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Symplectic and contact aspects of foliations and Anosov flows in dimension three

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Symplectic and contact aspects of foliations and Anosov flows in dimension three

Thomas Massoni

A Dissertation
Presented to the Faculty
of Princeton University
in Candidacy for the Degree
of Doctor of Philosophy

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Abstract

This dissertation studies the interactions between foliations and contact structures and the symplectic geometry of Anosov flows in dimension three. It is divided into two parts.

In the first part, we detail a new construction of codimension-one foliations on three-manifolds from suitable pairs of contact structures. This constitutes a converse result to a celebrated theorem of Eliashberg and Thurston about approximations of foliations by contact structures. Whereas foliations are rather rigid objects, this contact viewpoint reveals some surprising flexibility and provides new insight on the L-space conjecture.

The second part focuses on an important class of foliations arising from Anosov flows. We first show that three-dimensional Anosov flows can be entirely characterized in terms of symplectic geometry. As a result, we obtain new invariants for Anosov flows coming for Floer theory. We then explore the structure of these invariants and compute them explicitly in relevant cases. This is based on joint work with Kai Cieliebak, Oleg Lazarev, and Agustin Moreno, and it constitutes the first thorough investigation of Liouville manifolds beyond the Weinstein case. We finally sketch a proof that orbit equivalent Anosov flows define equivalent Liouville structures, implying that our Floer-theoretic invariants are topological invariants of Anosov flows.

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À mon grand-père.

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Chapter 0

Introduction

An effective way to study the global topology of a three-manifold is to investigate 2-dimensional structures attached to it. Two types of structures have been at the center of active research over the last decades: foliations, as *integrable* structures, and contact structures, as *maximally non-integrable* ones. Despite their apparent contrast, foliations and contact structures now appear as two faces of a more unified theory. This dissertation strengthens the relations between them, and explores some of the symplectic invariants associated with foliations. We now outline the content of this dissertation and refer the reader to the beginning of each chapter for a more detailed introduction.

Part I is based on a yet unpublished project and is divided into two chapters.

- In Chapter 1, we present a new construction of codimension-one foliations from pairs of contact structures in dimension three. This constitutes a converse result to a celebrated theorem of Eliashberg and Thurston about the approximation of foliations by contact structures. Under suitable hypotheses on the initial contact pair, we can ensure that the foliation we construct is *taut*. This allows us to characterize taut foliations entirely in terms of contact geometry. This viewpoint reveals surprising *flexible phenomena* and provides new insight on the *L-space conjecture*.

- Chapter 2 is entirely dedicated to the proof of a technical result needed in Chapter 1, and which might be of independent interest. We show that a *continuous* oriented plane field invariant under the flow of a suitable, possibly singular, smooth vector field is tangent to a *branching foliation* in the sense of Burago and Ivanov. The leaves of this branching foliation can then be separated to yield a genuine foliation. Our method of proof widely generalizes a strategy of Burago–Ivanov and could be applied to various other problems in three-dimensional dynamical systems.

Part II focuses on the symplectic geometry of *Anosov foliations* and *Anosov flows*.

- Chapter 3 is based on the article *Anosov flows and Liouville pairs in dimension three* [Mas23], accepted for publication in *Algebraic & Geometric Topology*. We establish a complete homotopy correspondence between three-dimensional Anosov flows and certain pairs of contact forms that we call *Anosov Liouville pairs*. We show a similar correspondence between projectively Anosov flows and bi-contact structures, extending the work of Mitsumatsu and Eliashberg–Thurston. As a consequence, every Anosov flow on a closed oriented three-manifold M gives rise to a Liouville structure on $\mathbb{R} \times M$ which is well-defined up to homotopy, and which only depends on the homotopy class of the Anosov flow. Our results also provide a new perspective on the classification problem of Anosov flows in dimension three.

- Chapter 4 is based on the article *Floer theory of Anosov flows in dimension three* [Cie+22] joint with Kai Cieliebak, Oleg Lazarev, and Agustin Moreno. We study the symplectic geometry of Anosov Liouville domains, via the wrapped Fukaya category, which we expect to be a powerful invariant of Anosov flows. The Lagrangian cylinders over the simple closed orbits span a natural A_∞-subcategory, the *orbit category* of the flow. We show that it does not satisfy Abouzaid's generation criterion; it is moreover "very large", in the sense that is not split-generated by any strict sub-family. This is in contrast with the Weinstein case, where critical points

of a Morse function play the role of the orbits. For the domain corresponding to the suspension of a linear Anosov diffeomorphism on the torus, we show that there are no closed exact Lagrangians which are either orientable, projective planes or Klein bottles. By contrast, in the case of the geodesic flow on a hyperbolic surface of genus $g \geq 2$ (corresponding to the McDuff example), we construct an exact Lagrangian torus for each embedded closed geodesic, thus obtaining at least $3g - 3$ tori which are not Hamiltonian isotopic to each other. For these two prototypical cases of Anosov flows, we explicitly compute the symplectic cohomology of the associated domains, as well as the wrapped Floer cohomology of the Lagrangian cylinders, and several pair-of-pants products.

- In Chapter 5, we give a detailed sketch of the proof that two orbit equivalent Anosov flows on a three-manifold M induce exact symplectomorphic Anosov Liouville structures on $\mathbb{R} \times M$. The main difficulty is that, by definition, an orbit equivalence is a *homeomorphism* and is not necessarily smooth. The key technical step is a careful approximation scheme for the orbit equivalence which might be of independent interest. As a consequence, the symplectic invariants studied in Chapter 4 are invariants of *orbit equivalence classes* of Anosov flows.

Part I

Taut foliations and contact pairs in dimension three

0.1 Introduction to Part I

Good things come in pairs.

Chinese proverb

Throughout the first part of this dissertation, M denotes a closed, connected, oriented three-dimensional manifold. The term foliation will always refer to a *codimension one* foliation. All the foliations and contact structures under consideration will be *(co)oriented*.

The main result of this part is a construction of foliations from pairs of contact structures. This can be regarded as a converse result to a celebrated theorem of Eliashberg and Thurston [ET98].

Main Theorem. *Let (ξ_-, ξ_+) be a positive contact pair on M, with a positively transverse vector field v. Assume either that one of $\xi_\pm$ is tight, or that they are everywhere transverse. Then there exists an aspherical $\mathcal{C}^0$-foliation $\mathcal{F}$ on M transverse to v.*

In particular, $T\mathcal{F}$ and $\xi_\pm$ are homotopic as oriented plane fields. If the vector field v is volume preserving, then the foliation $\mathcal{F}$ is *taut*. This allows us to completely characterize the existence of taut foliations on 3-manifolds purely in terms of contact geometry. Therefore, this construction can be regarded as the first step of a program to address the *L-space conjecture* from the point of view of contact topology.

0.1.1 Overview and motivations

0.1.1.1 Foliations and contact structures

We review some definitions and basic properties of (codimension-one) foliations and contact structures on 3-manifolds.

Foliations. A 2-dimensional foliation $\mathcal{F}$ on a 3-dimensional manifold M is, loosely speaking, a decomposition of M by surfaces, or *leaves*, which locally modelled on the standard foliation by horizontal planes on $\mathbb{R}^3$. Alternatively, a smooth foliation may be described in terms of its tangent plane field $T\mathcal{F}$. Assuming that it is cooriented, it can be realized as the kernel of a smooth, nowhere vanishing 1-form α satisfying

$$\alpha \wedge d\alpha = 0. \tag{0.1.1}$$

Reciprocally, any smooth nowhere vanishing 1-form satisfying (0.1.1) defines a cooriented foliation $\mathcal{F}$ with $T\mathcal{F} = \ker \alpha$ in a unique way by the Frobenius integrability theorem.

Foliations are among the most important structures on 3-manifolds and their theory is extremely rich. We refer to the encyclopedic books [CC00; CC03] and [Cal07] for a particularly exhaustive account of this theory and for the standard definitions. We wish to think about foliations as *fragile* objects which cannot be modified easily because of the integrability condition; a small perturbation of the 1-form α typically does not satisfy (0.1.1) anymore.

In this first part, we will mostly consider foliations and plane fields which are not smooth but only continuous. Let us recall the important

Definition 0.1.1. *A continuous plane field η on M is*

- ***Locally integrable*** *if for every point $p \in M$, there exists a (germ of) $\mathcal{C}^1$ surface tangent to η passing through p,*
- ***Integrable*** *if there exists a topological foliation $\mathcal{F}$ on M with $\mathcal{C}^1$-immersed leaves tangent to η,*
- ***Uniquely integrable*** *if it is locally integrable, and at each point the germ of integral surface is unique.*

We have the trivial implications

$$\text{Uniquely integrable} \implies \text{Integrable} \implies \text{Locally integrable},$$

but the converse implications are not true in general. Even one dimension lower, there exists continuous (Hölder regular) vector fields on $\mathbb{R}^2$ which are *not* tangent to foliations, or are tangent to many *distinct* foliations, see [BF03]. However, for Lipschitz continuous plane fields (or vector fields), these three definitions are equivalent by the Picard–Lindelöf–Cauchy–Lipschitz theorem. Besides, a given integrable continuous plane field might be tangent to many distinct foliations.

We are interested in foliations whose leaves are tangent to a continuous plane field:

Definition 0.1.2. *A topological foliation $\mathcal{F}$ on M is a $\mathcal{C}^0$-**foliation** if its leaves are smoothly immersed surfaces tangent to a global continuous plane field on M.*

Essentially, this is the lowest regularity class for which $T\mathcal{F}$ is defined as a continuous plane field. However, $\mathcal{C}^0$-foliations have *no transverse regularity*. By a theorem of Calegari [Cal01], any topological foliation on a 3-manifold is topologically isotopic to a $\mathcal{C}^0$-foliation. However, a $\mathcal{C}^0$-foliation can typically *not* be upgraded to a smooth—or even $\mathcal{C}^1$—foliation in a natural way.

Not every foliation on a 3-manifold is interesting. Indeed, Thurston showed that for every homotopy class of plane field on a closed 3-manifold, there exists a smooth foliation whose tangent plane field is in that class, see [Thu76]. However, these foliations have *Reeb components* by construction, making them too flexible to effectively measure the topology of the ambient manifold. Recall that a Reeb component of a foliation is a solid torus whose boundary is a leaf and such that the restriction of the foliation to the interior of the solid torus is a foliation by planes transverse to the core, see [Cal07, Example 4.7]. A foliation without Reeb components is called **Reebless**.

Another type of foliations was brought to prominence by the work of Thurston and Gabai: the class of *taut foliations*.

Definition 0.1.3. *A (smooth) foliation $\mathcal{F}$ on M is* ***(everywhere) taut*** *if for every $p \in M$, there exists an immersed loop in M transverse to $\mathcal{F}$ and passing through p.*

There exist many equivalent definitions of tautness, see [Cal07] for a sample. This definition still makes sense for $\mathcal{C}^0$-foliations. It can be adapted for topological foliations in different yet *nonequivalent* ways, see [CKR19]. However, all these definitions are equivalent up to a $\mathcal{C}^0$-small isotopy. While taut foliations are Reebless, the converse is not true in general but it holds on atoroidal manifolds. Importantly, the existence of a Reebless or taut foliation imposes strong restrictions on the topology and geometry of the ambient manifold M. For instance, Novikov's theorem implies that if M carries a Reebless foliation $\mathcal{F}$, then the leaves of $\mathcal{F}$ are π_1-injective and the loops transverse to $\mathcal{F}$ are non-contractible, see [Cal07]. This implies that M is irreducible and its universal cover is $\mathbb{R}^3$.

One could think of Reebless and taut foliations as *brittle* objects, fragile and rigid at the same time. While it is known by the work of Gabai [Gab83] that every closed irreducible 3-manifold with positive first Betti number carries a taut foliation, it is still an fundamental open problem to determine which *rational homology spheres* carry a Reebless or a taut foliation. The *L-space conjecture* described below provides some remarkable insight on that question.

Contact structures. A contact structure ξ on M is a smooth, *maximally non integrable* plane field. It can be (locally) represented as the kernel of a smooth 1-form α satisfying

$$\alpha \wedge d\alpha \neq 0 \tag{0.1.2}$$

pointwise. Since M is oriented and ξ is assumed to be cooriented, we can find a globally defined contact form α that satisfies either $\alpha \wedge d\alpha > 0$ or $\alpha \wedge d\alpha < 0$ everywhere on M. This sign does not depend on the choice of α. In the first case, ξ is called **positive** and in the second case, it is called **negative**. We further assume that α agrees with the coorientation of ξ, in the sense that it evaluates positively on vectors positively transverse to ξ.

By the Darboux theorem, a (positive) contact structure is locally modelled on the *standard* contact structure on $\mathbb{R}^3$ defined by the 1-form

$$\alpha_{\text{std}} \coloneqq r^2 d\theta + dz.$$

Here, (r, θ, z) denote the standard cylindrical coordinates on $\mathbb{R}^3$. Moreover, as opposed to foliations, contact structures are always smooth and quite *elastic*: a small $\mathcal{C}^1$ perturbation of α is still a contact form, and by Gray stability theorem, the resulting contact structure is contact isotopic to the original one. We refer to the book [Gei09] for a thorough introduction to contact geometry and the standard definitions.

As for foliations, not every contact structure on a 3-manifold is interesting. Mirroring Thurston's existence theorem for foliations, Eliashberg showed that every homotopy class of oriented plane fields on M contains a contact structure, which is *overtwisted*. A contact structure ξ is **overtwisted** if there exists an embedded disk in M which is transverse to ξ except along its boundary and its center. It is **tight** otherwise, and **universally tight** if its universal cover is tight. Overtwisted disks are to contact structures what Reeb components are to foliations, as they make contact structures flexible. On the other hand, tight contact structures are more rigid and more scarce.

The choice of a contact form α induces a **Reeb vector field** R on M uniquely

determined by

$$\alpha(R) = 1, \qquad d\alpha(R, \cdot\,) = 0.$$

A contact structure is **hypertight** if it admits a contact form whose corresponding Reeb vector field has no contractible closed orbit. By a theorem of Hofer [Hof93], hypertight contact structures are universally tight, but the converse does not hold in general.[1]

As opposed to Reebless or taut foliations, we would like to think of tight contact structures as *ductile* objects, both supple and rigid. Moreover, the classification problem for tight contact structures is perhaps more reasonable than its counterpart for taut foliations.[2] A *coarse classification* was established by [CGH09], and a fine classification is known for a number of 3-manifolds. Moreover, 3-dimensional contact topology benefits from an arsenal of tools and invariants, from convex surfaces to pseudoholomorphic curves.

Recently, Colin and Honda [CH20] showed a contact counterpart to Gabai's existence theorem for taut foliations: every closed irreducible 3-manifold with positive first Betti number carries a hypertight contact structure. However, determining which rational homology spheres carry a (hyper)tight contact structure still remains an important open problem.

0.1.1.2 Eliashberg–Thurston approximation theorem

At first glance, foliations and contact structures appear to be diametrically opposed notions—compare (0.1.1) with (0.1.2). However, the two theories share many key features, such as the dichotomy between *flexibility* and *rigidity*. A first breakthrough

[1] The standard tight contact structure on S^3 is universally tight but not hypertight.

[2] It is not clear to us which equivalence relation on $\mathcal{C}^0$-foliations to consider, whereas the answer is clear for contact structures: contact isotopy!

revealing a deep connection between foliations and contact structures was obtained by Eliashberg and Thurston:

Theorem (Eliashberg–Thurston [ET98])**.** *Let $\mathcal{F}$ be a $\mathcal{C}^2$ aspherical foliation on M. Then $T\mathcal{F}$ can be $\mathcal{C}^0$-approximated by positive and negative contact structures. Moreover, if $\mathcal{F}$ is taut, then its contact approximations are universally tight.*

Here, $\mathcal{F}$ is **aspherical** if none of its leaves is homeomorphic to a sphere. By the Reeb Stability Theorem, $\mathcal{F}$ is aspherical if and only if it is not homeomorphic to the product foliation by spheres on $S^1 \times S^2$, see [CC00, Theorem 6.1.6].

Tautness can be characterized by the existence of a closed 2-form ω that evaluates positively on leaves. If $\mathcal{F}$ is taut, Eliashberg and Thurston constructed from such a 2-form a symplectic structure on $[-1, 1] \times M$, which is a *weak symplectic filling* of $(M, \xi_-) \sqcup (M, \xi_+)$ where $\xi_\pm$ are negative and positive contact approximations of $\mathcal{F}$. They then appealed to a theorem of Eliashberg and Gromov to conclude that ξ_- and ξ_+ are universally tight.

This result provides an efficient way to construct (tight) contact structures by approximating (taut) foliations. It has been extended and generalized in various directions; we now summarize some of them.

$\mathcal{C}^0$-foliations. Eliashberg–Thurston theorem was generalized to $\mathcal{C}^0$-foliations independently by Kazez–Roberts [KR17] and Bowden [Bow16a].

Reebless foliations. Colin [Col02] showed that every $\mathcal{C}^2$ aspherical Reebless foliation $\mathcal{F}$ can be approximated by universally tight contact structures. This has been strengthened by Bowden who showed that every $\mathcal{C}^0$ small contact approximation of $\mathcal{F}$ is universally tight [Bow16b], and that Colin's result holds for $\mathcal{C}^0$-foliations [Bow16a].[3]

[3]However, it is not known if every sufficiently small contact approximation of a Reebless $\mathcal{C}^0$-foliation is universally tight.

Foliations without invariant transverse measures. More recently, Zung showed in [Zun24] that a $\mathcal{C}^2$ foliation without invariant transverse measures $\mathcal{F}$ can be approximated by contact structures whose Reeb vector fields are transverse to $\mathcal{F}$. Since $\mathcal{F}$ is taut, these contact approximations are hypertight. Moreover, Zung (implicitly) shows that $\mathcal{F}$ can be approximated by contact structures ξ_- and ξ_+, negative and positive, respectively, supporting a *Liouville pair*: there exist contact forms $\alpha_\pm$ for $\xi_\pm$ such that the 1-form

$$\lambda \coloneqq e^{-s}\alpha_- + e^s\alpha_+ \tag{0.1.3}$$

on $\mathbb{R}_s \times M$ is a *Liouville form*, i.e., $d\lambda$ is a nondegenerate 2-form. We call the pair (α_-, α_+) a **Liouville pair** supported by (ξ_-, ξ_+).

In view of Zung's theorem and in order to simplify the terminology, we propose the following definition:

Definition 0.1.4. *A $\mathcal{C}^0$-foliation if **hypertaut** if it has no invariant transverse measures.*

Hypertautness can be characterized by the existence of an *exact* 2-form $\omega = d\beta$ that evaluates positively on the leaves, see Lemma 1.4.3 below.

Anosov foliations. An interesting class of hypertaut foliations arises as the weak-(un)stable foliations of *Anosov flows*, see for instance the encyclopedic book [FH19] or the more accessible notes [Bar17]. In [Hoz22], Hozoori showed a correspondence between Anosov flows on 3-manifolds and a special class of Liouville pairs, that we call **Anosov Liouville pairs**. We upgraded this result in [Mas23] (reproduced as Chapter 3) to a *homotopy equivalence* between the space of Anosov flows and the space of Anosov Liouville pairs. The latter structures can be thought of as very specific approximations of Anosov foliations that remember the hyperbolic nature of the flow.

Uniqueness of contact approximations. In [Vog16], Vogel proved that for most $\mathcal{C}^2$ foliations without torus leaves, all the (positive) contact structures in a sufficiently small neighborhood of $T\mathcal{F}$ are contact isotopic. Therefore, the contact approximations of Eliashberg–Thurston are unique for these foliations. However, some foliations on $\mathbb{T}^3$ have infinitely many pairwise non isotopic contact approximations.

Approximation versus deformation. A *contact deformation* of a $\mathcal{C}^0$-foliation $\mathcal{F}$ is a $\mathcal{C}^0$-continuous 1-parameter family of plane fields (ξ_t), $t \in [0,1]$, where $\xi_0 = T\mathcal{F}$, and for $t > 0$, ξ_t is a contact structure. While $\mathcal{C}^2$ foliations 'with enough holonomy' can be deformed into contact structures (see [ET98, Proposition 2.9.2]), linear foliations without closed leaves on $\mathbb{T}^3$ do *not* admit contact deformations (see [Vog16, Corollary 9.11]). In contrast, Etnyre showed that any (cooriented) contact structure on a 3-manifold is a smooth deformation of a foliation, see [Etn07]. However, the foliations constructed by Etnyre *always* have Reeb components.[4]

0.1.1.3 A dictionary between foliations and contact pairs

While S^3 carries a (unique) tight contact structure, namely the standard one, it does not carry a Reebless foliation. Therefore, there is no hope to construct a geometrically relevant foliation from a *single* (tight) contact structure. Instead, one should consider a *pair* of contact structures with opposite orientation.

Definition 0.1.5. *A* ***contact pair*** *on M is a pair (ξ_-, ξ_+) where ξ_- is a negative contact structure and ξ_+ is a positive contact structure. Both are cooriented. A* ***bicontact structure*** *is a contact pair (ξ_-, ξ_+) such that ξ_- and ξ_+ are everywhere transverse. A contact pair is* ***positive*** *if there exists a vector field v positively transverse to ξ_- and ξ_+.*

[4] Etnyre starts with an *open book decomposition* adapted to the contact structure, and inserts a suitable Reeb component at each component of the biding of the open book.

Alternatively, a contact pair (ξ_-, ξ_+) is positive if and only if at every point $p \in M$ with $\xi_-(p) = \xi_+(p)$, these two planes have the same orientation (and opposite coorientations). A bicontact structure is automatically positive. Positive contact pairs were introduced by Colin and Firmo in [CF11]. See also [CH20] for a summary (in English!) of Colin–Firmo's paper.[5]

Eliashberg–Thurston theorem and its variations naturally produce positive contact pairs as approximations of foliations. We shall think of the positive and negative contact approximations, and the way they interact, as capturing some information about the original foliation. We will need both of them to be able to 'reconstruct' the foliation, or rather produce a new one that is closely related to it.

To each adjective characterizing the rigidity class of a foliation—*Reebless, taut, hypertaut, Anosov*—we associate an adjective for positive contact pairs. Notice the implications:

$$\text{Anosov} \implies \text{Hypertaut} \implies \text{Taut} \implies \text{Reebless}.$$

Definition 0.1.6. *Let (ξ_-, ξ_+) be a contact pair.*

- *It is **(universally) tight** of both $\xi_\pm$ are (universally) tight.*
- *It is **strongly tight** if for every point $p \in M$, there exists a closed loop positively transverse to both $\xi_\pm$ and passing through p.*
- *It is **Liouville** if there exists contact forms $\alpha_\pm$ for $\xi_\pm$ such that (α_-, α_+) is a Liouville pair, i.e., the 1-form λ on $\mathbb{R} \times M$ defined by (0.1.3) is a Liouville form.*
- *It is **Anosov Liouville** if there exists contact forms $\alpha_\pm$ for $\xi_\pm$ such that both (α_-, α_+) and $(-\alpha_-, \alpha_+)$ are Liouville pairs.*

As we will see, the following implications hold:

[5]Watch out: in [CH20], the authors use the terminology *contact pair* to mean *positive contact pair*!

Anosov Liouville $\implies$ Liouville $\implies$ Strongly tight $\implies$ Universally tight positive.

Motivated by the previous results, we propose the following dictionary between foliations and positive contact pairs:

Foliations	**Positive contact pairs**
Reebless	Universally tight
Taut	Strongly tight
Hypertaut	Liouville
Anosov	Anosov Liouville

Table 1: A dictionary between foliations and positive contact pairs

An aspherical foliation satisfying a property in the left column is approximated by a positive contact pair with the corresponding property in the right column.[6] With the notable exception of the (universally) tight case, we will show that a contact pair satisfying a property in the right column 'integrates' to a $\mathcal{C}^0$-foliation with the corresponding property in the left column. In particular, a 3-manifold $M \neq S^1 \times S^2$ carries a taut $\mathcal{C}^0$-foliation *if and only if* it carries a strongly tight contact pair; see Corollary D below.

0.1.1.4 The L-space conjecture

It is an open problem to determine which irreducible rational homology spheres carry a taut foliation. The famous L-space conjecture provides a hypothetical answer:

Conjecture (L-space conjecture [BGW13; Juh15])**.** *Let M be an irreducible rational homology 3-sphere. The following are equivalent:*

1. *M carries a cooriented taut foliation,*

2. *$\pi_1(M)$ is left-orderable,*

[6]Strictly speaking, we do not know if Zung's result for hypertaut foliations holds in the $\mathcal{C}^0$ setting. However, a hypertaut $\mathcal{C}^0$-foliation approximates to a *Liouville fillable* contact pairs, see Proposition 1.4.4.

3. M is not an L-space.

The second item can be thought of an algebraic characterization, while the third one comes from gauge theory. Here, a 3-manifold M is an L-space if its 'hat' Heegaard Floer homology is as small as possible, namely

$$\operatorname{rank}\widehat{HF}(M) = |H(M, \mathbb{Z})|,$$

or alternatively if its reduced Heegaard Floer homology vanishes, $HF^{\mathrm{red}}(M) = 0$. Lens spaces are examples of L-spaces. We refer to [Juh15] for an overview of Heegaard Floer homology. In this dissertation, we are mostly interested in the relation between items 1 and 3 of the L-space conjecture. We do not know how the (non) orderability of $\pi_1(M)$ could relate to contact pairs.

In [OS04], Ozsváth and Szabó proved the implication $1 \implies 3$ by showing that the *contact class* of a contact approximation of the foliation provided by the Eliashberg–Thurston theorem is nonzero in HF^{red}. Interestingly, contact structures seem to have better interactions with gauge theory than foliations.[7]

One key difficulty to reverse Oszváth–Szabó result and show that non-L-space irreducible rational homology spheres admit taut foliations is that very few general strategies are available to construct taut foliations. To our knowledge, all the known cases where the equivalence $1 \iff 3$ holds were obtain by rather ad hoc arguments, by considering a class of 3-manifolds and independently determining which are L-spaces and which admit a taut foliation. See for instance [BC17; Ras17; RR17; Han+20] for all graph manifolds, [Dun20] for a large census of hyperbolic manifolds, and [Kri20; Kri23; San23b; San23a; San23c] for 3-manifolds obtained by Dehn surgeries on suitable knots

[7] In [Zha22], Zhang associates to any *smooth* hypertaut foliation $\mathcal{F}$ two nonzero invariants $c_{\pm}(\mathcal{F})$ in monopole Floer homology, without invoking the Eliashberg–Thurston theorem. However, this approach is still symplectic in nature as it relies on the construction of a suitable exact symplectic structure on $\mathbb{R} \times M$ depending on $\mathcal{F}$. This construction also requires a high degree of regularity for the foliation.

and links in S^3. Moreover, these approaches are quite combinatorial and perhaps a bit miraculous in nature. Instead, we propose a general program to construct Reebless and taut foliations that remains closer to the argument of Oszváth–Szabó. Before that, let us state and discuss our main results.

0.1.2 Statement of results

0.1.2.1 A new construction of foliations

In [CF11], the authors construct a *locally integrable* plane field from a *normal* positive contact pair. This technical condition can always be achieved after some *non generic* deformation of the pair. In Section 1.1, we explain how to generalize this construction for *any* contact pair, removing the normality condition. This will be important for some of our applications, in particular for Theorem E below.

To a contact pair (ξ_-, ξ_+), we will associate a *canonical* vector field X satisfying $X \in \xi_- \cap \xi_+$, and vanishing exactly on the set $\Delta \coloneqq \{p \in M \mid \xi_-(p) = \xi_+(p)\}$. As a first step to obtain a foliation $\mathcal{F}$ from the contact pair, we construct a natural candidate for $T\mathcal{F}$:

Theorem A. *Let (ξ_-, ξ_+) be <u>any</u> positive contact pair on M and let $(\phi_t)_t$ be the flow of the associated vector field X.*

1. *The plane fields $\xi^t_\pm$ defined by*

$$\xi^t_\pm(\phi_t(x)) \coloneqq d\phi_t[\xi_\pm(x)] \tag{0.1.4}$$

converge uniformly to a continuous plane field η_u as t goes to $+\infty$.

2. *For a <u>generic</u> pair (ξ_-, ξ_+), η_u is locally integrable on M.*

Moreover, η_u depends continuously on (ξ_-, ξ_+).

Here, *generic* means that there exists an unspecified comeagre set of positive contact pairs for which 2 holds. We call η_u the **(weak-)unstable plane field** of the contact pair (ξ_-, ξ_+). It is only continuous, and it is not necessarily *uniquely* integrable, even for bicontact structures; see for instance [ET98, Example 2.2.9]. However, we adapt the strategy of [BI08] to show:

Theorem B. *Let (ξ_-, ξ_+) be a positive contact pair on M. If either*

- *ξ_- or ξ_+ is tight, or*
- *ξ_- and ξ_+ intersect along a generic link of saddle singularities (see Section 1.1.3),*

then the unstable plane field η_u can be $\mathcal{C}^0$-approximated by aspherical $\mathcal{C}^0$-foliations.

This immediately implies the Main Theorem from the beginning of the introduction. Let us briefly sketch the proof of this result. By Theorem A, we can *canonically* associate to a generic positive contact pair (ξ_-, ξ_+) a pair (X, η_u), where X is a smooth vector field and η_u is a continuous, locally integrable plane field tangent to X and invariant under the flow of X. We call such a pair a **polarized vector field**. By generically modifying (ξ_-, ξ_+), we can further simplify the dynamics of X and precisely understand the behavior of η_u near the singularities of X. We then adapt the methods of [BI08] to show that η_u is tangent to a *branching foliation*, namely, a collection of immersed, 'maximal' surfaces in M which cover M, and which may intersect but cannot topologically cross. The whole Chapter 2 of this dissertation is dedicated to this very technical and subtle construction. We finally appeal to a result in [BI08] to separate the leaves of the branching foliation and obtain a genuine foliation, at the expense of a $\mathcal{C}^0$-small perturbation of η_u. This solves Problem 5.3 from [CH20].

In the special case where (ξ_-, ξ_+) is a bicontact structure, X is a **projectively (or conformally) Anosov flow** (see [ET98; Mit95]) and η_u coincides with the weak-unstable bundle of X. Our theorem implies that, without even modifying (ξ_-, ξ_+), it is tangent to a branching foliation which can further be approximated by $\mathcal{C}^0$-foliations.

With Theorem B in hand, it is now relatively easy to show:

Theorem C. *If (ξ_-, ξ_+) is a strongly tight contact pair, then η_u can be $\mathcal{C}^0$-approximated by taut aspherical $\mathcal{C}^0$-foliations.*

Combined with the results of [CKR19] and the ($\mathcal{C}^0$ version of) Eliashberg–Thurston theorem, we obtain:

Corollary D. *If $M \neq S^1 \times S^2$, then M carries a taut (topological) foliation if and only if it carries a strongly tight contact pair.*

In Section 1.4.2, we show that a Liouville contact pair is automatically positive and supports a Liouville pair whose Reeb vector fields are both transverse to η_u. This implies:

Theorem E. *If the contact pair (ξ_-, ξ_+) supports a Liouville pair, then η_u can be $\mathcal{C}^0$-approximated by hypertaut $\mathcal{C}^0$-foliations. Moreover, $\xi_\pm$ are hypertight.*

Notice that supporting a Liouville pair is a $\mathcal{C}^1$ open condition for contact pairs. However, unlike strong tightness, it is *not* a $\mathcal{C}^0$ open condition. Besides, it is quite surprising to us that Liouville contact pairs are *automatically* hypertight; it is not true that a Liouville fillable contact structure is hypertight in general.

It turns out that the unstable plane field of a Liouville contact pair is quite rigid. For instance, it is $\mathcal{C}^1$ for Anosov Liouville pairs (see Chapter 3) and uniquely integrable in certain cases:

Theorem F (Corollary 1.4.15)**.** *Let (ξ_-, ξ_+) be a Liouville contact pair on M. If either*

- *$\xi_\pm$ are everywhere transverse, or*
- *$\xi_\pm$ intersect along a generic link of source singularities (see Section 1.1.3),*

then η_u is uniquely integrable. Moreover, it is tangent to a hypertaut $\mathcal{C}^0$-foliation.

Here, the Liouville condition imposes strong geometric and dynamical restrictions on η_u which force unique integrability. Perhaps more is true:

Question 0.1.7. *For a generic Liouville contact pair, is η_u uniquely integrable?*

Our techniques also provide a precise description of the *skeleton* of Liouville pairs on 3-manifolds:

Proposition G (Proposition 1.4.8)**.** *Let (α_-, α_+) be a Liouville pair on M. The skeleton of the associated Liouville structure λ on $V = \mathbb{R}_s \times M$ defined by* (0.1.3) *is of the form*

$$\mathfrak{skel}(V, \lambda) = \mathrm{graph}(\sigma) \coloneqq \big\{(\sigma(x), x) \mid x \in M\big\}$$

for some continuous function $\sigma : M \to \mathbb{R}$.

In particular, the skeleton is a continuous, separating hypersurface homeomorphic to M. Interestingly, it is not smooth in general.

0.1.2.2 Application: transverse surgeries on taut foliations

Contact structures are more malleable objects than foliations. As an application of our machinery, we obtain that the existence of taut foliations is preserved under "large slope" surgeries along transverse links. For simplicity, we only discuss the case of knots, but our results immediately generalize to links.

Let $K \subset M$ be a framed knot. If $s \in \mathbb{Q}$ is a rational number, we denote by $M_K(s)$ the manifold obtained from M by performing a Dehn surgery of slope s along K.

If $\mathcal{F}$ is a foliation on M transverse to K, performing a nontrivial Dehn surgery along K typically destroys $\mathcal{F}$, in the sense that $\mathcal{F}$ does not induce a foliation on $M_K(s)$. It rather induces a singular foliation, which is singular along the image of K in $M_K(s)$. One could instead perform a *turbulization*[8] (see [CC00, Example 3.3.11])

[8]Sometimes spelled *turbularization*.

along K by spinning the leaves of $\mathcal{F}$ along K hence creating a torus leaf around K, and inserting a Reeb component after surgery. Of course, this procedure *never* yields a taut foliation.

However, it is much easier to perform a surgery on an approximating contact pair and obtain a strongly tight contact pair on $M_K(s)$, provided that s is large enough. Combined with Theorem C, this implies:

Theorem H. *Let $\mathcal{F}$ be a taut aspherical $\mathcal{C}^0$-foliation on M, and $K \subset M$ be a framed knot transverse to $\mathcal{F}$. There exists $s_0 = s_0(K) \geq 0$ such that for every rational number $s \in \mathbb{Q}$ satisfying $|s| \geq s_0$, $M_K(s)$ carries a taut $\mathcal{C}^0$-foliation $\mathcal{F}'$. Moreover, the image of K in $M_K(s)$ is transverse to $\mathcal{F}'$.*

See Section 1.3.2 for a detailed proof. As a corollary, we obtain a generalization of the main result of [LR14], with a somewhat easier and more natural proof.

Let $K \subset S^3$ be a nontrivial knot. We denote by $\mathcal{S}_K \subset \mathbb{Q}$ the set of rational slopes $s \in \mathbb{Q}$ such that there exists a taut $\mathcal{C}^0$-foliation on $S^3 \setminus K$ which intersects the boundary of tubular neighborhood of K transversally along a nonsingular foliation of slope s. By a celebrated theorem of Gabai [Gab87], $\mathcal{S}_K$ contains 0. By [LR14, Theorem 1.1], $\mathcal{S}_K$ contains a neighborhood of 0. We actually get:

Corollary I. *$\mathcal{S}_K$ is an open subset of $\mathbb{Q}$.*

We obtain this corollary as a rather brutal application of Theorem H for which we have little control on the number s_0 in such generality. Instead, it would be interesting to construct a concrete contact pair adapted to K and carefully analyze the slopes of the characteristic foliations of the contact structures on the boundary of a (potentially large) solid torus containing K. As a starting point, one could consider the case *fibered knots*. There is already an extensive literature on taut foliations in fibered knot complements, see [Rob01a; Rob01b; Kri20; Kri23]. Moreover, in view of the L-space

conjecture and Heegaard Floer homology computations for Dehn surgeries on knots in S^3 (see [Kro+07; RR17]), one expects the following:

Conjecture. *If K is knot in S^3 with Seifert genus $g > 0$, then either $\mathcal{S}_K = \mathbb{Q}$, $\mathcal{S}_K = (-\infty, 2g-1) \cap \mathbb{Q}$, or $\mathcal{S}_K = (-2g+1, +\infty) \cap \mathbb{Q}$.*

However, to our knowledge, it is not known if $\mathcal{S}_K \subset \mathbb{Q}$ is an interval in general.

0.1.3 Discussion and further directions

0.1.3.1 Towards a contact L-space conjecture

While we are able to construct taut foliations from strongly tight contact pairs, our methods do not easily extend to tight contact pairs. However, we expect the following to be true:

Conjecture 1. *If $M \neq S^1 \times S^2$, then M carries a Reebless $\mathcal{C}^0$-foliation if and only if it carries a tight positive contact pair.*

We give some evidence for this conjecture in Section 1.3.1. The main difficulty comes from the lack of regularity of the unstable plane field η_u, and the fact that Reeblessness is *not* a natural property for plane fields (see the *phantom Reeb components* from [CKR19]). Notice that this conjecture would immediately imply that on an *atoroidal* closed 3-manifold, the existence of a tight positive contact pair is equivalent to the existence of a taut foliation.

A contact pair can always be deformed into a positive one when one of its contact structures is overtwisted, see Proposition 1.1.4 below. However, there is little hope to construct geometrically interesting foliations from such a pair. Upgrading tight contact pairs to positive ones seems much more difficult. At the very least, we should further assume that the contact structures are homotopic as oriented plane fields, but this might not be enough. Indeed, Lin recently remarked that the existence

of a taut foliation on a rational homology sphere imposes more constraints on the (monopole) Floer homology of the manifold, see [Lin23].[9] In particular, HM_* has a *direct* $\mathbb{F}$*-summand*, where $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$. Moreover, the *contact invariants* of the contact approximations of the foliation have to pair to 1, for the natural perfect pairing

$$\langle \cdot, \cdot \rangle : HM_*(M) \otimes HM_*(-M) \longrightarrow \mathbb{F}$$

induced by the Poincaré duality isomorphism $HM^*(M) \cong HM_*(-M)$. This motivates the following

Definition 0.1.8. *A contact pair* (ξ_-, ξ_+) *on* M *is* ***algebraically tight*** *if the contact invariants* $c(\xi_\pm) \in HM(\mp M)$ *satisfy*

$$\big\langle c(\xi_-), c(\xi_+) \big\rangle = 1.$$

Notice that if (ξ_-, ξ_+) is algebraically tight, then ξ_- and ξ_+ are tight and homotopic as oriented plane fields. Indeed, the contact class of an overtwisted contact structure vanishes, and the grading of the contact class corresponds to the homotopy class of the contact structure (as an oriented plane field). We propose:

Conjecture 2. *Let* (ξ_-, ξ_+) *be an algebraically tight contact pair on* M*. Then* (ξ_-, ξ_+) *is homotopic through contact pairs to a positive one.*

Here, we think of the contact classes as potential *obstructions* to deforming a tight contact pair into a positive one. One possible approach to this conjecture would be to consider the set $\Delta_- \subset M$ where ξ_- and ξ_+ coincides with *opposite* orientations. It is generically an embedded link which is null-homologous if ξ_- and ξ_+ are homotopic. Perhaps one could study the link Floer or sutured Floer homology of the pair (M, Δ_-) and use the algebraic tightness hypothesis to precisely understand the topology of Δ_-.

[9] It is known that (the various flavors of) monopole Floer homology and Heegaard Floer homology are isomorphic, so we simply refer to these invariants as 'Floer homology'.

Additionally, there could be 'elementary moves' that would inductively simplify Δ_-. Some obstruction to performing these moves could be determined by Floer-theoretic invariants of the contact pair.

It remains to understand how to construct suitable tight contact structures on rational homology spheres. This is of course a very delicate problem, but perhaps more tractable than constructing taut foliations.

Question 0.1.9 (Contact realization)**.** *Assume that M is an irreducible rational homology sphere.*

- *Which nonzero elements in the Floer homology of M can be realized as the contact class of a (necessarily tight) contact structure?*
- *If the Floer homology of M has a direct $\mathbb{F}$-summand, does M carry an algebraically tight contact pair?*

Assuming the previous two conjectures, a positive answer to the second item of this question would imply:

Conjecture 3. *If M is an irreducible rational homology sphere such that $HM_*(M)$ has a direct $\mathbb{F}$-summand, then M carries a Reebless $\mathcal{C}^0$-foliation.*

0.1.3.2 Variations and refinements

We conclude this introduction with some potential generalizations of our main result.

Uniqueness of the foliation. While the polarized vector field (X, η_u) associated to a positive contact pair (ξ_-, ξ_+) is *canonical*, the (branching) foliation tangent to η_u that we construct is not. Indeed, it depends on a certain number of choices that are carefully described in Chapter 2. Understanding how the foliation depends on these choices seems a completely impractical problem. In particular, we have a very limited

understanding of the topology of the leaves of the foliation. One could think of our approach as sacrificing some knowledge about the leaves in order to gain extra flexibility that ultimately allows us to integrate η_u. This shall be contrasted with *branched surfaces theory*, a well-established approach to construct foliations and laminations in dimension three. Besides, a given continuous plane field might be tangent to many distinct foliations which have drastically different holonomy properties. However, η_u 'remembers' many of the geometric properties of the contact pair.

Parametric version. To simplify the construction of a (branching) foliation tangent to η_u, we impose some generic conditions on (ξ_-, ξ_+) that make the dynamics of X more tractable. We believe that the overall strategy of Chapter 2 is quite robust and some of these conditions may be weakened. In particular, we ask:

Question 0.1.10. *Let (ξ_-^t, ξ_+^t), $t \in [0,1]$, be smooth path of positive contact pairs on M, and assume that one of $\xi_\pm^0$ is tight. Does there exist a path of $\mathcal{C}^0$-foliations $(\mathcal{F}_t)$, $t \in [0,1]$, such that $T\mathcal{F}_t$ is uniformly $\mathcal{C}^0$-close to η_u^t?*

It is not absolutely clear to us which topology to consider on the space of $\mathcal{C}^0$-foliations, since those are not determined by their tangent plane field. To answer this question, one would have to precisely understand the type of singularities that the associated vector field X can develop in a generic 1-parameter family, and adapt the various choices made in the non-parametric construction to this setting.

Version with boundary. We now assume that M is compact and has a nonempty boundary. Some of our constructions can be generalized to this setting. In particular, if (ξ_-, ξ_+) is a positive contact pair such that the associated vector field X is positively transverse to ∂M, then the construction of η_u goes through and yields a locally integrable plane field provided that the contact pair is generic. Moreover, the methods of Chapter 2 should extend to this setting so that Theorem B generalizes to manifolds

with boundary. It would be interesting to first prescribe a 1-dimensional (branching) foliation tangent to $\eta_u \cap \partial M$ along ∂M and construct a (branching) foliation on M which restricts to the chosen one along ∂M. This way, one could potentially control the boundary slope of the foliation.

Chapter 1

From contact pairs to foliations

This first Chapter contains the proofs of the main theorems stated in the introduction of Part I, assuming a technical result whose proof will be deferred to Chapter 2.

1.1 Contact pairs revisited

In this section, we extend the main construction of [CF11] and we prove Theorem A. It will be important to avoid using the normal form for contact pairs from [CF11], as the deformation to a normal contact pair is *not generic*.

1.1.1 The setup

We first recall the following

Definition 1.1.1. *A **contact pair** on M is a pair of cooriented contact structures (ξ_-, ξ_+), negative and positive, respectively. A **bicontact structure** is a contact pair (ξ_-, ξ_+) such that $\xi_\pm$ are everywhere transverse.*

Let (ξ_-, ξ_+) be a contact pair. We write

- Δ_+ for the set of $p \in M$ such that $\xi_-(p) = \xi_+(p)$ as oriented planes,

- Δ_- for the set of $p \in M$ such that $\xi_-(p) = -\xi_+(p)$ as oriented planes,

and we let $\Delta \coloneqq \Delta_- \sqcup \Delta_+$. The contact pair (ξ_-, ξ_+) is **positive** (resp. **negative**) if $\Delta_- = \varnothing$ (resp. $\Delta_+ = \varnothing$). Note that Δ has empty interior, since $\xi_\pm$ define opposite orientations on M.

Remark 1.1.2. *A contact pair (ξ_-, ξ_+) is positive if and only if there exists a (smooth) vector field v which is positively transverse to both ξ_- and ξ_+. In that case, ξ_- and ξ_+ are homotopic as oriented plane fields, and their first Chern classes $c_1(\xi_-)$ and $c_1(\xi_+)$ agree. For a generic contact pair (ξ_-, ξ_+), Δ is a smoothly embedded link. The Poincaré duals to the homology classes represented by $\Delta_\pm$ (with suitable orientations) are given by*

$$\mathrm{PD}[\Delta_\pm] = c_1(\xi_+) \pm c_1(\xi_-) \in H^2(M; \mathbb{Z}).$$

In particular, those are even *classes which only depend on the homotopy classes of $\xi_\pm$. If (ξ_-, ξ_+) can be deformed through contact pairs into a positive contact pair, then Δ_- is null-homologous in M.*

Assume now that Δ_- is null-homologous in M. Then ξ_- and ξ_+ are homotopic as plane fields if and only if the linking number $\mathrm{lk}(\Delta_-, \Delta_+)$ *of Δ_+ with Δ_- vanishes. This number is defined as the algebraic intersection number of Δ_+ (suitably oriented) with any oriented embedded surface $\Sigma \subset M$ with $\partial\Sigma = \Delta_-$. Let us choose an arbitrary Riemannian metric on g as well as a trivialization τ_- of TM in which ξ_- is constant. Identifying plane fields with their positive unit normal vectors, ξ_- becomes the constant map $M \to S^2$ equal to the North pole p_+ of S^2, and ξ_+ becomes a map $\nu_+ : M \to S^2$. The South pole p_- of S^2 corresponds to $-\xi_-$. We then have*

$$\Delta_\pm = \nu_+^{-1}(p_\pm).$$

Since $c_1(\xi_+) = c_1(\xi_-)$, ξ_- and ξ_+ are homotopic as oriented plane fields over the 2-skeleton of M. The obstruction to extending the homotopy over 3-cells is measured by

a Hopf invariant *for ν_+ with value in $\mathbb{Z}$. By an elementary version of the Pontryagin–Thom construction, this number equals the linking number* $\mathrm{lk}(\Delta_-, \Delta_+)$.

To summarize, ξ_- and ξ_+ are homotopic as oriented plane fields if and only if $[\Delta_-] = 0$ *and* $\mathrm{lk}(\Delta_-, \Delta_+) = 0$.

As discussed in the introduction, it is a central question to understand when a contact pair can be deformed to a positive one:

Question 1.1.3. *Let (ξ_-, ξ_+) be a contact pair on M such that ξ_- and ξ_+ are homotopic as oriented plane fields. Can (ξ_-, ξ_+) be deformed into a positive contact pair? Are there further Floer-theoretic obstructions?*

The following proposition, suggested to us by Vincent Colin, gives an affirmative answer to this question when one of $\xi_\pm$ is *overtwisted*:

Proposition 1.1.4. *Let ξ_+ be a positive contact structure on M.*

1. *There exists an overtwisted negative contact structure ξ_- on M such that (ξ_-, ξ_+) is a positive contact pair.*

2. *Let ξ_- be an overtwisted negative contact structure on M which is homotopic to ξ_+ as an oriented plane field. There exists path of negative contact structures $(\xi_-^\tau)_{0\leq\tau\leq 1}$ such that $\xi_-^0 = \xi_-$, and (ξ_-^1, ξ_+) is a positive contact pair.*

Proof. By Etnyre [Etn07], there exists a smooth foliation $\mathcal{F}$ on M such that ξ_+ is a deformation of $T\mathcal{F} = \eta$, in the sense that there exists a (smooth) 1-parameter family of smooth oriented plane fields $(\xi_+^\tau)_{0\leq\tau\leq 1}$ such that $\xi_+^0 = \eta$, $\xi_+^1 = \xi_+$, and ξ_+^τ is a (positive) contact structure for $\tau > 0$. Moreover, the foliation $\mathcal{F}$ has Reeb components by construction, and is approximated by *overtwisted* negative contact structures.[1]

[1]This essentially follows from an observation in [Etn07, Section 4]: an approximation of $\mathcal{F}$ by a negative contact structure can equivalently be seen as an approximation by a positive contact structure after reversing the orientation of M; in that case, the leaves of $\mathcal{F}$ coming from the pages of the open book spiral towards the Reeb components in a *counterclockwise* manner, which implies overtwistedness.

Therefore, there exist an overtwisted negative contact structure ξ'_- and $0 < \tau \leq 1$ such that (ξ'_-, ξ^τ_+) is a positive contact pair. Applying Gray's stability yields 1. If ξ_- is a negative contact structure as in 2, then ξ'_- and ξ_- are two overtwisted negative contact structures which are homotopic as plane fields. Eliashberg's classification of overtwisted contact structures [Eli89] implies that they are contact isotopic, and 2 follows easily. □

Of course, Question 1.1.3 is much more interesting (and difficult!) in the the case where both $\xi_\pm$ tight.

We now investigate some basic quantities associated with contact pairs. Let dvol be a volume form on M and $\alpha_\pm$ be contact forms for $\xi_\pm$ satisfying

$$\alpha_+ \wedge d\alpha_+ = -\alpha_- \wedge d\alpha_- = \text{dvol}.$$

There exists a unique (smooth) vector field X on M such that

$$\alpha_\pm(X) = 0, \qquad \iota_X \text{dvol} = \alpha_- \wedge \alpha_+.$$

Away from Δ, X is non-vanishing and spans the transverse intersection $\xi_- \pitchfork \xi_+$ with the correct orientation, and X vanishes along Δ. Note that X does not depend on the choice of dvol, so it only depends on the contact structures $\xi_\pm = \ker \alpha_\pm$. There exist smooth functions $f_0, g_0 : M \to \mathbb{R}$ defined by

$$d(\alpha_- \wedge \alpha_+) = f_0 \,\text{dvol},$$
$$\langle \alpha_-, \alpha_+ \rangle = g_0 \,\text{dvol},$$

where

$$\langle \alpha_-, \alpha_+ \rangle \coloneqq \alpha_- \wedge d\alpha_+ + \alpha_+ \wedge d\alpha_-.$$

The function g_0 does not depend on the choice of dvol, but the function f_0 does, since it is nothing more than the divergence of X for dvol:

$$\mathcal{L}_X \mathrm{dvol} = f_0 \, \mathrm{dvol}.$$

If dvol' is another volume form on M, there exists a unique function $\rho : M \to \mathbb{R}$ such that

$$\mathrm{dvol}' = e^{\rho} \, \mathrm{dvol}$$

and one easily checks that

$$f_0' = X \cdot \rho + f_0,$$

where f_0' is the divergence of X for dvol'. However, f_0 does not depend on dvol along Δ. Moreover:

Lemma 1.1.5. *Along Δ_+, $f_0 \geq 2$ and along Δ_-, $f_0 \leq -2$.*

Proof. Along Δ_+, there exists $u : \Delta_+ \to \mathbb{R}_{>0}$ such that

$$\alpha_+(p) = -u_+(p)\alpha_-(p).$$

Hence, if $p \in \Delta_+$,

$$f_0(p) = u_+(p) + \frac{1}{u_+(p)} \geq 2.$$

The case of Δ_- is similar. □

In particular, the divergence of X (for any volume form or metric) along Δ_+ (resp.

Δ_-) is positive (resp. negative). We also define smooth functions $g_\pm : M \to \mathbb{R}$ by

$$\alpha_- \wedge d\alpha_+ = g_+ \,\mathrm{dvol},$$
$$\alpha_+ \wedge d\alpha_- = g_- \,\mathrm{dvol},$$

so that $g_0 = g_- + g_+$ and $f_0 = g_- - g_+$. One easily checks that

$$\mathcal{L}_X \alpha_- = g_- \,\alpha_- + \alpha_+, \tag{1.1.1}$$
$$\mathcal{L}_X \alpha_+ = \alpha_- - g_+ \,\alpha_+. \tag{1.1.2}$$

Indeed, these formulae hold away from Δ since $\mathcal{L}_X \alpha_\pm$ vanish along X and can be written as linear combinations of $\alpha_\pm$. The coefficients can be determined by wedging $\mathcal{L}_X \alpha_\pm$ with $\alpha_\pm$. Since Δ has empty interior and all of the quantities involved are smooth, these formulae hold along Δ as well.

Remark 1.1.6. *Let $p \in \Delta_+$. Since X is tangent to ξ_+ (and ξ_-), the linearization d_pX of X at p has its image contained in $\xi_+(p)$. Indeed:*

$$\alpha_+ \circ d_pX = \mathcal{L}_X \alpha_+(p) = d_p(\alpha_+(X)) + d\alpha_+\big(X(p), \cdot\,\big) = 0.$$

Let $L_pX : \xi_+(p) \to \xi_+(p)$ denote the restriction of d_pX to $\xi_+(p)$. Then

$$\mathrm{tr}(L_pX) = \mathrm{tr}(d_pX) = \mathrm{div}_p X = f_0(p) > 2$$

by Lemma 1.1.5, hence $\mathrm{rank}(L_pX) \in \{1, 2\}$. *If* $\mathrm{rank}(L_pX) = 2$, *then the same holds for any nearby $p' \in \Delta_+$ and the constant rank theorem implies that Δ_+ is a 1-dimensional embedded submanifold of M near p. Moreover, Δ_+ is transverse to ξ_+ near p. If* $\mathrm{rank}(L_pX) = 1$, *the situation is more complicated. Certainly, Δ_+ is 'at most 2-dimensional' near p. It might be a 2-dimensional embedded submanifold near p: the*

contact pair defined by the contact forms

$$\alpha_\pm = \pm dz - y\,dx$$

on $\mathbb{R}^3$ *with standard coordinates* (x, y, z) *has* $\Delta_+ = \{y = 0\}$. *However, if* $\operatorname{rank}\big(d_pX\big) = 2$, *then* Δ_+ *is* 1*-dimensional near* p *and* tangent *to* $\xi_+(p)$ *at* p.

Besides, for a generic *contact pair* (ξ_-, ξ_+), $\operatorname{rank}\big(d_pX\big) = 2$ *at every* $p \in \Delta_+$, *and* $\operatorname{rank}(L_pX) = 2$ *at all but finitely many points in* Δ_+. *It follows that* Δ_+ *is smoothly embedded link in* M *that is transverse to* ξ_+ *away from finitely many isolated points. The same properties hold for* Δ_-.

1.1.2 Invariant plane fields

While Δ is a smoothly embedded link in M for a generic contact pair (ξ_-, ξ_+), we won't need to make such an assumption in the following proposition which generalizes [CF11, Proposition 2.4]. We fix an auxiliary volume form dvol on M which determines preferred of contact form $\alpha_\pm$ for $\xi_\pm$ as in the previous section.

Proposition 1.1.7. *Let* (ξ_-, ξ_+) *be* <u>any</u> *contact pair on* M.

- *There exists a unique continuous plane field* η_u *on* $M \setminus \Delta_-$ *satisfying*

  1. X *is tangent to* η_u *and* η_u *is invariant under the flow of* X,
  2. *Away from* Δ, $\eta_u \setminus \{0\}$ *lies in the cone*

  $$C_+ \coloneqq \big\{v \in T_pM \mid \alpha_\pm(v) > 0\big\} \cup \big\{v \in T_pM \mid \alpha_\pm(v) < 0\big\}$$

  and along Δ_+, η_u *coincides with* $\xi_\pm$.

Moreover, there exists a continuous 1*-form* α_s *on* M *such that* $\ker \alpha_s = \eta_u$ *on*

$M \setminus \Delta_-$, $\alpha_s = 0$ on Δ_-, α_s is differentiable along X and

$$\mathcal{L}_X \alpha_s = r_s \, \alpha_s,$$

where $r_s : M \to \mathbb{R}$ is continuous function defined everywhere on M. Moreover, there exists a continuous 1-form α_u on M such that $\ker \alpha_u = \eta_s$ on $M \setminus \Delta_+$, $\alpha_u = 0$ on Δ_+, α_u is differentiable along X and

$$\mathcal{L}_X \alpha_u = r_u \, \alpha_u,$$

where $r_u : M \to \mathbb{R}$ is a continuous function defined everywhere on M.

- *Similarly, there exists a unique continuous plane field η_s on $M \setminus \Delta_+$ satisfying*

  1. *X is tangent to η_s and η_s is invariant under the flow of X,*
  2. *Away from Δ, $\eta_s \setminus \{0\}$ lies in the cone*

$$C_- := \left\{ v \in T_pM \mid \pm \alpha_\pm(v) > 0 \right\} \cup \left\{ v \in T_pM \mid \pm \alpha_\pm(v) < 0 \right\}$$

  and along Δ_-, η_s coincides with $\xi_\pm$.

Moreover, there exists a continuous 1-form α_s on M such that $\ker \alpha_s = \eta_u$ on $M \setminus \Delta_-$, $\alpha_s = 0$ on Δ_-, α_s is differentiable along X and

$$\mathcal{L}_X \alpha_s = r_s \, \alpha_s,$$

where $r_s : M \to \mathbb{R}$ is continuous function defined everywhere on M.

The 1-forms α_u and α_s can be chosen so that the assignment

$$(\xi_-, \xi_+) \longmapsto (\alpha_s, \alpha_u)$$

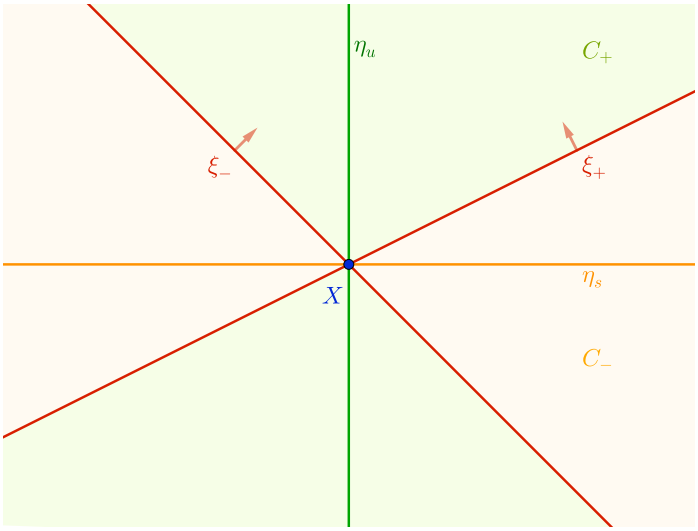


Figure 1.1: Away from Δ: the contact structures $\xi_\pm$ with their respective coorientations, the cones $C_\pm$, and the plane fields η_u and η_s. The vector field X points toward the reader.

defines a continuous map from the space of (smooth) positive contact pairs on M endowed with the $\mathcal{C}^\infty$ topology to the space of pairs of continuous 1-forms on M, endowed with the $\mathcal{C}^0$ topology. We can further assume that

$$\alpha_s \wedge \alpha_u = \alpha_- \wedge \alpha_+, \qquad r_s < r_u.$$

We call η_u (resp. η_s) the **(weak-)unstable** (resp. **(weak-)stable**) distribution of X. We call r_u (resp. r_s) the **expansion rate** in the weak-unstable (resp. weak-stable) direction of X (with respect to the specific choices of α_u and α_s). The plane fields η_u and η_s are both locally (but not necessarily uniquely) integrable on $M \setminus \Delta$. See Figure 1.1 for a sketch of the relative positions of $\xi_\pm$, η_u and η_s away from Δ.

Remark 1.1.8. *If $\Delta = \varnothing$, i.e., if (ξ_-, ξ_+) is a bicontact structure, we recover the result of Eliashberg-Thurston [ET98] and Mitsumatsu [Mit95] which states that the flow Φ of X is a conformally/projectively Anosov. In that case, as their names suggest, η_u and η_s are the weak-unstable and weak-stable distributions of Φ, which are locally integrable but not necessarily uniquely integrable, see [ET98, Example 2.2.9]. In the more general*

setting where $\Delta \neq \varnothing$, we could call the flow Φ of X a ***singular projectively Anosov flow****.*

Proof of Proposition 1.1.7. We look for a 1-forms α_u and α_s of the form

$$\alpha_u = e^{-\sigma_u}\alpha_- + e^{\sigma_u}\alpha_+ \tag{1.1.3}$$

$$\alpha_s = e^{-\sigma_s}\alpha_- - e^{\sigma_s}\alpha_+ \tag{1.1.4}$$

where $\sigma_u, \sigma_s : M \to \mathbb{R}$ are continuous functions which are continuously differentiable along X. These 1-forms have to satisfy

$$\alpha_s \wedge \mathcal{L}_X \alpha_s = \alpha_u \wedge \mathcal{L}_X \alpha_u = 0,$$

which after some elementary computations is equivalent to

$$X \cdot \sigma_u = \sinh(2\sigma_u) + \frac{1}{2} g_0, \tag{1.1.5}$$

$$X \cdot \sigma_s = -\sinh(2\sigma_s) + \frac{1}{2} g_0. \tag{1.1.6}$$

For $p \in M$, we consider the ODE

$$(E_p): \quad \dot{y}(t) = \sinh(2y(t)) + \frac{1}{2} g_0\big(\phi_X^t(p)\big),$$

where ϕ_X^t denotes the flow of X. After embedding M in some $\mathbb{R}^N$ by the Whitney embedding theorem, we obtain a system of ODEs of the form of the ones studied in Appendix 1.A, with

$$F(p, y, t) \coloneqq \sinh(2y) + \frac{1}{2} g_0(\phi_X^t(p)).$$

This function F is easily seen to satisfy conditions (C1) and (C2) from Appendix 1.A, so Lemma 1.A.1 applies and we define $\sigma_u(p)$ as the unique initial value of (E_p) such that the corresponding maximal solution is defined on $\mathbb{R}$ and is bounded. We readily

get that $t \mapsto \sigma_u(\phi_X^t(p))$ is differentiable and is the unique bounded solution to (E_p) with initial value $\sigma_u(p)$, so σ_u is differentiable along X and solves (1.1.5). Note that on Δ, σ_u is given by

$$\sigma_u(p) = -\frac{1}{2}\sinh^{-1}\left(\frac{g_0(p)}{2}\right).$$

A similar argument shows that the assignment $(\xi_-, \xi_+) \mapsto \sigma_u \in \mathcal{C}^0(M)$ is continuous. Indeed, it is enough to show that if (ξ_-^n, ξ_+^n), $n \geq 0$, is a sequence of positive contact pairs on M converging to (ξ_-, ξ_+), then the corresponding sequence of functions σ_u^n, $n \geq 0$, converges to σ_u uniformly. The corresponding sequence of functions g_0^n, $n \geq 0$, converges to g_0 and the corresponding sequence of vector fields X_n, $n \geq 0$, converges to X, both in the $\mathcal{C}^\infty$ topology. Therefore, we obtain a sequence of functions $F_n : M \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined by

$$F_n(p, y, t) \coloneqq \sinh(2y) + \frac{1}{2} g_0^n\big(\phi_{X_n}^t(p)\big)$$

which converges to F in the $\mathcal{C}^\infty_{\text{loc}}$ topology. One easily checks that the hypotheses of Lemma 1.A.2 are satisfied, hence σ_u^n converges to σ_u uniformly. As a result, α_u depends continuously on (ξ_-, ξ_+).

We proceed similarly for σ_s by replacing X with $-X$.

Away from Δ_+, α_u does not vanish and defines a continuous plane field η_s. Moreover, α_u vanishes along Δ_+. Indeed, there exists a smooth function $u_+ : \Delta_+ \to \mathbb{R}_{>0}$ such that for $p \in \Delta_-$,

$$\alpha_+(p) = -u_+(p)\, \alpha_-(p).$$

This implies that if $p \in \Delta_+$,

$$g_0(p) = u_+(p) - \frac{1}{u_+(p)},$$

and since

$$g_0(p) = e^{-2\sigma_u(p)} - e^{2\sigma_u(p)},$$

we obtain that

$$e^{\sigma_u(p)} = \frac{1}{\sqrt{u_+(p)}},$$

hence

$$\alpha_u(p) = \sqrt{u_+(p)}\,\alpha_-(p) - \frac{1}{\sqrt{u_+(p)}} u_+(p)\,\alpha_-(p) = 0.$$

Similarly, α_s does not vanish away from Δ_- and defines a continuous plane field η_u on $M \setminus \Delta_-$, while it vanishes on Δ_-. Analogous computations show that $\eta_u = \xi_\pm$ along Δ_+ and $\eta_s = \xi_\pm$ along Δ_-.

The expansion rates r_u and r_s can now be computed, using (1.1.1), (1.1.2) and (1.1.5), (1.1.6):

$$r_u = \cosh(2\sigma_u) + \frac{1}{2} f_0, \tag{1.1.7}$$

$$r_s = -\cosh(2\sigma_s) + \frac{1}{2} f_0 \tag{1.1.8}$$

Therefore, r_u and r_s are continuous on M, and

$$r_u - r_s = \cosh(2\sigma_u) + \cosh(2\sigma_s) \geq 2.$$

Finally,

$$\alpha_s \wedge \alpha_u = 2\cosh(\sigma_u - \sigma_s)\,\alpha_- \wedge \alpha_+,$$

but we can replace α_u and α_s with

$$\alpha'_u := \frac{1}{\sqrt{2\cosh(\sigma_u - \sigma_s)}}\,\alpha_u,$$

$$\alpha'_s := \frac{1}{\sqrt{2\cosh(\sigma_u - \sigma_s)}}\,\alpha_s,$$

so that $\alpha'_s \wedge \alpha'_u = \alpha_- \wedge \alpha_+$ and $r'_u - r'_s = r_u - r_s \geq 2$. □

Remark 1.1.9. *It can be shown that $r_u = 0$ along Δ_- and $r_s = 0$ along Δ_+. This is independent of the choice of α_u and α_s.*

1.1.3 Local integrability

To ensure local integrability of η_u along Δ_+ and of η_s along Δ_-, we will consider contact pairs satisfying some natural and *generic* conditions.

Definition 1.1.10. *The contact pair (ξ_-, ξ_+) is **regular** if the following conditions hold.*

(R1) The singular set $\Delta \subset M$ is a smoothly embedded link,

(R2) There exist finitely many points q_i, $1 \leq i \leq n$, such that Δ has a quadratic contact with $\xi_\pm$ at q_i, and away from these points, Δ is transverse to $\xi_\pm$,

(R3) At every point of Δ, the linearization of X has rank 2.

It follows from standard transversality results that regular contact pairs form a comeagre set in the space of contact pairs. In particular, it is a dense subset by Baire's theorem. Writing $Q \coloneqq \{q_i\}_{1 \leq i \leq n}$, it follows that points in $\Delta \setminus Q$ are of two types, depending on the sign of the determinant of the differential of X in the normal direction:

- Either this determinant is positive, and the singularity is a *source* (on Δ_+) or a *sink* (on Δ_-), or
- This determinant is negative, and the singularity is a *saddle*.

Hence, we can write

$$\Delta \setminus Q = \Delta_{\mathrm{so}} \sqcup \Delta_{\mathrm{si}} \sqcup \Delta_{\mathrm{sa}},$$

where Δ_{so} (resp. Δ_{si}, Δ_{sa}) is the disjoint union of open intervals in Δ corresponding to source (resp. sink, saddle) points.

Note at every point $p \in \Delta_+$, the linearization of X is tangent to $\eta_u(p) = \xi_+(p)$ by Remark 1.1.6. Similarly, at every point $p \in \Delta_-$, the linearization of X is tangent to $\eta_s(p)$.

The next Proposition generalizes [CF11, Proposition 2.5] by removing the normality assumption. We also fill some gaps and clarify some subtleties.

Proposition 1.1.11. *Let (ξ_-, ξ_+) be a regular contact pair. Then η_u is locally integrable on $M \setminus \Delta_-$, and η_s is locally integrable on $M \setminus \Delta_+$.*

Proof. We show that η_u is locally integrable at every point $p \in M \setminus \Delta_-$. We distinguish four cases.

Case 0: $p \in M \setminus \Delta$**.** Then $X(p) \neq 0$ and local integrability of η_u at p is standard, see [CF11, Proposition 2.5].

Case 1: $p \in \Delta_{\mathrm{so}}$**.** By the Unstable Manifold Theorem, there exists a germ of smooth surface $S^u = S^u(p)$ passing through p such that X is tangent to S^u and $T_p S^u = \eta_u(p) = \xi_\pm(p)$. Moreover, there exist smooth coordinates (x, y, z) near p in which the linearization of X at 0 writes

$$X_1 = ax\partial_x + by\partial_y,$$

where $a, b > 0$. Applying [Tak74, Theorem (1.13)], and noting that X has a line of singularities passing through 0, we obtain that X is *topologically* equivalent to the vector field

$$X' = x\partial_x + y\partial_y.$$

In these last coordinates, $S^u \subset \{z = 0\}$.

We claim that η_u is tangent to S^u in a neighborhood of p. Indeed, if $\eta_u(p')$ and $T_{p'}S^u$ disagree at a point $p' \in S^u$ near p, we consider a small disk D passing through p' and transverse to X. The intersection of η_u with this disk yields a continuous vector field on D, for which we can find a small flow line γ passing through p'. Because of the above (topological) normal form for X near p, there exists a point $p'' \in \gamma$ which is contained in the unstable manifold $\widetilde{S}^u$ of a point $\widetilde{p} \in \Delta$ distinct from p. Under the backward flow of X, p' converges to p and p'' converges to $\widetilde{p}$. This implies that the curve obtained by flowing γ along X for a sufficiently negative time cannot be everywhere tangent to η_u, which is a contradiction.

As a result, $TS^u = \eta_u$ along S^u, and S^u is a germ of integral surface of η_u around p. This also shows that η_u is *uniquely* integrable at p, and the characteristic foliations of $\xi_\pm$ coincide on S^u and are directed by X. The previous paragraph also implies the following property: if $S \subset M \setminus \Delta$ is an connected immersed surface tangent to η_u and if the backward flow line of X passing through some $p' \in S$ converges to $p \in \Delta_{\text{so}}$, then S is contained in the unstable manifold of X at p, i.e., the saturation of $S^u(p)$ by the flow of X. This property will be crucial in Case 3 below.

Case 2: $p \in \Delta_{\text{sa}}$. The strategy of [CF11, Proof of Proposition 2.5] can be adapted to this more general setting.[2] By the Invariant Manifolds Theorem, there exist smooth coordinates (x, y, z) near p such that

- The linearization X_1 of X at 0 is

$$X_1 = ax\partial_x - by\partial_y,$$

  where $a, b > 0$.[3]

[2] In [CF11], the authors use that at a saddle point, X is conjugated to its linearization; however, this conjugation is only *continuous* in general!

[3] Since X has positive divergence along Δ_+, we even have $b < a$. Unlike in [CF11], this won't be relevant in our proof.

- Δ is contained in $\{x = y = 0\}$,
- The weak unstable manifold of X is contained in $\{y = 0\}$ and the strong unstable manifold of X at 0 is contained in $\{y = z = 0\}$,
- The weak stable manifold of X is contained in $\{x = 0\}$ and the strong stable manifold of X at 0 is contained in $\{x = z = 0\}$.

It follows that in these coordinates, $\eta_u(0) = \text{span}\{\partial_x, \partial_y\}$. We can also find small constants $\epsilon, \delta_x, \delta_y > 0$ such that the following properties hold. Both η_u and X are transverse to the disks

$$D_x^{\pm} \coloneqq \left\{x = \pm\epsilon,\ y^2 + z^2 \leq \delta_x^2\right\}$$
$$D_y^{\pm} \coloneqq \left\{y = \pm\epsilon,\ x^2 + z^2 \leq \delta_y^2\right\}$$

and all the flow lines of X starting on $D_y^{\pm}$ either intersect $D_x^{\pm}$ or converge to Δ. By flowing $D_y^{\pm}$ along X, we obtain four smooth maps

$$f^{i,j} : D_y^i \cap \{jx > 0\} \longrightarrow D_x^j,$$

where $i, j \in \{-, +\}$. Moreover, $df^{i,j}$ send the intersection of D_y^i with η_u to the intersection of D_x^j with η_u, and if we write

$$p_x^{\pm} \coloneqq (\pm\epsilon, 0, 0), \qquad p_y^{\pm} \coloneqq (0, \pm\epsilon, 0),$$

then

$$\lim_{p \to p_y^i} f^{i,j}(p) = p_x^j$$

for every $i, j \in \{-, +\}$. The latter can be seen by *topologically* conjugating X with its linearization at 0, see [Tak74, Theorem (1.13)]. See Figure 1.2 for a lower dimensional sketch.

Since η_u intersects $D_y^{\pm}$ along a continuous vector field, we can find small integral curves $\gamma_y^{\pm} : (-\delta, \delta) \to D_y^{\pm}$ tangent to η_u and with $\gamma_y^{\pm}(0) = p_y^{\pm}$, where $0 < \delta < \delta_y$. Flowing these curves along X, i.e., applying the maps $f^{i,j}$, we obtain four curves

$$\gamma_x^{i,j} : I^i \longrightarrow D_x^j,$$

where $I^- = (-\delta, 0)$ and $I^+ = (0, \delta)$, which are both tangent to η_u. Moreover,

$$\lim_{t \to 0} \gamma_x^{i,j}(t) = p_x^j,$$

so the union of $\gamma_x^{i,j}$ and $\gamma_x^{-i,j}$ can be continuously extended to a curve

$$\gamma_x^j : (-\delta, \delta) \longrightarrow D_x^j$$

with $\gamma_x^j(0) = p_x^j$. The key point is that since the $\gamma_x^{\pm}$'s are continuous, continuously differentiable away from 0, and their derivatives extend continuously at 0, these curves are in fact $\mathcal{C}^1$.

The union of all of the flow lines of X starting from $\gamma_y^{\pm}$ produce two open surfaces $S^{\pm}$ such that

$$S \coloneqq S^- \cup \left\{ |x| \leq \epsilon,\ y = z = 0 \right\} \cup S^+$$

is a surface which can be written as the graph of a continuous function $f : U \subset \mathbb{R}^2 \to \mathbb{R}$ near 0. Here, U denotes an open neighborhood of 0 in $\mathbb{R}^2$. Note that $S \cap D_x^{\pm} = \gamma_x^{\pm}$. By the previous paragraph, f is differentiable away from 0 and its graph is tangent to η_u. The differentiability of f along $\left\{ |x| \leq \epsilon,\ y = z = 0 \right\} \setminus \{0\}$ follows from the differentiability of $\gamma_x^{\pm}$. Moreover, f has (vanishing) partial derivatives at 0, and its partial derivatives are continuous on U. As a result, f is $\mathcal{C}^1$ on U and is everywhere tangent to η_u, so S is a local integral surface for η_u near 0.

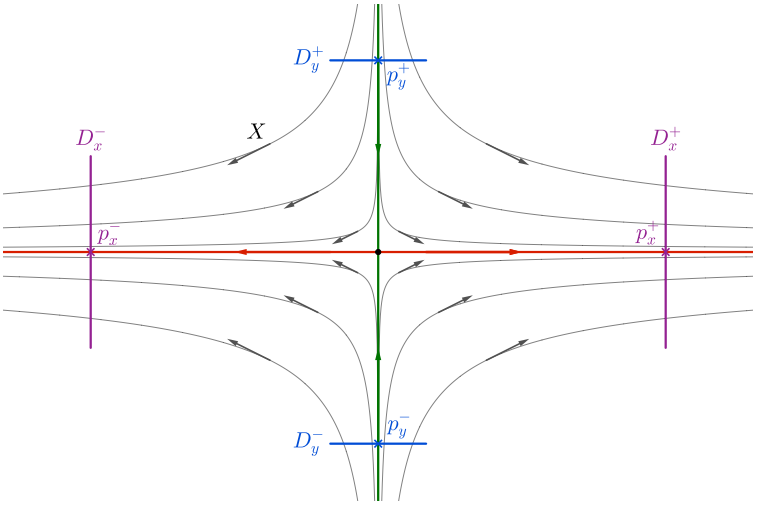


Figure 1.2: The 2-dimensional analogue of Case 2.

Case 3: $p \in Q$**.** This is the most delicate case. For the sake of clarity, we divide it into three steps.

- *Step 0 : setting things up.* We start with a *topological* description of the flow lines of X near p. Following [CF11, Proof of Lemme 2.2], there exist smooth coordinates (x, y, z) around p such that the linearization of X at 0 is of the form

$$X_1 = ax\partial_x + z\partial_y,$$

  where $a > 0$, and Δ is included in $\{x = 0,\ z + y^2 = 0\}$. We can further assume that the strong unstable manifold of X at 0 is included in $\{y = z = 0\}$. By [Tak74, Theorem (1.11)], for every $k \geq 1$, there exists a germ S^c of center manifold of class $\mathcal{C}^k$ passing through 0 which is tangent to the plane $\{x = 0\}$ at 0, and such that X is tangent to S^c along it. Let us take $k = 10$ for safety. We denote by $Y = Y(y, z)$ the restriction of X to $\{x = 0\}$, which is of class $\mathcal{C}^k$ and has vanishing 1-jet at 0. By [Tak74, Theorem (1.13)], X is *topologically* equivalent near 0 to the vector field

$$X' = ax\partial_x + z\partial_y + Y(y, z).$$

Notice that S^c contains Δ. Writing $Z(y,z) = z\partial_y + Y(y,z)$, Z is a $\mathcal{C}^k$ vector field in the (y,z)-plane whose linearization at 0 is $z\partial_y$. Moreover, we can assume that in these new coordinates, Δ is the line $\{x = z = 0\}$. This implies that for every y sufficiently small, $Y(y,0) \cong 0$, so we can write $Y(y,z) = zQ(y,z)$, where Q is a $\mathcal{C}^{k-1}$ vector field vanishing at $(0,0)$. We consider the vector field

$$G(y,z) \coloneqq \partial_z + Q(y,z),$$

so that

$$Y = z\left(\partial_y + Q(y,z)\right).$$

Here, G is mostly horizontal. Moreover, the points in $\{y > 0, z = 0\}$ correspond to singularities of X in Δ_{so}, while the points in $\{y < 0, z = 0\}$ correspond to singularities of X in Δ_{sa}. Therefore, the flow lines of G passing through a point $(y,0)$ with $y < 0$ are *topologically* negatively transverse to the y-axis, while the flow lines of G passing through $(y,0)$ with $y > 0$ are topologically positively transverse to the y-axis. One easily deduces that the flow line of G passing through $(0,0)$ only intersects the y-axis at $(0,0)$ and stays in the half-plane $\{z > 0\}$ away from $(0,0)$. Moreover, every flow line of G passing through $(y,0)$ for $y < 0$ sufficiently close to 0 intersects $\{y > 0, z = 0\}$. Those flow lines corresponds to connections from Δ_{sa} to Δ_{so} in a neighborhood of p. The other stable branches of Δ_{sa} near p all escape the chosen coordinate neighborhood of p.

Since G has a unique flow line passing through 0, Y has a unique flow line converging to 0 in positive time, and a unique flow line converging to 0 in negative time. In summary, we obtain that

- There exists a unique flow line of X converging to 0, and
- The set of the points near 0 which are converging to 0 under the backward

flow of X is a *topological* surface with boundary, whose boundary contains 0 and is contained in $\{y = z = 0\}$.

Unfortunately, we cannot conclude that this half-surface is tangent to η_u since we don't know that it is $\mathcal{C}^1$ yet.

We now restrict our attention to the box

$$\mathsf{B}_\epsilon \coloneqq \left\{|x| \leq \epsilon,\ |y| \leq \epsilon + \sqrt{\epsilon},\ |z| \leq \epsilon\right\}$$

for $\epsilon > 0$ small enough. Still from [CF11, Proof of Lemme 2.2], for ϵ sufficiently small, X is transverse to the vertical faces

$$F_x^\pm \coloneqq \{x = \pm\epsilon\} \cap \mathsf{B}_\epsilon, \qquad F_y^\pm \coloneqq \{y = \pm(\epsilon + \sqrt{\epsilon})\} \cap \mathsf{B}_\epsilon,$$

and enters B_ϵ along the face F_y^- and exits along the other vertical faces. Here, X has source singularities along $\Delta \cap \{y > 0\}$ and saddle singularities along $\Delta \cap \{y < 0\}$. Moreover, every backward flow line of X starting on $\partial\mathsf{B}_\epsilon$ either converges to a point in Δ, or exits $\partial\mathsf{B}_\epsilon$ through the face F_y^- or the top face $F_z^+ \coloneqq \{z = \epsilon\} \cap \mathsf{B}_\epsilon$.

We can find continuous simple curves $\gamma_u^\pm \subset F_x^\pm$ starting at $x^\pm \coloneqq \{x = \pm\epsilon,\ y = z = 0\}$ and ending on the interior of $F_x^\pm \cap \{z = -\epsilon\}$, the bottom edge of $F_x^\pm$, such that under the flow of $-X$, every point in $\gamma_u^\pm \setminus \{x^\pm\}$ converges to a point in $\Delta \cap \{y > 0\}$. In other words, points in $\gamma_u^\pm \setminus \{x^\pm\}$ lie in the intersection of the unstable disks of the source points in $\Delta \cap \{y > 0\}$ with $F_x^\pm$. Similarly, we can find a continuous simple curve $\gamma_b \subset F_y^-$ intersecting the interior of the bottom edge of F_y^- such that every point in γ_b converges to a point in $\Delta \cap \{y < 0\}$ under the flow of X, except one of its endpoints y_b which converges to 0. In other words, every point of γ_b lies in the stable manifold of a saddle point in Δ, or in the stable branch of X at 0. Notice that γ_b is contained in the intersection of S^c with F_y^-.

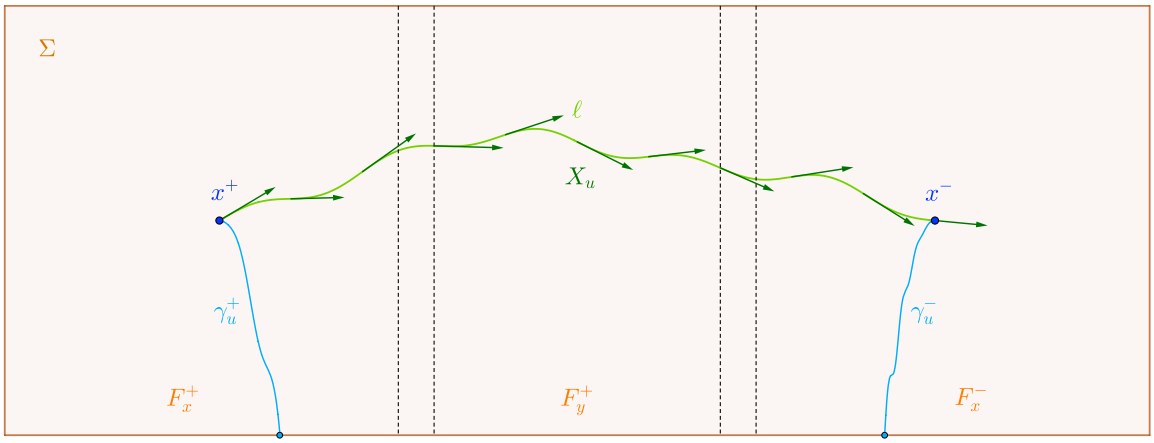


Figure 1.3: The surface Σ and a pre-lasso ℓ.

We define a smoothing Σ of the union of the exit vertical faces $F_x^- \cup F_y^+ \cup F_x^+$ as follows. We first consider a $\mathcal{C}^0$-small smoothing $\widetilde{c}$ of the piecewise linear planar curve

$$c \coloneqq \left\{x = -\epsilon,\ |y| \leq \epsilon + \sqrt{\epsilon}\right\} \cup \left\{|x| \leq \epsilon,\ y = \epsilon + \sqrt{\epsilon}\right\} \cup \left\{x = \epsilon,\ |y| \leq \epsilon + \sqrt{\epsilon}\right\}$$

such that $\widetilde{c}$ coincides with c away from a small neighborhood of the corners of c and is contained in the rectangle $\left\{|x| \leq \epsilon,\ |y| \leq \epsilon + \sqrt{\epsilon}\right\}$. We then set

$$\Sigma \coloneqq \left\{(x, y, z) \in \mathsf{B}_\epsilon \mid (x, y) \in \widetilde{c},\ |z| \leq \epsilon\right\}.$$

An appropriate choice of $\widetilde{c}$ ensures that X is positively transverse to Σ, and the curves $\gamma_u^\pm$ lie on Σ. If the curve $\widetilde{c}$ is parametrized by the variable $s \in [-1, 1]$, we obtain coordinates (s, z) on Σ in which Σ is simply the rectangle $[-1, 1] \times [-\epsilon, \epsilon]$. We will assume that x^+ and γ_u^+ are contained in $[-1, 0) \times [-\epsilon, \epsilon]$, and x^- and γ_u^- are contained in $(0, 1] \times [-\epsilon, \epsilon]$; see Figure 1.3.

In the next steps, we construct a germ of integral surface for η_u passing through 0 as follows. We construct two (germs of) integral surfaces with boundary $S^\pm \subset \mathsf{B}_\epsilon$, such that the unstable manifold of X at 0 is contained in the boundaries of $S^\pm$,

and such that $S = S^- \cup S^+$ is the graph of a continuous function near 0. Then, the same argument as in Case 2 will show that S is $\mathcal{C}^1$.

- *Step 1: construction of S^-.* We adapt the reasoning of Case 2. Choosing ϵ sufficiently small, the center manifold S^c intersects the face F_y^- along a $\mathcal{C}^1$ curve γ_c; this curve divides F_y^- into two halves, and contains the curve γ_b. The specific form of the linearization of X at 0, and the topological normal form from Step 0 imply the following property: if $(x_n)_n$ is a sequence of points in $F_y^- \setminus \gamma_c$ converging to y_b from the $x > 0$ (resp. $x < 0$) side at a sufficiently small asymptotic slope with respect to the plane $\{z = 0\}$, then for n large enough, the flow line of X passing through x_n exits B_ϵ along F_x^+ (resp. F_x^-), and the intersection of this flow line with the corresponding face converges to x^+ (resp. x^-) when n goes to ∞. Since η_u is transverse to S^c at 0, it intersects γ_c transversally at y_b along the face F_y^-. We can further shrink ϵ to make the angle between $\eta_u(y_b)$ and the plane $\{z = 0\}$ sufficiently small. As in Case 2, we then choose a flow line of $\eta_u \cap F_y^-$ passing through y_b. We flow it along X to obtain a surface whose union with $\{y = z = 0\}$, denoted by S^-, is a surface with boundary tangent to η_u.

- *Step 2: construction of S^+.* The idea of the construction of S^+ is to find a $\mathcal{C}^1$ curve ℓ on Σ starting at x^- and ending at x^+ so that ℓ is tangent to η_u, and every point on ℓ converges to 0 under the backward flow of X. We will refer to such a curve ℓ as a *lasso*. We first construct a *pre-lasso*, i.e., a curve satisfying the same assumptions as a lasso except for the condition on the convergence to 0; see Figure 1.3 We then construct a lasso from a suitable sequence of pre-lassos. The integral surface S^+ will be obtained by flowing a lasso along the backward flow of X.

  The intersection of η_u with Σ defines a continuous vector field X_u on Σ which is almost horizontal (its vertical coordinate is negligible with respect to its horizontal coordinate). Co-orienting η_u so that ∂_z is positively normal to η_u, this vector field

essentially points "from x^+ to x^-". The key fact is the following: any integral curve of X_u on Σ passing through x^+ or x^- is disjoint from $\gamma_u^+ \setminus \{x^+\}$ and $\gamma_u^- \setminus \{x^-\}$. Otherwise, a point on such a curve would converge to a point $p_s \in \Delta_{\text{so}}$, and flowing this curve along the backward flow of X would produce an integral surface for η_u passing through p_s. By Case 1, we know that there exists a unique such integral surface near p_s, namely, the unstable manifold of X at p_s. The topological normal form from Step 0 implies that our surface cannot contain $x_\pm$, a contradiction. We then argue that there exists a flow line of X_u passing through x^+ which exits Σ along

$$\partial_v^- \Sigma \coloneqq \partial\Sigma \cap \left\{x = -\epsilon,\ y = -(\epsilon + \sqrt{\epsilon})\right\}.$$

Indeed, if such a flow line γ exits $\partial\Sigma$ along bottom boundary

$$\partial_b \Sigma \coloneqq \partial\Sigma \cap \{z = -\epsilon\},$$

then it would intersect the unstable manifold of a source point, which is prohibited by the previous argument. If every such γ exits Σ along the interior of its top boundary

$$\partial_t \Sigma \coloneqq \partial\Sigma \cap \{z = \epsilon\},$$

we consider the set $\mathcal{S}$ of values of $s \in (-1, 1)$ such that (s, ϵ) is an exit point of such a flow line γ, and we define $s_{\text{sup}} \coloneqq \sup \mathcal{S}$. Picking a sequence $s_n \in \mathcal{S}$ converging to s_{sup}, we consider a sequence of flow lines γ_n for X_u all starting at x^+ and exiting Σ at (s_n, ϵ). This sequence of curves is uniformly Lipschitz, since X_u is continuous, so up to passing to a subsequence, we can assume that $(\gamma_n)_n$ converges uniformly to a curve γ_{sup} which is still a flow line of X_u starting at x^+,[4] and which exits Σ at $(s_{\text{sup}}, \epsilon)$. By assumption, we have $s_{\text{sup}} < 1$. There exists a small $\delta > 0$ such that the

[4]This is because a $\mathcal{C}^1$ function $\gamma : I \to \mathbb{R}$ defined on an interval containing 0 is a solution to the ODE $\dot{y}(t) = F(y(t))$ if and only if it satisfies $y(t) = y(0) + \int_0^t F(y(s))\, ds$ for every $t \in I$.

point $x' \coloneqq (s_{\sup}, \epsilon - \delta)$ escapes B_ϵ when flown backward along X, and escapes Σ along its top or lateral boundary when flown along X_u. We now consider a flow line γ' of $-X_u$ starting at x'. It cannot intersect $\gamma_{\sup}$, since otherwise, we could merge a portion of $\gamma_{\sup}$ with a portion of γ' and obtain a flow line of X_u starting at x^+ and escaping Σ along a point on the top boundary further right than $(s_{\sup}, \epsilon)$, or on the lateral boundary of Σ, and both of these excluded by hypothesis. Therefore, γ' has to pass below x^+ and intersect γ_u^+. This is also excluded, since some point on γ' converges to 0 under the backward flow of X by the topological normal form of X near p, which contradicts a previous argument.

To summarize, we now know that there exists a flow line of X_u passing through x^+ and which exits Σ on its lateral boundary $\partial_v^- \Sigma$. Such a flow line is not allowed to intersect γ_u^-, so it necessarily passes above x^-. We consider the set $\mathcal{Z}$ of $z \in (-\epsilon, \epsilon)$ such that there exists a flow line of X_u starting at x^+, passing through (s^-, z), and escaping Σ along $\partial_v^- \Sigma$. Here, $x^- = (s^-, z^-)$. Similarly as before, we define $z_{\inf} \coloneqq \inf \mathcal{Z}$, and we can obtain a flow line $\gamma_{\inf}$ starting at x^+ and passing through $(s^-, z_{\inf})$. We know that $z_{\inf} \geq z^-$. If $z_{\inf} > z^-$, arguing as before, we could obtain a flow line γ'' of $-X_u$ starting at x^- and intersecting γ_u^+, which is impossible. Therefore, $z_{\inf} = z^-$ and $\gamma_{\inf}$ passes through x^-. In other terms, we have shown the existence of a pre-lasso.

We now explain how to construct a lasso, i.e., a pre-lasso such that all of its points converge to 0 under the backward flow of X. Let $\mathcal{L}$ be the set of (unparametrized) pre-lassos $\ell \subset \Sigma$, i.e., the set of $\mathcal{C}^1$ curves on Σ tangent to X_u starting at x^+ and ending at x^-. We can assume without loss of generality that $z^+ = -1/2$ and $z^- = 1/2$. For $\ell \in \mathcal{L}$, we consider its intersection point $(0, z_0)$ with the vertical line $\{s = 0\} \cap \Sigma$. Arguing as before, by taking the infimum of such values of z_0, we obtain a pre-lasso ℓ_0 passing through a point $(0, z_0)$ which converges to 0 under the backward flow of X. Arguing inductively, we can construct a sequence $(\ell_n)_n$ of

pre-lassos such that for every n and every dyadic number $\mathfrak{d}$ of height at most n in $[-1/2, 1/2]$, the intersection point of ℓ_n with $\{s = \mathfrak{d}\} \cap \Sigma$ converges to 0 under the backward flow of X, and ℓ_m intersects $\{s = \mathfrak{d}\} \cap \Sigma$ at this same point when $m \geq n$. Passing to a subsequence, these pre-lassos converge to a pre-lasso ℓ_∞ such that a dense subset of its points converge to 0 under the backward flow of X. Since the points on ℓ not converging to 0 form an open subset of ℓ, we obtain that ℓ_∞ is a lasso.

As explain before, we obtain a surface S^+ by flowing a lasso ℓ under the backward flow of X. The boundary of S^+ is then $\ell \cup (\{y = z = 0\} \cap \mathsf{B}_\epsilon)$, and S^+ is everywhere tangent to η_u.

Finally, the union S of S^- and S^+ inside of B_ϵ is a continuous surface which can be written as the graph of a continuous function f. Arguing as in Case 2, by looking the intersection of this surface with $F_x^\pm$ near $x^\pm$, we conclude that f is in fact $\mathcal{C}^1$, so S is an integral surface for η_u passing through 0. □

Remark 1.1.12. *Handling Case* 3 *is particularly tricky because we are not aware of a sufficiently nice* $\mathcal{C}^1$ *normal form for* X *near a quadratic point* $p \in Q$.

Remark 1.1.13. *It is worth noting that in [CF11], the authors assume that* (ξ_-, ξ_+) *is in normal form to show the existence of* (η_u, η_s)*, as they need to show that the limits of* $\xi_\pm$ *along the flow of* X *exist. With the present method, we show the existence without any assumption on* (ξ_-, ξ_+)*. Then, even though [CF11] also uses the normal form to show local integrability along* Δ*, what is really needed for the proof is a precise understanding of the qualitative behavior of* X *near* Δ*. This can be achieved under some genericity assumption on* (ξ_-, ξ_+) *only, the point being that a plane field invariant under a possibly singular but sufficiently nice vector field tangent to it is locally integrable, even at the singularities.*

The following lemma provides a more refined description of the integral surfaces of η_u passing through a quadratic point $p \in Q$:

Lemma 1.1.14. *Let $p \in Q$ be a quadratic contact point. There exists a unique flow line $\gamma_s = \gamma_s(p)$ of X converging to p. Moreover, there exists a unique (germ of) surface with boundary $S^u = S^u(p)$ whose boundary contains p and is contained in the unstable manifold of X at p, and such that X is tangent to S^u. The plane field η_u is tangent to S^u and is uniquely integrable along S^u in the following sense. If (S, q) is a germ of integral surface of η_u at $q \in M$, then*

- *If q lies in the interior of S^u, then $S \subset S^u$.*
- *If q lies in the boundary of S^u, then the unstable branches of X at p divide S into two surfaces with boundary, one of which is entirely contained in S^u.*

Therefore, every (germ of) integral surface S of η_u at p is the union of two surfaces with boundary S^+ and S^-, where

- *The boundary of $S^\pm$ contains p and is contained in the unstable manifold of X at p,*
- *S^+ is included in the saturation of S^u by the flow of X,*
- *S^- contains γ_s.*

Hence, as a germ at p, S is entirely determined by its intersection with a small disk transverse to γ_s.

We call $\gamma_s(p)$ the **stable branch** of X at p, and $S^u(p)$ the **unstable half-disk** of X at p.

Proof. In the proof of Proposition 1.1.11, we saw that there is a unique flow line of X converging to p, and the flow lines converging to p in backward times form

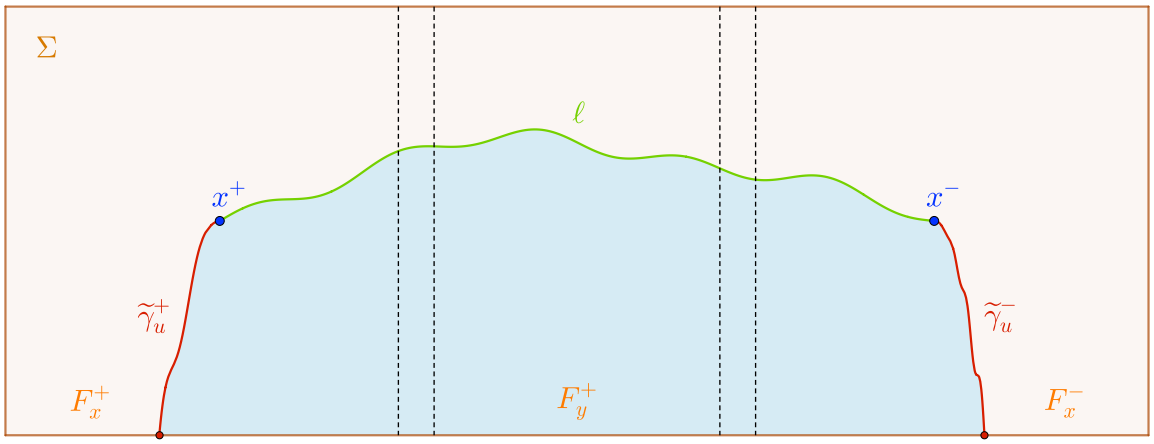


Figure 1.4: The surface Σ and its unique lasso ℓ. The curves $\widetilde{\gamma}_u^{\pm}$ are the intersection of Σ with the unstable manifolds of the saddle points on Δ. The blue region is the intersection of Σ with the unstable manifolds of the source points on Δ. All of the points in the orange region leave the box B_ϵ when flowed backward along X.

a topological half-disk containing p and the stable branches of X at p. We also constructed a $\mathcal{C}^1$ half-disk near p tangent to η_u satisfying these properties, so these two half-disks coincide. In particular, this unstable half-disk is $\mathcal{C}^1$. Moreover, the lasso ℓ constructed in the proof of Proposition 1.1.11 is unique, and $y_b \in F_y^-$ is the unique point in F_y^- converging to 0 under the flow of X. See Figure 1.4 for a detailed picture of Σ.

We now show that η_u is uniquely integrable along S^u in the sense of the statement of the lemma. Otherwise, there would exist a point $p_0 \in \ell \subset \Sigma$ and a small integral curve γ_0 of X_u containing p_0 which is not contained in ℓ. After deleting a portion of γ_0 and without loss of generality, we can assume that $\gamma_0 \cap \ell = \{p_0\}$, p_0 is the left boundary point of γ_0, and $p_0 \neq x^-$. Note that γ_0 necessarily lies above ℓ, as points below ℓ lie in the unstable manifolds of source singularity in Δ. Moreover, is γ_0 is sufficiently short, all of the points in $\gamma_0 \setminus \{p_0\}$ are flown to the back face F_y^- of the box B_ϵ under the backward flow of X. The intersection points of F_y^- with these flow lines form a continuous curve γ_0^- which is tangent to $\eta_u \cap F_y^-$, and one of its ends converges to y_b. It follows that $\gamma_0^- \cup \{y_b\}$ can be described near y_b as the graph of a $\mathcal{C}^1$ curve. However, by Step 1 in Case 3 of the proof of Proposition 1.1.11, the (forward)

flow lines of X starting at points in γ_0^- near $\{y_b\}$ would intersect $\partial \mathsf{B}_\epsilon$ away from ℓ, which is a contradiction.

The description of an integrable surface S near p then follows from the proof of Proposition 1.1.11. □

Remark 1.1.15. *In the proof of Proposition 1.1.11, we showed that there is a unique germ of integral surface for η_u at a source point $p \in \Delta_{\mathrm{so}}$, and a germ of integral surface at saddle point $p \in \Delta_{\mathrm{sa}}$ is entirely determined by its intersection with small disks transverse to the unstable (or stable) branches of X at p.*

1.1.4 Proof of Theorem A

First of all, the plane fields $\xi_\pm^t$ defined by (0.1.4) converge pointwise to (not necessarily continuous) plane fields $\eta_\pm^\infty$ when t goes to $+\infty$. This can be obtained from the standard argument explained in [CF11, Section 2.4]. We choose an arbitrary Riemannian metric g on M and we denote by ν the orthogonal plane field to X defined away from Δ. We can then measure the angle $\theta_\pm^t(p)$ between $\xi_\pm(p) \cap \nu(p)$ and $\xi_\pm^t(p) \cap \nu(p)$, which satisfy

$$-\pi < \theta_-^t \leq \theta_+^t < \pi.$$

The contact conditions for $\xi_\pm$ imply

$$\frac{d}{dt}\theta_+^t(p) > 0, \qquad \frac{d}{dt}\theta_-^t(p) < 0,$$

for $p \in M \backslash \Delta$. Moreover, if $p \in \Delta$, then for every $t \in \mathbb{R}$, $\xi_+^t(p) = \xi_-^t(p) = \xi_+(p) = \xi_-(p)$. This shows that the limits $\eta_\pm^\infty$ exist. If γ is a (possibly trivial) flow line of X, then by definition, the restrictions of $\eta_\pm^\infty$ to γ are invariant under the flow of X. In particular, they are differentiable along X. Moreover, $\eta_\pm^\infty$ are "sandwiched" between ξ_- and ξ_+ as in Proposition 1.1.7. The proof of the latter readily implies that $\eta_\pm^\infty$ coincide with

η_u along γ, hence these plane fields coincide on M. Now that we know that $\eta_\pm^\infty = \eta_u$ is continuous, we can argue as in [CF11, Lemme 3.1] and apply Dini's theorem to show that the convergence of $\xi_\pm^t$ to η_u is uniform. The first item of Theorem A is proved, and the second item immediately follows from Proposition 1.1.11. □

1.2 Constructing foliations from contact pairs

In this section, we prove Theorem B from the introduction, up to a (very) technical result (Theorem 1.2.1) whose proof occupies Chapter 2.

1.2.1 Polarized vector fields

Definition 1.2.1. *Let X be a smooth, possibly singular vector field on M. A **polarization** of X is a continuous, cooriented plane field η which contains X and is invariant under the flow of X. We call the pair (X, η) a **polarized vector field**.*[5]

In the previous sections, we saw that a positive contact pair (ξ_-, ξ_+) gives rise to a polarized vector field (X, η_u) which depends continuously on (ξ_-, ξ_+).

By analogy with Definition 1.1.10, we say that a polarized vector field (X, η) on M is **regular** if the following conditions are satisfied:

1. The singular set Δ of X is a smoothly embedded link,

2. The linearization of X has rank 2 along Δ,

3. Δ is transverse to η away from a finite set of points $Q \subset \Delta$ where it has "quadratic tangencies" with η, in the following sense. At each $p \in Q$, there exist

[5]This terminology is motivated by the following observation: away from the singular set of X, η defines an oriented line subbundle of the plane bundle $TM/\langle X \rangle$ which is invariant under the flow if X. After choosing a (locally defined) transverse plane field to X, this line bundle can be identifies with an oriented line field transverse to X.

smooth coordinates (x, y, z) in which the linearization of X at 0 is of the form

$$X_1 = ax\partial x + z\partial_y,$$

and $\eta(0) = \text{span}\{\partial_x, \partial_y\}$. Moreover, $\Delta = \{x = 0, z = \pm y^2\}$ near p in these coordinates.

4. (X, η) satisfies the conclusions of Lemma 1.1.14 at these quadratic points.

The proof of Proposition 1.1.11 implies that η is locally integrable. Similarly, we have a decomposition

$$\Delta = \Delta_{\text{so}} \sqcup Q \sqcup \Delta_{\text{sa}}$$

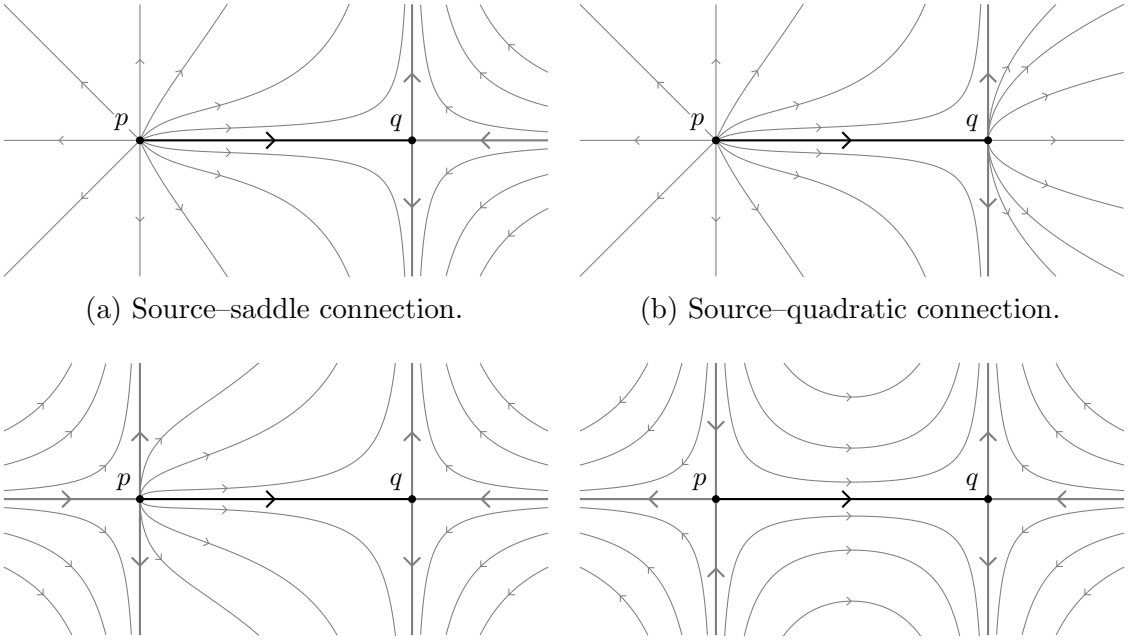
in terms of source, quadratic, and saddle singularities.

The analysis of the different types of singularities of X along Δ in the proofs of Proposition 1.1.11 and Lemma 1.1.14 readily implies the following lemma which will be useful in Chapter 2.

Lemma 1.2.2. *There exists a tubular neighborhood $\mathcal{N}$ of Δ, made of a disjoint union of tubular neighborhoods of the connected components of Δ, and satisfying the following property. If a nontrivial flow line γ of X is entirely contained in $\mathcal{N}$, then γ connects a source singularity to a saddle singularity, both in the same component of Δ.*

To simplify some of our proofs, especially in Chapter 2, we will make further generic assumptions on the contact pairs under consideration and their associated polarized vector fields. Recall that a **connection** of a vector field X is a nontrivial flow line γ converging to singularities of X in positive and negative times.

Definition 1.2.3. *Let (X, η) be a regular polarized vector field. A connection γ of X between singularities $\lim_{\pm\infty} \gamma =: p_\pm \in \Delta$ is **admissible** if it is of one of the four following types:*



(a) Source–saddle connection.

(b) Source–quadratic connection.

(c) Quadratic–saddle connection.

(d) Saddle–saddle connection.

Figure 1.5: The four types of admissible connections.

(A1) p_- *is a source singularity and* p_+ *is a saddle singularity,*

(A2) p_- *is a source singularity and* p_+ *is a quadratic singularity,*

(A3) p_- *is a quadratic singularity,* p_+ *is a saddle singularity, and* γ *is not an unstable branch of* p_-*,*

(A4) Both $p_\pm$ *are saddle singularities,* $p_- \neq p_+$*, and* $p_\pm$ *are not connected to any other saddle singularities.*

The fours types of admissible connections are depicted in Figure 1.5. We will also require a strengthening of case (A4) above, which will only become relevant in Section 2.3.3 of Chapter 2.

Definition 1.2.4. *A* ***broken triple saddle connection*** *is a sequence* $(\gamma_0, \ldots, \gamma_{n+1})$*,* $n \geq 0$*, of pairwise distinct unoriented connections of* X *such that*

- γ_0 *and* γ_{n+1} *are saddle-saddle connections,*

- *For every $0 \leq i \leq n$, γ_i and γ_{i+1} share an endpoint,*
- *For every $1 \leq i \leq n$, γ_i belongs to the unstable manifold of a source or quadratic singularity of X.*

Definition 1.2.5. *A regular polarized vector field (X, η) is **admissible** if X only has admissible connections, and no broken triple saddle connections. A regular positive contact pair (ξ_-, ξ_+) is **admissible** if its associated polarized vector field (X, η_u) is admissible.*

The next lemma follows from standard general position arguments applied to the stable and unstable manifolds of the singularities of X:

Lemma 1.2.6. *Any regular positive contact pair can be (smoothly) approximated by an admissible one with the same singular set Δ. Therefore, admissible positive contact pairs form a comeagre subspace of the space positive contact pairs.*

We will need further assumptions on the contact pairs under consideration to prevent the existence of *disks of tangency* for their associated polarized vector fields.

