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Linear and nonlinear wave equations on black hole spacetimes

Luk, Jonathan Winghong

Princeton, NJ : Princeton University

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Linear and nonlinear wave equations on black hole spacetimes

Jonathan Winghong Luk

A Dissertation
Presented to the Faculty
of Princeton University
in Candidacy for the Degree
of Doctor of Philosophy

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Adviser: Igor Rodnianski

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Abstract

In this thesis, I study three problems related to the linear and nonlinear wave equations on black hole spacetimes. These problems are motivated by the nonlinear stability of Kerr spacetime.

First, I prove that sufficiently regular solutions to the wave equation $\Box_g \Phi = 0$ on the exterior of the Schwarzschild black hole obey the estimates $|\Phi| \leq C_\delta (t^*)^{-\frac{3}{2}+\delta}$ and $|\partial_t \Phi| \leq C_\delta (t^*)^{-2+\delta}$ on a compact region of r, including inside the black hole region.

Second, I prove that sufficiently regular solutions to the wave equation $\Box_g \Phi = 0$ on the exterior of the sufficiently slowly rotating Kerr black hole also obey the estimates $|\Phi| \leq C_\delta (t^*)^{-\frac{3}{2}+\delta}$. The first two results are proved with the help of a new vector field commutator that is analogous to the scaling vector field on Minkowski spacetime. This result improves the known decay rates in the region of finite r and along the event horizon.

Third, I study a semilinear equation with derivatives satisfying a null condition on slowly rotating Kerr spacetimes. I prove that given suciently small initial data, the solution exists globally in time and decays with a quantitative rate to the trivial solution. The proof uses the robust vector field method and in particular makes use of the improved decay rates obtained in the first and second results.

Acknowledgements

First and foremost, I would like to thank my advisor, Professor Igor Rodnianski, for introducing me to this subject, suggesting to me many fascinating problems, and sharing with me many insights. I would also like to thank him for his constant encouragement.

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Last but certainly not least, I would like to thank my family. To my parents for their love and nurture, and for having instilled in me a love of knowledge. To my brother Jeremy, for always being a great friend. Finally, to my wife Cindy, for always being so loving, supportive and encouraging.

Previous Publications and Presentations

The content of this thesis is contained in the previously published works:

- J. Luk. Improved Decay for Solutions to the Linear Wave Equation on a Schwarzschild Black Hole, arXiv:0906.5588, *Annales Henri Poincare*, 11:805-880, 2010.
- J. Luk. A Vector Field Method Approach to Improved Decay for Solutions to the Wave Equation on a Slowly Rotating Kerr Black Hole, arXiv:1009.0671, *to appear in Analysis and PDE.*
- J. Luk. The Null Condition and Global Existence for Nonlinear Wave Equations on Slowly Rotating Kerr Spacetimes, arXiv:1009.4109, *to appear in Journal Eur. Math. Soc.*

This work has been presented at the following conferences/seminars:

- October 2009, Special Session on General Relativity and Related PDEs, AMS Sectional Meeting, Boca Raton.
- January 2010, Special Session on Mathematical Challenges of Relativity, AMS Joint Mathematics Meetings, San Francisco.
- November 2010, Analysis Seminar, University of North Carolina at Chapel Hill.
- April 2011, Scattering and Spectral Theory Seminar, Purdue University.

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Chapter 1

Introduction

A major open problem in general relativity is that of the nonlinear stability of Kerr spacetimes. These spacetimes are stationary axisymmetric asymptotically flat black hole solutions to the vacuum Einstein equations

$$R_{\mu\nu} = 0$$

in $3+1$ dimensions. They are parametrized by two parameters (M, a), representing respectively the mass and the specific angular momentum of a black hole. It is conjectured that Kerr spacetimes are stable. In the framework of the initial value problem, the stability of Kerr would mean that for any solution to the vacuum Einstein equations with initial data close to the initial data of a Kerr spacetime, its maximal Cauchy development has an exterior region that approaches a nearby, but possibly different, Kerr spacetime.

Kerr spacetimes have a one-parameter subfamily of spacetimes known as Schwarzschild spacetimes for which $a = 0$. The Schwarzschild metric in the so-called exterior region can be expressed as

$$g = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\sigma_{\mathbb{S}^2},$$

where $d\sigma_{\mathbb{S}^2}$ denotes the standard metric on the unit sphere. In the context of the problem of the nonlinear stability of Kerr, it is conjectured that a spacetime that is close to Schwarzschild initially will approach a Kerr spacetime that is also close to Schwarzschild, i.e., $a \ll M$.

To tackle the nonlinear stability of Kerr spacetimes, it is important to first understand the linear waves

$$\Box_g \Phi = 0$$

on the exterior region of Kerr spacetimes. In view of the expectation that spacetimes initially close to Schwarzschild will eventually settle to Kerr spacetimes with $a \ll M$, it is reasonalbe to first study the linear wave equation on Schwarzschild spacetimes and very slowly rotating Kerr spacetimes. This can be compared with the nonlinear stability of Minkowski spacetime whose proof requires a robust understanding of the quantitative decay of the solutions to the linear wave equation [10], [34].

The first and second results in this thesis concerns the robust, quantitative decay rates of the solutions to the linear wave equation on Schwarzschild and very slowly rotating Kerr black hole respectively. The pointwise decay of the solutions to the linear wave equation on Schwarzschild background is proved in [18], [5]. In particular, Dafermos-Rodnianski proved a decay rate of $|\Phi| \leq C \left(\max\{1, v\}\right)^{-1}$ everywhere in the exterior region, including along the event horizon [18]. (Here, the null variable v will be defined later and is useful to capture the decay rate along the event horizon.)

This decay rate is sharp as one approaches null infinity along the wave zone $\{\frac{t}{4} \leq r \leq \frac{t}{2}\}$. However, this decay rate is non-integrable and creates certain difficulties when applied to study nonlinear problems. In view of the problem of the nonlinear stability of Kerr, it would be desirable to prove an improved decay rate away from the wave zone. Our first result in this thesis is such an improvement for the wave equation on Schwarzschild spacetimes. We improve the decay rate in a finite region of r, including the decay rate along the event horizon and inside the black hole region.

Theorem 1.0.1. *[35] For arbitrarily small $\delta > 0$, $|\Phi| \leq C_{\delta,r_b,R} (\max\{1, v\})^{-\frac{3}{2}+\delta}$ in the region $\{r_b \leq r \leq R\}$ for any $r_b > 0$ and $R > 2M$.*

The pointwise decay of solutions to the linear wave equation on very slowly rotating Kerr spacetimes was first proved in [16] and subsequently and independently by [2]. Both results were proved via the vector field method. In particular, a decay rate of $|\Phi| \leq C (\max\{1, t^*\})^{-1+C|a|}$ is proved in [16]. The t^* variable is defined so that the decay rate along the event horizon is also captured. This has later been improved in [17] to $|\Phi| \leq C (\max\{1, t^*\})^{-1}$. As in the case for Schwarzschild, this cannot be improved as one approaches null infinity along the wave zone but nevertheless can be improved if we restrict to a finite region of r. In fact, a Price law type [50] of $|\Phi| \leq C (\max\{1, t^*\})^{-3+\delta}$ has been proved using Fourier analytic methods. Nevertheless, such methods are unstable and it is desirable in view of the nonlinear problem to obtain a proof of the improved decay rate using the more robust vector field method. In [36], we extended Theorem 1.0.1 to sufficiently slowly rotating Kerr spacetimes, i.e., Kerr spacetimes with $a \ll M$.

Theorem 1.0.2. *[36] For arbitrarily small $\delta > 0$, there exists $a_0 = a_0(\delta)$ such that on Kerr spacetime with mass M and specific angular momentum a, $|\Phi| \leq C_{a,\delta,r_b,R} (\max\{1, t^*\})^{-\frac{3}{2}+\delta}$ in the region $\{r_b \leq r \leq R\}$ for any $r_b > M - \sqrt{M^2 - a^2}$ and $R > M + \sqrt{M^2 - a^2}$.*

The second theorem in particular includes the result of the first. We, however, prefer to prove them as separate theorems because firstly that there are certain simplifications in the proof of the Schwarzschild case and the argument would be more transparent in this setting. Secondly, it is useful to prove certain estimates first on Schwarzschild spacetimes and then to prove the analogous estimates on slowly rotating Kerr spacetimes by resorting to stability arguments.

Our proof of the improved decay rates in Theorems 1.0.1 and 1.0.2 introduces a new vector field commutator S that is analogous to the scaling vector field in

Minkowski spacetime. We will show that for solutions to $\Box_g \Phi = 0$, $\Box_g (S\Phi)$ decays sufficiently towards spatial infinity and only grows mildly towards event horizon. We then prove energy estimates for $S\Phi$ with the help of (a slightly modified version of) the energy estimates of Φ in [18], [16]. This will enable us to prove the decay of $S\Phi$. With this decay, we follow Klainerman-Sideris [30] to improve the decay rate for $\partial_t \Phi$. We also introduce a novel method to improve the decay rates for Φ and its spatial derivatives.

After a sufficiently robust proof of quantitative decay rates for solutions to the linear wave equation is understood, a natural next question is to study the small data problem for a model nonlinear equation on a fixed black hole background. In the case of the nonlinear stability of Minkowski spacetime, the semilinear equation satisfying a null condition has served as a good model problem.

One of the main difficulties in proving the nonlinear stability of Minkowski spacetime arises from the fact that the decay of the linear wave equation on Minkowski spacetime is non-integrable. In 4+1 or higher dimension, the decay of the solution to the linear wave equation is sufficiently fast and the solution is integrable in time. This allows one to prove global existence of small data for nonlinear wave equations with any quadratic dertivative nonlinearity [27]. This would in particular give a simple proof of the stability of Minkowski spacetime in high dimensions using wave coordinates [8]. However, in 3+1 dimensions, which is also the dimension of physical relevance, the decay rate is insufficient to prove the global existence of solutions and indeed a counter-example is known [24]. Nevertheless, if the quadratic nonlinearity satisfies the null condition defined by Klainerman, it has been proved independently by Christodoulou [9] and Klainerman [28] that any solutions to sufficiently small initial data are global in time. This turns out to be an important model problem for the stability of Minkowski spacetimes, whose proof requires an analogous special structure of the nonlinearity.

We consider the analogous problem of small data global existence for semilinear equations on Kerr spacetime. We define an appropriate notion of the null condition of Kerr spacetimes and consider $\Box_{g_K}\Phi = F(D\Phi)$ where F satisfies the null condition. The improved decay rates in Theorem 1.0.1, 1.0.2 suggest that Φ has sufficient decay away from the wave zone. While the decay is non-integrable in the wave zone $\{\frac{t^*}{4} \leq r \leq \frac{t^*}{2}\}$, this set approaches the same set in Minkowski spacetime as $t^* \to \infty$ due to the asymptotic flatness of Kerr spacetime. Therefore, one expects that with a null condition similar to that on the Minkowski spacetime, a similar global existence result holds. Indeed, we have

Theorem 1.0.3. *[37] Consider $\Box_{g_K}\Phi = F(D\Phi)$ where F satisfies the null condition (see Section 5.3). Then for any initial data that are sufficiently small, the solution exists globally in time.*

This thesis will be organized as follows: In the next chapter, we will review some related works. In Chapter 3, we will introduce Schwarzschild and Kerr spacetimes and describe their geometry. In particular, we will emphasize the geometric features that is relevant to the properties of the solutions to wave equations. In Chapter 4, we will introduce the vector field method, which will form the basis of the analysis in the whole thesis. In Chapter 5, we will then be able to give precise statements of the three main theorems. In Chapters 6 and 7, we will outline the main ideas of the proof. First, we outline the ideas for proving a similar result on Minkowski spacetime. Then we point out some main ideas for the proof of our main results. Chapters 8, 9, 10 contain the proofs of theorems 1.0.1, 1.0.2 and 1.0.3 respectively. The content of these chapters are mostly contained in the previously published works [35], [36] and [37] respectively.

Chapter 2

Related Works

2.1 Decay of Solutions to Linear Wave Equations on Black Hole Spacetimes

We first consider previous works related to linear waves on Schwarzschild spacetimes. The problem of the uniform boundedness of solutions to $\Box_g \Phi = 0$ on the exterior of Schwarzschild occupied the physics community for some time. The first mathematically rigorous result was obtained by Wald [53] for solutions vanishing on the bifurcate sphere ($\mathcal{H}^+ \cap \mathcal{H}^-$ in Figure 1). Kay-Wald [26] later removed this restriction and proved the uniform boundedness of a more general class of solutions. They used the energy conservation law given by using ∂_t as a multiplier as well as the Killing fields $\{\partial_t, \Omega_i\}$ as commutators. The decay rates

$$|\Phi| \leq C v_+^{-1}, \quad \forall r \geq 2M, \tag{2.1}$$

$$|r\Phi| \leq C_R u_+^{-\frac{1}{2}}, \quad \forall r \geq R,$$

where $v_+ = \max\{v, 1\}$, $u_+ = \max\{u, 1\}$ and C_R depends only on an appropriate norm of the initial data, for sufficiently regular solutions to $\Box_g \Phi = 0$, were proved by

Dafermos-Rodnianski in [18]. We note that the decay rate (2.1) holds in the entire exterior region of Schwarzschild spacetimes, including along the event horizon. In addition to the vector fields in [53],[26], their approach employed several other (non-Killing!) vector fields. One is an analog of the Morawetz vector field Z in Minkowski spacetime. It has an associated nonnegative quantity which we will call the conformal energy. It has weights similar to that of the conformal energy on Minkowski spacetime so that its boundedness would imply a local energy decay. Another is a vector field of the form $X = f(r^*)\,\partial_{r^*}$. The construction of this vector field was motivated by [33]. Unlike other multipliers, X is constructed so that $K^{X,w^X}(\Phi)$ (instead of $J^{X,w^X}(\Phi)$) can be controlled. This is used to estimate some energy quantity integrated over spacetime, in particular error terms from the "conservation law" of the conformal energy. The estimates of X are iterated together with that of Z to achieve the boundedness of the conformal energy. This then implies the decay of Φ away from the event horizon. The estimate associated to X can be thought of as an *integrated in time* local energy decay. It was extensively studied in [1],[3],[5],[14],[18],[23].

In addition, [18] introduced a new - red shift vector field - which takes advantage of the geometry of the event horizon and is used crucially in proving the decay rate close to and along the event horizon. This vector field is one of the few stable features of the Schwarzschild spacetime. As we will see later, it will make a crucial appearance in this article to achieve the improved decay rate along the event horizon.

The study of pointwise decay was carried out independently by Blue-Sterbenz [5]. They showed a similar quantitative decay result for initial data vanishing on the bifurcate sphere, with a decay rate that is weaker than [18] along the event horizon. In the proof they used analogues of the vector fields Z and X but not the vector field Y. Strichartz estimates for solutions of the wave equation on Schwarzschild background were shown in [23].

The decay rates for the solutions to the linear wave equation on Schwarzschild

spacetime has then been improved to $|\Phi| \leq C v_+^{-\frac{3}{2}+\delta}$ in [35], which is contained in the first part of this thesis. In this work, the vector fields Z, X, and Y are used. In addition, a vector field commutator S that is analogous to the Minkowskian scaling vector field is used to obtain the improved decay rate.

Considerable attention has also been given to the problem of decay of solutions of the wave equations on the Schwarzschild spacetime restricted to a *fixed* spherical harmonic Φ_ℓ arising in the decomposition $\Phi(t, r, \omega) = \sum_\ell \Phi_\ell(r, t) Y_\ell(\omega)$, $\omega \in \mathbb{S}^2$. Such results for a fixed spherical harmonic have been obtained in [12], [21], [31], [38]. The ideas in studying fixed spherical harmonic have later been extend to prove a Price's law type local decay [22]. However, past experience shows that such refined decay estimates may not be stable for nonlinear problems.

Decay for general solutions to the wave equation on sufficiently slowly rotating Kerr spacetimes was first proved by Dafermos-Rodnianski [16] with a quantitative rate of $|\Phi| \leq C(t^*)^{-1+Ca}$. A similar result was later obtained by [2] using a physical space construction to obtain an integrated decay estimate. In all of [51], [16] and [2], the integrated decay estimate is proved and plays an important role. All proofs of such estimates rely heavily on the separability of the wave equation, or equivalently, the existence of a Killing tensor on Kerr spacetime. In a recent work [17], Dafermos-Rodnianski shows that assuming the integrated decay estimate (non-degenerate up to the event horizon if it exists) and boundedness for the wave equation on an asymptotically flat spacetime, the decay rate $|\Phi| \leq C(t^*)^{-1}$ holds. This in particular improves the rates in [16] and [2]. In a similar framework, but assuming in addition exact stationarity, Tataru [50] proved a local decay rate of $(t^*)^{-3}$ using Fourier-analytic methods. This applies in particular to Schwarzschild spacetimes and sufficiently slowly rotating Kerr spacetimes. As remarked before, refined estimates proved using Fourier analytic methods may not be sufficiently robust for nonlinear applications. A robust local decay rate of $(t^*)^{-\frac{3}{2}+Ca}$ was then proved in [36], which is

contained in this thesis. Subsequently, a different proof of a local decay rate of $(t^*)^{-3}$ using the fundamental solution of the Minkowskian wave equation was achieved in [41]. The program for studying linear stability is finally completed by Dafermos and Rodnianski who proved the decay of solutions to the wave equation on the full range of sub-extremal Kerr spacetimes $a < M$ [20], [19].

We refer the readers to Sections 3 and 4 in [16] and the introduction to [20] for further references on this problem.

2.2 Nonlinear Wave Equations on Black Hole Spacetime

For nonlinear equations, global existence for the equation with power nonlinearity $\Box_{g_k}\Phi = \pm|\Phi|^p\Phi$ was initiated in [43] and [44], in which the large data subcritical defocusing case of $p = 2$ is studied. Later, there have been much work on the small data problem in which the sign of the nonlinearity is not important, and that the dispersive properties of the linear equation plays a crucial role. Global existence was proved for small radial data for $p > 3$ on Reissner-Nordstrom spacetime [13] and for general small data vanishing on the bifurcate sphere for $p > 2$ [5] on Schwarzschild spacetime. Global existence was also proved for $p = 4$ on Schwarzschild spacetime with general data that has small non-degenerate energy [23]. This was extended to the case of sufficiently slowly rotating Kerr spacetime in [52]. Counterexample is known for the case $0 < p < \sqrt{2}$ [7]. To our knowledge, the work [37] which is contained in this thesis is the first work on semilinear equations with derivatives on black hole spacetimes.

2.3 Nonlinear Wave Equation satisfying a Null Condition

Semilinear wave equation satisfying the null condition has a long history. In 4+1 or higher dimension, the decay of the linear wave equation is sufficiently fast for one to prove global existence of small data for nonlinear wave equations with any quadratic nonlinearity [27]. However, in 3+1 dimensions, which is also the dimension of physical relevance, the decay rate is only sufficient to prove the almost global existence of solutions [25]. Indeed, a counter-example is known [24] for the equation

$$\Box_m \Phi = (\partial_t \Phi)^2.$$

Nevertheless, if the quadratic nonlinearity satisfies the null condition defined by Klainerman, it has been proved independently by Christodoulou [9] and Klainerman [28] that any solutions to sufficiently small initial data are global in time. There have been an extensive literature on extensions and variations of the original results, including the cases of the multiple-speed system and the exterior domains ([46], [47], [39], [40]).

Chapter 3

Geometry of Schwarzschild and Kerr Spacetimes

3.1 Schwarzschild Spacetime

Schwarzschild spacetime is the spherically symmetric asymptotically flat solution to the vacuum Einstein equations. The Schwarzschild metric in the exterior region $\{r > 2M\}$ is

$$g = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\sigma_{\mathbb{S}^2},$$

where $d\sigma_{\mathbb{S}^2}$ denotes the standard metric on the unit sphere. It is easy to observe from the metric that the vector field ∂_t is Killing and it is orthogonal to the hypersurfaces $t =$ constant. Spacetimes with this property are called static. It is also manifestly spherically symmetry and therefore has a basis of Killing vector fields Ω_i generating the symmetry. Moreover, Schwarzschild spacetimes are asymptotically flat. This means that the metric approaches the flat metric as we go to spatial infinity ($r \to \infty$).

Synge [49] and Kruskal [32] showed that the Schwarzschild metric can be extended past $r = 2M$ as a solution to the vacuum Einstein equations. Its maximal

development is usually described by a Penrose diagram, which depicts a conformal compactification of the 4 dimensional manifold quotiented out by spherical symmetry (Figure 1). In this diagram, the coordinate system $(t, r > 2M, \omega \in \mathbb{S}^2)$ with the metric described above represents the region I, which we will call from now on the exterior region. In the nonlinear stability problem, it is this region that is conjectured to be stable. Extended beyond $r = 2M$, the Schwarzschild spacetime contains a black hole (region II in the diagram). Physically, an observer outside the black hole region cannot receive signals emitted inside the black hole. The null hypersurface $r = 2M$ separating the exterior region I and the black hole is known as the event horizon $\mathcal{H}^+$.

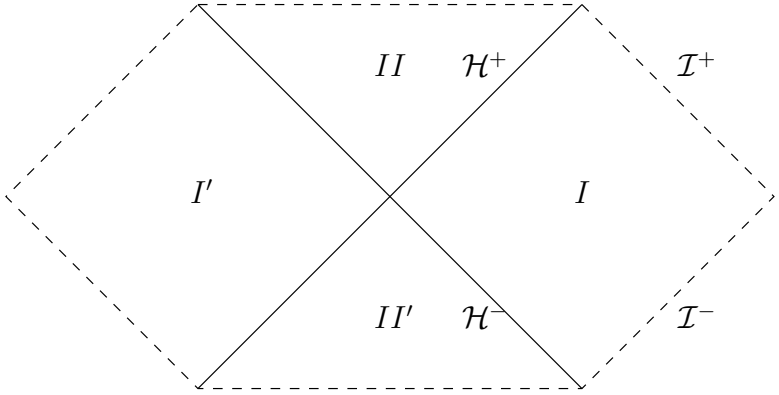


Figure 3.1: Schwarzschild spacetime

We return to the discussion of the exterior region of the Schwarzschild black hole. For notational convenience, we let

$$\mu = \frac{2M}{r}.$$

We denote as r^* the Regge-Wheeler tortoise coordinate

$$r^* = r + 2M \log (r - 2M) - 3M - 2M \log M.$$

In these coordinates, the Schwarzschild metric in the exterior region is given by

$$g = -(1-\mu)\,dt^2 + (1-\mu)\,dr^{*2} + r^2 d\sigma_{\mathbb{S}^2}.$$

Notice that in the above equation we have used both r^* and r. Here, and below, we think of r^* as the coordinate and r as a function on $\mathcal{Q}$, with $r(q) = \sqrt{\frac{Area(q)}{4\pi}}$, i.e. the physical radius of the 2-sphere under which the metric is symmetric. The coordinate r^* is $+\infty$ at spatial and null infinity; $-\infty$ at the event horizon and 0 at $r = 3M$. The set $\{r = 3M\}$ is known as the photon sphere. On this set trapping occurs: there exist null geodesics that lie in this set. In particular, these geodesics neither cross the event horizon nor approach null infinity. This suggests, via geometrical optics considerations, that one has to lose derivatives while proving energy estimates. We will return to this point in Section 3.3.

We notice that as in the coordinates (t, r, ω), ∂_t and Ω are Killing in the (t, r^*, ω) coordinates.

We also define the retarded and advanced Eddington-Finkelstein coordinates u and v by

$$t = v + u, r^* = v - u.$$

At the event horizon $\mathcal{H}^+$, $u = +\infty$. At future null infinity $\mathcal{I}^+$, $v = +\infty$. Notice that in these coordinates, the metric is given by

$$-4(1-\mu)\,dudv + r^2 d\sigma_{\mathbb{S}^2}.$$

In particular, this shows that ∂_u, ∂_v in this coordinate system are null.

In the following we are also going to consider the coordinate system (v, r, ω), where v and r are defined as above. We will only use this coordinate system when considering the region near the event horizon or inside the black hole. When there is no confusion,

we will not specify the coordinate system when we use the notation ∂_v. In the (v, r, ω) coordinate system the metric is

$$g = -(1-\mu)dv^2 + 2drdv + r^2 d\sigma_{\mathbb{S}^2}.$$

Notice that the coordinate v is originally defined only for $r > 2M$. However, as the metric in the (r, v) coordinate system is non-singularity, we can extend the v coordinate to $r > 0$. We will also refer to u inside the black hole region with $u = v - 2r - 4M\log(r-2M)$ defined as it is in the exterior region. Notice that u is not defined on the event horizon.

3.2 Kerr Spacetime

The Kerr spacetime is axisymmetric and stationary solution to the vacuum Einstein equations. It is parametrized by two parameters (M, a) representing the mass and specific angular momentum respectively. The Kerr metric in the Boyer-Lindquist coordinates takes the following form:

$$\begin{aligned} g_K = &-\left(1 - \frac{2M}{r\left(1+\frac{a^2\cos^2\theta}{r^2}\right)}\right)dt^2 + \frac{1+\frac{a^2\cos^2\theta}{r^2}}{1-\frac{2M}{r}+\frac{a^2}{r^2}}dr^2 + r^2\left(1+\frac{a^2\cos^2\theta}{r^2}\right)d\theta^2 \\ &+ r^2\left(1+\frac{a^2}{r^2}+\left(\frac{2M}{r}\right)\frac{a^2\sin^2\theta}{r^2\left(1+\frac{a^2\cos^2\theta}{r^2}\right)}\right)\sin^2\theta d\phi^2 \\ &- 4M\frac{a\sin^2\theta}{r\left(1+\frac{a^2\cos^2\theta}{r^2}\right)}dtd\phi. \end{aligned} \tag{3.1}$$

In this thesis, we will consider Kerr spacetimes with a small. It can then be thought of as a small perturbation of Schwarzschild spacetimes because by setting $a = 0$, we retrieve the metric for Schwarzschild spacetimes:

$$g_S = -\left(1-\frac{2M}{r_S}\right)dt_S^2 + \left(1-\frac{2M}{r_S}\right)^{-1}dr_S^2 + r_S^2 d\theta^2 + r_S^2\sin^2\theta d\phi^2.$$

The Cauchy development of Kerr spacetimes can be described schematically by taking a two dimensional slice as in the following diagram:

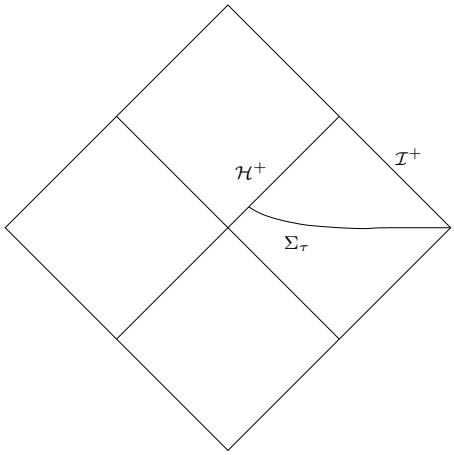


Figure 3.2: Kerr spacetime

Notice that (3.1) represents the metric on the exterior region (the right hand side in the diagram). In the coordinate system above, this is the region $\{r \geq r_+\}$, where r_+ is the larger root of $\Delta = r^2 - 2Mr + a^2$. This is the region that we will study. We foliate the exterior region of the Kerr spacetime by hypersurfaces Σ_τ as depicted in the diagram. A precise definition of the hypersurface Σ_τ will be given in in Section 9.1.3.

