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Symplectic Vortex Equation and Adiabatic Limits

Xu, Guangbo

Princeton, NJ : Princeton University

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Symplectic vortex equation and adiabatic limits

Guangbo Xu

A Dissertation
Presented to the Faculty
of Princeton University
in Candidacy for the Degree
of Doctor of Philosophy

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by the Department of
Mathematics
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Abstract

This thesis consists of four parts, on four separate topics in the study of the symplectic vortex equation and their adiabatic limits.

In the first part, we constructed a compactification of the moduli space of twisted holomorphic maps with Lagrangian boundary condition. It generalizes the compactness theorem of Mundet-Tian in the case of closed Riemann surfaces to the case of bordered Riemann surfaces, and it is the first step in developing the open-string analogue of Mundet-Tian's program.

In the second part, we studied the Morse theory of Lagrange multipliers, which is based on a joint work with Stephen Schecter. We also considered two adiabatic limits by varying a real parameter λ in this theory, which result in two different homology group. Via the homotopy provided by the variation of λ we prove that the two homology groups are isomorphic.

In the third part, we considered a $U(1)$-gauged linear σ-model and its low-energy adiabatic limits. Via adiabatic limits, we managed to classify all affine vortices in the target space $\mathbb{C}^N$ with diagonal $U(1)$-action and identify their moduli spaces, which generalizes Taubes' result for $N = 1$. This also gives a precise meaning of the "point-like instantons" described by Witten. We also computed the associated quantum Kirwan map by compactifying the moduli space.

In the fourth part, we introduce a new type of equations. It is a generalization of Witten's equation for a quasi-homogeneous polynomial W, by coupling a gauge field. The purpose of this generalization is to realize the geometric Landau-Ginzburg/Calabi-Yau correspondence predicted in string theory. This part is based on a work in progress joint with Gang Tian.

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Dedicated to My Past Self

Contents

Abstract iii

Acknowledgements iv

1 Introduction **1**

1.1 Background 1

1.2 Gauged σ-model and symplectic vortex equation 2

1.3 Preliminaries 3

2 A compactification of moduli space of twisted holomorphic maps with Lagrangian boundary condition **14**

2.1 Introduction to this chapter 14

2.2 Preliminaries for this chapter 16

2.3 Twisted holomorphic pairs from bordered curves with Lagrangian boundary condition 18

2.4 Twisted holomorphic pairs over semi-infinite cylinders with small energy density . . 24

2.5 Reflection of twisted pairs from strips with Lagrangian boundary condition 26

2.6 Definition of (c, L)-stable twisted holomorphic maps 37

2.7 Bounding the number of bubbles 47

2.8 Definition of the convergence 53

2.9 Compactness 60

3 Morse theory for Lagrange multipliers and adiabatic limits **74**

3.1 Introduction to this chapter 74

3.2 Morse homology and Morse-Smale-Witten complex of $(\mathcal{F}, g_\lambda)$ 78

3.3 Adiabatic limit $\lambda \to \infty$ 83

3.4 Fast-slow system associated to the $\lambda \to 0$ limit 91

3.5 Adiabatic limit $\lambda \to 0$ 100

3.6 Basic Morse theory and the chain complex $\mathcal{C}^0$ 114

4 Affine vortices in the adiabatic limits of gauged linear σ-model 119

4.1 Introduction to this chapter 119

4.2 Affine vortices 121

4.3 Degenerations of affine vortices 126

4.4 Quantum Kirwan map 133

4.5 The adiabatic limit of $U(1)$-vortices 134

4.6 Classification of $U(1)$-vortices and its moduli spaces 143

5 Vortex eqution and Landau-Ginzburg/Calabi-Yau correspondence 155

5.1 Introduction to this chapter 155

5.2 The Witten equation and the vortex equation 156

5.3 Energy identity 159

5.4 Application in gauged linear σ-model 160

5.5 Interior compactness 166

A 171

A.1 Geometric singular perturbation theory 171

A.2 Choosing the center manifold 177

Chapter 1

Introduction

1.1 Background

The development of theoretical physics in the twentieth century, most notably quantum field theory and string theory, has dramatically reshaped the study of differential geometry. To understand the beautiful and fanscinating mathematical predictions physicists made, mathematicians had to develop new machineary and lauguage to make rigorous statement and argument, instead of using the ill-defined path integral and many mysterious physics arguments. The most common feature of the language mathematicians used is the appearance of nonlinear ellitptic partial differential equations obtained from variational methods, and the moduli spaces of their solutions. The most exciting results were the construction of certain "invariants", obtained by doing intersection theory on these moduli spaces, which provide a lot of sophisticated topological and geometric information.

There are two basic models in quantum field theory, the σ-model and gauge theory. The most successful mathematical achievements in these two directions include Donaldson theory and Seiberg-Witten theory (which are for gauge theory in dimension 4), and Gromov-Witten theory (which is for σ-model in dimension 2, with target space a symplectic manifold).

In Donaldson theory, one considers a $SU(2)$-principal bundle P over a smooth 4-manifold X, and consider self-dual or anti-self-dual Yang-Mills connections on P (called instantons). This is a non-linear elliptic partial differential equation modulo gauge transformation. In Seiberg-Witten theory, one considers a 4-dimensional Spin_c-manifold, and the solutions to the Seiberg-Witten monopole equation. This is also an elliptic equation modulo gauge transformation. The topology of their moduli spaces of solutions contains a lot of nontrivial information of the differential topology of

the original four-manifolds, which are wrapped up to be the celebrated Donaldson invariants and Seiberg-Witten invariants. It is important to notice that the work of Uhlenbeck set up the analytical foundation on these striking development.

In Gromov-Witten theory, one consider maps from a Riemann surface into a symplectic manifold. With a choice of a suitable almost complex structure, the minimizers of the Dirichelet energy of maps are solutions to the nonlinear Cauchy-Riemann equation, which provides a natural class of minimal surfaces. On the other hand, string theory can be viewed as a quantum field theory if regarding the world-sheet as the domain of the σ-model. It was Gromov who first adopted this point-of-view, and proved many fantastic theorems in symplectic geometry. After that, a lot of far-reaching results have been proved or conjectured, most notably, the definition of Gromov-Witten invariants and Floer homology, as well as the various versions of "mirror symmetry".

There are still many other physics achievements which await mathematicians' feedback. Here in this dissertation, we consider a "hybrid" theory of gauge theory and σ-model, in dimension 2. The coupled theory has its natural connections with the previous two theories: for example, the gauged σ-model in dimension 2 can be related to gauge theory in dimension 4 via "dimensional reduction"; the adiabatic limits in gauged σ-model also reduces to an un-gauged σ-model. To go down to the earth of mathematics, the remaining of this introduction is written as a general preliminary, which are prerequisite for the four topics in this dissertation.

1.2 Gauged σ-model and symplectic vortex equation

In the situation of σ-model, if we have in addition a compact Lie group G acting on the target manifold M, which preserves all structures we need, then instead of considering maps from the space-time (which is always a Riemann surface Σ here), the fields should be "twisted". This means, for a principal G-bundle $P \to \Sigma$, a field consists of two components, a section of the associated bundle $P \times_G M$ and a G-connection on P. In addition, to write down the kinetic energy, the inclusion of gauge field is necessary.

The benefit of twisting a gauge field is manifest, in the case when M is a vector space: there are no nonconstant holomorphic curves in $\mathbb{C}^n$, but there are plenty of holomorphic sections of a vector bundle. For this case of linear actions on complex vector spaces, it has been studied intensively (see [25], [3], [4], [20], [18], [19]).

The rigorous formulation of this theory for general Hamiltonian actions was first given independently by Ignasi Mundet [37] and Cieliebak-Gaio-Salamon [8]. After it was introduced, a lot of

progress has been made and many important results have been proved. In certain cases, the so-called Hamiltonian Gromov-Witten invariant has been defined (see [37], [7]); Mundet and Tian are carrying out a program defining the quantum ring structure on the equivariant cohomology, which will be an S^1-equivariant analog of the quantum cohomology (see [38]); Gaio-Salamon considered certain adiabatic limit of the twisted holomorphic map which relates the Hamiltonian Gromov-Witten invariants of the symplectic manifold and the usual Gromov-Witten invariants of a symplectic quotient with respect to the group action (see [17]); Frauenfelder considered twisted holomorphic maps over the infinite strip and proved the Arnold-Givental conjecture for certain type of symplectic manifolds and Lagrangian submanifolds (see [14]); Ziltener and Woodward considered the vortex equation over the complex plane and define a quantum version of the Kirwan map (see [61], [57]).

It is natural to include a potential term on the target space M but this point-of-view in σ-model hasn't been fully adopted by mathematicians. Recently, Fan-Jarvis-Ruan developed a quantum singularity theory (see [10], [11], [9]) based on Witten's idea ([54]). It can be viewed as a σ-model with the target space a complex singularity, which is defined by a potential function $W : \mathbb{C}^n \to \mathbb{C}$, usually a quasi-homogeneous polynomial. One of the topic in this dissertation is to set up a mathematical theory which includes a potential function in the gauged σ-model, over a general target manifold M (generalizing $\mathbb{C}^n$).

1.3 Preliminaries

Let M be a $2n$-dimensional smooth manifold. A **symplectic structure** is a 2-form $\omega \in \Omega^2(M)$ satisfying

- it is closed, i.e. $d\omega = 0$;
- it is nondegenerate, i.e. $\omega^n \in \Omega^{2n}(M)$ is a nowhere vanishing $2n$-form on M.

A $2n$-dimensional manifold M together with a symplectic structure ω is called a **symplectic manifold** and is usually denoted by (M, ω).

An (embedded) **Lagrangian submanifold** of (M, ω) is an embedded submanifold $L \subset M$ of dimension n, such that $\omega|_L = 0 \in \Omega^2(L)$.

An **almost complex structure** on M is a section $J \in \Gamma(\mathrm{End}TM)$ such that $J^2 = -\mathrm{Id}_{TM}$. J is said to be compatible with a symplectic structure ω (or ω-compatible), if the tensor field defined

by

$$\omega(\cdot, J\cdot) \in \Gamma(T^*M \otimes T^*M) \tag{1.1}$$

is a Riemannian metric on M.

Let Σ be a Riemann surface and let $j_\Sigma : T\Sigma \to T\Sigma$ be the almost complex structure on Σ. A smooth map $u : \Sigma \to M$ is said to be (J, j_Σ)-holomorphic (or J-holomorphic if j_Σ is understood), if the differential of u is complex linear with resepct to J and j_Σ, i.e.,

$$du \circ j_\Sigma = J \circ du. \tag{1.2}$$

In general, for any smooth map $u : \Sigma \to M$, we define

$$\overline{\partial} u := \frac{1}{2}\left(du + J \circ du \circ j_\Sigma\right) \tag{1.3}$$

as the $(0,1)$-part of the differential du.

Let G be a compact Lie group of dimension k and let $\mathfrak{g}$ be its Lie algebra and $\exp : \mathfrak{g} \to G$ be the exponential map. Suppose G acts smoothly on M. For any vector $\xi \in \mathfrak{g}$, the **infinitesimal action** of ξ is the vector field $X_\xi \in \Gamma(TM)$ defined by

$$(X_\xi f)(p) := \left.\frac{d}{dt}\right|_{t=0} f\left(\exp(t\xi)p\right) \tag{1.4}$$

for any point $p \in M$ and any smooth function f defined near p. The action is called a **Hamiltonian action**, if there exists a smooth map (called a moment map of this action) μ from M to the dual vector space $\mathfrak{g}^*$ of $\mathfrak{g}$, such that for any $\xi \in \mathfrak{g}$,

$$\iota_{X_\xi}\omega = d\left(\mu \cdot \xi\right) \in \Omega^1(M). \tag{1.5}$$

Any two moment maps μ_1, μ_2 for the same action differ by a central element $\tau \in \mathfrak{g}^*$ on each connected component of M.

Let $P \to \Sigma$ be a G-principal bundle over Σ. Then the right G-action on P induces a family of vertical vector fields X_ξ on P for $\xi \in \mathfrak{g}$. A G-connection on P is a section $A \in \Omega^1(P, \mathfrak{g})$ satisfying

- $\iota_{X_\xi} A = \xi$;
- $(R_g)^* A = \mathrm{Ad}_g A$.

The space of smooth connections is denoted by $\mathcal{A}(P)$, which is an affine space.

The space of sections of the associated bundle $Y := P \times_G M$ can be identified with the space of G-equivariant maps

$$\mathcal{S}(Y) := \left\{ s \in C^\infty(P, M) \mid s(ph) = h^{-1}s(p),\ \forall h \in G,\ p \in P \right\}.$$

The group of (smooth) gauge transformations of P is

$$\mathcal{G}(P) := \left\{ g \in C^\infty(P, G) \mid g(ph) = h^{-1}g(p)h,\ \forall h \in G,\ p \in P \right\},$$

which is the group of automorphisms of P (as a G-principal bundle).

$\mathcal{G}(P)$ acts naturally on $\mathcal{A}(P) \times \mathcal{S}(Y)$ (on the right), by

$$g^* (A, s) = (g^* A, s \circ g).$$

1.3.1 J-holomorphic curves and the Gromov-Witten invariants

The Gromov-Witten invariants for a compact symplectic manifold (M, ω) is a family of linear maps, parametrized by nonnegative integers (g, n), and homology class $A \in H_2(M; \mathbb{Z})$,

$$GW_{g,n}^{M,A} : (H^*(M; \mathbb{Q}))^{\otimes n} \otimes H^* \left(\overline{\mathcal{M}}_{g,n}; \mathbb{Q} \right) \to \mathbb{Q}. \tag{1.6}$$

The first mathematical definition was given by Ruan ([41]) in genus zero and M semi-positive, and then was generalized by Ruan-Tian ([42], [43])for arbitrary genus for semi-positive M. For general compact symplectic manifold, it was defined by Li-Tian ([31]) and Fukaya-Ono ([16]).

We briefly describe the construction of Gromov-Witten invariants. Choose an ω-compatible almost complex structure J, consider all smooth maps u from a genus g Riemann surfaces Σ with n marked points to M such that u is holomorphic with respect to J and that the cycle u represent the homology class A. Consider the moduli space of all such objects,

$$\mathcal{M}_{g,n}(M, A, J). \tag{1.7}$$

Although in some cases, by choosing generic J, we can make this moduli space a smooth manifold, but it is in general noncompact. The noncompactness comes from two situations: one is the bubbling off phenomenon, the other is that the domain curve can degenerate to a nodal curve who represents

a point in the boundary of $\overline{\mathcal{M}}_{g,n}$. By adding new objects, called "stable maps" (introduced by Kontsevich in [29]), and defining suitable topology consistent with the bubbling and degenerating process, we can compactify the moduli space by

$$\overline{\mathcal{M}}_{g,n}(M,A,J) \supset \mathcal{M}_{g,n}(M,A,J). \tag{1.8}$$

The "virtual technique" developed by Li-Tian ([31]), Fukaya-Ono ([16]), Hofer-Wysocki-Zehnder ([23]) allows us to define a virtual fundamental class

$$\left[\overline{\mathcal{M}}_{g,n}(M,A,J)\right]^{vir} \in H_*\left(\overline{\mathcal{M}}_{g,n}(M,A,J);\mathbb{Q}\right). \tag{1.9}$$

On the other hand, we have natural maps

$$\begin{array}{ccc} \overline{\mathcal{M}}_{g,n}(M,A,J) & \xrightarrow{\text{ev}} & M^n \\ \pi\big\downarrow & & \\ \overline{\mathcal{M}}_{g,n} & & \end{array} \tag{1.10}$$

where ev is the "evaluation map", which maps any (equivalence class of) stable map to its image at the n marked points, and π is the "forgetful map", which maps any stable map to its underlying stable curve by forgetting the map and stablizing the domain.

Then the Gromov-Witten invariants of (M,ω) can be defined by

$$GW_{g,n}^{M,A}(\alpha_1,\ldots,\alpha_n;\beta) = \left\langle \text{ev}^*(\alpha_1,\ldots,\alpha_n)\cup\pi^*\beta, \left[\overline{\mathcal{M}}_{g,n}(M,A.J)\right]^{vir}\right\rangle. \tag{1.11}$$

In particular, using the genus-zero, three-point invariants, one can define a "quantum deformation" of the usual cohomology ring of M. For simplicity, we assume that (M,ω) is monotone, i.e., there exists a real number $c>0$ such that $c_1(TM) = c[\omega]$. We also assume that $H_2(M;\mathbb{Z})$ is torsion-free and there is an additive basis $A_1,\ldots,A_m$ of $H_2(M;\mathbb{Z})$ such that the homology class of any holomorphic sphere in M is of the form $d_1A_1+\cdots+d_mA_m$ with $d_i\geq 0$. Then we choose the Novikov ring Λ to be the polynomial ring $\mathbb{R}[q_1,\ldots,q_m]$ with $\deg q_i = 2c_1(A_i)$. The quantum cohomology ring of M is a ring with underlying abelian group

$$QH^*(M;\Lambda) := H^*(M;\mathbb{Q})\otimes_{\mathbb{Q}}\Lambda. \tag{1.12}$$

Choose an additive basis $\{e_\nu\}$ of $H^*(M;\mathbb{Q})$ over $\mathbb{Q}$, the multiplication is defined, for every $a,b\in$

$H^*(M;\mathbb{Q})$,

$$a *_q b := \sum_{A=d_1A_1+\cdots+d_mA_m} \sum_{\nu_1,\nu_2} GW_{0,3}^{M,A}(a,b,e_{\nu_1}) g^{\nu_1\nu_2} e_{\nu_2} \otimes q_1^{d_1}\cdots q_m^{d_m} \tag{1.13}$$

and extended in a Λ-linear way to $H^*(M;\Lambda)$. Here $(g^{\nu_1\nu_2})$ is the inverse matrix of the intersection matrix $\left(g_{\nu_1\nu_2} := \int_M e_{\nu_1} \cup e_{\nu_2}\right)$. This makes $QH^*(M;\Lambda)$ a graded algebra over Λ, which is called the *(small) quantum cohomology* of M.

For example, for the case $M = \mathbb{P}^n$ and ω the Fubini-Study form, $\Lambda = \mathbb{R}[q]$ with $\deg q = 2(n+1)$ and the quantum cohomology ring of $\mathbb{P}^n$ is

$$QH^*(\mathbb{P}^n;\Lambda) = \mathbb{R}[c,q]/\langle c^{n+1} = q\rangle.$$

1.3.2 The symplectic vortex equation

Suppose we have a Hamiltonian G-action on the symplectic manifold (M,ω) with a moment map $\mu : M \to \mathfrak{g}^*$ and suppose J is a G-invariant, ω-compatible almost complex structure on M. Let's choose a biinvariant metric on $\mathfrak{g}$ which induces an isomorphism $\mathfrak{g} \simeq \mathfrak{g}^*$. Let (Σ, j_Σ) be a Riemann surface and let $\Omega_\Sigma \in \Omega^2(\Sigma)$ be a smooth area form.

Let P be a G-principal bundle over Σ. Let $Y := P \times_G M$ and $\pi_Y : Y \to \Sigma$ be the natural projection. Let $T^VY \subset TY$ be the subbundle of vertical tangent vectors. If we choose a G-invariant almost complex structure J, then J induces a complex structure J^V on the vector bundle T^VY.

Any G-connection A on P induces a splitting $TY = \pi_Y^*T\Sigma \oplus T^VY$, which induces (together with J) an almost complex structure $J(A)$ on Y as the direct sum $\pi_Y^*j_\Sigma \oplus J^V$.

Let $u : \Sigma \to Y$ be a section of the bundle $Y \to \Sigma$, which is equivalent to an anti-equivariant map $\Phi : P \to M$. The covariant derivative d_Au of u with respect to A is given by

$$d_Au = \pi_A \circ du \in \Omega^1\left(\Sigma, u^*T^VY\right),$$

where $\pi_A : TY \to T^VY$ is the projection induced by the splitting given by A.

Since $T\Sigma$ and T^VY are both complex vector bundles, we can define the operator $\overline{\partial}_A$ by taking the $(0,1)$-part of d_A. More precisely, define

$$\overline{\partial}_Au = \frac{1}{2}\left(d_Au + J^V \circ d_Au \circ j_\Sigma\right) \in \Omega^{0,1}\left(\Sigma, u^*T^VY\right).$$

Definition 1.3.1. A **twisted holomorphic map** (or a gauged holomorphic map) from Σ to M is a triple (P, A, u), where $P \to \Sigma$ is a smooth G-principal bundle, A is a smooth G-connection on P and u is a smooth section of the associated bundle $Y := P \times_G M \to \Sigma$, satisfing the following **symplectic vortex equation**:

$$\begin{cases} \overline{\partial}_A u &= 0; \\ \Lambda F_A + \mu(u) &= 0. \end{cases} \tag{1.14}$$

Here $\Lambda : \Omega^2(\Sigma) \to \Omega^0(\Sigma)$ is the contraction with respect to the area form Ω_Σ; and the second equation makes sense because μ takes values in $\mathfrak{g}$ via the identification $\mathfrak{g} \simeq \mathfrak{g}^*$.

If we have a local trivialization $P|_U = U \times G$ and a local holomorphic coordinate $z = s + it$ on $U \subset \Sigma$, then u corresponds to a map $\phi : U \to M$ and $A = d + \Phi ds + \Psi dt$, the equation (1.14) reads

$$\begin{cases} \partial_s \phi - X_\Phi(\phi) + J(\phi)\left(\partial_t \phi - X_\Psi(\phi)\right) &= 0; \\ \left(\partial_s \Psi - \partial_t \Phi + [\Phi, \Psi]\right) ds dt + \mu(\phi)\Omega_\Sigma &= 0. \end{cases} \tag{1.15}$$

Definition 1.3.2. We say that two solutions (P, A, u) and (P', A', u') are **equivalent**, if there is a bundle isomorphism $\rho : P' \to P$ which lifts the identity map on Σ, such that $\rho^*(A, u) = (A', u')$.

Denote the left-hand-side of (1.14) by $\mathcal{F}(A, u) \in \Omega^{0,1}(u^* T^V Y) \oplus \Omega^0(\mathrm{ad}P)$. It is easy to check that, for any gauge transformation $g \in \mathcal{G}(P)$, $\mathcal{F}(g^*(A, u)) = g^*(\mathcal{F}(A, u))$. Therefore we say that the equation (1.14) is **gauge invariant**.

The energy of a twisted holomorphic map (P, A, u) is given by the Yang-Mills-Higgs functional

$$\mathcal{YMH}(P, A, u) = \frac{1}{2}\left(\|F_A\|_{L^2}^2 + \|\mu(u)\|_{L^2}^2 + \|d_A u\|_{L^2}^2\right). \tag{1.16}$$

Here the L^2-norms are defined with respect to the Riemannian metric on M determined by ω and J, and the Riemannian metric on Σ determined by Ω_Σ and j.

If the domain Σ is a closed Riemann surface, then the tuple (P, A, u) defines a fundamental class $[P, A, u] \in H_2^G(M; \mathbb{Z})$ which is a topological invariant. It is defined as follows. The principal bundle $P \to \Sigma$ induces a classifying map $c : \Sigma \to BG$ to the classifying space of G. It lifts to a map $\widetilde{c} : P \to EG$ of G-bundles, where $EG \to BG$ is the universal G-bundle. Hence it induces a map $\widetilde{c}_X : P \times_G X \to EG \times_G X =: X_G$. The equivariant (co)homology of X is defined to be the usual

(co)homology of X_G. Then the composition $\widetilde{c}_X \circ u : \Sigma \to X_G$ induces the homology class

$$[P, A, u] := (\widetilde{c}_X \circ u)_* [\Sigma] \in H_2 (X_G; \mathbb{Z}) = H_2^G(X; \mathbb{Z}). \tag{1.17}$$

On the other hand, in the equivariant de Rham theory of X, the equivariant differential form $\omega - \mu \in \Omega_G^2(X)$ is equivariantly closed. Hence it defines a cohomology class

$$[\omega - \mu] \in H_G^2(X). \tag{1.18}$$

We have the following energy identity.

Proposition 1.3.3. *(cf. [7, Proposition 2.2]) If Σ is a compact Riemann surface, then for any triple (P, A, u), we have*

$$\mathcal{YMH}(P, A, u) = \langle [\omega - \mu], [(P, A, u)] \rangle + \left\| \overline{\partial}_A u \right\|_{L^2}^2 + \frac{1}{2} \left\| \Lambda F_A + \mu(u) \right\|_{L^2}^2 \tag{1.19}$$

Proof. Recall that we have the equivariant symplectic form $\omega - \mu \in \Omega_G^2(X)$. And for the fibration $Y \to \Sigma$ and any G-connection A, we have the minimal coupling form

$$\omega_A \in \Omega^2(Y).$$

And we have

$$\int_\Sigma u^* \omega_A = \langle [\omega - \mu \otimes t], [(P, A, u)] \rangle. \tag{1.20}$$

We can write down ω_A locally. Take any local trivialization of $P \to \Sigma$, say $\phi_U : P|_U \to U \times G$, it induces a trivialization $Y|_U \simeq U \times X$. With respect to this trivialization, we can write $A = d + \alpha$, with $\alpha \in \Omega^1(U, \mathfrak{g})$. Let $\pi_2 : U \times X \to X$ be the projection. Then the form ω_A is written in this chart as $\pi_2^* \omega_X - d(\mu \cdot \alpha)$.

Let $w = s + it$ be a holomorphic coordinate on U and suppose $\Omega_\Sigma = \nu ds \wedge dt$. Write $\alpha =$

$d + \alpha_s ds + \alpha_t dt$, we have

$$\begin{aligned}
\nu \left| d_A u \right|^2 &= \left| \frac{\partial u}{\partial s} + X_{\alpha_s}(u) \right|^2 + \left| \frac{\partial u}{\partial t} + X_{\alpha_t}(u) \right|^2 \\
&= \left| \frac{\partial u}{\partial s} + X_{\alpha_s}(u) + J\left(\frac{\partial u}{\partial t} + X_{\alpha_t}(u) \right) \right|^2 - 2 \left\langle \frac{\partial u}{\partial s} + X_{\alpha_s}(u), J\left(\frac{\partial u}{\partial t} + X_{\alpha_t}(u) \right) \right\rangle \\
&= 2\nu \left| \overline{\partial}_A u \right|^2 + 2\omega \left(\frac{\partial u}{\partial s} + X_{\alpha_s}(u), \frac{\partial u}{\partial t} + X_{\alpha_t}(u) \right) \\
&= 2\nu \left| \overline{\partial}_A u \right|^2 + 2\nu\Lambda \left(u^* \omega_A + \mu \cdot F_A \right).
\end{aligned}$$

Therefore

$$\frac{1}{2} \left| d_A u \right|^2 \Omega_\Sigma = u^* \omega_A + \left| \overline{\partial}_A u \right|^2 \Omega_\Sigma + \mu(u) \cdot F_A. \tag{1.21}$$

Integrating over Σ and by (1.20)

$$\frac{1}{2} \left\| F_A \right\|_{L^2} + \frac{1}{2} \left\| d_A u \right\|_{L^2}^2 = \left\| \overline{\partial}_A u \right\|_{L^2}^2 + \int_\Sigma u^* \omega_A + \frac{1}{2} \left\| F_A + \mu(u) \Omega_\Sigma \right\|_{L^2}^2 - \frac{1}{2} \left\| \mu(u) \right\|_{L^2}^2. \tag{1.22}$$

□

This calculation implies that the equation (1.14) is a natural equation on pairs (A, u) whose solutions minimize (locally) the Yang-Mills-Higgs functional. This is similar to the fact that in the σ-model in an almost Kähler manifold, solutions to the nonlinear Cauchy-Riemann equation minimize the Dirichlet energy.

1.3.3 Hamitonian Gromov-Witten invariants

Hamiltonian Gromov-Witten (HGW) invariants are suggested by Mundet-Tian which are supposed to be an equivariant analogue of the Gromov-Witten invariants and its definition is also the (virtual) integration over the moduli space of twisted holomorphic maps. The inputs for HGW are equivariant cohomology classes rather than usual cohomology classes, and the evaluation maps are twisted. This means, over the moduli space $\mathcal{M}_{g,k}(M)$ of genus g twisted holomorphic maps with k marked points into M, to each marked point a G-principal bundle over the moduli space is associated. They are called the Poincaré bundles

$$P_i \to \mathcal{M}_{g,k}(M).$$

There are natural evaluation maps

$$ev_i : P_i \to M$$

which are G-equivariant. Hence it defines a pull-back

$$ev_i^* : H_G^*(M;\mathbb{Q}) \to H^*(\mathcal{M}_{g,k}(M);\mathbb{Q}).$$

While the forgetful map is still valid

$$\pi : \mathcal{M}_{g,k}(M) \to \overline{\mathcal{M}}_{g,k}.$$

Hence formally we can define the Hamiltonian Gromov-Witten invariants, for any equivariant homology class $A \in H_2^G(M;\mathbb{Z})$,

$$HGW_{g,k}^{M,A} : (H_G^*(M;\mathbb{Q}))^{\otimes k} \times H^*\left(\overline{\mathcal{M}}_{g,k};\mathbb{Q}\right) \to \mathbb{Q}$$

if using the virtual technique properly.

Although the full invariants (which allows complex structure of Σ to vary) are expected to be defined (for the case $G = S^1$, [38] through the compactness result in [39]), here we fix the domain curve Σ with fixed marked points, which represent a point in the smooth part of $\overline{\mathcal{M}}_{g,k}$. In this case, it has been defined in various cases, for example, [37], [7]. We denote by [pt] the cohomology class Poincaré dual to the cycle of a single point in $\overline{\mathcal{M}}_{g,n}$.

A beautiful results about the relation between HGW and GW is obtained in [17]. Assume that M is a noncompact, symplectic aspherical manifold, convex at infinity, and suppose the moment map is proper, and the symplectic quotient is a smooth monotone symplectic manifold. Then for $\Sigma = S^2$, we have

Theorem 1.3.4. *[17] For cohomology classes $\alpha_i \in H_G^*(M)$ with $\deg\alpha_i < 2N$, where N is the minimal Chern number of the symplectic quotient $\overline{M}$; let $A \in H_2^G(M)$ be a homology class which is the image of $\overline{A} \in H_2\left(\overline{M}\right)$ via the natural map $H_2\left(\overline{M}\right) \to H_2^G(M)$. Then we have*

$$HGW_{S^2,k}^{M,A}\left(\alpha_1,\ldots,\alpha_k;[\mathrm{pt}]\right) = GW_{S^2,k}^{\overline{M},\overline{A}}\left(\kappa(\alpha_1),\ldots,\kappa(\alpha_k);[\mathrm{pt}]\right).$$

Here $\kappa : H_G^*(M) \to H^*\left(\overline{M}\right)$ is the classical Kirwan map.

1.3.4 Example: holomorphic N-pairs

In this subsection we give an example of twisted holomorphic maps. In this example, one can see the relation between symplectic vortex equation and complex algebraic geometry. This example will also be used in Chapter 4.

Let $M = \mathbb{C}^k$ and let $G = U(k)$ which acts on $\mathbb{C}^k$ via the standard linear action. For the symplectic form

$$\omega = \sum_{i=1}^{k} dx_i \wedge dy_i = \frac{\sqrt{-1}}{2} \sum_{i=1}^{k} dz_i \wedge d\overline{z}_i \tag{1.23}$$

a moment map is

$$\mu(z_1, \ldots, z_k) = -\frac{\sqrt{-1}}{2} \left(\sum_{i=1}^{k} z_i \otimes \overline{z}_i^T - \tau I_k \right) \in \mathfrak{u}(k) \simeq \mathfrak{u}(k)^*. \tag{1.24}$$

If (P, A, u) is a twisted holomorphic map from Σ to $\mathbb{C}^k$, then the associated bundle $E := P \times_{U(k)} \mathbb{C}^k$ is a complex vector bundle with a Hermitian metric such that P is the unitary frame bundle of E. The $(0,1)$-component of A defines a holomorphic structure on E and A is then the Chern connection determined by $\overline{\partial}_A$ and the Hermitian metric. The section $u : P \to \mathbb{C}^k$ corresponds to a holomorphic section of the bundle $\left(E, \overline{\partial}_A\right)$.

We can also describe similar objects in homomorphic category. A pair $(\mathcal{E}, u)$, where $\mathcal{E} \to \Sigma$ is a holomorphic vector bundle of rank k and $u \in H^0(\mathcal{E})$ is a holomorphic section, is called a rank-k **holomorphic pair**.

More generally, we make take N copies of $\mathbb{C}^k$ and $U(k)$ acts on the N copies in a diagonal way, and the moment map is the sum of the N moment maps. In this case, a twisted holomorphic map corresponds to a rank k holomorphic vector bundle $\mathcal{E}$ with N holomorphic sections, which is called a rank-k holomorphic N-pair.

These two kinds of objects in the two different categories are closely related. For simplicity we only consider the abelian case, i.e., $k = 1$. We can take $\tau = 1$ without loss of generality. A holomorphic N-pair $(\mathcal{L}; \varphi_1, \ldots, \varphi_N)$ is called **stable**, if at least one of φ_j is nonzero.

Theorem 1.3.5. *(cf. [4]) For any compact Riemann surface Σ with any smooth area form Ω_Σ with* $\mathrm{Area}\Omega_\Sigma > 4\pi d$*, for any stable rank 1 holomorphic N-pair $(\mathcal{L}; \varphi_1, \ldots, \varphi_N)$ over Σ with* $\deg\mathcal{L} = d$*, there exists a unique smooth Hermitian metric H which solves the vortex equation, i.e., the following*

equation is satisfied:

$$F_H - \frac{\sqrt{-1}}{2}\left(\sum_{j=1}^{N}|\varphi_j|_H^2 - 1\right)\Omega_\Sigma = 0. \tag{1.25}$$

Here F_H is the Chern connection of $(\mathcal{L}, H)$.

Because the choice of a Hermitian metric is equivalent to fix a unitary gauge orbit inside a complex gauge orbit in the space of unitary connections, it is easy to see that (1.25) is equivalent to the second equation of (1.14) specialized to this case. In fact this is a special case of the celebrated *Hitchin-Kobayashi correspondence*. In [36], Mundet i Riera considered this problem for arbitrary Kähler manifold acted by a reductive Lie group and obtained his general Hitchin-Kobayashi correspondence.

Chapter 2

A compactification of moduli space of twisted holomorphic maps with Lagrangian boundary condition

2.1 Introduction to this chapter

This chapter is based on the paper [58]. We specialize to the case that the target manifold (X, ω) a compact symplectic manifold and $G = S^1$.

We consider the moduli space of solutions to (1.14) modulo gauge tranformation (with energy bounded above by a finite number). As in Gromov-Witten theory, the moduli space is in general noncompact. The noncompactness of the moduli space is caused by the bubbling-off phenomenon. Then, as in [37], we can use "stable" objects to compactify the moduli space.

However, one shall consider the case when the conformal structure of the curve (with marked points) varies in the Deligne-Mumford space $\overline{\mathcal{M}}_{g,n}$. This is necessary, in particular, if we want to study the fusion rule of the invariants and define quantum cup product on $H^*_{S^1}(X)$ (in which we have to work with $\overline{\mathcal{M}}_{0,4}$). In this situation, there is a new phenomenon absent in Gromov-Witten theory. Namely, when a node on the curve is forming, the two sides of the node don't always connect in the image of the section, but might be disjoint and connected by a gradient line of the function $h = -i\mu$, when the connection has a "critical holonomy" around the node. A similar phenomenon already appears in the work of Chen and Tian on harmonic maps (see [5]). To define the objects in

the compactification (i.e. stable maps) and to prove the compactness theorem, one needs to study delicately the behavior of twisted holomorphic maps from conformally long cylinders. The details were given in [39].

In this chapter we consider twisted holomorphic maps from bordered Riemann surfaces, and prove the compactness theorem of the corresponding moduli space. One imposes suitable boundary conditions so that the system (1.14) is elliptic. The boundary condition for u is that it should map the boundary of the Riemann surface to a compact Lagrangian submanifold L. Since there is an S^1-ambiguity of the value of u, L should be S^1-invariant. The boundary condition for the connection A is essentially a gauge fixing condition (e.g. Coulomb gauge), so doesn't appear explicitly.

Compared with the case of closed Riemann surfaces, there are two more types of degenerations we have to consider. Namely, the shrinking of a boundary circle and shrinking of an arc. In the first case, we have to study behaviors of twisted holomorphic maps on annuli $\Delta(1, r) = \{r \leq |z| \leq 1\}$ as $r \to 0$; this is called a type-2 boundary node in our context, and this is essentially the same as in the case of closed Riemann surfacc. In the second case, which we call the type-1 boundary nodes, we have to study behaviors of twisted holomorphic maps on long strips. To treat this case, we used a local doubling argument in a Weinstein neighborhood of the Lagrangian submanifold, and apply the results in [39] for cylinders.

This chapter is organized as follows. In Section 2.2, we include some preliminaries for the specific situations we are dealing with, in contrast to the general case in the introduction. In particular, we recall several theorems proved in [39] which are important in the proof of their compactness theorem. In Section 2.3, we consider twisted holomorphic maps from bordered Riemann surfaces with Lagrangian boundary condition, and prove that the boundary singularities essentially can be removed. In Section 2.4 and 2.5, we prove several results which treat two types of boundary degenerations respectively. In Section 2.6, we give the definition of stable twisted holomorphic maps in the bordered case. In Section 2.7, we show that in our moduli space, there are only finitely many different topological types of the underlying curve, if the energy is bounded by some constant. In Section 2.8 we define the notion of convergence in the moduli space and in Section 2.9 we prove the compactness theorem.

2.2 Preliminaries for this chapter

2.2.1 Notations and conventions

We identify the group S^1 with the set of complex numbers with modulus 1, and sometimes identify S^1 with $\mathbb{R}/2\pi\mathbb{Z}$ via the map $\mathbb{R}/2\pi\mathbb{Z} \ni \theta \mapsto e^{i\theta} \in S^1$.

We identify $\mathrm{Lie}S^1 \simeq i\mathbb{R} \simeq \left(\mathrm{Lie}S^1\right)^*$. The pairing between $i\mathbb{R}$ and $\mathrm{Lie}S^1 \simeq i\mathbb{R}$ is given by the $\langle ia, ib\rangle = ab (a, b \in \mathbb{R})$.

Throughout this chapter, (X, ω) will be a compact symplectic manifold. We will fix an effective Hamiltonian S^1 action on X with a moment map $\mu : X \to i\mathbb{R}$. For any $\lambda \in i\mathbb{R}$, we define

$$X^\lambda = \left\{ x \in X \,\middle|\, e^{2\pi\lambda} \cdot x = x \right\}.$$

We denote by $\mathcal{X} \in \Gamma(TX)$ the infinitesimal action of i. Define the real valued function $h = -i\mu = \langle i, \mu\rangle$.

We also fix an S^1-invariant almost complex structure I on X, compatible with ω in the sense that $g := \omega(\cdot, I\cdot)$ is a Riemannian metric on X. Unless otherwise mentioned, any statement involving a metric on X will implicitly refer to the metric g. The gradient of h with respect to g is then equal to $I\mathcal{X}$.

Let d be the distance function on X defined by the metric g. Define the pseudodistance $\mathrm{dist}_{S^1}(x, y) := \inf_{\theta\in S^1} d(x, \theta \cdot y)$. If $A, B \subset X$ are subsets, define $\mathrm{diam}_{S^1} A := \sup_{x,y\in A} \mathrm{dist}_{S^1} d(x, y)$ and $\mathrm{dist}_{S^1}(A, B) := \inf_{x\in A, y\in B} \mathrm{dist}_{S^1}(x, y)$. We denote by d_{Hauss} the Hausdorff distance between subsets:

$$d_{\mathrm{Hauss}}(A, B) := \sup_{x\in A} d(x, B) + \sup_{y\in B} d(y, A).$$

We assume that $L \subset X$ is an embedded compact Lagrangian submanifold, which is invariant under the S^1 action. So $\mathcal{X}|_L \subset \Gamma(TL)$ and $\mu|_L$ is locally constant.

A *curve* will be a 1-dimensional complex manifold, which may have at worst nodal singularities, not necessarily compact, may or may not have boundary.

There is a positive number $\epsilon_X = \epsilon_{X,I,\omega,L}$ such that for any nontrivial I-holomorphic sphere or nontrivial I-holomorphic disk with boundary in L, its energy is at least ϵ_X.

We will use the letter P to denote an S^1-principal bundle over some base B. The letter A will denote a connection, or a connection 1-form on P. A section of the associated bundle $P \times_{S^1} X$ will be viewed either as a map from B to $P \times_{S^1} X$, or as an equivariant map from P to X. In the first

case, we will use the letter φ and in the second case we use Φ. The correspondence $\Phi \leftrightarrow \varphi$ will be used implicitly. If we trivialize P locally over $U \subset B$, then the covariant derivative of A can be written as $d + \alpha$, where α is an $i\mathbb{R}$-valued 1-form on U, and the section can be viewed as a map $\phi : U \to X$. The correspondence $A \leftrightarrow \alpha$ and $\varphi \leftrightarrow \phi$ will also be used everywhere in this chapter.

Since S^1 is abelian, the group of gauge transformations on P, $\mathcal{G}(P)$, is identified with the space of smooth maps from C to S^1. Suppose $h : C \to S^1$ is a C^1-map. Then its action on the pair (A, φ) is given by

$$h^*(A, \varphi) = \left(A + d \log h, h^{-1}(\varphi)\right).$$

The letter Σ will denote a smooth Riemann surface, with or without boundary. The letter C will denote a more general curve, possibly with nodal singularities. $\mathbf{z}$ (resp. $\mathbf{x}$) will denote an **interior** (resp. **boundary**) marked point set, and its elements will be denoted by $z^1, \ldots$(resp. $x^1, \ldots$). $\mathbf{w}$ will denote the set of nodal points of C.

2.2.2 Meromorphic connections

Let $\mathbb{D}^*$ be the punctured unit disk and $P \to \mathbb{D}$ be an S^1-principal bundle. Then P is trivial but the set of homotopy classes of trivializations $T(P)$ has a natural structure on $\mathbb{Z}$-torsor[1]. Let A be a connection on P. We say that A is a **meromorphic connection** if its curvature F_A is bounded with respect to the standard metric on $\mathbb{D}$. Let $\epsilon > 0$ and let $\gamma_\epsilon \in S^1$ be the holonomy of A around the circle of radius ϵ centered at the origin.

Lemma 2.2.1. *[39, Lemma 5.1]*

1. *The limit* $\mathrm{Hol}(A, 0) := \lim_{\epsilon \to 0} \gamma_\epsilon$ *exists.*

2. *Any* $\tau \in T(P)$ *has a representative with respect to which the covariant derivative associated to* A *can be written in polar coordinates* (r, θ) *as*

$$d_A = d + \alpha + \lambda d\theta,$$

where α *is a smooth form with values in* $i\mathbb{R}$ *on* $\mathbb{D}^*$ *which extends continuously to* $\mathbb{D}$. $\lambda \in i\mathbb{R}$ *satisfies* $\mathrm{Hol}(A, 0) = e^{2\pi\lambda}$. *Moreover, the number* λ *depends on* A *and* τ *but not on the representative of* τ.

[1] Recall that if Γ is a group, a Γ-torsor is a set T with a free left action of Γ.

Let $(\Sigma, \mathbf{z})$ be a marked smooth closed Riemann surface. Let $P \to \Sigma \setminus \mathbf{z}$ be an S^1-principal bundle and A is a connection on P. We say that A is a **meromorphic connection** if its curvature F_A is bounded with respect to any smooth metric on Σ.

2.2.3 Critical residues

Let $F \subset X$ be the fixed point set of the S^1-action. Each connected component $F' \subset F$ is an embedded submanifold of X, and the S^1 action on X induces an action on the normal bundle $N_{F'} \to F'$, which splits in weights as $N = \oplus_{\chi \in \mathbb{Z}} N_\chi$. Define

$$\text{weight}(F') := \{\chi \in \mathbb{Z} : N_\chi \neq 0\}\,,\ \text{weight}(X) := \bigcup_{F' \subset F} \text{weight}(F') \subset \mathbb{Z}.$$

The set of critical residues is equal to

$$\Lambda_{cr} = \{\lambda \in i\mathbb{R} \mid \exists\ w \in \text{weight}(X)\ s.t.\ w\lambda \in i\mathbb{Z}\}\,.$$

2.2.4 Twisted holomorphic maps from cylinders

The key point in proving the compactness theorem in [39] is to study the behavior of the twisted holomorphic maps when a circle in the underlying curves shrink to a node, or equivalently, to understand the behavior of twisted holomorphic maps from long cylinders. Because the statements of the theorems in [39] are long and technical, we don't repeat them but refer the readers to [39, Section 4] for details. In the remaining sections we will also state analogues of some of these theorems, whose proofs are either omitted, or can be compared with their counterparts in [39].

2.3 Twisted holomorphic pairs from bordered curves with Lagrangian boundary condition

Definition 2.3.1. A bordered Riemann surface is a connected compact 2-manifold with nonempty boundary equipped with a complex structure.

A bordered Riemann surface Σ is canonically oriented by the complex structure. Its boundary $\partial\Sigma$ is the disjoint union of its (finitely many) connected componets B^i, with induced orientation. A bordered Riemann surface is topologically a sphere with $g \geq 0$ handles and with $h > 0$ disks removed. Such a bordered Riemann surface is said to be of type (g, h).

Definition 2.3.2. Let h be a positive integer, g, n be nonnegative integers, and $\overrightarrow{m} = (m^1, \ldots, m^h)$ be an h-tuple of nonnegative integers. A **marked bordered Riemann surface of type** (g, h) **with** $(n, \overrightarrow{m})$ **marked points** is an $(h+3)$-tuple

$$\left(\Sigma, \mathbf{B}; \mathbf{z}; \mathbf{x}^1, \ldots, \mathbf{x}^h\right)$$

where

- Σ is a bordered Riemann surface with type (g, h);
- $\mathbf{B} = (B^1, \ldots, B^h)$, where the B^i's are connected components of $\partial\Sigma$ oriented by the complex structure;
- $\mathbf{z} = (z_1, \ldots, z_n)$ is an n-tuple of distinct points in Σ^0, where are called **interior marked points**;
- $\mathbf{x^i} = (x^i_1, \ldots, x^i_{m^i})$ is an m^i-tuple of distinct points on the circle B_i. All the x^i_j's are call **boundary marked points**.

Remark 2.3.3. The moduli space which we will compactify is the one of isomorphism classes of twisted holomorphic maps from bordered Riemann surfaces of given type (g, h) with $(n, \overrightarrow{m})$ marked points. But in proving our compactness theorem, we won't distinguish boundary marked points on different boundary components. So we simplify the notations as follows: Σ will be a smooth Riemann surface with boundary $\partial\Sigma$. $\mathbf{z} = \{z_1, \ldots, z_k\} \subset \Sigma \setminus \partial\Sigma$ is the set of **interior** marked points and $\mathbf{x} = \{x_1, \ldots, x_l\} \subset \partial\Sigma$ is the set of **boundary** marked points.

Let's fix a conformal metric on Σ with volume form Ω. Recall that $L \subset X$ is an S^1-invariant Lagrangian submanifold.

Definition 2.3.4. Let $P \to \Sigma \setminus \mathbf{x} \cup \mathbf{z}$ be an S^1-principal bundle and $Y = P \times_{S^1} X$. Let φ be a section of Y. We say that φ maps the boundary $\partial\Sigma$ into L if $\varphi\left(\partial\Sigma \setminus \mathbf{x}\right) \subset P \times_{S^1} L$. Let A be a connection on P. We say that the tuple (P, A, φ) is holomorphic if $\overline{\partial}_A \varphi = 0$ wherever A and φ is defined. If both conditions holds, we call such a tuple (P, A, φ) a holomorphic tuple with boundary mapped into L.

We say that two tuples (P, A, φ) and (P', A', φ') over $(\Sigma, \mathbf{x} \cup \mathbf{z})$ are isomorphic, if there is a bundle isomorphism $g : P \to P'$ such that $g^* A' = A$ and $g^* \varphi' = \varphi$.

Example 2.3.5. Let $X = \mathbb{C}^N$ with the standard Kähler structure. Let S^1 act on X by complex multiplication. Then

$$\mu(z_1, \ldots, z_N) = -\frac{i}{2}\left(\sum_{i=1}^{N} |z_i|^2 - 1\right)$$

is a moment map. The symplectic quotient is isomorphic to the projective space $\mathbb{P}^{N-1}$ with homogeneous coordinates $[z_1, \ldots, z_N]$. A class of Lagrangian tori are given by

$$\overline{L} := \left\{[z_1, \ldots, z_N] \in \mathbb{P}^{N-1} \mid |z_1| : |z_2| : \cdots : |z_N| = c_1 : c_2 : \cdots : c_N\right\}$$

for $c_i > 0$, $c_1^2 + \cdots + c_N^2 = 1$. Then each L lifts to an invariant Lagrangian

$$L := \left\{(z_1, \ldots, z_N) \in \mathbb{C}^N \mid |z_i| = c_i,\ i = 1, \ldots, N\right\}.$$

Any holomorphic disk $u : \mathbb{D} \to \mathbb{C}^N$ with $u(\partial\mathbb{D}) \subset L$ is given by the Blaschke product (see for example [6], [56])

$$u_0(z) = \left(c_i \prod_{j=1}^{\mu_i} \frac{z - \alpha_j^{(i)}}{1 - \overline{\alpha_j^{(i)}} z}\right)_{i=1,\ldots,N}$$

for arbitrary $\mu_i \in \mathbb{Z}_{\geq 0}$ and $\alpha_j^{(i)} \in \mathrm{Int}\mathbb{D}$.

Consider the standard metric on $\mathbb{D}$. By the main theorem of [50], for any such holomorphic disk, there exists a unique smooth function

$$h : \mathbb{D} \to \mathbb{R},\ h|_{\partial\mathbb{D}} \equiv 0$$

such that

$$\Delta h + \frac{1}{2}\left(e^{-2h}|u_0|^2 - 1\right) = 0.$$

This means the gauge transformed pair

$$(A, u) := \left(d - \sqrt{-1} * dh, e^{-h} u_0\right)$$

satisfies the vortex equation on $\mathbb{D}$ for the standard area form. The boundary condition is still satisfied by u.

A **meromorphic** connection A on P is a smooth connection with F_A bounded with respect to the chosen metric. Because of the existence of nontrivial limit holonomy at interior marked points, one

can't extend the connection over those points. However, up to gauge transformation, a holomorphic tuple (P, A, φ) can be extended smoothly over boundary marked points, if they satisfy the vortex equation or flat connection equation. More precisely, we will prove the following theorem in this section.

Theorem 2.3.6. *Let $(\Sigma, \mathbf{x} \cup \mathbf{z})$ be a marked bordered Riemann surface. Let Ω be a smooth volume form on Σ and $c \in i\mathbb{R}$. Let (P, A, φ) be a holomorphic tuple with boundary mapped into L. If in addition we have either $\Lambda F_A + \mu(\varphi) = c$ or $F_A = 0$ on $\Sigma \setminus \mathbf{x} \cup \mathbf{z}$, where $c \in i\mathbb{R}$, then there exists a holomorphic tuple (P', A', φ') over $\Sigma \setminus \mathbf{z}$ with boundary mapped in L and a smooth isomorphism $g : P'|_{\Sigma \setminus \mathbf{x} \cup \mathbf{z}} \to P$ such that $g^*(A, \varphi) = (A', \varphi')|_{\Sigma \setminus \mathbf{x} \cup \mathbf{z}}$. Moreover, any two such extensions (P', A', φ') and (P'', A'', φ'') are isomorphic over $\Sigma \setminus \mathbf{z}$.*

The rest of the section is devoted to proving this theorem. In Subsection 2.3.1, we show that we can extend the bundle and the connection; in Subsection 2.3.2, we apply the removal of singularity theorem of [24] to extend the section; in Subsection 2.3.3 we prove a regularity theorem to show that the extension is smooth and in Subsection 2.3.4 the uniqueness of the extension.

After the theorem is proved, we will assume every twisted holomorphic map on a bordered Riemann surface is defined and smooth over all boundary marked points.

2.3.1 Extension of the bundle and the connection

We first consider the model case. Let $\mathbb{D} \subset \mathbb{C}$ be the closed unit disk and $\mathbb{D}^*$ be the punctured unit disk. Let $\mathbb{D}_+ = \mathbb{D} \cap \overline{\mathbb{H}}$ be the closed half disk and $\mathbb{D}^*_+ := \mathbb{D}^* \cap \overline{\mathbb{H}}$ be the punctured half disk. All of these spaces are endowed with the standard conformal structure. Then $\mathbb{D}^*_+$ is conformal to $[0, \infty) \times [0, \pi]$.

Let $P \to \mathbb{D}^*_+$ be the trivial S^1-bundle and A be a meromorphic connection on P. Then P extends uniquely to the trivial bundle over B, which is, in this subsection, denoted by $\widetilde{P}$.

Under the cylindrical coordinate $(t, \theta) \in \mathbb{D}^*_+$, write $d_A = d + \alpha = d + \alpha_t dt + \alpha_\theta d\theta$. Then there exists a (smooth) gauge transformation $g_1 : \mathbb{D}^*_+ \to S^1$ such that $d_{g_1^* A} = d + \alpha'_t dt$. Then define a continous 1-form $\widetilde{\alpha}$ on $\mathbb{D}^* \simeq [0, \infty) \times S^1$:

$$\widetilde{\alpha} = \widetilde{\alpha}_t(t, \theta) dt = \begin{cases} \alpha'_t(t, \theta) dt, & \text{if } \theta \in [0, \pi], \\ \alpha'_t(t, 2\pi - \theta) dt, & \text{if } \theta \in [\pi, 2\pi]. \end{cases}$$

Then $\widetilde{\alpha}$ gives a continuous connection on the trivial bundle $\widetilde{P} = \mathbb{D}^* \times S^1$. Note that the residue

of this connection at the origin is zero, hence by the proof of Lemma 2.2.1 there is a continuous gauge transformation $g_2 : \mathbb{D}^* \to S^1$, which is homotopically trivial, and whose restriction to each half disk is smooth, such that $\widetilde{\alpha} - d\log g_2$ extends to a continuous 1-form on $\mathbb{D}$. Set $g = g_2 \cdot g_1$, then the covariant derivative of g^*A is written as $d_{g^*A} = d + \alpha_0$ with α_0 a continuous 1-form on $\mathbb{D}_+$ which is smooth on $\mathbb{D}_+^*$.

In general, consider $P \to \Sigma \setminus \mathbf{x} \cup \mathbf{z}$ with a meromorphic connection A. Then P extends uniquely to a bundle $\widetilde{P} \to \Sigma \setminus \mathbf{z}$. We can do as above to find gauge transformations near each boundary marked point and glue them together to get a global gauge transformation $g : \Sigma \setminus \mathbf{x} \cup \mathbf{z} \to S^1$, then extend the connection g^*A to a continuous one on $\widetilde{P}$.

2.3.2 Extension of the section

Now let (P, A, φ) be a holomorphic tuple with P and A already extended over the boundary marked points as above, and φ maps $\partial\Sigma \setminus \mathbf{x}$ into L.

Near each boundary marked point x, we can identify (P, A, φ) with a twisted holomorphic pair (α, ϕ) on a punctured half disk $\mathbb{D}_+^*$. So it suffices to prove only for the model case. The 1-form α induces a *continuous* almost complex structure I_α on $\mathbb{D}_+ \times X$. Let $\widetilde{\phi} : \mathbb{D}_+^* \to \mathbb{D}_+ \times X$ be given by $\widetilde{\phi}(x) = (x, \phi(x))$. Then $\widetilde{\phi}$ is I_α-holomorphic and $\widetilde{\phi}(\mathbb{D}_+^* \cap \mathbb{R}) \subset (\mathbb{D} \cap \mathbb{R}) \times L$. We want to apply the theorem of Ivashkovich-Shevchishin [24] on the removal of boundary singularity for continuous almost complex structures with Lagrangian boundary condition.

To apply the theorem, we need the following definition:

Definition 2.3.7. (cf. [24, Definition 2.4]) Let (Y, h, J) be a Riemannian manifold equipped with a continuous almost complex structure. $W \subset Y$ is a totally real submanifold with respect to J. $A \subset W$ is a subset. We say that W is **h-uniformly totally real along A with a lower angle** $\alpha > 0$ if for any $w \in A$ and $\xi \in T_wW$, $\xi \neq 0$, the angle between $J\xi$ and T_wW is no less than α.

Theorem 2.3.8. *Let (Y, h, J) be a Riemannian manifold equipped with a continuous almost complex structure J. $W \subset Y$ is a totally real submanifold with respect to J. $A \subset W$ is a subset such that W is h-uniformly totally real along A with a lower angle $\alpha > 0$. Let $u : \mathbb{D}_+^* \to Y$ be a J-holomorphic map with $u(\mathbb{D}_+^* \cap \mathbb{R}) \subset A$. Suppose that J is h-uniformly continuous on $u(\mathbb{D}_+^*)$ and the closure of $u(\mathbb{D}_+^*)$ is h-complete; the energy of u is finite. Then u extends to the origin $0 \in \mathbb{D}_+$ as a $W^{1,p}$ map for some $p > 2$.*

Take a Riemannian metric g on X such that $TL \perp I(TL)$ and let h be the product metric on $Y = \mathbb{D} \times X$. In order to apply the above theorem for Y, $W = (\mathbb{D}_+ \cap \mathbb{R}) \times L$ for the almost complex

structure I_α, we only need to check that W is h-uniformly totally real along $\widetilde{\phi}(\mathbb{D}_+^* \cap \mathbb{R})$ with respect to I_α. Indeed, for $(x, y) \in (\mathbb{D}_+^* \cap \mathbb{R}) \times L$ and $(v, Y) \in T_{(x,y)}(\mathbb{D}_+^* \times X)$,

$$I_\alpha(v, Y) = (jv, IY + i(\alpha(jv)\mathcal{X} - \alpha(v)I\mathcal{X}))\,.$$

Suppose $|\alpha| < M$ on $\mathbb{D}_+$ since it is continuous. Let (s, t) be the Euclidean coordinate on $\mathbb{D}$ with s the real coordinate. $\alpha_s = \alpha(\partial_s)$, $\alpha_t = \alpha(\partial_t)$. Take two vectors $\xi_i = (a_i\partial_s, Y_i)$, $Y_i \in T_wL$, $i = 1, 2$. If $a_1 = 0$, then $I\xi_1 \perp \xi_2$. If $a_1 \neq 0$, then

$$\begin{aligned}
|\cos \angle_h(I_\alpha\xi_1, \xi_2)| = \frac{|\langle I_\alpha\xi_1, \xi_2\rangle|}{\|I_\alpha\xi_1\| \cdot \|\xi_2\|} &= \frac{|\langle (a_1\partial_t, IY_1 + ia_1(\alpha_t\mathcal{X} - \alpha_s I\mathcal{X})), (a_2\partial_s, Y_2)\rangle|}{(\|a_1\partial_t\|^2 + \|IY_1 + a_1 i(\alpha_t\mathcal{X} - \alpha_s I\mathcal{X})\|^2)^{1/2} \cdot \|\xi_2\|} \\
&= \frac{|a_1\alpha_t| \cdot |\langle \mathcal{X}, Y_2\rangle|}{(|a_1|^2 + \|a_1\alpha_t\mathcal{X}\|^2 + \|Y_1 - ia_1\alpha_s\mathcal{X}\|^2)^{1/2} \cdot \|\xi_2\|} \leq \frac{|a_1\alpha_t| \cdot \|\mathcal{X}\| \cdot \|Y_2\|}{(|a_1|^2 + \|a_1\alpha_t\mathcal{X}\|^2)^{1/2} \cdot \|\xi_2\|} \\
&\leq \frac{|\alpha_t|\|\mathcal{X}\|}{(1 + |\alpha_t|^2\|\mathcal{X}\|^2)^{1/2}} \leq \left(\frac{M^2\|\mathcal{X}\|_{L^\infty}^2}{1 + M^2\|\mathcal{X}\|_{L^\infty}^2}\right)^{1/2} < 1.
\end{aligned}$$

Thus we have proved W is h-uniformly totally real along $\widetilde{\phi}(\mathbb{D}_+^* \cap \mathbb{R})$ with a positive lower angle. Then we can apply Theorem 2.3.8 to this case and ϕ extends to a map of class $W^{1,p}$. Hence the tuple (P, A, Φ) is extended to a $W^{1,p}$ section on $\Sigma \setminus \mathbf{z}$.

2.3.3 Regularity for twisted holomorphic pairs

Now let Σ be a bordered Riemann surface with a smooth volume form Ω, $\mathbf{z} \subset \Sigma$ is a subset of interior marked points. Let P be an S^1-principal bundle over $\Sigma \setminus \mathbf{z}$ and $Y = P \times_{S^1} X$.

Theorem 2.3.9. *Suppose $p > 2$ and A is a connection on P of class $W^{1,p}$ and $\varphi \in W^{1,p}$ is a A-holomorphic section of Y and φ maps $\partial\Sigma$ into L. If (A, φ) satisfies either the vortex equation $\Lambda F_A + \mu(\varphi) = c$ or $F_A = 0$, both in the L^p sense, then there exists a gauge transformation g of class $W^{2,p}$ such that $g^*(A, \varphi)$ is smooth.*

Proof. This is a simple generalization of [7, Theorem 3.1] to the case of a bordered Riemann surface. The method is essentially the same, i.e. using the local slice theorem (see [52]) and elliptic bootstrapping. □

Thus, let (P, A, φ) be a holomorphic tuple over $(\Sigma, \mathbf{x} \cup \mathbf{z})$ which satisfies the boundary condition and either the vortex equation or $F_A = 0$. *A priori* A is a continuous connection on P and φ is of class $W^{1,p}$. Then the vortex equation or $F_A = 0$ implies that F_A is of class $W^{1,p}$ on $\Sigma \setminus \mathbf{z}$. Then by Poincaré lemma, we can find a $W^{2,p}$ gauge transformation h such that h^*A is actually of class $W^{1,p}$.

Then we can apply the above theorem to get regularity, i.e. there exists a $W^{2,p}$ gauge tranformation such that $g^*(A,\varphi)$ is smooth. But both (A,φ) and $g^*(A,\varphi)$ are smooth on $\Sigma \setminus \mathbf{x} \cup \mathbf{z}$, which implies that g is smooth on $\Sigma \setminus \mathbf{x} \cup \mathbf{z}$. Hence, every tuple on $\Sigma \setminus \mathbf{z} \cup \mathbf{x}$ is gauge equivalent to a smooth tuple on $\Sigma \setminus \mathbf{z}$ by a smooth gauge transformation.

2.3.4 Uniqueness

Suppose we have two tuples (P', A', φ'), (P'', A'', φ'') over $\Sigma \setminus \mathbf{z}$ which are both the extension of the original (P, A, φ). Then there exists a smooth isomorphism $g : P'|_{\Sigma\setminus\mathbf{x}\cup\mathbf{z}} \to P''|_{\Sigma\setminus\mathbf{x}\cup\mathbf{z}}$ such that on $\Sigma \setminus \mathbf{x} \cup \mathbf{z}$, $g^*(A'', \varphi'') = (A', \varphi')$. So for each $x_j \in \mathbf{x}$, take a local trivialization of P', P'' near x_j, with respect to which $d_{A'} = d + \alpha'$, $d_{A''} = d + \alpha''$, g is an S^1-valued function. One has,

$$\alpha' - d\log g = \alpha''.$$

But $d(\alpha' - \alpha'') = 0$, in the weak sense (hence in the usual sense), since $F_{A'} = F_{A''}$. So there exists a smooth function $h : \Sigma \setminus \mathbf{z} \to i\mathbb{R}$ such that $\alpha' - \alpha'' = dh$. So $\log g = h + C$ and g extends smoothly over x_j, which is still denoted by g. So $g^*(P', A') = (P'', A'')$. And $g^*\varphi' = \varphi''$ by the uniqueness of the removal of singularity Theorem 2.3.8.

2.4 Twisted holomorphic pairs over semi-infinite cylinders with small energy density

One type of boundary degenerations of bordered curves is the shrinking of a boundary circle to a point. Up to conformal transformation, it is equivalent to the process that the lengths of boundary cylinders tends to infinity. In the case of twisted holomorphic maps with boundary in L, such degenerations are almost the same as that treated in [39].

For any positive real number N, set $C_N^+ = [0, N+2) \times S^1$, $Z_N^+ = [0, N] \times S^1$. Let P be a principal bundle over C_N^+, and let A be a connection on P. We say that a given trivialization puts A in **temporal gauge** if with respect to this trivialization, the covariant derivative d_A takes the form $d + ad\theta$, where $a : C_N^+ \to i\mathbb{R}$ is a function. We say that the trivialization is in **balanced temporal gauge** if the restriction of a to the boundary $\{0\} \times S^1$ is equal to some constant $\lambda \in i\mathbb{R}$.

We give C_N^+ and Z_N^+ the standard product metrics and conformal structures. We denote by (t, θ) the coordinate of points in C_N^+. Let $\Omega = f dt \wedge d\theta$ be a volume form on C_N^+, and let η be a positive

real number. We say that Ω is **exponentially η-bounded** if $|f| + |\nabla f| < \eta e^{t-N}$.

We have the following analogue for [39, Theorem 4.1, 4.3, 4.4].

Theorem 2.4.1. *For any noncritical $\lambda \in i\mathbb{R} \setminus \Lambda_{cr}$, there exist positive real numbers $K_3, \sigma_3, \epsilon_3$ depending continuously on λ with the following properties. Let $N > 0$ be any real number and let $(\alpha, \phi) : C_N^+ \to X$ be a twisted holomorphic pair satisfying $\|\alpha - \lambda d\theta\|_{L^\infty(C_N^+)} < \epsilon_3$, $\|d\alpha\|_{L^\infty(C_N^+)} < \epsilon_3$, $\|d_\alpha \phi\|_{L^\infty(C_N)} < \epsilon_3$. Then for any $(t, \theta) \in Z_N^+$, we have*

$$|d_\alpha \phi(t, \theta)| \leq K_3 e^{\sigma_3 (t-N)} \|d_\alpha \phi\|_{L^\infty(Z_N^+)}.$$

In particular this implies that

$$\operatorname{diam}_{S^1} \left(\phi(Z_N^+) \right) < K_3 \|d_\alpha \phi\|_{L^\infty(Z_N^+)}.$$

Theorem 2.4.2. *Fix some number $c \in i\mathbb{R}$. There exists positive real numbers $\epsilon_4, \eta_0, \sigma_4, K_4$ with the following property. Let $N > 0$ be a real number and let $(\alpha, \phi) : C_N^+ \to X$ be a twisted holomorphic pair. Suppose that α is in balanced temporal gauge and that there exists some critical value $\lambda \in \Lambda_{cr}$ such that*

$$\|\alpha - \lambda d\theta\|_{L^\infty} < \epsilon_4, \ \|d_\alpha \phi\|_{L^\infty} < \epsilon_4.$$

Then there exist maps $\psi : [0, N] \to X^\lambda$ and $\phi_0 : Z_N^+ \to TX$ satisfying, for any $(t, \theta) \in Z_N^+$, $\phi_0(t, \theta) \in T_{e^{-\lambda\theta}\psi(t)} X$,

$$\phi(t, \theta) = e^{-\lambda\theta} \exp_{\psi(t)} \left(e^{\lambda\theta} \phi_0(t, \theta) \right), \text{ and } \int e^{\lambda\theta} \phi_0(t, \theta) = 0.$$

where exp *is the exponential map of the metric on X. Furthermore,*

1. *If Ω is an exponentially η-bounded volume form on C_N^+ for some $\eta < \eta_0$, and the vortex equation $\Lambda d\alpha + \mu(\phi) = c$ is satisfied, then for any $(t, \theta) \in Z_N^+$*

$$|\phi_0(t, \theta)| < K_4 e^{\sigma_4 (t-N)}.$$

2. *If $d\alpha = 0$, then for any $(t, \theta) \in Z_N^+$,*

$$|\phi_0(t, \theta)| < K_4 e^{\sigma_4 (t-N)}.$$

Theorem 2.4.3. *The same K_4, σ_4 satisfy the following property. Let $N > 0$ be a real number and let $(\alpha, \phi) : C_N^+ \to X$ be a twisted holomorphic pair satisfying the hypothesis of the above theorem for the ϵ_4. Assume furthermore that α is in balanced temporal gauge, so that $\alpha = ad\theta$ and $a|_{0\times S^1}$ is equal to some $\lambda_0 \in i\mathbb{R}$ with $|\lambda_0 - \lambda| < \epsilon_4$.*

1. *Suppose Ω is an exponentially η-bounded volume form on C_N^+ and that the vortex equation $\Lambda d\alpha + \mu(\phi) = c$ is satisfied. Then for any $t \in [0, N]$,*

$$|\psi'(t) + i(\lambda_0 - \lambda)\nabla h(\psi(t))| < K_4 e^{\sigma_4(t-N)}.$$

2. *Suppose that $d\alpha = 0$. Then for any $t \in [0, N]$,*

$$|\psi'(t) + i(\lambda_0 - \lambda)\nabla h(\psi(t))| < K_4 e^{\sigma_4(t-N)}.$$

These theorems can be proved the same way as proving [39, Theorem 4.1, Theorem 4.3, Theorem 4.4]. We omit the proof.

