* of a local ring A is an ultraproduct of a family A_x ($x \in X$) where each $A_x = A$.

Note that an ultrapower of a local ring A is canonically an A -algebra via the diagonal map.

The proof of the next lemma works without the local assumption with the general definition of ultraproduct, see Remark A.11. In the splinter case,

it holds for another reason, see Corollary [A.23](#).

Lemma 6.2. *Let A_x ($x \in X$) be a family of local rings, and $A = (\prod_{x \in X} A_x)_{\mathfrak{M}}$ be an ultraproduct. If each A_x is a splinter (resp. an MDS whose total fraction ring has Krull dimension 0, a derived splinter), then A is a splinter (resp. an MDS, a derived splinter).*

Proof. $\prod_x A_x$ is a splinter (resp. an MDS, a derived splinter) by Lemma [3.6](#).

Note that A_x is reduced by Lemma [3.3](#). Let K_x be the total fraction ring of A_x . By Lemma [2.6](#), in the MDS case $\prod_x K_x$ is the total fraction ring of $\prod_x S_x$ and $\dim \prod_x K_x = 0$. The result now follows from Lemma [3.8](#). $\square$

The next lemma also holds without the local assumption (Corollary [A.13](#)). We give an algebraic proof in the local case, which, this time, is of the same nature as the proof of Łoś's Theorem [A.12](#).

Lemma 6.3. *An ultraproduct of a family of local domains is an integral domain.*

Proof. Let $A = (\prod_{x \in X} A_x)_{\mathfrak{M}}$ be an ultraproduct of a family of local domains A_x ($x \in X$). We must show if $a = (a_x)_{x \in X}$ and $b = (b_x)_{x \in X}$ are two elements in $\prod_{x \in X} A_x$ such that $ab = 0$, then there exists an $s \notin \mathfrak{M}$ such that $sa = 0$ or $sb = 0$.

Let $U = \{x \in X \mid a_x = 0\}$ and $V = \{x \in X \mid b_x = 0\}$, so $U \cup V = X$ as A is an integral domain. Let $e_U \in \prod_{x \in X} A_x$ be the idempotent defined by $(e_U)_x = 1$ when $x \in U$ and $(e_U)_x = 0$ when $x \notin U$ and define e_V similarly.

Then $(1 - e_U)(1 - e_V) = 0$, so either $e_U \notin \mathfrak{M}$ or $e_V \notin \mathfrak{M}$. But $e_U a = 0 = e_V b$.

We win. □

Chapter 7

Regular residue extensions

Lemma 7.1. *Let $A \rightarrow B$ be a regular homomorphism of Noetherian rings. Assume that A is local and strictly Henselian. Then there exists an ultrapower of A that admits an A -algebra map from B .*

Proof. Write $B = \operatorname{colim}_i B_i$ where each B_i is smooth over A , possible by Popescu's Theorem 2.1. For each i , there exists an A -algebra map $B_i \rightarrow A$, as A is strictly Henselian (cf. [Stacks, Tag 055U]).

Let C be a finitely generated A -subalgebra of B . Then $C \subseteq B$ factors through some B_i , see [Stacks, Tag 00Q0]. Thus C also admits an A -algebra map $C \rightarrow A$. We conclude by [Sch10, Theorem 7.1.1]. $\square$

Remark 7.2. To get a proof with only algebraic constructions and no ultrafilter, one can apply the idea of Remark A.11. Instead of forming the ultrapower as in the proof of [Sch10, Theorem 7.1.1], one write down the

idempotents e_U and localize at a maximal ideal containing all $1 - e_U$. Details left to the reader.

Lemma 7.3. *Let $(A, \mathfrak{m}) \rightarrow (B, \mathfrak{n})$ be a flat homomorphism of Noetherian local rings with $\mathfrak{n} = \mathfrak{m}B$. Then any A -algebra map from B to an ultrapower of A is pure.*

Proof. Let our ultrapower be $A' = (A^X)_{\mathfrak{M}}$. We know from [Stacks, Tags 05CZ and 05CY] that A^X is flat over A . Then A' is flat over A . Since B is also flat over A , we have

$$\mathrm{Tor}_1^B(A', B/\mathfrak{m}B) = \mathrm{Tor}_1^A(A', A/\mathfrak{m}) = 0.$$

Since $B/\mathfrak{m}B = B/\mathfrak{n}$ is a field, the $\mathfrak{n}$ -adic completion C of the B -algebra A' is flat over B by [Stacks, Tag 0AGW]. Now note that $\mathfrak{M}$ lies above $\mathfrak{m}$ by [Stacks, Tag 0AME]. Therefore $C/\mathfrak{m}C = A'/\mathfrak{m}A'$ (cf. [Stacks, Tag 05GG]) is nonzero and C is faithfully flat over B . Therefore $B \rightarrow C$ is pure and so is $B \rightarrow A'$. $\square$

Theorem 7.4. *Let $(A, \mathfrak{m}) \rightarrow (B, \mathfrak{n})$ be a regular homomorphism of Noetherian local rings with $\mathfrak{n} = \mathfrak{m}B$. If A is a splinter (resp. a birational derived splinter, a derived splinter), so is B . In particular, the completion of a local G -ring that is a splinter (resp. a birational derived splinter, a derived splinter) is a splinter (resp. a birational derived splinter, a derived splinter).*

Proof. There is a commutative diagram

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ A^{sh} & \longrightarrow & B^{sh} \end{array}$$

where the vertical maps are strict Henselization, see [Stacks, Tag 04GU]. The rings A^{sh} and B^{sh} are Noetherian by [Stacks, Tag 06LJ]; and $A \rightarrow B^{sh}$ is regular by [Stacks, Tags 07EP and 07QI]; thus $A^{sh} \rightarrow B^{sh}$ is regular by Lemma 2.2.

By flat descent (Lemma 3.10 or Proposition 3.11) it suffices to show B^{sh} a splinter (resp. a birational derived splinter, a derived splinter). By Corollary 5.2 or Corollary 5.6, we may therefore assume A strictly Henselian.

By Lemmas 7.1 and 7.3, there exists a pure ring map $B \rightarrow A'$, where A' is an ultrapower of A . Then A' is a splinter (resp. a birational derived splinter, a derived splinter), by Lemma 6.2. We conclude by Lemma 3.10 (resp. Proposition 3.11) in the splinter (resp. derived splinter) case. In the birational derived splinter case, note that A is normal (Lemma 3.3), hence an integral domain, so A' is an integral domain (Lemma 6.3). Thus $B \rightarrow A'$ sends nonzerodivisors to nonzerodivisors, and we conclude by Proposition 3.11. $\square$

Corollary 7.5. *Let A be a Noetherian local G -ring that is a splinter (resp. a birational derived splinter, a derived splinter). Then any separable residue extension (Definition 2.3) of A is a splinter (resp. a birational derived splinter).*

ter, a derived splinter).

Proof. In view of Lemma 2.4, this is the special case of Theorem 7.4 where A is a G-ring. $\square$

Chapter 8

Openness in prime characteristic and regular ascent

8.1 Trace ideals

We use the notations in [DT21, §2.3] (Frobenius pushforward), and in our §2.6 (trace ideal).

Definition 8.1 (= [DT21, Definition 3.1.1]). An ideal $\mathfrak{a}$ of an $\mathbf{F}_p$ -algebra A is *uniformly F -compatible* if for all $e \in \mathbf{Z}_{>0}$ and all A -linear maps $\varphi : F_*^e A \rightarrow A$, we have $\varphi(F_*^e \mathfrak{a}) \subseteq \mathfrak{a}$.

Lemma 8.2 (cf. [DT21, Lemma 3.2.3]). *Let A be an $\mathbf{F}_p$ -algebra and X a qcqs A -scheme. Then the ideal $\mathfrak{t}(X/A)$ is uniformly F -compatible.*

Proof. Let $f : X \rightarrow \mathrm{Spec}(A)$ be the structural map and consider the following

commutative diagram of schemes

$$\begin{array}{ccc} X & \xrightarrow{F^e} & X \\ f \downarrow & & f \downarrow \\ \mathrm{Spec}(A) & \xrightarrow{F^e} & \mathrm{Spec}(A) \end{array}$$

where F^e is the e -th iterated absolute Frobenius of X or $\mathrm{Spec}(A)$. The map $\mathcal{O}_X \rightarrow F_*^e \mathcal{O}_X$ induces a map $Rf_* \mathcal{O}_X \rightarrow Rf_* F_*^e \mathcal{O}_X = F_*^e Rf_* \mathcal{O}_X$, the identity because F^e is affine. If $\eta : Rf_* \mathcal{O}_X \rightarrow A$ is a map in $D(A)$, and $\varphi : F_*^e A \rightarrow A$ is an A -linear map, then the composition

$$\eta' : Rf_* \mathcal{O}_X \rightarrow Rf_* F_*^e \mathcal{O}_X = F_*^e Rf_* \mathcal{O}_X \xrightarrow{F_*^e \eta} F_*^e A \xrightarrow{\varphi} A$$

is also a map in $D(A)$. If the image of η in A is a then it is clear that the image of η' in A is $\varphi(F_*^e a)$, so $\varphi(F_*^e \mathfrak{t}(X/A)) \subseteq \mathfrak{t}(X/A)$ as desired. $\square$

Consider the following conditions on an $\mathbf{F}_p$ -algebra A .

