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Interior regularity for area-minimizing currents

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Interior regularity for area-minimizing currents

Anna Skorobogatova

A Dissertation
Presented to the Faculty
of Princeton University
in Candidacy for the Degree
of Doctor of Philosophy

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Abstract

In this thesis, we study the interior regularity of area-minimizing integral currents and codimension 1 mod(p) area-minimizing currents. For m-dimensional area-minimizing integral currents of arbitrary codimension, we strengthen Almgren's celebrated sharp Hausdorff dimension estimate of $m-2$ for the singular set, first improving it to a local upper Minkowski dimension estiamte of $m-2$. We then further demonstrate that the singular set is in fact $(m-2)$-rectifiable and the tangent cone is unique at $\mathcal{H}^{m-2}$-a.e. point. The latter is based on a series of joint works with Camillo De Lellis and Paul Minter.

For m-dimensional area-minimizing mod(p) currents of codimension 1, we demonstrate $(m-2)$-rectifiability of the interior singularities with flat tangent cones. In combination with works of Naber-Valtorta, Minter-Wickramasekera and De Lellis-Hirsch-Marchese-Spoloar-Stuvard, this provides a detailed structural characterization of the interior singular set in this setting.

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Chapter 1

Introduction

The question of existence and regularity of critical points for the area functional has been studied in great depth by many, dating back to Lagrange in the 1700s, and has deep connections to numerous problems in the calculus of variations, geometric measure theory and PDE. To begin with, one must establish a suitable framework in which to develop a reasonable existence theory. However, such frameworks usually give rise to a weak notion of solution, which no longer can be assumed to be regular. The regularity theory for general critical points is still largely not well-understood. However, if one considers minimizers, the question becomes much more approachable. There are several suitable weak notions of an (oriented) area minimizing surface spanning a given boundary; two such notions are *integral currents* and *mod*(p) *currents*, for a given positive integer p. The former allows for "surfaces" to have integer multiplicities, while the latter only allows for multiplicities modulo p. In both frameworks, surfaces can have singularities, where the surface is not locally a smooth submanifold (with multiplicity) of its ambient space.

1.1 Integral currents

Currents are continuous linear functionals the space of compactly supported differential m-forms. This duality provides a well-defined weak notion of boundary as an $(m-1)$-

dimensional current and area (or mass). m-dimensional integral currents are m-dimensional currents that have finite boundary mass and whose associated canonical vector measure has m-rectifiable support and integer-valued density $\mathcal{H}^m$-a.e.[1] In this way, integral currents provide a natural generalization to m-dimensional integer multiplicity smooth oriented manifolds with boundary, and indeed are contained in the weak-$*$ closure of the space of such manifolds. In this framework, Federer and Fleming established existence of a homological minimizer, thus providing a suitable setting in which to analyze the singularities of minimizers. Consider an area-minimizing m-dimensional integral current T in a sufficiently regular $(m+\bar{n})$-dimensional embedded submanifold Σ of $\mathbb{R}^{m+n} \equiv \mathbb{R}^{m+\bar{n}+l}$. Namely,

$$\mathbf{M}(T) \leq \mathbf{M}(T + \partial S)$$

for any $(m+1)$-dimensional integral current S supported in Σ.

We define the interior *regular set* of T to be all points around which $\mathrm{spt}T$ can locally be expressed as the graph of a sufficiently regular map. Namely,

$$\mathrm{Reg}\, T := \left\{ p \in \mathrm{spt}T \setminus \mathrm{spt}(\partial T) \,:\, \exists R, \alpha > 0 \text{ s.t. } \mathrm{spt}T \cap \mathbf{B}_R(p) \subset \mathbb{R}^{m+n} \text{ is a } \mathrm{C}^{1,\alpha} \text{ submanifold} \right\}$$

Classical elliptic regularity theory allows the regularity of T in $\mathrm{Reg}(T)$ to be bootstrapped to the maximal regularity allowed by the ambient manifold. Consequently, we define the interior *singular set* of T to be

$$\mathrm{Sing}(T) := \mathrm{spt}(T) \setminus (\mathrm{spt}(\partial T) \cup \mathrm{Reg}(T)).$$

When the codimension $\bar{n}$ is one, the Hausdorff dimension of $\mathrm{Sing}(T)$ was shown to be at most $m-7$ (with purely isolated singularities when $m=7$) by Federer [28], with important contributions from Fleming, De Giorgi, Almgren and Simons for the lower dimensional cases

[1] See, e.g. [27] for the precise definitions.

(see [2,8,9,29,48]). It was further shown by Simon in [46] that the singular set is $(m-7)$-rectifiable. More recently, the work of Naber & Valtorta further established local $(m-7)$-dimensional Minkowski content estimates on the interior singular set. The sharpness of the dimension estimate in codimension one was verified by Bombieri, De Giorgi and Guisti in [5], who demonstrated that Simons' cone

$$C(\tfrac{1}{\sqrt{2}}\mathbb{S}^3 \times \tfrac{1}{\sqrt{2}}\mathbb{S}^3) = \left\{ x \in \mathbb{R}^8 : x_1^2 + x_2^2 + x_3^2 + x_4^2 = x_5^2 + x_6^2 + x_7^2 + x_8^2 \right\} \subset \mathbb{R}^8,$$

which has an isolated singularity at the origin, is area minimizing.

When $\bar{n} \geq 2$, the seminal work of Almgren [3] provides an upper Hausdorff dimension bound of $m-2$ for the interior singular set. The sharpness of this can be easily seen from considering any holomorphic variety of complex codimension one (real codimension two) with a singular branch point at the origin, such as

$$\left\{ (z,w) \in \mathbb{C}^2 : z^2 = w^3 \right\}.$$

Almgren's original proof has since been simplified by De Lellis and Spadaro in the series of works [18,20–22].

When $m = 2$, Chang [6] built upon Almgren's work to demonstrate that interior singularities are isolated. This was made more transparent by De Lellis, Spadaro and Spolaor in the works [23–25], where the authors further extend the result to two-dimensional semicalibrated currents and spherical intersections of 3-dimensional area-minimizing cones.

Understanding the size and structure of the singularities of T goes hand in hand with understanding the possible *tangent cones* at singularities. In light of the monotonicity of mass ratios[2], for any $x \in \mathrm{spt}T$, up to extracting a subsequence, any sequence of rescalings $T_{x,r_k} := (\frac{\cdot - x}{r_k})_\sharp T \llcorner \mathbf{B}_1$ converges weakly-$*$ to an area-minimizing *tangent cone*. However, a priori one cannot necessarily expect that a certain tangent cone is the unique one at a

[2] see e.g. [44]

given point $x \in \mathrm{Sing}(T)$. Establishing uniqueness of tangent cones is a question of great independent interest. When $\bar{n} = 1$, this was shown to be true up to $\mathcal{H}^{m-7}$-a.e. in the interior, in the work [46]. In higher codimension, White [52] showed that the tangent cone is unique at every interior point of a two-dimensional area-minimizing integral current T, in any codimension.

A key issue arising in the study of the interior singularities of area-minimizing integral currents stems from the presence of *flat singularities*, at which there exists a tangent cone supported in an m-dimensional plane. Such singularities indeed exist when $\bar{n} \geq 2$, as exemplified by the aforementioned examples of holomorphic subvarieties with branching singularities, and are the obstruction to an ε-regularity result such as that of Allard [1] or De Giorgi [8], both of which are only valid at flat singularities where the Hausdorff density of T remains near constant.

1.2 mod(p) currents

Similarly to the preceding section, suppose that T is an m-dimensional integer rectifiable current [3] contained in an $(m+\bar{n})$-dimensional sufficiently regular Riemannian submanifold Σ of $\mathbb{R}^{m+n}$ without boundary. Let $p \geq 2$ be a given integer, and define the equivalence relation $\sim_p$ by $T \sim_p S$ if $T = S \mod(p)$, where the latter is in the sense of the flat metric with mod(p) coefficients.[4] In this framework, we consider T to be a representative in its mod(p) equivalence class.

Analogously to integral currents, the area-minimizing property is defined is defined homologically. More precisely, we say that T is *area-minimizing* mod(p) in relatively open subset $\Omega \subset \Sigma$ if it has least mass in its mod(p) homology class within $\Omega \cap \Sigma$, namely

$$\mathbf{M}(T) \leq \mathbf{M}(T + S),$$

[3] an m dimensional current with m-rectifiable support and integer density $\mathcal{H}^m$-a.e.
[4] See, for instance, [27].

for each m-dim. integer rectifiable S in $\Omega \cap \Sigma$ with $[S]_p = [\partial R]_p$, where $[S]_p$ denotes the mod(p) equivalence class of a representative S. We may define the interior regular and singular set of T in exactly the same way as for integral currents.

Understanding the size and the structure of the singularities of T in this setting was a problem first studied by Federer [28] in the case $p = 2$, namely, unoriented surfaces. There, it was shown that the Hausdorff dimension of the singular set is at most $m-2$, while in [45,46], Simon subsequently improved this to $(m-2)$-rectifiability and local finiteness of $(m-2)$-dimensional Hausdorff measure. Furthermore, Taylor [51] handled the case when $m = 2$ and $p = 3$ in three-dimensional ambient Euclidean space, achieving a groundbreaking structural result demonstrating that the only singularities models are superpositions of three half-planes meeting along an axis at angles of $\frac{2\pi}{3}$, and the singularities are locally a $C^{1,\alpha}$ perturbation of this.

As for integral currents, flat singularities of T are the biggest obstruction to understanding the interior regularity properties of T. When p is odd, in the work [15] of De Lellis, Hirsch, Marchese and Stuvard, the authors demonstrate that the singular set of T is $(m-1)$-rectifiable. The key is that although flat singularities of T *can* arise, T has integer Hausdorff density at such singularities and thus when p is odd, the density there must be *strictly smaller than* $\frac{p}{2}$. Thus, such flat singularities can be dealt with inductively on the density Q, and we need not consider them; we only handle the highest density singular points with $Q = \frac{p}{2}$.

When the codimension $\bar{n} = 1$, the work of White [53] tells us that any point $x \in \text{spt}T$ with a flat tangent cone $k[\![\pi]\!]$ with integer multiplicity $k \in \left(-\frac{p}{2}, \frac{p}{2}\right)$ is necessarily regular. When p is odd, this tells us that in codimension one, there are no flat singular points. Making crucial use of this result, Taylor's structure theorem was successfully generalized in codimension one by De Lellis, Hirsch, Marchese, Spolaor and Stuvard in [13], where it was demonstrated that when p is odd, outside of an $(m-2)$-rectifiable set, the singular set of T is locally a $C^{1,\alpha}$ $(m-1)$-dimensional submanifold, with a singularity model consiting of a superposition of

m-dimensional half-spaces meeting in an $(m-1)$-dimensional axis.

However, when p is even, one cannot rule out the appearance of flat singular points of density $Q = \frac{p}{2}$. A prototypical example is as follows.

Example 1.2.1 (cf. Figure 1.1 below)**.** Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the map $f \equiv 0$ and let $g : \mathbb{R}^2 \to \mathbb{R}$ be a (non-trivial) solution of the minimal surface equation

$$\operatorname{div}\left(\frac{\nabla g}{\sqrt{1+|\nabla g|^2}}\right) = 0,$$

with $g(0) = \nabla g(0) = 0$. Let T be the two-dimensional area-minimizing current mod(4) given by

$$T := [\![\operatorname{graph}(f) \cup \operatorname{graph}(g)]\!],$$

with alternating orientations in the regions between the intersections of the two graphs, to ensure that $[\partial T]_4 \llcorner \mathbf{B}_1 = 0$.

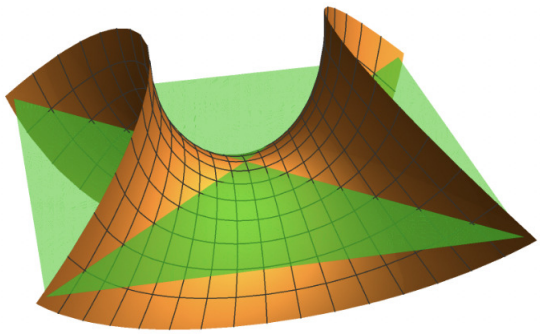


Figure 1.1: The graphs of f and g in Example 1.2.1.

The origin is an isolated flat singular point of density 2 here, meanwhile the curve segments $(\operatorname{graph}(f) \cap \operatorname{graph}(g)) \setminus \{0\}$ consist of singular points of density 2, at each of which there is a (unique) blow-up that is the union of 4 half-planes meeting in a line, with all of the orientations directed towards this line, to ensure that it is counted with multiplicity 4, and thus does not create a boundary mod(4).

In the recent works [40], [35] and [36], it was shown that at under the assumption that $\bar{n} = 1$ and with $p = 2Q$ for $Q \in \mathbb{N}$, at all flat singular Q-points of T the flat tangent cone is unique and there is a polynomial decay rate of T towards this flat tangent cone. In [40], the authors also establish Q-valued $C^{1,\alpha}$ multi-sheeted graphicality of T locally around such points. In [35], it is additionally shown that the Hausdorff dimension of $\mathfrak{F}_Q(T)$ is at most $m-2$, and up to a set of Hausdorff dimension at most $m-2$, the singular set of T is locally a $C^{1,\alpha}$ $(m-1)$-dimensional submanifold (cf. the above discussion in the case when p is odd). Note that these results heavily rely on the codimension one assumption, which allows one to classify the possible degrees of homogeneity of solutions to the linearized problem (see [35, Proposition 2.9]). Unlike in the case when p is odd, however, the authors in [35] were unable to easily establish $(m-2)$ rectifiability of the lower-dimensional part of the singular set, due to the presence of flat singularities.

1.3 Organization of results

In Chapter 2 of this thesis, we build on the works of Almgren and De Lellis & Spadaro to demonstrate that the Hausdorff dimension estimate can be sharpened to a local upper Minkowski dimension bound of $m-2$.[5]

In Chapter 4 we further improve this to $(m-2)$-rectifiability of the interior singular set, together with $\mathcal{H}^{m-2}$-a.e. uniqueness of tangent cones. The former is the natural analogue of the result of De Lellis, Spadaro & Spolaor in higher dimension, while the latter is the natural analogue of White's result in higher dimension.

In Chapter 5, we exploit the excess decay established in [36, 40] to demonstrate the $(m-2)$-rectifiability for the flat singular set of T when $\bar{n} = 1$ and p is even. Combined with the results mentioned above, this demonstrates that the interior singular set of T consists of a countable union of $C^{1,\alpha}$ $(m-1)$-dimensional embedded submanifolds, up to an $(m-2)$-

[5]The content of this chapter has been published in Communications of Pure and Applied Mathematics, in the article titled "An upper Minkowski dimension estimate for the interior singular set of area minimizing currents".

rectifiable set.[6]

1.4 Notation and preliminaries

m, n, Q will denote positive integers. Any additional assumptions on these will be clearly stated. We introduce the following list of frequently used notation throughout this thesis. In some of the chapters, we will further provide additional individual notation lists that are specific to that chapter.

$\mathcal{A}_Q(\mathbb{R}^k)$	the space of Q-tuples of vectors in $\mathbb{R}^k$ (see [18] for more details);
$\mathbf{B}_r(p)$	the $(m+n)$-dimensional Euclidean ball of radius r centered at p;
$\mathcal{B}_r(z)$	the geodesic ball of radius r centered at z on a given center manifold (see [21] for more details);
ω_m	the m-dimensional Hausdorff measure of the m-dimensional unit disk $B_1(\pi)$;
$\mathcal{H}^s$	the s-dimensional Hausdorff measure, $s \geq 0$;
d_H	the Hausdorff metric on the space of compact subsets of $\mathbb{R}^{m+n}$;
$\mathrm{W}^{k,p}(\Omega; \mathcal{A}_Q)$	the space of Q-valued p-integrable Sobolev maps with p-integrable distributional derivatives up to order $k \in \mathbb{N}$ on Ω;
π, ϖ	m-dimensional planes (i.e. affine subspaces) in $\mathbb{R}^{m+n}$;
$B_r(z, \pi)$	the m-dimensional Euclidean ball of radius r and center z in the m-dimensional plane π. If clear from context, center and/or plane omitted;
$E^\perp$	The orthogonal complement of E in $\mathbb{R}^{m+n}$
$\mathbf{C}_r(z, \pi)$	the infinite $(m+n)$-dimensional Euclidean cylinder $B_r(z, \pi) + \pi^\perp$

[6] The content of this chapter has been published in International Mathematical Research Notices, in the article titled "Rectifiability of Flat Singular Points for Area-Minimizing mod $2Q$ Hypercurrents".

with center z, radius r in direction $\pi^\perp$;

$\iota_{z,r}$ the scaling map $w \mapsto \dfrac{w-z}{r}$ around the center z;

τ_z the translation map $w \mapsto w+z$;

$f_\sharp$ the push-forward under the map f;

$E_{z,r}$ the rescaling $(\iota_{z,r})_\sharp E$ of the set E;

$T_{x,r}$ the pushforward $(\iota_{x,r})_\sharp T$ of a current T under the rescaling map $\iota_{x,r}$;

$T_p\mathcal{N}$ the tangent plane to the manifold $\mathcal{N}$ at the point $p \in \mathcal{N}$;

$\mathbf{T}_F$ the current $\sum_{i\in\mathbb{N}} \sum_{j=1}^{Q} (f_i^j)_\sharp [\![M_i]\!]$ induced by the push-forward of a Q-valued map $F : M \to \mathcal{A}_Q(\mathbb{R}^{m+n})$ on a Borel set $M \subset \mathbb{R}^m$ with decomposition $F|_{M_i} = \sum_{j=1}^{Q} [\![f_i^j]\!]$, $M = \sqcup_i M_i$ as in [20, Lemma 1.1] (see [20, Section 1.1] for a more detailed definition);

$E \Subset F$ The set E is compactly contained in the set F, namely, $\overline{E} \subset F$;

$A \simeq_\square B$ The quantities A and B are comparable, namely, $c_1 A \le B \le c_2 B$ for some positive constants c_1, c_2 dependent on the quantities $\square$;

$\Theta(T,p)$ the m-dimensional Hausdorff density of T at a given point p;

$\mathbf{p}_\pi$ the orthogonal projection to the m-plane $\pi \subset \mathbb{R}^{m+n}$;

$\mathcal{I}^m(X)$ the space of integral m-dimensional varifolds on X;

$\mathcal{F}(V_1,V_2)$ the flat distance between V_1 and V_2;

$\mathbf{F}(V_1,V_2)$ the varifold distance (induced by the weak-$*$ topology) between V_1 and V_2.

$\mathcal{F}_K^p(S,T)$ the flat distance modulo p between the m-dimensional integral flat chains S,T with compact support in K (see, for example [27, Section 4.2.26] or [15]);

$\partial^p[T]$ the boundary modulo p of $[T]$, defined by $\partial^p[T] \coloneqq [\partial T]$;

$\mathrm{spt}^p(T)$ the support mod p of $T \in \mathscr{R}_m$, defined by $\mathrm{spt}^p(T) := \bigcap_{S=T \bmod(p)} \mathrm{spt}S$;

We refer the reader to a standard text such as [27] or [44] for the relevant geometric measure theory background. For further details on Q-valued maps and related concepts, we recommend [18] and [20].

We will frequently be using the notion of *tilt excess* throughout this thesis. We recall the definition here.

Definition 1.4.1. *Given any m-dimensional plane π, denote by $\vec{\pi}$ the unit m-vector orienting this m-plane. Then for any cylinder $\mathbf{C}_r(x, \varpi)$, the tilt excess of T in $\mathbf{C}_r(x, \varpi)$ is defined by*

$$\mathbf{E}(T, \mathbf{C}_r(x, \varpi), \pi) := \frac{1}{2\omega_m r^m} \int_{\mathbf{C}_r(x,\varpi)} |\vec{T} - \vec{\pi}|^2 \, \mathrm{d}\|T\| \qquad \text{for any } m\text{-plane } \pi \subset \mathbb{R}^{m+n};$$

$$\mathbf{E}(T, \mathbf{C}_r(x, \varpi)) := \inf_{m\text{-planes } \pi} \mathbf{E}(T, \mathbf{C}_r(x, \varpi), \pi),$$

The definition of the tilt excess $\mathbf{E}(T, \mathbf{B}_r(x), \pi)$ in a ball $\mathbf{B}_r(x)$ with respect to a plane π and the optimal tilt excess $\mathbf{E}(T, \mathbf{B}_r(x))$ in such a ball are analogously defined.

Chapter 2

An upper Minkowski dimension estimate for the interior singular set of area minimizing currents

2.1 Introduction and Main Results

We will be working with the following core assumption throughout:

Assumption 2.1.1. T is an m-dimensional integral current in Σ, where Σ is an $(m+\bar{n})$-dimensional embedded submanifold of class C^{3,ε_0} in $\mathbb{R}^{m+n} \equiv \mathbb{R}^{m+\bar{n}+l}$, where $\varepsilon_0 \in (0,1)$ is a fixed constant. Assume that T is area-minimizing in Σ and that $\bar{n} \geq 2$.

In order to study the interior singularities of T, it will be convenient to stratify based on the number of dimensions of translational symmetry. Following Almgren (see also [54]), the set $\mathrm{spt}(T) \setminus \mathrm{spt}(\partial T)$ can be stratified through

$$\mathcal{S}^{(k)}(T) := \left\{ x \in \mathrm{spt}(T) \setminus \mathrm{spt}(\partial T) : \begin{smallmatrix} \text{any tangent cone of } T \text{ at } x \text{ has no more} \\ \text{than } k \text{ dimensions of translational invariance} \end{smallmatrix} \right\}, \tag{2.1}$$

where $k = 0, 1, \ldots, m$. In particular

$$\mathcal{S}^{(0)}(T) \subset \mathcal{S}^{(1)}(T) \subset \cdots \subset \mathcal{S}^{(m-1)}(T) \subset \mathcal{S}^{(m)}(T) = \operatorname{spt}(T) \setminus \operatorname{spt}(\partial T)\,.$$

Almgren's argument (which can be seen as a suitable generalization of Federer's reduction argument, cf. [28]) showed that

$$\dim_{\mathcal{H}}\left(\mathcal{S}^{(k)}(T)\right) \leq k\,.$$

In their recent groundbreaking work [43], Naber and Valtorta further proved that $\mathcal{S}^{(k)}(T)$ is k-rectifiable.

The results of this chapter are an initial step towards better understanding the problem of how to determine the structure of the singular set; in particular, the flat singular points $p \in \mathcal{S}^{(m)}(T) \setminus \mathcal{S}^{(m-1)}(T)$. Before we state our main result, let us introduce some common notation. We will use $\overline{\dim}_M(E)$ to denote the upper Minkowski dimension of a bounded set $E \subset \mathbb{R}^{m+n}$. Namely,

$$\overline{\dim}_M(E) := \inf\left\{ s \geq 0 \,:\, \limsup_{\delta \downarrow 0} \delta^s N(\delta, E) < \infty \right\},$$

where $N(\delta, E)$ denotes the smallest number of open balls of radius δ required to cover E.

For $Q \in \mathbb{N}$, we let

$$\operatorname{Sing}_{\geq Q}(T) := \operatorname{Sing}(T) \cap D_{\geq Q}(T)$$

where

$$D_{\geq Q}(T) := \{\, p \in \operatorname{spt}(T) \setminus \operatorname{spt}(\partial T) \,:\, \Theta(T, p) \geq Q \,\}\,.$$

When we are referring to singular points of density exactly Q, we will use the notation $\operatorname{Sing}_Q(T)$. We are now ready to state our main result, which is the following:

Theorem 2.1.2. *For T and Σ as in Assumption 2.1.1 and $Q \in \mathbb{N} \setminus \{0\}$,*

$$\overline{\dim}_M(\mathrm{Sing}_{\geq Q}(T) \cap \overline{\Omega}) \leq m - 2$$

for any $\Omega \Subset \mathbb{R}^{m+n} \setminus (\partial T \cup \mathrm{Sing}_{\geq Q+1}(T))$.

This provides a more refined dimension bound on the singular set than the existing $(m-2)$ Hausdorff dimension estimate due to Almgren [3]. However, we are not able to control the $(m-2)$-dimensional Minkowski content via the methods here; see Chapter 4 for partial progress in this direction.

Notice that in order to prove Theoreom 2.1.2, it suffices to establish the estimate

$$\overline{\dim}_M(\mathrm{Sing}_{[Q,Q+\varepsilon]}(T) \cap \overline{\Omega}) \leq m - 2, \qquad \text{for any } \varepsilon > 0,$$

where

$$\mathrm{Sing}_{[Q,Q+\varepsilon]}(T) := \mathrm{Sing}(T) \cap D_{[Q,Q+\varepsilon]}(T),$$

and

$$D_{[Q,Q+\varepsilon]}(T) := \{\, p \in \mathrm{spt}(T) \setminus \mathrm{spt}(\partial T) \,:\, Q \leq \Theta(T,p) \leq Q + \varepsilon \,\}.$$

This is because we have the following consequence of the quantitative stratification of Naber & Valtorta in [43]:

Lemma 2.1.3. *For any $\varepsilon > 0$, the following holds. For T and Σ as in Assumption 2.1.1 and $Q \in \mathbb{N} \setminus \{0\}$, the set*

$$\mathrm{Sing}_{\geq Q+\varepsilon}(T) \cap \overline{\Omega}$$

is $(m-3)$-rectifiable and has finite upper $(m-3)$-Minkowski content, for any $\Omega \Subset \mathbb{R}^{m+n} \setminus (\partial T \cup \mathrm{Sing}_{\geq Q+1}(T))$.

This allows us to ignore contribution from singularities of all non-integer densities above an arbitrarily small threshold. We defer the proof of this to Section 2.10. By virtue of the

Allard-Almgren ε-Regularity Theorem, in the case $Q = 1$ we have

$$\mathrm{Sing}_1(T) \subset \mathrm{Sing}_{\geq 1+\varepsilon}(T)$$

for any $\varepsilon > 0$. Thus, in the special case of singular points with multiplicities $Q \in [1, 2)$, we immediately deduce the following stronger result:

Proposition 2.1.4. *For any $\Omega \Subset \mathbb{R}^{m+n} \setminus (\partial T \cup \mathrm{Sing}_{\geq 2}(T))$, the set*

$$\mathrm{Sing}_{\geq 1}(T) \cap \overline{\Omega}$$

is $(m-3)$-rectifiable and has finite upper $(m-3)$-Minkowski content.

2.2 Key preliminary results

Before stating the important preliminary results needed to prove Theorem 2.2.3, we remind ourselves that the goal is to prove that for any given $\alpha > 0$, the upper $(m-2+\alpha)$-Minkowski content of $\mathrm{Sing}(T)$ vanishes:

$$\limsup_{r \downarrow 0} N(\mathrm{Sing}(T), r) r^{m-2+\alpha} = 0.$$

We argue by contradiction; suppose this is not true. As in the proof of the Hausdorff dimension bound, we would like to blow up around an arbitrary $\mathcal{H}^{m-2+\alpha}$-density point $p \in \mathrm{Sing}T$ and combine this with the existence of a flat tangent cone there to reach a contradiction. The problem is that the Minkowski contents do not behave desirably under Hausdorff convergence of sets, and in fact are not even measures so are not particularly easy to work with in general. The Hausdorff measure itself is also not necessarily lower-semicontinuous with respect to Hausdorff convergence, but to avoid issues with this one can relax it to the premeasure $\mathcal{H}^s_\infty$. Null sets are preserved under this replacement, so this does

not affect our contradiction argument.

Thus, we first reduce the problem of estimating the upper Minkowski content of $\mathrm{Sing}(T)$ to one of estimating the Hausdorff dimension. Of course, since the latter is a cruder dimension estimate, we pay the price of having to bound the Hausdorff dimension uniformly for every set in an appropriate compact family. A crucial tool that we will require is Almgren's Work Raccoon Lemma, which allows us to work with the Hausdorff measure instead of the Minkowski content. To the author's knowledge, this result is originally due to Almgren and a version of it is likely contained within [3], but first formally appeared in [53, Theorem 5.1]. However, in both of these references, it was used to establish Hausdorff dimension estimates for blow-up limits and as far as the author is aware, it has not yet been formally observed that this result also gives a Minkowski dimension bound.

Lemma 2.2.1 (Work Raccoon Lemma)**.** *Suppose that we have a non-trivial class $\mathscr{C}$ of compact subsets of $\overline{\mathbf{B}}_1 \subset \mathbb{R}^{m+n}$ such that*

(a) $\mathscr{C}$ is closed under rescalings, namely, for each $K \in \mathscr{C}$, $x \in \overline{\mathbf{B}}_1$ and $0 < r \leq 1$,

$$K_{x,r} \cap \overline{\mathbf{B}}_1 := (\iota_{x,r})_\sharp K \cap \overline{\mathbf{B}}_1 \in \mathscr{C},$$

(b) $\mathscr{C}$ is closed under Hausdorff convergence.

Then the family of exponents

$$\mathscr{A}(\mathscr{C}) := \{\, \alpha \geq 0 \,:\, \mathcal{H}^\alpha(K) = 0 \text{ for every } K \in \mathscr{C} \,\}$$

is an open half-line (α_0, ∞).

Moreover, for every $0 < r \leq 1$, the minimal number $N(K, r)$ of balls of radius r required to cover K satisfies

$$N(K, r) r^\alpha \leq C(\alpha) \qquad \text{for every } \alpha \in \mathscr{A}(\mathscr{C}). \tag{2.2}$$

We postpone the proof of this Lemma to Section 2.9, and will henceforth assume its validity.

We will henceforth fix an arbitrary set $\Omega \Subset \mathbb{R}^{m+n} \setminus (\partial T \cup \mathrm{Sing}_{\geq Q+1}(T))$. We now seek an appropriate family of sets to which the Work Raccoon Lemma applies. Recall that, as for the Hausdorff dimension estimate, we are interested in blowing up around points $x \in \mathrm{Sing}_{[Q,Q+\varepsilon]} T \cap \Omega$, in order to understand the size of this singular set. This motivates choosing the compact family of sets to be the smallest one that contains blow-up sequences of the singular set and is closed under Hausdorff convergence and rescalings of the form in (a) of the lemma. We formulate this more precisely as follows.

First of all, note that by finding a large enough ball containing Ω, we may, up to rescaling and translation, assume that $\Omega = \mathbf{B}_{1/2}$. Now, given $\varepsilon \in (0,1)$, let

$$A(\varepsilon) := \mathbf{B}_1 \setminus \mathrm{Sing}_{\geq Q+\varepsilon/2}(T). \tag{2.3}$$

By the upper-semicontinuity of the density, this is an open set, and so can be written as an (increasing) countable union of compact sets. Fix any one of these compact sets $\tilde{K} = \tilde{K}(\varepsilon)$ and let

$$K^0(\varepsilon, \tilde{K}) := \tilde{K}(\varepsilon) \cap \mathrm{Sing}_{\geq Q}(T).$$

Hence, in order to prove the Minkowski dimension estimate in Theorem 2.1.2, it suffices to establish it for $K^0(\varepsilon, \tilde{K})$, since $\overline{\mathbf{B}}_{1/2} \cap \mathrm{Sing}_{\geq Q+\varepsilon}(T)$ is a compact subset of $\mathbf{B}_1 \setminus \mathrm{Sing}_{\geq Q+\varepsilon/2}(T)$ and so is contained in one of these sets $\tilde{K}$. Our candidate for the family to which we wish to apply the Work Raccoon Lemma is then

$$\mathscr{C}(\varepsilon, \tilde{K}) := \left\{ K \text{ compact}, K \subset K^\infty : \begin{array}{c} K^0_{x_k, r_k}(\varepsilon, \tilde{K}) \cap \overline{\mathbf{B}}_1 \xrightarrow{d_H} K^\infty, \\ x_k \in \mathbb{R}^{m+n},\ r_k \in (0,1] \text{ convergent} \end{array} \right\}.$$

Indeed, we can say the following:

Proposition 2.2.2. *Given $\varepsilon \in (0,1)$ and $\tilde{K}(\varepsilon)$ as defined above, the collection $\mathscr{C}(\varepsilon, \tilde{K})$*

contains $K^0(\varepsilon, \tilde{K})$ *and satisfies the assumptions of Lemma* 2.2.1.

The proof of this Proposition is also contained in Section 2.9. It remains to show that α_0 as in Lemma 2.2.1 satisfies $\alpha_0 \leq m-2$ for this compact family $\mathscr{C}(\varepsilon, \tilde{K})$.

First of all, by extracting a subsequence, we may assume that given any $K \in \mathscr{C}(\varepsilon, \tilde{K})$, we have

$$x_k \to x \qquad \text{and} \qquad r_k \to r,$$

for the corresponding blow-up scales and centers.

Notice that if $r > 0$, then

$$K^\infty \subset \mathrm{Sing}(T_{x,r}) \cap \overline{\mathbf{B}}_1,$$

since $\overline{\mathbf{B}}_{r_k}(x_k) \xrightarrow{d_H} \overline{\mathbf{B}}_r(x)$. Thus in this case the existing dimension estimate due to Almgren [3] tells us that $\mathcal{H}^{m-2+\alpha}(K^\infty) = 0$ for each $\alpha > 0$. So we may assume that $r_k \downarrow 0$.

The main result Theorem 2.1.2 can now be concluded from the following:

Theorem 2.2.3. *There exists* $\varepsilon \in (0,1)$ *sufficiently small such that the following holds for any compact* $\tilde{K}(\varepsilon) \subset A(\varepsilon)$.

For any K^∞ *in the family* $\mathscr{C}(\varepsilon, \tilde{K})$ *of Proposition* 2.2.2*, we have the following dichotomy. Either*

(a) there exists an m*-dimensional area minimizing integral current* S *in* $\mathbb{R}^{m+n}$ *such that* $K^\infty \subset \mathrm{Sing}(S)$*;*

(b) there exists an m*-plane* $\pi_\infty \subset \mathbb{R}^{m+n}$ *and a non-trivial* Dir*-minimizer* $u : \pi_\infty \equiv \mathbb{R}^m \times \{0\} \supset B_{3/2} \to \mathcal{A}_Q$ *with*

$$K^\infty \subset \Delta_Q(u) := \left\{ z \in B_{3/2} \,:\, u(z) = Q[\![0]\!] \right\}.$$

In particular, applying Almgren's dimension estimate in case (a) *and* [18, Proposition 3.22]

in case (b), *we arrive at*

$$\mathcal{H}^{m-2+\alpha}(K^\infty) = 0 \qquad \textit{for every } \alpha > 0.$$

Unless otherwise mentioned, we will henceforth fix an arbitrary choice of $\varepsilon > 0$ and a compact set $\tilde{K}(\varepsilon) \subset A(\varepsilon)$, and will omit dependencies on ε and $\tilde{K}$ for all the sets defined above. The following section is dedicated to reducing ourselves to the second alternative (b) in Theorem 2.2.3, by characterizing the blow-up limit along a sequence with varying centers.

2.3 Reduction to Theorem 2.2.3(b)

We hope to use Almgren's center manifold construction to prove the above theorem. Unfortunately, in our reduction to merely establishing a uniform Hausdorff dimension bound of $m-2$ for our whole family of compact sets $\mathscr{C}$, we have been forced to include blow-ups with varying centers. Therefore it is necessary to check that singularities still persist along such diagonal blow-up sequences.

Firstly, for any $K^\infty \in \mathscr{C}$, we would like to ensure that the blow-up centers are singular Q-points of T, so that the sheets of our current are not collapsing around the centers. This will be important in order to ensure that our blow-up limit is non-trivial.

Proposition 2.3.1. *Let T and Σ be as in Assumption 2.1.1. Then for any $K^\infty \in \mathscr{C}$ as in Theorem 2.2.3, we may assume that the centers of the corresponding blow-ups $K^0_{x_k,r_k}$ satisfy*

$$x_k \in K^0.$$

Proof. Consider the original sequences of centers $x_k \in \mathbb{R}^{m+n}$ and blow-up scales $r_k \downarrow 0$. We may without loss of generality assume that (for k sufficiently large) $K^0 \cap \overline{\mathbf{B}}_{2r_k}(x_k) \neq \emptyset$, so we can find points $z_k \in K^0 \cap \overline{\mathbf{B}}_{2r_k}(x_k)$. But now we can instead consider the blow-ups

$K^0_{z_k,3r_k} \cap \overline{\mathbf{B}}_1$, the support of which contain $\mathrm{spt}(K^0_{x_k,r_k}) \cap \overline{\mathbf{B}}_1$. For k sufficiently large,

$$K^0 \cap \overline{\mathbf{B}}_{3r_k}(z_k) \subset A,$$

for $A = A(\varepsilon)$ as in (2.3). This completes the proof, after replacing $\tilde{K}$ by a slightly larger compact set contained in A if necessary. □

Notice that since K^0 is compact, Proposition 2.3.1 allows us to assume that for any $K \in \mathscr{C}$, the corresponding blow-up centers x_k satisfy $x_k \to x \in K^0$ up to subsequence. In particular, we have

$$Q \leq \Theta(T, x) \leq Q + \varepsilon. \tag{2.4}$$

Remark 2.3.2. Observe that (2.4) holds because we are blowing up a *compact* subset of $\mathrm{Sing}_{[Q,Q+\varepsilon]}(T)$. Without this, up to the author's knowledge, it remains unknown whether it is possible to rule out the situation that $\Theta(T, x_k) \leq Q + \varepsilon$ for each k, but $\Theta(T, x) > Q + \varepsilon$, even under the assumption that $x_k \in \mathrm{Sing}(T)$. We will see the importance of (2.4) when we investigate the structure of these blow-ups in the limit as $k \to \infty$.

We will henceforth assume the property (2.4) for any given $K \in \mathscr{C}$. Another key assumption that we impose throughout will be the following:

Assumption 2.3.3. Let T and Σ be as in Assumption 2.1.1. Given $K \in \mathscr{C}$ and $x = \lim_{k\to\infty} x_k$, we may assume that $\Sigma \cap \mathbf{B}_{7\sqrt{m}}(x)$ is the graph of a C^{3,ε_0} function $\Psi_p : \mathrm{T}_p\Sigma \cap \mathbf{B}_{7\sqrt{m}}(x) \to \mathrm{T}_p\Sigma^\perp$ for every $p \in \Sigma \cap \mathbf{B}_{7\sqrt{m}}(x)$. We may further assume that

$$\boldsymbol{c}(\Sigma) := \sup_{p \in \Sigma \cap \mathbf{B}_{7\sqrt{m}}(x)} \|D\Psi_p\|_{C^{2,\varepsilon_0}} \leq \bar{\varepsilon},$$

where $\bar{\varepsilon}$ will be determined later. This in particular gives us the following uniform control on the second fundamental form of Σ:

$$\mathbf{A} := \|A_\Sigma\|_{C^0(\Sigma)} \leq C_0 \boldsymbol{c}(\Sigma).$$

Unfortunately, due to the fact that the centers of the blow-ups are varying, we cannot necessarily deduce that the tangent current obtained in the blow-up limit is a cone; the monotonicity formula is only valid for a fixed center. Nevertheless, we are still able to deduce the following about the limit:

Proposition 2.3.4. *Suppose that T and Σ satisfy Assumption 2.1.1. Then we may choose $\varepsilon > 0$ sufficiently small such that the following holds for any $K^\infty \in \mathscr{C}(\varepsilon, \tilde{K})$ with corresponding blow-up centers $x_k \in K^0(\varepsilon, \tilde{K})$ and blow-up scales $r_k \downarrow 0$, satisfying Assumption 2.3.3 for $\bar{\varepsilon} = \bar{\varepsilon}(x) > 0$ small enough.*

There exists an m-dimensional area minimizing integral current S in $\mathbb{R}^{m+n}$, such that up to subsequence,

$$T_{x_k, r_k} \overset{*}{\rightharpoonup} S, \qquad \|T_{x_k, r_k}\|(\mathbf{B}) \longrightarrow \|S\|(\mathbf{B}) \qquad \text{for any open ball } \mathbf{B}, \tag{2.5}$$

and one of the following two alternatives holds:

(a) $K^\infty \subset \mathrm{Sing}_{\geq Q}(S)$;

(b) $S = Q[\![\pi_\infty]\!]$ for some m-dimensional plane π_∞.

Remark 2.3.5. An important consequence of this proposition is that the limiting set K^∞ is determined by the structure of a tangent current obtained as a weak-$*$ limit of the blow-ups. Indeed, we demonstrate in the proof that

$$K^\infty \subset D_{\geq Q}(S).$$

The first possibility is that the tangent current $\mathrm{Reg}_{\geq Q} S = \emptyset$, which leads to the conclusion of case (a). This case covers the first alternative in Theorem 2.2.3, where the singularities persist to the tangent S. The conclusion of the Minkowski dimension estimate in Theorem 2.1.2 then follows immediately from Almgren's dimension estimate [3].

In case (b), there is a flat tangent current. This can be considered as a first order approximation for T near x, which captures the regular behaviour of T there, meanwhile the singularities vanish in the limit. We thus need to show persistence of singularities at the level of a harmonic approximation to T at an infinitesimal scale around x. This approximation is constructed in such a way that the regular behaviour is absorbed into the domain, and so singularities do indeed persist.

Proof. First of all, by the definition of K^0, x is an interior point of T and (2.4) tells us that we can find $\delta > 0$ sufficiently small such that

$$\frac{\|T\|(\mathbf{B}_\delta(x))}{\omega_m \delta^m} \leq Q + 2\varepsilon. \tag{2.6}$$

We claim that for any open ball $\mathbf{B}_R \subset \mathbb{R}^{m+n}$ centered at the origin, we have

$$\sup_k \|T_{x_k,r_k}\|(\mathbf{B}_R) \leq Q + 3\varepsilon.$$

Indeed, due to the pointwise convergence of x_k to x,

$$\|T\|(\mathbf{B}_R(x_k)) \longrightarrow \|T\|(\mathbf{B}_R(x)).$$

for all but a countable collection of radii R. This can be easily verified by approximating the mass of T on balls via a pairing with appropriate m-forms, and exploiting the continuity property of T. Combining this with (2.6) and the fact that T is m-dimensional, and by using monotonicity formula around x_k when k is large enough such that $Rr_k < \delta$, we have

$$\frac{\|T_{x_k,r_k}\|(\mathbf{B}_R)}{\omega_m R^m} = \frac{T(\mathbf{B}_{Rr_k}(x_k))}{\omega_m (Rr_k)^m} \leq e^{C\mathbf{A}(\delta - Rr_k)} \frac{\|T\|(\mathbf{B}_\delta(x_k))}{\omega_m \delta^m} \leq \frac{\|T\|(\mathbf{B}_\delta(x))}{\omega_m \delta^m} + 3\varepsilon,$$

provided that we choose $\bar{\varepsilon}$ sufficiently small, only dependent on x.

Since x is an interior point for T, we may further choose k large enough such that

$$\partial T_{x_k,r_k} \llcorner \mathbf{B}_R = 0, \qquad \text{for } k \text{ sufficiently large.} \tag{2.7}$$

Thus, the Federer-Fleming Compactness Theorem (see, for example [44, Theorem 32.2]) for normal currents tells us that we can extract a subsequence and an m-dimensional area minimizing integral current S on $\mathbb{R}^{m+n}$ for which

$$T_{x_k,r_k} \overset{*}{\rightharpoonup} S \qquad \text{as Radon measures.}$$

Moreover, $\partial S = 0$, due to (2.7) and the continuity of the boundary operator with respect to weak-$*$ convergence.

To establish the convergence of masses, firstly observe that the lower-semicontinuity of the mass tells us that for any ball $\mathbf{B}$,

$$\|S\|(\mathbf{B}) \leq \liminf_{k\to\infty} \|T_{x_k,r_k}\|(\mathbf{B}).$$

Now suppose that for some ball $\mathbf{B}$ we have $\|S\|(\mathbf{B}) < \liminf_{k\to\infty} \|T_{x_k,r_k}\|(\mathbf{B})$. For k sufficiently large $\partial S \llcorner \mathbf{B} = \partial T_{x_k,r_k} \llcorner \mathbf{B} = 0$, so this contradicts the area minimizing property of T and all its rescalings (see). We can therefore extract a subsequence for which we indeed have

$$\|T_{x_k,r_k}\|(\mathbf{B}) \longrightarrow \|S\|(\mathbf{B}).$$

In particular, we deduce that for any $R > 0$,

$$\frac{\|S\|(\mathbf{B}_R)}{\omega_m R^m} \leq Q + 4\varepsilon. \tag{2.8}$$

Furthermore, one can show that

$$K^\infty \subset D_{\geq Q}(S). \tag{2.9}$$

Indeed, given any $y \in K^\infty$ and any $\rho > 0$, the local convergence of the masses (2.5) tells us that for any $\eta > 0$, we can find k sufficiently large such that

$$\begin{aligned}\frac{\|S\|(\mathbf{B}_\rho(y))}{\omega_m \rho^m} &\geq \frac{\|T_{x_k,r_k}\|(\mathbf{B}_\rho(y))}{\omega_m \rho^m} - \eta \\ &= \frac{\|T\|(x_k + \mathbf{B}_{\rho r_k}(r_k y))}{\omega_m (\rho r_k)^m} - \eta.\end{aligned}$$

Taking $\rho \downarrow 0$, we deduce that

$$\Theta(S, y) \geq \Theta(T, x_k + r_k y) - 2\eta \quad \text{for } k \text{ sufficiently large.}$$

But now, since $x_k + r_k y \to x$, we can exploit the upper-semicontinuity of the density to deduce that $\Theta(S, y) \geq Q - 2\eta$. Note that here we are using the fact that $y \in K^\infty$, which, by definition, means that for k sufficiently large, $\Theta(T, x_k + r_k y) \geq \eta$. Since η is arbitrary, we conclude that $\Theta(S, y) \geq Q$. However, we cannot deduce that $\Theta(S, y) \leq Q + c\varepsilon$ for some constant $c > 0$; the density might increase in the limit.

Thus, in order to try and determine the size of K^∞, it makes sense to investigate the structure of the set of high multiplicity points in $\mathrm{spt}(S)$. There are two possibilities: either there exists a regular point

$$z_0 \in \mathrm{Reg}_{\geq Q}\, S,$$

or all points in $D_{\geq Q}(S)$ are singular. If the latter holds then (2.9) tells us that we must necessarily have $K^\infty \subset \mathrm{Sing}_{\geq Q}(S)$ and so we are in case (a).

Suppose on the other hand that there exists a point $z_0 \in \mathrm{Reg}_{\geq Q}\, S$. First of all, let us assume that $\Theta(S, z_0) = Q$. We want to show that this implies S is flat. We plan to use Allard's Regularity Theorem [1, Section 8] to see that $S = Q[\![\Gamma]\!]$ for some smooth surface Γ, and that Γ is flat at both infinity and locally around z_0, so is in fact an m-dimensional plane. In order to see this, we will need to apply Allard to $\frac{S}{Q}$. However, we do not even

know that this object makes sense a priori. Thus, define

$$E := \{\, w \in \operatorname{Reg} S \,:\, \Theta(S, w) = Q \,\},$$

and let $\tilde{S} := S \llcorner E$. By definition, E is a $C^{1,\alpha}$-submanifold of Σ for some $\alpha \in (0,1)$. We proceed to show that $\tilde{S} = S$.

Notice that $\overline{E} \setminus E \subset \operatorname{Sing}(S)$. Indeed, given any $z \in \overline{E} \setminus E$, we know that $\Theta(S, z) \geq Q$ by upper-semicontinuity of the density. If $z \in \operatorname{Reg} S$, then in fact we could further conclude that $\Theta(S, z) = Q$, because there is an open neighbourhood of z on which the density agrees with $\Theta(S, z)$, but also $z_k \to z$ for some sequence $\{z_k\} \subset E$.

Again exploiting the dimension estimate [3] of Almgren on the singular set of S, we deduce that

$$\mathcal{H}^{m-2+\alpha}(\overline{E} \setminus E) = 0 \qquad \forall \alpha > 0.$$

But recall that $\partial S = 0$ and E is a smooth manifold, so

$$\operatorname{spt}(\partial \tilde{S}) = \operatorname{spt}(S \llcorner \partial E) \subset \overline{E} \setminus E,$$

and hence

$$\mathcal{H}^{m-2+\alpha}(\operatorname{spt}(\partial \tilde{S})) = 0 \qquad \forall \alpha > 0.$$

However, $\partial \tilde{S}$ is an $(m-1)$-dimensional integral flat chain, so by a well-known result of Federer (see [27, Theorem 4.2.14]), we deduce that

$$\partial \tilde{S} = 0 = \partial S,$$

and moreover $\tilde{S}$ is area minimizing, since S is. Now since $\Theta(\tilde{S}, \cdot) \equiv Q$, we can consider

$$\bar{S} := \frac{\tilde{S}}{Q}.$$

By virtue of (2.8), for any $R > 0$ we have

$$\begin{aligned} \frac{\|\bar{S}\|(\mathbf{B}_R(z_0))}{R^m} &\leq \frac{\|\bar{S}\|(\mathbf{B}_{R+|z_0|})}{R^m} \\ &\leq \frac{\omega_m(R+|z_0|)^m(Q+4\varepsilon)}{QR^m} \\ &= \omega_m\Big(1+\frac{|z_0|}{R}\Big)^m\Big(1+\frac{4\varepsilon}{Q}\Big). \end{aligned} \tag{2.10}$$

Hence, we have a uniform upper bound over $R > 0$ on the masses $\|\bar{S}_{z_0,R}\|(\mathbf{B}_1)$. When combining with the zero boundary condition, this allows us to again use the Federer-Fleming Compactness Theorem to extract a sequence $R_k \uparrow \infty$ and an m-dimensional area minimizing cone (with vertex at the origin) $\bar{S}_\infty$ such that $\bar{S}_{z_0,R_k} \overset{*}{\rightharpoonup} \bar{S}_\infty$ and $\|\bar{S}_{z_0,R_k}\|(\mathbf{B}_1) \longrightarrow \|\bar{S}_\infty\|(\mathbf{B}_1)$.

The fact that this *tangent at infinity* $\bar{S}_\infty$ is a cone follows from the monotonicity formula, in the same way as for a tangent at a fixed point.

By considering (2.10) at scales $R_k \uparrow \infty$, we have

$$\|\bar{S}_\infty\|(\mathbf{B}_1) \leq \omega_m\Big(1+\frac{4\varepsilon}{Q}\Big).$$

Since $\Theta(\bar{S}_\infty, \cdot) \equiv \Theta(\bar{S}, \cdot) \equiv 1$, we can choose $\varepsilon > 0$ sufficiently small in order to apply Allard's Regularity Theorem and deduce that $\bar{S}_\infty$ is a $C^{1,\alpha}$-graph locally near the origin. Since $\bar{S}_\infty$ is a cone, we conclude that

$$\bar{S}_\infty = [\![\pi]\!] \quad \text{for some } m\text{-plane } \pi.$$

Note that we can conclude this for $\bar{S}_\infty$ regardless of the center that we take for the blow-down limit.

Now we have two possibilities. If $\Theta(S, z_0) = Q$, we can combine this with the regularity of $\bar{S}$ near z_0 to deduce that in fact $\bar{S} = [\![\pi \cap E]\!]$ and hence $\tilde{S} = Q[\![\pi \cap E]\!]$. Indeed, since

$\|\bar{S}_\infty\|(\mathbf{B}_1) = \omega_m$, we have

$$\lim_{r\downarrow 0} \frac{\|\bar{S}\|(\mathbf{B}_r(z_0))}{\omega_m r^m} = 1 = \lim_{k\to\infty} \frac{\|\bar{S}\|(\mathbf{B}_{R_k}(z_0))}{\omega_m R_k^m}, \tag{2.11}$$

so the monotonicity formula allows us to conclude.

If instead $\Theta(S, z_0) \geq Q + 1$, then (2.11) instead becomes

$$\lim_{r\downarrow 0} \frac{\|\bar{S}\|(\mathbf{B}_r(z_0))}{\omega_m r^m} = \frac{Q+1}{Q} > 1 = \lim_{k\to\infty} \frac{\|\bar{S}\|(\mathbf{B}_{R_k}(z_0))}{\omega_m R_k^m}.$$

This, however, contradicts the monotonicity formula, so cannot occur.

It remains to check that $\tilde{S} = S$. Due to (2.8), we have

$$\|S - \tilde{S}\|(\mathbf{B}_R) \leq 4\varepsilon\omega_m R^m \qquad \text{for any } R > 0.$$

Thus, for any $y \in \operatorname{spt}(S - \tilde{S})$ and any $R > 0$, we have

$$\|S - \tilde{S}\|(\mathbf{B}_R(y)) \leq \omega_m \varepsilon (R + |y|)^m,$$

as well as

$$\|S - \tilde{S}\|(\mathbf{B}_R(y)) \geq \omega_m R^m,$$

simply because the density at any point in the support of an integral current is at least 1. Taking $R > |y|$ and then $\varepsilon < \frac{1}{2}$ (if this is not already the case), we indeed have $\tilde{S} = S$. □

We have therefore successfully reduced the problem to that of deducing that the second alternative of Theorem 2.2.3 holds true whenever we are in case (b) of Proposition 2.3.4. The remainder of this chapter will be dedicated to proving that this is indeed true.

Applying Propositions 2.3.1 and 2.3.4 to the sequences x_k and r_k, we can assume that the weak-$*$ limit S of our blow-up sequence is an m-plane with multiplicity Q. In summary, we henceforth make the following assumptions:

Assumption 2.3.6. Let T and Σ be as in Assumption 2.1.1 and let $\bar{\varepsilon}, \varepsilon > 0$ be given by Proposition 2.3.4. We assume that we have an arbitrary fixed set $K^\infty \in \mathscr{C}(\varepsilon, \tilde{K})$ satisfying Assumption 2.3.3. For this set K^∞, we have associated blow-up centers x_k, and scales $r_k \downarrow 0$ such that for any arbitrarily fixed $y \in K^\infty$, there is a sequence of points y_k with

(i) $x_k, x \in K^0, \qquad x_k \to x,$

(ii) $y_k \in K^0_{x_k,r_k} \cap \mathbf{B}_1, \qquad y_k \to y,$

(iii) $\mathbf{E}(T, \mathbf{B}_{r_k}(x_k)) = \mathbf{E}(T_{x_k,r_k}, \mathbf{B}_1) \longrightarrow 0 \qquad$ as $k \to \infty$.

The assumption (iii) is a consequence of the weak-$*$ convergence and the convergence of masses. Indeed, for any open $(m+n)$-dimensional ball B and any m-plane π, we have:

$$\begin{aligned}\mathbf{E}(T_{x_k,r_k}, \mathbf{B}, \pi) &= \frac{1}{2|\mathbf{B}|}\int_{\mathbf{B}} |\vec{T}_{x_k,r_k} - \vec{\pi}|^2 \, \mathrm{d}\|T_{x_k,r_k}\| \\ &= \frac{\|T_{x_k,r_k}\|(\mathbf{B})}{|\mathbf{B}|} - \frac{1}{|\mathbf{B}|}\int_{\mathbf{B}} \langle \vec{T}_{x_k,r_k}, \vec{\pi}\rangle \, \mathrm{d}\|T_{x_k,r_k}\|,\end{aligned}$$

and so, taking $k \to \infty$, we arrive at

$$\mathbf{E}(T_{x_k,r_k}, \mathbf{B}, \pi) \longrightarrow \mathbf{E}(S, \mathbf{B}, \pi) = \frac{\|S\|(\mathbf{B})}{|\mathbf{B}|} - \frac{1}{|\mathbf{B}|}\int_{\mathbf{B}} \langle \vec{S}, \vec{\pi}\rangle \, \mathrm{d}\|S\|. \tag{2.12}$$

Before we continue, we must first ensure that our setup will allow us to approximate our current well by a Q-valued map at the scales of the blow-up procedure (see [19]). In view of this, we will assume the following:

Assumption 2.3.7. Let T and Σ be as in Assumption 2.1.1, and suppose that Assumptions 2.3.6 and 2.3.3 hold. If we replace ε from Assumption 2.3.6 by $\min\{\varepsilon, \varepsilon_3\}$, then for each x_k we can find an m-plane π_k such that

$$\mathbf{p}^\sharp_{\pi_k} T \llcorner \mathbf{C}_{\frac{11\sqrt{m}r_k}{2}}(x_k, \pi_k) = Q[\![B_{\frac{11\sqrt{m}r_k}{2}}(x_k, \pi_k)]\!], \qquad \partial T \llcorner \mathbf{C}_{\frac{11\sqrt{m}r_k}{2}}(x_k, \pi_k) = 0.$$

From now on, we will work under Assumptions 2.1.1-2.3.7.

2.4 Overview of the remaining argument

Let us now discuss the overall approach towards proving the second alternative in Theorem 2.2.3. Our goal is to show that $K^\infty \subset \Delta_Q(u)$, where u is some Dir-minimizing Q-valued map defined over (an open ball in) an m-dimensional plane. We *would like* to argue as follows:

Step 1. Approximate our blow-up sequence T_{x_k,r_k} by 'almost' Dir-minimizing $W^{1,2}$-maps u_k parameterized over m-dimensional planes;

Step 2. renormalize and recenter the u_k appropriately so that, up to subsequence, we have strong convergence to some non-trivial Dir-minimizer u in L^2_{loc};

Step 3. Check if u is a viable candidate for the map in Theorem 2.2.3.

This proposed scheme, however, *does not work*. The main problem is that the limit u could potentially be trivial. Thus, the singularities at x_k and y_k will not have persisted in the limit. Such a phenomenon would occur whenever T has a prominent regular part of higher polynomial order coming into contact with its branched singular structure, locally around the blow-up centers. This regular behaviour would dominate in the limit, thus leading to disappearance of singularities.

We overcome this problem via the center manifold construction of Almgren, originally used to estimate the Hausdorff dimension of the singular set. Center manifolds provide a good replacement for m-planes as the objects on which we build our approximations in (1) of the above desired scheme. The reason for this is that the center manifolds capture the regular behaviour of T around the points x_k. We may then construct graphical approximations over the normal bundles of the center manifolds; these graphical approximations should only capture the singular behaviour of T locally.

We can then instead choose u to be an appropriately normalized limit of these graphs over the center manifolds. The singularities should now persist to this limit; uniform bounds on

the *frequency function* along our sequence of approximating graphs will allow us to conclude this. This will allow us to use the information about the size of the singular set for Dir-minimizers to achieve the claimed dimension bound.

Section 2.5 is dedicated to the set up of our sequence of center manifolds. In Section 2.6, we will then discuss the frequency function and its key properties, including the uniform bounds that are then proved in Section 2.7. We conclude with the final persistence of singularities argument in Section 2.8.

2.5 Almgren's Center Manifold Construction

2.5.1 The Refining Procedure

In order to build the sequence of center manifolds along the scales of our blow-up sequence, we require the refining procedure and Whitney decomposition from [21, Proposition 1.11] for each T_{x_k,r_k}, constructed via stopping time criteria based on the size of the excess and the height of T_{x_k,r_k}.

In view of [22], we will be taking this sequence over small intervals containing the scales r_k. These intervals will detect the scales at which the current T stops being sufficiently flat (and thus when a given center manifold no longer approximates T sufficiently well), allowing us to rescale and build a new center manifold and normal approximation which will approximate T better at the new scale. We restate these necessary preliminary results here for the convenience of the reader:

Fix $w \in \operatorname{spt}(T) \setminus \operatorname{spt}(\partial T)$ and $r < \frac{1}{6\sqrt{m}} \operatorname{dist}(w, \operatorname{spt}(\partial T))$. Define

$$T' := T_{w,r} \llcorner \mathbf{B}_{6\sqrt{m}}, \qquad \Sigma' := \iota_{w,r}(\Sigma).$$

We will later choose w to be our blow-up centers x_k and the corresponding r to be scales close to r_k.

Let $\mathscr{C} = \bigcup_{j\in\mathbb{N}} \mathscr{C}^j$ be the collection of all closed dyadic m-dimensional subcubes of $[-4,4]^m \subset \pi_0$. Namely, $\mathscr{C}^j$ consists of all cubes inside $[-4,4]^m$ of side length 2^{1-j} with vertices in $2^{1-j}\mathbb{Z}$. Given $L \in \mathscr{C}^k$, let $\ell(L) := 2^{1-j}$ denote the side length of L. We call $J \in \mathscr{C}^j$ an ancestor of L if $L \subset J$ and we call J a parent of L (or L a child of J) if furthermore $J \in \mathscr{C}^{j-1}$. We denote the center of L by x_L.

Given a cube $L \in \mathscr{C}$, Assumption 2.3.7 allows us to find $y_L \in \mathrm{Span}\{\pi_L\}^\perp$ such that

$$p_L := (x_L, y_L) \equiv x_L + y_L \in \mathrm{spt}(T').$$

Notice that the choice of y_L is not unique (since we assume $Q > 1$), but for each L we fix a choice.

Moreover, for any $M_0 > 0$, let $r_L := M_0\sqrt{m}\ell(L)$ and let $\mathbf{B}_L := \mathbf{B}_{64r_L}(p_L)$.

Let $\hat{\pi}_L$ be the optimal plane for the excess of T' in $\mathbf{B}_L$, namely

$$\mathbf{E}(T', \mathbf{B}_L) = \mathbf{E}(T', \mathbf{B}_L, \hat{\pi}_L).$$

Since we wish to remain to also remain within Σ when approximating T graphically, we further find a plane $\pi_L \subset T_{p_L}\Sigma$ such that

$$|\pi_L - \hat{\pi}_L| = \inf_{\pi\in T_{p_L}\Sigma} |\pi - \hat{\pi}_L|.$$

or equivalently, $\mathbf{E}(T', \mathbf{B}_L, \pi_L) = \inf_{\pi\in T_{p_L}\Sigma} \mathbf{E}(T', \mathbf{B}_L, \pi)$. The planes π_L will orient the cylinders in which we construct the local Lipschitz approximations from [19, Theorem 2.4]. This will be discussed in more detail in Section 2.5.2.

Fix $N_0 \in \mathbb{N}$. We now construct our refining procedure using a Whitney decomposition of $[-4,4]^m \subset \pi_0$ as follows. We first choose constants $\beta_2 = 4\delta_2$ as in [21, Assumption 1.8], and fix constants C_e and C_h to be determined later with the dependencies of [21, Assumption 1.9].

Given $j \geq N_0$, we set up three subfamilies of cubes $\mathscr{W}_e^j$, $\mathscr{W}_h^j$, $\mathscr{W}_n^j \subset \mathscr{W}$ and place cubes

$L \in \mathscr{C}$ into these families inductively, starting from $j = N_0$, as follows:

$$\text{Let } L \in \mathscr{W}_e^j \text{ if } \mathbf{E}(T', \mathbf{B}_L) > C_e \boldsymbol{m}_0 \ell(L)^{2-2\delta_2}; \tag{EX}$$

$$\text{Let } L \in \mathscr{W}_h^j \text{ if } L \notin \mathscr{W}_e^j \text{ and } \mathbf{h}(T', \mathbf{B}_L) > C_h \boldsymbol{m}_0^{\frac{1}{2m}} \ell(L)^{1+\beta_2}; \tag{HT}$$

$$\text{Let } L \in \mathscr{W}_n^j \text{ if } L \notin \mathscr{W}_e^j \cup \mathscr{W}_h^j \text{ but } L \text{ intersects an element of } \mathscr{W}^{j-1}. \tag{NN}$$

Here

$$\boldsymbol{m}_0 := \max\{\boldsymbol{c}(\Sigma)^2, \mathbf{E}(T', \mathbf{B}_{6\sqrt{m}}) \leq \varepsilon_2^2,$$

and ε_2 is a small positive constant to be determined later, with the dependencies of [21, Assumption 1.9]. If one of these conditions holds for L, then STOP refining. Define

$$\mathscr{W}^j := \mathscr{W}_e^j \cup \mathscr{W}_h^j \cup \mathscr{W}_n^j \subset \mathscr{C}^j \qquad \text{and} \qquad \mathscr{W} := \bigcup_{j \geq N_0} \mathscr{W}^j.$$

Otherwise, place L in the family $\mathscr{S}^j \subset \mathscr{C}^j$ and apply the above stopping conditions to the children cubes $L' \in \mathscr{C}^{j+1}$ of L. Furthermore, place all cubes $L \in \mathscr{C}^j$ with $j < N_0$ in $\mathscr{S} := \bigcup_{j \geq N_0} \mathscr{S}^j$. In other words, $\mathscr{S}$ is characterized by

$$J \in \mathscr{S} \qquad \Longleftrightarrow \qquad J \in \mathscr{C} \text{ and either } J \text{ has a child } L \in \mathscr{W} \text{ or } \ell(J) > 2^{1-N_0}.$$

Continue this refining procedure inductively.

Observe that given any $x \in [-4, 4]^m \subset \pi_0$, this refinement procedure either stops on some $L \in \mathscr{W}$ containing x, or we can continue refining indefinitely around x. We denote by Γ the set of points where the latter occurs:

$$\boldsymbol{\Gamma} := [-4, 4]^m \setminus \bigcup_{L \in \mathscr{W}} L = \bigcap_{j \geq N_0} \bigcup_{L \in \mathscr{S}^j} L.$$

Notice that for any $x \in \Gamma$, there exists a sequence of cubes $L_j \in \mathscr{S}^j$ with $x \in L_j$. Moreover,

we make the obvious but important observation that

$$L \in \mathscr{W}^j,\ j > N_0 \quad \Longrightarrow \quad \text{the parent cube of } L \text{ lies in } \mathscr{S}^{j-1}. \tag{2.13}$$

We recall the following result from [21], which tells us that the above construction indeed gives a Whitney decomposition.

Proposition 2.5.1 (Whitney decomposition, [21], Proposition 1.11)**.** *Provided that Assumptions 1.8 and 1.9 from [21] hold and ε_2 is sufficiently small, we can conclude that $(\mathbf{\Gamma}, \mathscr{W})$ is a Whitney decomposition of $[-4,4]^m \subset \pi_0$. More precisely,*

$$\mathbf{\Gamma} \cup \bigcup_{L \in \mathscr{W}} L = [-4,4]^m \text{ and } \mathbf{\Gamma} \text{ does not intersect any cube } L \in \mathscr{W}; \tag{w1}$$

$$\text{any pair of cubes } L,\ L' \in \mathscr{W} \text{ have disjoint interiors}; \tag{w2}$$

$$\text{if } L,\ L' \in \mathscr{W} \text{ and } L \cap L' \neq \emptyset, \text{ then } \frac{1}{2}\ell(L) \leq \ell(L') \leq 2\ell(L). \tag{w3}$$

These conditions further imply that

$$\operatorname{sep}(\mathbf{\Gamma}, L) := \inf\{\, |x-y| \,:\, x \in L, y \in \mathbf{\Gamma} \,\} \geq 2\ell(L) \quad \text{for every } L \in \mathscr{W}. \tag{2.14}$$

Moreover, for any choice of M_0 and N_0, there exists $C^* = C^*(M_0, N_0)$ such that whenever $C_e \geq C^*$ and $C_h \geq C^* C_e$, we have

$$\mathscr{W}^j = \emptyset \qquad \text{for every } j \leq N_0 + 6, \tag{2.15}$$

and

$$\mathbf{E}(T', \mathbf{B}_J) \leq C_e \boldsymbol{m}_0 \ell(J)^{2-2\delta_2}, \quad \mathbf{h}(T', \mathbf{B}_J) \leq C_h \boldsymbol{m}_0^{\frac{1}{2m}} \ell(J)^{1+\beta_2} \quad \text{for } J \in \mathscr{S}; \tag{2.16}$$

$$\mathbf{E}(T', \mathbf{B}_L) \leq C \boldsymbol{m}_0 \ell(L)^{2-2\delta_2}, \quad \mathbf{h}(T', \mathbf{B}_L) \leq C \boldsymbol{m}_0^{\frac{1}{2m}} \ell(L)^{1+\beta_2} \quad \text{for } L \in \mathscr{W}, \tag{2.17}$$

where $C = C(m, n, Q, \beta_2, \delta_2, M_0, N_0, C_e, C_h)$.

Note that we do not require an upper bound on the separation in terms of the size of the cube, despite the fact that this additional condition is usually included in the criteria for a Whitney decomposition. The choices of M_0 and N_0 will be determined by Propositions 2.5.10 and 2.5.12. When combined with the stopping conditions, observe that (2.17) tells us that for any $L \in \mathscr{W}$, the excess of T' over the ball $\mathbf{B}_L$ is comparable to $\boldsymbol{m}_0 \ell(L)^{2-2\delta_2}$ and the height is comparable to $\boldsymbol{m}_0^{\frac{1}{2m}} \ell(L)^{1+\beta_2}$.

The first two conditions (w1) and (w2) are an automatic consequence of the construction. The validity of the remaining conditions relies on the observation (2.13) mentioned above and a quantitative control [21, Proposition 4.1] on the tilting of the optimal planes for the excess. We omit the details, and instead refer the reader to [21, Section 4].

2.5.2 The Center Manifold $\mathcal{M}$

Given any cube $L \in \mathscr{S} \cup \mathscr{W}$, we wish to consider Almgren's strong Lipschitz approximation f_L from [19, Theorem 2.4] locally for $T' \llcorner \mathbf{C}_{32r_L}(p_L, \pi_L)$. Let us recall this result here (with the proof omitted), for the benefit of the reader.

Theorem 2.5.2 (Almgren's strong approximation, [19], Theorem 2.4)**.** *There exist constants* $C, \gamma_1, \varepsilon_1 > 0$ *(depending on* $m, n, \bar{n}, Q$*) with the following property. Assume that* T' *is area minimizing, satisfies* [19, Assumption 2.1] *in the cylinder* $\mathbf{C}_{4r}(p)$ *and* $E = \mathbf{E}(T', \mathbf{C}_{4r}(p), \pi) < \varepsilon_1$ *for some* m*-plane* $\pi \subset \mathbb{R}^{m+n}$*. Then, there is a map* $f : \pi \supset \mathcal{B}_r(p) \to \mathcal{A}_Q(\pi^\perp)$ *with* $\mathrm{spt}(f) \subset \Sigma$ *and a closed set* $K \subset B_r(p)$ *such that*

$$
\begin{aligned}
&\mathrm{Lip}(f) \leq C E^{\gamma_1},\\
&\mathbf{G}_f \llcorner (K \times \pi^\perp) = T' \llcorner (K \times \pi^\perp) \quad \text{and} \quad |B_r(p) \setminus K| \leq C E^{\gamma_1} (E + r^2 \mathbf{A}^2) r^m,\\
&\left| \|T'\|(\mathbf{C}_{\sigma r}(p)) - Q\omega_m (\sigma r)^m - \frac{1}{2} \int_{B_{\sigma r}(p)} |Df|^2 \right| \leq C E^{\gamma_1} (E + r^2 \mathbf{A}^2) r^m \quad \forall \sigma \in (0,1).
\end{aligned}
$$

If in addition $\boldsymbol{h}(T', \mathbf{C}_{4r}(p), \pi) := \sup\left\{ |\mathbf{p}^{\perp}(w) - \mathbf{p}^{\perp}(z)| \,:\, w, z \in \operatorname{spt}(T') \cap \mathbf{C}_{4r}(p) \right\} \leq r$, *then*

$$\operatorname{osc}(f) \leq C\mathbf{h}(T', \mathbf{C}_{4r}(p), \pi) + C(E^{1/2} + r\mathbf{A})r. \tag{2.18}$$

In addition, let us recall that provided the excess is sufficiently small, there is the following persistence of Q-points from the current to the strong Lipschitz approximation f in Theorem 2.5.2.

Theorem 2.5.3 (Persistence of Q-points I, [19], Theorem 2.7)**.** *For every* $\hat{\delta}, C^* > 0$, *there is* $\bar{s} \in \left(0, \frac{1}{2}\right)$ *such that, for every* $s < \bar{s}$, *there exists* $\hat{\varepsilon}(s, C^*, \hat{\delta}) > 0$ *with the following property. If* T' *is as in Theorem 2.5.2,* $E := \mathbf{E}(T', \mathbf{C}_{4r}(p), \pi) < \hat{\varepsilon}$, $r^2\mathbf{A}^2 \leq C^*E$ *and* $\Theta(T, (w_0, z_0)) = Q$ *for some* $(w_0, z_0) \in (\pi \times \pi^{\perp}) \cap \mathbf{C}_{\frac{r}{2}}(p, \pi)$, *then the approximation* f *of Theorem 2.5.2 satisfies*

$$\int_{B_{sr}(w_0)} \mathcal{G}(f, Q\llbracket \boldsymbol{\eta} \circ f \rrbracket)^2 \leq \hat{\delta} s^m r^{m+2} E. \tag{2.19}$$

We omit the proof here, but we make the following important observation, that will be useful at a later stage. Notice that (2.19) can in fact be improved to the height bound

$$\int_{B_{sr}(w_0)} \mathcal{G}(f, Q\llbracket \boldsymbol{\eta} \circ f \rrbracket)^2 \leq C s^{m+\gamma} r^{m+2} E, \tag{2.20}$$

where $\gamma > 0$ is a geometric constant given by [50, Lemma 5.2] and C is independent of s. This improved estimate is in fact established the alternative proof [50, Section 5.3, proof of Proposition 5.1], which crucially uses the improved height bound [50, Theorem 1.5].

From now on, we refer to such an approximation f_L as a π_L-*approximation*. In order to construct the π_L-approximations, the assumptions of Theorem 2.5.2 must be satisfied. Namely, we require that $\mathbf{E}(T', \mathbf{C}_{32r_L}(p_L, \pi_L))$ remains below a given small threshold, independently of the choice of L.

Let $\rho_\lambda := \lambda^{-m}\rho(\frac{\cdot}{\lambda})$, $\lambda > 0$, denote the L^1-invariant rescaling of a smooth bump function $\rho \in C_c^\infty(B_1(0, \pi_L))$ with $\int \rho = 1$ and $\int |z|^2 \rho(z)\, \mathrm{d}z = 0$. Define the *average of the sheets* $\boldsymbol{\eta} \circ f$

of a Q-valued map $f = \sum_i [\![f_i]\!]$ to be $\boldsymbol{\eta} \circ f := \frac{1}{Q} \sum_i f_i$. Then we have the following:

Lemma 2.5.4 ([21], Lemma 1.15)**.** *Let the assumptions of Proposition 2.5.1 hold, and assume $C_e \geq C^*$ and $C_h \geq C^* C_e$. For any choice of the other parameters, if ε_2 is sufficiently small, then the current $T' \llcorner \mathbf{C}_{32r_L}(p_L, \pi_L)$ satisfies the assumptions of Theorem 2.5.2 for any $L \in \mathscr{W} \cup \mathscr{S}$.*

Moreover, if $f_L : B_{8r_L}(p_L, \pi_L) \to \mathcal{A}_Q(\pi_L^\perp)$ is a π_L-approximation, denote by $\hat{h}_L : B_{7r_L}(p_L, \pi_L) \to \pi_L^\perp$ its smoothed average $\hat{h}_L := (\boldsymbol{\eta} \circ f_L) * \rho_{\ell(L)}$ and by $\bar{h}_L$ the map $\mathbf{p}_{T_{p_L}\Sigma}(\hat{h}_L)$, which takes values in the plane $\varpi_L := T_{p_L}\Sigma \cap \pi_L^\perp$. If we let h_L be the map $z \mapsto h_L(z) := (\bar{h}_L(z), \boldsymbol{\Psi}_{p_L}(x, \bar{h}_L(z))) \in \varpi_L \times T_{p_L}\Sigma^\perp$, then there is a smooth map $g_L : B_{4r_L}(p_L, \pi_0) \to \pi_0^\perp$ such that $\mathbf{G}_{g_L} = \mathbf{G}_{h_L} \llcorner \mathbf{C}_{4r_L}(p_L, \pi_0)$.

We refer to the maps h_L and g_L as *tilted L-interpolating functions* and *L-interpolating functions*, respectively. Thus, the π_L-approximations are indeed well-defined for all cubes $L \in \mathscr{W} \cup \mathscr{S}$, provided that we choose ε_2 sufficiently small (dependent on ε_1).

We omit the proof of Lemma 2.5.4; see [21] for this. Roughly speaking, the latter part of this Lemma tells us that for each cube $L \in \mathscr{W} \cup \mathscr{S}$, we can take the π_L-approximation f_L and average the sheets, smooth out and correct the image to lie within Σ, to give a smooth map that can be reparameterized over the original reference plane π_0. This is possible due to the control [21, Proposition 4.1] over the tilting of the optimal planes; we refer the reader to [21, Lemma B.1, Proposition 4.2] for the details.

We are now ready to patch together these L-interpolating functions over $\mathscr{P}^j := \mathscr{S}^j \cup \bigcup_{i=N_0}^j \mathscr{W}^i$ for each fixed level j in our partition of $[-4, 4]^m \subset \pi_0$. More precisely, let $\vartheta \in C_c^\infty\left(\left[-\frac{17}{16}, \frac{17}{16}\right]\right)$ and given any $L \in \mathscr{P}^j$, define $\vartheta_L := \vartheta(\frac{\cdot - x_L}{\ell(L)})$. Then take the corresponding partition of unity function

$$\hat{\vartheta}_{L,j} := \frac{\vartheta_L}{\sum_{L \in \mathscr{P}^j} \vartheta_L}.$$

This is well-defined and indeed forms a partition of unity over $L \in \mathscr{P}^j$, because of the nature

of the cubes L. Now consider

$$\hat{\varphi}_j := \sum_{L \in \mathscr{P}^j} \hat{\vartheta}_{L,j} g_L \qquad \text{on } (-4,4)^m \subset \pi_0.$$

We have almost finished constructing the center manifold; it remains to once again correct these interpolations $\hat{\varphi}_j$ to ensure that the images remain within Σ. To be more precise, we let $\bar{\varphi}_j$ be the function constructed from the $\bar{n}$ components of $\hat{\varphi}_j$ that live within $T_0\Sigma \cap \pi_0^\perp$ and we define the *glued interpolation*

$$\varphi_j(y) := (\bar{\varphi}_j(y), \mathbf{\Psi}_0(y, \bar{\varphi}_j(y))), \qquad y \in (-4,4)^m \subset \pi_0.$$

This iterative construction over $j \in \mathbb{N}$ enables us to get compactness in the C^3-topology; leading to the existence of the center manifold. This is summarized as follows:

Proposition 2.5.5 (Existence of the center manifold, [21], Theorem 1.17)**.** *Assume that the hypotheses of Lemma 2.5.4 hold and let* $\kappa := \min\{\frac{\varepsilon_0}{2}, \frac{\beta_2}{4}\}$. *Provided that* ε_2 *is sufficiently small, for any choice of the other parameters we have*

(i) $\|D\varphi_j\|_{C^{2,\kappa}} \leq C\boldsymbol{m}_0^{1/2}$ *and* $\|\varphi_j\|_{C^0} \leq C\boldsymbol{m}_0^{\frac{1}{2m}}$ *for* $C = C(m, n, Q, \beta_2, \delta_2, M_0, N_0, C_e, C_h)$*;*

(ii) if $L \in \mathscr{W}^i$ *and* H *is a cube concentric to* L *with* $\ell(H) = \frac{9}{8}\ell(L)$*, then* $\varphi_j \equiv \varphi_k$ *on* H *for any* $j, k \geq i + 2$*;*

(iii) $\varphi_j \xrightarrow{C^3} \boldsymbol{\varphi}$*, and* $\mathcal{M} := \mathrm{Gr}(\boldsymbol{\varphi}|_{(-4,4)^m})$ *is a* $C^{3,\kappa}$*-submanifold of* Σ*.*

See [21, Section 4.4] for a proof of this. The general idea is that the operations of smoothing out and averaging the sheets preserve the 'almost harmonicity' of the graphical approximations for T', resulting in uniform elliptic estimates over $j \in \mathbb{N}$ for *single-valued* maps over π_0. Note that the optimal regularity of the center manifold is governed by the regularity of Σ. We only require $C^{3,\kappa}$-regularity in order to get the necessary variational estimates for the $\mathcal{M}$-normal approximations appearing in Section 2.5.3; this was noticed

by De Lellis & Spadaro in their simplified proof of Almgren's dimension estimate[3]. In conclusion, we may define:

Definition 2.5.6 (center manifold)**.** *We call $\mathcal{M}$ the center manifold for T' relative to π_0. Letting $\mathbf{\Phi}(y) := (y, \boldsymbol{\varphi}(y))$, we define the* contact set *to be $\mathbf{\Phi}(\mathbf{\Gamma})$, where the pair $(\mathbf{\Gamma}, \mathscr{W})$ is the Whitney decomposition associated to $\mathcal{M}$.*

Intuitively, the contact set is the part of $\mathcal{M}$ constructed above the places in $[-4,4]^m$ where we are able to refine indefinitely as a result of both excess and the height remaining sufficiently small at every scale. On the contact set, we expect that

$$T' \llcorner \mathbf{\Phi}(\Gamma) \equiv Q[\![\mathcal{M}]\!] \llcorner \mathbf{\Phi}(\Gamma).$$

Given $L \in \mathscr{W}$, let H be the cube concentric to L with $\ell(H) = \frac{17}{16}\ell(L)$, and define

$$\mathcal{L} = \mathcal{L}(L) := \mathbf{\Phi}\left(H \cap [-7/2, 7/2]^m\right)$$

to be the *Whitney region* of L on $\mathcal{M}$.

2.5.3 The $\mathcal{M}$-Normal Approximations

Now that we have constructed our center manifold $\mathcal{M}$, we need to better approximate T' in those regions where $\mathcal{M}$ is not a sufficiently good approximation for T'; namely, away from the contact set. We do this by building an approximating graph for T' over the normal bundle of the center manifold. This is what we will refer to as an *$\mathcal{M}$-normal approximation*. Before we state the precise definition, we need to introduce the notion of an orthogonal projection map to $\mathcal{M}$:

Assumption 2.5.7. Let $\mathbf{U}$ be the 1-tubular neighbourhood of $\mathcal{M}$, namely

$$\mathbf{U} := \left\{ z \in \mathbb{R}^{m+n} \,:\, \exists!\ w =: \mathbf{p}(z) \in \mathcal{M} \text{ with } |z-w| < 1 \text{ and } z - w \in (T_w\mathcal{M})^\perp \right\}.$$

Then, for any choice of the other parameters, we choose ε_2 to be small enough such that $\mathbf{p} : \mathbf{U} \to \mathcal{M}$ extends to a $C^{2,\kappa}$ map on $\overline{\mathbf{U}}$ and $\mathbf{p}^{-1}(w) = w + \overline{(B_1(0, (T_w\mathcal{M})^\perp))}$ for every $w \in \mathcal{M}$. For each $w \in \mathcal{M}$, $\mathbf{p}^\perp(w, \cdot) : \mathbb{R}^{m+n} \to \mathbb{R}^{m+n}$ will denote the orthogonal projection onto $(T_x\mathcal{M})^\perp$.

Definition 2.5.8. *An $\mathcal{M}$-normal approximation of T' is a pair $(\mathcal{K}, F)$ such that $F : \mathcal{M} \to \mathcal{A}_Q(\mathbf{U})$ is Lipschitz and has the form*

$$F(z) = \sum_{i=1}^{Q} [\![z + N_i(z)]\!], \quad N_i(z) \in (T_z\mathcal{M})^\perp \text{ and } z + N_i(z) \in \Sigma \text{ for every } z \text{ and } i,$$

and $\mathcal{K} \subset \mathcal{M}$ is closed, contains $\mathbf{\Phi}\Big(\mathbf{\Gamma} \cap \Big[-\frac{7}{2}, \frac{7}{2}\Big]^m\Big)$ and $\mathbf{T}_F \llcorner \mathbf{p}^{-1}(\mathcal{K}) = T \llcorner \mathbf{p}^{-1}(\mathcal{K})$.

We will often abuse notation by referring to the *normal part* of F

$$N = \sum_i [\![N_i]\!] : \mathcal{M} \to \mathcal{A}_Q(\mathbb{R}^{m+n})$$

as the $\mathcal{M}$-normal approximation.

We want to see that it is possible to construct an $\mathcal{M}$-normal approximation for T' in a way that the approximation still behaves well outside of the contact set. We want to have *local estimates* on our $\mathcal{M}$-normal approximations on each Whitney region $\mathcal{L}$ in terms of the size of the corresponding cube L. Indeed this can be done by again using the properties of the π_L-approximations f_L from Theorem 2.5.2. We recall the estimates here, for the benefit of the reader:

Theorem 2.5.9 ($\mathcal{M}$-normal approximation)**.** *Let $\gamma_2 := \frac{\gamma_1}{4}$, where γ_1 is as in Theorem 2.5.2. Then, under the hypotheses of* [21, Theorem 1.17]*, for ε_2 sufficiently small there exists an $\mathcal{M}$-normal approximation $(\mathcal{K}, F)$ such that the following estimates hold on every Whitney region $\mathcal{L}$ associated to a cube $L \in \mathscr{W}$, with constants $C = C(m, n, Q, \beta_2, \delta_2, M_0, N_0, C_e, C_h)$:*

$$\mathrm{Lip}(N|_{\mathcal{L}}) \leq C\boldsymbol{m}_0^{\gamma_2}\ell(L)^{\gamma_2}, \qquad \|N|_{\mathcal{L}}\|_{C^0} \leq C\boldsymbol{m}_0^{\frac{1}{2m}}\ell(L)^{1+\beta_2}, \tag{2.21}$$

$$|\mathcal{L} \setminus \mathcal{K}| + \|\mathbf{T}_F - T'\|(\mathbf{p}^{-1}(\mathcal{L})) \leq C\boldsymbol{m}_0^{1+\gamma_2}\ell(L)^{m+2+\gamma_2}, \tag{2.22}$$

$$\int_{\mathcal{L}} |DN|^2 \leq C\boldsymbol{m}_0\ell(L)^{m+2-2\delta_2}. \tag{2.23}$$

Moreover, given any $a > 0$ and any Borel subset $\mathcal{V} \subset \mathcal{L}$, we have

$$\int_{\mathcal{V}} |\boldsymbol{\eta} \circ N| \leq C\boldsymbol{m}_0\big(\ell(L)^{m+3+\frac{\beta}{3}} + a\ell(L)^{2+\frac{\gamma_2}{2}}|\mathcal{V}|\big) + \frac{C}{a}\int_{\mathcal{V}} \mathcal{G}\big(N, Q[\![\boldsymbol{\eta} \circ N]\!]\big)^{2+\gamma_2}. \tag{2.24}$$

For this $\mathcal{M}$-normal approximation, the set $\mathcal{K}$ consists of those points in $\mathcal{M}$ above which (with respect to $\mathbf{p}$) T is graphical.

The main idea behind the proof of this theorem is to show that T can be approximated by a Lipschitz map that comes from the graphical approximation in Theorem 2.5.2, but is reparameterized over $\mathcal{M}$. This is again first done locally, over the Whitney regions $\mathcal{L}$ of the cubes L; see [20, Theorem 5.1]. One can easily check the compatibility of these local graphical approximations over the region $\mathcal{K}$ where they agree with T'. To construct F away from $\mathcal{K}$, we follow these steps:

Step 1. First extend F to a Lipschitz map (with little increase in the Lipschitz constant) $\bar{F}$ taking values in $\mathcal{A}_Q(\mathbf{U})$, using [18, Theorem 1.7];

Step 2. then modify $\bar{F}$ to take the form $\hat{F}(z) = \sum_i [\![z + \hat{N}_i(z)]\!]$ with $\hat{N}_i(z) \in (T_z\mathcal{M})^\perp$ for each z, by replacing $\bar{N}(z)$ with the projection

$$\hat{N}(z) = \sum_i [\![\mathbf{p}^\perp(z, \bar{N}_i(z))]\!]$$

to the normal bundle;

Step 3. finally, let $\varpi_z := T_z\Sigma \cap (T_z\mathcal{M})^\perp$ and replace each component $\hat{N}_i(z)$ with $\mathbf{p}_{\varpi_z}(\hat{N}_i(z))$ to reach the desired extension $F(z) = \sum_i [\![z + N_i(z)]\!]$ with $N_i(z) \in (T_z\mathcal{M})^\perp$ and $z + N_i(z) \in \Sigma$ for each z.

When combined with the verification of numerous technical details, this scheme indeed gives the result of Theorem 2.5.9; see [21, Section 6.2] for a full proof.

2.5.4 Properties of Center Manifolds

Let us now recall some important properties of center manifolds from [21] that will be useful in the succeeding sections. We omit the proofs here.

We begin with the following *splitting before tilting* phenomenon, which tells us that whenever we stop our refining procedure on a cube $L \in \mathscr{W}_e$, the lower excess bound passes through to the $\mathcal{M}$-normal approximation N and is comparable to the Dirichlet energy of N.

Proposition 2.5.10 (Splitting before tilting, [21], Proposition 3.4)**.** *There are functions $C_1 = C_1(\delta_2)$, $C_2 = C_2(M_0, \delta_2)$ such that if $M_0 \geq C_1$, $C_e \geq C_2$, if the hypotheses of Theorem 2.5.9 hold and if ε_2 is chosen sufficiently small, then the following holds.*

Suppose that for some center manifold $\mathcal{M}$ with corresponding rescaled current T', we find a cube $L \in \mathscr{W}_e$, $q \in \pi_0$ with $\operatorname{dist}(L, q) \leq 4\sqrt{m}\ell(L)$ and $\Omega := \mathbf{\Phi}(B_{\ell(L)/4}(q, \pi_0))$. Then

$$C_e \boldsymbol{m}_0 \ell(L)^{m+2-2\delta_2} \leq \ell(L)^m \mathbf{E}(T', \mathbf{B}_L) \leq C \int_\Omega |DN|^2,$$
$$\int_{\mathcal{L}} |DN|^2 \leq C\ell(L)^m \mathbf{E}(T', \mathbf{B}_L) \leq C_3 \ell(L)^{-2} \int_\Omega |N|^2.$$

Here, the constants C and C_3 are dependent on $\beta_2, \delta_2, M_0, N_0, C_e, C_h$.

We will further require the following persistence of Q-points result from a current T' to the $\mathcal{M}$-normal approximation when we stop refining due to large excess.

Proposition 2.5.11 (Persistence of Q-points II, [21, Proposition 3.6])**.** *Assume the hypotheses of Proposition 2.5.10 hold. For every $\eta_2 > 0$, there are constants $\bar{s}, \bar{\ell} > 0$, depending upon η_2, β_2, δ_2, M_0, N_0, C_e and C_h, such that, if ε_2 is sufficiently small, then the following*

holds. If $L \in \mathscr{W}_e$, $\ell(L) \le \bar{\ell}$, $\Theta(T', p) = Q$ *and* $\operatorname{dist}(\mathbf{p}_{\pi_0}(p), L) \le 4\sqrt{m}\ell(L)$, *then*

$$(\bar{s}\ell(L))^{-m} \int_{\mathcal{B}_{\bar{s}\ell(L)}(\mathbf{p}(p))} \mathcal{G}\left(N, Q[\![\boldsymbol{\eta} \circ N]\!]\right)^2 \le \frac{\eta_2}{\ell(L)^{m-2}} \int_{\mathcal{B}_{\ell(L)}(\mathbf{p}(p))} |DN|^2.$$

Another vital consequence of the center manifold construction is the following result. It tells us that if we are able to immediately restart a new center manifold after stopping the previous one (because the excess has remained sufficiently small), then we can compare the two center manifolds.

Proposition 2.5.12 (Comparison of center manifolds, [21], Proposition 3.7)**.** *There is a geometric constant* C_0 *and a constant* $\bar{c}_s = \bar{c}_s(\beta_2, \delta_2, M_0, N_0, C_e, C_h) > 0$ *with the following property.*

Assume the hypotheses of Proposition 2.5.10, $N_0 \ge C_0$ *and* ε_2 *is sufficiently small. If for some* $\bar{r} \in (0, 1)$,

(a) $\ell(L) \le c_s\rho$ *for every* $\rho > \bar{r}$ *and every* $L \in \mathscr{W}$ *with* $L \cap B_\rho(0, \pi_0) \neq \emptyset$;

(b) $\mathbf{E}(T', \mathbf{B}_{6\sqrt{m}\rho}) < \varepsilon_2$ *for every* $\rho > \bar{r}$;

(c) there is a cube $L \in \mathscr{W}$ *with* $\ell(L) \ge c_s\bar{r}$ *and* $L \cap \overline{B}_{\bar{r}}(0, \pi_0) \neq \emptyset$,

then

(i) the current $T'' := (T')_{0,\bar{r}} \llcorner \mathbf{B}_{6\sqrt{m}}$ *and the submanifold* $\Sigma'' := \iota_{0,\bar{r}}(\Sigma') \cap \mathbf{B}_{7\sqrt{m}}$ *satisfy all of the necessary assumptions for Proposition 2.5.1 for some new plane* $\tilde{\pi}_0$ *in place of* π_0;

(ii) for the center manifold $\widetilde{\mathcal{M}}$ *of* T'' *relative to* $\tilde{\pi}_0$ *and the* $\widetilde{\mathcal{M}}$*-normal approximation* $\widetilde{N}$, *we have*

$$\int_{\widetilde{\mathcal{M}} \cap \mathbf{B}_2} |\widetilde{N}|^2 \ge \bar{c}_s \max\{\mathbf{E}(T'', \mathbf{B}_{6\sqrt{m}}), \boldsymbol{c}(\Sigma'')^2\}. \tag{2.25}$$

Remark 2.5.13. Let us also recall the height bound [21, Theorem A.1], which tells us that for T' satisfying all of the previous assumptions, as long as $E := \mathbf{E}(T', \mathbf{C}_r(w, \pi)) < \varepsilon$ for ε sufficiently small and some m-plane π, one can decompose T' into $k \geq 1$ pairwise disjoint horizontal 'strips' parallel to π over $B_r(w, \pi)$, each one of height $rC_0E^{\frac{1}{2m}}$.

Furthermore, if $\Theta(T', w) \geq Q$ then we necessarily have $k = 1$, and so this realizes as the following height bound for T':

$$\mathbf{h}(T', \mathbf{C}_r(w, \pi)) \leq CE^{\frac{1}{2m}} r^{1+\beta_2} \qquad \text{for any } r > 0 \text{ sufficiently small.}$$

In fact, the improved height bound [50, Theorem 1.5] tells us that around any point $w \in \mathrm{spt}(T')$ with $\Theta(T', w) \geq Q$, there is the sharp height bound

$$\mathbf{h}(T', \mathbf{C}_r(w, \pi)) \leq CE^{1/2} r + C\mathbf{A}r^2 \qquad \text{for any } r > 0 \text{ sufficiently small.} \tag{2.26}$$

Since this result uses the Hardt-Simon inequality, note that the improved height bound only holds around points of density Q (or higher). This allows us to improve the height bound in Theorem 2.5.9 to the following:

Lemma 2.5.14. *Suppose that T' satisfies all of the prior assumptions with ε_2 sufficiently small, with corresponding center manifold $\mathcal{M}$ and $\mathcal{M}$-normal approximation N. Then for any $w \in \mathrm{spt}(T')$ with $\Theta(T', w) \geq Q$ and any $L \in \mathscr{W}_e$ with $\mathrm{dist}(\mathbf{p}_{\pi_0}(\mathbf{p}(w)), L) \leq 4\sqrt{m}\ell(L)$, we have*

$$\int_{\mathcal{L}} |N|^2 \leq C\ell(L)^{m+4-2\delta_2} \boldsymbol{m}_0, \tag{2.27}$$

for any $L \in \mathscr{W}$.

This Lemma will be used in the final contradiction argument within the proof of the lower frequency bound of Theorem 2.6.8.

Proof. The proof relies on passing the improved height bound (2.26) for the current T' through to the graphical approximations.

By virtue of a scaled version of the estimate [50, Lemma 1.8] that gives the improved height bound (2.26), for $L \in \mathscr{W} \cup \mathscr{S}$ we have

$$\mathbf{h}(T', \mathbf{C}_{32r_L}(p_L, \pi_L)) \leq Cr_L \mathbf{E}(T', \mathbf{C}_{32r_L}(p_L, \pi_L))^{1/2} + Cr_L^2 \mathbf{A} \leq C\boldsymbol{m}_0^{1/2} r_L^{2-\delta_2},$$

as long as we choose ε_3 sufficiently small. Via the oscillation bound (2.18) in Theorem 2.5.2 for the graphical approximations f_L, this in turn gives

$$\operatorname{osc}(f_L) \leq C\boldsymbol{m}_0^{1/2} r_L^{2-\delta_2} \qquad \text{on } B_{8r_L}(x_L, \pi_L).$$

We may now substitute this improved bound into the computations for the estimates on the $\mathcal{M}$-normal approximations in [21, Section 6] to conclude the desired improvement on the existing C^0 and L^2 estimates on the $\mathcal{M}$-normal approximations in Theorem 2.5.9. We omit the details. □

2.5.5 Intervals of Flattening

We are now ready to take a sequence of center manifolds and corresponding $\mathcal{M}$-normal approximations along our blow-up sequence of currents T_{x_k, r_k}. We begin by introducing *intervals of flattening* around the blow-up centers x_k. These will contain the scales r_k, but will be chosen to detect the scales at which we need to replace an existing graphical approximation with a new, improved one.

Given any fixed $k \in \mathbb{N}$, and $\varepsilon_3 \in (0, \varepsilon_2)$, set

$$\mathcal{R}_k := \left\{ r \in (0,1] \,:\, \mathbf{E}(T, \mathbf{B}_{6\sqrt{m}r}(x_k)) \leq \varepsilon_3^2 \right\}.$$

Notice that $\mathcal{R}_k$ is closed under left-hand limits. Now construct a family $\mathcal{F}_k = \{I_j^{(k)}\}_j$ of intervals $I_j^{(k)} = \left(s_j^{(k)}, t_j^{(k)}\right]$ inductively as follows. Let $t_0^{(k)}$ be the largest element of $\mathcal{R}_k$.

Given $t_j^{(k)}$, define

$$T_{k,j} := T_{x_k,t_j^{(k)}} \llcorner \mathbf{B}_{6\sqrt{m}}, \quad \Sigma_{k,j} := \iota_{x_k,t_j^{(k)}}(\Sigma) \cap \mathbf{B}_{7\sqrt{m}}.$$

Let $\pi_{k,j}$ be the m-plane satisfying

$$\mathbf{E}(T_{k,j}, \mathbf{B}_{6\sqrt{m}}, \pi_{k,j}) = \mathbf{E}(T_{k,j}, \mathbf{B}_{6\sqrt{m}})$$

for each k, j. Namely, $\pi_{k,j}$ is the plane π_0 for the rescaled current $T' = T_{k,j}$.

Let $\mathcal{M}_{k,j}$ be the center manifold from Section 2.5.2 for $T' = T_{k,j}$ and $\Sigma = \Sigma_{k,j}$ with respect to the m-plane $\pi_{k,j}$. Let $N^{(k,j)}$ be the corresponding $\mathcal{M}_{k,j}$-normal approximation.

We denote by $\boldsymbol{\varphi}_{k,j} : \pi_{k,j} \supset [-4,4]^m \to \pi_{k,j}^{\perp}$ the map from Proposition 2.5.5, whose graph is $\mathcal{M}_{k,j}$, meanwhile $\boldsymbol{\Phi}_{k,j}(z) := (x, \boldsymbol{\varphi}_{k,j}(z))$ is the corresponding parameterization of $\mathcal{M}_{k,j}$ over $\pi_{k,j}$. We use the usual notation $\mathbf{p}_{\pi_{k,j}}$ for the orthogonal projection to the plane $\pi_{k,j}$, meanwhile $\mathbf{p}_{k,j}$ (or simply $\mathbf{p}$ when there is no ambiguity) will be the projection map from Assumption 2.5.7 to $\mathcal{M}_{k,j}$. We will use the notation $\mathcal{B}$ for the geodesic balls on our center manifolds.

Consider the Whitney decomposition $\mathscr{W}^{(k,j)}$ of bad cubes from Proposition 2.5.1 for $T_{k,j}$ and define

$$s_j^{(k)} := t_j^{(k)} \max\Big(\big\{ c_s^{-1}\ell(L) : L \in \mathscr{W}^{(k,j)} \text{ and } c_s^{-1}\ell(L) \geq \operatorname{dist}(0,L) \big\} \cup \{0\} \Big),$$

where $c_s := \frac{1}{64\sqrt{m}}$.

Thus, for the center manifold $\mathcal{M}_{k,j}$, we call $(s_j^{(k)}, t_j^{(k)}]$ the (centered) interval of flattening around x_k.

If $s_j^{(k)} = 0$, then stop refining. Otherwise, let $t_{j+1}^{(k)}$ be largest element of $\mathcal{R}_k \cap \big(0, s_j^{(k)}\big]$, and

continue as above. By construction, our stopping conditions are characterized as follows:

$$\text{If } s_j^{(k)} > 0 \text{ and } \bar{r}_j^{(k)} := \frac{s_j^{(k)}}{t_j^{(k)}}, \text{ then there is a cube } L \in \mathscr{W}^{(k,j)} \text{ with} \tag{Stop}$$

$$\ell(L) = c_s \bar{r}_j^{(k)} \quad \text{and} \quad L \cap \overline{B}_{\bar{r}_j^{(k)}}(0, \pi_{k,j}) \neq \emptyset.$$

$$\text{If } \rho > \bar{r}_j^{(k)} = \frac{s_j^{(k)}}{t_j^{(k)}}, \text{ then} \tag{Go}$$

$$\ell(L) < c_s \rho \quad \text{for every } L \in \mathscr{W}^{(k,j)} \text{ with } L \cap B_\rho(0, \pi_{k,j}) \neq \emptyset.$$

Moreover, letting

$$\boldsymbol{m}_0^{(k,j)} := \max\{\boldsymbol{c}(\Sigma_{k,j})^2, \mathbf{E}(T_{k,j}, \mathbf{B}_{6\sqrt{m}})\},$$

we recall the following additional vital properties of the intervals of flattening.

Proposition 2.5.15 ([22], Proposition 2.2)**.** *Assuming $\varepsilon_3 > 0$ is sufficiently small, the following holds for any centered interval of flattening on any center manifold $\mathcal{M}_{k,j}$:*

(i) $s_j^{(k)} < \frac{t_j^{(k)}}{2^5}$ *and each family $\mathcal{F}_k$ is either countable with $t_j^{(k)} \downarrow 0$ as $j \to \infty$, or finite with $s_j^{(k)} = 0$ for the largest j;*

(ii) $\bigcup_j I_j^{(k)} = \mathcal{R}_k$*, and for each k sufficiently large, we can find an interval $\big(s_{j(k)}^{(k)}, t_{j(k)}^{(k)}\big]$ around x_k for which*

$$r_k \in \Big(\frac{s_j^{(k)}}{t_j^{(k)}}, 1\Big];$$

(iii) if $r \in \big(\frac{s_j^{(k)}}{t_j^{(k)}}, 3\big)$ and $J \in \mathscr{W}_n^{(k,j)}$ intersects $B := \mathbf{p}_{\pi_{k,j}}\big(\mathcal{B}_r(\mathbf{\Phi}_{k,j}(0))\big)$, then J is in the domain of influence $\mathscr{W}_n^{(k,j)}(H)$ (see [21, Definition 3.3]) of a cube $H \in \mathscr{W}_e^{(k,j)}$, with

$$\ell(H) \leq 3c_s r, \qquad \max\{\mathrm{dist}(H, B), \mathrm{dist}(H, J)\} \leq 3\sqrt{m}\ell(H) \leq \frac{3r}{16};$$

(iv) $\mathbf{E}(T_{k,j}; \mathbf{B}_r) \leq C_0 \varepsilon_3^2 r^{2-2\delta_2}$ *for every $r \in \big(\frac{s_j^{(k)}}{t_j^{(k)}}, 3\big)$;*

(v) $\sup\{\operatorname{dist}(x,\mathcal{M}_{k,j}) : x \in \operatorname{spt}(T_{k,j}) \cap \mathbf{p}_{k,j}^{-1}\big(\mathcal{B}_r(\mathbf{\Phi}_{k,j}(0))\big)\} \leq C_0[\boldsymbol{m}_0^{(k,j)}]^{\frac{1}{2m}} r^{1+\beta_2}$ *for every* $r \in \big(\frac{s_j^{(k)}}{t_j^{(k)}}, 3\big)$.

In particular, (i) tells us that $I_j^{(k)} \neq \emptyset$ for each k, j. We will need analogous stopping criteria for scales around other points $w \neq x_k$. In particular, we will be interested in considering the points $w = y_k$, when we show persistence of Q-points. We hence introduce the following definition:

Definition 2.5.16 (Non-centered intervals of flattening)**.** *Let $\mathcal{M}$ be a center manifold with a given centered interval of flattening $(s,t]$ and corresponding rescaled current T'. Let $w \in \operatorname{spt}(T') \cap \big(\mathbf{B}_{6\sqrt{m}} \setminus \{0\}\big)$. Then, let $\tilde{t} = \tilde{t}(w)$ be the largest element of*

$$\big\{ r \in \big(0, \operatorname{dist}(w, \partial\mathbf{B}_t)\big] : \mathbf{E}(T', \mathbf{B}_{6\sqrt{m}r}(w)) \leq \varepsilon_3^2 \big\},$$

and let

$$\tilde{s} = \tilde{s}(w) := \tilde{t} \max\Big(\big\{ c_s^{-1}\ell(L) : L \in \mathscr{W} \text{ and } c_s^{-1}\ell(L) \geq \operatorname{dist}(\mathbf{p}(w), L) \big\} \cup \{0\} \Big),$$

we call $(\tilde{s}, \tilde{t}]$ the (non-centered) interval of flattening around w (corresponding to $(s,t]$).

Observe that conditions (Stop) and (Go) still hold for any such non-centered interval of flattening, only for balls centered at $\mathbf{p}_{\pi_{k,j}}(w)$. Moreover, the appropriate analogues of the conclusions of Proposition 2.5.15 hold, but we will not use this so we do not discuss this in detail here.

In many of the following arguments, we will be taking diagonal sequences of center manifolds and intervals of flattening around varying centers. Thus, it will be beneficial to simplify notation as follows.

Whenever we have a diagonal sequence of blow-ups $T_{k,j(k)}$ for T, $\mathcal{M}_{k,j(k)}$ is a diagonal sequence of center manifolds with corresponding $\mathcal{M}_{k,j(k)}$-normal approximations $N^{(k,j(k))}$,

centered flattening intervals $(s^{(k)}_{j(k)}, t^{(k)}_{j(k)}]$ and rescaled currents $T_{k,j(k)}$, we will use the notation

$$T_k := T_{k,j(k)}, \qquad \mathcal{M}_k := \mathcal{M}_{k,j(k)}, \qquad N^{(k)} := N^{(k,j(k))}, \qquad \Sigma_k := \Sigma_{k,j(k)},$$

for the blow-ups of T and the associated center manifolds, normal approximations and rescaled ambient manifolds. We will use the notation

$$s_k := s^{(k)}_{j(k)}, \qquad t_k := t^{(k)}_{j(k)},$$

for the endpoints of the interval of flattening corresponding to $\mathcal{M}_k$, and we let

$$\boldsymbol{m}^{(k)}_0 := \boldsymbol{m}^{(k,j(k))}_0, \qquad \boldsymbol{\Phi}_k := \boldsymbol{\Phi}_{k,j(k)}, \qquad \boldsymbol{\varphi}_k := \boldsymbol{\varphi}_{k,j(k)}, \qquad \mathbf{p}_k := \mathbf{p}_{k,j(k)}.$$

We will also use the notation

$$\pi_k := \pi_{k,j(k)}, \qquad \mathscr{W}^{(k)} := \mathscr{W}^{(k,j(k))}, \qquad \mathscr{S}^{(k)} := \mathscr{S}^{(k,j(k))},$$

when discussing the Whitney decompositions for $\mathcal{M}_k$. We are now in a position to introduce the frequency function and investigate its properties around Q-points of T.

Remark 2.5.17. We will make frequent use of the fact that the defining cube L for the stopping condition (Stop) will necessarily arise either because of (EX) or (NN) when the intervals of flattening are being constructed around x_k. Indeed, as already observed in Remark 2.5.13, the property $\Theta(T, x_k) \geq Q$ combined with the height bound [21, Theorem A.1] if one chooses the constant C_h appropriately, it must necessarily be the case that $k = 1$ in [21, Theorem A.1]. Thus, indeed (HT) never occurs.

2.6 The Frequency Function

As previously mentioned, this section is dedicated to uniformly bounding the frequency function for our $\mathcal{M}_{k,j}$-normal approximations $N^{(k,j)}$. Morally, we expect the frequency function to capture the 'dominant frequency of oscillation' for each map $N^{(k,j)}$ at a prescribed scale around $\mathbf{p}_{k,j}(x_k)$.

Before continuing, let us recall the ultimate goal. We would like to take a sequence of center manifolds $\mathcal{M}_k$ approximating our current T at scales r_k around x_k. We wish to show that the corresponding $\mathcal{M}_k$-normal approximations $N^{(k)}$ (after normalizing appropriately) converge to the graph of a non-trivial Dir-minimizer u in a sufficiently strong sense to ensure persistence of Q-points. This will in turn enable us to conclude the alternative (b) in Theorem 2.2.3. In order to conclude that the limit u is non-trivial and that singularities persist, we will need to establish both upper and lower uniform bounds on the frequency function.

We begin by recalling Almgren's frequency function for (multi-valued) Dir-minimizers on an open subset of $\mathbb{R}^m$; this, along with its key properties, will play an important role in the proof of Theorem 2.6.8 and the concluding persistence of Q-points argument in Section 2.8.

Let $\Omega \subset \mathbb{R}^m$ be an open subset, and let $u : \Omega \to \mathcal{A}_Q$. For $r \in (0, \operatorname{dist}(y, \partial\Omega))$, we define

$$D_u(y,r) := \int_{B_r(y)} |Du|^2, \quad H_u(y,r) := \int_{\partial B_r(y)} |u|^2, \quad I_u(y,r) := \frac{rD_u(y,r)}{H_u(y,r)}. \tag{2.28}$$

Note that these quantities are well-defined for Dir-minimizers; see [18, Remark 3.14]. Moreover, the frequency $r \mapsto I_u(y,r)$ is monotone non-decreasing in r, so the limit

$$I_0(y) = \lim_{r \downarrow 0} I_u(y,r)$$

is well-defined. We refer the reader to [18, Section 3.4] for further details on the frequency function for Dir-minimizers and its properties.

A particularly important property of Dir-minimizers is the way their L^2-height scales on

balls centered around Q-points:

Proposition 2.6.1. *Suppose that $\Omega \subset \mathbb{R}^m$ open and that $u \in W^{1,2}(\Omega; \mathcal{A}_Q)$ is* Dir*-minimizing. If for some $y \in \Omega$, $u(y) = Q[\![0]\!]$ and $I_0 := \lim_{r\downarrow 0} I_u(y,r)$ denotes the frequency of u at y, then $I_0 > 0$ and for every $\rho < r < \operatorname{dist}(y, \partial\Omega)$, we have*

$$\int_{B_\rho(y)} |u|^2 \leq \left(\frac{\rho}{r}\right)^{m+2I_0} \int_{B_r(y)} |u|^2. \tag{2.29}$$

Conversely, if for some $y \in \Omega$ and some $c > 0$ one of the following alternatives holds:

(a)

$$\int_{B_\rho(y)} |u|^2 \leq C\rho^{m+2c} \qquad \text{for every } \rho < \operatorname{dist}(y, \partial\Omega) \text{ sufficiently small;}$$

(b) $I_0 \geq c$,

then $u(y) = Q[\![0]\!]$.

Remark 2.6.2. In fact, the alternatives (a) and (b) are equivalent; this can easily be checked via a contradiction argument.

The proof of this can be inferred from [18, Corollary 3.18], but we repeat it at the end of this section for convenience.

Definition 2.6.3 (Almgren's frequency function)**.** *Let $\mathcal{M}$ be a center manifold with corresponding $\mathcal{M}$-normal approximation N and rescaled current T', let $w \in \operatorname{spt}(T')$, and let $(s,t]$ be an interval of flattening around w.*

Let $r \in \left(\frac{s}{t}, R\right]$ be an arbitrary scale. Consider a Lipschitz cutoff $\phi : [0,\infty) \to [0,1]$ that is identically 1 on $[0, \frac{1}{2}]$, and vanishes on $[1,\infty)$. For the rest of this thesis we will fix a

convenient specific choice of ϕ, given by

$$\phi(t) = \begin{cases} 1 & \text{for } 0 \le t \le \frac{1}{2} \\ 2 - 2t & \text{for } \frac{1}{2} \le t \le 1 \\ 0 & \text{otherwise}. \end{cases} \tag{2.30}$$

Define

$$\begin{aligned} \mathbf{D}_N(\mathbf{p}(w), r) &:= \int_{\mathcal{M}} \phi\left(\frac{d(p - \mathbf{p}(w))}{r}\right) |DN|^2(p)\, \mathrm{d}p, \\ \mathbf{H}_N(\mathbf{p}(w), r) &:= -\int_{\mathcal{M}} \phi'\left(\frac{d(p - \mathbf{p}(w))}{r}\right) \frac{|N|^2(p)}{d(p - \mathbf{p}(w))}\, \mathrm{d}p, \\ \mathbf{I}_N(\mathbf{p}(w), r) &:= \frac{r\mathbf{D}_N(\mathbf{p}(w), r)}{\mathbf{H}_N(\mathbf{p}(w), r)}, \end{aligned}$$

where $d(q)$ denotes the geodesic distance between q and $\mathbf{\Phi}(0)$ on $\mathcal{M}$.

It will later be convenient to use this regularized version of Almgren's frequency function for Dir-minimizers also. Given a multi-valued Dir-minimizer u a point x in its domain Ω, and a scale $r \in (0, \operatorname{dist}(x, \partial\Omega))$, we let $\mathbf{I}_u(x, r)$ denote the regularized frequency of u at scale r centered at x, defined via the same Lipschitz cutoff as in Definition 2.6.3. One may easily check that for Dir-minimizers u, $r \mapsto e^{Cr}\mathbf{I}_u(x, r)$ is monotone non-decreasing for a suitable choice of $C = C(m) > 0$. In particular, we may define $\mathbf{I}_u(x, 0) := \lim_{r\downarrow 0} \mathbf{I}_u(x, r)$.

Let

$$\mathbf{\Omega}_N(\mathbf{p}(w), r) = \max\{\log \mathbf{I}_N(\mathbf{p}(w), r), \log c_0\},$$

for c_0 to be determined later. Observe that the above functions are regularized versions of the analogous functions defined for Q-valued harmonic maps (see (2.28)), and so it is easy to check that they are absolutely continuous.

When it is clear from context, we will omit N and/or w from the notation for the above quantities. Moreover, whenever we take center manifolds $\mathcal{M}_{k,j}$ at varying scales $[s_j^{(k)}, t_j^{(k)}]$

with varying centers x_k, we will let

$$\mathbf{D}_{k,j}(r) := \mathbf{D}_{N^{(k,j)}}(\mathbf{p}_{k,j}(x_k), r), \quad \mathbf{H}_{k,j}(r) := \mathbf{H}_{N^{(k,j)}}(\mathbf{p}_{k,j}(x_k), r), \quad \mathbf{I}_{k,j}(r) := \frac{r\mathbf{D}_{k,j}(r)}{\mathbf{H}_{k,j}(r)}.$$

For diagonal blow-up sequences as discussed in Section 2.5.5 with varying centers $\mathbf{p}_k(w_k) := \mathbf{p}_{k,j(k)}(w_k)$ (where w_k are not necessarily x_k) we will let

$$\mathbf{D}_k(r) := \mathbf{D}_{N^{(k,j(k))}}(\mathbf{p}_k(w_k), r), \quad \mathbf{H}_k(r) := \mathbf{H}_{N^{(k,j(k))}}(\mathbf{p}_k(w_k), r), \quad \mathbf{I}_k(r) := \frac{r\mathbf{D}_k(r)}{\mathbf{H}_k(r)}. \quad (2.31)$$

We use the analogous notation for $\mathbf{\Omega}$ also, when necessary.

Before we continue, we henceforth fix the following useful notation. Given an arbitrary center manifold $\mathcal{M} = \mathcal{M}_{k,j}$ with a corresponding rescaled current $T' = T_{x_k, t_j^{(k)}}$ and a point $w \in \operatorname{spt}(T')$, we define

$$R = R(w) := \operatorname{dist}(\mathbf{p}(w), \partial B_3). \quad (2.32)$$

This regularized frequency function exhibits a convenient *almost monotonicity* property, which can be compared with the monotonicity [18, Theorem 3.15] of the frequency for Dir-minimizers:

Theorem 2.6.4. *There exists a bounded function $f : [0, \infty) \to \mathbb{R}$ with $\lim_{r \downarrow 0} f(r) = 0$ such that the following holds.*

Suppose that we have a center manifold $\mathcal{M} = \mathcal{M}_{k,j}$ with a corresponding rescaled current $T' = T_{x_k, t_j^{(k)}}$, $\mathcal{M}$-normal approximation $N = N^{(k,j)}$ and an interval of flattening $(s, t]$ around any center $\mathbf{p}(w)$. Then

$$\mathbf{\Omega}_N(\mathbf{p}(w), a) \leq \mathbf{\Omega}_N(\mathbf{p}(w), b) + f(b) \qquad \text{for every } [a, b] \subset \left[\frac{s}{t}, R\right].$$

Remark 2.6.5. Note that the function f is *independent* of the choice of center manifold, choice of point around which we take the interval, and the interval itself. In fact, it will

become evident in the proof that one can take $f(r) = Cr^\gamma$ for some $\gamma = \gamma(m, n, Q) > 0$, and $C = C(m, n, Q, c_0) > 0$. Moreover, observe that this almost-monotonicity estimate tells us that

$$\mathbf{I}_N(\mathbf{p}(w), a) \leq e^{f(b)} \max\{c_0, \mathbf{I}_N(\mathbf{p}(w), b)\}. \tag{2.33}$$

The proof of this theorem is the same as that of [22, Theorem 3.2], with merely a sharper final estimate. Nevertheless, we reproduce the argument at the end of this section.

As explained above, we would firstly like to show uniform upper bounds on the frequencies $\mathbf{I}_{k,j}$, independently of the center:

Theorem 2.6.6. *There exists $J \in \mathbb{N}$ such that the following holds.*

For the center manifolds $\mathcal{M}_{k,j}$ with corresponding normal approximations $N^{(k,j)}$, rescaled current $T_{k,j} = T_{x_k, t_j^{(k)}}$ and centered flattening intervals $\left(s_j^{(k)}, t_j^{(k)}\right]$ around x_k, we have

$$\sup_{k,j} \sup_{r \in \left(\frac{s_j^{(k)}}{t_j^{(k)}}, \frac{3}{2^{J-1}}\right]} \mathbf{I}_{k,j}(r) < +\infty, \tag{2.34}$$

and

$$\sup_{k,j} \sup_{r \in (\frac{3}{2^{J-1}}, 3]} \min\left\{ \mathbf{I}_{k,j}(r), \frac{r^2 \int_{\mathcal{B}_r} |DN^{(k,j)}|^2}{\int_{\mathcal{B}_r} |N^{(k,j)}|^2} \right\} < +\infty. \tag{2.35}$$

In particular, given any point $w_k \in \mathrm{spt}(T_{k,j}) \cap \overline{\mathbf{B}}_1$ for each center manifold $\mathcal{M}_{k,j}$ and corresponding non-centered interval of flattening $\left[\tilde{s}_j^{(k)}, \tilde{t}_j^{(k)}\right)$ around w_k, we also have

$$\sup_{k,j} \sup_{r \in \left(\frac{\tilde{s}_j^{(k)}}{\tilde{t}_j^{(k)}}, \delta_{k,j}\right]} \mathbf{I}_{k,j}(r) < \infty$$

and

$$\sup_{k,j} \sup_{r \in (\delta_{k,j}, R_{k,j}]} \min\left\{ \mathbf{I}_{k,j}(r), \frac{r^2 \int_{\mathcal{B}_r} |DN^{(k,j)}|^2}{\int_{\mathcal{B}_r} |N^{(k,j)}|^2} \right\} < +\infty,$$

where

$$\delta_{k,j} := \mathrm{dist}\left(\mathbf{p}_{k,j}(w_k), \partial \mathcal{B}_{\frac{3}{2^{J-1}}}\right),$$

and $R_{k,j}$ is as in (2.32) for $\mathbf{p}_{k,j}(w_k)$.

For the version of this result for a sequence of center manifolds with a fixed center, see [22, Theorem 5.1].

Theorem 2.6.6 will allow us to disregard the possibility that as we blow up, the mass of T accumulates around $\partial \mathbf{B}_{R_{k,j}t_j^{(k)}}(w_k)$, creating a locally trivial blow-up limit. Indeed, this kind of phenomenon could only occur if the frequency were unbounded at the scales $t_j^{(k)}$, due to 'infinite order of collapsing' at these scales.

Note that for the larger scales $r \in \left(\frac{3}{2^{J-1}}, 3\right]$ around x_k on our center manifolds, we are satisfied with a cruder estimate of the form (2.35). This is because it suffices to have a uniform frequency bound up to some uniform scale independent of k and j, which in itself gives us the desired conclusion.

We will be using this upper frequency bound in the form of the following *reverse Sobolev inequality* (see [22, Corollary 5.3]), which will be a key tool for our concluding argument.

Corollary 2.6.7. *There exists a constant $C > 0$ such that the following holds. Suppose that $\mathcal{M} = \mathcal{M}_{k,j}$ is any center manifold with corresponding rescaled current $T' = T_{k,j}$ and $\mathcal{M}$-normal approximation $N = N^{(k,j)}$. Then for any point $w \in \operatorname{spt}(T')$ with $\Theta(T', w) \geq Q$ and any interval of flattening $[s, t)$ around $\mathbf{p}(w)$, for every $r \in \left(\frac{s}{t}, 1\right]$, there is $\bar{s} \in \left(\frac{R}{2}r, Rr\right]$ such that*

$$\int_{\mathcal{B}_{\bar{s}}(\mathbf{p}(w))} |DN|^2 \leq \frac{C}{r^2} \int_{\mathcal{B}_{\bar{s}}(\mathbf{p}(w))} |N|^2,$$

In the case where $w = x_k$ for the center manifold $\mathcal{M}_{k,j}$, the proof is in fact *exactly* the same as that for a fixed center in [22, Corollary 5.3]. This is because we recenter our planes $\pi_{k,j}$ in such a way that $\mathbf{p}_{k,j}(x_k) = 0$. For non-centered intervals of flattening, the argument is still analogous, since we are once again making use of the upper frequency bound of Theorem 2.6.6, which is independent of the center. We thus omit the proof here.

We also have the following uniform lower frequency bound for all scales sufficiently close to where we stop the center manifolds.

Theorem 2.6.8. *There exists $c_0 > 0$ and $\eta > 0$ such that the following holds.*

One has the following lower bound on the frequency for any center manifold $\mathcal{M}$ with corresponding $\mathcal{M}$-normal approximation N, rescaled current T', and any point $w \in \mathrm{spt}(T')$ with $\Theta(T', w) \geq Q$ that is the center of the interval of flattening $(s, t]$:

$$\mathbf{I}_N\Big(\mathbf{p}(w), \frac{r}{t}\Big) > c_0,$$

for every $r \in (s, R]$.

Remark 2.6.9. Note that w need not be the center of the center manifold construction. This will be important, since we will later choose w to be $\mathbf{p}_{k,j(k)}(y_k)$ in order to show that the Q-points persist in the blow-up limit.

Remark 2.6.10. Observe that the claim of Theorem 2.2.3 is stronger than the blow-up argument used in [22] and [3] to show the inheritence of singularities of the limiting map. There, for a fixed blow-up center x, the contradiction relies on a delicate capacitary argument. More precisely, after obtaining a non-trivial limiting Dir-minimizer u as above, the authors there conclude that Hausdorff limit K^∞ of the sets $\mathrm{Sing}_Q(T_k)$ for $T_k := T_{x,r_k} \llcorner \mathbf{B}_{6\sqrt{m}}$ satisfies

$$\mathcal{H}^{m-2+\alpha}\left(K^\infty \setminus \{\text{points sufficiently close to } \mathrm{Sing}_Q(u)\}\right) \geq \eta.$$

This may then be passed through to T_k via upper semi-continuity:

$$\mathcal{H}^{m-2+\alpha}_\infty\left(\mathrm{Sing}_Q(T_k) \setminus \{\text{points close to } \mathrm{Sing}_Q(u)\}\right) \geq \eta. \tag{2.36}$$

However, for any point $p \in \mathrm{Sing}_Q(T_k)$, there exists $\rho = \rho(p)$ arbitrarily small and $x_p \in \mathcal{M}_k$ near p such that:

$$c(p, \alpha)\|N^{(k)}\|_{L^2(\mathcal{B}_1(x_p))}\rho^{m-2+\alpha} \leq \int_{\mathcal{B}_\rho(x_p)} |DN^{(k)}|^2.$$

In other words, there is a quantitative control on the frequency from below, coming from

the persistence of singularities from T_k to $N^{(k)}$. One can now use this estimate to apply a covering argument to $\mathrm{Sing}_Q(T_k)$, contradicting (2.36).

Unfortunately, such an argument is insufficient in our case, since we require a dimension estimate for all of K^∞, rather than just those points that are far from $\mathrm{Sing}_Q(u)$. The easiest way to demonstrate the desired dimension estimate for K^∞ is hence by showing the set containment claimed in Theorem 2.2.3, for which the lower frequency bound of Theorem 2.6.8 is crucial. It will allow us to get an improvement on previous persistence of singularities arguments.

Before we proceed to prove the above results, let us show the following local estimate for the Dirichlet energy of the $\mathcal{M}$-normal approxiamtions at scales within the flattening intervals, which will come in useful later.

Proposition 2.6.11. *Suppose that ε_3 is sufficiently small. Then for any choice of center manifold $\mathcal{M}$ with corresponding rescaled current T', $\mathcal{M}$-normal approximation N and any interval of flattening $(s,t]$ around $w \in \mathrm{spt}(T')$, the following estimate holds for every $r \in \left(\frac{s}{t}, R\right]$:*

$$\mathbf{D}_{\mathbf{p}(w)}(r) \leq \int_{\mathcal{B}_r(\mathbf{p}(w))} |DN|^2 \leq C\boldsymbol{m}_0 r^{m+2-2\delta_2},$$

where $C = C(m, n, Q, \beta_2, \delta_2, M_0, N_0, C_e, C_h)$ is independent of the choice of center manifold, flattening interval, and center $\mathbf{p}(w)$.

Proof. We use the same notation as in Section 2.5. Choose ε_3 sufficiently small such that for each $r \in \left(\frac{s}{t}, R\right]$, the geodesic ball $\mathcal{B}_r(\mathbf{p}(w)))$ is contained in $\mathbf{\Phi}(\overline{B}_{\frac{R}{2}r}(\mathbf{p}_{\pi_0}(w)))$. Fix a radius r in this range. The first inequality is trivial due to the nature of the map ϕ. We can cover $\overline{B}_{\frac{R}{2}r}(\mathbf{p}_{\pi_0}(w))$ by $\mathbf{\Gamma}$ and the concentric cubes $\frac{17}{16}L$ of some subcollection $\mathscr{F}$ of cubes $L \in \mathscr{W}$. Note that since $\overline{B}_{\frac{R}{2}r}(\mathbf{p}_{\pi_0}(w)) \setminus \mathbf{\Gamma}$ is relatively compact in $[-4,4]^m$, this subcollection must be finite (and its cardinality can be controlled independently of r).

Now by construction, $N \equiv 0$ on $\mathbf{\Gamma} \cap \overline{B}_{\frac{R}{2}r}(\mathbf{p}_{\pi_0}(w))$. Moreover, for each $L \in \mathscr{W}$ with $L \cap \overline{B}_{\frac{R}{2}r}(\mathbf{p}_{\pi_0}(w)) \neq \emptyset$, the condition (Go) tells us that $\ell(L) < \frac{R}{2}c_s r$. Hence, by the estimates

on the $\mathcal{M}$-normal approximation established in Theorem 2.5.9, we deduce that

$$\int_{\mathbf{\Phi}(\overline{B}_{Rr/2}(\mathbf{p}_{\pi_0}(w)))} |DN|^2 \leq \sum_{L\in\mathscr{F}} \int_{\mathcal{L}} |DN|^2 \leq c \sum_{L\in\mathscr{F}} \boldsymbol{m}_0 \ell(L)^{m+2-2\delta_2} \leq c\boldsymbol{m}_0 r^{m+2-2\delta_2}.$$

The result follows. □

We now prove the characterization of Q-points in Proposition 2.6.1.

Proof of Proposition 2.6.1. We will henceforth omit the dependency of the quantities in (2.28) on u and y, for simplicity. Moreover, we will assume without loss of generality that $y = 0$. We begin by showing that as long as $H(r) > 0$,

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\bigg|_{\tau=r} \log\left(\frac{H(\tau)}{\tau^{m-1}}\right) = \frac{2I(r)}{r}. \tag{2.37}$$

Firstly, note that $H \in W^{1,1}$ since $u \in W^{1,2}$. Moreover, the distributional derivative of $|u|^2$ coincides with its approximate differential, so the chain rule [18, Proposition 2.8] applies and thus one can compute H'. These facts justify the elementary computation (at almost-every $\tau > 0$) that

$$\frac{\mathrm{d}}{\mathrm{d}\tau} \log\left(\frac{H(\tau)}{\tau^{m-1}}\right) = \frac{H'(\tau)}{H(\tau)} - \frac{m-1}{\tau},$$

with

$$\begin{aligned}
H'(\tau) &= \frac{\mathrm{d}}{\mathrm{d}\tau} \int_{B_\tau\setminus B_{\tau/2}} \frac{|u(z)|^2}{|z|}\,\mathrm{d}z \\
&= \frac{\mathrm{d}}{\mathrm{d}\tau} \int_{B_1\setminus B_{1/2}} \tau^{m-1}\frac{|u(\tau z)|^2}{|z|}\,\mathrm{d}z \\
&= 2\tau^{m-1}\int_{\frac{1}{2}}^1 s^{-1}\int_{\partial B_s}\sum_{i=1}^Q \langle u_i(\tau z), \nabla_z u_i(\tau z)\rangle\,\mathrm{d}\mathcal{H}^{m-1}(z)\,\mathrm{d}s + \frac{(m-1)}{\tau}H(\tau) \\
&= 2\tau^{m}\int_{\frac{1}{2}}^1 \int_{\partial B_s}\sum_{i=1}^Q \langle u_i(\tau z), \partial_\nu u_i(\tau z)\rangle\,\mathrm{d}\mathcal{H}^{m-1}(z)\,\mathrm{d}s + \frac{(m-1)}{\tau}H(\tau) \\
&= 2\int_{B_\tau\setminus B_{\tau/2}}\sum_{i=1}^Q \langle u_i, \partial_\nu u_i\rangle + \frac{(m-1)}{\tau}H(\tau)
\end{aligned}$$

$$= 2D(\tau) + \frac{m-1}{\tau} H(\tau).$$

From this differential identity, we infer that

$$\frac{\mathrm{d}}{\mathrm{d}\tau} \log\left(\frac{H(\tau)}{\tau^{m-1}}\right) = 2\frac{I(\tau)}{\tau}.$$

By continuity, (2.37) follows when we evaluate τ at some $r > 0$.

Integrating (2.37) from ρ to r and using the fact that I is monotone increasing, we have

$$\log\left(\frac{H(r)}{r^{m-1}}\right) - \log\left(\frac{H(\rho)}{\rho^{m-1}}\right) = 2\int_\rho^r \frac{I(\tau)}{\tau}\,\mathrm{d}\tau \geq 2I_0 \log\left(\frac{r}{\rho}\right), \tag{2.38}$$

or equivalently,

$$\begin{aligned}\frac{1}{\rho^{m-1+2I_0}} \int_{B_\rho \setminus B_{\rho/2}} \frac{|u(z)|^2}{|z|}\,\mathrm{d}z &= \frac{H(\rho)}{\rho^{m-1+2I_0}} \\ &\leq \frac{H(r)}{r^{m-1+2I_0}} \\ &= \frac{1}{r^{m-1+2I_0}} \int_{B_r \setminus B_{r/2}} \frac{|u(z)|^2}{|z|}\,\mathrm{d}z.\end{aligned}$$

More generally, for any $k \in \mathbb{N}$, this rescales as

$$\frac{2^{k(m-1+2I_0)}}{\rho^{m-1+2I_0}} \int_{B_{\rho/2^k} \setminus B_{\rho/2^{k+1}}} \frac{|u(z)|^2}{|z|}\,\mathrm{d}z \leq \frac{2^{k(m-1+2I_0)}}{r^{m-1+2I_0}} \int_{B_{r/2^k} \setminus B_{r/2^{k+1}}} \frac{|u(z)|^2}{|z|}\,\mathrm{d}z,$$

and thus

$$\frac{1}{\rho^{m+2I_0}} \int_{B_{\rho/2^k} \setminus B_{\rho/2^{k+1}}} |u(z)|^2\,\mathrm{d}z \leq \frac{1}{r^{m+2I_0}} \int_{B_{r/2^k} \setminus B_{r/2^{k+1}}} |u(z)|^2\,\mathrm{d}z.$$

Summing over k, this yields

$$\frac{1}{\rho^{m+2I_0}} \int_{B_\rho} |u(z)|^2\,\mathrm{d}z \leq \frac{1}{r^{m+2I_0}} \int_{B_r} |u(z)|^2\,\mathrm{d}z,$$

as required. It remains to show that $I_0 > 0$, but this follows easily from the inequality

$$H(r) \leq CrD(r) \qquad \text{for every } r \in \Big(0, \frac{1}{2}\operatorname{dist}(0, \partial\Omega)\Big),$$

which comes from the Hölder-regularity [18, Theorem 3.9] of Q-valued $W^{1,2}$-maps and the fact that $u(0) = Q[\![0]\!]$.

Now we show the converse. Suppose that $I_0 \geq c$ for some $c > 0$, but that $u(0) \neq Q[\![0]\!]$. Then, $u(0) = T \in \mathcal{A}_Q$ for some T with $d(T) > 0$, where $d(T)$ is defined as in [18, Definition 3.4]. However, the lower frequency bound allows us to conclude that the assumptions of [18, Proposition 3.6] hold for $\Omega = B_r$, for $r > 0$ sufficiently small. This would enable us to decompose u locally around 0 into two simpler functions. One of these must necessarily take a value other than zero at the origin, thus contradicting the height decay which comes as a consequence of the lower frequency bound. Indeed, (2.38) and the succeeding computations tell us that we must have

$$\frac{1}{\rho^{m+2c}} \int_{B_\rho} |u|^2 \leq \frac{1}{r^{m+2c}} \int_{B_r} |u|^2 \qquad \text{for any } \rho < r < \frac{1}{2}\operatorname{dist}(0, \partial\Omega).$$

The argument under assumption (a) follows analogously. □

We shall now prove the almost-monotonicity in Theorem 2.6.4. We must first introduce following quantities, centered at any point $w \in \operatorname{spt}(T')$, with corresponding $\mathcal{M}$-normal approximation N:

$$\begin{aligned}
\mathbf{E}_N(\mathbf{p}(w), r) &:= -\int_{\mathcal{M}} \phi'\left(\frac{d(p - \mathbf{p}(w))}{r}\right) \sum_{i=1}^{Q} \langle N_i(p), \partial_r N_i(p)\rangle \, \mathrm{d}p, \\
\mathbf{G}_N(\mathbf{p}(w), r) &:= -\int_{\mathcal{M}} \phi'\left(\frac{d(p - \mathbf{p}(w))}{r}\right) |\partial_r N|^2(p) \; d(p - \mathbf{p}(w)) \, \mathrm{d}p, \\
\mathbf{\Sigma}_N(\mathbf{p}(w), r) &:= \int_{\mathcal{M}} \phi\left(\frac{d(p - \mathbf{p}(w))}{r}\right) |N|^2(p) \, \mathrm{d}p,
\end{aligned}$$

Note that for a given point w, the quantities $\mathbf{E}_N(\mathbf{p}(w), \cdot)$, $\mathbf{G}_N(\mathbf{p}(w), \cdot)$ and $\mathbf{\Sigma}_N(\mathbf{p}(w), \cdot)$ are

all classically differentiable. By considering the inner and outer variations of the current $\mathbf{T}_F$ along convenient choices of vector fields, the following estimates arise:

Proposition 2.6.12. *For every $c_0 > 0$ and every γ_3 sufficiently small, there exists $C = C(c_0, \gamma_3, m, n, Q) > 0$ such that if $\varepsilon_3 \in (0, \varepsilon_2)$ sufficiently small, the following holds.*

Let $\mathcal{M}$ be a center manifold with corresponding rescaled current T', $\mathcal{M}$-normal approximation N, interval of flattening $[s,t]$ centered at $x = \mathbf{p}(w)$ and any $[a,b] \subset \left[\frac{s}{t}, R\right]$ with $\mathbf{I} \geq c_0$ on $[a,b]$. Omitting the dependency on N and x for $\mathbf{D}_N(x,\cdot), \mathbf{H}_N(x,\cdot)$ and all related quantities for simplicity, we have the following for any $r \in [a,b]$:

$$\left|\mathbf{H}'(r) - \frac{m-1}{r}\mathbf{H}(r) - \frac{2}{r}\mathbf{E}(r)\right| \leq C\mathbf{H}(r), \tag{2.39}$$

$$\left|\mathbf{D}(r) - \frac{1}{r}\mathbf{E}(r)\right| \leq C\mathbf{D}(r)^{1+\gamma_3} + C\varepsilon_3^2\mathbf{\Sigma}(r), \tag{2.40}$$

$$\begin{aligned}\left|\mathbf{D}'(r) - \frac{m-2}{r}\mathbf{D}(r) - \frac{2}{r^2}\mathbf{G}(r)\right| \\ \leq C\big(\mathbf{D}(r) + \mathbf{D}(r)^{\gamma_3}\mathbf{D}'(r) + \frac{1}{r}\mathbf{D}(r)^{1+\gamma_3}\big),\end{aligned} \tag{2.41}$$

$$\mathbf{\Sigma}(r) + r\mathbf{\Sigma}'(r) \leq Cr^2\mathbf{D}(r) \leq Cr^{m+2}\varepsilon_3^2. \tag{2.42}$$

We refer the reader to [22] for the proof. We are now ready to prove the almost-monotonicity.

Proof of Theorem 2.6.4. First of all, notice that since $\mathbf{\Sigma}' \geq 0$ (due to the nature of ϕ), (2.42) tells us that also $\mathbf{\Sigma}(r),\ r\mathbf{\Sigma}'(r) \leq Cr^2\mathbf{D}(r)$.

Note that we may assume that $\mathbf{I} > c_0$ on $[a,b]$, since $\mathbf{I}(a) > c_0$ (otherwise there is nothing to prove) and if $\mathbf{I}(r) \leq c_0$ at some point $b' \in (a,b)$, we can replace b by b'. Fix γ_3 and ε_3 small enough so that the above estimates hold. By (2.40), (2.42) and the fact that $r \leq 3$, we have

$$\begin{aligned}\mathbf{D}(r) &\leq \frac{\mathbf{E}(r)}{r} + C\mathbf{D}(r)^{1+\gamma_3} + \varepsilon_3^2\mathbf{\Sigma}(r) \\ &\leq \frac{\mathbf{E}(r)}{r} + C\varepsilon_3^2\mathbf{D}(r)\left[r^{\gamma_3 m} + r^2\right]\end{aligned}$$

$$\leq \frac{\mathbf{E}(r)}{r} + C\varepsilon_3^2 \mathbf{D}(r).$$

Rearranging and decreasing ε_3 further if necessary, we conclude that

$$\frac{\mathbf{D}(r)}{2} \leq \frac{\mathbf{E}(r)}{r}.$$

An analogous argument yields that

$$\frac{\mathbf{D}(r)}{r} \leq 2\mathbf{D}(r),$$

so $\mathbf{E} > 0$ on (a, b) and $\frac{1}{\mathbf{E}(r)}$ is thus a well-defined quantity. Now let

$$\mathbf{F}(r) := \frac{1}{\mathbf{D}(r)} - \frac{r}{\mathbf{E}(r)}.$$

Let us now estimate the derivative of $\mathbf{\Omega}$, in the hope that we can then integrate to obtain the almost-monotonicity. Observe that

$$\begin{aligned}
-\mathbf{\Omega}'(r) &= -\frac{\mathbf{I}'(r)}{\mathbf{I}(r)} \\
&= -\frac{\mathbf{H}(r)}{r\mathbf{D}(r)} \left[\frac{\mathbf{D}(r)}{\mathbf{H}(r)} + \frac{r\mathbf{D}'(r)}{\mathbf{H}(r)} - \frac{r\mathbf{D}(r)\mathbf{H}'(r)}{\mathbf{H}(r)^2} \right] \\
&= -\frac{1}{r} - \frac{\mathbf{D}'(r)}{\mathbf{D}(r)} + \frac{\mathbf{H}'(r)}{\mathbf{H}(r)} \\
&= -\frac{1}{r} + \frac{\mathbf{H}'(r)}{\mathbf{H}(r)} - \mathbf{D}'(r)\mathbf{F}(r) - \frac{r\mathbf{D}'(r)}{\mathbf{E}(r)}.
\end{aligned}$$

Now let us further estimate each of the terms on the right-hand side. By (2.39), we have

$$\frac{\mathbf{H}'(r)}{\mathbf{H}(r)} \leq \frac{m-1}{r} + \frac{2\mathbf{E}(r)}{r\mathbf{H}(r)} + C.$$

Furthermore, by (2.39) and the comparability of $\mathbf{D}(r)$ and $\frac{\mathbf{E}(r)}{r}$, we can achieve the bound

$$\begin{aligned}|\mathbf{F}(r)| &= \left|\frac{r}{\mathbf{D}(r)\mathbf{E}(r)}\left(\mathbf{D}(r) - \frac{1}{r}\mathbf{E}(r)\right)\right| \\ &\leq C\frac{r\left(\mathbf{D}(r)^{1+\gamma_3} + \mathbf{\Sigma}(r)\right)}{\mathbf{D}(r)\mathbf{E}(r)} \\ &\leq C\left[\mathbf{D}(r)^{\gamma_3 - 1} + \frac{\mathbf{\Sigma}(r)}{\mathbf{D}(r)^2}\right].\end{aligned}$$

Combining this estimate with (2.41), (2.42), and again by comparing $\frac{\mathbf{E}(r)}{r}$ to $\mathbf{D}(r)$, we conclude that

$$\begin{aligned}-\frac{r\mathbf{D}'(r)}{\mathbf{E}(r)} &\leq \frac{r\mathbf{D}(r)}{\mathbf{E}(r)}\left(C - \frac{m-2}{r}\right) - \frac{2\mathbf{G}(r)}{r\mathbf{E}(r)} + C\frac{r\mathbf{D}(r)^{\gamma_3}\mathbf{D}'(r) + \mathbf{D}(r)^{1+\gamma_3}}{\mathbf{E}(r)} \\ &\leq C - \frac{m-2}{r} + C\frac{\mathbf{D}(r)|\mathbf{F}(r)|}{r} - \frac{2\mathbf{G}(r)}{r\mathbf{E}(r)} \\ &\qquad + C\left[\mathbf{D}(r)^{\gamma_3 - 1}\mathbf{D}'(r) + \frac{\mathbf{D}(r)^{\gamma_3}}{r}\right] \\ &\leq -\frac{m-2}{r} - \frac{2\mathbf{G}(r)}{r\mathbf{E}(r)} \\ &\qquad + C\left[1 + \frac{\mathbf{D}(r)^{\gamma_3}}{r} + \frac{\mathbf{\Sigma}(r)}{r\mathbf{D}(r)} + \mathbf{D}(r)^{\gamma_3 - 1}\mathbf{D}'(r) + r^{\gamma_3 m - 1}\right] \\ &\leq -\frac{m-2}{r} - \frac{2\mathbf{G}(r)}{r\mathbf{E}(r)} + C\left[1 + \mathbf{D}(r)^{\gamma_3 - 1}\mathbf{D}'(r) + r^{\gamma_3 m - 1}\right].\end{aligned}$$

Now observe that Cauchy-Schwartz (applied first to the inner product, then to the integral) yields

$$\mathbf{E}(r)^2 \leq \mathbf{G}(r)\mathbf{H}(r),$$

and thus

$$\frac{2\mathbf{E}(r)}{r\mathbf{H}(r)} \leq \frac{2\mathbf{G}(r)}{r\mathbf{E}(r)}.$$

Bringing everything together, we obtain

$$-\mathbf{\Omega}'(r) \leq -\frac{1}{r} + \frac{m-1}{r} + \frac{2\mathbf{E}(r)}{r\mathbf{H}(r)} - \frac{m-2}{r} - \frac{2\mathbf{G}(r)}{r\mathbf{E}(r)}$$

$$
\begin{aligned}
&\quad + C\left[1 + \mathbf{D}(r)^{\gamma_3 - 1}\mathbf{D}'(r) + \frac{\mathbf{\Sigma}(r)\mathbf{D}'(r)}{\mathbf{D}(r)^2} + r^{\gamma_3 m - 1}\right] \\
&\leq C\left[1 + \mathbf{D}(r)^{\gamma_3 - 1}\mathbf{D}'(r) + \frac{\mathbf{\Sigma}(r)\mathbf{D}'(r)}{\mathbf{D}(r)^2} + r^{\gamma_3 m - 1}\right] \\
&\leq C\left[1 + \partial_r\left[\mathbf{D}(r)^{\gamma_3}\right] + \frac{\mathbf{\Sigma}(r)\mathbf{D}'(r)}{\mathbf{D}(r)^2} + r^{\gamma_3 m - 1}\right].
\end{aligned}
$$

Integrating over the interval $[a, b]$, we deduce that

$$
\begin{aligned}
\mathbf{\Omega}(a) - \mathbf{\Omega}(b) &\leq C\left[b + \mathbf{D}(b)^{\gamma_3} - \mathbf{D}(a)^{\gamma_3} - \int_a^b \mathbf{\Sigma}(r)\partial_r\left(\frac{1}{\mathbf{D}(r)}\right)\,\mathrm{d}r\right] \\
&\leq C\left[b + \mathbf{D}(b)^{\gamma_3} + \frac{\mathbf{\Sigma}(a)}{\mathbf{D}(a)} - \frac{\mathbf{\Sigma}(b)}{\mathbf{D}(b)} + \int_a^b \frac{\mathbf{\Sigma}'(r)}{\mathbf{D}(r)}\,\mathrm{d}r\right] \\
&\leq C\left[b + \mathbf{D}(b)^{\gamma_3} + \frac{\mathbf{\Sigma}(a)}{\mathbf{D}(a)} + \int_a^b r\,\mathrm{d}r\right] \\
&\leq C\left[b + b^{\gamma_3 m} + b^2\right].
\end{aligned}
$$

Note that $C = C(Q, m, n, c_0, \gamma_3)$. Thus, setting

$$
f(r) = Cr^{\min\{\gamma_3 m, 1\}},
$$

for C with the above dependencies, the proof is complete. □

2.7 Uniform bounds on frequency function

We now prove Theorem 2.6.6. Before we proceed, we recall the following result from [22], which infers height decay from excess decay for the $\mathcal{M}$-normal approximations:

Lemma 2.7.1 ([22], Lemma 5.2)**.** *Using the notation of Section 2.5.5, suppose that $\{\mathcal{M}_k\}_k$ is a sequence of center manifolds with corresponding $\mathcal{M}_k$-normal approximations $N^{(k)}$, rescaled currents $T_k = T_{x_k, t_k}$ and centered intervals of flattening $(s_k, t_k]$ around x_k. Suppose that for*

some sequence $\eta_k \in \left(\frac{s_k}{t_k}, 3\right]$, *we have*

$$\int_{\mathcal{B}_{\eta_k}(\mathbf{p}_k(x_k))\setminus\mathcal{B}_{\frac{\eta_k}{2}}(\mathbf{p}_k(x_k))} |N^{(k)}|^2 \longrightarrow 0 \qquad as\ k \to \infty..$$

Then we must necessarily have $\mathbf{E}(T_k, \mathbf{B}_{\eta_k}(x_k)) \longrightarrow 0$.

Although we have varying centers, the proof is analogous to that in [22]. We omit the details here.

2.7.1 Proof of the upper frequency bound

We are now in a position to show the uniform upper frequency bound of Theorem 2.6.6. Note that by [22, Theorem 5.1], we know that this result holds when all of the blow-ups have the same fixed center. The argument for varying centers and non-centered flattening intervals is essentially the same, since the estimates used are independent of the center. Nevertheless, we repeat it here for clarity, with some minor modifications that simplify the argument slightly.

Proof of Theorem 2.6.6. Observe that the second part of this Theorem follows immediately from the first part, so we just need to prove the uniform boundedness around the points x_k that we center our center manifolds around.

Suppose, for a contradiction, that the statement of the theorem is false. First of all, suppose that (2.34) fails. Then, for an arbitrary choice of $J \in \mathbb{N}$, up to subsequence (using the notation in Section 2.5.5) we can find scales $\frac{\rho_k}{t_k} \in \left(\frac{s_k}{t_k}, \frac{3}{2^{J-1}}\right]$ around x_k such that

$$\lim_{k\to\infty} \mathbf{I}_k\left(\frac{\rho_k}{t_k}\right) = \infty. \tag{2.43}$$

By the almost-monotonicity (2.33) of the frequency, we may assume that $\rho_k = \frac{3t_k}{2^{J-1}}$ for each $k \in \mathbb{N}$. There are three possibilities to consider, based on the size of the scales along which the frequency blows up. Due to Proposition 2.6.11, unboundedness of the frequency along given scales corresponds to the collapsing of the sheets of $N^{(k)}$ along these scales. Either

(a) $\limsup_k t_k > 0$;

(b) $\limsup_k t_k = 0$, but there exists a subsequence (not relabelled) along which each starting scale t_k comes immediately after a flattening interval $(\bar{s}_k, \bar{t}_k] := (s^{(k)}_{j(k)-1}, t^{(k)}_{j(k)-1}]$, centered at the same point x_k; namely $t_k = \bar{s}_k$;

(c) $\limsup_k t_k = 0$ and for every k (sufficiently large) we have $t_k < \bar{s}_k$.

Let us begin with case (a). Extract a subsequence for which $\frac{3t_k}{2^{J-1}} \geq \eta$ for some $\eta > 0$. By Proposition 2.6.11, for every $r \in \left(\frac{s_k}{t_k}, 3\right]$, the Dirichlet energy estimate

$$\mathbf{D}_k(r) \leq C\boldsymbol{m}_0^{(k)} r^{m+2-2\delta_2} \leq C\varepsilon_3^2. \tag{2.44}$$

holds. Hence, the unboundedness of $\mathbf{I}_k\left(\frac{3}{2^{J-1}}\right)$ tells us that for

$$\mathcal{A}_J := \mathcal{B}_{\frac{3}{2^{J-1}}}(\mathbf{p}_k(x_k)) \setminus \mathcal{B}_{\frac{3}{2^{J-2}}}(\mathbf{p}_k(x_k)),$$

we have

$$\int_{\mathcal{A}_J} |N^{(k)}|^2 \longrightarrow 0 \quad \text{as } k \to \infty.$$

We can now use Lemma 2.7.1 to deduce that

$$\mathbf{E}\big(T_k, \mathbf{B}_{\frac{3}{2^{J-1}}}\big) = \mathbf{E}\big(T, \mathbf{B}_{\frac{3t_k}{2^{J-1}}}(x_k)\big) \longrightarrow 0.$$

Since $\frac{3t_k}{2^{J-1}} \geq \eta$ and $x_k \to x$, we can thus conclude that

$$\mathbf{E}(T_{x,\eta}, \mathbf{B}_1) = \mathbf{E}(T, \mathbf{B}_\eta(z)) \leq c \liminf_{k\to\infty} \mathbf{E}\big(T, \mathbf{B}_{\frac{3t_k}{2^{J-1}}}(x_k)\big) = 0.$$

However, this contradicts the fact that $x \in \mathrm{Sing}_{\geq Q}(T)$.

We now move on to case (b). In this case, the excess is still below the threshold ε_3^2 at scale $\bar{s}_k = t_k$, but we must restart the center manifold because it no longer serves as a good

approximation for the current at that scale.

As a result of the condition (Go) in the flattening procedure, we encounter a cube $L \in \mathscr{W}^{(k,j(k)-1)}$ of size $\ell(L) \geq c_s \frac{\bar{s}_k}{\bar{t}_k} =: c_s \bar{r}_k$ with $L \cap \overline{B}_{\bar{r}_k} \neq \emptyset$ at which the refinement for $\mathcal{M}_k$ stops. We then restart the center manifold $\mathcal{M}_k$ and the $\mathcal{M}_k$-normal approximation $N^{(k)}$ immediately at this scale, since the total excess remains below the threshold ε_3^2, but locally near the center x_k the excess becomes too large.

The splitting observed at the stopping scale of the previous center manifold must then propagate to the new starting scale, so by the comparison of center manifolds in Proposition 2.5.12 and the control on the Dirichlet energy in Proposition 2.6.11 (at scale R), we deduce that

$$\int_{\mathcal{B}_R} |N^{(k)}|^2 \geq \bar{c} \int_{\mathcal{B}_R} |DN^{(k)}|^2,$$

where $\bar{c}$ is a geometric constant, independent of k and of the center x_k.

It remains to deal with the technicality that in our definition of the frequency, we are considering the L^2-mass of $N^{(k)}$ on *annuli* rather than balls. This is where the choice of J becomes important. We now revert back to our full collection of center manifolds. For any $J \in \mathbb{N}$ and any $k, j \in \mathbb{N}$, we may apply Hölder's inequality and a Sobolev embedding for $q \in [2, 2^*]$ if $m > 2$ and any $q \in [2, \infty)$ otherwise, to deduce that

$$\begin{aligned}\int_{\mathcal{B}_{\frac{3}{2^J}}} |N^{(k,j)}|^2 &\leq \left[\mathcal{H}^m(\mathcal{B}_{\frac{3}{2^J}})\right]^{1-\frac{2}{q}} \left[\int_{\mathcal{B}_{\frac{3}{2^J}}} |N^{(k,j)}|^q\right]^{\frac{2}{q}} \\ &\leq C 2^{-Jm\left(1-\frac{2}{q}\right)} \left[\int_{\mathcal{B}_{\frac{3}{2^J}}} |N^{(k,j)}|^2 + \int_{\mathcal{B}_{\frac{3}{2^J}}} |DN^{(k,j)}|^2\right].\end{aligned}$$

Here, we have again used the uniform curvature bounds in Proposition 2.5.5(i) for $\mathcal{M}_{k,j}$ to compare the volume of geodesic balls in $\mathcal{M}_{k,j}$ with that of Euclidean ones. We can thus

choose J sufficiently large, *independently of both* k *and* j, such that

$$\int_{\mathcal{B}_3 \setminus \mathcal{B}_{\frac{3}{2^J}}} |N^{(k,j)}|^2 \geq c \int_{\mathcal{B}_3} |DN^{(k,j)}|^2.$$

By decomposing this large annulus dyadically, we can find a small annulus $\mathcal{A}_\ell := \mathcal{B}_{\frac{3}{2^\ell}} \setminus \mathcal{B}_{\frac{3}{2^{\ell+1}}}$ for some $\ell = \ell(k,j) \leq J-1$ with

$$\mathbf{H}_{k,j}\Big(\frac{3}{2^\ell}\Big) \geq \int_{\mathcal{A}_\ell} |N^{(k,j)}|^2 \geq \frac{c}{2^J} \int_{\mathcal{B}_3} |DN^{(k,j)}|^2 \geq \frac{c}{2^J} \mathbf{D}_{k,j}\Big(\frac{3}{2^\ell}\Big), \tag{2.45}$$

where c is a geometric constant. Combining this with the almost-monotonicity of the frequency, we thus have $\mathbf{I}_k\big(\frac{3}{2^\ell}\big) \leq C\mathbf{I}_k\big(\frac{3}{2^{J-1}}\big) \leq C(J)$, contradicting (2.43).

This brings us to the final case (c), where for every k sufficiently large, we consistently restart the center manifolds around x_k because the excess exceeds the threshold ε_3^2:

$$\mathbf{E}(T_k, \mathbf{B}_{6\sqrt{m}\eta_k}(x_k)) > \varepsilon_3^2 \qquad \text{for some } \eta_k \in \left(1, \min\left\{3, \frac{\bar{s}_k}{t_k}\right\}\right].$$

But then Lemma 2.7.1 tells us that

$$\liminf_{k\to\infty} \mathbf{H}_k(\eta_k) > 0.$$

Combining this with the uniform boundedness of $\mathbf{D}_k(\eta_k)$ from Proposition 2.6.11 and the almost-monotonicity of the frequency, we achieve uniform bounds for $\mathbf{I}_k$ on $\big(\frac{s_k}{t_k}, \eta_k\big]$. Since $\eta_k \geq 1$ for every k, we can further choose J such that $\frac{3}{2^{J-1}} \leq 1$. This once again gives the desired contradiction and so (2.34) indeed holds for J sufficiently large.

Finally, we check that (2.35) for the choice of J given by the validity of (2.34). Here, we simply observe that by (2.45), for any $r \in \big(\frac{3}{2^{J-1}}, 3\big]$ we have

$$\int_{\mathcal{B}_r} |N^{(k,j)}|^2 \geq \mathbf{H}_{k,j}\Big(\frac{3}{2^{J-1}}\Big) \geq c(J) \int_{\mathcal{B}_3} |DN^{(k,j)}|^2 \geq c(J) r^2 \mathbf{D}_{k,j}(r),$$

where the presence of the factor r^2 can be verified by scaling. $\square$

2.7.2 Proof of the lower frequency bound

We now demonstrate the uniform lower frequency bound of Theorem 2.6.8, which will be vital for the persistence of singularities in the blow-up limit along the scales r_k.

Firstly, we require the following result, which tells us that frequency decay forces excess:

Lemma 2.7.2. *There exists ε_3 sufficiently small such that the following holds. Using the notation of Section 2.5.5, suppose $\mathcal{M}_k$ is a sequence of center manifolds corresponding to rescaled currents $T_k = T_{x_k,t_k}$ satisfying Assumptions 2.1.1-2.3.7, with given intervals of flattening $(s_k, t_k]$ around $w_k \in D_{[Q,Q+\varepsilon]}T_k \cap \overline{\mathbf{B}}_1$. Suppose that there exists a sequence of scales $\frac{\rho_k}{t_k} \in \left(\frac{s_k}{t_k}, R_k\right]$ for which*

$$\lim_{k\to\infty} \mathbf{I}_k\Big(\frac{\rho_k}{t_k}\Big) = 0$$

Then

$$\lim_{k\to\infty} \mathbf{E}\big(T_k, \mathbf{B}_{\rho_k/t_k}(w_k)\big) = 0.$$

Here $\mathbf{I}_k$ and all related quantities are centered around $\mathbf{p}_k(w_k)$.

Proof. We argue in an analogous way to that in the proof of Lemma 2.7.1. Once again, we may assume that the scales $\eta_k := \frac{\rho_k}{t_k}$ satisfy $\limsup_k \eta_k \geq \delta > 0$. Firstly, we distinguish between two possibilities. Either

(a) $\liminf_k \mathbf{H}_k(\delta) = 0$;

(b) $\liminf_k \mathbf{H}_k(\delta) > 0$.

In the first scenario, we immediately apply Lemma 2.7.1 to conclude. So let us assume that (b) holds (up to subsequence). We again argue by contradiction. If the statement of the lemma is false, then the almost-monotonicity of the frequency tells us that

$$\mathbf{I}_k(\delta) \longrightarrow 0 \qquad \text{but} \qquad \mathbf{E}\big(T_k, \mathbf{B}_{\rho_k/t_k}(w_k)\big) \geq c(\delta) > 0. \tag{2.46}$$

Combining this with (b), we have

$$\mathbf{D}_k(\delta) = \int_{\mathcal{B}_\delta(\mathbf{p}_k(w_k))} |DN^{(k)}|^2 \longrightarrow 0. \tag{2.47}$$

Projecting $\mathcal{B}_\delta(\mathbf{p}_k(w_k)) \subset \mathcal{M}_k$ to the reference plane π_k, we may assume that $\varepsilon_3 > 0$ is small enough to ensure that $B_{\frac{15}{16}\delta}(\mathbf{p}_{\pi_k}(w_k)) \subset \mathbf{p}_{\pi_k}(\mathcal{B}_\delta(\mathbf{p}_k(w_k)))$. Consider the family of Whitney cubes $\mathscr{W}^{(k)}$ on π_k. If no cube in this family intersects $B_{\frac{15}{16}\delta}(\mathbf{p}_{\pi_k}(w_k))$, then by construction, $N^{(k)} \equiv 0$ there, which we have assumed is no the case.

Thus, for each $k \in \mathbb{N}$ we may select $L_k \in \mathscr{W}^{(k)}$ to be the largest cube that intersects $B_{\frac{15}{16}\delta}(\mathbf{p}_{\pi_k}(w_k))$. Letting $d_k := \ell(L_k) < c_s\delta$, we proceed to show that $d_k \downarrow 0$.

We may assume that each cube L_k is in $\mathscr{W}_e^{(k)}$. Indeed, the fact that $\Theta(T_k, w_k) \geq Q$ ensures that the only stopping condition is (EX), up to replacing any cube in $\mathscr{W}_n^{(k)}$ by a nearby one in $\mathscr{W}_e^{(k)}$ of a comparable size. The fact that we do not stop refining due to (HT) can be ensured by choosing the constant C_h appropriately, due to the height bound [21, Theorem A.1]; see Remark 2.5.17.

Now since $L_k \cap B_{\frac{15}{16}\delta}(\mathbf{p}_{\pi_k}(w_k)) \neq \emptyset$ and $\delta > c_s^{-1}\ell(L_k)$, we may choose a ball $B^k \subset B_{\frac{15}{16}\delta}(\mathbf{p}_{\pi_k}(w_k))$ of radius $\sqrt{m}d_k$ with $\operatorname{dist}(B^k, L_k) \leq \sqrt{m}d_k$.

Since $L_k \in \mathscr{W}_e^{(k)}$, then the center q_k of B^k satisfies $\operatorname{dist}(L_k, q_k) \leq 4\sqrt{m}d_k$, and so we may use the splitting before tilting of Proposition 2.5.10 to deduce that

$$\int_{\mathcal{B}_\delta(\mathbf{p}_k(w_k))} |DN^{(k)}|^2 \geq \int_{\mathbf{\Phi}_k(B_{d_k/4}(q_k))} |DN^{(k)}|^2 \geq c\boldsymbol{m}_0^{(k)} d_k^{m+2-2\delta_2}, \tag{2.48}$$

where $c = c(m, n, Q, \beta_2, \delta_2, M_0, N_0, C_e, C_h)$. Combining this with (2.47), we deduce that $d_k \longrightarrow 0$. We then proceed exactly as in the proof of Lemma 2.7.1 to extract an m-dimensional area minimizing integral current S that is a weak-$*$ limit of the currents T_k, with $S = Q\llbracket\pi\rrbracket$ for some m-plane π, contradiction our large excess assumption. □

Let us now prove Theorem 2.6.8.

Proof of Theorem 2.6.8. We will once again prove this by contradiction. We again adopt the notation in Section 2.5.5, and we let $\mathbf{I}_k := \mathbf{I}_{\mathbf{p}_k(w_k),N^{(k)}}$ and we use analogous notation for all related quantities. If the statement of the theorem is false, we can find rescaled currents $T_k = T_{x_k,t_k} \llcorner \mathbf{B}_{6\sqrt{m}}$, center manifolds $\mathcal{M}_k$, $\mathcal{M}_k$-normal approximations $N^{(k)}$, and flattening intervals $(s_k, t_k]$ around $\mathbf{p}_k(w_k)$ for $w_k \in D_{[Q,Q+\varepsilon]}T_k$ such that the frequency $\mathbf{I}_k$ satisfies

$$\mathbf{I}_k\left(\frac{\rho_k}{t_k}\right) \longrightarrow 0 \qquad \text{for some } \frac{\rho_k}{t_k} \in \left(\frac{s_k}{t_k}, R\right]. \tag{2.49}$$

with

$$R := \liminf_k \operatorname{dist}(\mathbf{p}_k(w_k), \partial\mathcal{B}_3(\mathbf{p}_k(x_k)).$$

Observe that $R \geq 2$, since $w_k \in \mathbf{B}_1(x_k)$. Up to selecting a subsequence, we may further assume that $w_k \to w$. We now have two possible cases to consider; either

(a) $s_k > 0$ for every k and $\limsup_{k\to\infty} \frac{s_k}{\rho_k} > 0$;

(b) $s_k > 0$ for every k and $\limsup_{k\to\infty} \frac{s_k}{\rho_k} = 0$;

(c) $s_k = 0$ eventually.

Compactness argument. In both cases, we rescale the blow-ups of the current so that we are blowing up at the scales ρ_k instead of t_k. Let $\frac{\bar{s}_k}{t_k} \in \left(\frac{R\rho_k}{2t_k}, \frac{R\rho_k}{t_k}\right]$ be the scale at which the reverse Sobolev inequality of Corollary 2.6.7 holds for $r = \frac{\rho_k}{t_k}$. Then let $\bar{r}_k := \frac{2\bar{s}_k}{Rt_k} \in \left(\frac{\rho_k}{t_k}, \frac{2\rho_k}{t_k}\right]$. Let

$$\bar{T}_k := (\iota_{w_k,\bar{r}_k})_\sharp T_k = \left((\iota_{w_k,\bar{r}_k t_k})_\sharp T\right) \llcorner \mathbf{B}_{\frac{6\sqrt{m}}{\bar{r}_k}}, \quad \bar{\Sigma}_k := \iota_{w_k,\bar{r}_k}(\Sigma_k), \quad \overline{\mathcal{M}}_k := \iota_{w_k,\bar{r}_k}(\mathcal{M}_k),$$

and let

$$\bar{\boldsymbol{m}}_0^{(k)} := \max\{\boldsymbol{c}(\bar{\Sigma}_k)^2, \mathbf{E}(\bar{T}_k, \mathbf{B}_{6\sqrt{m}})\}.$$

Since the ambient manifolds $\bar{\Sigma}_k$ converge in $C^{3,\varepsilon_0}_{\mathrm{loc}}$ to $T_w\Sigma \equiv \mathbb{R}^{m+\bar{n}} \times \{0\}$, we have $\boldsymbol{c}(\bar{\Sigma}_k)^2 \xrightarrow{k\to\infty}$

0. Moreover, we we have $6\sqrt{m}\bar{r}_k \in \left(\frac{12\sqrt{m}\rho_k}{t_k}, \frac{24\sqrt{m}\rho_k}{t_k}\right]$, so Lemma 2.7.2 tells us that

$$\mathbf{E}(\bar{T}_k, \mathbf{B}_{6\sqrt{m}}) \leq C\mathbf{E}\big(T_k, \mathbf{B}_{\frac{24\sqrt{m}\rho_k}{t_k}}(w_k)\big) \longrightarrow 0. \tag{2.50}$$

Thus, $\bar{\boldsymbol{m}}_0^{(k)} \to 0$.

By an analogous argument to that in the proof of Proposition 2.3.4, we may thus extract a subsequence for which $T_k \overset{*}{\rightharpoonup} \pi_\infty$ for some m-dimensional plane π_∞. We will let $\mathbf{p}_\infty$ denote the orthogonal projection to this plane. We will use the notation $\bar{\mathbf{p}}_k$ for the projection map in Assumption 2.5.7 for the rescaled center manifold $\overline{\mathcal{M}}_k$.