2.5 Reflection of twisted pairs from strips with Lagrangian boundary condition

Another typical degeneration of bordered curves is the shrinking of an arc to a point, which is conformally equivalent to the forming of long strips. In this section we treat this type of degenerations in twisted holomorphic maps with boundary in L. We will see that in the limit, the two sides of the boundary node actually connect.

Indeed, if there is a global anti-holomorphic involution on (X, I) with fixed point set equal to L, then we can use the reflection principle to reduce the problem to the cylinderical case. Moreover, the holonomy along the central circle of the cylinder will be always trivial because the two half circles cancel with each other by the reflection. So by [39, Theorem 4.3, 4.4], the gradient flow lines will be degenerate and the two limit orbits of the long cylinders will coincide in the limit. In the general case, even though there is no global involution, the image of the strip lies in a small neighborhood of the Lagrangian L and there is still a local involution which allows us to apply the reflection principle.

2.5.1 A tubular neighborhood of the Lagrangian submanifold and local charts

Let N_L be the normal bundle of L, regarded as the subbundle of normal vectors of TX. Let $N_{L,\delta} \subset N_L$ be the subset consisting of all normal vectors with length less than δ. It has a natural reflection map $\sigma : N_{L,\delta} \to N_{L,\delta}$ given by $\sigma(x,v) = (x,-v)$. By basic differential geometry we know that for δ small enough, the exponential map $\exp : N_{L,\delta} \to X$ given by $(x,v) \mapsto \exp_x v$ is a diffeomorphism onto a tubular neighborhood of L in X, and $\mathrm{dist}_g(\exp_x v, L) = |v|$. In this section we fix a small δ_0, and identify N_{L,δ_0} with a tubular neighborhood of L by the exponential map. Since the S^1 action preserves the Riemannian metric, $N_L \to L$ is an equivariant bundle, and the infinitesimal action $\mathcal{X}$ is invariant under the reflection σ.

The Riemannian metric gives the Levi-Civita connection ∇^{LC}. Together with the orthogonal projection $TX|_L \to N_L$, ∇^{LC} induces a connection on the normal bundle $\pi : N_L \to L$. Then this connection induces the "horizontal-vertical" splitting $TN_L \simeq \pi^*TL \oplus \pi^*N_L$. The almost complex structure I gives a bundle isomorphism $I : TL \to N_L$. By the splitting this isomorphism gives an almost complex structure on N_{L,δ_0}, which is denoted by $\widetilde{I}_0$. Then $\widetilde{I}_0$ may be different from the original I. But they coincide on L (or the 0-section). Moreover, there exists a $K_0 > 0$, depending only on X, I and g, such that for any $(x,v) \in N_{L,\delta_0}$,

$$\left\| I(x,v) - \widetilde{I}_0(x,v) \right\|_g \leq K_0|v| = K_0 \mathrm{dist}_g(\exp_x v, L). \tag{2.1}$$

For any $y \in L$ and a small δ with $0 < \delta < \delta_0$, denoted by $B_L(y,\delta)$ the δ-ball in L centered at y. Take a normal coordinate on $B_L(y,\delta)$ to be $(x^1,\ldots,x^n)$ and ∂_i be the vector field $\partial/\partial x^i$. Define

$$\mathbb{C}^n \simeq \mathbb{R}^n \oplus i\mathbb{R}^n \ni \left((x^1,\ldots,x^n),(v^1,\ldots,v^n)\right) \mapsto \left((x^1,\ldots,x^n), \sum_i v^i (I\partial_i)\right) \in N_L.$$

This map is a diffeomorphism on a small neighborhood of the origin. Then for δ small enouth, we define $f_{y,\delta}$ to be the inverse of the above map restricted on $\pi^{-1}(B_L(y,\delta)) \cap N_{L,\delta}$. Let $\mathbb{C}^n$ be endowed with the standard complex structure I_0 and the standard metric g_0. One has that

$$f_{y,\delta}^{-1}(\mathbb{R}^n) = B_L(y,\delta), \ \left(f_{y,\delta}^{-1}\right)^* \widetilde{I}_0\Big|_{\mathbb{R}^n} = I_0|_{\mathbb{R}^n}, \ f_{y,\delta} \circ \sigma = \sigma_0 \circ f_{y,\delta},$$

where σ_0 is the conjugation of $\mathbb{C}^n$. Since $f_{y,\delta}$ is isometric at y, one has that for any ϵ with $0 < \epsilon < 1/2$,

there exists $\delta = \delta(\epsilon, y) > 0$ such that

$$\left\| g_0 - \left(f_{y,\delta}^{-1}\right)^* g \right\|_{L^\infty} < \epsilon, \text{ and } 1/2 \leq \|df\| \leq 2,$$

with respect to the metrics g on $N_{L,\delta}$. By taking $\delta(\epsilon, y)$ small one can also assume that the image of f is contained in the unit ball of $\mathbb{C}^n$.

Remark 2.5.1. Since $\left(f_{y,\delta}^{-1}\right)^* \left(\widetilde{I}_0\right)\Big|_{\mathbb{R}^n} = I_0|_{\mathbb{R}^n}$, by smoothness, for any $y \in L$ and $\delta < \delta_0$, there exists $K_{y,\delta} > 0$ such that on the image of $f_{y,\delta}$

$$\left\|(f_{y,\delta}^{-1})^* \widetilde{I}_0 - I_0\right\| \leq K_{y,\delta}|\mathbf{v}|, \ \left\|D\left((f_{y,\delta}^{-1})^* g\right)\right\| \leq K_{y,\delta}$$

with respect to the standard metric g_0 on $\mathbb{C}^n$. Here $\mathbf{v}$ denotes the vector $(v^1, \ldots, v^n)$.

2.5.2 Reflection

Let $S_N^+ = (-N-2, N+2) \times [0, \pi]$ and $S_N^- = (-N-2, N+2) \times [\pi, 2\pi]$ be the strips. Let $C_N = S_N^+ \cup S_N^-$ be the cylinder glued from the two parts. Denote by τ the natural reflection of C_N, given by $\tau(t, \theta) = (t, 2\pi - \theta)$. We say that a map $f : C_N \to N_{L,\delta}$ (resp. a differential form α on C_N) is **reflection-symmetric** if $f \circ \tau = \sigma \circ f$ (resp. $\tau^* \alpha = \alpha$).

In this section, we will do the following operations on each twisted holomorphic pair (α, ϕ) defined on S_N^+ with $\|d_\alpha \phi\|_{L^\infty}$ small enough. First, take a gauge transformation g_1 on S_N^+ such that $g_1^* \alpha$ only has dt component. Second, "reflect" the pair to get a twisted pair $(\tilde{\alpha}, \tilde{\phi})$ defined on the cylinder C_N. Third, use another gauge transformation g_2 on C_N such that $g_2^* \alpha$ is in balanced temporal gauge and such that $g_2^* \phi$ is reflection-symmetric. The resulting pair will NOT be holomorphic with respect to the original almost complex structure, but holomorphic with respect to a family of almost complex structures on N_{L,δ_0} which we will construct.

(1) The first gauge transformation. Let $(\alpha, \phi) : S_N^+ \to X$ be a twisted holomorphic pair with $\phi(\partial S_N^+) \subset L$ and $\|d_\alpha \phi\|_{L^\infty}$ small enough. Then $\phi(S_N^+) \subset N_{L,\delta_0}$.

Suppose $\alpha = \alpha_t dt + \alpha_\theta d\theta$. Then there exists a gauge transformation $g_1 : S_N^+ \to S^1$ such that $g_1^* \alpha = a dt$. Set $(\alpha', \phi') = g_1^*(\alpha, \phi)$.

(2) Reflect the pair (α', ϕ')**.** We define a pair $(\tilde{\alpha}, \tilde{\phi})$ on C_N as follows:

$$\widetilde{\alpha} = \begin{cases} a(t, \theta) dt, & \text{if } \theta \in [0, \pi]; \\ a(t, 2\pi - \theta) dt, & \text{if } \theta \in [\pi, 2\pi], \end{cases} \quad \widetilde{\phi}(t, \theta) = \begin{cases} \phi(t, \theta), & \text{if } \theta \in [0, \pi]; \\ \sigma(\phi(t, 2\pi - \theta)), & \text{if } \theta \in [\pi, 2\pi]. \end{cases}$$

Write $\widetilde{\alpha} = \widetilde{\alpha}_t dt$, then $\widetilde{\alpha}_t$ is continuous.

(3) Construct a family of almost complex structures on N_{L,δ_0}. Choose δ_1 to be smaller than the injective radius of g. Take a cut-off function $\rho : [0,\infty) \to \mathbb{R}$ such that $\rho \geq 0$, $\rho(t) = 1$ for $t \in [0, \delta_1/2]$ and $\rho(t) = 0$ for $t \geq \delta_1$. Now we define a *continuous* family of almost complex structures on N_{L,δ_0}, parametrized by points in C_N. For each $z \in S_N^+$, set $y = \pi(\phi'(z)) \in L$. $I(\phi'(z))$ is an almost complex structure on the vector space $T_{\phi'(z)}N_L \simeq T_yL \oplus N_{L,y}$, where the splitting is given by the connection on N_L. Trivialize TL, N_L on $B_L(y,\delta_1)$ by parallel transport along radial directions with respect to the Levi-Civita connection, then $I(\phi'(z))$ defines a complex structure of the bundle $(TL \oplus N_L)|_{B_L(y,\delta_1)}$. By the splitting $TN_L \simeq \pi^*TL \oplus \pi^*N_L$, this defines an almost complex structure on $N_{L,\delta_0} \cap \pi^{-1}(B(y,\delta_1))$, which is denoted by I_z^*. Recall that we have the other almost complex structure $\widetilde{I}_0$ on N_{L,δ_0}. Take $\rho_z : L \to \mathbb{R}$ to be $\rho_z(y') = \rho(d(y,y'))$. Define

$$I_z = \left(1 - \rho_z I_z^* \widetilde{I}_0\right) \widetilde{I}_0 \left(1 - \rho_z I_z^* \widetilde{I}_0\right)^{-1}$$

(it easy to check that $\left(1 - \rho_z I_z^* \widetilde{I}_0\right)$ is invertible). I_z coincides with I_z^* in $\pi^{-1}(B(y,\delta_1/2)) \cap N_{L,\delta_0}$ and coincides with $\widetilde{I}_0$ on $N_{L,\delta_0} - \pi^{-1}(B(y,\delta_1)) \cap N_{L,\delta_0}$. Moreover, if $z \in \partial S_N^+$ then $\phi'(z) \in L$ and it is easy to see that $I_z = \widetilde{I}_0$. Then for $z \in S_N^-$, set $I_z = -d\sigma \circ I_{\tau(z)} \circ d\sigma$. And we see that I_z is a continuous family of almost complex structures on N_{L,δ_0}.

It is easy to check that the pair $\left(\widetilde{\alpha}, \widetilde{\phi}\right)$ is I_z-holomorphic, i.e. for any $z \in C_N$, we have

$$\partial_t \widetilde{\phi}(z) - i\widetilde{\alpha}_t(z)\mathcal{X}\left(\widetilde{\phi}(z)\right) = I_z \partial_\theta \widetilde{\phi}(z).$$

Moreover, since all the structures we used here are S^1-invariant, we see that if we apply a gauge tranformation to the pair $\left(\widetilde{\alpha}, \widetilde{\phi}\right)$, then I_z is transformed by the same gauge tranformation in a natural way.

Also, by (2.1), one easily obtain that for any $z \in C_N$,

$$\left\| I_z - \widetilde{I}_0 \right\|_{L^\infty(g)} \leq K_0 \mathrm{dist}_g\left(\widetilde{\phi}(z), L\right). \tag{2.2}$$

(4) Construct the second gauge transformation. Set $b(t,\theta) = \int_0^t \widetilde{\alpha}_t(s,\theta)ds$, $g_2(t,\theta) = \exp(b(t,\theta))$. Since $\tilde{\alpha}$ is reflection-symmetric, so is g_2, i.e. $g_2(t,\theta) = g_2(t, 2\pi - \theta)$. So

$$g_2^*\tilde{\alpha} = -\left(\int_0^t \frac{\partial \widetilde{\alpha}_t}{\partial \theta} dt\right) d\theta$$

is in balanced temporal gauge, $g_2^*\widetilde{\alpha}|_{\{0\}\times S^1} = 0$ and $g_2^*\widetilde{\phi}$ is reflection-symmetric. Now set $\widetilde{I}_z = g_2(z)_* \circ I_z \circ (g_2(z)^{-1})_*$. Then $\widetilde{I}_z$ and $g^*\widetilde{\phi}$ satisfy (2.2) and the pair $g_2^*\left(\widetilde{\alpha}, \widetilde{\phi}\right)$ is $\widetilde{I}_z$-holomorphic.

With notations revised, we summarize that, from the original I-holomorphic pair $(\alpha, \phi) : S_N^+ \to X$, we get a pair $\left(\widetilde{\alpha}, \widetilde{\phi}\right) : C_N \to N_{L,\delta_0}$ and a continuous family of almost complex structures I_z on N_{L,δ_0} with the following properties:

- $\widetilde{\alpha}$ is in balanced temporal gauge, $\widetilde{\alpha}|_{\{0\}\times S^1} = 0$;
- $\widetilde{\phi}$ is reflection-symmetric;
- $\left(\widetilde{\alpha}, \widetilde{\phi}\right)$ is I_z-holomorphic;
- I_z satisfies (2.2).

We want to point out that the pair $\left(\widetilde{\alpha}, \widetilde{\phi}\right)$ we constructed is not necessarily smooth, but smooth on both S_N^+ and S_N^-.

Moreover, if the original pair satisfies the vortex equation, then the "upper half" of the new pair $\left(\widetilde{\alpha}, \widetilde{\phi}\right)$ also satisfies the vortex equation since we only did gauge transformations on the upper half. Define a function $\widetilde{\mu} : C_N \to i\mathbb{R}$ as follows: $\widetilde{\mu}(z) = \mu(\phi(z))$ if $z \in S_N^+$ and $\widetilde{\mu}(z) = 2c - \mu(\phi(\tau(z)))$ if $z \in S_N^-$. We can also construct a continuous volume form $\widetilde{\Omega}$ on C_N by reflection. The new pair satisfies the equation

$$\Lambda d\widetilde{\alpha} + \widetilde{\mu} = c.$$

Similarly, if the original pair is flat on S_N^+(i.e. $d\alpha = 0$), then $d\widetilde{\alpha} = 0$ on C_N. In either case, we have the estimate

$$\|d\widetilde{\alpha}\|_{L^\infty} \leq C.$$

The family I_z depends on the original pair (α, ϕ). We will prove theorems which state that for a given family I_z, certain conditions hold for all I_z-holomorphic pairs in balanced temporal gauge, including in particular, the pair $\left(\tilde{\alpha}, \tilde{\phi}\right)$.

2.5.3 An analogue of [39, Theorem 4.3]

Let K_0 be the one in (2.1).

Theorem 2.5.2. *Fix some $c \in i\mathbb{R}$. There exist positive real numbers ϵ, η_0, σ, K with the following property. Let $N > 0$ be a real number and let I_z be a continuous family of almost complex structures on N_{L,δ_0}, parametrized by $z \in C_N$. Let $(\alpha, \phi) : C_N \to N_{L,\delta_0}$ be a pair such that*

- *ϕ is reflection-symmetric;*
- *α is in balanced temporal gauge and $\alpha|_{\{0\}\times S^1} = 0$;*
- *$\|\alpha\|_{L^\infty} < \epsilon,\ \|d_\alpha \phi\|_{L^\infty} < \epsilon$.*

Then there exists maps $\psi : [-N, N] \to N_{L,\delta_0}$ and $\phi_0 : Z_N \to TX$ satisfying that for any $(t,\theta) \in Z_N$, $\phi_0(t,\theta) \in T_{\psi(t)}X$,

$$\phi(t,\theta) = \exp_{\psi(t)}(\phi_0(t,\theta)), \int_0^{2\pi} \phi_0(t,\theta) d\theta = 0.$$

Moreover, if (α,ϕ) is I_z-holomorphic and $\left\| I_z - \widetilde{I_0} \right\| \leq K_0 \mathrm{dist}(\phi(z), L)$, and that

$$|d\alpha| \leq \eta_0 e^{|t|-N}, \tag{2.3}$$

then for any $(t,\theta) \in Z_N$,

$$|\phi_0(t,\theta)| \leq K e^{\sigma(|t|-N)}.$$

Remark 2.5.3. In Theorem 4.3 of [39], they distinguish two cases, $d\alpha = 0$ or (α,ϕ) satisfies the vortex equation. But in order to prove the estimate, we only need (2.3).

Proof. The existence of ψ and ϕ_0 follows by applying the implicit function theorem. In the following we prove the rest of the theorem.

First we prove a lemma in a local chart. Denote $C = (-2,3) \times S^1$ and $Z = [-1,2] \times S^1$, $Z_I = [-1,0] \times S^1$, $Z_{II} = [0,1] \times S^1$, $Z_{III} = [1,2] \times S^1$. Let X be a smooth vector field on $\mathbb{C}^n$ such that $\sigma_* X = X$, where σ is the conjugation of $\mathbb{C}^n$.

Lemma 2.5.4. *For any $r > 0$ and $K > 0$ there exists an $\epsilon > 0$ with the following property. Let I_z be a continuous family of linear complex structures on $\mathbb{C}^n$ parametrized by points on C. Let g be a Riemannian metric on $\mathbb{C}^n$ and $\mathcal{X}$ be a smooth vector field on $B(0,r)$, satisfying*

- *$\|I_z - I_0\|_{L^\infty} < \epsilon$, $\|g_0 - g\|_{L^\infty} < \epsilon$, $\|Dg\|_{L^\infty} \leq K$, $\|\mathcal{X}\|_{L^\infty} \leq K$.*

Let $(\alpha,\phi) : C \to \mathbb{C}^n$ be a pair, which satisfies

- *α in balanced temporal gauge with $\alpha|_{\{0\}\times S^1} = 0$;*
- *$\partial_t \phi(z) = I_z(\partial_\theta \phi(z) - i\alpha_\theta(z)\mathcal{X}(\phi(z)))$;*
- *ϕ is reflection-symmetric, i.e., $\phi \circ \tau = \sigma \circ \phi$;*
- *$\|I_z - I_0\|_{L^\infty} \leq K\,|\mathrm{Im}\phi(z)|$;*

- $\|\alpha\|_{L^\infty} < \epsilon$, $\|d_\alpha \phi\|_{L^\infty} < \epsilon$, $\phi(C) \subset B(0, 2r)$.

Let ψ, ϕ_0 be defined as in Theorem 2.5.2 and suppose that $\|d\alpha\|_{L^2} < \epsilon \|\phi_0\|_{L^2}$. Then in terms of g, we have

$$\|\phi_0\|^2_{L^2(Z_{II})} \leq \frac{1}{3}\left(\|\phi_0\|^2_{L^2(Z_I)} + \|\phi_0\|^2_{L^2(Z_{III})}\right).$$

We will use the following two lemmas proved in [39].

Lemma 2.5.5 (Lemma 11.4,[39]). *Let $\phi : C \to \mathbb{C}^n$ be a holomorphic map and denote $\phi_{av}(t, \theta) := \phi(t, \theta) - \frac{1}{2\pi}\int_0^{2\pi} \phi(t, \nu) d\nu$. Then*

$$\|\phi_{av}\|^2_{L^2(Z_{II})} \leq \frac{1}{5}\left(\|\phi_{av}\|^2_{L^2(Z_I)} + \|\phi_{av}\|^2_{L^2(Z_{III})}\right)$$

where the norms are computed with respect to the standard metric on $\mathbb{C}^n$.

Lemma 2.5.6 (Lemma 11.5, [39]). *For any $K > 0$, there exist constants $\epsilon > 0$ with the following property: let g_0 be the standard metric on $\mathbb{C}^n$ and g be another Riemannian metric on $\mathbb{C}^n$ satisfying $K^{-1} g_0 \leq g \leq K g_0$ and $\|Dg\|_{L^\infty(\mathbb{C}^n, g_0)} \leq K$. Let $x \in \mathbb{C}^n$ and let $\gamma_0 : S^1 \to T_x\mathbb{C}^n$ be a smooth map satisfying $\int \gamma_0 = 0$ and $\sup|\gamma_0| < \epsilon$. Define $\gamma(\theta) = \exp_x^g \gamma_0(\theta)$ and*

$$\gamma_{av}(\theta) = \gamma(\theta) - \frac{1}{2\pi}\int_0^{2\pi} \gamma(\nu) d\nu.$$

Then $0.9\|\gamma_0\|_{L^2} \leq \|\gamma_{av}\|_{L^2} \leq 1.1\|\gamma_0\|_{L^2}$ and $0.9 \sup|\gamma_0| \leq \sup|\gamma_{av}| \leq 1.1 \sup|\gamma_0|$.

Remark 2.5.7. It is easy to see that for any $M > 0$, there exists $K_M > 0$ such that, if we replace the condition $\sup|\gamma_0| < \epsilon$ by $\sup|\gamma_0| \leq M$, then we have $\sup|\gamma_{av}| \leq K_M \sup|\gamma_0|$.

Proof of Lemma 2.5.4. Suppose the lemma is false, then there exists a sequence $\epsilon_u \to 0$, and a sequence of $I_{u,z}$, g_u, α_u, ϕ_u, $\mathcal{X}_u$ which satisfy the following conditions:

- $\|I_{u,z} - I_0\|_{L^\infty} < \epsilon_u$, $\|g_u - g_0\|_{L^\infty} < \epsilon_u$, $\|Dg_u\|_{L^\infty} \leq K$, $\|\mathcal{X}_u\|_{L^\infty} \leq K$;
- α_u is in balanced temporal gauge with $\alpha_u|_{\{0\}\times S^1} = 0$;
- $\partial_t \phi_u(z) = I_{u,z}\left(\partial_\theta \phi_u(z) - i\alpha_{u,\theta}(z)\mathcal{X}_u(\phi_u(z))\right)$;
- ϕ_u is reflection-symmetric;
- $\|I_{u,z} - I_0\|_{L^\infty} \leq K\,|\mathrm{Im}\phi_u(z)|$;
- $\|\alpha_u\|_{L^\infty} < \epsilon_u$, $\|d_{\alpha_u}\phi_u\|_{L^\infty} < \epsilon_u$, $\phi_u(C) \subset B(0, 2r)$;

- $\|d\alpha_u\|_{L^2} < \epsilon_u \|\phi_{u,0}\|_{L^2}$;
- $\|\phi_{u,0}\|^2_{L^2(Z_{II})} > \frac{1}{3}\left(\|\phi_{u,0}\|^2_{L^2(Z_I)} + \|\phi_{u,0}\|^2_{L^2(Z_{III})}\right)$.

Then $\|d\phi_u\|_{L^\infty} \leq \|d_{\alpha_u}\phi_u\|_{L^\infty} + \|\alpha_u\|_{L^\infty}\|\mathcal{X}_u\|_{L^\infty} \to 0$. This implies that $\|\phi_{u,0}\|_{L^\infty} \to 0$. Hence by Lemma 2.5.6, for big enough u, one has

$$\|\phi_{u,av}\|^2_{L^2(Z_{II})} > \frac{1}{4}\left(\|\phi_{u,av}\|^2_{L^2(Z_I)} + \|\phi_{u,av}\|^2_{L^2(Z_{III})}\right); \tag{2.4}$$

and

$$\frac{\|d\alpha_u\|_{L^2}}{\|\phi_{u,av}\|_{L^2}} \to 0. \tag{2.5}$$

Denote $x_u = \int_{S^1}\phi_u(0,\theta)d\theta$. By the assumption that ϕ is reflection-symmetric, one has that $x_u \in \mathbb{R}^n$. Set $\xi_u = \phi_u - x_u$. Then $\|d\xi_u\|_{L^2} > 0$, for otherwise one would get $\xi_{u,av} = \phi_{u,av} = 0$, which contradicts with (2.4).

Claim. There exists a constant $M > 0$ such that for all u, $\|\xi_u\|_{L^2} \leq M\|d\xi_u\|_{L^2}$, $\|\phi_{u,av}\|_{L^2} \leq M\|d\phi_{u,av}\|_{L^2}$ and $\|\alpha_u\|_{L^2} \leq M\|d\alpha_u\|_{L^2}$, where the norms are taken with respect to the metric g_0.

Proof of the claim. Using the Fourier expansion $\xi_u = \sum_k a_{u,k}(t)e^{ik\theta}$. The condition $\int_{S^1}\xi_u(0,\theta)d\theta = 0$ implies that $a_{u,0}(0) = 0$. While $d\xi_u = dt\sum_k a'_{u,k}(t)e^{ik\theta} + d\theta\sum_k a_{u,k}(t)ike^{ik\theta}$. Hence

$$\int_{-2}^{3}|a_{u,0}(t)|^2\,dt = \int_{-2}^{3}\left|\int_0^t a'_{u,0}(s)ds\right|^2 dt \leq \int_{-2}^{3}|t|\left(\int_0^t\left|a'_{u,0}(s)\right|^2 ds\right)dt \leq 100\int_{-2}^{3}\left|a'_{u,0}(t)\right|^2 dt.$$

So

$$\begin{aligned}\|\xi_u\|^2_{L^2} &= \sum_{k\neq 0}\int_{-2}^{3}|a_{u,k}(t)|^2\,dt + \int_{-2}^{3}|a_{u,0}(t)|^2\,dt \\ &\leq \sum_{k\neq 0}\int_{-2}^{3}|a_{u,k}(t)ik|^2\,dt + 100\int_{-2}^{3}\left|a'_{u,0}(t)\right|^2 dt \leq 100\|d\xi_u\|^2_{L^2}\end{aligned}$$

Since $\int_{S^1}\phi_{u,av}(0,\theta) = 0$ and $\int_{S^1}\alpha_u(0,\theta) = 0$, using the same method, we obtain the other two estimates. □

Claim. We have $\left\|\overline{\partial}_{I_0}\xi_u\right\|_{L^2} / \|d\xi_u\|_{L^2} \to 0$ as u goes to infinity.

Proof of the claim. We have $\partial_t \phi_u(z) = I_{u,z}\left(\partial_\theta \phi_u - i\alpha_u \mathcal{X}_u(\phi_u(z))\right)$. Hence we have

$$\overline{\partial}_{I_0}\xi_u = \partial_t \xi_u - I_0 \partial_\theta \xi_u = (I_{u,z} - I_0)\partial_\theta \xi_u - i\alpha_u I_{u,z}\mathcal{X}_u(\phi_u(z)),$$

$$\begin{aligned}
\left\|\overline{\partial}_{I_0}\xi_u\right\|_{L^2} &\leq \left\|I_{u,z} - I_0\right\|_{L^2} \left\|\partial_\theta \xi_u\right\|_{L^\infty} + \left\|i\alpha_u\right\|_{L^2} \left\|I_{u,z}\mathcal{X}_u\right\|_{L^\infty} \\
&\leq K \left\|\mathrm{Im}\phi_u(z)\right\|_{L^2} \left\|\partial_\theta \phi_u\right\|_{L^\infty} + 2KM \left\|d\alpha_u\right\|_{L^2} \leq K \left\|\mathrm{Im}\xi_u\right\|_{L^2} \left\|d\phi_u\right\|_{L^\infty} + 2KM\epsilon_u \left\|\phi_{u,0}\right\|_{L^2} \\
&\leq K \left\|\xi_u\right\|_{L^2} \left\|d\phi_u\right\|_{L^\infty} + 3KM\epsilon_u \left\|\phi_{u,av}\right\|_{L^2} \leq KM\epsilon_u \left\|d\xi_u\right\|_{L^2} + 2KM^2\epsilon \left\|d\phi_{u,av}\right\|_{L^2} \\
&\leq \epsilon_u \left(KM \left\|d\xi_u\right\|_{L^2} + 2KM^2 \left\|d\xi_{u,av}\right\|_{L^2}\right).
\end{aligned}$$

It is easy to see that $\left\|d\xi_u\right\|_{L^2}$ can contral $\left\|d\xi_{u,av}\right\|_{L^2}$. Hence the claim is proved. □

Now define $\xi'_u = \xi_u / \|d\xi_u\|_{L^2}$. Since $\int_{S^1} \xi'_u(0,\theta)d\theta = 0$ and $\|d\xi'_u\|_{L^2} = 1$, the L^2-norm of ξ'_u is uniformly bounded. Hence we have a uniform $W^{1,2}$-bound of ξ'_u. Then since the inclusion $W^{1,2} \subset L^2$ is compact, there exists a subsequence, which is still denoted by ξ'_u, converging in L^2 to some $\xi \in L^2(C, \mathbb{C}^n)$. By the above claim, ξ is a weak solution of $\overline{\partial}\xi = 0$ and hence holomorphic in the interior of C. By Garding's inequality $\|\xi'_u - \xi\|_{W^{1,2}} \leq \text{const.}(\|\overline{\partial}(\xi'_u - \xi)\|_{L^2} + \|\xi'_u - \xi\|_{L^2})$, ξ'_u converges to ξ in $W^{1,2}$ and hence $\|d\xi\|_{L^2} = 1$ and hence not constant.

Since $\xi'_u \to \xi$ in L^2, it follows that $\xi'_{u,av} \to \xi_{av}$ in L^2. Since $\xi'_{u,av}$ is a rescaling of $\xi_{u,av}$, $\xi_{u,av} = \phi_{u,av}$ and $g_u \to g_0$, inequality (2.4) implies that

$$\|\xi_{av}\|^2_{L^2(Z_{II})} \geq \frac{1}{4}\left(\|\xi_{av}\|^2_{L^2(Z_I)} + \|\xi_{av}\|^2_{L^2(Z_{III})}\right).$$

But this will contradicts Lemma 2.5.5 since ξ is non-constant. □

Then we have the corollary in N_{L,δ_0} instead of $\mathbb{C}^n$.

Corollary 2.5.8. *There exists an $\epsilon > 0$ with the following property. Let I_z be a continuous family of almost complex structures on N_{L,δ_0} parametrized by $z \in C$. Let $(\alpha, \phi) : C \to N_{L,\delta_0}$ be an I_z-holomorphic pair, which satisfies:*

- *α is in balanced temporal gauge with $\alpha|_{\{0\}\times S^1} = 0$;*
- *ϕ is reflection-symmetric;*
- *$\left\|I_z - \widetilde{I}_0\right\|_{L^\infty} \leq K_0 \mathrm{dist}_g(\phi(z), L)$;*
- *$\|\alpha\|_{L^\infty} < \epsilon$, $\|d_\alpha \phi\|_{L^\infty} < \epsilon$.*

Let ψ, ϕ_0 be defined as in Theorem 2.5.2 and suppose that $\|d\alpha\|_{L^2} < \epsilon\|\phi_0\|_{L^2}$. Then we have

$$\|\phi_0\|^2_{L^2(Z_{II})} \leq \frac{1}{3}\left(\|\phi_0\|^2_{L^2(Z_I)} + \|\phi_0\|^2_{L^2(Z_{III})}\right).$$

Proof. First, take δ small enough such that for each $y \in L$, the image of $f_{y,\delta}$ lies in the unit ball. Recall the definition of $K_{y,\delta}$ in (2.5.1). Set $K'_y = 3(K_0 + K_{y,\delta} + \|\mathcal{X}\|_{L^\infty})$ and apply Lemma 2.5.4 for $r = 1$ and $K = K'_y$, one gets an $\epsilon_{y,\delta}$. Then for each $y \in L$, take $\delta(y,\epsilon) < \delta$ such that (2.5.1) holds for the $\epsilon = \epsilon_{y,\delta}$.

Then $\{B_L(y,\delta(y))\}_{y\in L}$ gives an open cover of L and it has a finite subcover $\{B_L(y_i,\delta_i)\}$. Now, there exists $\epsilon' > 0$ such that for any pair (α,ϕ) with $\|\alpha\|_{L^\infty} < \epsilon'$, $\|d_\alpha\phi\|_{L^\infty} < \epsilon'$, the image of ϕ must lie in one of the $B_L(y_i,\delta_i)$'s. Take $\epsilon = \min(\epsilon', \epsilon_{y_i,\delta_i})$. Now we check that this ϵ satisfies the required condition. Suppose (α,ϕ) is a pair and I_z is a family of almost complex structures which satisfy the hypothesis. Then the image ϕ is contained in some $B_L(y_i,\delta_i)$ and hence it descends to a problem in $\mathbb{C}^n$. But by our definition of K'_{y_i}, ϵ and $\delta(y)$, the pair (α,ϕ) and I_z satisfies the hypothesis of Lemma 2.5.4 and hence one has the desired inequality. □

Then, using the same method as in [39], we prove Theorem 2.5.2. □

2.5.4 An analogue of [39, Theorem 4.4]

Now let's consider the following case. Let I_z be a continuous family of almost complex structures on N_{L,δ_0}, parametrized by $z \in C_N$, and let $(\alpha,\phi) : C_N \to N_{L,\delta_0}$ be an I_z-holomorphic pair. Let ϵ be that in the previous theorem and $\|\alpha\|_{L^\infty} < \epsilon$, $\|d_\alpha\phi\|_{L^\infty} < \epsilon$. This implies the existence of ψ and ϕ_0. Furthermore, by implicit function theorem, for any $(t,\theta) \in C_N$,

$$|\phi_0(t,\theta)|, |\nabla\phi_0(t,\theta)| \leq \text{const.} \sup_{\nu\in S^1} |\nabla\phi(t,\nu)|.$$

Theorem 2.5.9. *There exist positive constants K, σ with the following property. Let $N > 0$ be a real number and $(\alpha,\phi) : C_N \to N_{L,\delta_0}$ be an I_z-holomorphic pair satisfying the hypothesis of the previous theorem. If we have*

$$|d\alpha| \leq \eta_0 e^{|t|-N},$$

then for any $t \in [-N,N]$

$$|\psi'(t)| \leq Ke^{\sigma(|t|-N)}.$$

We first prove a lemma.

Lemma 2.5.10. *Let $K, M > 0$ be real numbers, let I_z be a continuous family of complex structures on $\mathbb{C}^n$, g be a Riemannian metric on $\mathbb{C}^n$ and $\mathcal{X}$ be a symmetric vector field on $\mathbb{C}^n$. Let $V \subset \mathbb{C}^n$ be a compact subset. Let $\phi : Z \to \mathbb{C}^n$ be a reflection-symmetric map satisfying: $\phi(Z) \subset V$; for each $z \in Z$, $\|I_z - I_0\| \leq K|\mathrm{Im}\phi(z)|$ and*

$$\partial_t \phi(z) = I_z \left(\partial_\theta \phi(z) - i\alpha \mathcal{X}(\phi(z))\right)$$

for some function $\alpha : Z \to i\mathbb{R}$. Suppose that $\|\nabla\phi\|_{L^\infty}$ is small enough so that $\psi : [-1, 2] \to \mathbb{C}^n$ and $\phi_0 : Z \to \mathbb{C}^n$ can be defined as in Theorem 2.5.2, and that

$$|\phi_0|, |\partial_\theta \phi_0|, |\partial_t \phi_0| \leq M. \tag{2.6}$$

Then for any $t \in (0, 1)$ one has

$$|\psi'(t)| \leq K' \left(\sup_{Z_t} |\phi_0| + \sup_{Z_t} |\alpha| \right) \tag{2.7}$$

for some constant K' independent of ϕ.

Proof. The proof is almost the same as that of [39, Lemma 12.1]. We have $\psi([0, 1]) \subset \tilde{V} = \{\exp_x^g v | x \in V, |v| \leq M\}$. Let $B \subset \mathbb{C}^n$ be the closed ball of radius M, let $E \times \tilde{V} \to \mathbb{C}^n$ be the exponential map $E(x, v) = \exp_x^g v$. Let $F = D_x E$ and $G = D_v E$. Since the domain is compact, the second derivatives of E are uniformly bounded. Hence there exists K_1 such that for any $(x, v) \in \tilde{V} \times B$,

$$|F(x, v) - \mathrm{Id}| \leq K_1 |v|, \ |G(x, v) - \mathrm{Id}| \leq K_1 |v|. \tag{2.8}$$

Taking partial derivatives with respect to t in the equation $\exp_\psi \phi_0 = \phi$, one obtains

$$F(\psi, \phi_0)\psi' + G(\psi, \phi_0)\partial_t \phi_0 = \partial_t \phi.$$

Writing $F = \mathrm{Id} + F - \mathrm{Id}$ and $G = \mathrm{Id} + G - \mathrm{Id}$, and integrating over S^1 of the above equation, making use of the fact that $\int \phi_0(t, \theta) d\theta = 0$ and (2.6), (2.8), it yields

$$\left| \psi'(t) - \frac{1}{2\pi} \int \partial_t \phi(t, \theta) d\theta \right| \leq K_2 \sup_{Z_t} |\phi_0|. \tag{2.9}$$

On the other hand,

$$\partial_t\phi(z) = I_z\left(\partial_\theta\phi(z) - i\alpha\mathcal{X}(\phi(z))\right) = (I_z - I_0)\left(\partial_\theta\phi(z)\right) + I_0\partial_\theta\phi(z) - i\alpha I_z\mathcal{X}(\phi(z)). \tag{2.10}$$

We also have

$$\partial_\theta\phi = G(\psi,\phi_0)\partial_\theta\phi_0 \Rightarrow |\partial_\theta\phi| \leq K_3 \sup|\partial_\theta\phi_0| \leq K_3 M.$$

Integrating (2.10) over S^1 and using the above estimate, we see that there exist K_4 and K_5 such that

$$\left|\int_{S^1}\partial_t\phi(t,\theta)d\theta\right| \leq K_4\left(\sup_{Z_t}|\mathrm{Im}(\phi)| + \sup_{Z_t}|\alpha|\right)$$
$$\leq K_4\left(\sup_{Z_t}|\phi_{av}| + \sup_{Z_t}|\alpha|\right) \leq K_5\left(\sup_{Z_t}|\phi_0| + \sup_{Z_t}|\alpha|\right).$$

Here $|\mathrm{Im}\phi| \leq |\phi_{av}|$ uses the fact that ϕ is reflection-symmetric; the last inequality uses the estimate in the Remark 2.5.7. Hence combining (2.9) and above, one obtains (2.7). □

Proof of theorem 2.5.9. It follows word by word from the proof of [39, Theorem 4.4]. □

Corollary 2.5.11. *Let $c \in i\mathbb{R}$. Then there exist real numbers $\epsilon_5, \sigma_5, \eta_0, K_5 > 0$ such that for any positive $N > 0$ and any I-holomorphic pair $(\alpha,\phi) : C_N^+ \to X$, which satisfies*

$$|d\alpha| \leq \eta_0 e^{|t|-N},\ \|d_\alpha\phi\|_{L^\infty} \leq \epsilon_5,$$

one has

$$\mathrm{diam}_{S^1}\phi\left(S^+_{N/2}\right) \leq K_5 e^{-\sigma_5 N}.$$

2.6 Definition of (c, L)-stable twisted holomorphic maps

2.6.1 Prestable bordered Riemann surfaces

Definition 2.6.1. [32, Definition 3.4] Let (x,y) be the coordinate on $\mathbb{C}^2$, and $A(x,y) = (\overline{x},\overline{y})$ be the complex conjugation. A **node** on a bordered Riemann surface is a singularity isomorphic to one of the following:

1. $(0,0) \in \{xy = 0\}$ (interior node);

2. $(0,0) \in \left\{x^2 - y^2 = 0\right\}/A$ (boundary node of type 1);

3. $(0,0) \in \left\{x^2+y^2=0\right\}/A$ (boundary node of type 2);

(So near a boundary node of type 1 (resp. type 2), the boundary of the bordered Riemann surface is the fixed point set of A, which is a neighborhood of the origin in $\mathbb{R}\times\{0\}\cup\{0\}\times\mathbb{R}$ (resp. a single point).

A **nodal bordered Riemann surface** is a singular bordered Riemann surface whose singularities are nodes. A **prestable bordered Riemann surface** is either a smooth bordered Riemann surface or a nodal bordered Riemann surface.

The **complex double** of a prestable bordered Riemann surface C is a prestable Riemann surface without boundary, which is intuitively, gluing C and $\overline{C}$ (same curve with opposite complex structure) along the corresponding boundary components. So for a boundary node of C, which is locally isomorphic to $\{x^2 \pm y^2 = 0\}/A$, there is a unique (interior) node of its complex double which is locally isomorphic to $\{x^2 \pm y^2 = 0\}$.

Let g be a nonnegative integer and h a positive integer. A *smooth* bordered Riemann surface C is said to be **of type** (g,h) if C is topologically a sphere attached with g handles and h disks removed. A nodal bordered Riemann surface is of type (g,h) if it is a degeneration of a smooth bordered Riemann surface of the same type. The reader may refer to [32] for the precise definition.

Let C be a prestable bordered Riemann surface, let $\mathbf{z}\subset C$ (resp. $\mathbf{x}\subset\partial C$) be a finite subset of interior (resp. boundary) points which is disjoint from nodes. Then the tuple $(C,\mathbf{x}\cup\mathbf{z})$ is called a **marked nodal bordered Riemann surface**. There is a unique curve $\hat{C}$ which is the disjoint union of finitely many components which are either smooth marked bordered Riemann surface or marked smooth Riemann surface without boundary, together with a holomorphic map $\hat{\pi}:\hat{C}\to C$. This map is called the **normalization** of C. $\hat{\pi}$ is 1-1 away from interior nodes and boundary nodes of type 1; The preimage of every interior node or boundary node of type 1 is two points.

2.6.2 Meromorphic connections on prestable marked bordered Riemann surfaces

Now let $(C,\mathbf{x}\cup\mathbf{z})$ be a prestable marked bordered Riemann surface. Let $\hat{\pi}:\hat{C}\to C$ be the normalization map. Denote by $\mathbf{w}$ the set of nodes of C, with $\mathbf{w}^{int}$, $\mathbf{w}^{b1}$, $\mathbf{w}^{b2}$ the subsets of different types of nodes, respectively. Given an S^1-principal bundle P over $\hat{C}\setminus\hat{\pi}^{-1}\left(\mathbf{z}\cup\mathbf{w}^{int}\cup\mathbf{w}^{b2}\right)$.

Definition 2.6.2. A meromorphic connection on P is a connection A on P such that F_A is bounded with respect to any smooth metric on $\hat{C}$ and that for every $w\in\mathbf{w}^{int}$ with preimages $y,y'\in\hat{C}$, we

have

$$\mathrm{Hol}(A, y) = \mathrm{Hol}(A, y')^{-1}.$$

2.6.3 Gluing data at interior nodes and boundary nodes of type 1

Now let $(C, \mathbf{x} \cup \mathbf{z})$ be a smooth marked bordered Riemann surface, let $P \to C \setminus \mathbf{z}$ be an S^1-principal bundle and let A be a meromorphic connection on P. Let $z \in \mathbf{z}$ and let S_z be the quotient space of $T_z C \setminus \{0\}$ by the action of $\mathbb{R}_+$ given by scalar multiplication. Choose a conformal metric on C and let $S_\epsilon \subset C$ be the circle of radius ϵ centered at z, and denote (P_ϵ, A_ϵ) be the restriction of (P, A) to S_ϵ. Let $P_z \to S_z$ be the S^1-principal bundle whose fibre over the class of $s \in T_z C - \{0\}$ is the set of A-covariantly constant sections of P defined over the ray $\{\exp ts | t \in (0, \delta)\}$ for some small δ. Parallel transport in radial directions gives the canonical isomorphism of bundles $\iota_\epsilon : P_z \to P_\epsilon$ for small enough ϵ and the connections $\iota_\epsilon^* A_\epsilon$ converges to a connection A_z on P_z. The pair (P_z, A_z) is independent of the conformal metric on C. We call (P_z, A_z) the **limit pair** of (P, A) at z.

Now let $(C, \mathbf{x} \cup \mathbf{z})$ be a marked nodal bordered Riemann sufrace and let $\hat{\pi} : \hat{C} \to C$ be the normalization. If $w \in \mathbf{w}^{int}$ with preimages y, y'. Then the circles $S_y, S_{y'}$ have natural structures of S^1-torsor, which are induced by the complex structure on C. Define $G_w(C)$ to be the set of all isomorphisms of S^1-torsors

$$g_C : S_y \to \overline{S_{y'}},$$

where $\overline{S_{y'}}$ is $S_{y'}$ with the inverse S^1-torsor structure.

Suppose P is an S^1-principal bundle over $\hat{C} \setminus \hat{\pi}^{-1}\left(\mathbf{z} \cup \mathbf{w}^{int} \cup \mathbf{w}^{b2}\right)$, and A is a meromorphic connection on P.

Let (P_y, A_y) and $(P_{y'}, A_{y'})$ be the limit pairs of (P, A) at y, y' respectively. Then define the set of **gluing data** at $w \in \mathbf{w}^{int}$ to be the set of pairs (g_C, g_P), where $g_C \in G_w(C)$ and $g_P : P_y \to g_C^* P_{y'}$ is an isomorphism of S^1-bundles such that $g_P^* g_C^* A_{y'} = A_y$.

Now suppose $w' \in \mathbf{w}^{b1}$ is a boundary node of type 1 with preimages y, y'. Note that P is defined over y and y'. Then we define the set of gluing data at w' to be $\mathrm{Iso}(P_y, P_{y'})$, the set of isomorphisms between the two fibres.

2.6.4 Stable marked bordered curves and the moduli space

Let $(C, \mathbf{x} \cup \mathbf{z})$ be a prestable marked bordered Riemann surface. We say that a point in C is exceptional if it is either a node or a marked point. Let $\hat{\pi} : \hat{C} \to C$ be the normalization. The exceptional points in $\hat{C}$ is by definition the preimages of the exceptional points in C. An irreducible

component D of C is called unstable, if $\pi^{-1}(D)$ satisfies one of the following conditions:

- it is a sphere with 2 or less exceptional points;
- it is a torus with no exceptional points;
- it is a disk with 1 interior exceptional point and no exceptional points on the boundary;
- it is a disk with no interior exceptional points and 2 or less exceptional points on the boundary;
- it is a "half-torus" with no exceptional points, i.e. its *complex double* is a 2-torus without exceptional points.

If no component of $\hat{C}$ is unstable, then we say that C is **stable**.

We say that $(g, h, n, \overrightarrow{m})$ is in the *stable domain*, if a smooth bordered Riemann surface of genus g and h boundary components, with $(n, \overrightarrow{m})$-marked points is stable. For the moduli space $\mathcal{M}_{(g,h),(n,\overrightarrow{m})}$ of smooth bordered Riemann surfaces of genus g and h boundary components, with $(n, \overrightarrow{m})$-marked points, and its compactification $\overline{\mathcal{M}}_{(g,h),(n,\overrightarrow{m})}$ of stable curves, the reader may refer to [32]. From now on we fix a $(g, h, n, \overrightarrow{m})$ in the stable domain.

2.6.5 Collapsing and stablization maps

Let $(C, \mathbf{x} \cup \mathbf{z})$ be a prestable marked bordered Riemann surface. A **collapsing map** is a quotient map $\pi^{co} : C \to C'$, where $(C', \mathbf{x}' \cup \mathbf{z}')$ is also a prestable marked bordered Riemann surface, and π^{co} is quotienting some of the unstable components of C. The marked point set $\mathbf{x}' \cup \mathbf{z}'$ is naturally induced from $\mathbf{x} \cup \mathbf{z}$, such that $\pi^{co} : \mathbf{x} \to \mathbf{x}'$, $\pi^{co} : \mathbf{z} \to \mathbf{z}'$ are bijections.

Now for a given collapsing map $\pi^{co} : C \to C'$. Then each node of C' falls in one of the following cases:

(1) If w is an interior node, one can find sequences of points of $\hat{C}$, $(y_0, \ldots, y_{d-1})$ and $(y'_1, \ldots, y'_d)$, which satisfy the following properties:

- the points $\{y_0, \ldots, y_{d-1}\}$ and $\{y'_1, \ldots, y'_d\}$ are all different;
- $\hat{\pi}(y_0)$ and $\hat{\pi}(y'_d)$ belong to components on which π^{co} is a local bijection;
- each pair $\{\hat{\pi}(y_j), \hat{\pi}(y'_j)\}$ is contained in a sphere bubble $O_j \subset C$ for each $1 \leq j \leq d-1$;
- for each j we have $\hat{\pi}(y_j) = \hat{\pi}(y'_{j+1})$;
- we have $\pi^{co}\left(\hat{\pi}(y_j)\right) = \pi^{co}\left(\hat{\pi}(y'_j)\right) = w$ for each j.

We call the sequences $(y_0, \ldots, y_{d-1})$, $(y'_1, \ldots, y'_d)$ an **ordered list of connecting exceptional points over** w.

(2) If w' is a boundeary node of type 1, one can find sequences of points of $\partial\hat{C}$, $(y_0, \ldots, y_{d-1})$ and $(y'_1, \ldots, y'_d)$, which satisfies the following properties:

- the points $\{y_0, \ldots, y_{d-1}\}$ and $\{y'_1, \ldots, y'_d\}$ are all different;
- $\hat{\pi}(y_0)$ and $\hat{\pi}(y'_d)$ belong to components on which π^{co} is a local bijection;
- each pair $\{\hat{\pi}(y_j), \hat{\pi}(y'_j)\}$ is contained in a disk bubble $D_j \subset C$ for each $1 \leq j \leq d-1$;
- for each j we have $\hat{\pi}(y_j) = \hat{\pi}(y'_{j+1})$;
- we have $\pi^{co}(\hat{\pi}(y_j)) = \pi^{co}\left(\hat{\pi}(y'_j)\right) = w'$ for each j.

We call the sequences $(y_0, \ldots, y_{d-1})$, $(y'_1, \ldots, y'_d)$ an **ordered list of connecting exceptional points over** w'.

(3) If w'' is a boundary node of type 2, one of the following two cases must hold.

Case 1. One can find a sequence of points of $\hat{C}$, $(y_0, \ldots, y_{d-1})$ and $(y'_1, \ldots, y'_{d-1})$ which satisfy the following properties:

- the points $\{y_0, \ldots, y_{d-1}\}$ and $\{y'_1, \ldots, y'_{d-1}\}$ are all different;
- $\hat{\pi}(y_0)$ belong to a component of C on which π^{co} is a local bijection, and $\hat{\pi}(y_{d-1})$ is a boundary node of type 2;
- each pair $\{\hat{\pi}(y_j), \hat{\pi}(y_j)\}$ is contained in a sphere bubble $B_j \subset C$ for each $1 \leq j \leq d-1$;
- for each j with $0 \leq j \leq j-2$ we have $\hat{\pi}(y_j) = \hat{\pi}(y'_{j+1})$;
- we have $\pi^{co}(\hat{\pi}(y_j)) = \pi^{co}\left(\hat{\pi}(y'_j)\right) = w''$ for each j.

Case 2. One can find a sequence of points of $\hat{C}$, $(y_0, \ldots, y_{d-1})$ and $(y'_1, \ldots, y'_d)$ which satisfy the following properties:

- the points $\{y_0, \ldots, y_{d-1}\}$ and $\{y'_1, \ldots, y'_d\}$ are all different;
- $\hat{\pi}(y_0)$ belong to a component of C on which π^{co} is a local bijection;
- each pair $\left\{\hat{\pi}(y_j), \hat{\pi}(y'_j)\right\}$ is contained in a sphere bubble O_j for $1 \leq j \leq d-1$ and y'_d is contained in a disk bubble O_d;

- for each j with $0 \leq j \leq j-1$ we have $\hat{\pi}(y_j) = \hat{\pi}\left(y'_{j+1}\right)$;
- we have $\pi^{co}(\hat{\pi}(y_j)) = \pi^{co}\left(\hat{\pi}(y'_j)\right) = w''$ for each j.

In either case, we call the corresponding list of points an **ordered list of connecting exceptional points over** w''. The difference is that in the first case a disk is collapsed and in the second case there is no disk.

In particular, the collapsing map which collapses all the unstable components of C is called the **stablization map**, and denoted by $\pi^{st} : (C, \mathbf{x} \cup \mathbf{z}) \to (C^{st}, \mathbf{x}^{st} \cup \mathbf{z}^{st})$. In this case, the components of C that are contracted by π^{st} are called **bubble components** (since there is no unstable torus component), and others are called **principal components**. A bubble component is either a sphere or a disk, which is called respectively a **sphere bubble** or a **disk bubble**. The points y_j or y'_j appeared above are called **connecting exceptional points**. The **connecting nodes** are the images of the connecting exceptional points under the map $\hat{\pi}$. The nodes which are not connecting are called **tree nodes**. We call the bubbles O_i or D_j the **connecting bubbles** over w. We call **tree bubbles** those bubbles which are not connecting.

2.6.6 Volume forms

Let g, n be nonnegative integers with $2g + n \geq 3$ and let $\overline{\mathcal{M}}_{g,n}$ be the moduli space of stable curves and let $f : \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$ be the forgetful map. Then we can choose a smooth (in the orbifold sense) Hermitian metric h on the universal curve $\overline{\mathcal{M}}_{g,n+1}$, which gives a conformal metric $h_{(C,\mathbf{x})}$ on each stable curve $(C, \mathbf{x})$ with $[(C, \mathbf{x})] \in \overline{\mathcal{M}}_{g,n}$ and the metric is invariant under the automorphism group of $(C, \mathbf{x})$. We call $h_{(C,\mathbf{x})}$ the **canonical metric** on $(C, \mathbf{x})$ and its volume form the **canonical volume form** on $(C, \mathbf{x})$. From now on we fix such an h for each pair (g, n).

Now let $(C, \mathbf{x} \cup \mathbf{z})$ be a marked stable bordered Riemann surface. Then it has a **complex double**, which is a stable curve $\left(C_{\mathbb{C}}, \mathbf{z}_{\mathbb{C}} = \left\{z_i^{\pm}, x_j\right\}\right)$ with an anti-holomorphic involution τ such that $\tau(z_i^+) = \tau(z_i^-)$ and $\tau(x_j) = x_j$ (if a boundary node of C is modelled on $\{x^2 \pm y^2 = 0\}/A$ (see Definition 2.6.1), then the anti-holomorphic involution is modelled on the complex conjugation A on $\{x^2 \pm y^2 = 0\}$). Then the metric $\frac{1}{2}\left(h_{(C_{\mathbb{C}},\mathbf{z}_{\mathbb{C}})} + \tau^* h_{(C_{\mathbb{C}},\mathbf{z}_{\mathbb{C}})}\right)$ is a metric which descends to a metric on the original bordered curve C. We denoted it by $h_{(C,\mathbf{x}\cup\mathbf{z})}$ and call it the **canonical metric** on $(C, \mathbf{x} \cup \mathbf{z})$ and the volume form induced from the canonical metric and the almost complex structure on C is called the **canonical volume form** on $(C, \mathbf{x} \cup \mathbf{z})$.

In the following, for any prestable curve C with stablization C^{st}, we will use metrics on each irreducible components of C as: if D is a principal component, then use the metric induced from

the canonical metric $h_{C^{st}}$; if D is a bubble component, then use an arbitrary smooth metric.

2.6.7 Definition of (c, L)-STHM's

Pick an element $c \in i\mathbb{R}$, and $(g, h, n, \overrightarrow{m})$ in the stable domain. A (c, L)**-stable twisted holomorphic map** from a bordered curve of genus g, with h boundary components and $(n, \overrightarrow{m})$ marked points is a tuple

$$\mathcal{C} = ((C, \mathbf{x} \cup \mathbf{z}), (P, A, \varphi), G, \{\mathcal{T}_w\})$$

where

1. $(C, \mathbf{x} \cup \mathbf{z})$ is a marked prestable bordered Riemann surface and the isomorphism class of its stabilization belongs to $\overline{\mathcal{M}}_{(g,h),(n,\overrightarrow{m})}$.
2. P is an S^1-principal bundle over $\hat{C} \setminus \hat{\pi}^{-1}\left(\mathbf{z} \cup \mathbf{w}^{int} \cup \mathbf{w}^{b2}\right)$, where $\hat{\pi} : \hat{C} \to C$ is the normalization map.
3. A is a meromorphic connection on P.
4. φ is a smooth section of the bundle $P \times_{S^1} M$, such that $\varphi(x) \in P \times_{S^1} L$ for any x in a boundary circle (but not boundary nodes of type 2) of C.
5. $G = \{G_w\}$ is a choice of gluing data at each node $w \in \mathbf{w}^{int} \cup \mathbf{w}^{b1}$.
6. For each interior connecting node $z \in C$ with preimage $\left\{y_j, y_j'\right\}$ in C', $\mathcal{T}_z$ is a pair of sections $\mathcal{T}_y : S_y \to P_y \times_{S^1} \mathcal{T}(X)$ and $\mathcal{T}_{y'} : S_{y'} \to P_{y'} \times_{S^1} \mathcal{T}(X)$ satisfying $\mathcal{T}_y = \mathcal{T}_{y'} \circ g_z$.
7. For each boundary node $w'' \in C$ of type 2, $\mathcal{T}_{w''}$ is a section $\mathcal{T}_{w''} : S_{w''} \to P_{w''} \times_{S^1} \mathcal{T}(X)$.

The tuple $\mathcal{C}$ must satisfy the following conditions.

(1) The section is holomorphic. The section φ satisfies the equation

$$\overline{\partial}_A \varphi = 0.$$

(2) Vortex equation. For each *principal component* $D \subset C$, let ν_D (resp. A_D, ϕ_D) be the restriction of $s^* v_{(C^{st}, \mathbf{x}^{st} \cup \mathbf{z}^{st})}$ (resp. A, ϕ), where $s : (C, \mathbf{x} \cup \mathbf{z}) \to (C^{st}, \mathbf{x}^{st} \cup \mathbf{z}^{st})$ is the stabilization map; then

$$\Lambda F_{A_D} + \mu(\varphi_D) = c.$$

(3) Flatness on bubbles. The restriction of A to each bubble component is flat.

(4) Critical holonomy at interior marked points. For any interior marked point $z \in \mathbf{z}$, the holonomy $\mathrm{Hol}(A, z)$ is critical.

(5) Finite energy. $\|d_A\varphi\|_{L^2(C)} < \infty$.

Let $y \in \hat{C}$ be the preimage of an interior node or a boundary node of type 2 of C in the normalization. We denote by $S_\epsilon \subset \hat{C}$ the circle of radius ϵ centered at y, and by $\iota_\epsilon : P_y \to P|_{S_\epsilon}$ be the natural isomorphism given by parallel transport with respect to A. Then the section $\varphi_\epsilon = \iota_\epsilon^* \varphi$ converges in C^0-topology, as $\epsilon \to 0$, to a section φ_y of $P_y \times_{S^1} X$ satisfying $d_{A_y}\varphi_y = 0$. We call (P_y, A_y, φ_y) the **limit triple** of (P, A, φ) at y.

(6) Stabilizers of monotone chains of gradient segments.

• Take any interior connecting node $w \in \mathbf{w}$, let y, y' be its preimage in the normalization, and let $\mathcal{T}_y : P_y \to \mathcal{T}(X)$, $\mathcal{T}_{y'} : P_{y'} \to \mathcal{T}(X)$ be the equivariant maps with values in the set of chains of gradient segments. Then $\mathcal{T}_y$ and $\mathcal{T}_{y'}$ are covariantly constant, i.e., take $\mathcal{T}_y$ for example, if we identify S_y with S^1 and trivialize $P_y \to S_y$ such that $d_{A_y} = d + \lambda d\theta$, then the corresponding map $\mathcal{T}_y : S_y \to \mathcal{T}(X)$ satisfies $\mathcal{T}_y(\theta) = e^{\lambda\theta}\mathcal{T}_y(0)$. Here λ is the residue at y with respect to the trivialization. This implies that the images of $\mathcal{T}_y$ and $\mathcal{T}_{y'}$ are contained in X^λ. Another consequence is that if λ is not critical, then the maps $\mathcal{T}_y$, $\mathcal{T}_{y'}$ are degenerate.

• For any boundary node of type 2 $w'' \in \mathbf{w}^{b2}$, the section $\mathcal{T}_{w''} : P_{w''} \to \mathcal{T}(X)$ is covariantly constant.

(7) Matching condition and monotonicity at interior connecting nodes.

Let $w \in C^{st}$ be an interior node. Pick an ordered list of connecting exceptional points over w, $(y_0, \ldots, y_{d-1})$, $(y'_1, \ldots, y'_d)$. For each j, denote by (P_j, A_j, φ_j) and (P'_j, A'_j, φ'_j) the limit triples of (P, A, φ) at the points y_j and y'_j respectively. Let $\Phi_j : P_j \to X$ and $\Phi'_j : P'_j \to X$ be the equivariant maps corresponding to φ_j and φ'_j respectively. Suppose that y_{j-1} and y'_j are the preimages of a connecting node $z_j \in C$. Then each $\mathcal{T}_{z_j}$ is a pair of maps $\mathcal{T}_{j-1} : P_j \to \mathcal{T}(X)$ and $\mathcal{T}'_j : P_j \to \mathcal{T}(X)$. Recall that there are maps $b, e : \mathcal{T}(X) \to X$ given by the beginning and end of gradient segments. Finally, let $g_j : P_j \to P'_{j+1}$ be the isomorphism given by the gluing data at z_j. Then the matching condition requires that either for all j, $\Phi_{j-1} = b \circ \mathcal{T}'_j \circ g_{j-1}$ and $\Phi'_j = e \circ \mathcal{T}'_j$ or the same equations hold with b and e switched for every j.

(8) Matching condition and monotonicity at boundary connecting nodes of type 2.

Let $w'' \in C^{st}$ be a boundary node of type 2. There are two possibilities on the connecting bubbles over w'' in C. The first is a sequence of bubbles $O_1, \ldots, O_d$ with $O_1, \ldots, O_{d-1}$ are sphere bubbles, and O_d is a disk bubble. Then we have an ordered list of connecting exceptional points, $(y_0, \ldots, y_{d-1})$, $(y'_1, \ldots, y'_d)$. With the notations used in the above condition, we should also have

$\Phi_{j-} = b \circ \mathcal{T}'_j \circ g_{j-1}$ and $\Phi'_j = e \circ \mathcal{T}'_j$ or the same equations with b and e switched. The second possibility is a sequence of sphere bubbles $O_1, \ldots, O_{d-1}$ with O_{d-1} has a boundary node of type 2. Then using the above notation, we have either, for $1 \leq j \leq d-1$, we have $\Phi_{j-1} = b \circ \mathcal{T}'_j \circ g_{j-1}$ and $\Phi'_j = e \circ \mathcal{T}'_j$, and $\Phi_{d-1} = b \circ \mathcal{T}_{d-1}$, with the **boundary condition** $e \circ \mathcal{T}_{d-1} \in L$; or the same equations with b and e switched (see Figure 2.1).

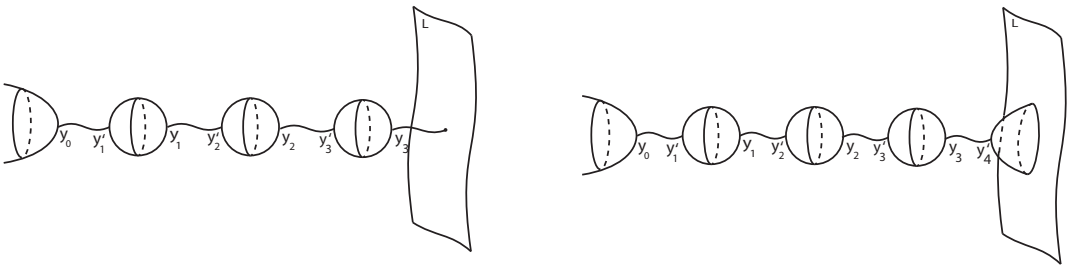

Figure 2.1: Chains of gradient segments and connecting bubbles at a type-2 boundary node in two cases respectively

(9) Matching conditions at interior tree nodes.

Let $w \in C$ be an interior tree node and let $y, y' \in \hat{C}$ be its preimages in the normalization. Let (P_y, A_y, ϕ_y) and $(P_{y'}, A_{y'}, \phi_{y'})$ be the limit triples, and let $g_w : P_y \to P_{y'}$ be the isomorphism of bundles given by the gluing data at w. Denote by $\Phi_y : P_y \to X$ and $\Phi'_y : P'_y \to X$ be the corresponding equivariant maps respectively, then we have $\Phi_y = \Phi'_y \circ g_w$. This implies that the limit orbits of φ at y and y' are the same.

(10) Matching conditions at boundary nodes of type 1

Let $w' \in C$ be a boundary node of type 1 with y, y' the preimages in the normalization. Let $g_{w'} : P_y \to P_{y'}$ be isomorphism of fibres given by the gluing data at w'. Let $\Phi_y : P_y \to L$ and $\Phi'_y : P_{y'} \to L$ be the corresponding equivariant maps. Then $\Phi_y = \Phi_{y'} \circ g_{w'}$.

(11) Stability condition.

If a bubble component $D \subset C$ is unstable, then the restriction of $d_A\varphi$ to D is not identically zero.

Remark 2.6.3. The last stability condition in the definition doesn't automatically guarantee that the automorphism group of a (c, L)-STHM is finite. We have to put a restriction on the pair (X, L) and the value c to have that. However, this is not related to the compactness issue.