Definition 1.2.7. *A **disk of tangency** for a polarized vector field (X, η) is an immersion $f : D \to M$ of the open 2-disk in M such that*

- *f is tangent to η,*
- *f extends to a continuous map $\bar{f} : \overline{D} \to M$ whose restriction to $\partial\overline{D} \cong S^1$ maps to a closed orbit of X.*

The presence of such disks could obstruct some of the main constructions in Chapter 2 below. However, if one of the contact structures in the pair is tight, or if $\Delta_{\text{so}} = \varnothing$ (see the hypothesis of Theorem B), then these disks cannot exist:

Proposition 1.2.8. *Let (ξ_-, ξ_+) be an admissible positive contact pair. If its associated polarized vector field (X, η_u) has a disk of tangency, then $\Delta_{\text{so}} \neq \varnothing$ and both ξ_- and ξ_+ are overtwisted.*

Proof. Let $f : D \to M$ be a disk of tangency that extends to $\bar{f} : \overline{D} \to M$, and let γ be the closed orbit of X such that $\bar{f}\big(\partial\overline{D}\big) \subset \gamma$.

Notice that $\bar{f}$ is an immersion as well: intersect γ with a small transversal surface Σ and consider the intersection of $f(D)$ with Σ. It is a collection of flow lines of the continuous vector field X_u obtained by intersecting η_u with Σ. These flow lines converge to the point $p_0 = \gamma \cap \Sigma$, so they extend in a $\mathcal{C}^1$ way at that point. This implies that $\bar{f}$ is a $\mathcal{C}^1$ immersion tangent to η_u.

Let $\widetilde{X}$ denote the pullback of X to $\overline{D}$. It is a continuous, uniquely integrable vector field tangent to $\partial\overline{D}$ with isolated singularities in D. These singularities are of three types: source (index $+1$), birth-death (index 0), or saddle (index -1). By the Poincaré–Hopf theorem, there must be a source singularity in D, i.e., $f(D) \cap \Delta_{\mathrm{so}} \neq \varnothing$.

Let us temporarily assume that $\bar{f}$ is an embedding and write $\mathcal{D} \coloneqq \bar{f}\big(\overline{D}\big)$. Since $L \coloneqq \partial\mathcal{D}$ is Legendrian for both ξ_- and ξ_+, and since X is tangent to L and to η_u, the Thurston–Bennequin numbers $\mathrm{tb}_\pm(L)$ of L for $\xi_\pm$ are both zero. Since a Legendrian knot K bounding an embedded disk in a tight contact 3-manifold has $\mathrm{tb}(K) \leq -\chi(\overline{D}) = -1$ by [Ben83], this implies that ξ_- and ξ_+ are both overtwisted.

We now explain how to reduce to the case where $\bar{f}$ is an embedding. We will repeatedly use the Poincaré–Bendixson theorem (see [Tes12, Theorem 7.16]) and the following fact: if a domain in D is bounded by a closed orbit of $\widetilde{X}$, then it contains a singularity of $\widetilde{X}$. This can be seen as an easy consequence of the hairy ball theorem.

Let us consider the oriented graph $G = (V, E)$ on D whose vertices are the singularities of $\widetilde{X}$ and whose oriented edges are the flow lines of $\widetilde{X}$ connecting two singularities; see Figure 1.6. Notice that these flow lines are stable branches of the saddle and birth-death singularities, since there are no sinks, and there are finitely many of them. This implies that G is a finite graph. We now argue that G is a tree. We first show by contradiction that G has no cycle. If G has a cycle, we choose an innermost one and we denote it by c. We also consider $\overline{D}_c \subset D$ the closed domain

bounded by c in D. Let δ be an edge in c connecting the singularity p to the singularity q. We distinguish three cases:

1. *p and q are saddles.* By the admissibility condition, we have $p \neq q$ and the two unstable branches of q cannot converge to singularities in positive time. One of these two branches is contained inside of $\overline{D}_c$. It cannot limit to a closed orbit as the latter would bound a domain in which $\widetilde{X}$ has no singularity. Therefore, the Poincaré–Bendixson theorem implies that it limits to a connected, *strict* subgraph G' of G included in $\overline{D}_c$ with at least one edge. Moreover, the graph G' has the following property: if δ' is an edge in G' limiting to a singularity z, which is a saddle or a birth-death, then at least one of the two unstable branches of z is also an edge in G'. This readily implies that G' contains a cycle, since there are finitely many singularities. Recall that c is an innermost cycle, so G' necessarily contains c, and in particular it contains δ, which is impossible since q is a saddle.

2. *p is a source or birth-death and q is a saddle.* None of the two unstable branches of q can be an edge of the cycle c, since the admissibility condition would imply that it converges to another saddle singularity, and this situation is prohibited by the first case. Therefore, the stable branch of q different than δ is an edge in c. If the unstable branch of q contained in $\overline{D}_c$ does not limit to a singularity, the argument of the first case applies and leads to a contradiction. This unstable branch could limit to a saddle z in $\overline{D}_c$, but the unstable branches of z cannot limit to singularities, and we similarly obtain a contradiction.

3. *p is a source and q is a birth-death.* The unstable branches of q cannot converge to singularities by admissibility. Therefore, there is an unstable flow line in the unstable half-disk of q, different than the unstable branches, which is an edge in c, possibly equal to δ if $p = q$. Once again, the behavior of the unstable branch of q contained in $\overline{D}_c$ leads to a contradiction.

To summarize, we have shown that the graph G is a forest, namely, a disjoint union of trees. Notice that since there is no cycle of singularities, every flow line of $\widetilde{X}$ either converges to a singularity or to a closed orbit. This allows us to restrict to the case where D does not contain any nontrivial closed orbit of $\widetilde{X}$. We can argue by induction on the number of connected components of G. Indeed, if G_0 denotes a connected component of G, then every leaf of G_0 admits an unstable flow line (unstable branch or a flow line contained in the unstable manifold) which is not contained in G. This flow line must limit to a closed orbit γ of $\widetilde{X}$. If γ bounds a disk D_γ disjoint from G_0, we restrict f to D_γ, reducing the number of connected components of the graph of singularities of $\widetilde{X}$. Otherwise, every closed orbit of $\widetilde{X}$ contained in D_γ bounds a disk disjoint from G_0. If such an orbit exists, we restrict f to the disk it bounds, reducing the number of connected components. Otherwise, we restrict f to D_γ, and we are done.

We now show that G is connected. Notice that the argument in previous paragraph readily implies that $\widetilde{X}$ has contracting holonomy along ∂D. The Poincaré–Bendixson theorem also implies that all of the stable branches of singularities are edges in G. Let $G_1, \ldots, G_n$, $n \geq 1$, denote the connected components of G. For every $1 \leq i \leq n$, there is an open neighborhood $\mathcal{N}_i$ of G_i in D with smooth boundary such that $\widetilde{X}$ is outward transverse to $\partial\mathcal{N}_i$. We can further assume that these neighborhood are pairwise disjoint. To construct such neighborhoods, we consider the union of small bands around each edge in G and small disks around the vertices. The conditions on G imply that we can modify this neighborhood so that $\widetilde{X}$ becomes outward transverse to its boundary. We also consider a slightly smaller disk $D' \subset D$ such that $\widetilde{X}$ is outward transverse to $\partial D'$. If $n \geq 2$, we would obtain a nonsingular vector field on $D' \setminus \bigcup_{1 \leq i \leq n} \mathcal{N}_i$ which is impossible for degree reasons. Therefore, G is connected and D' retracts onto G under the backward flow of $\widetilde{X}$.

We now show that f is injective. It is enough to show that the restriction of f

to the vertices of G is injective. Indeed, if two distinct points $p, q \in D$ are such that $f(p) = f(q)$, then our previous arguments imply that they lie on two distinct flow lines of $\widetilde{X}$ which converge to two singularities in negative time. These singularities are necessarily distinct since f is an immersion. Therefore, we obtain two distinct vertices of G with the same image under f. We define an equivalence relation $\sim$ on the vertices of G by

$$p \sim q \iff f(p) = f(q).$$

Since f is an immersion, this relation extends to the edges of G. We denote the quotient graph by $H \coloneqq G/\sim$ and the associated quotient map by $\tilde{f} : G \to H$. The graph H can also be described as the graph in M whose vertices are the singularities $f(p)$, $p \in V(G)$, and the edges are the flow lines of X connecting them. It is then easy to see that $\tilde{f}$ is a covering map in the graph theoretical sense. Since H is covered by the *finite* tree G, H itself a tree and $\tilde{f}$ is an isomorphism. This implies that $f_{|G}$ is injective, thus f is injective.

We are left to show that the restriction of $\bar{f}$ to $\partial\overline{D}$ is injective. If not, then this map winds around the closed orbit γ at least twice. With the above notations, the intersection of the transversal Σ with $\bar{f}(\overline{D})$ is a finite collection of flow lines of X_u which only intersect at p_0. If γ_u^0 is such a flow line, we consider its image γ_u^1 under the return map of X on Σ. By our assumptions, γ_u^0 and γ_u^1 are distinct and only intersect at p_0. After choosing coordinates (x, y) on Σ in which X_u is almost flat, we can represent γ_u^0 and γ_u^1 as graphs of $\mathcal{C}^1$ functions defined for $x \leq 0$ and passing through $(0, 0) \cong p_0$. Without loss of generality, we can assume that $\gamma_u^0 \leq \gamma_u^1$. We now consider the iterate images of γ_u^0 under the return map and obtain a sequence of flow lines γ_u^i, $i \geq 0$, satisfying $\gamma_u^i \leq \gamma_u^{i+1}$ for every $i \geq 0$. Here, we are using that η_u is cooriented and invariant under the flow of X. The γ_u^i's have to be pairwise disjoint, which is impossible since $\bar{f}_{|\partial\overline{D}}$ winds around γ finitely many times. $\square$

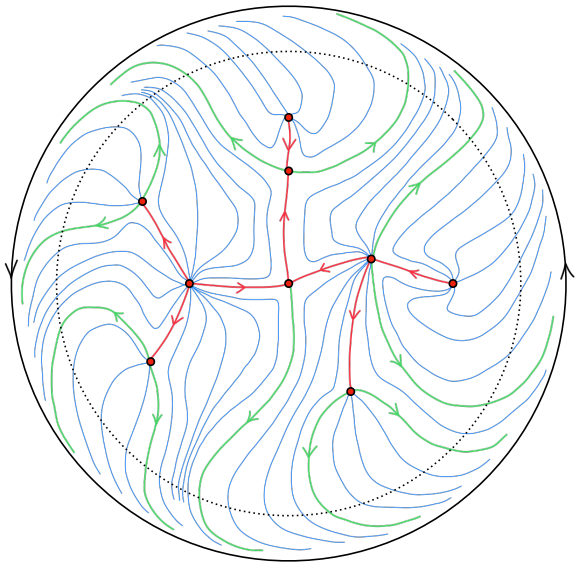


Figure 1.6: A tree on a leaf. In red: the tree G. In green: the stable branches limiting to the boundary orbit. In blue: flow lines of $\widetilde{X}$. The outer circle is a closed orbit of $\widetilde{X}$, the dotted circle is transverse to $\widetilde{X}$.

Remark 1.2.9. *Disks of tangency should be compared to the* (twisted) disks of contact *from [GO89], and the* sink disks *from [Li02], which play a crucial role in the theory of branched surfaces and laminations. The presence of such disks obstructs the construction of a lamination from a branched surface, and it is often a delicate combinatorial problem to modify a branched surface and remove those disks. While it can be particularly hard to certify that a given branched surface is sink disk free, or if a lamination extends to a foliation in its complement, Proposition 1.2.8 provides a very natural and tractable way to preclude the existence of disks of tangency.*

1.2.2 Proof of Theorem B

Let (ξ_-, ξ_+) be a positive contact pair satisfying the hypotheses of Theorem B. After a generic $\mathcal{C}^\infty$-small perturbation away from Δ, which induces a small $\mathcal{C}^0$ perturbation of η_u, we can assume that this contact pair is admissible by Lemma 1.2.6. By Proposition 1.2.8, the polarized flow (X, η_u) has no disks of tangency. We now appeal to the

Technical Theorem 1.2.1. *Let (X, η) be an admissible polarized vector field on M without disks of tangency. For every $\epsilon > 0$, there exists a $\mathcal{C}^0$-foliation $\mathcal{F}_\epsilon$ such that $d_{\mathcal{C}^0}(\eta, T\mathcal{F}_\epsilon) < \epsilon$. Moreover, there exists a continuous map $h_\epsilon : M \to M$ such that for every leaf L of $\mathcal{F}_\epsilon$, the restriction $h_{\epsilon|L} : L \to M$ is a $\mathcal{C}^1$ immersion tangent to η.*

The proof of this highly nontrivial result is deferred to Chapter 2 and will occupy the whole second half of this paper.

Let $\mathcal{F}$ be a $\mathcal{C}^0$-foliation approximating η_u together with a continuous map $h : M \to M$ as above. By [KR19, Theorem 4.1] (see also [Bow16a, Remark 3.3]), we can further assume that the leaves of $\mathcal{F}$ are smoothly immersed. We show by contradiction that $\mathcal{F}$ has no spherical leaves. Otherwise, we readily obtain an embedded sphere S in M whose tangent bundle is homotopic to $\eta_{u|S}$, and since $\xi_\pm$ are homotopic to η_u,

$$|\langle e(\xi_\pm), [S]\rangle| = |\langle e(\eta_u), [S]\rangle| = \chi(S) = 2.$$

However, by Bennequin–Eliashberg's inequality (see [Eli92]),

$$\langle e(\xi), [S]\rangle = 0$$

for a tight contact structure ξ, hence both $\xi_\pm$ are overtwisted. By applying the map h, we also obtain a $\mathcal{C}^1$-immersed sphere $f : S^2 \to M$ tangent to η_u, and pulling back X along f, we get a (continuous, uniquely integrable) vector field $\widetilde{X}$ on S^2 with isolated singularities which are sources (index $+1$), saddles (index -1), or birth-death singularities (index 0). The Poincaré–Hopf theorem readily implies that $\widetilde{X}$ has at least one source singularity, so $\Delta_{\text{so}} \neq \varnothing$. This contradicts the hypotheses of Theorem B, and finishes the proof. □

Remark 1.2.10. *Technically speaking, the hypothesis that (X, η) has no disk of tangency is not absolutely necessary for Theorem 1.2.1 to hold. It could be weaken,*

e.g., by requiring that every disk of tangency is contained in an open subset of M on which η is uniquely integrable. However, we do not know if the latter condition is achieved for the polarized vector field of a generic positive contact pair.

1.3 Strongly tight contact pairs

1.3.1 Tautness and tightness

The following definition already appears in [CF11] and [CH20].

Definition 1.3.1. *A contact pair (ξ_-, ξ_+) on M is **strongly tight** if for all $p, q \in M$, there exists a smooth arc from p to q positively transverse to both $\xi_\pm$.*

A strongly tight contact pair is necessarily positive. It will be convenient do reformulate this definition in various ways that are similar to some of the many characterizations of taut foliations. In Appendix 1.B, we generalize some of these definitions to (continuous) plane fields, which might also be of independent interest.

Proposition 1.3.2. *Let (ξ_-, ξ_+) be a positive contact pair on M. The following are equivalent:*

1. *(ξ_-, ξ_+) is strongly tight,*
2. *For every $p \in M$, there exists a smooth loop positively transverse to $\xi_\pm$ passing through p,*
3. *There exists a smooth closed 2-form ω such that $\omega_{|\xi_\pm} > 0$,*
4. *There exists a smooth volume preserving vector field positively transverse to $\xi_\pm$,*
5. *η_u is a taut plane field (see Definition 1.B.1).*

In particular, $\xi_\pm$ are weakly semi-fillable, hence universally tight.

Proof. The implications

$$1 \implies 2 \implies 3 \implies 4 \implies 5$$

are standard. The last implication follows from Proposition 1.B.3. If η_u is taut, since the distributions $\xi_\pm^t$ defined by (0.1.4) converge uniformly to η_u as t goes to $+\infty$, it is not hard to show that $\left(\xi_-^T, \xi_+^T\right)$ is strongly tight for T sufficiently large; see the proof of [CF11, Proposition 3.7]. Hence, (ξ_-, ξ_+) is strongly tight as well. Weak semi-fillability follows from the argument in [ET98, Corollary 3.2.8]. □

The third or fourth characterization immediately imply that strong tightness is an open condition in the $\mathcal{C}^0$ topology. Moreover, if $\mathcal{F}$ is a taut $\mathcal{C}^0$-foliation on M and if $\xi_\pm$ are positive and negative contact structures on M which are sufficiently $\mathcal{C}^0$-close to $T\mathcal{F}$, then (ξ_-, ξ_+) is strongly tight. Reciprocally, we have:

Theorem 1.3.3 (Theorem C)**.** *Let (ξ_-, ξ_+) be a strongly tight contact pair. Then η_u can be $\mathcal{C}^0$-approximated by taut $\mathcal{C}^0$-foliations.*

Proof. The strong tightness of (ξ_-, ξ_+) implies that η_u is a taut plane field by Proposition 1.3.2. Moreover, $\xi_\pm$ are both tight so Theorem B applies. Since (everywhere) tautness is an open condition for continuous plane fields, $\mathcal{C}^0$-foliations sufficiently close to η_u are also taut. □

Remark 1.3.4. *In [CF11, Proposition 3.6], it is shown that strongly tight contact pairs can be deformed into positive pairs without quadratic singularities, i.e., for which $\Delta = \Delta_+$ is transverse to $\xi_\pm$* everywhere*. It would be interesting to better understand the structure of η_s, the stable plane field of the pair, near Δ where it 'vanishes'.*

The strong tightness assumption on the pair (ξ_-, ξ_+) is quite restrictive and hard to work with in practice. A natural weakening would be to only assume that (ξ_-, ξ_+) is a positive contact pair and that both $\xi_\pm$ are tight (in particular, Theorem B applies). In

that case, Colin–Firmo [CF11, Théorème 3.3] showed that *if η_u is uniquely integrable*, then it is tangent to a foliation without Reeb components with nullhomologous core, and moreover M carries a Reebless foliation. The unique integrability assumption is crucial for their argument and is unfortunately impractical. As a plausible generalization of this result, we propose:

Conjecture 1.3.5. *If (ξ_-, ξ_+) is a* tight *positive contact pair, then η_u can be $\mathcal{C}^0$-approximated by a $\mathcal{C}^0$-foliation without nullhomologuous Reeb components, and M carries a Reebless $\mathcal{C}^0$-foliation.*

This conjecture would have the following important consequences:

- A closed oriented 3-manifold carries a Reebless foliation if and only if it carries a positive pair of tight contact structures.
- An atoroidal closed oriented 3-manifold carries a taut foliation if and only if it carries a positive pair of tight contact structures.
- On an atoroidal rational homology sphere, the unstable plane field of a positive pair of tight contact structures can be $\mathcal{C}^0$-approximated by taut $\mathcal{C}^0$-foliations.
- An atoroidal rational homology sphere carries a pair of transverse taut foliations if and only if it carries a tight bicontact structure.

As noted in [CF11], under the hypotheses of Conjecture 1.3.5, Bennequin's inequality for transverse curves implies that *no closed curve transverse to η_u can bound an embedded disk in M*. Unfortunately, this property is *not* open in the $\mathcal{C}^0$ topology! If it were, it would be possible to approximate η_u by a foliation $\mathcal{F}$ with trivial meridional holonomy along its Reeb components. Following the strategy of the proof of [CF11, Théorème 3.3], it would then be possible to remove the Reeb components of $\mathcal{F}$ and get a Reebless foliation $\mathcal{F}'$. Of course, the homotopy class of $T\mathcal{F}'$ might be different than the one of η_u. It seems more credible that some foliations approximating η_u might have

Reeb components with nontrivial meridional holonomy, and some might not (compare this to the *phantom Reeb components* from [CKR19]). In Appendix 1.D, we prove a two dimensional result showing that closed transverse curves are the only obstructions to finding foliations by circle, see Proposition 1.D.1. This serves as evidence for the above conjecture and might be of independent interest.

Interestingly, Etnyre [Etn07] shows that a Reeb component with *nontrivial* meridional holonomy can be approximated by a contact pair (ξ_-, ξ_+) where one of $\xi_\pm$ is universally tight while the other one is overtwisted. Moreover, a Reeb component with *trivial* meridional holonomy can be approximated by a pair of *universally tight* contact structures:

Example 1.3.6. *On $\mathbb{R}^2 \times S^1$ with coordinates (r, θ, z), where (r, θ) denote standard polar coordinates on $\mathbb{R}^2$, we define*

$$\alpha \coloneqq (1 - r^2)dz + rdr, \tag{1.3.1}$$

$$\beta \coloneqq \frac{1}{2}r^2 d\theta. \tag{1.3.2}$$

These are both smooth, nowhere vanishing 1-forms. The kernel of α is tangent to a smooth foliation $\mathcal{F}$ on $\mathbb{R}^2 \times S^1$ for which $D^2 \times S^1$ is a Reeb component. *Moreover,*

$$\begin{aligned} \langle \alpha, \beta \rangle &\coloneqq \alpha \wedge d\beta + \beta \wedge d\alpha \\ &= rdr \wedge d\theta \wedge dz \\ &> 0, \end{aligned}$$

and

$$\alpha \wedge d\alpha = \beta \wedge d\beta = 0,$$

so $\mathcal{F}$ can be linearly deformed *into two contact structures with opposite signs*

$$\xi_{\pm}^{\tau} \coloneqq \ker\left(\pm\alpha + \tau\beta\right)$$

for any $\tau > 0$, and $(\xi_{-}^{\tau}, \xi_{+}^{\tau})$ is a positive contact pair. Both $\xi_{\pm}^{\tau}$ are universally tight *since their lifts to $\mathbb{R}^2 \times \mathbb{R}$ are contactomorphic to the standard positive and negative tight contact structures on $\mathbb{R}^3$, respectively. This can be seen by considering a (z, θ)-invariant foliation by planes $\mathcal{G}$ on $\mathbb{R}^3$ which coincides with (the lift of) $\mathcal{F}$ near $\{r = 0\}$ and is transverse to $\mathcal{F}$ elsewhere (see Figure 1.7). The characteristic foliation of $\xi_{\pm}^{\tau}$ along a leaf of $\mathcal{G}$ has a unique singularity at 0, is radial near $\{r = 0\}$, and is diffeomorphic to the standard radial foliation on $\mathbb{R}^2$. Therefore, after applying a (z, θ)-equivariant diffeomorphism of $\mathbb{R}^3$ which maps $\mathcal{G}$ to the standard foliation of $\mathbb{R}^3$ by horizontal planes, and maps the characteristic foliations of ξ_{+}^{τ} along the leaves of $\mathcal{G}$ to radial foliations on the horizontal planes, the contact structure ξ_{+}^{τ} becomes*

$$\widetilde{\xi}_{+}^{\tau} \coloneqq \ker\left(dz + f_{\tau}(r)d\theta\right),$$

where $f_{\tau} : \mathbb{R}_{r\geq 0} \to \mathbb{R}_{\geq 0}$ vanishes at $r = 0$ and is positive elsewhere. It easily follows that $\widetilde{\xi}_{+}^{\tau}$ is contactomorphic to the standard tight contact structure $\ker\left(dz + r^2 d\theta\right)$ *on $\mathbb{R}^3$. A similar argument applies to ξ_{-}^{τ}.*

This shows that on an open 3-manifold, some foliations with Reeb components can *be linearly deformed into a positive pair of universally tight contact structures.*

1.3.2 Transverse surgeries

We now show that strongly tight contact pairs are preserved under transverse surgeries of sufficiently large slopes. Theorem H will follow easily.

Proposition 1.3.7. *Let (ξ_{-}, ξ_{+}) be a strongly tight contact pair on M and $K \subset M$ be*

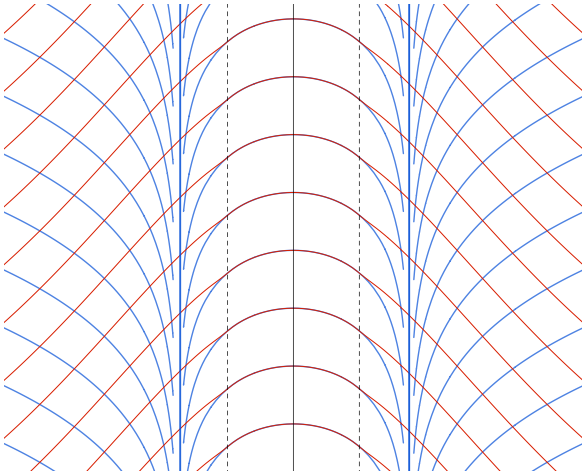


Figure 1.7: A θ-slice of the foliations $\mathcal{F}$ (in blue) and $\mathcal{G}$ (in red).

a knot positively transverse to $\xi_\pm$ and equipped with some framing. There exists $s_0 > 0$ such that for any rational number s satisfying $|s| \geq s_0$, $M_K(s)$ carries a strongly tight contact pair (ξ^s_-, ξ^s_+). Moreover, the image of K in $M_K(s)$ is positively transverse to $\xi^s_\pm$.

Proof. After possibly perturbing K to make it disjoint from Δ, there exists a small tubular neighborhood $\mathcal{N}_K \cong S^1 \times D^2$ of $K \cong S^1 \times \{0\}$ such that

- The vector field ∂_t is positively transverse to $\xi_\pm$ in $\mathcal{N}_K$, and the framing of K in the standard coordinates on $\mathcal{N}_K$ becomes trivial,
- For every $t \in S^1$, X is tangent to $\{t\} \times D^2$ and

$$\xi_\pm \cap T\big(\{t\} \times D^2\big) = \langle X \rangle.$$

We can apply [CF11, Proposition 3.5] to find a new positive pair (ξ'_-, ξ'_+) with associated vector field X' and singular set Δ' such that

- $\xi'_\pm$ is isotopic to $\xi_\pm$ rel $M \setminus \mathcal{N}_K$,
- The vector field ∂_t is positively transverse to $\xi'_\pm$ in $\mathcal{N}_K$,

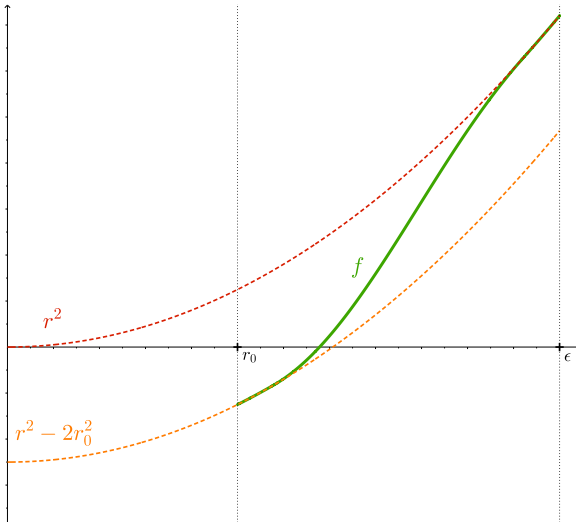


Figure 1.8: The function f.

- For every $t \in S^1$, the characteristic foliation of $\xi'_\pm$ on $\{t\} \times D^2$ is spanned by X' and has two singularities, a source and a saddle, in canceling position.

It follows that Δ' coincides with Δ outside of $\mathcal{N}_K$, and $\Delta' \cap \mathcal{N}_K$ has two parallel components, made of source and saddle singularities, respectively. We can further arrange that the new source component coincides with K and that there exists a small $\epsilon > 0$ such that in $S^1 \times D^2_\epsilon \subset S^1 \times D^2 \cong \mathcal{N}_K$, $\xi'_\pm$ is the kernel of the 1-form

$$\alpha^0_\pm \coloneqq \pm dt + r^2 d\theta,$$

where (r, θ) denotes the polar coordinates on D^2. Notice that (ξ'_-, ξ'_+) is still strongly tight.

We now modify $\xi^0_\pm \coloneqq \ker \alpha^0_\pm$ in $S^1 \times D^2_\epsilon$ relative to $T_\epsilon \coloneqq S^1 \times \partial D_\epsilon$ and perform an *admissible contact surgery* on the new contact structures simultaneously. To this end, we consider a sufficiently small radius $0 < r_0 < \epsilon$ such that $r_0^2 \in \mathbb{Q}$, and we write $r_0^2 = p/q$ for some coprime positive integers $p, q \in \mathbb{Z}_{>0}$. Along $T_{r_0} \coloneqq S^1 \times \partial D_{r_0}$, the characteristic foliation of ξ^0_+ has slope $s \coloneqq -1/r_0^2 = -q/p$. We modify ξ^0_- inside of $S^1 \times D_\epsilon$ so that its characteristic foliation also has slope s on T_{r_0}. We choose a smooth

function $f : [r_0, \epsilon] \to \mathbb{R}$ satisfying

- $f' > 0$,
- $f(r) = r^2$ near ϵ,
- $f(r) = r^2 - 2r_0^2$ near r_0.

See Figure 1.8. We then define

$$\widetilde{\alpha}_-^0 \coloneqq -dt + f(r)d\theta$$

inside of $S^1 \times ([r_0, \epsilon] \times S^1)$, which coincides with α_-^0 near T_ϵ. It is a negative contact form whose kernel "twists more" than the one of α_-^0. Moreover, its characteristic foliation on T_{r_0} coincides with the one of α_+^0. It is then easy to perform a *contact cut* (see [Ler01; BE13]) along T_{r_0} for both α_+^0 and $\widetilde{\alpha}_-^0$. This is possible since their moment maps for the S^1-action on a neighborhood of T_{r_0} given by a translation of slope s in the (θ, t) direction coincide. After this operation, we obtain two contact structures $\xi_\pm^s$ on the Dehn filling of $S^1 \times ([r_0, \epsilon] \times S^1)$ of slope s along T_{r_0}, which coincide with $\xi'_\pm$ near T_ϵ. After extending them by $\xi'_\pm$ outside of $S^1 \times D_\epsilon$, we obtain two contact structures, still denoted by $\xi_\pm^s$, on $M_K(s)$. If r_0 is sufficiently small, i.e., if $|s|$ is sufficiently large, then (ξ_-^s, ξ_+^s) is strongly tight in $S^1 \times D_\epsilon$. Since (ξ'_-, ξ'_+) is strongly tight in M, it is also strongly tight in $M \setminus (S^1 \times D_\epsilon)$ and (ξ_-^s, ξ_+^s) is strongly tight in $M_K(s)$. To perform a surgery of large positive slope, we can apply a similar argument to ξ'_-, or equivalently, reverse the orientation of M and consider the positive contact pair (ξ'_+, ξ'_-). □

Proof of Theorem H. Let $\mathcal{F}$ be a taut $\mathcal{C}^0$-foliation on M different from the standard foliation on $S^1 \times S^2$, and let K be a framed knot transverse to $\mathcal{F}$. By [KR17, Theorem 1.2], $T\mathcal{F}$ can be approximated by a strongly tight contact pair (ξ_-, ξ_+) such that K is positively transverse to $\xi_\pm$. By Proposition 1.3.2, for any $s \in \mathbb{Q}$ such that $|s|$ is

sufficiently large, $M_K(s)$ carries a strongly tight contact pair (ξ'_-, ξ'_+) such that K', the image of K after surgery, is positively transverse to $\xi'_\pm$. This implies that K' is transverse to η'_u, the unstable plane field of (ξ'_-, ξ'_+). Applying Theorem C, we readily obtain a taut $\mathcal{C}^0$-foliation $\mathcal{F}'$ in $M_K(s)$ such that K' is transverse to $\mathcal{F}'$. □

Proof of Corollary I. Note that $s \in \mathcal{S}_K$ if and only if $S^3_K(s)$ admits a taut $\mathcal{C}^0$-foliation transverse to K'_s, the image of K after surgery. If $s = p/q$, for coprime p and q (or $p = 0$ and $q = 1$ if $s = 0$), the choice of a framing for K'_s is equivalent to the choice of integers $u, v \in \mathbb{Z}$ such that $up - vq = 1$, which is also equivalent to the choice of a matrix in $\mathrm{SL}(2, \mathbb{Z})$ representing a mapping class of the torus $T^2 = S^1 \times S^1$ sending the meridian curve (the second component) to a curve of slope s. This matrix can be written as

$$A = \begin{pmatrix} u & q \\ v & p \end{pmatrix},$$

and sends a curve of slope $s' = p'/q'$ to a curve of slope

$$r = \frac{vq' + pp'}{uq' + qp'} = s \cdot \frac{s' + v/p}{s' + u/q}$$

if $s \neq 0$, and of slope

$$r = -\frac{1}{u + s'}$$

otherwise (in which case $v = -1$). In both cases, r converges to s as s' goes to $\pm\infty$, and every slope r sufficiently close to s can be written as such for some $s' \in \mathbb{Q}$ with $|s'|$ sufficiently large. Since a s-slope surgery on a nontrivial knot in S^3 never produces $S^1 \times S^2$ by Gabai's *Property R* theorem [Gab87] (and some elementary homological considerations for $s \neq 0$), the proof now easily follows from Theorem H. □

1.4 Liouville pairs

1.4.1 Elementary properties

We recall that a Liouville pair is a pair of contact forms (α_-, α_+) on M, negative a positive, respectively, such that the 1-form

$$\lambda = e^{-s}\alpha_- + e^s\alpha_+$$

is a positive Liouville form on $\mathbb{R}_s \times M$. In other terms, $\omega \coloneqq d\lambda$ is a non-degenerate 2-form on $\mathbb{R} \times M$, and $\omega \wedge \omega$ is a positive volume form.

If (α_-, α_+) is a Liouville pair and $\sigma : M \to \mathbb{R}$ is a smooth function, then $(e^{-\sigma}\alpha_-, e^{\sigma}\alpha_+)$ is also a Liouville pair. We say that these two Liouville pairs are **equivalent**. Up to equivalence, we can always assume that (α_-, α_+) is **balanced**, i.e.,

$$\alpha_+ \wedge d\alpha_+ = -\alpha_- \wedge d\alpha_-.$$

In that case, we denote by

$$\mathrm{dvol} \coloneqq \alpha_+ \wedge d\alpha_+$$

the associated volume form. Writing

$$d(\alpha_- \wedge \alpha_+) = f_0 \,\mathrm{dvol}$$

as before, where $f_0 \to \mathbb{R}$ is a smooth function, elementary calculations show that the Liouville condition is equivalent to

$$f_0 > -2. \tag{1.4.1}$$

The following lemma was already observed in [Mas23], see Chapter 3.

Lemma 1.4.1. *Let (α_-, α_+) be a Liouville pair on M. Then the underlying contact pair $(\xi_-, \xi_+) = (\ker \alpha_-, \ker \alpha_+)$ is positive, and $r_u > 0$ (see Proposition 1.1.7 for the notation).*

Proof. The fact that (ξ_-, ξ_+) is positive immediately follows from Lemma 1.1.5 and inequality (1.4.1). For α_u defined by (1.1.3), the expression (1.1.7) together with inequality (1.4.1) imply $r_u > 0$. □

If (ξ_-, ξ_+) is a contact pair and (α_-, α_+) is a Liouville pair such that $(\ker \alpha_-, \ker \alpha_+) = (\xi_-, \xi_+)$, we say that (ξ_-, ξ_+) **supports** (α_-, α_+). The following lemma shows that such a Liouville pair (if it exists) is unique up to homotopy.

Lemma 1.4.2. *Let (ξ_-, ξ_+) be a contact pair. The space of Liouville pairs (α_-, α_+) supported by (ξ_-, ξ_+) is either empty or contractible.*

Proof. Let (α_-, α_+) be a Liouville pair for (ξ_-, ξ_+). We can assume that it is balanced. It defines a volume form dvol and a vector field X. It is enough to show that the space of *balanced* Liouville pairs for (ξ_-, ξ_+) is contractible. Any such Liouville pair can be written as

$$\alpha'_\pm = e^\sigma \alpha_\pm$$

for some smooth function $\sigma : M \to \mathbb{R}$. Moreover, the Liouville condition is equivalent to

$$2X \cdot \sigma + f_0 > -2. \tag{1.4.2}$$

As a result, the space of balanced Liouville pairs for (ξ_-, ξ_+) is diffeomorphic to the space of functions σ satisfying (1.4.2). The latter is obviously convex, hence contractible. □

1.4.2 From hypertaut foliations to Liouville pairs

The following lemma is well-known for $\mathcal{C}^2$ foliations by the work of Sullivan [Sul76]. It is extended in [CC00] to $\mathcal{C}^0$-foliations; see [CC00, Lemma 10.2.6] and [CC00, Theorem 10.2.12].

Lemma 1.4.3. *Let $\mathcal{F}$ be a cooriented $\mathcal{C}^0$-foliation on M. The following are equivalent:*

1. *$\mathcal{F}$ carries no (nontrivial) invariant transverse measures,*

2. *There exists a smooth 1-form β such that*

$$d\beta_{|T\mathcal{F}} > 0.$$

We say that $\mathcal{F}$ is a **hypertaut** $\mathcal{C}^0$-foliation if it satisfies these properties. Equivalently, $\mathcal{F}$ is hypertaut if for some/any Riemannian metric g on M, there exists a smooth nonsingular *zero-flux* vector field v transverse to $\mathcal{F}$. Here, the *flux* of v through a closed oriented surface $\Sigma \subset M$ is defined as

$$\mathrm{Flux}(v)[\Sigma] \coloneqq \int_\Sigma \langle v, \nu_\Sigma \rangle \, \omega_\Sigma,$$

where ν_Σ is the positive normal vector to Σ and ω_Σ is the area form on Σ induced by g.

The next proposition is a version of [ET98, Corollary 3.2.5].

Proposition 1.4.4. *Let $\mathcal{F}$ be a hypertaut $\mathcal{C}^0$-foliation on M. Then $\mathcal{F}$ can be $\mathcal{C}^0$-approximated by contact structures ξ_- and ξ_+, negative and positive, respectively, such that there exists a Liouville structure on $[-1,1] \times M$ which is an exact filling of $\big(-M, \xi_-\big) \sqcup \big(M, \xi_+\big)$.*

Proof. Let β be a smooth 1-form on M such that $d\beta$ dominates $\mathcal{F}$. Since $\mathcal{F}$ has no closed (spherical) leaf, it can be approximated by negative and positive contact

structures ξ_- and ξ_+ by [KR17]. We can further assume that $d\beta_{|\xi_\pm} > 0$. Let α be a continuous 1-form defining $T\mathcal{F}$, and $\widetilde{\alpha}$ be a smoothing of α satisfying $\widetilde{\alpha} \wedge d\beta > 0$. For $\epsilon > 0$, we consider the the 1-form

$$\lambda \coloneqq \epsilon t\widetilde{\alpha} + \beta$$

on $[-1,1]_t \times M$. If ϵ is sufficiently small, then $\omega \coloneqq d\lambda$ is symplectic dominates $\xi_\pm$ on $\{\pm 1\} \times M$. We now appeal to [Eli91, Proposition 3.1] (see also [Eli04, Proposition 4.1]) to modify λ near $\{\pm 1\} \times M$ and obtain the desired Liouville filling. □

It is not clear to us how to recover a hypertaut foliation from a Liouville fillable positive contact pair. It is also unclear whether the Liouville structure obtained from a hypertaut $\mathcal{C}^0$-foliation by the previous proposition is unique up to homotopy (the contact approximations of $\mathcal{F}$ might not be unique). However, for $\mathcal{C}^2$ hypertaut foliations, it is possible to construct a more specific type of Liouville structure, namely, one induced by a Liouville pair. This construction relies on the following proposition which is the main result of Jonathan Zung's article [Zun24]:

Proposition 1.4.5 ([Zun24])**.** *If $\mathcal{F}$ is a $\mathcal{C}^2$ hypertaut foliation, then there exist 1-forms α and β such that*

$$\ker \alpha = T\mathcal{F}, \qquad \alpha \wedge d\beta > 0, \qquad \beta \wedge d\alpha \geq 0.$$

The first inequality simply means that $d\beta$ is a dominating 2-form for $\mathcal{F}$.

Corollary 1.4.6. *If $\mathcal{F}$ is a hypertaut $\mathcal{C}^2$ foliation, then there exists a Liouville pair (α_-, α_+) on M such that the contact structures $\xi_\pm = \ker \alpha_\pm$ are $\mathcal{C}^0$-close to $T\mathcal{F}$.*

Proof. For $\epsilon > 0$, we define

$$\alpha_\pm \coloneqq \epsilon\beta \pm \alpha, \tag{1.4.3}$$

where α and β are as in Proposition 1.4.5. Following [ET98], we write

$$\langle \alpha, \beta \rangle \coloneqq \alpha \wedge d\beta + \beta \wedge d\alpha.$$

One easily computes

$$\begin{aligned} \alpha_+ \wedge d\alpha_+ &= \epsilon \langle \alpha, \beta \rangle + O(\epsilon^2), \\ \alpha_- \wedge d\alpha_- &= -\epsilon \langle \alpha, \beta \rangle + O(\epsilon^2), \\ d(\alpha_- \wedge \alpha_+) &= -2\epsilon \, d(\alpha \wedge \beta). \end{aligned}$$

For ϵ small enough, $\alpha_\pm$ are contact forms with opposite orientations. Moreover, (α_-, α_+) is a Liouville pair if and only if

$$-d(\alpha_- \wedge \alpha_+) < 2\sqrt{-(\alpha_+ \wedge d\alpha_+) \cdot (\alpha_- \wedge d\alpha_-)},$$

which is equivalent to

$$d(\alpha \wedge \beta) < \langle \alpha, \beta \rangle + O(\epsilon).$$

This condition is satisfied for ϵ small enough, since

$$d(\alpha \wedge \beta) = \beta \wedge d\alpha - \alpha \wedge d\beta < \beta \wedge d\alpha + \alpha \wedge d\beta = \langle \alpha, \beta \rangle.$$

The 1-forms $\alpha_\pm$ might only be $\mathcal{C}^1$, but they can easily be smoothed to yield a smooth Liouville pair. □

1.4.3 Proof of Theorem E

Let (ξ_-, ξ_+) be a contact pair on M supporting a Liouville pair (α_-, α_+). By Proposition 1.4.7 below, we can assume that the Reeb vector field of α_+ is positively transverse to η_u, i.e., $d\alpha_{+|\eta_u} > 0$. Since $\xi_\pm$ are (strongly) semi-fillable, they are both tight. By

Theorem B, η_u can be $\mathcal{C}^0$-approximated by a $\mathcal{C}^0$-foliation $\mathcal{F}$, and we can further arrange that $d\alpha_{+|T\mathcal{F}} > 0$. By Lemma 1.4.3, $\mathcal{F}$ is hypertaut.

Proposition 1.4.7. *Let (α_-, α_+) be a Liouville pair on M, with underlying positive contact pair (ξ_-, ξ_+) and unstable plane field η_u. Then (α_-, α_+) is equivalent to a Liouville pair (α'_-, α'_+) whose Reeb vector fields $R'_\pm$ are both transverse (with opposite signs) to η_u.*

Proof. As usual, we assume that (α_-, α_+) is balanced, and we use the notations of the proof of Proposition 1.1.7. By (1.1.3) and (1.1.4), we can write

$$\alpha_- = \frac{1}{2\cosh(\sigma_u - \sigma_s)} \left(e^{\sigma_s}\alpha_u + e^{\sigma_u}\alpha_s\right), \tag{1.4.4}$$

$$\alpha_+ = \frac{1}{2\cosh(\sigma_u - \sigma_s)} \left(e^{-\sigma_s}\alpha_u - e^{-\sigma_u}\alpha_s\right). \tag{1.4.5}$$

Let $\sigma : M \to \mathbb{R}$ be some smooth function and $(\alpha'_-, \alpha'_+) \coloneqq (e^{-\sigma}\alpha_-, e^{\sigma}\alpha_+)$. We compute $\alpha_s\big(R'_\pm\big)$ in terms of σ. From the identities

$$\alpha'_\pm\big(R'_\pm\big) = 1, \tag{1.4.6}$$

$$\mathcal{L}_X\alpha'_\pm\big(R'_\pm\big) = 0, \tag{1.4.7}$$

we get (after some slightly tedious computations)

$$\alpha_s\big(R'_+\big) = -\frac{\overbrace{2e^{\sigma_u-\sigma}\cosh(\sigma_u-\sigma_s)}^{>0}}{X\cdot(\sigma_u-\sigma_s)+r_u-r_s}\left(X\cdot(\sigma-\sigma_s) - X\cdot\ln\cosh(\sigma_u-\sigma_s)+r_u\right), \tag{1.4.8}$$

$$\alpha_s\big(R'_-\big) = -\frac{2e^{\sigma-\sigma_u}\cosh(\sigma_u-\sigma_s)}{\underbrace{-X\cdot(\sigma_u-\sigma_s)+r_u-r_s}_{>0}}\left(X\cdot(\sigma-\sigma_s) + X\cdot\ln\cosh(\sigma_u-\sigma_s)-r_u\right). \tag{1.4.9}$$

From (1.1.5), (1.1.6), (1.1.7), and (1.1.8), one easily computes

$$\pm X \cdot \big(\sigma_u - \sigma_s\big) + r_u - r_s = e^{\pm 2\sigma_u} + e^{\pm 2\sigma_s} > 0.$$

Furthermore, we claim that

$$-X \cdot \ln\cosh(\sigma_u - \sigma_s) + r_u > 0.$$

Indeed, by (1.1.5) and (1.1.6),

$$\begin{aligned} X \cdot \ln\cosh(\sigma_u - \sigma_s) &= \tanh(\sigma_u - \sigma_s)\, X \cdot (\sigma_u - \sigma_s) \\ &= \tanh(\sigma_u - \sigma_s)\big(\sinh(2\sigma_u) + \sinh(2\sigma_s)\big) \\ &= 2\tanh(\sigma_u - \sigma_s)\big(\sinh(\sigma_u + \sigma_s)\cosh(\sigma_u - \sigma_s)\big) \\ &= 2\sinh(\sigma_u - \sigma_s)\sinh(\sigma_u + \sigma_s) \\ &= \cosh(2\sigma_u) - \cosh(2\sigma_s). \end{aligned}$$

Here, we used the classical formulae

$$\begin{aligned} \sinh(x) + \sinh(y) &= 2\sinh\Big(\frac{x+y}{2}\Big)\cosh\Big(\frac{x-y}{2}\Big) \\ 2\sinh(x)\sinh(y) &= \cosh(x+y) - \cosh(x-y). \end{aligned}$$

Therefore, using (1.1.7),

$$\begin{aligned} -X \cdot \ln\cosh(\sigma_u - \sigma_s) + r_u &= \cosh(2\sigma_s) - \cosh(2\sigma_u) + \cosh(2\sigma_u) + f_0/2 \\ &= \cosh(2\sigma_s) + f_0/2 \\ &> 0, \end{aligned}$$

since $f_0 > -2$. Let us fix some $\epsilon > 0$ such that

$$-X \cdot \ln\cosh(\sigma_u - \sigma_s) + r_u > \epsilon.$$

By Lemma 1.C.1, we can choose σ so that

$$|X \cdot (\sigma - \sigma_s)|_{\mathcal{C}^0} < \epsilon.$$

This ensures

$$\alpha_s\big(R'_+\big) < 0,$$
$$\alpha_s\big(R'_-\big) > 0,$$

so $R'_\pm$ are both transverse to $\eta_u = \ker \alpha_s$ as desired. Note that we do not need σ to be $\mathcal{C}^0$-close to σ_s. □

1.4.4 The skeleton

For a general Liouville manifold (V, λ) with Liouville vector field Z and Liouville flow (ϕ_Z^t), the **skeleton** is defined as

$$\mathfrak{skel}(V, \lambda) \coloneqq \bigcup_{\substack{K \subset V \\ \text{compact}}} \bigcap_{t>0} \phi_Z^{-t}(K).$$

In this section, we determine the geometric structure of the skeleton of a Liouville pair, defined as the skeleton of the associated Liouville structure. This result applies in particular to *Anosov Liouville structures*, defined in [Mas23] (see Chapter 3).

Proposition 1.4.8. *Let (α_-, α_+) be a Liouville pair on M. The skeleton of the*

associated Liouville structure

$$\lambda = e^{-s}\alpha_- + e^s\alpha_+$$

on $V = \mathbb{R}_s \times M$ *is of the form*

$$\mathfrak{skel}(V, \lambda) = \mathrm{graph}(\sigma) = \big\{(\sigma(p), p) \mid p \in M\big\}$$

for some continuous function $\sigma : M \to \mathbb{R}$*. Moreover,* σ *is continuously differentiable along the vector field* X *associated with the contact pair* $(\xi_-, \xi_+) = (\ker \alpha_-, \ker \alpha_+)$*, and the restriction of the Liouville flow to the skeleton is orbit equivalent to the flow of* X*.*[6]

Proof. First of all, we can assume that (α_-, α_+) is balanced, after applying a graphical diffeomorphism $\varphi : (s, p) \mapsto (s + \sigma_0(p), p)$ for a suitable smooth function $\sigma_0 : M \to \mathbb{R}$. Then, with the previous notations, the Liouville vector field for λ is given by

$$Z(s, p) = \frac{1}{\cosh(2s) + f_0(p)/2} \left(\big(\sinh(2s) + g_0(p)/2 \big)\, \partial_s + X(p) \right).$$

Writing

$$F(s, p) = \sinh(2s) + g_0(p)/2,$$
$$\widetilde{Z} = F\partial_s + X,$$

the vector fields Z and $\widetilde{Z}$ share the same skeleton. The flow line of $\widetilde{Z}$ starting at (s_0, p_0) is given by

$$\phi^t_{\widetilde{Z}}(s_0, p_0) = \big(\sigma(t), \phi^t_X(p_0)\big),$$

[6]Two flows are orbit equivalent if there exists a homeomorphism sending the oriented flow lines of the first to oriented flow lines of the second, possibly with a different parametrization.

where σ satisfies the ODE

$$E_{(s_0,p_0)} : \begin{cases} \dot{\sigma}(t) = F\big(\sigma(t), \phi_X^t(p_0)\big), \\ \sigma(0) = s_0. \end{cases}$$

By definition, $(s_0, p_0) \in \mathfrak{skel} = \mathfrak{skel}(V, \lambda)$ if and only if the corresponding maximal solution to $E_{(s_0,p_0)}$ is bounded. Lemma 1.A.1 implies that for every $p_0 \in M$, there exists a unique $\sigma(p_0) \in \mathbb{R}$ such that $(\sigma(p_0), p_0) \in \mathfrak{skel}$; with the notation of the proof of Proposition 1.1.7, $\sigma = \sigma_u$, which is continuous and continuously differentiable along X. Therefore, under the balancing condition, $\mathfrak{skel} = \mathrm{graph}(\sigma_u)$. Finally, the restriction of the Liouville flow to the skeleton is clearly orbit equivalent to the flow of X because of the form of $\widetilde{Z}$. □

Remark 1.4.9. *We see from the proof that the regularity of the skeleton, i.e., the regularity of the function σ, is the same as the regularity of the function σ_u. If X is Anosov, then it is as regular as the weak-stable bundle of X, which is always $\mathcal{C}^1$ but not necessarily $\mathcal{C}^2$.*

Proposition 1.4.8 shows in particular that the skeleton of the Liouville structure associated to a Liouville pair is a codimension one topological submanifold homeomorphic to M. It would be interesting to investigate the converse:

Question 1.4.10. *Let λ_0 be a Liouville structure on $V = M \times \mathbb{R}$. If $\mathfrak{skel}(V, \lambda_0) \subset V$ is a codimension one compact embedded topological submanifold, does there exist a Liouville pair (α_-, α_+) on M with associated Liouville structure λ on V such that (V, λ_0) and (V, λ) are exact symplectomorphic?*

1.4.5 Semi-Anosov flows and unique integrability

In this section, we define and study a certain type of 3-dimensional flows whose behavior interpolates between projectively Anosov and Anosov flows.

Definition 1.4.11. *Let $\Phi = (\phi_t)_t$ be a smooth flow on M generated by a vector field X, and $\Lambda \subset M$ be a Φ-invariant compact subset disjoint from the singular set of X. A **semi-Anosov splitting** for Φ on Λ is a continuous, Φ-invariant splitting*

$$T_\Lambda M = E^{ws} \oplus E^u$$

along Λ, where E^u is 1-dimensional, E^{ws} is 2-dimensional, $X \in E^{ws}$, and for some/any metric g on M, there exist constants $C > 0$ and $0 < a < b$ such that

- *For every $v \in E^u$ and $t \geq 0$,*

$$\|d\phi_t(v)\| \geq Ce^{bt}\|v\|,$$

- *For every $v \in E^{ws}$ and $t \geq 0$,*

$$\|d\phi_t(v)\| \leq Ce^{at}\|v\|.$$

*We call E^u the **(strong-)unstable bundle** of Φ, and E^{ws} the **weak-stable bundle** of Φ. We also call $E^{wu} \coloneqq \langle X \rangle \oplus E^u$ the **weak-unstable bundle** of Φ. We call Φ **semi-Anosov** if it is nonsingular and admits a semi-Anosov splitting on M*

It is easy to see that semi-Anosov flows are projectively Anosov (and without contracting closed orbits), but not necessarily Anosov. This raises the following

Question 1.4.12. *Is there a 3-manifold M carrying a semi-Anosov flow, but no Anosov flow?*

The next proposition is very similar to the Anosov case, and the proof is essentially the same.

Proposition 1.4.13. *If Φ admits a semi-Anosov splitting on Λ, then E^u is uniquely integrable and its leaves are smooth.*

Proof. It follows from a version of the Hadamard-Perron theorem [FH19, Theorem B.5.2] which implies a version of the Unstable Manifold theorem [FH19, Theorem 6.1.1]. □

We now establish a connection between Liouville pairs and semi-Anosov flows.

Proposition 1.4.14. *Let (α_-, α_+) be a Liouville pair with associated vector field X, and let $\Lambda \subset M$ be a compact subset disjoint from Δ and invariant by X. Then the flow of X admits a semi-Anosov splitting along Λ.*

Proof. Since $\Delta \cap \Lambda = \varnothing$, η_s and η_u are defined on Λ, and there exists continuous, non-vanishing 1-forms α_s, α_u such that $\ker \alpha_{u/s} = \eta_{s/u}$ and

$$\mathcal{L}_X \alpha_{s/u} = r_{s/u} \alpha_{s/u},$$

where $r_s < r_u$ and $r_u > 0$. We let $E^{ws} \coloneqq \eta_s$ and we want to find a suitable 1-dimensional, Φ-invariant subbundle $E^u \subset \eta_u$. On $N_X \coloneqq TM/\langle X \rangle$ restricted to Λ, we have a dominated splitting

$$N_X = \overline{\eta}_s \oplus \overline{\eta}_u,$$

where $\overline{\eta}_{s/u} \coloneqq \eta_{s/u}/\langle X \rangle$. Dualizing (α_s, α_u), we get continuous sections $\overline{e}_{s/u}$ of $\overline{\eta}_{s/u}$ and we define a (continuous) metric $\overline{g}$ on N_X for which $(\overline{e}_s, \overline{e}_u)$ is orthonormal. Moreover, there exist constants $C > 0$ and $0 < a < b$ such that

- For every $v \in \overline{\eta}_u$ and $t \in \mathbb{R}$,

$$\|d\phi_t(v)\| \geq Ce^{bt}\|v\|,$$

- For every $v \in \overline{\eta}_s$ and $t \in \mathbb{R}$,

$$\|d\phi_t(v)\| \leq Ce^{at}\|v\|.$$

The proof of [Doe87, Proposition 1.1] readily implies that there exists a Φ-invariant 1-dimensional subbundle $E^u \subset \eta_u$ defined along Λ satisfying the desired conditions. $\square$

We conclude with the following result, which is essentially the only instance in which we can ensure that η is a uniquely integrable plane field:

Corollary 1.4.15. *Let (ξ_-, ξ_+) be a regular contact pair on M supporting a Liouville pair (α_-, α_+). If Δ has no saddle singularities, then η_u is uniquely integrable, and its integral foliation is a hypertaut $\mathcal{C}^0$-foliation.*

Proof. If $\Delta = \varnothing$, combining Proposition 1.4.13, Proposition 1.4.14 and Proposition 1.4.7, η_u is uniquely integrable, its integral foliation is a $\mathcal{C}^0$-foliation, and there exists a smooth contact form α'_+ whose Reeb vector field is transverse to $\mathcal{F}$. Equivalently, $d\alpha'_+$ is positive on the leaves of $\mathcal{F}$, which implies hypertautness.

If $\Delta \neq \varnothing$ and $\Delta = \Delta_{\mathrm{so}}$, we consider a small tubular neighborhood U of Δ such that X is transverse to ∂U and outward pointing. We then define

$$\mathcal{V} \coloneqq \bigcup_{t>0} \phi_t(U), \qquad \Lambda \coloneqq M \setminus \mathcal{V},$$

so that Λ is compact and invariant under the flow of X. By the proof of Proposition 1.1.11, η_u is uniquely integrable on $\mathcal{V}$, since it is uniquely integrable near Δ. Moreover, the reasoning of the previous paragraph shows that η_u is uniquely integrable on Λ as well. $\square$

Unfortunately the hypothesis of this corollary are quite restrictive. We ask:

Question 1.4.16. *Can the condition $\Delta_{\mathrm{sa}} = \varnothing$ in Corollary 1.4.15 be removed? To show this, it would be sufficient to find a uniquely integrable vector field X_u defined away from Δ such that $\eta_u = \mathrm{span}\{X, X_u\}$.*

Appendix

1.A Key ODE lemma

Let $D \subset \mathbb{R}^N$ be a compact subset of $\mathbb{R}^N$ for some integer N. We consider a smooth function

$$\begin{aligned} F: \quad D \times \mathbb{R} \times \mathbb{R} &\longrightarrow \mathbb{R} \\ (x, y, t) &\longmapsto F(x, y, t) = F_x(y, t) \end{aligned}$$

satisfying

(C1) There exists $\epsilon > 0$ such that

$$\partial_y F_x \geq \epsilon,$$

(C2) There exists $C > 0$ such that for every $(x, y, t) \in D \times \mathbb{R} \times \mathbb{R}$,

$$\big|F_x(y, t) - F_x(y, 0)\big| \leq C.$$

We consider the following family of ODEs parametrized by $x \in D$:

$$(E_x): \begin{cases} \dot{y}(t) = F_x\big(y(t), t\big), \\ y(0) = y_0. \end{cases}$$

In Chapter 1, we repeatedly make use of the following elementary lemma.

Lemma 1.A.1. *For every $x \in D$, there exists a unique $y_0(x) \in \mathbb{R}$ such that the*

maximal solution to (E_x) *with initial value* $y_0(x)$ *is defined on* $\mathbb{R}$ *and is bounded. Moreover, the map*

$$x \in D \mapsto y_0(x) \in \mathbb{R}$$

is continuous.

Proof. We first show the existence and uniqueness of $y_0 = y_0(x)$ for a fixed $x \in D$. We will write $F(y,t) = F_x(y,t)$ for simplicity. The solutions to (E_x) satisfy the two key properties:

(P1) If $y_0 < \widetilde{y}_0$, and $y, \widetilde{y} : [0,T] \to \mathbb{R}$ are solutions to (E_x) with initial values y_0 and $\widetilde{y}_0$, respectively, then

$$\forall t \in [0,T], \quad \widetilde{y}(t) - y(t) \geq (\widetilde{y}_0 - y_0)\, e^{\epsilon t}. \tag{1.A.1}$$

(P2) There exists $A > 0$ such that if $y : [-T,T] \longrightarrow \mathbb{R}$ is a solution of (E_x), then

$$y(t) > A \Longrightarrow \dot{y}(t) > 1,$$
$$y(t) < -A \Longrightarrow \dot{y}(t) < -1.$$

Property (P1) easily follows from condition (C1). Under the hypothesis of (P1), we readily have

$$\forall t \in [0,T], \quad \widetilde{y}(t) > y(t).$$

Then, writing $z(t) \coloneqq \widetilde{y}(t) - y(t)$, we have

$$\begin{aligned}\dot{z}(t) &= F(\widetilde{y}(t), t) - F(y(t), t) \\ &= \int_{y(t)}^{\widetilde{y}(t)} \partial_y F(y, t)\, dy \\ &\geq \epsilon(\widetilde{y}(t) - y(t)) \\ &= \epsilon z(t),\end{aligned}$$

and (1.A.1) follows.

On the other hand, property (P2) easily follows from conditions (C1) and (C2). By condition (C1), $F(\,\cdot\,, 0) : \mathbb{R} \to \mathbb{R}$ is a strictly increasing diffeomorphism and there exists $A > 0$ such that

$$\begin{aligned}\forall y > A, \quad F(y, 0) > C + 1, \\ \forall y < -A, \quad F(y, 0) < -C - 1.\end{aligned}$$

Then, if $y(t) > A$,

$$\begin{aligned}\dot{y}(t) &= F(y, t) - F(y, 0) + F(y, 0) \\ &\geq -C + F(y, 0) \\ &> -C + C + 1 \\ &= 1.\end{aligned}$$

The other implication holds similarly. Here, we can further assume that A is independent of x by simply considering

$$A \coloneqq \epsilon^{-1} \left(1 + C + \sup_{x \in D} |F_x(0, 0)|\right).$$

Notice that a solution y of (E_x) is defined for all negative times, and $|y|$ is bounded by $\max(A, |y_0|)$ on $\mathbb{R}_{\leq 0}$. However, such a solution might be unbounded on $\mathbb{R}_{\geq 0}$, and might blow up in finite time. We define

- $I_+ = I_+(x) \subset \mathbb{R}$ the set of $y_0 \in \mathbb{R}$ such that the solution y to (E_x) with initial value y_0 is *not* bounded from above on $\mathbb{R}_{\geq 0}$,
- $I_- = I_-(x) \subset \mathbb{R}$ the set of $y_0 \in \mathbb{R}$ such that the solution y to (E_x) with initial value y_0 is *not* bounded from below on $\mathbb{R}_{\geq 0}$.
- $I_0 = I_0(x) \subset \mathbb{R}$ the set of $y_0 \in \mathbb{R}$ such that the solution y to (E_x) with initial value y_0 is bounded on $\mathbb{R}_{\geq 0}$.

By definition, we have

$$\mathbb{R} = I_- \sqcup I_0 \sqcup I_+.$$

By property (P2), if $y_0 \in I_\pm$ then the corresponding solution y diverges to $\pm\infty$, possibly in finite time. Moreover, I_+ is of the form $[a_+, +\infty)$ or $(a_+, +\infty)$, I_- is of the form $(-\infty, a_-]$ of $(-\infty, a_-)$, and property (C1) implies that I_0 contains at most one point. It is therefore enough to show that $I_\pm$ are open. Let $y_0 \in I_+$, and let y be the corresponding maximal solution, defined on $(-\infty, T)$ for some $0 < T \leq +\infty$. By definition of I_+, there exists $0 < t_0 < T$ such that $y(t_0) > A$. By the continuous dependence of solutions to an ODE with respect to the initial value, there exists $\delta > 0$ such that for every $\widetilde{y}_0 \in (y_0 - \delta, y_0 + \delta)$, the solution $\widetilde{y}$ to (E_x) with initial value $\widetilde{y}_0$ also exists up to time at least t_0, and $\widetilde{y}(t_0) > A$. Therefore, by property (P2), $\widetilde{y}$ diverges to $+\infty$ (possibly in finite time), and $\widetilde{y}_0 \in I_+$. It follows that I_+ is open, and I_- is open for similar reasons.

We now have a well-defined map $x \in D \mapsto y_0(x)$, and we are left to show that it is continuous. Let $x \in D$. If $\widetilde{y}_0 > y_0(x)$, then $\widetilde{y}_0 \in I_+(x)$ and the maximal solution $\widetilde{y}$ to (E_x) with initial value $\widetilde{y}_0$ diverges to $+\infty$, possibly in finite time. Hence, there exists

$t_0 > 0$ such that $\widetilde{y}(t_0)$ is defined and $\widetilde{y}(t_0) > A + 1$. By the continuous dependence of solutions to a family of ODEs with respect to a parameter (the dependence on the parameter x is smooth), there exists an open neighborhood U of x in D such that for every $\widetilde{x} \in U$, every maximal solution $\widetilde{y}$ to the ODE $(E_{\widetilde{x}})$ with initial value $\widetilde{y}_0$ is defined up to time at least t_0 and satisfies $\widetilde{y}(t_0) > A$. Therefore, by property (P2), $\widetilde{y}$ diverges to $+\infty$ and $\widetilde{y}_0 \in I_+(\widetilde{x})$, so $\widetilde{y}_0 > y_0(\widetilde{x})$. To summarize, we have shown that for every $x \in D$ and every $\widetilde{y}_0 > y_0(x)$, there exists an open neighborhood U of x in D such that for every $\widetilde{x} \in U$, $\widetilde{y}_0 > y_0(\widetilde{x})$. This means that y_0 is upper semicontinuous, and it is lower semicontinuous for similar reasons. □

The same methods also show that the map $x \in D \mapsto y_0(x)$ depends continuously on F. More precisely, let F as before, and let $(F_n)_{n\geq 1}$ be a sequence of smooth functions $F_n : D \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfying the following:

- For every $n \geq 1$, F_n satisfies conditions (C1) and (C2) for the same constants ϵ, C as F,
- $(F_n)_{n\geq 1}$ converges to F in $\mathcal{C}^\infty_{\mathrm{loc}}$.

We denote by (E^n_x) the family of ODEs corresponding to F_n.

Lemma 1.A.2. *With the above notations, let $y^n_0 : x \in D \mapsto \mathbb{R}$ denote the continuous map given by Lemma 1.A.1 for F_n, $n \geq 1$. Then the sequence $(y^n_0)_{n\geq 1}$ converges uniformly to y_0 in the $\mathcal{C}^0$ topology.*

Proof. For $n \geq 1$, we denote by $y_n(x,t)$ the unique bounded maximal solution of (E^n_x), so that $y^n_0(x) = y_n(x,0)$. By the proof of Lemma 1.A.1, there exists a constant $A > 0$, depending on ϵ and C but *not on x nor n*, such that

$$\forall n \geq 1,\ \forall t \in \mathbb{R}, \qquad |y_n(x,t)| \leq A.$$

Let us assume by contradiction that $(y_0^n)_{n\geq 1}$ does not converge to y_0. Up to passing to a subsequence, we can assume that there exists a sequence $(x_n)_{n\geq 1}$, $x_n \in D$, such that

- $(x_n)_{n\geq 1}$ converges to some $x_\infty \in D$,
- $\left(y_0^n(x_n)\right)_{n\geq 1}$ converges to some $y_\infty \in \mathbb{R}$,
- $y_\infty \neq y_0(x_\infty)$.

Without loss of generality, we assume that $y_\infty > y_0(x_\infty)$. By the proof of Lemma 1.A.1, the solution $\widetilde{y} = \widetilde{y}(t)$ of (E_{x_∞}) starting at y_∞ diverges to $+\infty$, possibly in finite time, so there exists $t_0 > 0$ such that $\widetilde{y}(t_0) = A + 1$. Since F_n converges to F in $\mathcal{C}^\infty_{\mathrm{loc}}$, the continuous dependence of solutions of a family of ODEs with respect to parameters implies that for $n \geq 1$ sufficiently large, $y_n(x_n, t_0) > A$, a contradiction. □

1.B Taut plane fields

In this appendix, M still denotes a closed, oriented 3-manifold. We extend the definition of tautness to arbitrary continuous cooriented plane fields on M and show some equivalent characterizations. This generalization is motivated by the following two observations:

1. There exist several *non-equivalent* notions of tautness for $\mathcal{C}^0$-foliations; see [CKR19]. The most robust one, *everywhere tautness*, can be phrased uniquely in terms of the tangent plane field of the foliation, i.e., it does not refer to the leaves.
2. For the purpose of translating properties of foliations into properties of contact structures and vice versa, it is convenient work directly at the level of plane fields.

Definition 1.B.1. *A cooriented continuous plane field η on M is* ***(everywhere) taut*** *if for every $p \in M$, there exists a smooth closed curve transverse to η and passing through p.*

Definition 1.B.2. *A* ***dead-end component*** *for η is a $\mathcal{C}^1$, codimension-0 submanifold with boundary $N \subset M$ such that ∂N is tangent to η and the co-orientation of η points inwards along ∂N.*

The next proposition is well-known if η is (uniquely) integrable, and most of the proof extends to the general case.

Proposition 1.B.3. *Let η be a cooriented continuous plane field on M. The following are equivalent:*

1. *η is taut,*
2. *For all points $p, q \in M$, there exists a smooth curve from p to q positively transverse to η,*
3. *There exists a (smooth, or $\mathcal{C}^1$) closed 2-form ω such that $\omega_{|\eta} > 0$,*
4. *There exists a (smooth, or $\mathcal{C}^1$) volume preserving vector field transverse to η,*
5. *There exists a continuous Riemannian metric on M and a (smooth, or $\mathcal{C}^1$) closed 2-form ω which calibrates η,*
6. *There are no dead-end components for η.*

In particular, a surface tangent to η is a stable minimal surface for a Riemannian metric as in 5.

Proof. The implications

$$2 \Longrightarrow 1 \Longrightarrow 3 \Longleftrightarrow 4 \Longleftrightarrow 5 \Longrightarrow 6$$

are standard and follow the same proofs as in the (uniquely) integrable case. We show *6* $\Longrightarrow$ *2*

Let $p_0 \in M$ and denote by $A_+ \coloneqq A_+(p_0)$ the set of points $q \in M$ such that there exists a smooth curve from p_0 to q positively transverse to η. Let us assume that $A_+ \neq M$. We show that $N \coloneqq \overline{A_+}$ is a dead-end component for η.

Since η is continuous, every point $q \in M$ has a neighborhood $U \cong D^2 \times (-1,1)$ with smooth local coordinates (x,y,z) in which η is given by

$$\eta = \ker\left(dz + f dx + g dy\right),$$

for continuous functions $f, g : D^2 \times (-1,1) \to \mathbb{R}$ with $f(0) = g(0) = 0$ and $|f|, |g| \leq 1/10$. We call such a neighborhood U a *standard neighborhood of* q. Moreover, if $q \in A_+$ and $\gamma : [0,1] \to M$ is a smooth curve from p to q transverse to η, we can find a standard neighborhood of q in which the image of γ is contained in $\{x = y = 0,\ z \leq 0\}$. It follows that the upper open cone in U centered at $(0,0,-1/2)$ defined by

$$z > \sqrt{x^2 + y^2} - 1/2$$

is contained in A_+ and contains $q \cong (0,0,0)$. This shows that A_+ is open.

Since $A_+ \neq M$ by assumption and M is connected, $\partial N = \partial A_+ = \overline{A_+} \setminus A_+ \neq \varnothing$. Let $q \in \partial N$ and U be a standard neighborhood of q as above. Since ∂_z is positively transverse to η in this neighborhood, there exists a map $h : D^2 \to (-1,1)$ such that

$$N \cap U = \{h(x,y) \leq z\}.$$

This map can be explicitly defined as

$$h(x,y) \coloneqq \inf\left\{z \in (-1,1)\ :\ (x,y,z) \in A_+ \cap U\right\}$$

and is lower semi-continuous. For $p \in U$, we consider the upper and lower half-cones

$$C_{\pm}(p) \coloneqq p + \left\{ \sqrt{x^2 + y^2} \leq \pm z \right\}.$$

We have the following elementary properties:

- If $p \in A_+ \cap U$ then $C_+(p) \subset A_+$,
- If $p \in U \setminus A_+$ then $C_-(p) \cap A_+ = \varnothing$.

These imply that if $p \in \partial N$ then $C_-(p) \cap N = \varnothing$ and $\operatorname{int} C_+(p) \subset N$. Therefore, h is 1-Lipschitz, hence differentiable almost everywhere. We now show that h is $\mathcal{C}^1$ and tangent to η. Without loss of generality, we can assume that h is differentiable at $(0,0)$. Note $\eta(0,0,0) = \operatorname{span}\{\partial_x, \partial_y\}$, $h(0,0) = 0$, and $|\partial_x h(0,0)|, |\partial_y h(0,0)| \leq 1$. These properties are satisfied *in any standard neighborhood of* q, which allows us to change the z-direction (after possibly shrinking U). Therefore, $\partial_x h(0,0) = \partial_y h(0,0) = 0$ and (the graph of) h is tangent to η at $(0,0)$. Similarly, h is tangent to η at every point where it is differentiable. Since η is continuous and h is Lipschitz, h is $\mathcal{C}^1$ and tangent to η everywhere.

As a result, N is a dead-end component for η. □

The fourth characterization of tautness immediately implies that it is an open condition in the $\mathcal{C}^0$ topology. Moreover, the sixth characterization generalizes the well-known fact that a $\mathcal{C}^1$-foliation without spherical and toroidal leaves is taut:

Corollary 1.B.4. *If η has no closed integral surfaces which are spheres or tori, then it is taut.*

Proof. Let us assume that η is not taut and let N be a dead-end component. We write

$$N = \bigsqcup_{1 \leq i \leq k} \Sigma_i,$$

where the $\Sigma_i's$ are closed connected orientable surfaces tangent to η. There exists a smooth vector field positively tansverse to η, which restricts to a nowhere vanishing vector field on N transverse to ∂N and inward pointing. By [Pug68] (see also [Goo75, Proposition 1.2]),

$$\sum_{i=1}^{k} \chi(\Sigma_i) = 0,$$

which is impossible if all of the Σ_i's have genus bigger or equal than 2. □

1.C Approximation lemma

In this appendix, we prove a technical lemma that was used in the proof of Proposition 1.4.7. This is a generalization of [Hoz22, Lemma 4.3] (see also [Mas23, Appendix A], reproduced below) which might be of independent interest. For this lemma only, we do not need to assume that M is 3-dimensional.

Lemma 1.C.1. *Let X be a smooth vector field on a closed manifold M. Let $f : M \to \mathbb{R}$ be a continuous function which is continuously differentiable along X, i.e., $X \cdot f$ is defined on M and continuous. For every $\epsilon > 0$, there exists a smooth function $\widetilde{f} : M \to \mathbb{R}$ such that*

$$\left|f - \widetilde{f}\right|_{\mathcal{C}^0} \leq \epsilon, \qquad \left|X \cdot (f - \widetilde{f})\right|_{\mathcal{C}^0} \leq \epsilon.$$

Proof. Let

$$\Delta \coloneqq \{x \in M : X(x) = 0\}$$

be the singular set of X, which is a compact subset of M. We proceed in 3 steps.

- *Step 1 : smoothing near Δ.* We fix some arbitrary metric g on M. For $\delta > 0$, let $U_\delta(\Delta)$ be the δ-neighborhood of Δ for this metric. Recall the following standard fact: there exists some constant $C_0 > 0$ such that for every sufficiently small $\delta > 0$,

there exists a smooth cutoff function $\varphi_\delta : U_\delta(\Delta) \to [0,1]$ such that

- $\varphi_\delta \equiv 1$ on $U_{\delta/10}(\Delta)$,
- $\varphi_\delta \equiv 0$ on $U_\delta(\Delta) \setminus U_{9\delta/10}(\Delta)$,
- $|d\varphi_\delta| \leq \frac{C_0}{\delta}$.[7]

Moreover, there exists a constant $C_1 > 0$ such that for every $x \in M$,

$$|X(x)| \leq C_1 \operatorname{dist}(x, \Delta).^{8}$$

It follows that there exists a constant $C_2 > 0$ such that for every $\delta > 0$,

$$|X \cdot \varphi_\delta| \leq C_2.$$

Let $f_0 : M \to \mathbb{R}$ be a smooth function so that

$$|f - f_0|_{\mathcal{C}^0} \leq \epsilon.$$

Since $X \cdot f = X \cdot f_0 = 0$ on Δ, there exists $\delta > 0$ such that

$$|X \cdot (f - f_0)|_{\mathcal{C}^0(U_\delta(\Delta))} \leq \epsilon.$$

We now write $U_0 \coloneqq U_\delta(\Delta)$, $V_0 \coloneqq U_{\delta/10}(\Delta) \subset U_0$ and $\psi_0 = \varphi_\delta$ for such a $\delta > 0$. It follows from the previous paragraph that

$$|X \cdot (\psi_0 f - \psi_0 f_0)|_{\mathcal{C}^0(U_0)} \leq |X \cdot \psi_0|_{\mathcal{C}^0} |f - f_0|_{\mathcal{C}^0} + |X \cdot (f - f_0)|_{\mathcal{C}^0(U_0)} \lesssim \epsilon,$$

[7] To see this, first embed M in some $\mathbb{R}^N$ and mollify a suitable Lipschitz cutoff function constructed from the distance function to the image of Δ.

[8] First show this on a sufficiently small neighborhood of Δ on which X is very close to its linearization, then enlarge C' so that it holds everywhere.

where the symbol $\lesssim$ means "less or equal than up to a constant factor that only depends on M, X and g". We define $\widetilde{f_0} \coloneqq \psi_0 f_0$.

- *Step 2: smoothing away from* Δ. As a consequence of the proof of [Hoz22, Lemma 4.3] (see also [Mas23, Appendix A], reproduced below), there exists
  - A finite collection $\mathcal{U} = (U_i)_{1 \leq i \leq N}$ of open subsets of M covering $M \setminus V_0$ (made of flow boxes for X),
  - A family of cutoff functions $(\varphi_i)_{1 \leq i \leq N}$ subordinate to $\mathcal{U}$ such that on $M \setminus V_0$,

$$\sum_{i=1}^{N} \varphi_i = 1,$$

    where $\varphi_i : M \to [0,1]$,
  - Smooth functions $f_i : M \to \mathbb{R}$, $1 \leq i \leq N$, such that

$$\left|\varphi_i f - f_i\right|_{\mathcal{C}^0} \leq \frac{\epsilon}{N}, \qquad \left|X \cdot \left(\varphi_i f - f_i\right)\right|_{\mathcal{C}^0} \leq \frac{\epsilon}{N}.$$

  For $1 \leq i \leq N$, we define $\psi_i \coloneqq (1 - \psi_0)\varphi_i$, so that $(\psi_i)_{0 \leq i \leq N}$ is a partition of unity subordinate to the open cover $\mathcal{U}' \coloneqq (U_i)_{0 \leq i \leq N}$ of M. We also write $\widetilde{f_i} \coloneqq (1 - \psi_0) f_i$ for $1 \leq i \leq N$, so that

$$\left|\psi_i f - \widetilde{f_i}\right|_{\mathcal{C}^0} \leq \frac{\epsilon}{N}, \qquad \left|X \cdot \left(\psi_i f - \widetilde{f_i}\right)\right|_{\mathcal{C}^0} \lesssim \frac{\epsilon}{N}$$

- *Step 3 : putting things together.* Finally, we define

$$\widetilde{f} \coloneqq \sum_{i=0}^{N} \widetilde{f_i},$$

and we readily obtain

$$\left|f-\widetilde{f}\right|_{\mathcal{C}^0} \leq \sum_{i=0}^{N} \left|\psi_i f-\widetilde{f_i}\right|_{\mathcal{C}^0} \lesssim \left|\psi_0 f-\widetilde{f_0}\right|_{\mathcal{C}^0}+\epsilon \lesssim \epsilon,$$
$$\left|X\cdot\left(f-\widetilde{f}\right)\right|_{\mathcal{C}^0} \leq \sum_{i=0}^{N} \left|X\cdot\left(\psi_i f-\widetilde{f_i}\right)\right|_{\mathcal{C}^0} \lesssim \left|X\cdot\left(\psi_0 f-\widetilde{f_0}\right)\right|_{\mathcal{C}^0}+\epsilon \lesssim \epsilon.$$

□

1.D Periodic solutions to ODEs on the cylinder

This appendix concerns the topology of periodic orbits of *continuous* vector fields on the cylinder. In particular, we give a necessary and sufficient condition for the existence of a foliation by circles tangent to (an approximation of) such a vector field. This serves as evidence for Conjecture 1.3.5, which would require a version of this result for continuous plane fields on a thickened 2-torus $\mathbb{T}^2\times(-1,1)$. Our result only applies to the intersection of such a plane field with an annulus transverse to the core torus $\mathbb{T}^2\times\{0\}$.

Let

$$\begin{array}{rccc} F: & \mathbb{R}\times\mathbb{R} & \longrightarrow & \mathbb{R} \\ & (x,t) & \longmapsto & F(x,t) \end{array}$$

be a *continuous*, bounded function satisfying

$$\forall x,t\in\mathbb{R},\quad F(x,t+1)=F(x,t),$$

so that F naturally induces a continuous function $\overline{F}: C\to\mathbb{R}$ on the cylinder $C\coloneqq\mathbb{R}\times\mathbb{R}/\mathbb{Z}$. The corresponding vector field X on C is defined by

$$X(x,t)\coloneqq\overline{F}(x,t)\,\partial_x+\partial_t$$

We are interested in the solutions to the ODE

$$(E): \begin{cases} \dot{y}(t) = F\big(y(t), t\big), \\ y(t_0) = y_0, \end{cases}$$

where $(t_0, y_0) \in \mathbb{R} \times \mathbb{R}$, or equivalently, in the curves on C tangent to X.

Clearly, if there exists a smooth closed curve transverse to X, then X cannot be tangent to a foliation by circles on C, and the same is true for any vector field sufficiently $\mathcal{C}^0$-close to X. Reciprocally,

Proposition 1.D.1. *Assume that there exist no smooth closed curves transverse to X. Then X can be $\mathcal{C}^0$-approximated by continuous vector fields tangent to foliations by circles on C.*

Proof. A simple closed curve γ on C tangent to X can be written as the graph of a ($\mathcal{C}^1$) function $f : S^1 \to \mathbb{R}$. Hence, we can naturally identify Γ with a subset of $\mathcal{C}^0(S^1, \mathbb{R})$. We endow Γ with the $\mathcal{C}^0$ topology and the natural partial order inherited from $\mathcal{C}^0(S^1, \mathbb{R})$. Notice that $\Gamma \subset \mathcal{C}^0(S^1, \mathbb{R})$ is closed, and the bounded subsets of Γ are relatively compact. This easily follows from the integral formulation of (E), the fact that the curves in Γ are K-Lipschitz, where $K \coloneqq \sup |F|$, and Ascoli–Arzelà theorem.

We construct a subset $\Gamma_{\text{tot}} \subset \Gamma$ such that $\leq$ restricts to a total order on Γ_{tot}, and

$$\bigcup_{\gamma \in \Gamma_{\text{tot}}} \gamma = C. \tag{1.D.1}$$

For that purpose, we will use following claim which will be proved later:

Claim. For every point $p \in C$, there exists $\gamma \in \Gamma$ passing through p.

One can easily deduce the following properties by suitably cutting a pasting curves in Γ:

- If $\gamma \in \Gamma$ and $p \in C \setminus \Gamma$ lies above (resp. below) γ, there exists $\gamma' \in \Gamma$ passing through p such that $\gamma \leq \gamma'$ (resp. $\gamma' \leq \gamma$).

- If $\gamma_0, \gamma_1 \in \Gamma$, $\gamma_0 \leq \gamma_1$, and $p \in C$ is in the interior of the domain in C bounded by γ_0 and γ_1, then there exists $\gamma \in \Gamma$ passing through p satisfying $\gamma_0 \leq \gamma \leq \gamma_1$.

Let $\mathcal{D} \subset \mathbb{R} \times S^1$ be a dense countable set. Using the above two bullet points, we can inductively construct a subset $\Gamma_{\mathcal{D}} \subset \Gamma$ on which $\leq$ restricts to a total order, and such that

$$\mathcal{D} \subset \bigcup_{\gamma \in \Gamma_{\mathcal{D}}} \gamma.$$

We now define

$$\Gamma_{\text{tot}} \coloneqq \overline{\Gamma_{\mathcal{D}}} \subset \Gamma,$$

the closure of $\Gamma_{\mathcal{D}}$ in Γ for the $\mathcal{C}^0$ topology. It is easy to check that $\leq$ restricts to a total order on Γ_{tot}, and that (1.D.1) is satisfied. In the terminology of [BI08], Γ_{tot} is a *branching foliation* tangent to X. The methods of [BI08, Section 7] can easily be adapted to show the following modified statement of [BI08, Theorem 7.2]: for every $\epsilon > 0$, there exists a continuous vector field X_ϵ on C such that $|X_\epsilon - X|_{\mathcal{C}^0} \leq \epsilon$, and X_ϵ is tangent to a foliation by circles.

We are left to prove the Claim stated above. Let $p \in C = \mathbb{R} \times \mathbb{R}/\mathbb{Z}$. Without loss of generality, we can assume $p = (0, 0)$. We will slightly abuse notations and write X for the vector field $F(x,t)\partial_x + \partial_t$ on the strip $Z \coloneqq \mathbb{R} \times [0, 1]$. Let $I \subset \mathbb{R}$ be the set of real numbers x such that there exists a curve in Z tangent to X from $(0, 0)$ to $(x, 1)$. By Kneser's theorem,[9] I is a closed interval. It contains 0 if and only if there exists $\gamma \in \Gamma$ passing through $(0, 0)$. Otherwise, we can assume without loss of generality that $m \coloneqq \min I > 0$, and we construct a closed transversal to X, contradicting our original hypothesis. Let γ_m be the curve in Γ from $(0, 0)$ to $(m, 1)$ which is minimal

[9] See [Har02], Chapter II, Theorem 4.1.

with respect to $\leq$.[10] For every $n \geq 0$, we let

$$X_n \coloneqq \big(F - 2^{-n}\big)\partial_x + \partial_t = X - 2^{-n}\partial_x,$$

and we denote by δ_n the minimal curve tangent to X_n starting at $(0,0)$. Note that for every $n \geq 0$, δ_n is (negatively) transverse to X, and $\delta_n \leq \gamma_m$. Let $x_n \in \mathbb{R}$ be such that δ_n intersects $\mathbb{R} \times \{1\}$ at $\{x_n\} \times \{1\}$. It is easy to see that $\lim_n x_n = m$. Indeed, $x_n \leq m$ and $(\delta_n)_n$ converges uniformly to a curve starting at $(0,0)$ and tangent to X. In particular, there exists $N \geq 0$ such that $x_N > 0$. It is then easy to modify δ_N near $\mathbb{R} \times \{1\}$ to obtain a smooth curve $\widetilde{\delta}_N$ from $(0,0)$ to $(0,1)$ and which is (negatively) transverse to X. Therefore, we have constructed a closed transversal to X, a contradiction. $\square$

[10] See [Har02], Chapter III, Theorem 2.1.

Chapter 2

Branching foliations

In this part, (X, η) denotes a polarized vector field on M which is admissible (see Definition 1.2.5) and without disks of tangency (see Definition 1.2.7). We also choose some arbitrary Riemannian metric $\boldsymbol{g}$ on M.

The goal of this part is to prove the Technical Theorem 1.2.1. It is completely independent of Chapter 1 and does not rely on any notion from contact or symplectic topology.

Because of the lack of smoothness and unique integrability of η, we will not be able to construct a foliation tangent to η directly. However, we will construct a weaker structure tangent to η called a *branching foliation*.

Branching foliations are defined in [BI08, Definition 4.1]. Roughly speaking, those are collections of 'maximal' surfaces which are allowed to touch but not to cross, and whose union is M. We recall the precise definition for the convenience of the reader.

Definition 2.0.1. *A **leaf collection** tangent to η is a set of maps $\mathscr{F} = \{f_i : \Sigma_i \to M\}_{i \in I}$ indexed by a set I such that for every $i \in I$,*

- *Σ_i is a smooth surface (i.e., a 2-dimensional manifold without boundary),*
- *$f_i : \Sigma_i \to M$ is a $\mathcal{C}^1$ immersion tangent to η.*

A leaf collection $\mathscr{F}$ is a ***branching foliation*** *if*

- *For every $i \in I$, the length metric for the intrinsic $\mathcal{C}^0$ Riemannian metric $\boldsymbol{g}_i \coloneqq f_i^* \boldsymbol{g}$ on Σ_i is complete,*
- *For every $i, j \in I$, f_i and f_j have no topological crossings (see [BI08, Definition 4.1]),*
- *The leaves of $\mathscr{F}$ cover M:*

$$\bigcup_{i \in I} f_i(\Sigma_i) = M.$$

Notice that this definition does not depend on the choice of the metric $\boldsymbol{g}$ since M is compact. Crucially, branching foliations can be $\mathcal{C}^0$ perturbed into genuine foliations by separating the leaves without introducing crossings. More precisely, the combination of [BI08, Lemma 7.1] and [BI08, Theorem 7.2] imply the following

Theorem 2.0.2. *If η is tangent to a branching foliation, then it can be $\mathcal{C}^0$-approximated by a $\mathcal{C}^0$-foliation $\mathcal{F}$. Moreover, there exists a continuous map $h : M \to M$ such that for every leaf $L \in \mathcal{F}$, the restriction $h_{|L} : L \to M$ is a $\mathcal{C}^1$ immersion tangent to η.*

We can further assume that h is close to the identity, hence of degree 1 and surjective, and that the leaf collection

$$\mathscr{F} \coloneqq \left\{ h_{|L} : L \to M \;\middle|\; L \in \mathcal{F} \right\}$$

is a branching foliation tangent to η. However, we will not need these additional properties.

We are now left to show that η is tangent to a branching foliation. To that end, we will essentially follow the strategy of [BI08] and we explain how to adapt their proof to our setup. Recall that the structure of interest is a pair (X, η) where

- X is a smooth vector field which plays the role of E^s from [BI08],

- η is a continuous plane field containing X and invariant under its flow, which plays the role of E^{cs} in [BI08].

However, there are two major differences between our vector field X and the vector field E^s from [BI08]: unlike E^s, X may have close orbits and singularities. A careful inspection of the proofs in [BI08] reveals that the presence of closed orbits (without singularities) is not a serious issue. However, the presence of singularities dramatically complicates the arguments and requires more general notions of *surfaces*, *patches*, *prefoliations*, etc. Furthermore, the hypothesis that (X, η) has no disks of tangency will be essential, as these disks could obstruct the construction of prefoliations, and more specifically the construction of *upper enveloping surfaces*. The (forward) structure of these surfaces is also more involved in our singular setting, and we will need to be particularly careful about their behavior near Δ.

2.1 General construction

Because of the presence of saddle and quadratic singularities, we will need to consider more general surfaces in M tangent to η than the ones in [BI08]. In particular, we will consider surfaces with corners with suitable local forms near Δ. We will call these surfaces *tiles*. We then generalize the main technical concept from [BI08], namely, the one of a *prefoliation*. These structures are intermediate steps in the construction of a genuine branching foliations, in the sense that their leaves are possibly non-maximal tiles. Prefoliations, and their weaker version defined below, also come with the structure of a (pre)order at every point of M, allowing to distinguish which tiles are above and below others, even though they geometrically coincide. The main technical—and most difficult!—step is to extend a (weak) prefoliation by suitably enlarging its tiles and adding new ones, while also extending the order structure. See Proposition 2.1.11 below, whose proof will occupy Sections 2.2, 2.3, and 2.4. As

in [BI08], this extension is particularly nontrivial because of the global behavior of the vector field X (see [BI08, Section 6.1], in particular Figures 1 and 2).

2.1.1 Tiles

We now fix a neighborhood $\mathcal{N}$ of Δ as in Lemma 1.2.2. We say that smooth local coordinates (x, y, z) around a point $p \in M$ are **adapted** to η if ∂_z is positively transverse to η and $\eta(0) = \operatorname{span}\{\partial_x, \partial_y\}$ in these coordinates. Every point $p \in M$ admits such local coordinates. If moreover $p \in \Delta_{\mathrm{sa}}$, then the projections of the stable and unstable branches of X at p on the (x, y)-plane determine four local quadrants at 0 in $\mathbb{R}^2$. If $p \in Q$, then the projections of the unstable branches of X at p on the (x, y)-plane determine two local half-planes at 0 in $\mathbb{R}^2$, one of which corresponds to the unstable half-disk of p. The projection of the stable branch at p divides the other half-plane into two quadrants.

Definition 2.1.1 (See [BI08], Definition 4.1)**.** *A* ***surface*** *in M is a $\mathcal{C}^1$ immersion $f : U \to M$, where U is a connected surface with boundary and convex and concave corners.*

The rest of [BI08, Definition 4.1] is still valid for this more general notion of surface.

Definition 2.1.2 (See [BI08], Definition 4.2)**.** *An* ***η-surface*** *is a surface $f : U \to M$ tangent to η, invariant under the flow of X, and such that U is homeomorphic to a simply connected subset of $\mathbb{R}^2$.*

The invariance condition means that the flow of the pullback $\widetilde{X} := f^*X$ of X to U is complete, or that the surface "consists of whole trajectories of X". Here, $\widetilde{X}$ is a continuous and uniquely integrable vector field on U. We call a nonsingular flow line of $\widetilde{X}$ a **X-line** on U. Since (X, η) has no disks of tangency, $\widetilde{X}$ has no nontrivial closed orbit on U, and X-lines are homeomorphic to $\mathbb{R}$.

Definition 2.1.3. *Let $f : U \to M$ be an η-surface, and D_0 be a connected open subset of U. If $p \in \overline{f(D_0)}$, we say that f is* ***locally graphical near*** *p* ***over*** *D_0 if there*

exists a small open ball B centered at p with smooth coordinates (x, y, z) adapted to η, a connected domain $D \subset \mathbb{R}^2$, a diffeomorphism $\varphi : D \to D_0 \cap f^{-1}(B)$ and a map $\widetilde{f} : D \to \mathbb{R}$ such that $f \circ \varphi : D \to B \subset \mathbb{R}^3$ is the graph of $\widetilde{f}$ over $D \subset \mathbb{R}^2$ in the following sense:

$$\forall (x, y) \in D, \quad f \circ \varphi(x, y) = \big(x, y, \widetilde{f}(x, y)\big) \in \mathbb{R}^3.$$

*Furthermore, we say that f has a **puncture** at p in D_0 if D is the complement of 0 in a neighborhood of 0, and f has a **slit** at p in D_0 if D is the complement of a $\mathcal{C}^1$ embedded curve starting at 0 in a neighborhood of 0, after possibly shrinking B in each case.*

Notice that if D_0 is a sufficiently small disk centered at $a \in U$, then f is automatically locally graphical near $f(a)$ over D_0, since f is an immersion tangent to η. The local behavior of f near a can be described as follows.

1. If $f(a) \in \Delta$. Then either a is in the interior of U, hence 0 is in the interior of D, or a is a boundary point or a corner of U, and the same holds for 0 in D. In the latter case, we further distinguish three cases.

   (a) $f(a) \in \Delta_{\text{so}}$. The intersection of D with a small open neighborhood of 0 is a sector bounded by two embedded $\mathcal{C}^1$ curves meeting at 0, corresponding to two flow lines of X emanating from a.

   (b) $f(a) \in \Delta_{\text{sa}}$. Recall that the projections of the stable and unstable branches at $f(a)$ on the (x, y)-plane in the above coordinates determine four (local) quadrants at 0. If a is a boundary point of U, the intersection of D with a small open neighborhood V of 0 is exactly the intersection of the union of two consecutive quadrants with V. If a is a convex corner of U, then $D \cap V$ is exactly the intersection of one of the quadrants with V. If a is a concave

corner of U, then $D \cap V$ is exactly the intersection of three consecutive quadrants with V.

(c) $f(a) \in Q$. Recall that the projections of the unstable branches at $f(a)$ on the (x, y)-plane determine two (local) half-planes at 0, and the projection of the stable branch divides one of these half-planes into two quadrants. The other half-plane corresponds to the projection of the unstable half-disk of $f(a)$. The description of the different possibilities for $D \cap V$ when a is a boundary point or a corner of U is left as an exercise to the reader.

2. If $f(a) \notin \Delta$, meaning that $X(f(a)) \neq 0$, we can further assume that X becomes ∂_x in the above local coordinates. By the invariance of η under X, $\widetilde{f} : D \to \mathbb{R}$ only depends on y. Note that a cannot be a corner of U. If a is in the interior of U, then D can be chosen of the form $(-\epsilon, \epsilon)^2$, and if a is a boundary point of U, D can be chosen of the form $(-\epsilon, \epsilon) \times [0, \epsilon)$ or $(-\epsilon, \epsilon) \times (-\epsilon, 0]$, where $\epsilon > 0$ is sufficiently small. We can further distinguish two cases:

   (a) The orbit of X passing through $f(a)$ is not periodic. In that case, f maps the flow line of $\widetilde{X}$ in U passing through a to the flow line of X in M passing through $f(a)$ bijectively.

   (b) The orbit of X passing through $f(a)$ is periodic. In that case, the flow line of $\widetilde{X}$ in U passing through a is not closed, since U is simply connected and (X, η) has no disks of tangency. The restriction of f to this flow line covers the closed orbit of X in M passing through $f(a)$ infinitely many times, in positive and negative times.

Notice that f sends the corners of U to Δ, and $f^{-1}(\Delta)$ is a discrete subset of U. Moreover, f sends the boundary components of $U \cap f^{-1}(\Delta)$ to entire flow lines of X.

As in [BI08], we can consider the **completion** of an η-surface $f : U \to M$. The completion $\overline{U}$ is the completion of U for the intrinsic metric on U, and f uniquely

extends to a continuous map $\bar{f} : \overline{U} \to M$. However, the completion of an η-surface in our singular setting is not necessarily an η-surface, since $\overline{U}$ might fail to be a smooth surface and $\bar{f}$ might fail to be an immersion. This occurs when f is not locally graphical near a singularity $p \in \Delta$ or has a slit at p. Because of this issue, we need to be particularly careful about the way the η-surfaces we consider intersect $\mathcal{N}$ and behave near Δ.

If $p \in \Delta$ and D_0 is a connected component of $f^{-1}(\mathcal{N})$ such that $p \in \overline{f(D_0)}$, we call D_0 a **singular region** of f for p.

Definition 2.1.4. *Let $f : U \to M$ be an η-surface.*

- *f is **well-cornered** if for every $p \in \Delta$ and every singular region $D_0 \subset U$ for p, f is locally graphical near p over D_0 and has no puncture nor slit at p in D_0.*
- *f is **unstable-complete** if for every $a \in U$,*
  1. *If $f(a)$ lies in the unstable disk of a source singularity $p \in \Delta_{\text{so}}$, then there exists an open subset $V \subset U$ containing the point a such that the restriction $f_{|V} : V \to M$ is a diffeomorphism onto this unstable disk.*
  2. *If $f(a)$ lies in the interior of the unstable half-disk of a quadratic singularity $q \in Q$, then there exists an open subset $V \subset U$ containing the point a such that the restriction $f_{|V} : V \to M$ is a diffeomorphism onto the interior of this unstable half-disk.*
- *f is an **η-tile**, or **tile** for simplicity, if it is well-cornered and unstable-complete.*

More concretely, a tile f has the following behavior near a singularity $p \in \Delta$:

- If $p \in \Delta_{\text{so}}$, then the domain D from Definition 2.1.3 contains a neighborhood of 0.

- If $p \in \Delta_{\text{sa}}$ and if 0 is not in the interior of D, then after possibly shrinking B, the interior of D is either the interior of a quadrant, or the interior of the union of two or three adjacent quadrants.

- If $p \in Q$ and if 0 is not in the interior of D, then after possibly shrinking B, the interior of D is the interior of a quadrant, the interior of the union of the two quadrants, the interior of the (projection of the) unstable half-plane, or the interior of the union of this half-plane with one quadrant.

We only have described the possibilities for the *interior* of D (intersected with a neighborhood of 0). The precise description of D is left as an exercise to the reader.

Lemma 2.1.5 (See [BI08], Lemma 4.3)**.** *The completion $\bar{f} : \overline{U} \to M$ of a tile $f : U \to M$ is a tile. Moreover, $\overline{U} \setminus U$ is contained in $\partial\overline{U}$.*

Proof. If $a \in \overline{U} \setminus U$ is such that $\bar{f}(a) \notin \Delta$, then the proof of [BI08, Lemma 4.3] shows that a is in the set-theoretic boundary of $\overline{U}$ and near a, $\overline{U}$ is obtained from U by adding a X-line. Otherwise, if $\bar{f}(a) \in \Delta$, then the effect of the completion near a corresponds to taking the closure of the domain D from Definition 2.1.3. Then $\overline{U}$ differs from U near a by adding the point a together with boundary components which are X-lines. It easy follows that $\overline{U}$ is surface with boundary and corners, and that $\bar{f}$ is a tile. The details of the proof are left to the reader. □

We can now extend the definitions in [BI08, Section 4] to tiles as follows. An **edge** of a tile $f : U \to M$ is a connected component of $\overline{U} \setminus \bar{f}^{-1}(\Delta)$. A point in $\bar{f}^{-1}(\Delta) \cap \partial\overline{U}$ is a **boundary singularity**. It is either a corner of $\overline{U}$, or a point between two edges which are X-lines with opposite orientations. The latter will be called a **fake corner**. Notice $\bar{f}$ sends fake corners to points in $Q \cup \Delta_{\text{sa}}$. An edge or boundary singularity (corner or fake corner) is **proper** if it is contained in U, and **improper** otherwise. Two edges are **adjacent** if they meet at a corner or at a fake corner. As in [BI08],

an edge has a forward or backward coorientation. This notion can be extended to (genuine) corners, but not to fake corner. We warn the reader that we will use the terminology *forward/backward corner* in a different way below.

The definition [BI08, Definition 4.4] also extends in a straightforward way: a **marked tile** is a pair (f, a) where $f : U \to M$ is a tile and $a \in U$. It is a **forward (resp. backward)** marked tile if the marking a belongs to a forward (resp. backward) edge or corner, and **passing** marked tile if the marking a belongs to the interior of U. Unlike in [BI08], there is no natural notion of "forward or backward half-tile" at a point $a \in U$. For instance, a could belong to a X-line connecting two singularities in the interior of U, and this X-line would not divide U into two half-surfaces.

2.1.2 Weak prefoliations

Similarly as in [BI08, Definition 4.5], if $\mathscr{A}$ is a collection of tiles and $p \in M$, we set

$$\mathscr{A}_p \coloneqq \{(f, a) \mid f \in \mathscr{A},\ f(a) = p\}$$

and

$$\mathscr{A}_\star \coloneqq \bigsqcup_{p \in M} \mathscr{A}_p.$$

If $f \in \mathscr{A}$, we will typically denote a corresponding element in $\mathscr{A}_\star$ by $f_\star$.

The definition of the *geometric order* in [BI08, Definition 4.6] extends in a straightforward way to a collection $\mathscr{A}$ of tiles. The notion of *patch* from Definition 4.7 of [BI08] is harder to adapt and is not crucial for our argument.

We now introduce a natural equivalence relation on marked tiles.

Definition 2.1.6. *Two marked tiles $f_\star = (f : U \to M, a)$ and $g_\star = (g : V \to M, b)$ are **equivalent** if there there exists a pointed diffeomorphism $\varphi : (U, a) \to (V, b)$ such that $g \circ \varphi = f$. In that case, we write $f_\star \approx g_\star$.*

It is straightforward to check that $\approx$ is an equivalence relation on marked tiles. Notice that if $(f, a) \approx (g, b)$, then $f(a) = g(b)$, and the diffeomorphism φ as above is unique, since f is an immersion and U is connected.

Remark 2.1.7. *It might happen that a single tile $f : U \to M$ admits two distinct markings $a, b \in U$ such that $(f, a) \approx (f, b)$. In that case, it is easy to check that the corresponding diffeomorphism φ has no fixed points and acts properly discontinuously on U. Therefore, the quotient $\Sigma := U/\varphi$ has a natural manifold structure and f induces an immersion $\Sigma \to M$ which is tangent to η. In other words, f is a nontrivial cover a surface tangent to η, but this surface is not a tile since Σ is not simply connected. Furthermore, if U only has one proper edge, then Σ has a closed boundary component which is mapped to a closed orbit of X in M.*

Definition 2.1.8 (See [BI08], Definition 4.8)**.** *A **weak prefoliation** $(\mathscr{A}, \precsim)$ is a collection $\mathscr{A}$ of tiles together with a partial preorder $\precsim$ on $\mathscr{A}_\star$ satisfying the following axioms.*[1]

0. Two marked tiles $f_\star \in \mathscr{A}_p$ and $g_\star \in \mathscr{A}_q$ are comparable by $\precsim$ if and only if $p = q$. Moreover, if $f_\star \precsim g_\star$ and $g_\star \precsim f_\star$ then $f_\star \approx g_\star$.

1. The preorder $\precsim$ is a refinement of the geometric order in the following sense: if $f_\star \precsim g_\star$, then $g_\star$ is locally above $f_\star$.

2. The preorder $\precsim$ is coherent along the intersection of tiles in the following sense: if $\gamma : [0, 1] \to M$ is continuous path, and if $\gamma_1 : [0, 1] \to U_1$ and $\gamma_2 : [0, 1] \to U_2$ are lifts of γ to tiles $f_1 : U_1 \to M$ and $f_2 : U_2 \to M$, respectively, then $(f_1, \gamma_1(0)) \precsim (f_2, \gamma_2(0))$ if and only if $(f_1, \gamma_1(1)) \precsim (f_2, \gamma_2(1))$.

In Axiom *0*, we *do not* assume that $f_\star \precsim g_\star$ if $f_\star \approx g_\star$. In other words, two equivalent tiles might be strictly comparable by $\precsim$, while two nonequivalent tiles

[1] A preorder on a set is a transitive and reflexive binary relation. It is an order if and only if it is antisymmetric. Dropping the antisymmetry requirement makes weak prefoliations more flexible than the prefoliations from [BI08]. See also the next footnote.

are necessarily strictly comparable by $\precsim$. We write $f_\star \simeq g_\star$ if $f_\star \precsim g_\star$ and $g_\star \precsim f_\star$, and $f_\star \prec g_\star$ if $f_\star \precsim g_\star$ and $f_\star \not\simeq g_\star$. If $f_\star \simeq g_\star$, we say that $f_\star$ and $g_\star$ are **order-equivalent**.[2]

Axioms *0* and *1* readily imply that if $g_\star \in \mathscr{A}_p$ is strictly locally above $f_\star \in \mathscr{A}_p$, then $f_\star \prec g_\star$. Combined with Axiom *2*, this implies that there are no topological crossings between tiles in $\mathscr{A}$; see the discussion after [BI08, Definition 4.8].

Axiom *2* can be modified by further requiring that γ is $\mathcal{C}^1$, tangent to η, and either tangent or transverse to X. A slight adaptation of the discussion after [BI08, Definition 4.8] shows that this seemingly weaker version of Axiom *2* is in fact equivalent to it. Indeed, if $\gamma : [0,1] \to M$ is only continuous, then γ_1 can be approximated by a piecewise $\mathcal{C}^1$ curve $\widetilde{\gamma}_1$ in U with the same endpoints as γ_1, which intersects the same X-lines and singularities as γ_1, and whose $\mathcal{C}^1$ segments are either tangent or transverse to $\widetilde{X}$.[3] Therefore, $\widetilde{\gamma} \coloneqq f \circ \widetilde{\gamma}_1$ admits a lift $\widetilde{\gamma}_2 : [0,1] \to U_2$ to f_2 with the same endpoints as γ_2.

A weak prefoliation is **forward (resp. backward)** if its tiles only have edges with forward (resp. backward) coorientations, and it is **complete** if its tiles are all complete. It **covers** M if for every $p \in M$, $\mathscr{A}_p \neq \varnothing$.

Example 2.1.9. *A branching foliation $\mathscr{F}$ has a natural (weak) prefoliation structure, constructed in the proof of [BI08, Theorem 7.2]. We give a slightly different construction. First of all, the leaves in $\mathscr{F}$ are obviously tiles. Now, if $f_\star = (f : U \to M, a)$ and $g_\star = (g : V \to M, b)$ are two marked leaves in $\mathscr{F}_p$, $p \in M$, we define the relation $\precsim_p^{\mathscr{F}}$ as follows: $f_\star \precsim_p^{\mathscr{F}} g_\star$ if for every (continuous, or $\mathcal{C}^1$) path $\gamma : [0,1] \to U$ starting at $\gamma(0) = a$ which lifts to a path $\widetilde{\gamma}$ to $g_\star$, $(g, \widetilde{\gamma}(1))$ is locally above $(f, \gamma(1))$. Since f and g have no topological crossings, they are comparable by $\precsim_p$. Moreover, it is not hard to*

[2] Obviously, $\precsim$ induces a genuine order on the quotient $\mathscr{A}_\star / \simeq$. Alternatively, we could circumvent the lack of antisymmetry of $\precsim$ by considering a more general definition of tiles, allowing their domains to be arbitrary surfaces different than the closed disk and the sphere.

