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The structure of graphs with no cycles of length 0 (mod 3)

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The structure of graphs with no cycles of length 0 (mod 3)

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Presented to the Faculty
of Princeton University
in Candidacy for the Degree
of Doctor of Philosophy

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Abstract

We examine the structure of graphs that have no cycles of length 0 (mod 3). We show that, if G is a simple 2-connected graph with no cycles of length 0 (mod 3), then G has two adjacent degree two vertices or G has two nonadjacent degree two vertices with the same neighborhood. Using this result, it follows that if G is a simple graph with no cycles of length 0 (mod 3), then for every induced subgraph H of G, the modularity of H, defined as the number of even independent sets minus the number of odd independent sets, is either -1, 0, or 1.

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To Barbara and Stephen Gauthier

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Chapter 1

Introduction

In the 1970s Erdős considered, for each pair of integers $0 \leq a < b$ such that there are infinitely many even integers congruent to a modulo b, the graphs that do not contain a cycle of length a modulo b and asked whether there exists an upper bound (as a function of a and b) for the average degree of such a graph. [6] Robertson showed the existence of an upper bound in the case $a = 0$, $b = 3$. [6] In 1988, Barefoot, Clark, Douthett, Entringer, and Fellows showed that every subdivision of a 3-connected cubic graph with at least ten vertices has a cycle of length 0 (mod 3). The authors also proved that any graph that is a subdivision of an n-vertex graph graph with at least $3n - 5$ edges has a cycle of length 0 (mod 3) and conjectured that every graph with minimum degree at least 3 contains a 0 (mod 3) cycle. [1] The conjecture was answered affirmatively by Chen and Saito, who showed that if G contains at most one vertex of degree less than three, then G contains a cycle of length 0 (mod 3). As a consequence, every graph with at least $2n - 3$ edges has a cycle of length 0 (mod 3), a bound that is tight due to $K_{2,n-2}$. [4] An immediate consequence is that every graph with no 0 (mod 3) cycle is 2-degenerate; hence, 3-colorable.

We can consider a similar problem about graphs with no *induced* cycle of length 0 (mod 3). Obviously, there is no chance of an upper bound for the average degree of such a graph (since every induced cycle in $K_{n,n}$ has length 4). In 2014, Bonamy, Charbit, and Thomassé showed that simple graphs with no induced cycle of length 0 (mod 3) have bounded chromatic number. [2] It is open whether or not these graphs are 3-colorable. It is also open whether, for any nonempty simple graph G not containing an induced cycle of length 0 (mod 3), there is an edge e such that $G \setminus \{e\}$ also does not have an induced cycle of length 0 (mod 3). But there is a fascinating connection between the two conjectures:

Theorem 1.0.1. *Suppose that for every simple nonempty graph G not containing an induced cycle of length* 0 *(mod* 3*), there is an edge e such that $G \setminus \{e\}$ also does not have an induced cycle of length* 0 *(mod* 3*). Then, every simple graph with no induced cycle of length* 0 *(mod* 3*) is* 3*-colorable.*

Proof. Suppose not. Let G be a minimal counterexample: a graph with no induced cycle of length 0 (mod 3) yet not 3-colorable. Then, by hypothesis, there exists an edge $xy \in E(G)$ such that $G \setminus xy$ has no induced cycle of length 0 (mod 3). By minimality of G, we know that $G \setminus \{xy\}$ is 3-colorable. Let $c\colon V(G) \setminus \mathbb{Z}_3$ be a proper coloring of $G \setminus \{xy\}$.

The idea is to modify the coloring one step at a time until x and y get different colors, at which point adding xy preserves the proper 3-coloring. For any proper 3-coloring c of $G \setminus \{xy\}$, the c-digraph on $G \setminus \{xy\}$ is formed by directing each edge from u to v such that $c(v) = c(u) + 1$. We claim that the c-digraph has no directed cycles. Suppose for a contradiction that it does; choose the shortest directed cycle, $v_1 v_2 \dots v_k$. This cycle is induced in the undirected graph $G \setminus e$. Moreover, starting from v_1, by the orientation, we have $c(v_2) = c(v_1) + 1$, $c(v_3) = c(v_1) + 2$, and so on, until $c(v_k) = c(v_1) + (k-1)$. Then, $c(v_1) = c(v_1) + k$. Therefore, k is congruent to 0 (mod 3). But we have just found an induced cycle of length 0 (mod 3) in $G \setminus \{xy\}$, which gives us a contradiction.

Hence, for any proper 3-coloring c of $G \setminus \{xy\}$, there is a vertex v that, in the c-digraph on $G \setminus \{xy\}$, is reachable from x and has outdegree zero. Thus, v has color $c(v)$ but all neighbors have color $c(v) - 1$. Now produce a 3-coloring c' from c by increasing the color on v by one. Notice that c' is a proper coloring, and the c'-digraph is formed from the c-digraph by reversing all edges incident to v. If $v = x$, then $c'(x) \neq c'(y)$, and we can add xy back to get a proper 3-coloring of G. Otherwise, in the c'-digraph, there is one less vertex reachable from x. So, we continue this process until x itself is a vertex with outdegree zero, at which point we change its color to make x and y have different colors. □

Finally, graphs with no induced cycles of length 0 (mod 3) arise when consider what we will call unimodular graphs. If G is a graph, we define the **modularity** $m(G)$ to be the number of even-sized independent sets minus the number of odd-sized independent sets. Equivalently, $m(G)$ is the evaluation of the independence polynomial

$$f_G(x) = \sum_{S \subseteq V(G); S \text{ independent}} x^{|S|}$$

at $x = -1$. We call a graph G **unimodular** if for every induced subgraph H of G, we have $m(H) \in \{-1, 0, 1\}$. As we will see, $m(C_n) \in \{-1, 0, 1\}$ if and only if n is not congruent to 0 (mod

3). So every unimodular graph has no induced cycles of length 0 (mod 3). It is open whether this is a complete characterization of unimodular graphs.

Our main results are the following:

Theorem 1.0.2. *Let G be a nonempty simple 2-connected graph with no cycles of length 0 (mod 3). Then, there exist two adjacent vertices of degree 2, or there exist two nonadjacent vertices of degree 2 with the same neighborhood.*

Theorem 1.0.3. *Let G be a simple graph with no cycles of length 0 (mod 3). Then, G is unimodular.*

Our tool is a swamp: a graph whose edges are weighted with the elements of $\mathbb{Z}_3$ such that no cycle has zero weight. Theorem 1.0.2 holds if every swamp of a 3-connected graph has an edge with weight 0. We consider the structure of simple graphs that have swamps (swampable graphs). We first establish some basic tools, including a construction of swamps on graphs with feedback sets of size at most 2 and special vertex feedback triples (a minor-closed class of graphs we call $3VFS$), the notion of a regular pair, and the structure of swamps with a regular pair. We then check the swampability of small graphs, confirming most of the computer-proven results in [1]. We then show that no graph with a K_5-minor is swampable, and then show that every nonplanar swampable graph other than the Wagner graph V_8 belongs to $3VFS$. Then, we proceed to swampable planar graphs. We show that swampable planar graphs with a prism plus a jump subdivision contain a zero-weight edge. Then, we also show that swampable planar graphs with a prism subdivision contain a zero-weight edge. Using Dirac's theorem [5], we then show that a swampable planar graph that does not contain a prism subdivision must be a wheel or $K_5 \setminus \{e\}$, both of which have two-vertex feedback sets. Finally, we show that every 3-connected graph in $3VFS$ has an edge with weight 0.

The third chapter describes results about modularity, the most significant one of which for our purposes is that the modularity of a graph with a three-edge path whose two internal vertices have degree two is minus the modularity of the graph formed by deleting the internal vertices path, identifying the ends if they were nonadjacent and deleting them otherwise. This will allow us to use Theorem 1.0.2 to deduce Theorem 1.0.3.

Chapter 2

The structure of graphs with no $0 \bmod 3$ cycles

2.1 Definitions

2.1.1 Standard notation

We mostly follow Bondy and Murty [3] for basic notation. All graphs are finite and may have loops and multiple edges. We denote the vertices and edges of G by $V(G)$ and $E(G)$. We denote an edge joining $u, v \in V(G)$ by uv. In non-simple graphs, the notation $\mu(u, v)$ denotes the number of edges joining u and v.

Let G be a graph, let $uv \in E(G)$, and let H be formed by deleting uv and adding a vertex x and edges ux and vx. We say that H is formed from G by **subdividing** uv to make a new vertex x, and, conversely, G is formed by H by **suppressing** the degree-2 vertex x. We say that H is a **subdivision** of G if there exists an injection $f\colon V(G) \to V(H)$ (each vertex in the image of f will be called a **corner vertex**; the other vertices are **series vertices**) and a collection of pairwise internally disjoint **branches** $\{P_{f(u)f(v)} : uv \in E(G)\}$ where each P_{ab} is an ab-path, such that every edge and every series vertex is on some branch. Equivalently, H is a subdivision of G if it can performed by repeatedly subdividing edges. If H_1 and H_2 are subdivisions of G, we say that the subdivisions **agree** if the corresponding functions f_1 and f_2 are equal.

If G is a graph and $X \subseteq V(G)$, the **induced subgraph** $G[X]$ is formed by deleting $V(G) \setminus X$.

A **path** is a walk with no repeated vertices or edges. We use the notation ab-path to denote a

path whose ends are a and b. If P is a path and $x, y \in V(P)$, the subpath of P whose ends are x and y is called the xy-segment of P, denoted $P[x, y]$. (The segment is the trivial path if $x = y$). Just as with intervals, we will use open brackets to exclude endpoints: for example, $P(x, y] = P[x, y] \setminus \{x\}$.

A **separation** of G is an ordered pair (A, B), where A and B are subgraphs of G, $E(A)$ and $E(B)$ partition $E(G)$, and $V(A) \cup V(B) = V(G)$. It is **nontrivial** if $V(A) \setminus V(B)$ and $V(B) \setminus V(A)$ are both nonempty. The **order** of a separation (A, B) is $|V(A) \cap V(B)|$. Notice that deleting $V(A) \cap V(B)$ produces a disconnected graph. For a positive integer k, a graph G is **k-connected** if there is no nontrivial separation of G of order less than k. As a special exception, however, the complete graph K_n is k-connected if and only if $k < n$.

If G and H are graphs, we say that H is a **minor** of G if it can be formed from a subgraph of G by contracting zero or more edges. A **model** of H in G is a collection $\{X_v \colon v \in V(H)\}$ of nonempty pairwise disjoint subsets of $V(G)$ such that for all $v \in V(G)$, $G[X_v]$ is connected and for all $u, v \in V(G)$ connected by k edges, there are at least k edges from X_u to X_v. It is straightforward to note that G contains H as a minor if and only if there is a model of H in G.

2.1.2 Swamps

If Γ is an abelian group, we say that a **Γ-weighted graph** is a graph G with a function $w \colon E(G) \to \Gamma$. If (G, w) is a Γ-weighted graph and $W = e_1 \dots e_k$ is a walk of G, the **weight** of W, denoted $w(W)$, is given by

$$w(W) = \sum_{i=1}^{k} w(e_i)$$

Given two weighted graphs (G, w) and (G', w'), a **weighted graph isomorphism** is a pair of functions (f, g) such that f is a graph isomorphism from G to G', g is an automorphism of Γ (which, in the case that $\Gamma = \mathbb{Z}_3$, means $g(x) \equiv \pm x$), and $w'(e) = g(w(f^{-1}(e)))$. We say that a $\mathbb{Z}_3$-weighted graph (G, w) is a **swamp** if no cycle of G has weight zero. If $f \colon A \to B$ is a function, and $C \subseteq A$, then $f|_C \colon C \to B$ is the function defined by $f|_C(x) = f(x)$ for all $x \in C$.

The following fact about $\mathbb{Z}_3$ is useful:

Lemma 2.1.1. *Let $a, b, c, d \in \mathbb{Z}_3$ be such that $a \neq b$ and $c \neq d$. Then, $\{a+c, a+d, b+c, b+d\} = \mathbb{Z}_3$.*

Proof. We have $b = a+1$ or $a = b+1$, and likewise $d = c+1$ or $c = d+1$. Without loss of generality, we may assume $b = a + 1$ and $d = c + 1$. Then,

$$\{a + c, a + d, b + c, b + d\} = \{a + c, a + c + 1, a + c + 1, a + c + 2\} = \mathbb{Z}_3$$

as desired. □

The following straightforward results will be useful:

Lemma 2.1.2. *If (G, w) is a swamp, then $(G, -w)$ is a swamp.*

Lemma 2.1.3. *If (G, w) is a swamp, the edges e with zero weight are a forest.*

Lemma 2.1.4. *If (G, w) is a swamp and H is a subgraph of G, then $\left(H, w|_{E(H)}\right)$ is a swamp.*

Lemma 2.1.5. *Let G be a graph. Let H be formed by subdividing edge xy of G to form a new vertex z. Let $w, w' \colon E(H) \to \mathbb{Z}_3$ be weight functions such that $w'(xz) + w'(zy) = w(xy)$ and $w'(e) = w(e)$ for $e \in E(H) \setminus \{xz, zy\}$. Then, (G, w) is a swamp if and only if (H, w') is a swamp.*

We say that G is swampable if there is a weight function w such that (G, w) is a swamp. Notice that if G is swampable, then it follows from Lemma 2.1.5 that every subdivision of a subgraph of G is swampable as well. It does not immediately follow, however, that if G is swampable and H is a minor of G, then H is swampable.

However, the following fact is true:

Lemma 2.1.6. *Let (G, w) be a swamp and let $F \subseteq E(G)$ be chosen such that $w(f) = 0$ for all $f \in F$. Then, $\left(G/F, w|_{E(G)\setminus F}\right)$ is a swamp.*

Proof. It suffices to prove the lemma when F is a single edge xy. Then, we want to show that $\left(G/xy, w|_{E(G)\setminus\{xy\}}\right)$ is a swamp. Let z be the vertex formed by contracting xy. Suppose for a contradiction there a zero-weight cycle C. If C does not contain z, then C is also a cycle in G, and $w(C) = 0$, contradicting that (G, w) is a swamp. So C must contain z. Before contracting, the edges of C either formed a cycle C' in G or an xy path P in G. In the former case, we have $w(C') = w(C) = 0$, contradicting that G is a swamp; in the latter, $w(P \cup \{xy\}) = w(P) = w(C) = 0$, again yielding a contradiction. This proves $\left(G/xy, w|_{E(G)\setminus\{xy\}}\right)$ is a swamp. □

The following lemma is a consequence:

Lemma 2.1.7. *Let (G, w) be a swamp, and let F be the forest of zero-weight edges. Let $S_1, \ldots, S_r$ be a partition of $V(G)$ such that each S_i is the vertex set of a component of F (in particular, if v is not incident to any edge of F, then $\{v\} = S_i$ for some i). Then, for all $1 \leq i < j \leq r$, there exists $a_{ij} \in \{1, 2\}$ such that $w(e) = a_{ij}$ for all e with one end in S_i and the other in S_j.*

Proof. All edges between S_i and S_j in G are parallel edges in G/F, and they all have nonzero weight. Since any two form a cycle, they must all have the same nonzero weight. □

We can also dispose of the question of swampability of non-simple graphs:

Lemma 2.1.8. *Let G be a graph, and H be formed from G by deleting all loops and removing all but one edge in each set of parallel edges. Then, G is swampable if and only if there is a swamp (H, w) where, for every pair of adjacent vertices $u, v \in V(H)$, $w(uv) \neq 0$ if $\mu_G(u, v) > 1$.*

Proof. ($\Longrightarrow$). Let (G, w) be a swamp. Then, for every pair $u, v \in V(G)$ of distinct vertices with $\mu(u, v) > 1$, there is exactly one nonzero weight k_{uv} on edges joining u and v. Now, for distinct adjacent $u, v \in V(G)$, set $w'(uv) = w(uv)$ if $\mu(u, v)$, else $w'(uv) = k_{uv}$. It is straightforward to verify that (H, w') is a swamp.

($\Longleftarrow$). If (H, w) is a swamp, then it is easy to see that, for $e \in E(G)$, taking $w'(e) = 1$ for e a loop, else $w'(e) = w(uv)$ where u and v are the ends of e, gives a swamp (G, w'). □

Because of Lemma 2.1.8, we will henceforth assume that all our graphs are loopless. (We will allow parallel edges since it can be helpful.)

Finally, we can dispose of low-connectivity cases:

Lemma 2.1.9. *Let G be a graph that is not 2-connected, and let w be a weight function. Then (G, w) is a swamp if and only if $(H, w|_{E(H)})$ is a swamp for each block H of G.*

Proof. The forward direction is trivial. The reverse direction holds because every cycle is in a block. □

Lemma 2.1.10. *Let G be a graph that is 2-connected with a nontrivial separation (A_1, A_2) of order 2. Let $\{x, y\} = V(A_1) \cap V(A_2)$. Let w be a weight function. Then (G, w) is a swamp if and only if there exists $i \in \{1, 2\}$ such that $(A_i, w|_{E(A_i)})$ is a swamp with all xy-paths having weight k, and $(A_{3-i} \cup \{e\}, w')$ is a swamp, where e is an added edge from x to y and w' is a weight function on $A_{3-i} \cup \{e\}$ satisfying $w'(e) = k$ and $w'(f) = w(f)$ for $f \neq e$.*

Proof. ($\Longrightarrow$). We first want to show that there exists an i for which all xy-paths in A_i have the same weight. (Notice that A_1 and A_2 both contain an xy-path since the separation is nontrivial and G is 2-connected.) If not, then we can find xy-paths in A_1 of weight k and ℓ, and we can find xy-paths in A_2 of weight m and n, where $k \neq \ell$ and $m \neq n$. Combining either path from A_1 with either path from A_2 produces a cycle whose weight is the sum of each path's weight; this gives us cycles in G having weight $k + m$, $k + n$, $\ell + m$, and $\ell + n$. By Lemma 2.1.1, at least one of these cycles has weight zero, contradicting that (G, w) is a swamp. Hence, one side of the separation (say A_1) has all paths the same length (say k). We then have $(A_i, w|_{E(A_i)})$ is a swamp by Lemma 2.1.4.

Also, we claim $(A_2 \cup \{e\}, w')$ is a swamp. Using any xy-path in A_1, we have that a subdivision of $A_2 \cup \{e\}$ is a subgraph of G, with the appropriate weights. Then, applying Lemmas 2.1.4 and 2.1.5 gives us that $(A_2 \cup \{e\}, w')$ is a swamp, as desired.

($\Longleftarrow$). Suppose for a contradiction that G is not a swamp. We may assume without loss of generality $i = 1$. Let C be a zero-weight cycle. If C is entirely contained in A_1 or A_2, then it produces a zero-weight cycle in A_1 or A_2, contrary to our hypothesis. So C must intersect both A_1 and A_2. We can therefore divide C into an xy-path P_1 in A_1 and an xy-path P_2 in A_2. But now, $w(P_1) = k$, by hypothesis, so rerouting C to use e instead of P_1 gives us $w'(C) = w(C)$ The cycle is still zero-weight, contradicting that $(A_2 \cup \{e\}, w')$ is a swamp. □

2.1.3 Jumps

Let G be a graph and H a subgraph of G. If $a, b \in V(H)$, an ab-**jump** in G with respect to H is an ab-path in G internally disjoint from H. If H is a subdivision of a 3-connected graph J, a jump from a corner vertex $v \in V(H)$ to a branch P of H is a jump from v to a series vertex of P. Similarly, if P and Q are distinct branches of H, a PQ-jump is a uv-jump where u and v are series vertices of P and Q, respectively. Often, we will state the jumps in terms of J, not H. So if we have a vertex $v \in V(J)$ and $e \in E(J)$, we mean by a ve-jump a vP-jump in G with respect to H, where P is the branch corresponding to e. Likewise, if $e, f \in E(J)$ are distinct, an ef-jump is a PQ jump where P and Q are the branches corresponding to e and f.

Suppose that G is a simple 3-connected graph, and G contains a subgraph H that is a subdivision of a simple 3-connected graph J. We want to choose H to be a subdivision of J that makes working with the graph easy. We say that H is a **direct subdivision** of J if each branch of H induces a path in G. For each C component of $G \setminus V(H)$, we say that C is **local** if there is an edge $xy \in E(J)$ such that every vertex of H adjacent to a vertex of C is x, y, or on the xy-branch.

Our goal is to find a subdivision H' of J with the same corner vertices such that H' is a direct subdivision and there are no local components (we call such a subdivision a **clean** subdivision). We first make a direct subdivision:

Lemma 2.1.11. *If G is a simple graph that contains a subdivision H of a simple graph J, then G contains a direct subdivision H' of J that agrees with H.*

Proof. For each branch P from x to y, replace it with a shortest xy-path P' in $G[V(P)]$. Because P' is a shortest path, it is induced. Moreover, each branch is replaced by a branch that is contained with the vertices of the old branch, so we maintain that we have a subdivision of J. The subdivision

H' we get is a direct subdivision that agrees with H. □

Lemma 2.1.12. *If G is a 3-connected simple graph that contains a direct subdivision of a 3-connected simple graph J, then G contains a clean subdivision H' of J that agrees with H.*

Proof. By induction on the number of vertices of $V(G) \setminus H$ that are in local components. If no vertices are in local components, then H is clean. Otherwise, choose a local component C of H, and let $P = v_0 v_1 \dots v_k$ be the branch of H (corresponding to edge $xy \in E(J)$) on which all neighbors of C lie. We claim there exist $0 \leq a < b < c \leq k$ such that v_b is adjacent to a vertex in a nonlocal component and there is a local component with vertices adjacent to v_a and v_c. Suppose not. Let $i_1 < \dots < i_j$ be the indices of vertices that are adjacent to nonlocal components. Then, for every local component, all of its neighbors on P are in one segment from $P[v_0, v_{i_1}]$, $P[v_{i_1}, v_{i_2}]$, and so on, through $P[v_{i_j}, v_k]$. But then, the ends of that segment are a cutset in G. This gives us a contradiction.

So, now we can choose a local component C' of H on the same branch P and $0 \leq a < b < c \leq k$ such that a is the smallest index of a neighbor of C', c is the largest index of a neighbor of C', and v_b is adjacent to a nonlocal component. Take P' to be a shortest path in C' from v_a to v_c. Replace P by $P[v_0, v_a] \cup P' \cup P[v_c, v_k]$. Then, P' is still an induced path (by our choice of a and c), and it does not interfere with any other branches. Moreover, we have reduced the number of vertices in a local component, added vertices to a nonlocal component, and left all other components of $G \setminus V(H)$ unchanged. So, this contradicts our choice of H.

Thus, a clean subdivision H' of J exists. □

Theorem 2.1.13. *Let G be a simple 3-connected graph, and let J be a simple 3-connected graph. Let H be a clean subdivision of J. Then, if $G \neq H$, one of the following holds:*

1. *If e is an edge of J such that the corresponding branch of H has a series vertex, then there exists a jump in G with respect to H between e and another edge of J or between e and a nonincident vertex of J.*

2. *Otherwise, H is a subgraph of G and one of the following holds:*

   (a) *H is a subgraph of G, and there exists a jump in G with respect to H between two nonadjacent vertices of H, or*

   (b) *H is a subgraph of G, $v_1 v_2 v_3$ is a triangle in H, and there is $u \in V(G) \setminus V(H)$ such that there is a 3-fan from u to $\{v_1, v_2, v_3\}$ internally disjoint from H.*

Proof. First, suppose H has a series vertex (that is, there is a branch of H that is not just an edge). Let uv be an edge of J where the corresponding branch (call it P) has a series vertex. Since G is 3-connected, we have $G \setminus \{u, v\}$ is connected. So, there is a path from a series vertex of P to some $w \in V(H) \setminus V(P)$. This w could be a series vertex of H (in which case we get from uv to a different edge of J), or it could be a corner vertex (giving us a jump between uv and a nonincident vertex).

Now suppose that H has no series vertices. We may assume $H = J$ is a proper subgraph of G. There are two cases. If $V(G) = V(H)$, then there is an edge of G not contained in H, which is a jump between two vertices of J. Otherwise, there is a vertex $u \in V(G) \setminus V(H)$. By Menger's theorem, there is a 3-fan from u to $V(H)$ internally disjoint from H. Let v_1, v_2, v_3 be the ends of the fan in $V(H)$, and let P_i be the uv_i-path in the fan. If any two of v_1, v_2, v_3 are nonadjacent (say v_1v_2), then $P_1 \cup P_2$ is a v_1v_2-jump. Otherwise, the last case holds. □

We can consider what happens to H and J after applying Theorem 2.1.13 and adding the appropriate jump to create a subgraph H' (and we may assume it is a clean subdivision of J). We can verify that H' is a subdivision of a 3-connected graph J'. And J' is formed from J as follows:

1. Adding a jump between distinct $e, f \in E(J)$ corresponds to subdividing e and f and adding an edge between the new series vertices.

2. Adding a jump from a vertex $v \in V(J)$ to a nonincident edge $e \in E(J)$ corresponds to subdividing e and adding an edge from the new series vertex to v.

3. Adding a jump between vertices of J corresponds to simply adding the edge.

4. Adding a vertex with three paths to a triangle corresponds to adding a new vertex to J whose neighborhood is a triangle.

We say that a 3-connected graph J' constructed as above is an **augmentation** of J.

From Theorem 2.1.13, we get the following:

Theorem 2.1.14. *Let G and J be simple 3-connected graphs such that G contains a subdivision of J. Then, G is isomorphic to J or there exists an augmentation J' of J such that G contains J' as a subdivision.*

2.2 The three-vertex feedback set construction

In this section, we discuss a method of constructing swamps from graphs that have special vertex feedback triples.

If G is a graph, a **vertex feedback set** is a set $S \subseteq V(G)$ such that $G \setminus S$ is a forest. If G is a graph, we say that $(a_1, a_2, b) \subseteq V(G)$ is a **special vertex feedback triple** if there exists a partition (A, B) of $V(G)$ such that $G[A]$ is a forest after deleting all edges from a_1 to a_2, $G[B]$ is a forest, $a_1, a_2 \in A$, $b \in B$, and for every edge from A to B, at least one end is in $\{a_1, a_2, b\}$. There is an alternative characterization that is easier to work with when finding special vertex feedback triples. With respect to an ordered triple (a_1, a_2, b), we will say that a component C of $G \setminus \{a_1, a_2, b\}$ is **type 1** if it has at most one edge to a_1 and at most one edge to a_2, and it is **type 2** if it has at most one edge to b.

Lemma 2.2.1. *Let G be a graph and let a_1, a_2, b be distinct vertices in G. Then (a_1, a_2, b) is a special vertex feedback triple if and only if for every component C of $G \setminus \{a_1, a_2, b\}$, C is a tree that is type 1 or type 2, and further there is at most one component that is not of type 2 and has an edge to both a_1 and a_2.*

Proof. ($\Longrightarrow$) Notice that, when we construct the partition (A, B) of $V(G)$, we must have that every component of $G \setminus \{a_1, a_2, b\}$ is completely within A or completely within B. Therefore, each such component C must be a tree. If it belongs to A, then it is type 1 (or else we get a cycle). If it belongs to B, then it is type 2. Moreover, there is at most one component belonging to A with edges to both a_1 and a_2 (as we can use two such components to create a cycle in A).

($\Longleftarrow$) Let A be the union of $\{a_1, a_2\}$ and all components not of type 2. Let B be the union of $\{b\}$ and all type 2 components. We claim that (A, B) verifies that (a_1, a_2, b) is a special vertex feedback triple. By construction, since each component of $G \setminus \{a_1, a_2, b\}$ is placed entirely in A or in B, we have that $\{a_1, a_2, b\}$ meet all edges from A to B. It remains to verify that A and B are forests. First, we see that B is a forest because every component has at most one edge to b, and so B is formed by taking the union of trees and adding a single vertex with at most one edge to each tree—hence, B is a forest. Now we check A is a forest. Notice that every component of $G \setminus \{a_1, a_2, b\}$ placed in A is type 1. By a similar argument as to B, $A \setminus \{a_1\}$ and $A \setminus \{a_2\}$ are both forests. Therefore, if A is not a forest, there must be a cycle using both a_1 and a_2. The cycle must run through two components of $G \setminus \{a_1, a_2, b\}$ placed in A. But those two components both have an edge to a_1 and an edge to a_2, contradicting our hypothesis. Therefore, A is a forest, and (A, B) demonstrates that (a_1, a_2, b) is a special vertex feedback triple. □

We can construct swamps on graphs with a special vertex feedback triple, according to the following two theorems. The first handles the case where a_1 and a_2 are nonadjacent; the second handles the case where a_1 and a_2 are adjacent.