3.3 Trapping, Redshift and Superradiance

The geometric features of the Kerr spacetime is reflected in the properties of the solutions to the wave equations. In particular, the methods that are used to understand the solutions to the wave equation on black hole spacetimes are based on the understanding of the geometry of the spacetimes. In this context, trapping, redshift and superradiance play a particularly important role.

Schwarzschild and Kerr spacetimes have trapped null geodesics, i.e. null geodesics that neither approach null infinity or cross the event horizon. Geometric optics con-

siderations suggest that one can construct solutions near these geodesics that do not decay after a long time. The remedy is that null geodesics are unstable. By carefully understanding the trapped set, one can prove a integrated local decay estimates. Nevertheless, the presense of the trapped set dictates that such estimates much lose derivatives [45]. This necessarily complicates the arguments to understand the behavior of linear and nonlinear waves.

On Schwarzschild spacetime, the trapped geodesics lie on the hypersurface $\{r = 3M\}$. The fact that trapped geodesics only lie on a hypersurface allows a proof of the integrated local decay estimates using the vector field method with carefully chosen vector fields that degenerate at $\{r = 3M\}$. The case of Kerr spacetime is more complicated. In particular, the trapped set does not lie in a codimension 1 hypersurface in physical space. In fact, it can be shown that an integrated local decay estimates cannot be proved using a vector field [1]! Neverthless and remarkably, the geodesic flow on Kerr spacetime is completely integrable [6]. This allows a precise understanding of trapping which would be important in studying the decay properties of the solutions to the wave equation.

The event horizon, which is ruled by null geodesics, would a priori seem like a trapped set. However, Dafermos-Rodnianski [18] showed the geometry of the event horizon is actually favorable for proving the decay for the solutions to the linear wave equation. In particular, [18] introduced a new - red shift vector field - which takes advantage of the geometry of the event horizon and is used crucially in proving the decay rate close to and along the event horizon. This vector field has been used to prove boundedness and decay of solutions to the wave equation on Schwarzschild and Kerr spacetimes [18], [15], [16], [20], [19]. This vector field will make crucial appearances in the thesis in understanding the decay of the solution to the wave equation near the event horizon.

In Kerr spacetime with $a > 0$, the geometric feature of superradiance poses ad-

ditional challenge to proving decay for solutions to the wave equation. This refers to the fact that the Killing vector field ∂_{t^*} is spacelike near the event horizon. As a result, the conserved energy given by the vector field ∂_{t^*} is not positive definite and does not control the solution.

In the particular case $a \ll M$ that is considered in this thesis, superradiance only occur in a small neighborhood of the event horizon. Thus a can be chosen to be sufficiently small so that the superradiant region is contained in the region where one can apply the red shift estimates alluded to above. Moreover, for $a \ll M$, the superradiant region is disjoint from the trapped region. This allows us to control all the error terms using the red shift estimates and the integrated local decay estimates.

3.4 Kerr Coordinates and Frames

For different calculations in the sequel, it would be useful to introduce a few different coordinate systems or frames on Kerr spacetimes.

3.4.1 Boyer-Lindquist Coordinates

We recall from above the coordinates (3.1) that we introduced earlier:

$$\begin{aligned} g_K = &- \left(1 - \frac{2M}{r\left(1+\frac{a^2\cos^2\theta}{r^2}\right)}\right) dt^2 + \frac{1+\frac{a^2\cos^2\theta}{r^2}}{1-\frac{2M}{r}+\frac{a^2}{r^2}} dr^2 + r^2\left(1+\frac{a^2\cos^2\theta}{r^2}\right) d\theta^2 \\ &+ r^2\left(1+\frac{a^2}{r^2}+\left(\frac{2M}{r}\right)\frac{a^2\sin^2\theta}{r^2\left(1+\frac{a^2\cos^2\theta}{r^2}\right)}\right)\sin^2\theta d\phi^2 \\ &- 4M\frac{a\sin^2\theta}{r\left(1+\frac{a^2\cos^2\theta}{r^2}\right)} dt d\phi. \end{aligned} \tag{3.2}$$

The coordinates in (3.1) are not regular at the event horizon $\mathcal{H}^+ = \{r = r_+\}$. It will be helpful in the sequel to use different coordinate systems on Kerr spacetimes. From now on we will call the coordinate system on which the metric (3.1) is defined the

Kerr (t, r, θ, ϕ) coordinates.

3.4.2 The Kerr (t, r^*, θ, ϕ) Coordinates

We define a new coordinate system - the Kerr (t, r^*, θ, ϕ) coordinates by

$$\frac{dr^*}{dr} = \frac{r^2 + a^2}{\Delta},$$

where $\Delta = r^2 - 2Mr + a^2$ is zero at the event horizon. In this coordinate system, the metric looks like

$$\begin{aligned} g_K = &- \left(1 - \frac{2M}{r(1 + \frac{a^2 \cos^2\theta}{r^2})}\right) dt^2 + \Delta \left(r^2 + a^2\right)^2 \left(r^2 + a^2 \cos^2\theta\right) dr^{*2} \\ &+ r^2 \left(1 + \frac{a^2 \cos^2\theta}{r^2}\right) d\theta^2 + r^2 \left(1 + \frac{a^2}{r^2} + (\frac{2M}{r}) \frac{a^2 \sin^2\theta}{r^2(1 + \frac{a^2 \cos^2\theta}{r^2})}\right) \sin^2\theta d\phi^2 \\ &- 4M \frac{a \sin^2\theta}{r(1 + \frac{a^2 \cos^2\theta}{r^2})} dt d\phi. \end{aligned}$$

Notice that since the definition of r^* depends only on r, it would be unambiguous to talk about the vector ∂_t.

3.4.3 Schwarzschild Coordinates on Kerr

In order to compare calculations on Kerr spacetimes to calculations on Schwarzschild spacetimes, it is helpful to exhibit a diffeomorphism between the two. We do so by defining an explicit map between the coordinate functions (t, r, θ, ϕ) on a Kerr spacetime and the coordinate functions $(t_S, r_S, \theta_S, \phi_S)$ on a Schwarzschild spacetime with the same mass. These will be defined differently near and away from the event

horizon. Take

$$\chi(r) = \begin{cases} 1 & r \leq r_Y^- - \frac{r_Y^- - r_+}{2} \\ 0 & r \geq r_Y^- - \frac{r_Y^- - r_+}{4} \end{cases},$$

where r_+, as above, is the larger root of $\Delta = r^2 - 2Mr + a^2$ and $r_Y^- > r_+$ is a constant to be determined later. With this $\chi(r)$, we can then define

$$r_S^2 - 2Mr_S = r^2 - 2Mr + a^2,$$

$$t_S + \chi(r_S)2M\log(r_S - 2M) = t + \chi(r)h(r), \quad \text{where } \frac{dh(r)}{dr} = \frac{2Mr}{r^2 - 2Mr + a^2},$$

$$\theta_S = \theta,$$

$$\phi_S = \phi + \chi(r)P(r), \quad \text{where } \frac{dP(r)}{dr} = \frac{a}{r^2 - 2Mr + a^2}.$$

Then, by identifying $(t_S, r_S, \theta_S, \phi_S)$ with the corresponding coordinate functions on Schwarzschild spacetimes, we have a diffeomorphism between Kerr spacetimes and Schwarzschild spacetimes. This coordinate system will be used and will be called the Schwarzschild $(t_S, r_S, \theta_S, \phi_S)$ coordinates on Kerr spacetimes. Once we have this diffeomorphism, we can put any system of Schwarzschild coordinates on Kerr spacetimes. These include the Schwarzschild $(t_S^*, r_S, \theta_S, \phi_S)$ coordinates, where $t_S^* = t_S + \chi(r_S)2M\log(r_S - 2M)$ and r_S, θ_S, ϕ_S are defined as above. We also define $t^* = t_S^* = t_S + \chi(r_S)2M\log(r_S - 2M)$ and use the Kerr (t^*, r, θ, ϕ^*) coordinates. Notice that $\partial_{t^*} = \partial_{t_S^*}$.

It is common to denote on Schwarzschild spacetimes $\mu = \frac{2M}{r_S}$. We will take the same notation on Kerr spacetimes, with the understand that it is always defined with respect to the Schwarzschild r_S coordinates. In particular $(1 - \mu)$ approaches 0 as $r \to r_+$ (the event horizon).

Another system of Schwarzschild coordinates can be defined by considering two coordinate charts on the standard unit 2-sphere and introducing a system of coor-

dinates (x_S^A, x_S^B) on each of them. We then define the Schwarzschild $(t_S^*, r_S, x_S^A, x_S^B)$ coordinates in the obvious manner. Using this coordinate system and the diffeomorphism as above, we have, for small a:

$$|(g_K)_{\alpha\beta} - (g_S)_{\alpha\beta}| \leq \epsilon r^{-2}. \tag{3.3}$$

This smallness assumption will be used throughout this thesis.

3.4.4 Null Frame near Event Horizon

Some extra cancellations for the estimates near the event horizon are best captured using a null frame. Hence we define a null frame $\{\hat{V}, \hat{Y}, E_1, E_2\}$ in the region $r \leq r_Y^-$, where r_Y^- is to be determined later. On the event horizon,

$$V = \partial_{t^*} + \frac{a}{2Mr_+}\partial_{\phi^*}$$

is the Killing null generator. A direct computation shows that it satisfies

$$\nabla_V V = \kappa V,$$

where κ is a strictly positive number on the event horizon. We want to extend V to a null frame. On the event horizon, define $\hat{Y}$ first on a 2-sphere given by a fixed t^* to be null, orthogonal to the 2-sphere and require that $g_K(V, \hat{Y}) = -2$. Define also locally an orthonormal frame $\{E_1, E_2\}$ tangent to the fixed 2-sphere. Notice that in the sequel, we will only need to work with a local null frame. We then extend this null frame off the fixed 2-sphere on the event horizon (with $\hat{V}|_{\mathcal{H}} = V$) by requiring

$$\nabla_{\hat{Y}}\hat{Y} = \nabla_{\hat{Y}}\hat{V} = \nabla_{\hat{Y}}E_A = 0, \tag{3.4}$$

where $A \in \{1, 2\}$. Then extend this null frame using the isomorphisms generated by V. We notice that the above equations hold everywhere. If we choose r_Y^- close enough to r_+, we still have, by Taylor's Theorem,

$$\nabla_{\hat{V}} \hat{V} = \kappa \hat{V} + b^Y \hat{Y} + b^1 E_1 + b^2 E_2, \tag{3.5}$$

where, in $r_+ \leq r \leq r_Y^-$, κ is a strictly positive function bounded away from 0 and $|b^\alpha| \leq C\,(1 - \mu)$.

In Schwarzschild spacetime, consider the frame on $\mathcal{S}^2$ given by $\{\frac{1}{r_S^2 \sin\theta_S}\partial_{\phi_S}, \frac{1}{r_S}\partial_{\theta_S}\}$. Then we get

$$\hat{V} = (1+\mu)\partial_{t_S^*} + (1-\mu)\partial_{r_S}, \hat{Y} = \partial_{t_S^*} - \partial_{r_S}, E_1 = \frac{1}{r_S}\partial_{\theta_S}, E_2 = \frac{1}{r_S \sin\theta_S}\partial_{\phi_S}$$

Since we consider Kerr spacetimes on which the metric is close to that on a Schwarzschild spacetime, we have that, in (t^*, r, θ, ϕ^*) coordinates, the null frame can be expressed as

$$\hat{V} = (1+\mu)\partial_{t^*} + (1-\mu)\partial_r + O_1(\epsilon)\partial,$$

$$\hat{Y} = \partial_{t^*} - \partial_r + O_1(\epsilon)\partial,$$

$$E_1 = \frac{1}{r}\partial_\theta + O_1(\epsilon)\partial,$$

$$E_2 = \frac{1}{r\sin\theta}\partial_{\phi^*} + O_1(\epsilon)\partial.$$

Alternatively, if we write E_α, where $\alpha = 1, 2, 3, 4$, for the null frame, we have

$$(1+\mu)\partial_{t^*} + (1-\mu)\partial_r = \hat{V} + O_1(\epsilon)E_\alpha,$$

$$\partial_{t^*} - \partial_r = \hat{Y} + O_1(\epsilon)E_\alpha,$$

$$\partial_\theta = rE_1 + O_1(\epsilon)E_\alpha,$$

$$\partial_{\phi^*} = r \sin\theta E_2 + O_1(\epsilon) E_\alpha.$$

We also define the vector fields $\hat{V}, \hat{Y}, E_1, E_2$ outside $\{r \leq r_Y^-\}$ by requiring them to be compactly supported in $\{r \leq r_Y^+\}$ (for some r_Y^+ to be determined) and invariant under the one parameter families of isometries generated by ∂_{t^*} and ∂_{ϕ^*}. Notice that in particular there is no requirement that the vector fields form a null frame in the region $\{r_Y^- \leq r \leq r_Y^+\}$.

Chapter 4

The Vector Field Method

Our proof of Theorem 1.0.1, 1.0.2 and 1.0.3 are based on the vector field method. The vector field method is known to give robust quantitive decay estimates and has in particular culminated in the proof of the nonlinear stability of the Minkowski spacetime.

In this chapter, we introduce the basic elements of the vector field method. This includes the use of vector field multipliers and vecotr field commutators.

The use of vector field multipliers allows us to derive identities and conservation laws of the currents associated to appropriately chosen vector fields. The use of vector field commutators allows us to get higher order and weighted higher order estimates. In particular, a main idea in this thesis is to introduce an appropriately chosen "scaling vector field" as a commutator that allows us to prove improved decay estimates which can then be applied to the nonlinear problem.

In this chapter, we also perform some basic calculations for applying the vector field method in Schwarzschild and Kerr spacetimes. These calculations will be used in latter chapters.

4.1 Vector Field Multipliers and Basic Identities for Currents

We consider the conservation laws for Φ satisfying $\Box_g \Phi = 0$. Define the energy-momentum tensor

$$T_{\mu\nu} = \partial_\mu \Phi \partial_\nu \Phi - \frac{1}{2} g_{\mu\nu} \partial^\alpha \Phi \partial_\alpha \Phi.$$

We note that $T_{\mu\nu}$ is symmetric and the wave equation implies that

$$D^\mu T_{\mu\nu} = 0.$$

Given a vector field V^μ, we define the associated currents

$$J^V_\mu (\Phi) = V^\nu T_{\mu\nu} (\Phi),$$

$$K^V (\Phi) = {}^{(V)}\pi_{\mu\nu} T^{\mu\nu} (\Phi),$$

where ${}^{(V)}\pi_{\mu\nu}$ is the deformation tensor defined by

$${}^{(V)}\pi_{\mu\nu} = \frac{1}{2} (D_\mu V_\nu + D_\nu V_\mu).$$

In particular, $K^V (\Phi) = {}^{(V)}\pi_{\mu\nu} = 0$ if V is Killing. Since the energy-momentum tensor is divergence-free,

$$D^\mu J^V_\mu (\Phi) = K^V (\Phi).$$

We also define the modified currents

$$J^{V,w}_\mu (\Phi) = J^V_\mu (\Phi) + \frac{1}{8} \left(w \partial_\mu \Phi^2 - \partial_\mu w \Phi^2 \right),$$

$$K^{V,w} (\Phi) = K^V (\Phi) + \frac{1}{4} w \partial^\nu \Phi \partial_\nu \Phi - \frac{1}{8} \Box_g w \Phi^2.$$

Then

$$D^\mu J^{V,w}_\mu(\Phi) = K^{V,w}(\Phi).$$

We integrate by parts with this in the region $\mathcal{R}$ that is bounded in the future by Σ_1, and bounded in the past by Σ_1. Denoting the future-directed normal to Σ_1 by $n^\mu_{\Sigma_1}$ (and the future-directed normal to Σ_0 by $n^\mu_{\Sigma_0}$), we have

Proposition 4.1.1.

$$\int_{\Sigma_1} J^V_\mu(\Phi)\, n^\mu_{\Sigma_1} + \iint_{\mathcal{R}} K^V(\Phi) = \int_{\Sigma_0} J^V_\mu(\Phi)\, n^\mu_{\Sigma_0},$$

and

$$\int_{\Sigma_1} J^{V,w^V}_\mu(\Phi)\, n^\mu_{\Sigma_1} + \iint_{\mathcal{R}} K^{V,w^V}(\Phi) = \int_{\Sigma_0} J^{V,w^V}_\mu(\Phi)\, n^\mu_{\Sigma_0}.$$

In particular, if V is Killing, the first identity can be considered as a conservation law. This is a manifestation of Noether's Theorem, which states that a differentiable one-parameter family of symmetries give rise to a conservation law.

One can similarly define the above quantities for the inhomogeneous wave equation $\Box_g \Phi = F$. In this case, the energy-momentum is no longer divergence free. Instead, we have

$$D^\mu T_{\mu\nu} = F\partial_\nu \Phi.$$

In this case,

$$D^\mu J^V_\mu(\Phi) = K^V(\Phi) + FV^\nu \partial_\nu \Phi.$$

For the modified current,

$$D^\mu J^{V,w}_\mu(\Phi) = K^{V,w}(\Phi) + \frac{1}{4} Fw\Phi + FV^\nu \partial_\nu \Phi.$$

Proposition 4.1.2.

$$\int_{\Sigma_1} J^V_\mu(\Phi)\, n^\mu_{\Sigma_1} + \iint_{\mathcal{R}} K^V(\Phi) = \int_{\Sigma_0} J^V_\mu(\Phi)\, n^\mu_{\Sigma_0} - \iint_{\mathcal{R}} F V^\nu \partial_\nu \Phi.$$

and

$$\int_{\Sigma_1} J^{V,w^V}_\mu(\Phi)\, n^\mu_{\Sigma_1} + \iint_{\mathcal{R}} K^{V,w^V}(\Phi) = \int_{\Sigma_0} J^{V,w^V}_\mu(\Phi)\, n^\mu_{\Sigma_0} + \iint_{\mathcal{R}} \left(-\frac{1}{4} F w \Phi - F V^\nu \partial_\nu \Phi \right).$$

In the case of Schwarzschild spacetimes, there are two choices of Σ_i that we will use. The first is to choose Σ_i to be $t = \textit{constant}$ slices. The second choice is for estimates near the event horizon. In this case, $\mathcal{R} = \{v_0 \leq v \leq v_1, t \geq t_0\}$, $\Sigma_0 = \{v = v_0, t \geq t_0\} \cup \{v_0 \leq v \leq v_1, t = t_0\}$ and $\Sigma_1 = \{v = v_1, t \geq t_0\} \cup \{v_0 \leq v \leq v_1, u = \infty\}$ (See Figure). In particular, Σ_1 includes part of the event horizon.

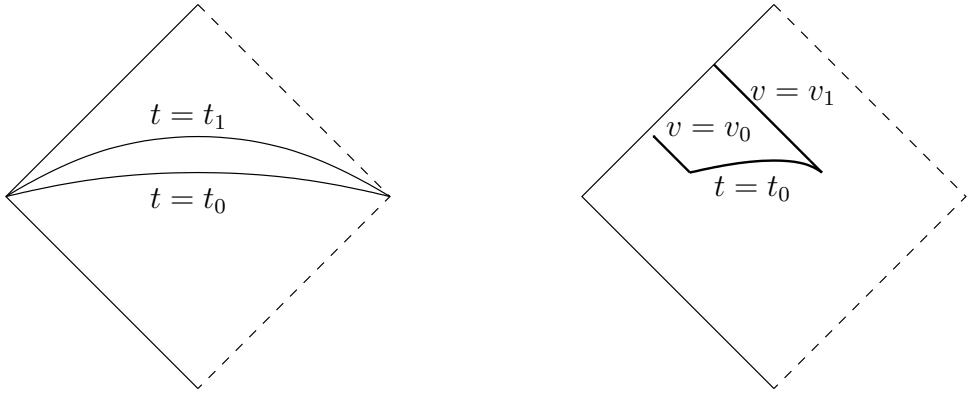


Figure 4.1: Regions of integration

In the Kerr case, we will take a more unified approach. Recall from the previous chapter that t^* is defined to be $t^* = t + \chi(r)h(r)$, where $\frac{dh(r)}{dr} = \frac{2Mr}{r^2 - 2Mr + a^2}$. A fixed $t^* = \tau$ slice, which we will also denote as Σ_τ is depicted in Figure 3.2. In applying the

above identities for the vector field currents for the Kerr spacetimes, we will take Σ_1 to be $\Sigma_\tau \cup \mathcal{H}^+(\tau', \tau)$, where $\mathcal{H}^+(\tau', \tau)$ is the part of the event horizon with $\tau' \leq t^* \leq \tau$. We will take Σ_0 to be $\Sigma_{\tau'}$. The region $\mathcal{R}$ bounded by Σ_0 and Σ_1 will be denoted as $\mathcal{R}(\tau', \tau)$.

4.1.1 Schwarzschild Computations

In the case of wave equation on Schwarzschild background, we can compute the energy momentum tensor explicitly in local coordinates $\left(t, r^*, x^A, x^B\right)$ or equivalently $\left(u, v, x^A, x^B\right)$, where x^A, x^B is an orthonormal basis on $\mathbb{S}^2$.

$$T_{uu}\left(\Phi\right) = \left(\partial_u \Phi\right)^2,$$

$$T_{vv}\left(\Phi\right) = \left(\partial_v \Phi\right)^2,$$

$$T_{uv}\left(\Phi\right) = \left(1-\mu\right)\left|\not{\nabla}\Phi\right|^2,$$

$$T_{AA}\left(\Phi\right) + T_{BB}\left(\Phi\right) = \left|\not{\nabla}\Phi\right|^2 - \partial^\alpha \Phi \partial_\alpha \Phi.$$

As a result,

$$\begin{aligned} K^V\left(\Phi\right) =& \frac{1}{4\left(1-\mu\right)}\left(\left(\partial_u\Phi\right)^2 \partial_v\left(V_v\left(1-\mu\right)^{-1}\right) + \left(\partial_v\Phi\right)^2 \partial_u\left(V_u\left(1-\mu\right)^{-1}\right)\right. \\ &\left. + \left|\not{\nabla}\Phi\right|^2\left(\partial_u V_v + \partial_v V_u\right)\right) - \frac{1}{2r}\left(V_u - V_v\right)\left(\left|\not{\nabla}\Phi\right|^2 - \partial^\alpha\Phi\partial_\alpha\Phi\right). \end{aligned}$$

4.1.2 Vector Field Multipliers under Metric Perturbations

If we consider Kerr spacetimes such that ϵ is small, vector fields multipliers that are defined in the Schwarzschild (t^*, r, x^A, x^B) coordinates or Schwarzschild $(t, r \geq r_Y^-, x^A, x^B)$ coordinates are stable. We can consider a fixed vector field defined on the differentiable structure of a Schwarzschild exterior and compare the currents obtained using the Schwarzschild metric and the Kerr metric.

Proposition 4.1.3. *Consider either the Schwarzschild $(t_S^*, r_S, x_S^A, x_S^B)$ coordinates or $(t_S, r_S \geq r_Y^-, x_S^A, x_S^B)$ coordinates. Suppose V is a vector field defined on either of these coordinates.*

$$|(J_S^{V,w^V})_\mu(\Phi)\, n_{\Sigma_\tau}^\mu - (J_K^{V,w^V})_\mu(\Phi)\, n_{\Sigma_\tau}^\mu| \leq C\epsilon r^{-2} \max_\alpha |V^\alpha| (\partial\Phi)^2,$$

$$|K_S^{V,w^V}(\Phi) - K_K^{V,w^V}(\Phi)| \leq C\epsilon r^{-2} \left((\sum_{k=0}^{1} \max_\alpha |\partial^k V^\alpha| + |w|)(\partial\Phi)^2 + \sum_{m=1}^{2} |\partial^m w| \Phi^2 \right)$$

4.2 Vector Field Commutators

A second part of the vector field method is to chosen appropriate vector fields as commutation. On one hand, this provides us with higher order energy, which is necessarily for studying the nonlinear problem. More importantly, this commutators can be chosen to have weights in t and r that can be a tool to prove decay of the solution to the wave equation. The decay rates that can be proved by this method turns out to be very robust and can be applied to the nonlinear problem.

4.2.1 Scaling Vector Field on Black Hole Spacetimes

One of the central ideas in this thesis is to introduce a vector field commutator on Schwarzschild and Kerr black holes that is an analogue of the scaling vector field in Minkowski spacetime. This allows us to improve the decay rate of the solution to the linear wave equation as well as to use this improved decay rate for the nonlinear problem.

In the sequel, the construct fo the scaling vector field is different in the Schwarzschild and Kerr cases. Nevertheless, the main idea is a construction such that

$$S = t^* \partial_{t^*} + h(r) r \partial_r,$$

where $h(r) \to r$ as $r \to \infty$. With this construction, the commutator $[\Box, S]$ has the following properties:

- First, due to the stationarity of the spacetimes, $[\Box, S]$ is independent of t^*. This allows us get estimates for $S\Phi$, which has a t^* weight after estimating error terms that do not grow with t^*. This would allow us to improve the decay rate of the solution to the wave equation.