[3] To see that, one can cover U with open disks which are either flow boxes for $\widetilde{X}$ or "standard" neighborhoods of the isolated singularities of $\widetilde{X}$.

show that $f_\star \precsim_p^{\mathscr{F}} g_\star$ *and* $g_\star \precsim_p^{\mathscr{F}} f_\star$ *if and only if* $f_\star \approx g_\star$*, so axiom* 0 *of Definition 2.1.8 is satisfied. Axioms* 1 *and* 2 *immediately follow from the definition of* $\precsim^{\mathscr{F}}$*.*

Definition 2.1.10 (See [BI08], Definition 4.9)**.** *Let* $\mathscr{A}$ *be a weak prefoliation. An* ***extension*** *of* $\mathscr{A}$ *is the data of*

- *A weak prefoliation* $\mathscr{B}$*,*
- *An map* $i : \mathscr{A} \to \mathscr{B}$*, called an* ***extension map****,*
- *For every* $f \in \mathscr{A}$ *with* $f : U \to M$ *and* $i(f) : U' \to M$*, an embedding* $\varphi_f : U \to U'$ *satisfying* $i(f) \circ \varphi_f = f$*,*

such that i *preserves the preorders. More precisely, the map* i *induces a map* $i_\star : \mathscr{A}_\star \to \mathscr{B}_\star$ *defined by*

$$i_\star(f_\star) \coloneqq \big(i(f), \varphi_f(a)\big),$$

where $f_\star = (f, a) \in \mathscr{A}$*. We require that the following holds:*

$$\forall f_\star, g_\star \in \mathscr{A}_\star, \quad f_\star \precsim^{\mathscr{A}} g_\star \iff i_\star(f_\star) \precsim^{\mathscr{B}} i_\star(g_\star).$$

Unfortunately, Proposition 4.11, Proposition 4.12 and Proposition 4.13 from [BI08] do not individually extend in a straightforward way to our setup. Instead, we will combine modified versions of these statements and show:

Proposition 2.1.11. *Let* $\mathscr{A}$ *be a complete and backward (resp. forward) weak prefoliation. There exists an extension* $\mathscr{A} \to \mathscr{B}$ *where* $\mathscr{B}$ *is complete, forward (resp. backward), and covers* M*.*

The proof of this proposition occupies the next four sections. We now show how it implies the Technical Theorem 1.2.1.

Proof of the Technical Theorem 1.2.1. Similarly as in [BI08], there is $r_0 > 0$ such that every ball of radius r_0 centered at a point in $M \setminus \mathcal{N}$ admits coordinates (x, y, z) in

which $X = \partial_x$, and the angle between η and $\mathrm{span}\{\partial_x, \partial_y\}$ is at most 10^{-10}. We can rescale the metric $\boldsymbol{g}$ so that $r_0 \geq 100$.

Applying Proposition 2.1.11 inductively, we construct a sequence of complete weak prefoliations $(\mathscr{A}^n)_{n\geq 0}$ such that

- $\mathscr{A}^0 = \varnothing$,
- $\mathscr{A}^1$ is forward and covers M,
- For every $n \geq 0$, $\mathscr{A}^{n+1}$ is an extension of $\mathscr{A}^n$, with an extension map denoted by $i^n : \mathscr{A}^n \to \mathscr{A}^{n+1}$,
- For every $k \geq 0$, $\mathscr{A}^{2k}$ is backward and $\mathscr{A}^{2k+1}$ is forward.

For every $f \in \mathscr{A}^1$, we consider the sequence of tiles $(f^n)_{n\geq 1}$ inductively defined by

$$f^1 \coloneqq f, \qquad \forall n \geq 0,\ f^{n+1} = i^n(f^n).$$

If $f^n : U_n \to M$, then by definition there exists a sequence of embeddings

$$U_1 \xrightarrow{\varphi^1} U_2 \xrightarrow{\varphi^2} \dots \xrightarrow{\varphi^{n-1}} U_n \xrightarrow{\varphi^n} U_{n+1} \xrightarrow{\varphi^{n+1}} \dots$$

such that for every $n \geq 1$, $f^{n+1} \circ \varphi^n = f^n$. Taking the colimit of the above diagram yields a surface U_∞, a sequence of embeddings $(\psi^n : U_n \to U_\infty)_{n\geq 1}$, and an immersion $f^\infty : U_\infty \to M$ such that for every $n \geq 1$, $f^\infty \circ \psi^n = f^n$ and $\psi^{n+1} \circ \varphi^{n+1} = \varphi^n$. In particular, f^∞ is tangent to η. If $a_\infty \in U_\infty$, then for $n \geq 1$ sufficiently large, there exists $a_n \in U_n$ such that $\psi^n(a_n) = a_\infty$. We say that the marked tile $(f^n, a_n) \in \mathscr{A}^n_\star$ is

an *ancestor* of (f^∞, a_∞). For $1 \leq m \leq n$, we write

$$\begin{aligned}
\varphi_n^m &\coloneqq \varphi^{n-1} \circ \cdots \circ \varphi^m, \\
U_n^m &\coloneqq \varphi_n^m(U_m) \subset U_n, \\
U_\infty^m &\coloneqq \psi^m(U_m), \\
a_n^m &\coloneqq \varphi_n^m(a_m) \in U_n.
\end{aligned}$$

We claim that the surface f^∞ is open and complete. Indeed, let $a_\infty \in U_\infty$ and (f^n, a_n) be an ancestor of (f^∞, a_∞). If $f^n(a_n) \notin \Delta$, then $a_{n+1} = \varphi^n(a_n)$ is in the interior of U_{n+1}, since either a_n is in the interior of U_n, or it is in an edge of U_n and a_{n+1} is not in an edge of U_{n+1} because these surfaces have edges with opposite coorientations. If $f^n(a_n) \in \Delta$ is a source, then it is already in the interior of U_n, and if it is a saddle or a quadratic singularity, then $a_{n+2} = a_{n+2}^n$ is in the interior of U_{n+2} by a similar argument as above. In both cases, $a_\infty = \psi^{n+1}(a_{n+1}) = \psi^{n+2}(a_{n+2})$ is in the interior of U_∞. To show completeness, it is enough to show that every closed ball for the intrinsic metric on U_∞ is compact, by the Hopf–Rinow–Cohn-Vossen Theorem [BBI01, Theorem 2.5.28]. The key observation is that if $a_\infty \in U_\infty$ and (f^n, a_n) is an ancestor of (f^∞, a_∞), then for every $k \geq 1$, the point $a_{n+3k} = a_{n+3k}^n \in U_{n+3k}$ is at distance at least k from the boundary of U_{n+3k}. For $k = 1$, we can argue as follows. If $\gamma : [0,1] \to \overline{U}_{n+3}$ is a piecewise $\mathcal{C}^1$ (or rectifiable) curve from a_{n+3} to an edge of U_{n+3}, and if $t_0 \in [0,1]$ is the last time such that $\gamma(t_0) \in \partial U_{n+3}^n$, then there exists $t_1 \in [t_0, 1]$ such that $f^{n+3}(\gamma(t_1)) \notin \mathcal{N}$.[4] The endpoints of γ belong to (the closures of) X-lines with the same coorientations in $\overline{U_{n+3}} \setminus U_n^{n+3}$, and the normalization of r_0 implies that the length of either $\gamma_{|[t_0,t_1]}$ or $\gamma_{|[t_1,1]}$ is at least 1, so the length of γ is at least 1. The general case $k \geq 1$ can be obtained by induction. Therefore, if $B = B(a_\infty, r) \subset U_\infty$ is a closed intrinsic ball of radius $r > 0$ centered at a_∞, then $B \subset U_\infty^N$ for a sufficiently

[4]For this to be true, we crucially need to apply the forward of backward extensions thrice, because of the possible presence of connections between two saddles.

large $N \geq 1$, and $B_N = \left(\psi^N\right)^{-1}(B) \subset U_N$ coincides with a closed intrinsic ball of radius r in U_N. By [BBI01, Theorem 2.5.28], B_N is compact since U_N is complete, so $B = \psi^N(B_N)$ is compact as well.

We now define

$$\mathscr{A}^\infty \coloneqq \left\{ f^\infty \mid f \in \mathscr{A}^1 \right\}.$$

For every $p \in M$, $\mathscr{A}^\infty_p \neq \varnothing$ by our hypothesis. If $\precsim^n$ denotes the preorder on $\mathscr{A}^n_\star$, we define a preorder $\precsim^\infty$ on $\mathscr{A}^\infty_\star$ as follows: if $f^\infty_\star, g^\infty_\star \in \mathscr{A}^\infty_\star$, we set $f^\infty_\star \precsim^\infty g^\infty_\star$ if and only if there exists $n \geq 1$ and ancestors $f^n_\star, g^n_\star \in \mathscr{A}^n_\star$ of $f^\infty_\star$ and $g^\infty_\star$, respectively, satisfying $f^n_\star \precsim^n g^n_\star$. It is straightforward to check that $\precsim^\infty$ is well-defined and that $(\mathscr{A}^\infty, \precsim^\infty)$ is a weak prefoliation. In particular, two surfaces $f^\infty, g^\infty \in \mathscr{A}^\infty$ have no topological crossings. Therefore, we have shown that $\mathscr{A}^\infty$ is a branching foliation tangent to η, and Theorem 2.0.2 finishes the proof. □

Remark 2.1.12. *Let us define a* ***partial branching foliation*** *in the same way as a branching foliation, without the condition that the leaves cover M. Then, Example 2.1.9 readily shows that a partial branching foliation induces a complete and backward/forward weak prefoliation. Proposition 2.1.11 and the previous proof immediately imply the following: for any partial branching foliation $\mathscr{F}^\circ$, there exists a branching foliation $\mathscr{F}$ with $\mathscr{F}^\circ \subset \mathscr{F}$.*

2.2 Upper enveloping tiles

In this section, we extend the notions of *upper enveloping surface* and *forward envelope* from [BI08, Section 6.1] to our setup. We will assume that the reader is familiar with the notion of a one-dimensional prefoliation on the plane from [BI08, Section 5]. Our proof of the existence of forward envelopes essentially follows the strategy of [BI08], but we find convenient to phrase it in a slightly more abstract way.

2.2.1 Preliminaries

Let $(\mathscr{A}, \precsim)$ be a (possibly empty) weak prefoliation which is backward and complete. Our goal is to extend the tiles in $\mathscr{A}$ to tiles with only forward edges. The first step is to define new neighboring tiles, the *forward envelopes*, which play the role of "upper neighbors" of the existing tiles (or are new tiles if $\mathscr{A}$ was empty). In the next sections, we extend the definition of forward envelopes to our setup. In the section after, we extend the preorder to these tiles. In the section after, we glue the tiles in $\mathscr{A}$ to their newly created neighbors, and take the completion. We then extend the preorder one last time. Before getting to the heart of the matter, we need to make some remarks and introduce some terminology.

The flow of X induces a natural $\mathbb{R}$-action on $\mathscr{A}_\star$ by "sliding the marked point" along a X-line. More precisely, for every $f_\star = (f : U \to M, a) \in \mathscr{A}_\star$ and $t \in \mathbb{R}$, we define the **t-translation** of $f_\star$ by

$$f_\star^t := \big(f, \widetilde{\phi}^t(a)\big),$$

where $\widetilde{\phi}^t$ denotes the flow of $\widetilde{X} = f^*X$. For every $p \in M$, we have

$$f_\star \in \mathscr{A}_p \iff f_\star^t \in \mathscr{A}_{\phi^t(p)},$$

where ϕ^t denotes the flow of X. Therefore, this operation extends to a bijection

$$\Phi^t : \mathscr{A}_p \to \mathscr{A}_{\phi^t(p)}$$

which obviously defines a $\mathbb{R}$-action on $\mathscr{A}_\star$.

If $p \in M \setminus \Delta$ and $D \subset M$ is an open 2-disk transverse to X centered at p, then $\mathscr{A}$ induces a *one-dimensional prefoliation* (in the sense of [BI08, Section 5.2]) on D for

the (continuous) vector field obtained as the trace of η on D. More precisely, if X_η denotes this vector field, then the intersections of tiles in $\mathscr{A}$ with D is a collection $\mathcal{A}$ of flow lines of X_η, and for every $q \in D$, the preorder $\precsim$ induces an total order on the set of flow lines of X_η passing through q as in [BI08, Section 5.2] (see also the beginning of [BI08, Section 5.1]). We call the data of $\mathcal{A}$ equipped with this order the **trace** of $\mathscr{A}$ on D.

We say that a singularity $p \in \Delta_{\mathrm{sa}} \cup Q$ is **surrounded by unstable manifolds** if either

- Every stable branch at p intersects the unstable manifold of a source or quadratic singularity, or
- $p \in \Delta_{\mathrm{sa}}$, p is connected to $q \in \Delta_{\mathrm{sa}}$, and three of the stable branches of p and q intersect unstable manifolds of source or quadratic singularities.

We define the **unstable locus** $M^{\mathrm{un}} \subset M$ as the union of interiors of unstable manifolds of source and quadratic singularities, together with the singularities in $\Delta_{\mathrm{sa}} \cup Q$ surrounded by unstable manifolds and their unstable branches.

We define the **regular locus** of M as $M^{\mathrm{reg}} \coloneqq M \setminus \Delta$. We further partition the singularity locus Δ into two sets

$$\Delta^{\mathrm{good}} \coloneqq \Delta \cap M^{\mathrm{un}}, \qquad \Delta^{\mathrm{bad}} \coloneqq \Delta \setminus \Delta^{\mathrm{good}},$$

and we define the **good locus** by $M^{\mathrm{good}} \coloneqq M \setminus \Delta^{\mathrm{bad}} = M^{\mathrm{reg}} \cup \Delta^{\mathrm{good}}$. Roughly speaking, this is the set of points at which it will make sense to define upper enveloping tiles.

Finally, we define the **pure locus** of M as $M^{\mathrm{pure}} \coloneqq M^{\mathrm{reg}} \setminus M^{\mathrm{un}}$. It is foliated by the flow lines of X disjoint from the unstable locus. We say that such a flow line is **pure**. Notice that we have

$$M^{\mathrm{good}} = M^{\mathrm{un}} \sqcup M^{\mathrm{pure}}.$$

Let $p \in \Delta_{\text{sa}} \cup Q$. If no stable branch of X at p is in M^{pure}, we say that p is a **un** saddle singularity. If exactly one stable branch of X at p is contained in M^{pure}, we say that p is a **quadratic-like** singularity. We denote by $\Delta_{\text{sa}}^{\text{pure}}$ the set of pure saddle singularities of X, and by Q^* the set of quadratic-like singularities of X.

We now describe the behavior of η on M^{un}.

Lemma 2.2.1. *The plane field η is uniquely integrable on M^{un}.*

Proof. We need to show that η is uniquely integrable at the singularities surrounded by unstable manifolds and along their unstable branches. This will immediately follow from the two slightly stronger claims:

1. *If there exists a connection γ from $p \in \Delta_{\text{so}} \cup Q$ to $q \in \Delta_{\text{sa}} \cup Q$, then η is uniquely integrable at q and along the unstable branches at q on the side of γ.*

2. *If moreover there exists a connection γ' from q to $q' \in \Delta_{\text{sa}}$, then η is uniquely integrable at q' and along the unstable branch at q' on the side of the quadrant determined by γ and γ'.*

The meaning of the underlined parts will be clarified in the proof. Let S^u denote the interior of the unstable manifold of X at p.

For the first claim, let S be a small open $\mathcal{C}^1$ disk tangent to η and passing through a point q_0 lying on one of the unstable branches γ^u at q, and let S_γ be the closure of the connected component of $S \setminus \gamma_u$ on the side of γ. This side can be characterized as follows: if D is a small smooth disk in M transverse to X centered at a point $p_0 \in \gamma$ close to q, then the points in S_γ sufficiently close to γ^u all get sent to D or to q under the negative flow of X, and the corresponding flow lines stay close to $\gamma \cup \{q\} \cup \gamma^u$. Let $\widehat{S}_\gamma$ be the union of the saturation of S_γ by the flow of X with γ, so that $\widehat{S}_\gamma$ is a $\mathcal{C}^1$ surface with boundary tangent to η. Since $\widehat{S}_\gamma$ intersects S^u along γ, and η is uniquely integrable there, the interior of $\widehat{S}_\gamma$ is entirely contained S^u. Therefore, S_γ is

contained in $S^u \cup \gamma^u$. If $q_0 = q$, then S_γ is defined in a similar way, and its interior contains point in γ, which belong to the unstable manifold of p. This implies that S_γ is contained in the union of S^u with q and the two unstable branches at q. The desired unique integrability property of η follows.

For the second claim, γ^u will denote the unstable branch at q' lying on the side determined by γ and γ'. Similarly, let S be a small open $\mathcal{C}^1$ disk tangent to η and passing through a point $q'_0 \in \gamma^u$, and let $S_{\gamma,\gamma'}$ be the closure of the connected component of $S \setminus \gamma^u$ in the side determined by γ and γ'. This side can be characterized as follows: if D is as before, then the points in $S_{\gamma,\gamma'}$ sufficiently close to γ^u all get sent to D or to q' under the negative flow of X, and these flow lines stay close to $\gamma \cup \{q\} \cup \gamma' \cup \{q'\} \cup \gamma^u$. Let $\widehat{S}_{\gamma,\gamma'}$ be the union of the saturation of $S_{\gamma,\gamma'}$ by the flow of X with γ', so that $\widehat{S}_{\gamma,\gamma'}$ is a $\mathcal{C}^1$ surface with boundary and corners tangent to η. Since $\widehat{S}_{\gamma,\gamma'}$ intersects γ' and lies on the side of γ' determined by γ, the previous point implies that $\widehat{S}_{\gamma,\gamma'}$ is entirely contained in $S^u \cup \gamma' \cup \gamma^u$, hence $S_{\gamma,\gamma'}$ is contained in $S^u \cup \gamma^u$. Finally, if $q'_0 = q'$, we denote by $S_{\gamma,\gamma'}$ the closure of the connected component of $S \setminus \left(\gamma' \cup \gamma^u\right)$ lying in the quadrant at q' determined by γ and γ'. Arguing as before, we deduce that $S_{\gamma,\gamma'}$ is entirely contained in $S^u \cup \gamma' \cup \{q'\} \cup \gamma^u$. The desired unique integrability property of η follows. □

Therefore, η induces a foliation $\mathcal{F}^{\mathrm{un}}$ on M^{un}. Notice that M^{un} is invariant under the flow of X, and contains no nontrivial closed orbits of X (independently of the hypothesis that (X, η) has no disk of tangency).

Lemma 2.2.2. *Let $f : L \to M$ be a leaf of η in M^{un}. Its universal cover $\widetilde{f} : \widetilde{L} \to M$ is a tile. We call it an* ***unstable tile****.*

Proof. By construction, f and $\widetilde{f}$ are made of unstable manifolds of source singularities and interiors of unstable manifolds of quadratic singularities, glued together along unstable connections of saddle or quadratic singularities surrounded by them. One

easily deduces, using the Poincaré–Bendixson theorem and the Hopf theorem, that L is not a sphere, hence $\widetilde{L}$ is homeomorphic to a simply connected subset of $\mathbb{R}^2$. Moreover, $\widetilde{f}$ is unstable-complete by definition. We are left to show that $\widetilde{f}$ is well-cornered at every singularity in $\Delta_{\text{sa}} \cup Q$. It is clearly the case at a singularity surrounded by unstable manifolds, by construction. Let $p \in \Delta_{\text{sa}}$ be surrounded by unstable manifolds, and γ be a connection from p to $q \in \Delta_{\text{sa}}$ which is *not* surrounded by unstable manifolds. Then $\widetilde{f}$ is well-cornered at q, since every singular region $D_0 \subset \widetilde{L}$ for q contains an open half-disk D_0' mapping to a half-disk D in M containing a portion of γ, and whose 'straight' part of the boundary is contained in the union of q and its unstable branches.

Recall that X only has *admissible* connections. It is now enough to show the following claim: if $g : U \to M$ is a $\mathcal{C}^1$ parametrization of the (interior of) the unstable manifold of singularity $p \in \Delta_{\text{so}} \cup Q$, then g is well-cornered. We will only treat the case of a source singularity, the case of a quadratic singularity being similar. Let $q \in \Delta_{\text{sa}} \cup Q$, and let $D_0 \subset U$ be a singular region of g for q. Since all of the X-lines in g converge to $g^{-1}(p)$ in negative time, D_0 cannot intersect X-lines corresponding to the unstable branches at q. Therefore, if D_0 contains a X-line corresponding to a connection from p to q, then D_0 contains a small open half-disk D_0' similar as before. Otherwise, D_0 contains a small open quadrant C_0 mapped to an open quadrant C in M whose 'corner' boundary is sent to the union of q with two consecutive unstable and stable branches. In both cases, g is well-cornered at q over D_0. □

2.2.2 Forward curves

To construct and study forward envelopes, it will be crucial to understand the 'forward' structure of the tiles under consideration with respect to the pullback of the vector field X. The forward structures of surfaces in [BI08] is much easier to understand due to the absence of singularities of X. In the context of this Chapter, we will need to carefully subdivide tiles into various regions, and extend the definition of

forward curves. A crucial step is to understand how two forward curves with the same endpoints are related to each other, which is achieved in Lemma 2.2.7 below. We also collect some technical results that will be useful later.

Definition 2.2.3. *Let $f : U \to M$ be a tile.*

- *The **regular locus** of f is the open subset $U^{\mathrm{reg}} := f^{-1}(M^{\mathrm{reg}}) \subset U$. It is foliated by X-lines.*
- *The **good locus** of f is the open subset $U^{\mathrm{good}} := f^{-1}(M^{\mathrm{good}}) \subset U$.*
- *The **unstable locus** of f is the open subset $U^{\mathrm{un}} := f^{-1}(M^{\mathrm{un}}) \subset U$. A connected component of U^{un} is called an **unstable component** of f.*
- *The **pure locus** of f is the subset $U^{\mathrm{pure}} := f^{-1}(M^{\mathrm{pure}}) \subset U$. A X-line in U^{pure} is called **pure**.*

One can easily show that the unstable components of a tile are simply connected. The proof is left as an exercise for the reader. We now we fix a marked tile $f_\star = (f : U \to M, a)$ for the rest of this section.

Definition 2.2.4. *A **forward (resp. backward) curve** in U is a $\mathcal{C}^1$ immersed curve $\gamma : [0,1] \to U$ such that for every $t \in [0,1]$, either*

1. *$\gamma(t) \in U^{\mathrm{un}}$, or*
2. *$\gamma(t) \in U^{\mathrm{reg}}$ and $\dot\gamma(t)$ is positively (resp. negatively) transverse to the X-line passing through $\gamma(t)$.*

*Furthermore, we say that γ is **strictly forward (resp. backward)** if it satisfies condition 2 above at every $t \in [0,1]$.*

We emphasize that we impose no constraint on the part of a forward curve which is contained in U^{un}. Outside of that part, the curve has to be transverse to the X-lines

it meets. If $\gamma : [0,1] \to U$ is a curve, we denote by $\check{\gamma} : [0,1] \to U$ the corresponding reversed curve defined by $\check{\gamma}(t) \coloneqq \gamma(-t)$, $t \in [0,1]$. Obviously, γ is forward if and only if $\check{\gamma}$ is backward.

The following lemma will be particularly useful in Section 2.3 below and serves as a warm-up.

Lemma 2.2.5. *Let $\gamma : [0,1] \to U$ be a forward curve.*

1. *If $\ell \subset U^{\mathrm{pure}}$ is a pure X-line, then γ intersects ℓ at most once.*

2. *If $\gamma(0) = \gamma(1)$, then γ is entirely contained in an unstable component of f.*

Proof. We show 1 by contradiction. Without loss of generality, let us assume that $\gamma(0) \in \ell$ and $\gamma(1) \in \ell$. We can further modify γ so that $\gamma(0) = \gamma(1)$ and $\gamma'(0) = \gamma'(1)$, by flowing γ along $\widetilde{X}$ near its starting point, since γ is positively transverse to ℓ at $t = 0$. After a small perturbation, we can further assume that γ has finitely many (and transverse) self-intersections. The latter can be resolved (whether they occur in U^{un} or U^{pure}) in order to obtain a finite collection of *embedded* forward curves. We can therefore assume that γ is embedded, and bounds a domain $D \subset U$ homeomorphic to a closed disk. We now analyze the possible behaviors of ℓ in D. Observe that γ cannot intersect ℓ except at $x = \gamma(0) = \gamma(1)$, so one of the two connected components of $\ell \setminus \{x\}$ is entirely contained in D. Assume first the 'negative' component is contained in D. Then by Poincaré–Bendixson, ℓ necessarily converges in negative times to a singularity $x_0 \in U$ contained in the interior of D. Moreover, since ℓ is pure, then x_0 is not a source and at least one stable branch of $\widetilde{X}$ at x_0 lies in U^{pure}. The latter cannot intersect γ since γ is forward, so it must converge in negative times to a saddle singularity x_1 (and x_0 is a saddle as well). Once again, one of the stable branches at x_1 must lie in U^{pure}, and intersect γ nor converge to a singularity in negative times, a contradiction. The same argument applies when the 'positive' component of $\ell \setminus \{x\}$ is contained in D by considering unstable branches instead of stable branches.

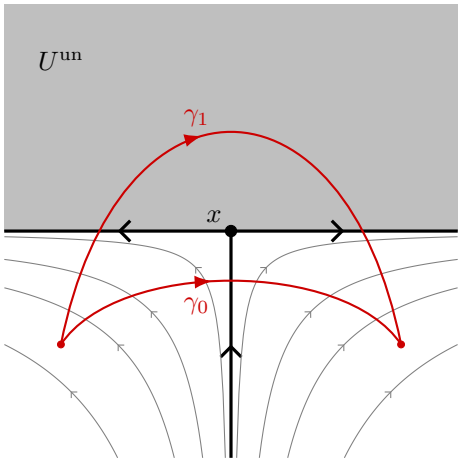


Figure 2.2.1: A (full) T-move.

The second item follows immediately, as a forward curve not entirely contained in U^{un} must intersect a pure X-line. □

The previous lemma readily implies that a forward curve is homotopic relative to U^{pure} (hence homotopic through forward curves) to an *embedded* one.

Definition 2.2.6. *Let $\gamma_0 : [0,1] \to U$ and $\gamma_1 : [0,1] \to U$ be two forward curves with the same endpoints, namely, $\gamma_0(0) = \gamma_1(0)$ and $\gamma_0(1) = \gamma_1(1)$.*

- *We say that γ_0 and γ_1 are **forward-homotopic** if they are homotopic relative to their endpoints through forward curves.*
- *We say that γ_1 is obtained from γ_0 by a **full T-move** if these two curves only intersect at their endpoints and bound a compact domain D in U containing exactly one singularity of $\widetilde{X}$. We further require that this singularity is quadratic-like, and γ_0 intersects its pure stable branch. See Figure 2.2.1.*
- *We say that γ_1 is obtained from γ_0 by a **(local) T-move** if there exists a segment $I \subset [0,1]$ such that γ_0 and γ_1 coincide on $[0,1] \setminus I$, and $\gamma_{1|I}$ is obtained from $\gamma_{0|I}$ by a full T-move.*

Notice that two forward-homotopic curves intersect the same pure X-lines and the same unstable components in U. Two forward curves that differ from a T-move across

a singularity x intersect the same pure X-lines except for the pure stable branch and unstable branches at x, and they intersect the same unstable components in U except for the unstable component at $f(x) \in Q^*$.

Two forward curves with the same endpoints are not necessarily forward-homotopic, but the are *up to finitely many local T-moves*:

Lemma 2.2.7. *Let γ_0 and γ_1 be two forward curves in U with the same endpoints. There exists forward curves $\widetilde{\gamma}_0$ and $\widetilde{\gamma}_1$ such that*

- *For $i \in \{0, 1\}$, γ_i and $\widetilde{\gamma}_i$ are forward-homotopic,*
- *$\widetilde{\gamma}_0$ and $\widetilde{\gamma}_1$ differ by a finite sequence of local T-moves.*

Proof. After applying forward-homotopies, we can assume that γ_0 and γ_1 are embedded and intersect at finitely many points. We can further reduce to the case where γ_0 and γ_1 only intersect at their endpoints. Let D be the compact domain in U bounded by γ_0 and γ_1. It contains finitely many singularities of $\widetilde{X}$. We will show the desired statement by induction on the number of these singularities.

If D contains no singularity of $\widetilde{X}$, then every X-line in the interior of $D \cap U^{\text{pure}}$ must intersect both γ_0 and γ_1 by Poincaré–Bendixson. If there is no such X-line, then (the interiors of) γ_0 and γ_1 are contained in the same unstable component and the result is immediate. Otherwise, since D is compact, we can find an open neighborhood U_0 of $D \cap U^{\text{pure}}$ foliated by X-lines intersecting γ_0 and γ_1 once each, and such that $D \setminus U_0$ intersects finitely many unstable components of U. We then use the flow of $\widetilde{X}$ to forward-homotope γ_0 to a forward curve $\widetilde{\gamma}_0$ which coincides with γ_1 on U_0, and we further homotope $\widetilde{\gamma}_0$ on each of the finitely many components of $D \setminus U_0$ to match γ_1.

Let us now assume that $x \in D$ is a singularity of $\widetilde{X}$. We analyze the three different possible cases.

- **Case 1:** $f(x) \in \Delta_{\text{so}}$. Then at least one X-line in the unstable manifold of x intersects γ_0 or γ_1, say γ_0. We can forward-homotope γ_0 to $\widetilde{\gamma}_0$ so that $\widetilde{\gamma}_0$ and γ_1

bound a compact domain $D' \subset D$ which does not contain x, hence reducing the number of singularities of $\widetilde{X}$ between the two forward curves.

- **Case 2:** $f(x) \in Q$. If $x \in U^{\mathrm{un}}$, we argue as in Case 1. Otherwise, the stable branch at x lies in U^{pure} and must intersect γ_0 or γ_1, say γ_0. We then perform a local T-move on γ_0 along this stable branch to reduce the number of singularities of $\widetilde{X}$ in D.

- **Case 3:** $f(x) \in \Delta_{\mathrm{sa}}$. If $x \in U^{\mathrm{un}}$, then we can argue as before. Otherwise, at least one stable branch at x lies in U^{pure} and intersects γ_0 or γ_1, say γ_0. If the other stable branch at x lies in U^{un}, then $f(x) \in Q^*$ and we argue as in Case 2. Otherwise, it lies in U^{pure} and either intersects γ_0, or converge in negative times to another saddle singularity x' in D. In the latter case, at least one stable branch at x' is contained in U^{pure}, and it must intersect γ_0. In both cases, one of the two unstable branches at x is 'trapped' in a domain bounded by γ_0 and by stable and unstable branches at x and at x' if it exists. If this unstable branch converges to another saddle singularity, the unstable branches of the latter are also trapped and cannot converge to singularities. However, this is impossible by Poincaré–Bendixson.

In all three cases, we constructed a forward-homotopy of γ_0 or γ_1, or performed some local T-move, reducing the number of singularities between γ_0 and γ_1. Note that for embedded forward curves, T-moves and forward-homotopies commute, so these operations can be performed in the order prescribed by the statement. □

Let us now consider the following situation. Assume that there exists two saddle singularities x_0 and x_1 in U such that

- There exists a connection γ from x_0 to x_1,

- One stable branch at x_0 is contained in U^{pure} while the other is contained in U^{un},

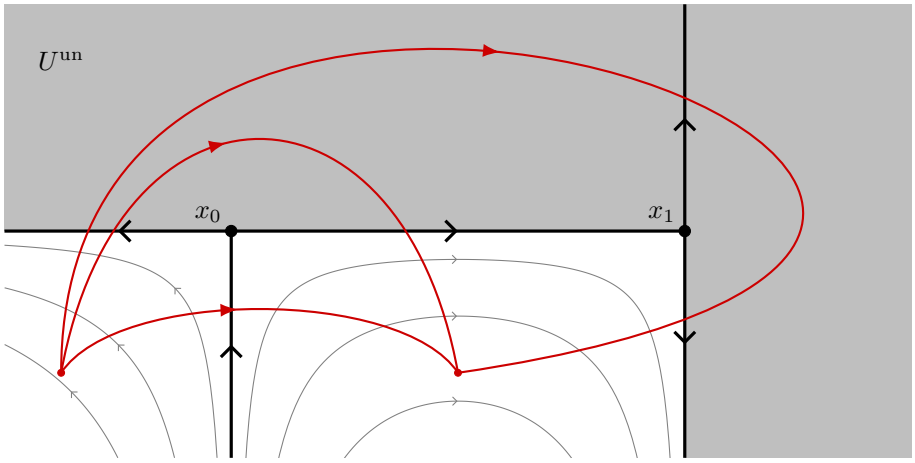


Figure 2.2.2: A double T-move.

- The stable branch at x_1 different than γ is contained in U^{un}.

Then one can perform a sequence of two consecutive T-moves, one across x_0 followed by another one across x_1, as depicted in Figure 2.2.2. We call this operation a **double T-move**. By our assumptions on X, there cannot be longer sequences of T-moves.

Let now $g_\star = (g : V \to M, b)$ be another marked tile satisfying $g(b) = f(a)$. The following lemma will be useful in the next sections.

Lemma 2.2.8. *Let $\gamma_0, \gamma_1 : [0,1] \to U$ be two forward curves such that $\gamma_0(0) = \gamma_1(0) = a$ and $\gamma_0(1) = \gamma_1(1)$. If both γ_0 and γ_1 admit lifts $\widetilde{\gamma}_0$ and $\widetilde{\gamma}_1$ to $g_\star$, then $\widetilde{\gamma}_0(1) = \widetilde{\gamma}_1(1)$.*

Proof. If a forward curve γ is obtained from γ_0 by a T-move across some singularity $x \in U$, we say that it is a *negative* T-move if γ intersects the pure stable branch at x. In that case, γ lifts to $g_\star$ to a curve $\widetilde{\gamma}$ with $\widetilde{\gamma}(1) = \widetilde{\gamma}_0(1)$. Here, we used that g has no slits.

The result is clear when γ_0 and γ_1 are forward-homotopic. In general, it is a consequence of the following immediate refinement of Lemma 2.2.7: there exists a forward curve γ in U such that for every $i \in \{0,1\}$, γ is obtained from γ_i by performing a forward-homotopy followed by finitely many *negative* T-moves. Therefore, γ lifts to $g_\star$ to a curve $\widetilde{\gamma}$ satisfying $\widetilde{\gamma}_0(1) = \widetilde{\gamma}(1) = \widetilde{\gamma}_1(1)$. □

In the next sections, we will need a strengthening of the unstable-complete condition from Definition 2.1.4:

Definition 2.2.9. *A tile $f : U \to M$ is **strongly unstable-complete** if for every connected component V of $U^{\mathrm{un}} \subset U$, there exists an unstable tile $g : W \to M$ and a diffeomorphism $\varphi : V \to W$ such that $g \circ \varphi = f$.*

Notice that a complete and forward tile is automatically strongly unstable-complete.

2.2.3 Forward envelopes

We consider a (possibly empty) weak prefoliation $(\mathscr{A}, \precsim^{\mathscr{A}})$ which is backward and complete. If $p \in M$, a **section** of $(\mathscr{A}_p, \precsim_p^{\mathscr{A}})$ is a subset $\Sigma_0 \subset \mathscr{A}_p$ satisfying: for all $f_\star \in \Sigma_0$ and $g_\star \in \mathscr{A}_p$, if $g_\star \precsim_p^{\mathscr{A}} f_\star$ then $g_\star \in \Sigma_0$. Typical examples of sections are:

- $\Sigma_0 = \varnothing$,
- $\Sigma_0 = \{g_\star \in \mathscr{A}_p \mid g_\star \precsim_p^{\mathscr{A}} f_\star\}$, where $f_\star \in \mathscr{A}_p$.

We now extend the definition of an *upper enveloping surface* from [BI08]:

Definition 2.2.10 (See [BI08], Definition 6.1)**.** *An **upper enveloping tile** is a pair (f, Σ), where $f : U \to M$ is a tile and $\Sigma : U \to \mathcal{P}(\mathscr{A}_\star)$ is a map which assigns a section $\Sigma(a) \subset \mathscr{A}_{f(a)}$ to every $a \in U$, such that the following conditions hold.*

*0. Either f has exactly one proper edge e_0, and this edge is a pure X-line with forward coorientation, or f has no proper edge but has a distinguished unstable component U_0. In the first case, we call e_0 the **initial edge** of f and we say that f is of **pure type**, and in the second case, we call U_0 the **initial unstable component** of f and we say that f is of **unstable type**. In both cases, we refer to e_0 or U_0 as the **initial locus** of f, and we denote it by I_0.*

1. f is strongly unstable-complete, in the sense of Definition 2.2.9.

2. *Every point in* U^{good} *can be reached from the initial edge or unstable component by a forward curve.*

3. *Let* $a \in U^{\mathrm{reg}}$ *and* $D \subset M$ *be a smooth open* 2*-disk transverse to* X *and containing* $f(a)$*. Let* $\gamma : [0,1] \to U$ *be a strictly forward curve starting at* a *and contained in* $f^{-1}(D)$*, and let* $\mathcal{A}$ *be the trace of* $\mathscr{A}$ *on* D*. We require that the curve* $f \circ \gamma$ *on* D *is contained in the one-dimensional upper envelope of* $(f(a), \Sigma(a) \cap D)$ *with respect to* $\mathcal{A}$ *(as defined in [BI08, Section 5.1]). Moreover, we require that for every* $t \in [0,1]$*,* $\Sigma(\gamma(t))$ *is the shadow of* $(f(a), \Sigma(a) \cap D)$ *at* $f(\gamma(t))$ *(as defined in [BI08, Section 5.1]).*

4. Σ *is coherent along* X*-lines, in the following sense: for every* $a \in U$ *and* $t \in \mathbb{R}$*,* $\Sigma\big(\widetilde{\phi}^t(a)\big) = \Phi^t\big(\Sigma(a)\big)$*.*

5. Σ *is coherent at singularities, in the following sense. Let* $a \in f^{-1}(\Delta)$ *and* $g_\star \in \mathscr{A}_{f(a)}$*, where* $g_\star = (g : U \to M, b)$*, and assume that there exist*

   - *a* X*-line* γ *in* U *converging to the point* a *in positive (resp. negative) times, and*
   - *two points* $a' \in \gamma$ *and* $b' \in V$ *such that* $f(a') = g(b')$ *and the* X*-line in* V *passing through* b' *converges to* b *in positive (resp. negative) times.*

   Then the following holds:

$$g_\star \in \Sigma(a) \iff (g, b') \in \Sigma(a').$$

These conditions easily imply that Σ is coherent along unstable components in the following sense. If $a, a' \in U$ belong to the same unstable component of f, then for every $g_\star = (g : V \to M, b) \in \mathscr{A}_{f(a)}$, $b \in V^{\mathrm{un}}$ and there is a unique $b' \in V^{\mathrm{un}}$ in the

same unstable component as b such that $g(b') = f(a')$. In that case, we have:

$$(g, b) \in \Sigma(a) \iff (g, b') \in \Sigma(a').$$

Moreover, if $a_0 \in U$ belongs to the initial locus I_0 of f, then the map Σ is entirely determined by $\Sigma(a_0) \subset \mathscr{A}_{f(a_0)}$. If $p = f(a_0)$ and $\Sigma(a_0) = \Sigma_0$, we say that the upper enveloping tile $F = (f, \Sigma)$ **starts at** (p, Σ_0). Notice that f is of pure type if and only if $p \in M^{\text{pure}}$, and f is of unstable type if and only if $p \in M^{\text{un}}$. It will be convenient to include the choice of an initial *basepoint* a_0 and consider *pointed* upper enveloping tiles $F_\circ = (f, \Sigma, a_0)$. We will typically denote upper enveloping tiles by capital letters, such as F for (f, Σ), G for (g, Σ'), etc. and triples (f, Σ, a_0) as above by $F_\circ$.[5] Following [BI08], we call the map Σ the **shadow** of F.

For the rest of this section, we fix a point $p \in M^{\text{good}}$ and a section $\Sigma_0 \subset \mathscr{A}_p$. We denote by $\mathscr{E}_p(\Sigma_0)$ the collection of pointed upper enveloping tiles starting at (p, Σ_0).

The following lemma is a generalization of [BI08, Lemma 6.2]. Although the main idea is the same, the proof is much more complicated in our setup.

Lemma 2.2.11. *There exists an upper enveloping tile starting at* (p, Σ_0).

Proof. If $p \in M^{\text{un}}$, we simply consider the unstable tile f passing through p with an appropriate basepoint, and we extend Σ_0 in the only possible way (recall that η is uniquely integrable along unstable tiles).

If $p \in M^{\text{pure}}$, we construct an upper envelope starting at p in 3 steps.

- *Step 1: constructing a forward strip.* By definition, there is a pure flow line of X passing through p. Let $D \subset M$ be a small open smooth disk transverse to X and passing through p, and $\mathcal{A}$ be the trace of $\mathscr{A}$ on D. It is a two-dimensional prefoliation in the sense of [BI08, Section 5.1]. Let $f_0 : [0, \epsilon) \to D$ be the (forward)

[5] We choose the subscript $\circ$ and not $\star$ to emphasize that the basepoint lies in the initial locus I_0 of the tile, and plays a different role than a marked point.

upper envelope for $(p, \Sigma_0 \cap D)$ with respect to $\mathcal{A}$, with shadow $\overline{\Sigma}_0$, defined up to some $\epsilon > 0$. As opposed to [BI08, Lemma 6.2], we do not impose any lower bound on ϵ. Let now $f : [0, \epsilon) \times \mathbb{R} \to M$ be the extension of f_0 under the flow of X, defined by $f(x, t) \coloneqq \varphi_X^t(f_0(x))$. The one-dimensional shadow $\overline{\Sigma}_0$ extends to a shadow Σ for f. Notice that f is not quite a tile yet, since it might fail to be strongly unstable-complete and well-cornered. To remedy this, we will further extend f.

- *Step 2: unstable-completion.* Since the unstable locus of f is open, there exists countably many open intervals $I_n \subset [0, \epsilon)$, $n \in \mathbb{N}$, such that this unstable locus is exactly the disjoint union of the subsets $I_n \times \mathbb{R}$. Therefore, there exists countably many closed intervals (possibly reduced to a point) $J_n \subset [0, \epsilon)$, $n \in \mathbb{N}$, such that the pure locus (complement of the unstable locus) is the disjoint union of the subsets $J_n \times \mathbb{R}$. We choose J_0 such that $0 \in J_0$. For every $n \in \mathbb{N}$, there exists an unstable tile $g_n : V_n \to M$ and a $\mathcal{C}^1$ map $\varphi_n : I_n \times \mathbb{R} \to V_n$ satisfying $g_n \circ \varphi_n = f_{|I_n \times \mathbb{R}}$. If I_n is of the form (a, b) with $b < \epsilon$, then g_n is injective: the curve $t \in (a, b) \mapsto \varphi_n(t, 0)$ in V_n does not intersect the same X-line twice, as it reaches two distinct edges of V_n on both sides. Therefore, φ_n is an embedding. If $b = \epsilon$, g_n might fail to be injective, e.g., if the transverse disk D 'spirals' around a source singularity. However, one can easily shrink ϵ (or modify D) so that g_n becomes an embedding. Recall that the completion of an unstable tile does not have closed boundary components, thanks to our hypothesis that (X, η) has no disk of tangency.

  We denote by $\widetilde{f} : \widetilde{U} \to M$ the extension of f obtained by gluing the g_n's to f along the φ_n's. There is a natural embedding $\varphi : [0, \epsilon) \times \mathbb{R} \hookrightarrow \widetilde{U}$ satisfying $\widetilde{f} \circ \varphi = f$. By construction, $\widetilde{U}$ is homeomorphic to a simply connected subset of $\mathbb{R}^2$, and $\widetilde{f}$ is strongly unstable-complete. The shadow Σ extends in a unique way to a shadow $\widetilde{\Sigma}$ for $\widetilde{f}$, and the conditions of Definition 2.2.10 are all satisfied. However, we need to extend $\widetilde{f}$ a bit further in order to make it well-cornered.

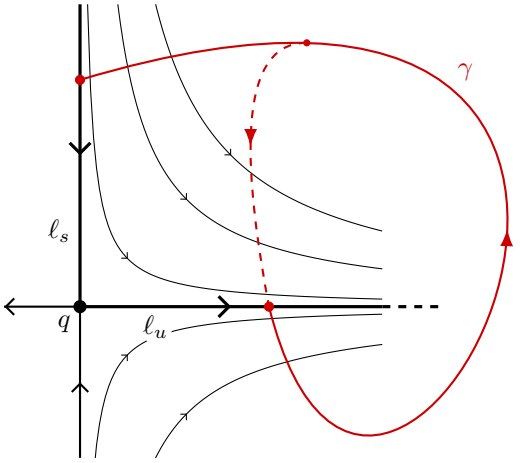


Figure 2.2.3: Sliding γ near its endpoint in Case 1.

- *Step 3: filling the slits.* Let $q \in \Delta_{\mathrm{sa}} \cup Q$, not surrounded by unstable manifolds, and $D_0 \subset \widetilde{U}$ be a singular region of $\widetilde{f}$ for q. Notice that if $\widetilde{f}$ fails to be graphical near q over D_0, or has a puncture or a slit at q in D_0, then one of the two cases happen:

  – **Case 1:** There exist two X-lines intersecting D_0 and mapping to opposite stable branches at q.

  – **Case 2:** There exist two X-lines intersecting D_0 and mapping to opposite unstable branches at q.

  We claim that Case 1 never happens. Otherwise, we could find a third X-line ℓ_u intersecting D_0 and mapping to an unstable branch at q. Since q is not surrounded by unstable manifolds, ℓ_u and at least one of the stable X-lines lie $\widetilde{U}^{\mathrm{pure}}$. We denote the latter one by ℓ_s. By definition of $\widetilde{f}$, there exists an embedded and strictly forward curve γ in $\widetilde{U}$ intersecting both ℓ_s and ℓ_u *positively*. Let us assume without loss of generality (up to reverting X and considering backward curves instead) that γ starts on ℓ_u and ends on ℓ_s. We then 'slide' γ near its endpoint along ℓ_s towards q and then along ℓ_u to obtain a *closed* strictly forward curve intersecting ℓ_u, which is impossible by Lemma 2.2.5. See Figure 2.2.3

  We now focus on Case 2. There are two subcases depending on whether $q \in \Delta_{\mathrm{sa}}$

or $q \in Q$. If $q \in \Delta_{\mathrm{sa}}$, the previous case implies that D_0 intersects a unique X-line mapping to a stable branch at q, and this X-line lies in $\widetilde{U}^{\mathrm{un}}$. Therefore, if $\widetilde{f}$ is not well-cornered at q over D_0, then it has a slit corresponding to the other stable branch at q. Similarly, if $q \in Q$ and if $\widetilde{f}$ is not well-cornered at q over D_0, then arguing as before shows that no X-line intersecting D_0 is mapped to the stable branch at q, and $\widetilde{f}$ has a slit corresponding to the stable branch at q. Note that it might happen that one of the unstable branches at q is the initial edge of $\widetilde{f}$, i.e., the X-line corresponding to $f_{|\{0\}\times\mathbb{R}}$. In that case, it is easy to see that $\widetilde{f}$ is well-cornered at q over D_0.

To summarize, in both subcases, if $\widetilde{f}$ is not well-cornered at q over D_0, then it has a slit corresponding to a stable branch of q, and the other stable branch lies in M^{un}; see Figure 2.2.4. Moreover, this stable branch might possibly be connected to another singularity q_- in negative times, in which case $q_- \in \Delta_{\mathrm{sa}}$.

We now explain how to fill the slits to make $\widetilde{f}$ well-cornered, and how to extend $\widetilde{\Sigma}$. With the previous notations, let $V \subset \widetilde{U}$ be the saturation of D_0 by X-lines. It is an open subset homeomorphic to an open disk. Clearly, there exists an embedding $\widehat{\varphi}_{q,D_0} = \widehat{\varphi} : V \hookrightarrow \widehat{V}$ where $\widehat{V}$ is homeomorphic to an open disk and $\widehat{\varphi}(V) \subset \widehat{V}$ is the complement of a proper embedding of $[0,\infty)$ into $\widehat{V}$, and such that $\widetilde{f}_{|V}$ extends on $\widehat{V}$ to a map $\widehat{f}_{q,D_0}$ satisfying:

- $\widehat{f}_{q,D_0} \circ \widehat{\varphi} = \widetilde{f}_{|V}$,
- $\widehat{f}_{q,D_0}$ is an injective immersion of an open disk into M tangent to η whose image contains q.

We can now glue $\widetilde{f}$ and $\widehat{f}_{q,D_0}$ along V to remove the slit at q over D_0. Performing this operation at every singularity with a slit, we obtain an immersion $\widehat{f} : \widehat{U} \to M$ tangent to η, where $\widehat{U}$ is homeomorphic to a simply connected domain of $\mathbb{R}^2$. We further have to extend $\widetilde{\Sigma}$ to $\widehat{U}$ so that the properties of Definition 2.2.10 are

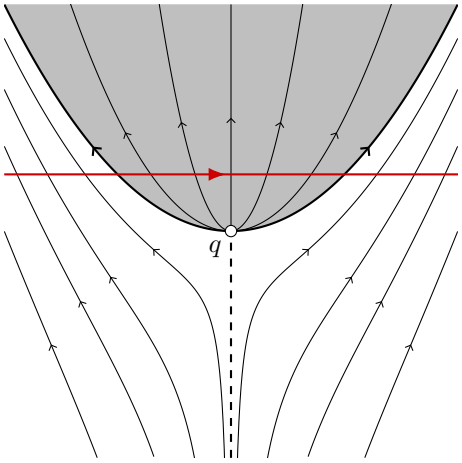


Figure 2.2.4: A slit at q. The red line corresponds to the forward curve induced by f_0.

satisfied. We fix q and D_0 as before, so that $\widetilde{f}$ has a simple slit at q over D_0, and we will view $D_0 \subset V \subset \widehat{V}$ as subsets of $\widehat{U}$. Let $x_0 \in \widehat{V}$ lying on the X-line mapped to the unstable branch ℓ_s at q, and let $p_0 \coloneqq \widehat{f}(x_0)$. Let $D \subset M$ be a small embedded disk in M passing through p_0 and transverse to X. Let $\gamma_0 : [-1, 1] \to \widehat{V}$ be a strictly forward curve with $\gamma_0(0) = x_0$ and such that $\widehat{f} \circ \gamma_0$ parametrized the intersection of D with $\widehat{f}\big(\overline{D}_0\big)$, where $\overline{D}_0$ denotes the closure of D_0 in $\widehat{U}$. We have to show that $\widehat{f} \circ \gamma_0$ is contained in the one-dimensional upper envelope of $\big(\widehat{f} \circ \gamma_0(-1), \widetilde{\Sigma}(\gamma_0(-1)) \cap D\big)$ in D, and that $\widetilde{\Sigma}(\gamma_0(t)) \cap D$ coincides with its shadow at $\widehat{f} \circ \gamma_0(t)$ for every $t \in [-1, 1] \setminus \{0\}$. In particular, this provides the desired extension of $\widetilde{\Sigma}$ at x_0, by considering the shadow of the former upper envelope at that point. To show the claim, we first observe that since $\mathscr{A}$ is backward and complete by hypothesis, a tile in $\mathscr{A}$ passing through q is of one of the following forms near q: a quadrant with backward boundary and convex corner at q, three quadrants (if $q \in \Delta_{\mathrm{sa}}$) or the union of a quadrant and a half plane (if $q \in Q$) with backward boundary and concave corner at q, or it contains q in the interior of its domain; see Figure 2.2.5. This induces a partition of $\mathscr{A}_q$ into three subsets $\mathscr{A}_q^{\mathrm{cvx}}$, $\mathscr{A}_q^{\mathrm{ccv}}$, $\mathscr{A}_q^{0}$ depending on whether a marked tile has a convex corner, concave corner, or no corner at q. Moreover, let p_+ be points in $\mathcal{N}$ in the right unstable

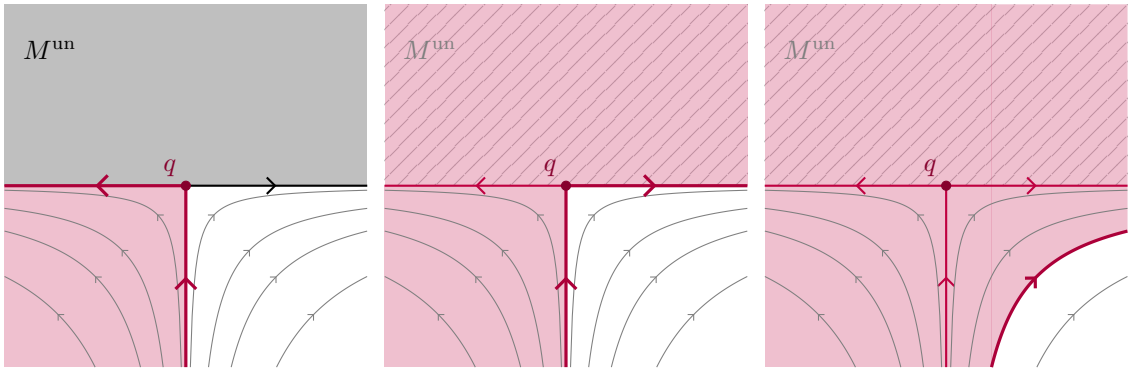
branches at q, and let x_+ be the preimage in $\widehat{V}$ of p_+ by $\widehat{f}$. Then 'sliding the marked point along stable/unstable branches at q' induces natural injective maps $\varsigma_0 : \mathscr{A}_{p_0} \to \mathscr{A}_q$ and $\varsigma_+ : \mathscr{A}_{p_+} \to \mathscr{A}_q$, which preserve the preorder $\precsim$. Moreover, because of the form of the tiles in $\mathscr{A}_q$, ς_0 is bijective, and ς_+ is a bijection onto $\mathscr{A}_q^0 \sqcup \mathscr{A}_q^{\mathrm{ccv}}$. Let $\gamma_+ : [0,1] \to \widehat{V}$ be an embedded strictly forward curve in $\widehat{V}$ from x_+ to $\gamma_0(1)$. If D_+ is an embedded disk in M transverse to X and containing $\widehat{f} \circ \gamma_+$, then $\widehat{f} \circ \gamma_+$ is contained in the one-dimensional upper envelope for $\big(p_+, \widetilde{\Sigma}(x_+) \cap D_+\big)$, and $\widetilde{\Sigma}(\gamma_+(t)) \cap D_+$ coincides with its shadow at $\widehat{f} \circ \gamma_+(t)$ for every $t \in [0,1]$. We define a section $\widehat{\Sigma}_+ \subset \mathcal{A}_{p_0} = \mathscr{A}_{p_0} \cap D$ as the smallest section of $\mathcal{A}_{p_0}$ (with respect to the inclusion) containing $\widetilde{\Sigma}_+ \coloneqq \varsigma_0^{-1} \circ \varsigma_+\big(\widetilde{\Sigma}(x_+)\big) \cap D$.[6] Notice that $\widehat{\Sigma}_+ \setminus \widetilde{\Sigma}_+$ only contains curves with a right boundary at p_0, corresponding to tiles which have a convex corner at q, i.e., whose image under ς_0 lie in $\mathscr{A}_q^{\mathrm{cvx}}$. Therefore, the invariance of the various structures of interest under the flow of X implies the curve $\gamma_{0|[0,1]}$ is contained in the one-dimensional upper envelope of $\big(p_0, \widehat{\Sigma}_+\big)$ in D and its shadow at $\widehat{f} \circ \gamma_0(t)$ for $t \in (0,1]$ is exactly $\widetilde{\Sigma}(\gamma_0(t)) \cap D$. Now, if $\widehat{\Sigma}_- \subset \mathcal{A}_{p_0}$ denotes the shadow at p_0 of the one-dimensional upper envelope of $\big(\widehat{f} \circ \gamma_0(-1), \widetilde{\Sigma}(\gamma_0(-1)) \cap D\big)$ in D, then $\widehat{\Sigma}_+ \subset \widehat{\Sigma}_-$, and $\widehat{\Sigma}_- \setminus \widehat{\Sigma}_+$ only contains curves with a right boundary at p_0. This implies that $\gamma_{0|(0,1]}$ is contained in the latter upper envelope, and its shadow at $\widehat{f} \circ \gamma_0(t)$ for $t \in (0,1]$ is $\widetilde{\Sigma}(\gamma_0(t)) \cap D$, as desired.

Moreover, if $x \in \widehat{V}$ is the preimage of q under $\widehat{f}$, there is a unique way to extend $\widehat{\Sigma}$ at x so that condition 5 of Definition 2.2.10 is satisfied.

Unfortunately, $\widehat{f}$ might still fail to be well-cornered, as new slits can appear in the extension process. This situation arises exactly when $\widetilde{f}$ has a slit corresponding to a stable branch of a singularity q which is connected to another singularity $q_- \in \Delta_{\mathrm{sa}}$

[6] Concretely, if $(A, \leq)$ is a preordered set and $\Sigma \subset A$, then the smallest section of A containing Σ is

$$\widehat{\Sigma} = \bigcup_{s \in \Sigma} \{a \in A \mid a \leq s\}.$$



(a) A tile in $\mathscr{A}_q^{\mathrm{cvx}}$. (b) A tile in $\mathscr{A}_q^{\mathrm{ccv}}$. (c) A tile in $\mathscr{A}_q^{0}$.

Figure 2.2.5: The three types of tiles in $\mathscr{A}_q$.

in negative times, and one of the stable branches at q_- lies in M^{un}. In that case, filling the slit at q creates a slit at q_- corresponding to the other stable branch at q_-. Our admissibility conditions on X guarantee that the latter stable branch is not connected to another singularity of X. We can then apply the previous argument at these 'secondary slits' to further extend $\left(\widehat{f}, \widehat{\Sigma}\right)$ to a pair $\left(\widehat{f}, \widehat{\Sigma}\right)$, where $\widehat{f}$ is now a *tile*. By construction, all the conditions of Definition 2.2.10 are satisfied, hence $\left(\widehat{f}, \widehat{\Sigma}\right)$ is an upper enveloping tile starting at (p, Σ_0).

□

We now define a natural preorder, the *extension preorder*, on $\mathscr{E}_p(\Sigma_0)$.

Definition 2.2.12. *Let $F_\circ = (f : U \to M, \Sigma, a_0)$ be a pointed upper enveloping tile. An **extension** of F is a pointed upper enveloping tile $G_\circ = (g : V \to M, \Sigma', b_0)$ together with an embedding $\varphi : U \to V$ satisfying*

- $g \circ \varphi = f$,
- $\varphi(a_0) = b_0$,
- *For every $a \in U$, $\Sigma'(\varphi(a)) = \Sigma(a)$.*

If φ is a diffeomorphism, we say that $F_\circ$ and $G_\circ$ are ***equivalent****. Otherwise, we say that $G_\circ$ is a* ***strict extension*** *of $F_\circ$. We call φ the* ***extension morphism****, and we write $\varphi : F_\circ \hookrightarrow G_\circ$, or simply $F_\circ \hookrightarrow G_\circ$.*

If F and G are upper enveloping tiles without basepoints, we say that G ***is an extension of*** *F (resp. F and G are* ***equivalent****) if there exist basepoints a_0 and b_0 for F and G, respectively, such that (G, b_0) is an extension of (F, a_0) (resp. (F, a_0) and (G, b_0) are equivalent).*

Observe that the extension morphism φ is *unique*: if φ and ψ are two extension morphisms from $F_\circ$ to $G_\circ$, then the set $\{a \in U \mid \varphi(a) = \psi(a)\}$ is nonempty and closed, and it is open since $g \circ \varphi = g \circ \psi$ and g is an immersion. This implies that if $\varphi : F_\circ \hookrightarrow G_\circ$ and $\psi : G_\circ \hookrightarrow F_\circ$, then ϕ and ψ are diffeomorphism which are inverse of each other. In particular, $F_\circ$ and $G_\circ$ are equivalent. This shows that preorder $\hookrightarrow$ on $\mathscr{E}_p(\Sigma_0)$ is a partial order up to equivalence.

We will be interested in the maximal elements for this preorder. By the following lemma, those are exactly the upper enveloping tiles without backward edges:

Lemma 2.2.13. *An upper enveloping tile admits a strict extension if and only if it has an edge with backward coorientation.*

Proof. Let $F = (f : U \to M, \Sigma)$ be an upper enveloping tile with a choice of initial basepoint $a_0 \in I_0$.

Let us first assume that $F_\circ = (F, a_0)$ admits a strict extension $\varphi : F_\circ \hookrightarrow G_\circ$, with $G_\circ = (g : V \to M, \Sigma', b_0)$. We identify U with the subset $\varphi(U) \subset V$. Since the extension is strict, the latter inclusion is strict. Let $x \in V \setminus U$, and γ be a forward curve from a_0 to x in V. Let $C_0 \subset [0, 1]$ be the connected component of $\gamma^{-1}(U)$ containing 0, and $t_0 \coloneqq \sup C_0$. Since U is open in V, $x_0 \coloneqq \gamma(t_0) \notin U$. Moreover, $x_0 \notin V^{\mathrm{un}}$, since f and g are strongly unstable-complete, so $x_0 \in V^{\mathrm{pure}}$. If ℓ denotes the pure X-line passing through x_0, then ℓ corresponds to a backward (improper) edge of f.

We now assume that F has an improper backward edge corresponding to a flow line γ of X in M^{pure}. Let $p \in \gamma$. By condition 3 of Definition 2.2.10, Σ induces a section $\Sigma_p \subset \mathscr{A}_p$ in a natural way, by extending the one-dimensional envelope and its shadow at p. By Lemma 2.2.11, we consider an upper enveloping tile $H_\circ = (h : W \to M, \Sigma'', c_0)$ starting at (p, Σ_p). We can then extend F across γ by attaching H to obtain an immersion $g : V \to M$ tangent to η together with a map $\Sigma' : V \to \mathcal{P}(\mathscr{A}_\star)$ extending Σ and Σ''. If γ is not connected to a singularity of X in positive or negative times, then one easily checks that (g, Σ') is an upper enveloping tile, and that it is a strict extension of F. In general, g might have *slits*. This happens precisely when γ is part of an improper *concave* corner of f, and/or γ is connected in negative times to a quadratic-like singularity $q \in Q^*$. The methods from Step 3 in the proof of Lemma 2.2.11 can be adapted to fill these slits (potentially in two steps for 'double slits') to obtain the desired extension of F. The (tedious yet straightforward) details are left to the reader. □

The previous lemma motivates the following

Definition 2.2.14 (See [BI08], Definition 6.10)**.** *A* ***forward envelope*** *is an upper enveloping tile such that all of its edges have forward coorientation.*

The construction of a forward envelope in our setup will be more delicate than in [BI08]. A crucial step is to show that the preorder $\hookrightarrow$ on $\mathscr{E}_p(\Sigma_0)$ is *directed*.

Lemma 2.2.15. *Two pointed upper enveloping tiles $F_\circ$ and $G_\circ$ starting at (p, Σ_0) admit a common extension. More precisely, there exist a pointed upper enveloping tile $H_\circ$ starting at (p, Σ_0) and two extension morphisms $\varphi : F_\circ \hookrightarrow H_\circ$ and $\psi : G_\circ \hookrightarrow H_\circ$.*

Proof. Let us write $F_\circ = (f : U \to M, \Sigma, a_0)$ and $G_\circ = (g : V \to M, \Sigma', b_0)$. We can first extend $F_\circ$ and $G_\circ$ so that both f and g are *T-complete*, in the following sense: if U contains a X-line mapped to a stable branch of a singularity which is either a quadratic singularity, or a saddle singularity whose other stable branch lies in

M^{un}, then U contains the corresponding unstable tile as well. For that, we apply the strategy of Lemma 2.2.13 at the improper backward edges of f and g corresponding to such stable branches, and we apply it a second time in the situations corresponding to double T-move.

We denote by $\mathcal{U}$ the collection of open subset $U' \subset U$ such that

- U' is *forward*, namely, for every point in $x \in U' \cap U^{\mathrm{good}}$, there exists a forward curve from a_0 to x contained in U',
- There exists an embedding $\iota : U' \hookrightarrow V$ such that $\iota(a_0) = b_0$ and $g \circ \iota = f_{|U'}$.

The embedding ι is then uniquely determined as follows: if $\gamma : [0,1] \to U$ is forward curve in U' from a_0 to x, then it lifts to $g_\star$ to a forward curve $\widetilde{\gamma} : \iota \circ \gamma$ in V from b_0 to $\iota(x)$. Since such a lift is unique, any other embedding ι' as in the second item would satisfy $\iota'(x) = \iota(x)$.

We argue that $\mathcal{U}$ is not empty. It is obvious if $p \in M^{\mathrm{un}}$, since f and g are strongly unstable-complete. If $p \in M^{\mathrm{pure}}$, it follows from condition 3 in Definition 2.2.10, since 1-dimensional upper envelopes are unique (see also [BI08, Corollary 6.4]). Moreover, Lemma 2.2.8 readily implies that $\mathcal{U}$ is stable under taking unions. We then define

$$U_g \coloneqq \bigsqcup_{U' \in \mathcal{U}} U',$$

with a corresponding embedding $\iota : U_g \to M$. Note that U_g is saturated by the flow of $\widetilde{X}$ and strongly unstable-complete. We make the following claim, which will be proved later.

Claim. *If γ is a forward curve in U from x_0 to x_1, then*

1. *If $x_0 \in U_g$, then $x_1 \in U_g$ if and only if $\gamma \subset U_g$,*
2. *If $x_1 \in U_g$, then $x_0 \in U_g$ and $\gamma \subset U_g$.*

Specializing at $x_0 = a_0$, this together with the conditions in Definition 2.2.10 readily imply that f and g have the same shadow on U_g, in the sense that $\Sigma' \circ \iota = \Sigma_{|U_g}$.

We construct a common extension of $F_\circ$ and $G_\circ$ by gluing f and g along U_g via ι. More precisely, we define

$$W \coloneqq \big(U \sqcup V\big) / \sim_\iota,$$

where $x \sim_\iota y$ if and only if $x \in U_g$, $y \in V$, and $y = \iota(x)$. We have natural injective maps $\varphi : U \hookrightarrow M$ and $\psi : V \hookrightarrow M$, together with a unique map $h : W \to M$ satisfying $h \circ \varphi = f$ and $h \circ \psi = g$. We extend Σ and Σ' to a map $\Sigma'' : W \to \mathcal{P}\big(\mathscr{A}_\star\big)$ satisfying $\Sigma'' \circ \varphi = \Sigma$ and $\Sigma'' \circ \psi = \Sigma'$.

So far, W is only defined as a topological space. We show that is has a natural structure of 2-manifold, possibly with boundary, such that φ and ψ are smooth, and h is a $\mathcal{C}^1$ immersion tangent to η. Note that W is simply connected by the Seifert–Van Kampen theorem.

We now analyze how f and g overlap near a point $x \in \partial U_g \subset U$. We first assume that $\underline{x \in U^{\mathrm{pure}}}$, and we denote by ℓ the X-line passing through x. Let us consider a small neighborhood of O_q of $q \coloneqq f(x)$ in M such that the connected component of $f^{-1}(O_q)$ containing x is diffeomorphic to an open disk. Denoting this connected component by U_x, we can further assume that $U_x \subset U^{\mathrm{reg}}$. We will show that $U_x \cap U_g$ is the left open half-disk in U_x bounded by ℓ, and that the connected component of $g^{-1}(O_q)$ intersecting $V_x \coloneqq \iota(U_x \cap U_g)$ is precisely V_x. In particular, g has an improper edge bounding the side of V_x corresponding to the flow line of X passing through q. Informally, g is entirely contained in f near x, and we can use f to define a chart for W near $\varphi(x)$.

To prove this, we consider two points $x_0, x_1 \in U_x$, where x_0 is to the left of ℓ and x_1 is to the right of ℓ, as well as a (strictly) forward curve γ from x_0 to x_1 contained in U_x. Since $x \in \partial U_g$, there must exist points along γ arbitrarily close to x belonging to U_g, and item 2 of the claim above implies that $x_0 \in U_g$. Moreover, if $x_1 \in U_g$, then

by item 1 of the claim above, γ would be entirely contained in U_g, contradicting that $x \in \partial U_g \subset U \setminus U_g$. Therefore, we obtained that $x_0 \in U_g$ and $x_1 \notin U_g$. This imply the desired description for $U_x \cap U_g$, and the description of g easily follows.

We now assume that $x \in f^{-1}(\Delta)$. Then $f(x) \in \Delta_{\mathrm{sa}} \cup Q$. With the above notations, the previous argument can be adapted to show that $U_x \cap U_g$ is an open quadrant or the interior of the union of two or three consecutive quadrants near x, and g is entirely contained in f near x. The case of two opposite open forward quadrants can be ruled out by considering two forward curves in U_g starting at a_0 and ending in these two quadrants. These curves can be extended passed consecutive stable and unstable branches at x in order to obtain two forward curves from a_0 to a common $x' \in U$. None of the stable branches at x are in U^{un}, since otherwise U_g would contain two consecutive quadrants at x. Moreover, one of these stable branches is not connected to any other singularity, and we assume that it is the stable branch involved in the previous extension. Since our two forward curves are related by forward-homotopies and T-moves, and that the chosen stable branch cannot be involved in a T-move, it must intersect both forward curves. A contradiction quickly follows by analyzing which portions of the forward curves are contained in U_g. The other possible cases for the shape of U_g at x are easier to investigate, and the details are left to the reader.

We can define a collection of open subsets $\mathcal{V}$ of V in the same manner as for U, and the corresponding maximal element V_f coincides with $\iota(U_g)$, with associated embedding $\iota^{-1} : \iota(U_g) \hookrightarrow V$. By symmetry, a similar description of the overlap between g and f holds near $\partial V_f \subset V$. Therefore, there exists a (unique) smooth structure on W making φ and ψ smooth and for which h is $\mathcal{C}^1$.

From the above arguments, it is easy to check that h is an upper enveloping tile starting at (p, Σ_0). If $c_0 \coloneqq \varphi(a_0) = \psi(b_0)$, then $H_\circ \coloneqq (h : W \to M, \Sigma'', c_0)$ is a common extension of $F_\circ$ and $G_\circ$, via φ and ψ.

We finish the proof by showing the claim. We first make the following observation:

if γ_0 and γ_1 are two forward curves in U with the same endpoints, then γ_0 is contained in U_g if and only if γ_1 is. Indeed, by Lemma 2.2.7, γ_0 and γ_1 differ by a finite sequence of T-moves and a forward isotopy. Since the latter preserves the set of pure X-lines and unstable components intersected by a forward curve, we can reduce to the case where γ_0 and γ_1 differ by a full T-move. Then, the desired equivalence easily follows from the conditions in Definition 2.2.10 and the fact that f and g are T-complete and without slits.

Now, with the notations from the claim, we first assume that $x_0 \in U_g$ and we denote by γ_0 a forward curve in U_g from a_0 to x_0. The concatenation $\widetilde{\gamma}_1 \coloneqq \gamma_0 * \gamma$ is then a forward curve from a_0 to x_1. If $\gamma \subset U_g$, then γ lifts to $(g, \iota(x_0))$, and $\widetilde{\gamma}_1$ lifts to $g_\star = (g, b_0)$. In that case, by conditions 0, 1 and 3 of Definition 2.2.10, there exists a neighborhood U' of γ which is an element of $\mathcal{U}$, hence $\gamma \subset U_g$. We now assume that $x_1 \in U_g$, and we show that $\gamma \subset U_g$, hence $x_0 \in U_g$ as well. Let γ_1 be a forward curve in U_g from a_0 to x_1, and γ_0 be a forward curve in U from a_0 to x_0. We still write $\widetilde{\gamma}_1 = \gamma_0 * \gamma$. Since γ_1 and $\widetilde{\gamma}_1$ are two forward curves from a_0 to x_1 and $\gamma_1 \subset U_g$, the previous paragraph implies that $\widetilde{\gamma}_1 \subset U_g$, hence $\gamma \subset U_g$ as desired. □

We can now prove the main result of this section:

Proposition 2.2.16 (See [BI08], Lemma 6.11)**.** *There exists a forward envelope starting at (p, Σ). It is unique up to equivalence.*

Proof. By the previous observations, we only need to show that $\mathscr{E}_p(\Sigma_0)$ admits a maximal element with respect to the extension preorder. Indeed, a maximal element is a forward envelope starting at (p, Σ_0), and it is unique up to equivalence since the preorder is directed. In particular, it is a greatest element.

We consider the disjoint union

$$\underline{U} \coloneqq \bigsqcup_{\substack{F_\circ \in \mathscr{E}_p(\Sigma_0), \\ f : U \to M}} U,$$

which ranges over all the pointed upper enveloping surfaces $F_\circ = (f : U \to M, \Sigma, a_0)$ starting at (p, Σ_0). We denote the component corresponding to $F_\circ$ by $U_{F_\circ}$. To avoid set-theoretic issues, we can restrict to the ones with $U \subset \mathbb{R}^2$. By definition, every tile is equivalent to one whose domain is a subset of $\mathbb{R}^2$.

There is a natural function $\underline{f} : \underline{U} \to M$ whose restriction on the component corresponding to $F_\circ = (f : U \to M, \Sigma, a_0)$ is f. We define an equivalence relation $\equiv$ on $\underline{U}$ as follows: if $a \in U_{F_\circ}$ and $b \in U_{G_\circ}$, then $a \equiv b$ if and only if $G_\circ$ is an extension of $F_\circ$ with extension morphism φ, and $\varphi(a) = b$. By the uniqueness of the extension morphism (see above), it is easy to check that $\equiv$ is a well-defined equivalence relation on $\underline{U}$. Moreover, if $a \equiv b$ then $\underline{f}(a) = \underline{f}(b)$. Therefore, $\underline{f}$ induces a map

$$f^\infty : U^\infty \to M, \qquad U^\infty := \underline{U}/\equiv .$$

Here, we endow U^∞ with the quotient topology.

We now show that U^∞ is a 2-dimensional manifold, possibly with boundary. By definition, U^∞ admits a smooth atlas for which f^∞ is a $\mathcal{C}^1$ immersion tangent to η, and U^∞ is saturated by the flow of X. Note that U^∞ has a distinguished point a_0^∞ corresponding to the class of the initial point of any element in $\mathscr{E}_p(\Sigma_0)$, and every point in U^∞ can be connected to a_0^∞ by a continuous path. Moreover, for every $F_\circ = (f : U \to M, \Sigma, a_0) \in \mathscr{E}_p(\Sigma_0)$, there is a natural smooth embedding $q_{F_\circ} : U \to U^\infty$ satisfying $f^\infty \circ q_{F_\circ} = f$. Lemma 2.2.15 easily implies that U^∞ is Hausdorff and satisfies the following property : for every compact $K \subset U^\infty$, there exists $F_\circ \in \mathscr{E}_p(\Sigma_0)$ and a compact $K' \subset U$ such that $K = q_{F_\circ}(K')$. One deduces that U^∞ is simply connected and aspherical, since every $U_{F_\circ}$ is. Observe that if $p \in M^{\mathrm{un}}$ then U^∞ has no boundary. Otherwise, U^∞ has a unique boundary component and it contains a_0^∞. Furthermore, U^∞ admits a (continuous) Riemannian metric (by either pulling back the Riemannian metric $\boldsymbol{g}$ on M along f^∞, or patching together the

intrinsic Riemannian metrics on the $U_{F_\circ}$'s). Since it is Hausdorff, it is metrizable and since it is connected, it is second-countable; see [Spi99, Appendix A, Theorem 1].[7] Therefore, we have shown that U^∞ is a smooth surface, possibly with boundary, which embeds in to $\mathbb{R}^2$.

The η-surface $f^\infty : U^\infty \to M$ is obviously strongly unstable-complete. One easily checks that it is well-cornered, using that every path $\gamma : [0,1] \to U^\infty$ admits a lift $\widetilde{\gamma} : [0,1] \to U_{F_\circ}$ for some $F_\circ \in \mathscr{E}_p(\Sigma_0)$, i.e., it satisfies $q_{F_\circ} \circ \widetilde{\gamma} = \gamma$. Therefore, f^∞ is a tile.

The collection of shadows of elements $F_\circ \in \mathscr{E}_p(\Sigma_0)$ naturally induces a shadow Σ^∞ for f^∞. It is uniquely characterized by the following property : for every $F_\circ \in \mathscr{E}_p(\Sigma_0)$, $\Sigma^\infty \circ q_{F_\circ} = \Sigma$, where Σ denotes the shadow of $F_\circ$. It is straightforward to check that $(f^\infty, \Sigma^\infty)$ is an upper enveloping tile, using that any path in U^∞ can be lifted to a path in some $U_{F_\circ}$. Therefore, $F_\circ^\infty \coloneqq (f^\infty, \Sigma^\infty, a_0^\infty)$ is an element of $\mathscr{E}_p(\Sigma_0)$. By construction, $F_\circ^\infty$ is an extension of every $F_\circ \in \mathscr{E}_p(\Sigma_0)$, which means that it is a greatest element for the extension order. □

2.3 Extending the order

Let $\mathscr{U}$ denote the collection of forward envelopes starting at the pairs (p, Σ_0) where

- $p \in M^{\text{good}}$, and p **does not belong to the pure stable branch of a quadratic-like singularity,**[8]
- $\Sigma_0 \subset \mathscr{A}_p$ is any section.

To avoid set-theoretic issues, we can restrict our attention to the forward envelopes whose domains are subsets of $\mathbb{R}^2$, which is always the case up to equivalence.

[7] We are grateful to Ryan Unger for pointing out this reference to us.

[8] This condition is not strictly necessary, but it will greatly simplify the construction of the preorder on $\mathscr{U}$ by drastically reducing the number of cases to consider in several proofs.

We will extend the results of [BI08, Section 6.2] to our setup. More precisely, we will define a preorder $\precsim$ on $\mathscr{U}_\star$, making $(\mathscr{U}, \precsim)$ a weak prefoliation. This preorder has to satisfy the following checklist:

- It is compatible with the geometric order and coherent along intersections,
- It extends to the set of *completed* forward envelopes $\overline{\mathscr{U}}$,
- Every marked tile in $\mathscr{A}_\star$ with marking on an edge has an *upper neighbor* (see [BI08, Definition 4.10] and Section 2.4.2.2 below) in $\overline{\mathscr{U}}$.

Unfortunately, these requirements are not enough the entirely specify $\precsim$ and make it a total relation. We will adapt the methods of [BI08] and define $\precsim$ by successive 'layers', depending on how the forward envelopes intersect. This will crucially rely on the *forward structure* with respect to X on forward envelopes, which essentially allows us to reduce the construction to a 1-dimensional problem. However, this forward structure is more complicated in our setup because of the presence of singularities.

2.3.1 Forward order on pure X-lines

Let $f : U \to M$ be a tile.

Definition 2.3.1. *We denote by $\mathcal{L} = \mathcal{L}_f$ the set of pure X-lines of f (see Definition 2.2.3). We define a (strict) preorder $\lhd$ on $\mathcal{L}$ as follows: if $\ell_0, \ell_1 \in \mathcal{L}$, then $\ell_0 \lhd \ell_1$ if and only if there exists a forward curve from ℓ_0 to ℓ_1. We denote by $\trianglelefteq$ the corresponding (nonstrict) preorder.*

Lemma 2.2.5 immediately implies that $\trianglelefteq$ is a *partial order*. We call it the **forward order** on $\mathcal{L}$.

Recall that two forward-homotopic curves intersect the same X-lines in $\mathcal{L}$. If they differ by a T-move across a singularity $x \in U$, the sets of X-lines in $\mathcal{L}$ that they intersect differ as follows. One intersects the stable branch $\ell_s \in \mathcal{L}$ of x but not its

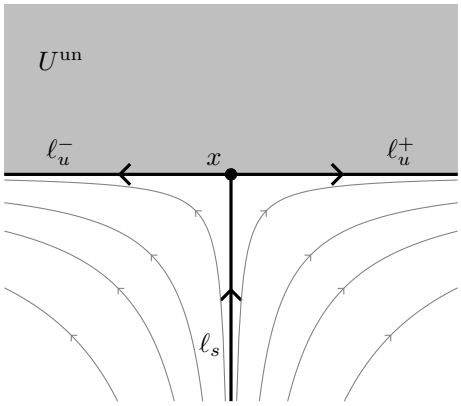


Figure 2.3.1: A special triple.

unstable branches $\ell_u^-, \ell_u^+ \in \mathcal{L}$, and the other intersects ℓ_u^- and ℓ_u^+ but not ℓ_s. Otherwise, they intersect the same X-lines in $\mathcal{L} \setminus \{\ell_s, \ell_u^-, \ell_u^+\}$. Here, we chose our notations so that $\ell_u^- \lhd \ell_u^+$, see Figure 2.3.1. We call $\left(\ell_s, \ell_u^-, \ell_u^+\right)$ a **special triple** in $\mathcal{L}$. The next lemma is an easy consequence of Lemma 2.2.5. The proof is left to the reader.

Lemma 2.3.2. *We have the following properties:*

1. *ℓ_s and $\ell_u^\pm$ are not comparable with respect to $\trianglelefteq$.*

2. *If $\ell \in \mathcal{L}$, then*

$$\begin{aligned} \ell_s \lhd \ell &\iff \ell_u^+ \lhd \ell, \\ \ell \lhd \ell_s &\iff \ell \lhd \ell_u^-, \\ \ell_u^- \trianglelefteq \ell \trianglelefteq \ell_u^+ &\iff \ell \in \{\ell_u^-, \ell_u^+\}. \end{aligned}$$

Notice that if f is the underlying tile of a forward envelope with a (necessarily unique and forward) proper edge, then this edge corresponds to *the smallest* element of $(\mathcal{L}, \trianglelefteq)$. If f has no edge and has an initial unstable component U_0, then the minimal elements of $(\mathcal{L}, \trianglelefteq)$ are exactly the X-lines in $\partial U_0 \subset U$ with backward coorientation with respect to U_0.

2.3.2 Backward domains of overlap

In this section, we adapt the definitions from [BI08, Section 6.2.1]. Our goal is to precisely describe how marked forward envelopes overlap in the backward region determined by the marked point. This will be crucial in the definition of the preorder on forward envelopes in the next section. As in [BI08, Section 6.2], this preorder will depend on how two marked forward envelopes overlap in this backward region.

2.3.2.1 Backward domain of an envelope

We fix a point $p \in M^{\mathrm{good}}$ together with a marked forward envelope $f_\star = (f : U \to M, a) \in \mathscr{U}_p$, which implicitly comes with the additional data of a shadow Σ and an initial locus I_0.

We denote by $\mathcal{I}_{f_\star} \subset \mathcal{L}_f$ the set of all X-lines in $\mathcal{L}_f$ intersected by the *backward* curves starting at a, or alternatively, by the forward curves from I_0 to a. If f is of pure type and a lies on the initial edge e_0 of f, we set $\mathcal{I}_{f_\star} \coloneqq \{e_0\}$ instead. This set is well-defined by the second item of Definition 2.2.10. It is empty exactly when f is of unstable type and $a_0 \in I_0 = U_0$.

Because of the presence of special triples in $\mathcal{I}_{f_\star}$, this set is not necessarily an interval for the preorder $\trianglelefteq$, unlike in [BI08]. To remedy this, we define an equivalence relation $\sim_T$ on $\mathcal{I}_{f_\star}$. We first define a homogeneous relation $\sim_T^{\mathrm{pre}}$ as follows: if $\ell_0, \ell_1 \in \mathcal{I}_{f_\star}$, then $\ell_0 \sim_T^{\mathrm{pre}} \ell_1$ if and only if $\ell_0 = \ell_1$, or ℓ_0 and ℓ_1 belong to the same special triple. We then define $\sim_T$ as the transitive closure of $\sim_T^{\mathrm{pre}}$. Notice that a X-line ℓ belongs to at most two distinct special triples, in which case ℓ coincides with the positive or negative unstable branch of one and with the stable branch of the other. The **reduction** of $\mathcal{I}_{f_\star}$ is defined by

$$\mathcal{I}_{f_\star}^{\mathrm{red}} \coloneqq \mathcal{I}_{f_\star} / \sim_T .$$

By Lemma 2.3.2, the partial order $\trianglelefteq$ naturally induces a partial order $\trianglelefteq^{\mathrm{red}}$ on $\mathcal{I}_{f_\star}^{\mathrm{red}}$,

and we have:

Lemma 2.3.3. *The poset* $\left(\mathcal{I}_{f_\star}^{\mathrm{red}}, \trianglelefteq^{\mathrm{red}}\right)$ *embeds into* $([0,1], \leq)$ *as a closed subset.*

Proof. Let $\gamma : [0,1] \to U$ be a forward curve from I_0 to a, and let $C \coloneqq \gamma^{-1}(U^{\mathrm{pure}})$. Then C is a closed subset of $[0,1]$ in bijection with the set of pure X-lines intersected by γ. Let $\lambda : C \hookrightarrow \mathcal{I}_{f_\star}$ be the increasing map sending $t \in C$ to the X-line passing through $\gamma(t)$. It induces a nondecreasing map $\lambda^{\mathrm{red}} : C \to \mathcal{I}_{f_\star}^{\mathrm{red}}$ which is surjective by Lemma 2.2.7. We define an equivalence relation $\sim$ on C as follows: if $t, t' \in C$, then $t \sim t'$ if and only if $\lambda^{\mathrm{red}}(t) = \lambda^{\mathrm{red}}(t')$. One easily checks that if $t < t'$ and $t \sim t'$, then $\{t, t'\} \subset \partial C$ and $(t, t') \cap C$ is either empty or a single point, depending on whether $\lambda(t)$ and $\lambda(t')$ are negative and positive branches of a special triple or of a double special triple, respectively. The order $\leq$ induces an order $\leq^{\mathrm{red}}$ on the quotient $C^{\mathrm{red}} \coloneqq C/\sim$. Since λ^{red} induces an isomorphism between $\left(C^{\mathrm{red}}, \leq^{\mathrm{red}}\right)$ and $\left(\mathcal{I}_{f_\star}^{\mathrm{red}}, \trianglelefteq^{\mathrm{red}}\right)$, it is enough to show that $\left(C^{\mathrm{red}}, \leq^{\mathrm{red}}\right)$ embeds into $([0,1], \leq)$ as a closed subset.

Let $\mathcal{U}$ be the collection of connected components of $[0,1] \setminus C$ of the form (t, t') with $t \sim t'$. We consider the closed set

$$\widehat{C} \coloneqq [0,1] \setminus \bigcup_{U \in \mathcal{U}} U \subset [0,1].$$

Notice that $C \subset \widehat{C}$ is closed, and $\sim$ extends to $\widehat{C}$ so that $\leq^{\mathrm{red}}$ extends to a total order on $\widehat{C}^{\mathrm{red}} \coloneqq \widehat{C}/\sim$, and the induced map $C^{\mathrm{red}} \to \widehat{C}^{\mathrm{red}}$ is (strictly) order preserving. Moreover, the image of the latter map is closed for the order topology. We can loosely think of quotienting $\widehat{C}$ by $\sim$ as 'collapsing the gaps' between the connected components of $\widehat{C}$. Recall the following characterization due to Cantor (see [Jec03, Theorem 4.3] for the unbounded case): every totally ordered set which is bounded, dense, separable, and complete is either a singleton or isomorphic to $([0,1], \leq)$. It is straightforward to check that $\left(\widehat{C}^{\mathrm{red}}, \leq^{\mathrm{red}}\right)$ satisfies these properties, hence embeds into $([0,1], \leq)$, as a closed subset. The composition $C^{\mathrm{red}} \to \widehat{C}^{\mathrm{red}} \to [0,1]$ yields the

desired embedding. □

A priori, the embedding of Lemma 2.3.3 depends on the choice of forward curve. However, one can easily show that a different forward curve induces the same embedding up to post-composition with an increasing homeomorphism of $[0,1]$. Alternatively, one can appeal to the following elementary fact: if a poset $(I, \preceq)$ embeds into $([0,1], \leq)$ as a closed subset, then this embedding is continuous for the order topology, and it is unique up to post-composition with an increasing homeomorphism of $[0,1]$.[9]

We now fix an embedding as in the previous lemma, and we define $I_{f_\star} \subset [0,1]$ to be the image of $\mathcal{I}_{f_\star}^{\mathrm{red}}$ under this embedding. We denote by $\mathcal{D}_{f_\star}$ the domain in U defined as the union of all of the X-lines in $\mathcal{I}_{f_\star}$, together with all of the unstable components of f crossed by the forward curves from I_0 to a. We call $\mathcal{D}_{f_\star}$ the **backward domain** of $f_\star$.[10]

2.3.2.2 Overlap of two envelopes

Let us consider a second marked forward envelope $g_\star = (g : V \to M, b) \in \mathscr{U}_p$ with corresponding poset $\mathcal{I}_{g_\star} \subset \mathcal{L}_g$ and backward domain $\mathcal{D}_{g_\star} \subset V$. If $\gamma : [0,1] \to U$ is a *backward* curve starting from a, a **lift** of γ to $g_\star$ is a curve $\widetilde{\gamma} : [0,1] \to V$ starting from b satisfying $f \circ \gamma = g \circ \widetilde{\gamma}$. If it exists, such a lift is unique and is a backward curve in V. We define $\mathcal{I}_{f_\star \cap g_\star} \subset \mathcal{I}_{f_\star}$ to be the set of all X-lines in $\mathcal{L}_{f_\star}$ intersected by some backward curve starting at a which lifts to $g_\star$. If g is of pure type and b belongs to the initial edge e_0' of g, we set $\mathcal{I} \coloneqq \{e_0'\}$ instead. We define the **backward domain of overlap** $\mathcal{D}_{f_\star \cap g_\star} \subset \mathcal{D}_{f_\star}$ to be the union of all of the X-lines in $\mathcal{I}_{f_\star \cap g_\star}$ together with

[9] If $\iota : I \to [0,1]$ is such an embedding with closed image C, then ι is open so $\iota^{-1} : C \to I$ is continuous. Since C is compact and the order topology on I is Hausdorff, ι^{-1} is a homeomorphism, hence ι is continuous. If $\iota' : I \to [0,1]$ is another embedding with closed image C', we simply extend $\iota' \circ \iota^{-1} : C \to [0,1]$ to an increasing homeomorphism $\phi : [0,1] \to [0,1]$ such that ϕ is affine on the connected components of $[0,1] \setminus C$, and $\iota' = \phi \circ \iota$.

[10] Beware: this domain has a more complicated topology than the domain D_X from [BI08, Section 6.2.1]. Because of the way we defined it, $\mathcal{D}_{f_\star}$ might not be simply connected as it can have punctures. This is not gonna be relevant for our arguments, and these missing points can be added for psychological comfort.

all of the unstable components of f crossed by the aforementioned backward curves lifting to $g_\star$. Notice that $\mathcal{I}_{f_\star \cap g_\star}$ is an *upward section* of $\mathcal{I}_{f_\star}$, in the following sense: for all $\ell_0 \in \mathcal{I}_{f_\star \cap g_\star}$ and $\ell_1 \in \mathcal{I}_{f_\star}$, if $\ell_0 \trianglelefteq \ell_1$ then $\ell_1 \in \mathcal{I}_{f_\star \cap g_\star}$. The same is true for the reduction

$$\mathcal{I}^{\mathrm{red}}_{f_\star \cap g_\star} \coloneqq \mathcal{I}_{f_\star \cap g_\star} / \sim_T, \qquad \mathcal{I}^{\mathrm{red}}_{f_\star \cap g_\star} \subset \mathcal{I}^{\mathrm{red}}_{f_\star}.$$

Let $I_{f_\star \cap g_\star} \subset I_{f_\star}$ denote the image of $\mathcal{I}^{\mathrm{red}}_{f_\star}$ under the chosen embedding from Lemma 2.3.3. Notice that $I_{f_\star \cap g_\star}$ can be empty, for instance if g is of unstable type and b belongs to the initial unstable component of g. We refer to the situation $I_{f_\star \cap g_\star} = \varnothing$ as case (I0). Otherwise, if $I_{f_\star \cap g_\star} \neq \varnothing$, we define

$$m \coloneqq \inf I_{f_\star \cap g_\star} \in I_{f_\star},$$

and we distinguish several cases:

(I1) $m \in I_{f_\star \cap g_\star}$, and $I_{f_\star \cap g_\star} = [m, 1] \cap I_{f_\star}$. Then $\mathcal{I}_{f_\star \cap g_\star}$ is of one of the following three types:

  (i) $\mathcal{I}_{f_\star \cap g_\star} = \big\{\ell \in \mathcal{I}_{f_\star} \mid \ell_m \trianglelefteq \ell\big\}$, where $\ell_m \in \mathcal{I}_{f_\star}$. Here, ℓ_m is the least element of $\mathcal{I}_{f_\star \cap g_\star}$. If ℓ_m is the negative unstable branch of a special triple in $\mathcal{I}_{f_\star}$, then it corresponds to the initial edge of $g_\star$.[11]

  (ii) $\mathcal{I}_{f_\star \cap g_\star} = \{\ell_s\} \cup \big\{\ell \in \mathcal{I}_{f_\star} \mid \ell_u^- \trianglelefteq \ell\big\}$, where $(\ell_s, \ell_u^-, \ell_u^+)$ is a special triple in $\mathcal{I}_{f_\star}$. Here, ℓ_s and ℓ_u^- are the two minimal elements of $\mathcal{I}_{f_\star \cap g_\star}$.

  (iii) $\mathcal{I}_{f_\star \cap g_\star} = \{\ell_s, \ell_s'\} \cup \big\{\ell \in \mathcal{I}_{f_\star} \mid \ell_u'^- \trianglelefteq \ell\big\}$, where $(\ell_s, \ell_u^-, \ell_u^+)$ and $(\ell_s', \ell_u'^-, \ell_u'^+)$ are a special triples in $\mathcal{I}_{f_\star}$ with $\ell_s' = \ell_u^-$. Here, ℓ_s, ℓ_s' and $\ell_u'^-$ are the three minimal elements of $\mathcal{I}_{f_\star \cap g_\star}$.

In these three cases, the minima of $\mathcal{I}_{f_\star}$ might be forward edges of an unstable

[11] Recall that we exclude the case where ℓ_m is the stable branch of a quadratic-like singularity and corresponds to the initial edge of $g_\star$.

component of U contained in $\mathcal{D}_{f_\star \cap g_\star}$, except in case (i) if ℓ_m corresponds to the initial edge of $g_\star$.

(I2) $m \notin I_{f_\star \cap g_\star}$, and $I_{f_\star \cap g_\star} = (m, 1] \cap I_{f_\star}$. In that case, there exists a (possibly non unique) $\ell_m \in \mathcal{I}_{f_\star}$ such that $\mathcal{I}_{f_\star \cap g_\star} = \left\{\ell \in \mathcal{I}_{f_\star} \mid \ell_m \lhd \ell\right\}$. We can further distinguish three cases:

(i) ℓ_m does not belong to a special triple in $\mathcal{I}_{f_\star}$, and ℓ_m is the infimum of $\mathcal{I}_{f_\star \cap g_\star}$ in $\mathcal{I}_{f_\star}$,

(ii) There exists a special triple $(\ell_s, \ell_u^-, \ell_u^+)$ such that ℓ_u^+ is not the stable branch of another special triple, and $\ell_m \in \{\ell_s, \ell_u^+\}$. In that case, $\mathcal{I}_{f_\star \cap g_\star}$ has exactly two maximal lower bounds in $\mathcal{I}_{f_\star}$, namely ℓ_s and ℓ_u^+.

(iii) There exists a special triple $(\ell_s, \ell_u^-, \ell_u^+)$ such that ℓ_u^+ is the stable branch of another special triple $(\ell_s', \ell_u'^-, \ell_u'^+)$, and $\ell_m \in \{\ell_s, \ell_s', \ell_u^+\}$. In that case, $\mathcal{I}_{f_\star \cap g_\star}$ has exactly three maximal lower bounds in $\mathcal{I}_{f_\star}$, namely ℓ_s, ℓ_s', and $\ell_u'^+$.

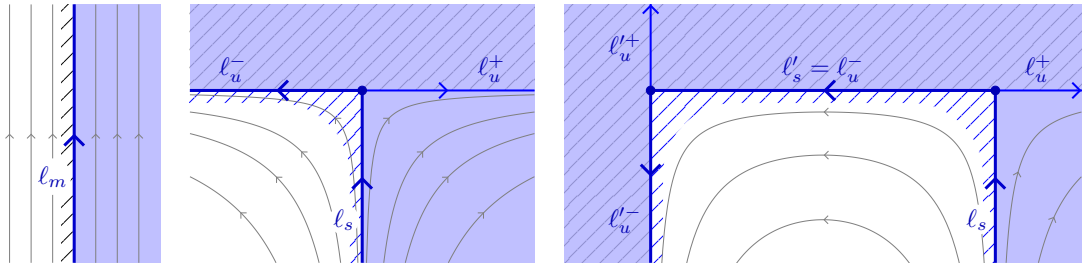
Notice that the case $\mathcal{I}_{f_\star \cap g_\star} = \{\ell_s\} \cup \left\{\ell \in \mathcal{I}_{f_\star} \mid \ell_u^+ \trianglelefteq \ell\right\}$, where $(\ell_s, \ell_u^-, \ell_u^+)$ is a special triple with $\ell_s, \ell_u^+ \in \mathcal{I}_{f_\star}$, cannot occur. Indeed, if both ℓ_s and ℓ_u^+ are in $\mathcal{I}_{f_\star \cap g_\star}$, then the unstable locus in U corresponding to this special triple is in $\mathcal{D}_{f_\star \cap g_\star}$, since forward envelopes have at most one proper edge,[12] and it follows that ℓ_u^- is also in $\mathcal{I}_{f_\star \cap g_\star}$. Other cases can be ruled out by similar arguments. This discussion implies:

Lemma 2.3.4. *Exactly one of the following holds:*

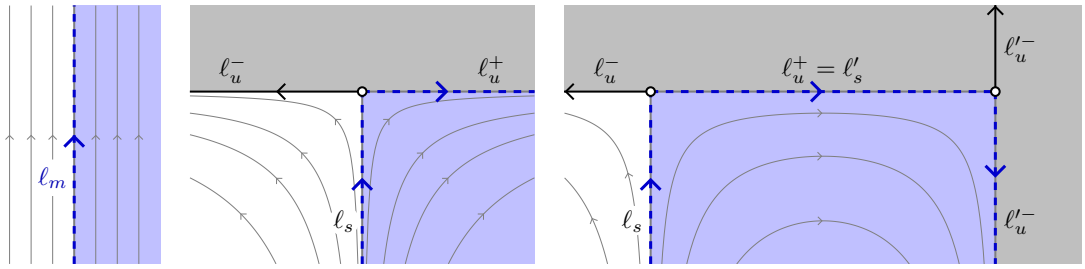
(I0) $\mathcal{I}_{f_\star \cap g_\star} = \varnothing$.

(I1) $\mathcal{I}_{f_\star \cap g_\star}$ *has one, two, or three minimal elements.*

[12]We also implicitly use the unique integrability properties of η established in the proof of Lemma 2.2.1.



(a) Case (I1) (i). (b) Case (I1) (ii). (c) Case (I1) (iii).



(d) Case (I2) (i). (e) Case (I2) (ii). (f) Case (I2) (iii).

Figure 2.3.2: Double overlaps. In each picture, the whole rectangle corresponds to $\mathcal{D}_{f_\star}$, and the blue region corresponds to $\mathcal{D}_{f_\star \cap g_\star}$. The blue hatched regions are possible unstable regions contained in $\mathcal{D}_{f_\star}$. Thick blue lines belong to $\mathcal{D}_{f_\star}$, while dashed blue lines do not. The gray regions are unstable regions in $\mathcal{D}_{f_\star}$.

(I2) $\mathcal{I}_{f_\star \cap g_\star}$ *has no minimal element but has one, two, or three maximal lower bounds in* $\mathcal{I}_{f_\star}$.

Moreover, a unique minimal element of $\mathcal{I}_{f_\star \cap g_\star}$ *is a least element, and a unique maximal lower bound of* $\mathcal{I}_{f_\star \cap g_\star}$ *in* $\mathcal{I}_{f_\star}$ *is an infimum.*

Figure 2.3.2 summarizes all the possibilities for these cases, as well as the corresponding domains $\mathcal{D}_{f_\star \cap g_\star}$.

There is a natural (strictly) order preserving map $\alpha_{f_\star \cap g_\star} : \mathcal{I}_{f_\star \cap g_\star} \to \mathcal{I}_{g_\star}$ defined as follows. If $p \in M^{\text{pure}}$ and ℓ is the X-line in U passing through a, we define $\alpha_{f_\star \cap g_\star}(\ell)$ as the X-line in V passing through b. Otherwise, if $\ell \in \mathcal{I}_{f_\star \cap g_\star}$ and $\gamma : [0,1] \to U$ is a forward curve from ℓ to a which admits a lift $\widetilde{\gamma} : [0,1] \to V$ to g with $\widetilde{\gamma}(1) = b$, then $\alpha_{f_\star \cap g_\star}(\ell)$ is the X-line in V passing through $\widetilde{\gamma}(0)$. By Lemma 2.2.8, this definition does not depend on the choice of the forward curve γ. By construction, $\alpha_{f_\star \cap g_\star}$ is strictly order preserving.

There is also a natural map $\varphi_{f_\star \cap g_\star} : \mathcal{D}_{f_\star \cap g_\star} \to \mathcal{D}_{g_\star}$ satisfying

$$g \circ \varphi_{f_\star \cap g_\star} = f_{|\mathcal{D}_{f_\star \cap g_\star}}, \tag{2.3.1}$$

defined as follows. If $p \in M^{\mathrm{pure}}$ and $x \in \mathcal{D}_{f_\star \cap g_\star}$ belongs to the X-line passing through a, then there exists a unique $t \in \mathbb{R}$ such that $x = \phi^t_{f^*X}(a)$ and $\varphi_{f_\star \cap g_\star}(x) \coloneqq \phi^t_{g^*X}(b)$. Otherwise, if $x \in \mathcal{D}_{f_\star \cap g_\star}$ and $\gamma : [0,1] \to U$ is a forward curve from x to a and $\widetilde{\gamma} : [0,1] \to V$ is a lift of γ to g ending at b, then we define $\varphi_{f_\star \cap g_\star}(x) \coloneqq \widetilde{\gamma}(0)$. Lemma 2.2.8 readily implies that $\varphi_{f_\star \cap g_\star}$ is well-defined and independent of the choice of γ. It satisfies the equality 2.3.1 by construction.

Exchanging $g_\star$ and $f_\star$, we similarly have a set $\mathcal{I}_{g_\star \cap f_\star} \subset \mathcal{I}_{g_\star}$ and a domain $\mathcal{D}_{g_\star \cap f_\star} \subset \mathcal{D}_{g_\star}$, as well as maps $\alpha_{g_\star \cap f_\star} : \mathcal{I}_{g_\star \cap f_\star} \to \mathcal{I}_{g_\star}$ and $\varphi_{g_\star \cap f_\star} : \mathcal{D}_{g_\star \cap f_\star} \to \mathcal{D}_{g_\star}$. By definition, the image of $\varphi_{f_\star \cap g_\star}$ is contained in $\mathcal{D}_{g_\star \cap f_\star}$, and $\varphi_{g_\star \cap f_\star} \circ \varphi_{f_\star \cap g_\star} = \mathrm{id}_{\mathcal{D}_{f_\star \cap g_\star}}$. Similarly, $\varphi_{f_\star \cap g_\star} \circ \varphi_{g_\star \cap f_\star} = \mathrm{id}_{\mathcal{D}_{g_\star \cap f_\star}}$, hence $\varphi_{f_\star \cap g_\star}$ is a diffeomorphism between $\mathcal{D}_{f_\star \cap g_\star}$ and $\mathcal{D}_{g_\star \cap f_\star}$. Likewise, $\alpha_{f_\star \cap g_\star}$ induces an order preserving bijection between $\mathcal{I}_{f_\star \cap g_\star}$ and $\mathcal{I}_{g_\star \cap f_\star}$ whose inverse is given by $\alpha_{g_\star \cap f_\star}$.

Notice that if $\mathcal{I}_{f_\star \cap g_\star} = \varnothing$, then $p \in M^{\mathrm{un}}$ and $U_c = \mathcal{D}_{f_\star \cap g_\star} \subset U$ is an unstable component of f. Similarly, if $\mathcal{I}_{f_\star \cap g_\star}$ has a least element ℓ_m which is not a forward boundary component of $\mathcal{D}_{f_\star \cap g_\star}$ in $\mathcal{D}_{f_\star}$, then there exists an unstable component $U_c \subset \mathcal{D}_{f_\star \cap g_\star}$ such that ℓ_m is a backward boundary component of U_c. In both cases, we say that U_c is the **initial unstable component of overlap** of $f_\star$ and $g_\star$.

2.3.2.3 Initial corners

Let $q \in \Delta_{\mathrm{sa}} \cup Q$. Recall that q is a **pure** saddle singularity if no stable branch of X at q is in M^{pure}, and q is a **quadratic-like** singularity if exactly one stable branch of X at q is contained in M^{pure}. The set of pure saddle singularities of X is denoted by $\Delta^{\mathrm{pure}}_{\mathrm{sa}}$, and the set of quadratic-like singularities of X is denoted by Q^*.

If $q \in \Delta_{\text{sa}}^{\text{pure}} \cup Q^*$, a **corner at** q is a pair $c = (\gamma_s, \gamma_u)$, where γ_s and γ_u are a stable and unstable branches of X at q, respectively, both contained in M^{pure}. We denote by $\mathfrak{C}_q$ the set of corners at q, and $\mathfrak{C}$ the set of all corners. Note that $\mathfrak{C}_q$ has four elements if $q \in \Delta_{\text{sa}}^{\text{pure}}$, and two elements if $q \in Q^*$.

A corner $c = (\gamma_s, \gamma_u) \in \mathfrak{C}$ is **forward** (resp. **backward**) is γ_u lies to the right (resp. to the left) of γ_s. To make sense of this orientation, we consider the (unit) vector $v_s, v_u \in \eta(q)$ defined as

$$v_s \coloneqq \lim_{t \to +\infty} \frac{\dot{\gamma}_s(t)}{|\dot{\gamma}_s(t)|}, \qquad\qquad v_u \coloneqq \lim_{t \to -\infty} \frac{\dot{\gamma}_u(t)}{|\dot{\gamma}_u(t)|}.$$

If (v_s, v_u) is positively (resp. negatively) oriented, with respect to the orientation on $\eta(q)$, then γ_u is to the left (resp. to the right) of γ_s. We denote by $\mathfrak{C}_q^+ \subset \mathfrak{C}_q$ the set of forward corners at q, by $\mathfrak{C}^+ \subset \mathfrak{C}$ the set of all corners. Note that $\mathfrak{C}+_q$ has two elements if $q \in \Delta_{\text{sa}}^{\text{pure}}$, and one element if $q \in Q^*$. Moreover, there is a canonical 'projection' map

$$\mathfrak{p} : \mathfrak{C}^+ \to \Delta_{\text{sa}}^{\text{pure}} \cup Q^*.$$

We now refine the description of case (I2) above for the description of the backward overlap of $f_\star$ and $g_\star$. We assume that $\mathcal{I}_{f_\star \cap g_\star}$ (hence $\mathcal{I}_{g_\star \cap f_\star}$) satisfies case (I2) of Lemma 2.3.4. We say that a corner $c \in \mathfrak{C}^+$ is the **initial (forward) corner of overlap** of $f_\star$ and $g_\star$ if, after possibly switching $f_\star$ and $g_\star$, there exists a maximal lower bounds $\ell_s \in \mathcal{I}_{f_\star}$ of $\mathcal{I}_{f_\star \cap g_\star}$ and a maximal lower bound $\ell_u \in \mathcal{I}_{g_\star}$ of $\mathcal{I}_{g_\star \cap f_\star}$ such that $c = \big(f \circ \ell_s, g \circ \ell_u\big)$. A more subtle situation can occur. If $c = (\gamma_s, \gamma_u) \in \mathfrak{C}_q^+$ and $c' = (\gamma_s', \gamma_u') \in \mathfrak{C}_{q'}^+$ are two forward corners such that $\gamma_u = \gamma_s' =: \gamma_0$, in which case q and q' are saddle singularities connected by γ_u, we say that (c, c') is the **initial (forward) double corner of overlap** of $f_\star$ and $g_\star$ if, after possibly switching $f_\star$ and $g_\star$, $\mathcal{I}_{f_\star \cap g_\star} \subset \mathcal{I}_{f_\star}$ has an infimum ℓ_s, $\mathcal{I}_{g_\star \cap f_\star} \subset \mathcal{I}_{g_\star}$ has an infimum ℓ_u', and $\gamma_s = f \circ \ell_s$ and $\gamma_u' = g \circ \ell_u'$. Then, after possibly switching $f_\star$ and $g_\star$, one of the following cases

holds:

(C1) Both $\mathcal{I}_{f_\star \cap g_\star} \subset \mathcal{I}_{f_\star}$ and $\mathcal{I}_{g_\star \cap f_\star} \subset \mathcal{I}_{g_\star}$ satisfy case (I2) (i) above, and have respective infima $\ell_{f_\star}$ and $\ell_{g_\star}$. Then, either

(i) $f_\star$ and $g_\star$ have an initial corner of overlap $c = \big(\ell_{f_\star}, \ell_{g_\star}\big)$, or

(ii) $f_\star$ and $g_\star$ have an initial double corner of overlap (c, c'), where $c = \big(f \circ \ell_{f_\star}, \gamma\big)$ and $c' = \big(\gamma, g \circ \ell_{g_\star}\big)$ for some saddle-saddle connection γ of X.