Define

$$\bar{N}^{(k)} : \overline{\mathcal{M}}_k \to \mathbb{R}^{m+n}, \qquad \bar{N}^{(k)}(p) := \frac{1}{\bar{r}_k} N^{(k)}(\mathbf{p}_k(w_k) + \bar{r}_k p),$$

and let

$$v_k := \frac{\bar{N}^{(k)} \circ \mathbf{e}_k}{\mathbf{h}_k}, \qquad v_k : \pi_k \supset B_R \to \mathcal{A}_Q(\mathbb{R}^{m+n}),$$

where $\mathbf{e}_k$ is the exponential map at $p_k := \frac{\boldsymbol{\Phi}_k(0)}{\bar{r}_k} \in \overline{\mathcal{M}}_k$ defined on $B_R \subset \pi_\infty \equiv T_{p_k}\overline{\mathcal{M}}_k$ and $\mathbf{h}_k := \|\bar{N}_k\|_{L^2(\mathcal{B}_{\frac{R}{2}})}$.

Notice that

$$\mathbf{p}_k(w_k) \longrightarrow \mathbf{p}_\infty(w) \qquad \text{as } k \to \infty.$$

This decay of the excesses at our chosen scales $\frac{\rho_k}{t_k}$ further allows us to conclude that

$$\overline{\mathcal{M}}_k \longrightarrow \pi_\infty \quad \text{in } C^{3,\frac{\kappa}{2}}(\mathbf{B}_{\frac{4}{3}R}), \qquad \mathbf{e}_k \longrightarrow \mathrm{id} : \pi_\infty \supset B_R \to B_R \quad \text{in } C^{2,\frac{\kappa}{2}}.$$

The former is an easy consequence of the estimates in Proposition 2.5.5, whereas the latter is a technical result coming from the regularity of $\overline{\mathcal{M}}_k$; see [22, Proposition A.4] and the proof of [21, Lemma 6.1] for more details.

Now we may pass the reverse Sobolev inequality that holds at scale $\frac{\bar{s}_k}{t_k}$ around w_k to the

maps v_k:

$$\begin{aligned}
\int_{B_{\frac{R}{2}}} |v_k|^2 &\geq C\mathbf{h}_k^{-2} \int_{\mathcal{B}_{\frac{R}{2}}} |\bar{N}^{(k)}|^2 \, \mathrm{d}\mathcal{H}^m \\
&= \mathbf{h}_k^{-2}\bar{r}_k^{-2} \int_{\mathcal{B}_{\frac{R}{2}}} |N^{(k)}(\mathbf{p}_k(w_k) + \bar{r}_k p)|^2 \, \mathrm{d}\mathcal{H}^m(p) \\
&= \mathbf{h}_k^{-2}\bar{r}_k^{-m-2} \int_{\mathcal{B}_{\frac{R\bar{r}_k}{2}}(\mathbf{p}_k(w_k))} |N^{(k)}|^2 \, \mathrm{d}\mathcal{H}^m \\
&= \mathbf{h}_k^{-2}\bar{r}_k^{-m-2} \int_{\mathcal{B}_{\frac{\bar{s}_k}{\bar{t}_k}}(\mathbf{p}_k(w_k))} |N^{(k)}|^2 \, \mathrm{d}\mathcal{H}^m \\
&\geq C\mathbf{h}_k^{-2}\bar{r}_k^{-m} \int_{\mathcal{B}_{\frac{\bar{s}_k}{\bar{t}_k}}(\mathbf{p}_k(w_k))} |DN^{(k)}|^2 \\
&\geq C \int_{B_{\frac{R}{2}}} |Dv_k|^2.
\end{aligned}$$

Here and in all that follows, we use the notation $\mathcal{B}$ for both the balls on the original center manifolds $\mathcal{M}_k$ and the rescaled and recentered ones $\overline{\mathcal{M}}_k$. It should always be clear from context which of these the ball lies in. Thus,

$$\limsup_k \|v_k\|_{W^{1,2}(B_{\frac{R}{2}})} < \infty,$$

and so by the Banach-Alaoglu weak-$*$ compactness theorem and the reflexivity of the space $W^{1,2}$, we can extract a subsequence for which

$$v_k \rightharpoonup v \quad \text{in } W^{1,2}(B_{\frac{R}{2}}; \mathcal{A}_Q(\mathbb{R}^{m+n})).$$

Furthermore, Rellich-Kondrachov tells us that

$$v_k \longrightarrow v \quad \text{strongly in } L^2(B_{\frac{R}{2}}; \mathcal{A}_Q).$$

Now let us analyze the properties of this limit v. First of all, clearly v is non-trivial, since

our choice of normalization tells us that $\|v\|_{L^2(B_{\frac{R}{2}})} = 1$.

Moreover, the average of the sheets of v vanishes. Indeed, arguing as in [22, (7.4)], the estimate (2.24) in Theorem 2.5.9 for the average of the sheets of $N^{(k)}$ allows us to deduce that

$$\begin{aligned}\int_{\mathcal{B}_{\frac{R\bar{r}_k}{2}}(\mathbf{p}_k(w_k))} |\boldsymbol{\eta}\circ N^{(k)}| &\leq C\bar{\boldsymbol{m}}_0^{(k)}\bar{r}_k\sum_i \ell(L_i)^{m+2+\frac{\gamma_2}{2}} \\ &\quad + \frac{C}{\bar{r}_k}\int_{\mathcal{B}_{\frac{R\bar{r}_k}{2}}(\mathbf{p}_k(w_k))} \mathcal{G}(N^{(k)}, Q[\![\boldsymbol{\eta}\circ N^{(k)}]\!])^2 \\ &\leq \frac{C}{\bar{r}_k}\int_{\mathcal{B}_{R\bar{r}_k}(\mathbf{p}_k(w_k))} |N^{(k)}|^2,\end{aligned}$$

where $\{L_i\}_{i\in\mathbb{N}}$ is the family of cubes corresponding to the collection of cube-ball pairs $\mathscr{L} = \{(L, B(L))\}$ from [22, Section 4.1] that intersect $\mathbf{p}_{\pi_k}\big(\mathcal{B}_{\frac{R\bar{r}_k}{2}}(\mathbf{p}_k(w_k))\big)$. Here, the final constant C on the right-hand side depends on R and the quantities stated in Theorem 2.5.9. The final inequality follows from the arguments of [22, Section 4.1] and the reverse Sobolev inequality of Corollary 2.6.7. Rescaling these estimates and using the C^0-control from Theorem 2.5.9 for $\bar{N}^{(k)}$ combined with the excess decay (2.50), we thus have

$$\begin{aligned}\int_{B_{\frac{R}{2}}} |\boldsymbol{\eta}\circ v_k| &\leq C\mathbf{h}_k^{-1}\int_{\mathcal{B}_{\frac{R}{2}}} |\boldsymbol{\eta}\circ \bar{N}^{(k)}| \\ &= \mathbf{h}_k^{-1}\bar{r}_k^{-m-1}\int_{\mathcal{B}_{\frac{R\bar{r}_k}{2}}(\mathbf{p}_k(w_k))} |\boldsymbol{\eta}\circ N^{(k)}| \\ &\leq C\mathbf{h}_k^{-1}\bar{r}_k^{-m-2}\int_{\mathcal{B}_{R\bar{r}_k}(\mathbf{p}_k(w_k))} |N^{(k)}|^2 \\ &\leq C\mathbf{h}_k^{-1}\int_{\mathcal{B}_R} |\bar{N}^{(k)}|^2 \\ &\leq C\mathbf{h}_k \\ &\leq [\bar{\boldsymbol{m}}_0^{(k)}]^{\frac{1}{2m}}\big[\mathcal{H}^m(\mathcal{B}_R(\bar{\mathcal{M}}_k))\big]^{1/2} \\ &\leq C[\bar{\boldsymbol{m}}_0^{(k)}]^{\frac{1}{2m}+\frac{1}{4}} \longrightarrow 0.\end{aligned}$$

Combining this with the strong L^2-convergence of v_k to v, we indeed have $\boldsymbol{\eta} \circ v \equiv 0$ almost-everywhere on $B_{\frac{R}{2}}$.

Finally, one can show that v is Dir-minimizing, and that in addition v_k converges to v strongly in $W^{1,2}_{\text{loc}}$. This is proved by contradiction, using the existence of a better competitor function for the Dirichlet energy to build a suitable competitor current for $\bar{T}_k$, thus contradicting its area minimizing property. The argument may be found in [22, Section 7.3], and is omitted here.

In summary, we have shown that the maps v_k converge strongly in $W^{1,2}_{\text{loc}} \cap L^2$ on $B_{\frac{R}{2}} \subset \pi_\infty$ to a map v such that

- v is non-trivial;
- $\boldsymbol{\eta} \circ v \equiv 0$ almost-everywhere on $B_{\frac{R}{2}}$;
- v is Dir-minimizing on $B_{\frac{R}{2}}$.

Moreover, the assumption (2.49), rescaled appropriately, tells us that

$$\bar{\mathbf{I}}_k\left(\frac{1}{2}\right) \leq C\bar{\mathbf{I}}_k\left(\frac{\rho_k}{\bar{r}_k t_k}\right) \leq C\mathbf{I}_k\left(\frac{\rho_k}{t_k}\right) \xrightarrow{k\to\infty} 0, \tag{2.51}$$

where $\bar{\mathbf{I}}_k := \mathbf{I}_{0,\bar{N}^{(k)}}$.

We now argue slightly differently for each of the three cases. We will henceforth let $\bar{\rho}_k := \frac{\rho_k}{t_k}$.

Case (a). In this case, the stopping condition (Stop) for the intervals of flattening and Lemma 2.5.14 tell us that

$$\int_{\mathcal{B}_{\sigma\frac{s_k}{t_k}}} |N^{(k)}|^2 \leq C\boldsymbol{m}_0^{(k)}\left(\frac{s_k}{t_k}\right)^{m+4-2\delta_2},$$

for a suitable geometric constant σ. Moreover, the (localized) Sobolev embedding [18, Propo-

sition 2.11] and the fact that (up to subsequence) $\rho_k \in (s_k, Cs_k]$ tells us that

$$\int_{\mathcal{B}_{\bar{\rho}_k}} |N^{(k)}|^2 \leq C \int_{\mathcal{B}_{\frac{\sigma s_k}{t_k}}} |N^{(k)}|^2 + C\bar{\rho}_k^2 \int_{\mathcal{B}_{\frac{\bar{\rho}_k}{2}}} |DN^{(k)}|^2. \tag{2.52}$$

Now the excess splitting of Proposition 2.5.10 at scale $\frac{s_k}{t_k}$ and the comparability of s_k and ρ_k gives

$$\int_{\mathcal{B}_{\bar{\rho}_k}} |DN^{(k)}|^2 \geq c\boldsymbol{m}_0^{(k)} \bar{\rho}_k^{m+2-2\delta_2}.$$

Together with (2.52) this yields

$$\int_{\mathcal{B}_{\bar{\rho}_k}} |N^{(k)}|^2 \leq C\bar{\rho}_k^2 \int_{\mathcal{B}_{\frac{\bar{\rho}_k}{2}}} |DN^{(k)}|^2 \leq C\bar{\rho}_k^2 \mathbf{D}_k(\bar{\rho}_k). \tag{2.53}$$

Finally, one may replace the left-hand side of (2.53) with $\mathbf{H}_k(\bar{\rho}_k)$ via the following estimate:

$$-\int_{\mathcal{M}} \frac{1}{d(x)} \phi'\left(\frac{d(x)}{\bar{\rho}_k}\right) |N^{(k)}|^2 \, \mathrm{d}x = 2 \int_{\mathcal{B}_{\bar{\rho}_k} \setminus \mathcal{B}_{\frac{\bar{\rho}_k}{2}}} \frac{1}{d(x)} |N^{(k)}|^2 \, \mathrm{d}x \leq C\bar{\rho}_k^{-1} \int_{\mathcal{B}_{\bar{\rho}_k}} |N^{(k)}|^2.$$

This, however, is in contradiction with (2.51).

Case (b). This case follows similarly to case (a), only now we have to propagate the lower frequency bound observed at the stopping scale s_k up to the blow-up scale ρ_k. In order to do this, we

Proceeding as in case (a) only at scale $\frac{10s_k}{t_k}$, we conclude that

$$\mathbf{I}_k\left(\frac{10s_k}{t_k}\right) \geq \eta,$$

where η is a small geometric constant. Since (up to subsequence) $\frac{s_k}{\rho_k} \downarrow 0$, the strong convergence in turn implies that $\mathbf{I}_v(0) > 0$. This is however inconsistent with (2.51), which guarantees that $\mathbf{I}_v\left(\frac{1}{2}\right) = 0$, since $\mathbf{I}_v$ is almost-monotone.

Case (c). In this case, we have $s_k = 0$ eventually along our sequence, namely, so we

cannot simply exploit the splitting before tilting as in the previous case. Instead, we will use the blow-up procedure described above, and demonstrate that the frequency of the limiting Dir-minimizer is strictly positive at scale 0.

First of all, observe that since $s_k = 0$ for all k sufficiently large, by Proposition 2.5.15(v) we must have that $w_k \in \mathbf{\Phi}_k(\mathbf{\Gamma}_k)$. In particular, the limiting Dir-minimizer v satisfies $v(0) = Q[\![0]\!]$. Invoking Proposition 2.6.1, we conclude that $I_v(0) = \mathbf{I}_v(0) > 0$. On the other hand, (2.51) tells us that $\mathbf{I}_v\left(\frac{1}{2}\right) = 0$, which contradicts the monotonicity of the frequency for v.

□

2.8 Persistence of Q-points

We now construct our limiting Dir-minimizer using our blow-up sequence with varying centers x_k, which we assume converge to some point x. Recall that we wish to blow up at the scales r_k around x_k, along which T has a flat tangent cone as in Proposition 2.3.4. Thus, we need to take a diagonal sequence of center manifolds with the intervals of flattening $(s_k, t_k] \ni r_k$ centered at x_k, and rescale them appropriately so that the reverse Sobolev inequality of Corollary 2.6.7 holds at scale 1, analogously to that in the preceding section. As usual, we let $\mathcal{M}_k$ denote the center manifold at scale t_k around x_k, with corresponding current $T_k = T_{x_k,t_k} \llcorner \mathbf{B}_{6\sqrt{m}}$.

Let $\frac{\bar{s}_k}{t_k} \in \left(\frac{3r_k}{2t_k}, \frac{3r_k}{t_k}\right]$ be the scale at which the reverse Sobolev inequality holds for $r = \frac{r_k}{t_k}$. Then let $\bar{r}_k := \frac{2\bar{s}_k}{3t_k} \in \left(\frac{r_k}{t_k}, \frac{2r_k}{t_k}\right]$. Let

$$\bar{T}_k := (\iota_{0,\bar{r}_k})_\sharp T_k = \left((\iota_{x_k,\bar{r}_k t_k})_\sharp T\right) \llcorner \mathbf{B}_{\frac{6\sqrt{m}}{\bar{r}_k}}, \qquad \bar{\Sigma}_k := \iota_{0,\bar{r}_k}\Sigma_k, \qquad \overline{\mathcal{M}}_k := \iota_{0,\bar{r}_k}\mathcal{M}_k,$$

and let

$$\bar{\boldsymbol{m}}_0^{(k)} \leq \max\{\boldsymbol{c}(\bar{\Sigma})^2, \mathbf{E}(\bar{T}_k, \mathbf{B}_{6\sqrt{m}})\}.$$

Since the ambient manifolds $\bar{\Sigma}_k$ converge in $C^{3,\varepsilon_0}_{\mathrm{loc}}$ to $T_x\Sigma \cong \mathbb{R}^{m+\bar{n}} \times \{0\}$, for k sufficiently

large we have

$$\boldsymbol{c}(\bar{\Sigma}_k)^2 \longrightarrow 0.$$

We will use the notation $\bar{\mathbf{p}}_k$ for the projection map in Assumption 2.5.7 for the rescaled center manifold $\overline{\mathcal{M}}_k$. Moreover, $6\sqrt{m}\bar{r}_k \in \left(\frac{12\sqrt{m}\rho_k}{t_k}, \frac{24\sqrt{m}\rho_k}{t_k}\right]$, so Lemma 2.7.2 tells us that

$$\mathbf{E}(\bar{T}_k, \mathbf{B}_{6\sqrt{m}}) \leq C\mathbf{E}\big(T_k, \mathbf{B}_{\frac{24\sqrt{m}r_k}{t_k}}\big) \longrightarrow 0. \tag{2.54}$$

Thus, $\bar{\boldsymbol{m}}_0^{(k)} \to 0$.

We now argue analogously to that in the compactness argument in the proof of Theorem 2.6.8. Let π_∞ be the plane defining the flat tangent cone of T_k as in Proposition 2.3.4. We will let $\mathbf{p}_\infty$ denote the orthogonal projection to this plane.

Define

$$\bar{N}^{(k)} : \overline{\mathcal{M}}_k \to \mathbb{R}^{m+n}, \qquad \bar{N}^{(k)}(p) := \frac{1}{\bar{r}_k} N^{(k)}(\bar{r}_k p),$$

and let

$$u_k := \frac{\bar{N}^{(k)} \circ \mathbf{e}_k}{\mathbf{h}_k}, \qquad u_k : \pi_k \supset B_3 \to \mathcal{A}_Q(\mathbb{R}^{m+n}),$$

where $\mathbf{e}_k$ is the exponential map at $p_k := \frac{\mathbf{\Phi}_k(0)}{\bar{r}_k} \in \overline{\mathcal{M}}_k$ defined on $B_3 \subset \pi_k = T_{p_k}\overline{\mathcal{M}}_k$ and $\mathbf{h}_k := \|\bar{N}^{(k)}\|_{L^2(\mathcal{B}_{3/2})}$. As before, this decay of the excesses at our chosen scales $\frac{r_k}{t_k}$ further allows us to conclude that

$$\overline{\mathcal{M}}_k \longrightarrow \pi_\infty \quad \text{in } C^{3,\frac{\kappa}{2}}(\mathbf{B}_{\frac{4}{3}}), \qquad \mathbf{e}_k \longrightarrow \mathrm{id} : \pi_\infty \supset B_3 \to B_3 \quad \text{in } C^{2,\frac{\kappa}{2}}.$$

Now we pass the reverse Sobolev inequality that holds at scale $\frac{\bar{s}_k}{t_k}$ around x_k to the maps u_k:

$$\int_{B_{3/2}} |u_k|^2 \geq C \int_{B_{3/2}} |Du_k|^2.$$

Once again, this allows us to use both weak-$*$ compactness and the Rellich-Kondrachov Compact Embedding Theorem to deduce that there exists a map $u \in W^{1,2}(B_{3/2}; \mathcal{A}_Q)$ such

that up to subsequence,

$$u_k \longrightarrow u \quad \text{strongly in } L^2(B_{3/2}; \mathcal{A}_Q).$$

Combining this with the reverse Sobolev inequality, we can in fact further improve this strong convergence to that in $W^{1,2}_{\text{loc}}(B_{3/2}; \mathcal{A}_Q)$.

Moreover, by arguing exactly as in the proof of the lower frequency bound, u satisfies the following properties:

- u is non-trivial;
- $\boldsymbol{\eta} \circ u \equiv 0$ almost-everywhere on $B_{3/2}$;
- u is Dir-minimizing on $B_{3/2}$.

It remains to show that whenever $y_k \to y$, with $y_k \in \mathbf{B}_{\bar{r}_k} \cap \text{Sing}_{[Q,Q+\varepsilon]}(T_k)$, we have $u(y) = Q[\![0]\!]$. We already know that y_k is a Q-point for T_k, but this does not ensure that the excess should decay quickly enough close to y_k; we might need stop our center manifolds at scales close to these points. Even in the case where we can refine indefinitely around each y_k, it is still necessary to check that this behaviour persists in the limit as we take $k \to \infty$. We consider cases depending on the non-centered intervals of flattening for $\mathcal{M}_k$ around $\mathbf{p}_k(y_k)$. For $\tilde{s}_k := s(y_k)$ as in Definition 2.5.16, there are 3 possibilities:

(a) There is a subsequence of indices k for which $\tilde{s}_k = 0$; namely, $\mathbf{p}_k(y_k)$ which lie in the contact sets $\boldsymbol{\Phi}_k(\boldsymbol{\Gamma}_k)$ for the center manifolds $\mathcal{M}_k$,

(b) There is a subsequence of indices k with $\frac{\tilde{s}_k}{r_k} \downarrow 0$; namely, we stop refining around $\mathbf{p}_{\pi_k}(y_k)$ at scales that decay relative to the blow-up scales r_k,

(c) We have $\liminf_k \frac{\tilde{s}_k}{r_k} > 0$; we stop refining around $\mathbf{p}_{\pi_k}(y_k)$ at scales comparable to the blow-up scales.

Note that these points y_k are *not* the centers for our center manifold constructions; they are simply points lying in $\overline{\mathbf{B}}_1$ that are 'captured' by T_k. Let $\bar{y}_k := \frac{y_k}{\bar{r}_k} \in \mathbf{B}_1 \cap \mathrm{Sing}_{[Q,Q+\varepsilon]}(\bar{T}_k)$ and $\bar{y} = \lim_k \bar{y}_k$.

First of all, observe that case (a) was already handled in the proof of Theorem 2.6.8 (see case (a) therein). For the remaining two cases, the idea is as follows. We wish to establish persistence of Q-points at the blow-up scales r_k, by exploiting the uniform frequency bound of Theorem 2.6.8.

Unfortunately, the scales r_k may be very far from the scales at which we stop refining around y_k and inherit the behaviour from T_k. These stopping scales are those at which we may use the lower frequency bound to show that the singularities persist. We then propagate this persistence up to the larger scales r_k that we are interested in, via the almost-monotonicity of Theorem 2.6.4.

2.8.1 Case (b)

In this case, we proceed as follows. Observe that Theorem 2.6.8 guarantees that

$$c_0 \leq \mathbf{I}_{\bar{N}^{(k)}}\left(\bar{\mathbf{p}}_k(\bar{y}_k), r\right) \leq C_0 \qquad \text{for every } r \in \left(\frac{\tilde{s}_k}{\bar{r}_k t_k}, R\right],$$

for $R := \liminf_k \mathrm{dist}(\bar{\mathbf{p}}_k(\bar{y}_k), \partial\mathcal{B}_3)$, provided that we take k sufficiently large. Now the $W^{1,2}_{\mathrm{loc}}$-convergence of u_k to u allows us to establish convergence of the frequencies

$$\mathbf{I}_{\bar{N}^{(k)}}(\bar{\mathbf{p}}_k(\bar{y}_k), r) \longrightarrow \mathbf{I}_u(\bar{y}, r) \qquad \text{for each } r \in \left[\limsup_k \frac{\tilde{s}_k}{\bar{r}_k t_k}, R\right).$$

Combining this with the fact that (up to subsequence) $\frac{\tilde{s}_k}{r_k} \downarrow 0$ and $\bar{r}_k t_k \in (r_k, 2r_k]$, we deduce that

$$c_0 \leq \mathbf{I}_u(\bar{y}, r) \leq C_0 \qquad \text{for every } r \in (0, R].$$

We may now once again use Proposition 2.6.1 to reach the desired conclusion.

2.8.2 Case (c)

In this case, (up to subsequence) the scales $\tilde{s}_k$ at which we stop refining around $\mathbf{p}_k(y_k)$ are comparable to the blow-up scales r_k. Take the cubes $L_k \in \mathscr{W}^{(k)}$ with $\ell(L_k) \geq c_s \frac{\tilde{s}_k}{t_k}$ and $\operatorname{dist}(L_k, \mathbf{p}_{\pi_k}(y_k)) \leq \frac{\tilde{s}_k}{t_k}$.

We claim that $u(\bar{y}) = Q[\![0]\!]$. In this case, we are unable to exploit the uniform lower bound of Theorem 2.6.8, since this only tells us that $I_u(\bar{y}, \cdot)$ has a positive lower bound at some scale comparable to $\liminf_{k\to\infty} \frac{\tilde{s}_k}{r_k} > 0$. We instead argue as follows.

Consider the strong Lipschitz approximations $f_k : B_{8r_{L_k}}(p_{L_k}, \pi_{L_k}) \to \mathcal{A}_Q(\pi_{L_k}^{\perp})$ from Theorem 2.5.2 for T_k. Take $g_k : \mathcal{M}_k \to \mathcal{A}_Q(\mathbb{R}^{m+n})$ to be the reparameterization N from [20, Theorem 5.1] of $f = f_k$ to the center manifolds. We may now rescale and normalize g_k in the same way as the $\mathcal{M}_k$-normal approximations $N^{(k)}$. Namely, we define $\bar{g}_k : \overline{\mathcal{M}}_k \to \mathcal{A}_Q(\mathbb{R}^{m+n})$ and $v_k : \pi_k \supset B_3 \to \mathcal{A}_Q(\mathbb{R}^{m+n})$ by

$$\bar{g}_k(p) := \frac{1}{\bar{r}_k} g_k(\bar{r}_k p), \qquad v_k := \frac{\bar{g}_k \circ \mathbf{e}_k}{\mathbf{h}_k}.$$

Then, for k sufficiently large, we have

$$\begin{aligned}
\int_{B_{3/2}} \mathcal{G}(v_k, u_k)^2 &\leq C\mathbf{h}_k^{-2} \int_{\mathcal{B}_{3/2}} \mathcal{G}(\bar{g}_k, \bar{N}^{(k)})^2 \\
&\leq C\mathbf{h}_k^{-2} \bar{r}_k^{-m-2} \int_{\mathcal{B}_{3/2\bar{r}_k}} \mathcal{G}(g_k, N^{(k)})^2 \\
&\leq C\mathbf{h}_k^{-2} \bar{r}_k^{-m-2} \int_{\mathbf{\Phi}_k(B_{3/2\bar{r}_k} \setminus K_k)} \mathcal{G}(g_k, N^{(k)})^2,
\end{aligned}$$

where $\mathbf{\Phi}_k$ is the map parameterizing the center manifold $\mathcal{M}_k$ over the plane π_k, as given in Definition 2.5.6. Let $E_k := \mathbf{E}(T_k, \mathbf{C}_{32r_{L_k}}(p_{L_k}), \pi_{L_k})$. Notice that since $r_{L_k} \simeq_m \ell(L_k) \simeq_m \bar{r}_k$, we have $E_k \to 0$. Furthermore, since $\Theta(T_k, y_k) \geq Q$, we necessarily have $L_k \in \mathscr{W}_e^{(k)}$. Therefore, the stopping condition (EX), the splitting before tilting in Proposition 2.5.10,

and the improved height bound in Lemma (2.5.14) tell us that

$$E_k \simeq_m \boldsymbol{m}_0^{(k)} \left(\frac{r_k}{t_k}\right)^{2-2\delta_2} \simeq_m \left(\frac{r_k}{t_k}\right)^{-m-2} \int_{\mathcal{B}_{\frac{\sigma r_k}{t_k}}(\mathbf{p}_k(y_k))} |N^{(k)}|^2 \simeq_m \mathbf{h}_k^2, \tag{2.55}$$

for some choice of $\sigma > 0$ that is independent of k. We may now use the L^∞-estimates (2.18) and (2.21), the height bounds (2.17), the $C^{3,\kappa}$-estimate (i) of Proposition 2.5.5, and the bound in Theorem 2.5.2 for the sets K_k over which $T_k \llcorner \mathbf{C}_{32r_{L_k}}(p_{L_k}, \pi_{L_k})$ is not graphical, to achieve the estimate

$$\int_{\mathbf{\Phi}_k(B_{3/2\bar{r}_k}\setminus K_k)} \mathcal{G}(g_k, N^{(k)})^2 \leq C\bar{r}_k^{2+2\beta_2} E_k^{\frac{1}{m}} \left|B_{3/2\bar{r}_k} \setminus K_k\right| \leq C\bar{r}_k^{m+2+2\beta_2} E_k^{\frac{1}{m}+1+\gamma_1}.$$

Combining this with the above control on the L^2-distance between v_k and u_k, the comparability (2.55) of the L^2 height and the excess, and the excess decay $E_k \longrightarrow 0$, we deduce that

$$\int_{B_{3/2}} \mathcal{G}(v_k, u_k)^2 \leq C\bar{r}_k^{2\beta_2} E_k^{\frac{1}{m}+\gamma_1} \longrightarrow 0.$$

Due to the convergence of u_k to u, this further tells us that v_k converges in $L^2(B_{3/2}; \mathcal{A}_Q)$ on to the same Dir-minimizer u. It remains to obtain persistence of Q-points for the sequence of maps v_k. Indeed, since $\Theta(T_k, y_k) \geq Q$, Theorem 2.5.3(2.20) tells us that there exists $\gamma > 0$ for which

$$\int_{B_{s\bar{r}_k}(\mathbf{p}_{\pi_{L_k}}(y_k))} \mathcal{G}(f_k, Q\llbracket \boldsymbol{\eta} \circ f_k \rrbracket)^2 \leq Cs^{m+\gamma}\bar{r}_k^{m+2} E_k,$$

for every $s > 0$ sufficiently small. Applying the estimates in [20, Theorem 5.1], rescaling, making use of Theorem 2.5.2(2.18) and (2.55), and using the control [21, Proposition 4.1(iv)] on the tilting between the planes π_{L_k} and π_k, we have

$$\begin{aligned}\int_{B_s(\mathbf{p}_{\pi_k}(\bar{y}_k))} |v_k|^2 &\leq C\mathbf{h}_k^{-2}\bar{r}_k^{-m-2} \int_{B_{s\bar{r}_k}(\mathbf{p}_{\pi_k}(y_k))} |g_k \circ \mathbf{\Phi}_k|^2 \\ &\leq C\mathbf{h}_k^{-2}\bar{r}_k^{-m-2} \int_{B_{s\bar{r}_k}(\mathbf{p}_{\pi_{L_k}}(y_k))} \mathcal{G}(f_k(p), Q\llbracket \boldsymbol{\varphi}_k(p) \rrbracket)^2\end{aligned}$$

$$\leq C\mathbf{h}_k^{-2}\bar{r}_k^{-m-2}\int_{B_{s\bar{r}_k}(\mathbf{p}_{\pi_{L_k}}(y_k))}\mathcal{G}(f_k(p),Q[\![\boldsymbol{\eta}\circ f_k]\!])^2$$
$$+C\mathbf{h}_k^{-2}s^m\bar{r}_k^{-2}\operatorname{Lip}(f_k)^2\|\boldsymbol{\varphi}_k\|_{C^1}^2\|g_k\circ\boldsymbol{\Phi}_k\|^2_{L^\infty(B_{\bar{r}_k}(\mathbf{p}_{\pi_k}(y_k)))}$$
$$\leq Cs^{m+\gamma}+Cs^m\bar{r}_k^{2\beta_2}E_k^{2\gamma_1+\frac{1}{m}},$$

for every $s<s_0$, where s_0 is independent of k. Taking $k\to\infty$, again using the excess decay, and combining with the strong L^2-convergence of v_k to v, we have

$$\int_{B_s(\bar{y})}|v|^2\leq Cs^{m+\gamma}.$$

Thus, applying Proposition 2.6.1, we are able to conclude that $I_v(\bar{y},0)\geq\frac{\gamma}{2}$ and that $v(\bar{y})=Q[\![0]\!]$ as expected.

2.9 Proof of the results in Section 2.2

Let us prove the preliminary results contained in Section 2.2. We begin with the proof of Lemma 2.2.1. The proof of all but (2.2) appears in [53], but nevertheless we repeat the argument here.

Before we begin with the proof, let us make the following important observation. Assuming the validity of Lemma 2.2.1,

$$\mathscr{A}(C)=\left\{\alpha\geq 0:\limsup_{r\downarrow 0}\sum_{K\in\mathscr{C}}r^\alpha N(K,r)=0\right\}. \tag{2.56}$$

Indeed, if $\alpha\in\mathscr{A}(\mathscr{C})$, then the Work Raccoon Lemma 2.2.1 allows us to choose $\beta\in(\alpha_0,\alpha)$, so we have

$$N(K,r)r^\alpha\leq r^{\alpha-\beta}N(K,r)r^\beta\leq C(\beta)r^{\alpha-\beta}\to 0\qquad\text{as } r\downarrow 0.$$

Conversely, if

$$\limsup_{r\downarrow 0}\sup_{K\in\mathscr{C}} r^{\alpha}N(K,r)=0,$$

then given any $K\in\mathscr{C}$, for any $\delta>0$ we can find a covering $\mathscr{U}_\delta$ of K by $N(K,\delta)$ open $(m+n)$-dimensional balls of radius δ. Then

$$\sum_{\mathbf{B}\in\mathscr{U}_\delta}(\operatorname{diam}\mathbf{B})^{\alpha}=2\delta^{\alpha}N(K,\delta).$$

Taking $\delta\downarrow 0$, the claim follows from the definition of the α-dimensional Hausdorff measure.

In view of the above, it is therefore crucial to prove that the half-line is *open* to make the identification (2.56). Hence, in order to estimate the upper Minkowski content uniformly over the compact family $\mathscr{C}$, it suffices to estimate the Hausdorff dimension uniformly.

Proof of Lemma 2.2.1. The fact that $\mathscr{A}(\mathscr{C})$ is an upper half-line is clear, since it is the intersection of upper half-lines, so to establish the first statement of the theorem, we just needs to show that it is open. Fix $\beta\in\mathscr{A}(\mathscr{C})$. We may assume that $\beta>0$, since $0\notin\mathscr{A}(\mathscr{C})$. Indeed, the closure under rescalings about all possible centers guarantees that $\mathcal{H}^0(K)=0$ for every $K\in\mathscr{C}$ only if $\mathscr{C}=\emptyset$.

Given any $K\in\mathscr{C}$ (recall that K is compact), we can then find a finite cover $\mathscr{U}(K):=\{\mathbf{B}_{r_i}(x_i)\}_{i=1}^N$ of K by open $(m+n)$-dimensional balls such that

$$\sum_{i=1}^{N} r_i^{\beta}<\frac{1}{2}.$$

Letting

$$\mathcal{W}(K):=\bigcup_{i=1}^{N}\mathbf{B}_{r_i}(x_i),$$

we know that we can find a finite collection of sets $K_1,...,K_M$ such that for any $K\in\mathscr{C}$, there exists $j\in\{1,...,M\}$ such that $K\subset\mathcal{W}(K_j)$. This is due to the compactness of $\mathscr{C}$ with respect to the Hausdorff topology.

Now write

$$\mathscr{U}(K_j) = \{\mathbf{B}_{r_{j,i}}(z_{j,i})\}_{i=1}^{N(K_j)}$$

to denote the balls from $\mathcal{W}(K_j)$. Since our collection is finite, we can certainly find $\alpha < \beta$ such that

$$\sum_{i=1}^{N(K_j)} r_{j,i}^{\alpha} \leq \frac{1}{2}$$

for each j.

We will show that $\alpha \in \mathscr{A}(\mathscr{C})$ by inductively replacing each cover $\mathscr{U}(K_j)$ by improved covers with reduced overlaps, which satisfy an α-dimensional packing condition. At each stage we will replace a ball in the cover by a collection of smaller balls that cover K more efficiently. We will later see how to further improve this efficient covering to one where the balls have comparable radii at each stage, in order to deduce that the stronger property (2.2) holds for the exponent α.

Fix any $K \in \mathscr{C}$, and select K_j such that $K \subset \mathcal{W}(K_j)$. At the first stage of our subdivision, let $\mathscr{U}_1 = \mathscr{U}(K_j)$.

Assume that we have constructed $\mathscr{U}_\ell$. Take any $\mathbf{B}_r(x) \in \mathscr{U}_\ell$. Choose $j = j(x,r)$ such that the blow-up $K_{x,r}$ satisfies

$$K_{x,r} \cap \mathbf{B}_1 \subset \mathcal{W}(K_{j(x,r)}),$$

with corresponding family

$$\mathscr{U}(K_{j(x,r)}) = \{\mathbf{B}_{r_{j(x,r),i}}(z_{j(x,r),i})\}$$

of balls in $\mathcal{W}(K_{j(x,r)})$. Now we scale these balls back down to the correct size for K itself, to get a family

$$\mathscr{S}(x,r) := \{\mathbf{B}_{r r_{j(x,r),i}}(x + r z_{j(x,r),i})\}_i.$$

Then we have

$$\sum_{\mathbf{B}_\rho(w)\in\mathscr{S}(x,r)} \rho^\alpha = \sum_i r^\alpha r_{j(x,r),i}^\alpha \leq \frac{1}{2} r^\alpha. \tag{2.57}$$

So indeed, this provides a more efficient covering of the portion of K inside $\mathbf{B}_r(x)$, than $\mathbf{B}_r(x)$ itself. Doing this for each ball within the family $\mathscr{U}_\ell$, we may thus naturally define

$$\mathscr{U}_{\ell+1} := \bigcup_{\mathbf{B}_r(x)\in\mathscr{U}_\ell} \mathscr{S}(x,r).$$

Summing the packing estimate (2.57) over the entire family $\mathscr{U}_{\ell+1}$ yields

$$\sum_{\mathbf{B}_\tau(x)\in\mathscr{U}_{\ell+1}} \tau^\alpha = \sum_{\mathbf{B}_r(x)\in\mathscr{U}_\ell} \sum_{B_\rho(w)\in\mathscr{S}(x,r)} \rho^\alpha \leq \frac{1}{2} \sum_{\mathbf{B}_r(x)\in\mathscr{U}_\ell} r^\alpha.$$

Hence, by our choice of $\mathscr{U}_1$,

$$\sum_{\mathbf{B}_\tau(x)\in\mathscr{U}_{\ell+1}} \tau^\alpha \leq \frac{1}{2^{\ell+1}}.$$

We can continue refining our cover in this way indefinitely, taking $\ell \uparrow \infty$, which tells us that $\mathcal{H}^\alpha(K) = 0$.

It remains to show the uniform boundedness of the upper Minkowski contents over the compact family $\mathscr{C}$. We will use the above efficient covering for the sets in our family and will amend it appropriately using compactness to further ensure that the radii of the balls are comparable at each stage. This additional improvement will be precisely what we need in order to obtain the strengthened packing estimate (2.2).

First of all, find the sizes of the smallest and largest balls among all the covers for all of the sets K_j, $j = 1, ..., M$:

$$\bar{r} := \min\left\{ r_{j,i} : \mathbf{B}_{r_{j,i}}(z_{j,i}) \in \mathscr{U}(K_j) \right\}_{i,j},$$
$$\bar{R} := \max\left\{ r_{j,i} : \mathbf{B}_{r_{j,i}}(z_{j,i}) \in \mathscr{U}(K_j) \right\}_{i,j}.$$

Notice that by construction, we necessarily have $\bar{R} \leq 2^{-\frac{1}{\alpha}} < 1$. For $K \in \mathscr{C}$ fixed as

before, let $\rho(\ell)$, $R(\ell)$ denote the radii of the smallest and largest balls in $\mathscr{U}_\ell$ respectively, we have

$$\rho(\ell) \geq \bar{r}\rho(\ell-1), \tag{2.58}$$

with equality whenever we refine one of the balls in our covering with a rescaled cover that includes a rescaled ball of minimal size (among all the $\mathscr{U}(K_j)$), between stage $\ell-1$ and ℓ.

We want to ensure that $R(\ell)$ is comparable to $\rho(\ell)$. If this is not the case, namely if

$$\frac{R(\ell)}{\rho(\ell)} > \bar{r}^{-1},$$

then we subdivide all the large balls $\mathbf{B}_\tau(w) \in \mathscr{U}_\ell$ with $\tau > \bar{r}^{-1}\rho(\ell)$, several times if necessary, to obtain a new covering $\mathscr{U}_\ell'$ such that

- $\sum_{\mathbf{B}_\tau(w)\in\mathscr{U}_\ell'} \tau^\alpha \leq \frac{1}{2^\ell}$,
- the radius of the largest ball is at most $\bar{R}^k R(\ell) \leq \bar{r}^{-1}\rho(\ell)$ for $k \in \mathbb{N}$ sufficiently large.

Note that the radius of the smallest ball in the cover remains the same in this procedure, since for any ball that we subdivide, the new balls will have radius at least $\rho(\ell)\bar{r}^{-1} > \rho(\ell)$.

Thus (not relabelling the new covers), we may assume that

$$R(\ell) < \bar{r}^{-1}\rho(\ell) \qquad \text{for each } \ell.$$

Combining this with (2.58), we further have

$$\rho(\ell) \in [\bar{r}^\ell, \bar{r}^{\ell-1}).$$

We claim that this new even more tightly packed family of coverings gives the packing condition for our arbitrary fixed compact set $K \in \mathscr{C}$. Choose any $r \in (0,1]$. Find the

smallest $\ell \in \mathbb{N}$ such that $R(\ell) < r$, namely,

$$R(\ell) < r \leq R(\ell - 1).$$

We want to consider a covering by balls of radius exactly r, so we may enlarge all the balls in our cover $\mathscr{U}_\ell$ to concentric ones with radius r that still cover K. This in particular tells us that

$$N(K, r) \leq \#\mathscr{U}_\ell.$$

We may now exploit the comparability of the sizes of all of these balls. Namely, we obtain

$$\begin{aligned} N(K,r)r^\alpha &\leq N(K,r)R(\ell-1)^\alpha \\ &\leq N(K,r)\bar{r}^{-\alpha}\rho(\ell-1)^\alpha \\ &\leq \bar{r}^{-\alpha-1}N(K,r)\rho(\ell)^\alpha \\ &\leq C(\alpha)\sum_{\mathbf{B}_\tau(w)\in\mathscr{U}_\ell}\tau^\alpha \\ &\leq C(\alpha). \end{aligned}$$

This concludes the proof. □

Let us now prove the statement of Proposition 2.2.2, which allows us to apply the Work Raccoon Lemma 2.2.1 to the family $\mathscr{C}(\varepsilon, \tilde{K})$.

Proof of Proposition 2.2.2. For ease of notation, we will omit dependencies on ε and $\tilde{K}$. Clearly $\mathscr{C}$ contains K^0, since one can take $x_k \equiv 0$ and $r_k \equiv 1$. The closure under rescaling property is trivial to check by simply rescaling the convergent sequence accordingly. The closure under Hausdorff convergence is proven as follows:

Suppose that $\{K_j\}_j \subset \mathscr{C}$, with $K_j^\infty \supset K_j \xrightarrow{d_H} K$. For each j, we obtain a sequence $K^0_{x_{j,k},r_{j,k}}$ with

$$K^0_{x_{j,k},r_{j,k}} \cap \overline{\mathbf{B}}_1 \xrightarrow{d_H} K_j^\infty.$$

But then for each j, we can find $k(j)$ large enough and a subset $\tilde{K}^0_j \subset K^0_{x_{j,k(j)},r_{j,k(j)}}$ such that

$$d_H(\tilde{K}^0_j \cap \overline{\mathbf{B}}_1, K_j) \leq \frac{1}{2^j}.$$

Combining this with the convergence of K_j to K, we conclude that

$$d_H(\tilde{K}^0_j \cap \overline{\mathbf{B}}_1, K) \longrightarrow 0 \qquad \text{as } j \to \infty.$$

In particular, the diagonal subsequence $\{K^0_{x_{j,k(j)},r_{j,k(j)}}\}_j$ of blow-ups can be taken for K, in order to see that it indeed lies in $\mathscr{C}(\varepsilon, \tilde{K})$. □

2.10 Non-integer multiplicity points

We give a proof of the $(m-3)$-rectifiability and (local) upper Minkowski content bound for those singular points of T that have non-integer multiplicities. Before we begin the proof of Lemma 2.1.3, let us recall some notation from [43]. Recalling the definition of k-th stratum $\mathcal{S}^{(k)}(T)$ as in Section 2.1, we will henceforth use the terminology *k-symmetric* for $k \in \{0, \ldots, m\}$ to refer to a cone that has k dimensions of translational invariance. For fixed $\rho > 0$ and any $\delta > 0$, the quantitative *k-th δ-stratum* $\mathcal{S}^k_\delta(T, \rho)$ at scale r is given by

$$\mathcal{S}^k_\delta(T,\rho) := \left\{ x \in \operatorname{spt}(T) \setminus \operatorname{spt}(\partial T) : \begin{array}{l} \mathbf{F}\big((\mathbf{e}_x)_\sharp(T_{x,s}) \llcorner B_1, V \llcorner B_1\big) \geq \delta s \ \ \forall s \in (0,\rho) \text{ and} \\ \text{any area-min. } (k+1)\text{-symm. cone } V \in \mathcal{I}^m(T_x\Sigma) \end{array} \right\}$$

where $\mathbf{e}_x$ is the exponential map $\mathbf{e}_x : T_x\Sigma \to \Sigma$, $B_1 \subset T_x\Sigma$. Recall that we call a varifold $V \in \mathcal{I}^m(\mathbb{R}^{m+\bar{n}})$ a *k-symmetric cone* if $(\iota_{0,r})_\sharp V = V$ for every $r > 0$ and if there exists a k-plane $\pi \subset \mathbb{R}^{m+\bar{n}}$ such that $(\tau_y)_\sharp V = V$ for each $y \in \pi$.

Note that for area minimizing currents without boundary, one can replace the varifold metric in the above definition with the flat metric, since the two are then equivalent.

The result [43, Theorem 1.4] then states that given any $\rho, \delta > 0$ the k-dimensional upper

Minkowski content of the stratum $\mathcal{S}^k_\delta(T,\rho)$ is locally bounded and $\mathcal{S}^k_\delta(T,\rho)$ is k-rectifiable.

Proof of Lemma 2.1.3. Fix $\varepsilon > 0$, $Q \in \mathbb{N}\setminus\{0\}$ and any Ω as in the statement of the lemma. Let

$$E := \mathrm{Sing}_{\geq Q+\varepsilon}(T) \cap \overline{\Omega}.$$

Clearly E is compact, due to the upper-semicontinuity of the density. We will proceed to show that

$$E \subset \mathcal{S}^{m-3}_\delta(T,\rho) \qquad \text{for some scale } \rho \text{ and proximity threshold } \delta > 0.$$

The idea is to show that if T is sufficiently $\mathbf{F}$-close to an area minimizing $(m-2)$-symmetric cone $V \in \mathcal{I}^m(T_x\Sigma)$ somewhere, this forces the mass ratio of T to remain sufficiently close to an integer value (since the density of V is always an integer). This will be inconsistent with our assumption that we are restricted to points of non-integer density. More specifically, we claim that there exists a scale $\rho > 0$ and a parameter $\eta \in (0,1)$ for which

$$\|T\|(\mathbf{B}_s(p)) \leq (Q+1-\eta)\omega_m s^m \qquad \text{for every } p \in E \text{ and every } s \in (0,\rho). \tag{2.59}$$

Indeed, if this is not the case then we can extract a sequence of centers $p_k \in E$ and scales $s_k, \eta_k \downarrow 0$ for which this fails, namely

$$\|T\|(\mathbf{B}_{s_k}(p_k)) > (Q+1-\eta_k)\omega_m s_k^m.$$

Up to subsequence (not relabelled), we may assume that p_k converges pointwise to some singular Q-point $p \in E$. The monotonicity formula and the convergence of the masses $\|T\|(\mathbf{B}_r(p_k)) \to \|T\|(\mathbf{B}_r(p))$ for all but a countable number of radii $r > 0$ then tells us that

for any such $r > 0$,

$$\frac{\|T\|(\mathbf{B}_r(p))}{\omega_m r^m} = \lim_{k\to\infty} \frac{\|T\|(\mathbf{B}_r(p_k))}{\omega_m r^m} \geq \liminf_{k\to\infty} e^{C\mathbf{A}(s_k - r)} \frac{\|T\|(\mathbf{B}_{s_k}(p_k))}{\omega_m s_k^m} \geq e^{-C\mathbf{A}r}(Q+1).$$

In particular, $\Theta(T,p) \geq Q+1$, which contradicts the fact that $p \in E$.

Now we may combine (2.59) with the properties of points in $\mathcal{S}_\delta^{m-3}(T,\rho)$ to reach a contradiction as follows. Observe that the following criteria holds:

(a) If $V \in \mathcal{I}^m(T_x\Sigma)$ is an $(m-2)$-symmetric area minimizing cone, then necessarily $\Theta(S,0) = \frac{\|S\|(\mathbf{B}_1)}{\omega_m}$ is a positive integer;

(b) One can choose $\delta > 0$ sufficiently small such that if $\mathbf{F}((\mathbf{e}_x)_\sharp(T_{x,s}) \llcorner B_1, V \llcorner B_1) < \delta s$, then

$$\big|\|T\|(\mathbf{B}_s(x)) - \|S\|(B_1)\big| < \min\Big\{\frac{\eta}{2}, \frac{\varepsilon}{4}\Big\} s^m \qquad \text{for any } x \in \mathbb{R}^{m+n};$$

(c) By the monotonicity formula, for any $x \in E$ and any $s < \rho$ sufficiently small we have

$$\|T\|(\mathbf{B}_s(x)) \geq e^{-C\mathbf{A}s}\Theta(T,x)\omega_m s^m \geq \big(Q + \frac{\varepsilon}{2}\big)\omega_m s^m.$$

The second property follows from the fact that

$$\mathbf{F}(V_1, V_2) = \mathcal{F}(V_1, V_2) + \mathbf{F}(|V_1|, |V_2|) \qquad \text{for any } V_1, V_2 \in \mathcal{I}^m.$$

Thus, choosing $\delta > 0$ sufficiently small as in (b), if there is a point $x \in E$ for which $\mathbf{F}((\mathbf{e}_x)_\sharp(T_{x,s}) \llcorner B_1, V \llcorner B_1) < \delta s$ for some $s < \rho$ and some $(m-2)$-symmetric area minimizing cone $V \in \mathcal{I}^m(T_x\Sigma)$, then

$$Q + \frac{\varepsilon}{4} \leq \frac{\|S\|(\mathbf{B}_s)}{\omega_m s^m} \leq Q + 1 - \frac{\eta}{2}.$$

This, however, contradicts (a), so the claimed inclusion $E \subset \mathcal{S}_\delta^{m-2}(T,\rho)$ indeed holds true

for $\rho > 0$ as in (2.59) and $\delta > 0$ as in (b). $\square$

Chapter 3

The fine structure of the singular set of area-minimizing integral currents: Singularity degree and rectifiability of singularities with degree > 1

3.1 Introduction

Suppose that T is an m-dimensional integral current in a complete smooth Riemannian manifold Σ. We assume that T is area-minimizing in some (relatively) open $\Omega \subset \Sigma$.[1] We use the same notation and termoniology as in Chapter 2.

In this chapter and the succeeding one, we prove the following counterpart of Simon's rectifiability theorem (see [46]) in higher codimension. This strengthens Almgren's dimension estimate [3] and the Minkowski dimension estimate established in Chapter 2.

Theorem 3.1.1. *Let T be an m-dimensional area-minimizing current in a C^{3,κ_0} complete Riemannian manifold of dimension $m + \bar{n} \geq m + 2$, with $\kappa_0 > 0$. Then* $\mathrm{Sing}(T)$ *is* $(m-2)$-

[1] Namely, we take only competitors supported in Ω.

rectifiable and there is a unique tangent cone at $\mathcal{H}^{m-2}$*-a.e.* $q \in \mathrm{Sing}(T)$.

As already mentioned, Theorem 3.1.1 can be improved in the case of $m = 2$, in which it is known that the singularities are isolated, see [6] and [23–25]. Moreover, the uniqueness of tangent cones in the latter case is known since the work [53] of White in the eighties.

On the other hand, the work of Liu [37] demonstrates that one cannot hope for area-minimizing surfaces of high codimension to avoid having fractal singularities if the metric on the ambient space is merely smooth rather than real analytic. Indeed, inspired by the recent work of Simon [47] in the setting of stable minimal hypersurfaces, it is shown that with a smooth ambient metric and sufficiently high codimension, one can prescribe arbitrary fractal singularities of codimension at least two for high codimension area-minimizers. Moreover, in the succeeding work [38], Liu further verifies that some such singularities can be made to persist under generic perturbations of the ambient metric. The singularities produced in these works always lie in the lower strata $\mathcal{S}^{(k)}(T)$, $k \leq m-2$, however, so the question of prescribing lower dimensional fractal sets of points with flat tangent cones remains open. Furthermore, it may be possible to improve the rectifiability statement if one takes a less stringent definition of $\mathrm{Sing}(T)$, because the examples of [37] are locally *immersed* submanifolds.

Our techniques do not show that $\mathrm{Sing}(T)$ has locally finite $\mathcal{H}^{m-2}$-measure but merely that we may decompose the flat singularities into countably many explicitly-defined pieces, each of which has locally finite measure. Local finiteness of $\mathcal{H}^{m-2}$-measure could be expected, and the general uniqueness of tangent cones remains widely open.

3.1.1 Flat singularities

The main issue is to establish the $(m-2)$-rectifiability of those singular points where at least one tangent cone is flat, since the remaining portion of the singular set is, by [43], $(m-2)$-rectifiable. However, in Chapter 4, we independently establish the $(m-2)$-rectifiability of the singular points with non-flat tangent cones as a consequence of the results therein.

Recall that we refer to a point $p \in \mathrm{Sing}(T)$ with at least one flat tangent cone is called a *flat singularity*. We know from the constancy theorem (cf. [27]) that a flat tangent cone at a point q must be an oriented m-dimensional plane counted with a positive integer multiplicity Q. The latter is indeed the density of the current at q and Allard's celebrated regularity theorem [1] guarantees that if $Q = 1$ the point is regular. We emphasize that the striking difference in complexity between the codimension one case and the case of higher codimension hinges on the fact that, in higher codimension, flat singular points might exist, while they cannot in codimension one. The latter phenomenon is due to the local characterization of integral hypercurrents as superpositions of boundaries of Caccioppoli sets (cf. [44, Theorem 27.6, Corollary 27.8]), which is very specific to the codimension one setting. The typical examples of area minimizers with flat singular points in higher codimension are branching singularities of holomorphic subvarieties of Kähler manifolds. Note moreover that the uniqueness of the tangent cone is a priori not known at flat singular points, even under the stronger assumption that *all* tangent cones at the considered point are flat.

In this chapter we begin with introducing a suitable notion of "singularity degree" of T at flat singular points. This is a real parameter which will be then used to suitably subdivide the set of flat singular points of T. It provides a way to make sense of the infinitesimal order of contact between T and the average of its sheets at flat singularities.

Example 3.1.2. We illustrate the intuition behind the singularity degree in the example of a holomorphic curve in $\mathbb{C}^2$, defined by

$$\Lambda := \{(w - h(z))^Q = z^p k(z) : (z, w) \in \mathbb{C}^2\}\,.$$

In this example we require that:

- $p > Q \geq 2$ are coprime integers;
- h and k are holomorphic functions;

- $k(0) \neq 0$.

Recall that, by Federer's classical theorem, Λ (with the standard orientation given by the complex structure) induces a 2-dimensional integral area-minimizing current $T = [\![\Lambda]\!]$ in $\mathbb{R}^4 \cong \mathbb{C}^2$. Since p is not a multiple of Q and the latter is strictly larger than 1, the origin is an interior singular point of T. Moreover, since p and Q are coprime and p is larger than Q, the (unique) tangent cone to T at 0 is given by $Q[\![\{w = 0\}]\!]$. In this particular example our notion of singularity degree of T at the flat singular point 0 gives the number p/Q. In particular, note that it ignores any possible lower degrees in the Taylor expansion of the holomorphic function h.

3.1.2 Singularity degree

A priori we have very little knowledge of the structure of the singularities at a general flat singular point of an area-minimizing current of arbitrary dimension and codimension. Thus, our definition of singularity degree will necessarily be somewhat involved. In particular, given a flat singular point q, we will first identify a suitable analytical definition of singularity degree for a given infinitesimal sequence $\{r_k\}$ of blow-up scales along which the rescaled currents T_{q,r_k} (cf. Section 3.2 for the definition) converge to a flat tangent cone. These numbers, which might depend on $\{r_k\}$, will be called *singular frequency values*, cf. Definition 3.2.6. The singularity degree of T at a flat singular point x will then be defined as the infimum of the singular frequency values at x, cf. Definition 3.2.8. We will prove a series of interesting properties related to the singularity degree, among which we select the following three:

(i) we will show that the singularity degree is necessarily at least 1, due to the Hardt-Simon inequality and we will show that the singular frequency values all coincide with the singularity degree, i.e. they are the same number, independent of the subsequence, cf. Theorem 3.2.9;

(ii) for each infinitesimal blow-up scale we will, up to extraction of a subsequence, identify

a suitable rescaled limit, which will be an homogeneous multivalued function and whose degree of homogeneity is indeed the singularity degree, cf. Theorem 3.2.10(i);

(iii) when the singularity degree is strictly larger than 1 we will show that the (flat) tangent cone at x is unique and the current decays to it polynomially fast, cf. Theorem 3.2.10(iv).

We will then show that the set of flat singular points where the singularity degree is strictly larger than 1 is $(m-2)$-rectifiable, while in Chapter 4 we will complete the proof by showing that the set of flat singular points where the singularity degree is 1 is $\mathcal{H}^{m-2}$ negligible. Concerning the uniqueness of the tangent cone, in this chapter we show that it is unique at flat singular points where the singularity degree is strictly larger than 1, while Chapter 4 will complete the proof by showing $\mathcal{H}^{m-2}$-a.e. uniqueness.

The three properties (i)-(ii)-(iii), together with the BV estimate of Proposition 3.6.2, will be fundamental in establishing the proof of Theorem 3.1.1.

3.1.3 Comparison with the work of Krummel & Wickramasekera

At the same time this work was being finished, Krummel & Wickramasekera were independently completing a program also establishing Theorem 3.1.1 (see [32–34]). Here we take a moment to discuss the differences and similarities between the two programs, each point addressing a key aspect of each of the three papers in each of the programs. One underlying theme in both programs is to relate structural properties of the singular set to the rate of decay of the current at certain points to its tangent cone.

- In both approaches a monotonicity formula plays an important role in the first step. In our approach, Almgren's monotonicity formula enters to associate to flat singular points (namely, singular points at which at least one tangent cone is supported on a plane) a real number, referred to as the singularity degree, which takes values at least 1. This number is morally the infinitesimal homogeneity of the current relative

to the average of its "sheets" (the role of which is played by center manifolds which are possibly varying with the scale). A byproduct is that, when the singularity degree is strictly larger than 1, the rate of decay to the tangent plane is at least a power law. This is accomplished in the present chapter. In their approach, Krummel & Wickramasekera define a "planar frequency function" at the level of the current (see [33]), whose definition does not require the introduction of a center manifold, and show that it satisfies a suitable approximate monotonicity whenever the current is decaying to a plane on some interval of radii about a given point. Using this, they prove a certain decomposition theorem holds for the singular set, namely that locally about points of density Q (for given $Q \in \mathbb{Z}_{\geq 1}$), the singular set splits into two disjoint sets, namely a relatively closed set (denoted in [33] by $\mathcal{B}$) where the current is decaying with a power law at all scales to a tangent plane with a fixed lower bound on the decay rate, and a set which satisfies a uniform weak approximation property. The latter set could still contain flat singular points. In our approach the analogous set to $\mathcal{B}$ would be the flat singular Q-points of singularity degree at least $1 + \delta$ in some appropriately small ball. Strictly speaking the two sets do not coincide because the set $\mathcal{B}$ in [33] has some uniform control in the prefactor of the power-lay decay to the unique flat tangent. This uniform control could possibly be achieved by making some of our arguments more quantitative.

- In both cases, one exploits the power law decay rate at each "good" flat singular point (i.e. points where the singularity degree is strictly larger than 1 in our setting, whilst for Krummel & Wickramasekera it is the subset $\mathcal{B}$ described above), in order to prove $(m-2)$-rectifiability for this subset. For our program, this is achieved in the latter part of this chapter, whilst for Krummel & Wickramasekera this is achieved in forthcoming work [34]. However, in Krummel & Wickramasekera's work, the construction of a center manifold is only needed to study flat singular points where not only is the tangent plane unique, but additionally the current is decaying at least quadratically to

this tangent plane. In such a setting, the center manifold construction is much simpler (one does not need to deal with intervals of flattening or changing center manifolds as described in Section 3.2, for example). The reason for this is that they are able to study the set of flat singular points in the set $\mathcal{B}$ described above at which the decay rate to the tangent plane is a power law with order strictly less than 2 via their planar frequency function.

- In both approaches one must also deal with "slowly decaying" flat singular points; in our works this is when the decay value is exactly 1 and for Krummel & Wickramasekera these points are contained in the second set of their decomposition theorem described above. This part is highly non-trivial, and in both programs it is shown that the relevant set is $\mathcal{H}^{m-2}$-null. For us, this is addressed in Chapter 4 and for Krummel & Wickramasekera this is handled in [32].

It should be noted that aside from the definition of our singularity degree a priori requiring center manifolds (which are a posteriori not necessary in the slow decay case), the order of the last two points above is irrelevant for concluding the program. One could conduct them in either order, and indeed in our case the last point above is chronologically the last step whilst in Krummel & Wickramasekera's program it is the second step.

One difference between the two sets of works is that our results are all in the general setting of a sufficiently smooth ambient Riemannian manifold, whilst the statements of [32–34] are in the Euclidean setting. However, we believe that this is also just a technical matter and not a substantial difference.

Two other differences have already been pointed out above:

(i) Whilst Krummel & Wickramasekera show that the set of singular points without a power law decay rate of some fixed small order to a unique tangent plane is $\mathcal{H}^{m-2}$-null, we show that the set of points with singularity degree exactly equal to 1 is $\mathcal{H}^{m-2}$-null. The former corresponds to points where our singularity degree is between 1 and $1+\delta$,

for a sufficiently small choice of $\delta > 0$.

(ii) Whilst Krummel & Wickramasekera get a uniform decay estimate for their set $\mathcal{B}$, we do not pursue this for the corresponding set in our approach and we instead subdivide it in a countable unions of sets for which the rate and the starting scale for the decay is uniform.

The combination of (i) and (ii) allow Krummel & Wickramasekera to achieve the additional conclusion that in fact the set of flat singular points in a sufficiently small neighborhood U of a point of density Q can be decomposed into the union of finitely many sets, say $F_1 \cup \ldots F_N$, each of which has locally finite $\mathcal{H}^{m-2}$ measure. In fact they show that $\mathcal{B}$ enjoys the latter structure while the flat singularities in its complement form an $\mathcal{H}^{m-2}$-null set. We caution the reader that this decomposition does not yield the finiteness of the measure of the whole set of flat singular points in U because the sets F_i are not apriori closed.

This raises the natural question of whether our approach is also amenable to yield similar conclusions. We in fact do not believe that (i) is a substantial obstacle for our approach and we think that it is possible to achieve an analogous statement. Concerning point (ii) we also believe that a suitable refinement of our argument can achieve a uniform decay estimate directly for the Q-points with singularity degree at least $1+\delta$ in a sufficiently small neighborhood of a point of density Q. These considerations are obviously influenced by the insight learned from the works of Krummel & Wickramasekera.

Provided one can prove the analogous statements to (i) and (ii) in our case (or using the estimates of Krummel and Wickramasekera in combination with our techniques, when the ambient is the Euclidean space), our methods herein would yield the conclusion that the flat singular Q-points with singularity degree at least $1+\delta$ can be decomposed into two sets with locally finite $\mathcal{H}^{m-2}$ measure and that the flat singular points in its complement form an $\mathcal{H}^{m-2}$-null set. In fact, since in our work we use a modification of the Naber-Valtorta approach, these two sets would have locally finite $(m-2)$-dimensional Minkowski content. In order to tackle the question of whether the entire set of Q-points with singularity degree

at least $1+\delta$ has locally finite Minkowski content, one would need instead to suitably modify the arguments in this chapter in order to tackle low frequency and high frequency points at the same time, a task which is certainly more challenging.

Finally, in the forthcoming work [34], Krummel & Wickramasekera additionally establish the existence of a unique non-zero (multi-valued) Dirichlet-minimizing tangent function at $\mathcal{H}^{m-2}$-a.e. flat singular point of the current. This is inherently different from our approach, given that one major point of the Naber-Valtorta technique is being able to tackle the rectifiability question without addressing the uniqueness of the tangent functions.

3.2 Main statements

In this section we define the singular frequency values and the singularity degree and give the main statements. We follow heavily the notation and terminology of the papers [21, 22] and from now on we will always make the following assumption.

Assumption 3.2.1. T is an m-dimensional integral current in $\Sigma \cap \Omega$ with $\partial T \llcorner \Omega = 0$, where Ω is an open set of $\mathbb{R}^{m+n} = \mathbb{R}^{m+\bar{n}+l}$ and Σ is an $(m+\bar{n})$-dimensional embedded submanifold of class C^{3,κ_0} with $\kappa_0 > 0$. T is area-minimizing in $\Sigma \cap \Omega$ and $\bar{n} \geq 2$. $0 \in \Omega$ is a flat singular point of T and $Q \in \mathbb{N} \setminus \{0, 1\}$ is the density of T at 0.

Since our statements are invariant under dilations, we can also assume that

Assumption 3.2.2. T and Σ satisfy Assumption 3.2.1 with $\Omega = \mathbf{B}_{7\sqrt{m}}$ and $\Sigma \cap \mathbf{B}_{7\sqrt{m}}(p)$ is the graph of a C^{3,κ_0} function $\Psi_p : \mathrm{T}_p\Sigma \cap \mathbf{B}_{7\sqrt{m}}(p) \to \mathrm{T}_p\Sigma^{\perp}$ for every $p \in \Sigma \cap \mathbf{B}_{7\sqrt{m}}$. Moreover

$$\boldsymbol{c}(\Sigma) \coloneqq \sup_{p \in \Sigma \cap \mathbf{B}_{7\sqrt{m}}} \|D\Psi_p\|_{\mathrm{C}^{2,\kappa_0}} \leq \bar{\varepsilon},$$

where $\bar{\varepsilon}$ is a small positive constant which will be specified later.

In particular the following uniform control on the second fundamental form A_Σ of $\Sigma \cap$

$\mathbf{B}_{7\sqrt{m}}$ holds:

$$\mathbf{A} := \|A_\Sigma\|_{C^0(\Sigma\cap\mathbf{B}_{7\sqrt{m}})} \leq C_0\boldsymbol{c}(\Sigma) \leq C_0\bar{\varepsilon}.$$

Following [22, Section 2] we introduce appropriate disjoint intervals $]s_j, t_j] \subset]0,1]$, called *intervals of flattening*, the union of which contains[2] those radii r such that the tilt excess $\mathbf{E}(T, \mathbf{B}_{6\sqrt{m}r})$ falls below a positive fixed threshold ε_3^2. Arguing as in [22, Section 2] for each rescaled current T_{0,t_j} and rescaled ambient manifold Σ_{0,t_j} we follow the algorithm detailed in [21] to produce a *center manifold* $\mathcal{M}$ and an appropriate multivalued map $N : \mathcal{M} \to \mathcal{A}_Q(\mathbb{R}^{m+n})$. The latter takes values in the normal bundle of $\mathcal{M}$ and gives an efficient approximation of the current T_{0,t_j} in $\mathbf{B}_3 \setminus \mathbf{B}_{s_j/t_j}$. Given a point x in a given center manifold $\mathcal{M}$, we will use the notation $\mathcal{B}_r(x)$ for the geodesic ball $\mathbf{B}_r(x) \cap \mathcal{M}$.

For technical reasons, however, we will use a slightly different definition for the parameter $\boldsymbol{m}_0$ in [21, Assumption 1.3]. Our $\boldsymbol{m}_0$, which we denote by $\boldsymbol{m}_{0,j}$ to underline the dependence on j, is defined as

$$\boldsymbol{m}_{0,j} := \max\{\mathbf{E}(T_{0,t_j}, \mathbf{B}_{6\sqrt{m}}), \bar{\varepsilon}^2 t_j^{2-2\delta_2}\}\,, \tag{3.1}$$

where $\delta_2 > 0$ is the parameter in [21, Assumption 1.8]. It can be readily checked that this change is of no consequence for the conclusions of [21, 22], the relevant point is that, because of simple scaling considerations, $\mathbf{c}(\Sigma_{0,t_j}) \leq \boldsymbol{m}_{0,j}$, therefore all the estimates claimed in [21, 22] are valid with our different choice of parameter $\boldsymbol{m}_{0,j}$, provided we choose it to fall below the same threshold ε_3 as in [22]. In light of this, we will henceforth make the following assumption.

Assumption 3.2.3. T and Σ satisfy Assumption 3.2.2. The parameter $\bar{\varepsilon}$ is chosen small enough so that $\boldsymbol{m}_{0,0} \leq \varepsilon_3^2$.

Before proceeding we record a fact proved in [22], which is however not explicitly stated there.

[2] It is not necessarily true that the inequality $\mathbf{E}(T, \mathbf{B}_{6\sqrt{m}r}) \leq \varepsilon_3^2$ holds for all $r \in]s_j, t_j]$. However the inequality certainly holds at all $r = t_j$, while for the remaining radii in the interval holds up to a suitably fixed constant C, cf. [22].

Lemma 3.2.4. *Suppose that T and Σ are as in Assumption 3.2.3. If $\{j_i\} \subset \mathbb{N}$ is the set of indices such that $t_{j_i} < s_{j_i-1}$, then either the latter is finite (i.e. $\bigcup_j]s_j, t_j]$ contains some open interval $]0, \rho[$), or*

$$\liminf_i \mathbf{E}(T_{0,t_{j_i}}, \mathbf{B}_{6\sqrt{m}}) \geq \varepsilon_3^2 \,. \tag{3.2}$$

The argument is taken from [22, Proof of Theorem 5.1], where the statement is shown in a step in the proof of the theorem. We thus omit the proof here.

Since we will repeatedly use it throughout the rest of the chapter, it is convenient to introduce the following terminology.

Definition 3.2.5. *Let T and Σ be as in Assumption 3.2.1. A* blow-up sequence of radii $\{r_k\}$ *is a vanishing sequence of positive real numbers such that T_{0,r_k} converges to a flat tangent cone.*

Of course a similar concept can be introduced by considering a different flat singular point x instead of the origin. In that case we will say that the sequence is a *blow-up sequence at the flat singular point x*.

Note that, having fixed a blow-up sequence $\{r_k\}$, for every k sufficiently large there is a unique $j(k)$ such that $r_k \in]s_{j(k)}, t_{j(k)}]$. By a rotation of coordinates, we may assume that the m-dimensional planes π_k over which we parameterize the center manifolds $\mathcal{M}_{j(k)}$ are identically equal to the same fixed plane $\pi_0 \equiv \mathbb{R}^m \times \{0\} \subset T_0\Sigma$. We use the following shorthand notations:

- T_k and Σ_k for the rescaled currents $T_{0,t_{j(k)}} \llcorner \mathbf{B}_{6\sqrt{m}}$ and ambient manifolds $\Sigma_{0,t_{j(k)}}$;
- $\mathcal{M}_k$ and N_k for the corresponding center manifolds and normal approximations of T_k;
- $\boldsymbol{\varphi}_k$ for map parameterizing the center manifold $\mathcal{M}_k$ over $B_3(\pi_0)$;
- $\boldsymbol{m}_0^{(k)}$ for the real numbers $\boldsymbol{m}_{0,j(k)}$ defined in (3.1).

3.2.1 Compactness procedure

Let T satisfy Assumption 3.2.3 and let $\frac{\bar{s}_k}{t_{j(k)}} \in \left]\frac{3r_k}{2t_{j(k)}}, \frac{3r_k}{t_{j(k)}}\right]$ be the scale at which the reverse Sobolev inequality [22, Corollary 5.3] holds for $r = \frac{r_k}{t_{j(k)}}$. Then let $\bar{r}_k := \frac{2\bar{s}_k}{3t_{j(k)}} \in \left]\frac{r_k}{t_{j(k)}}, \frac{2r_k}{t_{j(k)}}\right]$. We in turn define the corresponding rescalings of the currents T_k, as well as those of the ambient manifolds Σ_k and the center manifolds:

$$\bar{T}_k := (\iota_{0,\bar{r}_k})_\sharp T_k = \left((\iota_{0,\bar{r}_k t_{j(k)}})_\sharp T\right) \llcorner \mathbf{B}_{\frac{6\sqrt{m}}{\bar{r}_k}}, \qquad \bar{\Sigma}_k := \iota_{0,\bar{r}_k}(\Sigma_k), \qquad \overline{\mathcal{M}}_k := \iota_{0,\bar{r}_k}(\mathcal{M}_k),$$

together with the maps $\mathbf{\Phi}_k(x) := (x, \boldsymbol{\varphi}_k(\bar{r}_k x))$ parameterizing the graphs of the rescaled center manifolds Define

$$\bar{N}_k : \overline{\mathcal{M}}_k \to \mathbb{R}^{m+n}, \qquad \bar{N}_k(p) := \frac{1}{\bar{r}_k} N_k(\bar{r}_k p),$$

and let

$$u_k := \frac{\bar{N}_k \circ \mathbf{e}_k}{\mathbf{h}_k}, \qquad u_k : \pi_k \supset B_3 \to \mathcal{A}_Q(\mathbb{R}^{m+n}),$$

where $\mathbf{e}_k$ is the exponential map at $p_k := \frac{\mathbf{\Phi}_k(0)}{\bar{r}_k} \in \overline{\mathcal{M}}_k$ defined on $B_3 \subset \pi_k \simeq T_{p_k}\overline{\mathcal{M}}_k$ and $\mathbf{h}_k := \|\bar{N}_k\|_{L^2(\mathcal{B}_{3/2})}$. The reverse Sobolev inequality of [22, Corollary 5.3] gives a uniform control on the $W^{1,2}$ norm of u_k on $B_{3/2}(0, \pi_k)$ (which denotes the unit disk of π_k centered at 0 and with radius 3/2).

Then, following the proof of [22, Theorem 6.2], there exists a subsequence (not relabeled) a limiting m-plane π_0 and a Dir-minimizing map $u \in W^{1,2}(B_{3/2}(0,\pi_0); \mathcal{A}_Q(\pi_0^\perp))$ with $\boldsymbol{\eta} \circ u = 0$ and $\|u\|_{L^2(B_{3/2})} = 1$, such that (after we apply a suitable rotation to map π_k onto π)

$$u_k \longrightarrow u \quad \text{strongly in } W^{1,2}_{\text{loc}} \cap L^2. \tag{3.3}$$

Recall Almgren's frequency function $I_u(x,r)$ for multi-valued Dir-minimizers, defined in (2.28) in the preceding chapter. It is often more convenient to work with a smoother version

of the frequency function, which has more robust convergence properties. Following [21] we consider a Lipschitz cut-off function $\phi : [0,\infty) \to [0,1]$ which vanishes identically for t sufficiently large, equals 1 for t sufficiently small and is monotone nonincreasing. Recycling the previously introduced notation, let

$$\begin{aligned} D_u(x,r) &:= \int |Du(y)|^2 \phi\left(\frac{|y-x|}{r}\right) dy\,, \\ H_u(x,r) &:= -\int \frac{|u(y)|^2}{|y-x|} \phi'\left(\frac{|y-x|}{r}\right) dy\,, \\ I_u(x,r) &:= \frac{r D_u(x,r)}{H_u(x,r)}\,. \end{aligned}$$

The same computations showing the monotonicity of Almgren's frequency function for Dir-minimizers apply to the latter smoothed variant (cf. for instance [21, Section 3]; note that Almgren's frequency function corresponds, formally, to the choice $\phi = \mathbf{1}_{[0,1]}$). Moreover, it can be readily checked that all these smoothed frequency functions are constant when the map is radially homogeneous, and this constant is the degree of homogeneity of the map. It follows then from the arguments in [18, Section 3.5] that the limit

$$I_u(x,0) = \lim_{r \downarrow 0} I_u(x,r)$$

is independent of the weight ϕ. As in Chapter 2, we will fix a convenient specific choice of ϕ, given by (2.30). When $x = 0$, we will omit the dependency on x for I and related quantities, and will merely write $I_u(r)$.

Definition 3.2.6. *Any map u as defined by the above compactness procedure, recentered at $x \in \mathfrak{F}(T)$, is called a* fine blow-up *limit along the sequence r_k at x. The set*

$$\mathcal{F}(T,x) := \{\, I_u(0) \,:\, u \text{ is a fine blow-up at } x \text{ along some } r_k \downarrow 0 \,\}\,,$$

is the set of singular frequency values of T at x.

Remark 3.2.7. In the rest of the notes we will often omit the adjective "singular". The reason for using the adjective "fine" is that later on we will also introduce a notion of *coarse* blow-up, cf. Definition 3.3.1.

Definition 3.2.8. *The* singularity degree *of T at a flat singular point x is defined as*

$$\mathrm{I}(T,x) := \inf\{\alpha : \alpha \in \mathcal{F}(T,x)\}\,.$$

3.2.2 Main results

We are now in a position to state the main results of this chapter. Firstly, we demonstrate that the singularity degree indeed defines the order of contact of T with the average of its sheets around a given flat singular point.

Theorem 3.2.9. *Assume that T satisfies Assumption 3.2.3. Then $\mathrm{I}(T,0) \geq 1$ and $\mathcal{F}(T,0) = \{\mathrm{I}(T,0)\}$, i.e. there is one unique frequency value for T at 0 and it coincides with the singularity degree.*

However, our analysis delivers a number of additional pieces of information. We report them here even though some statements will need notions which will be only introduced in the next sections.

Theorem 3.2.10. *Under the same assumptions of Theorem 3.2.9 the following holds:*

(i) *All fine blow-ups are radially homogeneous and their homogeneity degree is $\mathrm{I}(T,0)$.*

(ii) *If $s_{j_0} = 0$ for some j_0, then $\lim_{r\downarrow 0} \mathbf{I}_{N_{j_0}}(r) = \mathrm{I}(T,0)$ (see below for the definition of $\mathbf{I}_{N_{j_0}}$).*

(iii) *If $\{s_j\}$ is infinite, then the functions $\mathbf{I}_{N_j}$ converge uniformly to $\mathrm{I}(T,0)$ if $\mathrm{I}(T,0) > 1$, while, when $\mathrm{I}(T,0) = 1$, $\lim_{k\to\infty} \mathbf{I}_{N_{j(k)}}(\frac{r_k}{t_{j(k)}}) = \mathrm{I}(T,0) = 1$ for every blow-up sequence r_k (recall that $j(k)$ is such that $r_k \in]s_{j(k)}, t_{j(k)}]$).*

(iv) *If $\mathrm{I}(T,0) > 1$, then $T_{0,r}$ converge polynomially fast to a unique flat tangent cone as $r \downarrow 0$.*

(v) *If* $\mathrm{I}(T,0) > 2-\delta_2$, *then* $s_{j_0} = 0$ *for some* j_0.

(vi) *If* $\mathrm{I}(T,0) < 2-\delta_2$ *then* $\{s_j\}$ *is infinite and* $\inf_j \frac{s_j}{t_j} > 0$.

The final main result of this chapter is the following.

Theorem 3.2.11. *Let* T *be as in Theorem 3.1.1 and* $Q \in \mathbb{N} \setminus \{1,2\}$ *Then the set*

$$\mathfrak{F}_{Q,>1}(T) := \{p \in \mathfrak{F}(T) : \Theta(T,p) = Q \quad \textit{and} \quad \mathrm{I}(T,p) > 1\}$$

is $(m-2)$*-rectifiable.*

In Chapter 4 (see Theorem 4.1.2) we will then complete the proof of the rectifiability of $\mathrm{Sing}(T)$ by showing that the following holds.

Theorem 3.2.12. *Let* T *and* Q *be as in Theorem 3.2.11. Then* $\mathfrak{F}_{Q,1}(T) := \mathfrak{F}_Q(T) \setminus \mathfrak{F}_{Q,>1}(T)$ *is an* $\mathcal{H}^{m-2}$*-null set.*

Notice that, while Theorem 3.2.11 and Theorem 3.2.12 together imply that the tangent cone is unique at $\mathcal{H}^{m-2}$-a.e. point $p \in \mathfrak{F}(T)$, this does not answer the question of the uniqueness of tangent cones at $\mathcal{H}^{m-2}$-a.e. point $p \in \mathcal{S}^{(m-2)}(T)$, as claimed in Theorem 3.1.1. The latter statement does not follow from the analysis of Naber and Valtorta in [43]; there, the authors only handle points $p \in \mathcal{S}^{(m-2)}(T)$ that are uniformly bounded away from $\mathfrak{F}(T)$. However, we address this in Chapter 4, and indeed the $\mathcal{H}^{m-2}$-a.e. uniqueness of tangent cones is a simple byproduct of the tools which we introduce therein.

3.2.3 Frequency function

We end the section by recalling introducing the ϕ-regularized frequency function of the normal approximation of T from Chapter 2. Recalling the function $\phi : [0,\infty[$ of (2.30), for a given center manifold $\mathcal{M}$ with corresponding $\mathcal{M}$-normal approximation $N : \mathcal{M} \to$

$\mathcal{A}_Q(\mathbb{R}^{m+n})$, the ϕ-regularized frequency function $\mathbf{I}_N(x,r)$ of N at a center point $x \in \mathcal{M}$ and scale $r > 0$ is defined analogously to that in Chapter 2, namely

$$\mathbf{I}_N(x,r) := \frac{r\mathbf{D}_N(x,r)}{\mathbf{H}_N(x,r)},$$

where

$$\mathbf{D}_N(x,r) := \int_{\mathcal{M}} |DN|^2 \phi\left(\frac{d(y,x)}{r}\right) \mathrm{d}y, \quad \mathbf{H}_N(x,r) := -\int_{\mathcal{M}} \frac{|\nabla_y d(y,x)|^2}{d(y,x)} |N|^2 \phi'\left(\frac{d(y,x)}{r}\right) \mathrm{d}y.$$

Here d is the geodesic distance on the center manifold $\mathcal{M}$, and we simply write $d(y)$ for the geodesic distance $d(0,y)$. We let $\mathbf{p}$ denote the orthogonal projecction to $\mathcal{M}$, as defined in [21, Assumption 2.1]. Since we will often take the above quantities to be centered at $x = 0$, we will omit the implicit dependency on x most of the time.

A major starting point in this chapter is the fact that the frequency function is bounded away from infinity and 0 (independently of the choice of center manifold and corresponding normal approximation), see Theorem 2.6.6 and Theorem 2.6.8.

3.3 The Hardt-Simon inequality and coarse blow-ups

3.3.1 Coarse blow-ups

Consider a blow-up sequence $\{r_k\}_k$ at the flat singular point 0 and let:

- T_{0,r_k} be the corresponding rescaled currents;
- Σ_{0,r_k} be the corresponding rescaled manifolds.

Without loss of generality we can assume that T_{0,r_k} converges to $Q[\![\pi_0]\!]$ with $\pi_0 = \mathbb{R}^m \times \{0\}$. For $r_k := \frac{r_k}{t_{j(k)}}$, where $]s_{j(k)}, t_{j(k)}]$ is the interval of flattening containing r_k, let $M > 0$ be large enough[3] such that $\mathbf{B}_L \subset \mathbf{C}_{4M\bar{r}_k}$ for any $L \in \mathscr{W}^{(k)}$ with $L \cap \overline{B}_{\bar{r}_k}(0,\pi_0) \neq \emptyset$ (cf. [21] for the

[3] In fact, $M = 1/2$ suffices

definitions). Consider further a sequence of planes π_k with the property that π_k optimizes the tilt excess of T_{0,r_k} in $\mathbf{B}_{8M}$ and consider that

$$\mathbf{E}(T_{0,r_k}, \mathbf{C}_{4M}, \pi_k) \leq \mathbf{E}(T_{0,r_k}, \mathbf{B}_{8M}) =: E_k \to 0\,, \tag{3.4}$$

and define $\mathbf{A}_k := \mathbf{A}_{\Sigma_{0,r_k}}$. Clearly we must have $\pi_k \to \pi_0$. By applying a rotation which is infinitesimally close to the identity we can map π_k to π_0. We then push forward the current T_{0,r_k} under this rotation so that we can assume $\pi_k = \pi_0$, while, with a slight abuse of notation, we keep using T_{0,r_k} and Σ_{0,r_k} for the rotated objects.

If $k_0 \in \mathbb{N}$ is large enough, we can ensure that

$$E_k + \mathbf{A}_k^2 < \min\left\{\varepsilon_1, \frac{1}{2}\right\} \qquad \text{for every } k \geq k_0, \tag{3.5}$$

where ε_1 is the threshold in [19, Theorem 2.4]. We can therefore let $f_k : B_1(0,\pi_0) \to \mathcal{A}_Q(\pi_0^\perp)$ be the strong Lipschitz approximation of [19, Theorem 2.4] for T_{0,r_k} and define the rescaled maps

$$\bar{f}_k := \frac{f_k}{E_k^{1/2}}\,. \tag{3.6}$$

We will make the additional assumption that

$$\mathbf{A}_k^2 \leq C r_k^2 = o(E_k)\,. \tag{3.7}$$

It then follows from [19] that, up to subsequences,

(i) $\bar{f}_k$ converges strongly in $L^2 \cap W^{1,2}_{\mathrm{loc}}(B_1(0,\pi_0))$ to a Dir-minimizing map $\bar{f} : B_1(0,\pi_0) \to \mathcal{A}_Q(\pi_0^\perp)$,

(ii) $\bar{f}$ takes values in the orthogonal complement to π_0 in $T_0\Sigma$,

(iii) $\bar{f}(0) = Q\llbracket 0 \rrbracket$.

Note that there is no guarantee that the blow-up is nontrivial: the nontriviality of $\bar{f}$ is in fact equivalent (cf. [19]) to

$$\liminf_{k\to\infty} \frac{\mathbf{E}(T_{0,r_k}, \mathbf{C}_\rho, \pi_k)}{E_k} \geq \bar{c} > 0 \tag{3.8}$$

for some $\rho > 0$ and some $\bar{c}$.

Definition 3.3.1. *A Dir-minimizing map $\bar{f}$ as above will be called a* coarse *blow-up (at 0). Its average free part is given by the map*

$$v(x) := \sum_i [\![\bar{f}_i(x) - \boldsymbol{\eta} \circ \bar{f}(x)]\!]\,. \tag{3.9}$$

We say that $\bar{f}$ is nontrivial if it does not vanish identically.

Obviously, if we focus our attention on some other flat singular point q, an obvious modification of the above procedure defines a notion of *coarse blow-up at q*. Our main claim for coarse blow-ups, which (as already pointed out) is a consequence of the Hardt-Simon inequality, is the following.

Theorem 3.3.2. *Let T be as in Assumption 3.2.3, $\bar{f}$ be a nontrivial coarse blow-up, and v be its average-free part. Then $I_{\bar{f}}(0) \geq 1$ and, if v does not vanish identically, $I_v(0) \geq 1$.*

In this section we prove Theorem 3.3.2.

3.3.2 Closure under rescalings

Before coming to the proof of Theorem 3.3.2 we need the following elementary observation, which verifies that the property of being a coarse blow-up is closed under normalized L^2 limits.

Lemma 3.3.3. *Let T be as in Assumption 3.2.3 and $\bar{f}$ be a nontrivial coarse blow-up. Let*

$\rho_j \downarrow 0$ be any vanishing sequence, let

$$D(j) := \int_{B_{\rho_j}} |D\bar{f}|^2,$$

and define the rescaled maps $\bar{f}_j(x) := (\rho_j^{2-m} D(j))^{-1/2} \bar{f}(\rho_j x)$. If $\bar{f}_\infty$ is the L^2 limit of any subsequence of $\{\bar{f}_j\}$ on B_1, then $\bar{f}_\infty$ is (up to a nonzero multiplicative factor) also a nontrivial coarse blow-up.

Proof. Let r_k be a blow-up sequence with the property that the maps $\bar{f}_k$ defined in the previous section converge to $\bar{f}$ and fix constants $\bar{\rho}$ and $\bar{c}$ so that (3.8) holds. We consider a sequence $r'_j := \rho_j r_{k(j)}$ and $\pi_j := \pi_{k(j)}$ and we will show that, for an appropriate choice of $k(j)$, the following holds:

(a) r'_j is a blow-up sequence, i.e. T_{0,r'_j} converges to $Q\llbracket\pi_0\rrbracket$;

(b) $E_j := \mathbf{E}(T_{0,r'_j}, \mathbf{C}_4, \pi_j)$ converges to 0;

(c) The conditions (3.7) and (3.8) hold for this new blow-up sequence;

(d) If f_j are the approximating maps given by [19, Theorem 2.4], then $E_j^{-1/2} f_j$ converges (up to subsequences) to $\lambda \bar{f}_\infty$ for some nonzero scalar λ.

The argument is a classical diagonal one and in order to deal efficiently will all the conditions, it is useful to decouple the two indices and introduce the radii $r_{j,k} := \rho_j r_k$. We introduce then the corresponding excess $E_{j,k} := \mathbf{E}(T_{0,r_{j,k}}, \mathbf{C}_4, \pi_k)$ and $\mathbf{A}_{j,k} := \mathbf{A}_{\Sigma_{0,r_{j,k}}}$. Combining the estimates of [19] with (3.7) we immediately see that there are two positive constants $\tilde{c}^+$ and $\tilde{c}^-$ such that

$$c^- \leq \liminf_{j\to\infty} \frac{E_{j,k}\rho_j^m}{E_k D(j)} \leq \limsup_{j\to\infty} \frac{E_{j,k}\rho_j^m}{E_k D(j)} \leq c^+ . \tag{3.10}$$

Moreover, obvious scaling arguments show that $\mathbf{A}_{j,k} = \rho_j^2 \mathbf{A}_k$. It is then pretty obvious that the conditions corresponding to (a), (b), and (c) above hold for any sequence $\{\rho_{j,k}\}_j$ once we keep k fixed. Observe also that for (c) we can choose constants which are independent of

k: the radius $\bar\rho$ can in fact be taken equal to $\frac{1}{2}$, while the constant $\bar c$ will depend only upon c^-. In particular, for any sequence $\{k(j)\}_j$ which converges to infinity sufficiently fast, (a), (b), and (c) will hold.

Next we apply, as above, suitable rotations and assume that all the planes π_k coincide with π_0 (without changing notation for the various objects introduced). We consider the rescaled maps

$$\tilde f_{j,k}(x) := \rho_j^{-1} f_k(\rho_j x)$$

and let instead $f_{j,k} : B_1(0,\pi_k) \to \mathcal{A}_Q(\pi_k^\perp)$ be the Lipschitz approximations which are given by [19, Theorem 2.4] applied to $T_{0,r_{j,k}}$. Observe that, by the estimates in [19, Theorem 2.4],

$$\lim_{j\to\infty} E_{k,j}^{-1/2} \|\mathcal{G}(f_{j,k}, \tilde f_{j,k})\|_{L^2} = 0\,.$$

On the other hand, for every fixed k, the limit of $E_{k,j}^{-1}\tilde f_{j,k}$ is clearly a scalar multiple $\lambda(k)$ of $\bar f_k$, and it is easy to see that this scalar multiple has a fixed range $[\lambda^-, \lambda^+]$ for positive constants $\lambda^\pm$ depending upon $c^\pm$ and upon the constant $\bar c$ in condition (3.8) for r_k. It follows therefore that (d) holds for any $k(j)$ which diverges sufficiently fast. □

3.3.3 Proof of Theorem 3.3.2

Recalling [18, Theorem 3.19], the frequency value α at 0 of any Dir-minimizer f is a positive number and by [18, Corollary 3.18], we have that

$$\lim_{\rho\to 0} \rho^{2-2\bar\alpha-m} \int_{B_\rho} |Df|^2 = \infty \qquad \forall \bar\alpha > \alpha\,, \tag{3.11}$$

$$\lim_{\rho\to 0} \rho^{2-2\bar\alpha-m} \int_{B_\rho} |Df|^2 = 0 \qquad \forall \bar\alpha < \alpha\,. \tag{3.12}$$

On the other hand, since the Dirichlet energy of $D\bar f$ is the sum of the Dirichlet energies of $\boldsymbol{\eta}\circ\bar f$ and its average free part v, for any coarse blow-up $\bar f$ we conclude that $I_{\bar f}(0) \le \min\{I_v(0), I_{\boldsymbol{\eta}\circ\bar f}(0)\}$. Recall that $\boldsymbol{\eta}\circ\bar f$ is a classical harmonic function and hence $I_{\boldsymbol{\eta}\circ\bar f}$ is a

positive integer. Thus, in order to prove that $I_v(0) \geq 1$, it suffices to show that $I_{\bar{f}}(0) \geq 1$. Introduce now

$$\bar{f}_r := r^{\frac{m-2}{2}} \frac{f(rx)}{\sqrt{\mathrm{Dir}(f, B_r)}}\,.$$

and apply Lemma 3.3.3 to conclude that, if there is a coarse blow-up $\bar{f}$ with $\alpha = I_{\bar{f}}(0)$, then there is a coarse blow-up which is α-homogeneous.

In this second part we prove that, if $\bar{f}$ is an α-homogeneous coarse blow-up, then necessarily $\alpha \geq 1$. This is in fact the same argument used in [55, Proposition 3.10] and we report it for the reader's convenience. Consider thus such a coarse blow-up and fix a blow-up sequence r_k leading to it, according to the procedure explained above. In order to simplify our notation we denote by T_k the current T_{0,r_k}.

First of all, recall that the error from the monotonicity formula for mass ratios gives the estimate

$$\int_{\mathbf{B}_4} \frac{1}{|q|^m} \left| \frac{q^\perp}{|q|} \right|^2 \mathrm{d}\|T_k\|(q) \leq C E_k + C \mathbf{A}_k^2\,. \tag{3.13}$$

See, for example, [50] for a derivation of this. The only subtlety compared to the classical literature (cf. for instance [44]) is that the usual derivation of the above estimate is reduced to the one for varifolds with bounded mean curvature. The latter is not good enough for us because it would give a linear dependence on $\mathbf{A}_k$, rather than a quadratic one. The quadratic improvement, which is possible using the stronger information that our current induces a *stationary varifold* in a Riemannian submanifold, is remarked in [19, Appendix A].

As described in the procedure leading to coarse blow-ups we rotate the currents suitably so that $\pi_k = \pi_0$. We next pass the inequality (3.13) to the Lipschitz approximations f_k given by [19, Theorem 2.4]. We let $\sum_i \llbracket (f_k)_i \rrbracket$ be a (measurable) selection for the f_k as in [18, Theorem 0.4]. We then write

$$\int_{K_k} \sum_i \frac{|\big(x + (f_k)_i(x)\big)^\perp|^2}{|x + (f_k)_i(x)|^{m+2}}\,\mathrm{d}x \leq C(E_k + \mathbf{A}_k^2) \leq C E_k, \tag{3.14}$$

where $K_k \subset B_1 \subset \pi_k$ is the (closed) domain over which the graph of the Lipschitz approximation f_k coincides with the current T_k (cf. [19, Theorem 0.4]. Note that, for the point $q = x + (f_k)_i(x) \in K_k \times \pi_k^\perp$, $q^\perp$ denotes the orthogonal projection of q to $(T_q \mathbf{G}_f)^\perp$, where $\mathbf{G}_f$ (the current induced by the graph of the multivalued function f) is defined as in [20, Definition 1.10].

However, since f_k is Lipschitz, and hence differentiable almost-everywhere by Rademacher's Theorem [18, Theorem 1.3], we can formally compute

$$\frac{\partial}{\partial r}\left(\frac{(f_k)_i(x)}{|x|}\right) = \frac{\partial}{\partial r}\left(\frac{x + (f_k)_i(x)}{|x|}\right) = \frac{\partial_r\left(x + (f_k)_i(x)\right)}{|x|} - \frac{x + (f_k)_i(x)}{|x|^2}. \tag{3.15}$$

Since the first term on the left-hand side belongs to $T_q \mathbf{G}_f$ at $q = x + (f_k)_i(x)$, we have

$$\left|\left[\frac{\partial}{\partial r}\left(\frac{(f_k)_i(x)}{|x|}\right)\right]^\perp\right|^2 = \frac{\left|[x + (f_k)_i(x)]^\perp\right|^2}{|x|^4}.$$

Combining this with (3.14), we have

$$\int_{K_k} \sum_i \frac{|x|^4}{|x + (f_k)_i(x)|^{m+2}} \left|\left[\frac{\partial}{\partial r}\left(\frac{(f_k)_i(x)}{|x|}\right)\right]^\perp\right|^2 \mathrm{d}x \leq C E_k\,. \tag{3.16}$$

We next wish to estimate the tangential component of the right-hand side of (3.15) as follows:

$$\left|\left[\frac{\partial}{\partial r}\left(\frac{(f_k)_i(x)}{|x|}\right)\right]^\parallel\right|^2 \leq \|\mathbf{p}_{\vec{T}_k(q)} - \mathbf{p}_{\pi_0}\|^2 \left|\frac{\partial}{\partial r}\left(\frac{(f_k)_i(x)}{|x|}\right)\right|^2 \leq C E_k^\beta \left|\frac{\partial}{\partial r}\left(\frac{(f_k)_i(x)}{|x|}\right)\right|^2,$$

where we have used that, at the point $q = x + (f_k)_i(x)$ of interest, the tangent to the current coincides with the tangent to $\mathbf{G}_f$, and the distance of the latter to π_0 can be estimated with the Lipschitz constant of f_k (cf. [19, Theorem 2.4]). Writing

$$\left|\frac{\partial}{\partial r}\left(\frac{(f_k)_i(x)}{|x|}\right)\right|^2 = \left|\left[\frac{\partial}{\partial r}\left(\frac{(f_k)_i(x)}{|x|}\right)\right]^\perp\right|^2 + \left|\left[\frac{\partial}{\partial r}\left(\frac{(f_k)_i(x)}{|x|}\right)\right]^\parallel\right|^2$$

we immediately conclude

$$\left|\frac{\partial}{\partial r}\left(\frac{(f_k)_i(x)}{|x|}\right)\right|^2 \leq 2\left|\left[\frac{\partial}{\partial r}\left(\frac{(f_k)_i(x)}{|x|}\right)\right]^{\perp}\right|^2$$

as soon as E_k is sufficiently small. Hence, by (3.16), we conclude

$$\int_{K_k}\sum_i \frac{|x|^4}{|x+(f_k)_i(x)|^{m+2}}\left|\frac{\partial}{\partial r}\left(\frac{(f_k)_i(x)}{|x|}\right)\right|^2 \,\mathrm{d}x \leq CE_k\,. \tag{3.17}$$

Next, consider $\bar{f}_k := E_k^{-1/2} f_k$ and infer, from (3.17) the estimate

$$\int_{\bigcap_{j\geq k_0} K_j\setminus B_\rho}\sum_i \frac{|x|^4}{|x+E_k^{1/2}(\bar{f}_k)_i(x)|^{m+2}}\left|\frac{\partial}{\partial r}\left(\frac{(\bar{f}_k)_i(x)}{|x|}\right)\right|^2 \,\mathrm{d}x \leq C\,,$$

for any $k \geq k_0$ and $\rho > 0$. Recall that:

- $\bar{f}_k$ converges strongly in $W^{1,2}(B_{1/2})$ to $\bar{f}$;
- The height bound of [50] implies that $\|\bar{f}_k\|_\infty$ is uniformly bounded.

We can thus pass into the limit in k to conclude

$$\int_{(B_{1/2}\setminus B_\rho)\cap\bigcap_{j\geq k_0} K_j}\sum_i \frac{1}{|x|^{m-2}}\left|\frac{\partial}{\partial r}\left(\frac{(\bar{f})_i(x)}{|x|}\right)\right|^2 \,\mathrm{d}x \leq C\,.$$

By choosing a fast converging subsequence, we can assume that the series $\sum |B_1 \setminus K_j|$ is summable. Therefore, let $k_0 \uparrow \infty$ and $\rho \downarrow 0$ we get

$$\int_{B_{1/2}}\sum_i \frac{1}{|x|^{m-2}}\left|\frac{\partial}{\partial r}\left(\frac{(\bar{f})_i(x)}{|x|}\right)\right|^2 \,\mathrm{d}x \leq C \tag{3.18}$$

Since $\bar{f}$ is α-homogeneous we have

$$\bar{f}_i(x) = |x|^\alpha \bar{f}_i\left(\frac{x}{|x|}\right),$$

and so

$$\frac{\partial}{\partial r}\left(\frac{\bar{f}_i(x)}{|x|}\right) = (\alpha-1)|x|^{\alpha-2}\bar{f}_i\left(\frac{x}{|x|}\right).$$

Inserting in (3.18) and passing to polar coordinates we conclude

$$(\alpha-1)^2\int_{\partial B_1}|\bar{f}|^2\int_0^{1/2} s^{-1-2(1-\alpha)}\,\mathrm{d}s \le C\,.$$

The latter inequality implies immediately $\alpha \geq 1$, and thus completes the proof.

3.4 Comparison of coarse and fine blow-ups

In this section we compare fine and coarse blow-ups at scales which are comparable to the left endpoints of a sequence of intervals of flattening. The main conclusion is that the average-free parts of coarse blow-ups are scalar multiples of fine blow-ups. More precisely we have the following proposition.

Proposition 3.4.1. *Let T be as in Assumption 3.2.3. Let r_k be a blow-up sequence at the origin and assume that*

$$\liminf_{k\to\infty}\frac{s_{j(k)}}{r_k} > 0\,. \tag{3.19}$$

Then (3.7) holds and we can consider a coarse blow-up $\bar{f}$ (generated by a subsequence) according to Section 3.3.1 and a fine blow-up u (generated by a further subsequence) according to the procedure detailed in Section 3.2.1. If we denote by v the average-free part of $\bar{f}$, then there is a real number $\lambda > 0$ such that $v = \lambda u$.

Remark 3.4.2. In general, without assumption (3.19) it might be that (3.7) does not hold and that we cannot, therefore, define a coarse blow-up. Even if we were to assume (3.7), but not (3.19), we could at best infer that $v = \lambda u$ for some $\lambda \geq 0$, but not that λ is necessarily positive. Easy examples for the latter behavior can be constructed using holomorphic curves of $\mathbb{C}^2$ of the form $\{(z,w) : (w-h(z))^Q = z^p\}$, for a nontrivial holomorphic h with $h(0) =$

$h'(0) = 0$ and a fraction $\frac{p}{Q}$ which is noninteger and larger than the order of vanishing of h at 0.

An obvious corollary of the latter proposition is that, under the above assumptions, v is necessarily nontrivial and that $I_v(0) = I_u(0)$.

3.4.1 Nontriviality and homogeneity of coarse blow-ups

If we combine it with Theorem 3.2.10(vi), Proposition 3.4.1 has the following further consequence, which will be useful later.

Corollary 3.4.3. *Let T be as in Assumption 3.2.1, let $\delta_2 > 0$ be the parameter in [21, Assumption 1.8] and assume the singularity degree $\mathrm{I}(T, 0)$ is strictly smaller than $2 - \delta_2$. Then any coarse blow-up $\bar{f}$ at 0 is nontrivial, $\mathrm{I}(T, 0)$-homogeneous, and has average 0 (so in particular $\bar{f} = v$ for the average-free part v).*

Moreover, for every $\gamma > 2(\mathrm{I}(T, 0) - 1)$, we have

$$\liminf_{r \downarrow 0} \frac{\mathbf{E}(T, \mathbf{B}_r)}{r^\gamma} > 0 \tag{3.20}$$

and there exists a radius r_0 (which depends on the current T) such that

$$\mathbf{E}(T, \mathbf{B}_r) \geq \frac{r^\gamma}{s^\gamma} \mathbf{E}(T, \mathbf{B}_s) \qquad \forall r < s < r_0 \,. \tag{3.21}$$

Proof. It follows directly from Proposition 3.4.1 and from Theorem 3.2.10(vi) that the average-free part of any coarse blow-up at 0 is nontrivial and is $\mathrm{I}(T, 0)$-homogeneous. We therefore just need to show that the average vanishes.

First of all observe that, if $\{\bar{f}_k\}$ is any family of coarse blow-ups, then $\|\bar{f}_k\|_{W^{1,2}(B_1)}$ is uniformly bounded and any limit $\bar{f}_\infty$ of any subsequence is also a coarse blow-up. Since every such $\bar{f}_\infty$ must have an average-free part which is nontrivial and $\mathrm{I}(T, 0)$-homogeneous,

it follows immediately that there is a positive number $\omega > 0$ such that

$$\int_{B_1} |D\bar{u}|^2 \geq \omega > 0$$

whenever $\bar{u}$ is the average-free part of a coarse blow-up $\bar{f}$. In particular, since the coarse blow-up $\bar{f}$ is itself nontrivial, we also conclude the existence of some constant $\Omega > 0$ such that

$$\int_{B_1} |D(\boldsymbol{\eta} \circ \bar{f})|^2 \leq \Omega \int_{B_1} |D\bar{u}|^2 \tag{3.22}$$

for every coarse blow-up $\bar{f}$, its average free part $\bar{u}$, and its average $\boldsymbol{\eta} \circ \bar{f}$.

Consider now the sequence $r_k \downarrow 0$ which generates any coarse blow-up $\bar{f}$ and let π_k be an optimal plane so that

$$\mathbf{E}(T, \mathbf{B}_{8Mr_k}, \pi_k) = \mathbf{E}(T, \mathbf{B}_{8Mr_k}) =: E_k \to 0.$$

The nontriviality property (3.22), the Taylor expansion of the area functional and (3.7) can be easily used to show that

$$\mathbf{E}(T, \mathbf{B}_{8Mr_k}) \leq C\mathbf{E}(T, \mathbf{B}_{r_k}, \pi_k) \leq C^2\mathbf{E}(T, \mathbf{B}_{r_k})\,.$$

From the above, if $\tilde{\pi}_k$ is an optimal plane such that $\mathbf{E}(T, \mathbf{B}_{r_k}, \tilde{\pi}_k) = \mathbf{E}(T, \mathbf{B}_{r_k})$, then $|\pi_k - \tilde{\pi}_k| \leq C\mathbf{E}(T, \mathbf{B}_{r_k})$ and thus

$$\mathbf{E}(T, \mathbf{B}_{8Mr_k}, \tilde{\pi}_k) \leq C\mathbf{E}(T, \mathbf{B}_{r_k})$$

However, observe as well that for any constant C fixed, the sequence Cr_k also generates (up to possibly extract a subsequence) a coarse blow-up: in fact the excess must go to 0 (because the currents T_{0,Cr_k} converges to the same tangent cone as T_{0,r_k}, which thus must be flat) and $\mathbf{E}(T, \mathbf{B}_{8CMr_k}) \geq C^{-m}\mathbf{E}(T, \mathbf{B}_{8Mr_k})$, so that (3.7) holds for the sequence Cr_k as well.

Letting π_k' be a plane with

$$E_k' := \mathbf{E}(T, \mathbf{B}_{16Mr_k}) = \mathbf{E}(T, \mathbf{B}_{16Mr_k}, \pi_k),$$

we have

$$|\pi_k - \pi_k'| = o((E_k')^{1/2})$$

and

$$C^{-1} \leq \liminf_k \frac{E_k}{E_k'} \leq \limsup_k \frac{E_k}{E_k'} \leq C\,,$$

so for k sufficiently large we can apply [19, Theorem 2.4] to T_{0,r_k} in $\mathbf{B}_{16M}$ relative to the plane π_k to get a Lipschitz approximation $g_k : B_{8M}(0,\pi_k) \to \mathcal{A}_Q((\pi_k)^\perp)$ in the cylinder $\mathbf{C}_{8M}(0,\pi_k)$, as in the algorithm detailed in Section 3.3.1 to generate another coarse blow-up $\bar g$ (along the sequence $2r_k$). This new Lipschitz approximation g_k coincides with f_k on $B_1(0,\pi_k)$, except for a set whose $\mathcal{H}^m$-measure is estimated by $o(E_k')$. In particular the rescaled functions $\bar g_k = (E_k')^{-\frac{1}{2}} g_k$ converge to a Dir-minimizing function $\bar g$ over $B_{8M}(0,\pi_0)$ which coincides with $\bar f$ on $B_1(0,\pi_0)$.

Next, we observe that

$$D(\boldsymbol{\eta}\circ\bar f)(0) = \frac{1}{\omega_m}\int_{B_1(0,\pi_0)} D(\boldsymbol{\eta}\circ\bar f) = \frac{1}{\omega_m 2^m}\int_{B_2(0,\pi_0)} D(\boldsymbol{\eta}\circ\bar g) = D(\boldsymbol{\eta}\circ\bar g)(0)\,,$$

by harmonicity of the two functions $\boldsymbol{\eta}\circ\bar f$ and $\boldsymbol{\eta}\circ\bar g$. But we then must have $D(\boldsymbol{\eta}\circ\bar f)(0) = D(\boldsymbol{\eta}\circ\bar g)(0) = 0$, otherwise we can use the Taylor expansion of [20] to contradict the optimality of the plane π_k.

But in fact, we may repeat this argument with the scale $16Mr_k$ replaced by $2^{j+3}Mr_k$ for every fixed $j \in \mathbb{N}$, and the same conclusions apply, with a constant depending on j, for E_k' replaced with $\mathbf{E}(T, \mathbf{B}_{2^{j+3}Mr_k})$.

In particular, if $r_k \downarrow 0$ is a sequence which generates a coarse blow-up $\bar f$, then $2^j r_k$ generates a coarse blow-up $\bar g_j$ with the property that $\bar f(x) = \lambda_j \bar g_j(2^{-j}x)$ for some positive

nonzero number λ_j.

Next, denote by $\bar u$ the average-free part of $\bar f$ and by v_j the average-free part of $\bar g_j$. Observe that $D\bar u$ and $D\bar v_j$ are $(\mathrm{I}(T,0)-1)$-homogeneous, while $D(\boldsymbol{\eta}\circ\bar f)$ and $D(\boldsymbol{\eta}\circ\bar g_j)$ are classical harmonic functions with $D(\boldsymbol{\eta}\circ\bar f)(0)=D(\boldsymbol{\eta}\circ\bar g_j)(0)=0$ and $\boldsymbol{\eta}\circ\bar g_j(0)=\boldsymbol{\eta}\circ\bar f(0)=0$, in particular $I_{\boldsymbol{\eta}\circ\bar g_j}(0)\geq 2$. Therefore, we observe that

$$\begin{aligned}\frac{\int_{B_1}|D(\boldsymbol{\eta}\circ\bar f)|^2}{\int_{B_1}|D\bar u|^2}&=\frac{\int B_{2^{-j}}|D(\boldsymbol{\eta}\circ\bar g_j)|^2}{\int_{B_{2^{-j}}}|D\bar v_j|^2}\\&\leq\frac{2^{-j(2I_{\boldsymbol{\eta}\circ\bar g_j}(0)-2)}\int_{B_1}|D(\boldsymbol{\eta}\circ\bar g_j)|^2}{2^{-j(2\mathrm{I}(T,0)-2)}\int_{B_1}|D\bar v_j|^2}\\&\leq\frac{2^{2j(\mathrm{I}(T,0)-2)}\int_{B_1}|D(\boldsymbol{\eta}\circ\bar g_j)|^2}{\int_{B_1}|D\bar v_j|^2}\end{aligned}$$

On the other hand the bound (3.22) is valid also for $\bar g_j$ and $\bar v_j$ in place of $\bar f$ and $\bar u$, because $\bar g_j$ is a coarse blow-up and $\bar v_j$ is its average-free part. In particular, recalling that $\mathrm{I}(T,0)<2-\delta_2$ we conclude

$$\frac{\int_{B_1}|D(\boldsymbol{\eta}\circ\bar f)|^2}{\int_{B_1}|D\bar u|^2}\leq 2^{-2\delta_2 j}\Omega\,.$$

Since Ω is fixed, j an arbitrary integer, and δ_2 a positive number, we immediately conclude that $D(\boldsymbol{\eta}\circ\bar f)\equiv 0$ and $\boldsymbol{\eta}\circ\bar f$ is a constant. On the other hand recall that, since $\Theta(T,0)=Q$, $\bar f(0)=Q[\![0]\!]$, and in particular $\boldsymbol{\eta}\circ\bar f(0)=0$. We thus have proved that $\boldsymbol{\eta}\circ\bar f\equiv 0$.

Next observe that the arguments detailed so far have also the following outcome. If $r_k\downarrow 0$ is a sequence such that $\mathbf{E}(T,\mathbf{B}_{r_k})\to 0$, then

$$\lim_{r\downarrow 0}\frac{\mathbf{E}(T,\mathbf{B}_{r/2})}{\mathbf{E}(T,\mathbf{B}_r)}=2^{-(\mathrm{I}(T,0)-1)}\,.$$

Fix now any $\gamma<\mathrm{I}(T,0)-1$. The above implies the following: there is $\bar r>0$ and $\bar E>0$ such that:

- If $r < \bar{r}$ and $\mathbf{E}(T, \mathbf{B}_r) < \bar{E}$, then

$$\frac{\mathbf{E}(T, \mathbf{B}_{r/2})}{\mathbf{E}(T, \mathbf{B}_r)} \geq 2^{-\gamma}\,.$$

We next distinguish two cases. We consider the following set

$$\mathcal{R} := \{0 < r < \bar{r} : \mathbf{E}(T, r) < 2^{-1}\bar{E}\}\,,$$

which can be easily checked to be open if $\bar{r}$ is sufficiently small. We then argue differently dependingon whether $\mathcal{R}$ contains a neighborhood of the origin or not (and notice that, when $\mathrm{I}(T, 0) > 1$, we are certainly in the first case). If it contains a neighborhood of the origin, then there is $\tilde{r} > 0$ such that

$$\frac{\mathbf{E}(T, \mathbf{B}_{r/2})}{\mathbf{E}(T, \mathbf{B}_r)} \geq 2^{-\gamma} \qquad \forall r < \tilde{r}\,.$$

In particular, if we let $\tilde{c} := \inf\{\mathbf{E}(T, \mathbf{B}_r) : \frac{\tilde{r}}{2} \leq r < \tilde{r}\} > 0$, iterating the inequality above at all dyadic scales we achieve

$$\mathbf{E}(T, \mathbf{B}_r) \geq \tilde{c}\left(\frac{r}{2\tilde{r}}\right)^{\gamma}\,.$$

If it does not contain the origin then let $\mathcal{R} = \bigcup_k]r_k^-, r_k^+[$ where $r_{k+1}^+ < r_k^-$ and both are infinite sequences of infinitesimal numbers. Then, $\mathbf{E}(T, \mathbf{B}_{r_k^+}) = \bar{E}$ and, up to subsequences, T_{0,r_k^+} converges to a cone C which is nonplanar and such that $\mathbf{E}(C, \mathbf{B}_\rho) = \frac{\bar{E}}{2}$ for every ρ. It follows in particular that there exists k_0 such that

$$\frac{\bar{E}}{4} \leq \mathbf{E}(T, \mathbf{B}_r) < \bar{E} \qquad \forall r \in \bigcup_{k \geq k_0} \left]\tfrac{r_k^+}{2}, r_k^+\right[\,.$$

In particular, arguing as above we conclude

$$\mathbf{E}(T, \mathbf{B}_r) \geq \frac{\bar{E}}{4} \left(\frac{r}{2r_k^+} \right)^\gamma \qquad \forall r \in \bigcup_{k \geq k_0}]r_k^-, r_k^+[\,,$$

while

$$\mathbf{E}(T, \mathbf{B}_r) \geq \bar{E} \qquad \forall r < r_{k_0}^+ \quad \text{s.t. } r \notin \bigcup_{k \geq k_0}]r_k^-, r_k^+[\,.$$

The combination of these two facts give that

$$\liminf_{r \downarrow 0} \frac{\mathbf{E}(T, \mathbf{B}_r)}{r^\gamma} > 0$$

and thus concludes the proof of (3.20). □

3.4.2 Reparametrization

An important tool for proving the Proposition 3.4.1 is the following lemma, where we follow the notation and techniques introduced in [20].

Lemma 3.4.4. *There are constants $\kappa(m, n, Q) > 0$ and $C(m, n, Q)$ with the following property. Consider:*

- *A Lipschitz map $g : \mathbb{R}^m \supset B_2 \to \mathcal{A}_Q(\mathbb{R}^n)$ with $\|g\|_{C^0} + \mathrm{Lip}\,(g) \leq \kappa$;*
- *A C^2 function $\boldsymbol{\varphi} : B_2 \to \mathbb{R}^n$ with $\boldsymbol{\varphi}(0) = 0$ and $\|D\boldsymbol{\varphi}\|_{C^1} \leq \kappa$;*
- *The function $f(x) = \sum_i \llbracket \boldsymbol{\varphi}(x) + g_i(x) \rrbracket$ and the manifold $\mathcal{M} := \{(x, \boldsymbol{\varphi}(x))\}$;*
- *The maps $N, F : \mathcal{M} \cap \boldsymbol{C}_{3/2} \to \mathcal{A}_Q(\mathbb{R}^{m+n})$ given by [20, Theorem 5.1], satisfying $F(p) = \sum_i \llbracket p + N_i(p) \rrbracket$, $N_i(p) \perp T_p\mathcal{M}$, and $\boldsymbol{T}_F \llcorner \boldsymbol{C}_{5/4} = \boldsymbol{G}_f \llcorner \boldsymbol{C}_{5/4}$.*

If we denote by $\tilde{g}$ the multivalued map $x \mapsto \tilde{g}(x) = \sum_i \llbracket (0, g_i(x)) \rrbracket \in \mathcal{A}_Q(\mathbb{R}^{m+n})$, then

$$\mathcal{G}(N(\boldsymbol{\varphi}(x)), \tilde{g}(x)) \leq C \|D\boldsymbol{\varphi}\|_{C^0} (\|g\|_{C^0} + \|D\boldsymbol{\varphi}\|_{C^0}) \qquad \forall x \in B_1 \,. \tag{3.23}$$

Proof. We fix a point $x \in B_1$, denote by $p \in \mathcal{M}$ the point $p = (x, \boldsymbol{\varphi}(x))$ and let $N(x) = \sum_i \llbracket q_i \rrbracket$ and $g(x) = \sum_i \llbracket p_i \rrbracket$. We fix a measurable selection for the function g, so that we can write $g = \sum_i \llbracket g_i \rrbracket$ and a corresponding measurable selection for f, where $f_i = \boldsymbol{\varphi} + g_i$. According to [20, Lemma 5.4], the set of points $\{q_i\}$ can be determined as follows. If we let $\varkappa$ be the orthogonal complement of $T_p\mathcal{M}$, then $\{q_i\}$ is given by the intersection of $p + \varkappa$ with the support of the current $\mathbf{G}_f$ (i.e. the set-theoretic graph of f). This means that there are points $x_1, \ldots, x_Q$ such that

$$q_i = (x_i, f_{j(i)}(x_i)) = (x_i, \boldsymbol{\varphi}(x_i) + g_{j(i)}(x_i)) \,,$$

where $j : \{1, \ldots, Q\} \to \{1, \ldots, Q\}$ is some unknown function. Observe that

$$|x_i - x| \leq C|q_i - p||\varkappa - \varkappa_0|$$

where $\varkappa_0$ denotes the vertical plane $\{0\} \times \mathbb{R}^n$. We therefore easily conclude the estimate

$$|x_i - x| \leq C\|N\|_{C^0}\|D\boldsymbol{\varphi}\|_0 \,.$$

Since however $\|N\|_{C^0} \leq C(\|g\|_{C^0} + \|\boldsymbol{\varphi}\|_{C^0}) \leq C(\|g\|_{C^0} + \|D\boldsymbol{\varphi}\|_{C^0})$, clearly

$$|x_i - x| \leq C\|D\boldsymbol{\varphi}\|_{C^0}(\|g\|_{C^0} + \|\boldsymbol{\varphi}\|_{C^0}) \,. \tag{3.24}$$

Given the Lipschitz bound on g we conclude that there is a $\pi(i)$ such that

$$|g_{j(i)}(x_i) - g_{\pi(i)}(x)| \leq C\|\boldsymbol{\varphi}\|_{C^0}(\|g\|_{C^0} + \|D\boldsymbol{\varphi}\|_{C^0}) \,. \tag{3.25}$$

If $\pi : \{1, \ldots, Q\} \to \{1, \ldots, Q\}$ were injective, we would immediately conclude (3.23). While this might generally not be the case, it certainly is when $Q = 1$, hence establishing the estimate in this particular case.

For the general case we argue by induction. Assume therefore to have fixed Q and to have proved the estimate valid for maps which are Q'-valued for every $Q' < Q$. Consider now the following alternatives:

(a) the diameter of the set $\{g_i(x)\}$ is smaller than $\|D\boldsymbol{\varphi}\|_{C^0}(\|g\|_{C^0} + \|D\boldsymbol{\varphi}\|_{C^0})$;

(b) the diameter of the set $\{g_i(x)\}$ is larger.

In the first case we have

$$|g_{j(i)}(x_i) - g_i(x)| \leq |g_{j(i)}(x_i) - g_{\pi(i)}(x)| + |g_{\pi(i)}(x) - g_i(x)| \leq (C+1)\|D\boldsymbol{\varphi}\|_{C^0}(\|g\|_{C^0} + \|D\boldsymbol{\varphi}\|_{C^0})\,.$$

In the second case we set $d := \|D\boldsymbol{\varphi}\|_{C^0}(\|g\|_{C^0} + \|\boldsymbol{\varphi}\|_{C^0})$ and recall [18, Proposition 1.6]: if the Lipschitz constant of g is smaller than a constant depending only on C, Q, and n, the map g decomposes, in the ball $B_{2d}(x)$ into two Lipschitz Q_i-valued maps with $Q_1 + Q_2 = Q$. In particular we can use the inductive assumption to get (3.23). □

3.4.3 Comparison estimates

In order to prove Proposition 3.4.1, (3.23) will be combined with two important estimates comparing the Lipschitz approximation and the normal approximation over the relevant center manifold.

The first estimate is the following control on the L^2 height of a normal approximation in terms of the tilt excess.

Lemma 3.4.5. *Under the assumptions of Proposition 3.4.1, the estimate (3.7) holds. Moreover, the following holds.*

(i) Let $\mathbf{h}_k$ be as in Section 3.2.1 for the scales r_k. Then we have

$$0 < \liminf_{k\to\infty} \frac{\mathbf{h}_k^2}{E_k} \leq \limsup_{k\to\infty} \frac{\mathbf{h}_k^2}{E_k} < \infty\,. \tag{3.26}$$

(ii) Let f_k be as in Section 3.3.1 and consider the map $\bar{\boldsymbol{\varphi}}_k$ on $B_2(0,\pi_k)$ whose graph coincides with the center manifold $(\mathcal{M}_{j(k)})_{0,r_k/t_{j(k)}}$ over the cylinder $\mathbf{C}_{3/2}(0,\pi_k)$. Then we have

$$\int_{B_{3/2}} |\bar{\boldsymbol{\varphi}}_k - \boldsymbol{\eta}\circ f_k|^2 = o(E_k)\,. \tag{3.27}$$

Proof. We fix r_k as in the statement and, upon extraction of a further subsequence, we assume the existence of

$$\lim_{k\to\infty} \frac{r_k}{s_{j(k)}} := \tilde{c} \in]0,+\infty[\,.$$

Observe that $\tilde{c}$ is at least 1. It is convenient to introduce the rescaled radii $\bar{r}_k := \frac{r_k}{t_{j(k)}}$ and $\bar{s}_{j(k)} := \frac{s_{j(k)}}{t_{j(k)}}$. Recalling the stopping condition which defines $s_{j(k)}$ in [22, Section 2.1], there is a cube $L_k \in \mathscr{W}^{j(k)}$ with $L_k \cap \mathbf{B}_{\bar{s}_{j(k)}}$ and $\ell(L_k) = c_s \bar{s}_{j(k)}$ for the specific geometric constant $c_s = \frac{1}{64\sqrt{m}}$. Observe that, since 0 is a point of density Q for the current T, [21, Proposition 3.1] implies that L_k cannot belong to $\mathscr{W}_h^{j(k)}$. If $L_k \in \mathscr{W}_n^{j(k)}$, we may apply [21, Corollary 3.2] to find a nearby cube $L_k' \in \mathscr{W}_e^{j(k)}$ of comparable size. Thus, we may assume that $L_k \in \mathscr{W}_e^{j(k)}$. We can thus apply [21, Proposition 3.4] to conclude

$$\boldsymbol{m}_{0,j(k)} \ell(L_k)^{2-2\delta_2} \le C\mathbf{E}(T_{0,t_{j(k)}}, \mathbf{B}_{L_k})\,.$$

Recalling however that the cylinder $\mathbf{C}_{4M\bar{r}_k}(0,\pi_k)$ contains $\mathbf{B}_{L_k}$, as well as our amended definition of $\boldsymbol{m}_{0,j(k)}$, we immediately conclude that

$$\begin{aligned} E_k &:= \mathbf{E}(T_{0,r_k}, \mathbf{C}_{4M}, \pi_k) \ge \mathbf{E}(T_{0,t_{j(k)}}, \mathbf{B}_{L_k}) \ge C^{-1}\boldsymbol{m}_{0,j(k)}\ell(L_k)^{2-2\delta_2} \\ &\ge C^{-1} c_s^2 \bar{\varepsilon}^2 t_{j(k)}^{2-2\delta_2} \frac{s_{j(k)}^{2-2\delta_2}}{t_{j(k)}^{2-2\delta_2}} = C^{-1} c_s^2 \bar{\varepsilon}^2 s_{j(k)}^{2-2\delta_2}\,. \end{aligned}$$

In light of the comparability of $s_{j(k)}$ and r_k, it thus follows immediately that

$$\liminf_{k\to\infty} \frac{E_k}{r_k^{2-2\delta_2}} > 0\,, \tag{3.28}$$

which in turn immediately implies (3.7). In addition, rescaling by $t_{j(k)}$ and again using the definition of $\boldsymbol{m}_{0,j(k)}$, we have

$$E_k \geq \tilde{C}^{-1} \boldsymbol{m}_{0,j(k)} \bar{r}_k^{2-2\delta_2} , \tag{3.29}$$

where $\tilde{C}$ is independent of k (it is not, however, a geometric constant, namely it might depend on the blow-up sequence that we fixed at the beginning).

Next, observe that by Theorem 2.6.8, we have

$$\mathbf{h}_k^2 \leq \frac{C}{\bar{r}_k^{m+1}} \mathbf{H}_{N_{j(k)}}(2\bar{r}_k) \leq \tilde{C} \bar{r}_k^{-m} \mathbf{D}_{N_{j(k)}}(2\bar{r}_k) ,$$

where $\tilde{C}$ is independent of k. On the other hand we recall (see for instance [22, Remark 3.4]) that $\mathbf{D}_{N_{j(k)}}(2\bar{r}_k) \leq C \boldsymbol{m}_{0,j(k)} \bar{r}_k^{m+2-2\delta_2}$. We thus conclude that

$$\mathbf{h}_k^2 \leq C \boldsymbol{m}_{0,j(k)} \bar{r}_k^{2-2\delta_2}$$

and we achieve the right-hand inequality of (3.26) when combining the above with (3.29).

As for the left-hand inequality of (3.26), first recall that, by [21, Proposition 3.4] we also have the opposite inequality

$$\mathbf{h}_k^2 \geq \tilde{C}^{-1} \bar{r}_k^{-m-2} \int_{\mathcal{L}_k} |N_{j(k)}|^2 \geq \tilde{C}^{-1} \boldsymbol{m}_0^{(k)} \bar{r}_k^{2-2\delta_2} , \tag{3.30}$$

where $\mathcal{L}_k$ is the Whitney region corresponding to L_k. On the other hand recall that π_k optimizes the tilt excess of T_{0,r_k} in $\mathbf{B}_{8M}$, which implies that it optimizes the tilt excess of $T_{0,t_{j(k)}}$ in $\mathbf{B}_{8M\bar{r}_k}$. Because of the condition $s_{j(k)} \leq r_k \leq \bar{c} s_{j(k)}$, we can find a cube $H \in \mathscr{S}^{j(k)} \cup \mathscr{W}^{j(k)}$ with the property that $\mathbf{B}_{32M\bar{r}_k} \supset \mathbf{B}_H \supset \mathbf{B}_{8M\bar{r}_k}$. Due to [21, Proposition 1.11], we thus must have

$$\begin{aligned} \mathbf{E}(T_{0,t_{j(k)}}, \mathbf{B}_{8M\bar{r}_k}, \pi_k) &\leq \mathbf{E}(T_{0,t_{j(k)}}, \mathbf{B}_{8M\bar{r}_k}, \pi_H) \leq C \mathbf{E}(T_{0,t_{j(k)}}, \mathbf{B}_H, \pi_H) \\ &\leq C \boldsymbol{m}_{0,j(k)} \ell(H)^{2-2\delta_2} \leq C \boldsymbol{m}_{0,j(k)} \bar{r}_k^{2-2\delta_2} . \end{aligned}$$

Combining this with the height bound [21, Theorem A.1] on $T_{0,t_{j(k)}}$, we can write

$$E_k = \bar{r}_k^{-m}\mathbf{E}(T_{0,t_{j(k)}}, \mathbf{C}_{4M\bar{r}_k}) \leq C\mathbf{E}(T_{0,t_{j(k)}}\mathbf{B}_{8M\bar{r}_k}, \pi_k) \leq C\boldsymbol{m}_{0,j(k)}\bar{r}_k^{2-2\delta_2}\,. \tag{3.31}$$

It thus follows immediately from (3.30) and (3.31) that $\liminf_k \frac{\mathbf{h}_k^2}{E_k} > 0$.

We now address the last part of the lemma, namely statement (ii). First of all we rescale the graphs of $\bar{\boldsymbol{\varphi}}_k$ and of f_k using an homothety of center 0 and ratio $\bar{r}_k$. We denote by $\bar{\boldsymbol{\varphi}}_k^r := \bar{r}_k^{-1}\bar{\boldsymbol{\varphi}}_k(\bar{r}_k\cdot)$ and $f_k^r := \bar{r}_k^{-1}f_k(\bar{r}_k\cdot)$ the corresponding maps and note that the desired estimate is equivalent to

$$\bar{r}_k^{-m-2}\int_{B_{3\bar{r}_k/2}(0,\pi_k)} |\bar{\boldsymbol{\varphi}}_k^r - \boldsymbol{\eta}\circ f_k^r|^2 = o(E_k)\,,$$

and given the estimate (3.29), it suffices to show

$$\int_{B_{3\bar{r}_k/2}(0,\pi_k)} |\bar{\boldsymbol{\varphi}}_k^r - \boldsymbol{\eta}\circ f_k^r|^2 = E_k^{1/2}o(\boldsymbol{m}_{0,j(k)}^{1/2}\bar{r}_k^{m+3-\delta_2})\,. \tag{3.32}$$

Consider now the plane $\pi_0(j(k))$ which served as reference to construct the center manifold $\mathcal{M}_{j(k)}$. It is easy to see that $|\pi_0(j(k)) - \pi_k| \leq C\boldsymbol{m}_{0,j(k)}^{1/2} \leq C\bar{\varepsilon}$ for some geometric constant (see [21, Proposition 4.1]). Since nothing else will be used about $\pi_0(j(k))$, except that it serves as reference to construct the center manifold $\mathcal{M}_{j(k)}$, in order to simplify our notation we will simply denote it by π_0, even though the plane does depend on k.

We now consider all the cubes $H \in \mathscr{W}^{j(k)}$ which intersect $\mathbf{B}_{2\bar{r}_k}$ and denote such collections by $\mathscr{C}^{(k)}$. For each $H \in \mathscr{C}^{(k)}$ we consider a cylinder $\mathbf{C}_{2C\ell(H)}(q_H, \pi_H)$, where C is a geometric constant (which will be specified later) and q_H is the center of the cube H. We then consider the cylinder $\mathbf{C}_{C\ell(H)}(q_H, \pi_k)$ and, given that the height of T_{0,r_k} over π_k converges to 0, conclude that the set $(\operatorname{gr}(\bar{\boldsymbol{\varphi}}_k^r) \cup \operatorname{gr}(\bar{f}_k^r)) \cap \mathbf{C}_{C\ell(H)}(q_H, \pi_k)$ is contained in $\mathbf{C}_{2C\ell(H)}(q_H, \pi_H)$. Further, let $\boldsymbol{\Phi}_k(\boldsymbol{\Gamma}_k)$ be the contact set of the current $T_{0,t_{j(k)}}$ and the center manifold $\mathcal{M}_{j(k)}$, as defined in [21, Definition 1.18], and denote by Γ_k its projection onto the plane π_k. Finally, it will also

be convenient to define the point $\bar{q}_L$ as the orthogonal projection onto π_k of q_L.

If C is a geometric constant sufficiently large (e.g. $10\sqrt{m}$ suffices, provided $\bar{\varepsilon}$ is small enough), then the set Γ_k and the disks $B_{C\ell(H)}(\bar{q}_H, \pi_k)$ cover the disk $B_{3\bar{r}_k/2}(0, \pi_k)$. It will be convenient to devise a slightly delicate cover, made of pairwise disjoint Borel sets, with the following algorithm. We enumerate the disks $B_{C\ell(H)}(\bar{q}_H, \pi_k)$ as B^i, $i \in \{1, 2, \ldots\} = \mathbb{N} \setminus \{0\}$ and set $F_0 := \Gamma_k \cap B_{3\bar{r}_k/2}$ and define inductively $F_{j+1} := B_{j+1} \setminus \bigcup_{i \leq j} F_i$.

Next, for each H we recall the approximating Lipschitz map f_H of [21, Definition 1.13 & Lemma 1.15] and let $\bar{f}_H$ be the reparametrization of $\operatorname{gr}(f_H) \cap \mathbf{C}_{C\ell(H)}(q_H, \pi_k)$ as a graph over the disk $B_{C\ell(H)}(\bar{q}_H, \pi_k)$, according to [20, Proposition 5.2]. We are now going to define a good set $G \subset B_{3\bar{r}_k/2}$ as follows

- $G \cap F_0$ consists of those points $q \in F_0$ where $\bar{f}_k(q) = Q\llbracket \bar{\boldsymbol{\varphi}}_k^r \rrbracket$;
- For each $j > 0$, $G \cap F_j$ consists of those points $q \in F_j$ where $\bar{f}_k$ coincides with $\bar{f}_H$ for the corresponding H such that $B_{C\ell(H)}(\bar{q}_H, \pi_k) = B^j$.

Observe that

$$B_{3\bar{r}_k/2} \setminus G \subset \underbrace{\mathbf{p}_{\pi_k}((\operatorname{spt}(T_{0,t_{j(k)}}) \setminus \operatorname{gr}(\bar{f}_k^r)) \cap \mathbf{C}_{3\bar{r}_k/2}(0, \pi_k))}_{=:\Xi_k^1} \cup \underbrace{\mathbf{p}_{\pi_k}(\operatorname{spt}(T_{0,t_{j(k)}} \setminus \mathbf{T}_{F_{j(k)}}) \cap \mathbf{C}_{3\bar{r}_k/2}(0, \pi_k)))}_{=:\Xi_k^2}.$$

On the other hand, recalling that $\mathbf{A}_k^2 = o(E_k)$, we can use [19, Theorem 2.4] to estimate

$$\begin{aligned} |\Xi_k^1| &\leq \mathcal{H}^m(\operatorname{spt}(T_{0,t_{j(k)}}) \setminus \operatorname{gr}(\bar{f}_k^r)) \cap \mathbf{C}_{3\bar{r}_k/2}(0, \pi_k)) \\ &\leq \bar{r}_k^m \mathcal{H}^m(\operatorname{spt}(T_{0,r_k}) \setminus \operatorname{gr}(f_k)) \cap \mathbf{C}_M(0, \pi_k)) = \bar{r}_k^m O(E_k^{1+\gamma_1}). \end{aligned}$$

As for Ξ^2_k, we instead use the analogous estimates for each f_H to get

$$\begin{aligned}|\Xi^2_k| &\leq \sum_{H\in\mathscr{C}^{(k)}} \mathcal{H}^m(\mathrm{spt}((T_{0,t_{j(k)}} \setminus \mathrm{gr}\, f_H)) \cap \mathbf{C}_{C\ell(H)}(q_H,\pi_k))\\ &\leq \sum_{H\in\mathscr{C}^{(k)}} \mathcal{H}^m((T_{0,t_{j(k)}} \setminus \mathrm{gr}\, f_H)) \cap \mathbf{C}_{2C\ell(H)}(q_H,\pi_H))\\ &\leq \sum_{L\in\mathscr{C}^{(k)}} \ell(H)^m(\boldsymbol{m}_{0,j(k)}\ell(H)^{2-2\delta_2})^{1+\gamma_1} \leq \boldsymbol{m}^{1+\gamma_1}_{0,j(k)}\bar{r}_k^{m+2+\gamma_1/2}\end{aligned}$$

(we recall here that the constant γ_1 is fixed in [19], while δ_2 is chosen later in [21, Assumption 1.8] and satifies $(2-2\delta_2)(1+\gamma_1)\leq 1+\gamma_1/2$).

On the other hand,

$$\|\bar{f}^r_k\|_{C^0(B_{3\bar{r}_k})} \leq C\mathbf{h}(T_{0,t_{j(k)}}, \mathbf{C}_{3\bar{r}_k}(0,\pi_k)) = C\bar{r}_k\mathbf{h}(T_{0,r_k}, \mathbf{C}_3(0,\pi_k)) \leq C\bar{r}_k E_k^{1/2}\,,$$

where in the latter inequality we have used the information that 0 is a point of density Q point of T and the classical L^∞ bound of Allard, cf. [50]. Moreover, recalling that $\|N_{j(k)}\|^2_{L^2(\mathcal{L})} \leq C\boldsymbol{m}^{1/2}_{0,j(k)}\ell(L)^{m+4-2\delta_2}$, we infer in particular the existence of at least one point $x\in \mathbf{p}_{\pi_k}(\mathcal{L})$ and $y\in\pi_k^\perp$ such that $(x,y)\in\mathrm{spt}(T_{0,t_{j(k)}})$ and

$$|\bar{\boldsymbol{\varphi}}^r_k(x)-y| \leq C\boldsymbol{m}^{1/2}_{0,j(k)}\bar{r}_k^{2-\delta_2}\,,$$

which in turn leads to the bound $|\bar{\boldsymbol{\varphi}}^r_k(x)| \leq C(\boldsymbol{m}^{1/2}_{0,j(k)}+E_k^{1/2})\bar{r}_k \leq CE_k^{1/2}\bar{r}_k$. Note that $\bar{\boldsymbol{\varphi}}^r_k$ is Lipschitz, with a constant uniformly controlled in k. We thus conclude that

$$\|\bar{f}^r_k\|_{C^0(B_{3\bar{r}_k})} + \|\bar{\boldsymbol{\varphi}}^r_k\|_{C^0(B_{3\bar{r}_k})} \leq CE_k^{1/2}\bar{r}_k\,. \tag{3.33}$$

In particular, combining the latter estimate with $|B_{3\bar{r}_k/2}\setminus G| \leq CE_k\bar{r}_k^m$, we conclude that

$$\int_{B_{3\bar{r}_k/2}\setminus G} |\bar{\boldsymbol{\varphi}}^r_k - \boldsymbol{\eta}\circ\bar{f}^r_k|^2 \leq CE_k^2\bar{r}_k^{m+2}\,. \tag{3.34}$$

Considering that on $G \cap F_0$ the functions $\bar{\boldsymbol{\varphi}}^r_k$ and $\boldsymbol{\eta} \circ \bar{f}^r_k$ coincide, we are left to estimate

$$\sum_{j\geq 1} \int_{G\cap F_j} |\bar{\boldsymbol{\varphi}}^r_k - \boldsymbol{\eta} \circ \bar{f}^r_k|^2 \leq E_k^{1/2} \bar{r}_k \sum_{j\geq 1} \int_{B_{C\ell(H)}(\bar{q}_H, \pi_k)} |\bar{\boldsymbol{\varphi}}^r_k - \boldsymbol{\eta} \circ \bar{f}_H| \,. \tag{3.35}$$

We now wish to estimate each integral in the above summation by changing coordinates to the reference plane π_H for each $H \in \mathscr{C}^{(k)}$. Denote by $\boldsymbol{\varphi}_H$ the function whose graph over $B_{2C\ell(H)}(q_H, \pi_H)$ coincides with $\mathcal{M}_{j(k)}$ (which, we recall, is the graph of $\bar{\boldsymbol{\varphi}}^r_k$ over an appropriate subset of π_k). We likewise introduce $\mathbf{f}_H$ which is the function over $B_{2C\ell(H)}(q_H, \pi_H)$ whose graph coincides with the graph of $\boldsymbol{\eta} \circ \bar{f}_H$. Applying [21, Lemma B.1(b)] we can then estimate

$$\int_{B_{C\ell(H)}(\bar{q}_H, \pi_k)} |\bar{\boldsymbol{\varphi}}^r_k - \boldsymbol{\eta} \circ \bar{f}_H| \leq C \int_{B_{2C\ell(H)}(q_H, \pi_H)} |\boldsymbol{\varphi}_H - \mathbf{f}_H| \,.$$

Let us now estimate

$$\int_{B_{2C\ell(H)}(q_H, \pi_H)} |\boldsymbol{\varphi}_H - \mathbf{f}_H| \leq \int_{B_{2C\ell(H)}(q_H, \pi_H)} |\boldsymbol{\varphi}_H - \boldsymbol{\eta} \circ f_H| + \int_{B_{2C\ell(H)}(q_H, \pi_H)} |\boldsymbol{\eta} \circ f_H - \mathbf{f}_H| \,. \tag{3.36}$$

In order to handle the second integral we wish to estimate $|\pi_k - \pi_H|$, since we will be using C^0-estimates on f_H here. First of all we compare the tilt between π_k and $\pi_{H'}$ for the ancestor H' of H with the smallest side length such that $\mathbf{B}_{H'} \supset \mathbf{B}_{8M\bar{r}_k}$. Observe that $\ell(H') \leq C\bar{r}_k$. Since $\pi_{H'}$ optimizes the tilt excess of $T_{0,t_{j(k)}}$ in $\mathbf{B}_{H'}$, while π_k optimizes the tilt excess of the same current over $\mathbf{B}_{8M\bar{r}_k}$, a simple comparison argument (cf. for instance [21, Proof of (4.5)]), implies

$$|\pi_k - \pi_{H'}| \leq C(\mathbf{E}(T_{0,t_{j(k)}}, \mathbf{B}_{8M})^{1/2} + \mathbf{E}(T_{0,t_{j(k)}}, \mathbf{B}'_H)^{1/2}) \leq C E_k^{1/2} + C\boldsymbol{m}_{0,j(k)}^{1/2} \bar{r}_k^{1-\delta_2} \,.$$

On the other hand, by [21, Proposition 4.1] we have

$$|\pi_H - \pi_{H'}| \leq C\boldsymbol{m}_{0,j(k)}^{1/2} \bar{r}_k^{1-\delta_2}$$

and we thus reach

$$|\pi_k - \pi_H| \leq CE_k^{1/2} + C\boldsymbol{m}_{0,j(k)}^{1/2}\bar{r}_k^{1-\delta_2} \leq CE_k^{1/2}\,.$$

We can now employ [21, Lemma 5.6] to estimate

$$\int_{B_{2C\ell(H)}(q_H,\pi_H)} |\boldsymbol{\eta}\circ f_H - \mathbf{f}_H| \leq C(\|f_H\|_{C^0(B_{2C\ell(H)}(q_H,\pi_H))} + E_k^{1/2})(\mathrm{Dir}\,(f_H) + \ell(H)^m E_k)\,.$$

Recall that $\|f_H\|_{C^0(B_{2C\ell(H)}(q_H,\pi_H)} \leq \boldsymbol{m}_{0,j(k)}^{1/2m}\ell(H)^{1+\beta_2}$, while $\mathrm{Dir}\,(f_H) \leq \boldsymbol{m}_{0,j(k)}\ell(H)^{m+2-2\delta_2} \leq \ell(H)^m E_k$. We thus easily conclude that

$$\int_{B_{2C\ell(H)}(q_H,\pi_H)} |\boldsymbol{\eta}\circ f_H - \mathbf{f}_H| \leq C\ell(H)^m E_k^{1+1/2m}\,.$$

We now come to the first integral in the right hand side of (3.36). First of all we recall the tilted interpolating function h_H of [21, Definition 1.16] and observe that, by construction, $\boldsymbol{\varphi}_H$ and h_H coincide in a neighborhood of q_H. Now recall that, by [21, Proposition 4.4] $\|Dh_H\| \leq C\boldsymbol{m}_{0,j(k)}^{1/2}$. Since moveover $\|D^2\varphi_H\|$ is controlled by the second fundamental form of $\mathcal{M}_{j(k)}$, which in turn is bounded by $\boldsymbol{m}_{0,j(k)}^{1/2}$, we easily see that the estimate $\|D^2\varphi_H\| \leq C\boldsymbol{m}_{0,j(k)}^{1/2}$ holds as well. In particular, using a second order Taylor expansion on a point where $\boldsymbol{\varphi}_H - h_H$ and its derivative both vanish (to gain an extra factor of $\ell(H)^2$) we can estimate

$$\int_{B_{2C\ell(H)}(q_H,\pi_H)} |\boldsymbol{\varphi}_H - \boldsymbol{\eta}\circ f_H| \leq C\boldsymbol{m}_{0,j(k)}^{1/2}\ell(H)^{m+2} + \int_{B_{2C\ell(H)}(q_H,\pi_H)} |h_H - \boldsymbol{\eta}\circ f_H|\,.$$

Finally we can use [21, Proposition 5.2] to estimate

$$\int_{B_{2C\ell(H)}(q_H,\pi_H)} |h_H - \boldsymbol{\eta}\circ f_H| \leq C\boldsymbol{m}_0^{(k)}\ell(H)^{m+3+\beta_2}\,.$$

In summary, we have reached the estimate

$$\int_{B_{C\ell(H)}(\bar{q}_H,\pi_k)} |\bar{\boldsymbol{\varphi}}_k^r - \boldsymbol{\eta}\circ \bar{f}_H| \leq C\boldsymbol{m}_{0,j(k)}^{1/2}\ell(H)^{m+2}\,.$$

Inserting this into (3.35) and decomposing into cubes H, we then get

$$\int_G |\bar{\boldsymbol{\varphi}}_k^r - \boldsymbol{\eta} \circ \bar{f}_k^r|^2 \leq C\bar{r}_k E_k^{1/2} \boldsymbol{m}_{0,j(k)}^{1/2} \sum_{H \in \mathscr{C}^{(k)}} \ell(H)^{m+2} \leq C E_k^{1/2} \boldsymbol{m}_{0,j(k)}^{1/2} \bar{r}_k^{m+3} .$$

The latter, together with (3.34), gives finally (3.32) and completes the proof of the lemma.

□

Proof of Proposition 3.4.1. We wish to compare

$$\tilde{N}_k := \frac{N_k \circ \boldsymbol{\varphi}_k}{\mathbf{h}_k} \qquad \text{and} \qquad v_k := \frac{\sum_i [\![(f_k)_i - \boldsymbol{\eta} \circ f_k]\!]}{E_k^{1/2}} ,$$

in particular we wish to show that they have the same L^2 limit, up to a scalar constant. Since both sequences are converging to respective Dir-minimizing maps, it suffices to compare the maps $\tilde{N}_k$ and v_k on some nonempty open set; we will do it on B_1 for simplicity.

First of all we replace $\boldsymbol{\eta} \circ f_k$ with the parameterizing map $\boldsymbol{\varphi}_k$ for $\mathcal{M}_k$ in v_k to give a map $\hat{v}_k$ given by

$$\hat{v}_k = \frac{\sum_i [\![(f_k)_i - \boldsymbol{\varphi}_k]\!]}{E_k^{1/2}} ,$$

since Lemma 3.4.5 implies that

$$\lim_{k \uparrow \infty} \int_{B_{3/2}} \mathcal{G}(v_k, \hat{v}_k)^2 = 0 .$$

Recalling [19],

$$|\mathbf{p}_k((\mathrm{spt}(T_{0,r_k}) \setminus \mathrm{gr}\,(f_k) \cup \mathrm{gr}\, f_k \setminus \mathrm{spt}(T_{0,r_k}) \cap \mathbf{C}_{3/2})| = o(E_k) .$$

Next, introduce the map $\mathbf{F}(p) := \sum_i [\![(N_k)_i(p) + p]\!]$ on $\mathcal{M}_k$ and let $f_k^1 : B_2(0, \pi_k) \to \mathcal{A}_Q(\pi_k^\perp)$ be the map whose graph coincides with the current $\mathbf{T}_F \cap \mathbf{C}_2(0, \pi_k)$. By [21, Theorem 2.4]

and [22, Section 4.2 & Corollary 5.3],

$$|\mathbf{p}_{\pi_k}((\mathrm{gr}\,(f_k^1) \setminus \mathrm{spt}(T_{0,r_k}) \cup \mathrm{spt}(T_{0,r_k}) \setminus \mathrm{gr}\,(f_k^1)) \cap \mathbf{C}_{3/2}| = o(E_k)\,.$$

In particular, if we consider the map

$$\hat{v}_k^1 = \frac{\sum_i [\![(f_k^1)_i - \boldsymbol{\varphi}_k]\!]}{E_k^{1/2}}$$

we have that $|\{\hat{v}_k^1 \neq \hat{v}_k\}| \to 0$, and using that both have a uniform bound on the Dirichlet energy, we conclude that

$$\lim_{k\to\infty} \int_{B_{3/2}} \mathcal{G}(\hat{v}_k^1, \hat{v}_k)^2 = 0\,.$$

We also take advantage of Lemma 3.4.5 to assume, up to extraction of a subsequence (not relabeled), that $E_k/\mathbf{h}_k^2$ converges to some finite constant $\lambda > 0$. We are therefore left to show that the maps $\tilde{N}_k$ and

$$\hat{v}_k^2 = \frac{\sum_i [\![(f_k^1)_i - \boldsymbol{\varphi}_k]\!]}{\mathbf{h}_k^{1/2}}$$

have the same limit. We now wish to apply Lemma 3.4.4 to the maps N_k. We observe that the map g in Lemma 3.4.4 can be taken to be the map g_k defined by

$$g_k := \sum_i [\![(f_k^1)_i - \boldsymbol{\varphi}_k]\!]\,.$$

Moreover, observe that $\|D\boldsymbol{\varphi}_k\|_{C^0}$ converges to 0. If we had a uniform bound on $\|g_k\|_{C^0}$ in terms of $\mathbf{h}_k$ we could then apply Lemma 3.4.4 to complete the proof. Given that we only have the bound $\|g_k\|_{L^2} \leq C\mathbf{h}_k$ we need to overcome this issue. We use the following simple argument. We fix a truncation parameter $\bar{M}$ and introduce the truncation

$$g_k^{\bar{M}} := \sum_i [\![(g_k)_i^{\bar{M}}]\!]$$

where the maps $(g_k)_i^{\bar{M}}$ are defined by replacing each component $(\xi_i)_j(x)$ of the vector $(g_k)_i(x)$ with $\max\{-\bar{M}, \min\{(\xi_i)_j(x), \bar{M}\}\}$. By the Sobolev embedding and the uniform $W^{1,2}$ bound on g_k it is easy to see that

$$\lim_{\bar{M}\to\infty} \sup_k \mathbf{h}_k^{-2} \int \mathcal{G}(g_k, g_k^{\bar{M}})^2 = 0\,.$$

Likewise, after defining the maps $N_k^{\bar{M}}$ as those corresponding to $g_k^{\bar{M}}$ in the same way as N_k corresponds to g_k, we see as well

$$\lim_{\bar{M}\to\infty} \sup_k \mathbf{h}_k^{-2} \int \mathcal{G}(N_k, N_k^{\bar{M}})^2 = 0\,.$$

We can now apply Lemma 3.4.4 to conclude that the limit (in k) of $\mathbf{h}_k^{-1} N_k^{\bar{M}} \circ \varphi_k$ and the limit of $g_k^{\bar{M}}$ coincides on B_1. Letting $\bar{M} \to \infty$ we then reach the desired conclusion. □

3.5 Frequency bound for fine blow-ups

In this section we prove the lower bound for the frequency values, which we equivalently restate as follows for the reader's convenience.

Theorem 3.5.1. *Suppose that T and Σ are as in Assumption 3.2.3 and let u be a fine blow-up. Then $I_u(0) \geq 1$.*

In order to show the theorem, we fix a blow-up sequence $\{r_k\}$ which generates the fine blow-up u through the procedure described in Section 3.2.1 and for each k sufficiently large we choose the interval of flattening $]s_{j(k)}, t_{j(k)}]$ which contains the radius r_k. We can then reduce the proof, up to extraction of a subsequence, to three different cases. In the first case we assume that there are finitely many intervals of flattening and hence (up to subsequence), there is a positive integer J such that:

$$s_J = 0 \qquad \text{and} \qquad \{r_k\}_k \subset]0, t_J]\,. \tag{3.37}$$

In the remaining three cases we assume that there are infinitely many intervals of flattening and that (up to subsequence) one of the following mutually exclusive conditions hold:

$$\lim_k \frac{s_{j(k)}}{r_k} > 0 \tag{3.38}$$

$$\lim_k \frac{s_{j(k)}}{r_k} = 0. \tag{3.39}$$

The proof will take advantage of a first coarse lower bound proved in Theorem 2.6.8, which in turn can be combined with the monotonicity computations in [22] to give a suitable almost-monotonicity formula for $\mathbf{I}_N$ (see also Chapter 2). We summarize these conclusions in the following theorem.

Theorem 3.5.2. *Let T, Σ be as in Assumption 3.2.3 and consider any center manifold $\mathcal{M}_j$ and any normal approximation N_j for a given interval of flattening $]s_j, t_j]$ at 0. Then,*

$$\mathbf{I}_{N_j}(r) \geq c_0 \qquad \forall r \in \left]\frac{s_j}{t_j}, 3\right] , \tag{3.40}$$

$$\mathbf{I}_{N_j}(a) \leq e^{Cb^\alpha}\mathbf{I}_{N_j}(b) \qquad \forall]a, b] \subset \left]\frac{s_j}{t_j}, 3\right] , \tag{3.41}$$

where $\alpha = \alpha(Q, m, n) > 0$, while c_0 and C are positive numbers which depend on T (but not on j).

3.5.1 Proof of Theorem 3.5.1 under assumption (3.37)

We let $\mathcal{M}$ be the center manifold related to the interval of flattening $]0, t_J]$, with corresponding normal approximation N. Since we are in the case with a single center manifold, we omit the dependency on N for $\mathbf{I}$ and related quantities. Observe that, by Theorem 3.5.2,

$$\mathbf{I}(a) \leq e^{Cb^\alpha}\mathbf{I}(b) \qquad \forall 0 < a \leq b < 3$$

and in particular we immediately see that

$$c_0 \leq \limsup_{r\downarrow 0} \mathbf{I}(r) \leq \liminf_{r\downarrow 0} \mathbf{I}(r) < +\infty\,.$$

So the limit $I_0 := \lim_{r\downarrow 0} \mathbf{I}(r)$ exists and it is positive and finite. It follows from the strong convergence of u_k from the definition of u being a fine blowup, that $I_u(r)$ is identically equal to I_0, and thus $I_0 = I_u(0)$. Therefore it just suffices to show that $I_0 \geq 1$. On the other hand, by [22, Proposition 3.5], we readily see that

$$\left| \frac{d}{dr} \log \frac{\mathbf{H}(r)}{r^{m-1}} - \frac{2\mathbf{I}(r)}{r} \right| \leq \frac{C\mathbf{I}(r)}{r^{1-\gamma}}\,,$$

for suitable constants C and $\gamma > 0$. In particular, for every $\varepsilon > 0$, the inequalities

$$\frac{2I_0 - \varepsilon}{r} \leq \frac{d}{dr} \log \frac{\mathbf{H}(r)}{r^{m-1}} \leq \frac{2I_0 + \varepsilon}{r}$$

hold as soon as r is smaller than a suitable scale $r(\varepsilon) > 0$. Integrating the latter differential inequality, we immediately conclude that

$$\liminf_{r\downarrow 0} \frac{\mathbf{H}(r)}{r^{m-1+2I_0+\varepsilon}} > 0$$

for every $\varepsilon > 0$. Combined with the inequality $\frac{r\mathbf{D}(r)}{\mathbf{H}(r)} = \mathbf{I}(r) \geq c_0$, we also conclude that

$$\liminf_{r\downarrow 0} \frac{\mathbf{D}(r)}{r^{m+2(I_0-1)+\varepsilon}} > 0\,.$$

On the other hand, due to the estimate [22, (3.4)] and the fact that $s_J = 0$, we must have

$$\mathbf{D}(r) \leq Cr^{m+2-2\delta_2}$$

where δ_2 is the small positive constant of [21, Assumption 1.8]. Comparing this with the previous asymptotic estimate, we conclude in particular that

$$2(I_0 - 1) \geq 2 - 2\delta_2 \,,$$

and since $2\delta_2 \leq \frac{1}{4m}$, we immediately get that $I_0 > 1$ (in fact it turns out that I_0 is rather close to 2, in this case).

3.5.2 Proof of Theorem 3.5.1 under assumption (3.38)

In this case we can apply Proposition 3.4.1 to a suitable subsequence of $\{r_k\}_k$, not relabeled, and find a coarse blow-up f whose average-free part v has the property that $v = \lambda u$ for some positive number λ. In particular $I_u(0) = I_v(0)$ and from Theorem 3.3.2 we conclude $I_u(0) \geq 1$.

3.5.3 Proof of Theorem 3.5.1 under assumption (3.39)

We fix a blow-up sequence $\{r_k\}_k$ and a corresponding fine blow-up u. One crucial property that we will use is that, because of the convergence of the maps u_k from Section 3.2.1 to the fine blow-up u, for every positive $\rho < 1$ we have

$$I_u(\rho) = \lim_{k\to\infty} \mathbf{I}_{N_{j(k)}} \left(\frac{\rho r_k}{t_{j(k)}} \right) \tag{3.42}$$

In particular, since

$$\mathbf{E}(T, \mathbf{B}_{s_{j(k)}}) = \mathbf{E}(T_{0,t_{j(k)}}, \mathbf{B}_{s_{j(k)}/t_{j(k)}}) \leq C\boldsymbol{m}_{0,j(k)} \frac{s_{j(k)}^{2-2\delta_2}}{t_{j(k)}^{2-2\delta_2}} \,,$$

we conclude that $\mathbf{E}(T, \mathbf{B}_{s_{j(k)}}) \to 0$. So $s_{j(k)}$ is itself a blow-up sequence, and we can apply the previous section to infer that, for any u' coarse blow-up generated by a subsequence, we have $I_{u'}(0) \geq 1$. In particular, since along this subsequence of $\{s_{j(k)}\}$ we have comparability of the

coarse and fine blow-ups due to Proposition 3.4.1, we can use the corresponding convergence (3.42) to infer that

$$\liminf_{k\to\infty} \mathbf{I}_{N_{j(k)}}\left(\frac{s_{j(k)}}{t_{j(k)}}\right) \geq 1\,.$$

Fix now an arbitrary small parameter $\delta > 0$. Our goal is to show that there is $\bar\rho > 0$ such that

$$\liminf_{k\to\infty} \mathbf{I}_{N_{j(k)}}\left(\frac{\rho r_k}{t_{j(k)}}\right) \geq 1-2\delta \qquad \forall \rho \in \left]\frac{s_{j(k)}}{r_k}, \bar\rho\right[\,. \tag{3.43}$$

Knowing (3.43) and (3.42), we would then infer that $I_u(\rho) \geq 1-2\delta$ for every positive $\rho < \bar\rho$, which in turn would imply $1-2\delta \leq I_u(0)$. The arbitrariness of δ then tells us that $I_u(0) = 0$.

In order to achieve (3.43), choose first k_0 large enough so that

$$\mathbf{I}_{N_{j(k)}}\left(\frac{s_{j(k)}}{t_{j(k)}}\right) \geq 1-\delta \qquad \forall k \geq k_0\,.$$

Next, because of (3.41) we can choose $\sigma > 0$ small enough (independent of k) with the property that

$$\mathbf{I}_{N_{j(k)}}(r) \geq 1-2\delta \qquad \forall r \in \left]\frac{s_{j(k)}}{t_{j(k)}}, \sigma\right], \quad \forall k \geq k_0\,.$$

Since however $r_k \leq t_{j(k)}$, while $\lim_{k\to\infty} \frac{s_{j(k)}}{r_k} = 0$, for any fixed positive $\rho < \sigma$ and for every k large enough we may conclude that $\frac{\rho r_k}{t_{j(k)}}$ must belong to the interval $[\frac{s_{j(k)}}{t_{j(k)}}, \sigma]$. This implies (3.43) with $\bar\rho = \sigma$ and thus completes the proof.

3.6 Frequency BV estimate

This section is dedicated to establishing a (quantitative) control on the radial variations of the frequency, which is crucial for proving Theorem 3.2.10.

We begin by defining the *universal frequency function*, which makes sense of the frequency continuously along all blow-up scales where it is possible to construct a center manifold for T.

Definition 3.6.1 (Universal frequency function)**.** *Suppose that T is as in Assumption 3.2.1 and let $\{]s_k, t_k]\}_{k=j_0}^{J}$ be a sequence of intervals of flattening with coinciding endpoints (i.e. such that $s_k = t_{k+1}$ for $k = j_0, \dots, J-1$), with corresponding center manifolds $\mathcal{M}_k$ and $\mathcal{M}_k$-normal approximations N_k. For $r \in]s_J, t_{j_0}]$, define*

$$\mathbf{I}(r) \coloneqq \mathbf{I}_{N_k}\left(\frac{r}{t_k}\right)\chi_{]s_k,t_k]}(r),$$
$$\mathbf{D}(r) \coloneqq \mathbf{D}_{N_k}\left(\frac{r}{t_k}\right)\chi_{]s_k,t_k]}(r),$$
$$\mathbf{H}(r) \coloneqq \mathbf{H}_{N_k}\left(\frac{r}{t_k}\right)\chi_{]s_k,t_k]}(r).$$

Unfortunately, unlike for the linearized problem, we do not have monotonicity of the frequency but merely almost monotonicity. Nevertheless, we can hope to control the variation of the negative part of the radial derivative for the frequency function. The main result of this section is the following proposition. We will use the convention that, given a BV function f of one variable, $\left[\frac{df}{dr}\right]_{\pm}$ will denote the positive and negative parts of its distribiutional derivatives, while $\|\mu\|_{TV}$ denotes the total variation of a measure μ on its domain of definition.

Proposition 3.6.2. *There exists $\gamma_4 = \gamma_4(m, n, Q) > 0$ and $C = C(m, n, Q)$ such that the following holds. Suppose that T satisfies Assumption 3.2.3. Let $\{]s_k, t_k]\}_{k=j_0}^{J}$ be intervals of flattening for T around 0 with coinciding endpoints. Then we have $\log(\mathbf{I}+1) \in \mathrm{BV}([s_J, t_{j_0}])$, with the quantitative total variation measure estimate*

$$\left|\left[\frac{d\log(\mathbf{I}+1)}{dr}\right]_{-}\right|\left(]s_J, t_{j_0}]\right) \leq C\sum_{k=j_0}^{J}\boldsymbol{m}_{0,k}^{\gamma_4}. \tag{3.44}$$

Moreover, for any $]a, b]$ which is contained in a single interval of flattening $]s_k, t_k[$ we have the improved estimate

$$\left|\left[\frac{d\log(\mathbf{I}+1)}{dr}\right]_{-}\right|\left(]a, b]\right) \leq C\left(\frac{b}{t_k}\right)^{\gamma_4}\boldsymbol{m}_{0,k}^{\gamma_4}. \tag{3.45}$$

Remark 3.6.3. In the proof of Theorem 3.11.4 appearing later in this chapter, we will need the BV estimate of Proposition 3.6.2 for a different definition of the universal frequency function, for which the intervals of flattening $]s_j, t_j]$ are choosing differently. We point out that, the crucial ingredients needed in proving the above estimates are the following:

(a) The estimate in each open interval holds because for each $r \in]\frac{s_j}{t_j}, 1]$ the sidelength $\ell(L)$ of any cube $L \in \mathscr{W}^{(k)}$ which intersects $B_r(0, \pi_k)$ is no larger than $c_s r$ for a fixed constant $c_s = \frac{1}{64\sqrt{m}}$.

(b) The estimate at the jumps holds because there is one cube $L \in \mathscr{W}^{(k)}$ which intersects $B_{s_j/t_j}(0, \pi_k)$ and has sidelength $\ell(L) \geq c_s \frac{s_j}{t_j}$.

While in (a) we cannot afford a similar control with a constant larger than c_s, in (b) we can afford a constant $\bar{c}_s$ smaller than c_s, at the price that the constant C in the estimate (3.44) will then depend on how small $\bar{c}_s$ is.

In order to prove Proposition 3.6.2, we will require a number of preliminary results, the proofs of which we will defer until later.

3.6.1 Auxiliary results for Proposition 3.6.2

First of all, we recall some key variational identities and estimates from Chapter 2 (see also [22]) for any normal approximation of T.

Let $]s, t]$ be an interval of flattening for T around 0 with corresponding center manifold $\mathcal{M}$ and $\mathcal{M}$-normal approximation N. We define the quantities

$$\begin{aligned}
\mathbf{E}_N(r) &:= -\frac{1}{r} \int_{\mathcal{M}} \phi' \left(\frac{d(y)}{r} \right) \sum_i N_i(y) \cdot DN_i(y) \nabla d(y) \, \mathrm{d}y \,, \\
\mathbf{G}_N(r) &:= -\frac{1}{r^2} \int_{\mathcal{M}} \phi' \left(\frac{d(y)}{r} \right) \frac{d(y)}{|\nabla d(y)|^2} \sum_i |DN_i(y) \cdot \nabla d(y)|^2 \, \mathrm{d}y \,, \\
\mathbf{\Sigma}_N(r) &:= \int_{\mathcal{M}} \phi \left(\frac{d(y)}{r} \right) |N(y)|^2 \, \mathrm{d}y \,.
\end{aligned}$$

We thus have the following.

Lemma 3.6.4. *There exist $\gamma_4(m,n,Q) > 0$ and $C(m,n,Q) > 0$ such that the following holds. Suppose that T, Σ satisfy Assumption 3.2.3 and let $]s,t]$ be an interval of flattening for T around 0 with corresponding center manifold $\mathcal{M}$ and $\mathcal{M}$-normal approximation N. Let $\boldsymbol{m}_0$ be as in (3.1) for $]s,t]$. Then $\mathbf{D}_N$ and $\mathbf{H}_N$ are absolutely continouous on $]\frac{s}{t},3]$ and for a.e. r we have*

$$\begin{aligned}
&\partial_r \mathbf{D}_N(r) = -\int_{\mathcal{M}} \varphi'\left(\frac{d(y)}{r}\right)\frac{d(y)}{r^2}|DN(y)|^2 \ \mathrm{d}y\\
&\partial_r \mathbf{H}_N(r) - \frac{m-1}{r}\mathbf{H}(x,r) = O(\boldsymbol{m}_0)\mathbf{H}_N(r) + 2\mathbf{E}_N(r),\\
&|\mathbf{D}_N(r) - \mathbf{E}_N(r)| \leq \sum_{j=1}^{5} |\operatorname{Err}_j^o| \leq C\boldsymbol{m}_0^{\gamma_4}\mathbf{D}_N(r)^{1+\gamma_4} + C\boldsymbol{m}_0\boldsymbol{\Sigma}_N(r),\\
&\left|\partial_r \mathbf{D}_N(r) - (m-2)r^{-1}\mathbf{D}_N(r) - 2\mathbf{G}_N(r)\right| \leq 2\sum_{j=1}^{5}|\operatorname{Err}_j^i| + C\boldsymbol{m}_0\mathbf{D}_N(r)\\
&\qquad \leq Cr^{-1}\boldsymbol{m}_0^{\gamma_4}\mathbf{D}_N(r)^{1+\gamma_4} + C\boldsymbol{m}_0^{\gamma_4}\mathbf{D}_N(r)^{\gamma_4}\partial_r\mathbf{D}_N(r) + C\boldsymbol{m}_0\mathbf{D}_N(r),
\end{aligned}$$

where Err_j^o and Err_j^i are as in [11, Proposition 9.8, Proposition 9.9].