- The requirement that $h(r) \to r$ as $r \to \infty$ implies that S tends to the Minkowski scaling vector field as $r \to \infty$. Since Schwarzschild and Kerr are asymptotically Minkowskian, and S is conformally Killing, we can conclude that $[\Box, S] \to 2\Box$ as $r \to 0$. A more precise understanding of this decay rate would allow us to control the error terms arising from the commutation with S.

4.2.2 Vector Field Commutators under Metric Perturbations

Some computations are easier in Schwarzschild spacetime than in Kerr spacetime. In the sequel, we will often consider consider a fixed vector field on the differentiable structure of the Schwarzschild exterior. We now show that for such vector fields, the commutators with $\Box_{g_S}$ and $\Box_{g_K}$ are close to each other as long as a is chosen to be sufficiently small:

Proposition 4.2.1. *Consider either the Schwarzschild $(t_S^*, r_S, x_S^A, x_S^B)$ coordinates or $(t_S, r_S \geq r_Y^-, x_S^A, x_S^B)$ coordinates. Suppose V is a vector field defined on either of these coordinates. Then*

$$|[\Box_{g_K} - \Box_{g_S}, V]\Phi| \leq C\epsilon r^{-2}(\sum_{m=1}^{2}\sum_{k=1}^{2}\max_{\alpha}|\partial^m V^\alpha||\partial^k \Phi|),$$

where ∂ is the coordinate derivative for the coordinate system on which V is defined.

Proof. We rewrite

$$\Box_{g_S} = g_S^{\alpha\beta}\partial_\alpha\partial_\beta + \eta_S^\alpha\partial_\alpha,$$

$$\Box_{g_K} = g_K^{\alpha\beta}\partial_\alpha\partial_\beta + \eta_K^\alpha\partial_\alpha.$$

Using $|(g_K)_{\alpha\beta}-(g_S)_{\alpha\beta}| \leq \epsilon r^{-2}$ and $|\partial_\gamma((g_K)_{\alpha\beta}-(g_S)_{\alpha\beta})| \leq \epsilon r^{-2}$, we have $|\sqrt{-\det g_K}-\sqrt{-\det g_S}| \leq \epsilon r^{-2}$ and $|\partial_\alpha(\sqrt{-\det g_K}-\sqrt{-\det g_S})| \leq \epsilon r^{-2}$. Therefore,

$$\sup_{\alpha,\beta}|g_S^{\alpha\beta}-g_K^{\alpha\beta}| + \sup_\alpha|\eta_S^\alpha-\eta_K^\alpha| \leq C\epsilon r^{-2}.$$

Therefore,

$$\begin{aligned}
&|[\Box_{g_K}-\Box_{g_S}, V]\Phi| \\
\leq&|(g_K^{\alpha\beta}-g_S^{\alpha\beta})(\partial_\alpha\partial_\beta V^\gamma)\partial_\gamma\Phi| + 2|(g_K^{\alpha\beta}-g_S^{\alpha\beta})\partial_\alpha V^\gamma\partial_\beta\partial_\gamma\Phi| + |(\eta_S^\alpha-\eta_K^\alpha)(\partial_\alpha V^\gamma)\partial_\gamma\Phi| \\
\leq&C\epsilon r^{-2}(\sum_{m=1}^{2}\sum_{k=1}^{2}\sup_\alpha|\partial^m V^\alpha||\partial^k\Phi|).
\end{aligned}$$

□

Chapter 5

Precise Statements of the Main Theorems

After the introduction to Schwarzschild and Kerr geometry and the vector field method, we can now give more precise statements of the main theorems 1.0.1, 1.0.2 and 1.0.3.

5.1 Precise Statement of Theorem 1.0.1

We prove both pointwise decay and energy decay for solutions of $\Box_g \Phi = 0$. From this point onwards, we assume $t_* > 1$, $v_* > 1$.

Define

$$
\begin{aligned}
E_0(\phi) =& \int_{\{r\geq 2M\}} \left(\sum_{k=0}^{3} J_\mu^N\left(\Omega^k\phi\right) n_{t_0}^\mu + \sum_{k=0}^{2} J_\mu^{Z,w^Z}\left(\Omega^k\phi\right) n_{t_0}^\mu \right) dVol_{t_0} \\
&+ \int_{\{r_b\leq r\leq r_0\}} \sum_{k=0}^{3} J_\mu^N\left(\Omega^k\phi\right) n_{v_0}^\mu dVol_{v_0}, \quad \text{where } r_0 > 2M \text{ is to be picked,} \\
E_1(\phi) =& E_0(S\phi) + \sum_{m=0}^{1}\sum_{k=0}^{4-m} E_0\left(\partial_t^m\Omega^k\phi\right), \\
E_2(\phi) =& \sum_{m=0}^{2}\sum_{k=0}^{2-m} E_1\left(\partial_t^m\Omega^k\phi\right), \\
E_3(\phi) =& \sum_{m=0}^{1}\sum_{k=0}^{1} E_0\left(\partial_t^m\Omega^k\phi\right) + E_1(\phi), \\
E_4(\phi) =& \sum_{m=0}^{2}\sum_{k=0}^{4-m} E_0\left(\partial_t^m\Omega^k\phi\right) + \sum_{m=0}^{2}\sum_{k=0}^{2-m} E_1\left(\partial_t\Omega^k\phi\right).
\end{aligned}
$$

Theorem 5.1.1. *Suppose Φ is a solution to the wave equation on the Schwarzschild spacetime, i.e., $\Box_g\Phi = 0$. Then for any $\delta > 0$ and any $0 \leq r_b \leq 2M \leq R < \infty$*

1. *Pointwise Decay of Φ*

$$
|\Phi(v_*, r)| \leq C_{\delta,r_b,R} v_*^{-\frac{3}{2}+\delta} E_2^{\frac{1}{2}}(\Phi) \qquad \text{for } r_b \leq r \leq R.
$$

2. *Pointwise Decay of Derivatives of Φ*

$$
|\nabla\Phi(v_*, r)| \leq C_{\delta,r_b,R} v_*^{-\frac{3}{2}+\delta} \sum_{k+m\leq 1} E_2^{\frac{1}{2}}\left(\partial_t^m\Omega^k\Phi\right) \qquad \text{for } r_b \leq r \leq R,
$$

   where ∇ denotes any derivatives.

3. *Decay of Non-degenerate Energy in the Region $2M \leq r \leq R$*

$$\begin{aligned}
&\int_{r_1^*}^{R} \left(\Phi(t_*)^2 + \left(\nabla\Phi\left(t_*\right)\right)^2\right) dVol_{t_*} \\
&+ \int_{\{-\infty<r^*\leq r_1^*\}} \left(\frac{1}{1-\mu}\left(\partial_u\Phi\right)^2 + \left(1-\mu\right)\left|\not\nabla\Phi\right|^2\right) dAdu_{\{v=v_*\}} \\
\leq & C_{\delta,r_1^*,R} \min\left\{t_*, v_*\right\}^{-3+\delta} \sum_{k+m\leq 1} E_1\left(\partial_t^m\Omega^k\Phi\right),
\end{aligned}$$

for any r_1 satisfying $r_1^* \leq R^*$.

4. *Decay of Non-degenerate Energy in the Region $r_b \leq r \leq 2M$ in (v, r, ω) Coordinates*

$$\int_{v_*}^{v_*+1} \left(\Phi^2 + \left(\nabla\Phi\right)^2\right) dAdv_r \leq C_{\delta,r_b} v_*^{-3+\delta} \sum_{k+m\leq 1} E_1\left(\partial_t^m\Omega^k\Phi\right),$$

for any $r_b \leq r \leq 2M$.

Remark 1. *The integral in statement 3 represents the part of non-degenerate energy restricted to the region $r \leq R$ (See Section 3.5). It should be compared with the corresponding part of the (degenerate) energy generated by the vector field $T = \frac{\partial}{\partial t}$*

$$\int_{r_1^*}^{R^*} \left(\nabla\Phi\left(t_*\right)\right)^2 dVol_{t_*} + \int_{\{2M\leq r^*\leq r_1^*\}} \left(\left(\partial_u\Phi\right)^2 + \left(1-\mu\right)\left|\not\nabla\Phi\right|^2\right) dAdu_{\{v=v_*\}}.$$

The non-degeneracy is more apparent if we write the second integral in the (v, r, ω) coordinates, which up to some constant is:

$$\int_{\{2M\leq r^*\leq r_1^*\}} \left(\left(\partial_r\Phi\right)^2 + \left|\not\nabla\Phi\right|^2\right) dAdr_{\{v=v_*\}}.$$

For the time derivatives, we have better decay estimates both in the sense that we have a better decay rate and have a larger region of spacetime on which the estimates

hold:

Theorem 5.1.2. *Suppose Φ is a solution to the wave equation on the Schwarzschild spacetime, i.e., $\Box_g \Phi = 0$. Then for any $\delta > 0$ and $r_b > 0$*

1. *Pointwise Decay of $\partial_t \Phi$*

$$|\partial_t \Phi(v^*)| \leq C_{\delta, r_b} v_*^{-2+\delta} E_4^{\frac{1}{2}}(\Phi) \qquad \text{for } r_b \leq r \leq 2M \text{ or } r^* \leq \frac{t_*}{2}.$$

2. *Decay of Non-degenerate Energy of $\partial_t \Phi$ in the Region $r^* \leq \frac{t_*}{2}$*

$$\begin{aligned} &\int_{r_1{}^*}^{\frac{t_*}{2}} \left((\partial_t \Phi(t_*))^2 + (\nabla \partial_t \Phi(t_*))^2 \right) dVol_{t_*} \\ &+ \int_{\{-\infty < r^* \leq r_1^*\}} \left(\frac{1}{1-\mu} (\partial_u \partial_t \Phi)^2 + (1-\mu) |\not\nabla \partial_t \Phi|^2 \right) dA du_{\{v = v_*\}} \\ \leq & C_{\delta, r^*} \min\{t_*, v_*\}^{-4+\delta} E_3(\Phi). \end{aligned}$$

   for any r_1^*.

3. *Decay of Non-degenerate Energy of $\partial_t \Phi$ in the Region $r_b \leq r \leq 2M$ in (v, r, ω) Coordinates*

$$\int_{v_*}^{v_*+1} \left((\partial_t \Phi)^2 + (\nabla \partial_t \Phi)^2 \right) dA dv_r \leq C_{\delta, r_b} v_*^{-4+\delta} E_3(\Phi),$$

   for any $r_b \leq r \leq 2M$.

We would like to point out that the pointwise decay rates in both theorems apply to region of finite r including inside the black hole.

5.2 Precise Statement of Theorem 1.0.2

With the currents defined, we can state our main theorem.

Theorem 5.2.1. *Suppose $\Box_{g_K}\Phi = 0$. Then for all $\eta > 0$, $R > r_+$ and all $M > 0$ there exists a_0 such that the following estimates hold in the region $\{r_+ \leq r \leq R\}$ on Kerr spacetimes with (M, a) for which $a \leq a_0$:*

1. *Improved Decay of Non-degenerate Energy*

$$\begin{aligned}
&\sum_{j=0}^{\ell} \int_{\Sigma_\tau \cap \{r \leq R\}} \left(D^j \Phi\right)^2 \\
\leq & C_R \tau^{-3+\eta} \left(\sum_{m=0}^{\ell+2} \int_{\Sigma_{\tau_0}} J_\mu^{Z+CN, w^Z} \left(\partial_{t^*}^m S \Phi\right) n_{\Sigma_{\tau_0}}^\mu \right. \\
&\qquad \left. + \sum_{m+k+j \leq \ell+5} \int_{\Sigma_{\tau_0}} J_\mu^{Z+CN, w^Z} \left(\partial_{t^*}^m \hat{Y}^k \tilde{\Omega}^j \Phi\right) n_{\Sigma_{\tau_0}}^\mu \right).
\end{aligned}$$

2. *Improved Pointwise Decay*

$$\begin{aligned}
&\sum_{j=0}^{\ell} \left|D^j \Phi\right| \\
\leq & C_R \tau^{-\frac{3}{2}+\eta} \left(\sum_{m=0}^{\ell+4} \int_{\Sigma_{\tau_0}} J_\mu^{Z+CN, w^Z} \left(\partial_{t^*}^m S \Phi\right) n_{\Sigma_{\tau_0}}^\mu \right. \\
&\qquad \left. + \sum_{m+k+j \leq \ell+7} \int_{\Sigma_{\tau_0}} J_\mu^{Z+CN, w^Z} \left(\partial_{t^*}^m \hat{Y}^k \tilde{\Omega}^j \Phi\right) n_{\Sigma_{\tau_0}}^\mu \right)^{\frac{1}{2}}.
\end{aligned}$$

Here the vector fied commutators S, $\tilde{\Omega}$ and $\hat{Y}$ will be defined in Section 9.2. The vector field N will be defined in Section 9.3; the vector field Z with the modifying function w^Z will be defined in Section 9.5.

Remark 2. *We will show that although $J_\mu^{Z, w^Z}(\Phi) n_{\Sigma_{t^*}}^\mu$ is not always nonnegative, $J_\mu^{Z+CN, w^Z}(\Phi) n_{\Sigma_{t^*}}^\mu$ is nonnegative for sufficiently large C. Hence all the energy quantities in the Theorem are nonnegative.*

Remark 3. *Since we have the improved decay of the non-degenerate energy, the above Theorem can be extended beyond the event horizon. Analogous to Theorem 5.1.1, for any $r_b \in (r_-, r_+)$, where r_- is the smaller root of $\Delta = r^2 - 2Mr - a^2$, the theorem holds up to $r \geq r_b$ for D understood as a regular derivative inside the black hole, and with the constant depending also on r_b. The proof is similar to that for 5.1.1 and will be omitted.*

Remark 4. *Moreover, we can get a further improved estimates for $\partial_{t^*}\Phi$ as in Theorem 5.1.2. However, in this case, we need to be more careful about the cutoff and elliptic estimates. Since the proof is similar in spirit with that of Theorem 5.1.2 and it will not be used for the nonlinear problem, it will be omitted.*

5.3 Precise Statement for Theorem 1.0.3

In order to give a precise statement for theorem 1.0.3, we need to define the null condition on Kerr spacetimes. An important observation for this problem is that while the decay of the solution to the linear wave equation is nonintegrable in time, some "good" derivatives of the solution decay better and is in particular integrable. We need to define a null condition that guarantees that one of the two derivatives in the quadratic nonlinear is a "good" derivative. Thus in order to define the null condition, we first need a notion of "good" and "bad" derivatives.

We will define the null condition using the Schwarzschild coordinates on Kerr spacetimes $(t_S, r_S, \theta_S, \phi_S)$. Define

$$r_S^* = r_S + 2M\log(r_S - 2M) - 3M - 2M\log M,$$

$$u = \frac{1}{2}(t_S - r_S^*),$$

$$v = \frac{1}{2}(t_S + r_S^*).$$

Notice that in Schwarzschild spacetime, the vector fields ∂_u and ∂_v are null.

We define in coordinates

$$\underline{L} = \partial_u, \text{ in the } (u, v, \theta_S, \phi_S) \text{ coordinates,}$$

$$L = 2\partial_{t^*} + \chi(r)\frac{a}{Mr_+}\partial_{\phi^*} - \underline{L}$$

We can now define the "good" and "bad" derivatives. Define

$$\not{\nabla} \in \{\frac{1}{r}\partial_\theta, \frac{1}{r}\partial_\phi\},$$

$$\overline{D} \in \{L, \frac{1}{r}\partial_\theta, \frac{1}{r}\partial_\phi\},$$

$$D \in \{\frac{1}{1-\mu}\underline{L}, L, \frac{1}{r}\partial_\theta, \frac{1}{r}\partial_\phi\}.$$

Notice that D spans the whole tangent space and that we always have $[D, \partial_{t^*}] = 0$.

We now define the null condition. On Minkowski spacetime, the classical null condition can be defined geometrically by requiring the nonlinearity to have the form

$$A^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi,$$

where A satisfies $A^{\mu\nu}\xi_\mu\xi_\nu = 0$ whenever ξ is null. On Kerr spacetime, we would like to define a notion of the null condition that includes this geometric notion. This is also because many physically relevant semilinear equations satisfy this condition. On the other hand, in order to prove the global existence result, we need to use the vector fields that capture the good derivative. We would therefore like to define the null condition using D and $\overline{D}$. In particular, we want the nonlinearity to have at least one good, i.e., $\overline{D}$, derivative. This on its own is however inconsistent with the geometric null condition. We therefore allow a term in the quadratic nonlinearity that does not have a good derivative but decays in r.

Definition 5.3.1. *Consider the nonlinearity $F(\Phi, D\Phi, t^*, r, \theta, \phi^*)$. We say that F satisfies the null condition if*

$$F = \Lambda_0(\Phi, t^*, r, \theta, \phi^*)D\Phi\overline{D}\Phi + \Lambda_1(\Phi, t^*, r, \theta, \phi^*)D\Phi D\Phi + \mathcal{C}(\Phi, D\Phi, t^*, r, \theta, \phi^*),$$

where

$$|D_\Phi^{i_1}\partial_{t^*}^{i_2}\partial_r^{i_3}\partial_\theta^{i_4}\partial_{\phi^*}^{i_5}\Lambda_j| \leq C(t^*)^{-i_2}r^{-i_3} \quad \text{for } i_1+i_2+i_3+i_4+i_5 \leq 16, j = 0, 1$$

and

$$|D_\Phi^{i_1}\partial_{t^*}^{i_2}\partial_r^{i_3}\partial_\theta^{i_4}\partial_{\phi^*}^{i_5}\Lambda_1| \leq C(t^*)^{-i_2}r^{-1-i_3} \quad \text{for } i_1+i_2+i_3+i_4+i_5 \leq 16 \text{ and } r \geq \frac{9t^*}{10}$$

and $\mathcal{C}$ denotes a polynomial that is at least cubic in $D\Phi$ (with coefficients in $\Phi, t^*, r, \theta, \phi^*$) satisfying

$$|D_\Phi^{i_1}\partial_{t^*}^{i_2}\partial_r^{i_3}\partial_\theta^{i_4}\partial_{\phi^*}^{i_5}\mathcal{C}| \leq C(t^*)^{-i_2}r^{-i_3}\sum_{s=3}^{S}|D\Phi|^s \quad \text{for } i_1+i_2+i_3+i_4+i_5 \leq 16$$

Remark 5. *The null condition is a special structure for the quadratic nonlinearity. We note that in our case, the restriction is necessary only for $r \geq \frac{9t^*}{10}$. Moreover, higher order terms should give better estimates and do not need any special structure.*

We are now in a position to state our Main Theorem precisely.

Theorem 5.3.2. *Consider the equation*

$$\Box_g\Phi = F(\Phi, D\Phi, t^*, r, \theta, \phi^*), \tag{5.1}$$

where F satisfies the null condition. There exists an ϵ such that if the initial data of

Φ *satisfies*

$$\sum_{i+j+k=16} \int_{\Sigma_{\tau_0}} J_\mu^{Z+CN,w^Z} \left(\hat{Y}^k \partial_{t^*}^i \tilde{\Omega}^j \Phi \right) n_{\Sigma_{\tau_0}}^\mu$$
$$+ \sum_{i+j+k=16} \int_{\Sigma_{\tau_0}} J_\mu^{Z+CN,w^Z} \left(\hat{Y}^k S \partial_{t^*}^i \tilde{\Omega}^j \Phi \right) n_{\Sigma_{\tau_0}}^\mu \leq \epsilon$$

and

$$\sum_{\ell=0}^{13} r|D^\ell \Phi(\tau_0)| + r|D^\ell S\Phi(\tau_0)| \leq \epsilon.$$

Then Φ *exists globally in time. Moreover, for all* $\eta > 0$*, we can take a sufficiently small such that the solution* Φ *obeys the decay estimate*

$$|\Phi| \leq C\epsilon r^{-1} u^{-\frac{1}{2}} (t^*)^\eta, |D\Phi| \leq C\epsilon r^{-1} u^{-1} (t^*)^\eta, |\overline{D}\Phi| \leq C\epsilon r^{-1} (t^*)^{-1+\eta} \text{ for } r \geq R, \text{ and}$$

$$|\Phi| \leq C_\delta \epsilon (t^*)^{-\frac{3}{2}+\eta} r^\delta, |D\Phi| \leq C_\delta \epsilon (t^*)^{-\frac{3}{2}+\eta} r^{-\frac{1}{2}+\delta} \text{ for } r \leq \frac{t^*}{4}.$$

Here, as in Theorem 5.2.1, the vector fied commutators S*,* $\tilde{\Omega}$ *and* $\hat{Y}$ *will be defined in Section 9.2. The vector field* N *will be defined in Section 9.3; the vector field* Z *with the modifying function* w^Z *will be defined in Section 9.5.*

We specialize to a particular case which resembles better the classical null condition [28].

Theorem 5.3.3. *Consider the equation*

$$\Box_g \Phi = \Gamma(\Phi) A^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi, \tag{5.2}$$

where A *satisfies* $A^{\mu\nu}\xi_\mu\xi_\nu = 0$ *whenever* $\xi \in TK$ *is null. If the initial data of* Φ

satisfies

$$\sum_{i+j+k=16} \int_{\Sigma_{\tau_0}} J_\mu^{Z+CN,w^Z} \left(\hat{Y}^k \partial_{t^*}^i \tilde{\Omega}^j \Phi \right) n_{\Sigma_{\tau_0}}^\mu$$
$$+ \sum_{i+j+k=16} \int_{\Sigma_{\tau_0}} J_\mu^{Z+CN,w^Z} \left(\hat{Y}^k S \partial_{t^*}^i \tilde{\Omega}^j \Phi \right) n_{\Sigma_{\tau_0}}^\mu \leq \epsilon$$

and

$$\sum_{\ell=0}^{13} r|D^\ell \Phi(\tau_0)| + r|D^\ell S\Phi(\tau_0)| \leq \epsilon.$$

Then the conclusion of Theorem 5.3.2 holds.

The above formulation is geometric and independent of the choice of coordinates. We note that this condition is obviously satisfied by the wave map equation in the intrinsic formulation.

Chapter 6

The Case of Minkowski Spacetime

Before proceeding to outlining the proof of the main theorems, we first outline how a similar result can be proved on Minkowski spacetime. Both the improved decay results for the linear wave equation and the small data global existence result for the nonlinear wave equation have their roots in the analogous theorems on Minkowski spacetime. These results have been proved via the vector field method first in the works of Klainerman [27], [28].

By now many variants of the proof is known. The main idea of the proof uses vector fields that generate symmetries or conformal symmetries of Minkowski spacetime as multipliers and commutators as described in the Chapter 4. However, as we move from Minkowski spacetime to the black hole spacetimes, one of the main difficulties is that Schwarzschild and Kerr spacetimes have much fewer symmetries and this severely limits the number of vector fields that can be used. In the following, we outline an approach to prove the analogous results on Minkowski spacetime using a very limited set of vector fields. This will pave the way to understand our main results on Schwarzschild and Kerr spacetimes.

6.1 Quantitative Decay Rates for Solutions to the Linear Wave Equation

We first discuss some decay results for the linear wave equation on Minkowski spacetimes. We would like to especially highlight techniques that are relevant to our result. In Minkowski space $\mathbb{R}^{3,1}$, the solutions to the wave equation with initial conditions $\Phi(t=0,x)=\Phi_0$ and $\partial_t\Phi(t=0,x)=\Phi_1$ can be written as

$$\Phi(t,x)=\frac{1}{4\pi t^2}\left(\partial_t\int_{\mathbb{S}^2}t\Phi_0(x+ty)\,dA(y)+\int_{\mathbb{S}^2}t\Phi_1(x+ty)\,dA(y)\right). \tag{6.1}$$

This formula implies immediately that $|\Phi(t,x)|\leq\frac{C}{t_+}$, where $t_+=\max\{t,1\}$. This decay is optimal in the variable t. However, improved decay can be seen in the null coordinates $v=\frac{1}{2}(t+r)$ and $u=\frac{1}{2}(t-r)$, where $r^2=\sum_{i=0}^{3}x_i^2$. In particular, (6.1) implies the strong Huygens' Principle, asserting that Φ with compactly supported initial data is compactly supported in the variable u. Therefore, denoting $v_+=\max\{v,1\}$, $u_+=\max\{u,1\}$, we have in particular

$$|\Phi|\leq\frac{C_N}{v_+u_+^N},\quad\forall N\geq 0.$$

If we just focus on the region $\{r\leq\frac{t}{2}\}$, where $t\sim v\sim u$, the decay can be written as

$$|\Phi|\leq\frac{C_N}{t_+^N},\quad\forall N\geq 0.$$

However, the use of the representation formula (6.1) is not available on perturbations of the Minkowski spacetime. In the proof of the nonlinear stability of Minkowski spacetime [10], [34], a more robust understanding of the decay of the linear waves was necessary. This was achieved by the vector field method. Let Φ be a solution to the linear wave equation on Minkowski spacetime, $\Box_m\Phi=0$. Taking the Killing vector

field ∂_t as a multiplier, we derive the energy conservation law by Proposition 4.1.1

$$\int \left((\partial_t \Phi)^2 + \sum_{i=1}^{3} (\partial_{x_i} \Phi) \right) dx_t = \int \left((\partial_t \Phi)^2 + \sum_{i=1}^{3} (\partial_{x_i} \Phi)^2 \right) dx_{t_0}. \tag{6.2}$$

Taking the Killing vector fields $\partial \in \{\partial_t, \partial_{x_i}\}$ as commutators, we see that the energy conservation law (6.2) can be applied to $\partial\Phi$ and we can control the L^2 norm of the derivatives of Φ of orders 1 and 2. Then using a Sobolev-type inequality $||\Phi||_{L^\infty(\mathbb{R}^3)} \leq C||\Phi||^{\frac{1}{2}}_{\dot{H}^1(\mathbb{R}^3)}||\Phi||^{\frac{1}{2}}_{\dot{H}^2(\mathbb{R}^3)}$ (which holds for compactly supported functions), uniform boundedness of the solutions to the wave equation can be proved. The Killing vector fields Ω_i generating the spherical symmetry can also be used as commutators. This is especially useful because compared to the angular derivatives, Ω_i has an extra factor of r, i.e., $\Omega \sim r\not{\nabla}$. This allows one to prove in [29] that for Φ decaying sufficiently fast at spatial infinity:

$$|\Phi| \leq \frac{C}{r} \sum_{m=1}^{2} \sum_{k=0}^{2} ||\partial_r^m \Omega^k \Phi||_{L^2(\mathbb{R}^3)},$$

which implies a decay in the region $\{r > \frac{t}{2}\}$:

$$|\Phi| \leq \frac{C}{r} \leq \frac{C}{v_+}$$

after applying (6.2) to $\Omega^k\Phi$.