(C2) Exactly one of $\mathcal{I}_{f_\star \cap g_\star}$ or $\mathcal{I}_{g_\star \cap f_\star}$ satisfies case (I2) (i) while the other satisfies case (I2) (ii). More precisely,

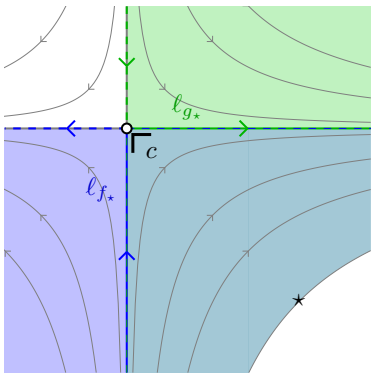
(i) $\mathcal{I}_{f_\star \cap g_\star}$ has an infimum ℓ_s in $\mathcal{I}_{f_\star}$, while $\mathcal{I}_{g_\star \cap f_\star}$ has exactly two maximal lower bounds ℓ'_s and ℓ'^+_u in $\mathcal{I}_{g_\star}$, such that $c = \big(f \circ \ell_s, g \circ \ell'_s\big)$ and $c' = \big(g \circ \ell'_s, g \circ \ell'^+_u\big)$ form a double corner. In that case, c is the initial corner of overlap of $f_\star$ and $g_\star$.

(ii) $\mathcal{I}_{f_\star \cap g_\star}$ has exactly two maximal lower bounds ℓ_s and ℓ^+_u in $\mathcal{I}_{f_\star}$, while $\mathcal{I}_{g_\star \cap f_\star}$ has an infimum ℓ'^+_u in $\mathcal{I}_{g_\star}$, such that $c = \big(f \circ \ell_s, f \circ \ell^+_u\big)$ and $c' = \big(f \circ \ell'^+_u, g \circ \ell'^+_u\big)$ form a double corner. In that case, c' is the initial corner of overlap of $f_\star$ and $g_\star$.

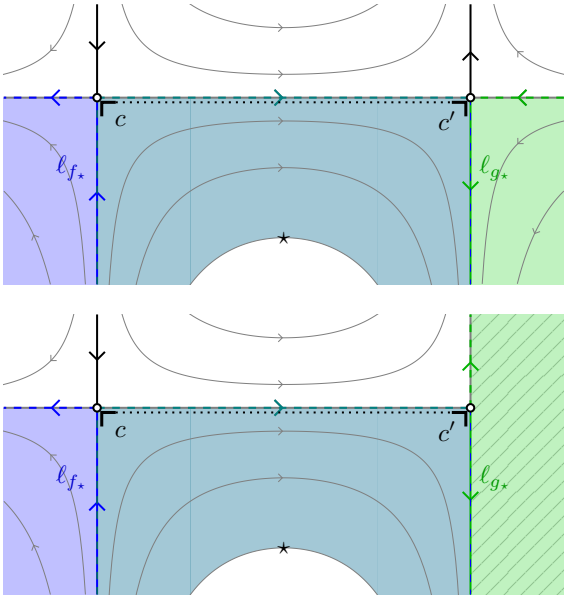
See Figure 2.3.3 for illustrations of all these cases.

We now describe how the shadows of $f_\star$ and $g_\star$ interact at an initial corner. This will be relevant in the definition of the extended preorder on $\mathscr{U}$. We begin with some definition.

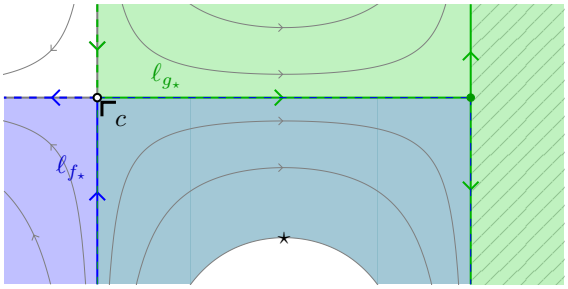
If $c = (\gamma_s, \gamma_u) \in \mathfrak{C}^+_q$ is a forward corner at q, then for any points $q_s \in \gamma_s$ and $q_u \in \gamma_u$, there exist well-defined order preserving map $\varsigma_s : \mathscr{A}_{q_s} \hookrightarrow \mathscr{A}_q$ and $\varsigma_u : \mathscr{A}_{q_u} \hookrightarrow \mathscr{A}_q$ obtained by sliding the marked point along the stable and unstable branch at c, respectively, to the corner point q. Here, we use that $\mathscr{A}$ is complete. Such maps already appeared in the proof of Lemma 2.2.11. If c is the initial corner of overlap of



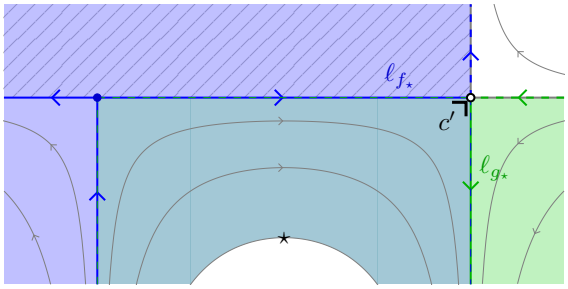
(a) Case (C1) (i).



(b) Case (C1) (ii). The right stable branch at c' might be in M^{pure} (top) or in M^{un} (bottom).



(c) Case (C2)(i).



(d) Case (C2)(ii).

Figure 2.3.3: Possible cases for initial (double) corners of overlap. The regions in blue correspond to $\mathcal{D}_{f_\star}$, the regions in green correspond to $\mathcal{D}_{g_\star}$, and their region of overlap is in teal. The thinly hatched regions are in M^{un}.

$f_\star$ and $g_\star$, and $\ell_s \in \mathcal{I}_{f_\star}$ and $\ell_u \in \mathcal{I}_{g_\star}$ are the corresponding 'entrance X-lines' of $f_\star$ and $g_\star$ along c, respectively, we choose arbitrary points $x_s \in \ell_s$ and $x_u \in \ell_u$ so that $q_s := f(x_s) \in \gamma_s$ and $q_u := g(x_u) \in \gamma_u$. We then set

$$\Sigma(c) := \varsigma_s\big(\Sigma(x_s)\big) \subset \mathscr{A}_q,$$
$$\Sigma'(c) := \varsigma_u\big(\Sigma'(x_u)\big) \subset \mathscr{A}_q.$$

Note that these sets are independent of the points x_s and x_u by axiom 4 of Defini-

tion 2.2.10. We then define $\widehat{\Sigma}(c) \subset \mathscr{A}_q$ and $\widehat{\Sigma}'(c) \subset \mathscr{A}_q$ as the smallest sections of $\mathscr{A}_q$ (with respect to the inclusion) containing $\Sigma(c)$ and $\Sigma'(c)$, respectively. Since those are two sections, and the preorder on $\mathscr{A}_q$ is total, one is included in the other. If $\widehat{\Sigma}(c) = \widehat{\Sigma}'(c)$, we say that $f_\star$ and $g_\star$ are **geometrically coherent at** c. Otherwise, we say that they are **geometrically noncoherent**.

We extend this definition to a double corner (c, c'). In the above notations, we consider points $q_s \in \gamma_s$, $q'_u \in \gamma'_u$, and $q_0 \in \gamma_0$. Sliding the marked points as before induces order preserving maps organized as follows:

$$\begin{array}{ccccc} \mathscr{A}_q & \xhookleftarrow{\varsigma_u} & \mathscr{A}_{q_0} & \xhookrightarrow{\varsigma'_s} & \mathscr{A}_{q'} \\ {\scriptstyle\varsigma_s}\uparrow & & & & \uparrow{\scriptstyle\varsigma'_u} \\ \mathscr{A}_{q_s} & & & & \mathscr{A}_{q'_u} \end{array}$$

We then consider $x_s \in \ell_s$, $x'_u \in \ell'_u$, and we define

$$\Sigma(c) \coloneqq \varsigma_s\big(\Sigma(x_s)\big) \subset \mathscr{A}_q,$$
$$\Sigma'(c') \coloneqq \varsigma'_u\big(\Sigma'(x'_u)\big) \subset \mathscr{A}_{q'}.$$

As before, these sets do not depend on the choices of x_s and x'_u. Denoting $\widehat{\Sigma}(c) \subset \mathscr{A}_q$ and $\widehat{\Sigma}'(c') \subset \mathscr{A}_{q'}$ the smallest sections containing $\Sigma(c)$ and $\Sigma'(c')$, respectively, we define

$$\Sigma_0(c) \coloneqq \varsigma_u^{-1}\big(\widehat{\Sigma}(c)\big) \subset \mathscr{A}_{q_0},$$
$$\Sigma'_0(c') \coloneqq (\varsigma'_s)^{-1}\big(\widehat{\Sigma}'(c')\big) \subset \mathscr{A}_{q_0},$$

and we denote by $\widehat{\Sigma}_0(c)$ and $\widehat{\Sigma}'_0(c')$ the smallest sections of $\mathscr{A}_{q_0}$ containing each of them. If $\widehat{\Sigma}_0(c) = \widehat{\Sigma}'_0(c')$, we say that $f_\star$ and $g_\star$ are **geometrically coherent at** (c, c'). Otherwise, we say that they are **geometrically noncoherent**.

2.3.2.4 Triple overlaps

We finally consider a third marked forward envelope $h_\star = (h : W \to M, c) \in \mathscr{U}_p$ with corresponding poset $\mathcal{I}_{h_\star} \subset \mathcal{L}_h$ and backward domain $\mathcal{D}_{h_\star} \subset W$. We define the "triple intersections" by:

$$\mathcal{I}_{f_\star \cap g_\star \cap h_\star} \coloneqq \mathcal{I}_{f_\star \cap g_\star} \cap \mathcal{I}_{f_\star \cap h_\star} \subset \mathcal{I}_{f_\star},$$
$$\mathcal{D}_{f_\star \cap g_\star \cap h_\star} \coloneqq \mathcal{D}_{f_\star \cap g_\star} \cap \mathcal{D}_{f_\star \cap h_\star} \subset \mathcal{D}_{f_\star}.$$

Lemma 2.2.8 easily implies

Lemma 2.3.5 (see [BI08], Lemma 6.12)**.** *The following identities hold:*

$$\alpha_{f_\star \cap g_\star}\big(\mathcal{I}_{f_\star \cap g_\star \cap h_\star}\big) = \mathcal{I}_{g_\star \cap f_\star \cap h_\star}, \qquad \varphi_{f_\star \cap g_\star}\big(\mathcal{D}_{f_\star \cap g_\star \cap h_\star}\big) = \mathcal{D}_{g_\star \cap f_\star \cap h_\star},$$
$$\alpha^{-1}_{f_\star \cap g_\star}\big(\mathcal{I}_{g_\star \cap h_\star}\big) = \mathcal{I}_{f_\star \cap g_\star \cap h_\star}, \qquad \varphi^{-1}_{f_\star \cap g_\star}\big(\mathcal{D}_{g_\star \cap h_\star}\big) = \mathcal{D}_{f_\star \cap g_\star \cap h_\star},$$
$$\alpha_{g_\star \cap h_\star} \circ \alpha_{f_\star \cap g_\star}(\ell) = \alpha_{f_\star \cap h_\star}(\ell), \qquad \varphi_{g_\star \cap h_\star} \circ \varphi_{f_\star \cap g_\star}(x) = \varphi_{f_\star \cap h_\star}(x),$$

for every $\ell \in \mathcal{I}_{f_\star \cap g_\star \cap h_\star}$ and $x \in \mathcal{D}_{f_\star \cap g_\star \cap h_\star}$.

Notice that by Lemma 2.3.3, either $I_{f_\star \cap g_\star \cap h_\star} = I_{f_\star \cap g_\star}$ or $I_{f_\star \cap g_\star \cap h_\star} = I_{f_\star \cap h_\star}$ or equivalently, either $I_{f_\star \cap g_\star} \subset I_{f_\star \cap h_\star}$ or $I_{f_\star \cap h_\star} \subset I_{f_\star \cap g_\star}$. Recall that none of the forward envelopes we consider have an initial edge corresponding to the pure stable branch of a quadratic-like singularity. In view of the cases (I0), (I1) and (I2) above, one easily deduces:

Lemma 2.3.6. *Up to permuting $g_\star$ and $h_\star$, the following inclusions hold:*

$$\mathcal{I}_{f_\star \cap g_\star} \subset \mathcal{I}_{f_\star \cap h_\star}, \qquad \mathcal{D}_{f_\star \cap g_\star} \subset \mathcal{D}_{f_\star \cap h_\star} \tag{2.3.2}$$

2.3.3 The preorder between envelopes

We are now ready to define a weak prefoliation structure on the collection $\mathscr{U}$ of forward envelopes. As in [BI08, Section 6.2], we construct a preorder $\precsim$ on $\mathscr{U}_\star$ in several steps. We first define a total order $\precsim_p$ on $\mathscr{U}_p$ at every point $p \in M^{\mathrm{good}}$ in five steps, before extending it to every $p \in M$.

2.3.3.1 Definition of the preorder

Let $f_\star, g_\star \in \mathscr{U}_p$ be two marked forward envelopes passing through $p \in M^{\mathrm{good}}$, with respective domains U and V, and shadows Σ and Σ'.

The geometric relation $\prec^{\mathrm{geo}}$. This relation corresponds to the relation $>_1$ from [BI08, Section 6.2.2], and is defined in a similar way. We set $f_\star \prec_p^{\mathrm{geo}} g_\star$ if there exists $a_0 \in \mathcal{D}_{f_\star \cap g_\star}$ with $b_0 = \varphi_{f_\star \cap g_\star}(a_0)$ such that we either have

(a) $\Sigma(a_0) \subsetneq \Sigma'(b_0)$, or

(b) (g, b_0) is locally strictly above (f, a_0).

The inclusion relation $\prec^{\mathrm{inc}}$. This relation corresponds to the relation $>_2$ from [BI08, Section 6.2.2] and is defined in a similar way. We say that $f_\star$ is a **strict subenvelope** of $g_\star$ if $f_\star$ and $g_\star$ are not comparable by $\prec_p^{\mathrm{geo}}$, and $\mathcal{D}_{f_\star} = \mathcal{D}_{f_\star \cap g_\star}$ but $\mathcal{D}_{g_\star \cap f_\star} \subsetneq \mathcal{D}_{g_\star}$. In this situation, $f_\star$ and $g_\star$ are comparable by $\prec^{\mathrm{inc}}$ and we set $f_\star \prec_p^{\mathrm{inc}} g_\star$.

The relation at corners $\prec^{\mathrm{cor}}$. This relation has no equivalent in [BI08] and is a feature of the presence of saddle and quadratic singularities. We start with some definitions.

Definition 2.3.7. *A collection of forward corners $\mathfrak{c} \subset \mathfrak{C}^+$ is **admissible** if it satisfies the following conditions.*

1. *$\mathfrak{c}$ contains exactly one forward corner at each pure saddle singularity, and no corner at quadratic-like singularities,*

2. *If (c, c') is a forward double-corner, then $\{c, c'\} \cap \mathfrak{c} = \varnothing$.*

Choice 1. *The relation $\prec^{\mathrm{cor}}$ depends on the choice of an admissible collection of forward corners $\mathfrak{c} \subset \mathfrak{C}^+$.*

This choice will be crucial later when we extend the weak prefoliation structure to completed envelopes, see Section 2.3.3.3 below.

We first assume that $f_\star$ and $g_\star$ are not comparable by $\prec^{\mathrm{geo}} \cup \prec^{\mathrm{inc}}$ and have an initial corner of overlap c. After possibly renaming them, we further assume that they meet along c in the way described in Section 2.3.2.3. Then, we set $f_\star \prec_p^{\mathrm{cor}} g_\star$ if one of the following holds:

1. $f_\star$ and $g_\star$ are geometrically noncoherent at c, and $\widehat{\Sigma}(c) \subsetneq \widehat{\Sigma}'(c)$,

2. $f_\star$ and $g_\star$ are geometrically coherent at c, and $c \in \mathfrak{c}$.

Otherwise, we set $g_\star \prec_p^{\mathrm{cor}} f_\star$.

We now assume that $f_\star$ and $g_\star$ are not comparable by $\prec^{\mathrm{geo}} \cup \prec^{\mathrm{inc}}$ and have an initial double corner of overlap (c, c') where they meet in the way described in Section 2.3.2.3. Then, we set $f_\star \prec_p^{\mathrm{cor}} g_\star$ if $f_\star$ and $g_\star$ are geometrically noncoherent at c, and $\widehat{\Sigma}_0(c) \subsetneq \widehat{\Sigma}'_0(c)$. Otherwise, we set $g_\star \prec_p^{\mathrm{cor}} f_\star$.

The relation along unstable components $\prec^{\mathrm{un}}$. This relation has no equivalent in [BI08] and is a feature of the presence of source and quadratic singularities. We need to introduce some terminology and make some auxiliary choices before defining it.

Let $L \hookrightarrow M$ be a leaf of η in M^{un}. A **forward boundary component** of L is a connected component of the complement the backward edges of L in ∂L. We

denote by $\mathcal{B}_L^+$ the set of forward boundary components of L. A forward boundary component might be closed or might contain one or two corners. Moreover, a boundary component might contain two consecutive flow lines of X with opposite orientations, corresponding to pure unstable branches $\gamma_u^\pm$ at a quadratic-like singularity q. We say that the pure stable branch γ_s at q is a **spike** for L if it is connected to a (pure) saddle singularity in negative times. In that case, q is a saddle singularity as well. Since X has no broken triple saddle connections (see Definition 1.2.4), L has *at most one* spike. The forward boundary component γ_u^- of L will be called a **distinguished boundary component** and will be denoted by $b_L^\dagger \in \mathcal{B}_L^+$. A total order on $\mathcal{B}_L^+$ for which $b_L^\dagger$ is maximal (if it exists) will be called **admissible**. We define $\mathfrak{O}_L^+$ as the collection of all admissible total orders on $\mathcal{B}_L^+$, and we set

$$\mathfrak{O}^+ \coloneqq \prod_L \mathfrak{O}_L^+,$$

where the product runs over all the leaves of η in M^{un}. The admissibility condition will become relevant in the proof of Lemma 2.3.12, to ensure transitivity of the full preorder.

Choice 2. *The relation $\prec^{\mathrm{un}}$ depends on the choice of an collection of admissible orders $\mathfrak{o} \in \mathfrak{O}^+$.*

Let $U_c \subset \mathcal{D}_{f_\star}$ be an unstable component of f which is not initial if f is of unstable type, and let $h \coloneqq f_{|U_c}$ be the corresponding unstable tile, which covers a leaf $L \hookrightarrow M$ of η in M^{un}. An **entrance edge** of $f_\star$ to U_c is a forward edge of U_c which belongs to $\mathcal{D}_{f_\star}$. It exists since U_c is not initial, but it is not necessarily unique. However, there are at most three of those. If there are two, then they correspond to consecutive stable and unstable branches of a singularity of X, and if there are three, then they correspond to a succession of a stable branch of a saddle singularity q, a connection between q and another saddle singularity q', and an unstable branch of q'. Such an

entrance edge projects to $\overline{L}$ via the universal covering map, and to a forward boundary component $b_{f_\star} \in \mathcal{B}_L^+$. Obviously, the latter is independent on the choice of entrance edge.

We set $f_\star \prec_p^{\mathrm{un}} g_\star$ if $f_\star$ and $g_\star$ are not comparable by $\prec_p^{\mathrm{geo}} \cup \prec_p^{\mathrm{inc}} \cup \prec_p^{\mathrm{cor}}$, and the following holds. There exists an initial unstable component of overlap $U_c \subset \mathcal{D}_{f_\star \cap g_\star}$ between $f_\star$ and $g_\star$ corresponding to a leaf $L \hookrightarrow M$ as above, together with forward boundary components $b_{f_\star}, b_{g_\star} \in \mathcal{B}_L^+$ induced by an entrance edge of $f_\star$ to U_c and of $g_\star$ to $\varphi_{\mathcal{D}_{f_\star \cap g_\star}}(U_c)$, respectively, which satisfy $b_{f_\star} <^{\mathfrak{o}} b_{g_\star}$. Here, $<^{\mathfrak{o}}$ denotes the total order on $\mathcal{B}_L^+$ coming from $\mathfrak{o} \in \mathfrak{O}^+$. Notice that $b_{f_\star}$ and $b_{g_\star}$ are necessarily distinct, since $f_\star$ and $g_\star$ don't have an initial (double) corner of overlap.

The X-lines orientation relation $\prec^{\mathrm{or}}$. This relation corresponds to the relation $>_3$ from [BI08, Section 6.2.4]. We define $f_\star \prec_p^{\mathrm{or}} g_\star$ if $f_\star$ and $g_\star$ are not comparable by $\prec_p^{\mathrm{geo}} \cup \prec_p^{\mathrm{inc}} \cup \prec_p^{\mathrm{cor}} \cup \prec_p^{\mathrm{un}}$, and the following holds. There exist maximal lower bounds $\ell_{f_\star} \in \mathcal{I}_{f_\star}$ and $\ell_{g_\star} \in \mathcal{I}_{g_\star}$ of $\mathcal{I}_{f_\star \cap g_\star} \subset \mathcal{I}_{f_\star}$ and $\mathcal{I}_{g_\star \cap f_\star} \subset \mathcal{I}_{g_\star}$, respectively, and two forward curves $\gamma_{f_\star} : [0,1] \to U$ and $\gamma_{g_\star} : [0,1] \to V$ such that

- $\gamma_{f_\star}(0) \in \ell_{f_\star}$ and $\gamma_{g_\star}(0) \in \ell_{g_\star}$,
- $\gamma_{f_\star}$ and $\varphi_{g_\star \cap f_\star}\big(\gamma_{g_\star|(0,1]}\big)$ are disjoint,
- There exists a pure X-line $\ell \in \mathcal{I}_{f_\star \cap g_\star}$, and real numbers $s < t$ such that $\ell(s) \in \gamma_{f_\star}$ and $\varphi_{f_\star \cap g_\star}\big(\ell(t)\big) \in \gamma_{g_\star}$.

For short, we use the orientation on the pure X-lines near $\ell_{f_\star}$ and $\ell_{g_\star}$ to define the relation in this situation, see Figure 2.3.4.

Strictly speaking, this definition depends on an overall choice of orientation on $\mathbb{R}$: we could have replaced the condition $s < t$ with $t < s$ in the above third bullet.

Combining the five relations. We are now in the position to define a relation $\precsim_p$ on $\mathscr{U}_p$ as follows. First, we set

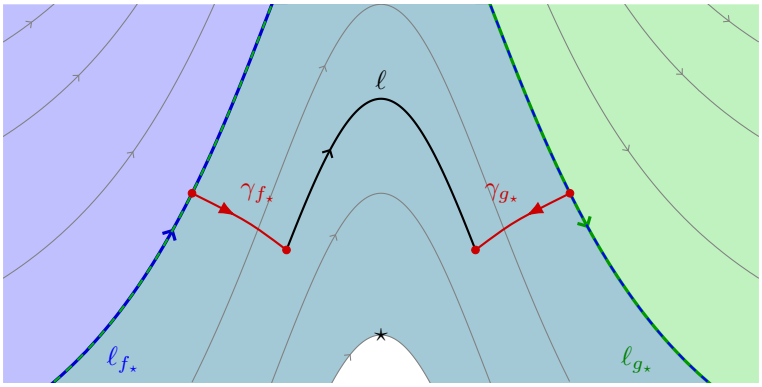


Figure 2.3.4: Overlap of $f_\star$ and $g_\star$ for $\prec^{\mathrm{or}}$. The blue region corresponds to $\mathcal{D}_{f_\star}$, the green region corresponds to $\mathcal{D}_{g_\star}$, and their region of overlap is in teal.

$$f_\star \prec_p g_\star \text{ if and only if } f_\star \prec_p^{\heartsuit} g_\star \text{ for some } \heartsuit \in \{\mathrm{geo}, \mathrm{inc}, \mathrm{cor}, \mathrm{un}, \mathrm{or}\}.$$

We say that $f_\star$ and $g_\star$ are **forward-equivalent** if there exists an equivalence $\varphi : f_\star \xrightarrow{\sim} g_\star$ satisfying $\Sigma' \circ \varphi = \Sigma$. In particular, $f_\star \approx_p g_\star$. Then, we define:

$$f_\star \precsim_p g_\star \text{ if and only if } f_\star \prec_p g_\star, \text{ or } f_\star \text{ and } g_\star \text{ are forward-equivalent.}$$

So far, the relation $\precsim_p$ is only defined for $p \in M^{\mathrm{good}}$. We extend it to every $p \in M$ as follows: $f_\star \precsim_p g_\star$ if and only if there exists $q \in M^{\mathrm{good}}$ and a continuous path $\gamma : [0,1] \to M$ with $\gamma(0) = p$, $\gamma(1) = q$, which lifts to both $f_\star$ and $g_\star$, and $f_{\gamma\star} \precsim_q g_{\gamma\star}$. Here, $f_{\gamma\star}$ and $g_{\gamma\star}$ are obtained from $f_\star$ and $g_\star$ by sliding the marked points along γ in the obvious way.

The collection of homogeneous relations $\{\precsim_p\}_{p\in M}$ naturally induces a homogeneous relation $\precsim$ on $\mathscr{U}_\star$.

2.3.3.2 Proof of the weak prefoliation axioms

This section is dedicated to the proof of

Proposition 2.3.8. *The set $\mathscr{U}$ of forward envelopes equipped with the relation $\precsim$ is a weak prefoliation.*

We split the proof into the following four lemmas.

Lemma 2.3.9 (Definition 2.1.8, Axiom 2)**.** *The relation $\precsim$ is coherent along intersections.*

Proof. The relation $\precsim$ is trivially coherent along unstable components and along pure X-lines by definition. It is also coherent along forward curves for the same reasons as in the proof of [BI08, Lemma 6.15]. We are left to show that it is coherent at singularities in M^{bad}. If $p \notin M^{\mathrm{good}}$ and $f_\star \in \mathscr{U}_p$, then the marked point of $f_\star$ lies in the interior of the domain of f. Therefore, coherence of $\precsim$ along intersections contained in M^{good} immediately implies coherence at p. □

Lemma 2.3.10 (Definition 2.1.8, Axiom 0)**.** *For every $p \in M$, $\precsim_p$ it total. Moreover, for all $f_\star, g_\star \in \mathscr{U}_\star$, $f_\star \simeq g_\star$ if and only if $f_\star$ and $g_\star$ are forward-equivalent.*

Proof. First assume $p \in M^{\mathrm{good}}$, and consider $f_\star, g_\star \in \mathscr{U}_p$ with our usual notations for their domains, shadows, and marked points.

If $f_\star$ and $g_\star$ are comparable by $\prec_p^{\mathrm{geo}}$, say $f_\star \prec_p^{\mathrm{geo}} g_\star$, then $f_\star$ and $g_\star$ are not equivalent and $g_\star \not\prec_p^{\mathrm{geo}} f_\star$. Indeed, arguing as in [BI08, Section 6.2.2] by considering forward curves from/to a_0, it is easy to see that for every $x \in \mathcal{D}_{f_\star \cap g_\star}$, $\Sigma(x) \subset \Sigma'\big(\varphi_{f_\star \cap g_\star}(x)\big)$ and (f, x) is not locally strictly above $\big(g, \varphi_{f_\star \cap g_\star}(x)\big)$.

If $f_\star$ and $g_\star$ are not comparable by $\prec_p^{\mathrm{geo}}$, then for every $x \in \mathcal{D}_{f_\star \cap g_\star}$, $\Sigma(x) = \Sigma'\big(\varphi_{f_\star \cap g_\star}(x)\big)$ and (f, x) and $\big(g, \varphi_{f_\star \cap g_\star}(x)\big)$ locally coincide.[13] We now analyze how $f_\star$ and $g_\star$ 'first meet' along $\mathcal{D}_{f_\star \cap g_\star}$ depending on which case from Lemma 2.3.4 holds:

(I0) $\mathcal{I}_{f_\star \cap g_\star} = \varnothing$. In that case, $f_\star$ and $g_\star$ have an initial unstable component of overlap $U_c \subset \mathcal{D}_{f_\star \cap g_\star}$.

If both U_c and $\varphi\big(U_c\big)$ correspond to the initial unstable components of $f_\star$ and $g_\star$, respectively, then $f_\star$ and $g_\star$ are forward-equivalent by Proposition 2.2.16.

[13] Notice that they automatically locally coincide in the forward direction based on properties of forward envelopes as soon as their shadows coincide.

If U_c corresponds the initial unstable component of $f_\star$, but $\varphi\big(U_c\big)$ does not correspond to the initial unstable component of $g_\star$, then $f_\star$ and $g_\star$ are comparable by $\prec^{\text{inc}}$ and $f_\star \prec^{\text{inc}} g_\star$. The case where U_c and $\varphi\big(U_c\big)$ are swapped is similar.

Otherwise, $f_\star$ and $g_\star$ are comparable by $\prec^{\text{un}}$, since U_c does not correspond to their initial unstable components and they don't have an initial (double) corner of overlap. Obviously, either $f_\star \prec_p^{\text{un}} g_\star$ of $g_\star \prec_p^{\text{un}} f_\star$ holds, but not both.

(I1) $\mathcal{I}_{f_\star \cap g_\star}$ *has minimal elements.* If $f_\star$ and $g_\star$ have an initial (double) corner of overlap, then they are comparable by $\prec_p^{\text{cor}}$, and only one of $f_\star \prec_p^{\text{cor}} g_\star$ or $g_\star \prec_p^{\text{cor}} f_\star$ holds. We now assume that they do not have an initial (double) corner of overlap.

If they have an initial unstable component of overlap, we proceed as in the last step of the previous item. Otherwise, $f_\star$ and $g_\star$ are comparable by $\prec_p^{\text{inc}}$. Indeed, if $\mathcal{I}_{f_\star \cap g_\star}$ has a unique minimal element, which is a minimum, then this minimum corresponds to the initial edge of either $f_\star$ or $g_\star$, depending on whether it is the minimum of $\mathcal{I}_{f_\star}$ or not. If $\mathcal{I}_{f_\star \cap g_\star}$ has strictly more than one minimal element, then it is easy to see that $g_\star \prec_p^{\text{inc}} f_\star$.

(I2) $\mathcal{I}_{f_\star \cap g_\star}$ *has no minimal elements.* Observe that if $f_\star$ and $g_\star$ are comparable by $\prec^{\text{cor}} \cup \prec_p^{\text{inc}} \cup \prec_p^{\text{un}}$, then $\mathcal{I}_{f_\star \cap g_\star}$ is either empty or has a minimal element. Therefore, we need to show that $f_\star$ and $g_\star$ are comparable by $\prec_p^{\text{or}}$. For that, we consider a forward curve $\gamma_0 : [0,1] \to U$ from the initial component of f to a, and a forward curve $\gamma_1 : [0,1] \to V$ from the initial component of g to b. There exist $t_0, t_1 \in [0,1]$ such that $\gamma_0(t_0)$ and $\gamma_1(t_1)$ belong X-lines $\ell_0 \subset U$ and $\ell_1 \subset V$ which are maximal lower bounds for $\mathcal{I}_{f_\star \cap g_\star} \subset \mathcal{I}_{f_\star}$ and $\mathcal{I}_{g_\star \cap f_\star} \subset \mathcal{I}_{g_\star}$, respectively. For $t > t_0$, $\gamma_0(t) \in \mathcal{D}_{f_\star \cap g_\star}$ and for $t > t_1$, $\gamma_1(t) \in \mathcal{D}_{g_\star \cap f_\star}$. Moreover, there exists $\epsilon > 0$ such that $\gamma_{f_\star} \coloneqq \gamma_{0[t_0, t_0+\epsilon)}$ and $\gamma_{g_\star} \coloneqq \gamma_{1[t_1, t_1+\epsilon)}$ are strictly forward, and $\gamma_{f_\star}$ and $\varphi_{g_\star \cap f_\star}\big(\gamma_{g_\star|(t_1, t_1+\epsilon)}\big)$ are disjoint. Since $f_\star$ and $g_\star$ do not have an initial unstable domain of overlap, there exists either

0. A pure X-line ℓ intersecting both $\gamma_{f_\star}$ and $\gamma_{g_\star}$,

1. A special triple $(\ell_s, \ell_u^-, \ell_u^+)$ such that ℓ_s intersects one of $\gamma_{f_\star}$ and $\gamma_{g_\star}$ and ℓ_u^- or ℓ_u^+ intersects the other,

2. Two special triples $(\ell_s, \ell_u^-, \ell_u^+)$ and $(\ell_s', \ell_u'^-, \ell_u'^+)$ with $\ell_s' \in \{\ell_u^-, \ell_u^+\}$ such that ℓ_s intersects one of $\gamma_{f_\star}$ and $\gamma_{g_\star}$ and $\ell_u'^-$ or $\ell_u'^+$ intersects the other.

In cases 1 and 2, we can modify $\gamma_{f_\star}$ or $\gamma_{g_\star}$ by performing suitable T-moves to reduce to case 0 while keeping the two curves disjoint. This implies that $f_\star$ and $g_\star$ are comparable by $\prec_p^{\mathrm{or}}$. It remains to show that not both $f_\star \prec_p^{\mathrm{or}} g_\star$ and $g_\star \prec_p^{\mathrm{or}} f_\star$ can hold simultaneously. If both inequalities hold, then after performing suitable T-moves and forward isotopies, we would obtain the following situation: there exists two forward curves $\gamma_{f_\star}$ and $\gamma_{g_\star}$ as above, together with two pure X-lines or pure 'broken X-lines' (as in cases 1 and 2 above), one from $\gamma_{f_\star}$ to $\gamma_{g_\star}$ and the other one from $\gamma_{g_\star}$ to $\gamma_{f_\star}$. This would imply the existence of a pure X-line which is 'trapped' (in positive or negative times) in a compact region of U, which is impossible. The details are left to the reader.

Finally, we assume that $p \notin M^{\mathrm{good}}$. In particular, $p \in \Delta_{\mathrm{sa}} \cup Q$, and the marked points of $f_\star$ and $g_\star$ belong the the interior of their domains of definition. We can slide these marked points along a stable or unstable branch at p to obtain marked tiles $(f, a'), (g, b') \in \mathscr{U}_{p'}$ where $p' \in M^{\mathrm{good}}$. Therefore, (f, a') and (g, b') are comparable by $\precsim_{p'}$ by the above, so $f_\star$ and $g_\star$ are comparable by $\precsim_p$ by definition. Finally, if $f_\star \precsim_p g_\star$ and $g_\star \precsim_p f_\star$, then the same holds at some nearby $p' \in M^{\mathrm{good}}$ by sliding the marked point a and b to nearby a' and b', hence $(f, a') \approx_{p'} (g, b')$. Sliding the marked points back yields $f_\star \approx_p g_\star$. □

Lemma 2.3.11 (Definition 2.1.8, Axiom 1)**.** *For all $f_\star, g_\star \in \mathscr{U}_\star$, if $f_\star \precsim g_\star$, then $g_\star$ is locally above $f_\star$.*

Proof. The proof is essentially the same as the one of [BI08, Lemma 6.13]. Let $p \in M$ and $f_\star, g_\star \in \mathscr{U}_p$. Assume first that $p \in M^{\mathrm{good}}$. If $f_\star$ and $g_\star$ are comparable by $\prec_p^{\mathrm{geo}}$, say $f_\star \prec_p^{\mathrm{geo}} g_\star$, then we connect the point a_0 given by the definition to the marked point a of f by a forward curve. Properties of upper enveloping tiles (see Definition 2.2.10) and one-dimensional envelopes (see [BI08, Section 5]) immediately imply that $g_\star$ is locally above $f_\star$. If $f_\star$ and $g_\star$ are not comparable by $\prec_p^{\mathrm{geo}}$, then $f_\star$ is not locally strictly above $g_\star$, and since one of these two forward tiles is locally above the other (for the same reasons as in [BI08, Lemma 6.5]), then $g_\star$ is locally above $f_\star$. If $p \notin M^{\mathrm{good}}$ and $f_\star$ is locally strictly above $g_\star$, then there exists $p' \in M^{\mathrm{good}}$ on one of the stable or unstable branches at p such that if $(f, a'), (g, b') \in \mathscr{U}_{p'}$ denote the marked tiles obtained by sliding the marked points of $f_\star$ and $g_\star$, respectively, then (f, a') is locally strictly above (g, b'). Therefore, $(g, b') \prec_{p'}^{\mathrm{geo}} (f, a')$, hence $g_\star \prec f_\star$. □

We finish the proof of Proposition 2.3.8 with the trickiest property:

Lemma 2.3.12. *The relation $\precsim$ is transitive.*

Proof. Let $p \in M$. Notice that if $f_\star, g_\star, h_\star \in \mathscr{U}_p$ and $f_\star$ and $g_\star$ are forward-equivalent, then $f_\star \precsim_p h_\star$ if and only if $g_\star \precsim_p h_\star$, by definition. Therefore, in view of Lemma 2.3.10, transitivity of $\precsim_p$ is equivalent to the statement: *every triple of elements in $\mathscr{U}_p$ admits an extremum (maximum or minimum).* Moreover, by Lemma 2.3.9, it is enough to consider the case $p \in M^{\mathrm{good}}$. By Lemma 2.3.6, we can assume that $\mathcal{I}_{f_\star \cap g_\star} \subset \mathcal{I}_{f_\star \cap h_\star}$ and $\mathcal{D}_{f_\star \cap g_\star} \subset \mathcal{D}_{f_\star \cap h_\star}$. We enumerate all the possible cases for the order between $f_\star$ and $g_\star$.

1. $f_\star$ and $g_\star$ are comparable by $\prec^{\mathrm{geo}}$. Assume that there exist $a_0 \in \mathcal{D}_{f_\star \cap g_\star}$ and $b_0 = \varphi_{f_\star \cap g_\star}(a_0)$ such that $\Sigma_f(a_0) \neq \Sigma_g(b_0)$, and consider $c_0 \coloneqq \varphi_{f_\star \cap h_\star}(a_0) \in \mathcal{D}_{h_\star}$. This is well-defined since $a_0 \in \mathcal{D}_{f_\star \cap g_\star} \subset \mathcal{D}_{f_\star \cap h_\star}$. Then, among the three sections $\Sigma_f(a_0)$, $\Sigma_g(b_0)$ and $\Sigma_h(c_0)$ of $\mathscr{U}_p$, there must exist a *strict* maximum or minimum for the inclusion, implying that the corresponding marked tiles is an extremum in

$\{f_\star, g_\star, h_\star\}$. Otherwise, there exists a_0 and b_0 as above such that (g, b_0) is locally strictly above (f, a_0). In that case, $p \in M^{\text{pure}}$. Intersecting $f_\star$, $g_\star$ and $h_\star$ with a small embedded disk in M passing through p and transverse to X, we obtain three 1-dimensional envelopes for the restriction of $\mathscr{A}$ to that disk, two of which are strictly comparable by the geometric order. We further consider two cases.

(a) If $c_0 = \varphi_{f_\star \cap h_\star}(a_0)$ does not belong to the initial edge of h (if it exists), then one of the three 1-dimensional envelopes is strictly below or above the two others, and the corresponding marked forward envelope is an extremum in $\{f_\star, g_\star, h_\star\}$.

(b) Otherwise, $f_\star$ and $g_\star$ 'begin to overlap' along the initial edge of $h_\star$. If $\Sigma_h(c_0) \neq \Sigma_f(a_0)$, then since $\Sigma_f(a_0) = \Sigma_g(b_0)$ by assumption, $h_\star$ is comparable by $\prec^{\text{geo}}$ with both $f_\star$ and $g_\star$. Moreover, it is either below or above both of them for $\prec^{\text{geo}}$, so $h_\star$ is an extremum of $\{f_\star, g_\star, h_\star\}$. If $\Sigma_h(c_0) = \Sigma_f(a_0) = \Sigma_g(b_0)$, then $h_\star$ is comparable by $\prec^{\text{inc}}$ with both $f_\star$ and $g_\star$, and $h_\star$ is similarly an extremum of $\{f_\star, g_\star, h_\star\}$.

2. $f_\star$ and $g_\star$ are comparable by $\prec^{\text{inc}}$. We first assume that $g_\star$ is a strict subenvelope of $f_\star$; in particular, $\mathcal{D}_{g_\star} = \mathcal{D}_{g_\star \cap f_\star}$. Notice that $\mathcal{D}_{h_\star \cap g_\star} \subset \mathcal{D}_{h_\star \cap f_\star}$ by Lemma 2.3.5. By the previous case, we can assume that $g_\star$ and $h_\star$ are not comparable by $\prec^{\text{geo}}$. We also have $\mathcal{D}_{g_\star} = \mathcal{D}_{g_\star \cap h_\star}$, so either $g_\star$ and $h_\star$ are order-equivalent, or $g_\star$ is a strict subenvelope of $h_\star$. Hence, either $f_\star$ or $g_\star$ is an extremum in $\{f_\star, g_\star, h_\star\}$. We now assume that $f_\star$ is a strict subenvelope of $g_\star$. Hence, $\mathcal{D}_{f_\star} = \mathcal{D}_{f_\star \cap g_\star} \subset \mathcal{D}_{f_\star \cap h_\star} = \mathcal{D}_{f_\star}$. In particular, $\mathcal{D}_{g_\star \cap f_\star} = \mathcal{D}_{g_\star \cap h_\star}$ and $\mathcal{D}_{h_\star \cap f_\star} = \mathcal{D}_{h_\star \cap g_\star}$, and by case 1 above, we can assume that $f_\star$ and $g_\star$ are not comparable by $\prec^{\text{geo}}$, and $f_\star$ and $h_\star$ are not comparable by $\prec^{\text{geo}}$. In this situation, $f_\star$ is a (not necessarily strict) subenvelope of $g_\star$ and $h_\star$, so $\{f_\star, g_\star, h_\star\}$ has an extremum by the previous argument.

3. $f_\star$ and $g_\star$ are comparable by $\prec^{\text{cor}}$. We consider three cases.

(a) $f_\star$ and $g_\star$ have a simple initial corner of overlap c, which is not connected to

another forward corner. Since we do not consider forward envelopes starting at the pure stable branch of a quadratic-like singularity, c is a pure corner. We consider two subcases.

i. $\mathcal{D}_{f_\star \cap g_\star} \subsetneq \mathcal{D}_{f_\star \cap h_\star}$. Then c is also the initial corner of overlap of $g_\star$ and $h_\star$. In that case, $\mathcal{D}_{g_\star \cap h_\star} = \mathcal{D}_{g_\star \cap f_\star}$ so by case 1 above, we can assume that $g_\star$ and $h_\star$ are not comparable by $\prec^{\mathrm{geo}}$, hence they are comparable by $\prec^{\mathrm{cor}}$. Let $a_0 \in U$ be a point on the X-line corresponding to the branch at c bounding $\mathcal{D}_{f_\star \cap g_\star}$ in $\mathcal{D}_{f_\star}$, and $c_0 \coloneqq \varphi_{f_\star \cap h_\star}(a_0)$. Note that $f_\star$ and $h_\star$ play a symmetric role here. If $\Sigma_f(a_0) \neq \Sigma_h(c_0)$, then $\widehat{\Sigma}_f(c) \neq \widehat{\Sigma}_h(c)$ (see Section 2.3.2.3 for the notation), and either $f_\star$ or $h_\star$ is an extremum of $\{f_\star, g_\star, h_\star\}$. Otherwise, $\widehat{\Sigma}_f(c) = \widehat{\Sigma}_h(c)$, and the order of $f_\star$ with $g_\star$ for $\prec^{\mathrm{cor}}$ is the same as the one of $h_\star$ with $g_\star$, implying that $g_\star$ is an extremum of $\{f_\star, g_\star, h_\star\}$.

ii. $\mathcal{D}_{f_\star \cap g_\star} = \mathcal{D}_{f_\star \cap h_\star}$. By case 1 above, we can assume that $f_\star$ and $h_\star$ are not comparable by $\prec^{\mathrm{geo}}$, and moreover they are not comparable by $\prec^{\mathrm{inc}}$. We consider three further subcases.

- $f_\star$ and $h_\star$ are comparable by $\prec^{\mathrm{cor}}$. Then c is also the initial corner of overlap between $f_\star$ and $g_\star$, and $\mathcal{D}_{g_\star \cap f_\star} \subsetneq \mathcal{D}_{g_\star \cap h_\star}$ so we can appeal to case 3(a)i above.
- $f_\star$ and $h_\star$ are comparable by $\prec^{\mathrm{un}}$. Then one easily sees that $h_\star$ and $g_\star$ are also comparable by $\prec^{\mathrm{un}}$ and share the same initial unstable domain of overlap as $h_\star$ and $f_\star$. Since the entrance edges of $f_\star$ and $g_\star$ lie on the same forward boundary component of the leaf corresponding to that unstable component, but the entrance edge(s) of $h_\star$ necessarily lies on a different forward boundary component, the order of $f_\star$ with $h_\star$ for $\prec^{\mathrm{un}}$ is the same as the one of $g_\star$ with $h_\star$, implying that $h_\star$ is an extremum of $\{f_\star, g_\star, h_\star\}$.
- $f_\star$ and $h_\star$ are comparable by $\prec^{\mathrm{or}}$. Then $g_\star$ and $h_\star$ are comparable by $\prec^{\mathrm{or}}$ as well, and it follows from the definition of $\prec^{\mathrm{or}}$ and the behavior of X-lines near a forward corner that the order of $f_\star$ with $h_\star$ for $\prec^{\mathrm{or}}$ is the same as the

one of $g_\star$ with $h_\star$, implying that $h_\star$ is an extremum of $\{f_\star, g_\star, h_\star\}$.

(b) $f_\star$ and $g_\star$ have a simple initial corner of overlap c, which is connected to another forward corner c'. We again consider two subcases.

i. *c' is a corner at a pure saddle singularity.* We further assume that $f_\star$ and $h_\star$ are comparable by $\prec^{\mathrm{cor}}$ and c' is their initial corner of overlap, as all the other cases can be treated as in case 3(a).

Up to permuting $f_\star$, $g_\star$ and $h_\star$, we can assume that (c, c') is a double corner and $f_\star$ and $h_\star$ meet along (c, c'). We will use the notations of Section 2.3.2.3. We consider the sections $\widehat{\Sigma}_f(c), \widehat{\Sigma}_g(c) \subset \mathscr{A}_q$, $\widehat{\Sigma}_g(c'), \widehat{\Sigma}_h(c') \subset \mathscr{A}_{q'}$, and $\Sigma_g(b_0), \widehat{\Sigma}_{f,0}(c), \widehat{\Sigma}_{h,0}(c') \subset \mathscr{A}_{q_0}$. Here, b_0 lies on the boundary of $\mathcal{D}_{g_\star \cap f_\star} \subset \mathcal{D}_{g_\star}$ and $g(b_0) = q_0$. Tracing through the definitions of these sets, one easily shows:

$$\widehat{\Sigma}_f(c) \subset \widehat{\Sigma}_g(c) \implies \widehat{\Sigma}_{f,0}(c) \subset \Sigma_g(b_0), \tag{2.3.3}$$

$$\widehat{\Sigma}_g(c) \subset \widehat{\Sigma}_f(c) \iff \Sigma_g(b_0) \subset \widehat{\Sigma}_{f,0}(c), , \tag{2.3.4}$$

and similarly,

$$\widehat{\Sigma}_h(c') \subset \widehat{\Sigma}_g(c') \implies \widehat{\Sigma}_{h,0}(c') \subset \Sigma_g(b_0),$$

$$\widehat{\Sigma}_g(c') \subset \widehat{\Sigma}_h(c') \iff \Sigma_g(b_0) \subset \widehat{\Sigma}_{h,0}(c).$$

This immediately implies the following:

- If $\widehat{\Sigma}_f(c) \subset \widehat{\Sigma}_g(c)$ and $\widehat{\Sigma}_g(c') \subsetneq \widehat{\Sigma}_h(c')$, then $\widehat{\Sigma}_{f,0}(c) \subsetneq \widehat{\Sigma}_{h,0}(c')$.
- If $\widehat{\Sigma}_g(c) \subset \widehat{\Sigma}_f(c)$ and $\widehat{\Sigma}_h(c') \subset \widehat{\Sigma}_g(c')$, then $\widehat{\Sigma}_{f,0}(c) \subset \widehat{\Sigma}_{h,0}(c')$.

By the definition of $\prec^{\mathrm{cor}}$, we have:

$$\widehat{\Sigma}_h(c) \subset \widehat{\Sigma}_g(c) \implies h_\star \prec^{\mathrm{cor}} g_\star,$$
$$\widehat{\Sigma}_g(c) \subset \widehat{\Sigma}_f(c) \implies g_\star \prec^{\mathrm{cor}} f_\star, \tag{2.3.5}$$
$$\widehat{\Sigma}_{h,0}(c) \subset \widehat{\Sigma}_{f,0}(c) \implies h_\star \prec^{\mathrm{cor}} f_\star. \tag{2.3.6}$$

It is easy to conclude that in all the possible cases for the inclusions between $\widehat{\Sigma}_f(c)$ and $\widehat{\Sigma}_g(c)$, and between $\widehat{\Sigma}_g(c)$ and $\widehat{\Sigma}_h(c)$, $\{f_\star, g_\star, h_\star\}$ has an extremum.

ii. *c' is a corner at a quadratic-like singularity.* If (c', c) is a double forward corner, all the possible cases can be treated as 3(a), so we assume that (c, c') is a double forward corner. For the same reasons, we assume that $f_\star$ and $h_\star$ are comparable by $\prec^{\mathrm{cor}}$ and (c, c') is their initial corner of overlap, so that $\mathcal{D}_{g_\star \cap f_\star} \subseteq \mathcal{D}_{g_\star \cap h_\star}$. Here, $f_\star$ and $g_\star$ intersect as in case (C2)(i) of Section 2.3.2.3, see Figure 2.3.3c, while $f_\star$ and $h_\star$ intersect as in case (C1) (ii), see Figure 2.3.3b (in particular the bottom one).

We use the same notations as before. We also consider points $y_0 \in \mathcal{D}_{g_\star}$ and $z_0 \in \mathcal{D}_{h_\star}$ such that $\varphi_{h_\star \cap g_\star}(z_0) = y_0$, and z_0 lies on the X-line corresponding to the minimum of $\mathcal{I}_{h_\star \cap g_\star}$ in $\mathcal{I}_{h_\star}$, so that $h(z_0)$ belongs to the right unstable branch of the singularity at c'.

The implications (2.3.3) and (2.3.4) still hold, as well as

$$\Sigma_h(z_0) \subset \Sigma_g(y_0) \implies \widehat{\Sigma}_{h,0}(c') \subset \Sigma_g(b_0).$$

The implications (2.3.5) and (2.3.6) hold for similar reasons, as well as

$$\Sigma_h(z_0) \subset \Sigma_g(y_0) \implies h_\star \prec g_\star.$$

Indeed, if $\Sigma_h(z_0) \subsetneq \Sigma_g(y_0)$, then $g_\star$ and $h_\star$ are comparable by $\prec^{\mathrm{geo}}$ and $h_\star \prec^{\mathrm{geo}} g_\star$.

If $\Sigma_h(z_0) = \Sigma_g(y_0)$, then $g_\star$ and $h_\star$ are not comparable by $\prec^{\mathrm{geo}}$, and two possibilities arise. Either z_0 belong to the initial edge of $h_\star$, and $h_\star \prec^{\mathrm{inc}} g_\star$, or $g_\star$ and $h_\star$ are comparable by $\prec^{\mathrm{un}}$ and their initial unstable domain of overlap corresponds to the unstable tile L at q' (recall that $f_\star$ and $h_\star$ do not meet along c). Moreover, the connection in $c \cap c'$ is a spike for L, and the entrance edge for $g_\star$ is the distinguished boundary component of L. By our conventions for admissible orders on unstable tiles, we have $h_\star \prec^{\mathrm{un}} g_\star$.

Examining all the possible inclusions between $\widehat{\Sigma}_f(c)$ and $\widehat{\Sigma}_g(c)$, and between $\Sigma_g(y_0)$ and $\Sigma_h(z_0)$, it is easy to show that $\{f_\star, g_\star, h_\star\}$ always has an extremum.

(c) $f_\star$ and $g_\star$ have an initial double corner of overlap (c, c'). The cases where $h_\star$ is comparable by $\prec^{\mathrm{geo}}$, $\prec^{\mathrm{inc}}$, or $\prec^{\mathrm{cor}}$ with either $f_\star$ or $g_\star$ can be treated as in 3(a) or 3(b), up to permutation. In the remaining cases, $h_\star$ is comparable by $\prec^{\mathrm{un}}$, or by $\prec^{\mathrm{or}}$, with both $f_\star$ and $g_\star$, and the strategy of case 3(a)ii applies.

4. $f_\star$ and $g_\star$ are comparable by $\prec^{\mathrm{un}}$. The only case that has not been treat yet, up to permutation, is when $f_\star$, $g_\star$, and $h_\star$ are all comparable by $\prec^{\mathrm{un}}$ and share the same initial unstable domain of overlap. In this situation, transitivity immediately follows from the definition of $\prec^{\mathrm{un}}$.

5. $f_\star$ and $g_\star$ are comparable by $\prec^{\mathrm{or}}$. The only case that has not been treated yet, up to permutation, is when $f_\star$, $g_\star$, and $h_\star$ are all comparable by $\prec^{\mathrm{or}}$, and $\mathcal{D}_{f_\star \cap g_\star} = \mathcal{D}_{f_\star \cap h_\star}$, $\mathcal{D}_{g_\star \cap f_\star} = \mathcal{D}_{g_\star \cap h_\star}$. Without loss of generality, let us assume that $f_\star \prec^{\mathrm{or}} g_\star$ and $g_\star \prec^{\mathrm{or}} h_\star$. We consider forward curves $\gamma_{f_\star}$, $\gamma_{g_\star}$, and $\gamma_{h_\star}$ as in the definition of $\prec^{\mathrm{or}}$; by item (I2) in the proof of Lemma 2.3.10, we can assume that there exists a pure X-line $\ell_0 \in \mathcal{I}_{f_\star \cap g_\star}$ from $\gamma_{f_\star}$ to $\gamma_{g_\star}$, and a pure X-line $\ell_1 \in \mathcal{I}_{g_\star \cap h_\star}$ from $\gamma_{g_\star}$ to $\gamma_{h_\star}$. Let $\ell_1' \in \mathcal{I}_{f_\star \cap g_\star}$ denote $\alpha_{g_\star \cap f_\star}(\ell_1)$. If $\ell_1' = \ell_0$, we readily obtain a pure X-line from $\gamma_{f_\star}$ to $\gamma_{h_\star}$ and $f_\star \prec^{\mathrm{or}} h_\star$. Otherwise, $\varphi_{g_\star \cap f_\star}(\gamma_{g_\star|(0,1]})$ induces a strictly forward curve intersecting ℓ_0 and ℓ_1', so either $\ell_0 \lhd \ell_1'$ or $\ell_1' \lhd \ell_0$. Let us consider the case

$\ell_0 \lhd \ell_1'$ first. Using Lemma 2.2.7, it is easy to show that either ℓ_0' intersects $\gamma_{f_\star}$, or is part of a broken X-line intersecting $\gamma_{f_\star}$ as in cases 1 and 2 in of item (I2) in the proof of Lemma 2.3.10. It follows that $f_\star \prec^{\mathrm{or}} h_\star$. The case $\ell_0 \lhd \ell_1'$ can be treated similarly by considering $\ell_0' \coloneqq \alpha_{f_\star \cap g_\star}(\ell_0)$ and ℓ_1 instead, and showing that ℓ_0' is part of a (broken) pure X-line intersecting $\gamma_{h_\star}$.

□

2.3.3.3 Completed forward envelopes

In this section, we extend the previously constructed weak prefoliation structure on $\mathscr{U}$ to the collection of *completed* forward envelopes $\overline{\mathscr{U}}$. Notice that since $\mathscr{U}$ covers M^{good} (minus the stable branches at quadratic-like singularities), $\overline{\mathscr{U}}$ covers M.

Let $f, g \in \mathscr{U}$ be forward envelopes (with our usual notations), with completions $\bar{f}, \bar{g} \in \overline{\mathscr{U}}$, and $a \in \overline{U}$ and $b \in \overline{V}$ such that $\bar{f}(a) = \bar{g}(b) =: p$. We write $\bar{f}_\star = (\bar{f}, a)$ and $\bar{g}_\star = (\bar{g}, b)$.

If $p \in M^{\mathrm{good}}$, then the extension of $\precsim_p$ to $\overline{\mathscr{U}}_p$ is entirely determined by axioms 1 and 2 of Definition 2.1.8. Indeed, a either belongs to U or to a forward edge of $\overline{U}$, and similarly for b. Near p, $\bar{f}_\star$ and $\bar{g}_\star$ locally look like a disk or a half-disk bounded by the flow line of X passing through p. Therefore, we can distinguish two cases:

1. $\bar{f}_\star$ and $\bar{g}_\star$ locally coincide. Then we can slide the marked points along a (possibly trivial) small curve γ from p to $p' \in M$ such that the new marked points satisfy $a' \in U$ and $b' \in V$. Then, we set $\bar{f}_\star \dot{\precsim}_p \bar{g}_\star$ if $(f, a') \precsim_{p'} (g, b')$, and $\bar{g}_\star \dot{\precsim}_p \bar{f}_\star$ otherwise. Notice that this does not depend on the choice of γ, since f and g intersect along a path-connected set near p.

2. One of $\bar{f}_\star$ and $\bar{g}_\star$ is locally strictly above the other, since f and g cannot topologically cross. We set $\bar{f}_\star \dot{\precsim}_p \bar{g}_\star$ if $\bar{g}_\star$ is locally strictly above $\bar{f}_\star$, and $\bar{g}_\star \dot{\precsim}_p \bar{f}_\star$ otherwise.

The previous arguments can easily be adapted to extend $\precsim_p$ to $\overline{\mathscr{U}}_p$ for $p \in Q^*$, because of the shape of forward tiles near a quadratic-like singularities, see Figure 2.2.5. Crucially, the interiors of two such tiles intersect along a *path-connected* set near p when they locally coincide. It is then straightforward to check that $\dot{\precsim}_p$ is an extension of $\precsim_p$ to $\overline{\mathscr{U}}_p$ and that the axioms of Definition 2.1.8 are satisfied on $M \setminus \Delta_{\mathrm{sa}}^{\mathrm{pure}}$.

The extension of $\precsim_p$ to $\overline{\mathscr{U}}_p$ for $p \in \Delta_{\mathrm{sa}}^{\mathrm{pure}}$ is more subtle. We consider two cases:

1. $\bar{f}_\star$ and $\bar{g}_\star$ intersect along a stable or unstable branch at p. We then slide the marked points along this branch to obtain $(\bar{f}, a'), (\bar{g}, b') \in \overline{\mathscr{U}}_{p'}$ and we set $\bar{f}_\star \dot{\precsim}_p \bar{g}_\star$ if $(\bar{f}, a') \dot{\precsim}_{p'} (\bar{g}, b')$, and $\bar{g}_\star \dot{\precsim}_p \bar{f}_\star$ otherwise. Notice that $\dot{\precsim}_{p'}$ is already defined.

2. $\bar{f}_\star$ and $\bar{g}_\star$ do not intersect along the stable and unstable branches at p. In that case, they form two opposite (forward) quadrants at p, and they only intersect at p near p. Let $\Sigma_f^{\llcorner} \subset \mathscr{A}_p$ denotes the set of marked tiles $h_\star \in \mathscr{A}_p$ such that either $h_\star$ is locally geometrically below f, or they locally coincide (relative to the domain of f) and after sliding the marked points inside the interiors of the domain of f and h, $(h, z') \in \Sigma_f(x')$. Here, x' and z' denote the slid marked points of f and h, respectively, satisfying $f(x') = h(z')$. We further denote by $\widehat{\Sigma}_f^{\llcorner} \subset \mathscr{A}_p$ the section generated by $\Sigma_f^{\llcorner}$. We similarly define $\widehat{\Sigma}_g^{\llcorner} \subset \mathscr{A}_p$. If $\widehat{\Sigma}_f^{\llcorner} \subsetneq \widehat{\Sigma}_g^{\llcorner}$ (resp. $\widehat{\Sigma}_g^{\llcorner} \subsetneq \widehat{\Sigma}_f^{\llcorner}$), we set $\bar{f}_\star \dot{\precsim}_p \bar{g}_\star$ (resp. $\bar{g}_\star \dot{\precsim}_p \bar{f}_\star$). This corresponds to the 'geometrically coherent' case in the definition of $\prec^{\mathrm{cor}}$.

   Furthermore, if there exists $\bar{h}_\star \in \overline{\mathscr{U}}_p$ such that

   - $\bar{f}_\star$ and $\bar{h}_\star$ intersect as in case 1 and $\bar{f}_\star \dot{\precsim}_p \bar{h}_\star$ (resp. $\bar{h}_\star \dot{\precsim}_p \bar{f}_\star$),
   - $\bar{g}_\star$ and $\bar{h}_\star$ intersect as in case 1 and $\bar{h}_\star \dot{\precsim}_p \bar{g}_\star$ (resp. $\bar{g}_\star \dot{\precsim}_p \bar{h}_\star$),

   then necessarily $\widehat{\Sigma}_f^{\llcorner} \subset \widehat{\Sigma}_g^{\llcorner}$ (resp. $\widehat{\Sigma}_g^{\llcorner} \subset \widehat{\Sigma}_f^{\llcorner}$). If moreover $\widehat{\Sigma}_f^{\llcorner} = \widehat{\Sigma}_g^{\llcorner}$, then we set $\bar{f}_\star \dot{\precsim}_p \bar{g}_\star$ (resp. $\bar{g}_\star \dot{\precsim}_p \bar{f}_\star$).

If $\widehat{\Sigma}^{\llcorner}_f = \widehat{\Sigma}^{\llcorner}_g$ and no such $\bar{h}_\star \in \overline{\mathscr{U}}_p$ exists, we set $\bar{f}_\star \dot{\precsim}_p \bar{g}_\star$ if $\bar{f}_\star$ is bounded by the forward corner in $\mathfrak{c}$, and $\bar{g}_\star \dot{\precsim}_p \bar{f}_\star$ otherwise.[14]

It is not a priori clear that $\dot{\precsim}$ is coherent at $p \in \Delta_{\mathrm{sa}}^{\mathrm{pure}}$ in case 1, since f and g can locally intersect along a *disconnected* set. This corresponds to the situation that we now describe. Let us pick local coordinates (x, y, z) about p adapted to η such that the stable branches at p are contained in $\{x = 0\}$, and the unstable branches at p are contained in $\{y = 0\}$. In these coordinates, we identify f and g with the graphs of two functions, still denoted by f and g, over domains $D_f, D_g \subset \mathbb{R}^2_{x,y}$. After possibly shrinking the coordinate neighborhood of p, we further assume

$$D_f = \{-\epsilon < x < 0\} \cup \{-\epsilon < y < 0\},$$
$$D_g = \{0 < x < \epsilon\} \cup \{0 < y < \epsilon\},$$

for some $\epsilon > 0$. Moreover, f and g continuously extend to 0 along $\{x = 0\} \cup \{y = 0\}$. Writing

$$Q_- := \{0 < x < \epsilon, -\epsilon < y < 0\},$$
$$Q_+ := \{-\epsilon < x < 0, 0 < y < \epsilon\},$$

we have

$$D_f \cap D_g = Q_- \sqcup Q_+,$$

and this set is *disconnected*. We choose arbitrary points $x_- \in Q_-$ and $x_+ \in Q_+$, and we set $f \prec_\pm g$ if one of the following holds:

(a) $f_{|Q_\pm} \leq g_{|Q_\pm}$ and $f_{|Q_\pm} \neq g_{|Q_\pm}$,

(b) $f_{|Q_\pm} = g_{|Q_\pm}$, and $(f, x_\pm) \prec (g, x_\pm)$.

[14] Alternatively, we could use the forward corner at p not in $\mathfrak{c}$. This choice has to be the same for all completed forward envelopes at p, but it does not need to be the same for all pure saddle singularities.

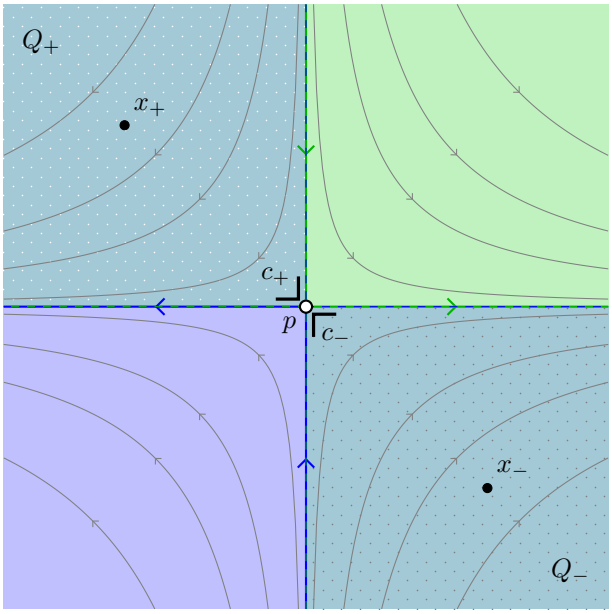


Figure 2.3.5: Overlap of f and g near p. The blue region corresponds to D_f, the green region corresponds to D_g, and their overlap is in teal. The quadrant with gray dots represents Q_- while the quadrant with white dots represents Q_+.

Otherwise, we set $g \prec_\pm f$. In that case, $g_{|Q_\pm} \leq f_{|Q_\pm}$, and $(g, x_\pm) \prec (f, x_\pm)$ if $f_{|Q_\pm} = g_{|Q_\pm}$, since f and g are forward envelopes and $(f, x_\pm) \not\approx (g, x_\pm)$. Moreover, when $f_{|Q_\pm} = g_{|Q_\pm}$, $(f, x_\pm)$ and $(g, x_\pm)$ are comparable by $\prec^{\mathrm{geo}}$ or $\prec^{\mathrm{cor}}$, and in he latter case their initial corner over overlap $c_\pm$ bounds $Q_\pm$. See Figure 2.3.5. The coherence at p of the relation $\dot{\precsim}$ between $\bar{f}$ and $\bar{g}$ can be stated as follows:

Lemma 2.3.13. *$f \prec_- g$ if and only if $f \prec_+ g$.*

Proof. Without loss of generality, we assume $c_+ \in \mathfrak{c}$. By sliding marked points to the corners and completing, we obtain sections $\widehat{\Sigma}_f(c_\pm), \widehat{\Sigma}_g(c_\pm) \subset \mathscr{A}_p$ as in Section 2.3.2.3. Because of the shape of backward tiles passing though p and of the flow of X near p, one easily shows

$$\widehat{\Sigma}_f(c_-) = \widehat{\Sigma}_f(c_+), \qquad \widehat{\Sigma}_g(c_-) = \widehat{\Sigma}_g(c_+). \tag{2.3.7}$$

Moreover, if $\widehat{\Sigma}_f(c_-) \subset \widehat{\Sigma}_g(c_-)$ then $f_{|Q_-} \leq g_{|Q_+}$ (and similarly for f and g swapped and for c_+). Indeed, we consider two cases.

1. There exists $y_- \in Q_-$ such that $\Sigma_f(y_-) \neq \varnothing$. Then every backward tile in $\Sigma_f(y_-)$ intersects $\{x = y = 0\}$ and is geometrically below f. If it passes through 0, then it induces an element in $\widehat{\Sigma}_f(c) \subset \widehat{\Sigma}_g(c)$, and this tile is also geometrically below g over Q_-, implying $f(y'_-) \leq g(y'_-)$ for every $y'_- \in Q_-$ in the forward region of Q_- containing y_- by the definition of one-dimensional envelopes (see [BI08, Section 5]). Moreover, for every $y'_- \in Q_-$ in the backward region of Q_- containing y_-, $\Sigma_f(y'_-) \neq \varnothing$ and the same argument shows that $f(y_-) \leq g(y_-)$. Therefore, $f_{|Q_-} \leq g_{|Q_+}$.

2. Otherwise, $\Sigma_f(y_-) = \varnothing$ for every $y_- \in Q_-$. Then no tile in $\widehat{\Sigma}_f(c_-)$ intersects $Q_- \times (-\epsilon, \epsilon)$, and f is necessarily geometrically below g over Q_-. Indeed, $f_{|Q_-}$ coincides with the restriction to Q_- of (the underlying tile of) the forward envelope starting at $\big((\epsilon/2, 0), \varnothing\big)$ which is geometrically the lowest such tile.

We now show:

$$\widehat{\Sigma}_g(c_\pm) \subset \widehat{\Sigma}_f(c_\pm) \implies g \prec_\pm f, \tag{2.3.8}$$

$$\widehat{\Sigma}_f(c_\pm) \subsetneq \widehat{\Sigma}_g(c_\pm) \implies f \prec_\pm g, \tag{2.3.9}$$

which will immediately imply the desired result. To show (2.3.8), we assume $\widehat{\Sigma}_g(c_\pm) \subset \widehat{\Sigma}_f(c_\pm)$ and we consider the order for f and g with respect to $\prec_-$.

1. $f_{|Q_-} \neq g_{|Q_-}$. Then since $g_{|Q_-} \leq f_{|Q_-}$ by the above argument, $g \prec_- f$.

2. $f_{|Q_-} = g_{|Q_-}$. Then $f_\star = (f, x_-)$ and $g_\star = (g, x_-)$ are comparable by either $\prec^{\mathrm{geo}}$ or $\prec^{\mathrm{cor}}$. First, for every $y_- \in Q_-$, $\Sigma_g(y_-) \subset \Sigma_f(y_-)$. Otherwise, there would exist a marked backward tile $a_\star \in \Sigma_g(y_-) \setminus \Sigma_f(y_-)$ which would geometrically coincide with f and g over Q_-, since $f \leq a \leq g$ over Q_- by elementary properties of one-dimensional envelopes and their shadows (see [BI08, Section 5]). However, sliding the marked point of $a_\star$ to the singularity at c_- would yield a marked tile

in $\widehat{\Sigma}_g(c_-) \setminus \widehat{\Sigma}_f(c_-)$, a contradiction. This implies that if $f_\star$ and $g_\star$ are comparable by $\prec^{\mathrm{geo}}$, then $g_\star \prec^{\mathrm{geo}} f_\star$. Otherwise, $f_\star$ and $g_\star$ are comparable by $\prec^{\mathrm{cor}}$ and since $c_- \notin \mathfrak{c}$, we have $g_\star \prec^{\mathrm{cor}} f_\star$ independently of the geometric (in)coherence of $f_\star$ and $g_\star$ at c_-.

Notice that the same argument applies to $\prec_+$, since $c_+ \in \mathfrak{c}$ so the order $\prec^{\mathrm{cor}}$ for geometrically coherent forward envelopes meeting at c_+ compared to c_- is reversed. Here, the choice of admissible corners 'twisting' $\prec^{\mathrm{cor}}$ in the geometrically coherent case is essential!

The proof of (2.3.9) is the same up to swapping f and g, and the geometrically coherent case for $\prec^{\mathrm{cor}}$ does not arise. □

It is now rather straightforward to check that the collection of relations $\{\dot{\precsim}_p\}_{p\in M}$ induces a relation $\dot{\precsim}$ on $\overline{\mathscr{U}}_\star$ which is coherent along intersections. This implies that $\dot{\precsim}$ satisfies the axioms of Definition 2.1.8 since $\precsim$ does; the details are left to the reader. To summarize, we obtain:

Proposition 2.3.14. *The set $\overline{\mathscr{U}}$ of completed forward envelopes equipped with $\dot{\precsim}$ is a weak prefoliation. It extends $(\mathscr{U}, \precsim)$ and covers M.*

2.4 Proof of Proposition 2.1.11

Let $\left(\mathscr{A}, \precsim^{\mathscr{A}}\right)$ be a (possibly empty) complete and backward weak prefoliation. In the previous sections, we constructed a complete and forward weak prefoliation $\left(\overline{\mathscr{U}}, \dot{\precsim}\right)$ which covers M. Our construction depends on auxiliary choices that are omitted from the notations. To construct an extension $\mathscr{A} \hookrightarrow \mathscr{B}$, we proceed in two steps.

1. First, we construct a weak prefoliation structure on $\mathscr{A} \cup \overline{\mathscr{U}}$ such that the inclusion $\mathscr{A} \hookrightarrow \mathscr{A} \cup \overline{\mathscr{U}}$ is an extension.

2. Then, we construct an extension $\mathscr{A} \cup \overline{\mathscr{U}} \hookrightarrow \mathscr{B}$ by gluing the edges and corners of tiles in $\mathscr{A}$ to suitable completed forward envelopes (the *upper neighbors*), while keeping copies of completed forward envelopes.

2.4.1 First extension

We define a relation $\precsim^{\cup}$ on $\mathscr{A} \cup \overline{\mathscr{U}}$ by adapting [BI08, Section 6.13] and the proof of [BI08, Proposition 4.12]. It restricts to $\precsim^{\mathscr{A}}$ on $\mathscr{A}_\star$ and to $\dot{\precsim}$ on $\overline{\mathscr{U}}_\star$. We define $\precsim_p^{\cup}$ in two steps. The first one is of a geometric nature, while the second one is purely formal.

2.4.1.1 Step 1: geometric extension

Let $p \in M$, $f_\star \in \mathscr{A}_p$, and $\bar{u}_\star \in \overline{\mathscr{U}}_p$. Here, $\bar{u}$ is the completion of a forward envelope $u \in \mathscr{U}$, $u : U \to M$, with shadow Σ.

We first use the shadow Σ and the geometric order to define a relation $\precsim_p^{\cup\circ}$:

- If the marked point $a \in \overline{U}$ of $\bar{u}_\star$ belongs to U, we set $f_\star \precsim_p^{\cup\circ} \bar{u}_\star$ if $f_\star \in \Sigma(a)$, and $\bar{u}_\star \precsim_p^{\cup\circ} f_\star$ otherwise.

- If $a \in \overline{U} \setminus U$ and one of $f_\star$ or $\bar{u}_\star$ is locally strictly above the other, their relation with respect to $\precsim_p^{\cup\circ}$ is defined as to match the geometric order.

We then extend $\precsim^{\cup\circ}$ along intersections at certain boundary components of $\overline{U}$ in a coherent way. More precisely, suppose that there exists a continuous path $\gamma : [0,1] \to \overline{U}$ starting at $a \in \overline{U} \setminus U$ which lifts to $f_\star$ so that f and $\bar{u}$ are comparable by $\precsim^{\cup\circ}$ at $p' = f \circ \gamma(1)$ by Step 1, after sliding the marked points along γ and its lift. We define the relation between $f_\star$ and $\bar{u}_\star$ with respect to $\precsim_p^{\cup\circ}$ as to match the one with respect to $\precsim_{p'}^{\cup\circ}$ after sliding the marked points. Notice that this step applies if the marked point of $\bar{u}_\star$ belongs to an edge connected to the initial edge of u via a (double) corner, and this (double) corner lifts to f.

This defines a relation $\precsim^{\cup\circ}$ on $\mathscr{A}_\star \cup \overline{\mathscr{U}}_\star$ that restricts to $\precsim^{\mathscr{A}}$ on $\mathscr{A}_\star$ and to $\dot{\precsim}$ on $\overline{\mathscr{U}}_\star$. By definition, it is coherent along intersections and is compatible with the geometric order. Moreover, it satisfies the following 'pseudo-transitivity' property:

Lemma 2.4.1. *Let $a_\star, b_\star, c_\star \in \mathscr{A}_\star \cup \overline{\mathscr{U}}_\star$. If $a_\star \precsim^{\cup\circ} b_\star$, $b_\star \precsim^{\cup\circ} c_\star$, and $a_\star$ and $c_\star$ are comparable by $\precsim^{\cup\circ}$, then $a_\star \precsim^{\cup\circ} c_\star$.*

Proof. Let $p \in M$ and consider $a_\star, b_\star, c_\star \in \mathscr{A}_p \cup \overline{\mathscr{U}}_p$ such that $a_\star \precsim_p^{\cup\circ} b_\star$, $b_\star \precsim_p^{\cup\circ} c_\star$, and $a_\star$ and $c_\star$ are comparable by $\precsim_p^{\cup\circ}$. The case $p \in M^{\mathrm{un}}$ is immediate. We then assume that $p \in M^{\mathrm{pure}}$ and we consider three cases, depending on the nature of the flow line γ of X passing through p.

1. *γ is not connected to a singularity of X.* Notice that if $f_\star \in \mathscr{A}_p$ and $\bar{u}_\star \in \overline{\mathscr{U}}_p$ are comparable by $\precsim_p^{\cup\circ}$, and if they only overlap along γ, then the marked point of $\bar{u}_\star$ necessarily belongs to the initial edge of u. Then, by considering all the possible cases depending on whether the three marked tiles belong to $\mathscr{A}_p$ or $\overline{\mathscr{U}}_p$, it is easy to check that $a_\star \precsim_p^{\cup\circ} c_\star$.

2. *γ is connected to a single singularity of X, which is not itself connected to a saddle singularity.* This case is slightly more complicated because of the following: if $f_\star \in \mathscr{A}_p$ and $\bar{u}_\star \in \overline{\mathscr{U}}_p$ are comparable by $\precsim_p^{\cup\circ}$, and if they only overlap along a (concave or convex) corner c containing γ, then the marked point of $\bar{u}_\star$ belongs to one of the edges at c, but not necessarily to the initial edge of u.

   Let us consider the following particular case: $a_\star = \bar{u}_\star \in \overline{\mathscr{U}}_p$, $b_\star = f_\star \in \mathscr{A}_p$, and $c_\star = \bar{v}_\star \in \overline{\mathscr{U}}_p$. Moreover, the marked point of $f_\star$ belongs to a concave corner on in the boundary of f, the marked point of $\bar{u}_\star$ belongs to the domain of u, but the marked point of $\bar{v}_\star$ does not belong to the domain of v. Arguing as in the proof of Lemma 2.4.4, we have that $\bar{u}_\star$ is locally geometrically below $\bar{v}_\star$. If they locally coincide, we can slide the marked points inside the forward domain of overlap of u and v, and compare them there. They would be comparable by either $\prec^{\mathrm{geo}}$, or

by $\prec^{\mathrm{cor}}$ and would be geometrically noncoherent at their initial corner of overlap. Both situations yield $\bar{u}_\star \dot{\precsim} \bar{v}_\star$ as desired.

The other cases can be treated in a similar way.

3. *γ is connected to a saddle singularity of X which is itself connected to another saddle singularity.* This case is more tedious than the previous one, but can be treated in a similar way. The details are left to the reader.

We are now left with the case where p is a pure saddle or a quadratic-like singularity. This includes the case of a saddle connected to another saddle singularity.

Let us consider the following particular case: $p \in \Delta_{\mathrm{sa}}^{\mathrm{pure}}$ is not connected to another saddle singularity, $a_\star = \bar{u}_\star \in \overline{\mathscr{U}}_p$, $b_\star = f_\star \in \mathscr{A}_p$, and $c_\star = \bar{v}_\star \in \overline{\mathscr{U}}_p$. Moreover, the marked point of $f_\star$ belongs to a concave corner on in the boundary of f, and $\bar{u}_\star$ and $\bar{v}_\star$ overlap with $f_\star$ near p along two different edges. Then using the notations of Section 2.3.3.3, $f_\star \in \widehat{\Sigma}_v^{\llcorner} \setminus \widehat{\Sigma}_u^{\llcorner}$, hence $\widehat{\Sigma}_u^{\llcorner} \subsetneq \widehat{\Sigma}_v^{\llcorner}$ and $\bar{u}_\star \precsim_p^{\cup} \bar{v}_\star$.

The remaining cases can be treated in a similar way, and the details are left to the reader. □

2.4.1.2 Step 2: abstract extension

Even after the previous step, the relation $\precsim_p^{\cup\circ}$ between $f_\star$ and $\bar{u}_\star$ might still remain undefined if the marked point of $f_\star$ and $\bar{u}_\star$ both belong to certain edges, or if they are singularities at certain corners intersecting along edges only. We proceed in three steps:

1. Let us first assume that p belongs to a pure flow line of X which is not connected to any singularity of X. We extend $\precsim^{\cup\circ}$ at p by the following abstract procedure:

   - If there exists $a_\star \in \mathscr{A}_p \cup \overline{\mathscr{U}}_p$ such that $f_\star \precsim_p^{\cup\circ} a_\star$ and $a_\star \precsim_p^{\cup\circ} \bar{u}_\star$, we set $f_\star \precsim_p^{\cup} \bar{u}_\star$.

- Otherwise, we set $\bar{u}_\star \precsim_p^{\cup} f_\star$.

By Lemma 2.4.1, this definition ensures that $\precsim_p^{\cup}$ is a total preorder on $\mathscr{A}_p \cup \overline{\mathscr{U}}_p$.

2. We now assume that p belongs to a connection between two saddle singularities $q_\pm \in \Delta_{\mathrm{sa}}$. By sliding the marked points towards $q_\pm$ along this connection, we obtain embeddings

$$\varsigma_\pm : \mathscr{A}_p \cup \overline{\mathscr{U}}_p \longrightarrow \mathscr{A}_{q_\pm} \cup \overline{\mathscr{U}}_{q_\pm}$$

which preserve $\precsim^{\cup\circ}$. We modify the previous abstract procedure to extend $\precsim^{\cup\circ}$ on $\mathscr{A}_p \cup \overline{\mathscr{U}}_p$. With the same notations as before, we set $f_\star \precsim_p^{\cup} \bar{u}_\star$ if one of the following holds:

- There exists $a_\star \in \mathscr{A}_p \cup \overline{\mathscr{U}}_p$ such that $f_\star \precsim_p^{\cup\circ} a_\star$ and $a_\star \precsim_p^{\cup\circ} \bar{u}_\star$.
- There exists $a_\star^\pm \in \mathscr{A}_{q_\pm} \cup \overline{\mathscr{U}}_{q_\pm}$ such that $\varsigma_\pm\big(f_\star\big) \precsim_{q_\pm}^{\cup\circ} a_\star^\pm$ and $a_\star^\pm \precsim_{q_\pm}^{\cup\circ} \varsigma_\pm\big(\bar{u}_\star\big)$.

Otherwise, we set $\bar{u}_\star \precsim_p^{\cup} f_\star$. We further coherently extend $\precsim^{\cup}$ along the flow line of X passing through p and at $q_\pm$ by sliding the marked points along this flow line. As in the previous item, Lemma 2.4.1 ensures that $\precsim_p^{\cup}$ is a total preorder on $\mathscr{A}_p \cup \overline{\mathscr{U}}_p$.

3. Finally, we assume that $p \in \Delta_{\mathrm{sa}} \cup Q$ is a saddle or quadratic singularity. We use the same abstract procedure as in the first item to extend $\precsim^{\cup}$ at p, taking into account the extension in step 2 if p is connected to another saddle singularity. As before, this yields a total preorder $\precsim_p^{\cup}$ on $\mathscr{A}_p \cup \overline{\mathscr{U}}_p$. We then extend $\precsim^{\cup}$ by coherence along the pure stable and unstable branches at p (no extension is needed along a connection to another saddle singularity).

It is now rather straightforward to prove:

Lemma 2.4.2. *The set $\mathscr{A} \cup \overline{\mathscr{U}}$ equipped with the relation $\precsim^{\cup}$ is a weak prefoliation which extends $\left(\mathscr{A}, \precsim^{\mathscr{A}}\right)$.*

Proof. Left to the reader. □

2.4.2 Second extension

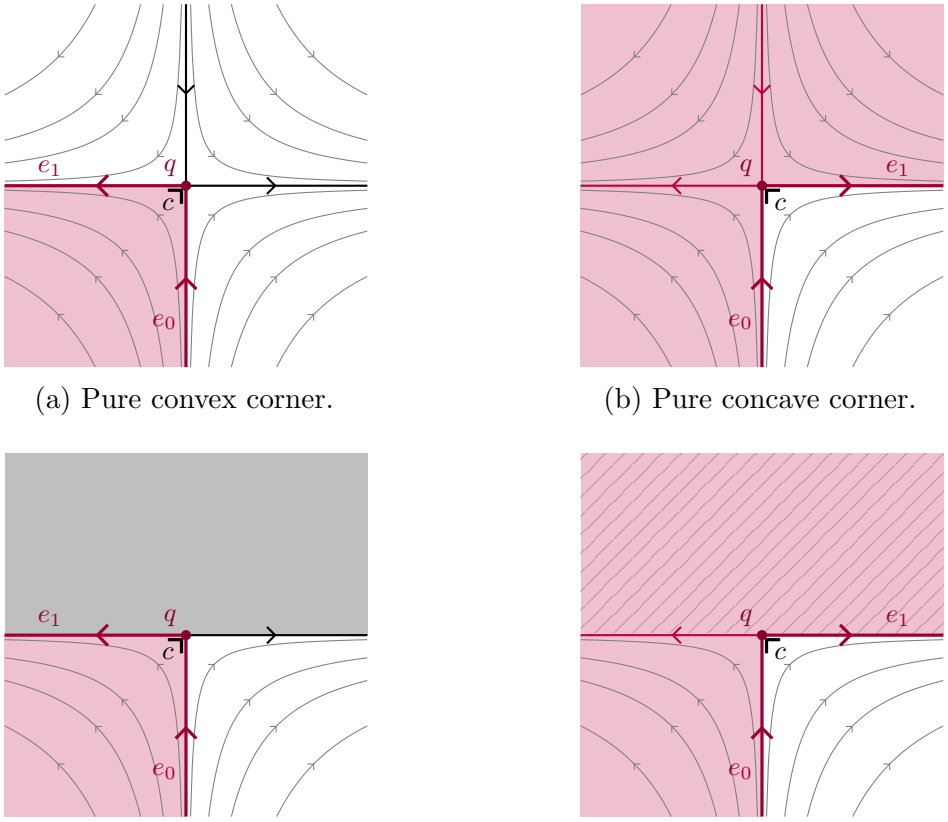
We now define an extension $\mathscr{A} \cup \overline{\mathscr{U}} \hookrightarrow \mathscr{B}$ by adapting the strategy of the proof of [BI08, Proposition 4.11]. To that extent, we define a similar notion of *upper neighbors* along the edges of tiles in $\mathscr{A}$ (see [BI08, Definition 4.10]), and we extend every such tile by attaching its upper neighbors to its boundary components. We also keep copies of tiles in $\overline{\mathscr{U}}$ to ensure that $\mathscr{B}$ covers M. We then define a weak prefoliation structure on $\mathscr{B}$ in a purely formal way. As before, we will need to treat the corners with particular care.

2.4.2.1 Typology of backward corners

Let $f : U \to M$ be a complete backward tile. We denote by $\mathcal{B}$ the set of boundary components of U, and by $\mathcal{E}$ the set of edges of U. A boundary component of U contains between one and three edges. Two edges $e_0, e_1 \in \mathcal{E}$ are **adjacent** if $c = (f(e_0), f(e_1))$ is a (backward or forward) corner in M. Note the importance of the order. We further distinguish two cases:

- The singularity at c is a pure saddle singularity. If c is a backward (resp. forward) corner, we say that (e_0, e_1) is a **pure convex (resp. concave) corner** of U (or f).

- The singularity at c is a quadratic-like singularity. If c is a backward (resp. forward) corner, we say that (e_0, e_1) is an **unstable convex (resp. concave) corner** of U (or f).



(a) Pure convex corner.

(b) Pure concave corner.

(c) Unstable convex corner.

(d) Unstable concave corner.

Figure 2.4.1: The four types of corners of backward tiles.

These four types of backward corners are illustrated in Figure 2.4.1.

If (e_0, e_1) is a pure forward or unstable (convex or concave) corner of f, we say that (e_0, e_1) is an **adjoining pair of edges**. If $e_2 \in \mathcal{E}$ and both (e_0, e_1) and (e_1, e_2) are adjoining pairs of edges, we call (e_0, e_1, e_2) an **adjoining triple of edges**. We define a **boundary row** of U (or f) as either

1. A single edge which is not part of an adjoining pair,
2. An adjoining pair of edges which is not part of an adjoining triple, or
3. An adjoining triple of edges.

We denote by $\mathcal{R}$ the set of boundary rows of f. Since every edge is included in a row, and every row is included in a boundary component, we have natural surjections $\mathcal{E} \twoheadrightarrow \mathcal{R} \twoheadrightarrow \mathcal{B}$.

We also define a collection of **distinguished edges** as follows. Each boundary row r contains exactly one distinguished edge, and moreover,

- If r is an adjoining pair of edges forming a forward corner c, then the last edge of r is distinguished if $c \notin \mathfrak{c}$, and the first one is if $c \in \mathfrak{c}$. Recall that $\mathfrak{c}$ denotes the chosen admissible collection of forward corners in the definition of the weak prefoliation structure on $\mathscr{U}$.
- If r is an adjoining triple of edges, the last edge of r is distinguished.

We denote by $\mathcal{E}^\bullet \subset \mathcal{E}$ the set of distinguished edges of f. A row made of a single edge will be called a **trivial row**, in which case this edge is automatically distinguished.

We now explain the relevance of these definitions. Any complete forward tile intersecting a distinguished edge of f will also intersect all the other edges of f in the same boundary row. Moreover, if it starts at such an edge, it will not intersect edges of f in different boundary rows. We leave it to the reader to draw the corresponding pictures. Therefore, we have to be careful to define upper neighbors at edges of f that are 'coherent' along corners at adjoining edges.

2.4.2.2 Upper neighbors

Let $p \in M^{\mathrm{pure}}$, $f_\star \in \mathscr{A}_p$, $\bar{u}_\star \in \mathscr{U}_p$, and assume that the marked points of $f_\star$ and $\bar{u}_\star$ belong to edges of the respective domains for f and $\bar{u}$. Following [BI08, Definition 4.10], we say that $\bar{u}_\star$ is an **upper neighbor** of $f_\star$ in $\big(\mathscr{A} \cup \overline{\mathscr{U}}, \precsim^{\cup}\big)$ if $f_\star \prec_p^{\cup} \bar{u}_\star$, and $\bar{u}_\star$ is the lowest such element in $\mathscr{A}_p \cup \overline{\mathscr{U}}_p$, i.e.,

$$\forall g_\star \in \mathscr{A}_p \cup \overline{\mathscr{U}}_p, \quad f_\star \prec_p^{\cup} g_\star \implies \bar{u}_\star \precsim_p^{\cup} g_\star. \tag{2.4.1}$$

Lemma 2.4.3. *With the previous notations, $f_\star$ admits an upper neighbor which is unique up to order-equivalence.*

Proof. The uniqueness immediately follows from the definition. To show existence, we consider the section $\Sigma_0 \subset \mathscr{A}_p$ defined by

$$\Sigma_0 \coloneqq \left\{ g_\star \in \mathscr{A}_p \mid g_\star \precsim_p^{\mathscr{A}} f_\star \right\}. \tag{2.4.2}$$

If p does not belong to the pure stable branch of a quadratic-like singularity, we define $\bar{u}_\star$ as the forward envelope starting at $\big(p, \Sigma_0\big)$ and with marked point corresponding to the initial basepoint of $\bar{u}$. We emphasize that this forward envelope might *not* be the upper neighbor of $f_\star$. However, we claim that if the marked point of $f_\star$ belong to the *distinguished edge* of a boundary row r, then $\bar{u}_\star$ is the upper neighbor of $f_\star$. To prove this, we consider three cases.

1. r is a trivial row. By definition of $\precsim^\cup$ in Step 1 of the previous section, $f_\star \precsim^\cup \bar{u}_\star$. Then, let $g_\star \in \mathscr{A}_p \cup \overline{\mathscr{U}}_p$ such that $f_\star \prec^\cup g_\star$. We consider two cases:

   - $g_\star \in \mathscr{A}_p$. Then $g_\star \notin \Sigma_0$, so $\bar{u}_\star \precsim_p^\cup g_\star$ by definition.
   - $g_\star = \bar{v}_\star \in \overline{\mathscr{U}}_p$. By definition, there exists $\bar{w}_\star \in \overline{\mathscr{U}}_p$ such that $f_\star \precsim_p^\cup \bar{w}_\star \precsim_p^\cup \bar{v}_\star$, and the marked point z of $\bar{w}$ belongs to the domain of w. We must have $f_\star \in \Sigma_w(z)$, otherwise $\bar{w}_\star$ would be locally geometrically below $f_\star$, so $\Sigma_0 \subset \Sigma_w(z)$. If this inclusion is strict, then $u_\star \prec_p^{\mathrm{geo}} w_\star$ and $\bar{u}_\star \dot{\precsim}_p \bar{w}_\star$, hence $\bar{u}_\star \precsim_p^\cup \bar{w}_\star \precsim_p^\cup \bar{v}_\star$. If $\Sigma_0 = \Sigma_w(z)$, then $u_\star \prec_p^{\mathrm{inc}} w_\star$, hence $\bar{u}_\star \precsim_p^\cup \bar{v}_\star$ as well.

2. $r = (e_0, e_1)$ is an adjoining pair of edges. We first assume that the corresponding corner c is a pure forward corner. Let $g_\star \in \mathscr{A}_p \cup \overline{\mathscr{U}}_p$ such that $f_\star \prec^\cup g_\star$. As in the previous case, $g_\star \in \mathscr{A}_p$ implies $\bar{u}_\star \prec_p^\cup g_\star$. The case $g_\star \in \overline{\mathscr{U}}_p$ is slightly more complicated. As before, there exists $\bar{w}_\star \in \overline{\mathscr{U}}_p$ such that $f_\star \precsim_p^\cup \bar{w}_\star \precsim_p^\cup \bar{v}_\star$. However, the marked point if $\bar{w}_\star$ might not belong to the domain of w. If it does, then the previous argument still implies $\bar{u}_\star \precsim_p^\cup \bar{w}_\star$. Otherwise, we slide the marked point of $\bar{w}_\star$ along c, and the new marked point z' now belongs to the domain of w. We also

slide the marked point of $f_\star$ along c to get a new tile $f'_\star$ passing through $p' \coloneqq w(z')$. We also define the corresponding section

$$\Sigma'_0 \coloneqq \left\{ g'_\star \in \mathscr{A}_{p'} \mid g'_\star \precsim^{\mathscr{A}}_{p'} f'_\star \right\}$$

so that $\Sigma'_0 \subset \Sigma_w(z')$. If this inclusion is strict, then $\widehat{\Sigma}_u(c) \subsetneq \widehat{\Sigma}_w(c)$, implying $\bar{u}_\star \precsim^{\cup}_p \bar{w}_\star$. If $\Sigma'_0 = \Sigma_w(z')$, then the definitions of $\prec^{\mathrm{cor}}$ and distinguished edges imply $\bar{u}_\star \precsim^{\cup}_p \bar{w}_\star$ as well.

The cases where c is a (forward or backward) quadratic-like corner is simpler since no forward envelope starts at the pure stable branch of the corresponding singularity. The details are left to the reader.

3. $r = (e_0, e_1, e_2)$ is an adjoining triple of edges. There are multiple cases to consider depending on the nature of the corners corresponding to (e_0, e_1) and to (e_1, e_2), but all of these cases can be treated as in case 2. The details are left to the reader.

If the marked point of $f_\star$ does not belong to a distinguished edge, we consider the previously constructed upper neighbor at (a point on) the distinguished edge in r, and we slide the marked point along the corner(s) to obtain a marked tile at p. By the previous arguments, it is easy to check that the latter is the upper neighbor of $f_\star$. □

We emphasize that by construction, the upper neighbors of $f \in \mathscr{A}$ satisfy the following coherence condition along boundary rows. Let (e_0, e_1) be an adjoining pair of edges for f forming a corner c in M, and $x_0 \in e_0$, $x_1 \in e_1$ be two boundary marked points. We denote by $\bar{u}^0_\star$ and $\bar{u}^1_\star$ the upper neighbors of (f, x_0) and (f, x_1), respectively. Then:

Lemma 2.4.4. *Up to order-equivalence, $\bar{u}^0_\star$ and $\bar{u}^1_\star$ differ by sliding the marked point along c.*

Therefore, each boundary row $r \in \mathcal{R}$ of f determines a completed forward envelope $\bar{u}^r \in \overline{\mathscr{U}}$, unique up to forward-equivalence, such that the upper neighbor of f for a marked point on an edge of r is $\bar{u}^r$, with a suitable marked point on the boundary component of the initial edge of u^r. Importantly, gluing $\bar{u}^r$ to f along r yields a tile.

Remark 2.4.5. *Assume that $p \in M^{\mathrm{pure}}$ belongs to a closed orbit γ of X with period $T > 0$, and $\bar{u}_\star \in \overline{\mathscr{U}}_p$ is the upper neighbor of $f_\star \in \mathscr{A}_p$. In that case, p belongs to the initial edge of u. Assume that $\bar{u}_\star^T \simeq_p \bar{u}_\star$, where we use the notation from Section 2.2.1 for the T-translation. Then $u_\star^T \simeq u_\star$ and $\Phi^T\big(\Sigma_0\big) = \Sigma_0$, where Σ_0 is defined by* (2.4.2)*. It immediately follows that $f_\star^T \simeq_p f_\star$. This implies that the upper neighbor of a tile $f_\star$ in $\mathscr{A}_p$ is order-equivalent to its T-translated if and only if $f_\star$ is. However, it is possible that $f_\star^T \approx_p f_\star$ without $f_\star^T \simeq_p f_\star$ (f is geometrically covering itself, but not from the viewpoint of the preorder), in which case we might not even have $\bar{u}_\star^T \approx_p \bar{u}_\star$. This is because our preorders contain* strictly more *information than the geometric order.*

2.4.2.3 Gluing upper neighbors

We now define a complete and backward weak prefoliation $\mathscr{B}$ extending $\mathscr{A} \cup \overline{\mathscr{U}}$ by gluing the upper neighbors to the backward edges of the tiles in $\mathscr{A}$, while keeping copies of the tiles in $\overline{\mathscr{U}}$. More precisely, to each tile $f \in \mathscr{A}$ we associate a new tile $\widetilde{f}$ defined as follows. If f has no edge, we set $\widetilde{f} = f$. Otherwise, for each $e \in \mathcal{E}^\bullet$, we choose a point $x_e \in e$ and write $p_e \coloneqq f(x_e)$. Then, we denote by $\bar{u}_e \in \overline{\mathscr{U}}_{p_e}$ the upper neighbor of (f, x_e). It can be glued to f along the row of edges containing e. Recall that by Lemma 2.4.4 above, the upper neighbor of f at adjoining edges to e coincides with $\bar{u}_e$ with a slid marked point. We then define $\widetilde{f}$ as the complete and forward tile obtained by gluing *all* such upper neighbors along the (rows of) edges of f, without repetitions. By construction, there is a natural embedding φ_f of the domain of f into the domain of $\widetilde{f}$ as in Definition 2.1.10.

We now define

$$\widetilde{\mathscr{A}} \coloneqq \{\widetilde{f} \mid f \in \mathscr{A}\}, \qquad \mathscr{B} \coloneqq \widetilde{\mathscr{A}} \cup \overline{\mathscr{U}}.$$

It comes with a natural map $i : \mathscr{A} \cup \overline{\mathscr{U}} \to \mathscr{B}$ which restricts to the identity on $\overline{\mathscr{U}}$. By construction, the tiles in $\mathscr{B}$ are complete and forward, and they cover M. It remains to define a weak prefoliation structure $\precsim^{\mathscr{B}}$ on $\mathscr{B}$ making i an extension. To this end, we proceed as in the proof of [BI08, Proposition 4.11], in three steps.

1. We first define the restriction of $\precsim^{\mathscr{B}}$ to $\widetilde{\mathscr{A}}$. Let $\widetilde{f}_\star \in \widetilde{\mathscr{A}_\star}$, where $f \in \mathscr{A}$. We consider two cases:

   (a) The marked point belongs to the domain of f. This induces a marked tile $f_\star \in \mathscr{A}_\star$.

   (b) The marked point does not belong to the domain of f, but to the domain of a completed forward envelope $\bar{u} \in \overline{\mathscr{U}}$. This induces a marked tile $\bar{u}_\star \in \overline{\mathscr{U}}_\star$.

   This allows us to define a 'restriction' map $\rho : \widetilde{\mathscr{A}_\star} \to \mathscr{A}_\star \cup \overline{\mathscr{U}}_\star$ as

   - $\rho\big(\widetilde{f}_\star\big) = f_\star$ in case (a),
   - $\rho\big(\widetilde{f}_\star\big) = \bar{u}_\star$ in case (b).

   Then, we define $\precsim^{\mathscr{B}}$ on $\widetilde{\mathscr{A}}$ as the pullback of $\precsim^{\cup}$ along ρ:

   $$\forall \widetilde{f}_\star, \widetilde{g}_\star \in \widetilde{\mathscr{A}_\star}, \quad \widetilde{f}_\star \precsim^{\mathscr{B}} \widetilde{g}_\star \iff \rho\big(\widetilde{f}_\star\big) \precsim^{\cup} \rho\big(\widetilde{g}_\star\big).$$

2. We now define $\precsim^{\mathscr{B}}$ between elements $\widetilde{f}_\star \in \widetilde{\mathscr{A}_\star}$ and $\bar{v}_\star \in \overline{\mathscr{U}}_\star$. If $\rho\big(\widetilde{f}_\star\big) = f_\star \in \mathscr{A}_\star$, we define the order between $\widetilde{f}_\star$ and $\bar{v}_\star$ for $\precsim^{\mathscr{B}}$ to match the one between $f_\star$ and $\bar{v}_\star$ for $\precsim^{\cup}$. If $\rho\big(\widetilde{f}_\star\big) = \bar{u}_\star \in \overline{\mathscr{U}}_\star$, we set

   $$\widetilde{f}_\star \precsim^{\mathscr{B}} \bar{v}_\star \iff \bar{u}_\star \mathrel{\dot{\precsim}} \bar{v}_\star,$$
   $$\bar{v}_\star \precsim^{\mathscr{B}} \widetilde{f}_\star \iff \bar{v}_\star \mathrel{\dot{\prec}} \bar{u}_\star.$$

That way, if $\bar{u}_\star \simeq \bar{v}_\star$, then $\widetilde{f}_\star \prec^{\mathscr{B}} \bar{v}_\star$.

3. Finally, the restriction of $\precsim^{\mathscr{B}}$ on $\overline{\mathscr{U}}_\star$ is defined to agree with $\dot{\precsim}$.

By virtue of the properties of upper neighbors, it is rather mechanical exercise to check:

Lemma 2.4.6. *The set $\mathscr{B}$ equipped with the relation $\precsim^{\mathscr{B}}$ is a weak prefoliation, and the map $i : \mathscr{A} \cup \overline{\mathscr{U}} \to \mathscr{B}$ is an extension.*

Proof. Left to the reader. □

2.4.3 Finishing the proof

We conclude the proof of Proposition 2.1.11. Starting from a complete and backward weak prefoliation $\mathscr{A}$, the desired forward extension $\mathscr{A} \hookrightarrow \mathscr{B}$ is obtained as the composition of the extensions $\mathscr{A} \hookrightarrow \mathscr{A} \cup \overline{\mathscr{U}}$ from Lemma 2.4.2 and $\mathscr{A} \cup \overline{\mathscr{U}} \hookrightarrow \mathscr{B}$ from Lemma 2.4.6. Finally, the case where $\mathscr{A}$ is *forward* instead of backward is obtained by reversing the (co)orientation of η and applying the previous results. □

Part II

Symplectic geometry of 3-dimensional Anosov flows

Chapter 3

Anosov flows and Liouville pairs in dimension three

3.1 Introduction

Throughout this Chapter, M denotes a closed, oriented, smooth manifold of dimension three. We will always assume that the Anosov and projectively Anosov flows on M under consideration are **oriented**, i.e., their stable and unstable foliations are oriented. This can always be achieved by passing to a suitable double cover of M. For simplicity, we will only consider smooth (i.e., $\mathcal{C}^\infty$) flows, as we are primarily interested in smooth contact and symplectic structures. Our main results hold for (projectively) Anosov flows generated by $\mathcal{C}^1$ vector fields with minor changes. Moreover, the structural stability of $\mathcal{C}^1$ Anosov vector fields [Rob75] ensures that any Anosov flow generated by a $\mathcal{C}^1$ vector field is topologically equivalent to a smooth Anosov flow, and these two flows are dynamically identical. The definitions and basic properties of Anosov and projectively Anosov flows are recalled in Section 3.3.1.

The notion of Anosov flow, originally introduced by Anosov [Ano63; Ano67] as a generalization of the geodesic flow on hyperbolic manifold, plays a central role

in the theory of smooth dynamical systems. The interplay between the dynamical and topological properties of Anosov flows is particularly rich and striking in dimension three. We refer to the nice survey [Bar17] for many relevant results and references, and to the book [FH19] for a more complete exposition. Eliashberg and Thurston [ET98], and independently Mitsumatsu [Mit95], introduced the more general concept of a *conformally/projectively Anosov flow* on three-manifolds, and established a correspondence between such flows and *bicontact structures*, i.e., transverse pairs of contact structures with opposite orientations. Recently, Hozoori [Hoz22] extended this correspondence to Anosov flows, and showed that (oriented) Anosov flows can be completely characterized in terms of bicontact structures admitting a pair of contact forms satisfying a natural symplectic condition. More precisely, Hozoori showed the following

Theorem ([Hoz22], Theorem 1.1)**.** *Let Φ be a non-singular flow on a closed oriented 3-manifold M, generated by a vector field X. Φ is oriented Anosov if and only if there exist transverse contact structures ξ_- and ξ_+, negative and positive, respectively, and contact forms α_- and α_+ for ξ_- and ξ_+, respectively, such that the 1-forms*

$$(1-t)\alpha_- + (1+t)\alpha_+ \qquad \textit{and} \qquad -(1-t)\alpha_- + (1+t)\alpha_+$$

are positively oriented Liouville forms on $[-1,1]_t \times M$.

Recall that a Liouville form on a smooth manifold with boundary V is a 1-form λ such that $\omega = d\lambda$ is symplectic, i.e., non-degenerate, and the Liouville vector field Z defined by $\omega(Z, \cdot\,) = \lambda$ is outward-pointing along the boundary of V. The pair (V, λ) is called a **Liouville domain**. The above theorem shows in particular that an Anosov flow on a 3-manifold M (under some suitable orientability assumptions recalled in Definition 3.3.1) gives rise to a Liouville structure on $[-1,1] \times M$ which is *not Weinstein*, since the latter manifold has a non-trivial third homology group and

disconnected boundary. It is natural to ask the following

Questions. *How do the Liouville structures constructed by Hozoori depend on the underlying Anosov flow? More precisely,*

(1) For a given Anosov flow Φ, is the space of pairs of contact forms (α_-, α_+) as in the previous Theorem path-connected?

(2) Does a path of Anosov flows induce a path of Liouville structures on $[-1, 1] \times M$?

(3) Does every bicontact structure (ξ_-, ξ_+) supporting an Anosov flow admit a pair of contact forms (α_-, α_+) as in the previous Theorem?

Here, we say that a bicontact structure (ξ_-, ξ_+) supports a non-singular flow generated by a vector field X if $X \in \xi_- \cap \xi_+$ (in the more precise Definition 3.2.1, we also add a condition on the orientations of $\xi_\pm$).

In the present Chapter, we give a complete answer to these questions and upgrade Hozoori's correspondence to a *homotopy equivalence* between the space of Anosov flows on M, and a space of suitable pairs of contact forms on M. To that extent, we will consider a *different* condition on the pair (α_-, α_+) than the one in Hozoori's theorem, and we first show

Theorem J. *Let Φ be a non-singular flow on a closed oriented 3-manifold M, generated by a vector field X. Φ is oriented Anosov if and only if there exists a pair of contact forms (α_-, α_+) on M such that $X \in \ker \alpha_- \cap \ker \alpha_+$, and the 1-forms*

$$e^{-s}\alpha_- + e^s\alpha_+ \qquad \text{and} \qquad -e^{-s}\alpha_- + e^s\alpha_+$$

are positively oriented Liouville forms on $\mathbb{R}_s \times M$.

In the terminology of [MNW13, Definition 1], we say that a pair of contact forms

(α_-, α_+) on a manifold M is a **Liouville pair** if the 1-form

$$\lambda \coloneqq e^{-s}\alpha_- + e^s\alpha_+$$

is a positively oriented Liouville form on $\mathbb{R}_s \times M$. By positively oriented, we mean that the volume form $d\lambda \wedge d\lambda$ is compatible with the natural orientation on $\mathbb{R} \times M$ induced by the natural orientation on $\mathbb{R}$ and the orientation on M.