We omit the proof of Lemma 3.6.4 here, since it involves a mere repetition of the arguments in the proofs of [22, Proposition 3.5] (see also [11, Proposition 9.5, Proposition 9.10]), combined with the observation that the constants may be optimized to depend on appropriate powers of $\boldsymbol{m}_0$. As a consequence of the estimates in Lemma 3.6.4, we have the following quantitative almost-monotonicity for the frequency in each interval of flattening.

Corollary 3.6.5. *There exist $\gamma_4(m,n,Q) > 0$ and $C(m,n,Q) > 0$ such that the following holds. Suppose that T, Σ, $]s,t]$, $\mathcal{M}$, N, and $\boldsymbol{m}_0$ are as in Lemma 3.6.4. Then $\mathbf{I}_N$ is absolutely continuous on $]\frac{s}{t},r]$ and for a.e. r we have*

$$\partial_r \mathbf{I}_N(r) \geq -C(1+\mathbf{I}_N(r))\boldsymbol{m}_0^{\gamma_4}\left(1 + \frac{\mathbf{D}_N(r)^{\gamma_4}}{r} + \mathbf{D}_N(r)^{\gamma_4-1}\partial_r\mathbf{D}_N(r)\right).$$

In addition to the above control on the frequency variations within each interval of flattening, we will also need to control the jumps of the frequency between successive intervals of flattening. In order to establish this, we will require the following intermediate results.

Lemma 3.6.6 (Expansion of excess)**.** *There exists a dimensional constant $C = C(m, n, Q) > 0$ such that the following holds. Let T, Σ be as in Assumption 3.2.3 and let $\mathcal{M}$ be a center manifold for T with $\mathcal{M}$-normal approximation N. Let $r \in]0, 1]$ and let $f : B_r(0, \pi) \to \mathcal{A}_Q(\pi^\perp)$ be a Lipschitz map with $\mathrm{Lip}(f) \leq c$. Let $\boldsymbol{\varphi}_r$ be a parameterizing map for $\mathcal{M}$ over π. Then we have*

$$\begin{aligned}\Big|\int_{\mathbf{C}_r(0,\pi)} |\vec{\mathbf{G}}_f - \vec{\mathcal{M}} \circ \mathbf{p}|^2 \phi\left(\frac{|\mathbf{p}_\pi(z)|}{r}\right) \,\mathrm{d}\|\mathbf{G}_f\|(z) - \int_{B_r(0,\pi)} \mathcal{G}\big(Df, Q[\![D\boldsymbol{\varphi}_r]\!]\big)^2 \,\mathrm{d}y\Big| \\ \leq C \int_{B_r(0,\pi)} (|Df|^4 + |D\boldsymbol{\varphi}_r|^4)\phi\left(\frac{|y|}{r}\right) \,\mathrm{d}y \\ + C \int_{\mathbf{C}_r(0,\pi)} \Big|\vec{\mathcal{M}}(\mathbf{p}(z)) - \vec{\mathcal{M}}\big(\boldsymbol{\varphi}_r(\mathbf{p}_\pi(z))\big)\Big| \,\mathrm{d}\|\mathbf{G}_f\|(z).\end{aligned}$$

An important consequence of Lemma 3.6.6 is the following comparability between the Dirichlet energy of N at a given scale, with that of Lipschitz approximations over suitable planes. We will henceforth take $\gamma_2 > 0$ to be as in [21]. Note that we may ensure that $\gamma_4 \leq \gamma_2$.

Corollary 3.6.7. *There exists a dimensional constant $C = C(m, n, Q) > 0$ such that the following holds. Let T, Σ satisfy Assumption 3.2.3. Let $]s, t]$ be an interval of flattening for T around 0 with corresponding center manifold $\mathcal{M}$ and $\mathcal{M}$-normal approximation N, let $\boldsymbol{m}_0$ be as in (3.1) for $]s, t]$ and let π be the plane used to define $\boldsymbol{\varphi}$ in the center manifold algorithm of [21]. Let $f : B_1(0, \pi) \to \mathcal{A}_Q(\pi^\perp)$ be a π-approximation for $T_{0,t}$ in $\mathbf{C}_4(0, \pi)$ according to [19] and for $\bar{r} = \frac{s}{t}$, let $f_L : B_{8r_L}(p_L, \pi_L) \to \mathcal{A}_Q(\pi_L^\perp)$ be a π_L-approximation for $T_{0,t}$ corresponding to a Whitney cube L as in [22, Section 2.1 (Stop)]. Let $\pi_{\bar{r}}$ be such that $\mathbf{E}(T_{0,t}, \mathbf{B}_{6\sqrt{m}\bar{r}}) = \mathbf{E}(T_{0,t}, \mathbf{B}_{6\sqrt{m}\bar{r}}, \pi_{\bar{r}})$ and let $B^L := B_{8r_L}(p_L, \pi_L)$. Let $f_{\bar{r}} : B_{\bar{r}}(0, \pi_{\bar{r}}) \to \mathcal{A}_Q(\pi_{\bar{r}}^\perp)$ be the map reparameterizing $\mathrm{gr}\,(f_L)$ as a graph over $\pi_{\bar{r}}$ and let $\boldsymbol{\varphi}_{\bar{r}}, \boldsymbol{\varphi}_L$ be the maps*

reparameterizing of $\mathrm{gr}(\boldsymbol{\varphi})$ *as graph over* $\pi_{\bar{r}}, \pi_L$ *respectively. Then we have*

$$\begin{aligned}\left|\int_{B_1(0,\pi)} \mathcal{G}(Df, Q[\![D\boldsymbol{\varphi}]\!])^2 \phi\left(|y|\right)\, \mathrm{d}y - \int_{\mathbf{B}_1\cap\mathcal{M}} |DN|^2 \phi\left(d(y)\right)\, \mathrm{d}y\right| \\ \leq C\int_{B_1(0,\pi)} (|Df|^4 + |D\boldsymbol{\varphi}|^4)\, \mathrm{d}y + C\boldsymbol{m}_0^{1+\gamma_2} + C\int_{\mathbf{B}_1\cap\mathcal{M}} (|\mathbf{A}_{\mathcal{M}}|^2|N|^2 + |DN|^4) \\ + C\int_{\mathbf{C}_1(0,\pi)} \left|\vec{\mathcal{M}}(\mathbf{p}(z)) - \vec{\mathcal{M}}\big(\boldsymbol{\varphi}(\mathbf{p}_\pi(z))\big)\right| \mathrm{d}\|\mathbf{G}_f\|(z),\end{aligned} \tag{3.46}$$

and

$$\begin{aligned}\left|\int_{B_{\bar{r}}(0,\pi_{\bar{r}})} \mathcal{G}(Df_{\bar{r}}, Q[\![D\boldsymbol{\varphi}_{\bar{r}}]\!])^2 \phi\left(\frac{|y|}{\bar{r}}\right)\, \mathrm{d}y - \int_{\mathbf{B}_{\bar{r}}\cap\mathcal{M}} |DN|^2 \phi\left(\frac{d(y)}{\bar{r}}\right)\, \mathrm{d}y\right| \\ \leq C\int_{B_{\bar{r}}(0,\pi_{\bar{r}})} (|Df_{\bar{r}}|^4 + |D\boldsymbol{\varphi}_{\bar{r}}|^4)\, \mathrm{d}y + C\int_{B^L} (|Df_L|^4 + |D\boldsymbol{\varphi}_L|^4)\, \mathrm{d}y \\ + C\boldsymbol{m}_0^{1+\gamma_2}\bar{r}^{m+2+\gamma_2} + C\int_{\mathcal{B}^L} (|\mathbf{A}_{\mathcal{M}}|^2|N|^2 + |DN|^4) \\ + C\int_{\mathbf{C}_{\bar{r}}(0,\pi_{\bar{r}})} \left|\vec{\mathcal{M}}(\mathbf{p}(z)) - \vec{\mathcal{M}}\big(\boldsymbol{\varphi}(\mathbf{p}_{\pi_{\bar{r}}}(z))\big)\right| \mathrm{d}\|\mathbf{G}_{f_{\bar{r}}}\|(z),\end{aligned} \tag{3.47}$$

where $\mathbf{A}_{\mathcal{M}}$ *is the second fundamental form of* $\mathcal{M}$ *and* γ_2 *is as in* [21].

We will in addition require the following comparison between the gradients of the parameterizing maps of consecutive center manifolds in the procedure [22, Section 2.1].

Lemma 3.6.8. *There exists a constant* $C = C(m, n, Q) > 0$ *such that the following holds. Suppose that* T, Σ *satisfy Assumption 3.2.3. Let* $\mathcal{M}_{k-1}, \mathcal{M}_k$ *be successive center manifolds for* T *with respective normal approximations* N_{k-1}, N_k, *associated to the respective intervals of flattening* $]t_k, t_{k-1}]$ *and* $]t_{k+1}, t_k]$, *as defined in Section 3.2. Assume that* $\mathbf{E}(T, \mathbf{B}_{6\sqrt{m}t_k}, \pi_k) = \mathbf{E}(T, \mathbf{B}_{6\sqrt{m}t_k})$ *for some plane* π_k *and let* $\tilde{\boldsymbol{\varphi}}_{k-1}$ *be a map reparametrizing* $\mathrm{gr}\,(\boldsymbol{\varphi}_{k-1})$ *as a graph over* π_k. *Letting* $\tilde{\boldsymbol{\varphi}}_k \coloneqq \tilde{\boldsymbol{\varphi}}_{k-1}\left(\frac{t_k}{t_{k-1}}\cdot\right)$, *we have*

$$\int_{B_1} |D\boldsymbol{\varphi}_k - D\tilde{\boldsymbol{\varphi}}_k|^2 \leq C\boldsymbol{m}_{0,k}^{3/2}. \tag{3.48}$$

and

$$\int_{B_2} |\boldsymbol{\varphi}_k - \tilde{\boldsymbol{\varphi}}_k|^2 \leq C\boldsymbol{m}_{0,k}\,. \tag{3.49}$$

Finally, we will need the following control on the difference between the projection $\mathbf{p}(z)$ of a point z to a center manifold $\mathcal{M}$, and the image under $\boldsymbol{\varphi}$ of the planar projection $\mathbf{p}_{\pi_0}(z)$:

Lemma 3.6.9. *There exists a constant $C = C(m, n, Q) > 0$ such that the following holds. Suppose that T, $\mathcal{M}$, $\boldsymbol{m}_0$, $\bar{r}$, f, $f_{\bar{r}}$, π, $\pi_{\bar{r}}$, $\boldsymbol{\varphi}_{\bar{r}}$ are as in Corollary 3.6.7. Then we have*

$$\int_{\mathbf{C}_{\bar{r}}(0,\pi_{\bar{r}})} \left|\vec{\mathcal{M}}(\mathbf{p}(z)) - \vec{\mathcal{M}}\big(\boldsymbol{\varphi}_{\bar{r}}(\mathbf{p}_{\pi_{\bar{r}}}(z))\big)\right| \mathrm{d}\|\mathbf{G}_{f_{\bar{r}}}\|(z) \leq C\bar{r}^{m+1}\boldsymbol{m}_0^{1+\gamma_2}, \tag{3.50}$$

$$\int_{\mathbf{C}_1(0,\pi)} \left|\vec{\mathcal{M}}(\mathbf{p}(z)) - \vec{\mathcal{M}}\big(\boldsymbol{\varphi}(\mathbf{p}_{\pi}(z))\big)\right| \mathrm{d}\|\mathbf{G}_{f}\|(z) \leq C\boldsymbol{m}_0^{1+\gamma_2}. \tag{3.51}$$

3.6.2 Proof of Proposition 3.6.2

We now have all of the relevant tools to prove the frequency variation estimate (3.44). We start with the preliminary observation that $\mathbf{I}$ is absolutely continuous on each interval $]s_k, t_k[$, while it might have jump discontinuities at the points $s_k = t_{k+1}$.

First, we control the jumps of $\mathbf{I}$ at these points. Letting $\mathbf{D}_k := \mathbf{D}_{N_k}$, $\mathbf{H}_k := \mathbf{H}_{N_k}$, and letting $\bar{\mathbf{D}}_k(r) := r^{-(m-2)}\mathbf{D}_k(r)$, $\bar{\mathbf{H}}_k(r) := r^{-(m-1)}\mathbf{H}_k(r)$ denote the corresponding scale-invariant quantities, we claim that we have the estimate

$$\left|\mathbf{I}(t_k^+) - \mathbf{I}(t_k^-)\right| = \left|\frac{\bar{\mathbf{D}}_{k-1}\left(\frac{t_k}{t_{k-1}}\right)}{\bar{\mathbf{H}}_{k-1}\left(\frac{t_k}{t_{k-1}}\right)} - \frac{\bar{\mathbf{D}}_k(1)}{\bar{\mathbf{H}}_k(1)}\right| \leq C\boldsymbol{m}_{0,k}^{\gamma_2}(1 + \mathbf{I}(t_k^+)). \tag{3.52}$$

Rearranging and using the triangle inequality, it suffices to demonstrate that

$$\left|\frac{\bar{\mathbf{D}}_{k-1}\left(\frac{t_k}{t_{k-1}}\right) - \bar{\mathbf{D}}_k(1)}{\bar{\mathbf{H}}_k(1)}\right| \leq C\boldsymbol{m}_{0,k}^{\gamma_2}, \tag{3.53}$$

$$\bar{\mathbf{D}}_{k-1}\left(\frac{t_k}{t_{k-1}}\right)\left|\frac{1}{\bar{\mathbf{H}}_{k-1}\left(\frac{t_k}{t_{k-1}}\right)} - \frac{1}{\bar{\mathbf{H}}_k(1)}\right| \leq C\boldsymbol{m}_{0,k}\mathbf{I}_{k-1}\left(\frac{t_k}{t_{k-1}}\right). \tag{3.54}$$

Before we proceed, given π_k such that $\mathbf{E}(T, \mathbf{B}_{6\sqrt{m}t_k}) = \mathbf{E}(T, \mathbf{B}_{6\sqrt{m}t_k}, \pi_k)$ let us introduce the Lipschitz approximation $f_k : B_3 \subset \pi_k \to \mathcal{A}_Q(\pi_k^\perp)$ of [19, Theorem 2.4] for $T_{0,t_k} \llcorner \mathbf{B}_{6\sqrt{m}}$ and the map $\tilde{f}_{k-1} \coloneqq f_{t_k/t_{k-1}} : B_{t_k/t_{k-1}}(0, \pi_k) \to \mathcal{A}_Q(\pi_k^\perp)$ from Corollary 3.6.7 with $\bar{r} = \frac{t_k}{t_{k-1}}$. We let $\tilde{\boldsymbol{\varphi}}_{k-1}$, $\tilde{\boldsymbol{\varphi}}_k$ be as in Lemma 3.6.8 and let $\tilde{f}_k \coloneqq \tilde{f}_{k-1}\big(\frac{t_k}{t_{k-1}} \cdot\big)$. We additionally introduce the measures $\mathrm{d}\mu_{k-1}(y) \coloneqq \phi_k\left(\frac{t_{k-1}}{t_k} d(y)\right) \mathrm{d}y$ and $\mathrm{d}\mu(y) \coloneqq \phi\left(d(y)\right) \mathrm{d}y$, where $\mathrm{d}y$ is the m-dimensional Lebesgue measure on π_k. We also define the balls $\mathcal{B}^{k-1} \coloneqq \mathbf{B}_{t_k/t_{k-1}} \cap \mathcal{M}_{k-1}$, $B^{k-1} \coloneqq B_{t_k/t_{k-1}}(0, \pi_k)$ and the cylinder $\mathbf{C}^{k-1} \coloneqq \mathbf{C}_{t_k/t_{k-1}}(0, \pi_k)$.

We begin with the estimate (3.53). Comparing both terms with the corresponding linearized quantity (cf. Corollary 3.6.7) and rescaling appropriately we have

$$\begin{aligned}
&\left|\bar{\mathbf{D}}_{k-1}\left(\frac{t_k}{t_{k-1}}\right) - \bar{\mathbf{D}}_k(1)\right| \\
&\quad \leq \left(\frac{t_k}{t_{k-1}}\right)^{-(m-2)} \left|\int_{\mathcal{B}^{k-1}} |DN_{k-1}|^2 \,\mathrm{d}\mu_{k-1} - \int_{B^{k-1}} \mathcal{G}(D\tilde{f}_{k-1}, Q[\![D\tilde{\boldsymbol{\varphi}}_{k-1}]\!])^2 \,\mathrm{d}\mu_{k-1}\right| \\
&\qquad + \left|\int_{\mathbf{B}_1 \cap \mathcal{M}_k} |DN_k|^2 \,\mathrm{d}\mu - \left(\frac{t_k}{t_{k-1}}\right)^{-(m-2)} \int_{B^{k-1}} \mathcal{G}(D\tilde{f}_{k-1}, Q[\![D\tilde{\boldsymbol{\varphi}}_{k-1}]\!])^2 \,\mathrm{d}\mu_{k-1}\right| \\
&\quad = \left(\frac{t_k}{t_{k-1}}\right)^{-(m-2)} \left|\int_{\mathcal{B}^{k-1}} |DN_{k-1}|^2 \,\mathrm{d}\mu_{k-1} - \int_{B^{k-1}} \mathcal{G}(D\tilde{f}_{k-1}, Q[\![D\tilde{\boldsymbol{\varphi}}_{k-1}]\!])^2 \,\mathrm{d}\mu_{k-1}\right| \\
&\qquad + \left|\int_{\mathbf{B}_1 \cap \mathcal{M}_k} |DN_k|^2 \,\mathrm{d}\mu - \int_{B_1(0,\pi_k)} \mathcal{G}(D\tilde{f}_k, Q[\![D\tilde{\boldsymbol{\varphi}}_k]\!])^2 \,\mathrm{d}\mu\right|.
\end{aligned}$$

Now we may use Lemma 3.6.8 to replace $\tilde{\boldsymbol{\varphi}}_k$ with $\boldsymbol{\varphi}_k$, yielding

$$\begin{aligned}
&\left|\bar{\mathbf{D}}_{k-1}\left(\frac{t_k}{t_{k-1}}\right) - \bar{\mathbf{D}}_k(1)\right| \\
&\quad \leq \left(\frac{t_k}{t_{k-1}}\right)^{-(m-2)} \left|\int_{\mathcal{B}^{k-1}} |DN_{k-1}|^2 \,\mathrm{d}\mu_{k-1} - \int_{B^{k-1}} \mathcal{G}(D\tilde{f}_{k-1}, Q[\![D\tilde{\boldsymbol{\varphi}}_{k-1}]\!])^2 \,\mathrm{d}\mu_{k-1}\right| \\
&\qquad + \left|\int_{\mathbf{B}_1 \cap \mathcal{M}_k} |DN_k|^2 \,\mathrm{d}\mu - \int_{B_1(0,\pi_k)} \mathcal{G}(D\tilde{f}_k, Q[\![D\boldsymbol{\varphi}_k]\!])^2 \,\mathrm{d}\mu\right| + C\boldsymbol{m}_{0,k}^{1+\gamma_2}.
\end{aligned}$$

We are now in a position to make use of Corollary 3.6.7, combined with the observation that $\tilde{f}_k$ is still a valid π_k-approximation for T_{0,t_k} in $\mathbf{C}_4(0, \pi_k)$ as in [19], since f_{k-1} is a π_{k-1}-

approximation for $T_{0,t_{k-1}}$ and we have the estimates [21, Proposition 4.1] on the tilting of π_k relative to π_{k-1}. This gives

$$\begin{aligned}\left|\bar{\mathbf{D}}_{k-1}\left(\frac{t_k}{t_{k-1}}\right)-\bar{\mathbf{D}}_k(1)\right| \leq C\left(\frac{t_k}{t_{k-1}}\right)^{-(m-2)}\Bigg(&\int_{B^{k-1}}(|Df_{k-1}|^4+|D\tilde{\boldsymbol{\varphi}}_{k-1}|^4)\,\mathrm{d}y\\&+\int_{B^{L_k}}(|Df_{L_k}|^4+|D\boldsymbol{\varphi}_{L_k}|^4)\,\mathrm{d}y\\&+\int_{\mathcal{B}^{L_k}}(|\mathbf{A}_{\mathcal{M}_{k-1}}|^2|N_{k-1}|^2+|DN_{k-1}|^4)\\&+\int_{\mathbf{C}^{k-1}}\left|\vec{\mathcal{M}}_{k-1}(\mathbf{p}(z))-\vec{\mathcal{M}}_{k-1}\big(\boldsymbol{\varphi}(\mathbf{p}_{\pi_k}(z))\big)\right|\mathrm{d}\|\mathbf{G}_{f_{k-1}}\|(z)\Bigg)\\&+C\Bigg(\boldsymbol{m}_{0,k-1}^{1+\gamma_2}\left(\frac{t_k}{t_{k-1}}\right)^{4+\gamma_2}+\int_{B_1(0,\pi_k)}(|D\tilde{f}_k|^4+|D\boldsymbol{\varphi}|^4)\,\mathrm{d}y\\&+\int_{\mathbf{B}_1\cap\mathcal{M}_k}(|\mathbf{A}_{\mathcal{M}_k}|^2|N_k|^2+|DN_k|^4)\\&+\int_{\mathbf{C}_1(0,\pi_k)}\left|\vec{\mathcal{M}}(\mathbf{p}(z))-\vec{\mathcal{M}}\big(\boldsymbol{\varphi}(\mathbf{p}_{\pi}(z))\big)\right|\mathrm{d}\|\mathbf{G}_f\|(z)+\boldsymbol{m}_{0,k}^{1+\gamma_2}\Bigg).\end{aligned}$$

Lemma 3.6.9 thus yields

$$\begin{aligned}\left|\bar{\mathbf{D}}_{k-1}\left(\frac{t_k}{t_{k-1}}\right)-\bar{\mathbf{D}}_k(1)\right| \leq C\left(\frac{t_k}{t_{k-1}}\right)^{-(m-2)}\Bigg(&\int_{B^{k-1}}(|Df_{k-1}|^4+|D\tilde{\boldsymbol{\varphi}}_{k-1}|^4)\,\mathrm{d}y\\&+\int_{B^{L_k}}(|Df_{L_k}|^4+|D\boldsymbol{\varphi}_{L_k}|^4)\,\mathrm{d}y\\&+\int_{\mathcal{B}^{L_k}}(|\mathbf{A}_{\mathcal{M}_{k-1}}|^2|N_{k-1}|^2+|DN_{k-1}|^4)\Bigg)\\&+C\Bigg(\boldsymbol{m}_{0,k-1}^{1+\gamma_2}\left(\frac{t_k}{t_{k-1}}\right)^{3}+\int_{B_1(0,\pi_k)}(|D\tilde{f}_k|^4+|D\boldsymbol{\varphi}|^4)\,\mathrm{d}y\\&+\int_{\mathbf{B}_1\cap\mathcal{M}_k}(|\mathbf{A}_{\mathcal{M}}|^2|N|^2+|DN|^4)+\boldsymbol{m}_{0,k}^{1+\gamma_2}\Bigg).\end{aligned}$$

We may now control $\boldsymbol{m}_{0,k-1}$ for $T_{0,t_{k-1}}$ in terms of the tilt excess $\mathbf{E}(T_{0,t_{k-1}},\mathbf{B}_{L_k})$, which is in

turn controlled by $\boldsymbol{m}_{0,k}$ of T_{0,t_k}:

$$\boldsymbol{m}_{0,k-1}\left(\frac{t_k}{t_{k-1}}\right)^{2-2\delta_2} \leq C\boldsymbol{m}_{0,k}. \tag{3.55}$$

This, in combination with the estimates [19, Theorem 2.4] and [21, Theorem 1.17, Theorem 2.4, Corollary 2.5] allows us to conclude that

$$\left|\bar{\mathbf{D}}_{k-1}\left(\frac{t_k}{t_{k-1}}\right) - \bar{\mathbf{D}}_k(1)\right| \leq C\boldsymbol{m}_{0,k}^{1+\gamma_2}.$$

Since the comparison of center manifolds [21, Proposition 3.7] gives $\bar{\mathbf{H}}_k(1) \geq c\boldsymbol{m}_{0,k}$ for some dimensional constant $c > 0$, the estimate (3.53) follows.

Let us now prove (3.54). First of all, observe that

$$\bar{\mathbf{D}}_{k-1}\left(\frac{t_k}{t_{k-1}}\right)\left|\frac{1}{\bar{\mathbf{H}}_{k-1}\left(\frac{t_k}{t_{k-1}}\right)} - \frac{1}{\mathbf{H}_k(1)}\right| = \frac{\mathbf{I}_{k-1}\left(\frac{t_k}{t_{k-1}}\right)}{\mathbf{H}_k(1)}\left|\mathbf{H}_k(1) - \bar{\mathbf{H}}_{k-1}\left(\frac{t_k}{t_{k-1}}\right)\right|.$$

To estimate the difference between the L^2-heights, we may one again compare both with the height of the corresponding Lipschitz approximations over the averages of their sheets:

$$\begin{aligned}
&\left|\bar{\mathbf{H}}_{k-1}\left(\frac{t_k}{t_{k-1}}\right) - \bar{\mathbf{H}}_k(1)\right| \\
&\quad\leq \left(\frac{t_k}{t_{k-1}}\right)^{-(m-1)}\left|\int_{\mathcal{B}^{k-1}} |N_{k-1}|^2 \,\mathrm{d}\mu_{k-1} - \int_{B^{k-1}} \mathcal{G}(f_{k-1}, Q[\![\tilde{\boldsymbol{\varphi}}_{k-1}]\!])^2 \,\mathrm{d}\mu_{k-1}\right| \\
&\qquad + \left|\int_{\mathbf{B}_1\cap\mathcal{M}_k} |N_k|^2 \,\mathrm{d}\mu - \left(\frac{t_k}{t_{k-1}}\right)^{-(m-1)}\int_{B^{k-1}} \mathcal{G}(f_{k-1}, Q[\![\tilde{\boldsymbol{\varphi}}_{k-1}]\!])^2 \,\mathrm{d}\mu_{k-1}\right| \\
&\quad= \left(\frac{t_k}{t_{k-1}}\right)^{-(m-1)}\left|\int_{\mathcal{B}^{k-1}} |N_{k-1}|^2 \,\mathrm{d}\mu_{k-1} - \int_{B^{k-1}} \mathcal{G}(f_{k-1}, Q[\![\tilde{\boldsymbol{\varphi}}_{k-1}]\!])^2 \,\mathrm{d}\mu_{k-1}\right| \\
&\qquad + \left|\int_{\mathbf{B}_1\cap\mathcal{M}_k} |N_k|^2 \,\mathrm{d}\mu - \int_{B_1} \mathcal{G}(\tilde{f}_k, Q[\![\tilde{\boldsymbol{\varphi}}_k]\!])^2 \,\mathrm{d}\mu\right|.
\end{aligned}$$

Now let $\tilde{g}_{k-1}$, $\tilde{g}_k$ be as in Lemma 3.4.4 for $\tilde{\boldsymbol{\varphi}}_{k-1}, f_{k-1}$ and $\boldsymbol{\varphi}_k, \tilde{f}_k$ respectively and let $A^{k-1} := B^{k-1} \setminus \frac{1}{2}B^{k-1}$, $A^k := B_1(0,\pi_k) \setminus B_{1/2}(0,\pi_k)$. The reverse triangle inequality and

the estimate (3.23) (combined with an appropriate rescaling) then allow us to deduce that

$$\begin{aligned}\left|\bar{\mathbf{H}}_{k-1}\left(\frac{t_k}{t_{k-1}}\right)-\bar{\mathbf{H}}_k(1)\right| &\leq C\left(\frac{t_k}{t_{k-1}}\right)^{-(m-1)}\int_{A^{k-1}}\mathcal{G}(N_{k-1}(\tilde{\boldsymbol{\varphi}}_{k-1}(y)),\tilde{g}_{k-1}(y))^2\,\mathrm{d}y\\ &\quad+\int_{A^k}\mathcal{G}(N_k(\boldsymbol{\varphi}_k(y)),\tilde{g}_k(y))^2\,\mathrm{d}y\\ &\leq C\left(\frac{t_k}{t_{k-1}}\right)^5\left(\|D\tilde{\boldsymbol{\varphi}}_{k-1}\|_{C^0}^4+\|D\tilde{\boldsymbol{\varphi}}_{k-1}\|_{C^0}^2\boldsymbol{m}_{0,k-1}\right)+C\|D\boldsymbol{\varphi}_k\|_{C^0}^4.\end{aligned}$$

The estimates in [21, Theorem 1.17, Proposition 3.4] then give

$$\left|\bar{\mathbf{H}}_{k-1}\left(\frac{t_k}{t_{k-1}}\right)-\bar{\mathbf{H}}_k(1)\right|\leq C\boldsymbol{m}_{0,k}^2.$$

Again using that $\bar{\mathbf{H}}_k(1)\geq c\boldsymbol{m}_{0,k}$, we further have

$$\frac{\mathbf{I}_{k-1}\left(\frac{t_k}{t_{k-1}}\right)}{\bar{\mathbf{H}}_k(1)}\leq C\boldsymbol{m}_{0,k}^{-1}\mathbf{I}_{k-1}\left(\frac{t_k}{t_{k-1}}\right).$$

The desired estimate follows immediately, and thus we are able to conclude that (3.52) holds.

From (3.52) we immediately conclude

$$\sum_{k=j_0}^{J}(\log(\mathbf{I}(t_k^+)+1)-\log(\mathbf{I}(t_k^-)+1))_-\leq C\sum_{k=j_0}^{J}\boldsymbol{m}_0^{\gamma_4}\,. \tag{3.56}$$

Indeed, if $\mathbf{I}(t_k^+)\geq\mathbf{I}(t_k^-)$, then $(\log(\mathbf{I}(t_k^+)+1)-\log(\mathbf{I}(t_k^-)+1))_-=0$, while otherwise, in light of the elementary inequality $\log(w)\leq w-1$ for $w>0$, we have

$$(\log(\mathbf{I}(t_k^+)+1)-\log(\mathbf{I}(t_k^-)+1))_-=\log(\mathbf{I}(t_k^-)+1)-\log(\mathbf{I}(t_k^+)+1)\leq\frac{\mathbf{I}(t_k^-)-\mathbf{I}(t_k^+)}{\mathbf{I}(t_k^+)+1}$$

and we can just sum (3.52) recalling that $\gamma_2\geq\gamma_4$ and $\boldsymbol{m}_0\leq 1$.

We next wish to control the absolutely continuous part of $\left[\frac{d\log(\mathbf{I}+1)}{dr}\right]_-$. Here, we exploit the almost-monotonicity in Corollary 3.6.5. We argue on each interval $]s_k,t_k[$ and will

henceforth let ∂_r denote differentiation in the variable $\frac{t}{t_k}$. Note that $\partial_r = t_k \partial_t$. Due to Corollary 3.6.5, for almost-every $t \in]t_{k+1}, t_k]$ we have

$$\begin{aligned}(\log(\mathbf{I}+1))'(t) &= \frac{1}{t_k}\partial_r \mathbf{I}_k\left(\frac{t}{t_k}\right)\left(1+\mathbf{I}_k\left(\frac{t}{t_k}\right)\right)^{-1} \\ &\geq -\frac{C}{t_k}\boldsymbol{m}_{0,k}^{\gamma_4}\left[1+\left(\frac{t}{t_k}\right)^{-1}\mathbf{D}_k\left(\frac{t}{t_k}\right)^{\gamma_4}+\mathbf{D}_k\left(\frac{t}{t_k}\right)^{\gamma_4-1}\mathbf{D}_k'\left(\frac{t}{t_k}\right)\right].\end{aligned}$$

We are now ready to introduce a monotone function $\boldsymbol{\Omega}$ which will help us close the estimate. First of all we let $\psi_k(t) := \frac{C}{t_k}\boldsymbol{m}_{0,k}^{\gamma_4}\mathbf{1}_{]t_{k+1},t_k]}(t)$ and let the absolutely continuous part of the derivative of $\boldsymbol{\Omega}$ be

$$\boldsymbol{\Omega}'(t) := \sum_{k=j_0}^{J}\psi_k(t)\left[1+\left(\frac{t}{t_k}\right)^{-1}\mathbf{D}_k\left(\frac{t}{t_k}\right)^{\gamma_4}+\mathbf{D}_k\left(\frac{t}{t_k}\right)^{\gamma_4-1}\mathbf{D}_k'\left(\frac{t}{t_k}\right)\right].$$

Next we consider the "jump measure"

$$\mu^j := C\sum_{k=j_0}^{J}\boldsymbol{m}_{0,k}^{\gamma_4}\delta_{t_k}.$$

Hence we set $\boldsymbol{\Omega}(s_J) = 0$ and define $\boldsymbol{\Omega}$ by integration, setting its distributional derivative to be $\mu^j + \boldsymbol{\Omega}'\mathcal{L}^1$. Observe that $\boldsymbol{\Omega}$ is monotone: μ^j is obviously a non-negative measure, while $\boldsymbol{\Omega}'$ is a non-negative function since both $\mathbf{D}_k$ and $\mathbf{D}_k'$ are non-negative (recall the explicit formula for the latter). On the other hand the estimates proved so far obviously show that $\log(\mathbf{I}+1)+\boldsymbol{\Omega}$ is non-decreasing. In particular it immediately follows that

$$\begin{aligned}\left|\left[\frac{d\log(\mathbf{I}+1)}{dr}\right]_-\right|\big(]s_J, t_{j_0}]\big) &\leq \left|\left[\frac{d\boldsymbol{\Omega}}{dr}\right]_+\right|\big(]s_J, t_{j_0}]\big) = \boldsymbol{\Omega}(t_{j_0})-\boldsymbol{\Omega}(s_J) \\ &\leq C\sum_{k=j_0}^{J}\boldsymbol{m}_{0,k}^{\gamma_4}+\sum_{k=j_0}^{J}\int_{s_k}^{t_k}(\boldsymbol{\Omega}')_+(t)\,\mathrm{d}t\end{aligned}$$

$$\leq C\sum_{k=j_0}^{J}\boldsymbol{m}_{0,k}^{\gamma_4}+C\sum_{k=j_0}^{J}\boldsymbol{m}_{0,k}^{\gamma_4}\int_{\frac{s_k}{t_k}}^{1}\left(1+s^{\gamma_4 m-1}+\partial_s(\mathbf{D}_k(s)^{\gamma_4})+s\right)\,\mathrm{d}s$$
$$\leq C\sum_{k=j_0}^{J}\boldsymbol{m}_{0,k}^{\gamma_4}+C\sum_{k=j_0}^{J-1}\boldsymbol{m}_{0,k}^{\gamma_4}\left(s+s^{\gamma_4 m}+\mathbf{D}_k(s)^{\gamma_4}\right)\Big|_{s=\frac{s_k}{t_k}}^{1}\leq C\sum_{k=j_0}^{J}\boldsymbol{m}_{0,k}^{\gamma_4}\,.$$

3.6.3 Proofs of auxiliary results from Section 3.6.1

Proof of Lemma 3.6.6. We will argue as in [20, Section 3.1], making use of the multiple-valued area formula. Consider

$$\begin{aligned}E&\coloneqq\int_{\mathbf{C}_r(0,\pi)}|\vec{\mathbf{G}}_f-\vec{\mathcal{M}}\circ\mathbf{p}|^2\phi\left(\frac{|\mathbf{p}_\pi(z)|}{r}\right)\,\mathrm{d}\|\mathbf{G}_f\|(z)\\&=2\int_{\mathbf{C}_r(0,\pi)}\phi\left(\frac{\mathbf{p}_\pi(p)|}{r}\right)\,\mathrm{d}\|\mathbf{G}_f\|(p)-2\int_{\mathbf{C}_r(0,\pi)}\langle\vec{\mathbf{G}}_f,\vec{\mathcal{M}}\circ\mathbf{p}\rangle\phi\left(\frac{|\mathbf{p}_\pi(z)|}{r}\right)\,\mathrm{d}\|\mathbf{G}_f\|(z).\end{aligned}$$

By the Q-valued area formula [20, Corollary 1.11], we have

$$\begin{aligned}2\int_{\mathbf{C}_r(0,\pi)}\phi\left(\frac{|\mathbf{p}_\pi(z)|}{r}\right)\,\mathrm{d}\|\mathbf{G}_f\|(z)&=2Q\int_{B_r(0,\pi)}\phi\left(\frac{|y|}{r}\right)\,\mathrm{d}y\\&\quad+\int_{B_r(0,\pi)}\left(|Df|^2\phi\left(\frac{|\mathbf{p}_\pi(y)|}{r}\right)+O(|Df|^4)\right)\,\mathrm{d}y.\end{aligned}$$

Meanwhile, for $\vec{\xi}=(e_1+D\boldsymbol{\varphi}|_{\mathbf{p}_\pi(y,f_i(y))}e_1)\wedge(e_m+D\boldsymbol{\varphi}|_{\mathbf{p}_\pi(y,f_i(y))}e_m)$ and $v_k^i=e_k+Df_i|_y e_k$, $w_k^i=e_k+D\boldsymbol{\varphi}|_{\mathbf{p}_\pi(y,f_i(y))}e_k$, we have

$$\begin{aligned}&2\int_{\mathbf{C}_r(0,\pi)}\langle\vec{\mathbf{G}}_f,\vec{\mathcal{M}}\circ\mathbf{p}\rangle\phi\left(\frac{|\mathbf{p}_\pi(z)|}{r}\right)\,\mathrm{d}\|\mathbf{G}_f\|(z)\\&\qquad=2\int_{\mathbf{C}_r(0,\pi)}\langle\vec{\mathbf{G}}_f(z),\vec{\mathcal{M}}\big(\boldsymbol{\varphi}(\mathbf{p}_\pi(z))\big)\rangle\phi\left(\frac{|\mathbf{p}_\pi(p)|}{r}\right)\,\mathrm{d}\|\mathbf{G}_f\|(z)\\&\qquad\quad+2\int_{\mathbf{C}_r(0,\pi)}\langle\vec{\mathbf{G}}_f(z),\Big(\vec{\mathcal{M}}(\mathbf{p}(z))-\vec{\mathcal{M}}\big(\boldsymbol{\varphi}(\mathbf{p}_\pi(z))\big)\Big)\rangle\phi\left(\frac{|\mathbf{p}_\pi(z)|}{r}\right)\,\mathrm{d}\|\mathbf{G}_f\|(z)\\&\qquad=\frac{2}{|\xi|}\int_{B_r(0,\pi)}\phi\left(\frac{|y|}{r}\right)\sum_i\langle v_1\wedge\cdots\wedge v_m,w_1\wedge\cdots\wedge w_m\rangle\,\mathrm{d}y\\&\qquad\quad+2\int_{\mathbf{C}_r(0,\pi)}\langle\vec{\mathbf{G}}_f(z),\Big(\vec{\mathcal{M}}(\mathbf{p}(z))-\vec{\mathcal{M}}\big(\boldsymbol{\varphi}(\mathbf{p}_\pi(z))\big)\Big)\rangle\phi\left(\frac{|\mathbf{p}_\pi(z)|}{r}\right)\,\mathrm{d}\|\mathbf{G}_f\|(z)\end{aligned}$$

$$= \frac{2}{|\xi|}\int_{B_r(0,\pi)} \phi\left(\frac{|y|}{r}\right)\sum_i \det B^i \,\mathrm{d}y$$
$$+ 2\int_{\mathbf{C}_r(0,\pi)} \langle \vec{\mathbf{G}}_f(z), \left(\vec{\mathcal{M}}(\mathbf{p}(z)) - \vec{\mathcal{M}}\big(\boldsymbol{\varphi}(\mathbf{p}_\pi(z))\big)\right)\rangle \phi\left(\frac{|\mathbf{p}_\pi(z)|}{r}\right) \,\mathrm{d}\|\mathbf{G}_f\|(z),$$

where $B^i_{jk} = \delta_{jk} + \langle Df_i|_y e_j, D\boldsymbol{\varphi}|_{\mathbf{p}_\pi(y,f_i(y))}e_k\rangle$. Expanding out the first term, we have

$$\frac{1}{|\xi|}\det B^i = \left(1 - \frac{1}{2}|D\boldsymbol{\varphi}|^2 + O(|D\boldsymbol{\varphi}|^4)\right)\left(1 + Df_i : D\boldsymbol{\varphi} + O(|Df|^2|D\boldsymbol{\varphi}|^2)\right).$$

Thus, we have

$$\begin{aligned}
E &= \int_{B_r(0,\pi)} |Df|^2\phi\left(\frac{|y|}{r}\right)\,\mathrm{d}y + Q\int_{B^L}|D\boldsymbol{\varphi}|^2\phi\left(\frac{|y|}{r}\right)\,\mathrm{d}y - 2\sum_i\int (Df_i : D\boldsymbol{\varphi})\phi\left(\frac{|y|}{r}\right)\,\mathrm{d}y \\
&\quad + O\left(\int_{B_r(0,\pi)}(|Df|^4 + |D\boldsymbol{\varphi}|^4 + |Df|^2|D\boldsymbol{\varphi}|^2\right) \\
&\quad + O\left(\int_{\mathbf{C}_r(0,\pi)}\left|\left\langle \vec{\mathbf{G}}_f(z), \left(\vec{\mathcal{M}}(\mathbf{p}(z)) - \vec{\mathcal{M}}\big(\boldsymbol{\varphi}(\mathbf{p}_\pi(z))\big)\right)\right\rangle\right| \mathrm{d}\|\mathbf{G}_f\|(z)\right) \\
&= \int_{B_r(0,\pi)} \mathcal{G}(Df, Q[\![D\boldsymbol{\varphi}]\!])^2\phi\left(\frac{|y|}{r}\right)\,\mathrm{d}y + O\left(\int_{B_r(0,\pi)}(|Df|^4 + |D\boldsymbol{\varphi}|^4\right) \\
&\quad + O\left(\int_{\mathbf{C}_r(0,\pi)}\left|\vec{\mathcal{M}}(\mathbf{p}(z)) - \vec{\mathcal{M}}\big(\boldsymbol{\varphi}(\mathbf{p}_\pi(z))\big)\right| \mathrm{d}\|\mathbf{G}_f\|(z)\right).
\end{aligned}$$

□

Proof of Corollary 3.6.7. It suffices to prove (3.47), since the argument for (3.46) is analogous (in fact it is easier since one does not need to reparameterize the graphical approximation from the cube L to the plane $\pi_{\bar{r}}$). Let us begin with the corresponding estimate for f_L. Letting F be as in [20, Assumption 3.1] for the normal approximation N and letting $\mathbf{C}^L \coloneqq \mathbf{C}_{32r_L}(p_L, \pi_L)$ and $\mathcal{B}^L \coloneqq \mathbf{B}_{64r_L}(p_L)\cap\mathcal{M}$, we have

$$\begin{aligned}
\int_{\mathbf{C}^L}|\vec{\mathbf{G}}_{f_L} - \vec{\mathcal{M}}\circ\mathbf{p}|^2\,\mathrm{d}\|\mathbf{G}_{f_L}\| &\le \int_{\mathbf{C}^L}|\vec{T} - \vec{\mathcal{M}}\circ\mathbf{p}|^2\,\mathrm{d}\|T\| + C\|T - \mathbf{G}_f\|(\mathbf{C}^L) \\
&\le \int_{\mathbf{p}^{-1}(\mathcal{B}^L)}|\vec{\mathbf{T}}_F - \vec{\mathcal{M}}\circ\mathbf{p}|^2\,\mathrm{d}\|\mathbf{T}_F\| + C\|T - \mathbf{G}_{f_L}\|(\mathbf{C}^L)
\end{aligned}$$

$$+ C\|T - \mathbf{T}_F\|(\mathbf{p}^{-1}(\mathcal{B}^L \setminus \mathcal{K}))\,,$$

where $\mathcal{K} \subset \mathcal{M}$ is the set over which T (in fact the slices $\langle T, \mathbf{p}, p\rangle$) coincides with $\mathbf{T}_F$ (i.e. the corresponding slices $\langle \mathbf{T}_F, \mathbf{p}, p\rangle$, which in fact are the currents $\sum_i \llbracket F_i(p)\rrbracket = \sum_i \llbracket p + N_i(p)\rrbracket$).

Applying (a localized version of) [20, Proposition 3.4], we have

$$\int_{\mathbf{C}^L} |\vec{\mathbf{G}}_{f_L} - \vec{\mathcal{M}} \circ \mathbf{p}|^2 \, \mathrm{d}\|\mathbf{G}_{f_L}\| \leq \int_{\mathcal{B}^L} |DN|^2 \, \mathrm{d}y + C\|T - \mathbf{G}_{f_L}\|(\mathbf{C}^L) + C\|T - \mathbf{T}_F\|(\mathbf{p}^{-1}(\mathcal{B}^L \setminus \mathcal{K})) + C\int_{\mathcal{B}^L} (|\mathbf{A}_{\mathcal{M}}|^2|N|^2 + |DN|^4).$$

Let us now control $\|T - \mathbf{G}_{f_L}\|$ and $\|T - \mathbf{T}_F\|$. To do this, we make use of the estimates in [19, Theorem 2.4] and [21], combined with a Vitali covering of $\mathcal{B}^L \setminus \mathcal{K}$ by Whitney regions $\mathcal{L}(L')$ and the height bound in [21, Proposition 4.1], to deduce that

$$\int_{\mathbf{C}^L} |\vec{\mathbf{G}}_{f_L} - \vec{\mathcal{M}} \circ \mathbf{p}|^2 \, \mathrm{d}\|\mathbf{G}_{f_L}\| \leq \int_{\mathbf{B}_L \cap \mathcal{M}} |DN|^2 \, \mathrm{d}y + C\boldsymbol{m}_0^{1+\gamma_1}\ell(L)^{m+2+\gamma_1} + C\boldsymbol{m}_0^{1+\gamma_2}\ell(L)^{m+2+\gamma_2} + C\int_{\mathcal{B}^L} (|\mathbf{A}_{\mathcal{M}}|^2|N|^2 + |DN|^4).$$

It remains to replace f_L with $f_{\bar{r}}$ inside $B_{\bar{r}}(0, \pi_{\bar{r}})$, but this is trivial since $\mathbf{G}_{f_L} \equiv \mathbf{G}_{f_{\bar{r}}} \llcorner \mathbf{C}_{\bar{r}}(0, \pi_{\bar{r}})$. Combining this with the fact that $\mathrm{spt}(\mathbf{G}_{f_{\bar{r}}}) \cap \mathbf{C}_{\bar{r}}(0, \pi_{\bar{r}}) \subset \mathrm{spt}(\mathbf{G}_{f_{\bar{r}}}) \cap \mathbf{C}^L$ and Lemma 3.6.6, the result follows. □

Proof of Lemma 3.6.8. Let $\eta \in \mathrm{C}^\infty_c(B_2)$ be a Lipschitz cutoff with $\eta \equiv 1$ on B_1. Integrating by parts and using the estimates in [21, Theorem 1.17], we have

$$\begin{aligned}\int_{B_1} |D\boldsymbol{\varphi}_k - D\tilde{\boldsymbol{\varphi}}_k|^2 &\leq \int_{B_2} |D\boldsymbol{\varphi}_k - D\tilde{\boldsymbol{\varphi}}_k|^2\eta \\ &= -\int_{B_2} (\boldsymbol{\varphi}_k - \tilde{\boldsymbol{\varphi}}_k)\eta\Delta(\boldsymbol{\varphi}_k - \tilde{\boldsymbol{\varphi}}_k) - \int_{B_2\setminus B_1} D\eta \cdot (\boldsymbol{\varphi}_k - \tilde{\boldsymbol{\varphi}}_k)D(\boldsymbol{\varphi}_k - \tilde{\boldsymbol{\varphi}}_k) \\ &\leq C\left(\boldsymbol{m}_{0,k}^{\frac{1}{2}} + \frac{t_k}{t_{k-1}}\boldsymbol{m}_{0,k-1}^{1/2}\right)\int_{B_2} |\boldsymbol{\varphi}_k - \tilde{\boldsymbol{\varphi}}_k|.\end{aligned}$$

In particular, taking into account (3.55), it suffices to prove (3.49). To that end, consider a Lipschitz approximation $f_k : B_3(0,\pi_k) \to \mathcal{A}_Q(\mathbb{R}^n)$ as in [19, Theorem 2.4] for the current T_{0,t_k} in the cylinder $\mathbf{C}_{12}(0,\pi_k)$, where the tilt excess is bounded by $C\boldsymbol{m}_{0,k}$. We claim that

$$\int_{B_2} |\boldsymbol{\varphi}_k - \boldsymbol{\eta}\circ f_k| \leq C\boldsymbol{m}_{0,k}\,, \tag{3.57}$$

$$\int_{B_2} |\tilde{\boldsymbol{\varphi}}_k - \boldsymbol{\eta}\circ f_k| \leq C\boldsymbol{m}_{0,k}\,, \tag{3.58}$$

and obviously (3.49) will follow from the latter.

First of all we observe that, since the tilt between the planes π_k and π_{k-1} is controlled by $\boldsymbol{m}_{0,k}^{1/2}$ due to [21, Proposition 4.1], all the estimates of [19, Theorem 2.4] apply to the map $\bar{f}_k : B_{5/2}(0,\pi_{k-1})$ which parametrizes graphically $\mathbf{G}_{f_k}$ in the cylinder $\mathbf{C}_{5/2}(0,\pi_{k-1})$. Setting $\bar{\boldsymbol{\varphi}}_k := \boldsymbol{\varphi}_{k-1}(\frac{t_k}{t_{k-1}}\cdot)$, (3.58) will actually follow from

$$\int_{B_2(0,\pi_{k-1})} |\bar{\boldsymbol{\varphi}}_k - \boldsymbol{\eta}\circ \bar{f}_k| \leq C\boldsymbol{m}_{0,k} \tag{3.59}$$

combined with [21, Lemma 5.6, Lemma B.1].

The argument leading to (3.59) is entirely analogous to the one leading to (3.57), with the only difference that instead of a control with $\boldsymbol{m}_{0,k}$ it leads to a control with

$$\boldsymbol{m}_{0,k} + \left(\frac{t_k}{t_{k-1}}\right)^{2-2\delta_2} \boldsymbol{m}_{0,k-1}\,.$$

However the latter is once again controlled by $C\boldsymbol{m}_{0,k}$ because of (3.55).

We now come to the proof of (3.57). We recall the algorithm leading to the construction of $\boldsymbol{\varphi}_k$. In particular, B_2 is covered by the union of contact set $\boldsymbol{\Gamma}$ and the Whitney cubes $L \in \mathscr{W}$ described in [21, Section 1]. We discard the cubes which are not intersecting B_2 and denote the family of remaining ones by $\mathscr{W}'$. Since the sidelength of each such cube is at most 2^{-N_0}, we can assume that each cube $L \in \mathscr{W}'$ is fully contained within $B_3(0,\pi_k)$, where f_k is

defined. We can then estimate

$$\int_{B_2} |\boldsymbol{\varphi}_k - \boldsymbol{\eta} \circ f_k| \leq \int_{\boldsymbol{\Gamma} \cap B_2} |\boldsymbol{\varphi}_k - \boldsymbol{\eta} \circ f_k| + \sum_{L \in \mathscr{W}'} \int_L |\boldsymbol{\varphi}_k - \boldsymbol{\eta} \circ f_k| \,. \tag{3.60}$$

Before coming to the estimates of each integrand in the above sums, we record the following important consequence of [19, Theorem 2.4] and [21, Theorem 1.17]:

$$\|\boldsymbol{\varphi}_k - \boldsymbol{\eta} \circ f_k\|_{C^0} \leq C \boldsymbol{m}_{0,k}^{\gamma} \,, \tag{3.61}$$

for $\gamma = \min\{\frac{1}{2m}, \gamma_1\}$, where $\gamma_1 > 0$ is as in [19, Theorem 2.4]. We moreover let $K \subset B_3(0, \pi_k)$ be the set of [19, Theorem 2.4] for f_k, namely the set over which, loosely speaking, the graph of f_k coincides with the current T_{0,t_k}.

In order to estimate the first integrand in the sum on the right-hand side of (3.60), observe that the identity

$$T_{0,t_k} \llcorner (\boldsymbol{\Gamma} \times \pi_k^{\perp}) = Q[\![\mathbf{G}_{\boldsymbol{\varphi}_k}]\!]$$

follows from [21, Corollary 2.2]. In particular $\boldsymbol{\varphi}_k \equiv \boldsymbol{\eta} \circ f_k$ on $\boldsymbol{\Gamma} \cap K$ and so we can estimate

$$\int_{\boldsymbol{\Gamma} \cap B_2} |\boldsymbol{\varphi}_k - \boldsymbol{\eta} \circ f_k| \leq |B_3 \setminus K| \|\boldsymbol{\varphi}_k - \boldsymbol{\eta} \circ f_k\|_{C^0} \leq C \boldsymbol{m}_{0,k}^{1+2\gamma} \,. \tag{3.62}$$

As for the remaining summands in the right hand side of (3.60), we introduce the plane of reference π_L of [21, Definition 1.14], the π_L-approximation f_L of Lemma [21, Lemma 1.15], and the tilted interpolating function h_L and the interpolating function g_L of [21, Definition 1.16]. We start by appealing to [21, Proposition 4.4(v)& Theorem 1.17(ii)] to estimate

$$\int_L |\boldsymbol{\varphi}_k - g_L| \leq C \boldsymbol{m}_{0,k} \ell(L)^{m+3+\beta_2/3} \,. \tag{3.63}$$

Next, let f_L' and $(\boldsymbol{\eta} \circ f_L)'$ be the functions defined on L and taking values, respectively, on $\mathcal{A}_Q(\pi_L^{\perp})$ and $\pi_L^{\perp}$, whose graphs coincide with the graphs of f_L and $\boldsymbol{\eta} \circ f_L$ on $L \times \pi_k^{\perp}$. We first

use [21, Lemma B.1(b)] to estimate

$$\int_L |g_L - (\boldsymbol{\eta} \circ f_L)'| \le C \int_{B_{2\sqrt{m}\ell(L)}(p_L,\pi_L)} |h_L - \boldsymbol{\eta} \circ f_L|, \tag{3.64}$$

where p_L is the center of L, while by [21, Proposition 5.2], we have

$$\int_{B_{2\sqrt{m}\ell(L)}(p_L,\pi_L)} |h_L - \boldsymbol{\eta} \circ f_L| \le C\boldsymbol{m}_{0,k}\ell(L)^{m+3+\beta_2}\,. \tag{3.65}$$

In addition, [21, Lemma 5.6] gives us the estimate

$$\int_L |(\boldsymbol{\eta} \circ f_L)' - \boldsymbol{\eta} \circ (f_L')| \le C\boldsymbol{m}_{0,k}\ell(L)^{m+3+\beta_2/2}\,. \tag{3.66}$$

Putting (3.63), (3.64), (3.65), and (3.66) together we then reach

$$\int_L |\boldsymbol{\varphi}_k - \boldsymbol{\eta} \circ (f_L')| \le C\boldsymbol{m}_{0,k}\ell(L)^{m+3+\gamma}\,, \tag{3.67}$$

for some $\gamma > 0$. Next, observe that by [21, Lemma 1.15] there is a set $K_L' \subset L$ such that in $K_L' \times \pi_k^\perp$, the current T coincides with the graph of f_L' and such that

$$|L \setminus K_L'| \le C\boldsymbol{m}_{0,k}^{1+\gamma_1}\ell(L)^m \tag{3.68}$$

It thus turns out that f_L' and f_k coincide over $K_L' \cap K$. In particular we can estimate

$$\int_L |\boldsymbol{\eta} \circ f_L' - \boldsymbol{\eta} \circ f_k| \le C(|L \setminus K| + |L \setminus K_L'|)\boldsymbol{m}_{0,k}^{\gamma_1} \le C\boldsymbol{m}_{0,k}^{1+2\gamma_1}\ell(L)^m\,, \tag{3.69}$$

which combined with (3.67) gives

$$\int_L |\boldsymbol{\varphi}_k - \boldsymbol{\eta} \circ f_k| \le C\boldsymbol{m}_{0,k}\ell(L)^{m+3+\gamma} + C\boldsymbol{m}_{0,k}^{1+2\gamma_1}\ell(L)^m\,. \tag{3.70}$$

Since the collection $\mathscr{W}'$ consists of disjoint cubes contained in B_3, we can sum (3.70) over

$L \in \mathscr{W}'$ to reach

$$\sum_{L\in\mathscr{W}'} \int_L |\boldsymbol{\varphi}_k - \boldsymbol{\eta}\circ f_k| \le C\boldsymbol{m}_{0,k} + C\boldsymbol{m}_{0,k}^{1+2\gamma_1} \le C\boldsymbol{m}_{0,k}\,. \tag{3.71}$$

Clearly, (3.60), (3.62), and (3.71) imply (3.57) and thus complete the proof. □

Proof of Lemma 3.6.9. We begin with the estimate (3.50). Due to the fact that $\|\boldsymbol{\varphi}_{\bar r}\|_{C^2} \le C\boldsymbol{m}_0^{1/2}$ and the estimates in [19, Theorem 2.4], we have

$$\begin{aligned}\int_{\mathbf{C}_{\bar r}(0,\pi_{\bar r})} \left|\vec{\mathcal{M}}(\mathbf{p}(z)) - \vec{\mathcal{M}}\big(\boldsymbol{\varphi}_{\bar r}(\mathbf{p}_{\pi_{\bar r}}(z))\big)\right| \mathrm{d}\|\mathbf{G}_f\|(z) &\le C\boldsymbol{m}_0^{1/2} \int_{\mathbf{C}_{\bar r}(0,\pi_{\bar r})} |\mathbf{p} - \boldsymbol{\varphi}_{\bar r}\circ \mathbf{p}_{\pi_{\bar r}}|\, \mathrm{d}\|\mathbf{G}_f\| \\ &\le C\boldsymbol{m}_0^{1/2} \int_{K\times\pi_{\bar r}^\perp} |\mathbf{p} - \boldsymbol{\varphi}_{\bar r}\circ \mathbf{p}_{\pi_{\bar r}}|\, \mathrm{d}\|T\| \\ &\quad + C\bar r^{m+1}\boldsymbol{m}_0^{1+\gamma_1}.\end{aligned}$$

Now by the definition of the scale $\bar r$, we may use the height bound [21, Corollary 2.2], the estimates in [21, Proposition 4.1] and to deduce that

$$\begin{aligned}\int_{K\times\pi_{\bar r}^\perp} |\mathbf{p} - \boldsymbol{\varphi}_{\bar r}\circ \mathbf{p}_{\pi_{\bar r}}|\, \mathrm{d}\|T\| &\le \int_{K\times\pi_{\bar r}^\perp} |\mathbf{p}(z) - z|\, \mathrm{d}\|T\|(z) \\ &\quad + \int_{K\times\pi_{\bar r}^\perp} |z - \boldsymbol{\varphi}_{\bar r}\circ \mathbf{p}_{\pi_{\bar r}}(z)|\, \mathrm{d}\|T\|(z) \\ &\le C\bar r^{m+1+\beta_2}\boldsymbol{m}_0^{1/2+1/2m}.\end{aligned}$$

This gives the claimed estimate (3.50). The estimate (3.51) follows analogously, only at unit scale and via the cover of B_1 with Whitney cubes of $\mathscr{W}$ and the coincidence region $\boldsymbol{\Gamma}$, as in [22, Section 4]. □

3.6.4 Frequency relative to different center manifolds

While this completes the proof of the desired BV bound, we wish to isolate one more general version of the estimates on the "jumps" of the frequency function at the endpoint scales t_j,

only this time, we want to compare the frequency functions at comparable scales, relative to two center manifolds with different centers. This is needed in the covering arguments in Chapter 3, in order to gain local Minkowski content bounds on each piece of the flat singular set in the subdivision therein. It follows directly from the above arguments, after observing that we are just using the presence of a "stopping cube" in one of the two center manifolds construction, at the desired scale, which is not "too small", together with the fact that at all larger scales there are no stopping cubes which are too large. We are in addition using the fact that all constants in the estimates on the center manifold and the associated normal approximation are independent of the center point of the construction (cf. [49]).

Lemma 3.6.10. *Consider T and Σ as in Assumption 3.2.1, let z and w be such that $\Theta(T,z) = \Theta(T,w) = Q$ and let $r \leq r_0, r_1$ be three positive numbers such that:*

(a) T_{z,r_0} falls under the Assumptions of [21, Theorem 1.17] and $\boldsymbol{\varphi}_0 : [-4,4]^m \supset \pi_0 \to \pi_0^\perp$ is the graphical map describing the center manifold $\mathcal{M}_0$ constructed in that theorem applied to T_{z,r_0}.

(b) T_{w,r_1} falls under the Assumptions of [21, Theorem 1.17] and $\boldsymbol{\varphi}_1 : [-4,4]^m \supset \pi_1 \to \pi_1^\perp$ is the graphical map describing the center manifold $\mathcal{M}_1$ constructed in that theorem applied to T_{w,r_1}.

(c) For the families of Whitney cubes $\mathscr{W}_0$ and $\mathscr{W}_1$ of [21, Definition 1.10] used in the construction of the respective center manifolds, we have

$$\ell(L) < c_s \rho \qquad \forall \rho \in \left[\frac{r}{r_0}, 4\right] \qquad \forall L \in \mathscr{W}_0 \text{ s.t. } L \cap B_\rho(0,\pi_0) \neq \emptyset \tag{3.72}$$

$$\ell(L) < c_s \rho \qquad \forall \rho \in \left[\frac{r}{r_1}, 4\right] \qquad \forall L \in \mathscr{W}_1 \text{ s.t. } L \cap B_\rho(0,\pi_1) \neq \emptyset\,, \tag{3.73}$$

where c_s is the geometric constant of [22, Section 2].

Define

$$\bar{c}_s := \max\{\ell(L) : L \in \mathscr{W}_0^e \text{ and } L \cap B_{r/r_0}(0,\pi_0) \neq \emptyset\}$$

and let N_0 *and* N_1 *be the graphical approximations of* T_{z,r_0} *on* $\mathcal{M}_0$ *and* T_{w,r_1} *on* $\mathcal{M}_1$ *respectively. Consider the points* $x_1 = (0, \boldsymbol{\varphi}_1(0)) \in \mathcal{M}_0$ *and* $x_0 = (\boldsymbol{p}_{\pi_0}(r_1^{-1}(w-z)), \boldsymbol{\varphi}_0(r_0^{-1}(w-z))) \in \mathcal{M}_1$. *Then we have (cf.* (3.52)*) the estimate*

$$|\mathbf{I}_{N_0}(x_0, r_0^{-1}r)) - \mathbf{I}_{N_1}(x_1, r_1^{-1}r)| \leq \bar{C}\boldsymbol{m}_0^{\gamma_2}(1 + \mathbf{I}_{N_0}(x_0, r_0^{-1}r))\,,$$

where the constant $\bar{C}$ *depends on* m, n, $\bar{n}$, Q, *and* $\bar{c}_s$.

3.7 Proof of Theorem 3.2.9: the case $\mathrm{I}(T,0) > 1$

The goal of this section is to prove that the singular frequency value is unique when $\mathrm{I}(T,0) > 1$. The proof will also show that the tangent cone is then a unique flat plane and that the rescaled currents converge polynomially fast to it. In particular this section will settle Theorem 3.2.10(iv), but also Theorem 3.2.10(i),(ii)&(iii) when $\mathrm{I}(T,0) > 1$.

Proposition 3.7.1. *Let* T *be as in Theorem 3.2.9. Then the conclusions* (i)-(iv) *of Theorem 3.2.10 hold whenever* $\mathrm{I}(T,0) > 1$.

In fact, since it will be useful in the remainder of this thesis, we record a consequence of our analysis which is more quantitative.

Proposition 3.7.2. *Let* T *be as in Theorem 3.2.9. For every* $I_0 > 1$ *there are positive constants* $C(m,n,Q)$ *and* $\alpha(I_0,m,n,Q)$ *with the following property. If* 0 *is a flat singular point at which* $\mathrm{I}(T,0) \geq I_0$, *then there is a radius* $r_0 = r_0(T) > 0$ *(which also implicitly*

depends on the center point, which we are here assuming is the origin) such that

$$\mathbf{E}(T, \mathbf{B}_r) \leq C \left(\frac{r}{r_0}\right)^{\alpha} \max\{\mathbf{E}(T, \mathbf{B}_{r_0}), \bar{\varepsilon}^2 r_0^{2-2\delta_2}\} \qquad \forall r < r_0 \,. \tag{3.74}$$

Moreover, we can choose α to be any number which satisfies the inequalities $\alpha < 2 - 2\delta_2$ and $\alpha < 2(\mathrm{I}(T, 0) - 1)$, at the price of a constant C which depends also upon α.

Before coming to the proof of the proposition we state the following technical fact which will prove to be very useful.

Lemma 3.7.3. *Let T be as in Theorem 3.2.9. If there are infinitely many intervals of flattening, then*

$$\liminf_{k \to \infty} \mathbf{E}(T, \mathbf{B}_{6\sqrt{m}t_k}) = 0$$

and hence

$$\liminf_{k \to \infty} \boldsymbol{m}_{0,k} = 0$$

Proof. The second conclusion is an obvious consequence of the first. In order to prove the first take a sequence r_j such that $r_j \to 0$ and $\mathbf{E}(T, \mathbf{B}_{6\sqrt{m}r_j}) \to 0$. Then r_j belongs to some interval of flattening $]s_{k(j)}, t_{k(j)}]$. We claim that

$$\lim_{j \to \infty} \mathbf{E}(T, \mathbf{B}_{6\sqrt{m}s_{k(j)}}) = 0 \,, \tag{3.75}$$

which clearly would imply $s_{k(j)} = t_{k(j)+1}$ and hence the conclusion of the lemma.

Up to extraction of a further subsequence, we distinguish two cases:

(i) If $\frac{s_{k(j)}}{r_j} \to 0$, since

$$\mathbf{E}(T, \mathbf{B}_{6\sqrt{m}s_{k(j)}}) \leq C \left(\frac{s_{k(j)}}{t_{k(j)}}\right)^{2-2\delta_2} \boldsymbol{m}_{0,k(j)} \leq C \left(\frac{s_{k(j)}}{r_j}\right)^{2-2\delta_2} \varepsilon_3^2 \,,$$

we conclude immediately that (3.75) holds.

(ii) If $\inf_j \frac{s_{k(j)}}{r_j} = \sigma > 0$, we then estimate

$$\mathbf{E}(T, \mathbf{B}_{6\sqrt{m}s_{k(j)}}) \leq \sigma^{-m}\mathbf{E}(T, \mathbf{B}_{6\sqrt{m}r_j})$$

and again (3.75) follows immediately.

□

We will also need the following two facts about Dir-minimizing functions. For the first one we refer to [18], while the second is a well-known fact about classical harmonic functions and can be proved, for instance, using the expansion into spherical harmonics.

Lemma 3.7.4. *If $u : \mathbb{R}^m \supset B_1 \to \mathcal{A}_Q(\mathbb{R}^n)$ is a Dir-minimizing function with $I_u(0) = I_0$, then*

$$\int_{B_\rho} |Du|^2 \leq \rho^{m+2I_0-2} \int_{B_1} |Du|^2 \qquad \forall \rho < 1\,. \tag{3.76}$$

Lemma 3.7.5. *If $w : \mathbb{R}^m \supset B_1 \to \mathbb{R}^n$ is a classical harmonic function, then*

$$\int_{B_\rho} |Dw - Dw(0)|^2 \leq \rho^{m+2} \int_{B_1} |Dw|^2 \qquad \forall \rho < 1\,. \tag{3.77}$$

In other words, after subtracting an optimal affine map, the frequency (at zero scale) of a classical harmonic map must be at least two. In particular, we can draw the following simple corollary.

Corollary 3.7.6. *Let $u : \mathbb{R}^m \supset B_1 \to \mathcal{A}_Q(\mathbb{R}^n)$ be Dir-minimizing. Then*

$$\int_{B_\rho} \mathcal{G}(Du, Q\llbracket D(\boldsymbol{\eta} \circ u)(0)\rrbracket)^2 \leq \rho^{m-2+2\min\{I_u(0),2\}} \int_{B_1} |Du|^2 \qquad \forall \rho < 1\,. \tag{3.78}$$

Proof of Proposition 3.7.1. From now on we assume that $\mathrm{I}(T, 0) > 1$. The main point will be to show the following decay property:

(Dec) There are $\varepsilon = \varepsilon(T) \in]0, \varepsilon_3]$, $\alpha = \alpha(I_0, m, n, Q) > 0$ and $\kappa \in \mathbb{N}$ such that, if

$$\mathbf{E}(T, \mathbf{B}_{6\sqrt{m}t_k}) < \varepsilon^2$$

and $k \geq \kappa$, then:

(a) The intervals of flattening $]s_k, t_k],]s_{k+1}, t_{k+1}], \ldots,]s_{k+\kappa}, t_{k+\kappa}]$ satisfy $s_{k+j-1} = t_{k+j}$ for $j = 1, \ldots, \kappa$.

(b) $\boldsymbol{m}_{0,k+\kappa} \leq \left(\frac{s_{k+\kappa}}{t_k}\right)^{\alpha} \boldsymbol{m}_{0,k}$.

Before coming to the proof of (Dec), observe that thanks to Lemma 3.7.3, there is at least one integer $k_0 \in \mathbb{N}$ which satisfies the small excess assumption of the decay and since it can be iterated, we may use (Dec) to conclude that

$$\boldsymbol{m}_{0,k_0+j\kappa} \leq \left(\frac{t_{k_0+j\kappa}}{t_{k_0}}\right)^{\alpha} \boldsymbol{m}_{0,k_0} \leq \varepsilon_3^2 \left(\frac{t_{k_0+j\kappa}}{t_{k_0}}\right)^{\alpha} \qquad \forall j \in \mathbb{N}\,.$$

On the other hand, when we have intervals of flattening with coinciding endpoints $s_k = t_{k+1}$, we can iterate the estimate

$$\boldsymbol{m}_{0,k+1} \leq C \left(\frac{t_{k+1}}{t_k}\right)^{2-2\delta_2} \boldsymbol{m}_{0,k} \leq C\boldsymbol{m}_{0,k}\,,$$

for $C = C(m, n, Q) > 0$, to conclude that indeed

$$\boldsymbol{m}_{0,k} \leq C \left(\frac{t_k}{t_{k_0}}\right)^{\alpha} \qquad \forall k \geq k_0\,. \tag{3.79}$$

We then also recall

$$\mathbf{E}(T, \mathbf{B}_r) \leq C \left(\frac{r}{t_k}\right)^{2-2\delta_2} \boldsymbol{m}_{0,k} \qquad \forall r \in [t_{k+1}, t_k]\,.$$

Combined with (3.79), we infer the geometric decay of the excess with $r_0 = t_{k_0}$, which

implies immediately the uniqueness of the tangent cone and the polynomial convergence of the rescalings (i.e. point (iv) of Theorem 3.2.10).

Note moreover that, from (3.79), the fact that $t_k \downarrow 0$ at least geometrically fast and the frequency BV estimate of the previous section, we conclude the existence of the limit

$$I_0 = \lim_{r \downarrow 0} \mathbf{I}(r)\,,$$

where $\mathbf{I}$ is the universal frequency function. This immediately implies that every fine blow-up is I_0-homogeneous, which in turn gives all the other conclusions of the proposition.

It therefore remains to show (Dec). First of all we choose $\alpha < \min 2\{\mathrm{I}(T,0) - 1, 1 - \delta_2\}$. The choice of κ will be more complicated, while those of k_0 and ε are subordinate to κ. We therefore fix κ at the moment, without specifying its choice, and treat it as a constant in order to obtain the choice of k_0 and ε. We start by showing that the first point (a) of (Dec) holds and to this effect we impose that k_0 is sufficiently large so that

$$\bar{\varepsilon}^2 t_{k_0}^{2-2\delta_2} \leq \varepsilon^2. \tag{3.80}$$

Next we recall that

$$\mathbf{E}(T, \mathbf{B}_{6\sqrt{m}s_k}) \leq C \left(\frac{s_k}{t_k}\right)^{2-2\delta_2} \boldsymbol{m}_{0,k} \leq C \boldsymbol{m}_{0,k} = C \max\{\bar{\varepsilon}^2 t_k^{2-2\delta_2}, \varepsilon^2\} \leq C\varepsilon^2\,,$$

for each $k \geq k_0$, where C is a geometric constant, independent of ε. In particular, if we choose ε sufficiently small, we conclude that $\mathbf{E}(T, \mathbf{B}_{6\sqrt{m}s_k}) \leq \varepsilon_3^2$, which in turn forces $t_{k+1} = s_k$. Observe also that $\boldsymbol{m}_{0,k+1} \leq C\boldsymbol{m}_{0,k}$, where the latter is the same constant of the previous estimate. In particular, as long as $t_{k+i+1} = s_{k+i}$ for $i \in \{0, \dots, j\}$, we get $\mathbf{E}(T, \mathbf{B}_{6\sqrt{m}s_j}) \leq C^j \boldsymbol{m}_{0,k}$. Since this must be repeated κ times, under the assumption that $C^{\kappa_0}\varepsilon^2 \leq \varepsilon_3^2$, we get by induction that $t_{k+j+1} = s_{k+j}$ and $\boldsymbol{m}_{k+j+1} \leq C\boldsymbol{m}_{k+j} \leq C^{j+1}\boldsymbol{m}_{0,k}$.

We next show the second point (b) of (Dec). First of all we observe that it suffices to

show

$$\mathbf{E}(T, \mathbf{B}_{6\sqrt{m}s_{k+\kappa-1}}) \leq \left(\frac{s_{k+\kappa-1}}{t_k}\right)^{\alpha} \boldsymbol{m}_{0,k}\,. \tag{3.81}$$

In fact, if $\boldsymbol{m}_{0,k} = \bar{\varepsilon}^2 t_k^{2-2\delta_2}$, since $2-2\delta_2 > \alpha$, we then have

$$\begin{aligned}\boldsymbol{m}_{0,k+\kappa_0} &= \max\{\mathbf{E}(T, \mathbf{B}_{6\sqrt{m}s_{k+\kappa-1}}), \bar{\varepsilon}^2 s_{k+\kappa-1}^{2-2\delta_2}\} \leq \left(\frac{s_{k+\kappa-1}}{t_k}\right)^{\alpha} \bar{\varepsilon}^2 t_k^{2-2\delta_2} \\ &= \left(\frac{s_{k+\kappa-1}}{t_k}\right)^{\alpha} \boldsymbol{m}_{0,k}\,.\end{aligned}$$

But if $\boldsymbol{m}_{0,k} = \mathbf{E}(T, \mathbf{B}_{6\sqrt{m}t_k})$, then $\mathbf{E}(T, \mathbf{B}_{6\sqrt{m}t_k}) \geq \bar{\varepsilon}^2 t_k^{2-2\delta_2}$ and hence again

$$\bar{\varepsilon}^2 s_{k+\kappa-1}^{2-2\delta_2} \leq \left(\frac{s_{k+\kappa-1}}{t_k}\right)^{\alpha} \mathbf{E}(T, \mathbf{B}_{6\sqrt{m}t_k}) \leq \left(\frac{s_{k+\kappa-1}}{t_k}\right)^{\alpha} m_{k+\kappa}\,.$$

Towards (3.81), we first argue as for the proof of point (i) of Theorem (3.2.10) to estimate

$$\mathbf{E}(T, \mathbf{B}_{6\sqrt{m}s_{k+\kappa-1}}) \leq C^{\kappa} \left(\frac{s_{k+\kappa-1}}{t_k}\right)^{2-2\delta_2} \boldsymbol{m}_{0,k}\,. \tag{3.82}$$

Since κ and C are fixed and $2-2\delta_2 > \alpha$, then clearly (3.81) follows if $\frac{s_{k+\kappa-1}}{t_\kappa}$ is sufficiently small. We are thus left to prove (3.81) under the addititional assumption that

$$\frac{s_{k+\kappa-1}}{t_k} \geq \rho_\ell > 0\,, \tag{3.83}$$

where ρ_ℓ is a fixed constant which depends on κ. Next, recall that $\frac{s_k}{t_k} \leq 2^{-5}$ by [22, Proposition 2.2]. We therefore infer that $s_{k+\kappa-1} \leq 2^{-5\kappa} t_k$. In fact κ will be chosen large enough so that the ratio $\frac{s_{k+\kappa-1}}{t_k}$ is sufficiently small, a condition which we specify here by

$$\frac{s_{k+\kappa-1}}{t_k} \leq \rho_u\,. \tag{3.84}$$

The claim is now that, for an appropriate choice of ρ_u (which in turn fixes the choice of κ and of ρ_ℓ), once ε and k_0^{-1} are sufficiently small, then (3.81) holds. Towards this we argue by

contradiction and assume that, no matter how small we choose ε and how large we choose k_0 (satisfying (3.80)), there is always a choice of $k \geq k_0$ for which (3.81) fails. This implies the existence of a sequence $t_k \downarrow 0$ with the property that

$$\boldsymbol{m}_{0,k} \downarrow 0 \qquad \text{and} \qquad \mathbf{E}(T, \mathbf{B}_{6\sqrt{m}s_{k+\kappa-1}}) > \left(\frac{s_{k+\kappa-1}}{t_k}\right)^{\alpha} \boldsymbol{m}_{0,k}\,, \tag{3.85}$$

while

$$\rho_\ell \leq \frac{s_{k+\kappa-1}}{t_k} \leq \rho_u \tag{3.86}$$

We now choose the radius r_k so that $8Mr_k = 6\sqrt{m}t_k$, where M is the constant of (3.4). We will assume that κ is large enough so that $r_k \geq s_{k+\kappa-1}$. Observe that we can now apply Proposition 3.4.1 and generate the coarse blow-up $\bar{f} : B_M \to \mathcal{A}_Q$ along the scales r_k, which is Dir-minimizing. In light of the comparability of the scales r_k and $s_{k+\kappa-1}$, the average-free part v of $\bar{f}$ is, up to a positive scalar multiple, a fine blow-up u, and we thus infer that $I_v(0) = I_u(0) \geq \mathrm{I}(T, 0)$. We can then apply Corollary 3.7.6 to infer that

$$\frac{1}{\sigma^m}\int_{B_\sigma} \mathcal{G}(D\bar{f}, Q[\![D(\boldsymbol{\eta}\circ\bar{f}(0))]\!])^2 \leq C\left(\frac{\sigma}{M}\right)^{2\alpha}\frac{1}{M^m}\int_{B_M}|D\bar{f}|^2\,.$$

We can now use the Taylor expansion of the tilt excess in [20] to infer that, for all $\sigma \in [\rho_\ell, \rho_u]$,

$$\mathbf{E}(T, \mathbf{B}_{6\sqrt{m}\sigma t_k}) \leq 8^m\sigma^{2\alpha}\mathbf{E}(T, \mathbf{B}_{6\sqrt{m}t_k}) + C(\mathbf{E}(T, \mathbf{B}_{6\sqrt{m}t_k} + t_k^2\mathbf{A}^2)^{1+\gamma}\,.$$

Since $\mathbf{A}t_k^2$ is controlled by $\boldsymbol{m}_{0,k}$, we easily conclude that, once we choose ρ_u small enough so that $8^m\rho_u^{2\alpha} \leq \frac{1}{2}\rho_u^\alpha$ and choose k large enough and ε small enough so that

$$C(\mathbf{E}(T, \mathbf{B}_{6\sqrt{m}t_k} + t_k^2\mathbf{A}^2)^{1+\gamma} \leq C\boldsymbol{m}_{0,k}^{1+\gamma} \leq C\varepsilon^{2\gamma}\boldsymbol{m}_{0,k} \leq \frac{1}{2}\rho_\ell^\alpha\boldsymbol{m}_{0,k}\,,$$

we achieve

$$\max_{[\rho_\ell\leq\sigma\leq\rho_u]}\sigma^{-\alpha}\mathbf{E}(T, \mathbf{B}_{6\sqrt{m}\sigma t_k}) \leq \boldsymbol{m}_{0,k}$$

for all k sufficiently large. However this is in contradiction with (3.85) and (3.84).

Observe that the threshold ε in (Dec) may be made independent of T (and the center point, which it also implicitly depends on). This may be done by replacing the above contradiction compactness argument with one in which a sequence of currents T_k and varying centers x_k are taken. However, in order to do this one must also verify that the conclusion of Proposition 3.4.1 holds for "diagonal" coarse and fine blow-ups taken along such a varying sequence of currents and centers. This is indeed true, but we omit the details here, since this is unnecessary for the remainder of our arguments. □

3.8 Proof of Theorem 3.2.9: the case $\mathrm{I}(T, 0) = 1$

In this section we complete the proof of Theorem 3.2.9 by handling the case $\mathrm{I}(T, 0) = 1$. We will moreover complete the proof of the points (i), (ii), and (iii) in Theorem 3.2.10.

Proposition 3.8.1. *Let T be as in Theorem 3.2.9. Then the conclusions (i), (ii) & (iii) of Theorem 3.2.10 hold whenever $\mathrm{I}(T, 0) = 1$.*

A key ingredient in the proof is a decay lemma which is a refinement of the one used in the proof of Proposition 3.7.1:

Lemma 3.8.2. *Let T be as in Theorem 3.2.9. For every $\gamma > 0$ and every $\eta > 0$ there are $\varepsilon > 0$ and $\rho > 0$ with the following property. Assume $]a, b]$ is an interval of radii such that*

(a) $0 < a < b \leq \rho$;

(b) $\mathbf{E}(T, \mathbf{B}_{6\sqrt{m}r}) \leq \varepsilon$ *for all* $a \leq r \leq b$;

(c) $\mathbf{I}(r) \geq 1 + \gamma$ *for all* $a \leq r \leq b$.

Consider the intervals of flattening $]s_{\bar{k}+\bar{j}}, t_{\bar{k}+\bar{j}}] \cup]s_{\bar{k}+\bar{j}-1}, t_{\bar{k}+\bar{j}-1}] \cup \ldots \cup]s_{\bar{k}}, t_{\bar{k}}]$ covering $]a, b]$

with the property that $t_{\bar{k}+\bar{j}} = s_{\bar{k}+\bar{j}-1}$, $\ldots$, $t_{\bar{k}+1} = s_{\bar{k}}$ *are contained in* $]a, b]$. *Then*

$$\sum_{i=1}^{\bar{j}} \boldsymbol{m}_{0,\bar{k}+i}^{\gamma_4} \leq \eta \,.$$

Proof. Observe that $\boldsymbol{m}_{0,\kappa+i} \leq \varepsilon^2$ for $i \geq 1$ just by assumption. Since by assumption we know that $\boldsymbol{m}_{0,k} \leq \bar{\varepsilon}^2$, it suffices to prove the decay of (Dec) as long as $k + \kappa \leq \bar{k} + \bar{j} - L$ where L is a fixed natural number. In the argument by contradiction leading to the proof of (Dec) we are thus also allowed to assume that L gets arbitrarily large, which in turn means that $\frac{t_k}{a_k}$ tends to infinity (where $]a_k, b_k]$ are corresponding intervals as above). In particular, notice that in the argument given for (Dec) the key point was to infer that the average-free part of the coarse blow-up v has $I_v(0) = I_u(0)$ for some fine blow-up u while $I_u(0) > 1$. In our situation the bound $I_u(0) \geq \mathrm{I}(T, 0)$ just gives $I_u(0) \geq 1$. On the other hand, using the fact that $\frac{a_k}{t_k} \to 0$ and our assumption that $\mathbf{I}(r) \geq 1 + \gamma$ for all $r \in]a_k, t_k]$, we can use the convergence of the frequency function to conclude

$$I_u(\rho) = \lim_{k \to \infty} \mathbf{I}(\rho r_k) \geq 1 + \gamma$$

for an arbitrary positive ρ. This in turn gives $I_u(0) \geq 1 + \gamma$. □

Proof of Proposition 3.8.1. As we have already argued at the start of the proof of Proposition 3.7.1, the key is in fact to prove the second part of Theorem 3.2.10(iii). We thus assume that there is some other blow-up sequence $r_k \to 0$ with the property that $\mathbf{I}(r_k) \to 1 + 2\gamma$ for some $\gamma > 0$. Our aim is then to show that this leads to a contradiction. We apply Lemma 3.8.2 from the previous section with some parameter $\eta > 0$ which will be chosen later. Fix the corresponding $\varepsilon > 0$ and $\rho > 0$ given by Lemma 3.8.2 and consider the set

$$\mathcal{R} := \left\{ r \in]0, \rho[\,:\, \mathbf{E}(T, \mathbf{B}_{6\sqrt{m}r}) \leq \varepsilon^2 \quad \text{and} \quad \mathbf{I}^+(r) \geq 1 + \gamma \right\} ,$$

(since the universal frequency function has jumps, at the jump points we let $\mathbf{I}^+(r)$ be the

right-hand limit). We might later need to choose ε even smaller than that prescribed by Lemma 3.8.2; the only property needed is that the conclusion of the Lemma still applies.

Observe that $\mathcal{R}$ cannot contain a neighborhood of the origin, otherwise we would have $\mathbf{I}(r) \geq 1+\gamma$ for all r sufficiently small, which in turn would imply that, if u is any fine blow-up, then

$$I_u(\rho) \geq 1+\gamma \qquad \forall \rho > 0\,.$$

This shows that $I_u(0) \geq 1+\gamma$ for every fine blow-up, in turn implying that $\mathrm{I}(T,0) \geq 1+\gamma$. On the other hand $\mathcal{R}$ must have 0 as an accumulation point, namely $\mathcal{R}$ consists of countably many disjoint intervals, which might or might not include any of their endpoints. We enumerate these intervals in order of decreasing scales, and for each one we consider its interior $]a_k, b_k[$. Note that $r_\ell \in]a_{k(\ell)}, b_{k(\ell)}[$ for all ℓ sufficiently large, due to the nature of our chosen sequence of blow-up scales.

Now notice that the intervals $]a_k, b_k[$ are contained within the full collection of intervals of flattening $]s_j, t_j]$ (with the tilt excess threshold $\bar{\varepsilon}$). Thus, we can find a sequence of radii $\tilde{\rho}_k > b_k$ approaching b_k asymptotically, with $]b_k, \tilde{\rho}_k] \cap \mathcal{R} = \emptyset$, such that one of the following two possibilities holds:

(a) there are $\rho_k \in]b_k, \tilde{\rho}_k]$ with $\mathbf{E}(T, \mathbf{B}_{6\sqrt{m}\rho_k}) > \varepsilon^2$ for infinitely many k;

(b) for infinitely many k the inequalities $\mathbf{E}(T, \mathbf{B}_{6\sqrt{m}r}) \leq \varepsilon^2$ and $\mathbf{I}(r) < 1+\gamma$ hold for all r in the interval $]b_k, \tilde{\rho}_k]$.

We first argue that, if ε is chosen sufficiently small, (a) cannot happen. We argue by contradiction; if this is not true, a subsequence of $T_{0,\tilde{\rho}_k}$ (and thus of T_{0,b_k}), not relabeled, must be converging to a cone which is not flat. We denote it by C. Repeat now the procedure above for each $\varepsilon = \frac{1}{j}$ and assume that for each we find a corresponding sequence $b_{k,j}$, with the property that $T_{0,b_{k,j}}$ is converging to a non-flat cone C_j. Letting $]s_{\ell(k,j)}, t_{\ell(k,j)}]$ denote the

interval of flattening containing $b_{k,j}$, clearly we first have

$$\lim_{k\to\infty} \frac{s_{\ell(k,j)}}{t_{\ell(k,j)}} \geq c(j) > 0 \qquad \forall j \in \mathbb{N},$$

for some constant $c(j)$ which depends only on C_e and δ_2 of the tilt excess stopping condition in the center manifold construction (cf. [21]) and on $\varepsilon = \frac{1}{j}$, just using that

$$\mathbf{E}(T, \mathbf{B}_r) \leq CC_e \left(\frac{r}{t_{k,j}}\right)^{2-2\delta_2} \bar{\varepsilon}^2 \forall r \in]s_{k,j}, t_{k,j}[$$

while $b_{k,j} \in]s_{k,j}, t_{k,j}]$ and

$$\mathbf{E}(T, \mathbf{B}_{b_{k,j}}) \geq \varepsilon_j^2 \,.$$

On the other hand because of the convergence of $T_{0,t_{k,j}}$ to the cone C_j we have

$$\lim_{k\to\infty} \frac{\mathbf{E}(T, \mathbf{B}_{t_{k,j}})}{\mathbf{E}(T, \mathbf{B}_{s_{k,j}})} = 1 \,.$$

In turn this implies, again because of the tilt excess stopping condition in the center manifold construction, that

$$\liminf_{k\to\infty} \frac{s_{\ell(k,j)}}{t_{\ell(k,j)}} \geq c > 0$$

for a constant c which this time is independent of j. In particular for any sequence $k(j) \uparrow \infty$ which explodes sufficiently fast we have

$$\lim_{j\to\infty} \frac{s_{\ell(k(j),j)}}{t_{\ell(k(j),j)}} \geq \frac{c}{2} > 0 \,.$$

We can therefore apply Proposition 3.4.1 and Corollary 3.4.3 to any such $b_{k(j),j}$ and infer that the corresponding fine and coarse blow-ups coincide.

We now argue that at least one such coarse blow-up has to be 1-homogeneous. First of all, for each k and j we denote by $f_{k,j}$ the Lipschitz approximation of the current $T_{0,b_{k,j}}$ given

by [19, Theorem 2.4] and by $\bar{f}_{k,j}$ its normalization $f_{k,j}/E_{k,j}^{\frac{1}{2}}$, where $E_{k,j} \coloneqq \mathbf{E}(T_{0,b_{k,j}}, \mathbf{B}_{6\sqrt{m}})$ as in Section 3.3.1.

Observe next that by our definition of the endpoints $b_{k,j}$, for each fixed j we have

$$E_{k,j} \overset{k\to\infty}{\longrightarrow} \mathbf{E}(C_j, \mathbf{B}_{6\sqrt{m}}) = \varepsilon_j^2.$$

For every fixed j we then conclude that the sequence of maps $\{\bar{f}_{k,j}\}_k$ are equi-Lipschitz and we can assume they converge uniformly to some map $\bar{f}_j$, up to subsequence (not relabeled). Moreover, this map is actually the limit of $\tilde{f}_{k,j} \coloneqq j f_{k,j} = \varepsilon_j^{-1} f_{k,j}$. Recall however that $\bar{f}_{k,j}$ has a uniform $W^{1,2}$ bound, which is independent of both k and j (unlike $\tilde{f}_{k,j}$, where it clearly depends on j). This bound is thus valid for $\bar{f}_j$ too and we can assume it converges, up to subsequences, strongly in L^2 to some $W^{1,2}$ map $\bar{f}$. By taking a suitable diagonal sequence, and noting that $C^{-1}E_{k,j} \leq \mathbf{E}(T_{0,b_{k,j}}, \mathbf{B}_{8M}) \leq CE_{k,j}$, the latter can be assumed to be (up to a scalar multiple $\lambda > 0$) the coarse blow-up generated by the sequence $b_{k(j),j}$.

Now [19, Theorem 2.4] guarantees the existence of a compact set $K_{k,j} \subset B_1$ over which the graph of $f_{k,j}$ coincides with the current $T_{0,b_{k,j}}$ and enjoying the estimate $|B_1 \setminus K_{k,j}| \leq Cj^{-2(1+\beta)}$ for some constants C and β. Recall that in the supports of $T_{0,b_{k,j}} \llcorner \overline{\mathbf{B}}_{5\sqrt{m}}$ converge in Hausdorff distance to the support of $C_j \llcorner \overline{\mathbf{B}}_{5\sqrt{m}}$.

Denote by λ_j^a the "anisotropic rescaling map" which maps $(x, y) \in \pi_0 \times \pi_0^\perp$ into (x, jy)), where we assume that π_0 is the plane over which we are considering the graphical approximations $f_{k,j}$ of $T_{0,b_{k,j}}$ (up to a rotation we can indeed assume that the plane is a given fixed one). Now, $\mathbf{G}_{\tilde{f}_{k,j}} \llcorner K_{k,j} \times \pi_0^\perp = (\lambda_j^a)_\sharp T_{0,b_{k,j}} \llcorner K_{k,j} \times \pi_0^\perp$. On the other hand, for each fixed j, the currents $(\lambda_j^a)_\sharp T_{0,b_{k,j}}$ converge to the current $(\lambda_j^a)_\sharp C_j$ (the convergence is in the sense of currents, but it also implies the local Hausdorff convergence of the supports, given that j is fixed). Let K_j be the Hausdorff limit as $k \to \infty$ of the compact sets $K_{k,j}$. By the uniform convergence of the functions $\bar{f}_{k,j}$ to $\bar{f}_j$ (as $k \to \infty$, with j fixed) it is easy to see that $\mathbf{G}_{\bar{f}_j} \llcorner K_j \times \pi_0^\perp = (\lambda_j^a)_\sharp C_j \llcorner K_j \times \pi_0^\perp$.

Next, observe that $(\lambda_j^a)_\sharp C_j$ is still a cone. Thus $\bar{f}_j$ coincides with a 1-homogeneous function over K_j. Observe also that $|K_j| \geq \limsup_k |K_{k,j}|$ and therefore $|B_1 \setminus K_j| \leq C j^{-2(1+\beta)}$. Since $|B_1 \setminus K_j| \downarrow 0$ it is easy to conclude that $\bar{f}$, which is the L^2 limit of $\bar{f}_j$, must in fact be 1-homogeneous.

Having concluded that the coarse blow up $\bar{f}$ is 1-homogeneous, we immediately infer that the average-free part is 1-homogeneous as well, which means that the fine blow-up is too. This however would be incompatible with the fact that $\mathbf{I}^-(b_{k(j),j}) \geq 1 + \gamma$.

We thus fix now a choice of ε sufficiently small which forces the alternative (b). Recall that the frequency BV bound gives that $|\mathbf{I}^-(b_k) - \mathbf{I}^+(b_k)| \leq C\varepsilon^{\gamma_4}$, which, combined with the fact that $\mathbf{I}^+(b_k) \leq 1 + \gamma$ in turn implies that

$$\mathbf{I}^-(b_k) \leq 1 + \frac{3}{2}\gamma\,, \tag{3.87}$$

once we take ε small enough. We now wish to show that $\left\| \left[\frac{d\mathbf{I}}{dr}\right]_- \right\|_{\mathrm{TV}(]a_k, b_k[)}$ can be made arbitrarily small, by choosing η and ε correspondingly small and k sufficiently large. This would imply that $\mathbf{I}$ has to be below $1 + \frac{7}{4}\gamma$ on all $]a_k, b_k[$ with k sufficiently large, thereby concluding the proof (since all but finitely many elements of the initial blow-up sequence r_k, on which $\mathbf{I}(r_k) \to 1 + 2\gamma$, must in fact be contained in $\mathcal{R}$, while we just showed that in a neighborhood of 0 relative to $\mathcal{R}$ the value of the universal frequency function is strictly below $1 + 2\gamma$). Let $]s_{j(k)}, t_{j(k)}]$ be the interval of flattening containing b_k. Using Lemma 3.8.2 and the BV estimate of Proposition 3.6.2, we already have that the desired estimate

$$\left\| \left[\frac{d\mathbf{I}}{dr}\right]_- \right\|_{\mathrm{TV}(]a_k, s_{j(k)}[)} \leq \eta \qquad \text{if } s_{j(k)} > a_k,$$

provided that ε is again chosen sufficiently small. Note that, even though the estimate is for $\log(\mathbf{I} + 1)$, we know apriori that $\mathbf{I}$ is bounded, so we can invert the log and get a an estimate for $\left\| \left[\frac{d\mathbf{I}}{dr}\right]_- \right\|_{\mathrm{TV}(]a'_k, b_k[)}$ as in (3.44). The only caveat is that the constant C in the right hand side of (3.44) will now depend upon $\|\mathbf{I}\|_\infty$ if we replace the left hand side with

$\left\| \left[\frac{d\mathbf{I}}{dr}\right]_- \right\|_{\mathrm{TV}}$. However, we only need a constant C which is independent of the radii, though it might depend on T.

We therefore set $a'_k := \max\{a_k, s_{j(k)}\}$ and we wish to show that $\left\| \left[\frac{d\mathbf{I}}{dr}\right]_- \right\|_{\mathrm{TV}(]a'_k, b_k[)}$ can be assumed arbitrarily small, provided ε is chosen wisely and k is sufficiently large. We observe that now $]a'_k, b_k[$ is contained in a single interval of flattening, and that the almost monotonicity estimate on the absolutely continuous part of frequency (3.45) gives

$$\left\| \left[\frac{d\mathbf{I}}{dr}\right]_- \right\|_{\mathrm{TV}(]a'_k, b_k[)} \leq C \left(\frac{a'_k}{t_{j(k)}}\right)^{\gamma_4} \boldsymbol{m}_{0,j(k)} .$$

Now, $\boldsymbol{m}_{0,j(k)}$ is at most $\bar{\varepsilon}^2$, and thus, if the ratio $\frac{a'_k}{t_{j(k)}}$ is sufficiently small we reach the desired threshold. We can therefore assume that

$$\frac{a'_k}{t_{j(k)}} \geq \bar{c} > 0$$

for some constant $\bar{c}$. With the latter lower bound at disposal it is simple to see that $\boldsymbol{m}_{0,j(k)}$ can be made arbitrarily small choosing ε small and k large. In fact, if we choose $\varepsilon = \frac{1}{i}$ and $k(i) \uparrow \infty$, we find that $T_{0,b_{k(i)}}$ converges to a flat plane, which in turn shows that $\mathbf{E}(T, \mathbf{B}_{6\sqrt{m}t_{j(k(i))}})$ must converge to 0. □

3.9 Proof of Theorem 3.2.10(v)&(vi)

Now, we prove the last two statements of Theorem 3.2.10.

3.9.1 The case $\mathrm{I}(T, 0) < 2 - \delta_2$

Choose $\alpha \in]\mathrm{I}(T, 0) - 1, 1 - \delta_2[$. Since all coarse and fine blow-ups are $I(T, 0)$-homogeneous, a simple compactness argument yields the following corollary.

(ND) There are $\varepsilon > 0$ and $\rho > 0$ such that, if $r < \rho$ and $\mathbf{E}(T, \mathbf{B}_{6\sqrt{m}\rho}) \leq \varepsilon$, then

$$\int_{\mathbf{B}_{\rho/2} \cap \mathcal{M}_j} |DN_j|^2 \geq 2^{-(m+2\alpha-2)} \int_{\mathbf{B}_\rho \cap \mathcal{M}_j} |DN_j|^2 \tag{3.88}$$

where $]s_j, t_j] \ni \rho$.

From (3.88) we immediately infer that the intervals of flattening cannot be finite. Indeed suppose this is not the case and let J be such that $s_J = 0$. Observe that under this assumption there is a unique flat tangent cone to T: indeed the center manifold $\mathcal{M}_J$ contains the origin and $Q[\![T_0\mathcal{M}_J]\!]$ is the unique tangent cone to T. We thus conclude $\mathbf{E}(T, \mathbf{B}_{6\sqrt{m}r}) \to 0$ as $r \downarrow 0$. In particular (3.88) must hold for all $\rho \leq \bar{\rho}$ for some positive $\bar{\rho}$ and we immediately conclude that there is a positive constant C such that

$$\int_{\mathcal{M}_J \cap \mathbf{B}_\rho} |DN_J|^2 \geq C^{-1} \rho^{m+2\alpha-2} \qquad \forall \rho < \bar{\rho}\,.$$

On the other hand, in light of [22, Remark 3.4] we also have

$$\int_{\mathcal{M}_J \cap \mathbf{B}_\rho} |DN_J|^2 \leq C \boldsymbol{m}_{0,J} \left(\frac{\rho}{t_j}\right)^{m+2-2\delta_2}.$$

This however forces the condition $\alpha - 1 \geq 1 - \delta_2$, which gives a contradiction. There are therefore infinitely many intervals of flattening $]s_j, t_j]$.

Now assume for a contradiction that, up to subsequence (not relabelled), we have

$$\lim_{j \to \infty} \frac{s_j}{t_j} = 0.$$

If $\mathbf{E}(T, \mathbf{B}_{6\sqrt{m}t_j})$ does not converge to 0 as $j \to \infty$, then, up to subsequence, we can assume that T_{0,t_j} converges to a cone C. Clearly, by definition, $\boldsymbol{m}_{0,j} = \mathbf{E}(T, \mathbf{B}_{6\sqrt{m}t_j})$ for j large enough, and moreover $\boldsymbol{m}_{0,j} \to \mathbf{E}(C, \mathbf{B}_{6\sqrt{m}})$. On the other hand, for every fixed $\rho > 0$

sufficiently small, we can pass into the limit in the inequality

$$\mathbf{E}(T_{0,t_j}, \mathbf{B}_\rho) \leq C\rho^{2-2\delta_2} \boldsymbol{m}_{0,j} \,,$$

which is valid for those infinitely many j's such that $\frac{s_j}{t_j} < \rho$, and conclude

$$\mathbf{E}(C, \mathbf{B}_\rho) \leq C\rho^{2-2\delta_2} \mathbf{E}(C, \mathbf{B}_{6\sqrt{m}}) \,,$$

which is impossible because the radial invariance of C guarantees that $\mathbf{E}(C, \mathbf{B}_\rho)$ is constant in ρ.

We have thus concluded that $\mathbf{E}(T, \mathbf{B}_{6\sqrt{m}t_j})$ converges to 0. In particular, so does $\boldsymbol{m}_{0,j}$. We thus conclude that, for every j sufficiently large, the inequality $\mathbf{E}(T, \mathbf{B}_{6\sqrt{m}\rho}) \leq \varepsilon^2$ must be valid for all $\rho \in [s_j, t_j]$. This however can be combined with (3.88) to deduce that

$$\int_{\mathcal{M}_j \cap \mathbf{B}_{s_j}} |DN_j|^2 \geq C^{-1} \left(\frac{s_j}{t_j}\right)^{m+2\alpha-2} \int_{\mathcal{M}_j \cap \mathbf{B}_{t_j}} |DN_j|^2 \,.$$

On the other hand using [21, Proposition 3.4] we immediately get

$$\int_{\mathcal{M}_j \cap \mathbf{B}_{t_j}} |DN_j|^2 \geq C^{-1} \boldsymbol{m}_{0,j} \,.$$

In particular we conclude

$$\int_{\mathcal{M}_j \cap \mathbf{B}_{s_j}} |DN_j|^2 \geq C^{-1} \left(\frac{s_j}{t_j}\right)^{m+2\alpha-2} \boldsymbol{m}_{0,j} \,.$$

But, as for the case already discussed above, this is at odds with the reverse inequality

$$\int_{\mathcal{M}_j \cap \mathbf{B}_{s_j}} |DN_j|^2 \leq C \left(\frac{s_j}{t_j}\right)^{m+2-2\delta_2} \boldsymbol{m}_{0,j}$$

when $\frac{s_j}{t_j}$ is allowed to become too small.

3.9.2 The case $\mathrm{I}(T,0) > 2 - \delta_2$

In this case we fix $\alpha \in]1-\delta_2, \mathrm{I}(T,0)-1[$. Note that in this case we know that the intervals of flattening cover a neighborhood of 0 and thus we can infer, again using the compactness and the fact that fine blow-ups are all $\mathrm{I}(T,0)$-homogeneous, the following decay lemma:

(D) There is $\rho > 0$ such that, if $r < \rho$, then

$$\int_{\mathbf{B}_{\rho/2}\cap\mathcal{M}_|} |DN_j|^2 \leq 2^{-(m+2\alpha-2)} \int_{\mathbf{B}_\rho\cap\mathcal{M}_j} |DN_j|^2 \tag{3.89}$$

when $]s_j, t_j] \ni \rho$.

This immediately implies that, if the intervals of flattening are infinitely many, then they must satisfy

$$\liminf_j \frac{s_j}{t_j} > 0\,.$$

To see this, we in fact argue by contradiction as above, using this time [21, Proposition 3.4], to infer that

$$\int_{\mathcal{M}_j\cap\mathbf{B}_{s_j}} |DN_j|^2 \geq C^{-1} \left(\frac{s_j}{t_j}\right)^{m+2-2\delta_2} \boldsymbol{m}_{0,j}\,, \tag{3.90}$$

while iterating (D) we instead would get

$$\int_{\mathcal{M}_j\cap\mathbf{B}_{s_j}} |DN_j|^2 \leq C \left(\frac{s_j}{t_j}\right)^{m+2\alpha-2} \boldsymbol{m}_{0,j}\,,$$

which this time is a contradiction because it would force $\alpha - 1 \leq 1 - \delta_2$ if $\frac{s_j}{t_j}$ is allowed to become too small, which does not hold.

We can now argue as in the proof of Proposition 3.7.1 to obtain, for every fixed κ large enough and every k sufficiently large (depending on κ), a decay of type

$$\mathbf{E}(T, \mathbf{B}_{6\sqrt{m}s_{k+\kappa}}) \leq C \left(\frac{s_{k+\kappa}}{t_k}\right)^{2\alpha} \mathbf{E}(T, \mathbf{B}_{6\sqrt{m}t_k}) + Ct_k^2\,.$$

It is not difficult to see that, if κ is chosen large enough, an iteration of this inequality

(combined with the information that $\liminf \frac{s_j}{t_j} > 0$) gives a decay of type

$$\mathbf{E}(T, \mathbf{B}_{6\sqrt{m}r}) \leq Cr^{2\beta} \tag{3.91}$$

for every $\beta < \alpha$. In particular we can choose $\beta > 2 - \delta_2$, and therefore conclude that, for a sufficiently large j, we must have $\boldsymbol{m}_{0,j} = \bar{\varepsilon}^2 t_j^{2-2\delta_2}$. But then (3.91) would imply

$$\mathbf{E}(T, \mathbf{B}_{6\sqrt{m}r}) \leq C s_j^{2\beta-(2-2\delta_2)} \left(\frac{s_j}{t_j}\right)^{2-2\delta_2} \boldsymbol{m}_{0,j} \leq C s_j^{\beta+\delta_2} \left(\frac{s_j}{t_j}\right)^{2-2\delta_2} \boldsymbol{m}_{0,j}\,. \tag{3.92}$$

But of course the latter is at odds with (3.90) when s_j is sufficiently small. This reaches a contradiction and thus shows that there could not be infinitely many intervals of flattening.

3.10 Rectifiability of points with $\mathrm{I}(T, \cdot) > 1$

We are now in a position to use the results of the preceding sections to begin the proof of Theorem 3.1.1. When $\mathrm{I}(T, p) > 1$, the tangent cone to p is a unique plane and that T has an order of contact with it which is at least $\min\{\mathrm{I}(T, p), 2\}$. In particular at any point $p \in \mathfrak{F}(T)$ where the tangent cone might be non-unique, the value of $\mathrm{I}(T, p)$ is necessarily 1. In the remainder of this chapter, we prove Theorem 3.2.11.

A fundamental tool to prove Theorem 3.2.11 is the technique developed by Naber and Valtorta in [42] to tackle the rectifiability of the singular set of harmonic maps between manifolds. In [16], De Lellis, Marchese, Spadaro, & Valtorta, showed that these techniques can be adapted to prove the $(m-2)$-rectifiability of the singular set of multiple-valued Dir-minimizing functions. The latter are the functions pioneered by Almgren in his big regularity paper [3] in order to study the "linearization" of the area functional for area-minimizing currents locally around flat singularities. In the remainder of this chapter we combine the results and estimates of the preceding sections with those of [21, 22], allowing us to suitably adapt the computations and arguments contained within [16], leading to the

rectifiability of $\mathfrak{F}_{Q,>1}(T)$. This is more transparent when proving $(m-2)$-rectifiability for the portion of points of $\mathfrak{F}_{Q,>1}(T)$ at which the singularity degree is above $2-\delta$ for a suitable small threshold $\delta > 0$, locally giving rise to a single graphical approximation for T which is suitably close to being a multiple-valued Dir-minimizer and is defined on a single *center manifold* domain that passes through all other such nearby singularities.

However, in general, we cannot necessarily hope for such a convenient graphical approximation for T, due to the presence of ever-changing graphical approximations for T locally around the points $p \in \mathfrak{F}_{Q,>1}(T)$ with $\mathrm{I}(T,p) < 2-\delta$, with corresponding domains that do not necessarily pass through the nearby points in $\mathfrak{F}_{Q,>1}(T)$. In this chapter, we will only discuss the details of the proof for the points with singularity degree above $2-\delta$. This will be followed by simply an outline of how to treat the points p with singularity degree $\mathrm{I}(T,p) \in (1, 2-\delta)$; we refer the reader to [17] for the omitted details.

It is worth pointing out that in our arguments, we subdivide $\mathfrak{F}_{Q,>1}(T)$ into countably many pieces (depending on the scale at which T is sufficiently close to an m-dimensional plane with multiplicity Q, and the decay rate towards this plane for the rescalings of T around a given point, which is determined by $\mathrm{I}(T,\cdot)$. Each of these pieces further has locally finite $(m-2)$-dimensional upper Minkowski content, but the subdivision prevents us from making the same conclusion for the entirety of $\mathfrak{F}_{Q,>1}(T)$ in Theorem 3.2.11.

3.11 Preliminaries and main results

We refer the reader to Section 3.2 to recall the definition of intervals of flattening, fine blow-ups, singularity degree and all associated terminology and notation.

We recall the following key result, which is a combination of Theorem 3.2.9 and Theorem 3.2.10, both of which were proved in the preceding sections.

Theorem 3.11.1 (Uniqueness of the frequency value)**.** *Assume that T satisfies Assumption 3.2.1 and that $x \in \mathrm{Sing}_{Q,f}(T)$. Then $\mathrm{I}(T,x) \geq 1$ and $\mathcal{F}(T,x) = \{\mathrm{I}(T,x)\}$. Moreover:*

(i) if $\mathrm{I}(T,x) > 1$*, then there is a unique flat tangent cone* π_0 *and* $T_{x,r}$ *converges to it polynomially fast;*

(ii) if $\mathrm{I}(T,x) > 2-\delta_2$*, then there are finitely many intervals of flattening at* x *(and in particular, one center manifold which passes through* x*).*

More precisely, the polynomial decay in Theorem 3.11.1 can be stated in the following way, which will come in useful later, cf. Proposition 3.7.2.

Proposition 3.11.2. *Let* T *be as in Theorem 3.11.1 and let* $p \in \mathfrak{F}_Q(T)$ *with* $\mathrm{I}(T,p) > 1$*. For any* $0 < \mu < \min\{\mathrm{I}(T,p)-1, 1-\delta_2\}$*, there exists* $C(\mathrm{I}(T,p), m, n, Q, \mu) > 0$ *and* $\tau_0(T,p) > 0$ *such that for every* $r < s < \tau_0$ *we have*

$$\mathbf{E}(T, \mathbf{B}_r(p)) \leq C \left(\frac{r}{s}\right)^{2\mu} \max\{\mathbf{E}(T, \mathbf{B}_s(p)), \bar{\varepsilon}^2 s^{2-2\delta_2}\}\,. \tag{3.93}$$

3.11.1 First subdivision

We are now ready to subdivide Theorem 3.2.11 into two main parts, based on the singularity degree.

Theorem 3.11.3. *Let* T *be as in Theorem 3.2.11 and* $Q \in \mathbb{N} \setminus \{0,1\}$*. Then the set*

$$\mathfrak{F}^h_Q := \{p \in \mathfrak{F}_Q(T) : \mathrm{I}(T,p) \geq 2-\delta_2\} \tag{3.94}$$

is $m-2$*-rectifiable.*

Theorem 3.11.4. *Let* T *be as in Theorem 3.2.11,* $Q \in \mathbb{N} \setminus \{0,1\}$*. Then*

$$\mathfrak{F}^l_Q := \{p \in \mathfrak{F}_Q(T) : 1 < \mathrm{I}(T,p) < 2-\delta_2\} \tag{3.95}$$

is $m-2$*-rectifiable.*

Here, we will only only provide full details the proof of Theorem 3.11.3, where we are able to locally find a single center manifold passing through the nearby points in $\mathfrak{F}_Q^h$.

The proof of Theorem 3.11.4 begins in Section 3.19, but we mostly provide merely statements of the required intermediate results, which are all in analogue with their counterparts from the proof of Theorem 3.11.3. We refer the reader to [17, Part 2] for more details of proof of Theorem 3.11.4.

3.12 Proof of Theorem 3.11.3: reduction to a single center manifold

3.12.1 Choice of δ_3, second subdivision and new center manifold

Suppose that T is as in Assumption 3.2.1 and suppose that $x \in \mathfrak{F}_Q^h$. We start by introducing a parameter δ_3 which is slightly larger than δ_2 as prescribed in [21, Assumption 1.8], while the remaining parameter β_2 will obey the same relation $\beta_2 = 4\delta_2$. The choice of δ_2 within [21, Assumption 1.8] is engineered so that a finite number of strict inequalities involving the dimension m and the parameters γ_1 and β_2 hold. These inequalities are then used to show that the estimates in [21, Theorem 1.17 & Theorem 2.4] hold for the positive exponents κ and γ_2 therein, which determine the regularity of the center manifold and the corresponding normal approximation. Decreasing δ_3 to δ_2 will then just make these exponents smaller. Likewise, [21, Proposition 3.4, Proposition 3.5, Proposition 3.6 & Proposition 3.7] will hold with slightly changed values of the constants involved in the estimates.

Now choose μ with the property that $1 - \delta_3 < \mu < 1 - \delta_2 \leq \mathrm{I}(T, x) - 1$ and let us invoke Proposition 3.11.2 for this choice of μ. Observe in particular that for every point $x \in \mathfrak{F}_Q^h$ the decay (3.93) holds with a constant C which is now fixed for all radii $r < r_0(x)$. In particular, for a small positive constant $\tilde{\varepsilon}$, to be specified later, $\mathfrak{F}_Q^h$ can be subdivided into a countable

union of sets $\mathbf{S}_j$, defined by

$$\mathbf{S}_j := \{p \in \mathfrak{F}^h_Q : \mathbf{E}(T, \mathbf{B}_{j^{-1}}(p)) \leq \tilde{\varepsilon}^2 \text{ and } (3.93) \text{ holds in } \mathbf{B}_r(p) \ \forall r < j^{-1}\} \tag{3.96}$$

Clearly, we have $\mathbf{S}_j \subset \mathbf{S}_{j+1}$, and thus Theorem 3.11.3 is reduced to proving $(m-2)$-rectifiability for each $\mathbf{S}_j$ with j large enough. This will be accomplished in the following statement.

Theorem 3.12.1. *There exists $\tilde{\varepsilon}(Q, m, n) > 0$ such that the following holds. Suppose that T is as in Assumption 3.2.1, let j be sufficiently large and let $p \in \mathbf{S}_j$. Let $\delta_3 > 0$ be as described above, set $r_0 := \frac{1}{6\sqrt{m}j}$ and define $\boldsymbol{m}_0 := \max\{\mathbf{E}(T_{p,r_0}, \mathbf{B}_{6\sqrt{m}}), \bar{\varepsilon} r_0^{2-2\delta_2}\} \leq \tilde{\varepsilon}^2$. Then*

(i) T_{p,r_0} satisfies the assumptions of the statements in [21] and [22] where the center manifold $\mathcal{M}_0$ is constructed using the parameter δ_3 in place of δ_2, and $\boldsymbol{m}_0$ as defined above.

(ii) The rescaling $\iota_{p,r_0}(\mathbf{S}_j) \cap \overline{\mathbf{B}}_{6\sqrt{m}}$ is contained in $\mathcal{M}_0$ and therefore so is its closure $\mathbf{S}$;

(iii) $\mathbf{S}$ is $(m-2)$-rectifiable and has the $(m-2)$-dimensional Minkowski content bound

$$|\mathbf{B}_r(\mathbf{S})| \leq Cr^2, \tag{3.97}$$

for a positive constant $C = C(m, n, \bar{n}, T, j)$.

The aim of this section is to prove the conclusion (i) and (ii) in Theorem 3.12.1. But indeed we will prove a stronger form of (ii), namely that $\mathbf{S}$ belongs to what is called, in [21], the "contact set" (denoted by $\mathbf{\Phi}(\mathbf{\Gamma})$; see [21]) between $\mathcal{M}_0$ and T_{p,r_0}. Points in the latter set enjoy better properties than a generic point in $\mathcal{M}_0 \cap \mathrm{spt}(T_{p,r_0})$. More precisely, the $\mathcal{M}$-normal approximation N is "almost" Dir-minimizing at all scales around such points, and as a consequence a series of important integral identities are valid up to small errors. These facts play a pivotal role in the rest of the paper, dedicated to prove (iii), and therefore we will record them below in Lemma 3.12.3. In fact we will need a suitable refinement, namely

that these identities are valid at any point $x \in \mathcal{M}_0$ and at any scale that is larger than a small scale comparable to the distance of x to $\mathbf{S}$, cf. Lemma 3.12.6.

Let us now show how Proposition 3.11.2 yields the result of Theorem 3.12.1(i).

3.12.2 Proof of Theorem 3.12.1(i) and of $\mathbf{S} \subset \mathbf{\Phi}(\mathbf{\Gamma}) \subset \mathcal{M}_0$

Let $p \in \mathfrak{F}^h_Q$ and let $r_0 := \frac{1}{6\sqrt{m}j}$. First of all observe that, by Proposition 3.11.2, for every $\eta > 0$, if j is chosen large enough (possibly larger than that corresponding to the scale where Proposition 3.11.2 applies), then $\mathbf{E}(T_{p,r_0}, \mathbf{B}_{6\sqrt{m}r}) < \eta r^{2-2\delta_2}$ for every $r \leq 1$. In particular, the scale 1 satisfies whichever smallness condition might be required by the modifications of the statements in [21, 22] for T_{p,r_0} with δ_3 in place of δ_2. Let now π_0 be the plane which is used as reference to apply the algorithm in [21, Section 1] and construct the center manifold $\mathcal{M}_0$. Moreover, consider any cube $L \in \mathscr{C}$, for the family $\mathscr{C}$ of dyadic cubes of π_0 defined in [21, Section 1] and let c_s be the constant in [22, (2.5)]. Obviously, if η is chosen sufficiently small, $\mathbf{E}(T_{p,r_0}, \mathbf{B}_L) < C_e \boldsymbol{m}_0 \ell(L)^{2-2\delta_3}$ and in particular L cannot belong to $\mathscr{W}^e$. On the other hand, by [21, Proposition 3.1], it also cannot belong to $\mathscr{W}^h$. Thus, by [22, Proposition 2.2(iii)], L cannot belong to $\mathscr{W}^n$ either. It follows therefore that the condition [22, (Stop), Section 2.1] is never met, and hence $s_0 = 0$. The above choice of η in turn determines how large j must be. This therefore implies that for j sufficiently large and $p \in \mathbf{S}_j$, the origin must lie in $\mathbf{\Phi}(\mathbf{\Gamma}) \subset \mathcal{M}_0$ for T_{p,r_0}.

Fix now a point $q \in \mathbf{S}$ and consider its projection $x = \mathbf{p}_{\pi_0}(q)$. The very same argument implies immediately that x cannot belong to any $L \in \mathscr{W}$ and it is, therefore, a subset of the set $\mathbf{\Gamma}$ of [21, Definition 1.17].

In fact we want to record a stronger consequence of the decay of the excess of Proposition 3.11.2.

Corollary 3.12.2. *Let T, p be as in Theorem 3.12.1 and assume that j is large enough so that the conclusion (i) of Theorem 3.12.1 applies. For every fixed $\bar{c}_s > 0$, the following is true, after further increasing j if necessary. For every $q \in \mathbf{S}$ and for every $r \leq 1$, every cube*

L which intersects $B_r(q, \pi_0)$ satisfies $\ell(L) \leq \bar{c}_s r$.

3.12.3 Frequency function, almost-monotonicity and frequency lower bound

From now on we will fix $\bar{c}_s$ arbitrarily (to be determined later; cf. Lemma 3.12.6) and assume that $j \in \mathbb{N}$ is fixed large enough such that the conclusions of Corollary 3.12.2 hold, that $0 \in \mathbf{S}_j$, and we will use the notation $\mathcal{M}$ and N for the center manifold $\mathcal{M}_0$ and the normal approximation N_0 (for the current T_{0,r_0}, with $r_0 = \frac{1}{6\sqrt{m}j}$, on the interval of flattening $]0, 1]$). In light of the above reasoning, we will henceforth work under the assumption that Theorem 3.12.1(i) applies and that $\mathbf{S} \subset \mathbf{\Phi}(\mathbf{\Gamma}) \subset \mathcal{M}$.

We can now introduce the pivotal object of our analysis, the (regularized) frequency function for any given $\mathcal{M}$-normal approximation N of T as in Assumption 3.2.1. Let ϕ be defined as above and let $d : \mathcal{M} \times \mathcal{M} \to \mathbb{R}^+$ be the geodesic distance on $\mathcal{M}$. We will repeatedly use the following properties of d, which are consequences of the $C^{3,\kappa}$-estimates on the center manifold $\mathcal{M}$ (we refer the reader to [21] and [11]):

(i) $d(x, y) = |x - y| + O\left(\boldsymbol{m}_0^{\frac{1}{2}}|x - y|^2\right)$,

(ii) $|\nabla_y d(x, y)| = 1 + O\left(\boldsymbol{m}_0^{\frac{1}{2}} d(x, y)\right)$,

(iii) $\nabla_y^2(d^2) = g + O(\boldsymbol{m}_0 d)$, where g is the metric induced on $\mathcal{M}$ by the Euclidean ambient metric.

We then introduce the following functions:

$$\begin{aligned}
\mathbf{D}(x, r) &\coloneqq \int_{\mathcal{M}} |DN|^2 \phi\left(\frac{d(y, x)}{r}\right) \, \mathrm{d}y \,, \\
\mathbf{H}(x, r) &\coloneqq -\int_{\mathcal{M}} \frac{|\nabla_y d(y, x)|^2}{d(y, x)} |N|^2 \phi'\left(\frac{d(y, x)}{r}\right) \, \mathrm{d}y \\
\mathbf{I}(x, r) &\coloneqq \frac{r\mathbf{D}(x, r)}{\mathbf{H}(x, r)} \,.
\end{aligned}$$

Note that we will often omit the implicit dependency on N of $\mathbf{I}$ and related quantities, since we are considering one single fixed normal approximation N throughout. When it is necessary to highlight such dependence, we will write $\mathbf{I}_N$, $\mathbf{D}_N$ and $\mathbf{H}_N$. We refer the reader to [22] or [11] for more details on the above quantities. Moreover, since in practically all the computations the derivative of d is taken in the variable which is the same as the integration variable, in all such cases we will write instead ∇d.

Analogously to Section 3.6.1, we define

$$\begin{aligned}
\mathbf{E}(x,r) &:= -\frac{1}{r}\int_{\mathcal{M}} \phi'\left(\frac{d(x,y)}{r}\right) \sum_i N_i(y)\cdot DN_i(y)\nabla d(x,y)\, \mathrm{d}y\,,\\
\mathbf{G}(x,r) &:= -\frac{1}{r^2}\int_{\mathcal{M}} \phi'\left(\frac{d(x,y)}{r}\right) \frac{d(x,y)}{|\nabla d(x,y)|^2} \sum_i |DN_i(y)\cdot \nabla d(x,y)|^2\, \mathrm{d}y\,,\\
\mathbf{\Sigma}(x,r) &:= \int_{\mathcal{M}} \phi\left(\frac{d(x,y)}{r}\right) |N(y)|^2\, \mathrm{d}y\,.
\end{aligned}$$

The first key point is that the variational identities, which we recall from Lemma 3.6.4, but recentered around $x \in \mathbf{\Phi}(\mathbf{\Gamma})$.

Lemma 3.12.3. *There exists $\gamma_4(m,n,Q) > 0$ sufficiently small and a constant $C(m,n,Q) > 0$ such that the following holds. Suppose that the conclusions of the previous sections apply to T_{0,r_0}, $\mathcal{M}$ and N and that $\tilde{\varepsilon}$ in Theorem 3.12.1 is sufficiently small. Then for any $x \in \mathbf{\Phi}(\mathbf{\Gamma})$ and any $r \in]0,1]$, we have the following identities*

$$\partial_r \mathbf{D}(x,r) = -\int_{\mathcal{M}} \phi'\left(\frac{d(x,y)}{r}\right)\frac{d(x,y)}{r^2}|DN(y)|^2 \ \ \mathrm{d}y \tag{3.98}$$

$$\partial_r \mathbf{H}(x,r) - \frac{m-1}{r}\mathbf{H}(x,r) = O(\boldsymbol{m}_0)\mathbf{H}(x,r) + 2\mathbf{E}(x,r), \tag{3.99}$$

$$|\mathbf{D}(x,r) - \mathbf{E}(x,r)| \le \sum_{j=1}^{5} |\operatorname{Err}_j^o| \le C\boldsymbol{m}_0^{\gamma_4}\mathbf{D}(x,r)^{1+\gamma_4} + C\boldsymbol{m}_0\mathbf{\Sigma}(x,r), \tag{3.100}$$

$$\begin{aligned}
\left|\partial_r \mathbf{D}(x,r) - (m-2)r^{-1}\mathbf{D}(x,r) - 2\mathbf{G}(x,r)\right| &\le 2\sum_{j=1}^{5} |\operatorname{Err}_j^i| + C\boldsymbol{m}_0\mathbf{D}(x,r) \\
&\le Cr^{-1}\boldsymbol{m}_0^{\gamma_4}\mathbf{D}(x,r)^{1+\gamma_4} + C\boldsymbol{m}_0^{\gamma_4}\mathbf{D}(x,r)^{\gamma_4}\partial_r\mathbf{D}(x,r) + C\boldsymbol{m}_0\mathbf{D}(x,r),
\end{aligned} \tag{3.101}$$

where Err_j^o *and* Err_j^i *are as in* [11, Proposition 9.8, Proposition 9.9].

Observe that the validity of the estimates on all scales and at all points $x \in \mathbf{\Phi}(\mathbf{\Gamma})$ uses the fact that, for any $q \in \mathbf{\Gamma}$, any Whitney cube $L \in \mathscr{W}$ which intersects the disk $B_r(q, \pi_0)$ is no larger than $c_s r$, where c_s is as in [22, (2.5)].

As an immediate consequence, arguing as in Corollary 3.6.5 we obtain that

$$\frac{d}{dr} \log(1 + \mathbf{I}(x, r)) \geq -C \boldsymbol{m}_0^{\gamma_4} \tag{3.102}$$

for $\gamma_4 > 0$ as in Lemma 3.12.3 and $C(m, n, Q) > 0$. In turn we can exploit the latter monotonicity to obtain the following corollary.

Corollary 3.12.4. *Let* T_{0,r_0}, $\mathcal{M}$, N, C, *and* γ_4 *be as in Lemma 3.12.3. Then for any* $x \in \mathbf{\Phi}(\mathbf{\Gamma})$ *we have the following:*

(i) $\mathbf{I}(x, 0) = \lim_{r \downarrow 0} \mathbf{I}(x, r)$ *exists and moreover* $x \mapsto \mathbf{I}(x, 0)$ *is upper semicontinuous;*

(ii) The unique tangent cone to T *at* x *is* $Q\llbracket T_x \mathcal{M} \rrbracket$*;*

(iii) x *is a flat singular point and* $\mathbf{I}(x, 0) \geq 2 - \delta_3$.

Proof. Points (i) and (ii) are obvious from the monotonicity of $\log(1 + \mathbf{I}(x, r))$. As for point (iii) a simple unique continuation argument using the fact that the singular set of T has dimension 2 shows that no point $x \in \mathbf{\Phi}(\mathbf{\Gamma})$ can be a regular point because otherwise in a neighborhood of it the current would just coincide with the center manifold. The estimate on $\mathbf{I}(x, 0)$ is instead achieved using the excess decay at x. □

We also observe that an obvious compactness argument gives a uniform bound for the frequency $\mathbf{I}(x, 4)$ as x varies in $\mathbf{B}_1 \cap \mathcal{M}$. In particular, given the validity of the monotonicity of $\log(1 + \mathbf{I})$, we can infer the following upper bound

$$\mathbf{I}(x, r) \leq \Lambda \qquad \forall x \in \mathbf{S}\,, \forall r \in]0, 4]\,. \tag{3.103}$$

On the other knowing that $\lim_{r\downarrow 0}\mathbf{I}(x,r)\geq 2-\delta_3$ for all $x\in\mathbf{\Phi}(\mathbf{\Gamma})$, it just suffices to choose $\mathbf{m}_0$ to be sufficiently small to gain a positive lower bound on $\mathbf{I}(x,r)$ at all scales.

Corollary 3.12.5. *Let T_{0,r_0}, $\mathcal{M}$, N, C, and γ_4 be as in Theorem 3.12.1 and let $\tilde{\varepsilon}$ therein be sufficiently small. Then there is a constant $\Lambda>0$ (depending on T) such that*

$$\frac{3}{2}\leq\mathbf{I}(x,r)\leq\Lambda\qquad\forall x\in\mathbf{\Phi}(\mathbf{\Gamma}),\ \forall r\in]0,4]\,. \tag{3.104}$$

3.12.4 Almost monotonicity at points x close to S

In general the estimates of Lemma 3.12.3 are not valid at *every* point $x\in\mathcal{M}$ and *every* scale r. As remarked, the condition for their validity at scale r is that $B_r(0,\mathbf{p}_{\pi_0}(x))$ does not intersect cubes of $\mathscr{W}$ which have side length larger than $c_s r$. We may ensure that this holds at all scales $r\in]0,1]$ for any point in $\mathbf{S}$, but cannot hope to achieve this at all scales $r\in]0,1]$ at points that are not in the contact set $\mathbf{\Phi}(\mathbf{\Gamma})$. Unfortunately, we need to consider points outside of the contact set when taking spatial variations. However, for an arbitrary small constant $\eta>0$, we can leverage Corollary 3.12.2 to establish the validity of the desired estimates at any given point $x\in\mathcal{M}\cap\mathbf{B}_1$ and every scale r larger than $\eta d(x,\mathbf{S})$ (where, as usual $d(x,\mathbf{S})=\min\{d(x,y):y\in\mathbf{S}\}$).

Lemma 3.12.6. *Suppose that the conclusions of the previous sections apply to T_{0,r_0}, $\mathcal{M}$ and N with $r_0=\frac{1}{6\sqrt{m}j}$ and that $\tilde{\varepsilon}$ in Theorem 3.12.1 is sufficiently small. For every fixed $\eta>0$, there exists a choice of $\bar{c}_s$ in Corollary 3.12.2 such that if j is larger than the corresponding threshold therein, then all the estimates of Lemma 3.12.3 (and hence that of (3.102) and Corollary 3.12.5) hold for every $x\in\mathcal{M}$ and every $r\in]\eta d(x,\mathbf{S}),1]$ by possibly adjusting the constants.*

The proof of Lemma 3.12.6 is entirely analogous to that of the fact that $\mathbf{S}\subset\mathbf{\Phi}(\mathbf{\Gamma})$, only taking $r\in]\eta d(x,\mathbf{S}),1]$ and observing that given $c_s>0$ as in [22, (2.5)], any cube $L\in\mathscr{C}$ with $L\cap B_r(q,\pi_0)\neq\emptyset$ and $\ell(L)>c_s r>\bar{c}_s\eta d(x,\mathbf{S})$ would in turn satisfy $L\cap B_{d(x,\mathbf{S})+r}(\tilde{q},\pi_0)\neq\emptyset$

for some $\mathbf{S} \ni \tilde{x} = \mathbf{p}_{\pi_0}(\tilde{q})$, contradicting the conclusion of Corollary 3.12.2 for $\bar{c}_s = \frac{c_s}{1+\frac{1}{\eta}}$.

3.12.5 Main reduction

Using an obvious covering argument and up to translations and dilations, we can always substitute T with T_{0,r_0}, so we can now summarize a set of assumptions which we will henceforth make until otherwise stated.

Assumption 3.12.7. For some fixed (yet as small as desired) positive constants ε_4, η, and some fixed $I_0 = 2 - \delta_3$ (yet as close to $2 - \delta_2$ as desired) the following holds.

(i) T satisfies Assumption 3.2.1, Assumption 3.2.2 and $\mathrm{I}(T, 0) \geq I_0$.

(ii) There is one interval of flattening $]0, 1]$ around 0 with corresponding $\boldsymbol{m}_{0,0} \equiv \boldsymbol{m}_0 \leq \varepsilon_4^2$.

(iii) If $\mathcal{M}$ is the corresponding center manifold, then $\mathbf{S} := \{x \in \mathfrak{F}_Q(T) : \mathbf{I}(x, 0) \geq I_0\} \cap \overline{\mathbf{B}}_1$ is a closed subset of $\mathcal{M}$.

(iv) For every $x \in \overline{\mathbf{B}}_1 \cap \mathcal{M}$, the estimates of Lemma 3.12.3, (3.102) and the upper bound on the frequency of Corollary 3.12.5 are valid for all radii $r \in]\eta\, d(x, \mathbf{S}), 1]$ (and hence for all radii $r \in]0, 1]$ when $x \in \mathbf{S}$).

Note that in particular, the choice of ε_4 in turn determines the final choice of $\tilde{\varepsilon}$ in Theorem 3.12.1 and the parameter $\bar{\varepsilon}$ in Assumption 3.2.2.

3.13 Almost-monotonicity and comparability of errors

In this section we establish some further consequences of Lemmas 3.12.3 and 3.12.6, (3.102) and Corollary 3.12.5. The estimates of this section will greatly simplify many subsequent computations.

Lemma 3.13.1. *For any fixed $\eta > 0$ and $\Lambda > 0$ as in (3.103), if ε_4 is chosen sufficiently small, then the following holds for any T as in Assumption 3.12.7, every $x \in \mathcal{M} \cap \mathbf{B}_1$ and any $\rho, r \in]\eta d(x, \mathbf{S}), 4]$.*

$$C^{-1} \leq \mathbf{I}(x,r) \leq \Lambda \tag{3.105}$$

$$\Lambda^{-1} r \mathbf{D}(x,r) \leq \mathbf{H}(x,r) \leq C r \mathbf{D}(x,r) \tag{3.106}$$

$$\mathbf{\Sigma}(x,r) \leq C r^2 \mathbf{D}(x,r) \tag{3.107}$$

$$\mathbf{E}(x,r) \leq C \mathbf{D}(x,r) \tag{3.108}$$

$$\rho^{1-m} \mathbf{H}(x,\rho) = r^{1-m} \mathbf{H}(x,r) \exp\left(-C \int_\rho^r \mathbf{I}(x,s) \frac{\mathrm{d}s}{s} - O(\boldsymbol{m}_0)(r-\rho) \right) \tag{3.109}$$

$$\mathbf{H}(x,r) \leq C \mathbf{H}(x, \tfrac{r}{4}) \tag{3.110}$$

$$\mathbf{H}(x,r) \leq C r^{m+3-2\delta_2} \tag{3.111}$$

$$\mathbf{G}(x,r) \leq C r^{-1} \mathbf{D}(x,r) \tag{3.112}$$

$$|\partial_r \mathbf{D}(x,r)| \leq C r^{-1} \mathbf{D}(x,r) \tag{3.113}$$

$$|\partial_r \mathbf{H}(x,r)| \leq C \mathbf{D}(x,r)\,, \tag{3.114}$$

where the constant C depends on I_0, Λ, and η, but not on ε_4. In particular:

$$|\mathbf{D}(x,r) - \mathbf{E}(x,r)| \leq C \boldsymbol{m}_0^{\gamma_4} r^{\gamma_4} \mathbf{D}(x,r) \tag{3.115}$$

$$|\partial_r \mathbf{D}(x,r) - (m-2) r^{-1} \mathbf{D}(x,r) - 2\mathbf{G}(x,r)| \leq C \boldsymbol{m}_0^{\gamma_4} r^{\gamma_4 - 1} \mathbf{D}(x,r) \tag{3.116}$$

$$\partial_r \mathbf{I}(x,r) \geq -C \boldsymbol{m}_0^{\gamma_4} r^{\gamma_4 - 1}\,. \tag{3.117}$$

From now we will work under the assumptions that the parameters allow for the conclusions of Lemma 3.13.1 to hold.

Assumption 3.13.2. T, I_0, δ_2, δ_3, ε_4, Λ and η are as in Assumption 3.12.7. In addition the parameter ε_4 is small enough, compared to I_0, Λ, and η, so that the estimates of Lemma 3.13.1 are valid.

3.13.1 Proof of Lemma 3.13.1

We begin with (3.105). The upper bound was already established. To achieve the lower bound, we proceed via contradiction, following a similar argument to that in [49].

Indeed, suppose that the lower bound in (3.105) fails. Then, one can find a sequence of currents T_k satisfying Assumption 3.12.7 with vanishing $\varepsilon_4 = \varepsilon_{4,k} \downarrow 0$ and extract a sequence of points $x_k \in \mathcal{M}_k \cap \mathbf{B}_1$ with corresponding normal approximations N_k and scales $r_k \in]\eta d(x, \mathbf{S}_{T_k}), 4]$ such that

$$\mathbf{I}_{N_k}(x_k, r_k) \longrightarrow 0.$$

In particular, this means that there exist points $y_k \in \mathbf{S}_{T_k}$ with $d(y_k, x_k) < \frac{1}{\eta} r_k$. Recentering around x_k, rescaling by $\bar{r}_k$ as in Section 3.2.1 and taking a normalized limit, we conclude that (up to subsequence) there exists a limiting Dir-minimizer $u : \pi_\infty \supset B_1 \to \mathcal{A}_Q(\pi_\infty^\perp)$ with

(i) $I_u(0, 1) = 0$,

(ii) $u(\bar{y}) = Q[\![0]\!]$ and $I_u(\bar{y}, 0) \geq I_0 = 2 - \delta_3$,

where $\bar{y}$ is the subsequential limit of $\frac{y_k - x_k}{\bar{r}_k}$. However, (i) implies that $D_u(0, 1) = 0$, which, combined with (ii) tells us that $u \equiv Q[\![0]\!]$ on B_1. This, however, contradicts the lower frequency bound in (ii).

The inequalities (3.106) are clearly just an alternative way of writing (3.105), while the estimate (3.108) is merely a consequence of (3.100) and (3.107) combined.

To obtain the equation (3.109), we first observe that (3.99) and (3.100) together yield the estimate

$$\begin{aligned} \partial_r \left(\log r^{1-m} \mathbf{H}(x, r) \right) &= \frac{\partial_r \mathbf{H}(x, r)}{\mathbf{H}(x, r)} - \frac{m - 1}{r} \\ &\leq \frac{2}{r} \mathbf{I}(x, r) + C \boldsymbol{m}_0 + \frac{2\mathbf{E}(x, r)}{\mathbf{H}(x, r)}. \end{aligned}$$

We then apply (3.108) and integrate between scales ρ and r to conclude. Setting $\rho = \frac{r}{4}$ and invoking the upper frequency bound in (3.105) clearly further implies (3.110).

To see that the L^2-height decay (3.111) holds, one may simply cover

$$\mathbf{p}_{\pi_0}\left((\mathbf{B}_r(x) \setminus \overline{\mathbf{B}}_{r/2}(x)) \cap \mathcal{M}\right)$$

by a family of disjoint Whitney cubes L with $\ell(L) \leq 2r$, and apply the estimate [21, Theorem 2.4 (2.3)] on each Whitney region $\mathcal{L}$ for each of these cubes L (see [22, Remark 3.4] for the corresponding estimate on $\mathbf{D}(x,r)$).

The inequality (3.112) follows immediately from the definition of $\mathbf{G}$, combined with the observation that $d(x,y) \leq r$ whenever $\phi'\left(\frac{d(x,y)}{r}\right) > 0$. Similarly, the bound (3.113) follows directly from the identity (3.98) and again the fact that $d(x,y) \leq r$.

Finally, the estimate (3.114) follows from (3.99) and the upper bounds in (3.106) and (3.108). The estimates (3.115)-(3.117) are an obvious consequence of the preceding estimates (3.105)-(3.114), the estimates in Lemma 3.12.3 and (3.102).

3.14 Spatial variations

In this section we will control how much N deviates from being homogeneous on average between two scales, in terms of the frequency pinching. The latter is defined as follows:

Definition 3.14.1. *Suppose that T, $\mathcal{M}$ and N are as in Assumption 3.13.2. For $x \in \mathbf{B}_1 \cap \mathcal{M}$ and any $\eta\, d(x,\mathbf{S}) < \rho \leq r \leq 1$, define the frequency pinching $W^r_\rho(x)$ between scales ρ and r by*

$$W^r_\rho(x) := |\mathbf{I}(x,r) - \mathbf{I}(x,\rho)|.$$

We begin with the following important proposition.

Proposition 3.14.2. *Suppose that T, $\mathcal{M}$, N are as in Assumption 3.13.2 and γ_4 is as in Lemma 3.12.3. There exists $C = C(m,n,Q,\Lambda) > 0$ and $\beta = \beta(m,n,Q,\Lambda) > 0$ such that the following estimate holds for every $x \in \mathbf{B}_1 \cap \mathcal{M}$ and for every pair ρ, r with $4\eta\, d(x,\mathbf{S}) < \rho \leq$*

$r < 1$. *If we define*

$$\mathcal{A}^{2r}_{\frac{\rho}{4}}(x) \coloneqq \left(\mathbf{B}_{2r}(x) \setminus \overline{\mathbf{B}}_{\frac{\rho}{4}}(x)\right) \cap \mathcal{M}$$

then

$$\begin{aligned}\int_{\mathcal{A}^{2r}_{\frac{\rho}{4}}(x)} \sum_i \left| DN_i(y) \frac{d(x,y)\nabla d(x,y)}{|\nabla d(x,y)|} - \mathbf{I}(x, d(x,y)) N_i(y) |\nabla d(x,y)| \right|^2 \frac{\mathrm{d}y}{d(x,y)} \\ \leq C\mathbf{H}(x,2r) \left(W^{4r}_{\frac{\rho}{8}}(x) + \boldsymbol{m}_0^{\gamma_4} r^{\gamma_4} \right) \log\left(\frac{16r}{\rho}\right).\end{aligned}$$

Remark 3.14.3. We warn the reader that, even though much of our computations are based on the ones of the work [16], we will use a different convention concerning derivatives of the maps N_i. In [16], the authors use the unusual convention $v \cdot DN_i$ for the directional derivative of the map N_i in direction v (where N_i is regarded as a map taking values into $\mathbb{R}^{m+n}$ and the derivative is understood as the vector consisting of the directional derivatives of each component of N_i). Here we will often stick to the more standard convention $\partial_v N_i$ or $DN_i v$.

We will also require the following control on variations of the frequency in terms of frequency pinching.

Lemma 3.14.4. *Suppose that T, $\mathcal{M}$ and N be as in Assumption 3.13.2 and let γ_4 be as in Lemma 3.12.3. Let $x_1, x_2 \in \mathbf{B}_1 \cap \mathcal{M}$ with $d(x_1, x_2) \leq \frac{r}{4}$, where r is such that $8\eta \max\{d(x_1, \mathbf{S}), d(x_2, \mathbf{S})\} < r \leq 1$. Then there exist $C = C(m, n, Q, \Lambda) > 0$ such that for any $z, y \in [x_1, x_2]$, we have*

$$|\mathbf{I}(y,r) - \mathbf{I}(z,r)| \leq C \left[\left(W^{4r}_{\frac{r}{8}}(x_1) \right)^{\frac{1}{2}} + \left(W^{4r}_{\frac{r}{8}}(x_2) \right)^{\frac{1}{2}} + \boldsymbol{m}_0^{\frac{\gamma_4}{2}} r^{\frac{\gamma_4}{2}} \right] \frac{d(z,y)}{r}.$$

In order to prove the latter, we will also need the following additional variation estimates and identities.

Lemma 3.14.5. *Let T, $\mathcal{M}$ and N be as in Assumption 3.13.2 and let $x \in \mathbf{B}_1 \cap \mathcal{M}$. Let $\eta\, d(x, \mathbf{S}) < \rho < r \leq 1$, and let v be a continuous vector field on $\mathcal{M}$. We have*

$$\partial_v \mathbf{D}(x,r) = -\frac{2}{r}\int \phi'\left(\frac{d(x,y)}{r}\right)\sum_i \partial_{\nu_x} N_i(y)\cdot \partial_v N_i(y)\,\mathrm{d}y + O\left(\boldsymbol{m}_0^{\gamma_4}\right) r^{\gamma_4-1}\mathbf{D}(x,r)$$
$$\partial_v \mathbf{H}(x,r) = -2\sum_i \int_{\mathcal{M}} \frac{|\nabla d(x,y)|^2}{d(x,y)}\phi'\left(\frac{d(x,y)}{r}\right)\langle \partial_v N_i(y), N_i(y)\rangle\,\mathrm{d}y\,.$$

3.14.1 Proof of Proposition 3.14.2

Since the center x here is fixed, we will suppress the dependency on x for $\mathbf{I}$ and all related quantities, for simplicity. By the estimates in Lemma 3.13.1 and the fact that $|\nabla d(x,y)| = 1 + O\big(\boldsymbol{m}_0^{\frac{1}{2}}|y-x|\big)$, we have

$$\begin{aligned}
W_{\frac{\rho}{4}}^{4r}(x) &= \int_{\frac{\rho}{4}}^{4r} \partial_s \mathbf{I}(s)\,\mathrm{d}s = \int_{\frac{\rho}{4}}^{4r} \frac{\partial_s[s\mathbf{D}(s)]}{\mathbf{H}(s)} - \frac{s\mathbf{D}(s)\partial_s\mathbf{H}(s)}{\mathbf{H}(s)^2}\,\mathrm{d}s \\
&\geq 2\int_{\frac{\rho}{4}}^{4r} \frac{s\mathbf{G}(s) - \mathbf{I}(s)\mathbf{D}(s)}{\mathbf{H}(s)}\,\mathrm{d}s - C\boldsymbol{m}_0^{\gamma_4}\int_{\frac{\rho}{4}}^{4r}\left(\frac{\mathbf{D}(s)^{1+\gamma_4} + s\partial_s[\mathbf{D}(s)^{1+\gamma_4}]}{\mathbf{H}(s)} + \mathbf{I}(s)\right)\,\mathrm{d}s \\
&= \underbrace{2\int_{\frac{\rho}{4}}^{4r} \frac{s\mathbf{G}(s) - 2\mathbf{I}(s)\mathbf{E}(s) + s^{-1}\mathbf{I}(s)^2\mathbf{H}(s)}{\mathbf{H}(s)}\,\mathrm{d}s}_{=:I} - C\boldsymbol{m}_0^{\gamma_4}\int_{\frac{\rho}{4}}^{4r}\frac{s^{\gamma_4}\mathbf{I}(s)\mathbf{D}(s)}{\mathbf{H}(s)}\,\mathrm{d}s \\
&\quad - C\boldsymbol{m}_0^{\gamma_4}\int_{\frac{\rho}{4}}^{4r}\left(\frac{\mathbf{D}(s)^{1+\gamma_4} + s\partial_s\left[\mathbf{D}(s)^{1+\gamma_4}\right]}{\mathbf{H}(s)} + \mathbf{I}(s)\right)\,\mathrm{d}s.
\end{aligned}$$

We can rewrite the integral I as

$$\begin{aligned}
I &= \int_{\frac{\rho}{4}}^{4r}\frac{1}{s\mathbf{H}(s)}\int -\phi'\left(\frac{d(x,y)}{s}\right)\frac{1}{d(x,y)}\left(\sum_j \left|DN_j\cdot\frac{d(x,y)\nabla d(x,y)}{|\nabla d(x,y)|}\right|^2\right. \\
&\qquad \left. -2\mathbf{I}(s)\sum_j N_j\cdot\left(DN_j\cdot\frac{d(x,y)\nabla d(x,y)}{|\nabla d(x,y)|}\right) + \mathbf{I}(s)^2|N(y)|^2|\nabla d(x,y)|^2\right)\,\mathrm{d}y\,\mathrm{d}s \\
&= \int_{\frac{\rho}{4}}^{4r}\frac{1}{s\mathbf{H}(s)}\int -\phi'\left(\frac{d(x,y)}{s}\right)\frac{\xi(x,y,s)}{d(x,y)}\,\mathrm{d}y\,\mathrm{d}s,
\end{aligned}$$

where

$$\xi(x,y,s) = \sum_j \left| DN_j \frac{d(x,y)\nabla d(x,y)}{|\nabla d(x,y)|} - \mathbf{I}(s) N_j(y) |\nabla d(x,y)| \right|^2 .$$

Combining this again with the estimates in Lemma 3.13.1, we thus arrive at the inequality

$$\begin{aligned} W^{4r}_{\frac{\rho}{4}}(x) &\geq 2 \int_{\frac{\rho}{4}}^{4r} \frac{1}{s\mathbf{H}(s)} \int_{\mathcal{A}^s_{\frac{s}{2}}(x)} \frac{\xi(x,y,s)}{d(x,y)} \, \mathrm{d}y \, \mathrm{d}s \\ &\quad - C\boldsymbol{m}_0^{\gamma_4} \int_{\frac{\rho}{4}}^{4r} \left(s^{\gamma_4 - 1} \mathbf{I}(s) + \frac{\mathbf{D}(s)^{1+\gamma_4}}{\mathbf{H}(s)} + \frac{s\mathbf{D}(s)^{\gamma_4} \partial_s \mathbf{D}(s)}{\mathbf{H}(s)} \right) \, \mathrm{d}s \\ &\geq 2 \int_{\frac{\rho}{4}}^{4r} \frac{1}{s\mathbf{H}(s)} \int_{\mathcal{A}^s_{\frac{s}{2}}(x)} \frac{\xi(x,y,s)}{d(x,y)} \, \mathrm{d}y \, \mathrm{d}s - C\boldsymbol{m}_0^{\gamma_4}(r^{\gamma_4} - \rho^{\gamma_4}). \end{aligned}$$

Now consider

$$\zeta(x,y) \coloneqq \sum_j \left| DN_j(y) \frac{d(x,y)\nabla d(x,y)}{|\nabla d(x,y)|} - \mathbf{I}(d(x,y)) N_j(y) |\nabla d(x,y)| \right|^2 .$$

The triangle inequality and the Cauchy-Schwartz inequality yields

$$\zeta(x,y) \leq 2\xi(x,y,s) + 2|\mathbf{I}(s) - \mathbf{I}(d(x,y))|^2 |N(y)|^2 \leq 2\xi(x,y,s) + CW^s_{d(x,y)}(x)|N(y)|^2.$$

We now proceed to estimate the pinching $W^s_{d(x,y)}(x)$ in terms of the pinching $W^{4r}_{\frac{\rho}{8}}(x)$. Observe that the almost monotonicity of the frequency (3.117) tells us that for any $\eta\, d(x,\mathbf{S}) < s < t \leq 1$, we have

$$\mathbf{I}(s) \leq \mathbf{I}(t) + C\boldsymbol{m}_0^{\gamma_4} t^{\gamma_4}. \tag{3.118}$$

This yields

$$W^s_{d(x,y)}(x) \leq W^{4r}_{\frac{\rho}{4}}(x) + C\boldsymbol{m}_0^{\gamma_4} r^{\gamma_4} - C\boldsymbol{m}_0^{\gamma_4} s^{\gamma_4}. \tag{3.119}$$

Moreover, we have

$$\int_{\mathcal{A}^{2r}_{\frac{\rho}{4}}(x)} \int_{d(x,y)}^{2d(x,y)} \frac{1}{s^2\mathbf{H}(s)}\zeta(x,y)\,\mathrm{d}s\,\mathrm{d}y \geq \frac{1}{\mathbf{H}(2r)} \int_{\mathcal{A}^{2r}_{\frac{\rho}{4}}(x)} \zeta(x,y) \int_{d(x,y)}^{2d(x,y)} \frac{1}{s^2}\,\mathrm{d}s\,\mathrm{d}y$$
$$\geq \frac{1}{2\mathbf{H}(2r)} \int_{\mathcal{A}^{2r}_{\frac{\rho}{4}}(x)} \frac{\zeta(x,y)}{d(x,y)}\,\mathrm{d}y.$$

Combining this with the preceding estimates, we therefore arrive at

$$\begin{aligned}
\log\left(\frac{16r}{\rho}\right) W^{4r}_{\frac{\rho}{8}}(x) &\geq C\int_{\frac{\rho}{4}}^{4r} \frac{1}{s\mathbf{H}(s)} \int_{\mathcal{A}^s_{\frac{s}{2}}(x)} \frac{\zeta(x,y)}{d(x,y)}\,\mathrm{d}y\,\mathrm{d}s - C\boldsymbol{m}_0^{\gamma_4} r^{\gamma_4} \log\left(\frac{16r}{\rho}\right) - C\boldsymbol{m}_0^{\gamma_4} r^{\gamma_4} \\
&\geq C\int_{\mathcal{A}^{2r}_{\frac{\rho}{4}}(x)} \int_{d(x,y)}^{2d(x,y)} \frac{1}{s^2\mathbf{H}(s)}\zeta(x,y)\,\mathrm{d}s\,\mathrm{d}y - C\boldsymbol{m}_0^{\gamma_4} r^{\gamma_4} \log\left(\frac{16r}{\rho}\right) - C\boldsymbol{m}_0^{\gamma_4} r^{\gamma_4} \\
&\geq \frac{C}{\mathbf{H}(2r)} \int_{\mathcal{A}^{2r}_{\frac{\rho}{4}}(x)} \frac{\zeta(x,y)}{d(x,y)}\,\mathrm{d}y - C\boldsymbol{m}_0^{\gamma_4} r^{\gamma_4} \log\left(\frac{16r}{\rho}\right) - C\boldsymbol{m}_0^{\gamma_4} r^{\gamma_4} \\
&\geq \frac{C}{\mathbf{H}(2r)} \int_{\mathcal{A}^{2r}_{\frac{\rho}{4}}(x)} \frac{\zeta(x,y)}{d(x,y)}\,\mathrm{d}y - C\boldsymbol{m}_0^{\gamma_4} r^{\gamma_4} \log\left(\frac{16r}{\rho}\right) - C\boldsymbol{m}_0^{\gamma_4} r^{\gamma_4}.
\end{aligned}$$

Rearranging, this yields the claimed estimate.

3.14.2 Proof of Lemma 3.14.5

Observe that we have

$$\partial_v \mathbf{D}(x,r) = \int \phi'\left(\frac{d(x,y)}{r}\right) \frac{\nabla d(x,y)}{r} \cdot v(y) |DN(y)|^2\,\mathrm{d}y. \tag{3.120}$$

Consider the vector field $X_i(p) = Y(\mathbf{p}(p))$ where

$$Y(y) \coloneqq \phi\left(\frac{d(x,y)}{r}\right) v. \tag{3.121}$$

Then

$$\mathrm{div}_{\mathcal{M}}\, Y = \phi'\left(\frac{d(x,y)}{r}\right) \frac{\nabla d(x,y)}{r} \cdot v - \langle Y, H_{\mathcal{M}}\rangle, \tag{3.122}$$

and

$$D_{\mathcal{M}}Y = \frac{1}{r}\phi'\left(\frac{d(x,y)}{r}\right) v(y)\otimes \nu_x(y) + -\phi\left(\frac{d(x,y)}{r}\right)\sum_j \mathbf{A}_{\mathcal{M}}(e_j, v)\otimes e_j, \tag{3.123}$$

where $\nu_x(y) \coloneqq \nabla d(x,y)$ and $\{e_j\}$ form an orthonormal frame of $\mathcal{M}$, with $e_1 = v$. Thus, testing [22, (3.25)] with this vector field and using the decay in Proposition 3.11.2 yields

$$\begin{aligned}\partial_v \mathbf{D}(x,r) &= \int_{\mathcal{M}} |DN|^2 \operatorname{div}_{\mathcal{M}} Y \, \mathrm{d}y + O(\boldsymbol{m}_0^{\frac{1}{2}}) r^{\frac{1-\delta_2}{2}} \mathbf{D}(x,r) \\ =&\frac{2}{r}\int \phi'\left(\frac{d(x,y)}{r}\right)\sum_i \langle \partial_{\nu_x} N_i(y), \partial_v N_i(y)\rangle \, \mathrm{d}y + O(\boldsymbol{m}_0^{\frac{1}{2}}) r^{\frac{1-\delta_2}{2}} \mathbf{D}(x,r) + \sum_{j=1}^{5} \widetilde{\mathrm{Err}}_j^i,\end{aligned} \tag{3.124}$$

where $\widetilde{\mathrm{Err}}_j^i$ are the inner variational errors in [22, (3.19), (3.26), (3.27), (3.28)], but for our new choice of vector field Y. We estimate them analogously to [22, Section 4], again using the excess decay of Proposition 3.11.2 to get improved scaling, combined with an analogous estimate to (3.116), to obtain

$$\begin{aligned}\sum_{j=1}^{5} |\widetilde{\mathrm{Err}}_j^i| &\leq C\boldsymbol{m}_0^{\gamma_4} r^{-1} \mathbf{D}(x,r)^{1+\gamma_4} + C\boldsymbol{m}_0^{\gamma_4}\mathbf{D}(x,r)^{\gamma_4}\partial_r \mathbf{D}(x,r) \\ &\leq C\boldsymbol{m}_0^{\gamma_4} r^{\gamma_4 - 1}\mathbf{D}(x,r)\end{aligned} \tag{3.125}$$

Note that in order to get these estimates we require $r > \eta\, d(x,\mathbf{S})$, to ensure that $B_r(q,\pi_0)$ does not intersect any cube $L \in \mathscr{W}$ with $\ell(L) > c_s r$ (cf. Lemma 3.12.6). The identity for $\partial_v \mathbf{H}(x,r)$ is merely a computation, identical to that in the proof of [16, Proposition 3.1].

3.14.3 Proof of Lemma 3.14.4

Let z, y be as in the statement of the lemma and let x lie in the line segment $[z, y]$. For a continuous vector field v, the chain rule yields

$$\partial_v \mathbf{I}(x,r) = \frac{r\partial_v \mathbf{D}(x,r)}{\mathbf{H}(x,r)} - \frac{\mathbf{I}(x,r)\partial_v \mathbf{H}(x,r)}{\mathbf{H}(x,r)}.$$

We may now proceed as in the proof of [16, Theorem 4.2]. Nevertheless, we repeat the argument here for clarity. Let μ_x be the measure with density

$$\mathrm{d}\mu_x = -\frac{|\nabla d(x,y)|}{d(x,y)}\phi'\left(\frac{d(x,y)}{r}\right)\,\mathrm{d}y,$$

and let

$$v(y) = d(x_1,x_2)\frac{\nabla d(x_1,y)}{|\nabla d(x_1,y)|} \qquad \eta_x(y) \coloneqq d(x,y)\frac{\nabla d(x,y)}{|\nabla d(x,y)|} = \frac{d(x,y)}{|\nabla d(x,y)|}\nu_x(y).$$

Then by Lemma 3.14.5, for every $r \in]\eta d(x,\mathbf{S}), 1]$ we have

$$\begin{aligned}\partial_v \mathbf{I}(x,r) = \frac{2}{\mathbf{H}(x,r)}\int_{\mathcal{M}}\sum_i \langle \partial_{\eta_x} N_i, \partial_v N_i\rangle\,\mathrm{d}\mu_x + C\boldsymbol{m}_0^{\gamma_4} r^{\gamma_4}\mathbf{I}(x,r)\\ - 2\frac{\mathbf{I}(x,r)}{\mathbf{H}(x,r)}\int_{\mathcal{M}}|\nabla d|\sum_i\langle \partial_v N_i, N_i\rangle\,\mathrm{d}\mu_x.\end{aligned}$$

We can now write

$$\begin{aligned}\partial_v N_i(y) &= DN_i(y)d(x_1,y)\frac{\nabla d(x_1,y)}{|\nabla d(x_1,y)|} - DN_i(y)d(x_2,y)\frac{\nabla d(x_2,y)}{|\nabla d(x_2,y)|}\\ &= \partial_{\eta_{x_1}}N_i(y) - \partial_{\eta_{x_2}}N_i(y)\\ &= \underbrace{\big(\partial_{\eta_{x_1}}N_i(y) - \mathbf{I}(x_1,d(x_1,y))N_i(y)\big)}_{=:\mathcal{E}_{1,i}} - \underbrace{\big(\partial_{\eta_{x_2}}N_i(y) - \mathbf{I}(x_2,d(x_2,y))N_i(y)\big)}_{=:\mathcal{E}_{2,i}}\\ &\quad + \underbrace{\mathbf{I}(x_1,d(x_1,y)) - \mathbf{I}(x_2,d(x_2,y))}_{=:\mathcal{E}_3}N_i(y).\end{aligned}$$

Thus, we have

$$\begin{aligned}\partial_v \mathbf{I}(x,r) &= \frac{2}{\mathbf{H}(x,r)}\int_{\mathcal{M}}\sum_i \langle \partial_{\eta_x} N_i, \mathcal{E}_{1,i}-\mathcal{E}_{2,i}\rangle \,\mathrm{d}\mu_x - 2\frac{\mathbf{I}(x,r)}{\mathbf{H}(x,r)}\int_{\mathcal{M}}|\nabla d|\sum_i \langle N_i, \mathcal{E}_{1,i}-\mathcal{E}_{2,i}\rangle \,\mathrm{d}\mu_x \\ &\quad + \frac{2}{\mathbf{H}(x,r)}\int_{\mathcal{M}}\mathcal{E}_3\sum_i \langle \partial_{\eta_x} N_i, N_i\rangle - 2\frac{\mathbf{I}(x,r)}{\mathbf{H}(x,r)}\int_{\mathcal{M}}|\nabla d|\mathcal{E}_3\sum_i |N_i|^2 \,\mathrm{d}\mu_x \\ &\quad + C\boldsymbol{m}_0^{\gamma_4} r^{\gamma_4}\mathbf{I}(x,r) \\ &= \frac{2}{\mathbf{H}(x,r)}\int_{\mathcal{M}}\sum_i \langle \partial_{\eta_x} N_i, \mathcal{E}_{1,i}-\mathcal{E}_{2,i}\rangle \,\mathrm{d}\mu_x - 2\frac{\mathbf{I}(x,r)}{\mathbf{H}(x,r)}\int_{\mathcal{M}}|\nabla d|\sum_i \langle N_i, \mathcal{E}_{1,i}-\mathcal{E}_{2,i}\rangle \,\mathrm{d}\mu_x \\ &\quad + \frac{2\mathcal{E}_3}{\mathbf{H}(x,r)}\left(\int_{\mathcal{M}}\sum_i \langle \partial_{\eta_x} N_i, N_i\rangle \,\mathrm{d}\mu_x - r\mathbf{D}(x,r)\right) \\ &\quad + C\boldsymbol{m}_0^{\gamma_4} r^{\gamma_4}\mathbf{I}(x,r).\end{aligned}$$

With the aim to establish control in terms of frequency pinching at the endpoints x_1 and x_2, we can now rewrite $\mathcal{E}_3$ as

$$\begin{aligned}\mathcal{E}_3 &= (\mathbf{I}(x_1, d(x_1,y)) - \mathbf{I}(x_1,r)) + (\mathbf{I}(x_1,r) - \mathbf{I}(x_2,r)) + (\mathbf{I}(x_2,r) - \mathbf{I}(x_2,d(x_2,y))) \\ &\leq W_r^{d(x_1,y)}(x_1) + W_{d(x_2,y)}^r(x_2) + \mathbf{I}(x_1,r) - \mathbf{I}(x_2,r).\end{aligned}$$

This, combined with the Cauchy-Schwartz inequality, Lemma 3.12.3, an analogous almost monotonicity estimate to (3.119) and the uniform upper bound on the frequency (3.105) tells us that

$$\begin{aligned}|\partial_v \mathbf{I}(x,r)| &\leq C\left[\int\sum_i \left(|\mathcal{E}_{1,i}|^2 + |\mathcal{E}_{2,i}|^2\right)\,\mathrm{d}\mu_x\right]^{\frac{1}{2}}\left(\frac{1}{\mathbf{H}(x,r)}\left[\int\sum_i |\partial_{\eta_x}N_i|^2\,\mathrm{d}\mu_x\right]^{\frac{1}{2}} + \frac{\mathbf{I}(x,r)}{\mathbf{H}(x,r)^{\frac{1}{2}}}\right) \\ &\quad + C\boldsymbol{m}_0^{\gamma_4} r\frac{|\mathbf{I}(x_1,r)-\mathbf{I}(x_2,r)|}{\mathbf{H}(x,r)}\left(\mathbf{D}(x,r)^{1+\gamma_4} + \boldsymbol{\Sigma}(x,r)\right) \\ &\quad + C\boldsymbol{m}_0^{\gamma_4} r\frac{|W_r^{d(x_1,y)}(x_1)| + |W_{d(x_2,y)}^r(x_2)|}{\mathbf{H}(x,r)}\left(\mathbf{D}(x,r)^{1+\gamma_4} + \boldsymbol{\Sigma}(x,r)\right) \\ &\quad + C\boldsymbol{m}_0^{\gamma_4} r^{\gamma_4}\mathbf{I}(x,r).\end{aligned}$$

Now invoking Proposition 3.14.2, for $\ell = 1, 2$ we have

$$\begin{aligned}\int \sum_i |\mathcal{E}_{\ell,i}|^2 \, \mathrm{d}\mu_x &= -\int \sum_i |\mathcal{E}_{\ell,i}|^2(y) \frac{|\nabla d(x,y)|}{d(x,y)} \phi'\left(\frac{d(x,y)}{r}\right) \, \mathrm{d}y \\ &\leq C\mathbf{H}(x_\ell, 2r)(W^{4r}_{\frac{r}{8}}(x_\ell) + \boldsymbol{m}_0^{\gamma_4} r^{\gamma_4}).\end{aligned}$$

Thus, when combined with the upper bounds in (3.105) and (3.107), we conclude that

$$\begin{aligned}|\partial_v \mathbf{I}(x,r)| &\leq C\left[(W^{4r}_{\frac{r}{8}}(x_1) + \boldsymbol{m}_0^{\gamma_4} r^{\gamma_4})^{\frac{1}{2}} + (W^{4r}_{\frac{r}{8}}(x_2) + \boldsymbol{m}_0^{\gamma_4} r^{\gamma_4})^{\frac{1}{2}}\right] \\ &\quad + C\boldsymbol{m}_0^{\gamma_4} r^{\gamma_4} \mathbf{I}(x,r)(\mathbf{D}(x,r)^{\gamma_4} + r^2 \mathbf{D}(x,r)) + C\boldsymbol{m}_0^{\gamma_4} r^{\gamma_4}. \\ &\leq C\left[W^{4r}_{\frac{r}{8}}(x_1)^{\frac{1}{2}} + W^{4r}_{\frac{r}{8}}(x_2)^{\frac{1}{2}} + \boldsymbol{m}_0^{\frac{\gamma_4}{2}} r^{\frac{\gamma_4}{2}}\right] + C\boldsymbol{m}_0^{\gamma_4} r^{\gamma_4}.\end{aligned}$$

Integrating this inequality over the geodesic segment $[z, y] \subset \mathcal{M}$ and using the estimates in Lemma 3.13.1, the result follows.

3.15 Quantitative splitting

In what follows we will need to consider affine subspaces spanned by families of vectors. For this reason it will be useful to introduce the following notation. Given an ordered set of points $X = \{x_0, x_1, \ldots, x_k\}$ we will denote by $V(X)$ the affine subspace spanned by $\{x_1 - x_0, x_2 - x_0, \ldots, x_k - x_0\}$ and centered at x_0, namely

$$V(X) = x_0 + \mathrm{span}(\{(x_1 - x_0), (x_2 - x_0), \ldots, (x_k - x_0)\}). \tag{3.126}$$

We will now show that approximate homogeneity implies the existence of an approximate spine in given directions. We begin with the following definition.