To achieve decay of Φ in $\{r \leq \frac{t}{2}\}$, one can use the conformally Killing vector field $Z = (t^2 + r^2)\,\partial_t + 2tr\partial_r$ introduced by Morawetz [42]. In this case, $K^Z(\Phi) \neq 0$. Nevertheless, by defining $w^Z = 2t$, $K^{Z,w^Z}(\Phi) = 0$ and therefore $\int J_0^{Z,w^Z}(\Phi)\,dx_t$ is a conserved quantity. Moreover, some algebraic manipulation would show

$$\int J_0^{Z,w^Z}(\Phi)\,dx_t \geq c \int \left(v^2 (\partial_v \Phi)^2 + u^2 (\partial_u \Phi)^2 + \left(v^2 + u^2\right) \left(\frac{\Phi^2}{r^2} + |\not{\nabla}\Phi|^2 \right) \right) dx,$$

where $\nabla\!\!\!/$ denotes the angular derivatives. The conserved nonnegative quantity

$$\int J_0^{Z,w^Z}(\Phi)\, dx_t$$

is known as the conformal energy. For the region $\{r \leq \frac{t}{2}\}$, notice that the boundedness of the conformal energy implies a local energy decay

$$\int_{\{r\leq\frac{t}{2}\}} \left(\frac{\Phi^2}{r^2} + (\partial_v \Phi)^2 + (\partial_u \Phi) + |\nabla\!\!\!/ \Phi|^2 \right) dx_t \leq \frac{C}{t_+}.$$

After considering the equations $\Box_m \left(\partial^k \Phi\right) = 0$, Sobolev embedding would imply the pointwise decay $|\Phi| \leq \frac{C}{t_+}$, for $r \leq \frac{t}{2}$. Notice that in this region t_+ is comparable to v_+. Therefore, we have in the whole of Minkowski spacetime

$$|\Phi| \leq \frac{C}{v_+}.$$

The above program of using the vector fields ∂_t and Z to prove decay of the solutions to wave equations has been carried out for black hole spacetimes [18], [16], [20], [5], [2]. The implementation is much more difficult even in the case of Schwarzschild because ∂_t becomes null at the event horizon and more seriously, Z is very far from being conformally Killing. The case of Kerr is even more intricate due to superradiance and the complicated trapping. Nevertheless, we take the above as our starting point. In the following, we discuss how one can improve the decay rates for solutions of the linear wave equation on Minkowski spacetime. Our approach on the black hole spacetime is inspired by the case on Minkowski space.

Klainerman-Sideris [30] showed that more decay can be achieved in the interior region $\{r \leq \frac{t}{2}\}$ for the derivatives of Φ. They used the scaling vector field $S = t\partial_t + r\partial_r$ as a commutator. Notice that S is conformally Killing and $[\Box_m, S] = 2\Box_m$. In particular, if one has $\Box_m \Phi = 0$, then $\Box_m (S\Phi) = S\Box_m \Phi + 2\Box_m \Phi = 0$. Therefore, any

decay results that hold for Φ also hold for $S\Phi$. Klainerman-Sideris [30] showed that

$$\begin{aligned}&\sum_{\partial\in\{\partial_t,\partial_{x_i}\}}||u_+\partial\partial_t\Phi||_{L^2(\mathbb{R}^3)}\\ \leq& C\sum_{\partial\in\{\partial_t,\partial_{x_i}\}}\left(||\partial S\Phi||_{L^2(\mathbb{R}^3)}+||\partial\Phi||_{L^2(\mathbb{R}^3)}+||\partial^2\Phi||_{L^2(\mathbb{R}^3)}+||\partial\Omega\Phi||_{L^2(\mathbb{R}^3)}\right).\end{aligned}$$

By cutting off appropriately and using the local energy decay estimates,

$$||\partial\partial_t\Phi||_{L^2\left(\{r\leq\frac{t}{2}\}\right)}\leq\frac{C}{t_+^2}\quad\text{in }\{r\leq\frac{t}{2}\}$$

since $\frac{1}{u_+}\leq\frac{C}{t_+}$ in this region. Again, using the Sobolev-type inequality above, one shows that $|\partial_t\Phi|\leq\frac{C}{t_+^2}$ in $\{r\leq\frac{t}{2}\}$. The other derivatives can also be estimated first by elliptic estimates and then the Sobolev inequality, since $||u_+\partial_t^2\Phi||_{L^2(\mathbb{R}^2)}=||u_+\Delta\Phi||_{L^2(\mathbb{R}^2)}$ by the linear wave equation. Therefore

$$|\partial\Phi|\leq\frac{C}{t_+^2}\quad\text{in }\{r\leq\frac{t}{2}\}.$$

We remark that in [30], the improved decay in $\{r\leq\frac{t}{2}\}$ can also be proved for the function Φ itself by inverting the Laplacian. This cannot be done on Schwarzschild spacetimes, it is impossible to invert the Laplacian for functions that do not vanish on the bifurcate sphere ($\mathcal{H}^+\cap\mathcal{H}^-$). Instead, we will introduce a novel method (outlined in the next chapter) to obtain improved decay. We note that this new method allows us only to get an improved decay rate of $t^{-\frac{3}{2}+\delta}$, which nevertheless is sufficient for the application in the semilinear problem.

6.2 Proof of Small Data Global Existence for the Semilinear Wave Equation satisfying the Null Condition

We now turn to the outline of the proof of the analogue of theorem 1.0.3 in Minkowski spacetime. From the previous section, we know that for Φ begin a solution to the linear wave equation on Minkowski spacetime,

$$\int_{\Sigma_t} J^T_\mu(\Phi) n^\mu_{\Sigma_t}, \int_{\Sigma_t} J^{Z,w^Z}_\mu(\Phi) n^\mu_{\Sigma_t}$$

are conserved in time. By commuting the equation with ∂, Ω and S, the above conservation laws would also allow us to prove decay. In particular, we have a decay of $|\Phi| \leq Ct^{-1}$, with an improvement in $\{r \leq \frac{t}{2}\}$ given by the commutation with S. Notice that we do not need to commute with the Lorentz boost, which lacks an analogue in Schwarzschild and Kerr spacetimes.

We now study how this decay rate can be used for the nonlinear problem. The main idea is to prove the above conservation and decay estimates in a bootstrap setting, showing that the decay to the linear wave equation is sufficiently strong that the nonlinear terms can be treated as error. In this framework, the decay of t^{-1} is borderline and since the decay rate is better when $r \leq \frac{t}{2}$, the difficulty arises when dealing with terms in the region $r \geq \frac{t}{2}$. Furthermore, in order to achieve this decay of t^{-1} it is imperative to show that $\int_{\Sigma_t} J^T_\mu(\Phi) n^\mu_{\Sigma_t}$ is uniformly bounded in time. We now show a heuristic argument. With the inhomogeneous term, the conservation law for the energy now has the error term:

$$\int_{\Sigma_t} J^T_\mu(\Phi) n^\mu_{\Sigma_t} \leq \int_{\Sigma_{t_0}} J^T_\mu(\Phi) n^\mu_{\Sigma_{t_0}} + \left(\int_{t_0}^t \left(\int_{\Sigma_t} (\Box_m \Phi)^2 \right)^{\frac{1}{2}} dt \right)^2,$$

and that for the conformal energy has the error term:

$$\int_{\Sigma_t} J_\mu^{Z,w^Z}(\Phi) n_{\Sigma_t}^\mu \leq \int_{\Sigma_{t_0}} J_\mu^{Z,w^Z}(\Phi) n_{\Sigma_{t_0}}^\mu + \left(\int_{t_0}^t \left(\int_{\Sigma_t} (t^2 + r^2) \left(\Box_m \Phi \right)^2 \right)^{\frac{1}{2}} dt \right)^2$$

Since $\Box_m \Phi$ is quadratic in $D\Phi$, we can use Holder's inequality on the inside integral to control one term in L^2 and one in L^∞. However, since on the linear level $D\Phi$ is bounded in L^2 and decays as t^{-1} in L^∞, the inhomogeneous term for the estimate for the energy is controlled by

$$\left(\int_{t_0}^t (t)^{-1} (\int_{\Sigma_t} J_\mu^T(\Phi) n_{\Sigma_t}^\mu)^{\frac{1}{2}} dt \right)^2 .$$

This is barely insufficient to show that the energy is bounded. We therefore need to make use of the null condition. The null condition would allow one to prove

$$\int (D\Phi \overline{D} \Phi)^2 \leq C t^{-2} \int_{\Sigma_t} J_\mu^{Z,w^Z}(\partial^k \Phi) n_{\Sigma_t}^\mu \tag{6.3}$$

In order to prove this estimate, we observe that in the conformal energy, the good derivatives (∂_v, $\not{\nabla}$) has better decay rates. In order to use this, we then need to control the conformal energy. Using again the null condition, the inhomogeneous term in the conservation law for the conformal energy can be bounded by

$$\left(\int_{t_0}^t t^{-1} (\int_{\Sigma_t} J_\mu^{Z,w^Z}(\Phi) n_{\Sigma_t}^\mu)^{\frac{1}{2}} dt \right)^2 .$$

This would not be sufficient to prove that the conformal energy is bounded, but is sufficient to prove that it grows no faster than t^η for sufficiently small data. This in turn would be sufficient to prove the boundedness of the energy and obtain all the necessary decay rates. In practice, the argument is more complicated as we need to control the higher order energy and conformal energy to obtain the decay rates.

Chapter 7

Main Ideas of the Proof

7.1 Quantitative Decay Rates for Solutions to the Linear Wave Equation

We now discuss the main ideas behind the proof of Theorems 1.0.1 and 1.0.2. We introduce a vector field $S = t\partial_t + r^*\partial_{r^*}$ which is analogous to the scaling vector field in Minkowski spacetime. Since Schwarzschild spacetimes are asymptotically flat, S is still an "asymptotic conformal symmetry" generating an "asymptotic almost conservation law". However, the error terms away from spacelike infinity are in general large. To see this more concretely, we recall that on Minkowski spacetimes, $\Box_m \Phi = 0$ implies $\Box_m (S\Phi) = 0$. This does not hold in Schwarzschild and Kerr spacetimes. Nevertheless, for $\Box_g \Phi = 0$, we still have a (schematic) equation $\Box_g (S\Phi) = h(r)(\nabla\Phi + \nabla^2\Phi)$ with $h \to 0$ as $r \to \infty$. The strategy is then to go through and sharpen the argument in Dafermos-Rodnianski [18], [16] and control the error terms that arise from $\Box_g (S\Phi) \neq 0$.

For simplicity of the exposition, we define $\Psi = S\Phi$. We will also focus the discussion on the Schwarzschild case. We would like to prove energy estimates for Ψ similar to those for Φ that are established in [18], except for a loss of an arbitrarily

small power of t. A key estimate that we would like to obtain is:

$$\int_{t_1}^{t_2}\int_{-\frac{t}{2}}^{\frac{t}{2}}\int_{\mathbb{S}^2}\left(\Psi^2+(\nabla\Psi)^2\right)\chi\left(r^*\right)dVol\leq C_\delta t_1^{-2+\delta}, \tag{7.1}$$

where χ is some weight and $t_1\leq t_2\leq(1.1)\,t_1$. A similar estimate is available with Ψ replaced by Φ from [18], [16]. In order to prove this, we argue in a similar fashion. We want to show, using the vector field X, that for t_1, t_2 as above

$$\int_{t_1}^{t_2}\int_{-\frac{t}{2}}^{\frac{t}{2}}\int_{\mathbb{S}^2}\left(\Psi^2+(\nabla\Psi)^2\right)\chi\left(r^*\right)dVol\leq C_\delta t_1^{-2+\delta}\{\text{conf. energy}(\Psi)\},$$

where the conformal energy is the current of the vector field Z on the boundary $\{t=t_i\}$. We then hope to show

$$\begin{aligned}&\{\text{conf. energy}(\Psi)\text{ at }t_2\}\\ \leq& C\left(\{\text{conf. energy}(\Psi)\text{ at }t_1\}+\int_{t_1}^{t_2}\int_{-\frac{t}{2}}^{\frac{t}{2}}\int_{\mathbb{S}^2}t\left(\Psi^2+(\nabla\Psi)^2\right)\chi\left(r^*\right)dVol\right).\end{aligned}$$

We then iterate two inequalities to obtain (7.1) as in [18]. In the case of Kerr, this is further complicated by the extra error term that grows with $(t^*)^2$.

The main difficulty in actually carrying out the above procedure is that each step is only true modulo some error terms that need not be small. These are error terms arising from the fact that Ψ does not satisfy the homogeneous wave equation, but only satisfies an inhomogeneous wave equation, which schematically can be thought of as $\Box_g\Psi = h(r^*)\left(\nabla\Phi+\nabla^2\Phi\right)$. $h(r^*)$ is not small but has the property that it decays as $r^*\to\infty$. This is a manifestation of the fact that Schwarzschild and Kerr are asymptotically flat and S is therefore an asymptotic conformal symmetry. If one applies the vector field method to this equation, one would generate an error term of the form

$$\int_{t_1}^{t_2}\int_{-\infty}^{\infty}\int_{\mathbb{S}^2}V^\mu\partial_\mu\Psi h(r^*)\left(\nabla\Phi+\nabla^2\Phi\right)dVol, \tag{7.2}$$

for the vector field multipliers V. If V is bounded, we can apply Cauchy-Schwarz and control (7.2) by

$$\left(\int_{t_1}^{t_2}\int_{-\infty}^{\infty}\int_{\mathbb{S}^2}(\nabla\Psi)^2\,dVol\right)^{\frac{1}{2}}\left(\int_{t_1}^{t_2}\int_{-\infty}^{\infty}\int_{\mathbb{S}^2}\tilde{h}\,(r^*)\left((\nabla\Phi)^2+(\nabla^2\Phi)^2\right)dVol\right)^{\frac{1}{2}} \quad (7.3)$$

We control the first factor by some energy quantities of Ψ which we are in the process of proving. They are set up so that we can estimate them with a bootstrap argument. We would then like to control the second factor by the integrated local decay estimates for Φ proved in [18] and [16].

There are two main difficulties in carrying out this procedure. Firstly, due to the presence of trapping, the integrated local energy decay estimate necessarily degenerates. Thus we cannot estimate the first factor in (7.3) by the integrated local energy decay estimate. Instead, we need to either integrate by parts in (7.2) or to ensure that the derivative of Ψ that appears in (7.3) is a derivative controllable by the integrated local energy decay estimates. Secondly, we need to control terms such that V^μ grows with t. In order to estimate these terms, we consider separately $\{r \leq \frac{t}{2}\}$ and $\{r \geq \frac{t}{2}\}$. For $r \leq \frac{t}{2}$, we use that fact that the integrated estimates for Φ actually decays with t. For $r \geq \frac{t}{2}$, we carefully track the decay rate of $h(r^*)$ in r^*, which now provides decay in t.

In [30], the estimates for Ψ are used to prove the decay for $\partial_t\Phi$ in Minkowski spacetime. We show that it is possible to argue similarly to prove the decay for $\partial_t\Phi$ in Schwarzschild spacetimes. The case for Kerr follows similarly but we will omit its proof. Recall that in [30], one then proceeds with elliptic estimates to prove the decay for other derivatives. However, on black hole spacetimes, if we are to prove an L^2 elliptic estimate, we are bound to have some lower order terms involving only one derivative of Φ. These terms cannot be controlled by the estimates of Ψ and therefore we are unable to use a similar method to prove the decay of the the spatial

derivatives of Φ.

Therefore, we introduce a new method, based on a novel application of S, to prove the decay for the function Φ as well as its derivatives in spatial directions. This uses the estimate (7.1) and the analogy of (7.1) for Ψ,

$$\int_{t_1}^{t_2} \int_{r_1^*}^{r_2^*} \left(\Phi^2 + (\partial_{r^*}\Phi)^2\right) dVol \leq Ct_1^{-2}, \tag{7.4}$$

for $t_1 \leq t_2 \leq (1.1)\, t_1$. (7.4) implies that there exists a time $\tilde{t} \in [t_1, t_2]$ such that

$$\int_{r_1^*}^{r_2^*} \left(\Phi^2 + (\partial_{r^*}\Phi)^2\right) dVol_{\tilde{t}} \leq C\tilde{t}^{-3}.$$

In order to show that the same holds for any t, we note that S is strictly timelike on a compact set of r^*. Therefore we can integrate in the direction of S from the slice $\tilde{t}$ to a generic slice t. This integration would not give an extra factor of t precisely because we already have the estimates for $\Psi = S\Phi$. After controlling the spacetime terms by (7.1) and (7.4), we show that for any t,

$$\int_{r_1^*}^{r_2^*} \left(\Phi^2 + (\partial_{r^*}\Phi)^2\right) dVol_t \leq C_\delta t^{-3+\delta}.$$

We use Sobolev Embedding to get the pointwise decay estimate for Φ and its derivatives (for $r_1^* \leq r^* \leq r_2^*$) after commuting with an appropriate number of Killing vector fields. We note in particular that in this proof, it is unnecessary to invert the Laplacian on Schwarzschild spacetime to prove the decay of Φ.

The argument above gives the decay of Φ in a compact region of space that is also away from the event horizon. In order to prove that Φ also decays along the event horizon, we use the red-shift vector field introduced in [18]. In fact, we can show using the red-shift estimates that any decay estimate proved on a compact region of r^* can be propagated to the event horizon, giving rise to a decay estimate

of the same rate. Moreover, using an identical argument with the red-shift vector field, which we also include for the sake of completeness, the decay estimate can be propagated to slightly beyond the event horizon into the black hole region. Once we have an estimate slightly beyond the event horizon, we can easily prove the same decay estimate anywhere inside the black hole region by taking advantage of the geometry of the region. This strategy for controlling the scalar field inside the black hole region was first used in [11] and [12] in the nonlinear setting, see also subsequent [23] in the linear setting.

7.2 Proof of Small Data Global Existence for the Semilinear Wave Equation satisfying the Null Condition

The discussion from the previous section allows us to improve the decay rate for $r \leq R$. Together with appropriate commutations with $\tilde{\Omega}$, we can in fact obtain an improved decay rate in the region $r \leq \frac{t}{2}$. Once we have a method to prove the extra decay in this region, the main difficulty for studying the semilinear problem is in controlling the error terms in the region $r \geq \frac{t}{2}$. To do this, we need to make use of the null condition and perform estimates as in Section 6.2.

One of the most serious difficulties that is different from the case of Minkowski spacetime arise from the fact that Kerr spacetimes contain trapped null geodesics. As a consequence, any decay results at the linear level must involve a loss of derivatives. This is manifested in the degeneracy of the integrated decay estimate near $r = 3M$. We note, however, that on the linear level the non-degenerate energy can be proved to be bounded without any loss of derivatives. We therefore prove energy bounds that is consistent with the linear scenario. We would try to prove on the highest level

of derivatives only a boundedness result and begin to prove decay results on the level of fewer derivatives. However, as we will see, the nonlinear effect comes into play and it is not possible to prove even the boundedness of the non-degenerate energy at the highest level of derivatives. We can nevertheless show that it is bounded by $(t^*)^\eta$. On the level of one less derivative, we can prove that the conformal energy grows no faster than $\tau^{1+\eta}$. Using this fact as we prove the estimates for the non-degenerate energy, we can show that at this level of derivatives, the non-degenerate energy is bounded. This is crucial for obtaining the necessary borderline decay of $(t^*)^{-1}$ in $r \geq \frac{t^*}{2}$, thus allowing us to close the bootstrap argument. Trapping would also cause a loss in derivatives when controlling the error terms arising from the commutation with S. To tackle this problem, we would commute with S only once. With this approach, we would not have an improved decay for $DS\Phi$ in $r \leq \frac{t^*}{2}$. Nevertheless, we can show that the bootstrap can be closed. Here we make use of the fact that as we close the assumptions for $S\Phi$, we are at a level of derivatives of Φ such that the local energy flux decays.

Chapter 8

Improved Decay Estimates on Schwarzschild Spacetime

In this chapter, we prove Theorem 1.0.1 (or more precisely, Theorems 5.1.1 and 5.1.2.

8.1 Notations

Before proceeding, we would like to first define the notations used for the coordinates and volume form.

For the r, r^* coordinates, we always use * to denote the Regge-Wheeler tortoise coordinate of the same point.

For the t coordinates:
t_0 denotes the time slice on which the initial data is posed.
t_* denotes the time slice on which we would like to control the solution.
t_i denotes dyadic time slices (which will be defined in Section 4).
t denotes a generic time slice.
We assume $t_0, t_*, t_i, t > 0$.

For volume forms:

$dVol$ denotes the spacetime volume form, $dVol = r^2 (1-\mu)\, dA dr^* dt$.

$dVol_t$ denotes the volume form on a time slice, $dVol_t = r^2\sqrt{1-\mu} dA dr^*$.

$dVol_v$ denotes a volume form on a v slice, $dVol_v = r^2\sqrt{1-\mu} dA du$.[1]

dA denotes the volume form on the standard sphere of radius 1.

Whenever we write $\int$ without integration limits, it denotes the integration over "whole space" that is appropriate for the volume form.

8.2 Vector Fields

8.2.1 Vector Field Multiplier T

Define $T = \partial_t$. Recall that T is Killing. Therefore,

$$K^T(\Phi) = 0.$$

In the following, we will consider this current on a constant t-slice.

One computes that in local coordinates

$$J^T_\mu(\Phi)\, n^\mu_t = \frac{1}{2\sqrt{1-\mu}} \left((\partial_t \Phi)^2 + (\partial_{r^*}\Phi)^2 + (1-\mu) |\not\nabla \Phi|^2 \right),$$

where n^μ_t is the normal to a t-slice.

[1] Most of the time it is clear from context whether we are integrating over a t or v slice. We will specify in the case of possible ambiguity, for example $dVol_{\{t=2v-r^*\}}$ is the volume form on a t slice, where t has the specified value.

8.2.2 Vector Field Multiplier X

Define $X = f(r^*)\partial_{r^*}$. In the following we will use different functions f. One computes that

$$\begin{aligned} K^X(\Phi) =& \frac{f'(r^*)(\partial_{r^*}\Phi)^2}{1-\mu} + \frac{1}{2}|\not\nabla\Phi|^2\left(\frac{2-3\mu}{r}\right)f(r^*) \\ & - \frac{1}{4}\left(2f'(r^*) + \frac{4(1-\mu)}{r}f(r^*)\right)\partial^\alpha\Phi\partial_\alpha\Phi. \end{aligned}$$

We consider the modified current using $w^X = 2f'(r^*) + \frac{4(1-\mu)}{r}f(r^*)$. Then

$$\begin{aligned} K^{X,w^X} =& \frac{f'(r^*)(\partial_{r^*}\Phi)^2}{1-\mu} + \frac{1}{2}|\not\nabla\Phi|^2\left(\frac{2-3\mu}{r}\right)f(r^*) - \frac{1}{8}\Box_g w^X\Phi^2 \\ =& \frac{f'(r^*)(\partial_{r^*}\Phi)^2}{1-\mu} + \frac{1}{2}|\not\nabla\Phi|^2\left(\frac{2-3\mu}{r}\right)f(r^*) \\ & - \frac{1}{4}\left(\frac{1}{1-\mu}f'''(r^*) + \frac{4}{r}f''(r^*) + \frac{\mu}{r^2}f'(r^*) - \frac{2\mu}{r^3}(3-4\mu)f(r^*)\right)\Phi^2, \\ J^{X,w^X}_\mu n^\mu_t =& \frac{1}{\sqrt{1-\mu}}f(r^*)\partial_t\Phi\partial_{r^*}\Phi + \frac{1}{2\sqrt{1-\mu}}\left(f'(r^*) + \frac{2(1-\mu)}{r}f(r^*)\right)(\partial_t\Phi)\Phi, \end{aligned} \tag{8.1}$$

where n^μ_t is the normal to a t-slice.

The vector field X is constructed to control a spacetime integral by the boundary terms, i.e., one hopes to control the integral of $K^{X,w^X}(\Phi)$ by the integral of $J^{X,w^X}_\mu(\Phi)n^\mu_t$. In order for this to be useful, we need $K^X(\Phi)$ to be everywhere positive. Such vector fields are constructed in [18] using spherical harmonic decomposition. In particular, it was shown in [18] that there exists a family of vector fields $X_l = f_l(r^*)\partial_{r^*}$ for $l \geq 0$ such that for any function Φ (not necessarily satisfying the wave equation), if we write out the spherical harmonic decomposition $\Phi = \sum_{l=0}^{\infty}\Phi_l$, $K^{X_l,w^{X_l}}(\Phi_l) \geq 0$.