Warning. *At first glance, Theorem J seems almost identical to Hozoori's theorem. However, we warn the reader that the condition on (α_-, α_+) that we consider is* ***different*** *than Hozoori's one. Indeed, there exist pairs of contact forms (α_-, α_+) which are Liouville pairs as defined above, but such that*

$$(1-t)\alpha_- + (1+t)\alpha_+$$

is ***not*** *a Liouville form on $[-1,1]_t \times M$; see Lemma 3.5.6. It turns out that our condition enjoys some nice symmetries (see Lemma 3.2.4) which make it much easier to work with. For instance, our notion of Liouville pair is easier to characterize than Hozoori's one (compare Lemma 3.2.7 which involves a single equation between three quantities, and Lemma 3.5.2 which involves two independent equations between four quantities). More importantly,* ***we do not know if our main results (Theorem K and Theorem L below) are true for Hozoori's notion of Liouville pair.*** *The corresponding computations are much more complicated because of their lack of symmetry.*

Theorem J motivates the following

Definition 3.1.1. *An* ***Anosov Liouville pair*** *(AL pair for short) on an oriented 3-manifold M is a pair of contact forms (α_-, α_+) such that both (α_-, α_+) and $(-\alpha_-, \alpha_+)$ are Liouville pairs. We denote by $\mathcal{AL} \coloneqq \mathcal{AL}(M) \subset \Omega^1(M) \times \Omega^1(M)$ the space of*

Anosov Liouville pairs on M.

Notice that we do not assume that $\xi_\pm \coloneqq \ker \alpha_\pm$ are transverse, since this is implied by the Liouville conditions; see Proposition 3.2.9. By Theorem J, the intersection $\xi_- \cap \xi_+$ is spanned by an Anosov vector field. A positive time reparametrization of an Anosov flow remains Anosov, and we denote by $\mathcal{AF} \coloneqq \mathcal{AF}(M)$ the space of smooth oriented Anosov flows on M up to positive time reparametrization. Alternatively, $\mathcal{AF}$ can be viewed as the space of smooth unit Anosov vector fields on M for an arbitrary Riemannian metric on M, or the space of smooth 1-dimensional oriented foliations spanned by Anosov vector fields on M, together with some extra orientation data. Hence, there is a natural continuous *intersection map*

$$\begin{array}{rccc} \mathcal{I}: & \mathcal{AL} & \longrightarrow & \mathcal{AF} \\ & (\alpha_-, \alpha_+) & \longmapsto & \ker \alpha_- \cap \ker \alpha_+ \end{array}$$

which sends an AL pair to the 1-dimensional (oriented) distribution obtained by intersecting the underlying contact structures. Here, we endow the spaces $\mathcal{AL}$ and $\mathcal{AF}$ with the $\mathcal{C}^\infty$ topology. Denoting by $\mathcal{BC}$ the space of smooth bicontact structures on M and by $\mathbb{P}\mathcal{AF}$ the space of smooth oriented projectively Anosov flows on M up to positive time reparametrization, we have a similar intersection map

$$\begin{array}{rccc} \mathbb{P}\mathcal{I}: & \mathcal{BC} & \longrightarrow & \mathbb{P}\mathcal{AF} \\ & (\xi_-, \xi_+) & \longmapsto & \xi_- \cap \xi_+ \end{array}$$

as well as a *kernel map*

$$\begin{array}{rccc} \underline{\ker}: & \mathcal{AL} & \longrightarrow & \mathcal{BC} \\ & (\alpha_-, \alpha_+) & \longmapsto & (\ker \alpha_-, \ker \alpha_+) \end{array}$$

The main results of this Chapter, answering the Questions (1), (2) and (3) above,

can be summarized as follows.

Theorem K. *The maps in the commutative diagram*

$$\begin{array}{ccc} \mathcal{AL} & \xrightarrow{\underline{\ker}} & \mathcal{BC} \\ \downarrow{\scriptstyle \mathcal{I}} & & \downarrow{\scriptstyle \mathbb{P}\mathcal{I}} \\ \mathcal{AF} & \hookrightarrow & \mathbb{P}\mathcal{AF} \end{array}$$

satisfy the following properties.

- $\mathcal{I}$ *and* $\mathbb{P}\mathcal{I}$ *are acyclic Serre fibrations (Theorem 3.4.8 and Theorem 3.4.9).*
- $\underline{\ker}$ *is an acyclic Serre fibration onto its image (Theorem 3.4.13).*
- *The inclusion* $\underline{\ker}(\mathcal{AL}) \subset \mathbb{P}\mathcal{I}^{-1}\big(\mathcal{AF}\big)$ *is strict in general (Theorem 3.3.15), but it is a homotopy equivalence (Theorem 3.4.15).*

Recall that an acyclic Serre fibration is a Serre fibration which is also a weak homotopy equivalence, or equivalently, whose fibers are weakly contractible. All the topological spaces under consideration have the homotopy type of a CW complex (see the beginning of Section 3.4), so these acyclic Serre fibrations are homotopy equivalences by the Whitehead theorem. Unpacking the notations,

- $\underline{\ker}(\mathcal{AL})$ is the space of bicontact structures (ξ_-, ξ_+) admitting contact forms α_-, α_+ such that (α_-, α_+) is an AL pair,
- $\mathbb{P}\mathcal{I}^{-1}\big(\mathcal{AF}\big)$ is the space of bicontact structures supporting an Anosov flow.

We emphasize that the top row in the diagram of Theorem K *only involves concepts from contact and symplectic geometry.* This enables us to identify projectively Anosov flows with bicontact structures, and Anosov flows with bicontact structures satisfying a quantitative constraint, coming from the existence of a suitable pair of contact forms. Moreover, the space of AL pairs for a fixed underlying bicontact structure is

(weakly) contractible if non-empty. Hence, AL pairs can be thought of as *auxiliary data attached to bicontact structures.*

Our results can be summarized by the following slogan:

The topological properties of the spaces $\mathcal{AF}$, $\mathbb{P}\mathcal{AF}$ and the inclusion $\mathcal{AF} \subset \mathbb{P}\mathcal{AF}$ can be translated into topological properties of the spaces $\mathcal{AL}$, $\mathcal{BC}$, and the map $\underline{\ker} : \mathcal{AL} \to \mathcal{BC}$, and vice versa.

One important missing piece in this correspondence between Anosov dynamics and contact topology is the mirror notion of *topological* or *orbit equivalence* of flows in the contact world.

Definition 3.1.2. *Two Anosov flows $\Phi = \{\phi^t\}$ and $\Psi = \{\psi^t\}$ on M are **topologically equivalent**, or **orbit equivalent**, if there exist a homeomorphism $h : M \to M$ and a continuous map $\tau : \mathbb{R} \times M \to \mathbb{R}$ such that $\tau(t, x) \geq 0$ for $t \geq 0$, and*

$$\psi^{\tau(t,x)} = h \circ \phi^t \circ h^{-1}(x)$$

for every $t \in \mathbb{R}$ and $x \in M$.

In other words, the topological equivalence h sends the oriented trajectories of ϕ onto the oriented trajectories of ψ, but does not necessarily preserves the parametrization. The structural stability of Anosov flows with smooth dependence on parameters [LMM86, Theorem A.1] implies that two smooth Anosov flows which are homotopic through smooth Anosov flows are topologically equivalent through a topological equivalence which is isotopic to the identity. We do not know if the converse is true.

Question 3.1.3. *If two (smooth) Anosov flows on M are topologically equivalent (via a topological equivalence which is merely continuous), what can be said about the spaces of AL pairs supporting them? How to characterize topological equivalence in terms of AL pairs?*

It is not clear to us how the (hyper)tight contact structure $\xi_\pm$ associated with a Anosov flow behave under topological equivalence. Solving these questions could have a significant impact in the understanding of Anosov flows from the perspective of contact geometry. For instance, a fundamental problem in 3-dimensional Anosov dynamics is the following

Question 3.1.4 ([Bar17])**.** *On a closed* 3*-manifold, are there finitely many Anosov flows up to topological equivalence?*

It is known by the work of Colin, Giroux and Honda [CGH09] that an atoroidal 3-manifold carries finitely many isotopy classes of *tight* contact structures. Although toroidal (and irreducible) 3-manifolds can carry infinitely many isotopy classes of tight contact structures, all of them can be obtained from finitely many contact structures by performing *Lutz twists* along suitable tori; see [CGH09]. The authors also show that there are finitely many tight contact structures for a prescribed *Giroux torsion*, up to isotopy and Dehn twists. Since the contact structures defined by (Anosov) Liouville pairs are by definition exactly semi-fillable, they are strongly fillable [Eli04, Corollary 1.4], hence they have zero Giroux torsion [Gay06, Corollary 3]. This observation plays an essential role in the recent solution of Question 3.1.4 for the class of $\mathbb{R}$*-covered* Anosov flows [BM23; Mar23].

We hope that this *coarse classification* of tight contact structures together with our homotopy correspondence could lead to important results in the classification of Anosov flows on 3-manifolds. To this end, it is crucial to understand the following

Question 3.1.5. *Let* (α_-, α_+) *be an AL pair on* M*. Fixing* α_+*, what can be said about the Anosov flow supported by an AL pair* (α'_-, α_+)*, where* α'_- *is isotopic to* α_-*?*

The main difficulty here is that a path $(\alpha_-^t)_{t\in[0,1]}$ of contact forms from $\alpha_-^0 = \alpha_-$ to $\alpha_-^1 = \alpha'_-$ might not induce a path of bicontact structures, as $\xi_-^t = \ker \alpha_-^t$ and $\xi_+ = \ker \alpha_+$ might fail to be transverse for some $t \in (0, 1)$. Even if transversality

holds, (α_-^t, α_+) might fail to be an AL pair. Nevertheless, one could try to analyze the failure of these properties for a generic path $(\alpha_-^t)_t$, and apply suitable modifications to it. We wish to explore this direction in future work.

A closely related question, already raised by Hozoori [Hoz22, Question 7.2] is the following.

Question 3.1.6. *Let Φ_0 and Φ_1 be two Anosov flows on M and assume that they are homotopic through* projectively *Anosov flows. Equivalently, assume that there exist two AL pairs (α_-^0, α_+^0) and (α_-^1, α_+^1) supporting Φ_0 and Φ_1, respectively, such that their underlying bicontact structures are homotopic (through bicontact structures). Are Φ_0 and Φ_1 homotopic through Anosov flows, i.e., are (α_-^0, α_+^0) and (α_-^1, α_+^1) homotopic through AL pairs? Are Φ_0 and Φ_1 topologically equivalent?*

From the point of view of Liouville geometry, it is natural to weaken the definition of AL pairs as follows.

Definition 3.1.7. *A Liouville pair (α_-, α_+) on M is a* ***weak Anosov Liouville pair*** *(wAL pair for short) if it satisfies the following two conditions.*

(1) The contact plane fields $\xi_\pm \coloneqq \ker \alpha_\pm$ are everywhere transverse,

(2) The intersection $\xi_- \cap \xi_+$ is spanned by an Anosov vector field.

An ***Anosov Liouville structure*** *(AL structure for short) on $V = \mathbb{R}_s \times M$ is a pair (ω, λ) where $\omega = d\lambda$ is a symplectic form and*

$$\lambda = e^{-s}\alpha_- + e^s\alpha_+$$

for a weak Anosov Liouville pair (α_-, α_+). We call the triple (V, ω, λ) an ***Anosov Liouville manifold****.*

An Anosov flow Φ is ***supported*** *by the AL structure (ω, λ) if the vector field X generating Φ satisfies $X \in \xi_- \cap \xi_+$.*

Note that the definition of wAL pairs *does* make reference to the underlying Anosov flow, as opposed to AL pairs. By Theorem J, there is an inclusion $\mathcal{AL} \subset \mathcal{AL}^w$, where $\mathcal{AL}^w$ denotes the space of wAL pairs on M. This inclusion is strict in general. The map $\mathcal{I}$ naturally extends to a map $\mathcal{I}^w : \mathcal{AL}^w \to \mathcal{AF}$, and similarly to the first bullet of Theorem K, we have:

Theorem L. *The map $\mathcal{I}^w : \mathcal{AL}^w \to \mathcal{AF}$ is an acyclic Serre fibration, hence a homotopy equivalence.*

Corollary L. *Let Φ_0 and Φ_1 be two Anosov flows on M, supported by AL structures (ω_0, λ_0) and (ω_1, λ_1), respectively. If Φ_0 and Φ_1 are homotopic through Anosov flows, then (ω_0, λ_0) and (ω_1, λ_1) are homotopic through AL structures, and (V, ω_0, λ_0) and (V, ω_1, λ_1) are exact symplectomorphic.*

Here, an exact symplectomorphism $\psi : (V, \omega_0, \lambda_0) \to (V, \omega_1, \lambda_1)$ is a diffeomorphism such that $\psi^*\lambda_1 = \lambda_0 + df$ for some smooth function $f : V \to \mathbb{R}$. In Corollary L, we can further assume that df has compact support.

Proof of Corollary L. If Φ_0 and Φ_1 are homotopic through Anosov flows, Theorem L provides a *continuous* path of smooth AL structures from (ω_0, λ_0) to (ω_1, λ_1). This path can be smoothed while ensuring the existence of some number $A > 0$ such that the corresponding Liouville vector fields are all transverse to $\{\pm A\} \times M$. Then, [CE12, Proposition 11.8] provides an exact symplectomorphism ψ such that $\psi^*\lambda_1 - \lambda_0$ is compactly supported. □

Anosov Liouville manifolds have numerous interesting invariants coming from Floer theory, e.g., symplectic cohomology and wrapped Fukaya category. As an important consequence of Corollary L, these are *invariants of the underlying Anosov flow*, and only depend on its homotopy class in the space of Anosov flows. Some of these invariants are studied in detail Chapter 4 which is based on joint work [Cie+22] with

Kai Cieliebak, Oleg Lazarev, and Agustin Moreno. To our knowledge, this is the first thorough analysis of symplectic invariants of non-Weinstein Liouville manifolds.

One can also consider Liouville pairs (α_-, α_+) whose underlying contact planes are everywhere transverse. We call such pairs **transverse Liouville pairs**. They correspond to particular projectively Anosov flow that we call **semi-Anosov flows**, see Remark 3.3.12 below. General Liouville pairs (without the transversality assumption) are more complicated to understand, but their underlying contact planes can only intersect *positively*, see Remark 3.2.10 below. In the terminology of [CF11], they constitute **positive contact pairs**.

These geometric structures are summarized in the following diagram; the ones in blue are the main protagonists of this Chapter. Liouville pairs and positive contact pairs will were already investigated in Chapter 1 of this dissertation.

$$\begin{array}{ccccccc} \{\text{AL pairs}\} & \subset & \{\text{Weak AL pairs}\} & \subset & \{\text{Transv. Liouville pairs}\} & \subset & \{\text{Liouville pairs}\} \\ & \searrow^{\ker} & & & \downarrow{\scriptstyle \ker} & & \downarrow{\scriptstyle \ker} \\ & & & & \{\text{Bicontact str.}\} & \subset & \{\text{Pos. contact pairs}\} \end{array}$$

3.2 Anosov Liouville pairs

3.2.1 Preliminary definitions

If X is a non-singular vector field on M, we write

$$N_X \coloneqq TM/\langle X \rangle.$$

An orientation on M naturally determines an orientation on the plane bundle $N_X \to M$. We denote by $\pi : TM \to N_X$ the quotient map. There is a correspondence between n-forms α on M satisfying $\iota_X \alpha = 0$ and n-forms $\overline{\alpha}$ on N_X. Moreover, a vector field

Y on M induces a section $\overline{Y} \coloneqq \pi(Y)$ on N_X. The operator $\mathcal{L}_X$, the Lie derivative along X, naturally induces an operator, still denoted by $\mathcal{L}_X$, on sections of N_X and on n-forms on N_X.

Definition 3.2.1. *A **bicontact structure** on an oriented 3-manifold M is a pair of co-oriented contact structures (ξ_-, ξ_+) such that ξ_- is negative, ξ_+ is positive and ξ_- and ξ_+ are transverse everywhere.*

*A non-singular flow Φ on M generated by a vector field X is **supported** by a bicontact structure (ξ_-, ξ_+) if $X \in \xi_- \cap \xi_+$, and the following orientation compatibility condition holds. Let $\overline{\xi}_\pm \subseteq N_X$ be the image of $\xi_\pm$ under the quotient map $\pi : TM \to N_X$. The orientations on M, $\xi_\pm$ and X induce natural orientations on N_X and $\overline{\xi}_\pm$. We require that the orientation on N_X coincides with the one on $\overline{\xi}_- \oplus \overline{\xi}_+$ (see Figure 3.2.1 and Figure 3.3.1).*

Similarly, Φ is supported by a (weak) Anosov Liouville pair (α_-, α_+) if it is supported by the bicontact structure $(\xi_-, \xi_+) = (\ker \alpha_-, \ker \alpha_+)$.

Note that the definitions of bicontact structures and (weak) Anosov Liouville pairs still make sense if $\xi_\pm$, or $\alpha_\pm$, are merely $\mathcal{C}^1$. We will always assume that bicontact structures and (weak) Anosov Liouville pairs are smooth unless stated otherwise. Bicontact structures and (weak) Anosov Liouville pairs obviously constitute open subsets of the space of pairs of 2-plane fields on M and the space of pairs of 1-forms on M, respectively, since they are defined by open conditions.

If (ξ_-, ξ_+) is a bicontact structure supporting a non-singular flow $\Phi = \{\phi^t\}$, then the bicontact structure obtained from (ξ_-, ξ_+) by reversing the coorientations of both ξ_- and ξ_+ supports Φ as well. Reversing the coorientation of ξ_- or ξ_+ only yields a bicontact structure supporting the reversed flow $\Phi^{-1} = \{\phi^{-t}\}$.

It is easy to deduce from Theorem J the very well-known

Corollary 3.2.2. *The space of (smooth, $\mathcal{C}^1$) Anosov vector fields on M is open in the $\mathcal{C}^1$ topology.*

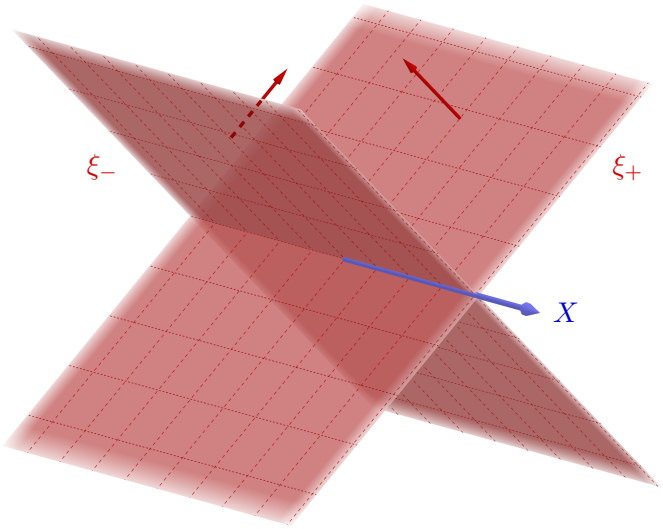


Figure 3.2.1: Coorientation convention for bicontact structures supporting a vector field or a flow.

Proof. Let X be an Anosov vector field on M and (α_-, α_+) be an AL pair supporting X. We choose a 1-form θ such that $\theta(X) \equiv 1$. If X' is another vector field which is sufficiently $\mathcal{C}^1$-close to X, the pair (α'_-, α'_+) defined by

$$\alpha'_\pm \coloneqq \alpha_\pm - \frac{\alpha_\pm(X')}{\theta(X')}\theta$$

is an AL pair supporting X' and by Theorem J, X' is Anosov. □

If (α_-, α_+) is an AL pair on M and $\sigma : M \to \mathbb{R}$ is a smooth function, it follows from the definition that

$$\sigma \cdot (\alpha_-, \alpha_+) \coloneqq \left(e^{-\sigma}\alpha_-, e^{\sigma}\alpha_+\right)$$

is also an AL pair that defines the same bicontact structure as (α_-, α_+). These two AL pairs will be called **equivalent**. This defines an action of $\mathcal{C}^\infty(M, \mathbb{R})$ on the space of AL pairs.

Definition 3.2.3. *A pair of contact forms (α_-, α_+) on M, negative and positive,*

respectively, is ***balanced*** *if*

$$\alpha_+ \wedge d\alpha_+ = -\alpha_- \wedge d\alpha_-.$$

In other words, (α_-, α_+) is balanced if $\alpha_\pm$ define opposite volume forms on M.

Lemma 3.2.4. *Two equivalent AL pairs on M define Liouville isomorphic Liouville structures on $\mathbb{R} \times M$. Any AL pair on M is equivalent to a (unique) balanced one.*

Proof. Let (α_-, α_+) be an AL pair on M and $\lambda \coloneqq e^{-s}\alpha_- + e^s\alpha_+$ be the corresponding Liouville form. If $\sigma \in \mathcal{C}^\infty(M, \mathbb{R})$ and $\lambda' \coloneqq e^{-(s+\sigma)}\alpha_- + e^{s+\sigma}\alpha_+$, the diffeomorphism

$$\begin{array}{rccc} \Psi: & \mathbb{R} \times M & \longrightarrow & \mathbb{R} \times M \\ & (s, x) & \longmapsto & (s - \sigma(x), x) \end{array}$$

satisfies $\Psi^*\lambda' = \lambda$. Moreover, if $f : M \to \mathbb{R}_{>0}$ is such that

$$\alpha_- \wedge d\alpha_- = -f\,\alpha_+ \wedge d\alpha_+,$$

then $\sigma \cdot (\alpha_-, \alpha_+)$ is balanced if and only if $\sigma = \frac{1}{4}\ln f$. □

As a straightforward application of Gray's stability theorem and the above lemma, we have the following

Lemma 3.2.5. *Let (α_-, α_+) be an AL pair and let $\xi_+ \coloneqq \ker \alpha_+$. If $\xi'_+ = \ker \alpha'_+$ is a contact structure homotopic to ξ_+, then there exists a path of AL pairs (α^t_-, α^t_+), $t \in [0, 1]$, such that $(\alpha^0_-, \alpha^0_+) = (\alpha_-, \alpha_+)$ and $\alpha^1_+ = \alpha'_+$.*

Definition 3.2.6. *A pair of contact forms (α_-, α_+) on M, negative and positive, respectively, is* ***closed*** *if $\alpha_- \wedge \alpha_+$ is a closed 2-form.*

It is straightforward to check that a closed pair (α_-, α_+) is an AL pair (see also Lemma 3.2.7 below). As we will see in Proposition 3.3.13, closed AL pairs are in

correspondence with volume preserving Anosov flows.

3.2.2 Elementary properties of Anosov Liouville pairs

The notion of Anosov Liouville pair can be conveniently characterized in the following way, which only involves the forms and their exterior differentials.

Lemma 3.2.7. *Let (α_-, α_+) be a pair of 1-forms on M. We write*

$$\alpha_+ \wedge d\alpha_+ = f_+ \,\mathrm{dvol},$$
$$\alpha_- \wedge d\alpha_- = -f_- \,\mathrm{dvol},$$
$$d(\alpha_- \wedge \alpha_+) = f_0 \,\mathrm{dvol},$$

where dvol *is any volume form on M and $f_\pm, f_0 : M \to \mathbb{R}$ are smooth functions. Then (α_-, α_+) is an AL pair if and only if $f_\pm > 0$, and*

$$f_0^2 < 4 f_- f_+. \tag{3.2.1}$$

Proof. Following [MNW13, Lemma 9.4], (α_-, α_+) is a Liouville pair if and only if for all constants $C_-, C_+ \geq 0$ with $(C_-, C_+) \neq (0, 0)$,

$$(C_+\alpha_+ - C_-\alpha_-) \wedge (C_+ d\alpha_+ + C_- d\alpha_-) > 0,$$

which is equivalent to

$$C_+^2 f_+ + C_- C_+ f_0 + C_-^2 f_- > 0.$$

Applying this fact to (α_-, α_+) and $(-\alpha_-, \alpha_+)$, we obtain that (α_-, α_+) is an AL pair if and only if $f_\pm > 0$ and for every $x \in \mathbb{R}$,

$$x^2 f_+ + x f_0 + f_- > 0,$$

which is equivalent to (3.2.1) by the quadratic formula. □

Remark 3.2.8. *The proof also shows that* (α_-, α_+) *is a Liouville pair if and only if* $f_\pm > 0$ *and* $-f_0 < 2\sqrt{f_- f_+}$.

We now use this criterion to show some natural geometric properties of Anosov Liouville pairs.

Proposition 3.2.9. *Let* (α_-, α_+) *be an Anosov Liouville pair. Then it defines a bicontact structure* $(\xi_-, \xi_+) = (\ker \alpha_-, \ker \alpha_+)$. *Moreover, if* $X \in \xi_- \cap \xi_+$ *is a nowhere vanishing vector field and* $R_\pm$ *is the Reeb vector field of* $\alpha_\pm$, *then* $\{X, R_-, R_+\}$ *is a basis at every point of* M.

Proof. We first show that ξ_- and ξ_+ intersect transversally everywhere. Assume by contradiction that there exist a point $x \in M$ and two linearly independent vectors $X, Y \in T_x M$ such that $\alpha_\pm(X) = \alpha_\pm(Y) = 0$. In what follows, all the quantities will be implicitly evaluated at this point x. We can assume without loss of generality that $d\alpha_+(X, Y) > 0$ and $\mathrm{dvol}(X, Y, R_+) = 1$. We compute

$$\begin{aligned}
\alpha_+ \wedge d\alpha_+(X, Y, R_+) &= d\alpha_+(X, Y) = f_+, \\
\alpha_- \wedge d\alpha_-(X, Y, R_+) &= \alpha_-(R_+) d\alpha_-(X, Y) = -f_-, \\
\alpha_- \wedge d\alpha_+(X, Y, R_+) &= \alpha_-(R_+) d\alpha_+(X, Y), \\
\alpha_+ \wedge d\alpha_-(X, Y, R_+) &= d\alpha_-(X, Y),
\end{aligned}$$

hence

$$\begin{aligned}
f_0^2 - 4 f_- f_+ &= (d\alpha_-(X, Y))^2 - 2\alpha_-(R_+) d\alpha_-(X, Y) d\alpha_+(X, Y) \\
&\qquad + \alpha_-(R_+)^2 (d\alpha_+(X, Y))^2 + 4\alpha_-(R_+) d\alpha_-(X, Y) d\alpha_+(X, Y) \\
&= (d\alpha_-(X, Y) + \alpha_-(R_+) d\alpha_+(X, Y))^2 \\
&\geq 0,
\end{aligned}$$

contradicting (3.2.1).

For the second part, we write

$$\alpha_- \wedge d\alpha_+ = g_+ \,\mathrm{dvol},$$
$$\alpha_+ \wedge d\alpha_- = g_- \,\mathrm{dvol},$$

where $g_\pm : M \to \mathbb{R}$ are smooth functions (note that $f_0 = g_- - g_+$) and we compute

$$\begin{aligned}
&\alpha_+ \wedge d\alpha_+(X, R_+, \cdot\,) = -d\alpha_+(X, \cdot\,) = f_+ \,\mathrm{dvol}(X, R_+, \cdot\,),\\
&\alpha_- \wedge d\alpha_-(X, R_+, \cdot\,) = -\alpha_-(R_+) d\alpha_-(X, \cdot\,) + d\alpha_-(X, R_+)\alpha_- = -f_- \,\mathrm{dvol}(X, R_+, \cdot\,),\\
&\alpha_- \wedge d\alpha_+(X, R_+, \cdot\,) = -\alpha_-(R_+) d\alpha_+(X, \cdot\,) = g_+ \,\mathrm{dvol}(X, R_+, \cdot\,),\\
&\alpha_+ \wedge d\alpha_-(X, R_+, \cdot\,) = -d\alpha_-(X, \cdot\,) + d\alpha_-(X, R_+)\alpha_+ = g_- \,\mathrm{dvol}(X, R_+, \cdot\,).
\end{aligned}$$

Let us assume that $\mathrm{dvol}(X, R_-, R_+) = 0$ at a point $x \in M$. In what follows, all the quantities will be implicitly evaluated at this point x. Plugging in R_- in the first two of the four equations above yields

$$d\alpha_-(X, R_+) = d\alpha_+(X, R_-) = 0.$$

Note that X and R_+ are not colinear since $\alpha_+(X) = 0$ and $\alpha_+(R_+) = 1$. The last two of the four equations above imply $\alpha_-(R_+) \neq 0$ and

$$f_+ = \frac{1}{\alpha_-(R_+)} g_+,$$
$$f_- = -\alpha_-(R_+) g_-.$$

Finally,

$$\begin{aligned} f_0^2 - 4f_-f_+ &= \big(g_- - g_+\big)^2 + 4g_-g_+ \\ &= \big(g_- + g_+\big)^2 \\ &\geq 0, \end{aligned}$$

contradicting (3.2.1). □

Remark 3.2.10. *A (non-Anosov) Liouville pair may not define a bicontact structure, namely* $\xi_- = \ker \alpha_-$ *and* $\xi_+ = \ker \alpha_+$ *may not be transverse everywhere. Nevertheless, the first part of the proof can easily be adapted to show that at a point where* ξ_- *and* ξ_+ *coincide, their orientations coincide (and their coorientations are opposite). In the terminology of [CF11],* (ξ_-, ξ_+) *is a* positive pair *of contact structures. After a generic perturbation of* α_- *and/or* α_+*, the singular set* $\Delta \coloneqq \{x \in M : \xi_-(x) = \xi_+(x)\}$ *is a smoothly embedded link in* M*. Moreover, it can be shown that* $f_0 > 0$ *along* Δ*, so the Liouville condition of Remark 3.2.8 is largely satisfied. We refer to Chapter 1 for detailed proofs of these facts and a thorough investigation of general Liouville pairs.*

For any AL pair (α_-, α_+), if X (or dvol) is chosen so that $\mathrm{dvol}(X, R_-, R_+) = 1$, then

$$\begin{aligned} f_+ &= d\alpha_+(X, R_-) = \mathcal{L}_X \alpha_+(R_-), \\ f_- &= d\alpha_-(X, R_+) = \mathcal{L}_X \alpha_-(R_+), \\ g_+ &= \alpha_-(R_+) f_+, \\ g_- &= -\alpha_+(R_-) f_-. \end{aligned}$$

Moreover, if (α_-, α_+) is balanced, i.e., if $f_+ = f_-$, the condition (3.2.1) becomes

$$|\alpha_-(R_+) + \alpha_+(R_-)| < 2. \tag{3.2.2}$$

In fact, (balanced) AL pairs can be completely characterized by their Reeb vector fields.

Proposition 3.2.11. *Let (α_-, α_+) be a pair of contact forms on M, negative and positive, respectively, and assume that it is balanced. Then it is an AL pair if and only if* (3.2.2) *is satisfied.*

Proof. We only have to show that under these hypothesis, the conclusions of Proposition 3.2.9 are satisfied, since these imply that $g_+ = \alpha_-(R_+)f_+$ and $g_- = -\alpha_+(R_-)f_-$ and Lemma 3.2.7 concludes the proof.

Assume first that ξ_- and ξ_+ are not transverse at a point $x \in M$. With the same notations as in the proof of Proposition 3.2.9, similar computations show that at this point,

$$\begin{aligned}
&\alpha_+ \wedge d\alpha_+(X, Y, R_+) = d\alpha_+(X, Y) = f_+,\\
&\alpha_- \wedge d\alpha_-(X, Y, R_+) = \alpha_-(R_+)d\alpha_-(X, Y) = -f_-,\\
&\alpha_+ \wedge d\alpha_+(X, Y, R_-) = \alpha_+(R_-)d\alpha_+(X, Y) = f_+ \operatorname{dvol}(X, Y, R_-),\\
&\alpha_- \wedge d\alpha_-(X, Y, R_-) = d\alpha_-(X, Y) = -f_- \operatorname{dvol}(X, Y, R_-),
\end{aligned}$$

hence by the first and third equalities,

$$\operatorname{dvol}(X, Y, R_-) = \alpha_+(R_-),$$

and by the second and fourth equalities,

$$\alpha_-(R_+)\alpha_+(R_-) = 1,$$

contradicting (3.2.2) by the inequality of arithmetic and geometric means.

Assuming now that $\operatorname{dvol}(X, R_-, R_+) = 0$ at a point $x \in M$, the proof of Proposi-

tion 3.2.9 showed that at this point,

$$d\alpha_-(X, R_+) = d\alpha_+(X, R_-) = 0,$$

and

$$g_+ = \alpha_-(R_+) f_+.$$

Similarly,

$$\begin{aligned}
\alpha_+ \wedge d\alpha_+(X, R_-, \cdot\,) &= -\alpha_+(R_-) d\alpha_+(X, \cdot\,) = f_+ \operatorname{dvol}(X, R_-, \cdot\,), \\
\alpha_- \wedge d\alpha_+(X, R_-, \cdot\,) &= -d\alpha_+(X, \cdot\,) = g_+ \operatorname{dvol}(X, R_-, \cdot\,),
\end{aligned}$$

hence

$$f_+ = \alpha_+(R_-) g_+.$$

Once again, we obtain that

$$\alpha_-(R_+)\alpha_+(R_-) = 1,$$

contradicting (3.2.2). □

3.3 From Anosov flows to Anosov Liouville pairs and back

In this section, we adapt the proof of [Hoz22, Theorem 1.1] to the setting of Anosov Liouville pairs as defined in the Introduction.

3.3.1 Anosov and projectively Anosov flows

We recall the definitions of Anosov and projectively Anosov flows with an emphasis on our orientation conventions, and recast them in terms of the existence of suitable

1-forms.

Definition 3.3.1. *Let $\Phi = \{\phi^t\}_{t\in\mathbb{R}}$ be a flow on M generated by a non-singular $\mathcal{C}^1$ vector field X.*

- *Φ is **Anosov** if there exists a continuous invariant **hyperbolic splitting***

$$TM = \langle X\rangle \oplus E^s \oplus E^u \tag{3.3.1}$$

where E^s, E^u are 1-dimensional bundles such that for some (any) Riemannian metric g on M, there exist constants $C, a > 0$ such that for all $v \in E^s$ and $t \geq 0$,

$$\|d\phi^t(v)\| \leq Ce^{-at}\|v\|,$$

and for all $v \in E^u$ and $t \geq 0$,

$$\|d\phi^t(v)\| \geq Ce^{at}\|v\|.$$

E^s and E^u are called the (strong) stable and unstable bundles of Φ, respectively.

- *Φ is **projectively Anosov** if there exists a continuous invariant splitting*

$$TM/\langle X\rangle = N_X = \overline{E}^s \oplus \overline{E}^u \tag{3.3.2}$$

where $\overline{E}^s, \overline{E}^u$ are 1-dimensional bundles such that for some (any) Riemannian metric $\overline{g}$ on N_X, there exist constants $C, a > 0$ such that for all unit vectors $v_s \in \overline{E}^s, v_u \in \overline{E}^u$, and $t \geq 0$,

$$\|d\phi^t(v_u)\| \geq Ce^{at}\,\|d\phi^t(v_s)\|.$$

*Such a splitting is called a **dominated splitting**. We denote by $E^{ws} \coloneqq \pi^{-1}\big(\overline{E}^s\big)$*

and $E^{wu} \coloneqq \pi^{-1}\big(\overline{E}^u\big)$ *the weak-stable and weak-unstable bundles of* Φ*, respectively.*

- *In both cases, if the constant* C *can be chosen to be* 1*, the corresponding metrics* g *and* $\overline{g}$ *are called* ***adapted*** *to* Φ*.*

- *The Anosov (resp. projectively Anosov) flow* Φ *is* ***oriented*** *if* E^s *and* E^u *are oriented (resp.* $\overline{E}^s$ *and* $\overline{E}^u$ *are oriented) and their orientations are compatible with the splitting* (3.3.1) *(resp. the splitting* (3.3.2)*).*

Anosov flows are projectively Anosov, with dominated splitting $N_X = \pi\big(E^s\big) \oplus \pi\big(E^u\big)$. Every three dimensional (projectively) Anosov flow admits a smooth adapted metric, see [FH19, Proposition 5.1.5].[1] Anosov famously showed that the weak and strong stable/unstable bundles of an Anosov flow are *uniquely integrable*. Moreover, E^{ws} and E^{wu} integrate into *taut foliations* $\mathcal{F}^{ws}$ and $\mathcal{F}^{wu}$, respectively. This is not true for projectively Anosov flows; see [ET98, Example 2.2.9].

In the rest of this Chapter, we implicitly assume that all of the Anosov and projectively Anosov flows under consideration are oriented.

The following definition appears in [Hoz22, Definition 3.11], see also [Hoz22, Proposition 3.12].

Definition 3.3.2. *Let* Φ *be a projectively Anosov flow on* M *generated by a vector field* X *and* $\overline{g}$ *be a Riemannian metric on* N_X*. The* ***expansion rates*** *in the stable and unstable directions for* $\overline{g}$ *are continuous functions* $r_s, r_u : M \to \mathbb{R}$ *defined by*

$$r_s \coloneqq \left.\frac{\partial}{\partial t}\right|_{t=0} \ln \|d\phi^t(\overline{e}_s)\|, \qquad r_u \coloneqq \left.\frac{\partial}{\partial t}\right|_{t=0} \ln \|d\phi^t(\overline{e}_u)\|,$$

where $\overline{e}_s$ *and* $\overline{e}_u$ *are unit sections of* $\overline{E}^s$ *and* $\overline{E}^u$*, respectively, which are continuous*

[1] The proof is given for Anosov flows but it easily generalizes to projectively Anosov flows in dimension three; note that it is not sufficient to integrate an arbitrary metric along the flow for a large time!

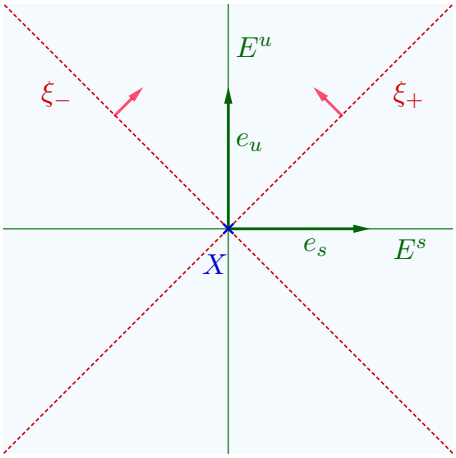


Figure 3.3.1: Orientation convention for the Anosov splitting. There is a similar picture for the dominated splitting. The vector field X points toward the reader, watch out.

and continuously differentiable along the flow Φ.[2] *Moreover,*

$$\mathcal{L}_X \overline{e}_s = -r_s \overline{e}_s, \qquad \mathcal{L}_X \overline{e}_u = -r_u \overline{e}_u.$$

The Lie derivative above means the following: if $e_{s,u}$ is a vector field on M which is a lift of $\overline{e}_{s,u}$ with the same regularity, the quantity $\mathcal{L}_X \overline{e}_{s,u} \coloneqq \pi\big(\mathcal{L}_X e_{s,u}\big)$ is a section of $\overline{E}^{s,u}$ which is independent of the choice of the lift. Here, $\mathcal{L}_X e_{s,u}$ denotes the usual Lie derivative along X, defined by

$$\mathcal{L}_X e_{s,u} \coloneqq \left.\frac{\partial}{\partial t}\right|_{t=0} \big(\phi^t\big)^* e_{s,u}.$$

Notice that this involves the *pullback* along ϕ^t, and thus the differential of the *inverse* of the flow, which explains the presence of negative signs in the previous formulae.

Many natural quantities (e.g., functions, vector fields, 1-forms, Riemannian metrics) defined for (projectively) Anosov flows are continuous and can be upgraded to quantities which are continuous *and* continuously differentiable along the flow by considering

[2] In particular, the function $t \mapsto d\phi^t(\overline{e}_{s,u})$ is differentiable and has positive norm, so $r_{s,u}$ is well-defined.

the averaging

$$\frac{1}{T}\int_0^T \left(\phi^t\right)^*_ dt$$

for some $T > 0$. Nevertheless, these quantities may not be $\mathcal{C}^1$. It is therefore natural to consider the following spaces (only the cases $k = 0, 1$ and $n = 0, 1$ will be relevant for us).

Definition 3.3.3. *Le X be a smooth, non-singular vector field on M, and $k \geq 0$ be a non-negative integer.*

- *A n-form α on M is of class $\mathcal{C}^k_X$ if α is differentiable along X, and both α and $\mathcal{L}_X\alpha$ are of class $\mathcal{C}^k$. We denote by $\Omega^n_{X,k}$ the space of n-forms on M of class $\mathcal{C}^k_X$ satisfying $\iota_X\alpha = 0$ (which is vacuous for $n = 0$, i.e., for functions). We also denote by $\Omega^n_X = \Omega^n_{X,\infty} \subset \Omega^n$ the space of smooth n-forms satisfying $\iota_X\alpha = 0$.*
- *On $\Omega^n_{X,k}$, there is a natural norm defined by*

$$|\alpha|_{\mathcal{C}^k_X} \coloneqq |\alpha|_{\mathcal{C}^k} + |\mathcal{L}_X\alpha|_{\mathcal{C}^k},$$

making $\left(\Omega^n_{X,k}, |\cdot|_{\mathcal{C}^k_X}\right)$ a Banach space. Ω^n_X is naturally a Fréchet space as a closed subspace of Ω^n

These definitions naturally extend to sections of N_X and n-forms on N_X.

In Appendix 3.A, we show some density results for these spaces which are particularly useful when dealing with (projectively) Anosov flows generated by $\mathcal{C}^1$ vector fields and can be used to bypass Hozoori's delicate approximation techniques in [Hoz22, Section 4]. The results in Appendix 3.A are not needed (except in the proof of Theorem 3.4.4) if we restrict our attention to smooth Anosov flows in view of Lemma 3.3.5 below.

In dimension three, the definitions of Anosov and projectively Anosov flows can be rephrased in terms of the existence of certain 1-forms. The following lemma is

essentially an adaptation of results of Mitsumatsu [Mit95] and Hozoori [Hoz22; Hoz23].

Lemma 3.3.4. *Let Φ be a smooth, non-singular flow on M generated by a vector field X, then*

(1) Φ is oriented projectively Anosov if and only if there exist $(\alpha_s, \alpha_u) \in \Omega^1_{X,0} \times \Omega^1_{X,0}$ and continuous functions $r_u, r_s : M \to \mathbb{R}$ such that

$$\overline{\alpha}_s \wedge \overline{\alpha}_u > 0, \qquad \mathcal{L}_X \alpha_s = r_s\, \alpha_s, \qquad \mathcal{L}_X \alpha_u = r_u\, \alpha_u,$$

and $r_s < r_u$. Here, $\overline{\alpha}_s$ and $\overline{\alpha}_u$ denote the 1-forms on N_X induced by α_s and α_u, respectively.

(2) Φ is oriented Anosov if and only if there exist α_s, α_u, r_s, r_u as above such that $r_s < 0 < r_u$.

(3) Φ is oriented volume preserving Anosov if and only if there exist α_s, α_u, r_s, r_u as above such that $r_u + r_s = 0$.

Moreover, $\ker \alpha_u = E^{ws}$ *and* $\ker \alpha_s = E^{wu}$.

Proof. (1) essentially follows from [Hoz22, Proposition 3.15]. We recall the main arguments. If Φ is projectively Anosov, we can choose an adapted metric and unit vector fields $\overline{e}_s$ and $\overline{e}_u \in N_X$ of class $\mathcal{C}^0_X$ such that $\overline{e}_s$ spans $\overline{E}^s$, $\overline{e}_u$ spans $\overline{E}^u$ and $(\overline{e}_s, \overline{e}_u)$ is positively oriented. The inequality $r_s < r_u$ follows from the definition of a dominated splitting and the fact that the metric is adapted. If $(\overline{\alpha}_s, \overline{\alpha}_u)$ denotes the dual basis of $(\overline{e}_s, \overline{e}_u)$, it induces a pair $(\alpha_s, \alpha_u) \in \Omega^1_{X,0} \times \Omega^1_{X,0}$, and the relations $\mathcal{L}_X \overline{e}_s = -r_s \overline{e}_s$ and $\mathcal{L}_X \overline{e}_u = -r_u \overline{e}_u$ imply the desired relations for α_s and α_u. Reciprocally, if (α_s, α_u) is such a pair, we define $(\overline{e}_s, \overline{e}_u)$ as the dual basis of $(\overline{\alpha}_s, \overline{\alpha}_u)$ and we easily check that it yields a projectively Anosov splitting of N_X. This is essentially because for $\star \in \{s, u\}$, $\phi^T_* \overline{e}_\star = \exp\left(\int_0^T r_\star \circ \phi^t \, dt\right) \overline{e}_\star$ and $r_s < r_u$, where $\Phi = \{\phi^t\}$. In particular, we have

$\ker \alpha_s = E^{wu}$ and $\ker \alpha_u = E^{ws}$ since these bundles are uniquely determined by the flow.

(2) follows from (1) and [Hoz22, Proposition 3.17].

The forward direction of (3) follows from the proof of [Mit95, Theorem 3]. Indeed, assuming that Φ is volume preserving Anosov, we can arrange that $r_u + r_s = 0$ in the following way. If dvol is a (smooth) volume form preserved by Φ, then $\tau \coloneqq \iota_X \text{dvol}$ is a non-degenerate 2-form on N_X invariant under Φ. There exists an adapted metric g of class $\mathcal{C}^0_X$ for which the Anosov splitting is orthogonal and the volume form for the induced metric $\overline{g}$ on N_X is precisely τ.[3] Hence, if e_s and e_u are $\mathcal{C}^0_X$ unit vector fields spanning E^s and E^u, respectively, then $\tau(\overline{e}_s, \overline{e}_u) = 1$. Differentiating this equality along X yields $r_u + r_s = 0$ as desired, and we obtain (α_s, α_u) by dualization as before. For the reverse direction, $r_s < 0 < r_u$ since $r_s < r_u$ and $r_s = -r_u$, so Φ is Anosov by (2). Moreover, if θ is a smooth 1-form satisfying $\theta(X) \equiv 1$, then dvol $\coloneqq \alpha_s \wedge \alpha_u \wedge \theta$ is a $\mathcal{C}^0_X$ volume form preserved by X and by [LMM86, Corollary 2.1], this volume form is smooth. □

It is well-known that in dimension three, the regularity of the weak-stable and weak-unstable bundles of a smooth (even $\mathcal{C}^2$) Anosov flow are $\mathcal{C}^1$. We have the following

Lemma 3.3.5. *If Φ is Anosov and smooth, we can further assume that α_s, α_u, r_s and r_u as in Lemma 3.3.4 are $\mathcal{C}^1$, i.e., α_s and α_u are $\mathcal{C}^1_X$.*

Proof. By [Has94, Corollary 1.8], E^{ws} and E^{wu} are $\mathcal{C}^1$ and an adapted metric can always be assumed to be smooth, so the construction in Lemma 3.3.4 yields $\mathcal{C}^1$ 1-forms α_s and α_u. The $\mathcal{C}^1$ regularity of r_s and r_u follows from a trick of Simić [Sim97]. First, let us choose a $\mathcal{C}^1$ 1-form α_u such that $\ker \alpha_u = E^{ws}$ and $\mathcal{L}_X \alpha_u = r_u \, \alpha_u$, where r_u is

[3]The induced metric $\overline{g}$ is the pushforward of the restriction of g to $E^s \oplus E^u$ along the projection $E^s \oplus E^u \to N_X$, which is an isomorphism. Concretely, if $\overline{v}_1, \overline{v}_2$ are vectors in N_X with lifts $v_1, v_2 \in E^s \oplus E^u$, then $\overline{g}(\overline{v}_1, \overline{v}_2) = g(v_1, v_2)$. Since X is orthogonal to $E^s \oplus E^u$ for g, the latter quantity does not depend on the choice of such lifts.

continuous and positive. Fix a smooth vector field Z positively transverse to E^{ws}, so that $f \coloneqq \alpha_u(Z) > 0$. Here, f is $\mathcal{C}^1$ and can be approximated by a smooth function $\widetilde{f} > 0$ so that $h \coloneqq \frac{\widetilde{f}}{f}$ is $\mathcal{C}^1$-close to 1. Setting $\widetilde{\alpha}_u \coloneqq h\alpha_u$, $\widetilde{\alpha}_u$ is $\mathcal{C}^1$ and satisfies $\mathcal{L}_X\widetilde{\alpha}_u = \widetilde{r}_u\,\widetilde{\alpha}_u$, where $\widetilde{r}_u$ is $\mathcal{C}^0$-close to r_u and can be assumed to be positive. We now show that $\widetilde{r}_u$ is $\mathcal{C}^1$. Indeed, $\widetilde{\alpha}_u(Z) = \widetilde{f}$ and

$$\begin{aligned}\widetilde{r}_u\widetilde{f} &= \left(\mathcal{L}_X\widetilde{\alpha}_u\right)(Z)\\ &= \mathcal{L}_X\left(\widetilde{\alpha}_u(Z)\right) - \widetilde{\alpha}_u\left(\mathcal{L}_X Z\right),\\ &= X\cdot\widetilde{f} - \widetilde{\alpha}_u\left(\mathcal{L}_X Z\right),\end{aligned}$$

and the last quantity is $\mathcal{C}^1$ since $X\cdot\widetilde{f}$ and $\mathcal{L}_X Z$ are smooth and $\widetilde{\alpha}_u$ is $\mathcal{C}^1$. The same argument applies to α_s. □

Remark 3.3.6. *The proof actually shows more. Since $\widetilde{r}_u = u + \widetilde{\alpha}_u(V)$ for some smooth function u and some smooth vector field V, and $\mathcal{L}_X\widetilde{\alpha}_u = \widetilde{r}_u\,\widetilde{\alpha}_u$, an immediate induction argument shows that for every integer $n \geq 0$, $\mathcal{L}_X^n\widetilde{\alpha}_u$ and $\mathcal{L}_X^n\widetilde{r}_u$ exist and are $\mathcal{C}^1$, where $\mathcal{L}_X^n \coloneqq \mathcal{L}_X \circ \cdots \circ \mathcal{L}_X$ denotes the Lie derivative along X iterated n times. In fact, it is well known that in our setting, the individual leaves of the weak-(un)stable foliation are smooth (see [LMM86, Lemma 2.1]).*

Remark 3.3.7. *The same argument works for smooth projectively Anosov flow whose weak-stable and weak-unstable distributions are $\mathcal{C}^1$. However, there are known examples of smooth projectively Anosov flows in dimension three whose weak distributions are not $\mathcal{C}^1$; see [ET98, Example 2.2.9].*

We call a pair of 1-forms (α_s, α_u) as in Lemma 3.3.4 (1) (resp. (2), (3)) a **defining pair** for the projectively Anosov (resp. Anosov, volume preserving Anosov) flow Φ. We further require defining pairs for (volume preserving) Anosov flows to be $\mathcal{C}^1$. We also impose the following conditions on orientations:

- The orientation on E^{ws}, induced by the orientation of X and the orientation on E^s or $\overline{E}^s$ which implicitly comes with Φ, agrees with the one induced by α_u,
- The orientation on E^{wu}, induced by the orientation of X and the orientation on E^u or $\overline{E}^u$ which implicitly comes with Φ, agrees with the one induced by $-\alpha_s$.[4]

Concretely, these properties mean that if $\star \in \{s, u\}$ and $\overline{e}_\star \in \overline{E}^\star$ forms an oriented basis, then $\alpha_\star(\overline{e}_\star) > 0$.

We denote by $\mathcal{D}_\Phi$ the space of defining pairs for Φ endowed with the $\mathcal{C}^0_X$ topology in the projectively Anosov case, and with the $\mathcal{C}^1$ topology in the (volume preserving) Anosov case.

Lemma 3.3.8. *The space $\mathcal{D}_\Phi$ of defining pairs for a projectively Anosov (resp. Anosov, volume preserving Anosov) flow Φ with its corresponding topology is contractible.*

Proof. Let us fix a defining pair $(\alpha_s, \alpha_u) \in \mathcal{D}_\Phi$ for a projectively Anosov flow Φ generated by a vector field X. If $(\alpha'_s, \alpha'_u) \in \mathcal{D}_\Phi$ is any other defining pair, then $\ker \alpha_u = \ker \alpha'_u$ and $\ker \alpha_s = \ker \alpha'_s$, and the orientations on these spaces agree. Hence, there exist (unique) functions $\rho_s, \rho_u : M \to \mathbb{R}$ of class $\mathcal{C}^0_X$ such that $\alpha'_u = e^{\rho_u}\alpha_u$ and $\alpha'_s = e^{\rho_s}\alpha_s$, and they satisfy

$$r'_u - r'_s = X \cdot (\rho_u - \rho_s) + r_u - r_s > 0. \tag{3.3.3}$$

Here, r'_u and r'_s are such that $\mathcal{L}_X \alpha'_s = r'_s\, \alpha'_s$ and $\mathcal{L}_X \alpha'_u = r'_u\, \alpha'_u$.

Reciprocally, if $\rho_s, \rho_u : M \to \mathbb{R}$ are functions as above satisfying (3.3.3), then $(\alpha'_s, \alpha'_u) \coloneqq (e^{\rho_s}\alpha_s, e^{\rho_u}\alpha_u)$ is also a defining pair for Φ.

[4]The somewhat strange minus sign is explained by the following remark. In the Euclidean plane $\mathbb{R}^2$ with its standard oriented basis (e_1, e_2), the dual basis (e_1^*, e_2^*) induces coorientations on the x and y-axis corresponding to the natural orientation on the x-axis and to the *opposite* of the natural orientation on the y-axis.

It follows that $\mathcal{D}_\Phi$ is homeomorphic to

$$\mathcal{R} \coloneqq \big\{(\rho_s, \rho_u) \in \mathcal{C}^0_X \times \mathcal{C}^0_X : X \cdot (\rho_u - \rho_s) + r_u - r_s > 0\big\},$$

and $\mathcal{R}$ is obviously convex, hence contractible. The proof for Anosov flows and volume preserving Anosov flows is similar. The condition on ρ_s and ρ_u becomes

$$X \cdot \rho_u + r_u > 0 \qquad \textit{and} \qquad X \cdot \rho_s + r_s < 0 \tag{3.3.4}$$

if Φ is Anosov, and

$$X \cdot (\rho_u - \rho_s) + r_u - r_s > 0 \qquad \textit{and} \qquad X \cdot (\rho_u + \rho_s) = 0 \tag{3.3.5}$$

if Φ is volume preserving Anosov. Both conditions (3.3.4) and (3.3.5) are convex in (ρ_s, ρ_u). □

Remark 3.3.9. *If Φ is a (projectively, volume preserving) Anosov flow generated by a vector field X, $f : M \to \mathbb{R}_{>0}$ is a positive function and Φ' is the (projectively, volume preserving) Anosov flow generated by fX, then $\mathcal{D}_\Phi = \mathcal{D}_{\Phi'}$. Indeed, if $(\alpha_s, \alpha_u) \in \mathcal{D}_\Phi$ then for $\star \in \{s, u\}$,*

$$\mathcal{L}_{fX}\alpha_\star = f r_\star\, \alpha_\star.$$

Since the stable/unstable bundles of Φ' are the same as the ones of Φ, (α_s, α_u) is a defining pair for Φ' with expansion rates $r'_{s,u} = f r_{s,u}$.

Therefore, there is a well-defined notion of defining pairs for (projectively, volume preserving) oriented Anosov line distributions.

3.3.2 From Anosov flows to Anosov Liouville pairs

Throughout this section, we assume that Φ is a smooth Anosov flow on M and we construct an AL pair supporting Φ, proving the first part of Theorem J. We choose a $\mathcal{C}^1$ defining pair $(\alpha_s, \alpha_u) \in \mathcal{D}_\Phi$ as in Lemma 3.3.4(2). Following [Hoz22, Section 4], we define

$$\alpha_- \coloneqq \alpha_u + \alpha_s, \qquad \alpha_+ \coloneqq \alpha_u - \alpha_s. \tag{3.3.6}$$

Note that $\alpha_\pm$ is of class $\mathcal{C}^1_X$, and the orientation compatibility conditions of Definition 3.2.1 are satisfied. Let dvol be the $\mathcal{C}^1$ volume form on M defined by $\alpha_s \wedge \alpha_u = \iota_X \mathrm{dvol}$. We will make use of the elementary identities:

$$\begin{aligned}
\alpha_s \wedge d\alpha_u &= -r_u \,\mathrm{dvol},\\
\alpha_u \wedge d\alpha_s &= r_s \,\mathrm{dvol},\\
\alpha_s \wedge d\alpha_s &= 0,\\
\alpha_u \wedge d\alpha_u &= 0.
\end{aligned}$$

The first one follows from:

$$\iota_X\big(\alpha_s \wedge d\alpha_u\big) = -\alpha_s \wedge \iota_X d\alpha_u = -\alpha_s \wedge \mathcal{L}_X \alpha_u = -r_u\, \alpha_s \wedge \alpha_u = -r_u \,\mathrm{dvol},$$

and the three others can be obtained by similar computations. We easily deduce:

$$\begin{aligned}
\alpha_+ \wedge d\alpha_+ &= (r_u - r_s)\,\mathrm{dvol},\\
\alpha_- \wedge d\alpha_- &= -(r_u - r_s)\,\mathrm{dvol},\\
d(\alpha_- \wedge \alpha_+) &= 2(r_u + r_s)\,\mathrm{dvol}.
\end{aligned}$$

Since $r_s < 0 < r_u$, α_- and α_+ are contact forms and the criterion of Lemma 3.2.7 is satisfied.[5] Therefore, (α_-, α_+) is a $\mathcal{C}^1$ AL pair supporting Φ.

Definition 3.3.10. *A **standard AL pair** supporting Φ is a $\mathcal{C}^1$ AL pair obtained by the previous construction. We denote by $\mathcal{AL}^{\mathrm{std}}_\Phi$ the space of these AL pairs, endowed with the $\mathcal{C}^1$ topology.*

There is an obvious (linear) homeomorphism between $\mathcal{D}_\Phi$ and $\mathcal{AL}^{\mathrm{std}}_\Phi$ induced by the map $(s, u) \mapsto (u + s, u - s)$. Since $\mathcal{D}_\Phi$ is contractible by Lemma 3.3.8, $\mathcal{AL}^{\mathrm{std}}_\Phi$ is contractible as well.

A standard AL pair (α_-, α_+) is not necessarily smooth by definition; it is smooth exactly when the weak-stable and weak-unstable foliations of Φ are smooth, which is a quite restrictive situation.[6] Nevertheless, any pair of smooth 1-forms (α'_-, α'_+) sufficiently $\mathcal{C}^1$-close to (α_-, α_+) and satisfying $\alpha'_\pm(X) = 0$ is a smooth AL pair supporting Φ. This shows the forward implication in Theorem J.

Let $R_\pm$ denote the Reeb vector fields of $\alpha_\pm$, defined by

$$\alpha_\pm(R_\pm) = 1,$$
$$d\alpha_\pm(R_\pm, \cdot\,) = 0.$$

Rewriting these equations in terms of α_s and α_u, and using the equalities

$$\iota_X d\alpha_s = \mathcal{L}_X \alpha_s = r_s\, \alpha_s,$$
$$\iota_X d\alpha_u = \mathcal{L}_X \alpha_u = r_u\, \alpha_u,$$

[5] Note that $\lambda \coloneqq e^{-s}\alpha_- + e^s\alpha_+$ is a Liouville form if and only if $r_u > 0$.

[6] By [Ghy93, Theorem 4.7], the smoothness of the weak-(un)stable implies that Φ is topologically equivalent to an *algebraic Anosov flow*, i.e., the suspension of an Anosov diffeomorphism of the 2-torus, or the geodesic flow on a closed hyperbolic surface, up to finite cover.

one easily computes:

$$\alpha_s(R_-) = \frac{r_u}{r_u - r_s} > 0, \qquad \alpha_u(R_-) = \frac{-r_s}{r_u - r_s} > 0,$$
$$\alpha_s(R_+) = \frac{-r_u}{r_u - r_s} < 0, \qquad \alpha_u(R_+) = \frac{-r_s}{r_u - r_s} > 0.$$

Therefore,

- R_- is positively transverse to E^{ws} and E^{wu},
- R_+ is positively transverse to E^{ws} and negatively transverse to E^{wu},

and this remains true for a smoothing of (α_-, α_+) as above.[7] Since $\mathcal{F}^{ws}$ is a *taut* foliation, we obtain that $R_\pm$ has no contractible closed Reeb orbit, thus $\xi_\pm = \ker \alpha_\pm$ is *hypertight*. This was already observed by Hozoori [Hoz22, Theorem 1.11].

3.3.3 From Anosov Liouville pairs to Anosov flows

We now turn to the second part of the proof of Theorem J. Let us assume that Φ is a smooth non-singular flow on M generated by a vector field X and suppose that it is supported by an AL pair (α_-, α_+). By Proposition 3.2.9, (α_-, α_+) defines a bicontact structure (ξ_-, ξ_+) supporting X, so Φ is projectively Anosov and there exists a dominated splitting $N_X \cong \overline{E}^s \oplus \overline{E}^u$ as in Definition 3.3.1. We shall construct a defining pair (α_s, α_u) as in Lemma 3.3.4(2), implying that Φ is Anosov.

The proof of [ET98, Proposition 2.2.6] (see also [Hoz22, Remark 3.10]) shows that $\xi_\pm$ is everywhere transverse to E^{ws} and E^{wu}. By our orientation conventions, there exist two continuous functions $\sigma_s, \sigma_u : M \to \mathbb{R}$ such that

$$\ker \left\{ e^{-\sigma_u} \alpha_- + e^{\sigma_u} \alpha_+ \right\} = E^{ws},$$
$$\ker \left\{ e^{-\sigma_s} \alpha_- - e^{\sigma_s} \alpha_+ \right\} = E^{wu}.$$

[7] These transversality properties for the Reeb vector fields are a key feature of Anosov flows and are not satisfied for projectively Anosov flows which are not Anosov, see [Hoz22, Theorem 6.3].

Note that σ_u and σ_s are also continuously differentiable along X.[8] Indeed, if $\overline{e}_s$ is any vector field of class $\mathcal{C}^0_X$ spanning $\overline{E}^s \subset N_X$, then

$$e^{\sigma_u}\alpha_+(\overline{e}_s) + e^{-\sigma_u}\alpha_-(\overline{e}_s) = 0,$$

hence

$$\sigma_u = \frac{1}{2}\ln\left(-\frac{\alpha_-(\overline{e}_s)}{\alpha_+(\overline{e}_s)}\right),$$

and this quantity is continuously differentiable along X; the same argument applies to σ_s.

We define[9]

$$\alpha_u \coloneqq \frac{1}{2\sqrt{\cosh(\sigma_u - \sigma_s)}}\big(e^{-\sigma_u}\alpha_- + e^{\sigma_u}\alpha_+\big),$$
$$\alpha_s \coloneqq \frac{1}{2\sqrt{\cosh(\sigma_u - \sigma_s)}}\big(e^{-\sigma_s}\alpha_- - e^{\sigma_s}\alpha_+\big),$$

so that

$$\ker\alpha_u = E^{ws},$$
$$\ker\alpha_s = E^{wu},$$

and

$$\alpha_- = \frac{1}{\sqrt{\cosh(\sigma_u - \sigma_s)}}\big(e^{\sigma_s}\alpha_u + e^{\sigma_u}\alpha_s\big), \tag{3.3.7}$$
$$\alpha_+ = \frac{1}{\sqrt{\cosh(\sigma_u - \sigma_s)}}\big(e^{-\sigma_s}\alpha_u - e^{-\sigma_u}\alpha_s\big). \tag{3.3.8}$$

Note that α_u and α_s are continuously differentiable along X, and since E^{ws} and E^{wu}

[8] We cannot assume that they are $\mathcal{C}^1$ yet, since we do not know that Φ is Anosov!

[9] The seemingly strange conformal factors will greatly simplify some computations later, in particular the inequality (3.4.1) in the proof of Lemma 3.4.7.

are invariant under Φ, there exist continuous functions $r_s, r_u : M \to \mathbb{R}$ such that

$$\mathcal{L}_X \alpha_s = r_s \, \alpha_s,$$
$$\mathcal{L}_X \alpha_u = r_u \, \alpha_u.$$

Moreover,

$$\alpha_- \wedge \alpha_+ = 2 \, \alpha_s \wedge \alpha_u,$$

so $\overline{\alpha}_s \wedge \overline{\alpha}_u > 0$. We are left to show that $r_s < 0 < r_u$, which will follow from Lemma 3.2.7. Let dvol be the unique volume form on M such that $\alpha_s \wedge \alpha_u = \iota_X \, \mathrm{dvol} =: \tau$.

Lemma 3.3.11. *With the same notations as in Lemma 3.2.7, we have*

$$f_+ = \frac{e^{-(\sigma_s+\sigma_u)}}{\cosh(\sigma_u - \sigma_s)} \left(X \cdot (\sigma_u - \sigma_s) + r_u - r_s \right), \tag{3.3.9}$$
$$f_- = \frac{e^{(\sigma_s+\sigma_u)}}{\cosh(\sigma_u - \sigma_s)} \left(-X \cdot (\sigma_u - \sigma_s) + r_u - r_s \right), \tag{3.3.10}$$
$$f_0 = 2(r_u + r_s). \tag{3.3.11}$$

Proof. Although $\alpha_\pm$ are smooth, the quantities α_s, α_u, σ_s and σ_u are not $\mathcal{C}^1$ so we cannot compute $d\alpha_\pm$ directly by differentiating from (3.3.7) and (3.3.8). However, these quantities are differentiable along X and the functions f_0, f_- and f_+ can be computed from

$$\alpha_+ \wedge \mathcal{L}_X \alpha_+ = -f_+ \, \tau,$$
$$\alpha_- \wedge \mathcal{L}_X \alpha_- = f_- \, \tau,$$
$$\mathcal{L}_X \alpha_- \wedge \alpha_+ + \alpha_- \wedge \mathcal{L}_X \alpha_+ = f_0 \, \tau.$$

Moreover, the quantities $\mathcal{L}_X \alpha_\pm$ *can* be computed from (3.3.7) and (3.3.8) by differen-

tiating along X. The calculations are left to the reader. □

Since $f_\pm > 0$, (3.3.9) and (3.3.10) imply

$$0 \leq |X \cdot \sigma| < r_u - r_s,$$

where $\sigma \coloneqq \sigma_u - \sigma_s$, and the inequality $f_0^2 < 4f_-f_+$ gives

$$(r_u + r_s)^2 \leq \cosh^2(\sigma)(r_u + r_s)^2 < (r_u - r_s)^2 - \big(X \cdot \sigma\big)^2 \leq (r_u - r_s)^2,$$

yielding $r_s < 0 < r_u$ as desired. This concludes the proof of Theorem J.

Remark 3.3.12. *Similar computations (and Lemma 3.A.2) show that if Φ is a non-degenerate flow on M, the following are equivalent:*

(1) Φ is supported by a transverse Liouville pair (α_-, α_+),

(2) Φ is projectively Anosov and admits a defining pair (α_s, α_u) with $r_u > 0$.

*Note that in case, the Reeb vector fields for the standard construction of Section 3.3.2 are still transverse to the weak-unstable bundle of Φ, but are not necessarily transverse to the weak-stable bundle of Φ. We wish to call Φ a **semi-Anosov flow**. Our techniques would also show that the space of semi-Anosov flows on M is homotopy equivalent to the space of transverse Liouville pairs on M.*

3.3.4 Volume preserving Anosov flows

Volume preserving Anosov flows, i.e., Anosov flows preserving a volume form, constitute a remarkable class of Anosov flows. They are topologically transitive, in the sense that they admit a dense orbit. A deep theorem of Asaoka [Asa08] implies that on closed 3-manifold, every transitive Anosov flow is topologically equivalent to a volume

preserving one. In this section, we show some striking connections between volume preserving Anosov flows and Anosov Liouville pairs.

Proposition 3.3.13. *Let Φ be a smooth non-singular flow on M. Then Φ is a volume preserving Anosov flow if and only if it is supported by a closed AL pair.*

Proof. Let us first assume that Φ preserves a (smooth) volume form dvol, and let $\tau \coloneqq \iota_X \mathrm{dvol}$. Note that τ is *closed*. Let (ξ_-, ξ_+) be any bicontact structure supporting Φ, and $\alpha_\pm$ two contact forms such that $\ker \alpha_\pm = \xi_\pm$. There exists a smooth positive function $\kappa : M \to \mathbb{R}_{>0}$ such that

$$\alpha_- \wedge \alpha_+ = \kappa\, \tau.$$

The positivity of κ follows from our conventions on the coorientations of bicontact structures. Then, $\left(\alpha_-, \frac{1}{\kappa}\alpha_+\right)$ is a closed pair as in Definition 3.2.6, and it is automatically an AL pair in view of Lemma 3.2.7 since the corresponding function f_0 vanishes.

Let us now assume that Φ is supported by a closed AL pair (α_-, α_+). By Theorem J, Φ is Anosov. Let θ be any smooth 1-form on M satisfying $\theta(X) \equiv 1$, where X is the vector field generating Φ, and define $\mathrm{dvol} \coloneqq \alpha_- \wedge \alpha_+ \wedge \theta$. It is easy to check that it is a volume form, and

$$\begin{aligned} \mathcal{L}_X \mathrm{dvol} &= \mathcal{L}_X(\alpha_- \wedge \alpha_+) \wedge \theta + \alpha_- \wedge \alpha_+ \wedge \mathcal{L}_X \theta \\ &= \alpha_- \wedge \alpha_+ \wedge d\theta(X, \cdot\,) \\ &= 0, \end{aligned}$$

hence Φ preserves a smooth volume form. □

Remark 3.3.14. *A special class of closed AL pairs is given by Geiges pairs, defined*

in [MNW13, Section 8.5] as pairs of contact forms (α_-, α_+) satisfying

$$\alpha_+ \wedge d\alpha_+ = -\alpha_- \wedge d\alpha_- > 0,$$
$$\alpha_+ \wedge d\alpha_- = \alpha_- \wedge d\alpha_+ = 0.$$

Geiges pairs are called (-1)-Cartan structures in [Hoz23], and they are shown to be in correspondence with volume preserving Anosov flows. Here, we note that (α_-, α_+) is a $\mathcal{C}^1$ Geiges pair if and only if $(\alpha_- - \alpha_+, \alpha_- + \alpha_+)$ is a defining pair for the underlying volume preserving Anosov flow. As a result, the space of Geiges pairs supporting a given flow is contractible. Not every (smooth) volume preserving Anosov flow is supported by a smooth (or even $\mathcal{C}^2$) Geiges pair, as it would imply that the weak-stable and weak-unstable bundles are $\mathcal{C}^2$, so the flow would be smoothly equivalent to an algebraic Anosov flow; see [Ghy92, Théorème A].

The previous proof shows more: for a volume preserving Anosov flow, *any* supporting bicontact structure can be realized as the kernel of an AL pair. Surprisingly, this is a characteristic feature of volume preserving Anosov flows.

Theorem 3.3.15. *Let Φ be a smooth Anosov flow on M. Then Φ preserves a volume form if and only if for every (smooth) supporting bicontact structure (ξ_-, ξ_+), there exists an AL pair (α_-, α_+) such that $\xi_\pm = \ker \alpha_\pm$.*

Proof. The forward direction follows from the first part of the proof of Proposition 3.3.13. Let us assume that every (smooth) bicontact structure (ξ_-, ξ_+) supporting Φ is defined by a (smooth) AL pair, and let us fix a defining pair (α_s, α_u) for Φ with associated expansion rates r_s and r_u as in Lemma 3.3.4. Let $A > 0$ be a positive real number and $\{\alpha_u^n\}_{n\in\mathbb{N}}$ and $\{\alpha_s^n\}_{n\in\mathbb{N}}$ be sequences of smooth 1-forms converging to α_u

and α_s, respectively, in the $\mathcal{C}^1$ topology. For every $n \in \mathbb{N}$, we define

$$\alpha_+^n \coloneqq \alpha_u^n - e^{-A}\alpha_s^n,$$
$$\alpha_-^n \coloneqq \alpha_u^n + e^{A}\alpha_s^n.$$

We also let

$$\alpha_+ \coloneqq \alpha_u - e^{-A}\alpha_s,$$
$$\alpha_- \coloneqq \alpha_u + e^{A}\alpha_s.$$

Then, for n sufficiently large (depending on A), (α_-^n, α_+^n) defines a (smooth) bicontact structure (ξ_-^n, ξ_+^n) supporting Φ. By assumption, there exists a smooth positive function $f_n : M \to \mathbb{R}_{>0}$ such that $(\alpha_-^n, f_n\alpha_+^n)$ is an AL pair defining (ξ_-^n, ξ_+^n). Lemma 3.2.7 will imply the following

Claim. *For every $\epsilon > 0$, there exists a smooth function $h_\epsilon : M \to \mathbb{R}$ such that*

$$|X \cdot h_\epsilon + r_u + r_s| \leq \epsilon. \tag{3.3.12}$$

Assuming the Claim for now, it follows that if θ is a smooth 1-form such that $\theta(X) \equiv 1$, then for every closed orbit γ of X,

$$\int_\gamma (r_u + r_s)\,\theta = 0.$$

If the flow is transitive, a classical theorem of Liv

sic implies that there exists a continuous function $h : M \to \mathbb{R}$ which is differentiable along X and satisfies

$$X \cdot h + r_u + r_s = 0.$$

Writing $\mathrm{dvol}' \coloneqq e^h \,\mathrm{dvol} = e^h\, \alpha_s \wedge \alpha_u \wedge \theta$, $\mathcal{L}_X \mathrm{dvol}' = 0$ so Φ preserves a positive continuous measure, and by [LMM86, Corollary 2.1], this measure is a smooth volume form.

It turns out that the condition (3.3.12) *implies* that the flow is transitive. We have not been able to find a proof of this fact in the literature. We refer to Appendix 3.B for a proof using the theory of Sinaï–Ruelle–Bowen measures.

We now prove the Claim. Let $\epsilon > 0$ and choose $A > 0$ such that

$$\frac{\sup_M (r_u - r_s)}{\cosh(A)} \leq \epsilon/3.$$

Let e_s and e_u be $\mathcal{C}^1$ vector fields satisfying

$$\begin{aligned} \alpha_s(e_s) &= 1, & \alpha_s(e_u) &= 0, \\ \alpha_u(e_s) &= 0, & \alpha_u(e_u) &= 1, \end{aligned}$$

so that $\alpha_s \wedge \alpha_u(e_s, e_u) = \mathrm{dvol}(X, e_s, e_u) = 1$. Since $(\alpha_-^n, f_n \alpha_+^n)$ is an AL pair for n large enough, Lemma 3.2.7 implies the inequality

$$\left| X \cdot \ln f_n + \frac{d(\alpha_-^n \wedge \alpha_+^n)(X, e_s, e_u)}{\alpha_-^n \wedge \alpha_+^n(e_s, e_u)} \right| < \frac{2\sqrt{-\left(\alpha_-^n \wedge d\alpha_-^n(X, e_s, e_u)\right) \cdot \left(\alpha_+^n \wedge d\alpha_+^n(X, e_s, e_u)\right)}}{|\alpha_-^n \wedge \alpha_+^n(e_s, e_u)|}.$$

One computes

$$\begin{aligned} \lim_{n\to\infty} \frac{d(\alpha_-^n \wedge \alpha_+^n)(X, e_s, e_u)}{\alpha_-^n \wedge \alpha_+^n(e_s, e_u)} &= r_u + r_s, \\ \lim_{n\to\infty} \frac{2\sqrt{-\left(\alpha_-^n \wedge d\alpha_-^n(X, e_s, e_u)\right) \cdot \left(\alpha_+^n \wedge d\alpha_+^n(X, e_s, e_u)\right)}}{|\alpha_-^n \wedge \alpha_+^n(e_s, e_u)|} &= \frac{r_u - r_s}{\cosh(A)}, \end{aligned}$$

by first replacing $\alpha_\pm^n$ by $\alpha_\pm$ and then writing these expressions in terms of α_s and α_u. We obtain that for n large enough such that the above two sequences are $\epsilon/3$-close to

their limits,

$$|X \cdot \ln f_n + r_u + r_s| \leq \frac{r_u - r_s}{\cosh(A)} + \epsilon/3 + \epsilon/3 \leq \epsilon,$$

hence $h_\epsilon \coloneqq \ln f_n$ satisfies the required inequality and the Claim is proved. □

Remark 3.3.16. *The proof can be adapted to show that if every bicontact structure supporting* Φ *is realized as the kernel of a Liouville pair, then the determinant of the Poincaré return map for every closed orbit of* Φ *is bigger or equal than* 1*. We expect that this should also imply that* Φ *is volume preserving.*

3.4 Spaces of Anosov Liouville pairs and bicontact structures

This section is dedicated to the proof of Theorem K from the Introduction. We first describe the main strategy in a more general setting. Let E and B be topological spaces and $f : E \to B$ be a continuous map. We can assume that E and B have the homotopy type of CW complexes. This is the case for the spaces we consider (e.g., $\mathcal{AL}$, $\mathcal{BC}$, $\mathcal{AF}$, $\mathbb{P}\mathcal{AF}$, etc.) as they are open subsets of Fréchet spaces.[10] To show that f is a homotopy equivalence, it is enough to show that it is a Serre fibration with (weakly) contractible fibers. However, it seems rather hard to show that the maps we care about (e.g., $\mathcal{I}$, $\mathbb{P}\mathcal{I}$) satisfy a homotopy lifting property, as this would require a careful understanding of how the stable and unstable bundles depend on the (projectively) Anosov flow. Instead, we choose a more indirect approach: we first show that these maps have contractible fibers, and we then show that they are *topological submersions*.

[10]Indeed, Fréchet spaces are absolute neighborhood retracts (ANRs) by a theorem of Dugundji; an open subset of an ANR is an ANR; every ANR has the homotopy type of a CW complex by a theorem of Milnor and Whitehead. However, it is known that an open subset of an infinite dimensional Fréchet space is *not* a CW complex.

Definition 3.4.1. *$f : E \to B$ is a **topological submersion** if it is surjective, and for every $x \in E$, there exists a neighborhood U of x in E such that if we write $y \coloneqq f(x)$ and $V \coloneqq U \cap f^{-1}(y)$, there exists a homeomorphism $U \xrightarrow{\sim} f(U) \times V$ making the following diagram commute:*

$$\begin{array}{ccc} U & \xrightarrow{\quad\sim\quad} & f(U) \times V \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle \mathrm{pr}_1} & \\ & f(U) & \end{array}$$

Here, pr_1 denotes the projection onto the first factor.

Fiber bundles are topological submersions, but the converse is not true since the product structure of topological submersions is only *local on the domain* and is not "uniform in the fibers"; see Figure 3.4.1 below. However, we have:

Lemma 3.4.2. *If $f : E \to B$ is a topological submersion with (weakly) contractible fibers, then f an acyclic Serre fibration.*

Proof. Since projections are open and openness is a local property, f is open. By [Mei02, Lemma 6], f is a *homotopic submersion* (see [Mei02, Definition 1]), also known as a *Serre micro-fibration* [Gro86]. The result then follows from [Mei02, Corollary 13] (see also [Wei05, Lemma 2.2]). □

3.4.1 Contractibility of fibers

In this section, we show:

Theorem 3.4.3. *Let Φ be a smooth Anosov flow on M. The spaces of AL pairs and weak AL pairs supporting Φ are both contractible.*

We also show a similar result for projectively Anosov flows:

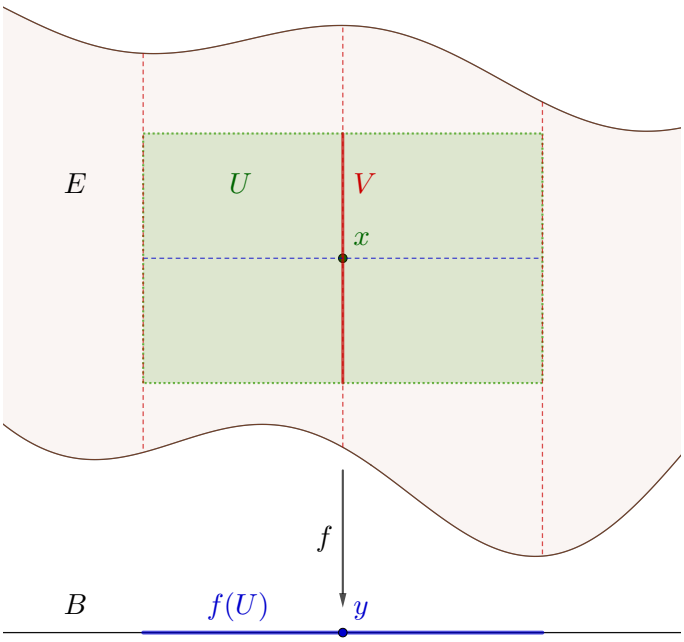


Figure 3.4.1: A topological submersion.

Theorem 3.4.4. *Let Φ be a smooth projectively Anosov flow on M. The space of bicontact structures supporting Φ is contractible.*

We obtain a version for volume preserving Anosov flows as well:

Theorem 3.4.5. *Let Φ be a smooth volume preserving Anosov flow on M. The space of closed AL pairs supporting Φ is contractible.*

If Φ is a smooth Anosov flow on M,

- $\mathcal{AL}_\Phi$ denotes the space of smooth AL pairs on M supporting Φ, endowed with the $\mathcal{C}^\infty$ topology,
- $\mathcal{AL}^1_\Phi$ denotes the space of $\mathcal{C}^1$ AL pairs on M supporting Φ, endowed with the $\mathcal{C}^1$ topology.

Smooth AL pairs supporting Φ form a dense subset of $\mathcal{AL}^1_\Phi$. Recall that $\mathcal{D}_\Phi$ denotes the space of $\mathcal{C}^1$ defining pairs on M for Φ, and $\mathcal{AL}^{\text{std}}_\Phi \subset \mathcal{AL}^1_\Phi$ denotes the space of $\mathcal{C}^1$

standard AL pairs supporting Φ, both endowed with the $\mathcal{C}^1$ topology. Theorem 3.4.3 will be a consequence of the contractibility of $\mathcal{AL}^{\text{std}}_\Phi$ and the following two lemmas.

Lemma 3.4.6. *The natural map $\mathcal{AL}_\Phi \to \mathcal{AL}^1_\Phi$ is a homotopy equivalence.*

Proof. This follows from some standard facts in algebraic topology. We will use that homotopy equivalences are *local* in the following sense:

Fact. *A continuous map $f : X \to Y$ between topological spaces is a homotopy equivalence if there exists a numerable open cover $\mathcal{U}$ of Y satisfying*

(1) $\mathcal{U}$ is stable under finite intersections,

(2) For every $U \in \mathcal{U}$, $f : f^{-1}(U) \to U$ is a homotopy equivalence.

See [Die71, Theorem 1] for a proof of this fact. Recall that an open cover is *numerable* if it admits a subordinate partition of unity. In our context, covers are automatically numerable since all the spaces under consideration are metrizable. We can simply cover $\mathcal{AL}^1_\Phi$ by sufficiently small open $\mathcal{C}^1$ balls and refine this cover by taking all possible finite intersections. These balls are convex as subsets of the space of pairs of $\mathcal{C}^1$ 1-forms on M, and so are finite intersections thereof, so all of the open subsets in our cover are contractible. Since smooth AL pairs supporting Φ are dense in $\mathcal{AL}^1_\Phi$, every open $\mathcal{C}^1$ ball in $\mathcal{AL}^1_\Phi$ intersects $\mathcal{AL}_\Phi$. The intersection of such a ball with $\mathcal{AL}_\Phi$ is also convex as a subset of $\Omega^1(M) \times \Omega^1(M)$, and so are finite intersections of such balls with $\mathcal{AL}_\Phi$. Thus, the conditions (1) and (2) of the Fact are trivially satisfied. □

Lemma 3.4.7. *$\mathcal{AL}^{\text{std}}_\Phi$ is a strong deformation retract of $\mathcal{AL}^1_\Phi$.*

Proof. Let $(\alpha_-, \alpha_+) \in \mathcal{AL}^1_\Phi$. As in Section 3.3.3, there exist functions $\sigma_s, \sigma_u : M \to \mathbb{R}$

satisfying

$$\ker\left\{e^{-\sigma_u}\alpha_- + e^{\sigma_u}\alpha_+\right\} = E^{ws},$$
$$\ker\left\{e^{-\sigma_s}\alpha_- - e^{\sigma_s}\alpha_+\right\} = E^{wu}.$$

If e_s and e_u are $\mathcal{C}^1$ vector fields such that $\overline{e}_s$ spans $\overline{E}^s$ and $\overline{e}_u$ spans $\overline{E}^u$, we can write

$$\sigma_s = \frac{1}{2}\ln\left(\frac{\alpha_+(e_u)}{\alpha_-(e_u)}\right), \qquad \sigma_u = \frac{1}{2}\ln\left(-\frac{\alpha_-(e_s)}{\alpha_+(e_s)}\right),$$

so σ_s and σ_u are $\mathcal{C}^1$. Moreover, the map $\mathcal{S} : (\alpha_-, \alpha_+) \mapsto (\sigma_s, \sigma_u)$ is continuous in the $\mathcal{C}^1$ topology. As before, we define

$$\alpha_u \coloneqq \frac{1}{2\sqrt{\cosh(\sigma_u - \sigma_s)}}\big(e^{-\sigma_u}\alpha_- + e^{\sigma_u}\alpha_+\big),$$
$$\alpha_s \coloneqq \frac{1}{2\sqrt{\cosh(\sigma_u - \sigma_s)}}\big(e^{-\sigma_s}\alpha_- - e^{\sigma_s}\alpha_+\big).$$

The computations of Section 3.3.3 show that $(\alpha_s, \alpha_u) \in \mathcal{D}_\Phi$. Therefore, we obtain a continuous map $\mathcal{D} : (\alpha_-, \alpha_+) \mapsto (\alpha_s, \alpha_u)$. We now define a strong deformation retraction $r : \mathcal{AL}^1_\Phi \times [0,1] \to \mathcal{AL}^1_\Phi$. Let $(\alpha_-, \alpha_+) \in \mathcal{AL}^1_\Phi$, and $(\sigma_s, \sigma_u) = \mathcal{S}(\alpha_-, \alpha_+)$ and $(\alpha_s, \alpha_u) = \mathcal{D}(\alpha_-, \alpha_+)$ as before. For $t \in [0,1]$, we define

$$\alpha_-^t \coloneqq \frac{1}{\sqrt{\cosh((1-t)\sigma)}}\big(e^{(1-t)\sigma_s}\alpha_u + e^{(1-t)\sigma_u}\alpha_s\big),$$
$$\alpha_+^t \coloneqq \frac{1}{\sqrt{\cosh((1-t)\sigma)}}\big(e^{-(1-t)\sigma_s}\alpha_u - e^{-(1-t)\sigma_u}\alpha_s\big),$$

where $\sigma = \sigma_u - \sigma_s$. We then have

- $\big(\alpha_-^0, \alpha_+^0\big) = (\alpha_-, \alpha_+)$,
- $\big(\alpha_-^1, \alpha_+^1\big) = \big(\alpha_u + \alpha_s, \alpha_u - \alpha_s\big) \in \mathcal{AL}^{\mathrm{std}}_\Phi$,
- If $(\alpha_-, \alpha_+) \in \mathcal{AL}^{\mathrm{std}}_\Phi$, then $\big(\alpha_-^t, \alpha_+^t\big) = (\alpha_-, \alpha_+)$ for every $t \in [0,1]$.

We claim that $\big(\alpha_-^t, \alpha_+^t\big)$ is an AL pair for every $t \in [0,1]$. Indeed, by Lemma 3.2.7 and Lemma 3.3.11, it is enough to show the inequality

$$\cosh^2\big((1-t)\sigma\big)(r_u + r_s)^2 + (1-t)^2\big(X \cdot \sigma\big)^2 < (r_u - r_s)^2. \tag{3.4.1}$$

It holds for $t = 0$ since (α_-, α_+) is an AL pair, and the left-hand side is obviously a non-increasing function of t, so it holds for every $t \in [0,1]$.

We finally define $r\big((\alpha_-, \alpha_+), t\big) \coloneqq \big(\alpha_-^t, \alpha_+^t\big)$ so that $r : \mathcal{AL}^1_\Phi \times [0,1] \to \mathcal{AL}^1_\Phi$ is continuous, and by the three bullets above, it is a strong deformation retraction of $\mathcal{AL}^1_\Phi$ onto $\mathcal{AL}^{\mathrm{std}}_\Phi$. □

Proof of Theorem 3.4.3. For AL pairs, combine Lemma 3.3.8, Lemma 3.4.6 and Lemma 3.4.7. For weak AL pairs, the argument can be modified as follows. Lemma 3.4.6 can be easily adapted to the case of weak AL pairs. Lemma 3.4.7 can be adapted to show that the space of $\mathcal{C}^1$ weak AL pairs supporting a smooth Anosov flow Φ deformation retracts onto the space of pairs of the form $\alpha_\pm = \alpha_u \mp \alpha_s$, where (α_s, α_u) satisfies the conditions of a defining pair for Φ *without the condition* $r_s < 0$ (but still satisfies $r_s < r_u$ and $0 < r_u$, see Remark 3.3.12). The latter space is convex, hence contractible. □

Proof of Theorem 3.4.4. We only sketch how to modify the proof of Theorem 3.4.3 and we leave the details to the interested reader.

First of all, we shall introduce the space of $\mathcal{C}^0_X$ bicontact structures. Those are continuous pairs of codimension 1 distributions (ξ_-, ξ_+) which are continuously differentiable along X and which are defined by some 1-forms $\alpha_-, \alpha_+ \in \Omega^0_X$ satisfying

$$\overline{\alpha}_- \wedge \overline{\alpha}_+ > 0, \qquad \overline{\alpha}_- \wedge \mathcal{L}_X\overline{\alpha}_- < 0, \qquad \overline{\alpha}_+ \wedge \mathcal{L}_X\overline{\alpha}_+ > 0, \tag{3.4.2}$$

as forms on N_X.

For the purpose of the proof, we choose an arbitrary $\mathcal{C}^0_X$ vector field e_u such that $\overline{e}_u$ spans $\overline{E}^u \subset N_X$ and defines the prescribed orientation. This is equivalent to choosing a 1-form $\alpha_u \in \Omega^1_{X,0}$ such that $\ker\alpha_u = E^{ws}$ as oriented 2-plane fields, with the normalization $\alpha_u(e_u) \equiv 1$.

We denote by $\mathcal{BC}_\Phi$ (resp. $\mathcal{BC}^0_\Phi$) the space of smooth (resp. $\mathcal{C}^0_X$) bicontact structures supporting Φ. We write $\mathcal{BC}^{\mathrm{std}}_\Phi \subset \mathcal{BC}^0_\Phi$ for the space of *standard* bicontact structures supporting Φ, of the form $\big(\ker\big(\alpha_u + \alpha_s\big), \ker\big(\alpha_u - \alpha_s\big)\big)$, where α_u is fixed by the condition $\alpha_u(e_u) \equiv 1$.

Lemma 3.4.6 can be easily adapted to show that the natural map $\mathcal{BC}_\Phi \to \mathcal{BC}^0_\Phi$ is a homotopy equivalence, using Lemma 3.A.2.

Lemma 3.4.7 can be modified as follows. For $(\xi_-, \xi_+) \in \mathcal{BC}^0_\Phi$, we denote by (α_-, α_+) the unique pair of 1-forms in Ω^0_X satisfying $\ker\alpha_\pm = \xi_\pm$ and $\alpha_\pm(e_u) = 1$. We define a $\mathcal{C}^0_X$ function $\sigma : M \to \mathbb{R}$ by

$$\ker\left\{e^{-\sigma}\alpha_- + e^{\sigma}\alpha_+\right\} = E^{ws},$$

so that

$$\alpha_u = \frac{1}{2\cosh(\sigma)}\big(e^{-\sigma}\alpha_- + e^{\sigma}\alpha_+\big),$$

and we define

$$\alpha_s \coloneqq \frac{1}{2\cosh(\sigma)}\big(\alpha_- - \alpha_+\big)$$

so that $\alpha_s \in \Omega^1_{X,0}$, and

$$\ker\alpha_s = E^{wu}.$$

This readily implies that

$$\alpha_- = \alpha_u + e^{\sigma}\alpha_s,$$

$$\alpha_+ = \alpha_u - e^{-\sigma}\alpha_s.$$

Writing $\mathcal{L}_X \alpha_u = r_u \alpha_u$ and $\mathcal{L}_X \alpha_s = r_s \alpha_s$, where $r_s, r_u : M \to \mathbb{R}$ are continuous, the last two inequalities in (3.4.2) are equivalent to

$$|X \cdot \sigma| < r_u - r_s.$$

Moreover,

$$\alpha_- \wedge \alpha_+ = 2\cosh(\sigma)\alpha_s \wedge \alpha_u.$$

This shows that (α_s, α_u) is a defining pair for Φ that satisfies $\alpha_u(e_u) \equiv 1$. For $t \in [0,1]$, we define

$$\alpha_-^t \coloneqq \alpha_u + e^{(1-t)\sigma}\alpha_s, \qquad \xi_-^t \coloneqq \ker \alpha_-^t$$
$$\alpha_+^t \coloneqq \alpha_u - e^{-(1-t)\sigma}\alpha_s, \qquad \xi_+^t \coloneqq \ker \alpha_+^t.$$

These formulae define a strong deformation retraction of $\mathcal{BC}^0_\Phi$ onto $\mathcal{BC}^{\mathrm{std}}_\Phi$. Moreover, $\mathcal{BC}^{\mathrm{std}}_\Phi$ is homeomorphic to the space of defining pairs (α_s, α_u) for Φ satisfying $\alpha_u(e_u) \equiv 1$, and one easily checks that this space is contractible. □

Proof of Theorem 3.4.5. The result essentially follows from Theorem 3.4.4. Let dvol be a smooth volume form preserved by Φ and $\tau \coloneqq \iota_X \mathrm{dvol}$. If (α_-, α_+) is a closed AL pair supporting Φ, there exists a smooth positive function $\kappa \coloneqq M \to \mathbb{R}_{>0}$ such that

$$\alpha_- \wedge \alpha_+ = \kappa\, \tau.$$

Moreover, $X \cdot \kappa = 0$ and since Φ is topologically transitive, κ is constant. One easily checks that the space of closed AL pairs supporting Φ is homotopy equivalent to the space of balanced pairs of contact forms (α_-, α_+) supporting Φ and satisfying $\alpha_- \wedge \alpha_+ = \tau$. We denote this space by $\mathcal{BC}^\tau_\Phi$. There is a natural continuous map $\mathcal{K} : \mathcal{BC}^\tau_\Phi \to \mathcal{BC}_\Phi$, obtained by taking kernels, which is surjective by Theorem 3.3.15.

One easily checks that $\mathcal{K}$ is injective and that it is a homeomorphism. Theorem 3.4.4 finishes the proof. □

3.4.2 Homotopy equivalences

Let us recall some notations.

- $\mathcal{AL}$ denotes the space of smooth AL pairs on M,
- $\mathcal{AF}$ denotes the space of smooth Anosov flows on M, up to positive time reparametrization,
- $\mathbb{P}\mathcal{AF}$ denotes the space of smooth projectively Anosov flows on M, up to positive time reparametrization.

Recall that there is a continuous map

$$\begin{array}{rccc} \mathcal{I}: & \mathcal{AL} & \longrightarrow & \mathcal{AF} \\ & (\alpha_-, \alpha_+) & \longmapsto & \ker \alpha_- \cap \ker \alpha_+ \end{array}$$

where we identify an oriented 1-dimensional distribution with any flow spanned by it. Similarly, there is a continuous map

$$\begin{array}{rccc} \mathbb{P}\mathcal{I}: & \mathcal{BC} & \longrightarrow & \mathbb{P}\mathcal{AF} \\ & (\xi_-, \xi_+) & \longmapsto & \xi_- \cap \xi_+ \end{array}$$

In this section, we show the main theorems of this Chapter:

Theorem 3.4.8. *The map $\mathcal{I}$ is an acyclic Serre fibration.*

Our argument can easily be adapted to the case of projectively Anosov flows (this result might already be known to some experts):

Theorem 3.4.9. *The map $\mathbb{P}\mathcal{I}$ is an acyclic Serre fibration.*

Remark 3.4.10. *With more work, it is possible to show that $\mathcal{I}$ is* ***shrinkable****: it is homotopy equivalent over $\mathcal{AF}$ to* id : $\mathcal{AF} \to \mathcal{AF}$*. Concretely, this means that there exists a section s of $\mathcal{I}$ such that $s \circ \mathcal{I}$ is* fiberwise homotopic *to* id*. This implies that the space sections of $\mathcal{I}$ is non-empty and contractible. To prove this statement, one would need to upgrade the results of Section 3.4.1 to hold* in family over $\mathcal{AF}$*. A key ingredient is [LMM86, Lemma 2.1], which shows that for smooth Anosov flows, the Anosov splitting depends continuously on the flow. We are not aware of a similar result for projectively Anosov flows.*

We will need the following

Lemma 3.4.11. *$\mathcal{I}$ is a topological submersion.*

Proof. By Theorem J, $\mathcal{I}$ is surjective. We fix some auxiliary Riemannian metric g on M and identify $\mathcal{AF}$ with the space of unit Anosov vector fields on M. Let $\boldsymbol{\alpha}^0 = (\alpha^0_-, \alpha^0_+) \in \mathcal{AL}$ and let X be the unit vector field generating $\mathcal{I}(\boldsymbol{\alpha}^0)$, whose flow is denoted by Φ. We choose an arbitrary smooth 1-form θ such that $\theta(X) \equiv 1$. For a unit vector field X' sufficiently close to X (so that $\theta(X') > 0$) and $\boldsymbol{\alpha} = (\alpha_-, \alpha_+) \in \mathcal{AL}_\Phi$, we define

$$\alpha'_\pm \coloneqq \alpha_\pm - \frac{\alpha_\pm(X')}{\theta(X')}\,\theta,$$

so that $\alpha'_\pm(X') = 0$. Since $\mathcal{AL} \subset \Omega^1 \times \Omega^1$ is open, we can find an open neighborhood $\mathcal{N}_{\boldsymbol{\alpha}^0}$ of $\boldsymbol{\alpha}^0$ in $\mathcal{AL}_\Phi$ and an open neighborhood $\mathcal{N}_\Phi$ of Φ in $\mathcal{AF}$ such that the map

$$\begin{aligned} \psi : \ \mathcal{N}_\Phi \times \mathcal{N}_{\boldsymbol{\alpha}^0} &\longrightarrow \mathcal{AL} \\ \big(X', \boldsymbol{\alpha}\big) &\longmapsto \boldsymbol{\alpha}' \end{aligned}$$

is well-defined. It is continuous and satisfies $\mathcal{I} \circ \psi = \mathrm{pr}_1$. Moreover, the restriction of ψ to $\{X\} \times \mathcal{N}_{\boldsymbol{\alpha}^0}$ is the inclusion $\mathcal{N}_{\boldsymbol{\alpha}^0} \subset \mathcal{AL}$. One easily checks that ψ is injective,

has open image and has an inverse given by $\psi^{-1}(\boldsymbol{\alpha}') \coloneqq \big(X', \boldsymbol{\alpha}\big)$ where

$$\alpha_\pm \coloneqq \alpha'_\pm - \alpha'_\pm(X)\,\theta,$$

and X' is the unit vector field spanning $\mathcal{I}\big(\boldsymbol{\alpha}'\big)$. Therefore, ψ^{-1} is a local trivialization of $\mathcal{I}$ around $\boldsymbol{\alpha}^0$. □

Proof of Theorem 3.4.8. By Theorem 3.4.3 and Lemma 3.4.11, $\mathcal{I}$ is a topological submersion with contractible fibers, hence an acyclic Serre fibration by Lemma 3.4.2. □

Proof of Theorem L. Lemma 3.4.11 and its proof hold verbatim for $\mathcal{I}^w$ so the previous proof applies to $\mathcal{I}^w$ as well. □

Proof of Theorem 3.4.9. By Theorem 3.4.4, we already know that the fibers of $\mathbb{P}\mathcal{I}$ are contractible so it is enough to adapt Lemma 3.4.11. It can be done by choosing an auxiliary smooth vector field Z, depending on an initial choice of $\big(\xi^0_-, \xi^0_+\big) \in \mathcal{BC}$, which is positively transverse to $\xi^0_\pm$ and satisfies $\theta(Z) \equiv 0$. We can uniquely choose contact forms $\alpha^0_\pm$ for $\xi^0_\pm$ by imposing $\alpha_\pm(Z) \equiv 1$. The proof of Lemma 3.4.11 can be reproduced with minor changes to provide a suitable trivialization near $\big(\xi^0_-, \xi^0_+\big)$. □

3.4.3 The kernel map

Recall that we have a continuous map

$$\begin{array}{rccc} \underline{\ker}: & \mathcal{AL} & \longrightarrow & \mathcal{BC} \\ & (\alpha_-, \alpha_+) & \longmapsto & (\ker \alpha_-, \ker \alpha_+), \end{array}$$

where the spaces $\mathcal{AL}$ and $\mathcal{BC}$ are endowed with the $\mathcal{C}^\infty$ topology.

Lemma 3.4.12. *The map* $\underline{\ker}$ *is open.*

Proof. It easily follows from the openness of $\mathcal{AL}$ and $\mathcal{BC}$ in the space of smooth 1-forms on M and the space of smooth plane fields on M, respectively, and the following elementary fact. If $\mathring{\Omega}^1 \subset \Omega^1$ denotes the space of nowhere vanishing 1-forms on M and Π denotes the space of smooth cooriented plane fields on M, the natural map

$$\begin{array}{rccc} \ker: & \mathring{\Omega}^1 & \longrightarrow & \Pi \\ & \alpha & \longmapsto & \ker \alpha \end{array}$$

is open (for the $\mathcal{C}^\infty$ topology on the domain and target). Indeed, after trivializing the tangent bundle of M and fixing an auxiliary Riemannian metric, we can identify Π with the space of smooth maps $M \to S^3$ (via the unit normal vector) and $\mathring{\Omega}^1$ with the space of smooth maps $M \to \mathbb{R}^3 \setminus \{0\}$, so that ker becomes the composition with the standard projection $\mathbb{R}^3 \setminus \{0\} \cong \mathbb{R} \times S^3 \to S^3$. Ultimately, ker boils down to the projection $\mathcal{C}^\infty(M, \mathbb{R}) \times \mathcal{C}^\infty(M, S^3) \to \mathcal{C}^\infty(M, S^3)$ onto the second factor, which is clearly open. □

Theorem 3.4.13. *The map* $\underline{\ker}$ *is an acyclic Serre fibration onto its image.*

Proof. As before, by Lemma 3.4.2, it is enough to show the following properties.

(1) The fibers of $\underline{\ker}$ over its image are contractible.

(2) $\underline{\ker}$ is a topological submersion onto its image.

We can simplify the situation by restricting to the space $\mathcal{AL}^b$ of *balanced* AL pairs, since there is a homeomorphism

$$\begin{array}{rccc} \vartheta: & \mathcal{C}^\infty(M, \mathbb{R}) \times \mathcal{AL}^b & \xrightarrow{\sim} & \mathcal{AL} \\ & \big(\sigma, (\alpha_-, \alpha_+)\big) & \longmapsto & \big(e^{-\sigma}\alpha_-, e^{\sigma}\alpha_+\big), \end{array}$$

and $\underline{\ker}$ is compatible with this homeomorphism in the obvious way.

Let us consider a bicontact structure (ξ_-, ξ_+) defined by a balanced AL pair (α_-, α_+), and let $\mathrm{dvol} \coloneqq \alpha_+ \wedge d\alpha_+$. We also consider a vector field $X \in \xi_- \cap \xi_+$ normalized so that $\alpha_- \wedge \alpha_+ = \iota_X \mathrm{dvol}$.

To show (1), note that any other balanced AL pair defining (ξ_-, ξ_+) is of the form $(e^\sigma \alpha_-, e^\sigma \alpha_+)$ for a smooth function $\sigma : M \to \mathbb{R}$ satisfying

$$|2X \cdot \sigma + f_0| < 2,$$

where we use the notation of Lemma 3.2.7. By assumption, $|f_0| < 2$, so the space of σ such that $(e^\sigma \alpha_-, e^\sigma \alpha_+)$ is a balanced AL pair defining (ξ_-, ξ_+) is convex, hence contractible.

To show (2), we consider an open neighborhood $\mathcal{V}$ of (α_-, α_+) in $\mathcal{AL}^b$. We can find a smaller neighborhood $\mathcal{V}' \subset \mathcal{V}$ such that for every $(\alpha'_-, \alpha'_+) \in \mathcal{V}'$, the pair

$$\left(\widetilde{\alpha}'_-, \widetilde{\alpha}'_+\right) \coloneqq \left(\frac{1}{\sqrt{f'}}\alpha'_-, \frac{1}{\sqrt{f'}}\ \alpha'_+\right)$$

is in $\mathcal{V}$, where

$$\alpha'_\pm \wedge d\alpha'_\pm = \pm f' \,\mathrm{dvol}.$$

Note that $\left(\widetilde{\alpha}'_-, \widetilde{\alpha}'_+\right)$ is a balanced AL pair satisfying $\widetilde{\alpha}'_+ \wedge d\widetilde{\alpha}'_+ = \mathrm{dvol}$. Since $\underline{\ker}$ is open by the previous lemma, $\mathcal{U}' \coloneqq \underline{\ker}(\mathcal{V}') \subset \mathcal{BC}$ is an open neighborhood of (ξ_-, ξ_+). Let $\widetilde{\mathcal{V}}' \subset \mathcal{V}$ be the subspace of elements of the form $\left(\widetilde{\alpha}'_-, \widetilde{\alpha}'_+\right)$ for $(\alpha'_-, \alpha'_+) \in \mathcal{V}'$. One easily checks that $\underline{\ker} : \widetilde{\mathcal{V}}' \to \mathcal{U}'$ is injective and open. It is surjective by definition, hence it is a homeomorphism. By the previous paragraph, there is an open neighborhood of (α_-, α_+) in $\underline{\ker}^{-1}\{(\xi_-, \xi_+)\} \cap \mathcal{AL}^b$ homeomorphic to

$$\Sigma_\epsilon \coloneqq \{\sigma : M \to \mathbb{R} : |X\ \cdot \sigma| < \epsilon\}$$

for some small $\epsilon > 0$. Therefore, after possibly shrinking ϵ, the map

$$\begin{array}{ccc} \widetilde{\mathcal{V}}' \times \Sigma_\epsilon & \longrightarrow & \mathcal{AL}^b \\ \big((\widetilde{\alpha}'_-, \widetilde{\alpha}'_+), \sigma\big) & \longmapsto & \big(e^\sigma \widetilde{\alpha}'_-, e^\sigma \widetilde{\alpha}'_+\big) \end{array}$$

induces a local trivialization of $\mathcal{AL}^b \to \underline{\ker}\,(\mathcal{AL})$ around (α_-, α_+).

This proves that $\underline{\ker}$ restricted to $\mathcal{AL}^b$ is a topological submersion with contractible fibers, and the same holds for $\underline{\ker}$ on $\mathcal{AL}$ via the homeomorphism ϑ. □

Remark 3.4.14. *Since* $\underline{\ker}$ *is open, its image has the homotopy type of a CW complex.*

There is an inclusion $\underline{\ker}\big(\mathcal{AL}\big) \subset \mathbb{P}\mathcal{I}^{-1}\big(\mathcal{AF}\big)$ which is strict according to Theorem 3.3.15.[11] In more concrete terms, there exist bicontact structures supporting Anosov flows which cannot be represented as the kernel of a AL pairs. Nevertheless, we have:

Theorem 3.4.15. *The inclusion* $\underline{\ker}\big(\mathcal{AL}\big) \subset \mathbb{P}\mathcal{I}^{-1}\big(\mathcal{AF}\big)$ *is a homotopy equivalence.*

Proof. It immediately follows from Theorem 3.4.8, Theorem 3.4.9, Theorem 3.4.13 and the commutative diagram

$$\begin{array}{ccccc} \mathcal{AL} & \xrightarrow{\sim} & \underline{\ker}\big(\mathcal{AL}\big) \subset \mathbb{P}\mathcal{I}^{-1}\big(\mathcal{AF}\big) & \hookrightarrow & \mathcal{BC} \\ & \searrow^{\mathcal{I}}_{\sim} & \downarrow{\scriptstyle \mathbb{P}\mathcal{I}}\wr & \lrcorner & \downarrow{\scriptstyle \mathbb{P}\mathcal{I}}\wr \\ & & \mathcal{AF} & \hookrightarrow & \mathbb{P}\mathcal{AF} \end{array}$$

Note that the corestriction of $\mathbb{P}\mathcal{I}$ over $\mathcal{AF}$ is also an acyclic Serre fibration, and all the spaces in this diagram have the homotopy type of CW complexes. □

[11] Indeed, any volume preserving Anosov flow can be perturbed near a closed orbit in such a way that the new Poincaré return map for this orbit has determinant different than 1, so the flow is not volume preserving anymore.

3.5 Linear Liouville pairs

As explained in the Introduction, there exist other possible definitions for Liouville pairs. The following one is used by some authors (e.g., [Mit95; Hoz22]).

Definition 3.5.1. *A pair of contact forms (α_-, α_+) on M is a **linear Liouville pair** if the 1-form*

$$(1-t)\alpha_- + (1+t)\alpha_+$$

on $[-1,1]_t \times M$ is a positively oriented Liouville form.