Theorem 2.2.2. *Let G be a graph with special vertex feedback triple (a_1, a_2, b) and corresponding partition (A, B). Suppose a_1 is not adjacent to a_2. Let $k \in \{1, -1\}$ be given. Let w be given as follows:*

1. *$w(e) = k$ if e goes from A to B.*
2. *$w(e) = 0$ otherwise.*

Then (G, w) is a swamp.

Proof. Let Z be the set of all edges from A to B. Without loss of generality, we assume $w(e) = 1$ for all $e \in Z$. Suppose for a contradicting there exists a weight-zero cycle C. Then C crosses the cut between A and B an even number of times, and this number must be positive, because A and B are forests. Hence, the number of edges in Z used in C is even and a multiple of three, so C uses at least six edges in Z. But every edge in Z is incident to a_1, a_2, or b, and the cycle can use at most two edges incident to each of those vertices. Therefore, the cycle uses exactly six edges in Z, each of which has exactly one end in $\{a_1, a_2, b\}$. Let x and y be the neighbors of b in C, then $x, y \in A \setminus \{a_1, a_2\}$. Then, proceeding from b to y, the cycle cannot go to a_1 or a_2 (because both their neighbors are in B), nor can it cross the cut (since every crossing edge is incident to a_1, a_2, or b). This gives us a contradiction. □

The following constructions are immediate corollaries of the special vertex feedback triple construction:

Theorem 2.2.3. *Let G be a graph with special vertex feedback triple (a_1, a_2, b) and corresponding partition (A, B), with a_1 adjacent to a_2. Let $k \in \{1, -1\}$ be given. Let w be given as follows:*

1. *$w(e) = -k$ if e goes from a_1 to a_2.*
2. *$w(e) = k$ if e goes from A to B.*
3. *$w(e) = 0$ otherwise.*

Then (G, w) is a swamp.

Proof. Subdivide every edge from a_1 to a_2 and place the new vertices in B, and give weight k to both new edges. Then a_1 and a_2 no longer being adjacent, apply Theorem 2.2.2. □

A special case of the special vertex feedback triple construction occurs when G has a two-vertex feedback set. If $\{a_1, a_2\}$ is a two-vertex feedback set of G, then $G \setminus \{a_1, a_2\}$ is a forest. Hence,

by adding a dummy isolated vertex b to get a new graph G', we get (a_1, a_2, b) is a special vertex feedback triple of G'. Therefore, by applying Theorem 2.2.2 or Theorem 2.2.3, we have the following:

Theorem 2.2.4. *Let G be a graph with two-vertex feedback set $\{a, b\}$. Let $k \in \{1, -1\}$ be given. Let w be as follows:*

1. *$w(e) = 0$ or $w(e) = -k$ (ad lib.) if $e = ab$.*
2. *$w(e) = k$ if e has exactly one end in $\{a, b\}$.*
3. *$w(e) = 0$ otherwise.*

Then (G, w) is a swamp.

We will define $3VFS$ to be the class of all graphs with a vertex feedback set of size at most two or a special vertex feedback triple. It is immediate from the definition that $3VFS$ is closed under taking minors. And every such graph is swampable by Theorems 2.2.2, 2.2.3, and 2.2.4. Therefore we have the following:

Theorem 2.2.5. *The class $3VFS$ is a minor-closed class of swampable graphs.*

2.3 Regular and irregular pairs

Let (G, w) be a swamp, and H a subgraph of G. We say that the unordered pair $\{a, b\} \in V(H)$ is **regular** if $w(P) = k$ for all ab-paths of H; otherwise, it is **irregular**. We say that an irregular pair $\{a, b\}$ is **highly irregular** if for every $k \in \mathbb{Z}_3$ there is an ab-path P such that $w(P) = k$.

Let G be a graph and H a subgraph of G, and let k be a positive integer. Let $(v_1, \ldots, v_k)$ and $(w_1, \ldots, w_k)$ be k-tuples of vertices in H such that, for all $1 \leq i < j \leq k$, $v_i \neq v_j$, $v_i \neq w_j$, and $w_i \neq w_j$. We say that $(P_1, \ldots, P_k)$ is a **k-tuple of $(v_1, \ldots, v_k; w_1, \ldots, w_k)$-linkage paths** in G with respect to H if,

1. for all $1 \leq i \leq k$, either $v_i = w_i$ and P_i is the trivial path, or $v_i \neq w_i$ and P_1 is a $v_i w_i$-jump in G with respect to H, and
2. for all $1 \leq i < j \leq k$, P_i and P_j are completely disjoint.

If G is a graph, we say that (H_1, H_2) is an **$(a, b; c, d)$-linkable decomposition** if the following hold:

1. H_1 and H_2 are edge-disjoint subgraphs of G.

2. a and b are vertices of H_1, and c and d are vertices of H_2, and a, b, c, d are distinct except that possibly $a = c$ or $b = d$.

3. If $v \in V(H_1) \cap V(H_2)$, then $v = a = c$ or $v = b = d$.

4. There exists a pair (P_1, P_2) of $(a, b; c, d)$-linkage paths in G with respect to $H_1 \cup H_2$.

We have another useful lemma:

Lemma 2.3.1. *Let (G, w) be a swamp, and let H be a subgraph of G. Suppose $\{a, b\}$ is an irregular pair in H. Let $c \in V(G) \setminus V(H)$, and let P be an ac-path internally disjoint from H. Then, $\{c, b\}$ is irregular in $H \cup P$.*

Proof. Since $\{a, b\}$ is irregular in H, there are two ab-paths with different lengths in a. Since P is internally disjoint from H, prepending P to those two ab-paths gives two cb paths with different lengths, as desired. □

Theorem 2.3.2. *Let (G, w) be a swamp and (H_1, H_2) an $(a, b; c, d)$-linkable decomposition of G. Then $\{a, b\}$ is regular in H_1 or $\{c, d\}$ is regular in H_2.*

Proof. Suppose for a contradiction that both pairs are irregular. Then, there exist ab-paths in H_1 with two different weights, say k and ℓ, and there exist cd-paths in H_2 with two different weights, say m and n. Let (P_1, P_2) be a pair of linkage paths of the linkage decomposition. Let $p = w(P_1) + w(P_2)$. Then, by Lemma 2.1.1, we have $-p$ is equal to one of $k + m$, $k + n$, $\ell + m$, and $\ell + n$. Without loss of generality, suppose $-p = k + m$. Let P_3 be a weight-k ab-path in H_1, and P_4 be a weight-m cd path in H_2. Then, concatenating P_3, P_2, P_4 and P_1 gives a weight-zero cycle. This contradicts that (G, w) is a swamp. □

The following lemma is useful for deducing edge weights in a swamp once we have a known irregular pair. It is a special case of Theorem 2.3.2:

Lemma 2.3.3. *Let (G, w) be a swamp with three internally disjoint ab-paths P_1, P_2, P_3. Suppose $w(P_1) = 0$. Then $w(P_2) = w(P_3) = k$ for some $k \in \{-1, 1\}$.*

Proof. Since $P_1 \cup P_2$ forms a cycle that is not zero-weight, we have $w(P_2) = k$ for some $k \in \{-1, 1\}$. Likewise, $w(P_3)$ is nonzero. But then $w(P_2) + w(P_3)$ must be nonzero as well, so $w(P_3) = w(P_2) = k$, as desired. □

Theorem 2.3.4. *Let (G, w) be a swamp with a, b, c distinct vertices on a common cycle C. Then, if $\{a, b\}$ is regular and $\{a, c\}$ is regular, we have $w(P) = 0$, where P is the bc-subpath of $C \setminus \{a\}$. As a corollary, $\{b, c\}$ is irregular.*

Proof. Partition C into three edge-disjoint paths, P_1, P_2, P_3, such that P_1 is an ab-path, P_2 is a bc-path, and P_3 is an ac-path. Since $\{a, b\}$ is regular, we have $w(P_1) = w(P_2) + w(P_3)$. Since $\{a, c\}$ is regular, we have $w(P_3) = w(P_2) + w(P_1)$. These two equations imply $2w(P_2) = 0$, so $w(P_2) = 0$, as desired.

The corollary follows since having all three pairs regular implies $w(P_1) = w(P_2) = w(P_3) = 0$, so C is a weight-zero cycle. □

There is an extension to a cycle that has four vertices, each consecutive pair of which is regular:

Lemma 2.3.5. *Let (G, w) be a swamp with a, b, c, d distinct vertices, in order, on a common cycle C. Then $\{a, b\}$, $\{b, c\}$, $\{c, d\}$, and $\{d, a\}$ are not all regular in C.*

Proof. Suppose for a contradiction all four pairs are regular. Then partition C into four-edge disjoint paths: an ab-path P_1, a bc-path P_2, a cd-path P_3, and a da-path P_4. By regularity, we have $w(P_i) = \left(\sum_{j=1}^{4} w(P_j)\right) - w(P_i)$. Adding together, we get $\sum_{i=1}^{4} w(P_i) = 3\sum_{i=1}^{4} w(P_i)$. Hence, $\sum_{i=1}^{4} w(P_i) = 0$, so C is zero-weight, contradicting that (G, w) is a swamp. □

Another useful application allows us to find a zero-weight edge when there are two triangles sharing a vertex and the other two vertices are linkable outside the triangle:

Lemma 2.3.6. *Let (G, w) be a swamp and let H_1, H_2 be cycles. Let a, b_1, c_1, b_2, c_2 be distinct vertices such that $V(H_1) \cap V(H_2) = \{a\}$, $b_1, c_1 \in V(H_1)$, $b_2, c_2 \in V(H_2)$ such that for each choice of $u \in \{b_1, c_1\}$, $v \in \{b_2, c_2\}$, there is a uv-path in G internally disjoint from $H_1 \cup H_2$. Then, there exists $i \in \{1, 2\}$ such that $\{a, b_i\}$ and $\{a, c_i\}$ are both regular in H_i (and hence, by Theorem 2.3.4, $w(P) = 0$ where P is the b_ic_i path of H_i not containing a).*

Proof. Suppose not. Then, there is a $u \in \{b_1, c_1\}$ such that $\{a, u\}$ is irregular in H_1, and there is a $v \in \{b_2, c_2\}$ such that $\{a, v\}$ is irregular in H_2. But then we use the uv-path to get a contradiction of Theorem 2.3.2. □

There is an analogous result with a similar proof:

Lemma 2.3.7. *Let (G, w) be a swamp and let H_1, H_2 be disjoint cycles. Let $a_1, b_1, c_1 \in V(H_1)$, $a_2, b_2, c_2 \in V(H_2)$ and let P be an a_1a_2-path internally disjoint from $H_1 \cup H_2$ such that for each choice of $u \in \{b_1, c_1\}$, $v \in \{b_2, c_2\}$, there is a uv-path in G internally disjoint from $H_1 \cup H_2 \cup P$. Then, there exists $i \in \{1, 2\}$ such that $\{a_i, b_i\}$ and $\{a_i, c_i\}$ are both regular in H_i (and hence, by Theorem 2.3.4, $w(P) = 0$ where P is the b_ic_i path of H_i not containing a).*

We end this section with another useful result about highly-irregular pairs:

Lemma 2.3.8. *Let (G, w) be a swamp, and let H be a subgraph of G. Suppose $\{a, b\} \subseteq V(H)$ is a highly irregular pair. Then, there is no ab-path internally disjoint from H.*

Proof. Suppose for a contradiction there is such a path P_1. Choose an ab-path P_2 in H such that $w(P_2) = -w(P_1)$. Then P_1 and P_2 together form a weight-zero cycle, contradicting that (G, w) is a swamp. □

2.4 The structure of 3-connected graphs with a regular pair

In this section, we prove the following theorem:

Theorem 2.4.1. *Let (G, w) be a swamp. Let $\{a, b\}$ be a regular pair of G, such that all ab-paths have weight k. Suppose that $G \cup \{ab\}$ is 3-connected. Then, the following hold:*

1. *$G \setminus \{a, b\}$ is a tree T, with $w(e) = 0$ for all $e \in T$;*
2. *$k \neq 0$;*
3. *$w(e) = k$ for all edges from a to b;*
4. *$w(e) = -k$ otherwise.*

Lemma 2.4.2. *Let G be a graph and let $a, x, y \in V(G)$ be distinct vertices. Let P_1 be an ax-path, and let P_2 be an ay-path. Then, $V(P_1) \cap V(P_2) = \{a\}$, or there exists a vertex $z \in V(G) \setminus \{x, y, a\}$, and a 3-fan from z to $\{a, x, y\}$.*

Proof. If $V(P_1) \cap V(P_2) = \{a\}$, we are done. Otherwise choose z to be the first vertex on P_2 that is also on P_1 as P_2 is traversed from z to a. Then, it is immediate from the construction that $P_2[y, z]$, the $P_1[a, z]$, and the $P_1[z, x]$ are the desired 3-fan. □

Lemma 2.4.3. *Let (G, w) be a swamp. Let $\{a, b\}$ be a regular pair of G. Let $x, y \in V(G) \setminus \{a, b\}$ be disjoint. Let $P_1, \ldots, P_4, Q$ be paths as follows:*

1. *P_1 is an ax-path,*
2. *P_2 is a bx-path,*
3. *P_3 is an ay-path,*
4. *P_4 is a by-path,*

5. *Q is an xy-path,*

such that all five paths are internally disjoint except that P_1 may intersect P_3 and P_2 may intersect P_4. Then $w(Q) = 0$.

Proof. We first prove the case where P_1 and P_3 are internally disjoint. Notice that $P_1 \cup Q \cup P_3$ determines a cycle. If $\{a, x\}$ or $\{a, y\}$ is irregular in $P_1 \cup Q \cup P_3$, then $\{a, b\}$ is irregular in G by Lemma 2.3.1. Thus, $\{a, x\}$ and $\{a, y\}$ must both be regular. By Theorem 2.3.4, we have $w(Q) = 0$, as desired.

Now suppose P_1 and P_3 are not internally disjoint. Applying Lemma 2.4.2, we get that, in $P_1 \cup P_3$ there is a vertex z with a 3-fan from z to $\{a, x, y\}$. Let P_1' be the zx-path, P_3' the zy-path, and R the az-path. Then, in $G \setminus (V(R) \setminus \{z\})$, we have that $\{z, b\}$ is regular (by Lemma 2.3.1) and $(P_1', P_2, P_3', P_4, Q)$ satisfy the properties of the lemma, and P_1' and P_3' are internally disjoint. So, again, $w(Q) = 0$. □

Lemma 2.4.4. *Let (G, w) be a swamp. Let $\{a, b\}$ be a regular pair of G. Let $x, y \in V(G) \setminus \{a, b\}$ be disjoint. Let $P_1, \ldots, P_4, Q$ be paths as follows:*

1. *P_1 is an ax-path,*
2. *P_2 is a bx-path,*
3. *P_3 is an ay-path,*
4. *P_4 is a by-path,*
5. *Q is an xy-path,*

such that all five paths are internally disjoint except that P_1 may intersect P_3 and P_2 may intersect P_4. Suppose further that, for every $v \in V(Q)$, there exists either a va-path internally disjoint from $V(Q) \cup V(P_2) \cup V(P_4)$ or a vb-path internally disjoint from $V(Q) \cup V(P_1) \cup V(P_3)$. Then, for every edge $e \in E(Q)$, we have $w(e) = 0$.

Proof. Label the vertices of Q in order $x = v_1, \ldots, v_n = y$. We want to show $w(v_i v_{i+1}) = 0$. Let A be the set of all vertices $v \in V(Q)$ for which there exists a va-path internally disjoint from $V(Q) \cup V(P_2) \cup V(P_4)$; let B be the set of all vertices $v \in V(Q)$ for which there exists a vb-path internally disjoint from $V(Q) \cup V(P_1) \cup V(P_3)$. By hypothesis, $A \cup B = V(Q)$.

There are now two cases to check.

1. Suppose v_i and v_{i+1} are both in A or both in B; without loss of generality, we may assume they are both in A. Let P_1' and P_3' be the v_ia- and $v_{i+1}a$-paths, and let $P_2' = P_2 \cup Q[v_1, v_i]$ and $P_4' = P_4 \cup Q[v_{i+1}, v_n]$. Then, applying Lemma 2.4.3 with $P_1', \ldots, P_4'$ and edge v_iv_{i+1}, we get $w(v_iv_{i+1}) = 0$, as desired.

2. Suppose v_i and v_{i+1} are in different sets; without loss of generality, we may assume $v_i \in A$ and $v_{i+1} \in B$. Let P_1' be the v_ia-path, and let P_4' be the $v_{i+1}b$-path. Then take $P_2' = P_2 \cup Q[v_1, v_i]$ and $P_3' = P_3 \cup Q[v_n, v_i + 1]$. Again, apply Lemma 2.4.3 with $P_1', \ldots, P_4'$ and edge v_iv_{i+1} to get $w(v_iv_{i+1}) = 0$.

This completes the proof. □

Lemma 2.4.5. *Let (G, w) be a swamp. Let $\{a, b\}$ be distinct vertices in G. Let C be a cycle in $G \setminus \{a, b\}$, and let $\{c, d, e\} \subseteq V(C)$ such that there exist the following paths, each internally disjoint from $\{a, b\} \cup V(C)$:*

1. *An ac-path P_1.*
2. *A bd-path P_2.*
3. *An ae-path P_3.*
4. *An be-path P_4.*

Further, suppose P_1 is disjoint from P_2 and P_4, and P_2 is disjoint from P_3. Then, $\{a, b\}$ is irregular in G.

Proof. By Theorem 2.3.4, at least one of $\{c, d\}, \{c, e\}, \{d, e\}$ is irregular in C. In each case, we can apply Lemma 2.3.1 to show $\{a, b\}$ is irregular in G, as follows:

1. If $\{c, d\}$ is irregular in C, use P_1 and P_2.
2. If $\{c, e\}$ is irregular in C, use P_1 and P_4.
3. If $\{d, e\}$ is irregular in C, use P_2 and P_3. □

Lemma 2.4.6. *Let (G, w) be a swamp. Let $\{a, b\}$ be disjoint vertices in G. Let C be a cycle in $G \setminus \{a, b\}$, and let $\{c, d, e, f\} \subseteq V(C)$ be vertices on C, in order. Let $P_1, \ldots, P_4$ be paths, each internally disjoint from $\{a, b\} \cup V(C)$:*

1. *An ac-path P_1.*

2. *An ad-path P_2.*

3. *A be-path P_3.*

4. *A bf-path P_4.*

Further, suppose all four paths are disjoint except that P_1 and P_2 may intersect, and P_3 and P_4 may intersect. Then $\{a, b\}$ is irregular in G.

Proof. It suffices to consider the case when $V(P_1) \cap V(P_2) = \{a\}$ and $V(P_3) \cap V(P_4) = \{b\}$. The general case follows from applying Lemma 2.4.2.

Write the cycle C as the union of four paths $Q_1, \ldots, Q_4$, where Q_1 is a cd-path, Q_2 a de-path, Q_3 an ef-path, Q_4 an fc-path. By Lemma 2.4.3, we have $w(Q_1) = w(Q_3) = 0$. Also, $\{d, e\}$ is regular in C (else we apply Lemma 2.3.1 to get $\{a, b\}$ is irregular in G). Therefore, $w(Q_2) = w(Q_4) = k$ for some $k \in \{-1, 1\}$.

We also see that $w(P_1) = w(P_2) = \ell$ and $w(P_3) = w(P_4) = m$ (or else $\{a, b\}$ is irregular, as desired), where $\ell, m \in \{-1, 1\}$. But we can now form cycles of weight $k + \ell$ ($P_1Q_4Q_3Q_2P_2$) and $k + m$ ($P_4Q_4Q_1Q_2P_3$). So $k = \ell = m$. But now $P_1Q_4P_4P_3Q_2P_2$ is a zero-weight cycle. Hence, we get a contradiction, so $\{a, b\}$ must be irregular. □

Lemma 2.4.7. *Let (G, w) be a swamp. Let $\{a, b\}$ be a regular pair of G. Suppose that $G \cup \{ab\}$ is 3-connected. Then every edge of $G \setminus \{a, b\}$ has weight zero.*

Proof. Let $xy \in E(G \setminus \{a, b\})$ be given. We claim there exist the following paths:

1. An ax-path P_1,

2. A bx-path P_2,

3. An ay-path P_3,

4. A by-path P_4,

such that the paths are internally disjoint from each other and from the path xy, except that P_1 may intersect P_3 and P_2 may intersect P_4. If the claim holds, then $w(xy) = 0$ by Lemma 2.4.3.

Suppose that there is no possible choice of paths $P_1, \ldots, P_4$. Choose a vertex $z \in (V(G) \setminus \{x, y\})$ and pairwise internally disjoint paths P_1, P_2, Q satisfying the following:

1. P_1 is an ax-path and P_2 is a bx-path,

2. $z \in V(P_1) \cup V(P_2)$, and

3. Q is an yz-path.

and subject to that, choose z and the paths to minimize the following:

1. if $z \in V(P_1) \setminus \{x\}$, the number of vertices on the path $P_1[a, z]$,

2. if $z \in V(P_2) \setminus \{x\}$, the number of vertices on the path $P_2[b, z]$.

The vertex and set of paths exist because there is a path in $G \setminus \{x\}$ from y to $(V(P_1) \cup V(P_2)) \setminus \{x\}$. Now assume without loss of generality that $z \in V(P_1) \setminus \{x\}$. Consider any 2-fan from y to $(V(P_1) \cup V(P_2)) \setminus \{x\}$ (such fans exist by the fan lemma). If exactly one endpoint of the fan is in $V(P_2) \setminus \{x\}$, then, we can extend the fan to get paths P_3 and P_4, giving a contradiction. Suppose both endpoints of the fan are in $V(P_2) \setminus \{x\}$. If Q is internally disjoint from both of the fan paths, we discard one fan path and add Q to get a 2-fan with exactly one endpoint in $V(P_2) \setminus \{x\}$, and again we have a contradiction. Otherwise, take the first intersection of Q with a fan path as Q is traversed from z, then reroute the fan path to use Q instead. This again creates a 2-fan with exactly one endpoint in $V(P_2) \setminus \{x\}$, so we have a contradiction. Hence, every 2-fan from y to $(V(P_1) \cup V(P_2)) \setminus \{x\}$ has both endpoints on $V(P_1) \setminus \{x\}$. By a similar rerouting argument with Q, there is a 2-fan from y to $\{z, c\}$, where c is a vertex on $P_1[z, x]$. Let R_1 be the yz-path in the fan, and let R_2 be the yc-path in the fan.

Now, because $\{z, x\}$ is not a cutset, there must be a jump from $(V(R_1) \cup V(R_2) \cup V(P_1(z, x))$ to $(V(P_1[a, z)) \cup V(P_2) \setminus \{z, x\}$. No such jump comes from $V(R_1) \cup V(R_2)$. If its other end were on $V(P_2)$, we could reroute to get a 2-fan with one end in $V(P_2) \setminus \{x\}$. If its other end were on $V(P_1[a, z))$, we could reroute to get a path from y to a vertex on $V(P_1[a, z))$, which contradicts our choice of Q and z. Hence, the jump S must come from some $w \in V(P_1(z, x))$. If the jump has its other end on $V(P_1[a, z))$, then, calling the other end z', we choose $P_1' = P_1[a, z'] \cup S \cup P_1[w, x]$, and $Q' = R_1 \cup P_1[z, z']$, and the new choice of z' and Q' contradicts minimality. So, the jump has its other end on $V(P_2) \setminus \{x\}$. We have two cases. If $w \in P_1(z, c]$, then there exist four paths that satisfy Lemma 2.4.5 for the cycle $\{xy\} \cup R_2 \cup P_1[c, x]$ (with the paths coming from $\{c, x, y\}$, contradicting that $\{a, b\}$ is regular. Otherwise, $w \in P_1(c, x)$, and then, using the $\{xy\} \cup R_2 \cup P_1[c, x]$ (with the paths coming from $\{c, x, y, z\}$), we have four paths satisfying Lemma 2.4.6, again contradicting that $\{a, b\}$ is regular.

Therefore, we must have paths $P_1, \ldots, P_4$, and then $w(xy) = 0$. It immediately follows that $G \setminus \{a, b\}$ has all edges weight zero, so no cycle is possible. □

With the lemma proved, the main theorem of this section is straightforward:

Proof of Theorem 2.4.1. By Lemma 2.4.7, and the fact that $G \setminus \{a, b\}$ is connected, we have $G \setminus \{a, b\}$ is a tree T, with $w(e) = 0$ for all $e \in T$. Also, it is trivial that $w(e) = k$ for each edge e joining a and b. Since $G \setminus \{b\}$ is 2-connected, there are at least two edges from a to $G \setminus \{b\}$. If two different edges from a to $G \setminus \{b\}$ had different weight, then, using the tree, we could get two different weight paths from a to b, contradicting that $\{a, b\}$ is regular. Also, in $G \setminus \{a, b\}$, every pair of vertices is connected by a zero-weight path. Thus, all edges from a to $G \setminus \{b\}$ have the same nonzero weight, ℓ. Similarly, all edges from b to $G \setminus \{a\}$ have the same nonzero weight, m. We have $\ell + m = k$, and neither ℓ nor m is zero. Thus, $\ell = m = -k$, as desired. □

The following corollary is immediate from applying Lemma 2.1.5:

Lemma 2.4.8. *Let (G, w) be a swamp. Let $\{a, b\}$ be a regular pair of G, such that all ab-paths have weight k. Suppose that $G \cup ab$ is a subdivision of a 3-connected graph, and a and b are corner vertices. Then, the following hold:*

1. *$G \setminus \{a, b\}$ is a tree T, and for every branch P of $G \setminus \{a, b\}$, $w(P) = 0$.*
2. *$k \neq 0$;*
3. *$w(P) = k$ for all branches P from a to b;*
4. *$w(P) = -k$ for all branches P with exactly one end in $\{a, b\}$.*

2.5 Swampability of small graphs

In this section, we will consider the possible swamps for small graphs: K_4, K_5, $K_{3,3}$, the prism, the cube, the octahedron, and the seven-vertex graph formed by deleting a vertex from the cube and adding a triangle on its neighborhood (which we will call the **Y-Δ cube**). These results will be useful in later proofs, since if (G, w) is a swamp, and G contains a subdivision of one of these graphs, then w must respect the possible swamps of the small graph (or, if the small graph is not swampable, then we contradict that G is swampable).

2.5.1 K_4 swamps

Theorem 2.5.1. *Let (K_4, w) be a swamp where K_4 has vertex set $\{v_1, v_2, v_3, v_4\}$. Then, up to weighted graph isomorphism, w satisfies one of the following:*

1. *$w(v_1v_2) = 0$, $w(v_3v_4) = -1$, $w(e) = 1$ for $e \notin \{v_1v_2, v_3v_4\}$.*

2. $w(v_1v_2) = w(v_3v_4) = 0$, $w(e) = 1$ *for* $e \notin \{v_1v_2, v_3v_4\}$.

3. $w(v_1v_2) = w(v_2v_3) = 0$, $w(v_1v_3) = -1$, $w(e) = 1$ *for* e *incident to* v_4.

Proof. Let F be the forest of weight-zero edges in the swamp (K_4, w) We prove the following:

1. F is claw-free. Let v be the vertex in F with degree 3. The sum of the weights of any two edges not incident to v is not zero, because they can be completed to a cycle by adding edges in F. Hence, all edges not incident to v have the same nonzero weight. But then they form a weight-zero 3-cycle.