2.6.8 Isomorphisms of (c, L)-STHM's

Take two (c, L)-STHM's

$$\mathcal{C}_i = ((C_i, \mathbf{x}_i \cup \mathbf{z}_i), (P_i, A_i, \varphi_i), G_i, \{\mathcal{T}_{w,i}\}), i = 1, 2.$$

We say that they are **isomorphic**, if there is an isomorphism $f : (C_1, \mathbf{x}_1 \cup \mathbf{z}_1) \to (C_2, \mathbf{x}_2 \cup \mathbf{z}_2)$ of prestable curves and an isomorphism $g : P_1 \to f^*P_2$ of S^1-principal bundles satisfying the following conditions:

(1) $g^*f^*A_2 = A_1$, $g^*f^*\varphi_2 = \varphi_1$ in the usual sense;

(2)item (see the definition of isomorphisms in [38])for each interior node $w_1 \in \mathbf{w}^{int} \subset C_1$, denote $w_2 = f(w_1)$. Then we can lift f to the normalization

$$\hat{f} : \hat{C}_1 \to \hat{C}_2.$$

Choose preimages y_1, y_1' of w_1 and y_2, y_2' of w_2 in their normalizations such that $\hat{f}(y_1) = y_2$, $\hat{f}(y_1') = y_2'$. Let (P_{y_j}, A_{y_j})(resp. $(P_{y_j'}, A_{y_j'})$) be the limit pairs of (P_j, A_j) at y_j(resp. y_j'). Let $g_w : P_{y_1} \to P_{y_2}$ and $g_w' : P_{y_1'} \to P_{y_2'}$ be the maps induced by g. The gluing data of $\mathcal{C}_j$ gives $\gamma_j : P_{y_j} \to P_{y_j'}$.

• If the limit holonomy of A_j at y_j is *generic*, then we require that the following diagram commutes

$$\begin{array}{ccc} P_{y_1} & \xrightarrow{g_w} & P_{y_2} \\ {\scriptstyle\gamma_1}\downarrow & & \downarrow{\scriptstyle\gamma_2} \\ P_{y_1'} & \xrightarrow{g_w'} & P_{y_2'}. \end{array}$$

• If the holonomy of A_j at y_j is *critical*, then we require that there exists $t \in \mathbb{R}$ such that the following diagram commutes

$$\begin{array}{ccc} P_{y_1} & \xrightarrow{g_w} & P_{y_2} \\ {\scriptstyle\gamma_1}\downarrow & & \downarrow{\scriptstyle\gamma_2 \circ \eta_{y_2}(t)} \\ P_{y_1'} & \xrightarrow{g_w'} & P_{y_2'}. \end{array}$$

Here $P_{y_2} \simeq S_{y_2} \times S^1$ and $\eta_{y_2} : P_{y_2} \times \mathbb{R} \to P_{y_2}$ is the flow given by the limit connection on P_{y_2}.

(3) For each boundary node of type 1 $w_1 \in \mathbf{w}^{b1}$, denote $w_2 = f(w_1)$. We require that there exists

$\gamma \in S^1$ in the stablizer of $\varphi_2(y_2)$ such that

$$\begin{array}{ccc} P_{y_1} & \xrightarrow{g_w} & P_{y_2} \\ {\scriptstyle \gamma_1}\downarrow & & \downarrow{\scriptstyle \gamma_2 \circ \gamma} \\ P_{y_1'} & \xrightarrow{g_w'} & P_{y_2'} \end{array}$$

commutes.

(4) Using the above notations, for each pair of chains of gradient segments $\mathcal{T}_{y_1}$, $\mathcal{T}_{y_2}$, we require that

$$\mathcal{T}_{y_1} = \mathcal{T}_{y_2} \circ g_w.$$

Remark 2.6.4. As Ignasi Mundet told the author, this definition seems unnatural in the first glance, because the relaxation from the natural but more rigid definition allows us to have nice structure of moduli space.

2.6.9 Yang-Mills-Higgs functionals

We give the definition of the Yang-Mills-Higgs functional for a (c, L)-STHM, which plays same role as energy in the theory of pseudoholomorphic curves.

Definition 2.6.5. Let $\mathcal{C}$ be a (c, L)-STHM. Its Yang-Mills-Higgs functional is defined to be

$$\mathcal{YMH}_c(\mathcal{C}) := \sum_{D \subset C} \|d_A \varphi\|^2_{L^2(D)} + \sum_{D \subset C_p} \left(\|F_A\|^2_{L^2(D)} + \|\mu(\varphi) - c\|^2_{L^2(D)} \right),$$

where the first summation is taken over all components of C while the second is taken over all principal components.

2.7 Bounding the number of bubbles

In this subsection, we will prove the following theorem.

Theorem 2.7.1. *For a given tuple $(g, h, n, \overrightarrow{m})$ in the stable domain and $K > 0$, there is a number $f(g, h, n, \overrightarrow{m}, K)$ such that for any c-stable twisted holomorphic maps with boundary $\mathcal{C}$ with the underlying curve $(C^{st}, \mathbf{x}^{st} \cup \mathbf{z}^{st}) \in \overline{\mathcal{M}}_{(g,h),(n,\overrightarrow{m})}$ and $\mathcal{YMH}_c(\mathcal{C}) \leq K$, the number of bubbles of C is no greater than $f(g, h, n, \overrightarrow{m}, K)$.*

In the case of pseudoholomorphic curves, the bounding of bubbles is relatively easy because the energy of nontrivial bubbles has a positive lower bound. But in the case of twisted holomorphic

curves, we can have bubbles of arbitrarily small energy. Then to find a bound, we have to consider such bubbles of small energy.

2.7.1 Canonical transports

Recall that in [39, Subsection 5.5], the notion of canonical transport is defined as follows. For any compact smooth genus zero curve B with $\mathbf{y} = \{y_1, y_2\} \subset B$, we have a choice of maps $f_{y_j} : B \setminus \mathbf{y} \to S_{y_j}$, $j = 1, 2$ (up to reparametrizing by an automorphism of $B \setminus \mathbf{y}$) and a canonical isomorphism of S^1-torsors $f_{y_0,y_1} : S_{y_0} \to \overline{S_{y_1}}$. Then to any S^1-principal bundle $P \to B \setminus \mathbf{y}$ and any flat connection A on P, we can associate bundle maps

$$g_{y_j} : P \to P_{y_j} \ (j = 1, 2), \ g_{y_0,y_1} : P_{y_0} \to P_{y_1}$$

which cover $f_{y_j}, j = 1, 2$ and f_{y_0,y_1} respectively. The maps g_{y_j} and g_{y_0,y_1} are called the **canonical transport** of (P, A).

Now we will extend the notion of canonical transport to disks. Consider the unit disk $\mathbb{D}$ and $0 \in \mathbb{D}$ the center and $1 \in \partial\mathbb{D}$. Set $\gamma : [0, 1] \to \mathbb{D}$ to be $\gamma(t) = t$ and $\gamma'(0) = u_0$. Then for any disk D with an interior point z_0, for any $z \in D \setminus \{z_0\}$, there is a unique biholomorphism $f : \mathbb{D} \to D$ such that $f(0) = z_0$ and z belongs to the image of $f \circ \gamma$. Set $g_{z_0} : D \setminus \{z_0\} \to S_{z_0}$ by $g_{z_0}(z) = [df(u_0)]$. If we have an S^1-principal bundle P over $D \setminus \{z_0\}$ with a flat connection A on P, then the parallel transport along $f \circ \gamma$ gives a morphism

$$g_{z_0} : P \to P_{z_0}. \tag{2.11}$$

We call the map g_{z_0} the **canonical transport** of P. In particular, if with respect to some trivialization $A = d + \lambda d\theta$. Then the trivialization gives $P \simeq \mathbb{D}^* \times S^1 \simeq [0, \infty) \times S^1 \times S^1$ and $P_{z_0} \simeq S^1 \times S^1$. Then g_{z_0} is just the projection $(t, \theta_1, \theta_2) \mapsto (\theta_1, \theta_2)$.

2.7.2 Twisted bubbles

Let $\mathbf{y} \subset S^2$ be a finite subset. We call a **twisted sphere bubble** over $(S^2, \mathbf{y})$ a triple (P, A, φ) consisting an S^1-principal bundle $P \to S^2 \setminus \mathbf{y}$, a flat meromorphic connection A on P and a section $\varphi : S^2 \setminus \mathbf{y} \to P \times_{S^1} X$ satisfying $\overline{\partial}_A \varphi = 0$. Similarly, let $\mathbf{y} \subset D$ be a finite subset in the interior of the unit disk. We call a **twisted disk bubble** over $(D, \mathbf{y})$ a triple (P, A, φ) consisting an S^1-principal bundle $P \to D \setminus \mathbf{y}$, a flat meromorphic connection A on P and a section φ of $P \times_{S^1} X$ satisfying

$\varphi(\partial D) \subset P \times_{S^1} L$ and $\overline{\partial}_A \varphi = 0$. A twisted sphere (or disk) bubble (P, A, φ) is **trivial** if the covariant derivative $d_A \varphi$ is identically zero.

For any $\epsilon > 0$ smaller than half the minimal distance δ_{cr} between two different elements in Λ_{cr}, define the sets $\Lambda_{cr}^{\epsilon,+}$ and $\Lambda_{cr}^{\epsilon,-}$ as

$$\Lambda_{cr}^{\epsilon,\pm} = \Lambda_{cr} \pm (0, \epsilon) = \{\lambda_{cr} \pm \delta | \lambda_{cr} \in \Lambda_{cr}, i\delta \in (0, \epsilon)\},$$

and denote $\Lambda_{gen}^{\epsilon} = i\mathbb{R} \setminus (\Lambda_{cr}^{\epsilon,+} \cup \Lambda_{cr}^{\epsilon,-})$.

In the usual Gromov-Witten theory we have a nonzero lower bound on the energy of nonconstant holomorphic spheres so it is easy to bound the number of bubbles once we have total energy bound. In the case of twisted holomorphic maps we don't have such lower bound. However, [39, Theorem 6.2] provides a key results in bounding the number of twisted sphere bubbles in the closed case.

We now state and prove an analogue for twisted disk bubbles.

Theorem 2.7.2. *Let $z_0 \in D$ be an interior point. Let ϵ be smaller than the ϵ_4 given in Theroem 2.4.2 and $\frac{1}{2}\delta_{cr}$. Then there exists $\delta > 0$ with the following property. Suppose that (P, A, φ) is a twisted disk bubble on (D, z_0) satisfying $\|d_A \varphi\|_{L^2} < \delta$. Take some local trivialization of P on $D \setminus \{z_0\}$. Then φ corresponds to some map $\phi : D \setminus \{z_0\} \to X$. Let λ be the corresponding residue of A at z_0.*

Case A. *If $\lambda \in \Lambda_{cr}$ then (P, A, φ) is trivial;*

Case B. *Otherwise, by Theorem 4.2 in [39], the following exists*

$$\phi(z_0) = \lim_{z \to z_0} \phi(z) \in F_k.$$

Then we have

1. *if $\lambda \in \Lambda_{cr}^{\epsilon,+}$, then $h(F_k) \leq h(L)$, with equality only if (P, A, φ) is trivial;*
2. *if $\lambda \in \Lambda_{cr}^{\epsilon,-}$, then $h(F_k) \geq h(L)$, with equality only if (P, A, φ) is trivial;*
3. *if $\lambda \in \Lambda_{gen}^{\epsilon}$, then (P, A, φ) is trivial.*

Let (P_{z_0}, A_{z_0}) be the limit pair of (P, A) at z_0. When case B holds, there is a covariantly constant flat map $\mathcal{T}_{z_0} : P_{z_0} \to \mathcal{T}^{\infty}(X)$ such that for any $p \in P_{z_0}$, we have

$$\mathcal{T}_{z_0}(p) = \{\phi(z_0)\} \cup \{\phi(q) | q \in P,\ g_{z_0}(q) = p\},$$

where $g_{z_0} : P \to P_{z_0}$ is the canonical transport (2.11).

Proof. The proof is similar to that of [39, Theorem 6.2]. Indeed, suppose first that $\lambda = ip/q \in \Lambda_{cr}$ for relatively prime integers p, q. Take a covering $\pi : D \to D$ of degree q totally ramified at z_0. Then π^*A has trivial holonomy around z_0. Since π^*A is flat, we can extend π^*P and π^*A to a bundle $P' \to D$ with a flat connection A'. We can take a trivialization $P \simeq D \times S^1$ with respect to which A' is trivial. Consider the induced trivialization of π^*P, we can view the section $\pi^*\phi$ as an I-holomorphic map $\phi : D \setminus \{z_0\} \to X$ of bounded energy. By the removal of singularity theorem, ϕ extends to a holomorphic disk $\widetilde{\phi} : D \to X$ satisfying $\left\|d\widetilde{\phi}\right\|_{L^2} = q \left\|d_A\varphi\right\|_{L^2}$. This is a standard result in Gromov-Witten theory that there is a lower bound of energy of nontrivial holomorphic disks, which depends only on ω and I. Since the critical residues contains finite many classes modulo integers, we can take δ small enough so that the lemma holds for all critical λ.

Now consider the case that λ is not critical. Fix a conformal isomorphism between $D - \{z_0\}$ and the semi-infinite cylinder $C^+ = [0, +\infty) \times S^1$, and take the latter the standard coordinate (t, θ) in such a way that as t goes to $+\infty$, (t, θ) approaches to the point z_0. Then we can identify $S_{z_0} \simeq S^1$ in such a way that the map $f_{z_0} \to S_{z_0}$ is the projection $(t, \theta) \mapsto \theta$.

Fix a triple (P, A, φ) and take a trivialization $P \times C^+ \times S^1$ with respect to which $d_A = d + \lambda d\theta$, since A is flat.

Suppose $\lambda \in \Lambda_{cr}^{\epsilon,+}$, so there exists $\lambda_{cr} \in \Lambda_{cr}$ such that $i(\lambda - \lambda_{cr}) \in (0, \epsilon)$. By a standard argument in Gromov-Witten theory (see [34, Proposition B.4.9]), there exists $\delta > 0$ such that $\|d_\alpha\phi\|_{L^2} < \delta$ implies that $\|d_\alpha\phi\|_{L^\infty} < \epsilon$. Then we can apply Theorem 2.4.2 and Theorem 2.4.3 to the pair (α, ϕ). Hence there exists $\psi : \mathbb{R}_{\geq 0} \to X^{\lambda_{cr}}$ and $\phi_0 : \mathbb{R}_{\geq 0} \times S^1 \to TX$ such that

$$\phi(t, \theta) = e^{-\lambda_{cr}\theta} \exp_{\psi(t)} \left(e^{\lambda_{cr}\theta}\phi_0(t, \theta)\right).$$

Take some $t \in \mathbb{R}_{\geq 0}$ and $N > t$. Apply Theorem 2.4.2, 2.4.3 to $[0, N] \times S^1 \subset C^+$ together with $d\alpha = 0$ implies that

$$|\phi_0(t, \theta)| \leq K_4 e^{\sigma_4(t-N)}, \ |\psi'(t) + i(\lambda - \lambda_{cr})\nabla h(\psi(t))| \leq K_4 e^{\sigma_4(t-N)}.$$

Making $N \to \infty$, we deduce that $\phi_0(t, \theta) = 0$, and $\psi'(t) = -i(\lambda - \lambda_{cr})\nabla h(\psi(t))$. Since $\phi(\{0\} \times S^1) \subset L$, we have $\psi(0) \in L$ and $\psi(t)$ is a downward gradient flow line of $h = -i\mu$. So we have $h(F_k) \leq h(L)$ and equality holds only if ϕ is trivial. The case when $\lambda \in \Lambda_{cr}^{\epsilon,-}$ is treated the same way.

Finally suppose $\lambda \in \Lambda_{gen}^{\epsilon}$. By Theorem 2.4.1, there exists real numbers K', σ', ϵ' such that its

statements hold for any $\lambda \in \Lambda^{\epsilon}_{gen}$ (since we have the continuous dependence of the constants on λ). Then we have the estimate

$$|d_\alpha \phi| \leq K_3 e^{\sigma_3 (t-N)} \|d_\alpha \phi\|_{L^\infty} .$$

Let N go to infinity, we see that the pair (α, ϕ) is in fact trivial.

The map $\mathcal{T}_{z_0} : P_{z_0} \to \mathcal{T}^\infty(X)$ is given by the trajectory $\psi(t)$ i.e. $\mathcal{T}_{z_0}(\theta_1, \theta_2) = e^{-\lambda_{cr}\theta_2}\psi(t)$. Here θ_1 is the coordinate on the base S_{z_0} and θ_2 is the coordinate on the fibre of P_{z_0}. It is easy to check that ψ coincides with the one given in the statement of the theorem. In particular, $\psi(0) \in L$. □

2.7.3 Proof of Theorem 2.7.1

Let Γ be the dual graph of C, so the set of vertices of Γ is the set of irreducible components of C, and for any pair of vertices v', v'' corresponding to components C', C'' (it is possible that $v' = v''$), there are as many edges connecting v' and v'' as there are many nodes in C at which C' and C'' meet. All subgraphs $\Gamma' \subset \Gamma$ which we shall mention will be saturated, that is, if v' and v'' are vertices of Γ', then all edges in Γ connecting v' and v'' are also contained in Γ'. We call a vertex of Γ a tree (resp. connecting) vertex if it corresponds to a tree (resp. connecting) bubble.

Denote by Λ^{σ}_{cr} the set of sums of finitely many elements in Λ_{cr}. This is a diskrete subgroup of $\mathbb{R}$. Then we can take ϵ small such that $\Lambda^{\sigma}_{cr} \subset \Lambda_{cr} \cup \Lambda^{\epsilon}_{gen}$.

Each sphere (resp. disk) tree vertex of Γ belongs to a maximal connected subgraph $T \subset \Gamma$ consisting of entirely of sphere (resp. disk) tree vertices, and T is a tree. We denote the collection of all trees of sphere tree vertices $T^s_1, \ldots, T^s_p$ and the collection of all trees of disk tree vertices $T^d_1, \ldots, T^d_q$. Each tree has a unique vertex, called the **root**, which is connected by an edge to the complement of this tree.

We slightly generalize [39, Lemma 6.3] as follows.

Lemma 2.7.3. *Let T be one of the trees $T^s_1, \ldots, T^s_p$.*

1. *For any sphere bubble $B \subset C$ corresponding to a vertex in some T^s_i, the residue of A at any node in B belongs to Λ^{σ}_{cr}.*
2. *If moreover, the root of T^i_s is connected by an edge to a connecting vertex or a disk tree vertex, then the limit holonomy of A at any node in B is trivial.*

Proof. The same as the proof of [39, Lemma 6.3] □

Proof of Theorem 2.7.1. We first show that, the total number of sphere tree vertices is bounded. Let $\{T^s_i\}_{i=1}^m$ be the set of trees in Γ consisting of sphere bubbles. We say that a vertex $v \in T^s_i$ is

stable if its degree *as a vertex of* Γ $\deg_\Gamma(v) \geq 3$. Since T_i^s is a tree, it has $|T_i^s| - 1$ edges and has a unique edge connecting to its complement. Hence we have

$$\sum_{v \in T_i^s} \deg_\Gamma(v) = 1 + \sum_{v \in T_i^s} \deg_{T_i^s}(v) = 1 + 2\left(|T_i^s| - 1\right).$$

Hence

$$2|T_i^s| = \sum_{v \in T_i^s} \deg_\Gamma(v) + 1 \geq 3\#\{v \in T_i^s \text{ stable}\} \Rightarrow \#\{v \in T_i^s \text{ unstable}\} \leq \frac{1}{3}|T_i^s|.$$

Summing up for $i = 1, \ldots, m$, one obtains that

$$\sum_{i=1}^{m} \#\{v \in T_i^s \text{ unstable}\} \leq \frac{1}{3}\sum_{i=1}^{m} |T_i^s|.$$

Among those unstable vertices, there are at most $|\mathbf{x}|$ contains interior marked points, while the other vertices really correspond to the unstable bubbles in the usual sense. The energy of each of those "real" unstable vertices is at least δ, due to the lemma. Hence we have

$$\delta\left(\frac{1}{3}\sum_{i=1}^{m} |T_i^s| - |\mathbf{x}|\right) \leq K.$$

This gives a upper bounded of the number of sphere tree vertices, which is denoted by N_1 temporarily.

The total number of connecting sphere bubbles is also bounded, according to the proof of [39, Theorem 6.1]. We denote this number by N_2. Note that this step uses the monotonicity property of [39, Theorem 6.2].

Now we bounded the number of disk bubbles. Let $T_j^d \subset \Gamma$ be a tree of disk tree bubbles. Then each vertex $v \in T_j^d$ has no interior marked point, or it will be stable. Also, if v has an interior node w, then v is the root of some T_i^s, so the residue at w belongs to Λ_{cr}^σ. But $\Lambda_{cr}^\sigma \subset \Lambda_{cr} \cup \Lambda_{gen}^\epsilon$. Hence by the above theorem, each vertex of T_j^d contributes at least δ to the energy. Hence the total number of disk tree vertices is bounded.

The only remaining bubbles are the disk connecting bubbles, each of which belongs to two cases, which are diskribed in Section 2.6.5. One is a single connecting bubble with an interior node, but there is at most one such bubble for each boundary node of type 2 of the stablization; the other kind is a chain of connecting bubbles, but it is easy to see that, the holonomy at each interior exceptional point belonging to this chain is trivial, hence each bubble in the sequence contributes at least δ to

the energy. This gives an upper bounde to the total number of bubbles of this kind. □

2.8 Definition of the convergence

2.8.1 Deformation of prestable curves

Let $(C, \mathbf{x}\cup\mathbf{z})$ be a prestable curve and h be a conformal metric. Let $\hat{\pi} : \hat{C} \to C$ be the normalization map and denote $\mathbf{y} = \hat{\pi}^{-1}(\mathbf{w}) = \hat{\pi}^{-1}(\mathbf{w}^{int}) \cup \hat{\pi}^{-1}(\mathbf{w}^{b1}) \cup \hat{\pi}^{-1}(\mathbf{w}^{b2}) =: \mathbf{y}^{int} \cup \mathbf{y}^{b1} \cup \mathbf{y}^{b2}$. We have an anti-holomorphic involution τ on the complex double $C_{\mathbb{C}}$, which induces, for each point $y \in \partial\hat{C}$, an anti-holomorphic involution $\tau : T_y\hat{C} \to T_y\hat{C}$. Denote by $\left(T_y\hat{C}\right)^{\mathbb{R}}$ to be the subspace of all τ-invariant vectors.

For each $y \in \mathbf{y}^{int}$, take a neighborhood of $U_y \subset \hat{C}$ such that there exists a biholomorphism $\xi_y : D(\epsilon) \to U_y$ with the disk $D(\epsilon) \subset T_y\hat{C}$ centered at 0 of radius ϵ; for each $y \in \mathbf{y}^{b1}$, take a neighborhood U_y small enough such that there exists a biholomorphism $\xi_y : D(\epsilon)/\tau \to U_y$, where $D(\epsilon) \subset T_y\hat{C}$ and τ the involution on $T_y\hat{C}$; for each $y \in \mathbf{y}$ with $\hat{\pi}(y)$ a boundary node of type 2, take a neighborhood U_y small enough such that there exists a biholomorphism $\xi_y : D(\epsilon) \to U_y$ where $D(\epsilon) \subset T_y\hat{C}$. For each $x \in \mathbf{x}$ and $z \in \mathbf{z}$, also choose a small neighborhood $B_x \subset \hat{C}$ of $\hat{\pi}^{-1}(x)$ and $B_z \subset \hat{C}$ of $\hat{\pi}^{-1}(z)$. Assume that these neighborhoods are pairwise disjoint.

Denote by J the complex structure of $\hat{C}$ and J' be another complex structure which coincides with J on each $\bigcup(U_y \cup B_x \cup B_z)$. Such complex structures are called $(C, \mathbf{x}\cup\mathbf{z})$-admissible. Take, for each $w \in \mathbf{w}^{int}$ with preimages $y, y' \in \hat{C}$, an element $\delta_w \in T_y\hat{C} \otimes_{\mathbb{C}} T_{y'}\hat{C}$ with $|\delta_z| < \epsilon^2$; for each $w' \in \mathbf{w}^{b1}$ of type 1 with preimages $y, y' \in \hat{C}$, an element $\delta_{w'} \in (T_y\hat{C} \otimes_{\mathbb{C}} T_{y'}\hat{C})^{\mathbb{R}}$ with $|\delta_{w'}| < \epsilon^2$; for each $w'' \in \mathbf{w}^{b2}$, with preimage $y \in \hat{C}$, a real number $\delta_{w''}$ with $0 \le \delta_{w''} < \epsilon$. The collection of numbers $\delta = \{\delta_w\}$ is called a set of **smoothing parameters** of C.

Now from the original curve with an admissible complex structure J' and a set of smoothing parameters δ, we can construct a new curve $C(J', h, \delta)$ as follows:

1. First replace the complex structure J by J' on the normalization $\hat{C}$;

2. For each interior node $\mathbf{w}^{int}$ with $\hat{\pi}^{-1}(w) = \{y, y'\}$, remove the two subsets $\xi_y\left(D(\frac{|\delta_w|}{\epsilon})\right)$ and $\xi_{y'}\left(D(\frac{|\delta_w|}{\epsilon})\right)$ from $\hat{C}$, and for each pair of elements $u \in D(\epsilon) \setminus D(\frac{|\delta_w|}{\epsilon}) \subset T_y\hat{C}$ and $v \in D(\epsilon) \setminus D(|\delta_w|/\epsilon) \subset T_{y'}\hat{C}$ with $u \otimes v = \delta_w$, identify their images $\xi_y(u)$ and $\xi_{y'}(v)$;

3. For each boundary node $w' \in \mathbf{w}^{b1}$ with $\hat{\pi}^{-1}(w') = \{y, y'\}$, remove the sets $\xi_y\left(D(\frac{|\delta_{w'}|}{\epsilon})/\tau\right)$ and $\xi_{y'}\left(D(\frac{|\delta_{w'}|}{\epsilon})/\tau\right)$ from $\hat{C}$, and for each pair of elements $[u] \in \left(D(\epsilon) \setminus D(\frac{|\delta_{w'}|}{\epsilon})\right)/\tau \subset T_y\hat{C}/\tau$

and $[v] \in \left(D(\epsilon) \setminus D(\frac{|\delta_{w'}|}{\epsilon})\right)/\tau \subset T_{y'}\hat{C}/\tau$ identify the images $\xi_y([u])$ and $\xi_{y'}([v])$ if there are representatives $u \in T_y\hat{C}$ and $v \in T_{y'}\hat{C}$ such that $u \otimes v = \delta_{w'}$;

4. For each $w'' \in \mathbf{w}^{b2}$, remove the open disk $\xi_{w''}(D(\delta_{w''}))$.

Then for every node $w \in \mathbf{w}$, define the "neck" $N_w(\delta_w) \subset C(J', \delta)$ to be the following subset in different cases:

1. if $w \in \mathbf{w}^{int}$, then $N_w(\delta_w)$ is the image by ξ_y of the annulus $D(\epsilon) \setminus D\left(\frac{|\delta_w|}{\epsilon}\right)$. $N_w(\delta_w)$ is conformal to the cylinder

$$N_w(\delta_w) \simeq [\log|\delta_w| - \log\epsilon, \log\epsilon] \times S^1.$$

2. if $w' \in \mathbf{w}^{b1}$, then $N_{w'}(\delta_{w'})$ is the image by ξ_y of $\left(D(\epsilon) \setminus D(\frac{|\delta_{w'}|}{\epsilon})\right)/\tau$. $N_{w'}(\delta_{w'})$ is conformal to the strip

$$N_{w'}(\delta_{w'}) \simeq [\log|\delta_{w'}| - \log\epsilon, \log\epsilon] \times [0, 1].$$

3. if $w'' \in \mathbf{w}^{b2}$, then $N_{w''}(\delta_{w''})$ is the annulus $\xi_{w''}(D(\epsilon) \setminus D(\delta_{w''}))$, which is conformal to

$$N_{w''}(\delta_{w''}) \simeq [\log|\delta_{w''}|, \log\epsilon] \times S^1.$$

A set of smoothing parameters $\{\delta_w\}$ is *small enough* if each $|\delta_w|$ is small enough. Given a compact subset $K \subset C - \mathbf{w}$, if the smoothing parameters $\{\delta_w\}$ is small enough so that $\xi_y(D(|\delta_w|/\epsilon))$ and $\xi_{y'}(D(|\delta_w|/\epsilon))$ are disjoint from K, then there is a canonical inclusion $\iota_K : K \to C(J', \delta)$. In particular, there is a canonical inclusion of the marked points $\mathbf{x} \cup \mathbf{z}$ inside any small deformation $C(J', \delta)$.

2.8.2 Definition of the convergence

In this subsection we define the notion of convergence of (c, L)-stable twisted holomorphic maps. Some part of this long definition is exactly the same as the corresponding part in [39, Subsection 7.4], for which we don't give full details for simplicity.

Let the following be a sequence of (c, L)-stable twisted holomorphic maps

$$\{\mathcal{C}_u = ((C_u, \mathbf{x}_u \cup \mathbf{z}_u), (P_u, A_u, \phi_u), G_u, \{\mathcal{T}_{u,w}\})\}_{u \in \mathbb{N}}.$$

Let $\mathcal{C} = ((C, \mathbf{x} \cup \mathbf{z}), (P, A, \phi), G, \{\mathcal{T}_w\})$ be a (c, L)-stable twisted holomorphic map. We say that the isomorphism classes $[\mathcal{C}_u]$ converge to $[\mathcal{C}]$, if one can write $\mathbb{N} = U_1 \cup \cdots U_n$ in such a way that for each j, the sequence $\{[C_{u'}]\}_{u' \in U_j}$ converges strictly to $[\mathcal{C}]$. While a sequence $\{[\mathcal{C}_u]\}$ **converges strictly** to $[\mathcal{C}]$ means the following:

• There is a $(C, \mathbf{x} \cup \mathbf{z})$-admissible complex structure J_u, smoothing parameters $\delta_u = \{\delta_{u,w}\}$ and a continuous map

$$\xi_u : C_u \to C(J_u, \delta_u) = C(J_u, h, \delta_u)$$

which can be written as $\xi_u = \xi'_u \circ \pi_u^{co}$ where $\pi_u^{co} : C_u \to C_u^0$ is a collapsing map which collapses some of the unstable connecting bubbles and $\xi'_u : C_u^0 \to C(J_u, \delta_u)$ is a biholomorphism of nodal curves.

• We assume that for each node $w \in C$, either all $\delta_{u,w}$ vanish or none of them vanishes (this is why we call this notion *strict* convergence). In the first case we say that w is an **old node** and in the second case we say that w is a **new node**.

• We require that each old interior node w of C and old boundary node w'' of type 2 of C, there are index sets B_w and $B_{w''}$ *indepdent* of u, such that there are collections of unstable connecting bubbles $\{O_{u,b,w}\}_{b \in B_w}$ and $\{O_{u,b,w''}\}_{b \in B_{w''}}$ which are those bubbles that π_u^{co} collapses. Also, π_u^{co} collapses a disk at w'' for all u, or doesn't collapse a disk at w'' for any u.

• Given the previous data, the pair (P_u, A_u) on C_u naturally induces a pair, which we still denote by (P_u, A_u), on the curve C_u^0. For any exhaustion $K_1 \subset \cdots \subset K_l \subset \cdots$ of the smooth locus of $C \setminus \mathbf{z}$ by compact subsets, any choice of conformal metric h on C, the following convergence properties are satisfied as $u \to \infty$. (For each K_l in this sequence, the set $\iota_u(K_l) \subset C_u^0$ are identified for u sufficiently large.)

(1) Convergence of the marked curves. The admissible complex structure J_u converges to J in the C^∞-topology on each compact subset $K_l \subset C$; for each node w, $\lim_{u \to \infty} \delta_{u,w} = 0$; the sequence of marked point sets $\mathbf{x}_u \cup \mathbf{z}_u$ converges to $\mathbf{x} \cup \mathbf{z}$.

(2) Convergence of the connections and sections away from nodes and interior marked points. For each K_l and any big enough u (so that the inclusion $\iota_{K_l} : K_l \to C(J_u, \delta_u)$ is defined) there exists an isomorphism of principal bundles

$$\rho_{u,l} : P|_{K_l} \to \iota_{K_l}^* P_u$$

such that $\rho_{u,l}^* \iota_{K_l}^* A_u$ converges to A and $\rho_{u,l}^* \iota_{K_l}^* \phi_u$ converges to ϕ on K_l in C^∞. This implies that the limit holonomy of A_u around each interior marked point converges to that of A. We can assume

that if $l < l'$, then

$$\rho_{u,l} = \rho_{u,l'}|_{K_l}$$

and we will write ρ_u instead of $\rho_{u,l}$ in the following. We can also assume that for an old boundary node w of type 1, with preimages y, y' in $\hat{C}$, the isomorphisms $\rho_{u,l}$ can be extended over y and y', for any u.

(3) Convergence near the interior marked points. For each interior marked point $z_u \in \mathbf{z}_u$ with $\lim z_u \to z \in \mathbf{z}$, take $D(z_u, \epsilon)^* \subset C_u$ the punctured disk centered at z_u of radius ϵ and $D(z, \epsilon)^*$, identify biholomorphically $D(z_u, \epsilon)^* \simeq \mathbb{R}_{\geq 0} \times S^1 \simeq D(z, \epsilon)^*$. For each $\tau > 0$ define $N_z^\tau = [\tau, \infty) \times S^1$, then we have

$$\lim_{\tau \to \infty} \left(\limsup_{u \to \infty} \operatorname{diam}_{S^1} \phi_u(N_z^\tau) \right) = 0.$$

(4) Convergence near the new interior nodes. Take some new interior node $w \in \mathbf{w}^{int}$, define for each $\tau > 0$ and big enough u the truncation of the neck $N_w(\delta_{u,w})$

$$N_u^\tau(w) = [\ln|\delta_{u,w}| - \ln\epsilon + \tau, \ln\epsilon - \tau] \times S^1 \subset N_w(\delta_{u,w}) = [\ln|\delta_{u,w}| - \ln\epsilon, \ln\epsilon] \times S^1.$$

The condition for the behavior of the sequence near the new node w is the same as item (4) in the definition in [39, Subsection 7.4].

(5) Convergence near the new boundary nodes of type 1. Take such a node $w' \in C$, and define for each τ the truncation for big enough u

$$N_u^\tau(w') = \left[\ln\left(\frac{|\delta_{u,w'}|}{\epsilon} \right) + \tau, \ln\epsilon - \tau \right] \times [0, \pi] \subset N_{w'}(\delta_{u,w'}) = \left[\ln\left(\frac{|\delta_{u,w'}|}{\epsilon} \right), \ln\epsilon \right] \times [0, \pi].$$

-*Limit behavior of the gluing data.* Let y, y' be the preimages of w' in the normalization of C and for any small enough ϵ, let $C_{y,\epsilon}, C_{y',\epsilon}$ be the semi-circles of radius ϵ centered at y, y' respectively. Let $P_{y,\epsilon}$ (resp. $P_{y',\epsilon}$) be the restriction of P to $C_{y,\epsilon}$ (resp. $C_{y',\epsilon}$) and let $\iota_\epsilon : P_y \times [0, \pi] \to P_{y,\epsilon}$ and $\iota'_\epsilon : P_{y'} \times [0, \pi] \to P_{y',\epsilon}$ be the isomorphisms given by radial parallel transport. We can assume that for big enough u there are $\tau_{u,\epsilon}, \tau'_{u,\epsilon} \in \mathbb{R}$ and identifications $C_{y,\epsilon} \simeq \{\tau_{u,\epsilon}\} \times [0, \pi] \subset N_u$ and $C_{y',\epsilon} \simeq \{\tau'_{u,\epsilon}\} \times [0, \pi]$, such that

$$[0, \pi] \xrightarrow{\exp_y} C_{y,\epsilon} \to \{\tau_{u,\epsilon}\} \times [0, \pi] \xrightarrow{p_2} [0, \pi]$$

is the identity map and the corresponding map for y' coincides with $r : \theta \mapsto \pi - \theta$. We define $P_{u,y,\epsilon}$ as the restriction of P_u to $C_{y,\epsilon}$ and define similarly $P_{u,y',\epsilon}$. Parallel transport along lines $[\tau_{u,\epsilon}, \tau'_{u,\epsilon}] \times [0,\pi] \subset N_u$ using the connection A_u gives an isomorphism of bundles $g_{u,w',\epsilon} : P_{u,y,\epsilon} \to P_{u,y',\epsilon}$. Hence we have a diagram which may *not* be commutative:

$$\begin{array}{ccccc} P_y \times [0,\pi] & \xrightarrow{\iota_\epsilon} & P_{y,\epsilon} & \xrightarrow{\rho_u} & P_{u,y,\epsilon} \\ {\scriptstyle g_{w'}\times r}\downarrow & & & & \downarrow{\scriptstyle g_{u,w',\epsilon}} \\ P_{y'} \times [0,\pi] & \xrightarrow{\iota'_\epsilon} & P_{y',\epsilon} & \xrightarrow{\rho_u} & P_{u,y',\epsilon}, \end{array}$$

where $g_{w'}$ is the gluing data at w given by G. Define $\gamma_{u,w',\epsilon}(\theta) : P_y \to P_{y'}$ to be the restriction of $(\iota'_\epsilon)^{-1} \circ \rho_u^{-1} \circ g_{u,w',\epsilon} \circ \rho_u \circ \iota_\epsilon$ to $P_y \times \{\theta\}$ and take any distance function d on $\mathrm{Iso}(P_y, P_{y'})$, we required that

$$\lim_{\epsilon \to 0} \left(\limsup_{u \to \infty} d(g_{w'} \circ \gamma, \gamma_{u,w',\epsilon}(\theta)) \right) = 0$$

for all $\theta \in [0,\pi]$ and some γ in the stablizer of $\varphi(w') \in L$.

-*Limit behavior of the section.* We require that

$$\lim_{\tau \to \infty} \left(\limsup_{u \to \infty} \mathrm{diam}_{S^1} \phi_u \left(N_u^\tau(w') \right) \right) = 0. \tag{2.12}$$

(6) Convergence near the new boundary nodes of type 2. If w'' is a boundary node of type 2 in C, then $N_{w''}(\delta_{u,w''})$ is conformal to the long cylinder $N_u = [0, 2\ln\epsilon - \ln|\delta_{w''}|] \times S^1 = [0, r_u] \times S^1$. Then trivialize P_u over N_u with respect to which A_u is in balance temporal gauge, and the pair (A_u, φ_u) can be viewed as a pair (α_u, ϕ_u) with α_u a 1-form and $\phi_u : N_u \to X$. The following must hold:

-*Limit behavior of the connection.* Choose appropriately the balanced temporal gauge for all u, the restriction of α_u to $\{0\} \times S^1 \subset N_u$ is equal to $(\lambda + \lambda_u) d\theta$, for λ independent of u and $\lambda_u \to 0$. Also, the limit holonomy of A around w is equal to $e^{\pm 2\pi\lambda}$.

-*Limit behavior of the section.* The trivialization of P_u on N_u which puts A_u in balanced temporal gauge, induces a trivialization of $P_w = S^1 \times S^1$. Then the section $\mathcal{T}_w$ becomes a map $T : S^1 \to \mathcal{T}(X)$. Denote $N_u^\tau = S_u^\tau \times S^1 = [0, r_u - \tau] \times S^1$. We require that for any $\theta \in S^1$,

$$\lim_{\tau \to \infty} \left(\limsup_{u \to \infty} d_{Hauss} \left(\varphi_u(S_u^\tau \times \{\theta\}), T(\theta) \right) \right) = 0,$$

and either

$$\lim_{\tau\to\infty}\left(\limsup_{u\to\infty} d(\varphi_u(0,\theta), b(T(\theta))) + d(\varphi(r_u-\tau,\theta), e(T(\theta)))\right) = 0$$

or the same formula holds switching b and e (in one case $i\lambda_u$ is positive for large u and in the other case $i\lambda_u$ is negative for large u).

-*Limit behavior of the energy density.* We have

$$\lim_{\tau\to\infty}\limsup_{u\to\infty}\|d_{A_u}\phi_u\|_{L^\infty(N_u^\tau)} = 0.$$

(7) Convergence near the old interior nodes. We require for each old interior node $w \in C$ the same condition as in the condition (5) or (6) (depending on whether w is a connecting or tree node) in the definition of convergence of c-STHM in [39, Section 7.4].

(8) Convergence near the old boundary nodes of type 1. If a boundary node $w' \in C$ is of type 1, we require the following conditions to hold:

-*Limit behavior of the gluing data.* Since the isomorphism $\rho_{u,l}$ can be extended over y and y', which are the preimages of w' in the normalization $\hat{C}$, the gluing data of P_u over w' can be viewed as an element of $\mathrm{Iso}(P_y, P_{y'})$. We require that the sequence converges to that of the gluing data of P at w' modulo the stablizer of $\varphi(y)$.

-*Limit behavior of the connection and the section.* Let y and y' be the preimages of w in the normalization C'. Then they can be identified with the the preimages of w in the normalization of C_u, for every u. Take U_y (resp. $U_{y'}$) a small neighborhood of y (resp. y') in the normalization C'. Then it can be identified with a small neighborhood of y (resp. y') in C'_u. We require that the connections and the sections converges on U_y and $U_{y'}$ both in C^∞-topology.

(9) Convergence near the old boundary nodes of type 2. If a boundary node $w'' \in C$ is of type 2, then it must be a connecting node. Let $B_{w''} = \{b_1, \ldots, b_{d-1}\}$ be the index set of vanishing bubbles over w''. Denote $O_j = O_{u,b_j}$ and assume that O_{d-1} contains a boundary component of C_u. O_{d-1} might be sphere with a boundary node of type 2, in which case we denote y_j, y'_j to be the nodes on O_j with y_j on the side of boundary; or O_{d-1} is a disk bubble, in which case we denote y_j, y'_j to be the node on O_j for $j < d-1$. Then we require the following conditions to hold (note that in this case we don't have requirement on the gluing data, and the following holds either π_u^{co} collapses a disk at w'' or not):

-*The vanishing bubbles do vanish.* For each $b_j \in B_{w''}$ the energy $\|d_{A_u}\phi_u\|_{L^2(O_j)}$ converges to 0.

-*Limit behavior of the connections.* The covariant derivative d_{A_u} can be written as $d+\alpha_u$ where α_u is in temporal gauge in N_{y_0} and there is some $\lambda_u \in i\mathbb{R}$ such that $\lim_{t\to\infty} \alpha_u|_{N_{y_0}(t,\theta)} = \lambda_u d\theta$. And $\lambda_u \to \lambda$ as $u \to \infty$, where λ is a logarithm of the limit holonomy of A around w. Finally, if $B_z \neq \emptyset$, then $\lambda \in \Lambda^{cr}$ and for large u we have $\lambda_u \notin \Lambda_{cr}$.

-*Limit behavior of the sections.* We distinguish two cases.

1. Suppose that λ_u is generic for big enough u. Then there is a collection of equivariant and covariantly constant maps (with respect to the limit connection $A_{w''}$ on $P_{w''}$)

$$\mathcal{T}_{w'',0}, \dots, \mathcal{T}_{w'',d-1} : P_y \to \mathcal{T}(X),$$

such that $\mathcal{T}_{w''} = \cup_{j=0}^{d-1} \mathcal{T}_{w'',j}$ and the following holds.

We have trivialization $t_{u,w''} : P_{u,w''} \to S^1 \times S^1$ such that the diagram commutes

$$\begin{array}{ccc} P_u|_{N_{w''}} & \xrightarrow{\tau^{A_u}} & P_{u,w''} \\ {\scriptstyle \tau^P_{u,w''}}\downarrow & & \downarrow{\scriptstyle t_{u,w''}} \\ N_{w''} \times S^1 & \longrightarrow & S^1 \times S^1 \end{array}$$

where $\tau_{u,w''}$ is the given trivialization of P_u and the bottom map is the projection: $N_{w''} \times S^1 \simeq R_{\geq 0} \times S^1 \times S^1 \to S^1 \times S^1$. Under the trivialization $\tau^P_{u,w''}$, we can view φ_u on $N_{w''}$ as a map $\phi_{u,w''} : N_{w''} \to X$. Now let

$$T_{u,w'',\epsilon}(\theta) = \mathcal{T}_{w'',0} \circ \iota_\epsilon \circ \rho_u^{-1} \circ \iota_{u,\epsilon} \circ t^{-1}_{u,w''}(\theta, 1)$$

be a map $T_{u,w'',\epsilon} : S^1 \to \mathcal{T}(X)$. Here $\iota_{u,\epsilon}$ and ι_ϵ are bundle maps defined by parallel transports, and $\rho_u : P \to P_u$ is the isomorphism on smooth locus. We require that for any $\theta \in S^1$, we have

$$\lim_{\epsilon\to 0} \limsup_{\tau\to\infty} \left(\limsup_u d_{Hauss}(\phi_{u,w''}(S^\tau_u \times \{\theta\}), T_{u,w'',\epsilon}(\theta)) \right) = 0.$$

On the other hand, the energy of vanishing bubbles converges to 0 hence each vanishing bubble shrink to a map $\mathcal{T}^{new}_j : P'_{u,y_j} \to \mathcal{T}(X)$ for $j = 1, \dots, d-1$. More precisely, we define the isomorphism of bundles

$$\gamma_{u,j,\epsilon} := g_{u,j} \circ \cdots \circ g'_{u,1} \circ g_{u,1} \circ \iota^{-1}_{u,\epsilon} \circ \rho_u \circ \iota_\epsilon : P_{w''} \to P'_{u,j}.$$

Here $g_{u,j} : P_{u,y_{j-1}} \to P_{u,y'_j}$ is the gluing data, and $g'_{u,j} : P_{u,y'_j} \to P_{u,y_j}$ is the canonical

transport on the bubble $O_{u,j}$ we defined in Section 2.7.1. We require that for each j and $p \in P_{w''}$,

$$\lim_{\epsilon\to 0}\left(\limsup_{u\to\infty} d\left(\mathcal{T}_j^{new}(\gamma_{u,j,\epsilon}(p)), \mathcal{T}_{w'',j}(p)\right)\right) = 0.$$

2. Suppose that $\lambda_u = \lambda \in \Lambda_{cr}$ for big enough u. Then there is no vanishing bubble. And we have equivariant maps $\mathcal{T}_{u,w''} : P_{w''} \to \mathcal{T}(X)$. We require first that

$$\lim_{\tau\to\infty}\left(\limsup_{u\to\infty} \mathrm{diam}_{S^1}\varphi_u(N^\tau_{w''})\right) = 0.$$

Let $\mathcal{T}_{w''} : P_{w''} \to \mathcal{T}(X)$ be the chain of gradient segments of the limit $\mathcal{C}$. We require that for any $p \in P_w$, we have

$$\lim_{\epsilon\to 0}\left(\limsup_{u\to\infty} d\left(\mathcal{T}_{u,w''}\circ\iota_{u,\epsilon}^{-1}\circ\rho_u\circ\iota_\epsilon^{-1}(p), \mathcal{T}_{w''}(p)\right)\right) = 0.$$

Here ι_ϵ is the isomorphism between $P_{w''}$ and $P_{w'',\epsilon}$ given by parallel transport with respect to A and $\iota_{u,\epsilon}$ is the isomorphism between $P_{u,w''}$ and $P_{u,w'',\epsilon}$ given by parallel transport with respect to A_u. ρ_u is the isomorphism between $P_{w'',\epsilon}$ and $P_{u,w'',\epsilon}$.

(10) Limit behavior of the energy density. We have

$$\lim_{\tau\to\infty}\limsup_{u\to\infty}\|d_{A_u}\phi_u\|_{L^\infty(N^\tau_{w''})} = 0.$$

2.9 Compactness

We will prove the following theorem in this section.

Theorem 2.9.1. *Let g, h, n be nonnegative integers and $\overrightarrow{m}$ be a nonnegative vector such that $(g, h, n, \overrightarrow{m})$ is in the stable domain. Let $K > 0$ be any positive real number and let $c \in i\mathbb{R}$. Let $\{\mathcal{C}_u\}$ be a sequence of (c, L)-STHM's, of genus g, with h boundary components and $(n, \overrightarrow{m})$-marked, satisfying $\mathcal{YMH}_c(\mathcal{C}_u) \le K$. Then there is a subsequence $\{\mathcal{C}_{u_j}\}$ such that the sequence of isomorphism classes $\{[\mathcal{C}_{u_j}]\}$ converges to $[\mathcal{C}]$ for a c-stable twisted holomorphic maps with boundary.*

The remaining of this section is devoted to its proof. The strategy of the proof is very much in parallel with that of the proof of [39, Theorem 8.1]. In the following, we will take subsequences of the original sequence of (c, L)-STHM's many but finite times, and still index the subsequences by the same symbols. The situations near interior marked points and interior nodes are covered in the

proof of [39, Theorem 8.1], so we omit many details in the corresponding steps.

2.9.1 Getting the first limit curve

Let $(C_u, \mathbf{x}_u \cup \mathbf{z})$ be the prestable curve underlying $\mathcal{C}_u$. By Theorem 2.7.1, the number of bubbles of each C_u is uniformly bounded so we can assume (by taking a subsequence) that all curves $(C_u, \mathbf{x}_u \cup \mathbf{z}_u)$ have the same combinatorial type (including the topological type of C_u and the distribution of marked points into different components).

We want to add new marked points (as few as possible, in some sense) such that the prestable curve becomes stable. Let $\mathbf{x}_u^0 \subset C_u$ be a list of boundary points such that: 1) $\mathbf{x}_u^0$ is disjoint from boundary nodes and $\mathbf{x}_u$; 2) every component of $(C_u, \mathbf{x}_u \cup \mathbf{x}_u^0 \cup \mathbf{z}_u)$, which has at least a circle boundary component, is stable; 3) $\mathbf{x}_u^0$ has as few elements as possible, for each u. Then let $\mathbf{z}_u^0 \subset C_u$ be a list of interior points such that: 1) $\mathbf{z}_u^0$ is disjoint from interior nodes and $\mathbf{z}_u$; 2) $(C_u, \mathbf{x}_u \cup \mathbf{x}_u^0 \cup \mathbf{z}_u \cup \mathbf{z}_u^0)$ is stable; 3) $\mathbf{z}_u^0$ has as few elements as possible, for each u. Denote $\mathbf{x}_u' = \mathbf{x}_u \cup \mathbf{x}_u^0$ and $\mathbf{z}_u' = \mathbf{z}_u \cup \mathbf{z}_u^0$. The each element in the new sequence $(C_u, \mathbf{x}_u' \cup \mathbf{z}_u')$ has the same combinatorial type.

Then by taking a subsequence, we can assume that the isomorphism class of $(C_u, \mathbf{x}_u' \cup \mathbf{z}_u')$ converges to the isomorphism class of a stable curve $(C', \mathbf{x}' \cup \mathbf{z}')$ in $\overline{\mathcal{M}}_{g,h,n,\vec{m}}$ and we can assume that the set $\mathbf{x}'$ (resp. $\mathbf{z}'$) contains the limit $\mathbf{x}$ (resp. $\mathbf{z}$) of the sequence $\mathbf{x}_u$ (resp. $\mathbf{z}_u$). We call points in $\mathbf{x}$ (resp. $\mathbf{z}$) **original boundary (resp. interior) marked points**. In the following, the lists of marked points $\mathbf{x}_u'$ and $\mathbf{z}_u'$ may increase, and the limit curve C' may change accordingly.

2.9.2 Adding tree bubbles, the first step

We will proceed as in [39, Section 8.3]. Take an exhaustion $K_1 \subset K_2 \subset \cdots$ of the smooth locus of $C' \setminus \mathbf{x} \cup \mathbf{z}$ by compact subsets. And for each $K = K_l$ in the sequence, since $(C_u, \mathbf{x}_u' \cup \mathbf{z}_u')$ converges to $(C', \mathbf{x}' \cup \mathbf{z}')$, we can assume that there is a canonical inclusion

$$\iota_{K,u} : K \to C_u$$

with image K_u. We will do the following operations inductively from $l = 1$.

• If $\sup_{K_u} |d_{A_u}\varphi_u|$ is bounded as $u \to \infty$, then we do nothing. We call this situation that **no tree bubble appears in** K_l.

• If $\sup_{K_u} |d_{A_u}\varphi_u| = s_u$ is not bounded, then pick for each u a point $z_u^1 \in K_u$ such that $|d_{A_u}\varphi_u|$ attains its maximum at z_u^1. By passing to a subsequence, there are two possibilities, according to which we will proceed differently.

Case A. $\lim_{u\to\infty} z_u^1$ lies in the interior of C'. Then pick an $z_u^2 \in C_u$ at distance s_u^{-1} from z_u^1. Set $\mathbf{z}_u'' = \mathbf{z}_u' \cup \{z_u^1, z_u^2\}$ and $\mathbf{x}_u'' = \mathbf{x}_u'$.

Case B. $\lim_{u\to\infty} z_u^1 = x^1 \in \partial C'$. Then take a holomorphic local coordinate $f : U_{x^1} \to D(\epsilon) \subset \mathbb{H}$ such that $f(x^1) = 0$. There are two subcases (the division into the following two cases is independent of the choice of the local coordinate): a). $\lim_{u\to\infty} \left(s^u \mathrm{Im} z_u^1\right) = \infty$. Then take z_u^2 in the interior of C_u with distance s_u^{-1} from z_u^1. Take $\mathbf{z}_u'' = \mathbf{z}_u' \cup \{z_u^1, z_u^2\}$ and $\mathbf{x}_u'' = \mathbf{x}_u'$; b). $\lim_{u\to\infty} \left(s^u \mathrm{Im} z_u^1\right) < \infty$. Take $x_u^1 = f^{-1}\left(\mathrm{Re} f(z_u^1)\right)$ and $x_u^2 \in \partial C_u$ with distance s_u^{-1} from x_u^1. Take $\mathbf{x}_u'' = \mathbf{x}_u' \cup \{x_u^1, x_u^2\}$ and $\mathbf{z}_u'' = \mathbf{z}_u'$.

In any case, we get a new sequence of stable curves $(C_u, \mathbf{x}_u'' \cup \mathbf{z}_u'')$. By taking a subsequence, we can assume that the sequence converges to a stable curve $(C'', \mathbf{x}'' \cup \mathbf{z}'')$. In Case A, C'' is the union of C' with a sphere bubble B connected at a new interior node $z_0 = \lim_{u\to\infty} z_u^1 \in K$, the two point set $\{z_u^1, z_u^2\}$ converges to two new marked points on B; in Case B-a), C'' is the union of C' with a sphere bubble B and a disk bubble D, where D is connected to C' through $z_0 = \lim_{u\to\infty} z_u^1 \in K$ and B is connected to B through some z_1, the two point set $\{z_u^1, z_u^2\}$ converges to two new marked points on B; in Case B-b), C'' is the union of C' with a disk bubble D, connected at $z_0 = \lim_{u\to\infty} z_u^1 \in K$, and the two point set $\{x_u^1, x_u^2\}$ converges to two boundary marked points on D.[2]

In any case, we can prove as in [39] that the triple induced on B (in Case A and Case B-a) or on D (in Case B-b)) has positive energy and hence at least ϵ_X.

We call this situation that **a nontrivial tree bubble appears in** K_l.

The induction process is, if no tree bubble appears in K_l, then we go to K_{l+1}; if a nontrivial tree bubble appears in K_l, then we replace $(C', \mathbf{x}_u' \cup \mathbf{z}_u')$ by $(C'', \mathbf{x}_u'' \cup \mathbf{z}_u'')$, and modify $K_1, K_2, \ldots$, as follows, and start the process again from the modified K_1.

- Choose a descending sequence $\epsilon_i \to 0$. For any $y \in C'$, let $B(y, \epsilon_i) \subset C'$ be the open ball centered at y of radius ϵ_i.
- In Case A, replace K_i by $(K_i \cup B) \setminus B(z_0, \epsilon_i)$;
- in Case B-a), replace K_i by $(K_i \cup D \cup B) \setminus B(z_1, \epsilon_i)$;
- in Case B-b), replace K_i by $(K_i \cup D) \setminus B(z_0, \epsilon_i)$.

This induction process must stops at finite times, which means the following. We have added finitely many marked points on $(C_u', \mathbf{x}_u' \cup \mathbf{z}_u')$ in the same pattern for all u, such that the new sequence

[2]There is a special case, i.e., if z_u^1 coincides with some newly added marked point subsection 2.9.1, or converges to some new marked points in 2.9.1. But we can choose the new marked points wisely to avoid this happen.

of stable marked curves has a subsequence (still denoted by the same symbols) converging to some stable curve $(C'', \mathbf{x}'' \cup \mathbf{z}'')$, such that for any compact subset K of the smooth locus of $C'' \setminus \mathbf{x}'' \cup \mathbf{z}''$, which is identified with $K_u \subset C'_u$, we have

$$\limsup_{u\to\infty} \sup_{K_u} |d_{A_u}\varphi_u| < \infty.$$

Or in other words, no tree bubble appears in K_l for all l.

We replace C' by C'', $\mathbf{x}'_u$ by $\mathbf{x}''_u$, $\mathbf{z}'_u$ by $\mathbf{z}''_u$, $\mathbf{x}'$ by $\mathbf{x}''$ and $\mathbf{z}'$ by $\mathbf{z}''$. We can still distinguish original marked points from the new marked points.

2.9.3 Adding tree bubbles, the second step

Bubbling off at new boundary nodes of type 1

We first consider boundary nodes. If $w' \in C'$ is a new boundary node of type 1. Then there is a long strip $N_{u,w'} = [\ln|\delta_{u,w'}| - \ln\epsilon, \ln\epsilon] \times [0,\pi] =: [p_u, q_u] \times [0,\pi]$ which can be identified with a subset of C_u for each u. Denote $N_u^\tau := [p_u + \tau, q_u - \tau] \times [0,\pi]$. Suppose that

$$\limsup_{\tau\to\infty} \left(\limsup_{u\to\infty} \|d_{A_u}\varphi_u\|_{L^\infty(N_u^\tau)} \right) = \infty$$

where we also use the standard cylindrical metric. Then we can pick $z_u^1 \in N_u$ such that

$$d(z_u^1, \{p_u, q_u\} \times [0,\pi]) \to \infty,\ s_u = \left|d_{A_u}\varphi_u(z_u^1)\right| \to \infty$$

and s_u is equal to the supreme of $|d_{A_u}\varphi_u|$ over the disk centered at z_u^1 of radius 1. We distinguish the following cases (by taking subsequences):

1. $\lim_{u\to\infty} s_u d(z_u^1, \partial N_u) = \infty$. Then take z_u^2 lying in the interior of N_u with distance s_u^{-1} to z_u^1 and $\mathbf{z}''_u = \mathbf{z}'_u \cup \{z_u^1, z_u^2\}$ and $\mathbf{x}''_u = \mathbf{x}'_u$;

2. $\lim_{u\to\infty} s_u d(z_u^1, \partial N_u) < \infty$. Suppose $z_u^1 = (a_u, \theta_u)$. We can assume that $\theta_u \to 0 (\theta_u \to \pi$ is similar). Then take $x_u^1 = (a_u, 0)$ and $x_u^2 = (a_u + s_u^{-1}, 0)$; set $\mathbf{x}''_u = \mathbf{x}'_u \cup \{x_u^1, x_u^2\}$ and $\mathbf{z}''_u = \mathbf{z}'_u$.

In any case, a subsequence of the curves C_u with the new set of marked points will converge to some stable curve C''. In the first case, C'' containes a sphere tree bubble on which the limit of $\{z_u^1, z_u^2\}$ are attached.

In the second case, C'' is the union of C' with two disk bubbles, one of which is connecting and one of which is a tree bubble; the tree bubble has two boundary marked points which are limits of x_u^1 and x_u^2.

Each of the tree bubble in the two cases contributes $\geq \epsilon_X$ to the total energy. So if we update the set of marked points as what we did before and repeat the process, then we must stop at finite times.

Bubbling off at old boundary nodes of type 1 and original boundary marked points

This case is much similar to the previous one. The only difference is that instead of having a strip of finite length, we will have a semiinfinite strip $[0, \infty) \times [0, \pi]$.

Bubbling off at new boundary nodes of type 2

Now consider new boundary nodes of type 2. Let $w'' \in C'$ be such a node. Then the "neck" of w'' in C_u is a cylinder conformal to $N_{u,w''} = [0, r_u] \times S^1$ with $r_u \to \infty$ and $S(u) = \{r_u\} \times S^1$ the boundary circle that shrink to w''. For each $\tau > 0$, denote $N_u^\tau := N_{u,w''}^\tau = [\tau, r_u] \times S^1$. Assume that

$$\limsup_{\tau \to \infty} \left(\limsup_{u \to \infty} \sup_{N_u^\tau} |d_{A_u} \varphi_u| \right) = \infty.$$

Then, we can take a sequence of points $z_u^1 = (t_u, \theta_u) \in N_u$, such that $t_u \to \infty$ and $s_u = |d_{A_u} \varphi_u(z_u)| \to \infty$ as $u \to \infty$. By taking a subsequence, we consider the following two cases:

1. If $\lim_{u \to \infty} s_u(r_u - t_u) = \infty$, then take another sequence of points $z_u^2 \in N_u$ such that $d(z_u, z_u^2) = s_u^{-1}$, set $\mathbf{z}_u'' = \mathbf{z}_u' \cup \{z_u^1, z_u^2\}$ and $\mathbf{x}_u'' = \mathbf{x}_u'$.

2. If $\lim_{u \to \infty} s_u(r_u - t_u) < \infty$, then take another sequence of points $x_u^1 = (r_u, \theta_u), x_u^2 = (r_u, \theta_u + s_u^{-1}) \in S(u)$. Set $\mathbf{z}_u'' = \mathbf{z}_u'$ and $\mathbf{x}_u'' = \mathbf{x}_u' \cup \{x_u^1, x_u^2\}$.

Then the new sequence of curves $(C_u, \mathbf{x}_u'' \cup \mathbf{z}_u'')$ will subsequentially converge to some stable curve $(C', \mathbf{x}'' \cup \mathbf{z}'')$: in the first case, C' will contain a sphere bubble on which the limits of z_u^1 and z_u^2 are attached (see Figure 2.2); in the second case, C' will contain a disk bubble on which the limits of $x_u^1$and x_u^2 are attached, and a disk connecting bubble (see Figure 2.3).

Bubbling off at old boundary nodes of type 2

Let $w'' \in C'$ be an old boundary node of type 2. Bubbling off at w'' is the same as the bubbling off at original interior marked points which was treated in [39, Subsection 8.4].

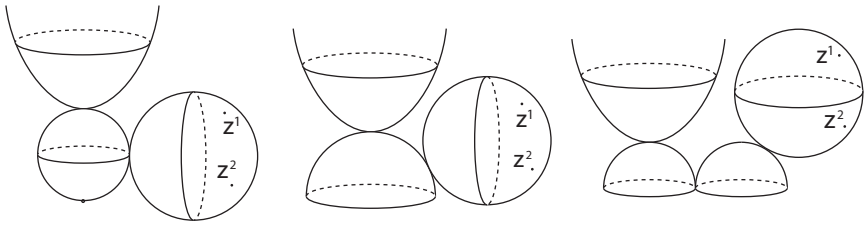

Figure 2.2: Limit curves when bubbling-off a sphere at a new type-2 boundary node, three different cases

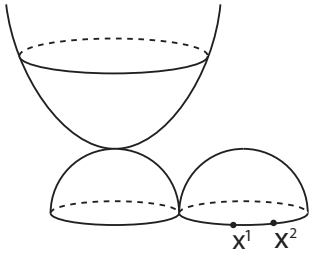

Figure 2.3: Limit curve when bubbling-off a disk bubble at a new type-2 boundary node

Tree bubbles, summary

There can still be tree bubbles which arises near old interior nodes, original interior marked points and new interior nodes. But they can be treated in the same way as in [39, Subsection 8.4] and we omit this part.

We now summarize what we have done after all the steps in this subsection. We can modify the exhausting sequence of compact subsets and repeat the process in Section 2.9.2. Then all these stop after finite times and we have found every possible tree bubble. This means the: by taking a subsequence and adding new marked points, we get a new sequence of stable curves $(C_u, \mathbf{x}_u \cup \mathbf{z}_u)$, which converges to $(C, \mathbf{x} \cup \mathbf{z})$. We have

1. For each compact subset $K \subset C \setminus \mathbf{x} \cup \mathbf{z} \cup \mathbf{w}$, which is identified with a compact subset of each C_u, we have
$$\limsup_{u \to \infty} \|d_{A_u} \varphi_u\|_{L^\infty(K_u)} < \infty.$$
Here the norm is taken with respect to the conformal metric on K.

2. Let $\{K_l\}_{l \geq 1}$ exhaust $C \setminus \mathbf{x} \cup \mathbf{z} \cup \mathbf{w}$. The complement of K_l in C_u is a union of "cylindrical-like" pieces, which is denoted by $N_{u,l}$. Then we have that
$$\limsup_{l \to \infty} \left(\limsup_{u \to \infty} \|d_{A_u} \varphi_u\|_{L^\infty(N_{u,l})} \right) < \infty.$$

Here the norm is taken with respect to the cylindrical metric.

2.9.4 Connecting bubbles appear

The matching conditions at various types of nodes are fulfilled only when the energy densities at the neck regions are very small. The bubbling-off of tree bubbles cannot guarentee this condition so we have to find more connecting bubbles.

At interior nodes

For each new interior node $w \in C'$, we do the same operation as in [39, Subsection 8.5]. Let's just briefly explain what happens.

Let N_u be the neck associated to the new node w. Its conformal length goes to infinity as $u \to \infty$, and we use d to denote the distance function for the cylindrical metric on N_u. For ϵ smaller than the ϵ_1 in [39, Theorem 4.1] and ϵ_2 in [39, Theorem 4.3], we define an **ϵ-bubbling list** to be a sequence of lists $\{(z_1^u, \dots, z_r^u)\}_u$ which satisfies

1. $z_j^u \in N_u$;
2. for each $i \neq j$, $d(z_i^u, z_j^u) \to \infty$ and $d(z_i^u, \partial N_u) \to \infty$ as $u \to \infty$;
3. for each j, $\liminf_{u\to\infty} \left|d_{A_u}\varphi_u(z_j^u)\right| \geq \epsilon$.

Then we can prove the following lemma:

Lemma 2.9.2. *[39, Lemma 8.2] There is a constant $\kappa_b > 0$ depending only on X, ω, I, such that if $\{(z_1^u, \dots, z_r^u)\}_u$ is an ϵ-bubbling list, then $r \leq \kappa_b \sup_u \mathcal{YMH}_c(\mathcal{C}_u)$.*

Then for each new interior node $w \in C'$, we take an ϵ-bubbling list of maximal length. If the list is empty, then we do nothing. Otherwise, add to the marked point sets $\mathbf{z}'_u$ the points $z_1^u, \dots, z_r^u$. Passing to a subsequence, we may assume that the new sequence of stable curves converges to some stable curve $(C'', \mathbf{x}'' \cup \mathbf{z}'')$, by which we replace $(C', \mathbf{x}' \cup \mathbf{z}')$.

We do a similar process for each old interior node, each old boundary node of type 2, and each original interior marked point of C'.

At type-1 boundary nodes

Now we consider new boundary nodes of type 1. Let $w \in C'$ be such a node. $N_u := [p_u, q_u] \times [0, \pi]$ be the corresponding neck in C_u. We assume that N_u is contained in a principal component of

$(C_u, \mathbf{x}_u \cup \mathbf{z}_u)$. Let the volume form on N_u be $f_u dt \wedge d\theta$. Then for any $l > 0$, there is a constant κ_l independent of u such that

$$\left|\nabla^l f_u(z)\right| \leq \kappa_l e^{-\min\{t-p_u, q_u - t\}}.$$

Take any trivialization of P_u over N_u, and write $d_{A_u} = d + \alpha_u$. Since (α_u, ϕ_u) satisfies the vortex equation, we have

$$d\alpha_u = f_u(c - \mu(\phi_u))dt \wedge d\theta.$$

Let $\epsilon > 0$ be smaller than the ϵ_5 in Corollary 2.5.11. We define an **ϵ-bubbling list** to be a sequence of lists $\{(y_1^u, \ldots, y_r^u)\}_u$ satisfying

1. $y_j^u \in N_u$ and $\min\left\{d(y_j^u, \{p_u\} \times [0,\pi]), d(x_j^u, \{q_u\} \times [0,\pi])\right\} \to \infty$ as $u \to \infty$;
2. for each $i \neq j$, $d(y_i^u, y_j^u) \to \infty$ as $u \to \infty$;
3. for each j we have $\liminf_{u\to\infty} \left|d_{\alpha_u}\phi_u(y_j^u)\right| \geq \epsilon$.