Condition 8.3. A is essentially of finite type over a Noetherian F -pure local ring R , and $R \rightarrow A$ is F -pure.

Condition 8.4. A is either F -finite, or essentially of finite type over a Noetherian local G-ring R .

Theorem 8.5 (cf. [DT21, Theorem 4.3.1]). *Let A be a ring. Assume that A satisfies Condition 8.3, or satisfies Condition 8.4 and is F -pure.*

Then the set of ideals $\Sigma_A := \{\mathfrak{t}(X/A) \mid X \text{ proper over } A\}$ is finite.

Proof. If A is F -finite, then A is F -split since A is Noetherian and F -pure. Therefore the number of uniformly F -compatible ideals of A is finite, see [DT21, Proposition 3.4.1], so we conclude by Lemma 8.2 that Σ_A is finite. If A is essentially of finite type over a Noetherian local G-ring, then since A is F -pure, there exists a faithfully flat ring map $A \rightarrow B$ where B is Noetherian, F -finite, and F -pure by [Mur21, Theorem 3.4] (cf. the proof of [Mur21, Corollary 3.5]) and Σ_B is finite by what we just proved. For $\mathfrak{t}(X/A) \in \Sigma_A$, we have $\mathfrak{t}(X/A)B = \mathfrak{t}(X_B/B) \in \Sigma_B$ (see §2.6; this is why we need properness), so the finiteness of Σ_B implies that of Σ_A .

Finally, when A satisfies Condition 8.3, we can similarly find a faithfully flat ring map $R \rightarrow R'$ where R' is Noetherian, F -finite, and F -pure, since the completion of R is F -pure [Has10, Lemma 3.26]. By [Has10, Proposition 2.4] $R' \otimes_R A$ is F -pure, and is Noetherian and F -finite since it is essentially of finite type over R' . The rest of the argument above carries verbatim. $\square$

8.2 Openness of loci

Lemma 8.6. *Let A be a ring that satisfies Condition 8.4. Then A is quasi-excellent. Moreover, the normal and F -pure loci of $\mathrm{Spec}(A)$ are open.*

Proof. By a theorem of Kunz, F -finite Noetherian rings are excellent, see [Mat80, Theorem 108]. Noetherian local G-rings are quasi-excellent, and the property of being quasi-excellent is preserved by extensions essentially of finite type, see [Mat80, (34.A)]. We got quasi-excellence. Thus the normal

locus is open by, for example, [EGA IV₂, Corollaire 6.13.5].

Openness of F -pure locus is trivial in the F -finite case (as pure is equivalent to split, see [Stacks, Tag 058L]) and is [Mur21, Corollary 3.5] in the other case. $\square$

Lemma 8.7. *Let A be a ring that satisfies Condition 8.3. Then the normal locus of A is open.*

Proof. For every $\mathfrak{p} \in \operatorname{Spec}(A)$ such that $A_{\mathfrak{p}}$ is normal, since A is Noetherian, there exists $f \in A, f \notin \mathfrak{p}$ such that A_f is an integral domain. We may thus assume from the outset that A is an integral domain.

For a Noetherian integral domain C put

$$\Sigma_{\nu}(C) = \{\mathfrak{t}(D/C) \mid D \text{ is a finite extension of } C \text{ in the fraction field of } C\}.$$

It is elementary to see that $\Sigma_{\nu}(C_P) = \{IC_P \mid I \in \Sigma_{\nu}(C)\}$ for all $P \in \operatorname{Spec}(C)$. Also, if $C \subseteq D$ is an extension of integral domains of the same fraction field, then $C \rightarrow D$ splits as a map of C -modules if and only if $C = D$. One sees that the normal locus of a Noetherian integral domain C is always $\operatorname{Spec}(C) \setminus \bigcup_{\mathfrak{a} \in \Sigma_{\nu}(C)} V(\mathfrak{a})$.

By Theorem 8.5, $\Sigma_{\nu}(A)$ is finite. This concludes the proof. $\square$

Corollary 8.8. *Let A be a ring that satisfies Condition 8.3. Then the normalization of A is finite over A .*

Proof. By [Has10, Proposition 2.4] A is F -pure, in particular reduced. Thus by Lemma 8.7 A_f is normal for some nonzerodivisor $f \in A$. Moreover, for all

$\mathfrak{p} \in \operatorname{Spec}(A)$, the completion $A_{\mathfrak{p}}^{\wedge}$ is F -pure (see for example [Has10, Lemma 3.26]), thus reduced, so the normalization of $A_{\mathfrak{p}}$ is finite [Stacks, Tag 032Y]. Our result now follows from [EGA IV₂, Proposition 6.13.6], which is stated for domains but works for reduced rings as well. $\square$

Remark 8.9 (cf. [DT22, Example 4.0.3]). The previous two results (and thus Theorem 8.5) fail for a general F -pure Noetherian ring, and in fact, fails even for a general F -pure Noetherian locally excellent ring.

For a counterexample, we let k be an algebraically closed field of characteristic p , and let $(A_0, \mathfrak{m}_0)$ be a k -algebra essentially of finite type over k that is a non-normal F -pure integral domain. For instance we can take $A_0 = k[x, y]_{(x, y)} / (xy + x^3 + y^3)$, so $A_0^{\wedge} \cong k[[x, y]] / (xy)$. Then by [Hoc73b], there is a Noetherian integral domain A which has local rings at maximal ideals of the form $A_0 \otimes_k K$ where K/k is a field extension (thus is F -pure and excellent), and has non-open normal locus, and thus the normalization of A is not finite [EGA IV₂, Proposition 6.13.2]. [Hoc73b] also shows that none of the loci $\operatorname{Spl}(A)$, $\operatorname{MDS}(A)$, and $\operatorname{DS}(A)$ are open.

Before we state and prove the next result, we remind the reader that a splinter or a derived splinter containing $\mathbf{F}_p$ is always F -pure.

Theorem 8.10. *Let A be a ring that satisfies Condition 8.3 or Condition 8.4. Then the locus of prime ideals $\mathfrak{p}$ of A such that $A_{\mathfrak{p}}$ is a splinter (resp. an F -pure birational derived splinter, a derived splinter) is open.*

Proof. Note that if A satisfies Condition 8.3 then A is F -pure, see [Has10,

Proposition 2.4]. If A satisfies Condition 8.4, we may assume A F -pure by Lemma 8.6. Thus in both cases, we need to show the loci $\text{Spl}(A)$, $\text{MDS}(A)$, and $\text{DS}(A)$ open.

By Lemmas 3.3 and 3.5 these loci are all contained in the normal locus of A , so we may assume A normal by Lemmas 8.6 or 8.7.

We have shown that Σ_A is finite (Theorem 8.5), thus for $\square = \text{FS}$, M , or PS the set

$$\Sigma_{\square} := \{\mathfrak{t}(f) \mid f \text{ is a(n) } \square\text{-morphism to } \text{Spec}(A)\} \subseteq \Sigma_A$$

is finite. For every $\mathfrak{p} \in \text{Spec}(A)$ and every $\square$ -morphism $g : Y \rightarrow \text{Spec}(A_{\mathfrak{p}})$, g extends to a(n) $\square$ -morphism $f : X \rightarrow \text{Spec}(A)$, see Lemmas 2.11 and 2.12, and $\mathfrak{t}(f)A_{\mathfrak{p}} = \mathfrak{t}(g)$, see §2.6. On the other hand, a localization of a(n) $\square$ -morphism is a(n) $\square$ -morphism (in the case $\square = \text{M}$ use Lemma 2.10). We conclude that the splinter (resp. birational derived splinter, derived splinter) locus is

$$\begin{aligned} & \{\mathfrak{p} \in \text{Spec}(A) \mid \mathfrak{a}A_{\mathfrak{p}} = A_{\mathfrak{p}}, \forall \mathfrak{a} \in \Sigma_{\square}\} \\ &= \bigcap_{\mathfrak{a} \in \Sigma_{\square}} (\text{Spec}(A) \setminus V(\mathfrak{a})) \end{aligned}$$

for $\square = \text{FS}$ (resp. M , PS). This locus is open since $\Sigma_{\square}$ is finite. $\square$

Theorem 8.11. *Let $A \rightarrow B$ be a regular homomorphism of Noetherian $\mathbf{F}_p$ -algebras. If A is a splinter (resp. an F -pure birational derived splinter, a*

derived splinter), so is B .