2. F has at least one edge. If not, then w gives us a (not proper) 2-coloring of the edges of G. Because G has no weight-zero cycle, the coloring has no monochromatic triangles, nor any 4-cycles with two edges of each color.

   Pick v_1. We may assume $w(v_1v_2) = w(v_1v_3) = k$. If $w(v_1v_4) = k$, then all other edges must have color $-k$, which creates a monochromatic triangle. Hence, $w(v_1v_4) = -k$. Also, $w(v_2v_3) = -k$, since otherwise $v_1v_2v_3$ would be a monochromatic triangle. Now, we have $v_1v_3v_2v_4$ is a cycle with two edges of weight $-k$ and one edge of weight k; the other edge, v_2v_4 must therefore have weight $-k$. This forces $w(v_3v_4) = k$. But now $v_1v_3v_4v_2$ is a cycle with two edges of weight k and two of weight $-k$—a contradiction.

3. F has no three-edge path. Suppose $w(v_1v_2) = w(v_2v_3) = w(v_3v_4)$. Then, any two of the remaining three edges can completed to a cycle using F. Hence, they must all have the same weight, k. But now $v_3v_1v_4v_2$ is a zero-weight cycle.

4. If F has a two-edge matching, case 2 holds. Since F has no three-edge path, we have that F has two components, each a single edge. Then all other edges have the same nonzero weight by Lemma 2.1.7. This is a swamp by Theorem 2.2.2 (with a special vertex feedback triple consisting of the two ends of one weight-zero edge and any third vertex).

5. If F is a two-edge path $v_1v_2v_3$, then case 3 holds. By lemma 2.1.7, all edges incident to v_4 have the same nonzero weight, say k. Then, $w(v_1v_3)$ is not zero (since $v_1v_2v_3$ would be a zero-weight cycle), nor is it k (since $v_1v_3v_4$ would be a zero-weight cycle), so its weight is $-k$. This is a swamp by Theorem 2.2.3, with special vertex feedback triple (v_1, v_3, v_4).

6. If F has a single edge, say v_1v_2, then case 1 holds. By Lemma 2.1.7, there exist $k, \ell \in \{1, 2\}$ such that $k = w(v_1v_3) = w(v_2v_3)$ and $\ell = w(v_1v_4) = w(v_2v_4)$. Also, since $v_1v_3v_2v_4$ is a cycle, we have $k + \ell \neq 0$, so $k = \ell$. Finally, we have $w(v_3v_4) \neq 0$ by hypothesis, nor does $w(v_3v_4) = k$

(since $v_1v_3v_4$ would be a zero-weight cycle), so $w(v_3v_4) = -k$. This is also a swamp, as verified by Theorem 2.2.4 applied to the two-vertex feedback set $\{v_3, v_4\}$.

With all possible cases for F exhausted, the theorem holds. □

For any $k \geq 3$, the **wheel graph** W_k is formed by taking a length k cycle and adding a vertex adjacent to every vertex on the cycle. Before moving on to the swampability of other graphs, we first prove this useful lemma about nonzero swamps.

Lemma 2.5.2. *For all $k \geq 3$, if (W_k, w) is a swamp, there exists an edge $e \in E(W_k)$ such that $w(e) = 0$.*

Proof. By induction on k. The base case ($k = 3$) is Theorem 2.5.1. Now, suppose $k \geq 4$ is given and the proposition holds for $k - 1$. Suppose for a contradiction the proposition is false for k; that is, there is a weight function $w\colon W_k \to \mathbb{Z}_3 \setminus \{0\}$ such that (W_k, w) is a swamp. Label the vertices of W_k as $v_0, v_1, \ldots, v_k$ such that v_0 is adjacent to v_i for all i and $v_1 \ldots v_k$ is a cycle. Now, for every i, deleting the edge v_0v_i produces a subdivision of W_{k-1}. By Lemma 2.1.5 and the induction hypothesis, we must have $w(v_{i-1}v_i) + w(v_iv_{i+1}) = 0$ (where the indices wrap around if necessary). Adding over all i gives us

$$2\left(w(v_1v_2) + \cdots + w(v_{k-1}v_k) + w(v_kv_1)\right) = 0$$

and therefore, the cycle $v_1 \ldots v_k$ is zero-weight. This contradicts that we have a swamp and completes the inductive step. □

2.5.2 K_5 is not swampable

Theorem 2.5.3. *K_5 is not swampable.*

Proof. Suppose for a contradiction that (K_5, w) is a swamp. We show that every two-edge matching in K_5 has at least one weight-zero edge. Let ab, cd be a matching, and let e be the unsaturated vertex. Define $H_1 = G[\{a, b, e\}]$ and $H_2 = G[\{c, d, e\}]$. By Lemma 2.3.6, either $\{a, e\}$ and $\{b, e\}$ are regular in H_1, or $\{c, e\}$ and $\{d, e\}$ are regular in H_2. Without loss of generality, assume $\{a, e\}$ and $\{b, e\}$ are both regular in H_1. By Theorem 2.3.4, we have $w(ab) = 0$, as desired.

Since every two-edge matching of K_5 has at least one weight-zero edge, and $E(K_5)$ can be partitioned into five two-edge matchings, we have that K_5 has at least five weight-zero edges. But the weight-zero edges form a forest (Theorem 2.1.3), and a forest subgraph of K_5 has at most four edges. Hence, K_5 is not swampable. □

2.5.3 $K_{3,3}$ swamps

Theorem 2.5.4. *Let $K_{3,3}$ have a bipartition $\{a_1, a_2, a_3\}$ and $\{b_1, b_2, b_3\}$, and let $(K_{3,3}, w)$ be a swamp. Then, up to weighted graph isomorphism, one of the following holds:*

1. *The function w satisfies $w(e) = 0$ if e is incident to a_1 and $w(e) = 1$ otherwise.*
2. *The function w satisfies $w(e) = 0$ if $e \in \{a_1b_1, a_1b_2, a_2b_3, a_3b_3\}$ and $w(e) = 1$ otherwise.*

Proof. Let F be the forest of zero-weight edges of $K_{3,3}$. We prove the following propositions:

1. F has no three-edge path. Suppose F has a three-edge path $v_1v_2v_3v_4$. Because there are edge disjoint paths from v_2 to v_4 and v_1 to v_3, and v_1v_4 is an edge, we get that $K_{3,3}$ has a K_4 subdivision, three of whose edges are in F. By Lemma 2.1.5, this implies K_4 has a swamp with a three-edge path of zero-weight edges, contradicting Theorem 2.5.1.

2. If F has a claw, then case 1 holds. Suppose (A, B) is a bipartition of $K_{3,3}$ where $A = \{a_1, a_2, a_3\}$ and $B = \{b_1, b_2, b_3\}$, and where $w(a_1b_i) = 0$ for all $i \in \{1, 2, 3\}$. Since F has no three-edge path, we have that F has exactly three edges, all in the claw. By Lemma 2.1.7, there exist $k, \ell \in \{1, 2\}$ such that $w(a_2b_i) = k$ and $w(a_3b_i) = \ell$ for all $i \in \{1, 2, 3\}$. Moreover, since $a_2b_1a_2b_2$ is a cycle of weight $2(k + \ell)$, we must have $k = \ell$. This is precisely case 1. This is a swamp according to Theorem 2.2.4 with two-vertex feedback set $\{a_2, a_3\}$.

3. If F has a two-edge path but not a claw, then case 2 holds. Let (A, B) be a bipartition of $K_{3,3}$ where $A = \{a_1, a_2, a_3\}$ and $B = \{b_1, b_2, b_3\}$, and where $w(a_1b_1) = w(a_1b_2) = 0$ and $w(a_1b_3) = k \in \{1, 2\}$. Since F has no 3-edge path and no claw, one component of F has vertex set $\{a_1, b_1, b_2\}$. By Lemma 2.1.7, there exist $\ell, m \in \{1, 2\}$ such that $w(a_2b_1) = w(a_2b_2) = \ell$ and $w(a_3b_1) = w(a_3b_2) = m$. Since $a_2b_1a_3b_2$ is a cycle whose weight is $2(\ell + m)$, we have $\ell + m \neq 0$, so $\ell = m$.

   Now, since the cycle $a_1b_3a_3b_1$ is not zero-weight, we have $k + w(b_3a_3) + \ell \neq 0$. Also, since $a_1b_3a_3b_2a_2b_1$ is not zero-weight, we have $k + w(b_3a_3) + 3\ell \neq 0$, or equivalently, $k + w(b_3a_3) \neq 0$. Hence, $k + w(b_3a_3) = \ell$, so $w(b_3a_3) = \ell - k$. Similarly, $w(b_3a_2) = \ell - k$. Now, consider the cycle $a_2b_2a_3b_3$, whose weight is $\ell + \ell + (\ell - k) + (\ell - k) = \ell + k$. Since $\ell + k \neq 0$ and both ℓ and k are nonzero, we have $\ell = k$. This implies $w(b_3a_2) = w(b_3a_3) = 0$, and we verify case 2 holds. This is a swamp as per Theorem 2.2.2 with special vertex feedback triple (b_1, b_2, b_3).

4. F is not a matching. Suppose, for a contradiction, that F is a matching (F may be empty). Let (A, B) be a bipartition of $K_{3,3}$ where $A = \{a_1, a_2, a_3\}$, $B = \{b_1, b_2, b_3\}$ and $F = \{a_ib_i \colon 1 \leq$

$i \leq |F|\}$. We have four cases based on the number of edges in F.

(a) $|F| = 3$. By Lemma 2.1.7, there exist $k, \ell, m \in \{1, 2\}$ such that $w(a_1b_2) = w(a_2b_1) = k$, $w(a_1b_3) = w(a_3b_1) = \ell$, and $w(a_2b_3) = w(a_3b_2) = m$. Also, since $a_1b_2a_3b_3a_2b_1$ is a cycle with weight $2k + 2m$, we have $k = m$. Likewise, $a_1b_3a_2b_2a_3b_1$ is a cycle with weight $2\ell + 2m$, so $\ell = m$. Therefore, all six edges not in F have the same weight, k. But $a_1b_2a_3b_1a_2b_3$ is a cycle with weight $6k = 0$. This gives a contradiction.

(b) $|F| = 2$. By Lemma 2.1.6, we have that G/F is a swamp of K_4 with no zero-weight edges. This contradicts Theorem 2.5.1.

(c) $|F| = 1$. By Lemma 2.1.6, we have that G/F is a swamp of W_4 with no zero-weight edges. This contradicts Lemma 2.5.2.

(d) $|F| = 0$. Notice that for $i \in \{1, 2, 3\}$, deleting a_ib_i produces a subdivision of K_4 in which all edges not incident to a_i or b_i are not subdivided. By suppressing the degree two vertices, we get a swamp of K_4 in which the subdivided edges were a perfect matching. Hence, the only possibilities are cases 1 and 2 of Theorem 2.5.1. In both those cases, however, we have that the edges incident to a zero-weight edge have the same nonzero weight. Hence, the weight of all edges not incident to a_i or b_i are the same. In particular, we have

$$w(a_1b_1) = w(a_1b_2) = w(a_2b_1) = w(a_2b_2)$$

and

$$w(a_2b_2) = w(a_2b_3) = w(a_3b_2) = w(a_3b_3)$$

and further

$$w(a_3b_3) = w(a_3b_1) = w(a_1b_3) = w(a_1b_1)$$

So all nine edges have the same nonzero weight. This gives us a contradiction, however, because $a_1b_1a_2b_2a_3b_3$ is a zero-weight cycle.

□

It is very useful to find $K_{3,3}$ swamps because all edges with nonzero weight have the same weight. This lemma will be useful:

Lemma 2.5.5. *Let G be a graph and let H_1, H_2 be subgraphs of G that are isomorphic to a subdivision of $K_{3,3}$. Suppose further that $H_1 \cap H_2$ includes a cycle containing four branch vertices of*

both H_1 and H_2. Then there exists a $k \in \{-1, 1\}$ such that if P is a branch of H_1 or of H_2, then $w(P) \in \{0, k\}$.

Proof. Apply Theorem 2.5.4 to H_1 and H_2 respectively. We have that there exist $k_1, k_2 \in \{-1, 1\}$ such that all branches in H_i have weight 0 or k_i. There is at least one branch in the given 4-cycle that has nonzero weight (because otherwise the cycle would be zero-weight). This branch is a branch in both H_1 and H_2, so its weight must be k_1 and k_2. So $k_1 = k_2$. This gives us the desired result. □

2.5.4 Prism swamps

In this section, we prove the following about the swamps of the prism graph $K_3 \square K_2$. We let $K_3 \square K_2$ be labeled such that a_1, a_2, a_3 and b_1, b_2, b_3 are disjoint cliques, and $a_i b_i$ is an edge for $i \in \{1, 2, 3\}$. If $1 \leq i < j \leq 3$, the edges $a_i a_j$ and $b_i b_j$ are said to be **corresponding**.

Theorem 2.5.6. *Suppose $(K_3 \square K_2)$, labeled as above, is a swamp. Then, up to isomorphism, one of the following holds:*

1. *$w(e) = 0$ if $e \in \{a_1a_2, a_1b_1, a_2b_2, a_3b_3\}$, $w(e) = -1$ if $e = b_1b_2$, $w(e) = 1$ otherwise.*
2. *$w(e) = 0$ if $e \in \{a_1a_2, a_1b_1, a_3b_3, b_2b_3\}$, $w(e) = 1$ otherwise.*
3. *$w(e) = 0$ if $e \in \{a_1a_2, a_1b_1, b_1b_3\}$, $w(e) = 1$ otherwise.*
4. *$w(e) = 0$ if $e \in \{a_1a_2, b_1b_3\}$, $w(e) = 1$ if $e \in \{a_1b_1, a_1a_3, a_2a_3\}$, $w(e) = -1$ otherwise.*
5. *$w(e) = 0$ if $e = a_1a_2$, $w(e) = 1$ if $e \in \{a_2a_3, a_1a_3, b_1b_2\}$, $w(e) = -1$ otherwise.*

In particular, the graph has no two zero-weight edges in a triangle.

Proof. We first show that one triangular face has two regular pairs. For each choice of $1 \leq i < j \leq 3$, we have by Theorem 2.3.2 $\{a_i a_j\}$ is regular in the triangle $a_1a_2a_3$, or $\{b_i b_j\}$ is regular in the triangle $b_1b_2b_3$. Hence, one triangle has two regular pairs, which means that one edge is zero-weight and the other two have equal non-zero weight (see Theorem 2.3.4). Now without loss of generality, assume $w(a_1a_2) = 0$ and $w(a_1a_3) = w(a_2a_3) = 1$. From applying Lemma 2.3.3 to the paths a_1a_2, $a_1a_3a_2$, and either of $a_1b_1b_2a_2$ and $a_1b_1b_3b_2a_2$, we have the following equation:

$$2 - w(a_1b_1) - w(a_2b_2) = w(b_1b_2) = w(b_1b_3) + w(b_2b_3) \tag{2.1}$$

Also, we have $w(b_1b_2) \neq 0$ since otherwise we have b_1b_2 is irregular in triangle $b_1b_2b_3$, contradicting Theorem 2.3.2.

We now have cases based on the a_ib_i edges that are zero-weight.

1. Suppose a_1b_1 and a_2b_2 are zero-weight. Then, by (2.1), we get $w(b_1b_2) = 2$. Also, we can apply Lemma 2.3.3 to get that $a_1a_3b_3b_2$ and $a_2a_3b_3b_1$ have weight 2. This implies $w(a_3b_3) + w(b_1b_3) = w(a_3b_3) + w(b_2b_3) = 1$. We deduce $2w(a_3b_3) + w(b_1b_3) + w(b_2b_3) = 2$. But (2.1) gives $w(b_1b_3) + w(b_2b_3) = 2$. So, $w(a_3b_3) = 0$ and $w(b_1b_3) = w(b_2b_3) = 1$. This gives us case 1. It is a swamp by Theorem 2.2.3 because (b_1, b_2, a_3) is a special vertex feedback triple.

2. Suppose a_1b_1 and a_3b_3 are zero-weight, and we are not in the previous case (so a_2b_2 is not zero-weight). Since $w(a_2b_2) + w(b_1b_2) = 2$ and neither edge is zero-weight, we have $w(a_2b_2) = w(b_1b_2) = 1$. Applying Lemma 2.3.3 to a_1b_1, $a_1a_2b_2b_1$, and $a_1a_3b_3b_1$, we get that $w(a_1a_3) + w(a_3b_3) + w(b_1b_3) = 2$. Therefore, $w(b_1b_3) = 1$. Then, by (2.1), we get $w(b_2b_3) = 0$. This is case 2. It is a swamp by Theorem 2.2.2 with vertex feedback triple (b_1, a_2, a_3).

3. Suppose only one edge is zero-weight. We claim it is not a_3b_3. Suppose for a contradiction that $w(a_3b_3) = 0$. Then, applying Lemma 2.3.3 to paths a_3b_3, $a_3a_1b_1b_3$, and $a_3a_2b_2b_3$, we have there exists $k \in \{0, 1\}$ such that $w(a_1b_1) + w(b_1b_3) = w(a_2b_2) + w(b_2b_3) = k$. But $k \neq 0$, for otherwise $a_1b_1b_3b_2a_2$ is a zero-weight cycle. Also, by (2.1), $w(b_1b_2) = 2 - w(a_1b_1) - w(a_2b_2)$, which is not zero. Therefore, at least one of $w(a_1b_1)$ and $w(a_2b_2)$ is -1. Without loss of generality, suppose $w(a_1b_1) = -1$. Then $w(b_1b_3) = -1$, and $w(b_1b_2) = -w(a_2b_2)$. But now $a_2a_3b_3b_1b_2$ is a cycle with weight

$$1 + 0 - 1 - w(a_2b_2) + w(a_2b_2) = 0$$

a contradiction. Therefore, $w(a_3b_3) \neq 0$. So exactly one of a_1b_1 or a_2b_2 is zero-weight. Without loss of generality, suppose $w(a_2b_2) = 0$. By (2.1), we get $w(a_1b_1) + w(b_1b_2) = 2$, and neither edge is zero-weight. So $w(a_1b_1) = w(b_1b_2) = 1$. Using Lemma 2.3.3 we get $w(a_3b_3)+w(b_3b_2) = 1$ (from $a_1a_2b_2$, $a_1b_1b_2$, and $a_1a_3b_3b_2$). Also, from (2.1), we have $w(b_1b_3) + w(b_2b_3) = 1$, so $w(b_1b_3) = w(a_3b_3) = 1 - w(b_2b_3)$. But $w(a_3b_3) \neq 0$ by hypothesis, and since $a_1b_1b_3a_3$ has weight $2 + w(b_1b_3) + w(a_3b_3)$, we conclude $w(a_3b_3) \neq 2$. So $w(a_3b_3) = w(b_1b_3) = 1$, and $w(b_2b_3) = 0$. This is case 3. It is a swamp by Theorem 2.2.4 with two-vertex feedback set $\{a_3, b_1\}$.

4. Now suppose that no a_ib_i edge is zero-weight, and further that $w(a_1b_1) \neq w(a_2b_2)$. Without loss of generality, we may assume $w(a_1b_1) = -1$ and $w(a_2b_2) = 1$. By (2.1), we conclude $w(b_1b_2) = -1$ and $w(b_1b_3) + w(b_2b_3) = -1$. Using Lemma 2.3.3 with paths $a_2b_2b_1$, $a_2a_1b_1$,

and $a_2a_3b_3b_1$, we get $w(a_3b_3) + w(b_1b_3) = 1$. There are two possibilities based on $w(a_3b_3)$. If $w(a_3b_3) = -1$, then $w(b_1b_3) = -1$, and $w(b_2b_3) = 0$. This is case 4. If $w(a_3b_3) = 1$, then $w(b_1b_3) = 0$, and $w(b_2b_3) = -1$, which is also isomorphic to case 4. We now verify that case 4 is a swamp. It suffices to show that for all $1 \leq i < j \leq 3$, deleting a_ia_j and suppressing the ends gives a swamp of K_4 as described in Theorem 2.5.1, since it would imply that we would have to use every edge in the triangle $a_1a_2a_3$ in the zero-weight cycle, but the triangle is not a zero-weight cycle. This is so (deleting a_1a_2 gives case 2, deleting a_1a_3 or a_2a_3 gives case 3). Therefore, case 4 is a swamp.

5. For the final case, suppose $w(a_1b_1) = w(a_2b_2)$ and neither edge is zero-weight, nor is a_3b_3 zero-weight. Notice that if $w(a_1b_1) = w(a_2b_2) = 1$, then (2.1) gives $w(b_1b_2) = 0$—a contradiction. So $w(a_1b_1) = w(a_2b_2) = -1$, and $w(b_1b_2) = 1$. Now apply Lemma 2.3.3 with paths $a_2b_2b_1$, $a_2a_1b_1$, and $a_2a_3b_3b_1$ to get $w(a_3b_3) + w(b_1b_3) = 1$. Apply the lemma with paths $a_1b_1b_2$, $a_1a_2b_2$, and $a_1a_3b_3b_2$, and we get $w(a_3b_3) + w(b_2b_3) = 1$. So $w(b_1b_3) = w(b_2b_3)$; by (2.1), the weights are both -1. Then $w(a_3b_3) = -1$. This is case 5. Like in the previous case, we can verify this to be a swamp by considering every $1 \leq i < j \leq 3$ and checking that deleting a_ia_j and suppressing the ends gives a K_4 swamp under Theorem 2.5.1. Deleting a_1a_2 gives case 3; deleting a_1a_3 or a_2a_3 (they are symmetric) gives case 1. So case 5 is a swamp.

Since we have exhausted all possible weights on the a_ib_i edges, we conclude that the five swamps we have found are a complete list up to isomorphism. □

2.5.5 The cube is not swampable

The cube graph is a cubic graph on eight vertices $v_1, \ldots, v_8$ such that $v_1, \ldots, v_4$ and $v_5, \ldots, v_8$ are two 4-cycles, and v_iv_{i+4} are edges for $i = 1, 2, 3, 4$. We will call the four-edge matching formed from the cut between two disjoint faces a **parallel matching**.

Theorem 2.5.7. *The cube graph is not swampable.*

Proof. First we show that no two zero-weight edges are incident to the same vertex. Suppose not; by symmetry we may assume $w(v_1v_2) = w(v_2v_3) = 0$. Then, deleting v_4v_8 and suppressing both ends, we get a prism swamp in which two edges in a triangle are zero-weight. This contradicts Theorem 2.5.6. Hence, the zero-weight edges are a matching. We claim that if one edge of a face is zero-weight, the opposite edge is also zero-weight. That is, without loss of generality, $w(v_1v_2) = 0$. We claim $w(v_3v_4) = 0$. Suppose not. By deleting v_3v_7 and suppressing both ends, we get a

prism with a triangular face, with edge weights zero (from v_1v_2), $w(v_2v_3)$, and $w(v_3v_4) + w(v_4v_1)$. Likewise, deleting v_4v_8 and suppressing both ends gives a prism with triangular face weights zero, $w(v_2v_3)+w(v_3v_4)$, and $w(v_4v_1)$. Notice by Theorem 2.5.6, we have $w(v_2v_3)$ and $w(v_2v_3)+w(v_3v_4)$ are both nonzero, and $w(v_3v_4)$ is nonzero by hypothesis, so $w(v_2v_3) = w(v_3v_4)$. Also, $w(v_3v_4) = w(v_4v_1)$. But now the weight of cycle $v_1v_2v_3v_4$ is $3w(v_2v_3) = 0$, contradicting that we have a swamp.

Hence, there are only two possible cases for the zero-weight edges, which we handle in turn:

1. The zero-weight edges are a parallel matching. Without loss of generality, assume this is the matching. Notice that deleting any zero-weight edge and suppressing the ends gives a prism, which must be of type 1 in Theorem 2.5.6. In particular, the two faces with no zero-weight edges, $v_1v_2v_3v_4$ and $v_5v_6v_7v_8$, have different weights. Thus, every time an edge is deleted, the fourth zero-weight edge appears on the same face; without loss of generality, say that this face is $v_1v_2v_3v_4$. Now, notice that when v_1v_5 is deleted and the ends suppressed, the only way to get a zero-weight edge on the triangle is by having $w(v_4v_1) + w(v_1v_2) = 0$. Likewise, deleting v_3v_7 and suppressing, the only way to get a zero-weight edge is to have $w(v_2v_3) + w(v_3v_4) = 0$. But together, these two equations imply $v_1v_2v_3v_4$ is zero-weight—a contradiction.

2. There are no zero-weight edges. We claim that every parallel matching has all edges the same weight. Assume without loss of generality the parallel matching is $\{v_iv_{i+4} : i \in \{1, 2, 3, 4\}\}$. Then, for each deleting v_iv_{i+4} and suppressing the ends gives a prism swamp. Any zero-weight edges in the swamp must have come from suppression, so they only appear on triangles, and if there are two zero-weight edges, they are corresponding. This rules out cases 1 through 4 of Theorem 2.5.6. Hence, case 5 is the only possibility. So the other three edges of the parallel matching have the same weight. We can do this for every parallel matching. These weights are also all equal (or else we get a face with alternating 1 and -1 weights, giving a zero-weight cycle). So all edges have the same nonzero weight. But the cube has a six-edge cycle, which will have zero-weight. This gives a contradiction.

Therefore, the cube is not swampable. □

2.5.6 The octahedron is not swampable

The octahedron graph is a six vertex 4-regular shown in Figure 2.1

Theorem 2.5.8. *Let G be the octahedron, shown in Figure 2.1. Then G is not swampable.*

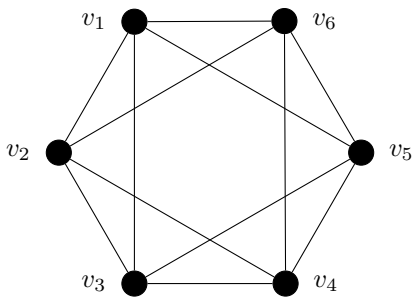

Figure 2.1: The octahedron.

Proof. Suppose for a contradiction that (G, w) is a swamp. Notice that deleting v_1v_2, v_3v_4, and v_5v_6 yields a prism with triangular faces $v_1v_3v_5$ and $v_2v_4v_6$. By Theorem 2.5.6, at least one edge in a triangular face has weight 0. Without loss of generality, suppose $w(v_1v_3) = 0$. Then $\{v_1, v_3\}$ is irregular in triangle $v_1v_2v_3$. By Theorem 2.3.2, in triangle $v_4v_5v_6$, we have $\{v_4, v_5\}$, $\{v_4, v_6\}$, and $\{v_5, v_6\}$ are all regular. This, though, contradicts Theorem 2.3.4. Thus, (G, w) is not a swamp. □

2.5.7 The Y-Δ cube is not swampable

As described above, the Y-Δ cube is formed by taking a cube, deleting a vertex, and adding a triangle on its neighborhood. It is shown in Figure 2.2.

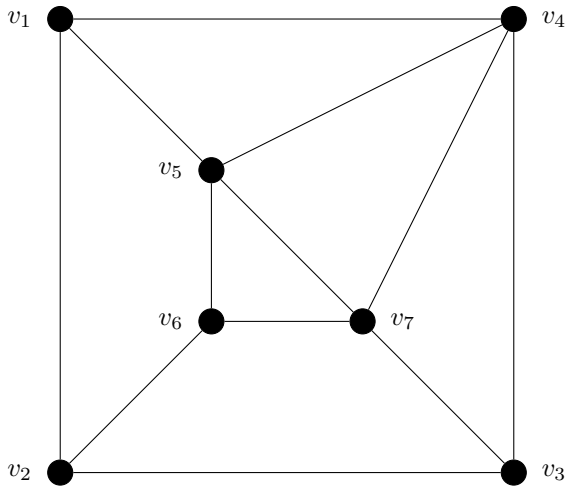

Figure 2.2: The Y-Δ cube.