Definition 3.15.1. *We say that a set $X = \{x_0, x_1, \ldots, x_k\} \subset \mathbf{B}_r(x)$ is ρr-linearly indepen-*

dent if

$$d(x_i, V(\{x_0, \dots, x_{i-1}\})) \geq \rho r \qquad \text{for all } i = 1, \dots, k$$

We say that a set $F \subset \mathbf{B}_r(x)$ ρr*-spans a* k*-dimensional affine subspace* V *if there is a* ρr*-linearly independent set of points* $X = \{x_i\}_{i=0}^k \subset F$ *such that* $V = V(X)$*.*

The following lemma gives a quantitative notion of the existence of an approximate spine in $\mathbf{S}$, provided that N is (quantitatively) almost-homogeneous about an $(m-2)$-dimensional linear space.

Lemma 3.15.2. *Suppose that* T*,* $\mathcal{M}$*,* N *are as in Assumption 3.13.2, let* $x \in \mathbf{S}$ *and let* $\rho, \tilde{\rho}, \bar{\rho} \in]0, 1]$ *be given. There exists* $\varepsilon = \varepsilon_{3.15.2}(m, n, Q, \Lambda, \rho, \tilde{\rho}, \bar{\rho}) \in]0, \varepsilon_4^2]$ *such that the following holds. Suppose that for some* $r > 0$*,*

$$\max\{\mathbf{E}(T, \mathbf{B}_{2r}(x)), \bar{\varepsilon}(2r)^{2-2\delta_2}\} \leq \varepsilon.$$

Let $X = \{x_i\}_{i=0}^{m-2} \subset \mathbf{B}_r(x) \cap \mathbf{S}$ *be a* ρr*-linearly independent set of points with*

$$W_{\tilde{\rho} r}^{2r}(x_i) < \varepsilon \qquad \text{for each } i.$$

Then $\mathbf{S} \cap (\mathbf{B}_r \setminus \mathbf{B}_{\bar{\rho} r}(V(X))) = \emptyset$*.*

Proof. We argue by contradiction. Without loss of generality, assume $x = 0$. Suppose that the statement is false. Then there exists sequences $\varepsilon_k \downarrow 0$, $r_k \downarrow 0$ and corresponding sequences of center manifolds $\mathcal{M}_k$ and normalized normal approximations $\bar{N}_k$ with $\mathbf{H}_{\bar{N}_k}(0, 1) = 1$ for T_{0, r_k}. Moreover, for $\mathbf{S}_k := \mathbf{S}(T_{0, r_k})$, there is a sequence of $(m-1)$-tuples of points $X_k := \{x_{k,0}, x_{k,1}, \dots, x_{k,m-2}\} \subset \mathbf{B}_1 \cap \mathbf{S}_k$ such that

(i) X_k is ρ-linearly independent for some $\rho \in]0, 1]$;

(ii) $W_{\tilde{\rho}}^2(\bar{N}_k, x_{k,i}) \leq \varepsilon_k \to 0$ as $k \to \infty$ for some $\tilde{\rho} \in]0, 1]$;

(iii) there exists a point $y_k \in \mathbf{S}_k \cap \mathbf{B}_1 \setminus \mathbf{B}_{\bar{\rho}}(V(X_k)))$.

We can thus use the compactness argument from Section 3.2.1 to conclude that

(1) $\mathcal{M}_k \longrightarrow \pi_\infty$ in $C^{3,\kappa}$;

(2) $\bar{N}_k \circ \mathbf{e}_k \longrightarrow u$ in L^2 and in $W^{1,2}_{\mathrm{loc}}$, where u is a Dir-minimizer with $\boldsymbol{\eta} \circ u \equiv 0$;

(3) X_k converges pointwise to $X_\infty = \{x_0, \ldots, x_{m-2}\} \subset \pi_\infty$;

(4) y_k converge pointwise to $y \in \pi_\infty \cap \bar{\mathbf{B}}_1 \setminus \mathbf{B}_{\tilde{\rho}}(V(X_\infty))$ with $u(y) = Q\llbracket 0 \rrbracket$.

Denote by $\Delta_Q(u)$ the set of points $y \in \pi_\infty$ such that $u(y) = Q\llbracket \boldsymbol{\eta} \circ u(y) \rrbracket = Q\llbracket 0 \rrbracket$. Due to the dichotomy [18, Proposition 3.22], we know that

$$\dim_{\mathcal{H}}(\Delta_Q(u) \cap \bar{B}_1) \leq m - 2\,. \tag{3.127}$$

Indeed $H_u(0,1) = 1$ and $\boldsymbol{\eta} \circ u \equiv 0$, so u cannot be identically equal to $Q\llbracket \boldsymbol{\eta} \circ u \rrbracket$. Moreover, $H_u(y,\tau) > 0$ for every $\tau \in (0,1)$ and every $y \in B_1$, since otherwise we would contradict the dimension estimate (3.127). This, in combination with (ii) tells us that

$$I_u(x_i, \tilde{\rho}) = I_u(x_i, 2) \qquad \text{for } i = 0, \ldots, m-2.$$

The monotonicity of the (regularized) frequency for Dir-minimizers then tells us that u is α_i-homogeneous about the center x_i in the annulus $B_2(x_i) \setminus B_{\tilde{\rho}}(x_i) \subset \pi_\infty$, for some $\alpha_i > 0$. We can then extend u to an α_i-homogeneous function about x_i on π_∞; call it v_i. Observe that for any $z \neq x_i$, there is a neighbourhood $U_z \subset \pi_\infty$ of z on which v_i is a Dir-minimizer (by using a scaling argument and the fact that v_i agrees with a Dir-minimizer on $B_2(x_i) \setminus B_{\tilde{\rho}}(x_i) \subset \pi_\infty$).

This allows us to apply the unique continuation result [16, Lemma 6.9] to conclude that $u = v_i$ on $B_{64} \setminus \{x_i\}$, and hence $u = v_i$ on B_{64}. By iteratively applying [16, Lemma 6.8], we may thus conclude that $\alpha_i = \alpha$ for each $i = 0, \ldots, m-2$, and that $u \equiv Q\llbracket 0 \rrbracket$ on the $(m-2)$-dimensional plane $V(X_\infty) = x_0 + \mathrm{span}\{(x_{m-2} - x_0), \ldots, (x_1 - x_0)\}$. In other words, u is an α-homogeneous function in two variables about the $(m-2)$-dimensional plane $V(X_\infty)$.

Since $y \notin V(X_\infty)$ and $u(y) = Q[\![0]\!]$ but u is α-homogeneous, this implies that $u \equiv Q[\![0]\!]$ on $x_0 + \mathrm{span}\{x_{m-2} - x_0, \dots, x_1 - x_0, y - x_0\}$. This however contradicts the dimension estimate on $\Delta_Q(u)$, thus allowing us to conclude. □

The following lemma tells us that it suffices to have approximate homogeneity on a linearly independent set of points, in order to conclude approximate homogeneity in the entire affine subspace spanned by these points.

Lemma 3.15.3. *Suppose that T, $\mathcal{M}$ and N are as in Assumption 3.13.2, let $x \in \mathbf{S}$ and let $\rho, \tilde{\rho}, \bar{\rho} \in]0, 1]$ be given. Then for any given $\delta > 0$, there exists $\varepsilon = \varepsilon_{3.15.3} > 0$, dependent on $m, n, Q, \Lambda, \rho, \tilde{\rho}, \bar{\rho}, \delta$, such that the following holds. Suppose that for some $r > 0$,*

$$\max\{\mathbf{E}(T, \mathbf{B}_{2r}(x)), \bar{\varepsilon}(2r)^{2-2\delta_2}\} \leq \varepsilon.$$

Let $X = \{x_i\}_{i=0}^{m-2} \subset \mathbf{B}_r(x) \cap \mathbf{S}$ be a ρr-linearly independent set of points with

$$W^{2r}_{\tilde{\rho} r}(x_i) < \varepsilon \qquad \text{for every } i.$$

Then for every $y_1, y_2 \in \mathbf{B}_r(x) \cap \mathbf{B}_{\varepsilon r}(V(X)) \cap \mathbf{S}$ and for every $r_1, r_2 \in [\bar{\rho} r, r]$ we have

$$|\mathbf{I}(y_1, r_1) - \mathbf{I}(y_2, r_2)| \leq \delta\,.$$

Proof. We again argue by contradiction. Without loss of generality, assume $x = 0$. Suppose that the statement is false. Then there exists sequences $\varepsilon_k \downarrow 0$, $r_k \downarrow 0$ and corresponding sequences of center manifolds $\mathcal{M}_k$ and normalized normal approximations $\bar{N}_k$ with $\mathbf{H}_{\bar{N}_k}(0, 1) = 1$ for T_{0,r_k} and a sequence of $(m-1)$-tuples of points $X_k := \{x_{k,0}, x_{k,1}, \dots, x_{k,m-2}\} \subset \mathbf{B}_1 \cap \mathbf{S}_k$ (where $\mathbf{S}_k := \mathbf{S}(T_{0,r_k})$) such that

(i) X_k is ρ-linearly independent for some $\rho > 0$;

(ii) $W^2_{\tilde{\rho}}(\bar{N}_k, x_{k,i}) \leq \varepsilon_k \to 0$ as $k \to \infty$ for some $\tilde{\rho} > 0$;

(iii) there are points $y_{k,1},\ y_{k,2} \in \mathbf{B}_1 \cap \mathbf{B}_{\varepsilon_k}(V(X_k)) \cap \mathbf{S}_k$ and corresponding scales $r_{k,i} \in [\bar{\rho}, 1]$ such that

$$|\mathbf{I}_k(y_{k,1}, r_{k,1}) - \mathbf{I}_k(y_{k,2}, r_{k,2})| \geq \delta > 0,$$

where $\mathbf{I}_k \coloneqq \mathbf{I}_{\bar{N}_k}$.

We can thus use the compactness argument from Section 3.2.1 to conclude that

(1) $\mathcal{M}_k \longrightarrow \pi_\infty$ in $C^{3,\kappa}$;

(2) $\bar{N}_k \circ \mathbf{e}_k \longrightarrow u$ in L^2 and in $W^{1,2}_{\mathrm{loc}}$, where u is a Dir-minimizer with $\boldsymbol{\eta} \circ u \equiv 0$;

(3) X_k converges pointwise to $X_\infty = \{x_0, \ldots, x_{m-2}\}$;

(4) $y_{k,i}$ converge pointwise to y_i and the respective scales $r_{k,i}$ converge to scales $r_i \in [\bar{\rho}, 1]$ for $i = 1, 2$.

Arguing as in the proof of Lemma 3.15.2, we can deduce that $u \equiv Q[\![0]\!]$ on $V(X_\infty) = x_0 + \mathrm{span}\{x_{m-2} - x_0, \ldots, x_1 - x_0\}$, with $I_u(y, \tau) \equiv \alpha > 0$ for any $y \in V(X_\infty)$ and any $\tau > 0$. However, since $y_{k,1},\ y_{k,2} \in \mathbf{S}_k$ and $r_{k,i} \in [\bar{\rho}, 1]$, we also have $\mathbf{I}_k(y_{k,i}, r_{k,i}) \to I_u(y_i, r_i)$ for $i = 1, 2$, so (iii) contradicts the homogeneity of u. □

3.16 Flatness control

In this section we proceed to estimate the "mean flatness" in a ball for a measure μ supported in $\mathbf{S}$, in terms of a $(m-2)$-dimensional μ-weighted average of the frequency pinching (plus a suitable lower order term). We introduce therefore the so called Jones' β_2 coefficients, which will give us a suitable tool to measure the mean flatness of μ at a given scale around a given point.

Definition 3.16.1. *Given a Radon measure μ in $\mathbb{R}^{m+n}$, and $k \in \{0, 1, \ldots, m+n-1\}$, we define the k-dimensional Jones' β_2 coefficient of μ as*

$$\beta^k_{2,\mu}(x,r) \coloneqq \inf_{\text{affine } k\text{-planes } L} \left[r^{-k} \int_{\mathbf{B}_r(x)} \left(\frac{\operatorname{dist}(y,L)}{r} \right)^2 \, \mathrm{d}\mu(y) \right]^{1/2}.$$

The pivotal estimate of this section is thus the following proposition.

Proposition 3.16.2. *There exist thresholds $\eta(m) > 0$, $\varepsilon(\Lambda, m, n, Q, \eta)$, $\alpha_0 = \alpha_0(\Lambda, m, n, Q) > 0$ and $C(\Lambda, m, n, Q) > 0$ such that the following holds. Suppose that T, $\mathcal{M}$ and N satisfy Assumption 3.13.2 with parameters ε_4 and η smaller than the above thresholds. Suppose that μ is a finite non-negative Radon measure with $\operatorname{spt}(\mu) \subset \mathbf{S}$. Then for all $r \in]0,1]$ and every $x_0 \in \mathbf{B}_{r/8} \cap \mathbf{S}$ we have*

$$[\beta^{m-2}_{2,\mu}(x_0, r/8)]^2 \leq \frac{C}{r^{m-2}} \int_{\mathbf{B}_{r/8}(x_0)} W^{4r}_{r/8}(x) \, \mathrm{d}\mu(x) + C\boldsymbol{m}_0^{\alpha_0} r^{-(m-2-\alpha_0)} \mu(\mathbf{B}_{r/8}(x_0)).$$

In order to prove this proposition, we will require the following lemma, the proof of which is omitted here and can be found in [16].

Lemma 3.16.3 ([16], Lemma 5.4)**.** *Let $\Omega \subset \mathbb{R}^m$ be a connected open set and suppose that $u : \Omega \to \mathcal{A}_Q(\mathbb{R}^n)$ is a Dir-minimizer. Assume there is a ball $B \subset \Omega$ and a system of coordinates $x_1, \ldots, x_m$ such that $u\big|_B$ is a function of x_1 only. Then u is a function of only x_1 on all of Ω.*

3.16.1 Proof of Proposition 3.16.2

We may assume that $\mu(\mathbf{B}_{r/8}(x_0)) > 0$, else the claimed inequality is trivial. Since it will be convenient for us to restrict ourselves to $(m-2)$-dimensional affine subspaces L of $T_{x_0}\mathcal{M}$ in Definition 3.16.1, we begin with some basic linear algebra to simplify the mean flatness that

we would like to control. Let

$$\bar{x}_{x_0,r} \coloneqq \frac{1}{\mu(\mathbf{B}_{r/8}(x_0))} \int_{\mathbf{B}_{r/8}(x_0)} x \, \mathrm{d}\mu(x) \tag{3.128}$$

denote the barycenter of μ in $\mathbf{B}_{r/8}(x_0) \cap T_{x_0}\mathcal{M}$, where $\mathbf{p}_{x_0}$ is the orthogonal projection of $\mathbb{R}^{m+n}$ onto $T_{x_0}\mathcal{M}$. Following the approach of [43] and [16, Section 5], we may consider the symmetric bilinear form $b_{x_0,r} : T_{x_0}\mathcal{M} \times T_{x_0}\mathcal{M} \to \mathbb{R}$ defined by

$$\begin{aligned} b_{x_0,r}(v,w) &\coloneqq \int_{\mathbf{B}_{r/8}(x_0)} ((\mathbf{p}_{x_0}(x) - \mathbf{p}_{x_0}(\bar{x}_{x_0,r})) \cdot v) \, ((\mathbf{p}_{x_0}(x) - \mathbf{p}_{x_0}(\bar{x}_{x_0,r})) \cdot w) \ \mathrm{d}\mu(x) \\ &= \int_{\mathbf{B}_{r/8}(x_0)} ((x - \bar{x}_{x_0,r}) \cdot v) \, ((x - \bar{x}_{x_0,r}) \cdot w) \ \mathrm{d}\mu(x), \end{aligned} \tag{3.129}$$

and we diagonalize $b_{x_0,r}$. This yields an orthonormal basis $\{v_i\}_{i=1}^m$ of eigenvectors and a corresponding family of eigenvalues $0 \leq \lambda_m \leq \cdots \leq \lambda_1$ for the linear map

$$\begin{aligned} T(v) &\coloneqq \int_{\mathbf{B}_{r/8}(x_0)} ((\mathbf{p}_{x_0}(x) - \mathbf{p}_{x_0}(\bar{x}_{x_0,r})) \cdot v) \, \mathbf{p}_{x_0}(x) \, \mathrm{d}\mu(x) \\ &= \int_{\mathbf{B}_{r/8}(x_0)} ((x - \bar{x}_{x_0,r}) \cdot v) \, \mathbf{p}_{x_0}(x) \, \mathrm{d}\mu(x), \end{aligned} \tag{3.130}$$

which diagonalize $b_{x_0,r}$. Namely, $T(v_i) = \lambda_i v_i$ and $b_{x_0,r}(v_i, v_i) = \lambda_i$.

This yields the characterization

$$\inf_{\substack{\text{affine } (m-2)\text{-planes} \\ L \subset T_{x_0}\mathcal{M}}} \left(\frac{r}{8}\right)^{-(m-2)} \int_{\mathbf{B}_{r/8}(x_0)} \left(\frac{\operatorname{dist}(y, L)}{r}\right)^2 \mathrm{d}\mu(y) = (r/8)^{-m} \left(\lambda_{m-1} + \lambda_m\right). \tag{3.131}$$

We therefore conclude that

$$[\beta_{2,\mu}^{m-2}(x_0, r/8)]^2 \leq (r/8)^{-m} \left(\lambda_{m-1} + \lambda_m\right) \leq 2 \, (r/8)^{-m} \, \lambda_{m-1} \, . \tag{3.132}$$

Moreover the $(m-2)$-planes minimizing the left hand side of (3.131) are those of the form $L = x_0 + \operatorname{span}\{v_1, \ldots, v_{m-2}\}$, for any choice of orthonormal basis as above.

Fix now any $z \in \mathbf{B}_{2r}(x_0) \setminus \mathbf{B}_{r/4}(x_0) \cap \mathcal{M}$. Following [16], we would like to differentiate the map N at the point z along the vector v_j. However the latter vector is an element of $T_{x_0}\mathcal{M}$ and not an element of $T_z\mathcal{M}$. In order to find a suitable element of $T_z\mathcal{M}$ we consider the geodesic segment connecting x_0 and z on $\mathcal{M}$ and the parallel transport along it. This gives a well-defined linear map $\boldsymbol{\ell}_z : T_{x_0}\mathcal{M} \to T_z\mathcal{M}$. This map is, in fact, the differential $d\mathbf{e}_{x_0}|_\zeta$ of the exponential map $\mathbf{e}_{x_0}$ at the point $\zeta = \mathbf{e}_{x_0}^{-1}(z)$.

Since

$$\int_{\mathbf{B}_{r/8}(x_0)} ((x - \bar{x}_{x_0,r}) \cdot v_j)\mathbf{p}_{x_0}(z - x)\, \mathrm{d}\mu(x) = -\lambda_j v_j$$

and

$$\int_{\mathbf{B}_{r/8}(x_0)} (x - \bar{x}_{x_0,r}) \cdot v_j)\, \mathrm{d}\mu(x) = 0,$$

for each $j = 1, \dots, m$, $i = 1, \dots, Q$ and any fixed $\alpha > 0$ (to be determined later) we have

$$\begin{aligned}
&-\lambda_j DN_i(z) \cdot \boldsymbol{\ell}_z(v_j) = \\
&DN_i(z) \cdot \boldsymbol{\ell}_z(-\lambda_j v_j) = DN_i(z) \cdot \boldsymbol{\ell}_z \left(\int_{\mathbf{B}_{r/8}(x_0)} ((x - \bar{x}_{x_0,r}) \cdot v_j)\mathbf{p}_{x_0}(z - x)\, \mathrm{d}\mu(x) \right) \\
&\qquad\qquad - \alpha N_i(z) \int_{\mathbf{B}_{r/8}(x_0)} ((x - \bar{x}_{x_0,r}) \cdot v_j)\, \mathrm{d}\mu(x) \\
&\qquad = \int_{\mathbf{B}_{r/8}(x_0)} (x - \bar{x}_{x_0,r}) \cdot v_j \left[DN_i(z) \cdot \boldsymbol{\ell}_z(\mathbf{p}_{x_0}(z - x)) - \alpha N_i(z) \right] \mathrm{d}\mu(x).
\end{aligned}$$

Now observe that

$$\left| \boldsymbol{\ell}_z(\mathbf{p}_{x_0}(z - x)) - \frac{d(x,z)\nabla d(x,z)}{|\nabla d(x,z)|} \right| \leq \boldsymbol{m}_0^{1/2} r^2 \,.$$

We therefore reach the (approximate) identity

$$\begin{aligned}
-\lambda_j DN_i(z) \cdot \boldsymbol{\ell}_z(v_j) = &\int_{\mathbf{B}_{r/8}(x_0)} ((x - \bar{x}_{x_0,r}) \cdot v_j) \left[DN_i(z) \cdot \frac{d(x,z)\nabla d(x,z)}{|\nabla d(x,z)|} - \alpha N_i(z) \right] \mathrm{d}\mu(x) \\
&+ O\left(\boldsymbol{m}_0^{1/2} r^2 \right) |DN_i(z)| \int_{\mathbf{B}_{r/8}(x_0)} |(x - \bar{x}_{x_0,r}) \cdot v_j| \,.
\end{aligned}$$

We now square both sides, sum over the components i of the Q-valued map N, and use

the Cauchy-Schwartz inequality and the estimates on the distance d. Letting $w(x,z) \coloneqq \frac{d(x,z)\nabla d(x,z)}{|\nabla d(x,z)|}$, we thus have

$$\begin{aligned}
&\lambda_j^2 \sum_i |DN_i(z)\cdot \boldsymbol{\ell}_z(v_j)|^2 \\
&\quad \leq C \int_{\mathbf{B}_{r/8}(x_0)} |(x-\bar{x}_{x_0,r})\cdot v_j|^2 \,\mathrm{d}\mu(x) \int_{\mathbf{B}_{r/8}(x_0)} \sum_i |DN_i(z)\cdot w(x,z) - \alpha N_i(z))|^2 \,\mathrm{d}\mu(x) \\
&\qquad + C\boldsymbol{m}_0 r^4 \mu(\mathbf{B}_{r/8}(x_0)) |DN(z)|^2 \int_{\mathbf{B}_{r/8}(x_0)} |(x-\bar{x}_{x_0,r})\cdot v_j|^2 \,\mathrm{d}\mu(x)
\end{aligned}$$

Recalling the definition

$$\lambda_j = b_{x_0,r}(v_j,v_j) = \int_{\mathbf{B}_{r/8}(x_0)} |(x-\bar{x}_{x_0,r})\cdot v_j|^2 \,\mathrm{d}\mu(x),$$

we thus achieve

$$\begin{aligned}
\lambda_j \sum_i |DN_i(z)\cdot \boldsymbol{\ell}_z(v_j)|^2 \leq C \int_{\mathbf{B}_{r/8}(x_0)} \sum_i |DN_i(z)\cdot w(x,z) - \alpha N_i(z))|^2 \,\mathrm{d}\mu(x) \\
+ C\boldsymbol{m}_0 r^4 \mu(\mathbf{B}_{r/8}(x_0)) |DN(z)|^2. \qquad (3.133)
\end{aligned}$$

In what follows we will use the shorthand notation

$$|DN(z)\cdot v|^2 = \sum_i |DN_i(z)\cdot v|^2.$$

We now set $\mathcal{A}^{2r}_{r/4}(x_0) \coloneqq \mathbf{B}_{2r}(x_0)\setminus \mathbf{B}_{r/4}(x_0)\cap \mathcal{M}$ and use (3.132) (plus the ordering $0\leq \lambda_m \leq \lambda_{m-1}\leq \ldots \leq \lambda_1$) to get the following inequality:

$$\begin{aligned}
&[\beta^{m-2}_{2,\mu}(x_0,r/8)]^2 \int_{\mathcal{A}^{2r}_{r/4}(x_0)} \sum_{j=1}^{m-1} |DN(z)\cdot \boldsymbol{\ell}_z(v_j)|^2 \,\mathrm{d}z \\
&\qquad \leq C\frac{\lambda_{m-1}}{r^m} \int_{\mathcal{A}^{2r}_{r/4}(x_0)} \sum_{j=1}^{m-1} |DN(z)\cdot \boldsymbol{\ell}_z(v_j)|^2 \,\mathrm{d}z
\end{aligned}$$

$$\leq \frac{C}{r^m} \int_{\mathcal{A}^{2r}_{r/4}(x_0)} \sum_{j=1}^{m} \lambda_j |DN(z) \cdot \boldsymbol{\ell}_z(v_j)|^2 \, \mathrm{d}z$$

We hence use (3.133) to get

$$\begin{aligned}
&[\beta^{m-2}_{2,\mu}(x_0, r/8)]^2 \int_{\mathcal{A}^{2r}_{r/4}(x_0)} \sum_{j=1}^{m-1} |DN(z) \cdot \boldsymbol{\ell}_z(v_j)|^2 \, \mathrm{d}z \\
&\quad \leq C\boldsymbol{m}_0 r^{4-m} \mu(\mathbf{B}_{r/8}(x_0)) \int_{\mathbf{B}_{2r}(x_0)\cap\mathcal{M}} |DN|^2 \\
&\qquad + Cr^{-m} \int_{\mathcal{A}^{2r}_{r/4}(x_0)} \int_{\mathbf{B}_{r/8}(x_0)} \sum_i |DN_i(z) \cdot w(x,z) - \alpha N_i(z))|^2 \, \mathrm{d}\mu(x) \, \mathrm{d}z \\
&\quad \leq C\boldsymbol{m}_0 r^{4-m} \mu(\mathbf{B}_{r/8}(x_0)) \int_{\mathbf{B}_{2r}(x_0)\cap\mathcal{M}} |DN|^2 \\
&\qquad + Cr^{-m} \int_{\mathcal{A}^{2r}_{r/4}(x_0)} \int_{\mathbf{B}_{r/8}(x_0)} \sum_i |DN_i(z) \cdot w(x,z) - \alpha N_i(z))|^2 \, \mathrm{d}\mu(x) \, \mathrm{d}z \\
&\quad \leq C\boldsymbol{m}_0 r^{4-m} \mu(\mathbf{B}_{r/8}(x_0)) \int_{\mathbf{B}_{2r}(x_0)\cap\mathcal{M}} |DN|^2 \\
&\qquad + Cr^{-m} \underbrace{\int_{\mathcal{A}^{2r}_{r/4}(x_0)} \int_{\mathbf{B}_{r/8}(x_0)} \sum_i |w(x,z) \cdot DN_i(z) - \mathbf{I}(x, |z-x|) N_i(z))|^2 \, \mathrm{d}\mu(x) \, \mathrm{d}z}_{=:\mathcal{R}(x_0,r)} \\
&\qquad + Cr^{-m} \int_{\mathcal{A}^{2r}_{r/4}(x_0)} \int_{\mathbf{B}_{r/8}(x_0)} |\mathbf{I}(x, |z-x|) - \alpha|^2 |N(z)|^2 \, \mathrm{d}\mu(x) \, \mathrm{d}z.
\end{aligned}$$

Firstly, observe that Fubini's theorem and the estimates in Proposition 3.14.2 and Lemma 3.13.1 tell us that for $r \in]0, 1]$,

$$\mathcal{R}(x_0, r) \leq Cr \int_{\mathbf{B}_{r/8}(x_0)} \mathbf{H}(x, 2r)(W^{2r}_{r/4}(x) + \boldsymbol{m}_0^{\gamma_4} r^{\gamma_4}) \, \mathrm{d}\mu(x) \, .$$

Now fix any such r and choose

$$\alpha := \frac{1}{\mu(\mathbf{B}_{r/8}(x_0))} \int \mathbf{B}_{r/8}(x_0) \mathbf{I}(y, r) \, \mathrm{d}\mu(y).$$

We may hence once again use the triangle inequality to write

$$\begin{aligned}
&\int_{\mathcal{A}^{2r}_{r/4}(x_0)} \int_{\mathbf{B}_{r/8}(x_0)} |\mathbf{I}(x,|z-x|)-\alpha|^2 |N(z)|^2 \,\mathrm{d}\mu(x)\,\mathrm{d}z \\
&\qquad \leq C \int_{\mathcal{A}^{2r}_{r/4}(x_0)} \int_{\mathbf{B}_{r/8}(x_0)} |W^r_{|z-x|}(x)|^2 |N(z)|^2 \,\mathrm{d}\mu(x)\,\mathrm{d}z \\
&\qquad\qquad + C \int_{\mathcal{A}^{2r}_{r/4}(x_0)} \int_{\mathbf{B}_{r/8}(x_0)} |\mathbf{I}(x,r)-\alpha|^2 |N(z)|^2 \,\mathrm{d}\mu(x)\,\mathrm{d}z \\
&\qquad =: (I)+(II).
\end{aligned}$$

We estimate the two terms on the right-hand side separately. For (I), we may use the almost-monotonicity (3.117) combined with (3.105) to conclude that

$$\begin{aligned}
(I) &\leq Cr\mathbf{H}(x_0,2r)\left(\int_{\mathbf{B}_{r/8}(x_0)} \left[W^{2r}_{r/4}(x)\right]^2 \,\mathrm{d}\mu(x) + \boldsymbol{m}_0^{2\gamma_4} r^{2\gamma_4} \mu(\mathbf{B}_{r/8}(x_0))\right) \\
&\leq Cr\mathbf{H}(x_0,2r)\left(\int_{\mathbf{B}_{r/8}(x_0)} W^{2r}_{r/4}(x) \,\mathrm{d}\mu(x) + \boldsymbol{m}_0^{2\gamma_4} r^{2\gamma_4} \mu(\mathbf{B}_{r/8}(x_0))\right).
\end{aligned}$$

Meanwhile, to control (II) we use Lemma 3.14.4, (3.110) and (3.111) to deduce that for $\eta = \eta(m) > 0$ sufficiently small,

$$\begin{aligned}
(II) &\leq \frac{Cr\mathbf{H}(x_0,2r)}{\mu(\mathbf{B}_{r/8}(x_0))} \int_{\mathbf{B}_{r/8}(x_0)} \int_{\mathbf{B}_{r/8}(x_0)} (W^{4r}_{r/8}(x) + W^{4r}_{r/8}(y) + \boldsymbol{m}_0^{\gamma_4} r^{\gamma_4}) \,\mathrm{d}\mu(x)\,\mathrm{d}\mu(y) \\
&\leq Cr\mathbf{H}(x_0,2r)\left(\int_{\mathbf{B}_{r/8}(x_0)} W^{4r}_{r/8}(x) \,\mathrm{d}\mu(x) + \boldsymbol{m}_0^{\gamma_4} r^{\gamma_4} \mu(\mathbf{B}_{r/8}(x_0))\right).
\end{aligned}$$

Taking into account that

$$r\int_{\mathbf{B}_{2r}(x_0)\cap\mathcal{M}} |DN|^2 \leq C\mathbf{H}(x_0,2r)\,,$$

it thus remains to check that

$$\int_{\mathcal{A}^{2r}_{r/4}(x_0)} \sum_{j=1}^{m-1} |DN(z) \cdot \boldsymbol{\ell}_z(v_j)|^2 \, \mathrm{d}z \geq c(\Lambda) \frac{\mathbf{H}(x_0, 2r)}{r}, \tag{3.134}$$

for some $C(\Lambda) > 0$. We prove this by contradiction. The inequality is scaling invariant, so by rescaling and recentering, we may assume that $r = 1$ and $x_0 = 0$. If (3.134) fails, then we can extract a sequence of currents T_k with corresponding center manifolds $\mathcal{M}_k$ and corresponding normalized normal approximations $\bar{N}_k$ with $\int_{\mathbf{B}_1 \cap \mathcal{M}_k} |D\bar{N}_k|^2 \leq C\Lambda$ and $\int_{\mathbf{B}_2 \setminus \overline{\mathbf{B}}_1 \cap \mathcal{M}_k} |\bar{N}_k|^2 = 1$, for which

- $\mathcal{M}_k \to \pi_\infty$ (taking $\boldsymbol{m}_0^{(k)}\} \leq \varepsilon_k^2 \to 0$),
- $\boldsymbol{\eta} \circ \bar{N}_k \to 0$,
- $\bar{N}_k(y_k) = Q\llbracket 0 \rrbracket$ for some $y_k \in \mathbf{B}_{1/8} \cap \mathcal{M}_k$ (since $\mu_{T_k}(\mathbf{B}_{r/8}) > 0$),

but

$$\int_{\mathbf{B}_2 \setminus \overline{\mathbf{B}}_1 \cap \mathcal{M}_k} \sum_{j=1}^{m-1} |D\bar{N}_k(z) \cdot \boldsymbol{\ell}_z^k(v_j^k)|^2 \longrightarrow 0,$$

for some choice of orthonormal vectors $\{v_1^k, \ldots, v_{m-1}^k\}$, where $\boldsymbol{\ell}_z^k$ is the linear map $\boldsymbol{\ell}_z : T_0\mathcal{M}_k \to T_z\mathcal{M}_k$. Up to subsequence, we can find a limiting Dir-minimizer $u : \pi_\infty \supset B_2 \to \mathcal{A}_Q(\mathbb{R}^{m+n})$ with

- $\int_{B_2 \setminus \overline{B}_1} |u|^2 = 1$,
- $\int_{B_1} |Du|^2 \leq C\Lambda$,
- $\boldsymbol{\eta} \circ u \equiv 0$,
- $u(y) = Q\llbracket 0 \rrbracket$ for some $y \in B_{1/8}$,

and

$$\int_{B_2 \setminus \overline{B}_1} \sum_{j=1}^{m-1} |Du(z) \cdot v_j|^2 = 0$$

for some choice of orthonormal directions $\{v_1, \dots, v_{m-1}\}$ (where each v_j is the limit of v_j^k; note also that, since $\mathcal{M}_k$ converge to the flat plane π_∞, the maps $\boldsymbol{\ell}_z^k$ converge to the identity map from π_∞ onto itself).

Proceeding as in the proof of [16, Proposition 5.3], we deduce that u is a function of only one variable on $B_2 \setminus \overline{B}_1$, and hence on the whole of B_2 due to Lemma 3.16.3. Since $u(y) = Q[\![0]\!]$, we deduce that $\dim_{\mathcal{H}}(\Delta_Q u) \geq m-1$, which is a contradiction, since u is non-trivial.

3.17 Rectifiability of $\mathfrak{F}_Q^h$

This section is dedicated to proving the rectifiability of $\mathbf{S}$ in claim (iii) of Theorem 3.12.1. In order to do this, we make use of the following result from [12], which we re-state here for the convenience of the reader.

Theorem 3.17.1. *Let $k \in \mathbb{N}$ be an integer with $k < m+n$. Suppose that $E \subset \mathbb{R}^{m+n}$ is a Souslin set that is non σ-finite with respect to $\mathcal{H}^k$. Then there exists a closed subset $F \subset E$ with $0 < \mathcal{H}^k(F) < \infty$ that is purely k-unrectifiable.*

Unlike the arguments in [42], here we do not require the Minkowski content bound (3.97) in Theorem 3.12.1(iii) to conclude rectifiability. This is because we may replace the covering arguments of Naber and Valtorta by Theorem 3.17.1, together with the existence of a Frostman measure supported on each finite $\mathcal{H}^{m-2}$-measure subset of $\mathbf{S} \cap \mathbf{B}_1$ in order to provide us with the necessary a priori measure bound on balls. We may then appeal to the work of Azzam and Tolsa [4], where the rectifiability of a Radon measure comes from the control of its β_2-square function (see (3.136)). However, we provide a proof of the bound (3.97) in Section 3.18 nevertheless, since we believe that this is of independent interest.

3.17.1 Proof of rectifiability in Theorem 3.12.1(iii)

If $\mathbf{S} \cap \bar{\mathbf{B}}_1$ is σ-finite with respect to $\mathcal{H}^{m-2}$, but not rectifiable, then it is a classical fact that $\mathbf{S}$ contains a purely $(m-2)$-unrectifiable closed set F with $0 < \mathcal{H}^{m-2}(F) < \infty$ (see e.g. [39, Theorem 15.6]). On the other hand, if $\mathbf{S} \cap \mathbf{B}_1$ is not σ-finite, then we can appeal to Theorem 3.17.1 to again find such a closed subset F.

Let μ be a Frostman measure supported on F (see [39, Theorem 8.17], for example). Namely, μ is a nontrivial non-negative Radon measure with $\mathrm{spt}(\mu) \subset F$ and

$$\mu(\mathbf{B}_r(x)) \leq r^{m-2} \quad \forall x\,,\ \forall r \leq 1. \tag{3.135}$$

In light of the characterization of rectifiability in [4, Theorem 1.1], it suffices to prove that for every $y \in \mathbf{B}_1$ and every $t \leq \frac{1}{32}$ we have

$$\int_{\mathbf{B}_t(y)} \int_0^t [\beta_{2,\mu}^{m-2}(z,s)]^2 \frac{\mathrm{d}s}{s}\, \mathrm{d}\mu(z) < \infty. \tag{3.136}$$

Indeed this would imply that the support of μ is rectifiable, but since such support has finite and positive $\mathcal{H}^{m-2}$ measure and it is contained in F, we reach a contradiction. We appeal to Proposition 3.16.2: we are integrating z with respect to μ, so necessarily $z \in \mathbf{S}$ when z is in the support of μ and thus the estimate applies.

We have

$$\begin{aligned}\int_{\mathbf{B}_t(y)} \int_0^t [\beta_{2,\mu}^{m-2}(z,s)]^2 \frac{\mathrm{d}s}{s}\, \mathrm{d}\mu(z) \leq C \int_{\mathbf{B}_t(y)} \int_0^t s^{-(m-1)} \int_{\mathbf{B}_s(z)} W_s^{32s}(w)\, \mathrm{d}\mu(w)\, \mathrm{d}s\, \mathrm{d}\mu(z) \\ + C\boldsymbol{m}_0^{\alpha_0} \int_{\mathbf{B}_t(y)} \int_0^t \frac{\mu(\mathbf{B}_s(z))}{s^{m-1-\alpha_0}}\, \mathrm{d}s\, \mathrm{d}\mu(z).\end{aligned}$$

Thus, via analogous estimates to those in Step 4 in the proof of Lemma 3.18.1, addition-

ally invoking the bound (3.135), we deduce that

$$\begin{aligned}\int_{\mathbf{B}_t(y)}\int_0^t [\beta_{2,\mu}^{m-2}(z,s)]^2\frac{\mathrm{d}s}{s}\,\mathrm{d}\mu(z) &\le C\int_{\mathbf{B}_{2t}(y)}\int_0^t W_s^{32s}(w)\frac{\mathrm{d}s}{s}\,\mathrm{d}\mu(w)\\ &\quad + C\boldsymbol{m}_0^{\alpha_0}\int_{\mathbf{B}_t(y)}\int_0^t s^{-1+\alpha_0}\,\mathrm{d}s\,\mathrm{d}\mu(z)\\ &\le Ct^{m-2}W_0^{1/8} + Ct^{m-2+\min\{\alpha_0,\beta\}}\boldsymbol{m}_0^{\alpha_0}\\ &\le C(m,n,Q,\Lambda).\end{aligned}$$

This yields the desired contradiction, completing the proof.

3.18 Minkowski content bound in Theorem 3.12.1

Here, we provide the proofs of the Minkowski content bound (3.97) of Theorem 3.12.1(iii). This will be done by combining the estimate of Proposition 3.16.2 with an iterative covering technique borrowed from [42]. We start with the following covering lemma.

Lemma 3.18.1. *Let $\rho \le 1/100$, let $\sigma < \tau < \frac{1}{8}$ and let $\eta > 0$ be a fixed number smaller than the threshold of Proposition 3.16.2. There exists $\varepsilon_4 = \varepsilon_4(\Lambda, m, n, Q) > 0$ sufficiently small such that the following holds. Suppose that T is as in Assumption 3.13.2 for these choices of η and ε_4. Let $x_0 \in \mathbf{S} \cap \mathbf{B}_{1/8}$, let $D \subset \mathbf{S} \cap \mathbf{B}_\tau(x_0)$ and let $U := \sup_{y\in D}\mathbf{I}(y,\tau)$.*

Then there exists $\delta = \delta_{3.18.1}(m,n,Q,\Lambda,\rho) > 0$, a dimensional constant $C_R = C_R(m) > 0$ and a finite cover of D by balls $\mathbf{B}_{r_i}(x_i)$ such that

(a) $r_i \ge 10\rho\sigma$;

(b) $\sum_i r_i^{m-2} \le C_R\tau^{m-2}$;

(c) For every i, either $r_i \le \sigma$ or

$$F_i := D \cap \mathbf{B}_{r_i}(x_i) \cap \{\, y : \mathbf{I}(y,\rho r_i) \in (U-\delta, U+\delta)\,\} \subset \mathbf{B}_{\rho r_i}(V_i),$$

for some $(m-3)$-dimensional subspace $V_i \subset \mathbb{R}^{m+n}$.

The parameters ε_4 and η of Assumption 3.13.2 are first chosen small enough so that we can apply Proposition 3.16.2. Then, ε_4 is further decreased if necessary, to ensure that $\boldsymbol{m}_0^{\alpha_0}$ falls below a desired small dimensional constant, in order to absorb a suitable error term. Lemma 3.18.1 will in turn be used to prove the following second efficient covering result, analogous to [16, Proposition 7.2], where the parameter ρ will be chosen smaller than a geometric constant depending only on m.

Proposition 3.18.2. *Fix η as in Lemma 3.18.1 and let $\tau < \frac{1}{8}$. There exist $\delta = \delta(\Lambda, m, n, Q)$, $\varepsilon_4(\Lambda, m, n, Q, \delta) > 0$ and a dimensional constant $C_V = C_V(m) \geq 1$ such that the following holds.*

Assume that T is as in Assumption 3.13.2 for the above choices of η and ε_4. Suppose that $x_0 \in \mathbf{S} \cap \mathbf{B}_{1/8}$ and let $D \subset \mathbf{S} \cap \mathbf{B}_\tau(x_0)$ and $U := \sup_{y \in D} \mathbf{I}(y, \tau)$. Then, for every $s \in]0, \tau[$, there exists a finite cover of D by balls $\mathbf{B}_{r_i}(x_i)$ with $r_i \geq s$ and a decomposition of D into sets $A_i \subset D$ such that

(a) $A_i \subset D \cap \mathbf{B}_{r_i}(x_i)$;

(b) $\sum_i r_i^{m-2} \leq C_V \tau^{m-2}$;

(c) For every i we have either $r_i = s$ or

$$\sup_{y \in A_i} \mathbf{I}(y, r_i) \leq U - \delta.$$

3.18.1 Proof of Lemma 3.18.1

Without less of generality we assume that $x_0 = 0$.

Step 1: Inductive procedure. We inductively construct special families $\mathscr{C}(k)$, $k = \{0, ...\kappa\}$,

where $\kappa = -\lfloor \log_{10\rho}(8\sigma) \rfloor$, consisting of covers by balls of D, such that

$$\mathbf{B}_r(x) \in \mathscr{C}(k) \implies r = \frac{(10\rho)^j}{8} \text{ for some } j \in \{0, \dots, k\}\,.$$

The procedure goes as follows. At the starting step $\mathscr{C}(0) = \{\mathbf{B}_{1/8}\}$. Suppose now that we have already constructed the cover $\mathscr{C}(k)$. Take a ball $\mathbf{B}_r(x) \in \mathscr{C}(k)$. If $r = \frac{(10\rho)^j}{8}$ with $j < k$, place $\mathbf{B}_r(x)$ into $\mathscr{C}(k+1)$.

If $r = \frac{(10\rho)^k}{8}$, for fixed $\delta > 0$ (to be determined later) consider the set

$$F(\mathbf{B}_r(x)) \coloneqq D \cap \mathbf{B}_r(x) \cap \{\, y : \mathbf{I}(y, \rho r) \in (U - \delta, U + \delta) \,\}\,.$$

Notice that this is the set of points in $D \cap \mathbf{B}_r(x)$ at which the frequency is pinched by at most δ. We then have two possibilities:

keep $\mathbf{B}_r(x)$ in $\mathscr{C}(k+1)$ if $F(\mathbf{B}_r(x))$ does not ρr-span an $(m-2)$-dimensional affine subspace; (K)

discard $\mathbf{B}_r(x)$ from $\mathscr{C}(k+1)$ if $F(\mathbf{B}_r(x))$ ρr-spans an $(m-2)$-dimensional affine subspace $V = V(\mathbf{B}_r(x)) \subset \mathbb{R}^{m+n}$. (D)

Observe that if (K) holds, then there is an $(m-3)$-dimensional space V such that $F(\mathbf{B}_r(x)) \subset \mathbf{B}_{\rho r}(V)$, i.e. $B_r(x)$ satisfies condition (c) of the statement of the lemma.

We next wish to apply Lemma 3.15.2 with $\rho = \tilde{\rho}$. Choosing $\delta \leq \frac{\varepsilon_{3.15.2}}{2}$, we may conclude that within every ball $\mathbf{B}_r(x) \in \mathscr{C}(k)$ for which (D) holds,

$$D \cap \mathbf{B}_r(x) \subset \mathbf{B}_{\rho r}(V).$$

We may thus replace

$$\{\mathbf{B}^i\}_i := \left\{ \mathbf{B}_{\frac{(10\rho)^k}{8}}(x) \in \mathscr{C}(k) : \text{(D) holds for } \mathbf{B}_{\frac{(10\rho)^k}{8}}(x) \right\}$$

with a collection $\mathscr{F}(k+1)$ of balls $\{\widetilde{\mathbf{B}}^i\}_i$ with radius $\frac{(10\rho)^{k+1}}{8}$ that cover

$$D \cap \bigcup_i \mathbf{B}_{\frac{\rho(10\rho)^k}{8}}(V(\mathbf{B}^i)).$$

We may moreover ensure that the concentric balls $\frac{1}{5}\widetilde{\mathbf{B}}^i$ are pairwise disjoint, and that their centers are contained in $D \cap \mathbf{B}_r(x) \cap \bigcup_i (V(\mathbf{B}^i))$. Letting $\mathscr{B}(k+1)$ be a Vitali subcover of

$$\{\, \mathbf{B}_r(x) \in \mathscr{C}(k) : \text{(K) holds} \,\},$$

and letting

$$\mathscr{C}(k+1) := \mathscr{F}(k+1) \cup \mathscr{B}(k+1),$$

we have a new cover of D.

Step 2: Frequency pinching. Before we continue, we first show that the following frequency pinching estimate holds: for any $\xi > 0$, we can choose $\delta > 0$ sufficiently small (dependent on ρ and ξ) such that either

$$\mathscr{C}(\kappa) = \{\mathbf{B}_{1/8}\} \qquad \text{or} \qquad \mathbf{I}\left(x, \frac{\rho r}{5}\right) \in [U-\xi, U+\xi] \quad \forall\, \mathbf{B}_r(x) \in \mathscr{C}(k),\ k = 0, \dots, \kappa. \tag{3.137}$$

Indeed, if we do not stop refining immediately, then for any ball $\mathbf{B}_r(x) \in \mathscr{C}(k)$ we have $r = \frac{(10\rho)^{j+1}}{8}$ for some $j+1 \leq k$. Thus, by construction, we know that there exists a ball $\mathbf{B}'$ of radius $\frac{(10\rho)^j}{8}$ satisfying (D); namely $F(\mathbf{B}')$ $\rho\frac{(10\rho)^j}{8}$-spans an $(m-2)$-dimensional affine subspace V, and that $x \in D \cap V \cap \mathbf{B}'$. There must hence exist at least one other point $z \in F(\mathbf{B}') \cap V$. We now wish to apply Lemma 3.15.3 with $\rho = \tilde{\rho}$ and $\bar{\rho} = \frac{\rho}{5}$: provided that

$\delta \leq \min\{\delta_{3.15.3}, \frac{\xi}{2}\}$, we have

$$\left|\mathbf{I}\left(x, \frac{\rho r}{5}\right) - \mathbf{I}(z, r)\right| \leq \frac{\xi}{2}.$$

Since $z \in F(\mathbf{B}')$, this yields the second alternative in (3.137).

Step 3: Discrete $(m-2)$-dimensional measures and coarse packing estimate. It remains to check that the covering $\mathscr{C}(\kappa)$ satisfies the conditions (a)-(c). Since κ is the smallest integer such that $\frac{(10\rho)^\kappa}{8} \leq \sigma$, the conditions (a) and (c) are a trivial consequence of the construction of the inductive covering.

Hence, we just need to verify that (b) holds; namely, that

$$\sum_i s_i^{m-2} \leq C_R,$$

where $\mathscr{C}(\kappa) = \{\mathbf{B}_{5s_i}(x_i)\}_i$. For this, we will make use of [42, Theorem 3.4]. With this in mind, we introduce the discrete measures

$$\mu := \sum_i s_i^{m-2}\delta_{x_i}, \qquad \mu_s := \sum_{i:s_i\leq s} s_i^{m-2}\delta_{x_i}.$$

First of all, note that $\mu_s \equiv 0$ for every $s < \bar{r} := \frac{1}{5}\frac{(10\rho)^\kappa}{8}$, due to the construction of our covering.

We proceed to inductively show that

$$\mu_s(\mathbf{B}_s(x)) \leq C_R s^{m-2} \quad \text{for every } x \in \mathbf{B}_\tau \text{ and every } s = \bar{r}2^j, \quad j = 0, \dots, J := \log_2\left(\frac{\bar{r}}{8}\right) - 4. \tag{3.138}$$

The base case is trivially true, since

$$\mu_{\bar{r}}(\mathbf{B}_{\bar{r}}(x)) = N(x, \bar{r})\bar{r}^{m-2},$$

where $N(x, \bar{r}) := \#\{\, i : s_i = \bar{r}\,\}$. This is a dimensional constant, since we have a Vitali

cover.

Suppose that (3.138) holds for $0, \dots, j$, for some $j < J$. Leting $r \coloneqq \bar{r}2^j$, we will first of all show that

$$\mu_{2r}(\mathbf{B}_{2r}(x)) \leq C(m) C_R (2r)^{m-2}, \tag{3.139}$$

for some dimensional constant $C(m)$. This follows by simply subdividing μ_{2r} into

$$\mu_{2r} = \mu_r + \sum_{i : r < s_i \leq 2r} s_i^{m-2} \delta_{x_i} =: \mu_r + \tilde{\mu}_r,$$

combined with the observation that $\mathbf{B}_{2r}(x)$ can be covered by at most $C(m)$ balls $\mathbf{B}_r(x_i)$, on each of which we may use the inductive assumption, meanwhile

$$\tilde{\mu}_r(\mathbf{B}_{2r}(x)) \leq \bar{N}(x, 2r)(2r)^{m-2} \leq C(m)(2r)^{m-2}, \tag{3.140}$$

where $\bar{N}(x, 2r) \coloneqq \#\{\, i \,:\, r < s_i \leq 2r \,\}$.

Step 4: Inductive packing estimate We will now improve the coarse bound (3.139) to the estimate

$$\mu_{2r}(\mathbf{B}_{2r}(x)) \leq C_R (2r)^{m-2}, \tag{3.141}$$

where C_R is the dimensional constant coming from [42, Theorem 3.4].

Let $\bar{\mu} \coloneqq \mu_{2r} \llcorner \mathbf{B}_{2r}(x)$. We claim that

$$\int_{\mathbf{B}_t(y)} \int_0^t [\beta_{2,\bar{\mu}}^{m-2}(z, s)]^2 \frac{\mathrm{d}s}{s} \, \mathrm{d}\bar{\mu}(z) < \delta_0^2 t^{m-2} \qquad \forall y \in \mathbf{B}_{2r}(x), \ \forall t \in (0, 2r], \tag{3.142}$$

where $\delta_0 = \delta_0(m) > 0$ is as in [42, Theorem 3.4] (denoted by simply δ therein).

Firstly, notice that Proposition 3.16.2 (coupled with the fact that $x_i \in \mathbf{S}$) tells us that there exists $\alpha_0 > 0$ (as in the statement of the proposition) such that we have

$$[\beta_{2,\bar{\mu}}^{m-2}(x_i, s)]^2 \leq \frac{C}{s^{m-2}} \int_{\mathbf{B}_s(x_i)} \bar{W}_s(z) \, \mathrm{d}\bar{\mu}(z) + C \boldsymbol{m}_0^{\alpha_0} \frac{\bar{\mu}(\mathbf{B}_s(x_i))}{s^{m-2-\alpha_0}} \qquad \forall s \in (0, 2r], \tag{3.143}$$

where $\bar{W}_s(x_i) := W_s^{32s}(x_i)\mathbf{1}_{s>s_i}$, since the balls $\mathbf{B}_{s_i}(x_i)$ are pairwise disjoint.

Let us first deal with the second term on the right-hand side of (3.143). Consider first the case $r < t \leq 2r$. Then, due to (3.139), we have

$$\begin{aligned}\bar{N}(y,2r)(2r)^{m-2}\int_{\mathbf{B}_t(y)}\int_0^t \frac{\bar{\mu}(\mathbf{B}_s(x_i))}{s^{m-2-\alpha_0}}\frac{\mathrm{d}s}{s}\,\mathrm{d}\bar{\mu}(z)&\\ \leq \boldsymbol{m}_0^{\alpha_0}\big(C(m)+\bar{N}(y,2r)\big)(C_R(m))^2(2r)^{2(m-2)}t^{-(m-2-\alpha_0)}&\\ \leq C(m)\boldsymbol{m}_0^{\alpha_0}t^{m-2+\alpha_0}.&\end{aligned}$$

In the case $t \leq r$, we first of all notice that if there exists $x_i \in \mathbf{B}_t(y)$ with $s_i \geq 3t$, then there are no other points $x_j \in \mathbf{B}_{2t}(y)$ since the balls $\mathbf{B}_{s_i}(x_i)$ are pairwise disjoint. This in turn implies that $\beta_{2,\bar{\mu}}^{m-2}(z,s) = 0$ for every $z \in \mathbf{B}_t(y)$ and every $s \leq t$.

Thus, we may assume that $s_i < 3t$ for every $x_i \in \mathbf{B}_t(y)$, in which case we can use the inductive assumption, combined with the fact that $\mu_{2r} = \mu_t + \tilde{\mu}_t$ to conclude that

$$\int_{\mathbf{B}_t(y)}\int_0^t C\boldsymbol{m}_0^{\alpha_0}\frac{\bar{\mu}(\mathbf{B}_s(x_i))}{s^{m-2-\alpha_0}}\frac{\mathrm{d}s}{s}\,\mathrm{d}\bar{\mu}(z) \leq C_R(m)C(m)\boldsymbol{m}_0^{\alpha_0}t^{m-2+\alpha_0}.$$

Here we have used that $\mathbf{B}_t(y)$ can be covered by at most $\bar{N}(y,3t)$ balls of radius $s_i \in [t,3t)$, so we get an estimate analogous to (3.140) for $\tilde{\mu}_t$. Thus, by decreasing ε_4 further if necessary, so that $\boldsymbol{m}_0^{\alpha_0}$ is small enough (dependent only on m), we can indeed ensure that (3.142) holds.

To control the frequency term $W_s^{32s}(x_i)$ on the right-hand side of (3.143), we proceed in almost exactly the same way as in the proof of [16, Lemma 7.3]. Nevertheless, we repeat the argument here for the convenience of the reader.

Let $t \in (0,2r]$. Due to the inductive assumption (3.138) and Fubini's theorem, we have

$$\begin{aligned}\int_{\mathbf{B}_t(y)}\int_0^t \frac{1}{s^{m-2}}\int_{\mathbf{B}_s(z)}\bar{W}_s(w)\,\mathrm{d}\bar{\mu}(w)\frac{\mathrm{d}s}{s}\,\mathrm{d}\bar{\mu}(z)&\\ = \int_0^t \frac{1}{s^{m-2}}\int_{\mathbf{B}_t(y)}\int_{\mathbf{B}_s(z)}\bar{W}_s(w)\,\mathrm{d}\bar{\mu}(w)\,\mathrm{d}\bar{\mu}(z)\frac{\mathrm{d}s}{s}&\end{aligned}$$

$$
\begin{aligned}
&= \int_0^t \frac{1}{s^{m-2}} \int_{\mathbf{B}_t(y)} \int_{\mathbf{B}_s(z)} \bar{W}_s(w)\, \mathrm{d}\mu_s(w)\, \mathrm{d}\mu_s(z) \frac{\mathrm{d}s}{s} \\
&= \int_0^t \frac{1}{s^{m-2}} \int_{\mathbf{B}_{s+t}(y)} \bar{W}_s(w) \int_{\mathbf{B}_s(w)} \mathrm{d}\mu_s(z)\, \mathrm{d}\mu_s(w) \frac{\mathrm{d}s}{s} \\
&= \int_0^r \frac{1}{s^{m-2}} \int_{\mathbf{B}_{s+t}(y)} \bar{W}_s(w) \int_{\mathbf{B}_s(w)} \mathrm{d}\mu_s(z)\, \mathrm{d}\mu_s(w) \frac{\mathrm{d}s}{s} \\
&\quad + \int_r^t \frac{1}{s^{m-2}} \int_{\mathbf{B}_{s+t}(y)} \bar{W}_s(w) \int_{\mathbf{B}_s(w)} \mathrm{d}\mu_s(z)\, \mathrm{d}\mu_s(w) \frac{\mathrm{d}s}{s}.
\end{aligned}
$$

For the first term on the right-hand side, we use the inductive assumption and another application of Fubini's theorem to conclude that

$$
\int_0^r \frac{1}{s^{m-2}} \int_{\mathbf{B}_{s+t}(y)} \bar{W}_s(w) \int_{\mathbf{B}_s(w)} \mathrm{d}\mu_s(z)\, \mathrm{d}\mu_s(w) \frac{\mathrm{d}s}{s} \leq C_R \int_{\mathbf{B}_{2t}(y)} \int_0^r \bar{W}_s(w) \frac{\mathrm{d}s}{s}\, \mathrm{d}\mu_t(w).
$$

Meanwhile, to estimate the second term on the right-hand side, we use the coarse bound (3.140) to obtain

$$
\int_r^t \frac{1}{s^{m-2}} \int_{\mathbf{B}_{s+t}(y)} \bar{W}_s(w) \int_{\mathbf{B}_s(w)} \mathrm{d}\mu_s(z)\, \mathrm{d}\mu_s(w) \frac{\mathrm{d}s}{s} \leq C(m) \int_{\mathbf{B}_{4r}(y)} \int_r^t \bar{W}_s(w) \frac{\mathrm{d}s}{s}\, \mathrm{d}\mu_t(w).
$$

In conclusion, we have

$$
\int_{\mathbf{B}_t(y)} \int_0^t \frac{1}{s^{m-2}} \int_{\mathbf{B}_s(z)} \bar{W}_s(w)\, \mathrm{d}\bar{\mu}(w) \frac{\mathrm{d}s}{s}\, \mathrm{d}\bar{\mu}(z) \leq C(m) \int_{\mathbf{B}_{2t}(y)} \int_0^t \bar{W}_s(w) \frac{\mathrm{d}s}{s}\, \mathrm{d}\mu_t(w). \quad (3.144)
$$

We may now estimate the total frequency pinching between scale 0 and t as follows. Letting N be the smallest integer such that $2^N s_i \geq t$, the almost-monotonicity of the frequency at all scales $s \in]0, 1]$ around x_i yields

$$
\begin{aligned}
&\int_{s_i}^t W_s^{32s}(x_i) \frac{\mathrm{d}s}{s} \\
&\quad \leq \sum_{j=0}^{N} \int_{2^j s_i}^{2^{j+1} s_i} W_s^{32s}(x_i) \frac{\mathrm{d}s}{s}
\end{aligned}
$$

$$
\begin{aligned}
&\leq \sum_{j=0}^{N} \left[(1 + C\boldsymbol{m}_0^{\alpha_0}(2^{j+6}s_i)^{\beta})\mathbf{I}(x_i, 2^{j+6}s_i) - (1 - C\boldsymbol{m}_0^{\alpha_0}(2^j s_i)^{\beta})\mathbf{I}(x_i, 2^j s_i)\right] \int_{2^j s_i}^{2^{j+1}s_i} \frac{\mathrm{d}s}{s} \\
&\leq \log 2 \sum_{j=0}^{N} W_{2^j s_i}^{2^{j+6}s_i}(x_i) + C\boldsymbol{m}_0^{\alpha_0} t^{\beta} \\
&= \log 2 \sum_{\ell=0}^{5} \sum_{j=0}^{N} W_{2^{j+\ell} s_i}^{2^{j+\ell+1}s_i}(x_i) + C\boldsymbol{m}_0^{\alpha_0} t^{\beta} \\
&= \log 2 \sum_{\ell=0}^{5} W_{2^{\ell} s_i}^{2^{\ell+N+1}s_i}(x_i) + C\boldsymbol{m}_0^{\alpha_0} t^{\beta} \\
&\leq 6 W_{s_i}^{1/8}(x_i) \log 2 + C\boldsymbol{m}_0^{\alpha_0} t^{\beta} \\
&\overset{(3.137)}{\leq} 6\xi \log 2 + C\boldsymbol{m}_0^{\alpha_0} t^{\beta}.
\end{aligned}
$$

A combination of the inductive assumption for $t \leq r$ and the coarse bound (3.139) tells us that $\mu_t(\mathbf{B}_{2t}(y)) \leq C(m)t^{m-2}$. Thus, we conclude that

$$
\int_{\mathbf{B}_{2t}(y)} \int_0^t \bar{W}_s(w) \frac{\mathrm{d}s}{s} \, \mathrm{d}\mu_t(w) \leq C(m)\xi t^{m-2} + C\boldsymbol{m}_0^{\alpha_0} t^{m-2+\beta}.
$$

By choosing ε_4 even smaller to decrease $\boldsymbol{m}_0^{\alpha_0}$ further if necessary, and taking ξ sufficiently small (which relies on choosing δ small enough), we successfully establish the tighter inductive packing estimate (3.141).

3.18.2 Proof of Proposition 3.18.2

We may once again assume that $x_0 = 0$. First, let ε_4 be as in Lemma 3.18.1; we will later choose it to be smaller if necessary. We will apply Lemma 3.18.1 iteratively to build families of "stopping time regions", where a new covering is built within a large ball on which we stopped the previous covering procedure early. We will show that the iteration can be stopped after finitely many steps, at which point we will have packed the singularities tightly enough to obtain the desired conclusion. Fix $\tau < \frac{1}{8}$ arbitrarily, and for now also fix the parameter ρ arbitrarily; it will be determined later. We first apply Lemma 3.18.1 with

our fixed choice of τ and $\sigma = s$.

This yields a cover $\mathcal{C}(0) := \{\mathbf{B}_{r_i}(x_i)\}$. We can subdivide this cover into the 'good' balls $\mathcal{G}(0) := \{\, \mathbf{B}_{r_i}(x_i) \,:\, r_i \leq s \,\}$ and the 'bad' balls $\mathcal{B}(0) := \{\, \mathbf{B}_{r_i}(x_i) \,:\, r_i > s \,\}$.

Construct a new cover $\mathcal{C}(1)$ of D as follows. Place all balls in $\mathcal{G}(0)$ into $\mathcal{C}(1)$. For each ball $\mathbf{B}_{r_i}(x_i) \in \mathcal{B}(0)$, Lemma 3.18.1 (c) tells us that all points $y \in D \cap \mathbf{B}_{r_i}(x_i)$ with $\mathbf{I}(y, \rho r_i) \in (U - \delta_{3.18.1}, U + \delta_{3.18.1})$ are contained in a ρr_i-tubular neighbourhood of V_i for some $(m-3)$-dimensional affine space $V_i \subset \mathbb{R}^{m+n}$.

We may thus cover $\mathbf{S} \cap \mathbf{B}_{\rho r_i}(V_i) \cap \mathbf{B}_{r_i}(x_i)$ by at most $C(m)\rho^{-(m-3)}$ balls $\{\mathbf{B}^{i,k}\}_{k=1}^{N(i)}$ of radius $2\rho r_i$, centered at points in $\mathbf{S}$.

For any given index i, there are now two possibilities; either

(i) $2\rho r_i < s$: then we include both $\mathbf{B}_{r_i}(x_i)$ and the balls $\{\mathbf{B}^{i,k}\}_k$ into $\mathcal{C}(1)$ and stop refining for this i;

(ii) $2\rho r_i \geq s$: then we apply Lemma 3.18.1 to each ball $\mathbf{B}^{i,k}$ for this fixed i (with $\tau = 2\rho r_i$ and $\sigma = s$), yielding a new cover of $\mathbf{B}^{i,k}$ by balls. We place both $\mathbf{B}_{r_i}(x_i)$ and these new balls (for each $k = 1, \ldots, N(i)$) into the new cover $\mathcal{C}(1)$.

We can then iterate this procedure inductively, only at each stage k letting

$$\mathcal{B}(k) = \left\{\, \mathbf{B}_{r_i}(x_i) \in \mathcal{C}(k) \,:\, s < r_i \leq 2\rho r_j \text{ for some } \mathbf{B}_{r_j}(x_j) \in \mathcal{C}(k-1) \,\right\},$$

until after finitely many steps of the iteration, we obtain a cover $\mathcal{C}(\ell)$ where the radius of every ball is no larger than s. Note that $\ell = \ell(\rho, s)$, and that as long as we choose $\rho \leq \frac{1}{2C(m)}$, we have

$$\sum_{\mathbf{B}_{r_i}(x_i) \in \mathcal{C}(\ell)} r_i^{m-2} \leq 2 \sum_{\mathbf{B}_{r_j}(x_j) \in \mathcal{C}(0)} r_j^{m-2} \leq 2C_R.$$

Now if there are any balls of radius $r_i < s$ in our covering $\mathcal{C}(\ell)$, we may replace them with concentric balls of radius s; since $\rho = \rho(m)$ is now fixed, this would only increase the packing estimate (b) by a factor of $C(m)$, since no ball can be smaller than $10\rho s$.

Now let $\delta := \delta_{3.18.1}(m, n, Q, \Lambda, \rho)$ for our choice of $\rho = \rho(m)$. Making use of the almost-monotonicity (3.117) of the frequency and the uniform upper frequency bound (3.105), for any given $\tau < \frac{1}{8}$, any $y \in \mathbf{B}_\tau(x) \cap \mathbf{S}$ and any $\rho < \tau$ we have

$$\mathbf{I}(y, \rho) \leq \mathbf{I}(y, \tau) + C\boldsymbol{m}_0^{\gamma_4}\tau^{\gamma_4} \leq U + C\boldsymbol{m}_0^{\gamma_4}\tau^{\gamma_4} \leq U + C\varepsilon_4^{2\gamma_4}.$$

Thus, choosing ε_4 smaller if necessary, we may ensure that $\mathbf{I}(y, \rho) < U + \delta$ for every $y \in \mathbf{B}_\tau(x)$ and every $\rho < \tau$.

Finally, if $\mathbf{B}_{r_i}(x_i) \in \mathcal{G}(\ell) \cup \mathcal{B}(\ell)$, let $A_i := D \cap \mathbf{B}_{r_i}(x_i)$. On the other hand, if $\mathbf{B}_{r_i}(x_i) \in \mathcal{B}(k)$ for $k \leq \ell - 1$, let

$$A_i' := (D \cap \mathbf{B}_{r_i}(x_i)) \setminus \bigcup_{\substack{\mathbf{B}_{r_j}(x_j) \subset \mathbf{B}_{r_i}(x_i) \\ \mathbf{B}_{r_j}(x_j) \in \mathcal{B}(k+1)}} \mathbf{B}_{r_j}(x_j). \tag{3.145}$$

Observe that in each A_i' in (3.145), we necessarily have

$$\sup_{y \in A_i'} \mathbf{I}(y, \rho r_i) \leq U - \delta,$$

due to our choice of τ. Thus, since $\mathbf{I}(y, \rho r_i) \geq c(m)\mathbf{I}(y, r_i)$, we may replace $\mathbf{B}_{r_i}(x_i)$ with a collection of at most $C(m)\rho^{-m}$ balls of radius ρr_i that covers A_i', which again only increases the packing estimate (b) by a dimensional constant. We may then let $A_j := A_i' \cap \mathbf{B}_{r_j}(x_j)$ for each ball $\mathbf{B}_{r_j}(x_j)$ in this cover of A_i'.

3.18.3 Proof of the bound (3.97) in Theorem 3.12.1(iii)

With the latter proposition at hand we are in a position to conclude the Minkowski content bound in Theorem 3.12.1(iii) from Proposition 3.18.2. The proof of this is almost identical to the proof of [16, Theorem 2.5], but we sketch it here nevertheless.

Let us first establish the upper Minkowski content bound. Let τ be as in Proposition 3.18.2, and cover $\mathbf{B}_{1/8}$ by a family $\mathcal{F}_0$ of at most $N(m)$ balls of radius τ. Due to (3.105),

we have

$$U(x) \coloneqq \sup\{\, \mathbf{I}(y,\tau) \,:\, y \in \mathbf{S} \cap \mathbf{B}_\tau(x) \,\} \leq \Lambda.$$

Fix a ball $\mathbf{B}_\tau(x) \in \mathcal{F}_0$. Applying Proposition 3.18.2 with an arbitrary fixed choice of $s \in]0,\tau[$ and $D = \mathbf{S} \cap \mathbf{B}_\tau(x)$, we get a resulting decomposition of $\mathbf{S} \cap \mathbf{B}_\tau(x)$ into sets $\{A_i\}_i$, with $A_i \subset \mathbf{B}_{s_i}(x_i)$. Now consider the collection

$$\mathcal{B}_0(x) \coloneqq \{\, \mathbf{B}_{s_i}(x_i) \,:\, s_i > s \,\}.$$

Notice that for every ball $\mathbf{B}_{s_i}(x_i) \in \mathcal{B}_0(x)$, we have

$$\sup\{\, \mathbf{I}(y,s_i) \,:\, y \in A_i \,\} \leq U(x) - \delta_{3.18.2}.$$

We can now once again cover each such A_i by $N(m)$ balls of radius τs_i, and then for each such ball $\mathbf{B}_{\tau s_i}(x)$, apply Proposition 3.18.2 again to $D = A_i \cap \mathbf{B}_{\tau s_i}(x)$, with the same fixed s to yield a new decomposition $\{A_{i,j}\}_j$ of each A_i, with corresponding balls $\{\mathbf{B}_{s_{i,j}}(x_{i,j})\}$ for which we have

$$\sum_{i,j} s_{i,j}^{m-2} \leq C(\tau,m) C_V \sum_i s_i^{m-2} \leq C(\tau,m)^2 C_V^2.$$

In addition, observe that either $s_{i,j} = s$ or

$$\sup\{\, \mathbf{I}(y,s_{i,j}) \,:\, y \in A_{i,j} \,\} \leq U - 2\delta_{3.18.2}.$$

In the latter case, we repeat the above refined covering step. Iterating this procedure, for each $k \in \mathbb{N}$ we can find a decomposition $\{A_i^{(k)}\}_i$ with a corresponding covering by balls $\{\mathbf{B}_{s_i^{(k)}}(x_i^{(k)})\}_i$ such that

$$\sum_i [s_i^{(k)}]^{m-2} \leq C(\tau,m)^k C_V^k, \tag{3.146}$$

and for which

$$\sup\left\{\, \mathbf{I}(y,s_i^{(k)}) \,:\, y \in A_i^{(k)} \,\right\} \leq U - k\delta_{3.18.2}.$$

Thus, this inductive procedure terminates after finitely many steps, and so we end up with a cover of D_0 by balls of radius exactly s, for which the $(m-2)$-dimensional upper Minkowski content bound (3.146) is indeed a dimensional constant.

3.19 Proof of Theorem 3.11.4: subdivision of low frequency points

We are now in a position to prove Theorem 3.11.4. It will be useful to decompose the set $\mathfrak{F}_Q^l$ into a countable collection of pieces as follows. First of all, we may write

$$\mathfrak{F}_Q^l \cap \bar{\mathbf{B}}_1 = \bigcup_{K\in\mathbb{N}} \mathfrak{S}_K,$$

for

$$\mathfrak{S}_K := \{y \in \mathfrak{F}_Q(T) : 2 - \delta_2 - 2^{-K} \geq \mathrm{I}(T,y) \geq 1 + 2^{-K}\} \cap \mathbf{B}_1\,.$$

Moreover, we can further decompose each subset $\mathfrak{S}_K$ as $\mathfrak{S}_K = \bigcup_{J\in\mathbb{N}} \mathfrak{S}_{K,J}$, but before introducing the latter decomposition, we have the following immediate consequence of Proposition 3.11.2, using the fact that $\mathrm{I}(T,x) \geq 1 + 2^{-K}$ for every $x \in \mathfrak{S}_K$.

Corollary 3.19.1. *Let T be as in Proposition 3.11.2 and let $\mu := 2^{-K-1}$. For every $x \in \mathfrak{S}_K$ there exists $r_u(x) > 0$ such that*

$$\mathbf{E}(T, \mathbf{B}_r(x)) \leq \left(\frac{r}{s}\right)^{2\mu} \max\{\mathbf{E}(T, \mathbf{B}_s(x)), \bar{\varepsilon}^2 s^{2-2\delta_2}\} \qquad \forall r < s < r_u\,. \tag{3.147}$$

On the other hand, appealing to Corollary 3.4.3, we can use the upper bound $\mathrm{I}(T,x) \leq 2 - \delta_2 - 2^{-K}$ to derive a lower bound for the excess. More precisely

Corollary 3.19.2. *Let T be as in Proposition 3.11.2 and let $\nu := 1 - \delta_2 - 2^{-K-1}$. For every*

$x \in \mathfrak{S}_K$ there is $r_l(x)$ such that

$$\mathbf{E}(T, \mathbf{B}_r(x)) \geq r^{2\nu} \qquad \forall r < r_l \tag{3.148}$$

and

$$\mathbf{E}(T, \mathbf{B}_r(x)) \geq \left(\frac{r}{s}\right)^{2\nu} \mathbf{E}(T, \mathbf{B}_s(x)) \qquad \forall r < r_l\,. \tag{3.149}$$

Remark 3.19.3. Observe that, since $\nu > \mathrm{I}(T, x) - 1$, by choosing r_l sufficiently small we have eliminated constants in the left hand side of both inequalities.

Notice that if $r < \min\{r_l(x), r_u(x)\}$, (3.147) can be simplified further because, by (3.148), $2\nu \geq 2 - \delta_2$ and $\bar{\varepsilon} \leq 1$, the maximum in the right hand side of (3.147) is achieved by $\mathbf{E}(T, \mathbf{B}_s(x))$.

We now further subdivide $\mathfrak{S}_K$ into pieces in order to achieve uniformity of the thresholds r_l and r_u for every point in each fixed piece. Strictly speaking, we would like to define $\mathfrak{S}_{K,J}$ to be those points $x \in \mathfrak{S}_K$ for which the upper and lower estimates (3.147), (3.148), and (3.149) hold for all radii $r \leq J^{-1}$. However, for technical reasons it will be convenient to have a closed set in our definition. We thus define $\mathfrak{S}_{K,J}$ as follows.

Definition 3.19.4. *Let T be as in Theorem 3.19.6 and let ε_5 be a small positive constant which will be specified later. For every $K \in \mathbb{N} \setminus \{0\}$ define $\mu = \mu(K) := 2^{-K-1}$ and $\nu := \nu(K) := 1 - \delta_2 - 2^{-K-1}$. We define $\mathfrak{S}_{K,J}$ (which implicitly also depends on ε_5) for $K, J \in \mathbb{N} \setminus \{0\}$ as those points $x \in \operatorname{spt}(T) \cap \overline{\mathbf{B}}_1$ for which*

$$\mathbf{E}(T, \mathbf{B}_r(x)) \leq \left(\frac{r}{s}\right)^{2\mu(K)} \mathbf{E}(T, \mathbf{B}_s(x)) \qquad \forall r \leq s \leq \frac{6\sqrt{m}}{J} \tag{3.150}$$

$$\mathbf{E}(T, \mathbf{B}_r(x)) \geq \left(\frac{r}{s}\right)^{2\nu(K)} \mathbf{E}(T, \mathbf{B}_s(x)) \qquad \forall r \leq s \leq \frac{6\sqrt{m}}{J} \tag{3.151}$$

$$\mathbf{E}(T, \mathbf{B}_{6\sqrt{m}J^{-1}}) \leq \varepsilon_5^2 \tag{3.152}$$

$$\mathbf{E}(T, \mathbf{B}_r(x)) \geq r^{2\nu(K)} \qquad \forall r \leq \frac{6\sqrt{m}}{J}\,. \tag{3.153}$$

We record here some immediate corollaries of our overall discussion.

Proposition 3.19.5. *Let T be as in Theorem 3.11.4 and define $\mathfrak{S}_{K,J}$ as in Definition 3.19.4. Then*

(i) $\mathfrak{F}^l_Q \subset \bigcup_{J\geq J_0, K\geq K_0} \mathfrak{S}_{K,J}$ *for every* K_0 *and* J_0.

(ii) Each $\mathfrak{S}_{K,J}$ *is closed and contained in* $\mathfrak{F}_Q(T)$.

(iii) $1 + 2^{-K} \leq \mathrm{I}(T, x) \leq 2 - \delta_2 - 2^{-K}$ *for all* $x \in \mathfrak{S}_{K,J}$.

Indeed, point (ii) is a simple exercise in measure theory, while the claims (i) and (iii) are obvious consequences of Corollary 3.19.1 and Corollary 3.19.2.

We now can observe that by translating, rescaling and intersecting with smaller closed balls, we can, without loss of generality, set the parameter J to be equal to 1. In particular the validity of Theorem 3.11.4 can be reduced to the following.

Theorem 3.19.6. *There exists $\varepsilon_5(m, n, Q) > 0$ such that the following holds. Let T be as in Theorem 3.11.4. Then the closed set $\mathfrak{S} := \mathfrak{S}_{K,1}$ (which, recall, depends on ε_5) is $(m-2)$-rectifiable and has the $(m-2)$-dimensional Minkowski content bound*

$$|\mathbf{B}_r(\mathfrak{S})| \leq Cr^2, \tag{3.154}$$

for a positive constant $C = C(m, n, \bar{n}, T, K)$.

From now on we may further assume that K is fixed, and so we will drop both subscripts from $\mathfrak{S}_{K,1}$.

3.20 Universal frequency function and radial variations

The argument to prove Theorem 3.19.6 follows a strategy which has many similarities with that used to prove Theorem 3.12.1. The major difference is that, unlike in the proof of

Theorem 3.11.3, we cannot hope to find a single center manifold passing through all points $p \in \mathfrak{S}$. To get around this, we decompose the interval $]0,1]$ into countably many sub-intervals whose endpoints are given by a suitable geometric sequence and construct a center manifold for each of them, hence use it to compute a corresponding frequency function. These sub-intervals will be comparable to the intervals of flattening from Section 3.2.1, but suitably adapted to fit in with the covering arguments in Section 3.17. We may in turn construct center manifolds and corresponding normal approximations associated to these intervals, analogously to that in Section 3.2.1. We then define a corresponding universal frequency function, adapted to these intervals. Using estimates from Section 3.6, we are still able to bound the total variation of the universal frequency function quantitatively, in terms of the excess at the starting scale. To that end, the lower and upper bounds on the excess provided in the previous section will be key ingredients.

3.20.1 Center manifolds

We now fix a constant $\gamma = \gamma(m,n,Q) > 0$ whose choice will specified later. Given a point $x \in \mathfrak{S}$ and a geometric blow-up sequence $\{\gamma^k\}_k$ for some $\gamma \in]0,1/2]$, we apply the procedure in [21, Theorem 1.17] to T_{x,γ^k} and define a corresponding center manifold $\mathcal{M}_{x,k}$. It follows from (3.150) and (3.152) that, because of our choice of $\tilde{\varepsilon}$, the theorem is indeed applicable. It turns out that we in fact need to guarantee that $\mathbf{E}(T, \mathbf{B}_{6\sqrt{m}\gamma^k})$ is even smaller because we need to adjust the parameters in [21, Assumption 1.9] in order ensure applicability of Proposition 3.20.1 below and of other similar statements. This however follows from the fact that we are free to choose $\tilde{\varepsilon}$ small enough.

We next notice that it follows from (3.153) that we may replace the blow-up procedure in Section 3.2.1 with the intervals $]\gamma^{k+1}, \gamma^k]$ in place of $]s_k, t_k]$, and with $\boldsymbol{m}_{0,k}$ therein defined instead by

$$\boldsymbol{m}_{x,k} = \mathbf{E}(T_{x,\gamma^k}, \mathbf{B}_{6\sqrt{m}}) \,. \tag{3.155}$$

We next denote by $N_{x,k}$ the corresponding normal approximation for T_{x,γ^k} as constructed in [21, Theorem 2.4].

We will henceforth denote by d the geodesic distance on the center manifold $\mathcal{M}_{x,k}$. In reality this is a collection of functions $d = d_{x,k}$ which depend on the points x and the integer k, but since this dependence is not important, we will omit it. We now record two relevant facts. One is a consequence of adjusting suitably one of the parameters in [21, Assumption 1.8], while the other is a consequence of the lower bound (3.151). In order to state these facts, we denote by $\pi_{x,k}$ the plane used to construct the graphical parametrization $\boldsymbol{\varphi}_{x,k}$ of the center manifold in [21, Theorem 2.4] and by $\mathscr{W}^{x,k}$ the collection of Whitney cubes in [21, Definition 1.10]. Note that the center manifold $\mathcal{M}_{x,k}$ does not necessarily contain the origin $0 = \iota_{x,\gamma^{-k}}(x)$. However we use the point $(0, \boldsymbol{\varphi}_{x,k}(0)) \in \pi_{x,k} \times \pi_{x,k}^{\perp}$ as a proxy for it and we will denote it by $p_{x,k}$.

Proposition 3.20.1. *Let γ and η be two fixed constants and let $c_s = \frac{1}{64\sqrt{m}}$ be as in* [22, Section 2]. *Upon choosing the number N_0 in* [21, Assumption 1.9] *sufficiently large and adjusting accordingly the constants C_e, C_h and ε_2 in* [21, Assumpion 1.9] *we can ensure that*

(a) For every $w \in \mathcal{M}_{x,k}$ and every radius r such that $\eta\gamma \leq r \leq 3$ the largest cube $L \in \mathscr{W}^{x,k}$ intersecting the disk $B_r(\mathbf{p}_{\pi_{x,k}}(w), \pi_{x,k})$ satisfies $\ell(L) \leq c_s r$.

There is a positive constant $\bar{c}_s \leq c_s$ depending upon K and all the various parameters in [21, Assumption 1.9], *with the exception of ε_2 in there, such that*

(b) $B_\gamma(0, \pi_{x,k})$ intersects a cube $L \in \mathscr{W}^{x,k}$ with $\ell(L) \geq \bar{c}_s\gamma$, which violates the excess condition (EX) of [21, Definition 1.10].

Proof. Point (a) is merely a consequence of the fact that $\ell(L) \leq 2^{-N_0-6}$, which comes from the construction of the center manifold (see [21, Proposition 1.11]). As for the second point, (3.151) and (3.155) together imply that some cube intersecting $B_\gamma(0, \pi_{x,k})$ of sidelength comparable to γ must violate the conditions (EX) of [21, Definition 1.10]. □

3.20.2 Frequency functions

Next, for each center manifold $\mathcal{M}_{x,k}$ we define a corresponding frequency function. We just proceed as in Section 2.6, in particular we choose the cut-off ϕ as in (2.30) and set

$$\begin{aligned}
\mathbf{D}_{x,k}(w,r) &:= \int_{\mathcal{M}_{x,k}} |DN_{x,k}(z)|^2 \phi\left(\frac{d(w,z)}{r}\right) \,\mathrm{d}z\,,\\
\mathbf{H}_{x,k}(w,r) &:= -\int_{\mathcal{M}_{x,k}} \frac{|\nabla d(w,z)|^2}{d(w,z)} |N_{x,k}(z)|^2 \phi'\left(\frac{d(w,z)}{r}\right) \,\mathrm{d}z\\
\mathbf{I}_{x,k}(w,r) &:= \frac{r\mathbf{D}_{x,k}(w,r)}{\mathbf{H}_{x,k}(w,r)}\,.
\end{aligned}$$

We moreover define the quantities

$$\begin{aligned}
\mathbf{E}_{x,k}(w,r) &:= -\frac{1}{r}\int_{\mathcal{M}_{x,k}} \phi'\left(\frac{d(w,z)}{r}\right) \sum_i (N_{x,k})_i(z)\cdot D(N_{x,k})_i(z)\nabla d(w,z)\; dz,\\
\mathbf{G}_{x,k}(w,r) &:= -\frac{1}{r^2}\int_{\tilde{\mathcal{M}}_{x,k}} \phi'\left(\frac{d(w,z)}{r}\right) \frac{d(w,z)}{|\nabla d(w,z)|^2} \sum_i |D(\tilde{N}_{x,k})_i(z)\cdot \nabla d(w,z)|^2,\\
\mathbf{\Sigma}_{x,k}(w,r) &:= \int_{\tilde{\mathcal{M}}_{x,k}} \phi\left(\frac{d(w,z)}{r}\right) |\tilde{N}_{x,k}(z)|^2.
\end{aligned}$$

that are pivotal for the almost monotonicity of the frequency function $\mathbf{I}_{x,k}$. For the following lemma the arguments are the same as those given for Lemma 3.12.3 and its strengthening Lemma 3.12.6.

Lemma 3.20.2. *There exists $\varepsilon(m,n,Q)$ such that for any $\varepsilon_5 \in]0,\varepsilon]$, there exists $\gamma_4(m,n,Q) > 0$ sufficiently small and a constant $C(m,n,Q) > 0$ such that the following estimates hold for every $x \in \mathfrak{S}$, any $w \in \mathcal{M}_{x,k}$ and every radius $r \in]\eta\gamma, 4]$:*

$$\partial_r \mathbf{D}_{x,k}(w,r) = -\int_{\mathcal{M}_{x,k}} \varphi'\left(\frac{d(w,z)}{r}\right) \frac{d(w,z)}{r^2} |DN_{x,k}(z)|^2 \; dz \tag{3.156}$$

$$\partial_r \mathbf{H}_{x,k}(w,r) - \frac{m-1}{r}\mathbf{H}_{x,k}(w,r) = O(\boldsymbol{m}_{x,k})\mathbf{H}_{x,k}(w,r) + 2\mathbf{E}_{x,k}(w,r), \tag{3.157}$$

$$|\mathbf{D}_{x,k}(w,r) - \mathbf{E}_{x,k}(w,r)| \le \sum_{j=1}^{5} |\operatorname{Err}_j^o| \le C\boldsymbol{m}_{x,k}^{\gamma_4}\mathbf{D}_{x,k}(w,r)^{1+\gamma_4} + C\boldsymbol{m}_{x,k}\mathbf{\Sigma}_{x,k}(w,r), \tag{3.158}$$

$$
\begin{aligned}
&\left|\partial_r \mathbf{D}_{x,k}(w,r) - (m-2)r^{-1}\mathbf{D}_{x,k}(w,r) - 2\mathbf{G}_{x,k}(w,r)\right| \qquad (3.159)\\
&\quad \leq 2\sum_{j=1}^{5} |\operatorname{Err}_j^i| + C\boldsymbol{m}_{x,k}\mathbf{D}_{x,k}(w,r)\\
&\quad \leq Cr^{-1}\boldsymbol{m}_{x,k}^{\gamma_4}\mathbf{D}_{x,k}(w,r)^{1+\gamma_4} + C\boldsymbol{m}_{x,k}^{\gamma_4}\mathbf{D}_{x,k}(w,r)^{\gamma_4}\partial_r\mathbf{D}_{x,k}(w,r) + C\boldsymbol{m}_{x,k}\mathbf{D}_{x,k}(w,r),
\end{aligned}
$$

where Err_j^o *and* Err_j^i *are as in* [11, Proposition 9.8, Proposition 9.9].

3.20.3 Universal frequency function and total variation estimate

We are now in a position to introduce the universal frequency function adapted to our situation. Here, we amend Definition 3.6.1 accordingly.

Definition 3.20.3 (Universal frequency function adapted to $\{\gamma^j\}_j$)**.** *For* $r \in]\gamma^{k+1}, \gamma^k]$, $x \in \mathfrak{S}$ *and* $p_{x,k} = (0, \boldsymbol{\varphi}_{x,k}(0))$, *define*

$$
\begin{aligned}
\mathbf{I}(x,r) &\coloneqq \mathbf{I}_{x,k}\left(p_{x,k}, \frac{r}{\gamma^k}\right),\\
\mathbf{D}(x,r) &\coloneqq \mathbf{D}_{x,k}\left(p_{x,k}, \frac{r}{\gamma^k}\right),\\
\mathbf{H}(x,r) &\coloneqq \mathbf{H}_{x,k}\left(p_{x,k}, \frac{r}{\gamma^k}\right).
\end{aligned}
$$

We recall the following BV estimate on the universal frequency function, which, although originally stated for the universal frequency as defined in Section 3.6, also holds for the universal frequency function in Definition 3.20.3.

Proposition 3.20.4. *There exists* $\bar{\varepsilon}(m,n,Q) \in]0,\varepsilon]$ *such that for any* $\varepsilon_5 \in]0,\bar{\varepsilon}]$, *there exists* $C = C(m,n,Q,\gamma)$ *such that the following holds for every* $x \in \mathfrak{S}$*:*

$$
\left\| \left[\frac{d\log(1+\mathbf{I}(x,\cdot))}{dr} \right]_- \right\|_{\mathrm{TV}([0,1])} \leq C\sum_k [\boldsymbol{m}_{x,k}]^{\gamma_4}, \qquad (3.160)
$$

We observe that the estimate of the total variation on each open interval $]\gamma^{k+1}, \gamma^k[$ is just

using Lemma 3.20.2. As for the proof given in Section 3.6 to estimate the jumps

$$|\log(1+\mathbf{I}(x,\gamma^k))^+ - \log(1+\mathbf{I}(x,\gamma^k)^-|$$

observe that the crucial ingredient which allows us to apply the same argument in Section 3.6 is given by Proposition 3.20.1(b).

3.20.4 Upper and lower bounds on the frequency

As an immediate consequence of the total variation estimate we infer the existence of an upper bound for the frequency $\mathbf{I}(x,r)$. We also infer the existence of the limit $\mathbf{I}(x,0) = \lim_{r\downarrow 0}\mathbf{I}(x,r) = \mathrm{I}(T,x) \geq 1+2^{-K}$. In turn, choosing $\tilde{\varepsilon}$ sufficiently small we infer the following:

Corollary 3.20.5. *For $\bar{\varepsilon}$ as in Proposition 3.20.4 and any $\varepsilon_5 \in]0,\bar{\varepsilon}]$, the following holds:*

$$1+2^{-K-1} \leq \mathbf{I}(x,r) \leq 2 \qquad \forall x \in \mathfrak{S}, \forall r \in]0,1]\,.$$

Hence, a simple contradiction argument also guarantees a similar control for points sufficiently close to $\mathfrak{S}$ at the appropriate scale.

Corollary 3.20.6. *There exists $\varepsilon^* \in]0,\bar{\varepsilon}]$ such that for any $\varepsilon_5 \in]0,\varepsilon^*]$ and any $x \in \mathfrak{S}$, there is a positive constant $C_0(\gamma,\eta,m,n,Q)$, such that the following holds for every $w \in \mathcal{M}_{x,k}$ and every $r \in]\eta\gamma,4]$:*

$$C_0^{-1} \leq \mathbf{I}_{x,k}(w,r) \leq C_0\,.$$

Using the latter lower bound we can remove the logarithm from the monotonicity estimate.

Corollary 3.20.7. *For any $\varepsilon_5 \in]0,\varepsilon^*]$, x and r as in Corollary 3.20.6 we have*

$$\partial_r \mathbf{I}_{x,k}(w,r) \geq -C\tilde{m}_{x,k}^{\gamma_4} - C\boldsymbol{m}_{x,k}^{\gamma_4}\frac{\mathbf{D}_{x,k}(w,r)^{\gamma_4}}{r} - C\boldsymbol{m}_{x,k}^{\gamma_4}\mathbf{D}_{x,k}(w,r)^{\gamma_4-1}\partial_r\mathbf{D}_{x,k}(w,r)$$

$$- C\boldsymbol{m}_{x,k}^{2\gamma_4}\left(\frac{\mathbf{D}_{x,k}(w,r)^{\gamma_4}}{r}+r\right).$$

The estimates in Lemma 3.20.2 and Corollary 3.20.7 can be simplified to the following, which is an analogue of Lemma 3.13.1.

Lemma 3.20.8. *Suppose that T, ε_5, γ_4, x, $\mathcal{M}_{x,k}$ and $N_{x,k}$ are as in Corollary 3.20.7. Then there exist constants C dependent on K, γ, and η, but not on x, k or ε_5, such that the following estimates hold for every $w \in \mathcal{M}_{x,k} \cap \mathbf{B}_1$ and any $\rho, r \in]\eta\gamma, 4]$.*

$$C^{-1} \leq \mathbf{I}_{x,k}(w,r) \leq C \tag{3.161}$$

$$C^{-1} r\mathbf{D}_{x,k}(w,r) \leq \mathbf{H}_{x,k}(w,r) \leq Cr\mathbf{D}_{x,k}(w,r) \tag{3.162}$$

$$\mathbf{\Sigma}_{x,k}(w,r) \leq Cr^2\mathbf{D}_{x,k}(w,r) \tag{3.163}$$

$$\mathbf{E}_{x,k}(w,r) \leq C\mathbf{D}_{x,k}(w,r) \tag{3.164}$$

$$\frac{\mathbf{H}_{x,k}(w,\rho)}{\rho^{m-1}} = \frac{\mathbf{H}_{x,k}(w,r)}{r^{m-1}} \exp\left(-C\int_\rho^r \mathbf{I}_{x,k}(w,s)\frac{\mathrm{d}s}{s} - O(\boldsymbol{m}_{x,k})(r-\rho)\right) \tag{3.165}$$

$$\mathbf{H}_{x,k}(w,r) \leq C\mathbf{H}_{x,k}(w,\tfrac{r}{4}) \tag{3.166}$$

$$\mathbf{H}_{x,k}(w,r) \leq Cr^{m+3-2\delta_2} \tag{3.167}$$

$$\mathbf{G}_{x,k}(w,r) \leq Cr^{-1}\mathbf{D}_{x,k}(w,r) \tag{3.168}$$

$$|\partial_r \mathbf{D}_{x,k}(w,r)| \leq Cr^{-1}\mathbf{D}_{x,k}(w,r) \tag{3.169}$$

$$|\partial_r \mathbf{H}_{x,k}(w,r)| \leq C\mathbf{D}_{x,k}(w,r)\,,. \tag{3.170}$$

In particular:

$$|\mathbf{D}_{x,k}(w,r) - \mathbf{E}_{x,k}(w,r)| \leq C\boldsymbol{m}_{x,k}^{\gamma_4} r^{\gamma_4}\mathbf{D}_{x,k}(w,r) \tag{3.171}$$

$$\left|\partial_r \mathbf{D}_{x,k}(w,r) - \frac{m-2}{r}\mathbf{D}_{x,k}(w,r) - 2\mathbf{G}_{x,k}(w,r)\right| \leq C\boldsymbol{m}_{x,k}^{\gamma_4} r^{\gamma_4-1}\mathbf{D}_{x,k}(w,r) \tag{3.172}$$

$$\partial_r \mathbf{I}_{x,k}(w,r) \geq -C\boldsymbol{m}_{x,k}^{\gamma_4} r^{\gamma_4-1}\,. \tag{3.173}$$

The proof of Lemma 3.20.8 follows from Lemma 3.20.2 and Corollary 3.20.7 in exactly

the same way as Lemma 3.13.1 follows from Lemma 3.12.3, (3.102) and Corollary 3.12.5. We leave the details to the reader.

3.21 Spatial variations

We begin by defining frequency pinching in this setting, using the universal frequency.

Definition 3.21.1. *Let T and $\mathfrak{S}$ be as in Theorem 3.19.6, let $x \in \mathfrak{S}$ and assume $\mathcal{M}_{x,k}$ and $N_{x,k}$ are as in Section 3.20.1. Consider $w \in \mathcal{M}_{x,k} \cap \mathbf{B}_1$ and a corresponding point $y = x + \gamma^k w$. Let ρ, r be two radii which satisfy the inequalities*

$$\eta\gamma^{k+1} \leq \rho \leq r < 4\gamma^k \,. \tag{3.174}$$

We define the frequency pinching $W^r_\rho(x,k,y)$ around y between the scales ρ and r

$$W^r_\rho(x,k,y) \coloneqq \left|\mathbf{I}_{x,k}\Big(w,\gamma^{-k}r\Big) - \mathbf{I}_{x,k}\Big(w,\gamma^{-k}\rho\Big)\right| .$$

Our first key spatial variational result is the following.

Proposition 3.21.2. *Assume T and $\mathfrak{S}$ are as in Theorem 3.19.6 and let γ_4 be as in Lemma 3.20.2. Let $x \in \mathfrak{S}$ and $k \in \mathbb{N}$. Then there exists $C = C(m,n,Q,K,\gamma,\eta)$ such that, for any $w \in \mathcal{M}_{x,k} \cap \mathbf{B}_1$ and any radii satisfying $4\eta\gamma^{k+1} \leq \rho \leq r < 2\gamma^k$, the following holds. Let $y = x + \gamma^k w$ and let $\mathcal{A}^{2r}_{\frac{\rho}{4}}(w) \coloneqq \Big(\mathbf{B}_{2r/\gamma^k}(w) \setminus \overline{\mathbf{B}}_{\frac{\rho}{4\gamma^k}}(w)\Big) \cap \mathcal{M}_{x,k}$. Then*

$$\begin{aligned}\int_{\mathcal{A}^{2r}_{\frac{\rho}{4}}(w)} \sum_i \left| D(N_{x,k})_i(z)\frac{d(w,z)\nabla d(w,z)}{|\nabla d(w,z)|} - \mathbf{I}_{x,k}(w,d(w,z))(N_{x,k})_i(z)|\nabla d(w,z)|\right|^2 \frac{\mathrm{d}z}{d(w,z)} \\ \leq C\mathbf{H}_{x,k}\left(w,\frac{2r}{\gamma^k}\right)\left(W^{4r}_{\frac{\rho}{8}}(x,k,y) + \boldsymbol{m}^{\gamma_4}_{x,k}\left(\frac{r}{\gamma^k}\right)^{\gamma_4}\right)\log\left(\frac{16r}{\rho}\right).\end{aligned}$$

Given the estimates in Lemma 3.20.8, the proof is entirely analogous to that of the corresponding Proposition 3.14.2. We omit the details here, and simply refer the reader to

[17].

We will also require the following control on spatial variations of the frequency in terms of frequency pinching:

Lemma 3.21.3. *Let γ_4 be as in Lemma 3.20.2, T and $\mathfrak{S}$ as in Theorem 3.19.6, $x \in \mathfrak{S}$ and $k \in \mathbb{N}$. Let $x_1, x_2 \in \mathbf{B}_1 \cap \mathcal{M}_{x,k}$, $y_i = x + \gamma^k x_i$ and $d(x_1, x_2) \leq \gamma^{-k} \frac{r}{4}$, where r is such that*

$$8\eta\, \gamma^{k+1} < r \leq \gamma^k\,.$$

Then there exists $C = C(m, n, Q, \gamma, \eta) > 0$ such that for any $z_1, z_2 \in [x_1, x_2]$, we have

$$\begin{aligned}&\left|\mathbf{I}_{x,k}\left(z_1, \frac{r}{\gamma^k}\right) - \mathbf{I}_{x,k}\left(z_2, \frac{r}{\gamma^k}\right)\right| \\ &\qquad \leq C\left[\left(W^{4r}_{r/8}(x,k,y_1)\right)^{\frac{1}{2}} + \left(W^{4r}_{r/8}(x,k,y_2)\right)^{\frac{1}{2}} + \boldsymbol{m}^{\frac{\gamma_4}{2}}_{x,k}\left(\frac{r}{\gamma^k}\right)^{\frac{\gamma_4}{2}}\right]\frac{\gamma^k d(z_1,z_2)}{r}\,.\end{aligned}$$

The proof of this lemma is much like that of Lemma 3.14.4 can be found in [17] and is also omitted here. Note that it relies on spatial variation esstimates analogous to those in Lemma 3.14.5, but for $\mathbf{D}_{x,k}$ and $\mathbf{H}_{x,k}$.

3.22 Quantitative splitting

This section parallells the analogous one for the case of high frequency points. In what follows we will need to consider affine subspaces spanned by families of vectors. Recall the definition of the affine subspace $V(X)$ spanned by an ordered set of points $X = \{x_0, x_1, \ldots, x_k\}$ $\{x_1 - x_0, x_2 - x_0, \ldots, x_k - x_0\}$ and centered at x_0, as in (3.126), and the notions of ρr-linear independence and ρr-spanning from Definition 3.15.1.

The following is an analogue of Lemma 3.15.2, inferring a quantitative existence of an approximate spine in $\mathfrak{S}$, assuming quantitative approximate-homogeneity of $N_{x,j(k)}$.

Lemma 3.22.1. *Let T and $\mathfrak{S}$ be as in Theorem 3.19.6 and let $\rho, \bar{\rho} \in]0,1]$, $\tilde{\rho} \in]\eta, 1]$ be given.*

If ε_5 in Definition 3.19.4 is smaller than a suitable threshold $\tilde{\varepsilon}(m, n, Q, \gamma, K, \rho, \tilde{\rho}, \bar{\rho}) \leq \varepsilon^*$, then the following holds. Let $x \in \mathfrak{S}$ and $\gamma^{k+1} \leq r \leq \gamma^k$. Consider points $X = \{x_i\}_{i=0}^{m-2} \subset \mathbf{B}_r(x) \cap \mathfrak{S}$ which satisfy the following contitions:

- *X is a ρr-linearly independent set;*
- *if y_i denotes the closest point to x_i with the property that $\gamma^{-k}(y_i - x)$ belongs to $\mathcal{M}_{x,k}$, then*

$$W_{\tilde{\rho}r}^{2r}(x, j(k), y_i) < \tilde{\varepsilon} \qquad \text{for each } i.$$

Then $\mathfrak{S} \cap (\mathbf{B}_r \setminus \mathbf{B}_{\bar{\rho}r}(V(X))) = \emptyset$.

The argument is similar to that given for Lemma 3.15.2. We omit the proof here, and refer the reader to [17] for the details.

We will also require the following lemma, which controls spatial and radial frequency variations via frequency pinching in $(m-2)$ independent directions, and is the analogue of Lemma 3.15.3.

Lemma 3.22.2. *Let T and $\mathfrak{S}$ be as in Theorem 3.19.6 and let $\rho, \bar{\rho} \in]0, 1]$, $\tilde{\rho} \in]\eta, 1]$ be given. For any $\delta > 0$, if ε_5 is smaller than a suitable threshold $\tilde{\varepsilon}(m, n, Q, \gamma, K, \rho, \tilde{\rho}, \bar{\rho}, \delta) \leq \varepsilon^*$, then the following holds. Let $x \in \mathfrak{S}$ and $\gamma^{k+1} \leq r \leq \gamma^k$. Consider points $X = \{x_i\}_{i=0}^{m-2} \subset \mathbf{B}_r(x) \cap \mathfrak{S}$ which satisfy the following contitions:*

- *X is a ρr-linearly independent set;*
- *if y_i denotes the closest point to x_i with the property that $\gamma^{-k}(y_i - x)$ belongs to $\mathcal{M}_{x,k}$, then*

$$W_{\tilde{\rho}r}^{2r}(x, j(k), y_i) < \tilde{\varepsilon} \qquad \text{for each } i.$$

Then for every $z_1, z_2 \in \mathbf{B}_r(x) \cap \mathbf{B}_{\tilde{\varepsilon}r}(V(X))$ and for every $r_1, r_2 \in [\bar{\rho}, 1]$, if w_i denotes the closest point to $\gamma^{-k}(z_i - x)$ that belongs to $\mathcal{M}_{x,k}$, the following estimate holds:

$$|\mathbf{I}_{x,k}(w_1, r_1) - \mathbf{I}_{x,k}(w_2, r_2)| \leq \delta\,.$$

The proof of the Lemma follows the same arguments of the proof of Lemma 3.15.3 and the adjustments needed are the same as those employed in the proof of Lemma 3.22.1.

3.23 Flatness control

This section is the counterpart of Section 3.16. We wish to state a counterpart of Proposition 3.16.2. Note however an important difference: points in the set $\mathfrak{S}$ are *not* necessarily contained in the rescaling of the (scaled and translated) center manifold $x + \gamma^k \mathcal{M}_{x,k}$. For this reason we introduce a suitable "projection".

Definition 3.23.1. *Consider the center manifold $\mathcal{M}_{x,k}$ and denote by $\pi_{x,k}$ the reference plane used to construct in* [21, Theorem 1.17] *the map $\boldsymbol{\varphi} : \pi_{x,k} \supset [-4,4]^m \to \pi_{x,k}^\perp$ such that* $\mathrm{gr}\,(\boldsymbol{\varphi}) = \mathcal{M}_{x,k}$. *For any $p \in \mathbf{B}_{\gamma^k}(x)$ let*

$$q = (\mathbf{p}_{\pi_{x,k}}(\gamma^{-k}(p-x)), \boldsymbol{\varphi}(\mathbf{p}_{\pi_{x,k}}(\gamma^{-k}(p-x)))) \in \mathcal{M}_{x,k}\,.$$

Finally we let $\mathbf{p}_{x,k} : \mathbf{B}_{\gamma^k}(x) \to (x + \gamma^k \mathcal{M}_{x,k})$ be the map $p \mapsto x + \gamma^k q$.

Proposition 3.23.2. *There are $\alpha_0 = \alpha_0(\Lambda, m, n, Q) > 0$, $\eta = \eta(m) \in]0, \frac{1}{8}[$, $C(\Lambda, m, n, Q, \gamma) > 0$ with the following property. Assume ε_5 in Definition 3.19.4 is smaller than a threshold which depends on m, n, Q, K, η, and let T and $\mathfrak{S}$ be as in Theorem 3.19.6. Suppose that μ is a finite non-negative Radon measure with $\mathrm{spt}(\mu) \subset \mathfrak{S}$ and let $x_0 \in \mathfrak{S}$. Then for all $r \in \left]8\eta\gamma^{k+1}, \gamma^k\right]$ we have*

$$[\beta_{2,\mu}^{m-2}(x_0, r/8)]^2 \leq C r^{-(m-2)} \int_{\mathbf{B}_{r/8}(x_0)} W_{r/8}^{4r}(x_0, k, \mathbf{p}_{x_0,k}(x))\ \mathrm{d}\mu(x)$$
$$+ C \boldsymbol{m}_{x_0,k}^{\alpha_0} r^{-(m-2-\alpha_0)} \mu(\mathbf{B}_{r/8}(x_0)).$$

Proof. Let $\nu := (\mathbf{p}_{x_0,k})_\sharp \mu \llcorner \mathbf{B}_{r/8}(x_0)$. This measure is indeed supported in $x + \gamma^k \mathcal{M}_{x,k}$ and

we can therefore apply the same argument of Proposition 3.16.2. Observe moreover that

$$\int W^{4r}_{r/8}\left(x_0,k,y\right)\ \mathrm{d}\nu(y)$$

would then be, by definition,

$$r^{-(m-2)}\int_{\mathbf{B}_{r/8}(x_0)} W^{4r}_{r/8}\left(x_0,k,\mathbf{p}_{x_0,k}(x)\right)\ \mathrm{d}\mu(x)\,.$$

We can thus write the estimate

$$\begin{aligned}[\beta^{m-2}_{2,\nu}(x_0,r/8)]^2 \leq Cr^{-(m-2)}&\int_{\mathbf{B}_{r/8}(x_0)} W^{4r}_{r/8}\left(x_0,k,\mathbf{p}_{x_0,k}(x)\right)\ \mathrm{d}\mu(x)\\ &+C\boldsymbol{m}^{\alpha_0}_{x_0,k}r^{-(m-2-\alpha_0)}\nu(\mathbf{B}_{r/8}(x_0)),\end{aligned}$$

using Proposition 3.16.2 after rescaling all the objects by γ^{-k}, for the choice of α_0 therein. Notice that in order to retain the validity of Proposition 3.16.2 with ν, we are using Lemma 3.20.8, Lemma 3.21.3 and 3.21.2. However, since the distance between $x\in\mathfrak{S}\cap\mathbf{B}_{r/8}(x_0)$ and $x+\gamma^k\mathcal{M}_{x,k}$ is controlled by $\boldsymbol{m}^{1/2}_{x_0,k}r$, we have the estimate

$$[\beta^{m-2}_{2,\mu}(x_0,r/8)]^2 \leq 2[\beta^{m-2}_{2,\nu}(x_0,r/8)]^2+C\boldsymbol{m}_{x_0,k}r^{-(m-2)}\mu(\mathbf{B}_{r/8}(x_0))\,.$$

Recall next that by Proposition 3.11.2, we have $\boldsymbol{m}_{x_0,k}\leq C\gamma^{\bar{\alpha}k}$ for some positive exponent $\bar{\alpha}$ (depending on Λ,m,n,Q). Thus, since $\gamma^k\leq(8\eta\gamma)^{-1}r$, we easily conclude that

$$\boldsymbol{m}_{x_0,k}r^{-(m-2)}\mu(\mathbf{B}_{r/8}(x_0))\leq C\boldsymbol{m}^{1/2}_{x_0,k}r^{-(m-2-\alpha_0)}\mu(\mathbf{B}_{r/8}(x_0))$$

up to further decreasing the previous choice of the exponent α_0 if necessary. Combining this with the observation that $\nu(\mathbf{B}_{r/8}(x_0))\leq C\mu(\mathbf{B}_{r/8}(x_0))$, the conclusion of Proposition 3.23.2 follows immediately. □

3.24 Proof of Theorem 3.19.6

Given Proposition 3.23.2, Lemma 3.22.1 and Lemma 3.22.2, this follows via a completely analogous procedure to that outlined in Section 3.17.1 and Section 3.18.3. We refer the reader to [17] for more details.

Chapter 4

The fine structure of the singular set of area-minimizing integral currents: Rectifiability of degree 1 points and $\mathcal{H}^{m-2}$-a.e uniqueness of tangent cones

4.1 Introduction and main results

Let T be an m-dimensional integral current in a complete smooth $(m+\bar{n})$-dimensional Riemannian manifold Σ. We assume that T is area-minimizing in some (relatively) open $\Omega \subset \Sigma$, and we use the same notation introduced in the previous chapters. Building on the results in Chapter 3, this chapter is devoted to completing the proof of Theorem 3.1.1.

In the preceding chapter, we demonstrated the following.

Theorem 4.1.1. *Let T be an m-dimensional area-minimizing integral current in a C^{3,κ_0} complete Riemannian manifold of dimension $m+\bar{n} \geq m+2$, with $\kappa_0 > 0$. Then $\mathfrak{F}_{Q,>1} := \mathfrak{F}_Q(T) \cap \{\mathrm{I}(T,\cdot) > 1\}$ is $(m-2)$-rectifiable and the tangent cone is unique at every $p \in \mathfrak{F}_{Q,>1}$.*

In the present Chapter, we handle the rectifiability question for the remaining part of

$\mathfrak{F}_Q(T)$, as well as the $\mathcal{H}^{m-2}$-a.e. uniqueness of tangent cone question in the remaining portion of the singular set. More precisely, we prove the following:

Theorem 4.1.2. *Let T be as in Theorem 4.1.1. The following holds:*

(i) The set $\mathfrak{F}_{Q,1}(T) := \mathfrak{F}_Q(T) \cap \{\mathrm{I}(T,\cdot) = 1\}$ is $\mathcal{H}^{m-2}$-null;

(ii) T has a unique tangent cone at $\mathcal{H}^{m-2}$-a.e. $p \in \mathcal{S}^{(m-2)}(T)$.

4.2 Preliminaries and notation

4.2.1 Notation

We will be using the same basic notation as in previous chapters; we refer the reader to the introductions therein for a reminder. The plane π will be omitted from cylinders and disks $B_r(p,\pi)$ if clear from the context; if the center p is omitted, then it is assumed to be the origin. For example, $\mathbf{C}_r := \mathbf{C}_r(0,\pi)$ for $r > 0$ if the choice of π is clear from the context. The Hausdorff distance between two subsets A and B of $\mathbb{R}^{m+\bar{n}}$ will be denoted by $\mathrm{dist}(A,B)$.

The scalar product between vectors $v, w \in \mathbb{R}^{m+n}$ is denoted by $v \cdot w$ and likewise for product of matrices we will use $A \cdot B$. $|\cdot|$ will denote the Euclidean norm of vectors and the Hilbert-Schmidt norm of matrices. For subsets E of $\mathbb{R}^m$ we will use the shorthand $|E|$ for their Lebesgue measure. This convention will often also be used for the $\mathcal{H}^m$ measures of subsets E of affine m-dimensional subspaces of $\mathbb{R}^{m+n}$.

Since our statements are local and invariant under dilations and translations, we will work with the underlying Assumption 3.2.2 from Chapter 3 here. We will additionally assume that $m \geq 3$ throughout this chapter.

Following the notation of Chapter 3, we let $\mathbf{A}_\Sigma$ denote the C^0 norm of the second fundamental form A_Σ of Σ in $\mathbf{B}_{7\sqrt{m}}$. In particular, under Assumption 3.2.2, we have

$$\mathbf{A}_\Sigma \coloneqq \|A_\Sigma\|_{C^0(\Sigma \cap \mathbf{B}_{7\sqrt{m}})} \leq C_0 \boldsymbol{c}(\Sigma) \leq C_0 \bar{\varepsilon}.$$

We will often drop the subscript as the underlying ambient manifold will mostly be clear from the context.

We recall from Definition 1.4.1 the *oriented tilt-excess* $\mathbf{E}(T, \mathbf{B}_r(p), \pi)$, $\mathbf{E}(T, \mathbf{C}_r(p, \pi_0), \pi)$ and the respective optimal quantities $\mathbf{E}(T, \mathbf{C}_r(p, \pi_0))$ and $\mathbf{E}(T, \mathbf{B}_r(p))$ over all oriented m-dimensional planes π.

The tilt-excess is morally a planar L^2 "gradient" excess. We will shortly also introduce a notion of planar L^2 "height" excess.

We also refer to Chapter 3 for the notion of a *coarse blow-up* of T at p (and all related terminology and notation), which, roughly speaking, under the assumption that $r^2\mathbf{A}^2 \ll \mathbf{E}(T, \mathbf{B}_{6\sqrt{m}r}(p)) \ll 1$ along a given family of scales r, gives rise to a Dir-minimizing Q-valued function over $B_1(0, \pi_0)$ for some m-dimensional plane π_0 as a subsequential normalized limit. This Dir-minimizer is a normalized limit of graphs that approximate the rescaled current $T_{p,\rho}$ in the domain $\mathbf{C}_1(0, \pi_0) \cap \mathbf{B}_4$ with a mass error which is $o(\mathbf{E}(T, \mathbf{B}_{6\sqrt{m}}))$, along with other related error estimates which are detailed in Chapter 3 and [19].

The properties of $\mathfrak{F}_{Q,1}(T)$ that will be most useful to us here are contained within the following proposition, which is a consequence of the analysis in Chapter 3 (more precisely, see Proposition 3.4.1 and Corollary 3.4.3).

Proposition 4.2.1. *Let T, Σ and $\mathbf{A}$ be as in Assumption 3.2.2. For every $p \in \mathfrak{F}_{Q,1}(T)$ and any sequence $r_k \downarrow 0$ with the property that $\mathbf{E}_k := \mathbf{E}(T, \mathbf{B}_{6\sqrt{m}r_k}(p)) \downarrow 0$ the following holds:*

(i) For $\mathbf{A}_k := \mathbf{A}_{\Sigma_{p,r_k}}$ we have $\lim_{k\to\infty} \mathbf{E}_k^{-1} r_k^{2-2\delta_2} \mathbf{A}_k^2 = 0$;

(ii) Any coarse blow-up $\bar{f}$ generated by a subsequence of $\{r_k\}$ has positive Dirichlet energy, is 1-homogeneous, and satisfies $\boldsymbol{\eta} \circ \bar{f} \equiv 0$.

4.2.2 Main decay theorem

To prove Theorem 3.1.1, we will show that a key excess decay theorem holds under the assumption that T is much closer to a cone with exactly $m-2$ directions of translation

invariance than it is to any m-dimensional plane. Before coming to the statement of this theorem, let us first introduce suitable notions of L^2 height excess of T relative to such cones. We begin by defining the cones of interest.

Definition 4.2.2. *For every fixed integer $Q \geq 2$ we denote by $\mathscr{C}(Q)$ those subsets of $\mathbb{R}^{m+n}$ which are unions of $N \leq Q$ m-dimensional planes $\pi_1, \ldots, \pi_N$ satisfying the following properties:*

(i) $\pi_i \cap \pi_j$ is the same $(m-2)$-dimensional plane V for every pair (i,j) with $i < j$;

(ii) Each plane π_i is contained in the same $(m+\bar{n})$-dimensional plane ϖ.

If $p \in \Sigma$, then $\mathscr{C}(Q,p)$ will denote the subset of $\mathscr{C}(Q)$ for which $\varpi = T_p\Sigma$.

$\mathscr{P}$ and $\mathscr{P}(p)$ will denote the subset of those elements of $\mathscr{C}(Q)$ and $\mathscr{C}(Q,p)$ respectively which consist of a single plane; namely, with $N = 1$. For $\mathbf{S} \in \mathscr{C}(Q) \setminus \mathscr{P}$, the $(m-2)$-dimensional plane V described in (i) above is referred to as the spine *of $\mathbf{S}$ and will often be denoted by $V(\mathbf{S})$.*

We are now in a position to introduce the *conical L^2 height excess* between T and elements in $\mathscr{C}(Q)$.

Definition 4.2.3. *Given a ball $\mathbf{B}_r(q) \subset \mathbb{R}^{m+n}$ and a cone $\mathbf{S} \in \mathscr{C}(Q)$, we define the* one-sided conical L^2 height excess of T relative to $\mathbf{S}$ in $\mathbf{B}_r(q)$, *denoted $\hat{\mathbf{E}}(T, \mathbf{S}, \mathbf{B}_r(q))$, by*

$$\hat{\mathbf{E}}(T, \mathbf{S}, \mathbf{B}_r(q)) := \frac{1}{r^{m+2}} \int_{\mathbf{B}_r(q)} \operatorname{dist}^2(p, \mathbf{S})\, d\|T\|(p).$$

At the risk of abusing notation, we further define the corresponding reverse one-sided excess *as*

$$\hat{\mathbf{E}}(\mathbf{S}, T, \mathbf{B}_r(q)) := \frac{1}{r^{m+2}} \int_{\mathbf{B}_r(q) \cap \mathbf{S} \setminus \mathbf{B}_{ar}(V(\mathbf{S}))} \operatorname{dist}^2(x, \operatorname{spt}(T))\, d\mathcal{H}^m(x)\,,$$

where $a = a(m)$ is a dimensional constant, to be specified later. We subsequently define the

two-sided conical L^2 height excess *as*

$$\mathbb{E}(T, \mathbf{S}, \mathbf{B}_r(q)) := \hat{\mathbf{E}}(T, \mathbf{S}, \mathbf{B}_r(q)) + \hat{\mathbf{E}}(\mathbf{S}, T, \mathbf{B}_r(q))\,.$$

We finally introduce the planar L^2 height excess *which is given by*

$$\mathbf{E}^p(T, \mathbf{B}_r(q)) = \min_{\pi \in \mathscr{P}(q)} \hat{\mathbf{E}}(T, \pi, \mathbf{B}_r(q))\,.$$

Let us now state our main excess decay theorem. This is similar in spirit to the excess decay theorem originally seen in [45, Lemma 1], however a crucial difference with the present setting is that in [45, Lemma 1] there is a built-in multiplicity one assumption which in particular rules out branch point singularities a priori. Our excess decay theorem in contrast is in a higher multiplicity setting, and therefore is closer (and indeed our proof follows a similar pattern) to the excess decay theorems first seen in [56, Section 13] (see also [31, Lemma 5.6 and Lemma 12.1] and [40, Theorem 2.1]).

Theorem 4.2.4 (Excess Decay Theorem)**.** *For every $Q, m, n, \bar{n}$, and $\varsigma > 0$, there are positive constants $\varepsilon_0 = \varepsilon_0(Q, m, n, \bar{n}, \varsigma) \leq \frac{1}{2}$, $r_0 = r_0(Q, m, n, \bar{n}, \varsigma) \leq \frac{1}{2}$ and $C = C(Q, m, n, \bar{n})$ with the following property. Assume that*

(i) T and Σ are as in Assumption 3.2.2;

(ii) $\|T\|(\mathbf{B}_1) \leq (Q + \frac{1}{2})\omega_m$;

(iii) There is $\mathbf{S} \in \mathscr{C}(Q, 0) \setminus \mathscr{P}(0)$ such that

$$\mathbb{E}(T, \mathbf{S}, \mathbf{B}_1) \leq \varepsilon_0^2 \mathbf{E}^p(T, \mathbf{B}_1) \tag{4.1}$$

and

$$\mathbf{B}_{\varepsilon_0}(\xi) \cap \{p : \Theta(T, p) \geq Q\} \neq \emptyset \qquad \forall \xi \in V(\mathbf{S}) \cap \mathbf{B}_{1/2}\,; \tag{4.2}$$

(iv) $\mathbf{A}^2 \leq \varepsilon_0^2 \mathbb{E}(T, \mathbf{S}', \mathbf{B}_1)$ for any *$\mathbf{S}' \in \mathscr{C}(Q, 0)$.*

Then there is a $\mathbf{S}' \in \mathscr{C}(Q,0) \setminus \mathscr{P}(0)$ *such that*

(a) $\mathbb{E}(T, \mathbf{S}', \mathbf{B}_{r_0}) \leq \varsigma \mathbb{E}(T, \mathbf{S}, \mathbf{B}_1)$

(b) $\dfrac{\mathbb{E}(T, \mathbf{S}', \mathbf{B}_{r_0})}{\mathbf{E}^p(T, \mathbf{B}_{r_0})} \leq 2\varsigma \dfrac{\mathbb{E}(T, \mathbf{S}, \mathbf{B}_1)}{\mathbf{E}^p(T, \mathbf{B}_1)}$

(c) $\operatorname{dist}^2(\mathbf{S}' \cap \mathbf{B}_1, \mathbf{S} \cap \mathbf{B}_1) \leq C\mathbb{E}(T, \mathbf{S}, \mathbf{B}_1)$

(d) $\operatorname{dist}^2(V(\mathbf{S}) \cap \mathbf{B}_1, V(\mathbf{S}') \cap \mathbf{B}_1) \leq C \dfrac{\mathbb{E}(T, \mathbf{S}, \mathbf{B}_1)}{\mathbf{E}^p(T, \mathbf{B}_1)}$.

4.2.3 Structure of the chapter

The majority of this chapter will be dedicated to the proof of Theorem 4.2.4. We begin with a refined $L^2 - L^\infty$ height bound in Section 4.3, which will be a key tool here. However, we believe that this height bound is of interest in itself, and would have a number of other applications.

Section 4.4 will contain the proof of the Excess Decay Theorem 4.2.4. The proof consists of several key parts, modeled off the foundational works of Simon ([45]) and Wickramasekera ([56]); the reader familiar with these works will find many of the arguments in this part similar in spirit to those seen in these, and we will make the effort to point out the similarities and differences throughout the work to aid the reader. Section 4.5 contains some key results regarding the relative positioning and angles between pairs of planes in cones in $\mathscr{C}(Q) \setminus \mathscr{P}$, followed by Section 4.6, which introduces effective graphical approximations for T over such cones. In Section 4.7, we then use the preceding two sections to demonstrate that we may replace the initial cone $\mathbf{S}$ in Theorem 4.2.4 with a *balanced* cone (cf. Definition 4.6.4). In Section 4.8 we reduce the proof of Theorem 4.2.4 to an a priori much weaker decay statement, in particular one where we can assume that the two-sided excess is much smaller than the *minimal* angle in the cone (as opposed to the maximal angle, which is morally what (4.1)

says). Section 4.9 is then dedicated to the *Simon estimates*, including the non-concentration estimates, at the spine of the cone in $\mathscr{C}(Q) \setminus \mathscr{P}$, after which, in Section 4.10 we demonstrate the excess decay conclusion at the level of the *linearized problem* of multiple-valued Dirichlet minimizers. Section 4.4 is then concluded with Section 4.11, in which we put together everything from the previous sections to conclude the proof of Theorem 4.2.4.

In Section 4.12 we use Theorem 4.2.4 combined with a covering procedure, analogously to that done by Simon ([45]), to prove Theorem 3.1.1.

4.3 $L^2 - L^\infty$ height bound

The aim of this section is to prove a generalization of Allard's tilt-excess and L^∞ estimates (see [1, Section 8]). Allard's original work, in the context of stationary varifolds, bounds the L^∞ distance and L^2 tilt excess from a single plane with the L^2 distance to it. Here, we will restrict ourselves to the much smaller class of area-minimizing currents, with the additional benefit being that are able to control the L^∞ distance and L^2 tilt excess from a *finite* collection of mutually disjoint planes by the L^2 distance to the union of the planes.

4.3.1 Main statements

For the remainder of this section, we make the following additional assumption.

Assumption 4.3.1. T, Σ, and $\mathbf{A}$ are as in Assumption 3.2.2. For some oriented m-dimensional plane $\pi_0 \subset \mathbb{R}^{m+\bar{n}}$ passing through the origin and some positive integer Q, we have

$$(\mathbf{p}_{\pi_0})_\sharp T \llcorner \mathbf{C}_2(0, \pi_0) = Q\llbracket B_2(\pi_0)\rrbracket\,,$$

and $\|T\|(\mathbf{C}_2) \leq (Q + \frac{1}{2})\omega_m 2^m$.

The main result of this section is the following.

Theorem 4.3.2 (L^∞ and tilt-excess estimates)**.** *For every $1 \le r < 2$, Q, and N, there is a positive constant $\bar{C} = \bar{C}(Q, m, n, \bar{n}, N, r) > 0$ with the following property. Suppose that T, Σ, $\mathbf{A}$ and π_0 are as in Assumption 4.3.1, let $p_1, \ldots, p_N \in \pi_0^\perp$ be distinct points, and set $\boldsymbol{\pi} := \bigcup_i p_i + \pi_0$. Let*

$$E := \int_{\mathbf{C}_2} \operatorname{dist}^2(p, \boldsymbol{\pi})\, d\|T\|(p)\,. \tag{4.3}$$

Then

$$\mathbf{E}(T, \mathbf{C}_r, \pi_0) \le \bar{C}(E + \mathbf{A}^2) \tag{4.4}$$

and, if $E \le 1$,

$$\operatorname{spt}(T) \cap \mathbf{C}_r \subset \{p : \operatorname{dist}(p, \boldsymbol{\pi}) \le \bar{C}(E^{1/2} + \mathbf{A})\}\,. \tag{4.5}$$

A translation followed by a simple scaling argument clearly gives corresponding estimates when 0 is replaced by an arbitrary center and the scales r and 2 are replaced by two arbitrary radii $\rho < R$. We additionally record the following consequence of Theorem 4.3.2, which will be used frequently throughout this chapter.

Corollary 4.3.3. *For each pair of positive integers Q and N, there is a positive constant $\delta = \delta(Q, m, n, \bar{n}, N)$ with the following properties. Assume that:*

(i) T, Σ, and $\mathbf{A}$ are as in Assumption 3.2.2;

(ii) T is area-minimizing in Σ and for some positive $r \le \frac{1}{4}$ and $q \in \operatorname{spt}(T) \cap \mathbf{B}_1$ we have $\partial T \llcorner \mathbf{C}_{4r}(q) = 0$, $(\mathbf{p}_{\pi_0})_\sharp T = Q[\![B_{4r}(q)]\!]$, and $\|T\|(\mathbf{C}_{2r}(q)) \le \omega_m(Q + \frac{1}{2})(2r)^m$;

(iii) $p_1, \ldots, p_N \in \mathbb{R}^{m+n}$ are distinct points with $\mathbf{p}_{\pi_0}(p_i) = q$ and $\varkappa := \min\{|p_i - p_j| : i < j\}$;

(iv) $\pi_1, \ldots, \pi_N$ are oriented planes passing through the origin with

$$\tau := \max_i |\pi_i - \pi_0| \le \delta \min\{1, r^{-1}\varkappa\}\,; \tag{4.6}$$

(v) Upon setting $\boldsymbol{\pi} = \bigcup_i (p_i + \pi_i)$*, we have*

$$(r\mathbf{A})^2 + (2r)^{-m-2} \int_{\mathbf{C}_{2r}(q)} \operatorname{dist}^2(p, \boldsymbol{\pi}) d\|T\| \leq \delta^2 \min\{1, r^{-2}\varkappa^2\} . \tag{4.7}$$

Then $T \llcorner \mathbf{C}_r(q) = \sum_{i=1}^N T_i$ *where*

(a) Each T_i *is an integral current with* $\partial T_i \llcorner \mathbf{C}_r(q) = 0$*;*

(b) $\operatorname{dist}(q, \boldsymbol{\pi}) = \operatorname{dist}(q, p_i + \pi_i)$ *for each* $q \in \operatorname{spt}(T_i)$*;*

(c) $(\mathbf{p}_{\pi_0})_\sharp T_i = Q_i \llbracket B_r(q) \rrbracket$ *for some non-negative integer* Q_i*.*

Proof of Corollary 4.3.3

First of all, by scaling we can assume that $r = 1$ and up to translation we can assume $q = 0$. We then introduce

$$E := \int_{\mathbf{C}_2} \operatorname{dist}^2(p, \boldsymbol{\pi}) d\|T\|(p) .$$

Next let $\bar{\boldsymbol{\pi}} := \bigcup_i (p_i + \pi_0)$ and observe that

$$\bar{E} := \int_{\mathbf{C}_2} \operatorname{dist}^2(p, \bar{\boldsymbol{\pi}}) d\|T\|(p) \leq CE + C\tau^2 \leq C\delta^2 \min\{1, \varkappa^2\} ,$$

for a constant $C = C(m, n, \bar{n}, Q)$ which we stress is independent of δ and $\varkappa$. Since we also have $\mathbf{A}^2 \leq \delta^2 \varkappa^2$, we can apply Theorem 4.3.2 to conclude that

$$\operatorname{spt}(T) \cap \mathbf{C}_{3/2} \subset \{p : \operatorname{dist}(p, \bar{\boldsymbol{\pi}}) \leq C_1 \delta \varkappa\} ,$$

where $C_1 = C_1(m, n, \bar{n}, Q, N)$ is again independent of δ and $\varkappa$. In particular if we choose $\delta = \delta(m, n, \bar{n}, Q, N) > 0$ so that $C_1 \delta < 1/4$, we have

$$\operatorname{spt}(T) \cap \mathbf{C}_{3/2} \subset \{p : \operatorname{dist}(p, \bar{\boldsymbol{\pi}}) < 4^{-1} \varkappa\} .$$

Due to the definition of $\varkappa$, $\{p : \mathrm{dist}(p, \bar{\boldsymbol{\pi}}) < 4^{-1}\varkappa\}$ consists of N disjoint open sets $U_i := \{p : \mathrm{dist}(p - p_i, \pi_0) < \varkappa/4\}$. We then set $T_i := T \llcorner \mathbf{C}_{3/2} \cap U_i$; clearly each T_i is integral, has no boundary in $\mathbf{C}_{3/2}$, and $\sum_i T_i = T \llcorner \mathbf{C}_{3/2}$. It follows also that the T_i's have disjoint support and so each of them is area-minimizing. If we further take $\delta < 1/8$ such that $C_1\delta < 1/8$, we can in addition ensure that for each point $p \in U_i$,

$$\mathrm{dist}(p, p_i + \pi_i) \leq \frac{\varkappa}{4}, \tag{4.8}$$

$$\mathrm{dist}(p, p_j + \pi_j) \geq \frac{\varkappa}{2} \qquad \forall j \neq i\,. \tag{4.9}$$

and thus (b) is certainly satisfied. Moreover, by [21, Lemma 1.6], observe that $(\mathbf{p}_{\pi_0})_\sharp T_i = Q_i[\![B_1]\!]$ for some $Q_i \in \mathbb{Z}$; clearly we must have $\sum_i Q_i = Q$. Again applying Theorem 4.3.2 we have

$$\mathbf{E}(T, \mathbf{C}_1, \pi_0) \leq C\delta^2\,.$$

In particular, if δ is chosen small enough, by [19, Theorem 2.4] we can ensure the existence of a subset $K \subset B_1(\pi_0)$ of positive measure with the property that for each $x \in K$, the slice $\langle T, \mathbf{p}_{\pi_0}, x\rangle$ (see [44] for the definition and properties of the slicing map) given by $\sum_j k_j \delta_{\xi_j}$ for some finite collection of positive integers k_j and points $\xi_j(x) \in \pi_0^\perp$. Fix any such point x and observe that

$$\langle T_i, \mathbf{p}_{\pi_0}, x\rangle = \sum_{\xi_j \in U_i} k_j \delta_{\xi_j}$$

while

$$Q_i = \sum_{\xi_j \in U_i} k_j\,.$$

It follows immediately that Q_i cannot be negative, which completes the proof. □

4.3.2 Preliminaries

Let us collect all the required preliminary results for the proof of Theorem 4.3.2.

Oriented and non-oriented tilt-excess

Given an m-dimensional plane π and a cylinder $\mathbf{C} = \mathbf{C}_r(q, \pi)$, recall that the *non-oriented tilt excess* is given by

$$\mathbf{E}^{no}(T, \mathbf{C}) := \frac{1}{2\omega_m r^m} \int_{\mathbf{C}} |\mathbf{p}_{T(x)} - \mathbf{p}_\pi|^2 \, d\|T\|(x) \,, \tag{4.10}$$

where $T(x)$ denotes the (approximate) tangent plane to T at x. This is more generally defined for integral varifolds and does not take into consideration the orientation of T and π, in contrast with the oriented tilt excess $\mathbf{E}(T, \mathbf{C})$ defined previously.

It is obvious that $|\mathbf{p}_\alpha - \mathbf{p}_\beta| \leq C|\vec{\alpha} - \vec{\beta}|$ for a geometric constant $C = C(m, n)$ and for every pair of oriented planes, so $\mathbf{E}^{no}(T, \mathbf{C}) \leq C\mathbf{E}(T, \mathbf{C})$. On the other hand, the opposite inequality is only true if $|\vec{\alpha} - \vec{\beta}|$ is sufficiently small, for instance, if it is no larger than 1. In particular $\vec{\alpha}$ and $\vec{\beta}$ could be opposite orientations for the same linear subspace: in that case $|\vec{\alpha} - \vec{\beta}| = 2$ while $|\mathbf{p}_\alpha - \mathbf{p}_\beta| = 0$.

Nonetheless, for area-minimizing currents as in Assumption 3.2.2, the nonoriented excess controls the oriented excess. The idea of the argument is borrowed from [15, Theorem 16.1], but we repeat it here for clarity. A simple scaling and translation argument, which is left to the reader, gives a corresponding estimate on any pair of parallel concentric cylinders.

Proposition 4.3.4. *For every $1 \leq r < 2$ there is a constant $\bar{C} = \bar{C}(Q, m, n, \bar{n}, 2 - r)$ such that, if T , Σ and $\mathbf{A}$ are as in Assumption 4.3.1, then*

$$\mathbf{E}(T, \mathbf{C}_r) \leq \bar{C}(\mathbf{E}^{no}(T, \mathbf{C}_2) + \mathbf{A}^2) \,. \tag{4.11}$$

Proof. Since $\mathbf{E}(T, \mathbf{C}_r) \leq \frac{1}{\omega_m r^m} \|T\|(\mathbf{C}_r)$, we can without loss of generality assume that

$$\mathbf{E}^{no}(T, \mathbf{C}_2) + \mathbf{A}^2 \leq \delta$$

for some fixed constant δ, as long as in the end we choose $\delta = \delta(Q, m, n, \bar{n}, 2 - r) > 0$.

Moreover, for any $\eta > 0$, if δ is chosen to be sufficiently small depending on η, we can also assume without loss of generality that $\mathbf{E}(T, \mathbf{C}_{1+r/2}) \leq \eta$ for another fixed constant η: indeed, arguing by contradiction, if this were not true for some $\eta > 0$, we could find a sequence of $\delta_k \downarrow 0$ and a sequence of T_k with $\mathbf{E}^{no}(T_k, \mathbf{C}_2) \leq \delta_k \downarrow 0$ yet $\mathbf{E}(T_k, \mathbf{C}_{1+r/2}) > \eta$ for all k. If the supports of the currents are equibounded, then this would give that the T_k converge, locally in mass in $\mathbf{C}_2$, to a union of planes which are parallel to π_0 (counted with a suitable multiplicity), which in turn would imply $\mathbf{E}(T_k, \mathbf{C}_{1+r/2}) \to 0$, giving the desired contradiction. If the supports are not equibounded, one can resort to the height bound [21, Theorem A.1] to decompose each T_k into a disjoint finite sum of area-minimizing currents T_k^j, each of which have equibounded supports (after translation). One can then argue as above for T_k^j, for each j.

We thus assume

$$\Lambda_0 := \mathbf{E}(T, \mathbf{C}_{1+r/2}) + \mathbf{A}^2 \leq \eta\,, \tag{4.12}$$

for some η whose choice will be given later (depending only on $Q, m, n, \bar{n}, 2-r$). We also assume

$$\mathbf{A}^2 \leq \mathbf{E}(T, \mathbf{C}_{1+r/2})\,, \tag{4.13}$$

otherwise the estimate we are looking for is trivially true. In particular $\Lambda_0 \leq 2\mathbf{E}(T, \mathbf{C}_{1+r/2})$.

Set $r_0 := 1 + \frac{r}{2}$. Recalling Almgren's strong Lipschitz approximation ([19, Theorem 1.4]), provided η is sufficiently small (depending on $Q, m, n, \bar{n}, 2-r$), there is a set $K \subset B_{r_0/4}(\pi_0)$ and a Lipschitz Q-valued map $f : B_{r_0/4} \to \mathcal{A}_Q(\pi_0^\perp)$ such that

- $\|T\|((B_{r_0/4} \setminus K) \times \pi_0^\perp) \leq C\Lambda_0^{1+2\gamma}$;
- $\mathrm{Lip}\,(f) \leq C\Lambda_0^{2\gamma}$;
- $\mathbf{G}_f \llcorner (K \times \pi_0^\perp) = T \llcorner (K \times \pi_0^\perp)$.

γ and C are fixed positive constants depending on $Q, m, n, \bar{n}$. At the price of making C larger, we can achieve the same estimates with $B_{r_0 - C\Lambda_0^{2\gamma}}$ in place of $B_{r_0/4}$; this is achieved with a

simple covering argument, applying the Lipschitz approximation theorem in a collection of $C\Lambda_0^{-m\beta}$ cylinders whose cross-sections are disks in π_0 of radius Λ_0^β which cover $B_{r_0-C\Lambda_0^\gamma}$, for a suitable choice of $\beta = \beta(\gamma, m)$ (for a detailed argument, see [15, Proposition 16.2]).

Moreover, if we assume that η is small enough, by halving the exponents 2γ to γ we can assume that the constant C in all the above estimates (including in the radius of $B_{r_0-C\Lambda_0^{2\gamma}}$) is at most $\frac{1}{4}$. More precisely, for that fixed $C = C(Q, m, n, \bar{n})$ we have $C\Lambda_0^{2\gamma} = C\Lambda_0^\gamma \cdot \Lambda_0^\gamma \leq (C\eta^\gamma)\Lambda_0^\gamma$, so if we choose η small enough so that $C\eta^{\gamma/2} \leq 1/4$, we guarantee this.

We now set

$$r_1 := r_0 - \Lambda_0^\gamma \,.$$

Summarizing we have an approximation on B_{r_1}, which we still denote by f, such that

- $\|T\|((B_{r_1} \setminus K) \times \pi_0^\perp) \leq \frac{1}{4}\Lambda_0^{1+\gamma}$;
- $\mathrm{Lip}\,(f) \leq \Lambda_0^\gamma$;
- $\mathbf{G}_f \llcorner (K \times \pi_0^\perp) = T \llcorner (K \times \pi_0^\perp)$.

If η is small enough (which guarantees smallness of $\mathrm{Lip}(f)$, and hence comparability between the oriented and non-oriented tilt-excess of $\mathbf{G}_f$), we can estimate

$$\mathbf{E}(\mathbf{G}_f \llcorner (K \times \pi_0^\perp), \mathbf{C}_{r_1}) \leq C\mathbf{E}^{no}(\mathbf{G}_f \llcorner (K \times \pi_0^\perp), \mathbf{C}_{r_1}) \leq C\mathbf{E}^{no}(T, \mathbf{C}_2)$$

so that, in particular

$$\mathbf{E}(T, \mathbf{C}_{r_1}) \leq C\mathbf{E}^{no}(T, \mathbf{C}_2) + \frac{1}{2}\Lambda_0^{1+\gamma} \,.$$

If $\Lambda_0^{1+\gamma} \leq \mathbf{E}(T, \mathbf{C}_{r_1})$ we are then done, provided that $r_1 > r$, which may be achieved by taking η sufficiently small. Otherwise we must have

$$\mathbf{E}(T, \mathbf{C}_{r_1}) \leq \Lambda_0^{1+\gamma} \,.$$

We now iterate the above argument, inductively setting

$$r_k := r_{k-1} - \Lambda_{k-1}^{\gamma} \tag{4.14}$$

$$\Lambda_k := \mathbf{E}(T, \mathbf{C}_{r_k}) \leq \Lambda_{k-1}^{1+\gamma}. \tag{4.15}$$

We stop at a certain step if we have the desired estimate and $r_k \geq r$.

Since $\Lambda_k \leq \Lambda_0^{(1+\gamma)^k} \leq \eta^{(1+\gamma)^k}$ and $r_0 - r > 0$, provided that η is small enough, the inequality $r_k \geq r$ is always guaranteed, no matter how large k is. Therefore, if the procedure never stops, we conclude that $\mathbf{E}(T, \mathbf{C}_r) = 0$. But of course in this case the sought-after inequality is trivially true. This completes the proof. □

The case $N = 1$

Observe that the difficulty of Theorem 4.3.2 lies in the case when $N \geq 2$. Indeed, when Q is arbitrary and $N = 1$, the L^2 tilt-excess estimate (4.4) is an immediate consequence of the work of Allard in [1] (see also [10, Proposition 4.1]), combined with Proposition 4.3.4. Regarding the L^∞ estimate (4.5), this is also contained within [1] when the ambient space is Euclidean (and thus $\mathbf{A} = 0$). More generally the general techniques in [1] would give a linear dependence of the estimate in $\mathbf{A}$: to see that in our case this can be improved to a quadratic the reader can consult, for instance, [50]. From now on we will therefore assume the following.

Proposition 4.3.5. *The conclusions of Theorem 4.3.2 holds for $N = 1$ (and Q arbitrary).*

Combinatorial lemmas

We will make use of the following combinatorial lemmas. The first one is essentially a consequence of [18, Lemma 3.8], but since we need additional information we provide a self-contained proof.

Lemma 4.3.6. *Fix any positive $\bar{\delta} \leq \frac{1}{2}$ and a natural number $N \geq 2$. There is a constant $\bar{C} = \bar{C}(\bar{\delta}, N)$ with the following property. Given a set (of distinct points) $P \subset \mathbb{R}^n$ with cardinality N, there is a subset $P' \subset P$ of cardinality at least two such that:*

(i) $\max\{|q-p| : q, p \in P'\} \leq \bar{C} \min\{|q-p| : q, p \in P',\ q \neq p\}$;

(ii) $\mathrm{dist}(p, P') \leq \bar{\delta} \min\{|q_1 - q_2| : q_1, q_2 \in P',\ q_1 \neq q_2\}$ *for every* $p \in P$;

(iii) the points in $P \setminus P'$ can be ordered as $\{p_1, \dots, p_J\}$ so that, setting $P_0 = P$ and $P_j := P \setminus \{p_1, \dots, p_j\}$, the following property holds:

$$\mathrm{dist}(p_j, P_j) = \min\{|p-q| : p, q \in P_{j-1},\ p \neq q\}\,.$$

Lemma 4.3.7. *Fix any $0 < \delta \leq \frac{1}{2}$ and $\varepsilon > 0$. Given a set (of distinct points) $P \subset \mathbb{R}^n$ with cardinality $N \geq 2$, we can find a nonempty subset $\tilde{P} \subset P$ such that*

(i) $\mathrm{dist}(p, \tilde{P}) \leq \delta^{-1}(1+\delta^{-1})^{N-2}\varepsilon$ *for every* $p \in P$;

(ii) Either $\tilde{P}$ is a singleton, or $\max\{\varepsilon, \mathrm{dist}(p, \tilde{P})\} \leq \delta \min\left\{ |q_1 - q_2| \,:\, q_1, q_2 \in \tilde{P},\ q_1 \neq q_2 \right\}$ *for any $p \in P$.*

Lemma 4.3.8. *Consider a set $P = \{p_1, \dots, p_N\} \subset \mathbb{R}^n$ of $N \geq 2$ distinct points and set $M := \max_{i,j}\{|p_i - p_j|\}$. Then we can decompose $P = P_1 \cup P_2$ into two disjoint non-empty sets such that*

$$\min\{|p_1 - p_2| : p_1 \in P_1, p_2 \in P_2\} \geq \frac{M}{2^{N-2}}\,.$$

Proof of Lemma 4.3.6

First of all, note that if $N = 2$ the conclusion of the lemma is trivially true for $P' = P$, so we may assume $N \geq 3$. Then we define inductively the sets P_j, with $j = \{0, \dots, N-2\}$ and the elements p_j in the following way. $P_0 = P$ and, for every $j \geq 1$, we select $p_j \in P_{j-1}$ with

the property that

$$M_{j-1} := \max\{|p-q| : p,q \in P_{j-1}\} = \max\{|p-q| : p,q \in P_{j-1} \setminus \{p_j\}\}$$
$$m_{j-1} := \min\{|p-q| : p,q \in P_{j-1},\ p \neq q\} \leq \min\{|p-q| : p,q \in P_{j-1} \setminus \{p_j\},\ p \neq q\}$$

Since the cardinality of P_{j-1} is always at least three it is clear that such an element exists: we first select p,q such that $|p-q| = \min\{|a-b| : a \neq b \in P_{j-1}\}$ and then p',q' such that $|p'-q'| = \max\{|a-b| : a,b \in P_{j-1}\}$. If the pairs (p,q) and (p',q') are the same, then obviously all points in P_{j-1} are equidistant and we can define p_j to be any of them. Otherwise the two pairs have at most one element in common, and we define p_j to be an element in $\{q,p\} \setminus \{q',p'\}$. We then let $P_j := P_{j-1} \setminus \{p_j\} = P \setminus \{p_1,\dots,p_j\}$. By construction, we have

$$m_{j-1} = \operatorname{dist}(p_j, P_{j-1} \setminus \{p_j\}) = \operatorname{dist}(p_j, P_j)\,, \tag{4.16}$$

while at the same time $M_j = M_0$ and $m_j \geq m_{j-1}$. Observe in particular that the requirements (iii) is satisfied if we terminate this process at any stage P_J.

Now let $\lambda = \lambda(\bar\delta) \geq 1$ be a large number whose choice will be specified later and let J be the smallest number j such that $M_j \leq \lambda^{N-2-j} m_j$. Since P_{N-2} consists of two elements, $M_{N-2} = m_{N-2}$ and thus such a number J exists. Obviously $P' = P_J$ satisfies (i) with $C = \lambda^{N-2}$. We claim that for (ii) to hold we just need λ to be large enough. Note that

$$\min\{|p-q| : p,q \in P',\ p \neq q\} = m_J \geq \lambda^{-(N-2-J)} M_J\,,$$

while, for every $p \in \{p_1,\dots,p_J\}$, by our definition of J and by (4.16), we have

$$\begin{aligned}\operatorname{dist}(p,P') &\leq \sum_{i=1}^{J} \operatorname{dist}(p_i,P_i) = \sum_{i=1}^{J} m_{i-1} \leq M_J \lambda^{-(N-2-J)} \sum_{k\geq 0} \lambda^{-k} \\ &= \frac{M_J \lambda^{-(N-2-J)}}{\lambda - 1} \leq \frac{m_J}{\lambda - 1}\,.\end{aligned}$$

In particular, it suffices to choose $\frac{1}{\lambda-1} = \bar{\delta}$.

Proof of Lemma 4.3.7

If $\varepsilon \leq \delta \min\{|q_1 - q_2| : q_1, q_2 \in P,\, q_1 \neq q_2\}$, we then set $\tilde{P} = P$. Otherwise, pick a pair of points $p_0, p_1 \in P$ not satisfying this property and set $P_1 := P \setminus \{p_1\}$. By construction, $\operatorname{dist}(p_1, P_1) \leq \delta^{-1}\varepsilon$. If $N = 2$, $\tilde{P} := P_1$ contains a single element and hence (ii) holds. On the other hand, if $N \geq 3$ and $\delta^{-1}\varepsilon \leq \delta \min\{|q_1 - q_2| : q_1, q_2 \in P_1,\ q_1 \neq q_2\}$, we define $\tilde{P} = P_1$, which satisfies (ii) because $\operatorname{dist}(p_1, P_1) \leq \delta^{-1}\varepsilon$. Otherwise we discard yet another point p_2 from P in the same way as above, with P_1 in place of P, and set $P_2 := P_1 \setminus \{p_2\}$. This time, we notice that

$$\operatorname{dist}(p_2, P_2) \leq \delta^{-2}\varepsilon\,,$$

while

$$\operatorname{dist}(p_1, P_2) \leq \operatorname{dist}(p_1, P_1) + \operatorname{dist}(p_2, P_2) \leq (\delta^{-1} + \delta^{-2})\varepsilon\,.$$

Iterating this procedure, we generate a family of sets P_j, $j = 1, \ldots, J$. We stop if either P_J is a singleton, or if

$$\delta^{-1}(1 + \delta^{-1})^{J-1}\varepsilon \leq \delta \min\{|q_1 - q_2| : q_1, q_2 \in P_J,\ q_1 \neq q_2\}\,.$$

For every point $p \in P$, since $\operatorname{dist}(p_j, P_j) \leq \delta^{-2}(1 + \delta^{-1})^{j-2}$ for each $j \geq 2$, we therefore inductively have

$$\operatorname{dist}(p, P_J) \leq \sum_{j=1}^{J} \operatorname{dist}(p_j, P_j) \leq \delta^{-1}(\delta^{-1} + 1)^{J-1}\varepsilon\,.$$

In particular, when we stop the set $\tilde{P} = P_J$ satisfies (ii). Since P_{N-1} would necessarily be a singleton, we must stop at a $J \leq N = 1$, which makes the estimate (i) trivially true.

Proof of Lemma 4.3.8

When $N = 2$ and $P = \{p_1, p_2\}$, we can clearly just let $P_j := \{p_j\}$ for $j = 1, 2$. We now argue by induction on N. Fix $N \geq 3$, and suppose the conclusion of the lemma holds for $N' \leq N-1$. Select $p \in P$ such that $\max\{|q - q'| : q, q' \in P \setminus \{p\}\} = M = \max\{|q - q'| : q, q' \in P\}$. By our inductive hypothesis, we can then decompose $P \setminus \{p\}$ into $P'_1 \cup P'_2$ such that

$$\min\{|p_1 - p_2| : p_1 \in P'_1, p_2 \in P'_2\} \geq \frac{M}{2^{N-3}}\,.$$

Let $p_i \in P'_i$ be such that $\mathrm{dist}(p, p_i) = \mathrm{dist}(p, P'_i)$ for $i = 1, 2$. Since $\mathrm{dist}(p_1, p_2) \geq \frac{M}{2^{N-3}}$, at least one among $\mathrm{dist}(p_i, p)$ has to be $\geq \frac{M}{2^{N-2}}$. Assuming, upon relabeling, that the latter happens for $i = 1$, we set $P_1 = P'_1$ and $P_2 = P'_2 \cup \{p\}$, proving the conclusion of the Lemma.

Well-separated case

Let us first demonstrate the validity of Theorem 4.3.2 under the assumption that the planes in $\boldsymbol{\pi}$ are well-separated.

Lemma 4.3.9. *For every $1 \leq \bar{r} < 2$ there is a constant $\sigma_1 = \sigma_1(Q, m, n, \bar{n}, 2 - \bar{r}) > 0$ with the following property. Let T be as in Assumption 4.3.1, while $p_1, \ldots, p_N \in \pi_0^{\perp}$ are distinct points and $\boldsymbol{\pi} := \bigcup_i p_i + \pi_0$. Assume that*

$$E \leq \sigma_1 \min\{H, 1\}^{m+2}\,, \tag{4.17}$$

where E is as in (4.3) and $H := \min\{|p_i - p_j| : i \neq j\}$. Then

$$\mathrm{spt}(T) \cap \mathbf{C}_{\bar{r}} \subset \{q : \mathrm{dist}(q, \boldsymbol{\pi}) \leq \tfrac{H}{4}\}, \tag{4.18}$$

and, in particular, all the conclusions of Theorem 4.3.2 hold.

First of all, observe that it suffices to prove (4.18) in order to conclude all the conclusions

of Theorem 4.3.2. Indeed, if we set $T_i := T \llcorner \mathbf{C}_{\bar r} \cap \{q : \mathrm{dist}(q, p_i + \pi_0) \le \frac{H}{4}\}$, we obtain the decomposition $T \llcorner \mathbf{C}_{\bar r} = T_1 + \cdots + T_N$ with

(i) $\partial T_i \llcorner \mathbf{C}_{\bar r} = 0$;

(ii) $\mathrm{spt}(T_i) \cap \mathrm{spt}(T_j) = \emptyset$ for $i \neq j$;

(iii) $\mathrm{dist}(q, \boldsymbol{\pi}) = \mathrm{dist}(q, p_i + \pi_0)$ for every $q \in \mathrm{spt}(T_i)$.

In particular, given $r \in [1, 2)$, for $\bar r \in (r, 2)$ the conclusions of Theorem 4.3.2 can be drawn by using the case $N = 1$ applied to each T_i.

The proof of Lemma 4.3.9 is based on the following yet simpler lemma.

Lemma 4.3.10. *For every $1 \le \bar r < 2$ there is a constant $\sigma_2 = \sigma_2(Q, m, n, \bar n, 2 - \bar r) > 0$ with the following property. Let T, $\boldsymbol{\pi}$, E, and H be as in Lemma 4.3.9, but instead of (4.17) assume that*

$$E \le \sigma_2 \qquad \text{and} \qquad H \ge 1\,. \tag{4.19}$$

Then (4.18) and all the conclusions of Theorem 4.3.2 hold in $\mathbf{C}_{\bar r}$.

Proof. Observe that since $H \ge 1$, by Chebyshev's inequality we have

$$\|T\|(\mathbf{C}_{\bar r} \cap \{\mathrm{dist}(\cdot, \boldsymbol{\pi}) \ge \tfrac{H}{8}\}) \le 64\sigma_2\,.$$

On the other hand, for $\rho := \min\{2 - \bar r, \frac{H}{8}\}$, if $\mathrm{spt}(T) \cap \{\mathrm{dist}(\cdot, \boldsymbol{\pi}) \ge \frac{H}{4}\}$ is not empty the monotonicity formula would give

$$\|T\|(\mathbf{C}_{\bar r} \cap \{\mathrm{dist}(\cdot, \boldsymbol{\pi}) \ge \tfrac{H}{8}\}) \ge C^{-1}\rho^m$$

for some dimensional constant $C > 0$, which yields a contradiction for a sufficiently small choice of σ_2. □

Proof of Lemma 4.3.9. Recall that we just need to prove (4.18). If $H \geq 1$ the claim follows from Lemma 4.3.10. If $H \leq 1$, we just need to show that

$$\mathrm{spt}(T) \cap \mathbf{C}_{\bar{r}H}(q) \cap \{\mathrm{dist}(\cdot, \boldsymbol{\pi}) > \tfrac{H}{4}\} = \emptyset$$

whenever the cylinder $\mathbf{C}_{2H}(q)$ is contained in the original cylinder $\mathbf{C}_2$. But then it suffices to consider the current $T_{q,H} := (\lambda_{q,H})_\sharp T$ for the map $\lambda_{q,H}(x) := \frac{x-q}{H}$ and to apply Lemma 4.3.10 to $T_{q,H}$ and $\lambda_{q,H}(\boldsymbol{\pi})$: the pair falls under the assumptions provided $\sigma_1 \leq \sigma_2$, after an obvious scaling argument. □

We record another two simple observations which will be useful in the sequel. They are both proven in exactly the same way as Lemma 4.3.10.

Lemma 4.3.11. *For every $1 \leq r < 2$, there is a constant $\sigma_3 = \sigma_3(Q, m, n, \bar{n}, 2-r) > 0$ with the following property. Let T and $\boldsymbol{\pi}$ be as in Lemma 4.3.10, but instead of (4.19) assume only that*

$$E \leq \sigma_3 \,. \tag{4.20}$$

Then $\mathrm{spt}(T) \cap \mathbf{C}_r \subset \{x : \mathrm{dist}(x, \boldsymbol{\pi}) \leq 1\}$.

Lemma 4.3.12. *For every $1 \leq r < 2$, there is a constant $\bar{\sigma} = \bar{\sigma}(Q, m, n, \bar{n}, N, 2-r) > 0$ such that the following holds. Let T and $\boldsymbol{\pi}$ be as in Lemma 4.3.10, but instead of (4.19) assume that $M := \max_{i,j}\{|p_i - p_j|\} \geq 1$ and*

$$E \leq \bar{\sigma} M^2 \,. \tag{4.21}$$

Then $\mathrm{spt}(T) \cap \mathbf{C}_r \subset \{x : \mathrm{dist}(x, \boldsymbol{\pi}) \leq 2^{-N} M\}$.

Combining Lemma 4.3.12 with the combinatorial Lemma 4.3.8, we get the following separation lemma.

Lemma 4.3.13. *Under the assumptions of Lemma 4.3.12, there is a decomposition* $\boldsymbol{\pi} = \boldsymbol{\pi}^1 \cup \boldsymbol{\pi}^2$ *and* $T \llcorner \mathbf{C}_r = T_1 + T_2$ *such that*

(i) The sets $\boldsymbol{\pi}^1, \boldsymbol{\pi}^2$ *are disjoint, non-empty, and unions of a subset of the planes in* $\boldsymbol{\pi}$*;*

(ii) $\partial T_i \llcorner \mathbf{C}_r = 0$ *for* $i = 1, 2$*;*

(iii) $\operatorname{spt}(T_1) \cap \operatorname{spt}(T_2) = \emptyset$*;*

(iv) $\operatorname{dist}(q, \boldsymbol{\pi}) = \operatorname{dist}(q, \boldsymbol{\pi}^i)$ *for every* $q \in \operatorname{spt}(T_i)$ *and each* $i = 1, 2$*.*

Arguing as in the proof of Lemma 4.3.9, we can draw the following further conclusion.

Lemma 4.3.14. *For every* $1 \leq r < 2$*, there is a constant* $\tilde{\sigma} = \tilde{\sigma}(Q, m, n, \bar{n}, N, 2 - r) > 0$ *such that the following holds. Let* T *and* $\boldsymbol{\pi}$ *be as in Lemma 4.3.9, but instead of (4.17), for* $M := \max_{i,j}\{|p_i - p_j|\}$ *assume that*

$$E \leq \tilde{\sigma} \min\{M, 1\}^{m+2}.$$

Then there is a decomposition $\boldsymbol{\pi} = \boldsymbol{\pi}^1 \cup \boldsymbol{\pi}^2$ *and* $T \llcorner \mathbf{C}_r = T_1 + T_2$*, as in Lemma 4.3.13.*

4.3.3 Proof of tilt-excess estimate

The aim of this section is to prove (4.4). This will be done independently to the proof of L^∞ estimate (4.5); we defer the latter to the next section. We begin with two "approximate" estimates on the oriented tilt-excess.

First lemma

In this lemma we derive a first approximate estimate.

Lemma 4.3.15. *For every pair of radii* $1 \leq r < R \leq 2$ *there are constants* $\bar{C} = \bar{C}(Q, m, n, \bar{n}, R - r) > 0$ *and* $\gamma = \gamma(Q, m, n, \bar{n}) > 0$ *such that the following holds. Let* T*,* Σ *and* $\mathbf{A}$ *be as in*

Assumption 4.3.1, suppose that $p_1, \dots, p_N \in \pi_0^\perp$ are distinct points, and let $\boldsymbol{\pi} := \bigcup_i p_i + \pi_0$. Assume that E is as in (4.3) and let $H := \min\{|p_i - p_j| : i \neq j\}$. Then

$$\mathbf{E}(T, \mathbf{C}_r) \leq \bar{C}(E + \mathbf{A}^2) + \bar{C}\left(\frac{E}{H^2}\right)^\gamma \mathbf{E}(T, \mathbf{C}_R) + \bar{C}\mathbf{E}(T, \mathbf{C}_R)^{1+\gamma}. \tag{4.22}$$

Proof. We start by defining a suitable vector field $X : \mathbb{R}^{m+n} \to \pi_0^\perp$. In fact, we will actually define $\bar{X} : \pi_0^\perp \to \pi_0^\perp$, and then set $X(x) := \bar{X}(\mathbf{p}^\perp_{\pi_0}(x))$. Firstly, define $\tilde{X}$ on the union of the disks $B_{H/4}(p_i, \pi_0^\perp)$ by taking $\tilde{X}(y) := (y - p_i)$ on each $B_{H/4}(p_i, \pi_0^\perp)$. It is simple to check that the Lipschitz constant of $\tilde{X}$ can be bounded by 3; indeed, if y_1, y_2 lie in disks $B_{H/4}(p_{i_1}, \pi_0^\perp)$ and $B_{H/4}(p_{i_2}, \pi_0^\perp)$ respectively, then

$$|p_{i_1} - p_{i_2}| \leq |y_1 - y_2| + H/2 \leq |y_1 - y_2| + |p_{i_1} - p_{i_2}|/2 \quad \Longrightarrow \quad |p_{i_1} - p_{i_2}| \leq 2|y_1 - y_2|.$$

The desired Lipschitz bound of 3 follows immediately. We can thus use the Kirszbraun theorem to extend $\tilde{X}$ to a vector field $\bar{X} : \pi_0^\perp \to \pi_0^\perp$ which has the same Lipschitz constant; this determines $\bar{X}$, and in turn determines X. In particular, we deduce that X has the following properties:

(i) X takes values in $\pi_0^\perp$;

(ii) $|X(x)| \leq 3\,\mathrm{dist}(x, \boldsymbol{\pi})$ and $|\nabla_v X(x)| \leq 3|v|$;

(iii) $\nabla_v X = 0$ for every $v \in \pi_0$;

(iv) $X(x) = \mathbf{p}^\perp_{\pi_0}(x - p_i)$ if $\mathrm{dist}(x, p_i + \pi_0) \leq \frac{H}{4}$.

Note that by regularizing $\bar{X}$ via convolution, we can obtain a smooth vector field with the same properties, except that (iv) will hold in a slightly smaller tubular neighborhood of $\boldsymbol{\pi}$. We will thus ignore the regularity issues and use X as a test in the first variation formula for T.