*The pair (α_-, α_+) is a **linear Anosov Liouville pair** (ℓAL pair for short) if both (α_-, α_+) and $(-\alpha_-, \alpha_+)$ are linear Liouville pairs.*

Note that for this definition, $[-1,1] \times M$ is a Liouville *domain* instead of a Liouville *manifold*. In this section, we study some similarities and differences between Liouville pairs and linear Liouville pairs. In particular, we show that those are two different notions (Lemma 3.5.6). Moreover, a pair a contact forms which is both a Liouville pair and a linear Liouville pair defines Liouville structures in two different ways, and we show that they are homotopic (Proposition 3.5.8). We believe that this result is relevant since all of the natural constructions of Liouville pairs we are aware of satisfy both definitions. The linear formulation might be more convenient in some situations. The results in this section are independent from the main results of this Chapter.

3.5.1 Elementary properties

The results in Section 3.2.2 can be adapted to ℓAL pairs. First of all, ℓAL pairs can be characterized in the following way (see Lemma 3.2.7):

Lemma 3.5.2. *Let (α_-, α_+) be a pair of contact forms on M. We write*

$$\begin{aligned}
\alpha_+ \wedge d\alpha_+ &= f_+ \,\mathrm{dvol},\\
\alpha_- \wedge d\alpha_- &= -f_- \,\mathrm{dvol},\\
\alpha_- \wedge d\alpha_+ &= g_+ \,\mathrm{dvol},\\
\alpha_+ \wedge d\alpha_- &= g_- \,\mathrm{dvol},
\end{aligned}$$

where dvol *is any volume form on M and $f_\pm, g_\pm : M \to \mathbb{R}$ are smooth functions. Then (α_-, α_+) is a ℓAL pair if and only if*

$$|g_-| < f_- \qquad \text{and} \qquad |g_+| < f_+. \tag{3.5.1}$$

Proof. The pair (α_-, α_+) is a linear Liouville pair if and only if for every $t \in [-1, 1]$,

$$(\alpha_+ - \alpha_-) \wedge (d\alpha_+ + d\alpha_- + t(d\alpha_+ - d\alpha_-)) > 0,$$

which is equivalent to

$$f_+ + f_- + g_- - g_+ + t(f_+ - f_- - g_- - g_+) > 0.$$

This inequality is satisfied for every $t \in [-1, 1]$ if and only if it is satisfied for $t = -1$ and $t = 1$, which is equivalent to $g_+ < f_+$ and $-g_- < f_-$. □

Similarly to Proposition 3.2.9, we also have:

Proposition 3.5.3. *Let (α_-, α_+) be a ℓAL pair. Then it defines a bicontact structure $(\xi_-, \xi_+) = (\ker \alpha_-, \ker \alpha_+)$. Moreover, if $X \in \xi_- \cap \xi_+$ is a nowhere vanishing vector field, then (X, R_-, R_+) is a basis at every point of M.*

Proof. To see that ξ_- and ξ_+ are everywhere transverse, we argue by contradiction

and assume that they coincide at a point $x \in M$. With the same notations as in the proof of 3.2.9, we readily get

$$\begin{aligned} f_+ &= d\alpha_+(X,Y), \\ f_- &= -\alpha_-(R_+)d\alpha_-(X,Y), \\ g_+ &= \alpha_-(R_+)d\alpha_+(X,Y), \\ g_- &= d\alpha_-(X,Y), \end{aligned}$$

hence

$$|g_-g_+| = |\alpha_-(R_+)d\alpha_-(X,Y)d\alpha_+(X,Y)| = f_-f_+,$$

contradicting (3.5.1).

Now, assuming that $\mathrm{dvol}(X, R_-, R_+) = 0$ at a point $x \in M$, the computations in the proof of Proposition 3.2.9 show

$$\begin{aligned} f_- &= -\alpha_-(R_+)g_-, \\ f_+ &= \frac{1}{\alpha_-(R_+)}g_+, \end{aligned}$$

hence

$$|g_-g_+| = f_-f_+,$$

contradicting (3.5.1) once again. □

Remark 3.5.4. *A main difference between Anosov Liouville pairs as in Definition 3.1.1 and linear Anosov Liouville pairs as in Definition 3.5.1 is that there does not seem to be a natural action of $\mathcal{C}^\infty(M,\mathbb{R})$ on ℓAL pairs. Moreover, we do not know if there is a natural modification making a ℓAL pair balanced.*

ℓAL pairs can also be characterized by their Reeb vector fields (see Proposition 3.2.11):

Proposition 3.5.5. *Let (α_-, α_+) be a pair of contact forms on M, negative and positive, respectively. Then it is a ℓAL pair if and only if*

$$|\alpha_-(R_+)| < 1 \qquad \textit{and} \qquad |\alpha_+(R_-)| < 1. \tag{3.5.2}$$

Proof. If (α_-, α_+) is a ℓAL pair, then Proposition 3.5.3 and the computations in the proof of Proposition 3.2.9 imply

$$\begin{aligned} g_+ &= \alpha_-(R_+)\, f_+, \\ g_- &= -\alpha_+(R_-)\, f_-, \end{aligned}$$

and (3.5.2) follows from (3.5.3).

Now, assuming that (α_-, α_+) satisfies (3.5.2), it is enough to prove that the conclusions of Proposition 3.5.3 are satisfied. This follows exactly from the proof of Proposition 3.2.11. □

Combining Proposition 3.2.11 and Proposition 3.5.5, we obtain that any *balanced* ℓAL pair is an AL pair. The converse is *not* true by the following lemma. It also implies that the two definitions of Liouville pairs (Definition 3.1.1 and Definition 3.5.1) are *different*:

Lemma 3.5.6. *Every smooth volume preserving Anosov flow on M admits a supporting balanced AL pair which is <u>not</u> a ℓAL pair, and whose underlying bicontact structure is <u>not</u> defined by a ℓAL pair.*

Proof. Let (α_s, α_u) be a defining pair for a volume preserving Anosov flow $\Phi = \{\phi^t\}$. For $A \geq 1$, we define

$$\begin{aligned} \alpha_- &\coloneqq e^{-A}\alpha_u + e^{A}\alpha_s, \\ \alpha_+ &\coloneqq e^{A}\alpha_u - e^{-A}\alpha_s. \end{aligned}$$

If dvol is such that $\iota_X \mathrm{dvol} = \alpha_s \wedge \alpha_u$, where X is the vector field generating the flow, then one easily computes

$$f_\pm = r_u - r_s = 2r_u,$$
$$g_\pm = -2\sinh(2A)r_u,$$

so (α_-, α_+) is a $\mathcal{C}^1$ closed balanced AL pair, but it is not a ℓAL pair since $|g_\pm| > f_\pm$. This remains true for a suitable smoothing of (α_-, α_+).

Let us assume for simplicity that the pair (α_-, α_+) as above is smooth. We show that there are no functions $h_\pm : M \to \mathbb{R}_{>0}$ such that $(h_-\alpha_-, h_+\alpha_+)$ is a ℓAL pair. Indeed, let us assume by contradiction that such functions exist. By Lemma 3.5.2 they would satisfy the following inequalities:

$$|\cosh(2A)h_+ X \cdot h_- - \sinh(2A)r_u h_- h_+| < r_u h_-^2,$$
$$|\cosh(2A)h_- X \cdot h_+ + \sinh(2A)r_u h_- h_+| < r_u h_+^2.$$

Writing $\rho_\pm \coloneqq 1/h_\pm$, these are equivalent to

$$|X \cdot \rho_- + \tanh(2A)r_u\rho_-| < \frac{r_u}{\cosh(2A)}\rho_+,$$
$$|X \cdot \rho_+ - \tanh(2A)r_u\rho_+| < \frac{r_u}{\cosh(2A)}\rho_-.$$

Fixing a point $x \in M$, we define $y_\pm : \mathbb{R} \to \mathbb{R}_{>0}$ by

$$y_\pm(t) \coloneqq \rho_\pm \circ \phi^t(x).$$

There exists $C > 0$ such that $0 < y_\pm < C$. Moreover, these functions satisfy

$$|\dot{y}_- + a y_-| < \epsilon y_+,$$
$$|\dot{y}_+ - a y_+| < \epsilon y_-,$$

where

$$a(t) \coloneqq \tanh(2A)\, r_u \circ \phi^t(x) > 0,$$
$$\epsilon(t) \coloneqq \frac{r_u \circ \phi^t(x)}{\cosh(2A)} > 0.$$

Since $A \geq 1$, we have $a > 2\epsilon$. It follows that for every $T > 0$,

$$\begin{aligned}\int_0^T a y_+ \, dt &\leq \int_0^T \epsilon y_- \, dt + y_+(T) + y_0(T) \\ &\leq \frac{1}{2}\int_0^T a y_- \, dt + 2C \\ &\leq \frac{1}{2}\int_0^T \epsilon y_+ \, dt + 3C \\ &\leq \frac{1}{4}\int_0^T a y_+ \, dt + 3C,\end{aligned}$$

hence

$$\int_0^T a y_+ \, dt \leq 4C.$$

However, $a y_+$ is bounded from below by some positive constant, which contradicts the previous inequality for T large enough.

If (α_-, α_+) is only $\mathcal{C}^1$, this strategy still applies to a suitable smoothing of (α_-, α_+) which is sufficiently $\mathcal{C}^1$-close to (α_-, α_+). □

Remark 3.5.7. *We also expect that there exist (unbalanced) ℓAL pairs which are not AL pairs, but the construction seems more delicate.*

3.5.2 Induced Liouville structures

The "standard construction" of Section 3.3.2 yields a pair of contact forms which is both an AL pair and a ℓAL pair (after smoothing). If (α_-, α_+) is a pair of contact forms which is both a Liouville pair and a linear Liouville pair, we can consider two Liouville structures on $\mathbb{R}_s \times M$:

(1) The *completion* $\widehat{\lambda}_{\text{lin}}$ of the Liouville domain $[-1,1]_t \times M$ with the "linear" Liouville form

$$\lambda_{\text{lin}} \coloneqq (1-t)\alpha_- + (1+t)\alpha_+,$$

(2) The Liouville structure induced by the "exponential" Liouville form

$$\lambda_{\text{exp}} \coloneqq e^{-s}\alpha_- + e^s\alpha_+.$$

Here, the completion of a Liouville domain V is obtained by attaching to ∂V the symplectization $[0,\infty) \times \partial V$ of the contact structure at the boundary. This procedure yields an open manifold $\widehat{V}$ with controlled geometry at infinity. See [CE12, Section 11.1] for a precise definition. The next proposition shows in particular that (1) and (2) above produce equivalent Liouville structures on $\mathbb{R} \times M$.

Proposition 3.5.8. *Let (α_-, α_+) be a pair of contact forms which is both a Liouville pair and a linear Liouville pair. The Liouville structures $\widehat{\lambda}_{\text{lin}}$ and λ_{exp} on $\mathbb{R} \times M$ are Liouville homotopic.*

Proof. We choose an arbitrary volume form dvol and we define $f_\pm$, $g_\pm$ as in Lemma 3.5.2. We also choose $A > 0$ such that $f_\pm < A$ and $|g_\pm| < A$. We proceed in 3 steps.

Step 1 : extending λ_{lin}. Let $\epsilon > 0$. We choose a smooth function $\phi = \phi_\epsilon : \mathbb{R} \to \mathbb{R}_{\geq 0}$ satisfying

- $\phi(s) = 0$ for $s \leq -1 - \epsilon$,

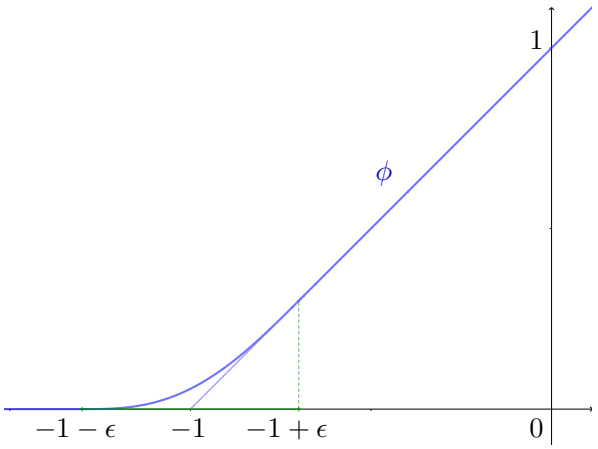


Figure 3.5.1: The function ϕ.

- $\phi(s) > 0$ for $s > -1-\epsilon$,
- $\phi(s) = 1+s$ for $s \geq -1+\epsilon$,
- ϕ is non-decreasing and convex.

We claim that for ϵ sufficiently small, the 1-form

$$\lambda_0 \coloneqq \phi(-s)\alpha_- + \phi(s)\alpha_+$$

on $\mathbb{R}_s \times M$ is Liouville. On $[-1+\epsilon, 1-\epsilon] \times M$, it coincides with λ_{lin} which is Liouville. On $(-\infty, -1-\epsilon] \times M$, it coincides with $(1-s)\alpha_-$ which is Liouville since α_- is a negative contact form. On $[1+\epsilon, \infty) \times M$, it coincides with $(1+s)\alpha_+$ which is Liouville since α_+ is positive contact form. If $s \in [1-\epsilon, 1+\epsilon]$, we compute:

$$\begin{aligned} d\lambda_0 \wedge d\lambda_0 &= \big\{(1+s)\big(f_+ - \phi'(-s)g_+\big) + \phi(-s)\big(\phi'(-s)f_- + g_-\big)\big\}\, ds \wedge \text{dvol} \\ &= F\, ds \wedge \text{dvol}. \end{aligned}$$

Note that

$$0 \leq \phi(-s) \leq \epsilon, \qquad 0 \leq \phi'(-s) \leq 1,$$

hence

$$F \geq (2-\epsilon)\min\{f_+, f_+ - g_+\} + 2\epsilon A,$$

where $f_+ > 0$ and $f_+ - g_+ > 0$ by Lemma 3.5.2, thus $F > 0$ for ϵ small enough. The case $s \in [-1-\epsilon, -1+\epsilon]$ is similar.

Step 2: λ_0 is Liouville homotopic to $\widehat{\lambda}_{\text{lin}}$. We will use the following elementary fact: if two Liouville structures on a manifold with boundary V are Liouville homotopic, then their completions on $\widehat{V}$ are Liouville homotopic; see [CE12, Lemma 11.6]. By definition, $([-1-\epsilon, 1+\epsilon] \times M, \lambda_0)$ is a Liouville domain whose completion is exactly $(\mathbb{R} \times M, \lambda_0)$. Moreover, if ϵ is small enough, then $([-1-\epsilon, 1+\epsilon] \times M, \lambda_0)$ and $([-1+\epsilon, 1-\epsilon] \times M, \lambda_0) = ([-1+\epsilon, 1-\epsilon] \times M, \lambda_{\text{lin}})$ are Liouville domains which are Liouville homotopic (after identifying $[-1-\epsilon, 1+\epsilon]$ and $[-1+\epsilon, 1-\epsilon]$), and $([-1+\epsilon, 1-\epsilon] \times M, \lambda_{\text{lin}})$ and $([-1, 1] \times M, \lambda_{\text{lin}})$ are also Liouville homotopic (after identifying $[-1+\epsilon, 1-\epsilon]$ and $[-1, 1]$). This shows that $([-1-\epsilon, 1+\epsilon] \times M, \lambda_0)$ and $([-1, 1] \times M, \lambda_{\text{lin}})$ are Liouville homotopic (after identifying $[-1-\epsilon, 1+\epsilon]$ with $[-1, 1]$), and so are their completions.

Step 3: λ_0 is Liouville homotopic to λ_{exp}. For $\tau \in [0, 1]$, we set

$$\psi_\tau(s) \coloneqq \tau e^s + (1-\tau)\phi(s),$$

and

$$\lambda_\tau \coloneqq \psi_\tau(-s)\alpha_- + \psi_\tau(s)\alpha_+.$$

The family $\{\lambda_\tau\}_{\tau \in [0,1]}$ interpolates between λ_0 and $\lambda_1 = e^{-s}\alpha_- + e^s\alpha_+$. It is enough to show that for every $\tau \in (0, 1)$, $d\lambda_\tau \wedge d\lambda_\tau$ is a positive volume form on $\mathbb{R} \times M$. By symmetry, it is enough to show it for $s \geq 0$. The computation of $d\lambda_\tau \wedge d\lambda_\tau$ reveals

that the latter is equivalent to

$$f_+ + a_\tau(s)g_- - b_\tau(s)g_+ + a_\tau(s)b_\tau(s)f_- > 0, \tag{3.5.3}$$

where

$$a_\tau(s) \coloneqq \frac{\psi_\tau(-s)}{\psi_\tau(s)}, \qquad b_\tau(s) = \frac{\psi'_\tau(-s)}{\psi'_\tau(s)}.$$

It is easy to check that for $\tau \in (0,1)$ and $s \geq 0$, $0 \leq a_\tau(s) \leq 1$ and $0 \leq b_\tau(s) \leq 1$. Since (α_-, α_+) is both an exponential and a linear Liouville pair, we have that for every $a \in [0,1]$,

$$f_+ + ag_- - ag_+ + a^2 f_- > 0,$$
$$f_+ + ag_- - g_+ + af_- > 0,$$

so for every $a \in [0,1]$ and $b \in [a,1]$, we have

$$f_+ + ag_- - bg_+ + abf_- > 0. \tag{3.5.4}$$

By compactness, there exists $\delta > 0$, only depending on (α_-, α_+), such that for every $a \in [0,1]$ and $b \in [a-\delta, 1]$, the inequality (3.5.4) is satisfied. We claim that for every $\tau \in (0,1)$ and $s \geq 0$, we have

$$b_\tau(s) - a_\tau(s) \geq -\epsilon. \tag{3.5.5}$$

Indeed, fixing $\tau \in (0,1)$, we consider two cases.

Case 1. If $s \in [0, 1-\epsilon) \cup [1+\epsilon, \infty)$,

$$\psi_\tau(s) \geq \psi'_\tau(s),$$
$$\psi'_\tau(-s) \geq \psi_\tau(-s),$$

and (3.5.5) follows trivially since the left-hand side is non-negative.

Case 2. If $s \in [1-\epsilon, 1+\epsilon)$,

$$\psi_\tau(s) = \tau e^s + (1-\tau)(1+s) \geq 1,$$
$$\psi'_\tau(s) = \tau e^s + (1-\tau) \geq 1,$$

and we compute

$$\psi'_\tau(-s)\psi_\tau(s) = \tau^2 + \tau(1-\tau)\left\{(1+s)e^{-s} + e^s\phi'(-s)\right\} + (1-\tau)^2(1+s)\phi'(-s),$$
$$\psi'_\tau(s)\psi_\tau(-s) = \tau^2 + \tau(1-\tau)\left\{e^s\phi(-s) + e^{-s}\right\} + (1-\tau)^2\phi(-s),$$

hence

$$\psi'_\tau(-s)\psi_\tau(s) - \psi'_\tau(s)\psi_\tau(-s) = (1-\tau)\Big\{\tau\Big(\overbrace{e^s\big(\phi'(-s) - \phi(-s)\big) + se^{-s}}^{(1)}\Big) + (1-\tau)\Big(\underbrace{(1+s)\phi'(-s) - \phi(-s)}_{(2)}\Big)\Big\}.$$

Since $\phi'(-s) \geq 0$ and $0 \leq \phi(-s) \leq \epsilon$,

$$(1) \geq -e^{1+\epsilon}\epsilon + (1-\epsilon)e^{-(1+\epsilon)},$$
$$(2) \geq -\epsilon.$$

For ϵ small enough, say $\epsilon \leq 1/100$, $(1) \geq 0$, and (3.5.5) follows.

This shows that for ϵ small enough, only depending on (α_-, α_+), the inequal-

ity (3.5.3) is satisfied for every $\tau \in (0,1)$ and $s \geq 0$. The case $s \leq 0$ can be treated similarly. □

Remark 3.5.9. *As mentioned in the introduction, we do not know if Theorem 3.4.3 and Theorem 3.4.8 are also true for linear Anosov Liouville pairs. The proof of Theorem 3.4.8 would immediately adapt to the linear case, provided that the space of ℓAL pairs supporting a given flow is (weakly) contractible. Our attempts at proving this fact for ℓAL pairs were fruitless because of the complexity and the lack of symmetry of the equations we obtained.*

Appendix

3.A Smoothing lemmas

This appendix concerns useful smoothing lemmas which are required to extend the results of this Chapter to Anosov flows generated by $\mathcal{C}^1$ vector fields, as their weak-stable and weak-unstable bundles are not necessarily $\mathcal{C}^1$. The approach can also be used to bypass Hozoori's delicate approximation techniques in [Hoz22, Section 4]. We state the results in greater generality than needed. M now denotes a closed n-dimensional manifold ($n \geq 1$) and X denotes a non-singular vector field on M of class $\mathcal{C}^k$, $1 \leq k \leq \infty$ (without any Anosovity condition). We fix an arbitrary auxiliary metric on M.

The first smoothing lemma follows from [Hoz22, Lemma 4.3] and the regular approximation of differentiable functions by smooth ones.

Lemma 3.A.1. *Let $f : M \to \mathbb{R}$ be a continuous function which is continuously differentiable along X. Then for every $\epsilon > 0$, there exists a smooth function $f^\epsilon : M \to \mathbb{R}$ satisfying*

$$|f^\epsilon - f|_{\mathcal{C}^0} \leq \epsilon \qquad \text{and} \qquad |X \cdot f^\epsilon - X \cdot f|_{\mathcal{C}^0} \leq \epsilon.$$

In other words, with the notations of Definition 1.1.10, $\mathcal{C}^\infty$ is dense in $\mathcal{C}^0_X$. The same holds with $\mathcal{C}^\ell_X$ in place of $\mathcal{C}^0_X$, for $0 \leq \ell \leq k-1$. We will need a similar result

for 1-forms on M.

Lemma 3.A.2. *The space of $\mathcal{C}^k$ 1-forms on M vanishing along X is dense in $\Omega^1_{X,\ell}$ for $0 \leq \ell \leq k-1$.*

Proof. This is a straightforward adaptation of the proof of [Hoz22, Lemma 4.3].

By compactness of M, we can find a positive real number $\tau > 0$ and a finite collection $\{(U_i, V_i, \phi_i)\}_{1 \leq i \leq N}$ where

- $V_i \subset U_i \subset M$ are open subsets of M,
- $\{V_i\}_{1 \leq i \leq N}$ is a covering of M,
- $\phi_i : U_i \to (-2\tau, 2\tau)_t \times D$ is a $\mathcal{C}^k$ diffeomorphism such that $\phi_i(V_i) = (-\tau, \tau) \times D$ and $d\phi_i(X) = \partial_t$. Here, D denotes the open unit disk in $\mathbb{R}^{n-1}$.

Such a collection can be obtained by taking a finite collection of sufficiently small flow-boxes for X covering M.

Let $\{\psi_i\}_{1 \leq i \leq N}$ be a partition of unity subordinate to the open covering $\{V_i\}_{1 \leq i \leq N}$. For every i, the support of ψ_i is contained in V_i so we can find $0 < r < 1$ such that the support of $\psi_i \circ \phi_i^{-1}$ is contained in $(-\tau, \tau) \times D_r$, where D_r denotes the open disk of radius r.

Let $h : \mathbb{R} \to \mathbb{R}_{\geq 0}$ be a smooth bump function satisfying

- For $|t| \leq \tau$, $h(t) = 1$,
- For $|t| \geq 2\tau$, $h(t) = 0$,
- h is non-decreasing on $(-\infty, 0)$ and non-increasing on $(0, \infty)$.

Let $\alpha \in \Omega^1_{X,\ell}$. By definition, $\alpha = \sum_{i=1}^N \psi_i \alpha$. For every i, we write $\alpha_i \coloneqq \psi_i \alpha$ and $\alpha_i' \coloneqq (\phi_i)_* \alpha_i$. We have reduced the problem to a single flow-box $(-2\tau, 2\tau) \times D$.

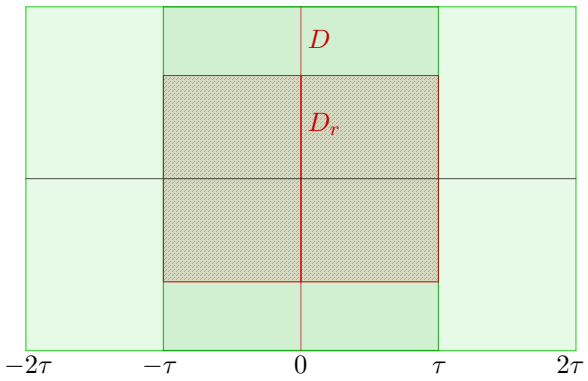


Figure 3.A.1: The nested open sets in the argument. The support of α_i' is contained in the hashed region.

Let $\epsilon > 0$. In what follows, the symbol "$\lesssim$" means "less or equal than up to a constant factor that does not depend on ϵ". For a fixed i, let β_i^0 be a smooth 1-form on D with support contained in D_r satisfying

$$\left|\beta_i^0 - \alpha'_{i|\{0\}\times D}\right|_{\mathcal{C}^\ell} \leq \epsilon.$$

By definition, $\mathcal{L}_{\partial t}\alpha_i'$ is $\mathcal{C}^\ell$ and vanishes along ∂_t. Therefore, we can find a smooth 1-form η_i on $(-2\tau, 2\tau) \times D$ with support contained in $(-\tau, \tau) \times D_r$ satisfying

$$\eta_i(\partial_t) = 0,$$
$$|\eta_i - \mathcal{L}_{\partial t}\alpha_i'|_{\mathcal{C}^\ell} \leq \epsilon.$$

We can extend β_i^0 to a smooth 1-form β_i on $(-2\tau, 2\tau) \times D$ by setting

$$\beta_i(\partial_t) \coloneqq 0,$$
$$\mathcal{L}_{\partial t}\beta_i \coloneqq \eta_i.$$

Finally, we define $\beta_i' \coloneqq h\beta_i$. This is a smooth 1-form with support in $(-2\tau, 2\tau) \times D_r$.

Note that at a point $(t,x) \in (-2\tau, 2\tau) \times D$, we have

$$\left(\beta_i - \alpha_i'\right)_{(t,x)} = \left(\beta_i^0 - \alpha_{i|\{0\}\times D}'\right)_x + \int_0^t \left(\eta_i - \mathcal{L}_{\partial t}\alpha_i'\right)_{(s,x)} ds,$$

so

$$\begin{aligned} |\beta_i - \alpha_i'|_{\mathcal{C}^\ell} &\leq \left|\beta_i^0 - \alpha_{i|\{0\}\times D}'\right|_{\mathcal{C}^\ell} + 2\tau \left|\eta_i - \mathcal{L}_{\partial t}\alpha_i'\right|_{\mathcal{C}^\ell} \lesssim \epsilon, \\ |\mathcal{L}_{\partial t}\beta_i - \mathcal{L}_{\partial t}\alpha_i'|_{\mathcal{C}^\ell} &= |\eta_i - \mathcal{L}_{\partial t}\alpha_i'|_{\mathcal{C}^\ell} \leq \epsilon. \end{aligned}$$

Since the support of α_i' is contained in $(-\tau, \tau) \times D_r$, we readily get

$$\left|\beta_{i|((-2\tau,-\tau)\cup(\tau,2\tau))\times D}\right|_{\mathcal{C}^\ell} \lesssim \epsilon.$$

Moreover,

- On $((-2\tau, -\tau) \cup (\tau, 2\tau)) \times D$,

$$\begin{aligned} \beta_i' - \alpha_i' &= h\beta_i, \\ \mathcal{L}_{\partial t}\beta_i' - \mathcal{L}_{\partial t}\alpha_i' &= (\partial_t h)\beta_i, \end{aligned}$$

- On $(-\tau, \tau) \times D$,

$$\begin{aligned} \beta_i' - \alpha_i' &= \beta_i - \alpha_i', \\ \mathcal{L}_{\partial t}\beta_i' - \mathcal{L}_{\partial t}\alpha_i' &= \eta_i - \mathcal{L}_{\partial t}\alpha_i', \end{aligned}$$

and we obtain

$$\begin{aligned} |\beta_i' - \alpha_i'|_{\mathcal{C}^\ell} &\lesssim \epsilon, \\ |\mathcal{L}_{\partial t}\beta_i' - \mathcal{L}_{\partial t}\alpha_i'|_{\mathcal{C}^\ell} &\lesssim \epsilon. \end{aligned}$$

Finally, we define

$$\beta := \sum_i \phi_i^* \beta_i',$$

so that β is a $\mathcal{C}^k$ 1-form on M satisfying $\beta(X) = 0$, and

$$|\beta - \alpha|_{\mathcal{C}^\ell} \leq \sum_{i=1}^N |\phi_i^* \beta_i' - \phi_i^* \alpha_i'|_{\mathcal{C}^\ell} \lesssim \sum_{i=1}^N |\beta_i' - \alpha_i'|_{\mathcal{C}^\ell} \lesssim \epsilon,$$
$$|\mathcal{L}_X \beta - \mathcal{L}_X \alpha|_{\mathcal{C}^\ell} \leq \sum_{i=1}^N |\phi_i^* (\mathcal{L}_{\partial t} \beta_i') - \phi_i^* (\mathcal{L}_{\partial t} \alpha_i')|_{\mathcal{C}^\ell} \lesssim \sum_{i=1}^N |\mathcal{L}_{\partial t} \beta_i' - \mathcal{L}_{\partial t} \alpha_i'|_{\mathcal{C}^\ell} \lesssim \epsilon.$$

This finishes the proof. □

3.B Almost volume preserving Anosov flows

In this appendix, we prove a technical result used in the proof of Theorem 3.3.15. Let us recall the setup. Φ is a smooth Anosov flow on a closed oriented 3-manifold M, generated by a vector field X. For an adapted metric g, $r_u > 0$ and $r_s < 0$ denote the expansion rates in the unstable and stable directions respectively. The divergence of X for this metric is simply $\mathrm{div}_g X = r_u + r_s$. We say that Φ is **almost volume preserving** if it satisfies one of the following equivalent conditions (compare with (3.3.12)):

(C1) For every $\epsilon > 0$, there exists a smooth function $f_\epsilon : M \to \mathbb{R}$ satisfying

$$|\mathrm{div}_g X + X \cdot f_\epsilon| \leq \epsilon.$$

(C2) For every $\epsilon > 0$, there exists a smooth volume form dvol_ϵ on M satisfying

$$|\mathrm{div}_\epsilon X| = \left| \frac{\mathcal{L}_X \mathrm{dvol}_\epsilon}{\mathrm{dvol}_\epsilon} \right| \leq \epsilon,$$

where $\mathrm{div}_\epsilon X$ denotes the divergence of X with respect to dvol_ϵ.

Proposition 3.B.1. *If Φ is a smooth almost volume preserving Anosov flow on M, then Φ is volume preserving.*

Proof. As noted in the proof of Theorem 3.3.15, it is enough to show that Φ is topologically transitive, i.e., Φ has a dense orbit. We closely follow the strategy from [Mic22] that relies on some key properties of Sinaï–Ruelle–Bowen (SRB) measures for Anosov diffeomorphisms. Here, the time-one map $\phi = \phi^1$ of Φ is not Anosov since it is neither contracting nor expanding in the direction of the flow. Nevertheless, the main arguments of [Mic22] can be adapted to our context with only minor modifications.

A Borel probability measure μ on M preserved by ϕ is a *SRB measure* if it has absolutely continuous conditional measures on unstable manifolds, see [LY85, Definition 1.4.2]. Here, the (strong) unstable manifolds of Φ and ϕ coincide. By [CY05, Theorem C], ϕ admits a SRB measure. Let g be a Riemannian metric on M adapted to Φ (so that $r_s < 0 < r_u$) and μ be any Borel probability measure preserved by ϕ. For μ-a.e. $x \in M$ and for unit vectors $v_u \in E^u(x)$ and $v_s \in E^u(x)$, the *Lyapunov exponents*

$$\Lambda_u(x) \coloneqq \lim_{n\to\pm\infty} \frac{1}{n} \ln \|d\phi^n(v_u)\| = \lim_{n\to\pm\infty} \frac{1}{n} \int_0^n r_u \circ \phi^s(x)\, ds,$$
$$\Lambda_s(x) \coloneqq \lim_{n\to\pm\infty} \frac{1}{n} \ln \|d\phi^n(v_s)\| = \lim_{n\to\pm\infty} \frac{1}{n} \int_0^n r_s \circ \phi^s(x)\, ds,$$

exist, are finite, independent of the metric, and satisfy $\Lambda_s(x) < 0 < \Lambda_u(x)$. The third Lyapunov exponent corresponding to the direction of the flow is 0.

The hypothesis on Φ implies that $\Lambda_u + \Lambda_s = 0$ μ-a.e. Indeed, let $\epsilon > 0$ and choose a smooth (hence bounded) function $f_\epsilon : M \to \mathbb{R}$ such that $|r_u + r_s + X \cdot f_\epsilon| < \epsilon$. Then

μ-a.e., we have

$$\begin{aligned}|\Lambda_u + \Lambda_s| &\leq \limsup_{n\to\infty} \frac{1}{n} \int_0^n |r_u + r_s + X \cdot f_\epsilon| \circ \phi^s \, ds + \limsup_{n\to\infty} \frac{1}{n} \left| \int_0^n (X \cdot f_\epsilon) \circ \phi^s \, ds \right| \\ &\leq \epsilon + \limsup_{n\to\infty} \frac{1}{n} |f_\epsilon \circ \phi^n - f_\epsilon| \\ &= \epsilon.\end{aligned}$$

By [LY85, Theorem A], μ is a SRB measure for ϕ if and only if it satisfies the *Pesin entropy formula*

$$h_\mu(\phi) = \int_M \Lambda_u \, d\mu.$$

Assuming that μ is a SRB measure for ϕ, it follows that

$$h_\mu\big(\phi^{-1}\big) = \int_M -\Lambda_s \, d\mu,$$

so since $-\Lambda_s$ corresponds to the unstable Lyapunov exponent of ϕ^{-1}, μ is also a SRB measure for ϕ^{-1}. By [Led84, Théorème (5.5)], μ is absolutely continuous with respect to the Lebesgue measure on M (induced by some Riemannian metric). If Ω denotes the non-wandering set of Φ, then $\mu(\Omega) = 1$ so the Lebesgue measure of Ω is positive. Smale's spectral decomposition for Ω states that Ω can be partitioned into finitely many *basic sets* $\Omega_1, \ldots, \Omega_N$. Those are compact invariant subsets on which Φ is transitive. It follows that there exists a basic set Ω_i with positive Lebesgue measure. By [BR75, Corollary 5.7 (b)], $\Omega_i = M$ and Φ is transitive. This concludes the proof. □

Remark 3.B.2. *The proof can be adapted to show that any almost volume preserving Anosov flow of class $\mathcal{C}^2$ on any closed manifold is volume preserving. In higher dimension, Λ_u should be replaced by the sum of the positive Lyapunov exponents*

(counted with multiplicity) and Λ_s by the sum of the negative ones.

Remark 3.B.3. *The main result of [Mic22] asserts that an Anosov* diffeomorphism *of class $\mathcal{C}^2$ on a closed manifold M satisfying that at every periodic point, the Poincaré return map has determinant one is volume preserving. The same result remains true for an Anosov flow whose Poincaré return map at every closed orbit has determinant one. As in the proof in [Mic22], this condition and Anosov's* shadowing property *imply that $\Lambda_u + \Lambda_s = 0$ μ-a.e., which is enough to conclude.*

Chapter 4

Floer theory of Anosov flows in dimension three

4.1 Introduction

The study of Lagrangian submanifolds is a central topic in symplectic geometry. These submanifolds (equipped with extra structure and adjectives) are usually packaged as the objects of an A_∞-category, the *Fukaya category*, whose morphisms consist of suitable Floer chain complexes. These are endowed with operations which are based on counts of punctured disks solving Floer's equation, and underlie the rich algebraic structure of the theory. This category is central to the far-reaching homological mirror symmetry conjecture of Kontsevich, a bridge between symplectic and algebraic geometry, which has its origins in string theory.

On the symplectic side, the notion of a *Weinstein manifold*, introduced by Eliashberg and Gromov [EG91] based on a construction of Weinstein [Wei91], gives a particularly rich class of examples of symplectic manifolds, all of which are modelled on the cylindrical end over a contact manifold at infinity. The corresponding theory is the symplectician's approach to Morse theory. The interplay between the geometry

and the dynamics imposes strong topological restrictions on a Weinstein domain, i.e., its homotopy type is that of a CW-complex of at most half the dimension. In particular, except in dimension two, its boundary is always non-empty and connected. The (wrapped) Fukaya category of a Weinstein domain admits a finite collection of building blocks, e.g., it is generated by the unstable manifolds of the critical points of the chosen Morse–Lyapunov function [Cha+22; GPS24]. This is a fundamental structural result for the underlying symplectic geometry of these examples.

On the other hand, there exist examples of exact symplectic manifolds with contact boundary (i.e., Liouville domains) which do not admit a Weinstein structure, and which are especially interesting from a dynamical perspective. Instances of such were constructed in [McD91; Gei95; Mit95; MNW13], all of the form $V = [-1, 1] \times M$ for some closed manifold M, and in particular with disconnected contact-type boundary

$$\partial V = (M_+, \xi_+ = \ker \alpha_+) \sqcup (M_-, \xi_- = \ker \alpha_-),$$

with $M_\pm := \{\pm 1\} \times M$.

4.1.1 Bridge to Anosov theory

We focus our attention on 4-dimensional examples. If a closed oriented 3-manifold M admits an smooth Anosov flow, then $[-1, 1] \times M$ carries a Liouville structure with contact-type boundary by [Hoz22, Theorem 1.1]. This construction generalizes work of Eliashberg–Thurston [ET98] and Mitsumatsu [Mit95, Theorem 3], and builds a bridge between Anosov theory and *bi-contact topology*. The direction of the flow spans the 1-dimensional intersection $\xi_- \cap \xi_+$ of the contact structures at each end. Moreover, the Liouville structure obtained this way only depends on the underlying Anosov flow up to Liouville homotopy, and a 1-parameter family of Anosov flows induces a 1-parameter family of Liouville structures, see [Mas23, Theorem 3]. As a

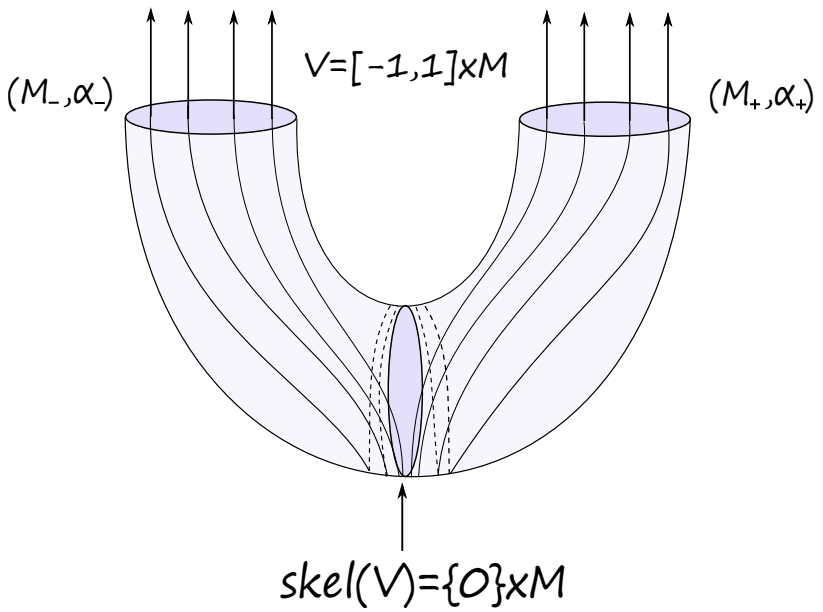
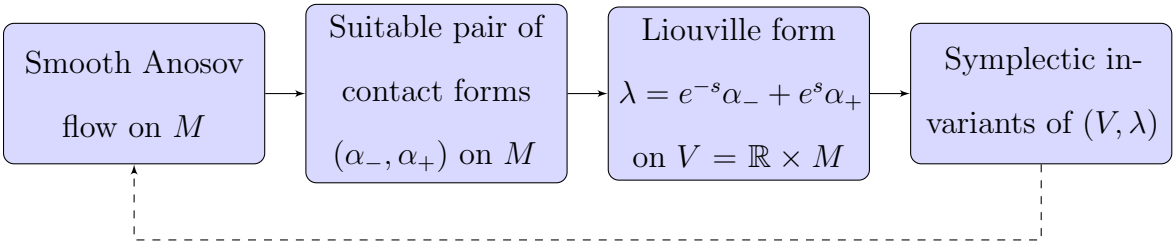


Figure 4.1.1: A Liouville domain of the form $V = [-1, 1] \times M$. For the McDuff and torus bundle domains described below, V retracts onto the skeleton $\mathrm{skel}(V) = M_0 = \{0\} \times M$ under the negative Liouville flow. The Liouville flow is tangent to M_0 and is Anosov on M_0. For the more general constructions in [Hoz22; Mas23], the skeleton is the graph of a $\mathcal{C}^1$ but not necessarily smooth function on M_0; in fact, the regularity of the skeleton is related to the regularity of the weak-stable foliation [Hoz].

result, all the symplectic invariants (e.g., symplectic cohomology, Rabinowitz Floer cohomology, wrapped Fukaya category) of these Liouville structures are *invariants of the underlying Anosov flow*, only depending on its homotopy class in the space of Anosov flows. One may restrict the focus to the class of *smooth* Anosov flows, as Anosov's structural stability implies that every $\mathcal{C}^1$ Anosov flow can be smoothen without changing its homotopy class in the space of Anosov flows, and the new flow is topologically equivalent to the original one. In this Chapter, we initiate a systematic exploration of this bridge between symplectic geometry and Anosov theory, via the study of the wrapped Fukaya categories of the associated Liouville structures. The constructions mentioned above can be summarized as follows:



The first two arrows are constructed in [Hoz22] (and [Mas23] for this specific type of Liouville form; see Chapter 3). The present Chapter focuses on the last box of the flowchart. The dashed arrow represents the bridge to Anosov theory, i.e., the fact that the symplectic invariants are invariants of the flow. Below, we will explicitly compute relevant invariants for the two simplest examples of Anosov flows: suspensions of Anosov diffeomorphisms on the torus, and geodesic flows of hyperbolic surfaces.

4.1.2 The orbit category of Anosov flows

The Liouville structures we consider on $V = \mathbb{R} \times M$ are of the form $\lambda = e^{-s}\alpha_- + e^s\alpha_+$, where α_- and α_+ are two contact forms on M with opposite orientations. Their underlying contact structures $\xi_- = \ker \alpha_-$ and $\xi_+ = \ker \alpha_+$ are transverse and their intersection $\xi_- \cap \xi_+$ is spanned by an Anosov vector field (see Definition 4.2.1). We will call (V, λ) obtained this way an *Anosov Liouville manifold*, and the underlying domain $([-1, 1] \times M, \lambda)$ an *Anosov Liouville domain*.[1]

We may find an abundance of simple closed orbits for the Anosov flow generated by this vector field, and any such orbit Λ is Legendrian for both contact structures. Therefore,

$$\mathcal{L}_\Lambda = \mathbb{R} \times \Lambda \subset V = \mathbb{R} \times M$$

is a strictly exact Lagrangian (i.e., $\lambda|_{\mathcal{L}_\Lambda} \equiv 0$) with Legendrian boundary at infinity in

[1] In this Chapter all Liouville manifolds will be of finite type, i.e., completions of Liouville domains. Therefore, we can freely go back and forth between Liouville domains and manifolds, with the boundary of the Liouville domain corresponding to the ideal boundary (at infinity) of the corresponding Liouville manifold.

$\partial_\infty V = \{\pm\infty\} \times M$.

Next, we consider the full A_∞-subcategory $\mathcal{W}_0(V)$ of the wrapped Fukaya category $\mathcal{W}(V)$ whose objects are the Lagrangians $\{\mathcal{L}_\Lambda\}$; this subcategory contains all information pertaining to the orbits of the Anosov flow. We call $\mathcal{W}_0(V)$ the *orbit category* of the Anosov flow, as we may think of it as the set of simple closed orbits of the flow endowed with the structure of an A_∞-category.[2]

In the case where the skeleton is $M_0 = \{0\} \times M$, the Lagrangian $\mathcal{L}_\Lambda$ is contained in the unstable manifold of the orbit $\Lambda \subset M_0$, i.e., $\mathcal{L}_\Lambda$ retracts to Λ under the negative Liouville flow. This phenomenon generalizes to the case where the skeleton is the graph of a $\mathcal{C}^1$ function $f : M \to \mathbb{R}$ over M_0, with Λ replaced by the graph of f over Λ. Note that in the Weinstein case, a critical point of the Morse function can be viewed as an orbit of the Liouville flow, and the corresponding Lagrangian co-core is the whole unstable manifold. By analogy to the Weinstein case, it is therefore natural to ask whether the collection $\{\mathcal{L}_\Lambda\}$ of these Lagrangians recovers all of the wrapped Fukaya category of V, i.e., whether $\mathcal{W}(V)$ lies in the split-closure of $\mathcal{W}_0(V)$.

For this purpose, one would usually try to appeal to Abouzaid's generation criterion [Abo10], which ensures split-generation under the hypothesis that the *open-closed map*[3]

$$\mathcal{OC} : \mathrm{HH}_{*-2}(\mathcal{W}(V)) \to SH^*(V)$$

hits the unit when restricted to the orbit category $\mathcal{W}_0(V)$. Here, $SH^*(V)$ denotes the symplectic cohomology of V, and $\mathrm{HH}_*(\mathcal{W}(V))$, the Hochschild homology of $\mathcal{W}(V)$. There is always a splitting $SH^*(V) = SH^*_c(V) \oplus SH^*_{nc}(V)$ as $\mathbb{Z}$-modules, where $SH^*_c(V)$ and $SH^*_{nc}(V)$ denote the summands of symplectic cohomology generated by the contractible and non-contractible Hamiltonian orbits, respectively. For a 4-dimensional Anosov Liouville manifold $V = \mathbb{R} \times M$, the contact forms $\alpha_\pm$ at

[2]The objects in $\mathcal{W}(V)$ and $\mathcal{W}_0(V)$ are geometric, i.e., Lagrangians. Concretely, we do not consider the derived category obtained by passing to the split-closure plus the idempotent completion.

[3]We use the grading conventions from [Gan19; GPS20], which differ from [Abo10].

both components of the boundary at infinity can be chosen *hypertight* (see [Hoz22, Theorem 1.11]), i.e., they admit no contractible Reeb orbits, which implies that $SH_c^*(V) \cong H^*(M)$. In particular, $SH_c^*(V)$ contains the unit, lying in the degree zero part $H^0(M)$. We shall prove the following, which is in contrast to the Weinstein case:

Theorem 4.1.1 (Open-closed map)**.** *Let $V = \mathbb{R} \times M$ be a 4-dimensional Anosov Liouville manifold. Let $\mathcal{W}_0(V)$ be the orbit category of the Anosov flow, generated by its simple closed orbits. Then $\mathcal{W}_0(V)$ does **not** satisfy Abouzaid's generation criterion, i.e., the restriction of the open-closed map to $\mathcal{W}_0(V)$,*

$$\mathcal{OC}_0 : \mathrm{HH}_{*-2}(\mathcal{W}_0(V)) \to SH^*(V),$$

*does **not** hit the unit.*

More precisely, there is a splitting $\mathrm{HH}_*(\mathcal{W}_0(V)) = \mathrm{HH}_*^c \oplus \mathrm{HH}_*^{nc}$ such that $\mathcal{OC}_0$ splits as a sum of two maps

$$\mathcal{OC}_0^c : \mathrm{HH}_{*-2}^c \to SH_c^*(V) \cong H^*(M),$$
$$\mathcal{OC}_0^{nc} : \mathrm{HH}_{*-2}^{nc} \to SH_{nc}^*(V),$$

and

$$\mathrm{Im}(\mathcal{OC}_0^c) \subseteq H^2(M;\mathbb{Z}) \oplus H^3(M;\mathbb{Z}).$$

Moreover, there is an isomorphism

$$\mathrm{HH}_*^c \cong \bigoplus_{\Lambda} W_*$$

where $W_* := \mathrm{HH}_*(C^*(S^1))$, the Hochschild homology of the singular cochain dg-algebra of the circle, has infinite rank and is supported in degrees 0 and 1, and the sum runs over the simple closed orbits of the Anosov flow. Hence, the kernel of $\mathcal{OC}_0$ has infinite

rank.

In fact, more is true. The following theorem says that the subcategory $\mathcal{W}_0(V)$ is "maximally non-finitely split-generated". Loosely speaking, it is very large.

Theorem 4.1.2 (Non-finite split-generation)**.** *Let $\mathcal{A}$ be a full A_∞-subcategory of the orbit category $\mathcal{W}_0(V)$, generated by a collection $\mathcal{C}$ of simple closed orbits of the flow. If $L = \mathcal{L}_\Lambda$ is a Lagrangian cylinder corresponding to an orbit $\Lambda \notin \mathcal{C}$, then L is* ***not*** *split-generated by $\mathcal{A}$. In particular,*

1. *Any two Lagrangians $\mathcal{L}_\Lambda, \mathcal{L}_{\Lambda'}$ with $\Lambda \neq \Lambda'$ are* ***not*** *quasi-isomorphic in $\mathcal{W}(V)$,*
2. *$\mathcal{W}_0(V)$ is* ***not*** *split-generated by finitely many objects,*
3. *$\mathcal{W}_0(V)$ is* ***not*** *homologically smooth.*

Here, recall that an A_∞-category is *homologically smooth* if the diagonal bi-module is perfect, i.e., it is split-generated by tensor products of left and right Yoneda modules (see, e.g., [Gan19]). We remark that, although failing to satisfy Abouzaid's criterion, the family $\{\mathcal{L}_\Lambda\}$ might still constitute a split-generating collection for $\mathcal{W}(V)$. If it does split-generate, the above results would imply that the whole category $\mathcal{W}(V)$ is not homologically smooth, and the total open-closed $\mathcal{OC}$ map fails to be an isomorphism. This would be in stark contrast with the Weinstein case, in which it is an isomorphism, a fact originally envisioned by Seidel in his ICM address [Sei02] (see also [Sei09]), and which follows from [Abo10; Gan13; Cha+22; GPS24]. In fact, we do not know how to construct other exact Lagrangians which do not lie in the split-closure of $\mathcal{W}_0(V)$.

Remark 4.1.3. *The following remarks are in order.*

1. *The main geometric input in the proof of Theorems 4.1.1, 4.1.2, and 4.1.11 is the Topological Disk Lemma 4.3.1, which implies that for suitable Floer data, a punctured Floer disk with at least one nonconstant chord at its inputs must*

have a nonconstant chord at its output. This lemma has algebraic consequences beyond the ones stated here, cf. the discussion in §4.7. The proof of this lemma relies crucially on the tautness of the Anosov weak-stable foliation.

2. *The collection $\{\mathcal{L}_\Lambda\}$ constitutes a fairly general class of Lagrangians in V. As mentioned earlier, the $\mathcal{L}_\Lambda$'s are strictly exact, namely the Liouville form vanishes on them, hence the Liouville vector field is everywhere tangent to them. In fact, in the cases where skel$(V) = \{0\} \times M$ is smooth (e.g., for the McDuff and torus bundle domains defined below), any connected, strictly exact Lagrangian in V which is transverse to the skeleton must be one of the $\mathcal{L}_\Lambda$'s. Indeed, any such Lagrangian must intersect the skeleton along a closed orbit and must be invariant under the Liouville flow which is proper away from the skeleton. Furthermore, one could use Proposition 3.5 of [HHL23] to show that any exact Lagrangian which is strictly exact near the skeleton (but not necessarily elsewhere) is actually generated by a finite sub-collection of $\{\mathcal{L}_\Lambda\}$.*

3. *More generally, one can ask whether an arbitrary exact Lagrangian, not necessarily strictly exact near the skeleton, is (split-)generated by the $\mathcal{L}_\Lambda$'s. Assuming the skeleton is smooth, there are no restrictions on the loop arising from a transverse intersection of a (local, non-strictly exact) Lagrangian with the skeleton, beyond the fact that the loop is* non-characteristic*, i.e., nowhere tangent to the* 1*-dimensional characteristic distribution. In particular, any homotopy class of loops is achieved as such a transverse intersection. If one could use the Anosov flow to flow an arbitrary non-characteristic loop in the skeleton to a periodic orbit, then one would have generation for an arbitrary (non-strictly) exact Lagrangian in V. Indeed, this is one approach to proving generation in the Weinstein case, where a generic point in the skeleton converges to a maximum of the Weinstein Morse function [Cha+22]. The difficulty here is dynamical: it*

is not true that arbitrary non-characteristic loops converge to closed orbits under the Anosov flow.

4.1.3 Closed Lagrangians

A further natural question concerns the existence of nontrivial *closed* Lagrangians. We shall address this for some concrete examples, which we now describe.

McDuff domains. The first example of a Liouville domain of the form $(V = [-1,1] \times M, \lambda)$ is due to McDuff [McD91], where M is the unit cotangent bundle of a hyperbolic surface Σ, and V is a subdomain of the cotangent bundle of Σ with symplectic form twisted with a magnetic field. It corresponds to an Anosov Liouville domain where the Anosov flow is (a rotated version of) the geodesic flow on Σ, a prototypical example of Anosov flow. The skeleton M has $\mathrm{SL}(2,\mathbb{R})$-geometry. We will refer to these examples as the *McDuff domains*. The Lagrangians $\mathcal{L}_\Lambda$ described above correspond to the positive conormal bundle of a closed oriented geodesic γ; we denote the corresponding orbit Λ by Λ_γ (the unit conormal lift of γ), and the corresponding Lagrangian by $\mathcal{L}_\gamma$. In Section 4.4.1, we will show the following:

Theorem 4.1.4 (McDuff domains: closed exact Lagrangians)**.** *In every McDuff domain, there exist* $3g-3$ *pairwise disjoint exact Lagrangian tori in distinct homotopy classes, where* g *denotes the genus of* Σ*.*

Indeed, we construct an embedded exact Lagrangian torus $\mathbb{T}_\gamma$ for each closed embedded geodesic $\gamma \subset \Sigma$. Taking $3g-3$ disjoint closed geodesics in a pair of pants decomposition of Σ, we obtain the above. The torus $\mathbb{T}_\gamma$ may be constructed from two Hamiltonian isotopic copies of $\mathcal{L}_\gamma$ by S^1-equivariant versions of Polterovich surgeries, although we provide a simpler and more explicit construction. We also show that the torus $\mathbb{T}_\gamma$ is generated by $\mathcal{L}_\gamma$ as an element of the wrapped Fukaya category, see Lemma 4.4.4. It is also possible to associate *non-exact* (but weakly exact) Lagrangian

tori to each closed embedded geodesic by a much simpler construction; see Section 4.2.2.

Torus bundle domains. The torus bundle domains were first described by Geiges [Gei95] and independently by Mitsumatsu [Mit95]. They correspond to the case of Sol geometry. As a smooth manifold, they are of the form $[-1,1]\times M$, where M is a $\mathbb{T}^2$-bundle over S^1 whose monodromy is given by a hyperbolic matrix $A \in \mathrm{SL}(2,\mathbb{Z})$. These are the Anosov Liouville domains with Anosov flow given by the suspension of A, viewed as an Anosov diffeomorphism of the torus. The Lagrangians $\mathcal{L}_\Lambda$ arise from periodic orbits $\mathcal{O}$ of A; we denote the corresponding orbit Λ by $\Lambda_\mathcal{O}$, and the corresponding Lagrangian by $\mathcal{L}_\mathcal{O}$. The Reeb vector fields of $\alpha_\pm$ on $M_\pm$ give linear flows on each $\mathbb{T}^2$-fiber with slope varying from fiber to fiber.[4]

Theorem 4.1.5 (Torus bundle domains: closed exact Lagrangians)**.** *In every torus bundle domain, there are no closed exact Lagrangian submanifolds which are either orientable, projective planes, or Klein bottles.*

However, similarly to the McDuff domains, the torus bundle domains admit non-exact but weakly exact Lagrangian tori, namely the $\mathbb{T}^2$-fibers.

The McDuff domains (and finite quotients thereof corresponding to unit cotangent bundles of hyperbolic orbifold surfaces) and the torus bundle domains correspond to *algebraic Anosov flows*. Those are the flows generated by Anosov vector fields which are invariant vector fields on manifold quotients of the Lie groups $\widetilde{\mathrm{SL}}\big(2,\mathbb{R}\big)$ (the universal cover of $\mathrm{SL}\big(2,\mathbb{R}\big)$) and Sol^3 (the semi-direct product of $\mathbb{R}$ with $\mathbb{R}^2$ for the action $z\cdot(x,y)=(e^z x, e^{-z}y)$), respectively. Algebraic Anosov flows enjoy several important properties which are relevant in the construction of their associated Liouville structures: they are *volume preserving*, and their stable and unstable foliations are *smooth*. By classical theorems of Ghys [Ghy92; Ghy93], those are essentially the only (volume preserving) Anosov flows with smooth Anosov splitting in dimension 3.

[4] Breen and Christian [BC21] recently proved that the *stabilization* of this domain is Weinstein.

4.1.4 Symplectic cohomology and Rabinowitz Floer cohomology

We are also interested in computing relevant symplectic invariants for Anosov Liouville domains, as they are homotopy invariants for the Anosov flow. We begin with their symplectic cohomology and Rabinowitz Floer cohomology, which can be studied in arbitrary dimension. The hypertightness of the contact boundary, which follows from the tautness of the weak-stable foliation of the Anosov flow for Anosov Liouville domains, plays a crucial role in the computation. Under a suitable assumption on the free homotopy classes of the closed Reeb orbits, we can entirely compute the symplectic cohomology *as a ring*. We exploit the relations between symplectic cohomology and Rabinowitz Floer cohomology to drastically simplify the study of the ring structure on SH^* and reduce it to the ring structure on the Rabinowitz Floer cohomology of the two contact boundary components. The latter are independent of the Liouville filling and can be computed independently of each other.[5]

Theorem 4.1.6 (Rabinowitz Floer cohomology and symplectic cohomology rings)**.** *Let $V = [-1, 1] \times M$ be a Liouville domain of dimension $2n$ such that each boundary component $M_\pm = \{\pm 1\} \times V$ is hypertight. Then its Rabinowitz Floer cohomology splits as a ring,*

$$RFH^*(V) \cong RFH^*(M_-) \oplus RFH^*(M_+).$$

Moreover, its symplectic cohomology splits as a $\mathbb{Z}$-module,

$$SH^*(V) = SH^*_-(V) \oplus SH^*_0(V) \oplus SH^*_+(V),$$

where $SH^*_\pm(V)$ is generated by closed Reeb orbits on $M_\pm$ and $SH^*_0(V) \cong H^*(M)$ is

[5]Note that in the case of Anosov Liouville domains, the skeleton is *not* a stable Hamiltonian hypersurface. It is not possible to apply the usual neck-stretching argument to study holomorphic curves crossing it.

generated by critical points on V.

Assume in addition that the free homotopy classes of positively or negatively parametrized closed Reeb orbits on M_+ are distinct from those on M_-. Then the summands $SH^*_\pm(V)$ and $SH_0(V)$, as well as $SH^*_{0-}(V) = SH^*_-(V) \oplus SH^*_0(V)$ and $SH^*_{0+}(V) = SH^*_0(V) \oplus SH^*_+(V)$, are subrings given by

- $SH^*_0(V) \cong H^*(M)$ *with the cup product;*
- $SH^*_\pm(V) \cong RFH^*_{>0}(M_\pm)$*;*
- $SH^*_{0\pm}(V) \cong RFH^*_{\geq 0}(M_\pm)$*.*

Here $RFH^*_{>0}(M_\pm)$ and $RFH^*_{\geq 0}(M_\pm)$ are the positive and nonnegative action parts of Rabinowitz Floer cohomology, respectively, with the product described in [CO18].

Remark 4.1.7. *Some remarks are in order.*

1. *The assumption on hypertightness for $M_\pm$ holds for all examples of Liouville domains $V = [-1,1] \times M$ that we are aware of. We do not know if the assumption on free homotopy classes of Reeb orbits holds for any Anosov Liouville domain. In Lemma 4.2.7, we verify that it holds for McDuff domains and torus bundle domains.*

2. *We call symplectic/Rabinowitz Floer* cohomology *what is called homology in the papers [CFO10; CO18] and grade it by n minus the Conley–Zehnder index, i.e., degree $*$ is replaced by $n - *$.*

3. *Here, we use the non S^1-equivariant version of symplectic cohomology. If $c_1(V) = 0$ (which is the case in our examples, see Lemma 4.2.4) it has an integer grading, and we fix such a choice.*

4. *In some cases, similar arguments also impose strong restrictions on the higher structures (e.g., A_∞, L_∞-structure) on the symplectic cochain complex. How-*

ever, these can be quite complicated in general, and involve curves having inputs/outputs from both M_+ and M_-.

5. *Under the assumption on the homotopy classes of Reeb orbits, Theorem 4.1.6 implies a rather curious algebraic description for SH^*: $A_0 := SH^*_0$ and $A_\pm := SH^*_{0\pm}$ are sub-$\mathbb{Z}$-algebras of $A := SH^*$, $I_\pm := SH^*_\pm \subset A_\pm \subset A$ are* ideals *such that $I_- \cap I_+ = 0$ and the quotients $A_\pm/I_\pm$ are isomorphic to A_0. Therefore, SH^* is the fiber product of $\mathbb{Z}$-algebras*

$$\begin{array}{ccc} A & \longrightarrow & A_- \\ \downarrow & \lrcorner & \downarrow \\ A_+ & \longrightarrow & A_0 \end{array}$$

where the maps $A_\pm \to A_0$ are the quotient maps by $I_\pm$. This corresponds to "gluing the affine schemes $\mathrm{Spec}(A_-)$ *and* $\mathrm{Spec}(A_+)$ *along the subscheme* $\mathrm{Spec}(A_0)$*".*

For McDuff domains and torus bundle domains, which satisfy the assumptions of Theorem 4.1.6 by Lemma 4.2.7 below, the ring structures can be described more explicitly:

Corollary 4.1.8 (Symplectic cohomology rings of McDuff domains)**.** *Let $V = [-1,1]\times M$ be a McDuff domain modeled on $M = S^*\Sigma$. Then the splitting in Theorem 4.1.6 becomes*

$$SH^*(V) \cong tH^*(M)[t] \oplus H^*(M) \oplus H_{2-*}(\mathcal{L}^{nc}\Sigma),$$

where t is a variable of degree zero representing the S^1-fibre and $\mathcal{L}^{nc}\Sigma$ is the space of non-contractible loops on Σ. The subrings are the following:

- *$SH^*_{0-}(V) \cong H^*(M)[t]$ and $SH^*_-(V) \cong tH^*(M)[t]$ with product as polynomial rings;*

- $SH^*_{0+}(V) \cong \check{H}^{\geq 0}_{2-*}(\mathcal{L}\Sigma)$, *the nonnegative action part of Rabinowitz loop homology with product described in [CHO22];*

- $SH^*_+(V) \cong H_{2-*}(\mathcal{L}^{nc}\Sigma)$ *with the loop product.*

Corollary 4.1.9 (Symplectic cohomology of torus bundle domains)**.** *Let $V = [-1,1] \times M$ be a torus bundle domain. Then the splitting in Theorem 4.1.6 becomes*

$$SH^*(V) \cong \bigoplus_{\Gamma} H^*\big(S^1\big) \oplus H^*(M) \oplus \bigoplus_{\Gamma} H^*\big(S^1\big),$$

where Γ is in bijection with $\mathbb{Q} \cap [0,1)$ and corresponds to the $\mathbb{T}^2$-fibers along which the $\alpha_\pm$-orbits have rational slope (i.e., the rational lines inside the cone of Figure 4.6.5 below).

Remark 4.1.10 (Torus bundle domains: products)**.** *The product in SH^* of two $\alpha_\pm$-orbits γ_1, γ_2 lying in two different $\mathbb{T}^2$-fibers with respective rational slopes s_1, s_2 (measured in a fundamental domain $\mathbb{T}^2 \times [0,\nu)$ of the torus bundle, see below for the construction) can only be a linear combination of orbits lying in $\mathbb{T}^2$-fibers with specific slopes determined by s_1 and s_2. However, we do not know if there are any Floer solutions contributing to this product.*

4.1.5 Wrapped Floer cohomology

The following is an open string version of Theorem 4.1.6; unlike in the closed string case, which holds in more generality, the proof here requires the Topological Disk Lemma 4.3.1 and hence the result holds only for Anosov Liouville domains. For the purpose of exposition, the explicit computations for the McDuff domains and torus bundle domains are carried out in Section 4.6.

Theorem 4.1.11 (Wrapped Floer cohomology)**.** *Let $V = [-1,1] \times M$ be a 4-*

dimensional Anosov Liouville domain. Consider the $\mathbb{Z}$-algebra

$$A = \bigoplus_{\Lambda,\Lambda'} HW^*(\mathcal{L}_\Lambda, \mathcal{L}_{\Lambda'}),$$

where the sum runs over pairs of simple closed orbits of the Anosov flow.

1. *We have a natural splitting of A as a $\mathbb{Z}$-module*

$$A = I_- \oplus A_0 \oplus I_+,$$

where

$$A_0 := \bigoplus_{\Lambda} HW_0^*(\mathcal{L}_\Lambda, \mathcal{L}_\Lambda),$$

$$I_\pm := \bigoplus_{\Lambda,\Lambda'} HW_\pm^*(\mathcal{L}_\Lambda, \mathcal{L}_{\Lambda'}),$$

with $HW_0^*(\mathcal{L}_\Lambda, \mathcal{L}_\Lambda) \cong H^*\big(S^1; \mathbb{Z}\big)$ generated by intersection points of $\mathcal{L}_\Lambda$ with a perturbation of itself, and $HW_\pm^*(\mathcal{L}_\Lambda, \mathcal{L}_{\Lambda'})$ generated by chords from Λ to Λ' in $M_\pm$.

2. *If we let*

$$A_\pm := A_0 \oplus I_\pm,$$

then A_0 and $A_\pm$ are subrings of A, $I_\pm$ are ideals of A such that $I_- \cap I_+ = 0$, and the product on

$$A_0 = \bigoplus_{\Lambda} H^*(\mathcal{L}_\Lambda; \mathbb{Z}) \cong \bigoplus_{\Lambda} H^*(S^1; \mathbb{Z})$$

is the direct sum of the cup product on each $H^*(S^1; \mathbb{Z})$.

Remark 4.1.12. *The $\mathbb{Z}$-algebra A satisfies the same fiber product description as in Remark 4.1.7 (5) in terms of the $\mathbb{Z}$-algebras $A_\pm$, A_0 and the ideals $I_\pm$.*

A word on gradings and coefficients. Anosov Liouville manifolds have vanishing first Chern class since their symplectic tangent bundle is trivial (Lemma 4.2.4), so

their wrapped Fukaya category comes with a natural $\mathbb{Z}$-grading. All of the Lagrangians that we consider are spin, and the Lagrangian cylinders have a canonical spin structure. Throughout this Chapter, we will use $\mathbb{Z}$ coefficients unless stated otherwise (we will need Novikov coefficients when dealing with weakly exact Lagrangians); our results remain valid over $\mathbb{Z}/2\mathbb{Z}$ or $\mathbb{R}$.

4.2 Non-Weinstein Liouville domains

In this section, we review the construction, described by Mitsumatsu [Mit95] and generalized by Hozoori [Hoz22], of Liouville domains with disconnected contact-type boundary associated to an Anosov flow on a 3-manifold. In particular, we will describe the McDuff domains and the torus bundle domains in more detail. In Chapter 3, based on [Mas23], we introduced the notion of *Anosov Liouville structure*, a generalization of the previous constructions, and shows a correspondence (up to homotopy) between these structures and Anosov flows on 3-manifolds. This gives rise to a well-defined class of *Anosov Liouville manifolds* for which our main results apply.

4.2.1 Anosov Liouville manifolds

We will only sketch the main construction and refer to [Hoz22] and Chapter 3 for a more complete exposition. Consider $\phi_t : M \to M$ a smooth Anosov flow generated by a smooth non-singular vector field X on a closed oriented 3-manifold M. Let

$$TM = \langle X \rangle \oplus E^s \oplus E^u \tag{4.2.1}$$

be the Anosov splitting, i.e., it is a *continuous*, $d\phi_t$-invariant splitting of TM such that ϕ_t is exponentially contracting on E^s (the stable subbundle) and exponentially expanding on E^u (the unstable subbundle). More precisely, for some (any) Riemannian metric g on M, there exist constants $C > 0$ and $\mu > 0$ such that for every $v \in E^s$ and

$t \in \mathbb{R}$,

$$\|d\phi_t(v)\| \leq Ce^{-\mu t}\|v\|,$$

and for every $v \in E^u$ and $t \in \mathbb{R}$,

$$\|d\phi_t(v)\| \geq Ce^{\mu t}\|v\|.$$

The metric g is said to be *adapted* to ϕ_t if the constant C above equals 1. An Anosov flow on a closed manifold always admits a smooth adapted Riemannian metric, see [FH19, Proposition 5.1.5].

We will further assume that E^s and E^u are oriented, and that the orientation of M coincides with the orientation of the splitting (4.2.1). This can always be achieved after passing to a suitable double cover of M. We will call such a flow an *oriented* Anosov flow. Denote

$$E^{ws} := \langle X \rangle \oplus E^s, \qquad E^{wu} := \langle X \rangle \oplus E^u$$

the weak-stable and weak-unstable subbundles, respectively. Even though E^s and E^u are only continuous in general, a classical result of Hasselblatt implies that for a smooth Anosov flow in dimension 3, E^{ws} and E^{wu} are of class $\mathcal{C}^1$. They integrate to codimension one foliations $\mathcal{F}^{ws}$ and $\mathcal{F}^{wu}$ called the weak-stable and weak-unstable foliations, respectively, and they are both *taut foliations*.[6]

For an adapted metric g, let e_s and e_u be two unit vector fields of class $\mathcal{C}^1$ spanning

[6]A codimension one foliation on a closed manifold is *taut* if every leaf intersects a transverse circle, i.e., a closed loop transverse to the foliation. By Novikov's theorem, transverse circles to a taut foliation are non-contractible (see [Cal07, Theorem 4.37], noting that taut foliations have no Reeb components). As explained below, there is a Reeb vector field everywhere transverse to $\mathcal{F}^{ws}$. This vector field preserves a (contact) volume form, so the tautness of $\mathcal{F}^{ws}$ follows from [Cal07, Theorem 4.29]. Similarly, $\mathcal{F}^{wu}$ is taut as it is the weak stable foliation of the reverse of the Anosov flow.

E^s and E^u, respectively. There exist continuous functions $r_s, r_u : M \to \mathbb{R}$ such that

$$[X, e_s] = -r_s e_s, \qquad [X, e_u] = -r_u e_u,$$

and since the metric is adapted, $r_s < 0 < r_u$. The functions r_s and r_u are called the *expansion rates* of the flow in the stable and unstable direction, respectively (see [Hoz22, Section 3]). These quantities depend on the choice of the metric.

By duality, there exist 1-forms α_s and α_u of class $\mathcal{C}^1$ with $\ker \alpha_u = E^{ws}, \ker \alpha_s = E^{wu}$, $\alpha_s(e_s) = \alpha_u(e_u) = 1$, and satisfying

$$\mathcal{L}_X \alpha_s = r_s\, \alpha_s, \quad \mathcal{L}_X \alpha_u = r_u\, \alpha_u.$$

One easily checks that

$$\alpha_- := \alpha_u + \alpha_s, \quad \alpha_+ := \alpha_u - \alpha_s, \tag{4.2.2}$$

are 1-forms of class $\mathcal{C}^1$ defining two contact structures $\xi_\pm = \ker \alpha_\pm$ intersecting transversely along $\langle X \rangle$. Moreover, the 1-form λ on $V = \mathbb{R}_s \times M$ given by

$$\lambda := e^{-s}\alpha_- + e^s \alpha_+$$

is a Liouville form (i.e., the *continuous* 2-form $\omega := d\lambda$ is non-degenerate), whose Liouville vector field has a positive ∂_s component for $s > 0$, a negative ∂_s component for $s < 0$, and is tangent to $\{0\} \times M$. Moreover, the Reeb vector fields $R_\pm$ of $\alpha_\pm$ satisfy

$$\alpha_u(R_+) = \alpha_u(R_-) = \frac{-r_s}{r_u - r_s} > 0,$$

hence they are both positively transverse to the weak-stable foliation $\mathcal{F}^{ws}$, so $\alpha_\pm$ are

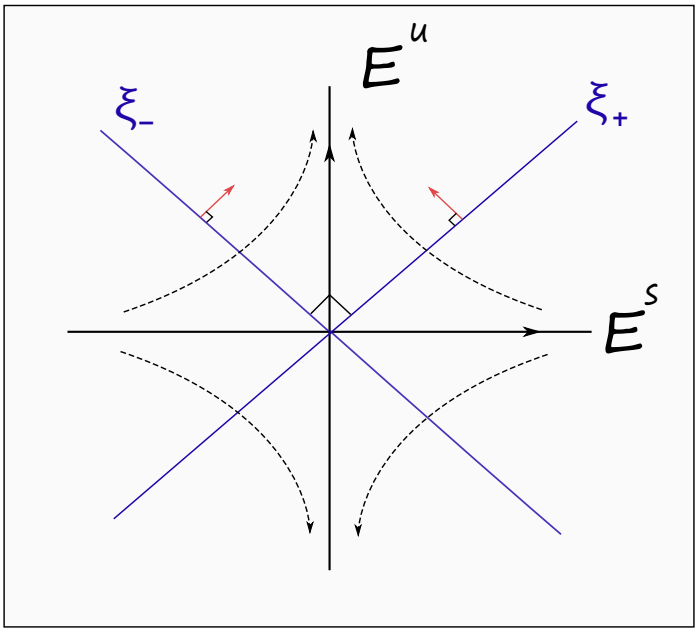


Figure 4.2.1: The Anosov splitting of $N = TM/\langle X \rangle$. The contact structures $\xi_\pm$, when flowed along X, approach E^u. The red arrows show the co-orientations.

hypertight by the tautness of $\mathcal{F}^{ws}$. We call this construction of a ($\mathcal{C}^1$) Anosov Liouville structure the *standard construction*.

At this point, λ is only of class $\mathcal{C}^1$. It can be smoothen to obtain a Liouville structure on the domain $[-A, A] \times M$, for some arbitrary $A > 0$. Being more careful, it is possible to smoothen $\alpha_\pm$ and obtain $\widetilde{\alpha}_\pm$ such their contact structures $\widetilde{\xi}_\pm = \ker \widetilde{\alpha}_\pm$ still intersect transversally along $\langle X \rangle$, and their Reeb vector fields are still positively transverse to $\mathcal{F}^{ws}$, see Chapter 3. The latter property will be crucial in the proofs of Theorem 4.1.1 and Theorem 4.1.2.

This construction fits in the more general framework of the following

Definition 4.2.1. *An* Anosov Liouville structure *on* $\mathbb{R} \times M$ *is a Liouville structure induced by a smooth Liouville form* λ *of the form*

$$\lambda = e^{-s}\alpha_- + e^s \alpha_+ \tag{4.2.3}$$

where (α_-, α_+) *is a pair of contact forms satisfying the following two conditions.*

1. *The contact plane fields* $\xi_\pm = \ker \alpha_\pm$ *are transverse everywhere.*

2. *The line distribution* $\xi_- \cap \xi_+$ *is spanned by an Anosov vector field.*

The pair (ξ_-, ξ_+) constitutes a *bi-contact structure* on M, i.e., a transverse pair of contact plane fields with opposite orientations, meaning that $\alpha_\pm \wedge d\alpha_\pm$ induce volume forms on M with opposite orientations. See Figure 4.2.1.

We say that the Anosov Liouville structure *supports* an Anosov flow ϕ_t if the vector field X generating ϕ_t satisfies $X \in \xi_- \cap \xi_+$ (and the orientations are compatible). The following theorem is proved in Chapter 3.

Theorem 4.2.2. *For any smooth oriented Anosov flow* ϕ_t *on a closed oriented* 3*-manifold* M*, the space of Anosov Liouville structures on* $V = \mathbb{R} \times M$ *supporting* ϕ_t *is non-empty and contractible. Moreover, the map that sends an Anosov Liouville structure to the underlying Anosov flow (up to positive time reparametrization) is a Serre fibration with contractible fibers, hence a homotopy equivalence.*

It follows that any Anosov Liouville structure is Liouville homotopic to one such that the Reeb vector fields of the contact boundaries are both *positively transverse to some taut foliation* (the weak-stable foliation of the underlying Anosov flow). Moreover, the above theorem implies that all the symplectic invariants (symplectic cohomology, wrapped Fukaya category, etc.) of an Anosov Liouville manifold/domain are in fact *invariants of the underlying Anosov flow.* Importantly, the orbit category only depends on the Anosov flow but not on the specific choice of Anosov Liouville structures, since two such structures are homotopic *through Anosov Liouville structures supporting the same flow.* Therefore, the cylinders over the closed orbits of the flow are exact Lagrangians throughout the Liouville homotopy.

Remark 4.2.3. *(cf. footnote 1). In this Chapter, we work with Liouville domains and Liouville manifolds. Anosov Liouville manifolds are finite type Liouville manifolds, and hence are Liouville isomorphic to the completions of Liouville domains obtained*

by truncating them. Conversely, some of our examples are Liouville domains (like the McDuff and torus bundle domains), whose completions are Anosov Liouville manifolds.

We use Anosov Liouville manifolds in Definition 4.2.1 because an Anosov vector field has a canonically associated Liouville manifold, up to exact symplectomorphism. Due to various choices (like the metric and the smoothing), there is not a canonical Liouville domain one could associate to an Anosov flow (and thereby obtain quantitative symplectic invariants from it).

By the following lemma, the wrapped Fukaya category of an Anosov Liouville manifold has a natural $\mathbb{Z}$-grading.

Lemma 4.2.4. *The symplectic tangent bundle (TV, ω) of an Anosov Liouville manifold is trivial. Therefore, $c_1(TV, \omega) = 0$.*

Proof. Let X be a non-singular vector field generating the underlying Anosov flow and θ be a smooth 1-form on M satisfying $\theta(X) = 1$. We define two vector fields X_s and X_θ on $V = \mathbb{R}_s \times M$ by

$$\omega(\cdot, X_s) = ds, \qquad \omega(\cdot, X_\theta) = \theta.$$

From $\omega = d\lambda = ds \wedge (e^s\alpha_+ - e^{-s}\alpha_-) + e^{-s}d\alpha_- + e^s d\alpha_+$, we get

$$\big(e^s\alpha_+ - e^{-s}\alpha_-\big)(X_s) = \omega(\partial_s, X_s) = 1,$$

so replacing θ with $\theta - \theta(X_s)\,(e^s\alpha_+ - e^{-s}\alpha_-)$, we can assume that $\theta(X_s) = 0$. Then, one easily checks that $\{\partial_s, X_s, X, X_\theta\}$ is a symplectic trivialization of (TV, ω). □

As mentioned in the Introduction, an Anosov Liouville manifold $(V = \mathbb{R} \times M, \lambda)$ supporting an Anosov flow ϕ_t admits many interesting *exact* Lagrangian submanifolds obtained from closed orbits of the flow in the following way. Let $\Lambda \subset M$ be a simple closed orbit of ϕ_t and let $\mathcal{L}_\Lambda := \mathbb{R} \times \Lambda \subset V$. Since $T\Lambda \subset \xi_- \cap \xi_+$, Λ is *Legendrian* for

both ξ_- and ξ_+, and $\lambda_{|\mathcal{L}_\Lambda} \equiv 0$, so $\mathcal{L}_\Lambda$ is a *strictly exact* Lagrangian submanifold of V. In particular, the Liouville vector field X_λ is tangent to $\mathcal{L}_\Lambda$, and $\mathcal{L}_\Lambda$ is cylindrical. The boundary at infinity of $\mathcal{L}_\Lambda$ is simply given by two copies of Λ, one in each component of the boundary at infinity of V. Since ϕ_t possesses infinitely many simple closed orbits, we obtain infinitely many objects in the wrapped Fukaya category of (V, λ). These Lagrangians are obviously spin.

It is also worth noting that for an Anosov Liouville structure λ obtained from the standard construction, the leaves of the weak-stable foliation $\mathcal{F}^{ws} \subset \{0\} \times M$ are also strictly exact Lagrangians, i.e., λ vanishes on E^{ws} along $M_0 = \{0\} \times M$. Indeed, $\lambda = 2\alpha_u$ along M_0 by (4.2.2). However, due to classical results in Anosov theory [Ver74], the Lagrangians leaves of $\mathcal{F}^{ws}$ are copies of either $\mathbb{R}^2$ or a cylinder (depending on whether they contain a closed orbit or not). In particular they are not closed submanifolds, or properly embedded with Legendrian boundary, and so do *not* define elements in the wrapped Fukaya category. Besides, the regularity of the foliation $\mathcal{F}^{ws}$ is no better than $\mathcal{C}^1$ in general. Nevertheless, if it is smooth, it is possible to smoothen λ in such a way that the leaves of $\mathcal{F}^{ws}$ are still strictly exact Lagrangians.

Remark 4.2.5. *In [Mit95] and [Hoz22], λ is defined as*

$$(1-t)\alpha_- + (1+t)\alpha_+ \tag{4.2.4}$$

on $[-1,1]_t \times M$, yielding a Liouville domain rather than a Liouville manifold. We warn the reader that there are (at least) two notions of Liouville pairs *in the literature, one using a linear interpolation between α_+ and α_- as in* (4.2.4)*, and the other using the exponential interpolation as in* (4.2.3)*. Although closely related, these two notions are* different*: there exist pairs of contact forms (α_-, α_+) such that $e^{-s}\alpha_- + e^s\alpha_+$ is a Liouville form on $\mathbb{R}_s \times M$, but $(1-t)\alpha_- + (1+t)\alpha_+$ is* not *a Liouville form on $[-1,1]_t \times M$, see [Mas23, Lemma 5.2]. In the present Chapter, we stick to the*

exponential version.

Remark 4.2.6. *In Chapter 3, building upon the work of Hozoori [Hoz22], we defined a space of pairs of contact forms called* Anosov Liouville pairs. *Those are pairs of contact forms (α_-, α_+) such that both (α_-, α_+) and $(-\alpha_-, \alpha_+)$ are Liouville pairs (for the exponential version of the definition* (4.2.3)*). The space of Anosov Liouville pairs is homotopy equivalent to the space of (smooth) Anosov flows on a three manifold. Anosov Liouville pairs give rise to Anosov Liouville structures, but the converse is not necessarily true. Besides, the definition of Anosov Liouville pairs makes no reference to Anosov flows, as opposed to Anosov Liouville structures. In that sense, Anosov Liouville pairs are purely contact topological objects.*

4.2.2 McDuff domains

We now review the construction due to McDuff [McD91], corresponding to manifolds with $\mathrm{SL}(2,\mathbb{R})$ geometry. Let Σ be a closed orientable surface of genus $g \geq 2$. Fix a hyperbolic metric on Σ and let σ be the area form associated to the metric, with total area $-\chi(\Sigma)$. Consider the twisted cotangent bundle

$$(T^*\Sigma, \omega_\sigma = d\lambda_{\mathrm{can}} + \pi^*\sigma),$$

where $\pi : T^*\Sigma \to \Sigma$ is the projection and $\lambda_{\mathrm{can}} = p\,dq$ is the canonical Liouville form. For $0 < a < 1 < b$, consider the subdomain

$$(V, \omega) = (\{p \in T^*\Sigma : a \leq |p| = r \leq b\}, \omega_\sigma|_V) \subset (T^*\Sigma, \omega_\sigma).$$

Since $\pi^*\sigma$ is exact away from the zero section, ω is an exact symplectic form on V, and one checks that the associated Liouville vector field points outwards at each boundary component. Therefore, (V, ω) is a Liouville domain of the form $([a, b] \times S^*\Sigma, d\lambda_\sigma)$,

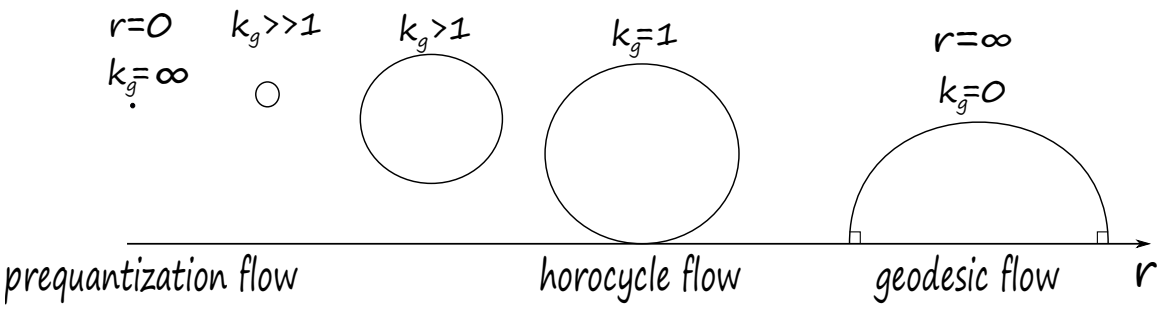


Figure 4.2.2: The characteristic flow at each r-slice M_r, viewed in the universal cover $\mathbb{H}$.

where $\lambda_\sigma = \lambda_{\text{can}} + \lambda_{\text{pre}}$, with $d\lambda_{\text{pre}} = \pi^*\sigma$. The 1-form λ_{pre} is a prequantization form, i.e., a connection for the principal S^1-bundle $\pi : S^*\Sigma \to \Sigma$. The canonical contact structure is $\xi_{\text{can}} = \ker \alpha_{\text{can}}$ on $S^*\Sigma \subset T^*M$, and $\xi_{\text{pre}} = \ker \alpha_{\text{pre}}$ is the S^1-invariant prequantization contact structure on $S^*\Sigma \to \Sigma$, where we denote $\alpha_{\text{can}} = \lambda_{\text{can}}|_{S^*\Sigma}$ the standard contact form on $S^*\Sigma$, and $\alpha_{\text{pre}} = \lambda_{\text{pre}}|_{S^*\Sigma}$.

We may understand this construction in terms of *ideal Liouville domains* [MNW13, Section 4.2]. Observe that $\lambda_\sigma = r\alpha_{\text{can}} + \alpha_{\text{pre}}$, where $r \in (0, +\infty)$. If we denote

$$M_r := \{p \in T^*\Sigma : |p| = r\},$$

we have

$$\omega_\sigma|_{M_r} = rd\alpha_{\text{can}} + d\alpha_{\text{pre}} = r\left(d\alpha_{\text{can}} + \frac{1}{r}d\alpha_{\text{pre}}\right).$$

It follows that the characteristic line field of M_r is the kernel of $d\alpha_{\text{can}} + \frac{1}{r}d\alpha_{\text{pre}}$, whose integral curves correspond to *magnetic geodesics* of geodesic curvature $k_g \equiv 1/r$. This interpolates between the vertical flow generated by the S^1-fiber R_{pre} (the Reeb vector field of α_{pre}), and the geodesic flow generated by R_{can} (the Reeb vector field of α_{can}), going through the horocycle flow at $r = 1$ (which is not a Reeb flow). See Figure 4.2.2.

Identifying $S^*\Sigma$ with $\text{PSL}(2,\mathbb{R})/\pi_1(\Sigma)$, the contact forms α_{can} and α_{pre} can also be obtained as the quotient of two invariant contact forms on $\text{PSL}(2,\mathbb{R})$, see [Gei95]

for more details. Alternatively, α_{can} and α_{pre} lift to contact forms $\widetilde{\alpha}_{\text{can}}$ and $\widetilde{\alpha}_{\text{pre}}$ on the unit cotangent bundle $S^*\mathbb{H}$ of the hyperbolic half-plane $\mathbb{H}$. The usual $\mathrm{PSL}(2,\mathbb{R})$ action on $\mathbb{H}$ induces an action on $T\mathbb{H}$, which induces an action on $T^*\mathbb{H}$ using the musical isomorphism $T\mathbb{H} \cong T^*\mathbb{H}$ for the hyperbolic metric, and this action can be restricted to $S^*\mathbb{H} \subset T^*\mathbb{H}$. Here, we use the bundle metric on $T^*\mathbb{H}$ induced by the hyperbolic bundle metric on $T\mathbb{H}$ by the musical isomorphism. The forms $\widetilde{\alpha}_{\text{can}}$ and $\widetilde{\alpha}_{\text{pre}}$ are both invariant under this $\mathrm{PSL}(2,\mathbb{R})$ action on $S^*\mathbb{H}$. Using the standard coordinates $(x,y) \cong x+iy = z$ on $\mathbb{H}$ and writing $(p_x,p_y) \cong p_x + ip_y = p \in \mathbb{C}$ for the dual coordinates, the bundle metric on $T^*\mathbb{H} \cong \mathbb{H}\times\mathbb{C}$ is simply $y^2|dp|^2$. If $\varphi \in \mathbb{R}/2\pi\mathbb{Z}$ denotes the angular coordinate on the fibers for this metric, we have

$$\widetilde{\alpha}_{\text{can}} = \frac{1}{y}\big(\cos(\varphi)dx + \sin(\varphi)dy\big), \tag{4.2.5}$$

$$\widetilde{\alpha}_{\text{pre}} = \frac{1}{y}dx + d\varphi. \tag{4.2.6}$$

Note that $d\widetilde{\alpha}_{\text{pre}} = \frac{dx\wedge dy}{y^2}$ is the hyperbolic volume form on $\mathbb{H}$. Here we normalize the prequantization form so that it integrates to 2π along the fibers. It will be useful to write these forms using the *Fermi coordinates* (r,t) on $\mathbb{H}$ for the geodesic $\gamma : t \mapsto ie^t$ (see, e.g., [Hum97]). Here, $r,t \in \mathbb{R}$ and in these coordinates, the hyperbolic metric becomes $dr^2 + \cosh(r)^2dt^2$. The usual (x,y) coordinates are related to the Fermi coordinates by

$$(x,y) = \left(\tanh(r)e^t, \frac{1}{\cosh(r)}e^t\right).$$

For a fixed $t_0 \in \mathbb{R}$, the points in $\mathbb{H}$ with Fermi coordinates (r,t_0) for $r \in \mathbb{R}$ describe the geodesic in $\mathbb{H}$ orthogonal to γ and passing through $\gamma(t_0)$. The Fermi coordinates also induce an angular coordinate $\theta \in \mathbb{Z}/2\pi\mathbb{Z}$ on the fibers of $S^*\mathbb{H} \cong \mathbb{R}^2\times S^1$,[7] and

[7]The correspondence between φ and θ is given by $\theta = \varphi + \arctan\sinh(r)$.

the forms $\widetilde{\alpha}_{\text{can}}$ and $\widetilde{\alpha}_{\text{pre}}$ become

$$\widetilde{\alpha}_{\text{can}} = \cos(\theta)dr + \cosh(r)\sin(\theta)dt, \tag{4.2.7}$$

$$\widetilde{\alpha}_{\text{pre}} = \sinh(r)dt + d\theta. \tag{4.2.8}$$

Recall that $\mathrm{PSL}(2,\mathbb{R})$ is generated by the matrices

$$T_\tau := \begin{pmatrix} 1 & \tau \\ 0 & 1 \end{pmatrix}, \quad S := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

where $\tau \in \mathbb{R}$. The action of T_τ on $S^*\mathbb{H}$ in the usual coordinates is given by

$$T_\tau \cdot (z, \varphi) = (z + \tau, \varphi),$$

and the action of S in Fermi coordinates is given by

$$S \cdot (r, t, \theta) = (-r, -t, \theta + \pi).$$

It easily follows that the forms $\widetilde{\alpha}_{\text{can}}$ and $\widetilde{\alpha}_{\text{pre}}$ are invariant under the $\mathrm{PSL}(2,\mathbb{R})$ action on $S^*\mathbb{H}$ and descend to 1-forms on any manifold quotient of $S^*\mathbb{H}$, including $S^*\Sigma$. Moreover, writing $\alpha_+ = \alpha_{\text{can}}$ and $\alpha_- = \alpha_{\text{pre}}$, it follows from the above formulas that

$$\alpha_+ \wedge d\alpha_+ = -\alpha_- \wedge d\alpha_- > 0, \tag{4.2.9}$$

$$\alpha_- \wedge d\alpha_+ = \alpha_+ \wedge d\alpha_- = 0, \tag{4.2.10}$$

where the orientation of $S^*\Sigma$ comes from the canonical contact form α_{can} (i.e., the fibers of $S^*\Sigma$ are *negatively* oriented). In the terminology of [MNW13], (α_-, α_+) is a *Geiges pair*. One can easily check that the Reeb vector fields $R_\pm$ satisfy $R_\pm \in \xi_\mp$. The (ideal) Liouville form $\lambda_\sigma = r\alpha_+ + \alpha_-$ does not quite match Definition 4.2.1, but

it can be modified as follows. For $0 < a < 1 < b$, we consider the 1-parameter family of 1-forms $\{\lambda_t\}_{t\in[0,1]}$ on $[a, b] \times M$ defined by

$$\lambda_t := r\alpha_+ + \left(\frac{t}{r} + 1 - t\right)\alpha_-.$$

For $t = 0$, it is simply λ_σ and for $t = 1$, it corresponds to $e^{-s}\alpha_- + e^s\alpha_+$ on $[\ln a, \ln b] \times M$ after the change of variable $r = e^s$. It easily follows from (4.2.9) and (4.2.10) that $\{\lambda_t\}_t$ is a Liouville homotopy. This deformation allows us to view the McDuff domain as $(V = [-1, 1] \times M, \lambda = e^{-s}\alpha_- + e^s\alpha_+)$.

Alternatively, one can also write

$$\lambda_\sigma = r\alpha_+ + \alpha_- = \sqrt{r}\left(\sqrt{r}\alpha_+ + \frac{1}{\sqrt{r}}\alpha_-\right) = e^s\big(e^s\alpha_+ + e^{-s}\alpha_-\big) = e^s\lambda,$$

where we let $\sqrt{r} = e^s$. Note that the Liouville vector fields of λ and $e^s\lambda$ are positively proportional to each other, and therefore share the same skeleton. Using the Geiges relations (4.2.9) and (4.2.10), one can show that the Liouville vector field of λ is of the form

$$X_\lambda = \tanh(2s)\partial_s - \frac{1}{\cosh(2s)}X,$$

where X is a suitable non-singular vector field on M satisfying $X \in \xi_- \cap \xi_+$. In particular, X is tangent to the underlying Anosov flow and the skeleton of λ is exactly $\{0\} \times M$. Hence, the Liouville vector field X_λ is tangent to the skeleton where it coincides with (the opposite of) the underlying Anosov flow. This implies that the skeleton of λ_σ is exactly $S^*\Sigma \subset T^*\Sigma \setminus 0_\Sigma$.

More generally, one can take further manifold quotients of $S^*\mathbb{H}$ to obtain unit cotangent bundles of hyperbolic orbifold surfaces.

The underlying Anosov flow. For the McDuff domains, the underlying Anosov flow generated by $\xi_- \cap \xi_+$ may be described as follows. Let $G \in \xi_- \subset TM$ be the

vector field generating the geodesic flow, and let X be the Anosov vector field spanning $\xi_- \cap \xi_+$. If we take a complex structure J on Σ which is compatible with the metric and the orientation, then we can lift it to ξ_- so that $JG = X$, which is compatible with orientations. Alternatively, we can view J as a diffeomorphism $J : M \to M$ whose differential satisfies $J_*G = X$, and so if $\Phi : \mathbb{R} \times M \to M$ denotes the geodesic flow, then the flow of X is $\Psi_t = J \circ \Phi_t \circ J^{-1}$. Concretely, this means that the time-t flow of X starting at $(q, v) \in M$ is $(q, v) \mapsto \Psi_t(q, v) = J\Phi_t(q, J^{-1}v)$. Since the geodesic flow preserves angles, we see that an orbit of X through $(q, v) \in M$ is the positive conormal lift of the geodesic $\gamma(t) = \pi(\Phi_t(q, J^{-1}v)) \in \Sigma$ that passes through $(q, J^{-1}v)$, where $\pi : M \to \Sigma$ is the bundle projection.

McDuff domains: exact Lagrangians. The Lagrangians described in the Introduction have a precise description in this case, as the positive halves of the conormal bundles of oriented geodesics. Indeed, consider a closed orbit $\Lambda_\gamma \subset M$ of the Anosov flow, where $\gamma = \pi(\Lambda_\gamma)$ denotes its geodesic footprint in Σ. As we already noted, Λ_γ is Legendrian for both contact structures $\xi_\pm$, hence

$$\mathcal{L}_\gamma = [-1, 1] \times \Lambda_\gamma \subset V = [-1, 1] \times M$$

is an exact Lagrangian which has Legendrian boundary in ∂V. By construction, we have

$$\mathcal{L}_\gamma = \left\{ p \in \gamma^* T^*\Sigma : p(\dot{\gamma}) = 0,\ \det\left(\dot{\gamma}, p^\sharp\right) > 0 \right\} \subset T^*\Sigma \setminus 0_\Sigma$$

the positive conormal bundle of the oriented geodesic $\gamma \subset \Sigma$, where $p^\sharp$ is the image of p under the musical isomorphism $\sharp : T^*\Sigma \to T\Sigma$.

McDuff domains: weakly exact Lagrangian tori. A Lagrangian $L \subset V$ is *weakly exact* if $\omega \cdot \pi_2(V, L) = 0$. While there are no weakly exact closed Lagrangians in the standard end of the McDuff domain (see Lemma 4.4.6), we can find weakly exact Lagrangian tori in the prequantization end, viewed as the positive symplectization

$([0,+\infty) \times S^*\Sigma, d(e^s\alpha_{\text{pre}}))$ of the prequantization form. These tori then lie in the McDuff manifold away from the skeleton. Indeed, consider an embedded geodesic $\gamma \subset \Sigma$, and let

$$T_\gamma = S^*\Sigma|_\gamma \to \gamma$$

viewed as an S^1-bundle over γ. To see that T_γ is Lagrangian, we can take a collar neighbourhood $\mathcal{N}(\gamma) = \gamma \times [-\epsilon, \epsilon] \subset \Sigma$ of γ with coordinates (r,t), so that the prequantization form on $S^*\mathcal{N}(\gamma) = S^1 \times \mathcal{N}(\gamma)$ looks like $\alpha_{\text{pre}} = \sinh(r)dt + d\theta$. Then one readily sees that $d(e^s\alpha_{\text{pre}})$ vanishes on T_γ, which is parametrized by (t,θ) in these coordinates. Since γ and the S^1-fiber are non-contractible in V, we see that $\pi_2(V, T_\gamma) = 0$, which implies that T_γ is weakly exact; note that T_γ is not exact, as α_{pre} is not exact along T_γ. However, in Section 4.4.1, we will construct an exact Lagrangian torus in the isotopy class of T_γ.

4.2.3 Torus bundle domains

We now describe the torus bundle domains, which correspond to closed quotients with Sol geometry.

On $\mathbb{R}^3$ with coordinates (x,y,z), we consider the pair of 1-forms

$$\alpha_\pm := \pm e^z dx + e^{-z} dy. \tag{4.2.11}$$

These form a Geiges pair (they satisfy (4.2.9) and (4.2.10)) whose Reeb vector fields are

$$R_\pm = \frac{1}{2}\left(\pm e^{-z}\partial_x + e^z \partial_y\right). \tag{4.2.12}$$

Moreover, $\alpha_\pm(R_\mp) = \alpha_\pm(\partial_z) = 0$ and so $\xi_\pm = \ker \alpha_\pm = \langle R_\mp, \partial_z \rangle$. On $\mathbb{R}^4 = \mathbb{R}_s \times \mathbb{R}^3$,

we consider the 1-form

$$\begin{aligned}\lambda_0 :=& e^{-s}\alpha_- + e^s\alpha_+ \\ =& 2\sinh(s)e^z dx + 2\cosh(s)e^{-z}dy.\end{aligned}$$

An elementary computation shows that $\omega_0 := d\lambda_0$ is an exact symplectic form on $\mathbb{R}^4$. The Liouville vector field for λ_0 is

$$X_0 = \tanh(2s)\partial_s - \frac{1}{\cosh(2s)}\partial_z. \tag{4.2.13}$$

Let $A \in \mathrm{SL}(2,\mathbb{Z})$ be a positive hyperbolic matrix (satisfying $\mathrm{tr}(A) > 2$). Then there exists $P \in \mathrm{SL}(2,\mathbb{R})$ and $\nu \in \mathbb{R} \setminus \{0\}$ such that

$$PAP^{-1} = \begin{pmatrix} e^\nu & 0 \\ 0 & e^{-\nu} \end{pmatrix} =: D_\nu$$

and we have a commuting diagram

$$\begin{array}{ccc} \mathbb{R}^2/\mathbb{Z}^2 \times \mathbb{R} & \xrightarrow{\phi} & \mathbb{R}^2/\mathbb{Z}^2 \times \mathbb{R} \\ \downarrow{\scriptstyle P\times\mathrm{id}} & & \downarrow{\scriptstyle P\times\mathrm{id}} \\ \mathbb{R}^2/P(\mathbb{Z}^2) \times \mathbb{R} & \xrightarrow{\psi} & \mathbb{R}^2/P(\mathbb{Z}^2) \times \mathbb{R} \end{array}$$

with the maps $\phi(v,z) = (Av, z-\nu)$ and $\psi(v,z) = (D_\nu v, z-\nu)$. The 1-forms $\alpha_\pm$ are invariant under translation by the lattice $P\big(\mathbb{Z}^2\big) \subset \mathbb{R}^2_{x,y}$ as well as under the map ψ, so they descend to the mapping torus $M = ((\mathbb{R}^2/P(\mathbb{Z}^2) \times \mathbb{R}_z)/\psi$. Hence, λ_0 descends to a Liouville form λ on $V = \mathbb{R}_s \times M$. The expression (4.2.13) for the Liouville vector field is still valid in this quotient, so $\mathbb{R}\times M$ is a Liouville domain with disconnected end, and $[-1,1]\times M$ is a Liouville domain with disconnected boundary. The expression (4.2.13) also implies that the skeleton of (V,λ) is $\{0\}\times M$. The Liouville vector field is tangent

to it, where it corresponds to (the opposite of) the underlying Anosov flow. Note that by (4.2.12), $R_\pm$ are tangent to the $\mathbb{T}^2$-fibers and restrict to linear flows on them.

By the diagram above, M is diffeomorphic (via $P \times \mathrm{id}$) to the $\mathbb{T}^2$-bundle over S^1 with monodromy A and the underlying Anosov flow is simply the suspension of the Anosov diffeomorphism of the torus induced by A. In this model, the Reeb vector fields take the form

$$R_\pm(z) = \frac{1}{2}\left(\pm e^{-z} v_x + e^z v_y\right) \tag{4.2.14}$$

with the eigenvectors $v_x = P^{-1}\partial_x$ and $v_y = P^{-1}\partial_y$ of A to the eigenvalues $e^{\pm\nu}$.

Torus bundle domains: exact Lagrangians. The Lagrangians $\mathcal{L}_\Lambda$ described in the Introduction also appear in the torus bundle domains, and admit a more explicit description. Indeed, the hyperbolic matrix $A \in \mathrm{SL}(2,\mathbb{Z})$ naturally acts on $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$, and this action has a lot of periodic points. A finite orbit $\mathcal{O}$ for this action gives rise to a closed orbit $\Lambda = \Lambda_\mathcal{O}$ for the vector field ∂_z on M. $\Lambda_\mathcal{O}$ is then a closed, connected Legendrian submanifold of $(M, \xi_\pm)$ and

$$\mathcal{L}_\mathcal{O} = \mathcal{L}_\Lambda = [-1,1] \times \Lambda_\mathcal{O} \subset V$$

is an exact cylindrical Lagrangian submanifold of (V, λ).

Torus bundle domains: Lagrangian torus fibration. As will be shown, (V, λ) has no closed exact Lagrangian tori, yet it does have interesting non-exact but weakly exact Lagrangian tori. Namely, the torus fibration $M \to S^1$ gives rise to a torus fibration $V \to [-1,1] \times S^1$ whose fibers are Lagrangian. There is no clear analogue of these in the McDuff domains, as the corresponding plane distribution spanned by $\{R_-, R_+\}$ is contact.

Let $\mathbb{T}_{s,z}$ denote the fiber over $(s,z) \in [-1,1] \times S^1$. The fibration homotopy

exact sequence implies that $\pi_2(V, \mathbb{T}_{s,z}) = 0$, so these tori are weakly exact. Their Floer homology is well-defined *over the Novikov ring* $\Lambda_{\mathbb{Z}}$,[8] invariant under Hamiltonian isotopies, and straightforward to compute: $HF^*(\mathbb{T}_{s,z}) = H^*(\mathbb{T}^2; \mathbb{Z}) \otimes \Lambda_{\mathbb{Z}}$, and $HF^*(\mathbb{T}_{s,z}, \mathbb{T}_{s',z'}) = 0$ if $(s, z) \neq (s', z')$. Therefore, these Lagrangian tori are pairwise non-Hamiltonian isotopic. Moreover, the Floer homology of $\mathcal{L}_{\mathcal{O}}$ with any $\mathbb{T}_{s,z}$ is independent of (s, z), given by

$$HF^*(\mathcal{L}_{\mathcal{O}}, \mathbb{T}_{s,z}) = \bigoplus_{p \in \mathcal{O}} \Lambda_{\mathbb{Z}} \cdot p,$$

since $\mathcal{L}_{\mathcal{O}} \cap \mathbb{T}_{s,z} = \mathcal{O}$ and it is easy to show that there are no non-trivial Floer strips between these intersection points. Indeed, consider a Floer strip $u : D \to \mathbb{R} \times M$ joining p to p' with lower boundary u_- on $\mathcal{L}_{\mathcal{O}}$ and upper boundary u_+ on $\mathbb{T}_{s,z}$. Its composition v with the projection $\mathbb{R} \times M \to M$ has lower boundary v_- on $\Lambda_{\mathcal{O}}$ and upper boundary v_+ on $\mathbb{T}_{s,z}$. Note that the projection $\pi : M \to S^1$ maps $\mathbb{T}_{s,z}$ to one point and restricts to a finite covering $\Lambda_{\mathcal{O}} \to S^1$. Thus, $\pi \circ v_-$ is a contractible loop in S^1, and therefore the endpoints of its lift v_- agree, $p = p'$. It follows that v_+ is a contractible loop at p in $\mathbb{T}_{s,z}$, so u is topologically trivial, and therefore constant, and is not counted in the differential.

4.2.4 Homotopy classes of Reeb orbits

The following lemma will be useful for computing the product structure on the symplectic cohomology of the McDuff and torus bundle domains.

Lemma 4.2.7. *Let $V = [-1, 1] \times M$ be a McDuff domain or a torus bundle domain. Then the free homotopy classes of positively or negatively parametrized closed Reeb orbits on M_+ are distinct from those on M_-.*

[8] The Novikov ring $\Lambda_{\mathbb{Z}}$ is defined as the subring of $\mathbb{Z}[\![\mathbb{R}]\!]$ whose elements are of the form $\sum_i n_i t^{a_i}$, where $n_i \in \mathbb{Z}$, $a_i \in \mathbb{R}$, and $a_1 < a_2 < \ldots$, $a_i \to +\infty$.

Proof. Consider first a McDuff domain modeled on $S^*\Sigma$, where Σ is a closed oriented surface of genus $g \geq 2$. Closed Reeb orbits in M_+ are lifts of closed geodesics on Σ, so their free homotopy classes project non-trivially onto Σ, in particular they are nontrivial. The free homotopy classes of positively or negatively parametrized closed Reeb orbits in M_- are nonzero multiples of the fibre class, so they are also nontrivial and distinct from the classes of Reeb orbits in M_+.