Theorem 2.5.9. *Let G be the Y-Δ cube, shown in Figure 2.2. Then G is not swampable.*

Proof. Suppose for a contradiction that (G, w) is a swamp. Consider the triangles $v_1v_4v_5$ and

$v_3v_4v_7$. They satisfy Lemma 2.3.6, so either $\{v_1, v_4\}$ and $\{v_5, v_4\}$ are regular in $v_1v_4v_5$, or $\{v_3, v_4\}$ and $\{v_7, v_4\}$ are regular in $v_3v_4v_7$. By symmetry, we may assume the former is the case. So $w(v_1v_5) = 0$ and $w(v_1v_4) = w(v_4v_5) = k$ for $k \in \{-1, 1\}$. Now, in triangle $v_5v_6v_7$, we have by Theorem 2.3.2 that $\{v_5, v_6\}$ and $\{v_5, v_7\}$ are regular. So by Theorem 2.3.4, we have $w(v_6v_7) = 0$ and $w(v_5v_6) = w(v_5v_7) = \ell$ for $\ell \in \{-1, 1\}$. Now, $\{v_6, v_7\}$ is irregular in triangle $v_5v_6v_7$, so we use Theorem 2.3.2 to conclude that, in triangle $v_3v_4v_7$, we have $\{v_3, v_7\}$ and $\{v_4, v_7\}$ are regular. Therefore, $w(v_3v_4) = 0$ and $w(v_3v_7) = w(v_4v_7) = m$ for $m \in \{-1, 1\}$. Since $w(v_3v_4) = 0$, we have $\{v_3, v_4\}$ is irregular in cycle $v_1v_2v_3v_4$, so by Theorem 2.3.2, we have $\{v_4, v_7\}$ is regular in triangle $v_4v_5v_7$. So $w(v_4v_5) + w(v_5v_7) = w(v_4v_7)$. This gives us $k + \ell = m$. Since all three weights are nonzero, we ahve $k = \ell = -m$. And so

$$w(v_1v_4) = w(v_4v_5) = w(v_5v_6) = w(v_5v_7) = k$$

and

$$w(v_4v_7) = w(v_3v_7) = -k$$

Now, we see that $w(v_5v_7) = k$ but $w(v_5v_4) + w(v_4v_7) = k - k = 0$. So $\{v_5, v_7\}$ is irregular in triangle $v_4v_5v_7$. Also since $w(v_6v_7) = 0$, we have $\{v_6, v_7\}$ is irregular in cycle $v_2v_3v_7v_6$. But this contradicts Theorem 2.3.2. So (G, w) is not a swamp. □

2.6 Swampable graphs with a V_8 subdivision

Our next step is to consider the graph V_8 (called the **Wagner graph**), defined as the cubic graph constructed by taking a cycle $v_1 \dots v_8$ (each edge of which we will call a rim edge) and adding edges v_iv_{i+4} for $i \in \{1, \dots, 4\}$ (we will call these added edges diameters). Throughout this section, the indices will be assumed to be taken modulo 8. The Wagner graph is shown in Figure 2.3.

It is useful to have the following alternative characterization of a V_8 subdivision.

Lemma 2.6.1. *Let G be a graph with two disjoint 4-cycles, $v_1v_2v_3v_4$ and $v'_1v'_2v'_4v'_3$, and a matching $\{v_iv'_i : i \in \{1, 2, 3, 4\}\}$, and no other vertices or edges. Then, G is isomorphic to V_8.*

Proof. Take the vertices in the order $v_1, v_4, v'_4, v'_2, v_2, v_3, v'_3, v'_1$. □

In this section, we prove the following theorem:

Theorem 2.6.2. *Let G be a simple 3-connected swampable graph with a V_8 subdivision. Then, G is isomorphic to V_8.*

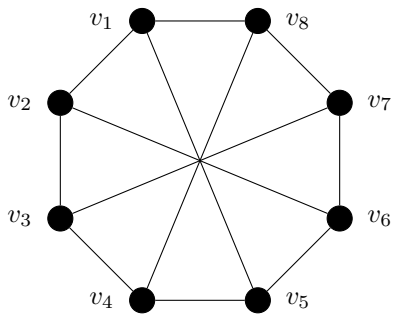

Figure 2.3: The Wagner graph V_8.

We start by describing the possible swamps of V_8 itself.

Theorem 2.6.3. *Let (V_8, w) be a swamp. Then, up to swamp isomorphism:*

1. $w(e) = 1$ *if e is a diameter,*
2. $w(v_1v_8) = w(v_4v_5) = 1$, *and*
3. $w(e) = 0$ *for all other edges e.*

Proof. We first show there exists $k \in \{-1, 1\}$ such that $w(e) \in \{0, k\}$ for all edges e.

Notice that for $i = 1, \ldots, 4$, deleting the diameter v_iv_{i+4} produces a $K_{3,3}$ subdivision, say H_i. Notice that H_i and H_{i+1} contain a 4-cycle in the intersection. So, we can apply Lemma 2.5.5 repeatedly to conclude there exists $k \in \{-1, 1\}$ such that that all branches in each H_i have weight 0 or k. Since every edge in V_8 is a branch in some H_i, every edge has weight 0 or k. Without loss of generality, we may assume $k = 1$. Also, notice that every two consecutive rim edges are a branch in some H_i, so there are no two consecutive rim edges each with weight 1.

We now claim that every diameter has weight 1. Suppose not; say $w(v_1v_5) = 0$. Now, if we delete v_7v_8 and suppress both ends, we have a prism with one triangular face $v_1v_4v_5$. Since $w(v_1v_5) = 0$, by Theorem 2.5.6, we have $w(v_4v_5) \neq 0$. So $w(v_4v_5) = 1$. Next, we delete v_2v_3 and suppress both ends, giving us a prism with one triangular face $v_1v_5v_6$. Since $w(v_1v_5) = 0$, by Theorem 2.5.6, we conclude $w(v_5v_6) \neq 0$. So $w(v_5v_6) = 1$. But v_4v_5 and v_5v_6 are two consecutive rim edges with weight 1, which we showed to be impossible. Hence, every diameter has weight 1.

Now, for each i, we have $v_iv_{i+4}v_{i+5}v_{i+1}$ is a cycle. Both diameters in the cycle have weight 1, and since the cycle is not zero-weight, we conclude $w(v_iv_{i+1}) = w(v_{i+4}v_{i+5})$. Because the rim edges do not form a zero-weight cycle, at least one of them has weight 1; without loss of generality, suppose $w(v_1v_8) = 1$. Then $w(v_4v_5) = 1$, and since no two consecutive rim edges have weight 1,

$w(v_1v_2) = w(v_3v_4) = w(v_5v_6) = w(v_7v_8) = 0$. Also, notice that $v_1v_2v_3v_4v_5$ is a cycle of $-1+w(v_2v_3)$, so $w(v_2v_3) = 0$, and $w(v_6v_7) = 0$. Thus, we have shown that w satisfies the conditions of the lemma. We need to show that (V_8, w) is a swamp. Let $A = \{v_1, \ldots, v_4\}$ and $B = \{v_5, \ldots, v_8\}$ Notice that A and B induce trees, all edges of which have weight 0. Also, there are exactly six edges from A to B, all of which have weight 1. So a cycle that has zero-weight crosses the cut between A and B either zero or six times. The former is impossible because A and B induce trees; the latter is impossible because the cycle would have to include v_1v_5, v_5v_4, v_4v_8, and v_8v_1, which is already a complete cycle. Hence, (V_8, w) is a swamp. □

To continue with the proof of Theorem 2.6.2, we show that we cannot add any jumps and maintain that we have a swampable graph. We tackle the jumps in the following order:

1. Jumps between non-adjacent vertices,
2. Jumps between a vertex and a non-incident edge, and
3. Jumps between edges.

Lemma 2.6.4. *Let (V_8, w) be a swamp. Every pair of non-adjacent vertices is highly irregular. In particular, if G is a proper simple spanning supergraph of V_8, then G is not swampable.*

Proof. By swamp isomorphism, we may assume V_8 and w satisfy Theorem 2.6.3. We now need to find paths of three different lengths between every pair of nonadjacent vertices. By symmetry, this reduces to five pairs of vertices. Table 2.1 exhibits three paths of different lengths between each pair of vertices.

Vertex pair	Weight 0 path	Weight 1 path	Weight 2 path
$\{v_1, v_3\}$	$v_1v_2v_3$	$v_1v_8v_4v_5v_6v_7v_3$	$v_1v_8v_4v_3$
$\{v_1, v_4\}$	$v_1v_2v_3v_4$	$v_1v_5v_6v_2v_3v_7v_8v_4$	$v_1v_5v_4$
$\{v_1, v_6\}$	$v_1v_8v_4v_5v_6$	$v_1v_5v_6$	$v_1v_5v_4v_8v_7v_3v_2v_6$
$\{v_1, v_7\}$	$v_1v_8v_4v_5v_6v_7$	$v_1v_8v_7$	$v_1v_8v_4v_5v_6v_2v_3v_7$
$\{v_2, v_7\}$	$v_2v_1v_5v_4v_8v_7$	$v_2v_1v_8v_7$	$v_2v_1v_5v_4v_3v_7$

Table 2.1: Showing that nonadjacent vertices in the unique V_8 swamp are highly irregular.

So every pair of nonadjacent vertices is highly irregular. The remainder of the lemma follows from Lemma 2.3.8. □

Next, we turn to jumps between vertices and non-incident edges. We will first prove the following lemma:

Lemma 2.6.5. *Let G be a graph and w a weight function. Let H be a subgraph of G that is a subdivision of a 3-connected graph J. Let $u, v \in V(J)$ and $e \in V(J)$ be given such that e is incident to u but not v, and there is a jump P in G from v to e. Let Q be the branch corresponding to edge e. Suppose w is a weight function of G such that $(H, w|_H)$ is a swamp and $\{u, v\}$ are highly irregular in $H \setminus S$, where S is the set of internal vertices of Q. Then (G, w) is not a swamp.*

Proof. Let x be the vertex in Q such that P is a vx-jump. Then $Q \cup P[x, u]$ is a uv-jump in $H \setminus S$. But this contradicts Lemma 2.3.8. □

Lemma 2.6.6. *Let (V_8, w) be a swamp, labeled as in Theorem 2.6.3. Then, for every pair (v, e) where $v \in V(V_8)$, $e \in E(V_8)$ and e is not incident to v, there is an end u of e such that u and v are highly irregular in $V_8 \setminus e$.*

Proof. See Table 2.2. □

v	e	u	Weight 0 path	Weight 1 path	Weight 2 path
v_1	v_2v_3	v_2	v_1v_2	$v_1v_5v_4v_8v_7v_6v_2$	$v_1v_8v_7v_6v_2$
v_1	v_3v_4	v_3	$v_1v_2v_3$	$v_1v_8v_4v_5v_6v_7v_3$	$v_1v_8v_7v_3$
v_1	v_4v_5	v_4	$v_1v_2v_3v_4$	$v_1v_5v_6v_2v_3v_7v_8v_4$	$v_1v_8v_4$
v_1	v_5v_6	v_6	$v_1v_5v_4v_8v_7v_6$	$v_1v_8v_7v_6$	$v_1v_5v_4v_8v_7v_3v_2v_6$
v_1	v_6v_7	v_7	$v_1v_5v_4v_8v_7$	$v_1v_8v_7$	$v_1v_8v_4v_5v_6v_2v_3v_7$
v_1	v_7v_8	v_7	$v_1v_8v_4v_5v_6v_7$	$v_1v_5v_6v_7$	$v_1v_8v_4v_5v_6v_2v_3v_7$
v_1	v_2v_6	v_2	v_1v_2	$v_1v_8v_4v_5v_6v_7v_3v_2$	$v_1v_5v_4v_3v_2$
v_1	v_3v_7	v_3	$v_1v_2v_3$	$v_1v_8v_4v_5v_6v_2v_3$	$v_1v_5v_4v_3$
v_1	v_4v_8	v_4	$v_1v_2v_3v_4$	$v_1v_8v_7v_3v_2v_6v_5v_4$	$v_1v_5v_4$
v_2	v_3v_4	v_3	v_2v_3	$v_2v_1v_5v_4v_8v_7v_3$	$v_2v_6v_7v_3$
v_2	v_4v_5	v_5	$v_2v_3v_4v_8v_1v_5$	$v_2v_1v_8v_7v_6v_5$	$v_2v_6v_7v_3v_4v_8v_1v_5$
v_2	v_5v_6	v_5	$v_2v_3v_4v_8v_1v_5$	$v_2v_1v_5$	$v_2v_6v_7v_3v_4v_8v_1v_5$
v_2	v_6v_7	v_7	$v_2v_1v_5v_4v_8v_7$	$v_2v_1v_8v_7$	$v_2v_6v_5v_1v_8v_4v_3v_7$
v_2	v_7v_8	v_8	$v_2v_6v_7v_3v_4v_8$	$v_2v_1v_8$	$v_2v_6v_7v_3v_4v_5v_1v_8$
v_2	v_1v_8	v_1	v_2v_1	$v_2v_6v_7v_3v_4v_5v_1$	$v_2v_6v_5v_1$
v_2	v_1v_5	v_1	v_2v_1	$v_2v_6v_7v_3v_4v_8v_1$	$v_2v_6v_7v_8v_1$
v_2	v_3v_7	v_3	v_2v_3	$v_2v_6v_7v_8v_1v_5v_4v_3$	$v_2v_6v_5v_4v_3$
v_2	v_4v_8	v_4	$v_2v_3v_4$	$v_2v_6v_7v_8v_1v_5v_4$	$v_2v_6v_5v_4$

Table 2.2: Paths verifying Lemma 2.6.5.

The last step is to consider edge-to-edge jumps. Consider V_8 without weights on the edges. Up to isomorphism, there are eight jumps:

1. v_1v_8 to v_1v_2 (consecutive rim edges),
2. v_1v_8 to v_2v_3 (rim edges 2 apart),
3. v_1v_8 to v_3v_4 (rim edges 3 apart),

4. v_1v_8 to v_4v_5 (opposite rim edges),

5. v_1v_8 to v_1v_5 (rim edge to incident diameter),

6. v_1v_8 to v_2v_6 (rim edge to nonincident diameter),

7. v_1v_5 to v_2v_6 (consecutive diameters), and

8. v_1v_5 to v_3v_7 (nonconsecutive diameters).

We will call an edge-to-edge jump type n (where $1 \leq n \leq 8$) according to the number of the jump on the list above. We first notice that four of the jumps produce cubes:

Lemma 2.6.7. *Augmenting V_8 by jumps of type 2, 3, 6, and 7 yields a cube subdivision.*

Proof. If we have a jump from v_1v_8 to v_2v_3, then deleting v_7v_8 and suppressing both ends yields a cube. If we have a jump from v_1v_8 to v_3v_4, then letting x be the end of the jump on v_3v_4, deleting xv_4 and suppressing both ends gives a cube. If we have a jump from v_1v_8 to v_2v_6, deleting v_5v_6 and suppressing gives a cube. Finally, if we have a jump from v_1v_5 to v_2v_6, then we can delete v_2v_3 and suppress both ends to get a cube. □

Next, we handle jumps of type 1 and 5 in a similar way.

Lemma 2.6.8. *Augmenting V_8 by a jump of type 1 yields a graph G that is not swampable.*

Proof. Suppose for a contradiction that (G, w) is a swamp. Without loss of generality, let the jump be from v_1v_8 to v_1v_2. Let x be the end of the jump on v_1v_8 and y the end on v_1v_2. Notice that deleting the jump and suppressing yields V_8. In particular, this gives us that every diameter has weight 1, and one of the following holds:

1. $w(v_2v_3) = w(v_6v_7) = 0$ or

2. $w(v_3v_4) = w(v_7v_8) = 0$.

The cases are symmetric, so assume $w(v_2v_3) = w(v_6v_7) = 0$. Then, in the cycle $v_2v_3v_7v_6$ (call it C), we have $\{v_2, v_3\}$ and $\{v_7, v_6\}$ are both irregular. We claim that all three pairs in the triangle v_1xy (call it T) are regular. Notice that (T, C) is a $(v_1, x; v_6, v_7)$-linkable decomposition, a $(v_1, y; v_3, v_2)$-linkable decomposition, and an $(x, y; v_3, v_2)$-linkable decomposition. By Theorem 2.3.2, all three pairs of vertices in T are regular. But this contradicts Theorem 2.3.4. □

Lemma 2.6.9. *Augmenting V_8 by a jump of type 5 yields a graph G that is not swampable.*

Proof. Suppose for a contradiction that (G, w) is a swamp. Without loss of generality, let the jump be from v_1v_8 to v_1v_5. Let x be the end of the jump on v_1v_8 and y be the end of the jump on v_1v_5. Notice that deleting the jump and suppressing yields V_8. In particular, this gives us that every diameter has weight 1, and one of the following holds:

1. $w(v_2v_3) = w(v_6v_7) = 0$ or
2. $w(v_3v_4) = w(v_7v_8) = 0$.

The cases are symmetric, so assume $w(v_2v_3) = w(v_6v_7) = 0$. Then, in the cycle $v_2v_3v_7v_6$ (call it C), we have $\{v_2, v_3\}$ and $\{v_7, v_6\}$ are both irregular. We claim that all three pairs in the triangle v_1xy (call it T) are regular. Notice that (T, C) is a $(v_1, x; v_2, v_3)$-linkable decomposition, a $(v_1, y; v_2, v_3)$-linkable decomposition, and an $(x, y; v_7, v_6)$-linkable decomposition. By Theorem 2.3.2, all three pairs of vertices in T are regular. But this contradicts Theorem 2.3.4. □

The last two cases are a type 4 jump and a type 8 jump. Adding a type 4 jump produces the 10-vertex Möbius ladder, shown in Figure 2.4. Adding a type 8 jump produces the Petersen graph, shown in Figure 2.5.

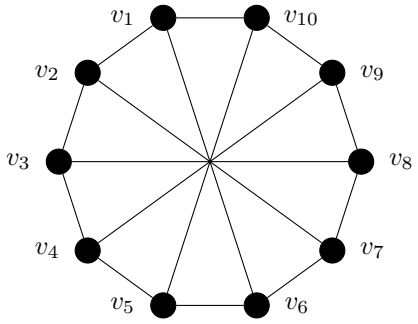

Figure 2.4: The Möbius ladder on ten vertices.

Neither graph is swampable.

Lemma 2.6.10. *Let G be the Möbius ladder on ten vertices, shown in Figure 2.4. Then G is not swampable.*

Proof. Suppose for a contradiction that (G, w) is a swamp. We define diameters and rim edges of G analogously to V_8. Notice that deleting any diameter yields a subdivision of V_8 where the remaining four diameters of G are diameters of V_8. Applying Theorem 2.6.3, we conclude that each of the remaining diameters of G has the same nonzero weight. Thus, we conclude all five diameters

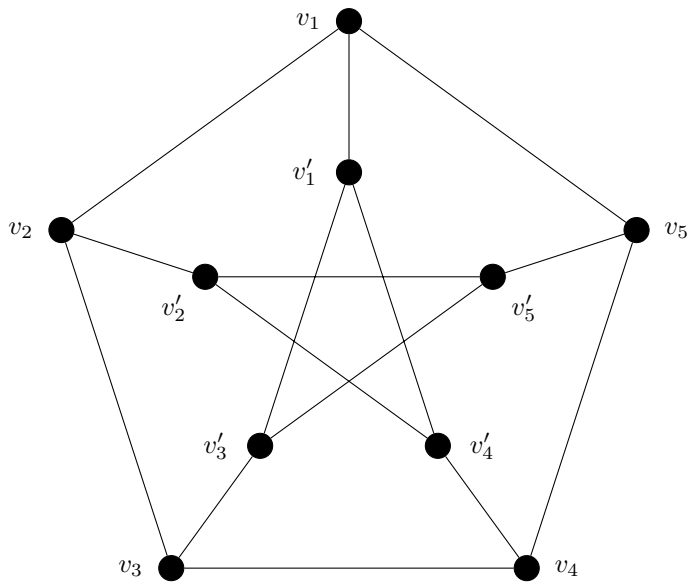

Figure 2.5: The Petersen graph.

of G have the same nonzero weight, which we may assume to be 1. Moreover, from the five V_8 subdivisions we produce, we conclude that every pair of opposite edges v_iv_{i+1} and $v_{i+5}v_{i+6}$ has the same weight, and the weight of the rim edges totals -1. Now consider the cycle

$$v_1v_6v_7v_2v_3v_8v_9v_4v_5v_{10}$$

It uses all five diameters and exactly one edge from each pair of opposite rim edges. The five diameters each have weight 1, giving a total weight -1, and the five rim edges have weight exactly half that of the outside cycle, so the five rim edges have weight 1. Thus, the total weight of the cycle is 0, which contradicts that (G, w) is a swamp. □

Lemma 2.6.11. *Let G be the Petersen graph, shown in Figure 2.5. Then G is not swampable.*

Proof. Consider deleting v_1v_2 and suppressing both ends; this yields V_8. Then, $v_3v_3'v_5'v_4v_3$ is a cycle (call it C) in the resulting V_8, and by Theorem 2.6.3, every 5-cycle has three consecutive zero-weight edges. However, notice that in the Petersen graph, $v_1v_1'v_3'v_5'v_5$ is another 5-cycle (call it C'), and there is a perfect matching between $V(C)$ and $V(C')$. So, in particular, if we take two consecutive zero-weight edges of C, we can construct a prism subdivision of the Petersen graph where both those edges are in the same triangle. But this is impossible by Theorem 2.5.6. So the Petersen graph is not swampable. □

We finish with the proof of Theorem 2.6.2:

Proof of Theorem 2.6.2. Suppose G is a simple 3-connected graph with a V_8 subdivision, and G is not isomorphic to V_8. By Theorem 2.1.14, G contains a V_8 augmentation (call it H) as a subdivision. Since V_8 is triangle-free, the augmentation must be formed by adding a jump to V_8. Adding a jump between distinct edges gives a graph that is not swampable by Lemmas 2.6.7, 2.6.8, 2.6.9, 2.6.10, and 2.6.11. Adding a jump between an edge and a nonincident vertex gives a graph that is not swampable by Lemma 2.6.5. Adding a jump between nonadjacent vertices gives a graph that is not swampable by Lemma 2.6.4. So, in any event, H is not swampable. This contradicts that G is swampable. □

2.7 Non-swampability of graphs with K_5 minor

In this section, we prove the following:

Theorem 2.7.1. *Let G be a graph with a K_5 minor. Then, G is not swampable.*

Our starting point is Theorem 2.5.3, which demonstrates that K_5 itself is not swampable.

To prove Theorem 2.7.1, we will suppose that the theorem is false and assume G is a minimal counterexample. We say that G is **K_5-minor-minimal** if G has no vertices with degree at most 2 and no proper subgraph of G has a K_5 minor.

Lemma 2.7.2. *Let G be a K_5-minor-minimal graph. Then G has the following properties:*

1. *$V(G)$ can be partitioned into $(X_1, \ldots, X_5)$ such that, for all $1 \leq i \leq 5$, $G[X_i]$ is a tree.*
2. *For all $1 \leq i < j \leq 5$, there is a unique pair $(u, v) \in X_i \times X_j$ such that u is adjacent to v.*
3. *For all $1 \leq i \leq 5$, if v is a leaf of $G[X_i]$, then there exist $1 \leq j < k \leq 5$, with $i \neq j$, $i \neq k$, such that v has neighbors in X_j and X_k.*
4. *For all $1 \leq i \leq 5$, $G[X_i]$ is a single vertex with exactly one neighbor in each other X_j for $j \neq i$, or $G[X_i]$ is a copy of K_2, with each vertex having exactly one neighbor in each of exactly two other X_j.*

If G is a K_5-minor-minimal graph, we will use $X_1, \ldots, X_5$ to denote the sets into which $V(G)$ is partitioned that satisfy Lemma 2.7.2. With those sets determined, we say that $\{u, v\}$ is a **split pair** if there is $1 \leq i \leq 5$ such that $\{u, v\} = X_i$. From the lemma, it is immediate that a vertex is in a split pair if it has degree 3, and if a vertex is not in a split pair, it has degree 4.

It is straightforward to verify that any two nonadjacent vertices in G are connected by three internally disjoint paths, so G is 3-connected.

We say that a 3-connected graph G is **internally-4-connected** if every 3-vertex cutset of G is an independent set separating a single vertex from the rest of the graph. Notice that for G to be internally-4-connected, no vertex in a triangle in G can contain a degree 3 vertex.

The point of considering K_5-minor-minimal graphs that are internally 4-connected is to apply the structure theorem proved by Maharry and Robertson about internally-4-connected graphs with no V_8-subdivision.

Theorem 2.7.3 ([8]). *Let G be an internally-4-connected graph with no V_8-subdivision. Then, one of the following holds:*

1. *G is planar.*
2. *There exist adjacent $x, y \in V(G)$ such that $G \setminus \{x, y\}$ is a cycle.*
3. *G has a vertex cover with at most four vertices.*
4. *G is isomorphic to the line graph of $K_{3,3}$.*
5. *$|V(G)| \leq 7$.*

Therefore, if G is a swampable K_5-minor-minimal internally-4-connected graph with at least eight vertices, then G has a V_8-subdivision (which, necessarily being proper, contradicts Theorem 2.6.2), or one of the five cases of Theorem 2.7.3 hold. Cases 1 and 5 are clearly impossible. Case 4 is impossible as well because the line graph of $K_{3,3}$ is not K_5-minor-minimal (in fact, it contains a K_5 subdivision). In cases 2 and 3, Maharry and Robertson point out that G must contain a cube subdivision ([8]), but the cube is not swampable by Theorem 2.5.7.

We can use Theorem 2.7.3 to dispose of the swampability of many of the larger K_5-minor-minimal graphs.

Lemma 2.7.4. *Suppose that G is a triangle-free K_5-minor-minimal graph. Then G is internally-4-connected.*

Proof. Suppose not. Since G is 3-connected and triangle-free, the only obstacle to being internally-4-connected is for there to exist a partition (A, B, S) of $V(G)$ such that $|S| \leq 3$, $|A|, |B| \geq 2$, and no edges go from A to B. Choose k and number $X_1, \ldots, X_5$ such that $X_i \cap S = \emptyset$ if and only if $1 \leq i \leq k$. Note $k \geq 2$. The union $\bigcup_{i=1}^{k} X_i$ is connected, so it belongs to A or to B. Without loss of

generality, suppose $\bigcup_{i=1}^{k} X_i \subseteq A$. Now every vertex in B is part of an X_i for $i > k$. For each such vertex $v \in X_i \cap B$, its neighbors are in at least three different X_j, none of which has index at most k. Therefore, we have $k = 2$, and each vertex B has exactly one neighbor in X_3, X_4, and X_5. Since only X_1 and X_2 are disjoint from S, we can label the vertices in S as s_3, s_4, s_5, where $s_i \in X_i$. For each i for which $X_i \cap B$ is nonempty, let b_i be the single vertex in the intersection. We claim that, where distinct b_i and b_j exist, that b_i is adjacent to b_j. This is so because there is only one edge between X_i and X_j, and it must be incident to both b_i and b_j. From this, we conclude that $|B| = 2$ (since G is triangle-free). Without loss of generality, suppose b_3 and b_4 exist but b_5 does not. Then, the only vertex in X_5 that could be adjacent to b_3 is s_5, so $b_3 s_5$ is an edge. Likewise, $b_4 s_5$ is an edge. But b_3 and b_4 are adjacent, so we get a triangle $b_3 b_4 s_5$—contradicting that G is triangle-free. □

Finally, we note that if Theorem 2.7.1 is false, we can find a K_5-minor-minimal with the fewest vertices as a counterexample. Therefore, we have

Lemma 2.7.5. *Suppose G is a K_5-minor-minimal graph, and w is a weight function such that (G, w) is swampable. Suppose no K_5-minor-minimal graph H with $|V(H)| < |V(G)|$ is swampable. Then, for every edge e joining vertices of a split pair in G, we have $w(e) \neq 0$.*

Proof. If $w(e) = 0$, we could contract e to get a smaller K_5-minor-minimal graph that is still swampable (Lemma 2.1.6). □

With these preliminary results proven, we tackle all possible K_5-minor-minimal graphs and show they are not swampable, in ascending order of the number of vertices they contain. The simplest case—K_5 itself—is not swampable by Theorem 2.5.3.