Proof. Note that a splinter is F -pure, since the Frobenius map is integral and induces a bijective map of spectra. Note also that F -purity localizes and ascends along regular maps, see [Has10, Proposition 2.4], so in the dévissage below A is always F -pure and so is B .

By Popescu's Theorem 2.1, and by the limit result Lemma 4.1 or Theorem 4.5, we may assume B is a smooth A -algebra.

We must show $\mathrm{Spl}(B) = \mathrm{Spec}(B)$ (resp. $\mathrm{MDS}(B) = \mathrm{Spec}(B)$, $\mathrm{DS}(B) = \mathrm{Spec}(B)$), see Corollary 3.9 or Corollary 3.12. By Corollary 5.2 or Theorem 5.4, and [Stacks, Tag 054L], we may assume B a polynomial ring over A , and by induction on the number of variables, it suffices to show $A[Y]$ a splinter (resp. a birational derived splinter, a derived splinter), where Y is an indeterminate.

By Corollary 3.9 or Corollary 3.12, we may assume $(A, \mathfrak{m}, k)$ local, and by induction on $\dim A$, that $A[Y]$ is a splinter (resp. a birational derived splinter, a derived splinter) outside $Z := V(\mathfrak{m}A[Y])$. We may also assume k separably closed, in particular infinite, by taking the strict Henselization of A . Here we used Corollary 5.2 or Corollary 5.6, and Lemma 3.10 or Proposition 3.11, and the fact that the strict Henselization of a Noetherian local ring is a Noetherian local ring, see [Stacks, Tag 06LJ].

Let $W = \mathrm{Spl}(A[Y])$ (resp. $\mathrm{MDS}(A[Y])$). Thus W is open in $\mathrm{Spec}(A[Y]) = \mathbf{A}_A^1$ by Theorem 8.10. On the other hand, W is stable under translations by all $a \in S$ (in fact all ring automorphisms of $A[Y]$); and by Theorem 7.4,

$A[Y]_{\mathfrak{m}_A[Y]}$ is a splinter (resp. a birational derived splinter, a splinter). Thus $Z \cap W$ is a non-empty open subset of Z stable under translations by all $a \in A$, thus corresponds to a non-empty open subset of $\mathbf{A}_k^1$ stable under translations by all $a \in k$. Such a set must be the whole of $\mathbf{A}_k^1$ since k is infinite. Therefore $Z \cap W = Z$ and $W = \operatorname{Spec}(A[Y])$ as desired. $\square$

Remark 8.12. Assume both A and B are complete local and $A \rightarrow B$ is local. Let k (resp. l) be the residue field of A (resp. B). If l/k is separable [Stacks, Tag 0300], we can argue as follows to avoid taking limit, hence avoiding derived category techniques in Chapter 4.2. By [Stacks, Tags 03C3 and 05GH] we can find a residue extension (Definition 2.3) $A \rightarrow A_l$ that realizes $k \rightarrow l$. A_l is then formally smooth over A by [Stacks, Tag 07PM] and thus maps to B . Now it is clear that B is isomorphic to a power series algebra over A , thus a residue extension of a localization of a polynomial algebra over A . By Lemma 2.4 this residue extension is regular. Then by Theorem 7.4 we are reduced to the case of a polynomial algebra.

However, in general l/k is not necessarily separable. A counterexample is given by $A = k = \mathbf{F}_p(t)$, $B = A[X]_{(X^p - t)}^\wedge$, so $l = \mathbf{F}_p(t^{1/p})$.

Remark 8.13. For $B = A[Y]_{\mathfrak{m}_A[Y]}$, the map $A \rightarrow B$ actually satisfies the statement of Theorem 5.4 for M-morphisms, so we do not need the full force of Theorem 7.4 in the case of birational derived splinters. This is because an M-morphism $Z \rightarrow \operatorname{Spec}(B)$ is dominated by a blowup of an ideal $J \subseteq B$; and if $J = IB$ for an ideal $I \subseteq A[Y]$, then the base change of the blowup

of $\text{Spec}(A)$ of the ideal generated by the coefficients of elements of I to B dominates Z . This construction appears in [LT81] immediately before its Theorem 2.1; details omitted.

Chapter 9

Splinters in equal characteristic zero

9.1 Equivalence of notions

We will show that under very mild conditions, there is only one interesting notion of splinter for $\mathbf{Q}$ -algebras, see Proposition 9.2 below.

Let $A \rightarrow B$ be a ring map. The multiplication of an element b of B is an A -module endomorphism of B . If B is a free module over A , then said endomorphism has a well-defined trace, which we denote by $\mathrm{Tr}(b)$. This defines an A -linear map $\mathrm{Tr} : B \rightarrow A$ that restricts to the multiplication by d on A . In general, if $f : X \rightarrow S$ is a finite locally free morphism of schemes, then there is a trace map $\mathrm{Tr} : f_*\mathcal{O}_X \rightarrow \mathcal{O}_S$, in particular for a finite locally free ring map $A \rightarrow B$ we get $\mathrm{Tr} : B \rightarrow A$. This construction is compatible

with arbitrary base change.

Before we proceed, we note that for a reduced ring A , an integral ring map $A \rightarrow B$ is injective if and only if it induces a surjective map of spectra. In particular injectivity is stable under arbitrary base change $A \rightarrow A'$ provided A' is reduced.

Lemma 9.1. *Let A be a reduced ring, $A \rightarrow B$ an injective, finite, and finitely presented ring map. Let $s \in A$ be a nonzerodivisor. Assume that $A_s \rightarrow B_s$ is finite locally free, so we have a trace map $\text{Tr} : B_s \rightarrow A_s$.*

Then for all $b \in B$, $\text{Tr}(b/1) \in A_s$ is integral over A .

Proof. It is clear that by taking a sufficiently large subring of A , we may assume A of finite type over $\mathbf{Z}$, in particular has finitely many minimal primes. Then an element of A_s is integral over A if and only if its image in $(A/\mathfrak{p})_s$ is integral over $A/\mathfrak{p}$ for all minimal primes $\mathfrak{p}$ of A . Thus we may assume A an integral domain, and the result follows from Galois theory. $\square$

Proposition 9.2. *Let A be a $\mathbf{Q}$ -algebra and let K be the total fraction ring of A . If $\dim K = 0$, then the followings hold.*

- (1) *A is a splinter if and only if A is reduced and integrally closed in K .*
- (2) *A is a derived splinter if and only if A is an MDS.*

In particular, a Noetherian normal ring (resp. birational derived splinter) that contains $\mathbf{Q}$ is a splinter (resp. a derived splinter).

Remark 9.3. If A is an integral domain, the proof of (1) is much shorter, see [Hoc73a, Lemmas 2 and 3].

Proof. We first show (1). “Only if” is noted in Lemma 3.3. Assume A is reduced and integrally closed in K . Let $\mathrm{Spec}(B) \rightarrow \mathrm{Spec}(A)$ be an FS-morphism, in other words, $A \rightarrow B$ is finite, finitely presented, and injective. We want $A \rightarrow B$ splits.

Since $\dim K = 0$, every module over K is flat by [Stacks, Tag 092F]. In particular $B \otimes_A K$ is a finite locally free K -module. Since A is integrally closed in K , every clopen subset of $\mathrm{Spec}(K)$ comes from a clopen subset of $\mathrm{Spec}(A)$, so we may assume $B \otimes_A K$ a locally free K -module of constant rank $d > 0$. We may therefore find a nonzero divisor $s \in A$ such that B_s a locally free A_s -module of rank d .

By Lemma 9.1, $\mathrm{Tr} : B_s \rightarrow A_s$ induces a map $\mathrm{Tr}|_B : B \rightarrow A$. The map $\frac{1}{d} \mathrm{Tr}|_B$ splits $A \rightarrow B$.