Lemma 2.9.3. *There is some constant $\kappa_d > 0$ depending only on X, ω and I, such that for any ϵ-bubbling list $\{(y_1^u, \ldots, y_r^u)\}$, $r \leq \kappa_b \mathcal{YMH}_c(C_u)$.*

Proof. We can claim that there exists $\delta > 0$ such that if $(\alpha, \phi) : S \to X$ is a twisted holomorphic pair from the unit square S to X with , then

$$\|\alpha\|_{W^{3,p}(S)} \leq \delta,\ \|d_\alpha \phi\|_{L^2(S)} \leq \delta \Rightarrow \|d_\alpha \phi\|_{L^\infty(S)} < \epsilon.$$

If the claim is not true, then there exists a sequence of pairs $(\alpha_j, \phi_j) : S \to X$ such that $\|\alpha_j\|_{W^{3,p}} \to 0$, $\left\|d_{\alpha_j}\phi_j\right\|_{L^2} \to 0$ with $\left\|d_{\alpha_j}\phi_j\right\|_{L^\infty} \geq \epsilon$. By [39, Lemma 3.4], passing to a subsequence, ϕ_j converges to some $\phi : S \to X$ in $W^{2,p}$-topology. Hence $\|d\phi\|_{L^2} = 0$, i.e. $d\phi \equiv 0$. But $d_{\alpha_j}\phi_j \to d\phi$ in C^0-topology, this contradicts with $\left\|d_{\alpha_j}\phi_j\right\|_{L^\infty} \geq \epsilon$.

Now suppose that $\{(y_1^u, \ldots, y_r^u)\}_u$ is an ϵ-bubbling list and let $S_u^j \subset N_u$ be a unit square that contains y_j^u. It follows from the definition of ϵ-bubbling lists that for u big enough, S_u^j are pairwise disjoint. We then can deduce that $\|\alpha_u\|_{W^{3,p}(S_u^j)} \to 0$ for each j. Then by the previous claim, we must have $\|d_{\alpha_u}\phi_u\|_{L^2(S_u^j)} \geq \delta$. Hence $\kappa_d = \delta^{-1}$ is qualified for this lemma. □

Then let $\{(y_1^u, \ldots, y_r^u)\}_u$ be an ϵ-bubbling list of maximal length. If the list is empty, then we do nothing. If not, suppose $y_j^u = (a_j^u, \theta_j^u)$ and set $\mathbf{x}_u'' = \mathbf{x}_u' \cup \left\{x_j^u = (a_j^u, 0) | 1 \leq j \leq r\right\}$. Passing to a subsequence, we may assume that the sequence $(C_u, \mathbf{x}_u'' \cup \mathbf{z}_u')$ converges to some stable curve

$(C'', \mathbf{x}'' \cup \mathbf{z}')$. The curve C'' obtained from C' by adding a sequence of disk connecting bubbles at w (see Figure 2.4).

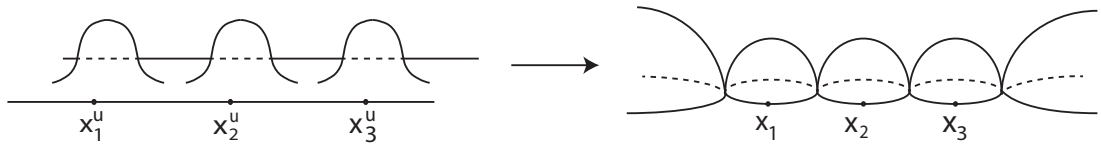

Figure 2.4: An ϵ-bubbling list at a new type-1 boundary node

We repeat this process for each new boundary node of type 1, and do a similar operation to each old boundary nodeof type 1. Finally, we do a similar operation to each original boundary marked point.

At type-2 boundary nodes

Let w'' be a new boundary node of type 2 and $N_u = [0, r_u] \times S^1$ be the corresponding neck in C_u. Let (α_u, ϕ_u) be the corresponding twisted pair of (A_u, φ_u) under some local trivialization. We define an **ϵ-bubbling list** to be a sequence of lists $\{(z_1^u, \ldots, z_s^u)\}_u$ such that

1. $z_j^u = (t_j^u, \theta_j^u) \in N_u$, $t_1^u < t_2^u < \ldots < t_r^u$ and $\lim_{u\to\infty} t_j^u = \infty$;

2. for each $i \neq j$, $t_i^u - t_j^u \to \infty$;

3. for each j we have $\liminf_{u\to\infty} \left|d_{\alpha_u}\phi_u(z_j^u)\right| \geq \epsilon$.

In the same way, we have a uniform upper bound of the number r for an ϵ-bubbling list at any new boundary node of type 2.

Now let $\{(z_1^u, \ldots, z_r^u)\}_u$ be an ϵ-bubbling list of maximal length. If $r = 0$, then we do nothing. Otherwise, we distinguish two cases (by taking a subsequence).

• $\lim_{u\to\infty} r_u - t_r^u = \infty$. Then we take two points $x_{r+1}^u, x_{r+1}^{u'} \in \{r_u\} \times S^1 \subset N_u$ such that $d(x_{r+1}^u, x_{r+1}^{u'}) = 1$. Then add to the marked point set $\mathbf{z}_u'$ the points $z_1^u, \ldots, z_r^u$ and add to $\mathbf{x}_u'$ the points $x_{r+1}^u, x_{r+1}^{u'}$. Then a subsequence of the new sequence of stable curves converges to some stable curve $(C'', \mathbf{z}'' \cup \mathbf{x}'')$, which looks like Figure 2.5.

• $\lim_{u\to\infty} r_u - t_r^u < \infty$. Then take $x_r^u = (r_u, \theta_r^u) \in N_u$ and $x_r^{u'} = (r_u, \theta_r^u + 1)$. Add to the marked point set $\mathbf{z}_u'$ the points $z_1^u, \ldots, z_{r-1}^u$ and add to $\mathbf{x}_u'$ the points $x_r^u, x_r^{u'}$. Then a subsequence of the new sequence of stable curves converges to some stable curve $(C'', \mathbf{z}'' \cup \mathbf{x}'')$, which looks like Figure 2.6.

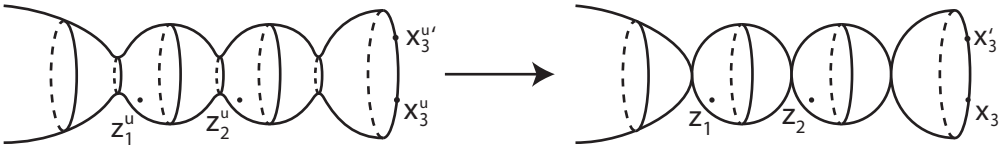

Figure 2.5: An ϵ-bubbling list at a new type-2 boundary node with $r = 2$

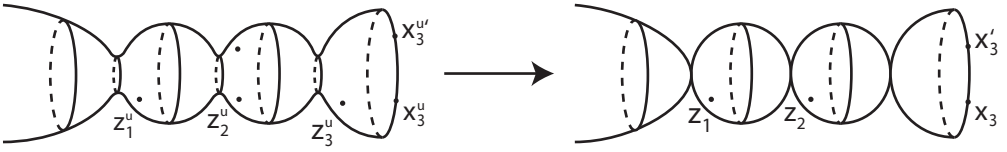

Figure 2.6: An ϵ-bubbling list at a new boundary node with $r = 3$

2.9.5 Constructing the limit, first part

After all the above process, we obtain a stable curve $(C', \mathbf{x}' \cup \mathbf{z}')$. We have $\mathbf{x} \subset \mathbf{x}'$ (resp. $\mathbf{z} \subset \mathbf{z}'$) which are the limit $\mathbf{x}_u$ (resp. $\mathbf{z}_u$). We may assume that for any u, there is a $(C', \mathbf{x} \cup \mathbf{z})$-admissible complex structure I_u and smoothing parameters $\{\delta_u\}$, and biholomorphism $\xi_u' : C_u \to C'(I_u, \delta_u)$. Moreover, I_u converges to I on C'.

Constructing the bundle, the connection and the section

Take a compact set $K \subset C' \setminus \mathbf{z}$ disjoint from nodes, denote $\iota_{K,u} : K \to C'(I_u, \delta_u)$ the canonical inclusion with image K_u. Since we have applied the previous processes, we can assume that

$$\sup_u \sup_{K_u} |F_{A_u}| < \infty, \quad \sup_u \sup_{K_u} |d_{A_u} \varphi_u| < \infty.$$

Then standard arguments can be applied to deduce that there is an S^1-principal bundle P over the smooth locus of $C' \setminus \mathbf{z}$, a smooth connection A on P, a smooth section of $P \times_{S^1} X$ and bundle isomorphisms $\rho_{K,u} : P|_K \to \iota_{K,u}^* P_u|_{K_u}$, such that $\rho_{K,u}|_{K'} = \rho_{K',u}$ for $K' \subset K$ and $\rho_{K,u}^* \iota_{K,u}^* (A_u, \varphi_u)$ converges to (A, φ) in $C^\infty(K)$.

The convergence implies that $\overline{\partial}_A \varphi = 0$, and that on each principal components of C', the pair (A, φ) satisfies the equation

$$\Lambda F_A + \mu(\varphi) = c,$$

where Ω is the canonical volume form on $(C^{st}, \mathbf{x}^{st} \cup \mathbf{z}^{st})$. It follows that the curvature of A is bounded (since μ is bounded) and hence A is a meromorphic connection. On the other hand, the restriction of A to each bubble component of C' is flat. Moreover, $\|d_A\varphi\|_{L^2}$ is finite.

By the removal of boundary singularity (Theorem 2.3.6), we can assume that the triple (P, A, φ) is extended to each pair y and y', where $\{y, y'\}$ is the preimage of any boundary node w' of type 1.

We have to delete the new marked points, which may violate the stability because then there might be unstable connecting bubbles with zero energy. Let $\{O_b\}$ be the collection of those bubbles in C' and define C to be the quotient $C'/\cup O_b$. The marked point set $\mathbf{x} \cup \mathbf{z}$ descends to a list of points in C via the quotient map. Then $(C, \mathbf{x} \cup \mathbf{z})$ is a prestable curve. This will be the support of our limit. The triple (P, A, φ) descends to a triple over $(C, \mathbf{x} \cup \mathbf{z})$, which we denote by the same symbols.

Constructing the gluing data

The method of constructing gluing data on [39, p. 1165-1166] can be used here to construct gluing data at new interior nodes and new boundary nodes of type 1. For new interior nodes, it is the same as in [39]. So we only define the gluing data at new boundary nodes of type 1.

Recall the definition of $\gamma_{u,w,\epsilon}(\theta) : P_y \to P'_y$, which only requires the convergence of the underlying curve. Let d be a distance function on $\mathrm{Iso}(P_y, P_{y'})$. By the convergence of the connection, we have that

$$\lim_{\epsilon\to 0}\left(\sup_{0<\epsilon_1<\epsilon_2<\epsilon}\limsup_{u\to\infty} d(\gamma_{u,w,\epsilon_1}(\theta), \gamma_{u,w,\epsilon_2}(\theta))\right) = 0.$$

By Stokes' theorem, we also have that for $\theta_1, \theta_2 \in [0, \pi]$,

$$\lim_{\epsilon\to 0}\left(\limsup_{u\to\infty} d(\gamma_{u,w,\epsilon}(\theta_1), \gamma_{u,w,\epsilon}(\theta_2))\right) = 0.$$

Thus there exists a unique isomorphism $g_w : P_y \to P_{y'}$ such that

$$\lim_{\epsilon\to 0}\left(\limsup_{u\to\infty} d(g_w, \gamma_{u,w,\epsilon}(\theta))\right) = 0$$

for all $\theta \in [0, \pi]$. We define the gluing data at the node w to be g_w, and the convergence condition for gluing data at the new node w is automatically satisfied.

2.9.6 Constructing the limit, second part

We construct chains of gradient segments here. The construction of chains of gradient segments at interior nodes is given in [39, Section 8.7, 8.8]. Then we only consider the case at type-2 boundary nodes.

Let w'' be such a node of C. If w'' is a new node, and let $\lambda \in i\mathbb{R}$ be the residue of A at w'', with respect to some trivialization. Then we distinguish two cases:

• $\lambda \notin \Lambda_{cr}$. Then the limit orbit of ϕ at w'' is in the fixed point set. Let $N_u = [0, r_u] \times S^1$ be the neck which shrink to w'' and $N_u^\tau = [\tau, r_u] \times S^1$. Theorem 2.4.1 implies that

$$\lim_{\tau \to \infty} \left(\limsup_{u \to \infty} \operatorname{diam}_{S^1}(\varphi_u(N_u^\tau)) \right) = 0.$$

Then we can define the chain of gradient segment to be the degenerate one.

• $\lambda \in \Lambda_{cr}$. In this case the chain of gradient segments is given essentially by Theorem 2.4.2, 2.4.3 and [39, Theorem 4.5], which is the same as the construction in [39, Section 8.7].

If w'' is an old node, then the construction of the chain of gradient segments at w'' is just a tautology of the definition of the convergence of (c, L)-STHM's near old type-2 boundary nodes.

It is easy to check that one end of the chain of gradient segments lies in L, which satisfies our boundary condition in this case.

2.9.7 The limit is a (c, L)-STHM

So far we have constructed every ingredient necessary for a (c, L)-STHM. However, the definitions of (c, L)-STHM's and their convergence are so long that they deserve careful checking. We check in this subsection that the object we have constructed is indeed a (c, L)-STHM and the readers can check themselves that this (c, L)-STHM is indeed the limit of the subsequence we have chosen.

The first 3 conditions of being a (c, L)-STHM are automatically satisfied by the limit $\mathcal{C}$. The finite energy condition and the stability condition are also established when we construct the underlying curve of $\mathcal{C}$. The condition the the holonomis at interior marked points being critical is covered in [39].

Monotonicity of chains of gradient segments

The monotonicity states that, for an interior node or type-2 boundary node $w \in C^{st}$, the chains of gradient segments connecting the connecting bubbles should be all downward or all upward. In the

interiod node case, it has a proof in [39, Section 8.7]; in the case of type-2 boundary node, it follows from a similar estimate. We omit the details.

Stablizer of chains of gradient segments

This condition is also automatically satisfied when the monotone chains of gradient segments appear.

Matching conditions

Among various matching conditions, the only one which is not quite clear is that at boundary nodes of type 1.

If $w' \in C$ is an old boundary node of type 1, then the convergence $\rho_{K,u}^* \iota_{K,u}^* (A_u, \phi_u)$ to (A, ϕ) can be extended over the preimages of w' in the normalization of C. By passing to a subsequence, we can also assume that the gluing data at w' converges. Hence the matching condition at w' is preserved in the limit.

Now suppose $w' \in C$ is a new boundary node of type 1. Let y, y' be its preimages in the normalization C'. Recall the setting and the diagram

$$\begin{array}{ccccc} P_y \times [0,\pi] & \xrightarrow{\iota_\epsilon} & P_{y,\epsilon} & \xrightarrow{\rho_u} & P_{u,y,\epsilon} \\ \downarrow{\scriptstyle g_{w'} \times r} & & & & \downarrow{\scriptstyle g_{u,w',\epsilon}} \\ P_{y'} \times [0,\pi] & \xrightarrow{\iota'_\epsilon} & P_{y',\epsilon} & \xrightarrow{\rho_u} & P_{u,y',\epsilon}. \end{array}$$

The gluing data at w' is defined to be the limit

$$g_{w'} := \lim_{\epsilon \to 0} \left(\lim_{u \to \infty} \gamma_{u,w',\epsilon}(\theta) \right).$$

Let $\phi_{y,\epsilon}$ (resp. $\phi_{y',\epsilon}$) to be the restriction of ϕ to $P_{y,\epsilon}$ (resp. $P_{y',\epsilon}$). We have that for any $p \in P_y$ and $\theta \in [0, \pi]$

$$\lim_{\epsilon \to 0} d\left(\phi_y(p), \phi_{y,\epsilon} \circ \iota_\epsilon(p, \theta)\right) = 0 \tag{2.13}$$

and that $\lim_{u \to \infty} \phi_{u,y,\epsilon} \circ \rho_u \to \phi_{y,\epsilon}$. Moreover,

$$\lim_{\epsilon \to 0} \limsup_{u \to \infty} d\left(\mathrm{Im}\phi_{u,y,\epsilon}, \mathrm{Im}\phi_{u,y',\epsilon}\right) = 0.$$

Hence for any $p \in P_y$, $\theta \in [0, \pi]$,

$$
\begin{aligned}
& d\left(\phi_y(p), \phi_{y'}(g_w(p))\right) \\
\leq\ & d\left(\phi_y(p), \phi_{y,\epsilon} \circ \iota_\epsilon(p,\theta)\right) + d\left(\phi_{y,\epsilon} \circ \iota_\epsilon(p,\theta), \phi_{u,y,\epsilon} \circ \rho_u \circ \iota_\epsilon(p,\theta)\right) \\
+\ & d\left(\phi_{u,y,\epsilon} \circ \rho_u \circ \iota_\epsilon(p,\theta), \phi_{u,y',\epsilon} \circ g_{u,z,\epsilon} \circ \rho_u \circ \iota_\epsilon(p,\theta)\right) \\
+\ & d\left(\phi_{u,y',\epsilon} \circ g_{u,z,\epsilon} \circ \rho_u \circ \iota_\epsilon(p,\theta), \phi_{y',\epsilon} \circ \rho_u^{-1} \circ g_{u,z,\epsilon} \circ \rho_u \circ \iota_\epsilon(p,\theta)\right) \\
+\ & d\left(\phi_{y',\epsilon} \circ \rho_u^{-1} \circ g_{u,z,\epsilon} \circ \rho_u \circ \iota_\epsilon(p,\theta), \phi_{y'} \circ \gamma_{u,w,\epsilon}(p,\theta)\right) + d\left(\phi_{y'} \circ \gamma_{u,w,\epsilon}(p,\theta), \phi_{y'} \circ g_w(p)\right).
\end{aligned}
$$

Apply $\limsup_{u\to\infty}$ to the right hand side, one obtains

$$
\begin{aligned}
& d\left(\phi_y(p), \phi_{y'}(g_w(p))\right) \\
\leq\ & d\left(\phi_y(p), \phi_{y,\epsilon} \circ \iota_\epsilon(p,\theta)\right) + \limsup_{u\to\infty} d\left(\phi_y \circ \iota_\epsilon(p,\theta), \phi_{u,y',\epsilon} \circ g_{u,z,\epsilon} \circ \rho_u \circ \iota_\epsilon(p,\theta)\right) \\
+\ & \limsup_{u\to\infty} d\left(\phi_{y',\epsilon} \circ \rho_u^{-1} \circ g_{u,z,\epsilon} \circ \rho_u \circ \iota_\epsilon(p,\theta), \phi_{y'} \circ \gamma_{u,w,\epsilon}(p,\theta)\right) \\
+\ & \limsup_{u\to\infty} d\left(\phi_{y'} \circ \gamma_{u,w,\epsilon}(p,\theta), \phi_{y'} \circ g_w(p)\right) \\
\leq\ & d\left(\phi_y(p), \phi_{y,\epsilon} \circ \iota_\epsilon(p,\theta)\right) + d\left(\phi_y \circ \iota_\epsilon(p,\theta), \phi_{y'}(P_{y',\epsilon})\right) + d\left(\phi_{y'}(P_{y',\epsilon}), \phi_{y'}(P_{y'})\right).
\end{aligned}
$$

Then let $\epsilon \to 0$, we see that $\phi_y = \phi_{y'} \circ g_w$.

Chapter 3

Morse theory for Lagrange multipliers and adiabatic limits

3.1 Introduction to this chapter

Hamiltonian Floer homology was invented by Andreas Floer, based on Witten's viewpoint on Morse theory [53], to study the fixed points of a Hamiltonian diffeomorphism ϕ on a symplectic manifold (M, ω). The Floer theory of a generic Hamiltonian diffeomorphism (or Hamiltonian isotorpy, Hamiltonian path) can also be viewed as a Morse perturbation of the corresponding Morse-Bott theory: pseudoholomorphic curves. It is then natural to consider gauged σ-model in this setting, by imposing a generic, G-invariant Hamiltonian perturbation on the symplectic vortex equation. This was originally proposed in [8].

While it is an ongoing work of the author to develop this theory in full generality, it is interesting to notice that some special phenomenon of this theory can be viewed in a much simpler model. The adiabatic limits of the Floer homology group in this theory should coincide with the Floer homology group of the symplectic quotient at the level $\mu = 0$ with the Hamiltonian descended from upstairs. This will be an analogue of the result in [17]. On the other hand, one may wonder about the limit in the opposite direction. For example, in the setting of Hamiltonian Gromov-Witten invariants (or gauged Gromov-Witten invariants), González-Woodward studied this limit in genus zero case (see [21]).

However, since it is an infinite-dimensional and equivariant theory, it is too complicated to see clearly about the limiting behavior of this theory. Instead, we consider a finite dimensional model

theory, which still contains a real paramter and the variation of the parameter can give two different limit theory. This is described as follows.

3.1.1 Basic set-up

The Lagrange multiplier function

Let M be a compact manifold. Suppose f and μ are Morse functions on M, and 0 is a regular value of μ. Then we consider the Lagrange multiplier function

$$\begin{array}{rccc} \mathcal{F}: & M \times \mathbb{R} & \to & \mathbb{R}, \\ & (x, \eta) & \mapsto & f(x) + \eta\mu(x). \end{array}$$

The critical point set of $\mathcal{F}$ is

$$\mathrm{Crit}(\mathcal{F}) = \{(x, \eta) \mid \mu(x) = 0, df(x) + \eta d\mu(x) = 0\},$$

and there is a bijection

$$\begin{array}{ccc} \mathrm{Crit}(\mathcal{F}) & \simeq & \mathrm{Crit}\left(f|_{\mu^{-1}(0)}\right), \\ (x, \eta) & \mapsto & x. \end{array}$$

We consider the Morse theory of $\mathcal{F}$. Take a Riemannian metric g on M and the standard Euclidean metric e on $\mathbb{R}$. Denote by g_1 the product metric $g \oplus e$ on $M \times \mathbb{R}$. Then the gradient vector field of $\mathcal{F}$ with respect to g_1 is

$$\nabla\mathcal{F}(x, \eta) = (\nabla f + \eta\nabla\mu, \mu(x)).$$

The differential equation for the negative gradient flow of $\mathcal{F}$ is

$$\begin{cases} x' + \nabla f(x) + \eta\nabla\mu(x) = 0; \\ \eta' + \mu(x) = 0. \end{cases} \tag{3.1}$$

Let $p_\pm = (x_\pm, \eta_\pm) \in \mathrm{Crit}(\mathcal{F})$, and let $\mathcal{M}(p_-, p_+)$ be the moduli space of trajectories of the negative gradient flow that approach $p_\pm$ as $t \to \pm\infty$ respectively. Then for generic choice of data (f, μ, g),

this moduli space will be a smooth manifold with dimension

$$\dim \mathcal{M}(p_-, p_+) = \text{index}(p_-) - \text{index}(p_+) - 1.$$

When the dimension is zero, we can count the number of elements of the corresponding moduli space and use the counting to define the Morse-Smale-Witten complex.

Adiabatic limits

We can show that if we rescale the metric on the $\mathbb{R}$-part, by $g_\lambda := g \oplus \lambda^{-2} e$, $\lambda \in \mathbb{R}^+$ we get the same homology groups, for generic λ for which the chain complex is defined. The system (3.1) is replaced by

$$\begin{cases} x' + (\nabla f(x) + \eta \nabla \mu(x)) &= 0; \\ \eta' + \lambda^2 \mu(x) &= 0. \end{cases} \tag{3.2}$$

We then consider the limits of the complex for $(\mathcal{F}, g_\lambda)$ as λ approaches ∞ and zero. Orbits in the two limits will be completely different geometric objects, but the counting of them gives the same homology groups. Indeed, in the limit $\lambda \to \infty$, orbits will converge to those of the negative gradient flow of the pair $(f|_{\mu^{-1}(0)}, g|_{\mu^{-1}(0)})$ on the hypersurface $\mu^{-1}(0) \subset M$. This implies that for generic λ, the homology of the chain complex for $(\mathcal{F}, g_\lambda)$ will be isomorphic to the singular homology of $\mu^{-1}(0)$. Hence if the $\lambda \to 0$ limit gives a well-defined chain complex, its homology gives a new perspective to look at the homology of $\mu^{-1}(0)$.

Sketch of the proof

Let's fix two critical points $p_-,\ p_+ \in \text{Crit}\mathcal{F}$, and consider λ-trajectories from p_- to p_+ (which are solutions to (3.2) which are asymptotic to $p_\pm$ as $t \to \pm\infty$) for different λ. The energy of such any such trajectory is $\mathcal{F}(p_-) - \mathcal{F}(p_+)$, which is independent of λ.

For large λ, because of the finiteness of energy, one can show that all λ-trajectories from p_- to p_+ will be confined *a priori* in a small neighborhood of $\mu^{-1}(0) \times \mathbb{R}$. In fact, we show that these orbits will be close to the orbits of the negative gradient flow of $f|_{\mu^{-1}(0)}$. This gives the correspondence between the Morse-Smale-Witten chain complex for $(\mathcal{F}, g_\lambda)$ for large λ, and the complex for $\big(f|_{\mu^{-1}(0)}, g|_{\mu^{-1}(0)}\big)$.

On the other hand, when $\lambda \to 0$, trajectories from p_- to p_+ will converge to some "singular" trajectory. This part of the story is a typical example of fast-slow systems studied in geometric

singular perturbation theory. There is only one slow variable, η. We will show that for index $(p_-, \mathcal{F})-$ index $(p_+, \mathcal{F}) = 1$, if $\lambda > 0$ is small enough, then near each singular trajectory from p_- to p_+, there exists a unique λ-trajectory from p_- to p_+. This gives a correspondence between the Morse-Smale-Witten chain complex of $(\mathcal{F}, g_\lambda)$ for λ small, and a complex defined by counting those singular trajectories.

The method we use for the $\lambda \to 0$ limit is similar to the one used in [2]. However, in [2], the slow manifolds are normally hyperbolic, so one can use the classical Exchange Lemma [26]. On the other hand, one cannot avoid the existence of points on the slow manifold where normal hyperbolicity is lost. To overcome this difficulty, one needs a more general exchange lemma, which is provided in [45]. In Appendix A.1 we review facts from geometric singular perturbation theory that are used here.

3.1.2 Organization of this chapter

In Section 3.2, we review basic facts about the Morse-Smale-Witten complex, and we construct this complex CM^λ for the function $\mathcal{F}$ and the metric g_λ.

In Section 3.3, we consider the large-scale limit $\lambda \to \infty$ of our theory. We prove a compactness theorem which shows that for λ large, all λ-trajectories are close to a negative gradient flow trajectory of the restriction $(\overline{f}, \overline{g}) := \big(f|_{\mu^{-1}(0)}, g|_{\mu^{-1}(0)}\big)$, and a "gluing" theorem, which shows that the moduli space of λ-trajectories is homeomorphic to the moduli space of negative gradient flow trajectories of the restricted pair $(\overline{f}, \overline{g})$.

In Section 3.4 and Section 3.5, we consider the small-scale limit $\lambda \to 0$ of our theory. We also prove a pair of theorems in this case similar to what we have done in Section 3.3. The situation is much more complicated: the limiting trajectories will be singular, and the method we used to prove the gluing theorem is by applying Schecter's General Exchange Lemma.

In Section 3.6, we sketch various wall-crossing phenomena of the complex $\mathcal{C}^0$ when adding to μ a constant c, to recover some elementary facts about Morse theory.

3.1.3 Conventions

We shall always assume that M, f, μ and g are C^∞. A property will be called *generic* if it is true on a countable intersection of open dense sets in the Fréchet space of C^∞ functions. We will claim that generic data satisfy many nice properties which we will use, but we don't bother ourself in proving these statements. In fact, they can be proved for generic C^k-data for some finite integer k.

All chain complexes will be chain complexes of $\mathbb{Z}_2$-modules, hence no orientation issues will be considered. We shall use the notation $\mathbb{R}_+ = (0, \infty)$, $\mathbb{Z}_+ = \{n \in \mathbb{Z} \mid n \geq 0\}$, and we use $\text{index}(p, f)$ to denote the Morse index for p as a critical point of f. We denote by $\text{Crit}(f)$ the critical point set of any smooth function f.

We use the symbols with a bar, as $\overline{f}$, $\overline{g}$ to denote the restriction of f, g to the hypersurface $\mu^{-1}(0)$.

3.2 Morse homology and Morse-Smale-Witten complex of $(\mathcal{F}, g_\lambda)$

3.2.1 Morse homology

For the topological aspects of Morse theory, there is the classical book [35]. Here we will adopt the viewpoint of Witten [53], which is by now standard.

Let M be a smooth manifold without boundary. A smooth function $f : M \to \mathbb{R}$ is a *Morse function* if its differential df is a transverse section of the cotangent bundle T^*M. This implies that the critical points of f are discrete, and by the Morse lemma near each $p \in \text{Crit}(f)$, there exists a local coordinate chart $(x_1, \ldots, x_n)$ such that

$$f(x) = f(p) - \sum_{i=1}^{k_p} x_i^2 + \sum_{i=k_p+1}^{n} x_i^2.$$

The integer k_p is well-defined and is called the *Morse index* of the critical point p. We denote this as

$$\text{index}\,(p, f) = k_p \in \mathbb{Z}_+,$$

and we denote the set of critical points of f of index k by $\text{Crit}_k(f)$ Now for any complete Riemannian metric g on M, the gradient vector field ∇f is the dual of the 1-form df. We denote by $\phi_t : M \to M$ the flow of $-\nabla f$. For each $p \in \text{Crit}(f)$, the unstable and stable manifolds are defined by

$$W^u(p) = \left\{ x \in M | \lim_{t \to -\infty} \phi_t(x) = p \right\}, \quad W^s(p) = \left\{ x \in M | \lim_{t \to +\infty} \phi_t(x) = p \right\}$$

and

$$\dim W^u(p) = \text{index}\,(p,f),\ \dim W^s(p) = n - \text{index}\,(p,f).$$

The pair (f,g) is called *Morse-Smale* if for any two critical points p, q, $W^u(p)$ and $W^s(q)$ intersect transversely in M. This condition condition holds for generic (f,g) (see for example, [47]). For $p \neq q$, there is a free $\mathbb{R}$-action on $W^u(p) \cap W^s(q)$ by time translation along the flow.

Then we define the *Morse-Smale-Witten chain complex* of $\mathbb{Z}_2$-modules for a Morse-Smale pair (f,g),

$$CM(f,g) = (CM_*, \partial)\,,\ CM_k = \bigoplus_{p \in \text{Crit}_k(f)} \mathbb{Z}_2\langle p \rangle,$$

where $\partial : CM_k \to CM_{k-1}$ is given by

$$\partial p = \sum_{q \in \text{Crit}_{k-1}(f)} n_{p,q} \cdot q,\ n_{p,q} = \#\left[(W^u(p) \cap W^s(q))/\mathbb{R}\right] \in \mathbb{Z}_2. \tag{3.3}$$

One can show that $\#\left[(W^u(p) \cap W^s(q))/\mathbb{R}\right]$ is finite and $\partial \circ \partial = 0$ (under the Palais-Smale condition on f, see [47]), hence $CM(f,g)$ is a chain complex. Its homology group is called the *Morse homology* of (f,g) with $\mathbb{Z}_2$-coefficients, denoted by $HM_*(f,g;\mathbb{Z}_2)$. In the case where M is compact, this homology is independent of the choice of Morse-Smale pair (f,g) and is actually isomorphic to the singular homology of M.

3.2.2 Morse homology and the analytic setting

The book [47] give a comprehensive treatment of the analytical perspective on finite-dimensional Morse theory. Here we give a brief review.

Let p and q be two critical points of f, and consider the nonlinear ODE with asymptotic condition

$$x' + \nabla f(x) = 0,\ \lim_{t \to -\infty} x(t) = p,\ \lim_{t \to +\infty} x(t) = q. \tag{3.4}$$

The space of solutions to this boundary value problem can be identified with $W^u(p) \cap W^s(q)$ by identifying the solution $x(t)$ with the point $x(0)$.

Indeed, the space of solutions to such a boundary value problem can be viewed as a finite-dimensional submanifold of an infinite-dimensional Banach manifold. We consider the Banach man-

ifold $\mathcal{B}$ of those $W^{1,2}_{loc}$ maps $x : \mathbb{R} \to M$ for which there exists $T > 0$ such that for $t \in [T, +\infty)$ (respectively $t \in (-\infty, -T]$), $d(x(t), q)$ (respectively $d(x(t), p)$) is less than the injectivity radius of (M, g) and $\exp_q^{-1}(x(t)) \in W^{1,2}([T, +\infty), T_qM)$ (respectively $\exp_p^{-1}(x(t)) \in W^{1,2}((-\infty, -T], T_pM)$. There is a Banach space bundle $\mathcal{E} \to \mathcal{B}$ whose fibre over $x : \mathbb{R} \to M$ is $L^2(\mathbb{R}, x^*TM)$. There is a smooth section

$$\begin{array}{rccc} \mathcal{S}: & \mathcal{B} & \to & \mathcal{E} \\ & x & \mapsto & x' + \nabla f(x) \end{array}$$

whose zero locus is exactly the moduli space $\widetilde{\mathcal{M}}(p, q) \simeq W^u(p) \cap W^s(q)$, the space of solutions to (3.4). This is a Fredholm section, which means that the linearization of $\mathcal{S}$ at each $x \in \mathcal{S}^{-1}(0)$, which is given by

$$\begin{array}{rccc} D\mathcal{S}_x: & T_x\mathcal{B} & \to & \mathcal{E}_x \\ & V & \mapsto & \nabla_t V + \nabla_V(\nabla f) \end{array}$$

is a Fredholm operator with Fredholm index equal to $\text{index}\,(p) - \text{index}\,(q)$.

In order ensure that $\widetilde{\mathcal{M}}(p, q)$ is a smooth manifold, we need that the linearization of $\mathcal{S}$ is surjective along $\mathcal{S}^{-1}(0)$. In this case, we say that $\widetilde{\mathcal{M}}(p, q) = \mathcal{S}^{-1}(0)$ is *transverse*. This condition is equivalent to transversality of $W^u(p)$ and $W^s(q)$.

$\widetilde{\mathcal{M}}(p, q)$ has a free $\mathbb{R}$-action by time translation, so we can define $\mathcal{M}(p, q) = \widetilde{\mathcal{M}}(p, q)/\mathbb{R}$, the space of orbits from p to q. The compactness of $\mathcal{M}(p, q)$ is an issue, especially when M is noncompact. A control on the behavior of a solution $x : \mathbb{R} \to M$ is its *energy*

$$E(x) = \int_{\mathbb{R}} |\nabla f(x(t))|^2 \, dt = f(p) - f(q),$$

which only depends on the values of f at p and q. The energy is essential for many useful estimates. An assumption used to yield compactness is the Palais-Smale condition on f (which is automatic when M is compact): any sequence $x_i \in M$ for which $f(x_i)$ is uniformly bounded and $|\nabla f(x_i)| \to 0$ has a convergent subsequence. Under the transversality and Palais-Smale assumptions, if $\text{index}(q, f) = \text{index}\,(p, f) - 1$, then $\mathcal{M}(p, q)$ is finite; if $\text{index}\,(q, f) = \text{index}\,(p, f) - 2$, then $\mathcal{M}(p, q)$ can be compactified to be a smooth one-dimensional manifold with boundary; the boundary points correspond to the broken orbits from p to q that pass through another critical point r with $\text{index}\,(r, f) = \text{index}\,(p, f) - 1$. These two facts imply that the boundary operator (3.3) is well-defined

and $\partial \circ \partial = 0$.

3.2.3 The Morse-Smale-Witten complex of $(\mathcal{F}, g_\lambda)$

For the pair $(\mathcal{F}, g_\lambda)$ defined in Section 3.1, some modifications to the construction of the Morse-Smale-Witten complex are required. We shall assume

(A1) M is a compact manifold.

(A2) f and μ are Morse functions on M.

(A3) 0 is a regular value of μ and $f|_{\mu^{-1}(0)}$ is Morse.

The metric g on M and the Morse function μ satisfying (A3) are arbitrary and fixed. We shall make further assumptions, always denoted (Ak), that are true for generic f. Alternatively, we could fix the Morse functions f; the assumptions are then true for generic (μ, g).

In each of our results, we shall assume without comment that all assumptions (Ak) made up to that point hold.

Lemma 3.2.1. *Under the assumption (A3), the Lagrange multiplier $\mathcal{F}$ is a Morse function on $M \times \mathbb{R}$ and for any $p = (x_p, \eta_p) \in \mathrm{Crit}(\mathcal{F})$, $\mathrm{index}(p, \mathcal{F}) = \mathrm{index}(x_p, f|_{\mu^{-1}(0)}) + 1$.*

Proof. Let $p \in \mathrm{Crit}(\mathcal{F})$. Since 0 is a regular value of μ, there exists a local coordinate chart $(x_1, \ldots, x_n)$ near x_p such that $\mu(x_1, \ldots, x_n) = x_n$ and p has coordinates $(0, \ldots, 0)$. Then we see that the second derivative of $\mathcal{F}$ at p with respect to the coordinates $(x_1, \ldots, x_n, \eta)$ is of the form

$$D^2\mathcal{F}(0) = \begin{pmatrix} A & b & 0 \\ b^T & c & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Here A is equal to the second derivative of $\overline{f}$ at x_p with respect to the coordinates $(x_1, \ldots, x_{n-1})$ on $\mu^{-1}(0)$. So this implies that $D^2\mathcal{F}$ is nondegenerate at p, and its determinant is equal to $-\det A$. Deforming b by tb and let $t \to 0$, it is still the case for any t. Therefore it implies that the matrix has one more positive and one more negative eigenvalues than A. □

Now we consider the unstable and stable manifolds of critical points of $\mathcal{F}$. Since we are dealing with the family of metrics g_λ, we want transversality of unstable and stable manifolds not just for a single g_λ, but also for the family. Let ϕ_t^λ denote the flow of (3.2) for the given value of λ, let

$p_-, p_+ \in \mathrm{Crit}(\mathcal{F})$, and let I be an interval in λ-space. Define the subsets

$$W^u(p_-, \lambda) = \left\{ p \in M \times \mathbb{R} \mid \lim_{t \to -\infty} \phi_t^\lambda(p) = p_- \right\},$$
$$W^s(p_+, \lambda) = \left\{ p \in (M \times \mathbb{R} \mid \lim_{t \to +\infty} \phi_t^\lambda(p) = p_+ \right\},$$
$$W^u(p_-, I) = \{(p, \lambda) \in (M \times \mathbb{R}) \times I \mid p \in W^u(p_-, \lambda)\},$$
$$W^s(p_+, I) = \{(p, \lambda) \in (M \times \mathbb{R}) \times I \mid p \in W^s(p_+, \lambda)\};$$

$$\widetilde{\mathcal{M}}^I(p_-, p_+) = W^u(p_-, I) \cap W^s(p_+, I),\ \mathcal{M}^I(p_-, p_+) = \widetilde{\mathcal{M}}^I(p_-, p_+)/\mathbb{R}.$$

We assume:

(A4) For all $p_-, p_+ \in \mathrm{Crit}(\mathcal{F})$, $W^u(p_-, \mathbb{R}_+)$ and $W^s(p_+, \mathbb{R}_+)$ intersect transversely in $M \times \mathbb{R}$.

Assumption (A4) implies that for all but discretely many $\lambda \in \mathbb{R}_+$, $W^u(p_-, \lambda)$ and $W^s(p_+, \lambda)$ are transverse. We denote by $\Lambda^{reg} \subset \mathbb{R}_+$ the subset of λ's for which $\widetilde{\mathcal{M}}^\lambda(p_-, p_+)$ is transverse for all $p_-, p_+ \in \mathrm{Crit}(\mathcal{F})$.

Since the manifold $M \times \mathbb{R}$ is noncompact, we also need the following result:

Lemma 3.2.2. *For each $\lambda \in \mathbb{R}_+$, the function $\mathcal{F}$ satisfies the Palais-Smale condition with respect to the metric g_λ, i.e., for any sequence $\widetilde{x}_i \in M \times \mathbb{R}$ such that $\mathcal{F}(\widetilde{x}_i)$ is bounded and $|\nabla \mathcal{F}(\widetilde{x}_i)|_{g_\lambda} \to 0$, there exists a convergent subsequence of $\widetilde{x}_i$.*

Proof. Suppose $\widetilde{x}_i = (x_i, \eta_i)$. Since M is compact, we may assume that $\eta_i \to +\infty$ or $-\infty$. The condition that $\mathcal{F}(\widetilde{x}_i)$ is bounded implies that $\mu(x_i) \to 0$; but since 0 is a regular value of μ, $\nabla \mathcal{F}(\widetilde{x}_i) = \nabla f(x_i) + \eta_i \nabla \mu(x_i)$ cannot have arbitrary small norm. □

Now, for any $\lambda \in \Lambda^{reg}$, we can define the associated Morse-Smale-Witten complex of $(\mathcal{F}, g_\lambda)$, which is denoted by $CM^\lambda = CM(\mathcal{F}, g_\lambda)$. Its homology is denoted by HM^λ. All CM^λ have the same generators and gradings, but the boundary operator ∂^λ may change when λ crosses one of the exceptional values in $\mathbb{R}_+ \setminus \Lambda^{reg}$. To show that the homology HM^λ is independent of λ, we have

Lemma 3.2.3. *For any $L > 0$, there exists $K_L > 0$ such that for any $p_\pm \in \mathrm{Crit}(\mathcal{F})$, $W^u(p_-, (0, L]) \cap W^s(p_+, (0, L]) \subset M \times [-K_L, K_L]$.*

Proof. Suppose this statement doesn't hold, then there exists $p_-, p_+ \in \mathrm{Crit}(\mathcal{F})$, a sequence λ_i converging to $\lambda_\infty \in [0, L]$ and $(x_i, \eta_i) \in W^u(p_-, \lambda_i) \cap W^s(p_+, \lambda_i)$ such that $\lim_{i \to \infty} |\eta_i| = +\infty$. Suppose for each i there are solutions $\widetilde{x}_i = (x_i, \eta_i) : \mathbb{R} \to M \times \mathbb{R}$ to (3.2) for $\lambda = \lambda_i$ such that

$(x_i, \eta_i) = \widetilde{x}_i(0)$. Then since λ_i are uniformly bounded, for any $R > 0$, we have $|\eta_i|\,|_{[-R,R]} \to \infty$ uniformly.

On one hand, we have

$$\mathcal{F}(p_-) \geq \mathcal{F}(x_i, \eta_i) = f(x_i) + \eta_i \mu(x_i) \geq \mathcal{F}(p_+)$$

which implies that $\mu(x_i) \to 0$ uniformly on $[-R, R]$. Then, by the definition of the energy of the solution $\widetilde{x}_i$, we see

$$\mathcal{F}(p_+) - \mathcal{F}(p_-) \geq \int_{-R}^{R} \|\nabla f(x_i(t)) + \eta_i(t) \nabla \mu(x_i(t))\|^2 \, dt \to \infty$$

which is impossible. □

Corollary 3.2.4. *For any $\lambda_1, \lambda_2 \in \Lambda^{reg}$, there is a canonical isomorphism $\Phi_{\lambda_1,\lambda_2} : HM^{\lambda_1} \to HM^{\lambda_2}$ such that for $\lambda_1, \lambda_2, \lambda_3 \in \Lambda^{reg}$, $\Phi_{\lambda_2,\lambda_3} \circ \Phi_{\lambda_1,\lambda_2} = \Phi_{\lambda_1,\lambda_3}$.*

Proof. For any $\lambda_1, \lambda_2 \in \Lambda^{reg}$, $\lambda_1 < \lambda_2$, we can compare the two complexes CM^{λ_1} and CM^{λ_2} as in the case of compact manifolds, thanks to the compactness provided by the previous lemma. More precisely, we can either use the continuation principle, or the bifurcation analysis as done in [12] to show that the two chain complexes have isomorphic homology. Note that as we vary λ from λ_1 to λ_2, the critical point set is fixed so there is no "birth-death" of critical points. What may happen is for some $\lambda \in (\lambda_1, \lambda_2)$, there may exists a trajectory between two different critical points with the same Morse index. □

Remark 3.2.5. In general, the Morse homology will be very sensitive to the Morse-Smale pair if the underlying manifold is noncompact (see for example [28]).

3.3 Adiabatic limit $\lambda \to \infty$

In this section we study the limiting behavior of λ-trajectories and the chain complex $\mathcal{C}^\lambda$ as we vary λ such that $\lambda \to 0$. We expect that the λ-trajectories approach to trajectories of $\left(\overline{f}, \overline{g}\right)$ in $\mu^{-1}(0)$ and CM^λ is isomorphic to the Morse-Smale-Witten complex $CM\left(\overline{f}, \overline{g}\right)$. So natually we assume

(A5) $(\overline{f}, \overline{g})$ is Morse-Smale.

The main theorem of this section is:

Theorem 3.3.1. *For $\lambda \in \Lambda^{reg}$ sufficiently large, the complex CM^λ is isomorphic to the Morse-Smale-Witten complex $CM(\overline{f}, \overline{g})$ with grading shifted by one.*

Corollary 3.3.2. *For any $\lambda \in \Lambda^{reg}$, there is a canonical isomorphism*

$$\Phi_{\lambda,+\infty} : HM_*^\lambda \to H_{*-1}(\mu^{-1}(0); \mathbb{Z}_2).$$

Remark 3.3.3. There are other ways to relate the two chain complexes. For example, in the appendix of [15] Frauenfelder had a different approach by deforming the function f in the normal direction of $\mu^{-1}(0)$.

3.3.1 Proof of Theorem 3.3.1

For any $\widetilde{x} = (x, \eta) \in \widetilde{\mathcal{M}}^\lambda(p_-, p_+)$, we have

$$E = \mathcal{F}(p_-) - \mathcal{F}(p_+) = -\int_{-\infty}^{+\infty} \left\langle \nabla^\lambda \mathcal{F}(\widetilde{x}(t)), \widetilde{x}'(t) \right\rangle_\lambda dt = \|x'\|_{L^2}^2 + \lambda^2 \|\mu(x)\|_{L^2}^2 .$$

So $\|\mu(x)\|_{L^2}^2 \leq \lambda^{-2} E$. On the other hand, one has

$$\left\| \frac{d}{dt} \mu(x(t)) \right\|_{L^2} \leq \|d\mu\|_{L^\infty} \|x'(t)\|_{L^2} .$$

So by Sobolev embedding $W^{1,2} \to C^0$, for any $\epsilon > 0$, there exists $\Lambda_\epsilon > 0$ such that for $\lambda > \Lambda_\epsilon$ and $\widetilde{x} \in \widetilde{\mathcal{M}}^\lambda(p_-, p_+)$, we have $x(t) \in U_\epsilon = \mu^{-1}((-\epsilon, \epsilon))$.

Now, consider the Banach manifold $\mathcal{B} = \mathcal{B}^{1,2}$ of $W^{1,2}_{loc}$-maps $\widetilde{x} = (x, \eta)$ from $\mathbb{R} \to U_\epsilon \times \mathbb{R}$ such that $\widetilde{x}$ is asymptotic to $\left(p_\pm, \eta_{p_\pm}\right) \in \mathrm{Crit}(\mathcal{F})$ at $\pm\infty$ in the following sense: there exists $R > 0$ and $\widetilde{W}_- \in W^{1,2}((-\infty, -R], T_{p_-} M \oplus \mathbb{R})$ and $\widetilde{W}_+ \in W^{1,2}([R, +\infty), T_{p_+} M \oplus \mathbb{R})$ such that $\widetilde{x}|_{(-\infty,-R]} = \exp_{p_-} \widetilde{W}_-$, $\widetilde{x}|_{[R,+\infty)} = \exp_{p_+} \widetilde{W}_+$. Consider $\mathcal{E} \to \mathcal{B}$ the Banach space bundle whose fibre over $\widetilde{x}$ is $\mathcal{E}_{\widetilde{x}} = L^2(x^* TM \oplus \mathbb{R})$. For $\lambda > \Lambda_\epsilon$, consider the Fredholm section $\widetilde{S}^\lambda : \mathcal{B} \to \mathcal{E}$ given by

$$\widetilde{S}^\lambda(x, \eta) = \left(\frac{dx}{dt} + \nabla f + \eta \nabla \mu, \frac{d\eta}{dt} + \lambda^2 \mu(x) \right).$$

Then $\widetilde{\mathcal{M}}^\lambda(p_-,p_+)=\left(\widetilde{S}^\lambda\right)^{-1}(0)$. The linearization of $\widetilde{\mathcal{S}}^\lambda$ at $\widetilde{x}=(x,\eta)\in\mathcal{B}$ is given by

$$\begin{array}{rccc}\widetilde{\mathcal{D}}_{\widetilde{x}}: & T_{\widetilde{x}}\mathcal{B} & \to & \mathcal{E}_{\widetilde{x}} \\ & \begin{pmatrix} V \\ H \end{pmatrix} & \mapsto & \begin{pmatrix} \nabla_t V+\nabla_V(\nabla f+\eta\nabla\mu)+H\nabla\mu \\ \frac{dH}{dt}+\lambda^2 d\mu(V) \end{pmatrix}.\end{array}$$

On U_ϵ, there is the line bundle $L\subset TM$ generated by $\nabla\mu$, and denote by $L^\perp$ the orthogonal complement with respect to the Riemannian metric g. Then, for any $\widetilde{x}$, we can decompose the domain and target space of $\widetilde{D}_{\widetilde{x}}$ as

$$T_{\widetilde{x}}\mathcal{B}\simeq W_L(\widetilde{x})\oplus W_T(\widetilde{x}),\ W_L(\widetilde{x})=W^{1,2}(x^*L\oplus\mathbb{R}),\ W_T(\widetilde{x})=W^{1,2}(x^*L^\perp); \tag{3.5}$$

$$\mathcal{E}_{\widetilde{x}}\simeq\mathcal{E}_L(\widetilde{x})\oplus\mathcal{E}_T(\widetilde{x}),\ \mathcal{E}_L(\widetilde{x})=L^2(x^*L\oplus\mathbb{R}),\ \mathcal{E}_T(\widetilde{x})=L^2(x^*L^\perp). \tag{3.6}$$

We rescale the norms on $W_L(\widetilde{x})$ and $\mathcal{E}_L(\widetilde{x})$ as follows. We identify $(h_1\nabla\mu,h_2)\in W_L(\widetilde{x})$ with $(h_1,h_2)\in W^{1,2}\oplus W^{1,2}$ and define

$$\|(h_1,h_2)\|_{W_\lambda}=\lambda\|h_1\|_{L_2}+\|h_1'\|_{L^2}+\|h_2\|_{L^2}+\lambda^{-1}\|h_2'\|_{L^2}; \tag{3.7}$$

and for $(h_1\nabla\mu,h_2)\in\mathcal{E}_L(\widetilde{x})$, identify it with $(h_1,h_2)\in L^2\oplus L^2$ and define

$$\|(h_1,h_2)\|_{L_\lambda}=\|h_1\|_{L^2}+\lambda^{-1}\|h_2\|_{L^2}. \tag{3.8}$$

We leave the norms on their complements unchanged, and use W_λ and L_λ to denote the norms on $T_{\widetilde{x}}\mathcal{B}$ and $\mathcal{E}_{\widetilde{x}}$ respectively.

Now we describe the limit objects. There exists a smooth function $\widetilde{\eta}:U_\epsilon\to\mathbb{R}$ defined by the condition

$$\langle\nabla\mu(x),\nabla f(x)+\widetilde{\eta}(x)\nabla\mu(x)\rangle=0.$$

Then $\nabla f+\widetilde{\eta}\nabla\mu$ is a smooth vector field whose restriction to $\mu^{-1}(0)$ is the gradient of $\overline{f}=f|_{\mu^{-1}(0)}$ with respect to the restriction of the Riemannian metric. We denote by $\widetilde{\mathcal{N}}^\infty(p_-,p_+)$ the space of solutions of the negative gradient flow of $\overline{f}$, and by $\mathcal{N}^\infty(p_-,p_+)$ the quotient space by identifying

reparametrizations, and by $\overline{\mathcal{N}}^{\infty}(p_-, p_+)$ the compactified moduli space by adding broken orbits.

Proposition 3.3.4. *Suppose $\lambda_\nu \to \infty$ and $\widetilde{x}_\nu = (x_\nu, \xi_\nu) \in \widetilde{\mathcal{M}}^{\lambda_\nu}(p_-, p_+)$. Then there is a subsequence, still indexed by ν, and a broken trajectory $\mathcal{Y} = ([y_i])_{i=1}^n \in \overline{\mathcal{N}}^{\infty}(p_-, p_+)$ and a constant $c_0 > 0$ such that*

1. *$\|(\mu(x_\nu), \eta_\nu - \widetilde{\eta}(x_\nu))\|_{W_\lambda} \leq c_0 \lambda_\nu^{-1}$;*
2. *There exists $t_{1,\nu}, t_{2,\nu}, \ldots, t_{n,\nu} \in \mathbb{R}$ such that $x_\nu(t_{i,\nu} + \cdot)$ converges to y_i in C^1_{loc}-topology;*

Proof. Apply $d\mu$ to the equation $x_\nu'(t) + \nabla f(x_\nu(t)) + \eta_\nu(t)\nabla\mu(x_\nu(t)) = 0$, we obtain

$$\begin{aligned} 0 = \frac{d}{dt}\mu(x_\nu(t)) + \langle \nabla\mu(x_\nu(t)), \nabla f(x_\nu(t)) + \eta_\nu(t)\nabla\mu(x_\nu(t))\rangle \\ = \frac{d}{dt}\mu(x_\nu(t)) + \lambda_\nu |\nabla\mu|^2 \left(\frac{1}{\lambda_\nu}(\eta_\nu(t) - \widetilde{\eta}(x_\nu(t)))\right). \end{aligned} \tag{3.9}$$

Also we have

$$\frac{d}{dt}\left(\frac{1}{\lambda_\nu}(\eta_\nu(t) - \widetilde{\eta}(x_\nu(t)))\right) + \lambda\mu(x_\nu(t)) = -\frac{1}{\lambda_\nu}\frac{d}{dt}\widetilde{\eta}(x_\nu(t)). \tag{3.10}$$

Consider the linear operator

$$\begin{array}{rccc} D: & W^{1,2}(\mathbb{R}, \mathbb{R}^2) & \to & L^2(\mathbb{R}, \mathbb{R}^2) \\ & (f_1, f_2) & \mapsto & \left(\frac{d}{dt}f_1 + \lambda |\nabla\mu|^2 f_2, \frac{d}{dt}f_2 + \lambda f_1\right) \end{array}.$$

If regarded as an unbounded operator from L^2 to L^2, then it is bounded from below by $c\lambda$ for some constant c. Then (3.9) and (3.10) imply that

$$\|\mu(x_\nu)\|_{L^2} + \lambda_\nu^{-1}\|\eta_\nu - \widetilde{\eta}(x_\nu)\|_{L^2} \leq \frac{c}{\lambda_\nu^2}\left\|\frac{d}{dt}\widetilde{\eta}(x_\nu)\right\|_{L^2} \leq c\lambda_\nu^{-2}.$$

Also, D has a uniformly bounded right inverse, so

$$\|\mu(x_\nu)\|_{W^{1,2}} + \lambda_\nu^{-1}\|\eta_\nu - \widetilde{\eta}(x_\nu)\|_{W^{1,2}} \leq c\lambda_\nu^{-1}.$$

This implies the first claim of this proposition. Moreover, the Sobolev embedding $W^{1,2} \to C^0$ implies in particular that η_ν is uniformly bounded since $\widetilde{\eta}(x_\nu)$ is. It follows that $|x_\nu'|$ is uniformly bounded.

Now we identify $U_\epsilon \simeq \mu^{-1}(0) \times (-\epsilon, \epsilon)$ such that the projection to the second component is equal

to μ. Then we can write

$$x_\nu(t) = (\overline{x}_\nu(t), \mu(x_\nu(t)))\,.$$

Projecting to the first factor and using the fact that $|x'_\nu|$ is uniformly bounded, we see that there exists $K > 0$ independent of ν, such that

$$\left|\overline{x}'_\nu(t) + \nabla\overline{f}\,((\overline{x}_\nu(t)))\right| \leq K|\mu(x_\nu(t))|.$$

This implies that a subsequence of $\overline{x}_\nu$ converges to a broken trajectory for the induced function $\overline{f}$ in C^0_{loc}-topology. More precisely, there exists $t_{1,\nu}, \ldots, t_{k,\nu} \in \mathbb{R}$ such that $\overline{x}_\nu(t_{i,\nu} + \cdot)$ converges to a trajectory y_i in $\mu^{-1}(0)$ in C^0_{loc}-topology.

Then, by the Sobolev embedding $W^{1,2} \to C^0$ and the C^0_{loc} convergence of $x_\nu(t_{i,\nu} + \cdot)$ to y_i, we see that $\eta_\nu(t_{i,\nu} + \cdot)$ converges to $\widetilde{\eta}(y_i)$ in C^0_{loc}. This implis that the convergence $x_\nu(t_{i,\nu} + \cdot)$ to y_i is in C^1_{loc}. □

Now we want to prove that, any $y \in \widetilde{\mathcal{N}}^\infty(p_-, p_+)$ can be approximated by $\widetilde{x} \in \widetilde{\mathcal{M}}^\lambda(p_-, p_+)$, so that in particular, when $\mathrm{index}(p_-, \mathcal{F}) - \mathrm{index}\,(p_+, \mathcal{F}) = 1$, there is a canonical one-to-one correspondence between $\mathcal{N}^\infty(p_-, p_+)$ and $\mathcal{M}^\lambda(p_-, p_+)$.

Since we have assumed that the restriction of (f, g) to $\mu^{-1}(0)$ is Morse-Smale, the trajectory y is transverse. Namely, the linearized operator

$$\begin{array}{rccc} \overline{D}_y: & W^{1,2}(y^*T\mu^{-1}(0)) & \to & L^2(y^*T\mu^{-1}(0)) \\ & V & \mapsto & \nabla_{y'}V + \nabla_V(\nabla f + \widetilde{\eta}\nabla\mu) \end{array}$$

is surjective and has a bounded right inverse $\overline{Q}_y$. In particular, we can choose $\overline{Q}_y$ such that

$$\mathrm{Im}\overline{Q}_y = \left\{V \in W^{1,2}(y^*T\mu^{-1}(0)) \mid g(V(0), W(0)) = 0\ \forall W \in \ker\overline{D}_y\right\}.$$

For any large λ, we take our approximate solution just to be

$$\widetilde{y}(t) = (y(t), \widetilde{\eta}(y(t)))\,.$$

Note that $y(t)$ converges to $p_\pm$ exponentially as $t \to \pm\infty$, so $\widetilde{y} \in \mathcal{B}$. Then denote the linearization

of $\widetilde{S}^\lambda$ at $\widetilde{y}$ by

$$\begin{array}{rccc} \widetilde{D}_{\widetilde{y}}: & W^{1,2}(y^*TM\oplus\mathbb{R}) & \to & L^2(y^*TM\oplus\mathbb{R}) \\ & \begin{pmatrix} V \\ h \end{pmatrix} & \mapsto & \begin{pmatrix} \nabla_{y'}V + \nabla_V(\nabla f + \widetilde{\eta}\nabla\mu) + h\nabla\mu \\ h' + \lambda^2 d\mu(V) \end{pmatrix}. \end{array}$$

Now we construct a right inverse to $\widetilde{D}_{\widetilde{y}}$ out of $\overline{Q}_y$. Note that, for any $\overline{V} \in W_T(\widetilde{y})$

$$\widetilde{D}_{\widetilde{y}}(\overline{V}) = \big(\nabla_t\overline{V} + \nabla_{\overline{V}}(\nabla f + \eta\nabla\mu), \lambda^2 d\mu(\overline{V})\big) = \big(\overline{D}_y(\overline{V}), 0\big) \in \mathcal{E}_T(\widetilde{y}).$$

Hence with respect to the decomposition (3.5) and (3.6), the linearized operator can be written as

$$\widetilde{D}_{\widetilde{y}} = \begin{pmatrix} \overline{D}_y & A_y \\ 0 & D'_{\widetilde{y}} \end{pmatrix}.$$

Here A_y is given by

$$A_y(h_1\nabla\mu, h_2) = h_1 W$$

where W is a smooth tangent vector field on $\mu^{-1}(0)$. Hence there exists $c_1 > 0$ such that

$$\|A_y(h_1\nabla\mu, h_2)\|_{L_\lambda} \le c_1 \|h_1\|_{L^2} \le c_1\lambda^{-1} \|h_1\|_{W_\lambda}.$$

Now we look at the operator $D'_{\widetilde{y}}$. After trivialize the bundle $\mathbb{R}\{\nabla\mu\}\oplus\mathbb{R}$ isometrically to $\mathbb{R}^2$, we see the operator $D'_{\widetilde{y}}$ is transformed into

$$\mathfrak{D}'_{\widetilde{y}} = \frac{d}{dt} + \begin{pmatrix} a(t) & \lambda b(t) \\ \lambda b(t) & 0 \end{pmatrix} =: \frac{d}{dt} + B(t).$$

It is easy to see that, for λ large, $B(t)$ has 1 positive eigenvalue and one negative eigenvalue, both of which are bounded away from zero by $c\lambda$, where c is a constant independent of λ. Hence we have a (unique)bounded right inverse

$$Q'_{\widetilde{y}} : \mathcal{E}_L(\widetilde{y}) \to W_L(\widetilde{y}),\ \big\|Q'_{\widetilde{y}}\big\| \le c_2$$

for some $c_2 > 0$. Then define

$$Q_{\widetilde{y}} = \begin{pmatrix} \overline{Q}_y & 0 \\ 0 & Q'_{\widetilde{y}} \end{pmatrix}$$

which serves the approximate right inverse.

Lemma 3.3.5. *There exists $\Lambda_0 > 0$, such that for all $\lambda > \Lambda_0$,*

$$\left\| \mathrm{Id} - \widetilde{D}_{\widetilde{y}} Q_{\widetilde{y}} \right\|_{L_\lambda} < \frac{1}{2}$$

with respect to the operator norm of the space L_λ.

Proof. Note that $\mathrm{Id} - \widetilde{D}_{\widetilde{y}} Q_{\widetilde{y}} = A_y \circ Q'_{\widetilde{y}}$ and

$$\left\| A_{\widetilde{y}} Q'_{\widetilde{y}}(h_1 \nabla\mu, h_2) \right\|_{L_\lambda} \leq \frac{c_1}{\lambda} \left\| Q'_{\widetilde{y}}(h_1 \nabla\mu, h_2) \right\|_{W_\lambda} \leq \frac{c_1 c_2}{\lambda} \left\| (h_1 \nabla\mu, h_2) \right\|_{L_\lambda}.$$

□

Hence $\widetilde{D}_{\widetilde{y}} Q_{\widetilde{y}}$ is invertible and a right inverse can be constructed as

$$\widetilde{Q}_{\widetilde{y}} = Q_{\widetilde{y}} \left(\widetilde{D}_{\widetilde{y}} Q_{\widetilde{y}} \right)^{-1} : L_\lambda \to W_\lambda.$$

And it is uniformly bounded by some constant which is independent of λ.

Now since we want to apply the implicit function theorem (cf. [34, Appendix A]), we need to trivialize the Banach manifold $\mathcal{B}$ and the Banach space bundle $\mathcal{E}$ locally near $\widetilde{y}$. We identify a neighborhood of $\widetilde{y}$ in $\mathcal{B}$ with a small ball (with respect to the W_λ-norm) centered at the origin of $T_{\widetilde{y}}\mathcal{B}$ as follows: using the identification $U_\epsilon \simeq \mu^{-1}(0) \times (-\epsilon, \epsilon)$, for any $\widetilde{W} = (W, a\nabla\mu, h) \in T_{\widetilde{y}}\mathcal{B}$, we identify it with the map $\rho(\widetilde{W}) = (\overline{\exp}_{\widetilde{y}} W, a, \widetilde{\eta}(\widetilde{y}) + h) \in \mu^{-1}(0) \times (-\epsilon, \epsilon) \times \mathbb{R}$ where $\overline{\exp}$ is the exponential map inside $\mu^{-1}(0)$. We trivialize $\mathcal{E}$ over such a neighborhood around $\widetilde{y}$ by parallel transport along the radial geodesics, which is denoted by $\Phi_{\widetilde{W}} : \mathcal{E}_{\rho(\widetilde{W})} \to \mathcal{E}_{\widetilde{y}}$. Then for $\widetilde{W} \in T_{\widetilde{y}}\mathcal{B}$ with $\left\| \widetilde{W} \right\|_{W_\lambda}$ small, denote by $\widetilde{D}_{\widetilde{W}} : T_{\widetilde{y}}\mathcal{B} \to \mathcal{E}_{\widetilde{y}}$ the linearization of $\Phi_{\widetilde{W}} \circ \widetilde{S}^\lambda(\rho(\widetilde{W}))$ at $\widetilde{W}$. We have

Lemma 3.3.6. *There exists $\epsilon_3, c_3 > 0$, independent of λ, such that for all $\widetilde{W} \in T_{\widetilde{y}}\mathcal{B}$ with $\left\| \widetilde{W} \right\|_{W_\lambda} < \epsilon_3$, one has*

$$\left\| \widetilde{D}_{\widetilde{W}} - \widetilde{D}_{\widetilde{y}} \right\| \leq c_3 \left\| \widetilde{W} \right\|_{W_\lambda}. \tag{3.11}$$

Proof. First consider the case when the $\mathbb{R}$-component of $\widetilde{W}$ is zero. We denote by $\widetilde{S}_1$ the first component of $\widetilde{S}^\lambda$, which is independent of λ. Then we see

$$\left\|D\widetilde{S}_1(\widetilde{W}) - D\widetilde{S}_1(\widetilde{y})\right\| \leq c\left\|\widetilde{W}\right\|_{L^\infty} \tag{3.12}$$

which is standard. Now because the $\mathbb{R}$-component of $\widetilde{W}$ is zero, we have $\left\|\widetilde{W}\right\|_{L^\infty} \leq c\left\|\widetilde{W}\right\|_{W_\lambda}$ by our definition of the W_λ-norm (3.7).

The second component of $\widetilde{S}^\lambda$ is denoted by $\widetilde{S}^\lambda_2$. Then we see in this case, for any $\widetilde{V} = (V, h) \in T_{\widetilde{y}}\mathcal{B}$

$$D\widetilde{S}^\lambda_2(\widetilde{W})(\widetilde{V}) = \lambda^2 h. \tag{3.13}$$

Hence the variation of the derivative of $\widetilde{S}^\lambda_2$ is always zero.