2.7.1 G has six vertices

Up to isomorphism, there is a unique K_5-minor-minimal graph with six vertices, shown in Figure 2.6:

Lemma 2.7.6. *The graph G shown in Figure 2.6 is not swampable.*

Proof. Suppose not; that is, (G, w) is a swamp. Deleting $v_2 v_3$ and $v_4 v_5$ gives us a swamp of $K_{3,3}$ (with bipartition of left vertices versus right vertices as shown in Figure 2.6). There are two possible swamps by Theorem 2.5.4, which we handle in turn:

1. Suppose the zero-weight edges are a claw. Without loss of generality, we may assume the vertices have been relabeled $\{a_1, a_2, a_3, b_1, b_2, b_3\}$ and w has been negated, if necessary, such

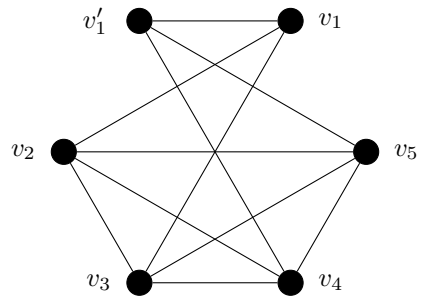

Figure 2.6: The unique K_5-minor-minimal graph with six vertices.

that case 1 of Theorem 2.5.4 holds. Notice that for $1 \leq i < j \leq 3$, the pair $\{b_i, b_j\}$ is highly irregular. It suffices to demonstrate this for $\{b_1, b_2\}$, where we can find paths $b_1a_1b_2$, $b_1a_2b_2$, and $b_1a_3b_3a_2b_2$ with weight 0, 2, and 1 respectively. But this contradicts Lemma 2.3.8 because the $K_{3,3}$ swamp came from deleting one edge on each side of the bipartition. So the zero-weight edges are not a claw.

2. Suppose the zero-weight edges form 2 two-edge paths. Without loss of generality, assume the vertices have been relabeled $\{a_1, a_2, a_3, b_1, b_2, b_3\}$ and w has been negated, if necessary, such that case 2 of Theorem 2.5.4 holds. Now consider what weight we must put on an edge with both ends on one side of the bipartition to avoid creating any zero-weight cycles. If the edge joins two vertices that are incident to only one zero-weight edge (that is, the edge is a_2a_3 or b_1b_2), then we can find paths of weight 0 and -1, so the edge must have weight -1. Otherwise, the edge has at one end a vertex adjacent to two zero-weight edges. Then, we can find paths of weight 1 and 0, so the weight of the edge must be 1.

   All that is left is to verify that we cannot add edges in this way to both sides of the bipartition. Notice that a_2 and a_3 are symmetric, as are b_1 and b_2, and, further, we can flip the entire bipartition. So there are only three cases to check:

   (a) The edges are a_1a_2 (weight 1) and b_1b_2 (weight -1). This gives a zero-weight cycle $a_1a_2b_2b_1a_3b_3$.

   (b) The edges are a_1a_2 and b_1b_3 (both weight 1). Then $a_1a_2b_2a_3b_1b_3$ is a zero-weight cycle.

   (c) The edges are a_2a_3 and b_1b_2 (both weight -1). Then $a_2a_3b_2b_1$ is a zero-weight cycle.

Hence, this contradicts that (G, w) is a swamp. □

2.7.2 G has seven vertices

Suppose G is a K_5-minor-minimal graph with seven vertices. Let $\{v_1, v_1'\}$ and $\{v_2, v_2'\}$ be the split pairs, such that v_1 is adjacent to v_2. There are two cases; either v_1 and v_2 have a common neighbor, or they do not. This gives us two possible K_5-minor-minimal graphs with seven vertices, up to isomorphism.

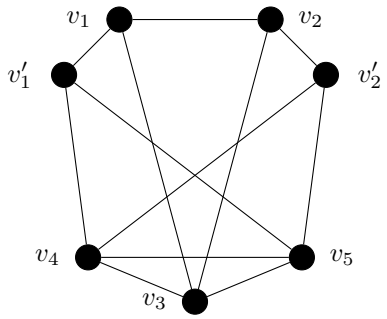

Figure 2.7: A K_5-minor-minimal graph with seven vertices, where v_1 and v_2 have a common neighbor.

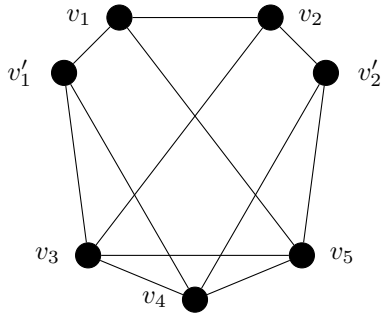

Figure 2.8: A K_5-minor-minimal graph with seven vertices, where v_1 and v_2 do not have a common neighbor.

Neither graph is swampable.

Lemma 2.7.7. *Let G be the graph shown in Figure 2.7. Then G is not swampable.*

Proof. Suppose for a contradiction that (G, w) is a swamp. Notice that we can find two $K_{3,3}$ subdivisions in G: delete v_4v_5 and then either delete v_3v_1 and suppress v_1, or delete v_3v_2 and suppress v_2. These subdivisions share the edges of the 4-cycle $v_1'v_5v_2'v_4$. Hence, when we apply Theorem 2.5.4 to each subdivision, we get that the nonzero weights are the same, by Lemma 2.5.5. Notice that every edge other than v_1v_2 is a part of at least one of the subdivisions. Hence, we may assume that $w(e) \in \{0, 1\}$ for all $e \in E(G) \setminus \{v_1v_2, v_4v_5\}$. We further have by Lemma 2.7.5

that $w(v_1v_1') = w(v_2v_2') = 1$. Notice that $v_1v_2v_2'$ is a branch in one of the $K_{3,3}$ subdivisions, so $w(v_1v_2) + w(v_2v_2')$ is 0 or 1. Thus, $w(v_1v_2)$ is 0 or 2.

Now we notice that we can form a prism subdivision with $v_1v_2v_3$ as a triangular face (delete v_3v_5 and v_4v_5 and suppress v_5). By Theorem 2.5.6, we have that at most one of the edges v_1v_2, v_1v_3, v_2v_3 has zero weight. The edge v_1v_3 cannot have zero weight, because then $w(v_1v_2) = 2$ and $w(v_2v_3) = 1$, giving us that $v_1v_2v_3$ is a zero-weight cycle. Likewise, v_2v_3 is no zero weight. So $w(v_1v_3) = w(v_2v_3) = 1$. Next, we show $w(v_1v_2) = 2$. Suppose not; that is, $w(v_1v_2) = 0$. Then $v_1'v_1v_2v_2'$ has weight -1, and $v_1'v_1v_3v_2v_2'$ has weight 1. So both $v_1'v_4v_2'$ and $v_1'v_5v_2'$ have weight 0, but they together produce a zero-weight cycle. Hence, $w(v_1v_2) = 2$. With these weights determined, we have that in triangle $v_1v_2v_3$, that $\{v_1, v_3\}$ and $\{v_2, v_3\}$ are both irregular. These can both be linked to triangle $v_3v_4v_5$. This gives us in triangle $v_3v_4v_5$ that $\{v_3, v_4\}$ and $\{v_3, v_5\}$ are both regular. By Theorem 2.3.4, we have $w(v_4v_5) = 0$. So, in triangle $v_1'v_4v_5$, we have $\{v_4, v_5\}$ is irregular. Using the linkage paths v_4v_3 and $v_5v_2'v_2$, we contradict Theorem 2.3.2. This contradicts the assumption that G was swampable. □

Lemma 2.7.8. *The graph G shown in Figure 2.8 is not swampable.*

Proof. Suppose for a contradiction that (G, w) is a swamp. By deleting v_3v_4 and v_4v_5 and suppressing v_4, we get a $K_{3,3}$ subdivision. By Theorem 2.5.4, we may assume that $w(e) \in \{0, 1\}$ for all e not incident to v_4 and further $w(v_1v_4) + w(v_4v_2') \in \{0, 1\}$.

Now consider the triangles $v_1'v_3v_4$ and $v_2'v_4v_5$. There exist paths $v_1'v_1v_5$, $v_1'v_1v_2v_2'$, $v_3v_2v_2'$, and v_3v_5, each internally disjoint from the triangles. We apply Lemma 2.3.6 to conclude that either $\{v_1', v_4\}$ and $\{v_3, v_4\}$ are both regular in $v_1'v_3v_4$, or $\{v_2', v_4\}$ and $\{v_5, v_4\}$ are regular in $v_2'v_4v_5$. Without loss of generality, suppose the former is true. Then $w(v_1'v_3) = 0$ and $w(v_1'v_4) = w(v_3v_4)$. Now, taking H_1 to be the triangle $v_1'v_4v_3$ and H_2 to be the cycle $v_1v_2v_2'v_5$, we have a $(v_1', v_3; v_1, v_5)$-linkable decomposition, as well as a $(v_1', v_3; v_1, v_2)$-linkable decomposition. Hence, in H_2, we have $\{v_1, v_5\}$ and $\{v_1, v_2\}$ are both regular by Theorem 2.3.2. By Theorem 2.3.4, we have $w(v_2v_2') + w(v_2'v_5) = 0$; as both edges have weight either 0 or 1, we conclude $w(v_2v_2') = w(v_2'v_5) = 0$. Also, $w(v_1v_2) = w(v_1v_5) = 1$.

Now we turn to triangle $v_2'v_5v_4$. Notice that G contains a prism subdivision with a triangular face $v_2'v_5v_4$. Hence, neither $v_2'v_4$ nor v_4v_5 has zero weight. So, they both have the same weight. Using Lemma 2.3.3 with paths $v_2'v_5$ $v_2'v_2v_1v_5$, and $v_2'v_4v_5$, we get $w(v_2'v_4) + w(v_4v_5) = -1$, so $w(v_2'v_4) = w(v_4v_5) = 1$. Since $w(v_1'v_4) + w(v_4v_2') \in \{0, 1\}$ and $w(v_1'v_4) \neq 0$, we have $w(v_1'v_4) = -1$, and also $w(v_3v_4) = -1$. Now that $v_1'v_4v_3$ is a path with weight 1, we know from Lemma 2.3.3 that $w(v_1'v_1) + w(v_1v_2) + w(v_2v_3) = 1$. Since $w(v_1v_2) = 1$, we have $w(v_1'v_1) + w(v_2v_3) = 0$; as neither

edge has weight -1, both must have weight 0. But now $v_3v'_1v_1v_5v_4$ is a zero-weight cycle. This contradicts that (G, w) is a swamp. □

2.7.3 G has eight vertices

Suppose G is a K_5-minor-minimal graph with eight vertices. Then, there are three split pairs, which we label $\{v_i, v'_i\}$ for $i \in \{1, 2, 3\}$, and two vertices of degree 4, which we call v_4 and v_5.

There are three possibilities for the edges between X_i and X_j for $1 \leq i < j \leq 3$. The first case is that they form a triangle, giving us the graph shown in Figure 2.9.

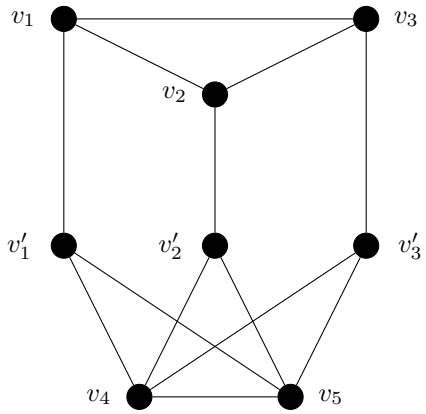

Figure 2.9: A K_5-minor-minimal graph with eight vertices, with a triangle of vertices in split pairs.

The second case is that there is a vertex in a split pair that has an edge to both other split pairs. In this case, we may assume without loss of generality that v_1 is adjacent to v_2 and v_3. Now there are three possibilities. First, we could have v_2 is adjacent to v'_3 (or, equivalently, v_3 is adjacent to v'_2. If that is the case, then $\{v'_1, v'_2\}$ is complete to $\{v_4, v_5\}$, and we may assume that v_3 is adjacent to v_4 and v'_3 is adjacent to v_5. But this graph, shown in Figure 2.10, consists of V_8 with an additional edge v_4v_5 (as verified by Lemma 2.6.1, the two 4-cycles being $v_1v_2v'_3v_3$ and $v'_1v_4v'_2v_5$). Thus, the graph is not swampable by Theorem 2.6.2.

The second possibility is that $v'_2v'_3$ is an edge. In that case, we may assume without loss of generality that v'_2 is adjacent to v_4. We can have either v'_3 is adjacent to v_4 or v'_3 is adjacent to v_5. This gives us two possible graphs, shown in Figures 2.11 and 2.12.

The last case is that there is no vertex in a split pair with an edge to both split pairs. Then, we may assume that v_1 is adjacent to v'_3 and v'_1 is adjacent to v_2. The edge between X_2 and X_3 can only go between v'_2 and v_3. So $v_1v'_1v_2v'_2v_3v'_3$ is a cycle. Then, one of v_4 and v_5 is adjacent to at least two of $\{v_1, v_2, v_3\}$. Therefore, we may assume v_4 is adjacent to v_1 and v_2; then v_5 is adjacent to v'_1

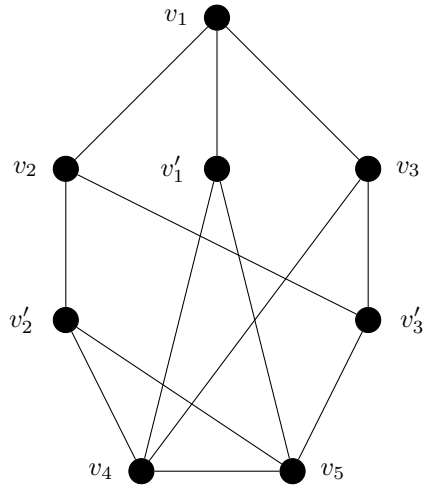

Figure 2.10: A K_5-minor-minimal graph with eight vertices, which contains V_8 as a proper subgraph.

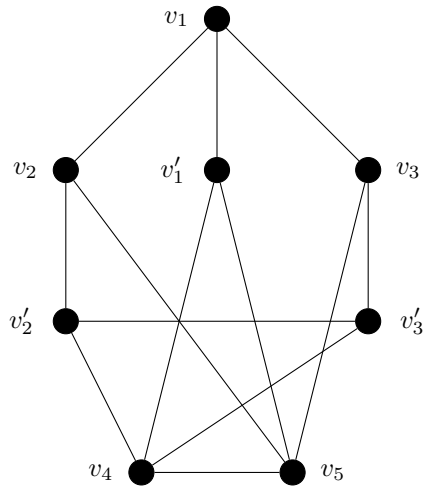

Figure 2.11: A K_5-minor-minimal graph with eight vertices, with cycle $v_1v_2v_2'v_3'v_2$, having both v_2' and v_3' adjacent to v_4.

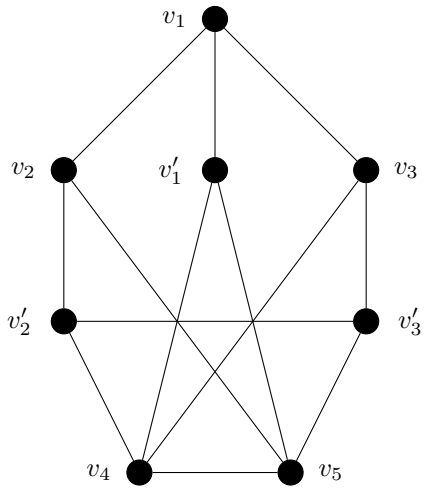

Figure 2.12: A K_5-minor-minimal graph with eight vertices, containing cycle $v_1v_2v_2'v_3'v_2$, where v_2' and v_3' have no common neighbor.

and v_2'. But if v_4 is adjacent to v_3, then the graph is triangle-free, and so we can apply Theorem 2.7.3 to conclude that G is definitely not swampable. Otherwise, v_4 is adjacent to v_3' which results in the graph in Figure 2.13.

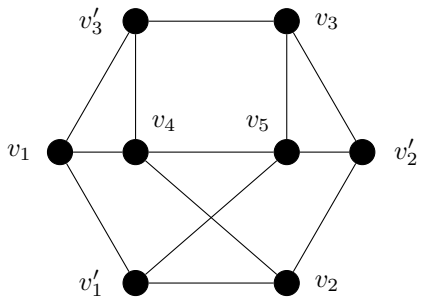

Figure 2.13: A K_5-minor-minimal graph with eight vertices, with a 6-cycle using the split pair vertices, and a triangle.

With the possible cases exhausted, we show that the graphs that we have not already shown to be not swampable are, in fact, not swampable.

Lemma 2.7.9. *Let G be the graph shown in Figure 2.9. Then G is not swampable.*

Proof. Suppose for a contradiction that (G, w) is a swamp. Then, by Theorem 2.3.4, in triangle $v_1v_2v_3$, there exists an irregular pair within the triangle $\{v_1, v_2, v_3\}$. Without loss of generality, we may assume $\{v_1, v_2\}$ is irregular. Then, by Theorem 2.3.2, in $G[\{v_1', v_2', v_3', v_4, v_5]$, we have $\{v_1', v_2'\}$ is regular. Let H be formed from $G[\{v_1', v_2', v_3', v_4, v_5]$ by suppressing v_3'; note that H has a double

edge connecting v_4 and v_5. Then, there is a swamp of H in which $\{v_1', v_2'\}$ is regular. However, this contradicts Theorem 2.4.1. □

Lemma 2.7.10. *Let G be the graph shown in Figure 2.11. Then G is not swampable.*

Proof. Suppose for a contradiction that (G, w) is a swamp. Notice that G has a $K_{2,3}$ subgraph, with bipartition $(\{v_1, v_5\}, \{v_2, v_1', v_3\})$. To this, we can add v_2', v_3', or v_4 to get three different $K_{3,3}$ subdivisions, all of which have the 4-cycle $v_1v_1'v_5v_3$ in common. By Lemma 2.5.5, we may assume that, in these three subdivisions, all branches have weight 0 or 1. Three such branches are the edges v_iv_i' for $i \in \{1, 2, 3\}$; those edges must have weight 1 by Lemma 2.7.5. Since $v_2'v_3'v_3$ is a branch of a $K_{3,3}$ subdivision, its weight is 0 or 1, and as $w(v_3v_3') = 1$, we have $w(v_2'v_3') \in \{0, -1\}$. This implies that $w(v_2v_2') + w(v_2'v_3') + w(v_3'v_3) \neq 0$.

Now consider the triangle $v_1'v_4v_5$. We claim $\{v_1', v_5\}$ is regular in the triangle. Suppose not. By Theorem 2.3.2, we have that in cycle $v_1v_2v_2'v_3'v_3$, that $\{v_1, v_2\}$ and $\{v_1v_3\}$ are both regular. By Theorem 2.3.4, we have $w(v_2v_2') + w(v_2'v_3') + w(v_3'v_3) = 0$, which gives us a contradiction. So $\{v_1', v_5\}$ is regular in triangle $v_1'v_4v_5$. In particular, $w(v_1'v_5) = 1$.

We turn now to triangles $v_1'v_4v_5$ and $v_2'v_3'v_4$. For each choice of $a \in \{v_1', v_5\}$ and $b \in \{v_2'v_3'\}$, there is an ab-path disjoint from the triangles. Applying Lemma 2.3.6 and the fact that $\{v_1', v_5\}$ is regular in triangle $v_1'v_4v_5$, we conclude that $\{v_2', v_4\}$ and $\{v_3', v_4\}$ are both regular in triangle $v_2'v_3'v_4$, so $\{v_2', v_3'\}$ is irregular in the triangle. By Theorem 2.3.2, we have in cycle $v_1v_1'v_5v_3$, that $\{v_1, v_2\}$ and $\{v_1', v_2\}$ are both regular. By Theorem 2.3.4, we conclude $w(v_1'v_5) + w(v_5v_2) = 0$. As neither edge has weight -1, we conclude $w(v_1'v_5) = w(v_5v_2) = 0$. But this contradicts that $w(v_1'v_5) = 1$. So (G, w) is not a swamp. □

Lemma 2.7.11. *Let G be the graph shown in Figure 2.12. Then G is not swampable.*

Proof. Suppose for a contradiction that (G, w) is a swamp. We will show that in the triangle $v_1'v_4v_5$, all three pairs of vertices are regular, which contradicts Theorem 2.3.4. First, suppose $\{v_4, v_5\}$ is not regular in the triangle. Then, $\{v_4, v_5\}$ is regular in H, where H is formed from G by deleting v_1', suppressing v_1, and deleting v_4v_5. Notice that H plus the edge v_4v_5 is $K_{3,3}$, which is 3-connected, and H has a cycle that missed both v_4 and v_5. This contradicts Theorem 2.4.1. So $\{v_4, v_5\}$ is regular in the triangle.

Now we must show the other two pairs are also regular. By symmetry, it suffices to prove $\{v_1'v_4\}$ is regular. Notice that deleting vertex v_1' and suppressing v_1 gives $K_{3,3}$. By Theorem 2.5.4, we may assume all edges in the $K_{3,3}$ have weight 0 or 1. Two such edges, v_2v_2' and v_3v_3', both have weight

1 by Lemma 2.7.5, and $w(v_2'v_3') \neq -1$. So now suppose $\{v_1'v_4\}$ is irregular in the triangle. Then, in the cycle $v_1v_2v_2'v_3'v_3$, we have $\{v_1, v_2'\}$ and $\{v_1, v_3\}$ are both regular by Theorem 2.3.2. Thus, by Theorem 2.3.4, we have $w(v_2'v_3') + w(v_3'v_3) = 0$. As $w(v_3'v_3) = 1$, we conclude $w(v_2'v_3') = -1$, giving us a contradiction. So $\{v_1', v_4\}$ is regular; likewise, $\{v_1', v_5\}$ is also regular. Since we have shown a triangle has all three pairs regular, we conclude (G, w) cannot be a swamp. □

Lemma 2.7.12. *Let G be the graph shown in Figure 2.13. Then G is not swampable.*

Proof. Suppose for a contradiction that (G, w) is a swamp. First, notice that by deleting v_4v_3' and v_5v_3, and suppressing v_3 and v_3', we get $K_{3,3}$. So, by Theorem 2.5.4, we may assume that every edge not incident to v_3 or v_3' has weight 0 or 1.

Consider the triangles $v_1v_4v_3'$ and $v_5v_2'v_3$. We claim that $\{v_1, v_3'\}$ and $\{v_4, v_3'\}$ are both regular in $v_1v_4v_3'$, or $\{v_5, v_3\}$ and $\{v_5, v_2'\}$ are both regular in $v_5v_2'v_3$. If not, we have two irregular pairs that we can link by linkage paths v_3v_3' and one of $v_1v_1'v_5$, $v_1v_1'v_2v_2'$, v_4v_5, or $v_4v_2v_2'$, contradicting Theorem 2.3.2. Hence, we may assume without loss of generality that $\{v_1, v_3'\}$ and $\{v_4, v_3'\}$ are both regular in $v_1v_3'v_4$. Thus, by Theorem 2.3.4, we have $w(v_1v_4) = 0$.

Now consider the 4-cycle $v_1'v_5v_2'v_2$. By Theorem 2.3.2, we have $\{v_1', v_2\}$ is regular in the cycle, as is $\{v_1', v_5\}$. Hence, by Theorem 2.3.4, we have $w(v_5v_2') + w(v_2'v_2) = 0$. Since neither edge has weight -1, we conclude both edges have weight 0. But $w(v_2'v_2) = 0$ contradicts Lemma 2.7.5. So no swamp is possible. □

2.7.4 G has nine vertices

Because of Theorem 2.7.3 and Lemma 2.7.4, we only need to work with K_5-minor-minimal nine-vertex graphs with a triangle. There are two cases: the triangle can have all vertices part of split pairs, or it may use two vertices of split pairs and the lone vertex that is not part of a split pair.

In the first case, we may suppose $v_1v_2v_3$ is a triangle. Then, v_1', v_2', and v_3' are all adjacent to v_5, and two of them are adjacent to the same vertex in X_4. Without loss of generality, we may assume v_1' and v_2' are adjacent to v_4, while v_3' and v_5 are adjacent to v_4'. The result is the graph shown in Figure 2.14.

Now suppose that there is no triangle of split pair vertices. In this case, let the neighbors of v_5 (the vertex that is not in a split pair) be v_1, v_2, v_3, and v_4. The triangle in G must consist of v_5 and two its neighbors; without loss of generality assume v_1 is adjacent to v_2. We have several subcases. First, we may have v_3 is adjacent to v_4. In this case, the remaining four edges only use v_i' vertices, so they are $v_1'v_3'$, $v_1'v_4'$, $v_2'v_3'$, and $v_2'v_4'$. The resulting graph is shown in Figure 2.15. The

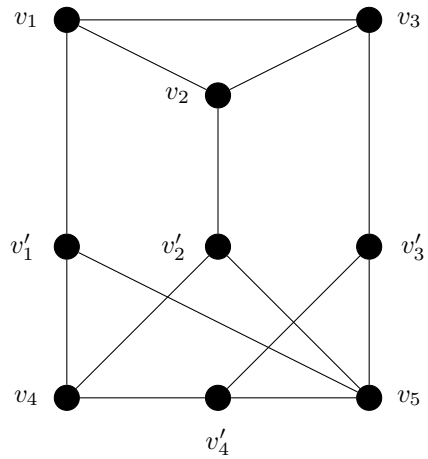

Figure 2.14: A K_5-minor-minimal graph with nine vertices, with a triangle of vertices in split pairs.

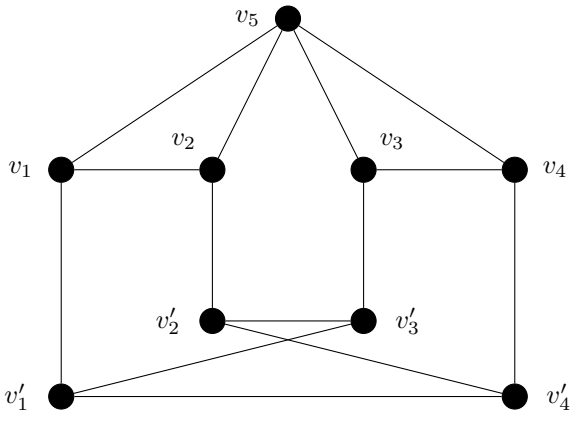

Figure 2.15: A K_5-minor-minimal graph with nine vertices, where v_5 is in two triangles.

second possibility is that the edge between X_3 and X_4 uses exactly one of v_3 and v_4. Without loss of generality, suppose v_3v_4' is an edge. Then v_3' is adjacent to both v_1' and v_2'. Now v_4 is adjacent to exactly one of v_1' and v_2'; without loss of generality, suppose v_4v_1' is an edge. So then $v_4'v_2'$ is an edge. This gives the graph shown in Figure 2.16. The graph, though, contains a V_8 subdivision: if we delete v_5v_2 and suppress v_2, then $v_1v_1'v_3'v_2'$ and $v_5v_4v_4'v_3$ are cycles satisfying Lemma 2.6.1. So the graph is not swampable.