In order to simplify our notation we introduce the set

$$G := \left\{ x \in \mathbf{C}_2 : \mathrm{dist}(x, \boldsymbol{\pi}) \leq \frac{H}{4} \right\} = \bigcup_i \left\{ x \in \mathbf{C}_2 : \mathrm{dist}(x, p_i + \pi_0) \leq \frac{H}{4} \right\}$$

and write G^c for its complement in $\mathbf{C}_2$. Because of (iv), (see e.g. [10, Lemma 4.2]) we have

$$\mathrm{div}_{\vec{T}(x)} X(x) = \frac{1}{2} |\mathbf{p}_{\pi_0} - \mathbf{p}_{T(x)}|^2 \qquad \forall x \in G \cap \mathrm{spt}(T), \tag{4.23}$$

while, because of (i), (iii), and the Lipschitz regularity of X,

$$|\mathrm{div}_{\vec{T}(x)} X(x)| \leq C |\mathbf{p}_{\pi_0} - \mathbf{p}_{T(x)}|^2 \qquad \forall x \in G^c \cap \mathrm{spt}(T)\,. \tag{4.24}$$

To see the latter, let $e_1, \ldots, e_m$ be an orthonormal base of the tangent plane to T at x and compute

$$\begin{aligned}
|\mathrm{div}_{\vec{T}(x)} X(x)| &\leq \sum_i |\nabla_{e_i} X(x) \cdot e_i| \overset{\text{(i)}}{=} \sum_i |\nabla_{e_i} X(x) \cdot (e_i - \mathbf{p}_{\pi_0}(e_i))| \\
&\overset{\text{(iii)}}{=} \sum_i |(\nabla_{e_i} X(x) - \nabla_{\mathbf{p}_{\pi_0}(e_i)} X(x)) \cdot (e_i - \mathbf{p}_{\pi_0}(e_i))| \\
&\leq \sum_i |e_i - \mathbf{p}_{\pi_0}(e_i)| |\nabla_{e_i - \mathbf{p}_{\pi_0}(e_i)} X(x)| \\
&\overset{\text{(ii)}}{\leq} 3 \sum_i |e_i - \mathbf{p}_{\pi_0}(e_i)|^2 \leq C |\mathbf{p}_{\vec{T}(x)} - \mathbf{p}_{\pi_0}|^2\,.
\end{aligned}$$

Let us now choose radii $r_1 < r_2$ so that $r_1 - r = r_2 - r_1 = R - r_2$ (i.e. $r_1 = (R + 2r)/3$, $r_2 = (2R + r)/3$). Let $\tilde{\chi} : \pi_0 \to \mathbb{R}$ be a smooth cut-off function which is identically 1 on $B_{r_1}(0, \pi_0)$, vanishes outside $\bar{B}_{r_2}(0, \pi_0)$, and satisfies $|D\tilde{\chi}| \leq 2/(r_2 - r_1)$. Extend $\tilde{\chi}$ vertically to get a function $\chi : \mathbb{R}^{m+n} \to \mathbb{R}$, namely $\chi(x) = \tilde{\chi}(\mathbf{p}_{\pi_0}(x))$. We now want to take $\chi^2 X$ as a test function in the first variation formula for (the stationary varifold associated to) T; since $\chi^2 X$ need not have compact support in the directions orthogonal to π, to justify this simply note that since χ is supported in the cylinder $\bar{\mathbf{C}}_{r_2}$, and since the support of T is

compact on any cylinder which is slightly smaller than $\mathbf{C}_2$, we can multiply $\chi^2 X$ by another test function which is translation invariant in the π_0 variables, is 1 in the ball of radius $2\sup\{\mathbf{p}^\perp_{\pi_0}(x) : x \in \mathrm{spt}\|T\| \cap \bar{\mathbf{C}}_{r_2}\}$ in the $\pi_0^\perp$ variables, and vanishes outside the ball of radius $3\sup\{\mathbf{p}^\perp_{\pi_0}(x) : x \in \mathrm{spt}\|T\| \cap \bar{\mathbf{C}}_{r_2}\}$; this then has compact support in all of $\mathbf{C}_2$ and agrees with $\chi^2 X$ on $\mathbf{C}_2 \cap \mathrm{spt}(T)$, therefore allowing us to take $\chi^2 X$ as a test function. Hence, we have

$$\begin{aligned}\int_{\mathbf{C}_2} \chi^2 \mathrm{div}_{\vec{T}(x)} X(x) d\|T\|(x) = &- \int_{\mathbf{C}_2} \chi^2 X(x) \cdot \vec{H}_T(x)\, d\|T\|(x) \\ &- 2\int_{\mathbf{C}_2} \chi(x) \nabla_{\vec{T}(x)} \chi(x) \cdot X(x)\, d\|T\|(x)\,,\end{aligned}$$

where $\vec{H}_T$ is the generalized mean curvature of T. Combining this with (4.23), (4.24), (ii), and the fact that $|\vec{H}_T(x)| \leq C\mathbf{A}$, we get

$$\begin{aligned}\int_{\mathbf{C}_2} \chi^2 |\mathbf{p}_{T(x)} - \mathbf{p}_{\pi_0}|^2\, d\|T\|(x) \leq C \int_{\mathbf{C}_{r_2} \cap G^c} &|\mathbf{p}_{T(x)} - \mathbf{p}_{\pi_0}|^2\, d\|T\|(x) \\ &+ C(E + \mathbf{A}^2) + 2\left| \int_{\mathbf{C}_2} \chi(x) \nabla_{\vec{T}(x)} \chi(x) \cdot X(x)\, d\|T\|(x) \right|.\end{aligned}$$

Observe next that in light of (i), $\nabla_{\vec{T}} \chi \cdot X = (\nabla_{\vec{T}} \chi - \nabla_{\vec{\pi}_0} \chi) \cdot X(x)$. We can thus estimate further

$$\begin{aligned}2\left| \int_{\mathbf{C}_2} \chi(x) \nabla_{\vec{T}(x)} \chi(x) \cdot X(x)\, d\|T\|(x) \right| &\leq \frac{C}{R-r} \int_{\mathbf{C}_{r_2}} \chi(x) |\mathbf{p}_{\pi_0} - \mathbf{p}_{T(x)}| |X(x)|\, d\|T\|(x) \\ &\leq \frac{CE}{(R-r)^2} + \frac{1}{4} \int_{\mathbf{C}_2} \chi^2(x) |\mathbf{p}_{\pi_0} - \mathbf{p}_{T(x)}|^2\, d\|T\|(x),\end{aligned}$$

where in the latter inequality we have used that $ab \leq \frac{a}{2\varepsilon} + \frac{\varepsilon b}{2}$ for any two non-negative numbers a and b and a suitably small choice of $\varepsilon > 0$, combined with (ii). In particular, we conclude that

$$\mathbf{E}^{no}(T, \mathbf{C}_{r_1}) \leq \bar{C}(E + \mathbf{A}^2) + C \int_{\mathbf{C}_{r_2} \cap G^c} |\mathbf{p}_{T(x)} - \mathbf{p}_{\pi_0}|^2\, d\|T\|(x)\,,$$

for $\bar{C}$ now also dependent on $R-r$. When combined with Proposition 4.3.4, we arrive at

$$\mathbf{E}(T,\mathbf{C}_r) \le \bar{C}(E+\mathbf{A}^2) + C\int_{\mathbf{C}_{r_2}\cap G^c} |\mathbf{p}_{T(x)} - \mathbf{p}_{\pi_0}|^2\, d\|T\|(x)\,.$$

where again $\bar{C} = \bar{C}(Q,m,n,\bar{n},R-r) > 0$ (we recall that $r_1 - r = (R-r)/3$ here). We now use (a scaled version of) Almgren's estimate [19, Theorem 7.1] (the first version of this estimate is in [3, Sections 3.24–3.26, Section 3.30(8)]), applied within $\mathbf{C}_{r_2}$, with the choice $A = \mathbf{p}_{\pi_0}(\mathbf{C}_{r_2}\cap G^c\cap \mathrm{spt}(T)) \equiv B_{r_2}(\pi_0)\cap \mathbf{p}_{\pi_0}(G^c\cap\mathrm{spt}(T))$, to conclude that

$$\begin{aligned}\mathbf{E}(T,\mathbf{C}_r) &\le \bar{C}(E+\mathbf{A}^2) + \bar{C}(\mathbf{E}(T,\mathbf{C}_R)+\mathbf{A}^2)^{1+\gamma} + \bar{C}|A|^{\gamma}(\mathbf{E}(T,\mathbf{C}_R)+\mathbf{A}^2)\\ &\le \bar{C}(E+\mathbf{A}^2) + \bar{C}\mathbf{E}(T,\mathbf{C}_R)^{1+\gamma} + \bar{C}\|T\|(G^c\cap\mathbf{C}_R)^{\gamma}\mathbf{E}(T,\mathbf{C}_R)\,,\end{aligned}$$

where we have used that $\mathbf{A}\le 1$ and $|A|\le \|T\|(G^c\cap\mathbf{C}_R)$ (here, $|A|$ denotes the measure of the set A).

However, observe that $\mathrm{dist}(x,\boldsymbol{\pi}) \ge \frac{H}{4}$ for all $x\in G^c$ and thus by Chebyshev we get

$$\|T\|(G^c\cap\mathbf{C}_R) \le \frac{CE}{H^2}\,,$$

which concludes the proof. □

Second lemma

Now, we iterate Lemma 4.3.15 to achieve a closer approximation of (4.4), but under the additional assumption that the height excess of T relative to the family of planes is much smaller than the minimal separation of the planes. We will remove this assumption in the next section.

Proposition 4.3.16. *For every pair of scales $1\le r<r_0<2$, there are constants $\bar{C} = \bar{C}(Q,m,n,\bar{n},N,r_0-r,2-r_0) > 0$ and $\sigma_4 = \sigma_4(Q,m,n,\bar{n},N,r_0-r,2-r_0) > 0$ with the*

following properties. Assume T, $\boldsymbol{\pi}$ and H are as Lemma 4.3.15. If in addition we have

$$E \leq \sigma_4 \min\{H^2, 1\}\,, \tag{4.25}$$

then

$$\mathbf{E}(T, \mathbf{C}_r) \leq \bar{C}(E + \mathbf{A}^2) + \bar{C}\left(\frac{E}{H^2}\right)\mathbf{E}(T, \mathbf{C}_{r_0})\,. \tag{4.26}$$

Proof. This estimate will be proved by a two-variable induction over Q and N. Our inductive assumption is that

(IH) *The estimate is valid for any pair $Q' \leq Q$ and $N' \leq N$ with $Q' + N' < Q + N$.*

By Proposition 4.3.5, the case $N = 1$ and arbitrary Q is always valid. Moreover, due to (4.25), we can assume without loss of generality that $H \leq 1$, since otherwise we can apply Lemma 4.3.10, yielding the conclusions of Theorem 4.3.2 in $\mathbf{C}_r$, which in particular trivially implies (4.26).

We can similarly assume that $M := \max\{|p_i - p_j|\} \leq 1$ and

$$\frac{E}{M^{m+2}} \geq \tilde{\sigma}. \tag{4.27}$$

Indeed, if $M > 1$ then we can apply Lemma 4.3.13 to decompose T into T_1, T_2 (as in the statement of the lemma) and appeal to the induction assumption (IH). Note that the validity of the hypothesis (4.25) remains unchanged under such a decomposition (as E can only decrease while H can only increase). We can apply (IH) since if both $T_1, T_2 \neq 0$, then necessarily $\Theta(T_i, \cdot) \leq Q - 1$ (and so Q is decreased), while if, say, $T_2 = 0$, then Q remains the same for T_1, but N decreases by at least 1. Thus, we may indeed assume that $M \leq 1$. Given this, one can additionally assume that (4.27) indeed holds via the same argument, except now invoking Lemma 4.3.14 instead of Lemma 4.3.13.

Now observe that for every given $\pi_i \in \boldsymbol{\pi}$ we can easily estimate

$$\int_{\mathbf{C}_2} \operatorname{dist}^2(x, \pi_i)\, d\|T\|(x) \leq 2M^2 + 2E\,;$$

Thus, using Proposition 4.3.5 (for any fixed plane $\pi_i \in \boldsymbol{\pi}$), we have for any $1 \leq R < 2$,

$$\mathbf{E}(T, \mathbf{C}_R) \leq \bar{C}(2M^2 + 2E + \mathbf{A}^2)\,. \tag{4.28}$$

where $\bar{C}$ also depends on $2 - R$. Under our current assumptions, the right-hand side of (4.28) will be at most $\bar{C}$, provided we take σ_4 sufficiently small (depending only on allowed parameters). In particular, combining this with (4.27), and the assumption $H \leq 1$ (so $E \leq 1$ also) gives

$$\mathbf{E}(T, \mathbf{C}_R) \leq \bar{C}\mathbf{A}^2 + \bar{C}E^{\frac{2}{m+2}} \leq \tilde{C}\mathbf{A}^2 + \bar{C}\left(\frac{E}{H^2}\right)^{\frac{2}{m+2}}\,,$$

where $\bar{C}$ now also depends on $\tilde{\sigma}$. Now taking $\gamma = \gamma(Q, m, n, \bar{n}) > 0$ as in Lemma 4.3.15, the above bound clearly implies

$$\mathbf{E}(T, \mathbf{C}_R)^\gamma \leq \bar{C}\mathbf{A}^{2\gamma} + \bar{C}\left(\frac{E}{H^2}\right)^{\frac{2\gamma}{m+2}}\,,$$

where now $\bar{C}$ also depends on γ. Using Lemma 4.3.15 and (4.25) we now infer that for any $1 \leq \rho < R < 2$ we have

$$\begin{aligned}\mathbf{E}(T, \mathbf{C}_\rho) &\leq \bar{C}(E + \mathbf{A}^2) + \tilde{C}\left[\left(\frac{E}{H^2}\right)^{\frac{2\gamma}{m+2}} + \mathbf{A}^{2\gamma}\right]\mathbf{E}(T, \mathbf{C}_R) \\ &\leq \bar{C}_1(E + \mathbf{A}^2) + \bar{C}_1\left(\sigma_4^{\frac{\gamma}{m+2}} + \mathbf{A}^{\frac{(m+3)\gamma}{m+2}}\right)\left(\frac{E}{H^2} + \mathbf{A}^2\right)^{\frac{\gamma}{m+2}}\mathbf{E}(T, \mathbf{C}_R)\end{aligned} \tag{4.29}$$

where $\bar{C}_1 = \bar{C}_1 C(Q, m, n, \bar{n}, \gamma, \tilde{\sigma}, R - \rho, 2 - R)$. By selecting σ_4 small enough we can ensure that

$$\bar{C}_1 \sigma_4^{\frac{\gamma}{m+2}} \leq \frac{1}{4}\,.$$

Next, if $\bar{C}_1 \mathbf{A}^{\frac{(m+3)\gamma}{m+2}} \leq \frac{1}{4}$ we infer that

$$\mathbf{E}(T, \mathbf{C}_\rho) \leq \bar{C}_1(E + \mathbf{A}^2) + \frac{1}{2}\left(\frac{E}{H^2} + \mathbf{A}^2\right)^{\frac{\gamma}{m+2}} \mathbf{E}(T, \mathbf{C}_R)\,. \tag{4.30}$$

Otherwise, if $\bar{C}_1 \mathbf{A}^{\frac{(m+3)\gamma}{m+2}} \geq \frac{1}{4}$, we can estimate $\mathbf{A}^{2\gamma} \leq \bar{C}_2 \mathbf{A}^2$ for some constant $\bar{C}_2$ with the same dependencies as $\bar{C}_1$. Then, noting that (4.28) in particular gives that $\mathbf{E}(T, \mathbf{C}_R) \leq \bar{C}$ (as remarked previously), from (4.29) we have (with the same choice of σ_4 mentioned previously)

$$\mathbf{E}(T, \mathbf{C}_\rho) \leq \bar{C}_3(E + \mathbf{A}^2) + \frac{1}{4}\left(\frac{E}{H^2}\right)^{\frac{\gamma}{m+2}} \mathbf{E}(T, \mathbf{C}_R)$$

with a worse constant $\bar{C}_3$ (with the same dependencies as $\bar{C}$). We can therefore assume (4.30) to be valid irrespective of the value of $\mathbf{A}$.

We are now in a position to iterate (4.30). Fix the smallest natural number $J = J(Q, m, n, \bar{n})$ such that $\frac{J\gamma}{m+2} \geq 1$ and apply (4.30) with $\rho = r_i$, $R = r_{i-1}$ for a sequence of radii r_i, where $r_J = r$, $r_{i-1} = r_i + \frac{r_0 - r}{J}$. This, in particular, fixes the size of the difference between radii used in all the inequalities used above, and in particular it gives a uniform bound for the constants above, hence fixing the choice of $\sigma_4 = \sigma_4(Q, m, n, \bar{n}, r_0 - r, 2 - r_0) > 0$.

If for some i we have

$$\left(\frac{E}{H^2} + \mathbf{A}^2\right)^{\frac{\gamma}{m+2}} \mathbf{E}(T, \mathbf{C}_{r_{i-1}}) \leq \mathbf{E}(T, \mathbf{C}_{r_i})$$

we then can absorb the second term in the right hand side of (4.30) into the left hand side to conclude that

$$\mathbf{E}(T, \mathbf{C}_{r_i}) \leq \bar{C}(E + \mathbf{A}^2)\,.$$

Given that $1 \leq r \leq r_i < 2$ we achieve the desired estimate (4.26) in this case. Otherwise, we must have

$$\mathbf{E}(T, \mathbf{C}_{r_i}) \leq \left(\frac{E}{H^2} + \mathbf{A}^2\right)^{\frac{\gamma}{m+2}} \mathbf{E}(T, \mathbf{C}_{r_{i-1}})$$

for all i, which leads us to

$$\mathbf{E}(T, \mathbf{C}_r) = \mathbf{E}(T, \mathbf{C}_{r_J}) \leq \left(\frac{E}{H^2} + \mathbf{A}^2 \right)^{\frac{J\gamma}{m+2}} \mathbf{E}(T, \mathbf{C}_{r_0}) \,. \tag{4.31}$$

Given that $\frac{J\gamma}{m+2} \geq 1$ (so we may write $\frac{J\gamma}{m+2} = 1 + \tilde{\gamma}$ for some $\tilde{\gamma} \geq 0$), and since $\frac{E}{H^2} \leq \sigma_4 \leq 1$, we again arrive at the desired estimates (4.26) (using again that $\mathbf{E}(T, \mathbf{C}_{r_0}) \leq \bar{C}$ from (4.28) to absorb the $\mathbf{A}^2$ factor from the parenthesis into the first term on the right-hand side of (4.26)). □

Proof of the tilt-excess estimate

We are finally in a position to prove (4.4). The goal is to exploit the combinatorial results in Section 4.3.2 to remove the hypothesis (4.25) in Proposition 4.3.16 by possibly replacing $\boldsymbol{\pi}$ with a smaller, refined sub-collection of planes $\bar{\boldsymbol{\pi}}$, and in turn conclude the tilt-excess estimate for $\bar{\boldsymbol{\pi}}$.

Let $\boldsymbol{\pi} := \bigcup_i (p_i + \pi_0)$ be as in the statement of Theorem 4.3.2. Fix a positive parameter $0 < \delta \leq 1/2$ (whose choice will be determined later) and apply Lemma 4.3.7 to $P = \{p_1, \ldots, p_N\}$ with this choice of δ and $\varepsilon = E^{1/2}$, yielding a subset $\tilde{P} \subset P$ obeying the conclusions of Lemma 4.3.7. Let $\tilde{\boldsymbol{\pi}} = \bigcup_{\tilde{p} \in \tilde{P}} (\tilde{p} + \pi_0) \subset \boldsymbol{\pi}$ be the corresponding union of parallel planes. By property (i) of Lemma 4.3.7 we then have

$$\tilde{E} := \int_{\mathbf{C}_2} \operatorname{dist}^2(q, \tilde{\boldsymbol{\pi}}) \, d\|T\|(q) \leq 2E + C \max_i \operatorname{dist}(p_i, \tilde{P})^2 \tag{4.32}$$

and hence

$$\tilde{E} \leq C(1 + \delta^{-2}(1 + \delta^{-1})^{2N-4})E \,. \tag{4.33}$$

If $\tilde{P}$ is a singleton, then we can apply Proposition 4.3.5 to T and $\bar{\boldsymbol{\pi}}$ to conclude (4.4) from (4.33), since we are then in the case $N = 1$ of Theorem 4.3.2.

We may thus henceforth assume that $\tilde{P}$ consists of at least two distinct points. In this

case, property (ii) of Lemma 4.3.7 gives that $E^{1/2} \leq \delta \tilde{H}$ and $\max_i \operatorname{dist}(p_i, \tilde{P}) \leq \delta \tilde{H}$, where $\tilde{H} := \min\{|\tilde{p} - \tilde{q}| : \tilde{p}, \tilde{q} \in \tilde{P}, \tilde{p} \neq \tilde{q}\}$, and thus combining this with (4.32), we get

$$\tilde{E} \leq C\delta^2 \tilde{H}^2 \,, \tag{4.34}$$

where the constant $C = C(Q, m, n, \bar{n})$ is independent of δ.

Fix another constant $0 < \bar{\delta} \leq 1/2$ (to be determined later). We now apply Lemma 4.3.6 with this $\bar{\delta}$ to further refine $\tilde{\boldsymbol{\pi}}$, finding a second subset $P_* \subset \tilde{P}$ with the properties listed in Lemma 4.3.6. By property (i) of Lemma 4.3.6, if we denote by $M_* := \max\{|p-q| : p, q \in P'\}$ and $H_* := \min\{|p' - q'| : p', q' \in P',\ p' \neq q'\}$, we achieve that

$$M_* \leq \bar{C}(\bar{\delta}, N) H_* \tag{4.35}$$

and, combining (ii) of Lemma 4.3.6 with (4.34), we have

$$E_* := \int_{\mathbf{C}_2} \operatorname{dist}^2(q, \boldsymbol{\pi}_*)\, d\|T\|(q) \leq 2\tilde{E} + C\bar{\delta}^2 H_*^2 \leq C(\delta^2 + \bar{\delta}^2) H_*^2 \,, \tag{4.36}$$

since $\tilde{H} \leq H_*$. Here, $C = C(Q, m, n, \bar{n})$ is a new constant which we stress is independent of $\bar{\delta}$, and $\boldsymbol{\pi}_* := \bigcup_{p_* \in P_*} (p_* + \pi_0)$ is the union of parallel planes corresponding to P_*. Let $J := \#(\tilde{P} \setminus P_*)$ and let $P_j \subset \tilde{P}$ be the collections of points given by property (iii) of Lemma 4.3.6; in particular, $P_0 = \tilde{P}$, $P_J = P_*$. Define radii $(r_j)_{j=0}^{J+2}$ by $r_{J+2} = 2$, $r_0 = r$, and $r_j - r_{j-1} = \frac{2-r}{J+2}$. We will apply various estimates on the tilt-excess between the radii $r_j < r_{j-1}$; note that the parameter σ_4 in Proposition 4.3.16 when applied at such scales obeys $\sigma_4 = \sigma_4\big(Q, m, n, \bar{n}, N, (2-r)/(J+2)\big)$ and so is now fixed (independent of j). One should note however that the dependence of σ_4 in the radius variable, $(2-r)/(J+2)$, only actually depends on a lower bound on the radius, which here is $(2-r)/N$, meaning $\sigma_4 = \sigma_4(Q, m, n, \bar{n}, N, 2-r)$. Recalling (4.34) and (4.36), we now choose δ and $\bar{\delta}$, depending

only on $Q, m, n, \bar{n}$, so that

$$E_* \le \sigma_4 H_*^2 \qquad \text{and} \qquad \tilde{E} \le \sigma_4 \tilde{H}^2 \,. \tag{4.37}$$

We may further assume that $H_*, \tilde{H} \le 1$, since otherwise we may apply Lemma 4.3.10 to reach the desired conclusion. Having fixed all the parameters, we may henceforth treat all constants depending on them as just $\bar{C} = \bar{C}(Q, m, n, \bar{n}, N, 2-r)$.

Now, applying Proposition 4.3.5 (or, more precisely, the $N = 1$ case of (4.4)) to T with $r = r_{J+1}$ and $p + \pi_0$ in place of π_0 for any $p \in P_*$, we get

$$\mathbf{E}(T, \mathbf{C}_{r_{J+1}}) \le \bar{C}(E_* + M_*^2 + \mathbf{A}^2) \le \tilde{C}(E_* + H_*^2 + \mathbf{A}^2)$$

using (4.35) in the last inequality (we stress that this is the only time we need to apply Proposition 4.3.5, namely to $P_* = P_J$ as it is the only time we have comparability between the maximum and minimum distances between the planes in $\boldsymbol{\pi}_j$, as defined below). If we now apply Proposition 4.3.16 with $\boldsymbol{\pi}_*$, r_J, r_{J+1} in place of $\boldsymbol{\pi}$, r, r_0 respectively, we get

$$\begin{aligned} \mathbf{E}(T, \mathbf{C}_{r_J}) \le \bar{C}(E_* + \mathbf{A}^2) + \bar{C}\left(\frac{E_*}{H_*^2}\right) \mathbf{E}(T, \mathbf{C}_{r_{J+1}}) &\le \bar{C}(E_* + \mathbf{A}^2) + \bar{C}\left(\frac{E_*}{H_*^2}\right)(E_* + H_*^2 + \mathbf{A}^2) \\ &\le \bar{C}(E_* + \mathbf{A}^2) \end{aligned} \tag{4.38}$$

where in the last inequality we have used the inequality $E_* \le \sigma_4 H_*^2$ from (4.37).

Next, for $j = 0, 1, \ldots, J$, define the collections $\boldsymbol{\pi}_j := \bigcup_{p \in P_j} (p + \pi_0)$ of parallel planes associated to the sets P_j; we remark that $P_J = P_*$. For each such J set

$$E_j := \int_{\mathbf{C}_2} \operatorname{dist}^2(q, \boldsymbol{\pi}_j) \, \|T\|(q)$$

and

$$H_j := \min\{|p - q| : p, q \in P_j, \, p \ne q\} \,.$$

Now observe that, for each $j = 0, 1, \ldots, J$, by property (iii) of Lemma 4.3.6, we have that

$$E_j \leq 2E_{j-1} + CH_{j-1}^2. \tag{4.39}$$

Now combining (4.38), (4.39) with $j = J$, and the fact that $E_* = E_J$, we have

$$\mathbf{E}(T, \mathbf{C}_{r_J}) \leq \bar{C}(E_{J-1} + H_{J-1}^2 + \mathbf{A}^2).$$

We now distinguish two possibilities. If $E_{J-1} \leq \sigma_4 H_{J-1}^2$, then we can apply Proposition 4.3.16 with $\boldsymbol{\pi}_{J-1}$, r_{J-1}, r_J in place of $\boldsymbol{\pi}$, r, r_0 to get

$$\mathbf{E}(T, \mathbf{C}_{r_{J-1}}) \leq \bar{C}(E_{J-1} + \mathbf{A}^2) + \bar{C}\left(\frac{E_{J-1}}{H_{J-1}^2}\right)\mathbf{E}(T, \mathbf{C}_{r_J}) \leq \bar{C}(E_{J-1} + \mathbf{A}^2).$$

On the other hand, if the opposite inequality holds, namely $H_{J-1}^2 \leq \sigma_4^{-1} E_{J-1}$, then we can estimate directly and see that

$$\mathbf{E}(T, \mathbf{C}_{r_{J-1}}) \leq \bar{C}\mathbf{E}(T, \mathbf{C}_{r_J}) \leq \bar{C}(E_{J-1} + \mathbf{A}^2)$$

where we remark that we have used in the first inequality that $r \geq 1$ here. So, we see that in either situation, this inequality holds.

Now iterate this argument, namely the one beginning after (4.39), we see that we get

$$\mathbf{E}(T, \mathbf{C}_{r_j}) \leq \bar{C}(E_j + \mathbf{A}^2)$$

for each $j = J, J-1, \ldots, 0$. In particular, taking $j = 0$, we get

$$\mathbf{E}(T, \mathbf{C}_r) \leq \bar{C}(E_0 + \mathbf{A}^2).$$

However, since $E_0 = \tilde{E}$ and $\tilde{E} \leq CE$ from (4.33), we reach the desired conclusion. □

4.3.4 Proof of L^∞ height bound

It remains to prove (4.5) of Theorem 4.3.2, which will be achieved by induction on N. As already observed, if $N = 1$ we know that Theorem 4.3.2 holds (for all Q) by Proposition 4.3.5. The core of the inductive argument is the following proposition.

Proposition 4.3.17. *Fix $N \geq 2$ and assume that, under the assumptions of Theorem 4.3.2, (4.5) holds for any $N' < N$ and any $Q' \leq Q$. Then it holds for N and Q.*

Clearly once we have shown this, Theorem 4.3.2 follows by induction.

A decay lemma

The crucial ingredient for Proposition 4.3.17 is the following L^2 height excess decay, which crucially relies on the tilt-excess estimate (4.4) that we have already established.

Lemma 4.3.18. *There are constants $\rho_0 = \rho_0(m, n, Q) > 0$ and $C = C(Q, m, n, \bar{n}) > 0$ such that, for every fixed $0 < \rho \leq \rho_0$, there are $\sigma_5 = \sigma_5(Q, m, n, \bar{n}, N, \rho) > 0$ and $0 < \beta_0 = \beta_0(Q, m, n, \bar{n}) < 1$ such that the following holds. Assume T, E, and $\boldsymbol{\pi}$ are as in Theorem 4.3.2 with $P = \{p_1, \ldots, p_N\}$ and that*

$$E + \mathbf{A}^2 \leq \sigma_5. \tag{4.40}$$

Then there is another set of points $P' := \{q_1, \ldots, q_{N'}\}$ with $N' \leq Q$ such that:

(A) $\operatorname{dist}(q_i, P) \leq C(E + \mathbf{A}^2)^{1/2}$ *for each i;*

(B) If we set $\boldsymbol{\pi}' := \bigcup(q_i + \pi_0)$, then

$$\int_{\mathbf{C}_{2\rho}} \operatorname{dist}^2(x, \boldsymbol{\pi}')\, d\|T\|(x) \leq \rho^{m+2\beta_0}(E + \mathbf{A}^2)\,. \tag{4.41}$$

Remark 4.3.19. Note that we need not have $N' \leq N$ in the conclusion of Lemma 4.3.18; the number of planes in the new collection $\boldsymbol{\pi}'$ may increase.

Proof. As usual constants denoted by C will depend only upon $Q, m, n, \bar{n}$ (their dependence on N can be reduced to a dependence on Q given that $N \in \{1, \dots, Q\}$). Recall that we have just shown the validity of the tilt-excess estimate (4.4), and thus (taking $r = 1$ in (4.4)), we have

$$\mathbf{E} := \mathbf{E}(T, \mathbf{C}_1) \leq C(E + \mathbf{A}^2)\,. \tag{4.42}$$

By choosing $\sigma_5 = \sigma_5(Q, m, n, \bar{n}, N) > 0$ sufficiently small, we can therefore ensure that both $\mathbf{E}$ and $\mathbf{A}$ are as small as we wish. We now subdivide into two cases, depending on the relative sizes of $\mathbf{E}$ and $\mathbf{A}$.

Case 1: $\mathbf{A}^3 \leq \mathbf{E}$. Here we apply the strong Lipschitz approximation theorem [19, Theorem 2.4]. By translating, we may without loss of generality assume that the origin belongs to $\mathrm{spt}(T)$ (and hence to the manifold Σ), that $\pi_0 = \mathbb{R}^m \times \{0\}$ (by rotating) and that $\Psi \equiv \Psi_0 : \mathbb{R}^{m+\bar{n}} \to \mathbb{R}^{n-\bar{n}}$ is the map parametrizing Σ graphically over $\mathbb{R}^{m+\bar{n}}$ in $\mathbf{C}_2$, as in Assumption 3.2.2. By [19, Remark 2.5], we have the estimates

$$\Psi(0) = 0 \quad \text{and} \quad \|D\Psi\|_{C^2} \leq C(\mathbf{E}^{1/2} + \mathbf{A})\,; \tag{4.43}$$

here, $C = C(m, n, \bar{n})$. By [19, Theorem 2.4] there exists constants $\gamma = \gamma(Q, m, n, \bar{n}) > 0$ and $\varepsilon = \varepsilon(Q, m, n, \bar{n}) > 0$ such that if $\mathbf{E} < \varepsilon$ (which can be guaranteed provided σ_5 is sufficiently small), then there is a multi-valued map $u : B_{1/4}(0, \pi_0) \to \mathbb{R}^{\bar{n}}$ such that $f = (u, \Psi(x, u))$ (using the notation of [19]) is a good approximation of $T \llcorner \mathbf{C}_{1/4}$, in the following sense:

(i) $\mathrm{Lip}\,(f) \leq C(\mathbf{E} + \mathbf{A}^2)^\gamma \leq C(E + \mathbf{A}^2)^\gamma$;

(ii) There is a closed set $K \subset B_{1/4}$ of measure at least $\frac{1}{2}|B_{1/4}|$ such that $\mathbf{G}_f \llcorner (K \times \mathbb{R}^n) = T \llcorner (K \times \mathbb{R}^n)$ and

$$\|T\|((B_{1/4} \setminus K) \times \mathbb{R}^n) \leq C(\mathbf{E} + \mathbf{A}^2)^{1+\gamma} \leq C(E + \mathbf{A}^2)^{1+\gamma}\,;$$

where $C = C(Q, m, n, \bar{n})$. Notice that in the above estimates, we have used (4.42) to get the

improved control in terms of the L^2 height excess. From the above properties, the estimate [19, Theorem 2.4(2.6)] and (4.43) we also see that

$$\int_{B_{1/4}} |Df|^2 \leq C \int_K |Df|^2 + C(\mathbf{E}+\mathbf{A}^2)^{1+\gamma} \leq C\mathbf{E} + C(\mathbf{E}+\mathbf{A}^2)^{1+\gamma} \overset{(4.40)}{\leq} C(\mathbf{E}+\mathbf{A}^2)\,. \quad (4.44)$$

Moreover, for every fixed η, if $\mathbf{E}$ is sufficiently small (depending on η), by [19, Theorem 2.6] there is a Dir-minimizing function $v : B_{1/4} \to \mathcal{A}_Q(\mathbb{R}^{\bar{n}})$ such that, if we set $g = (v, \Psi(x, v))$, then

$$\int_{B_{1/4}} \mathcal{G}(f,g)^2 \leq \eta\mathbf{E} \quad (4.45)$$

$$\int_{B_{1/4}} |Dg|^2 \leq \int_{B_{1/4}} |Df|^2 + \eta\mathbf{E} \overset{(4.44)}{\leq} C(\mathbf{E}+\mathbf{A}^2) \overset{(4.42)}{\leq} C(E+\mathbf{A}^2)\,. \quad (4.46)$$

Note that the condition $\mathbf{A} \leq \mathbf{E}^{\frac{1}{4}+\bar{\delta}}$ in [19, Theorem 2.6] is satisfied, with $\bar{\delta} = 1/12$ in this case, since we are assuming $\mathbf{A}^3 \leq \mathbf{E}$.

Observe now that, by a simple Chebyshev argument, for at least half of the points $x \in K \cap B_{1/8}$ we have the following property:

(a) if $f(x) = \sum_i Q_i \llbracket f_i(x) \rrbracket$ (with $f_i(x)$ distinct), for each i there is a $j(i)$ such that $|f_i(x) - p_{j(i)}| \leq CE^{1/2}$.

Hence, by another Chebyshev argument, combined with (4.45), we may find at least one such point $x \in K \cap B_{1/8}$ for which we have the corresponding property for $g(x)$ (in fact, we can find a set of positive measure on which this holds):

(b) if $g(x) = \sum_i \tilde{Q}_i \llbracket g_i(x) \rrbracket$ (with $g_i(x)$ distinct), for every i there is a $j(i)$ such that $|g_i(x) - p_{j(i)}| \leq CE^{1/2}$.

Now, combining (4.46) with the Hölder estimate [18, Theorem 3.9] for Dir-minimizing functions, we also conclude an analogous estimate at 0:

(c) if $g_i(0) = \sum_i Q_i^* [\![g_i(0)]\!]$ (with $g_i(0)$ distinct), for every i there is a $j(i)$ such that $|g_i(0) - p_{j(i)}| \leq C(E^{1/2} + \mathbf{A})$

Notice that since $x \in B_{1/8}$, $\mathrm{dist}(x, \partial B_{1/4}) \geq C^{-1}$ and so the choice of δ in [18, Theorem 3.9] only depends on n, m, Q). We now set $q_i := g_i(0)$; obviously the number of distinct points q_i is at most Q. Clearly, by (c) above, this choice of q_i obeys conclusion (A) of the lemma.

Observe that, by all the estimates carried over so far and by Lemma 4.3.11, for each $\rho \leq \frac{1}{8}$ and some $\alpha = \alpha(m, Q) \in]0, 1[$ we have

$$\begin{aligned}\int_{\mathbf{C}_{2\rho}} \mathrm{dist}^2(x, \boldsymbol{\pi}')\, d\|T\|(x) &\overset{\text{(ii)}}{\leq} C(E + \mathbf{A}^2)^{1+\gamma} + C\int_{B_{2\rho}} \mathcal{G}(f(x), g(0))^2\, dx \\ &\leq C\sigma_5^\gamma (E + \mathbf{A}^2) + C\eta(E + \mathbf{A}^2) + C\rho^{m+2\alpha}(E + \mathbf{A}^2)\,;\end{aligned}$$

where Lemma 4.3.11 was used in the first inequality, while in the second inequality we use (4.40), (4.45), the α-Hölder continuity of g from [18, Theorem 3.9], and (4.46).

Now choose $\beta_0 \in]0, \alpha/2]$, and choose $\rho_0 > 0$ with $C\rho_0^\alpha \leq \frac{1}{3}$. Then choose $\eta = \eta(\rho)$ such that $C\eta \leq \frac{1}{3}\rho^{m+2\beta_0}$. Then, given this choice of η, provided that we choose σ_5 sufficiently small so that our application of [19, Theorem 2.6] was valid with this η, and so that $C\sigma_5^\gamma \leq \frac{1}{3}\rho^{m+2\beta_0}$, the above estimate clearly then gives (B) of the lemma. This completes the proof of Case 1.

Case 2: In this case we have $\mathbf{E} \leq \mathbf{A}^3$. As in the previous case, using the same notation, we introduce the Lipschitz approximation $f = (u, \Psi(x, u))$ on $B_{1/4}(0, \pi_0)$. Observe that, since the graph of f coincides with the current T on $K \times \mathbb{R}^n \subset B_{1/4} \times \mathbb{R}^n$, we again have the first two inequalities of (4.44), only now we further estimate this as follows:

$$\int_{B_{1/4}} |Df|^2 \leq C\mathbf{E} + C(\mathbf{E} + \mathbf{A}^2)^{1+\gamma} \leq C\mathbf{A}^{2+2\gamma} \leq C\sigma_5^\gamma \mathbf{A}^2\,.$$

We then use the Poincaré inequality for Q-valued functions (see e.g. [18, Proposition 2.12 &

Proposition 4.9]) and Hölder's inequality to find a point $Y \in \mathcal{A}_Q$ such that

$$\int_{B_{1/4}} \mathcal{G}(f,Y)^2 \leq C \left(\int_{B_{1/4}} \mathcal{G}(f,Y)^{2^*} \right)^{2/2^*} \leq C \int_{B_{1/4}} |Df|^2 .$$

In particular we reach

$$\int_{B_{1/4}} \mathcal{G}(f,Y)^2 \leq C\sigma_5^\gamma \mathbf{A}^2 . \tag{4.47}$$

Write $Y = \sum_i Q_i[\![q_i]\!]$, where the q_i are distinct, and set $\boldsymbol{\pi}' := \bigcup_i (q_i + \pi_0)$. We then find that, for every $0 < \rho \leq \frac{1}{8}$, by Lemma 4.3.11, (4.47), and (ii), we have

$$\int_{\mathbf{C}_{2\rho}} \mathrm{dist}^2(x, \boldsymbol{\pi}')\, d\|T\|(x) \leq C\sigma_5^\gamma \mathbf{A}^2 + \|T\|((B_{1/4} \setminus K) \times \mathbb{R}^n) \leq C\sigma_5^\gamma \mathbf{A}^2 .$$

In particular, for every fixed ρ, we may fix β_0 to be an arbitrary positive dimensional constant and choose σ_5 small enough (dependent on β_0) to guarantee that $C\sigma_5^\gamma \leq \rho^{m+2\beta_0}$

$$\int_{\mathbf{C}_{2\rho}} \mathrm{dist}^2(x, \boldsymbol{\pi}')\, d\|T\|(x) \leq \rho^{m+2\beta_0} \mathbf{A}^2 .$$

This proves conclusion (B) of the lemma, in this case.

We are left with proving the estimate (A). Recall that, in light of (ii), we have $|B_{1/4} \setminus K| \leq \frac{1}{2}|B_{1/4}|$, and hence, via a similar Chebyshev argument as in Case 1, now using (4.47), we may estimate

$$\mathrm{dist}^2(q_i, \mathrm{spt} f(x)) \leq C\mathbf{A}^2 \qquad \forall i ,$$

for all x in a subset $K' \subset K$ whose measure is at least $\frac{1}{4}|B_{1/4}|$. On the other hand, on at least half of the set K' we also have by another Chebyshev argument

$$\max_{p \in \mathrm{spt} f(x)} \mathrm{dist}^2(p, \boldsymbol{\pi}) \leq CE .$$

Therefore, choosing a point $p \in \mathrm{spt} f(x)$ for x in this latter set and using the triangle

inequality we get

$$\operatorname{dist}^2(q_i, \boldsymbol{\pi}) \leq C(E + \mathbf{A}^2) \qquad \forall i\,,$$

which proves (A) and thus completes Case 2, which also completes the proof. □

Proof of Proposition 4.3.17

Having proved the L^2 height excess decay lemma, we are now in a position to prove the inductive step given in Proposition 4.3.17. Fix $1 \leq r < 2$ as in the statement of Theorem 4.3.2 and set

$$M := \max_{i,j}\{|p_i - p_j|\}\,.$$

We will show the existence of a constant $\sigma_6 = \sigma_6(Q, m, n, \bar{n}, N, r) > 0$ such that, if

$$E + \mathbf{A}^2 \leq \sigma_6 M^2\,, \tag{4.48}$$

then there is decomposition $\boldsymbol{\pi} = \boldsymbol{\pi}_1 \cup \boldsymbol{\pi}_2$ and $T \llcorner \mathbf{C}_{1+r/2} = T_1 + T_2$ such that

(i) The sets $\boldsymbol{\pi}_1, \boldsymbol{\pi}_2$ are disjoint and nonempty;

(ii) $\partial T_i \llcorner \mathbf{C}_{1+r/2} = 0$;

(iii) $\operatorname{spt}(T_1) \cap \operatorname{spt}(T_2) = \emptyset$;

(iv) $\operatorname{dist}(q, \boldsymbol{\pi}) = \operatorname{dist}(q, \boldsymbol{\pi}_i)$ for every $q \in \operatorname{spt}(T_i)$, for each $i = 1, 2$.

In particular, assuming such a decomposition, after rescaling $1 + r/2$ to scale 2 we can apply the inductive assumption to T_1 and T_2 (note that whilst we could have, e.g. $T_2 = 0$, i.e. Q remains fixed, however by (i) we then know that N strictly decreases) to conclude

$$\operatorname{spt}(T_i) \cap \mathbf{C}_r \subset \{\operatorname{dist}(\cdot, \boldsymbol{\pi}_i) \leq \bar{C}(E^{(i)} + \mathbf{A}^2)^{1/2}\}$$

where $\bar{C} = \bar{C}(Q, m, n, \bar{n}, N, r)$ and

$$E^{(i)} := \int_{\mathbf{C}_{1+r/2}} \mathrm{dist}^2(x, \boldsymbol{\pi}_i)\, d\|T_i\|(x)\,.$$

Given that obviously $E^{(1)}, E^{(2)} \leq CE$, we would have reached the desired estimate (4.5), provided that (4.48) holds. On the other hand, if the estimate (4.48) does not hold, i.e. $M^2 \leq \sigma_6^{-1}(E + \mathbf{A}^2)$, we can take an arbitrary plane $p_i + \pi_0$ from $\boldsymbol{\pi}$ and conclude that

$$\int_{\mathbf{C}_2} \mathrm{dist}^2(x, p_i + \pi_0)\, d\|T\|(x) \leq \bar{C}(E + \mathbf{A}^2)\,.$$

It would then follow from the case $N = 1$ of Theorem 4.3.2, namely Proposition 4.3.5, that

$$\mathrm{spt}(T) \cap \mathbf{C}_r \subset \{\mathrm{dist}(\cdot, p_i + \pi_0) \leq \bar{C}(E + \mathbf{A}^2)^{1/2}\} \subset \{\mathrm{dist}(\cdot, \boldsymbol{\pi}) \leq \bar{C}(E + \mathbf{A}^2)^{1/2}\}\,,$$

for any fixed plane $p_i + \pi_0 \in \boldsymbol{\pi}$, proving (4.5).

We are thus left to prove the existence of the decomposition satisfying (i)-(iv) under the assumption that (4.48) holds for a sufficiently small σ_6. We first use the combinatorial Lemma 4.3.8 to decompose $P \equiv \{p_1, \ldots, p_N\} = P_1 \cup P_2$, and consequently $\boldsymbol{\pi} = \boldsymbol{\pi}_1^* \cup \boldsymbol{\pi}_2^*$ for $\boldsymbol{\pi}_i^* = \bigcup_{p \in P_i}(p + \pi_0)$, with the property that P_1 and P_2 are disjoint non-empty sets satisfying

$$\mathrm{sep}\,(P_1, P_2) \equiv \min\{|p_1 - p_2| : p_1 \in P_1,\, p_2 \in P_2\} \geq \frac{M}{2^{N-2}}\,. \tag{4.49}$$

Our claim is that the desired decomposition will be achieved with the latter choice. In light of (4.49), this will follow immediately if we have that

$$\mathrm{spt}(T) \cap \mathbf{C}_{1+r/2} \subset \{x : \mathrm{dist}(x, \boldsymbol{\pi}) \leq 2^{-N} M\}\,. \tag{4.50}$$

Let us thus show that (4.50) indeed holds. Fix a point $x \in B_{1+r/2}(0, \pi_0)$ and the scaling factor $1 - r/2$. Let T^0 be the rescaled current $(\iota_{x,1-r/2})_\sharp T$. Likewise, introduce the sets

$\boldsymbol{\pi}_0 := \iota_{x,1-r/2}(\boldsymbol{\pi})$, $\Sigma^0 = \iota_{x,1-r/2}(\Sigma)$, the shorthand notation $\mathbf{A}_0$ for the supremum norm of the second fundamental form of Σ^0, and

$$E_0 := \int_{\mathbf{C}_2} \mathrm{dist}^2(y, \boldsymbol{\pi}_0)\, d\|T^0\|(y)\,.$$

Observe that $E_0 \leq (1-r/2)^{-m-2}E$, $\mathbf{A}_0^2 = (1-r/2)^2\mathbf{A}^2$. Setting $M_0 := (1-r/2)^{-1}M$, since $1-\frac{r}{2} \leq 1$ we see immediately that

$$E_0 + \mathbf{A}_0^2 \leq (1-r/2)^{-m}\sigma_6 M_0^2\,.$$

Let now $\bar{\sigma}$ be the threshold of Lemma 4.3.12. We start by choosing σ_6 small enough so that

$$\sigma_7 := (1-r/2)^{-m}\sigma_6 \leq \bar{\sigma}\,. \tag{4.51}$$

In particular, if $M_0 \geq 1$ we could apply Lemma 4.3.12 and achieve the estimate

$$\mathrm{spt}(T^0) \cap \mathbf{C}_1 \subset \left\{\mathrm{dist}(\cdot, \boldsymbol{\pi}_0) \leq 2^{-N}M_0\right\}\,.$$

Given that this holds for every $x \in B_{1+r/2}(0, \pi_0)$, it would then imply (4.50) and conclude the proof.

Otherwise, if $M_0 < 1$, we may fix $\rho = \rho_0 < 1$, with ρ_0 as in Lemma 4.3.18, and let σ_5 be the corresponding threshold from Lemma 4.3.18. We may further choose σ_7 (as defined in (4.51)) small enough such that $\sigma_7 \leq \sigma_5$. In particular, since $M_0 < 1$, we can apply Lemma 4.3.18 to T^0 and $\boldsymbol{\pi}_0$. Denote by $\boldsymbol{\pi}_0'$ the corresponding new group of planes and let

- $T^1 = (\iota_{0,\rho})_\sharp T^0$;
- $\Sigma^1 = \iota_{0,\rho}(\Sigma^0)$, with $\mathbf{A}_1$ the supremum norm of the second fundamental form of Σ^1;
- $\bar{\boldsymbol{\pi}}_1 = \iota_{0,\rho}(\boldsymbol{\pi}_0')$;

- $M_1 = \rho^{-1} M_0$;
- $\bar{E}_1 = \int_{\mathbf{C}_2} \operatorname{dist}^2(x, \bar{\boldsymbol{\pi}}_1)\, d\|T^1\|(x)$.

Observe that from (4.41) we have the inequality (with $\beta_0 < 1$ as in Lemma 4.3.18)

$$\bar{E}_1 + \mathbf{A}_1^2 \leq \rho^{2\beta_0 - 2}(E_0 + \mathbf{A}_0^2) \leq \rho^{2\beta_0 - 2}\sigma_7 M_0^2 = \rho^{2\beta_0}\sigma_7 M_1^2\,. \tag{4.52}$$

As $\rho < 1$, we can thus keep iterating this procedure with the same ρ, and along the iteration each $\bar{\boldsymbol{\pi}}_k$ consists of some number of planes, which could be larger than the initial number N, but it never exceeds Q. This in particular means that the parameter σ_7 remains fixed, dependent on only ρ, Q, N and dimensional constants. generating currents T^j, ambient manifolds Σ^j, families of parallel planes $\boldsymbol{\pi}'_j$ and $\bar{\boldsymbol{\pi}}_j = \iota_{0,\rho}(\boldsymbol{\pi}'_j)$ until we reach the first index k such that

$$M_k = \rho^{-k} M_0 \geq 1.$$

Let us now see how this iteration affects our original current T. Setting

$$\bar{E}_j := \int_{\mathbf{C}_2} \operatorname{dist}^2(x, \bar{\boldsymbol{\pi}}_j)\, d\|T^j\|(x)$$

we may conclude, analogously to (4.52), that

$$\bar{E}_j + \mathbf{A}_j^2 \leq \rho^{2\beta_0 j}\sigma_7 M_j^2\,.$$

Now define $\boldsymbol{\pi}_j := \iota_{0,\rho^j}(\boldsymbol{\pi}_0) = \iota_{0,\rho^j}(\iota_{x,(1-r/2)}(\boldsymbol{\pi}))$ and set

$$E_j := \int_{\mathbf{C}_2} \operatorname{dist}^2(x, \boldsymbol{\pi}_j)\, d\|T^j\|(x)\,.$$

We wish to estimate E_j and the distance of the new family of planes $\boldsymbol{\pi}'_j$ from the one $\boldsymbol{\pi}_j$

obtained by rescaling $\boldsymbol{\pi}_0$. We therefore introduce

$$d(j) := \max\{\mathrm{dist}(\pi, \boldsymbol{\pi}_j) : \pi \in \boldsymbol{\pi}'_j\}$$

Note that from (A) from Lemma 4.3.18 we have

$$d(0) \leq C(E_0 + \mathbf{A}_0^2)^{1/2} \leq C\sigma_7^{1/2} M_0 \,.$$

Subsequently, using the triangle inequality (to estimate the distance from $\boldsymbol{\pi}_1$ to $\bar{\boldsymbol{\pi}}_1$ and then from $\bar{\boldsymbol{\pi}}_1$ to $\boldsymbol{\pi}'_1$)

$$d(1) \leq \left(C\sigma_7^{1/2} M_0\right) \rho^{-1} + C(E_1 + \mathbf{A}_1^2)^{1/2} \leq C\sigma_7^{1/2} M_1 (1 + \rho^{\beta})$$

where we have used our previous bounds and the fact $M_1 = \rho^{-1} M_0$. Inductively, we achieve

$$d(j) \leq C\sigma_7^{1/2} M_j \sum_{i=0}^{j} \rho^{j\beta} \,;$$

we stress that C is the same each time we apply the lemma and so it is constant in j. In particular,

$$d(k) \leq C\sigma_7^{1/2} M_k \,.$$

Since

$$E_k \leq C d(k)^2 + C\bar{E}_k \,,$$

we conclude that

$$E_k \leq C\sigma_7 M_k^2 \,,$$

where the constant C depends only on $m, n, \bar{n}$, and Q. In particular, due to the fact that $M_k \geq 1$ by construction (and M_k is the maximal distance between the planes in $\boldsymbol{\pi}_k$), we may choose σ_7 smaller than a suitable geometric constant in order to apply Lemma 4.3.12,

concluding that

$$\operatorname{spt}(T^k) \cap \mathbf{C}_1 \subset \{\operatorname{dist}(\cdot, \boldsymbol{\pi}_k) \leq 2^{-N} M_k\}\,.$$

Rescaling this information in the original system of coordinates gives

$$\operatorname{spt}(T) \cap \mathbf{C}_{\rho^k(1-r/2)}(x) \subset \{\operatorname{dist}(\cdot, \boldsymbol{\pi}) \leq 2^{-N} M\}\,.$$

The arbitrariness of $x \in B_{1+r/2}(0, \pi_0)$ gives then (4.50) and hence concludes the proof. □

The proof of Theorem 4.3.2 is therefore complete.

4.4 Overview of proof of Theorem 4.2.4

We are now in a position to prove Theorem 4.2.4. The overall structure of the remainder of the chapter is as follows. In Section 4.5, we begin with a few technical preliminaries regarding the relative positioning of planes, and some combinatorial preliminaries which will be needed for the graphical parameterizations in the following section. In Section 4.6, we set up efficient graphical parameterizations for T over *balanced* cones in $\mathscr{C}(Q)$ (as defined in the preceding section), away from the spines of the cones and when the two-sided excess of T relative to $\mathbf{S}$ is much smaller than the *minimal angle* of the cone (such an assumption being used in this context is equivalent to that used by Wickramasekera [56, Section 10, Hypothesis $(\star\star)$]). This construction heavily relies on the height bound from Section 4.3, and given this our construction is a suitable adaptation of those seen in the work of Simon ([45]) and Wickramasekera ([56]), with added technical complications from working in arbitrary codimension and with area-minimizing currents. Section 4.7 is then dedicated to showing that we may assume that the cone $\mathbf{S}$ in Theorem 4.2.4 is balanced, allowing us to approximate T with a multi-valued graph over this cone away from its spine, as demonstrated in the previous section. In Section 4.8 we reduce the proof of Theorem 4.2.4 to an a priori much weaker weaker decay theorem which has a much stronger assumption on the size of the L^2

height excess of T relative to $\mathbf{S}$; the idea is to use an excess decay statement with multiple scales, analogous to that first seen in [56, Section 13]. In Section 4.9, we then show the non-concentration of excess and the Simon estimates at the spine of the cone $\mathbf{S}$; these are the key estimates first used in Simon's work ([45]) and then adapted to a degenerate setting, as we similar face here, by Wickramasekera ([56, Section 10]). We then display the analogue of Theorem 4.2.4 at the linearized level (for multi-valued Dir-minimizers) in Section 4.10. Finally, in Section 4.11, we conclude with a blow-up procedure, collecting all the previous arguments to yield a limiting Dir-minimizer that contradicts the results in Section 4.10 if we assume the failure of Theorem 4.2.4.

4.5 Relative position of pairs of planes

We begin with some elementary results on the relative positions of pairs of planes. These will be useful for results appearing later in this section regarding comparability of angle parameters for cones in $\mathscr{C}(Q)$ and multi-valued Dir-minimizers formed from superpositions of linear maps.

We start with a simple lemma in linear algebra.

Lemma 4.5.1. *Consider two m-dimensional linear subspaces α, β of $\mathbb{R}^N$ and the two quadratic forms $Q_1 : \alpha \to \mathbb{R}$, $Q_2 : \beta \to \mathbb{R}$ given by $Q_1(v) := \mathrm{dist}^2(v, \beta)$ and $Q_2(w) := \mathrm{dist}^2(w, \alpha)$. Then:*

(a) Q_1 and Q_2 have the same eigenvalues with the same multiplicities.

(b) For every orthonormal base $v_1, \ldots, v_m$ of eigenvectors of Q_1 there is a corresponding orthonormal base $w_1, \ldots, w_m$ of eigenvectors of Q_2 with the property that

$$Q_1(v_i) = Q_2(w_i) = 1 - (w_i \cdot v_i)^2 \,,$$

and $v_i \cdot w_j = 0$ for $i \neq j$.

(c) If $\alpha \cap \beta^\perp = \{0\}$ a choice of $w_1, \dots, w_m$ is given by $w_i = \frac{\mathbf{p}_\beta(v_i)}{|\mathbf{p}_\beta(v_i)|}$.

Before proving this, let us note that this lemma motivates the following definition:

Definition 4.5.2. *Given two m-dimensional linear subspaces α, β of $\mathbb{R}^N$ whose intersection has dimension $m-k$, we order the k positive eigenvalues $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_k$ of Q_1 as in Lemma 4.5.1, with the convention that the number of occurrences of the same real λ in the list equals its multiplicity as eigenvalue of Q_1. The Morgan angles of the pair α and β are the numbers $\theta_i(\alpha, \beta) := \arcsin \sqrt{\lambda_i}$ for $i = 1, \dots, k$.*

Of course we can define the Morgan angles between two intersecting affine planes of the same dimension by simply translating an intersection point to the origin.

The following is then an immediate corollary of Lemma 4.5.1 and basic linear algebra:

Corollary 4.5.3. *Suppose that α, β are two m-dimensional linear subspaces of $\mathbb{R}^N$. Denote by V their $(m-k)$-dimensional intersection and let $\theta_1 \le \cdots \le \theta_k$ be their Morgan angles. Then*

$$|v| \sin \theta_1 \le \operatorname{dist}(v, \beta) \le |v| \sin \theta_k \qquad \forall v \in V^\perp \cap \alpha \,, \tag{4.53}$$

and

$$\operatorname{dist}(\alpha \cap \mathbf{B}_1, \beta \cap \mathbf{B}_1) = \sup\{\operatorname{dist}(v, \beta) : v \in \alpha \cap \mathbf{B}_1\} = \sin \theta_k \,. \tag{4.54}$$

Indeed, this simply follows because $V^\perp \cap \alpha$ is the eigenspace of the quadratic form $Q_1 = \operatorname{dist}^2(\cdot, \beta)$ spanned by those eigenvectors which have positive eigenvalues, combined with the fact that the extremal eigenvalues of a quadratic form are realized through constrained optimization over the unit sphere. Note that the first equality in (4.54) is simply due to the fact that α and β are m-dimensional subspaces.

Proof of Lemma 4.5.1. We prove (b) and (a) at the same time. Moreover, since we give an explicit construction for the vectors w_i, the proof will also show (c). We start by observing that

$$Q_1(v) = |\mathbf{p}_\beta^\perp(v)|^2 = v \cdot (\mathbf{p}_\beta^\perp(v)) \,. \tag{4.55}$$

Now pick an orthonormal basis $\{v_1, \ldots, v_m\}$ of α which diagonalizes Q_1. Moreover, let us remark that we can extend Q_1 to a quadratic form $\bar{Q}_1$ on $\mathbb{R}^N$ by setting it identically 0 on $\alpha^\perp$. $\bar{Q}_1$ can be computed explicitly as

$$\bar{Q}_1(z) = z \cdot (\mathbf{p}_\alpha \circ \mathbf{p}_\beta^\perp \circ \mathbf{p}_\alpha(z)) \tag{4.56}$$

for a general $z \in \mathbb{R}^N$. In particular, each v_i must be an eigenvector of the symmetric matrix $\mathbf{p}_\alpha \circ \mathbf{p}_\beta^\perp \circ \mathbf{p}_\alpha$, which in turn implies that

$$\mathbf{p}_\alpha(\mathbf{p}_\beta^\perp(v_i)) = \lambda_i v_i \tag{4.57}$$

for some constant $0 \leq \lambda_i \leq 1$. Observe, moreover, that in case $\lambda_i = 1$, then necessarily

$$\mathbf{p}_\beta^\perp(v_i) = v_i \,.$$

Without loss of generality let us order the v_i to assume that $0 \leq Q_1(v_i) \leq Q_1(v_j) \leq 1$ for all $i \leq j$. Let us write k for the largest element of $\{1, \ldots, m\}$ for which $Q_1(v_k) < 1$. Observe that $\mathbf{p}_\beta(v_j) = 0$ for all $j > k$. For $i \leq k$, define $w_i := \frac{\mathbf{p}_\beta(v_i)}{|\mathbf{p}_\beta(v_i)|}$, and write $W = \mathrm{span}\{w_1, \ldots, w_k\}$. Select any orthonormal basis $\{w_{k+1}, \ldots, w_m\}$ of $W^\perp \cap \beta$. Since $\mathbf{p}_\beta(v_j) = 0$ for $j > k$, it is obvious that $W = \mathbf{p}_\beta(\alpha)$. In particular, it follows that $w_j \perp \alpha$ for every $j > k$, and so for $j > k$ we have $1 = Q_1(v_j) = Q_2(w_j)$ and $w_j \cdot v_i = 0$ for all i.

To handle the case $j \leq k$, note that, if $i \neq j$, then

$$\begin{aligned} v_i \cdot w_j &= |\mathbf{p}_\beta(v_j)|^{-1}(v_i \cdot \mathbf{p}_\beta(v_j)) = -|\mathbf{p}_\beta(v_j)|^{-1}(v_i \cdot \mathbf{p}_\beta^\perp(v_j)) \\ &= -|\mathbf{p}_\beta(v_j)|^{-1}(v_i \cdot \mathbf{p}_\alpha(\mathbf{p}_\beta^\perp(v_j))) \overset{(4.57)}{=} -\lambda_j |\mathbf{p}_\beta(v_j)|^{-1} v_i \cdot v_j = 0 \,. \end{aligned}$$

Next consider $v = \sum_{i=1}^{k} a_i v_i$ and compute

$$\begin{aligned} Q_1(v) &= |\mathbf{p}_\beta^\perp(v)|^2 = |v|^2 - |\mathbf{p}_\beta(v)|^2 \\ &= \sum_i a_i^2 - \left|\sum_{i=1}^{k} (w_i \cdot v) w_i\right|^2 = \sum_i a_i^2 - \left|\sum_{i=1}^{k} a_i (w_i \cdot v_i) w_i\right|^2 \\ &= \sum_i a_i^2 \left(1 - (w_i \cdot v_i)^2\right) - 2 \sum_{i<j\leq k} a_i a_j (w_i \cdot w_j)(w_i \cdot v_i)(w_j \cdot v_j)\,. \end{aligned}$$

Given that $Q_1(v) = \sum_i a_i^2 Q_1(v_i) = \sum_i a_i^2 (1 - (v_i \cdot w_i)^2)$ and the coefficients a_i can be chosen arbitrarily, we must then have

$$(w_i \cdot w_j)(w_i \cdot v_i)(w_j \cdot v_j) = 0$$

for every $1 \leq i < j \leq k$. In particular, since $(w_i \cdot v_i)(w_j \cdot v_j) \neq 0$ when $i < j \leq k$ we immediately infer that $w_i \cdot w_j = 0$, hence concluding that $w_1, \ldots, w_m$ is an orthonormal basis of β. Moreover, we have already seen that $Q_1(v_i, v_i) = 1 - (v_i \cdot w_i)^2$.

Next, fix $j \leq k$ and observe that, by construction, w_j is orthogonal to v_ℓ when $\ell > k$. We also already know that for $i, j \leq k$, $i \neq j$, that w_j is orthogonal to v_i. In particular, this gives that $\mathbf{p}_\alpha(w_j) = (v_j \cdot w_j) v_j$ for every j. Hence, given any $w = \sum_i b_i w_i \in \beta$, we have

$$Q_2(w) = |w|^2 - |\mathbf{p}_\alpha(w)|^2 = \sum_i \mu_i^2 (1 - (w_i \cdot v_i)^2).$$

This proves that $Q_1(v_i) = Q_2(w_i)$ and that w_i is an orthonormal basis of eigenvectors of Q_2, which completes the proof of (b) and also implies at the same time the statement (a). □

4.5.1 Rotating planes

In this section we use the material of the previous one to define "canonical rotations" which map equidimensional linear subspaces onto each other. These objects are not strictly necessary for our considerations, but they help streamlining some arguments.

Consider two linear subspaces α and β of $\mathbb{R}^{m+n}$ of the same dimension k and assume that

$$\alpha \cap \beta^{\perp} = \{0\}\,. \tag{4.58}$$

We then define a canonical element $R(\alpha, \beta) \in O(\mathbb{R}^{m+n})$ mapping α onto β in the following fashion. First of all consider the quadratic form $Q(v) = \mathrm{dist}^2(v, \beta)$ on α introduced in the previous section and let $v_1, \ldots, v_k$ be an orthonormal base which diagonalizes Q. Likewise consider the quadratic form $Q^{\perp}$ on $\alpha^{\perp}$ defined by $\mathrm{dist}^2(v, \beta^{\perp})$ and let $v_{k+1}, \ldots, v_{m+n}$ be an orthonormal base which diagonalizes $Q^{\perp}$. We then define $w_i := \frac{\mathbf{p}_{\beta}(v_i)}{|\mathbf{p}_{\beta}(v_i)|}$ for $1 \leq i \leq k$ and $w_i := \frac{\mathbf{p}^{\perp}_{\beta}(v_i)}{|\mathbf{p}^{\perp}_{\beta}(v_i)|}$ for $k+1 \leq i \leq m+n$. By Lemma 4.5.1, $w_1, \ldots, w_{m+n}$ is an orthonormal base of $\mathbb{R}^{m+n}$ and if we define $R(v_i) = w_i$ and extend R by linearity we clearly have an element of $O(\mathbb{R}^{m+n})$ with the property that $O(\alpha) = \beta$.

The properties of this map is then given in the following

Lemma 4.5.4. *Let α and β be two equidimensional linear subspaces such that $\alpha \cap \beta^{\perp} = \{0\}$. Then*

(a) $R = R(\alpha, \beta)$ is well-defined, i.e. its definition does not depend on the choice of the diagonalizing orthonormal bases of Q and $Q^{\perp}$;

(b) R is an element of $SO(m+n)$;

(c) $R(\alpha, \beta) = (R(\beta, \alpha))^{-1}$ and R is the identity on $\alpha \cap \beta$ and $\alpha^{\perp} \cap \beta^{\perp}$; in particular $R(\alpha, \alpha)$ is the identity;

(d) R depends continuously on α and β.

Proof. Concerning (a), the only ambiguity in the definition stems from the fact that the vectors $v_1, \ldots, v_{m+n}$ are not uniquely defined. Assume thus $v'_1, \ldots v'_m$ and $v'_{m+1}, \ldots, v'_{m+n}$ form two other orthonormal bases of Q and $Q^{\perp}$. Define w'_i accordingly. Moreover, without loss of generality, assume that the eigenvalues are the same for v_i and v'_i. If the eigenvalues

of Q are all distinct and those of $Q^\perp$ are also all distinct, then $v_i = \pm v_i'$ and hence $w_i = \pm w_i'$ and we see immediately that the definition of R does not depend on the choice of the bases. Assume otherwise that there are some eigenvalues with higher multiplicity. To fix ideas assume therefore that $v_j, \dots, v_\ell$ form a base of a maximal eigenspace W for a fixed eigenvalue λ of Q. Then $v_j', \dots, v_\ell'$ is also a base of the same eigenspace. Note however that, by the very definition of Q, $|\mathbf{p}_\beta(v)| = \sqrt{1-\lambda}|v|$ for every $v \in W$. In particular we see immediately that, computing v_k' as a linear combination $\sum_{j \le i \le \ell} a_{\ell i} v_i$, then $w_k' = \sum_{j \le i \le \ell} a_{\ell i} w_i$. Therefore R is well defined in this case as well.

Concerning (b), observe that $v_i \cdot w_i > 0$ for all i, and thus $v_1, \dots, v_{m+n}$ and $w_1, \dots, w_{m+n}$ have the same orientation. Concerning (c) observe first that $\alpha \cap \beta$ is the maximal eigenspace of Q for the eigenvalue 0, while $\alpha^\perp \cap \beta^\perp$ is the maximal eigenspace of Q for the eigenvalue 0. Thus $v_1, \dots, v_{m+n}$ contains an orthonormal base of the former and an orthonormal base for the latter and R is by definition the identity on these elements. Moreover, it turns out that, by Lemma 4.5.1, when defining $R(\beta, \alpha)$, we can choose $w_1, \dots w_k$ and $w_{k+1}, \dots, w_{m+n}$ as orthonormal bases diagonalizing $\mathrm{dist}^2(\cdot, \alpha)$ and $\mathrm{dist}^2(\cdot, \alpha^\perp)$. But then Lemma 4.5.1(b) implies that $v_i = \frac{\mathbf{p}_\alpha(w_i)}{|\mathbf{p}_\alpha(w_i)|}$ for $1 \le i \le m+n$, from which it immediately follows that $R(\beta, \alpha)$ maps w_i in v_i, and it is thus the inverse of $R(\alpha, \beta)$.

As for the continuous dependence, observe that if $\alpha_k \to \alpha$ and $\beta_k \to \beta$ and we fix $v_1^k, \dots, v_{m+n}^k$ with $v_1^k, \dots, v_m^k$ diagonalizing $\mathrm{dist}^2(\cdot, \beta_k)$ on α_k and $v_{m+1}^k, \dots, v_{m+n}^k$ diagonalizing $\mathrm{dist}^2(\cdot, \beta_k^\perp)$ on $\alpha_k^\perp$, then we can extract a subsequence for which v_i^k converge to v_i. It follows immediately that the orthonormal vectors $v_1, \dots, v_m$ diagonalize Q and $v_{m+1}, \dots, v_{m+n}$ diagonalize $Q^\perp$. Hence the algorithm given to determine $R(\alpha_k, \beta_k)$ shows immediately that the corresponding subsequence must converge to $R(\alpha, \beta)$. This completes the proof. □

4.5.2 Area-minimizers and Dir-minimizers

Of particular interest for us is the following consequence of F. Morgan's work [41, Theorem 2].

Lemma 4.5.5. *Let $\mathbf{S} \subset \mathbb{R}^{m+n}$ be the union of N distinct m-dimensional planes $\alpha_1, \ldots, \alpha_N$ with the property that, for every $i < j$, $\alpha_i \cap \alpha_j$ is the same $(m-2)$-dimensional plane V. If T is an m-dimensional integral area-minimizing current such that $\mathrm{spt}(T) = \mathbf{S}$, then the (two) Morgan angles of any pair α_i, α_j, $i \neq j$, coincide.*

This follows from [41], since Corollary 4.5.3 gives the equivalence of θ_1 and θ_2 with the angles in [41] once the planes are oriented accordingly.

We complement this with the following counterpart for Dir-minimizers:

Proposition 4.5.6. *Let $m, n \geq 2$. There are positive absolute constants c_1 and c_2 depending on n such that the following holds. Assume $u : \mathbb{R}^m \to \mathcal{A}_Q(\mathbb{R}^n)$ is a Dir-minimizing map such that $u(x) = \sum_i k_i [\![L_i(x)]\!]$ for distinct linear maps $L_i : \mathbb{R}^m \to \mathbb{R}^n$ and non-zero integers k_i with the property that $\max_i |L_i| \leq c_1$ and that, for every $i < j$, the kernel of $L_i - L_j$ is the same $(m-2)$-dimensional subspace V of $\mathbb{R}^m$ while all L_i vanish on V. Let π_i be the m-dimensional planes given by the graphs of the L_i's. Then, for every pair $i < j$ the Morgan angles $\theta_k = \theta_k(\pi_i, \pi_j)$, $k = 1, 2$, satisfy the inequality $\theta_2 \leq c_2 \theta_1$.*

We stress here that as the subspace V in Proposition 4.5.6 has dimension $m-2$, there are exactly two Morgan angles for each pair of planes π_i, π_j, $i \neq j$, which in the above are denoted $\theta_1(\pi_i, \pi_j)$, $\theta_2(\pi_i, \pi_j)$, which by definition obey $\theta_1(\pi_i, \pi_j) \leq \theta_2(\pi_i, \pi_j)$. In particular, Proposition 4.5.6 tells us that they are comparable to each other, while Lemma 4.5.5 tells us in the case the planes are area-minimizing, these two angles must in fact coincide. It is possible that this stronger result could hold for Dir-minimizing unions of planes also, however we will not need this here and so do not pursue this.

Proposition 4.5.6 will be derived from the following special case and some elementary linear algebra.

Lemma 4.5.7. *There is a geometric constant $\mu \geq 1$ with the following property. Let $u : \mathbb{R}^2 \to \mathcal{A}_2(\mathbb{R}^2)$ be a Dir-minimizing map such that $u(x) = [\![B(x)]\!] + [\![-B(x)]\!]$ for some linear*

$B : \mathbb{R}^2 \to \mathbb{R}^2$. *Then B is quasiconformal, in the sense that*

$$\max\{|B(x)| : |x| = 1\} \leq \mu \min\{|B(x)| : |x| = 1\}\,. \tag{4.59}$$

Proof. Without loss of generality we can assume $|B| = 1$, otherwise normalize B to ensure this. If the lemma were false, then there would be a sequence of Dir-minimizing maps $u_k(x) = [\![B_k(x)]\!] + [\![-B_k(x)]\!]$, for some sequence $B_k : \mathbb{R}^2 \to \mathbb{R}^2$ of linear maps with $|B_k| = 1$, for which there is a sequence e_k of unit vectors with $B(e_k) \to 0$. Extracting a subsequence, we can assume that $e_k \to e$ and that $B_k \to B$. B is thus a linear map with $|B| = 1$, and so in particular if we set $u(x) := [\![B(x)]\!] + [\![-B(x)]\!]$, then u does not vanish identically. However, the line $\{\lambda e : \lambda \in \mathbb{R}\}$ would be a line of singularities, while from [18, Proposition 3.20] we know that u is Dir-minimizing. In particular we would contradict Almgren's partial regularity theorem for Dir-minimizers, cf. [18, Theorem 0.11]. This contradiction proves the result. □

Given Lemma 4.5.7, the elementary linear algebra needed to prove Proposition 4.5.6 is the following:

Lemma 4.5.8. *There is a geometric constant $c = c(n) > 0$ with the following property. Consider two linear maps $L_1, L_2 : \mathbb{R}^2 \to \mathbb{R}^n$ with $|L_i| \leq c(n)$ and* $\operatorname{rank}(L_1 - L_2) = 2$. *If*

$$\sigma_1 = \min\{|B(e)| : |e| = 1\} \leq \sigma_2 = \max\{|B(e)| : |e| = 1\}$$

denote the two singular values of $B = \frac{1}{2}(L_1 - L_2)$, and π_1, π_2 are the two planes in $\mathbb{R}^{2+n}$ given by the graphs of L_1 and L_2 respectively, then

$$\frac{\sigma_2}{4} \leq \theta_2(\pi_1, \pi_2) \leq 4\sigma_2$$
$$\frac{\sigma_1}{4} \leq \theta_1(\pi_1, \pi_2) \leq 4\sigma_1$$

where $\theta_1(\pi_1,\pi_2) \le \theta_2(\pi_1,\pi_2)$ are the Morgan angles of the planes π_1, π_2.

Proof. Let $A := \frac{1}{2}(L_1 + L_2)$ and $B := \frac{1}{2}(L_1 - L_2)$. Fix $p \in \partial\mathbf{B}_1 \cap \pi_1$. Since in particular $p \in \pi_1$, we may find $x \in \mathbb{R}^2$ such that $p = (x, (A+B)(x))$. Note that $|x| \le 1$ as $|p| = 1$. Since $(x, (A-B)(x)) \in \pi_2$ we clearly have

$$\operatorname{dist}(p,\pi_2) \le |2B(x)| \le 2\max\{|B(e)| : |e| = 1\}\,. \tag{4.60}$$

Next, let $q \in \pi_2$ be the point of minimum distance in π_2 from p and let $\xi \in \mathbb{R}^2$ be so that $q = (\xi, L_2(\xi)) = (\xi, (A-B)(\xi))$. Then,

$$\operatorname{dist}^2(p,\pi_2) = |p-q|^2 = |x-\xi|^2 + |(A+B)(x) - (A-B)(\xi)|^2\,. \tag{4.61}$$

Set $\eta := x - \xi$ and rewrite the expression in (4.61) as

$$|\eta|^2 + |(A+B)(\eta+\xi) - (A-B)(\xi)|^2 = |\eta|^2 + |2B(\xi) + (A+B)(\eta)|^2.$$

Since η is the minimum point of the latter quadratic expression, we can differentiate it to find that η obeys

$$2\eta + 4(A+B)^T B(\xi) + 2(A+B)^T(A+B)(\eta) = 0.$$

In turn, we can solve this as

$$\eta = -2(\mathrm{Id} + (A+B)^T(A+B))^{-1}(A+B)^T(B(\xi)) = -2(\mathrm{Id} + L_1^T L_1)^{-1} L_1^T(B(\xi))\,.$$

Insert the latter in (4.61) to find

$$\operatorname{dist}^2(p,\pi_2) = 4|(\mathrm{Id} + L_1^T L_1)^{-1} L_1^T(B(\xi))|^2 + 4|\left(\mathrm{Id} - L_1(\mathrm{Id} + L_1^T L_1)^{-1} L_1^T\right)(B(\xi))|^2\,. \tag{4.62}$$

Observe now that the operator norm of $(\mathrm{Id} + L_1^T L_1)^{-1}$ is always at most 1: indeed, since $\mathrm{Id} + L_1^T L_1$ is self-adjoint and its smallest eigenvalue is at least 1, its inverse is self-adjoint and has positive eigenvalues no larger than 1. We can therefore estimate

$$\left|\left(\mathrm{Id} - L_1(\mathrm{Id} + L_1^T L_1)^{-1} L_1^T\right)(B(\xi))\right| \geq |B(\xi)| - |L_1|^2 |B(\xi)|\,.$$

In particular, if we choose the constant c sufficiently small, we conclude

$$\mathrm{dist}^2(p, \pi_2) \geq \frac{1}{2}|B(\xi)|^2\,. \tag{4.63}$$

But, again under the assumption that c is sufficiently small, and since $|q| \leq 1$ (as $\mathrm{dist}(p, 0) = 1$ and $0 \in \pi_2$), we have that $|\xi| \geq \frac{\sqrt{2}}{2}$. Hence we get

$$\mathrm{dist}(p, \pi_2) \geq \frac{1}{2}\min\{|B(e)| : |e| = 1\}\,. \tag{4.64}$$

Combining (4.60) and (4.64) and recalling that $\sigma_2 = \max\{|B(e)| : |e| = 1\}$ and $\sigma_1 = \min\{|B(e)| : |e| = 1\}$ are the two singular values of B, we conclude that

$$\frac{\sigma_1}{2} \leq \min\{\mathrm{dist}(p, \pi_2) : p \in \partial\mathbf{B}_1 \cap \pi_1\} \leq \max\{\mathrm{dist}(p, \pi_2) : p \in \partial\mathbf{B}_1 \cap \pi_1\} \leq 2\sigma_2\,. \tag{4.65}$$

Next let $e \in \mathbb{R}^2$ be a unit vector with $|B(e)| = \sigma_2$ and consider $v = (e, L_1(e))$ and $v' := \frac{v}{|v|} \in \pi_1 \cap \partial\mathbf{B}_1$. Decreasing c further if necessary, we have

$$\mathrm{dist}(v, \pi_2) \leq \frac{3\sqrt{2}}{4}\,\mathrm{dist}(v', \pi_2) \leq \frac{3\sqrt{2}}{4}\max\{\mathrm{dist}(p, \pi_2) : p \in \partial\mathbf{B}_1 \cap \pi_1\}\,. \tag{4.66}$$

Consider $q = \mathbf{p}_{\pi_2}(v)$ and let $\xi \in \mathbb{R}^2$ be such that $(\xi, L_2(\xi)) = q$. Note that

$$|\xi - e| \leq |v - q| \leq |\mathbf{p}_{\pi_2} - \mathbf{p}_{\pi_1}||v|\,.$$

In particular $|\xi - e| \leq C(|L_1| + |L_2|) \leq Cc$. On the other hand, recall that we have the inequality (4.63) but for v in place of p (since we did not use the fact that $p \in \partial\mathbf{B}_1$ to achieve this). In particular, provided c is sufficiently small we can write

$$\operatorname{dist}(v, \pi_2) \geq \frac{1}{\sqrt{2}}|B(e)| - \frac{1}{\sqrt{2}}|B(e-\xi)| \geq \frac{\sigma_2}{\sqrt{2}}(1 - |\xi - e|) \geq \frac{3\sigma_2}{4\sqrt{2}}\,. \tag{4.67}$$

We can now combine (4.66) and (4.67), yielding

$$\sigma_2 \leq 2\max\{\operatorname{dist}(p, \pi_2) : p \in \partial\mathbf{B}_1 \cap \pi_1\}\,. \tag{4.68}$$

In a similar fashion, fix now e such that $|B(e)| = \sigma_1$ and consider v and v' as above, for this choice of e. Arguing analogously to (4.60), it follows that

$$\min\{\operatorname{dist}(p, \pi_2) : p \in \pi_1 \cap \partial\mathbf{B}_1\} \leq \operatorname{dist}(v', \pi_2) \leq \operatorname{dist}(v, \pi_2) \leq 2|B(e)|\,, \tag{4.69}$$

because $(e, L_2(e)) \in \pi_2$. In particular we deduce that

$$\min\{\operatorname{dist}(p, \pi_2) : p \in \pi_1 \cap \partial\mathbf{B}_1\} \leq 2\sigma_1\,. \tag{4.70}$$

Summarizing, if we combine (4.65), (4.68), and (4.70), we arrive at

$$\frac{\sigma_1}{2} \leq \min\{\operatorname{dist}(p, \pi_2) : p \in \pi_1 \cap \partial\mathbf{B}_1\} \leq 2\sigma_1 \tag{4.71}$$

$$\frac{\sigma_2}{2} \leq \max\{\operatorname{dist}(p, \pi_2) : p \in \pi_1 \cap \partial\mathbf{B}_1\} \leq 2\sigma_2\,. \tag{4.72}$$

Recall now that, since we are dealing with two-dimensional planes, the variational definition of the eigenvalues of quadratic forms gives

$$\theta_1 = \arcsin(\min\{\operatorname{dist}(p, \pi_2) : p \in \partial\mathbf{B}_1 \cap \pi_1\})$$

$$\theta_2 = \arcsin(\max\{\mathrm{dist}(p,\pi_2) : p \in \partial \mathbf{B}_1 \cap \pi_1\})\,.$$

On the other hand $\max\{\mathrm{dist}(p,\pi_2) : p \in \partial \mathbf{B}_1 \cap \pi_1\}$ is controlled by $|L_1| + |L_2|$, hence if we choose the latter sufficiently small we get

$$\frac{1}{2}\min\{\mathrm{dist}(p,\pi_2) : p \in \partial \mathbf{B}_1 \cap \pi_1\} \leq \theta_1 \leq 2\min\{\mathrm{dist}(p,\pi_2) : p \in \partial \mathbf{B}_1 \cap \pi_1\}$$
$$\frac{1}{2}\max\{\mathrm{dist}(p,\pi_2) : p \in \partial \mathbf{B}_1 \cap \pi_1\} \leq \theta_2 \leq 2\max\{\mathrm{dist}(p,\pi_2) : p \in \partial \mathbf{B}_1 \cap \pi_1\}$$

Combining the latter inequalities with (4.71) and (4.72) we conclude the proof. □

We record the following corollary; it is not needed for now, but will be convenient for us to use later on.

Corollary 4.5.9. *There is a geometric constant $c = c(n) > 0$ with the following property. Consider two linear maps $L_1, L_2 : \mathbb{R}^m \to \mathbb{R}^n$ with the property that $\ker(L_1 - L_2)$ is $(m-2)$-dimensional, let $\lambda > 0$, and denote by α_i and β_i the m-dimensional planes given by the graphs of L_i and λL_i respectively, for $i = 1, 2$. If $\max\{\lambda, 1\}|L_i| \leq c(n)$, then*

$$\frac{\theta_1(\alpha_1,\alpha_2)}{\theta_2(\alpha_1,\alpha_2)} \leq 16^2 \frac{\theta_1(\beta_1,\beta_2)}{\theta_2(\beta_1,\beta_2)}\,.$$

Proof. For $i = 1, 2$, letting σ_i and $\sigma_i^{(\lambda)}$ denote the singular values of $\frac{1}{2}(L_1 - L_2)$ and $\frac{1}{2}(\lambda L_1 - \lambda L_2)$ respectively, we have $\sigma_i^{(\lambda)} = \lambda\sigma_i$, and so the inequality is an immediate corollary of Lemma 4.5.8. □

We can now prove Proposition 4.5.6.

Proof of Proposition 4.5.6. First of all observe that, for every choice of i and j, the map $v(x) = [\![L_i(x)]\!] + [\![L_j(x)]\!]$ is Dir-minimizing. It therefore suffices to prove the proposition when $Q = 2$ and we have two distinct linear maps L_1 and L_2.

Next consider the $(m-2)$-dimensional subspace W which is the kernel of $L_1 - L_2$, and let $A = \frac{1}{2}(L_1 + L_2)$ and $B = \frac{1}{2}(L_1 - L_2)$. Observe that since A is linear, the function

$$\tilde{v}(x) := v(x) \ominus A(x) \equiv [\![L_1(x) - A(x)]\!] + [\![L_2(x) - A(x)]\!] = [\![B(x)]\!] + [\![-B(x)]\!]$$

is also Dir-minimizing (see [18, Lemma 3.23]). Now, for any point $z \in \mathbb{R}^m$, let us write $z = (x, y) \in W^\perp \times W$. Since the image of B on $W^\perp$ lies in a 2-dimensional subspace, and the kernel of B is $(m-2)$-dimensional, we may quotient out the kernel of B and consider it as a function $W^\perp \to \mathbb{R}^2$. As the domain of B is then a 2-dimensional subspace, we can then apply Lemma 4.5.7 to conclude that (4.59) holds for B, and thus we can control the ratio of the two singular values of B by a geometric constant, μ. But then observe that the two Morgan angles of the planes π_1 and π_2 coincide with the Morgan angles of the 2-dimensional planes of $\mathbb{R}^4$ constructed above. We can therefore apply Lemma 4.5.8 to conclude the proof of Proposition 4.5.6. □

4.5.3 Separated regions, alignment, and shifting

Here we collect three lemmas of a different flavor, which all have to do with the geometry of a collection of planes which all intersect in a common $(m-2)$-dimensional subspace; they will be used later on at different stages, when proving Theorem 4.2.4.

The following lemma shows that, even in the absence of a comparison estimate between the Morgan angles for a given finite collection of planes, it is possible to find a sizeable region of one of the planes α where the minimum distance of a point in that region to the other planes is comparable to the minimum of the Hausdorff distances of these planes to α.

Lemma 4.5.10 (Separated region). *There is a constant $0 < c = c(m, N) < \frac{1}{2}$ with the following property. Suppose that $\alpha, \beta_1, \ldots, \beta_N$ are distinct m-dimensional subspaces of $\mathbb{R}^{m+n}$.*

Then there is a point $\xi \in \alpha \cap \partial \mathbf{B}_{1/2}$ with the property that

$$\min_i \inf\{\operatorname{dist}(\zeta, \beta_i) : \zeta \in \mathbf{B}_c(\xi) \cap \alpha\} \geq c \min_i \operatorname{dist}(\alpha \cap \mathbf{B}_1, \beta_i \cap \mathbf{B}_1)\,. \tag{4.73}$$

Proof. Following the notation of Lemma 4.5.1 and Definition 4.5.2, let us denote by $Q_i : \alpha \to \mathbb{R}$ the quadratic forms $Q_i(v) := \operatorname{dist}^2(v, \beta_i)$ and let e_i be an eigenvector corresponding to the its maximal eigenvalue λ^i and hence also to the largest Morgan angle θ^i of the pair (α, β_i). Complete e_i to an orthonormal basis which diagonalizes Q_i. For any $\zeta \in \alpha$ we thus have

$$\operatorname{dist}^2(\zeta, \beta_i) \geq (e_i \cdot \zeta)^2 \lambda^i = (e_i \cdot \zeta)^2 \sin^2(\theta^i)$$

Recalling that, by Corollary 4.5.3, $\sin(\theta^i) = \operatorname{dist}(\alpha \cap \mathbf{B}_1, \beta_i \cap \mathbf{B}_1)$, we conclude that

$$\operatorname{dist}(\zeta, \beta_i) \geq |e_i \cdot \zeta| \operatorname{dist}(\alpha \cap \mathbf{B}_1, \beta_i \cap \mathbf{B}_1)\,.$$

Therefore, we just need to find a vector $\xi \in \partial \mathbf{B}_{1/2} \cap \alpha$ with the property that

$$|\zeta \cdot e_i| \geq c \qquad \forall i, \forall \zeta \in \mathbf{B}_c(\xi) \cap \alpha$$

for $c = c(m, N)$ to be determined. Given the elementary estimate

$$|\zeta \cdot e_i| \geq |\xi \cdot e_i| - c\,,$$

for any such ζ and ξ, it therefore suffices to find $\xi \in \partial \mathbf{B}_{1/2} \cap \alpha$ such that

$$|\xi \cdot e_i| \geq 2c \qquad \forall i\,.$$

Now for each $i = 1, \ldots, N$, let $S_i := \{v \in \partial \mathbf{B}_{1/2} \cap \alpha : |v \cdot e_i| \leq 2c\}$ and observe that $\mathcal{H}^{m-1}(S_i) \leq C(m)c$ for some constant $C(m)$. In particular, $\mathcal{H}^m(\bigcup_i S_i) \leq NC(m)c$, and thus

if $c = c(m, N)$ is chosen sufficiently small, $\partial \mathbf{B}_{1/2} \setminus \bigcup_i S_i$ must have positive measure, and hence it contains at least one point ξ which obeys the desired conditions. □

As an immediate corollary of Lemma 4.5.10 above we have the following.

Corollary 4.5.11. *Let $\alpha, \beta_1, \ldots, \beta_N$ be a collection of m-dimensional distinct planes of $\mathbb{R}^{m+n}$ and set $\mathbf{S} = \bigcup_i \beta_i$. Then there is a constant $\bar{C} = \bar{C}(N, m) > 0$ with*

$$\min_i \operatorname{dist}(\alpha \cap \mathbf{B}_1, \beta_i \cap \mathbf{B}_1) \leq \bar{C} \operatorname{dist}(\alpha \cap \mathbf{B}_1, \mathbf{S} \cap \mathbf{B}_1) .$$

The following lemma concerns itself with the alignment of spines of pairs of cones in $\mathscr{C}(Q)$, cf. Definition 4.2.2, and will be of fundamental importance.

Lemma 4.5.12. *For every $M > 0$ and natural numbers m, n, and Q, there is a constant $\bar{C} = \bar{C}(M, m, n, Q) > 0$ with the following property. Assume that*

(i) $\mathbf{S}$ and $\mathbf{S}'$ consist of $2 \leq N, N' \leq Q$ m-dimensional distinct planes $\alpha_1, \ldots, \alpha_N$ and $\beta_1, \ldots, \beta_{N'}$, respectively;

(ii) The intersection of every pair $\alpha_i \neq \alpha_j$ is a single $(m-2)$-dimensional subspace $V(\mathbf{S})$ and the intersection of every pair $\beta_i \neq \beta_j$ is a single $(m-2)$-dimensional subspace $V(\mathbf{S}')$;

(iii) For every pair $\alpha_i \neq \alpha_j$ the two Morgan angles $\theta_1(\alpha_i, \alpha_j) \leq \theta_2(\alpha_i, \alpha_j)$ satisfy $\theta_2 \leq M\theta_1$.

Then

$$\operatorname{dist}(V(\mathbf{S}) \cap \mathbf{B}_1, V(\mathbf{S}') \cap \mathbf{B}_1) \leq \bar{C} \frac{\operatorname{dist}(\mathbf{S} \cap \mathbf{B}_1, \mathbf{S}' \cap \mathbf{B}_1)}{\min_i \operatorname{dist}(\mathbf{S} \cap \mathbf{B}_1, \alpha_i \cap \mathbf{B}_1)} . \tag{4.74}$$

Proof. We may assume without loss of generality that the planes α_i are ordered such that

$$\operatorname{dist}(\alpha_1 \cap \mathbf{B}_1, \alpha_2 \cap \mathbf{B}_1) = \max_{i \neq j} \operatorname{dist}(\alpha_i \cap \mathbf{B}_1, \alpha_j \cap \mathbf{B}_1) ,$$

and observe that

$$\min_i \operatorname{dist}(\mathbf{S} \cap \mathbf{B}_1, \alpha_i \cap \mathbf{B}_1) \leq \operatorname{dist}(\alpha_1 \cap \mathbf{B}_1, \alpha_2 \cap \mathbf{B}_1)\,. \tag{4.75}$$

On the other hand, because of Corollary 4.5.11 and since $\alpha_i \subset \mathbf{S}$ for all i, we can select β_i and β_j such that

$$\operatorname{dist}(\alpha_1 \cap \mathbf{B}_1, \beta_i \cap \mathbf{B}_1) \leq \bar{C} \operatorname{dist}(\mathbf{S} \cap \mathbf{B}_1, \mathbf{S}' \cap \mathbf{B}_1) \tag{4.76}$$

$$\operatorname{dist}(\alpha_2 \cap \mathbf{B}_1, \beta_j \cap \mathbf{B}_1) \leq \bar{C} \operatorname{dist}(\mathbf{S} \cap \mathbf{B}_1, \mathbf{S}' \cap \mathbf{B}_1)\,. \tag{4.77}$$

To simplify our notation we use V and V' in place of $V(\mathbf{S})$ and $V(\mathbf{S}')$. Because of the condition (iii) on the Morgan angles $\theta_k(\alpha_1, \alpha_2)$, using (4.53) and (4.54) we have

$$\operatorname{dist}(w, \alpha_2) \geq \bar{C}^{-1} \operatorname{dist}(\alpha_1 \cap \mathbf{B}_1, \alpha_2 \cap \mathbf{B}_1) |w - \mathbf{p}_V(w)| \qquad \forall w \in \alpha_1\,. \tag{4.78}$$

Fix now $v' \in V' \cap \mathbf{B}_1$ and observe that, since v' belongs to both β_i and β_j, due to (4.76) and (4.77) we must have

$$|\mathbf{p}_{\alpha_1}(v') - v'| = \operatorname{dist}(v', \alpha_1) \leq \bar{C} \operatorname{dist}(\mathbf{S} \cap \mathbf{B}_1, \mathbf{S}' \cap \mathbf{B}_1)$$

$$|\mathbf{p}_{\alpha_2}(v') - v'| = \operatorname{dist}(v', \alpha_2) \leq \bar{C} \operatorname{dist}(\mathbf{S} \cap \mathbf{B}_1, \mathbf{S}' \cap \mathbf{B}_1)\,.$$

In particular, from the triangle inequality and the fact $|\mathbf{p}_{\alpha_2}(v')| \leq 1$ one can then deduce

$$\operatorname{dist}(\mathbf{p}_{\alpha_1}(v'), \alpha_2) \leq \bar{C} \operatorname{dist}(\mathbf{S} \cap \mathbf{B}_1, \mathbf{S}' \cap \mathbf{B}_1)\,. \tag{4.79}$$

Now, let $w = \mathbf{p}_{\alpha_1}(v')$ and observe that

$$\operatorname{dist}(v', V) \leq |v' - w| + |w - \mathbf{p}_V(w)| \leq \bar{C} \operatorname{dist}(\mathbf{S} \cap \mathbf{B}_1, \mathbf{S}' \cap \mathbf{B}_1) + |w - \mathbf{p}_V(w)|\,, \tag{4.80}$$

while, by (4.78) and (4.79),

$$|w - \mathbf{p}_V(w)| \le \bar{C} \frac{\operatorname{dist}(w, \alpha_2)}{\operatorname{dist}(\alpha_1 \cap \mathbf{B}_1, \alpha_2 \cap \mathbf{B}_1)} \le \bar{C} \frac{\operatorname{dist}(\mathbf{S} \cap \mathbf{B}_1, \mathbf{S}' \cap \mathbf{B}_1)}{\operatorname{dist}(\alpha_1 \cap \mathbf{B}_1, \alpha_2 \cap \mathbf{B}_1)} \,. \tag{4.81}$$

Thus, given (4.75) and $\operatorname{dist}(\alpha_1 \cap \mathbf{B}_1, \alpha_2 \cap \mathbf{B}_1) \le 1$, (4.80) and (4.81) lead to

$$\operatorname{dist}(v', V) \le \bar{C} \frac{\operatorname{dist}(\mathbf{S} \cap \mathbf{B}_1, \mathbf{S}' \cap \mathbf{B}_1)}{\min_i \operatorname{dist}(\mathbf{S} \cap \mathbf{B}_1, \alpha_i \cap \mathbf{B}_1)} \,.$$

Observe however that v' is an arbitrary element of $V' \cap \mathbf{B}_1$, and we have thus reached

$$\sup\{\operatorname{dist}(v', V) : v' \in V' \cap \mathbf{B}_1)\} \le \bar{C} \frac{\operatorname{dist}(\mathbf{S} \cap \mathbf{B}_1, \mathbf{S}' \cap \mathbf{B}_1)}{\min_i \operatorname{dist}(\mathbf{S} \cap \mathbf{B}_1, \alpha_i \cap \mathbf{B}_1)} \,.$$

On the other hand, since V and V' have the same dimension, by Corollary 4.5.3 we infer that

$$\sup\{\operatorname{dist}(v', V) : v' \in V' \cap \mathbf{B}_1)\} = \operatorname{dist}(V \cap \mathbf{B}_1, V' \cap \mathbf{B}_1) \,,$$

hence concluding the desired statement. □

Finally, the following lemma gives a lower bound for the distance of z to $q + \mathbf{S}$ for many points $z \in \mathbf{S}$. Before stating it we introduce some useful terminology.

Definition 4.5.13. *A set $\Omega \subset \mathbb{R}^{m+n}$ is said to be invariant under rotation around a linear subspace V if $R(\Omega) = \Omega$ for any rotation R of $\mathbb{R}^{m+n}$ which fixes V.*

Lemma 4.5.14 (Shifting lemma)**.** *For every $M \ge 1$ and every open set $U \subset \mathbf{B}_1$ that is invariant under rotations around V, there is a constant $\bar{C} = \bar{C}(M, m, n, N, U) > 0$ with the following properties. Assume that $\mathbf{S} = \alpha_1 \cup \cdots \cup \alpha_N$ satisfies (i), (ii), and (iii) in Lemma 4.5.12, let $q \in \mathbf{B}_{1/2}$, and let $\boldsymbol{\mu}(\mathbf{S}) = \max_{i<j} \operatorname{dist}(\alpha_i \cap \mathbf{B}_1, \alpha_j \cap \mathbf{B}_1)$. Then there is an index $j \in \{1, \dots, N\}$ and a subset $\Omega \subset \alpha_j \cap U$ such that $\mathcal{H}^m(\Omega) \ge \bar{C}^{-1}$ and*

$$|\mathbf{p}^{\perp}_{\alpha_1}(q)| + \boldsymbol{\mu}(\mathbf{S}) |\mathbf{p}_{V^{\perp} \cap \alpha_1}(q)| \le \bar{C} \operatorname{dist}(z, q + \mathbf{S}) \qquad \forall z \in \Omega \,. \tag{4.82}$$

Remark 4.5.15. Observe that Lemma 4.5.14 can be scaled. Under the assumption that $\mathbf{B}_r$ replaces $\mathbf{B}_1$, $U_{0,r} \equiv \iota_{0,r}^{-1}(U)$ replaces U, and $q \in \mathbf{B}_{r/2}$, we can conclude the existence of a subset $\Omega \subset U_{0,r}$ with measure larger than $\bar{C}^{-1} r^m$ with the property that (4.82) holds. Under these assumptions the constant $\bar{C}$ can be taken to be the same as the one in Lemma 4.5.14.

In order to prove Lemma 4.5.14, we will need the following elementary result.

Lemma 4.5.16. *There is a dimensional constant $C_0 = C_0(m,n) > 0$ with the following property. Let $\mathbf{S}$, M, and q be as in Lemma 4.5.14. Then*

$$|\mathbf{p}^{\perp}_{\alpha_1}(q)| + \boldsymbol{\mu}(\mathbf{S})|\mathbf{p}_{V^{\perp}\cap\alpha_1}(q)| \leq C_0 M \max_i |\mathbf{p}^{\perp}_{\alpha_i}(q)|\,. \tag{4.83}$$

Proof. Let k be such that

$$|\mathbf{p}^{\perp}_{\alpha_k}(q)| = \max_i |\mathbf{p}^{\perp}_{\alpha_i}(q)|\,.$$

Thus, in particular, $|\mathbf{p}^{\perp}_{\alpha_1}(q)| \leq |\mathbf{p}^{\perp}_{\alpha_k}(q)|$, and so it remains to show that

$$\boldsymbol{\mu}(\mathbf{S})|\mathbf{p}_{V^{\perp}\cap\alpha_1}(q)| \leq C_0 M \max_i |\mathbf{p}^{\perp}_{\alpha_i}(q)|\,. \tag{4.84}$$

First of all pick j which maximizes $\operatorname{dist}(\alpha_j \cap \mathbf{B}_1, \alpha_k \cap \mathbf{B}_1)$, so that in particular

$$\operatorname{dist}(\alpha_j \cap \mathbf{B}_1, \alpha_l \cap \mathbf{B}_1) \leq \operatorname{dist}(\alpha_j \cap \mathbf{B}_1, \alpha_k \cap \mathbf{B}_1) \qquad \forall l\,.$$

Thus, the triangle inequality yields

$$\operatorname{dist}(\alpha_j \cap \mathbf{B}_1, \alpha_k \cap \mathbf{B}_1) \geq \frac{1}{2}\boldsymbol{\mu}(\mathbf{S})\,.$$

Next, recalling the definition of Morgan angles, observe that for every $w \in \alpha_j \cap V^{\perp}$ we have (using the above line, assumption (iii) in Lemma 4.5.12 and (4.54))

$$\boldsymbol{\mu}(\mathbf{S})|w| \leq C_0 M |\mathbf{p}^{\perp}_{\alpha_k}(w)|\,.$$

Now choose $w = \mathbf{p}_{V^\perp \cap \alpha_j}(q) = \mathbf{p}^\perp_V(q) - \mathbf{p}^\perp_{\alpha_j}(q)$. Since $\mathbf{p}^\perp_{\alpha_k} \circ \mathbf{p}^\perp_V = \mathbf{p}^\perp_{\alpha_k}$ (because $V \subset \alpha_k$) we may estimate

$$\begin{aligned}\boldsymbol{\mu}(\mathbf{S})|\mathbf{p}_{V^\perp \cap \alpha_j}(q)| &\le C_0 M|\mathbf{p}^\perp_{\alpha_k}(\mathbf{p}^\perp_V(q))| + C_0 M|\mathbf{p}^\perp_{\alpha_k}(\mathbf{p}^\perp_{\alpha_j}(q))| \\ &\le C_0 M|\mathbf{p}^\perp_{\alpha_k}(q)| + C_0 M|\mathbf{p}^\perp_{\alpha_j}(q)|\,.\end{aligned}$$

Using the maximality of k, we then reach

$$\boldsymbol{\mu}(\mathbf{S})|\mathbf{p}_{V^\perp \cap \alpha_j}(q)| \le C_0 M|\mathbf{p}^\perp_{\alpha_k}(q)|\,. \tag{4.85}$$

It remains to replace α_j with α_1 in the projection on the left-hand side of the above inequality. Since $\mathbf{p}_{V^\perp \cap \alpha_1} \circ \mathbf{p}_{\alpha_j} = \mathbf{p}_{V^\perp \cap \alpha_1} \circ \mathbf{p}_{V^\perp \cap \alpha_j}$, we have

$$\begin{aligned}|\mathbf{p}_{V^\perp \cap \alpha_1}(q)| &\le |\mathbf{p}_{V^\perp \cap \alpha_1}(\mathbf{p}_{V^\perp \cap \alpha_j}(q))| + |\mathbf{p}_{V^\perp \cap \alpha_1}(\mathbf{p}^\perp_{\alpha_j}(q))| \\ &\le |\mathbf{p}_{V^\perp \cap \alpha_j}(q)| + |\mathbf{p}^\perp_{\alpha_j}(q)| \\ &\le |\mathbf{p}_{V^\perp \cap \alpha_j}(q)| + |\mathbf{p}^\perp_{\alpha_k}(q)|\,.\end{aligned}$$

Combining the latter inequality with (4.85) and using that $\boldsymbol{\mu}(\mathbf{S}) \le 1$, we reach (4.84) and hence complete the proof of the lemma. □

We can now prove Lemma 4.5.14.

Proof of Lemma 4.5.14. We choose j such that $|\mathbf{p}^\perp_{\alpha_j}(q)| = \max_i |\mathbf{p}^\perp_{\alpha_i}(q)|$ and using Lemma 4.5.16 we aim at proving that

$$|\mathbf{p}^\perp_{\alpha_j}(q)| \le \bar{C}_1 \operatorname{dist}(z, q + \mathbf{S}) \qquad \forall z \in \Omega\,, \tag{4.86}$$

for some set $\Omega \subset U$ with $\mathcal{H}^m(\Omega) \ge \bar{C}_1^{-1}$, where the constant $\bar{C}_1$ is allowed to depend on U.

In fact it turns out that the constant $\bar{C}_1$ depends on

$$\gamma := \inf\{\mathcal{H}^m(U \cap \alpha) : V \subset \alpha \text{ and } \alpha \text{ is an } m\text{-dimensional subspace}\}\,. \tag{4.87}$$

It is not entirely obvious that γ is necessarily positive. Note, however, that if a point p belongs to U and another point q satisfies $\mathbf{p}_V(q) = \mathbf{p}_V(p)$ and $\operatorname{dist}(q, V) = \operatorname{dist}(p, V)$, then necessarily $q \in U$, in light of the rotational invariance of U. In particular there is an open subset $U' \subset V \times \mathbb{R}^+$ such that $U = \{p : (\mathbf{p}_V(p), \operatorname{dist}(p, V)) \in U'\}$. From this we conclude that, not only is γ positive, but in fact that $\mathcal{H}^m(U \cap \alpha)$ is exactly the same number for every m-dimensional plane which contains V.

Observe first that (4.86) is equivalent to

$$\min_i |\mathbf{p}^\perp_{\alpha_i}(z - q)| \geq \bar{C}_1^{-1} |\mathbf{p}^\perp_{\alpha_j}(q)| \qquad \forall z \in \Omega\,. \tag{4.88}$$

Now assume that the claim is false, namely that (4.88) fails for all $z \in U \cap \alpha_j$ with the exception of a set E of Hausdorff measure smaller than $\bar{C}_1^{-1}$. In particular by choosing $\bar{C}_1$ large enough, we can ensure that, for some $i \in \{1, \ldots, N\}$ the set $F_i \subset \alpha_j \cap U$ where

$$|\mathbf{p}^\perp_{\alpha_i}(z - q)| \ \leq \bar{C}_1^{-1} |\mathbf{p}^\perp_{\alpha_j}(q)| \qquad \forall z \in F_i \tag{4.89}$$

has measure at least $\frac{\gamma}{N+1}$. We then claim that

$$|\mathbf{p}^\perp_{\alpha_i}(q)| \leq C\gamma^{-4}\bar{C}_1^{-1} |\mathbf{p}^\perp_{\alpha_j}(q)|\,, \tag{4.90}$$

$$|\mathbf{p}^\perp_{\alpha_i}(z)| \leq C\gamma^{-4}\bar{C}_1^{-1} |\mathbf{p}^\perp_{\alpha_j}(q)| \qquad \forall z \in \mathbf{B}_1 \cap \alpha_j\,, \tag{4.91}$$

where the constant C depends only on m and N. Note that i is the index chosen such that $\mathcal{H}^m(F_i) \geq \frac{\gamma}{N+1}$, but the second estimate is claimed for every $z \in \mathbf{B}_1 \cap \alpha_j$, and the latter will in the end be used to get a contradiction.

In order to prove (4.90) and (4.91) consider first the set $F_i' := \mathbf{p}_{V^\perp}(F_i)$. The latter belongs to the 2-dimensional subspace $V^\perp \cap \alpha_j$ and the coarea formula implies immediately that

$$\mathcal{H}^2(F_i') \geq C^{-1}\mathcal{H}^m(F_i) \geq C^{-1}\gamma \tag{4.92}$$

for a positive dimensional $C = C(m, N)$. Choose next a vector $e_1 \in F_i'$ such that $|e_1| \geq \frac{1}{2}\sup\{|x| : x \in F_i'\}$ and observe that $|e_1| \geq C\sqrt{\gamma}$ because $4\pi|e_1|^2 \geq \mathcal{H}^2(F_i')$. Hence choose $e_2 \in F_i'$ such that

$$|e_2 - |e_1|^{-2}(e_2 \cdot e_1)e_1| \geq \frac{1}{2}\sup\{|x - |e_1|^{-2}(x \cdot e_1)e_1| : x \in F_i'\}$$

and observe that $\mathcal{H}^2(F_i') \leq 4|e_2 - |e_1|^{-2}(e_1 \cdot e_2)e_1|$, so that

$$||e_1|e_2 - |e_1|^{-1}(e_1 \cdot e_2)e_1| \geq C^{-1}\gamma^{3/2}\,.$$

We next define the linear map $\Phi : \mathbb{R}^2 \to V^\perp \cap \alpha_j$ by $(\lambda_1, \lambda_2) \mapsto \lambda_1 e_1 + \lambda_2 e_2$ and observe, by elementary geometry that $|\det \Phi| = ||e_1|e_2 - |e_1|^{-1}(e_1 \cdot e_2)e_1|$. In particular, $|\det \Phi^{-1}| \leq C\gamma^{-3/2}$ and, given that $|\Phi| \leq C$, we also have $|\Phi^{-1}| \leq C\gamma^{-3/2}$. Therefore $F_i'' := \Phi^{-1}(F_i')$ is contained in a disk of radius at most $C\gamma^{-3/2}$. Consider now the number

$$\mu := \sup\{|\lambda_1 + \lambda_2 - 1| : (\lambda_1, \lambda_2) \in F_i''\}$$

and notice that $\mathcal{H}^2(F_i'') \leq C\gamma^{-3/2}\mu$. Since $|\det \Phi| \leq 1$ we get $\mathcal{H}^2(F_i') \leq \mathcal{H}^2(F_i'')$, and thus, when combined with (4.92), we infer $\mu \geq C^{-1}\gamma^{5/2}$, and thus the existence of $(\lambda_1, \lambda_2) \in F_i''$ such that

$$|\lambda_1 + \lambda_2 - 1| \geq C^{-1}\gamma^{5/2}\,. \tag{4.93}$$

Observe now that, by the very definition of F_i', there are $v_1, v_2, v_3 \in V$ such that

$$e_1 + v_1 \in F_i$$
$$e_2 + v_2 \in F_i$$
$$\lambda_1 e_1 + \lambda_2 e_2 + v_3 \in F_i \, .$$

Since $V \subset \alpha_i$ we can write

$$\begin{aligned}(\lambda_1 + \lambda_2 - 1)\mathbf{p}^\perp_{\alpha_i}(q) &= \lambda_1 \mathbf{p}^\perp_{\alpha_i}(q - e_1) + \lambda_2 \mathbf{p}^\perp_{\alpha_i}(q - e_2) - \mathbf{p}^\perp_{\alpha_i}(q - (\lambda_1 e_1 + \lambda_2 e_2)) \\ =&\lambda_1 \mathbf{p}^\perp_{\alpha_i}(q - e_1 - v_1) + \lambda_2 \mathbf{p}^\perp_{\alpha_i}(q - e_2 - v_2) - \mathbf{p}^\perp_{\alpha_i}(q - (\lambda_1 e_1 + \lambda_2 e_2 + v_3)) \, .\end{aligned}$$

In particular we conclude

$$|\mathbf{p}^\perp_{\alpha_i}(q)| \leq C\bar{C}_1^{-1}\gamma^{-5/2}|\mathbf{p}^\perp_{\alpha_j}(q)| \, , \tag{4.94}$$

which is in fact stronger than (4.90). On the other hand we can also write

$$|\mathbf{p}^\perp_{\alpha_i}(e_k)| \leq |\mathbf{p}^\perp_{\alpha_i}(e_k - q)| + |\mathbf{p}^\perp_{\alpha_i}(q)| \qquad k = 1, 2$$

and hence we immediately conclude

$$|\mathbf{p}^\perp_{\alpha_i}(e_k)| \leq C\bar{C}_1^{-1}\gamma^{-5/2}|\mathbf{p}^\perp_{\alpha_j}(q)| \, . \tag{4.95}$$

In turn, using again the map Φ we can express any $z \in \mathbf{B}_1 \cap \alpha_j$ as $z = \lambda_1 e_1 + \lambda_2 e_2 + v$ for some coefficients $|\lambda_i| \leq C\gamma^{-3/2}$ and some vector $v \in V$. In particular we achieve (4.91) from (4.95).

Observe however that (4.91) can be equivalently written as

$$\mathrm{dist}(z, \alpha_i) \leq C\bar{C}_1^{-1}\gamma^{-4}|\mathbf{p}^\perp_{\alpha_j}(q)| \qquad \forall z \in \mathbf{B}_1 \cap \alpha_j \, , \tag{4.96}$$

and thus also as

$$\operatorname{dist}(\alpha_i \cap \mathbf{B}_1, \alpha_j \cap \mathbf{B}_1) \leq C\bar{C}_1^{-1}\gamma^{-4}|\mathbf{p}_{\alpha_j}^{\perp}(q)| \tag{4.97}$$

Since γ is fixed, $|q| \leq \frac{1}{2}$, and α_i and α_j have the same dimension, the latter estimate implies that, by choosing $\bar{C}_1$ appropriately large, we can assume that the linear subspaces α_i and α_j are sufficiently close. In particular, given that $|q| \leq \frac{1}{2}$, for an appropriate large choice of $\bar{C}_1$ the affine subspace $q + \alpha_i^{\perp}$ must intersect $\mathbf{B}_1 \cap \alpha_j$ at some point z. But then at that point z we would have

$$\mathbf{p}_{\alpha_i}^{\perp}(q - z) = q - z\,. \tag{4.98}$$

Since $z \in \alpha_j$ we must have $|q - z| \geq \operatorname{dist}(q, \alpha_j) = |\mathbf{p}_{\alpha_j}^{\perp}(q)|$. Hence

$$|\mathbf{p}_{\alpha_i}^{\perp}(q - z)| \geq |\mathbf{p}_{\alpha_j}^{\perp}(q)|\,.$$

On the other hand, combining (4.90) and (4.91) we get

$$|\mathbf{p}_{\alpha_i}^{\perp}(q - z)| \leq 2C\bar{C}_1^{-1}\gamma^{-4}|\mathbf{p}_{\alpha_j}^{\perp}(q)|\,.$$

Since C is a constant which depends only on m and N we can now choose $\bar{C}_1$ large enough, depending only on γ, m, and N so that $C\bar{C}_1^{-1}\gamma^{-4} < \frac{1}{2}$. But then the last two inequalities would be in contradiction, unless $\mathbf{p}_{\alpha_j}^{\perp}(q) = \mathbf{p}_{\alpha_i}^{\perp}(q - z) = 0$. In particular we conclude that q belongs to α_j. In this case, however, (4.86) holds trivially. □

4.6 Graphical approximations

In this section we start facing two of the geometric issues that complicate the proof of Theorem 4.2.4. In an ideal situation the cone $\mathbf{S}$ in the statement of the theorem consists of N m-dimensional planes $\alpha_1, \ldots, \alpha_N$ with the additional properties that, for any pair $\alpha_i \neq \alpha_j$,

(i) The Hausdorff distance between α_i and α_j is relatively larger than $\mathbb{E}(T, \mathbf{S}, \mathbf{B}_1)$;

(ii) The two Morgan angles $\theta_1(\alpha_i, \alpha_j)$, $\theta_2(\alpha_i, \alpha_j)$ formed by them are comparable.

Under these two additional assumptions we can hope to use the bounds of Theorem 4.3.2 and Corollary 4.3.3 to give a good approximation of T by graphs over the collection of planes α_i, after removing a small neighborhood of the spine $V(\mathbf{S})$.

Although there is no reason to assume (i) and (ii) for $\mathbf{S}$ a priori, we can hope to achieve them for a different cone $\mathbf{S}'$ without increasing the excess $\mathbb{E}(T, \mathbf{S}', \mathbf{B}_1)$ too much relative to the excess $\mathbb{E}(T, \mathbf{S}, \mathbf{B}_1)$. In light of this we first specify an algorithm that allows us to gain control on how much the excess increases when we discard planes of $\mathbf{S}$ until we achieve (i). This algorithm is summarized in the Pruning Lemma 4.6.2. For later use we want to iteratively apply Lemma 4.6.2 and keep track of the structure of the planes which have been discarded during this process; this is accomplished in Lemma 4.6.3. As for (ii), we will work under the assumption that it holds for now. Later, in Section 4.7.1, we will demonstrate that the new cone $\mathbf{S}'$ achieving (i) indeed additionally satisfies (ii).

In the remainder of this Section, we show how to gain a graphical approximation under a suitable quantification of (i) and (ii). We begin with a "crude approximation" in Section 4.6.3, followed by a more intricate "refined approximation" in Section 4.6.5; the latter will be needed later.

4.6.1 Pruning Lemma and Layer Subdivision

The main purpose of this section is to introduce two very useful elementary combinatorial lemmas with the aim discussed above.

We will always work under the following assumption. We will often use the terminology "plane" when referring to a linear subspace of $\mathbb{R}^{m+n}$.

Assumption 4.6.1. $\mathbf{S} \subset \mathbb{R}^{m+n}$ is an m-dimensional cone such that

(i) $\mathbf{S}$ is a union of N distinct m-dimensional planes $\alpha_1, \ldots, \alpha_N$;

(ii) for each pair $i \neq j$ the intersection $\alpha_i \cap \alpha_j$ is the same $(m-2)$-dimensional plane V, which we refer to as the *spine* of $\mathbf{S}$.

Note in particular that the cones in the class of $\mathscr{C}(p, Q)$ of Definition 4.2.2 fall under the latter assumption.

Our first technical lemma we call the *Pruning Lemma*. It has two main uses. One is to prove the second technical lemma (Lemma 4.6.3); we will explain the meaning behind that lemma when we get to it. The other use will be to "prune" a cone, throwing away some of its planes, and ultimately get that the excess relative to the pruned cone is sufficiently small relative to the minimal angle of the pruned cone, which is a crucical assumption for our graphical approximation results later.

Lemma 4.6.2 (Pruning lemma)**.** *Let $N \geq 2$, $D > 0$, and $0 < \delta \leq 1$. Set $\Gamma := \delta^{2-N}(N-1)!$ and $\varepsilon := (1+\Gamma)^{-1}\delta$. If*

(i) $\mathbf{S} = \alpha_1 \cup \cdots \cup \alpha_N$ is as in Assumption 4.6.1;

(ii) $D \leq \varepsilon \max_{i<j} \operatorname{dist}(\alpha_i \cap \mathbf{B}_1, \alpha_j \cap \mathbf{B}_1)$;

then there is a subcollection $I \subset \{1, \ldots, N\}$ consisting of at least 2 elements and satisfying the following requirements:

$$\max_j \min_{i \in I} \operatorname{dist}(\alpha_i \cap \mathbf{B}_1, \alpha_j \cap \mathbf{B}_1) \leq \Gamma D \tag{4.99}$$

$$D + \max_j \min_{i \in I} \operatorname{dist}(\alpha_i \cap \mathbf{B}_1, \alpha_j \cap \mathbf{B}_1) \leq \delta \min_{j,\ell \in I:\, j<\ell} \operatorname{dist}(\alpha_j \cap \mathbf{B}_1, \alpha_\ell \cap \mathbf{B}_1)\,. \tag{4.100}$$

$$\max_{i,j \in I:\, i<j} \operatorname{dist}(\alpha_i \cap \mathbf{B}_1, \alpha_j \cap \mathbf{B}_1) = \max_{i<j} \operatorname{dist}(\alpha_i \cap \mathbf{B}_1, \alpha_j \cap \mathbf{B}_1)\,. \tag{4.101}$$

Proof. Set $I(0) = \{1, \ldots, N\}$. If either (a) $N = 2$ or (b) $N \geq 3$ and

$$D \leq \delta \min_{i<j} \operatorname{dist}(\alpha_i \cap \mathbf{B}_1, \alpha_j \cap \mathbf{B}_1)\,, \tag{4.102}$$

then we select $I = I(0)$ and the proof is complete, since the left hand side of (4.99) is zero (and hence the left hand side of (4.100) equals D), while (4.101) is obvious. Observe also that, since $\varepsilon < \delta$, the condition (4.102) is implied by (ii) when $N = 2$.

Otherwise, we select indices ℓ_1 and ℓ_2 such that

$$\min_{i<j} \operatorname{dist}(\alpha_i \cap \mathbf{B}_1, \alpha_j \cap \mathbf{B}_1) = \operatorname{dist}(\alpha_{\ell_1} \cap \mathbf{B}_1, \alpha_{\ell_2} \cap \mathbf{B}_1)$$

and indices j_1 and j_2 such that

$$\max_{i<j} \operatorname{dist}(\alpha_i \cap \mathbf{B}_1, \alpha_j \cap \mathbf{B}_1) = \operatorname{dist}(\alpha_{j_1} \cap \mathbf{B}_1, \alpha_{j_2} \cap \mathbf{B}_1)\,.$$

Since $N \geq 3$ we can choose them so that $\{j_1, j_2\} \neq \{\ell_1, \ell_2\}$. In particular, we can pick $\ell(0) \in \{\ell_1, \ell_2\} \setminus \{j_1, j_2\}$. We then set $I(1) := I(0) \setminus \{\ell(0)\}$. Notice that

$$\max_{i<j\in I(1)} \operatorname{dist}(\alpha_i \cap \mathbf{B}_1, \alpha_j \cap \mathbf{B}_1) = \max_{i<j\in I(0)} \operatorname{dist}(\alpha_i \cap \mathbf{B}_1, \alpha_j \cap \mathbf{B}_1)\,,$$

while, by the assumption that (4.102) fails,

$$\min_{j\in I(1)} \operatorname{dist}(\alpha_{\ell(0)} \cap \mathbf{B}_1, \alpha_j \cap \mathbf{B}_1) < \delta^{-1} D\,.$$

Assuming that we have inductively selected sets $I(0), I(1), \dots I(s)$, we use the same procedure above to select a new subset $I(s+1) \subset I(s)$ by removing one element $\ell(s)$, provided that the cardinality of $I(s)$ is at least 3 and

$$D + \max_j \min_{i\in I(s)} \operatorname{dist}(\alpha_i \cap \mathbf{B}_1, \alpha_j \cap \mathbf{B}_1) > \delta \min_{i<j\in I(s)} \operatorname{dist}(\alpha_i \cap \mathbf{B}_1, \alpha_j \cap \mathbf{B}_1)\,. \tag{4.103}$$

Otherwise, we stop; clearly this process must terminate in finitely many steps. We denote by σ the index of the stopping step; note that $\sigma \leq N - 2$. We claim that $I = I(\sigma)$ satisfies the requirements of the lemma.

First of all we prove the inequality

$$\min_{j\in I(s)} \operatorname{dist}(\alpha_{\ell(s')}\cap\mathbf{B}_1, \alpha_j\cap\mathbf{B}_1) \le (s-s') \min_{j\in I(s)} \operatorname{dist}(\alpha_{\ell(s-1)}\cap\mathbf{B}_1, \alpha_j\cap\mathbf{B}_1) \quad \forall s' < s \le \sigma\,. \tag{4.104}$$

Note that if $s = s' + 1$ the inequality is in fact an obvious equality (just from how it is written). In particular, the claim holds when $s' = \sigma - 1$. We now assume that the claim holds for all $s' > s_0$ and will proceed to show it when $s' = s_0$, by induction. Fix $s > s_0$ and let $j_* \in I(s_0 + 1)$ be such that

$$\min_{j\in I(s_0+1)} \operatorname{dist}(\alpha_{\ell(s_0)} \cap \mathbf{B}_1, \alpha_j \cap \mathbf{B}_1) = \operatorname{dist}(\alpha_{\ell(s_0)} \cap \mathbf{B}_1, \alpha_{j_*} \cap \mathbf{B}_1)\,.$$

We first observe that, by the very definition of $\ell(s_0)$ and j_* we have

$$\operatorname{dist}(\alpha_{\ell(s_0)} \cap \mathbf{B}_1, \alpha_{j_*} \cap \mathbf{B}_1) \le \min_{j\in I(s)} \operatorname{dist}(\alpha_{\ell(s-1)} \cap \mathbf{B}_1, \alpha_j \cap \mathbf{B}_1)\,. \tag{4.105}$$

In particular if $j_* \in I(s)$, the inequality (4.104) is obvious.

Otherwise $j_* \notin I(s)$ and so $j_* = \ell(s_*)$ for some $s_0 < s_* < s$. In this case, using (4.105) and the fact that, by the inductive assumption,(4.104) holds for $s = s_*$, we write

$$\begin{aligned}
&\min_{j\in I(s)} \operatorname{dist}(\alpha_{\ell(s_0)} \cap \mathbf{B}_1, \alpha_j \cap \mathbf{B}_1) \\
\le\, &\operatorname{dist}(\alpha_{\ell(s_0)} \cap \mathbf{B}_1, \alpha_{j_*} \cap \mathbf{B}_1) + \min_{j\in I(s)} \operatorname{dist}(\alpha_{\ell(s_*)} \cap \mathbf{B}_1, \alpha_j \cap \mathbf{B}_1) \\
\le\, &\min_{j\in I(s)} \operatorname{dist}(\alpha_{\ell(s-1)} \cap \mathbf{B}_1, \alpha_j \cap \mathbf{B}_1) + (s - s_*) \min_{j\in I(s)} \operatorname{dist}(\alpha_{\ell(s-1)} \cap \mathbf{B}_1, \alpha_j \cap \mathbf{B}_1)\,.
\end{aligned}$$

In particular, since $s_* - s_0 \ge 1$, we have shown (4.104) for $s = s_0$. We thus conclude that (4.104) indeed holds by induction over s.

Next, note that (4.104) implies that

$$\max_j \min_{i\in I(s)} \operatorname{dist}(\alpha_i \cap \mathbf{B}_1, \alpha_j \cap \mathbf{B}_1) \le s \min_{i\in I(s)} \operatorname{dist}(\alpha_{\ell(s-1)} \cap \mathbf{B}_1, \alpha_i \cap \mathbf{B}_1) \tag{4.106}$$

for all s, by simply maximizing over all $s' < s$ on the left-hand side of (4.104), since $s - s' \leq s$. In particular, combined with (4.103), we must have

$$D + t \min_{i \in I(t)} \operatorname{dist}(\alpha_{\ell(t-1)} \cap \mathbf{B}_1, \alpha_i \cap \mathbf{B}_1) > \delta \min_{i \in I(t+1)} \operatorname{dist}(\alpha_{\ell(t)} \cap \mathbf{B}_1, \alpha_i \cap \mathbf{B}_1)$$

for $t = 1, \ldots, s-1$, with $s \leq \sigma$ fixed arbitrarily (note that the right-hand side of this expression equals that of (4.104), by definition of $\ell(t)$). Setting $d(t) := \min_{i \in I(t)} \operatorname{dist}(\alpha_{\ell(t-1)} \cap \mathbf{B}_1, \alpha_i \cap \mathbf{B}_1)$, we rewrite the above as the recursive inequality

$$\delta^{-1}(D + td(t)) > d(t+1)\,,$$

which, setting the convention $d(0) = 0$, can be assumed valid for $j = 0$ as well. We can thus iterate this to get

$$\begin{aligned} d(s) &\leq \delta^{-1} D \left(1 + \delta^{-1}(s-1) + \delta^{-2}(s-1)(s-2) + \cdots + \delta^{-(s-1)}(s-1)!\right) \\ &\leq \delta^{-1} D \left(s\delta^{-(s-1)}(s-1)!\right) \\ &= \delta^{-s} s! D\,. \end{aligned}$$

Since $s \leq N-2$, we get $d(s) \leq \delta^{2-N}(N-2)!D$, so combining with (4.106), for any $s \leq \sigma$ we have

$$\max_j \min_{i \in I(s)} \operatorname{dist}(\alpha_i \cap \mathbf{B}_1, \alpha_j \cap \mathbf{B}_1) \leq s d(s) \leq (N-2) d(s) \leq \delta^{2-N}(N-1)!D\,.$$

It is therefore the case that (4.99) holds with $I = I(\sigma)$.

Next, (4.100) certainly holds with $I = I(\sigma)$ by construction if $|I(\sigma)| \geq 3$, since then the procedure stopped due to the fact that (4.103) fails. We thus have to show that (4.100)

holds when $|I(\sigma)| = 2$. In this case observe that our procedure guarantees that

$$\max_{i<j\in I(\sigma)} \operatorname{dist}(\alpha_i \cap \mathbf{B}_1, \alpha_j \cap \mathbf{B}_1) = \max_{i<j} \operatorname{dist}(\alpha_i \cap \mathbf{B}_1, \alpha_j \cap \mathbf{B}_1)\,, \tag{4.107}$$

which shows that (4.101) holds in general. But then, as $|I(\sigma)| = 2$ we certainly have

$$\min_{i<j\in I(\sigma)} \operatorname{dist}(\alpha_i \cap \mathbf{B}_1, \alpha_j \cap \mathbf{B}_1) = \max_{i<j\in I(\sigma)} \operatorname{dist}(\alpha_i \cap \mathbf{B}_1, \alpha_j \cap \mathbf{B}_1)$$

and so combining this with (4.107), using assumption (ii) and the fact that we have already proved (4.99), we have

$$\begin{aligned} D + \max_j \min_{i\in I(\sigma)} \operatorname{dist}(\alpha_i \cap \mathbf{B}_1, \alpha_j \cap \mathbf{B}_1) \leq D + \Gamma D &= (1+\Gamma)D \\ &\leq (1+\Gamma)\varepsilon \max_{i<j} \operatorname{dist}(\alpha_i \cap \mathbf{B}_1, \alpha_j \cap \mathbf{B}_1) \\ &= \delta \min_{i<j\in I(\sigma)} \operatorname{dist}(\alpha_i \cap \mathbf{B}_1, \alpha_j \cap \mathbf{B}_1)\,, \end{aligned}$$

proving (4.100) in this case also (we have used that $\varepsilon = (1+\Gamma)^{-1}\delta$ here). □

We now iteratively apply Lemma 4.6.2 to a cone S comprised of planes $\{\alpha_1, \ldots, \alpha_N\}$ as in Assumption 4.6.1, getting a finite family of subcollections of the integers $\{1, \ldots, N\}$ leading to a family of simpler cones, in the following fashion. The collections of planes after each application of Lemma 4.6.2 can be thought of as "layers", which are subcollections of a starting one. Moreover, when we fix plane in a certain layer, the closest one among those of the previous layers is much closer than the minimum distance between any pair of planes belonging to these previous layers. What we end up with is that each plane we throw away when moving from one layer to the next is a distance comparable to the minimal distance in the original layer, whilst for the last layer we have comparability between the maximum and minimum distances of the planes.

Lemma 4.6.3 (Layer subdivision)**.** *For every integer $N \geq 2$ and every $0 < \delta \leq 1$, there is*

$\eta = \eta(\delta, N) > 0$ *with the following properties. Let* $\mathbf{S}$ *and* $\alpha_1, \ldots, \alpha_N$ *be as in Assumption 4.6.1. Then, there is* $\kappa \in \mathbb{N}$ *and subcollections* $I(0) \supsetneq I(1) \supsetneq \cdots \supsetneq I(\kappa)$ *of the integers* $\{1, \ldots, N\}$*, each of cardinality at least* 2 *and with* $I(0) = \{1, \ldots, N\}$*, so that the numbers*

$$m(s) := \min_{i<j\in I(s)} \operatorname{dist}(\alpha_i \cap \mathbf{B}_1, \alpha_j \cap \mathbf{B}_1) \tag{4.108}$$

$$d(s) := \max_{i\in I(0)} \min_{j\in I(s)} \operatorname{dist}(\alpha_i \cap \mathbf{B}_1, \alpha_j \cap \mathbf{B}_1) \tag{4.109}$$

$$M(s) := \max_{i<j\in I(s)} \operatorname{dist}(\alpha_i \cap \mathbf{B}_1, \alpha_j \cap \mathbf{B}_1) \tag{4.110}$$

satisfy the following requirements:

(i) $M(\kappa) = M(0)$*;*

(ii) $\eta M(\kappa) \le m(\kappa)$*;*

(iii) $d(s) \le \delta m(s)$ *and* $\eta d(s) \le m(s-1)$ *for every* $1 \le s \le \kappa$*;*

(iv) $m(s-1) \le \delta m(s)$ *for every* $1 \le s \le \kappa$*.*

Proof. Let $0 < \delta \le 1$. We fix $\eta > 0$ such that $\eta \le \varepsilon$, where ε is the constant in Lemma 4.6.2 corresponding to δ/N in place of δ. In particular, $N\eta \le \Gamma^{-1}$, where Γ is again as in Lemma 4.6.2 with δ replaced by δ/N.

If $\eta M(0) \le m(0)$, we set $\kappa = 0$ and obviously the Lemma holds. Otherwise, we inductively apply Lemma 4.6.2 with $D = m(s-1)$ to produce $I(s)$ from $I(s-1)$, as long as $\eta M(s-1) > m(s-1)$. We wish to check that the conclusions of the lemma hold for this particular choice of subcollections. The fact that the sequence of sets is strictly decreasing and each set has cardinality at least two are obvious. Property (i) is immediate from (4.101) of Lemma 4.6.2, and (iv) is immediate from (4.100), since we are taking $D = m(s-1)$. Property (ii) holds by construction, as the process must terminate in finite time; at worst,

when $\kappa = N-2$ and $|I(\kappa)| = 2$. Moreover, by (4.100) of Lemma 4.6.2 we have the inequality

$$\max_{i\in I(s-1)} \min_{j\in I(s)} \operatorname{dist}(\alpha_i \cap \mathbf{B}_1, \alpha_j \cap \mathbf{B}_1) \leq \frac{\delta}{N} m(s)\,.$$

for every $s = 1, \ldots, \kappa$ (recall that when generating $I(s)$, we are applying Lemma 4.6.2 with $I(s-1)$ in place of $\{1, \ldots, N\}$). But the triangle inequality then gives

$$d(s) \leq \frac{\delta}{N}(m(1) + \cdots + m(s))\,.$$

Since $m(t) \leq m(s)$ for all $1 \leq t < s$, and $\kappa \leq N-2$, we therefore must have $d(s) \leq \delta m(s)$ for every s, proving the first inequality in (iii). Finally, observe that by (4.99) of Lemma 4.6.2, we also have

$$\max_{i\in I(s-1)} \min_{j\in I(s)} \operatorname{dist}(\alpha_i \cap \mathbf{B}_1, \alpha_j \cap \mathbf{B}_1) \leq \Gamma m(s-1)$$

for $s = 1, \ldots, \kappa$, and thus by the same triangle inequality argument as above, we achieve $d(s) \leq N\Gamma m(s-1)$ for every s. Hence, recalling that $N\eta \leq \Gamma^{-1}$, we achieve the second inequality of (iii). □

4.6.2 Graphical parameterizations for T over $\mathbf{S}$

The aim of this section is to efficiently parameterize area-minimizing currents over cones $\mathbf{S}$ as in Definition 4.2.2 satisfying the additional pairwise Morgan angle comparability condition (ii) outlined in the introduction of Section 4.6. We recall that we are working under the Assumption 3.2.2 throughout. Moreover, recall the sets $\mathscr{C}(Q)$ and $\mathscr{P}$ as in Definition 4.2.2 and the L^2 based excesses of Definition 4.2.3.

The important concept of a "balanced cone" is given in the following definition.

Definition 4.6.4. *Let $\mathbf{S} \in \mathscr{C}(Q)$, $M \geq 1$, and let $\alpha_1, \ldots, \alpha_N$ be the N distinct m-dimensional*

planes forming $\mathbf{S}$. *We say that* $\mathbf{S}$ *is* M-balanced *if for every* $i \neq j$ *the inequality*

$$\theta_2(\alpha_i, \alpha_j) \leq M\theta_1(\alpha_i, \alpha_j) \tag{4.111}$$

holds for the two Morgan angles of the pair α_i, α_j.

Observe that the condition is empty when $\mathbf{S} \in \mathscr{P}$. Moreover, if we say that a cone $\mathbf{S} = \alpha_1 \cup \cdots \cup \alpha_N \in \mathscr{C}(Q)$ is M-balanced for some $M > 0$, we will implicitly be assuming that all the α_i are distinct. Loosely speaking, we are interested in balanced cones because two planes within a balanced cone can only degenerate to a single plane, and not to two planes meeting along an $(n-1)$-dimensional axis.

We will give two different graphical approximation results: a series of "crude approximation" results, followed by a series of "refined approximation" results.

4.6.3 Crude approximation statements

Here we will give the statements of the crude approximation results; in the next section, we will prove all of them. The key starting point is the following splitting lemma. We recall the notation

$$\boldsymbol{\sigma}(\mathbf{S}) := \min_{i<j} \operatorname{dist}(\alpha_i \cap \mathbf{B}_1, \alpha_j \cap \mathbf{B}_1).$$

Lemma 4.6.5 (Crude splitting)**.** *For every* $Q, m, n, \bar{n} \in \mathbb{N}$ *and every* $M > 0$, $\rho, \eta > 0$, *there are constants* $\delta = \delta(Q, m, n, \bar{n}, M, \rho, \eta) > 0$ *and* $\varrho = \varrho(Q, m, n, \bar{n}, M, \rho, \eta) > 0$ *with the following property. Let* T *be as in Assumption 3.2.2 with* $\|T\|(\mathbf{B}_4) \leq (Q + \frac{1}{2})4^m\omega_m$. *Assume that* $2 \leq N \leq Q$, *that* $\mathbf{S} = \alpha_1 \cup \cdots \cup \alpha_N \in \mathscr{C}(Q)$ *is* M*-balanced, and*

$$\int_{\mathbf{B}_4 \setminus \mathbf{B}_\rho(V)} \operatorname{dist}^2(p, \mathbf{S})\, d\|T\|(p) + \mathbf{A}^2 \leq \delta^2 \boldsymbol{\sigma}(\mathbf{S})^2 =: \delta^2\sigma^2\,, \tag{4.112}$$

where $V = V(\mathbf{S})$ *is the spine of* $\mathbf{S}$. *Then the following properties hold:*

(a) The sets $W_i := (\mathbf{B}_4 \setminus \bar{\mathbf{B}}_\rho(V)) \cap \{\operatorname{dist}(\cdot, \alpha_i) < \varrho\sigma\}$ *are pairwise disjoint;*

(b) $\mathrm{spt}(T) \cap \mathbf{B}_{4-\eta} \setminus \bar{\mathbf{B}}_{\rho+\eta}(V) \subset \bigcup_i W_i$.

From the above lemma, the tilt-excess bound from Theorem 4.3.2, and Almgren's strong Lipschitz approximation over planes ([19, Theorem 1.4]), we can then conclude the following, where we use heavily the notation of [19, Theorem 1.4] (in particular, if u is a Lipschitz multi-valued map, $\mathrm{gr}\,(u)$ will denote its set-theoretic graph and $\mathbf{G}_u$ the current naturally induced by it).

Proposition 4.6.6. *Let T, W_i, and all associated notation be as in Lemma 4.6.5. Consider for each $i \in \{1, \ldots, N\}$ the regions $\Omega_i := (\mathbf{B}_{4-2\eta} \cap \alpha_i) \setminus \bar{\mathbf{B}}_{\rho+\eta}(V)$ and $\mathbf{\Omega}_i := \mathbf{B}_{4-\eta} \cap \mathbf{p}_{\alpha_i}^{-1}(\Omega_i)$. Set $T_i := T \llcorner (W_i \cap \mathbf{\Omega}_i)$ and*

$$E_i := \int_{\mathbf{B}_4 \setminus \mathbf{B}_\rho(V)} \mathrm{dist}^2(p, \alpha_i)\, d\|T_i\|(p)$$

Then, there are non-negative integers $Q_1, \ldots, Q_N$ with $\sum_i Q_i \leq Q$ satisfying the following properties:

(a) $\partial T_i \llcorner \mathbf{\Omega}_i = 0$;

(b) α_i *can be appropriately oriented so that* $(\mathbf{p}_{\alpha_i})_\sharp T_i = Q_i [\![\Omega_i]\!]$;

(c) The following estimate holds

$$\mathrm{dist}^2(q, \alpha_i) \equiv |\mathbf{p}_{\alpha_i}^\perp(q)|^2 \leq C E_i + C\mathbf{A}^2 \qquad \forall q \in \mathrm{spt}(T_i) \cap \mathbf{\Omega}_i\,; \tag{4.113}$$

(d) For all i with $Q_i \geq 1$, there are Lipschitz multi-valued maps $u_i : \Omega_i \to \mathcal{A}_{Q_i}(\alpha_i^\perp)$ and closed sets $K_i \subset \Omega_i$ such that $\mathrm{gr}(u_i) \subset \Sigma$, $T_i \llcorner \mathbf{p}_{\alpha_i}^{-1}(K_i) = \mathbf{G}_{u_i} \llcorner \mathbf{p}_{\alpha_i}^{-1}(K_i)$, and the following estimates hold:

$$\|u_i\|_\infty^2 + \|Du_i\|_{L^2}^2 \leq C(E_i + \mathbf{A}^2) \tag{4.114}$$

$$\mathrm{Lip}\,(u_i) \leq C(E_i + \mathbf{A}^2)^\gamma \tag{4.115}$$

$$|\Omega_i \setminus K_i| + \|T\|(\mathbf{\Omega}_i \setminus \mathbf{p}_{\alpha_i}^{-1}(K_i)) \leq C(E_i + \mathbf{A}^2)^{1+\gamma}\,; \tag{4.116}$$

(e) $Q_i = 0$ *if and only if* $T_i = 0$*;*

(f) Finally, if in addition we have the "reverse excess" estimate

$$\int_{\mathbf{S}\cap\mathbf{B}_{4-2\eta}\setminus B_{\rho+2\eta}(V)} \mathrm{dist}^2(p, \mathrm{spt}(T))\, d\mathcal{H}^m(p) \leq \delta^2\sigma^2\,, \tag{4.117}$$

then $Q_i \geq 1$ *for every* i*.*

Here, $\gamma = \gamma(Q, m, n, \bar{n}) > 0$ *and* $C = C(Q, m, n, \bar{n}) > 0$*;*

In several situations we will need to ensure that $\sum_i Q_i = Q$ in Proposition 4.6.6. This conclusion needs however an additional assumption, as well as smallness of the tubular neighbourhood radius ρ of the spine. Depending on the situation we can guarantee this either by using (1) the existence of a point of large density; or (2) the existence of a region with sufficiently large mass.

Lemma 4.6.7. *There exists* $\rho_* = \rho_*(Q, m) \in (0, 1)$ *such that if we assume that* T *satisfies the assumptions of Lemma 4.6.5 with* $\rho \leq \rho_*$ *and either:*

(a) $\{\Theta(T, \cdot) \geq Q\} \cap \mathbf{B}_\varepsilon(0) \neq \emptyset$ *for a sufficiently small* $\varepsilon = \varepsilon(Q, m, n, \bar{n})$*; or*

(b) for some $C_* > 0$*,* ρ_* *is sufficiently small also depending on* C_**, and there is a closed set* $\Omega \subset \mathbf{B}_4$ *with non-empty interior that is invariant under rotation around* V *(c.f. Definition 4.5.13), for which* $\|T\|(\Omega) \geq (Q - \frac{1}{2})\mathcal{H}^m(\alpha_1 \cap \Omega)$*, and with* $\mathcal{H}^m(\alpha_1 \cap \Omega) \geq C_*$*.*

Then, if Q_i *is as in Proposition 4.6.6 and* δ *is sufficiently small, we have* $\sum_i Q_i = Q$*.*

We will only be applying Lemma 4.6.7 to specific choices of Ω, for which we may ensure that $\mathcal{H}^m(\alpha_1 \cap \Omega) \geq C_*$ always holds for a specific choice of $C_* = C_*(Q, m) > 0$. In particular a suitable choice of $\rho_* = \rho_*(Q, m)$ will work for both alternatives (a) and (b) above that

we care about, without any additional assumptions involving C_* in alternative (b). We will therefore fix this choice of radius $\rho_* = \rho_*(Q, m)$ for the rest of the paper.

Finally, we remark that all the statements above can be suitably scaled and translated to analogous statements where the initial domain $\mathbf{B}_4$ in Lemma 4.6.5 is replaced by an arbitrary ball $\mathbf{B}_{4r}(q)$.

An approximation statement analogous to Proposition 4.6.6 also holds in the much simpler setting in which $\mathbf{S}$ consists of a single plane. This case is somewhat special because we cannot identify a unique spine V and at the same time the number σ in Lemma 4.6.5 is ∞, hence the smallness condition (4.112) would be empty. For this reason we state the proposition separately even though it could be embedded as a special case of Proposition 4.6.6.

Proposition 4.6.8 (Crude approximation on a single plane)**.** *For every $Q, m, n, \bar{n} \in \mathbb{N}$ and $\rho, \eta > 0$, there exist $\delta = \delta(Q, m, n, \bar{n}, \rho, \eta) > 0$ and $C = C(Q, m, n, \bar{n})$ with the following property. Let T be as in Assumption 3.2.2 with $\|T\|(\mathbf{B}_4) \leq (Q + \frac{1}{2})4^m\omega_m$. Assume $\mathbf{S} = \alpha_1 \in \mathscr{P}$, $V \subset \alpha_1$ is an $(m-2)$-dimensional subspace, and*

$$E_1 + \mathbf{A}^2 := \int_{\mathbf{B}_4 \setminus B_\rho(V)} \operatorname{dist}(p, \mathbf{S})^2 \, d\|T\|(p) + \mathbf{A}^2 \leq \delta^2 \,. \tag{4.118}$$

Set $\Omega_1 := (\mathbf{B}_{4-2\eta} \cap \alpha_1) \setminus \bar{B}_{\rho+\eta}(V)$ and $\mathbf{\Omega}_1 := \mathbf{B}_{4-\eta} \cap \mathbf{p}_{\alpha_1}^{-1}(\Omega_1)$ and $T_1 := T \llcorner \mathbf{\Omega}_1$ Then, there is non-negative integer $Q_1 \leq Q$ such that the following holds:

(a) $\partial T_1 \llcorner \mathbf{\Omega}_1 = 0$

(b) α_1 can be appropriately oriented so that $(\mathbf{p}_{\alpha_1})_\sharp T_1 = Q_1 [\![\Omega_1]\!]$;

(c) The following estimate holds

$$\operatorname{dist}^2(q, \alpha_1) \equiv |q - \mathbf{p}_{\alpha_1}(q)|^2 \leq CE_1 + C\mathbf{A}^2 \qquad \forall q \in \operatorname{spt}(T_1) \cap \mathbf{\Omega}_1 \,; \tag{4.119}$$

(d) There is a Lipschitz multi-valued map $u_1 : \Omega_1 \to \mathcal{A}_{Q_1}(\alpha_1^\perp)$ and a closed set $K_1 \subset \Omega_1$

such that $\mathrm{gr}(u_1) \subset \Sigma$, $T_1 \llcorner \mathbf{p}_{\alpha_1}^{-1}(K_1) = \mathbf{G}_u \llcorner \mathbf{p}_{\alpha_1}^{-1}(K_1)$ *and the following estimates hold:*

$$\|u_1\|_\infty^2 + \|Du_1\|_{L^2}^2 \leq C(E_1 + \mathbf{A}^2) \tag{4.120}$$

$$\mathrm{Lip}\,(u_1) \leq C(E_1 + \mathbf{A}^2)^\gamma \tag{4.121}$$

$$|\Omega_1 \setminus K_1| + \|T\|(\mathbf{\Omega}_1 \setminus \mathbf{p}_{\alpha_1}^{-1}(K_1)) \leq C(E_1 + \mathbf{A}^2)^{1+\gamma}\,; \tag{4.122}$$

(e) $Q_1 = 0$ *if and only if* $T_1 = 0$*;*

(f) If $\rho \leq \rho_*$ *and one of conditions (a) and (b) in Lemma 4.6.7 holds then* $Q_1 = Q$*.*

Here, $\gamma = \gamma(Q, m, n, \bar{n}) > 0$ *and* $C = C(Q, m, n, \bar{n}) > 0$*.*

We also have an analogous scaled versions of the above where we replace $\mathbf{B}_4(0)$ by an arbitrary ball $\mathbf{B}_{4r}(p)$ and all quantities are scaled accordingly.

We now prove all of the above results.

4.6.4 Proofs of Crude Approximation Results

We will prove Lemma 4.6.5, Proposition 4.6.6, and Lemma 4.6.7. Proposition 4.6.8 then follows by the same arguments, and so we will leave the details to the reader.

Proof of Lemma 4.6.5. We remark that statement (a) of Lemma 4.6.5 holds as soon as ϱ is smaller than an suitable constant which depends only on M, ρ, η (and the number of planes as well as the dimensions). We fix any such ϱ, and claim that as soon as δ is small enough (b) holds as well.

Indeed, to this end we may argue by contradiction: consider a sequence T_k of integral currents and a sequence Σ_k of Riemannian submanifolds of $\mathbb{R}^{m+n}$ satisfying Assumption 3.2.2, together with cones $\mathbf{S}_k \in \mathscr{C}(Q)$ such that

(i) $\|T_k\|(\mathbf{B}_4) \leq (Q + \frac{1}{2})\omega_m 4^m$;

(ii) $\mathbf{S}_k = \alpha_1^k \cup \cdots \cup \alpha_{N(k)}^k \in \mathscr{C}(Q)$ is M-balanced, where $N(k) \leq Q$;

(iii) If we write $\sigma_k := \boldsymbol{\sigma}(\mathbf{S}_k)$ and

$$\mathbf{E}_k := \int_{\mathbf{B}_4 \setminus B_\rho(V(\mathbf{S}_k))} \mathrm{dist}^2(p, \mathbf{S}_k) d\|T_k\|(p)\,,$$

we have $\sigma_k^{-2}(E_k + \mathbf{A}_k^2) \to 0$, where $\mathbf{A}_k$ corresponds to Σ_k;

(iv) There are points $p_k \in \mathrm{spt}(T_k) \cap \mathbf{B}_{4-\eta} \setminus \overline{B}_{\rho+\eta}(V(\mathbf{S}_k))$ such that $\mathrm{dist}(p_k, \mathbf{S}_k) \geq \varrho\sigma_k$.

Observe that σ_k is a bounded sequence since $0 \leq \sigma_k \leq 1$. Hence, up to the extraction of a subsequence and after applying suitable rotations, we can assume that:

(v) $V(\mathbf{S}_k)$ is a fixed $(m-2)$-dimensional plane V and $N \equiv N(k) \leq Q$ is a fixed integer;

(vi) $\mathbf{S}_k$ converges, locally in Hausdorff distance, to $\mathbf{S} \in \mathscr{C}(Q)$ which is the union of $N' \leq N$ distinct planes α_i such that $\alpha_i \cap \alpha_j = V$ for all pairs $i < j$ (note that we could have $N' = 1$, in which case the latter condition here is vacuous);

(vii) T_k converges to an integral current T which is area-minimizing in $\mathbf{B}_4$ and obeys $\partial T = 0$;

(viii) $\mathrm{spt}(T) \cap \mathbf{B}_4 \setminus B_\rho(V) \subset \mathbf{S}$.

By the constancy theorem and the fact that $\partial T = 0$, it then follows that $T \llcorner \mathbf{B}_4 \setminus \overline{B}_\rho(V) = \sum_i \bar{Q}_i [\![\alpha_i]\!] \llcorner (\mathbf{B}_4 \setminus \overline{B}_\rho(V))$ where the $\bar{Q}_i$ are integers. Orienting the α_i suitably, we can assume that $\bar{Q}_i \geq 0$.

We also know that $\mathrm{spt}(T_k)$ converges locally in Hausdorff distance to $\mathrm{spt}(T)$. In particular, passing to a subsequence, the points p_k converge to some point p which must lie in one of the planes α_j which form $\mathbf{S}$. We denote by π_0 this latter plane and observe that clearly $|p| \leq 4 - \eta$ whilst $\mathrm{dist}(p, V) \geq \rho + \eta$. For each fixed i, the sequence of planes α_i^k must converge (locally in Hausdorff distance) to some plane of $\mathbf{S}$ and, again upon extraction of a suitable subsequence and relabelling, we may assume that there is an $N_0 \leq N$ with the property that α_i^k converges to π_0 when $i \leq N_0$, whilst it converges to some other plane of $\mathbf{S}$ when $i > N_0$. Clearly we know that $N_0 \geq 1$ by construction.

Now for a fixed parameter $\bar{\eta} > 0$, consider the currents $T'_k := T_k \llcorner ((\mathbf{B}_{4-\eta/2} \setminus \overline{B}_{\rho+\eta}(V)) \cap \{\mathrm{dist}(\cdot, \pi_0) < \bar{\eta}\})$ and the cones $\mathbf{S}'_k := \alpha^k_1 \cup \cdots \cup \alpha^k_{N_0}$. Observe that, if we choose $\bar{\eta}$ sufficiently small, the convergence properties outlined above imply that:

- $\partial T'_k = 0$ in $\mathbf{B}_{4-\eta/2} \setminus \overline{B}_{\rho+\eta}(V)$ for all k sufficiently large;
- $\mathrm{dist}(q, \mathbf{S}_k) = \mathrm{dist}(q, \mathbf{S}'_k)$ for all $q \in \mathrm{spt}(T'_k)$ and all k sufficiently large;
- T'_k converges to $\bar{Q}_j \llbracket \pi_0 \cap \mathbf{B}_{4-\eta/2} \setminus \overline{B}_{\rho+\eta}(V) \rrbracket$, where $1 \leq \bar{Q}_j \leq Q$ is as above.

Now consider the cylinder $\mathbf{C}_{4r}(p, \pi_0)$, where p is the limit point as above and r is a geometric constant. Observe that, upon choosing r suitably, in light of (iii), for all k sufficiently large we can apply Corollary 4.3.3 to T'_k in the cylinder $\mathbf{C}_{4r}(p, \pi_0)$ (which we may ensure is disjoint from $B_\rho(V)$ provided that we take $4r \leq \eta$). We can therefore decompose $T'_k \llcorner \mathbf{C}_r(p, \pi_0)$ as $T'_{k,1} + \cdots + T'_{k,N_0}$ with the property that

$$\mathrm{dist}(q, \alpha^k_i) = \mathrm{dist}(q, \mathbf{S}'_k) = \mathrm{dist}(q, \mathbf{S}_k) \qquad \forall q \in \mathrm{spt}(T'_{k,i})\,. \tag{4.123}$$

Moreover, for each fixed i, the currents $T'_{k,i}$ converge to $Q_i \llbracket B_r(p, \pi_0) \rrbracket$, for some non-negative integers Q_i which obey $\sum_i Q_i = \bar{Q}_j$. Consider now the points $p_{k,i} := \alpha^k_i \cap \mathbf{p}^{-1}_{\pi_0}(p)$ (which exist for all k sufficiently large) and the cylinders $\mathbf{C}^{k,i} := \mathbf{C}_{r/2}(p_{k,i}, \alpha^k_i)$. The latter cylinder is converging to $\mathbf{C}_{r/2}(p, \pi_0)$ and hence in particular, for k large enough, we know that $\mathbf{C}^{k,i} \cap \mathbf{B}_1 \subset \mathbf{C}_r(p, \pi_0)$. Hence, $\partial T'_{k,i} \llcorner \mathbf{C}^{k,i} = 0$ for k large enough because $\partial T'_{k,i} \llcorner \mathbf{C}_r(p, \pi_0) = 0$ and $\mathrm{spt}(T'_{k,i})$ are converging locally in Hausdorff distance to $\bar{B}_r(p, \pi_0)$. In particular, it follows that $(\mathbf{p}_{\alpha^k_i})_\sharp T'_{k,i} = Q'_i \llbracket B_{r/2}(p_{k,i}, \alpha^k_i) \rrbracket$ for some integers $Q'_i \geq 0$ and that $\|T'_{k,i}\|(\mathbf{C}^{k,i}) \to Q_i \omega_m 2^{-m} r^m$.

However, because of (4.123) we also have

$$\int_{\mathbf{C}^i} \mathrm{dist}^2(q, \alpha^k_i)\, d\|T'_{k,i}\|(q) \leq \mathbf{E}_k\,.$$

We can now apply (a scaled version of) the L^∞ estimate (4.5) from Theorem 4.3.2 to conclude that

$$\mathrm{dist}^2(q, \alpha_i^k) \leq Cr^{-m-2}\mathbf{E}_k + Cr^2\mathbf{A}^2 \leq C(r^{-m-2} + r^2)\delta^2\sigma_k^2$$

for all $q \in \mathrm{spt}(T'_{k,i}) \cap \mathbf{C}_{r/4}(p_{k,i}, \alpha_i^k)$. But the contradiction point p_k must be contained in one of the cylinders $\mathbf{C}_{r/4}(p_{k,i}, \alpha_i^k)$ for k sufficiently large, and it is also contained in the support of $T'_k = \sum_i T'_{k,i}$; we may pass to a subsequence to fix the i for which this is true. Hence, for k large enough we must have the estimate

$$\mathrm{dist}^2(p_k, \alpha_i^k) \leq C(r^{-m-2} + r^2)\delta^2\sigma_k^2$$

where i is now fixed. But since r is fixed, it suffices to choose δ small enough to ensure $C(r^{-m-2} + r^2)\delta^2 \leq \varrho^2/4$ and contradict (iv) above; notice that δ indeed has the correct dependencies. □

Proof of Proposition 4.6.6. First of all notice that the conclusions (a), (b), and the estimate (c) all follow from the arguments in the proof of Lemma 4.6.5. The estimates in point (d) of Proposition 4.6.6 then follow from the Lipschitz approximation of [19] and (4.4) of Theorem 4.3.2.

As for the conclusion (e), because for each i one has the identity

$$\|T_i\|(\mathbf{\Omega}_i) = Q_i|\Omega_i| + \frac{1}{2}\int_{\Omega_i} |\vec{T} - \vec{\alpha}_i|^2 \, d\|T_i\| \,,$$

if we again apply the tilt-excess estimate (4.4) from Theorem 4.3.2, when $Q_i = 0$ we must have

$$\|T_i\|(\mathbf{\Omega}_i) \leq C\delta^2\sigma^2 \,.$$

In particular, if r is the radius of the cylinders considered in the proof of Lemma 4.6.5, once δ is sufficiently small, the monotonicity formula guarantees that $\|T_i\|(\mathbf{B}_r(q)) = 0$ for every ball $\mathbf{B}_r(q)$ which is contained in $\mathbf{\Omega}_i$ (indeed, as soon as the mass ratio of T_i falls below 1, we

get a contradiction if T_i is not zero as $\Theta(T_i, q) \geq 1$ at every point $q \in \mathrm{spt}(T_i)$). This then implies conclusion (e).

As for (f), we easily conclude from the argument above that, if $Q_i = 0$, then the distance of any point $q \in \Omega_i$ to $\mathrm{spt}(T)$ must be at least $\min\{\varrho\sigma, r\}$. We thus infer

$$\int_{\mathbf{S}\cap\mathbf{B}_4\setminus B_\rho(V)} \mathrm{dist}^2(p, \mathrm{spt}(T))\, d\mathcal{H}^m(p) \geq c \min\{\varrho^2\sigma^2, r^2\}$$

for some geometric constant c. This obviously contradicts (4.117) if δ is small enough. This completes the proof of Proposition 4.6.6. □

Proof of Lemma 4.6.7. We argue by contradiction, starting in an identical manner to that seen in the proof of Lemma 4.6.5. Indeed, following the notation used there, it suffices to show the limiting current T which obeys

$$T \llcorner \mathbf{B}_4 \setminus \overline{B}_\rho(V) = \sum_i \bar{Q}_i [\![\alpha_i]\!] \llcorner (\mathbf{B}_4 \setminus \overline{B}_\rho(V))$$

satisfies $\sum_i \bar{Q}_i = Q$ for any $\rho \leq \rho_*(Q, m)$ sufficiently small, when we suppose that one of the hypotheses (a) or (b) holds along the contradiction sequence (for suitable $\varepsilon = \varepsilon_k \downarrow 0$ in the case of (a) and suitable sets Ω_k in the case of (b)).

First, note that from the monotonicity formula and convergence (namely that we have weak$-*$ convergence of the masses $\|T_k\|$ to $\|T\|$, where T_k are the currents as in the proof of Lemma 4.6.5) we know $\|T\|(\mathbf{B}_3) \leq (Q + \frac{1}{2})3^m\omega_m$, which for sufficiently small ρ evidently implies that we must have $\sum_i \bar{Q}_i \leq Q$. Thus, we just need to show that $\sum_i \bar{Q}_i \geq Q$ when we additional suppose the hypotheses from (a) or (b).

Note next that we can cover $\mathbf{B}_3 \cap V$ by $C\rho^{-(m-2)}$ balls of radius ρ; if we double the radius of each ball, we may then without loss of generality assume that they cover $\mathbf{B}_3 \cap B_\rho(V)$ also. But then the monotonicity formula for the mass ratios of T gives for any such ball $\mathbf{B}_i$ in this

cover,

$$\|T\|(\mathbf{B}_i) \leq C\rho^m \|T\|(\mathbf{B}_{7/2}) \leq C(Q,m)\rho^m$$

and so

$$\|T\|(\mathbf{B}_3 \cap B_\rho(V)) \leq C\rho^m \cdot \rho^{-(m-2)} = C\rho^2. \tag{4.124}$$

Let us first suppose that the alternative (a) holds. Taking a sequence $\varepsilon_k \downarrow 0$, for the sequence of currents T_k from the proof of Lemma 4.6.5, we have a sequence of points $p_k \in \mathbf{B}_{\varepsilon_k}$ with $\Theta(T_k, p_k) \geq Q$. Upper semi-continuity of the density guarantees that the limiting current satisfies $\Theta(T,0) \geq Q$, and so $\|T\|(\mathbf{B}_3) \geq 3^m Q\omega_m$, and hence $\|T\|(\mathbf{B}_3 \setminus \overline{B}_\rho(V)) \geq 3^m Q\omega_m - C\rho^2$. But then in light of the structure of T, this directly implies that

$$\sum_i \bar{Q}_i(3^m\omega_m - 3^{m-2}\omega_2\rho^2) \geq 3^m Q\omega_m - C\rho^2$$

and so since the $\bar{Q}_i$ are non-negative integers if $\rho \leq \rho_* = \rho_*(Q,m)$ is sufficiently small this evidently implies that $\sum_i \bar{Q}_i \geq Q$ in this case.

If instead (b) holds, then we have $\|T_k\|(\Omega_k) \geq (Q - \frac{1}{2})\mathcal{H}^m(\alpha_1^k \cap \Omega_k)$ for each k and subsets Ω_k satisfying the given assumptions in (b). Since $\mathcal{H}^m(\alpha_k^1 \cap \Omega_k) \geq C_*$, we may pass to a subsequence to ensure that $\mathcal{H}^m(\alpha_k^1 \cap \Omega_k) \to \tilde{C}_* \in [C_*, 4^m\omega_m]$. Again from the weak$-*$ convergence of the masses $\|T_k\|$ to $\|T\|$, using the rotational invariance of T around V in the region $\mathbf{B}_4 \setminus B_\rho(V)$, we readily get

$$\left(Q - \tfrac{1}{2}\right)\tilde{C}_* \leq \lim_{k\to\infty} \|T_k\|(\Omega_k) \leq \lim_{k\to\infty} \|T_k\|(\Omega_k \setminus B_\rho(V)) + C\rho^2 \leq \sum_i \bar{Q}_i \tilde{C}_* + C\rho^2.$$

Note that in the second inequality we use a mass bound analogous to the one in (4.124), which one may observe still holds for T_k in $\Omega_k \cap B_\rho(V)$ since Ω_k is closed and contained in $\mathbf{B}_4$. Thus, provided $\rho \leq \rho_* = \rho_*(Q,m,C_*)$ is sufficiently small, as $\sum_i \bar{Q}_i$ is always an integer this gives that $\sum_i \bar{Q}_i \geq Q$, completing the proof. □

4.6.5 Refined approximation

We now come to the second graphical approximation, which follows a much more refined procedure, using the layers introduced in Lemma 4.6.3. For the reader acquainted with [56], this should be compared to [56, Remark (3) in Section 8, and Section 10].

Assumption 4.6.9 (Assumptions for the refined approximation)**.** Suppose T and Σ are as in Assumption 3.2.2 and $\|T\|(\mathbf{B}_4) \leq 4^m(Q+\frac{1}{2})\omega_m$. Suppose $\mathbf{S} = \alpha_1 \cup \cdots \cup \alpha_N$ is a cone in $\mathscr{C}(Q) \setminus \mathscr{P}$ which is M-balanced, where $M > 0$ is a given fixed constant, and V is the spine of $\mathbf{S}$. For a sufficiently small constant $\varepsilon = \varepsilon(Q, m, n, \bar{n}, M)$ whose choice will be fixed in Assumption 4.6.11 below, suppose that $\{\Theta(T,\cdot) \geq Q\} \cap \mathbf{B}_\varepsilon(0) \neq \emptyset$ and (cf. Proposition 4.6.6) suppose that

$$\mathbb{E}(T, \mathbf{S}, \mathbf{B}_4) + \mathbf{A}^2 \leq \varepsilon^2 \boldsymbol{\sigma}(\mathbf{S})^2 \,. \tag{4.125}$$

Recall that $\mathbb{E}(T, \mathbf{S}, \mathbf{B}_4)$ is a two-sided L^2 excess between T and $\mathbf{S}$.

Whitney decomposition

Let L_0 be the closed cube in V with side-length $\frac{2}{\sqrt{m-2}}$ centered at 0 (if we make the identification $V = \mathbb{R}^{m-2}$, then we can write explicitly $L_0 = [-\frac{1}{\sqrt{m-2}}, \frac{1}{\sqrt{m-2}}]^{m-2}$) and consider the set

$$R := \{p : \mathbf{p}_V(p) \in L_0 \text{ and } 0 < |\mathbf{p}_{V^\perp}(p)| \leq 1\} \,. \tag{4.126}$$

We recall here that we are assuming $m \geq 3$ (cf. Assumption 3.2.2).

Obviously R is invariant under rotations around V. We next decompose R into a countable family of closed sets which are also invariant under rotations around V. Firstly, for every $\ell \in \mathbb{N}$ denote by $\mathcal{G}_\ell$ the collection of $(m-2)$-dimensional cubes in the spine V obtained by subdividing L_0 into $2^{\ell(m-2)}$ cubes of side-length $\frac{2^{1-\ell}}{\sqrt{m-2}}$, and we let $\mathcal{G} = \bigcup_\ell \mathcal{G}_\ell$. Note that we can generate $\mathcal{G}_{\ell+1}$ from $\mathcal{G}_\ell$ by bisecting every face of every cube in $\mathcal{G}_\ell$. We write L for a cube in $\mathcal{G}$, so $L \in \mathcal{G}_\ell$ for some $\ell \in \mathbb{N}$. When we want to emphasize the dependence of the integer ℓ on L we will write $\ell(L)$ and we will call it the *generation of* L. If $L \subset L'$ and

$\ell(L') = \ell(L) + 1$, we then call L' the *parent* of L, and L a *child* of L', while more generally, when $\ell(L') > \ell(L)$, we say that L' is an *ancestor* of L and L a *descendant* of L'.

For every $L \in \mathcal{G}_\ell$ we let

$$R(L) := \{p : \mathbf{p}_V(p) \in L \quad \text{and} \quad 2^{-\ell-1} \leq |\mathbf{p}_{V^\perp}(p)| \leq 2^{-\ell}\}\,.$$

Observe that $R(L_0)$ is not the region R (as $L_0 \in \mathcal{G}_0$ so $\ell(L_0) = 0$), but rather $R = \bigcup_{L\in\mathcal{G}} R(L)$.

For each $L \in \mathcal{G}_\ell$ we let $y_L \in V$ be its center and denote by $\mathbf{B}(L)$ the ball $\mathbf{B}_{2^{2-\ell(L)}}(y_L)$ (in $\mathbb{R}^{m+n}$) and by $\mathbf{B}^h(L)$ the set $\mathbf{B}(L) \setminus B_{\rho_* 2^{-\ell(L)}}(V)$, where ρ_* is as in Lemma 4.6.7. It will be convenient to consider slight enlargements of the sets $R(L)$. More precisely, given a positive number $1 \leq \lambda \leq \frac{3}{2}$ and $L \in \mathcal{G}_\ell$ we will denote by λL the cube concentric to L in V with side-length $\frac{\lambda 2^{1-\ell}}{\sqrt{m-2}}$ and $\lambda R(L)$ the set

$$\lambda R(L) := \{p : \mathbf{p}_V(p) \in \lambda L \quad \text{and} \quad \lambda^{-1}2^{-\ell-1} \leq |\mathbf{p}_{V^\perp}(p)| \leq \lambda 2^{-\ell}\}\,.$$

In fact, we will use some fixed choices of λ which depend only on the dimension and which are rather close to 1. Finally, an important role will be played by the "planar cross sections" of the sets $R(L)$ and $\lambda R(L)$, namely the intersections of these sets with the planes α_i forming the cone $\mathbf{S}$; these intersections will be denoted by L_i and λL_i, respectively.

The following elementary lemma, whose proof is left to the reader, summarizes some important geometric properties of the sets just introduced, and verifies that $\{R(L) : L \in \mathcal{G}\}$ is indeed a Whitney decomposition towards V. With a slight abuse of terminology we will talk about the interior of L and L_i meaning their interiors in the relative topology of V and α_i, respectively. In order to help the reader visualize the content of Lemma 4.6.10 we refer to Figure 4.1 below.