Next, we consider (2). A derived splinter in an MDS by definition. Assume A an MDS and let $f : X \rightarrow S = \mathrm{Spec}(A)$ be a PS-morphism. We want to show $A \rightarrow Rf_*\mathcal{O}_X$ splits. By Lemma 2.16 we may assume the base change $f_K : X_K \rightarrow \mathrm{Spec}(K)$ finite, thus finite locally free by [Stacks, Tag 092F]. Thus $f : X \rightarrow S$ is finite locally free over some $D(a)$ where $a \in A$ is a nonzerodivisor, see [Stacks, Tag 06AC]. Thus by [Stacks, Tag 0B49] there

exists a commutative diagram of schemes

$$\begin{array}{ccc} X' & \longrightarrow & X \\ f' \downarrow & & \downarrow f \\ S' & \xrightarrow{g} & S \end{array}$$

where S' is the blowup of S along a finitely generated ideal I with $V(I) \subseteq V(aA)$ set-theoretically, X' the strict transform of X , and f' is finite locally free. In particular $IK = K$, so $S' \rightarrow S$ is an M-morphism, and $X' = X$ after the base change to $\mathrm{Spec}(A_a)$. Thus f' is surjective, and dividing the trace map $\mathrm{Tr} : f'_* \mathcal{O}_{X'} \rightarrow \mathcal{O}_{S'}$ by the degree of f' (which is a locally constant function on S') we see $\mathcal{O}_{S'} \rightarrow f'_* \mathcal{O}_{X'}$ splits. On the other hand, $A \rightarrow Rg_* \mathcal{O}_{S'}$ splits since A is an MDS. Thus $A \rightarrow R(g \circ f')_* \mathcal{O}_{X'}$ splits, so $A \rightarrow Rf_* \mathcal{O}_X$ splits. $\square$

9.2 Quasi-excellent birational derived splinters

We will use the fact that every reduced quasi-excellent $\mathbf{Q}$ -scheme X admits a projective resolution of singularities, that is, a projective birational map $X' \rightarrow X$ with X' regular. See [Tem08].

Proposition 9.4 (cf. [Kov00]). *Let A be a quasi-excellent $\mathbf{Q}$ -algebra. Then the followings are equivalent.*

- (1) *A is a birational derived splinter.*

(2) *There exists a resolution of singularities $f : X \rightarrow \operatorname{Spec}(A)$ such that $A \rightarrow Rf_*\mathcal{O}_X$ splits.*

(3) *For any resolution of singularities $f : X \rightarrow \operatorname{Spec}(A)$, we have $A = Rf_*\mathcal{O}_X$.*

If the equivalent conditions above hold, we say A has rational singularities, cf. [Kol13, Definition 2.76]. In this case A is Cohen-Macaulay, see [Mur22, Proposition 7.1].

Proof. In all cases A is reduced. The existence of resolutions shows that (3) implies (1) and that (1) implies (2). It suffices to show (2) implies (3).

We make the following observation. Let $\mathfrak{p} \in \operatorname{Spec}(A)$ and let A' be the completion of $A_{\mathfrak{p}}$. Then $A_{\mathfrak{p}} \rightarrow A'$ is regular by definition, so A' is normal and if $f : X \rightarrow \operatorname{Spec}(A)$ is a resolution, then the base change $f' : X' := X \times_{\operatorname{Spec}(A)} \operatorname{Spec}(A') \rightarrow \operatorname{Spec}(A')$ is also a resolution, see for example [Stacks, Tags 0C22 and 033A]. Since $\mathfrak{p}$ was arbitrary, by flat base change (cf. §2.6) and descent, we may always assume A complete local. In particular, A has a dualizing complex [Stacks, Tag 0BFR] and is excellent.

Assume (2). [CR15] and the existence of projective resolutions show that for all resolutions $g : Y \rightarrow \operatorname{Spec}(A)$, $A \rightarrow Rg_*\mathcal{O}_Y$ splits. Since we have the Grauert-Riemenschneider vanishing theorem [Mur22, Theorem A] in this case, [Kol13, Corollary 2.75] shows that $Rg_*\mathcal{O}_Y = A$, as desired. $\square$

Corollary 9.5. *Let A be a quasi-excellent $\mathbf{Q}$ -algebra. Then the locus of primes $\mathfrak{p}$ where $A_{\mathfrak{p}}$ is a splinter (resp. a birational derived splinter, a derived*

splinter) is open.

Proof. The normal locus is open by [EGA IV₂, Corollaire 6.13.5], hence so is the splinter locus by Proposition 9.2. There is a resolution of singularities $f : X \rightarrow \operatorname{Spec}(A)$ by [Tem08], and by Proposition 9.4 the birational derived splinter locus is the complement of $V(\mathfrak{t}(f))$ (see §2.6), thus open. We conclude by Proposition 9.2. $\square$

Corollary 9.6. *Let $A \rightarrow B$ be a regular map of Noetherian $\mathbf{Q}$ -algebras. Assume that A is a G -ring and a birational derived splinter. Then B is a birational derived splinter.*

Proof. We may assume A local by Corollary 3.12. Then A is quasi-excellent, so there exists a resolution $f : X \rightarrow \operatorname{Spec}(A)$. Since $A \rightarrow B$ is regular, $f_B : X_B \rightarrow \operatorname{Spec}(B)$ is a resolution. We conclude by Proposition 9.4 and flat base change (§2.6). $\square$

Appendix A

First-order study of rings and splinters

An excellent reference is [\[Sch10\]](#).

A.1 First-order theory of rings

Formulas

An *atomic formula* $\varphi(x_1, \dots, x_n)$ of free variables $x_1, \dots, x_n$ is an expression

$$F(x_1, \dots, x_n) = G(x_1, \dots, x_n) \tag{A.1}$$

where F and G are polynomials in the variables $x_1, \dots, x_n$ with integer co-

efficients.

A (first-order) *formula* φ of free variables $x_1, \dots, x_n$ is an expression inductively obtained from atomic formulas using logical connectors and quantifiers. More precisely, if φ is a formula, the negation $\neg\varphi$ is a formula; if φ, ψ are formulas, the conjunction $\varphi \wedge \psi$, the disjunction $\varphi \vee \psi$, and the implication statement $\varphi \Rightarrow \psi$ are formulas; if $\varphi(x) = \varphi(x, x_1, \dots, x_n)$ is a formula of one more free variable, the existential statement $\exists x(\varphi(x))$ and universal statement $\forall x(\varphi(x))$ are formulas.

All formulas are obtained this way, that is, the set of formulas is defined to be the smallest set that contains the atomic formulas and is closed under the operations above.

A formula with no free variables is called a *sentence*.

Satisfaction

Given an atomic formula φ as in (A.1), we say *a ring A and elements $a_1, \dots, a_n$ of A satisfy φ* , or *A satisfies $\varphi(a_1, \dots, a_n)$* , if $F(a_1, \dots, a_n) = G(a_1, \dots, a_n)$.

For a formula φ and its negation $\neg\varphi$, we say *A satisfies $\neg\varphi(a_1, \dots, a_n)$* if *$A$ does not satisfy $\varphi(a_1, \dots, a_n)$* . Similar for the other logical connectors $\wedge, \vee$, and $\Rightarrow$. For quantifiers, we say *A satisfies $\exists x(\varphi(x, a_1, \dots, a_n))$* if there exists $a \in A$ such that *A satisfies $\varphi(a, a_1, \dots, a_n)$* . Similar for universal statements.

Now for every formula φ , we have a well-defined concept of satisfaction of φ by a ring A and elements $a_1, \dots, a_n$. When $n = 0$, i.e. φ is a sentence,

we say A *satisfies* φ if A and the unique 0-uple of elements of A satisfy φ .

Remark A.1. It is clear that the satisfaction of $\varphi \vee \psi$ is always the same as $\neg(\neg\varphi \wedge \neg\psi)$; $\varphi \Rightarrow \psi$ as $\neg\varphi \vee \psi$; $\forall x(\varphi(x))$ as $\neg\exists x(\neg\varphi(x))$.

Example A.2. A ring A is an integral domain if and only if it satisfies the sentence $\forall x \forall y (xy = 0 \Rightarrow (x = 0 \vee y = 0))$; A is reduced if and only if it satisfies the sentence $\varphi_{\text{red}} = \forall x (x^2 = 0 \Rightarrow x = 0)$.

Example A.3. Consider the formula

$$\varphi_{\text{gen}}^n(x_1, \dots, x_n, y) = \exists z_1 \dots \exists z_n (x_1 z_1 + \dots + x_n z_n = y)$$

where n is a non-negative integer. A ring A and elements $a_1, \dots, a_n, b$ satisfies φ_{gen}^n if and only if b lies in the ideal generated by $a_1, \dots, a_n$.

Substitution

Let $H_1, \dots, H_n$ be polynomials in the variables $y_1, \dots, y_m$ with integer coefficients. Given a formula $\varphi(x_1, \dots, x_n)$, we want to define the *substitution* $\varphi(H_1, \dots, H_n)$, which is a formula of free variables $y_1, \dots, y_m$.

First, assume φ is an atomic formula as in (A.1). Then $\varphi(H_1, \dots, H_n)$ is defined to be the atomic formula $F(H_1, \dots, H_n) = G(H_1, \dots, H_n)$ of free variables $y_1, \dots, y_m$.

We can now bring in logical connectors and quantifiers. For example, the formula $\exists x(\varphi(x))(H_1, \dots, H_n)$ is defined to be $\exists x(\varphi(x, H_1, \dots, H_n))$, where $x, H_1, \dots, H_n$ are viewed as polynomials in the variables $x, y_1, \dots, y_m$.