Now we consider the case $\widetilde{W} = (0, h_0)$ with $h_0 \in W^{1,2}(\mathbb{R})$. Then for any $\widetilde{W}_0 \in T_{\widetilde{y}}\mathcal{B}$ small, by our definition of our norms (3.7) and (3.8),

$$\begin{aligned}\left\|(\widetilde{D}_{\widetilde{W}+\widetilde{W}_0} - \widetilde{D}_{\widetilde{W}_0})(V,h)\right\|_{L_\lambda} &= \|h_0\nabla_V\nabla\mu\|_{L_\lambda} = \|h_0\nabla_V\nabla\mu\|_{L^2} \\ &\leq c\left\|\widetilde{W}\right\|_{W_\lambda}\|V\|_{L^2} \leq c\left\|\widetilde{W}\right\|_{W_\lambda}\|(V,h)\|_{W_\lambda}\end{aligned} \tag{3.14}$$

for some $c > 0$.

Combining (3.12), (3.13) and (3.14) we obtain (3.11). □

On the other hand, we see

$$\left\|\widetilde{S}^\lambda(\widetilde{y})\right\|_{L_\lambda} = \lambda^{-1}\left\|\frac{d}{dt}\widetilde{\eta}(y)\right\|_{L^2} \leq c_4\lambda^{-1}$$

for some constant $c_4 > 0$. By the implicit function theorem, there exists $\epsilon_4 > 0$ such that for sufficiently large λ, there exists a unique $\widetilde{W}_\lambda(y) \in T_{\widetilde{y}}\mathcal{B}$ satisfying

$$\left\|\widetilde{W}_\lambda(y)\right\|_{W_\lambda} \leq \epsilon_4,\ \widetilde{W}_\lambda(y) \in \mathrm{Im}Q_{\widetilde{y}},\ \widetilde{S}^\lambda(\exp_{\widetilde{y}}\widetilde{W}_\lambda(y)) = 0.$$

Moreover, there exists $c_5 > 0$ such that

$$\left\|\widetilde{W}_\lambda(y)\right\|_{W_\lambda} \leq \frac{c_5}{\lambda}.$$

We hence define the *gluing map* (for large Λ_0) to be

$$\begin{array}{rccc} \Phi: & [\Lambda_0,+\infty)\times\widetilde{\mathcal{N}}^\infty(p_-,p_+) & \to & \cup_{\lambda\geq\Lambda_0}\widetilde{\mathcal{M}}^\lambda(p_-,p_+) \\ & (\lambda,y) & \mapsto & \exp_{\widetilde{y}}\widetilde{W}_\lambda(y). \end{array}$$

Theorem 3.3.7. *For each $\lambda\in[\Lambda_0,+\infty)\cap\Lambda^{reg}$, the restriction of Φ to $\{\lambda\}\times\widetilde{\mathcal{N}}^\infty(p_-,p_+)$ is a homeomorphism onto $\widetilde{\mathcal{M}}^\lambda(p_-,p_+)$.*

Proof. To complete the proof of this theorem, it remains to prove the local surjectivity of the gluing map Φ. For simplicity, we only prove for the case when the Morse indices of p_- and p_+ differ by one, which is enough for our purpose. Suppose it is not the case, then there exists a sequence $\lambda_i\to\infty$ and $\widetilde{x}_i\in\widetilde{\mathcal{M}}^{\lambda_i}(p_-,p_+)$ which converges to some $\widetilde{y}\in\widetilde{\mathcal{N}}^\infty(p_-,p_+)$ as described by Proposition 3.3.4 (with $n=1$ and $t_{1,\nu}=0$) but don't lie in the range of Φ. Then, consider the hyperplanes $N(y(s))=(y'(s))^\perp\subset T_{y(s)}M$ for $s\in\mathbb{R}$. For each i, there exists $s_i\in\mathbb{R}$ such that $s_i\to 0$ and

$$\widetilde{x}_i(0)\in\left\{(\exp_{y(s_i)}V,\eta)\mid\eta\in\mathbb{R},\ V\in N(y(s_i)),\ \|V\|\leq\epsilon\right\}.$$

Then if we write $\widetilde{x}_i(t)=\exp_{\widetilde{y}(s_i+t)}\widetilde{W}_i(t)$, then $\widetilde{W}_i\in\mathrm{Im}Q_{\widetilde{y}(s_i+\cdot)}$ and $\left\|\widetilde{W}_i\right\|_{W_{\lambda_i}}\leq c_0\lambda_i^{-1}$ by Proposition 3.3.4. By the uniqueness part of the implicit function theorem, this implies that $\widetilde{x}_i=\Phi(\lambda_i,\widetilde{y}(s_i+\cdot))$, which contradicts with our assumption. □

Now Theorem 3.3.1 is an immediate consequence.

Remark 3.3.8. One can also prove this theorem using geometric singular perturbation theory (see the discussion in the paper [46] following this part, which is provided by Stephen Schecter).

3.4 Fast-slow system associated to the $\lambda\to 0$ limit

Now we consider the limit of (3.2) as $\lambda\to 0$. In this limit, (3.2) is a "fast-slow system" [26]. There is one slow variable, η. In this section, we will identify the slow manifold, and study the fast flow and slow equation, under appropriate generic assumptions. (This terminology is reviewed in Appendix A.1.) We will also relate various Morse indices.

3.4.1 The slow manifold

The *slow manifold* for (3.2) is the set of equilibria for $\lambda = 0$:

$$\mathcal{C}_{\mathcal{F}} = \{(x, \eta) \in M \times \mathbb{R} \mid \nabla f(x) + \eta \nabla \mu(x) = 0\}.$$

We assume:

(A6) $\mathrm{Crit}(f) \cap \mathrm{Crit}(\mu)$ is empty.

(A7) $\nabla f + \eta \nabla \mu$ is a transverse section of $\pi^* TM$, where $\pi : M \times \mathbb{R} \to M$ is the projection.

(A7) implies that $\mathcal{C}_{\mathcal{F}}$ is a 1-dimensional smooth submanifold of $M \times \mathbb{R}$. Indeed, it is noncompact and its end at infinity is asymptotic to $(\mathrm{Crit}\mu \times [R, +\infty)) \cup (\mathrm{Crit}\mu \times (-\infty, -R])$ for R large. This can be seen as follows. If there is a sequence $(x_i, \eta_i) \in \mathcal{C}_{\mathcal{F}}$ with $\lim_{i\to\infty} |\eta_i| = \infty$, then we see that $\nabla\mu(x_i) \to 0$ and a subsequence of x_i converging to some $y \in \mathrm{Crit}\mu$. Conversely, for any $y \in \mathrm{Crit}\mu$, let $S_\epsilon(y) \subset M$ denote the sphere of radius ϵ around y. Since by (A2) y is a nondegenerate critical point of μ, we see that for small $\epsilon > 0$, the map

$$S_\epsilon(y) \to S(T_y M), \ x \mapsto \frac{\nabla\mu(x)}{\|\nabla\mu(x)\|}$$

has degree ± 1. On the other hand, by (A6) we can use local coordinates near y in which the vector field ∇f is constant. Hence there exist a unique pair of points $(x_\epsilon^\pm, \eta_\epsilon^\pm)$, with $x_\epsilon^\pm \in S_\epsilon(p)$, such that

$$(x_\epsilon^\pm, \eta_\epsilon^\pm) \in \mathcal{C}_{\mathcal{F}}.$$

We can order the two points so that $\lim_{\epsilon\to 0} \eta_\epsilon^\pm = \pm\infty$.

For fixed η, denote the function $f_\eta = f + \eta\mu : M \to \mathbb{R}$. Then consider the function

$$\begin{array}{rccc} d_{\mathcal{C}} : & \mathcal{C}_{\mathcal{F}} & \to & \mathbb{R} \\ & p = (x, \eta) & \mapsto & \det \mathrm{Hess} f_\eta(x). \end{array}$$

We assume

(A8) 0 is a regular value of $d_{\mathcal{C}}$.

(A9) For any $p = (x, \eta) \in d_{\mathcal{C}}^{-1}(0)$, $\mu(x) \neq 0$. Therefore $d_{\mathcal{C}}^{-1}(0) \cap \mathrm{Crit}(\mathcal{F}) = \emptyset$.

We denote $\mathcal{C}_{\mathcal{F}}^{sing} = d_{\mathcal{C}}^{-1}(0)$. From (A8), as $(x, \eta) \in \mathcal{C}_{\mathcal{F}}$ crosses a point in $\mathcal{C}_{\mathcal{F}}^{sing}$, one and only one of the eigenvalues of $\mathrm{Hess} f_\eta(x)$ changes its sign.

By the implicit function theorem, $\mathcal{C}_{\mathcal{F}}$ can be smoothly parameterized by η near any point of $\mathcal{C}_{\mathcal{F}} \setminus \mathcal{C}_{\mathcal{F}}^{sing}$ as $x = x(\eta)$. On the other hand, near $\mathcal{C}_{\mathcal{F}}^{sing}$ we have the following result.

Proposition 3.4.1. *Let $p = (x_p, \eta_p) \in \mathcal{C}_{\mathcal{F}}^{sing}$. We can choose local coordinates $(x_1, \ldots, x_n)$ on M such that x_p corresponds to $(0, \ldots, 0)$, and near p, $\mathcal{C}_{\mathcal{F}}$ is parameterized by x_n. Moreover, for some $c \neq 0$,*

$$\eta = \eta_p + cx_n^2 + O(x_n^3).$$

Thus $\eta|_{\mathcal{C}_{\mathcal{F}}}$ has a nondegenerate critical point at p.

This is a standard result, but we shall give a proof in Subsection 3.4.3.

3.4.2 The fast flow

The fast flow $\Phi_t : M \times \mathbb{R} \to M \times \mathbb{R}$ of (3.2) is the flow on $M \times \mathbb{R}$ determined by (3.2) for $\lambda = 0$. The set of equillibria of the fast flow is just $\mathcal{C}_{\mathcal{F}}$, and along the fast flow, η is constant.

If $p = (x_p, \eta_p) \in \mathcal{C}_{\mathcal{F}}$, then x_p is a critical point of the function $f_{\eta_p} : M \to \mathbb{R}$. A *fast solution* from $p_- \in \mathcal{C}_{\mathcal{F}}$ to $p_+ \in \mathcal{C}_{\mathcal{F}}$ is a solution $\widetilde{x}(t)$ of (3.2) for $\lambda = 0$ such that $\lim_{t \to \pm\infty} \widetilde{x}(t) = p_\pm$. A *fast orbit* from p_- to p_+ is an equivalence class of fast solutions from p_- to p_+ modulo time translation. A fast solution or orbit is *trivial* if the orbit consists of a single point.

For $p = (x_p, \eta_p) \in \mathcal{C}_{\mathcal{F}}$, x_p is a nondegenerate critical point if and only if $p \in \mathcal{C}_{\mathcal{F}} \setminus \mathcal{C}_{\mathcal{F}}^{sing}$. Note that the Morse index of a nondegenerate critical point x of f_η can be defined as the number of negative eigenvalues of the matrix $\mathrm{Hess} f_\eta(x)$. We define the Morse index of a degenerate critical point the same way. We shall use the already introduced notation $\mathrm{index}\,(x_p, f_{\eta_p})$ to denote the Morse index in this sense of x_p as a critical point of f_{η_p}.

$\mathrm{Index}\,(x_p, f_{\eta_p})$ is locally constant on $\mathcal{C}_{\mathcal{F}} \setminus \mathcal{C}_{\mathcal{F}}^{sing}$. Near $p \in \mathcal{C}_{\mathcal{F}}^{sing}$, the values of $\mathrm{index}\,(x_p, f_{\eta_p})$ on the two branches differ by one, and $\mathrm{index}\,(x_p, f_{\eta_p})$ is the lower of these two numbers.

Transversality assumptions

On $\mathcal{C}_{\mathcal{F}} \setminus \mathcal{C}_{\mathcal{F}}^{sing}$, the fast flow is normally hyperbolic. For any subset β of $\mathcal{C}_{\mathcal{F}} \setminus \mathcal{C}_{\mathcal{F}}^{sing}$ define

$$W^u(\beta) = \left\{ (x, \eta) \in M \times \mathbb{R} \mid \lim_{t \to -\infty} \Phi_t(x, \eta) \in \beta \right\}$$

and

$$W^s(\beta) = \left\{ (x,\eta) \in M \times \mathbb{R} \mid \lim_{t\to+\infty} \Phi_t(x,\eta) \in \beta \right\}.$$

Note that in this section we always have $\lambda = 0$, so we will not use the notation of Subsection 3.2.3 to specify a value of λ. If β is connected, define $\text{index}\,(\beta)$ to be $\text{index}\,(x_p, f_{\eta_p})$ for any $p = (x_p, \eta_p)$ in β.

We assume

(A10) $W^u\left(\mathcal{C}_\mathcal{F} \setminus \mathcal{C}_\mathcal{F}^{sing}\right)$ and $W^s\left(\mathcal{C}_\mathcal{F} \setminus \mathcal{C}_\mathcal{F}^{sing}\right)$ intersect transversally in $M \times \mathbb{R}$.

(A11) For each $p = (x_p, \eta_p) \in \text{Crit}(\mathcal{F})$, the pair (f_{η_p}, g) is Morse-Smale on M.

Assumption (A10) can be thought of as a weak version of the Morse-Bott-Smale transversality condition (see [1]) for a Morse-Bott function and a metric. The relation between the two assumptions is that (A11) implies that the transversality in (A10) holds near certain points, but (A10) does not imply (A11).

There are two special types of nontrivial fast orbits that play important roles in our construction. We introduce them in the remainder of this subsection.

Handle-slides and cusp orbits

By (A10) we see, if $\beta_1, \beta_2 \subset \mathcal{C}_\mathcal{F} \setminus \mathcal{C}_\mathcal{F}^{sing}$ are connected, then

$$\dim\left(W^u(\beta_1) \cap W^s(\beta_2)\right) = \text{index}\,(\beta_1) - \text{index}\,(\beta_2) + 1.$$

In particular, when $\text{index}\,(\beta_1) = \text{index}\,(\beta_2)$, the dimension is one, so the intersection consists of diskrete nontrivial fast orbits. We call such an orbit a *handle-slide*. If a handle-slide is contained in $M \times \{\eta\}$, we say that a handle-slide happens at η.

We assume:

(A12) The composition $\pi : \text{Crit}(\mathcal{F}) \cup \mathcal{C}_\mathcal{F}^{sing} \hookrightarrow M \times \mathbb{R} \to \mathbb{R}$ is injective. Moreover, handle-slides happen at distinct values of η that are not in the image of π.

Now we look near a point $p = (x_p, \eta_p) \in \mathcal{C}_\mathcal{F}^{sing}$. All the eigenvalues of $\text{Hess} f_{\eta_p}(x_p)$ have absolute values great than a constant $a > 0$, except for the one that changes its sign at p. Then we define $W_a^u(p)$ to be the space of maps $u : (-\infty, 0] \to M$ satisfying

1. $(u(t), \eta_p) = \Phi_t(u(0), \eta_p)$ for $t \le 0$;

2. For sufficiently large T, for $t \in (-\infty, -T]$,

$$u(t) = \exp_{x_p} V(t), \quad V \in W^{1,2}_a((-\infty, -T], T_{x_p}M).$$

Here for any open subset $\Omega \subset \mathbb{R}$, the space $W^{1,2}_a(\Omega)$ is the space of functions f with $e^{a|t|}f \in W^{1,2}(\Omega)$.

Similarly we can define $W^s_a(p)$. By the map $u \mapsto u(0)$, $W^u_a(p)$ and $W^s_a(p)$ are naturally identified with smooth submanifolds of M, which are called the a-exponential unstable and stable manifolds of p, having dimensions

$$\dim W^u_a(p) = \text{index}\,(x_p, f_{\eta_p}),\ \dim W^s_a(p) = n - \text{index}\,(x_p, f_{\eta_p}) - 1.$$

Let $\mathring{W}^u(p) = W^u(p) \setminus W^u_a(p)$ and $\mathring{W}^s(p) = W^s(p) \setminus W^s_a(p)$. These manifolds are unions of orbits that approach p polynomially rather than exponentially. They are submanifolds of $M \times \{\eta_p\}$ of dimension $\dim W^u_a(p) + 1$ and $\dim W^s_a(p) + 1$. We assume

(A13) For each $p = (x_p, \eta_p) \in \mathcal{C}^{sing}_{\mathcal{F}}$ and $q = (x_q, \eta_q) \in \mathcal{C}_{\mathcal{F}}$ with $\eta_p = \eta_q = \eta$, $W^u_a(x_p)$ and $W^s(x_q)$ (respectively $W^s_a(x_p)$ and $W^u(x_q)$) intersect transversely in $M \times \{\eta\}$, $\mathring{W}^u(x_p)$ and $W^s(x_q)$ (respectively $\mathring{W}^s(x_p)$ and $W^u(x_q)$) intersect transversely in $M \times \{\eta\}$.

The reader may refer to [13] and [12] for this transversality in the case of Lagrangian intersections.

(A13) implies that for $p = (x_p, \eta_p) \in \mathcal{C}^{sing}_{\mathcal{F}}$ and $q = (x_q, \eta_q) \in \mathcal{C}_{\mathcal{F}} \setminus \mathcal{C}^{sing}_{\mathcal{F}}$, $W^u_a(p) \cap W^s(q) = \emptyset$ if $\text{index}\,(x_q, f_{\eta_q}) \ge \text{index}\,(x_p, f_{\eta_p}) - 1$; and $W^u(q) \cap W^s_a(p) = \emptyset$ if $\text{index}\,(x_p, f_{\eta_p}) \ge \text{index}\,(x_q, f_{\eta_q}) - 2$. For example, $W^u(q) \cap W^s_a(p) = \emptyset$ if

$$\begin{aligned} 0 \ge \dim W^u(q) + \dim W^s_a(p) - (n+1) &= \text{index}\,(x_q, f_{\eta_q}) + (n - \text{index}\,(x_p, f_{\eta_p}) - 1) - (n+1) \\ &= \text{index}\,(x_q, f_{\eta_q}) - \text{index}\,(x_p, f_{\eta_p}) - 2. \end{aligned}$$

When the inequalities are equalities, $W^u(p) \cap W^s(q) = \mathring{W}^u(p) \cap W^s(q)$ and $W^u(q) \cap W^s(p) = W^u(q) \cap \mathring{W}^s(p))$ consist of isolated orbits corresponding to solutions that approach p like $\frac{1}{t}$ rather than exponentially. We call such orbits *cusp orbits*; parametrized ones are *cusp solutions*. An argument similar to one in [12] shows that between p and q there are only finitely many cusp orbits.

Short fast orbits

By Proposition 3.4.1, near $p \in \mathcal{C}_{\mathcal{F}}^{sing}$, one can parametrize $\mathcal{C}_{\mathcal{F}}$ by $\gamma_p : (-\epsilon, \epsilon) \to \mathcal{C}_{\mathcal{F}}$ such that

$$\gamma_p(0) = p,\ \eta(t) = \eta_p \pm s^2. \tag{3.15}$$

We choose the orientation of γ_p such that $\text{index}\,(\gamma_p((-\epsilon, 0))) = \text{index}\,(\gamma_p((0, \epsilon))) + 1$. For ϵ small enough there exists a unique such parametrization γ and we call γ the *canonical parametrization* near p. Then, for ϵ small enough, there is a unique trajectory of the flow of $-\nabla f_{\eta(s)}$ from $\gamma(-s)$ to $\gamma(s)$ and they are all contained in a small neighborhood of x_p.

Lemma 3.4.2. *There exists $\epsilon_0 > 0$ such that for all $p \in \mathcal{C}_{\mathcal{F}}^{sing}$ and for $s \in (0, \epsilon_0)$ there exists a unique trajectory of the fast flow from $\gamma_p(-s)$ to $\gamma_p(s)$, where $\gamma_p : (-\epsilon_0, \epsilon_0) \to \mathcal{C}_{\mathcal{F}}$ is the canonical parametrization of $\mathcal{C}_{\mathcal{F}}$ near p.*

If $0 < s \leq \epsilon < \epsilon_0$, then we call a fast orbit from $\gamma_p(-s)$ to $\gamma_p(s)$ an *ϵ-short orbit.*

Lemma 3.4.3. *For any ϵ small enough, there exists $\delta = \delta(\epsilon) > 0$ such that, for all $\eta \in \mathbb{R}$, any trajectories of the flow of $-\nabla f_\eta$ which is not an ϵ-short trajectory has energy no less than δ.*

Proof. Denote $\mathcal{C}_{\mathcal{F}}^{sing,\epsilon} = \cup_{p \in \mathcal{C}_{\mathcal{F}}^{sing}} \gamma_p((-\epsilon, \epsilon))$. Because we have assumed that the map $\mathcal{C}_{\mathcal{F}}^{sing} \to M \times \mathbb{R} \to \mathbb{R}$ is injective, any fast trajectory both of whose beginning and end lie in $\mathcal{C}_{\mathcal{F}}^{sing,\epsilon}$ much be an ϵ-short trajectory. So suppose the lemma doesn't hold, then there exists $\epsilon > 0$ and a sequence of nontrivial fast trajectories $(y_k, \eta_k) \subset M \times \{\eta_k\}$ such that

$$\lim_{k \to \infty} f_{\eta_k}(y_k(-\infty)) - f_{\eta_k}(y_k(+\infty)) = 0 \tag{3.16}$$

and they are not ϵ-short ones. Hence without loss of generality, we may assume that for all k, $y_k(-\infty) \notin \mathcal{C}_{\mathcal{F}}^{sing,\epsilon}$.

1. $\lim_{k \to \infty} \eta_k = \pm\infty$, which implies that $y_k(\pm\infty)$ converges to points in $\text{Crit}(\mu)$. However, if we rescale the orbit by $z_k(t) = y_k(\eta_k^{-1} t)$, then z_k converges to a (possibly broken) nontrivial trajectory of the flow of $-\nabla\mu$, which contradicts with (3.16).

2. $\lim_{k \to \infty} \eta_k = \eta_\infty \in \mathbb{R}$. Then a subsequence of y_k converges to a broken trajectory of f_{η_∞}, which must be a constant one. Hence

$$\lim_{k \to \infty} (y_k(\pm\infty)) =: y_\infty \in \mathcal{C}_{\mathcal{F}} \setminus \mathcal{C}_{\mathcal{F}}^{sing,\epsilon}.$$

So for k large enough, $y_k(\pm\infty)$ lie in the same connected component of $\mathcal{C}_\mathcal{F} \setminus \mathcal{C}_\mathcal{F}^{sing,\frac{\epsilon}{2}}$. This is impossible because on each such component, distinct points have distinct values of η.

□

3.4.3 The slow equation

On $\mathcal{C}_\mathcal{F} \setminus \mathcal{C}_\mathcal{F}^{sing}$, which is parameterized by η, the slow equation is given by restricting the second equation of (3.2) to $\mathcal{C}_\mathcal{F} \setminus \mathcal{C}_\mathcal{F}^{sing}$ and dividing by λ^2:

$$\eta' = -\mu(x(\eta)). \tag{3.17}$$

Thus η' changes sign only when $\mu = 0$. We have

$$\mathrm{Crit}(\mathcal{F}) = \{(x, \eta) \in \mathcal{C}_\mathcal{F} \mid \mu(x) = 0\}.$$

Thus η' changes sign only at points of $\mathrm{Crit}(\mathcal{F})$. By (A9), η' does not change sign at points of $\mathcal{C}_\mathcal{F}^{sing}$.

Proposition 3.4.4. *Equilibria of (3.17) are hyperbolic.*

Proof. Let $p = (x_p, \eta_p)$ be an equilibrium of (3.17). Then $\mu(x_p) = 0$, so by (A3), $d\mu(x_p) \neq 0$. Hence we can choose local coordinates $(x_1, \ldots, x_n)$ near x_p such that $\mu(x_1, \ldots, x_n) = x_n$. Near p, $\mathcal{C}_\mathcal{F}$ can be defined without reference to the metric g by the equations

$$\frac{\partial f}{\partial x_i} + \eta\delta_{nj},\ j = 1, \ldots, n; \quad \delta_{ij} = \text{Kronecker delta}. \tag{3.18}$$

Let $e_n = \begin{pmatrix} 0 & \ldots & 0 & 1 \end{pmatrix}^T$, the nth standard unit basis vector in $\mathbb{R}^n$. Then the derivative of the system (3.18) is the $n \times (n+1)$ matrix

$$\begin{pmatrix} \mathrm{Hess} f(x) & e_n \end{pmatrix} = \begin{pmatrix} A & b & 0 \\ b^T & c & 1 \end{pmatrix}, \tag{3.19}$$

where A is an $(n-1) \times (n-1)$ matrix, and $(b^T, c) = \left(\frac{\partial^2 f}{\partial x_1 \partial x_n}, \cdots, \frac{\partial^2 f}{\partial x_{n-1} \partial x_n}, \frac{\partial^2 f}{\partial x_n^2}\right)$.

Since x_p is a critical point of $f|_{\mu^{-1}(0)}$, by (A5), A is invertible. The matrix (3.19) has full rank; a nonzero vector in its kernel is tangent at p to $\mathcal{C}_\mathcal{F}$. Since $p \in \mathcal{C}_\mathcal{F} \setminus \mathcal{C}_\mathcal{F}^{sing}$, such a vector has nonzero η-component, and one can check that it has nonzero x_n-component. Therefore near p, when $\mathcal{C}_\mathcal{F}$ is

parameterized by η, we have $x(\eta) = (y(\eta), x_n(\eta))$ with

$$x_n(\eta) = a(\eta - \eta_p) + \mathcal{O}(\eta - \eta_p)^2, \quad a \neq 0.$$

But then (3.17) becomes

$$\eta' = a(\eta - \eta_p) + \mathcal{O}(\eta - \eta_p)^2,$$

which proves the result. □

Next we diskuss the slow equation near points of $\mathcal{C}_{\mathcal{F}}^{sing}$. Recall that if p_0 is an equilibrium of $\dot{p} = h(p)$, h is C^s, $Dh(p_0)$ has m eigenvalues (counting multiplicity) with real part 0, and E is the corresponding m-dimensional invariant subspace of $\dot{v} = Dh(p_0)v$, then there is a C^s invariant manifold through p_0 and tangent there to E, called the *center manifold* of $\dot{p} = h(p)$ at p_0.

Proposition 3.4.5. *Let $p = (x_p, \eta_p) \in \mathcal{C}_{\mathcal{F}}^{sing}$. We can choose local coordinates $(x_1, \ldots, x_n)$ on M near x_p, with x_p corresponding to 0, in which*

$$D\nabla f(0) = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}, \quad A \text{ an invertible symmetric } (n-1) \times (n-1) \text{ matrix.} \tag{3.20}$$

The center manifold of (3.2) at p is parameterized by (x_n, η). There are numbers $c \neq 0$ and $d \neq 0$ such that the system (3.2), restricted to the center manifold, is

$$x_n' = c(\eta - \eta_p) + dx_n^2 + \ldots, \tag{3.21}$$

$$\eta' = -\lambda^2(\mu(x_p) + \ldots), \tag{3.22}$$

where ... indicates terms of higher order (and, in (3.21), other second-order terms). Therefore near p, $\mathcal{C}_{\mathcal{F}}$ is parameterize by x_n, and $\eta|\mathcal{C}_{\mathcal{F}}$ has a nondegenerate critical point at p.

Proof. (A7) implies that we can choose local coordinates $(x_1, \ldots, x_n)$ on M near x_p, with x_p corresponding to 0, in which (3.20) holds. In these coordinates, $\mu = \mu(x_p) + a^T x + \ldots$ with $a \in \mathbb{R}^n$. Write $x = (y, x_n)$ with $y \in \mathbb{R}^{n-1}$, and write $a = (b, c)$ with $b \in \mathbb{R}^{n-1}$ and $c \in \mathbb{R}$. Near p the system

(3.2) becomes

$$y' = Ay + b(\eta - \eta_p) + \ldots, \tag{3.23}$$

$$x_n' = c(\eta - \eta_p) + dx_n^2 + \ldots, \tag{3.24}$$

$$\eta' = -\lambda^2(\mu(x_p) + \ldots), \tag{3.25}$$

where $\ldots$ indicates higher-order terms (and, in (3.24), other second-order terms). (A7) implies that $c \neq 0$, and (A8) implies that $d \neq 0$. The linearization of (3.23)–(3.25) at $(y, x_n, \eta) = (0, 0, \eta_p)$ is

$$\begin{pmatrix} y' \\ x_n' \\ \eta' \end{pmatrix} \begin{pmatrix} A & 0 & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} y \\ x_n \\ \eta - \eta_p \end{pmatrix}.$$

The matrix has a two-dimensional generalized eigenspace for the eigenvalue 0 that is tangent to the center manifold of (3.23)–(3.25) at p. The center manifold is parameterized by (x_n, η), and to lowest order it is given by

$$y = -A^{-1}b(\eta - \eta_p)$$

The restriction of (3.23)–(3.25) to the center manifold is therefore given by (3.21)–(3.22). □

Proposition 3.4.1 is of course just a partial restatement of Proposition 3.4.5.

3.4.4 Three indices

In this subsection we describe the relation between three different Morse indices:

1. For any $p = (x_p, \eta_p) \in \mathcal{C}_{\mathcal{F}}$, we have $\operatorname{index}(x_p, f_{\eta_p})$, used in Subsection 3.4.2.
2. For $p = (x_p, \eta_p) \in \operatorname{Crit}(\mathcal{F})$, we can consider $\operatorname{index}(p, \mathcal{F})$.
3. If $p = (x_p, \eta_p) \in \operatorname{Crit}(\mathcal{F})$, then $x_p \in \operatorname{Crit}(f|_{\mu^{-1}(0)})$, and we can consider $\operatorname{index}(x_p, f|_{\mu^{-1}(0)})$.

We have already seen in Lemma 3.2.1 that $\operatorname{index}(p, \mathcal{F}) = \operatorname{index}(x_p, f|_{\mu^{-1}(0)}) + 1$ for $p = (x_p, \eta_p) \in \operatorname{Crit}(\mathcal{F})$. The best way to see the relation of the different indices is to look at the corresponding unstable manifolds.

Proposition 3.4.6. *Let $p = (x_p, \eta_p) \in \operatorname{Crit}(\mathcal{F})$.*

1. If p is a repeller of the slow equation, then $\operatorname{index}(p, \mathcal{F}) = \operatorname{index}(x_p, f_{\eta_p}) + 1$.

2. If p is an attractor of the slow equation, then $\operatorname{index}(p, \mathcal{F}) = \operatorname{index}(x_p, f_{\eta_p})$.

Proof. If p is a repeller of the slow equation, then for small $\lambda > 0$, the equilibrium p of (3.2) has one more positive eigenvalue than the equilibrium x_p of f_{η_p}. Since the index of a critical point of a Morse function h is the number of positive eigenvalues of the corresponding equilibrium of ∇h for any metric, (1) follows. (2) is similar. □

3.5 Adiabatic limit $\lambda \to 0$

3.5.1 Slow solutions, fast solutions, and fast-slow solutions

Let $p_-, p_+ \in \mathcal{C}_{\mathcal{F}}$.

Recall that a nontrivial fast solution from p_- to p_+ is a nonconstant solution $\widetilde{x}(t)$, $-\infty < t < \infty$, of (3.2) for $\lambda = 0$, such that $\lim_{t\to\pm\infty} \widetilde{x}(t) = p_\pm$.

Let $(x(\eta), \eta)$ parameterize the closure of a component of $\mathcal{C}_{\mathcal{F}} \setminus \mathcal{C}_{\mathcal{F}}^{sing}$. Let $I(\alpha_-, \alpha_+) \subset \mathbb{R}$ be the closed interval from α_- to α_+, with $\alpha_- \le \alpha_+$; we allow $\alpha_- = -\infty$, $\alpha_+ = +\infty$, and $\alpha_- = \alpha_+ \in \mathbb{R}$. Let $\eta(t)$, t in the interior of $I(\alpha_-, \alpha_+)$, be a solution of $\eta' = -\mu(x(\eta))$. If α_- (respectively α_+) is finite, we extend $\eta(t)$ continuously to α_- (respectively α_+). Let $p(t) = (x(\eta(t)), \eta(t))$, $t \in I(\alpha_-, \alpha_+)$, and let $p_\pm \in \mathcal{C}_{\mathcal{F}}$. Then $p(t)$ is a *slow solution from p_- to p_+* (for short, a *slow solution*) provided:

1. if α_- (respectively α_+) is finite, then $p(\alpha_-) = p_-$ (respectively $p(\alpha_+) = p_+$);
2. if $\alpha_- = -\infty$ (respectively $\alpha_+ = +\infty$), then $\lim_{t\to-\infty} p(t) = p_-$ (respectively $\lim_{t\to+\infty} p(t) = p_+$).

A slow solution or its orbit is *trivial* if the orbit is a single point in $\mathcal{C}_{\mathcal{F}}$.

A *fast-slow solution from p_- to p_+* of (3.2) (for short, a *fast-slow solution*) is a sequence

$$\mathscr{X} = (p_0, \sigma_1, p_1, \sigma_2, p_2, \ldots, p_{n-1}, \sigma_n, p_n) \tag{3.26}$$

such that:

1. $p_0 = p_-$, $p_n = p_+$.
2. Each $p_i \in \mathcal{C}_{\mathcal{F}}$.
3. Each σ_i is a nontrivial fast solution or a slow solution from p_{i-1} to p_i. Trivial slow solutions are allowed, but σ_1 and σ_n are not allowed to be trivial slow solutions.

4. Either (i) σ_i is slow for i even and fast for i odd, or (ii) σ_i is slow for i odd and fast for i even.

Trivial slow solutions are allowed to deal with the possibility that a fast solution to p is followed by a fast solution from p.

An orbit (fast orbit, slow orbit, or fast-slow orbit) is an equivalence class of solutions obtained by forgetting the parametrization (but remembering the orientation in which $\mathcal{F}$ decreases). In particular, a fast-slow orbit is denoted by $[\mathscr{X}]$ for $\mathscr{X}$ as in (3.26).

Remark 3.5.1. A fast-slow orbit defined above is similar to a "flow line with cascades" in Morse-Bott theory (see [14]), where the fast solutions correspond to "cascades". The difference is that the slow manifold $\mathcal{C}_{\mathcal{F}}$ is not everywhere normally hyperbolic, and we can have $p_i \in \mathcal{C}_{\mathcal{F}}^{sing}$.

In the remainder of this subsection, we will diskuss some properties of fast-slow orbits from $p_- \in \mathcal{C}_{\mathcal{F}}$ to $p_+ \in \mathcal{C}_{\mathcal{F}}$. We are particularly interested in the case $p_\pm \in \mathrm{Crit}\,(\mathcal{F})$ and $\mathrm{index}(p_-, \mathcal{F}) - \mathrm{index}(p_+, \mathcal{F}) = 1$.

The following proposition is obvious.

Proposition 3.5.2. *Consider a nontrivial slow solution $p(t)$, $t \in I(\alpha_-, \alpha_+)$, from p_- to p_+.*

1. *If $p_- \in \mathcal{C}_{\mathcal{F}}^{sing}$, then either (i) for all $t \in (\alpha_-, \alpha_+)$, $\mathrm{index}\,(x_{p(t)}, f_{\eta_{p(t)}}) = \mathrm{index}\,(x_-, f_{\eta_-})$, or (ii) for all $t \in (\alpha_-, \alpha_+)$, $\mathrm{index}\,(x_{p(t)}, f_{\eta_{p(t)}}) = \mathrm{index}\,(x_-, f_{\eta_-}) + 1$.*

2. *If $p_+ \in \mathcal{C}_{\mathcal{F}}^{sing}$, then either (i) for all $t \in (\alpha_-, \alpha_+)$, $\mathrm{index}\,(x_{p(t)}, f_{\eta_{p(t)}}) = \mathrm{index}\,(x_+, f_{\eta_+})$, or (ii) for all $t \in (\alpha_-, \alpha_+)$, $\mathrm{index}\,(x_{p(t)}, f_{\eta_{p(t)}}) = \mathrm{index}\,(x_+, f_{\eta_+}) + 1$.*

A slow solution $p(t)$ for $t \in I(\alpha_-, \alpha_+)$ is *regular* if it is nontrivial and satisfies:

1. If $p_- \in \mathcal{C}_{\mathcal{F}}^{sing}$, then for all $t \in (\alpha_-, \alpha_+)$, $\mathrm{index}\,(x_{p(t)}, f_{\eta_{p(t)}}) = \mathrm{index}\,(x_-, f_{\eta_-}) + 1$.

2. If $p_+ \in \mathcal{C}_{\mathcal{F}}^{sing}$, then for all $t \in (\alpha_-, \alpha_+)$, $\mathrm{index}\,(x_{p(t)}, f_{\eta_{p(t)}}) = \mathrm{index}\,(x_+, f_{\eta_+})$.

A slow orbit is regular if it is the orbit of a regular slow solution.

See Figure 3.1.

Proposition 3.5.3. *Let γ be either a slow or fast solution from $p_- = (x_-, \eta_-)$ to $p_+ = (x_+, \eta_+)$.*

1. *If γ is a regular slow solution, then:*

   (a) *If $p_- \in \mathcal{C}_{\mathcal{F}}^{sing}$, then $\mathrm{index}\,(x_+, f_{\eta_+}) = \mathrm{index}\,(x_-, f_{\eta_-}) + 1$.*

   (b) *Otherwise $\mathrm{index}\,(x_+, f_{\eta_+}) = \mathrm{index}\,(x_-, f_{\eta_-})$*

2. *If γ is a nontrivial fast solution (so $\eta_+ = \eta_-$), then:*

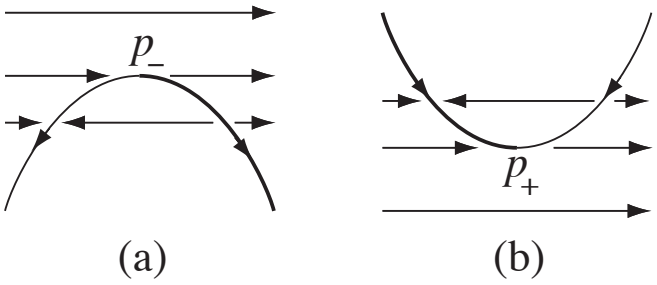

Figure 3.1: (a) Direction of slow flow is downward. The thick curve is a regular slow orbit that starts at p_-. Points on it have index one greater than p_-. (b) Direction of slow flow is downward. The thick curve is a regular slow orbit that ends at p_+. Points on it have the same index as p_+.

(a) At most one of $p_\pm$ *is in* $\mathrm{Crit}(\mathcal{F}) \cup \mathcal{C}^{sing}_{\mathcal{F}}$.

(b) $\mathrm{index}\,(x_+, f_{\eta_+}) \le \mathrm{index}\,(x_-, f_{\eta_-})$.

(c) If $p_+ \in \mathrm{Crit}(\mathcal{F}) \cup \mathcal{C}^{sing}_{\mathcal{F}}$ *or* $p_- \in \mathrm{Crit}(\mathcal{F})$, *then* $\mathrm{index}\,(x_+, f_{\eta_+}) \le \mathrm{index}\,(x_-, f_{\eta_-}) - 1$.

Proof. The result for slow solutions is immediate from their definition.

For fast solutions, (2)(a) follows from assumption (A12)

Let $p(t)$ be a fast solution, let $\mathrm{index}\,(x_-, f_{\eta_-}) = k$, and $\mathrm{index}\,(x_+, f_{\eta_+}) = l$. (Of course $\eta_+ = \eta_-$.)

If $p_+ \in \mathrm{Crit}(\mathcal{F}) \cup \mathcal{C}^{sing}_{\mathcal{F}}$, then (A11) or (A13) implies that $W^u(p_-)$ and $W^s(p_+)$ intersect transversally within $M \times \{\eta_-\}$. We have $\dim W^u(p_-) = k$, and $\dim W^s(p_+) = n - l$, so the intersection is nonempty provided $k + (n - l) \ge n + 1$, i.e., $l \le k - 1$.

If $p_- \in \mathrm{Crit}(\mathcal{F})$, the same argument applies. However, if $p_- \in \mathcal{C}^{sing}_{\mathcal{F}}$, then $\dim W^u(p_-) = k + 1$. We again have $\dim W^s(p_+) = n - l$, so the intersection is nonempty provided $(k+1)+(n-l) \ge n+1$, i.e., $l \le k$.

If neither p_- nor p_+ is in $\mathrm{Crit}(\mathcal{F}) \cup \mathcal{C}^{sing}_{\mathcal{F}}$, then we do not know that $W^u(p_-)$ and $W^s(p_+)$ intersect transversally within $M \times \{\eta_-\}$. However, by (A10), $(k + 1) + (n - l + 1) \ge n + 2$, so $l \le k$. □

Proposition 3.5.4. *Consider a fast-slow solution* $\mathscr{X} = (p_0, \sigma_1, p_1, \sigma_2, p_2, \ldots, p_{n-1}, \sigma_n, p_n)$ *from* $p_- = (x_-, \eta_-)$ *to* $p_+ = (x_+, \eta_+)$, *with neither in* $\mathcal{C}^{sing}_{\mathcal{F}}$ *and* $\mathrm{index}\,(x_+, f_{\eta_+}) \ge \mathrm{index}(x_-, f_{\eta_-}) = k$. *Then*

1. *For all* $i \ge 1$, $\mathrm{index}\,(x_{p_i}, f_{p_i}) = k$, *except for the case where* $p_i \in \mathcal{C}^{sing}_{\mathcal{F}}$ *and* σ_i *is a fast solution, in which case* $\mathrm{index}\,(x_{p_i}, f_{p_i}) = k - 1$. *In particular,* $\mathrm{index}\,(x_+, f_{\eta_+}) = k$.

2. *All slow solutions appearing in* $\mathscr{X}$ *are regular. In particular, they are nontrivial.*

3. *If* $p_i \in \mathrm{Crit}(\mathcal{F})$, *then* $i = 0$ *or* $i = n$. *In the first case,* $p_0 = p_- \in \mathrm{Crit}^-(\mathcal{F})$ *and* σ_1 *is slow; in the second case,* $p_n = p_+ \in \mathrm{Crit}^+(\mathcal{F})$ *and* σ_n *is slow.*

4. The fast solutions are handle-slides or cusp solutions.

Proof. Along a fast-slow solution, index $(x_{p_i}, f_{p_i}) \leq$ index $(x_{p_{i-1}}, f_{p_{i-1}})$, except along a regular slow solution for which $p_{i-1} \in \mathcal{C}_{\mathcal{F}}^{sing}$ and $p_i \in \mathcal{C}_{\mathcal{F}} \backslash \mathcal{C}_{\mathcal{F}}^{sing}$; in this case index $(x_{p_i}, f_{p_i}) =$ index $(x_{p_{i-1}}, f_{p_{i-1}})+$ 1. However, the index drops along the fast solution from p_{i-2} to p_{i-1}. Thus (1) holds; also, nontrivial slow solutions must be regular and (3) must hold, otherwise we would have index $(x_+, f_{\eta_+}) <$ index(x_-, f_{η_-}). (3) rules out fast solutions that start or end in Crit$(\mathcal{F})$; any remaining fast solutions are handle-slides or cusp solutions, so (4) holds. Since handle-slides and cusp solutions occur at different values of η, slow solutions must be nontrivial, which completes the proof of (2). □

Now we describe fast-slow solutions between two critical points whose indices differ by 1. Recall from Subsection 3.2.1 that $\text{Crit}_k(\mathcal{F})$ denotes the set of index k critical points of $\mathcal{F}$. Let $\text{Crit}_k^+(\mathcal{F})$ (respectively $\text{Crit}_k^-(\mathcal{F})$) denote the set of equilibria in $\text{Crit}_k(\mathcal{F})$ that are stable (respectively unstable) equilibria of the slow equation. Let $\text{Crit}^{\pm}(\mathcal{F}) = \cup_{k \geq 0} \text{Crit}_k^{\pm}(\mathcal{F})$.

Proposition 3.5.5. *Let $p_\pm = (x_\pm, \eta_\pm) \in \text{Crit}(\mathcal{F})$ with $p_- \in \text{Crit}_k(\mathcal{F})$, $p_+ \in \text{Crit}_{k-1}(\mathcal{F})$. A fast-slow solution $\mathscr{X} = (p_0, \sigma_1, p_1, \sigma_2, p_2, \ldots, p_{n-1}, \sigma_n, p_n)$ from p_- to p_+ has the following properties.*

1. *If σ_i is a fast solution from $p_{i-1} = (x_{p_{i-1}}, \eta_{p_{i-1}})$ to $p_i = (x_{p_i}, \eta_{p_i})$ (of course $\eta_{p_i} = \eta_{p_{i-1}}$), then* index $(x_{p_i}, f_{\eta_{p_i}}) =$ index $(x_{p_{i-1}}, f_{\eta_{p_{i-1}}})$, *unless:*

   (a) $p_i \in \mathcal{C}_{\mathcal{F}}^{sing}$, in which case index $(x_{p_i}, f_{\eta_{p_i}}) =$ index $(x_{p_{i-1}}, f_{\eta_{p_{i-1}}}) - 1$.

   (b) $i = 1$, $p_0 = p_- \in \text{Crit}_k^+(\mathcal{F})$, in which case σ_1 is fast (so $\eta_{p_1} = \eta_p$) and

   $$\text{index}\,(x_{p_1}, f_{\eta_{p_1}}) = \text{index}\,(x_-, f_{\eta_-}) - 1.$$

   (c) $i = n$, $p_n = p_+ \in \text{Crit}_{k-1}^-(\mathcal{F})$, in which case σ_n is fast (so $\eta_q = \eta_{p_{n-1}}$) and

   $$\text{index}\,(x_+, f_{\eta_+}) = \text{index}\,(x_{p_{n-1}}, f_{\eta_{p_{n-1}}}) - 1.$$

2. *All slow solutions appearing in $\mathscr{X}$ are regular. In particular, they are nontrivial.*

3. *For $i = 1, \ldots, n-1$, $p_i \notin \text{Crit}(\mathcal{F})$.*

4. *The fast solutions in $\mathscr{X}$ are handle-slides or cusp solutions, except for σ_1 when $p_- \in \text{Crit}_k^+(\mathcal{F})$ and σ_n when $p_+ \in \text{Crit}_{k-1}^-(\mathcal{F})$.*

In addition:

(I) Suppose $p_- \in \mathrm{Crit}_k^+(\mathcal{F})$ *and* $p_+ \in \mathrm{Crit}_{k-1}^+(\mathcal{F})$*. Then the odd* σ_i *are fast, and* n *is even.*

(II) Suppose $p_- \in \mathrm{Crit}_k^+(\mathcal{F})$ *and* $p_+ \in \mathrm{Crit}_{k-1}^-(\mathcal{F})$*. Then the odd* σ_i *are fast, and* n *is odd.*

(III) Suppose $p_- \in \mathrm{Crit}_k^-(\mathcal{F})$ *and* $p_+ \in \mathrm{Crit}_{k-1}^+(\mathcal{F})$*. Then the odd* σ_i *are slow, and* n *is odd.*

(IV) Suppose $p_- \in \mathrm{Crit}_k^-(\mathcal{F})$ *and* $p_+ \in \mathrm{Crit}_{k-1}^-(\mathcal{F})$*. Then the odd* σ_i *are slow, and* n *is even.*

Proof. We prove (1)–(4) separately for the cases (I) and (II). The proofs for cases (III) and (IV) are similar so we omit them. We remark that in one of the cases, case (III), we can have $n = 1$, which means $\mathscr{X}$ may contain a single slow solution and no fast solutions.

Proof in case (I). In this case, $\mathrm{index}\,(x_-, f_{\eta_-}) = k$ and $\mathrm{index}\,(x_+, f_{\eta_+}) = k-1$. Since p_- is an attractor for the slow equation, the first solution σ_1 must be fast, so the odd σ_i are fast. By Proposition 3.5.3 (2), $p_1 \notin \mathrm{Crit}(\mathcal{F}) \cup \mathcal{C}_{\mathcal{F}}^{sing}$ and $\mathrm{index}\,(x_{p_1}, f_{\eta_{p_1}}) \leq k-1$. Consider the portion of the fast-slow solution from p_1 to q. Then Proposition 3.5.4 implies that $\mathrm{index}\,(x_{p_1}, f_{\eta_{p_1}}) = k-1$, conclusion (2) of Proposition 3.5.5 holds, and $p_i \notin \mathrm{Crit}^{\pm}(\mathcal{F})$ for $i = 2, \ldots, n-1$ so (3) holds. Proposition 3.5.4 also implies that all fast solutions except the first are handle-slides or cusp solutions, so (4) holds.

Since p_+ is an attractor for the slow equation, *a priori* the last solution could be slow or fast. However, if the last solution were fast, from Proposition 3.5.3 (2) we would have $\mathrm{index}\,(x_+, f_{\eta_+}) < \mathrm{index}\,(x_{p_{n-1}}, f_{\eta_{p_{n-1}}}) = k-1$, contradiction. Therefore the last solution is slow, so n is even. Then (1) and (4) follow from Proposition 3.5.4.

Proof in case (II). As with case (I), $\mathrm{index}\,(x_-, f_{\eta_-}) = k$; σ_1 must be fast, so the odd σ_i are fast; $p_1 \notin \mathrm{Crit}(\mathcal{F}) \cup \mathcal{C}_{\mathcal{F}}^{sing}$; and $\mathrm{index}\,(x_{p_1}, f_{\eta_{p_1}}) \leq k-1$. Since p_+ is a repeller for the slow equation, $\mathrm{index}\,(x_+, f_{\eta_+}) = k-2$, and the last solution must be fast. Then $p_{n-1} \notin \mathrm{Crit}(\mathcal{F}) \cup \mathcal{C}_{\mathcal{F}}^{sing}$, and by Proposition 3.5.3 (2), $\mathrm{index}\,(x_{p_{n-1}}, f_{\eta_{p_{n-1}}}) \geq k-1$. As in the previous argument, for the portion of the fast-slow solution from p_1 to p_{n-1}, apply Proposition 3.5.4; we obtain (1)–(4). Since the first and last solutions are fast, n is odd. □

3.5.2 The chain complex

Now let $p, q \in \mathrm{Crit}(\mathcal{F})$. We denote by $\widetilde{\mathcal{N}}^0(p,q)$ the space of all fast-slow solutions from p to q and $\mathcal{N}^0(p,q)$ the space of all fast-slow orbits from p to q.

Proposition 3.5.6. *If* $p, q \in \mathrm{Crit}(\mathcal{F})$ *and* $\mathrm{index}\,(q, \mathcal{F}) = \mathrm{index}\,(p, \mathcal{F}) - 1$*, then* $\#\mathcal{N}^0(p,q) < +\infty$.

Proof. By Proposition 3.5.5, we only need to show that there are finitely many handle-slides or cusp orbits contained in $\mathcal{F}^{-1}([\mathcal{F}(q), \mathcal{F}(p)])$. But this follows from our description of $\mathcal{C}_{\mathcal{F}}$ at the beginning of Section 3.4 and the transversality assumption (A10), (A11), (A13). □

Now we define a chain complex $(\mathcal{C}^0, \partial^0)$ of $\mathbb{Z}_2$-modules. It has the same generators and gradings as CM^λ. Its boundary operator $\partial^0 : \mathcal{C}^0_k \to \mathcal{C}^0_{k-1}$ is defined by

$$\langle \partial^0 p, q \rangle = \#\mathcal{N}^0(p, q) \bmod 2.$$

The main theorem of this section and the essentially new thing we constructed here is:

Theorem 3.5.7. *$(\mathcal{C}^0, \partial^0)$ is a chain complex, i.e., $\partial^0 \circ \partial^0 = 0$. Moreover, the homology $H_k(\mathcal{C}^0, \partial^0)$ is canonically isomorphic to $H_{k-1}(\mu^{-1}(0), \mathbb{Z}_2)$.*

The proof of this theorem is carried out in the following way. Let $p, q \in \mathrm{Crit}(\mathcal{F})$. First we prove that for small $\lambda > 0$, any orbit in $\mathcal{M}^\lambda(p, q)$ is close to a fast-slow orbit in $\mathcal{N}^0(p, q)$ (the compactness theorem, Theorem 3.5.9). Then we restrict to the case $\mathrm{index}(p, \mathcal{F}) - \mathrm{index}(q, \mathcal{F}) = 1$ and show that for each small $\lambda > 0$, each orbit in $\mathcal{N}^0(p, q)$ is close to exactly one orbit in $\mathcal{M}^\lambda(p, q)$ (the gluing theorem, Theorem 3.5.14). The resulting one-to-one correspondence implies that $(\mathcal{C}^0, \partial^0)$ is isomorphic to CM^λ for small $\lambda \in \Lambda^{reg}$. Then by Corollary 3.3.2, we obtain the isomorphism.

3.5.3 The compactness theorem

Definition 3.5.8. Let $p, q \in \mathrm{Crit}(\mathcal{F})$. Let $\lambda_\nu \in \mathbb{R}^+$ be a sequence such that $\lim_{\nu\to\infty} \lambda_\nu = 0$. We say that a sequence of (parametrized) trajectory $\widetilde{x}_\nu \in \widetilde{\mathcal{M}}^{\lambda_\nu}(p, q)$ *converges* to a parametrized fast-slow solution $\mathscr{X} = (p_0, \sigma_1, p_1, \sigma_2, \ldots, p_{n-1}, \sigma_n, p_n) \in \widetilde{\mathcal{N}}^0(p, q)$, if the following holds

1. For any fast solution σ_i contained in $\mathscr{X}$, there exists $t_{i,\nu} \in \mathbb{R}$ such that the sequence of maps

$$\widetilde{x}_\nu(\cdot + t_{i,\nu}) = (x_\nu(\cdot + t_{i,\nu}), \eta_\nu(\cdot + t_{i,\nu}))$$

converges to σ_i in the C^∞_{loc}-topology.

2. For any slow solution σ_j contained in $\mathscr{X}$, there exists a sequence of intervals $I_{j,\nu} \subset \mathbb{R}$ such that

$$\lim_{\nu\to\infty} d(I_{j,\nu}, t_{i,\nu}) = \infty$$

for any $t_{i,\nu}$ satisfying the first condition for any fast solution σ_i, and such that

$$\lim_{\nu\to\infty} \widetilde{x}_\nu(I_{j,\nu}) = \sigma_j$$

in Hausdorff topology.

It is easy to see that the limit is unique in $\mathcal{N}^0(p,q)$ and it only depends on the sequence of orbits $[\widetilde{x}_\nu]$ but not the representatives. Moreover, the image of $\widetilde{x}_\nu$ will converge to the image of $\mathscr{X}$ in Hausdorff topology.

We would like to prove the following theorem in the remaining of this section.

Theorem 3.5.9. *Suppose $p,q \in \mathrm{Crit}(\mathcal{F})$ and $\lambda_\nu \to 0^+$ be a sequence of real numbers. Suppose $\widetilde{x}_\nu = (x_\nu, \eta_\nu) \in \widetilde{\mathcal{M}}^{\lambda_\nu}(p,q)$. Then there exists a subsequence (still indexed by ν), and a fast-slow solution $\mathscr{X} = (p_0, \sigma_1, p_1, \sigma_2, \cdots, p_{n-1}, \sigma_n, p_n) \in \widetilde{\mathcal{N}}^0(p,q)$, such that (x_ν, ξ_ν) converges to $\mathscr{X}$ in the sense of Definition 3.5.8.*

A key point in proving this theorem is to use the energy control to bound the number of pieces of fast trajectories appearing in the limit. This is similar to the proof of Gromov compactness theorem of J-holomorphic curves where there is a lower bound of energy for any nontrivial bubble. After finding all the fast trajectories in the limit, each adjacent two of them are connected by a slow trajectory. And before the first fast trajectory and after the last fast trajectory, one may or may not need to add nontrivial slow ones, depending on the types of p and q (i.e., repeller or attracter of the slow flow).

From now on the sequence $\widetilde{x}_\nu$ is given and we are free to take subsequences as many times as we want.

The limit set

For any subsequence ν' of ν and intervals $I_{\nu'} \subset \mathbb{R}$, set

$$L_\infty(\nu', I_{\nu'}) = \bigcap_{k\geq 1} \overline{\cup_{\nu'\geq k} \widetilde{x}_{\nu'}(I_{\nu'})} \subset M \times \mathbb{R}.$$

Lemma 3.5.10. *We have*

(I) $(x,\eta) \in L_\infty(\nu', I_{\nu'})$ if and only if there exists a subsequence ν'' of ν' and a sequence of numbers $t_{\nu''} \in I_{\nu''}$, such that $\lim_{\nu''\to\infty} \widetilde{x}_{\nu''}(t_{\nu''}) = (x,\eta)$.

(II) $L_\infty(\nu', I_{\nu'})$ is compact.

(III) If there exist $s_{\nu'} \in I_{\nu'}$ such that $\lim_{\nu'\to\infty} \widetilde{x}_{\nu'}(s_{\nu'})$ exists, then $L_\infty(\nu', I_{\nu'})$ is connected.

Proof. The first statement and the closedness is by definition, and compactness follows from Lemma 3.2.3. It remains to show the connectedness. If not, then there are two nonempty closed subset $L_1, L_2 \subset M \times i\mathbb{R}$ such that $L_\infty(\nu', I_{\nu'}) = L_1 \cup L_2$, and L_1, L_2 has a nonzero distance. Suppose $\lim_{\nu'\to\infty} \widetilde{x}_{\nu'}(s_{\nu'}) \in L_1$, then there exists a subsequence (still indexed by ν') and $w_{\nu'} \in I_{\nu'}$ such that $d(\widetilde{x}_{\nu'}(w_{\nu'}), L_2) \to 0$. Then there exists $w'_{\nu'} \in [s_{\nu'}, w_{\nu'}]$ such that $d(\widetilde{x}_{\nu'}(w'_{\nu'}), L_1) = d(\widetilde{x}_{\nu'}(w'_{\nu'}), L_2) > 0$ for the same subsequence. Then there is a subsequence of $\widetilde{x}_{\nu'}(w'_{\nu'})$ converges to a point "between" L_1 and L_2, which contradicts with $L_\infty(\nu', I_{\nu'}) = L_1 \cup L_2$. □

The following lemma is obvious.

Lemma 3.5.11. *suppose we have two sequence of numbers, $s_\nu < t_\nu$ with $\lim_{\nu\to\infty} \widetilde{x}_\nu(s_\nu) = a$, $\lim_{\nu\to\infty} \widetilde{x}_\nu(t_\nu) = b$. Then $\mathcal{F}(a) \geq \mathcal{F}(b)$. And for any $z \in L_\infty(\nu, [s_\nu, t_\nu])$, $\mathcal{F}(z) \in [\mathcal{F}(a), \mathcal{F}(b)]$.*

Identify all fast orbits

By the compactness of $L_\infty(\nu, \mathbb{R})$, and the fact that $\lambda_\nu \to 0$, for any sequence $t_\nu \in \mathbb{R}$, there exists a subsequence for which $\eta_\nu(\cdot + t_\nu)$ converges to a constant function η_1, on any finite interval. Then by usual argument of Morse theory, there is a subsequence of $\widetilde{x}_\nu(\cdot + t_\nu)$ converging to a fast solution $\widetilde{y}_1 := (y_1, \eta_1) : \mathbb{R} \to M \times \mathbb{R}$ with $\mathrm{Im}\widetilde{y}_1 \subset M \times \{\eta_1\}$, in C^∞_{loc}-topology. So $L_\infty(\nu, \mathbb{R})$ is the union of fast orbits (which could be constant orbits). We first prove that for a suitable subsequence it only contains finitely many nonconstant ones.

Proposition 3.5.12. *There exists a subsequence (still indexed by ν without loss of generality), and $t_{1,\nu}, \ldots, t_{n,\nu} \in \mathbb{R}$ (n could be zero), satisfying*

(I) $t_{1,\nu} < t_{2,\nu} < \cdots < t_{n,\nu}$, $\lim_{\nu\to\infty} |t_{i,\nu} - t_{j,\nu}| = \infty$ for all $i \neq j$.

(II) $\widetilde{x}_\nu(\cdot + t_{i,\nu})$ converges to a (nonconstant) fast solution $\sigma_i := (y_i, \eta_i) : \mathbb{R} \to M \times \mathbb{R}$ in the C^∞_{loc}-topology.

(III) For any sequence $s_\nu \in \mathbb{R}$ such that $\lim_{\nu\to\infty} |s_\nu - t_{i,\nu}| = \infty$ for all i, any convergent subsequence of $\widetilde{x}_\nu(s_\nu)$ has limit in $\mathcal{C}_\mathcal{F}$.

Proof. If $L_\infty(\nu, \mathbb{R}) \subset \mathcal{C}_\mathcal{F}$, then nothing has to be proven. Suppose it is not the case. Then we take $\delta, \epsilon > 0$ as in Lemma 3.4.3 and small enough, and $r > 0$ small enough such that

- All ϵ-short orbits are contained in $\cup_{w\in\mathcal{C}^{sing}_\mathcal{F}} B_r(w)$;

- All nontrivial fast orbits which are not ϵ-short has energy no less than δ;
- For each $w \in \mathcal{C}_{\mathcal{F}}^{sing}$, $\mathcal{C}_{\mathcal{F}} \cap B_{2r}(w)$ can be parametrized using the canonical parametrization as in (3.15), so that $\mathcal{F}$ restricted to both components of $\mathcal{C}_{\mathcal{F}} \cap B_{2r}(w) \setminus \{w\}$ is monotonic.

Let $N = \lfloor \frac{E}{\delta} \rfloor$, where $E = \mathcal{F}(p_-) - \mathcal{F}(p_+)$ is the total energy. Then it is easy to prove by induction, that there exists $k \leq N$ (could be zero), a subsequence ν' of the original sequence, a sequence of numbers $\{t_{i,\nu'}\}_{i=1,\dots,k}$ and fast solutions $\widetilde{y}_i := (y_i, \eta_i)$ satisfying

1. $\lim_{\nu' \to \infty} t_{i+1,\nu'} - t_{i,\nu'} = +\infty$, for $i = 1, \dots, k-1$;
2. $\lim_{\nu' \to \infty} \widetilde{x}_{\nu'}(t_{i,\nu'} + \cdot) = (y_i, \eta_i)$ in C^∞_{loc}-topology;
3. each $\widetilde{y}_i$ has energy $\geq \delta$.
4. Any other nonconstant fast orbit contained in $L_\infty(\nu', \mathbb{R})$ is an ϵ-short orbit.

We replace ν' by ν for simplicity. Let $N' = \#\mathcal{C}_{\mathcal{F}}^{sing}$. Then we can continue the induction to find all ϵ-short orbits, and we claim that the induction stops at finite time and we can find at most $N'(N+1)$ ϵ-short trajectories. Suppose not, then there exists a $w \in \mathcal{C}_{\mathcal{F}}^{sing}$ and a subsequence, still indexed by ν, and sequence of numbers

$$s_{1,\nu} < s_{2,\nu} < \cdots < s_{N+2,\nu}$$

such that for each j, $\widetilde{x}_\nu(s_{j,\nu} + \cdot)$ converges in C^∞_{loc} to an ϵ-short solution $\widetilde{z}_j := (z_j, \xi_j)$ whose orbit is contained in $B_r(w)$. And there exists $j \in \{1, \dots, N+1\}$ such that $L_\infty(\nu, [s_{j,\nu}, s_{j+1,\nu}])$ contains no orbits which are not ϵ-short. Then we take $[t_{j,\nu}, t_{j+1,\nu}] \subset [s_{j,\nu}, s_{j+1,\nu}]$ such that

$$\lim_{\nu \to \infty} \widetilde{x}_\nu(t_{j,\nu}) = \lim_{t \to +\infty} \widetilde{z}_j(t) =: a_+ \in \mathcal{C}_{\mathcal{F}}, \ \lim_{\nu \to \infty} \widetilde{x}_\nu(t_{j+1,\nu}) = \lim_{t \to -\infty} \widetilde{z}_{j+1}(t) =: b_- \in \mathcal{C}_{\mathcal{F}}$$

and denote $\lim_{t \to -\infty} \widetilde{z}_j(t) = a_-$, $\lim_{t \to +\infty} \widetilde{z}_{j+1}(t) = b_+$. Suppose $L_\infty(\nu, [t_{j,\nu}, t_{j+1,\nu}])$ contains some other ϵ-short orbit contained in $B_r(w)$, then it can contain only finitely many, because those orbits must start from the arc between a_- and b_- and ends at the arc between a_+ and b_+, whose energy has a nonzero lower bound. Therefore, by taking a subsequence if necessary, and chooing subintervals of $[t_{j,\nu}, t_{j+1,\nu}]$, we may assume that $T = L_\infty(\nu, [t_{j,\nu}, t_{j+1,\nu}])$ contains no other ϵ-short orbit which is also contained in $B_r(w)$. This implies that

$$T \subset \mathcal{C}_{\mathcal{F}} \cup \bigcup_{v \in \mathcal{C}_{\mathcal{F}}^{sing},\ v \neq w} B_r(v).$$

Indeed, T is contained in a connected component of the latter set, which is homotopy equivalent to a circle or $\mathbb{R}$.

Now by the fact that w is a nondegenerate critical point of $\mathcal{F}|_{\mathcal{C}_\mathcal{F}}$, we see that $w \notin T$, because otherwise by Lemma 3.5.11, $\mathcal{F}(w) \in [\mathcal{F}(a_+), \mathcal{F}(b_-)]$. But there are other points contained in $\mathcal{C}_\mathcal{F} \cap (B_{2r}(w) \setminus B_r(w))$ which is not in T, by the monotonicity requirement in Lemma 3.5.11. Then this contradicts with the connectedness of T.

□

Identify all slow orbits

One can not go from p to q only through fast orbits because $\eta_p \neq \eta_q$ according to our assumption (A12). Hence in the limit slow orbits appear. The identification of them is easy intuitively, because $\mathcal{C}_\mathcal{F}$ is 1-dimensional and our limit object should be connected.

Proposition 3.5.13. *Assume the sequence $\widetilde{x}_\nu$ and $\{t_{i,\nu}\}_{i=1,\dots,n}$ satisfy the condition of the last proposition. Let $t_{0,\nu} = -\infty$ and $t_{n+1,\nu} = +\infty$. Then there exists a subsequence (still indexed by ν), and slow orbits $\gamma_i \subset \mathcal{C}_\mathcal{F}$, $i = 0, \dots, n$ satisfying the following:*

(I) For each $i = 1, \dots, n$, γ_i is from $\widetilde{y}_i(+\infty)$ to $\widetilde{y}_{i+1}(-\infty)$.

(II) If $p \in \mathrm{Crit}^+(\mathcal{F})$, then γ_0 is the single point p; if $q \in \mathrm{Crit}^-(\mathcal{F})$, then γ_n is the single point q.

(III) For each $i = 1, \dots, n-1$, there exists intervals $(a_{i,\nu}, b_{i,\nu}) \subset (t_{i-1,\nu}, t_{i,\nu})$ such that

$$\lim_{\nu\to\infty} |a_{i,\nu} - t_{i-1,\nu}| = \lim_{\nu\to\infty} |b_{i,\nu} - a_{i,\nu}| = \lim_{\nu\to\infty} |t_{i,\nu} - b_{i,\nu}| = \infty$$

and $\widetilde{x}_\nu([a_{i,\nu}, b_{i,,\nu}])$ converges to γ_i in Hausdorff topology.