Lemma 4.6.10. *Consider the collection of cubes $\mathcal{G}$ introduced above and its elements L. Then the following properties hold:*

(i) *Given any pair of distinct $L, L' \in \mathcal{G}$ the interiors of $R(L)$ and $R(L')$ are pairwise disjoint and $R(L) \cap R(L') \neq \emptyset$ if and only if $L \cap L' \neq \emptyset$ and $|\ell(L) - \ell(L')| \leq 1$, while the interiors of L and L' are disjoint if $\ell(L) \leq \ell(L')$ and L' is not an ancestor of L.*

(ii) *The union of $R(L)$ ranging over all $L \in \mathcal{G}$ is the whole set R.*

(iii) *The diameters of the sets L, $R(L)$, λL, $\lambda R(L)$, L_i, λL_i, and $\mathbf{B}^h(L)$ are all comparable to $2^{-\ell(L)}$ and, with the exception of $L, \lambda L$, all comparable to the distance between an arbitrarily element within them and V; more precisely, any such diameter and distance is bounded above by $C2^{-\ell(L)}$ and bounded below by $C^{-1}2^{-\ell(L)}$ for some constant C which depends only on m and n.*

(iv) *There is a constant $C = C(m, n)$ such that, if $\mathbf{B}^h(L) \cap \mathbf{B}^h(L') \neq \emptyset$, then $|\ell(L) - \ell(L')| \leq C$ and $\operatorname{dist}(L, L') \leq C2^{-\ell(L)}$. In particular, for every $L \in \mathcal{G}$, the subset of $L' \in \mathcal{G}$ for which $\mathbf{B}^h(L)$ and $\mathbf{B}^h(L')$ have nonempty intersection is bounded by a constant.*

(v) *$\sum_{L \in \mathcal{G}_\ell} \mathcal{H}^{m-2}(L) = C(m)$ for any ℓ and therefore, for any $\kappa > 0$,*

$$\sum_{L \in \mathcal{G}} 2^{-(m-2+\kappa)\ell(L)} \leq C(\kappa, m)\,. \tag{4.127}$$

Layering and choice of the parameters

We now fix a $\bar{\delta} > 0$ (whose choice will specified below in Assumption 4.6.11) and apply the layering subdivision Lemma 4.6.3 with this $\bar{\delta}$ in place of δ therein, to identify a family of sub-cones $\mathbf{S} = \mathbf{S}_0 \supsetneq \mathbf{S}_1 \supsetneq \cdots \supsetneq \mathbf{S}_\kappa$ where $\mathbf{S}_k$ consists of the union of the planes α_i with $i \in I(k)$ for the set of indices $I(k)$ given by Lemma 4.6.3. We then distinguish two cases:

(a) if $\max_{i<j \in I(\kappa)} \operatorname{dist}(\alpha_i \cap \mathbf{B}_1, \alpha_j \cap \mathbf{B}_1) < \bar{\delta}$, we define an additional cone $\mathbf{S}_{\kappa+1}$ consisting of a single plane, given by the smallest index in $I(\kappa)$ and we set $\bar{\kappa} := \kappa + 1$ and $I(\bar{\kappa}) := \{\min I(\kappa)\}$;

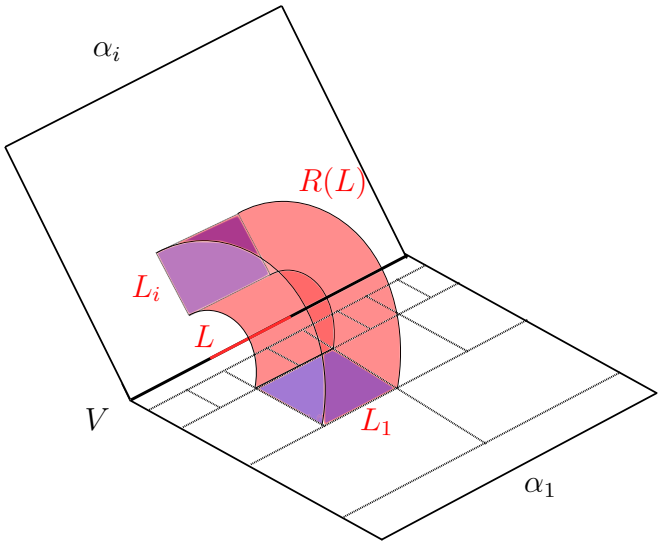


Figure 4.1: An illustration of the Whitney decomposition of R (illustrated on α_1). The subspace V is represented by the thick line joining the two planes, with a representation cube L showing, with the corresponding sets $R(L)$ (in red) and L_i (in blue). The set $R(L)$ is obtained by rotating the given cube L_1 around V in the ambient $(m+n)$-dimensional space, a portion of which is shown.

(b) otherwise, we select no smaller cone and set $\bar{\kappa} := \kappa$.

We next detail the choice of the various parameters involved in our discussion.

Assumption 4.6.11 (Heirarchy of parameters)**.** Firstly, we denote by δ^* the minimum of the parameters δ needed to ensure that Proposition 4.6.6, Lemma 4.6.7, and Proposition 4.6.8 are applicable to all the cones $\mathbf{S}_k$, $k \in \{0, 1, \dots, \bar{\kappa}\}$: note that all the $\mathbf{S}_k$ are M-balanced by construction and that therefore $\delta^* = \delta^*(m, n, \bar{n}, Q, M) > 0$, and in particular its choice does not depend on $\bar{\delta}$. Subsequently, we fix a parameter $\tau = \tau(m, n, \bar{n}, Q, M) > 0$ smaller than $\frac{\delta^*}{C}$ for some large constant $C = C(m, n, \bar{n}, Q) > 0$. The parameter $\bar{\delta}$ leading to the layering $\mathbf{S}_0 \supsetneq \mathbf{S}_1 \supsetneq \cdots \supsetneq \mathbf{S}_{\bar{\kappa}}$ is then chosen to be much smaller than τ; so $\bar{\delta} = \bar{\delta}(m, n, \bar{n}, Q, M) > 0$. In particular, $\bar{\delta} \leq \delta^*$. Finally, $\varepsilon = \varepsilon(m, n, \bar{n}, Q, \delta^*, \bar{\delta}, \tau) > 0$ will be chosen even smaller than $\bar{\delta}$.

Outer, central, and inner regions

We will next subdivide the cubes in $\mathcal{G}$ using the following criterion. In order to simplify our notation we introduce the shorthand:

$$\mathbf{E}(L,k) := 2^{(m+2)\ell(L)} \int_{\mathbf{B}^h(L)} \mathrm{dist}^2(q, \mathbf{S}_k)\, d\|T\|(q)\,.$$

For every $k \in \{0,1,\ldots,\bar{\kappa}\}$, recall that $I(k)$ gives the subset of $\{1,\ldots,N\}$ such that $\mathbf{S}_k = \bigcup_{i\in I(k)} \alpha_i$. If $I(k)$ consists of more than one element, we set

$$\mathbf{s}(k) := \min_{i<j\in I(k)} \mathrm{dist}(\alpha_i \cap \mathbf{B}_1, \alpha_j \cap \mathbf{B}_1)\,, \tag{4.128}$$

while we set $\mathbf{s}(k) := \bar{\delta}$ if $I(k)$ is a singleton.

Definition 4.6.12. *Let $L \in \mathcal{G}$. We say that:*

(i) L is an outer cube *if $\mathbf{E}(L',0) \le \tau^2 \mathbf{s}(0)^2$ for every ancestor L' of L (including L).*

(ii) L is a central cube *if it is not an outer cube and if $\min_k \mathbf{E}(L',k)/\mathbf{s}(k)^2 \le \tau^2$ for every ancestor L' of L (including L).*

(iii) L is an inner cube *if it is neither an outer nor a central cube, but its parent is an outer or a central cube.*

The corresponding families of cubes will be denoted by $\mathcal{G}^o$, $\mathcal{G}^c$, and $\mathcal{G}^{in}$, respectively. Observe that any cube $L \in \mathcal{G}$ is either an outer cube, or a central cube, or an inner cube, or a descendant of inner cube.

We correspondingly define three subregions of R:

- The *outer region*, denoted R^o, is the union of $R(L)$ for L varying over elements of $\mathcal{G}^o$.
- The *central region*, denoted R^c, is the union of $R(L)$ for L varying over elements of $\mathcal{G}^c$.

- Finally, the *inner region*, denoted R^{in}, is the union of $R(L)$ for L ranging over the elements of $\mathcal{G}$ which are neither outer nor central cubes.

Alternatively, the inner region can be defined as the union of $R(L)$ for L ranging over the inner cubes and their descendants. For a visual illustration of the subdivision and the corresponding cubes we refer to Figure 4.2.

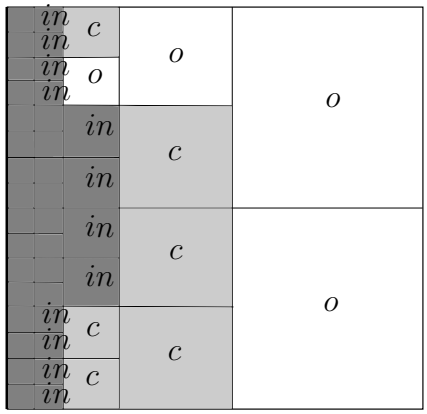


Figure 4.2: An example of a possible labeling of the cubes and of a corresponding subdivision of R. The outer region is white, while the central region is lightly shadowed and the inner region is shadowed. The labels o, c, and in identify rotationally invariant sets $R(L)$ corresponding to cubes L which are, respectively, outer, central, and inner cubes. Note that descendants of inner cubes *are not* inner cubes, even though the corresponding rotationally invariant regions are included in the inner region.

The following lemma will be pivotal to define our refined approximation.

Lemma 4.6.13. *Let T and $\mathbf{S}$ be as in Assumption 4.6.9 and assume the parameters $\bar{\delta}$, δ^*, τ, and ε satisfy Assumption 4.6.11 and the ratios $\frac{\varepsilon}{\bar{\delta}}$, $\frac{\bar{\delta}}{\tau}$, and $\frac{\tau}{\delta^*}$ are smaller than a constant $c = c(Q, m, n, \bar{n}) > 0$. Then*

(i) $L_0 \in \mathcal{G}^o$ and moreover, for every choice of τ and $\ell \in \mathbb{N}$ there is a constant $\bar{c} = \bar{c}(Q, m, n, \bar{n}, \tau, \ell) > 0$ such that, if $\varepsilon < \bar{c}$, then $\mathcal{G}_\ell \subset \mathcal{G}^o$.

(ii) For every $L \in \mathcal{G}^c$ there is an index $k(L) \in \{0, \dots, \bar{\kappa}\}$ such that

$$\mathbf{E}(L, k) \leq \tau^2 \mathbf{s}(k)^2 \tag{4.129}$$

for all $k \geq k(L)$ while for all $k < k(L)$ we have

$$\mathbf{E}(L,k) > \tau^2 \mathbf{s}(k)^2 \,. \tag{4.130}$$

(iii) For every $L \in \mathcal{G}^o$ we have that (4.129) holds for $k = 0$ (and thus for every k), whilst for every $L \in \mathcal{G}^{in}$ we have that (4.130) holds for every k.

(iv) There is a constant $\bar{C} = \bar{C}(Q, m, n, \bar{n}, \bar{\delta}, \delta^*, \tau) > 0$ such that

$$\mathbf{E}(L, k(L)) \leq \bar{C}\mathbf{E}(L,0) \qquad \forall L \in \mathcal{G}^c \tag{4.131}$$

$$1 \leq \bar{C}\mathbf{E}(L,0) \qquad \forall L \in \mathcal{G}^{in}. \tag{4.132}$$

(v) For every $L \in \mathcal{G}^o$, Proposition 4.6.6 is applicable to the current $T_{y_L, 2^{-\ell(L)}}$ and the cone $\mathbf{S}_0$, whilst for every $L \in \mathcal{G}^c$ either Proposition 4.6.6 or Proposition 4.6.8 is applicable to the current $T_{y_L, 2^{-\ell(L)}}$ and the cone $\mathbf{S}_{k(L)}$ (depending on whether $\mathbf{S}_{k(L)}$ consists of at least two planes or is a single plane).

Given Lemma 4.6.13 it is convenient to introduce the convention $k(L) = 0$ if $L \in \mathcal{G}^o$ and to set $\mathbf{E}(L) := \mathbf{E}(L, k(L))$. Note however that $k(L)$ might be indeed equal to 0 for several elements of $\mathcal{G}^c$ as well.

Proof. First of all, for any $L \in \mathcal{G}$ we have, since $\mathbf{B}^h(L) \subset \mathbf{B}_4$, that

$$\mathbf{E}(L,0) \leq C 2^{(m+2)\ell(L)} \mathbb{E}(T, \mathbf{S}, \mathbf{B}_4) \leq C 2^{(m+2)\ell(L)} \varepsilon^2 \mathbf{s}(0)^2 \,,$$

where the second inequality comes from (4.125), and where C depends only on m. In particular statement (i) in the Lemma follows immediately, as for any $L \in \mathcal{G}_\ell$ we have an upper bound on $\ell(L)$ and so choosing $C 2^{(m+2)\ell} \varepsilon^2 < \tau^2$ we get the desired inequality.

Regarding statement (ii), notice that

$$\mathbf{E}(L,k) \leq C\,\mathrm{dist}^2(\mathbf{S}_{k-1},\mathbf{S}_k) + C\mathbf{E}(L,k-1) \tag{4.133}$$

for each $k = 0,\dots,\bar{\kappa}$ and for some constant $C = C(m,n)$. Recalling that $\mathrm{dist}(\mathbf{S}_{k-1},\mathbf{S}_k) \leq \bar{\delta}\mathbf{s}(k)$ and that $\mathbf{s}(k-1) \leq \bar{\delta}\mathbf{s}(k)$ (by Lemma 4.6.3 (iii) and (iv)), we immediately get

$$\mathbf{E}(L,k) \leq C\left(1 + \frac{\mathbf{E}(L,k-1)}{\mathbf{s}(k-1)^2}\right)\bar{\delta}^2\mathbf{s}(k)^2\,.$$

In particular, if $\frac{\bar{\delta}}{\tau}$ is smaller than an appropriate geometric constant, we infer that the inequality $\mathbf{E}(L,k-1) \leq \tau^2\mathbf{s}(k-1)^2$ implies $\mathbf{E}(L,k) \leq \tau^2\mathbf{s}(k)^2$. But then statement (ii) holds if we let $k(L)$ be the smallest integer k for which $\mathbf{E}(L,k) \leq \tau^2\mathbf{s}(k)^2$, whose existence is guaranteed by the assumption that $L \in \mathcal{G}^c$. The same argument implies immediately the statement (iii) (as if $L \in \mathcal{G}^o$ then we know the minimum integer $k(L)$ above is $k(L) = 0$, whilst if $L \in \mathcal{G}^{in}$ the minimum over all k of the ratio $\mathbf{E}(L,k)/\mathbf{s}(k)^2$ is larger than τ^2).

Now recalling the second half of Lemma 4.6.3(iii), we have

$$\mathrm{dist}(\mathbf{S}_{k-1},\mathbf{S}_k) \leq \eta^{-1}\mathbf{s}(k-1)\,,$$

where η depends on N and $\bar{\delta}$. In particular, recalling (4.133), if $\mathbf{E}(L,k-1) > \tau^2\mathbf{s}(k-1)^2$ we conclude the inequality

$$\mathbf{E}(L,k) \leq \frac{C}{\eta^2}\mathbf{s}(k-1)^2 + C\mathbf{E}(L,k-1) \leq C(1+(\eta\tau)^{-2})\mathbf{E}(L,k-1)\,.$$

If $L \in \mathcal{G}^c$ we can apply the latter for $k = 1,\dots,k(L)$ to conclude (4.131). If $L \in \mathcal{G}^{in}$ we can apply it for every k, combined with (4.130), to conclude that

$$\mathbf{s}(\bar{\kappa})^2 \leq C\mathbf{E}(L,0)\,.$$

We now distinguish two cases:

- $I(\bar\kappa)$ consists of a single element; in this case $\mathbf{s}(\bar\kappa) = \bar\delta$.
- $I(\bar\kappa)$ consists of more than one element; in this case (from Lemma 4.6.3(ii) and the defining property (b) of $\bar\kappa$ in Section 4.6.5)

$$\mathbf{s}(\bar\kappa) \geq \eta \max_{i<j\in I(\bar\kappa)} d(\alpha_i \cap \mathbf{B}_1, \alpha_j \cap \mathbf{B}_1) \geq \eta\bar\delta\,.$$

In both cases we conclude that $\mathbf{s}(\bar\kappa)$ is bounded away from 0 and thus (4.132) holds, proving (iv).

As for (v), observe that for $L \in \mathcal{G}^o \cup \mathcal{G}^c$,

$$\int_{\mathbf{B}_4\setminus B_{\rho*}(V)} \mathrm{dist}^2(q, \mathbf{S}_{k(L)})\, d\|T_{y_L,2^{-\ell(L)}}\|(q) = \mathbf{E}(L, k(L)) \leq \tau^2 \mathbf{s}(k(L))^2\,,$$

where we have used (iii). Moreover, $\mathbf{s}(0) \leq \mathbf{s}(k)$ for all k and $\mathbf{A}^2 \leq \varepsilon^2\mathbf{s}(0)^2$ by (4.125). Hence, if we denote by $\mathbf{A}_L$ the supremum norm of the second fundamental form of the rescaled manifold $\Sigma_{y_L,2^{-\ell(L)}}$, we then have

$$\int_{\mathbf{B}_4\setminus B_{\rho*}(V)} \mathrm{dist}^2(q, \mathbf{S}_{k(L)})\, d\|T_{y_L,2^{-\ell(L)}}\|(q) + \mathbf{A}_L^2 \leq C(\tau^2 + 2^{-2\ell(L)}\varepsilon^2)\mathbf{s}(k)^2\,.$$

Recalling the definition of $\mathbf{s}(k)$ it suffices to guarantee that $C(\tau^2+\varepsilon^2) \leq (\delta^*)^2$ to guarantee that Proposition 4.6.6 is applicable in case $I(k)$ consists of more than one element. Otherwise we know that $\mathbf{s}(k) = \bar\delta$ and the same smallness condition guarantees the applicability of Proposition 4.6.8. This completes the proof. □

Local approximations

For each outer and central cube L, from Lemma 4.6.13(v) we apply Proposition 4.6.6 or Proposition 4.6.8 to the current $T_{y_L,2^{-\ell(L)}}$ and the cone $\mathbf{S}_{k(L)}$ (the choice of which proposition

to apply depends only on whether $|I(k(L))| > 1$ or not). We thus gain a corresponding Lipschitz approximation for $T_{y_L,2^{-\ell(L)}}$ on the planar domains $(\mathbf{B}_2 \setminus \overline{B}_{1/16}(V)) \cap \mathbf{S}_{k(L)}$. By translating and scaling back we gain corresponding Lipschitz approximations of the current T defined over the open domains $\Omega(L) := (\mathbf{B}_{2^{1-\ell(L)}}(y_L) \setminus \overline{B}_{2^{-4-\ell(L)}}(V)) \cap \mathbf{S}_{k(L)}$ of $\mathbf{S}_{k(L)}$.

We set $\Omega_i(L) := \Omega(L) \cap \alpha_i$ and denote by $u_{L,i}$ the corresponding multi-valued function given above and by $Q_{L,i}$ the number of values $u_{L,i}$ takes. We moreover denote by $\mathbf{\Omega}_i(L)$ the sets $2^{-\ell(L)}\mathbf{\Omega}_i + y_L$ and by $K_i(L)$ the sets $2^{-\ell(L)}K_{L,i} + y_L$, with $\mathbf{\Omega}_i$ and $K_{L,i}$ given by Proposition 4.6.6 (or Proposition 4.6.8) for $T_{y_L,2^{-\ell(L)}}$. These objects are defined only for the indices i belonging to the collection $I(k(L))$, but we may extend our notation to allow also the case $Q_{L,i} = 0$, in which the map $u_{L,i}$ does not exist (these will be the indices of planes not contained in $\mathbf{S}_{k(L)}$); this results in a slight abuse of our terminology and notation. The collection of corresponding multi-valued functions (ranging over all i) corresponding to a given L will be denoted by u_L, and we will call them "local approximations of T related to L". The following is the key proposition detailing the refined approximation.

Proposition 4.6.14 (Refined approximation)**.** *Let T, Σ and $\mathbf{S}$ be as in Assumption 4.6.9, and suppose that the assumptions of Lemma 4.6.13 hold. There exists $1 < \lambda = \lambda(m) \leq \frac{3}{2}$ and $\bar{C} = \bar{C}(Q, m, n, \bar{n}, \delta^*) > 0$, not depending on $\bar{\delta}$, τ, and ε, such that the local approximations $u_{L,i}$ satisfy the following properties.*

(i) For every fixed $i \in \{1, \ldots, N\}$, $Q_{L,i}$ is the same positive *integer for every $L \in \mathcal{G}^o$.*

(ii) $\sum_i Q_{L,i} = Q$ for every $L \in \mathcal{G}^o \cup \mathcal{G}^c$.

(iii) For every $L \in \mathcal{G}^o \cup \mathcal{G}^c$ we have $\mathrm{spt}(T) \cap \lambda R(L) \subset \bigcup_i \mathbf{\Omega}_i(L)$ *and*

$$2^{2\ell(L)}|q - \mathbf{p}_{\alpha_i}(q)|^2 \leq \bar{C}(\mathbf{E}(L) + 2^{-2\ell(L)}\mathbf{A}^2) \qquad \forall q \in \mathrm{spt}(T) \cap \mathbf{\Omega}_i(L)\,; \tag{4.134}$$

(iv) For $L \in \mathcal{G}^o \cup \mathcal{G}^c$, if we set

$$T_{L,i} := T \llcorner \mathbf{\Omega}_i(L) \cap \left\{ \operatorname{dist}(\cdot, \alpha_i) < \bar{C} 2^{-\ell(L)} (\mathbf{E}(L) + 2^{-2\ell(L)} \mathbf{A}^2)^{1/2} \right\}$$

(with $\bar{C}$ larger than the constant in the estimate (4.134)), then, for $K_i(L) = 2^{-\ell(L)} K_i + y_L \subset \Omega_i(L)$ as above,

$$T_{L,i} \llcorner \mathbf{p}_{\alpha_i}^{-1}(K_i(L)) = \mathbf{G}_{u_{L,i}} \llcorner \mathbf{p}_{\alpha_i}^{-1}(K_i(L)) \,,$$

$\operatorname{gr}(u_{L,i}) \subset \Sigma$, *and the following estimates hold:*

$$2^{2\ell(L)} \|u_{L,i}\|_\infty^2 + 2^{m\ell(L)} \|Du_{L,i}\|_{L^2}^2 \leq \bar{C} (\mathbf{E}(L) + 2^{-2\ell(L)} \mathbf{A}^2) \tag{4.135}$$

$$\operatorname{Lip}(u_{L,i}) \leq C (\mathbf{E}(L) + 2^{-2\ell(L)} \mathbf{A}^2)^\gamma \tag{4.136}$$

$$|\Omega_i(L) \setminus K_i(L)| + \|T_{L,i}\|(\mathbf{\Omega}_i(L) \setminus \mathbf{p}_{\alpha_i}^{-1}(K_i(L))) \leq \bar{C} 2^{-m\ell(L)} (\mathbf{E}(L) + 2^{-2\ell(L)} \mathbf{A}^2)^{1+\gamma} \,. \tag{4.137}$$

(v) For every $L \in \mathcal{G}^o \cup \mathcal{G}^c$, $\Theta(T, \cdot) \leq \max_i Q_{L,i} + \frac{1}{2}$ on $R(L)$. In particular, $\Theta(T, \cdot) \leq Q - \frac{1}{2}$ on $R(L)$ if $L \in \mathcal{G}^o$.

We will prove Proposition 4.6.14 in tandem with the next lemma. Consider a domain $U \subset \lambda R(L)$ which is invariant under rotations around V and set $U_i := \alpha_i \cap U$. We wish to replace $T \llcorner U$ with the union of the portions of the graphs of the functions $u_{L,i}$ lying over U_i. This will generate errors of two types. One type is due to the fact that $\operatorname{spt}(T) \cap U$ is not completely contained in the union of the graphs of the functions $u_{L,i}$: this will be taken care of by Proposition 4.6.14 above. A second type of error is due to the fact that, even though the estimate (4.134) holds, if we define

$$\tilde{U} := \bigcup_i \{ q : \mathbf{p}_{\alpha_i}(q) \in U_i \quad \text{and} \quad |q - \mathbf{p}_{\alpha_i}(q)| \leq \bar{C} 2^{-\ell(L)} (\mathbf{E}(L) + 2^{-2\ell(L)} \mathbf{A}^2)^{1/2} \} \tag{4.138}$$

(where $\bar{C}$ is assumed to be the constant of estimate (4.134)), there still is a difference between $\mathrm{spt}(T) \cap \tilde{U}$ and $\mathrm{spt}(T) \cap U$. The purpose of the following lemma is to estimate errors of this second type:

Lemma 4.6.15. *Under the assumptions of Lemma 4.6.13, consider $L \in \mathcal{G}^o \cup \mathcal{G}^c$, let $U \subset \lambda R(L)$ be a set invariant under rotations around V whose cross-sections $U_i = U \cap \alpha_i$ are Lipschitz open sets or the closures of Lipschitz open sets, and define $\tilde{U}$ as in (4.138). Then*

$$\begin{aligned} \|T\|(U \setminus \tilde{U}) + \|T\|(\tilde{U} \setminus U) \leq \bar{C} \left(\mathbf{E}(L) + 2^{-2\ell(L)}\mathbf{A}^2\right) 2^{-\ell(L)} \mathcal{H}^{m-1}(\partial U_i) \\ + \bar{C} 2^{-m\ell(L)} \left(\mathbf{E}(L) + 2^{-2\ell(L)}\mathbf{A}^2\right)^{1+\gamma} \end{aligned} \tag{4.139}$$

where the constant $\bar{C}$ depends on the parameters $m, n, \bar{n}, Q, \bar{\delta}$, and the Lipschitz regularity of the boundary of the rescaled cross-section $2^{\ell(L)} U_i$.

Remark 4.6.16. We will in fact only apply this Lemma to sets U whose cross-sections $U_i = U \cap \alpha_i$ have a limited number of shapes, up to the rescaling factor $2^{-\ell(L)}$. In particular, in these cases the corresponding constant $\bar{C}$ in the estimate depends only upon $m, n, \bar{n}, Q$, and $\bar{\delta}$.

Proof of Proposition 4.6.14 and of Lemma 4.6.15. We will prove the two statements at the same time. We start by showing that the estimates of the statements (iii) and (iv) of Proposition 4.6.14 hold. Fix $L \in \mathcal{G}^o \cup \mathcal{G}^c$, let $k = k(L)$, and $\ell = \ell(L)$. In order to simplify our notation we write $T' := T_{y_L, 2^{-\ell}}$ and $\Sigma' := \Sigma_{y_L, 2^{-\ell}}$. The supremum norm of the second fundamental form of Σ' is clearly $2^{-\ell}\mathbf{A}$. On the other hand, by scaling invariance,

$$\int_{\mathbf{B}_4 \setminus B_{\rho_*}(V)} \mathrm{dist}^2(q, \mathbf{S}_k)\, d\|T'\|(q) = \mathbf{E}(L)\,. \tag{4.140}$$

Let us assume that $I(k)$ consists of more than one element; in particular, by Lemma 4.6.13(v), we can apply Proposition 4.6.6 to T', Σ' and $\mathbf{S}_k$. The argument is entirely analogous when $I(k)$ consists of one element except instead we apply Proposition 4.6.8, and so we will just

focus on the former case. We consider the domains $\mathbf{\Omega}_i$ given by Proposition 4.6.6 and recall that from Lemma 4.6.5 it follows that

$$\mathrm{spt}(T') \cap \mathbf{B}_3 \setminus B_{1/16}(V) \subset \bigcup_i \mathbf{\Omega}_i \,,$$

In turn this rescales to the statement

$$\mathrm{spt}(T) \cap \mathbf{B}_{3\cdot 2^{-\ell}}(y_L) \setminus B_{2^{-\ell-4}}(V) \subset \bigcup_i \mathbf{\Omega}_i(L) \,.$$

For an appropriate choice of the constant λ (i.e. sufficiently close to 1),

$$\lambda R(L) \subset \mathbf{B}_{3\cdot 2^{-\ell}}(y_L) \setminus B_{2^{-\ell-4}}(V)$$

and so the first claim of Proposition 4.6.14(iii) follows. As for the height estimate in (iii), Proposition 4.6.6(c) and (4.140) gives

$$|q - \mathbf{p}_{\alpha_i}(q)|^2 \leq C(\mathbf{E}(L) + 2^{-2\ell}\mathbf{A}^2) \qquad \forall q \in \mathbf{\Omega}_i \cap \mathrm{spt}(T')$$

which scales to

$$|q - \mathbf{p}_{\alpha_i}(q)|^2 \leq C2^{-2\ell}(\mathbf{E}(L) + 2^{-2\ell}\mathbf{A}^2) \qquad \forall q \in \mathbf{\Omega}_i(L) \cap \mathrm{spt}(T) \,,$$

i.e. (4.134). This proves (iii).

The first two claims of Proposition 4.6.14(iv) follow directly from the corresponding claims in Proposition 4.6.6. As for the estimates, if we denote by u_i and K_i the multi-valued functions approximating $T'_i := T' \llcorner \mathbf{\Omega}_i \cap W_i$ and the corresponding "coincidence sets" given by Proposition 4.6.6, then, taking into account (4.140), we have the estimates

$$\|u_i\|^2_\infty + \|Du_i\|^2_{L^2} \leq C(\mathbf{E}(L) + 2^{-2\ell}\mathbf{A}^2)$$

$$\mathrm{Lip}\,(u_i) \leq C(\mathbf{E}(L) + 2^{-2\ell}\mathbf{A}^2)^{\gamma}$$

$$|\Omega_i \setminus K_i| + \|T_i'\|(\mathbf{\Omega}_i \setminus \mathbf{p}_{\alpha_i}^{-1}(K_i)) \leq C(\mathbf{E}(L) + 2^{-2\ell}\mathbf{A}^2)^{1+\gamma}\,.$$

The estimates in Proposition 4.6.14(iv) therefore follow from the obvious scaling relations

$$\|u_i\|_\infty = 2^\ell \|u_{L,i}\|_\infty$$

$$\|Du_i\|_\infty = \|Du_{L,i}\|_\infty$$

$$\|Du_i\|_{L^2} = 2^{m\ell/2}\|Du_{L,i}\|_{L^2}$$

$$|\Omega_i \setminus K_i| = 2^{m\ell}|\Omega_i(L) \setminus K_i(L)|$$

$$\|T_i'\|(\mathbf{p}_{\alpha_i}^{-1}(\Omega_i \setminus K_i)) = 2^{m\ell}\|T_{L,i}\|(\mathbf{p}_{\alpha_i}^{-1}(\Omega_i(L) \setminus K_i(L)))$$

which completes the proof of (iv).

Next, we will prove Proposition 4.6.14(i). We claim that $Q_{L,i} = Q_{L',i}$ if $L, L' \in \mathcal{G}^o$ and L' is the parent of L; notice that since $L_0 \in \mathcal{G}^o$ by Lemma 4.6.13(i), this will hold for any $i \in I(0)$, and thus for every $i \in \{1, \ldots, N\}$. First, notice that because of (4.134), it is easy to see that $T_{L,i}$ and $T_{L',i}$ coincide over $\mathbf{\Omega}_i(L) \cap \mathbf{\Omega}_i(L')$. In particular, the two currents $\mathbf{G}_{u_{L,i}}$ and $\mathbf{G}_{u_{L',i}}$ coincide over $\mathbf{p}_{\alpha_i}^{-1}(K_i(L) \cap K_i(L'))$. Clearly by (iv), namely (4.137), $K_i(L) \cap K_i(L')$ has positive measure provided $\mathbf{E}(L)$, $\mathbf{E}(L')$, and $\mathbf{A}$ are smaller than a geometric constant, and all these conditions can be ensured by choosing τ and ε smaller than a geometric constant (here it is crucial that the constant C in the estimates of statement (iv) does not depend on τ and ε). But once we know that $K_i(L) \cap K_i(L')$ has positive measure and the current graphs coincide over this set we clearly conclude that $Q_{L,i} = Q_{L',i}$.

Given this, since every parent of an element in $\mathcal{G}^o$ belongs to $\mathcal{G}^o$ (by definition of $\mathcal{G}^o$) we conclude from the above that $Q_{L,i} = Q_{L_0,i}$, which obviously implies the first claim of Proposition 4.6.14(i). The fact that all of them are positive follows from Proposition 4.6.6(f) once we assume ε is sufficiently small, because it will force $Q_{L_0,i} \geq 1$ for every i. Thus (i) is proved.

Next, because of (i), the conclusion of point (ii) holds for $L \in \mathcal{G}^o$ once we show $\sum_i Q_{L_0,i} = Q$. For the latter we can use Lemma 4.6.7 with assumption (a), since $y_{L_0} = 0$ and $\{\Theta(T,\cdot) \geq Q\} \cap \mathbf{B}_\varepsilon(0) \neq \emptyset$.

In order to prove (ii) when $L \in \mathcal{G}^c$ we are again going to argue inductively, showing that:

(A) If $\sum_i Q_{L',i} = Q$ for $L' \in \mathcal{G}^c \cup \mathcal{G}^o$, then $\sum_i Q_{L,i} = Q$ for every child $L \in \mathcal{G}^c \cup \mathcal{G}^o$ of L'.

In order to show (A) we will in fact use the result of Lemma 4.6.15, which we shall prove now.

Fix $L \in \mathcal{G}^c \cup \mathcal{G}^o$ and a set $U \subset \lambda R(L)$ which is invariant under rotations around V. Firstly, from what has been proved so far of Proposition 4.6.14, namely (iii) and (iv), we can conclude that $T \llcorner (U \cup \tilde{U}) = \sum_i T_{L,i} \llcorner (U \cup \tilde{U})$ and $\mathrm{spt}(T_{L,i}) \cap \mathrm{spt}(T_{L,j}) = \emptyset$ for all $i < j$ (which follows from (iv) provided τ and ε are sufficiently small, as then the neighbourhoods of the α_i in (iv) are all disjoint, and from the choice of λ, which ensures that $U \cap \tilde{U} \cap \mathrm{spt}(T)$ is contained in the union of $\mathbf{\Omega}_i(L)$). In particular, we have

$$\|T\|(U \Delta \tilde{U}) = \sum_i \|T_{L,i}\|(U \Delta \tilde{U})$$

where $U \Delta \tilde{U} := (U \setminus \tilde{U}) \cup (\tilde{U} \setminus U)$ is the symmetric difference of U and $\tilde{U}$. Thus we can again use the conclusion (iv) in Proposition 4.6.14 to estimate further

$$\|T\|(U \Delta \tilde{U}) \leq \sum_i \|\mathbf{G}_{u_{L,i}}\|(U \Delta \tilde{U}) + C 2^{-m\ell(L)} (\mathbf{E}(L) + 2^{-2\ell(L)} \mathbf{A}^2)^{1+\gamma} .$$

Consider now the set

$$\Delta_i := \mathbf{p}_{\alpha_i}(\mathrm{gr}\,(u_{L,i}) \cap (U \Delta \tilde{U}))$$

Since by choosing τ and ε small enough we can assume $\mathrm{Lip}(u_{L,i}) \leq 1$ (by (4.136)), it follows immediately from this this Lipschitz regularity that

$$\|\mathbf{G}_{u_{L,i}}\|(U \Delta \tilde{U}) \leq C Q_{L,i} |\Delta_i|$$

for some geometric constant C. Since $\sum_i Q_{L,i} \leq Q$ (note in particular that we do not need the equality $\sum_i Q_{L,i} = Q$ in this argument), in order to reach the estimate of Lemma 4.6.15 it suffices to show that

$$|\Delta_i| \leq C(\mathbf{E}(L) + 2^{-2\ell(L)}\mathbf{A}^2)2^{-\ell(L)}\mathcal{H}^{m-1}(\partial U_i)\,, \tag{4.141}$$

where C has the dependencies in the statement of the lemma (recall that $U_i = U \cap \alpha_i$). We will indeed show that

$$\operatorname{dist}(q, \partial U_i) \leq C2^{-\ell(L)}(\mathbf{E}(L) + 2^{-2\ell(L)}\mathbf{A}^2) \qquad \forall q \in \Delta_i\,.$$

In particular, using the Lipschitz regularity of ∂U_i, (4.141) follows immediately (by taking a suitable cover of ∂U_i and taking the tubular neighbourhood of each set in this cover with radius 2 times the above distance bound to cover Δ_i). In order to show (4.141), fix a point $x \in \Delta_i$ and observe that by definition there must be a point $p \in \operatorname{gr}(u_{L,i}) \cap (U \Delta \tilde{U})$ such that $x = \mathbf{p}_{\alpha_i}(p)$. Let $v = \mathbf{p}_V(p) = \mathbf{p}_V(x)$ and denote by σ the half-line in α_i which originates at v and contains x. On this half-line we denote by y the point such that

$$\operatorname{dist}(y, V) = |y - v| = \operatorname{dist}(p, V)\,, \tag{4.142}$$

and moreover as all points on this half-line project to v, we have

$$\mathbf{p}_V(y) = \mathbf{p}_V(p)\,. \tag{4.143}$$

Observe that $p \in U$ if and only if $y \in U$ (using that U is invariant under rotations about V and (4.142), (4.143)), which happens if and only if $y \in U_i$. On the other hand $p \in \tilde{U}$ if and only if $x = \mathbf{p}_{\alpha_i}(p) \in U_i$. Therefore one of the following two alternatives hold:

- $p \in U \setminus \tilde{U}$, and hence $y \in U_i$ but $x \notin U_i$

- $p \in \tilde{U} \setminus U$, and hence $x \in U_i$ but $y \notin U_i$.

In both cases the segment σ joining x and y must contain a point of ∂U_i and thus $\operatorname{dist}(x, \partial U_i) \leq |x-y|$. We therefore wish to estimate the latter.

To that end consider the triangle with vertices v, p, and x and let θ be the angle at v. We then have

$$|x-y| = |p-v|(1-\cos\theta) \leq C2^{-\ell(L)}(1-\cos\theta) \tag{4.144}$$

$$\sin\theta = \frac{|p-x|}{|p-v|} \leq C(\mathbf{E}(L) + 2^{-2\ell(L)}\mathbf{A}^2)^{1/2}\,. \tag{4.145}$$

Here, the upper bound on $|p-v|$ comes from (4.135) and the upper bound on the distance of U_i from V, while the lower bound on $|p-v|$ also comes from the lower bound on the distance of U_i to V, and the upper bound on $|p-x|$ comes from (4.135) (this is using that $p \in \operatorname{gr}(u_{L,i})$, but also holds more generally when $p \in \operatorname{spt}(T_{L,i})$). In particular as (4.145) tells us that θ is small (e.g. $\theta < \pi/3$ suffices), we get $\sin(\theta) \geq \theta/2$; as $1-\cos(\theta) \leq \theta^2/2$ is always true, combining these with (4.144) and (4.145) we establish the desired bound on $|x-y|$, hence completing the proof of (4.141) and so also the proof of Lemma 4.6.15.

Having proved Lemma 4.6.15 we now come to the proof of (A). Consider $L, L' \in \mathcal{G}^c \cup \mathcal{G}^o$, with L a child of L'. Consider the set $U := \lambda R(L) \cap \lambda R(L')$; clearly U is invariant under rotations around V. Under the assumption that $\sum_i Q_{L',i} = Q$, we can use Lemma 4.6.15 to prove

$$\|T\|(U) \geq (Q - \tfrac{1}{2})\mathcal{H}^m(U_1)\,. \tag{4.146}$$

for $U_1 := U \cap \alpha_1$ as before. Indeed, introduce the set $\tilde{U}$ for this choice of U as in Lemma 4.6.15 and note that

$$\|T\|(\tilde{U}) = \sum_i \|T_{L',i}\|(\tilde{U}) \geq \sum_i Q_{L',i}\mathcal{H}^m(U_i) = Q\mathcal{H}^m(U_1)$$

(observe that the set $\tilde{U}$ is formed by cylindrical domains with cross-sections U_i, and in

particular $(\mathbf{p}_{\alpha_i})_\sharp(T_{L',i}) \llcorner \tilde{U}) = Q_{L',i}[\![U_i]\!]$.

We can now use Lemma 4.6.15 to estimate

$$\|T\|(U) \geq \|T\|(\tilde{U}) - \|T\|(\tilde{U} \setminus U) \geq Q\mathcal{H}^m(U_1) - C2^{-\ell(L)}(\mathbf{E}(L) + 2^{-2\ell(L)}\mathbf{A}^2)\mathcal{H}^{m-1}(\partial U_1)\,.$$

Note that in principle the constant C appearing in the estimate depends on the region U (which in turn depends on L and L'), however the latter is determined by the cross-section which, after rescaling by $2^{\ell(L)}$ and translating, is the intersection of two shapes ranging in a finite number of possibilities (the number of which depends on λ). In particular, by Remark 4.6.16, we can assume that the constant C depends only on the constants m, n, $\bar{n}$, Q, $\bar{\delta}$ and λ (note however that we have already fixed $\lambda = \lambda(m)$). On the other hand we also get $2^{-\ell(L)}\mathcal{H}^{m-1}(\partial U_1) \leq C\mathcal{H}^m(U_1)$, with a constant C depending on m and λ. We thus conclude

$$\|T\|(U) \geq (Q - C(\mathbf{E}(L) + 2^{-2\ell(L)}\mathbf{A}^2))\mathcal{H}^m(U_1) \geq (Q - C(\tau^2 + \varepsilon^2))\mathcal{H}^m(U_1)\,.$$

Since the constant C is independent of both τ and ε, an appropriate smallness condition on these two parameters guarantees the validity of (4.146).

If we now consider the current $T_{y_L,2^{-\ell(L)}}$, the corresponding rescaled set $\Omega = 2^{\ell(L)}(U - y_L)$ satisfies requirement (b) of Lemma 4.6.7, which implies that $\sum_i Q_{L,i} = Q$, if τ is sufficiently small. While it is true that Lemma 4.6.7 imposes a smallness condition on τ which depends on the set Ω, it is easy to see that the latter varies among a fixed number of sets (since L is a child of L'), which depend on the relative position of L compared to L' because once λ is fixed, $\lambda R(L) \cap \lambda R(L')$ overlap for a fixed number of cubes L, L' always. In particular, there is a choice of smallness condition on τ which ensures the applicability of the lemma for every pair of cubes L and L' as in (A). Therefore to finish the proof, note that as (ii) holds for $\mathcal{G}^o$, it follows by induction from (A) that (ii) holds for every $L \in \mathcal{G}^c \cup \mathcal{G}^o$, as every cube L in this set is a descendent of a cube in $\mathcal{G}^o$, while their ancestors are all in $\mathcal{G}^c \cup \mathcal{G}^o$. So we are done with this part.

We finally prove (v). Fix a point $p \in R(L) \cap \operatorname{spt}(T)$ and note that it must belong to $\operatorname{spt}(T_{L,i})$ for some i. For $\rho = (\lambda - 1)\frac{2^{-\ell(L)-1}}{\sqrt{m-2}}$ we estimate

$$\begin{aligned}\|T_i\|(\mathbf{B}_\rho(p)) \leq \|T_i\|(\mathbf{C}_\rho(p,\pi_i)) &\leq Q_{L,i}\rho^m + C(\mathbf{E}(L) + 2^{-2\ell(L)}\mathbf{A}^2)^{1+\gamma}2^{-m\ell(L)} \\ &\leq (Q_{L,i} + C(\mathbf{E}(L) + 2^{-2\ell(L)}\mathbf{A}^2)^{1+\gamma})\rho^m\,.\end{aligned}$$

Then if ε and τ are small enough, we conclude the claim from (4.129) and the monotonicity formula. □

4.6.6 Coherent approximation on the outer region and first blow-up

For technical reasons, it is useful to have a single multi-valued approximation defined over the union of all L_i for L varying among the elements of $\mathcal{G}^o$. Proposition 4.6.18 below gives a precise statement of this, but we first introduce some notation.

Definition 4.6.17. *Let T, Σ, and $\mathbf{S}$ be as in Proposition 4.6.14. Fix $L \in \mathcal{G}^o$. We denote by $\mathscr{N}(L)$ the set of $L' \in \mathcal{G}^o$ such that $R(L) \cap R(L') \neq \emptyset$, and let*

$$\bar{\mathbf{E}}(L) := \max\{\mathbf{E}(L') : L' \in \mathscr{N}(L)\}\,.$$

Proposition 4.6.18 (Coherent outer approximation)**.** *Let T and $\mathbf{S}$ be as in Proposition 4.6.14. For every $i \in \{1,\dots,N\}$ we define*

$$R_i^o := \bigcup_{L\in\mathcal{G}^o} L_i \equiv \alpha_i \cap \bigcup_{L\in\mathcal{G}^o} R(L)$$

and let $Q_i := Q_{L_0,i}$. Then, there are Lipschitz multi-valued maps $u_i : R_i^o \to \mathcal{A}_{Q_i}(\alpha_i^\perp)$ and closed subsets $\bar{K}_i(L) \subset L_i$ satisfying the following properties.

(i) $\operatorname{gr}(u_i) \subset \Sigma$ *and* $T_{L,i} \llcorner \mathbf{p}_{\alpha_i}^{-1}(\bar{K}_i(L)) = \mathbf{G}_{u_i} \llcorner \mathbf{p}_{\alpha_i}^{-1}(\bar{K}_i(L))$ *for every $L \in \mathcal{G}^o$, where $T_{L,i}$ are*

defined as in Proposition 4.6.14;

(ii) The following estimates hold

$$2^{2\ell(L)}\|u_i\|^2_{L^\infty(L_i)} + 2^{m\ell(L)}\|Du_i\|^2_{L^2(L_i)} \le C(\bar{\mathbf{E}}(L) + 2^{-2\ell(L)}\mathbf{A}^2) \tag{4.147}$$

$$\|Du_i\|_{L^\infty(L_i)} \le C(\bar{\mathbf{E}}(L) + 2^{-2\ell(L)}\mathbf{A}^2)^\gamma \tag{4.148}$$

$$|L_i \setminus \bar{K}_i(L)| + \|T_{L,i}\|(\mathbf{p}_{\alpha_i}^{-1}(L_i \setminus \bar{K}_i(L))) \le C2^{-m\ell(L)}(\bar{\mathbf{E}}(L) + 2^{-2\ell(L)}\mathbf{A}^2)^{1+\gamma}\,, \tag{4.149}$$

where the constant C depends only upon $Q, m, n, \bar{n}$, and $\bar{\delta}$.

Proof. The ideas of the proof are borrowed from [18, Section 1.2.2] and [21, Section 6.2]. On the one hand there is a slight complication compared to the arguments borrowed from [18, Section 1.2.2] due to the fact that the regions $R(L)$ are not cubes; on other hand compared to the arguments borrowed from [21, Section 6.2] our situation is considerably simpler.

First observe that, since $L_i = R(L) \cap \alpha_i$, the set $\mathscr{N}(L)$ is equivalently described as those cubes $L' \in \mathcal{G}^o$ such that $L_i \cap L'_i \neq \emptyset$ for some i. In fact, given the invariance of $R(L)$ under rotations around V, if this is true for some i then it is true for all i, and so we can fix an arbitrary i. Notice that

(a) the cardinality of $\mathscr{N}(L)$ is bounded by a dimensional constant;

(b) $|\ell(L') - \ell(L)| \le 1$ for every $L' \in \mathscr{N}(L)$.

Indeed, (a) follows by the construction of the collection of cubes $\mathcal{G}$ and their associated sets $R(L)$, while (b) is a consequence of Lemma 4.6.10(i). For each cube $L \in \mathcal{G}^o$, consider the approximations $u_{L,i}$ and the coincidence sets $K_i(L)$ given by Proposition 4.6.14 and define

$$\bar{K}_i(L) := \bigcap_{L' \in \mathscr{N}(L)} K_i(L')\,.$$

Observe that for every $L' \in \mathscr{N}(L)$ we have $u_{L,i} = u_{L',i}$ on $\bar{K}_i(L)$. Note that

$$|L_i \setminus \bar{K}_i(L)| \leq \sum_{L' \in \mathscr{N}(L)} |\Omega_i(L') \setminus K_i(L')|$$

and

$$\begin{aligned}
&\|T_{L,i}\|(\mathbf{p}_{\alpha_i}^{-1}(L_i \setminus \bar{K}_i(L))) \\
&\qquad \leq \|T_{L,i}\|(\mathbf{p}_{\alpha_i}^{-1}(L_i \setminus K_i(L))) + \sum_{L' \in \mathscr{N}(L) \setminus \{L\}} \|T_{L,i}\|(\mathbf{p}_{\alpha_i}^{-1}(L_i \setminus K_i(L'))) \\
&\qquad \leq \|T_{L,i}\|(\mathbf{p}_{\alpha_i}^{-1}(L_i \setminus K_i(L))) + \sum_{L' \in \mathscr{N}(L) \setminus \{L\}} \|T_{L',i}\|(\mathbf{\Omega}_i(L') \setminus \mathbf{p}_{\alpha_i}^{-1}(K_i(L'))) \,,
\end{aligned}$$

and thus (4.149) follows from Proposition 4.6.14(iv) and (a) and (b) above.

Note that we may define $u_i := u_{L,i}$ on $\bar{K}_i(L)$, which is clearly well-defined. To complete the proof we need to extend u_i from $\bigcup_{L \in \mathcal{G}^o} \bar{K}_i(L)$ to a Lipschitz map on $R_i^o := \bigcup_{L \in \mathcal{G}^o} L_i$ which satisfies $\operatorname{gr}(u_i) \subset \Sigma$ and the estimates (4.147) and (4.148). Recall that we have the estimates

$$\|u_i\|_{L^\infty(\bar{K}_i(L))} \leq C 2^{-\ell(L)} (\mathbf{E}(L) + 2^{-2\ell(L)} \mathbf{A}^2)^{1/2} \tag{4.150}$$

$$\operatorname{Lip}(u_i|_{\bar{K}_i(L)}) \leq C (\mathbf{E}(L) + 2^{-2\ell(L)} \mathbf{A}^2)^{\gamma} \,, \tag{4.151}$$

by Proposition 4.6.14(iv), just because on the set $\bar{K}_i(L)$ the map u_i coincides with $u_{L,i}$.

Observe moreover that, if we consider the larger domains $\tilde{K}_i(L) := \bigcup_{L' \in \mathscr{N}(L)} \bar{K}_i(L)$, we still have estimates analogous to (4.150), (4.151), with $\bar{\mathbf{E}}(L)$ replacing $\mathbf{E}(L)$, namely

$$\|u_i\|_{L^\infty(\tilde{K}_i(L))} \leq C 2^{-\ell(L)} (\bar{\mathbf{E}}(L) + 2^{-2\ell(L)} \mathbf{A}^2)^{1/2} \tag{4.152}$$

$$\operatorname{Lip}(u_i|_{\tilde{K}_i(L)}) \leq C (\bar{\mathbf{E}}(L) + 2^{-2\ell(L)} \mathbf{A}^2)^{\gamma} \,, \tag{4.153}$$

Our aim is to show that we can find an extension u_i to $\bigcup_{L \in \mathcal{G}^o} L_i$ and use (4.152), (4.153) to

show that for each $L \in \mathcal{G}^o$ this extension satisfies

$$\|u_i\|_{L^\infty(L_i)} \leq C2^{-\ell(L)}(\bar{\mathbf{E}}(L) + 2^{-2\ell(L)}\mathbf{A}^2)^{1/2} \tag{4.154}$$

$$\mathrm{Lip}(u_i|_{L_i}) \leq C(\bar{\mathbf{E}}(L) + 2^{-2\ell(L)}\mathbf{A}^2)^{\gamma}\,, \tag{4.155}$$

The remaining claim, namely the L^2 bound on Du_i over L_i claimed in point (ii) of the proposition, is then an obvious consequence of (4.151), the bound on $|L_i \setminus \bar{K}_i(L)|$, and the fact that

$$\|Du_i\|^2_{L^2}(\bar{K}_i(L)) \leq C2^{-m\ell(L)}(\mathbf{E}(L) + 2^{-2\ell(L)}\mathbf{A}^2)\,,$$

the latter being again a consequence of $u_i|_{\bar{K}_i(L)} = u_{L,i}$.

In order to accomplish the latter task we first observe that we can ignore the requirement that $\mathrm{gr}\,(u_i) \subset \Sigma$. Indeed, fix the $\bar{n}$-dimensional subspace $\pi = \alpha_i^\perp \cap T_0\Sigma$ which is the orthogonal complement of α_i in $T_0\Sigma$ and let $\Psi : \mathbf{B}_7 \cap T_0\Sigma \to T_0\Sigma^\perp$ be the map whose graph describes Σ. Recall that $\|D^2\Psi\|_{C^0} \leq C\mathbf{A}$ and, since $D\Psi(0) = 0$, we conclude $\|D\Psi\|_{C^0} \leq C\mathbf{A}$ as well. It thus suffices to find an extension of the π-component u_i^π of the map u_i with the desired estimate and compose it with Ψ in the remaining components of $\alpha_i^\perp$ to find the desired extension of u_i (the formula for the latter map would then be $x \mapsto (u_i^\pi(x), \Psi(x, u_i^\pi(x))) \in \pi \times T_0\Sigma^\perp$).

Once we are allowed to ignore the above issue, we consider a cellular decomposition of the L_i's into 0-cells (the 0-skeleton), 1-cells attached to the 0-cells (the 1-skeleton), 2-cells, and so on, with the final m-dimensional cells being the interiors of the L_i's. This can be done canonically across all L, note indeed that each L_i is the product, in $V \times (V^\perp \cap \alpha_i)$, of the cube $L \subset V$ with the annulus $\{y \in V^\perp \cap \alpha_i : 2^{-\ell(L)-1} \leq |y| \leq 2^{-\ell(L)}\}$.

We denote by $\mathcal{S}_i$ the collection of i-cells in the above. We also slightly fatten each 0-cell p to open neighborhoods U^p so that the separation between any U^p and U^q with $p, q \in L_i$ is at least $c2^{-\ell(L)}$ for some dimensional constant $c > 0$. Now, for every 1-cell σ with endpoints $p, q \in \mathcal{S}_0$, we slightly fatten $\sigma \setminus (U^p \cup U^q)$ to an open set U^σ and again we take care that the

separation between two fattenings U^σ and U^τ for distinct 1-cells σ and τ contained in the same L_i is at least $c2^{-\ell(L)}$. We proceed in this way over all skeleta. See [21, Figure 1] for a graphical illustration of this.

We then find a Lipschitz extension of the maps u_i to the points 0-skeleton in the following fashion:

For each point P in the 0-skeleton, we let $N(P)$ be the union of L_i' for those cubes $L' \in \mathcal{G}^o$ which contain P. The extension is then done using [18, Theorem 1.7], which ensures that, after setting $K := \bigcup_{L\in\mathcal{G}^o} \bar{K}_i(L)$

$$\mathrm{Lip}(u_i|_{(K\cap N(P))\cup U^p}) \leq C\mathrm{Lip}(u_i|_{K\cap N(P)}) \tag{4.156}$$

$$\|u_i\|_{L^\infty((K\cap N(P))\cup U^p)} \leq C\|u_i\|_{L^\infty(K\cap N(P))}\,. \tag{4.157}$$

To extend to the 1-cells fix any $\sigma \in \mathcal{S}_1$ contained in some L_i with endpoints p and q, and let $N(\sigma)$ be the union of all the L_i' which intersect σ. Observe that, because U^p and U^q are far enough apart from one another, using (4.156) and (4.157) we get the estimates

$$\begin{aligned}
\mathrm{Lip}(u_i|_{(K\cap N(\sigma))\cup(U^p\cup U^q)}) &\leq C\mathrm{Lip}(u_i|_{K\cap N(\sigma)}) + C2^{-\ell(L)}\|u_i\|_{L^\infty(K\cap N(\sigma))}\\
\|u_i\|_{L^\infty((K\cap N(\sigma))\cup(U^p\cup U^q))} &\leq C\|u_i\|_{L^\infty(K\cap N(\sigma))}\,.
\end{aligned}$$

We then proceed and define u_i on each U^σ *separately* so that (again, using [18, Theorem 1.7])

$$\begin{aligned}
\mathrm{Lip}(u_i|_{(K\cap N(\sigma))\cup(U^p\cup U^q\cup U^\sigma)}) &\leq C\mathrm{Lip}(u_i|_{(K\cap N(\sigma))\cup(U^p\cup U^q)})\\
\|u_i\|_{L^\infty((K\cap N(\sigma))\cup(U^p\cup U^q\cup U^\sigma))} &\leq C\|u_i\|_{L^\infty((K\cap N(\sigma))\cup(U^q\cup U^p))}\,.
\end{aligned}$$

We proceed inductively in this fashion and after going over all the skeleta we finally arrive

at an extension which, defining $N(L_i)$ as the union of the L_i' over $L' \in \mathscr{N}(L)$, satisfies

$$\begin{aligned}
&\mathrm{Lip}(u|_{(K\cap N(L_i))\cup L_i)}) \leq C\mathrm{Lip}(u|_{K\cap N(L_i)}) + C2^{-\ell(L)}\|u_i\|_{L^\infty(K\cap N(L_i))}\,.\\
&\|u_i\|_{L^\infty((K\cap N(L_i))\cup L_i)} \leq C\|u_i\|_{L^\infty(K\cap N(L_i))}\,.
\end{aligned}$$

Given that $K\cap N(L_i) = \tilde{K}_i(L)$, we conclude the desired bounds from (4.152) and (4.153). □

4.6.7 Blow-up

We have thus constructed a collection of N multi-valued approximations on the outer region. From the estimates of the previous proposition and a suitable covering argument it is easy to see that, away from the spine V, their Dirichlet energies are controlled by the conical excess $\hat{\mathbf{E}}(T, \mathbf{S}, \mathbf{B}_4)$. Our next main goal is to show that, after we normalize them by $(\hat{\mathbf{E}}(T, \mathbf{S}, \mathbf{B}_4))^{1/2}$, they are close to a Dir-minimizing function. The control of the Dirichlet energy is still too crude for our final purpose, as it degenerates when we get closer to the spine: in order to gain a uniform control outside the spine we will need to use Simon's estimates; Section 4.9 will be devoted to this task. However, this "first blow-up" will be useful for two reasons: it is sufficient for the purpose of Section 4.7.1, where we "balance" the best approximating cone (recall the discussion at the beginning of Section 4.6), and it will ultimately be used in the final blow-up argument to prove Dir-minimality of the blow-up limit away from the spine. In the final blow-up argument, the better control on the Dirichlet energy will then be used to prove Dir-minimality across the spine, using the fact that $(m-2)$-dimensional subspaces have vanishing $W^{1,2}$-capacity.

Proposition 4.6.19 (First blow-up)**.** *Let T and $\mathbf{S}$ be as in Proposition 4.6.14 and assume the parameters δ^*, τ, and $\bar{\delta}$ are fixed. Then, for every $\sigma, \varsigma > 0$ there are constants $C = C(m, n, Q, \delta^*, \tau, \bar{\delta}) > 0$ and $\varepsilon = \varepsilon(m, n, Q, \delta^*, \tau, \bar{\delta}, \sigma, \varsigma) > 0$ such that the following holds:*

(i) $R \setminus B_\sigma(V)$ is contained in the outer region R^o (recall R is as in (4.126));

(ii) If u_i are the maps of Proposition 4.6.18 and $R_i := (R \setminus B_\sigma(V)) \cap \alpha_i$ then

$$\int_{R_i} |Du_i|^2 \leq C\sigma^{-2}\hat{\mathbf{E}}(T, \mathbf{S}, \mathbf{B}_4) + C\mathbf{A}^2\,. \tag{4.158}$$

(iii) If additionally $\mathbf{A}^2 \leq \varepsilon^2 \hat{\mathbf{E}}(T, \mathbf{S}, \mathbf{B}_4)$ and we set $v_i := \hat{\mathbf{E}}(T, \mathbf{S}, \mathbf{B}_4)^{-1/2} u_i$, then there is a map $w_i : R_i \to \mathcal{A}_{Q_i}(\alpha_i^\perp)$ which is Dir-minimizing and such that

$$d_{W^{1,2}}(v_i, w_i) \leq \varsigma\,, \tag{4.159}$$

where $d_{W^{1,2}}$ is the $W^{1,2}$ distance defined in [18].

Note that, crucially, while the parameter ε must be chosen suitably small depending on ς, the constant C in (4.158) is instead independent of it. As already mentioned, the control (4.158) is indeed sub-optimal and we will later be able to remove the factor σ^{-2}.

Remark 4.6.20. We wish to spent a few words on a procedure which will be used often in this chapter. Assume we have a sequence of currents T_k, manifolds Σ_k, and cones $\mathbf{S}_k$ satisfying the assumptions of Proposition 4.6.19 for some fixed choice of the parameters δ^*, τ, and $\bar{\delta}$. We assume that the cones $\mathbf{S}_k$ are converging locally in the sense of Hausdorff distance to some limiting $\mathbf{S}_\infty$, and fix some sequence of m-dimensional planes $\alpha_i^k \subset \mathbf{S}_k$ converging to a plane $\alpha_i^\infty \subset \mathbf{S}_\infty$. Assume moreover that $\hat{\mathbf{E}}_k = \hat{\mathbf{E}}(T_k, \mathbf{S}_k, \mathbf{B}_1)$ and $\hat{\mathbf{E}}(T_k, \mathbf{S}_k, \mathbf{B}_1)^{-1}\mathbf{A}_k^2$ are both converging to zero. Let u_i^k be the coherent outer approximations over the domains $R_{i,k}^o$ which consist of the outer regions R_k^o for each T_k intersected with the planes α_i^k. We would like to use Proposition 4.6.19 to extract a Dir-minimizing limit of suitable normalizations of the maps u_i^k, namely $v_i^k := E_k^{-1/2} u_i^k$ for some choice of normalization constants E_k satisfying $\hat{\mathbf{E}}_k \leq E_k$. One technical point is that the maps v_i^k are not defined on the same plane. In order to deal with issue apply a rotation and map α_i^k onto α_i^∞. In fact, even though this is not needed, it is convenient to choose the canonical rotations $R_k = R(\alpha_i^k, \alpha_i^\infty)$ of Lemma 4.5.4. We can then extract, up to extracting a subsequence, a limit v_i of $v_i^k \circ R_k^{-1}$; note that

the latter maps are defined over (subdomains of) the same plane. Observe that the rotations R_k do depend on i and thus we do not have a single canonical rotation which works for every i. However, we can also see from Lemma 4.5.4 that, if we consider the graphs of the maps v_i^k as subsets of $\mathbb{R}^{m+n}$, the latter are indeed converging to the graph of v_i. Our limiting object is, in that sense, canonical.

In the sequel, when we are referring to "the blow-up v_i of the maps v_i^k" we will assume that we have followed the above algorithm.

Proof. First of all we let σ be given and fixed. We then select ℓ so that $\frac{\sigma}{2} \leq 2^{-\ell} \leq \sigma$ and define

$$\mathcal{G}_{\leq \ell} := \bigcup_{j=0}^{\ell} \mathcal{G}_j \,.$$

We can then appeal to Lemma 4.6.13(i) to get that, if ε is small enough, then $\mathcal{G}_{\ell+1} \subset \mathcal{G}^o$. But by definition of outer cubes the latter implies in fact $\mathcal{G}_{\leq \ell+1} \subset \mathcal{G}^o$, which implies the first claim of the proposition.

Let $L \in \mathcal{G}_{\leq \ell}$, and observe that (b) in the proof of Proposition 4.6.18 tells us that for any $L' \in \mathscr{N}(L)$, we have $L' \in \mathcal{G}_{\leq \ell+1}$. Appealing to Proposition 4.6.18(ii) (namely, (4.147)) we have

$$\begin{aligned} \int_{L_i} |Du_i|^2 &\leq C 2^{-m\ell(L)} (\bar{\mathbf{E}}(L) + 2^{-2\ell(L)} \mathbf{A}^2) \\ &\leq C 2^{-(m+2)\ell(L)} \mathbf{A}^2 + C \sum_{L' \in \mathscr{N}(L)} 2^{-m\ell(L')} \mathbf{E}(L', 0) \,, \end{aligned}$$

where $\mathscr{N}(L)$ and $\bar{\mathbf{E}}(L)$ are as in Definition 4.6.17. Since the cardinality of $\mathscr{N}(L)$ is bounded by a geometric constant $C = C(m, n)$, we immediately conclude

$$\begin{aligned} \int_{R_i} |Du_i|^2 &\leq C \sum_{L \in \mathcal{G}_{\leq \ell+1}} 2^{-m\ell(L)} (\mathbf{E}(L, 0) + 2^{-2\ell(L)} \mathbf{A}^2) \\ &= C \sum_{L \in \mathcal{G}_{\leq \ell+1}} \left(2^{2\ell(L)} \int_{\mathbf{B}^h(L)} \mathrm{dist}^2(q, \mathbf{S}) \, d\|T\|(q) + 2^{-(m+2)\ell(L)} \mathbf{A}^2 \right) \end{aligned}$$

$$\leq C\sigma^{-2}\sum_{L\in\mathcal{G}^o}\int_{\mathbf{B}^h(L)}\mathrm{dist}^2(q,\mathbf{S})\,d\|T\|(q)+C\mathbf{A}^2\,,$$

where we have used (4.127) (with $\kappa=4$). We next observe that $\mathbf{B}^h(L)\subset\mathbf{B}_4$ for every $L\in\mathcal{G}$ and that, by Lemma 4.6.10(iv), for given any point $q\in\mathbf{B}_4$ the cardinality of elements $L\in\mathcal{G}$ for which $q\in\mathbf{B}^h(L)$ is bounded by a constant $C=C(m,n)$. We thus conclude (4.158).

All that remains to be proven is (iii). By the definition of $T_{L,i}$ as the restriction of T on a suitable open set, for each i there is an integral current T_i with the following properties:

- $\partial T_i=0$ on $\mathbf{p}_{\alpha_i}^{-1}\left(\bigcup_{L\in\mathcal{G}^o}L_i\right)$;
- T_i is area-minimizing;
- $T_i\llcorner\mathbf{p}_{\alpha_i}^{-1}(L_i)=T_{L,i}\llcorner\mathbf{p}_{\alpha_i}^{-1}(L_i)$.

In particular, if we define $K_i:=\bigcup_{L\in\mathcal{G}^o}\bar{K}_i(L)$, the argument leading to (4.158) and the estimates in Proposition 4.6.18 lead to the following:

$$T_i\llcorner\mathbf{p}_{\alpha_i}^{-1}(K_i\cap R_i)=\mathbf{G}_{u_i}\llcorner\mathbf{p}_{\alpha_i}^{-1}(K_i\cap R_i) \tag{4.160}$$

$$\|u_i\|^2_{L^\infty(R_i)}\leq C(\hat{\mathbf{E}}(T,\mathbf{S},\mathbf{B}_4)+\mathbf{A}^2) \tag{4.161}$$

$$\|Du_i\|_{L^\infty}\leq C(\sigma^{-2}\hat{\mathbf{E}}(T,\mathbf{S},\mathbf{B}_4)+\mathbf{A}^2)^\gamma \tag{4.162}$$

$$|R_i\setminus K_i|+\|T_i\|(\mathbf{p}_{\alpha_i}^{-1}(R_i\setminus K_i))\leq C(\sigma^{-2}\hat{\mathbf{E}}(T,\mathbf{S},\mathbf{B}_4)+\mathbf{A}^2)^{1+\gamma}\,. \tag{4.163}$$

We are thus in the same position to apply the arguments of [19] leading to [19, Theorem 2.6] in order to conclude point (iii) of Proposition 4.6.19. The reader will notice that the only obstruction to applying [19, Theorem 2.6] is that the domain of the map u_i given above is not a ball. However, the arguments only use the regularity of the boundary of the domain, and since the boundary of R_i is Lipschitz, those arguments apply here as well. □

4.7 Cone balancing

In this section we introduce a suitable procedure which allows us to pass from a possibly unbalanced cone to a balanced cone whilst only changing the excess by a constant. This procedure is done under the assumption that the two-sided L^2 height excess of T relative to a cone $\mathbf{S} \in \mathscr{C}(Q)$ is significantly smaller than the planar L^2 height excess of T. To make our exposition cleaner, we recall the notation

$$\boldsymbol{\sigma}(\mathbf{S}) := \min_{i<j} \operatorname{dist}(\alpha_i \cap \mathbf{B}_1, \alpha_j \cap \mathbf{B}_1)$$
$$\boldsymbol{\mu}(\mathbf{S}) := \max_{i<j} \operatorname{dist}(\alpha_i \cap \mathbf{B}_1, \alpha_j \cap \mathbf{B}_1)$$

where i, j in the minimum and maximum range over the indices of the planes in $\mathbf{S}$.

Proposition 4.7.1 (Cone balancing)**.** *Assume that T and Σ are as in Assumption 3.2.2, $\mathbf{B}_1 = \mathbf{B}_1(0) \subset \Omega$, $\mathbf{S} \in \mathscr{C}(Q)$, and $\alpha_1, \dots, \alpha_N$ are as in Definition 4.2.2. Then, there are constants $C = C(Q, m, n, \bar{n}) > 0$ and $\varepsilon_0 = \varepsilon_0(Q, m, n, \bar{n}) > 0$ with the following property. Assume that*

$$\mathbf{A}^2 \leq \varepsilon_0^2 \mathbb{E}(T, \mathbf{S}, \mathbf{B}_1) \leq \varepsilon_0^4 \mathbf{E}^p(T, \mathbf{B}_1)\,. \tag{4.164}$$

Then there is a subset $\{i_1, \dots, i_k\} \subset \{1, \dots, N\}$ with $k \geq 2$ such that, upon setting $\mathbf{S}' = \alpha_{i_1} \cup \cdots \cup \alpha_{i_k}$, the following holds:

(a) $\mathbf{S}'$ is C-balanced;

(b) $\mathbb{E}(T, \mathbf{S}', \mathbf{B}_1) \leq C\mathbb{E}(T, \mathbf{S}, \mathbf{B}_1)$;

(c) $\operatorname{dist}^2(\mathbf{S} \cap \mathbf{B}_1, \mathbf{S}' \cap \mathbf{B}_1) \leq C\mathbb{E}(T, \mathbf{S}, \mathbf{B}_1)$;

(d) $C^{-1}\mathbf{E}^p(T, \mathbf{B}_1) \leq \boldsymbol{\mu}(\mathbf{S})^2 = \boldsymbol{\mu}(\mathbf{S}')^2 \leq C\mathbf{E}^p(T, \mathbf{B}_1)$.

Using the Pruning Lemma (Lemma 4.6.2), the proof of Proposition 4.7.1 can be reduced to showing the following proposition, which roughly says that if the two-sided L^2 height

excess of T relative to $\mathbf{S}$ is significantly smaller than the minimal angle in the cone $\mathbf{S}$, then in fact $\mathbf{S}$ is already C-balanced for some constant C. Indeed, intuitively, since T is area-minimizing, if it is very close to $\mathbf{S}$ then we would expect the union of planes in $\mathbf{S}$ to roughly behave like an area-minimizer, and from Morgan's result (Lemma 4.5.5) we would expect $\mathbf{S}$ to be balanced.

Proposition 4.7.2. *Assume that T, Σ, and $\mathbf{S}$ are as in Proposition 4.7.1. Then there are constants $C = C(m, n, \bar{n}, N)$ and $\varepsilon = \varepsilon(m, n, \bar{n}, N)$ with the following property. If we additionally have that*

$$N \geq 2 \quad \textit{and} \quad \mathbf{A}^2 + \mathbb{E}(T, \mathbf{S}, \mathbf{B}_1) \leq \varepsilon^2 \boldsymbol{\sigma}(\mathbf{S})^2 \,, \tag{4.165}$$

then $\mathbf{S}$ is C-balanced.

Let us first show that Proposition 4.7.2 implies Proposition 4.7.1.

Proof of Proposition 4.7.1. Suppose that Proposition 4.7.2 holds, and let ε_* be the minimum over $N \leq Q$ of all the constants $\varepsilon = \varepsilon(m, n, \bar{n}, N)$ from Proposition 4.7.2. Then fix $\varepsilon_0 \leq \varepsilon_*$ to be determined later.

The hypotheses of Proposition 4.7.1 in particular give that $\mathbb{E}(T, \mathbf{S}, \mathbf{B}_1) \leq \varepsilon_0^2 \mathbf{E}^p(T, \mathbf{B}_1)$. Next we estimate

$$\begin{aligned} \mathbf{E}^p(T, \mathbf{B}_1) &\leq \int_{\mathbf{B}_1} \operatorname{dist}^2(x, \alpha_1) \, d\|T\|(x) \leq C\hat{\mathbf{E}}(T, \mathbf{S}, \mathbf{B}_1) + C \max_{i<j} \operatorname{dist}^2(\alpha_i \cap \mathbf{B}_1, \alpha_j \cap \mathbf{B}_1) \\ &\leq C\varepsilon_0^2 \mathbf{E}^p(T, \mathbf{B}_1) + C\boldsymbol{\mu}(\mathbf{S})^2 \,, \end{aligned}$$

where $C = C(m, n)$. In particular for ε_0 smaller than a geometric constant we get

$$\mathbf{E}^p(T, \mathbf{B}_1) \leq C\boldsymbol{\mu}(\mathbf{S})^2 \,, \tag{4.166}$$

where $C = C(m,n)$. Thus we also conclude

$$\mathbb{E}(T, \mathbf{S}, \mathbf{B}_1) \leq \underbrace{C\varepsilon_0^2}_{=:\eta} \boldsymbol{\mu}(\mathbf{S})^2 .$$

Let us now fix $\delta > 0$ (which will be determined later), and let Γ be as in Lemma 4.6.2 for this choice of δ and N. If we take $\eta^{1/2} = (1+\Gamma)^{-1}\delta$ in the above, we see that if $\varepsilon_0 = \varepsilon_0(m,n,N,\delta) > 0$ is sufficiently small, and $D := \mathbb{E}(T, \mathbf{S}, \mathbf{B}_1)^{1/2}$, then

$$D \leq (1+\Gamma)^{-1}\delta \boldsymbol{\mu}(\mathbf{S}) .$$

Thus we are in the situation to apply Lemma 4.6.2 with this choice of D. This yields a subset $I = \{i_1, \ldots, i_k\} \subset \{1, \ldots, N\}$, with $k \geq 2$, such that the corresponding planes $\{\alpha_{i_1}, \ldots, \alpha_{i_k}\}$ satisfy (4.99), (4.100), (4.101) of the Pruning Lemma. Now set $\mathbf{S}' := \alpha_{i_1} \cup \cdots \cup \alpha_{i_k}$; we claim that Proposition 4.7.1 holds with this $\mathbf{S}'$. We start by observing that, since $\boldsymbol{\mu}(\mathbf{S}) = \boldsymbol{\mu}(\mathbf{S}')$ (by (4.101)) and (4.166) holds, we just need

$$\boldsymbol{\mu}(\mathbf{S}')^2 \leq C\mathbf{E}^p(T, \mathbf{B}_1) \tag{4.167}$$

to complete the proof of (d). However, let us first prove (a)–(c).

Observe that conclusion (4.99) of the Pruning Lemma gives

$$\max_{j=1,\ldots,N} \min_{i \in I} \operatorname{dist}^2(\alpha_i \cap \mathbf{B}_1, \alpha_j \cap \mathbf{B}_1) \leq \Gamma^2 \mathbb{E}(T, \mathbf{S}, \mathbf{B}_1).$$

In particular, this will give condition (c) once we have chosen δ appropriately. Furthermore, observe the following consequence of this: for $q \in \operatorname{spt}(T) \cap \mathbf{B}_1$, suppose $\operatorname{dist}(q, \mathbf{S}') = \operatorname{dist}(q, \alpha_{i_j})$ for some $i_j \in I$. If $\operatorname{dist}(q, \mathbf{S}) = \operatorname{dist}(q, \alpha_{i_j})$, then we clearly have $\operatorname{dist}(q, \mathbf{S}') = \operatorname{dist}(q, \mathbf{S})$. Otherwise, we must have $\operatorname{dist}(q, \mathbf{S}) = \operatorname{dist}(q, \alpha_\ell)$ for some $\ell \notin I$. By the above

consequence of the Pruning Lemma however, there is some $i_{j_*} \in I$ for which

$$\operatorname{dist}^2(\alpha_\ell \cap \mathbf{B}_1, \alpha_{i_{j_*}} \cap \mathbf{B}_1) \leq \Gamma^2 \mathbb{E}(T, \mathbf{S}, \mathbf{B}_1).$$

In particular, we must have

$$\begin{aligned}\operatorname{dist}^2(q, \mathbf{S}') \leq \operatorname{dist}^2(q, \alpha_{i_{j_*}} \cap \mathbf{B}_1) &\leq 4 \operatorname{dist}^2(q, \alpha_\ell) + 4 \operatorname{dist}^2(\alpha_\ell \cap \mathbf{B}_1, \alpha_{i_{j_*}} \cap \mathbf{B}_1) \\ &\leq 4 \operatorname{dist}^2(q, \mathbf{S}) + 4\Gamma^2 \mathbb{E}(T, \mathbf{S}, \mathbf{B}_1).\end{aligned}$$

In either case, we see that for any $q \in \operatorname{spt}(T) \cap \mathbf{B}_1$,

$$\operatorname{dist}^2(q, \mathbf{S}') \leq 4 \operatorname{dist}^2(q, \mathbf{S}) + 4\Gamma^2 \mathbb{E}(T, \mathbf{S}, \mathbf{B}_1)$$

which evidently gives

$$\hat{\mathbf{E}}(T, \mathbf{S}', \mathbf{B}_1) \leq 4(1 + \Gamma^2(Q+1)\omega_m)\mathbb{E}(T, \mathbf{S}, \mathbf{B}_1).$$

This deals with controlling one term of the two-sided height excess $\mathbb{E}(T, \mathbf{S}', \mathbf{B}_1)$. However, controlling the other term is simple as $\mathbf{S}' \subset \mathbf{S}$, and so $\hat{\mathbf{E}}(\mathbf{S}', T, \mathbf{B}_1) \leq \hat{\mathbf{E}}(\mathbf{S}, T, \mathbf{B}_1)$. Combining we therefore get $\mathbb{E}(T, \mathbf{S}', \mathbf{B}_1) \leq C\mathbb{E}(T, \mathbf{S}, \mathbf{B}_1)$ where $C = C(Q, m, n, \delta)$; this proves conclusion (b) provided we choose $\delta = \delta(Q, m, n, \bar{n}) > 0$ in the end.

To show that conclusion (a) holds we will apply Proposition 4.7.2 to $\mathbf{S}'$. For this we must verify that the hypothesis (4.165) holds in this situation. Note that from (4.100) of the Pruning Lemma we have

$$\mathbb{E}(T, \mathbf{S}, \mathbf{B}_1) + \max_{j=1,\ldots,N} \min_{i \in I} \operatorname{dist}^2(\alpha_i \cap \mathbf{B}_1, \alpha_j \cap \mathbf{B}_1) \leq 2\delta^2 \min_{i<j \in I} \operatorname{dist}^2(\alpha_i \cap \mathbf{B}_1, \alpha_j \cap \mathbf{B}_1).$$

In particular, if we perform exactly the same bounds as above when we proved (b), except

replacing the estimate from (4.99) by the above, we would end up with

$$\hat{\mathbf{E}}(T, \mathbf{S}', \mathbf{B}_1) \leq 2\hat{\mathbf{E}}(T, \mathbf{S}, \mathbf{B}_1) + 8(Q+1)\delta^2 \min_{i<j\in I} \operatorname{dist}^2(\alpha_i \cap \mathbf{B}_1, \alpha_j \cap \mathbf{B}_1)$$
$$\leq (8Q+12)\delta^2 \min_{i<j\in I} \operatorname{dist}^2(\alpha_i \cap \mathbf{B}_1, \alpha_j \cap \mathbf{B}_1)$$

where in the second inequality we have used the fact $\mathbb{E}(T, \mathbf{S}, \mathbf{B}_1) \leq 2\delta^2 \min_{i<j\in I} \operatorname{dist}^2(\alpha_i \cap \mathbf{B}_1, \alpha_j \cap \mathbf{B}_1)$ again from the statement of (4.100) above. But again, since for the other half of the excess we have

$$\hat{\mathbf{E}}(\mathbf{S}', T, \mathbf{B}_1) \leq \hat{\mathbf{E}}(\mathbf{S}, T, \mathbf{B}_1) \leq \mathbb{E}(T, \mathbf{S}, \mathbf{B}_1) \leq 2\delta^2 \min_{i<j\in I} \operatorname{dist}^2(\alpha_i \cap \mathbf{B}_1, \alpha_j \cap \mathbf{B}_1)$$

we see that

$$\mathbb{E}(T, \mathbf{S}', \mathbf{B}_1) \leq (8Q+14)\delta^2 \min_{i<j\in I} \operatorname{dist}^2(\alpha_i \cap \mathbf{B}_1, \alpha_j \cap \mathbf{B}_1).$$

Hence, if we choose $\delta = \delta(Q, m, n, \bar{n}) > 0$ obeying $(8Q+14)\delta^2 < \epsilon_*/2$, we get that one part of the inequality in (4.165) holds for T and $\mathbf{S}'$. However, the other part of the inequality in (4.165) evidently follows, since by assumption we have

$$\mathbf{A}^2 \leq \varepsilon_0^2 \mathbb{E}(T, \mathbf{S}, \mathbf{B}_1) \leq 2\delta^2 \varepsilon_0^2 \min_{i<j\in I} \operatorname{dist}^2(\alpha_i \cap \mathbf{B}_1, \alpha_j \cap \mathbf{B}_1)$$

and thus we have that (4.165) holds for suitably chosen $\delta = \delta(Q, m, n, \bar{n}) > 0$. Hence, with this choice of δ we can apply Proposition 4.7.2 to see that $\mathbf{S}'$ is C-balanced for some $C = C(Q, m, n, \bar{n}) > 0$, which completes the proof of (a).

It remains to prove (d), namely, as already observed, (4.167). Fix any plane π and observe that there is one plane α_{i_1} from $\mathbf{S}'$ with $i_1 \in I$ (which without loss of generality by relabeling we can assume to be $i_1 = 1$) such that

$$\operatorname{dist}(\alpha_1 \cap \mathbf{B}_1, \pi \cap \mathbf{B}_1) \geq \frac{1}{2}\boldsymbol{\mu}(\mathbf{S}')\,.$$

Indeed, if not we would get a contradiction to the definition of $\boldsymbol{\mu}(\mathbf{S}')$ by the triangle inequality.

We next show that there is an element $p \in \alpha_1$, a radius $r(m,n) > 0$, and constant $C(m,n) > 0$ with the property that $\mathbf{B}_r(p) \subset \mathbf{B}_{3/4} \setminus B_{1/4}(V)$ and

$$\operatorname{dist}(q,\pi) \geq C^{-1}\boldsymbol{\mu}(\mathbf{S}') \qquad \forall q \in B_r(p,\alpha_1)\,. \tag{4.168}$$

Recall first that $\operatorname{dist}(p,\pi) = |\mathbf{p}_\pi^\perp(p)|$ and that, by linearity of the map $\mathbf{p}_\pi^\perp$ we have $|\mathbf{p}_\pi^\perp(p)| = |p| \cdot |\mathbf{p}_\pi^\perp(p/|p|)| = |p| \operatorname{dist}(p/|p|,\pi)$, and so

$$|\mathbf{p}_\pi^\perp(p)| \leq \operatorname{dist}(\alpha_1 \cap \mathbf{B}_1, \pi \cap \mathbf{B}_1)|p| \qquad \forall p \in \alpha_1\,. \tag{4.169}$$

Next choose any $v \in V^\perp \cap \alpha_1$ with $|v| = \frac{1}{2}$. Consider then the disk $B_{1/8}(v,\alpha_1)$ and inside this disk select a base $e_1, \dots e_m$ of α_1 with the property that any element $x \in \alpha_1 \cap \overline{\mathbf{B}}_1$ can be written as a linear combination $\sum_i \lambda_i e_i$ with $|\lambda_i| \leq C = C(m)$. It follows that, for some element e_i we must necessarily have

$$|\mathbf{p}_\pi^\perp(e_i)| \geq \frac{1}{mC} \operatorname{dist}(\alpha_1 \cap \mathbf{B}_1, \pi \cap \mathbf{B}_1)\,.$$

Indeed by Corollary 4.5.3 there is a vector $e \in \overline{\mathbf{B}}_1 \cap \alpha_1$ with $|\mathbf{p}_\pi^\perp(e)| = \operatorname{dist}(\alpha_1 \cap \mathbf{B}_1, \pi \cap \mathbf{B}_1)$. We can thus use the property above to write $e = \sum_i \lambda_i e_i$ and estimate

$$\operatorname{dist}(\alpha \cap \mathbf{B}_1, \pi \cap \mathbf{B}_1) = |\mathbf{p}_\pi^\perp(e)| \leq \sum_i |\lambda_i||\mathbf{p}_\pi^\perp(e_i)| \leq C \sum_i |\mathbf{p}_\pi^\perp(e_i)|\,.$$

Set then $p = e_i$ and choose the radius r to equal $\min\{\frac{1}{2mC}, \frac{1}{8}\}$. We can then use the last inequality and (4.169) to show that for all $q \in B_r(p,\alpha_1)$,

$$\operatorname{dist}(q,\pi) = |\mathbf{p}_\pi^\perp(q)| \geq |\mathbf{p}_\pi^\perp(p)| - |\mathbf{p}_\pi^\perp(q-p)| \geq \frac{1}{2mC} \operatorname{dist}(\alpha_1 \cap \mathbf{B}_1, \pi \cap \mathbf{B}_1) \geq \frac{1}{4mC}\boldsymbol{\mu}(\mathbf{S})\,.$$

This establishes (4.168).

In particular, if the parameter ε_0 is small enough, we can apply Lemma 4.6.5, Proposition 4.6.6, and Lemma 4.6.7 to $T_{0,1/4}$ and $\mathbf{S}'$ (note that we have already established that $\mathbf{S}'$ is C-balanced by the above and that the hypotheses of these results do hold here). In particular, let ϱ be the parameter in Lemma 4.6.5. If we consider the current

$$T' := T \llcorner \mathbf{B}_r(p) \cap \{\mathrm{dist}(\cdot, \alpha_1) \leq \varrho \boldsymbol{\sigma}(\mathbf{S}')\}$$

we will have that $\partial T' = 0$ in $\mathbf{B}_r(p)$ and that $\|T'\|(\mathbf{B}_r(p)) \geq C^{-1} r^m$ for some constant $C(m,n)$. Moreover, from (4.113) in Proposition 4.6.6, we have

$$\mathrm{dist}(q, \alpha_1) \leq C \hat{\mathbf{E}}(T, \mathbf{S}', \mathbf{B}_1)^{1/2} + C\mathbf{A} \qquad \forall q \in \mathbf{B}_r(p) \cap \mathrm{spt}(T')$$

for some constant $C = C(Q, m, n, \bar{n})$. Since $\hat{\mathbf{E}}^2(T, \mathbf{S}', \mathbf{B}_1) + \mathbf{A}^2 \leq C\varepsilon_0^2 \boldsymbol{\mu}(\mathbf{S}')^2$, if ε_0 is chosen sufficiently small we then easily conclude from the above and (4.168) that

$$\mathrm{dist}(q, \pi) \geq C^{-1} \boldsymbol{\mu}(\mathbf{S}') \qquad \forall q \in \mathbf{B}_r(p) \cap \mathrm{spt}(T') \,.$$

Squaring and integrating the latter inequality with respect to $d\|T'\|$, and using the lower bound $\|T'\|(\mathbf{B}_r(p)) \geq C^{-1} r^m$ we reach

$$\boldsymbol{\mu}(\mathbf{S}')^2 \leq C \hat{\mathbf{E}}(T, \pi, \mathbf{B}_1) \,.$$

However, since π is an arbitrary m-dimensional plane, this completes the proof of (4.167). □

We now come to the proof of Proposition 4.7.2. We first prove a key special case of it when $\boldsymbol{\mu}(\mathbf{S})$ and $\boldsymbol{\sigma}(\mathbf{S})$ are comparable.

Proposition 4.7.3. *Assume that T, Σ, and $\mathbf{S}$ are as in Proposition 4.7.1. Fix $\eta > 0$. Then, there exist constants $C = C(m, n, \bar{n}, N, \eta)$ and $\varepsilon = \varepsilon(m, n, \bar{n}, N, \eta) > 0$ with the following property. If (4.165) and the assumptions of Proposition 4.7.1 hold with this choice of ε, and*

moreover if

$$\eta \boldsymbol{\mu}(\mathbf{S}) \leq \boldsymbol{\sigma}(\mathbf{S}), \tag{4.170}$$

then $\mathbf{S}$ *is* C*-balanced.*

Notice in particular that this implies the $N = 2$ case of Proposition 4.7.2, as in that situation we have $\boldsymbol{\mu}(\mathbf{S}) = \boldsymbol{\sigma}(\mathbf{S})$.

Proof. We argue by contradiction. If the proposition were false, then we could find sequences $T_k, \Sigma_k, \mathbf{A}_k$ and $\mathbf{S}_k = \alpha_1^k \cup \cdots \cup \alpha_N^k$ as in Proposition 4.7.2 such that

$$\mathbf{A}_k^2 + \mathbb{E}(T_k, \mathbf{S}_k, \mathbf{B}_1) \leq \varepsilon_k^2 \boldsymbol{\sigma}(\mathbf{S}_k)^2$$

and for which (4.170) holds for each k, but (after relabelling the planes) the Morgan angles $\theta_1(\alpha_1^k, \alpha_2^k)$ and $\theta_2(\alpha_1^k, \alpha_2^k)$ obey

$$\frac{\theta_1(\alpha_1^k, \alpha_2^k)}{\theta_2(\alpha_1^k, \alpha_2^k)} \to 0. \tag{4.171}$$

We have two possibilities: either (a) $\limsup_{k\to\infty} \boldsymbol{\sigma}(\mathbf{S}_k) > 0$ or (b) $\limsup_{k\to\infty} \boldsymbol{\sigma}(\mathbf{S}_k) = 0$. In the case of (a) the situation is simple: we simply pass to a subsequence (which we do not relabel) for which $\lim_{k\to\infty} \boldsymbol{\sigma}(\mathbf{S}_k) > 0$, $T_k \to T_\infty$ as currents, where T_∞ is some m-dimensional area-minimizing integral current in $\mathbf{B}_1$, and $\mathbf{S}_k \to \mathbf{S}_\infty = \alpha_1^\infty \cup \cdots \cup \alpha_N^\infty$ for a collection of N *distinct* planes α_i^∞, locally in Hausdorff distance, with $\alpha_i^k \to \alpha_i^\infty$ for each $i = 1, \ldots, N$. By the assumption that $\mathbb{E}(T_k, \mathbf{S}_k, \mathbf{B}_1) \to 0$, we must have $\operatorname{spt}(T_\infty) \cap \mathbf{B}_1 = \mathbf{S}_\infty \cap \mathbf{B}_1$. Indeed, we first see that $\operatorname{spt}(T_\infty) \cap \mathbf{B}_1 \subset \mathbf{S}_\infty \cap \mathbf{B}_1$ and then the constancy theorem implies $T_\infty \llcorner \mathbf{B}_1 = \sum_i k_i [\![\alpha_i^\infty]\!]$ for some $k_i \in \mathbb{Z}$. The only obstruction to the equality $\operatorname{spt}(T_\infty) \cap \mathbf{B}_1 = \mathbf{S}_\infty \cap \mathbf{B}_1$ is then the vanishing of some of the coefficients k_i, which would come from orientation cancellation in the limit of the T_k; however, this would contradict the T_k being area-minimizing.

But then by Lemma 4.5.5, we would necessarily have $\theta_1(\alpha_i^\infty, \alpha_j^\infty) = \theta_2(\alpha_i^\infty, \alpha_j^\infty) > 0$ for each $i < j \in \{1, \ldots, N\}$, where the fact that these angles are non-zero follows from

the assumption that $\liminf_k \boldsymbol{\sigma}(\mathbf{S}_k) > 0$. But we clearly have $\theta_1(\alpha_1^k, \alpha_2^k) \to \theta_1(\alpha_1^\infty, \alpha_2^\infty)$ and $\theta_2(\alpha_1^k, \alpha_2^k) \to \theta_2(\alpha_1^\infty, \alpha_2^\infty)$ from the local Hausdorff distance convergence in $\mathbf{B}_1$, and hence we would have $\theta_1(\alpha_1^k, \alpha_2^k)/\theta_2(\alpha_1^k, \alpha_2^k) \to 1$, contradicting (4.171).

Now let us handle the case (b). Here, we can pass to a further subsequence to ensure that $\alpha_i^k \to \alpha_\infty$ for all $i = 1, \ldots, N$, locally in Hausdorff distance, for some m-dimensional plane α_∞. Since $\mathbb{E}(T_k, \mathbf{S}_k, \mathbf{B}_1) \to 0$ we therefore also have that $\hat{\mathbf{E}}(T_k, \alpha_\infty, \mathbf{B}_1) \to 0$. Note that we may assume without loss of generality that $\alpha_1^k = \alpha_\infty$ for all k: indeed, we can choose a sequence of rotations $q_k : \mathbb{R}^{m+n} \to \mathbb{R}^{m+n}$ with $q_k \to \mathrm{id}_{\mathbb{R}^{m+n}}$ and obeying $q_k(\alpha_1^k) = \alpha_\infty$, and then consider $q_k(\mathbf{S}_k)$ and $(q_k)_\sharp T_k$; note that $\boldsymbol{\sigma}(\mathbf{S}_k) = \boldsymbol{\sigma}(q_k(\mathbf{S}_k))$ (see the discussion in Section 4.5.1). Hence, for k sufficiently large, we can consider the (strong) Lipschitz approximations $f_k : B_{3/4}(0, \alpha_\infty) \to \mathcal{A}_Q(\alpha_\infty^\perp)$ of Almgren (see [19, Theorem 2.4]) for the T_k relative to α_∞ (here, the assumption $\hat{\mathbf{E}}(T_k, \alpha_\infty, \mathbf{B}_1) \to 0$ is sufficient in light of Allard's tilt-excess inequality, see for example [1, Proposition 4.1]). Now set

$$\bar{f}_k := \frac{f_k}{\boldsymbol{\sigma}(\mathbf{S}_k)}.$$

Note that because $\alpha_1^k = \alpha_\infty$ for all k, we have

$$\hat{\mathbf{E}}(T_k, \alpha_\infty, \mathbf{B}_1) \leq 4\hat{\mathbf{E}}(T_k, \mathbf{S}_k, \mathbf{B}_1) + C\boldsymbol{\mu}(\mathbf{S}_k)^2 \leq (4\varepsilon_k^2 + C\eta^{-1})\boldsymbol{\sigma}(\mathbf{S}_k)^2$$

and so, again using Allard's tilt-excess inequality, the estimates from Almgren's strong Lipschitz approximation (see again [19, Theorem 2.4]) give that $\bar{f}_k \to \bar{f}_\infty$ strongly in $W^{1,2}_{\mathrm{loc}}(\mathbf{B}_{3/4}) \cap L^2(\mathbf{B}_{3/4})$ for some Dir-minimizer $\bar{f}_\infty$. Note also that, if for each i, we denote by L_i^k the linear maps parameterizing the planes α_i^k over α_∞ (in particular, $L_1^k = 0$ by construction, although this will not play any role in the proof), then we necessarily have that $|L_i^k| \leq C\boldsymbol{\mu}(\mathbf{S}_k) \leq C\eta^{-1}\boldsymbol{\sigma}(\mathbf{S}_k)$ for some dimensional constant C, and so if we write

$$\bar{L}_i^k := \frac{L_i^k}{\boldsymbol{\sigma}(\mathbf{S}_k)},$$

then we can pass to a further subsequence to ensure also that $\bar{L}_i^k \to \bar{L}_i$ for some linear map $\bar{L}_i$ over α_∞. Clearly, for each $i \neq j$ we have $|\bar{L}_i - \bar{L}_j| \geq c > 0$ by definition of $\boldsymbol{\sigma}(\mathbf{S}_k)$, and so $\bar{L}_i \neq \bar{L}_j$ whenever $i \neq j$. Noting that

$$\int \operatorname{dist}^2(x, \mathbf{S}_k)\, d\|\mathbf{G}_{f_k}\|(x) \leq C\hat{\mathbf{E}}(T_k, \mathbf{S}_k, \mathbf{B}_1) + C\left(\mathbf{A}_k^2 + \hat{\mathbf{E}}(T_k, \alpha_\infty, \mathbf{B}_1)\right)^{1+\gamma} = o(\boldsymbol{\sigma}(\mathbf{S}_k)^2)$$

where $o(\boldsymbol{\sigma}(\mathbf{S}_k)^2)$ denotes a term which converges to zero as $k \to \infty$ when divided by $\boldsymbol{\sigma}(\mathbf{S}_k)^2$. Dividing both sides of the above inequality by $\boldsymbol{\sigma}(\mathbf{S}_k)^2$ and taking $k \to \infty$, we see that necessarily the graph of $\bar{f}_\infty$ is supported in the union of the graphs of $\bar{L}_i$, $i = 1, \ldots, N$, and so in particular we have

$$\bar{f}_\infty = \sum_i Q_i \llbracket \bar{L}_i \rrbracket$$

for some non-negative integers Q_i such that $\sum_i Q_i = Q$. We claim that necessarily both Q_1, Q_2 (and in fact all Q_i, but we won't need this to see the contradiction) are strictly positive. Indeed, once we have this we can complete the proof of the claim in the following manner: for every constant $\lambda > 0$ the map $\lambda \bar{f}_\infty$ is Dir-minimizing, and so if $Q_1, Q_2 > 0$ then, by Proposition 4.5.6, the planes $\bar{\alpha}_i$ that are the graphs of $\lambda \bar{L}_i$ are c_2-balanced for all λ sufficiently small, where c_2 is an absolute constant (note that necessarily the intersection of the $\bar{\alpha}_i$ is a $(m-2)$-dimensional subspace, as the intersection of the α_i^k is an $(m-2)$-dimensional subspace for all k, and so up to passing to a subsequence this subspace will converge to an $(m-2)$-dimensional subspace which is necessarily the intersection of the planes $\bar{\alpha}_i$). In particular, we have

$$\frac{\theta_1(\bar{\alpha}_1, \bar{\alpha}_2)}{\theta_2(\bar{\alpha}_1, \bar{\alpha}_2)} \geq c_2 > 0.$$

Fix such a constant λ for which the above is true. Now, let $\bar{\alpha}_i^k$ be the planes which are the graphs of $\lambda \bar{L}_i^k$, $i = 1, 2$. Shrinking λ if necessary (which can be done independently of k, as $|\bar{L}_i^k| \leq C$ for all k and some geometric constant C), we can infer from Corollary 4.5.9 that

we have

$$\frac{\theta_1(\bar\alpha_1^k,\bar\alpha_2^k)}{\theta_2(\bar\alpha_1^k,\bar\alpha_2^k)} \le 16^2 \frac{\theta_1(\alpha_1^k,\alpha_2^k)}{\theta_2(\alpha_1^k,\alpha_2^k)}.$$

But then our assumption (4.171) implies from this that

$$\frac{\theta_1(\bar\alpha_1^k,\bar\alpha_2^k)}{\theta_2(\bar\alpha_1^k,\bar\alpha_2^k)} \to 0\,.$$

Noting that

$$\frac{\theta_1(\bar\alpha_1^k,\bar\alpha_2^k)}{\theta_2(\bar\alpha_1^k,\bar\alpha_2^k)} \to \frac{\theta_1(\bar\alpha_1,\bar\alpha_2)}{\theta_2(\bar\alpha_1,\bar\alpha_2)} \ge c_2 > 0$$

this gives the desired contradiction.

Therefore, all that remains to show is that $Q_1, Q_2 > 0$. In order to prove this, we use Lemma 4.5.10 to find points $\xi_k \in \alpha_1^k \cap \partial\mathbf{B}_{1/2}$ and a positive constant $c_0 = c_0(m,N)$ with the property that

$$\min_{j>1} \inf\{\mathrm{dist}(\zeta,\alpha_j^k) : \zeta \in \mathbf{B}_{2c_0}(\xi_k) \cap \alpha_1^k\} \ge 2c_0 \min_{j>1} \mathrm{dist}(\alpha_1^k \cap \mathbf{B}_1, \alpha_j^k \cap \mathbf{B}_1)\,. \tag{4.172}$$

In particular, for $\delta = \delta(Q,m,n,\bar n,N)$ as in the Splitting Corollary (Corollary 4.3.3) and for $r = r(\delta,\eta)$ sufficiently small, the assumptions of Corollary 4.3.3 are satisfied in $\mathbf{C}_{4rc_0}(\xi_k)$.

Indeed, conditions (i), (ii), (iii) are clear. (iv) follows from (4.172), as it gives

$$\min_{j>1} \inf\{\mathrm{dist}(\zeta,\alpha_j^k) : \zeta \in \mathbf{B}_{2c_0}(\xi_k) \cap \alpha_1^k\} \ge 2c_0\boldsymbol{\sigma}(\mathbf{S}_k) \ge 2c_0\eta\boldsymbol{\mu}(\mathbf{S}_k)$$

which in turn gives (iv) if we chose $r = 2\delta c_0\eta$ with a possibly smaller constant c_0. Note indeed that in the left hand side of (iv) we are using the distance between oriented planes, however it is just a matter of choosing the right orientation for the planes to be able to conclude that left hand side of (iv) is bounded above by $C\boldsymbol{\mu}(\mathbf{S})$. Finally, condition (v) follows from the fact $\mathbf{A}_k^2 + \mathbb{E}(T_k,\mathbf{S}_k,\mathbf{B}_1) \le \varepsilon_k^2\boldsymbol{\sigma}(\mathbf{S}_k)^2$, combined with the bound $\boldsymbol{\mu}(\mathbf{S}_k) \le \delta\min\{1, r^{-1}\varkappa\}$ just established for the above choice of r.

From Corollary 4.3.3 we may conclude that $T \llcorner \mathbf{B}_{rc_0}(\xi_k)$ splits into currents T_i, $i = 1, \ldots, N$, with the T_i supported in small disjoint neighborhoods of $\alpha_i^k \cap \mathbf{C}_{rc_0}(\xi_k)$; in particular, T_1 and T_2 are supported in two small disjoint neighbourhoods of $\alpha_1^k \cap \mathbf{C}_{rc_0}(\xi_k)$ and $\alpha_2^k \cap \mathbf{C}_{rc_0}(\xi_k)$ respectively, with the property that $(\mathbf{p}_{\alpha_1^k})_\sharp T_i = Q_i(k)[\![B_{rc_0}(\xi_k)]\!]$ for *positive* integers $Q_i(k) \geq 1$. Upon extraction of a subsequence we can assume the $Q_i(k)$ equal positive integers Q_i' independent of k and it follows easily that $Q_i' = Q_i$. This concludes the proof. □

We now come to the proof of Proposition 4.7.2. At one point in the proof we will need the following proposition, which also plays a key role later on in our work. Its proof will be given in Section 4.10, where we collect several important facts regarding Dir-minimizers.

Proposition 4.7.4. *Assume $\Omega \subset \mathbb{R}^m$ is a Lipschitz domain, $V \subset \mathbb{R}^m$ is an $(m-2)$-dimensional plane, and $v \in W^{1,2}(\Omega; \mathcal{A}_Q(\mathbb{R}^n))$ is a map with the property that the restriction of v to $\Omega_\varepsilon := \Omega \setminus B_\varepsilon(V)$ is Dir-minimizing for every $\varepsilon > 0$. Then v is Dir-minimizing in Ω.*

Proof of Proposition 4.7.2. We argue by induction on N. The case $N = 2$ is already established by Proposition 4.7.3, since in that situation we have $\boldsymbol{\mu}(\mathbf{S}) = \boldsymbol{\sigma}(\mathbf{S})$.

So fix $N \geq 3$. We may assume inductively the validity of Proposition 4.7.2 for all $N' < N$. If $C(m, n, \bar{n}, N')$ and $\varepsilon = \varepsilon(m, n, \bar{n}, N')$ denote the corresponding constants, set

$$C_1 := \max_{N' \leq N-1} C(m, n, \bar{n}, N') \quad \text{and} \quad \varepsilon_1^* := \min_{N' \leq N-1} \varepsilon(m, n, \bar{n}, N').$$

Suppose however that the conclusion of the proposition fails for N. We may therefore find sequences T_k, Σ_k, $\mathbf{A}_k$, $\mathbf{S}_k = \alpha_1^k \cup \cdots \cup \alpha_N^k$ and $\varepsilon_k \downarrow 0$ with

$$\mathbf{A}_k^2 \leq \varepsilon_k^2 \mathbb{E}(T_k, \mathbf{S}_k, \mathbf{B}_1) \leq \varepsilon_k^4 \boldsymbol{\sigma}(\mathbf{S}_k)^2, \tag{4.173}$$

such that, up to relabelling the planes in $\mathbf{S}_k$, we have

$$\frac{\theta_1(\alpha_1^k, \alpha_2^k)}{\theta_2(\alpha_1^k, \alpha_2^k)} \to 0. \tag{4.174}$$

Now, we may assume that

$$\lim_{k\to\infty} \boldsymbol{\sigma}(\mathbf{S}_k)/\boldsymbol{\mu}(\mathbf{S}_k) = 0\,. \tag{4.175}$$

Indeed, if this were not true, then up to passing to a subsequence we may assume that $\boldsymbol{\sigma}(\mathbf{S}_k) \geq \eta^*\boldsymbol{\mu}(\mathbf{S}_k)$ for all k and some $\eta^* > 0$, at which point we may apply Proposition 4.7.3 to get that for all k sufficiently large, the cones $\mathbf{S}_k$ are C-balanced for some $C > 0$ independent of k. But then we would have $\theta_1(\alpha_1^k, \alpha_2^k)/\theta_2(\alpha_1^k, \alpha_2^k) \geq C^{-1} > 0$ for all k, in direct contradiction to (4.173).

Now apply the layer subdivision lemma (Lemma 4.6.3) with $\delta = 1$ to each $\mathbf{S}_k$ (the exact choice of δ is unimportant). This provides subcollections $\{1, \ldots, N\} = I_k(0) \supsetneq I_k(1) \supsetneq \cdots \supsetneq I_k(\kappa(k))$ satisfying properties (i)–(iv) of Lemma 4.6.3. Up to extracting a further subsequence (which we again do not relabel) we can assume that $\kappa(k)$ is independent of k, and moreover that all the sets of indices $I_k(l)$, $\ell = 0, 1, \ldots, \kappa$ are independent of k; we therefore label them $I(l)$. Since we are assuming now that $\boldsymbol{\sigma}(\mathbf{S}_k)/\boldsymbol{\mu}(\mathbf{S}_k) \to 0$, we necessarily have that $\kappa \geq 1$, as otherwise conclusion (ii) of Lemma 4.6.3 would give that $\eta\boldsymbol{\mu}(\mathbf{S}_k) \leq \boldsymbol{\sigma}(\mathbf{S}_k)$ for all k and for some fixed $\eta = \eta(N) > 0$ independent of k, giving a contradiction.

For each k and $l \in \{0, 1, \ldots, \kappa\}$, write $\mathbf{S}_k^{(l)} := \bigcup_{i\in I(\ell)} \alpha_i^k$ for the cone at the l^{th} layer of $\mathbf{S}_k$. Also write

$$\mathbf{d}_k^{(l)} := \max_{i\in I(0)} \min_{j\in I(l)} \operatorname{dist}(\alpha_i^k \cap \mathbf{B}_1, \alpha_j^k \cap \mathbf{B}_1).$$

From (iii) of Lemma 4.6.3, we know that

$$\frac{\boldsymbol{\sigma}(\mathbf{S}_k^{(\ell-1)})}{\mathbf{d}_k^{(l)}} \geq \eta > 0 \tag{4.176}$$

for all k and $\ell \in \{1, \ldots, \kappa\}$, where again $\eta = \eta(N) > 0$ is a fixed constant.

Suppose first that we have

$$\lim_{k\to\infty} \frac{\mathbf{d}_k^{(1)}}{\boldsymbol{\sigma}(\mathbf{S}_k^{(1)})} = 0\,. \tag{4.177}$$

Then we have $\mathbf{d}_k^{(1)} \leq \delta_k \boldsymbol{\sigma}(\mathbf{S}_k^{(1)})$ for some non-negative sequence $\delta_k \to 0$. We can thus estimate

$$\begin{aligned}\hat{\mathbf{E}}(T_k, \mathbf{S}_k^{(1)}, \mathbf{B}_1) &\leq \hat{\mathbf{E}}(T_k, \mathbf{S}_k, \mathbf{B}_1) + C(\mathbf{d}_k^{(1)})^2 \\ &\leq \varepsilon_k^2 \boldsymbol{\sigma}(\mathbf{S}_k)^2 + C\delta_k^2 \boldsymbol{\sigma}(\mathbf{S}_k^{(1)})^2 = (\varepsilon_k^2 + C\delta_k^2)\boldsymbol{\sigma}(\mathbf{S}_k^{(1)})^2 \,,\end{aligned} \tag{4.178}$$

where $C = C(m, n, Q)$ and we have used that $\boldsymbol{\sigma}(\mathbf{S}_k) \leq \boldsymbol{\sigma}(\mathbf{S}_k^{(1)})$, since $\mathbf{S}_k^{(1)} \subset \mathbf{S}_k$; this inclusion also gives

$$\hat{\mathbf{E}}(\mathbf{S}_k^{(1)}, T_k, \mathbf{B}_1) \leq \hat{\mathbf{E}}(\mathbf{S}_k, T_k, \mathbf{B}_1) \leq \varepsilon_k^2 \boldsymbol{\sigma}(\mathbf{S}_k)^2 \leq \varepsilon_k^2 \boldsymbol{\sigma}(\mathbf{S}_k^{(1)})^2 \,.$$

Thus, for all k sufficiently large, we will have that hypothesis (4.165) holds with the smallness threshold ε_1^*. This allows us to conclude, by the induction hypothesis, that $\mathbf{S}_k^{(1)}$ is C-balanced for some $C = C(m, n, \bar{n}, N)$ (in fact, $C = C_1$).

Now, we know from (4.177) that $\mathbf{d}_k^{(1)} \to 0$. Analogously to case (b) in the proof of Proposition 4.7.3, we perform a blow-up. This time, however, we will be normalizing by $\mathbf{d}_k$ and building graphical approximations over the (balanced) cones $\mathbf{S}_k^{(1)}$. Indeed, (4.178) tells us that the hypotheses of Proposition 4.6.14 (and therefore Propositions 4.6.18 and 4.6.19) hold for T_k relative to the balanced cone $\mathbf{S}_k^{(1)}$ for all k sufficiently large. Thus, T_k splits into a sum of disjoint currents, denoted by T_k^j, near each plane $\alpha_j^k \subset \mathbf{S}_k^{(1)}$.

Now choose any sequences $\varsigma_k \downarrow 0$ and $\sigma_k \downarrow 0$. Passing to a subsequence, we may apply Proposition 4.6.19 with σ_k, ς_k in place of σ, ς to T_k and $\mathbf{S}_k^{(1)}$ to find multi-valued Lipschitz maps u_k^j over the sets $R_i^k := (R \setminus B_{\sigma_k}(V)) \cap \alpha_i^k$ for $i \in I(1)$ with the property that, if $v_k^i := \hat{\mathbf{E}}(T_k, \mathbf{S}_k^{(1)}, \mathbf{B}_1)^{-1/2} u_k^i$, then

$$d_{W^{1,2}}(v_k^i, w_k^i) \leq \varsigma_k \tag{4.179}$$

where w_k^i are Dir-minimizing maps on R_i^k. So, if we set $\bar{u}_i^k := (\mathbf{d}_k^{(1)})^{-1} u_i^k$, then estimating similarly to (4.178), since $\boldsymbol{\sigma}(\mathbf{S}_k) \leq \mathbf{d}_k^{(1)}$, we have

$$\hat{\mathbf{E}}(T_k, \mathbf{S}_k^{(1)}, \mathbf{B}_1) \leq (\varepsilon_k^2 + C)(\mathbf{d}_k^{(1)})^2$$

meaning that we can blow-up via rescaling by $\mathbf{d}_k^{(1)}$ instead of $\hat{\mathbf{E}}(T_k, \mathbf{S}_k^{(1)}, \mathbf{B}_1)^{-1/2}$ (see the remark after Proposition 4.6.19). Indeed, the estimates (4.114)–(4.116), combined with the conclusion (ii) of Proposition 4.6.19, (4.179), and a diagonal argument give that $\bar{u}_i^k \to \bar{u}_j^\infty$ locally strongly in $W^{1,2}(\mathbf{B}_1(0) \setminus V)$ and locally uniformly in $\mathbf{B}_1(0) \setminus V$, for some $\bar{u}_j^\infty$ which is Dir-minimizing in $\mathbf{B}_{1-\varepsilon}(0) \setminus B_\rho(V)$ for every $0 < \varepsilon, \rho < 1/2$. Note also that, if for each i, we denote by L_i^k the linear maps parameterizing the planes $\alpha_i^k \in \mathbf{S}_k$ over the closest plane in $\mathbf{S}_k^{(1)}$, then we necessarily have that $|L_i^k| \leq C\mathbf{d}_k^{(1)}$ for some dimensional constant C. So, if we write

$$\bar{L}_i^k := (\mathbf{d}_k^{(1)})^{-1} L_i^k$$

then we can pass to a further subsequence to ensure also that $\bar{L}_i^k \to \bar{L}_i$ for some linear map $\bar{L}_i$ over some plane α_i^∞ in $\mathbf{S}_\infty^{(1)} := \lim_{k\to\infty} \mathbf{S}_k^{(1)}$ (note that $\mathbf{S}_\infty^{(1)}$ need not be a single plane). However, as $|L_i^k - L_j^k| \geq c\boldsymbol{\sigma}(\mathbf{S}_k)$ for some dimensional constant c (by definition of $\boldsymbol{\sigma}(\mathbf{S}_k)$) whenever L_i^k, L_j^k are defined over the same plane, from (4.176) we know that $|\bar{L}_i - \bar{L}_j| \geq c\eta > 0$, i.e. $\bar{L}_i \neq \bar{L}_j$ for $i \neq j \in I(0)$. Note now that

$$\int \operatorname{dist}^2(x, \mathbf{S}_k)\, d\|\mathbf{G}_{u_j^k}\|(x) \leq C\hat{\mathbf{E}}(T_k, \mathbf{S}_k, \mathbf{B}_1) + C\left(\mathbf{A}_k^2 + \hat{\mathbf{E}}(T_k, \mathbf{S}_k, \mathbf{B}_1)\right)^{1+\gamma} = o\left((\mathbf{d}_k^{(1)})^2\right)$$

where $\gamma = \gamma(Q, m, n, \bar{n}) > 0$ is as in Proposition 4.6.6. Thus, dividing both sides by $(\mathbf{d}_k^{(1)})^2$ and taking $k \to \infty$, we see that necessarily the graph of $\bar{u}_j^\infty$ is supported on the union of all the graphs of the maps $\bar{L}_i$ associated to the plane α_j. Letting J_j denote this collection of indices i for $j \in I(1)$, on $\Omega = (\mathbf{B}_1 \cap \alpha_j) \setminus V$ we have

$$\bar{u}_j^\infty = \sum_{i \in J_j} Q_i^j [\![\bar{L}_i]\!]$$

for some non-negative integers Q_i^j such that $\sum_{i,j} Q_i^j = Q$. Following a similar argument to the proof in Proposition 4.7.3, we can in fact show that $Q_i^j \geq 1$ for each i, j. Now, since $\bar{u}_j^\infty$ is Dir-minimizing in $\mathbf{B}_{1-\varepsilon} \setminus B_\rho(V)$ for every $\varepsilon, \rho > 0$, we may now apply Proposition 4.7.4

to conclude that $\bar{u}_j^\infty$ extends to a Dir-minimizer (which we still denote by $\bar{u}_j^\infty$) on $\mathbf{B}_{1/2} \cap \alpha_j$. If the unbalancing assumption from (4.174) has $1, 2 \in J_j$ for some j, then we can argue as in case (b) of Proposition 4.7.3 to arrive at a contradiction. If we have $1 \in J_j$ and $2 \in J_{j^*}$ for some $j \neq j^*$, we can again argue in the same way, except now using the fact that $\mathbf{S}_k^{(1)}$ is balanced, and so the planes α_j^k, $\alpha_{j^*}^k$ are balanced. Thus, in this situation we arrive at a contradiction.

To summarize, we have now shown that (4.177) cannot hold. From Lemma 4.6.3 (as $\delta = 1$) we know that $\mathbf{d}_k^{(1)} \leq \boldsymbol{\sigma}(\mathbf{S}_k^{(1)})$, and so we may therefore pass to a subsequence to ensure that

$$\mathbf{d}_k^{(1)} \geq c_1 \boldsymbol{\sigma}(\mathbf{S}_k^{(1)}) \tag{4.180}$$

for all k and for some constant $c_1 > 0$.

To progress, we now move to the next layer, namely the cones $\mathbf{S}_k^{(2)}$. We know from (4.176) that $\boldsymbol{\sigma}(\mathbf{S}_k^{(1)}) \geq \eta \mathbf{d}_k^{(2)}$. Now, if we have

$$\lim_{k \to \infty} \frac{\mathbf{d}_k^{(2)}}{\boldsymbol{\sigma}(\mathbf{S}_k^{(2)})} = 0 \tag{4.181}$$

we argue that we can follow an analogous argument to that above when (4.177) held, except now performing a blow-up of T_k relative to $\mathbf{S}_k^{(2)}$. Indeed, if (4.181) holds, then we have $\mathbf{d}_k^{(2)} \leq \delta_k' \boldsymbol{\sigma}(\mathbf{S}_k^{(2)})$, for some non-negative sequence $\delta_k' \to 0$. We can estimate as in (4.178) to find

$$\begin{aligned} \hat{\mathbf{E}}(T_k, \mathbf{S}_k^{(2)}, \mathbf{B}_1) &\leq \hat{\mathbf{E}}(T_k, \mathbf{S}_k, \mathbf{B}_1) + C(\mathbf{d}_k^{(2)})^2 \leq \varepsilon_k^2 \boldsymbol{\sigma}(\mathbf{S}_k)^2 + C(\delta_k')^2 \boldsymbol{\sigma}(\mathbf{S}_k^{(2)})^2 \\ &\leq (\varepsilon_k^2 + C(\delta_k')^2) \boldsymbol{\sigma}(\mathbf{S}_k^{(2)})^2 \end{aligned}$$

where again we have used $\boldsymbol{\sigma}(\mathbf{S}_k) \leq \boldsymbol{\sigma}(\mathbf{S}_k^{(2)})$ since $\mathbf{S}_k^{(2)} \subset \mathbf{S}_k$. Therefore, once again we get that the hypothesis (4.165) holds with the smallness threshold ε_1^* for all k sufficiently large, and so by the induction hypothesis we have that $\mathbf{S}_k^{(2)}$ is C-balanced for some $C = C(m, n, \bar{n}, N)$.

The only aspect of the above blow-up procedure which we need to check in this case is the separation of all the rescaled linear functions of the planes in $\mathbf{S}_k$ over $\mathbf{S}_k^{(2)}$. As we are rescaling our approximations by $\mathbf{d}_k^{(2)}$, we therefore need to show that $\frac{\boldsymbol{\sigma}(\mathbf{S}_k)}{\mathbf{d}_k^{(2)}}$ has a uniform lower bound. Indeed, combining (4.176) and (4.180), we have

$$\frac{\boldsymbol{\sigma}(\mathbf{S}_k)}{\mathbf{d}_k^{(2)}} = \frac{\boldsymbol{\sigma}(\mathbf{S}_k)}{\mathbf{d}_k^{(1)}} \cdot \frac{\mathbf{d}_k^{(1)}}{\boldsymbol{\sigma}(\mathbf{S}_k^{(1)})} \cdot \frac{\boldsymbol{\sigma}(\mathbf{S}_k^{(1)})}{\mathbf{d}_k^{(2)}} \geq \eta \cdot c_1 \cdot \eta = \eta^2 c_1 > 0\,. \tag{4.182}$$

This tells us that we may repeat the blow-up argument as above, with $\mathbf{d}_k^{(1)}$ replaced by $\mathbf{d}_k^{(2)}$ and $\mathbf{S}_k^{(1)}$ replaced with $\mathbf{S}_k^{(2)}$, to contradict the lack of balancing (4.174) of $\mathbf{S}_k$ when (4.181) holds.

Proceeding now inductively, we see that we must be able to find a subsequence with

$$\mathbf{d}_k^{(l)} \geq c_l \boldsymbol{\sigma}(\mathbf{S}_k^{(l)})$$

for all $l = 1, \ldots, \kappa$ and some constants $c_l > 0$. However, we can now show that this is in direct contradiction to the assumption (4.175). Indeed, an identical calculation from (4.182) gives

$$\frac{\boldsymbol{\sigma}(\mathbf{S}_k)}{\boldsymbol{\mu}(\mathbf{S}_k)} = \frac{\boldsymbol{\sigma}(\mathbf{S}_k^{(\kappa)})}{\boldsymbol{\mu}(\mathbf{S}_k)} \cdot \prod_{\ell=0}^{\kappa-1} \left(\frac{\boldsymbol{\sigma}(\mathbf{S}_k^{(\ell)})}{\mathbf{d}_k^{(\ell+1)}} \cdot \frac{\mathbf{d}_k^{(\ell+1)}}{\boldsymbol{\sigma}(\mathbf{S}_k^{(\ell+1)})} \right) \geq \frac{\boldsymbol{\sigma}(\mathbf{S}_k^{(\kappa)})}{\boldsymbol{\mu}(\mathbf{S}_k)} \cdot \eta^\kappa \prod_{l=1}^{\kappa} c_l\,,$$

where we recall $\mathbf{S}_k^{(0)} = \mathbf{S}_k$. But we know from Lemma 4.6.3(i) and (ii) that $\boldsymbol{\mu}(\mathbf{S}_k) = \boldsymbol{\mu}(\mathbf{S}_k^{(\kappa)})$ and $\eta \boldsymbol{\mu}(\mathbf{S}_k^{(\kappa)}) \leq \boldsymbol{\sigma}(\mathbf{S}_k^{\kappa})$. Hence, the above gives

$$\frac{\boldsymbol{\sigma}(\mathbf{S}_k)}{\boldsymbol{\mu}(\mathbf{S}_k)} \geq \eta^{\kappa+1} \prod_{l=1}^{\kappa} c_l > 0\,,$$

thus reaching the desired contradiction to (4.175), and completing the proof. □

4.8 Reduction of the main decay theorem

In this section, we reduce the proof of Theorem 4.2.4 to an a priori much weaker decay statement; Theorem 4.8.2 below. For this we will utilize the balancing result from Proposition 4.7.1. Recall that for a cone $\mathbf{S} \in \mathscr{C}(Q)$, we write

$$\boldsymbol{\sigma}(\mathbf{S}) := \min_{i<j} \operatorname{dist}(\alpha_i \cap \mathbf{B}_1, \alpha_j \cap \mathbf{B}_1), \qquad \boldsymbol{\mu}(\mathbf{S}) := \max_{i<j} \operatorname{dist}(\alpha_i \cap \mathbf{B}_1, \alpha_j \cap \mathbf{B}_1).$$

From now on, we make the following assumption regarding the (balancing) constant M:

Assumption 4.8.1. The constant $M \geq 1$ is chosen so that the application Proposition 4.7.1 yields an M-balanced cone. Thus, $M = M(Q, m, n, \bar{n})$.

Theorem 4.8.2 (Weak Excess Decay Theorem)**.** *Fix $Q, m, n, \bar{n}$ as before, and let $M \geq 1$ be as in Assumption 4.8.1. Fix also $\varsigma_1 > 0$. Then, there are constants $\varepsilon_1 = \varepsilon_1(Q, m, n, \bar{n}, \varsigma_1) \in (0, 1/2]$, $r_1^1 = r_1^1(Q, m, n, \bar{n}, \varsigma_1) \in (0, 1/2]$ and $r_1^2 = r_1^2(Q, m, n, \bar{n}, \varsigma_1) \in (0, 1/2]$, such that the following holds. Suppose that*

(i) *T and Σ are as in Assumption 3.2.2;*

(ii) *$\|T\|(\mathbf{B}_1) \leq (Q + \frac{1}{2})\omega_m$;*

(iii) *There is $\mathbf{S} \in \mathscr{C}(Q, 0)$ which is M-balanced, such that*

$$\mathbb{E}(T, \mathbf{S}, \mathbf{B}_1) \leq \varepsilon_1^2 \boldsymbol{\sigma}(\mathbf{S})^2 \tag{4.183}$$

and

$$\mathbf{B}_{\varepsilon_1}(\xi) \cap \{p : \Theta(T, p) \geq Q\} \neq \emptyset \qquad \forall \xi \in V(\mathbf{S}) \cap \mathbf{B}_{1/2}\,; \tag{4.184}$$

(iv) *$\mathbf{A}^2 \leq \varepsilon_1^2 \mathbb{E}(T, \tilde{\mathbf{S}}, \mathbf{B}_1)$ for every $\tilde{\mathbf{S}} \in \mathscr{C}(Q, 0)$.*

Then, there is $\mathbf{S}' \in \mathscr{C}(Q,0) \setminus \mathscr{P}(0)$ *such that for some* $i \in \{1,2\}$ *we have*

$$\mathbb{E}(T, \mathbf{S}', \mathbf{B}_{r_1^i}) \leq \varsigma_1 \mathbb{E}(T, \mathbf{S}, \mathbf{B}_1)\,. \tag{4.185}$$

The above theorem would appear to be considerably weaker than Theorem 4.2.4; not only are we assuming the cone $\mathbf{S}$ is balanced, but there is also a significant difference between the smallness assumption (4.1) and (4.183). In fact, (4.183) implies the second inequality in (4.1) for ε_1 suitably small (indeed, Proposition 4.6.6 then gives $\boldsymbol{\sigma}(\mathbf{S})^2 \leq C\mathbf{E}^p(T, \mathbf{B}_1)$), whilst when the cone $\mathbf{S}$ arises from Proposition 4.7.1 (as it will) the second inequality in (4.1) is equivalent, up to constants, to (from Proposition 4.7.1(d))

$$\mathbb{E}(T, \mathbf{S}, \mathbf{B}_1) \leq \varepsilon_0^2 \boldsymbol{\mu}(\mathbf{S})^2\,. \tag{4.186}$$

In order to show that in fact the above seemingly weaker statement implies Theorem 4.2.4, the idea is to first show that Theorem 4.8.2 implies a multiple radii decay version of Theorem 4.2.4, by first removing the necessary planes in the cone $\mathbf{S}$ to reach a balanced cone, and then inductively removing additional planes to obtain a cone (that is still balanced) for which assumption (4.183) holds. In doing so we reach an intermediate statement, namely an excess decay with finitely many scales, from which we then derive Theorem 4.2.4. This is the following, the idea of which is similar to that seen in [56, Section 13]:

Proposition 4.8.3 (Multiple Radii Excess Decay)**.** *Fix* $Q, m, n, \bar{n}$ *as before, and let* $M \geq 1$ *be as in Assumption 4.8.1. Let* $\bar{N} := Q(Q-1)$*. Fix* $\varsigma_2 > 0$*. Then, there exist positive constants* $\varepsilon_2, r_1, \ldots, r_{\bar{N}} \leq \frac{1}{2}$*, depending only on* $Q, m, n, \bar{n}$*, and* ς_2*, such that the following holds. Suppose that* $T, \Sigma, \mathbf{S}$ *are as in Theorem 4.2.4, i.e.:*

(i) T *and* Σ *are as in Assumption 3.2.2 and* $\mathbf{B}_1 \subset \Omega$*;*

(ii) $\|T\|(\mathbf{B}_1) \leq (Q + \frac{1}{2})\omega_m$*;*

(iii) *There is* $\mathbf{S} \in \mathscr{C}(Q,0) \setminus \mathscr{P}(0)$ *such that*

$$\mathbb{E}(T, \mathbf{S}, \mathbf{B}_1) \leq \varepsilon_2^2 \mathbf{E}^p(T, \mathbf{B}_1),$$

and that

$$\mathbf{B}_{\varepsilon_2}(\xi) \cap \{p : \Theta(T,p) \geq Q\} \neq \emptyset \qquad \forall \xi \in V(\mathbf{S}) \cap \mathbf{B}_{1/2}\,;$$

(iv) $\mathbf{A}^2 \leq \varepsilon_2^2 \mathbb{E}(T, \tilde{\mathbf{S}}, \mathbf{B}_1)$ *for every* $\tilde{\mathbf{S}} \in \mathscr{C}(Q,0)$.

Then, there is a $\mathbf{S}' \in \mathscr{C}(Q,0) \setminus \mathscr{P}(0)$ *and* $i \in \{1, \ldots, \bar{N}\}$ *such that:*

$$\mathbb{E}(T, \mathbf{S}', \mathbf{B}_{r_i}) \leq \varsigma_2 \mathbb{E}(T, \mathbf{S}, \mathbf{B}_1)\,. \tag{4.187}$$

We will then be able to show that Proposition 4.8.3 implies Theorem 4.2.4, which therefore demonstrates that the weak excess decay from Theorem 4.8.2 implies the stronger excess decay statement from Theorem 4.2.4. Let us begin by showing that Theorem 4.8.2 implies Proposition 4.8.3.

Proof of Proposition 4.8.3 from Theorem 4.8.2. Fix $T, \Sigma, \mathbf{S}$, and ς_2 as in the statement of Proposition 4.8.3.

Firstly, because of Proposition 4.7.1, if we take $\varepsilon_2 \leq \varepsilon_0$ where $\varepsilon_0 = \varepsilon_0(Q, m, n, \bar{n}) > 0$ is as in Proposition 4.7.1, then we can find a cone $\tilde{\mathbf{S}}$ which is M-balanced and obeys $\mathbb{E}(T, \tilde{\mathbf{S}}, \mathbf{B}_1) \leq M\mathbb{E}(T, \mathbf{S}, \mathbf{B}_1)$, where $M = M(Q, m, n, \bar{n})$ is as in Assumption 4.8.1. Thus, we have

$$\mathbb{E}(T, \tilde{\mathbf{S}}, \mathbf{B}_1) \leq M\mathbb{E}(T, \mathbf{S}, \mathbf{B}_1) \leq M\varepsilon_2^2 \mathbf{E}^p(T, \mathbf{B}_1)$$

In particular, this coupled with the other estimates in Proposition 4.7.1 gives that if we can prove the result for $\tilde{\mathbf{S}}$, then the result follows for $\mathbf{S}$, up to changing ς_2 by a factor of M. Thus, we may without loss of generality assume that $\mathbf{S}$ is M-balanced and $M^{-1}\mathbf{E}^p(T, \mathbf{B}_1) \leq \boldsymbol{\mu}(\mathbf{S}) \leq M\mathbf{E}^p(T, \mathbf{B}_1)$.

We would now like to apply Theorem 4.8.2. However, a priori, it may be the case that (4.183) does not hold for $\mathbf{S}$, as we are merely assuming that $\mathbb{E}(T, \mathbf{S}, \mathbf{B}_1) \leq \varepsilon_2^2 \mathbf{E}^p(T, \mathbf{B}_1)$. The remaining part of the argument deals with this difficulty; the price to pay is that the decay might occur at one of finitely many scales, but nonetheless the number of possible scales which are needed belong to a set of controlled cardinality.

We start by introducing, for every integer $k \in \{2, \dots, Q\}$, functions $\varepsilon_{(k)}(s)$, $r^1_{(k)}(s)$, and $r^2_{(k)}(s)$ of a parameter $s > 0$ as follows:

- If one takes $Q = k$ and $\varsigma_1 = s$ in Theorem 4.8.2, then $\varepsilon_{(k)}(s) := \varepsilon_1(k, m, n, \bar{n}, s)$, where the ε_1 is the constant from Theorem 4.8.2 with this choice of Q, ς_1, and $r^1_{(k)}(s) := r^1_1(k, m, n, \bar{n}, s)$, $r^2_{(k)}(s) := r^2_1(k, m, n, \bar{n}, s)$ are the corresponding radii, where r^1_1 and r^2_1 are as in Theorem 4.8.2.

In particular, k here is the number of planes forming the cone $\mathbf{S}$ to which Theorem 4.8.2 is applied, whilst s is the specified decay factor.

We now denote by N the number of planes forming $\mathbf{S}$. If $N = 2$, then $\boldsymbol{\mu}(\mathbf{S}) = \boldsymbol{\sigma}(\mathbf{S})$, and so from (4.186) we get that the assumptions of Theorem 4.8.2 hold (up to ε_2 changing by a constant), and so if we take ε_2 smaller than $\varepsilon_{(2)}(\varsigma_2)$, we see that we can apply Theorem 4.8.2. The corresponding radii from this application of Theorem 4.8.2 will be denoted by $r^1_{2,2}$ and $r^2_{2,2}$ (the first subscript here denotes the number of planes, N, in the cone $\mathbf{S}$, whilst the second subscript denotes the number of planes left at the point of application of Theorem 4.8.2 – see below), which are simply $r^1_{(2)}(\varsigma_2)$ and $r^2_{(2)}(\varsigma_2)$. In particular, the proof is complete in the case $Q = 2$.

From now on we therefore assume $N \geq 3$. We may take $\varepsilon_2 \leq \varepsilon_{(N)}(\varsigma_2)$. We then consider two cases. If we have

$$\mathbb{E}(T, \mathbf{S}, \mathbf{B}_1) \leq \varepsilon_{(N)}(\varsigma_2)^2 \boldsymbol{\sigma}(\mathbf{S})^2$$

then we can simply apply Theorem 4.8.2 to get the desired decay at one of the two radii

$r^1_{N,N} := r^1_{(N)}(\varsigma_2)$, $r^2_{N,N} := r^2_{(N)}(\varsigma_2)$. We can therefore assume that

$$\mathbb{E}(T, \mathbf{S}, \mathbf{B}_1) > \varepsilon_{(N)}(\varsigma_2)^2 \boldsymbol{\sigma}(\mathbf{S})^2. \tag{4.188}$$

We now follow the same idea in the argument for the Pruning Lemma (Lemma 4.6.2). Let $\mathbf{S} = \alpha_1 \cup \cdots \cup \alpha_N$, where the α_i are distinct m-dimensional planes. We can assume, upon relabelling, that

$$\operatorname{dist}(\alpha_1 \cap \mathbf{B}_1, \alpha_2 \cap \mathbf{B}_1) = \boldsymbol{\sigma}(\mathbf{S})$$

and that there are two indices $i_*, j_* \neq 1$ with

$$\operatorname{dist}(\alpha_{i_*} \cap \mathbf{B}_1, \alpha_{j_*} \cap \mathbf{B}_1) = \boldsymbol{\mu}(\mathbf{S}).$$

We now remove the plane α_1 and consider $\mathbf{S}_{N-1} := \alpha_2 \cup \cdots \cup \alpha_N$; note that $\mathbf{S}_{N-1}$ is still M-balanced, $V(\mathbf{S}_{N-1}) = V(\mathbf{S})$, $\boldsymbol{\mu}(\mathbf{S}) = \boldsymbol{\mu}(\mathbf{S}_{N-1})$, and

$$\operatorname{dist}^2(\mathbf{S} \cap \mathbf{B}_1, \mathbf{S}_{N-1} \cap \mathbf{B}_1) \leq \boldsymbol{\sigma}(\mathbf{S})^2 \leq \varepsilon_{(N)}(\varsigma_2)^{-2} \mathbb{E}(T, \mathbf{S}, \mathbf{B}_1);$$

these conditions imply that if we can show the result for $\mathbf{S}_{N-1}$, the corresponding statement for $\mathbf{S}$ follows.

Now observe that

$$\mathbb{E}(T, \mathbf{S}_{N-1}, \mathbf{B}_1) \leq C_0 \mathbb{E}(T, \mathbf{S}, \mathbf{B}_1) + C_0 \boldsymbol{\sigma}(\mathbf{S})^2 \leq C_0 \left(1 + \varepsilon_{(N)}(\varsigma_2)^{-2}\right) \mathbb{E}(T, \mathbf{S}, \mathbf{B}_1).$$

for some constant $C_0 = C_0(Q, m, n, \bar{n}) \geq 1$. Let us write $C_1^* := C_0(1 + \varepsilon_{(N)}(\varsigma_2)^{-2})$, and so $C_1^* = C_1^*(N, m, n, \bar{n}, \varsigma_2) > 1$, and so the above inequality is just

$$\mathbb{E}(T, \mathbf{S}_{N-1}, \mathbf{B}_1) \leq C_1^* \mathbb{E}(T, \mathbf{S}, \mathbf{B}_1).$$

Therefore, if

$$\mathbb{E}(T, \mathbf{S}, \mathbf{B}_1) \leq \varepsilon_{(N-1)}(\varsigma_2/C_1^*)^2 \sigma(\mathbf{S}_{N-1})^2 \tag{4.189}$$

and if we take $\varepsilon_2 \leq \varepsilon_{(N-1)}(\varsigma_2/C_1^*)/\sqrt{C_1^*}$ (which of course we may), then the above gives that we can apply Theorem 4.8.2 to the triple T, Σ, and $\mathbf{S}_{N-1}$ to conclude the desired decay statements in Proposition 4.8.3, for one of the radii

$$r^i_{N,N-1} := r^i_{(N-1)}(\varsigma_2/C_1^*), \qquad i \in \{1, 2\}.$$

If (4.189) is not true, i.e. if we cannot apply Theorem 4.8.2 as above, then we must have

$$\mathbb{E}(T, \mathbf{S}, \mathbf{B}_1) > \varepsilon_{(N-1)}(\varsigma_2/C_1^*)^2 \boldsymbol{\sigma}(\mathbf{S}_{N-1})^2. \tag{4.190}$$

We now wish to apply the above procedure inductively; indeed, suppose we have performed the above procedure K times, and at each stage we have not been able to apply Theorem 4.8.2. Thus, upon relabelling the planes in $\mathbf{S}$, for $k = 0, 1, \ldots, K$ we have cones $\mathbf{S}_{N-k} = \alpha_{k+1} \cup \cdots \cup \alpha_N$ which are all M-balanced and obey $V(\mathbf{S}_{N-k}) = V(\mathbf{S})$, $\boldsymbol{\mu}(\mathbf{S}_{N-k}) = \boldsymbol{\mu}(\mathbf{S})$. We also know that $\mathbf{S}_{N-(k+1)}$ is formed from $\mathbf{S}_{N-k}$ by removing a single plane, namely α_{k+1}, which is a plane in $\mathbf{S}_{N-k}$ achieving the minimal separation $\boldsymbol{\sigma}(\mathbf{S}_{N-k})$ and such that there are two other planes in $\mathbf{S}_{N-k}$ which achieve the maximal separation $\boldsymbol{\mu}(\mathbf{S}_{N-k})$ (this can be done as long as $N - k \geq 3$, i.e. $0 \leq k \leq N - 3$). The criterion along this sequence is that

$$\mathbb{E}(T, \mathbf{S}, \mathbf{B}_1) > \varepsilon_{(N-k)}(\varsigma_2/C_k^*)^2 \boldsymbol{\sigma}(\mathbf{S}_{N-k})^2$$

where $C_k^* = C_0(C_{k-1}^* + \varepsilon_{(N-(k-1))}(\varsigma_2/C_{k-1}^*)^{-2})$ is defined inductively for $k \geq 1$, with $C_0^* = 1$ and $C_0 = C_0(Q, m, n, \bar{n})$ is as above, and

$$\mathbb{E}(T, \mathbf{S}_{N-k}, \mathbf{B}_1) \leq C_k^* \mathbb{E}(T, \mathbf{S}, \mathbf{B}_1)\,;$$

for this procedure we need to assume $\varepsilon_2 \leq \varepsilon_{(N-k)}(\varsigma_2/C_k^*)/\sqrt{C_k^*}$ for each $k = 0, 1, \ldots, K$ (which of course we may). We also therefore have

$$\operatorname{dist}^2(\mathbf{S}_{N-(k+1)} \cap \mathbf{B}_1, \mathbf{S}_{N-k} \cap \mathbf{B}_1) \leq \boldsymbol{\sigma}(\mathbf{S}_{N-k})^2 \leq \varepsilon_{(N-k)}(\varsigma_2/C_k^*)^{-2}\mathbb{E}(T, \mathbf{S}, \mathbf{B}_1)$$

and thus

$$\begin{aligned}\mathbb{E}(T, \mathbf{S}_{N-(k+1)}, \mathbf{B}_1) &\leq C\mathbb{E}(T, \mathbf{S}_{N-k}, \mathbf{B}_1) + C\boldsymbol{\sigma}(\mathbf{S}_{N-k})^2 \\ &\leq C_0\mathbb{E}(T, \mathbf{S}_{N-k}, \mathbf{B}_1) + C_0\varepsilon_{(N-k)}(\varsigma_2/C_k^*)^{-2}\mathbb{E}(T, \mathbf{S}, \mathbf{B}_1)\end{aligned}$$

which in particular gives the inductive definition of C_k^*. Note that this procedure can occur at most $N-2$ times, and thus all the constants and smallness assumptions here only depend on $N, m, n, \bar{n}$, and ς_2, and so certainly we can choose $\varepsilon_2 = \varepsilon_2(N, m, n, \bar{n}, \varsigma_2)$ small enough so that the above procedure is guaranteed whenever we cannot apply Theorem 4.8.2.

Hence, for this choice of ε_2, if we are unable to apply Theorem 4.8.2 at all steps to conclude, the above process must eventually terminate when the cone is formed of exactly two planes, i.e. at $\mathbf{S}_2$. Since we are unable to apply Theorem 4.8.2 to $\mathbf{S}_2$, we must have

$$\mathbb{E}(T, \mathbf{S}, \mathbf{B}_1) > \varepsilon_{(2)}(\varsigma_2/C_{N-2}^*)^2\boldsymbol{\sigma}(\mathbf{S}_2)^2. \tag{4.191}$$

However, as $\mathbf{S}_2$ consists of two planes we know that $\boldsymbol{\sigma}(\mathbf{S}_2) = \boldsymbol{\mu}(\mathbf{S}_2)$, and moreover since by construction we know that $\boldsymbol{\mu}(\mathbf{S}_2) = \boldsymbol{\mu}(\mathbf{S})$, we see that (4.191) is equivalent to

$$\mathbb{E}(T, \mathbf{S}, \mathbf{B}_1) > \varepsilon_{(2)}(\varsigma_2/C_{N-2}^*)^2\boldsymbol{\mu}(\mathbf{S})^2. \tag{4.192}$$

However, we are assuming (see Proposition 4.8.3(iii)) that $\mathbb{E}(T, \mathbf{S}, \mathbf{B}_1) \leq \varepsilon_2^2\mathbf{E}^p(T, \mathbf{B}_1)$, which from (4.186) we know implies

$$\mathbb{E}(T, \mathbf{S}, \mathbf{B}_1) \leq C\varepsilon_2^2\boldsymbol{\mu}(\mathbf{S})^2$$

for some $C = C(Q, m, n, \bar{n}) > 0$. Thus, if we ensure that $C\varepsilon_2^2 < \frac{1}{2}\varepsilon_{(2)}(\varsigma_2/C^*_{N-2})^2$ also, then this would give a contradiction to (4.192); hence, for such a choice of ε_2 we see that once we reach $\mathbf{S}_2$, we in fact *must* be able to apply Theorem 4.8.2 in the above procedure.

We hence conclude that, for any fixed $N \leq Q$, if the threshold ε_2 satisfies all the (finitely many) smallness conditions above, then we have that Proposition 4.8.3 holds for one of the radii $r^1_{N,N}, r^2_{N,N}, r^1_{N,N-1}, r^2_{N,N-1}, \ldots, r^1_{N,2}, r^1_{N,2}$, of which there are $2(N-1)$ radii. In particular, if ε_2 satisfies all of these smallness conditions as we range N over $\{2, \ldots, Q\}$, then the above argument is valid for any cone $\mathbf{S} \in \mathscr{C}(Q, 0)$ comprised of $N \in \{2, \ldots, Q\}$ planes, i.e. for every $\mathbf{S} \in \mathscr{C}(Q, 0) \setminus \mathscr{P}$. Thus, Proposition 4.8.3 holds for every $\mathbf{S} \in \mathcal{C}(Q, 0) \setminus \mathscr{P}$ for one of the radii in the collection $\{r^i_{j,k}\}$, where $i \in \{1, 2\}$ and $2 \leq k \leq j \leq Q$. Thus, the number of possible decay scales is

$$\sum_{N=2}^{Q} 2(N-1) = Q(Q-1) = \bar{N}$$

as claimed in the statement of Proposition 4.8.3. This completes the proof. □

Now we will prove Theorem 4.2.4 from Proposition 4.8.3, thus reducing the proof of Theorem 4.2.4 to proving the weaker version, Theorem 4.8.2. First, we observe a corollary of the Proposition 4.7.1 and Proposition 4.6.14, the purpose of which is to control the excess of rescalings; it will be a useful tool in the proof of the reduction.

Corollary 4.8.4. *Suppose $\tilde{T}$ and $\tilde{\Sigma}$ satisfy Assumption 3.2.2, let $\tilde{\mathbf{S}} \in \mathscr{C}(Q, 0) \setminus \mathscr{P}(0)$, and write $\tilde{\mathbf{A}}^2$ for the supremum norm of the second fundamental form of $\tilde{\Sigma}$. Fix also a radius $\bar{r} \in (0, 1]$. Then, there are constants $\tilde{\varepsilon} = \tilde{\varepsilon}(Q, m, n, \bar{n}, \bar{r}) > 0$ and $\tilde{C} = \tilde{C}(Q, m, n, \bar{n}, \bar{r}) > 0$ such that the following holds. If we have*

$$\mathbb{E}(\tilde{T}, \tilde{\mathbf{S}}, \mathbf{B}_1) \leq \tilde{\varepsilon}^2 \mathbf{E}^p(\tilde{T}, \mathbf{B}_1)$$

and

$$\tilde{\mathbf{A}}^2 \leq \tilde{\varepsilon}^2 \mathbb{E}(\tilde{T}, \bar{\mathbf{S}}, \mathbf{B}_1) \qquad \forall \bar{\mathbf{S}} \in \mathscr{C}(Q, 0)\,,$$

then there is $\mathbf{S}' \in \mathscr{C}(Q,0) \setminus \mathscr{P}(0)$ *such that*

$$\mathbb{E}(\tilde{T}, \mathbf{S}', \mathbf{B}_r) \leq \tilde{C}\mathbb{E}(\tilde{T}, \tilde{\mathbf{S}}, \mathbf{B}_1) \qquad \forall r \in [\bar{r}, 1]. \tag{4.193}$$

Proof. Observe that for any $r \in [\bar{r}, 1]$ we clearly have

$$\hat{\mathbf{E}}(\tilde{T}, \tilde{\mathbf{S}}, \mathbf{B}_r) = r^{-m-2} \int_{\mathbf{B}_r} \operatorname{dist}^2(x, \tilde{\mathbf{S}})\, d\|\tilde{T}\|(x) \leq \bar{r}^{-m-2}\hat{\mathbf{E}}(\tilde{T}, \tilde{\mathbf{S}}, \mathbf{B}_1).$$

On the other hand, there is no obvious reason to have a bound of the form

$$\hat{\mathbf{E}}(\tilde{\mathbf{S}}, \tilde{T}, \mathbf{B}_r) \leq C(\bar{r})\mathbb{E}(\tilde{T}, \tilde{\mathbf{S}}, \mathbf{B}_1)$$

because in the integral defining the reverse excess $\hat{\mathbf{E}}(\tilde{\mathbf{S}}, \tilde{T}, \mathbf{B}_r)$ we omit a tubular neighbourhood of the spine $V(\tilde{\mathbf{S}})$ of radius ar, and thus there is no inclusion property of one domain of integration into the other when we are comparing two different scales.

To get around this problem, first note that, provided $\tilde{\varepsilon} \leq \varepsilon_0$ where $\varepsilon_0 = \varepsilon_0(Q, m, n, \bar{n})$ is as in Proposition 4.7.1, we can apply Proposition 4.7.1 to assume without loss of generality that $\tilde{\mathbf{S}}$ is M-balanced (M as in Assumption 4.8.1; all that changes is the two-sided excess increases by a factor of $M = M(Q, m, n, \bar{n})$).

Fix $\delta > 0$ (to be determined later). Now, apply the Pruning Lemma (Lemma 4.6.2) with this choice of δ and with $D = \mathbf{E}(\tilde{T}, \tilde{\mathbf{S}}, \mathbf{B}_1)^{1/2}$ to the (balanced) cone $\tilde{\mathbf{S}}$, in exactly the same way as was done in the proof of Proposition 4.7.1, to yield a new cone $\mathbf{S}'$. The latter is the union of some subset of the planes in $\tilde{\mathbf{S}}$, in particular it is M-balanced and obeys $V(\mathbf{S}') = V(\tilde{\mathbf{S}})$. Moreover it satisfies

$$\mathbb{E}(\tilde{T}, \mathbf{S}', \mathbf{B}_1) \leq C_\delta \mathbb{E}(\tilde{T}, \tilde{\mathbf{S}}, \mathbf{B}_1)$$

and

$$\mathbb{E}(\tilde{T}, \mathbf{S}', \mathbf{B}_1) \leq C\delta^2 \boldsymbol{\sigma}(\mathbf{S}')^2$$

where here $C_\delta = C_\delta(Q, m, n, \bar{n}, \delta)$ and $C = C(Q, m, n, \bar{n})$. In particular, we have

$$\tilde{\mathbf{A}}^2 + \mathbb{E}(\tilde{T}, \mathbf{S}', \mathbf{B}_1) \leq C(\tilde{\varepsilon}^2 + \delta^2)\boldsymbol{\sigma}(\mathbf{S}')^2.$$

Thus, we are in a situation to apply the refined approximation (Proposition 4.6.14) to $\tilde{T}$, $\mathbf{S}'$, provided we take $\tilde{\varepsilon}, \delta$ sufficiently small. The region we are interested in building the refined approximation over is $\mathbf{B}_1 \setminus B_{a\bar{r}/2}(V(\tilde{\mathbf{S}}))$, and so in particular the radius of this neighbourhood of the spine is fixed only depending on $\bar{r}$ and a. Thus, by Lemma 4.6.13, if we take $\tilde{\varepsilon}$ and δ smaller than a constant depending only on $Q, m, n, \bar{n}, \bar{r}$, we can ensure that $\mathbf{B}_1 \setminus B_{a\bar{r}/2}(V(\tilde{\mathbf{S}})) \subset R^o$, (recall that R^o is the outer region of the refined approximation). In particular, one can apply Proposition 4.6.14; the estimates therein (summed over the cubes $L \in \mathcal{G}^o \cup \mathcal{G}^c$ that intersect $\mathbf{B}_r \setminus B_{ar}(V(\tilde{\mathbf{S}}))$ give that

$$\hat{\mathbf{E}}(\mathbf{S}', \tilde{T}, \mathbf{B}_r) \leq \tilde{C}\mathbb{E}(\tilde{T}, \mathbf{S}', \mathbf{B}_1)$$

for any $r \in [\bar{r}, 1]$, where $\tilde{C} = \tilde{C}(Q, m, n, \bar{n}, \bar{r})$. Hence by choosing such an appropriate $\delta = \delta(Q, m, n, \bar{n}, \bar{r})$ and $\tilde{\varepsilon} = \tilde{\varepsilon}(Q, m, n, \bar{n}, \bar{r})$, we see that the proof is complete. □

The next lemma gives the final ingredient for our proof that Proposition 4.8.3 implies Theorem 4.2.4(a). The lemma provides a condition for which the rescalings of Corollary 4.8.4 also satisfy the conditions of Theorem 4.2.4.

Lemma 4.8.5. *Fix a radius $\bar{r} \in (0, \frac{1}{2}]$ and $\varepsilon_2 \in (0, 1)$. Then, there exists a positive number $\varepsilon_0 = \varepsilon_0(Q, m, n, \bar{n}, \bar{r}, \varepsilon_2)$ such that the following holds. Suppose that T, Σ, and $\mathbf{S}$ satisfy the assumptions in Theorem 4.2.4 with this choice of ε_0. Then, for every radius $r \in [\bar{r}, \frac{1}{2}]$ we*

have the following. If there is a cone $\mathbf{S}_r \in \mathscr{C}(Q,0)$ *which obeys*

$$\mathbb{E}(T, \mathbf{S}_r, \mathbf{B}_r) \leq \mathbb{E}(T, \mathbf{S}, \mathbf{B}_1), \tag{4.194}$$

and moreover

$$\mathbf{A}^2 r^2 \leq \varepsilon_0^2 \inf\{\mathbb{E}(T, \mathbf{S}', \mathbf{B}_r) : \mathbf{S}' \in \mathscr{C}(Q,0)\}, \tag{4.195}$$

then the rescalings $T_{0,r}$, $\Sigma_{0,r}$, *and* $\mathbf{S}_r$ *satisfy the assumptions (i)–(iii) of Theorem 4.2.4 (and so also Proposition 4.8.3 and Corollary 4.8.4) with* ε_2 *in place of* ε_0, *i.e.*

$$\mathbb{E}(T_{0,r}, \mathbf{S}_r, \mathbf{B}_1) \leq \varepsilon_2^2 \mathbf{E}^p(T_{0,r}, \mathbf{B}_1)$$

and

$$\mathbf{B}_{\varepsilon_2}(\xi) \cap \{p : \Theta(T_{0,r}, p) \geq Q\} \neq \emptyset \qquad \forall \xi \in V(\mathbf{S}_r) \cap \mathbf{B}_{1/2}.$$

Proof. We argue by contradiction. Fix ε_2 and $\bar{r}$. If the conclusion of the lemma is false, then we can find a sequence of currents T_k, manifolds Σ_k, cones $\mathbf{S}_k$, and $\varepsilon_0^k \downarrow 0$ as in Theorem 4.2.4 with ε_0^k in place of ε_0 for which the present lemma fails. Namely, there exist radii $r_k \in [\bar{r}, \frac{1}{2}]$ and cones $\mathbf{S}_{r_k} \equiv (\mathbf{S}_k)_{r_k}$ which obey

$$(\varepsilon_0^k)^{-2} r_k^2 \mathbf{A}_k^2 \leq \mathbb{E}(T_k, \mathbf{S}_{r_k}, \mathbf{B}_{r_k}) \leq \mathbb{E}(T_k, \mathbf{S}_k, \mathbf{B}_1)$$

yet either we have (as the inequality $\mathbf{A}_{0,r_k}^2 \leq \varepsilon_2^2 \mathbb{E}((T_k)_{0,r_k}, \mathbf{S}_{r_k}, \mathbf{B}_1)$ holds for all k sufficiently large by the above assumption)

$$\mathbb{E}(T_k, \mathbf{S}_{r_k}, \mathbf{B}_{r_k}) > \varepsilon_2^2 \mathbf{E}^p(T_k, \mathbf{B}_{r_k})$$

or

$$\mathbf{B}_{r_k \varepsilon_2}(\xi) \cap \{p : \Theta(T_k, p) \geq Q\} = \emptyset \quad \text{ for some } \xi \in V(\mathbf{S}_{r_k}) \cap \mathbf{B}_{r_k/2}.$$

In particular, since the assumptions of Theorem 4.2.4 hold for T_k and $\mathbf{S}_k$, points of density at least Q in T_k accumulate on the spines $V(\mathbf{S}_k)$. Thus, if we pass to a subsequence for which $V(\mathbf{S}_k)$ and $V(\mathbf{S}_{r_k})$ converge, the second condition above in fact gives that we must have

$$\lim_{k\to\infty} \operatorname{dist}(V(\mathbf{S}_{r_k}) \cap \mathbf{B}_1, V(\mathbf{S}_k) \cap \mathbf{B}_1) \geq \bar{r}\varepsilon_2/2 > 0 \tag{4.196}$$

whilst the first condition above gives

$$\liminf_{k\to\infty} \frac{\mathbb{E}(T_k, \mathbf{S}_{r_k}, \mathbf{B}_{r_k})}{\mathbf{E}^p(T_k, \mathbf{B}_{r_k})} \geq \varepsilon_2^2 > 0. \tag{4.197}$$

Without loss of generality, we can select planes π_k realizing $\mathbf{E}^p(T_k, \mathbf{B}_1)$, and, up to performing a rotation, we can assume they all coincide with the same fixed plane, π_0. We can also pass to a subsequence to ensure that the number N_k of planes forming $\mathbf{S}_k$ is a constant N (obeying $N \leq Q$), and similarly the number of planes forming $\mathbf{S}_{r_k}$ is a constant $\bar{N}$.

Let us first contradict (4.197). Note that we have

$$\mathbb{E}(T_k, \mathbf{S}_{r_k}, \mathbf{B}_{r_k}) \leq \mathbb{E}(T_k, \mathbf{S}_k, \mathbf{B}_1) \leq (\varepsilon_0^k)^2 \mathbf{E}^p(T_k, \mathbf{B}_1) \to 0\,.$$

and thus (4.197) tells us that we must have $\mathbf{E}^p(T_k, \mathbf{B}_{r_k}) \to 0$. In particular, up to subsequences, we know that the current T_k is converging to a limiting current T_∞ which coincides with an integer multiple of some plane π_∞ in $\mathbf{B}_{r_k}$. On the other hand, if we extract a converging subsequence, not relabelled, we have $\mathbf{S}_k \to \mathbf{S}_\infty$ in the local Hausdorff topology for some $\mathbf{S}_\infty \in \mathscr{C}(Q, 0)$. Hence we also conclude that $\operatorname{spt}(T) \cap \mathbf{B}_1 \equiv \mathbf{S}_\infty \cap \mathbf{B}_1$. This implies that the cone $\mathbf{S}_\infty \in \mathscr{C}(Q, 0)$ is in fact the plane π_∞. In particular we conclude $\mathbf{E}^p(T_k, \mathbf{B}_1) \to 0$.

We can therefore apply Almgren's (strong) Lipschitz approximation to the sequence T_k and perform a blow-up procedure. We normalize the Lipschitz approximations by

$$E_k^{1/2} = \mathbf{E}^p(T_k, \mathbf{B}_1)^{1/2}$$

and, upon possibly applying suitable rotations (see Remark 4.6.20), extract a Dir-minimizing function $f_\infty : B_{1/2}(\pi_\infty) \to \mathcal{A}_Q(\pi_\infty^\perp)$ in the blow-up limit. Moreover, since $\mathbb{E}(T_k, \mathbf{S}_k, \mathbf{B}_1) \leq (\varepsilon_0^k)^2 \mathbf{E}^p(T_k, \mathbf{B}_1)$, we see that if we describe the cones $\mathbf{S}_k$ as a union of graphs of linear maps $L_1^k, \ldots, L_N^k : \pi_\infty \to \pi_\infty^\perp$ then, as we have previously seen, upon extraction of a subsequence we will have that $E_k^{-1/2} L_i^k$ converges to some linear function L_i^∞ for each $i = 1, \ldots, N$. At least two of these linear functions are distinct, given that by Proposition 4.7.1(d) $\boldsymbol{\mu}(\mathbf{S})$ is comparable to $E_k^{1/2}$. But now, since $f_\infty = \sum_{i=1}^N \llbracket L_i^\infty \rrbracket$ and $r_k \leq \frac{1}{2}$, this would imply that $C^{-1} \leq \mathbf{E}^p(T_k, \mathbf{B}_{r_k}) / \mathbf{E}^p(T_k, \mathbf{B}_1) \leq C$ for some geometric constant $C > 0$, and hence we would have

$$\frac{\mathbb{E}(T_k, \mathbf{S}_{r_k}, \mathbf{B}_{r_k})}{\mathbf{E}^p(T_k, \mathbf{B}_{r_k})} \leq C \frac{\mathbb{E}(T_k, \mathbf{S}_{r_k}, \mathbf{B}_{r_k})}{\mathbf{E}^p(T_k, \mathbf{B}_1)} \leq C \frac{\mathbb{E}(T_k, \mathbf{S}_k, \mathbf{B}_1)}{\mathbf{E}^p(T_k, \mathbf{B}_1)} \leq C(\varepsilon_0^k)^2 \to 0$$

which is in direct contradiction to (4.197).

Now we contradict (4.196). We distinguish two cases here, depending on whether $\mathbf{E}^p(T, \mathbf{B}_1)$ converges to 0 or not. In the latter case we can assume, after extracting a suitable subsequence, that $\mathbf{S}_k$ converges to a non-planar cone $\mathbf{S}_\infty \in \mathscr{C}(Q, 0) \setminus \mathscr{P}$ in the local Hausdorff topology which is the support of an area-minimizing current. But then it would follow that $\mathbf{S}_{r_k}$ converges to the same cone. In particular $V(\mathbf{S}_{r_k}) \cap \mathbf{B}_1$ and $V(\mathbf{S}_k) \cap \mathbf{B}_1$ both converge to $V(\mathbf{S}_\infty) \cap \mathbf{B}_1$, contracting (4.196).

In the other case note that, since $\mathbb{E}(T_k, \mathbf{S}_{r_k}, \mathbf{B}_{r_k}) \leq (\varepsilon_0^k)^2 \mathbf{E}^p(T_k, \mathbf{B}_1)$, if we assume that the cones $\mathbf{S}_{r_k}$ are the union of $\bar{N}$ distinct linear maps $\bar{L}_i^k$, we see that $E_k^{-1/2} \bar{L}_i^k$ converge to a linear map $\bar{L}_i^\infty$ for each $i = 1, \ldots, \bar{N}$. Furthermore the union of the graphs of these limiting linear maps must coincide with the support of the map f_∞ found previously (indeed, we get from the above inequality that they must coincide on $\mathbf{B}_{\bar{r}}$, and hence on all of $\mathbf{B}_1$ as they are linear). But this is only possible if

$$\operatorname{dist}(V(\mathbf{S}_{r_k}) \cap \mathbf{B}_1, V(\mathbf{S}_k) \cap \mathbf{B}_1) \to 0$$

which contradicts (4.196). This completes the proof. □

We are now ready to complete the proof of Theorem 4.2.4 from Proposition 4.8.3. We first address the most important conclusion, which is the decay estimate in (a). We will see after how (b), (c), and (d) can be derived from this.

Proof of Theorem 4.2.4(a) from Proposition 4.8.3. We therefore fix T, Σ, $\mathbf{S}$, and ς as in the statement of Theorem 4.2.4, and fix parameters ε_0 and $r_0 < 1/2$ to be specified later.

Choose $\varsigma_2 = \frac{1}{2}$ in Proposition 4.8.3, and denote by $\varepsilon_2 = \varepsilon_2(Q, m, n, \bar{n})$ the corresponding constant from Proposition 4.8.3 for this choice of ς_2. Also denote by $r_1, \dots, r_{\bar{N}} \leq \frac{1}{2}$ the corresponding radii from Proposition 4.8.3 with this choice of ς_2. Write $\bar{r} := \min\{r_i : i = 1, \dots, \bar{N}\}$, which depends only on $Q, m, n, \bar{n}$.

Define

$$\mathbb{E}(T, \mathbf{B}_r) := \inf\{\mathbb{E}(T, \mathbf{S}, \mathbf{B}_r) : \mathbf{S} \in \mathscr{C}(Q, 0) \setminus \mathscr{P}(0)\}$$

and

$$\mathcal{A} := \{r \in [r_0, 1/2] : \varepsilon_0^{-2} \mathbf{A}^2 r^2 \geq \mathbb{E}(T, \mathbf{B}_r)\}.$$

We now distinguish three possibilities.

Case 1: $\mathcal{A} = \emptyset$. We then have $\varepsilon_0^{-2} r^2 \mathbf{A}^2 \leq \mathbb{E}(T, \mathbf{B}_r)$ for all $r \in [r_0, 1/2]$. If $\varepsilon_0 \leq \varepsilon_2$, then at scale 1 the assumptions of Theorem 4.2.4 give that we can apply Proposition 4.8.3 to $T, \Sigma, \mathbf{S}$ to get that there exists $\rho_1 \in [\bar{r}, 1/2]$ (indeed, $\rho_1 = r_i$ for some $i = 1, \dots, \bar{N}$, but this is not relevant) and a cone $\mathbf{S}_1 \in \mathscr{C}(Q, 0) \setminus \mathscr{P}(0)$ such that

$$\mathbb{E}(T, \mathbf{S}_1, \mathbf{B}_{\rho_1}) \leq \frac{1}{2} \mathbb{E}(T, \mathbf{S}, \mathbf{B}_1).$$

Let us assume (as we may) that $r_0 < \bar{r}$. Since $\mathcal{A} = \emptyset$, we therefore fall into the realm of Lemma 4.8.5, taking ε_2 and $\bar{r}$ to be the above constants. If ε_0 is smaller than the corresponding constant from Lemma 4.8.5 with this choice of ε_2 and $\bar{r}$ (and thus only depends on $Q, m, n, \bar{n}$), we see that Lemma 4.8.5 gives that T_{0,ρ_1}, Σ_{0,ρ_1}, and some cone $\mathbf{S}_1'$ (namely the cone achieving the infimum in $\mathbb{E}(T, \mathbf{B}_{\rho_1})$) obey the necessary conditions to apply Propo-

sition 4.8.3 again. Thus, we find a second radius ρ_2 obeying $\rho_2/\rho_1 \in [\bar{r}, 1/2]$ and a cone $\mathbf{S}_2 \in \mathscr{C}(Q,0) \setminus \mathscr{P}(0)$ obeying

$$\mathbb{E}(T, \mathbf{S}_2, \mathbf{B}_{\rho_2}) \leq \frac{1}{2}\mathbb{E}(T, \mathbf{S}_1, \mathbf{B}_{\rho_1}) \leq \frac{1}{4}\mathbb{E}(T, \mathbf{S}, \mathbf{B}_1).$$

We now keep iterating this procedure, until we arrive at a radius ρ_k so that the next radius ρ_{k+1} is smaller than r_0 (in particular we no longer necessarily have that $\varepsilon_0^{-2}\mathbf{A}^2\rho_{k+1}^2 \leq \mathbb{E}(T, \mathbf{B}_{\rho_{k+1}})$); as $\rho_{k+1}/\rho_k \in [\bar{r}, 1/2]$, when this happens we have $\bar{r} \leq \rho_{k+1}/\rho_k < r_0/\rho_k$. Since we can apply Corollary 4.8.4 to T_{0,ρ_k}, Σ_{0,ρ_k}, and $\mathbf{S}_k$ (again, using the fact that $\rho_k \notin \mathcal{A} = \emptyset$ and Lemma 4.8.5), provided ε_2 is smaller than the corresponding constant from Corollary 4.8.4 with this choice of $\bar{r}$ (which can of course be arranged), we therefore get the existence of a cone $\mathbf{S}' \in \mathscr{C}(Q,0) \setminus \mathscr{P}(0)$ such that (choosing $r^* = r_0/\rho_k \in [\bar{r}, 1]$ in Corollary 4.8.4, so that $r^*\rho_k = r_0$)

$$\mathbb{E}(T_{0,\rho_k}, \mathbf{S}', \mathbf{B}_{r^*}) \leq C\mathbb{E}(T_{0,\rho_k}, \mathbf{S}_k, \mathbf{B}_1)$$

where $C = C(Q, m, n, \bar{n})$ (indeed the constant C depends on $\bar{r}$, but at this stage the latter has been fixed as depending only upon Q, m, n, and $\bar{n}$). In particular, combined with the above iteration, the last inequality yields

$$\mathbb{E}(T, \mathbf{S}', \mathbf{B}_{r_0}) \leq C\mathbb{E}(T, \mathbf{S}_k, \mathbf{B}_{\rho_k}) \leq C \cdot 2^{-k}\mathbb{E}(T, \mathbf{S}, \mathbf{B}_1).$$

Observe however that as $\bar{r} < r_0/\rho_k < r_0/\bar{r}^k$, i.e. $r_0 > \bar{r}^{k+1}$, we have $k + 1 \geq \log(r_0)/\log(\bar{r})$. Hence, choosing $r_0 = r_0(Q, m, n, \bar{n})$ sufficiently small we can ensure that $k \geq \frac{1}{2}\left\lfloor \frac{\log(r_0)}{\log(\bar{r})} \right\rfloor$, and hence

$$\mathbb{E}(T, \mathbf{S}', \mathbf{B}_{r_0}) \leq C2^{-\frac{1}{2}\left\lfloor \frac{\log(r_0)}{\log(\bar{r})} \right\rfloor}\mathbb{E}(T, \mathbf{S}, \mathbf{B}_1). \tag{4.198}$$

Now we may further take r_0 small enough to make sure that $C2^{-\frac{1}{2}\left\lfloor \frac{\log(r_0)}{\log(\bar{r})} \right\rfloor} \leq \varsigma$ to deduce the conclusion (a) of Theorem 4.2.4.