Moreover, one has

$$\int_{\mathbb{S}^2}\left(\frac{\left(\partial_{r^*}\Phi_l\right)^2}{\left(1+|r^*|\right)^2(1-\mu)}+\frac{\Phi_l^2}{\left(1+|r^*|\right)^4(1-\mu)}\right)dA\leq C\int_{\mathbb{S}^2}K^{X_l,w^{X_l}}\left(\Phi_l\right)dA \tag{8.2}$$

for $l\geq 1$, where C can be picked to be independent of l, and

$$\int_{\mathbb{S}^2}\frac{\left(\partial_{r^*}\Phi_0\right)^2}{\left(1+|r^*|\right)^{1+\delta}r^2(1-\mu)}dA\leq C\int_{\mathbb{S}^2}K^{X_l,w^{X_l}}\left(\Phi_0\right)dA.$$

Moreover for this choice of X_l, the boundary terms are also controllable as shown in [18]:

$$\int K^{X_l,w^{X_l}}\left(\Phi_l\right)dVol\leq C\int J_\mu^T\left(\Phi\right)n_{t_0}^\mu dVol_{t_0}.$$

Remark 6. *We note that although $K^{X_l,w^{X_l}}(\Phi)$ is shown to be nonnegative everywhere, it has a weight in front of $|\nabla\!\!\!/\Phi|^2$ that degenerates at $r=3M$. Therefore, we cannot directly estimate the integral of $|\nabla\!\!\!/\Phi|^2$ by that of $K^{X_l,w^{X_l}}(\Phi)$. Instead, we will consider the equation $\Box_g(\Omega\Phi)=0$ and estimate the relevant quantities with $\int K^{X_l,w^{X_l}}(\Omega\Phi)\,dVol$. This loss of derivative is related to the trapping phenomenon that we mentioned in Section 1.1.*

In section 3 we will construct two more vector fields of this form. One will be a modified X_0 to control a weighted L^2-norm of the zeroth spherical harmonic and the other will be used to control the behavior at infinity.

8.2.3 Vector Field Multiplier Z

Define $Z=u^2\partial_u+v^2\partial_v$. This is the analogue of the conformal vector field in Minkowski spacetime. Like the case in Minkowski spacetime, it is used to show decay for the solution to the wave equation. One computes that

$$K^Z=-t|\nabla\!\!\!/\Phi|^2\left(\frac{1}{2}+\frac{\mu r^*}{4r}-\frac{r^*(1-\mu)}{2r}\right)-\frac{1}{4}\frac{2tr^*(1-\mu)}{r}\partial^\alpha\Phi\partial_\alpha\Phi.$$

We consider the modified current using $w^Z = \frac{2tr^*(1-\mu)}{r}$. Then

$$\begin{aligned}
K^{Z,w^Z} =& -t|\not\nabla\Phi|^2\left(\frac{1}{2}+\frac{\mu r^*}{4r}-\frac{r^*(1-\mu)}{2r}\right)-\frac{1}{8}\Box_g w^Z\Phi^2\\
=& -t|\not\nabla\Phi|^2\left(\frac{1}{2}+\frac{\mu r^*}{4r}-\frac{r^*(1-\mu)}{2r}\right)-\frac{t}{4}\mu r^{-2}\Phi^2\left(2+\frac{r^*(4\mu-3)}{r}\right),\\
J_\mu^{Z,w^Z}n_t^\mu =& \frac{1}{4\sqrt{1-\mu}}\left(u^2(\partial_u\Phi)^2+v^2(\partial_v\Phi)^2+(1-\mu)\left(u^2+v^2\right)|\not\nabla\Phi|^2\right.\\
&\left.+\frac{2tr^*(1-\mu)}{r}\Phi\partial_t\Phi-\frac{r^*(1-\mu)}{r}\Phi^2\right).
\end{aligned}$$

where n_t^μ is the normal to a t-slice.

It is shown in [18] that there exist r_1^*, r_2^* such that for $r^* \le r_1^*$ or $r^* \ge r_2^*$, $K^{Z,w^Z} \ge 0$. Moreover, it is shown that $\int J_\mu^{Z,w^Z} n_t^\mu dVol_t$ is everywhere non-negative.

More specifically, if we define $S = u\partial_u + v\partial_v$ and $\underline{S} = -u\partial_u + v\partial_v$,

$$\begin{aligned}
&\int J_\mu^{Z,w^Z} n_t^\mu dVol_t\\
=&\int \frac{1}{8\sqrt{1-\mu}}\left(\mu\left((S\Phi)^2+(\underline{S}\Phi)^2\right)+(1-\mu)\left(\left(S\Phi+\frac{r^*}{r}\Phi\right)^2+\left(\underline{S}\Phi+\frac{t}{r}\Phi\right)^2\right)\right.\\
&\left.+2(1-\mu)\left(u^2+v^2\right)|\not\nabla\Phi|^2\right)dVol_t.
\end{aligned}$$

8.2.4 Vector Field Multipliers Y', Y and N

Define $Y' = \frac{y_1(r^*)}{1-\mu}\partial_u + y_2(r^*)\partial_v$, where $y_1, y_2 > 0$ are supported in $r \le (1.2)r_0$, with $y_1 = 1$, $y_2 = 0$ at the event horizon and $y_1'(r^*) \sim y_2(r^*) \sim C(1-\mu)$ for $2M \le r \le r_0$.

Here we want to choose r_0 small enough so that

1. Y' is supported on $r < 3M$ (i.e. $(1.2)r_0 < 3M$),
2. $K^{Y'}(\Phi) \ge 0$ on $2M \le r \le r_0$,
3. $CK^{Y'}(\Phi) \ge \frac{1}{\sqrt{1-\mu}} J_\mu^{Y'}(\Phi) n^\mu_{\{v=const.\}}$ on $2M \le r \le r_0$.

The vector field Y' is designed to capture the red-shift effect at the event horizon [18]. Using the current $J^{Y'}$, we will not only produce estimates on constant t-slices, but also on constant v- slices. We will therefore record here all the relevant computations.

We have

$$K^{Y'} = \frac{(\partial_u \Phi)^2}{2(1-\mu)^2}\left(\frac{y_1\mu}{r} - y_1'\right) + \frac{(\partial_v \Phi)^2}{2(1-\mu)} y_2' + \frac{1}{2}|\not\nabla \Phi|^2 \left(\frac{y_1'}{1-\mu} - \frac{(y_2(1-\mu))'}{1-\mu}\right) - \frac{1}{r}\left(\frac{y_1}{1-\mu} - y_2\right)\partial_u\Phi\partial_v\Phi,$$

$$J_\mu^{Y'}(\Phi)\, n^\mu_{\{v=const.\}} = \frac{1}{2\sqrt{1-\mu}}\left(\frac{y_1}{1-\mu}(\partial_u\Phi)^2 + (1-\mu)\, y_2 |\not\nabla\Phi|^2\right),$$

$$J_\mu^{Y'}(\Phi)\, n^\mu_{\{t=const.\}} = \frac{1}{2\sqrt{1-\mu}}\left(\frac{y_1}{1-\mu}(\partial_u\Phi)^2 + y_2(\partial_v\Phi)^2 + (y_1 + (1-\mu)\, y_2)|\not\nabla\Phi|^2\right).$$

From this we see that if r_0 is chosen to be close enough to $2M$, requirements 2 and 3 can be satisfied.

We modify this vector field so that it has better bounds on constant t- slices.

Define $Y = Y' + \chi(r)\, T$, where $\chi(r)$ is a cutoff function with $\chi(r) = \begin{cases} 1 & r \le r_0 \\ 0 & r \ge (1.2)\, r_0 \end{cases}$.

Y has the following properties:

1. Y is supported on $r < (1.2)\, r_0$,
2. $K^Y = K^{Y'}$ on $r < r_0$,
3. $CK^Y(\Phi) \ge \frac{1}{\sqrt{1-\mu}} J_\mu^Y(\Phi)\, n^\mu_{\{v=const.\}}$ on $2M \le r \le r_0$.

On the region $2M \le r \le r_0$, we have

$$J_\mu^Y(\Phi)\, n^\mu_{\{v=const.\}} = \frac{1}{2\sqrt{1-\mu}}\left(\left(\frac{y_1}{1-\mu} + \frac{1}{2}\right)(\partial_u\Phi)^2 + (1-\mu)(y_2+1)|\not\nabla\Phi|^2\right),$$

$$J_\mu^Y(\Phi)\, n^\mu_{\{t=const.\}} = \frac{1}{2\sqrt{1-\mu}}\left(\left(\frac{y_1}{1-\mu} + \frac{1}{2}\right)(\partial_u\Phi)^2 + \left(y_2 + \frac{1}{2}\right)(\partial_v\Phi)^2 + \left(y_1 + (1-\mu)(y_2+1)\right)|\not\nabla\Phi|^2\right).$$

We argue without computation that for $r_0 \le r \le (1.2)\, r_0$,

$$|K^Y| \le C\frac{1}{\sqrt{1-\mu}} J_\mu^T(\Phi)\, n^\mu_{\{t=const.\}},$$

$$J_\mu^Y(\Phi)\, n^\mu_{\{v=const.\}} \leq C J_\mu^T(\Phi)\, n^\mu_{\{v=const.\}},$$

$$J_\mu^Y(\Phi)\, n^\mu_{\{t=const.\}} \leq C J_\mu^T(\Phi)\, n^\mu_{\{t=const.\}}.$$

This is true because $J_\mu^T(\Phi)\, n^\mu_{\{t=const.\}}$ controls every derivative of Φ while the terms in $J_\mu^T(\Phi)\, n^\mu_{\{v=const.\}}$ and $J_\mu^{Y'}(\Phi)\, n^\mu_{\{t=const.\}}$ contain only derivatives ∂_u and $\not\nabla$. Thus the only difference is the weights, which are functions of r and are harmless since r is bounded on this region. Extend Y to inside the black hole smoothly by the requirement that N is future directed and causal.

Define $N = T + Y$ for $r \geq 2M$. N is future directed and causal everywhere, thus $J_\mu^N(\Phi)\, n^\mu_{\{t=const.\}} \geq 0$. Away from the horizon, namely when $r \geq 1.2r_0$,

$$J_\mu^N(\Phi)\, n^\mu_{\{t=const.\}} = J_\mu^T(\Phi)\, n^\mu_{\{t=const.\}}.$$

However, as we approach the horizon,

$$J_\mu^N(\Phi)\, n^\mu_{\{t=const.\}} \sim J_\mu^Y(\Phi)\, n^\mu_{\{t=const.\}}$$

and thus $J_\mu^N(\Phi)\, n^\mu_{\{t=const.\}}$ gives a much stronger bound. We assume for our energy classes that the integral of $J_\mu^N(\Phi)\, n^\mu_{\{t=const.\}}$ is bounded initially and this clearly implies the boundedness for the corresponding integrals for J^T and J^Y initially. The flux corresponding to J^N should be thought of as a non-degenerate energy, which does not degenerate at the event horizon. This allows us to prove decay results along the event horizon.

Before introducing the vector field commutator S, we end this part on vector field multipliers by explicitly noting what each of the positive quantities bounds. Most of these are direct consequences of the expressions of the currents, except that for J^{Z,w^Z}, which requires some manipulation and is proved in [18].

Proposition 8.2.1. *1.*

$$\frac{1}{\sqrt{1-\mu}}\left(\left(\partial_{r^*}\Phi\right)^2+\left(\partial_t\Phi\right)^2+\left(1-\mu\right)\left|\not\nabla\Phi\right|^2\right)\leq CJ_\mu^T\left(\Phi\right)n_t^\mu,$$

2.

$$\begin{aligned}&\int_{-\infty}^{\infty}\int_{\mathbb{S}^2}\frac{1}{\sqrt{1-\mu}}\left(u^2\left(\partial_u\Phi\right)^2+v^2\left(\partial_v\Phi\right)^2+\left(1-\mu\right)\left(u^2+v^2\right)\left|\not\nabla\Phi\right|^2\right)dVol_t\\ \leq&C\int_{-\infty}^{\infty}\int_{\mathbb{S}^2}J_\mu^{Z,w^Z}\left(\Phi\right)n_t^\mu dVol_t,\end{aligned}$$

3.

$$\int_{-\infty}^{\infty}\int_{\mathbb{S}^2}\frac{1}{\sqrt{1-\mu}}\left(\left(1-\mu\right)\frac{\left(r^*\right)^2+t^2}{r^2}\Phi^2\right)dVol_t\leq C\int_{-\infty}^{\infty}\int_{\mathbb{S}^2}J_\mu^{Z,w^Z}\left(\Phi\right)n_t^\mu dVol_t,$$

4.

$$\int_{\mathbb{S}^2}\frac{\left(\partial_{r^*}\Phi\right)^2}{\left(1+\left|r^*\right|\right)^2r^{1+\delta}\left(1-\mu\right)}dA\leq C\int_{\mathbb{S}^2}\sum_l K^{X_l,w^{X_l}}\left(\Phi\right)dA,$$

5.

$$\int_{\mathbb{S}^2}\frac{\left|\not\nabla\Phi\right|^2}{\left(1+\left|r^*\right|\right)^4\left(1-\mu\right)}dA\leq C\int_{\mathbb{S}^2}\sum_l K^{X_l,w^{X_l}}\left(\Omega\Phi\right)dA.$$

8.2.5 Vector Field Commutator S

Define $S=t\partial_t+r^*\partial_{r^*}=v\partial_v+u\partial_u$.

This vector field, together with the usual Killing fields, will be commuted with $\Box_g$. We note that the vector field $t\partial_t+r\partial_r$ is conformally Killing on Minkowski with $[\Box_m,t\partial_t+r\partial_r]=2\Box_m$. Therefore, the commutator $[\Box_g,S]$ is expected to approach $2\Box_g$ towards spatial infinity, where the spacetime approaches Minkowski.

We set $\Psi=S\Phi$ and derive an equation for Ψ.

Proposition 8.2.2. *1.*

$$[\Box_g, S] = \left(2 + \frac{r^*\mu}{r}\right)\Box_g + \frac{2}{r}\left(\frac{r^*}{r} - 1 - \frac{2r^*\mu}{r}\right)\partial_{r^*} + 2\left(\left(\frac{r^*}{r} - 1\right) - \frac{3r^*\mu}{2r}\right)\not{\Delta}.$$

2.

$$\Box_g \Psi = g_1(r^*)\,\partial_{r^*}\Phi + g_2(r^*)\,\not{\Delta}\Phi,$$

$$\text{where } |g_1(r^*)|, \frac{|g_2(r^*)|}{r} \sim \begin{cases} \frac{(\log r)_+}{r^2} & r >> 2M \\ |r^*| & r \sim 2M \end{cases}, \ (\log r)_+ = \max\{\log r, 1\}.$$

Remark 7. *Equivalently, we write* $|g_1(r^*)|, \frac{|g_2(r^*)|}{r} \sim \frac{(1+|r^*|)(\log r)_+}{r^3}$.

Proof.

$$\begin{aligned}[-(1-\mu)^{-1}\partial_t^2, S] =& -2(1-\mu)^{-1}\partial_t^2 + r^*\partial_{r^*}(1-\mu)^{-1}\partial_t^2 \\ =& -2(1-\mu)^{-1}\partial_t^2 - \frac{r^*\mu}{r(1-\mu)}\partial_t^2,\end{aligned}$$

$$\begin{aligned}[(1-\mu)^{-1}\partial_{r^*}^2, S] =& 2(1-\mu)^{-1}\partial_{r^*}^2 - r^*\partial_{r^*}(1-\mu)^{-1}\partial_{r^*}^2 \\ =& 2(1-\mu)^{-1}\partial_{r^*}^2 + \frac{r^*\mu}{r(1-\mu)}\partial_{r^*}^2,\end{aligned}$$

$$\begin{aligned}[\frac{2}{r}\partial_{r^*}, S] =& \frac{2}{r}\partial_{r^*} + \frac{2r^*(1-\mu)}{r^2}\partial_{r^*} \\ =& \frac{4}{r}\partial_{r^*} + \left(\frac{2r^*(1-\mu)}{r^2} - \frac{2}{r}\right)\partial_{r^*},\end{aligned}$$

$$[\not{\Delta}, S] = \frac{2r^*(1-\mu)}{r}\not{\Delta},$$

$$\begin{aligned}&[\Box_g, S] \\ =& \left(2 + \frac{r^*\mu}{r}\right)\Box_g + \left(\frac{2r^*(1-\mu)}{r^2} - \frac{2}{r} - \frac{2r^*\mu}{r^2}\right)\partial_{r^*} + \left(\frac{2r^*(1-\mu)}{r} - 2 - \frac{r^*\mu}{r}\right)\not{\Delta} \\ =& \left(2 + \frac{r^*\mu}{r}\right)\Box_g + \frac{2}{r}\left(\frac{r^*}{r} - 1 - \frac{2r^*\mu}{r}\right)\partial_{r^*} + 2\left(\left(\frac{r^*}{r} - 1\right) - \frac{3r^*\mu}{2r}\right)\not{\Delta}.\end{aligned}$$

2. is immediate from 1. if we let

$$g_1(r^*) = \frac{2}{r}\left(\frac{r^*}{r} - 1 - \frac{2r^*\mu}{r}\right)$$

and

$$g_2(r^*) = 2\left(\left(\frac{r^*}{r} - 1\right) - \frac{3r^*\mu}{2r}\right).$$

□

8.3 Estimates for Φ

The following has been proved in [18] and is collected for later use.

Theorem 8.3.1 (Dafermos-Rodnianski)**.** *1.*

$$\int J_\mu^T(\Phi)\, n_{t_*}^\mu dVol_{t_*} = \int J_\mu^T(\Phi)\, n_{t_0}^\mu dVol_{t_0},$$

2.

$$\int \sum_l K^{X_l, w^{X_l}}(\Phi)\, dVol \leq C \int J_\mu^T(\Phi)\, n_{t_0}^\mu dVol_{t_0},$$

3.

$$\int J_\mu^Z(\Phi)\, n_{t_*}^\mu dVol_{t_*} \leq C E_0(\Phi),$$

4.

$$\int_{-\frac{t_*}{2}}^{\frac{t_*}{2}} J_\mu^T(\Phi)\, n_{t_*}^\mu dVol_{t_*} \leq C E_0(\Phi)\, t_*^{-2},$$

5.

$$\int_{t_1}^{t_2}\int_{-\frac{t}{2}}^{\frac{t}{2}} \sum_l K^{X_l, w^{X_l}}(\Phi)\, dVol \leq C E_0(\Phi)\, t_1^{-2},$$

where $t_1 \leq t_2 \leq (1.1)\, t_1$.

The following Hardy type inequality is also proved in [18] and will be used throughout this chapter.

Lemma 8.3.2.

$$\int (1+|r^*|)^{-2} \Phi^2 (1-\mu)^{-\frac{1}{2}} dVol_{t_0} \leq C \int (\partial_{r^*}\Phi)^2 (1-\mu)^{-\frac{1}{2}} dVol_{t_0}.$$

Remark 8. *This can be written equivalently in local coordinates as*

$$\int_{-\infty}^{\infty} \int_{\mathbb{S}^2} \frac{\Phi^2}{(1+|r^*|)^2} r^2 dA dr^* \leq C \int_{-\infty}^{\infty} \int_{\mathbb{S}^2} (\partial_{r^*}\Phi)^2 r^2 dA dr^*.$$

We construct a vector field X_0 to control the spacetime integral of Φ^2 itself.

Proposition 8.3.3.

$$\int \frac{\Phi^2}{(1+|r^*|)^4} dVol \leq C \int J_\mu^T(\Phi) n_{t_0}^\mu dVol_{t_0}.$$

Proof. We first notice that we already have control of a weighted L^2-norm of the non-zeroth spherical harmonics. This is because by (8.2),

$$\int_{\mathbb{S}^2} \frac{\Phi_l^2}{(1+|r^*|)^4 (1-\mu)} dA \leq C \int_{\mathbb{S}^2} K^{X_l, w^{X_l}}(\Phi_l) dA$$

for $l \geq 1$. This together with Theorem 8.3.1.2 would give

$$\int \frac{\Phi_l^2}{(1+|r^*|)^4 (1-\mu)} dVol \leq C \int J_\mu^T(\Phi) n_{t_0}^\mu dVol_{t_0}$$

for $l \geq 1$. So it suffices to consider the zeroth spherical harmonic.

Define $X_0 = f_0 \partial_{r^*}$, with $f_0(r^*) = f_0(r) = -\frac{M^3}{(1+4\mu^{-2})} = -\frac{\mu^3 r^3}{8(1+4\mu^{-2})}$.

Suppose we act with X_0 on the zeroth spherical harmonic of Φ_0.

Using (8.1),

$$\begin{aligned}
&K^{X_0,w^{X_0}}\left(\Phi_0\right)\\
=&\frac{f_0'\left(r^*\right)\left(\partial_{r^*}\Phi_0\right)^2}{1-\mu}+\frac{1}{2}|\not\nabla\Phi_0|^2\left(\frac{2-3\mu}{r}\right)f_0\left(r^*\right)\\
&-\frac{1}{4}\left(\frac{1}{1-\mu}f_0'''\left(r^*\right)+\frac{4}{r}f_0''\left(r^*\right)+\frac{\mu}{r^2}f_0'\left(r^*\right)-\frac{2\mu}{r^3}\left(3-4\mu\right)f_0\left(r^*\right)\right)\Phi_0^2\\
=&\frac{f_0'\left(r^*\right)\left(\partial_{r^*}\Phi_0\right)^2}{1-\mu}\\
&-\frac{1}{4}\left(\frac{1}{1-\mu}f_0'''\left(r^*\right)+\frac{4}{r}f_0''\left(r^*\right)+\frac{\mu}{r^2}f_0'\left(r^*\right)-\frac{2\mu}{r^3}\left(3-4\mu\right)f_0\left(r^*\right)\right)\Phi_0^2,
\end{aligned}$$

and

$$\begin{aligned}
&J_\mu^{X_0,w^{X_0}}\left(\Phi_0\right)n_t^\mu\\
=&\frac{1}{\sqrt{1-\mu}}f_0\left(r^*\right)\partial_t\Phi_0\partial_{r^*}\Phi_0+\frac{1}{2\sqrt{1-\mu}}\left(f_0'\left(r^*\right)+\frac{2\left(1-\mu\right)}{r}f_0\left(r^*\right)\right)\left(\partial_t\Phi_0\right)\Phi_0,
\end{aligned}$$

where we have used $\not\nabla\Phi_0=0$.

We would have to show firstly that $K^{X_0,w^{X_0}}\left(\Phi_0\right)\geq 0$ and controls Φ^2, and secondly that $J_\mu^{X_0,w^{X_0}}\left(\Phi_0\right)n^\mu$ is controllable by $J_\mu^T\left(\Phi_0\right)n^\mu$. We first compute the derivatives of f_0:

$$\begin{aligned}
f_0'\left(r^*\right)=&\left(1-\mu\right)\partial_r f_0\left(r\right)\\
=&\frac{\mu r^2\left(1-\mu\right)}{\left(1+4\mu^{-2}\right)^2}\geq 0\\
f_0''\left(r^*\right)=&\left(1-\mu\right)^2\partial_r^2 f_0+\frac{\mu\left(1-\mu\right)}{r}\partial_r f_0\\
=&\left(1-\mu\right)^2\left(-\frac{16r}{\mu\left(1+4\mu^{-2}\right)^3}+\frac{2\mu r}{\left(1+4\mu^{-2}\right)^2}\right)+\frac{\mu^2 r\left(1-\mu\right)}{\left(1+4\mu^{-2}\right)^2}\\
f_0'''\left(r^*\right)=&\left(1-\mu\right)^3\partial_r^3 f_0+\frac{3\mu\left(1-\mu\right)^2}{r}\partial_r^2 f_0+\frac{\mu^2\left(1-\mu\right)}{r^2}\partial_r f_0-\frac{2\mu\left(1-\mu\right)^2}{r^2}\partial_r f_0\\
=&\left(1-\mu\right)^3\left(\frac{384}{\mu^3\left(1+4\mu^{-2}\right)^4}-\frac{48}{\mu\left(1+4\mu^{-2}\right)^3}\right)\\
&+\frac{3\mu\left(1-\mu\right)^2}{r}\left(-\frac{16r}{\mu\left(1+4\mu^{-2}\right)^3}+\frac{\mu r}{\left(1+4\mu^{-2}\right)^2}\right)+\frac{\mu^2\left(1-\mu\right)\left(3\mu-2\right)}{\left(1+4\mu^{-2}\right)^2}.
\end{aligned}$$

A computation shows that

$$\begin{aligned}&\frac{1}{1-\mu}f_0'''+\frac{4}{r}f_0''+\frac{\mu}{r^2}f_0'-\frac{2\mu}{r^3}\left(3-4\mu\right)f_0\\=&-\frac{\mu^6\left(192+\mu\left(128+\mu\left(-784+\mu\left(464+\mu\left(-28+\mu\left(52+\mu\left(-3+4\mu\right)\right)\right)\right)\right)\right)\right)}{4\left(4+\mu^2\right)^4}.\end{aligned}$$

We need to show that

$$192+\mu\left(128+\mu\left(-784+\mu\left(464+\mu\left(-28+\mu\left(52+\mu\left(-3+4\mu\right)\right)\right)\right)\right)\right)\geq 0$$

for $0\leq\mu\leq 1$.

Case 1: $\frac{11}{20}\leq\mu\leq 1$

$192+128\mu-784\mu^2+464\mu^3=16\left(-12-20\mu+29\mu^2\right)\left(\mu-1\right)\geq 0.$

$52-3\mu+4\mu^2$ reaches its minimum at $\frac{3}{8}$. Hence, $52-3\mu+4\mu^2\geq\frac{823}{16}$.

$-28+\mu\left(52-3\mu+4\mu^2\right)\geq -28+\frac{11}{20}\frac{823}{16}\geq 0.$

Case 2: $0\leq\mu\leq\frac{11}{20}$

$464-28\mu+\frac{823}{16}\mu^2$ has negative discriminant, hence ≥ 0.

Also, for this range of μ, $192+128\mu-784\mu^2\geq 0$.

Therefore, $K^{X_0,w^{X_0}}\left(\Phi_0\right)\geq 0$. Moreover, $\Phi_0^2\leq CK^{X_0,w^{X_0}}\left(\Phi_0\right)$.