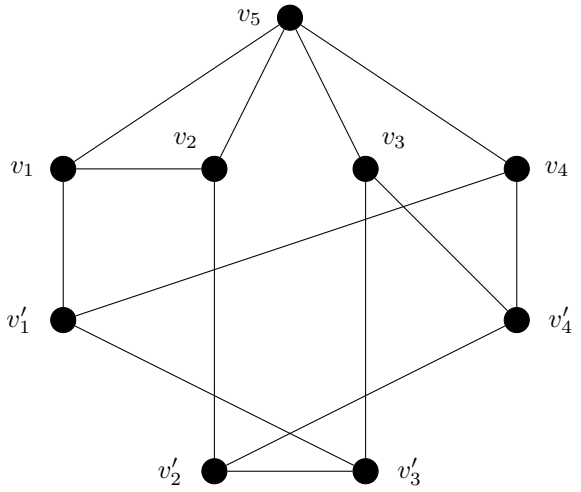

Figure 2.16: A K_5-minor-minimal graph with nine vertices, with a V_8 subdivision.

The last possibility is that the X_3 to X_4 edge is $v_3'v_4'$. In this case, one of v_3 and v_3' is adjacent to v_1' and the other is adjacent to v_2'; without loss of generality suppose v_3v_1' and $v_3'v_2'$ are edges. Notice now that $v_2'v_4'$ cannot be an edge, because it would form a triangle where each vertex is in a split pair. So v_4' must be adjacent to v_1', and v_4 is adjacent to v_2'. We get the graph shown in Figure 2.17. The graph contains a V_8 subdivision: if we delete v_2v_5 and suppress v_2, then $v_2'v_4v_4'v_3'$ and $v_1v_5v_3v_1'$ are cycles satisfying Lemma 2.6.1. So this graph is not swampable either.

So the only nine-vertex K_5-minor-minimal graphs to check for swampability are those shown in Figures 2.14 and 2.15. Neither is swampable, as shown in the following lemmas:

Lemma 2.7.13. *Let G be the graph shown in Figure 2.14. Then G is not swampable.*

Proof. Suppose for a contradiction that (G, w) is a swamp. We claim that, in triangle $v_1v_2v_3$ (call it T), both $\{v_1, v_3\}$ and $\{v_2, v_3\}$ are regular. As we can exchange (v_1, v_1') and (v_2, v_2') without breaking symmetry, it suffices to show $\{v_1, v_3\}$ is regular. Suppose not. Then by Theorem 2.3.2, $\{v_1', v_3'\}$ is regular in $G\backslash\{v_1, v_2, v_3\}$. Let H_1 be formed from $G\backslash\{v_1, v_2, v_3\}$ by suppressing v_2'. Then $H_1 \cup \{v_1'v_3'\}$ is a wheel with v_5 as its center, hence 3-connected, and $v_4v_4'v_5$ is a cycle. This contradicts Theorem

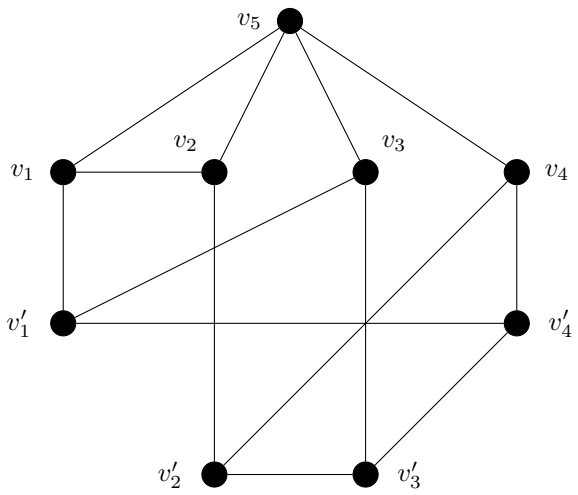

Figure 2.17: Another K_5-minor-minimal graph with nine vertices, also having a V_8 subdivision.

2.4.1. So $\{v_1, v_3\}$ and $\{v_2, v_3\}$ are both regular in T. By Theorem 2.3.4, we have $\{v_1, v_2\}$ is irregular in T and $w(v_1v_2) = 0$. Thus, $\{v_1', v_2'\}$ is regular in $G \setminus \{v_1, v_2, v_3\}$.

Let H_2 be formed from $G \setminus \{v_1, v_2, v_3, v_3'\}$ by suppressing v_4'. Now, by Theorem 2.4.1, we have $w(v_4v_4') + w(v_4'v_5) = 0$ and $w(v_1'v_4) = w(v_4v_2') = w(v_2'v_5) = w(v_5v_1') = k$ for some $k \in \{-1, 1\}$. Without loss of generality assume $k = 1$. Consider the triangle $v_3'v_4'v_5$ (call it T'), Notice that in the cycle $v_1'v_4v_2'v_5$ (call it C), we have $\{v_2', v_5\}$ is irregular. This irregular pair gives us that $\{v_3', v_5\}$ is regular in T' because (C, T') is also a $(v_2', v_5; v_3', v_5)$-linkable decomposition. Also, (T, T') is a $(v_1, v_2; v_4', v_5)$-linkable decomposition, so $\{v_4', v_5\}$ is regular in T'. Hence, in T', we have $w(v_3'v_4') = 0$ and $w(v_3'v_5) = w(v_4'v_5) = k$ for some $k \in \{-1, 1\}$. This gives us $w(v_4v_4') = -k$. Now delete $v_4'v_5$ and v_1v_2. We have a subdivision of $K_{3,3}$. One branch is $v_4v_4'v_3'$ with weight $-k$; another is v_5v_3' with weight k. This contradicts Theorem 2.5.4. So (G, w) is not a swamp. □

Lemma 2.7.14. *Let G be the graph shown in Figure 2.15. Then G is not swampable.*

Proof. Suppose for a contradiction that (G, w) is a swamp. Notice that $v_1v_2v_5$ and $v_3v_4v_5$ are triangles that satisfy Lemma 2.3.6, and the triangles are symmetric, so we may assume $\{v_1, v_5\}$ and $\{v_2, v_5\}$ are both regular in $v_1v_2v_5$ (making $\{v_1, v_2\}$ irregular in the triangle), so $w(v_1v_2) = 0$. Now, in $G \setminus \{v_1, v_2, v_5\}$, we have $\{v_1', v_2'\}$ is regular. Let H be formed from $G \setminus \{v_1, v_2, v_5\}$ by suppressing v_3 and v_4. Then, in H, we have $\{v_1', v_2'\}$ is regular, and we can apply Theorem 2.4.1 to show that $w(v_3'v_3) + w(v_3v_4) + w(v_4v_4') = 0$ and $w(v_1'v_3') = w(v_3'v_2') = w(v_2'v_4') = w(v_4'v_1') = k$ for some $k \in \{-1, 1\}$ (and we may assume $k = 1$). Let C be the cycle $v_1'v_3'v_2'v_4'$ and let T be the triangle $v_3v_4v_5$. Notice that (C, T) is a $(v_1', v_4'; v_5, v_4)$-linkable decomposition and also a $(v_1', v_3'; v_5, v_3)$-linkable

decomposition. Since $\{v_1', v_3'\}$ and $\{v_1', v_4'\}$ are both irregular in C, we conclude $\{v_3, v_5\}$ and $\{v_4, v_5\}$ are regular in T. So $\{v_3, v_4\}$ is irregular in T, and $w(v_3v_4) = 0$. Thus, $w(v_3'v_3) = -w(v_4'v_4)$. Now we can apply Lemma 2.3.3 to v_3v_4, $v_3v_3'v_1'v_4'v_4$, and $v_3v_5v_4$ to conclude that $w(v_3v_5) + w(v_5v_4) = -1$; since $w(v_3v_5) = w(v_5v_4)$, we get $w(v_3v_5) + w(v_5v_4) = 1$. Also, we apply the lemma to paths v_3v_4, $v_3v_5v_4$, and $v_3v_3'v_2'v_2v_1v_1'v_4'v_4$, getting

$$w(v_3v_3') + w(v_3'v_2') + w(v_2'v_2) + w(v_2v_1) + w(v_1v_1') + w(v_1'v_4') + w(v_4'v_4) = -1$$

This simplifies to

$$w(v_2'v_2) + w(v_1v_1') = 0$$

Finally, applying the lemma to paths v_1v_2, $v_1v_1'v_3v_2'v_2$, and $v_1v_5v_2$, we conclude that $w(v_1v_5) + w(v_5v_2) = -1$, and since both edges have the same weight, $w(v_1v_5) = w(v_5v_2) = 1$. By Lemma 2.7.5, no v_iv_i' edge has zero weight. Therefore, of v_1v_1' and v_2v_2', one edge has weight 1; the other has weight -1. Of v_3v_3' and v_4v_4', one edge has weight 1 and the other -1. Therefore, we may choose $i \in \{1, 2\}$, $j \in \{3, 4\}$ such that $w(v_iv_i') = 1$ and $w(v_jv_j') = -1$. Then $v_5v_iv_i'v_j'v_j$ is a zero-weight cycle. This contradicts that (G, w) is a swamp. □

2.7.5 G has ten vertices

If G is triangle-free, then we can apply Theorem 2.7.3 to conclude that G is not swampable. So the only graphs left to consider are ones with a triangle. The triangle necessarily has all its vertices in split pairs. Without loss of generality, suppose $v_1v_2v_3$ is a triangle. Then, one vertex in X_4 is adjacent to a vertex in X_5; without loss of generality, suppose $v_4'v_5'$ is the edge. There are two cases. First, v_4' and v_5' may have a common neighbor, which must be v_1', v_2', or v_3'. Without loss of generality, suppose it is v_1'. Then the resulting graph is shown in Figure 2.18. The second case is that v_4' and v_5' do not have a common neighbor. In that case, we may assume without loss of generality that $v_4'v_1'$ and $v_5'v_2'$ are edges. Then v_4v_2', v_4v_3', v_5v_1', and v_5v_3' are all edges. This gives us the graph shown in Figure 2.19. However, it contains a V_8 subdivision: after deleting v_2v_3 and suppressing both ends, we have the cycles $v_4'v_5'v_5v_1'$ and $v_4v_2'v_1v_3'$ satisfy Lemma 2.6.1. Thus, this graph is not swampable.

Thus, there is only one more K_5-minor-minimal graph to check for swampability: the graph shown in Figure 2.18.

Lemma 2.7.15. *Let G be the graph shown in Figure 2.18. Then G is not swampable.*

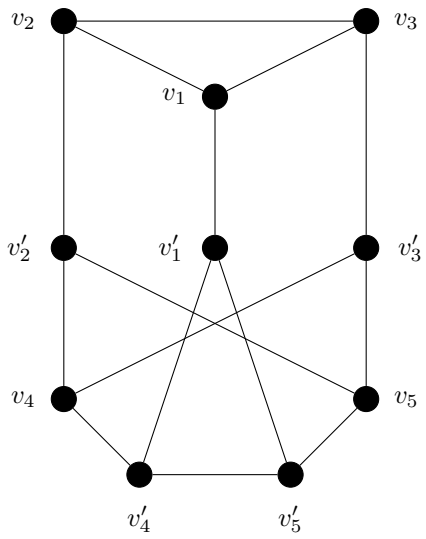

Figure 2.18: A K_5-minor-minimal graph with ten vertices, with two triangles of vertices in split pairs.

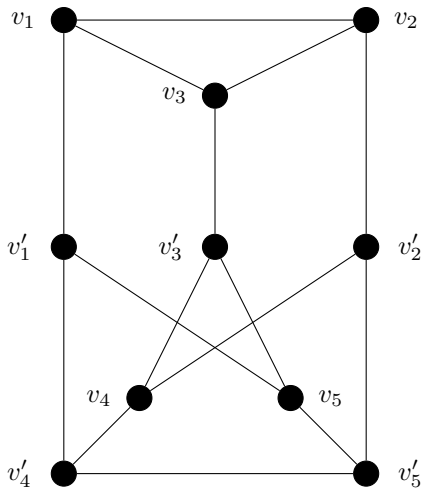

Figure 2.19: A K_5-minor-minimal graph with ten vertices, with one triangle of vertices in split pairs. It is not swampable because it contains a V_8 subdivision.

Proof. Suppose for a contradiction that (G, w) is a swamp. We claim that in triangle $v_1v_2v_3$, that both $\{v_1, v_2\}$ and $\{v_1, v_3\}$ are regular. Since swapping (v_2, v_2') and (v_3, v_3') gives an isomorphic graph, it suffices to show $\{v_1, v_2\}$ is regular. Suppose for a contradiction that it is irregular. Then, $\{v_1', v_2'\}$ is regular in $G \setminus \{v_1, v_2, v_3\}$. Notice that $G \setminus \{v_1, v_2, v_3\} \cup \{v_1'v_2'\}$ is a subdivided prism This, though, contradicts Lemma 2.4.8. So $\{v_1, v_2\}$ and $\{v_1, v_3\}$ are both regular in $v_1v_2v_3$. By Theorem 2.3.4, we have $\{v_2, v_3\}$ is irregular in triangle $v_1v_2v_3$. So, $\{v_2', v_3'\}$ is regular in $G \setminus \{v_1, v_2, v_3, v_1'\}$. Now, $G \setminus \{v_1, v_2, v_3, v_1'\} \cup \{v_2'v_3'\}$ is a subdivided K_4, with $v_4v_4'v_5'v_5$ as a branch. By Lemma 2.4.8, we have $w(v_4v_4') + w(v_4'v_5') + w(v_5'v_5) = 0$.

Let H_1 be formed by deleting v_1v_3 and $v_1'v_5'$ and suppressing all four ends. Let H_2 be formed by deleting v_1v_3 and $v_1'v_4'$ and suppressing all four ends. Notice that H_1 and H_2 are copies of $K_{3,3}$, and they share the 4-cycle $v_2'v_5v_3'v_4$. Moreover, v_4v_4' is an edge in H_1, and $v_5'v_5$ is an edge in H_2. By Lemma 2.5.5, we may assume that each of v_4v_4' and v_5v_5' has weight 0 or 1. However, by Lemma 2.7.5, neither edge has weight zero. So $w(v_4v_4') = w(v_5v_5') = 1$, and we deduce $w(v_4'v_5') = 1$. But then, in H_1, the branch $v_4'v_5'v_5$ has weight -1, but the edge v_4v_4' has weight 1, contradicting Theorem 2.5.4. Hence, (G, w) is not a swamp. □

Thus, having shown that all K_5-minor-minimal graphs are not swampable, we conclude that Theorem 2.7.1 is true.

2.8 Nonplanar swamps

In this section, we address the swampability of nonplanar graphs. We will prove the following theorem:

Theorem 2.8.1. *Let G be a simple nonplanar swampable graph, not isomorphic to V_8. Then G is in class $3VFS$.*

Since we have already shown that no graph with a K_5 minor is swampable, the only nonplanar graphs to check are ones with no K_5 minor. By Kuratowski's theorem, a nonplanar graph with no K_5 minor has a $K_{3,3}$ minor (equivalently, a $K_{3,3}$ subdivision). [7]

Before stating Wagner's theorem characterizing graphs with no K_5 minor, we must first define the k-sum of two graphs. If G and H are graphs, and $\{a_1, \ldots, a_k\} \subseteq V(G)$ and $\{b_1, \ldots, b_k\} \subseteq V(H)$ are cliques, the **k-sum** is formed by taking G and H, identifying each a_i and b_i as a new vertex c_i, and deleting zero or more edges with both ends in $\{c_1, \ldots, c_k\}$. Note that a 0-sum corresponds to taking a disjoint union.

We now state Wagner's theorem.

Theorem 2.8.2 ([9]). *Let G be a graph with no K_5 minor. Then G can be constructed by* 0-, 1-, 2-, *and* 3-*sums starting from planar graphs and V_8.*

In the Wagner construction, notice that if we ever take a (nontrivial) k-sum for $k < 3$, then the resulting graph has a cutset of size less than 3 that can never be repaired by further k-sums. Therefore, if we want to apply Wagner's theorem to get a 3-connected graph, we must only take 3-sums. Also, since V_8 is triangle-free, we cannot perform a 3-sum on V_8, so it cannot be used as a building block if we want to get a 3-connected graph.

Therefore, we have the following:

Theorem 2.8.3. *Let G be a* 3-*connected graph with no K_5 minor. Then G is isomorphic to V_8, or G is planar, or G can be constructed by* 3-*sums starting from planar graphs.*

Since we already know V_8 is swampable, all the remaining nonplanar graphs to be checked for swampability are constructed by 3-sums starting from planar graphs. In particular, we know that any such graph G has a 3-vertex cutset. We get the following lemma:

Lemma 2.8.4. *Let G be a simple* 3-*connected nonplanar graph with no K_5 minor that is not isomorphic to V_8. Then, there is a* 3-*vertex cutset $\{a_1, a_2, a_3\} \subseteq V(G)$ such that $G \setminus \{a_1, a_2, a_3\}$ has at least three components.*

Proof. By induction on $|V(G)|$. From Theorem 2.8.3 we have that there is a nontrivial separation (A, B) such that $|V(A) \cap V(B)| = 3$ and, letting A^+ be formed by adding an edge between nonadjacent vertices in $V(A) \cap V(B)$ (and similarly defining B^+), we have that neither A^+ nor B^+ contains a K_5 minor. Since G is nonplanar, it has a $K_{3,3}$ minor. We may also assume that neither A^+ nor B^+ has a $K_{3,3}$ minor: if one has a $K_{3,3}$ minor (say A^+), then we can apply induction to A^+ to get a 3-vertex cutset with at least three components; then, taking a 3-sum only changes one component. Hence, we have A^+ and B^+ contain neither K_5 nor $K_{3,3}$ as a minor, so they are both planar. Since having a $K_{3,3}$ minor is equivalent to having a $K_{3,3}$ subdivision, let $x_1, x_2, x_3, y_1, y_2, y_3$ be the corner vertices of the subdivision (where $(\{x_1, x_2, x_3\}, \{y_1, y_2, y_3\})$ is the bipartition of $K_{3,3}$).

We claim it is impossible for $V(A) \setminus V(B)$ and $V(B) \setminus V(A)$ to contain an x_i and a y_j (in some order). Suppose for a contradiction that $x_1 \in V(A) \setminus V(B)$ and $y_1 \in V(B) \setminus V(A)$. If $y_2, y_3 \in V(B)$, then all three vertices in $V(A) \cap V(B)$ are used (either by y_i or the x_1y_i-branch). This forces $x_2, x_3 \in V(B) \setminus V(A)$. But then, moving x_1 to a vertex in $V(A) \cap V(B) \setminus \{y_2, y_3\}$, we get a $K_{3,3}$ subdivision in B^+, but B^+ is planar. So, we may assume $y_2 \in V(A) \setminus V(B)$. Likewise, we may

assume $x_2 \in V(B) \setminus V(A)$. Now, so far, two vertices in $V(A) \cap V(B)$ must be used on the x_1y_1- and x_2y_2-branches. Placing x_3 uses up the last vertex in $V(A) \cap V(B)$ (either we choose the last vertex to be x_3, or an x_3y_i-path must cross $V(A) \cap V(B)$). But now there is nowhere to place y_3. This gives a contradiction.

So now assume that $x_1 \in V(A) \setminus V(B)$ (at least one corner vertex must be strictly on one side of the separation). This forces $y_1, y_2, y_3 \in V(A)$. Now, if $x_2, x_3 \in V(A)$, we have a $K_{3,3}$ subdivision in A^+, which we cannot have. So, we may assume $x_2 \in V(B) \setminus V(A)$. Then, $y_1, y_2, y_3 = V(A) \cap V(B)$, and this uses up $V(A) \cap V(B)$. So x_3 is in $V(A) \setminus V(B)$ or $V(B) \setminus V(A)$. Without loss of generality, assume $x_3 \in V(A) \setminus V(B)$. The subdivision gives us six internally disjoint paths: three paths P_1, P_2, P_3 from x_1 to $V(A) \cap V(B)$, and three paths Q_1, Q_2, Q_3 from x_3 to $V(A) \cap V(B)$.

Finally, we show that x_1 and x_3 are in separate components of $G \setminus (V(A) \cap V(B))$. Suppose not; then, there is a path from $(V(P_1) \cup V(P_2) \cup V(P_3)) \setminus (V(A) \cap V(B))$ to $(V(Q_1) \cup V(Q_2) \cup V(Q_3)) \setminus (V(A) \cap V(B))$. But then, we can contract to get a graph with x_1 and x_3 to each other and to all vertices in $V(A) \cap V(B)$. This, however, gives a K_5 minor in A^+. So x_1 and x_3 are in separate components of $G \setminus (V(A) \cap V(B))$, and x_2, being on the other side of the separation, is clearly in its own component as well. This gives us three desired components of $G \setminus (V(A) \cap V(B))$. □

Therefore, if G is a simple 3-connected nonplanar graph with no K_5 minor that is not isomorphic to V_8, then there is a 3-vertex cutset $\{a_1, a_2, a_3\}$ whose removal leaves at least three components C_1, C_2, C_3 (we will ignore the remainder for now). We will let $H_i = G[V(C_i) \cup \{a_1, a_2, a_3\}] \setminus \{a_ia_j \in E(G) \colon 1 \leq i < j \leq 3\}$.

The following claim about branch weights will be very useful:

Lemma 2.8.5. *There exists $k \in \{-1, 1\}$ such that for every component C_i and for every $z \in C_i$ and for every 3-fan from z to $\{a_1, a_2, a_3\}$ with paths P_1, P_2, P_3, we have $w(P_j) \in \{0, k\}$ for all $j \in \{1, 2, 3\}$.*

Proof. The key point is to fix 3-fans in two other components to $\{a_1, a_2, a_3\}$ and notice that adding any fan in C_i gives a $K_{3,3}$ subdivision. Moreover, every such subdivision agrees on a subdivided C_4 with four branch vertices (using the other components). The result follows from Lemma 2.5.5. □

Hereafter, we assume the lemma holds with $k = 1$.

First, we handle the case where H_i is a component for which a_j and a_k are irregular in $H_i \setminus \{a_\ell\}$, where j, k, ℓ are distinct. (This will always be the case when C_i has a cycle.)

Lemma 2.8.6. *Suppose that $\{a_1, a_2\}$ is irregular in $H_1 \setminus \{a_3\}$. Then $H_2 \cup H_3$ satisfies that $\{a_1, a_2\}$ is regular and $H_2 \cup H_3 \cup \{a_1a_2\}$ is 3-connected unless a_3 has degree 2, in which case suppressing a_3 leaves a 3-connected graph.*

Proof. We have that $H_2 \cup H_3$ satisfies that $\{a_1, a_2\}$ is regular by Theorem 2.3.2. So all that's left is to prove connectivity.

We prove this by showing that every pair $\{x, y\}$ of nonadjacent vertices in $H_2 \cup H_3 \cup \{a_1a_2\}$, not including a_3, has three internally-disjoint paths. Since G is 3-connected, we have three internally-disjoint xy-paths in G, but the problem is that the paths may have gone through C_1. We have several cases:

1. Suppose $x, y \in V(C_2)$. Notice that every path entering C_1 or C_3 uses up two of $\{a_1, a_2, a_3\}$, so there is only one such path. If no path goes through C_1, we are done. Otherwise, suppose a path does go through C_1. Then, no path goes through C_3, so we can reroute the path hitting C_1 to use C_3 instead.

2. Suppose $x \in V(C_2), y \in V(C_3)$. Then each path intersects $\{a_1, a_2, a_3\}$ exactly once, so none could have gone through C_1.

3. Suppose $x = a_1$, $y \in V(C_2)$. By Menger's theorem, there is a 3-fan from y to $\{a_1, a_2, a_3\}$. We can extend the ya_2-path using edge a_1a_2 and the ya_3-path using an a_3a_1-path in H_3. This gives us three internally-disjoint xy-paths.

Hence, this shows that $H_2 \cup H_3 \cup \{a_1, a_2\}$ is 3-connected except for possibly a_3 having degree 2, in which case, suppressing it gives a 3-connected graph. □

From Lemma 2.8.6, we can apply Theorem 2.4.1 to determine the structure of $H_2 \cup H_3$. All edges in $H_2 \cup H_3$ with exactly one end in $\{a_1, a_2\}$ are part of a 3-fan, so by Lemma 2.8.5, and our convention, their weights are all 1. Hence, all a_1a_2-paths in $H_2 \cup H_3$ have weight -1. Importantly, in $H_2 \cup H_3$, we get two different length paths between a_1 and a_3, depending on whether we go through a_2, and similarly with a_2 and a_3. Hence, we have the following by Theorem 2.3.2:

Lemma 2.8.7. *For $i = 1, 2$, $\{a_i, a_3\}$ is regular in $H_1 \setminus \{a_{3-i}\}$.*

We also have the following:

Lemma 2.8.8. *In H_1, there does not exist an a_1a_2-path of length 1.*

We will prove another useful fact about H_1:

Lemma 2.8.9. *Let C be a cycle in $H_1 \setminus \{a_1, a_2\}$. Choose $\{b_1, b_2, b_3\} \in V(C)\setminus$ such that $b_3 = a_3$ if $a_3 \in V(C)$ and there exist vertex-disjoint a_ib_i-paths P_i for $i \in \{1, 2, 3\}$ (where P_3 will be trivial if $b_3 = a_3$). Partition C into a b_1v-path Q_1, a vb_2-path Q_2, and a b_2b_1-path Q_3. Then, there exists $k \in \{-1, 1\}$ such that $w(Q_3) = 0$, $w(Q_1) = w(Q_2) = k$, and $w(P_1) = w(P_2) = k - 1$.*

Proof. By Lemma 2.8.7, we have that, in C, both $\{b_1, b_3\}$ and $\{b_3, b_2\}$ are regular. Hence, by Theorem 2.3.4, we conclude that $w(Q_3) = 0$ and $w(Q_1) = w(Q_2) = k$ for some $k \in \{-1, 1\}$. By Lemma 2.8.5, there exist $\ell, m \in \{0, 1\}$ such that $w(P_1) = \ell$ and $w(P_2) = m$. Notice that $P_1 \cup Q_3 \cup P_2$ determines an a_1a_2-path of weight $\ell + m$, which is not 1 by Lemma 2.8.8. Hence, $\ell = m$. Also, $P_1 \cup Q_1 \cup Q_2 \cup P_2$ determines an a_1a_2-path of weight $-k - \ell$, which is also not 1. So $k + \ell \neq -1$. Therefore, if $k = 1$, we must have $\ell = 0$; if $k = -1$, then $\ell = 1$. In any event, $\ell = k - 1$. The lemma follows. □

We get the following useful result from the lemma:

Lemma 2.8.10. *Let C be a cycle in $H_1 \setminus \{a_1, a_2, a_3\}$. Choose $\{b_1, b_2, b_3\} \in V(C)\setminus$ such that there exist vertex-disjoint a_ib_i-paths P_i for $i \in \{1, 2, 3\}$. There is no path in H_1 from $P_3 \setminus \{a_3\}$ to $(C \cup P_1 \cup P_2)\setminus\{a_1, a_2, b_3\}$. In particular, $\{a_1, a_2, b_3\}$ is a cutset in H_1 separating a_3 from $V(C)\setminus\{b_3\}$.*

Proof. By Lemma 2.8.9, we have $w(Q_3) = 0$, $w(Q_1) = w(Q_2) = k$, and $w(P_1) = w(P_2) = k - 1$ for some $k \in \{-1, 1\}$. Suppose for a contradiction that there is a path R in H_1 from $P_3 \setminus \{a_3\}$ to $(C \cup P_1 \cup P_2) \setminus \{a_1, a_2, a_3\}$. Let x be the end of R on $P_3 \setminus \{a_3\}$, and let y be the end of R on $(C \cup P_1 \cup P_2) \setminus \{a_1, a_2, a_3\}$. By symmetry, we may assume $y \in P_1 \cup Q_1 \cup Q_3$. Each case is handled in turn:

1. If $y = b_1$, then consider the cycle $Q_1 \cup R \cup P_3[x, b_3]$. Then, P_1, $P_2 \cup Q_2$, and $P_3[a_3, x]$ are paths satisfying Lemma 2.8.9. However, $w(P_1) = k - 1$ but $w(P_2) + w(Q_2) = -k - 1$, so they are not equal, as the lemma requires. This gives us a contradiction.