This gives, inductively, a well-defined formula $\varphi(H_1, \dots, H_n)$ of free variables $y_1, \dots, y_m$ for each φ .

If A is a ring and $b_1, \dots, b_m$ are elements in A , it is clear that

$$\begin{aligned} & A \text{ satisfies } \varphi(H_1, \dots, H_n)(b_1, \dots, b_m) \\ \iff & A \text{ satisfies } \varphi(H_1(b_1, \dots, b_m), \dots, H_n(b_1, \dots, b_m)). \end{aligned}$$

Example A.4. Consider the formula $\varphi_{\text{inv}}(w) = \varphi_{\text{gen}}^1(w, 1) = (\exists z)(wz = 1)$. Here, $\varphi_{\text{gen}}^1 = \varphi_{\text{gen}}^1(x, y)$ is the formula defined in Example A.3 of free variables x, y , and we substitute in the polynomials $H_1(w) = w$ and $H_2(w) = 1$ to make a formula of free variable w . For $a \in A$, A satisfies $\varphi_{\text{inv}}(a)$ if and only if a is invertible in A .

From here it is clear that a ring A is a field if and only if it satisfies the sentence $\varphi_{\text{fld}} = \forall x (x = 0 \vee \varphi_{\text{inv}}(x))$, and A is local if and only if it satisfies the sentence

$$\varphi_{\text{loc}} = \forall x \forall y (\varphi_{\text{inv}}(x + y) \Rightarrow (\varphi_{\text{inv}}(x) \vee \varphi_{\text{inv}}(y))).$$

Example A.5. Let

$$\varphi_{\text{nzd}}^n(x_1, \dots, x_n, y) = \forall z (\neg \varphi_{\text{gen}}^n(x_1, \dots, x_n, z) \Rightarrow \neg \varphi_{\text{gen}}^n(x_1, \dots, x_n, yz)),$$

where n is a non-negative integer and φ_{gen}^n is defined in Example A.3.

Then A and $(a_1, \dots, a_n, b)$ satisfy φ_{nzd}^n if and only if b is a nonzerodivisor

in the module $A/(a_1, \dots, a_n)$.

Therefore, elements $a_1, \dots, a_n$ form a regular sequence in A if and only if A and $a_1, \dots, a_n$ satisfy the formula

$$\varphi_{\text{regseq}}^n(x_1, \dots, x_n) := \neg \varphi_{\text{gen}}^n(x_1, \dots, x_n, 1) \wedge \bigwedge_{i=1}^n \varphi_{\text{nzdv}}^{i-1}(x_1, \dots, x_i).$$

From here it is clear that a ring A satisfies the sentence

$$\bigwedge_{1 \leq i < j \leq n} \forall x_1 \forall x_2 \dots \forall x_n (\varphi_{\text{regseq}}^n(x_1, \dots, x_n) \Rightarrow \varphi_{\text{regseq}}^n(x_1, \dots, x_{i-1}, x_j, x_{i+1}, \dots, x_{j-1}, x_i, x_{j+1}, \dots, x_n))$$

if and only if every regular sequence of length n is permutable in A .

Example A.6. There exists a formula $\varphi_{\text{simp}}^n(x_1, \dots, x_n, y)$ such that A and $(a_1, \dots, a_n, b)$ satisfy φ_{simp}^n if and only if the A -module $(a_1, \dots, a_n, b)/(a_1, \dots, a_n)$ is simple (build with φ_{gen} , details omitted). Now it is not difficult to see that for every $m \in \mathbf{Z}_{\geq 1}$ there exists a sentence $\varphi_{\ell=m}$ such that a ring A satisfies $\varphi_{\ell=m}$ if and only if A is (Artinian and) of length m .

A.2 Ultraproduct and Łoś's Theorem

We use the notions of filters and ultrafilters, see for example [Bou71, Chapitre I, §§6.1-6.5].

Definition A.7. Let X be a set. Let $\mathcal{U}$ be a family of subsets of X .

We say $\mathcal{U}$ a *filter* if $\mathcal{U}$

- (1) does not contain the empty subset;
- (2) is closed under finite intersections, in particular $X \in \mathcal{U}$; and
- (3) is closed under taking supersets (*i.e.*, if $A \subseteq B \subseteq X$ and $A \in \mathcal{U}$, then $B \in \mathcal{U}$).

We say $\mathcal{U}$ an *ultrafilter* if $\mathcal{U}$ further satisfies

- (4) for every $A \subseteq X$, either A or $X \setminus A$ belongs to $\mathcal{U}$.

Remark A.8. We do not exclude *principal* ultrafilters, that is, ultrafilters of the form $\mathcal{U}_x = \{A \subseteq X \mid x \in A\}$ for some $x \in X$. It is not difficult to see that for an ultrafilter $\mathcal{U}$, either $\mathcal{U}$ is principal, or X is infinite and $\mathcal{U}$ contains the *Fréchet filter*, *i.e.* the family of cofinite subsets of X .

The next lemma shows us, for example, on an infinite subset there are indeed non-principal ultrafilters.

Lemma A.9 ([Bou71, Chapitre I, §6.5]). *The followings hold.*

- (1) *Ultrafilters are exactly the filters maximal with respect to inclusion.*
- (2) *Every filter is contained in an ultrafilter. More generally, if a family of subsets of X satisfies Definition A.7(1)(2), then it is contained in an ultrafilter.*

Definition A.10. Let X be a set, $\mathcal{U}$ an ultrafilter on X . Let A_x ($x \in X$) be a family of sets.

The *ultraproduct* of A_x ($x \in X$) with respect to $\mathcal{U}$, denoted by $\prod_{x \in X} A_x / \mathcal{U}$, is the set $\prod_{x \in X} A_x$ modulo the relation $\sim_{\mathcal{U}}$ defined by $(a_x)_x \sim_{\mathcal{U}} (b_x)_x$ if and only if $\{x \in X \mid a_x = b_x\} \in \mathcal{U}$. Note that $\sim_{\mathcal{U}}$ is indeed an equivalence relation by Definition A.7(2). If all A_x are the same set A , the ultraproduct $A^X / \mathcal{U}$ is called the *ultrapower* of A with respect to $\mathcal{U}$. We denote it by $A^{\mathcal{U}}$.

If each A_x is a ring, then coordinate-wise addition and multiplication makes the ultraproduct $\prod_x A_x / \mathcal{U}$ a ring. For a ring A , an ultrapower $A^{\mathcal{U}}$ is canonically an A -algebra via the diagonal map.

Remark A.11. In the case each A_x is a ring, each $U \subseteq X$ determines an idempotent e_U by $(e_U)_x = 0$ when $x \notin U$ and $(e_U)_x = 1$ when $x \in U$. Denote by $J_{\mathcal{U}}$ the ideal generated by all $1 - e_U$ ($U \in \mathcal{U}$). Then it is easy to verify that $a \sim_{\mathcal{U}} b$ if and only if $a - b \in J_{\mathcal{U}}$, so the ultraproduct $\prod_{x \in X} A_x / \mathcal{U}$ is identified with $\prod_{x \in X} A_x / J_{\mathcal{U}}$, and therefore identified with the localization $S_{\mathcal{U}}^{-1} \prod_{x \in X} A_x$, where $S_{\mathcal{U}}$ is the multiplicative set generated by e_U ($U \in \mathcal{U}$).

The next theorem, in the general framework of model theory, is [BS66, Chapter 5, Theorem 2.1], or [Hod93, Theorem 9.5.1]. We give a proof in our case due to its utter importance.

Theorem A.12 (Łoś). *Let A_x ($x \in X$) be rings, $a_{1,x}, \dots, a_{n,x} \in A_x$. Let A be the ultraproduct of A_x with respect to an ultrafilter $\mathcal{U}$, a_j the element in A represented by $(a_{j,x})_x$. Let φ be a formula with n free variables. Then A*

and $a_1, \dots, a_n$ satisfy φ if and only if the set of all $x \in X$ such that A_x and $a_{1,x}, \dots, a_{n,x}$ satisfy φ lies in $\mathcal{U}$.

Proof. For atomic formulas this is clear since addition and multiplication are defined coordinate-wise.

Since formulas are built from atomic formulas using logical connectors and quantifiers, we only need to show the class of formulas that satisfy the theorem is closed under these operations.

If φ satisfies the theorem, so does $\neg\varphi$ by Definition [A.7\(1\)\(2\)\(4\)](#).

If φ, ψ are two formulas that satisfy the theorem, so is $\varphi \wedge \psi$ by Definition [A.7\(2\)\(3\)](#).

By Remark [A.1](#), we also have the cases of $\vee$ and $\Rightarrow$.