Proof. By the previous proposition, we may find sequences of intervals $I_{i,\nu} = [a_{i,\nu}, b_{i,\nu}] \subset \mathbb{R}$, $i = 1, \dots, n-1$ such that for each ν, $I_{i,\nu} \cap I_{j,\nu} = \emptyset$, for $i \neq j$, $b_{i,\nu} - a_{i,\nu} \to \infty$, and

$$\lim_{\nu\to\infty} \widetilde{x}_\nu(a_{i,\nu}) = \widetilde{y}_i(+\infty), \ \lim_{\nu\to\infty} \widetilde{x}_\nu(b_{i,\nu}) = \widetilde{y}_{i+1}(-\infty).$$

And take $I_{0,\nu} = (-\infty, a_{0,\nu}]$, $I_{n,\nu} = [b_{n,\nu}, +\infty)$ such that

$$\lim_{\nu\to\infty} \widetilde{x}_\nu(a_{0,\nu}) = \widetilde{y}_1(-\infty), \ \lim_{\nu\to\infty} \widetilde{x}_\nu(b_{n,\nu}) = \widetilde{y}_n(+\infty).$$

1. For each $i \in \{1, \dots, n\}$, set $y_{i,\pm} = \widetilde{y}_i(\pm\infty)$. And set $y_{0,+} = p$, $y_{n+1,-} = q$. For $i = 0, \dots, n$, let $\Gamma_i \subset \mathcal{C}_\mathcal{F}$ be the union of all slow orbits from $y_{i,+}$ to $y_{i+1,-}$. Since $\mathcal{C}_\mathcal{F}$ is a 1-dimensional manifold, Γ_i is either a single point, or is one nontrivial slow orbit, or is the union of two nontrivial slow orbits. We would like to prove that $L_\infty(\nu, I_{i,\nu}) \subset \Gamma_i$.

   If Γ_i is the union of two slow orbits, then Γ_i the whole connected component of it in $\mathcal{C}_\mathcal{F}$. By the connectedness of $L_\infty(\nu, I_{i,\nu})$ the claim is true.

   Suppose Γ_i is simply-connected and $L_\infty(\nu, I_{i,\nu})$ is also simply-connected. Suppose in this case the claim is not true, then there exists a subsequence (still indexed by ν) and $t_\nu \in I_{i,\nu}$ such that $\lim_{\nu\to\infty} \widetilde{x}_\nu(t_\nu) = z \in L_\infty(\nu, I_{i,\nu}) \setminus \Gamma_i$ and

$$\mathcal{F}(y_{i,+}) > \mathcal{F}(z) > \mathcal{F}(y_{i+1,-}).$$

   But since $L_\infty(\nu, [t_\nu, b_{i,\nu}])$ and $L_\infty(\nu, [a_{i,\nu}, t_\nu])$ are connected, either the former contains $y_{i,+}$ or the latter contains $y_{i+1,-}$, either of which contradicts Lemma 3.5.11.

   We claim that it is impossible that Γ_i is homeomorphic to a closed interval and $L_\infty(\nu, I_{i,\nu})$ is homeomorphic to a circle $C \subset \mathcal{C}_\mathcal{F}$. Suppose not, then Lemma 3.5.11 implies that $y_{i,+}$ and $y_{i+1,-}$ must be the absolute maximum and minimum of the function $\mathcal{F}$ restricted to this component C. The critical points of $\mathcal{F}|_C$ can be put in a cyclic order and adjacent to $y_{i,+}$, the two local minimum are $y_{i+1,-}$ and some $z \in C \setminus \Gamma_i$. Taking a subsequence and find $t_\nu \in I_{i,\nu}$ such that $\lim_{\nu\to\infty} \widetilde{x}_\nu(t_\nu) = z$. And consider the interval $J_{i,\nu} = [t_\nu, b_{i,\nu}]$. Then $L_\infty(\nu, J_{i,\nu})$ is homeomorphic to a closed interval and doesn't contain $y_{i,+}$. But this is impossible because then it must contain another local maximum z' between z and $y_{i+1,-}$ to keep the connectedness of $L_\infty(\nu, J_{i,\nu})$, while $\mathcal{F}(z') \notin [\mathcal{F}(z), \mathcal{F}(y_{i+1,0})]$, which contradicts with Lemma 3.5.11.

2. We can identify one slow orbit contained in Γ_i. More precisely, we would like to show that, there exists $\gamma_i \subset \Gamma_i$ a slow orbit from $y_{i,+}$ to $y_{i+1,-}$ and a subsequence ν' such that $L_\infty(\nu', I_{i,\nu'}) = \gamma_i$. Assume it is impossible, then $\Gamma_i = \gamma_i^1 \cup \gamma_i^2$ the union of two different (nonconstant) slow orbits, and we can assume that there exists a subsequence ν' such that for *any* further subsequence $\{\nu''\} \subset \{\nu'\}$, $\gamma_i^1 \subset L_\infty(\nu'', I_{i,\nu''})$.

   Now suppose there exists $\widetilde{z} \in L_\infty(\nu', I_{i,\nu'}) \cap \left(\Gamma_i \setminus \gamma_i^1\right)$. Then there exists a subsequence ν'', and $t^-_{\nu''}, t_{\nu''}, t^+_{\nu''} \in I_{i,\nu''}$, $t^-_{\nu''} < t_{\nu''} < t^+_{\nu''}$, such that

$$\lim_{\nu''\to\infty} \widetilde{x}_{\nu''}(t_{\nu''}) = \widetilde{z}, \ \lim_{\nu''\to\infty} \widetilde{x}_{\nu''}(t^\pm_{\nu''}) = \widetilde{z}^\pm \in \mathrm{Int}(\gamma_i^1)$$

and $\mathcal{F}(y_{i,+}) > \mathcal{F}(\widetilde{z}^-) > \mathcal{F}(\widetilde{z}) > \mathcal{F}(\widetilde{z}^+) > \mathcal{F}(y_{i+1,-})$.

Since

$$\left(\Gamma_i \setminus \gamma_i^1\right) \cap \mathcal{F}^{-1}\left(\left[\mathcal{F}(\widetilde{z}^-), \mathcal{F}(\widetilde{z}^+)\right]\right)$$

has a nonzero distance from γ_1, it means

$$L_\infty(\nu'', [t^-_{\nu''}, t^+_{\nu''}])$$

is diskonnected, which contradicts with Lemma 3.5.10. Hence we have $L_\infty(\nu', I_{i,\nu'}) = \gamma_i^1$.

3. So in summary, there exists a slow orbit γ_i from $y_{i,+}$ to $y_{i+1,-}$, and a subsequence (still indexed by ν), such that $L_\infty(\nu, I_{i,\nu}) = \gamma_i$. By the connectedness of $L_\infty(\nu, I_{i,\nu})$, this implies that $\widetilde{x}_\nu(I_{i,\nu})$ converges to γ_i in Hausdorff topology. (II) of this proposition simply follows from the requirement that $\mathcal{F}$ is decreasing along γ_0 and γ_n, and p is attracting and q is repelling in this case.

□

So we complete the proof of the compactness theorem.

3.5.4 Gluing

In this subsection we prove the following theorem.

Theorem 3.5.14. *Suppose $p \in \mathrm{Crit}_k(\mathcal{F})$, $q \in \mathrm{Crit}_{k-1}(\mathcal{F})$. Then, there exists $\epsilon_0 > 0$, such that for all $\lambda \in (0, \epsilon_0]$, there exists a bijection*

$$\Phi^\lambda : \mathcal{N}^0(p,q) \to \mathcal{M}^\lambda(p,q)$$

such that for any $\mathscr{X} \in \widetilde{\mathcal{N}}^0(p,q)$, there exist representatives $\widetilde{x}^\lambda \in \widetilde{\mathcal{M}}^\lambda(p,q)$ of $\Phi^\lambda([\mathscr{X}])$ such that $\{\widetilde{x}^\lambda\}$ converges to $\mathscr{X}$ in the sense of Definition 3.5.8.

Proof. Let $p \in \mathrm{Crit}_k(\mathcal{F})$, $q \in \mathrm{Crit}_{k-1}(\mathcal{F})$. Theorem 3.5.9 implies that for $\lambda > 0$ small, any orbit in $\mathcal{M}^\lambda(p,q)$ is close to a fast-slow orbit in $\mathcal{N}^0(p,q)$. We must show that for any fast-slow orbit from p to q, if $\lambda > 0$ is small, then there exists a unique orbit from p to q that lies near the given fast-slow orbit.

The proof consists of two parts. In the first part, we construct the gluing map Φ^λ, by using the Exchange Lemma (Lemma A.1.1) and General Exchange Lemma (Lemma A.1.2). In the second part, we show that Φ^λ is a bijection.

We remark that the Exchange Lemma and General Exchange Lemma (cf. Section A.1) are for C^r differential equations, and some regularity is lost in the derivative of the constructed solution with respect to ϵ. In our situation the differential equations are of class C^∞, so of course each solution is of class C^∞.

Now we start to construct the gluing map. We will frequently use Proposition 3.5.5 without citing it, and we will use the notation $W^u(p,\lambda)$, etc., from Subsection 3.2.3.

Let $\mathscr{X} = (p_0, \sigma_1, p_1, \ldots, p_{n-1}, \sigma_n, p_n)$ be a fast-slow solution from p to q. For small $\lambda > 0$, we will follow $W^u(p,\lambda)$ around the fast-slow solution until it meets $W^s(q,\lambda)$ transversally. Any orbit in $\mathcal{M}^\lambda(p,q)$ that is near the given fast-slow solution must be in the portion of $W^u(p,\lambda)$ that we follow; uniqueness is a consequence of the transversality at the end of the proof (Step 8).

We have $p \in \mathrm{Crit}_k^+(\mathcal{F}) \cup \mathrm{Crit}_k^-(\mathcal{F})$ and $q \in \mathrm{Crit}_{k-1}^+(\mathcal{F}) \cup \mathrm{Crit}_{k-1}^-(\mathcal{F})$.

Step 1. Suppose $p = (x_p, \eta_p) \in \mathrm{Crit}_k^+(\mathcal{F})$. Then index $(x_p, f_{\eta_p}) = k$, and σ_1 is a fast solution from p to $p_1 = (x_{p_1}, \eta_{p_1}) \in \mathcal{C}_\mathcal{F} \setminus \mathcal{C}_\mathcal{F}^{sing}$, with index $(x_{p_1}, f_{\eta_{p_1}}) = k-1$. By (A13), $W^u(p,0)$ meets $W^s(p_1,0)$ transversally within $M \times \{\eta_p\}$ along σ_1. Then $W^u(p,0)$ meets $W^s(\sigma_2,0)$ transversally in $M \times \mathbb{R}$ along σ_1. (More precisely, σ_2 should be replaced by a compact portion of the complete orbit corresponding to σ_2; we shall use this sort of abuse of notation throughout this section.) The dimension of the intersection is $\dim W^u(p,0) + \dim W^s(\sigma_2,0) - (n+1) = k + (n-(k-1)+1) - (n+1) = 1$.

Thus σ_1 is isolated in the intersection. The next orbit σ_2 is slow, and we have (i) $p_2 \in \mathcal{C}_\mathcal{F} \setminus \mathcal{C}_\mathcal{F}^{sing}$, or (ii) $p_2 \in \mathcal{C}_\mathcal{F}^{sing}$. In both cases, index $(x_{p_2}, f_{\eta_{p_2}}) = k-1$.

Step 2. After Step 1, in case (i), the Exchange Lemma [26] implies that for $\lambda > 0$ small, $W^u(p,\lambda)$, followed along the flow, becomes close to $W^u(\sigma_2,0)$ near p_2. By *close* we mean close in the C^s-topology for some large s, which decreases in the course of the proof; see Appendix A.1. Notice that the dimension of $W^u(\sigma_2,0)$ is $(k-1)+1 = k$ as it should be.

The next orbit σ_3 is fast, and we have (i) $p_3 \in \mathcal{C}_\mathcal{F} \setminus \mathcal{C}_\mathcal{F}^{sing}$, or (ii) $p_3 \in \mathcal{C}_\mathcal{F}^{sing}$. In case (i), index $(x_{p_3}, f_{\eta_{p_3}}) = k-1$; in case (ii), index $(x_{p_3}, f_{\eta_{p_3}}) = k-2$.

Step 3. After Step 1, in case (ii), let N_0 denote the center manifold of (3.2) for $\lambda = 0$ at p_2, which has dimension 2. The choice of center manifold is not unique; we may choose it to include the start of σ_3 (see Appendix A.2). By the Exchange Lemma, for $\lambda > 0$ small, $W^u(p,\lambda)$, followed along the flow, becomes close to $W^u(\sigma_2,0)$ where σ_2 enters N_0. Of course $W^u(\sigma_2,0)$ (dimension k) is transverse to $W^s(N_0,0)$ (dimension $(n-(k-1)-1)+2 = n-k+2$); $n-(k-1)-1$ is the number

of negative eigenvalues at p_2. The dimension of the intersection is 1. For small $\lambda > 0$, N_0 perturbs to a normally hyperbolic invariant manifold N_λ, and $W^u(p,\lambda)$ is transverse to $W^s(N_\lambda,\lambda)$.

The solution in $W^u(p,\lambda) \cap W^s(N_\lambda,\lambda)$ approaches a solution $p_2^\lambda(t)$ in N_λ that is initially near $\sigma_2 \cap N_0$. The system restricted to N_λ, $\lambda \geq 0$, has been analyzed in [30]; see Appendix A.1. The result is that $p_2^\lambda(t)$ leaves N_λ close to $\sigma_3 \cap N_0$. Then the General Exchange Lemma implies that $W^u(p,\lambda)$ becomes close to the restriction of $W^u(N_0,0)$ to $\sigma_3 \cap N_0$, i.e., $W^u(p_2,0)$, as $W^u(p,\lambda)$ exits a neighborhood of N_λ. $W^u(p_2,0)$ has dimension $(k-1)+1 = k$ as it should. The orbit σ_3 is fast, $p_3 \in \mathcal{C}_\mathcal{F} \setminus \mathcal{C}_\mathcal{F}^{sing}$, and index $(x_{p_3}, f_{\eta_{p_3}}) = k-1$.

Step 4. After Step 2, in case (i), $W^u(p,\lambda)$ is close to $W^u(\sigma_2,0)$ near p_2, and by (A10), $W^u(\sigma_2,0)$ is transverse to $W^s(\sigma_4,0)$ along σ_3. Continuation from here is like continuation after Step 1, described in Steps 2–3.

Step 5. After Step 2, in case (ii), we note that by (A13), $W^u(p_2,0)$ (dimension $k-1$) is transverse to $W^s(p_3,0)$ (dimension $n-(k-2)$) within $M \times \{\eta_{p_3}\}$ along σ_3. Let N_0 denote the center manifold of (3.2) for $\lambda = 0$ at p_3, which has dimension 2; we choose it to include the end of σ_3 (see Appendix A.2). Then $W^u(\sigma_2,0)$ (dimension k) is transverse to $W^s(N_0,0)$ (dimension $(n-(k-2)-1)+2 = n-k+3$) along σ_3. The intersection is 2-dimensional and consists of solutions that track an open set of solutions in N_0 around σ_3.

Step 6. After Step 3, we note that by (A13), $W^u(p_2,0)$ (dimension k) meets $W^s(p_3,0)$ (dimension $n-(k-1)$) transversally within $M \times \{\eta_{p_3}\}$ along σ_3. As in Step 1, $W^u(p_2,0)$ meets $W^s(\sigma_4,0)$ transversally in $M \times \mathbb{R}$ along σ_3, and $W^u(p,\lambda)$ is close to $W^u(p_2,0)$ near σ_3. The next orbit σ_4 is a slow solution from p_3 to (i) $p_4 \in \mathcal{C}_\mathcal{F} \setminus \mathcal{C}_\mathcal{F}^{sing}$, or (ii) $p_4 \in \mathcal{C}_\mathcal{F}^{sing}$. In both cases, index $(x_{p_4}, f_{\eta_{p_4}}) = k-1$. Continuation from here is like continuation after Step 1, described in Steps 2–3.

Step 7. After Step 5, for small $\lambda > 0$, the tracked orbits include an open set U_λ in N_λ, the perturbation of N_0, that lies above an open set U_0 in N_0 that contains $\sigma_4 \cap N_0$ in its interior. The General Exchange Lemma implies that $W^u(p,\lambda)$ is close to the restriction of $W^u(N_0,0)$ to U_0, i.e., to $W^u(\sigma_4 \cap N_0)$, as it exits a neighborhood of N_λ. Continuation from here is like continuation after Step 2, described in Steps 4–5.

Step 8. Continuation proceeds using analogs of the steps previously described until $W^u(p)$ arrives near p_{n-1}.

If $q \in \mathrm{Crit}_k^+(\mathcal{F})$, then σ_n is slow, and index $(x_{p_{n-1}}, f_{\eta_{p_{n-1}}}) =$ index $(x_q, f_{\eta_q}) = k-1$. $W^u(p,\lambda)$ arrives near p_n along the slow orbit σ_n, close to $W^u(\sigma_n,0)$, which has dimension k. $W^u(\sigma_n,0)$ is transverse to $W^s(\sigma_n,0)$, with dimension $n-(k-1)+1 = n-k+2$. Therefore $W^u(p,\lambda)$ is transverse to $W^s(q,\lambda)$, which is close to $W^s(\sigma_n,0)$. The intersection has dimension 1 and gives the solution

from p to q.

If $q \in \mathrm{Crit}_k^-(\mathcal{F})$, then σ_n is a fast orbit from $p_{n-1} \in \mathcal{C}_\mathcal{F} \setminus \mathcal{C}_\mathcal{F}^{sing}$ to $q \in \mathcal{C}_\mathcal{F} \setminus \mathcal{C}_\mathcal{F}^{sing}$; we have $\mathrm{index}\,(x_{p_{n-1}}, f_{\eta_{p_{n-1}}}) = k-1$ and $\mathrm{index}\,(x_q, f_{\eta_q}) = k-2$. By (A11), $W^u(p_{n-1}, 0)$ meets $W^s(q,0)$ transversally in $M \times \{\eta_q\}$ along σ_n. Therefore $W^u(\sigma_n, 0)$ meets $W^s(q,0)$ transversally in $M \times \mathbb{R}$ along σ_n. Since $W^u(p,\lambda)$ is close to $W^u(\sigma_n, 0)$, $W^u(p,\lambda)$ meets $W^s(q,\lambda)$ transversally in $M \times \mathbb{R}$ near σ_n. The intersection has dimension one and gives the solution from p to q.

Step 9. Suppose $p = (x_p, \eta_p) \in \mathrm{Crit}_k^-(\mathcal{F})$. Then $\mathrm{index}\,(x_p, f_{\eta_p}) = k-1$, and σ_1 is a slow solution from p to $p_1 = (x_{p_1}, \eta_{p_1})$. Either (i) $p_1 \in \mathcal{C}_\mathcal{F} \setminus \mathcal{C}_\mathcal{F}^{sing}$, or (ii) $p_1 \in \mathcal{C}_\mathcal{F}^{sing}$. In either case, $\mathrm{index}\,(x_{p_1}, f_{\eta_{p_1}}) = k-1$.

In case (i), $W^u(p,\lambda)$ is close to $W^u(\sigma_1, 0)$. Step 3 above describes how to continue from p_1. Case (ii) is left to the reader.

Now we show that the gluing map just constructed is a bijection. We first note that there exist small $\delta > 0$ and $\lambda_0 > 0$ such that for all $\lambda \in (0, \lambda_0)$, the orbit we constructed is the unique one in $\mathcal{M}^\lambda(p,q)$ such that its Hausdorff distance from the fast-slow orbit $[\mathscr{X}]$ is less than δ. This is a general fact about exchange lemma constructions; see for example [27], p. 1021.

Thus the gluing map is a bijection provided it is surjective. We claim that there exists $\lambda_0 > 0$ such that the gluing map is surjective for all $\lambda \in (0, \lambda_0)$. If not, then there exist a sequence $\lambda_i \to 0$ and a sequence of orbits $[\widetilde{x}_i] \in \mathcal{M}^{\lambda_i}(p,q)$ such that for each i, $[\widetilde{x}_i]$ is not in the image of Φ^{λ_i}. Then, by the compactness theorem (Theorem 3.5.9), without loss of generality, $[\widetilde{x}_i]$ converges to a fast-slow orbit $[\mathscr{Y}]$ in the sense of Definition 3.5.8. In particular, this sequence converges in the Hausdorff topology. Then for large i, $\Phi^{\lambda_i}([\mathscr{Y}])$, which is the unique orbit in $\mathcal{M}^{\lambda_i}(p,q)$ in a small Hausdorff neighborhood of $[\mathscr{Y}]$, must be $[\widetilde{x}_i]$. This contradiction proves the claim. □

3.6 Basic Morse theory and the chain complex $\mathcal{C}^0$

Let us recall the following basic properties of any Morse function $\mu : M \to \mathbb{R}$, where M is a compact manifold.

1. If $[c_1, c_2]$ contains no critical value of μ, then $\mu^{-1}((-\infty, c_1])$ is diffeomorphic to $\mu^{-1}((-\infty, c_2])$;

2. If (i) $c \in (c_1, c_2)$ is the only critical value of μ contained in $[c_1, c_2]$, (ii) there is a unique critical point x of μ such that $\mu(x) = c$, and (iii) x has Morse index k, then the sublevel set $\mu^{-1}((-\infty, c_2])$ is obtained from the sublevel set $\mu^{-1}((-\infty, c_1])$ by attaching a k-handle.

Thus a change in the level set of μ corresponds to a change in the sublevel set of μ. The chain complex $\mathcal{C}^0$ defined by counting fast-slow orbits gives an indirect way to obtain, at the homology level, this change of topology. In this section we will give a brief indication of how this works.

We replace μ by $\mu_c = \mu - c$ for any regular value c of μ, and denote the corresponding Lagrange multiplier by $\mathcal{F}_c : M \times \mathbb{R} \to \mathbb{R}$. Then $\mathrm{Crit}(\mathcal{F}_c)$ is just the critical point set of the restriction of f to the hypersurface $\mu^{-1}(c)$. On the other hand, the slow manifold $\mathcal{C}_{\mathcal{F}_c} = \mathcal{C}_{\mathcal{F}}$ is unchanged, and $\mathrm{Crit}(\mathcal{F}_c) = \mathcal{C}_{\mathcal{F}_c} \cap (\mu^{-1}(c) \times \mathbb{R})$. The fast flow is unchanged when we vary c, and the changes in generators and the slow flow can be seen transparently from the intersection of $\mathcal{C}_{\mathcal{F}}$ and $\mu^{-1}(c) \times \mathbb{R}$.

We have to consider various "wall-crossing" phenomena of the chain complex $\mathcal{C}^0(f, \mu - c)$ when we vary c, not only at the critical values of μ, but also at each c where an assumption we made to define the chain complex fails. We assume that for each $c \in \mathbb{R}$, if c is a critical value of μ, then $\mu^{-1}(c)$ contains only one critical point; if c is a regular value, then at most one of our assumptions is violated; and the violations happen at diskrete values of c. This assumption is satisfied for generic (f, μ, g).

3.6.1 When c does not cross a critical value

When $[c_1, c_2]$ contains no critical value of μ, $\mu^{-1}(c_1)$ and $\mu^{-1}(c_2)$ are diffeomorphic. We look at the chain complexes $\mathcal{C}^0(f, \mu - c_1)$ and $\mathcal{C}^0(f, \mu - c_2)$ in several cases.

1. Suppose that as c moves from c_1 to c_2, a degenerate critical point of $\mu|_{\mathcal{C}_{\mathcal{F}}}$ appears (in violation of Proposition 3.4.4) and then splits into nondegenerate critical points. Generically two nondegenerate critical points are produced as shown in Figure 3.2. Thus two new generators of $\mathcal{C}^0(f, \mu - c_2)$ are produced. We denote them by $p \in \mathrm{Crit}_k^+$ and $q \in \mathrm{Crit}_{k+1}^-$. Between them there is a slow trajectory γ.

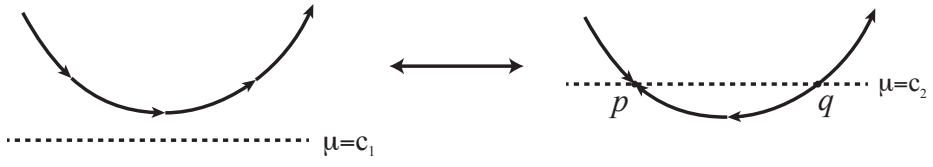

Figure 3.2: The dashed lines represent the hypersurfaces $\mu = c_1$ and $\mu = c_2$, one of which has two more intersection points with $\mathcal{C}_{\mathcal{F}}$ than the other.

We now prove that the two chain complexes are quasi-isomorphic. We denote the chain complex associated to the left picture by $(\{D_i\}_{i \geq 0}, \partial_D)$ and the chain complex associated to the right picture

by $(\{C_i\}_{i\geq 0}, \partial_C)$. Then there is a canonical inclusion

$$\iota : D_i \to C_i$$

and projection

$$\pi : C_i \to D_i$$

by forgetting $p \in C_k$ and $q \in C_{k+1}$. It is easy to see that $\partial_C(q) - p \in \mathrm{Im}\iota$. However, these two maps are not chain maps. The natural chain maps should be corrections of ι and π, where the correction terms appear because some fast-slow trajectory in the left picture gets blocked by the creation of p and q.

We define the corrected chain maps on generators of the chain complexes and extend it linearly over $\mathbb{Z}_2$, by

$$\begin{array}{rccl} \widetilde{\pi}: & C_i & \to & D_i \\ & x & \mapsto & \begin{cases} \pi(x), & x \neq p, q; \\ 0, & x = q; \\ \pi\left(\partial_C(q) - p\right), & x = p. \end{cases} \end{array}$$

and

$$\begin{array}{rccl} \widetilde{\iota}: & D_i & \to & C_i \\ & x & \mapsto & \begin{cases} \iota(x), & x \notin D_{k+1}; \\ \iota(x) - \langle \partial_D(x), p \rangle q, & x \in D_{k+1}. \end{cases} \end{array}$$

Now we check that both $\widetilde{\pi}$ and $\widetilde{\iota}$ are chain maps and they are homotopy inverses to each other. Indeed, this is exactly the same as the bifurcation analysis done in [12, Section 3] based on a gluing phenomenon. Here we have exactly the same gluing phenomenon so the chain maps have the same form.

2. Suppose that as c moves from c_1 to c_2, a critical point of $\mathcal{F}_c$ passes through the start or end of a handle-slide (in violation of (A5)). See Figure 3.3. It is easy to see that $\mathcal{C}^0(f, \mu - c_1)$ and $\mathcal{C}^0(f, \mu - c_2)$ are actually isomorphic.

3. Suppose that as c moves from c_1 to c_2, a critical point $p \in \mathrm{Crit}_k^-$ passes through a point $w \in \mathcal{C}_{\mathcal{F}}^{sing}$ (in violation of (A5)) and then becomes a critical point $q \in \mathrm{Crit}_k^+$; see Figure 3.4. If

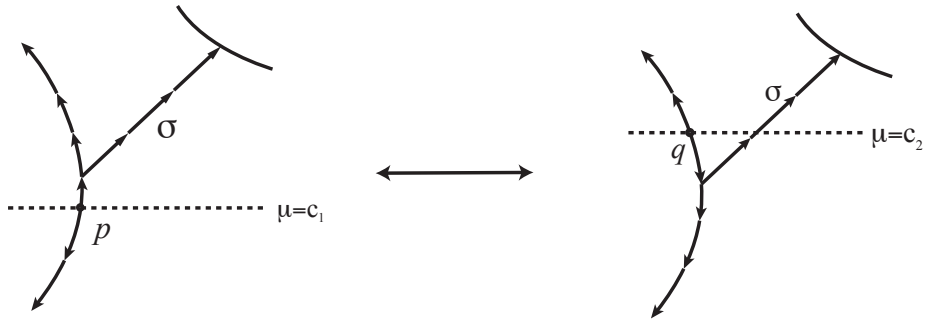

Figure 3.3: The dashed lines represent the hypersurfaces $\mu = c_1$ and $\mu = c_2$. σ is a handle-slide. This change replace a generator $p \in \mathrm{Crit}_k^-$ by another $q \in \mathrm{Crit}_k^-$ and actually gives isomorphic chain complexes.

there is a cusp trajectory σ_1 starting at w, it will contribute to the definition of $\partial^0(p)$ in the chain complex $\mathcal{C}^0(f, \mu - c_1)$. In $\mathcal{C}^0(f, \mu - c_2)$, the corresponding contribution to $\partial^0(q)$ is given by a fast trajectory σ_2 starting from q.

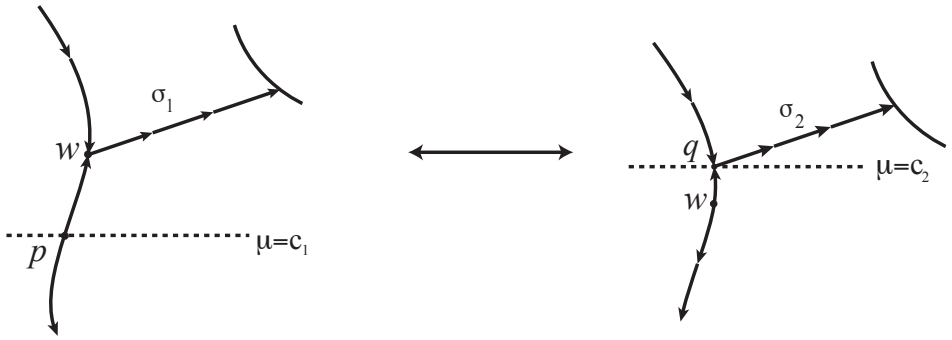

Figure 3.4: The dashed lines sides represent the hypersurfaces $\mu = c_1$ and $\mu = c_2$ respectively. σ_1 is a cusp trajectory starting from $w \in \mathcal{C}_{\mathcal{F}}^{sing}$, and σ_2 is a regular fast trajectory starting from q. The generator $p \in \mathrm{Crit}_k^-$ is replaced by $q \in \mathrm{Crit}_k^+$; the chain complexes are isomorphic.

3.6.2 When c crosses a critical value

Suppose that one point $c \in (c_1, c_2)$ is a critical value μ, with a single critical point x of index k of μ. Then topologically, $\mu^{-1}(c_1)$ and $\mu^{-1}(c_2)$ differ by attaching a k-handle, and the homology changes accordingly. We can recover this fact by comparing the chain complexes $\mathcal{C}^0(f, \mu - c_1)$ and $\mathcal{C}^0(f, \mu - c_2)$. In the generic case, one complex has two more generators, one of index $k+1$ and one

of index $n-k$. Figure 3.5 illustrates one case of this changing.

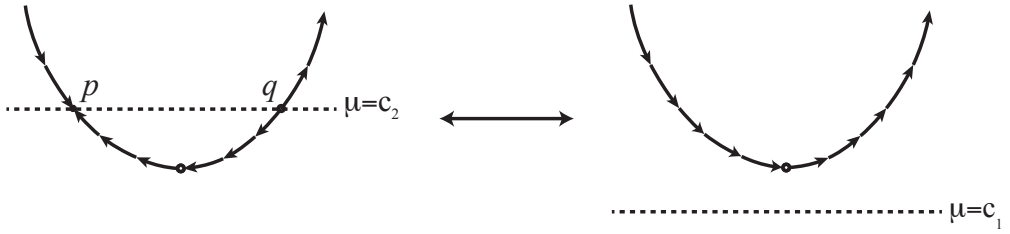

Figure 3.5: On the left: the left branch of $\mathcal{C}_{\mathcal{F}}$ has index $n-k$ (of the fast flow) and $\eta << 0$, and the right branch of $\mathcal{C}_{\mathcal{F}}$ has index k and $\eta >> 0$. It has two more generators p and q with degrees $n-k$ and $k+1$ respectively than the right picture.

Chapter 4

Affine vortices in the adiabatic limits of gauged linear σ-model

4.1 Introduction to this chapter

This chapter is on the geometry of the symplectic vortex equation over the complex plane and their moduli spaces. The vortex equation over $\mathbb{C}$ and the moduli spaces of solutions are the central objects in the project of Ziltener (cf. [61]) to define a quantum version of the Kirwan map for symplectic manifold X with Hamiltonian action by a compact Lie group G. Here we work with a concrete example, where, the target manifold is the vector space $\mathbb{C}^N$ acted by $U(1)$ via complex multiplication and the symplectic quotient of this action is the projective space $\mathbb{P}^{N-1}$.

Many years ago, in [48], Taubes gave the classification of finite energy vortices in the case where the target $X = \mathbb{C}$ with $U(1)$-action. The moduli space for each "vortex number" $d > 0$ is $\mathrm{Sym}^d\mathbb{C}$, the d-fold symmetric product of the complex plane. For target $\mathbb{C}^N$ with $N \geq 2$ and the diagonal $U(1)$-action there was no result of either construction of nontrivial solutions or classification, until very recently Venugopalan-Woodward ([51]) claim that, for target manifold a projective variety acted by a reductive Lie group (including our case) using heat flow method one can identify the solutions to the vortex equation and algebraic maps from $\mathbb{C}$ to the "quotient stack" X/G with certain condition on the asymptotic behavior at infinity.

For the special example considered in this note, we take a different approach which can be certainly extended to more general situations. This approach contains three basic ingredients. The first one is the adiabatic limit analysis for vortex equation over a compact Riemann surface Σ

with growing area form, most of which is provided by [17]. In particular, it implies that (in the symplectic aspherical case) as we grow the area form on Σ, the energy density of solutions blows up at most in the same rate as we enlarging the surface. The second ingredient is the Hitchin-Kobayashi correspondence for stable N-pairs over the compact Σ provided by Bradlow's theorem [4] (see Theorem 1.3.5). This correspondence works for any large area form on the domain curve, hence we can identify moduli spaces of solutions for different area forms with the same algebraic moduli space. The third ingredient is the observation that the energy concentration of a sequence of solutions has an algebraic description (in our example this means a sequence of holomorphic N-pairs develops a base point). With the understanding of these ingredients, we can manipulate the bubbling in the algebraic moduli space and construct all possible solutions.

This bubbling phenomenon may not be novel to physicists. For example in [55], Witten argued heuristically about the formation of such "point-like instantons". Here we reformulate it more precisely and give a more comprehensive treatment in the special case of $\mathbb{P}^{N-1}$-model.

It is worth pointing out that our approach is very elementary, and by looking at concrete examples, it helps understand the behavior of the vortices (for example, why they merge to the moment level surface and converge to holomorphic spheres in the symplectic quotient). One can also generalize this method to other cases, for example, toric manifolds and flag manifolds, both as symplectic quotients of Euclidean spaces.

A purpose of classifying affine vortices and identifying their moduli spaces is to compute the quantum version of the Kirwan map, which was proposed by D. Salamon and a rigorous definition relies on an ongoing project of Ziltener. The quantization of the Kirwan map is supposed to intertwine the Hamiltonian Gromov-Witten invariants of a Hamiltonian G-manifold and the usual Gromov-Witten invariants of its symplectic quotient (which extends Theorem to the case that α_i can have arbitrary degree) and defines a morphism of "cohomological field theory algebras" (see [40]). In the case we considered in this chapter, we can identify the moduli space and a natural compactification (we call it the Uhlenbeck compactification), which we can use to compute the quantum Kirwan map $\kappa_Q : H^*_{U(1)}\left(\mathbb{C}^N\right) \to QH^*\left(\mathbb{P}^{N-1}\right)$. We also identified the stable map compactification of the moduli space of nontrivial affine vortices of the lowest degree.

4.1.1 Organization of this chapter

The first half of this chapter is about general theory: Section 4.2 is a review of previous work of Ziltener on affine vortices and its relation with vortex equation over a compact Riemann surface via

the adiabatic limit. In Section 4.3 we consider the compactification of the moduli space of affine vortices, where we give some refinement of definitions of Ziltener. In Section 4.4 we review the (formal) definition of the quantum Kirwan map.

In the second half we restrict to the special case for $U(1)$-action on $\mathbb{C}^N$. In Section 4.5 we give a detailed description of the vortex bubbling phenomenon in the adiabatic limit. In Section 4.6 we use the adiabatic limit trick to give a classification of finite energy affine vortices which generalizes Taubes' classification for $N = 1$; we also identify its moduli space and compute the associated quantum Kirwan map.

4.2 Affine vortices

We survey some basic facts about vortex equation over $\mathbb{C}$ with area form $\Omega_{\mathbb{C}} = ds \wedge dt$. We call a solution to the vortex equation (1.14) in this case an *affine vortex*. All G-bundles over $\mathbb{C}$ are trivial and isomorphisms between them are all isotopic, so we will work solely with "the" trivial bundle $P = \mathbb{C} \times G$. So a connection A will be written canonically as $d+\alpha$ with $\alpha \in \Omega^1(\mathbb{C}, \mathfrak{g})$ and the section u corresponds canonically to a map $u : \mathbb{C} \to M$. A gauge transformation is then a map $g : \mathbb{C} \to G$.

The general theory for vortices over $\mathbb{C}$ initiated from the paper of Gaio-Salamon [17] and a lot of analytic framework has been settled down by Fabian Ziltener in [61]. The algebraic theory of these objects are also studied in [57].

First of all, we make several assumptions on the manifold (M, ω) and the action. They are satisfied, for example, in the case of Subsection 1.3.4.

We assume

1. (M, ω) is aspherical, i.e., for any embedded sphere $S^2 \subset M$,

$$\int_{S^2} \omega = 0.$$

2. The moment map μ is proper, and 0 is a regular value, such that the restriction of the G-action on $\mu^{-1}(0)$ is free.

3. There exists a G-invariant, ω-compatible almost complex structure J such that the tuple (M, ω, J, μ) is convex at infinity. This means there exists a proper G-invariant function f :

$M \to [0, \infty)$ and a constant $C > 0$ such that for any $(x, v) \in TM$ with $f(x) \geq C$,

$$\omega(\nabla_v \nabla f(x), J(x)v) - \omega(\nabla_{Jv} \nabla f(x), v) \geq 0,\ \omega(\mathcal{X}_{\mu(x)}, \nabla f) \geq 0.$$

We give two examples of affine vortices.

Example 4.2.1. An affine vortex is called trivial if it is equivalent to (A, u) where $A = d$ is the trivial connection on the trivial bundle, and $u : \mathbb{C} \to M$ is a constant map with value in $\mu^{-1}(0)$. In particular, an affine vortex is trivial if and only if it has zero energy.

Example 4.2.2. In the case $M = \mathbb{C}$, $U(1)$ acts on $\mathbb{C}$ by complex multiplication with moment map $\mu(z) = -\frac{\sqrt{-1}}{2}(|z|^2 - 1)$, Taubes (see [48], [25]) classified all planar vortices with finite energy. More precisely, for any "vortex number" $d > 0$ and for any d-tuple of unordered points $z_1, \ldots, z_d \in \mathbb{C}$, there is a unique solution (up to gauge) which is of the form

$$(A, u) = \left(d - \partial h + \overline{\partial} h, e^{-h}(z - z_1) \cdots (z - z_d)\right).$$

Here h is the unique solution to the *Kazdan-Warner equation* over $\mathbb{C}$:

$$\Delta h + \frac{1}{2}\left(e^{-2h} \prod_{j=1}^{d} |z - z_j|^2 - 1\right) = 0.$$

There are two natural classes of symmetry of $\mathbb{C}$, the translations and rotations, with respect to which the equation is invariant. But there exsits solutions which have infinitely many rotational symmetry, which will result in non-smooth moduli space. Hence we won't identify two solutions if they differ by a rotation.

Definition 4.2.3. An isomorphism from (A_1, u_1) to (A_2, u_2) is a pair (t, g), where $t : \mathbb{C} \to \mathbb{C}$ is a translation, which lifts naturally to a bundle map between the trivial G-bundles, and g is a gauge transformation $g : \mathbb{C} \to G$, such that $g^* t^* (A_1, u_1) = (A_2, u_2)$.

We have the regularity modulo gauge tranformation.

Proposition 4.2.4. *[59, Proposition D.2] For any solution (A, u) of class $W^{1,p}_{loc}$, there exists a gauge tranformation g of class $W^{2,p}_{loc}$ such that $g^*(A, u)$ is smooth.*

From now on any solution will be assumed to be smooth unless otherwise mentioned.

The next important property for affine vortices is its behavior near infinity. First we look at the decay of energy density.

Proposition 4.2.5. *[60, Corollary 4] Suppose (A,u) is an affine vortex with finite energy. Then for any $\epsilon>0$ there exists $C_\epsilon>0$ such that*

$$e_{A,u}(z)\leq C_\epsilon|z|^{-4+\epsilon},\ \forall|z|\geq 1.$$

Then we have the asymptotic behavior of vortices in suitable gauge. A solution (A,u) is **in radial gauge** if for $|z|$ large, $A=d+\eta(z)d\theta$.

Proposition 4.2.6. *[17, Proposition 11.1] If (A,u) is an affine vortex with finite energy in radial gauge with $A=d+\xi(z)d\theta$ for $|z|$ large. Then there exists a $W^{1,2}$-map $x:S^1\to\mu^{-1}(0)$ and an L^2-map $\eta:S^1\to\mathfrak{g}$ such that $x'(\theta)+\mathcal{X}_{\eta(\theta)}(x(\theta))=0$ and*

$$\lim_{r\to\infty}\sup_\theta d\left(u(re^{i\theta}),x(\theta)\right)=0,\ \lim_{r\to\infty}\int_0^{2\pi}\left|\xi(re^{i\theta})-\eta(\theta)\right|^2d\theta=0.$$

To state the next proposition we introduce some notations. Consider the standard embedding $\mathbb{C}\to S^2$. An **extension** of (A,u) is a triple $(\widetilde{P},\iota,\widetilde{u})$, where $\widetilde{P}\to S^2$ is a topological G-bundle, $\iota:\mathbb{C}\times G\to\widetilde{P}$ is a bundle map which descends to the inclusion $\mathbb{C}\to S^2$, and $\widetilde{u}:\widetilde{P}\to M$ is an equivariant continuous map such that $\widetilde{u}\circ\iota=u$.

Proposition 4.2.7. *[17][61] If $\mathcal{YMH}(A,u)<\infty$, then there exists a unique extension $(\widetilde{P},\iota,\widetilde{u})$ of (A,u) onto the sphere, such that $\widetilde{u}(\widetilde{P}_\infty)\subset\mu^{-1}(0)$, where $\widetilde{P}_\infty$ is the fibre of $\widetilde{P}$ at $\infty\in S^2$.*

The extension $(\widetilde{P},\iota,\widetilde{u})$ defines an equivariant homology class $\widetilde{u}_*[S^2]\in H_2^G(M;\mathbb{Z})$. Indeed, the Yang-Mills-Higgs functional of (A,u) is equal to the paring $\langle[\omega-\mu],\widetilde{u}_*[S^2]\rangle$, where $[\omega-\mu]$ is the equivariant cohomology class defined by the equivariantly closed two form $\omega-\mu$.

Identify u topologically with $v:\mathbb{D}\to M$ which extends continuously to the boundary. Then the extension of (A,u) to S^2 is given by $g_\infty:S^1\to G$ and $x_\infty\in\mu^{-1}(0)$ such that $v|_{\partial\mathbb{D}}(e^{i\theta})=g_\infty(e^{i\theta})x_\infty$. Trivialize v^*TM on $\mathbb{D}$, then the loop g_∞ is identifed with a loop of invertible matrices. The degree of $\det g_\infty$ is defined to be the **Maslov index** of the solution (A,u), denoted by $\mathrm{ind}_\mu(A,u)$ (see [61, Definition 2.6] for details).

4.2.1 Fredholm theory of affine vortices

Let z be the standard coordinate on $\mathbb{C}$. Let $\rho(z) = (1+|z|^2)^{\frac{1}{2}}$ for $z \in \mathbb{C}$. For $\delta \in \mathbb{R}$, $p > 1$, consider the Banach spaces (over complex numbers)

$$W^{k,p}_{\delta,\mathfrak{euc}} := \left\{ u \in W^{k,p}_{loc}(\mathbb{C}) \mid \rho^\delta u \in W^{k,p}(\mathbb{C}) \right\}.$$

And we denote $L^p_\delta = W^{0,p}_{\delta,\mathfrak{euc}}$.

For a smooth solution (A,u), the space of infinitesimal deformations is described as follows. Regard u as a smooth map from $\mathbb{C}$ to M. The connection A induces a connection on the bundle u^*TM and the bundle $T^*\mathbb{C}\otimes\mathfrak{g}$, both denoted by ∇^A. For $(V,\alpha) \in W^{1,p}_{loc}(\mathbb{C}, u^*TM \oplus T^*\mathbb{C}\otimes\mathfrak{g})$, define

$$|(V,\alpha)|_{W_{p,\delta}} := |V|_{L^\infty} + \left|\nabla^A V\right|_{L^p_\delta} + \left|\nabla^A \alpha\right|_{L^p_\delta} + |d\mu(V)|_{L^p_\delta} + |d\mu(JV)|_{L^p_\delta} + |\alpha|_{L^p_\delta}.$$

And define $W_{p,\delta} \subset W^{1,p}_{loc}$ be the subspace of vectors with finite $W_{p,\delta}$-norm. If $A = d + \alpha_0$, then the linearization of the gauge tranformation is $h \mapsto (-dh + [\alpha_0, h], -\mathcal{X}_h)$, whose adjoint is the map

$$\begin{array}{rcl} W_{p,\delta} & \to & L^p_\delta(\mathbb{C},\mathfrak{g}) \\ (V,\alpha) & \mapsto & -d^*\alpha - \Lambda[*\alpha_0,\alpha] - d\mu(JV) \end{array}$$

Then it is easy to see, the linearization at such (A,u) is a bounded linear operator

$$\begin{array}{rcl} D_{A,u}: \quad W_{p,\delta} & \to & L^p_\delta \\ (V,\alpha) & \mapsto & \begin{pmatrix} \left(\nabla^A V\right)^{0,1} + \frac{1}{2}(\nabla_V J)\circ d_A u \circ j + \mathcal{X}^{0,1}_\alpha \\ d\alpha + [\alpha_0,\alpha] + d\mu(V)\cdot dsdt \\ -d^*\alpha - \Lambda[*\alpha_0,\alpha] - d\mu(JV) \end{pmatrix} \end{array} \tag{4.1}$$

Proposition 4.2.8. *[61] There exists $p_0 > 2$ such that for all $p \in (2,p_0)$ and $\delta \in \left(1-\frac{2}{p}, 2-\frac{2}{p}\right)$, $D_{A,u}$ is Fredholm and* $\mathrm{index} D_{A,u} = 2\mathrm{ind}_\mu(A,u) + \dim M - 2\dim G$.

4.2.2 The adiabatic limit

For a compact Riemann surface Σ, fix a smooth area form $\Omega_\Sigma \in \Omega^2(\Sigma)$. Let $\lambda > 0$ be a real number. A **λ-twisted holomorphic map** from Σ to M (or a λ-vortex) is a solution to the vortex equation

(1.14) with the area form Ω_Σ replaced by $\lambda^2\Omega_\Sigma$, i.e.,

$$\begin{cases} \overline{\partial}_A u &= 0; \\ F_A + \lambda^2 \mu(u)\Omega_\Sigma &= 0. \end{cases} \tag{4.2}$$

Its energy is defined in the same way using the area form $\lambda^2\Omega_\Sigma$. If we agree that all Sobolev norms appearing in the following are taken with respect to the fixed area form Ω_Σ, then the energy of a λ-twisted holomorphic map is given by

$$\mathcal{YMH}^\lambda(P, A, u) = \frac{1}{2}\left(\|d_A u\|_{L^2}^2 + \lambda^{-2}\|F_A\|_{L^2}^2 + \lambda^2\|\mu(u)\|_{L^2}^2\right).$$

What is of interest is the limit process $\lambda \to \infty$, which is called the **adiabatic limit process**. If we fix the topological type of the vortex (which implies the uniform bound on the energy), then we see that $\|\mu(u)\|_{L^2} \to 0$ as $\lambda \to \infty$. Indeed, $\mu(u)$ will converge to zero except for finitely many points in Σ, and those points are where nontrivial affine vortices bubble off.

More precisely, suppose λ_k is a sequence of real numbers diverging to infinity, and (A_k, u_k) is a sequence λ_k-vortices (i.e., solutions to (4.2)). The energy density function for (A_k, u_k) will be

$$e_k(z) = \frac{1}{2}\left(|d_{A_k} u_k(z)|^2 + \lambda_k^2 |\mu(u_k(z))|^2\right).$$

If the sequence of functions e_k is not uniformly bounded, and suppose there exists a sequence of point $p_k \in \Sigma$ such that

$$\limsup_{k\to\infty} c_k := \limsup_{k\to\infty} e_k(p_k) = \limsup_{k\to\infty} \sup_\Sigma e_k = +\infty.$$

Then in [17] it was shown that in the following three possibilities (for suitable subsequences) corresponding bubbles will appear:

1. $\lim_{k\to\infty} \lambda_k^{-2} c_k = +\infty$. In this case, a nontrivial holomorphic sphere in M will bubble off.
2. $0 < \lim_{k\to\infty} \lambda_k^{-2} c_k < +\infty$. In this case, a nontrivial planar vortex will bubble off.
3. $\lim_{k\to\infty} \lambda_k^{-2} c_k = 0$. In this case, a nontrivial holomorphic sphere in the symplectic quotient will bubble off.

This is called the **bubbling zoology** of the adiabatic limit. Since we have assumed that (M, ω) is

aspherical, the first bubble type won't appear. In particular we have

$$\limsup_{k\to\infty} \lambda_k^{-2} \|e_k\|_{L^\infty} < \infty. \tag{4.3}$$

Remark 4.2.9. In the general situation, given an arbitrary sequence (A_i, u_i), we don't know *a priori* where the energy density will blow up and what type of bubbles may appear. But in the case of Kähler targets, using the algebraic description of vortices (e.g., stable N-pairs) provided by the Hitchin-Kobayashi correspondence, we observe that the cause of the energy concentration is governed in the algebraic side. Using this property we can manipulate the energy concentration for the adiabatic limit process and construct affine vortex bubbles. This is what we do in the last two sections of this chapter for the case $X = \mathbb{C}^N$.

4.3 Degenerations of affine vortices

In the case studied by Taubes, we have already seen one type of degeneration of affine vortices. Namely, vortices may "split": the relative distance between points in $\overrightarrow{z} \in \mathrm{Sym}^d\mathbb{C}$ may diverge to infinity, which means a sequence of vortices with vortex number d can split into up to d nontrivial vortices. There are two other types of degenerations. If the symplectic quotient $\overline{M}$ allows nontrivial holomorphic spheres, then the energy of a sequence of affine vortices can concentrate at infinity and a holomorphic sphere in $\overline{M}$ can bubble off. Another possibility is sphere bubbles in the interior. But since we have assumed that M is aspherical, this type of bubbles can only appear when marked points coming together, i.e., ghost bubbles.

The bubbling phenomenon has been studied in [61]. And more generally, if we allow arbitrarily many marked points, the we can use the language of *complexified multiplihedron* to describe the moduli of stable vortices and its topology, as done in [57]. Here we only consider the case of at most one interior marked point, hence no need to introduce the formal language.

4.3.1 Stable maps modelled on a rooted tree

A rooted tree $T = (V, E, R)$ is a tree $T = (V, E)$ with a distinguished vertex $R \in V$(the *root*). For a rooted tree T the edges are automatically oriented towards the root. We define the *depth* $d_T : V \to \mathbb{Z}_{\geq 0} \cup \{\infty\}$ such that

1. $d_T(R) = \infty$;

2. $v_1 E v_2 \in E \Longrightarrow d_T(v_1) + 1 = d_T(v_2)$ or $v_2 = R$;

3. If there is no v_1 such that $v_1 E v_2 \in E$, then $d_T(v_2) = 0$.

An n-labelling of a rooted tree T is a map $\rho : \{\alpha_1, \ldots, \alpha_n\} \to V$.

Definition 4.3.1. An $(n, 1)$-marked genus zero stable map to $\overline{M}$ modelled on an n-labelled rooted tree $(T, \rho) = (V, E, R; \rho)$ is a tuple

$$(\mathbf{u}, \mathbf{w}) := \left(\{u_{v_i}\}_{v_i \in V}, \{w_{i_1 i_2}\}_{v_{i_1} E v_{i_2} \in E}, \{w_{\alpha_j}\}_{j=1,\ldots,n}\right)$$

where

1. $u_{v_i} : \mathbb{C} \to \overline{M}$ is a holomorphic map with finite energy (hence extends to a holomorphic sphere);

2. $w_{i_1 i_2} \in \mathbb{C}$ and $w_{\alpha_j} \in \mathbb{C}$.

They are subject to the following conditions:

1. For each $v_{i_1} E v_{i_2} \in E$, $u_{v_{i_1}}(\infty) = u_{v_{i_2}}(w_{i_1 i_2}) \in \overline{M}$;

2. For each $v_{i_2} \in V$, the points $w_{i_1 i_2}$ for all $v_{i_1} E v_{i_2} \in E$ and z_{α_j} for all $\rho(\alpha_j) = v_{i_2}$ are all distinct;

3. If $u_{v_{i_2}}$ is a constant map, then $\#\{v_{i_1} \in V \mid v_{i_1} E v_{i_2} \in E\} + \#\{\alpha_j \mid \rho(\alpha_j) = v_{i_2}\} \geq 2$.

For each $v_i \in V$, we define $Z_{v_i} := \left\{w_{v_i' v_i} \in \mathbb{C} \mid v_i' E v_i \in E\right\} \cup \left\{w_{\alpha_j} \mid \rho(\alpha_j) = v_i\right\}$.

This refined notion of stable maps is used because the domain of an affine vortex has a canonical marked point ∞. Then each stable map $(\mathbf{u}, \mathbf{w})$ defined as above, there is a dinstinguished marked point ∞ on the component corresponding to R. For each edge $v_1 E v_2$, the coordinate of the node on the component corresponding to v_1 is automatically ∞. We can define various notions (such as isomorphisms and convergence) of stable maps as done in [34, Chapter 5], with the restriction that all tree maps should be maps between rooted trees, and we should only use affine linear transformations instead of arbitrary Möbius transformations.

4.3.2 $(0, 1)$-marked and $(1, 1)$-marked stable affine vortices

We now describe the objects we will use to compactify the moduli space of affine vortices diskussed in Section 4.2.

Definition 4.3.2. An **admissible pair of rooted trees** is a pair $\left(\widetilde{T}, T\right)$ where $\widetilde{T} = \left(\widetilde{V}, \widetilde{E}, R\right)$ is a rooted tree, $T \subset \widetilde{T}$ is a rooted subtree and we allow $T = \emptyset$ such that the following conditions hold

1. If $T = \emptyset$, then $\widetilde{T}$ has a single vertex R;

2. If $T = (V, E, R) \neq \emptyset$, then $\widetilde{V} \setminus V$ consists of vertices of depth zero;

A **labelled admissible pair of rooted trees** is an admissible pair of rooted trees $\left(\widetilde{T}, T\right)$ together with a vertex $V_0 \in \widetilde{V} \setminus V$, denoted by $\left(\widetilde{T}, T; V_0\right)$.

If $T \neq \emptyset$, then the set of edges $\beta_k E v_i \in \widetilde{E}$ for $\beta_k \in \widetilde{V} \setminus V$ and $v_i \in V$ induces a labelling $\rho_{\widetilde{T},T} : \widetilde{V} \setminus V \to V$, making $\left(T, \rho_{\widetilde{T},T}\right)$ a rooted tree with a $\#(\widetilde{V} \setminus V)$-labelling.

More generally, if $T \neq \emptyset$, and $T' \subset T$ is a rooted subtree (which is allowed to be empty), then denote by $\mathcal{B}_{\widetilde{T},T'} := \{B_1, \ldots, B_t\}$ the set of connected components of $\widetilde{T} \setminus T'$. Each element $B_l \in \mathcal{B}_{\widetilde{T},T'}$ represent a rooted tree $\widetilde{T}_{B_l}$ so that $(B_l, B_l \cap T)$ is an admissible pair of rooted trees. If $V_0 \in B_l$, then $(B_l, B_l \cap T; V_0)$ is a labelled admissible pair of rooted trees.

Definition 4.3.3. A $(0,1)$-marked stable affine vortex modelled on an admissible pair of rooted trees $\left(\widetilde{T}, T\right)$ is a tuple

$$\widetilde{\mathbf{W}} := \left((\mathbf{u}, \mathbf{w}) ; \{W_{\beta_k}\}_{\beta_k \in \widetilde{V} \setminus V}\right)$$

where

1. If $T \neq \emptyset$, then $(\mathbf{u}, \mathbf{w})$ is an $(s,1)$-marked genus zero stable map to $\overline{M}$ modelled on the labelled rooted tree $\left(T, \rho_{\widetilde{T},T}\right)$; if $T = \emptyset$ then $(\mathbf{u}, \mathbf{w})$ is empty.

2. **Stability.** For each $\beta_k \in \{\beta_1, \ldots, \beta_s\} = \widetilde{V} \setminus V$, W_{β_k} is a *nontrivial* affine vortex.

They are subject to the following constrains:

- If $T \neq \emptyset$, then for each $\beta_k \in \widetilde{V} \setminus V$, $ev_\infty(W_{\beta_k}) = ev_k(\mathbf{u}) \in \overline{M}$.

Definition 4.3.4. A $(1,1)$-marked stable affine vortex modelled on a labelled admissible pair of rooted trees $\left(\widetilde{T}, T; V_0\right)$ is a tuple

$$\widetilde{\mathbf{W}} := \left((\mathbf{u}, \mathbf{w}) ; \{W_{\beta_k}\}_{\beta_k \in \widetilde{V} \setminus V} ; \{z_0\}\right)$$

where

1. If $T \neq \emptyset$, then $(\mathbf{u}, \mathbf{w})$ is an $(s,1)$-marked genus zero stable map to $\overline{M}$ modelled on the labelled tree $\left(T, \rho_{\widetilde{T},T}\right)$; if $T = \emptyset$ then $(\mathbf{u}, \mathbf{w})$ is empty.

2. For each $\beta_k \in \{\beta_1, \ldots, \beta_s\} = \widetilde{V} \setminus V$, W_{β_k} is an affine vortex;

3. $z_0 \in \mathbb{C}$.

They are subject to the following constrains:

1. **Stability.** If W_{β_k} is trivial then $V_0 = \beta_k$.

2. If $T \neq \emptyset$, then for each $\beta_k \in \widetilde{V} \setminus V$, $ev_\infty(W_{\beta_k}) = ev_k(\mathbf{u}) \in \overline{M}$.

We will call the marked point z_0 an **interior marked point** of $\widetilde{\mathbf{W}}$.

The following picture illustrates a typical marked stable affine vortex. Here the "tear drops" represent the vortices W_k and the spheres represent the components mapped into $\mu^{-1}(0)$. A dotted component indicates that it has zero energy.

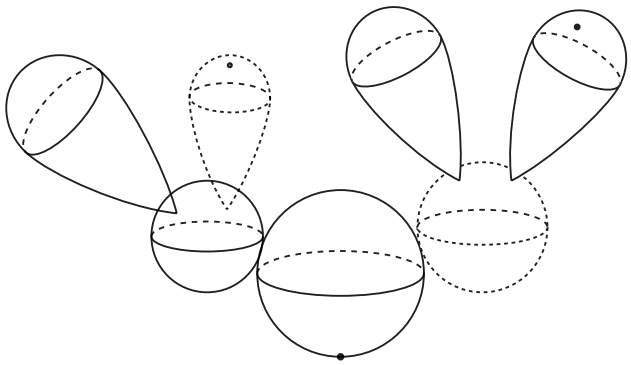

Figure 4.1: A typical marked stable affine vortex.

Definition 4.3.5. An isomorphism between two $(1,1)$-marked planar stable vortices

$$\widetilde{\mathbf{W}} := \left((\mathbf{u}, \mathbf{w}); \{W_{\beta_k}\}_{\beta_k \in \widetilde{V} \setminus V}; \{z_0\}\right)$$

and

$$\widetilde{\mathbf{W}}' := \left((\mathbf{u}', \mathbf{w}'); \{W'_{\beta_k}\}_{\beta_k \in \widetilde{V} \setminus V}; \{z'_0\}\right)$$

modelled on the same labelled admissible pair of rooted trees $\left(\widetilde{T}, T; V_0\right)$ is a tuple

$$\left(f, \{t_{\beta_k}, g_{\beta_k}\}_{\beta_k \in \widetilde{V} \setminus V}, \{\phi_{v_i}\}_{v_i \in V}\right)$$

where

1. $f : (\widetilde{T}, T; V_0) \to \left(\widetilde{T}, T; V_0\right)$ is an automorphism;

2. For each $\beta_k \in \widetilde{V} \setminus V$, t_{β_k} is a translation on $\mathbb{C}$ and $g_{\beta_k} : \mathbb{C} \to G$ is a gauge transformation;

3. For each $v_i \in V$, ϕ_{v_i} is an affine linear transformation and if $T \neq \emptyset$, then $(f|_T, \{\phi_{v_i}\}_{v_i \in V})$ is an isomorphism between $(\mathbf{u}, \mathbf{w})$ and $(\mathbf{u}', \mathbf{w}')$ as $(s, 1)$-marked genus zero stable maps modelled on the $\#\left(\widetilde{V} \setminus V\right)$-labelled rooted tree $\left(T, \rho_{\widetilde{T},T}\right)$.

They must satisfy the following conditions:

1. For each $\beta_k \in \widetilde{V} \setminus V$, $(t_{\beta_k}, g_{\beta_k})$ is an isomorphism from W_{β_k} to $W'_{f(\beta_k)}$ (see Definition 4.2.3);

2. $t_{V_0}(z'_0) = z_0$.

For $k = 0, 1$, we define the homology class of a $(k, 1)$-marked stable affine vortex $\widetilde{\mathbf{W}}$ to be the sum of the homology classes of each of its components, which is an element in $H_2^G(M; \mathbb{Z})$. This only depends on its isomorphism class. For $A \in H_2^G(M; \mathbb{Z})$ we denote by $\overline{\widetilde{\mathcal{M}}}^{\mathbb{A}}_{1,1}(M, A)$ be the category of all $(1, 1)$-marked stable affine vortices of homology class A and the morphisms are isomorphisms between the objects. Denote by $\overline{\mathcal{M}}^{\mathbb{A}}_{1,1}(M, A)$ the space of isomorphism classes of $(1, 1)$-marked stable affine vortices of homology class A. There is a well-defined evaluation map

$$ev_\infty : \overline{\mathcal{M}}^{\mathbb{A}}_{1,1}(M, A) \to \overline{M}.$$

4.3.3 Degeneration of affine vortices

Now we describe the topology on the moduli space of stable affine vortices. We first give the definition the convergence of a sequence of affine vortices to a stable affine vortex, which essentially coincide with the definition in [61].

By a theorem of Guillemin and Sternberg, there is a neighborhoof U_ϵ of $\mu^{-1}(0) \subset M$, which is (canonically) symplectomorphic to $\mu^{-1}(0) \times \mathfrak{g}^*_\epsilon$, where $\mathfrak{g}^*_\epsilon$ is an ϵ-ball of $\mathfrak{g}^*$ centered at the origin with respect to some biinvariant inner product, such that the moment map μ restricted to U_ϵ is equal to the projection onto $\mathfrak{g}^*_\epsilon$. Hence there is a well-defined map

$$\pi_\mu : U_\epsilon \to \overline{M}.$$

Definition 4.3.6. Let $W^\nu = (A^\nu, u^\nu)$ be a sequence of affine vortices and let

$$\widetilde{\mathbf{W}} = \left((\mathbf{u}, \mathbf{w}); \{W_{\beta_k}\}_{\beta_k \in \widetilde{V} \setminus V}\right)$$

be a $(0,1)$-marked stable affine vortex modelled on an admissible pair of rooted trees $\left(\widetilde{T},T\right)$. We say that the sequence $\{W^\nu\}$ converges to $\widetilde{\mathbf{W}}$, if $E:=\lim_{\nu\to\infty}E(W^\nu)<\infty$ exists, and

$$E=E(\widetilde{\mathbf{W}})$$

and for each $v_i\in V$ and $\beta_k\in\widetilde{V}\setminus V$, there exists affine linear transformations $\phi^\nu_{v_i}$, $\phi^\nu_{\beta_k}$ such that the following conditions are satisfied:

1. For each $\beta_k\in\widetilde{V}\setminus V$, $\phi^\nu_{\beta_k}$ is a translation, and there exists a sequence of gauge transformations $g^\nu_{\beta_k}:\mathbb{C}\to G$ such that $\left(g^\nu_{\beta_k}\right)^*\left(\phi^\nu_{\beta_k}\right)^*(A^\nu,u^\nu)$ converges to W_{β_k} uniformly in any compact subset of $\mathbb{C}$;

2. For each $v_i\in V$, the sequence affine linear transformation $\phi^\nu_{v_i}(y)=a^\nu_{v_i}y+b^\nu_{v_i}$ with $a^\nu_{v_i}$ converges to infinity;

3. For each $v_i\in V$, $\left(\phi^\nu_{v_i}\right)^*(\mu\circ u^\nu)$ converges to zero uniformly on any compact subset of $\mathbb{C}\setminus Z_{v_i}$, and the map $\left(\phi^\nu_{v_i}\right)^*(\pi_\mu\circ u^\nu)$ converges to u_{v_i} uniformly on any compact subset of $\mathbb{C}\setminus Z_{v_i}$.

4. If $T\neq\emptyset$, then for any $\beta_k\in\widetilde{V}\setminus V$, the sequence of affine linear transformations $\left(\phi^\nu_{\rho_{\widetilde{T},T}(\beta_k)}\right)^{-1}\circ\phi^\nu_{\beta_k}$ converges to the constant map w_{β_k} uniformly on any compact subset of $\mathbb{C}$;

5. For any $v_iEv_j\in E$, the sequence of affine linear tranformations $\left(\phi^\nu_{v_j}\right)^{-1}\circ\phi^\nu_{v_i}$ converges uniformly on any compact subset of $\mathbb{C}$ to the constant map $w_{v_iv_j}$.

Definition 4.3.7. Let $W^\nu=(A^\nu,u^\nu;z^\nu_0)$ be a sequence of $(1,1)$-marked affine vortices and let

$$\widetilde{\mathbf{W}}=\left((\mathbf{u},\mathbf{w});\{W_{\beta_k}\}_{\beta_k\in\widetilde{V}\setminus V};\{z_0\}\right)$$

be a $(1,1)$-marked stable affine vortex modelled on the labelled admissible pair of rooted trees $(\widetilde{T},T;V_0)$. We say that the sequence W^ν converges to $\widetilde{\mathbf{W}}$, if $E:=\lim_{\nu\to\infty}E(W^\nu)<\infty$ exists, and

$$E=E(\widetilde{\mathbf{W}})$$

and for each $v_i\in V$ and $\beta_k\in\widetilde{V}\setminus V$, there exists affine linear transformations $\phi^\nu_{v_i}$, $\phi^\nu_{\beta_k}$ such that the following conditions are satisfied:

1. For each $\beta_k\in\widetilde{V}\setminus V$, $\phi^\nu_{\beta_k}$ is a translation, and there exists a sequence of gauge transformations $g^\nu_{\beta_k}:\mathbb{C}\to G$ such that $\left(g^\nu_{\beta_k}\right)^*\left(\phi^\nu_{\beta_k}\right)^*(A^\nu,u^\nu)$ converges to W_{β_k} uniformly in any compact subset of $\mathbb{C}$;

2. $\lim_{\nu\to\infty}\left(\phi^{\nu}_{V_0}\right)^{-1}(z^{\nu}_0)=z_0$;

3. For each $v_i\in V$, the sequence affine linear transformation $\phi^{\nu}_{v_i}(y)=a^{\nu}_{v_i}y+b^{\nu}_{v_i}$ with $a^{\nu}_{v_i}$ converges to infinity;

4. For each $v_i\in V$, $\left(\phi^{\nu}_{v_i}\right)^{*}(\mu\circ u^{\nu})$ converges to zero uniformly on any compact subset of $\mathbb{C}\setminus Z_{v_i}$, and the map $\left(\phi^{\nu}_{v_i}\right)^{*}(\pi_{\mu}\circ u^{\nu})$ converges to u_{v_i} uniformly on any compact subset of $\mathbb{C}\setminus Z_{v_i}$.