2. If $y \in Q_1 \setminus \{b_1, a_3\}$, then we can apply Lemma 2.8.9 with the cycle C' determined by $R \cup Q_1[y, b_3] \cup P_3[b_3, x]$ and paths $P_1 \cup Q_1[b_1, y]$, $P_2 \cup Q_2$, and $P_3[a_3, x]$. Hence, we get $w(Q_1[y, b_3]) = 0$. So, in C', $\{y, b_3\}$ is irregular. But in $H_2 \cup H_3 \cup P_1 \cup P_2 \cup Q_3$, we have $\{b_1, b_2\}$ is irregular. This violates Theorem 2.3.2.

3. If $y \in P_1 \setminus \{a_1, b_1\}$, then we can apply Lemma 2.8.9 with the cycle determined by $R \cup P_1[y, b_1] \cup Q_1 \cup P_3[b_3, x]$ and paths $P_1[a_1, y]$, $P_3[a_3, x]$, and our choice of either $P_2 \cup Q_3$ or $P_2 \cup Q_2$. However, $w(Q_3) \neq w(Q_2)$, so at least one choice gives a path with a different weight from $P_1[a_1, y]$, contradicting the lemma.

4. If $y \in Q_3 \setminus \{b_1, b_2\}$, then in C, we have four consecutive regular pairs: $\{b_1, y\}$, $\{y, b_2\}$, $\{b_2, b_3\}$, and $\{b_3, b_1\}$. This contradicts Lemma 2.3.5.

□

The final step is to show that if $H_1 \setminus \{a_1, a_2\}$ contains a cycle C, we can construct a special vertex feedback triple (a_1, a_2, b_3), where b_3 is chosen to satisfy Lemma 2.8.10.

Theorem 2.8.11. *Let C be a cycle in $H_1 \setminus \{a_1, a_2\}$. Choose $\{b_1, b_2, b_3\} \in V(C)\setminus$ such that $b_3 = a_3$ if $a_3 \in V(C)$ and there exist vertex-disjoint a_ib_i-paths P_i for $i \in \{1, 2, 3\}$ (where P_3 will be trivial if $b_3 = a_3$). Then, (a_1, a_2, b_3) is a special vertex feedback triple.*

Proof. We know that H_1 contains the subdivision shown in Figure 2.20. By Lemma 2.1.12, we may

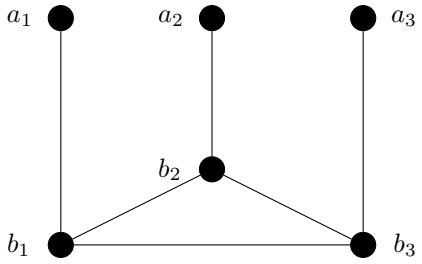

Figure 2.20: H_1 contains this subdivision, except that the edge a_3b_3 does not exist if $a_3 = b_3$. It becomes 3-connected after adding two vertices adjacent to $\{a_1, a_2, a_3\}$, obtained by taking a 3-fan from H_2 and H_3.

assume that H_1 contains a clean subdivision of the graph shown in Figure 2.20. For $i \in \{1, 2, 3\}$, let P_i be the a_ib_i-path in the subdivision (where P_3 may be trivial if $a_3 = b_3$. Let Q_1 be the b_1b_3-path, Q_2 be the b_2b_3-path, and Q_3 be the b_1b_2-path.

Let D_1 be the component of $H_1 \setminus \{a_1, a_2, b_3\}$ that contains $V(C) \setminus \{b_3\}$. We want to show that D_1 is a forest and there is at most one edge from D_1 to $\{a_1\}$ and at most one edge from D_1 to $\{a_2\}$; this implies that D_1 is a type 1 component.

By Lemma 2.8.9, there exists $k \in \{-1, 1\}$ such that $w(P_1) = w(P_2) = k-1$, $w(Q_1) = w(Q_2) = k$, and $w(Q_3) = 0$.

Now, in $G[V(D_1) \cup \{a_1, a_2, b_3\}]$, we claim only the following types of jumps are possible:

1. From b_3 anywhere, or

2. From a_1 to a_2.

We do this by proving the following (note the symmetry):

1. We cannot add a vertex x adjacent to $\{b_1, b_2, b_3\}$, because then b_1b_2x is a cycle and $\{a_1, a_2, x\}$ is not a cutset in H_1 separating a_3 from x, contradicting Lemma 2.8.10.

2. Likewise, for $i \in \{1, 2\}$, jumping from b_1 to x on b_2b_3 gives a cycle b_1b_2x, where $\{a_1, a_2, x\}$ is not a cutset in H_1 separating a_3 from x, again contradicting Lemma 2.8.10.

3. There is no jump from a_1b_1 to b_1b_2, b_2, or b_2b_3. If there were, we would get a cycle where $\{a_1, a_2, b_1\}$ is not a cutset in H_1 separating a_3 from b_1, contradicting Lemma 2.8.10.

4. There is also no jump from x on edge a_1b_1 to y on edge b_1b_3. If there were, we would get a cycle and Lemma 2.8.10 would force $\{a_1, a_2, y\}$ to be a cutset in H_1 separating a_3 from y, which it is not.

5. Next, we show there is no jump R from a_1 to b_2. Suppose there is. By Lemma 2.3.3, $w(R) = k + 1$. By Lemma 2.8.5, we have $k + 1 \neq -1$, so $k = -1$. So $w(R) = 0$, but $w(P_2) = 1$. Thus, we have found an a_1a_2-path having weight 1, contradicting Lemma 2.8.8.

6. Next, we claim there is no jump R from a_1 to x on b_1b_2. Because we can substitute R for P_i, we get $w(R) = k - 1$ also, and $w(Q_3[x, b_2]) = w(Q_3[b_1, x]) = 0$. If $k = 1$, we get a zero-weight cycle using $P_i \cup Q_3[b_1, x] \cup R$. If $k = 2$, we get a zero-weight cycle using $P_i \cup Q_1 \cup Q_2 \cup Q_3[b_2, x]$. Hence, this jump is impossible.

7. Now, we show there is no jump from a_1 to b_1b_3. This is because adding such a jump produces the same graph as adding a jump from a_1 to b_1b_2.

8. Next, we show there is no jump R from a_1 to x on b_2b_3. Since we can substitute R for P_1, we have $w(R) = k - 1$. Also, $w(Q_2[b_2, x]) = 0$ and $w(Q_2[x, b_3]) = k$. We have a cycle $P_1 \cup Q_3 \cup Q_2[b_2, x] \cup R$ having weight $-(k - 1)$, so $k = 2$. But then $P_1 \cup Q_1 \cup Q_2[b_3, x] \cup R$ is a cycle of weight $k + 1 = 0$. This is impossible.

9. Finally, we show there is no jump R from a_1 to x on a_2b_2. By Lemma 2.3.3, we have that $w(P_1) + w(R) + w(P_2[x, b_2]) = -k$, so $w(R) + w(P_2[x, b_2]) = k + 1$. But $w(P_{3-i}|_{a_{3-i}x}) + w(P_{3-i}[x, b_{3-i}]) = k - 1$. So $w(R) = w(P_{3-i}[a_{3-i}, x]) - 1$. However, both are 0 or 1 by Lemma 2.8.5. Thus, $w(R) = 0$ and $w(P_{3-i}[a_{3-i}, x]) = 1$. But those two paths together form an a_1a_2-path having weight 1, contradicting Lemma 2.8.8.

We have found all possible jumps. Now, we will show that D_1 is a forest by showing that every vertex in D_1 appears in the subdivision shown in Figure 2.20. Let $S = (V(P_1) \cup V(P_2) \cup V(Q_1) \cup$

$V(Q_2) \cup V(Q_3)$ Let F be a component of $D_1 \setminus S$. Notice that F is nonlocal and attaches to at least one vertex of D_1. Because of the jumps we have ruled out, the following cases for where F can attach are exhaustive, up to symmetry:

1. F can attach to b_3 and have at least two attachments on Q_3 (including possibly b_1 and b_2). Let x, y be the vertices on Q_3 adjacent to a vertex of F, with the vertices going in the order b_1xyb_2 (perhaps $x = b_1$ or $y = b_2$). There exists a 3-fan from some $v \in F$ to $\{x, y, b_3\}$ internally disjoint from the drawn paths; let the vx-path be R_1, the vy-path be R_2, and the vb_3-path be R_3. By Lemma 2.8.9, we have $w(R_1) = w(R_2) = \ell$ for some $\ell \in \{-1, 1\}$, $w(P_1)+w(Q_3[b_1, x]) = w(P_2) + w(Q_3[b_2, y]) = \ell - 1$, and $w(Q_3[x, y]) = 0$. Thus, $w(Q_3[b_1, x]) = w(Q_3[b_2, y]) = \ell - k$. But, now $\{x, b_3\}$ is regular in $Q_3[x, b_2] \cup Q_2 \cup R_1 \cup R_3$, and we have paths $Q_3[x, y]$, $Q_3[y, b_2] \cup Q_2$, R_1, R_3, and R_2 on which we can use Lemma 2.4.3 to conclude $w(R_2) = 0$. But this gives us a contradiction.

2. F can attach to b_3 and have at least two attachments on P_1 (including possibly a_1 and b_1). Let x, y be the vertices on P_1 adjacent to a vertex of F, with the vertices going in the order a_1xyb_1 (perhaps $x = a_1$ or $y = b_1$). There exists a 3-fan from some $v \in F$ to $\{x, y, b_3\}$ internally disjoint from the drawn paths; let the vx-path be R_1, the vy-path be R_2, and the vb_3-path be R_3. Consider the cycle C' formed by $Q_2 \cup Q_3 \cup P_1[b_1, y] \cup R_2 \cup R_3$. Now, with respect to this new cycle, R_1 is a forbidden jump (from v on the $R_2 \cup R_3$ branch to a_1 or the $P_1[a_1, y]$ branch). So it is impossible for the same reason that a jump from a_1 or a_1b_1 to b_1b_3 is impossible.

Therefore, all vertices of D_1 are represented in the subdivision shown in Figure 2.20. All that's left to add are edges, but only the edge a_1a_2 and edges from v. Thus, D_1 is a tree, and there is only one vertex adjacent to a_1 and one vertex adjacent to a_2. So D_1 is a type 1 component.

If $a_3 = b_3$, then this finishes the proof, as $G \setminus (V(H_1) \setminus \{a_3\})$ is a tree with at most one edge to a_3, so it is a type 2 component. Then, (a_1, a_2, b_3) is a special vertex feedback triple by Lemma 2.2.1.

Now, suppose $a_3 \neq b_3$. Then, we have additional components of $G \setminus \{a_1, a_2, b_3\}$ to consider. Let D_2 be the component of $G \setminus \{a_1, a_2, b_3\}$ containing a_3. Notice that D_2 contains all components of $G \setminus \{a_1, a_2, a_3\}$ besides C_1. We claim that each component of $(D_2 \cap C_1) \setminus V(P_3)$ attaches to only one vertex of P_3 (including a_3 and b_3). Suppose not; let F be a component that attaches to x and y on $V(P_3)$ such that b_3xya_3 appear on P_3 in that order (and maybe $b_3 = x$ or $y = a_3$). Then, F is not local, so it must attach to a_1 or a_2; assume without loss of generality it attaches to a_1. Then, there exists a 3-fan from some $v \in V(F)$ to $\{a_1, x, y\}$. Let R_1 be the a_1v-path, R_2 the xv-path, and R_3 the yv-path. Now, using cycle $R_2 \cup R_3 \cup P_3[y, x]$, we apply Lemma 2.8.9 to get that there exists $\ell \in \{-1, 1\}$

such that $w(R_1) = w(P_2) + w(Q_2) + w(P_3[b_3, x]) = \ell - 1$, $w(R_2) = 0$, and $w(R_3) = w(P_3[y, x]) = \ell$. Thus, $w(P_3[b_3, x]) = k + \ell$. But now, we have $w(R_1) + w(P_1) + w(Q_1) + w(P_3[b_3, x]) = -\ell - 2$, $w(R_3) + w(P_3[y, x]) = -\ell$, and $w(R_2) = 0$. This violates Lemma 2.3.3.

Therefore, every component of $(D_2 \cap C_1) \setminus V(P_3)$ attaches to exactly one vertex of P_3 (and it cannot be v because it belongs to the same component of $H_1 \setminus \{a_1, a_2, b_3\}$ as a_3). Thus, D_2 is a tree, and only one vertex is adjacent to v. So we have that D_2 is a type 2 component.

Lastly, we want to show the remaining components of $G \setminus \{a_1, a_2, b_3\}$ are trees and type 2 components. Let D_3 be any such component. First, we show D_3 is a tree. Suppose for a contradiction it has a cycle C'. Then, by Menger's theorem, there exist three vertex-disjoint paths from $V(C')$ to $\{a_1, a_2, b_3\}$. Let c_1 and c_2 be the vertices joined to a_1 and a_2, respectively. Applying Lemma 2.8.9, we get that $\{c_1, c_2\}$ is irregular in C'. But $\{b_1, b_2\}$ is irregular in C. This violates Theorem 2.3.2. Therefore, D_3 has no cycles.

We check now that D_3 has at most one vertex adjacent to b_3. Suppose not. Choose a pair of vertices $\{x, y\} \in V(D_3)$ adjacent to b_3. Since $G \setminus \{b_3\}$ is 2-connected, there is a pair of vertex disjoint paths from $\{x, y\}$ to $\{a_1, a_2\}$; without loss of generality, assume the paths are an xa_1-path (call it R_1) and a ya_2-path (call it R_2). Since D_3 induces a tree, there is a unique xy-path R_3 in D_3, which is internally disjoint from R_1 and R_2. Now apply Lemma 2.8.9 to the cycle xR_3yb_3. Then, there exists an $\ell \in \{-1, 1\}$ such that $w(R_1) = w(R_2) = \ell - 1$, $w(R_3) = 0$, and $w(xb_3) = w(yb_3) = \ell$. Now notice that $1 - k - \ell$ is the weight of both the cycle $P_1 \cup Q_3 \cup Q_2 \cup \{b_3x\} \cup R_1$ and the a_1a_2-path $P_1 \cup Q_3 \cup Q_2 \cup \{b_3x\} \cup R_3 \cup R_2$. Thus, $1 - k - \ell = -1$, so $k + \ell = -1$, and $k = \ell = 1$. But now $P_1 \cup Q_3 \cup P_2 \cup R_2 \cup R_3 \cup R_1$ is a zero-weight cycle, giving us a contradiction. So D_3 is a type 2 component.

Thus, we have found every component of $G \setminus \{a_1, a_2, b_3\}$ is a tree, and every tree is type 2, with the exception of D_1, which is type 1. So (a_1, a_2, b_3) is a special vertex feedback triple, as desired. □

We will prove one more lemma before proceeding to the proof of Theorem 2.8.1.

Lemma 2.8.12. *Let G be a 3-connected graph with a 3-vertex cutset $\{a_1, a_2, a_3\}$. Suppose T is a component of G that is a tree. Then, for every edge $xy \in E(T)$, there exist $1 \leq i < j \leq 3$ and $1 \leq k < \ell \leq 3$ such that $\{i, j, k, \ell\} = \{1, 2, 3\}$ and there exist internally disjoint xa_i-, xa_j-, ya_k-, and ya_ℓ-paths.*

Proof. Take a 3-fan from x to $\{a_1, a_2, a_3\}$. First, suppose one of the paths passes through y. Then, delete the edge xy from that path. So, up to symmetry, we have so far three internally disjoint paths: an xa_1-path, an xa_2-path, and a ya_3-path. Let z be the next vertex after y on the ya_3-path.

Take a path from y to $\{a_1, a_2\}$ in $G \setminus \{x, z\}$. Then, this path is the fourth path we need, and it is internally disjoint from the other three paths (since T is a tree). So in this case we are done. Second, suppose that none of the paths passes through y. Then, we have internally disjoint xa_i-paths for all $i \in \{1, 2, 3\}$. Take a 2-fan from y to $\{a_1, a_2, a_3\}$ in $G \setminus \{x\}$. Then, for one i for which we have found a ya_i-path, discard the xa_i-path. Again, we have four internally disjoint paths as desired. □

Proof of Theorem 2.8.1. By Lemma 2.8.4, there is a cutset $\{a_1, a_2, a_3\}$ whose deletion creates components $C_1, \ldots, C_k$ for $k \geq 3$. Let $H_i = G[V(C_i) \cup \{a_1, a_2, a_3\}]$. If, in any H_i, we can delete one a_i (say a_3) so that the other two vertices in the cutset are irregular, then by Lemma 2.8.6, $H_2 \cup \cdots \cup H_k$ has the structure of Theorem 2.4.1, with the exception that a_3 might have degree two. If $H_1 \setminus \{a_1, a_2\}$ has a cycle, then use Theorem 2.8.11. Otherwise, $H_1 \setminus \{a_1, a_2\}$ is a tree, as is $(H_2 \cup \cdots \cup H_k) \setminus \{a_1, a_2\}$. The trees both contain a_3. So $\{a_1, a_2\}$ is a two-vertex feedback set.

The second case is where every H_i has every pair from $\{a_1, a_2, a_3\}$ regular after the third vertex is deleted. This implies each C_i is a tree. If they are all trivial, then we just have $K_{3,n}$, which has a two-vertex feedback set. So consider a component that is nontrivial (say C_1). We first show that every edge has weight 0. Suppose $xy \in E(C_1)$. Then, by Lemma 2.8.12, we have distinct $i, j, k \in \{1, 2, 3\}$ and four internally disjoint paths: an xa_i-path P_1, a ya_i-path P_2, an xa_j-path P_3, and a ya_k-path P_4. By regularity and Lemma 2.3.1, we have in the cycle $P_2 \cup P_3 \cup \{xy\}$, that $\{x, a_i\}$ and $\{y, a_i\}$ are both regular. By Theorem 2.3.4, this gives us $w(xy) = 0$. Using Lemma 2.8.5, we conclude that every edge from a_i to C_1 has weight 0 or 1, but it only has weight 0 if it is the unique edge from a_i to C_1.

Now, for a nontrivial component (again, say C_1), we claim there are one or two vertices in $\{a_1, a_2, a_3\}$ to which C_1 has a zero-weight edge. Certainly not all three of $\{a_1, a_2, a_3\}$ have a zero-weight edge (we already found two internally-disjoint paths from some a_i to the end of any edge of C). To show that there is at least one vertex in $\{a_1, a_2, a_3\}$ with a zero-weight edge to C_1, suppose for a contradiction that this is not the case. Then, H_1 contains a subdivision of a claw with leaves $\{a_1, a_2, a_3\}$, all edges of the claw having weight 1. Now, arbitrarily choose a subdivision of a claw for two other components (say H_2 and H_3) This gives us a $K_{3,3}$ subdivision, and the swamp must satisfy Theorem 2.5.4. The only possible case is that we have a claw of zero-weight edges, in either H_2 or H_3 (say H_2). Then, H_3 has a claw subdivision with leaves $\{a_1, a_2, a_3\}$ and all edges having weight 1. But there is a four-edge path in H_1 from some a_i to a_j using only edges incident to $\{a_1, a_2, a_3\}$; the path therefore has weight 1. Completing it to a cycle using an $a_j a_i$-path in H_3 gives us a zero-weight cycle, contradicting that we have a swamp. So there are two cases. First,

suppose C_1 has a zero-weight edge to a_1 but not a_2 or a_3. By Theorem 2.5.4, we have for every other component, either a zero-weight edge to a_1 only or to a_2 and a_3 only (but only one component can be in the latter category). We claim that (a_2, a_3, a_1) is a special vertex feedback triple; every component is a tree that is type 2, unless it is the only component with zero-weight edges to a_2 and a_3, in which case it is type 1. Now suppose C_1 has a zero-weight edge to a_2 and a_3 but not a_1. Then, by Theorem 2.5.4, every other component has a zero-weight edge only to a_1. Again, (a_2, a_3, a_1) is a special vertex feedback triple, with every component other than C_1 being type 2 and C_1 being type 1. □

We conclude by showing that nonplanar 3-connected swamps have a zero-weight edge.

Theorem 2.8.13. *Let G be a 3-connected simple nonplanar graph, and let (G, w) be a swamp. Then there exists an edge e for which $w(e) = 0$.*

Proof. If G is isomorphic to V_8, then this is true by Theorem 2.6.3. So now suppose G is not isomorphic to V_8. By Lemma 2.8.4, there exists a cutset $\{a_1, a_2, a_3\}$ whose deletion leaves at least three components $C_1, C_2, \ldots, C_k$. Let $H_i = G[V(C_i) \cup \{a_1, a_2, a_3\}]$. If, in any H_i, we have a j for which $\{a_1, a_2, a_3\} \setminus \{a_j\}$ is irregular in $H_i \setminus \{a_j\}$, (say $j = 1$) then $H_2 \cup \cdots \cup H_k$ has $\{a_1, a_2\}$ regular, and it is 3-connected except that a_3 may have degree 2 (in which case suppressing it produces a 3-connected graph. So, we apply Theorem 2.4.1 to determine that $H_2 \cup \cdots \cup H_k \setminus \{a_1, a_2\}$ is a tree with all edges zero, except that one of the edges may have been subdivided to create a_3. This is only a problem if $H_2 \cup \cdots \cup H_k \setminus \{a_1, a_2\}$ has a single edge, which means that it has two vertices. Then, $k = 3$ and we have the structure shown in Figure 2.21. We have $w(a_1c_1) = w(c_1a_2) = w(a_2c_2) = w(c_2a_1)$,

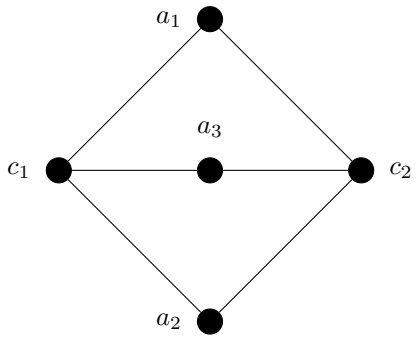

Figure 2.21: The only structure for $H_2 \cup H_3$ that does not immediately have a zero-weight edge.

and we may assume all those weights are 1. We also have $w(c_1a_3) + w(a_3c_2) = 0$. If either is zero, then we are done. So, we may assume, without loss of generality, that $w(c_1a_3) = 1$ and

$w(c_2a_3) = -1$. But now $a_1c_1a_3$ is a weight -1 path, $a_1c_2a_3$ is a weight 0 path, and $a_1c_2a_2c_1a_3$ is a weight 1 path. So $\{a_1, a_3\}$ is highly irregular in $H_2 \cup H_3$. But there is an a_1a_3-path in H_1. This contradicts Lemma 2.3.8. So we have a zero-weight edge.

Now assume that for every i and j that $H_i \setminus \{a_j\}$ has $\{a_1, a_2, a_3\} \setminus \{a_j\}$ regular. As in the proof of Theorem 2.8.1, if any component is nontrivial, it has a zero-weight edge. Otherwise, the graph is $K_{3,n}$, and it contains $K_{3,3}$ as a subgraph. We know that $K_{3,3}$ has a zero-weight edge by Theorem 2.5.4. This gives us a zero-weight edge in G, as desired. □

2.9 Planar graphs with prism plus a jump

The prism with a jump is shown in Figure 2.22.

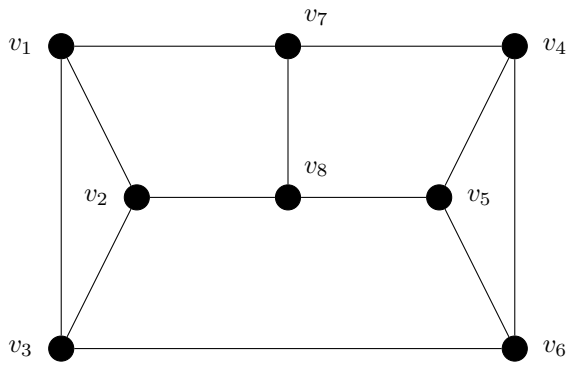

Figure 2.22: The prism plus a jump. Notice that (v_7, v_6, v_2) is a special vertex feedback triple, so this graph is swampable.

Theorem 2.9.1. *The prism plus a jump is swampable.*

Proof. Notice that (v_7, v_6, v_2) is a special vertex feedback triple. By Theorem 2.2.2, we have that the graph is swampable. □

We can say more about the possible swamps of prism plus a jump.

Lemma 2.9.2. *Let G be the prism with a jump as shown in Figure 2.22. Suppose (G, w) is a swamp. Then either $w(v_1v_2) = 0$ or $w(v_4v_5) = 0$.*

Proof. Apply Lemma 2.3.7 to triangles $v_1v_2v_3$ and $v_4v_5v_6$. □

Lemma 2.9.3. *Let G be the prism with a jump as shown in Figure 2.22. Suppose (G, w) is a swamp and $w(v_4v_5) = 0$. Then, either $w(v_1v_3) = 0$ or $w(v_2v_3) = 0$.*

Proof. Notice $\{v_4, v_5\}$ is irregular in triangle $v_4v_5v_6$. By Theorem 2.3.2, we have $\{v_1, v_2\}$ is regular in $v_1v_2v_3$. If $\{v_1, v_3\}$ is regular in $v_1v_2v_3$, then $w(v_2v_3) = 0$ by Theorem 2.3.4. Likewise, if $\{v_2, v_3\}$ is regular in $v_1v_2v_3$, then $w(v_1v_3) = 0$. So now assume both $\{v_1, v_3\}$ and $\{v_2, v_3\}$ are irregular in $v_1v_2v_3$. By Theorem 2.3.2, we have in cycle $v_7v_4v_5v_8$, that $\{v_8, v_5\}$ and $\{v_7, v_5\}$ are regular, giving us $w(v_7v_8) = 0$. But now $\{v_7, v_8\}$ is irregular in cycle $v_1v_7v_8v_2$, and $\{v_4, v_5\}$ is irregular in cycle $v_4v_5v_6$. This, though, contradicts Theorem 2.3.2. □

Lemma 2.9.4. *Let G be the prism with a jump as shown in Figure 2.22. Suppose (G, w) is a swamp. Then $w(v_7v_8) \neq 0$.*

Proof. Suppose for a contradiction that $w(v_7v_8) = 0$. By Lemma 2.9.2, we have either $w(v_1v_2) = 0$ or $w(v_4v_5) = 0$. We may assume $w(v_4v_5) = 0$. Then $\{v_7, v_8\}$ is irregular in cycle $v_1v_7v_8v_2$ and $\{v_4, v_5\}$ is irregular in triangle $v_4v_5v_6$. This contradicts Theorem 2.3.2. □

Theorem 2.9.5. *Let (G, w) be a swamp of a planar graph containing a subdivision of the prism with a jump as shown in Figure 2.22, and suppose the weights of the v_4v_5 and the v_1v_3 branch are both zero. Let H be formed by deleting the cycle with branches v_1v_2, v_2v_3, and v_3v_1. Then, in H, $\{v_7, v_6\}$ is a regular pair. Moreover, every edge on the v_4v_5 branch has weight zero.*

Proof. Since $\{v_1, v_3\}$ is irregular in the cycle with branches v_1v_2, v_2v_3, and v_3v_1, we have $\{v_7, v_6\}$ is regular in H by Theorem 2.3.2. Moreover, since G is planar, by Theorem 2.1.14, every internal vertex of the v_7v_6 branch is produced by adding a jump to another edge or a vertex other than v_7 or v_6; since the jump must be planar, it must land on the v_7v_4, v_7v_8, v_8v_5, v_5v_6, or v_4v_6 branches. Thus, in H, the v_7v_6 branch is a path satisfying Lemma 2.4.4. Thus, every edge on the branch has weight zero. □

2.10 Planar graphs with prism subdivision but no jump

In this section, we will prove the following:

Theorem 2.10.1. *Let G be a 3-connected planar graph with a prism subdivision, but not a subdivision of prism plus a jump. Suppose (G, w) is a swamp. Then there exists $e \in E(G)$ such that $w(e) = 0$.*

The graph shown in Figure 2.23 is formed by adding a vertex-to-non-triangle-edge jump.