Let $\varphi(y)$ be a formula of one more variable for which the theorem is true. Let U be the set of $x \in X$ such that A_x and $a_{1,x}, \dots, a_{n,x}$ satisfy $\exists y(\varphi(y))$. We must show that $U \in \mathcal{U}$ is equivalent to that A and $a_1, \dots, a_n$ satisfy $\exists y(\varphi(y))$.

First assume that A and $a_1, \dots, a_n$ satisfy $\exists y(\varphi(y))$. Then there exists $b \in A$ represented by $(b_x)_x \in \prod_{x \in X} A_x$ such that A and $b, a_1, \dots, a_n$ satisfy φ . Since the theorem is true for φ , the set of $x \in X$ such that A_x and $b_x, a_{1,x}, \dots, a_{n,x}$ satisfy φ lies in $\mathcal{U}$. By definition, these x all lie in the set U , so $U \in \mathcal{U}$ by Definition [A.7\(3\)](#).

Conversely, assume $U \in \mathcal{U}$. For each $x \in U$ fix b_x so that A_x and $b_x, a_{1,x}, \dots, a_{n,x}$ satisfy φ . For $x \notin U$ let $b_x = 0$. Taking into account Definition [A.7\(3\)](#) again, we see that for the element b represented by $(b_x)_x$,

A and $b, a_1, \dots, a_n$ satisfy φ , since the theorem is true for φ . Therefore A and $a_1, \dots, a_n$ satisfy $\exists y(\varphi(y))$.

By Remark A.1, we also have the case of the universal quantifier and the proof is finished. $\square$

From Example A.2 we get

Corollary A.13. *An ultraproduct of integral domains is an integral domain.*

We now consider an alternative construction of ultraproduct for local rings.

Theorem A.14. *Let $(A_x, \mathfrak{m}_x)$ ($x \in X$) be a family of local rings. Let $\mathcal{A} = \prod_{x \in X} A_x$ and $\mathfrak{M} = \prod_{x \in X} \mathfrak{m}_x$. Then the followings hold.*

- (1) $\mathfrak{M}$ is the Jacobson radical of $\mathcal{A}$.
- (2) For an ultrafilter $\mathcal{U}$ on X , let $J_{\mathcal{U}}$ be the ideal as in Remark A.11. Then $\mathfrak{M}_{\mathcal{U}} := \mathfrak{M} + J_{\mathcal{U}}$ is a maximal ideal, and every maximal ideal of $\mathcal{A}$ is of the form $\mathfrak{M}_{\mathcal{U}}$ for a unique $\mathcal{U}$;
- (3) In the notations of (2), the ultraproduct $\mathcal{A}/\mathcal{U}$ is canonically identified with the localization $\mathcal{A}_{\mathfrak{M}_{\mathcal{U}}}$.

Proof. (1) is clear, see for example [Stacks, Tag 0AME].

For (2), let $\mathcal{U}$ be an ultrafilter on X . Since the construction of $J_{\mathcal{U}}$ is functorial, $\mathcal{A}/\mathfrak{M}_{\mathcal{U}}$ is the ultraproduct of the fields $A_x/\mathfrak{m}_x$ with respect to $\mathcal{U}$.

By Łoś's Theorem (and Example A.4) $\mathcal{A}/\mathfrak{M}_{\mathcal{U}}$ is a field, so $\mathfrak{M}_{\mathcal{U}}$ is a maximal ideal. Conversely, given a maximal ideal $\mathfrak{N}$ of $\mathcal{A}$, consider the set

$$\mathcal{V} = \{V \subseteq X \mid 1 - e_V \in \mathfrak{N}\}$$

where e_V is as in Remark A.11. We shall verify that $\mathcal{V}$ is an ultrafilter and $\mathfrak{M}_{\mathcal{V}} = \mathfrak{N}$. Uniqueness follows from Lemma A.9, since, by construction, for an ultrafilter $\mathcal{V}'$ with $\mathfrak{M}_{\mathcal{V}'} = \mathfrak{N}$, we must have $\mathcal{V}' \subseteq \mathcal{V}$. We have $1 - e_{\emptyset} = 1 \notin \mathfrak{N}$, so $\mathcal{V}$ satisfies Definition A.7(1). We have $1 - e_{V \cap W} = 1 - e_V + e_V(1 - e_W)$ and $1 - e_X = 0$, so $\mathcal{V}$ satisfies Definition A.7(2) since $\mathfrak{N}$ is an ideal. Similar for Definition A.7(3) since $1 - e_V = (1 - e_V)(1 - e_W)$ when $W \subseteq V$. Finally, $(1 - e_V)(1 - e_{X \setminus V}) = 0$, so $\mathcal{V}$ satisfies Definition A.7(4) since $\mathfrak{N}$ is a prime ideal. Thus $\mathcal{V}$ is an ultrafilter. By construction, $J_{\mathcal{V}} \subseteq \mathfrak{N}$. Also $\mathfrak{M} \subseteq \mathfrak{N}$ by (1). Thus $\mathfrak{M}_{\mathcal{V}} \subseteq \mathfrak{N}$, and since $\mathfrak{M}_{\mathcal{V}}$ is maximal, $\mathfrak{M}_{\mathcal{V}} = \mathfrak{N}$, as desired.

Finally, we show (3). By (1)(2) we see $J_{\mathcal{U}}$ is contained in a unique maximal ideal $\mathfrak{M}_{\mathcal{U}}$, thus $\mathcal{A}/\mathcal{U} = \mathcal{A}/J_{\mathcal{U}}$ is local and equals $(\mathcal{A}/J_{\mathcal{U}})_{\mathfrak{M}_{\mathcal{U}}}$. Therefore it suffices to show $J_{\mathcal{U}}\mathcal{A}_{\mathfrak{M}_{\mathcal{U}}} = 0$. This follows from the fact that in a local ring, the only idempotent contained in the maximal ideal is 0. $\square$

Remark A.15. From the theorem we see an ultraproduct of a family of local rings is local. This also follows from Łoś's Theorem and Example A.4.

Remark A.16. At this point we can also easily see that an ultrapower of a Noetherian local ring needs not to be Noetherian. In fact, take any positive-dimensional Noetherian local ring $(A, \mathfrak{m})$ and take $a \in \mathfrak{m}$ that is not nilpotent.

Let $X = \mathbf{Z}_{\geq 1}$ and let $\mathcal{U}$ be an arbitrary non-principal ultrafilter on X . The element of $A^{\mathcal{U}}$ represented by $(a^n)_n$ then lies in all powers of the maximal ideal of $A^{\mathcal{U}}$ but is nonzero, so $A^{\mathcal{U}}$ is not Noetherian.

However, by Example A.6, an ultrapower of an Artinian ring is always an Artinian ring of the same length. Nonetheless, by the same example, an ultraproduct of a family of Artinian rings of unbounded lengths is not Artinian. If, say, the rings are local and their maximal ideals are square-zero, Łoś's Theorem tells that the maximal ideal of the ultraproduct is also square-zero. So the ultraproduct is not Noetherian either.

A.3 Finite and finitely presented algebras

Construction A.17. Let $\mathbf{n} = (n, d_1, \dots, d_n, m)$ be an $(n+2)$ -uple of positive integers. Let $\Lambda = \Lambda(\mathbf{n}) = \{\alpha = (\alpha_1, \dots, \alpha_n) \mid 0 \leq \alpha_i < d_i, \alpha_i \in \mathbf{Z}\}$ be an index set. Let $\mathbf{x} = \mathbf{x}(\mathbf{n}) = \{x_{ik} \mid 1 \leq i \leq n, 0 \leq k < d_i\}$ and $\mathbf{y} = \mathbf{y}(\mathbf{n}) = \{b_{j,\alpha} \mid 1 \leq j \leq m, \alpha \in \Lambda\}$ be two sets of free variables that we will use later to write formulas.

Let A be a ring, $\mathbf{a} = (a_{ik})_{1 \leq i \leq n, 0 \leq k < d_i}$ be elements of A , and $X_1, \dots, X_n$ be indeterminates. For each $i \in \{1, \dots, n\}$ let $H_i = H_i(\mathbf{a}) = a_{i0} + a_{i1}X_i + \dots + X_i^{d_i} \in A[X_i]$ be a monic polynomial of degree $d_i > 0$.

Let $B = B(\mathbf{a}) = A[X_1, \dots, X_n]/(H_1, \dots, H_n)$. Then B is a finite free A -algebra with a basis $X^\alpha = X_1^{\alpha_1} \dots X_n^{\alpha_n}$ ($\alpha \in \Lambda$).

Let $\mathbf{b} = (b_{j,\alpha})_{1 \leq j \leq m, \alpha \in \Lambda}$ be elements of A . Let $L_j = L_j(\mathbf{b}) = \sum_{\alpha \in \Lambda} b_{j,\alpha} X^\alpha \in$

B for $j = 1, 2, \dots, m$, and let $C = C(\mathbf{a}, \mathbf{b})$ be the finite and finitely presented A -algebra $B/(L_1, \dots, L_m)$.