It now remains only to control the boundary terms. Using Lemma 8.3.2 and Cauchy-Schwarz,

$$\begin{aligned}&\int J_\mu^{X_0,w^{X_0}}\left(\Phi_0\right)n^\mu dVol_t\\=&\int\frac{1}{\sqrt{1-\mu}}f_0\left(r^*\right)\partial_t\Phi_0\partial_{r^*}\Phi_0+\frac{1}{2\sqrt{1-\mu}}\left(f_0'\left(r^*\right)+\frac{2\left(1-\mu\right)}{r}f_0\left(r^*\right)\right)\left(\partial_t\Phi_0\right)\Phi_0 dVol_{t_0}\\\leq& C\int\frac{1}{\sqrt{1-\mu}}\left(\left(\partial_t\Phi_0\right)^2+\left(\partial_{r^*}\Phi_0\right)^2+\frac{1}{\left(1+|r^*|\right)^2}\Phi_0^2\right)dVol_{t_0}\\\leq& C\int\frac{1}{\sqrt{1-\mu}}\left(\left(\partial_t\Phi_0\right)^2+\left(\partial_{r^*}\Phi_0\right)^2\right)dVol_{t_0}\\\leq& C\int J_\mu^T\left(\Phi_0\right)n_{t_0}^\mu dVol_{t_0}.\end{aligned}$$

□

We would like to construct a vector field $\tilde{X} = \tilde{f}(r^*)\partial_{r^*}$ so as to improve the weights in r of the spacetime integral that can be controlled. More precisely, we have the following:

Proposition 8.3.4.

$$\int_{t_0}^{t_*}\int_1^\infty\int_{\mathbb{S}^2}\left(r^{-1-\delta}\left(\partial_{r^*}\Phi\right)^2+r^{-3-\delta}\Phi^2\right)dVol\leq C\int J_\mu^T(\Phi)\,n_{t_0}^\mu dVol_{t_0},$$

$$\int_{t_0}^{t_*}\int_1^\infty\int_{\mathbb{S}^2}r^{-1}|\nabla\!\!\!/\Phi|^2dVol\leq C\sum_{k=0}^{1}\int J_\mu^T\left(\Omega^k\Phi\right)n_{t_0}^\mu dVol_{t_0},$$

for $0<\delta<\frac{1}{2}$.

Remark 9. *The loss of derivative above is unnecessary because we are considering only a subregion of* $\{r^*>0\}$. *One can construct yet another variant of the vector field* X *to achieve the above estimate without any loss of derivatives. However, since this would not improve the regularity in our final result, it is not pursued here.*

Proof. Let $\tilde{X}=\tilde{f}(r^*)\partial_{r^*}$, where $\tilde{f}=\chi(r^*)\left(1-\frac{1}{(1+r^*)^\delta}\right)$ and χ is a cutoff function satisfying

$$\chi=\begin{cases}0 & r^*\leq 1\\ 1 & r^*\geq\max\{100,100M\}\end{cases}.$$

We recall (8.1):

$$\begin{aligned}K^{\tilde{X},w^{\tilde{X}}}(\Phi)=&\frac{\tilde{f}'(r^*)\left(\partial_{r^*}\Phi\right)^2}{1-\mu}+\frac{1}{2}|\nabla\!\!\!/\Phi|^2\left(\frac{2-3\mu}{r}\right)\tilde{f}(r^*)\\&-\frac{1}{4}\left(\frac{1}{1-\mu}\tilde{f}'''(r^*)+\frac{4}{r}\tilde{f}''(r^*)+\frac{\mu}{r^2}\tilde{f}'(r^*)-\frac{2\mu}{r^3}(3-4\mu)\tilde{f}(r^*)\right)\Phi^2,\\J_\mu^{\tilde{X},w^{\tilde{X}}}(\Phi)\,n_t^\mu=&\frac{1}{\sqrt{1-\mu}}\tilde{f}(r^*)\,\partial_t\Phi\partial_{r^*}\Phi+\frac{1}{2\sqrt{1-\mu}}\left(\tilde{f}'(r^*)+\frac{2(1-\mu)}{r}\tilde{f}(r^*)\right)(\partial_t\Phi)\,\Phi,\end{aligned}$$

Since we already have control of the spacetime integrals on a compact set using

Theorem 8.3.1 and Proposition 8.3.3, we only have to show that $K^{\tilde{X},w^{\tilde{X}}}(\Phi) \geq 0$ for $r^* \geq \max\{100, 100M\}$. For $r^* \geq \max\{100, 100M\}$, we have

$$\tilde{f}'(r^*) = \frac{\delta}{(1+r^*)^{1+\delta}}$$

$$\tilde{f}''(r^*) = -\frac{\delta(1+\delta)}{(1+r^*)^{2+\delta}}$$

$$\tilde{f}'''(r^*) = \frac{\delta(1+\delta)(2+\delta)}{(1+r^*)^{3+\delta}}.$$

Clearly, the coefficient of $(\partial_{r^*}\Phi)^2$ and $|\nabla\!\!\!/\Phi|^2$ in $K^{\tilde{X},w^{\tilde{X}}}(\Phi)$ is positive for

$$r^* \geq \max\{100, 100M\}.$$

We now study the coefficient of Φ^2 in $K^{\tilde{X},w^{\tilde{X}}}(\Phi)$ for $r^* \geq \max\{100, 100M\}$:

$$\begin{aligned}
&\frac{1}{1-\mu}\tilde{f}''' + \frac{4}{r}\tilde{f}'' + \frac{\mu}{r^2}\tilde{f}' - \frac{2\mu}{r^3}(3-4\mu)\tilde{f} \\
=&\frac{1}{1-\mu}\frac{\delta(1+\delta)(2+\delta)}{(1+r^*)^{3+\delta}} - \frac{4\delta(1+\delta)}{r(1+r^*)^{2+\delta}} + \frac{2M\delta}{r^2(1+r^*)^{1+\delta}} \\
&- \frac{12M}{r^3(1+r^*)^{\delta}} + \frac{32M^2}{r^5(1+r^*)^{\delta}} - \frac{2\mu}{r^3}(3-4\mu) \\
\leq&\frac{3\delta(1+\delta)(2+\delta)}{2(1+r^*)^{3+\delta}} - \frac{4\delta(1+\delta)}{r(1+r^*)^{2+\delta}} + \frac{2M\delta}{r^2(1+r^*)^{1+\delta}} - \frac{12M}{r^3(1+r^*)^{\delta}} + \frac{32M^2}{r^5(1+r^*)^{\delta}} \\
\leq&\frac{\delta(1+\delta)}{r(1+r^*)^{2+\delta}}\left(\frac{3\delta}{2}-1\right) + \frac{M}{r^3(1+r^*)^{\delta}}\left(2\delta - 12 + \frac{32}{100}\right) \\
<&0.
\end{aligned}$$

Hence $K^{\tilde{X},w^{\tilde{X}}}(\Phi) \geq 0$ for $r^* \geq \max\{100, 100M\}$.

Moreover, on this region of r^*, $\left(r^{-1-\delta}(\partial_{r^*}\Phi)^2 + r^{-3-\delta}\Phi^2 + r^{-1}|\nabla\!\!\!/\Phi|^2\right) \leq CK^{\tilde{X},w^{\tilde{X}}}(\Phi)$.

Finally, we have $\int J_\mu^{\tilde{X},w^{\tilde{X}}}(\Phi)\, n^\mu dVol_t \leq CJ_\mu^T(\Phi)\, n^\mu dVol_t$ using Lemma 8.3.2 and Cauchy-Schwarz exactly as in Proposition 8.3.3. □

Remark 10. *The weights in the Proposition are the same as those for Minkowski*

space. Since Schwarzschild is asymptotically flat, they are the expected weights.

Corollary 8.3.5. *In local coordinates, Theorem 8.3.1, Proposition 8.3.3 and 8.3.4 imply via Proposition 8.2.1 the following bounds:*

1. $\int \frac{1}{\sqrt{1-\mu}} \left(\left(\partial_{r^*}\Phi\right)^2 + \left(\partial_t \Phi\right)^2 + (1-\mu) \left|\nabla\!\!\!/ \Phi\right|^2 \right) dVol_{t_*} \leq C \int J_\mu^T (\Phi)\, n_{t_0}^\mu dVol_{t_0}$,
2. $\int \frac{1}{\sqrt{1-\mu}} \left(u^2 \left(\partial_u \Phi\right)^2 + v^2 \left(\partial_v \Phi\right)^2 + (1-\mu) \left|\nabla\!\!\!/ \Phi\right|^2 \right) dVol_{t_*} \leq C E_0 (\Phi)$,
3. $\int \sqrt{1-\mu} \frac{(r^*)^2+t^2}{r^2} \Phi^2 dVol_{t_*} \leq C E_0 (\Phi)\, t_*^{-2}$,
4. $\int_{t_1}^{t_2} \int_{-\frac{t}{2}}^{\frac{t}{2}} \frac{r^{1-\delta}(\partial_{r^*}\Phi)^2}{(1+|r^*|)^2(1-\mu)} + \frac{r^{1-\delta}\Phi^2}{(1+|r^*|)^4} dVol \leq C E_0 (\Phi)\, t_1^{-2}$, *where* $t_1 \leq t_2 \leq (1.1)\, t_1$,
5. $\int_{t_1}^{t_2} \int_{-\frac{t}{2}}^{\frac{t}{2}} \frac{r^3 |\nabla\!\!\!/ \Phi|^2}{(1+|r^*|)^4 (1-\mu)} dVol \leq C \sum_{k=0}^1 E_0 \left(\Omega^k \Phi\right) t_1^{-2}$, *where* $t_1 \leq t_2 \leq (1.1)\, t_1$.

8.4 Estimates for Ψ

In this section, we would like to imitate [18] and prove an analogue of Theorem 8.3.1. For technical reasons, however, we will need to lose an arbitrarily small power of t.

Theorem 8.4.1.
1. $\int J_\mu^T (\Psi)\, n_{t_*}^\mu dVol_{t_*} \leq C \left(\int J_\mu^T (\Psi)\, n_{t_0}^\mu dVol_{t_0} + E_1 (\Phi) \right)$,
2. $\int J_\mu^Z (\Psi)\, n_{t_*}^\mu dVol_{t_*} \leq C t_*^\delta E_1 (\Phi)$,
3. $\int_{\{-\frac{t_*}{2} \leq r^* \leq \frac{t_*}{2}\}} J_\mu^T (\Psi)\, n_{t_*}^\mu dVol_{t_*} \leq C E_1 (\Phi)\, t_*^{-2+\delta}$,
4. $\int_{t_1}^{t_2} \int_{-\frac{t}{2}}^{\frac{t}{2}} \sum_l K^{X_l, w X_l} (\Psi)\, dVol \leq C E_1 (\Phi)\, t_1^{-2+\delta}$, *where* $t_1 \leq t_2 \leq (1.1)\, t_1$,

The general strategy is as follows. We follow the argument in [18] but now in the conservation law for each of the vector fields, there is an extra error term which is a spacetime integral that looks like $\int_{t_0}^{t_*} V^\mu \partial_\mu \Psi \Box_g \Psi dVol$ (as well as an extra term $-\frac{1}{4} \int_{t_0}^{t_*} w \Psi \Box_g \Psi dVol$ for the modified currents). Very often, we need to show that this integral decays (or does not grow) with t_*, thus we need to "produce" some decay in t. We do this by splitting the domain of integration into three regions and estimating them separately:

1. For the region $\{\frac{t}{2} \leq r^* \leq \infty\}$, we use the fact that $\Box_g \Psi$ contains negative powers of r^*, (which is a consequence of the asymptotic flatness of Schwarzschild). In this region, negative powers of r^* can be estimated by negative powers of t.

2. For the region $\{-\frac{t}{2} \leq r^* \leq \frac{t}{2}\}$, we note that we have decay in the spacetime integral of Φ for each dyadic slab by Corollary 8.3.5.4, 8.3.5.5. We therefore estimate the integral on this region by that of $K^X(\Phi)$. Here, it is essential that we use the improved X estimates given by Proposition 8.3.4.

3. For the region $\{-\infty \leq r^* \leq -\frac{t}{2}\}$, we make use of the fact that there is an extra factor of $(1-\mu)^{\frac{1}{2}}$ in the spacetime volume form compared to the volume form on a time-slice (see Section 1.5). From the definition of r^*, we have $(1-\mu) \leq Ce^{cr^*}$, thus the factor of $(1-\mu)^{\frac{1}{2}}$ gives exponential decay in r^*, which translates to exponential decay in t in this region. Therefore, on this region, we first estimate on each time slice, and then carry out the integration in t.

Since we will often perform integration dyadically, we first set up the notation. We define a dyadic partition of $[t_0, t_*]$ by $t_0 \leq t_1 \leq ... \leq t_n = t_*$, where $t_i \leq (1.1)\, t_{i-1}$ and n is the minimal integer such that this can be done. In particular, $\log(t_* - t_0) \sim n$. We begin with the T estimate.

Proposition 8.4.2.

$$\int J_\mu^T(\Psi)\, n_{t_*}^\mu dVol_{t_*} \leq C \int J_\mu^T(\Psi)\, n_{t_0}^\mu dVol_{t_0} + C \sum_{k=0}^{2} E_0\left(\Omega^k \Phi\right).$$

Proof. The conservation law gives

$$\int J_\mu^T(\Psi)\, n_{t_*}^\mu dVol_{t_*} = \int J_\mu^T(\Psi)\, n_{t_0}^\mu dVol_{t_0} + \int \partial_t \Psi \Box_g \Psi dVol.$$

We split the error term into three parts and estimate them separately.

By Corollary 8.3.5.1,

$$
\begin{aligned}
&|\int_{\frac{t}{2}}^{\infty}\partial_t\Psi\Box_g\Psi dVol| \\
\leq& C\int_{t_0}^{t_*}\left(\int_{-\infty}^{\infty}\int_{\mathbb{S}^2}J_\mu^T(\Psi)\,n_t^\mu dVol_t\right)^{\frac{1}{2}} \\
&\qquad\times\left(\int_{\frac{t}{2}}^{\infty}\int_{\mathbb{S}^2}\frac{(\log r)_+^2}{r^4}\left(\left(\partial_{r^*}\Phi\right)^2+|\not\nabla\Omega\Phi|^2\right)dVol_t\right)^{\frac{1}{2}}dt \\
\leq& C\sup_{t_0\leq t\leq t_*}\left(\int_{-\infty}^{\infty}\int_{\mathbb{S}^2}J_\mu^T(\Psi)\,n_t^\mu dVol_t\right)^{\frac{1}{2}} \\
&\qquad\times\left(\sum_{k=0}^{1}\int_{-\infty}^{\infty}\int_{\mathbb{S}^2}J_\mu^T\left(\Omega^k\Phi\right)n_{t_0}^\mu dVol_{t_0}\right)^{\frac{1}{2}}\int_{t_0}^{t_*}t^{-\frac{3}{2}}dt \\
\leq& C\sup_{t_0\leq t\leq t_*}\left(\int_{-\infty}^{\infty}\int_{\mathbb{S}^2}J_\mu^T(\Psi)\,n_t^\mu dVol_t\right)^{\frac{1}{2}}\left(\sum_{k=0}^{1}\int_{-\infty}^{\infty}\int_{\mathbb{S}^2}J_\mu^T\left(\Omega^k\Phi\right)n_{t_0}^\mu dVol_{t_0}\right)^{\frac{1}{2}}
\end{aligned}
$$

For the middle region, we observe that By Corollary 8.3.5.4 and 8.3.5.5,

$$
\begin{aligned}
&|\int_{-\frac{t}{2}}^{\frac{t}{2}} \partial_t \Psi \Box_g \Psi dVol| \\
\leq & C \int_{t_0}^{t_*} \left(\int_{-\infty}^{\infty} \int_{\mathbb{S}^2} J_\mu^T(\Psi) n_t^\mu dVol_t \right)^{\frac{1}{2}} \\
& \qquad \times \left(\int_{-\frac{t}{2}}^{\frac{t}{2}} \int_{\mathbb{S}^2} \frac{(1+|r^*|)^2 (\log r)_+^2 (1-\mu)^{\frac{3}{2}}}{r^6} \left((\partial_{r^*}\Phi)^2 + |\not{\nabla}\Omega\Phi|^2 \right) dVol_t \right)^{\frac{1}{2}} dt \\
\leq & C \sup_{t_0 \leq t \leq t_*} \left(\int_{-\infty}^{\infty} \int_{\mathbb{S}^2} J_\mu^T(\Psi) n_t^\mu dVol_t \right)^{\frac{1}{2}} \\
& \qquad \times \left(\sum_{k=0}^{2} \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \left(\int_{-\frac{t}{2}}^{\frac{t}{2}} \sum_l K^{X_l, w^{X_l}} \left(\Omega^k \Phi \right) (1-\mu) dVol_t \right)^{\frac{1}{2}} dt \right) \\
\leq & C \sup_{t_0 \leq t \leq t_*} \left(\int_{-\infty}^{\infty} \int_{\mathbb{S}^2} J_\mu^T(\Psi) n_t^\mu dVol_t \right)^{\frac{1}{2}} \\
& \qquad \times \left(\sum_{k=0}^{2} \sum_{i=0}^{n-1} t_i^{\frac{1}{2}} \left(\int_{t_i}^{t_{i+1}} \int_{-\frac{t}{2}}^{\frac{t}{2}} \sum_l K^{X_l, w^{X_l}} \left(\Omega^k \Phi \right) dVol \right)^{\frac{1}{2}} \right) \\
\leq & C \sup_{t_0 \leq t \leq t_*} \left(\int_{-\infty}^{\infty} \int_{\mathbb{S}^2} J_\mu^T(\Psi) n_t^\mu dVol_t \right)^{\frac{1}{2}} \left(\sum_{k=0}^{2} E_0 \left(\Omega^k \Phi \right) \right)^{\frac{1}{2}} \left(\sum_{i=0}^{n-1} t_i^{-\frac{1}{2}} \right) \\
\leq & C \sup_{t_0 \leq t \leq t_*} \left(\int_{-\infty}^{\infty} \int_{\mathbb{S}^2} J_\mu^T(\Psi) n_t^\mu dVol_t \right)^{\frac{1}{2}} \left(\sum_{k=0}^{2} E_0 \left(\Omega^k \Phi \right) \right)^{\frac{1}{2}}
\end{aligned}
$$

By Corollary 8.3.5.1,

$$
\begin{aligned}
&\left|\int_{-\infty}^{-\frac{t}{2}} \partial_t \Psi \Box_g \Psi dVol\right| \\
\leq & C\int_{t_0}^{t_*}\left(\int_{-\infty}^{\infty}\int_{\mathbb{S}^2} J_\mu^T(\Psi) n_t^\mu dVol_t\right)^{\frac{1}{2}} \\
&\quad \times\left(\int_{-\infty}^{-\frac{t}{2}}\int_{\mathbb{S}^2}(1+|r^*|)^2(1-\mu)^{\frac{3}{2}}\left((\partial_{r^*}\Phi)^2+|\not\nabla\Omega\Phi|^2\right)dVol_t\right)^{\frac{1}{2}} dt \\
\leq & C\sup_{t_0\leq t\leq t_*}\left(\int_{-\infty}^{\infty}\int_{\mathbb{S}^2} J_\mu^T(\Psi) n_t^\mu dVol_t\right)^{\frac{1}{2}}\left(\sum_{k=0}^{1}\int_{-\infty}^{\infty}\int_{\mathbb{S}^2} J_\mu^T\left(\Omega^k\Psi\right) n_t^\mu dVol_t\right)^{\frac{1}{2}}\int_{t_0}^{t_*} e^{-ct}dt \\
\leq & C\sup_{t_0\leq t\leq t_*}\left(\int_{-\infty}^{\infty}\int_{\mathbb{S}^2} J_\mu^T(\Psi) n_t^\mu dVol_t\right)^{\frac{1}{2}}\left(\sum_{k=0}^{1}\int_{-\infty}^{\infty}\int_{\mathbb{S}^2} J_\mu^T\left(\Omega^k\Psi\right) n_t^\mu dVol_t\right)^{\frac{1}{2}}
\end{aligned}
$$

These together show that

$$
\begin{aligned}
&\int J_\mu^T(\Psi) n_{t_*}^\mu dVol_{t_*} \\
\leq & \int J_\mu^T(\Psi) n_{t_0}^\mu dVol_{t_0} + C\sup_{t_0\leq t\leq t_*}\left(\int J_\mu^T(\Psi) n_t^\mu dVol_t\right)^{\frac{1}{2}}\left(\sum_{k=0}^{2} E_0\left(\Omega^k\Phi\right)\right)^{\frac{1}{2}},
\end{aligned}
$$

which implies the Proposition with the following Lemma, taking $h_1(t)=0$ and $h_2(t)=\sum_{k=0}^{2} E_0\left(\Omega^k\Phi\right)$. □

Lemma 8.4.3. *Suppose $f(t)$ is continuous, $h_1(t)$, $h_2(t)$ are increasing and we have*

$$
f(t_*)\leq C\left(f(t_0)+h_1(t_*)+\sup_{t_0\leq t\leq t_*} f(t)^{\frac{1}{2}} h_2(t_*)^{\frac{1}{2}}\right),
$$

for all $t_*\geq t_0$.

Then

$$
f(t_*)\leq C(f(t_0)+h_1(t_*)+h_2(t_*)).
$$

Proof. Suppose $\sup_{t_0 \leq t \leq t_*} f(t)$ is achieved by $f\left(\tilde{t}\right)$ for some $t_0 \leq \tilde{t} \leq t_*$. Then

$$f\left(\tilde{t}\right) \leq C\left(f\left(t_0\right) + h_1\left(\tilde{t}\right) + f\left(\tilde{t}\right)^{\frac{1}{2}} h_2\left(\tilde{t}\right)^{\frac{1}{2}}\right).$$

$h_1(t)$, $h_2(t)$ increasing implies,

$$f\left(\tilde{t}\right) \leq C\left(f\left(t_0\right) + h_1\left(t_*\right) + f\left(\tilde{t}\right)^{\frac{1}{2}} h_2\left(t_*\right)^{\frac{1}{2}}\right).$$

Using Cauchy-Schwarz and subtracting $\frac{1}{2} f\left(\tilde{t}\right)$ from both sides,

$$f\left(\tilde{t}\right) \leq C(f\left(t_0\right) + h_1\left(t_*\right) + h_2\left(t_*\right)).$$

Clearly, $f(t_*) \leq \sup_{t_0 \leq t \leq t_*} f(t) = f\left(\tilde{t}\right)$. Hence we have the lemma. □

We then derive an X estimate. Here unlike in the case for Φ, in which the $\tilde{X}$ estimate was used to improve the already known estimate from X_l, we need to consider both of them at the same time.

Proposition 8.4.4.

$$\begin{aligned}
&\int_{t_0}^{t_*} |K^{\tilde{X}, w^{\tilde{X}}}(\Psi)| + \sum_l K^{X_l, w^{X_l}}(\Psi_l)\, dVol \\
\leq & C\left(\int J_\mu^T(\Psi)\, n_{t_*}^\mu dVol_{t_*} + \int J_\mu^T(\Psi)\, n_{t_0}^\mu dVol_{t_0}\right) + C t_0^{-2+\delta} \sum_{k=0}^{2} E_0\left(\Omega^k \Phi\right)
\end{aligned}$$

Remark 11. *The reader may ask why this Proposition gives decay for the error term while the statement of Proposition 8.4.2 does not. In fact, the proof of Proposition 8.4.2 is sufficient to show that the error term decays. However, we do not pursue this as it is unnecessary for later use.*

Proof. Decompose $\Psi = \sum_l \Psi_l$ into spherical harmonics.
Since Schwarzschild spacetimes are spherically symmetric, $\Box_g \Psi_l = g_1(r^*)\, \partial_{r^*} \Phi_l +$

$g_2(r^*)\nabla\!\!\!/(\Omega\Phi_l)$.

Notice that $K^{\tilde{X},w^{\tilde{X}}}(\Psi)$ is not everywhere positive. It is identically zero for $r^* \leq 1$ and as we have shown in the proof of Proposition 8.3.4, $K^{\tilde{X},w^{\tilde{X}}} \geq 0$ for $r^* \geq max\{100, 100M\}$. On the remaining (not necessarily positive) region $1 \leq r^* \leq max\{100, 100M\}$, we have $|K^{\tilde{X},w^{\tilde{X}}}(\Psi_l)| \leq CK^{X_l,w^{X_l}}(\Psi_l)$. (Notice that we have avoided the region around $r = 3M$ where this inequality is potentially problematic.) In particular, applying Proposition 2 for the vector field X, we have

$$
\begin{aligned}
&\int |K^{\tilde{X},w^X}(\Psi_l)|dVol + \int K^{X_l,w^{X_l}}(\Psi_l)\,dVol \\
\leq &\int K^{\tilde{X},w^{\tilde{X}}}(\Psi_l)\,dVol + (C+1)\int K^{X_l,w^{X_l}}(\Psi_l)\,dVol \\
= &\int J_\mu^{\tilde{X},w^{\tilde{X}}}(\Psi_l)\,n_{t_*}^\mu dVol_{t_*} - \int J_\mu^{\tilde{X},w^{\tilde{X}}}(\Psi_l)\,n_{t_0}^\mu dVol_{t_0} \\
&+ (C+1)\left(\int J_\mu^{X_l,w^{X_l}}(\Psi_l)\,n_{t_*}^\mu dVol_{t_*} - \int J_\mu^{X_l,w^{X_l}}(\Psi_l)\,n_{t_0}^\mu dVol_{t_0}\right) \\
&+ \frac{1}{4}\int\left(\tilde{f}' + \frac{2(1-\mu)}{r}\tilde{f}\right)\Psi_l\Box\Psi_l dVol - \int\left(\tilde{f}\partial_{r^*}\Psi_l\right)\Box\Psi_l dVol \\
&+ (C+1)\left(\frac{1}{4}\int\left(f_l' + \frac{2(1-\mu)}{r}f_l\right)\Psi_l\Box\Psi_l dVol - \int(f_l\partial_{r^*}\Psi_l)\Box\Psi_l dVol\right) \\
\leq &C\left(\int J_\mu^T(\Psi_l)\,n_{t_*}^\mu dVol_{t_*} + \int J_\mu^T(\Psi_l)\,n_{t_0}^\mu dVol_{t_0} + \int|\left(r^{-1}\Psi_l + \partial_{r^*}\Psi_l\right)\Box\Psi_l|dVol\right).
\end{aligned}
$$

We split the last term into three integrals and estimate them separately.