5. If $T\neq\emptyset$, then for any $\beta_k\in\widetilde{V}\setminus V$, the sequence of affine linear transformations $\left(\phi^{\nu}_{\rho_{\widetilde{T},T}(\beta_k)}\right)^{-1}\circ\phi^{\nu}_{\beta_k}$ converges to the constant map w_{β_k} uniformly on any compact subset of $\mathbb{C}$;

6. For any $v_iEv_j\in E$, the sequence of affine linear tranformations $\left(\phi^{\nu}_{v_j}\right)^{-1}\circ\phi^{\nu}_{v_i}$ converges uniformly on any compact subset of $\mathbb{C}$ to the constant map $w_{v_iv_j}$.

Now the notion of convergence in the space of planar stable vortices can be easily defined:

Definition 4.3.8. Let $\widetilde{\mathbf{W}}^{\nu}=\left((\mathbf{u}^{\nu},\mathbf{w}^{\nu});\left\{W^{\nu}_{\beta_k}\right\}_{\beta_k\in\widetilde{V}^{\nu}\setminus V^{\nu}};\{z^{\nu}_0\}\right)$ be a sequence of $(1,1)$-marked stable affine vortices modelled on a sequence of labelled admissible pair of rooted trees $\left(\widetilde{T}^{\nu},T^{\nu};V^{\nu}_0\right)$, and

$$\widetilde{\mathbf{W}}^{\infty}=\left((\mathbf{u}^{\infty},\mathbf{w}^{\infty});\{W^{\infty}_{\beta_k}\}_{\beta_k\in\widetilde{V}^{\infty}\setminus V^{\infty}};\{z^{\infty}_0\}\right)$$

be a $(1,1)$-marked stable affine vortex modelled on the labelled admissible pair of rooted trees $\left(\widetilde{T}^{\infty},T^{\infty};V^{\infty}_0\right)$. We say that the sequence $\left\{\widetilde{\mathbf{W}}^{\nu}\right\}$ converges to $\widetilde{\mathbf{W}}^{\infty}$ if the following condition holds:

1. If $T^{\infty}=\emptyset$, then for large ν, $T^{\nu}=\emptyset$ and $\widetilde{\mathbf{W}}^{\nu}$ converges to $\widetilde{\mathbf{W}}^{\infty}$ in the sense of Definition 4.3.7;

2. If $T^{\infty}\neq\emptyset$, then there exists a rooted subtree $T'\subset T^{\infty}$ such that

   (a) If $T'=\emptyset$ then for large ν, $T^{\nu}=\emptyset$;

   (b) If $T'\neq\emptyset$, then for large ν, $T^{\nu}\neq\emptyset$, $\#\left(\widetilde{V}^{\nu}\setminus V^{\nu}\right)=\#\mathcal{B}_{\widetilde{T}^{\infty},T'}$, and the sequence of genus zero stable maps $(\mathbf{u}^{\nu},\mathbf{w}^{\nu})$ converges to the stable map $\widetilde{\mathbf{W}}^{\infty}\Big|_{T'}$ which is modelled on the rooted tree T' with labelling $\rho_{\widetilde{T}^{\infty},T'}$;

   (c) For large ν, there is a bijection $\mathfrak{s}:\mathcal{B}_{\widetilde{T}^{\infty},T'}\to\widetilde{V}^{\nu}\setminus V^{\nu}$ and a unique B_0 such that $V^{\nu}_0=\mathfrak{s}(B_0)$; and such that for each $B_l\in\mathcal{B}_{\widetilde{T}^{\infty},T'}$, if $B_l=B_0$, then $\left(W^{\nu}_{V_0},z^{\nu}_0\right)$ converges to $\widetilde{\mathbf{W}}^{\infty}\Big|_{B_0}$ in the sense of Definition 4.3.7; if $B_l\neq B_0$, then $W^{\nu}_{\mathfrak{s}(B_l)}$ converges to $\widetilde{\mathbf{W}}^{\infty}\Big|_{B_l}$ in the sense of Definition 4.3.6.

Now we can state the compactness theorem of Ziltener. We specialize to the case where we have at most one interior marked points.

Theorem 4.3.9. *[61, Theorem 3] If $\widetilde{\mathbf{W}}^\nu$ is a sequence of $(k,1)$-marked stable affine vortices with $k=0$ or 1, and $\limsup_{\nu\to\infty} E\left(\widetilde{\mathbf{W}}^\nu\right) < \infty$, then there is a subsequence and a $(k,1)$-marked stable affine vortex $\widetilde{\mathbf{W}}^\infty$ such that the subsequence converges to $\widetilde{\mathbf{W}}^\infty$ in the sense of Definition 4.3.8.*

Example 4.3.10. Suppose we are in the case of *Example* 4.2.2. A sequence of vortices is prescribed by a sequence of points $\overrightarrow{z}^\nu \in \mathrm{Sym}^d\mathbb{C}$. Modulo translation, the bubbling is caused by the phenomenon that at least two points in $\overrightarrow{z}^\nu$ are separated infinitely far away. Then in the limit, $\overrightarrow{z}^\nu$ are divided into groups, points in the same group will stay within finite distance and points belonging to different groups will be infinitely far away. Hence the limit depends on a partition $d = d_1 + \ldots + d_r$, a curve $C \in \overline{\mathcal{M}}_{0,r+1}$, and for each $i \in \{1,\ldots,r\}$, an element $\overrightarrow{z}_i \in \mathrm{Sym}^{d_i}\mathbb{C}$ up to translation. We have the similar description if we add a marked point.

4.4 Quantum Kirwan map

4.4.1 The Poincaré bundle and the evaluation maps

Let's consider the moduli space of $(1,1)$-marked stable affine vortices. We call the component which contains the interior marked point the *primary component*.

Fix a homology class $A \in H_2^G(M)$. Consider the category of $(1,1)$-marked stable affine vortices, $\overline{\widetilde{\mathcal{M}}}^{\mathbb{A}}_{1,1}(M,A)$ with homology class equal to A. The morphism set between two objects is the set of isomorphisms $(f, \{t_{\beta_k}, g_{\beta_k}\})$ defined in Definition 4.3.5. Also consider the category of objects $\left(\widetilde{\mathbf{W}}, p\right)$ where $\widetilde{\mathbf{W}}$ is an object of $\overline{\widetilde{\mathcal{M}}}^{\mathbb{A}}_{1,1}(M,A)$ and $p \in S^1 \subset \mathbb{C}$ which is regarded as a point in the fibre at the marked point z_0 of the trivial principal bundle of the primary component. A morphism between $\left(\widetilde{\mathbf{W}}, p\right)$ and $\left(\widetilde{\mathbf{W}}', p'\right)$ is an isomorphism $(f, \{t_{\beta_k}, g_{\beta_k}\})$ between $\widetilde{\mathbf{W}}$ and $\widetilde{\mathbf{W}}'$ such that $p = p' g_{V_0}(z_0)$. Taking quotient modulo isomorphisms, we get a principal G-bundle over $\overline{\mathcal{M}}^{\mathbb{A}}_{1,1}(M,A)$, denoted by

$$\overline{\mathcal{P}}_0 \to \overline{\mathcal{M}}^{\mathbb{A}}_{1,1}(M,A).$$

The right G-action is induced from $\left(\widetilde{\mathbf{W}}, p\right) \cdot g = \left(\widetilde{\mathbf{W}}, pg\right)$. There is a well-defined equivariant evaluation induced from $\left(\widetilde{W}, p\right) \mapsto u_{V_0}(p) \in M$, denoted by

$$ev_0 : \overline{\mathcal{P}}_0 \to M.$$

This induces a map, with an abuse of notation, $ev_0 : \overline{\mathcal{M}}^{\mathbb{A}}_{1,1}(M,A) \to M_G := EG \times_G M$, to the Borel construction M_G of M, whose cohomology is the equivariant cohomology of M. Also, we have the

evaluation map $ev_\infty : \overline{\mathcal{M}}^{\mathbb{A}}_{1,1}(M,A) \to \overline{M}$ by evaluating the root component at infinity.

4.4.2 The quantum Kirwan map

We use the notations in Subsection 1.3.1 and assuming that $(\overline{M},\overline{\omega})$ is monotone. Suppose we have a natural orientation of the deformation complex of the equation (1.14) (over $\mathbb{C}$) with gauge tranformations, and there is a well-defined virtual fundamental class $\left[\overline{\mathcal{M}}^{\mathbb{A}}_{1,1}(M,A)\right]^{vir}$, then the quantum Kirwan map $\kappa_Q : H^*_G(M;\mathbb{R}) \to QH^*\left(\overline{M};\Lambda\right)$ is defined by

$$\kappa_Q(\alpha) = \sum_{A=d_1A_1+\cdots+d_mA_m} \sum_{\nu_1,\nu_2} \left\langle (ev_0)^*\alpha \cup (ev_\infty)^* e_{\nu_1}, \left[\overline{\mathcal{M}}^{\mathbb{A}}_{1,1}(M,A)\right]^{vir} \right\rangle \cdot g^{\nu_1\nu_2} e_{\nu_2} \otimes q_1^{d_1}\cdots q_m^{d_m}. \tag{4.4}$$

Note that if we specialize $\kappa_Q(\alpha)$ at $q_1 = \cdots = q_m = 0$, then the only contribution is given by $\overline{\mathcal{M}}^{\mathbb{A}}_{1,1}(M,0)$, which is homeomorphic to $\overline{M}$; the Poincaré bundle over this moduli is isomorphic to the G-bundle $\mu^{-1}(0) \to \overline{M}$. So the result will be the classical Kirwan map $\kappa(\alpha)$.

4.5 The adiabatic limit of $U(1)$-vortices

4.5.1 Holomorphic N-pairs and Hitchin-Kobayashi correspondence for rescaled area form

From now on, we consider concrete examples. More precisely, we work with $M = \mathbb{C}^N$ with the diagonal S^1-action, whose moment map is

$$\mu(z_1,\ldots,z_N) = -\frac{\sqrt{-1}}{2}\left(\sum_{j=1}^N |z_j|^2 - 1\right).$$

Here we identify $\mathrm{Lie}S^1 \simeq i\mathbb{R} \simeq \left(\mathrm{Lie}S^1\right)^*$. The symplectic quotient is the projective space $\mathbb{P}^{N-1}$. We assume $N \geq 2$.

Recall in Subsection 1.3.4, a degree d, rank 1, stable holomorphic N-pair over the Riemann surface Σ is a tuple $(\mathcal{L};\varphi_1,\ldots,\varphi_N)$ where $\mathcal{L} \to \Sigma$ is a degree d holomorphic line bundle and $\varphi_j \in H^0(\mathcal{L})$ such that at least one of them is nonzero. On the other hand, let $L \to \Sigma$ be a fixed smooth Hermitian line bundle of degree d; let Ω_Σ be a smooth area form on Σ. Let $\widetilde{\mathcal{M}}^\lambda_\Sigma\left(\mathbb{C}^N,L\right)$ be the space of all

solutions $(A;\phi_1,\ldots,\phi_N)$ to the following equation:

$$\begin{cases} \overline{\partial}_A\phi_j &= 0; \\ F_A &= \frac{\lambda^2\sqrt{-1}}{2}\left(\sum_{j=1}^N |\phi_j|^2 - 1\right)\Omega_\Sigma. \end{cases} \tag{4.5}$$

Here A is a unitary connection on L and ϕ_i are smooth sections of L. And let $\mathcal{M}_\Sigma^\lambda\left(\mathbb{C}^N, L\right)$ be the space of gauge equivalence classes of such solutions.

By the Hitchin-Kobayashi correspondence (Theorem 1.3.5), the moduli space of (isomorphism classes of) degree d rank 1 stable holomorphic N-pairs is homeomorphic to $\mathcal{M}_\Sigma^\lambda(\mathbb{C}^N, L)$ for any λ. In particular, we always regard a holomorphic line bundle $\mathcal{L}$ having the underlying smooth line bundle L with a holomorphic structure given by $\overline{\partial}_A$, the $(0,1)$-part of the connection A. Take $\lambda = 1$ and we represent a holomorphic N-pair to an element in $\widetilde{\mathcal{M}}_\Sigma^1(\mathbb{C}^N, L)$. Then for each $(B;\varphi_1,\ldots,\varphi_N) \in \widetilde{\mathcal{M}}_\Sigma^1(\mathbb{C}^N, L)$ and for each $\lambda > 0$, there exists a unique h_λ such that

$$\Lambda F_B - \sqrt{-1}\Delta h_\lambda - \frac{\lambda^2\sqrt{-1}}{2}\left(e^{-2h_\lambda}\sum_{j=1}^N |\varphi_j|^2 - 1\right) = 0. \tag{4.6}$$

The corresponding solution to (4.5) is given by

$$A = B + \sqrt{-1}d^c h_\lambda = B - \partial h_\lambda + \overline{\partial} h_\lambda,\ \phi_j = e^{-h_\lambda}\varphi_j$$

which is the result by applying the "purely imaginary gauge tranformation" e^{h_λ} to the tuple $(B;\varphi_1,\ldots,\varphi_N)$.

For any λ-vortex $(A;\phi_1,\ldots,\phi_N) \in \widetilde{\mathcal{M}}_\Sigma^\lambda(\mathbb{C}^N, L)$, the λ-energy density and λ-energy are given by

$$e_\lambda(z) := \sum_{j=1}^N |d_A\phi_j(z)|^2 + \frac{\lambda^2}{2}\left|\sum_{j=1}^N |\phi_j(z)|^2 - 1\right|^2$$

$$E_\lambda(A;\phi_1,\ldots,\phi_N) := \int_\Sigma e_\lambda(z)\Omega_\Sigma = 2\pi \mathrm{deg} L. \tag{4.7}$$

Let $(\mathcal{L}, \overrightarrow{\varphi}) = (\mathcal{L};\varphi_1,\ldots,\varphi_N)$ be a stable holomorphic N-pair. The *base locus* of $\overrightarrow{\varphi}$ is the intersection of the zeroes of φ_j, denoted by $Z \subset \Sigma$. The multiplicity m_p of $p \in Z$ is the minimum of

the multiplicities of p as a zero of φ_j. Then $\overrightarrow{\varphi}$ defines a holomorphic map

$$\begin{array}{rccc} [\overrightarrow{\varphi}]: & \Sigma \setminus Z & \to & \mathbb{P}^{N-1} \\ & z & \mapsto & [\varphi_1(z), \ldots, \varphi_N(z)] \end{array}$$

which extends uniquely to a holomorhic map from Σ to $\mathbb{P}^{N-1}$ by removal of singularity, which is still denoted by $[\overrightarrow{\varphi}]$. It is easy to see that

$$\deg [\overrightarrow{\varphi}] = \deg L - \sum_{p \in Z} m_p.$$

4.5.2 The adiabatic limit

We will study the behavior of a sequence of objects $(A_\lambda, u_\lambda) \in \widetilde{\mathcal{M}}^\lambda_\Sigma(\mathbb{C}^N, L)$ as $\lambda \to \infty$ and give a refined version of the bubbling zoology of the adiabatic limit. In particular, we will give an algebraic condition on the bubbling of nontrivial affine vortices.

From now on until the end of this section, we fix a sequence of stable holomorphic N-pairs, which, by the Hitchin-Kobayashi correspondence, can be identified with a sequence $(B_k; \overrightarrow{\varphi}_k) = (B_k; \varphi_{1,k}, \ldots, \varphi_{N,k}) \in \widetilde{\mathcal{M}}^1_\Sigma\left(\mathbb{C}^N, L\right)$. We assume that this sequence converges to $(B; \varphi_1, \ldots, \varphi_N) \in \widetilde{\mathcal{M}}^1_\Sigma\left(\mathbb{C}^N, L\right)$, whose base locus is denoted by $Z \subset \Sigma$. We also fix a sequence $\lambda_k \to \infty$, hence for each k let $h_k : \Sigma \to \mathbb{R}$ denote the unique solution to the equation (4.6) for the vortex $(B_k; \overrightarrow{\varphi}_k)$ and $\lambda = \lambda_k$. We denote $A_k := B_k - \partial h_k + \overline{\partial} h_k$ and $\phi_{j,k} := e^{-h_k} \varphi_{j,k}$.

For each $p \in \Sigma$, we choose a local holomorphic coordiante $\xi_p : B_{r_p} \to U_p$, where $r_p > 0$ and $B_{r_p} \subset \mathbb{C}$ is the radius r_p disk centered at $0 \in \mathbb{C}$, U_p is a neighborhood of $p \in \Sigma$. Up to rescaling, we can assume that $\xi_p^* \Omega_\Sigma = \sigma_p ds dt$, where (s, t) is the standard coordinates on $\mathbb{C}$ and $\sigma_p : B_{r_p} \to \mathbb{R}_+$ is a smooth function, such that $\sigma_p(0) = 1$. We also trivialize $\xi_p^* L$ smoothly over B_{r_p} by $\zeta_p : B_{r_p} \times \mathbb{C} \to \xi_p^* L$. We call such a triple $\left(U_p, B_{r_p}, \xi_p, \zeta_p\right)$ an admissible chart near p. Then any triple (L, A, u) over Σ can be pulled back by an admissible chart to B_{r_p}, to a triple $\left(B_{r_p} \times \mathbb{C}, d + \alpha; \phi\right)$ over B_{r_p}.

Then for any $q \in \xi_p\left(B_{\frac{r_p}{2}}\right) \subset U_p$ and for $r \in \left(0, \frac{r_p}{2}\right)$, the inclusion $B_r \simeq B_{\xi_p^{-1}(q)}(r) \subset B_{r_p}$ induces a chart near q, from the admissible chart $(U_p, B_{r_p}, \xi_p, \zeta_p)$. Then for any large number λ, we zoom in by $B_{\lambda r} \simeq B_r$. With the admissible chart near p understood, we abbreviate by

$$s^*_{q,\lambda}\left(A, u\right)$$

to be the pull-back object on the trivial bundle over $B_{\lambda r}$.

Lemma 4.5.1. *Suppose $p_k \in \Sigma$ is a sequence of points such that $\lim_{k\to\infty} p_k = p$ and $r_k > 0$ is a sequence of real numbers such that*

$$\lim_{k\to\infty} r_k = 0, \ \lim_{k\to\infty} R_k := \lim_{k\to\infty} \lambda_k r_k = +\infty. \tag{4.8}$$

Choosing an admissible chart centered at p. Then we obtain a sequence of pull-back pairs

$$(d + \alpha_k; \phi_k) := s^*_{p_k,\lambda_k} (A_k, u_k)$$

on B_{R_k}. Then, there exists a subsequence (still indexed by k), such that $(d + \alpha_k, \phi_k)$ converges in C^∞_{loc} on $\mathbb{C}$ to an affine vortex with finite energy. Note that the limit can be trivial, and we don't need to take gauge transformations.

Proof. Let's first see why we don't need to take gauge transformations. Indeed, with respect to the admissible chart write

$$B_k = d + \beta_k = d + \Phi_k dx + \Psi_k dy.$$

Abbreviate $s^*_k = s^*_{p_k,\lambda_k}$, we have

$$s^*_k \beta_k = \lambda_k^{-1} (s^*_k \Phi_k ds + s^*_k \Psi_k dt)$$

converges to zero in any compact subset $K \subset \mathbb{C}$, because the sequence of connections B_k converges on Σ. Now we see that

$$s^*_k A_k = d + s^*_k \beta_k - \sqrt{-1} s^*_k d^c h_k$$

with $d(s^*_k \beta_k - \sqrt{-1} s^*_k d^c h_k)$ uniformly bounded, $d^*(s^*_k \beta_k - \sqrt{-1} s^*_k d^c h_k) \to 0$. Hence there exists a subsequence of $s^*_k A_k$ (still indexed by k) converging in C^0_{loc}.

Then, on the product $B_{R_k} \times \mathbb{C}^N$, the connection $s^*_k A_k$ and the almost complex structures on $\mathbb{C}^N$ and B_{R_k} induces a sequence of almost complex structures J_k with respect to which $\overrightarrow{\phi}_k$ is holomorphic. J_k converges (in C^0) because $s^*_k A_k$ converges weakly in $W^{1,p}(K)$. The energy density of $\overrightarrow{\phi}_k$ is also uniformly bounded by (4.3) and (4.7). By the standard method, a subsequence of $\overrightarrow{\phi}_k$ converges to a section $\overrightarrow{\phi}_\infty$ which is holomorphic with respect to the connection $d + \alpha_\infty$ on $\mathbb{C}$.

And it is easy to check that the limit pair $\left(d+\alpha_\infty, \overrightarrow{\phi}_\infty\right)$ satisfies the vortex equation on $\mathbb{C}$ with respect to the area form $dsdt$. Finally it is standard to show that the limit is actually smooth and the convergence is in C^∞_{loc}. □

The above lemma is somehow a fact *a priori*, which will be used for many times to guarantee the existence of converging subsequence.

4.5.3 Convergence away from base locus

Proposition 4.5.2. *If the limit of the sequence of points p_k in Lemma 4.5.1 lies in $\Sigma \setminus Z$, then for any sequence $r_k > 0$ satisfying (4.8), the limit affine vortex of any convergent subsequence we obtained in Lemma 4.5.1 is trivial. In particular, the function $\mu_k = \sum_{j=1}^N |\phi_{j,k}|^2 - 1$ converges to zero in $C^0_{loc}(\Sigma \setminus Z)$.*

Proof. In Lemma 4.5.1, denote by $h'_k = s_k^* h_k$, then (4.6) implies

$$\Delta h'_k + \frac{1}{2}\left(e^{-2h'_k}\sum_{j=1}^N |s_k^*\psi_{j,k}|^2 - 1\right) + \frac{\sqrt{-1}}{\lambda_k^2} s_k^*\left(\Lambda F_{B_k}\right) = 0.$$

Because p is not in the base locus, the sequence of functions $\sum_{j=1}^N |s_k^*\psi_{j,k}|^2$ converges to a nonzero constant b uniformly on any compact subset of $\mathbb{C}$. Then the sequence h'_k converges to a solution to the following Kazdan-Warner equation on $\mathbb{C}$

$$\Delta h + \frac{1}{2}\left(be^{-2h} - 1\right) = 0.$$

But there is only one solution h which has asymptotic value $\frac{1}{2}\log b$, which is the constant. This implies that the vortex is trivial.

Then if there exists $p_k \in K$ such that $\lim_{k\to\infty} p_k = p \in K$ and $\lim_{k\to\infty} |\mu_k(p_k)| > 0$, this means the energy density blows up like λ_i^{-2} near p. Then as in [17], the process in Lemma 4.5.1 will produce a nontrivial affine vortex, which contradicts with the above. □

Then away from the base locus, the sequence of sections $\overrightarrow{\phi}_k$ will sink into the level set $\mu^{-1}(0)$. Then we expect that, by projecting to the quotient $\mu^{-1}(0)/S^1$, it will converges to a holomorphic map in $\mathbb{P}^{N-1}$. The precise meaning is described as follows.

For $\overrightarrow{z} \in \mathbb{C}^N$ with $|\mu(\overrightarrow{z})| < \frac{1}{2}$, we have the map

$$\begin{array}{ccc} \mu^{-1}\left(\sqrt{-1}(-\frac{1}{2},\frac{1}{2})\right) & \to & \mu^{-1}(0)\times\sqrt{-1}(-\frac{1}{2},\frac{1}{2}) \\ \overrightarrow{z} & \mapsto & \left(\left(\sum_{j=1}^N |z_j|^2\right)^{-\frac{1}{2}}\overrightarrow{z}, \mu(\overrightarrow{z})\right). \end{array}$$

This induces the map $\pi : \mu^{-1}\left(\sqrt{-1}(-\frac{1}{2},\frac{1}{2})\right) \to \mathbb{P}^{N-1}$ by projecting.

Proposition 4.5.3. *For any compact subset $K \subset \Sigma \setminus Z$, the sequence of maps $\pi \circ \overrightarrow{\phi}_k : K \to \mathbb{P}^{N-1}$ converges to the holomorphic map $\left[\overrightarrow{\psi}\right]|_K : K \to \mathbb{P}^{N-1}$.*

Proof. Indeed, observe that $\left(\pi \circ \overrightarrow{\phi}_k\right)|_K = \left[\overrightarrow{\psi}_k\right]|_K$. And since $\overrightarrow{\psi}_k$ converges to an N-pair which is base point free over K, the convergence is obvious. □

4.5.4 Bubbling at base points

Now for each $p \in Z$, take an admissible chart $(U_p, B_r, \xi_p, \zeta_p)$ near p. Also for each k, take a local holomorphic section e_k such that $e_k(0) = \zeta_p(0)(1) \in L_p$ and assume that $\lim_{k\to\infty} e_k$ converges to a smooth section of L over U_p.

With respect to the admissible chart,

$$B_k = d + \beta_k,\ \beta_k \in \Omega^1(U, i\mathbb{R}),\ B = d + \beta$$

and $\beta_k \to \beta$, $\psi_{j,k} \to \psi_j$ in C^∞-topology.

For each k, and $j = 1, \ldots, N$, denote by $Z_{j,k} = (\phi_{j,k})^{-1}(0) \cap U$. Each element $z_{j,k} \in Z_{j,k}$ has an associated multiplicity $\mathfrak{m}(z_{j,k}) \in \mathbb{Z}^+$. By taking a subsequence if necessary, we assume that there are finite sets $\mathfrak{Z}_j$ and bijections

$$\rho_k : \mathfrak{Z}_j \to Z_{j,k}$$

and maps $\mathfrak{m} : \mathfrak{Z}_j \to \mathbb{Z}_{>0}$ such that $\mathfrak{m}(\rho_k(\mathfrak{z}_j)) = \mathfrak{m}(\mathfrak{z}_j)$. (Note that $Z_{j,k}$ could be empty. This happens only if the section $\phi_{j,k}$ converges to zero on Σ.)

We then consider all sequences $\{p_k\} \subset U$ that converges to p, which are identified with a sequence of complex numbers $\{z_k\}$ converging to the origin. By taking a further subsequence if necessary, we assume that for each j, there is a subset $\mathfrak{W}_j \subset \mathfrak{Z}_j$ (associated to the sequence $\{z_k\}$) such that

$$\limsup_{k\to\infty} \lambda_k |z_k - \rho_k(\mathfrak{w}_j)| < +\infty, \forall \mathfrak{w}_j \in \mathfrak{W}_j;$$

$$\liminf_{k\to\infty} \lambda_k \left|z_k - \rho_k(\mathfrak{z}_j)\right| = +\infty, \forall \mathfrak{z}_j \in \mathfrak{Z}_j \setminus \mathfrak{W}_j.$$

We can write that for $z \in U$,

$$\phi_{j,k}(z) = e_k(z) \prod_{\mathfrak{z}_j \in \mathfrak{Z}_j} \left(z - \rho_k(\mathfrak{z}_j)\right)^{\mathfrak{m}(\mathfrak{z}_j)} f_{j,k}(z)$$

where $f_{j,k}$ is a nonvanishing holomorphic function. Then the assumption that $\psi_{j,k}$ converges to ψ_j implies that

$$f_j := \lim_{\nu\to\infty} f_{j,k}$$

exists.

For each j, denote

$$d_j := \sum_{\mathfrak{w}_j \in \mathfrak{W}_j} \mathfrak{m}(\mathfrak{w}_j) \geq 0.$$

Define

$$t_k := \max\left\{ \lambda_k^{-d_j} \left|f_{j,k}\left(z_k\right)\right| \prod_{\mathfrak{z}_j \in \mathfrak{Z}_j \setminus \mathfrak{W}_j} \left|z_k - \rho_k(\mathfrak{z}_j)\right|^{\mathfrak{m}(\mathfrak{z}_j)} : \mathfrak{W}_j \neq \emptyset \right\}$$

$$T_k := \max\left\{ \left|f_{j,k}\left(z_k\right)\right| \prod_{\mathfrak{z}_j \in \mathfrak{Z}_j} \left|z_k - \rho_k(\mathfrak{z}_j)\right|^{\mathfrak{m}(\mathfrak{z}_j)} : \mathfrak{W}_j = \emptyset \right\}$$

with the convention that $\max \emptyset = 0$.

Lemma 4.5.4. *If* $\liminf_{k\to\infty} \frac{T_k}{t_k} < +\infty$*, then a subsequence of the sequence* $s_k^*\left(A_k; \overrightarrow{\phi}_k\right)$ *converges to a nontrivial affine vortex; if* $\liminf_{k\to\infty} \frac{T_k}{t_k} = +\infty$*, then there is a subsequence* $s_k^*\left(A_k; \overrightarrow{\phi}_k\right)$ *converges to a trivial affine vortex.*

Proof. In the first case, we assume by taking a subsequence that there exists j_0 such that for all k,

$$t_k = \lambda_k^{-d_{j_0}} \left|f_{j_0,k}\left(z_k\right)\right| \prod_{\mathfrak{z}_{j_0} \in \mathfrak{Z}_{j_0} \setminus \mathfrak{W}_{j_0}} \left|z_k - \rho_k(\mathfrak{z}_{j_0})\right|^{\mathfrak{m}(\mathfrak{z}_{j_0})}.$$

And, if $\lim_{k\to\infty}\psi_{j_0,k}=0$, we may assume by taking a subsequence that

$$\lim_{k\to\infty}\frac{f_{j_0,k}}{|f_{j_0,k}(z_k)|}$$

converges to a nonvanishing holomorphic function on U.

Then we have

$$\begin{aligned}
s_k^*\phi_{j_0,k}(w) &= e^{\nu}(z)e^{-h_k\left(z_k+\lambda_k^{-1}w\right)}\prod_{\mathfrak{z}_{j_0}\in\mathfrak{Z}_{j_0}}\left(\frac{w}{\lambda_k}+(z_k-\rho_k(\mathfrak{z}_{j_0}))\right)^{\mathfrak{m}(\mathfrak{z}_{j_0})}f_{j_0,k}(z_k+\lambda_k^{-1}w)\\
&= e_k(z)e^{-h_k\left(z_k+\lambda_k^{-1}w\right)}\prod_{\mathfrak{w}_{j_0}\in\mathfrak{W}_{j_0}}\left(\frac{w}{\lambda_k}+(z_k-\rho_k(\mathfrak{w}_{j_0}))\right)^{\mathfrak{m}(\mathfrak{w}_{j_0})}\prod_{\mathfrak{z}_{j_0}\in\mathfrak{Z}_{j_0}\setminus\mathfrak{W}_{j_0}}\left(\frac{w}{\lambda_k}+(z_k-\rho_k(\mathfrak{z}_{j_0}))\right)^{\mathfrak{m}(\mathfrak{z}_{j_0})}f_{j_0,k}\\
&= e_k(z)e^{-h_k\left(z_k+\lambda_k^{-1}w\right)}t_k\prod_{\mathfrak{w}_{j_0}\in\mathfrak{W}_{j_0}}\left(w+\lambda_k\left(z_k-\rho_k(\mathfrak{w}_{j_0})\right)\right)^{\mathfrak{m}(\mathfrak{w}_{j_0})}\prod_{\mathfrak{z}_{j_0}\in\mathfrak{Z}_{j_0}\setminus\mathfrak{W}_{j_0}}\left(\frac{w}{\lambda_k(z_k-\rho^{\nu}(\mathfrak{z}_{j_0}))}+1\right)^{\mathfrak{m}(\mathfrak{z}_{j_0})}\\
&\qquad\cdot\frac{f_{j_0,k}(z_k+\lambda_k^{-1}w)}{|f_{j_0,k}(z_k)|}\prod_{\mathfrak{z}_{j_0}\in\mathfrak{Z}_{j_0}\setminus\mathfrak{W}_{j_0}}\left(\frac{z_k-\rho_k(\mathfrak{z}_{j_0})}{|z_k-\rho_k(\mathfrak{z}_{j_0})|}\right)^{\mathfrak{m}(\mathfrak{z}_{j_0})}.
\end{aligned}$$

Then by taking a subsequence, the product of the last four factors converges to a nonzero polynomial, uniformly on any compact subset of $\mathbb{C}$, which has zeroes at $\lim_{\nu\to\infty}\lambda_\nu\left(z^\nu-\rho^\nu(\mathfrak{w}_{j_0})\right)$ for each $\mathfrak{w}_{j_0}\in\mathfrak{W}_{j_0}$. Then by the *a priori* convergence of $s_k^*\phi_{j_0,k}$ (Lemma 4.5.1), this implies that the function

$$e^{-H_k(w)}:=e^{-h_k\left(z_k+\lambda_k^{-1}w\right)+\log t_k}$$

converges to a smooth function, denoted by H_0. Then, for all $j\neq j_0$,

$$\begin{aligned}
s_k^*\phi_{j,k}(w) &= e_k(z)e^{-h_k\left(z_k+\lambda_k^{-1}w\right)}\prod_{\mathfrak{z}_j\in\mathfrak{Z}_j}\left(\frac{w}{\lambda_k}+(z_k-\rho_k(\mathfrak{z}_j))\right)^{\mathfrak{m}(\mathfrak{z}_j)}f_{j,k}\\
&= e_k(z)f_{j,k}e^{-H_k(w)}t_k^{-1}\prod_{\mathfrak{w}_j\in\mathfrak{W}_j}\left(\frac{w}{\lambda_k}+(z_k-\rho_k(\mathfrak{w}_j))\right)^{\mathfrak{m}(\mathfrak{w}_j)}\prod_{\mathfrak{z}_j\in\mathfrak{Z}_j\setminus\mathfrak{W}_j}\left(\frac{w}{\lambda_k}+(z_k-\rho_k(\mathfrak{z}_j))\right)^{\mathfrak{m}(\mathfrak{z}_j)}\\
&= e_k(z)e^{-H_k(w)}\prod_{\mathfrak{w}_j\in\mathfrak{W}_j}(w+\lambda_k(z_k-\rho_k(\mathfrak{w}_j)))^{\mathfrak{m}(\mathfrak{w}_j)}\prod_{\mathfrak{z}_j\in\mathfrak{Z}_j\setminus\mathfrak{W}_j}\left(\frac{w}{\lambda_k(z_k-\rho_k(\mathfrak{z}_j))}+1\right)^{\mathfrak{m}(\mathfrak{z}_j)}\frac{f_{j,k}(z_k+\lambda_k^{-1}w)}{|f_{j,k}(z_k)|}\\
&\qquad\cdot\prod_{\mathfrak{z}_j\in\mathfrak{Z}_j\setminus\mathfrak{W}_j}(z_k-\rho_k(\mathfrak{z}_j))^{\mathfrak{m}(\mathfrak{z}_j)}\frac{|f_{j,k}(z_k)|}{\lambda_k^{d_j}t_k}.
\end{aligned}$$

Then taking a subsequence, we can assume that

$$a_j := \lim_{k\to\infty} \prod_{\mathfrak{z}_j \in \mathfrak{Z}_j \setminus \mathfrak{W}_j} (z_k - \rho_k(\mathfrak{z}_j))^{\mathfrak{m}(\mathfrak{z}_j)} \frac{|f_{j,k}(z_k)|}{\lambda_k^{d_j} t_k} \in \mathbb{C}$$

exists. Then taking a further subsequence, we have

$$\lim_{k\to\infty} s_k^* \phi_{j,k}(w) = e_\infty(0) H_0(w) a_j \prod_{\mathfrak{w}_j \in \mathfrak{W}_j} \left(w + \lim_{k\to\infty} \lambda_k (z_k - \rho_k(\mathfrak{w}_j)) \right)^{\mathfrak{m}(\mathfrak{w}_j)}.$$

Since we know that the limit section $\lim_{k\to\infty} s_k^* (\phi_{1,k}, \dots, \phi_{N,k})$ doesn't vanish identically, $H_0 \neq 0$ and hence the sequence of functions $\widetilde{h}_k := s_k^* h_k - \log t_k$ converges to a smooth function $\widetilde{h}$ on $\mathbb{C}$. And the limit vortex is nontrivial because at least for $j = j_0$, it is of the form

$$a_{j_0} e^{-\widetilde{h}(w)} \prod_{s=1}^{d_{j_0}} (w + \xi_s)$$

Now we look at the second case: $\liminf_{k\to\infty} \frac{T_k}{t_k} = +\infty$. Then we may assume that there exists a subsequence and j_0 such that

1. For each j, $\lim_{k\to\infty} \psi_{j,k} = 0 \Longrightarrow \frac{f_{j,k}}{|f_{j,k}(z_k)|}$ converges to a nonvanishing holomorphic function on U;

2. $\mathfrak{W}_{j_0} = \emptyset$ and
$$T_k = |f_{j_0,k}(z_k)| \prod_{\mathfrak{z}_{j_0} \in \mathfrak{Z}_{j_0}} |z_k - \rho_k(\mathfrak{z}_{j_0})|^{\mathfrak{m}(\mathfrak{z}_{j_0})}.$$

Then

$$\begin{aligned} s_k^* \phi_{j_0,k}(w) &= e_k(z) e^{-h_k(z)} f_{j_0,k} \prod_{\mathfrak{z}_{j_0} \in \mathfrak{Z}_{j_0}} \left(\frac{w}{\lambda_k} + (z_k - \rho(\mathfrak{z}_{j_0})) \right)^{\mathfrak{m}(\mathfrak{z}_{j_0})} \\ &= e_k(z) e^{-h_k(z)} T_k \prod_{\mathfrak{z}_{j_0} \in \mathfrak{Z}_{j_0}} \left(\frac{w}{\lambda_k (z_k - \rho_k(\mathfrak{z}_{j_0}))} + 1 \right)^{\mathfrak{m}(\mathfrak{z}_{j_0})} \frac{f_{j_0,k}}{|f_{j_0,k}(z_k)|} \prod_{\mathfrak{z}_{j_0} \in \mathfrak{Z}_{j_0}} \left(\frac{z_k - \rho_k(\mathfrak{z}_{j_0})}{|z_k - \rho_k(\mathfrak{z}_{j_0})|} \right)^{\mathfrak{m}(\mathfrak{z}_{j_0})}. \end{aligned}$$

(By taking a subsequence) the product of the last three factors converges to a nonzero constant, which implies that $e^{-h_k(z_k + \lambda_k^{-1} w)} T_k$ converges to a smooth function $H_0 : \mathbb{C} \to \mathbb{R}$.

Then by our assumption $\lim_{k\to\infty} \frac{T_k}{t_k} = \infty$, we can easily see that $s_k^* \phi_{j_0,k}$ dominates other $s_k^* \phi_{j,k}$. In particular, if $\mathfrak{W}_j \neq \emptyset$, then

$$\lim_{k\to\infty} s_k^* \phi_{j,k} = 0;$$

if $\mathfrak{W}_j = \emptyset$, then

$$\lim_{k\to\infty} s_k^* \phi_{j,k} = a_j H_0$$

for some constant a_j. Since the limit of $s_k^* (\phi_{1,k}, \ldots, \phi_{N,k})$ doesn't vanish identically, we see that the sequence of functions $h_k(z_k + \lambda_k^{-1} w) - \log T_k$ converges to a smooth function $\widetilde{h}$ on $\mathbb{C}$, which must be (by Lemma 4.5.1) the solution to the Kazdan-Warner equation

$$\Delta \widetilde{h} + \frac{1}{2}\left(e^{-2\widetilde{h}} \sum_{\mathfrak{W}_j = \emptyset} |a_j|^2 - 1 \right) = 0.$$

This implies that the limit vortex is trivial. □

From the proof we also see, if we replace z_k be z_k' with $\limsup_{k\to\infty} \lambda_k^{-1} |z_k - z_k'| < \infty$, then the nontrivial affine vortex we get from the first case will differ by a translation.

Remark 4.5.5. If we assume that $\varphi_j = \lim_{k\to\infty} \varphi_{j,k}$ is nonzero for each j, then from the proof of the above lemma, we see that a nontrivial affine vortex bubbles off if each $\phi_{j,k}$ contributes at least one zero so that they concentrate in a rate no slower than $\lambda_k \to \infty$ (the case where some $\varphi_j = 0$ is more subtle). Due to the energy quantization (i.e., the energy of nontrivial affine vortices is bounded from below), we can find all possible nontrivial affine vortex bubbles at the base point p. There might be some missing degrees, caused by the zeroes of $\phi_{j,k}$ which concentrate in a slower rate than λ_k^{-1}. This slower concentration means holomorphic spheres in $\mathbb{P}^{N-1}$ bubble off, and the bubbling happens at the "neck region" between different affine vortices and between affine vortices and the original domain Σ.

4.6 Classification of $U(1)$-vortices and its moduli spaces

We consider an arbitray affine vortex $\left(A, \overrightarrow{\phi}\right)$ with target $\mathbb{C}^N$ with finite energy.

Lemma 4.6.1. *There exists a complex gauge transformation $g = e^{h_1 + i h_2}$ such that $g^* A$ is the trivial connection. If g' is another complex gauge tranformation which also transform A to the trivial connection, then $g' = g e^f$ where f is an entire function.*

Proof. Suppose $A = d + \Phi ds + \Psi dt$. We first show that there exists a unitary gauge transformation

$g_1 = e^{if}$ such that

$$d^* \left(g_1^* A - d\right) = 0, \tag{4.9}$$

i.e., the Coulomb gauge condition. Indeed,

$$d^* \left(g_1^* A - d\right) = d^* \left(\left(\Phi - i\frac{\partial f}{\partial x}\right) dx + \left(\Psi - i\frac{\partial f}{\partial y}\right) dy\right) = i\Delta f - \frac{\partial \Phi}{\partial x} - \frac{\partial \Psi}{\partial y}.$$

Hence there exists f such that (4.9) holds. Then for $g_2 = e^h$ with h a real valued function,

$$g_2^* g_1^* A - d = -i\left(\frac{\partial h}{\partial x} dy - \frac{\partial h}{\partial y} dx\right) + \left(g_1^* A - d\right) = -i * dh + \left(g_1^* A - d\right) = -i * \left(dh - i * \left(g_1^* A - d\right)\right).$$

The existence of h such that $g_2^* g_1^* A = d$ follows from Poincaré lemma. □

Then, up to a unitary gauge transformation, we can assume that $\left(A, \overrightarrow{\phi}\right)$ is in Coulomb gauge, and $A = d - \partial h + \overline{\partial} h$ for some real valued function h and $\overline{\partial}_A = \overline{\partial} + \overline{\partial} h = e^{-h} \overline{\partial} e^h$. So $\overline{\partial}_A \phi_j = 0 \Leftrightarrow \overline{\partial}(e^h \phi_j) = 0$. The vortex equation is equivalent the following equation on real valued function $h : \mathbb{C} \to \mathbb{R}$

$$\Delta h + \frac{1}{2}\left(e^{-2h} \sum_{j=1}^{N} \left|\psi_j(z)\right|^2 - 1\right) = 0 \tag{4.10}$$

where ψ_j are entire functions.

Lemma 4.6.2. *If $E\left(A, \overrightarrow{\phi}\right) < \infty$, then $\psi_j = e^h \phi_j$ are polynomials and the Maslov index of $\left(A, \overrightarrow{\phi}\right)$ is equal to the maximum of the degrees of ψ_j.*

Proof. By Proposition 4.2.7 there exists $g : S^1 \to U(1)$ such that for each j and all $\theta \in S^1$,

$$\lim_{r \to +\infty} g(e^{-i\theta}) e^{-h(re^{i\theta})} \psi_j(re^{i\theta}) \to a_j \tag{4.11}$$

and $\sum_{j=1}^{N} |a_j|^2 = 1$. The Maslov index of the vortex is by definition the degree of g.

Then for each j with $a_j \neq 0$, (4.11) implies that ∞ is not an essential singularity of ψ_j. Hence ψ_j is a polynomial. Then it is easy to see that $a_j \neq 0 \Longrightarrow \deg \psi_j = d$. Moreover, if $a_l = 0$ and $a_j \neq 0$, then

$$\lim_{r \to \infty} \frac{\psi_l(re^{i\theta})}{\psi_j(re^{i\theta})} = \frac{a_l}{a_j} = 0$$

which implies that ψ_l is a polynomial with degree strictly less than d. □

Before we proceed, we consider the uniqueness of the solutions to the Kazdan-Warner equation (4.10). For given N polynomials $\{\psi_j\}_{1\leq j\leq N}$, suppose $\sum_{j=1}^{N}|\psi_j(z)|^2$ are asymptotic to $c^2|z|^{2d}$ as $|z|\to\infty$ for some $c>0$.

Lemma 4.6.3. *If $h_1, h_2 : \mathbb{C}\to\mathbb{R}$ are both smooth and solve equation (4.10) on $\mathbb{C}$ such that*

$$\lim_{|z|\to\infty}\left(e^{-2h_i(z)}\sum_{j=1}^{N}|\psi_j(z)|^2-1\right)|z|^{4-\epsilon}=0,\ i=1,2,$$

then $h_1=h_2$.

Proof. We have

$$\Delta(h_1-h_2)+\left(e^{-2h_1}-e^{-2h_2}\right)\sum_{j=1}^{N}|\psi_j|^2=0.$$

Then the L^2-pairing with h_1-h_2 gives

$$0=|d(h_1-h_2)|^2_{L^2}+\int_{\mathbb{C}}(h_1-h_2)\left(e^{-2h_2}-e^{-2h_1}\right)\sum_{j=1}^{N}|\psi_j|^2\geq 0.$$

So this is true only if $\psi_j\equiv 0$ or $h_1=h_2$. □

4.6.1 Construction of arbitrary vortices

We have seen that all finite energy affine vortices are obtained from the solution to (4.10) for ψ_j polynomials, and the solution, if exists, is unique. Now with our understanding of the adiabatic limit of equation (1.14) and the bubbling off phenomenon at the base locus (in the proof of Lemma 4.5.4), it is rather easy to construct solutions to (4.10) with any given set of polynomials.

Indeed, suppose we are given N polynomials $\psi_1,\ldots,\psi_N$, with $\psi_i=a_i(z-z_i^1)\cdots(z-z_i^{d_i})$. It is trivial to add zero polynomials hence we assume that $a_i\neq 0$. Let $d:=\max_{1\leq i\leq N}d_i$.

Now, consider the degree d line bundle $\mathcal{O}(d)\to\mathbb{P}^1$. Take the sequence $\lambda_k=k$. Consider the sequence of sections of $\mathcal{O}(d)$, where are sequences of degree d polynomials

$$\varphi_i^{(k)}(z)=k^{d_i-d}a_i(1-z)^{d-d_i}\prod_{j=1}^{d_i}\left(z-\frac{z_i^j}{k}\right).$$

Then for each k, $\left(\mathcal{O}(d); \varphi_1^{(k)}, \ldots, \varphi_N^{(k)}\right)$ is a rank 1 stable holomorphic N-pair over $\mathbb{P}^1$. By the Hitchin-Kobayashi correspondence, there exists a metric $H^{(k)}$ on $\mathcal{O}(d)$ which solves the vortex equation. Now the origin is a base point of the limit N-pairs $(\mathcal{O}(d); \phi_1, \ldots, \phi_N)$ with $\phi_i(z) = a_i z^d$ for $d_i = d$ and $\phi_i(z) = 0$ for $d_i < d$. As in the proof of Lemma 4.5.4, we take $p = p_k = 0$ and $r_k := \lambda_k^{-\frac{1}{2}} = k^{-\frac{1}{2}}$. It is easy to see that the sequence satisfies the criterion $\lim_{k\to\infty} \frac{T_k}{t_k} < \infty$ of Lemma 4.5.4. Hence by zooming in with a factor $\lambda_k = k$, we see a nontrivial affine vortex of Maslov index d bubbles off. Equivalently, we constructed the unique solution to the Kazdan-Warner equation (4.10) with the given polynomials ψ_j on $\mathbb{C}$.

4.6.2 Identify the moduli space

Denote

$$\widetilde{N}_d := \left\{ (a_{jl})_{1\le j\le N,\ 0\le l\le d} \in \mathbb{C}^{(d+1)N} \mid (a_{1d}, \ldots, a_{Nd}) \neq (0, \ldots, 0) \right\}.$$

This corresponds to the space of N polynomials as the input of (4.10). $\mathbb{C}^*$ acts on $\widetilde{N}_d$ freely and denote $N_d := \widetilde{N}_d/\mathbb{C}^*$. We will use $\mathbf{x}$ to denote a general point in N_d.

Corollary 4.6.4. *There is an orientation preserving homeomorphism*

$$\Phi_{HK}^d : N_d \to \mathcal{M}_{1,1}^{\mathbb{A}}\left(\mathbb{C}^N, d\right).$$

And for each $(A, u) \in \widetilde{\mathcal{M}}_{1,1}^{\mathbb{A}}\left(\mathbb{C}^N, d\right)$, the linearization $D_{A,u}$ defined in (4.1) is surjective.

Proof. We define $\widetilde{\Phi}_{HK}^d : N_d \to \widetilde{\mathcal{M}}_{0,1}^{\mathbb{A}}\left(\mathbb{C}^N, d\right)$ to be the map which assigns to $\mathbf{x} = [a_{jl}]_{1\le j\le N, 0\le l\le d}$ the affine vortex

$$\left(A = d - \partial h + \overline{\partial} h, u = e^{-h}\left(\psi_1, \ldots, \psi_N\right)\right)$$

where $\psi_j(z) = \sum_{l=0}^d a_{jl} z^l$ and h is the solution to (4.10) with the input $\psi_1, \ldots, \psi_N$. Then define $\Phi_{HK}^d(\mathbf{x})$ to be the equivalence class of $(1,1)$-marked affine vortex $\left(\widetilde{\Phi}_{HK}^d(\mathbf{x}), 0\right)$, where $0 \in \mathbb{C}$ is the interior marked point.

To show show that map is continuous, we need to show that the solution to the Kazdan-Warner equation depends continuously on the polynomials $\psi_1, \ldots, \psi_N$, which can be proved in a standard way. The orientation-preserving property then follows from a C^1-dependence on $\psi_1, \ldots, \psi_N$.

Now we show the surjectivity of the linearization map. First notice that the linearization of gauge tranformation is injective, because the image of u is not in the fixed point set of the S^1-action

on $\mathbb{C}^N$. Hence it suffices to prove that the map

$$D_{A,u}(V,\alpha) = \begin{pmatrix} \overline{\partial}_A V + \mathcal{X}_\alpha^{0,1}(u) \\ d\alpha + d\mu(V)dsdt \end{pmatrix}$$

is surjective, where $(V,\alpha) \in W_{p,\delta}$ which is defined in Subsection 4.2.1. Note that the space $W_\delta^{1,p}(\mathbb{C}^N) := \left\{ V \in W_{loc}^{1,p}(\mathbb{C}^N) \mid |V|_{L_\delta^p} + \left|\nabla^A V\right|_{L_\delta^p} < \infty \right\}$ and $W_\delta^{1,p}(\Lambda^{1,0}\mathbb{C})$ are contained in $W_{p,\delta}$ by obvious inclusions. Hence it suffices to show that the two maps

$$\begin{array}{rccc} \overline{\partial}_A : & W_\delta^{1,p}(\mathbb{C}^N) & \to & L_p^\delta(\Lambda^{0,1}\mathbb{C}\otimes\mathbb{C}^N) \\ & V & \mapsto & \overline{\partial}_A V \end{array} \tag{4.12}$$

$$\begin{array}{rccc} \overline{\partial} : & W_\delta^{1,p}(\Lambda^{1,0}\mathbb{C}) & \to & L_\delta^p(\Lambda^2\mathbb{C}) \\ & \alpha^{1,0} & \mapsto & \overline{\partial}\alpha^{1,0} \end{array} \tag{4.13}$$

both have dense range (they are not Fredholm).

To prove that (4.12) has dense range, it suffices to prove for $N = 1$. Then the operator is

$$\overline{\partial}_A f = \left(e^{-h}\overline{\partial}(e^h f)\right).$$

If its range is not dense, then there exists $g \in (L_\delta^p)^* = L_{-\delta}^q$ such that

$$\int_{\mathbb{C}} g dz \wedge \overline{\partial}_A f = 0, \ \forall f \in C_c^\infty.$$

This implies that $e^{-h}g$ is an entire function. But e^h grows like $|z|^d$ for $d \geq 0$. Hence $g = 0$ or g grows at least like $|z|^d$. For such a g in the latter case to lie in $L_{-\delta}^q$, we should have

$$-2 > q(d-\delta) \Longrightarrow \delta > d + 2 - \frac{2}{p} \geq 2 - \frac{2}{p}$$

which contradicts with our choice of p and δ. The operator (4.13) is equivalent to the map $f \mapsto \frac{\partial f}{\partial \overline{z}}$ so it is essentially the same as the case of (4.12) to deduce that it has dense range. □

Remark 4.6.5. In particular, the transversality of the linearization $D_{A,u}$ implies that the moduli space N_d is a correct one to define the quantum Kirwan map. The remaining is to show that the evaluation maps indeed give a pseudo-cycle. This can be seen by giving suitable compactifications

of the moduli space on which the evaluation maps extend continuously. In the last two sections, we will first give the stable map compactification of the lowest nontrivial moduli, and then construct a compactification (which we call Uhlenbeck compactification) for arbitrary degrees which allows us to extend the evaluation maps.

4.6.3 Stable map compactification of $\mathcal{M}^{\mathbb{A}}_{1,1}\left(\mathbb{C}^N,1\right)$

In this subsection we identify the compactification $\overline{\mathcal{M}}^{\mathbb{A}}_{1,1}\left(\mathbb{C}^N,1\right)$ defined in Subsection 4.3.2, which is the smallest nontrivial case. Depending on the type of degeneration, $\overline{\mathcal{M}}^{\mathbb{A}}_{1,1}\left(\mathbb{C}^N,1\right)$ is stratified as described in Figure 4.2: the open stratum is $\mathcal{M}^{\mathbb{A}}_{1,1}\left(\mathbb{C}^N,1\right)$; two lower strata of depth 1 are: 1) the stratum $\mathcal{T}_1$ corresponding to the moduli of degree 1 holomorphic spheres in $\mathbb{P}^{N-1}$ with one marked point and a ghost vortex attached to the sphere, which is the same as $\mathcal{M}_2\left(\mathbb{P}^{N-1},1\right)$ and 2) the stratum $\mathcal{T}_2$ corresponding to the moduli of $(0,1)$-marked degree 1 affine vortices with two ghost components attached at ∞; the lowest stratum $\mathcal{S}$ is the moduli of holomorphic spheres with two ghost components attached at ∞. Note that, points in each stratum only depend on the isomorphism class of their nontrivial components and the other ghost components are attached to the nontrivial components in a unique way.

We need to find the correct compactification of N_1. Note that $N_1=\mathcal{O}(1)^{\oplus N}$ where $\mathcal{O}(1)\to\mathbb{P}^{N-1}$ is the degree 1 line bundle. For $\mathbf{a}:=(a_1,\ldots,a_N)\neq 0$, we use the homogeneous coordinates $[a_1,\ldots,a_N]$ to denote a point in the base $\mathbb{P}^{N-1}$, $\mathbf{a}$ to denote a basis vector of the fibre of $\mathcal{O}(-1)$ over the point $[a_1,\ldots,a_N]$, and $\mathbf{a}^*$ the dual basis of the fibre of $\mathcal{O}(1)$ over $[a_1,\ldots,a_N]$. Then a point in N_1 will be denoted by $(b_1,\ldots,b_N)\mathbf{a}^*$.

There is a natural compactification of $\mathcal{O}(1)^{\oplus N}$ by the projective bundle

$$\mathbb{P}(\mathcal{E}):=\mathbb{P}\left(\mathcal{O}(1)^{\oplus N}\oplus\mathbb{C}\right)$$

where $\mathcal{O}(1)^{\oplus N}$ embeds into $\mathbb{P}(\mathcal{E})$ as

$$(b_1\mathbf{a}^*,\ldots,b_N\mathbf{a}^*)\mapsto[b_1\mathbf{a}^*,\ldots,b_N\mathbf{a}^*,1]\,.$$

We see that $\mathbb{P}(\mathcal{E})\setminus N_1=\mathbb{P}^{N-1}\times\mathbb{P}^{N-1}$. Remember that, the moduli space of genus zero, degree 1, 2-marked stable map to $\mathbb{P}^{N-1}$ is isomorphic to the blow up $Bl_\Delta\left(\mathbb{P}^{N-1}\times\mathbb{P}^{N-1}\right)$ along the diagonal

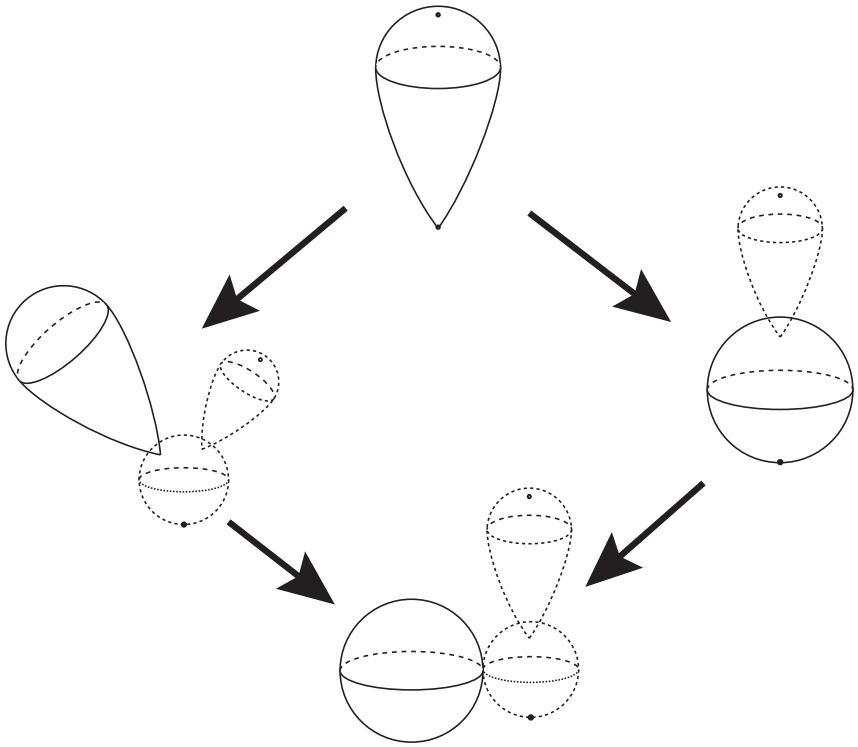

Figure 4.2: The stratification of $\overline{\mathcal{M}}^{\mathbb{A}}_{1,1}\left(\mathbb{C}^N, 1\right)$.

$\Delta \subset \mathbb{P}^{N-1} \times \mathbb{P}^{N-1}$. Regard $\Delta \subset \mathbb{P}(\mathcal{E})$ as a codimension N subvariety. We denote by

$$\overline{N}_1 := Bl_{\Delta}\mathbb{P}(\mathcal{E})$$

the blown-up. Denote the exceptional divisor by $E \simeq \mathbb{P}(N_{\Delta} \oplus \mathbb{C})$, where N_{Δ} is the normal bundle of Δ in $\mathbb{P}^{N-1} \times \mathbb{P}^{N-1}$. Now the space $\overline{N}_1$ can be stratified: the top stratum is just N_1; there are two strata of depth one, which are $T_1 := \mathbb{P}^{N-1} \times \mathbb{P}^{N-1} \setminus \Delta$ and $T_2 := E \setminus \mathbb{P}(N_{\Delta})$; there is a single lowest stratum of depth two, which is $S := \mathbb{P}(N_{\Delta})$. For each point $\mathbf{x} = ([a_1, \ldots, a_N], [a_1, \ldots, a_N]) \in \Delta$, the fibre of T_2 over $\mathbf{x}$ has the coordinates $[(v_1\mathbf{a}^*, \ldots, v_N\mathbf{a}^*), (-v_1\mathbf{a}^*, \ldots, -v_N\mathbf{a}^*), 1]$, where $(v_1, \ldots, v_N) \perp (a_1, \ldots, a_N)$ in $\mathbb{C}^N$ and hence

$$(v_1\mathbf{a}^*, \ldots, v_N\mathbf{a}^*) \in \mathrm{Hom}\left(\mathcal{O}(1)|_{[\mathbf{a}]}, \mathbf{a}^{\perp}\right) = T_{[\mathbf{a}]}\mathbb{P}^{N-1}.$$

Now we prove that the map Φ^1_{HK} defined in Corollary 4.6.4 extends to

$$\overline{\Phi}^1_{HK} : \overline{N}_1 \to \overline{\mathcal{M}}^{\mathbb{A}}_{1,1}\left(\mathbb{C}^N, 1\right)$$

which is a homeomorphism with respect to the Gromov convergence defined in Subsection 4.3.3 and respects the stratifications of the domain and the target.

We first define the extension as follows.

1. $\overline{\Phi}^1_{HK} : T_1 \to \mathcal{T}_1$ assigns to $[a_1, \ldots, a_N, b_1, \ldots, b_N]$ the $(1,1)$-marked stable affine vortex in $\mathcal{T}_1$ whose nontrivial component is the equivalent to the holomorphic sphere $z \mapsto [a_1 z + b_1, \ldots, a_N z + b_N]$.

2. $\overline{\Phi}^1_{HK}$ restricted to $T_2 = E \setminus \mathbb{P}(N_{\Delta})$ assigns each $[(v_1, \ldots, v_N)\mathbf{a}^*, -(v_1, \ldots, v_N)\mathbf{a}^*, 1]$ to the equivalence class of $(1,1)$-marked stable affine vortices in $\mathcal{T}_2$ whose nontrivial component is equivalent to the vortex $\widetilde{\Phi}^1_{HK}([a_1, \ldots, a_N, -v_1, \ldots, -v_N])$.

Now we have to prove

Lemma 4.6.6. *$\overline{\Phi}^1_{HK}$ is continuous.*

Proof. It is continuous on each stratum. Hence we take a sequence $\mathbf{x}_i \in N_1$ which converges to a point $\mathbf{x}_{\infty} \in \overline{N}_1 \setminus N_1$. We may write $\mathbf{x}_i = [(b_{1,i}, \ldots, b_{N,i})\,\mathbf{a}_i^*, w_i] \in \mathbb{P}\left(\mathcal{O}(1)^{\oplus N} \oplus \mathbb{C}\right)$ with $\mathbf{a}_i$, $\mathbf{b}_i$ unit vectors, $w_i \in \mathbb{C}^*$, and $\lim_{i\to\infty} \mathbf{a}_i = \mathbf{a}, \lim_{i\to\infty} \mathbf{b}_i = \mathbf{b}$, $\lim_{i\to\infty} w_i = 0$. Let $d_i := d_{\mathbb{P}^{N-1}}([\mathbf{a}_i], [\mathbf{b}_i])$.

We first observe that

1. $\mathbf{x}_\infty \in T_1 \Longrightarrow [\mathbf{a}] \neq [\mathbf{b}]$ in $\mathbb{P}^{N-1}$;

2. $\mathbf{x}_\infty \in T_2 \Longrightarrow [\mathbf{a}] = [\mathbf{b}]$ and $\lim_{i\to\infty} |w_i|^{-1} d_i < \infty$;

3. $\mathbf{x}_\infty \in S \Longrightarrow [\mathbf{a}] = [\mathbf{b}]$ and $\lim_{i\to\infty} |w_i|^{-1} d_i = \infty$.

1. In the first and the third case, we have

$$\lim_{i\to\infty} \inf_{z\in\mathbb{C}} \left|\mathbf{a}_i z + w_i^{-1}\mathbf{b}_i\right|^2 \geq \lim_{i\to\infty} |w_i|^{-2} d_i^2 = +\infty.$$

Then look at the Kazdan-Warner equation

$$\Delta_0 h_i + \frac{1}{2}\left(e^{-2h_i}\sum_{j=1}^{N}\left|a_{j,i}z + \epsilon_i^{-1}b_{j,i}\right|^2 - 1\right) = 0.$$

We denote

$$h_i'(z) := h_i(z) - \frac{1}{2}\log\left(\left|\overrightarrow{a}_i z + \epsilon_i^{-1}\overrightarrow{b}_i\right|^2\right).$$

Then

$$\Delta_0 h_i' + \frac{1}{2}(e^{-2h_i'} - 1) = -\frac{1}{2}\Delta_0 \log\left(\left|\overrightarrow{a}_i z + \epsilon_i^{-1}\overrightarrow{b}_i\right|^2\right).$$

Note that the right hand side converges to zero uniformly over $\mathbb{C}$, which implies that h_i' converges to zero.

2. In the first case, take the Möbius transformation

$$\begin{array}{rccc} g_i: & S^2 & \to & S^2 \\ & w & \mapsto & z = w_i^{-1}w. \end{array}$$

We see the sequence of maps

$$\pi_\nu\left(g_i^*(e^{-h_i}\phi_{j,i})\right)(w) = e^{-h_i(\epsilon_i^{-1}w)} w_i^{-1}(a_{j,i}w + b_{j,i}).$$

converges on $\mathbb{C} \subset S^2$ to

$$w \mapsto [a_1 w + b_1, \ldots, a_N w + b_N] \in \mathbb{P}^{N-1}$$

which extends to a nontrivial holomorphic map from $\mathbb{P}^1$ to $\mathbb{P}^{N-1}$ of degree 1. By Definition 4.3.8, this means that in the first case, the sequence $\Phi^1_{HK}(\mathbf{x}_i) \in \mathcal{M}^{\mathbb{A}}_{1,1}\left(\mathbb{C}^N, 1\right)$ converges to $\overline{\Phi}^1_{HK}(\mathbf{x}_\infty)$.

3. In the third case, for large i we write $\mathbf{b}_i = c_i\mathbf{a}_i + \mathbf{y}_i$ with $c_i \in \mathbb{C}$ and $\mathbf{y}_i \perp \mathbf{a}_i$. Then by the condition $\mathbf{x}_\infty \in S$, we have $\lim_{i\to\infty} d_{\mathbb{P}^{N-1}}([\mathbf{a}_i], [\mathbf{b}_i])^{-1}\mathbf{y}_i$ exists. Then take the sequence of Möbius transformations

$$z = g_i(w) = \frac{d_i}{w_i}w - c_i w_i^{-1}.$$

We see

$$\lim_{i\to\infty}[g_i^*\phi_{1,i}, \ldots, g_i^*\phi_{N,i}] = \lim_{i\to\infty}[a_{1,i}d_i w + y_{1,i}, \ldots, a_{N,i}d_i w + y_{N,i}] = [a_1 w + y_1, \ldots, a_N w + y_N]$$

which is a degree one holomorphic sphere in $\mathbb{P}^{N-1}$. Adding proper ghost components, we see that this means the sequence $\Phi^1_{HK}(\mathbf{x}_i)$ converges to $\overline{\Phi}^1_{HK}(\mathbf{x}_\infty)$.