Lemma A.18. *Use the notations in Construction A.17. There exists a formula $\psi_1^{\mathbf{n}} = \psi_1^{\mathbf{n}}(\mathbf{x}, \mathbf{y})$ such that for all rings A and all choices of elements $\mathbf{a}, \mathbf{b}$, A satisfy $\psi_1^{\mathbf{n}}(\mathbf{a}, \mathbf{b})$ if and only the A -module generated by $L_1, \dots, L_m$ is an ideal of B .*

Proof. By elementary calculation, one easily sees that there exist polynomials $D_{i,j,\alpha} \in \mathbf{Z}[\mathbf{x}, \mathbf{y}]$ (of degree at most 2) such that one always has

$$X_i L_j = G_{ij} H_i + \sum_{\alpha \in \Lambda} D_{i,j,\alpha}(\mathbf{a}, \mathbf{b}) X^\alpha$$

for $1 \leq i \leq n$ and $1 \leq j \leq m$, where $G_{ij} \in A[X_1, \dots, X_n]$. NOTE: in the final version explicit form of D and G will be given.

The A -module generated by $L_1, \dots, L_m$ is an ideal of B if and only if $\sum_{\alpha} D_{i,j,\alpha}(\mathbf{a}, \mathbf{b}) X^\alpha$ is an A -linear combination of $L_1, \dots, L_m$ for all i, j . Therefore, introducing a set of new free variables $\mathbf{z} = \{z_{ij}^l \mid 1 \leq i \leq n, 1 \leq j, l \leq m\}$, we see that

$$\exists \mathbf{z} \bigwedge_{i,j,\alpha} D_{i,j,\alpha}(\mathbf{x}, \mathbf{y}) = \sum_l z_{ij}^l y_{l,\alpha}$$

is what we want. □

Note that all finite and finitely presented algebras arise this way. More precisely:

Lemma A.19. *Let A be a ring and C' a finite and finitely presented A -algebra. Then there exist $\mathbf{n}, \mathbf{a}, \mathbf{b}$ as in Construction A.17 such that $C' \cong C(\mathbf{a}, \mathbf{b})$ and that A satisfies $\psi_1^{\mathbf{n}}(\mathbf{a}, \mathbf{b})$ as in Lemma A.18.*

Proof. Since C' is finite over A , we can find some $n, d_1, \dots, d_n$ and $\mathbf{a}$ and a surjection of A -algebras $B(\mathbf{a}) \rightarrow C'$. The kernel of this map is a finitely generated A -module by [Stacks, Tag 0564], so we can find an m and some $\mathbf{b}$ such that this ideal is generated by $L_1, \dots, L_m$ as an A -module, as desired. $\square$

Lemma A.20. *Use the notations in Construction A.17. There exist formulas $\psi_2^{\mathbf{n}} = \psi_2^{\mathbf{n}}(\mathbf{x}, \mathbf{y})$ and $\psi_3^{\mathbf{n}} = \psi_3^{\mathbf{n}}(\mathbf{x}, \mathbf{y})$ with the following properties.*

Let A be a ring and $\mathbf{a}, \mathbf{b}$ be elements. Assume that A satisfies $\psi_1(\mathbf{a}, \mathbf{b})$ as in Lemma A.18. Then

- (1) A satisfies $\psi_2(\mathbf{a}, \mathbf{b})$ if and only if $A \rightarrow C$ is injective; and*
- (2) A satisfies $\psi_3(\mathbf{a}, \mathbf{b})$ if and only if the map $A \rightarrow C$ splits as a map of A -modules.*

Proof. For (1), note that Lemma A.18 means that those L 's are relations for the module C as a quotient of the free module B . Thus injectivity is just to say that a nonzero $c \in A$ cannot be an A -linear combination of $L_1, \dots, L_m$. In other words, there exist no elements d_j ($1 \leq j \leq m$) such that $\sum_j d_j b_{j,0} = c$ and $\sum_j d_j b_{j,\alpha} = 0$ for all $\alpha \neq 0$. This is now easily seen to be described by a

formula, which we write out for the reader:

$$\neg \exists z \left(\neg(z = 0) \wedge \exists w_1 \dots \exists w_m \left(\sum_j w_j y_{j,0} = z \wedge \bigwedge_{\alpha \in \Lambda \setminus \{0\}} \sum_j w_j y_{j,\alpha} = 0 \right) \right).$$

Similarly, for (2), the existence of a splitting is to say that there exist elements $c_\alpha \in A$ ($\alpha \in \Lambda$), the images of X^α under the presumed splitting, with $c_0 = 1$ and $\sum_\alpha b_{j,\alpha} c_\alpha = 0$ for all j . We omit the formula. $\square$

Remark A.21. It can be inferred from the proof that the formulas ψ_2 and ψ_3 do not involve $\mathbf{x}$. We do not need this fact.

Corollary A.22. *There exists a set Ψ of sentences such that a ring A is a splinter if and only if it satisfies all sentences in Ψ .*

Proof. Notations as in Construction A.17 and Lemmas A.18 and A.20, write

$$\Psi = \{\varphi_{\text{red}}\} \cup \{\forall \mathbf{x}(\mathbf{n}) \forall \mathbf{y}(\mathbf{n}) (\psi_1^{\mathbf{n}} \wedge \psi_2^{\mathbf{n}} \Rightarrow \psi_3^{\mathbf{n}}) \mid \mathbf{n} \text{ varies}\}.$$

Here the sentence φ_{red} characterizes reducedness, see Example A.2. Ψ is what we want, see Lemmas 3.3 and A.19. $\square$

By Łoś's Theorem we have

Corollary A.23. *An ultraproduct of splinters is a splinter.*

Remark A.24. We cannot combine Łoś's Theorem and Corollary A.22 to conclude that if an ultraproduct is a splinter, then “most” of the factors are

splinters. This is the infinite intersection of the subsets of coordinates satisfying each formula in Ψ is usually not in the ultrafilter. For a counterexample, let $p_1, \dots, p_n, \dots$ be infinitely many primes and $A_1, \dots, A_n, \dots$ be normal integral $\mathbf{F}_{p_i}$ -algebras that are not splinters. Let $\mathcal{U}$ be a non-principal ultrafilter (see Remark A.8) on $X = \mathbf{Z}_{\geq 1}$ and let $A = \prod_i A_i / \mathcal{U}$. Via Łoś's Theorem one can show that A is a normal domain containing $\mathbf{Q}$, so A is a splinter by Proposition 9.2. Details omitted.

This example and Łoś's Theorem also tell us any set Ψ as in the statement of Corollary A.22 must be infinite.

A.4 Complement

Let $\mathcal{P}$ be a property of rings. We say $\mathcal{P}$ is a *first-order* property if there exists a set Ψ of sentences such that a ring A has the property $\mathcal{P}$ if and only if it satisfies all sentences in Ψ .

Example A.25. From Examples A.2, A.4, and A.6, and Corollary A.22, we see that the following properties of a ring A are first-order: A is reduced, A is an integral domain, A is a field, A is local, A is not Artinian (use $\Psi = \{\neg\varphi_{\ell=m} \mid m \in \mathbf{Z}_{\geq 1}\}$), A is a splinter.

The following properties are also first-order: A is a Bézout domain, A is a valuation ring, A is seminormal, A is absolutely weakly normal (see [Stacks, Tag 0EUL]), A is integrally closed in its total ring of fractions, A is absolutely integrally closed (see [Stacks, Tag 0DCM]), A is a Henselian local ring, A is a

strictly Henselian local ring. Proofs are left to the reader.

Example A.26. In Remark A.16, we see that an ultrapower of a Noetherian ring can be a non-Noetherian ring. Thus neither of the properties “ A is Noetherian” and “ A is not Noetherian” is first-order. Similarly, in the same remark we see an ultraproduct of a family of Artinian rings can be a non-Artinian ring, so the property “ A is Artinian” is not first-order.

However, an ultraproduct of a family of non-Noetherian rings is always non-Noetherian. To see this, let our ultraproduct be $A = \prod_{x \in X} A_x / \mathcal{U}$. Since no A_x is Noetherian, we can find elements $a_{1,x}, a_{2,x}, \dots$, of A_x , such that for all m and all x , $a_{m+1,x}$ is not in the ideal generated by $a_{1,x}, a_{2,x}, \dots, a_{m,x}$. Let α_j be the element of A represented by $(a_{j,x})_x$. Then from Example A.3 we see that for all m , $\alpha_{m+1,x}$ is not in the ideal generated by $\alpha_{1,x}, \alpha_{2,x}, \dots, \alpha_{m,x}$. Thus A is not Noetherian.