Consider next a torus bundle domain M associated to the hyperbolic monodromy matrix A. The lifts of the Reeb vector fields to $\mathbb{T}^2 \times \mathbb{R}$ are $\widetilde{R}_\pm = \frac{1}{2}\big(\pm e^{-z} v_x + e^z v_y \big)$, which are tangent to the torus fibers spanned by v_x, v_y (the chosen eigenvectors of A). Therefore, the closed $R_\pm$-orbits arise whenever $\widetilde{R}_\pm$ has rational slope. From this we see that the free homotopy classes of positively or negatively parametrized Reeb orbits in M_+ and M_- are nontrivial and distinct. □

4.3 Open-closed map and non split-generation

In this section, we prove Theorem 4.1.1 and Theorem 4.1.2 from the Introduction. Both will be consequences of the following elementary topological lemma:

Lemma 4.3.1 (Topological Disk Lemma)**.** *Let X be a $\mathcal{C}^1$ Anosov vector field on a closed 3-manifold M with oriented weak-stable foliation $\mathcal{F}^{ws}$. Then there exists no continuous map $u : \overline{\mathbb{D}} \to M$ satisfying the following conditions (see Figure 4.3.1):*

- *There exist $n \geq 1$ and $a_1 < b_1 < a_2 < \cdots < a_n < b_n < a_{n+1} = a_1$ cyclically ordered points on $\partial\overline{\mathbb{D}}$ such that u is $\mathcal{C}^1$ on $\partial\overline{\mathbb{D}} \setminus \{a_1, b_1, \ldots, a_n, b_n\}$,*
- *u is tangent to X on (a_i, b_i) for every $1 \leq i \leq n$,*
- *u is positively transverse to $\mathcal{F}^{ws}$ on (b_i, a_{i+1}) for every $1 \leq i \leq n$.*

Proof. We argue by contradiction. Let $u : \overline{\mathbb{D}} \to M$ be such a disk and let $\gamma := u_{|\partial\mathbb{D}}$ be its boundary loop. On each interval (a_i, b_i) of $\partial\overline{\mathbb{D}}$, γ is tangent to X and can be

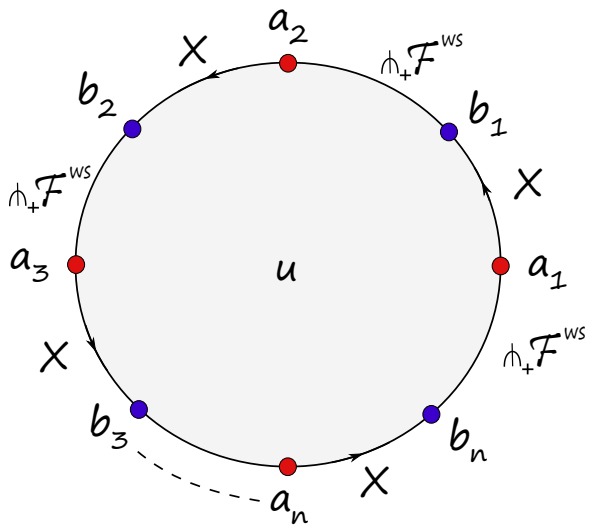


Figure 4.3.1: The tautness of the weak-stable foliation $\mathcal{F}^{ws}$ prevents the existence of disks as above.

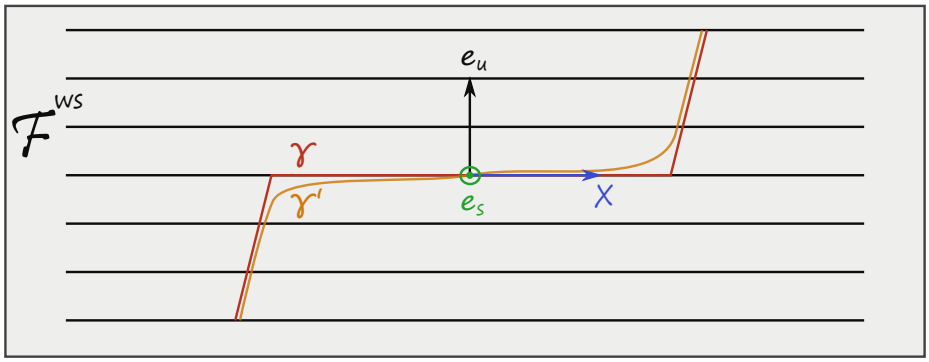


Figure 4.3.2: The smoothen pushoff γ' of γ, transverse to $\mathcal{F}^{ws}$.

modified so that it becomes positively transverse to $\mathcal{F}^{ws}$. Indeed, it suffices to push it along the weak-unstable foliation and smoothen it, see Figure 4.3.2. The new loop γ' obtained this way is still contractible and positively transverse to $\mathcal{F}^{ws}$ *everywhere*, which is impossible since $\mathcal{F}^{ws}$ is taut. □

Proof of Theorem 4.1.1. The models that we use for the wrapped Fukaya category, the symplectic cohomology and the open-closed map are the ones described in [Abo10] (see also [Gan13]). As explained in Section 4.2.1, we can assume that the Reeb vector field on $\partial_\infty V$ is positively transverse to $\mathcal{F}^{ws}$ on both components of $\partial_\infty V$, and this property will still hold after a small perturbation of the Liouville form on the

boundary (to ensure non-degeneracy of the Reeb chords and orbits). The generators of $CW^*(\mathcal{L}_\Lambda, \mathcal{L}_{\Lambda'})$ are given by:

1. Hamiltonian chords from (a small cylindrical Hamiltonian perturbation of) $\mathcal{L}_\Lambda$ to (a small cylindrical Hamiltonian perturbation of) $\mathcal{L}_{\Lambda'}$ for a Hamiltonian that is quadratic at infinity; these correspond to Reeb chords on the contact boundary, and

2. if $\Lambda = \Lambda'$, two intersection points between $\mathcal{L}_\Lambda$ and a small Hamiltonian perturbation of $\mathcal{L}_{\Lambda'}$ corresponding to a small perfect Morse function on the cylinder.

We can further assume that all the generators of type (2) lie in a fixed compact region of V, and all the generators of type (1) lie away from this region.

By definition, the underlying chain complex of $\mathrm{HH}_*(\mathcal{W}_0)$, denoted by $CC_* = CC_*(\mathcal{W}_0)$, is generated by elements of the form $a_1 \otimes a_2 \otimes \cdots \otimes a_d$ where $a_i \in CW^*(\mathcal{L}_{\Lambda_i}, \mathcal{L}_{\Lambda_{i+1}})$.[9] We define CC_*^c as the subgroup of CC_* generated by strings only involving intersection points as in (2) above, and CC_*^{nc} as the subgroup of CC_* generated by strings involving at least one chord as in (1) above. Recall that the differential of CC_* on a generator $a_1 \otimes \cdots \otimes a_d$ has the form:

$$\begin{aligned} d_{CC}(a_1 \otimes \cdots \otimes a_d) = &\sum_{i+j \leq d} \pm a_i \otimes \cdots \otimes a_{i+j} \otimes \mu^{d-j-1}(a_{i+j+1}, \ldots, a_d, a_1, \ldots, a_{i-1}) \\ &+ \sum_{i+j \leq d} \pm a_1 \otimes \cdots \otimes a_{i-1} \otimes \mu^{j+1}(a_i, \ldots, a_{i+j}) \otimes a_{i+j+1} \otimes \cdots \otimes a_d, \end{aligned} \tag{4.3.1}$$

where the μ^k's denote the A_∞ products on $\mathcal{W}$. We decompose the proof into four steps.

Step 1. CC_*^c and CC_*^{nc} are subcomplexes of CC_*.

[9] Some authors use the reverse ordering on the string of generators in order to simplify some sign computations. It won't be relevant here and we opt for the easiest ordering.

Let $a_1 \otimes \cdots \otimes a_d$ be a generator of CC^c_*. By definition, all of the a_i's are intersection points between small perturbations of a single $\mathcal{L}_\Lambda$. A maximum principle argument implies that the A_∞ product $\mu^d(a_1, \ldots, a_d)$ is a linear combination of intersection points only (no chords are involved). This shows that CC^c_* is a subcomplex of CC_*. Let's now assume that $a_1 \otimes \cdots \otimes a_d$ is a generator of CC^{nc}_*, so that one of the $a_i's$ is a chord. For simplicity, let's assume that a_1 is a chord. Then all of the terms in the right hand side of (4.3.1) involve either a_1 or an A_∞ product of a_1 with other generators. Hence, it is enough to establish the following property:

(C) The A_∞ product of generators $a_1, \ldots, a_d$, one of which is a chord, is a linear combination of chords (no self-intersections are involved).

Assume by contradiction that there exist generators $a_1, \ldots, a_d$, one of which is a chord, such that the product $\mu^d(a_1, \ldots, a_d) \in CW^*(\mathcal{L}_{\Lambda_1}, \mathcal{L}_{\Lambda_{d+1}})$ is not a linear combination of chords only (in particular, $\mathcal{L}_{\Lambda_1} = \mathcal{L}_{\Lambda_{d+1}}$). By definition of the A_∞ products, this implies that there exists a Floer disk with positive boundary punctures asymptotic to $a_1, \ldots, a_d$, and a negative boundary puncture asymptotic to a self-intersection point b. We can undo the perturbations of the Lagrangians $\mathcal{L}_{\Lambda_i}$ and reverse the small Hamiltonian isotopies, making the self-intersections disappear, and turn the asymptotics into boundary components. After projecting the new disk to M, we obtain a topological disk whose boundary is alternatively tangent to the Anosov flow (coming from the Lagrangian boundary condition) or to a Reeb orbit (coming from the asymptotics). Since the Reeb flow is positively transverse to $\mathcal{F}^{ws}$, this disk satisfies the assumptions of Lemma 4.3.1, which is impossible. This proves Property (C) and finishes this step.

Step 2. The image of the restriction of $\mathcal{OC}_0$ to HH^{nc}_* is contained in SH^{*+2}_{nc}.

Assume by contradiction that there exist generators $a_1, \ldots, a_d$, one of which is a chord, such that $\mathcal{OC}_0(a_1 \otimes \cdots \otimes a_d)$ is not in SC^{*+2}_{nc} (the subcomplex of SC^{*+2}

generated by non-contractible Hamiltonian orbits). By definition of the open-closed map, this implies that there exists a Floer disk with boundary punctures asymptotic to $a_1, \dots, a_d$ and an interior puncture asymptotic to a contractible closed Hamiltonian orbit x. We cap this puncture and perform the same modifications as in Step 1 to obtain a topological disk satisfying the assumptions of Lemma 4.3.1, which again is impossible.

Step 3. The image of the restriction of $\mathcal{OC}_0$ to HH_*^c is contained in $SH_c^{*+2} = H^{*+2}(M)$.

Since both components of the contact boundary of V are hypertight and remain hypertight after a small perturbation (since the Reeb vector fields on both components are positively transverse to $\mathcal{F}^{ws}$), for a careful choice of Floer data for the definition of SH^*, the subcomplex SC_c^* of SC^* coincides with the Morse chain complex of a Morse function on M, and all the generators of SC_{nc}^* are contained in the cylindrical ends of V. Recall that the generators of CC_*^c involve Lagrangian intersection points contained in a compact subset of V away from the cylindrical ends, so this step is a consequence of [CO18, Lemmas 2.2 and 2.3].

Step 4. The image of $\mathcal{OC}_0^c$ is contained in $H^2(M;\mathbb{Z}) \oplus H^3(M;\mathbb{Z})$.

Since the Lagrangians $\mathcal{L}_\Lambda$ are disjoint, HH_*^c decomposes as

$$\mathrm{HH}_*^c = \bigoplus_\Lambda \mathrm{HH}_*(C^*(\mathcal{L}_\Lambda)) \cong \bigoplus_\Lambda \mathrm{HH}_*(C^*(S^1)), \tag{4.3.2}$$

where $C^*(\mathcal{L}_\Lambda)$ is the dg-algebra of singular cochains on $\mathcal{L}_\Lambda$ and the direct sums run over the simple closed orbits of the Anosov flow. Here, we are using the A_∞ equivalence between the low energy part of the Floer A_∞-algebra $CW^*(\mathcal{L}_\Lambda, \mathcal{L}_\Lambda)$ with $C^*(\mathcal{L}_\Lambda)$ due to Abouzaid [Abo11b, Theorem 1] in the case of closed Lagrangians, which can easily be extended to our setting by the maximum principle. Finally, $W_* := \mathrm{HH}_*(C^*(S^1))$ is

supported in degrees 0 and 1,[10] and Step 4 follows. □

Remark 4.3.2. *With more work, we expect that it is possible to show that*

$$\mathrm{Im}(\mathcal{OC}_0^c) = \Gamma \oplus H^3(M;\mathbb{Z}),$$

where $\Gamma \subseteq H^2(M;\mathbb{Z})$ *is generated by the Poincaré duals of the closed orbits of the Anosov flow.*[11] *Indeed, for every simple closed orbit* Λ *of the flow, the composition*

$$H^*(\Lambda;\mathbb{Z}) = H^*(\mathcal{L}_\Lambda;\mathbb{Z}) \to \mathrm{HH}_*(C^*(\mathcal{L}_\Lambda)) \xrightarrow{\mathcal{OC}_0} SH_c^{*+2} = H^{*+2}(M;\mathbb{Z})$$

is the shriek map $H^*(\Lambda;\mathbb{Z}) \xrightarrow{i^!} H^{*+2}(M;\mathbb{Z})$ *induced by the inclusion* $\Lambda \subset M$ *and its image is exactly* $\langle \mathrm{PD}([\Lambda])\rangle \oplus H^3(M;\mathbb{Z})$. *Therefore,* $\Gamma \oplus H^3(M;\mathbb{Z})$ *is contained in the image of* $\mathcal{OC}_0$. *Moreover,* $\mathcal{OC}_0 : \mathrm{HH}_*(C^*(\mathcal{L}_\Lambda)) \to H^{*+2}(M;\mathbb{Z})$ *should factor as*

$$\mathrm{HH}_*(C^*(\mathcal{L}_\Lambda)) \to H^*(\mathcal{L}_\Lambda;\mathbb{Z}) \xrightarrow{i^!} H^{*+2}(M;\mathbb{Z}),$$

where the first map corresponds to a map $\mathrm{HH}_*(C^*(S^1)) \to H^*(S^1;\mathbb{Z})$ *obtained by composing the "Jones map"* $\mathrm{HH}_*(C^*(S^1)) \to H^*(LS^1,\mathbb{Z})$, *where* LS^1 *denotes the free loop space of* S^1, *with the "restriction to constant loops"* $H^*(LS^1;\mathbb{Z}) \to H^*(S^1;\mathbb{Z})$.

[10] There is an isomorphism of dg-algebras $C^*(S^1) \cong \mathbb{Z}[x]/x^2 =: R$ where $\deg(x) = 1$ and R has no differential. Hence, its Hochschild homology is the Hochschild homology of its underlying graded algebra. To compute it, we can use a graded version of the periodic resolution of R as a R^e-module, where $R^e := R \otimes_{\mathbb{Z}} R$ (see [Wei03, Exercise 9.1.4]). Setting $u := x \otimes 1 - 1 \otimes x \in R^e$ and $v := x \otimes 1 + 1 \otimes x \in R^e$, this resolution is

$$\ldots \xrightarrow{v} R^e \xrightarrow{u} R^e \xrightarrow{v} R^e \xrightarrow{u} R^e \xrightarrow{\mu} R \longrightarrow 0,$$

where $\mu : R^e \to R$ denotes the product. Here, we use *cohomological* grading conventions. After tensoring the resolution with R over R^e, we obtain the periodic complex

$$\ldots \xrightarrow{2x} R \xrightarrow{0} R \xrightarrow{2x} R \xrightarrow{0} R.$$

This complex computes $\mathrm{HH}_*(R,R)$ and is supported in degrees 0 and 1 (recall that we are computing the Hochschild homology of a *graded* algebra). Besides, $\mathrm{HH}_*(R,R)$ has infinite rank.

[11] By a theorem of Fried [Fri82], $\Gamma = H^2(M;\mathbb{Z})$ when the Anosov flow is transitive (in particular, when it is volume preserving).

Hence, the image of $\mathcal{OC}_0$ should be contained in $\Gamma \oplus H^3(M;\mathbb{Z})$. To establish such a factorization, one could degenerate the holomorphic annuli involved in the definition of $\mathcal{OC}_0$ to some Morse flow-trees by sending the Hamiltonian data to 0, and obtain a purely Morse-theoretic model for $\mathcal{OC}_0$. In this model, the desired factorization should be more manifest.

Remark 4.3.3 (Non-contractible component)**.** *It is possible to obtain information on the non-contractible summand $\mathcal{OC}_0^{nc} : \mathrm{HH}_{*-2}^{nc} \to SH_{nc}^*(V)$ of the open-closed map $\mathcal{OC}_0$, by using the fact that $\mathcal{OC}_0$ is a module map for suitable actions of $SH^*(V)$, coinciding with the pair of pants product on the target (see [RS17, Section 8]). Namely, consider the splitting*

$$SH^*(V) = \bigoplus_{\beta \in \pi_1^{\mathrm{free}}(V)} SH_\beta^*(V) = SH_c^*(V) \oplus SH_{nc}^*(V)$$

where $SH_\beta^*(V)$ is generated by Hamiltonian orbits in the free homotopy class β, and $SH_{nc}^*(V) = \bigoplus_{\beta \neq 0} SH_\beta^*(V)$. The pair of pants product satisfies

$$SH^*(V)_{nc} \cdot SH^*(V)_c \subseteq SH^*(V)_{nc}.$$

It follows that

$$SH_{nc}^*(V) \cdot Im(\mathcal{OC}_0^c) \subseteq Im(\mathcal{OC}_0^{nc}). \tag{4.3.3}$$

Moreover, using Lemma 4.3.1, one can explicitly check that the action of $SH^*(V)$ on HH_{*-2} is compatible with the splittings, i.e.,

$$SH_c^*(V) \cdot \mathrm{HH}_{*-2}^c \subseteq \mathrm{HH}_*^c,$$

$$SH_c^*(V) \cdot \mathrm{HH}_{*-2}^{nc} \subseteq \mathrm{HH}_*^{nc},$$

$$SH_{nc}^*(V) \cdot \mathrm{HH}_{*-2}^c \subseteq \mathrm{HH}_*^{nc}.$$

Then, we have

$$SH^*_{nc}(V) \cdot \ker \mathcal{OC}^c_0 \subseteq \ker \mathcal{OC}^{nc}_0. \tag{4.3.4}$$

Note that $SH^*_{nc}(V) = SH^*_-(V) \oplus SH^*_+(V)$, $SH^*_c(V) = H^*(M)$, *by hypertightness of the contact forms on* $M_\pm$. *Under the further assumption that the free homotopy classes of orbits in* M_+ *differ from those in* M_-, *by Theorem 4.1.6 we have*

$$SH^*_{\pm,\beta}(V) \cdot Im(\mathcal{OC}^c_0) \subseteq Im(\mathcal{OC}^{nc}_0) \cap SH^*_{\pm,\beta}(V),$$

where $SH^*_{\pm,\beta}(V)$ *is generated by orbits in* $M_\pm$ *in the non-trivial free homotopy class* $\beta \neq 0$. *Note that there are no invertible elements in* $SH^*_{nc}(V)$, *as* $SH^*_\pm(V)$ *are subrings by Theorem 4.1.6, so that we may not take inverses in the inclusions (4.3.3) and (4.3.4). If the assumption on homotopy classes is dropped, there could* a priori *be invertible elements in* $SH^*_{nc}(V)$. *If this is the case, these are necessarily not in the image of* $\mathcal{OC}^{nc}_0$, *as this is easily seen to contradict Theorem 4.1.1.*

Proof of Theorem 4.1.2. Let $\mathcal{A}$ be a full subcategory of $\mathcal{W}_0(V)$ corresponding to a collection $\mathcal{C}$ of simple closed orbits of the Anosov flow, and $L = \mathcal{L}_\Lambda$, for a simple closed orbit $\Lambda \notin \mathcal{C}$. Let $\mathcal{A}'$ be the full subcategory of $\mathcal{W}_0(V)$ generated by $\mathcal{A}$ and L, and $\mathcal{C}' := \mathcal{C} \cup \{\Lambda\}$. It follows from the proof of Theorem 4.1.1 that $\mathrm{HH}_*(\mathcal{A})$ splits as

$$\mathrm{HH}_*(\mathcal{A}) = \mathrm{HH}^c_*(\mathcal{A}) \oplus \mathrm{HH}^{nc}_*(\mathcal{A}),$$

and furthermore

$$\mathrm{HH}^c_*(\mathcal{A}) = \bigoplus_{\Lambda \in \mathcal{C}} W_*,$$

and similarly for $\mathrm{HH}_*(\mathcal{A}')$. Moreover, the natural map $\iota_* : \mathrm{HH}_*(\mathcal{A}) \to \mathrm{HH}_*(\mathcal{A}')$ induced by the inclusion $\mathcal{A} \subset \mathcal{A}'$ splits as a sum of a map $\mathrm{HH}^{nc}_*(\mathcal{A}) \to \mathrm{HH}^{nc}_*(\mathcal{A}')$ and the map $\mathrm{HH}^c_*(\mathcal{A}) \to \mathrm{HH}^c_*(\mathcal{A}) \oplus W_* = \mathrm{HH}^c_*(\mathcal{A}')$ given by the inclusion of the first factor due to the fact that $\mathcal{L}_\Lambda$ is disjoint from the other Lagrangians in $\mathcal{A}$. Hence, ι_* is not

an isomorphism since W_* is non-zero. If L were split-generated by $\mathcal{A}$, then $\mathcal{A}'$ would be split-generated by $\mathcal{A}$ and ι_* would be an isomorphism (see for instance [GPS20, Lemma 5.2.]).

(1) follows immediately since for $\mathcal{A} := \{\mathcal{L}_\Lambda\}$, $\mathcal{L}_{\Lambda'} \notin \mathcal{A}$ is not split-generated by $\mathcal{L}_\Lambda$ and is a fortiori not isomorphic to $\mathcal{L}_\Lambda$. For (2), if $\mathcal{A}$ is a finite collection of objects of $\mathcal{W}_0$, there exists an object $\mathcal{L}_\Lambda \in \mathcal{W}_0 \setminus \mathcal{A}$, thus $\mathcal{L}_\Lambda$ is not split-generated by $\mathcal{A}$. Finally, a homologically smooth A_∞-category whose morphism chain complexes are *cofibrant* is necessarily finitely split-generated[12] and (3) follows. □

4.4 Closed Lagrangian submanifolds

We now investigate the existence of closed (weakly) exact Lagrangians in the McDuff and torus bundle domains. Note that any closed oriented Lagrangian L in a four dimensional Liouville domain of the form $[-1, 1] \times M$ is necessarily a 2-torus. Indeed,

$$0 = [L] \cdot [L] = -\chi(L),$$

where the first equality follows from the (smooth) displaceability of L and the second equality follows from the Weinstein neighborhood theorem.

4.4.1 Closed Lagrangians in the McDuff domains

In this section, we show that the McDuff domains or manifolds contain *exact* embedded Lagrangian tori, one for each embedded closed geodesic. The idea behind the construction is due to Georgios Dimitroglou Rizell.

Let γ be an oriented simple closed geodesic in Σ of length ℓ and $S^1_\ell = \mathbb{R}/\ell\mathbb{Z}$.

[12]See [GPS20, Section 3.1.1] for the definition and relevance of the cofibrancy assumption; it is used in [GPS24, Lemma A.2] to show the A_∞ version of the Yoneda lemma, which implies that perfect modules are compact objects in the module category, see [GPS23, Section A.3]. Then, the argument of [Rou08, Lemma 3.14], which is stated for field coefficients, can easily be adapted to show finite split-generation for $\mathbb{Z}$ coefficients.

Lemma 4.4.1. *For $\delta > 0$ sufficiently small, there exists a neighborhood U_δ of $T^*\Sigma_{|\gamma} \setminus 0_\Sigma$ diffeomorphic to $V_\delta := T^*C_\delta \setminus 0_{C_\delta} \cong C_\delta \times (\mathbb{R}^2 \setminus \{0\})$, where $C_\delta = (-\delta, \delta) \times S^1_\ell$, with coordinates (r, t, x, y), in which the McDuff Liouville form becomes*

$$\lambda_C = x dr + \big(\cosh(r) y + \sinh(r)\big) dt + d\theta, \tag{4.4.1}$$

where θ denotes the angular coordinate in the (x, y)-plane.

Proof. For $\delta > 0$ sufficiently small, there exists a neighborhood of γ in Σ diffeomorphic to $C_\delta = (-\delta, \delta) \times S^1_\ell$ with Fermi coordinates (r, t) in which the hyperbolic metric writes $dr^2 + \cosh(r)^2 dt^2$, see Section 4.2.2. If (ρ, τ) denotes the dual coordinates of (r, t), the canonical Liouville form on T^*C_δ is $\lambda_{\text{can}} = \rho dr + \tau dt$. Changing the fiber coordinates to $(x, y) := \big(\rho, \frac{\tau}{\cosh(r)}\big)$, the bundle metric on T^*C_δ becomes the standard Euclidean metric $dx^2 + dy^2$ and by (4.2.8), the preqantization contact form writes $\alpha_{\text{pre}} = \sinh(r) dt + d\theta$, where θ is the angular coordinate in the (x, y)-plane. Recall that the McDuff Liouville form is defined as $\lambda_{\text{can}} + \alpha_{\text{pre}}$, so (4.4.1) follows. □

We now construct exact Lagrangian tori in (V_δ, λ_C) using the above coordinates. Let $\epsilon := \tanh(\delta) < 1$ and let $\beta = (f, g) : S^1 \to \mathbb{R}^2 \setminus \{0\}$ be a smooth Jordan curve satisfying the following assumptions:

1. β winds around 0 once positively,
2. β is contained in the strip $\mathbb{R} \times (-\epsilon, \epsilon)$,
3. The domain D bounded by β in $\mathbb{R}^2$ satisfies

$$\int_D \frac{dx \wedge dy}{1 - y^2} = 2\pi.$$

See Figure 4.4.1. We define an embedding $h : \mathbb{T}^2 \cong S^1_\ell \times S^1 \hookrightarrow V_\delta$ by

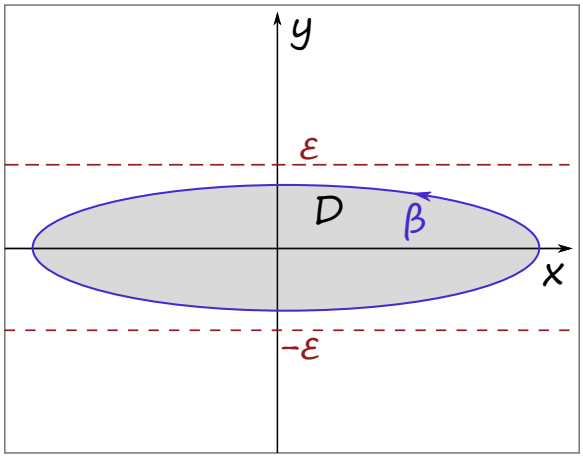


Figure 4.4.1: The Jordan curve β.

$$h(t_1, t_2) := \left(-\tanh^{-1}(g(t_2)), t_1, \beta(t_2)\right).$$

Note that since $|g(t_2)| < \epsilon < 1$, h is well-defined.

Lemma 4.4.2. *The embedding h is an exact Lagrangian embedding.*

Proof. A straightforward computation[13] shows

$$h^*\lambda_C = -(\beta \circ \pi_2)^*\mu + (\beta \circ \pi_2)^* d\theta,$$

where $\mu := \frac{x}{1-y^2}\, dy$ and $\pi_2 : S^1_\ell \times S^1 \to S^1$ is the projection onto the second factor. Note that the 1-form $h^*\lambda_C$ is closed, and it is exact if and only if its integral along $\{0\} \times S^1$ is 0. By Stokes' theorem, the latter is equivalent to

$$-\int_D d\mu + \int_\beta d\theta = -\int_D \frac{dx \wedge dy}{1-y^2} + 2\pi = 0,$$

which is exactly condition (3) above. □

Remark 4.4.3. *This construction yields an exact Lagrangian torus $\mathbb{T}_\gamma$ for every simple closed geodesic γ on Σ which is well defined up to exact Lagrangian isotopy.*

[13] Recall that $\frac{d}{dt}\tanh^{-1}(t) = \frac{1}{1-t^2}$.

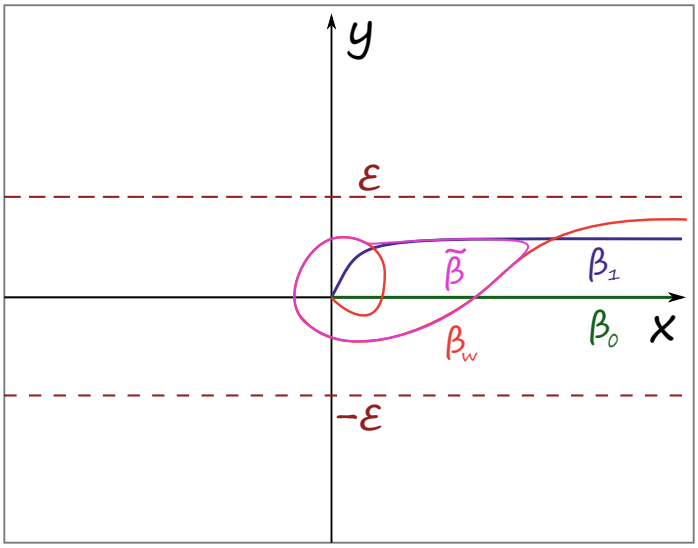


Figure 4.4.2: The curves $\beta_0, \beta_1, \beta_w, \widetilde{\beta}$.

Indeed, any choice of curve β satisfying the conditions (1), (2) and (3) above yields an exact Lagrangian torus and an isotopy of β through curves satisfying these conditions yields a exact Lagrangian isotopy. Moreover, Σ admits $3g-3$ simple closed geodesics that are pairwise disjoint, and the construction can be applied to each of those in such a way that the $3g-3$ exact Lagrangian tori that we obtain are pairwise disjoint. This construction can also be performed along an immersed *geodesic on Σ to obtain an* immersed *exact Lagrangian torus.*

The exact Lagrangian torus $\mathbb{T}_\gamma$ can also be obtained in the following way. Let $\mathcal{L}_\gamma$ be the positive conormal lift of γ in $T^*\Sigma \setminus 0_\Sigma$, which we know is an exact cylindrical Lagrangian. In the above coordinates, it is simply

$$\mathcal{L}_\gamma = \{(0,t,x,0) : t \in S^1_\ell, x \in \mathbb{R}_{>0}\}.$$

Note that for a choice of β in the construction of $\mathbb{T}_\gamma$ as in Figure 4.4.1, $\mathbb{T}_\gamma$ and $\mathcal{L}_\gamma$ intersect cleanly along a circle. For any smooth simple curve $\beta = (f,g) : \mathbb{R}_{>0} \to$

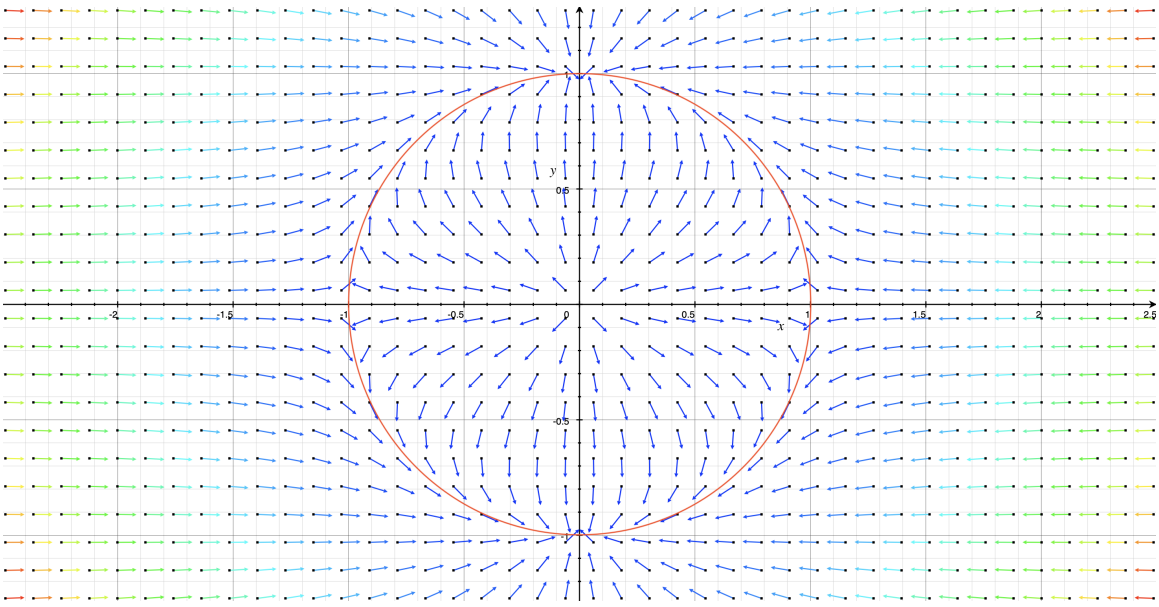


Figure 4.4.3: The vector field X. The locus where $\mathcal{L}_{\gamma;\beta}$ intersects the skeleton corresponds to the unit circle.

$\mathbb{R}^2 \setminus \{0\}$ contained in the strip $\mathbb{R} \times (-\epsilon, \epsilon)$, the subset

$$\mathcal{L}_{\gamma;\beta} := \big\{\big(-\tanh^{-1}(g(s)), t, \beta(s)\big) : s \in \mathbb{R}_{>0},\, t \in S^1_\ell\big\}$$

is an exact Lagrangian submanifold of (V_δ, λ_C). For $\beta_0(s) = (s, 0)$, we have $\mathcal{L}_\gamma = \mathcal{L}_{\gamma,\beta_0}$ as in Figure 4.4.2. Under suitable assumptions on the behavior of β for s near 0 and s near $+\infty$, $\mathcal{L}_{\gamma;\beta}$ is also cylindrical at infinity. To be more precise, let X be the vector field on $\mathbb{R}^2 \setminus \{0\}$ defined by

$$X(x, y) = x\big(1 - x^2 - 2y^2\big)\partial_x + y(1 - y^2)\,\partial_y.$$

Near 0, X essentially behaves like the radial vector field $x\partial_x + y\partial_y$, and for $x \to +\infty$, X essentially behaves like a horizontal vector field (colinear to ∂_x), see Figure 4.4.3. We further assume that

(4a) $\lim_{s\to 0} \beta(s) = 0$ and β is tangent to X for s near 0,

(4b) $\lim_{s\to+\infty} f(s) = +\infty$ and β is tangent to X for s sufficiently large.

Then $\mathcal{L}_{\gamma;\beta}$ is cylindrical at infinity. Indeed, it is enough to check that λ_C restricted to $\mathcal{L}_{\gamma;\beta}$ is zero for s near 0 and s near $+\infty$. On $\mathcal{L}_{\gamma;\beta}$, λ_C restricts to $\rho dr + d\theta$ and it vanishes if and only if the following equation is satisfied:

$$-\frac{fg'}{1-g^2} + \frac{fg' - gf'}{f^2+g^2} = 0.$$

The latter is equivalent to

$$g'f\left(1 - f^2 - 2g^2\right) - f'g(1-g^2) = 0$$

and it is satisfied by the assumptions (4a) and (4b), i.e., we get $g'f' - f'g' = 0$.

For a curve β_1 as in Figure 4.4.2, $\mathcal{L}_{\gamma;\beta_1}$ is Hamiltonian isotopic to $\mathcal{L}_\gamma$ and is disjoint from it. For β_w also as in Figure 4.4.2, $\mathcal{L}_{\gamma;\beta_w}$ is a positive wrapping of $\mathcal{L}_\gamma$ and intersects $\mathcal{L}_{\gamma;\beta_1}$ cleanly along two circles. These intersections can be resolved to obtain $\mathcal{L}_{\gamma;\widetilde{\beta}}$ for some closed curve $\widetilde{\beta}$. For an appropriate choice of wrapping, $\widetilde{\beta}$ satisfies the assumptions (1), (2) and (3) above. To summarize, we obtained $\mathbb{T}_\gamma$ from $\mathcal{L}_\gamma$ and a slight push-off of $\mathcal{L}_\gamma$, by wrapping $\mathcal{L}_\gamma$ passed its push-off and resolving the two S^1-families of intersections. This naturally suggests:

Lemma 4.4.4. *As an object of $\mathcal{W}(V)$, $\mathbb{T}_\gamma$ is generated by $\mathcal{L}_\gamma$.*

Proof. It is well-known that a (non-equivariant) Polterovich surgery induces a cone in the Fukaya category. However, this fact does not immediately extend to our S^1-equivariant setup. Instead, we will use Proposition 3.5 from [HHL23] which relies on *Lagrangian cobordisms*.

Let $R_0 > 1$, to be chosen large enough later, and $0 < r_0 < \epsilon/2$ small enough. We consider a smooth simple closed curve $\beta : \mathbb{R}/2\pi\mathbb{Z} \to \mathbb{R}^2 \setminus \{0\}$ defined in polar

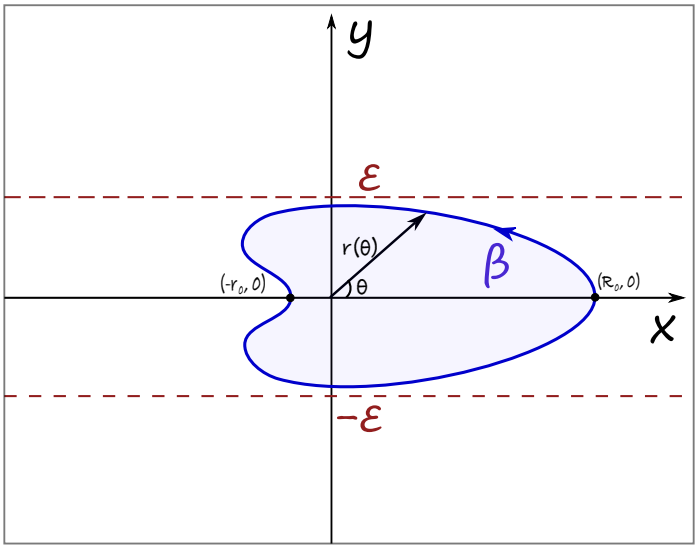


Figure 4.4.4: A special choice of curve β.

coordinates (r, θ) as $\beta(\theta) = (r(\theta), \theta)$,[14] where the function $r : \mathbb{R}/2\pi\mathbb{Z} \to \mathbb{R}_{>0}$ satisfies the following conditions:

(i) r is even, i.e., $r(-\theta) = r(\theta)$ for every $\theta \in \mathbb{R}/2\pi\mathbb{Z}$. In particular, $r'(0) = r'(\pi) = 0$.

(ii) $r(0) = R_0$, $r(\pi) = r_0$.

(iii) r is strictly decreasing on $[0, \pi]$.

(iv) $r''(0) < 0$ and $r''(\pi) > 0$.

(v) β satisfies conditions (2) and (3) above.

It is easy to construct such a function provided that $R_0 > 1$ is large enough; see Figure 4.4.4. By Lemma 4.4.2 above, we know that $\mathcal{L}_{\gamma;\beta}$ is an exact Lagrangian in V which is Hamiltonian isotopic to $\mathbb{T}_\gamma$. Moreover, any primitive $f : \mathcal{L}_{\gamma;\beta} \to \mathbb{R}$ of $\lambda_{|\mathcal{L}_{\gamma;\beta}}$ is induced by a function defined on β. Therefore, we assume that f is of the form $f : \beta \subset \mathbb{R}^2 \to \mathbb{R}$. We normalize it so that $f(\beta(0)) = f(R_0, 0) = 0$. By symmetry and

[14]Here, r denotes the radial coordinate in the (x, y)-plane, and is different from the coordinate on C_δ from Lemma 4.4.1.

since β satisfies condition (3), we have:

$$f(\beta(\pi)) - f(\beta(0)) = \int_0^\pi \beta^*(-\mu + d\theta) = -\frac{1}{2}\left(\int_0^{2\pi} \beta^*\mu\right) + \pi = -\frac{1}{2} \times 2\pi + \pi = 0,$$

hence $f(\beta(\pi)) = f(-r_0, 0) = 0$. Furthermore, one easily computes:

$$\begin{aligned}
\left.\frac{d}{d\theta}\right|_{\theta=0} f(\beta(\theta)) &= -\mu(\beta'(0)) + d\theta(\beta'(0)) = -R_0^2 + 1 < 0, \\
\left.\frac{d}{d\theta}\right|_{\theta=\pi} f(\beta(\theta)) &= -\mu(\beta'(\pi)) + d\theta(\beta'(\pi)) = -r_0^2 + 1 > 0.
\end{aligned}$$

Therefore, for fixed large R_0 and small r_0, we can choose a slightly smaller $R_0' < R_0$ and a slightly bigger $r_0' > r_0$ such that the following hold:

- The circle $C(0, R_0')$ centered at 0 and of radius R_0' intersects β transversally in exactly two points, p_0 and p_1,
- The circle $C(0, r_0')$ centered at 0 and of radius r_0' intersects β transversally in exactly two points, q_0 and q_1,
- $f(p_0) = f(q_0) =: c_0$, $f(p_1) = f(q_1) =: c_1$, and $c_0 < 0 < c_1$,
- There exists a Liouville domain $V_0 \subset V$ whose completion is exact symplectomorphic to V and such that in our coordinates,

$$U_\delta \cap V_0 = U_\delta \cap A\big(r_0', R_0'\big),$$

  where $A\big(r_0', R_0'\big)$ denotes the closed annulus centered at 0 of inner radius r_0' and outer radius R_0'.

For the Liouville domain V_0, we can simply consider the set of points in $T^*\Sigma$ with fiberwise radius between r_0' and R_0'. The intersection $A\big(r_0', R_0'\big) \cap \beta$ has two connected components, $\check{\beta}_0$ and $\check{\beta}_1$, with respective endpoints p_0 and q_0, and p_1 and

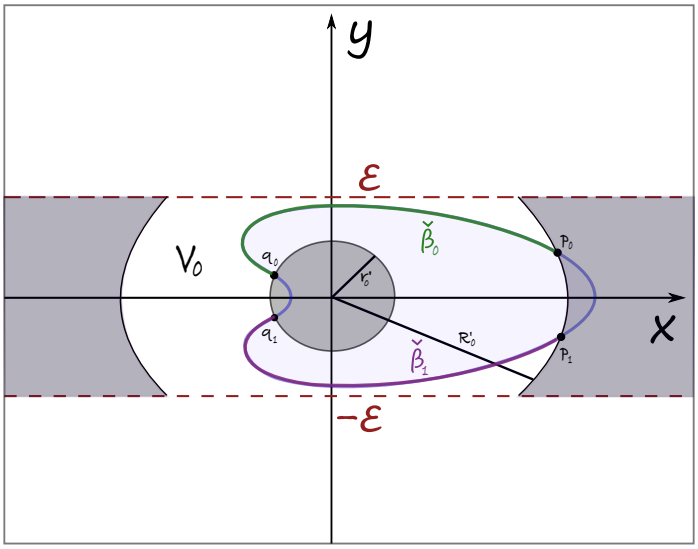


Figure 4.4.5: The Liouville domain V_0.

q_1. See Figure 4.4.5. They induce two exact Lagrangians with Legendrian boundaries $L_0 \coloneqq \mathcal{L}_{\gamma;\check{\beta}_0}$ and $L_1 \coloneqq \mathcal{L}_{\gamma;\check{\beta}_1}$ in V_0, whose completions are exact Lagrangian isotopic to $\mathcal{L}_\gamma$. We can then apply Proposition 3.5 from [HHL23] to deduce that $\mathbb{T}_\gamma$ is isomorphic in the pre-triangulated closure of $\mathcal{W}(V)$ to a cone $[L_1 \to L_0] \cong [\mathcal{L}_\gamma \to \mathcal{L}_\gamma]$, which concludes the proof. □

Notice that $\mathbb{T}_\gamma$ can be obtained from $\mathcal{L}_{\overline{\gamma}}$ by a similar procedure, where $\overline{\gamma}$ denotes γ with the opposite orientation. The previous proof can easily be adapted to show that $\mathbb{T}_\gamma$ is also generated by $\mathcal{L}_{\overline{\gamma}}$. Similarly, one can realize $\mathbb{T}_\gamma$ as a cone of $\mathcal{L}_\gamma$ and $\mathcal{L}_{\overline{\gamma}}$ in different ways. This implies that there exist non trivial relations between cones over $\mathcal{L}_\gamma$ and $\mathcal{L}_{\overline{\gamma}}$ in the wrapped Fukaya category.

Remark 4.4.5. *This construction remains valid for a closed curve β satisfying (1), (2) but not (3). The corresponding Lagrangian torus is incompressible, hence weakly exact, but not exact, and it is disjoint from the skeleton and contained in the prequantization end if β is contained in the open unit disk.*[15] *We expect these tori to*

[15] If $\mathbb{D}$ denotes the open unit disk in the plane,

$$\int_{\mathbb{D}} \frac{dx \wedge dy}{1-y^2} = 2\pi,$$

be Lagrangian (and perhaps Hamiltonian) isotopic to the ones from Section 4.2.2,[16] *and to be split-generated by $\mathcal{L}_\gamma$ in the wrapped Fukaya category of V with Novikov coefficients.*

All of the (weakly) exact Lagrangian tori coming from the previous construction either intersect the skeleton or are entirely contained in the component of the complement of the skeleton corresponding to the prequantization bundle end. In fact, the component of the complement of the skeleton corresponding to the cotangent bundle end cannot contain weakly exact Lagrangian tori:

Lemma 4.4.6. *If Σ is a closed orientable surface of genus $g \geq 2$, then $(T^*\Sigma \setminus 0_\Sigma, \lambda_{\text{can}})$ does not contain any weakly exact Lagrangian tori.*

Proof. We only sketch the main arguments and leave the details to the reader. Assume by contradiction that $T \subset (T^*\Sigma \setminus 0_\Sigma, \lambda_{\text{can}})$ is a weakly exact Lagrangian torus. The image of the morphism $i_T : \pi_1(T) \cong \mathbb{Z}^2 \to \pi_1(S^*\Sigma)$ induced by the inclusion $T \subset T^*\Sigma \setminus 0_\Sigma \cong \mathbb{R} \times S^*\Sigma$ is contained in a subgroup $H_\gamma = \langle \widetilde{\gamma}, f \rangle$ generated by the class of the lift of a loop $\gamma \subset \Sigma$ and the class f a fiber. Indeed, there is a short exact sequence

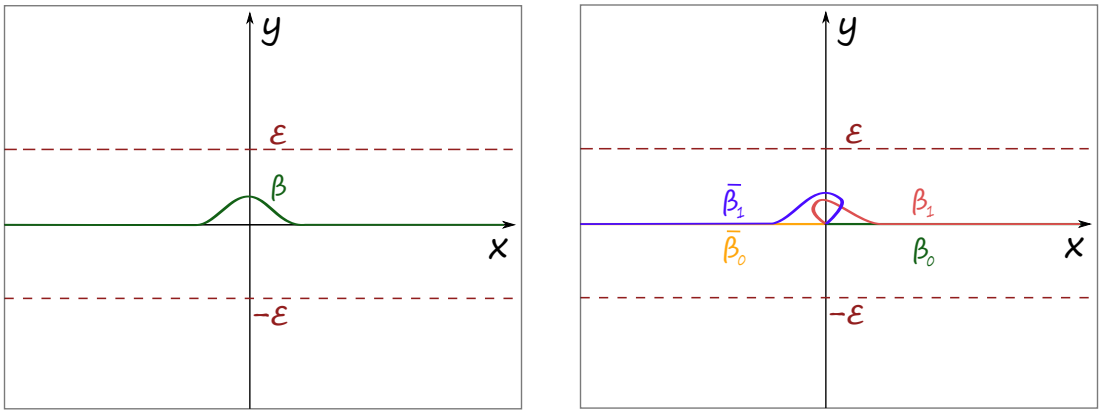
$$1 \longrightarrow \mathbb{Z} \xrightarrow{i_*} \pi_1(S^*\Sigma) \xrightarrow{\pi_*} \pi_1(\Sigma) \longrightarrow 1$$

induced by the homotopy long exact sequence for the fiber bundle $S^1 \overset{i}{\hookrightarrow} S^*\Sigma \overset{\pi}{\to} \Sigma$. Recall that a subgroup of $\pi_1(\Sigma)$ generated by at most two elements is either trivial, infinite cyclic, or free. Therefore, there exists a loop γ on Σ such that the image of the composition $\pi_* \circ i_T$ is contained in $\langle \gamma \rangle$ (where γ is arbitrary if the image is trivial). The short exact sequence readily implies that the image of i_T is contained in H_γ.

It follows from the preceding discussion that T lifts to a weakly exact Lagrangian torus in $(T^*C \setminus 0_C, \lambda_{\text{can}})$, where $C = \mathbb{R} \times S^1$. Here, $\{0\} \times S^1$ projects onto the

so $\mathcal{L}_{\gamma;\beta}$ is *never* exact when β is contained in $\mathbb{D}$.

[16] One could compare these tori by finding an explicit exact symplectomorphism between the symplectization of the prequantization contact form and the McDuff Liouville form in the prequantization end using Moser's trick. Unfortunately, the computations seem quite tedious.



(a) The curve β for the U-shaped Lagrangian. (b) Curves β_0, β_1 and their opposite $\overline{\beta}_0, \overline{\beta}_1$.

Figure 4.4.6: The relevant curves for the U-shaped Lagrangian.

curve γ. Closing up the cylinder C by quotienting the $\mathbb{R}$ direction by a sufficiently large translation, we obtain a *displaceable* weakly exact Lagrangian torus T' in $(T^*\mathbb{T}^2 \setminus 0_{\mathbb{T}^2}, \lambda_{\text{can}})$. This implies that $T' \subset T^*\mathbb{T}^2$ is diplaceable, and by a theorem of Chekanov [Che98] (see also [Oh97]), there exists $A > 0$ such that T' bounds a nontrivial J-holomorphic disk with energy less than A for any tame and convex almost complex structure J. Since T' is weakly exact in $T^*\mathbb{T}^2 \setminus 0_{\mathbb{T}^2}$, these disks necessarily intersect the zero section. Performing a neck-stretching near the zero section, we obtain a sequence of holomorphic disks converging to a holomorphic building of genus 0 such that one of its components is a holomorphic plane in $T^*\mathbb{T}^2$ asymptotic to a closed Reeb orbit and intersecting the zero section. This is impossible since the Reeb orbits in $S^*\mathbb{T}^2$ project to non-contractible curves on $\mathbb{T}^2$. □

We end this section with one last construction. We now take $\beta : \mathbb{R} \to \mathbb{R}^2 \setminus \{0\}$ to be as in Figure 4.4.6a, with $\beta(s) = (s, 0)$ for $|s| \geq 1$. The corresponding Lagrangian cylinder $\mathcal{L}_{\gamma;\beta}$ is exact and intersects the skeleton of V. It has two cylindrical ends in the cotangent bundle end of V which coincide with $\mathcal{L}_\gamma \sqcup \mathcal{L}_{\overline{\gamma}}$, where $\overline{\gamma}$ denotes γ with the opposite orientation. We call it a *U-shaped Lagrangian* associated with γ. It can also be obtained from $\mathcal{L}_\gamma$ and $\mathcal{L}_{\overline{\gamma}}$ by wrapping them in opposite ways in the prequantization end in order to create an clean intersection along a circle, and

then resolving this intersection. See Figure 4.4.6b for the corresponding picture from the point of view of the β curves, where the curve $\overline{\beta}_0(s) = (s, 0)$ for $s \in \mathbb{R}_{<0}$ gives $\mathcal{L}_{\overline{\gamma}} = \mathcal{L}_{\overline{\gamma};\overline{\beta}_0}$, and $\beta_1, \overline{\beta}_1$ are the wrappings of β_0 and $\overline{\beta}_0$ respectively. The proof of Lemma 4.4.4 can easily be adapted to show that, as an object of $\mathcal{W}(V)$, the U-shaped Lagrangian is generated by $\mathcal{L}_\gamma$ and $\mathcal{L}_{\overline{\gamma}}$.

4.4.2 Closed Lagrangians in the torus bundle domains

We now assume that M is a $\mathbb{T}^2$-bundle over S^1 with monodromy given by a positive hyperbolic matrix $A \in \mathrm{SL}(2, \mathbb{Z})$, $(V = \mathbb{R} \times M, \lambda)$ is a torus bundle manifold, and we show Theorem 4.1.5 from the Introduction.

It follows from the homotopy long exact sequence that $\pi_1(V) = \pi_1(M) \cong \mathbb{Z}^2 \rtimes_A \mathbb{Z}$ is the semi-direct product of $\mathbb{Z}^2$ with $\mathbb{Z}$ where $\mathbb{Z}$ acts on $\mathbb{Z}^2$ by A, and this group is torsion-free as it sits in the short exact sequence

$$0 \to \mathbb{Z}^2 \to \mathbb{Z}^2 \rtimes_A \mathbb{Z} \to \mathbb{Z} \to 0.$$

Recall that if K denotes the Klein bottle, $\pi_1(K) = \langle a, b \,|\, aba = b \rangle$.

Lemma 4.4.7. *Let $G := \mathbb{Z}^2 \rtimes_A \mathbb{Z}$ and $H = \mathbb{Z}^2$, $\mathbb{Z}/2\mathbb{Z}$ or $\langle a, b \,|\, aba = b \rangle$. If $\phi : H \to G$ is a group morphism then either $\phi(H) \subseteq \mathbb{Z}^2 \rtimes 0$ or $\phi(H)$ is a cyclic subgroup of G.*

Proof. The case $H = \mathbb{Z}/2\mathbb{Z}$ is trivial since G is torsion-free.

For the case $H = \langle a, b \,|\, aba = b \rangle$, let $\phi(a) := (v, n)$ and $\phi(b) := (w, m)$. The relation $aba = b$ imposes $n = 0$ and $v + A^m v = 0$. Since -1 is not an eigenvalue of A^m by the hyperbolicity of A, we get $v = 0$ so $\phi(a) = 0$. Therefore, $\phi(H)$ is the cyclic subgroup of G generated by $\phi(b)$.

Now let's assume that $H = \mathbb{Z}^2$. It suffices to prove that if $\phi : H \hookrightarrow G$ is injective then its image lies in $G_0 := \mathbb{Z}^2 \rtimes 0$. We identify H with a subgroup of G isomorphic to

$\mathbb{Z}^2$. Denoting $H_0 := H \cap G_0$, there is an injection $H/H_0 \hookrightarrow G/G_0 \cong \mathbb{Z}$ which implies that $H_0 \neq 0$. There are two cases left:

- Case 1: $H_0 \cong \mathbb{Z}$. Then H is generated by two elements $(v, 0)$ and (w, m) where $v \neq 0$ and $m \neq 0$. Since these elements commute, we get: $v + w = w + A^m v$, so $A^m v = v$ and 1 is an eigenvalue of A^m, but this contradicts the hyperbolicity of A.

- Case 2: $H_0 \cong \mathbb{Z}^2$. Then $[H : H_0]$ is finite and must equal 1 because of the injection $H/H_0 \hookrightarrow \mathbb{Z}$, so $H = H \cap H_0 \subseteq \mathbb{Z}^2 \rtimes 0$ as desired.

This concludes the proof. □

We now explicitly describe the cover V' of V associated with the cyclic subgroup generated by an element of the form $(v, 1) \in \mathbb{Z}^2 \rtimes_A \mathbb{Z}$, and the cover V'' of V associated with the normal subgroup $\mathbb{Z}^2 \rtimes 0 \subset \mathbb{Z}^2 \rtimes_A \mathbb{Z}$. To this end, we construct V' and V'' as quotients of $(\mathbb{R}^4, \lambda_0)$ and we show that those are exact symplectomorphic to subsets of suitable cotangent bundles.

Recall that $A \in \mathrm{SL}(2, \mathbb{Z})$ is a hyperbolic matrix and $P \in \mathrm{SL}(2, \mathbb{R})$ and $\nu \in \mathbb{R} \setminus \{0\}$ are such that

$$PAP^{-1} = \begin{pmatrix} e^{\nu} & 0 \\ 0 & e^{-\nu} \end{pmatrix} =: D_{\nu}.$$

For $v_0 \in \mathbb{R}^2$, let

$$\begin{aligned} \phi_{\nu,v_0} : \ \mathbb{R}^2 \times \mathbb{R} &\longrightarrow \mathbb{R}^2 \times \mathbb{R} \\ (v, z) &\longmapsto (D_{\nu} v + v_0, z - \nu), \end{aligned}$$

where $v = (x, y)$. The quotient $\mathbb{R}^3/\phi_{\nu,v_0}$ is an affine plane bundle over S^1 whose monodromy is given by $D_{\nu} + v_0$. The contact forms $\alpha_{\pm}$ are preserved by ϕ_{ν,v_0} so they descend to contact forms on $\mathbb{R}^3/\phi_{\nu,v_0}$, and λ_0 descends to a Liouville form λ' on $V' := \mathbb{R} \times \mathbb{R}^3/\phi_{\nu,v_0}$.

Lemma 4.4.8. *(V', λ') is exact symplectomorphic to $(T^*(\mathbb{R} \times S^1), \lambda_{\text{can}})$.*

Proof. We first trivialize the affine plane bundle $\mathbb{R}^3/\phi_{\nu,v_0} \to S^1 = \mathbb{R}/\nu\mathbb{Z}$. Let

$$\begin{aligned} \tilde{\theta}: \quad \mathbb{R}^2 \times \mathbb{R} &\longrightarrow \mathbb{R}^2 \times \mathbb{R} \\ (v, z) &\longmapsto (D_z^{-1} v + w_0, z), \end{aligned}$$

where $D_z := \begin{pmatrix} e^z & 0 \\ 0 & e^{-z} \end{pmatrix}$ and $w_0 := (I - D_\nu)^{-1} v_0$. This specific choice of w_0 ensures that $\tilde{\theta}(v, z - \nu) = \phi_{\nu,v_0} \circ \tilde{\theta}(v, z)$ so $\tilde{\theta}$ induces a diffeomorphism $\overline{\theta} : \mathbb{R}^2 \times S^1 \to \mathbb{R}^3/\phi_{\nu,v_0}$ and a diffeomorphism $\theta : \mathbb{R} \times \mathbb{R}^2 \times S^1 \to V'$. We then have

$$\begin{aligned} \theta^* \lambda' &= \sinh(s) e^z \theta^* dx + \cosh(s) e^{-z} \theta^* dy \\ &= \sinh(s) dx + \cosh(s) dy - (\sinh(s) x - \cosh(s) y)\, dz \\ &= d\left(\sinh(s) x + \cosh(s) y\right) \\ &\qquad - (\cosh(s) x + \sinh(s) y)\, ds - (\sinh(s) x - \cosh(s) y)\, dz. \end{aligned}$$

Let $f : (s, x, y, z) \mapsto \sinh(s) x + \cosh(s) y$ and consider the change of coordinates given by

$$\begin{aligned} \psi: \quad \mathbb{R} \times \mathbb{R}^2 \times S^1 &\longrightarrow \mathbb{R} \times \mathbb{R}^2 \times S^1 \\ (s, x, y, z) &\longmapsto \left(s, \begin{pmatrix} -\sinh(s) & \cosh(s) \\ -\cosh(s) & -\sinh(s) \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix}, z \right) = (s, a, b, z). \end{aligned}$$

In these new coordinates, we get

$$\left(\theta \circ \psi^{-1}\right)^* \lambda' = -d\left(f \circ \psi^{-1}\right) + a dz + b ds,$$

so after identifying $\mathbb{R} \times \mathbb{R}^2 \times S^1$ with $T^*(\mathbb{R} \times S^1)$, $\theta \circ \psi^{-1} : T^*(\mathbb{R} \times S^1) \to V'$ gives the desired exact symplectomorphism. □

The contact forms $\alpha_\pm$ and therefore the Liouville form λ_0 are translation invariant along the $\mathbb{R}^2_{x,y}$ direction. Quotienting by the lattice $P(\mathbb{Z}^2) \subset \mathbb{R}^2_{x,y}$ yields the exact symplectic manifold $(V'', \lambda'') = (\mathbb{R} \times (\mathbb{R}^2/P(\mathbb{Z}^2)) \times \mathbb{R}, \lambda'')$.

Lemma 4.4.9. *(V'', λ'') is exact symplectomorphic to a subset of $(T^*\mathbb{T}^2, \lambda_{\text{can}})$ disjoint from the zero section.*

Proof. First of all, P induces a diffeomorphism $\mathbb{R}^2/\mathbb{Z}^2 \xrightarrow{\sim} \mathbb{R}^2/P(\mathbb{Z}^2)$. We compute

$$\begin{aligned} P^*\lambda'' &= \sinh(s)e^{-z}P^*dx + \cosh(s)e^z P^*dy \\ &= P^T \begin{pmatrix} \sinh(s)e^{-z} \\ \cosh(s)e^z \end{pmatrix} \begin{pmatrix} dx \\ dy \end{pmatrix}. \end{aligned}$$

Let's consider the diffeomorphism

$$\begin{aligned} \psi_0 : \quad \mathbb{R}^2 &\longrightarrow \mathbb{R} \times \mathbb{R}_{>0} \\ (s, z) &\longmapsto \big(\sinh(s)e^{-z}, \cosh(s)e^z\big), \end{aligned}$$

which induces a diffeomorphism $\psi := P^T\psi_0 : \mathbb{R}^2 \xrightarrow{\sim} P^T(\mathbb{R} \times \mathbb{R}_{>0}) \subset \mathbb{R}^2_{a,b}$. We then have

$$\big(\psi^{-1}\big)^*\lambda'' = adx + bdy,$$

so after identifying $\mathbb{R}_a \times \mathbb{R}^2/\mathbb{Z}^2_{x,y} \times \mathbb{R}_b$ with $T^*\mathbb{T}^2$, ψ induces an exact symplectomorphism $(V'', \lambda'') \to (T^*\mathbb{T}^2 \setminus Z, \lambda_{\text{can}})$, where $Z \subset T^*\mathbb{T}$ is an closed subset containing the zero section. □

Since A preserves the lattice $\mathbb{Z}^2$, D_ν preserves the lattice $P(\mathbb{Z}^2)$. Then, ϕ_{ν,v_0} descends to a diffeomorphism $\overline{\phi}_{\nu,v_0}$ of $\mathbb{R}^2/P(\mathbb{Z}^2) \times \mathbb{R}$, or alternatively $P(\mathbb{Z}^2)$ induces a well-defined action on the fibers of $\mathbb{R}^3/\phi_{\nu,v_0} \to S^1$. After quotienting V' by the diffeomorphism induced by $\overline{\phi}_{\nu,v_0}$, or equivalently quotienting V'' by the action induced by $P(\mathbb{Z}^2)$, we obtain the torus bundle manifold $(V = \mathbb{R} \times M, \lambda)$.

Proof of Theorem 4.1.5. Let's assume by contradiction that there exists an embedded exact Lagrangian $L \hookrightarrow (V, \lambda)$ in the torus bundle domain that is either a torus, a projective plane or a Klein bottle. By Lemma 4.4.7, we can distinguish two cases:

- Case 1: the image of $\pi_1(L) \to \pi_1(V) = \mathbb{Z}^2 \rtimes_A \mathbb{Z}$ is contained in $\mathbb{Z}^2 \rtimes 0$. Then L lifts to an embedded exact Lagrangian surface $L \hookrightarrow (V'', \lambda'')$, where V'' is the cover of V corresponding to $\mathbb{Z}^2 \rtimes 0$. By Lemma 4.4.9, we obtain an exact Lagrangian surface in $(T^*\mathbb{T}^2, \lambda_{\text{can}})$ disjoint from the zero section, but this is impossible by a theorem of Gromov [Gro85].

- Case 2: the image of $\pi_1(L) \to \pi_1(V)$ is cyclic and is not contained in $\mathbb{Z}^2 \rtimes 0$. Then, it is generated by an element of the form (v, n) with $n \neq 0$. The cover of V corresponding to the subgroup $\langle (v, n) \rangle \subset \pi_1(V)$ is essentially of the form (V', λ') but with A replaced by A^n. By Lemma 4.4.8, we get a closed exact Lagrangian surface $L \hookrightarrow (T^*(\mathbb{R} \times S^1), \lambda_{\text{can}})$, but this is impossible by a theorem of Lalonde and Sikorav [LS91] asserting that if U is an open manifold then $(T^*U, \lambda_{\text{can}})$ has no closed exact Lagrangian submanifold.

This finishes the proof. □

4.5 Rabinowitz Floer cohomology and symplectic cohomology

In this section we prove Theorem 4.1.6 from the Introduction and its corollaries.

Let V be a Liouville domain of dimension $2n$. We denote by $SH^*(V)$ its symplectic cohomology and by $RFH^*(V)$ its Rabinowitz Floer cohomology [CFO10; CO18] (cf. the remarks after Theorem 4.1.6 on gradings). Recall from [CFO10] that there is a commutative diagram of exact sequences relating symplectic cohomology, Rabinowitz

Floer cohomology, and symplectic homology $SH^*(V, \partial V) \cong SH_{2n-*}(V)$, of the form[17]

$$\begin{array}{ccccccccccc}
\longrightarrow & SH^*(V,\partial V) & \xrightarrow{\varepsilon} & SH^*(V) & \xrightarrow{\iota} & RFH^*(V) & \xrightarrow{\pi} & SH^{*+1}(V,\partial V) & \longrightarrow \\
& \downarrow & & \uparrow & & \delta\uparrow & & \downarrow & \\
\longrightarrow & H^*(V,\partial V) & \xrightarrow{\varepsilon_0} & H^*(V) & \xrightarrow{\iota_0} & H^*(\partial V) & \xrightarrow{\pi_0} & H^{*+1}(V,\partial V) & \longrightarrow
\end{array}$$

Suppose now that ∂V is hypertight. Then $RFH^*(V) \cong RFH^*(\partial V)$ as a ring, where the latter is defined on the symplectization of ∂V; in particular, it does not depend on the filling V ([Ueb19]). If in addition ∂V has two components, it follows that the ring structure in Rabinowitz Floer cohomology splits nicely into two pieces.

Proposition 4.5.1 (Splitting of Rabinowitz Floer cohomology)**.** *Let $V = [-1, 1] \times M$ be a $2n$-dimensional Liouville domain such that each boundary component $M_\pm = \{\pm 1\} \times V$ is hypertight. Then we have a ring isomorphism*

$$RFH^*(V) \cong RFH^*(M_-) \oplus RFH^*(M_+).$$

In particular, the unit in $RFH^*(V)$ is of the form $1 = (1_-, 1_+)$. Moreover, the commutative diagram above becomes

$$\begin{array}{ccccccccc}
0 \longrightarrow & SH^*(V) & \xrightarrow{\iota} & RFH^*(M_-) \oplus RFH^*(M_+) & \xrightarrow{\pi} & SH^{*+1}(V,\partial V) & \longrightarrow 0 \\
& \uparrow & & \delta\uparrow & & \downarrow & \\
0 \longrightarrow & H^*(M) & \xrightarrow{\iota_0} & H^*(M) \oplus H^*(M) & \xrightarrow{\pi_0} & H^*(M) & \longrightarrow 0
\end{array}$$

where ι_0 is the diagonal embedding, and $\delta = \delta_- \oplus \delta_+$ is the direct sum of the corresponding maps $\delta_\pm : H^*(M_\pm) \to RFH^*(M_\pm)$.

Proof. The splitting $RFH^*(V) \cong RFH^*(M_-) \oplus RFH^*(M_+)$ follows from the preceding discussion. Next, let us compute the lower row in the commutative diagram for $V = I \times M$ with $I = [-1, 1]$. The long exact sequence of the pair $(I, \partial I)$ in singular

[17] Notice the directions of the vertical arrows.

cohomology is

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^0(I) & \xrightarrow{\iota_0} & H^0(\partial I) & \xrightarrow{\pi} & H^1(I,\partial I) & \longrightarrow & 0 \\ & & \| & & \| & & \| & & \\ 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{Z}\oplus\mathbb{Z} & \longrightarrow & \mathbb{Z} & \longrightarrow & 0 \end{array}$$

where the map $\mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}$ is the diagonal embedding. By naturality of the Künneth formula [Spa66, Ch. 5 Sec. 6 Thm. 1], the long exact sequence of the pair $(V, \partial V)$ is obtained by tensoring this one with $H^*(M)$,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^*(V) & \xrightarrow{\iota_0} & H^*(\partial V) & \xrightarrow{\pi} & H^{*+1}(V,\partial V) & \longrightarrow & 0 \\ & & \| & & \| & & \| & & \\ 0 & \longrightarrow & H^*(M) & \longrightarrow & H^*(M)\oplus H^*(M) & \longrightarrow & H^*(M) & \longrightarrow & 0 \end{array}$$

where ι_0 is the diagonal embedding. Since the map ε factors through $\varepsilon_0 = 0$, we obtain $\varepsilon = 0$ and thus the commutative diagram in the proposition. □

Proposition 4.5.1 provides an efficient tool for computing the product structure on symplectic cohomology in terms of that on Rabinowitz Floer cohomology:

Corollary 4.5.2 (Splitting of symplectic cohomology)**.** *Let $V = [-1,1] \times M$ be a $2n$-dimensional Liouville domain such that each boundary component $M_\pm = \{\pm 1\} \times V$ is hypertight, and the free homotopy classes of positively or negatively parametrized closed Reeb orbits on M_+ are distinct from those on M_-. With respect to the splitting as a $\mathbb{Z}$-module*

$$SH^*(V) \cong SH^*_-(V) \oplus SH^*_0(V) \oplus SH^*_+(V),$$

we denote the components of the product μ on $SH^*(V)$ by $\mu_c^{ab} := \pi_c \circ \mu|_{SH_a^*(V)\otimes SH_b^*(V)}$, where $a, b, c \in \{+,-,0\}$ and $\pi_c : SH^*(V) \to SH_c^*(V)$ is the natural projection.[18] *The only possibly nontrivial components of the product are the following:*

- *μ_0^{00}, the cup product on $SH_0^*(V) \cong H^*(M)$;*

[18] Here, $c \in \{+,-,0\}$ does not mean "contractible".

- $\mu_-^{--}, \mu_-^{-0}, \mu_-^{0-}$, *determined by the product on* $RFH^*(M_-)$*;*
- $\mu_+^{++}, \mu_+^{+0}, \mu_+^{0+}$, *determined by the product on* $RFH^*(M_+)$*.*

Proof. The hypothesis on the free homotopy classes implies that the injective ring map ι in Proposition 4.5.1 sends $SH_-^*(V)$ to $RFH^*(M_-)$ and $SH_+^*(V)$ to $RFH^*(M_+)$. Due to the splitting $RFH^*(V) \cong RFH^*(M_-) \oplus RFH^*(M_+)$ as a ring, this implies that μ cannot have components involving both $-$ and $+$. Since all orbits on $M_\pm$ are non-contractible, μ also cannot have components involving exactly two 0's. This leaves only the possibilities listed in the corollary as well as μ_0^{--} and μ_0^{++}.

To see that the latter two vanish, recall from Proposition 4.5.1 that ι sends $SH_0^*(V)$ to both summands via the composition

$$SH_0^*(V) \cong H^*(M) \xrightarrow{\iota_0} H^*(M) \oplus H^*(M) \xrightarrow{\delta} RFH^*(M_-) \oplus RFH^*(M_+),$$

where ι_0 is the diagonal map and $\delta = \delta_- \oplus \delta_+$. Consider $a, b \in SH_+^*(V)$ and write $\mu(a,b) = c_0 + c_+ \in SH_0^*(V) \oplus SH_+^*(V)$. Then $\iota\mu(a,b) = \delta_- c_0 + d_+$ for some $d_+ \in RFH^*(M_+)$. On the other hand, $\iota\mu(a,b) = \mu(\iota a, \iota b) \in RFH^*(M_+)$, which implies $\delta_- c_0 = 0$ and therefore $c_0 = \pi_0 c_0 = 0$. This shows $\mu_0^{++} = 0$, and $\mu_0^{--} = 0$ follows analogously.

By Morse cohomology, μ_0^{00} equals the cup product on $SH_0^*(V) \cong H^*(M)$. Since ι is an injective ring map, the remaining components are determined by $RFH^*(M_\pm)$ as stated. □

Proposition 4.5.1 and Corollary 4.5.2 together yield Theorem 4.1.6 from the Introduction. As a consequence, we now prove Corollaries 4.1.8 and 4.1.9.

Proof of Corollary 4.1.8. By Lemma 4.2.7, a McDuff domain $V = [-1,1] \times M$ with $M = S^*\Sigma$ satisfies the hypotheses of Corollary 4.5.2. So it only remains to compute the summands and the components of the product with respect to the $\mathbb{Z}$-module

splitting

$$SH^*(V) \cong SH^*_-(V) \oplus SH^*_0(V) \oplus SH^*_+(V).$$

To compute the components $\mu^{--}_-, \mu^{-0}_-, \mu^{0-}_-$, we perturb the Hamiltonian by a Morse function on the base Σ. Since $\pi_2(\Sigma) = 0$, the 3-punctured Floer spheres contributing to these components project onto gradient Y-trees in Σ connecting critical points, whose counts give the cup product on $H^*(\Sigma)$. A further perturbation by Morse functions on the S^1-fibres over critical points yields the desired form.

Since $RFH^*(S^*\Sigma, \xi_{\text{can}})$ is independent of the filling, we can compute it using its filling by the unit disk cotangent bundle $D^*\Sigma$, so it equals the Rabinowitz loop cohomology $\breve{H}^*(\mathcal{L}\Sigma)$ defined in [CHO22]. As shown there, its product is an amalgamation of the loop product on the positive action part, the cup product on the zero action part, and the Goresky–Hingston product on the negative action part. □

Proof of Corollary 4.1.9. Recall that the Reeb vector fields $R_\pm = \frac{1}{2}\big(\pm e^{-z}v_x + e^z v_y\big)$ are tangent to the torus fibers, spanned by v_x, v_y (two eigenvectors of A), where $z \in [0, \nu)$ (here, $e^{\pm\nu}$ are the eigenvalues of the hyperbolic monodromy matrix). Therefore the closed $R_\pm$-orbits arise whenever $R_\pm$ has rational slope, foliating the fibers corresponding to a subset Γ of $[0, \nu)$ in bijection with $\mathbb{Q} \cap [0, 1)$. Moreover, different fibers correspond to orbits in different homology classes, and so there is no Floer cylinder between them. Since for $z \in \Gamma$, the fiber over z is a Morse-Bott S^1-family, after introducing a Morse function on S^1 that breaks the symmetry, we see that $SH^*_\pm(V) = \bigoplus_\Gamma H^*(S^1)$. This concludes the proof. □

4.6 Wrapped Floer cohomology of Anosov Liouville domains

4.6.1 Computation of wrapped Floer cohomology

In this section, we prove the first part of Theorem 4.1.11 from the Introduction. We apply this to compute the wrapped Floer cohomology groups of the Lagrangians $\mathcal{L}_\Lambda$ in any Anosov Liouville domain, and in particular for the McDuff domains and the torus bundle domains. The products are studied below.

Proof of Theorem 4.1.11(1): splitting as $\mathbb{Z}$-modules. Similarly as in Theorem 4.1.6, for any Liouville domain of the form $V = [-1, 1] \times M$, and any two exact Lagrangians $L_1, L_2 \subset V$ with Legendrian boundary, the wrapped Floer cochain complex splits as a $\mathbb{Z}$-module as

$$CW^*(L_1, L_2) = CW^*_-(L_1, L_2) \oplus CW^*_0(L_1, L_2) \oplus CW^*_+(L_1, L_2).$$

Here, the first summand is generated by (Hamiltonian chords corresponding to) α_--chords, the second one by intersection points of L_1 and L_2 lying in the interior of V, and the third one by (Hamiltonian chords corresponding to) α_+-chords. By an open version of [CO18, Lemma 2.2, Lemma 2.3], there cannot be any Floer strips to an $\alpha_\pm$-chord coming from an $\alpha_\mp$-chord or an intersection point in the interior of V. This means that the wrapped differential is of the form

$$\partial_W = \begin{pmatrix} \partial^-_W & 0 & 0 \\ \partial^{-,0}_W & \partial^0_W & \partial^{+,0}_W \\ 0 & 0 & \partial^+_W \end{pmatrix}.$$

In the case of a 4-dimensional Anosov Liouville domain and L_1, L_2 of the form $\mathcal{L}_\Lambda, \mathcal{L}_{\Lambda'}$,

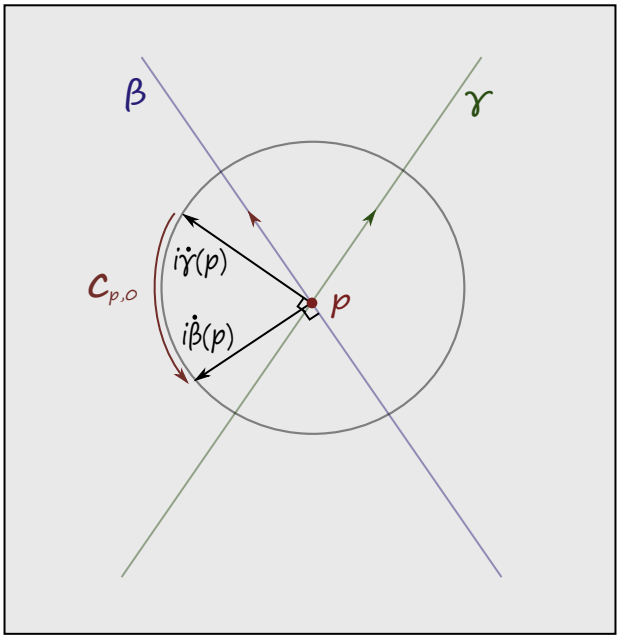


Figure 4.6.1: The chord $c_{p,0}$ from Λ_γ to Λ_β, for $p \in \gamma \cap \beta$, joining $i\dot{\gamma}(p)$ to $i\dot{\beta}(p)$ along the S^1-fiber.

for $\Lambda \neq \Lambda'$, recall that these Lagrangians are actually disjoint. Thus $CW_0^*(\mathcal{L}_\Lambda, \mathcal{L}_{\Lambda'}) = 0$, and therefore

$$HW^*(\mathcal{L}_\Lambda, \mathcal{L}_{\Lambda'}) = HW^*_-(\mathcal{L}_\Lambda, \mathcal{L}_{\Lambda'}) \oplus HW^*_+(\mathcal{L}_\Lambda, \mathcal{L}_{\Lambda'})$$

if $\Lambda \neq \Lambda'$. If $\Lambda = \Lambda'$, then ∂_W^0 can be identified with the Morse differential for $\mathcal{L}_\Lambda$, and Lemma 4.3.1 implies $\partial_W^{+,0} = \partial_W^{-,0} = 0$ (cf. property (C) in the proof of Theorem 4.1.1). □

As particular cases of Theorem 4.1.11(1), we have the following computations as $\mathbb{Z}$-modules.

Proposition 4.6.1 (McDuff domains: wrapped Floer cohomology)**.** *Consider a McDuff domain $V = [-1, 1] \times S^*\Sigma$. Let $\Lambda_\gamma, \Lambda_\beta \subset M = S^*\Sigma$ be the positive conormal lifts of oriented closed geodesics $\gamma, \beta \subset \Sigma$, and $\mathcal{L}_\gamma, \mathcal{L}_\beta$ the corresponding Lagrangians. We denote by $\overline{\beta}$ the geodesic β with opposite orientation.*

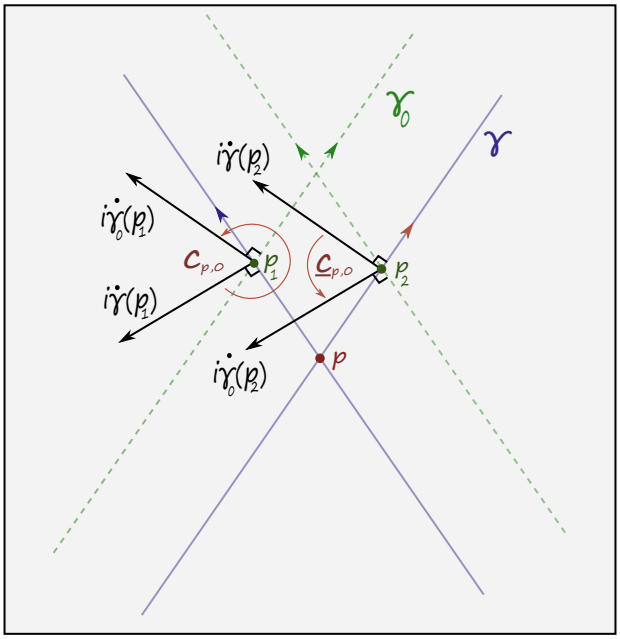


Figure 4.6.2: If $\gamma = \beta$ and p is a self-intersection of γ, it resolves into two intersections p_1 and p_2 between γ and a small push-off γ_0, respectively giving the chord $c_{p,0}$ and its complementary $\underline{c}_{p,0}$, both going from the conormal lifts Λ_γ to Λ_{γ_0}.

1. *If $\gamma \neq \beta, \overline{\beta}$, then as a $\mathbb{Z}$-module we have*

$$HW^*(\mathcal{L}_\gamma, \mathcal{L}_\beta) \cong H_*^{>0}(\mathcal{L}(\gamma,\beta)) \oplus \bigoplus_{p\in\gamma\cap\beta} \mathbb{Z}[t]\cdot c_p,$$

where $\mathcal{L}(\gamma,\beta)$ is the space of free paths in Σ from γ to β, $H_*^{>0}(\mathcal{L}(\gamma,\beta))$ is the homology of $\mathcal{L}(\gamma,\beta)$ relative to the constant paths, and $t^k c_p =: c_{p,k}$ is a generator corresponding to an intersection point $p \in \gamma\cap\beta$ obtained by the concatenation of paths $c_{p,k} = t^k * c_p$, where t is the S^1-fiber and $c_p = c_{p,0}$ is the shortest positively oriented path on the S^1-fiber going from Λ_γ to Λ_β (see Figure 4.6.1).

2. *If $\gamma = \beta$, then*

$$HW^*(\mathcal{L}_\gamma, \mathcal{L}_\gamma) \cong H_*^{>0}(\mathcal{L}(\gamma,\gamma)) \oplus H^*\big(S^1;\mathbb{Z}\big) \oplus \bigoplus_{p\in\Gamma_\gamma} \mathbb{Z}(t)\cdot c_p,$$

where Γ_γ denotes the set of self-intersections of γ, and $t^k c_p =: c_{p,k}$ is defined via

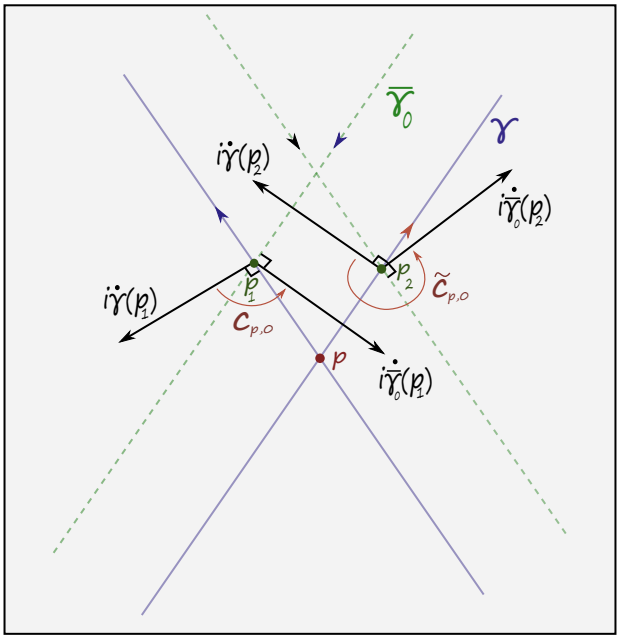


Figure 4.6.3: If $\beta = \overline{\gamma}$ and p is a self-intersection of γ, the inverse $\overline{\gamma}_0$ of a small push-off γ_0 intersects γ in p_1, p_2, and produces a chord $c_{p,0}$ from $i\dot{\gamma}(p_1)$ to $i\dot{\overline{\gamma}}_0(p_1) = -i\dot{\gamma}_0(p_1)$, and a chord $\widetilde{c}_{p,0}$ from $i\dot{\gamma}(p_2)$ to $i\dot{\overline{\gamma}}_0(p_2) = -i\dot{\gamma}_0(p_2)$.

the concatenation $c_{p,k} = t^k * c_p$ *for* $k \in \mathbb{Z}$ *(see Figure 4.6.2).*

3. *If* $\gamma = \overline{\beta}$*, then*

$$HW^*(\mathcal{L}_\gamma, \mathcal{L}_{\overline{\gamma}}) \cong H_*^{>0}(\mathcal{L}(\gamma,\gamma)) \oplus \bigoplus_{p \in \Gamma_\gamma} (\mathbb{Z}[t] \cdot c_p \oplus \mathbb{Z}[t] \cdot \widetilde{c}_p)\,,$$

where $t^k \widetilde{c}_p = \widetilde{c}_{p,k} = t^k * \widetilde{c}_p$*, and if* c_p *joins positively* $v \in S^*_p\Sigma$ *to* $w \in S^*_p\Sigma$*, then* $\widetilde{c}_p$ *joins positively* $-w$ *to* $-v$ *(see Figure 4.6.3).*

Proof. Assume first that $\gamma \neq \beta, \overline{\beta}$. We note that α_+-Reeb chords c_+ between Λ_γ and Λ_β project to Σ as *binormal geodesics paths* (or *binormal chords*) $c_+^\perp : [0,1] \to \Sigma$

between γ and β, i.e.,

$$\nabla_{\partial_t c_+^\perp} \partial_t c_+^\perp = 0,$$
$$c_+^\perp(0) \in \gamma, \qquad c_+^\perp(1) \in \beta,$$
$$\partial_t c_+^\perp(0) \perp \gamma, \qquad \partial_t c_+^\perp(1) \perp \beta.$$

There is a unique such binormal chord in each free homotopy class of paths between γ and β, i.e., in each connected component of $\mathcal{L}(\gamma, \beta)$. They are precisely the critical points of the length functional

$$L : \mathcal{P}(\gamma, \beta) \to \mathbb{R}, \qquad L(x) = \int_{[0,1]} |\dot{x}(t)| dt.$$

Note that since γ and β cannot possibly intersect tangentially, every $c_+^\perp \in \mathrm{crit}(L)$ has positive length, i.e., $L\big(c_+^\perp\big) > 0$. We may then adapt the analysis of [AS06], by considering suitable moduli spaces of Floer strips with mixed boundary conditions (see Figure 4.6.4), and conclude that

$$HW_+^*(\mathcal{L}_\gamma, \mathcal{L}_\beta) \cong H_*^{>0}(\mathcal{L}(\gamma, \beta)).$$

Similarly, α_--Reeb chords c_- between Λ_γ and Λ_β project to Σ as intersection points $p \in \gamma \cap \beta$. Since α_--orbits are all closed and correspond to fibers of the prequantization bundle, to each $p \in \gamma \cap \beta$ there corresponds a family of Reeb chords $c_{p,k}, k \in \mathbb{N} \cup \{0\}$, one for each iteration along the S^1-fiber, so that $c_{p,k} = t^k * c_p$ where t denotes the fiber and c_p the shortest positively oriented path along the fiber going from Λ_γ to Λ_β. Therefore,

$$CW_+^*(\mathcal{L}_\gamma, \mathcal{L}_\beta) = \bigoplus_{p \in \gamma \cap \beta} \bigoplus_{k=0}^{\infty} \mathbb{Z} \cdot c_{p,k} = \bigoplus_{p \in \gamma \cap \beta} \mathbb{Z}[t] \cdot c_p.$$

A Floer strip between two such chords projects to Σ as a bigon between γ and β,

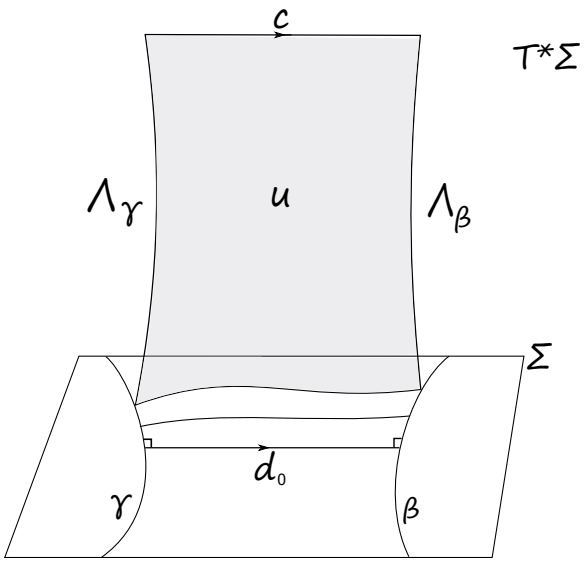


Figure 4.6.4: For c a α_+-chord between Λ_γ and Λ_β, $d_0 \in \mathrm{crit}(L)$ a binormal path between γ and β, and the Hamiltonian $H(q,p) = |p|^2/2$, the counts of H-Floer solutions $u : [0,+\infty) \times [0,1] \to T^*\Sigma$ with Lagrangian boundary in $\mathcal{L}_\gamma$, $\mathcal{L}_\beta$ and Σ, which approach c at infinity, and whose intersection with Σ lies in the unstable manifold of d_0 (with respect to the gradient flow of L), gives a chain map inducing an isomorphism $HW^*_+(\mathcal{L}_\gamma, \mathcal{L}_\beta) \cong H^{>0}_*(\mathcal{L}(\gamma,\beta))$.

which does not exist since geodesics minimize the length in their free homotopy class.

This implies

$$HW^*_+(\mathcal{L}_\gamma, \mathcal{L}_\beta) \cong CW^*_+(\mathcal{L}_\gamma, \mathcal{L}_\beta),$$

and the proposition follows for the case $\gamma \neq \beta, \overline{\beta}$. The remaining cases follow by perturbation of the self-intersections of γ, taking orientations into consideration, as depicted in Figures 4.6.2 and 4.6.3. □

Proposition 4.6.2 (Torus bundle domains: wrapped Floer cohomology)**.** *Consider a torus bundle domain $V = [-1,1] \times M$. Let $\mathcal{O}, \mathcal{O}' \subset \mathbb{T}^2$ be two periodic orbits of the hyperbolic monodromy matrix A, and $\mathcal{L}_\mathcal{O}, \mathcal{L}_{\mathcal{O}'}$ the corresponding Lagrangians. Then there is no differential in the wrapped Floer cochain complex between $\mathcal{L}_\mathcal{O}, \mathcal{L}_{\mathcal{O}'}$ (save the Morse differential if $\mathcal{O} = \mathcal{O}'$), and we have the following.*

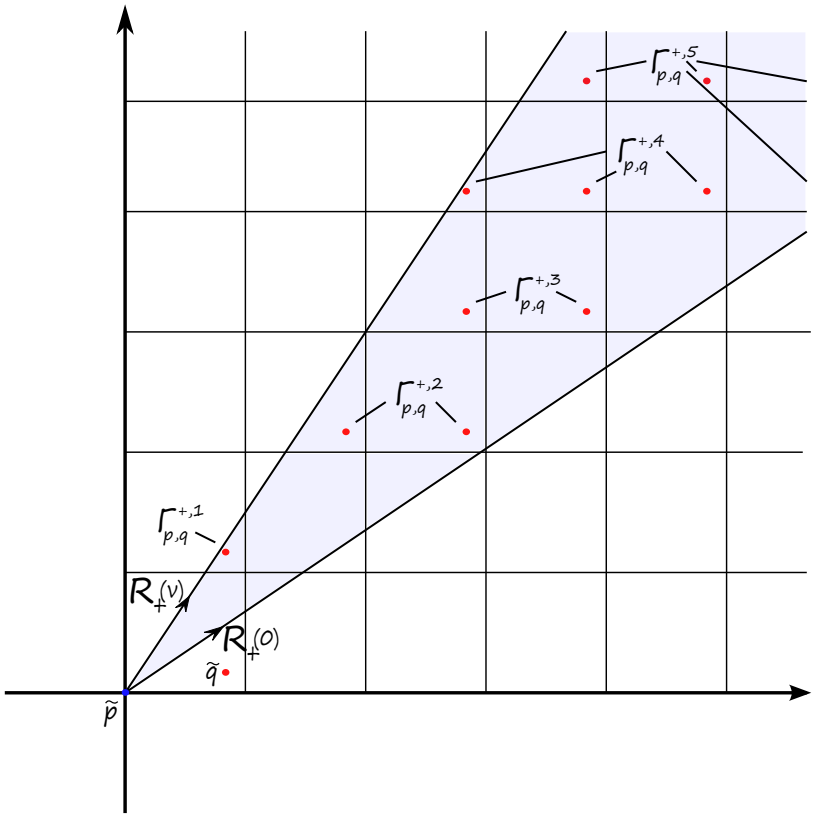


Figure 4.6.5: The linear path between $\tilde{p}$ and a translate of $\tilde{q}$ lying inside the cone $\mathbb{R}^{\geq 0}\langle v, w\rangle$, where $v = R_+(0), w = R_+(\nu)$, gives a α_+-chord between p and q, lying on a $\mathbb{T}^2$-fiber which depends on the slope of the path. In the figure, we depict the sets $\Gamma^{+,k}_{p,q} = \Gamma^{+,\leq k}_{p,q} \backslash \Gamma^{+,\leq k-1}_{p,q}$, where k is the box-length of the corresponding chords. There is a similar description for the "minus" end.

1. *If $\mathcal{O} \neq \mathcal{O}'$, then*

$$HW^*(\mathcal{L}_{\mathcal{O}}, \mathcal{L}_{\mathcal{O}'}) = \bigoplus_{p \in \mathcal{O}, q \in \mathcal{O}'} \left(\bigoplus_{z^+ \in \Gamma^+_{p,q}} \mathbb{Z} \cdot z^+ \oplus \bigoplus_{z^- \in \Gamma^-_{p,q}} \mathbb{Z} \cdot z^- \right),$$

where $\Gamma^{\pm}_{p,q}$ is the set of straight lines in $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ from p to q with directions in the positive cone spanned by the Reeb vector fields $R_{\pm}(0)$ and $R_{\pm}(\nu)$ from (4.2.14).

2. *If $\mathcal{O} = \mathcal{O}'$, then*

$$HW^*(\mathcal{L}_{\mathcal{O}}, \mathcal{L}_{\mathcal{O}}) = H^*\big(S^1; \mathbb{Z}\big) \oplus \bigoplus_{p,q \in \mathcal{O}} \left(\bigoplus_{z^+ \in \Gamma^+_{p,q}} \mathbb{Z} \cdot z^+ \oplus \bigoplus_{z^- \in \Gamma^-_{p,q}} \mathbb{Z} \cdot z^- \right).$$

Remark 4.6.3. *If $\tilde{p}, \tilde{q} \in \mathbb{R}^2$ are lifts of $p, q \in \mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$, then $\Gamma^{\pm}_{p,q}$ is in canonical one-to-one correspondence with the set of $\mathbb{Z}^2$-translates $\tilde{q} + (m, n)$ of $\tilde{q}$, with $m, n \in \mathbb{Z}$, which lie inside the positive cone based at $\tilde{p}$ and spanned by the Reeb vector fields $R_{\pm}(0)$ and $R_{\pm}(\nu)$ from* (4.2.14)*; see Figure 4.6.5.*

Note that $\Gamma^{\pm}_{p,q} = \bigcup_{k=0}^{\infty} \Gamma^{\pm,\leq k}_{p,q}$, where $\Gamma^{\pm,\leq k}_{p,q}$ is the set of translates $\tilde{q} + (m, n)$ with $|(m, n)| = \max\{|m|, |n|\} \leq k$. This filtration is equivalent (up to a factor of $\sqrt{2}$ and a bounded error) to the action filtration and $\#\Gamma^{\pm,\leq k}_{p,q}$ grows quadratically with k.

Proof. Put $\Lambda = \Lambda_{\mathcal{O}}$, $\Lambda' = \Lambda_{\mathcal{O}'}$ for two orbits $\mathcal{O}, \mathcal{O}'$ of the hyperbolic monodromy matrix. We will first show that the differential ∂_W vanishes identically if $\mathcal{O} \neq \mathcal{O}'$. As the differential has no mixed terms between the two boundary components of V, it suffices to check $\partial^+_W = 0$ (the argument for ∂^-_W is analogous). Given a Floer strip u between two α_+-chords c_1, c_2, we may first project it on M. Moreover, since it is simply connected, we may lift it to the cover $\mathbb{R}^2 \times \mathbb{R}$ of M, and then further project it to the $\mathbb{R}^2$-factor. This gives a loop $a = \widetilde{c}_1 * \widetilde{c}_2^{-1}$ which is obtained by concatenating two paths $\widetilde{c}_1$ and the opposite of $\widetilde{c}_2$, lifting c_1 and c_2, respectively, each of which is linear and follows the direction of the Reeb field R_+. Then, the only possibility is that

$\widetilde{c}_1 = \widetilde{c}_2$ This implies that c_1, c_2 have the same slope, hence lie in the same $\mathbb{T}^2$-fiber, and therefore $c_1 = c_2$. But then the strip is the trivial strip, which is not counted in the differential.

It remains to describe the generators of $HW^*(\mathcal{L}_{\mathcal{O}}, \mathcal{L}_{\mathcal{O}'})$. For this, note that each Reeb chord for $R_\pm$ from $\mathcal{O}$ to $\mathcal{O}'$ has a unique representative in $\mathbb{T}^2 \times [0, \nu)$ by a linear path from some $p \in \mathcal{O}$ to some $q \in \mathcal{O}'$ in direction $R_\pm(z)$ from (4.2.14), for some $z \in [0, \nu)$. This shows that the nonconstant generators are indexed by the filtered sets $\Gamma^{\pm}_{p,q}$ as described in the statement of the Theorem, for $p \in \mathcal{O}$ and $q \in \mathcal{O}'$. In the case $\mathcal{O} = \mathcal{O}'$ we have in addition the constant intersection points, which after a Morse perturbation give the $H^*\big(S^1; \mathbb{Z}\big)$ summand. This concludes the proof. □

4.6.2 The product structure

In this section, we study the product $\mathfrak{m} = \mathfrak{m}_2$ on the algebra

$$A = \bigoplus_{\Lambda, \Lambda'} HW^*(\mathcal{L}_\Lambda, \mathcal{L}_{\Lambda'}),$$

for any Anosov Liouville domain, and complete the proof of Theorem 4.1.11. We have a natural splitting of A as a $\mathbb{Z}$-module

$$A = I_- \oplus A_0 \oplus I_+,$$

where

$$I_\pm := \bigoplus_{\Lambda, \Lambda'} HW^*_\pm(\mathcal{L}_\Lambda, \mathcal{L}_{\Lambda'}), \qquad A_0 := \bigoplus_{\Lambda} HW^*_0(\mathcal{L}_\Lambda, \mathcal{L}_\Lambda),$$

and we let

$$A_\pm = A_0 \oplus I_\pm.$$

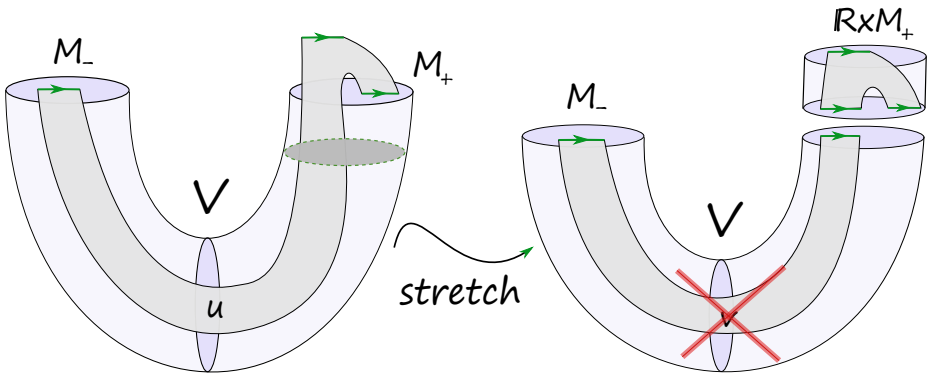


Figure 4.6.6: Stretching the neck gives a disk v with only positive ends, contradicting the Topological Disk Lemma 4.3.1.

Note that $A_0 = \bigoplus_\Lambda H^*\big(S^1;\mathbb{Z}\big)$. The following is an open-string analogue of the statement for symplectic cohomology of Theorem 4.1.6.

Proof of Theorem 4.1.11(2): the ring structure. The fact that the product on A_0 is as claimed follows from the standard fact that the isomorphism with Morse homology preserves the product structure.

To see the remaining claims, we argue as follows. Assume by contradiction that there exists a 3-punctured disk $u : \mathcal{D} \to V$ contributing to $\mathfrak{m}$ which has mixed ends (i.e., at least one asymptotic in M_+, and at least one in M_-), two of which are positive, and one which is negative. After stretching the neck along contact-type slices slightly to the interior of M_- and/or M_+, the resulting broken disk configuration contains a punctured disk v with only positive ends asymptotic to Reeb chords in the contact-type slices; see Figure 4.6.6. As in Step 1 of the proof of Theorem 4.1.1, the projection of v to M gives rise to a topological disk that contradicts Lemma 4.3.1. A product of two chords in M_+ or M_- with output in the constants is excluded similarly. This finishes the proof of Theorem 4.1.11 of the Introduction. □

McDuff domains: open-string products. We now focus on the algebra A corresponding to the McDuff domains. First, we compute the coefficients of $\mathfrak{m} : A \otimes A \to A$ on I_+, which is the free $\mathbb{Z}$-module with generators all the positive

length binormal chords between γ and β, for varying closed geodesics γ and β. Consider $\pi_0\mathcal{L}(\gamma,\beta)$, the set of free homotopy classes of paths from γ to β (which is a countable set). We identify elements in $\pi_0\mathcal{L}(\gamma,\beta)$ with their unique binormal chord representative, and view $CW^*_+(\mathcal{L}_\gamma,\mathcal{L}_\beta)$ as the free $\mathbb{Z}$-module over $\pi_0\mathcal{L}(\gamma,\beta)$. For c a binormal chord, we denote by $\overline{c}$ the binormal chord with opposite orientation (so that if $c \in \pi_0\mathcal{L}(\gamma,\beta)$, then $\overline{c} \in \pi_0\mathcal{L}(\beta,\gamma)$).

We first consider the product $\mathfrak{m}_+ = \mathfrak{m}|_{I_+\otimes I_+} : I_+ \otimes I_+ \to I_+$, defined by counting 3-punctured Floer disks in the completion $\mathbb{R}\times M$ of $V = [-1,1]\times M$ with two inputs and one output asymptotic to Hamiltonian chords near $M_+ = \{1\}\times M$. By the maximum principle, these Floer disks are contained in $\mathbb{R}_+ \times M$. By a deformation argument as in [BO09], their counts agree with the counts of index 1 holomorphic disks in the symplectization $\mathbb{R}_+ \times S^*\Sigma$ with two positive punctures and one negative puncture asymptotic to Reeb chords in the unit cotangent bundle $S^*\Sigma$. Similarly, we have a coproduct $\check{\mathfrak{m}}_+ : I_+ \to I_+ \otimes I_+$, counting index 1 holomorphic disks in $\mathbb{R}_+ \times S^*\Sigma$ with one positive puncture and two negative ones. We also have an open-string cobracket $\Delta : I_+ \to I_+ \otimes I_+$, defined as follows. Let $\gamma_0,\gamma_1,\gamma_2 \subset \Sigma$ be geodesics, and consider a binormal chord $c \in \pi_0\mathcal{L}(\gamma_0,\gamma_2)$. Let $x_1,\ldots,x_n$ be the intersection points of c with γ_1. At each x_i, c may be split into two (oriented) paths $p_1^i \in \mathcal{L}(\gamma_0,\gamma_1)$ and $p_2^i \in \mathcal{L}(\gamma_1,\gamma_2)$, and we let c_1^i and c_2^i be the corresponding binormal chord representatives. We define

$$\Delta^{\gamma_1}(c) := \sum_{i=1}^n c_1^i \otimes c_2^i$$

and $\Delta := \sum_\gamma \Delta^\gamma$. For c_1,c_2,c_3 binormal chords, we denote by $\langle \mathfrak{m}_+(c_1,c_2),c_3\rangle$ the coefficient in front of c_3 of $\mathfrak{m}_+(c_1,c_2)$. Similarly, we define $\langle \check{\mathfrak{m}}_+(c_1), c_2\otimes c_3\rangle$, and $\langle \Delta(c_1), c_2\otimes c_3\rangle$, as the coefficients of $\check{\mathfrak{m}}_+(c_1)$ (resp. $\Delta(c_1)$) in front of $c_2\otimes c_3$.

The following reduces the computation of $\mathfrak{m}_+$ to that of the string operation Δ:

Proposition 4.6.4. *For binormal chords c_1,c_2,c_3, the following formulas hold with*

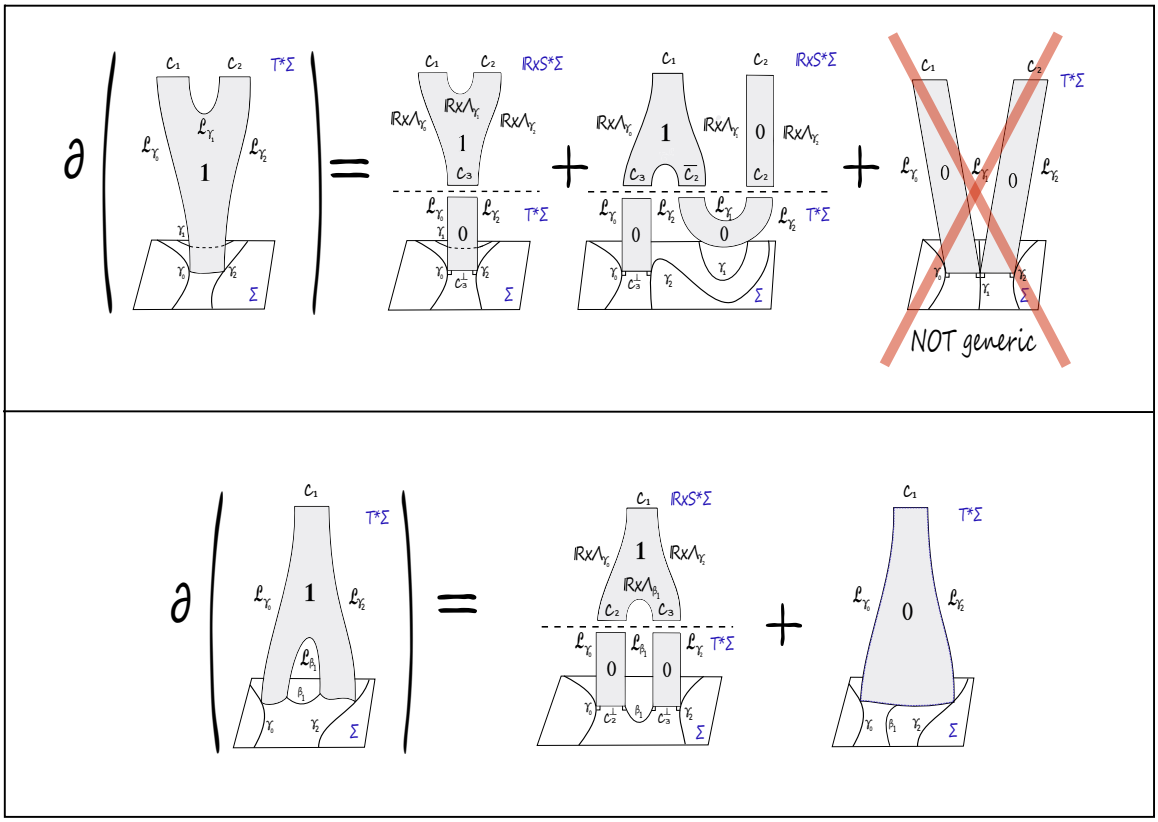


Figure 4.6.7: The degenerations of 1-dimensional moduli spaces of curves with Lagrangian boundary on the zero section gives the relations between the structure constants of $\mathfrak{m}_+$ and Δ.