Case 2: $\inf \mathcal{A} = r_0$. Note that $\mathcal{A}$ is certainly a closed set, and so in this case we have

$r_0 \in \mathcal{A}$. Hence, we have

$$\mathbb{E}(T, \mathbf{B}_{r_0}) \leq \varepsilon_0^{-2} \mathbf{A}^2 r_0^2 \leq r_0^2 \mathbb{E}(T, \mathbf{S}, \mathbf{B}_1)$$

where the second inequality comes from our assumption in Theorem 4.2.4. In particular, we can find $\mathbf{S}' \in \mathscr{C}(Q, 0) \setminus \mathscr{P}(0)$ with

$$\mathbb{E}(T, \mathbf{S}', \mathbf{B}_{r_0}) \leq 2\mathbb{E}(T, \mathbf{B}_{r_0}) \leq 2r_0^2 \mathbb{E}(T, \mathbf{S}, \mathbf{B}_1)$$

and so we just need to take $2r_0^2 < \varsigma$ to deduce conclusion (a) of Theorem 4.2.4.

Case 3: $\mathcal{A} \neq \emptyset$ yet $\inf \mathcal{A} > r_0$. In this case, let $r_I = \inf \mathcal{A} \in \mathcal{A}$. Let $\mathbf{S}_I \in \mathscr{C}(Q, 0) \setminus \mathcal{P}(0)$ be such that

$$\mathbb{E}(T, \mathbf{S}_I, \mathbf{B}_{r_I}) \leq 2\mathbb{E}(T, \mathbf{B}_{r_I}) \leq 2\varepsilon_0^{-2} \mathbf{A}^2 r_I^2. \tag{4.199}$$

Now observe that

$$\lim_{r \uparrow r_I} \mathbb{E}(T, \mathbf{S}_I, \mathbf{B}_r) = \mathbb{E}(T, \mathbf{S}_I, \mathbf{B}_{r_I})$$

Moreover, as $[r_0, r_I) \neq \emptyset$ and $[r_0, r_I) \cap \mathcal{A} = \emptyset$, we see that for all $r \in [r_0, r_I)$,

$$\varepsilon_0^{-2} r^2 \mathbf{A}^2 \leq \mathbb{E}(T, \mathbf{B}_r) \leq \mathbb{E}(T, \mathbf{S}_I, \mathbf{B}_r).$$

Taking $r \uparrow r_I$ in this expression, we deduce

$$\varepsilon_0^{-2} r_I^2 \mathbf{A}^2 \leq \mathbb{E}(T, \mathbf{S}_I, \mathbf{B}_{r_I}).$$

Moreover, from (4.199) and our assumption (4.1) we know

$$\mathbb{E}(T, \mathbf{S}_I, \mathbf{B}_{r_I}) \leq 2\varepsilon_0^{-2} \mathbf{A}^2 r_I^2 \leq 2r_I^2 \mathbb{E}(T, \mathbf{S}, \mathbf{B}_1). \tag{4.200}$$

Hence, we have all the necessary assumptions to apply Lemma 4.8.5, which guarantees that

we can apply Proposition 4.8.3 to T_{0,r_I}, Σ_{0,r_I}, and $\mathbf{S}_I$. We can now argue as in Case 1, but with r_I replacing 1 as a starting point of the iteration and with r_0/r_I as the lower endpoint. We therefore get the existence of an $\mathbf{S}' \in \mathscr{C}(Q,0) \setminus \mathscr{P}(0)$ with the property that (see (4.198))

$$\mathbb{E}(T, \mathbf{S}', \mathbf{B}_{r_0}) \leq C 2^{-\frac{1}{2}\frac{\log(r_0/r_I)}{\log(\bar{r})}} \mathbb{E}(T, \mathbf{S}_I, \mathbf{B}_{r_I}).$$

Combining this with (4.200) we arrive at

$$\mathbb{E}(T, \mathbf{S}', \mathbf{B}_{r_0}) \leq \left(2 r_I^2 C 2^{\frac{1}{2}\frac{\log(r_I)}{\log(\bar{r})}} \right) 2^{-\frac{1}{2}\frac{\log(r_0)}{\log(\bar{r})}} \mathbb{E}(T, \mathbf{S}, \mathbf{B}_1).$$

Noting that $2^{\frac{1}{2}\frac{\log(r_I)}{\log(\bar{r})}} = \left(\frac{1}{r_I}\right)^{\frac{\log(2)}{2\log(1/\bar{r})}} \leq r_I^{-\frac{1}{2}}$, the above becomes

$$\mathbb{E}(T, \mathbf{S}', \mathbf{B}_{r_0}) \leq \left(2C r_I^{3/2} \right) 2^{-\frac{1}{2}\frac{\log(r_0)}{\log(\bar{r})}} \mathbb{E}(T, \mathbf{S}, \mathbf{B}_1) \leq 2C \cdot 2^{-\frac{1}{2}\frac{\log(r_0)}{\log(\bar{r})}} \mathbb{E}(T, \mathbf{S}, \mathbf{B}_1).$$

Thus choosing r_0 small enough so that $2C \cdot 2^{-\frac{1}{2}\frac{\log(r_0)}{\log(\bar{r})}} \leq \varsigma$, we deduce (a) from Theorem 4.2.4. As the above three situations exhaust all possibilities, the proof is completed. □

To summarize: we have now reduced the proof of point (a) in Theorem 4.2.4 to proving the a priori much weaker result of Theorem 4.8.2. We will next address how points (b), (c), and (d) follow from (a). In fact, since it will prove to be useful also later when we get to the rectifiability statement in Theorem 3.1.1, we state here a more general lemma.

Lemma 4.8.6. *There is a constant $C = C(Q, m, n, \bar{n}) > 0$ with the following property. For every $r_0 > 0$, there exists $\varepsilon = \varepsilon(Q, m, n, \bar{n}, r_0) > 0$ such that if*

(i) T and Σ are as in Assumption 3.2.2;

(ii) $\|T\|(\mathbf{B}_1) \leq (Q + \frac{1}{2})\omega_m$; and

(iii) there is $\mathbf{S} \in \mathscr{C}(Q,0) \setminus \mathscr{P}(0)$ such that

$$\mathbf{A}^2 + \mathbb{E}(T, \mathbf{S}, \mathbf{B}_1) \leq \varepsilon^2 \mathbf{E}^p(T, \mathbf{B}_1) \ ; \tag{4.201}$$

then

$$C^{-1}\mathbf{E}^p(T,\mathbf{B}_1) \le \mathbf{E}^p(T,\mathbf{B}_{r_0}) \le C\mathbf{E}^p(T,\mathbf{B}_1)\,. \tag{4.202}$$

Moreover, there is a constant $\bar{C} = \bar{C}(Q,m,n,\bar{n},r_0)$ *such that, provided* ε *is chosen possibly smaller, if there is another cone* $\mathbf{S}' \in \mathscr{C}(Q,0)\backslash\mathscr{P}(0)$ *that also obeys*

$$\mathbb{E}(T,\mathbf{S}',\mathbf{B}_{r_0}) \le \varepsilon^2\mathbf{E}^p(T,\mathbf{B}_1)\,, \tag{4.203}$$

then

$$\mathrm{dist}^2(\mathbf{S}\cap\mathbf{B}_1,\mathbf{S}'\cap\mathbf{B}_1) \le \bar{C}(\mathbf{A}^2+\mathbb{E}(T,\mathbf{S},\mathbf{B}_1)+\mathbb{E}(T,\mathbf{S}',\mathbf{B}_{r_0})) \tag{4.204}$$

$$\mathrm{dist}^2(V(\mathbf{S})\cap\mathbf{B}_1,V(\mathbf{S}')\cap\mathbf{B}_1) \le \bar{C}\mathbf{E}^p(T,\mathbf{B}_1)^{-1}(\mathbf{A}^2+\mathbb{E}(T,\mathbf{S},\mathbf{B}_1)+\mathbb{E}(T,\mathbf{S}',\mathbf{B}_{r_0}))\,. \tag{4.205}$$

Proof. First of all we argue for (4.202). We start by demonstrating that

$$\boldsymbol{\mu}(\mathbf{S})^2 \le C\mathbf{E}^p(T,\mathbf{B}_1)\,. \tag{4.206}$$

In fact, fix π such that $\mathbf{E}^p(T,\mathbf{B}_1) = \hat{\mathbf{E}}(T,\pi,\mathbf{B}_1)$ and use Allard's L^2–L^∞ height bound to infer that

$$\mathrm{dist}^2(q,\pi) \le C\mathbf{E}^p(T,\mathbf{B}_1) + C\mathbf{A}^2 \qquad \forall q\in\mathrm{spt}(T)\cap\mathbf{B}_{1/2}\,. \tag{4.207}$$

Next, let α_i be one of the planes which form $\mathbf{S}$ and recall that

$$\int_{\alpha_i\cap\mathbf{B}_1\setminus B_a(V)} \mathrm{dist}^2(q,\mathrm{spt}(T))\,d\mathcal{H}^m(q) \le \mathbb{E}(T,\mathbf{S},\mathbf{B}_1)\,.$$

Using Chebyshev's inequality (applied to the measure $\mathcal{H}^m\llcorner\alpha_i$), we have that the set $F_i\subset\alpha_i\cap\mathbf{B}_{1/4}\setminus\mathbf{B}_a(V)$ of points $q\in\alpha_i\cap\mathbf{B}_{1/4}\setminus B_a(V)$ obeying

$$\mathrm{dist}^2(q,\mathrm{spt}(T)) \le \frac{C}{\mathcal{H}^m(\alpha_i\cap\mathbf{B}_{1/4}\setminus\mathbf{B}_a(V))}\mathbb{E}(T,\mathbf{S},\mathbf{B}_1) \tag{4.208}$$

satisfies $\mathcal{H}^m(F_i) \geq \left(1 - \frac{1}{C}\right) \mathcal{H}^m(\alpha_i \cap \mathbf{B}_{1/4} \setminus \mathbf{B}_a(V))$ and so for a choice of $C = C(m) > 0$ sufficiently large, there exist m linearly independent vectors $e_1, \ldots, e_m \in F_i$ with the property that every $v \in \alpha_i \cap \mathbf{B}_1$ can be written as $v = \sum_j \lambda_j e_j$ with $|\lambda_j| \leq C$.

Combining the above distance inequalities, for each $j \in \{1, \ldots, m\}$ we have

$$\operatorname{dist}^2(e_j, \pi) \leq C(\mathbb{E}(T, \mathbf{S}, \mathbf{B}_1) + \mathbf{A}^2 + \mathbf{E}^p(T, \mathbf{B}_1)) \leq C\mathbf{E}^p(T, \mathbf{B}_1)\,.$$

For each $v \in \alpha_i \cap \mathbf{B}_1$, using the identity $\operatorname{dist}(v, \pi) = |\mathbf{p}_\pi^\perp(v)|$, this in turn implies

$$\operatorname{dist}^2(v, \pi) \leq C\mathbf{E}^p(T, \mathbf{B}_1)\,.$$

We thus can use (4.54) of Corollary 4.5.3 to infer that

$$\operatorname{dist}^2(\alpha_i \cap \mathbf{B}_1, \pi \cap \mathbf{B}_1) \leq C\mathbf{E}^p(T, \mathbf{B}_1)\,, \tag{4.209}$$

which in turn implies (4.206).

Observe also that we have the inequality

$$\mathbf{E}^p(T, \mathbf{B}_1) \leq \hat{\mathbf{E}}(T, \alpha_1, \mathbf{B}_1) \leq C\hat{\mathbf{E}}(T, \mathbf{S}, \mathbf{B}_1) + C\boldsymbol{\mu}(\mathbf{S})^2 \leq C\varepsilon^2\mathbf{E}^p(T, \mathbf{B}_1) + C\boldsymbol{\mu}(\mathbf{S})^2\,,$$

and so in particular we conclude

$$C^{-1}\mathbf{E}^p(T, \mathbf{B}_1) \leq \boldsymbol{\mu}(\mathbf{S})^2 \leq C\mathbf{E}^p(T, \mathbf{B}_1)\,, \tag{4.210}$$

if ε is sufficiently small. Moreover, since we have not used any information other than the smallness of $\mathbb{E}(T, \mathbf{S}, \mathbf{B}_1) + \mathbf{A}^2$ with respect to $\mathbf{E}^p(T, \mathbf{B}_1)$, the same argument applies to $\mathbf{S}'$ and T_{0,r_0}, which allows us to conclude

$$C^{-1}\mathbf{E}^p(T, \mathbf{B}_{r_0}) \leq \boldsymbol{\mu}(\mathbf{S}')^2 \leq C\mathbf{E}^p(T, \mathbf{B}_{r_0})\,. \tag{4.211}$$

We next observe that from (4.206) (see also (4.209)) we immediately get the inequality on the right-hand side of (4.202). Indeed, we can write

$$\begin{aligned}\mathbf{E}^p(T,\mathbf{B}_{r_0}) &\le \hat{\mathbf{E}}(T,\pi,\mathbf{B}_{r_0}) \le Cr_0^{-2}\max_i \operatorname{dist}^2(\alpha_i\cap\mathbf{B}_{r_0},\pi\cap\mathbf{B}_{r_0}) + C\hat{\mathbf{E}}(T,\mathbf{S},\mathbf{B}_{r_0})\\ &\le C\mathbf{E}^p(T,\mathbf{B}_1) + Cr_0^{-m-2}\hat{\mathbf{E}}(T,\mathbf{S},\mathbf{B}_1) \le C(1+\varepsilon r_0^{-m-2})\mathbf{E}^p(T,\mathbf{B}_1)\end{aligned}$$

and so to conclude the right-hand inequality in (4.202) it suffices to choose ε small compared to r_0.

We next argue by contradiction for the left-hand inequality in (4.202). If the conclusion is false, then we could find a sequence of area-minimizing currents T_k, ambient manifolds Σ_k, cones $\mathbf{S}_k \in \mathscr{C}(Q,0)\setminus\mathscr{P}(0)$, and parameters $\varepsilon^k \downarrow 0$ for which the left-hand inequality in (4.202) fails for some fixed r_0, namely

$$\lim_{k\to\infty}\frac{\mathbf{E}^p(T_k,\mathbf{B}_{r_0})}{\mathbf{E}^p(T_k,\mathbf{B}_1)} < C_*^{-1}\,, \tag{4.212}$$

for some constant C_* which will be specified only later and which will turn out to be independent of r_0. Without loss of generality, we can select planes π_k with $\mathbf{E}^p(T_k,\mathbf{B}_1) = \hat{\mathbf{E}}(T_k,\pi_k,\mathbf{B}_1)$, and, up to performing a rotation, we can assume they all coincide with the same fixed plane, π_∞. We can also pass to a subsequence to ensure that the number N_k of planes forming $\mathbf{S}_k$ is a constant N (obeying $N\le Q$).

Observe that we must necessarily have

$$\mathbf{E}^p(T_k,\mathbf{B}_1)\to 0\,.$$

Indeed, we can assume $\mathbf{S}_k \to \mathbf{S}_\infty \in \mathscr{C}(Q,0)\setminus\mathscr{P}(0)$ locally in the Hausdorff topology, while T_k converges weakly to some area-minimizing current T_∞ and it follows easily that $\operatorname{spt}(T_\infty)\cap\mathbf{B}_1 = \mathbf{S}_\infty\cap\mathbf{B}_1$. If $\mathbf{E}^p(T_k,\mathbf{B}_1)$ does not converge to 0, then T_∞ is not supported in a plane, so it must be an area-minimizing integral cone which is not planar. But then we see immediately

that

$$\lim_{k\to\infty} \frac{\mathbf{E}^p(T_k, \mathbf{B}_1)}{\mathbf{E}^p(T_k, \mathbf{B}_{r_0})} = 1\,.$$

In particular this would be a contradiction if we impose $C_* \geq 1$ in (4.212).

We conclude then that $\mathbf{S}_k$ converges indeed to π_∞. We can therefore apply Almgren's (strong) Lipschitz approximation [19, Theorem 2.4] to the sequence T_k relative to the plane π_∞ and perform a blow-up procedure. We normalize the Lipschitz approximations by

$$E_k^{1/2} = \mathbf{E}^p(T_k, \mathbf{B}_1)^{1/2}$$

and, upon possibly applying suitable rotations (see Remark 4.6.20), extract a Dir-minimizing function $f_\infty : B_1(\pi_\infty) \to \mathcal{A}_Q(\pi_\infty^\perp)$ in the blow-up limit (the Dir-minimizing property follows from [19, Theorem 2.6]). Moreover, since $\mathbb{E}(T_k, \mathbf{S}_k, \mathbf{B}_1) \leq (\varepsilon^k)^2 \mathbf{E}^p(T_k, \mathbf{B}_1)$, we see that if we describe the cones $\mathbf{S}_k$ as a union of graphs of linear maps $L_1^k, \ldots, L_N^k : \pi_\infty \to \pi_\infty^\perp$ then upon extraction of a subsequence we will have that $E_k^{-1/2} L_i^k$ converges to some linear function L_i^∞ for each $i = 1, \ldots, N$, because of (4.206).

Observe that:

- L_i^∞ and L_j^∞ are not necessarily distinct for every $i \neq j$;
- However there is *at least one* pair of indices $i \neq j$ for which they are indeed distinct.

The second fact is a simple consequence of (4.210).

Up to reordering, we can assume that there are exactly $\bar{N} \geq 2$ distinct linear maps $L_1^\infty, \ldots, L_{\bar{N}}^\infty$ in the collection $\{L_i^\infty\}$. Consider then the cones $\mathbf{S}'_k \subset \mathbf{S}_k$ given by the graph of the linear maps $L_1^k, \ldots, L_{\bar{N}}^k$. It follows from the arguments outlined so far that

$$\lim_{k\to\infty} \frac{\hat{\mathbf{E}}(T_k, \mathbf{S}'_k, \mathbf{B}_1)}{\mathbf{E}^p(T_k, \mathbf{B}_1)} \to 0\,,$$

$$\liminf_{k\to\infty} \frac{\boldsymbol{\sigma}(\mathbf{S}'_k)}{\boldsymbol{\mu}(\mathbf{S}'_k)} > 0\,,$$

and

$$\lim_{k\to\infty} \frac{\boldsymbol{\mu}(\mathbf{S}_k)}{\boldsymbol{\mu}(\mathbf{S}'_k)} = 1\,.$$

Together with the fact that $C^{-1} \leq \frac{\mathbf{E}^p(T_k,\mathbf{B}_1)}{\boldsymbol{\mu}(\mathbf{S}_k)^2} \leq C$, we conclude that

$$\max_{1\leq i<j\leq \bar{N}} \int_{B_1} |L_i^\infty - L_j^\infty|^2 \geq 2C_*^{-1}$$

for a suitable choice of $C_* = C_*(Q, m, n, \bar{n}) > 0$.

We can now argue as in the proof of Proposition 4.7.3 to conclude that

$$f_\infty = \sum_{i=1}^{\bar{N}} Q_i [\![L_i^\infty]\!]$$

for some *positive* integers $Q_i \geq 1$.

Consider now for each k a plane π_k^1 such that $\hat{\mathbf{E}}(T_k, \pi_k^1, \mathbf{B}_{r_0}) = \mathbf{E}^p(T_k, \mathbf{B}_{r_0})$. By Allard's L^∞-L^2 bound, for each $v \in \pi_k^1 \cap \mathbf{B}_{r_0/2}$ there is a point $q \in \mathrm{spt}(T_k) \cap \mathbf{B}_{r_0}$ such that $\mathbf{p}_{\pi_k^1}(q) = v$ and

$$|v-p|^2 \leq Cr_0^2(\mathbf{E}^p(T_k, \mathbf{B}_{r_0}) + r_0^2\mathbf{A}_k^2) \leq Cr_0^{-m}\mathbf{E}^p(T_k, \mathbf{B}_1)\,.$$

However, by (4.207), $\mathrm{dist}^2(p, \pi_\infty) \leq C(\mathbf{E}^p(T_k, \mathbf{B}_1) + \mathbf{A}_k^2) \leq C\mathbf{E}^p(T_k, \mathbf{B}_1)$. It follows immediately that

$$\mathrm{dist}^2(\pi_k^1 \cap \mathbf{B}_{r_0}, \pi_\infty \cap \mathbf{B}_{r_0}) \leq Cr_0^{-m}\mathbf{E}^p(T_k, \mathbf{B}_1)$$

which in turn implies

$$\mathrm{dist}^2(\pi_k^1 \cap \mathbf{B}_1, \pi_\infty \cap \mathbf{B}_1) \leq Cr_0^{-m-2}\mathbf{E}^p(T_k, \mathbf{B}_1)\,.$$

Let $A_k : \pi_\infty \to \pi_\infty^\perp$ be the linear map whose graph gives π_k^1 and note that, from the above discussion, $\|A_k\|_{L^\infty(B_1(0,\pi_\infty))} \leq Cr_0^{-1-m/2}E_k^{1/2}$. In particular we can extract an L^2 limit A_∞

of the maps $E_k^{-1/2} A_k$, up to subsequences. Now the estimates in [19, Theorem 2.4] imply

$$\lim_{k\to\infty} \frac{\mathbf{E}^p(T_k, \mathbf{B}_{r_0})}{\mathbf{E}^p(T_k, \mathbf{B}_1)} = r_0^{-m-2} \int_{B_{r_0}} \mathcal{G}(f_\infty, Q[\![A_\infty]\!])^2 = \int_{B_1} \mathcal{G}(f_\infty, Q[\![A_\infty]\!])^2 \,,$$

where in the last equality we have used the 1-homogeneity of both f_∞ and A_∞. On the other hand by the triangle inequality,

$$\int_{B_1} \mathcal{G}(f_\infty, Q[\![A_\infty]\!])^2 \geq \frac{1}{2} \max_{1\leq i<j\leq \bar{N}} \int_{B_1} |L_i^\infty - L_j^\infty|^2 \geq C_*^{-1} \,.$$

Thus, we are in contradiction with (4.212), which concludes the proof of (4.202).

Step 1: Reduction via enlarging, balancing, and pruning. We will first show that one can without loss of generality assume also the following on $\mathbf{S}$ and $\mathbf{S}'$:

(B) $\mathbf{S}$ and $\mathbf{S}'$ are both M-balanced (with M the constant of Assumption 4.8.1);

(P) For a suitably small constant $\delta = \delta(Q, m, n, \bar{n}, r_0) > 0$ we have

$$\mathbf{A}^2 + \mathbb{E}(T, \mathbf{S}, \mathbf{B}_1) \leq \delta^2 \boldsymbol{\sigma}(\mathbf{S})^2 \tag{4.213}$$

$$r_0^2 \mathbf{A}^2 + \mathbb{E}(T, \mathbf{S}', \mathbf{B}_{r_0}) \leq \delta^2 \boldsymbol{\sigma}(\mathbf{S}')^2 \,. \tag{4.214}$$

For the present reduction step, we will show the existence of another cone $\mathbf{S}_1 \in \mathscr{C}(Q, 0)$ such that:

(i) (B) and (4.213) above hold for $\mathbf{S}_1$ in place of $\mathbf{S}$;

(ii) $V(\mathbf{S}) = V(\mathbf{S}_1)$;

(iii) $\mathbb{E}(T, \mathbf{S}_1, \mathbf{B}_1) \leq \bar{C}(\mathbb{E}(T, \mathbf{S}, \mathbf{B}_1) + \mathbf{A}^2)$, for a constant $\bar{C} = \bar{C}(Q, m, n, \bar{n}, \delta)$;

(iv) $\mathrm{dist}^2(\mathbf{S}_1 \cap \mathbf{B}_1, \mathbf{S} \cap \mathbf{B}_1)^2 \leq \bar{C}(\mathbb{E}(T, \mathbf{S}, \mathbf{B}_1) + \mathbf{A}^2)$;

We can then use the same argument to find a cone $\mathbf{S}_1'$ satisfying (i)–(iv) for T_{0,r_0} and $\mathbf{S}'$. Given this, it is easy to check that the assumptions of the lemma hold with $T, \mathbf{S}_1, \mathbf{S}_1'$ in

place of $T, \mathbf{S}, \mathbf{S}'$ (up to controlled constant factors), and that if one can prove the result for $T, \mathbf{S}_1, \mathbf{S}'_1$, then the result follows for $T, \mathbf{S}, \mathbf{S}'$. Thus, after Step 1 we will have reduced the proof to the case where we can also assume the validity of (B) and (P).

In order to accomplish the task of this step, we observe that, by choosing ε in (4.201) small enough, T, Σ, and $\mathbf{S}$ satisfy all the requirements of Proposition 4.7.1, except possibly $\mathbf{A}^2 \leq \varepsilon_0^2 \mathbb{E}(T, \mathbf{S}, \mathbf{B}_1)$, for the parameter $\varepsilon_0 = \varepsilon_0(Q, m, n, \bar{n})$ in Proposition 4.7.1. Let us assume for the moment that this does not hold, i.e.

$$\mathbb{E}(T, \mathbf{S}, \mathbf{B}_1) < \varepsilon_0^{-2} \mathbf{A}^2 \,. \tag{4.215}$$

In this case we wish to deform $\mathbf{S}$ to another cone $\mathbf{S}_e \in \mathscr{C}(Q, 0)$ such that $\mathbb{E}(T, \mathbf{S}_e, \mathbf{B}_1) < \varepsilon_0^{-2} \mathbf{A}^2$ and $V(\mathbf{S}_e) = V(\mathbf{S}) =: V$. This can be accomplished as follows. We consider a plane $\pi \in \mathscr{P}(0)$ containing V and observe that, because of (4.201), provided that we take $\varepsilon < \sqrt{N}\varepsilon_0$, we have

$$N\hat{\mathbf{E}}(T, \pi, \mathbf{B}_1) > \varepsilon_0^{-2} \mathbf{A}^2 \,. \tag{4.216}$$

Next enumerate the (distinct) planes forming $\mathbf{S}$ as $\alpha_1, \ldots, \alpha_N$, and for each $i = 1, \ldots, N$ fix a continuous path of planes $\alpha_i(t)$ in $T_0\Sigma$ connecting $\alpha_i(0) = \alpha_i$ and $\alpha_i(1) = \pi$, with the property that $V \subset \alpha_i(t)$ for all $t \in [0, 1]$. The existence of the path can be reduced to the existence of a continuous path for the cross sections $V^\perp \cap \alpha_i(t) \subset V^\perp \cap T_0\Sigma$ and the existence of the latter is a consequence of the path connectedness of the Grassmannian of 2-planes in $\mathbb{R}^{2+\bar{n}}$).

Next we want to avoid that along the path we have $\alpha_i(t) = \alpha_j(t)$ for some $i \neq j$ and some $0 < t < 1$. To that end, denote by X the set of m-dimensional subspaces of $T_0\Sigma$ which contain V, and consider its N-fold product $X \times \cdots \times X$. Let $Z \subset X \times \cdots \times X$ be the set of elements $(\beta_1, \beta_2, \ldots, \beta_N)$ such that $\beta_i = \beta_j$ for some $i \neq j$. We then need to show that $(X \times \cdots \times X \setminus Z) \cup \{(\pi, \ldots, \pi)\}$ is path connected. Note that Z can be written as the union of $Z_{ij} := \{(\beta_1, \ldots, \beta_N) : \beta_i = \beta_j\}$. On the other hand the dimension of Z_{ij} equals the

dimension of Z_{12}, while the latter can be written as $\Delta \times X \times \cdots \times X$ where $\Delta \subset X \times X$ is the diagonal $\{(\beta, \beta) : \beta \in X\}$. Hence the codimension of Z in $X \times \cdots \times X$ equals the codimension of the diagonal Δ in $X \times X$. Since the codimension of Δ in $X \times X$ is the dimension of X, which is strictly larger than 1, we conclude that the codimension of Z in $X \times \cdots \times X$ is at least 2. This in particular implies the path connectedness of the desired set, meaning we can assume our path obeys $\alpha_i(t) \neq \alpha_j(t)$ for all $i \neq j$ and all $t \in (0,1)$.

Now let $t \mapsto \mathbf{S}(t) = \alpha_1(t) \cup \cdots \cup \alpha_N(t)$, $t \in [0,1]$, denote the above continuous path from $\mathbf{S}$ to π. Observe that by definition,

$$t \mapsto \hat{\mathbf{E}}(T, \mathbf{S}(t), \mathbf{B}_1) \quad \text{is continuous for } 0 \leq t \leq 1, \tag{4.217}$$

$$t \mapsto \hat{\mathbf{E}}(\mathbf{S}(t), T, \mathbf{B}_1) \quad \text{is continuous for } 0 \leq t < 1, \tag{4.218}$$

and

$$\lim_{t \uparrow 1} \hat{\mathbf{E}}(\mathbf{S}(t), T, \mathbf{B}_1) = N \hat{\mathbf{E}}(\pi, T, \mathbf{B}_1)\,.$$

These properties, combined with (4.215) and (4.216), are enough to claim the existence of a $t_e \in [0,1)$ such that for $\mathbf{S}_e := \mathbf{S}(t_e)$ we have

$$\frac{1}{2}\varepsilon_0^{-2}\mathbf{A}^2 < \mathbb{E}(T, \mathbf{S}_e, \mathbf{B}_1) < \varepsilon_0^{-2}\mathbf{A}^2\,.$$

However, it might be that $\mathbf{S}_e$ is not an element of $\mathscr{C}(Q, 0)$; this can only happen if the intersection of two of the planes forming $\mathbf{S}_e$ is strictly bigger than V (and so is $(m-1)$-dimensional as they do not coincide). Denote by $Z' \subset X \times \cdots \times X$ the set of elements $(\beta_1, \ldots, \beta_N)$ such that $\dim(\beta_i \cap \beta_j) \geq m-1$ for some $i \neq j$. Z' is closed and has non-empty interior in $X \times \cdots \times X$. In particular, if we perturb the planes forming $\mathbf{S}_e$ slightly (within the N-fold product space $X \times \cdots \times X$), in light of the continuity of the map $\tilde{\mathbf{S}} \mapsto \mathbf{E}(T, \tilde{\mathbf{S}}, \mathbf{B}_1)$, we can ensure that the inequality above holds and that at the same time the new cone, which we will abuse of notation and still denote by $\mathbf{S}_e$, is an element of $\mathscr{C}(Q, 0)$.

We have now found a suitable cone $\mathbf{S}_e$ when the unfavourable assumption (4.215) holds. Notice that for such $\mathbf{S}_e$ we also have

$$\mathbb{E}(T, \mathbf{S}_e, \mathbf{B}_1) < \varepsilon_0^{-2}\mathbf{A}^2 \leq \varepsilon_0^{-2}\varepsilon^2\mathbf{E}^p(T, \mathbf{B}_1)$$

by (4.201), and thus if $\varepsilon^2 < \varepsilon_0^4$, we have that the assumptions of Proposition (4.7.1) hold for T and $\mathbf{S}_e$. If we are in the favourable situation where (4.215) does not hold, then we have

$$\mathbf{A}^2 \leq \varepsilon_0^2\mathbb{E}(T, \mathbf{S}, \mathbf{B}_1)\,,$$

and so we can simply set $\mathbf{S}_e := \mathbf{S}$, and again all the assumptions of Proposition 4.7.1 hold for T and $\mathbf{S}_e$.

Thus, we are in a position to apply Proposition 4.7.1 to T and $\mathbf{S}_e$ to find a cone $\mathbf{S}_1$ which satisfies (ii) and is M-balanced. However, it must be noticed that indeed Proposition 4.7.1 is proved by showing that $\mathbf{S}_1$ satisfies the smallness assumption (4.165) in Proposition 4.7.2, with the choice of $\varepsilon = \varepsilon(m, n, \bar{n}, N)$ therein. In particular, since the argument of Proposition 4.7.1 allows one to ensure that $\mathbf{S}_1$ satisfies (4.165) with ε replaced by any $\delta \leq \varepsilon$, up to further decreasing ε_0 (dependent on δ), we get exactly (4.213), and moreover we are still left with the freedom of choosing δ. Thus, $\mathbf{S}_1$ satisfies (i) and (ii).

By Proposition 4.7.1 we also know that

$$\mathbb{E}(T, \mathbf{S}_1, \mathbf{B}_1) \leq \bar{C}\mathbb{E}(T, \mathbf{S}_e, \mathbf{B}_1)\,,$$

where $\bar{C} = \bar{C}(Q, m, n, \bar{n}, \delta)$. On the other hand, by construction either $\mathbf{S}_e = \mathbf{S}$, or

$$\mathbb{E}(T, \mathbf{S}_e, \mathbf{B}_1) \leq \varepsilon_0^{-2}\mathbf{A}^2\,,$$

and in particular we infer (iii) (we stress that now $\varepsilon_0 = \varepsilon_0(Q, m, n, \bar{n}, \delta)$).

So far we have proved that (i), (ii), and (iii) hold. Property (iv) would follow from Proposition 4.7.1 in the case $\mathbf{S}_e = \mathbf{S}$, but unfortunately this may not be the case and so we will have to provide a more subtle argument. In fact, we will use the validity of (B) and (P) for $\mathbf{S}_1$ to prove (iv) even when $\mathbf{S}_e \neq \mathbf{S}$ (for δ chosen sufficiently small).

We first assume that $\delta = \delta(Q, m, n, \bar{n}) > 0$ is sufficiently small so that Lemma 4.6.5 and Proposition 4.6.6 apply to $T_{0,1/4}$ and $\mathbf{S}_1$. We will show that this implies

$$\operatorname{dist}^2(\mathbf{S}_1 \cap \mathbf{B}_1, \mathbf{S} \cap \mathbf{B}_1)^2 \leq C(\mathbb{E}(T, \mathbf{S}_1, \mathbf{B}_1) + \mathbb{E}(T, \mathbf{S}, \mathbf{B}_1) + \mathbf{A}^2)\,. \tag{4.219}$$

As observed, we would need to show this when $\mathbf{E}(T, \mathbf{S}, \mathbf{B}_1) < \varepsilon_0^{-2}\mathbf{A}^2$, but the argument is in fact more general and does not use the latter information. Combined with (iii), (4.219) proves (iv).

We enumerate the (distinct) planes $\beta_1, \ldots, \beta_{N'}$ forming $\mathbf{S}_1$, while we also recall the enumeration $\alpha_1, \ldots, \alpha_N$ for $\mathbf{S}$. Let $W_i \subset \mathbf{B}_4$ be the disjoint neighbourhoods of the planes β_i as in Lemma 4.6.5, and consider their homothetic rescalings $\tilde{W}_i := (W_i)_{0,4}$. Observe that $T \llcorner \mathbf{B}_{3/4} \setminus B_{1/32}(V)$ is supported in the union of the $\tilde{W}_i$ and let $T_i := T \llcorner \tilde{W}_i$. By Proposition 4.6.6(c) we know that

$$\operatorname{dist}(p, \beta_i) \leq C(\mathbb{E}(T, \mathbf{S}_1, \mathbf{B}_1) + \mathbf{A}^2)^{1/2} \qquad \text{for all } p \in \operatorname{spt}(T_i)\,. \tag{4.220}$$

For each fixed $i = 1, \ldots, N'$, consider a unit vector $e_1 \in V^\perp \cap \beta_i$, let $\xi_1 := \frac{e_1}{4}$, and define the disk $B_i := B_{1/32}(\xi_1, \beta_i)$. By Proposition 4.6.6(f) we know that $((\mathbf{p}_{\beta_i})_\sharp T_i) \llcorner (B_{1/2}(0, \beta_i) \setminus B_{1/16}(V)) = Q_i[\![(B_{1/2}(0, \beta_i) \setminus B_{1/16}(V)]\!]$ for some integer $1 \leq Q_i \leq Q$, and thus

$$\|T_i\|(\mathbf{p}_{\beta_i}^{-1}(B_i)) \geq c_0 > 0 \tag{4.221}$$

for some geometric constant $c_0(m,n,\bar{n})$. Since

$$\int \mathrm{dist}^2(p,\mathbf{S})\, d\|T_i\|(p) \le \mathbb{E}(T,\mathbf{S},\mathbf{B}_1)\,,$$

for a fixed constant $\tilde{C} = \frac{C_*}{c_0}$, with $C_* > 0$ to be determined, by Chebyshev's inequality and (4.221) we have

$$\|T_i\|(\{p \in \mathbf{p}_{\beta_i}^{-1}(B_i) : \mathrm{dist}^2(p,\mathbf{S}) > \tilde{C}\mathbb{E}(T,\mathbf{S},\mathbf{B}_1)\}) \le C_*^{-1}\|T_i\|(\mathbf{p}_{\beta_i}^{-1}(B_i))\,.$$

Thus, the set $E_i \subset \mathbf{p}_{\beta_i}^{-1}(B_i) \cap \mathrm{spt}\|T_i\|$ of points $p \in \mathbf{p}_{\beta_i}^{-1}(B_i)$ which obey

$$\mathrm{dist}^2(p,\mathbf{S}) \le \tilde{C}\mathbb{E}(T,\mathbf{S},\mathbf{B}_1)\,, \tag{4.222}$$

has $\|T_i\|(E_i) \ge (1 - 1/C_*)\|T_i\|(\mathbf{p}_{\beta_i}^{-1}(B_i))$.

We then use Proposition 4.6.6(iv) to estimate

$$\begin{aligned}
Q_i\mathcal{H}^m(B_i \setminus \mathbf{p}_{\beta_i}(E_i)) &\le C_0\|T_i\|(\mathbf{p}_{\beta_i}^{-1}(B_i \setminus \mathbf{p}_{\beta_i}(E_i))) + C_0(\mathbf{A}^2 + \mathbb{E}(T,\mathbf{S}_1,\mathbf{B}_1))^{1+\gamma} \\
&\le C_0\|T_i\|(\mathbf{p}_{\beta_i}^{-1}(B_i) \setminus E_i) + C_0(\mathbf{A}^2 + \mathbb{E}(T,\mathbf{S}_1,\mathbf{B}_1))^{1+\gamma} \\
&\le \frac{C_0}{C_*}\|T_i\|(\mathbf{p}_{\beta_i}^{-1}(B_i)) + C_0(\mathbf{A}^2 + \mathbb{E}(T,\mathbf{S}_1,\mathbf{B}_1))^{1+\gamma} \\
&\le \frac{C_0}{C_*}Q_i\mathcal{H}^m(B_i) + C_0(\mathbf{A}^2 + \mathbb{E}(T,\mathbf{S}_1,\mathbf{B}_1))^{1+\gamma}
\end{aligned}$$

where $C_0 = C_0(Q,m,n,\bar{n})$. Note however that $\mathbb{E}(T,\mathbf{S}_1,\mathbf{B}_1) \le C_1(\mathbb{E}(T,\mathbf{S},\mathbf{B}_1) + \mathbf{A}^2)$ for another constant $C_1 = C_1(Q,m,n,\bar{n},\delta)$. Hence, if $\mathbf{A}^2 + \mathbb{E}(T,\mathbf{S},\mathbf{B}_1)$ is sufficiently small and C_* sufficiently large compared to C_0, we get

$$\mathcal{H}^m(\mathbf{p}_{\beta_i}(E_i)) \ge \frac{1}{2}\mathcal{H}^m(B_i)\,.$$

For each point $p \in E_i$ we let $j(p) \in \{1,\dots,N\}$ be an index such that $\mathrm{dist}(p,\alpha_{j(p)}) =$

$\mathrm{dist}(p, \mathbf{S})$. We then define $F_{i,j} := \mathbf{p}_{\beta_i}(\{p \in E_i : j(p) = j\})$. Clearly from the above lower bound on $\mathcal{H}^m(\mathbf{p}_{\beta_i}(E_i))$, there must be an index $j_* \in \{1, \dots, N\}$ for which $\mathcal{H}^m(F_{i,j_*}) \geq \frac{1}{2N}\mathcal{H}^m(B_i)$. Fix now an arbitrary point w_1 in $F_{i,j*}$ and recall that, since it belongs to $B_i = B_{1/32}(\xi_1, \beta_i)$, $\frac{3}{8} \geq |w_1| \geq \frac{1}{8}$. Next,we let $r_* > 0$ and consider the open set

$$\Lambda(r_*) := \{w \in \beta_i : |w \cdot w_1|^2 > (1 - r_*^2)|w_1|^2|w|^2\}\,.$$

By the lower bound on the Hausdorff measure of F_{i,j_*} the set $F_{i,j*} \setminus \Lambda(r_*)$ must contain at least one point w_2 if we choose $r_* = r_*(m, N)$ appropriately.

Clearly w_2 enjoys as well the bound $\frac{3}{8} \geq |w_2| \geq \frac{1}{8}$. Moreover, since e_1 is orthogonal to V, the angles formed by the w_k and V must both be larger than a geometric constant. Finally, the sine of the angle formed by w_1 and w_2 is at least r_*. In particular we can complete the pair w_1, w_2 to a basis of β_i, via an orthonormal basis $v_1, \dots, v_{m-2}$ of V. Thus, any vector $v \in \mathbf{B}_1 \cap \beta_i$ can be written as a linear combination

$$v = \sum_i \lambda_i v_i + \lambda_{m-1} w_1 + \lambda_m w_2$$

for a choice of λ_i which satisfy $|\lambda_i| \leq C$ for some constant $C = C(m, N)$.

For the points w_1, w_2, if we let p_1, p_2 denote points in E_i which obey $j(p) = j_*$ and $\mathbf{p}_{\beta_i}(p_k) = w_k$ for $k = 1, 2$, it follows from (4.222) and (4.220) that, for $k = 1, 2$,

$$\begin{aligned} |\mathbf{p}^{\perp}_{\alpha_{j*}}(w_k)|^2 &= \mathrm{dist}^2(w_k, \alpha_{j_*}) \leq 2\,\mathrm{dist}^2(w_k, p_k) + 2\,\mathrm{dist}^2(p_k, \alpha_{j_*}) \\ &= 2\,\mathrm{dist}^2(p_k, \beta_i) + 2\,\mathrm{dist}^2(p_k, \mathbf{S}) \leq C(\mathbb{E}(T, \mathbf{S}_1, \mathbf{B}_1) + \mathbb{E}(T, \mathbf{S}, \mathbf{B}_1) + \mathbf{A}^2) \end{aligned}$$

where $C = C(Q, m, n, \bar{n}, \delta)$. In particular, using the linearity of $\mathbf{p}^{\perp}_{\alpha_{j*}}$ and the fact that $V \subset \alpha_{j_*}$, this shows that

$$\mathrm{dist}^2(v, \alpha_{j_*}) \leq C(\mathbb{E}(T, \mathbf{S}_1, \mathbf{B}_1) + \mathbb{E}(T, \mathbf{S}, \mathbf{B}_1) + \mathbf{A}^2) \qquad \forall v \in \beta_i \cap \mathbf{B}_1\,.$$

Summarizing our argument so far, for every plane β_i in $\mathbf{S}_1$ we can show the existence of a plane α_{j_*} in $\mathbf{S}$ such that

$$\mathrm{dist}^2(\beta_i \cap \mathbf{B}_1, \alpha_{j_*} \cap \mathbf{B}_1) \leq C(\mathbb{E}(T, \mathbf{S}_1, \mathbf{B}_1) + \mathbb{E}(T, \mathbf{S}, \mathbf{B}_1) + \mathbf{A}^2)\,. \tag{4.223}$$

We now would like to prove the converse; namely, if we fix any plane α_j forming $\mathbf{S}$, we look for a plane β_{i_*} in $\mathbf{S}_1$ satisfying (4.223) with i_*, j in place of i, j_*. This time, we choose a unit vector $f_1 \in V^\perp \cap \alpha_j$. We let $\zeta_1 = f_1/4$ and set $B_j := B_{1/32}(\zeta_1, \alpha_j)$. We then know that

$$\int_{B_j} \mathrm{dist}^2(q, \mathrm{spt}(T)) d\mathcal{H}^m(q) \leq \mathbb{E}(\mathbf{S}, T, \mathbf{B}_1)\,.$$

Once again applying Chebyshev's inequality, this time with $\mathcal{H}^m$, we get that if $F_j \subset B_j$ is the set of points $q \in B_j$ which obey

$$\mathrm{dist}^2(q, \mathrm{spt}(T)) \leq (2/\mathcal{H}^m(B_j))\mathbb{E}(T, \mathbf{S}, \mathbf{B}_1)\,, \tag{4.224}$$

then we have $\mathcal{H}^m(F_j) \geq \frac{1}{2}\mathcal{H}^m(B_j)$. Now for each $q \in F_j$ let $p(q) \in \mathrm{spt}(T)$ be such that $\mathrm{dist}(q, \mathrm{spt}(T)) = |q - p(q)|$. Thus, (4.224) tells us that if ε is chosen small enough, we can guarantee that $|q - p(q)| < 1/16$ and so in particular $p(q) \in \mathbf{B}_{3/4} \setminus \mathbf{B}_{1/32}(V)$. It thus follows that $p(q) \in \tilde{W}_i$ for some $i \in \{1, \ldots, N'\}$. For each such i, let $F_{j,i}$ be the set of points $q \in F_j$ for which at least one such $p(q)$ belongs to $\tilde{W}_i$. Arguing as before, it follows that for some i_* we have $\mathcal{H}^m(F_{j,i_*}) \geq \frac{1}{2N'}\mathcal{H}^m(B_j)$. We can now argue as above that F_{j,i_*} contains two appropriate points w_1, w_2, complete the pair with vectors in V to form an appropriate base of α_j and use this base to prove

$$\mathrm{dist}^2(\alpha_j \cap \mathbf{B}_1, \beta_{i_*} \cap \mathbf{B}_1) \leq C(\mathbb{E}(T, \mathbf{S}_1, \mathbf{B}_1) + \mathbb{E}(T, \mathbf{S}, \mathbf{B}_1) + \mathbf{A}^2)\,.$$

Combined with (4.223), this therefore completes the proof of (4.219) and hence the first step.

Step 2. Concluding from the reduction. Now that we have reduced to the setting where we may assume the validity of (B) and (P), we conclude the proof of the proposition by showing that the claimed conclusions hold under these additional assumptions, if we choose δ small enough. First of all we observe that, by (iv) in Step 1, (4.202), (4.210), and (4.211) we have

$$C^{-1}\mathbf{E}^p(T,\mathbf{B}_1) \le \min\{\boldsymbol{\mu}(\mathbf{S})^2, \boldsymbol{\mu}(\mathbf{S}')^2\} \le \max\{\boldsymbol{\mu}(\mathbf{S})^2, \boldsymbol{\mu}(\mathbf{S}')^2\} \le C\mathbf{E}^p(T,\mathbf{B}_1)\,. \tag{4.225}$$

So we can appeal to Lemma 4.5.12 to conclude (4.205) from (4.204). We will thus focus on proving (4.204).

The latter task is similar to the one accomplished at the end of Step 1; while it is more complicated by the fact that $V(\mathbf{S})$ and $V(\mathbf{S}')$ might be different (unlike $V(\mathbf{S})$ and $V(\mathbf{S}_1)$ in Step 1), we can now take advantage of (B) and (P) for both (whereas previously, we only knew that $\mathbf{S}_1$ was balanced). Indeed, by choosing δ small enough we can ensure, using Lemma 4.6.5, Proposition 4.6.6 and Corollary **??** for $\mathbf{S}'$, $T_{0,r_0/4}$, and $\mathbf{S}$, $T_{0,1/4}$, respectively, that the following situation holds:

(i) If we enumerate the planes $\beta_1,\dots,\beta_{N'}$ forming $\mathbf{S}'$ and set $V' := V(\mathbf{S}')$, then there are pairwise disjoint neighborhoods W_j' of $\beta_j \cap \mathbf{B}_{r_0/2} \setminus B_{r_0/16}(V')$, a constant $C = C(Q,m,n,\bar n) > 0$ and positive integers Q_j' such that

$$\operatorname{spt}(T) \cap \mathbf{B}_{r_0/2} \setminus B_{r_0/16}(V') \subset \bigcup_j W_j'\,; \tag{4.226}$$

$$\operatorname{dist}^2(p,\beta_j) \le C(\mathbb{E}(T,\mathbf{S}',\mathbf{B}_{r_0}) + \mathbf{A}^2 r_0^2) r_0^2 \qquad \forall p \in \operatorname{spt}(T) \cap W_j'\,; \tag{4.227}$$

$$\begin{aligned}(\mathbf{p}_{\beta_j})_\sharp(T \llcorner W_j' \cap \mathbf{p}_{\beta_j}^{-1}(B_{r_0/2}(0,\beta_j) \setminus B_{r_0/16}(V'))) \\ = Q_j'\llbracket B_{r_0/2}(0,\beta_j) \setminus B_{r_0/16}(V')\rrbracket\,.\end{aligned} \tag{4.228}$$

(ii) If we enumerate the planes $\alpha_1,\dots,\alpha_N$ forming $\mathbf{S}$ and set $V = V(\mathbf{S})$, then there are pairwise disjoint neighborhoods W_i of $\alpha_i \cap \mathbf{B}_{1/2} \setminus \mathbf{B}_{r_0/16}(V)$, constants $C = C(Q,m,n,\bar n) >$

0, $\bar{C} = \bar{C}(Q, m, n, \bar{n}, r_0) > 0$ and positive integers Q_i such that

$$\operatorname{spt}(T) \cap \mathbf{B}_{1/2} \setminus \mathbf{B}_{r_0/16}(V) \subset \bigcup_i W_i \tag{4.229}$$

$$\begin{aligned} \operatorname{dist}^2(p, \alpha_i) &\leq C(\mathbb{E}(T, \mathbf{S}, \mathbf{B}_1) + \mathbf{A}^2) \\ &\leq \bar{C}(\mathbb{E}(T, \mathbf{S}, \mathbf{B}_1) + \mathbf{A}^2 r_0^2) r_0^2 \qquad \forall p \in \operatorname{spt}(T) \cap W_i \cap \mathbf{B}_{r_0} \end{aligned} \tag{4.230}$$

$$\begin{aligned} &(\mathbf{p}_{\alpha_i})_\sharp (T \llcorner W_i \cap \mathbf{p}_{\alpha_i}^{-1}(B_{r_0}(0, \alpha_i) \setminus \mathbf{B}_{r_0/16}(V))) \\ &\qquad = Q_i [\![B_{r_0}(0, \alpha_i) \setminus \mathbf{B}_{r_0/16}(V)]\!] \,. \end{aligned} \tag{4.231}$$

We will now proceed to show that all these estimates imply, for each plane α_i in $\mathbf{S}$, the existence of a plane β_j in $\mathbf{S}'$ such that

$$\operatorname{dist}^2(\alpha_i \cap \mathbf{B}_1, \beta_j \cap \mathbf{B}_1) \leq \bar{C}(\mathbb{E}(T, \mathbf{S}, \mathbf{B}_1) + \mathbb{E}(T, \mathbf{S}', \mathbf{B}_1) + \mathbf{A}^2) \,. \tag{4.232}$$

Since the argument can be symmetrized to switch the roles of α_i and β_j, this would conclude the proof of (4.204). Without loss of generality we assume that $i = 1$ and we start by fixing a vector $e \in \alpha_1$ with $|e| = \frac{r_0}{4}$ such that $\mathbf{B}_{2c_0 r_0}(e) \subset \mathbf{B}_{r_0/2} \setminus (\mathbf{B}_{r_0/16}(V) \cup \mathbf{B}_{r_0/16}(V'))$, where $c_0 = c_0(m)$ is a positive dimensional constant. For each point $q \in B_{c_0 r_0}(e, \alpha_1)$, by (4.230) and (4.231) above we can find a point $p = p(q) \in \operatorname{spt}(T) \cap W_1$ such that

$$|p - q|^2 \leq \bar{C}(\mathbb{E}(T, \mathbf{S}, \mathbf{B}_1) + \mathbf{A}^2 r_0^2) r_0^2$$

and then by (4.226) we find W_j' (for some $j = j(q) \in \{1, \dots, N'\}$) such that $p \in W_j'$. In particular, setting $q_j := \mathbf{p}_{\beta_j}(p)$, we conclude from (4.227) that

$$|p - q_j|^2 \leq C(\mathbb{E}(T, \mathbf{S}, \mathbf{B}_1) + \mathbf{A}^2 r_0^2 + \mathbb{E}(T, \mathbf{S}', \mathbf{B}_{r_0})) r_0^2 \,.$$

So, for each point $q \in B_{c_0 r_0}(e, \alpha_1)$ we conclude that there is a plane $\beta_{j(q)}$ in $\mathbf{S}'$ such that

$$\operatorname{dist}(q, \beta_{j(q)})^2 \leq C(\mathbb{E}(T, \mathbf{S}, \mathbf{B}_1) + \mathbf{A}^2 r_0^2 + \mathbb{E}(T, \mathbf{S}', \mathbf{B}_{r_0}))r_0^2\,, \tag{4.233}$$

where $C = C(Q, m, n, \bar{n}, r_0) > 0$. Select now m linearly independent vectors $v_1, \ldots, v_m \in B_{c_0 r_0}(e, \alpha_1)$ with the property that, if we set $e_i := \frac{v_i}{r_0}$, then every vector $v \in \mathbf{B}_1 \cap \alpha_1$ is a linear combination

$$v = \sum_i \lambda_i e_i$$

with $|\lambda_i| \leq C$, for some constant C which depends only on c_0 and therefore only on m.

In light of the above argument, for each vector v_i we may find a plane $\beta_{j(i)}$ in $\mathbf{S}'$ such that

$$\operatorname{dist}^2(v_i, \beta_{j(i)}) \leq C(\mathbb{E}(T, \mathbf{S}, \mathbf{B}_1) + \mathbf{A}^2 r_0^2 + \mathbb{E}(T, \mathbf{S}', \mathbf{B}_{r_0}))r_0^2\,. \tag{4.234}$$

Now, we would like to achieve (4.234) but with *the same plane*, say $\beta_{j(1)}$, for each v_i, up to increasing the constant by a fixed amount. Suppose that $j(i) \neq j(1)$ for some $i > 1$. Consider the line segment $[v_1, v_i]$ and parameterize it with a constant speed curve $\gamma : [0, 1] \to [v_1, v_i]$ with $\gamma(0) = v_1$ and $\gamma(1) = v_i$. By continuity of $t \mapsto \operatorname{dist}(\gamma(t), \beta_{j(1)})$, for some $\sigma \in (0, 1]$ we have

$$\operatorname{dist}^2(\gamma(t), \beta_{j(1)}) \leq 2C(\mathbb{E}(T, \mathbf{S}, \mathbf{B}_1) + \mathbf{A}^2 r_0^2 + \mathbb{E}(T, \mathbf{S}', \mathbf{B}_{r_0}))r_0^2 \qquad \forall t \in [0, \sigma]\,.$$

Now let τ be the maximal number in $[0, 1]$ such that

$$\operatorname{dist}^2(\gamma(\tau), \beta_{j(1)}) \leq 2C(\mathbb{E}(T, \mathbf{S}, \mathbf{B}_1) + \mathbf{A}^2 r_0^2 + \mathbb{E}(T, \mathbf{S}', \mathbf{B}_{r_0}))r_0^2\,. \tag{4.235}$$

If $\tau = 1$ then we indeed arrive at (4.234) with $\beta_{j(i)}$ replaced by $\beta_{j(1)}$ and C replaced by $2C$. Otherwise, if $\tau < 1$, then we must have equality in (4.235). Since $\gamma(\tau) \in B_{c_0 r_0}(e, \alpha_1)$, (4.233)

implies that there must be another index $j' = j'(\tau) \neq j(1)$ such that

$$\mathrm{dist}^2(\gamma(\tau), \beta_{j'}) \leq C(\mathbb{E}(T, \mathbf{S}, \mathbf{B}_1) + \mathbf{A}^2 r_0^2 + \mathbb{E}(T, \mathbf{S}', \mathbf{B}_{r_0}))r_0^2 \,. \tag{4.236}$$

If $p_{j(1)} = \mathbf{p}_{\beta_{j(1)}}(\gamma(\tau))$ and $p_{j'} = \mathbf{p}_{\beta_{j'}}(\gamma(\tau))$ are the respective nearest point projections of $\gamma(\tau)$, and if we ensure that ε is taken to be small enough, we can guarantee that both points belong to $\mathbf{B}_{2c_0 r_0}(e)$. Recall that our choice of the vector e guarantees that this ball does not intersect $B_{r_0/16}(V')$. In particular, if we set $q_{j(1)} = p_{j(1)}/r_0$ and $q_{j'} = p_{j'}/r_0$ we find $q_{j(1)} \in \beta_{j(1)} \cap \mathbf{B}_1 \setminus B_{1/16}(V')$ and $q_{j'} \in \beta_{j'} \cap \mathbf{B}_1 \setminus B_{1/16}(V')$ are such that

$$|q_{j(1)} - q_{j'}|^2 \leq 6C(\mathbb{E}(T, \mathbf{S}, \mathbf{B}_1) + \mathbf{A}^2 r_0^2 + \mathbb{E}(T, \mathbf{S}', \mathbf{B}_{r_0}))\,.$$

Since the pair of planes have V' as common spine and are M-balanced, we can combine the above with Corollary 4.5.3 (namely, first use (4.54), then M-balanced, then the first inequality in (4.53), then the above) to conclude that

$$\mathrm{dist}^2(\beta_{j(1)} \cap \mathbf{B}_1, \beta_{j'} \cap \mathbf{B}_1) \leq C'(\mathbb{E}(T, \mathbf{S}, \mathbf{B}_1) + \mathbf{A}^2 r_0^2 + \mathbb{E}(T, \mathbf{S}', \mathbf{B}_{r_0})) \tag{4.237}$$

for some constant C' which depends on M and the previous constant C.

We now look at the point τ' which is the maximum in the set of $t \in [0, 1]$ for which

$$\mathrm{dist}^2(\gamma(t)), \beta_{j'})^2 \leq 2C(\mathbb{E}(T, \mathbf{S}, \mathbf{B}_1) + \mathbf{A}^2 r_0^2 + \mathbb{E}(T, \mathbf{S}', \mathbf{B}_{r_0}))r_0^2 \,, \tag{4.238}$$

where C is again the constant in (4.233). Now τ' must be strictly larger than τ, by (4.236). Moreover, if $\tau' < 1$ then

$$\mathrm{dist}^2(\gamma(\tau'), \beta_{j'}) = 2C(\mathbb{E}(T, \mathbf{S}, \mathbf{B}_1) + \mathbf{A}^2 r_0^2 + \mathbb{E}(T, \mathbf{S}', \mathbf{B}_{r_0}))r_0^2 \,. \tag{4.239}$$

In this case we can look at the closest plane $\beta_{j''}$ to $\gamma(\tau')$, which satisfies

$$\mathrm{dist}^2(\gamma(\tau'),\beta_{j''}) \leq C(\mathbb{E}(T,\mathbf{S},\mathbf{B}_1)+\mathbf{A}^2r_0^2+\mathbb{E}(T,\mathbf{S}',\mathbf{B}_{r_0}))r_0^2\,, \tag{4.240}$$

where C is the constant in (4.233). Obviously by construction $j'' \notin \{j(1), j'\}$. But then we can repeat the argument above with j' replacing $j(1)$ and j'' replacing j' and estimate

$$\mathrm{dist}^2(\beta_{j'}\cap\mathbf{B}_1,\beta_{j''}\cap\mathbf{B}_1) \leq C'(\mathbb{E}(T,\mathbf{S},\mathbf{B}_1)+\mathbf{A}^2r_0^2+\mathbb{E}(T,\mathbf{S}',\mathbf{B}_{r_0})) \tag{4.241}$$

with the same constant C' of (4.236). We can iterate this procedure until we reach the right endpoint $t=1$. Any time that we do not stop, we find a *new* plane in the collection forming $\mathbf{S}'$. Since there is a finite number N' of such planes, we conclude that the procedure stops after at most $N'-1$ steps. The last plane we find is a plane β_{κ_i} which satisfies (4.234) with $2C$ in place of C and β_{κ_i} in place of $\beta_{j(i)}$. On the other hand we have a chain of at most $N'-1$ inequalities of the form (4.241) between each pair of consecutive planes found during the procedure. In particular for every i we have

$$\mathrm{dist}^2(\beta_{j(1)}\cap\mathbf{B}_1,\beta_{\kappa_i}\cap\mathbf{B}_1) \leq (N'-1)^2C'(\mathbb{E}(T,\mathbf{S},\mathbf{B}_1)+\mathbf{A}^2r_0^2+\mathbb{E}(T,\mathbf{S}',\mathbf{B}_{r_0}))\,,$$

which combined with (4.234) gives that (4.234) holds with $\beta_{j(1)}$ in place of $\beta_{j(i)}$ with a larger constant C, for every i, which is what we wanted. Summarizing the discussion above, we have found a larger constant C such that, if we set $j=j(1)$, then

$$\mathrm{dist}^2(e_i,\beta_j) = |\mathbf{p}^\perp_{\beta_j}(e_i)|^2 \leq C(\mathbb{E}(T,\mathbf{S},\mathbf{B}_1)+\mathbf{A}^2r_0^2+\mathbb{E}(T,\mathbf{S}',\mathbf{B}_{r_0}))$$

for every i. Writing an arbitrary vector in $\mathbf{B}_1\cap\alpha_1$ in terms of the basis e_i as described above, we then conclude that

$$\mathrm{dist}^2(\alpha_1\cap\mathbf{B}_1,\beta_j\cap\mathbf{B}_1)^2 \leq C(\mathbb{E}(T,\mathbf{S},\mathbf{B}_1)+\mathbf{A}^2r_0^2+\mathbb{E}(T,\mathbf{S}',\mathbf{B}_{r_0}))\,.$$

This completes the proof of (4.232) and thus completes the proof of the lemma. □

Thus, to prove Theorem 4.2.4, now we just need to prove Theorem 4.8.2.

4.9 Estimates at the spine

In this section we address the pivotal estimates needed at the spine of the cone for the proof of Theorem 4.2.4, which are a suitable adaptation of the groundbreaking work of Simon in [45]. They will be crucial for demonstrating that in the end, after a blow-up procedure under the assumption that Q-points accumulate across the spine, the graphical approximations constructed in Proposition 4.6.19 will remain controlled as one approaches the spine, and will converge to a Dir-minimizer that has an $(m-2)$-dimensional subspace of Q-points. We start by detailing the assumptions which will be used through this section. Note that, in contrast to [13], we denote the estimates in Theorem 4.9.2, Corollary 4.9.3 and Proposition 4.9.4 collectively as *Simon's estimates*. Indeed, although the first occurrence of an estimate analogous to (4.243) below first appeared in the work of Hardt and Simon in [30], the framework therein is significantly simpler. The first appearance of the estimates (4.243), (4.244), (4.245) and (4.246), albeit in a multiplicity one setting, is in [45]; the corresponding estimates in a setting of higher multiplicity, as is the case in the present work, first appeared in [56]. We note that (4.247) is a more refined version of an estimate in [45], but in this form it appeared first in [56]. One important difference between our work and [45, 56] is that, even when we are at a fixed distance from the spine of the cone $\mathbf{S}$ and the excess of the current from $\mathbf{S}$ is very small, the current is not necessarily a graph, nor a multigraph (it is merely approximated suitably by the latter), and therefore we need to deal with appropriate additional error estimates coming from regions of non-graphicality.

Throughout this section, we will work under an analogous assumption to Assumption 4.6.9, but with possibly smaller parameters:

Assumption 4.9.1 (Assumptions for Simon's estimates)**.** Suppose T and Σ are as in Assumption 3.2.2 and $\|T\|(\mathbf{B}_4) \leq 4^m(Q+\frac{1}{2})\omega_m$. Suppose $\mathbf{S} = \alpha_1 \cup \cdots \cup \alpha_N$ is a cone in $\mathscr{C}(Q) \setminus \mathscr{P}$ which is M-balanced, where $M > 0$ is a given fixed constant, and V is the spine of $\mathbf{S}$. For a sufficiently small constant $\varepsilon = \varepsilon(Q, m, n, \bar{n}, M)$ smaller than the ε-threshold in Assumption 4.6.9, whose choice will be fixed by the statements of Theorem 4.9.2, Corollary 4.9.3, and Proposition 4.9.4 below, suppose that

$$\mathbb{E}(T, \mathbf{S}, \mathbf{B}_4) + \mathbf{A}^2 \leq \varepsilon^2 \boldsymbol{\sigma}(\mathbf{S})^2 \,. \tag{4.242}$$

We recall once again the notation

$$\boldsymbol{\sigma}(\mathbf{S}) := \min_{i<j} \operatorname{dist}(\alpha_i \cap \mathbf{B}_1, \alpha_j \cap \mathbf{B}_1)\,, \qquad \text{and} \qquad \boldsymbol{\mu}(\mathbf{S}) := \max_{i<j} \operatorname{dist}(\alpha_i \cap \mathbf{B}_1, \alpha_j \cap \mathbf{B}_1)\,.$$

Before stating the inequalities that we need, let us introduce the following short-hand notation for points $q \in \operatorname{spt}(T)$ at which the m-rectifiable set $\operatorname{spt}(T)$ has an approximate tangent plane $\pi(q)$ oriented by the simple m-vector $\vec{T}(q)$:

- $\mathbf{p}_{\vec{T}}$ and $\mathbf{p}^{\perp}_{\vec{T}}$ will denote the orthogonal projections onto $\pi(q)$ and its orthogonal complement $(\pi(q))^{\perp}$ respectively;
- $q_{\|}$ and $q^{\perp}$ will denote the vectors $\mathbf{p}_{\vec{T}}(q)$ and $\mathbf{p}^{\perp}_{\vec{T}}(q)$ respectively.

We now split the key estimates into three separate statements and we will proceed with their proofs afterwards.

Theorem 4.9.2 (Simon's error and gradient estimates)**.** *Assume T, Σ, and $\mathbf{S}$ are as in Assumption 4.9.1, suppose in addition that $\Theta(T, 0) \geq Q$ and set $r = \frac{1}{3\sqrt{m-2}}$. Then there is a constant $C = C(Q, m, n, \bar{n}, M) > 0$ and a choice of $\varepsilon = \varepsilon(Q, m, n, \bar{n}, M) > 0$ in Assumption 4.9.1 sufficiently small such that*

$$\int_{\mathbf{B}_r} \frac{|q^{\perp}|^2}{|q|^{m+2}}\, d\|T\|(q) \leq C(\mathbf{A}^2 + \hat{\mathbf{E}}(T, \mathbf{S}, \mathbf{B}_4)) \tag{4.243}$$

$$\int_{\mathbf{B}_r} |\mathbf{p}_V \circ \mathbf{p}_{\vec{T}}^{\perp}|^2 \, d\|T\| \leq C(\mathbf{A}^2 + \hat{\mathbf{E}}(T, \mathbf{S}, \mathbf{B}_4)) \,. \tag{4.244}$$

We refer to the first estimate (4.243) as "Simon's error estimate", since it is estimating the error in the monotonicity formula. Meanwhile, we refer to the second estimate (4.244) as "Simon's gradient estimate", since if T were the graph of a function, this would be an L^2 bound on the derivative of the function in the directions parallel to V.

An important corollary which will not require much additional work is the following.

Corollary 4.9.3 (Simon's non-concentration estimate)**.** *Assume T, Σ, $\mathbf{S}$ and r are as in Theorem 4.9.2. Then, there is a choice of $\varepsilon = \varepsilon(Q, m, n, \bar{n}, M)$ in Assumption 4.9.1, possibly smaller than that in Theorem 4.9.2, such that for every $\kappa \in (0, m+2)$,*

$$\int_{\mathbf{B}_r} \frac{\operatorname{dist}^2(q, \mathbf{S})}{|q|^{m+2-\kappa}} \, d\|T\|(q) \leq C_\kappa(\mathbf{A}^2 + \hat{\mathbf{E}}(T, \mathbf{S}, \mathbf{B}_4)) \,, \tag{4.245}$$

where here $C_\kappa = C_\kappa(Q, m, n, \bar{n}, M, \kappa)$.

This section will be concluded by deriving the following consequence of Corollary 4.9.3, which will require a subtle geometric consideration.

Proposition 4.9.4 (Simon's shift inequality)**.** *Assume T, Σ, and $\mathbf{S}$ are as in Assumption 4.9.1 and in addition $\{\Theta(T, \cdot) \geq Q\} \cap \mathbf{B}_\varepsilon(0) \neq \emptyset$. Then there is a radius $r = r(Q, m, n, \bar{n})$ and a choice of $\varepsilon = \varepsilon(Q, m, n, \bar{n}, M)$ in Assumption 4.9.1, possibly smaller than those in Theorem 4.9.2 and Corollary 4.9.3 such that for each $\kappa \in (0, m+2)$, there are constants $\bar{C}_\kappa = \bar{C}_\kappa(Q, m, n, \bar{n}, M, \kappa) > 0$ and $C = C(Q, m, n, \bar{n}, M)$ such that the following holds. If $q_0 \in \mathbf{B}_r(0)$ and $\Theta(T, q_0) \geq Q$, then*

$$\int_{\mathbf{B}_{4r}(q_0)} \frac{\operatorname{dist}^2(q, q_0 + \mathbf{S})}{|q - q_0|^{m+2-\kappa}} \, d\|T\|(q) \leq \bar{C}_\kappa(\mathbf{A}^2 + \hat{\mathbf{E}}(T, \mathbf{S}, \mathbf{B}_4)) \,. \tag{4.246}$$

$$|\mathbf{p}_{\alpha_1}^{\perp}(q_0)|^2 + \boldsymbol{\mu}(\mathbf{S})^2 |\mathbf{p}_{V^\perp \cap \alpha_1}(q_0)|^2 \leq C(\mathbf{A}^2 + \hat{\mathbf{E}}(T, \mathbf{S}, \mathbf{B}_4)) \,. \tag{4.247}$$

Remark 4.9.5. Observe that $\mathbf{p}_V^\perp = \mathbf{p}_{\alpha_1}^\perp + \mathbf{p}_{V^\perp \cap \alpha_1}$. In particular, when we know that $\mathbf{S}$ is at a fixed positive distance from any plane (namely, $\boldsymbol{\mu}(\mathbf{S})$ is bounded from below away from 0), then (4.247) gives a control on how far q_0 can be from the spine V. This particular case corresponds to what Simon proves in [45], while, as already mentioned, (4.247) is a refinement which appears first in the work [56] of Wickramasekera.

4.9.1 Proof of Simon's error and gradient estimates (Theorem 4.9.2)

Monotonicity formula

We begin by recalling the monotonicity formula for mass ratios, along with some consequences. For T as in Assumption 3.2.2 and $\rho \in (0, 4]$, the monotonicity formula reads

$$\int_{\mathbf{B}_\rho} \frac{|q^\perp|^2}{|q|^{m+2}}\, d\|T\|(q) = \frac{\|T\|(\mathbf{B}_\rho)}{\rho^m} - \omega_m \Theta(T, 0) - \frac{1}{m} \int_{\mathbf{B}_\rho} (q^\perp \cdot \vec{H}_T(q))(|q|^{-m} - \rho^{-m})\, d\|T\|(q)\,.$$

The mean curvature vector $\vec{H}_T(q)$ is defined as

$$\vec{H}_T(q) := \sum_{i=1}^m A_\Sigma(e_i, e_i)\,,$$

where A_Σ is the second fundamental form of Σ and $e_1, \ldots, e_m$ is an orthonormal base of the approximate tangent $\pi(q)$ to $\mathrm{spt}(T)$ at q.

We next assume that $\Theta(T, 0) \geq Q$, fix the multiplicities $Q_1, \ldots, Q_N$ given by Proposition 4.6.18 and observe that for ρ as above, we have

$$\sum_i Q_i \mathcal{H}^m(\alpha_i \cap \mathbf{B}_\rho) = Q\omega_m \rho^m \leq \omega_m \Theta(T, 0) \rho^m\,. \tag{4.248}$$

Fix $r = \frac{1}{3\sqrt{m-2}}$ as in the statement and note that any constants depending on r become in turn dimensional constants. We moreover fix a smooth, monotone non-increasing function

$\chi : [0,\infty) \to \mathbb{R}$ with $\chi \equiv 1$ on $[0,r]$ and $\chi \equiv 0$ on $[2r,\infty)$ and we introduce the function

$$\Gamma(t) := -\int_t^\infty \frac{d}{ds}(\chi(s)^2) s^m \, ds \,.$$

Recall the elementary equalities

$$\int_{\mathbf{B}_R} \chi(|q|)^2 \, d\mu(q) = \int_0^R \chi(t)^2 \frac{d}{dt}(\mu(\mathbf{B}_t)) \, dt \tag{4.249}$$

$$\int_{\mathbf{B}_R} \Gamma(|q|) \, d\mu(q) = \int_0^R \chi(t)^2 \frac{d}{dt}(t^m \mu(\mathbf{B}_t)) \, dt \,, \tag{4.250}$$

which are valid for any Radon measure μ with $\mu(\{0\}) = 0$ via Fubini's Theorem and the definition of the distributional derivative of the BV function $t \mapsto \mu(\mathbf{B}_t)$. We then multiply both sides of the monotonicity formula by ρ^m, differentiate the resulting identity in ρ, multiply by $\chi(\rho)^2$, integrate over $\rho \in [0,2r]$ and use (4.249), (4.250), and (4.248) (treating, for example, $\frac{|q^\perp|^2}{|q|^{m+2}} d\|T\|(q)$ as a Radon measure $d\mu$ as above). Since $\Gamma \geq C^{-1} \mathbf{1}_{[0,r]}$ for some constant $C(m,r) > 0$, we get

$$\begin{aligned} \int_{\mathbf{B}_r} \frac{|q^\perp|^2}{|q|^{m+2}} \, d\|T\|(q) \leq C \left[\int \chi^2(|q|) \, d\|T\|(q) - \sum_i Q_i \int_{\alpha_i} \chi^2(|q|) \, d\mathcal{H}^m(q) \right] \\ + C \underbrace{\int \frac{|\Gamma(|q|) q^\perp \cdot \vec{H}_T(q)|}{|q|^m} \, d\|T\|(q)}_{=: \text{(A)}} \,. \end{aligned} \tag{4.251}$$

We first demonstrate that the error term (A) coming from the mean curvature of T may be controlled by $\mathbf{A}^2$ and to that end we observe that

$$|\vec{H}_T(q)| \leq m\mathbf{A} \,, \tag{4.252}$$

$$|T_q\Sigma - T_0\Sigma| \leq C\mathbf{A}|q| \tag{4.253}$$

$$|\mathbf{p}^\perp_{T_q\Sigma}(q)| \leq C\mathbf{A}|q| \qquad \forall q \in \Sigma \,. \tag{4.254}$$

Indeed the first inequality is obvious by the very definition of $\vec{H}_T$, while the second and the third are a simple exercise in differential geometry and we leave them to the reader.

Observe that since T is area-minimizing in Σ, $\vec{H}_T(q)$ is orthogonal to $T_q\Sigma$, while $\pi(q) \subset T_q\Sigma$, so

$$q^\perp \cdot \vec{H}_T(q) = q \cdot \vec{H}_T(q) = \mathbf{p}^\perp_{T_q\Sigma}(q) \cdot \vec{H}_T(q)\,. \tag{4.255}$$

In particular, since $\mathrm{spt}(T) \subset \Sigma$, (4.252) and (4.254) imply

$$(\mathrm{A}) \leq C\mathbf{A}^2 \int_{\mathbf{B}_r} |q|^{1-m}\, d\|T\|(q) \leq C\mathbf{A}^2$$

where again $C = C(m, r) > 0$. We can therefore write

$$\int_{\mathbf{B}_r} \frac{|q^\perp|^2}{|q|^{m+2}}\, d\|T\|(q) \leq C\left[\int \chi^2(|q|)\, d\|T\|(q) - \sum_i Q_i \int_{\alpha_i} \chi^2(|q|)\, d\mathcal{H}^m(q)\right] + C\mathbf{A}^2. \tag{4.256}$$

We now wish to estimate the first term on the right hand side of (4.256) in terms of the excess $\mathbb{E}(T, \mathbf{S}, \mathbf{B}_4)$; to achieve this, we will test the first variation formula for both T and the current whose support is $\mathbf{S}$ with the vector field

$$X(q) = \chi(|q|)^2 \mathbf{p}^\perp_V(q)\,.$$

The first variation formula for T reads

$$\int \mathrm{div}_{\vec{T}}\, X(q)\, d\|T\|(q) = -\underbrace{\int X^\perp(q) \cdot \vec{H}_T(q)\, d\|T\|(q)}_{=:\,(\mathrm{B})}\,, \tag{4.257}$$

where we use the shorthand notation $X^\perp(q) = \mathbf{p}^\perp_{\vec{T}}(X(q))$ and

$$\mathrm{div}_{\vec{T}}\, X(q) = \sum_i \partial_{e_i} X(q) \cdot e_i \tag{4.258}$$

for any orthonormal base $e_1, \ldots, e_m$ of the approximate tangent $\pi(q)$ of $\mathrm{spt}(T)$.

Writing $x = \mathbf{p}_V^\perp(q)$, we can now use (4.252) to write

$$|X^\perp(q) \cdot \vec{H}_T(q)| = \chi(|q|)^2 |\mathbf{p}^\perp_{T_q\Sigma}(x) \cdot \vec{H}_T(q)| \leq m\mathbf{A}|\mathbf{p}^\perp_{T_q\Sigma}(x)|\,.$$

Observe next that $x = \mathbf{p}_V^\perp(q) = q - \mathbf{p}_V(q)$ and $V \subset T_0\Sigma$. Therefore for $q \in \mathrm{spt}(T)$ we can use (4.253) and (4.254) to estimate

$$|\mathbf{p}^\perp_{T_q\Sigma}(x)| \leq |\mathbf{p}^\perp_{T_q\Sigma}(q)| + |\mathbf{p}^\perp_{T_q\Sigma}(\mathbf{p}_V(q))| \leq C\mathbf{A}|q| + |\mathbf{p}^\perp_{T_q\Sigma} - \mathbf{p}^\perp_{T_0\Sigma}||\mathbf{p}_V(q)| \leq C\mathbf{A}|q|\,.$$

In particular

$$|(\mathrm{B})| \leq C\mathbf{A}^2 \int \chi(|q|)^2 |q|\, d\|T\|(q) \leq C\mathbf{A}^2\,,$$

which in turn leads to

$$\int \mathrm{div}_{\vec{T}} X(q)\, d\|T\|(q) = O(\mathbf{A}^2)\,. \tag{4.259}$$

For $x = \mathbf{p}_V^\perp(q)$, $q \in \mathrm{spt}(T)$, let $\mathbf{p}_{\vec{T}}(x) \equiv \mathbf{p}_{\pi(q)}(x)$ denote the orthogonal projection of x onto the approximate tangent plane $\pi(q)$ at q. We next compute $\mathrm{div}_{\vec{T}} X$:

$$\mathrm{div}_{\vec{T}}\, X(q) = 2\chi(|q|)\, \mathbf{p}_{\vec{T}}(x) \cdot \nabla\chi(|q|) + \chi(|q|)^2 \mathrm{div}_{\vec{T}} x\,.$$

Let us now compute $\mathrm{div}_{\vec{T}} x$. Complete $e_1, \ldots, e_m$ to an orthonormal basis $e_1, \ldots, e_m, \nu_1 \ldots, \nu_n$ of $\mathbb{R}^{m+n}$, and compute

$$\begin{aligned}
\mathrm{div}_{\vec{T}} x &= \sum_i e_i \cdot \partial_{e_i} x = \sum_i e_i \cdot \mathbf{p}_V^\perp(e_i) \\
&= m - \sum_i e_i \cdot \mathbf{p}_V(e_i) = m - \mathrm{tr}\,(\mathbf{p}_V) + \sum_j \nu_j \cdot \mathbf{p}_V(\nu_j) = 2 + \sum_j \nu_j \cdot \mathbf{p}_V(\nu_j)\,,
\end{aligned}$$

where the last equality follows from the fact that V is $(m-2)$-dimensional. Let us now

rewrite the sum on the right-hand side in a more convenient form. Observe that

$$\operatorname{tr}\left(\mathbf{p}_V \circ \mathbf{p}_{\vec{T}}^{\perp}\right) = \operatorname{tr}\left(\mathbf{p}_{\vec{T}}^{\perp} \circ \mathbf{p}_V \circ \mathbf{p}_{\vec{T}}^{\perp}\right),$$

and also, since $\mathbf{p}_V^T = \mathbf{p}_V$, $\mathbf{p}_{\vec{T}}^T = \mathbf{p}_{\vec{T}}$, and $\mathbf{p}_V \circ \mathbf{p}_V = \mathbf{p}_V$, we have

$$\begin{aligned}|\mathbf{p}_V \circ \mathbf{p}_{\vec{T}}^{\perp}|^2 &= \operatorname{tr}\left((\mathbf{p}_V \circ \mathbf{p}_{\vec{T}}^{\perp})^T \circ (\mathbf{p}_V \circ \mathbf{p}_{\vec{T}}^{\perp})\right) = \operatorname{tr}\left((\mathbf{p}_{\vec{T}}^{\perp})^T \circ \mathbf{p}_V^T \circ \mathbf{p}_V \circ \mathbf{p}_{\vec{T}}^{\perp}\right) \\ &= \operatorname{tr}\left(\mathbf{p}_{\vec{T}}^{\perp} \circ \mathbf{p}_V \circ \mathbf{p}_V \circ \mathbf{p}_{\vec{T}}^{\perp}\right) = \operatorname{tr}\left(\mathbf{p}_{\vec{T}}^{\perp} \circ \mathbf{p}_V \circ \mathbf{p}_{\vec{T}}^{\perp}\right),\end{aligned}$$

Thus, we deduce that

$$\sum_j \nu_j \cdot \mathbf{p}_V(\nu_j) = \operatorname{tr}\left(\mathbf{p}_V \circ \mathbf{p}_{\vec{T}}^{\perp}\right) = |\mathbf{p}_V \circ \mathbf{p}_{\vec{T}}^{\perp}|^2,$$

so in summary,

$$\operatorname{div}_{\vec{T}} X(q) = 2\chi(|q|)\, \mathbf{p}_{\vec{T}}(x) \cdot \nabla\chi(|q|) + \chi(|q|)^2(2 + |\mathbf{p}_V \circ \mathbf{p}_{\vec{T}}^{\perp}|^2). \tag{4.260}$$

Plugging this into (4.259) we then conclude that

$$\int \chi(|q|)^2(2 + |\mathbf{p}_V \circ \mathbf{p}_{\vec{T}}^{\perp}|^2)\, d\|T\|(q) \leq C\mathbf{A}^2 - \int 2\chi(|q|)\, \mathbf{p}_{\vec{T}}(x) \cdot \nabla\chi(|q|)\, d\|T\|(q). \tag{4.261}$$

Since $\mathrm{id} = \mathbf{p}_V + \mathbf{p}_V^{\perp} = \mathbf{p}_{\vec{T}} + \mathbf{p}_{\vec{T}}^{\perp}$ and $x = \mathbf{p}_V^{\perp}(q)$, we have

$$\begin{aligned}\mathbf{p}_{\vec{T}}(x) \cdot \nabla\chi(|q|) &= \mathbf{p}_{\vec{T}}(x) \cdot \mathbf{p}_V(\nabla\chi(|q|)) + \mathbf{p}_{\vec{T}}(x) \cdot \mathbf{p}_{V^\perp}(\nabla\chi(|q|)) \\ &= -\mathbf{p}_{\vec{T}}^{\perp}(x) \cdot \mathbf{p}_V(\nabla\chi(|q|)) + \mathbf{p}_{\vec{T}}(x) \cdot \mathbf{p}_{V^\perp}(\nabla\chi(|q|))\end{aligned} \tag{4.262}$$

On the other hand, once again using that $\mathbf{p}_V^T = \mathbf{p}_V$, we have

$$|\mathbf{p}_{\vec{T}}^{\perp}(x) \cdot \mathbf{p}_V(\nabla\chi(|q|))| = |(\mathbf{p}_V \circ \mathbf{p}_{\vec{T}}^{\perp})(\mathbf{p}_{\vec{T}}^{\perp}(x)) \cdot \nabla\chi(|q|)| \leq C|\mathbf{p}_V \circ \mathbf{p}_{\vec{T}}^{\perp}||\mathbf{p}_{\vec{T}}^{\perp}(x)|.$$

Introducing the short-hand notation $x^\perp$ for $\mathbf{p}_{\vec{T}}^\perp(x)$, as well as $\nabla_V\chi(|q|)$ and $\nabla_{V^\perp}\chi(|q|)$ for $\mathbf{p}_V(\nabla\chi(|q|))$ and $\mathbf{p}_{V^\perp}(\nabla\chi(|q|))$ respectively, we arrive at

$$\begin{aligned}-2\chi(|q|)\,\mathbf{p}_{\vec{T}}(x)\cdot\nabla\chi(|q|) &\le -2\chi(|q|)\,\mathbf{p}_{\vec{T}}(x)\cdot\nabla_{V^\perp}\chi(|q|) + C|\chi(|q|)||\mathbf{p}_V\circ\mathbf{p}_{\vec{T}}^\perp||x^\perp| \\ &\le -2\chi(|q|)\,\mathbf{p}_{\vec{T}}(x)\cdot\nabla_{V^\perp}\chi(|q|) + \frac{1}{2}\chi(|q|)^2|\mathbf{p}_V\circ\mathbf{p}_{\vec{T}}^\perp|^2 + C|x^\perp|^2\mathbf{1}_{\mathbf{B}_{2r}}\,.\end{aligned} \tag{4.263}$$

Inserting the latter inequality in (4.261) we arrive at

$$\begin{aligned}\int\chi^2(|q|)\left(1+\tfrac{1}{4}|\mathbf{p}_V\circ\mathbf{p}_{\vec{T}}^\perp|^2\right)\,d\|T\|(q) \le C\mathbf{A}^2 + C\int_{\mathbf{B}_{2r}}|x^\perp|^2 d\|T\| \\ -\int\chi(|q|)\,\mathbf{p}_{\vec{T}}(x)\cdot\nabla_{V^\perp}\chi(|q|)\,d\|T\|(q)\,.\end{aligned} \tag{4.264}$$

Consider next the current $S := \sum_i Q_i\llbracket\alpha_i\rrbracket$, where we recall that the Q_i are the multiplicities on the outer region. Observe that, since $X(q) = \chi(|q|)^2 x$, if $\{\Phi_t\}_t$ denotes the one-parameter family of diffeomorphisms generated by X, then $(\Phi_t)_\sharp S = S$ for each t, since $\mathbf{S}$ is invariant under rescalings in any direction in $V^\perp$. In particular, the first variation formula tells us that we must have

$$\int \mathrm{div}_{\vec{S}}X\,d\|S\| = 0\,.$$

Now we may repeat the computation above leading to (4.260), but with $\vec{S}$ in place of $\vec{T}$; notice however that (4.260) is in fact simpler because $\mathbf{p}_V\circ\mathbf{p}_{\vec{S}}^\perp = 0$. Also notice that (4.262) is simpler, because for $q\in\mathbf{S} = \mathrm{spt}(S)$ we have that x is tangent to S and therefore $\mathbf{p}_{\vec{S}}^\perp = 0$ whilst $\mathbf{p}_{\vec{S}}(x) = x$, meaning the first term on the right hand side of (4.262) vanishes and the second is simply $x\cdot\mathbf{p}_{V^\perp}(\nabla\chi(|q|))$. Thus, in place of (4.264) we get

$$\int\chi^2(|q|)\,d\|S\|(q) = -\int\chi(|q|)x\cdot\nabla_{V^\perp}\chi(|q|)\,d\|S\|(q)\,.$$

Subtracting this from (4.264) and rearranging, we arrive at

$$\begin{aligned}\int \chi^2(|q|)\,d\|T\|(q) - \int \chi^2(|q|)\,d\|S\|(q) \\ &\leq \int \chi^2(|q|)\,d\|T\|(q) - \int \chi^2(|q|)\,d\|S\|(q) + \frac{1}{4}\int \chi^2(|q|)|\mathbf{p}_V \circ \mathbf{p}_{\vec{T}}^{\perp}|^2\,d\|T\|(q) \\ &\leq C\mathbf{A}^2 + C\int_{\mathbf{B}_{2r}} |x^{\perp}|^2 d\|T\| + \int \chi(|q|)x\cdot\nabla_{V^{\perp}}\chi(|q|)\,d\|S\|(q) \\ &\quad - \int \chi(|q|)\,\mathbf{p}_{\vec{T}}(x)\cdot\nabla_{V^{\perp}}\chi(|q|)\,d\|T\|(q)\,. \end{aligned} \tag{4.265}$$

On the other hand, observe that for any function f on $\mathbb{R}^{m+n}$,

$$\int f(q)\,d\|S\|(q) = \sum_i Q_i \int_{\alpha_i} f(q)\,d\mathcal{H}^m(q)\,.$$

Therefore, combining (4.265) with our monotonicity formula estimate (4.256), we arrive at

$$\begin{aligned}\int_{\mathbf{B}_r} \frac{|q^{\perp}|^2}{|q|^{m+2}}\,d\|T\|(q) \leq C\mathbf{A}^2 + C\int_{\mathbf{B}_{2r}} |x^{\perp}|^2 d\|T\| + \sum_i Q_i \int_{\alpha_i} \chi(|q|)\,x\cdot\nabla_{V^{\perp}}\chi(|q|)\,d\mathcal{H}^m(q) \\ - \int \chi(|q|)\,\mathbf{p}_{\vec{T}}(x)\cdot\nabla_{V^{\perp}}\chi(|q|)\,d\|T\|(q)\,. \end{aligned} \tag{4.266}$$

Observe however that (4.256) gives

$$\int \chi^2(|q|)d\|T\|(q) - \int \chi^2(|q|)d\|S\|(q) \geq -C\mathbf{A}^2\,.$$

Thus, from (4.265) we may further infer that

$$\begin{aligned}\int_{\mathbf{B}_r} |\mathbf{p}_V \circ \mathbf{p}_{\vec{T}}^{\perp}|^2\,d\|T\| \leq C\mathbf{A}^2 + C\int_{\mathbf{B}_{2r}} |x^{\perp}|^2 d\|T\| + \sum_i Q_i \int_{\alpha_i} \chi(|q|)\,x\cdot\nabla_{V^{\perp}}\chi(|q|)\,d\mathcal{H}^m(q) \\ - \int \chi(|q|)\,\mathbf{p}_{\vec{T}}(x)\cdot\nabla_{V^{\perp}}\chi(|q|)\,d\|T\|(q)\,. \end{aligned} \tag{4.267}$$

Key estimates

The rest of the proof is dedicated to estimating the terms in the right hand side of (4.266) and, equivalently, (4.267). To that end we will use the refined graphical approximation of Proposition 4.6.14 and observe that r has been chosen so that $\mathbf{B}_{2r} \subset R \cup V$. Observe moreover that V is a negligible set in all the integrals appearing in the right and side of (4.266) and we will therefore ignore it. Recalling the inner, central, and outer regions (cf. Definition 4.6.12), we will split our task into four estimates , depending on whether we are integrating over the inner region (near the spine), or central or outer regions (the latter two will be coupled together):

$$\underbrace{\int_{R^{in}} |x^{\perp}|^2 d\|T\|}_{=:\,(\mathrm{C})} \leq C(\mathbf{A}^2 + \hat{\mathbf{E}}(T, \mathbf{S}, \mathbf{B}_4)) \tag{4.268}$$

$$\underbrace{\sum_j Q_j \int_{\alpha_j \cap R^{in}} \chi(|q|)\, |x \cdot \nabla_{V^{\perp}} \chi(|q|)|\, d\mathcal{H}^m(q) + \int_{R^{in}} \chi(|q|)\, |\mathbf{p}_{\vec{T}}(x) \cdot \nabla_{V^{\perp}} \chi(|q|)|\, d\|T\|(q)}_{=:\,(\mathrm{D})}$$

$$\leq C(\mathbf{A}^2 + \hat{\mathbf{E}}(T, \mathbf{S}, \mathbf{B}_4)) \tag{4.269}$$

$$\underbrace{\int_{R^o \cup R^c} |x^{\perp}|^2 d\|T\|}_{=:\,(\mathrm{E})} \leq C(\mathbf{A}^2 + \hat{\mathbf{E}}(T, \mathbf{S}, \mathbf{B}_4)) \tag{4.270}$$

$$\underbrace{\left| \sum_j Q_j \int_{\alpha_j \cap (R^o \cup R^c)} \chi(|q|)\, x \cdot \nabla_{V^{\perp}} \chi(|q|)\, d\mathcal{H}^m(q) - \int_{R^o \cup R^c} \chi(|q|)\, \mathbf{p}_{\vec{T}}(x) \cdot \nabla_{V^{\perp}} \chi(|q|)\, d\|T\|(q) \right|}_{=:\,(\mathrm{F})}$$

$$\leq C(\mathbf{A}^2 + \hat{\mathbf{E}}(T, \mathbf{S}, \mathbf{B}_4))\,. \tag{4.271}$$

Once we have established these four estimates, the result follows from (4.266) and (4.267). We stress that, whilst in (4.269) we do not care about subtracting the two terms as in (4.266), (4.267) (indeed, our estimates on the inner region will suffice there), it is important in the outer and central regions that we are subtracting the two terms, as in (4.271); the argument will exploit in a crucial way a cancellation effect due to the fact that this is a difference

between two nearly equal quantities (so, we do not wish to crudely estimate (F) by the sum of the two terms therein at any point, unlike in (D)).

Estimates in the inner region

This section is dedicated to prove the first two inequalities, (4.268) and (4.269). First of all observe that

$$|x^{\perp}|^2 \leq |x|^2 = \operatorname{dist}(q, V)^2 \tag{4.272}$$

Note that,

$$\chi(|q|)|x \cdot \nabla_{V^{\perp}} \chi(|q|)| = \chi(|q|)|x|^2 \frac{|\chi'(|q|)|}{|q|} \,. \tag{4.273}$$

On the other hand $\chi'(|q|) = 0$ if $|q| \leq r$ and $|\chi'(|q|)| \leq \frac{C}{r}$ otherwise; in particular we conclude that for $C = C(m) > 0$ we have

$$\chi(|q|)|x \cdot \nabla_{V^{\perp}} \chi(|q|)| \leq C \operatorname{dist}(q, V)^2 \,. \tag{4.274}$$

On the other hand

$$\chi(|q|)|\mathbf{p}_{\vec{T}}(x) \cdot \nabla_{V^{\perp}} \chi(|q|)| \leq \chi(|q|)|\mathbf{p}_{\vec{T}}(x) \cdot x| \frac{|\chi'(|q|)|}{|q|} = \chi(|q|)|\mathbf{p}_{\vec{T}}(x)|^2 \frac{|\chi'(|q|)|}{|q|}$$

and since $|\mathbf{p}_{\vec{T}}(x)| \leq |x|$, we can combine it with the second inequality of (4.272) to estimate

$$\chi(|q|)|\mathbf{p}_{\vec{T}}(x) \cdot \nabla_{V^{\perp}} \chi(|q|)| \leq C \operatorname{dist}(q, V)^2 \,. \tag{4.275}$$

Using the monotonicity formula for $y \in [-\frac{1}{\sqrt{m-2}}, \frac{1}{\sqrt{m-2}}]^{m-2} \subset V$ and $\rho \leq 2$ (which in particular tells us that $\|T\|(\mathbf{B}_\rho(y)) \leq C\rho^m$), we easily conclude that

$$\int_{\mathbf{B}_\rho(y)} (|x^{\perp}|^2 + \chi(|q|)|\mathbf{p}_{\vec{T}}(x) \cdot \nabla_{V^{\perp}} \chi(|q|)|) \, d\|T\|(q) \leq C\rho^{m+2} \,, \tag{4.276}$$

$$\sum_j Q_j \int_{\alpha_j \cap \mathbf{B}_\rho(y)} \chi(|q|)|x \cdot \nabla_{V^\perp} \chi(|q|)|\, d\mathcal{H}^m(q) \leq C\rho^{m+2}\,. \tag{4.277}$$

Consider now the cubes $L \in \mathcal{G}^{in}$ as in Definition 4.6.12 and enumerate them as $\{L_k\}_k$. Let $y_k := y_{L_k} \in V$ be the corresponding centers and let $\rho_k = 2^{2-\ell(L_k)}$ be the radii of the balls $\mathbf{B}(L_k)$ as defined in Section 4.6.5. Clearly $\{\mathbf{B}_{\rho_k}(y_k)\}$ is a covering of R^{in}. We now appeal to (4.132) in Lemma 4.6.13 and use (4.276), (4.277) with $y = y_k$ and $\rho = \rho_k$ to estimate

$$(\mathrm{C}) + (\mathrm{D}) \leq C \sum_k \rho_k^{m+2} \mathbf{E}(L_k, 0) = C \sum_k \int_{\mathbf{B}^h(L_k)} \mathrm{dist}^2(q, \mathbf{S})\, d\|T\|(q)\,. \tag{4.278}$$

Recall that, by Lemma 4.6.10(iv), the collection of sets $\mathbf{B}^h(L_k)$ has a control on their overlaps (each point in R belongs to at most $C(m)$ such sets). Therefore

$$(\mathrm{C}) + (\mathrm{D}) \leq C \int_{\mathbf{B}_4} \mathrm{dist}^2(q, \mathbf{S})\, d\|T\|(q)\,,$$

which gives (4.268) and (4.269).

Estimates in the central and outer regions

In this section we prove (4.270) and (4.271) and hence conclude the proof of Theorem 4.9.2. We first observe that, because of Lemma 4.6.13(iii), (iv) and Lemma 4.6.10(iv), (v) we have

$$\sum_{L \in \mathcal{G}^c \cup \mathcal{G}^o} 2^{-(m+2)\ell(L)} (\mathbf{E}(L) + 2^{-2\ell(L)} \mathbf{A}^2) \leq C(\hat{\mathbf{E}}(T, \mathbf{S}, \mathbf{B}_4) + \mathbf{A}^2)\,. \tag{4.279}$$

For each $L \in \mathcal{G}^c \cup \mathcal{G}^o$ we consider the approximations u_L given by Proposition 4.6.14. These approximations consist of $Q_{L,i}$-valued functions defined over the regions λL_i for some sub-collection of the planes $\{\alpha_1, \ldots, \alpha_N\}$ for each L. In order to simplify our notation, we do not keep track of these collections for different L.

The estimate (4.270) is less laborious than (4.271), thanks to the fact that we do not

need to exploit any cancellation effect. We can in particular write

$$(\mathrm{E}) \leq \sum_{L\in\mathcal{G}^c\cup\mathcal{G}^o} \int_{R(L)} |x^\perp|^2 \, d\|T\| \,. \tag{4.280}$$

Now, for each region $R(L)$ with $L \in \mathcal{G}^c \cup \mathcal{G}^o$, we can use that $|x^\perp|^2 \leq C2^{-2\ell(L)}$ on $\mathrm{spt}(T) \cap R(L)$, together with Proposition 4.6.14(iii) and (4.137), to estimate

$$\int_{R(L)} |x^\perp|^2 d\|T\| \leq \sum_i \int_{\Omega(L)} |x^\perp|^2 \, d\|\mathbf{G}_{u_{L,i}}\| + C2^{-(m+2)\ell(L)}(\mathbf{E}(L) + 2^{-2\ell(L)}\mathbf{A}^2) \,.$$

Using (4.279) we then conclude that

$$(\mathrm{E}) \leq \sum_{L\in\mathcal{G}^c\cup\mathcal{G}^o} \sum_i \int_{\Omega(L)} |x^\perp|^2 \, d\|\mathbf{G}_{u_{L,i}}\| + C(\hat{\mathbf{E}}(T, \mathbf{S}, \mathbf{B}_4) + \mathbf{A}^2) \,. \tag{4.281}$$

Consider now coordinates z on α_i and recall that, for a point $q = (z, (u_{L,i})_j(z))$, for some $j = 1, \ldots, Q_{L,i}$, in the graph of the $Q_{L,i}$-valued function $u_{L,i} = \sum_j [\![(u_{L,i})_j]\!]$, the vector $x^\perp$ is the projection of $x = \mathbf{p}_V^\perp(q)$ onto the orthogonal complement of the tangent plane $\vec{\mathbf{G}}_{u_{L,i}}$ to the graph of $(u_{L,i})_j$ at q. In particular, since at such a point q we have $\big|\mathbf{p}^\perp_{\vec{\mathbf{G}}_{u_{L,i}}} - \mathbf{p}^\perp_{\alpha_i}\big| \leq C|\nabla(u_{L,i})_j(z)|$, and moreover because $\mathbf{p}^\perp_{\alpha_i} \circ \mathbf{p}^\perp_V = \mathbf{p}^\perp_{\alpha_i}$, this yields (noting that $|x| \leq |q| \leq C2^{-\ell(L)}$)

$$\begin{aligned} |x^\perp| = \big|\mathbf{p}^\perp_{\vec{\mathbf{G}}_{u_{L,i}}}(x)\big| &\leq \big|\mathbf{p}^\perp_{\vec{\mathbf{G}}_{u_{L,i}}} - \mathbf{p}^\perp_{\alpha_i}\big| \cdot |x| + |\mathbf{p}^\perp_{\alpha_i}(\mathbf{p}^\perp_V(q))| \\ &\leq C|\nabla(u_{L,i})_j(z)| \cdot 2^{-\ell(L)} + |(u_{L,i})_j(z)| \end{aligned} \tag{4.282}$$

Now if we square this expression and integrate it, we get from the two bounds in (4.135) (to control the integrand) as well as the Lipschitz bound (4.136) (to control the Jacobian) that

$$\sum_i \int_{\Omega(L)} |x^\perp|^2 \, d\|\mathbf{G}_{u_{L,i}}\| \leq C \sum_{i,j} \int_{\Omega_i(L)} |\nabla(u_{L,i})_j(z)|^2 2^{-2\ell(L)} + |(u_{L,i})_j(z)|^2 d\|\mathbf{G}_{u_{L,i}}\|$$

$$\leq C2^{-(m+2)\ell(L)}(\mathbf{E}(L)+2^{-2\ell(L)}\mathbf{A}^2).$$

In particular, plugging the latter estimate in (4.281) and using again (4.279) we reach (4.270).

We now come to the proof of (4.271), which is more laborious. First of all, because $\|\mathbf{G}_{u_{L,i}}\|(\partial R(L))=0$ for each $L\in\mathcal{G}^c\cup\mathcal{G}^o$, we can use Proposition 4.6.14(iii), the estimate (4.137) and the estimate (4.275) to deduce that

$$\left|\int_{\partial R(L)}\chi(|q|)\,\mathbf{p}_{\vec{T}}(x)\cdot\nabla_{V^\perp}\chi(|q|)\,d\|T\|(q)\right|\leq C2^{-(m+2)\ell(L)}(\mathbf{E}(L)+2^{-2\ell(L)}\mathbf{A}^2)\,.$$

Thus, letting $(R(L))^\circ$ denote the interior of $R(L)$, we can once again use (4.279) to estimate

$$\begin{aligned}(\mathrm{F})\leq\sum_{L\in\mathcal{G}^c\cup\mathcal{G}^o}\Bigg|\sum_j Q_j\int_{\alpha_j\cap(R(L))^\circ}\chi(|q|)\,x\cdot\nabla_{V^\perp}\chi(|q|)\,d\mathcal{H}^m\\ -\int_{(R(L))^\circ}\chi(|q|)\,\mathbf{p}_{\vec{T}}(x)\cdot\nabla_{V^\perp}\chi(|q|)\,d\|T\|\Bigg|+C(\hat{\mathbf{E}}(T,\mathbf{S},\mathbf{B}_4)+\mathbf{A}^2)\,.\end{aligned}\tag{4.283}$$

We remove the boundary as above to ensure that our regions are disjoint. Now we want to straighten out the curved regions (using Lemma 4.6.15) to cylinders over disjoint cubes in our planes where we can pass to graphs. So, next, again taking into account (4.275), and then (recalling the notation $L_j=\alpha_j\cap R(L)$) using Lemma 4.6.15 with $U=(R(L))^\circ$ and $\tilde{U}=\bigcup_j\mathbf{p}_{\alpha_j}^{-1}(L_j)$, since $\mathcal{H}^{m-1}(\partial L_j)\leq C2^{-(m-1)\ell(L)}$, we can further estimate

$$\begin{aligned}&\left|\int_{(R(L))^\circ}\chi(|q|)\,\mathbf{p}_{\vec{T}}(x)\cdot\nabla_{V^\perp}\chi(|q|)\,d\|T\|-\sum_j\int_{\mathbf{p}_{\alpha_j}^{-1}(L_j)}\chi(|q|)\,\mathbf{p}_{\vec{T}}(x)\cdot\nabla_{V^\perp}\chi(|q|)\,d\|T_{L,j}\|\right|\\ &\quad\leq C2^{-(m+2)\ell(L)}(\mathbf{E}(L)+2^{-2\ell(L)}\mathbf{A}^2)\,.\end{aligned}$$

We can now yet again use (4.275), as well as the estimate, (4.137) of Proposition 4.6.14 to

further estimate

$$\Big|\sum_j \int_{\mathbf{p}_{\alpha_j}^{-1}(L_j)} \chi(|q|)\, \mathbf{p}_{\vec{T}}(x)\cdot \nabla_{V^\perp}\chi(|q|)\, d\|T_{L,j}\| \\ - \sum_j \int_{\mathbf{p}_{\alpha_j}^{-1}(L_j)} \chi(|q|)\, \mathbf{p}_{\vec{\mathbf{G}}_{u_{L,j}}}(x)\cdot \nabla_{V^\perp}\chi(|q|)\, d\|\mathbf{G}_{u_{L,j}}\|\Big| \\ \leq C 2^{-(m+2)\ell(L)}(\mathbf{E}(L) + 2^{-2\ell(L)}\mathbf{A}^2)$$

Combining the above estimates with (4.283) and again making use of (4.279), we arrive at

$$(\mathrm{F}) \leq \sum_{L\in\mathcal{G}^c\cup\mathcal{G}^o} \Big|\sum_j \Big[Q_j \int_{\alpha_j\cap(R(L))^\circ} \chi(|q|)\, x\cdot\nabla_{V^\perp}\chi(|q|)\, d\mathcal{H}^m \\ - \int_{\mathbf{p}_{\alpha_j}^{-1}(L_j)} \chi(|q|)\, \mathbf{p}_{\vec{\mathbf{G}}_{u_{L,j}}}(x)\cdot\nabla_{V^\perp}\chi(|q|)\, d\|\mathbf{G}_{u_{L,j}}\|\Big]\Big| \\ + C(\hat{\mathbf{E}}(T,\mathbf{S},\mathbf{B}_4) + \mathbf{A}^2)\,. \tag{4.284}$$

Now note that the multiplicities Q_j are the ones from the outer region, and so they do not necessarily match the multiplicities $Q_{L,j}$ of the multi-valued functions $u_{L,j}$ when $L\in\mathcal{G}^c$. However recall the computation (see (4.273)),

$$\chi(|q|)x\cdot\nabla_{V^\perp}\chi(|q|) = \frac{\chi(|q|)\chi'(|q|)}{|q|}|x|^2\,, \tag{4.285}$$

which since $x\in V^\perp$ shows that the integrand $\chi(|q|)x\cdot\nabla_{V^\perp}\chi(|q|)$ is invariant under rotations which keep the spine V fixed. Since for every $j,k\in\{1,\dots,N\}$ there is a rotation which maps α_j onto α_k and fixes V, the integral

$$\int_{\alpha_j\cap(R(L))^\circ} \chi(|q|)\, x\cdot\nabla_{V^\perp}\chi(|q|)\, d\mathcal{H}^m$$

is independent of the plane α_j. In particular, given that $\sum_j Q_{L,j} = \sum_j Q_j = Q$, we can in

fact write from (4.284) (using $L_j = \alpha_j \cap R(L)$

$$\begin{aligned}(\mathrm{F}) \le \sum_{L\in\mathcal{G}^c\cup\mathcal{G}^o}\sum_j \Bigg| Q_{L,j}\int_{L_j}\chi(|q|)\, x\cdot\nabla_{V^\perp}\chi(|q|)\,d\mathcal{H}^m \\ -\int_{\mathbf{p}_{\alpha_j}^{-1}(L_j)}\chi(|q|)\,\mathbf{p}_{\vec{\mathbf{G}}_{u_{L,j}}}(x)\cdot\nabla_{V^\perp}\chi(|q|)\,d\|\mathbf{G}_{u_{L,j}}\|\Bigg| \\ + C(\hat{\mathbf{E}}(T,\mathbf{S},\mathbf{B}_4)+\mathbf{A}^2) \quad (4.286)\end{aligned}$$

Next, following the same computation as for (4.275) we get

$$|\chi(|q|)\,\mathbf{p}_{\vec{\mathbf{G}}_{u_{L,j}}}(x)\cdot\nabla_{V^\perp}\chi(|q|)| \le C\,\mathrm{dist}(q,V)^2 \le C\cdot 2^{-2\ell(L)}\,,$$

and hence, through the usual Taylor expansion of the area functional for a multi-valued graph, for $k=1,\dots,Q_{L,j}$ letting $q_k \coloneqq (z,(u_{L,j})_k(z)) \equiv z+(u_{L,j})_k(z)\in\alpha_j\times\alpha_j^\perp$, we get

$$\begin{aligned}\Bigg|\int_{\mathbf{p}_{\alpha_j}^{-1}(L_j)}\chi(|q|)\,\mathbf{p}_{\vec{\mathbf{G}}_{u_{L,j}}}(x)\cdot\nabla_{V^\perp}\chi(|q|)\,d\|\mathbf{G}_{u_{L,j}}\| \\ -\int_{L_j}\sum_k\chi(|q_k|)\,\mathbf{p}_{\vec{\mathbf{G}}_{(u_{L,j})_k}}(\mathbf{p}_V^\perp(q_k))\cdot\nabla_{V^\perp}\chi(|q_k|)\,d\mathcal{H}^m(z)\Bigg| \\ \le C2^{-2\ell(L)}\int_{L_j}|Du_{L,j}|^2\,d\mathcal{H}^m\,.\end{aligned}$$

We can in particular use (4.286), (4.135), and (4.279) to obtain

$$\begin{aligned}(\mathrm{F}) \le \sum_{L\in\mathcal{G}^c\cup\mathcal{G}^o}\sum_j \Bigg|\int_{L_j}[Q_{L,j}\chi(|z|)\,\mathbf{p}_V^\perp(z)\cdot\nabla_{V^\perp}\chi(|z|) \\ -\sum_k\chi(|q_k|)\,\mathbf{p}_{\vec{\mathbf{G}}_{(u_{L,j})_k}}(\mathbf{p}_V^\perp(q_k))\cdot\nabla_{V^\perp}\chi(|q_k|)]\,d\mathcal{H}^m(z)\Bigg| \\ + C(\hat{\mathbf{E}}(T,\mathbf{S},\mathbf{B}_4)+\mathbf{A}^2) \quad (4.287)\end{aligned}$$

Now use the coordinates $z=(\zeta,\xi)\in V\times V^\perp$ for points $z\in\alpha_j$. Recalling (4.285), the first

integrand above is then given by

$$h(z) := Q_{L,j} \frac{\chi(|z|)\chi'(|z|)}{|z|} |\xi|^2 \,.$$

Meanwhile, we write the second integrand as

$$g(z) := \sum_{k=1}^{Q_{L,j}} \frac{\chi(|q_k|)\chi'(|q_k|)}{|q_k|} \mathbf{p}_{\vec{\mathbf{G}}_{(u_{L,j})_k}}(\mathbf{p}_V^\perp(q_k)) \cdot \mathbf{p}_V^\perp(q_k) \,.$$

Considering that $\chi'(|q|) = 0$ when $|q| \leq r$, for each $k = 1, \ldots, Q_{L,j}$ we can estimate (by Taylor expansion)

$$\left| \frac{\chi(|q_k|)\chi'(|q_k|)}{|q_k|} - \frac{\chi(|z|)\chi'(|z|)}{|z|} \right| \leq C 2^{2\ell(L)} |(u_{L,j})_k(z)|^2 \,.$$

On the other hand we have

$$\left| \mathbf{p}_{\vec{\mathbf{G}}_{(u_{L,j})_k}}(\mathbf{p}_V^\perp(q_k)) \cdot \mathbf{p}_V^\perp(q_k) \right| \leq C 2^{-2\ell(L)} \,.$$

In particular, if we define

$$\bar{g}(z) := \sum_{k=1}^{Q_{L,j}} \frac{\chi(|z|)\chi'(|z|)}{|z|} \mathbf{p}_{\vec{\mathbf{G}}_{(u_{L,j})_k}}(\mathbf{p}_V^\perp(q_k)) \cdot \mathbf{p}_V^\perp(q_k) \,,$$

we then get

$$|g(z) - \bar{g}(z)| \leq C |u_{L,j}(z)|^2 \,. \tag{4.288}$$

Next, recalling the definition of q_k, notice that

$$\mathbf{p}_V^\perp(q_k) = \xi + (u_{L,j})_k(z) \,,$$

On the other hand,

$$
\begin{aligned}
&|\mathbf{p}_{\vec{\mathbf{G}}_{(u_{L,j})_k}}(\mathbf{p}_V^{\perp}(q_k))\cdot\mathbf{p}_V^{\perp}(q_k)-|\xi|^2| \\
&\qquad=\left||\mathbf{p}_{\vec{\mathbf{G}}_{(u_{L,j})_k}}(\xi+(u_{L,j})_k)|^2-|\xi|^2\right| \\
&\qquad=\left||\mathbf{p}_{\vec{\mathbf{G}}_{(u_{L,j})_k}}(\xi)|^2-|\xi|^2\right|+|\mathbf{p}_{\vec{\mathbf{G}}_{(u_{L,j})_k}}((u_{L,j})_k)|^2 \\
&\qquad\qquad+2\left|\mathbf{p}_{\vec{\mathbf{G}}_{(u_{L,j})_k}}(\xi)\cdot\mathbf{p}_{\vec{\mathbf{G}}_{(u_{L,j})_k}}((u_{L,j})_k)\right| \\
&\qquad\le|\mathbf{p}^{\perp}_{\vec{\mathbf{G}}_{(u_{L,j})_k}}(\xi)|^2+|(u_{L,j})_k|^2+2|\mathbf{p}_{\vec{\mathbf{G}}_{(u_{L,j})_k}}(\xi)|\left|(\mathbf{p}_{\vec{\mathbf{G}}_{(u_{L,j})_k}}-\mathbf{p}_{\alpha_j})((u_{L,j})_k)\right| \\
&\qquad\le|\mathbf{p}^{\perp}_{\vec{\mathbf{G}}_{(u_{L,j})_k}}-\mathbf{p}^{\perp}_{\alpha_j}|^2|\xi|^2+|(u_{L,j})_k|^2+2|\xi|\cdot|Du_{L,j})_k||(u_{L,j})_k| \\
&\qquad\le|Du_{L,j})_k|^2 2^{-2\ell(L)}+|(u_{L,j})_k|^2+2^{1-\ell(L)}|Du_{L,j})_k||(u_{L,j})_k| \\
&\qquad\le C2^{-2\ell(L)}|Du_{L,j})_k|^2+C|(u_{L,j})_k|^2
\end{aligned}
$$

where we have used that $\mathbf{p}_{\alpha_j}((u_{L,j})_k)=0$ and $\mathbf{p}^{\perp}_{\alpha_j}(\xi)=0$. In particular, we arrive at

$$
|\bar{g}(z)-h(z)|\le C|u_{L,j}(z)|^2+2^{-2\ell(L)}|Du_{L,j}(z)|^2\,.
$$

Combining this last inequality and (4.288) with the bound (4.135), for each $L\in\mathcal{G}^c\cup\mathcal{G}^o$ and each index j enumerating the planes for the corresponding cone associated to L, we thus achieve

$$
\begin{aligned}
&\left|\int_{L_j}Q_{L,j}\chi(|z|)\,\mathbf{p}_{V^\perp}(z)\cdot\nabla_{V^\perp}\chi(|z|)\,d\mathcal{H}^m(z)\right. \\
&\qquad\left.-\sum_k\int_{L_j}\chi(|q_k|)\,\mathbf{p}_{\vec{\mathbf{G}}_{(u_{L,j})_k}}(\mathbf{p}_V^{\perp}(q_k))\cdot\nabla_{V^\perp}\chi(|q_k|)\,d\mathcal{H}^m(z)\right| \\
&\qquad\le C2^{-(m+2)\ell(L)}(\mathbf{E}(L)+2^{-2\ell(L)}\mathbf{A}^2)
\end{aligned}
$$

Together with (4.287) and (4.279), this completes the proof of (4.271) and thus the proof of the theorem. □

4.9.2 Proof of Simon's non-concentration estimate (Corollary 4.9.3)

We observe that Corollary 4.9.3 is a direct consequence of Theorem 4.9.2 and of the following lemma (after an appropriate rescaling to adjust the radius).

Lemma 4.9.6. *Let T, Σ and $\mathbf{S}$ be as in Assumption 4.9.1 with $\mathbf{B}_1 \subset \Omega$ and $\Theta(T,0) \geq Q$. Then we may choose ε sufficiently small in Assumption 4.9.1 such that for each $\kappa > 0$ we have*

$$\int_{\mathbf{B}_1} \frac{\operatorname{dist}^2(q,\mathbf{S})}{|q|^{m+2-\kappa}}\, d\|T\|(q) \leq C_\kappa \int_{\mathbf{B}_1} \frac{|q^\perp|^2}{|q|^{m+2}}\, d\|T\|(q) + C_\kappa(\hat{\mathbf{E}}(T,\mathbf{S},\mathbf{B}_4) + \mathbf{A}^2)\,, \tag{4.289}$$

where $C_\kappa = C_\kappa(Q,m,n,\bar{n},M,\kappa) > 0$.

Proof of Lemma 4.9.6. Fix $\kappa \in (0, m+2)$. We test the first variation identity (4.257) for T with the vector field

$$X(q) := \operatorname{dist}^2(q,\mathbf{S}) \underbrace{(\max\{r,|q|\}^{-m-2+\kappa} - 1)_+}_{=:f(q)} q\,,$$

where $r > 0$ and for a function g we define $g_+(q) \coloneqq \max\{g(q),0\}$. Note that f is supported in $\mathbf{B}_1 \setminus \mathbf{B}_r$. In order to estimate the integrand in (B) of (4.257), we first observe that

$$f(q)\operatorname{dist}^2(q,\mathbf{S}) \leq |q|^{\kappa-m}\,.$$

Recalling (4.255), (4.252), and (4.254), we have

$$|\vec{H}_T \cdot X^\perp(q)| \leq C\mathbf{A}^2|q|^{1+\kappa-m}\,.$$

Hence we can estimate

$$|(\mathrm{B})| \leq C\mathbf{A}^2 \int_{\mathbf{B}_1} |q|^{1+\kappa-m}\, d\|T\|(q) \leq C\mathbf{A}^2\,.$$

In particular we conclude that

$$\int \mathrm{div}_{\vec{T}} X \; d\|T\| = O(\mathbf{A}^2)\,. \tag{4.290}$$

We next compute

$$\mathrm{div}_{\vec{T}} X(q) = m\,\mathrm{dist}^2(q,\mathbf{S}) f(q) + f(q)\nabla\,\mathrm{dist}^2(q,\mathbf{S})\cdot q_{\|} + \mathrm{dist}^2(q,\mathbf{S})\nabla f(q)\cdot q_{\|}\,. \tag{4.291}$$

Moreover, we can explicitly compute

$$\begin{aligned}\nabla f(q)\cdot q_{\|} &= -(m+2-\kappa)|q|^{-m-4+\kappa}|q_{\|}|^2\mathbf{1}_{\mathbf{B}_1\setminus\mathbf{B}_r}(q)\\ &= -(m+2-\kappa)|q|^{-m-2+\kappa}\mathbf{1}_{\mathbf{B}_1\setminus\mathbf{B}_r}(q) + (m+2-\kappa)|q|^{-m-4+\kappa}|q^{\perp}|^2\mathbf{1}_{\mathbf{B}_1\setminus\mathbf{B}_r}(q)\,.\end{aligned} \tag{4.292}$$

On the other hand, using the 2-homogeneity of $\mathrm{dist}^2(q,\mathbf{S})$ we can likewise compute

$$\begin{aligned}\nabla\,\mathrm{dist}^2(q,\mathbf{S})\cdot q_{\|} &= \nabla\,\mathrm{dist}^2(q,\mathbf{S})\cdot q - \nabla\,\mathrm{dist}^2(q,\mathbf{S})\cdot q^{\perp}\\ &= 2\,\mathrm{dist}^2(q,\mathbf{S}) - \nabla\,\mathrm{dist}^2(q,\mathbf{S})\cdot q^{\perp}\,.\end{aligned}$$

Inserting this and (4.292) into (4.291) we reach

$$\begin{aligned}\mathrm{div}_{\vec{T}} X(q) &= \kappa\,\mathrm{dist}(q,\mathbf{S})^2|q|^{-m-2+\kappa}\mathbf{1}_{\mathbf{B}_1\setminus\mathbf{B}_r}(q)\\ &\quad + (m+2)r^{-m-2+\kappa}\,\mathrm{dist}^2(q,\mathbf{S})\mathbf{1}_{\mathbf{B}_r}(q) - (m+2)\,\mathrm{dist}^2(q,\mathbf{S})\mathbf{1}_{\mathbf{B}_1}(q)\\ &\quad + (m+2-\kappa)\,\mathrm{dist}^2(q,\mathbf{S})|q^{\perp}|^2|q|^{-m-4+\kappa}\mathbf{1}_{\mathbf{B}_1\setminus\mathbf{B}_r}(q) - \nabla\,\mathrm{dist}^2(q,\mathbf{S})\cdot q^{\perp} f(q)\,.\end{aligned} \tag{4.293}$$

In turn, inserting this in (4.290) we achieve

$$\kappa\int_{\mathbf{B}_1\setminus\mathbf{B}_r}\frac{\mathrm{dist}^2(q,\mathbf{S})}{|q|^{m+2-\kappa}}\,d\|T\|(q) \le C\mathbf{A}^2 + (m+2)\hat{\mathbf{E}}(T,\mathbf{S},\mathbf{B}_1)$$

$$+ \underbrace{\int \nabla \operatorname{dist}(\mathbf{S},q)^2 \cdot q^\perp f(q)\, d\|T\|(q)}_{=:\,(\mathrm{C})}\,. \tag{4.294}$$

On the other hand, using that $\operatorname{dist}(q,\mathbf{S})$ is 1-Lipschitz, we can estimate

$$\begin{aligned}
|(\mathrm{C})| &\le 2\int \operatorname{dist}(q,\mathbf{S})|q^\perp| f(q)\, d\|T\|(q)\\
&\le \kappa \int \operatorname{dist}^2(q,\mathbf{S}) f(q)\, d\|T\|(q) + \frac{1}{\kappa}\int |q^\perp|^2 f(q)\, d\|T\|(q)\\
&\le \kappa \int_{\mathbf{B}_1\setminus\mathbf{B}_r} \frac{\operatorname{dist}^2(q,\mathbf{S})}{|q|^{m+2-\kappa}}\, d\|T\|(q)\\
&\quad + \kappa r^{-m-2+\kappa}\int_{\mathbf{B}_r} \operatorname{dist}^2(q,\mathbf{S})\, d\|T\|(q) + \frac{1}{\kappa}\int_{\mathbf{B}_1}\frac{|q^\perp|^2}{|q|^{m+2}}\, d\|T\|(q)\,.
\end{aligned} \tag{4.295}$$

Inserting (4.295) into (4.294) we then reach

$$\int_{\mathbf{B}_1\setminus\mathbf{B}_r} \frac{\operatorname{dist}^2(q,\mathbf{S})}{|q|^{m+2-\kappa}}\, d\|T\|(q) \le C_\kappa \int_{\mathbf{B}_1}\frac{|q^\perp|^2}{|q|^{m+2}}\, d\|T\|(q) + C_\kappa(\mathbf{A}^2 + \hat{\mathbf{E}}(T,\mathbf{S},\mathbf{B}_1)) + C\frac{\|T\|(\mathbf{B}_r)}{r^{m-\kappa}}\,. \tag{4.296}$$

Letting $r\downarrow 0$ and using $\|T\|(\mathbf{B}_r)\le Cr^m$ we reach (4.289). □

4.9.3 Proof of Simon's shift inequality (Proposition 4.9.4)

In Lemma 4.9.7 below we will show the following inequality for each $\kappa\in(0,m+2)$, under the assumption that ε is chosen sufficiently small and that $\rho=\rho(m)$ is a dimensional constant:

$$\int_{\mathbf{B}_\rho(q_0)} \frac{\operatorname{dist}(q,q_0+\mathbf{S})^2}{|q-q_0|^{m+2-\kappa}}\, d\|T\|(q) \le C^\star_\kappa(\mathbf{A}^2 + \hat{\mathbf{E}}(T,\mathbf{S},\mathbf{B}_4) + |\mathbf{p}^\perp_{\alpha_1}(q_0)|^2 + \boldsymbol{\mu}(\mathbf{S})^2|\mathbf{p}_{V^\perp\cap\alpha_1}(q_0)|^2)\,, \tag{4.297}$$

for $C^\star_\kappa = C^\star_\kappa(Q,m,n,\bar n)$. Assuming the validity of this, our aim is therefore to show that we also have

$$|\mathbf{p}^\perp_{\alpha_1}(q_0)|^2 + \boldsymbol{\mu}(\mathbf{S})^2|\mathbf{p}_{V^\perp\cap\alpha_1}(q_0)|^2 \le C(\mathbf{A}^2 + \hat{\mathbf{E}}(T,\mathbf{S},\mathbf{B}_4))\,, \tag{4.298}$$

for $C=C(Q,m,n,\bar n)$. Observe that (4.297) and (4.298) together yield (4.246).

Fix now a scale $\bar r \le \frac{\rho}{2}$ (for ρ fixed as in (4.297)), whose choice will be specified later. Observe that, by assuming ε is small enough depending on $\bar r$, Proposition 4.6.14(v) guarantees that $\mathrm{dist}(q_0, V) \le \frac{\bar r}{2}$. We can now apply a scaled version of Lemma 4.5.14 to find an index j and a subset $\Omega \subset \alpha_j \cap \mathbf{B}_{\bar r}(\mathbf{p}_V(q_0)) \setminus \mathbf{B}_{\bar r/2}(V)$ with the property that

$$|\mathbf{p}^\perp_{\alpha_1}(q_0)|^2 + \boldsymbol{\mu}(\mathbf{S})^2 |\mathbf{p}_{V^\perp \cap \alpha_1}(q_0)|^2 \le \bar C\, \mathrm{dist}(z, q_0 + \mathbf{S})^2 \qquad \forall z \in \Omega \tag{4.299}$$

and $\mathcal{H}^m(\Omega) \ge \bar C^{-1} \bar r^m$ for some geometric constant $\bar C > 0$ (from the proof of Lemma 4.5.14, we know $\bar C = \bar C(Q, m, n, \bar n)$ due to the specific choice of $U = \mathbf{B}_{\bar r} \setminus \mathbf{B}_{\bar r/2}(V)$ here in Lemma 4.5.14; see also Remark 4.5.15). If we choose ε sufficiently small, depending on $\bar r$, by Proposition 4.6.14 we can find a further subset $\Omega' \subset \Omega$ with measure larger than $(2\bar C)^{-1} \bar r^m$ and such that over each $z \in \Omega'$ we can find a point p in the outer graphical approximation lying in $z + \alpha_j^\perp$ and in the support of the current T with the property that

$$\mathrm{dist}(z, q_0 + \mathbf{S}) \le \mathrm{dist}(p, q_0 + \mathbf{S}) + C \bar r^{-m/2} (\hat{\mathbf{E}}(T, \mathbf{S}, \mathbf{B}_4) + \bar r^2 \mathbf{A}^2)^{1/2}\,.$$

In particular we achieve

$$|\mathbf{p}^\perp_{\alpha_1}(q_0)|^2 + \boldsymbol{\mu}(\mathbf{S})^2 |\mathbf{p}_{V^\perp \cap \alpha_1}(q_0)|^2 \le \bar C\, \mathrm{dist}^2(p, q_0 + \mathbf{S}) + C \bar r^{-m} (\hat{\mathbf{E}}(T, \mathbf{S}, \mathbf{B}_4) + \bar r^2 \mathbf{A}^2) \tag{4.300}$$

for all p in the subset Ω'' of points $p \in \mathrm{spt}(T)$ which coincide with the outer graphical approximation of Proposition 4.6.14 restricted to the subset Ω'. For this set we clearly have $\|T\|(\Omega'') \ge \mathcal{H}^m(\Omega') \ge (2\bar C)^{-1} \bar r^m$. Observe moreover that, since $\Omega' \subset \mathbf{B}_{\bar r}(\mathbf{p}_V(q_0)) \subset \mathbf{B}_{3\bar r/2}(q_0)$, if ε is sufficiently small (depending on $\bar r$), then $\Omega'' \subset \mathbf{B}_{2\bar r}(q_0)$. We now integrate (4.300) over Ω'' with respect to $d\|T\|$ to find

$$\begin{aligned} |\mathbf{p}^\perp_{\alpha_1}(q_0)|^2 + \boldsymbol{\mu}(\mathbf{S})^2 |\mathbf{p}_{V^\perp \cap \alpha_1}(q_0)|^2 \le C \bar r^{-m} & \int_{\mathbf{B}_{2\bar r}(q_0)} \mathrm{dist}^2(p, q_0 + \mathbf{S})\, d\|T\|(p) \\ & + C \bar r^{-m} (\hat{\mathbf{E}}(T, \mathbf{S}, \mathbf{B}_4) + \bar r^2 \mathbf{A}^2)\,. \end{aligned}$$

We can in particular write

$$|\mathbf{p}^{\perp}_{\alpha_1}(q_0)|^2 + \boldsymbol{\mu}(\mathbf{S})^2|\mathbf{p}_{V^\perp\cap\alpha_1}(q_0)|^2 \leq C\bar{r}^{\frac{7}{4}}\int_{\mathbf{B}_{2\bar{r}}(q_0)}\frac{\mathrm{dist}^2(q,q_0+\mathbf{S})}{|q-q_0|^{m+7/4}}\,d\|T\|(q) + C\bar{r}^{-m}(\hat{\mathbf{E}}(T,\mathbf{S},\mathbf{B}_4)+\bar{r}^2\mathbf{A}^2)\,. \tag{4.301}$$

Given that the constant C is independent of the radius $\bar{r}$, for any fixed $\delta>0$ if we choose $\bar{r}=\bar{r}(Q,m,n,\bar{n},\delta)$ sufficiently small we achieve

$$|\mathbf{p}^{\perp}_{\alpha_1}(q_0)|^2 + \boldsymbol{\mu}(\mathbf{S})^2|\mathbf{p}_{V^\perp\cap\alpha_1}(q_0)|^2 \leq \delta\int_{\mathbf{B}_{2\bar{r}}(q_0)}\frac{\mathrm{dist}(q,q_0+\mathbf{S})^2}{|q-q_0|^{m+7/4}}\,d\|T\|(q) + C\bar{r}^{-m}(\hat{\mathbf{E}}(T,\mathbf{S},\mathbf{B}_4)+\bar{r}^2\mathbf{A}^2)\,.$$

We can now insert the latter in (4.297) (with $\kappa=\frac{1}{4}$) and, upon fixing $\delta=\delta(Q,m,n,\bar{n})$ small enough (recalling that $2\bar{r}\leq\rho$), conclude that

$$\begin{aligned}\int_{\mathbf{B}_{2\bar{r}}(q_0)}\frac{\mathrm{dist}(q,q_0+\mathbf{S})^2}{|q-q_0|^{m+7/4}}\,d\|T\|(q) &\leq \int_{\mathbf{B}_{\rho}(q_0)}\frac{\mathrm{dist}(q,q_0+\mathbf{S})^2}{|q-q_0|^{m+7/4}}\,d\|T\|(q)\\ &\leq C\bar{r}^{-m}(\bar{r}^2\mathbf{A}^2+C\hat{\mathbf{E}}(T,\mathbf{S},\mathbf{B}_4))\\ &\quad+\frac{1}{2}\int_{\mathbf{B}_{2\bar{r}}(q_0)}\frac{\mathrm{dist}(q,q_0+\mathbf{S})^2}{|q-q_0|^{m+7/4}}\,d\|T\|(q)\,.\end{aligned} \tag{4.302}$$

In particular we conclude

$$\int_{\mathbf{B}_{2\bar{r}}(q_0)}\frac{\mathrm{dist}(q,q_0+\mathbf{S})^2}{|q-q_0|^{m+7/4}}\,d\|T\|(q) \leq C\bar{r}^{-m}(\bar{r}^2\mathbf{A}^2+\hat{\mathbf{E}}(T,\mathbf{S},\mathbf{B}_4))\,,$$

for this fixed choice of $\bar{r}$ and inserting the latter into (4.301) we achieve (4.298).

We are left with the task of showing that (4.297) holds. This is accomplished in the following:

Lemma 4.9.7. *If T, Σ, and $\mathbf{S}$ are as in Assumption 4.9.1 and $\rho=\frac{1}{12(m-2)}$, then (4.297) holds for every point $q_0\in\mathbf{B}_\rho$ with the property that $\Theta(T,q_0)\geq Q$.*

Proof. First of all, observe the following elementary fact for all $q, q_0, \mathbf{S}$:

$$\operatorname{dist}(q, q_0 + \mathbf{S}) \le |\mathbf{p}^{\perp}_{\alpha_1}(q_0)| + C\boldsymbol{\mu}(\mathbf{S})|\mathbf{p}_{V^{\perp}\cap\alpha_1}(q_0)| + \operatorname{dist}(q, \mathbf{S})\,. \tag{4.303}$$

Fix $\bar{\rho} = \frac{1}{4\sqrt{m-2}}$. Next note that, assuming ε is sufficiently small, we can assume that $\mathbf{B}_{4\bar{\rho}} \setminus \mathbf{B}_{a\bar{\rho}/8}(V)$ is in the outer region R^o and moreover that $\operatorname{dist}(q_0, V) < a\bar{\rho}/8$, by Proposition 4.6.14(v). Thus, using (4.303) with q_0 as in the statement of the lemma, we gain the inequality

$$\mathbb{E}(T_{q_0,\bar{\rho}}, \mathbf{S}, \mathbf{B}_4) + \mathbf{A}^2 \le \underbrace{C_0(\hat{\mathbf{E}}(T, \mathbf{S}, \mathbf{B}_4) + \mathbf{A}^2 + |\mathbf{p}^{\perp}_{\alpha_1}(q_0)|^2 + \boldsymbol{\mu}(\mathbf{S})^2|\mathbf{p}_{V^{\perp}\cap\alpha_1}(q_0)|^2)}_{=: D^2}\,, \tag{4.304}$$

for some constant C_0 which is now independent of ε.

Note that, if $D^2 \le \bar{\varepsilon}\boldsymbol{\sigma}(\mathbf{S})^2$ for $\bar{\varepsilon} = \bar{\varepsilon}(Q, m, n, \bar{n}, M)$ which is the threshold needed to apply Corollary 4.9.3, then we could apply Corollary 4.9.3 with $T_{q_0,\bar{\rho}}$ in place of T and the desired inequality (4.297) would then follow.

To handle the general case we fix a suitable $\delta > 0$, which will be chosen depending on $\bar{\varepsilon}$. We wish to apply the Pruning Lemma 4.6.2 with this choice of δ and D. Let $\varepsilon_* = \varepsilon_*(\delta, N)$ be the threshold needed for the applicability of Lemma 4.6.2 and observe that we need to prove that

$$D^2 \le \varepsilon_*^2\boldsymbol{\mu}(\mathbf{S})^2\,. \tag{4.305}$$

Note that we can assume

$$C_0(\hat{\mathbf{E}}(T, \mathbf{S}, \mathbf{B}_4) + \mathbf{A}^2) \le \frac{\varepsilon_*^2}{2}\boldsymbol{\mu}(\mathbf{S})^2 \tag{4.306}$$

by choosing ε sufficiently small, depending on ε_* in addition to existing dependencies.

Next observe that for every fixed $\lambda > 0$, by choosing ε sufficiently small depending on λ, the set $\mathbf{B}_\rho \setminus \mathbf{B}_\lambda(V)$ will be contained in the outer region R^o. In particular, applying

Proposition 4.6.14(v) as before, it follows that $\mathrm{dist}(q_0, V) \le \lambda$ and thus for $\lambda \le \varepsilon_*/(2\sqrt{C_0})$ we gain

$$C_0 \boldsymbol{\mu}(\mathbf{S})^2 |\mathbf{p}_{V^\perp \cap \alpha_1}(q_0)|^2 \le \frac{\varepsilon_*^2}{4} \boldsymbol{\mu}(\mathbf{S})^2 \,. \tag{4.307}$$

Consider moreover the ball $\mathbf{B} := \mathbf{B}_{4\lambda}(\mathbf{p}_V(q_0))$ and estimate

$$\hat{\mathbf{E}}(T, \alpha_1, \mathbf{B}) \le C_1 \boldsymbol{\mu}(\mathbf{S})^2 + C_1 \hat{\mathbf{E}}(T, \mathbf{S}, \mathbf{B}) \le C_1 \boldsymbol{\mu}(\mathbf{S})^2 + C_1 \lambda^{-m-2} \hat{\mathbf{E}}(T, \mathbf{S}, \mathbf{B}_4) \,,$$

where $C_1 = C_1(m, n, \bar{n}, Q)$. In particular applying Allard's L^2–L^∞ bound for α_1 we conclude that

$$|\mathbf{p}_{\alpha_1}^\perp(q_0)|^2 \le C_1 \lambda^2 \boldsymbol{\mu}(\mathbf{S})^2 + C_1 \lambda^{-m} \hat{\mathbf{E}}(T, \mathbf{S}, \mathbf{B}_4) + C_1 \lambda^2 \mathbf{A}^2 \,.$$

We can now select λ so that $C_0 C_1 \lambda^2 \le \frac{\varepsilon_*^2}{16}$, yielding

$$C_0 |\mathbf{p}_{\alpha_1}^\perp(q_0)|^2 \le \frac{\varepsilon_*^2}{16} \boldsymbol{\mu}(\mathbf{S})^2 + C \varepsilon_*^{-m-2} \hat{\mathbf{E}}(T, \mathbf{S}, \mathbf{B}_4) + C \varepsilon_*^2 \mathbf{A}^2 \,.$$

We are now in the position to choose ε sufficiently small (depending on ε_*) so that

$$C_0 |\mathbf{p}_{\alpha_1}^\perp(q_0)|^2 \le \frac{\varepsilon_*^2}{8} \boldsymbol{\mu}(\mathbf{S})^2 \,. \tag{4.308}$$

Summing (4.306), (4.307), and (4.308) we obtain (4.305).

So we can now indeed apply Lemma 4.6.2 to find a cone $\mathbf{S}' \subset \mathbf{S}$, indexed as $\mathbf{S}' = \bigcup_{i \in I} \alpha_i$, with the properties that

$$D^2 + \max_j \min_{i \in I} \mathrm{dist}^2(\alpha_i \cap \mathbf{B}_1, \alpha_j \cap \mathbf{B}_1) \le \delta^2 \boldsymbol{\sigma}(\mathbf{S}')^2 \,,$$

and

$$\max_j \min_{i \in I} \mathrm{dist}^2(\alpha_i \cap \mathbf{B}_1, \alpha_j \cap \mathbf{B}_1) \le \Gamma^2 D^2 \tag{4.309}$$

for $\Gamma = \Gamma(\delta)$ given by Lemma 4.6.2. Now, since by (4.304) and the triangle inequality

$$\hat{\mathbf{E}}(T_{q_0,\bar{\rho}}, \mathbf{S}', \mathbf{B}_4) \leq CD^2 + C \max_j \min_{i \in I} \operatorname{dist}^2(\alpha_i \cap \mathbf{B}_1, \alpha_j \cap \mathbf{B}_1) \leq C\delta^2 \boldsymbol{\sigma}(\mathbf{S}')^2 \,, \tag{4.310}$$

where C is a constant independent of δ, and since $\mathbf{S}' \subset \mathbf{S}$ we have by (4.304) again that $\hat{\mathbf{E}}(\mathbf{S}', T_{q_0,\bar{\rho}}, \mathbf{B}_4) + \mathbf{A}^2 \leq C_0 D^2 \leq C\delta^2 \boldsymbol{\sigma}(\mathbf{S}')^2$, we can choose $\delta = \delta(Q, m, n, \bar{n}, M)$ small to achieve the applicability of Corollary 4.9.3 with $\mathbf{S}'$ replacing $\mathbf{S}$ and $T_{q_0,\bar{\rho}}$ replacing T. Given that $\rho = \frac{\bar{\rho}}{3\sqrt{m-2}}$ the estimate (4.245), together with (4.310) and (4.309), give

$$\int_{\mathbf{B}_\rho(q_0)} \frac{\operatorname{dist}^2(q, q_0 + \mathbf{S}')}{|q - q_0|^{m+2-\kappa}} \, d\|T\|(q) \leq C(\mathbf{A}^2 + \Gamma^2 D^2) \,. \tag{4.311}$$

Since however we have chosen δ and thus we have Γ fixed, we can treat the latter as a constant depending only on Q, m, n, $\bar{n}$, and M. Since $\mathbf{S} \supset \mathbf{S}'$, we trivially have $\operatorname{dist}(q, q_0 + \mathbf{S}) \leq \operatorname{dist}(q, q_0 + \mathbf{S}')$, and hence, given our definition of D^2, (4.311) implies (4.297). □

4.10 Linearization

The aim of this section is to collect some results on Dir-minimizing functions which will be pivotal to close the proof of Theorem 4.2.4. We follow the notation of [18].

Definition 4.10.1. *Let $Q, m,$ and n be positive integers. We denote by:*

- *$\mathscr{H}_1$ the space of 1-homogeneous locally Dir-minimizing functions $u : \mathbb{R}^m \to \mathcal{A}_Q(\mathbb{R}^n)$;*
- *$\mathscr{L}_1$ the subspace of $u \in \mathscr{H}_1$ which are invariant by translations along at least $(m-2)$-independent directions;*
- *$\mathscr{H}_1^0$ and $\mathscr{L}_1^0$ the subsets of $\mathscr{H}_1$ and $\mathscr{L}_1$ consisting of maps u such that $\boldsymbol{\eta} \circ u \equiv 0$.*

Observe that all these spaces are locally compact in the L^2_{loc} topology. In particular, all the minima appearing in the inequalities below (e.g. see the right hand side of (4.312)) are attained. We begin with a suitable decay lemma.

Theorem 4.10.2. *For every Q, m, n, and $\varepsilon > 0$, there is $\rho = \rho(Q, m, n, \varepsilon) \in (0, \frac{1}{2})$ with the following property. Assume that $u : \mathbb{R}^m \supset B_1 \to \mathcal{A}_Q(\mathbb{R}^n)$ is Dir-minimizing, that $u(0) = Q[\![0]\!]$, and $I_{0,u}(0) := \lim_{r\to 0} I_{0,u}(r) \geq 1$. Then*

$$\min_{v\in\mathscr{H}_1} \int_{B_r} \mathcal{G}(u,v)^2 \leq \varepsilon r^{m+2} \int_{B_1} |u|^2 \qquad \forall r \leq \rho\,. \tag{4.312}$$

If additionally for some constant $\vartheta > 0$ and for every $r \leq \frac{1}{2}$ there are $m-2$ points $z_1, \ldots, z_{m-2} \in B_r$ such that $u(z_i) = Q[\![0]\!]$, $I_{z_i,u}(0) \geq 1$ and $\det((z_i \cdot z_j)_{i,j}) \geq \vartheta r^{m-2}$, then

$$\min_{v\in\mathscr{L}_1} \int_{B_r} \mathcal{G}(u,v)^2 \leq \varepsilon r^{m+2} \int_{B_1} |u|^2 \qquad \forall r \leq \rho \tag{4.313}$$

(where ρ will depend also on ϑ). Finally, when $\boldsymbol{\eta} \circ u \equiv 0$ we have the equalities

$$\min_{v\in\mathscr{H}_1} \int_{B_r} \mathcal{G}(u,v)^2 = \min_{v\in\mathscr{H}_1^0} \int_{B_r} \mathcal{G}(u,v)^2\,, \tag{4.314}$$

$$\min_{v\in\mathscr{L}_1} \int_{B_r} \mathcal{G}(u,v)^2 = \min_{v\in\mathscr{L}_1^0} \int_{B_r} \mathcal{G}(u,v)^2\,. \tag{4.315}$$

Let us first remark that the identities (4.314) and (4.315) are simple consequences of the following elementary algebraic fact. Consider any pair $T = \sum_i [\![T_i]\!]$, $S = \sum_i [\![S_i]\!] \in \mathcal{A}_Q(\mathbb{R}^n)$ and define

$$T' = \sum_i [\![T_i - \boldsymbol{\eta}(T)]\!]\,, \qquad \text{and} \qquad S' = \sum_i [\![S_i - \boldsymbol{\eta}(S)]\!]\,,$$

then

$$\mathcal{G}(T,S)^2 = Q|\boldsymbol{\eta}(T) - \boldsymbol{\eta}(S)|^2 + \mathcal{G}(T',S')^2\,.$$

Given this then, for example, in (4.314), we would have for $v \in \mathscr{H}_1$,

$$\mathcal{G}(u,v)^2 = Q|\boldsymbol{\eta} \circ v|^2 + \mathcal{G}(u,v')^2$$

where $v' = v - \boldsymbol{\eta} \circ v$, meaning $\min_{v \in \mathscr{H}_1} \int_{B_r} \mathcal{G}(u,v)^2 \geq \min_{v \in \mathscr{H}^0_1} \int_{B_r} \mathcal{G}(u,v)^2$; the other inequality is of course trivial as we are minimizing over a larger set. Thus, to prove Theorem 4.10.2 we just need to prove (4.312) and (4.313). We will first show, using a compactness argument, that if the frequency of a Dir-minimizer is pinched between 1 and $1+\eta$ then the function is very close to be 1-homogeneous. We begin with the following intermediate lemma.

Lemma 4.10.3. *For every $Q, m, n, \bar{\varepsilon} > 0$, there is a constant $\eta = \eta(Q, m, n, \bar{\varepsilon}) > 0$ with the following property. Fix $r > 0$. Then if $u : B_r \to \mathcal{A}_Q(\mathbb{R}^n)$ is a Dir-minimizing function such that $u(0) = Q[\![0]\!]$, $I_{0,u}(0) \geq 1$ and $I_{0,u}(r) \leq 1 + \eta$, then*

$$\min_{v \in \mathscr{H}_1} \int_{B_r} \mathcal{G}(u,v)^2 \leq \bar{\varepsilon} r \int_{\partial B_r} |u|^2 \,. \tag{4.316}$$

If additionally for some constant $\vartheta > 0$ there are $m-2$ points $z_1, \ldots, z_{m-2} \in B_{r/2}$ such that $u(z_i) = Q[\![0]\!]$, $I_{z_i,u}(0) \geq 1$ and $\det((z_i \cdot z_j)_{i,j}) \geq \vartheta r^{m-2}$, then, under the assumption that the parameter η is small enough, depending also on the value of ϑ, we have

$$\min_{v \in \mathscr{L}_1} \int_{B_r} \mathcal{G}(u,v)^2 \leq \bar{\varepsilon} r \int_{\partial B_r} |u|^2 \,. \tag{4.317}$$

Proof. By scaling the domain and u, we can assume without loss of generality that $r = 1$ and $H_u(1) \coloneqq \int_{\partial B_1} |u|^2 = 1$. We then argue by contradiction and assume the statements to be false for some fixed $\bar{\varepsilon}$ (and ϑ) no matter how small η is. In particular we can set $\eta = \frac{1}{k}$ and select corresponding maps u_k satisfying $u_k(0) = Q[\![0]\!]$, $1 \leq I_{0,u_k}(0) \leq I_{0,u_k}(1) \leq 1 + \frac{1}{k}$, $H_{u_k}(1) = 1$ and

$$\min_{v \in \mathscr{H}_1} \int_{B_1} \mathcal{G}(u_k,v)^2 \geq \bar{\varepsilon} \,. \tag{4.318}$$

As for the second statement of the lemma, we know additionally

($\star$) the existence of points $z_1^k, \ldots, z_{m-2}^k \in B_{1/2}$ such that $u_k(z_i^k) = Q[\![0]\!]$, $I_{z_i^k,u_k}(0) \geq 1$ and $\det((z_i^k \cdot z_j^k)_{i,j}) \geq \vartheta$,

while

$$\min_{v\in\mathscr{L}_1}\int_{B_1}\mathcal{G}(u_k,v)^2\geq\bar\varepsilon\,. \tag{4.319}$$

As $\int_{B_1}|Du_k|^2=I_{0,u_k}(1)\cdot H_{u_k}(1)\leq 1+\frac{1}{k}$, we can appeal to the compact embedding of the $W^{1,2}(B_1;\mathcal{A}_Q(\mathbb{R}^n))$ in $L^2(B_1;\mathcal{A}_Q(\mathbb{R}^n))$ (cf. [18, Proposition 2.11]) to extract (in both cases) a subsequence, not relabeled, which converges strongly in L^2 to some u. In fact (up to extraction of another subsequence), we can also assume that $u_k|_{\partial B_1}\to u|_{\partial B_1}$ by the trace property of $W^{1,2}$ (cf. [18, Proposition 2.10]), while u is Dir-minimizing (cf. [18, Proposition 3.20]) and $u_k\to u$ strongly in $W^{1,2}_{\rm loc}(B_1)$. It also follows, from Hölder estimates (cf. [18, Theorem 3.9] that $u(0)=[\![0]\!]$. Also, from upper semi-continuity of the frequency, $I_{0,u}(0)\geq\limsup_{k\to\infty}I_{0,u_k}(0)\geq 1$, while $I_{0,u}(1)\leq\liminf_{k\to\infty}I_{0,u_k}(1)=1$. Hence, by the monotonicity of the frequency function (cf. [18, Theorem 3.15]) $I_{0,u}(r)\equiv 1$, which in turn implies that u is 1-homogeneous (cf. [18, Corollary 3.16]). In particular $u\in\mathscr{H}_1$ and so the L^2 convergence of u_k to u contradicts (4.318).

Under the additional assumption ($\star$) we can assume, up to extraction of a further subsequence, that $z_i^k\to z_i\in B_{1/2}$ and obviously $\det((z_i\cdot z_j)_{i,j})\rangle\geq\vartheta$. In particular the vectors v_i span an $(m-2)$-dimensional subspace V. Again by the Hölder continuity of u_k and upper semi-continuity of the frequency function we conclude $u(z_i)=Q[\![0]\!]$ and $I_{z_i,u}(0)\geq 1$. But because of the 1-homogeneity of u we have the same properties for $u(\sigma z_i)$ for every $\sigma\in(0,1]$, in particular, again by upper semi-continuity of the frequency, $I_{z_i,u}(0)\leq 1$. Arguing as in [18, Proof of Lemma 3.4], it follows that $u(x+\lambda z_i)=u(x)$ for every x, every i, and every $\lambda\in\mathbb{R}$. In particular $u\in\mathscr{L}_1$. By the strong L^2-convergence of u_k to u this contradicts (4.319). □

Proof of Theorem 4.10.2. We fix ε and ϑ as in the statement of the Theorem. Fix $\bar\varepsilon>0$ (to be determined, possibly smaller than ε), and let $\eta=\eta(Q,m,n,\bar\varepsilon)$ be the parameter given by Lemma 4.10.3 with this choice of $\bar\varepsilon$. We then define

$$\varrho:=\inf\{0\leq r\leq\tfrac{1}{2}:I_{0,u}(r)\geq 1+\eta\}\,.$$

In particular, we can apply Lemma 4.10.3 to infer, respectively in each case ((4.312) or (4.313)) that

$$\min_{v\in\mathscr{H}_1}\int_{B_r}\mathcal{G}(u,v)^2\le\bar{\varepsilon}r\int_{\partial B_r}|u|^2\qquad\forall r\le\varrho \tag{4.320}$$

$$\min_{v\in\mathscr{L}_1}\int_{B_r}\mathcal{G}(u,v)^2\le\bar{\varepsilon}r\int_{\partial B_r}|u|^2\qquad\forall r\le\varrho\,. \tag{4.321}$$

We know by the decay of the L^2 height in terms of the frequency (cf. [18, Corollary 3.18]) that

$$\int_{\partial B_r}|u|^2\le\left(\frac{r}{s}\right)^{m-1+2I_{0,u}(r)}\int_{\partial B_s}|u|^2\qquad\forall r\le s\le 1\,. \tag{4.322}$$

Moreover, we have

$$\min_{t\in[1/2,1]}\int_{\partial B_t}|u|^2\le C\int_{B_1}|u|^2\,, \tag{4.323}$$

for some constant $C=C(Q,m,n)$. Thus, we infer from (4.320) (respectively (4.321)), combined with (4.322) with s chosen to be $t\in[1/2,1]$ realizing the minimum and the fact that $I_{0,u}(r)\ge 1$ for all $r>0$, that

$$\min_{v\in\mathscr{H}_1}\int_{B_r}\mathcal{G}(u,v)^2\le C\bar{\varepsilon}r^{m+2}\int_{B_1}|u|^2\qquad\forall r\le\varrho \tag{4.324}$$

$$\min_{v\in\mathscr{L}_1}\int_{B_r}\mathcal{G}(u,v)^2\le C\bar{\varepsilon}r^{m+2}\int_{B_1}|u|^2\qquad\forall r\le\varrho\,. \tag{4.325}$$

This proves the result for $r\le\varrho$; however, ϱ is not a geometric constant (it depends on u), so we are not done. Now, by monotonicity of the frequency and the definition of ϱ we know that $I_{0,u}(r)\ge 1+\eta$ for every $r>\varrho$. So, again by (4.322) and (4.323) (choosing appropriate $t\in[1/2,1]$ again), for every $r\in(\varrho,1]$ we have

$$\int_{\partial B_s}|u|^2\le\left(\frac{s}{r}\right)^{m+1}\int_{\partial B_r}|u|^2\le s^{m+1}\left(\frac{r}{t}\right)^{2\eta}\int_{\partial B_t}|u|^2\le Cs^{m+1}r^{2\eta}\int_{B_1}|u|^2\qquad\forall s\le r\,.$$

Integrating the latter over $s \in [0, r]$ we get

$$\int_{B_r} |u|^2 \leq C r^{m+2+2\eta} \int_{B_1} |u|^2 \qquad \forall \varrho < r \leq 1\,.$$

Since, however, the function $v \equiv Q[\![0]\!]$ belongs to $\mathscr{L}_1 \subset \mathscr{H}_1$, we can combine the latter inequality with (4.324) (resp. (4.325)) to get

$$\min_{v \in \mathscr{H}_1} \int_{B_r} \mathcal{G}(u, v)^2 \leq C \max\{\bar{\varepsilon}, r^{2\eta}\} r^{m+2} \int_{B_1} |u|^2 \qquad \forall r \leq \frac{1}{2}\,, \tag{4.326}$$

and respectively

$$\min_{v \in \mathscr{L}_1} \int_{B_r} \mathcal{G}(u, v)^2 \leq C \max\{\bar{\varepsilon}, r^{2\eta}\} r^{m+2} \int_{B_1} |u|^2 \qquad \forall r \leq \frac{1}{2},\,. \tag{4.327}$$

We now choose first $\bar{\varepsilon}$ so that $C\bar{\varepsilon} \leq \varepsilon$, which in turn fixes the value of $\eta > 0$ determined by Lemma 4.10.3. Hence we can choose $\rho = \rho(Q, m, n, \varepsilon) > 0$ such that $C\rho^{2\eta} \leq \varepsilon$. Then the desired estimates (4.312) and (4.313) for $r \leq \rho$ follow. □

The second result we will need is a suitable "removability result" which we stated previously in Proposition 4.7.4. We recall the statement for the convenience of the reader.

Proposition 4.10.4. *Assume $\Omega \subset \mathbb{R}^m$ is a Lipschitz domain, $V \subset \mathbb{R}^m$ is an $(m-2)$-dimensional plane, and $v \in W^{1,2}(\Omega; \mathcal{A}_Q(\mathbb{R}^n))$ is a map with the property that the restriction of v to $\Omega_\varepsilon := \Omega \setminus \mathbf{B}_\varepsilon(V)$ is Dir-minimizing for every $\varepsilon > 0$. Then, v is Dir-minimizing in Ω.*

We remark an important subtlety: the set which is removed from Ω to get to Ω_ε is *not* compactly contained in Ω when $m \geq 3$.

We will in fact prove the following lemma, where we use the notion of p-capacity, $\mathrm{Cap}_p^m(A)$, of a set $A \subset \mathbb{R}^m$. We refer the reader to e.g. [26] for the definition and preliminaries.

Lemma 4.10.5. *Let $\Omega \subset \mathbb{R}^m$ be any bounded Lipschitz domain and $A \subset \mathbb{R}^m$ any set with $\mathrm{Cap}_2^m(A) = 0$. Then, for any two maps $v, u \in W^{1,2}(\Omega, \mathcal{A}_Q(\mathbb{R}^n))$ such that $v = u$ on $\partial\Omega$ and for any $\delta > 0$, we can find a third map $w \in W^{1,2}(\Omega, \mathcal{A}_Q(\mathbb{R}^n))$ which coincides with v on $\partial\Omega \cup (\Omega \cap U)$ for some neighborhood U of A and such that*

$$\int_\Omega |Dw|^2 \leq \int_\Omega |Du|^2 + \delta\,.$$

Let us first prove Proposition 4.10.4 from Lemma 4.10.5.

Proof of Proposition 4.10.4. First note that since V is an $(m-2)$-dimensional plane, it obeys $\mathrm{Cap}_2^m(V) = 0$ (note here $m > 2$, and so this follows because sets with finite $\mathcal{H}^{m-2}$-measure have vanishing 2-capacity, see [26]; the result still holds for $m = 2$ by using the classical logarithmic cut-off trick).

Suppose for contradiction that v is not Dir-minimizing in Ω. Then we can find a competitor $u \in W^{1,2}(\Omega; \mathcal{A}_Q(\mathbb{R}^n))$ with $v = u$ on $\partial\Omega$ and

$$\int_\Omega |Du|^2 < \int_\Omega |Dv|^2 - \delta$$

for some $\delta > 0$. Now apply Lemma 4.10.5 with this choice of v, u and δ to find a map $w \in W^{1,2}(\Omega; \mathcal{A}_Q(\mathbb{R}^n))$ which coincides with v on $\partial\Omega \cup (\Omega \cap \mathbf{B}_{\varepsilon_*}(V))$ for some $\varepsilon_* > 0$. In particular, we have

$$\int_\Omega |Dw|^2 \leq \int_\Omega |Du|^2 + \delta < \int_\Omega |Dv|^2.$$

However, this contradicts v being Dir-minimizing on Ω_{ε_*} (since $v = w$ on $\mathbf{B}_{\varepsilon_*}(V)$). □

All that remains is to prove Lemma 4.10.5.

Proof of Lemma 4.10.5. We fix Ω a Lipschitz domain and we let $v, u : \Omega \to \mathcal{A}_Q(\mathbb{R}^n)$ be any pair of $W^{1,2}(\Omega; \mathcal{A}_Q(\mathbb{R}^n))$ functions with $v|_{\partial\Omega} = u|_{\partial\Omega}$, where the boundary data is defined in the Sobolev trace sense (see [18, Section 2.2.2]). Consider now $\tilde{u} := \boldsymbol{\xi}_{BW} \circ u$ and $\tilde{v} :=$

$\boldsymbol{\xi}_{BW} \circ v$, where $\boldsymbol{\xi}_{BW} : \mathcal{A}_Q(\mathbb{R}^n) \to \mathbb{R}^N$ is the special bi-Lipschitz embedding of White, which is constructed via a modification of the original embedding of Almgren (see [6] or [18, Corollary 2.2]).

Let $z := \tilde{u} - \tilde{v}$ and extend it to be identically equal to 0 on $\mathbb{R}^m \setminus \Omega$. We claim that there is a sequence $\{\zeta_k\} \subset W^{1,2}(\mathbb{R}^m, \mathbb{R}^N)$ with the following properties:

- ζ_k vanishes identically on $(\mathbb{R}^m \setminus \Omega) \cup U_k(A)$, where $U_k(A)$ is some open neighborhood of A (which depends on k);
- $\|\zeta_k - z\|_{W^{1,2}} \to 0$ as $k \uparrow \infty$.

Assuming the existence of such a sequence $\{\zeta_k\}$ for now, we then let $w_k := \boldsymbol{\xi}_{BW}^{-1} \circ (\boldsymbol{\rho} \circ (\tilde{v} + \zeta_k))$, where $\boldsymbol{\rho}$ is Almgren's Lipschitz retraction of $\mathbb{R}^N$ onto $\boldsymbol{\xi}_{BW}(\mathcal{A}_Q(\mathbb{R}^N))$ (cf. [18, Chapter 2]). Observe that $\tilde{v} + \zeta_k = \tilde{v}$ on $\partial\Omega \cup (\Omega \cap U_k(A))$ and so $w_k = v$ on $\partial\Omega \cup (\Omega \cap U_k(A))$. Moreover

$$\int_\Omega |Dw_k|^2 = \int_\Omega |D(\boldsymbol{\rho} \circ (\tilde{v} + \zeta_k))|^2$$

and

$$\int_\Omega |Du|^2 = \int_\Omega |D\tilde{u}|^2 .$$

On the other hand $\tilde{v} + \zeta_k$ converges to $\tilde{v} + z = \tilde{v} + (\tilde{u} - \tilde{v}) = \tilde{u}$ strongly in $W^{1,2}(\mathbb{R}^m; \mathbb{R}^N)$ and, because $\boldsymbol{\rho}$ is Lipschitz and equal to the identity on $\boldsymbol{\xi}_{BW}(\mathcal{A}_Q(\mathbb{R}^n))$, $\boldsymbol{\rho} \circ (\tilde{v} + \zeta_k)$ converges strongly in $W^{1,2}$ to $\boldsymbol{\rho} \circ \tilde{u} = \tilde{u}$. In particular we conclude from this and the above two identities that the Dirichlet energy of w_k converges to the Dirichlet energy of u, and thus to conclude the result of the lemma we just need to take $w = w_k$ for k sufficiently large (depending on δ).

So now we just need to show the existence of ζ_k satisfying the desired properties. By the definition of Cap_2^m, given $\varepsilon > 0$, we may choose $\rho_\varepsilon \in C_c^\infty(B_\varepsilon(A))$ with $\rho_\varepsilon \geq \mathbf{1}_{B_{\varepsilon/2}(A)}$ and such that

$$\int |D\rho_\varepsilon|^2 \leq \mathrm{Cap}_2^m(A) + \varepsilon = \varepsilon.$$

In particular we can assume, by truncating ρ_ε, that it takes values between 0 and 1; clearly it also follows that $\rho_\varepsilon \to 0$ pointwise almost everywhere as $\varepsilon \downarrow 0$.

Next we choose M sufficiently large and consider the truncation function $z_M = (z_M^1, \dots, z_M^N)$ given by

$$z_M^i := \min\{z^i, M\}\mathbf{1}_{\{z^i > 0\}} + \max\{z^i, -M\}\mathbf{1}_{\{z^i \leq 0\}},$$

where we have written $z = (z^1, \dots, z^N)$. It is simple to see that $z_M \to z$ in $W^{1,2}$ as $M \to \infty$, and so for each $k \in \{1, 2, \dots\}$ we can select $M = M(k)$ so that $\|z_M - z\|_{W^{1,2}} \leq \frac{1}{2k}$. Observe also that z_M always vanishes outside of Ω. Next we take $\zeta_k := z_M(1 - \rho_{\varepsilon_k})$ for a sufficiently small ε_k to be chosen. Note indeed that for fixed M, $\|z_M \rho_\varepsilon\|_{L^2} \to 0$ as $\varepsilon \downarrow 0$ by dominated convergence. On the other hand

$$D(z_M(1 - \rho_\varepsilon)) - Dz_M = -\rho_\varepsilon Dz_M - z_M D\rho_\varepsilon$$

and $\|Dz_M \rho_\varepsilon\|_{L^2} \to 0$ as $\varepsilon \downarrow 0$, again by dominated convergence, while we have $\|z_M D\rho_\varepsilon\|_{L^2} \leq \sqrt{N}M\|D\rho_\varepsilon\|_{L^2}$ converges to 0 as well as long as we keep M fixed while $\varepsilon \downarrow 0$. Thus, $z_M(1 - \rho_\varepsilon) \to z_M$ in $W^{1,2}$, and so we can choose $\varepsilon = \varepsilon_k$ sufficiently small so that $\|z_M(1 - \rho_{\varepsilon_k}) - z_M\|_{W^{1,2}} \leq \frac{1}{2k}$. This is the choice of ε_k used to define ζ_k. Note then that $\|\zeta_k - z\|_{W^{1,2}} \leq 1/k \to 0$, and also that ζ_k vanishes identically on $(\mathbb{R}^m \setminus \Omega) \cup B_{\varepsilon_k/2}(A)$; this completes the proof. □

4.11 Final blow-up argument

In this section we complete the proof of Theorem 4.8.2, which we recall in turn implies the validity of Theorem 4.2.4, as demonstrated in Section 4.8.

4.11.1 Two regimes

The proof of Theorem 4.8.2 will be split into two cases, which will both be proved via a blow-up argument. We start by giving the detailed statements.

Proposition 4.11.1 (Collapsed decay). *For every $Q, m, n, \bar{n}$, and $\varsigma_1 > 0$ there are positive constants $\varepsilon_c = \varepsilon_c(Q, m, n, \bar{n}, \varsigma_1) \le 1/2$ and $r_c = r_c(Q, m, n, \bar{n}, \varsigma_1) \le 1/2$ with the following property. Assume that*

(i) T and Σ are as in Assumption 3.2.2, and $\|T\|(\mathbf{B}_1) \le \omega_m(Q + \frac{1}{2})$;

(ii) There is a cone $\mathbf{S} \in \mathscr{C}(Q, 0)$ which is M-balanced (with M as in Assumption 4.8.1), such that (4.183) and (4.184) hold with ε_c in place of ε_1, and in addition $\boldsymbol{\mu}(\mathbf{S}) \le \varepsilon_c$;

(iii) $\mathbf{A}^2 \le \varepsilon_c^2 \mathbb{E}(T, \tilde{\mathbf{S}}, \mathbf{B}_1)$ for every $\tilde{\mathbf{S}} \in \mathscr{C}(Q, 0)$.

Then, there is another cone $\mathbf{S}' \in \mathscr{C}(Q, 0) \setminus \mathscr{P}(0)$ such that

$$\mathbb{E}(T, \mathbf{S}', \mathbf{B}_{r_c}) \le \varsigma_1 \mathbb{E}(T, \mathbf{S}, \mathbf{B}_1)\,. \tag{4.328}$$

Proposition 4.11.2 (Non-collapsed decay). *For every $Q, m, n, \bar{n}$, $\varepsilon_c^\star > 0$, and $\varsigma_1 > 0$, there are positive constants $\varepsilon_{nc} = \varepsilon_{nc}(Q, m, n, \bar{n}, \varepsilon_c^\star, \varsigma_1) \le 1/2$ and $r_{nc} = r_{nc}(Q, m, n, \bar{n}, \varepsilon_c^\star, \varsigma_1) \le \frac{1}{2}$ with the following property. Assume that*

(i) T and Σ are as in Assumption 3.2.2 and $\|T\|(\mathbf{B}_1) \le \omega_m(Q + \frac{1}{2})$;

(ii) There is $\mathbf{S} \in \mathscr{C}(Q, 0)$ which is M-balanced (with M as in Assumption 4.8.1), such that (4.183) and (4.184) hold with ε_{nc} in place of ε_1 and in addition $\boldsymbol{\mu}(\mathbf{S}) \ge \varepsilon_c^\star$;

(iii) $\mathbf{A}^2 \le \varepsilon_{nc}^2 \mathbb{E}(T, \tilde{\mathbf{S}}, \mathbf{B}_1)$ for every $\tilde{\mathbf{S}} \in \mathscr{C}(Q, 0)$.