Example A.27. The properties “the Krull dimension of A is in S ” (S an arbitrary nonempty subset of $\mathbf{Z}_{\geq 1}$) and “ A has infinite Krull dimension” are not first-order. To see this, note that for a valuation ring A of value group Γ , the Krull dimension of A is the same as the rank of Γ . Take an A such that the rank of Γ is an element of S (or any finite rank Γ for the latter property). Consider a non-principal ultrapower $A^{\mathcal{U}}$ of A with index set $X = \mathbf{Z}_{\geq 1}$. We have seen that $A^{\mathcal{U}}$ is a valuation ring. Using, say, elements $(a^{n^m})_n$ where a is a nonzero, non-invertible element and $m \in \mathbf{Z}_{\geq 1}$, we see that the value group of $A^{\mathcal{U}}$ has infinite rank, so $A^{\mathcal{U}}$ has infinite Krull dimension.

For zero-dimensional rings, a non-principal ultraproduct of $k[t]/t^n$ ($n \in \mathbf{Z}_{\geq 1}$) also has infinite Krull dimension. One way to see this is to consider the ideals $\mathfrak{p}_m(\epsilon)$ ($\epsilon > 0$) consisting of elements represented by $(a_n)_n$ with $v_t(a_n)/n^{1/m} \geq \epsilon$ and $\mathfrak{p}_m = \cup_{\epsilon > 0} \mathfrak{p}_m(\epsilon)$. Details omitted. Thus the properties “the Krull dimension of A is in S ” ($S \subseteq \mathbf{Z}_{\geq 0}$) are not first-order. In both examples, it can be shown that the Krull dimension of the ultraproduct is $2^{\aleph_0}$.

However, the property $\mathcal{P}_{\leq m} =$ “every chain of finitely generated prime ideals in A has length at most m ” ($m \in \mathbf{Z}_{\geq 0}$) is indeed first-order. To see this, for each $(m+2)$ -uple of non-negative integers $n_0, \dots, n_{m+1}$, consider the property “there is no chain of primes $\mathfrak{p}_0 \subsetneq \dots \subsetneq \mathfrak{p}_{m+2}$ in A such that $\mathfrak{p}_{j+1}/\mathfrak{p}_j$ can be generated by n_j elements.” One can write a single sentence to characterize this property using φ_{gen} in Example A.3. Then the infinite set of such formulas with varying $n_0, \dots, n_{m+1}$ gives a set Ψ_m that characterizes $\mathcal{P}_{\leq m}$. For a Noetherian ring A , $\dim A \leq m$ if and only if A satisfies $\mathcal{P}_{\leq m}$. Moreover, for a Noetherian local ring A , $\dim A > m$ if and only if for all non-invertible elements $a_1, \dots, a_m$, $A/(a_1, \dots, a_m)$ is not Artinian. Using the constructions in Examples A.4 and A.6, we see that there exists a set Ψ'_m of sentences such that for a Noetherian local ring A , $\dim A > m$ if and only if A satisfies Ψ'_m , so $\dim A = m$ if and only if A satisfies $\Psi'_{m-1} \cup \Psi_m$. This shows that for a ring A , even one finitely generated over an algebraically closed field, the locus of primes $\mathfrak{p}$ such that $A_{\mathfrak{p}}$ satisfies a certain first-order property does not necessarily have good topological properties.

Appendix B

Weak (M-)derived splinters

We work in the derived ∞ -category $\mathcal{D}(A)$ of A -modules for a ring A , cf. [HA, §1.3.2 and §7.1]. This allows us to talk about colimit intelligently. In fact, we will not get out of the ∞ -category $\mathcal{D}^-(A)$ of right-bounded objects. Splitting in $\mathcal{D}(A)$ is the same as splitting in its homotopy category $D(A)$, the derived (1-)category of A -modules.

Definition B.1. Let A be a ring, $E \in \mathcal{D}(A)$ perfect, $F \in \mathcal{D}(A)$. We say a map $E \rightarrow F$ is *pure* if for every factorization $E \rightarrow K \rightarrow F$ where K is perfect, the map $E \rightarrow K$ splits.

Definition B.2. Let A be a ring. We say A is a weak M-derived splinter (resp. weak derived splinter) if for every M-morphism (resp. PS-morphism) $f : X \rightarrow \mathrm{Spec}(A)$, the map $A \rightarrow Rf_*\mathcal{O}_X$ is pure.

We use the shorthand WMDS (resp. WDS) for weak M-derived splinter

(resp. weak derived splinter). For Noetherian rings, we do not have anything new, see Lemma 4.2.

In the rest of this chapter, we indicate that the good properties we considered in Chapters 3–6 hold for the notions WMDS and WDS without Noetherian assumptions. We may need mild extra assumptions, for example the total fraction ring having Krull dimension 0.

Discussion B.3. By the proof of Lemma 3.3 we see a WMDS (resp. WDS) is reduced and integrally closed in its total fraction ring.

Discussion B.4. Note that a perfect complex over the direct product of a family of rings is the direct product of a family of perfect complexes over the factors. By the proof of Lemma 3.6, we see the direct product of a family of WMDSs (resp. WDSs) is a WMDS (resp. WDS).

Discussion B.5. If E is perfect and $(E \rightarrow F_i)_i$ is a system of maps, then $E \rightarrow \operatorname{colim}_i F_i$ is pure if and only if $E \rightarrow F_i$ is pure for each i . This is because a perfect testing object K is a compact object in $\mathcal{D}(A)$. From this and [Stacks, Tag 09ZR] we see that for a WDS A and a proper surjective morphism $f : X \rightarrow \operatorname{Spec}(A)$, $A \rightarrow Rf_* \mathcal{O}_X$ is pure.

Discussion B.6. For a map $E \rightarrow F$ where E is perfect, we can always write $(E \rightarrow F) = \operatorname{colim}_i (E \rightarrow F_i)$ where each F_i is perfect. Since splitting is preserved by base change, using Discussion B.5 we see if $E \rightarrow F$ is pure, then for all ring maps $A \rightarrow B$, $E \otimes_A^L B \rightarrow F \otimes_A^L B$ is pure. Therefore we see from the proof of Lemma 3.8 that if A is a WDS (resp. a WMDS whose total fraction ring has Krull dimension 0), then any localization $S^{-1}A$ is a

WDS (resp. WMDS). From the proof of Lemma 6.2, and Remark A.11, we see an ultraproduct of WDSs (resp. WMDSs whose total fraction ring has Krull dimension 0) is a WDS (resp. WMDS).

Discussion B.7. Purity is local, since the formation of $\mathrm{Hom}_{D(A)}(E, K)$ commutes with localization when E and K are perfect, cf. §2.6. This way we see that being a WMDS (resp. WDS) is a local property, *i.e.* Corollary 3.9 holds for the property of being a WMDS (resp. a WDS).

Discussion B.8. Materials in Chapter 4 shows that a direct limit of WDSs is a WDS. For WMDSs, note that if a reduced local ring A is integrally closed in its total fraction ring K , and $\dim K = 0$, then A is an integral domain. This is because K has no nontrivial idempotents, so $\mathrm{Spec}(K)$ is connected; but $\mathrm{Spec}(K)$ is also totally disconnected (see for example [Stacks, Tag 0905]), so $\mathrm{Spec}(K)$ is a singleton and K is a field. Thus materials in Chapter 4 also shows that for a direct system of WMDSs whose total fraction ring has Krull dimension 0, the colimit is a WDS. Now materials in Chapter 5 show that if A is a WDS (resp. a WMDS) whose total fraction ring has Krull dimension 0 and B is an inductively étale A -algebra, then B is a WDS (resp. WMDS). See Remark 5.5.

Discussion B.9. If $A \rightarrow B$ is a pure ring map (resp. a pure ring map that sends nonzerodivisors to nonzerodivisors) and B is a WDS (resp. a WMDS), so is A . To see this, Note that for all qcqs morphism $f : X \rightarrow \mathrm{Spec}(A)$ of schemes the map $B \rightarrow Rf_{B*}\mathcal{O}_{X_B}$ factors through $B \rightarrow Rf_*\mathcal{O}_X \otimes_A^L B$. Arguing as in the beginning of the proof of Proposition 3.11, we see that it

suffices to show for a map $A \rightarrow F$ in $\mathcal{D}(A)$, if $B \rightarrow F \otimes_A^L B$ is pure, so is $A \rightarrow F$. To show this, we may assume F perfect, so $B \rightarrow F \otimes_A^L B$ splits. The splitting induces a map $F \rightarrow B$ in $\mathcal{D}(A)$ sending $1 \in H^0(F)$ to 1. Such a map must factor through a finitely presented A -submodule M of B , and we know $A \rightarrow M$ splits, since $A \rightarrow B$ is pure. Thus $A \rightarrow F$ splits.

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