By Theorem 8.3.1.2,

$$\begin{aligned}
&\int_{t_0}^{t_*}\int_{\frac{t}{2}}^{\infty}\int_{\mathbb{S}^2}|\left(r^{-1}\Psi_l+\partial_{r^*}\Psi_l\right)\Box\Psi_l|dVol\\
\leq& C\int_{t_0}^{t_*}\int_{\frac{t}{2}}^{\infty}\int_{\mathbb{S}^2}r^{-1+\frac{\delta}{2}}\left(\log r\right)_+\left(r^{-\frac{3}{2}-\frac{\delta}{4}}|\Psi_l|+r^{-\frac{1}{2}-\frac{\delta}{4}}|\partial_{r^*}\Psi_l|\right)\\
&\qquad\times\left(r^{-\frac{1}{2}-\frac{\delta}{4}}\left(|\partial_{r^*}\Phi|+|\not\nabla\Omega\Phi|\right)\right)dVol\\
\leq& Ct_0^{-1+\frac{\delta}{2}}\left(\int_{t_0}^{t_*}\int_{\frac{t}{2}}^{\infty}\int_{\mathbb{S}^2}|K^{\tilde{X},w^{\tilde{X}}}\left(\Psi_l\right)|dVol\right)^{\frac{1}{2}}\\
&\qquad\times\left(\sum_{k=0}^{1}\int_{t_0}^{t_*}\int_{\frac{t}{2}}^{\infty}\int_{\mathbb{S}^2}|K^{\tilde{X},w^{\tilde{X}}}\left(\Omega^k\Phi_l\right)|dVol\right)^{\frac{1}{2}}\\
\leq& Ct_0^{-1+\frac{\delta}{2}}\left(\int_{t_0}^{t_*}\int_{-\infty}^{\infty}\int_{\mathbb{S}^2}|K^{\tilde{X},w^{\tilde{X}}}\left(\Psi_l\right)|dVol\right)^{\frac{1}{2}}\\
&\qquad\times\left(\sum_{k=0}^{1}\int_{t_0}^{t_*}\int_{-\infty}^{\infty}\int_{\mathbb{S}^2}|K^{\tilde{X},w^{\tilde{X}}}\left(\Omega^k\Phi_l\right)|dVol\right)^{\frac{1}{2}}\\
\leq&\frac{1}{4}\int_{t_0}^{t_*}\int_{-\infty}^{\infty}\int_{\mathbb{S}^2}|K^{\tilde{X},w^{\tilde{X}}}\left(\Psi_l\right)|dVol+Ct_0^{-2+\delta}\sum_{k=0}^{1}\int_{-\infty}^{\infty}\int_{\mathbb{S}^2}J_\mu^T\left(\Omega^k\Phi_l\right)n_0^\mu dVol_{t_0}.
\end{aligned}$$

By Theorem 8.3.1.5,

$$\begin{aligned}
&\int_{t_0}^{t_*}\int_{-\frac{t}{2}}^{\frac{t}{2}}\int_{\mathbb{S}^2}|\left(r^{-1}\Psi_l+\partial_{r^*}\Psi_l\right)\Box\Psi_l|dVol\\
\leq& C\int_{t_0}^{t_*}\int_{-\frac{t}{2}}^{\frac{t}{2}}\int_{\mathbb{S}^2}\left(r^{-\frac{3}{2}-\frac{\delta}{4}}|\Psi_l|+r^{-\frac{1}{2}-\frac{\delta}{4}}|\partial_{r^*}\Psi_l|\right)\left(r^{-\frac{1}{2}-\frac{\delta}{4}}\left(|\partial_{r^*}\Phi|+|\not\nabla\Omega\Phi|\right)\right)dVol\\
\leq& C\left(\int_{t_0}^{t_*}\int_{-\frac{t}{2}}^{\frac{t}{2}}\int_{\mathbb{S}^2}\left(|K^{\tilde{X},w^{\tilde{X}}}\left(\Psi_l\right)|+K^{X_l,w^{X_l}}\left(\Psi_l\right)\right)dVol\right)^{\frac{1}{2}}\\
&\qquad\times\left(\sum_{k=0}^{2}\int_{t_0}^{t_*}\int_{-\frac{t}{2}}^{\frac{t}{2}}\int_{\mathbb{S}^2}|K^{\tilde{X},w^{\tilde{X}}}\left(\Omega^k\Phi_l\right)|dVol\right)^{\frac{1}{2}}\\
\leq&\frac{1}{4}\int_{t_0}^{t_*}\int_{-\infty}^{\infty}\int_{\mathbb{S}^2}\left(|K^{\tilde{X},w^{\tilde{X}}}\left(\Psi_l\right)|+K^{X_l,w^{X_l}}\left(\Psi_l\right)\right)dVol+C\sum_{i=0}^{n-1}\sum_{k=0}^{2}t_i^{-2}E_0\left(\Omega^k\Phi_l\right)\\
\leq&\frac{1}{4}\int_{t_0}^{t_*}\int_{-\infty}^{\infty}\int_{\mathbb{S}^2}\left(|K^{\tilde{X},w^{\tilde{X}}}\left(\Psi_l\right)|+K^{X_l,w^{X_l}}\left(\Psi_l\right)\right)dVol+Ct_0^{-2}\sum_{k=0}^{1}E_0\left(\Omega^k\Phi_l\right).
\end{aligned}$$

By Theorem 8.3.1.1, Proposition 8.4.2 and Lemma 8.3.2,

$$\begin{aligned}
&\int_{t_0}^{t_*}\int_{-\infty}^{-\frac{t}{2}}\int_{\mathbb{S}^2}|\left(r^{-1}\Psi_l+\partial_{r^*}\Psi_l\right)\Box\Psi_l|dVol\\
\leq& C\int_{t_0}^{t_*}\int_{-\infty}^{-\frac{t}{2}}\int_{\mathbb{S}^2}|r^*|\left(|\Psi_l|+|\partial_{r^*}\Psi_l|\right)\left(|\partial_{r^*}\Phi_l|+|\nabla\!\!\!/\Omega\Phi_l|\right)dVol\\
\leq& C\int_{t_0}^{t_*}\left(\int_{-\infty}^{\infty}\left(\frac{1}{\left(1+|r^*|\right)^2}\Psi_l^2+\left(\partial_{r^*}\Psi_l\right)^2\right)dAdr^*\right)^{\frac{1}{2}}\\
&\times\left(\int_{-\infty}^{-\frac{t}{2}}\left(r^*\right)^4(1-\mu)\left(\left(\partial_{r^*}\Phi_l\right)^2+(1-\mu)\left|\nabla\!\!\!/\Omega\Phi_l\right|^2\right)dAdr^*\right)^{\frac{1}{2}}dt\\
\leq& C\int_{t_0}^{t_*}\left(\int_{-\infty}^{\infty}\left(\partial_{r^*}\Psi_l\right)^2dAdr^*\right)^{\frac{1}{2}}\left(\int_{-\infty}^{\infty}\left(\left(\partial_{r^*}\Phi_l\right)^2+(1-\mu)\left|\nabla\!\!\!/\Omega\Phi_l\right|^2\right)dAdr^*\right)^{\frac{1}{2}}e^{-ct}dt\\
\leq& C\sup_{t_0\leq t\leq t_*}\left(\int_{-\infty}^{\infty}J_\mu^T\left(\Psi_l\right)n_t^\mu dVol_t\right)^{\frac{1}{2}}\left(\sum_{k=0}^{2}\int_{-\infty}^{\infty}J_\mu^T\left(\Omega^k\Phi_l\right)n_t^\mu dVol_{t_0}\right)^{\frac{1}{2}}e^{-ct_0}\\
\leq& C\int J_\mu^T\left(\Psi_l\right)n_{t_0}^\mu dVol_{t_0}+Ct_0^{-2}\sum_{k=0}^{2}E_0\left(\Omega^k\Phi_l\right),
\end{aligned}$$

Subtract the terms with K from both sides and get

$$\begin{aligned}
&\int_{t_0}^{t_*}|K^{\tilde{X},w^{\tilde{X}}}\left(\Psi_l\right)|+K^{X_l,w^X}\left(\Psi_l\right)dVol\\
\leq& C\left(\int J_\mu^T\left(\Psi_l\right)n_{t_*}^\mu dVol_{t_*}+\int J_\mu^T\left(\Psi_l\right)n_{t_0}^\mu dVol_{t_0}\right)+Ct_0^{-2+\delta}\sum_{k=0}^{2}E_0\left(\Omega^k\Phi_l\right).
\end{aligned}$$

Sum over $l\geq 0$ to get the Proposition. □

We localize the estimates in the above Proposition to obtain decay as in [18].

Proposition 8.4.5. *Let $t_0\leq t_1\leq(1.1)\,t_0$, $|r_1^*|+|r_2^*|\leq\frac{t_0}{2}$. Then*

$$\begin{aligned}
&\int_{\mathcal{P}}|K^{\tilde{X},w^{\tilde{X}}}\left(\Psi\right)|+\sum_l K^{X_l,w^{X_l}}\left(\Psi_l\right)dVol\\
\leq& C\left(t_0^{-2}\int J_\mu^Z\left(\Psi\right)n_0^\mu dVol_{t_0}+t_0^{-2+\delta}\sum_{k=0}^{2}\sum_{m=0}^{1}E_0\left(\partial_t^m\Omega^k\Phi\right)\right),
\end{aligned}$$

where $\mathcal{P}=\{t_0\leq t\leq t_1,-\frac{t}{2}\leq r^*\leq\frac{t}{2}\}$ or $\mathcal{P}=\{t_0\leq t\leq t_1,r_1^*-(t_1-t)\leq r^*\leq$

$r_2^* + (t_1 - t)\}$.

Proof. Let $\chi = \begin{cases} 1 & |x| \leq 1 \\ 0 & |x| \geq 1.1 \end{cases}$. On $t = t_0$, let $\tilde{\Phi} = \chi\left(\frac{r^*}{0.65t_0}\right)\Phi$, $\partial_t\tilde{\Phi} = \chi\left(\frac{r^*}{0.65t_0}\right)\partial_t\Phi$ and solve for $\Box_g\tilde{\Phi} = 0$ for $t \geq t_0$.

Following [18], we have

$$\int_{-0.715t_0}^{0.715t_0} \frac{1}{\sqrt{1-\mu}}\Phi^2 dVol_{t_0} \leq \int J_\mu^Z(\Phi)\, n_{t_0}^\mu dVol_{t_0}.$$

This is true because of Proposition 8.2.1.2, 8.2.1.3 and an elementary one-dimensional estimate:

$$\int_{-a}^{a} |f(x)|^2 dx \leq Ca^2\left(\int_{-a}^{a} |\partial_x f(x)|^2 + \int_{-1}^{1} |f(x)|^2 dx\right),$$

for $a \geq 1$.[2]

Using this, we can estimate the current of $\tilde{\Phi}$:

$$\begin{aligned} &\int J_\mu^T\left(\tilde{\Phi}\right) n_{t_0}^\mu dVol_{t_0} \\ \leq &\int_{-0.715t_0}^{0.715t_0} J_\mu^T(\Phi)\, n_{t_0}^\mu dVol_{t_0} + Ct_0^{-2}\int_{-0.715t_0}^{0.715t_0} \frac{1}{\sqrt{1-\mu}}\Phi^2 dVol_{t_0} \\ \leq &Ct_0^{-2}\int J_\mu^Z(\Phi)\, n_{t_0}^\mu dVol_{t_0}. \end{aligned}$$

Similarly,

$$\int J_\mu^T\left(\Omega\tilde{\Phi}\right) n_{t_0}^\mu dVol_{t_0} \leq Ct_0^{-2}\int J_\mu^Z(\Omega\Phi)\, n_{t_0}^\mu dVol_{t_0}.$$

[2]One can prove this one-dimensional estimate by first considering $g = 0$ on $[-\frac{1}{2}, \frac{1}{2}]$ and the trivial bound $\int_{-a}^{a} |g(x)| dx \leq Ca\int_{-a}^{a} |\partial_x g(x)| dx$. Then one sets $g(x) = f(x)^2$ and use Cauchy-Schwarz to get $\int_{-a}^{a} |f(x)|^2 dx \leq Ca^2\int_{-a}^{a} |\partial_x f(x)|^2 dx$.. Finally, one cuts off $f(x)$ to be identically zero in $[-\frac{1}{2}, \frac{1}{2}]$.

Define $\tilde{\Psi} = S\tilde{\Phi}$. By Proposition 8.4.4,

$$\begin{aligned}
&\int_{\mathcal{P}} |K^{\tilde{X},w^{\tilde{X}}}\left(\tilde{\Psi}\right)| + \sum_l K^{X_l,w^{X_l}}\left(\tilde{\Psi}_l\right) dVol \\
\leq & C\left(\int J_\mu^T\left(\tilde{\Psi}\right) n_{t_1}^\mu dVol_{t_1} + \int J_\mu^T\left(\tilde{\Psi}\right) n_{t_0}^\mu dVol_{t_0}\right) + Ct_0^{-2+\delta}\sum_{k=0}^{2} E_0\left(\Omega^k\tilde{\Phi}\right).
\end{aligned}$$

The left hand side equals $\int_{\mathcal{P}} |K^{\tilde{X},w^{\tilde{X}}}(\Psi)| + \sum_l K^{X_l,w^{X_l}}(\Psi_l)\, dVol$ by finite speed of propagation.

$\int J_\mu^T\left(\tilde{\Psi}\right) n_{t_0}^\mu dVol_{t_0}$ can be estimated in a similar way as $\int J_\mu^T\left(\tilde{\Phi}\right) n_{t_0}^\mu dVol_{t_0}$. More specifically, we claim that

$$\int J_\mu^T\left(\tilde{\Psi}\right) n_{t_0}^\mu dVol_{t_0} \leq Ct_0^{-2}\left(\int J_\mu^Z(\Psi)\, n_{t_0}^\mu dVol_{t_0} + E_0(\Phi)\right).$$

To see this, we first note that

$$\tilde{\Psi} = \chi\left(\frac{r^*}{0.65t_0}\right)\Psi + \frac{r^*}{0.65t_0}\chi'\left(\frac{r^*}{0.65t_0}\right)\Phi.$$

Also note that on the support of $\tilde{\Psi}$, $\frac{|r^*|}{t_0} \leq C$. Therefore,

$$\begin{aligned}
&\int J_\mu^T\left(\tilde{\Psi}\right) n_{t_0}^\mu dVol_{t_0} \\
\leq & \int_{-0.715t_0}^{0.715t_0} J_\mu^T(\Psi)\, n_{t_0}^\mu dVol_{t_0} + Ct_0^{-2}\int_{-0.715t_0}^{0.715t_0}\frac{1}{\sqrt{1-\mu}}\Psi^2 dVol_{t_0} \\
&+ C\int_{-0.715t_0}^{0.715t_0} J_\mu^T(\Phi)\, n_{t_0}^\mu dVol_{t_0} + Ct_0^{-2}\int_{-0.715t_0}^{0.715t_0}\frac{1}{\sqrt{1-\mu}}\Phi^2 dVol_{t_0} \\
\leq & Ct_0^{-2}\left(\int J_\mu^Z(\Psi)\, n_{t_0}^\mu dVol_{t_0} + \int J_\mu^Z(\Phi)\, n_{t_0}^\mu dVol_{t_0}\right) \\
\leq & Ct_0^{-2}\left(\int J_\mu^Z(\Psi)\, n_{t_0}^\mu dVol_{t_0} + E_0(\Phi)\right).
\end{aligned}$$

We would now want to control $\int J_\mu^T\left(\tilde{\Psi}\right) n_{t_1}^\mu dVol_{t_1}$. Using the conservation law for T

and an integration by parts in t,

$$\begin{aligned}
&\int J_\mu^T\left(\tilde{\Psi}\right) n_{t_1}^\mu dVol_{t_1} \\
=&\int J_\mu^T\left(\tilde{\Psi}\right) n_{t_0}^\mu dVol_{t_0} - \int \partial_t\tilde{\Psi}\Box_g\tilde{\Psi}dVol \\
\leq&\int J_\mu^T\left(\tilde{\Psi}\right) n_{t_0}^\mu dVol_{t_0} + |\int \tilde{\Psi}\Box_g\left(\partial_t\tilde{\Psi}\right)dVol| + |\int \tilde{\Psi}\Box_g\tilde{\Psi}\sqrt{1-\mu}dVol_{t_0}| \\
&+|\int \tilde{\Psi}\Box_g\tilde{\Psi}\sqrt{1-\mu}dVol_{t_1}|.
\end{aligned}$$

We first estimate the spacetime error term in this expression. Using Proposition 8.2.1,

8.3.3, and 8.3.4,

$$
\begin{aligned}
&|\int \tilde{\Psi}\Box_g\left(\partial_t\tilde{\Psi}\right)dVol|\\
\leq&\int|\tilde{\Psi}\Box_g\left(S\left(\partial_t\tilde{\Phi}\right)\right)|dVol+\int|\tilde{\Psi}\Box_g\left(\partial_t\tilde{\Phi}\right)|dVol\\
\leq&C\int\frac{\left(1+|r^*|\right)\left(\log r\right)_+}{r^3}|\tilde{\Psi}|\left(|\partial_{r^*}\partial_t\tilde{\Phi}|+|\not{\nabla}\left(\partial_t\Omega\tilde{\Phi}\right)|\right)dVol\\
\leq&C\left(\int r^{\delta}\frac{r^{1-\frac{\delta}{4}}\tilde{\Psi}^2}{\left(1+|r^*|\right)^4}dVol\right)^{\frac{1}{2}}\left(\int\frac{\left(1+|r^*|\right)^6\left(\left(\partial_t\partial_{r^*}\tilde{\Phi}\right)^2+|\not{\nabla}\left(\Omega\partial_t\tilde{\Phi}\right)|^2\right)}{r^{7+\frac{\delta}{4}}}dVol\right)^{\frac{1}{2}}\\
\leq&Ct_0^{\frac{\delta}{2}}\left(\int|K^{\tilde{X},w^{\tilde{X}}}\left(\tilde{\Psi}\right)|+\sum_l K^{X_l}\left(\tilde{\Psi}_l\right)dVol\right)^{\frac{1}{2}}\\
&\times\left(\sum_{k=0}^{2}\int|K^{\tilde{X},w^{\tilde{X}}}\left(\partial_t\Omega^k\tilde{\Phi}\right)|+\sum_l K^{X_l,w^{X_l}}\left(\partial_t\Omega^k\tilde{\Phi}_l\right)dVol\right)^{\frac{1}{2}}\\
\leq&Ct_0^{\frac{\delta}{2}}\left(\int J_\mu^T\left(\tilde{\Psi}\right)n_{t_1}^\mu dVol_{t_1}+\int J_\mu^T\left(\tilde{\Psi}\right)n_{t_0}^\mu dVol_{t_0}+t_0^{-2+\delta}\sum_{k=0}^{2}E_0\left(\Omega^k\tilde{\Phi}\right)\right)^{\frac{1}{2}}\\
&\times\left(\sum_{k=0}^{2}\int J_\mu^T\left(\partial_t\Omega^k\tilde{\Phi}\right)n_{t_0}^\mu dVol_{t_0}\right)^{\frac{1}{2}}\\
\leq&\frac{1}{4}\int J_\mu^T\left(\tilde{\Psi}\right)n_{t_1}^\mu dVol_{t_1}+C\int J_\mu^T\left(\tilde{\Psi}\right)n_{t_0}^\mu dVol_{t_0}+Ct_0^{-2+\delta}\sum_{k=0}^{2}E_0\left(\Omega^k\tilde{\Phi}\right)\\
&+Ct_0^{-2+\delta}\left(\sum_{k=0}^{2}\int J_\mu^{Z,w^Z}\left(\partial_t\Omega^k\tilde{\Phi}\right)n_{t_0}^\mu dVol_{t_0}\right)\\
\leq&\frac{1}{4}\int J_\mu^T\left(\tilde{\Psi}\right)n_{t_1}^\mu dVol_{t_1}+C\int J_\mu^T\left(\tilde{\Psi}\right)n_{t_0}^\mu dVol_{t_0}+Ct_0^{-2+\delta}\sum_{m=0}^{1}\sum_{k=0}^{2}E_0\left(\partial_t^m\Omega^k\tilde{\Phi}\right)
\end{aligned}
$$

where at the third to last step we again used Proposition 8.4.4.

We estimate the boundary terms using Lemma 8.3.2 and Corollary 8.3.5.1

$$
\begin{aligned}
&|\int \tilde{\Psi}\Box_g\tilde{\Psi}\sqrt{1-\mu}dVol_{t_0}| \\
\leq& C\int (1+|r^*|)\, r^{-3}\,(\log r)_+\,\sqrt{1-\mu}|\tilde{\Psi}|\left(|\partial_{r^*}\tilde{\Phi}|+|\not{\nabla}\Omega\tilde{\Phi}|\right)dVol_{t_0} \\
\leq& C\left(\int (1+|r^*|)^{-2}\,\tilde{\Psi}^2\,(1-\mu)^{-\frac{1}{2}}\,dVol_{t_0}\right)^{\frac{1}{2}}\left(\int\left(\left(\partial_{r^*}\tilde{\Phi}\right)^2+|\not{\nabla}\Omega\tilde{\Phi}|^2\right)\sqrt{1-\mu}dVol_{t_0}\right)^{\frac{1}{2}} \\
\leq& C\int J_\mu^T\left(\tilde{\Psi}\right)n_{t_0}^\mu dVol_{t_0}+C\int\left(J_\mu^T\left(\tilde{\Phi}\right)+J_\mu^T\left(\Omega\tilde{\Phi}\right)\right)n_{t_0}^\mu dVol_{t_0}.
\end{aligned}
$$

The term for $t=t_1$ is done analogously, but with a more careful choice of constant.

$$
\begin{aligned}
&|\int \tilde{\Psi}\Box_g\tilde{\Psi}\sqrt{1-\mu}dVol_{t_1}| \\
\leq& C\int (1+|r^*|)\, r^{-3}\,(\log r)_+\,\sqrt{1-\mu}|\tilde{\Psi}|\left(|\partial_{r^*}\tilde{\Phi}|+|\not{\nabla}\Omega\tilde{\Phi}|\right)dVol_{t_1} \\
\leq& C\left(\int (1+|r^*|)^{-2}\,\tilde{\Psi}^2\,(1-\mu)^{-\frac{1}{2}}\,dVol_{t_1}\right)^{\frac{1}{2}}\left(\int\left(\left(\partial_{r^*}\tilde{\Phi}\right)^2+|\not{\nabla}\Omega\tilde{\Phi}|^2\right)\sqrt{1-\mu}dVol_{t_1}\right)^{\frac{1}{2}} \\
\leq& \frac{1}{4}\int J_\mu^T\left(\tilde{\Psi}\right)n_{t_1}^\mu dVol_{t_1}+C\int\left(J_\mu^T\left(\tilde{\Phi}\right)+J_\mu^T\left(\Omega\tilde{\Phi}\right)\right)n_{t_1}^\mu dVol_{t_1} \\
=& \frac{1}{4}\int J_\mu^T\left(\tilde{\Psi}\right)n_{t_1}^\mu dVol_{t_1}+C\int\left(J_\mu^T\left(\tilde{\Phi}\right)+J_\mu^T\left(\Omega\tilde{\Phi}\right)\right)n_{t_0}^\mu dVol_{t_0}.
\end{aligned}
$$

Combining these estimates and subtracting $\frac{1}{2}\int J_\mu^T\left(\tilde{\Psi}\right)n_{t_1}^\mu dVol_{t_1}$ on both sides, we get

$$
\begin{aligned}
&\int J_\mu^T\left(\tilde{\Psi}\right)n_{t_1}^\mu dVol_{t_1} \\
\leq& Ct_0^{-2}\int J_\mu^Z\left(\Psi\right)n_{t_0}^\mu dVol_{t_0}+Ct_0^{-2+\delta}\sum_{m=0}^{1}\sum_{k=0}^{2}E_0\left(\partial_t^m\Omega^k\tilde{\Phi}\right).
\end{aligned}
$$

It remains to control $\sum_{m=0}^{1}\sum_{k=0}^{2} E_0\left(\Omega^k\partial_t^m\tilde{\Phi}\right)$.

$$
\begin{aligned}
&\sum_{m=0}^{1}\sum_{k=0}^{2} E_0\left(\partial_t^m\Omega^k\tilde{\Phi}\right)\\
\leq& C\int_{-0.715t_0}^{0.715t_0}\left(\sum_{m=0}^{1}\sum_{k=0}^{5} J_\mu^N\left(\partial_t^m\Omega^k\tilde{\Phi}\right)n_{t_0}^\mu+\sum_{m=0}^{1}\sum_{k=0}^{4} J_\mu^Z\left(\partial_t^m\Omega^k\tilde{\Phi}\right)n_{t_0}^\mu\right)dVol_{\{t=t_0\}}\\
\leq& C\int_{-0.715t_0}^{0.715t_0}\left(\sum_{m=0}^{1}\sum_{k=0}^{5} J_\mu^N\left(\partial_t^m\Omega^k\Phi\right)n_{t_0}^\mu+\sum_{m=0}^{1}\sum_{k=0}^{4} J_\mu^Z\left(\partial_t^m\Omega^k\Phi\right)n_{t_0}^\mu\right)dVol_{\{t=t_0\}}\\
&+Ct_0^{-2}\int_{-0.715t_0}^{0.715t_0}\left(\sum_{m=0}^{1}\sum_{k=0}^{5}\left(\partial_t^m\Omega^k\Phi\right)^2+\sum_{m=0}^{1}\sum_{k=0}^{4} t_0^2\left(\partial_t^m\Omega^k\Phi\right)^2\right)r^2dAdr^*\\
\leq& C\sum_{m=0}^{1}\sum_{k=0}^{2} E_0\left(\partial_t^m\Omega^k\Phi\right).
\end{aligned}
$$

□

After establishing the X estimates, we turn to the Z estimates for Ψ.