Then, there is another cone $\mathbf{S}' \in \mathscr{C}(Q, 0) \setminus \mathscr{P}(0)$ such that

$$\mathbb{E}(T, \mathbf{S}', \mathbf{B}_{r_{nc}}) \le \varsigma_1 \mathbb{E}(T, \mathbf{S}, \mathbf{B}_1)\,. \tag{4.329}$$

Remark 4.11.3. In fact, for the proof of Proposition 4.11.1 and Proposition 4.11.2, instead of condition (iii) all we will need to assume is that $\mathbf{A}^2 \leq \varepsilon_c^2 \mathbb{E}(T, \mathbf{S}, \mathbf{B}_1)$ or $\mathbf{A}^2 \leq \varepsilon_{nc}^2 \mathbb{E}(T, \mathbf{S}, \mathbf{B}_1)$, respectively, for the cone $\mathbf{S}$ as in (ii) in each proposition, respectively.

Theorem 4.8.2 obviously follows from the two propositions above. In fact, being given $Q, m, n, \bar{n}$, and ς_1 as in Theorem 4.8.2, we first apply Proposition 4.11.1 and get ε_c and r_c. We then apply Proposition 4.11.2 with the same choice of $Q, m, n, \bar{n}$, and ς_1, and with $\varepsilon_c^\star = \varepsilon_c$, to get ε_{nc} and r_{nc}. It is then clear that Theorem 4.8.2 holds if we set $\varepsilon_1 := \min\{\varepsilon_c, \varepsilon_{nc}\}$, $r_2^1 := r_c$ and $r_2^2 := r_{nc}$.

Both propositions will be proved by a blow-up procedure, which will assume, by seeking a contradiction, that the statements fail. In the proof of Proposition 4.11.2 this means that $\varepsilon_c^\star$ is fixed while ε_{nc} will be taken to be arbitrarily small and so in particular $\boldsymbol{\mu}(\mathbf{S})$ will stay away from 0 while the ratio

$$\frac{\mathbb{E}(T, \mathbf{S}, \mathbf{B}_1)}{\boldsymbol{\sigma}(\mathbf{S})^2} \tag{4.330}$$

is arbitrarily small. We will call the latter a *Simon blow-up* (since in the work [45] the cones are uniformly non-collapsed, analogously to this case), and in this case we will take limits of suitable rescalings of the coherent outer approximations of Proposition 4.6.18.

In the proof of Proposition 4.11.1 we will instead assume that ε_c is arbitrarily small. This will mean that *both* the ratio in (4.330) and $\boldsymbol{\mu}(\mathbf{S})$ are arbitrarily small. In this case we will approximate the current in the outer region by reparameterizing the graphs of the coherent outer approximations over a single plane, which we can fix to be one of the planes forming $\mathbf{S}$, say α_1. We will call these the *transversal coherent approximations*, as opposed to the ones of the Simon blow-up, which will be called the *orthogonal coherent approximations*. By construction the transversal approximations are superpositions of multi-valued maps, each of them close to the linear map describing α_j as a graph over α_1. We will then subtract the latter linear map from the sheet of the corresponding multi-valued approximation and study the limits of appropriate rescalings. We will call this procedure a *Wickramasekera blow-up*

(since the work [56] is the first appearance of such a blow-up where the cones are collapsing to a high-multiplicity flat plane).

4.11.2 Transversal coherent approximation

In this section we work under the assumption of Proposition 4.11.1 and recast the coherent outer approximation of Proposition 4.6.18 as an approximation through a Lipschitz multi-valued function over a single plane (the latter function representing the cone $\mathbf{S}$). The choice of the plane is not really important as long as its Hausdorff distance from $\mathbf{S}$ in $\mathbf{B}_1$ is comparable to $\boldsymbol{\mu}(\mathbf{S})$; we may without loss of generality take it to be the first indexed plane α_1 forming $\mathbf{S}$. Given a multi-valued function $g = \sum_i \llbracket g_i \rrbracket$ and a single-valued function f with the same domain, we also use the notational shorthand $g \ominus f$ for

$$g \ominus f(x) \coloneqq \sum_i \llbracket g_i(x) - f(x) \rrbracket \,,$$

and similarly we define $g \oplus f$. We moreover recall the set $\mathscr{N}(L)$ of "neighbouring cubes" to L and the quantity $\bar{\mathbf{E}}(L)$ introduced in Definition 4.6.17 and introduce

$$\widetilde{\mathbf{E}}(L) := \max\{\bar{\mathbf{E}}(L') : L' \in \mathscr{N}(L)\} = \max\{\mathbf{E}(L'') : L'' \in \mathscr{N}(L'),\, L' \in \mathscr{N}(L)\}$$

i.e. we maximize the excess of neighbours of neighbours.

Proposition 4.11.4 (Transversal coherent approximation)**.** *Let T, Σ, and $\mathbf{S}$ be as in Proposition 4.6.14, let $\ell \in \mathbb{N}$ be the maximal number such that $\mathcal{G}_{\ell+2} \subset \mathcal{G}^o$, and consider the regions $\widetilde{R}_i^o \coloneqq \alpha_i \cap \bigcup_{L \in \mathcal{G}_{\leq \ell}} R(L)$. Let $u_i : R_i^o \to \mathcal{A}_{Q_i}(\alpha_i^\perp)$ be the Q_i-valued maps in Proposition 4.6.18. Under the additional assumption that $\boldsymbol{\mu}(\mathbf{S}) \leq c_0$ for a geometric constant $c_0 = c_0(m, n, \bar{n}) > 0$, the following holds.*

(a) Each plane α_i is the graph over α_1 of a linear map $A_i : \alpha_1 \to \alpha_1^\perp$ with $|A_i| \leq C\boldsymbol{\mu}(\mathbf{S})$ and $\ker(A_i) = V$.

(b) For each i there is a map $v_i : \widetilde{R}_1^o \to \mathcal{A}_{Q_i}(\alpha_1^\perp)$ with the property that

$$\mathbf{G}_{v_i} = \mathbf{G}_{u_i} \llcorner \mathbf{p}_{\alpha_1}^{-1}(\widetilde{R}_1^o)\,.$$

(c) If we let $v : \widetilde{R}_1^o \to \mathcal{A}_Q(\alpha_1^\perp)$ be the Q-valued function $v := \sum_i [\![v_i]\!]$ (note that $Q = \sum_{i=1}^N Q_i$), then

$$\|v\|_{L^\infty} + \|Dv\|_{L^2} \leq C\boldsymbol{\mu}(\mathbf{S}) \tag{4.331}$$

(d) If we let

$$K := \mathbf{p}_{\alpha_1}((\mathrm{spt}(T) \cap \mathbf{B}_{1/2} \cap \mathbf{p}_{\alpha_1}^{-1}(\widetilde{R}_1^o)) \setminus \mathrm{gr}\,(v))$$

then, for all $L \in \mathcal{G}_{\leq \ell}$,

$$|L_1 \setminus K| + \|T\|(\mathbf{p}_{\alpha_1}^{-1}(L_1 \setminus K)) \leq C2^{-m\ell(L)}(\widetilde{\mathbf{E}}(L) + 2^{-2\ell(L)}\mathbf{A}^2)^{1+\gamma}\,, \tag{4.332}$$

where L_1 is as in Proposition 4.6.18.

(e) If we let $w_i := v_i \ominus A_i : \tilde{R}_1^o \to \mathcal{A}_{Q_i}(\alpha_1^\perp)$, then

$$2^{2\ell(L)}\|w_i\|^2_{L^\infty(L_1)} + 2^{m\ell(L)}\|Dw_i\|^2_{L^2(L_1)} \leq C(\widetilde{\mathbf{E}}(L) + 2^{-2\ell(L)}\mathbf{A}^2) \tag{4.333}$$

$$\|Dw_i\|_{L^\infty(L_1)} \leq C(\widetilde{\mathbf{E}}(L) + 2^{-2\ell(L)}\mathbf{A}^2)^\gamma \tag{4.334}$$

Here, $C = C(Q, m, n, \bar{n})$.

Proof. Claim (a) is obvious. For the rest, all we are essentially doing is changing the coordinates of u_i, to give a parameterization over α_1 rather than α_i. Claims (b) and and the L^∞-estimate in (c) follow immediately from [20, Proposition 5.2]. The L^2-gradient estimate in (c) follows immediately from the L^2 bound in (4.333) and (a). Conclusion (d) follows immediately from the corresponding estimate in Proposition 4.6.18. As for (4.334), observe

that, for a.e. $x \in \tilde{R}_1^o$, $|Dw_i|(x)$ equals, up to a geometric constant, the following quantity:

$$\sup\{|\vec{\mathbf{G}}_{v_i}(x,y) - \vec{\alpha}_i| : (x,y) \in \mathrm{gr}\,(v_i)\}$$

while an analogous formula holds for $|Du_i|(x)$, with $x \in R_i^o$: since the graph of v_i equals that of u_i in a different coordinate system, the estimate (4.334) follows from the analogous one for u_i in Proposition 4.6.18. Likewise, $\|Dw_i\|^2_{L^2(L_1)}$ is equivalent, up to constants, to

$$\int_{\mathbf{p}_{\alpha_1}^{-1}(L_1)} |\vec{\mathbf{G}}_{v_i}(p) - \vec{\alpha}_i|^2 \, d\|\mathbf{G}_{v_i}\|(p)\,.$$

Thus from the corresponding inequality for u_i, we obtain the inequality

$$\|Dw_i\|^2_{L^2(L_1)} \leq C2^{-m\ell(L)}(\widetilde{\mathbf{E}}(L) + 2^{-2\ell(L)}\mathbf{A}^2)\,.$$

As for $\|w_i\|_{L^\infty(L_1)}$ it is easy to see that it is controlled by $\|u_i\|_{L^\infty(\lambda L_i)}$ for a choice of λ slightly larger than 1 (cf. the arguments in [20, Section 5]). This establishes (4.333) and so completes the proof. □

4.11.3 Non-concentration estimates

In this section we draw some important conclusions from Simon's estimates in Section 4.9. We will work under the following assumptions throughout this section.

Assumption 4.11.5. T and Σ are as in Assumption 3.2.2. $\mathbf{S} \in \mathscr{C}(Q,0) \setminus \mathscr{P}(0)$ is such that

$$\mathbf{A}^2 + \mathbb{E}(T, \mathbf{S}, \mathbf{B}_1) \leq \varepsilon^2 \boldsymbol{\sigma}(\mathbf{S})^2 \tag{4.335}$$

and

$$\mathbf{B}_\varepsilon(y) \cap \{\Theta(T,\cdot) \geq Q\} \neq \emptyset \quad \text{for all } y \in V(\mathbf{S}) \cap \mathbf{B}_1\,, \tag{4.336}$$

for some $\varepsilon > 0$, to be determined.

Next, consider recall the family of cubes $\mathcal{G}$, where $L \in \mathcal{G}$ has center point $y_L \in V$, as defined in Section 4.6.5.

Definition 4.11.6. *Let T, Σ, and $\mathbf{S}$ be as in Assumption 4.11.5. For every $L \in \mathcal{G}$ we let $\beta(L)$ be a point p in $\{\Theta(T,\cdot) \geq Q\}$ which minimizes the distance to y_L. This point will be called the* nail *of L.*

Of course, there might be more than one candidate for the nail of L, but under Assumption 4.11.5, one nail always exists as the set $\{\Theta(T,\cdot) \geq Q\}$ is closed by upper semi-continuity of density. We will henceforth just assume that some arbitrary choice for each given nail has been made. The main conclusion of this section is then the following.

Proposition 4.11.7. *There is a constant $C = C(m,n,\bar{n},Q)$ such that, for every $\varrho > 0$ there exists $\varepsilon = \varepsilon(Q,m,n,\bar{n},\varrho) > 0$ with the following property. Assume T, Σ, and $\mathbf{S}$ are as in Assumption 4.11.5 with this choice of ε, let r_* denote the radius of Proposition 4.9.4 and let $r = \frac{r_*}{4}$. Then*

$$\int_{\mathbf{B}_r} \frac{\mathrm{dist}^2(q,\mathbf{S})}{\max\{\varrho, \mathrm{dist}(q,V)\}^{3/2}}\, d\|T\|(q) \leq C(\hat{\mathbf{E}}(T,\mathbf{S},\mathbf{B}_1) + \mathbf{A}^2)\,. \tag{4.337}$$

Moreover, if $u_1,\ldots,u_N$ are the coherent approximations of Proposition 4.6.18, upon suitably modifying them (without relabelling) we can assume that they satisfy the following stronger estimates for each $j \in \{1,\ldots,N\}$:

$$\int_{(\mathbf{B}_r \cap \alpha_j)\setminus \mathbf{B}_\varrho(V)} \frac{|Du_j(z)|^2}{\mathrm{dist}(z,V)^{3/2}}\, dz \leq C(\hat{\mathbf{E}}(T,\mathbf{S},\mathbf{B}_1) + \mathbf{A}^2) \tag{4.338}$$

$$\sum_{i:\, 2^{-i-1} \geq \varrho} \sum_{L \in \mathcal{G}_i} \int_{L_j} \frac{|u_j(z) \ominus (\mathbf{p}^\perp_{\alpha_j}(\beta(L)))|^2}{\mathrm{dist}(z,V)^{7/2}}\, dz \leq C(\hat{\mathbf{E}}(T,\mathbf{S},\mathbf{B}_1) + \mathbf{A}^2)\,. \tag{4.339}$$

Finally, under the additional assumption that $\boldsymbol{\mu}(\mathbf{S}) \leq c_0$ for the constant $c_0 = c_0(m,n,\bar{n})$ of Proposition 4.11.4, the maps $w_1,\ldots,w_N$ and $A_1,\ldots,A_N$ defined therein satisfy the cor-

responding estimates

$$\int_{(\mathbf{B}_r\cap\alpha_1)\setminus\mathbf{B}_\varrho(V)} \frac{|Dw_j(z)|^2}{\operatorname{dist}(z,V)^{3/2}}\,dz \le C(\mathbf{A}^2+\hat{\mathbf{E}}(T,\mathbf{S},\mathbf{B}_1)) \tag{4.340}$$

$$\sum_{2^{-i-1}\ge\varrho}\sum_{L\in\mathcal{G}_i}\int_{L_1}\frac{|w_j(z)\ominus(\mathbf{p}^\perp_{\alpha_1}(\beta(L))+A_j(\mathbf{p}_{V^\perp\cap\alpha_1}(\beta(L))))|^2}{\operatorname{dist}(z,V)^{7/2}}\,dz \le C(\mathbf{A}^2+\hat{\mathbf{E}}(T,\mathbf{S},\mathbf{B}_1))\,. \tag{4.341}$$

Proof. We begin with the first estimate (4.337); we may assume that $\varrho<1$ is smaller than half the radius in Proposition 4.9.4, as if not the inequality follows trivially. We begin by estimating the part of the integral on the left-hand side that is over $\mathbf{B}_r\cap\mathbf{B}_\varrho(V)$. Cover $V\cap\mathbf{B}_r$ with $C\varrho^{-(m-2)}$ balls $\mathbf{B}_\varrho(y_i)$ of radius ϱ (of course, we can assume $\mathbf{B}_\varrho(y_i)\cap V\neq\emptyset$ for each i) and observe that $\{\mathbf{B}_{2\varrho}(y_i)\}_i$ covers $B_\varrho(V)\cap\mathbf{B}_r$. If (4.336) holds with $\varepsilon<\varrho$, then for each i we can find a point $p_i\in\mathbf{B}_{2\varrho}(y_i)$ such that $\Theta(T,p_i)\ge Q$. In particular we can estimate, using Proposition 4.9.4 with $\kappa=1/4$, centered at each p_i,

$$\begin{aligned}
\varrho^{-3/2}\int_{\mathbf{B}_r\cap\mathbf{B}_\varrho(V)}\operatorname{dist}^2(q,\mathbf{S})\,d\|T\|(q) &\le \varrho^{-3/2}\sum_i\int_{\mathbf{B}_r\cap B_{2\varrho}(y_i)}\operatorname{dist}^2(q,\mathbf{S})\,d\|T\|(q)\\
&\le C\varrho^{-3/2}\sum_i\int_{\mathbf{B}_r\cap\mathbf{B}_{2\varrho}(y_i)}\operatorname{dist}^2(q,p_i+\mathbf{S})\,d\|T\|(q)\\
&\qquad + C\varrho^{-3/2}\sum_i\varrho^m\big(|\mathbf{p}^\perp_{\alpha_1}(p_i)|^2+\boldsymbol{\mu}(\mathbf{S})^2|\mathbf{p}_{V^\perp\cap\alpha_1}(p_i)|^2\big)\\
&\le C\varrho^{m+7/4-3/2}\sum_i\int_{\mathbf{B}_r\cap\mathbf{B}_{4\varrho}(p_i)}\frac{\operatorname{dist}^2(q,p_i+\mathbf{S})}{|q-p_i|^{m+7/4}}\,d\|T\|(q)\\
&\qquad + C\varrho^{-(m-2)}\varrho^{m-3/2}(\hat{\mathbf{E}}(T,\mathbf{S},\mathbf{B}_1)+\mathbf{A}^2)\\
&\le C\varrho^{-(m-2)}\varrho^{m+1/4}(\hat{\mathbf{E}}(T,\mathbf{S},\mathbf{B}_1)+\mathbf{A}^2)+C\varrho^{1/2}(\hat{\mathbf{E}}(T,\mathbf{S},\mathbf{B}_1)+\mathbf{A}^2)\\
&\le C(\hat{\mathbf{E}}(T,\mathbf{S},\mathbf{B}_1)+\mathbf{A}^2)\,.
\end{aligned} \tag{4.342}$$

We next consider the region $\mathbf{B}_r\setminus\mathbf{B}_\varrho(V)$ and note that we can we cover it with the regions

$R(L)$ for $L \in \mathcal{G}$ such that $2^{-\ell(L)-1} \geq \varrho$. However, we only include in the cover the cubes L for which $R(L)$ have a nonempty intersection with $\mathbf{B}_r$: denote the latter family of cubes by $\mathscr{F}$. For each such cube we denote by $\beta(L)$ its nail as in Definition 4.11.6. Note in particular that, if $\varepsilon < \frac{\varrho}{4}$ in (4.336), then

$$C^{-2}2^{-\ell(L)} \leq C^{-1}|q-\beta(L)| \leq \mathrm{dist}(q,V) \leq C|q-\beta(L)| \leq C^2 2^{-\ell(L)} \qquad \forall q \in R(L)\,,$$

where C is a geometric constant. Again combining with Proposition 4.9.4 with $\kappa = \frac{1}{4}$, this allows us to estimate

$$\begin{aligned}
&\int_{\mathbf{B}_r \setminus \mathbf{B}_\varrho(V)} \frac{\mathrm{dist}^2(q,\mathbf{S})}{\mathrm{dist}(q,V)^{3/2}}\, d\|T\|(q) \\
&\qquad \leq C \sum_{i \leq -1-\log_2(\varrho)} \sum_{L \in \mathcal{G}_i \cap \mathscr{F}} 2^{3i/2} \int_{R(L)} \mathrm{dist}^2(q,\mathbf{S})\, d\|T\|(q) \\
&\qquad \leq C \sum_{i \leq -1-\log_2(\varrho)} \sum_{L \in \mathcal{G}_i \cap \mathscr{F}} 2^{3i/2} \int_{R(L)} \mathrm{dist}^2(q,\beta(L)+\mathbf{S})\, d\|T\|(q) \\
&\qquad\quad + C \sum_{i \leq -1-\log_2(\varrho)} \sum_{L \in \mathcal{G}_i \cap \mathscr{F}} 2^{3i/2} 2^{-mi} \big(|\mathbf{p}^\perp_{\alpha_1}(\beta(L))|^2 + \boldsymbol{\mu}(\mathbf{S})^2 |\mathbf{p}_{V^\perp \cap \alpha_1}(\beta(L))|^2 \big) \\
&\qquad \leq C \sum_{i \leq -1-\log_2(\varrho)} \sum_{L \in \mathcal{G}_i \cap \mathscr{F}} 2^{3i/2-mi-7i/4} \int_{R(L)} \frac{\mathrm{dist}^2(q,\beta(L)+\mathbf{S})}{|q-\beta(L)|^{m+7/4}}\, d\|T\|(q) \\
&\qquad\quad + C(\hat{\mathbf{E}}(T,\mathbf{S},\mathbf{B}_1) + \mathbf{A}^2) \sum_{i \leq -1-\log_2(\varrho)} 2^{3i/2-mi} \#\mathcal{G}_i \\
&\qquad \leq C(\hat{\mathbf{E}}(T,\mathbf{S},\mathbf{B}_1) + \mathbf{A}^2) \sum_{i \leq -1-\log_2(\varrho)} (2^{-i/4-mi} + 2^{3i/2-mi}) \#\mathcal{G}_i\,.
\end{aligned}$$

Since the cardinality $\#\mathcal{G}_i$ of $\mathcal{G}_i$ is bounded by $C2^{(m-2)i}$, it follows that

$$\int_{\mathbf{B}_r \setminus \mathbf{B}_\varrho(V)} \frac{\mathrm{dist}^2(q,\mathbf{S})}{\mathrm{dist}(q,V)^{3/2}}\, d\|T\|(q) \leq C(\hat{\mathbf{E}}(T,\mathbf{S},\mathbf{B}_1) + \mathbf{A}^2)\,. \tag{4.343}$$

Clearly (4.342) and (4.343) imply (4.337).

We next use the same family $\mathscr{F}$ of cubes to prove the remaining estimates. Fix any cube

$L \in \mathscr{F}$ and recall Proposition 4.6.14, but rather than approximating over $\mathbf{S}$, approximate the current over the shifted cone $\mathbf{S} + \beta(L)$ (which we can do by Proposition 4.9.4, namely (4.247)). We then gain an approximation $\tilde{u}_{L,i} : \mathbf{p}_{V^\perp \cap \alpha_i}(\beta(L)) + \lambda L_i \to \mathcal{A}_{Q_i}(\alpha_i^\perp)$ which satisfies the estimate

$$\|D\tilde{u}_{L,i}\|_{L^2}^2 \leq C2^{-m\ell(L)}(\hat{\mathbf{E}}(T, \mathbf{S} + \beta(L), \mathbf{B}_{C2^{-\ell(L)}}(\beta(L))) + 2^{-2\ell(L)}\mathbf{A}^2)\,.$$

Observe that, by Proposition 4.9.4, namely (4.246), we get [A:shouldn't this estimate go above, to justify getting the graphical approximation over the cubes relative to the shifted cone? Since we need the excess to the shifted cone to be small enough]

$$\hat{\mathbf{E}}(T, \mathbf{S} + \beta(L), \mathbf{B}_{C2^{-\ell(L)}}(\beta(L))) \leq C2^{\ell(L)/4}(\hat{\mathbf{E}}(T, \mathbf{S}, \mathbf{B}_1) + \mathbf{A}^2)\,. \tag{4.344}$$

(note that we can apply (4.246) provided $C2^{-\ell(L)} \leq r$, where $r = r(Q, m, n, \bar{n})$ is the radius from Proposition 4.9.4, i.e. provided $\ell(L)$ is larger than a constant depending on $Q, m, n, \bar{n}$. For the cubes L with $\ell(L)$ smaller than this constant, the inequality above follows more easily as we have an upper bound on $\ell(L)$). In particular we have

$$\|D\tilde{u}_{L,i}\|_{L^2}^2 \leq C2^{-m\ell(L)+\ell(L)/4}(\hat{\mathbf{E}}(T, \mathbf{S}, \mathbf{B}_1) + \mathbf{A}^2)\,.$$

Proposition 4.6.14, (4.247) and (4.344) also give the estimate

$$\|\tilde{u}_{L,i} \ominus (\mathbf{p}_{\alpha_i}^\perp(\beta(L)))\|_{L^\infty} \leq C2^{-7\ell(L)/4}(\hat{\mathbf{E}}(T, \mathbf{S}, \mathbf{B}_1) + \mathbf{A}^2)$$

We can now consider the map $\hat{u}_{L,i} := \tilde{u}_{L,i} \ominus (\mathbf{p}_{\alpha_i}^\perp(\beta(L)))$ over the domain $\mathbf{p}_{V^\perp \cap \alpha_i}(\beta(L)) + \lambda L_i$, which we can assume contains L_i, provided that ε is sufficiently small. Observe that we can use this new map in the algorithm within Proposition 4.6.18 in place of $u_{L,i}$ to construct a coherent approximation from $\tilde{u}_{L,i}$, since this new map satisfies the same estimates as above.

We can in turn use this new coherent approximation to construct a coherent transversal approximation as in Proposition 4.11.4. In particular, if we keep denoting these improved approximations by u_i and w_i without relabeling, they will satisfy the estimates

$$\int_{L_i} \frac{|Du_i|^2}{\mathrm{dist}(z,V)^{3/2}}\,dz \le C2^{-m\ell(L)+7\ell(L)/4}(\hat{\mathbf{E}}(T,\mathbf{S},\mathbf{B}_1)+\mathbf{A}^2) \tag{4.345}$$

$$\int_{L_1} \frac{|Dw_i|^2}{\mathrm{dist}(z,V)^{3/2}}\,dz \le C2^{-m\ell(L)+7\ell(L)/4}(\hat{\mathbf{E}}(T,\mathbf{S},\mathbf{B}_1)+\mathbf{A}^2) \tag{4.346}$$

$$\int_{L_i} \frac{|u_i \ominus (\mathbf{p}^{\perp}_{\alpha_i}(\beta(L)))|^2}{\mathrm{dist}(z,V)^{7/2}}\,dz \le C2^{-m\ell(L)+7\ell(L)/4}(\hat{\mathbf{E}}(T,\mathbf{S},\mathbf{B}_1)+\mathbf{A}^2) \tag{4.347}$$

$$\int_{L_1} \frac{|w_i \ominus (\mathbf{p}^{\perp}_{\alpha_1}(\beta(L)) + A_i(\mathbf{p}_{V^{\perp}\cap\alpha_1}(\beta(L))))|^2}{\mathrm{dist}(z,V)^{7/2}}\,dz \le C2^{-m\ell(L)+7\ell(L)/4}(\hat{\mathbf{E}}(T,\mathbf{S},\mathbf{B}_1)+\mathbf{A}^2)\,. \tag{4.348}$$

We now sum the latter estimates over $L \in \mathscr{F} \cap \mathcal{G}_i$ for $i \le -\log_2(\varrho) - 1$. Considering that the cardinality of $\mathcal{G}_i$ is $C2^{(m-2)i}$ we get (4.338)–(4.341). □

4.11.4 Blow-ups

We are now ready to define blow-up sequences for the contradiction argument in the two cases of Proposition 4.11.1 and Proposition 4.11.2. In both cases we have a sequence of currents T_k, cones $\mathbf{S}_k$, and manifolds Σ_k for which the assumption of their respective proposition fails, with a sequence of thresholds $\varepsilon_c = 1/k$ in the case of Proposition 4.11.1 and with $\varepsilon_{nc} = 1/k$ in the case of Proposition 4.11.2 (we stress that $\varepsilon^\star_c$ is a fixed number in the case of Proposition 4.11.2 and currently unrelated to ε_c!). In particular, we know that (cf. Remark 4.11.3)

$$\lim_{k\uparrow\infty}\Big(\frac{\mathbf{A}_k^2}{\mathbb{E}(T_k,\mathbf{S}_k,\mathbf{B}_1)} + \frac{\mathbb{E}(T_k,\mathbf{S}_k,\mathbf{B}_1)}{\boldsymbol{\sigma}(\mathbf{S}_k)^2}\Big) = 0 \tag{4.349}$$

while, denoting by $\beta^k(L)$ the nails of Definition 4.11.6 for T_k, we have

$$\lim_{k\uparrow\infty} |\beta^k(L) - y_L| = 0 \qquad \forall L \in \mathcal{G}\,. \tag{4.350}$$

Moreover, upon extracting a subsequence, we can assume that $\mathbf{S}_k$ consists of a fixed number $N(k) = N \in \{2, \dots, Q\}$ of distinct planes, which we denote by $\alpha_1^k, \dots, \alpha_N^k$. Applying suitable rotations, we can assume that $T_0\Sigma_k$, $V(\mathbf{S}_k)$, and α_1^k are independent of k. In particular, we will often use V in place of $V(\mathbf{S}_k)$ and α_1 in place of α_1^k.

The distinction between the case of Proposition 4.11.1 and Proposition 4.11.2 is that

(C) In the collapsed case (Proposition 4.11.1) we have $\boldsymbol{\mu}(\mathbf{S}_k) \to 0$;

(NC) In the non-collapsed case (Proposition 4.11.2) we have $\liminf_k \boldsymbol{\mu}(\mathbf{S}_k) \geq \varepsilon_c^\star > 0$, for some $\varepsilon_c^\star > 0$.

In particular, in the non-collapsed case we can assume in addition that:

(NC1) Up to relabelling, $\mathrm{dist}(\alpha_1^k \cap \mathbf{B}_1, \alpha_N^k \cap \mathbf{B}_1) = \boldsymbol{\mu}(\mathbf{S}_k)$;

(NC2) Each sequence α_i^k converges (in Hausdorff distance) to a plane α_i, and α_N is necessarily distinct from α_1 (but some of the other planes α_i could coincide).

In the collapsed case we consider the transversal coherent approximations w_i^k, while in the non-collapsed case we consider the orthogonal coherent approximations u_i^k. In both cases we assume that their domains of definition $\mathrm{Dom}\,(w_i^k)$ and $\mathrm{Dom}\,(u_i^k)$ contain, respectively, $(\mathbf{B}_r \cap \alpha_1) \setminus \mathbf{B}_{1/k}(V)$ and $(\mathbf{B}_r \cap \alpha_i^k) \setminus \mathbf{B}_{1/k}(V)$. We further consider the rescaled functions

$$\bar{u}_i^k := \frac{u_i^k}{\sqrt{\mathbb{E}(T_k, \mathbf{S}_k, \mathbf{B}_1)}} \tag{4.351}$$

$$\bar{w}_i^k := \frac{w_i^k}{\sqrt{\mathbb{E}(T_k, \mathbf{S}_k, \mathbf{B}_1)}}\,. \tag{4.352}$$

Observe that we have uniform estimates for the $W^{1,2}$ norms of $\bar{u}_i^k$ and $\bar{w}_k^i$ (from Proposition 4.11.7, which are even stronger as they have a weight). The maps w_i^k are also defined over

the same plane α_1. For the maps u_i^k there is the annoying technicality that they are not. In the latter case we refer to Remark 4.6.20 and assume they are, up to composition with a small rotation, defined on the same planes α_i.

In particular, up to extraction of a subsequence we can assume that the $\bar{u}_i^k$ and $\bar{w}_i^k$ converge strongly in $W^{1,2}$ locally away from V and strongly in L^2 on the entirety of B_r (due to the non-concentration estimates from Proposition 4.11.7) to $W^{1,2}$ maps $\bar{u}_i$ and $\bar{w}_i$ defined on $(\mathbf{B}_r \cap \alpha_i) \setminus V$ and $(\mathbf{B}_r \cap \alpha_1) \setminus V$ respectively (and taking values in $\mathcal{A}_Q(\alpha_i^\perp)$ and $\mathcal{A}_Q(\alpha_1^\perp)$ respectively). However, since V is an $(m-2)$-dimensional subspace, which has zero 2-capacity, the maps $\bar{u}_i$ and $\bar{w}_i$ belong indeed to $W^{1,2}(\alpha_i \cap \mathbf{B}_r)$ and $W^{1,2}(\alpha_1 \cap \mathbf{B}_r)$ respectively. In the collapsed case we also introduce the linear maps $A_i^k : \alpha_1 \to \alpha_1^\perp$ whose graphs describe the planes α_i^k, and can consider their rescalings

$$\bar{A}_i^k := \frac{A_i^k}{\boldsymbol{\mu}(\mathbf{S}_k)}\,. \tag{4.353}$$

Moreover, up to extraction of a further subsequence, we assume that $\bar{A}_i^k$ converges to some linear map $\bar{A}_i$.

We are now ready to state our main proposition concerning the properties of the limiting maps $\bar{u}_i$ and $\bar{w}_i$.

Proposition 4.11.8. *Let $\bar{u}_i^k$, $\bar{w}_i^k$ and their respective limits $\bar{u}_i$, $\bar{w}_i$ be as described above. The following holds in the non-collapsed case:*

(a) Each $\bar{u}_i$ is Dir-minimizing on $\mathbf{B}_r \cap \alpha_i$, for r as in Proposition 4.11.7.

(b) $\bar{u}_i = Q_i\llbracket \boldsymbol{\eta} \circ \bar{u}_i \rrbracket$ on V.

(c) $\boldsymbol{\eta} \circ \bar{u}_i(0) = 0$ and the frequency obeys $I_{0,\bar{u}_i}(0) \geq 1$. Moreover, $I_{y,\bar{u}_i \ominus \boldsymbol{\eta} \circ \bar{u}_i}(0) \geq 1$ for all $y \in V \cap \mathbf{B}_r$.

(d) There is a smooth function $\beta : V \to V^\perp$ such that $\boldsymbol{\eta} \circ \bar{u}_i = \mathbf{p}_{\alpha_i}^\perp(\beta)$ on V for every i. (Note that with a slight abuse of notation we are using the same letter identifying the

nails of Definition 4.11.6. In fact this function β is the trace of the limit of suitable normalizations of the nails.)

The following holds in the collapsed case:

(e) Each $\bar{w}_i$ is Dir-minimizing on $\mathbf{B}_r \cap \alpha_1$.

(f) $\bar{w}_i = Q_i\llbracket \boldsymbol{\eta} \circ \bar{w}_i \rrbracket$ on V.

(g) $\boldsymbol{\eta} \circ \bar{w}_i(0) = 0$ and the frequency $I_{0,\bar{w}_i}(0) \geq 1$. Moreover, $I_{y,\bar{w}_i \ominus \boldsymbol{\eta} \circ \bar{w}_i}(0) \geq 1$ for all $y \in V \cap \mathbf{B}_r$;

(h) There are smooth function $\beta^\perp : V \to \alpha_1^\perp$ and $\beta_\parallel : V \to V^\perp \cap \alpha_1$ such that $\boldsymbol{\eta} \circ \bar{w}_i = \beta^\perp + \bar{A}_i(\beta_\parallel)$ on V for all i.

The functions $\bar{u}_i$ and β also enjoy the estimates

$$\|\bar{u}_i\|_{W^{1,2}} + \|\beta\|_{C^2} \leq C \tag{4.354}$$

for some constant C which depends only on $Q, m, n, \bar{n}$ and $\varepsilon_c^\star$. The functions $\bar{w}_i$, $\beta^\perp$, and $\beta_\parallel$ enjoy the estimates

$$\|\bar{w}_i\|_{W^{1,2}} + \|\beta^\perp\|_{C^2} + \|\beta_\parallel\|_{C^2} \leq C \tag{4.355}$$

for some constant C which depends only on $Q, m, n, \bar{n}$.

Proof. We start with (a). Here we first appeal to Proposition 4.6.19 to prove the Dir-minimality of $\bar{u}_i$ away from the spine V. Observe that, if, along the sequence $\bar{u}_i^k$, the double-sided excess $\mathbb{E}(T_k, \mathbf{S}_k, \mathbf{B}_1)$ stays comparable to $\hat{\mathbf{E}}(T_k, \mathbf{S}_k, \mathbf{B}_1)$, this claim follows from Proposition 4.6.19(iii) because $\mathbf{A}^2$ is infinitesimal compared to $\mathbb{E}(T_k, \mathbf{S}_k, \mathbf{B}_1)$. The alternative left is that (up to subsequence) $\hat{\mathbf{E}}(T_k, \mathbf{S}_k, \mathbf{B}_1)/\mathbb{E}(T_k, \mathbf{S}_k, \mathbf{B}_1)$ is infinitesimal. In this case, by Proposition 4.6.19(ii), $D\bar{u}_i$ vanishes. Having established the Dir-minimality away from the spine, Proposition 4.10.4 allows us to conclude that it is Dir-minimizing on any $\Omega \subset\subset \mathbf{B}_r$; this proves (a).

Now let us show (b). We next observe that, by (4.247) (in which one can clearly replace α_1 by any other plane of $\mathbf{S}$), $|\mathbf{p}^{\perp}_{\alpha_i^k}(\beta^k(L))| \leq C\mathbb{E}(T_k, \mathbf{S}_k, \mathbf{B}_1)^{1/2}$ and, because of the lower bound on $\mu(\mathbf{S}_k)$, $|\mathbf{p}_{V^\perp \cap \alpha_i^k}(\beta^k(L))| \leq C\mathbb{E}(T_k, \mathbf{S}_k, \mathbf{B}_1)^{1/2}$, for each $i = 1, \dots, N$. In particular we easily conclude that

$$|\mathbf{p}^{\perp}_{V}(\beta^k(L))| \leq C\mathbb{E}(T_k, \mathbf{S}_k, \mathbf{B}_1)^{1/2}$$

for a constant $C = C(Q, m, n, \bar{n}, \varepsilon_c^\star)$. Thus we can assume that for every L, we can find $\bar{\beta}(L)$ such that

$$\lim_{k\to\infty} \frac{\mathbf{p}^{\perp}_{V}(\beta^k(L))}{\mathbb{E}(T_k, \mathbf{S}_k, \mathbf{B}_1)^{1/2}} = \bar{\beta}(L)\,.$$

It is notationally convenient to think of $\bar{\beta}$ as a piecewise constant function which is defined over $\mathbf{B}_r$ as being identically equal to $\bar{\beta}(L)$ on the set $R(L)$ (note that this is well-defined away from V and the overlaps of the regions $R(L)$, but this set has measure 0). In that way we can divide by $\mathbb{E}(T_k, \mathbf{S}_k, \mathbf{B}_1)$ in (4.339) and pass to the limit to conclude that

$$\int_{\mathbf{B}_r \cap \alpha_i} \frac{|\bar{u}_i(z) \ominus \mathbf{p}^{\perp}_{\alpha_i}(\bar{\beta})(z)|^2}{\operatorname{dist}(z, V)^{7/2}}\, dz < \infty\,. \tag{4.356}$$

But in particular, since $|\bar{u}_i(z) \ominus (\boldsymbol{\eta} \circ \bar{u}_i)(z)| \leq |\bar{u}_i(z) \ominus \mathbf{p}^{\perp}_{\alpha_i}(\bar{\beta})(z)|$ (this is just the trivial fact that the barycenter minimizes this quantity), we get

$$\int_{\mathbf{B}_r \cap \alpha_i} \frac{|\bar{u}_i(z) \ominus \boldsymbol{\eta} \circ \bar{u}_i(z)|^2}{\operatorname{dist}(z, V)^{7/2}}\, dz < \infty\,. \tag{4.357}$$

Given that the numerator is continuous, it must be identically 0 for $z \in V \cap \mathbf{B}_r$, otherwise the integral would diverge (recall that V has codimension 2 and thus the integral of $\operatorname{dist}(z, V)^{-s}$ diverges for every $s \geq 2$). In particular we have shown (b).

As for (c), recall first that we have the estimate of Corollary 4.9.3. Once again choosing $\kappa = \frac{1}{4}$, this is easily seen to imply the estimate

$$\int_{\mathbf{B}_r \cap \alpha_i} \frac{|\bar{u}_i(z)|^2}{|z|^{m+7/4}}\, dz < \infty\,.$$

Arguing as above, this clearly implies that $|\bar{u}_i(0)| = 0$ and hence the first claim of point (c). We next argue for the second part of the claim; here we resort to the full version of Corollary 4.9.3 for any $\kappa \in (0, m+2)$. Observe that it suffices to prove the latter claim $I_{y,\bar{u}_i \ominus \boldsymbol{\eta} \circ \bar{u}_i}(0) \geq 1$ for all $y \in V \cap \mathbf{B}_r$. Indeed, the fact that $I_{0,\bar{u}_i}(0) \geq 1$ follows immediately from this at $y = 0$, combined with the property $\boldsymbol{\eta} \circ \bar{u}_i(0) = 0$ that we have just proved. Fix a point $y \in V \cap \mathbf{B}_r$ and for each $k \in \mathbb{N}$, consider a point $q_k \in \mathbf{B}_{\varepsilon_k}(y)$ with $\Theta(T_k, q_k) \geq Q$ and fix any $\rho \in (0, r)$. It follows from Proposition 4.9.4 that, if $p \in \operatorname{gr}(u_i^k) \cap \operatorname{spt}(T_k)$ and $|\mathbf{p}_{\alpha_i}(p)| \geq \rho$, then $\operatorname{dist}(p, q_k) + \mathbf{S}_k) = \operatorname{dist}(p, q_k + \alpha_i^k)$ for all k sufficiently large (depending on ρ and κ) and in particular, if $K_{k,i} \subset \alpha_i^k$ denotes the set over which the graph of u_i^k coincides with the current T, then Proposition 4.9.4

$$\int_{(\mathbf{B}_r(y) \setminus \mathbf{B}_\rho(y)) \cap K_{k,i}} \frac{|u_i^k(z) \ominus \mathbf{p}^\perp_{\alpha_i^k}(q_k)|^2}{|z-y|^{m+2-\kappa}}\, dz \leq C(\kappa) \mathbb{E}(T_k, \mathbf{S}_k, \mathbf{B}_1)\,,$$

for every positive κ, and for k sufficiently large, depending on ρ, κ.

Again recall that $|u_i^k \ominus \boldsymbol{\eta} \circ u_i^k| \leq |u_i^k \ominus \mathbf{p}^\perp_{\alpha_i^k}(q_k)|$, and so in particular we conclude that

$$\int_{(\mathbf{B}_r(y) \setminus \mathbf{B}_\rho(y)) \cap K_{k,i}} \frac{|u_i^k(z) \ominus \boldsymbol{\eta} \circ u_i^k(z)|^2}{|z-y|^{m+2-\kappa}}\, dz \leq C(\kappa) \mathbb{E}(T_k, \mathbf{S}_k, \mathbf{B}_1)\,.$$

Dividing by $\mathbb{E}(T_k, \mathbf{S}_k, \mathbf{B}_1)$, taking the limiting in k and noting that the measure of $(\mathbf{B}_r \setminus \mathbf{B}_\rho) \cap (\alpha_i^k \setminus K_{k,i})$ converges to 0 (cf. (4.149)), we arrive at

$$\int_{(\mathbf{B}_r(y) \setminus \mathbf{B}_\rho(y)) \cap \alpha_i} \frac{|\bar{u}_i(z) \ominus \boldsymbol{\eta} \circ \bar{u}_i(z)|^2}{|z-y|^{m+2-\kappa}}\, dz \leq C(\kappa)\,.$$

On the other hand letting $\rho \downarrow 0$ we then get

$$\int_{\mathbf{B}_r(y) \cap \alpha_i} \frac{|\bar{u}_i(z) \ominus \boldsymbol{\eta} \circ \bar{u}_i(z)|^2}{|z-y|^{m+2-\kappa}}\, dz \leq C(\kappa) \tag{4.358}$$

Now let $\gamma := I_{y,\bar{u}_i \ominus \boldsymbol{\eta} \circ \bar{u}_i}(0)$. It follows from [18, Corollary 3.18] that for every $\gamma' > \gamma$ there is

a positive constant C (depending on γ' and $\bar{u}_i$) such that

$$\int_{\partial \mathbf{B}_\sigma(y)\cap\alpha_1} |\bar{u}_i \ominus \boldsymbol{\eta}\circ\bar{u}_i|^2 \geq C\sigma^{m-1+2\gamma'}\,,$$

for every $\sigma \in (0,r)$. Combined with (4.358), we conclude that

$$\int_0^r \sigma^{2\gamma'+\kappa-3}\,d\sigma < \infty$$

which implies that $2\gamma'+\kappa > 2$. Letting $\kappa \downarrow 0$ and $\gamma' \downarrow \gamma = I_{y,\bar{u}_i\ominus\boldsymbol{\eta}\circ\bar{u}_i}(0)$ we then conclude that $I_{y,\bar{u}_i\ominus\boldsymbol{\eta}\circ\bar{u}_i}(0) \geq 1$. Note that one could alternatively conclude that $I_{y,\bar{u}_i\ominus\boldsymbol{\eta}\circ\bar{u}_i}(0) \geq 1$ via the Hardt-Simon inequality, as done in [45].

We next come to point (d). Observe that if ζ is in $V^\perp$ then

$$|\zeta| \leq C(|\mathbf{p}^\perp_{\alpha_1}(\zeta)| + |\mathbf{p}^\perp_{\alpha_N}(\zeta)|)\,, \tag{4.359}$$

(where the constant C in particular depends on $\varepsilon^\star_c$). We next observe that we have (from (4.356), (4.357))

$$\int_{\mathbf{B}_r\cap\alpha_i} \frac{|\mathbf{p}^\perp_{\alpha_i}(\bar{\beta}) - \boldsymbol{\eta}\circ\bar{u}_i|^2}{\mathrm{dist}(z,V)^{7/2}}\,dz < \infty\,. \tag{4.360}$$

For each $\rho \in (0,r)$ consider the functions defined on V which result averaging $\bar{\beta}$ and $\boldsymbol{\eta}\circ\bar{u}_i$ respectively, over 2-dimensional disks in α_i (for any $i \in \{1,\ldots,N\}$) of radius ρ perpendicular to V and centered at $y \in V$:

$$\bar{\beta}_\rho(y) = \frac{1}{\pi\rho^2}\int_{\mathbf{B}_\rho(y)\cap(y+V^\perp)\cap\alpha_i} \bar{\beta}\,,$$
$$g_\rho(y) = \frac{1}{\pi\rho^2}\int_{\mathbf{B}_\rho(y)\cap(y+V^\perp)\cap\alpha_i} \boldsymbol{\eta}\circ\bar{u}_i\,.$$

We clearly have, from (4.360),

$$\int_{V\cap \mathbf{B}_{\sqrt{r^2-\rho^2}}} |\mathbf{p}^\perp_{\alpha_i}(\bar\beta_\rho) - g_\rho|^2 \le C\rho^{3/2}\,.$$

In particular $\mathbf{p}^\perp_{\alpha_i}(\bar\beta_\rho)$ converges, as $\rho\downarrow 0$, in $L^2(V\cap \mathbf{B}_r)$ to $\lim_{\rho\downarrow 0} g_\rho = \boldsymbol{\eta}\circ \bar u_i$. But then for any given $\rho\in(0,r)$,

$$\int_{V\cap \mathbf{B}_{\sqrt{r^2-\rho^2}}} |\mathbf{p}^\perp_{\alpha_i}(\bar\beta_\rho) - \mathbf{p}^\perp_{\alpha_i}(\bar\beta_{\rho'})|^2 \le C\rho^{3/2} + \int_{V\cap \mathbf{B}_{\sqrt{r^2-\rho^2}}} |g_\rho - g_{\rho'}|^2 \qquad \forall \rho'<\rho$$

Hence, recalling that $\bar\beta$ takes values in $V^\perp$, applying (4.359) and the above inequality with $i=1,N$, we conclude

$$\int_{V\cap \mathbf{B}_{\sqrt{r^2-\rho^2}}} |\bar\beta_\rho - \bar\beta_{\rho'}|^2 \le C\rho^{3/2} + \int_{V\cap \mathbf{B}_{\sqrt{r^2-\rho^2}}} |g_\rho - g_{\rho'}|^2 \qquad \forall \rho'<\rho\,.$$

In particular $\{\bar\beta_\rho\}_{\rho>0}$ converges strongly in $L^2(V\cap \mathbf{B}_r)$ as $\rho\downarrow 0$ to some function $\beta\in L^2(V\cap \mathbf{B}_r)$ and clearly $\mathbf{p}^\perp_{\alpha_i}(\beta) = \boldsymbol{\eta}\circ \bar u_i$ for each $i=1,\ldots,N$. The smoothness of β follows easily from the smoothness of the $\boldsymbol{\eta}\circ\bar u_i$, which in turn is obvious from the fact that they are the traces on V of classical harmonic functions. This proves (d).

Coming to the estimate (4.354) in the non-collapsed case, by construction (namely, the L^2 non-concentration estimates (4.338), (4.339)), $\|\bar u_i\|_{W^{1,2}}\le C$ for some constant $C = C(Q,m,n,\bar n)$, which in fact does not depend on $\varepsilon^\star_c$. Next, as $\mathbf{p}^\perp_{\alpha_i}(\beta) = \boldsymbol{\eta}\circ\bar u_i$ and the cone $\mathbf{S}$ is well-separated (cf. (4.359)), β is determined as a linear combination of the $\mathbf{p}^\perp_{\alpha_i}(\beta)$ for $i=1,N$, which in turn are the traces of $\boldsymbol{\eta}\circ\bar u_i$. The latter are harmonic functions and they enjoy the same $W^{1,2}$ bound on $\bar u_i$. Hence, the C^2 estimate follows from the classical theory of harmonic functions.

The conclusion (e) is more subtle than (a). As above, we wish to argue that $\bar w_i$ is Dir-minimizing on every $\Omega\subset\subset \mathbf{B}_r\cap\alpha_1\setminus V$ and invoke Proposition 4.10.4 to argue that therefore it is Dir-minimizing on $\mathbf{B}_r\cap\alpha_1$. First of all we assume, without loss of generality, that the

double-sided excess $\mathbb{E}(T_k, \mathbf{S}_k, \mathbf{B}_1)$ and the one sided excess $\hat{\mathbf{E}}(T_k, \mathbf{S}_k, \mathbf{B}_1)$ are comparable, since the alternative would be that the latter is infinitesimal compared to the former, and in that case $\bar{w}_i$ would vanish. Then, we notice that in order to argue for the minimality in $\Omega \subset\subset \mathbf{B}_r \cap \alpha_1 \setminus V$ we cannot invoke directly the argument of [19, Theorem 2.6]. However, since the mismatch in mass between the current T and the graphs of the multi-valued maps $\sum_i w_i^k \oplus A_i^k$ is controlled by $E_k^{1+\gamma} := \mathbb{E}(T_k, \mathbf{S}_k, \mathbf{B}_1)^{1+\gamma}$, the area-minimizing property of T and the arguments in [19, Section 5] shows the following minimality property for the map $w_i^k \oplus A_i^k$:

(Min) If $\tilde{w}_i^k$ is a Lipschitz map which coincides with w_i^k outside of Ω and its Lipschitz constant is controlled by 1, then

$$\|\mathbf{G}_{\tilde{w}_i^k \oplus A_i^k}\|(\Omega \times \alpha_1^\perp) \geq \|\mathbf{G}_{w_i^k \oplus A_i^k}\|(\Omega \times \alpha_1^\perp) - C E_k^{1+\gamma}\,. \tag{4.361}$$

Our aim is to show that, if $\bar{w}_i$ is not Dir-minimizing, then (4.361) is violated for k sufficiently large. Assuming it is not Dir-minimizing, the Lipschitz truncation and "cut-and-paste" arguments of [19, Theorem 2.6] show the existence of a sequence $\tilde{w}_i^k$ of multi-valued maps such that

(i) $\tilde{w}_i^k = w_i^k$ outside Ω;

(ii) $\mathrm{Lip}(\tilde{w}_i^k) \leq C E_k^\gamma$;

(iii) The Dirichlet energy of $\tilde{w}_i^k$ has the following gain:

$$\int_\Omega |D\tilde{w}_i^k|^2 \leq \int_\Omega |Dw_i^k|^2 - \vartheta E_k\,, \tag{4.362}$$

for some positive $\vartheta > 0$ independent of k.

We now wish to show that (i)-(iii) contradict (4.361). We let $\mathcal{A}(Du)$ be the area integrand for the graph of a single-valued function u. More precisely, if we denote by $\mathcal{M}(B)$ the set of

all $k \times k$ minors of the matrix B with $k \geq 2$, then[1]

$$\mathcal{A}(B) = \sqrt{1 + |B|^2 + \sum_{M \in \mathcal{M}(B)} \det(M)^2}\,.$$

We next use the notation $w_i^k = \sum_j [\![w_{i,j}^k]\!]$ and $\tilde{w}_i^k = \sum_j [\![\tilde{w}_{i,j}^k]\!]$. Using the area formula in [20, Corollary 1.11] we can compute

$$\|\mathbf{G}_{\tilde{w}_i^k \oplus A_i^k}\|(\Omega \times \alpha_1^\perp) = \int_\Omega \sum_j \mathcal{A}(A_i^k + D\tilde{w}_{i,j}^k)$$

and

$$\|\mathbf{G}_{w_i^k \oplus A_i^k}\|(\Omega \times \alpha_1^\perp) = \int_\Omega \sum_j \mathcal{A}(A_i^k + Dw_{i,j}^k)\,.$$

In particular, defining

$$\Delta := \|\mathbf{G}_{\tilde{w}_i^k \oplus A_i^k}\|(\Omega \times \alpha_1^\perp) - \|\mathbf{G}_{w_i^k \oplus A_i^k}\|(\Omega \times \alpha_1^\perp)\,,$$

we arrive at the expression

$$\Delta = \int_\Omega \left(\sum_j \mathcal{A}(A_i^k + D\tilde{w}_{i,j}^k) - \sum_j \mathcal{A}(A_i^k + Dw_{i,j}^k) \right)\,. \tag{4.363}$$

We now wish to make a Taylor expansion of the function $\mathcal{A}$ at the constant A_i^k. In particular we can write

$$\mathcal{A}(A_i^k + B) = \mathcal{A}(A_i^k) + \mathcal{L}_{k,i}(B) + \mathcal{Q}_{k,i}(B) + O(|B|^3)$$

where $\mathcal{L}_{k,i}$ are appropriate linear functions and $\mathcal{Q}_{k,i}$ are appropriate quadratic forms. However note that at $A_i^k = 0$ the quadratic form would be $\frac{|B|^2}{2}$, and given that $|A_i^k| \leq \boldsymbol{\mu}(\mathbf{S}_k)$ we

[1] Note that, when u is a scalar function, Du only has 1×1 minors, namely $\mathcal{M}(Du)$ is empty, and we would use the convention that the formula reduces to $\mathcal{A}(B) = \sqrt{1 + |B|^2}$. However in our case u is necessarily vector-valued.

can further write

$$\mathcal{A}(A_i^k + B) = \mathcal{A}(A_i^k) + \mathcal{L}_i^k(B) + \frac{|B|^2}{2} + O((\boldsymbol{\mu}(\mathbf{S}_k) + |B|)|B|^2)\,.$$

Now, inserting the latter expansion in our expression (4.363) and using that $\|D\tilde{w}_i^k\|_\infty + \|Dw_i^k\|_\infty \leq CE_k^\gamma$ and $\|D\tilde{w}_i^k\|_{L^2} + \|Dw_i^k\|_{L^2} \leq CE_k$ we arrive at

$$\begin{aligned}\Delta &= \int_\Omega \left(\sum_j \left(\mathcal{L}_i^k(D\tilde{w}_{i,j}^k) + \frac{|D\tilde{w}_{i,j}^k|^2}{2} \right) - \sum_j \left(\mathcal{L}_i^k(Dw_{i,j}^k) + \frac{|Dw_{i,j}^k|^2}{2} \right) \right) \\ &\quad + O((\boldsymbol{\mu}(\mathbf{S}_k) + E_k^\gamma)E_k) \\ &= Q_i \underbrace{\int_\Omega \mathcal{L}_i^k(D(\boldsymbol{\eta} \circ \tilde{w}_i^k - \boldsymbol{\eta} \circ w_i^k))}_{=:(\mathrm{L})} + \frac{1}{2} \underbrace{\int_\Omega (|D\tilde{w}_i^k|^2 - |Dw_i^k|^2)}_{=:(\mathscr{Q})} + O((\boldsymbol{\mu}(\mathbf{S}_k) + E_k^\gamma)E_k)\,,\end{aligned}$$

where we have used the linearity of $\mathcal{L}_i^k$. Now observe that the function

$$f_k := \boldsymbol{\eta} \circ \tilde{w}_i^k - \boldsymbol{\eta} \circ w_i^k$$

is a single-valued Lipschitz function that vanishes on $\partial\Omega$. On the other hand $\mathcal{L}_i^k$ is a linear function. In particular, if we write $f_k = (f_k^1, \ldots, f_k^n)$ for the components of the functions f_k, there are vectors $v_k^1, \ldots v_k^n \in \mathbb{R}^n$ (determined by $\mathcal{L}_i^k$) such that

$$(\mathrm{L}) = \sum_l \int_\Omega v_k^l \cdot \nabla f_k^l = \sum_l \int_\Omega \operatorname{div}(v_k^l f_k^l) = \sum_l \int_{\partial\Omega} f_k^l \nu \cdot v_k^l = 0\,,$$

where ν denotes the unit normal determined by the Stokes' orientation of $\partial\Omega$. Moreover, by (4.362) we have

$$(\mathscr{Q}) \leq -\vartheta E_k\,.$$

We can thus write

$$\Delta \leq -\frac{\vartheta}{2} E_k + O((\boldsymbol{\mu}(\mathbf{S}_k) + E_k^\gamma)E_k)\,.$$

Since $E_k + \boldsymbol{\mu}(\mathbf{S}_k) \to 0$, the latter clearly contradicts (4.361). This concludes the proof of (e).

The argument for (f) is entirely analogous to the argument for (b), where we use (4.341) in place of (4.339): indeed (4.341) leads to the conclusion that

$$\int_{\mathbf{B}_r \cap \alpha_1} \frac{|\bar{w}_i \ominus \boldsymbol{\eta} \circ \bar{w}_i|^2}{\operatorname{dist}(z, V)^{7/2}} < \infty \,,$$

which in turn clearly implies (f) using the continuity of $\bar{w}_i$. The argument for (g) is entirely analogous to the argument for (c): using Corollary 4.9.3 with $\kappa = \frac{1}{4}$ we derive

$$\int_{\mathbf{B}_r \cap \alpha_1} \frac{|\bar{w}_i(z)|^2}{|z|^{m+7/4}} \, dz < \infty \,,$$

which yields $\bar{w}_i(0) = \boldsymbol{\eta} \circ \bar{w}_i(0) = 0$, while Proposition 4.9.4 gives

$$\int_{\mathbf{B}_r(y) \cap \alpha_1} \frac{|\bar{w}_i \ominus \boldsymbol{\eta} \circ \bar{w}_i(z)|^2}{|z - y|^{m+2-\kappa}} \, dz \leq C(\kappa) \tag{4.364}$$

for $y \in V \cap \mathbf{B}_r$. We then use this in place of (4.358) to conclude the desired lower bound on the frequency.

We finally come to (h). Recall the notation $\beta^k(L)$ for the nail of L when the current is T_k. Recall that from (4.247) that

$$\begin{aligned} |\mathbf{p}^{\perp}_{\alpha_1}(\beta^k(L))| &\leq C\mathbb{E}(T_k, \mathbf{S}_k, \mathbf{B}_1)^{1/2} \,, \\ \boldsymbol{\mu}(\mathbf{S}_k)|\mathbf{p}_{V^{\perp} \cap \alpha_1}(\beta^k(L))| &\leq C\mathbb{E}(T_k, \mathbf{S}_k, \mathbf{B}_1)^{1/2} \,. \end{aligned}$$

Arguing as in (d), we assume, upon extraction of a subsequence, that there are $\bar{\beta}^{\perp}$ and $\bar{\beta}^{\|}$ such that

$$\begin{aligned} \frac{\mathbf{p}^{\perp}_{\alpha_1}(\beta^k(L))}{\mathbb{E}(T_k, \mathbf{S}_k, \mathbf{B}_1)^{1/2}} &\to \bar{\beta}^{\perp}(L) \\ \frac{\boldsymbol{\mu}(\mathbf{S}_k)\mathbf{p}_{V^{\perp} \cap \alpha_1}(\beta^k(L))}{\mathbb{E}(T_k, \mathbf{S}_k, \mathbf{B}_1)^{1/2}} &\to \bar{\beta}_{\|}(L) \,. \end{aligned}$$

In analogy with the argument for (d) we consider both functions as defined over $\mathbf{B}_r \cap \alpha_1$. Since $\bar{A}_i^k = \boldsymbol{\mu}(\mathbf{S}_k)^{-1} A_i^k$, by passing into the limit in (4.341) we get to, for each $i = 1, \dots, N$

$$\int_{\mathbf{B}_r \cap \alpha_1} \frac{|\bar{w}_i \ominus (\bar{\beta}^\perp + \bar{A}_i(\bar{\beta}_\parallel))|^2}{\operatorname{dist}(z, V)^{7/2}} \, dz < \infty \,. \tag{4.365}$$

Combining with (4.364), in turn these estimates imply

$$\int_{\mathbf{B}_r \cap \alpha_1} \frac{|\boldsymbol{\eta} \circ \bar{w}_i - (\bar{\beta}^\perp + \bar{A}_i(\bar{\beta}_\parallel))|^2}{\operatorname{dist}(z, V)^{7/2}} \, dz < \infty \,. \tag{4.366}$$

As $\bar{A}_1 = 0$, we immediately conclude that $\bar{\beta}^\perp = \boldsymbol{\eta} \circ \bar{w}_1$ on $V \cap \mathbf{B}_r$, and by the triangle inequality that

$$\int_{\mathbf{B}_r \cap \alpha_1} \frac{|\boldsymbol{\eta} \circ \bar{w}_i - (\boldsymbol{\eta} \circ \bar{w}_1 + \bar{A}_i(\bar{\beta}_\parallel))|^2}{\operatorname{dist}(z, V)^{7/2}} \, dz < \infty \,. \tag{4.367}$$

Next, notice that $\sum_i [\![\bar{A}_i]\!]$ is Dir-minimizing (indeed, this will simply be the usual blow-up of T_k relative to α_1). Moreover, because $\max_i |A_i^k| \geq C^{-1} \boldsymbol{\mu}(\mathbf{S}_k)$, we have that $\max_i |\bar{A}_i| \geq C^{-1} > 0$ for some constant $C = C(Q, m, n, \bar{n})$; we also know that this maximum is achieved by $\bar{A}_N$. Since $\bar{A}_1 = 0$, the map $[\![\bar{A}_N]\!] + [\![0]\!]$ is Dir minimizing and hence, by subtracting the average and rescaling by a factor 2, so is $[\![\bar{A}_N]\!] + [\![-\bar{A}_N]\!]$. In particular we can apply Lemma 4.5.7 and conclude that, if we let W be the image of $\bar{A}_N$, $\bar{A}_N : V^\perp \to W$ is invertible and its inverse B satisfies $|B| \leq C$ for some $C = C(Q, m, n, \bar{n})$.

We therefore have the identity $\bar{\beta}_\parallel = B(\bar{A}_N(\bar{\beta}_\parallel))$. For $y \in V \cap \mathbf{B}_r$, define now the functions $(\bar{\beta}_\parallel)_\rho$ and h_ρ as follows, analogously to those the proof of (d): for $y \in V \cap \mathbf{B}_r$,

$$(\bar{\beta}_\parallel)_\rho(y) = \frac{1}{\pi \rho^2} \int_{\mathbf{B}_\rho(y) \cap (y + V^\perp) \cap \alpha_1} \bar{\beta}_\parallel \,, \tag{4.368}$$

$$h_\rho(y) = \frac{1}{\pi \rho^2} \int_{\mathbf{B}_\rho(y) \cap (y + V^\perp) \cap \alpha_1} (\boldsymbol{\eta} \circ \bar{w}_i - \boldsymbol{\eta} \circ \bar{w}_1) \,. \tag{4.369}$$

Arguing as in the proof of (d) we use (4.367) to conclude that $\bar{A}_i((\bar{\beta}_\parallel)_\rho)$ converges in $L^2(V \cap \mathbf{B}_r)$ to the function $\boldsymbol{\eta} \circ \bar{w}_i - \boldsymbol{\eta} \circ \bar{w}_1 = \lim_{\rho \downarrow 0} h_\rho$, which in particular gives us the conclusion

that $\boldsymbol{\eta} \circ \bar{w}_N - \boldsymbol{\eta} \circ \bar{w}_1$ is in the image of $\bar{A}_N$. Hence, if we define $\beta_{\|} := B(\boldsymbol{\eta} \circ \bar{w}_N - \boldsymbol{\eta} \circ \bar{w}_1)$ and $\beta^{\perp} := \boldsymbol{\eta} \circ \bar{w}_1$, we see immediately that we have the identity

$$\boldsymbol{\eta} \circ \bar{w}_i = \boldsymbol{\eta} \circ \bar{w}_1 + \bar{A}_i(\beta_{\|}) = \beta^{\perp} + \bar{A}_i(\beta_{\|}) \, .$$

In order to conclude the proof we need to show the desired estimates (4.355) on $\|\bar{w}_i\|_{W^{1,2}}$, $\|\beta^{\perp}\|_{C^2}$, and $\|\beta_{\|}\|_{C^2}$. The first is obvious because $\|\bar{w}_i^k\|_{W^{1,2}}$ is bounded by a universal constant. The second is also obvious because $\beta^{\perp}$ is the restriction to V of the harmonic function $\boldsymbol{\eta} \circ \bar{w}_1$ whose $W^{1,2}$ norm is controlled by $\|\bar{w}_1\|_{W^{1,2}}$. Finally, the estimate on $\|\beta_{\|}\|_{C^2}$ follows from the same argument because $\beta_{\|} = B(\boldsymbol{\eta} \circ \bar{w}_N - \boldsymbol{\eta} \circ \bar{w}_1)$ and the norm of the linear map B is bounded by a universal constant. Thus, the proof of Proposition 4.11.8 is complete. □

4.11.5 Final argument

In this section we are going to show Proposition 4.11.1 and Proposition 4.11.2. We fix the decay scale ς_1 and we will show that this will be reached at a certain radius, r_c or r_{nc}, whether we are in the collapsed or non-collapsed setting, respectively, via a contradiction argument. We start with Proposition 4.11.1; we fix a contradiction sequence T_k, $\mathbf{S}_k$, and Σ_k as in the previous section and use Proposition 4.11.8(e)-(h) to extract the blow-up limits $\bar{A}_i$, $\bar{w}_i$, $\beta^{\perp}$, and $\beta_{\|}$. As before, without loss of generality we have rotated so that the planes α_1^k all coincide with the same plane α_1. We next observe that, by (e)-(h), the functions $\beta^{\perp}$ and $\beta_{\|}$ both equal 0 at the origin. We therefore linearize them and let $\gamma^{\perp}$ and $\gamma_{\|}$ be their respective linearizations at 0. Observe that the C^2-regularity of $\beta^{\perp}$ and $\beta_{\|}$ guarantees that

$$|\beta^{\perp}(y) - \gamma^{\perp}(y)| \leq C|y|^2 \tag{4.370}$$

$$|\beta_{\|}(y) - \gamma_{\|}(y)| \leq C|y|^2 \, . \tag{4.371}$$

The constant C depends only on the C^2 norm of $\beta^\perp$ and $\beta_\parallel$, which in turn is bounded by a constant depending only upon on $Q, m, n, \bar{n}$, by Proposition 4.11.8. Observe that $\gamma_\parallel : V \to V^\perp \cap \alpha_1$ and let $\gamma_\parallel^T : V^\perp \cap \alpha_1 \to V$ be its transpose. We build a skew-symmetric map of α_1 onto itself by mapping

$$V \oplus (V^\perp \cap \alpha_1) \ni y + x \mapsto \gamma_\parallel(y) - \gamma_\parallel^T(x)\,.$$

This skew-symmetric map generates a one-parameter family $R[t]$ of rotations of α_1, which we may extend to all of $\mathbb{R}^{m+\bar{n}}$ by setting it to be the identity on $\alpha_1^\perp$ and extended linearly. We next define the rotations

$$R_k := R\left[\frac{\mathbb{E}(T_k, \mathbf{S}_k, \mathbf{B}_1)^{1/2}}{\boldsymbol{\mu}(\mathbf{S}_k)}\right]$$

and observe that these rotations map α_1 and $\alpha_1^\perp$ onto themselves.

The rotated cones $\mathbf{S}'_k := R_k(\mathbf{S}_k)$ are thus a first step towards the cones which will have the desired decay at the radius r_c. Next consider the Dir-minimizing map $\bar{w}_i \ominus \boldsymbol{\eta} \circ \bar{w}_i$ and the (single-valued) harmonic function $\zeta_i := \boldsymbol{\eta} \circ \bar{w}_i - \gamma^\perp - \bar{A}_i(\gamma_\parallel)$, where the latter two linear maps are extended in the $V^\perp \cap \alpha_1$ directions as constant (in particular, being linear, they are harmonic). To the map $\bar{w}_i \ominus \boldsymbol{\eta} \circ \bar{w}_i$ we apply Theorem 4.10.2 (namely (4.313), which we can do by Proposition 4.11.8(g)): for a fixed δ, which will be chosen later, we find a radius $\bar{r} = \bar{r}(Q, m, n, \bar{n}, \delta)$ such that for every $\rho < \bar{r}$ we can find a 1-homogeneous Dir-minimizer $h_{i,\rho} \in \mathscr{L}_1$ with the property that

$$\int_{\mathbf{B}_\rho \cap \alpha_1} \mathcal{G}(\bar{w}_i \ominus \boldsymbol{\eta} \circ \bar{w}_i, h_{i,\rho})^2 \leq \delta \rho^{m+2}$$

(indeed, $\int_{\mathbf{B}_1 \cap \alpha_1} |\bar{w}_i|^2 \leq 1$ by construction). As for the classical harmonic part ζ_i, since $D\zeta_i(0) = 0$, we find a linear map ξ_i (namely, the linearization of ζ_i) which vanishes on V

such that

$$|\zeta_i(z) - \xi_i(z)| \leq C|z|^2 .$$

We fix now a radius $r_c < \bar{r}$. We are now ready to define a new sequence of cones $\mathbf{S}''_k$. We take the linear functions A^k_i whose graphs over α_1 give the planes α^k_i, hence the linear functions ξ_i, and construct the maps

$$A^k_i + \mathbb{E}(T_k, \mathbf{S}_k, \mathbf{B}_1)^{1/2}\xi_i ,$$

and then we add to them the multi-valued functions $\mathbb{E}(T_k, \mathbf{S}_k, \mathbf{B}_1)^{1/2}h_{i,\rho}$, and compose the resulting Q-valued function with R_k^{-1}. The formula for this Q-valued function over α_1 is thus given by

$$\sum_i (\mathbb{E}(T_k, \mathbf{S}_k, \mathbf{B}_1)^{1/2}h_{i,\rho} \oplus (A^k_i + \mathbb{E}(T_k, \mathbf{S}_k, \mathbf{B}_1)^{1/2}(\xi_i + \gamma^\perp))) \circ R_k^{-1} .$$

By construction, since the support of the graph of any element $h \in \mathscr{L}_1$ with $h(0) = Q[\![0]\!]$ lies in $\mathscr{C}(Q, 0)$, the graphs of these functions give new cones $\mathbf{S}''_k$ which belong to $\mathscr{C}(Q, 0)$. From the estimates that we have, we can check that, if $r_c = \rho \leq \bar{r}$

$$\lim_{k\to\infty} \frac{\mathbb{E}(T_k, \mathbf{S}''_k, \mathbf{B}_{r_c})}{\mathbb{E}(T_k, \mathbf{S}_k, \mathbf{B}_1)} \leq C\delta + Cr_c^2 ,$$

where C is just a geometric constant. In particular, we choose δ sufficiently small so that $C\delta \leq \frac{\varsigma_1}{2}$, which in turn fixes $\bar{r}$, and hence we fix $r_c \leq \bar{r}$ so that $Cr_c^2 \leq \frac{\varsigma_1}{4}$. With this choice we conclude that

$$\lim_{k\to\infty} \frac{\mathbb{E}(T_k, \mathbf{S}''_k, \mathbf{B}_{r_c})}{\mathbb{E}(T_k, \mathbf{S}_k, \mathbf{B}_1)} \leq \frac{3\varsigma_1}{4} .$$

Now, with this particular choice of r_c, which depends only on ς_1, we get for k large enough a contradiction to the absence of decay with factor ς_1. This proves Proposition 4.11.1.

The proof of Proposition 4.11.2 works in a very similar way. We again assume that ς_1 and $\varepsilon^\star_c$ are given, that r_{nc} is fixed, and that there is absence of decay by ς_1 for sequences

T_k, Σ_k, and $\mathbf{S}_k$. We then apply Proposition 4.11.8 and get the maps $\bar{u}_i$ and β. Arguing as above, we linearize the map β (which vanishes at 0) to a map γ. Again we note that

$$|\beta(y) - \gamma(y)| \leq C|y|^2 .$$

However, this time the constant C depends on $\varepsilon_c^\star$ as well as the C^2 norm of β. First of all we consider γ as a map from V to $V^\perp$, we let $\gamma^T : V^\perp \to V$ be its transpose, we again build the skew-symmetric map

$$V \oplus V^\perp \ni y + x \mapsto \gamma(y) - \gamma^T(x)$$

and hence we consider the one-parameter family of rotations $R[t]$ generated by it. We then introduce the multi-valued functions $\bar{u}_i \ominus \boldsymbol{\eta} \circ \bar{u}_i$ and the harmonic functions $\boldsymbol{\eta} \circ \bar{u}_i - \mathbf{p}^\perp_{\alpha_i}(\gamma)$, where we assume that γ is extended in the $V^\perp$ directions as a constant map.

As above, we fix $\delta > 0$ (whose choice will be specified later) and we appeal to Theorem 4.10.2 to find a threshold $\bar{r} = \bar{r}(Q, m, n, \bar{n}, \delta) > 0$ with the property that, for every $\rho < \bar{r}$ we can find a 1-homogeneous map $h_{i,\rho} \in \mathscr{L}_1$ (again, this map lies in $\mathscr{L}_1$ due to (4.313), which is applicable due to Proposition 4.11.8(c)) with the property that

$$\int_{\mathbf{B}_\rho \cap \alpha_i} \mathcal{G}(\bar{u}_i \ominus \boldsymbol{\eta} \circ \bar{u}_i, h_{i,\rho})^2 \leq \delta \rho^{m+2} .$$

Likewise we find a linear map ξ_i which vanishes on V and such that

$$|(\boldsymbol{\eta} \circ \bar{u}_i - \mathbf{p}^\perp_{\alpha_i}(\gamma))(z) - \xi_i(z)| \leq C|z|^2 ,$$

where the constant C depends again on $\varepsilon_c^\star$.

We are now ready to find the desired new cones. First of all we define

$$R_k := R[\mathbb{E}(T_k, \mathbf{S}_k, \mathbf{B}_1)^{1/2}]$$

and we consider the first adjustment to the cones as

$$\mathbf{S}'_k := R_k(\mathbf{S}_k)\,.$$

Next recall that along the sequence we are fixing α_1^k to be always the same plane (by applying a suitable rotation), and we are assuming that α_i^k converges to α_i. Moreover, for each k and $i \neq 1$, we fix a rotation $O_{k,i}$ which maps the approximating planes α_i^k onto α_i (where we use Lemma 4.5.4 to determine the $O_{k,i}$). In particular, we now consider the maps $\mathbb{E}(T_k, \mathbf{S}_k, \mathbf{B}_1)^{1/2}(h_{i,\rho} \oplus \xi_i)$ over α_i, compose them with $O_{k,i}^{-1} \circ R_k^{-1}$ on the right and with $R_k \circ O_{k,i}$ on the left.

We thus find the following multi-valued maps over $(\alpha_i^k)' = R_k(\alpha_i^k)$, namely

$$R_k \circ O_{k,i} \circ (\mathbb{E}(T_k, \mathbf{S}_k, \mathbf{B}_1)^{1/2}(h_{i,\rho} \oplus \xi_i)) \circ O_{k,i}^{-1} \circ R_k^{-1}\,.$$

The graphs of these maps give the new cone $\mathbf{S}''_k$, which are easily seen to belong to $\mathscr{C}(Q, 0)$. We now arrive at the same conclusion of the argument for Proposition 4.11.1, namely, under the assumption that $\rho = r_{nc} \leq \bar{r}$,

$$\lim_{k\to\infty} \frac{\mathbb{E}(T_k, \mathbf{S}''_k, \mathbf{B}_{r_{nc}})}{\mathbb{E}(T_k, \mathbf{S}_k, \mathbf{B}_1)} \leq C\delta + Cr_{nc}^2\,.$$

The only difference is that in this case the constants C depend upon $\varepsilon_c^\star$ as well. However, since the $\varepsilon_c^\star$ and ς_1 are both fixed, the constants above are also fixed, and we just choose δ so that $C\delta \leq \frac{\varsigma_1}{2}$. This in turn fixes $\bar{r}$, and we can further choose $r_{nc} \leq \bar{r}$ so that $Cr_{nc}^2 \leq \frac{\varsigma_1}{4}$. As in the proof of Proposition 4.11.1 we conclude that

$$\lim_{k\to\infty} \frac{\mathbb{E}(T_k, \mathbf{S}''_k, \mathbf{B}_{r_{nc}})}{\mathbb{E}(T_k, \mathbf{S}_k, \mathbf{B}_1)} \leq \frac{3\varsigma_1}{4}\,.$$

Since r_{nc} is now fixed, this now contradicts the assumption that there was no decay by a

factor ς_1 at that radius for any of the currents T_k. Thus the proof of Proposition 4.11.2 is complete. □

To summarize, we have now established Theorem 4.2.4.

4.12 Uniqueness of Tangent Cones and Rectifiability

4.12.1 Rectifiability

In this section we will use Theorem 4.2.4 to prove Theorem 4.1.2(i), namely that the set $\mathfrak{F}_{Q,1}$ has $\mathcal{H}^{m-2}$-measure zero. Before proceeding with the proof that Theorem 4.2.4 implies Theorem 4.1.2(i), we begin with some preliminaries.

First of all, we would like to show that the cones in the class $\mathscr{C}(Q,0) \setminus \mathscr{P}(0)$ are the "prevalent" fine blow-ups appearing in the compactness procedure in Chapter 3. This can be thought of as the analogue of [45, Lemma 2.4] and [31, Lemma 4.3] for the present setting, and is the key (and, in fact, the only) ingredient needed from the analysis in Chapter 3.

Lemma 4.12.1. *For each $\varepsilon \in (0,1]$, the following holds. Suppose that T and Σ are as in Assumption 3.2.2. Then, for each $p \in \mathfrak{F}_{Q,1}(T) \cap \mathbf{B}_1$ there exists $\bar\rho = \bar\rho(p,\varepsilon) > 0$ such that the following dichotomy holds for each $\rho \in (0,\bar\rho]$:*

(a) There exists $\mathbf{S} \in \mathscr{C}(Q,p) \setminus \mathscr{P}(p)$ with

$$(\rho\mathbf{A})^2 + \mathbb{E}(T,\mathbf{S},\mathbf{B}_\rho(p)) \leq \varepsilon^2 \mathbf{E}^p(T,\mathbf{B}_\rho(p));$$

(b) There exists an $(m-3)$-dimensional affine subspace $V \subset T_p\Sigma$ (depending on ρ) such that

$$\mathfrak{F}_{Q,1} \cap \mathbf{B}_\rho(p) \subset \{q : \mathrm{dist}(q,V) < \varepsilon\rho\}.$$

Remark 4.12.2. The subspace V in alternative (b) of Lemma 4.12.1 arises as either

- the spine of a non-flat area-minimizing cone;
- the set of multiplicity Q points of a 1-homogeneous Q-valued Dir-minimizer arising as a coarse blow-up at p (cf. Proposition 4.2.1).

Proof of Lemma 4.12.1. We argue by contradiction. Namely, suppose that there exists some $\varepsilon \in (0,1]$ for which the following holds. There exist T, Σ, a point $p \in \mathfrak{F}_{Q,1}(T) \cap \mathbf{B}_1$, and a corresponding sequence of scales $\bar\rho_k \downarrow 0$ such that both (a) and (b) fail on balls $\mathbf{B}_{\bar\rho_k}(p)$, for this choice of ε.

Up to extracting a further subsequence, we have two possibilities. Either $T_{p,\bar\rho_k}$ converges to a flat tangent cone in $\mathbf{B}_{6\sqrt{m}}$, namely $Q\llbracket\pi\rrbracket$ for some m-dimensional plane π, or it converges to a non-flat tangent cone T_∞. In the latter case we let V be the spine of T_∞, and note that either V is $(m-2)$-dimensional, and hence its support is a cone $\mathbf{S} \in \mathscr{C}(Q,0) \setminus \mathscr{P}(0)$, or the dimension of V is at most $m-3$. On the other hand, in the latter case $\{\Theta(T_k,\cdot) \geq Q\} \cap \overline{\mathbf{B}}_1$ converges in the sense of Hausdorff to a subset of $\{\Theta(T_\infty,\cdot) \geq Q\}$ (by upper semi-continuity of the density) and the latter set is in fact the spine V. Hence, in the latter case, alternative (b) of the lemma holds for all sufficiently large k, giving a contradiction. In the former case clearly alternative (a) holds for all sufficiently large k, again giving the desired contradiction (note that since $\lim_{k\to\infty} \mathbf{E}^p(T_{p,\bar\rho_k}, \mathbf{B}_{6\sqrt{m}}) > 0$ in this case, we clearly have $(\bar\rho_k \mathbf{A})^2 \leq \varepsilon^2 \mathbf{E}^p(T_{p,\bar\rho_k}, \mathbf{B}_{6\sqrt{m}})$ for all k sufficiently large).

So we are left with contradicting the case where $T_{p,\bar\rho_k}$ converges to a flat tangent cone; in this situation $\mathbf{E}^p(T, \mathbf{B}_{6\sqrt{m}\bar\rho_k}(p)) \downarrow 0$. Proposition 4.2.1 then tells us that

$$\mathbf{E}^p(T, \mathbf{B}_{6\sqrt{m}\bar\rho_k}(p))^{-1} \cdot \rho_k^2 \mathbf{A}_k^2 \to 0. \tag{4.372}$$

Now let π_k be a plane such that $\mathbf{E}^p(T, \mathbf{B}_{6\sqrt{m}\bar\rho_k}(p)) = \hat{\mathbf{E}}(T, \pi_k, \mathbf{B}_{6\sqrt{m}\bar\rho_k}(p))$ and, without loss of generality, we can rotate to assume that π_k is a fixed plane π for all k. We now consider a coarse blow-up f as defined in Section 3.3; we know that f is non-trivial and 1-homogeneous by Proposition 4.2.1. We then have two possibilities: either f is translation invariant in $m-2$

independent directions, or its spine has dimension at most $m-3$. In the former case, notice that in combination with the fact that $I_{0,f}(0) = 1$ (as f is 1-homogeneous), the support of the (multi-valued) graph of f is a superposition of planes $\mathbf{S} \in \mathscr{C}(Q,0) \setminus \mathscr{P}(0)$. In light of the estimates in [19, Theorem 2.4], combined with (4.372) and the strong L^2-convergence of the normalizations of the Lipschitz approximations of $T_{p,\bar\rho_k} \llcorner \mathbf{B}_{6\sqrt{m}}$ to f, we contradict (a).

Otherwise, consider the set Z of points z which are limits of $\mathbf{p}_\pi(p_k)$ with $\Theta(T_{p,\bar\rho_k}, p_k) = Q$ and $|p_k| \leq 1$. By [19, Theorem 2.7] $f(z) = Q\llbracket \boldsymbol{\eta} \circ f(z)\rrbracket$ for any such point z (this also follows by the Hardt–Simon inequality). Moreover, since by Proposition 4.2.1 we know that $\boldsymbol{\eta} \circ f \equiv 0$, we in fact have $f(z) = Q\llbracket 0\rrbracket$ at such z. By the 1-homogeneity of f and the upper semi-continuity of the frequency function, we know that $I_{z,f}(0) \leq 1$. On the other hand by the Hardt-Simon inequality, as in Section 3.3, $I_{z,f}(0) \geq 1$. Therefore, $I_{z,f}(0) = 1$ and hence f is translation invariant along any $z \in Z$. In particular $Z \subset V$, which, being at most $(m-3)$-dimensional, would imply that (b) would hold for $\rho = \bar\rho_k$ when k is large enough. This contradiction therefore proves the result. □

It will become convenient to subdivide $\mathfrak{F}_{Q,1}(T) \cap \mathbf{B}_1$ as follows.

Definition 4.12.3. *Fix $\varepsilon^\dagger > 0$. Suppose that T, Σ are as in Assumption 3.2.2. For $r > 0$, we let $\mathfrak{F}_{Q,1,r}(T)$ denote the set of all points $p \in \mathfrak{F}_{Q,1}(T) \cap \mathbf{B}_1$ for which the conclusions of Lemma 4.12.1 hold for all scales $\rho \in (0,r]$ with $\varepsilon^\dagger$ in place of ε, and moreover $\|T\|(\mathbf{B}_\rho(p)) \leq (Q + \frac{1}{4})\omega_m \rho^m$ for all $\rho \in (0,r]$.*

Notice that by Lemma 4.12.1 we may write

$$\mathfrak{F}_{Q,1}(T) \cap \mathbf{B}_1 = \bigcup_k \mathfrak{F}_{Q,1,2^{-k}}(T) \cap \mathbf{B}_1\,. \tag{4.373}$$

The fact that $\iota_{0,r}(\mathfrak{F}_{Q,1,r}(T)) \subset \mathfrak{F}_{Q,1,1}(T_{0,r})$ for any $r > 0$, combined with a translation, together imply that it suffices to prove Theorem 4.1.2(i) for $\mathfrak{F}_{Q,1,1}(T)$.

We will now proceed to show that a set satisfying a number of properties at every scale around every point is, up to a $\mathcal{H}^k$-negligible set, a countable union of k-dimensional $C^{1,\alpha}$

graphs. This is only one possible abstract formulation of what the arguments used by Simon in the proof of [45, Theorem 1] imply for general sets.

Assumption 4.12.4. $k \in \mathbb{N}$ and $\delta, \alpha, \varepsilon > 0$ are fixed. E is a Borel subset of $\mathbf{B}_2 \subset \mathbb{R}^{m+n}$ such that for every $p \in E \cap \mathbf{B}_1$ and every $r \in (0, 1 - |q|]$ there is a choice of a k-dimensional affine subspace $V(p, r)$ satisfying the following properties:

(1) $E \cap \mathbf{B}_r(p) \subset B_{\varepsilon r}(V(p, r))$.

(2) if $0 \leq a < b \leq 1 - |p|$ are radii for which the condition

$$V(p, r) \cap \mathbf{B}_{r/2} \subset B_{\delta r}(E) \tag{4.374}$$

holds for every $r \in (a, b]$, then the following estimate holds for every $a < s < r \leq b$:

$$|\mathbf{p}_{V(p,r)} - \mathbf{p}_{V(p,s)}| \leq \varepsilon \left(\frac{r}{b}\right)^{\alpha}. \tag{4.375}$$

We now state the result concerning how a set satisfying Assumption 4.12.4 decomposes into a union of a $\mathcal{H}^k$-negligible set and a countable union of k-dimensional $C^{1,\alpha}$ submanifolds:

Theorem 4.12.5. *Let $k \in \mathbb{N}$ and $\alpha, \delta > 0$ be fixed numbers. There exists $\varepsilon_s = \varepsilon_s(k, m, n, \delta) > 0$ such that, if E satisfies Assumption 4.12.4 with $\varepsilon \leq \varepsilon_s$, then $E \cap \mathbf{B}_{1/2}$ can be decomposed as a disjoint union $\tilde{E} \cup R$ where*

(i) $\mathcal{H}^k(\tilde{E}) = 0$;

(ii) $\tilde{E}$ consists of all points $p \in E \cap \mathbf{B}_{1/2}$ for which there is a sequence $\rho_k \downarrow 0$ violating condition (4.374);

(iii) R is contained in a countable union of $C^{1,\alpha}$ graphs (each defined on an open cube in a k-dimensional affine subspace of $\mathbb{R}^{m+n}$).

We omit the proof of Theorem 4.12.5, since it is a simple consequence of the arguments given in [45]. We next demonstrate how Theorem 4.1.2(i) follows from it. In order to do this, we require the following lemma (in place of [45, Lemma 1]), which is a simple consequence of Theorem 4.2.4. It verifies that not only does the ratio of double-sided excess and planar excess under the assumptions of Theorem 4.2.4, but in fact so does the maximum of this ratio and the ratio between $r^2\mathbf{A}^2$ and the planar excess.

Lemma 4.12.6. *There are positive constants $\varepsilon_{\mathrm{f}} = \varepsilon_{\mathrm{f}}(Q, m, n, \bar{n})$, $\theta = \theta(Q, m, n, \bar{n}) \le \frac{1}{2}$, and $C = C(Q, m, n, \bar{n})$ with the following property. Let T and Σ be as in Assumption 3.2.2 and $p \in \mathfrak{F}_{Q,1,1}(T)$. Assume $0 < r \le 2 - |p|$ and*

- *$\mathbf{S} \in \mathscr{C}(Q,p) \setminus \mathscr{P}(p)$ satisfies $\mathbb{E}(T, \mathbf{S}, \mathbf{B}_r(p)) = \inf\{\mathbb{E}(T, \bar{\mathbf{S}}, \mathbf{B}_r(p)) : \bar{\mathbf{S}} \in \mathscr{C}(Q,p)\}$;*
- *$\max\{\varepsilon_{\mathrm{f}}^{-2}\mathbf{A}^2 r^2, \mathbb{E}(T, \mathbf{S}, \mathbf{B}_r(p))\} \le \varepsilon_{\mathrm{f}}^2 \mathbf{E}^p(T, \mathbf{B}_r(p))$;*
- *$\mathbf{B}_{\varepsilon_{\mathrm{f}} r}(q) \cap \mathfrak{F}_{Q,1,1}(T) \neq \emptyset$ for every $q \in V(\mathbf{S}) \cap \mathbf{B}_{r/4}(p)$.*

Then there is a cone $\mathbf{S}' \in \mathscr{C}(Q,p) \setminus \mathscr{P}(p)$ such that

$$\frac{\max\{\varepsilon_{\mathrm{f}}^{-2}\mathbf{A}^2\theta^2 r^2, \mathbb{E}(T, \mathbf{S}', \mathbf{B}_{\theta r}(p))\}}{\mathbf{E}^p(T, \mathbf{B}_{\theta r}(p))} \le \frac{1}{4}\frac{\max\{\varepsilon_{\mathrm{f}}^{-2}\mathbf{A}^2 r^2, \mathbb{E}(T, \mathbf{S}, \mathbf{B}_r(p))\}}{\mathbf{E}^p(T, \mathbf{B}_r(p))}\,, \tag{4.376}$$

and

$$\operatorname{dist}^2(V(\mathbf{S}) \cap \mathbf{B}_r(p), V(\mathbf{S}') \cap \mathbf{B}_r(p)) \le C\frac{\max\{\varepsilon_{\mathrm{f}}^{-2}\mathbf{A}^2 r^2, \mathbb{E}(T, \mathbf{S}, \mathbf{B}_r(p))\}}{\mathbf{E}^p(T, \mathbf{B}_r(p))}\,. \tag{4.377}$$

Proof. Let $\varsigma = \frac{1}{8}$ in Theorem 4.2.4, and let $\varepsilon_{\mathrm{f}} \coloneqq \varepsilon_0$ and $\theta \coloneqq r_0$ be as in Theorem 4.2.4 for this choice of ς. The decay (4.376) is obvious by Theorem 4.2.4 if $\varepsilon_{\mathrm{f}}^{-2}\mathbf{A}^2 r^2 \le \mathbb{E}(T, \mathbf{S}, \mathbf{B}_r)$. In the other case we apply the continuity argument in Step 1 of Lemma 4.8.6 to find a cone $\mathbf{S}_e$ such that $\varepsilon_{\mathrm{f}}^{-2}\mathbf{A} r^2 \le \mathbb{E}(T, \mathbf{S}_e, \mathbf{B}_r) \le 2\varepsilon_{\mathrm{f}}^{-2}\mathbf{A}^2 r^2$ and obeying the conditions therein, at which point we may once again apply Theorem 4.2.4, now with $\varsigma = \frac{1}{16}$, and again choose $\varepsilon_{\mathrm{f}} \coloneqq \varepsilon_0$ and $\theta \coloneqq r_0$ to be as in Theorem 4.2.4 for this choice of ς. The conclusion (4.377) is an immediate consequence of Theorem 4.2.4(d), (4.376), and the reasoning above. □

We will also require the following lemma, which gives control on the tilting between different k-dimensional subspaces that a given set E satisfying Assumption 4.12.4 is bilaterally close to. It is analogous to [7, Lemma 5.13]. We thus refer the reader to the proof therein, and do not include it here.

Lemma 4.12.7. *Let $k \in \mathbb{N}$ and $\alpha, \delta > 0$ be fixed numbers. There are positive constants $C_0 = C_0(m,n,k)$, $\delta_0 = \delta_0(m,n,k)$, and $\varepsilon_0 = \varepsilon_0(m,n,k)$ such that the following holds. Assume $E \subset \mathbb{R}^{m+n}$, $p \in E$, and $r > 0$ are radii such that Assumption 4.12.4(1) and (4.374) hold for some $V = V(p,r)$ and $\varepsilon \leq \varepsilon_0$ and $\delta \leq \delta_0$. Assume moreover that V' is another k-dimensional affine subspace for which*

$$E \cap \mathbf{B}_r(p) \subset B_{\varepsilon r}(V')\,.$$

Then

$$|\mathbf{p}_V - \mathbf{p}_{V'}| \leq C_0 \varepsilon \tag{4.378}$$

and, if we set $\delta' := 2\delta + C_0\varepsilon$,

$$V' \cap \mathbf{B}_{r/2}(p) \subset B_{\delta' r}(E)\,. \tag{4.379}$$

We are now in a position to conclude the proof of the fact that $\mathfrak{F}_{Q,1}(T)$ is $\mathcal{H}^{m-2}$-null.

Proof of Theorem 4.1.2(i). Let $E = \mathfrak{F}_{Q,1,1}(T)$, let $k = m-2$, and in Definition 4.12.3 and fix $\varepsilon = \min\{\varepsilon_s, \varepsilon_{\mathrm{f}}\}$, where ε_{f} is the constant of Lemma 4.12.6 and ε_s is the constant of Theorem 4.12.5 Choose $\varepsilon^\dagger = \varepsilon$. We will proceed to verify that Assumption 4.12.4 holds for this choice of E and k, and with a choice of $\delta = \delta(\varepsilon)$ sufficiently small. First of all, notice that Lemma 4.12.1 implies that property (1) of Assumption 4.12.4 holds for each $p \in E \cap \mathbf{B}_1$, with $V(p,r)$ defined to be the spine of $\mathbf{S}$ (defined therein, which implicitly depends on p and r) if alternative (a) of Lemma 4.12.1 holds in $\mathbf{B}_r(p)$, and any $(m-2)$-dimensional affine subspace containing V of alternative (b) of Lemma 4.12.1 if that holds in $\mathbf{B}_r(p)$ instead.

To see that (2) of Assumption 4.12.4 holds, we proceed as follows. Suppose that (4.374) holds for a given point $p \in E \cap \mathbf{B}_1$ and for all scales $r \in (a, b]$, for a given pair of radii $a < b$ as in (2). Then for all such scales r, the alternative (a) of Lemma 4.12.1 must hold in $\mathbf{B}_r(p)$, and in addition, Lemma 4.12.7 tells us that provided that δ is chosen sufficiently small (depending on ε_{f}), all of the hypotheses of Lemma 4.12.6 hold with the ball $\mathbf{B}_{r/2}(p)$ replacing $\mathbf{B}_r(p)$. Now fix any such r and find $j \in \mathbb{N}$ such that $r \in (\theta^j b, \theta^{j-1} b]$. Applying Lemma 4.12.6 j times successively, starting from $\mathbf{B}_b(p)$, yields

$$\operatorname{dist}(V(\mathbf{S}_{j-1}) \cap \mathbf{B}_b(p), V(\mathbf{S}_j) \cap \mathbf{B}_b(p)) \leq C 2^{-j} \varepsilon_{\mathrm{f}} \leq \varepsilon \left(\frac{r}{b}\right)^{\alpha},$$

for an appropriate choice of $\alpha = \alpha(Q, m, n, \bar{n}, \theta)$, where $\mathbf{S}_j, \mathbf{S}_{j-1}$ are the cones $\mathbf{S}, \mathbf{S}'$ in Lemma 4.12.6 when r is replaced by $\theta^{j-1} r$. This verifies that property (2) of Assumption 4.12.4 holds for this choice of E, and thus allows us to apply Theorem 4.12.5 with the above choice of parameters. As observed above for each point $p \in R$ all of the hypotheses of Lemma 4.12.6 are satisfied in $\mathbf{B}_r(p)$ for each sufficiently small scale, so one may iteratively apply this lemma to deduce that any tangent cone at each such point p will be supported in a unique element $\mathbf{S} \in \mathscr{C}(Q, p) \setminus \mathscr{P}(p)$. This, however, is in contradiction with the fact that $p \in \mathfrak{F}_{Q,1,1}(T)$. Therefore, $R = \emptyset$ here. Combined with (4.373), this completes the proof. □

4.12.2 Uniqueness of tangent cones

Having proved the rectifiability, we now turn to the conclusion of Theorem 4.1.2, namely the $\mathcal{H}^{m-2}$-a.e. uniqueness of tangent cones. Recalling Theorem 3.2.10, we know that the tangent cone is unique at every flat singular point in $\mathfrak{F}_Q(T) \setminus \mathfrak{F}_{Q,1}(T)$, and given the previous section, it is therefore unique at $\mathcal{H}^{m-2}$-a.e. point in $\mathfrak{F}_Q(T)$. Since Q is arbitrary, it remains to show that the tangent cone is unique at $\mathcal{H}^{m-2}$-a.e. point $p \in \mathcal{S}^{(m-2)}$. Although countable $(m-2)$-rectifiability of $\mathcal{S}^{(m-2)} \setminus \mathcal{S}^{(m-3)}$ follows from [43], we will achieve this independently as an additional consequence of the arguments in this section, together with the $\mathcal{H}^{m-2}$-a.e.

uniqueness of tangent cones, following an argument analogous to that in the previous section. First of all recall that every point $p \in \mathcal{S}^{(m-2)} \setminus \mathcal{S}^{(m-3)}$ has integer density and that $\mathcal{S}^{(m-3)}$ has Hausdorff dimension at most $m-3$. Then we have the following counterpart of Lemma 4.12.1 above.

Lemma 4.12.8. *For each $\varepsilon \in (0,1]$, the following holds. Suppose that T and Σ are as in Assumption 3.2.2. For each $p \in \mathcal{S}^{(m-2)} \setminus \mathcal{S}^{(m-3)} \cap \mathbf{B}_1$ with $\Theta(T,p) = Q$, there exists $\bar\rho = \bar\rho(p,\varepsilon) > 0$ such that the following dichotomy holds for each $\rho \in (0,\bar\rho]$:*

(a) There exists $\mathbf{S} \in \mathscr{C}(Q,p) \setminus \mathscr{P}(p)$ and

$$(\rho\mathbf{A})^2 + \mathbb{E}(T,\mathbf{S},\mathbf{B}_\rho(p)) \leq \varepsilon^2 \mathbf{E}^p(T,\mathbf{B}_\rho(p));$$

(b) There exists an $(m-3)$-dimensional affine subspace $V \subset T_p\Sigma$ (depending on ρ) such that

$$\{\Theta(T,\cdot) \geq Q\} \cap \bar{\mathbf{B}}_\rho(p) \subset \{q : \operatorname{dist}(q,V) < \varepsilon\rho\}.$$

Proof. The proof is by contradiction and is much simpler than the one of Lemma 4.12.1 since in this case

$$\liminf_{\rho\downarrow 0} \mathbf{E}^p(T,\mathbf{B}_\rho(p)) > 0\,.$$

Therefore, any tangent cone to T at p either has an $(m-2)$-dimensional spine, and hence its support is an element of $\mathscr{C}(Q,p) \setminus \mathscr{P}(p)$, or it has an $(m-3)$-dimensional spine, leading to the desired contradiction in either case. □

We now subdivide $\mathcal{S}^{(m-2)} \setminus \mathcal{S}^{(m-3)}$ into countably many pieces (analogously to the subdivision of $\mathfrak{F}_{Q,1}(T)$ carried out in Definition 4.12.3) in the following way:

Definition 4.12.9. *Let $\varepsilon^\dagger > 0$ and $\delta > 0$ be fixed. Suppose that T, Σ are as in Assumption 3.2.2. For $r > 0$ we let $\mathcal{S}_{Q,\delta,r}(T)$ be the set of points $q \in (\mathcal{S}^{(m-2)} \setminus \mathcal{S}^{(m-3)}) \cap \mathbf{B}_1$ of density $\Theta(q,T) = Q$ such that*

- *The dichotomy of Lemma 4.12.8 applies at every scale* $\rho \in (0, r]$ *with* $\varepsilon = \varepsilon^\dagger$*;*
- $\mathbf{E}^p(T, \mathbf{B}_\rho) \geq \delta$ *for every* $\rho \in (0, r]$*.*

First observe that we have the decomposition

$$\mathcal{S}^{(m-2)} \setminus \mathcal{S}^{(m-3)} = \bigcup_{Q,j,\ell} \mathcal{S}_{Q,1/j,1/\ell}\,.$$

Thus, in order to conclude the $(m-2)$-rectifiability and $\mathcal{H}^{m-2}$-a.e. uniqueness of tangent cones, it suffices to prove that each piece $\mathcal{S}_{Q,1/j,1/\ell}$ in the above decomposition is rectifiable and the tangent cone is unique at $\mathcal{H}^{m-2}$-a.e. point $q \in \mathcal{S}_{Q,1/j,1/\ell}$. By scaling we assume $\ell = 1$, and we may without loss of generality further assume that $j = 1$, in order to simplify notation.

The proof of the $(m-2)$-rectifiability of $\mathcal{S}^{(m-2)} \setminus \mathcal{S}^{(m-3)}$ and uniqueness of tangent cones at $\mathcal{H}^{m-2}$-a.e. point $p \in \mathcal{S}^{(m-2)}$ will once again be concluded from Theorem 4.12.5. In order to do this, we first require the following analogue of Lemma 4.12.6, which is proved in exactly the same way.

Lemma 4.12.10. *There are positive constants* $\varepsilon_{\mathrm{nf}} = \varepsilon_{\mathrm{nf}}(Q, m, n, \bar{n})$*,* $\theta = \theta(Q, m, n, \bar{n}) \leq \frac{1}{2}$ *with the following property. Let* T *and* Σ *be as in Assumption 3.2.2 and let* $p \in \mathcal{S}_{Q,1,1}(T)$*. Assume* $0 < r \leq 2 - |p|$ *and*

- $\mathbf{S} \in \mathscr{C}(Q, p) \setminus \mathscr{P}(p)$ *satisfies* $\mathbb{E}(T, \mathbf{S}, \mathbf{B}_r(p)) = \inf\{\mathbb{E}(T, \bar{\mathbf{S}}, \mathbf{B}_r(p)) : \bar{\mathbf{S}} \in \mathscr{C}(Q, p)\}$*;*
- $\max\{\varepsilon_{\mathrm{nf}}^{-2} r^2 \mathbf{A}^2, \mathbb{E}(T, \mathbf{S}, \mathbf{B}_r(p))\} \leq \varepsilon_{\mathrm{nf}}^2 \mathbf{E}^p(T, \mathbf{B}_r(p)))$*;*
- $\mathbf{B}_\varepsilon(q) \cap \mathcal{S}_{Q,1,1}(T) \neq \emptyset$ *for every* $q \in V(\mathbf{S}) \cap \mathbf{B}_{r/4}(p)$*.*

Then there is a $\mathbf{S}' \in \mathscr{C}(Q, 0)$ *such that*

$$\frac{\max\{\varepsilon_{\mathrm{nf}}^{-2} \theta^2 r^2 \mathbf{A}^2, \mathbb{E}(T, \mathbf{S}', \mathbf{B}_{\theta r})\}}{\mathbf{E}^p(T, \mathbf{B}_{\theta r})} \leq \frac{1}{4} \frac{\max\{\varepsilon_{\mathrm{nf}}^{-2} r^2 \mathbf{A}^2, \mathbb{E}(T, \mathbf{S}, \mathbf{B}_r)\}}{\mathbf{E}^p(T, \mathbf{B}_r)}\,,$$

and

$$\operatorname{dist}^2(V(\mathbf{S}) \cap \mathbf{B}_r(p), V(\mathbf{S}') \cap \mathbf{B}_r(p)) \leq C \frac{\max\{\varepsilon_{\mathrm{f}}^{-2} \mathbf{A}^2 r^2, \mathbb{E}(T, \mathbf{S}, \mathbf{B}_r(p))}{\mathbf{E}^p(T, \mathbf{B}_r(p))} .$$

Proof of Theorem 4.1.2(ii) and rectifiability of $\mathcal{S}^{(m-2)} \setminus \mathcal{S}^{(m-3)}$. Let $E = \mathcal{S}_{Q,1,1}(T)$, let $k = m-2$, and in Definition 4.12.3 and fix $\varepsilon = \min\{\varepsilon_s, \varepsilon_{\mathrm{nf}}\}$, where $\varepsilon_{\mathrm{nf}}$ is the constant in Lemma 4.12.10 and ε_s is the constant in Theorem 4.12.5. Choose $\varepsilon^\dagger = \varepsilon$. Arguing exactly as in the proof of Theorem 4.1.2(i), with Lemma 4.12.8 and Lemma 4.12.10 applied in place of Lemma 4.12.1 and 4.12.6 respectively, we verify that Assumption 4.12.4 holds for this choice of E and k, and with a choice of $\delta = \delta(\varepsilon)$ sufficiently small. We may thus apply Theorem 4.12.5 with the above choice of parameters. At each point $p \in R$, all of the hypotheses of Lemma 4.12.10 are satisfied in $\mathbf{B}_r(p)$ for each sufficiently small scale $r > 0$. This in turn implies that the tangent cone is supported in a unique element $\mathbf{S} \in \mathscr{C}(Q, p) \setminus \mathscr{P}(p)$ at each such point, and the supports of the rescalings $T_{p,r}$ decay towards $\mathbf{S}$ with a power law for all r sufficiently small. But, because of this decay and the graphical approximation in Proposition 4.6.14 applied to $T_{p,r}$ and this unique $\mathbf{S}$, we can select r sufficiently small to see that the origin is in fact in the closure of the outer region. Conclusion (i) of Proposition 4.6.14 ensures that the multiplicities assigned to the planes in $\mathbf{S} \cap \mathbf{B}_\rho(0)$ with $\rho \leq r$ do not change with the radius. Thus, not only is the support of the tangent cone unique, but so is the tangent cone itself. Combined with (4.373), this completes the proof. □

Chapter 5

Rectifiability of flat singular points for area-minimizing mod$(2Q)$ hypercurrents

5.1 Introduction and main results

Throughout this chapter, we will make the following underlying assumption:

Assumption 5.1.1. $T \in \mathscr{R}_m(\mathbb{R}^{m+n})$ is an m-dimensional representative mod(p) in a C^{3,α_0} $(m+\bar{n})$-dimensional Riemannian submanifold $\Sigma \subset \mathbb{R}^{m+n}$ with $\alpha_0 \in (0,1)$. T is area-minimizing mod(p) in $\Sigma \cap \mathbf{B}_{7\sqrt{m}}$ for some open set $\mathbf{B}_{7\sqrt{m}} \subset \mathbb{R}^{m+n}$ containing 0 and $\partial^p[T] \llcorner \mathbf{B}_{7\sqrt{m}} = 0$ mod(p). Note that $\Theta(T,x) \in \left[1, \frac{p}{2}\right]$ for $\|T\|$-almost every x.

We may assume that $\Sigma \cap \mathbf{B}_{7\sqrt{m}}$ is the graph of a C^{3,α_0} function $\Psi_p : \mathrm{T}_p\Sigma \cap \mathbf{B}_{7\sqrt{m}} \to \mathrm{T}_p\Sigma^\perp$ for every $p \in \Sigma \cap \mathbf{B}_{7\sqrt{m}}$. We may further assume that

$$\boldsymbol{c}(\Sigma, \mathbf{B}_{7\sqrt{m}}) \coloneqq \sup_{p \in \Sigma \cap \mathbf{B}_{7\sqrt{m}}} \|D\Psi_p\|_{C^{2,\alpha_0}} \leq \bar{\varepsilon},$$

where $\bar{\varepsilon}$ will be determined later. This in particular gives us the following uniform control

on the second fundamental form A_Σ of Σ:

$$\mathbf{A}_\Sigma \coloneqq \|A_\Sigma\|_{C^0(\Sigma\cap\mathbf{B}_{7\sqrt{m}})} \leq C_0\boldsymbol{c}(\Sigma,\mathbf{B}_{7\sqrt{m}}) \leq C_0\bar{\varepsilon}.$$

We adopt the same notation and terminology as that in Chapters 3 and 4. We recall that the set of flat singular points with density $Q \in \mathbb{N}$ is denoted by $\mathfrak{F}_Q(T)$. The main result of this chapter is as follows:

Theorem 5.1.2. *Let T satisfy Assumption 5.1.1 (c.f. [15, Assumptions 17.5]) with $\bar{n} = 1$ and $p = 2Q$ for some $Q \in \mathbb{N}$. If the parameters in [15, Assumption 17.11] are chosen appropriately, then $\mathfrak{F}_Q(T)$ is $(m-2)$-rectifiable.*

This, in particular, implies that when $\bar{n} = 1$ and $p = 2Q$, up to an $(m-2)$-rectifiable set, the singular set of T is locally a $C^{1,\alpha}$ $(m-1)$-dimensional submanifold.

For $k = 0, \ldots, m$, we define the k-th stratum $\mathcal{S}^{(k)}(T)$ analogously to that for integral currents, see (2.1). When combined with the results of [35,36,40,43], we immediately obtain the following decomposition of the interior singular set in the case $\bar{n} = 1$.

Corollary 5.1.3. *Let T satisfy Assumption 5.1.1 (c.f. [15, Assumptions 17.5]) with $\bar{n} = 1$ and $p = 2Q$ for some $Q \in \mathbb{N}$. If the parameters in [15, Assumption 17.11] are chosen appropriately, then the interior singular set $\mathrm{Sing}(T)$ decomposes as a disjoint union*

$$\mathrm{Sing}(T) = \mathfrak{F}_Q(T) \sqcup (\mathcal{S}^{(m-1)}(T) \setminus \mathcal{S}^{(m-2)}(T)) \sqcup \mathcal{S}^{(m-2)}(T),$$

where $\mathfrak{F}_Q(T)$ and $\mathcal{S}^{(m-2)}(T)$ are $(m-2)$-rectifiable and $\mathcal{S}^{(m-1)}(T) \setminus \mathcal{S}^{(m-2)}(T)$ is a countable union of $(m-1)$-dimensional embedded $C^{1,\alpha}$ submanifolds of Σ, for some $\alpha = \alpha(Q,m) > 0$.

5.1.1 Connection with stable minimal hypersurfaces

Any T satisfying Assumption 5.1.1 induces a stable integral varifold. Moreover, the existing known regularity theory for stable integral varifolds of codimension one ($\bar{n} = 1$) appears to be

consistent with the known regularity theory for area-minimizing mod(p) hypersurfaces; see the works [56] and [40]. In the more general framework of codimension one m-dimensional stable integral varifolds, the work [31] of Krummel & Wickramasekera demonstrates that locally in the regions where there are no points of density 3 or higher, the interior flat singular set is $(m-2)$-rectifiable. However, to the knowledge of the author, such a result is still open for higher multiplicities, in such a framework.

5.1.2 Adaptation of techniques in Chapter 3

The proof of Theorem 5.1.2 follows the same strategy as that of Theorem 3.11.3. Note that the *singularity degree* $\mathrm{I}(T,x)$ for $x \in \mathfrak{F}_Q(T)$ defined therein corresponds to the frequency $\mathbf{I}(x,0)$ for the local multi-valued Lipschitz approximation N of T around x herein, in light of the decay estimate (2) of Proposition 5.2.1. This is the key aspect of the argument which relies on the assumption that $\bar{n} = 1$. The excess decay of Proposition 5.2.1 in turn enables one to maintain the same center manifold domain $\mathcal{M}$ and approximation N in the compactness procedure used to define the singularity degree, therefore avoiding the need for the more intricate adaptation of the rectifiable Reifenberg arguments in [42] with varying graphical approximations, as done in the proof of Theorem 3.11.4. The classification of frequencies $\mathbf{I}(x,0)$ in [35, Proposition 2.8] (see Proposition 5.3.1) therefore guarantees that the set $\mathfrak{F}_Q(T)$ is contained in the set $\mathfrak{F}^h_Q$ of Chapter 3. On the other hand, here N and may have zeros $\{x : N(x) = Q\llbracket 0 \rrbracket\}$ with frequency $\mathbf{I}(x,0) = 1$ which form $(m-1)$-dimensional sets that disconnect the domain of N, unlike in the framework of area-minimizing integral currents.

For the majority of the intermediate results herein, one may get around this issue by decomposing the domain into disjoint connected regions, in each of which the graphical approximation N is in fact a classical Q-valued Lipschitz map (see e.g. [14]). However, for the compactness and unique continuation results in Section 5.5 that lead to quantitative splitting conclusions, more care must be taken, and one must exploit the structure of *special Q-valued Dir-minimizers*, which arise as solutions to the linearized problem in this setting.

5.2 Notation and preliminaries

We refer the reader to the list of notation in Section 1.4 for frequently used notation.

We are now in a position to introduce some key notions that will be pivotal in this article. In order to study the behaviour of T around flat singularities, one needs Almgren's celebrated center manifold construction (see [3, 21]) and the linear theory of *special Q-valued maps* in this setting. We refer the reader to [14, 15] for the relevant background theory and notation in the mod(p) framework. We recall the definition (4.10) of the *non-oriented excess* of T.

We define the mod(p) excess $\mathbf{E}(T, \mathbf{B}_r(x), \pi)$ of T in $\mathbf{B}_r(x)$ with respect to the m-dimensional plane π, as well as $\mathbf{E}(T, \mathbf{B}_r(x))$ and $\mathbf{E}(T, \mathbf{C}_r(x))$ in $\mathbf{C}_r(x) = B_r(x, \pi) \times \pi^\perp$, in the same way as the tilt excess for integral currents, see Definition 1.4.1.

Note that although $\|T\| = \|T\|_p$ since T is a representative mod(p), this is no longer necessarily the case for $\|\mathbf{p}_\sharp T\|$ and $\|\mathbf{p}_\sharp T\|_p$, since $\mathbf{p}_\sharp T$ need not be a representative mod(p).

5.2.1 Reduction to a single center manifold

Following the procedure in Section 3.2, we introduce appropriate disjoint intervals $]s_j, t_j] \subset]0, 1]$, called *intervals of flattening*, the union of which identifies those radii r such that the non-oriented excess $\mathbf{E}^{no}(T, \mathbf{B}_{6\sqrt{m}r})$ falls below a positive fixed threshold ε_3^2. Arguing as in [15, Section 25] for each rescaled current T_{0,t_j} and rescaled ambient manifold Σ_{0,t_j} we produce a center manifold $\mathcal{M}_j$ and an appropriate multivalued map $N_j : \mathcal{M}_j \to \mathscr{A}_Q(\mathbb{R}^{m+n})$. The latter takes values in the normal bundle of $\mathcal{M}$ and gives an efficient approximation of the current T_{0,t_j} in $\mathbf{B}_3 \setminus \mathbf{B}_{s_j/t_j}$. We recall the following (non-oriented) tilt excess decay from [35, 36]:

Proposition 5.2.1 ([35, Proposition 2.3])**.** *Let $\delta_2 > 0$ be fixed as in* [15, Assumption 17.10]. *There exists $\varepsilon_1 = \varepsilon_1(\delta_2, m, Q)$ and a positive constant $C = C(\delta_2, m, Q)$ such that the following holds. Suppose that T satisfies Assumption 5.1.1 with $\bar{n} = 1$ and $p = 2Q$. Assume that*

$q \in \mathfrak{F}_Q(T)$ and that

$$\mathbf{E}^{no}(T, \mathbf{B}_\rho(q), \pi_q) + (\rho \mathbf{A})^2 < \varepsilon_1, \qquad \mathbf{C}_{\rho/4}(q, \pi_q) \cap \mathrm{spt}_p(T) \subset \mathbf{B}_{\rho/2}(q). \tag{5.1}$$

Then for every $r \leq \frac{\rho}{32}$, we have

$$\mathbf{E}^{no}(T, \mathbf{B}_r(q)) \leq C \left(\frac{r}{\rho}\right)^{2-2\delta_2} \left(\mathbf{E}^{no}(T, \mathbf{B}_\rho(q), \pi_q) + (\rho \mathbf{A})^2\right). \tag{5.2}$$

Moreover, by an obvious adaptation of the proof of [35, Proposition 2.4], we have the following.

Proposition 5.2.2 (cf. [35, Proposition 2.4])**.** *For every $\bar{c}_s > 0$, there is a constant $\tau_0 = \tau_0(m, Q, \bar{c}_s) > 0$ with the following property. Let T satisfy Assumption 5.1.1 with $\bar{n} = 1$, $p = 2Q$ and let $\delta_2 > 0$ be fixed as in [15, Assumption 17.10]. Then there exists a choice of parameters in [15, Assumption 17.11] with this choice of δ_2, such that if the assumptions [15, Assumption 17.5] hold and (5.1) holds with $\rho = 6\sqrt{m}$, then*

(a) The decay (5.2) holds for every $r \leq \tau_0$;

(b) $\mathfrak{F}_Q(T) \cap \mathbf{B}_{\tau_0}$ is a subset of $\mathbf{\Phi}(\mathbf{\Gamma}) \subset \mathcal{M}_0$;

(c) for each $q = (x_q, y_q) \in \mathfrak{F}_Q(T) \cap \mathbf{B}_{\tau_0}$, where $x_q \in \pi_0$, $y_q \in \pi_0^\perp$, we have

$$L \in \mathscr{W} \qquad \Longrightarrow \qquad \ell(L) < \bar{c}_s \, \mathrm{dist}(x_q, L).$$

Indeed, notice that in the statement [35, Proposition 2.4], one may replace $\frac{1}{64\sqrt{m}}$ with a choice of $\bar{c}_s > 0$ arbitrarily small, at the price of allowing the scale τ_0 to additionally depend on $\bar{c}_s$. We do not include the details here, and refer the reader to the proof therein.

Now fix $\bar{c}_s > 0$ (to be determined in Theorem 5.2.3 below) and let us decompose $\mathfrak{F}_Q(T)$

into a countable union of (nested) sets as follows:

$$\mathbf{S}_j := \left\{ q \in \mathfrak{F}_Q(T) : \text{the assumptions of Proposition 5.2.2 hold with } \bar{c}_s \text{ for } T_{q,\frac{1}{6\sqrt{m}j}} \right\}.$$

Now given any point $q \in \mathbf{S}_j$, we may apply Proposition 5.2.2 with this choice of $\bar{c}_s$ to $T_{q,\frac{1}{6\sqrt{m}j}}$ to reduce Theorem 5.1.2 to the following theorem, the first three conclusions of which are an immediate consequence of the above discussion (without any constraint on ε_4 or η). Note that here we use a slightly different definition of $\boldsymbol{m}_0$ to that in [15], but the arguments therein remain unchanged under such a replacement (up to possibly decreasing the parameter ε_2 therein).

Theorem 5.2.3. *There exists $\varepsilon_4 = \varepsilon_4(Q,m) > 0$, $\eta = \eta(Q,m) > 0$ such that for some $\bar{c}_s = \bar{c}_s(m,\eta) > 0$, the following holds. Let T satisfy Assumption 5.1.1 with $\bar{n} = 1$, $p = 2Q$ and let $\delta_2 > 0$ be fixed as in [15, Assumption 17.10]. Then for any $j \in \mathbb{N}$ and any $q \in \mathbf{S}_j$, letting $r_0 := \frac{\tau_0}{6\sqrt{m}j}$ for τ_0 as in Proposition 5.2.2 and $\boldsymbol{m}_0 := \mathbf{E}^{no}(T_{q,r_0}, \mathbf{B}_{6\sqrt{m}}) + (6\sqrt{m}\mathbf{A})^2 < \varepsilon_4^2$, there exists a choice of parameters in [15, Assumption 17.11] with this choice of δ_2, such that the following properties hold:*

(i) T_{p,r_0} satisfies the assumptions of the relevant statements in [15] (in place of T), where the center manifold $\mathcal{M}_0$ is constructed using δ_2 and $\boldsymbol{m}_0$ as defined above;

(ii) the decay (5.2) holds for T_{q,r_0} for all scales $r \le 1$;

(iii) the rescaling $\iota_{q,r_0}(\mathbf{S}_j) \cap \overline{\mathbf{B}}_{6\sqrt{m}}$, and thus also its closure $\mathbf{S}$ (for which we omit dependency on j), is contained in $\mathcal{M}_0$;

(iv) for every $x \in \mathcal{M}_0 \cap \mathbf{B}_{6\sqrt{m}}$ and for every $r \in]\eta d(x,\mathbf{S}), 1]$ (where $d(x,\mathbf{S}) = \min\{d(x,y) : y \in \mathbf{S}\}$), every cube L which intersects $B_r(q,\pi_0)$ satisfies $\ell(L) \le c_s r$, where $c_s = \frac{1}{64\sqrt{m}}$ is as in [15, (25.5)].

(v) $\mathbf{S}$ has finite upper $(m-2)$-dimensional Minkowski content and it is $(m-2)$-rectifiable.

Proof of Theorem 5.2.3(iv). For any point $x \in \mathbf{S}$, the conclusion follows immediately from conclusion (c) of Proposition 5.2.2 with $\bar{c}_s = \eta$. It remains to verify the conclusion at points outside of $\mathbf{S}$. Taking $r \in]\eta d(x, \mathbf{S}), 1]$ and c_s as in the statement of the theorem, observe that any cube $L \in \mathscr{C}$ with $L \cap B_r(q, \pi_0) \neq \emptyset$ and $\ell(L) > c_s r > \bar{c}_s \eta d(x, \mathbf{S})$ would in turn satisfy $L \cap B_{d(x,\mathbf{S})+r}(\tilde{q}, \pi_0) \neq \emptyset$ for some $\mathbf{S} \ni \tilde{x} = \mathbf{p}_{\pi_0}(\tilde{q})$, contradicting conclusion (c) of Proposition 5.2.2 for the choice $\bar{c}_s = \frac{c_s}{1+\frac{1}{\eta}}$. □

The remainder of this article will thus be dedicated to proving the conclusion (v) of Theorem 5.2.3, from which the conclusion of Theorem 5.1.2 follows immediately by an elementary covering argument. The conclusion of 5.2.3(v) will be given in Section 5.7.1, at the very end of the article.

Translating by q and rescaling so that r_0 is taken to be unit scale, and henceforth denoting $\mathcal{M}_0$ by simply $\mathcal{M}$, we may thus make the following assumptions from now on.

Assumption 5.2.4. For some fixed (yet arbitrary) positive constants ε_4 and η, the following holds.

(i) T satisfies Assumption 5.1.1 with $\bar{n} = 1$, $p = 2Q$ and $0 \in \mathfrak{F}_Q(T)$.

(ii) There is one interval of flattening $]0, 1]$ around 0 with corresponding $\boldsymbol{m}_{0,0} \equiv \boldsymbol{m}_0 := \mathbf{E}^{no}(T, \mathbf{B}_{6\sqrt{m}}) + (6\sqrt{m}\mathbf{A})^2 \leq \varepsilon_4^2$.

(iii) If $\mathcal{M}$ is the corresponding center manifold with normal approximation N, then $\mathbf{S} := \mathfrak{F}_Q(T) \cap \mathbf{B}_1$ is contained in the contact set $\mathbf{\Phi}(\mathbf{\Gamma})$ of $\mathcal{M}$ and the excess decay of Proposition 5.2.1 holds at each $q \in \mathbf{S}$, for all scales $r \in]0, 1]$ and $Q[\![T_q\mathcal{M}]\!] \llcorner \mathbf{B}_1$ is the unique flat tangent cone of T at any such q.

(iv) For every $x \in \mathbf{B}_1 \cap \mathcal{M}$, the conclusion (iv) of Theorem 5.2.3 is valid for all radii $r \in]\eta\, d(x, \mathbf{S}), 1]$ (and hence for all radii $r \in]0, 1]$ when $x \in \mathbf{S}$).

5.3 Frequency function and radial variations

Let us begin by introducing the (regularized) frequency function for the $\mathcal{M}$-normal approximation N of T as in Assumption 5.2.4. Let $\phi : [0, \infty[\to \mathbb{R}$ be the cutoff function defined in (2.30).

Let $d : \mathcal{M} \times \mathcal{M} \to [0, \infty[$ denote the geodesic distance on $\mathcal{M}$. We recall that d satisfies the properties (i)-(iii) of Section 3.12.3, which are consequences of the $C^{3,\kappa}$-estimates on the center manifold $\mathcal{M}$. We additionally recall the quantities $\mathbf{D}_N(x,r)$, $\mathbf{H}_N(x,r)$ and $\mathbf{I}_N(x,r)$ as defined in Section 3.2.3. We will often omit the dependency on N of these quantities, since we are considering one single fixed center manifold $\mathcal{M}$ and associated normal approximation N throughout. We refer the reader to [22] or [11] for more details on the above quantities and their basic properties. Moreover, since in practically all the computations, the derivative of d is taken in the variable which is the same as the intergration variable, in all such cases we will write instead ∇d.

We will also need to use the analogous quantities $D_u(x,r)$, $H_u(x,r)$, $I_u(x,r)$ for Dir-minimizing maps $u : \mathbb{R}^m \supset \Omega \to \mathscr{A}_Q$, defined as in Section 3.2.1.

In addition, we define the scale invariant quantities

$$\bar{\mathbf{H}}(x,r) \coloneqq r^{-(m-1)}\mathbf{H}(x,r), \qquad \bar{\mathbf{D}}(x,r) \coloneqq r^{-(m-2)}\mathbf{D}(x,r)$$

and we recall the quantities $\mathbf{E}(x,r)$, $\mathbf{G}(x,r)$ and $\mathbf{\Sigma}(x,r)$ as defined in Section 3.12.3.

We will frequently be using the variational identities of Lemma 3.12.3, Corollary 3.6.5 and Lemma 3.13.1, all of which hold verbatim here.

Note that the proofs work in the same way in this context, with the exception that the limiting Dir-minimizer in the proof of the lower frequency bound of (3.105) takes values in $\mathscr{A}_Q$, not $\mathcal{A}_Q$.

Before we go any further, we need to address the following. As well as knowing that $\mathbf{S} \subset \mathbf{\Phi}(\mathbf{\Gamma})$, we will need to know that the frequency $\mathbf{I}(x,0)$ is sufficiently large (namely,

above the threshold $2-\delta_2$) for every $x \in \mathbf{S}$. Indeed, this is the case; by studying the linearized problem that arises from a compactness procedure and ruling out the possibility that $\mathbf{I}(x,0)=1$ (since x is a *flat* singular point of T), one can show that $\mathbf{I}(x,0)$ is always a positive integer strictly larger than 1 for each $x \in \mathbf{S}$.

Proposition 5.3.1 ([35, Proposition 2.8]). *Let T, $\mathcal{M}$, N and γ_4 be as in Lemma 3.12.3. For every $x \in \mathbf{S}$ and any sequence of scales $r_k \downarrow 0$, the following properties hold (up to subsequence):*

(i) $\bar{N}_{x,r_k} := \frac{N(\mathbf{e}(x,r_k\cdot))}{\bar{\mathbf{D}}(x,r_k)^{1/2}}$ converges strongly in $W^{1,2}_{\mathrm{loc}}$ to an $\mathbf{I}(x,0)$-homogeneous Dir-minimizer $u : \pi_\infty \supset B_1 \to \mathscr{A}_Q(\mathbb{R})$ with $\eta \circ u = 0$ and $u(0) = Q\llbracket 0 \rrbracket$;

(ii) the frequency $\mathbf{I}(x,0)$ is a positive integer strictly larger than 1.

However, the above proposition does not guarantee that there are no points $y \in \mathbf{\Phi}(\mathbf{\Gamma})$ with $\mathbf{I}(y,0)=1$ nearby to points $x \in \mathbf{S}$ (in fact, we expect any $x \in \mathbf{S}$ to be an accumulation of such points y). With this information in mind, we let

$$\Delta_Q N := \{x \in \mathcal{M} : N(x) = Q\llbracket 0 \rrbracket\}, \qquad \Delta_Q^{2-\delta} N := \{x \in \mathcal{M} : N(x) = Q\llbracket 0 \rrbracket,\ \mathbf{I}(x,0) \geq 2-\delta\}.$$

We will use the same notation for Dir-minimizers $u : \mathbb{R}^m \supset \Omega \to \mathscr{A}_Q(\mathbb{R}^n)$. Thus, Proposition 5.3.1 enables us to say that for T, $\mathcal{M}$, N as in Assumption 5.2.4, we have $\mathbf{S} \subset \Delta_Q^{2-\delta} N$.

5.4 Spatial frequency variations

Here, we control how much N deviates from being homogeneous on average locally around a point x between two scales, in terms of the radial frequency variation at x between those scales. We recall the following terminology from Chapter 3.

Definition 5.4.1. *Suppose that T, $\mathcal{M}$ and N are as in Assumption 5.2.4. For $x \in \mathbf{B}_1 \cap \mathcal{M}$*

and any $\eta\, d(x,\mathbf{S}) < \rho \leq r \leq 1$, *define the* frequency pinching $W^r_\rho(x)$ *between* ρ *and* r *by*

$$W^r_\rho(x) \coloneqq |\mathbf{I}(x,r) - \mathbf{I}(x,\rho)|.$$

Proposition 5.4.2. *Suppose that* T, $\mathcal{M}$, N *are as in Assumption 5.2.4, let* γ_4 *be as in Lemma 3.12.3 and let* $\Lambda > 0$ *be as in (3.105). There exists* $C = C(m,n,Q,\Lambda) > 0$ *and* $\beta = \beta(m,Q,\Lambda) > 0$ *such that the following estimate holds for every* $x \in \mathbf{B}_1 \cap \mathcal{M}$ *and for every pair* ρ, r *with* $4\eta\, d(x,\mathbf{S}) < \rho \leq r < 1$. *If we define*

$$\mathcal{A}^{2r}_{\frac{\rho}{4}}(x) \coloneqq \left(\mathbf{B}_{2r}(x) \setminus \overline{\mathbf{B}}_{\frac{\rho}{4}}(x)\right) \cap \mathcal{M}$$

then

$$\begin{aligned}\int_{\mathcal{A}^{2r}_{\frac{\rho}{4}}(x)} \sum_i \left| DN_i(y) \frac{d(x,y)\nabla d(x,y)}{|\nabla d(x,y)|} - \mathbf{I}(x,d(x,y))N_i(y)|\nabla d(x,y)|\right|^2 \frac{\mathrm{d}y}{d(x,y)} \\ \leq C\mathbf{H}(x,2r)\left(W^{2r}_{\frac{\rho}{4}}(x) + \boldsymbol{m}_0^{\gamma_4} r^{\gamma_4} \log\left(\frac{4r}{\rho}\right)\right).\end{aligned}$$

We refer the reader to the proof of Proposition 3.14.2 for the proof of Proposition 5.4.2. Since we just require the estimates in Lemma 3.13.1 to prove this proposition, the proof remains completely unchanged in this setting. We will also require the following control on variations of the frequency in terms of frequency pinching.

Lemma 5.4.3. *Suppose that* T, $\mathcal{M}$ *and* N *be as in Assumption 5.2.4, let* γ_4 *be as in Lemma 3.12.3 and let* $\Lambda > 0$ *be as in (3.105). Let* $x_1, x_2 \in \mathbf{B}_1 \cap \mathcal{M}$ *with* $d(x_1,x_2) \leq \frac{r}{8}$, *where* r *is such that* $8\eta \max\{d(x_1,\mathbf{S}), d(x_2,\mathbf{S})\} < r \leq 1$. *Then there exists a constant* $C = C(m,Q,\Lambda) > 0$ *such that for any* $z, y \in [x_1,x_2]$, *we have*

$$|\mathbf{I}(y,r) - \mathbf{I}(z,r)| \leq C\left[\left(W^{4r}_{\frac{r}{8}}(x_1)\right)^{\frac{1}{2}} + \left(W^{4r}_{\frac{r}{8}}(x_2)\right)^{\frac{1}{2}} + \boldsymbol{m}_0^{\frac{\gamma_4}{2}} r^{\frac{\gamma_4}{2}}\right] \frac{d(z,y)}{r}.$$

The proof of Lemma 5.4.3 relies on the following additional variation estimates and

identities.

Lemma 5.4.4. *Let T, $\mathcal{M}$ and N be as in Assumption 5.2.4 and let $x \in \mathbf{B}_1 \cap \mathcal{M}$. Let $\eta\, d(x, \mathbf{S}) < \rho < r \leq 1$, and suppose that v is a vector field on $\mathcal{M}$. Then the following identities hold:*

$$\partial_v \mathbf{D}(x,r) = \frac{2}{r}\int \phi'\left(\frac{d(x,y)}{r}\right)\sum_i \partial_{\nu_x} N_i(y)\cdot \partial_v N_i(y)\,\mathrm{d}y + O\left(\boldsymbol{m}_0^{\gamma_4}\right) r^{\gamma_4 - 1}\mathbf{D}(x,r)$$
$$\partial_v \mathbf{H}(x,r) = -2\sum_i \int_{\mathcal{M}} \frac{|\nabla d(x,y)|^2}{d(x,y)}\phi'\left(\frac{d(x,y)}{r}\right)\langle \partial_v N_i(y), N_i(y)\rangle\,\mathrm{d}y\,.$$

Proof of Lemma 5.4.4. The proof of these estimates is entirely analogous to those in Lemma 3.14.5, so we omit many of the details. Indeed, notice that the identity (3.120) still holds here, by decomposing the domain of the integral in $\mathbf{D}(x,r)$ into the disjoint components of $\mathbf{B}_r(x) \cap \mathcal{M}_+$ and $\mathbf{B}_r(x) \cap \mathcal{M}_-$ as defined in [15], each of which is a relatively open set by [14, Corollary 2.8] (which also holds for a submanifold domain with $C^{3,\kappa}$-regularity). Note that the set $\mathbf{B}_r(x) \cap \mathcal{M}_0$ where $N = Q[\![0]\!]$ as defined in [15] (not to be confused with our former terminology of this form) has $\mathcal{H}^m$-measure zero.

Now we test [22, (3.25)] with the vector field $X_i(q) = Y(\mathbf{p}(q))$ for

$$Y(y) \coloneqq \phi\left(\frac{d(x,y)}{r}\right) v,$$

which satisfies the differential identities (3.122), (3.123), and exploit the excess decay of Proposition 5.2.1 to establish the same estimates on the inner variational errors $\widetilde{\mathrm{Err}}_j^i$ therein (see [15, (26.9), (26.16), (26.17), (26.18)]) for this vector field here.

The identity for $\partial_v \mathbf{H}(x,r)$ is once again a simple computation, identical to that in the proof of [16, Proposition 3.1], again combined with the domain decomposition into $\mathcal{M}_+$ and $\mathcal{M}_-$. □

Having established the validity of Lemma 5.4.4, the proof of Lemma 5.4.3 follows in exactly the same way as that of Lemma 3.14.4, yet again decomposing the domains of

integration in $\mathbf{D}(x,r)$, $\mathbf{H}(x,r)$ into $\mathcal{M}_+$ and $\mathcal{M}_-$ when taking spatial derivatives of $\mathbf{I}(x,r)$, and noting that the estimates remain unchanged.

5.5 Quantitative spine splitting

We will now demonstrate that under the assumption of approximate homogeneity around a collection of points spanning a given subspace, one achieves the existence of an approximate spine in that subspace, in a quantitative manner. We will be considering affine subspaces spanned by families of linearly independent vectors, so we introduce the following notation. Given an ordered set of points $X = \{x_0, x_1, \ldots, x_k\}$ we denote by $V(X)$ the affine subspace spanned by the vectors $\{x_1 - x_0, x_2 - x_0, \ldots, x_k - x_0\}$ and centered at x_0:

$$V(X) = x_0 + \operatorname{span}(\{(x_1 - x_0), (x_2 - x_0), \ldots, (x_k - x_0)\}) .$$

We recall the notions of *ρr-linear independence* and *ρr-spanning* from Definition 3.15.1.

5.5.1 Compactness and homogeneity

Before we proceed, we require the following invariance and unique continuation results, which are analogous to their counterparts in [16] in the framework of $\mathcal{A}_Q$-valued Dir-minimizers, but we must verify that the presence of codimension one zeros in this setting does not pose an obstruction to their validity.

Lemma 5.5.1. *Let $\Omega \subset \mathbb{R}^m$ be a connected open set and let $u : \Omega \to \mathscr{A}_Q(\mathbb{R}^n)$ be a continuous map that is homogeneous about two points x_1 and x_2, with respective homogeneities α_1 and α_2.*

Then $\alpha_1 = \alpha_2$ and u is invariant along the direction $x_2 - x_1$. Moreover, we have $x_1 + \operatorname{span}\{x_2 - x_1\} \subset \Delta_Q u$.

Lemma 5.5.2. *Let $\delta \in (0,1)$, let $\Omega \subset \mathbb{R}^m$ be a connected open set and suppose that $u_1, u_2 : \Omega \to \mathscr{A}_Q(\mathbb{R}^n)$ are two homogeneous maps such that*

(a) both u_1 and u_2 locally minimize the Dirichlet energy;

(b) there exists a non-empty open set $U \subset \Omega$ such that $u_1 \equiv u_2$ on U;

(c) for $j = 1, 2$ we have $\Delta_Q u_j \equiv \Delta_Q^{2-\delta} u_j$.

Then $u_1 \equiv u_2$ on Ω.

Proof of Lemma 5.5.1. The proof of this follows by a very similar reasoning to the proof of [16, Lemma 6.8], which is the analogous result for classical Q-valued Dir-minimizers. However, one has to check that the argument is unchanged by the presence of the regions $\Omega_\pm$, separated by the points $x \in \Delta_Q u$ with $I_u(x, 0) = 1$.

The homogeneity of u tells us that

$$u^\pm(x) = \sum_i \left[\!\!\left[|x - x_\ell|^{\alpha_\ell} u_i \left(\frac{x - x_\ell}{|x - x_\ell|} + x_\ell \right) \right]\!\!\right], \qquad x \in \Omega_\pm \sqcup \Omega_0 \text{ respectively}, \quad \ell = 1, 2.$$

Recall that that each of u^+ and u^- is a classical $\mathcal{A}_Q(\mathbb{R}^n)$-valued Dir-minimizer. We may thus extend each of these to $\mathbb{R}^m$ by homogeneity, and apply [16, Lemma 6.8] to each one individually. The conclusion follows immediately. □

Proof of Lemma 5.5.2. Observe that condition (c) tells us that for $j = 1, 2$, it holds that $\Omega \setminus \Delta_Q u_j$ is a affine subspace of dimension at most $m - 2$, in light of the homogeneity assumption, combined with the knowledge that all one-dimensional $\mathscr{A}_Q(\mathbb{R}^n)$-valued Dir-minimizers are locally superpositions of linear functions. Thus, u_1 and u_2 can be identified with classical (homogeneous) Q-valued Dir-minimizers that take values in $\mathcal{A}_Q$. This means that [16, Lemma 6.9] can be applied directly. □

The following lemma gives a quantitative notion of the existence of an approximate spine in $\mathbf{S}$, provided that N is (quantitatively) almost-homogeneous about an $(m-2)$-dimensional

submanifold of the center manifold. It is entirely analogous to Lemma 3.15.2, only posed in the mod(p) setting.

Lemma 5.5.3. *Suppose that T, $\mathcal{M}$, N are as in Assumption 5.2.4, let $x \in \mathbf{S}$ and let $\rho, \tilde{\rho}, \bar{\rho} \in]0,1]$ be given. There exists $\varepsilon = \varepsilon_{5.5.3}(m, Q, \Lambda, \rho, \tilde{\rho}, \bar{\rho}) \in]0, \varepsilon_4^2]$ such that the following holds. Suppose that for some $r > 0$,*

$$\mathbf{E}^{no}(T, \mathbf{B}_{2r}(x)) + (2r\mathbf{A})^2 \leq \varepsilon.$$

Suppose that $X = \{x_i\}_{i=0}^{m-2} \subset \mathbf{B}_r(x) \cap \mathbf{S}$ is a ρr-linearly independent set of points with

$$W_{\tilde{\rho} r}^{2r}(x_i) < \varepsilon \qquad \text{for each } i.$$

Then $\mathbf{S} \cap (\mathbf{B}_r \setminus \mathbf{B}_{\bar{\rho} r}(V(X))) = \emptyset$.

Proof. We prove this by contradiction. We may without any loss of generality assume that $x = 0$. Now, suppose that the statement of the lemma is false. Then we may find sequences $\varepsilon_k \downarrow 0$, $r_k \downarrow 0$ and corresponding sequences of center manifolds $\mathcal{M}_k$ and normalized normal approximations $\bar{N}_k$ with $\mathbf{H}_{\bar{N}_k}(0,1) = 1$ for T_{0,r_k}. Letting $\mathbf{S}_k \coloneqq \mathbf{S}(T_{0,r_k})$, we in addition have a sequence of $(m-1)$-tuples of points $X_k \coloneqq \{x_{k,0}, x_{k,1}, \ldots, x_{k,m-2}\} \subset \mathbf{B}_1 \cap \mathbf{S}_k$ such that

(i) X_k is ρ-linearly independent for some $\rho \in]0,1]$;

(ii) $W_{\tilde{\rho}}^2(\bar{N}_k, x_{k,i}) \leq \varepsilon_k \to 0$ as $k \to \infty$ for some $\tilde{\rho} \in]0,1]$;

(iii) there exists a point $y_k \in \mathbf{S}_k \cap (\mathbf{B}_1 \setminus \mathbf{B}_{\bar{\rho}}(V(X_k)))$.

We can thus proceed to use a compactness argument as in Proposition 5.3.1 (see, also [15, Section 28] or Section 3.2.1 in the integral currents framework) in order to deduce that

(1) $\mathcal{M}_k \longrightarrow \pi_\infty$ in $C^{3,\kappa}$;

(2) there exists a Dir-minimizer $u : \pi_\infty \supset B_1 \to \mathscr{A}_Q(\mathbb{R})$ with $\boldsymbol{\eta} \circ u \equiv 0$ such that $\bar{N}_k \circ \mathbf{e}_k \longrightarrow u$ in L^2 and in $W^{1,2}_{\text{loc}}$;

(3) The sequence X_k converges pointwise to $X_\infty = \{x_0, \dots, x_{m-2}\}$;

(4) The points y_k converge pointwise to $y \in \bar{B}_1 \setminus B_{\bar{\rho}}(V(X_\infty)) \subset \pi_\infty$ with $u(y) = Q[\![0]\!]$.

By [35, Theorem 3.6, Theorem 3.7] and a standard stratification argument, we know that $\dim_{\mathcal{H}}(\Delta_Q^{2-\delta} u) \leq m-2$, since $H_u(0,1) = 1$ and $\eta \circ u = 0$, so u cannot be a classical harmonic map with multiplicity Q. Moreover, $H_u(y,\tau) > 0$ for every $\tau \in (0,1)$, since otherwise we would contradict the dimension estimate on $\Delta_Q^{2-\delta} u$. This, in combination with (ii) tells us that

$$I_u(x_i, \tilde{\rho}) = I_u(x_i, 2) \geq 2 - \delta \qquad \text{for } i = 0, \dots, m-2.$$

The monotonicity of the regularized frequency as defined in Section 5.2 for $\mathscr{A}_Q$-valued Dir-minimizers then tells us that u is α_i-homogeneous about x_i within the annulus $B_2(x_i) \setminus B_{\tilde{\rho}}(x_i) \subset \pi_\infty$, for some $\alpha_i \geq 2$. Firstly, we may immediately deduce that $\alpha_i = \alpha$ for some fixed $\alpha \geq 2 - \delta$ by iteratively applying Lemma 5.5.1, and also that $\Delta_Q u = \Delta_Q^{2-\delta} u$. We may then extend u to an α-homogeneous function v about the $(m-2)$-dimensional affine subspace $V(X_\infty)$, in light of Lemma 5.5.2.

Since $y \notin V(X_\infty)$ and $u(y) = Q[\![0]\!]$, but u is α-homogeneous about $V(X_\infty)$, this implies that $u \equiv Q[\![0]\!]$ on $L := x_0 + \operatorname{span}\{x_{m-2} - x_0, \dots, x_1 - x_0, y - x_0\}$, and $I_u(\cdot, 0) \equiv \alpha \geq 2$ on the $(m-1)$-dimensional plane L. This however contradicts the dimension estimate on $\Delta_Q^{2-\delta} u$, thus allowing us to conclude. □

The following lemma, which is the mod(p) analogue of Lemma 3.15.3, tells us that it is enough to establish approximate homogeneity on a linearly independent set of points, in order to achieve approximate homogeneity in the affine subspace spanned by these points.

Lemma 5.5.4. *Suppose that T, $\mathcal{M}$ and N are as in Assumption 5.2.4, let $x \in \mathbf{S}$ and let $\rho, \tilde{\rho}, \bar{\rho} \in]0,1]$ be given. Then for any $\delta > 0$, there exists $\varepsilon = \varepsilon_{5.5.4} > 0$, dependent on $m, Q, \Lambda, \rho, \tilde{\rho}, \bar{\rho}, \delta$ for which the following property holds. Suppose that for some $r > 0$ we have*

$$\mathbf{E}^{no}(T, \mathbf{B}_{2r}(x)) + (2r\mathbf{A})^2 \leq \varepsilon.$$

In addition, suppose that $X = \{x_i\}_{i=0}^{m-2} \subset \mathbf{B}_r(x) \cap \mathbf{S}$ *is a* ρr*-linearly independent set of points with*

$$W_{\tilde{\rho}r}^{2r}(x_i) < \varepsilon \qquad \text{for every } i = 0, \dots, m-2.$$

Then for every $y_1, y_2 \in \mathbf{B}_r(x) \cap \mathbf{B}_{\varepsilon r}(V(X)) \cap \mathbf{S}$ *and for every* $r_1, r_2 \in [\bar{\rho}r, r]$ *we have*

$$|\mathbf{I}(y_1, r_1) - \mathbf{I}(y_2, r_2)| \leq \delta\,.$$

Proof. We will once again proceed to argue by contradiction. We again assume that $x = 0$ without loss of generality. If the statement of the lemma is false, we may find sequences $\varepsilon_k \downarrow 0$, $r_k \downarrow 0$ and corresponding sequences of center manifolds $\mathcal{M}_k$ and normalized normal approximations $\bar{N}_k$ with $\mathbf{H}_{\bar{N}_k}(0,1) = 1$ for T_{0,r_k}, with a sequence of $(m-1)$-tuples of points $X_k \coloneqq \{x_{k,0}, x_{k,1}, \dots, x_{k,m-2}\} \subset \mathbf{B}_1 \cap \mathbf{S}_k = \mathbf{B}_1 \cap \mathbf{S}(T_{0,r_k})$ such that

(i) The set X_k is ρ-linearly independent for some $\rho > 0$;

(ii) $W_{\tilde{\rho}}^2(\bar{N}_k, x_{k,i}) \leq \varepsilon_k \to 0$ as $k \to \infty$ for some $\tilde{\rho} > 0$;

(iii) there exist points $y_{k,1},\ y_{k,2} \in \mathbf{B}_1 \cap \mathbf{B}_{\varepsilon_k}(V(X_k)) \cap \mathbf{S}_k$ and corresponding scales $r_{k,i} \in [\bar{\rho}, 1]$ with

$$|\mathbf{I}_k(y_{k,1}, r_{k,1}) - \mathbf{I}_k(y_{k,2}, r_{k,2})| \geq \delta > 0,$$

where $\mathbf{I}_k \coloneqq \mathbf{I}_{\bar{N}_k}$.

We may now use an analogous compactness argument to that in the proof of Lemma 5.5.3 to conclude that, up to subsequence, we have

(1) $\mathcal{M}_k \longrightarrow \pi_\infty$ in $C^{3,\kappa}$;

(2) $\bar{N}_k \circ \mathbf{e}_k \longrightarrow u$ in L^2 and in $W^{1,2}_{\mathrm{loc}}$, where u is an $\mathscr{A}_Q$-valued Dir-minimizer with $\boldsymbol{\eta} \circ u \equiv 0$;

(3) the collections of points X_k converge pointwise to $X_\infty = \{x_0, \dots, x_{m-2}\}$;

(4) the points $y_{k,i}$ converge pointwise to y_i and the respective scales $r_{k,i}$ converge to $r_i \in [\bar{\rho}, 1]$ for $i = 1, 2$.

Proceeding as in the proof of Lemma 5.5.3, we arrive at the conclusion $u \equiv Q[\![0]\!]$ on $x_0 + \operatorname{span}\{x_{m-2} - x_0, \dots, x_1 - x_0\} = V(X_\infty)$ with the additional property that

$$I_u(x_i, \tilde{\rho}) = I_u(x_i, 2) \geq 2 - \delta \qquad \text{for } i = 0, \dots, m-2.$$

Thus, $I_u(y, \tau) \equiv \alpha \geq 2 - \delta$ for any $y \in V(X_\infty)$ and any $\tau > 0$. On the other hand, since $r_{k,i} \in [\bar{\rho}, 1]$ and $\bar{\rho} > \eta \min\{d(y_1, \mathbf{S}), \eta d(y_2, \mathbf{S})\}$, we additionally have $\mathbf{I}_k(y_{k,i}, r_{k,i}) \to I_u(y_i, r_i)$ for $i = 1, 2$, so the property (iii) is in contradiction with the homogeneity of u about $V(X_\infty)$. □

5.6 Jones' β_2 coefficient control

This section is dedicated to controlling the "mean flatness" in a ball for a given Radon measure μ supported in $\mathbf{S}$, in terms of an $(m-2)$-dimensional μ-weighted average of the frequency pinching, up to a lower order error term. Recall Definition 3.16.1 of the $(m-2)$-dimensional Jones' β_2 coefficient. It will enable us to measure the mean flatness of μ at a given scale around a given point. The main result of this section is the following, which yields the desired control on the β_2 coefficient of a measure supported in $\mathbf{S}$.

Proposition 5.6.1. *There exist thresholds $\eta = \eta_{5.6.1}(m) > 0$, $\varepsilon = \varepsilon_{5.6.1}(\Lambda, m, Q, \eta)$, $\alpha_0 = \alpha_0(\Lambda, m, Q) > 0$ and $C(\Lambda, m, Q) > 0$ such that the following holds. Suppose that T, $\mathcal{M}$ and N satisfy Assumption 5.2.4 with parameters $\varepsilon_4 \leq \varepsilon_{5.6.1}$ and $\eta \leq \eta_{5.6.1}$. Suppose that μ is a finite non-negative Radon measure with $\operatorname{spt}(\mu) \subset \mathbf{S}$. Then for all $r \in]0, 1]$ and every $x_0 \in \mathbf{B}_{r/8} \cap \mathbf{S}$ we have*

$$[\beta_{2,\mu}^{m-2}(x_0, r/8)]^2 \leq \frac{C}{r^{m-2}} \int_{\mathbf{B}_{r/8}(x_0)} W_{r/8}^{4r}(x) \, \mathrm{d}\mu(x) + C\boldsymbol{m}_0^{\alpha_0} r^{-(m-2-\alpha_0)} \mu(\mathbf{B}_{r/8}(x_0)).$$

The proof of Proposition 5.6.1 requires the following preliminary lemma regarding a characterization of $\mathscr{A}_Q$-valued Dir-minimizers that are $(m-1)$-invariant.

Lemma 5.6.2. *Let $A_{r,R}(\bar{z}) := B_R(\bar{z}) \setminus \bar{B}_r(\bar{z}) \subset \mathbb{R}^m$ and suppose that $u : B_R(\bar{z}) \to \mathscr{A}_Q(\mathbb{R}^n)$ is a non-trivial Dir-minimizer. Assume there is a ball $B \subset \Omega$ and a system of coordinates $x_1, \dots, x_m$ such that $u\big|_{A_{r,R}(\bar{z})}$ is a function of x_1 only. Then u is a function of only x_1 on all of $B_R(\bar{z})$.*

Moreover, one of the following two alternatives holds:

(i) $\Delta_Q u = \emptyset$;

(ii) there is a one-homogeneous Dir-minimizer $v : \mathbb{R} \to \mathscr{A}_Q(\mathbb{R}^n)$ such that $u(x) = v(x_1)$ for $x \in \Omega$.

Remark 5.6.3. Note that in case (ii) in Lemma 5.6.2, $\Delta_Q u = \{x_1 = c\}$ for some $c \in \mathbb{R}$ and $I_u(x, \tau) = 1$ for every $x \in \Delta_Q u$ and every $\tau > 0$.

Proof of Lemma 5.6.2. Fix a point $x \in \operatorname{Reg} u \cap A_{r,R}(\bar{z})$, with coordinates $x_1, \dots, x_m$. Then, by definition (see [14, Definition 10.1]) there must be a neighbourhood $U \ni x$ and a sign $\epsilon \in \{+1, -1\}$ such that

$$u(y) = \left(\sum_{i=1}^{Q} [\![u_i(y)]\!], \epsilon \right) \qquad \text{for } y \in U.$$

More precisely, this is because $\Delta_Q u$ is a relatively closed set of Hausdorff dimension at most $m-1$.

This in particular implies that we may identify $u\big|_U$ with a classical $\mathcal{A}_Q$-valued Dir-minimizer. Moreover, the invariance of u in $A_{r,R}(\bar{z})$ implies that $u_i(y) = (y_1 - c_i)v_i$ for some $c_i \in \mathbb{R}$, $v_i \in \mathbb{R}^n$, since one-dimensional classical $\mathcal{A}_Q$-valued Dir-minimizers necessarily have affine decompositions. In addition, for any $i \neq j$, either $u_i(x) \neq u_j(x)$ or $u_i \equiv u_j$ on U.

We may thus rewrite u in terms of distinct representatives as

$$u(y) = \left(\sum_{i=1}^{Q'} k_i [\![u_i(y)]\!], \epsilon \right) \qquad \text{for } y \in U,$$

where $k_i \in \mathbb{N}$, $\mathbb{N} \ni Q' \leq Q$ and $u_i(x) \neq u_j(x)$ for every $i \neq j$.

Now define

$$M(x) \coloneqq \sup\{y_1 > x_1 : u_i(y) = u_j(y) \text{ for some } i \neq j\},$$

with the convention that $M(x) = +\infty$ if this set is empty.

We will proceed to show that

$$u(y) = \left(\sum_{i=1}^{Q'} k_i [\![u_i(y)]\!], \epsilon \right) \qquad \text{for } y \in \{x_1 < y_1 < M(x)\} \cap B_R(\bar{z}). \tag{5.3}$$

Let W be the maximal open set in $\{x_1 < y_1 < M(x)\} \cap B_R(\bar{z})$ in which (5.3) holds. Note that $W \supset U$. If $W \neq \{x_1 < y_1 < M(x)\} \cap B_R(\bar{z})$, we can choose a point $\xi \in \{x_1 < y_1 < M(x)\} \cap B_R(\bar{z}) \cap \partial W$. Since $\xi_1 < M(x)$, we must have $u_i(\xi) \neq u_j(\xi)$ for every $i \neq j$. This means that we may apply the unique continuation for each single-valued Dir-minimizer u_i, to conclude that there is a neighbourhood $V \ni \xi$ on which (5.3) holds. This, however, contradicts the maximality of W, so indeed (5.3) holds on the entirety of $\{x_1 < y_1 < M(x)\} \cap B_R(\bar{z})$.

Now, notice that if $M(x) < \sup\{y_1 : y \in B_R(\bar{z})\}$, then

$$\{y_1 = M(x)\} \cap B_R(\bar{z}) \subset \{u = Q[\![0]\!]\}.$$

This is due to the fact that $\mathcal{H}^{m-1}\left(\{y_1 = M(x)\} \cap B_R(\bar{z})\right) > 0$, while $\dim_{\mathcal{H}}\left((B_R(\bar{z}))_{\pm} \cap \Delta_Q u\right) \leq m - 2$.

Similarly, let

$$L(x) \coloneqq \inf\{y_1 < x_1 : u_i(y) = u_j(y) \text{ for some } i \neq j\},$$

with the convention that $L(x) = -\infty$ if this set is empty.

Proceeding in exactly the same way as above, we conclude that

$$u(y) = \left(\sum_{i=1}^{Q'} k_i [\![u_i(y)]\!], \epsilon \right) \qquad \text{for } y \in \{L(x) \leq y_1 \leq M(x)\} \cap B_R(\bar{z}), \tag{5.4}$$

and

$$\{y_1 = L(x)\} \cap B_R(\bar{z}) \subset \{u = Q[\![0]\!]\}.$$

Moreover, notice that if $M(x) < +\infty$, then $L(x) = -\infty$, and if $L(x) > -\infty$ then $M(x) = +\infty$. This is due to the fact that u is non-trivial, and each u_i is affine on $\{L(x) < y_1 < M(x)\}$.

Now there are two possibilities; either one of $L(x), M(x)$ lies in $[\bar{z}_1 - R, \bar{z}_1 + R]$, or not. In the latter case, $B_r(\bar{z}) \subset \{L(x) \leq y_1 \leq M(x)\}$ and so it is immediate that the representation formula (5.4) holds in $B_R(\bar{z})$. In the former case, suppose without loss of generality that $L(x) \in [\bar{z}_1 - R, \bar{z}_1 + R]$. However, since on $A_{r,R}(\bar{z}) \cap \{y_1 < L(x)\}$, u remains a function of x_1 only, we may again exploit the affine structure of $\mathcal{A}_Q$-valued Dir-minimizers to conclude that the representation formula (5.4) holds in the entirety of $B_R(\bar{z})$. The dichotomy (i) or (ii) follows immediately in both cases. □

Remark 5.6.4. In fact, the proof of Lemma 5.6.2 demonstrates that the conclusion of the lemma holds true in any open, connected domain $\Omega \subset \mathbb{R}^m$ in place of $B_R(x_0)$, if u is a function of x_1 only on an open subset $\Omega' \subset \Omega$ that contains a point x with $u(x) = Q[\![0]\!]$. The author suspects that this more general version result may remain true even without the requirement that Ω' contains a point x with $u(x) = Q[\![0]\!]$. However, since such a more general version of the result is not required here, we do not pursue this here.

Proof of Proposition 5.6.1. We may assume that $\mu(\mathbf{B}_{r/8}) > 0$, since the desired estimate is otherwise trivial. The majority of this proof follows exactly as that of Proposition 3.16.2. Indeed, letting $\mathcal{A}^{2r}_{r/4}(x_0) \coloneqq (\mathbf{B}_{2r}(x_0) \setminus \mathbf{B}_{r/4}(x_0)) \cap \mathcal{M}$ and proceeding in exactly the same

manner as the proof therein, we arrive at the estimate

$$[\beta^{m-2}_{2,\mu}(x_0, r/8)]^2 \int_{\mathcal{A}^{2r}_{r/4}(x_0)} \sum_{j=1}^{m-1} |DN(z) \cdot \boldsymbol{\ell}_z(v_j)|^2 \, \mathrm{d}z$$
$$\leq Cr^{-(m-1)} \mathbf{H}(x_0, 2r) \left(\int_{\mathbf{B}_{r/8}(x_0)} W^{4r}_{r/8}(x) \, \mathrm{d}\mu(x) + \boldsymbol{m}_0^{\alpha_0} r^{\alpha_0} \mu(\mathbf{B}_{r/8}(x_0)) \right),$$

for $\alpha_0 > 0$ sufficiently small, where $\boldsymbol{\ell}_z : T_{x_0}\mathcal{M} \to T_z\mathcal{M}$ is the linear map that corresponds to the differential $d\mathbf{e}_{x_0}|_\zeta$ of the exponential map $\mathbf{e}_{x_0}$ at the point $\zeta = \mathbf{e}^{-1}_{x_0}(z)$.

It thus remains to check that

$$\int_{\mathcal{A}^{2r}_{r/4}(x_0)} \sum_{j=1}^{m-1} |DN(z) \cdot \boldsymbol{\ell}_z(v_j)|^2 \, \mathrm{d}z \geq c(\Lambda) \frac{\mathbf{H}(x_0, 2r)}{r}, \tag{5.5}$$

for some $C(\Lambda) > 0$. We prove this via a contradiction and compactness argument, as usual. By scaling and translation invariance of the claimed bound, we may assume that $r = 1$ and $x_0 = 0$. If (5.5) fails, then we can extract a sequence of currents T_k with $\boldsymbol{m}_0^{(k)} \leq \varepsilon_k^2 \to 0$, corresponding center manifolds $\mathcal{M}_k$ in $\mathbf{B}_1$, normalized normal approximations $\bar{N}_k$ with $\int_{\mathbf{B}_2 \setminus \bar{\mathbf{B}}_1 \cap \mathcal{M}_k} |\bar{N}_k|^2 = 1$ and $\int_{\mathbf{B}_1 \cap \mathcal{M}_k} |D\bar{N}_k|^2 \leq C\Lambda$, such that

- $\mathcal{M}_k \to \pi_\infty$,
- $\boldsymbol{\eta} \circ \bar{N}_k \to 0$,
- $\bar{N}_k(y_k) = Q\llbracket 0 \rrbracket$ for some $y_k \in \mathbf{B}_{1/8} \cap \mathcal{M}_k$ (since $\mu_{T_k}(\mathbf{B}_{r/8}) > 0$),

but with

$$\int_{\mathbf{B}_2 \setminus \mathbf{B}_{1/4} \cap \mathcal{M}_k} \sum_{j=1}^{m-1} |D\bar{N}_k(z) \cdot \boldsymbol{\ell}^k_z(v^k_j)|^2 \longrightarrow 0,$$

for some choice of orthonormal vectors $\{v^k_1, \ldots, v^k_{m-1}\}$. Up to subsequence, we can extract a limiting Dir-minimizer $u : \pi_\infty \supset B_2 \to \mathscr{A}_Q(\mathbb{R})$ with

- $\int_{B_2 \setminus \bar{B}_1} |u|^2 = 1$,

- $\int_{B_1} |Du|^2 \leq C\Lambda$,
- $\boldsymbol{\eta} \circ u \equiv 0$,
- $u(y) = Q[\![0]\!]$ for some $y \in B_{1/8}$,

but for which

$$\int_{B_2 \setminus \bar{B}_1} \sum_{j=1}^{m-1} |Du(z) \cdot v_j|^2 = 0$$

for orthonormal directions v_j which are the (pointwise) limit of the directions v_j^k. Thus, arguing as in the proof of [16, Proposition 5.3], we conclude that u is a function of only one variable on $B_2 \setminus \bar{B}_{1/4}$, and so Lemma 5.6.2 tells us that it is a function of only one variable the whole of B_2. Since $u(y) = Q[\![0]\!]$, we have $\dim_{\mathcal{H}}(\Delta_Q u) \geq m - 1$, which contradicts the fact that u is non-trivial. □

5.7 Coverings, Minkowski bound and rectifiability

Now that we have the desired bounds on the β_2 coefficients as in Proposition 5.6.1, we are in a position to conclude the result of Theorem 5.1.2. The conclusion is achieved via an iterative covering procedure, originally appearing in [42]. It has since then further been used in [16] in the context of classical multiple-valued Dirichlet-minimizing functions, and appeared in the arguments of Chapter 3.

The proofs of the results in this section are completely identical to those in Section 3.18, relying only on the preceding results, which have now been established in this context, in the previous sections of this article. Thus, the proofs are omitted here.

We begin with the following covering lemma, which is the analogue of Lemma 3.18.1.

Lemma 5.7.1. *Let $\rho \leq \frac{1}{100}$, let $\sigma < \tau < \frac{1}{8}$ and let $\eta = \eta_{5.6.1} > 0$. There exists $\varepsilon_4 = \varepsilon_4(\Lambda, m, Q, \alpha_0) > 0$ sufficiently small such that the following holds. Suppose that T is as in Assumption 5.2.4 for these choices of η and ε_4. Let $x \in \mathbf{S} \cap \mathbf{B}_{1/8}$, let $D \subset \mathbf{S} \cap \mathbf{B}_\tau(x)$ and let $U \coloneqq \sup_{y \in D} \mathbf{I}(y, \tau)$.*

Then there exists $\delta = \delta_{5.7.1}(m, Q, \Lambda, \rho) > 0$, *a dimensional constant* $C_R = C_R(m) > 0$ *and a finite cover of* D *by balls* $\mathbf{B}_{r_i}(x_i)$ *such that*

(a) $r_i \geq 10\rho\sigma$;

(b) $\sum_i r_i^{m-2} \leq C_R \tau^{m-2}$;

(c) For every i, *either* $r_i \leq \sigma$ *or*

$$F_i := D \cap \mathbf{B}_{r_i}(x_i) \cap \{y : \mathbf{I}(y, \rho r_i) \in (U - \delta, U + \delta)\} \subset \mathbf{B}_{\rho r_i}(V_i),$$

for some $(m-3)$*-dimensional subspace* $V_i \subset \mathbb{R}^{m+n}$.

Remark 5.7.2 (Heirarchy of parameters)**.** The parameters ε_4 and η of Assumption 5.2.4 are initially taken to be small enough so that we can apply Proposition 5.6.1. Then, ε_4 is further decreased if necessary, to ensure that $\boldsymbol{m}_0^{\alpha_0}$ falls below a desired small dimensional constant, in order to absorb a suitable error term (see the proof of Proposition 3.16.2). Lemma 5.7.1 will then be used to prove the following additional efficient covering result, entirely analogous to Proposition 3.18.2, where the parameter ρ will be chosen smaller than a geometric constant depending only on m.

Proposition 5.7.3. *Let* $\eta = \eta_{5.6.1} > 0$ *and let* $\varepsilon_4 > 0$ *be as in Lemma 5.7.1. There exist* $\delta = \delta_{5.7.3}(m, Q, \Lambda)$, *a scale* $\tau = \tau(m, Q, \Lambda, \delta) < \frac{1}{8}$ *and a dimensional constant* $C_V = C_V(m) \geq 1$ *such that the following holds.*

Assume that T *is as in Assumption 5.2.4 for these choices of* η *and* ε_4. *Suppose that* $x \in \mathbf{S} \cap \mathbf{B}_{1/8}$ *and let* $D \subset \mathbf{S} \cap \mathbf{B}_\tau(x)$ *and* $U := \sup_{y \in D} \mathbf{I}(y, \tau)$. *Then, for every* $s \in]0, \tau[$, *there exists a finite cover of* D *by balls* $\mathbf{B}_{r_i}(x_i)$ *with* $r_i \geq s$ *and a decomposition of* D *into sets* $A_i \subset D$ *such that*

(a) $A_i \subset D \cap \mathbf{B}_{r_i}(x_i)$;

(b) $\sum_i r_i^{m-2} \leq C_V \tau^{m-2}$;

(c) For every i we have either $r_i = s$ or

$$\sup_{y \in A_i} \mathbf{I}(y, r_i) \leq U - \delta.$$

5.7.1 Conclusion of Theorem 5.2.3(v)

The conclusion of Theorem 5.2.3(v) now follows from Proposition 5.7.3 by exactly the same reasoning as that in Section 3.17.

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