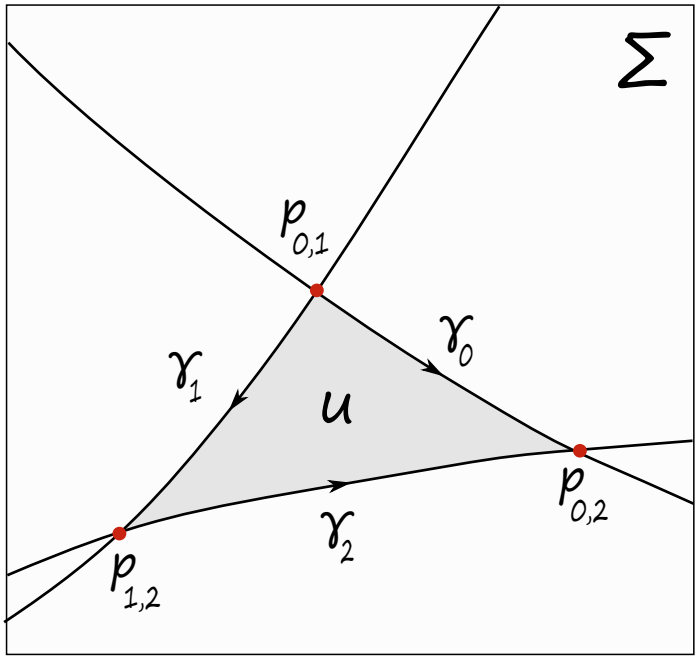


Figure 4.6.8: The product on I_- is given by counts of immersed geodesic triangles on Σ.

suitable signs:

$$\langle \mathfrak{m}_+(c_1, c_2), c_3\rangle = \pm\langle \breve{\mathfrak{m}}_+(c_1), c_3 \otimes \overline{c_2}\rangle \pm \langle \breve{\mathfrak{m}}_+(c_2), \overline{c_1} \otimes c_3\rangle,$$
$$\langle \breve{\mathfrak{m}}_+(c_1), c_2 \otimes c_3\rangle = \langle \Delta(c_1), c_2 \otimes c_3\rangle\,.$$

Proof. The proof is a straightforward adaptation of that in [CL09] (see also [Sch09]). One counts the boundary configurations in a 1-dimensional moduli space of curves with boundary on the zero section to obtain the above relations; see Figure 4.6.7. We omit further details. □

We now consider the product $\mathfrak{m}_- = \mathfrak{m}|_{I_-\otimes I_-} : I_- \otimes I_- \to I_-$, where I_- is generated by $c_{p,k}$ for intersection points p of geodesics γ, β for varying γ, β, and $k \in \mathbb{N} \cup \{0\}$. Again, after a deformation we may assume that $\mathfrak{m}_-$ counts index 1 holomorphic curves in the symplectization $\mathbb{R} \times S^*\Sigma$ of the prequantization circle bundle $S^*\Sigma$ with two positive punctures and one negative puncture. We can also arrange that the almost

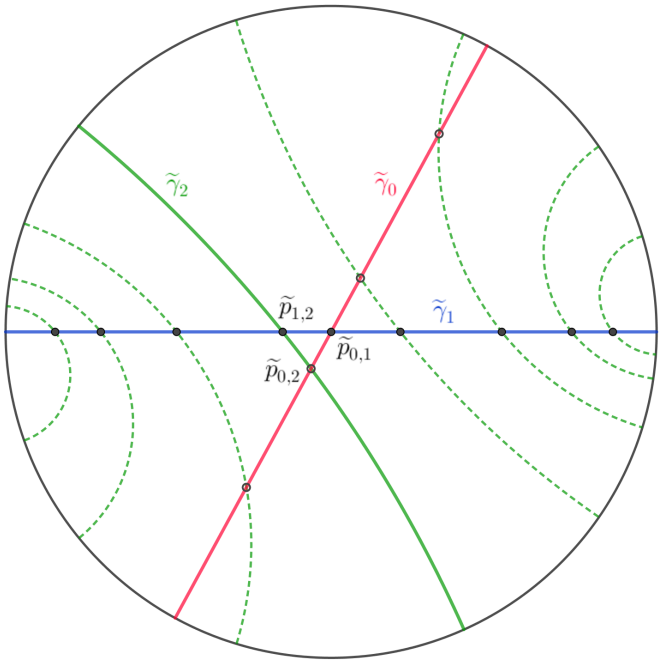


Figure 4.6.9: Geodesic triangles between lifts of γ_0, γ_1 and γ_2. The dashed geodesics are hyperbolic translates of $\widetilde{\gamma}_2$ along $\widetilde{\gamma}_1$. In this example, there are 4 geodesic triangles contributing to the product.

complex structure on J on $\mathbb{R} \times S^*\Sigma$ is such that the projection $\mathbb{R} \times S^*\Sigma \to \Sigma$ is (J, j)-holomorphic, where j is the complex structure on the Riemann surface Σ. For simplicity, we will only consider the product between three Lagrangian cylinders corresponding to geometrically distinct geodesics.

Proposition 4.6.5. *Let $\gamma_0, \gamma_1, \gamma_2$ be three geometrically distinct closed oriented geodesics on Σ intersecting at $p_{0,1} \in \gamma_0 \cap \gamma_1$, $p_{1,2} \in \gamma_1 \cap \gamma_2$ and $p_{0,2} \in \gamma_0 \cap \gamma_2$, together with $k_{0,1}, k_{1,2}, k_{0,2} \in \mathbb{N} \cup \{0\}$. The coefficient*

$$\mathfrak{m}_-^{012} := \left\langle \mathfrak{m}_-\left(t^{k_{0,1}} c_{p_{0,1}}, t^{k_{1,2}} c_{p_{1,2}}\right), t^{k_{0,2}} c_{p_{0,2}} \right\rangle \in \mathbb{Z}$$

is zero unless $k_{0,2} = k_{0,1} + k_{1,2}$, in which case it is equal to the number of immersed geodesic triangles with cyclically ordered vertices $p_{0,1}, p_{1,2}, p_{0,2}$ and edges on the γ_i's as in Figure 4.6.8, counted with appropriate signs.

Proof. Let $\mathfrak{T}$ be the moduli space of holomorphic triangles defining $\mathfrak{m}_-^{012}$, and $\mathcal{T}$ be the moduli space of immersed geodesic triangles on Σ as above. If $u \in \mathfrak{T}$ is a holomorphic triangle, it projects to a geodesic triangle on Σ with the required properties. This defines a natural map $\pi : \mathfrak{T} \to \mathcal{T}$. We will only show that π is a bijection and we leave it to the reader to work out the relevant orientations and signs.

We can lift $u \in \mathfrak{T}$ to a holomorphic triangle in $\mathbb{R} \times S^*\mathbb{H}$ whose boundaries lie in the Lagrangian cylinders over the unit conormals of lifts $\widetilde{\gamma}_i$ in $\mathbb{H}$ of the geodesics γ_i. By (4.2.6), the contact structure on $S^*\mathbb{H} \cong \mathbb{H} \times S^1$ becomes

$$\widetilde{\alpha}_{\text{pre}} = \frac{1}{y} dx + d\varphi,$$

where φ denotes S^1 coordinate. Under the biholomorphism $\mathbb{H} \cong \mathbb{D}$, it further becomes

$$\widetilde{\alpha}_{\text{pre}} = \frac{2r^2}{1 - r^2} d\theta + dt,$$

where (r, θ) are the usual polar coordinates on $\mathbb{D}$ and t denotes the S^1 coordinate on $\mathbb{D} \times S^1 \cong S^*\mathbb{D}$. Note that this is simply the S^1-contactization of the Liouville manifold $\left(\mathbb{D}, \lambda_{\mathbb{D}}\right)$ where $\lambda_{\mathbb{D}} := \frac{2r^2}{1-r^2} d\theta$. We can further lift u to the symplectization of the $\mathbb{R}$-contactization of $\left(\mathbb{D}, \lambda_{\mathbb{D}}\right)$, with boundary conditions on the Lagrangian cylinders L_i over suitable lifts of the unit conormals of the geodesics $\widetilde{\gamma}_i$. These Legendrian lifts in $\mathbb{D} \times \mathbb{R}$ only depend on the $k_{i,j}$'s up to a global translation in the t-direction (which coincides with the Reeb direction). This readily implies that $k_{0,2} = k_{0,1} + k_{1,2}$. By [Dim16, Theorem 2.1.], for a suitable choice of almost complex structure on $\mathbb{R} \times \mathbb{D} \times \mathbb{R}$, there is a one-to-one correspondence between

1. SFT-type holomorphic triangles in $\mathbb{R} \times \mathbb{D} \times \mathbb{R}$ between the L_i's, and
2. Holomorphic triangles in $\mathbb{D}$ between the $\widetilde{\gamma}_i$'s.

Since the $\widetilde{\gamma}_i$'s bound a unique holomorphic triangle in $\mathbb{D}$ (which is embedded and

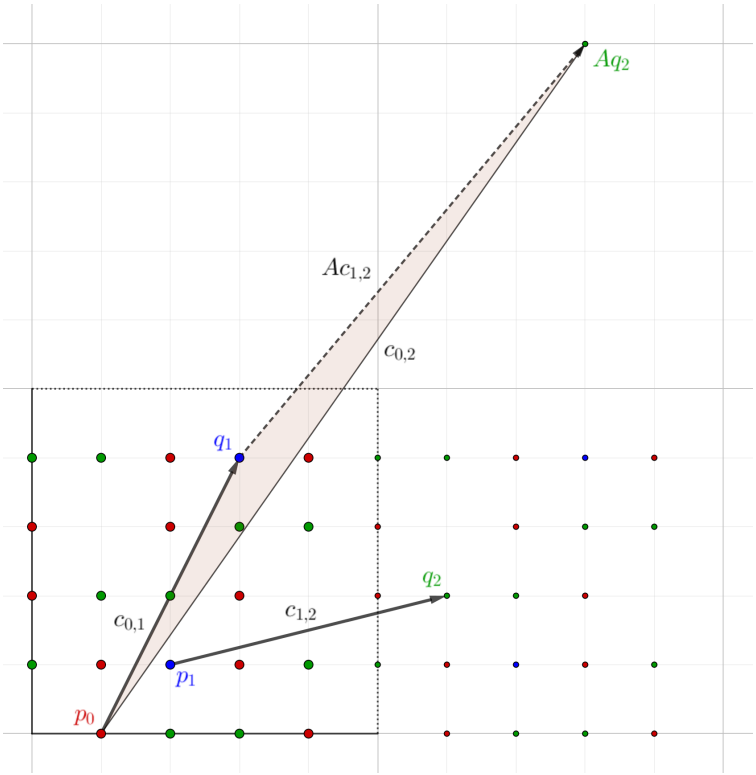


Figure 4.6.10: Torus bundle domain: three orbit $\mathcal{O}_0$ (red), $\mathcal{O}_1$ (blue) and $\mathcal{O}_2$ (green), together with Reeb chords $c_{0,1}$ and $c_{1,2}$, and a potential candidate for a holomorphic triangle contributing to the product $\mathfrak{m}\big(c_{0,1}, c_{1,2}\big)$.

cut-out transversally), this implies that π is injective. This also shows that π is surjective, as any immersed geodesic triangle $\Delta \in \mathcal{T}$ lifts to an embedded geodesic triangle $\widetilde{\Delta} \subset \mathbb{D}$ between lifts $\widetilde{\gamma}_i$ of the γ_i's. By the uniformization theorem, $\widetilde{\Delta}$ can be identified with a holomorphic map from a disk with three boundary punctures to $\mathbb{D}$, which is cut-out transversally. By the above correspondence, it lifts to a SFT-type holomorphic triangle between the Lagagrangians L_i which is cut-out transverally and which projects to a holomorphic triangle in $\mathbb{R} \times S^*\Sigma$ contributing to the coefficient $\mathfrak{m}_-^{012}$. □

Remark 4.6.6. *It can further be shown that the number of immersed geodesic triangles in $\mathcal{T}$ only depends on the length of γ_1, the angle between the γ_0 and γ_1 at $p_{0,1}$, the angle between γ_1 and γ_2 at $p_{1,2}$, and the distance between $p_{0,1}$ and $p_{1,2}$. To see it, it suffices to lift γ_0 and γ_1 to $\mathbb{D}$ such that $p_{0,1}$ is lifted to $0 \in \mathbb{D}$. The possible lifts of*

γ_2 all differ by the action of the hyperbolic translation of length $\ell(\gamma_1)$ along $\widetilde{\gamma}_1$, see Figure 4.6.9.

Remark 4.6.7 (Torus bundle domains: open-string products)**.** *In the case of a torus bundle domain, consider three periodic orbits $\mathcal{O}_0, \mathcal{O}_1, \mathcal{O}_2 \subset \mathbb{T}^2$ of the monodromy matrix corresponding to three closed orbits $\Lambda_0, \Lambda_1, \Lambda_2$ of the Anosov flow. Using the notations of Proposition 4.6.2 and Remark 4.6.3, consider a chord $c_{0,1} \in HW^*(\mathcal{L}_{\mathcal{O}_0}, \mathcal{L}_{\mathcal{O}_1})$ from $p_0 \in \mathcal{O}_0$ to $q_1 \in \mathcal{O}_1$ of slope $z_{0,1}^{\pm} \in \Gamma^{\pm}_{p_0,q_1}$, and a chord $c_{1,2} \in HW^*(\mathcal{L}_{\mathcal{O}_1}, \mathcal{L}_{\mathcal{O}_2})$ from $p_1 \in \mathcal{O}_1$ to $q_2 \in \mathcal{O}_2$ of slope $z_{1,2}^{\pm} \in \Gamma^{\pm}_{p_1,q_2}$. Recall that the slope is measured with respect to the fundamental domain $\mathbb{T}^2 \times [0, \nu)$, and we can view $z_{i,j}^{\pm}$ as elements of $[0, \nu)$. We don't necessarily have $q_1 = p_1$, but there exists $k_1 \in \mathbb{Z}$ such that $A^{k_1} p_1 = q_1$, and the set of such exponents is of the form $\mathcal{K}_1 = k_1 + |\mathcal{O}_1| \cdot \mathbb{Z}$. Note that the direction of $A^k c_{1,2}$ viewed as a chord in $\mathbb{T} \times [k\nu, (k+1)\nu)$ becomes $R_{\pm}\big(z_{1,2}^{\pm} + k\nu\big)$. It easily follows that for each $k \in \mathcal{K}_1$, there exists a unique chord $c_{0,2} \in HW^*(\mathcal{L}_{\mathcal{O}_0}, \mathcal{L}_{\mathcal{O}_2})$ from p_0 to $A^k q_2$ such that there exists a topological disk with three boundary punctures mapping to M with punctures asymptotic to $c_{0,1}$, $c_{1,2}$ and $c_{0,2}$, and boundary components in Λ_0, Λ_1 and Λ_2, see Figure 4.6.10. Here, we are using that the slope of the lift $\widetilde{R}_{\pm}$ of $R_{\pm}$ to $\mathbb{T}^2 \times \mathbb{R}_z$ is a monotonous function of $z \in \mathbb{R}$. Moreover, this topological disk is unique up to homotopy. However, we do not know whether some of these disks can be realized as Floer solutions contributing to the product $\mathfrak{m}_{\pm}\big(c_{0,1}, c_{1,2}\big)$.*

4.7 Discussion and future directions

We now list some questions that remain open or unexplored. Fix an Anosov Liouville domain $V = [-1, 1] \times M$. Recall that we have a distinguished family of exact cylindrical Lagrangians $\{\mathcal{L}_\Lambda\}_\Lambda$, indexed by the simple closed orbits of the corresponding Anosov flow on M, and $\mathcal{W}_0(V)$ (the *orbit category* of the flow) is the full subcategory of the wrapped Fukaya category on these objects.

Bridge to Anosov theory. As noted in the Introduction, the symplectic invariants of V are homotopy invariants of the Anosov flow, so it is natural to wonder whether they can be used to address questions in Anosov theory. In this direction, a fundamental question is to determine whether the symplectic invariants are in fact *topological* invariants of the flow, that is, invariants up to topological (i.e. orbital) equivalence.[19]

Question 4.7.1. *If two Anosov flows are topologically equivalent, are the associated Anosov Liouville structures exact symplectomorphic?*

In Chapter 4, we answer this question affirmatively and provide a detailed sketch of proof; the full details will appear elsewhere. This implies that all invariants which depend only on the Liouville isomorphism class of the associated Anosov Liouville structure are topological invariants of the flow. One can also wonder about the converse:

Question 4.7.2. *Conversely, if two Anosov Liouville structures are exact symplectomorphic, are the underlying Anosov flows topological equivalent?*

An affirmative answer to the above two questions would yield a complete dictionary between the symplectic geometry of Anosov Liouville domains, and the study of 3-dimensional Anosov flows. A possible approach to the second question consists of considering the skeletons of the corresponding Anosov Liouville structures, as the Liouville flow coincides with the Anosov flow along these, and try to determine whether they can be isotoped to each other, in order to obtain the desired equivalence.

Now, as the orbit category is only defined up to quasi-equivalence,[20] its quasi-equivalence class is then a homotopy invariant of the flow, and one might try to refine

[19]Two flows are topologically equivalent, or orbit equivalent, if there exists a *homeomorphism* sending the oriented trajectories of one to the oriented trajectories of the other, without necessarily preserving the parametrization. It follows from Anosov's structural stability that if two Anosov flows are homotopic through Anosov flows, then they are topologically equivalent.

[20]An A_∞-functor $F : \mathcal{A} \to \mathcal{B}$ between A_∞-categories is a quasi-equivalence if the cohomological functor $H^*F : H^*\mathcal{A} \to H^*\mathcal{B}$ is an equivalence of (ordinary) categories. More generally, two A_∞-categories are quasi-equivalent if there exists a zig-zag of quasi-equivalences between them.

this. Namely, as a particular case of the above questions, we can ask whether it recovers its topological equivalence class:

Question 4.7.3. *If two smooth Anosov flows are topologically equivalent, are their orbit categories quasi-equivalent? Reciprocally, if two smooth (transitive) Anosov flows have quasi-equivalent orbit categories, are they topologically equivalent?*

By a result of Barthelmé and Mann [BM23], for the case of $\mathbb{R}$-covered Anosov flows,[21] spectral rigidity reduces the above question to whether the quasi-equivalence class of the orbit category recovers the set of free homotopy classes represented by the closed orbits. As an interesting subclass of examples, this includes all *Reeb Anosov flows* (which coincide with skewed $\mathbb{R}$-covered Anosov flows up to topological equivalence, by a recent result of Théo Marty [Mar23].)

Along the same lines as discussed above, the following seems to be an open question in Anosov theory, communicated to us by Rafael Potrie.

Question 4.7.4. *If two Anosov flows are topologically homotopic (i.e., topologically equivalent via an equivalence which is homotopic to the identity), are they homotopic as Anosov flows?*

This question would admit an affirmative answer, provided one can approximate a given topological homotopy by a C^1-diffeomorphism, such that the weak-unstable distributions and the direction of the flow stay C^1-close. In the approach we envision for addressing Question 4.7.1 above, a similar approximation is needed, although in general it only yields C^0-closeness, which is not enough.

While we expect a positive answer to Question 4.7.4, theoretically one could try to detect a negative answer via the symplectic invariants of the associated domain, as they are homotopy invariants. For instance, it seems to be an open question

[21] An Anosov flow on a 3-manifold M is $\mathbb{R}$-covered if its stable *leaf space* (or equivalently, its unstable leaf space) is homeomorphic to $\mathbb{R}$. Here, the (un)stable leaf space is the quotient of the universal cover $\widetilde{M}$ of M by the relation "being on the same leaf of the lift $\widetilde{\mathcal{F}^\star}$ of $\mathcal{F}^\star$", for $\star \in \{u, s\}$.

whether, on a unit tangent bundle over a surface, the space of Riemannian metrics with Anosov geodesic flow is connected. There could potentially be examples of such metrics whose geodesic flows may not be connected, in the space of Anosov flows, to an Anosov flow coming from a metric with negative curvature (the space of which is well-known to be connected). Potential candidates are the surfaces embedded in $\mathbb{R}^3$ constructed by Donnay and Pugh [DP03], which necessarily have regions of positive and negative curvature. However, there is a unique topological equivalence class of Anosov flows on the unit tangent bundle of a surface [Ghy84], and therefore the symplectic invariants will *not* distinguish the Donnay–Pugh examples from the standard examples, if Question 4.7.1 holds true. Moreover, the set of connected components of the space of Anosov flows on the unit tangent bundle of a surface Σ has a subset that is in one-to-one correspondence with $H^1(\Sigma;\mathbb{Z})$, and is therefore infinite (see [Mat13]). But all of these examples are *smoothly* equivalent (via an equivalence which is *not* homotopic to the identity), and therefore have isomorphic symplectic invariants.

In this article, we have computed the symplectic invariants of the prototypical *algebraic* Anosov flows. In order to understand the invariants of examples that go beyond these basic cases, a starting point is to understand their behavior under surgeries. Indeed, Handel and Thurston [HT80], and Frank and Williams [FW80], gave the first examples of non-algebraic Anosov flows, via surgery methods. This was generalized by Goodman [Goo83] to a surgery along a curve in the vicinity of a periodic orbit. Moreover, Fried [Fri83] produced *topological* Anosov flows via Dehn surgery also along a periodic orbit. Recently, Shannon[Sha21] proved that Fried's surgery is topologically equivalent to Goodman's, while Salmoiraghi [Sal23; Sal24] has further developed surgery constructions for (projectively) Anosov flows using the supporting bi-contact structures, generalizing Goodman's. On the symplectic side, this raises the following question, which arose from conversations with Surena Hoozori:

Question 4.7.5. *What is the effect of Fried–Goodman surgery on the symplectic invariants (e.g., on the orbit category, or on the symplectic cohomology)?*

The spectral decomposition theorem of Smale [Sma67] applies in particular to Anosov flows, as they satisfy Axiom A, i.e., the non-wandering set of the flow can be decomposed as a finite union of compact invariant sets –the *basic sets*–, on each of which the flow is transitive. Brunella [Bru05] has further shown the existence of incompressible tori which are transverse to the flow and separate the basic sets in the spectral decomposition, by choosing a Lyapunov function for the flow. Transitive Anosov flows can also admit transverse tori, such as the famous Bonatti–Langevin example [BL94]. By analogy to the case of a suspension of a toral Anosov map, in which the fibers are incompressible transverse tori and also weakly exact Lagrangian after taking the product with $\mathbb{R}$, one can ask:

Question 4.7.6. *Do the Brunella tori of a non-transitive Anosov flow, and more generally tori transverse to an Anosov flow, give rise to weakly exact Lagrangian tori?*

The open-closed map. In Theorem 4.1.1, we showed that the contractible component $\mathcal{OC}_0^c$ of the open-closed map restricted to the orbit category is neither surjective nor injective (its kernel has in fact infinite rank). The non-contractible component $\mathcal{OC}_0^{nc}$ seems highly nontrivial and might contain interesting information about the underlying Anosov flow. It is natural to ask:

Question 4.7.7. *What are the image and kernel of the non-contractible component of the open-closed map* $\mathcal{OC}_0^{nc} : \mathrm{HH}_{*-2}^{nc} \to SH_{nc}^*(V)$*?*

We do not expect $\mathcal{OC}_0^{nc}$ to be injective in general (see Remark 4.3.3). When the underlying Anosov flow is transitive, its simple closed orbits form a dense subset of M. Hence, for every closed Reeb orbit $\gamma_\pm$, there exist a finite collection of Lagrangian cylinders $\mathcal{L}_{\Lambda_1}, \ldots, \mathcal{L}_{\Lambda_k}$ and Reeb chords $c_1^\pm : \Lambda_1 \to \Lambda_2, \ldots, c_k^\pm : \Lambda_k \to \Lambda_1$ such that $\gamma^\pm$

is freely homotopic to a loop obtained by concatenating the chords c_i's and paths in the $\mathcal{L}_{\Lambda_i}$'s in an alternating way. In other words, there are no obvious topological obstructions to the existence of holomorphic annuli in the chain level definition of $\mathcal{OC}_O^{nc}$ hitting a generator of SC_{nc}^* corresponding to $\gamma_\pm$. This raises the following

Question 4.7.8. *If the underlying Anosov flow is transitive, is $\mathcal{OC}_0^{nc}$ surjective?*

Split-generation. The following are questions pertaining to the wrapped Fukaya category $\mathcal{W}(V)$.

Question 4.7.9. *We have the following alternative. Either*

- *$\mathcal{W}_0(V)$ split-generates $\mathcal{W}(V)$, and therefore $\mathcal{W}(V)$ is not homologically smooth; or*
- *$\mathcal{W}_0(V)$ does not split-generate $\mathcal{W}(V)$, i.e., there exists an exact Lagrangian (closed or with cylindrical ends) in V that is not split-generated by the Lagrangian cylinders.*

Which one is true?

While we have not established split-generation, it is still possible that the Lagrangians $\mathcal{L}_\Lambda$ determine $\mathcal{W}(V)$ in the following weaker sense:

Question 4.7.10. *Is the restricted Yoneda functor*

$$\mathcal{W}(V) \to \mathcal{W}_0(V) - \text{Mod}$$

full and faithful?

Split-generation would imply an affirmative answer to this question, but the converse does not hold. For example, the zero-section $M \subset T^*M$ does not split-generate $\mathcal{W}(T^*M)$ (since it is a proper object in a non-proper A_∞-category), but if

M is simply-connected, then the restricted Yoneda embedding to $C^*(M) - \mathrm{Mod}$ is full and faithful by Koszul duality; see [Ada56; LS23].

Assume now that $V = [-1, 1] \times M$ is a torus bundle domain. Let $\mathcal{W}_0'(V)$ be the wrapped Fukaya category of V (with coefficients in the *Novikov ring* $\Lambda_{\mathbb{Z}}$) containing $\mathcal{W}_0(V) \otimes \Lambda_{\mathbb{Z}}$ and the relatively exact Lagrangian tori $\mathbb{T}_{s,z}$ given by the fibers of the torus fibration on V. The open-closed map $\mathcal{OC}_0 : \mathrm{HH}_{*-2}(\mathcal{W}_0(V)) \otimes \Lambda_{\mathbb{Z}} \to SH^*(V) \otimes \Lambda_{\mathbb{Z}}$ factors as

$$\begin{array}{ccc} \mathrm{HH}_{*-2}(\mathcal{W}_0(V)) \otimes \Lambda_{\mathbb{Z}} & \xrightarrow{\mathcal{OC}_0} & SH^*(V) \otimes \Lambda_{\mathbb{Z}} \\ \downarrow & \nearrow_{\mathcal{OC}_0'} & \\ \mathrm{HH}_{*-2}(\mathcal{W}_0'(V)) & & \end{array}$$

Recalling that $\mathcal{OC}_0$ does not hit the unit, we conjecture the following stronger claim:

Conjecture 4.7.11. *$\mathcal{OC}'$ does not hit the unit. Moreover, $\mathcal{W}_0'(V)$ is not homologically smooth.*

This conjecture would be true if the following question admits an affirmative answer:

Question 4.7.12. *Are the fibers $\mathbb{T}_{s,z}$ split-generated by the exact cylinders $\mathcal{L}_\Lambda$?*

We now discuss split-generation for the McDuff domain. As pointed out to us by Mohammed Abouzaid, the Lagrangian foliation by cotangent fibers on $T^*\Sigma$ restricts to a foliation on the McDuff domain by punctured cotangent fibers, which are still Lagrangian (note that the magnetic term in the twisted symplectic form is horizontal, and therefore vanishes along the fibers). Abouzaid [Abo11a] proved that a cotangent fiber generates the wrapped Fukaya category of $T^*\Sigma$. However, the punctured cotangent fibers in the McDuff domain are *not* exact Lagrangians, as their boundary on the prequantization end is a Reeb orbit and thus not Legendrian. It is conceivable that they form the objects of an enlarged category, by taking into account the fact that the geometry of a twisted cotangent bundle is geometrically bounded

(cf. [Gro23]). Assuming one can enlarge the wrapped Fukaya category to include such Lagrangians as objects, the following question is natural.

Question 4.7.13. *Do the punctured cotangent fibers generate an "enlarged" wrapped Fukaya category of the McDuff domain?*

Structure of the orbit category. For a general Anosov Liouville domain with orbit category $\mathcal{W}_0$, Theorem 4.1.11 implies that the cohomology category $H^*\mathcal{W}_0$ can be interpreted as a suitable fiber product of categories. Indeed, we can define A_∞-categories $\mathcal{W}_0^-$, $\mathcal{W}_0^+$ and $\mathcal{W}_0^c$ whose objects are the same as $\mathcal{W}_0$ and whose morphism chain complexes are generated by chords in M_- and interior intersection points, chords in M_+ and interior intersection points, and only interior intersection points, respectively. We can also define *non-unital* A_∞-categories $\mathcal{I}^-$ and $\mathcal{I}^+$ whose objects are the same as $\mathcal{W}_0$ and whose morphism chain complexes are generated by chords in M_- only and chords in M_+ only, respectively. There are faithful functors $\mathcal{I}^\pm \to \mathcal{W}_0^\pm$ making $H^*\mathcal{I}^\pm$ ideals in $H^*\mathcal{W}_0^\pm$, in the sense that composing a morphism in $H^*\mathcal{I}^\pm$ with any morphism in $H^*\mathcal{W}_0^\pm$ gives a morphism in $H^*\mathcal{I}^\pm$. Moreover, the quotients[22] $H^*\mathcal{W}_0^\pm/H^*\mathcal{I}^\pm$are isomorphic to $H^*\mathcal{W}_0^c$ and $H^*\mathcal{W}_0$ becomes the (strict) fiber product

$$\begin{array}{ccc} H^*\mathcal{W}_0 & \longrightarrow & H^*\mathcal{W}_0^- \\ \downarrow & \lrcorner & \downarrow \\ H^*\mathcal{W}_0^+ & \longrightarrow & H^*\mathcal{W}_0^c \end{array}$$

where the functors $H^*\mathcal{W}_0^\pm \to H^*\mathcal{W}_0^c$ are the quotients by $H^*\mathcal{I}^\pm$. Note that $H^*\mathcal{W}_0^c$ is isomorphic to a disjoint union

$$H^*\mathcal{W}_0^c = \bigsqcup_\Lambda \mathbb{Z}[x]/x^2$$

indexed by the simple closed orbits of the underlying Anosov flow.

[22]Here, we are considering the naive quotient at the level of morphisms. It is not a localization.

Similarly, $\mathcal{I}^{\pm} \subset \mathcal{W}_0^{\pm}$ are "A_∞-ideals", in the sense that any A_∞-product in $\mathcal{W}_0^{\pm}$ involving a morphism in $\mathcal{I}^{\pm}$ gives a morphism in $\mathcal{I}^{\pm}$, and the quotients $\mathcal{W}_0^{\pm}/\mathcal{I}^{\pm}$ are quasi-isomorphic to $\mathcal{W}_0^c$.[23]

Question 4.7.14. *Is $\mathcal{W}_0$ the homotopy fiber product of*

$$\begin{array}{ccc} & & \mathcal{W}_0^- \\ & & \downarrow \\ \mathcal{W}_0^+ & \longrightarrow & \mathcal{W}_0^c \end{array}$$

where the A_∞-functors $\mathcal{W}_0^{\pm} \to \mathcal{W}_0^c$ are induced by the quotients by $\mathcal{I}^{\pm}$?

The subtlety here is to understand products in $\mathcal{W}_0$ involving both chords in M_+ and in M_- as inputs. Using a similar neck-stretching argument as in the proof of Theorem 4.1.11 (2) and using Lemma 4.3.1, one can show that for given inputs, there exist suitable Floer data so that the corresponding product vanishes. We expect that a careful filtration argument should lead to a positive answer to the previous question. These (homotopy) fiber product descriptions of $\mathcal{W}_0$ and $H^*\mathcal{W}_0$ mirror the geometric construction of V as "gluing two symplectizations along the skeleton".

We expect that the *Rabinowitz wrapped Fukaya category* of Ganatra–Gao–Venkatesh (see [GGV22]) should also enjoy a natural splitting when restricted to the Lagrangian cylinders $\mathcal{L}_\Lambda$, that should mirror the splitting of Rabinowitz Floer homology in Theorem 4.1.6.

[23]Here, we are considering a quotient by a *non-full* and *non-unital* subcategory.

Chapter 5

Orbit equivalence of Anosov flows and Liouville structures

5.1 Introduction

5.1.1 Notations

Let M and M' be two closed oriented connected 3-manifolds. We consider two *oriented* smooth Anosov flows Φ and Φ' on M and M', generated by smooth vector fields X and X', respectively. We fix once and for all Riemannian metrics on M and M', and all the distances will be measures with respect to these metrics.

We denote by $\mathcal{P} \coloneqq \Gamma\big(\widetilde{\mathrm{Gr}_2}(TM)\big)$ the space of continuous, oriented plane fields on M, with the induced metric coming from the chosen Riemannian metric on M.

We will typically denote by $\mathcal{F}$ and $\mathcal{F}'$ the weak-unstable foliations of Φ and Φ'. Classically, these foliations are $\mathcal{C}^1$. We will write $\eta = T\mathcal{F}$ and $\eta' = T\mathcal{F}'$.

Recall that an **orbit equivalence** $h : M \to M'$ between Φ and Φ' is a homeomorphism sending flow lines of Φ to flow lines of Φ'. It is a classical fact that it sends the leaves of $\mathcal{F}$ to leaves of $\mathcal{F}'$. The orbit equivalence h is **oriented** if it is orientation preserving, preserves the orientation on flow lines, as well as the orientation on the

leaves of the weak-unstable foliations. Any orbit equivalence between Φ and Φ' can be made oriented, after possibly changing the orientation on M', on Φ', and on $\mathcal{F}'$.

By Chapter 3, Φ and Φ' induce standard *linear* Anosov Liouville structures on $V \coloneqq [-1,1] \times M$ and $V' \coloneqq [-1,1] \times M'$, which are well-defined up to exact symplectomorphism. We denote these Liouville structures λ and λ', respectively. We refer to Chapter 3 for the precise definitions of these structures.

5.1.2 Main results

We now assume that the Anosov flows Φ and Φ' are orbit equivalent, via an *oriented* orbit equivalence $h : M \to M'$. The main result of this chapter is

Theorem 5.1.1. *The Liouville domains (V, λ) and (V', λ') are exact symplectomorphic.*

This implies:

Corollary 5.1.2. *If (ξ_-, ξ_+) and (ξ'_-, ξ'_+) are bicontact structures supporting Φ and Φ', respectively, then $\xi_\pm$ and $\xi'_\pm$ are contactomorphic.*

Proof. First of all, we can assume that (ξ_-, ξ_+) and (ξ'_-, ξ'_+) support standard Anosov Liouville pairs, since the space of bicontact structures supporting a given (projectively) Anosov flow is contractible. By Theorem 5.1.1, these Anosov Liouville pairs induce exact symplectomorphic Liouville structures on $[-1,1] \times M$, hence their contact boundaries are contactomorphic, which implies the desired result. □

The proof of Theorem 5.1.1 has three key ingredients which are completely independent of each other.

Smoothing the orbit equivalence. The first ingredient is a careful smoothing result for the orbit equivalence h. Namely, we approximate h by a smooth diffeomorphism while keeping some control on η.

Proposition 5.1.3. *For every $\epsilon > 0$, there exists a smooth diffeomorphism $\widetilde{h} : M \to M'$ satisfying*

$$d_{\mathcal{C}^0}(h, \widetilde{h}) < \epsilon, \qquad d_{\mathcal{C}^0}\big(\eta, \widetilde{h}^* \eta'\big) < \epsilon.$$

There are two crucial observations in the proof of this proposition:

1. h sends $\mathcal{F} = \mathcal{F}_\Phi^{\mathrm{wu}}$ to $\mathcal{F}' = \mathcal{F}_{\Phi'}^{\mathrm{wu}}$, and both of these foliations are $\mathcal{C}^1$,
2. Continuous increasing function on $I = [0, 1]$ can easily be approximated (in parametric and relative ways) by smooth functions with *positive* derivative.

Uniqueness of contact approximations. The second ingredient is a variant of a deep result of Vogel [Vog16] concerning the uniqueness of contact approximations to suitable foliations.

Since $\mathcal{F}$ is an Anosov foliation, there exists a smooth 1-form β on M such that $d\beta$ is positive on the leaves of $\mathcal{F}$. Let $\mathcal{P}_+ \subset \mathcal{P}$ denote the space of oriented plane fields ζ on M satisfying $d\beta_{|\zeta} > 0$. Let (ξ_-, ξ_+) be the bicontact structure determined by λ. We can assume $\xi_\pm \in \mathcal{P}_+$.

Proposition 5.1.4 (Proposition 5.3.1)**.** *There exists a $\mathcal{C}^0$-neighborhood $\mathcal{N} \subset \mathcal{P}_+$ of η such that every positive (resp. negative) contact structure $\xi \in \mathcal{N}$ is contact homotopic to ξ_+ (resp. ξ_-) within $\mathcal{N}$.*

It will be crucial that the contact isotopy stay within $\mathcal{P}_+$.

Deformation of *exact* weak symplectic fillings. The third and last ingredient is a variation on a classical result of Eliashberg [Eli91; Eli04] concerning the deformation of exact weak symplectic fillings of contact manifolds.

Proposition 5.1.5 (Proposition 5.4.2)**.** *Let V be a compact manifold with boundary, $(\lambda_t)_{t\in[0,1]}$ be a path of 1-forms on V, and $(\xi_t)_{t\in[0,1]}$ be a path of contact structures on ∂V. Assume that*

- *For $i \in \{0,1\}$, λ_i is a Liouville form and* $\ker \lambda_{i|\partial V} = \xi_i$.
- *For every $t \in [0,1]$, $d\lambda_t$ is symplectic and $d\lambda_{t|\xi_t} > 0$ along ∂V.*

Then λ_0 and λ_1 are Liouville isotopic, hence exact symplectomorphic.

5.1.3 Proof of Theorem 5.1.1

We explain how to put our three ingredients together to prove our main result. All the structures under consideration will be $\mathcal{C}^1$, which is enough for our purposes.

By definition, we can choose λ to be a $\mathcal{C}^1$ Anosov Liouville structure of the form

$$\lambda = \beta + \tau\alpha,$$

where α and β are 1-forms of class $\mathcal{C}^1$ satisfying

$$\alpha \wedge d\alpha = 0, \tag{5.1.1}$$

$$\alpha \wedge d\beta > 0, , \tag{5.1.2}$$

and $\ker \alpha = T\mathcal{F} = \eta$. We make similar choices for λ'.

Let $\mathcal{N} \subset \mathcal{P}_+$ as in Proposition 5.1.4. By Proposition 5.1.3, there exists a diffeomorphism $\widetilde{h}$ of M such that $\widetilde{h}^*\eta' \in \mathcal{N}$. We can therefore assume that X and X' are both defined on M, and that $\eta' \in \mathcal{N}$. Since $\mathcal{N} \subset \mathcal{P}_+$, then for any $\zeta \in \mathcal{N}$, $d\beta_{|\zeta} > 0$. In particular,

$$\alpha' \wedge d\beta > 0. \tag{5.1.3}$$

After possibly replacing α with $C\alpha$ for $C \gg 1$ large enough, we can further assume that $\xi_\pm \coloneqq \ker(\beta \pm \alpha)$ satisfy $\xi_\pm \in \mathcal{N}$. We similarly rescale α' so that $\xi'_\pm \in \mathcal{N}$. By Proposition 5.1.4 there exist paths of contact structures $(\xi^t_\pm)_{t\in[0,1]}$ on M such that

- $\xi_\pm^0 = \xi_\pm$, $\xi_\pm^1 = \xi'_\pm$,
- For every $t \in [0,1]$, $d\beta_{|\xi_\pm^t} > 0$.

We now construct a path of $\mathcal{C}^1$ 1-forms on $V = [-1,1]_\tau \times M$ connecting λ and λ', and satisfying the assumptions of Proposition 5.1.5. Let $K \gg 1$ and $k(t) \coloneqq 1+(1-t)K$. For $t \in [0,1]$, we define:

$$\begin{aligned}
\lambda_t^1 &\coloneqq k(t)\beta + \tau\alpha, & \xi_\pm^{1,t} &\coloneqq \xi_\pm,\\
\lambda_t^2 &\coloneqq K\beta + \tau\alpha, & \xi_\pm^{2,t} &\coloneqq \xi_\pm^t,\\
\lambda_t^3 &\coloneqq K\beta + \tau\big[(1-t)\alpha + t\alpha'\big], & \xi_\pm^{3,t} &\coloneqq \xi'_\pm,\\
\lambda_t^4 &\coloneqq K\big[(1-t)\beta + t\beta'\big] + \tau\alpha', & \xi_\pm^{4,t} &\coloneqq \xi'_\pm,\\
\lambda_t^5 &\coloneqq k(1-t)\beta' + \tau\alpha', & \xi_\pm^{5,t} &\coloneqq \xi'_\pm.
\end{aligned}$$

Using (5.1.1), (5.1.2), and (5.1.3) above, it is easy to check that if K large enough, then for all $1 \leq i \leq 5$ and $t \in [0,1]$, $d\lambda_t^i$ is symplectic and dominates $\xi_\pm^{i,t}$ along $\partial V = \{\pm 1\} \times M$. Therefore, if $\big(\widetilde{\lambda}_t\big)_t$ denotes the (continuous) path obtained by concatenating $\big(\lambda_t^1\big)_t, \ldots, \big(\lambda_t^5\big)_t$, and $\big(\widetilde{\xi}_\pm^t\big)_t$ denotes the path obtained by concatenating $\big(\xi_\pm^{1,t}\big)_t, \ldots, \big(\xi_\pm^{5,t}\big)_t$, then $\big(\widetilde{\lambda}_t\big)_t$ and $\big(\widetilde{\xi}_\pm^t\big)_t$ satisfy the conditions of Proposition 5.1.5, hence $\widetilde{\lambda}_0 = \lambda$ and $\widetilde{\lambda}_t = \lambda'$ are Liouville homotopic, hence exact symplectomorphic. $\square$

5.2 Smoothing orbit equivalences

In this section, we sketch the proof of Proposition 5.1.3 from the introduction. This constitutes the main new ingredient in our strategy.

5.2.1 Preliminaries

Recall that h is orientation preserving, preserves the orientations of the flow lines of Φ and Φ', as well as the orientations of the leaves of $\mathcal{F}$ and $\mathcal{F}'$.

Definition 5.2.1. *A $\mathcal{C}^1$ coordinate system (x, y, z) near $p \in M$ is* ***adapted to*** Φ *if in these coordinates,*

$$X = \partial_x, \qquad \eta = \operatorname{span}\{\partial_x, \partial_y\}. \tag{5.2.1}$$

There exists a uniform constant $\delta_0 > 0$ such that every open ball of radius less than δ_0 in M admits coordinates adapted to Φ. There exists a similar constant δ_0' for Φ' on M'.

Notice that if (x, y, z) are coordinates near $p \in M$ adapted to Φ and (x', y', z') are coordinates near $p' \in M'$ adapted to Φ', then the orbit equivalence h is locally of the form

$$h(x, y, z) = \big(a(x, y, z), b(y, z), c(z)\big), \tag{5.2.2}$$

where a, b, and c are continuous functions. Moreover, $x \mapsto a(x, \cdot, \cdot)$, $y \mapsto b(y, \cdot)$ and $z \mapsto c(z)$ are *strictly increasing*.

To approximate h by a diffeomorphism, we will approximate the functions c, b, and a locally by $\mathcal{C}^1$ ones. Importantly, while these functions depend on the choices of adapted coordinates, their regularity class **does not**.

To approximate h *globally*, we will consider a sufficiently fine triangulation $\mathcal{T}$ of M together with adapted coordinates near each simplex of $\mathcal{T}$, and successively approximate the locally defined functions a, b, c by induction on the skeleton. The delicate part of the proof is to ensure the differential of $\widetilde{h}$, the smoothing of h, is invertible at every point of M.

5.2.2 Good covers

Let $0 < \delta < \delta_0$ and $\mathcal{T}$ be a triangulation of M. We say that $\mathcal{T}$ is δ-**fine** is each of its simplices is included in a ball of radius $\delta/2$. We now assume that $\mathcal{T}$ is δ-fine

and in general position with respect to $\mathcal{F}$ and X, which can always be achieved by considering a sufficiently fine subdivision of $\mathcal{T}$. For $0 \leq i \leq 3$, we will write

$$\mathcal{T}_i \coloneqq \big\{ \mathfrak{t} \in \mathcal{T} \;\big|\; \dim(\mathfrak{t}) = i \big\}.$$

A **δ-good cover** $\big(\mathcal{U}, \boldsymbol{\varphi}\big)$ of M adapted to $\mathcal{T}$ and Φ is a collection $\mathcal{U} = (U_\mathfrak{t})_{\mathfrak{t} \in \mathcal{T}}$ of open subsets of M indexed by the simplices of $\mathcal{T}$, together with a collection of diffeomorphisms $\boldsymbol{\varphi} = (\varphi_\mathfrak{t})_{\mathfrak{t} \in \mathcal{T}}$, $\varphi_\mathfrak{t} : U_\mathfrak{t} \to (0,1)^3$, such that for every $\mathfrak{t} \in \mathcal{T}$,

1. $\mathfrak{t} \subset U_\mathfrak{t}$, $\operatorname{diam}(U_\mathfrak{t}) < \delta$, and $\varphi_\mathfrak{t}$ defines coordinates on $U_\mathfrak{t}$ adapted to Φ,
2. If $\mathfrak{t}' \in \mathcal{T}$, then $U_\mathfrak{t} \cap U_{\mathfrak{t}'} \subset U_{\mathfrak{t} \cap \mathfrak{t}'}$,
3. If $\dim(\mathfrak{t}) \geq 1$, the set

$$N_\mathfrak{t} \coloneqq \varphi_\mathfrak{t} \left(U_\mathfrak{t} \cap \bigcup_{\mathfrak{t}' \in \partial \mathfrak{t}} U_{\mathfrak{t}'} \right) \subset (0,1)^3$$

contains the ℓ^1-neighborhood of radius $2w_\mathfrak{t}$ of a union of $\dim(\mathfrak{t})$ pairs of opposite faces of $[0,1]^3$, for some width $0 < w_\mathfrak{t} < 1/10$.

Moreover, for every $\mathfrak{t}' \in \mathcal{T}$ with $\dim(\mathfrak{t}') = \dim(\mathfrak{t})$, the set

$$\varphi_\mathfrak{t} \left(U_\mathfrak{t} \cap U_{\mathfrak{t}'} \right) \subset N_\mathfrak{t}$$

is contained in the ℓ^1-neighborhood of radius $w_\mathfrak{t}$ of $\partial [0,1]^3$.

See Figure 5.2.1 for a 2-dimensional illustration. Notice that one can modify the width $w_\mathfrak{t}$ by simply modifying the diffeomorphism $\varphi_\mathfrak{t}$. Good covers of M can easily be constructed by first considering a sufficiently fine and generic triangulation $\mathcal{T}$ of M, and then proceeding by induction on the skeleton of $\mathcal{T}$:

Lemma 5.2.2. *For every $0 < \delta < \delta_0$, there exists a δ-good cover of M adapted to Φ and to some triangulation of M.*

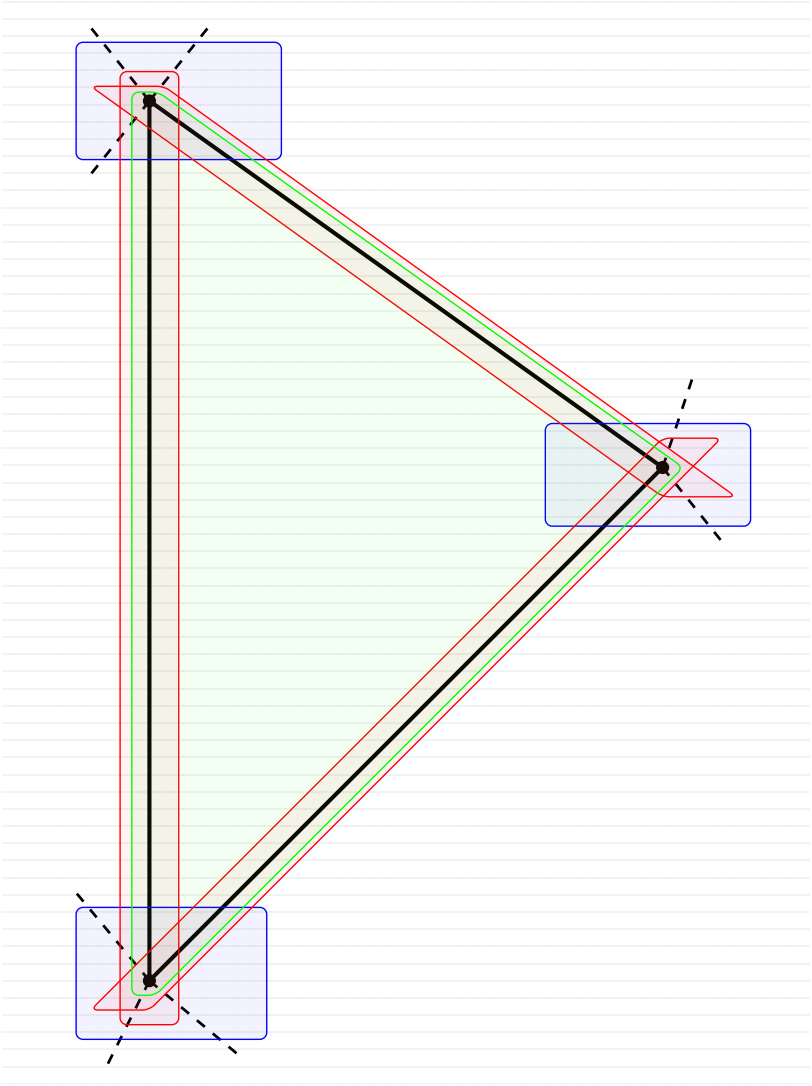


Figure 5.2.1: A 2-dimensional good cover adapted to a 1-dimensional foliation near a 2-simplex.

5.2.3 Approximations of increasing functions

In this section, we describe a 'toy h-principle' for continuous increasing functions from I to $\mathbb{R}$, also known as orientation preserving topological embeddings of the closed 1-dimensional disk into $\mathbb{R}$. The proofs, elementary yet a bit tedious, are left to the reader and will appear elsewhere.

Let $I = [0, 1]$, and $D^n = I^n$. Let $\mathcal{I}(I, \mathbb{R}) \subset \mathcal{C}^0(I, \mathbb{R})$ denote the space of strictly increasing continuous functions on I. For $1 \leq r \leq \infty$, let $\mathcal{C}^r_+(I, \mathbb{R}) \subset \mathcal{I}(I, \mathbb{R})$ denote

the space of $\mathcal{C}^r$ functions with positive derivative on I.

For $\delta \in (0, 1/2)$, let $N^n_\delta \subset D^n$ denote the δ-neighborhood of $\partial D^n \subset D^n$ for the standard ℓ^1 norm on D^n.

If $U \subset D^m \times D^n \times I$ is open and $f : U \to \mathbb{R}$ is $\mathcal{C}^r$, we set

$$|f|_{\mathcal{C}^{0|r|0}} \coloneqq \sup_{x \in D^m, t \in I} \big|f(x, \cdot, t)\big|_{\mathcal{C}^r}.$$

The next lemma is essentially a consequence of an elementary observation: for $1 \leq r \leq \infty$, $\mathcal{C}^r_+(I, \mathbb{R})$ is a *convex* and dense subset of $\mathcal{I}(I, \mathbb{R})$ for the $\mathcal{C}^0$ topology. It serves as a warm-up for the more technical Lemma 5.2.4 below.

Lemma 5.2.3 (Warm-up)**.** *Let $f \in \mathcal{I}(I, \mathbb{R})$, $r \geq 1$, $\delta \in (0, 1/4)$, and $\epsilon > 0$.*

1. *There exists $g \in \mathcal{C}^\infty_+(I, \mathbb{R})$ such that $g(0) = f(0)$, $g(1) = f(1)$, and $\big|f - g\big|_{\mathcal{C}^0} < \epsilon$.*
2. *Let $\bar{f} : [0, 2\delta) \cup (1 - 2\delta, 1] \to \mathbb{R}$ be a $\mathcal{C}^r$ map with positive derivative such that $\big|f - \bar{f}\big|_{\mathcal{C}^0} < \epsilon$ and $\bar{f}(\delta) < \bar{f}(1 - \delta)$. There exists $g \in \mathcal{C}^r_+(I, \mathbb{R})$ such that $g_{\big|_{[0,\delta)\cup(1-\delta,1]}} = \bar{f}_{\big|_{[0,\delta)\cup(1-\delta,1]}}$ and $\big|f - g\big|_{\mathcal{C}^0} < \epsilon$.*

We now give the full version of our 'toy h-principle' for increasing functions:

Lemma 5.2.4. *Let $m \geq 0$, $n \geq 1$, $r \geq 1$, $\epsilon > 0$, $\delta \in (0, 1/4)$, and $f : D^m \times I \to \mathbb{R}$ be a continuous map satisfying*

$$\forall x \in D^m, \quad f_x \in \mathcal{I}(I, \mathbb{R}).$$

- ***(Parametric)*** *There exists a smooth function $g : D^m \times I \to \mathbb{R}$ satisfying*

$$\forall x \in D^m, \quad g_x \in \mathcal{C}^\infty_+(I, \mathbb{R}),$$

and

$$\big|f - g\big|_{\mathcal{C}^0} < \epsilon.$$

- ***(Parametric and ∂I-relative)*** *Let $\bar{f} : D^m \times N^1_{2\delta} \to M$ be a $\mathcal{C}^r$ map satisfying*

$$\forall x \in D^m,\ \forall t \in N^1_{2\delta}, \qquad \partial_t \bar{f}_x(t) > 0, \quad \bar{f}_x(\delta) < \bar{f}_x(1-\delta),$$

and

$$\left|f - \bar{f}\right|_{\mathcal{C}^0} < \epsilon.$$

Then, there exists a $\mathcal{C}^r$ map $g : D^m \times I \to \mathbb{R}$ satisfying

$$\begin{aligned} \forall x \in D^m, &\quad g_x \in \mathcal{C}^r_+(I, \mathbb{R}), \\ \forall (x,t) \in D^m \times N^1_\delta, &\quad g(x,t) = \bar{f}(x,t), \end{aligned}$$

and

$$\left|f - g\right|_{\mathcal{C}^0} < \epsilon.$$

- ***(Parametric and ∂D^n-relative)*** *There exists a constant $C > 0$ depending on f, n, r, and δ, such that the following holds.*

  Let $\bar{f} : \left(D^m \times N^n_{2\delta}\right) \times I \to \mathbb{R}$ be a $\mathcal{C}^r$ map satisfying

$$\forall x \in D^m \times N^n_{2\delta}, \quad \bar{f}_x \in \mathcal{C}^r_+(I, \mathbb{R}),$$

  and

$$\left|f \circ \mathrm{pr}_1 - \bar{f}\right|_{\mathcal{C}^0} < \epsilon,$$

  where $\mathrm{pr}_1 : D^m \times D^n \to D^m$ denotes the projection onto the first disk.

  Then, there exists a $\mathcal{C}^r$ map $g : \left(D^m \times D^n\right) \times I \to \mathbb{R}$ satisfying

$$\begin{aligned} \forall x \in D^m \times N^n_\delta, &\quad g_x = \bar{f}_x, \\ \forall x \in D^m \times D^n, &\quad g_x \in \mathcal{C}^r_+(I, \mathbb{R}), \end{aligned}$$

and

$$\left|f \circ \mathrm{pr}_1 - g\right|_{\mathcal{C}^0} < \epsilon, \qquad \left|g\right|_{\mathcal{C}^{0|r|0}} < C\left(\epsilon + \left|\bar{f}\right|_{\mathcal{C}^{0|r|0}}\right).$$

- ***(Parametric and $\partial\big(D^n \times I\big)$-relative)*** *There exists a constant $C > 0$ (the same as above) depending on f, n, r, and δ such that the following holds.*

  Let $\bar{f} : D^m \times N^{n+1}_{2\delta} \to \mathbb{R}$ be a $\mathcal{C}^r$ map satisfying

$$\forall x \in D^m \times N^n_{2\delta}, \quad \bar{f}_x \in \mathcal{C}^r_+(I, \mathbb{R}),$$
$$\forall (x,t) \in D^m \times D^n \times N^1_{2\delta}, \quad \partial_t \bar{f}_x(t) > 0, \quad \bar{f}_x(\delta) < \bar{f}_x(1-\delta),$$

  and

$$\left|f \circ \mathrm{pr}_1 - \bar{f}\right|_{\mathcal{C}^0} < \epsilon.$$

  Then, there exists a $\mathcal{C}^r$ map $g : \big(D^m \times D^n\big) \times I \to \mathbb{R}$ satisfying

$$\forall (x,t) \in D^m \times N^{n+1}_{\delta}, \quad g(x,t) = \bar{f}(x,t),$$
$$\forall x \in D^m \times D^n, \quad g_x \in \mathcal{C}^r_+(I, \mathbb{R}),$$

  and

$$\left|f \circ \mathrm{pr}_1 - g\right|_{\mathcal{C}^0} < \epsilon, \qquad \left|g\right|_{\mathcal{C}^{0|r|0}} < C\left(\epsilon + \left|\bar{f}\right|_{\mathcal{C}^{0|r|0}}\right).$$

Remark 5.2.5. *For $n \geq 1$ and $r \geq 0$, let $\mathrm{Emb}^r_+(D^n, \mathbb{R}^n)$ denote the space of $\mathcal{C}^r$ orientation-preserving embeddings of the closed disk D^n into $\mathbb{R}^n$. Then for $n \geq 2$ and $r \geq 1$, there exists an element $f \in \mathrm{Emb}^0_+(D^n, \mathbb{R}^n)$ such that **no** $\mathcal{C}^0$ neighborhood $\mathcal{N}$ of f in $\mathrm{Emb}^0_+(D^n, \mathbb{R}^n)$ satisfies that $\mathcal{N} \cap \mathrm{Emb}^r_+(D^n, \mathbb{R}^n)$ is path connected for the $\mathcal{C}^r$ topology! In the case $n = 2$, the reader can think of a continuous embedding of the disk that looks like an infinite spiral near a point. However, two smooth embeddings $D^2 \hookrightarrow \mathbb{R}^2$ which are $\mathcal{C}^0$-close are isotopic via a $\mathcal{C}^0$-small isotopy (which is not $\mathcal{C}^1$-small).*

5.2.4 Proof of Proposition 5.1.3

Let $\epsilon > 0$. Without loss of generality, let us assume that $\epsilon < \delta_0'$.

Let $0 < \delta < \delta_0$ be such that the image of every ball of radius δ in M by h is contained in a ball a radius $\epsilon/2$ in M'. In particular, the latter balls admit coordinates adapted to Φ'.

By Lemma 5.2.2, let $\big(\mathcal{U}, \boldsymbol{\varphi}\big)$ be a good cover of M adapted to $\mathcal{F}$ and to a triangulation $\mathcal{T}$ of M. **These quantities <u>on the domain of h</u> will be fixed throughout the proof.**

For each simplex $\mathfrak{t} \in \mathcal{T}$, we fix coordinates on $h\big(U_\mathfrak{t}\big)$ adapted to Φ'. Defining

$$h_\mathfrak{t} \coloneqq h \circ \varphi_\mathfrak{t}^{-1} : (0,1)^3 \to M',$$

we can write in these coordinates

$$h_\mathfrak{t}(x,y,z) = \big(a_\mathfrak{t}(x,y,z), b_\mathfrak{t}(y,z), c_\mathfrak{t}(z)\big).$$

We will modify these coordinates <u>on the target of h</u> in the proof.

The distances between h and its successive approximations will be measured with respect to **fixed** metrics on M and M', and similarly for the distances between plane fields and line fields.

We now proceed in four steps.

Step 1: smoothing c. We first approximate h by a map $\widetilde{h}_1 : M \to M'$ so that in our chosen coordinates, $\widetilde{h}_{1,\mathfrak{t}} \coloneqq \widetilde{h}_1 \circ \varphi_\mathfrak{t}^{-1}$ is of the form

$$\widetilde{h}_{1,\mathfrak{t}}(x,y,z) = \big(a_\mathfrak{t}(x,y,z), b_\mathfrak{t}(y,z), \widetilde{c}_\mathfrak{t}(x,y,z)\big), \tag{5.2.3}$$

where $a_{\mathfrak{t}}$ and $b_{\mathfrak{t}}$ are as before, and $\widetilde{c}_{\mathfrak{t}}$ is $\mathcal{C}^1$ and is sufficiently $\mathcal{C}^0$-close to $c_{\mathfrak{t}}$. Moreover, we will ensure that $|\partial_x \widetilde{c}_{\mathfrak{t}}|$ and $|\partial_y \widetilde{c}_{\mathfrak{t}}|$ are sufficiently small, and $\partial_z \widetilde{c}_{\mathfrak{t}} > 0$ everywhere. This will imply that $\widetilde{h}_1$ sends $\mathcal{F}$ to a foliation $\mathcal{F}'_1$ which is $\mathcal{C}^1$-close to $\mathcal{F}'$, in the sense that $T\mathcal{F}'$ and $T\mathcal{F}'_1$ are $\epsilon/4$-close. Likewise, $\widetilde{h}$ sends the flow lines of Φ to the flow lines of a new flow Φ'_1 whose line field ℓ'_1 is $\epsilon/4$ -close to the line field ℓ' of Φ' in the $\mathcal{C}^0$ topology.

Let us pick some $0 < \epsilon_0 < \epsilon/4$, to be chosen sufficiently small later. We construct the functions $\widetilde{c}_{\mathfrak{t}}$ by induction on the skeleton of $\mathcal{T}$.

- *The* 0*-skeleton.* For every vertex $\mathfrak{t} \in \mathcal{T}_0$, we simply consider an ϵ_0-approximation $\widetilde{c}_{\mathfrak{t}} = \widetilde{c}_{\mathfrak{t}}(z)$ of $c(z)$ satisfying $\partial_z \widetilde{c}_{\mathfrak{t}} > 0$ given by item 1 of Lemma 5.2.3. Here, the variable z corresponds to the variable t in the lemma.

- *The* 1*-skeleton.* For every edge $\mathfrak{t} \in \mathcal{T}_1$, we apply item 2 of Lemma 5.2.3 with $\delta = w_{\mathfrak{t}}$, by first assuming that ϵ_0 is small enough so that the hypothesis are satisfied. This allows us to extend $\widetilde{c}_{\mathfrak{t}'}$, $\mathfrak{t}' \in \partial\mathfrak{t}$, to a $\mathcal{C}^1$ function $\widetilde{c}_{\mathfrak{t}} = \widetilde{c}_{\mathfrak{t}}(z)$ satisfying $\partial_z \widetilde{c}_{\mathfrak{t}} > 0$.[1]

- *The* 2*-skeleton.* We now consider a 2-face $\mathfrak{t} \in \mathcal{T}_2$. Notice that the approximation $\widetilde{c}_{\mathfrak{t}}$ is already prescribed on $N_{\mathfrak{t}} \subset (0,1)^3$, the $2w_{\mathfrak{t}}$-neighborhood of the union of two pairs of opposite faces of the cube $[-1,1]^3$. Moreover, $\widetilde{c}_{\mathfrak{t}}$ is ϵ_0-close to $c_{\mathfrak{t}}$ on $N_{\mathfrak{t}}$. Concretely, we have to smoothings of $c_{\mathfrak{t}}$ which agree near the boundary of $[0,1]$, and we want to interpolate between them. We can then apply the *parametric and* ∂I*-relative* approximation from Lemma 5.2.4 to extend $\widetilde{c}_{\mathfrak{t}}$ to a $\mathcal{C}^1$ map $\widetilde{c}_{\mathfrak{t}} : (0,1)^3 \to \mathbb{R}^3$, such that $\partial_z \widetilde{c}_{\mathfrak{t}} > 0$ and

$$|\widetilde{c}_{\mathfrak{t}} - c_{\mathfrak{t}}| < \epsilon_0, \qquad |\partial_x \widetilde{c}_{\mathfrak{t}}|_{\mathcal{C}^0} + |\partial_y \widetilde{c}_{\mathfrak{t}}|_{\mathcal{C}^0} < C\epsilon_0,$$

where the constant C depends on $c_{\mathfrak{t}}$, on $w_{\mathfrak{t}}$ and on the various coordinate systems,

[1] Strictly speaking, $\widetilde{c}_{\mathfrak{t}}$ is an extension of the restrictions of $\widetilde{c}_{\mathfrak{t}'}$ to a $w_{\mathfrak{t}}$-neighborhood of the boundary. May the reader excuse this abuse of language.

but not on ϵ_0.

- *The* 3*-skeleton.* Finally, we consider a 3-simplex $\mathfrak{t} \in \mathcal{T}$ and we apply the *parametric and* $\partial\big(D^2 \times I\big)$*-relative* approximation of Lemma 5.2.4 to extend $\widetilde{c}_{\mathfrak{t}}$, defined on a neighborhood of $\partial[-1,1]^3$, to $(0,1)^3$, such that $\partial_z \widetilde{c}_{\mathfrak{t}} > 0$ and

$$|\widetilde{c}_{\mathfrak{t}} - c_{\mathfrak{t}}| < \epsilon_0, \qquad |\partial_x \widetilde{c}_{\mathfrak{t}}|_{\mathcal{C}^0} + |\partial_y \widetilde{c}_{\mathfrak{t}}|_{\mathcal{C}^0} < C' \epsilon_0,$$

  where C' does not depend on ϵ_0.

The maps locally defined by (5.2.3) patch together into a continuous map $\widetilde{h}_1 : M \to M'$ satisfying

$$d_{\mathcal{C}^0}\big(h, \widetilde{h}_1\big) < \widetilde{C}\epsilon_0, \qquad d_{\mathcal{C}^0}\big(\ell', \ell'_1\big) < \widetilde{C}\epsilon_0, \qquad d_{\mathcal{C}^0}\big(\eta', \eta'_1\big) < \widetilde{C}\epsilon_0,$$

where the constant $\widetilde{C}$ does not depend on ϵ_0.[2] We then choose ϵ_0 small enough so that $\widetilde{C}\epsilon_0 < \epsilon/4$.

Step 2: smoothing b**.** For every $\mathfrak{t} \in \mathcal{T}$, we *change* the coordinates on the open set $\widetilde{h}_1(U_{\mathfrak{t}})$, so that they are now adapted to Φ'_1 and $\mathcal{F}'_1$, and $\widetilde{h}_{1,\mathfrak{t}}$ is of the form

$$\widetilde{h}_{1,\mathfrak{t}}(x,y,z) = \big(a_{\mathfrak{t}}(x,y,z), b_{\mathfrak{t}}(y,z), z\big).$$

This is possible since $\partial_z \widetilde{c}_{\mathfrak{t}} > 0$. Essentially, after the first step, the new homeomorphism $\widetilde{h}_1$ is 'transversally smooth' with respect to $\mathcal{F}$ and $\mathcal{F}'_1$.

We can then proceed as in Step 1 to approximate $b_{\mathfrak{t}}$ inductively on the skeleton of $\mathcal{T}$ to obtain $\mathcal{C}^1$ maps $\widetilde{b}_{\mathfrak{t}}(x,y,z)$ satisfying $\partial_y \widetilde{b}_{\mathfrak{t}}(x,y,z) > 0$. We can furthermore ensure that $|\partial_x \widetilde{b}_{\mathfrak{t}}| + |\partial_z \widetilde{b}_{\mathfrak{t}}|$ is sufficiently small. Notice that this requires the various versions of Lemma 5.2.4. These local approximations patch together into a map $\widetilde{h}_2 : M \to M'$

[2]Note that it depends on the derivatives of the various coordinate charts, among other things.

which is $\epsilon/4$-close to $\widetilde{h}_1$. By construction $\widetilde{h}_1$ still maps $\mathcal{F}$ to $\mathcal{F}'_1$, but it maps the flow lines of Φ'_1 to the flow lines of a flow Φ'_2 whose line field is $\epsilon/4$-close to the one of Φ'_1.

Step 3: smoothing a**.** For every $\mathfrak{t} \in \mathcal{T}$, we once again change the adapted coordinates on the open set $\widetilde{h}_2(U_\mathfrak{t})$, so that $\widetilde{h}_{2,\mathfrak{t}}$ is of the form

$$\widetilde{h}_{2,\mathfrak{t}}(x,y,z) = \big(a_\mathfrak{t}(x,y,z), y, z\big).$$

This is possible since $\partial_y \widetilde{b}_\mathfrak{t} > 0$. After the second step, $\widetilde{h}_2$ is transversally smooth with respect to $\mathcal{F}$ and $\mathcal{F}'_1$, as well as transversally smooth with respect to Φ and Φ'_1 relatively to the leaves of the previous foliations.

We once again apply the previous argument to approximate the $a_\mathfrak{t}$ by $\mathcal{C}^1$ functions $\widetilde{a}_\mathfrak{t}$ which are $\epsilon/4$-close to $a_\mathfrak{t}$, satisfy $\partial_x \widetilde{a}_\mathfrak{t} > 0$, and patch together into a $\mathcal{C}^1$ map $\widetilde{h}_3$. Notice that $\widetilde{h}_3$ maps $\mathcal{F}$ to $\mathcal{F}'_1$, and maps flow lines of Φ to flow lines of Φ'_2; this last steps only 'reparametrizes' $\widetilde{h}_2$ along the flow of Φ to make it $\mathcal{C}^1$.

Step 4: finishing the proof. To summarize, we obtain a $\mathcal{C}^1$ map $\widetilde{h}_3$ which satisfies

$$d^0_{\mathcal{C}}\big(\widetilde{h}_3, h\big) < \frac{3\epsilon}{4}, \qquad d^0_{\mathcal{C}}\big((\widetilde{h}_3)_*\eta, \eta'\big) < \frac{3\epsilon}{4}.$$

We now argue that for ϵ small enough, $\widetilde{h}_3$ is a diffeomorphism. Indeed, $\widetilde{h}_3$ is open, and it is closed since M is compact (and M' is Hausdorff), so it is surjective since M' is connected. It follows that $\widetilde{h}_3$ is a covering map. If ϵ is small enough, then h and $\widetilde{h}_3$ are homotopic. In particular, they have the same degree which is equal to one, so $\widetilde{h}_3$ is a $\mathcal{C}^1$ diffeomorphism. Finally, we approximate $\widetilde{h}_3$ by a smooth diffeomorphism $\widetilde{h} : M \to M'$ in the $\mathcal{C}^1$ topology. This is possible since $\mathrm{Diff}^\infty(M, M')$ is dense in $\mathrm{Diff}^1(M, M')$ for the $\mathcal{C}^1$ topology; any sufficiently $\mathcal{C}^1$-small approximation of $\widetilde{h}_3$ by a smooth *map* is a smooth diffeomorphism by the previous reasoning. This way, the

plane fields $\widetilde{h}_*\eta$ and η' are ϵ-close in the $\mathcal{C}^0$ topology. □

5.3 Uniqueness of contact approximations

Recall that $\mathcal{F} = \mathcal{F}_\Phi^{\mathrm{wu}}$ and $\eta = \eta_u = T\mathcal{F}$. Let $\mathcal{P}_+ \subset \mathcal{P}$ denote the space of oriented plane fields ζ on M satisfying $d\beta_{|\zeta} > 0$. Let (ξ_-, ξ_+) be the bicontact structure determined by λ. We can assume $\xi_\pm \in \mathcal{P}_+$.

Proposition 5.3.1. *There exists a $\mathcal{C}^0$-neighborhood $\mathcal{N} \subset \mathcal{P}_+$ of η such that every positive (resp. negative) contact structure $\xi \in \mathcal{N}$ is contact homotopic to ξ_+ (resp. ξ_-) within $\mathcal{N}$.*

Sketch of proof. Recall that $\mathcal{F}$ has no closed leaves and has linear holonomy. We explain how the proof of [Vog16, Theorem 5.1] can be adapted. It depends on an arbitrary choice of a rank 1 foliation $\mathcal{I}$ transverse to $\mathcal{F}$. Here, we choose $\mathcal{I}$ to be the integral foliation of $\ker d\beta$, so that a plane field ζ is (positively) transverse to $\mathcal{I}$ if and only if it belongs to $\mathcal{P}_+$. One of the conditions in the definition of Vogel's neighborhood is to be transverse to $\mathcal{I}$, see item (1) in [Vog16, Section 4C1].[3]

Besides, Vogel's theorem is written for $\mathcal{C}^2$-foliations, but $\mathcal{F}$ is only $\mathcal{C}^1$ with a $\mathcal{C}^1$ tangent plane field. However, the $\mathcal{C}^2$ hypothesis is only used to invoke Sacksteder's theorem [Vog16, Theorem 5.2] and ensure that every exceptional minimal set contains a curve with attracting linear holonomy, called a *Sacksteder curve*. Lemma 5.3.2 below immediately implies that every (exceptional) minimal set contains a closed orbit of X, which is such a Sacksteder curve.

With these modifications, the proof of [Vog16, Theorem 5.1] goes through and yields the desired neighborhood $\mathcal{N}$ for positive contact structures. The same holds for negative contact structures by changing the orientation on M. □

[3] Unfortunately, there are some gaps in Vogel's proof for this stronger statement. The details will appear elsewhere.

Lemma 5.3.2 (Folklore)**.** *Let L be a leaf of $\mathcal{F}$. Then $\overline{L}$ contains a closed orbit of Φ.*

Proof. It suffices to show that every leaf of $\mathcal{F} = \mathcal{F}^{\mathrm{wu}}$ intersects a cylindrical leaf of the *weak-stable* foliation $\mathcal{F}^{\mathrm{ws}}$. The latter leaf contains a closed orbit γ of Φ, and any other orbit of Φ on that leaf contains γ in its closure.

Let $\Omega \subset M$ denote the non-wandering set of Φ. By Anosov Closing Lemma [FH19, Theorem 5.3.11], the union of the closed orbits of Φ is a dense subset of Ω. By compactness of M, we can find some $\epsilon > 0$ such that if γ is a periodic orbit of Φ, then every weak-unstable leaf L of $\mathcal{F}$ which intersects the ϵ-neighborhood $U_\epsilon(\gamma)$ of γ also intersects the weak-stable leaf passing through γ. Let U denote the union of the open sets of the form $U_\epsilon(\gamma)$, for γ a closed orbit of Φ. Then $\Omega \subset U$, and every leaf L of $\mathcal{F}$ intersects U since it is saturated by Φ. This concludes the proof. □

5.4 Deformation of weak symplectic fillings

In this section, V be a compact oriented manifold with boundary $\partial V = M$ of arbitrary dimension.

The next lemma is a well-known result due to Eliashberg: a weak symplectic filling which is exact can be deformed near the boundary into a Liouville filling. It serves as a motivation and warm-up for Proposition 5.4.2 below.

Lemma 5.4.1. *Let (V, ω) be a weak symplectic filling of a closed contact manifold (M, ξ) such that ω is exact, and let λ be a primitive of ω. There exists a 1-parameter family $(\lambda_t)_{0 \leq t \leq 1}$ of 1-forms on V such that*

(a) $\lambda_0 = \lambda$ and $\lambda_{1|\partial V}$ is a contact form for ξ,

(b) For every $0 \leq t \leq 1$, $\omega_t \coloneqq d\lambda_t$ is symplectic and dominates ξ along ∂V.

Furthermore, we can assume that $(\lambda_t)_t$ is constant away from an arbitrarily small neighborhood of ∂V.

Proof. We essentially follow the proofs of [Eli91, Proposition 3.1] and [Eli04, Proposition 4.1] with some additional details. We write $\beta \coloneqq \lambda_{|\partial V}$. By assumption, $d\beta_{|\xi} > 0$ and there exists a (unique) contact form α for ξ such that $d\beta_{|\xi} = d\alpha_{|\xi}$. We define $\delta \coloneqq \beta - \alpha$, so that $\alpha \wedge d\delta = 0$. On $(0,1] \times M$, we consider the Liouville form

$$\widetilde{\lambda} \coloneqq t\alpha + \delta$$

and we write $\widetilde{\omega} \coloneqq d\widetilde{\lambda}$. We obtain two coisotropic embeddings of the presymplectic manifold $(M, d\alpha + d\beta)$: one as the boundary of $\big(V, \omega\big)$ and one as the boundary $\{1\} \times M$ of $\big((0,1] \times M, \widetilde{\omega}\big)$. By the local uniqueness of coisotropic embeddings [Got82], there exist

- A neighborhood $\mathcal{U}$ of $\{1\} \times M$ in $(0,1] \times M$,
- A neighborhood $\mathcal{V}$ of ∂V in V,
- A symplectomorphism $\psi : \big(\mathcal{U}, \widetilde{\omega}\big) \to \big(\mathcal{V}, \omega\big)$ such that $\psi_{|\{1\}\times M}$ coincides with the identification $M \cong \partial V$.

By shrinking $\mathcal{U}$ and $\mathcal{V}$, we can further assume that $\mathcal{U}$ is of the form $(1-\epsilon, 1] \times M$ for any sufficiently small $\epsilon > 0$. We obtain coordinates on a tubular neighborhood of ∂V in V in which λ becomes

$$t\alpha + \delta + \theta$$

for some closed 1-form θ satisfying $\theta_{|\partial V} = 0$. This readily implies that θ is exact, and we write $\theta = df$ for some function $f : (1-\epsilon, 1] \times M \to \mathbb{R}$. Let $\varphi_0 : (1-\epsilon, 1] \to [0,1]$ be a smooth non-increasing cutoff function such that $\varphi_0 \equiv 1$ near $1-\epsilon$ and $\varphi_0 \equiv 0$ on $(1-\epsilon/2, 1]$. We then define a 1-form $\lambda_{1/2}$ on V such that $\lambda_{1/2} = \lambda$ outside of $\mathcal{V}$, and

$$\lambda_{1/2} \coloneqq t\alpha + \delta + d(\varphi f)$$

in $\mathcal{V}$. Note that $d\lambda_{1/2} = d\lambda$, $\lambda_{1/2} = t\alpha + \delta$ near ∂V, and there is an obvious homotopy between λ and $\lambda_{1/2}$ satisfying condition (b). Let $C > 1$ and consider a smooth non-decreasing function $\varphi_1 : (1 - \epsilon/2, 1] \to [1, C]$ such that $\varphi_1 \equiv 1$ near $1 - \epsilon/2$ and $\varphi_1 \equiv C$ on $(1 - \epsilon/4, 1]$. We then define

$$\lambda_{3/4} \coloneqq t\varphi_1(t)\alpha + \delta$$

on $(1 - \epsilon/2, 1] \times M$ and extend it to the rest of V by $\lambda_{1/2}$. Since $\alpha \wedge d\delta = 0$, one easily checks that $d\lambda_{3/4}$ is symplectic and dominates ξ along ∂V. Once again, there is an obvious homotopy from $\lambda_{1/2}$ to $\lambda_{3/4}$ satisfying condition (b). Finally, let $\varphi_2 : (1 - \epsilon/4, 1] \to [0, 1]$ be a smooth non-increasing cutoff function such that $\varphi_2 \equiv 1$ near $1 - \epsilon/4$ and $\varphi_2 \equiv 0$ near 1. We define

$$\lambda_1 \coloneqq Ct\alpha + \varphi_2(t)\delta$$

and extend it to the rest of V by $\lambda_{3/4}$. For C large enough, $d\lambda_1$ is symplectic and λ_1 is homotopic to $\lambda_{3/4}$ through 1-forms satisfying condition (b). On ∂V, $\lambda_{1|\partial V} = C\alpha$ is a contact form for ξ. □

Proposition 5.4.2. *Let $(\lambda_t)_{t\in[0,1]}$ be a path of 1-forms on V, and $(\xi_t)_{t\in[0,1]}$ be a path of contact structures on ∂V. Assume that*

(a) For $i \in \{0, 1\}$, λ_i is a Liouville form and $\lambda_{i|\partial V}$ is a contact form for ξ_i.

(b) For every $t \in [0, 1]$, $d\lambda_t$ is symplectic and dominates ξ_t along ∂V.

Then λ_0 and λ_1 are Liouville homotopic, hence exact symplectomorphic.

Proof. First of all, we can assume that the path $(\xi_t)_t$ is constant and equal to a fixed contact structure ξ after composing $(\lambda_t)_t$ with an isotopy of V supported near ∂V.

The deformation of λ near ∂V in the proof of Lemma 5.4.1 can be performed in a parametric way, by first invoking a parametric version of the local uniqueness of

coisotropic embeddings. Moreover, if λ already restricts to a contact form for ξ on ∂V, then so will λ_t, $0 \leq t \leq 1$, since in that case $\delta = 0$. Applying this to the path $(\lambda_t)_t$, we obtain a family of 1-forms $(\lambda_{s,t})_{0 \leq s,t \leq 1}$ such that

- For every t, $\lambda_{0,t} = \lambda_t$,
- For every t, $(\lambda_{s,t})_s$ satisfies the conditions of Lemma 5.4.1,
- For every s, $\lambda_{s,0_{|\partial V}}$ and $\lambda_{s,1_{|\partial V}}$ are contact forms for ξ.

We obtain a Liouville homotopy between λ_0 and λ_1 by concatenating the Liouville homotopies $(\lambda_{s,0})_s$, $(\lambda_{1,t})_t$ and $(\lambda_{1,1-t})_t$. □

Therefore, an exact weak symplectic filling gives rise to a Liouville structure which is well-defined up to Liouville homotopy.

Let us give a more conceptual point of view on these results. Let $\mathfrak{L}(V)$ denote the space of Liouville structures on V. Let

$$\mathscr{L}(V) \coloneqq \pi_0\big(\mathfrak{L}(V)\big)$$

denote the set of Liouville structures on V up to homotopy. We also define a set of (homotopy classes of) a **pre-Liouville structures** on V;

Definition 5.4.3. *A **pre-Liouville structure** on V is a pair (λ, ξ), where $\lambda \in \Omega^1(V)$ and ξ is a contact structure on ∂V, such that $d\lambda$ is symplectic and dominates ξ along ∂V.*

We denote by $\mathfrak{p}\mathfrak{L}(V)$ the space of pre-Liouville structures on V. There is a natural continuous map

$$\begin{array}{rccc} \mathfrak{p}: & \mathfrak{L}(V) & \longrightarrow & \mathfrak{p}\mathfrak{L}(V) \\ & \lambda & \longmapsto & \big(\lambda, \ker \lambda_{|\partial V}\big). \end{array}$$

We also define

$$\wp\mathscr{L}(V) \coloneqq \pi_0\big(\mathfrak{p}\mathfrak{L}(V)\big)$$

as the set of homotopy classes of pre-Liouville structures on V. The map $\mathfrak{p}$ above naturally induces a map

$$\wp : \mathscr{L}(V) \longrightarrow \wp\mathscr{L}(V).$$

Notice that Lemma 5.4.1 means that $\wp$ is surjective while Proposition 5.4.2 means that $\wp$ is injective. Therefore, we get:

Corollary 5.4.4. *The map $\wp$ is bijective.*

Remark 5.4.5. *Our techniques can be extended to show that $\mathfrak{p}$ is a homotopy equivalence. The details are left to the interested reader.*

References

[AS06] Alberto Abbondandolo and Matthias Schwarz. "On the Floer homology of cotangent bundles". In: *Communications on Pure and Applied Mathematics* 59.2 (2006), pp. 254–316.

[Abo10] Mohammed Abouzaid. "A geometric criterion for generating the Fukaya category". In: *Publications Mathématiques de l'IHÉS* 112 (2010), pp. 191–240.

[Abo11a] Mohammed Abouzaid. "A Cotangent Fibre Generates the Fukaya Category". In: *Advances in Mathematics* 228.2 (2011), pp. 894–939.

[Abo11b] Mohammed Abouzaid. "A Topological Model for the Fukaya Categories of Plumbings". In: *Journal of Differential Geometry* 87.1 (2011), pp. 1–80.

[Ada56] John Frank Adams. "On the cobar construction". In: *Proceedings of the National Academy of Sciences of the United States of America* 42 (1956), pp. 409–412.

[Ano63] Vladimir Anosov. "Ergodic properties of geodesic flows on closed Riemannian manifolds of negative curvature". In: *Dokl. Akad. Nauk SSSR* 151.6 (1963), pp. 1250–1252.

[Ano67] Vladimir Anosov. "Geodesic flows on closed Riemannian manifolds of negative curvature". In: *Trudy Mat. Inst. Steklov* 90 (1967), p. 209.

[Asa08] Masayuki Asaoka. "On invariant volumes of codimension-one Anosov flows and the Verjovsky conjecture". In: *Inventiones Mathematicae* 174.2 (2008), p. 435.

[BE13] John Baldwin and John Etnyre. "Transverse surgery does not preserve tightness". In: *Mathematische Annalen* 357.2 (2013), pp. 441–468.

[Bar17] Thomas Barthelmé. *Anosov flows in dimension* 3. *Preliminary version.* 2017. URL: `http://www.crm.umontreal.ca/sms/2017/pdf/diapos/Anosov_flows_in_3_manifolds.pdf`.

[BM23] Thomas Barthelmé and Kathryn Mann. *Orbit equivalences of* $\mathbb{R}$*-covered Anosov flows and hyperbolic-like actions on the line (appendix with Jonathan Bowden).* To appear in *Geometry & Topology*. 2023. arXiv: `2012.11811`.

[Ben83] Daniel Bennequin. "Entrelacements et équations de Pfaff. (French)". In: *Third Schnepfenried geometry conference*. Vol. 107. Astérisque. Société Mathématique de France, 1983, pp. 87–161.

[BF03] Christian Bonatti and John Franks. *A Hölder continuous vector field tangent to many foliations*. 2003. arXiv: `0303291`.

[BL94] Christian Bonatti and Rémi Langevin. "Un exemple de flot d'Anosov transitif transverse à un tore et non conjugué à une suspension". In: *Ergodic Theory and Dynamical Systems* 14.4 (1994), pp. 633–43.

[BO09] Frédéric Bourgeois and Alexandru Oancea. "An exact sequence for contact- and symplectic homology". In: *Inventiones Mathematicae* 174.3 (2009), pp. 611–680.

[Bow16a] Jonathan Bowden. "Approximating C^0-foliations by contact structures". In: *Geometric and Functional Analysis* 26.5 (2016), pp. 1255–1296.

[Bow16b] Jonathan Bowden. "Contact perturbations of Reebless foliations are universally tight". In: *Journal of Differential Geometry* 104.2 (2016), pp. 219–237.

[BR75] Rufus Bowen and David Ruelle. "The ergodic theory of Axiom A flows". In: *Inventiones Mathematicae* 29.3 (1975), pp. 181–202.

[BC17] Steven Boyer and Adam Clay. "Foliations, orders, representations, L-spaces and graph manifolds". In: *Advances in Mathematics* 310 (2017), pp. 159–234.

[BGW13] Steven Boyer, Cameron McA Gordon, and Liam Watson. "On L-spaces and left-orderable fundamental groups". In: *Mathematische Annalen* 356 (2013), pp. 1213–1245.

[BC21] Joseph Breen and Austin Christian. *Mitsumatsu's Liouville domains are stably Weinstein*. 2021. arXiv: 2109.07615v1.

[Bru05] Marco Brunella. "Separating the basic sets of a nontransitive Anosov flow". In: *Bulletin of the London Mathematical Society* 25.5 (2005), pp. 487–490.

[BBI01] Dimitri Burago, Yuri Burago, and Sergei Ivanov. *A Course in Metric Geometry*. Vol. 33. Graduate Studies in Mathematics. American Mathematical Society, 2001. ISBN: 978-1-4704-6853-8.

[BI08] Dmitri Burago and Sergei Ivanov. "Partially hyperbolic diffeomorphisms of 3-manifolds with Abelian fundamental groups". In: *Journal of Modern Dynamics* 2.4 (2008), pp. 541–580.

[Cal01] Danny Calegari. "Leafwise smoothing laminations". In: *Algebraic & Geometric Topology* 1 (2001), pp. 579–587.

[Cal07] Danny Calegari. *Foliations and the Geometry of 3-Manifolds*. Oxford Mathematical Monographs. Oxford Science Publications, 2007. ISBN: 978-0-1985-7008-0.

[CC00] Alberto Candel and Lawrence Conlon. *Foliations I*. Vol. 23. Graduates Studies in Mathematics. American Mathematical Society, 2000. ISBN: 978-0-8218-0809-2.

[CC03] Alberto Candel and Lawrence Conlon. *Foliations II*. Vol. 60. Graduates Studies in Mathematics. American Mathematical Society, 2003. ISBN: 978-0-8218-0881-8.

[Cha+22] Baptiste Chantraine, Georgios Dimitroglou Rizell, Paolo Ghiggini, and Roman Golovko. "Geometric generation of the wrapped Fukaya category of Weinstein manifolds and sectors". In: *Annales Scientifiques de l'École Normale Supérieure (to appear)* (2022). arXiv: 1712.09126.

[Che98] Yuri Chekanov. "Lagrangian Intersections, Symplectic Energy, and Areas of Holomorphic Curves". In: *Duke Mathematical Journal* 95.1 (1998), pp. 213–26.

[CE12] Kai Cieliebak and Yakov Eliashberg. *From Stein to Weinstein and back: symplectic geometry of affine complex manifolds*. Vol. 59. Colloquium Publications. American Mathematical Society, 2012.

[CFO10] Kai Cieliebak, Urs Fauenfelder, and Alexandru Oancea. "Rabinowitz Floer homology and symplectic homology". In: *Annales Scientifiques de l'École Normale Supérieure* 43.4 (2010), pp. 957–1015.

[CHO22] Kai Cieliebak, Nancy Hingston, and Alexandru Oancea. *Poincaré duality for loop spaces*. 2022. arXiv: 2008.13161.

[CL09] Kai Cieliebak and Janko Latschev. "The role of string topology in symplectic field theory". In: *New perspectives and challenges in symplectic field theory*. CRM Proc. Lecture Notes. American Mathematical Society, 2009, pp. 113–146.

[Cie+22] Kai Cieliebak, Oleg Lazarev, Thomas Massoni, and Agustin Moreno. *Floer theory of Anosov flows in dimension three*. 2022. arXiv: 2211.07453.

[CO18] Kai Cieliebak and Alexandru Oancea. "Symplectic homology and the Eilenberg-Steenrod axioms. Appendix written jointly with Peter Albers". In: *Algebraic & Geometric Topology* 18.4 (2018), pp. 1953–2130.

[Col02] Vincent Colin. "Structures de contact tendues sur les variétés toroidales et approximation de feuilletages sans composante de Reeb". In: *Topology* 41.5 (2002), pp. 1017–1029.

[CF11] Vincent Colin and Sebastião Firmo. "Paires de Structures de Contact Sur Les Variétés de Dimension Trois. (French)". In: *Algebraic & Geometric Topology* 11.5 (2011), pp. 2627–53.

[CGH09] Vincent Colin, Emmanuel Giroux, and Ko Honda. "Finitude homotopique et isotopique des structures de contact tendues. (French)". In: *Publications Mathématiques de l'IHÉS* 109.1 (2009), pp. 245–293.

[CH20] Vincent Colin and Ko Honda. "Foliations, contact structures and their interactions in dimension three." In: *Surveys in 3-Manifold Topology and Geometry*. Vol. 25. Surveys in Differential Geometry. International Press, 2020, pp. 71–101.

[CKR19] Vincent Colin, William Kazez, and Rachel Roberts. "Taut foliations". In: *Communications in Analysis and Geometry* 27.2 (2019), pp. 357–375.

[CY05] William Cowieson and Lai-Sang Young. "SRB measures as zero-noise limits". In: *Ergodic Theory and Dynamical Systems* 25.4 (2005), pp. 1115–38.

[Die71] Tammo tom Dieck. "Partitions of unity in homotopy theory". In: *Compositio Mathematica* 23.2 (1971), pp. 159–67.

[Dim16] Georgios Dimitroglou Rizell. "Lifting pseudo-holomorphic polygons to the symplectisation of $P \times \mathbb{R}$ and applications". In: *Quantum Topology* 7.1 (2016), pp. 29–105.

[Doe87] Claus Doering. "Persistently transitive vector fields on three-dimensional manifolds". In: *Procs. on Dynamical Systems and Bifurcation Theory* 160 (1987), pp. 59–89.

[DP03] Victor Donnay and Charles Pugh. "Anosov geodesic flows for embedded surfaces". In: *Geometric methods in dynamics. II.* Vol. 287. Astérique. Société Mathématique de France, 2003, pp. 61–69.

[Dun20] Nathan Dunfield. "Floer homology, group orderability, and taut foliations of hyperbolic 3-manifolds." In: *Geometry & Topology* 24.4 (2020), pp. 2075–2125.

[Eli89] Yakov Eliashberg. "Classification of overtwisted contact structures on 3-manifolds". In: *Inventiones mathematicae* 89.3 (1989), pp. 623–637.

[Eli91] Yakov Eliashberg. "On symplectic manifolds with some contact properties". In: *Journal of Differential Geometry* 33.1 (1991), pp. 233–238.

[Eli92] Yakov Eliashberg. "Contact 3-manifolds twenty years since J. Martinet's Work". In: *Annales de l'Institut Fourier* 42.1-2 (1992), pp. 165–192.

[Eli04] Yakov Eliashberg. "A few remarks about symplectic filling". In: *Geometry & Topology* 8.1 (2004), pp. 277–293.

[EG91] Yakov Eliashberg and Mikhael Gromov. "Convex Symplectic Manifolds". In: *Proceedings of Symposia in Pure Mathematics* 52.2 (1991), pp. 135–162.

[ET98] Yakov Eliashberg and William Thurston. *Confoliations*. Vol. 13. University Lecture Series. American Mathematical Society, 1998. ISBN: 978-0-8218-0776-7.

[Etn07] John B Etnyre. “Contact structures on 3-manifolds are deformations of foliations”. In: *Mathematical Research Letters* 14.5 (2007), pp. 775–779.

[FH19] Todd Fisher and Boris Hasselblatt. *Hyperbolic Flows*. Zurich lectures in advanced mathematics. European Mathematical Society, 2019.

[FW80] John Franks and Bob Williams. “Anomalous Anosov flows”. In: *Global theory of dynamical systems (Proc. Internat. Conf., Northwestern Univ., Evanston, Ill., 1979)*. Vol. 819. Lecture Notes in Mathematics. Springer, Berlin, 1980, pp. 158–174.

[Fri82] David Fried. “Flow equivalence, hyperbolic systems and a new zeta function for flows”. In: *Commentarii Mathematici Helvetici* 57.1 (1982), pp. 237–59.

[Fri83] David Fried. “Transitive Anosov flows and pseudo-Anosov maps”. In: *Topology* 22.3 (1983), pp. 299–303.

[Gab83] David Gabai. “Foliations and the topology of 3-manifolds”. In: *Journal of Differential Geometry* 18.3 (1983), pp. 445–503.

[Gab87] David Gabai. “Foliations and the topology of 3-manifolds. III”. In: *Journal of Differential Geometry* 26.3 (1987), pp. 479–536.

[GO89] David Gabai and Ulrich Oertel. “Essential Laminations in 3-Manifolds”. In: *Annals of Mathematics* 130.1 (1989), pp. 41–73.

[Gan13] Sheel Ganatra. *Symplectic Cohomology and Duality for the Wrapped Fukaya Category*. 2013. arXiv: 1304.7312.

[Gan19] Sheel Ganatra. *Automatically generating Fukaya categories and computing quantum cohomology*. 2019. arXiv: 1605.07702.

[GGV22] Sheel Ganatra, Yuan Gao, and Sarah Venkatesh. *Rabinowitz Fukaya categories and the categorical formal punctured neighborhood of infinity*. 2022. arXiv: 2212.14863.

[GPS20] Sheel Ganatra, John Pardon, and Vivek Shende. "Covariantly Functorial Wrapped Floer Theory on Liouville Sectors". In: *Publications Mathématiques de l'IHÉS* 131.1 (2020), pp. 73–200.

[GPS23] Sheel Ganatra, John Pardon, and Vivek Shende. *Microlocal Morse theory of wrapped Fukaya categories*. To appear in *Annals of Mathematics*. 2023. arXiv: 1809.08807.

[GPS24] Sheel Ganatra, John Pardon, and Vivek Shende. "Sectorial descent for wrapped Fukaya categories". In: *Journal of the American Mathematical Society* 37 (2024), pp. 499–635.

[Gay06] David T Gay. "Four-dimensional symplectic cobordisms containing three-handles". In: *Geometry & Topology* 10 (2006), pp. 1749–1759.

[Gei95] Hansjörg Geiges. "Examples of Symplectic 4-Manifolds with Disconnected Boundary of Contact Type". In: *Bulletin of the London Mathematical Society* 27.3 (1995), pp. 278–280.

[Gei09] Hansjörg Geiges. *An Introduction to Contact Topology*. Vol. 109. Cambridge studies in advanced mathematics. Cambridge University Press, 2009. ISBN: 978-0-5116-1143-8.

[Ghy84] Étienne Ghys. "Flots d'Anosov sur les 3-variétés fibrées en cercles. (French)". In: *Ergodic Theory and Dynamycal Systems* 4.1 (1984), pp. 67–80.

[Ghy92] Étienne Ghys. "Déformations de flots d'Anosov et de groupes fuchsiens. (French)". In: *Annales de l'Institut Fourier* 42.1–2 (1992), pp. 209–47.

[Ghy93] Étienne Ghys. "Rigidité différentiable des groupes fuchsiens. (French)". In: *Publications Mathématiques de l'IHÉS* 78.1 (1993), pp. 163–85.

[Goo75] Sue Goodman. “Closed leaves in foliations of codimension one”. In: *Commentarii Mathematici Helvetici* 50.1 (1975), pp. 383–388.

[Goo83] Sue Goodman. “Dehn surgery on Anosov flows”. In: *Geometric dynamics (Rio de Janeiro, 1981)*. Vol. 1007. Lecture Notes in Mathematics. Springer, Berlin, 1983, pp. 300–307.

[Got82] Mark Gotay. “On coisotropic imbeddings of presymplectic manifolds”. In: *Proceedings of the American Mathematical Society* 84.1 (1982), pp. 111–114.

[Gro23] Yoel Groman. “Floer theory and reduced cohomology on open manifolds”. In: *Geometry & Topology* 27.4 (2023), pp. 1273–1390.

[Gro85] Mikhael Gromov. “Pseudo holomorphic curves in symplectic manifolds”. In: *Inventiones Mathematicae* 82.2 (1985), pp. 307–347.

[Gro86] Mikhael Gromov. *Partial Differential Relations*. Vol. 9. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge / A Series of Modern Surveys in Mathematics. Springer Berlin, Heidelberg, 1986. ISBN: 978-3-540-12177-0.

[HT80] Michael Handel and William Thurston. “Anosov flows on new three manifolds”. In: *Inventiones Mathematicae* 59.2 (1980), pp. 95–103.

[HHL23] Andrew Hanlon, Jeff Hicks, and Oleg Lazarev. *Relating categorical dimensions in topology and symplectic geometry*. 2023. arXiv: 2308.13677.

[Han+20] Jonathan Hanselman, Jacob Rasmussen, Sarah Dean Rasmussen, and Liam Watson. “L-spaces, taut foliations, and graph manifolds”. In: *Compositio Mathematica* 156.3 (2020), pp. 604–612.

[Har02] Philip Hartman. *Ordinary Differential Equations*. Second edition. Vol. 38. Classics in Applied Mathematics. SIAM, 2002. ISBN: 978-0-89871-510-1.

[Has94] Boris Hasselblatt. "Regularity of the Anosov splitting and of horospheric foliations". In: *Ergodic Theory and Dynamical Systems* 14.4 (1994), pp. 645–666.

[Hof93] Helmut Hofer. "Pseudoholomorphic curves in symplectizations with applications to the Weinstein conjecture in dimension three". In: *Inventiones Mathematicae* 114 (1993), pp. 515–563.

[Hoz22] Surena Hozoori. *Symplectic geometry of Anosov flows in dimension* 3 *and bi-contact topology*. 2022. arXiv: 2009.02768.

[Hoz23] Surena Hozoori. "On Anosovity, divergence and bi-contact surgery". In: *Ergodic Theory and Dynamical Systems* 43.10 (2023).

[Hoz] Surena Hozoori. Personal communication.

[Hum97] Christoph Hummel. *Gromov's Compactness Theorem for Pseudo-holomorphic Curves*. Vol. 151. Progress in Mathematics. Birkhäuser Verlag, Basel, 1997.

[Jec03] Thomas Jech. *Set theory*. 3rd ed. Vol. 13. Springer Monographs in Mathematics. Springer, 2003. ISBN: 978-3-642-07899-6.

[Juh15] András Juhász. "A survey of Heegaard Floer homology". In: *New ideas in low dimensional topology*. Vol. 56. Series on Knots and Everything. World Scientific, 2015, pp. 237–295.

[KR17] William Kazez and Rachel Roberts. "C^0 approximations of foliations". In: *Geometry & Topology* 21.6 (2017), pp. 3601–57.

[KR19] William Kazez and Rachel Roberts. "$C^{1,0}$ foliation theory". In: *Algebraic & Geometric Topology* 19.6 (2019), pp. 2763–2794.

[Kri20] Siddhi Krishna. "Taut foliations, positive 3-braids, and the L-space conjecture". In: *Journal of Topology* 13.3 (2020), pp. 1003–1033.

[Kri23] Siddhi Krishna. *Taut foliations, braid positivity, and unknot detection*. 2023. arXiv: 1809.03959.

[Kro+07] Peter Kronheimer, Tomasz Mrowka, Peter Ozsváth, and Zoltan Szabó. "Monopoles and lens space surgeries". In: *Annals of Mathematics* 165 (2007), pp. 457–546.

[LS91] François Lalonde and Jean-Claude Sikorav. "Sous-variétés lagrangiennes et lagrangiennes exactes des fibrés cotangents. (French)". In: *Commentarii Mathematici Helvetici* 66.1 (1991), pp. 18–33.

[LS23] Oleg Lazarev and Zachary Sylvan. "Prime-localized Weinstein subdomains". In: *Geometry & Topology* 27.2 (2023), pp. 699–737.

[Led84] François Ledrappier. "Propriétés ergodiques des mesures de Sinaï. (French)". In: *Publications Mathématiques de l'IHÉS* 59.1 (1984), pp. 163–88.

[LY85] François Ledrappier and Lai-Sang Young. "The metric entropy of diffeomorphisms. Part I: Characterization of measures satisfying Pesin's entropy formula". In: *The Annals of Mathematics* 122.3 (1985), p. 509.

[Ler01] Eugene Lerman. "Contact cuts". In: *Israel Journal of Mathematics* 124.1 (2001), pp. 77–92.

[Li02] Tao Li. "Laminar branched surfaces in 3–manifolds". In: *Geometry & Topology* 6.1 (2002), pp. 153–194.

[LR14] Tao Li and Rachel Roberts. "Taut foliations in knot complements". In: *Pacific Journal of Mathematics* 269.1 (2014), pp. 149–168.

[Lin23] Francesco Lin. *A remark on taut foliations and Floer homology*. 2023. arXiv: 2309.01222.

[LMM86] Rafael de la Llave, José Manuel Marco, and Roberto Moriyón. "Canonical perturbation theory of Anosov systems and regularity results for the Livsic cohomology equation". In: *The Annals of Mathematics* 123.3 (1986), pp. 537–611.

[Mar23] Théo Marty. *Skewed Anosov flows are orbit equivalent to Reeb-Anosov flows in dimension* 3. 2023. arXiv: 2301.00842.

[Mas23] Thomas Massoni. *Anosov flows and Liouville pairs in dimension three*. To appear in *Algebraic & Geometric Topology*. 2023. arXiv: 2211.11036.

[MNW13] Patrick Massot, Klaus Niederkrüger, and Chris Wendl. "Weak and strong fillability of higher dimensional contact manifolds". In: *Inventiones Mathematicae* 192.2 (2013), pp. 287–373.

[Mat13] Shigenori Matsumoto. "The space of (contact) Anosov flows on 3-manifolds". In: *Journal of Mathematical Sciences of The University of Tokyo* 20.2 (2013), pp. 445–460.

[McD91] Dusa McDuff. "Symplectic manifolds with contact type boundaries". In: *Inventiones Mathematicae* 103.3 (1991), pp. 651–671.

[Mei02] Gaël Meignez. "Submersions, fibrations and bundles". In: *Transactions of the American Mathematical Society* 354.9 (2002), pp. 3771–87.

[Mic22] Fernando Micena. "Some sufficient conditions for transitivity of Anosov diffeomorphisms". In: *Journal of Mathematical Analysis and Applications* 515.2 (2022).

[Mit95] Yoshihiko Mitsumatsu. "Anosov flows and non-Stein symplectic manifolds". In: *Annales de l'Institut Fourier* 45.5 (1995), pp. 1407–1421.

[Oh97] Yong-Geun Oh. "Gromov-Floer theory and disjunction energy of compact Lagrangian embeddings". In: *Mathematical Research Letters* 4.6 (1997), pp. 895–905.

[OS04] Peter Ozsváth and Zoltan Szabó. “Holomorphic disks and genus bounds”. In: *Geometry & Topology* 8.1 (2004), pp. 311–334.

[Pug68] Charles Pugh. “A Generalized Poincaré index formula”. In: *Commentarii Mathematici Helvetici* 7.3 (1968), pp. 217–226.

[RR17] Jacobb Rasmussen and Sarah Dean Rasmussen. “Floer simple manifolds and L-space intervals”. In: *Advances in Mathematics* 322.5 (2017), pp. 738–805.

[Ras17] Sarah Dean Rasmussen. “L-space intervals for graph manifolds and cables”. In: *Compositio Mathematica* 153.5 (2017), pp. 1008–1049.

[RS17] Alexander Ritter and Ivan Smith. “The monotone wrapped Fukaya category and the open-closed string map”. In: *Selecta Mathematica* 23.1 (2017), pp. 533–642.

[Rob01a] Rachel Roberts. “Taut foliations in punctured surface bundles, I”. In: *Proceedings of the London Mathematical Society* 82.3 (2001), pp. 747–768.

[Rob01b] Rachel Roberts. “Taut foliations in punctured surface bundles, II”. In: *Proceedings of the London Mathematical Society* 83.2 (2001), pp. 443–471.

[Rob75] Clark Robinson. “Structural stability of C^1 flows”. In: *Dynamical Systems–Warwick 1974*. Lecture Notes in Mathematics 468. Springer, Berlin, Heidelberg, 1975, pp. 262–275.

[Rou08] Raphaël Rouquier. “Dimensions of Triangulated Categories”. In: *Journal of K-Theory* 1.2 (2008), pp. 193–256.

[Sal23] Federico Salmoiraghi. *Goodman surgery and projectively Anosov flows*. 2023. arXiv: 2202.01328.

[Sal24] Federico Salmoiraghi. *Surgery on Anosov flows using bi-contact geometry*. 2024. arXiv: 2104.07109.

[San23a] Diego Santoro. *L-spaces, taut foliations and fibered hyperbolic two-bridge links*. 2023. arXiv: 2304.14914.

[San23b] Diego Santoro. *L-spaces, taut foliations and the Whitehead link*. To appear in *Algebraic & Geometric Topology*. 2023. arXiv: 2201.01211.

[San23c] Diego Santoro. *Taut foliations from knot diagrams*. 2023. arXiv: 2402.01225.

[Sch09] Eric Schoenfeld. "Higher Symplectic Field Theory Invariants for Cotangent Bundles of Surfaces". PhD thesis. Stanford University, 2009.

[Sei02] Paul Seidel. "Fukaya categories and deformations". In: *Proceedings of the International Congress of Mathematicians*. Vol. II. Higher Ed. Press, Beijing, 2002, pp. 351–360.

[Sei09] Paul Seidel. "Symplectic homology as Hochschild homology". In: *Algebraic geometry—Seattle 2005*. Vol. 80. Proceedings of Symposia in Pure Mathematics. American Mathematical Society, 2009, pp. 351–360.

[Sha21] Mario Shannon. *Hyperbolic models for transitive topological Anosov flows in dimension three*. 2021. arXiv: 2108.12000.

[Sim97] Slobodan Simić. "Codimension one Anosov flows and a conjecture of Verjovsky". In: *Ergodic Theory and Dynamical Systems* 17.5 (1997), pp. 1211–31.

[Sma67] Stephen Smale. "Differentiable dynamical systems". In: *Bulletin of the American Mathematical Society* 73.6 (1967), pp. 747–817.

[Spa66] Edwin Spanier. *Algebraic Topology*. McGraw-Hill, 1966.

[Spi99] Michael Spivak. *A Comprehensive Introduction to Differential Geometry*. 3rd ed. Vol. I. Publish or Perish, Inc, 1999. ISBN: 978-0-9140-9870-6.

[Sul76] Dennis Sullivan. “Cycles for the dynamical study of foliated manifolds and complex manifolds”. In: *Inventiones Mathematicae* 36.1 (1976), pp. 225–255.

[Tak74] Floris Takens. “Singularities of vector fields”. In: *Publications Mathématiques de l'IHÉS* 1.43 (1974), pp. 47–100.

[Tes12] Gerald Teschl. *Ordinary Differential Equations and Dynamical Systems*. Vol. 140. Graduate Studies in Mathematics. American Mathematical Society, 2012. ISBN: 978-0-8218-8328-0.

[Thu76] William Thurston. “Existence of codimension-one foliations”. In: *Annals of Mathematics* 104.2 (1976), pp. 249–268.

[Ueb19] Peter Uebele. “Periodic Reeb flows and products in symplectic homology”. In: *Journal of Symplectic Geometry* 17.4 (2019), pp. 1201–1250.

[Ver74] Alberto Verjovsky. “Codimension one Anosov flows”. In: *Bol. Soc. Mat. Mexicana* 19.2 (1974), pp. 49–77.

[Vog16] Thomas Vogel. “On the uniqueness of the contact structure approximating a foliation”. In: *Geometry & Topology* 20.5 (2016), pp. 2439–2573.

[Wei03] Charles Weibel. *Introduction to Homological Algebra*. Reprint. 1997. Cambridge University Press, 2003.

[Wei91] Alan Weinstein. “Contact Surgery and Symplectic Handlebodies”. In: *Hokkaido Mathematical Journal* 20.2 (1991), pp. 241–51.

[Wei05] Michael Weiss. “What does the classifying space of a category classify?” In: *Homology, Homotopy and Applications* 7.1 (2005), pp. 185–95.

[Zha22] Boyu Zhang. “Monopoles and foliations without holonomy-invariant transverse measure”. In: *Journal of Symplectic Geometry* 20.1 (2022), pp. 191–258.

[Zun24] Jonathan Zung. *Reeb flows transvere to foliations*. To appear in *Geometry & Topology*. 2024. arXiv: 2103.01325.