Lemma 2.10.2. *Let G be the prism plus a vertex-to-edge jump as shown in Figure 2.23. Let (G, w) be a swamp. Then either $w(a_1a_4) = 0$ or $w(b_2b_3) = 0$.*

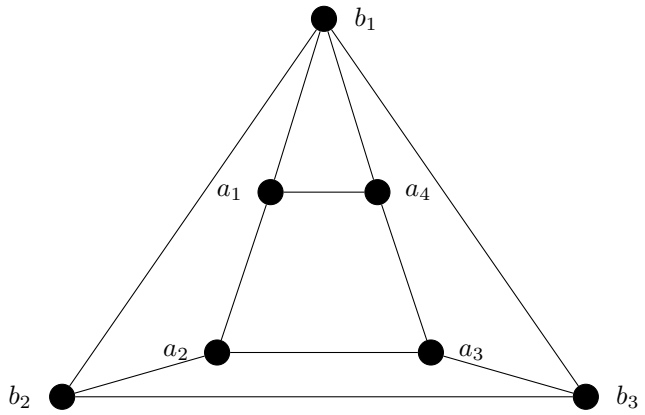

Figure 2.23: The prism plus a vertex-to-edge jump.

Proof. If $w(a_1a_4) = 0$ we are done. Otherwise, suppose not. Then, by Theorem 2.3.4, we have $\{a_1, b_1\}$ or $\{a_4, b_1\}$ is irregular in the triangle $a_1b_1a_4$. By symmetry, we may assume $\{a_1, b_1\}$ is irregular. Then, by Theorem 2.3.2, we have in cycle $a_2a_3b_3b_2$, that $\{a_2, b_2\}$ and $\{a_2, b_3\}$ are regular. By Theorem 2.3.4, this gives us $w(b_2b_3) = 0$, as desired. □

With this lemma, we can prove Theorem 2.10.1.

Proof of Theorem 2.10.1. Take a drawing of G and let C_1 and C_2 be disjoint faces. By Menger's theorem, there exist three vertex disjoint paths from $V(C_1)$ to $V(C_2)$. Let the paths be from $a_i \in V(C_1)$ to $b_i \in V(C_2)$ for $i \in \{1, 2, 3\}$. We may assume, by Lemma 2.1.12, that G contains a clean prism subdivision. We claim that the cleaning process preserves the six branches of the subdivision using C_1 and C_2. Notice each branch is already an induced path, so nothing changes when we proceed to a direct subdivision. To show that the cycle does not change when we reroute to eliminate local components, notice that rerouting the cycle enlarges the region bounded by the cycle. In particular, vertices that used to be on the cycle are now inside the region, and they can only be local components (the components are only adjacent to the new branch and its ends). Thus, C_1 and C_2 remain faces in the clean subdivision.

By Theorem 2.5.6, we have a zero edge on a triangular face. Without loss of generality, assume the a_1a_3 branch is zero-weight. If it is not subdivided, then we have our zero-weight edge, and we are done. If it is subdivided, there is a jump, which, by planarity and the fact that C_1 is a face, must go to a_1b_1, b_1, or b_1b_3. A jump to a_1b_1 gives prism plus a jump. A jump to b_1b_3 gives a cube, which is not swampable by Theorem 2.5.7. So the jump must go to b_1. Let the other end of the jump be a_4. Then, we have the graph shown in Figure 2.23. We have two cases. First, suppose

the a_1a_4 branch does not have zero weight. Then, the weight of the b_2b_3 branch (call it P) is zero by Theorem 2.10.2. If P is an edge, then it is zero-weight and we are done. Otherwise, there is an internal vertex b_4, and it has a jump to a_2 or a_3. Without loss of generality, suppose the jump is to a_2. Now, replace the a_2b_2-path by the a_2b_4-path, and apply Theorem 2.10.2. Then, $w(P[b_4b_3]) = 0$, and so $w(P[b_2b_4]) = 0$. If $P[b_2b_4]$ is a single edge, we are done. Otherwise, for every internal vertex b_5, the only jump is to a_2. Again, we replace the a_2b_2-path with the a_2b_5-path and apply Theorem 2.10.2 to get that $w(P[b_5b_3]) = 0$, so $w(P[b_2b_5]) = 0$. Thus, all the edges of $P[b_2b_4]$ are zero-weight.

So now suppose that the a_1a_4 branch is zero-weight. If it is a single edge, we have found a zero-weight edge. Otherwise, consider an internal vertex a_5; it must have a jump to b_1. If the weight of the a_1a_5 branch is nonzero, then we use the a_5b_1 jump instead of the a_4b_1 jump and apply Theorem 2.10.2 to get that the b_2b_3 branch is zero-weight. We apply the same arguments as in the previous case to find a zero-weight edge on the b_2b_3 branch. If this never happens, then every edge on the a_1a_4 branch is zero-weight, and again, this shows the existence of a zero-weight edge. □

2.11 Planar graphs with no prism subdivision

We finally reach the case of 3-connected planar graphs with no prism subdivision. By Dirac's theorem [5], these graphs are K_4, $K_5 \setminus \{e\}$, and the wheels. All three of these graphs have a two-vertex feedback set.

2.12 Swamps of graphs in $3VFS$

We conclude the chapter by proving the following about the swamps of 3-connected graphs in $3VFS$.

Theorem 2.12.1. *Let G be a 3-connected graph in $3VFS$ and let (G, w) be a swamp. Then there exists an edge e for which $w(e) = 0$.*

Proof. If G is nonplanar, then the result holds by Theorem 2.8.13. So we may assume G is planar.

There are two cases. First, suppose that G has a two-vertex feedback set $\{x, y\}$. Then $G \setminus \{x, y\}$ is a nontrivial tree. If the tree has three or more leaves, then each leaf is adjacent to x and y, and there is a non-leaf vertex in the tree with internally disjoint paths to the leaves. This gives us $K_{3,3}$, which contradicts that G is planar. So, we may assume $G \setminus \{x, y\}$ has exactly two leaves; it is a path $v_1 \dots v_n$. Then v_1 and v_n are both adjacent to x and y. First, suppose $n \geq 4$. If v_2 is adjacent to x and v_{n-1} is adjacent to y (or vice versa), then we have a prism subdivision in which each triangular face is not subdivided. By Theorem 2.5.6, at least one edge on a triangular face has zero weight.

Therefore, we may assume that v_2 and v_{n-1} are adjacent to x but not y. If y is adjacent to v_i for $3 \leq i \leq n-2$, then we can apply Lemma 2.3.6 with triangles xv_1v_2 and $xv_{n-1}v_n$ to get that $w(v_1v_2) = 0$ or $w(vn-1v_n) = 0$. Otherwise, y has three neighbors: v_1, v_n, and x, and every vertex v_i is adjacent to x. Thus, G is a wheel (with x as its center), and we apply Lemma 2.5.2 to find a zero-weight edge. If $n = 3$, then G is a 5-vertex graph with minimum degree 3, so either a wheel (which has a zero-weight edge) or $K_5 \setminus \{e\}$ (which, having K_4 as a subgraph, has a zero-weight edge by Theorem 2.5.1). If $n = 2$, then G is K_4, which has a zero-weight edge by Theorem 2.5.1. This completes the case where G has a two-vertex feedback set.

Now suppose that G has a special vertex feedback triple (a_1, a_2, b), with corresponding partition (A, B). If $A = \{a_1, a_2\}$, then $\{a_1, a_2\}$ is a two-vertex feedback set, and we are done. So suppose this is not the case. Then, every vertex in A has at most one neighbor in B (namely, b). Thus, any vertex in $A \setminus \{a_1, a_2\}$ has degree at least 2 in A. So we have that A is a tree with two leaves: a_1 and a_2; that is, an a_1a_2-path. In order for $\{a_1, a_2\}$ to not be a two-vertex feedback set, there must be a cycle in $B \setminus \{a_1, a_2\}$. Thus, $G[A]$ is a path with at least four vertices. Let x be the next vertex after a_1 on the a_1a_2-path, and let y be the last vertex before a_2. Then, x and y need to have degree at least 3, so they are both adjacent to b. If b is the only vertex in B, then $G \setminus \{a_1, b\}$ is a tree, and so $\{a_1, b\}$ is a two-vertex feedback set, and we are done. Note that every leaf of $G[B]$ is either b or a vertex adjacent to both a_1 and a_2. We now show that b is a leaf of $G[B]$. Suppose not. Then, there exist two internally disjoint paths from b to leaves, say c_1 and c_2. Now there is a subdivision of $K_{3,3}$ with bipartition $\{b, a_1, a_2\}$ and $\{x, c_1, c_2\}$, contradicting that G is planar. Therefore, b is a leaf. We claim that there is only one other leaf of $G[B]$. Suppose not. Then, there exist leaves c_1, c_2 distinct from b. There exists a vertex $d \in G[B]$ with a 3-fan B to $\{b, c_1, c_2\}$. But now we have a $K_{3,3}$ subdivision with bipartition $\{a_1, a_2, d\}$ and $\{b, c_1, c_2\}$, which shows G is nonplanar. Therefore $G[B]$ is a path with b at one end. Let c be the other end of the path; we have c is adjacent to a_1 and a_2. We have three cases. If a_1 and a_2 are both adjacent to b, then $A \cup \{c\}$ is a cycle, every vertex of which is adjacent to B. So G contains a wheel subgraph, which has a zero-weight edge by Lemma 2.5.2. Now suppose only one vertex (say a_1) is adjacent to b. Then a_2 is adjacent to another vertex in B aside from b and c. Let d be the next vertex after c on the cb-path. If d is adjacent to a_2, then G contains a prism with triangular faces a_1bx and cda_2, none of which are subdivided. So at least one edge is zero-weight by Theorem 2.5.6. If d is adjacent to a_1, then take z to be the next vertex after x on the a_1a_2-path in A (perhaps $z = y$). Then G contains a prism subdivision with triangular faces bxy and ea_1c, neither of which is subdivided; again we get a zero-weight edge. The third and final case is that b is adjacent to neither a_1 nor a_2. Again, $|B| \geq 3$, so let d be the next vertex after

c on the cb-path. Then, d is adjacent to a_1 or a_2, and the vertices are symmetric, so we may assume d is adjacent to a_1. Then a_1dc and xby are non-subdivided triangular faces of a prism subdivision of G; at least one edge is zero-weight by Theorem 2.5.6. So we are done. □

2.13 Finding adjacent degree two vertices

We finish the chapter with the proof of Theorem 1.0.2

Theorem 2.13.1. *Let G be a 3-connected swamp, and let (G, w) be a swamp. Then, there exists an edge $e \in E(G)$ having $w(e) = 0$.*

Proof. If G is in $3VFS$, the result follows from Theorem 2.12.1. If G is nonplanar, the result is due to Theorem 2.8.13. Otherwise, if G contains a prism plus a jump subdivision, this follows from Theorem 2.9.5. Otherwise, if G contains a prism subdivision, this follows from Theorem 2.10.1. Otherwise, G is a wheel or $K_5 \setminus \{e\}$, both of which are in $3VFS$. □

Proof of Theorem 1.0.2. By induction on $|V(G)|$. Suppose G is a nonempty simple 2-connected graph with no cycles of length 0 (mod 3). If the degree two vertices do not form an independent set, then we win. So, assume the degree two vertices form an independent set. Form H by suppressing all degree two vertices. Since G was simple, H is loopless. If H has parallel edges, say joining x and y, then, in G, we had xy as an edge and a vertex adjacent to both x and y (impossible, since G has no cycles of length 0 (mod 3)) or G had two degree two vertices with neighborhood $\{x, y\}$, which is one of the desired results, so we are done. Hence, we may assume H is simple. Since H is simple and G was 2-connected, we have that H is not isomorphic to K_2. If H is isomorphic to K_3, then G was a cycle, which is 2-regular, so any pair of adjacent vertices is a desired pair of adjacent degree two vertices. Define $w\colon E(H) \to \mathbb{Z}_3$ by $w(e) = -1$ if e was formed by suppressing an edge of G; $w(e) = 1$ otherwise. Since no edge of H has zero weight, H must not be 3-connected (this would violate Theorem 2.13.1). So there is a nontrivial separation (A, B) of H of order 2. Let $\{x, y\} = V(A) \cap V(B)$. Notice that since the minimum degree of H is at least 3, we have $|V(A) \setminus V(B)| \geq 2$ and $|V(B) \setminus V(A)| \geq 2$. By Lemma 2.1.10, one side of the separation (say A) is a swamp with all xy-paths having the same weight k. Now choose a nontrivial 2-separation (A', B') of H such that $V(A') \cap V(B')$ is regular in A' and, subject to that, minimizing $|V(A')| + |E(A')|$. Let $\{x', y'\} = V(A') \cap V(B')$. Then x' and y' are nonadjacent. If there is a cutvertex z separating x' and y', then $\{x', z\}$ is a cutset separating a subgraph of $A' \setminus \{y\}$ in which $\{x', z\}$ is regular, contradicting our choice of A'. So, by Menger's theorem, there are two internally disjoint $x'y'$-paths

P_1 and P_2 in A'. These paths are in the same component in $A' \setminus \{x', y'\}$ (otherwise, move one path to B' and reduce $|V(A')|$), so there exist $a \in V(P_1) \setminus \{x', y'\}$, $b \in V(P_2) \setminus \{x', y'\}$ and an ab-path P_3 in A' internally disjoint from $P_1 \cup P_2$. Now choose Q_1, Q_2, Q_3, c, d such that Q_1 and Q_2 are internally disjoint $x'y'$-paths in A', $c \in V(Q_1) \setminus \{x', y'\}$, $d \in V(Q_2) \setminus \{x', y'\}$ and Q_3 a cd-path internally disjoint from $Q_1 \cup Q_2$, and subject to this, minimize the length of Q_3. If Q_3 is an edge, it is zero-weight by Lemma 2.4.3, giving us a contradiction. Therefore, Q_3 has an internal vertex. But $\{c, d\}$ is not a cutset separating the internal vertices of Q_3 from the rest of the graph (or else we get a separation (A'', B'') with $Q_3 \subseteq A''$ $|V(A'')| < |V(A')|$, $V(A'') \cap V(B'') = \{c, d\}$, which is regular in A'' by Lemma 2.3.1). Thus, there is a path R from an internal vertex e of Q_3 to either Q_1 or Q_2, internally disjoint from $Q_1 \cup Q_2 \cup Q_3$. But then we can reroute the paths to get Q_1', Q_2', Q_3', c', d' with Q_3' shorter, contradicting our choice of paths. This gives us a contradiction.

So we conclude that the two adjacent degree two vertices exist. □

Chapter 3

Modularity

In this chapter, we apply Theorem 1.0.2 to prove Theorem 1.0.3.

Recall that the **modularity** of a graph G, denoted $m(G)$, is the number of even independent sets minus the number of odd independent sets. Equivalently, $m(G)$ is found by evaluating the **independence polynomial**

$$f_G(x) = \sum_{S \subseteq V(G); S \text{ independent}} x^{|S|}$$

at $x = -1$.

3.1 Basic modularity results

The null graph has modularity 1 (it has one independent set: the empty set). The single-vertex graph has modularity 0 (it has two independent sets: the empty set and the single vertex).

We can prove basic results about the independence polynomial that give us results about modularity:

Lemma 3.1.1. *If G is disconnected and $C_1, \ldots, C_k$ are its components, then*

$$f_G(x) = \prod_{i=1}^{k} f_{C_i}(x)$$

In particular, $m(G) = \prod_{i=1}^{k} m(C_i)$, and if G has an isolated vertex, then $m(G) = 0$.

Proof. Every independent set S of G corresponds to a k-tuple $(S_1, \ldots, S_k)$ where S_i is an independent

set in C_i. Therefore, the number of independent sets of size n is equal to

$$\sum_{(n_1,\ldots,n_k):n_1+\cdots+n_k=n} \prod_{i=1}^{k} a_{in_i}$$

where a_{ij} is the coefficient of x^j in $f_{C_i}(x)$. But this is the coefficient of x^n when multiplying the polynomials $f_{C_1}(x)f_{C_2}(x)\cdots f_{C_k}(x)$. This proves the lemma. □

The following result is immediate:

Lemma 3.1.2. *The disjoint union of a collection of unimodular graphs is unimodular.*

There is also a useful recurrence relation for the independence polynomial. For $v \in V(G)$, the **closed neighborhood** $N[v]$ is the set of v and all neighbors of v. So, if we want to consider all independent sets of G, we can fix v, take the independent sets not containing v (which correspond to independent sets in $G \setminus \{v\}$) and add the independent sets containing v (which correspond to independent sets in $G \setminus N[v]$). This shows the following recurrence.

Lemma 3.1.3. *Let G be a graph and v a vertex in G. Then*

$$f_G(x) = xf_{G\setminus N[v]}(x) + f_{G\setminus\{v\}}(x)$$

In particular

$$m(G) = m(G \setminus \{v\}) - m(G \setminus N[v])$$

We have a formula to compute the modularity of a graph with a cutvertex.

Lemma 3.1.4. *Let G be a connected graph. Let (A, B) be a nontrivial 1-separation of G with $A \cap B = \{v\}$. Then*

$$m(G) = m(A \setminus \{v\})m(B \setminus \{v\}) - m(A \setminus N[v])m(B \setminus \{v\})$$

Proof. Apply Lemma 3.1.3 to G with vertex v, and use the fact that $G \setminus \{v\}$ is the disjoint union of $A \setminus \{v\}$ and $B \setminus \{v\}$, and $G \setminus N[v]$ is the disjoint union of $A \setminus N[v]$ and $B \setminus N[v]$. Then apply Lemma 3.1.1. □

We use the formula of Lemma 3.1.4 to prove that the 1-sum of unimodular graphs is unimodular.

Lemma 3.1.5. *Let G be a connected graph. Let (A, B) be a nontrivial 1-separation of G with $A \cap B = \{v\}$. Then, if A and B are both unimodular, so is G.*

Proof. Let H be an induced subgraph of G. We want to show $m(H) \in \{-1, 0, 1\}$. If H is an induced subgraph of A or B, then we are done. Otherwise, if H does not contain v, it is the disjoint union of unimodular graphs, so it is unimodular. So, now suppose H is not an induced subgraph of A or B and contains v. Let $A' = A \cap H$ and $B' = B \cap H$; notice both A' and B' are unimodular. Then (A', B') is a nontrivial 1-separation of G with $A' \cap B' = \{v\}$. By Lemma 3.1.4, we have

$$m(H) = m(A' \setminus \{v\})m(B' \setminus \{v\}) - m(A' \setminus N[v])m(B' \setminus N[v])$$

Since A' and B' are unimodular, each of the four modularities on the right hand side is -1, 0, or 1. If any of them is zero, then $m(H)$ is the product of two numbers in $\{-1, 0, 1\}$, so it is in $\{-1, 0, 1\}$. So now, suppose that all four modularities are in $\{-1, 1\}$. Now notice that $m(A') = m(A' \setminus \{v\}) - m(A' \setminus N[v])$, and $m(A') \in \{-1, 0, 1\}$. Thus, we must have $m(A' \setminus \{v\}) = m(A' \setminus N[v])$. Likewise, $m(B' \setminus \{v\}) = m(B' \setminus N[v])$. But then this gives us $m(H) = 0$, which is what we desired. □

Lastly, we have a useful lemma about graphs with dominating vertices. If u, v are nonadjacent vertices in G, we say that v **dominates** u if every neighbor of u is a neighbor of v.

Lemma 3.1.6. *Let G be a graph with $u, v \in V(G)$ nonadjacent such that v dominates u. Then, $m(G) = m(G \setminus \{v\})$.*

Proof. Applying Lemma 3.1.3 to v, we have $m(G) = m(G \setminus \{v\}) - m(G \setminus N[v])$. However, in $G \setminus N[v]$, the vertex u is isolated. Thus, $m(G \setminus N[v]) = 0$ by Lemma 3.1.1 and $m(G) = m(G \setminus \{v\}$, as desired. □

3.2 Modularity of paths and cycles

We can use Lemma 3.1.3 to calculate the modularity of paths and cycles. Recall that the null graph has modularity 1 and the single-vertex graph has modularity 0.

We first calculate the modularity of paths:

Lemma 3.2.1. *The modularity of the n-vertex path P_n is given by*

$$m(P_n) = \begin{cases} 1 & \text{if } n \equiv 0, 5 \pmod 6 \\ 0 & \text{if } n \equiv 1, 4 \pmod 6 \\ -1 & \text{if } n \equiv 2, 3 \pmod 6 \end{cases}$$

Proof. By induction on n. The cases $n \leq 1$ are proved above. Now suppose we are given P_n with $n \geq 2$. Let v be a leaf of P_n. Then, $P_n \setminus \{v\}$ is isomorphic to P_{n-1}, and $P_n \setminus N[v]$ is isomorphic to P_{n-2}. Thus, we have by Lemma 3.1.3 that

$$m(P_n) = m(P_{n-1}) - m(P_{n-2})$$

and we verify this to be the case for all residue classes of $n \pmod 6$. □

Next, we compute the modularity of cycles:

Lemma 3.2.2. *For $n \geq 3$, the modularity of the n-vertex cycle C_n is given by*

$$m(C_n) = \begin{cases} 2 & \text{if } n \equiv 0 \pmod 6 \\ 1 & \text{if } n \equiv 1, 5 \pmod 6 \\ -1 & \text{if } n \equiv 2, 4 \pmod 6 \\ -2 & \text{if } n \equiv 3 \pmod 6 \end{cases}$$

In particular, C_n is unimodular if and only if n is not a multiple of 3.

Proof. Let $v \in V(C_n)$. Then $G \setminus \{v\}$ is isomorphic to P_{n-1}, while $G \setminus N[v]$ is isomorphic to P_{n-3}. Then $m(C_n) = m(P_{n-1}) - m(P_{n-3})$, and the formula for $m(C_n)$ follows immediately from Lemma 3.2.1. As a result, cycles with length 0 (mod 3) are definitely not unimodular, but cycles with length not 0 (mod 3) are unimodular (their modularity is in $\{-1, 0, 1\}$ and every proper induced subgraph is a disjoint union of possibly-trivial paths). □

3.3 Shortening three-edge paths with both internal vertices degree two

In this section, we will prove the following theorems:

Theorem 3.3.1. *Let G be a graph, let x and y be distinct nonadjacent vertices of G, and let P be a four-vertex path $xaby$, with $d(a) = d(b) = 2$. Let H be the graph formed from G by deleting a and b and identifying x and y as a new vertex z. Then, $m(G) = -m(H)$.*

Proof. We can apply Lemma 3.1.3 twice on G to x and y respectively. This gives us

$$m(G) = m(G \setminus \{x, y\}) - m(G \setminus (\{x\} \cup N[y])) - m(G \setminus (\{y\} \cup N[x])) + m(G \setminus (N[x] \cup N[y]))$$

Notice the second term is zero, because $\{a\}$ is an isolated vertex in $G \setminus (\{x\} \cup N[y])$. Likewise, the third term is also zero, because $\{b\}$ is an isolated vertex in $G \setminus (\{y\} \cup N[x])$. Finally, the first term leaves $\{a, b\}$ as a one-edge component, which has modularity -1. So

$$m(G \setminus \{x, y\}) = -m(G \setminus \{x, y, a, b\})$$

This leaves us with

$$m(G) = -m(G \setminus \{x, y, a, b\}) + m(G \setminus (N[x] \cup N[y]))$$

Now notice that $H \setminus \{z\} = G \setminus \{x, y, a, b\}$ and $H \setminus N[z] = G \setminus (N[x] \cup N[y])$. Thus,

$$m(G) = -m(H \setminus \{z\}) + m(H \setminus N[z]) = -m(H)$$

as desired. □

Theorem 3.3.2. *Let G be a graph, let x and y be distinct adjacent vertices of G, and let P be a four-vertex path $xaby$, with $d(a) = d(b) = 2$. Then, $m(G) = -m(G \setminus \{a, b, x, y\})$.*

Proof. The proof is similar to that of Theorem 3.3.1. Apply Lemma 3.1.3 on G to x. This gives us

$$m(G) = m(G \setminus \{x\}) - m(G \setminus N[x])$$

Then, apply it again on $G \setminus \{x\}$ to y, and we get

$$m(G) = m(G \setminus \{x, y\}) - m(G \setminus N[x]) - m(G \setminus N[y])$$

Notice that both $G \setminus N[x]$ and $G \setminus N[y]$ have an isolated vertex, so their modularities are both zero. Thus, $m(G) = m(G \setminus \{x, y\}$. Because $G \setminus \{x, y\}$ has $\{a, b\}$ as a one-edge component, we get that

$$m(G) = m(G \setminus \{x, y\}) = -m(G \setminus \{a, b, x, y\})$$

as desired. □

Now, we prove Theorem 1.0.3:

Proof of Theorem 1.0.3. Suppose for a contradiction the theorem is false. Choose a counterexample

G minimizing $|V(G)|$. Clearly G is nonempty. Thus, every proper induced subgraph of G is unimodular, but $m(G) \notin \{-1, 0, 1\}$. If G is not 2-connected, then apply Lemma 3.1.2 or 3.1.5 to get that G is unimodular, a contradiction. Thus, G is 2-connected. By Theorem 1.0.2, there exist two adjacent vertices a, b of degree 2, or there exist two nonadjacent vertices x, y of degree 2 with the same neighborhood. We will take the latter case first. In that case, x dominates y, so, by Lemma 3.1.6, we have $m(G) = m(G \setminus \{x\}) \in \{-1, 0, 1\}$, giving us a contradiction. Now, suppose we have two adjacent vertices a, b of degree 2. Let a's neighbors be x and b, and let b's neighbors be y and a. We have $x \neq y$, or else xab is a triangle, contradicting that G has no cycles of length 0 (mod 3). Thus, there is a path $xaby$ satisfying Theorem 3.3.1 or Theorem 3.3.2. Applying the theorem, there is a graph H for which $m(G) = -m(H)$, such that H has no cycle of length 0 (mod 3) and $|V(H)| < |V(G)|$. But since G is a minimal counterexample, we know that H is unimodular, so $m(H) \in \{-1, 0, 1\}$. But then $m(G) = -m(H) \in \{-1, 0, 1\}$, which gives us a contradiction. So every simple graph with no cycle of length 0 (mod 3) is unimodular. □

Chapter 4

Conclusion

In this dissertation, we have shown that simple 2-connected graphs with no 0 (mod 3) cycle have two adjacent degree two vertices or two nonadjacent degree two vertices with the same neighborhood. Then, using this result, we were able to show that all graphs with no 0 (mod 3) cycle are unimodular.

We can try using the tools used to analyze swamps and apply them to swamp-like objects (perhaps calling them induced-cycle swamps) that correspond to graphs with no 0 (mod 3) induced cycle. However, we need to distinguish edges having weight 1 (which may spoil an otherwise-zero-weight cycle) from edges having weight 0 or -1 (which can be safely deleted because we know they are actually subdivided). Some of the results, therefore, will not carry over to the induced cycle case. For example, Lemma 2.1.2 does not hold, because $K_{3,3}$ has no 0 (mod 3) induced cycle (so placing weight 1 on all edges gives an induced-cycle swamp), but subdividing every edge produces a graph with a 0 (mod 3) induced cycle (so placing weight 2 on all edges give an induced-cycle non-swamp). The $K_{3,3}$ example also shows that there is no hope of trying to find edges with a specific weight in a 3-connected induced-cycle-swampable graph. With this caveat in mind, though, many of the swamp analysis techniques we have demonstrated, such as regular pairs, can be adapted to induced-cycle swamps, but the applicability is rather limited, due to the likely presence of chords when assembling a cycle of length 0 (mod 3) from paths with appropriate lengths (mod 3).

Finally, we can try to prove the following conjecture:

Conjecture 4.0.3. *Every* 3*-connected swampable graph is in* $3VFS$ *or isomorphic to* V_8.

If the conjecture holds, then this would show that swampability is closed under taking minors, a result that is not immediately obvious from the definition of a swamp. Our results in this direction are promising (as illustrated in Theorem 2.8.1 for nonplanar swampable graphs), but the extension

of the proof to the remaining cases is difficult, though tantalizingly achievable.

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