4. In the second case, write $\mathbf{b}_i = c_i\mathbf{a}_i + \mathbf{y}_i$ with $c_i \in \mathbb{C}$, $\mathbf{y}_i \perp \mathbf{a}_i$. Since $\mathbf{x}_i = [(b_{1,i}, \ldots, b_{N,i})\mathbf{a}_i^*, w_i] \in \mathbb{P}\left(\mathcal{O}(1)^{\oplus N} \oplus \mathbb{C}\right)$ converge to $\mathbf{x}_\infty \in \mathbb{P}\left(N_\Delta \oplus \mathbb{C}\right) \setminus \mathbb{P}\left(N_\Delta\right)$, the limit

$$\mathbf{v} := \lim_{i\to\infty} w_i^{-1}\mathbf{y}_i \in \mathbf{a}^\perp \subset \mathbb{C}^N$$

exists. Then consider the sequence of translations

$$z = t_i(w) = w - c_i w_i^{-1}.$$

Then we see the sequence of polynomials

$$t_i^*\phi_{j,i} = a_{j,i}(w - c_i w_i^{-1}) + w_i^{-1}b_{j,i} = a_{j,i}w - w_i^{-1}y_{j,i}$$

converge to $a_j w - v_j$. By the continuous dependence of the solution to the Kazdan-Warner equation on the given N polynomials, we see that $t_i^*\widetilde{\Phi}^1_{HK}(\mathbf{x}_i)$ converge uniformly on any compact subset to the affine vortex $\widetilde{\Phi}^1_{HK}([a_1, \ldots, a_N, v_1, \ldots, v_N])$. Hence $\Phi^1_{HK}(\mathbf{x}_i)$ converge to $\overline{\Phi}^1_{HK}(\mathbf{x}_\infty)$.

□

4.6.4 The Uhlenbeck compactification of $\mathcal{M}_{1,1}^{\mathbb{A}}\left(\mathbb{C}^N, d\right)$ and the quantum Kirwan map

We define the Uhlenbeck compactification to be a quotient space of the stable map compactification, by only remembering the sum of the degrees of the components of the stable map which doesn't contain the marked point 0.

Proposition 4.6.7. *The Uhlenbeck compactification* $\overline{\mathcal{M}}_{1,1}^{\mathbb{A},U}\left(\mathbb{C}^N, d\right)$ *is homeomorphic to* $\mathbb{P}^{N(d+1)-1}$.

Proof. We see that we have a filtration $\mathbb{P}^{N(d+1)-1} \supset \mathbb{P}^{Nd-1} \supset \cdots \supset \mathbb{P}^{N-1}$ where the inclusion $\mathbb{P}^{Nk-1} \to \mathbb{P}^{N(k+1)-1}$ is given by

$$[a_{Nk}, a_{Nk-1}, \ldots, a_1] \mapsto [0, \ldots, 0, a_{Nk}, a_{Nk-1}, \ldots, a_1].$$

So $N_d = \mathbb{P}^{N(d+1)-1} \setminus \mathbb{P}^{Nd-1}$ and $\mathbb{P}^{N(d+1)-1} = \cup_{0\leq k\leq d} N_k$. We define the extension of Φ_{HK}^d

$$\overline{\Phi}_{HK}^{d,U} : \mathbb{P}^{N(d+1)-1} \to \overline{\mathcal{M}}_{1,1}^{\mathbb{A},U}\left(\mathbb{C}^N, d\right)$$

to be the map such that for $\mathbf{x} \in N_k \subset \mathbb{P}^{Nd-1}$, $\overline{\Phi}_{HK}^{d,U}(\mathbf{x})$ is the equivalence class of stable affine vortices whose primary component is equivalent to $\widetilde{\Phi}_{HK}^{k}(\mathbf{x})$. It remains to show that this map is continuous with respect to the degeneration of affine vortices.

Indeed, suppose $\mathbf{x}_i \in N_d$ and $\lim_{i\to\infty} \mathbf{x}_i = \mathbf{x}_\infty \in N_k$ for $k < d$. Represent $\mathbf{x}_\infty$ by N polynomials $\{\psi_{j,\infty}\}_{1\leq j\leq N}$ with maximal degree k. Without loss of generality, we can assume that $\mathbf{x}_i$ can be represented by N polynomials $\left\{\widetilde{\psi}_{j,i}\right\}_{1\leq j\leq N}$ such that $\deg\widetilde{\psi}_{j,i}$ is independent of i and $d_j := \deg\widetilde{\psi}_{j,i} \geq \deg\psi_{j,\infty} =: d_{j,\infty}$. Then $d_j > d_{j,\infty}$ implies that $d_j - d_{j,\infty}$ zeroes of $\widetilde{\psi}_{j,i}$ diverge to infinity. Hence we can write

$$\widetilde{\psi}_{j,i}(z) = \psi_{j,i}(z) \prod_{s=1}^{d_j - d_{j,\infty}} \left(1 - \frac{z}{w_{j,i,s}}\right)$$

with $\lim_{i\to\infty} \psi_{j,i} = \psi_{j,\infty}$, $\lim_{i\to\infty} |w_{j,i,s}| = \infty$. Then for each i there exists functions h_i solving the Kazdan-Warner equation

$$\Delta h_i + \frac{1}{2}\left(e^{-2h_i} \sum_{j=1}^{N} \left|\widetilde{\psi}_{j,i}\right|^2 - 1\right) = 0.$$

By the compactness theorem of Ziltener (Theorem 4.3.9), a subsequence of

$$\left(d - \partial h_i + \overline{\partial} h_i, e^{-h_i}\left(\widetilde{\psi}_{1,i}, \ldots, \widetilde{\psi}_{N,i}\right)\right)$$

converges to a $(1,1)$-marked stable affine vortex $\widetilde{\mathbf{W}}^{\infty}$; in particular, a subsequence converges uniformly on any compact subset of $\mathbb{C}$ to the primary component of $\widetilde{\mathbf{W}}^{\infty}$. This implies that for each j, $\left\{e^{-h_i}\widetilde{\psi}_{j,i}\right\}_i$ has a convergent subsequence. Since $\lim_{i\to\infty}\widetilde{\psi}_{j,i} = \psi_{j,\infty}$, this implies that h_i converges on $\mathbb{C}$ to a smooth function h_∞, which solves the equation

$$\Delta h_\infty + \frac{1}{2}\left(e^{-2h_\infty}\sum_{j=1}^{N}|\psi_{j,\infty}|^2 - 1\right) = 0.$$

This implies that the primary component of $\widetilde{\mathbf{W}}^{\infty}$ is equivalent to $\left(\widetilde{\Phi}^k_{HK}(\mathbf{x}_\infty), 0\right)$. Hence we have proved that any subsequence of $\Phi^d_{HK}(\mathbf{x}_i)$ has a subsequence converging to $\overline{\Phi}^{d,U}_{HK}(\mathbf{x}_\infty)$. This implies the continuity of $\overline{\Phi}^{d,U}_{HK}$. □

Proposition 4.6.8. *The line bundle associated to the Poincaré bundle $\overline{\mathcal{P}}^U_0 \to \overline{\mathcal{M}}^U_{1,1}\left(\mathbb{C}^N, d\right)$ is isomorphic to $\mathcal{O}(1) \to \mathbb{P}^{Nd-1}$.*

Proof. It suffices to check on each N_k. □

Now the evaluation ev_∞ doesn't extends to $\overline{\mathcal{M}}^U_{1,1}\left(\mathbb{C}^N, d\right)$, but it doesn't affect our computation of the Kirwan map. Indeed we can blow up $\mathbb{P}^{N(d+1)-1}$ along $\mathbb{P}^{Nd-1} = \cup_{0\le k\le d} N_k$ on which ev_∞ extends continuously. The blown-up is denoted by $\overline{N}^*_d$.

Now we compute the quantum Kirwan map, the result of which is of no surprise. $H^*_{U(1)}\left(\mathbb{C}^N\right)$ is generated by the universal first Chern class u of degree 2. For any $m \ge 0$, write $m = d_m N + r$ with $0 \le r \le N-1$. Denote by $c \in H^2\left(\mathbb{P}^{N-1}\right)$ the generator. Then by the definition (4.4)

$$\kappa_Q(u^m) = \sum_{d\ge 0}\sum_{0\le i\le N-1}\left\langle (ev_0)^*(u^m)\cup(ev_\infty)^*(c^i), \left[\overline{N}^*_d\right]\right\rangle \cdot c^{N-1-i}\otimes q^d$$
$$= \left\langle (ev_0)^*(u^m)\cup(ev_\infty)^*(c^{N-1-r}), \left[\overline{N}^*_{d_m}\right]\right\rangle \cdot c^r \otimes q^{d_m} = c^r\otimes q^{d_m} = \kappa_Q(u) *_q \kappa_Q(u) *_q \cdots *_q \kappa_Q(u).$$

Hence $\kappa_Q : H^*_{U(1)}(\mathbb{C}^N) \to QH^*(\mathbb{P}^{N-1}, \Lambda)$ is a ring homomorphism. It extends to a homomorphism

$$\kappa^\Lambda_Q : H^*_{U(1)}(\mathbb{C}^N, \Lambda) \to QH^*(\mathbb{P}^{N-1}, \Lambda)$$

linearly over Λ, with kernel generated by $q - u^N$.

Chapter 5

Vortex eqution and Landau-Ginzburg/Calabi-Yau correspondence

5.1 Introduction to this chapter

The Landau-Ginzburg/Calabi-Yau correspondence in string theory ([22], [33], [49]) relates the non-linear σ-model of Calabi-Yau hypersurfaces in quasi-projective spaces and the Landau-Ginzburg model whose superpotential is the defining polynomial for the hypersurface. Based on Witten's observation ([55]) the two theories should be different "phases" of the same theory.

For simplicity of this introduction, assume that W is a quintic homogeneous polynomials in five variables $x_1, \ldots, x_5$. Assuming W is nondegenerate, which means $dW(\mathbf{x}) = 0 \Longrightarrow \mathbf{x} = 0$, it defines a Calabi-Yau hypersurface Z_W in $\mathbb{P}^4$. The 2-dimensional nonlinear σ-model of Z_W has its correlation functions coincide with the counting of holomorphic curves inside Z_W. On the other hand, Fan-Jarvis-Ruan ([10], [11], [9]) recently defines a cohomological field theory for such W, based on constructing the moduli space of solutions to Witten's equation (see [54])

$$\overline{\partial} u_i + \overline{\frac{\partial W}{\partial u_i}} = 0, \ i = 1, \ldots, 5 \tag{5.1}$$

which describes certain quantum singularity theory associated to W (called the FJRW-theory). It has been long conjectured that there should be some "correspondence" between these two theory

and one should be able to state it and prove it in mathematics.

Witten's idea to relate these two theories is to consider a $U(1)$-gauged version of the equation (5.1) for the superpotential pW which has one additional complex variable p. Then the so-called Fayet-Iliopoulos D-term, which is a real number r, parametrizes a homotopy which connects the two theories as limits $r \to +\infty$ and $r \to -\infty$ respectively. Mathematically, it contains a birational transformation between two different symplectic quotients of a Hamiltonian $U(1)$-action, which are $\mathbb{C}^5/\mathbb{Z}_5$ and $\mathcal{O}(-5) \to \mathbb{P}^4$ respectively.

Though straightforward to follow Witten's idea, it is very complicated to formulate this approach mathematically. For example, to state (5.1) rigorously on a Riemann surface, Fan-Jarvis-Ruan introduces the notion of W-structures and W-curves, whose moduli space is a branched cover over the Deligne-Mumford space. And for the LG/CY correspondence, gauge theory comes into play and one needs to written down carefully how these complicated structures can be unified. Such a framework is one main concern of here.

Moreover, using this framework we hope that one can prove the LG/CY correspondence for hypersurfaces in the way proposed by Witten. The use of the vortex equation and its adiabatic limits should be the main idea of carrying out Witten's heuristic argument.

In this chapter, we first introduce a framework of studying a gauged σ-model with a superpotential W, which combines a gauged version of (5.1) and the symplectic vortex equation, for possibly nonabelian gauge group. Then we specialize to the case of gauged linear σ-model and LG/CY correspondence, where the gauge group becomes abelian. As the first technical results, we also prove an interior compactness of the moduli space of solutions to a perturbed equation for the linear case.

Remark 5.1.1. This chapter is based on a work in progress joint with Gang Tian, which we started in August 2012. We are aware that other people are also interested in this problem and working on it. In particular, we started to know that Fan-Jarvis-Ruan is working on the geometric LG/CY correspondence when the author of this thesis attended Jarvis' talk at a conference at University of Michigan in March 2013.

5.2 The Witten equation and the vortex equation

We start with very general setup. Let Σ be a smooth Riemann surface and $G^{\mathbb{C}}$ be a reductive Lie group whose maximal compact subgroup is G. Choosing a smooth area form Ω_Σ on Σ, and an Ad-invariant metric on the Lie algebra $\mathfrak{g}$ of G. Let K_Σ be the canonical bundle of Σ.

Let $W : X \to \mathbb{C}$ be a holomorphic function. W is called **quasi-homogeneous**, if there exists a

unique character $\rho : G^{\mathbb{C}} \to \mathbb{C}^*$ such that

$$W(gx) = \rho(g)W(x). \tag{5.2}$$

5.2.1 W-structures

Definition 5.2.1. A W-structure on a Riemann surface Σ is a holomorphic $G^{\mathbb{C}}$-bundle $P \to \Sigma$ and an isomorphism of holomorphic line bundles

$$\varphi : P \times_\rho \mathbb{C} \to K_\Sigma.$$

Two W-structures (P_i, φ_i), $i = 1, 2$ are isomorphic, if there exists an isomorphism $f : P_1 \to P_2$ such that $\varphi_2 \circ f = \varphi_1$.

Now suppose we are given a W-structure (P, φ), we can lift the superpotential W to the associated bundle $Y := P \times_{G^{\mathbb{C}}} X$ as follows. φ is equivalent to an invariant map

$$\widetilde{\varphi} : P \times \mathbb{C} \to K_\Sigma$$

which is linear in the $\mathbb{C}$-direction and which covers the identity map of Σ. We define a map

$$\begin{array}{rccc} \widetilde{\mathcal{W}} : & P \times X & \to & K_\Sigma \\ & (p, x) & \mapsto & W(x)\widetilde{\varphi}(p, 1). \end{array} \tag{5.3}$$

It is invariant because for any $g \in G^{\mathbb{C}}$,

$$\widetilde{\mathcal{W}}(pg, g^{-1}x) = W(g^{-1}x)\widetilde{\varphi}(pg, 1) = \rho(g)^{-1}W(x)\rho(g)\widetilde{\varphi}(p, 1) = \widetilde{\mathcal{W}}(p, x).$$

Hence it induces a section

$$\mathcal{W} \in \Gamma\left(Y, \pi^* K_\Sigma\right).$$

5.2.2 Gauge theory formulation

Let's consider a G-principal bundle Q_0 such that there exists a smooth isomorphism $Q_0 \times_\rho \mathbb{C} \simeq K_\Sigma$. Let $P_0 := Q_0 \times_G G^{\mathbb{C}}$. Let $\mathcal{A}(Q_0)$ be the space of G-connections on Q_0. Then for any $A \in \mathcal{A}(Q_0)$, the $(0, 1)$-part of A induces a holomorphic structure on P_0.

Consider

$$\mathcal{N}_W := \{A \in \mathcal{A}(Q_0) \mid \exists \text{ holomorphic } \varphi : P_0 \times_\rho \mathbb{C} \simeq K_\Sigma\}$$

and

$$\mathcal{E}_W := \{(A, \varphi) \mid A \in \mathcal{N}_W,\ \varphi : P_0 \times_\rho \mathbb{C} \to K_\Sigma\}.$$

Then $\mathcal{E}_W \to \mathcal{N}_W$ is a $\mathbb{C}^*$-bundle, which is trivializable.

The group of complex gauge transformations acts on $\mathcal{E}_W$ by

$$g^* (A, \varphi) = (g^* A, \rho(g)\varphi).$$

Then by (5.3), we obtained a family of superpotentials

$$\mathcal{W} \in \Gamma (\mathcal{E}_W \times Y, \pi^* K_\Sigma).$$

We denote W_A^φ the restriction of $\mathcal{W}$ to $\{(A, \varphi)\} \times Y$. In particular by (5.3) we see

$$W_{g^* A}^{g^* \varphi} = \rho(g) W_A^\varphi.$$

5.2.3 The equations

The Kähler metric on X induces a Hermitian metric on the vertical tangent bundle $T^V Y \to Y$. For any $(A, \varphi) \in \mathcal{E}_W$, we can take the vertical differential (without referring to any connection)

$$d^V W_A^\varphi \in \Gamma \left(Y, \pi^* K_\Sigma \otimes \left(T^V Y\right)^*\right).$$

The complex structure on Σ induces a $\mathbb{C}$-anti-linear isomorphism $K_\Sigma \simeq \Omega_\Sigma^{0,1}$, which is locally $dz \mapsto d\bar{z}$. The vertical Hermitian metric on $T^V Y$ induces a $\mathbb{C}$-anti-linear isomorphism $\left(T^V Y\right)^* \simeq T^V Y$. These two isomorphisms induces the **vertical gradient** of W_A^φ, which is

$$\nabla W_A^\varphi \in \Gamma \left(Y, \pi^* \Omega_\Sigma^{0,1} \otimes T^V Y\right).$$

Then the equation is on pairs $(A, \varphi, u) \in \mathcal{E} \times \Gamma(Y)$

$$\begin{cases} \overline{\partial}_A u + \nabla W_A^{\varphi}(u) &= 0; \\ \Lambda F_A + \mu(u) &= 0. \end{cases} \tag{5.4}$$

We explain each term in this equation. $\overline{\partial}_A u \in \Omega^{0,1}\left(\Sigma, u^* T^V Y\right)$ is the $(0,1)$-part of the differential of u with respect to the connection A; $\nabla W_A^{\varphi}(u) := u^* \nabla W_A^{\varphi} \in \Omega^{0,1}\left(\Sigma, u^* T^V Y\right)$, which only depends on the 0-jet of u; $\Lambda : \Omega^2(\Sigma) \to \Omega^0(\Sigma)$ is the contraction with respect to a smooth area form Ω_Σ on Σ; for the second equation to make sense, we need to specify a biinvariant metric on $\mathfrak{g}$ to identify $\mathfrak{g}$ with $\mathfrak{g}^*$.

Remark 5.2.2. If $\mathcal{W} \equiv 0$, we take can $\mathcal{E} = \mathcal{N} = \mathcal{A}$. The equation thus only depends on (A, u). In this case our equation is the same as the equation for twisted holomorphic maps in [8] and [37], where we don't have constrain on the holomorphic structure defined by A.

Remark 5.2.3. It is easy to see that the equation is invariant under unitary gauge transformations. However, since we use gradient to define the equation, and the Hermitian metric is not invariant under $G^{\mathbb{C}}$-action, the first equation of (3.9) is not invariant under complex gauge transformation. This is different from the case of Hamiltonian Gromov-Witten theory.

5.3 Energy identity

5.3.1 The Lagrangian

For any configuration (A, φ, u), we define its kinetic energy to be

$$E_{kin}(A, \varphi, u) := \frac{1}{2} \left\| d_A u \right\|^2_{L^2(\Sigma)} + \frac{1}{2} \left\| F_A \right\|^2_{L^2(\Sigma)}.$$

The potential energy of (A, φ, u) is defined to be

$$E_{pot}(A, \varphi, u) := \left\| \nabla W_A^{\varphi}(u) \right\|^2_{L^2(\Sigma)} + \frac{1}{2} \left\| \mu(u) \right\|^2_{L^2(\Sigma)}.$$

The total energy for (A, φ, u) is the sum of kinetic and potential energy, i.e.,

$$E(A, \varphi, u) = E_{kin}(A, \varphi, u) + E_{pot}(A, \varphi, u).$$

5.3.2 The energy identity

Now we compute the energy for any (A, φ, u). We temporarily write for short $W_A^\varphi = W$.

Theorem 5.3.1. *For any* $(A, \varphi, u) \in \mathcal{E}_W \times \mathcal{S}(Y)$,

$$E(A,\varphi,u) = \int_\Sigma u^*\omega_A + 2\mathrm{Im}\int_\Sigma \overline{\partial}(W\circ u) + \left\|\overline{\partial}_A u + \nabla W(u)\right\|_{L^2}^2 + \frac{1}{2}\left\|F_A + \mu(u)\Omega_\Sigma\right\|_{L^2}^2. \tag{5.5}$$

Proof. This formula follows from (1.22) and the following

$$\begin{aligned}\left|\overline{\partial}_A u\right|^2 + \left|\nabla W(u)\right|^2 &= \left|\overline{\partial}_A u + \nabla W(u)\right|^2 - 2\mathrm{Re}\left\langle \overline{\partial}_A u, \nabla W(u)\right\rangle \\ &= \left|\overline{\partial}_A u + \nabla W(u)\right|^2 + 2\mathrm{Im}\Lambda\left(d^V W(u)\cdot(\overline{\partial}_A u)\right) = \left|\overline{\partial}_A u + \nabla W(u)\right|^2 + 2\mathrm{Im}\Lambda\left(\overline{\partial}(W\circ u)\right).\end{aligned}$$

The last equality follows from the fact that W is holomorphic with respect to $\overline{\partial}_A$ and takes value in K_Σ. □

Remark 5.3.2. In the following applications, we consider the equation (5.4) over a punctured Riemann surface. Then we expect (prove) that the behavior of the section u near the punctures is nice, so the first two terms of the left-hand-side of (5.5) are equal to topological terms, which are a cohomological pairing and a "residue" respectively.

5.4 Application in gauged linear σ-model

5.4.1 Quasi-homogeneous polynomials and symmetries

Let $W_0 : \mathbb{C}^n \to \mathbb{C}$ be a nondegenerate quasi-homogeneous polynomial, written as

$$W_0 = \sum_{j=1}^{s} a_j x_1^{b_{j1}} \cdots x_n^{b_{jn}}$$

as the sum of monomials. By quasi-homogeneous, we mean that there exists a set of coprime natural numbers $d_1, \ldots, d_n, D$ such that for all $\lambda \in \mathbb{C}$,

$$W_0\left(\lambda^{d_1}x_1, \ldots, \lambda^{d_n}x_n\right) = \lambda^D W_0(x_1, \ldots, x_n). \tag{5.6}$$

The rational numbers $q_i := \frac{d_i}{D}$ are called the **weights** of the polynomial W_0.

By nondegenerate, we mean the following

1. W_0 contains no monomial of the form $x_i x_j$ for $i \neq j$;

2. The hypersurface defined by $W_0 = 0$ in weighted projective space is non-singular.

Let $G_{W_0} \subset U(1)^n$ be the group of elements $g \in U(1)^n$ such that $W_0 \circ g = W_0$. By nondegeneracy, G_{W_0} is a finite group and it always contains the cyclic group generated by

$$J := \left(e^{2\pi q_1 \sqrt{-1}}, \ldots, e^{2\pi q_n \sqrt{-1}}\right).$$

Let $K \subset U(1)^n$ be the compact subgroup

$$K := \left\{\exp\left(2\pi\sqrt{-1}t(d_1, \ldots, d_n)\right) \mid t \in \mathbb{R}\right\}.$$

Then $K \cap G_{W_0} = \langle J \rangle \simeq \mathbb{Z}_D$. Let $G_R \subset U(1)^n$ be the subgroup generated by K and G_{W_0}. Then topologically it is a finite cover of $K \simeq U(1)$.

Lemma 5.4.1. *The exponents of W_0 define s characters $\overrightarrow{b}_j := (b_{j1}, \ldots, b_{jn})$, $j = 1, \ldots, s$ of $U(1)^n$ and $G_R \subset U(1)^n$ is the subgroup where all $\overrightarrow{b}_j$ restrict to the same representation.*

Proof. On the Lie algebra level, since W_0 is nondegenerate, the subspace of the Lie algebra of $U(1)^n$ where the characters act in the same way is 1-dimensional, which is the Lie algebra of K. Hence the only remaining group elements are those $g \in U(1)^n$ on which $\overrightarrow{b}_j$ acts trivially. They are exactly the subgroup G_{W_0}. □

Let $\rho_R : G_R \to U(1)$ be this representation. Let $G_R^{\mathbb{C}}$ be the complexification of G_R and by abuse of notation use $\rho_R : G_R^{\mathbb{C}} \to \mathbb{C}^*$ to denote the complexified representation. Then it is easy to see that for $g \in G_R^{\mathbb{C}}$, $x \in \mathbb{C}^n$,

$$W_0(gx) = \rho_R(g) W_0(x),$$

which is a special case of (5.2).

5.4.2 W_0-structures

Original definition of Fan-Jarvis-Ruan

We recall the notion of W_0-structures and W_0-curves in [11]. For general W_0 the definition is very sophisticated. Here we only include the most intuitive part of the definition.

Let Σ be a smooth compact Riemann surface and let $\{p_1, \ldots, p_k\}$ be a subset of markings. Denote $\Sigma^* := \Sigma \setminus \{p_1, \ldots, p_k\}$. We glue orbifold charts to Σ^* to obtain an orbifold Riemann surface, which is still denoted by Σ.

Definition 5.4.2. A W_0-structure on Σ is a collection

$$\gamma := (\mathcal{L}_1, \ldots, \mathcal{L}_n; \phi_1, \ldots, \phi_s)$$

where $\mathcal{L}_i \to \Sigma$ are holomorphic orbi-line bundles, and

$$\phi_j : W_j(\mathcal{L}_1, \ldots, \mathcal{L}_n) \to K_{\log} \tag{5.7}$$

are isomorphisms of line bundles. $\{\phi_1, \ldots, \phi_s\}$ are subject to a constrain, i.e., for any $x \in \Sigma^*$, there exists a local coordinate z and trivializations of $\mathcal{L}_i$ such that

$$\phi_j\left(e_1^{\otimes b_{j1}} \otimes \cdots \otimes e_n^{b_{jn}}\right) = dz;$$

and near each puncture p_k, there exist a complex coordinate w centered at p_k and local holomorphic frame $e_1, \ldots, e_n$ such that

$$\phi_j\left(e_1^{\otimes b_{j1}} \otimes \cdots \otimes e_n^{b_{jn}}\right) = \frac{dw}{w}. \tag{5.8}$$

For the polynomial x^r, an x^r-structure is called an r-spin structure, which is essentially an r-th root of the log-canonical bundle.

An equivalent definition

Now we give an equivalent definition of W_0-structures, which fits in the general framework introduced in Subsection 5.2.1. This is partly inspired by the lecture of Tyler Jarvis in a conference at University of Michigan in March 2013, and this definition is probably already known to people.

Definition 5.4.3. A W_0-structure over Σ is a pair $\widetilde{\sigma} := (P_{\widetilde{\sigma}}, \varphi_{\widetilde{\sigma}})$, where $P_{\widetilde{\sigma}} \to \Sigma$ is a holomorphic $G_R^{\mathbb{C}}$-bundle, and

$$\varphi_{\widetilde{\sigma}} : \mathcal{L}_{\widetilde{\sigma}} := P_{\widetilde{\sigma}} \times_{\rho_R} \mathbb{C} \to K_{\log}$$

is a holomorphic isomorphism.

Now we show that this definition is equivalent to the definition of Fan-Jarvis-Ruan.

Indeed, if we have a pair $(P_{\widetilde{\sigma}}, \varphi_{\widetilde{\sigma}})$, then via the inclusion $\iota_R : G_R^{\mathbb{C}} \hookrightarrow (\mathbb{C}^*)^n$, we obtain the associated holomorphic line bundles $\mathcal{L}_1, \ldots, \mathcal{L}_n$. While we have $\overrightarrow{b}_j \circ \iota_R = \rho_R$, which implies that the isomorphism $\varphi_{\widetilde{\sigma}}$ actually coincides with the isomorphisms in (5.7) up to a $\mathbb{C}^*$-factor.

Conversely, if we are given a W_0-structure $(\mathcal{L}_1, \ldots, \mathcal{L}_n; \phi_1, \ldots, \phi_s)$, then the existence of the isomorphisms in (5.7) and Lemma 5.4.1 imply that the structure group of $\mathcal{L}_1 \oplus \cdots \oplus \mathcal{L}_n$ reduces from $(\mathbb{C}^*)^n$ to $G_R^{\mathbb{C}}$.

5.4.3 The gauged linear σ-model equations

In this subsection we apply the abstract construction to the case which is related to Witten's argument about Landau-Ginzburg/Calabi-Yau correspondence in [55].

For general W_0 we will have a gauge theory with gauge group $G = U(1) \times G_R$. The Lie algebra of G is isomorphic to that of $U(1)^2$, which is $\left(\sqrt{-1}\mathbb{R}\right)^2$. We denote by $e_0 = \sqrt{-1}\,(1,0)$ and $e_1 = \sqrt{-1}\,(0,1)$ the standard basis. We choose the metric so that e_0, e_1 form an orthonormal basis.

Let $X = \mathbb{C}^{n+1}$ be the target space. The $G^{\mathbb{C}}$-action on X is given as follows. The inclusion $\iota_R : G_R \hookrightarrow U(1)^n$ induces an action of the factor G_R on the last n coordinates of X. And we define

$$(g_0, g_1) \cdot (x_0, \ldots, x_n) = \left(g_0^{-D} x_0, \iota_R(g_1)\left(g_0^{d_1} x_1, \ldots, g_0^{d_n} x_n\right)\right). \tag{5.9}$$

With respect to the standard Kähler form

$$\omega_X := \frac{\sqrt{-1}}{2} \sum_{i=0}^{n} dx_i \wedge d\overline{x}_i$$

a moment map is (after identifying the Lie algebra with its dual)

$$\mu(x_0, x_1, \ldots, x_n) = -\frac{\sqrt{-1}}{2}\left(-D|x_0|^2 + \sum_{i=1}^{n} d_i\,|x_i|^2, \sum_{i=1}^{n} d_i\,|x_i|^2\right) \tag{5.10}$$

which is a proper function.

We consider the character ρ of G, which is trivial on the $U(1)$-factor and is equal to ρ_R on the G_R-factor.

Let $W_0 : \mathbb{C}^n \to \mathbb{C}$ be the quasi-homogeneous polynomial. Define

$$\begin{array}{rccc} W: & X & \to & \mathbb{C} \\ & (x_0, x_1, \ldots, x_n) & \mapsto & x_0 W_0(x_1, \ldots, x_n). \end{array}$$

Then W is quasi-homogeneous in the sense of (5.2) for the action (5.9) and for the character $\rho : G \to U(1)$.

We fix a W_0-structure $\widetilde{\sigma} = (P_{\widetilde{\sigma}}, \varphi_{\widetilde{\sigma}})$. By choosing a Hermitian metric on the line bundles, we fix a reduction of $P_{\widetilde{\sigma}}$ to a G_R-bundle, denoted by Q_R and a G_R-connection $A_{\widetilde{\sigma}}$. Now, let's consider a G-principal bundles Q_0 over Σ^*. It is equivalent to a Hermitian line bundle L_0 and a principal G_R-bundle, which we require to be (isomorphic to) Q_R. Let $P_R := Q_R \times_{G_R} G_R^{\mathbb{C}}$ be the complexified, smooth $G_R^{\mathbb{C}}$-bundle, which is canonically, smoothly isomorphic to $P_{\widetilde{\sigma}}$. L_0 is allowed to have orbifold structure over $\{p_1, \ldots, p_k\}$. Via the representation (5.9), the associated bundle to Q_0 is a vector bundle

$$V := V_0 \oplus V_1 \oplus \cdots V_n$$

where $V_0 = L_0^{-D}$.

Now we can use the gauge theory formulation as in Subsection 5.2.2. Any unitary connection on Q_0 gives the line bundles holomorphic structures. We denote

$$\widetilde{\mathcal{N}}_{\widetilde{\sigma}} := \left\{ A = (A_0, A_1) \in \mathcal{A}(Q_0) \mid \left(P_R, \overline{\partial}_{A_1}\right) \simeq P_{\widetilde{\sigma}} \right\}.$$

So for all $(A_0, A_1) \in \widetilde{\mathcal{N}}_{\widetilde{\sigma}}$, the A_1's belong to the same complex gauge orbit of $A_{\widetilde{\sigma}}$. More restrictively, we consider $\mathcal{N}_{\widetilde{\sigma}}$ consisting of pairs (A_0, A_1) with $A_1 = g^* A_{\widetilde{\sigma}}$ and g is any complex gauge transformation of P_R such that $g(p_k) = 1,\ k = 1, \ldots, t$.

On the other hand, if the W_0-structure $\widetilde{\sigma}$ gives n holomorphic line bundles $\mathcal{L}_1, \ldots, \mathcal{L}_n$ and the pair $(L_0, \overline{\partial}_0)$ defines a holomorphic line bundle $\mathcal{L}_0$. Then V becomes a holomorphic vector bundle with

$$V_0 \simeq \mathcal{L}_0^{-D},\ V_i \simeq \mathcal{L}_i \otimes \mathcal{L}_0^{d_i}.$$

Now we have a family of isomorphisms $\varphi_{A_1} : P_R \times_{\rho_R} \mathbb{C} \to K_{log}$ for $A_1 = g^* A_{\widetilde{\sigma}}$ and hence a family of holomorphic isomorphisms

$$\varphi_A : P_0 \times_\rho \mathbb{C} \to K_{\log},\ A \in \mathcal{N}_{\widetilde{\sigma}}.$$

Now we can write down the equation (5.4) in this special case. We consider pairs $(A, u) \in \mathcal{N}_{\widetilde{\sigma}} \times \Gamma(V)$. We use the moment map $\mu + \frac{\sqrt{-1}}{2}(r, R)$, where μ is the function in (5.10), and a *smooth* area form Ω on the closed Riemann surface Σ.

In particular, we may formally write the first equation of (5.4) as

$$\begin{cases} \overline{\partial} u_0 + \overline{W_0}(u_1, \dots, u_n) &= 0; \\ \overline{\partial} u_i + \overline{u_0 \frac{\partial W_0}{\partial x_i}}(u_1, \dots, u_n) &= 0. \end{cases} \tag{5.11}$$

But keep in mind that the W_0 above depends on gauge.

We may write the second equation of (5.4) as

$$\begin{cases} \Lambda F_0 - \frac{\sqrt{-1}}{2}\left(-D|x_0|^2 + \sum_{i=1}^n d_i |x_i|^2 - r\right) &= 0; \\ \Lambda F_1 - \frac{\sqrt{-1}}{2}\left(\sum_{i=1}^n d_i |x_i|^2 - R\right) &= 0. \end{cases} \tag{5.12}$$

Here F_0 is the curvature of L_0 and F_1 is the curvature on Q_R.

5.4.4 Low energy behavior

Note that in the linear case, the equivariant 2-form $\omega_X - \mu$ is exact. Hence the only topological term in the energy identity is

$$\frac{1}{4\pi}(r \mathrm{deg} L_0 + R \mathrm{deg} L). \tag{5.13}$$

Now we are going to vary some parameters in the theory. Recall that the original metric on $\mathfrak{g}$ is

$$(e^0)^2 + (e^1)^2 \in \mathrm{Sym}^2 \mathfrak{g}^*.$$

Now for $\lambda > 0$, we take a different metric

$$\lambda^{-2}(e^0)^2 + (e^1)^2$$

which rescales the e_0 direction.

On the other hand, we rescale the symplectic form from ω to $\epsilon\omega$. Then, the quadratic part of the moment map is rescaled by ϵ also. Now the energy is changed to be

$$E_{kin} = \frac{\epsilon}{2} \|d_A u\|_{L^2}^2 + \|F_+\|_{L^2}^2 + \lambda^{-2} \|F_0\|_{L^2}^2$$

and

$$E_{pot} = 2\epsilon^{-1} \left\| d^V W_A(u) \right\|_{L^2}^2 + \frac{\lambda^2}{8} \left\| \epsilon \left(-D|u_0|^2 + \sum_{i=1}^n d_i |u_i|^2 \right) - r \right\|_{L^2}^2 + \frac{1}{8} \left\| \epsilon \sum_{i=1}^n d_i |u_i|^2 - R \right\|_{L^2}^2 .$$

Here the norms are computed using the fixed data (the original metrics before varying them). While the sum is still equal to (5.13) plus a residue term, which could be uniformly bounded.

The adiabatic limit we are going to run is $\lambda \to \infty$, $\epsilon \to 0$. Because of $\epsilon \to 0$, we see that $d^V W_A(u) \to 0$ in the L^2-sense. For the case $r > 0$, we see that this corresponds to the Calabi-Yau phase; for the case $r < 0$, this corresponds to the Landau-Ginzburg phase.

Remark 5.4.4. This outline is due to Witten ([55]), and I learned about it from the videod lecture by David Morrison in 2012 at Simons Center of Stony Brook University. To actually realize the above idea, a lot more has to be done.

5.5 Interior compactness

In this section we consider the compactness of the moduli space of solutions to the equation (5.11) and (5.12) defined in the last section. We assume that the energy of solutions has a uniform bound. This is not true *a priori*, but after suitably perturbing the equation near the orbifold points, one should be able to prove such a bound.

Theorem 5.5.1. *For any sequence (A_i, u_i) of solutions to the equation (5.11) and (5.12), with*

$$\limsup_{i \to \infty} E(A_i, u_i) < \infty,$$

there exists a subsequence of (A_i, u_i) (still indexed by i for simplicity), a sequence of gauge transformations $g_i : \Sigma^* \to G$ and a solution (A, u) to (5.11) and (5.12) such that for any compact subset $K \subset \Sigma^*$, we have $g_i^*(A_i, u_i)|_K$ converges to $(A, u)|_K$ in C^∞-topology.

5.5.1 The mean value inequality

The quasi-homogeneity of W_0 implies that

$$W_0 = \sum_{i=1}^n q_i x_i \cdot \partial_i W_0, \ \left(q_i := \frac{d_i}{D} \right).$$

Then for the polynomial $W = x_0 W_0$, which is still quasi-homogeneous but not nondegenerate, for any $\lambda \in (0,1)$, we have

$$x_0 W_0 = \lambda x_0 W_0 + (1-\lambda) x_0 \sum_{i=1}^{n} q_i x_i \cdot \partial_i W_0.$$

Denote $q_0^\lambda = \lambda$, $q_i^\lambda = (1-\lambda) q_i$ for $i = 1, \ldots, n$.

Proposition 5.5.2. *Let Δ be the Laplacian on Σ^* associated to the smooth metric on Σ. Then there exist constants $B_0, B_1 > 0$ such that for any solution (A, u) to the equations (5.11) and (5.12), we have*

$$\Delta \left(\sum_{i=0}^{n} q_i^\lambda |u_i|^2 \right) \geq -B_0 - B_1 \left(\sum_{i=0}^{n} q_i^\lambda |u_i|^2 \right)^2. \tag{5.14}$$

Proof. We simplify our notations and write the perturbed Witten equation as

$$\overline{\partial} u + \nabla W(u) = 0.$$

Then we have

$$\sum_{i=0}^{n} q_i^\lambda \left\langle \overline{\partial} u_i, u_i \right\rangle = -\sum_{i=0}^{n} q_i^\lambda \left\langle \nabla_i W(u), u_i \right\rangle = -\overline{W(u)} \in \Omega^{0,1}.$$

Take ∂ on both sides, we obtain

$$\sum_{i=0}^{n} q_i^\lambda \left(\langle \partial \overline{\partial} u_i, u_i \rangle - \langle \overline{\partial} u_i, \overline{\partial} u_i \rangle \right) = -\overline{\overline{\partial} W(u)} = -\overline{d^V W \cdot \overline{\partial} u} = \overline{d^V W \cdot \nabla W(u)}.$$

Then we compute

$$\begin{aligned}
\Delta\left(\sum_{i=0}^{n} q_i^{\lambda}|u_i|^2\right) &= 2\sqrt{-1}\Lambda\partial\overline{\partial}\left(\sum_{i=0}^{n} q_i^{\lambda}|u_i|^2\right) = 2\sqrt{-1}\Lambda\sum_{i=0}^{n} q_i^{\lambda}\partial\left(\langle\overline{\partial}u_i, u_i\rangle + \langle u_i, \partial u_i\rangle\right) \\
&= 2\sqrt{-1}\Lambda\sum_{i=0}^{n} q_i^{\lambda}\left(\langle\partial\overline{\partial}u_i, u_i\rangle + \langle\partial u_i, \partial u_i\rangle - \langle\overline{\partial}u_i, \overline{\partial}u_i\rangle + \langle u_i, \overline{\partial}\partial u_i\rangle\right) \\
&= 2\sum_{i=0}^{n} q_i^{\lambda}\left(\left|\overline{\partial}u_i\right|^2 + |\partial u_i|^2\right) + 2\sqrt{-1}\Lambda\sum_{i=0}^{n} q_i^{\lambda}\left(\langle\partial\overline{\partial}u_i, u_i\rangle + \langle u_i, F_A u_i\rangle - \langle u_i, \partial\overline{\partial}u_i\rangle\right) \\
&= 2\sum_{i=0}^{n} q_i^{\lambda}\left(\left|\overline{\partial}u_i\right|^2 + |\partial u_i|^2\right) + 4\mathrm{Re}\sqrt{-1}\Lambda\sum_{i=0}^{n} q_i^{\lambda}\langle\partial\overline{\partial}u_i, u_i\rangle + 2\sqrt{-1}\Lambda\sum_{i=0}^{n} q_i^{\lambda}\langle u_i, F_A u_i\rangle \\
&= 2\sum_{i=0}^{n}\left[q_i^{\lambda}\left|\partial u_i\right|^2 + (2-q_i^{\lambda})\left|\overline{\partial}u_i\right|^2\right] + 2\sqrt{-1}\Lambda_c\sum_{i=0}^{n} q_i^{\lambda}\langle u_i, F_A u_i\rangle \geq 2\sqrt{-1}\Lambda_c\sum_{i=0}^{n} q_i^{\lambda}\langle u_i, F_A u_i\rangle.
\end{aligned} \tag{5.15}$$

Let F_0, F_1 be the two components of F_A. By the representation (5.9), the vortex equation and the form of the moment map, the last term in the above can be estimated as follows.

$$\begin{aligned}
2\sqrt{-1}\Lambda\sum_{i=0}^{n} q_i^{\lambda}\langle u_i, F_A u_i\rangle &= 2\sqrt{-1}\Lambda\left(q_0^{\lambda}\langle u_0, -DF_0 u_0\rangle + \sum_{i\geq 1} q_i^{\lambda}\langle u_i, d_i(F_1+F_0)u_i\rangle\right) \\
&= 2\sqrt{-1}\Lambda\left(\sum_{i\geq 1} d_i q_i^{\lambda}|u_i|^2 F_1 + \left[-q_0^{\lambda}D|u_0|^2 + \sum_{i\geq 1} q_i^{\lambda} d_i|u_i|^2\right]F_0\right) \\
&= \left(\sum_{i\geq 1}(1-\lambda)d_i q_i|u_i|^2\right)\left(\sum_{i\geq 1} d_i|u_i|^2 - R\right) \\
&+ \left(-\lambda D|u_0|^2 + \sum_{i\geq 1}(1-\lambda)q_i d_i|u_i|^2\right)\left(-D|u_0|^2 + \sum_{i\geq 1} d_i|u_i|^2 - r\right) \\
&\geq -B_0 - B_1\left(\sum_{i\geq 0} q_i^{\lambda}|u_i|^2\right)^2.
\end{aligned} \tag{5.16}$$

Here B_0 and B_1 are constants depending only on λ, the weights q_i and r, R. □

Remark 5.5.3. In the case that W_0 is homogeneous, we could choose λ suitably so that on the unperturbed region, the function $\sum_{i=0}^{n} q_i^{\lambda}|u_i|^2$ is subharmonic.

Lemma 5.5.4. *[44, Page 21] Let $\Omega \subset \mathbb{R}^2$ be an open subset, $e : \Omega \to \mathbb{R}_{\geq 0}$ is a smooth function. Suppose*

$$\Delta e \geq -B_0 - B_1 e^2.$$

Then for any closed disk $B_r \subset \Omega$ with center z_0,

$$\int_{B_r} e(z)|dz|^2 \leq \frac{\pi}{16B_1} \Longrightarrow e(z_0) \leq \frac{8}{\pi r^2}\int_{B_r} e(z)|dz|^2 + \frac{B_0 r^2}{2}.$$

5.5.2 Interior compactness

Now denote

$$Q(u) = \sum_{i=0}^{n} q_i^{\lambda}|u_i|^2.$$

Proposition 5.5.5. *For any compact subset $K \subset \Sigma^*$, there exists a constant $c_K > 0$ such that for any solution (A, u) to the equations (5.11) and (5.12), we have*

$$\sup_K Q(u) \leq c_K.$$

Proof. We can take compact subset $K' \subset \Sigma^*$ containing K properly, such that there exists $r_0 > 0$ such that for any $z \in K$, $B_{r_0}(z) \subset K'$. Then for any $z \in K$, $r < r_0$, by Cauchy-Schwarz, for some $c_1 > 0$

$$\int_{B_r(z)} Q(u)\Omega \leq c_1 r \left(\int_{B_r(z)} Q(u)^2\Omega\right)^{\frac{1}{2}} \leq c_1 r c_{K'}.$$

Here the last inequality follows from the fact that $|\mu(u)|_{L^2}$ is uniformly bounded with respect to the smooth metric and $Q(u)$ is a linear combination of the quadratic terms of μ.

Take $r_1 = \min\left\{r_0, \frac{\pi}{16B_1 c_1 c_{K'}}\right\}$, where B_1 is the one in (5.14). Then by the mean value inequality, we have for $z \in K$,

$$Q(u(z)) \leq \frac{8}{\pi r_1^2} c_1 r_1 c_{K'} + B_0 r_1^2 =: C_K.$$

□

Proof of interior compactness. The most crucial step has been established, i.e., Proposition 5.5.5, and the remaining argument is routine.

First, note that we have a compact slice $\mathcal{S} \subset \mathcal{N}_{\widetilde{\sigma}}$ of the complex gauge action, which is essentially the Jacobian of Σ. Hence for each i there exists a complex gauge transformation $g_i = \exp(h_i + \sqrt{-1}h_i')$ and $s_i \in \mathcal{S}$ such that $A_i = g_i^* s_i$. Then

$$F_{A_i} = F_{s_i} + \sqrt{-1}\Delta h_i.$$

Then the uniform bound on the L^2-norm of the curvature implies that

$$\|\Delta h_i\|_{L^2} \leq C$$

where Δ is the Laplacian on the closed surface. It follows that up to unitary gauge transformation, a subsequence of connections converges in $L^p(K)$ for any $p > 1$. We write $h_i = (h_i^0, h_i^1)$, where h_i^0 is the $U(1)$-component and h_i^1 is the G_R-component. Remember that we assumed that $h_i^1(p_k) = 0$ for orbifold points p_k. Then the elliptic estimate and Sobolev embedding imply that a subsequence of h_i^1 converges in $C^0(K)$.

Now, only h_i^1 is responsible for the change of the potential W, and hence the inhomogeneous term ∇W. The uniform bounds on $Q(u)$ and h_i^1 implies that the inhomogeneous term ∇W stays bounded over K. Therefore we have

$$\limsup_{i\to\infty} \|d_A u_i\|_{L^\infty(K)} < \infty.$$

Otherwise, sphere bubbling will happen; but it is impossible because the target space is $\mathbb{C}^{n+1}$.

Thus, we see that we have a subsequence (still indexed by i) such that u_i converges in $W^{1,p}(K)$ to a limit section u_∞ over K. Moreover, by the usual argument we can prove convergence in higher derivatives. The limit is constructed by taking an exhausting sequence of K, and by the diagonal construction. □

Appendix A

This appendix is copied from the paper [46], which is written by Stephen Schecter.

A.1 Geometric singular perturbation theory

Readers familiar with geometric singular perturbation theory may wish to skip most of this appendix; however, the paragraph before Theorem A.1.2 and that theorem may be unfamiliar, and the extension of that theorem described toward the end of this section, while just an observation, is new and essential to Chapter 3.

Let Z be a manifold, let $\dot{p} = f(p, \epsilon) = f_\epsilon(p)$ be a differential equation on Z with parameter ϵ (i.e., each f_ϵ is a section of TZ), let ϕ_t^ϵ be the flow, and let N_0 be a compact submanifold of Z that is invariant under the the flow of $\dot{p} = f_0(p)$. N_0 is called *normally hyperbolic* if there is a splitting of $TZ|N_0$, $TZ|N_0 = S \oplus U \oplus TN_0$, such that under $D\phi_t^0$, all vectors in S shrink at a faster exponential rate than any vector in TN_0, and under $D\phi_{-t}$, all vectors in U shrink at a faster exponential rate than any vector in TN_0. (There are less restrictive definitions, but this one suffices for our purposes.)

Normally hyperbolic invariant manifolds have stable and unstable manifolds with flow-preserved fibrations, and the whole structure persists under perturbation. This structure is most easily described in local coordinates.

Let us assume that $\dim N_0 = m$, $\dim Z = n+m$, and fibers of S (respectively U) have dimension k (respectively l), with $k + l = n$. Near a point of N_0 one can choose coordinates $p = \Phi(x, y, z, \epsilon)$, $(x, y, z, \epsilon) \in \Omega_1 \times \Omega_2 \times (-\epsilon_0, \epsilon_0)$, Ω_1 open subset of $\mathbb{R}^k \times \mathbb{R}^l$, Ω_2 an open subset of $\mathbb{R}^m$, such that,

for small ϵ, $\dot{p} = f(p, \epsilon)$ becomes

$$\dot{x} = A(x, y, z, \epsilon)x, \tag{A.1}$$

$$\dot{y} = B(x, y, z, \epsilon)y, \tag{A.2}$$

$$\dot{z} = h(z, \epsilon) + x^T C(x, y, z, \epsilon)y; \tag{A.3}$$

the matrices A, B, and C are $k \times k$, $l \times l$, and $k \times l$ respectively. One cannot assume that $A(0, 0, z, 0)$ has eigenvalues with negative real part or that $B(0, 0, z, 0)$ has eigenvalues with positive real part, since the exponential convergence of vectors under the linearized flow need only occur as $t \to \pm\infty$. Nevertheless, this is true in the examples below.

We list some facts and terminology.

1. If $\dot{p} = f(p, \epsilon)$ is C^{r+3}, the coordinate change can be chosen so that the new system is C^{r+1}.

2. For each ϵ, the subspaces $y = 0$, $x = 0$, and their intersection are invariant. For fixed ϵ, the set $\{(x, y, z) \mid x = 0 \text{ and } y = 0\}$ (dimension m) corresponds to part of a normally hyperbolic invariant manifold N_ϵ; the set $y = 0$ (dimension $m + k$) corresponds to part of the stable manifold of N_ϵ, $W^s(N_\epsilon)$; and the set $x = 0$ (dimension $m + l$) corresponds to part of the unstable manifold of N_ϵ, $W^u(N_\epsilon)$.

3. If $(x(t), 0, z(t))$ is a solution in $W^s(N_\epsilon)$, then $(0, 0, z(t))$ is a solution in N_ϵ; and if $(0, y(t), z(t))$ is a solution in $W^u(N_\epsilon)$, then $(0, 0, z(t))$ is again a solution in N_ϵ. Each solution in $W^s(N_\epsilon)$ (respectively $W^u(N_\epsilon)$) approaches exponentially a solution in N_ϵ as time increases (respectively decreases).

4. Given a point $p = (0, 0, z_0)$ in N_ϵ, the *stable* (respectively *unstable*) *fiber* of p is the set of all points $(x, 0, z_0)$ (dimension k) (respectively $(0, y, z_0)$ (dimension l)). For each t, the time-t map of the flow takes fibers to fibers; in this sense the fibration is flow-invariant. Solutions that start in the stable (respectively) unstable fiber of p approach the solution that starts at p exponentially at t increases (respectively decreases).

5. Given an $P_\epsilon \subset N_\epsilon$, we shall refer to the union of the stable (respectively unstable) fibers of points in P_ϵ as $W^s(N_\epsilon)$ (respectively $W^u(N_\epsilon)$) *restricted to* P_ϵ. If P_ϵ is invariant, $W^s(N_\epsilon)$ restricted to P_ϵ, for example, may be smaller than $W^s(P_\epsilon)$, since the latter may include solutions in N_ϵ that approach P_ϵ at a slower exponential rate, together with points in their stable fibers.

Examples:

1. Suppose N_0 is a compact manifold of equilibria of dimension m, and each equilibrium in N_0 has k eigenvalues with negative real part and l eigenvalues with positive real part. Then N_0 is a compact normally hyperbolic invariant manifold. The stable (respectively unstable) fiber of a point p in N_0 is just its stable (respectively unstable) manifold. In (A.3), $h(z, 0) \equiv 0$.

2. Suppose N_0 is a compact manifold with boundary of equilibria of dimension m, and each equilibrium in N_0 has k eigenvalues with negative real part and l eigenvalues with positive real part. N_0 does not fit the definition of a normally hyperbolic invariant manifold that we have given, but a coordinate system as above still exists, except that Ω_2 may have boundary and N_ϵ may be only locally invariant (i.e., solutions may leave through the boundary).

3. Suppose p_0 is a critical point of $\dot{p} = f_0(p)$ at which the linearization has k eigenvalues with negative real part, l eigenvalues with positive real part, and m eigenvalues with 0 real part. On a neighborhood of p_0, one can choose coordinates as above such that the system takes the form (A.1)–(A.3); this is the Center Manifold Theorem. Here the N_ϵ are locally invariant open manifolds.

4. Fast-slow systems. Consider the system

$$\dot{w} = f(w, z, \epsilon), \tag{A.4}$$

$$\dot{z} = \epsilon g(w, z, \epsilon), \tag{A.5}$$

with $w \in \mathbb{R}^n$, $z \in R^m$, and $0 \leq \epsilon < \epsilon_0$. System (A.4)–(A.5) is a *fast-slow system*; w is the fast variable and z is the slow variable. The fast limit is (A.4)–(A.5) with $\epsilon = 0$. If 0 is a regular value of f, then $\{(w, z) : f(w, z, 0) = 0\}$ is a manifold of dimension m called the *slow manifold*. The slow manifold is the set of equilibria of the fast system. Let N_0 be a compact subset of the slow manifold that is a manifold with boundary of dimension m. N_0 is normally hyperbolic (in the sense of Example 2) if there are numbers k and l with $k + l = n$ such that at each point $(w, z, 0)$ of N_0, $D_z f(w, z, 0)$ has k eigenvalues with negative real part and l eigenvalues with positive real part. By the Implicit Function Theorem, N_0 can be described as $w = \chi(z, 0)$. Then for $\epsilon > 0$ small there is a normally hyperbolic locally invariant manifold N_ϵ (as in Example 2) near N_0, given by $w = \chi(z, \epsilon)$.

The name "geometric singular perturbation theory" comes from the fact that after a rescaling

of time, (A.4)–(A.5) can be rewritten as

$$\epsilon \dot{w} = f(w, z, \epsilon), \tag{A.6}$$

$$\dot{z} = g(w, z, \epsilon), \tag{A.7}$$

which is a singularly perturbed system.

In Example 4, N_ϵ is given by an expansion $w = \chi_0(z) + \epsilon\chi_1(z) + \ldots$, and the system restricted to N_ϵ, with coordinate z, is given by

$$\dot{z} = \epsilon g(\chi_0(z) + \epsilon\chi_1(z) + \ldots, z, \epsilon) = \epsilon g(\chi_0(z), z, 0) + \mathcal{O}(\epsilon^2). \tag{A.8}$$

Thus, after division by ϵ, system (A.4)–(A.5) restricted to N_ϵ is given by $\dot{z} = g(\chi_0(z), z, 0) + \mathcal{O}(\epsilon)$. The differential equation $\dot{z} = g(\chi_0(z), z, 0)$ is called the *slow equation*. The expansion of N_ϵ, and hence of (A.4)–(A.5) restricted to N_ϵ, can be computed to any order using the invariance of N_ϵ. Similar calculations can be done in Examples 2 and 3. In the latter case, typically the center manifold at p_0 for $\epsilon = 0$ is not known explicitly, so it too must be computed as an expansion in a coordinate system near p_0.

We shall abuse terminology and refer to the N_ϵ as normally hyperbolic invariant manifolds in all the above situations.

Suppose we wish to follow an $(l+1)$-dimensional manifold of solutions M_ϵ as it passes near a normally hyperbolic invariant manifold N_ϵ in a manifold Z, where N_0 is a normally hyperbolic invariant manifold of equilibria. We choose coordinates so that the system is (A.1)–(A.3), with $h(z, \epsilon) = \epsilon\tilde{h}(z, \epsilon)$. We assume that near the point $(x^*, 0, z^*)$, the M_ϵ, $\epsilon \geq 0$, fit together to form a C^{r+1} manifold with boundary in $Z \times \mathbb{R}$; and we assume that M_0 meets $W^s(N_0)$ transversally in the solution through $(x^*, 0, z^*)$, which approaches the equilibrium $(0, 0, z^*)$ as $t \to \infty$. The system restricted to N_ϵ has the form $\dot{z} = \epsilon\tilde{h}(z, \epsilon)$; we assume $\tilde{h}(z^*, 0) \neq 0$. Let ψ_t denote the flow of $\dot{z} = \tilde{h}(z, 0)$, and let $z^\dagger = \psi_T(z^*)$ for some $T > 0$. Let $I = \{(\psi_t(z^*) : T - \delta < t < T + \delta\}$, a small interval around $z^\dagger$ in the orbit of z^* for $\dot{z} = \tilde{h}(z, 0)$. Let $J = \{0\} \times \{0\} \times I$. Let V be a small open set around $(0, 0, z^\dagger)$ in $W^u(N_0)$ restricted to J, which has dimension $l + 1$. Then we have:

Theorem A.1.1 (Exchange Lemma)**.** *For small $\epsilon > 0$, parts of M_ϵ fit together with V to form a C^r manifold in $Z \times \mathbb{R}$.*

See Figure A.1. Notice that for small $\epsilon > 0$, M_ϵ meets $W^s(N_\epsilon)$ transversally in a solution that tracks a solution in N_ϵ that starts at a point $(0, 0, z(\epsilon))$, with $z(\epsilon)$ near z^*. For $\epsilon > 0$, the orbit of

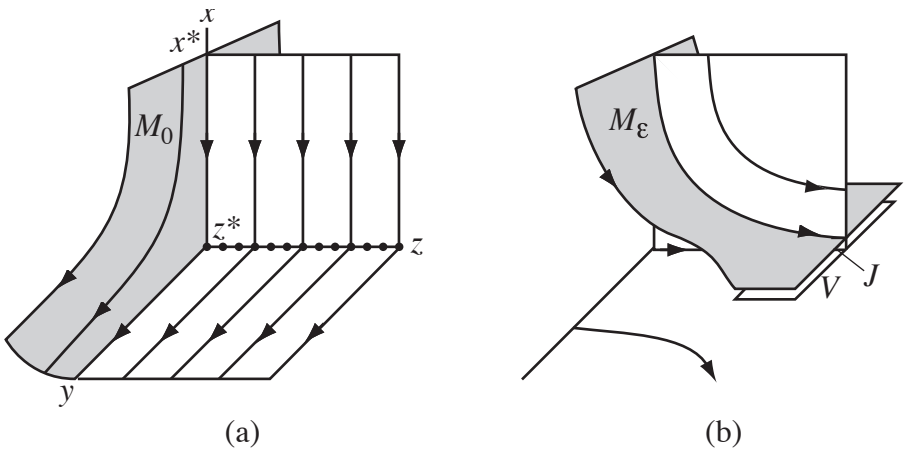

Figure A.1: The Exchange Lemma with $k = l = m = 1$: (a) $\epsilon = 0$, (b) $\epsilon > 0$.

$\dot{z} = \epsilon\tilde{h}(z_\epsilon)$ through z_ϵ equals the orbit of $\dot{z} = \tilde{h}(z, \epsilon)$ through $z(\epsilon)$. This orbit is close to the orbit of $\dot{z} = \tilde{h}(z, 0)$ through z^*. Thus we find that portions of the orbits tracked in N_ϵ for $\epsilon > 0$ limit on a curve J in N_0 that is not an orbit of the system (A.1)–(A.3) on N_0 (these orbits are just points). Nevertheless the M_ϵ for $\epsilon > 0$ become close to $W^u(N_0)$ restricted to J.

Now suppose we wish to track an $(l + s + 1)$-dimensional manifold of solutions M_ϵ, with $0 \leq s \leq m - 1$, as it passes near a normally hyperbolic invariant manifold N_ϵ in a manifold Z, where N_0 is a normally hyperbolic invariant manifold but does not consist of equilibria. (This occurs, for example, Step 3 of the proof of Theorem 3.5.14.) We again choose coordinates so that the system is (A.1)–(A.3). We choose a cross-section $\tilde{M}_\epsilon$ to the flow within M_ϵ, of dimension $l + s$ (so that M_ϵ is the union of orbits that start in $\tilde{M}_\epsilon$), and we assume that $\tilde{M}_0$ meets $W^s(N_0)$ transversally at $(x^*, 0, z^*)$. For small $\epsilon \geq 0$, $\tilde{M}_\epsilon$ meets $W^s(N_\epsilon)$ in a manifold Q_ϵ of dimension s; we assume that Q_ϵ projects regularly along the stable fibration to a submanifold P_ϵ of N_ϵ of dimension s. We also assume that for small $\epsilon \geq 0$, the vector field $(0, 0, h(x, \epsilon))$ is not tangent to P_ϵ. (Of course these assumptions follow from the corresponding ones at $\epsilon = 0$.) Finally, we assume that for $\epsilon > 0$, following P_ϵ along the flow for time $\mathcal{O}(\frac{1}{\epsilon})$ produces a submanifold P_ϵ^* of dimension $s + 1$ of N_ϵ, and the manifolds P_ϵ^* fit together with a submanifold P_0^* of N_0, of dimension $s + 1$, to form a C^{r+1} manifold in $Z \times \mathbb{R}$. This assumption typically requires that some solution of the system restricted to N_0 that starts in P_0 approaches an equilibrium. We emphasize that, analogous to the usual Exchange Lemma, P_0^* is not the result of following P_0 along the flow for $\epsilon = 0$. Let V be a small open neighborhood of P_0^* in $W^u(N_0)$ restricted to P_0^*, which has dimension $l + s + 1$.

Theorem A.1.2 (General Exchange Lemma)**.** *For small $\epsilon > 0$, parts of M_ϵ fit together with V to*

form a C^r manifold in $Z \times \mathbb{R}$.

Some technical hypotheses that are not relevant to this thesis have been omitted. We actually need a small generalization of Theorems A.1.1 and A.1.2, whose proofs are essentially the same.

We will use C^r-topology to measure submanifolds. If $S \subset X$ is a smooth submanifold, we say another C^r-submanifold S'(of the same dimension) is C^r-close to S, if with some smooth identification of the normal bundle of S with a tubular neighborhood of S, S' can be identified with a C^r-small section of the normal bundle.

In Theorem A.1.1, replace the assumption that near the point $(x^*, 0, z^*)$, the M_ϵ, $\epsilon \geq 0$, fit together to form a C^{r+1} manifold with boundary, with the assumption that $M_\epsilon \to M_0$ in the C^{r+1}-topology. The conclusion becomes that parts of M_ϵ converge to V in the C^r-topology.

In Theorem A.1.2, make the replacement in the assumptions just mentioned, and replace the assumption that the P^*_ϵ, $\epsilon \geq 0$, fit together to form a C^{r+1} manifold with the assumption that $P^*_\epsilon \to P^*_0$ in the C^{r+1}-topology. The conclusion again becomes that parts of M_ϵ converge to V in the C^r-topology.

These generalizations are required for the following reason. Consider the fast-slow system

$$\dot{z}_1 = -z_2 + z_1^2, \tag{A.9}$$

$$\dot{z}_2 = -\epsilon, \tag{A.10}$$

which is related to (3.21)–(3.22). See Figure A.2. For $\epsilon = 0$, any compact portion N_0 of the curve of equilibria $z_1 = -z_2^{\frac{1}{2}}$ is normally hyperbolic (in fact attracting). It perturbs to the normally hyperbolic manifold N_ϵ, on which system reduces to $\dot{y} = -\epsilon$. For small $\epsilon > 0$, a solution in or close to N_ϵ arrives in the region $\delta < z_1 < 2\delta$ ($\delta > 0$) along a curve given by $z_2 = \rho(z_1, \epsilon)$, $\delta < z_1 < 2\delta$; $\rho = \mathcal{O}(\epsilon^{\frac{2}{3}})$. As $\epsilon \to 0$, this curve, which in examples with greater dimension may be P^*_ϵ, approaches $z_2 = 0$, which in examples may be P^*_0, in the C^s-topology for any s, but does not fit together with $z_2 = 0$ to form a manifold with a high degree of differentiability in $z_1 z_2 \epsilon$-space.

A.2 Choosing the center manifold

Let us consider for concreteness a system in the form (A.1)–(A.3), with N_0 two-dimensional, and the equation on N_ϵ given by (A.9)–(A.10):

$$\dot{x} = A(x, y, z_1, z_2, \epsilon)x, \tag{A.11}$$

$$\dot{y} = B(x, y, z_1, z_2, \epsilon)y, \tag{A.12}$$

$$\dot{z}_1 = -z_2 + z_1^2 + x^T c_1(x, y, z_1, z_2, \epsilon)y, \tag{A.13}$$

$$\dot{z}_2 = -\epsilon + x^T c_2(x, y, z_1, z_2, \epsilon)y, \tag{A.14}$$

$(x, y) \in \mathbb{R}^k \times \mathbb{R}^l$. The system arises by center manifold reduction at the origin, so $A(0, 0, z_1, z_2, 0)$ has eigenvalues with negative real part ,and $B(0, 0, z_1, z_2, 0)$ has eigenvalues with positive real part. We wish to follow a manifold M_ϵ of dimension $l+1$ as it passes N_ϵ (i.e., $z_1 z_2$-space). M_0 meets $W^s(N_0)$ transversally at a point $(x^*, 0.z_1^*, 0)$ with $z_1^* < 0$; hence the intersection includes the semiorbit γ that starts at $(x^*, 0, z_1^*, 0)$, which approaches the origin as $t \to \infty$. We wish to replace the center manifold N_0 for $\epsilon = 0$ by one that contains γ. To do this, replace all semiorbits in Figure A.2 that start at points $(0, 0, z_1^*, z_2)$, $|z_2| < (z_1^*)^2$, with the semiorbits that start at $(x^*, 0, z_1^*, z_2)$. (The semiorbits that start at $(0, 0, z_1^*, z_2)$ and at $(x^*, 0, z_1^*, z_2)$ both approach $(0, 0, -z_2^{\frac{1}{2}}, z_2)$ as $t \to \infty$, and the second arrives tangent to $z_1 z_2$-space.) The result will be a new center manifold, in a smaller neighborhood of the origin, that contains γ. The differentiability class of this manifold will decrease as we move away from the origin. Now N_0 perturbs to new center manifold N_ϵ for $\epsilon > 0$.

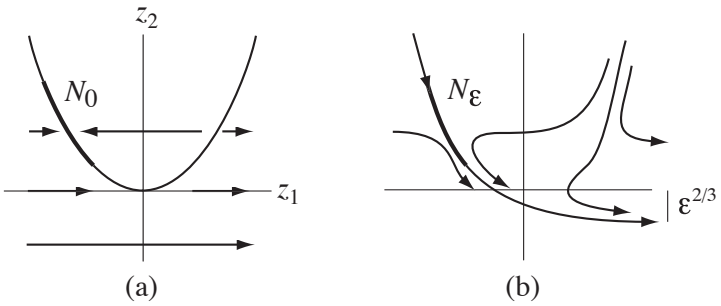

Figure A.2: Flow of (A.9)–(A.10): (a) $\epsilon = 0$, (b) $\epsilon > 0$.

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