Proposition 8.4.6.

$$
\begin{aligned}
&\int J_\mu^{Z,w^Z}(\Psi)\,n_{t_*}^\mu dVol_{t_*}\\
\leq& C\int J_\mu^{Z,w^Z}(\Psi)\,n_{t_0}^\mu dVol_{t_0}+C\sum_{k=0}^{1}\int_{t_0}^{t_*}\int_{r_1^*}^{r_2^*} t\sum_l K^{X_l,w^{X_l}}\left(\Omega^k\Psi_l\right)dVol\\
&+C\left(\int_{t_0}^{t_*}\int_{-\frac{t}{2}}^{\frac{t}{2}} t^{2+2\delta}\left(|K^{\tilde{X},w^{\tilde{X}}}(\Psi)|+\sum_l K^{X_l,w^{X_l}}(\Psi_l)\right)dVol\right)^{\frac{1}{2}}\\
&\qquad\times\left(\sum_{m=0}^{1}\sum_{k=0}^{2} E_0\left(\partial_t^m\Omega^k\Phi\right)\right)^{\frac{1}{2}}\\
&+Ct_*^\delta\sum_{k=0}^{2} E_0\left(\Omega^k\Phi\right).
\end{aligned}
$$

Proof. By Proposition 2 applied to the vector field Z,

$$\begin{aligned}
&\int J_\mu^{Z,w^Z}(\Psi)\, n_{t_*}^\mu dVol_{\{t=t_*\}}\\
=&\int J_\mu^{Z,w^Z}(\Psi)\, n_{t_0}^\mu dVol_{\{t=t_0\}} + \int K^{Z,w^Z}(\Psi)\, dVol - \int \frac{tr^*(1-\mu)}{2r}\Box_g\Psi dVol\\
&+\int \left(u^2\partial_u\Psi + v^2\partial_v\Psi\right)\Box_g\Psi dVol.
\end{aligned}$$

As remarked before, there exists r_1^*, r_2^* with $r_1^* < r_2^*$ such that $K^{Z,w^Z}(\Psi)$ is non-positive for $r^* \le r_1^*$ or $r^* \ge r_2^*$. Therefore,

$$\begin{aligned}
\int K^{Z,w^Z}(\Psi)\, dVol \le& \int_{r_1^*}^{r_2^*} K^{Z,w^Z}(\Psi)\, dVol\\
\le& C\int_{t_0}^{t_*}\int_{r_1^*}^{r_2^*} t\left(\Psi^2 + |\not\nabla\Psi|^2\right) dVol\\
\le& C\sum_{k=0}^{1}\int_{t_0}^{t_*}\int_{r_1^*}^{r_2^*} t\sum_l K^{X_l,w^{X_l}}\left(\Omega^k\Psi_l\right) dVol.
\end{aligned}$$

For the first error term, we again estimate by looking at three separate regions. By Proposition 8.2.1.3 and Corollary 8.3.5.1,

$$\begin{aligned}
&\left|\int_{t_0}^{t_*}\int_{\frac{t}{2}}^{\infty}\frac{tr^*(1-\mu)}{2r}\Psi\Box_g\Psi dVol\right|\\
\le& C\int_{t_0}^{t_*}\left(\int_{\frac{t}{2}}^{\infty}\int_{\mathbb{S}^2}\frac{t^2}{r^2}\Psi^2 r^2(1-\mu)\, dAdr^*\right)^{\frac12}\\
&\qquad\times\left(\int_{\frac{t}{2}}^{\infty}\int_{\mathbb{S}^2}\frac{(\log r)_+^2}{r^2}\left((\partial_{r^*}\Phi)^2 + |\not\nabla\Omega\Phi|^2\right) r^2 dAdr^*\right)^{\frac12} dt\\
\le& C\sup_{t_0\le t\le t_*}\left(\int J_\mu^{Z,w^Z}(\Psi)\, n_t^\mu dVol_t\right)^{\frac12}\left(\sum_{k=0}^{1}\int J_\mu^T\left(\Omega^k\Phi\right) n_{t_0}^\mu dVol_{\{t=t_0\}}\right)^{\frac12}\int_{t_0}^{t_*} t^{-1+\frac{\delta}{2}}dt\\
\le& Ct_*^{\frac{\delta}{2}}\sup_{t_0\le t\le t_*}\left(\int J_\mu^{Z,w^Z}(\Psi)\, n_t^\mu dVol_t\right)^{\frac12}\left(\sum_{k=0}^{1}E_0\left(\Omega^k\Phi\right)\right)^{\frac12}.
\end{aligned}$$

By Proposition 8.2.1.3, Corollary 8.3.5.4 and Corollary 8.3.5.5,

$$
\begin{aligned}
&|\int_{t_0}^{t_*}\int_{-\frac{t}{2}}^{\frac{t}{2}}\frac{tr^*\left(1-\mu\right)}{2r}\Psi\Box_g\Psi dVol|\\
\leq& C\int_{t_0}^{t_*}\left(\int_{-\frac{t}{2}}^{\frac{t}{2}}\int_{\mathbb{S}^2}\frac{t^2}{r^2}\Psi^2r^2\left(1-\mu\right)dAdr^*\right)^{\frac{1}{2}}\\
&\times\left(\int_{-\frac{t}{2}}^{\frac{t}{2}}\int_{\mathbb{S}^2}\frac{\left(1+|r^*|\right)^2\left(\log r\right)_+^2\left(1-\mu\right)^2}{r^4}\left(\left(\partial_{r^*}\Phi\right)^2+|\not\nabla\Omega\Phi|^2\right)r^2dAdr^*\right)^{\frac{1}{2}}dt\\
\leq& C\sup_{t_0\leq t\leq t_*}\left(\int J_\mu^{Z,w^Z}\left(\Psi\right)n_t^\mu dVol_t\right)^{\frac{1}{2}}\\
&\times\int_{t_0}^{t_*}\left(\int_{-\frac{t}{2}}^{\frac{t}{2}}\int_{\mathbb{S}^2}\frac{\left(\log r\right)_+^2}{r^2}\left(\left(\partial_{r^*}\Phi\right)^2+|\not\nabla\Omega\Phi|^2\right)r^2\left(1-\mu\right)dAdr^*\right)^{\frac{1}{2}}dt\\
\leq& C\sup_{t_0\leq t\leq t_*}\left(\int J_\mu^{Z,w^Z}\left(\Psi\right)n_t^\mu dVol_t\right)^{\frac{1}{2}}\\
&\times\left(\sum_{i=0}^{n-1}t_i^{\frac{1}{2}}\left(\int_{t_i}^{t_{i+1}}\int_{-\frac{t}{2}}^{\frac{t}{2}}\frac{\left(\log r\right)_+^2}{r^2}\left(\left(\partial_{r^*}\Phi\right)^2+|\not\nabla\Omega\Phi|^2\right)r^2\left(1-\mu\right)dAdr^*dt\right)^{\frac{1}{2}}\right)\\
\leq& C\sup_{t_0\leq t\leq t_*}\left(\int J_\mu^{Z,w^Z}\left(\Psi\right)n_t^\mu dVol_t\right)^{\frac{1}{2}}\left(\sum_{i=0}^{n-1}\sum_{k=0}^{2}t_i^{-\frac{1}{2}}E_0\left(\Omega^k\Phi\right)\right)^{\frac{1}{2}}\\
\leq& C\sup_{t_0\leq t\leq t_*}\left(\int J_\mu^{Z,w^Z}\left(\Psi\right)n_t^\mu dVol_t\right)^{\frac{1}{2}}\left(\sum_{k=0}^{2}E_0\left(\Omega^k\Phi\right)\right)^{\frac{1}{2}}.
\end{aligned}
$$

By Proposition 8.2.1.3 and Corollary 8.3.5.1 and using the fact that $(1-\mu) \leq Ce^{cr^*}$,

$$
\begin{aligned}
&|\int_{t_0}^{t_*}\int_{-\infty}^{-\frac{t}{2}} \frac{tr^*(1-\mu)}{2r}\Psi\Box_g\Psi dVol| \\
\leq& C\int_{t_0}^{t_*}\left(\int_{-\infty}^{-\frac{t}{2}}\int_{\mathbb{S}^2}\frac{t^2}{r^2}\Psi^2 r^2(1-\mu)\,dAdr^*\right)^{\frac{1}{2}} \\
&\times\left(\int_{-\infty}^{-\frac{t}{2}}\int_{\mathbb{S}^2}(r^*)^4\left((\partial_{r^*}\Phi)^2+(1-\mu)\left|\nabla\!\!\!/\Omega\Phi\right|^2\right)r^2(1-\mu)\,dAdr^*\right)^{\frac{1}{2}}dt \\
\leq& C\sup_{t_0\leq t\leq t_*}\left(\int J_\mu^{Z,w^Z}(\Psi)\,n_t^\mu dVol_t\right)^{\frac{1}{2}}\int_{t_0}^{t_*}e^{-ct}\left(\sum_{k=0}^{1}\int J_\mu^T\left(\Omega^k\Phi\right)n_t^\mu dVol_t\right)^{\frac{1}{2}}dt \\
\leq& C\sup_{t_0\leq t\leq t_*}\left(\int J_\mu^{Z,w^Z}(\Psi)\,n_t^\mu dVol_t\right)^{\frac{1}{2}}\left(\sum_{k=0}^{1}E_0\left(\Omega^k\Phi\right)\right)^{\frac{1}{2}}.
\end{aligned}
$$

The estimation of the second error term is slightly more involved because there is a factor of t^2 in the integrand. In particular, even near spacelike infinity, one needs to use estimates for the spacetime integral for Φ. We intend to estimate this term separately in three regions as above. However, for technical reasons, we will divide the regions slightly differently. Divide as usual the interval into $t_0 \leq t_1 \leq \cdots \leq t_n = t_*$. We then set the three regions to be $\bigcup_{i=0}^{n-1}\{t_i \leq t \leq t_{i+1}, r^* > \frac{t_i}{2}\}$, $\bigcup_{i=0}^{n-1}\{t_i \leq t \leq t_{i+1}, -\frac{t_i}{2} \leq r^* \leq \frac{t_i}{2}\}$, $\bigcup_{i=0}^{n-1}\{t_i \leq t \leq t_{i+1}, r^* < -\frac{t_i}{2}\}$.

In the region $\bigcup_{i=0}^{n-1}\{t_i \leq t \leq t_{i+1}, r^* > \frac{t_i}{2}\}$, we estimate one power of t by that in $J^{Z,w^Z}(\Phi)$ and the other is canceled with the decay in r. To achieve this we use

Proposition 8.2.1.3, Theorem 8.3.1.1, 8.3.1.2 and Proposition 8.3.4,

$$
\begin{aligned}
&|\sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}}\int_{\frac{t_i}{2}}^{\infty}\left(u^2\partial_u\Psi+v^2\partial_v\Psi\right)\Box_g\Psi dVol|\\
\leq& C\sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}}\left(\int_{\frac{t_i}{2}}^{\infty}\int_{\mathbb{S}^2}\left(u^2\left(\partial_u\Psi\right)^2+v^2\left(\partial_v\Psi\right)^2\right)r^2dAdr^*\right)^{\frac{1}{2}}\\
&\times\left(\int_{\frac{t}{2}}^{\infty}\int_{\mathbb{S}^2}\frac{(\log r)_+^2}{r^2}\left(\left(\partial_{r^*}\Phi\right)^2+|\not\nabla\Omega\Phi|^2\right)r^2dAdr^*\right)^{\frac{1}{2}}dt\\
\leq& C\sup_{t_0\leq t\leq t_*}\left(\int J_\mu^{Z,w^Z}\left(\Psi\right)n_t^\mu dVol_t\right)^{\frac{1}{2}}\\
&\times\left(\sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}}\left(\int_{\frac{t_i}{2}}^{\infty}\int_{\mathbb{S}^2}\frac{(\log r)_+^2}{r^2}\left(\left(\partial_{r^*}\Phi\right)^2+|\not\nabla\Omega\Phi|^2\right)r^2dAdr^*\right)^{\frac{1}{2}}dt\right)\\
\leq& C\sup_{t_0\leq t\leq t_*}\left(\int J_\mu^{Z,w^Z}\left(\Psi\right)n_t^\mu dVol_t\right)^{\frac{1}{2}}\\
&\times\left(\sum_{i=0}^{n-1}t_i^{\frac{1}{2}}\int_{t_i}^{t_{i+1}}\int_{\frac{t_i}{2}}^{\infty}\int_{\mathbb{S}^2}\frac{1}{r^{2-\frac{\delta}{2}}}\left(\left(\partial_{r^*}\Phi\right)^2+|\not\nabla\Omega\Phi|^2\right)r^2dAdr^*dt\right)^{\frac{1}{2}}\\
\leq& C\sup_{t_0\leq t\leq t_*}\left(\int J_\mu^{Z,w^Z}\left(\Psi\right)n_t^\mu dVol_t\right)^{\frac{1}{2}}\\
&\times\left(\sum_{i=0}^{n-1}t_i^{\frac{1}{2}}t_i^{-\frac{1}{2}+\frac{\delta}{2}}\int_{t_i}^{t_{i+1}}\int_{\frac{t_i}{2}}^{\infty}\int_{\mathbb{S}^2}\frac{1}{r^{1+\frac{\delta}{2}}}\left(\left(\partial_{r^*}\Phi\right)^2+|\not\nabla\Omega\Phi|^2\right)r^2dAdr^*dt\right)^{\frac{1}{2}}\\
\leq& C\sup_{t_0\leq t\leq t_*}\left(\int J_\mu^{Z,w^Z}\left(\Psi\right)n_t^\mu dVol_t\right)^{\frac{1}{2}}\\
&\times\left(\sum_{i=0}^{n-1}t_i^{\frac{\delta}{2}}\left(\int J_\mu^T\left(\Phi\right)n_{t_i}^\mu dVol_{t_i}+\int J_\mu^T\left(\Phi\right)n_{t_{i+1}}^\mu dVol_{t_{i+1}}\right)^{\frac{1}{2}}\right)\\
\leq& C\sup_{t_0\leq t\leq t_*}\left(\int J_\mu^{Z,w^Z}\left(\Psi\right)n_t^\mu dVol_t\right)^{\frac{1}{2}}\left(\int J_\mu^T\left(\Phi\right)n_{t_0}^\mu dVol_{t_0}\right)^{\frac{1}{2}}\left(\sum_{i=0}^{n-1}t_i^{\frac{\delta}{2}}\right)\\
\leq& Ct_*^{\frac{\delta}{2}}\sup_{t_0\leq t\leq t_*}\left(\int J_\mu^{Z,w^Z}\left(\Psi\right)n_t^\mu dVol_t\right)^{\frac{1}{2}}\left(\int J_\mu^T\left(\Phi\right)n_{t_0}^\mu dVol_{t_0}\right)^{\frac{1}{2}}.
\end{aligned}
$$

For the region $\bigcup_{i=0}^{n-1}\{t_i\leq t\leq t_{i+1},-\frac{t_i}{2}\leq r^*\leq\frac{t_i}{2}\}$, we first rewrite into (t,r^*)-coordinates and then perform an integration by parts in t. It is to avoid extra

boundary terms during this integration by parts that we have divided our regions differently from before. The reason that we perform this integration by parts is that instead of a spacetime integral term with $\partial_t \Psi$, we would prefer a term with Ψ, which can then be controlled by the integral of $|K^{\tilde{X},w^{\tilde{X}}}| + \sum_l K^{X_l,w^{X_l}}$.

$$
\begin{aligned}
&|\sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \int_{-\frac{t_i}{2}}^{\frac{t_i}{2}} \left(u^2 \partial_u \Psi + v^2 \partial_v \Psi\right) \Box_g \Psi dVol| \\
\leq & C \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \int_{-\frac{t_i}{2}}^{\frac{t_i}{2}} \int_{\mathbb{S}^2} |t r^* \partial_{r^*} \Psi \Box_g \Psi| r^2 \left(1-\mu\right) dA dr^* dt \\
&+ C| \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \int_{-\frac{t_i}{2}}^{\frac{t_i}{2}} \int_{\mathbb{S}^2} \left(t^2 + (r^*)^2\right) \partial_t \Psi \Box_g \Psi r^2 \left(1-\mu\right) dA dr^* dt| \\
\leq & C \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \int_{-\frac{t_i}{2}}^{\frac{t_i}{2}} \int_{\mathbb{S}^2} |t r^* \partial_{r^*} \Psi \Box_g \Psi| r^2 \left(1-\mu\right) dA dr^* dt \\
&+ C \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \int_{-\frac{t_i}{2}}^{\frac{t_i}{2}} \int_{\mathbb{S}^2} t^2 |\Psi \Box_g \left(\partial_t \Psi\right)| r^2 \left(1-\mu\right) dA dr^* dt \\
&+ C \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \int_{-\frac{t_i}{2}}^{\frac{t_i}{2}} \int_{\mathbb{S}^2} t |\Psi \Box_g \Psi| r^2 \left(1-\mu\right) dA dr^* dt \\
&+ C \sum_{i=0}^{n-1} \int_{-\frac{t_i}{2}}^{\frac{t_i}{2}} t^2 |\Psi \Box_g \Psi| \sqrt{1-\mu} dVol_{t_i} + C \sum_{i=0}^{n-1} \int_{-\frac{t_i}{2}}^{\frac{t_i}{2}} t^2 |\Psi \Box_g \Psi| \sqrt{1-\mu} dVol_{t_{i+1}} \\
\leq & C \int_{t_0}^{t_*} \int_{-\frac{t}{2}}^{\frac{t}{2}} \int_{\mathbb{S}^2} |t r^* \partial_{r^*} \Psi \Box_g \Psi| r^2 \left(1-\mu\right) dA dr^* dt \\
&+ C \int_{t_0}^{t_*} \int_{-\frac{t}{2}}^{\frac{t}{2}} \int_{\mathbb{S}^2} t^2 |\Psi \Box_g \left(\partial_t \Psi\right)| r^2 \left(1-\mu\right) dA dr^* dt \\
&+ C \int_{t_0}^{t_*} \int_{-\frac{t}{2}}^{\frac{t}{2}} \int_{\mathbb{S}^2} t |\Psi \Box_g \Psi| r^2 \left(1-\mu\right) dA dr^* dt + C \sum_{i=0}^{n} \int_{-\frac{t_i}{2}}^{\frac{t_i}{2}} t_i^2 |\Psi \Box_g \Psi| \sqrt{1-\mu} dVol_{t_i}.
\end{aligned}
$$

We now group this into three parts: firstly, the spacetime term that grows like t^2; secondly, the spacetime terms that grow like t; and finally, the boundary terms.

By Proposition 8.3.3, 8.3.4, Proposition 8.2.1.4, 8.2.1.5, Theorem 8.3.1.1 and 8.3.1.2,

$$
\begin{aligned}
&\int_{t_0}^{t_*}\int_{-\frac{t}{2}}^{\frac{t}{2}}\int_{\mathbb{S}^2} t^2|\Psi\Box_g\left(\partial_t\Psi\right)|r^2\left(1-\mu\right)dAdr^*dt\\
\leq& C\int_{t_0}^{t_*}\int_{-\frac{t}{2}}^{\frac{t}{2}}\frac{t^2\left(\log r\right)_+\left(1+|r^*|\right)|\Psi|\left(|\partial_{r^*}\Phi_t|+|\not\nabla\Omega\Phi_t|\right)}{r^3}\\
\leq& C\left(\int_{t_0}^{t_*}\int_{-\frac{t}{2}}^{\frac{t}{2}} t^{2+\delta}r^{\delta}\frac{r^{1-\frac{\delta}{4}}\Psi^2}{\left(1+|r^*|\right)^4}dVol\right)^{\frac{1}{2}}\\
&\times\left(\sum_{i=0}^{n-1}t_i^{1-\frac{\delta}{2}}\left(\int_{t_i}^{t_{i+1}}\int_{-\frac{t}{2}}^{\frac{t}{2}}\frac{\left(1+|r^*|\right)^6\left(\left(\partial_{r^*}\Phi_t\right)^2+|\not\nabla\Omega\Phi_t|^2\right)}{r^{7+\frac{\delta}{4}}}dVol\right)^{\frac{1}{2}}\right)\\
\leq& C\left(\int_{t_0}^{t_*}\int_{-\frac{t}{2}}^{\frac{t}{2}} t^{2+\delta}r^{\delta}\left(|K^{\tilde{X},w^{\tilde{X}}}\left(\Psi\right)|+\sum_l K^{X_l,w^{X_l}}\left(\Psi_l\right)\right)dVol\right)^{\frac{1}{2}}\\
&\times\left(\sum_{i=0}^{n-1}t_i^{-\frac{\delta}{2}}\right)\left(\sum_{m=0}^{1}\sum_{k=0}^{2}E_0\left(\partial_t^m\Omega^k\Phi\right)\right)^{\frac{1}{2}}\\
\leq& C\left(\int_{t_0}^{t_*}\int_{-\frac{t}{2}}^{\frac{t}{2}} t^{2+2\delta}\left(|K^{\tilde{X},w^{\tilde{X}}}\left(\Psi\right)|+\sum_l K^{X_l,w^{X_l}}\left(\Psi_l\right)\right)dVol\right)^{\frac{1}{2}}\\
&\times\left(\sum_{m=0}^{1}\sum_{k=0}^{2}E_0\left(\partial_t^m\Omega^k\Phi\right)\right)^{\frac{1}{2}}.
\end{aligned}
$$

By Proposition 8.3.3, 8.3.4, Proposition 8.2.1.4, 8.2.1.5, Corollary 8.3.5.4 and 8.3.5.5,

$$
\begin{aligned}
&\int_{t_0}^{t_*}\int_{-\frac{t}{2}}^{\frac{t}{2}}\int_{\mathbb{S}^2}|tr^*\partial_{r^*}\Psi\Box_g\Psi|r^2\left(1-\mu\right)dAdr^*dt\\
&+\int_{t_0}^{t_*}\int_{-\frac{t}{2}}^{\frac{t}{2}}\int t|\Psi\Box_g\Psi|r^2\left(1-\mu\right)dAdr^*dt\\
\leq& C\left(\int_{t_0}^{t_*}\int_{-\frac{t}{2}}^{\frac{t}{2}}\frac{t\left(\log r\right)_+\left(1+|r^*|\right)\left(|r^*\partial_{r^*}\Psi|+|\Psi|\right)\left(|\partial_{r^*}\Phi|+|\not{\nabla}\Omega\Phi|\right)}{r^3}dVol\right)\\
\leq& C\left(\int_{t_0}^{t_*}\int_{-\frac{t}{2}}^{\frac{t}{2}}t^\delta r^\delta\left(\frac{r^{1-\frac{\delta}{4}}\Psi^2}{\left(1+|r^*|\right)^4}+\frac{\left(\partial_{r^*}\Psi\right)^2}{r^{1+\frac{\delta}{4}}}\right)dVol\right)^{\frac{1}{2}}\\
&\times\left(\sum_{i=0}^{n-1}t_i^{1-\frac{\delta}{2}}\left(\int_{t_i}^{t_{i+1}}\int_{-\frac{t}{2}}^{\frac{t}{2}}\frac{\left(1+|r^*|\right)^6\left(\left(\partial_{r^*}\Phi\right)^2+|\not{\nabla}\Omega\Phi|^2\right)}{r^{7+\frac{\delta}{4}}}dVol\right)^{\frac{1}{2}}\right)\\
\leq& C\left(\int_{t_0}^{t_*}\int_{-\frac{t}{2}}^{\frac{t}{2}}t^\delta r^\delta\left(|K^{\tilde{X},w^{\tilde{X}}}\left(\Psi\right)|+\sum_l K^{X_l,w^{X_l}}\left(\Psi_l\right)\right)dVol\right)^{\frac{1}{2}}\\
&\times\left(\sum_{k=0}^{2}E_0\left(\Omega^k\Phi\right)\right)^{\frac{1}{2}}\left(\sum_{i=0}^{n-1}t_i^{-\frac{\delta}{2}}\right)\\
\leq& C\left(\int_{t_0}^{t_*}\int_{-\frac{t}{2}}^{\frac{t}{2}}t^{2\delta}\left(|K^{\tilde{X},w^{\tilde{X}}}\left(\Psi\right)|+\sum_l K^{X_l,w^{X_l}}\left(\Psi_l\right)\right)dVol\right)^{\frac{1}{2}}\left(\sum_{k=0}^{2}E_0\left(\Omega^k\Phi\right)\right)^{\frac{1}{2}}.
\end{aligned}
$$

By Proposition 8.2.1.1, 8.2.1.3 and Theorem 8.3.1.4,

$$
\begin{aligned}
&\sum_{i=0}^{n}\int_{-\frac{t_i}{2}}^{\frac{t_i}{2}}t_i^2|\Psi\Box_g\Psi|\sqrt{1-\mu}dVol_{t_i}\\
\leq& C\left(\sum_{i=0}^{n}t_i\left(\int J_\mu^{Z,w^Z}\left(\Psi\right)n_{t_i}^\mu dVol_{t_i}\right)^{\frac{1}{2}}\left(\sum_{k=0}^{1}\int_{-\frac{t_i}{2}}^{\frac{t_i}{2}}J_\mu^T\left(\Omega^k\Phi\right)n_{t_i}^\mu dVol_{t_i}\right)^{\frac{1}{2}}\right)\\
\leq& C\sup_{t_0\leq t\leq t_*}\left(\int J_\mu^{Z,w^Z}\left(\Psi\right)n_t^\mu dVol_t\right)^{\frac{1}{2}}\left(\sum_{k=0}^{1}E_0\left(\Omega^k\Phi\right)\right)^{\frac{1}{2}}\left(\sum_{i=0}^{n}1\right)\\
\leq& Ct_*^{\frac{\delta}{2}}\sup_{t_0\leq t\leq t_*}\left(\int J_\mu^{Z,w^Z}\left(\Psi\right)n_t^\mu dVol_t\right)^{\frac{1}{2}}\left(\sum_{k=0}^{1}E_0\left(\Omega^k\Phi\right)\right)^{\frac{1}{2}}.
\end{aligned}
$$

These together give:

$$
\begin{aligned}
&|\int_{t_0}^{t_*}\int_{-\frac{t}{2}}^{\frac{t}{2}}\left(u^2\partial_u\Psi+v^2\partial_v\Psi\right)\Box\Psi dVol|\\
\leq& C\left(\int_{t_0}^{t_*}t^{2+2\delta}\left(|K^{\tilde{X},w^{\tilde{X}}}(\Psi)|+\sum_l K^{X_l,w^{X_l}}(\Psi_l)\right)dVol\right)^{\frac{1}{2}}\sum_{m=0}^{1}\sum_{k=0}^{2}E_0\left(\partial_t^m\Omega^k\Phi\right)^{\frac{1}{2}}\\
&+Ct_*^{\frac{\delta}{2}}\sup_{t_0\leq t\leq t_*}\left(\int J_\mu^{Z,w^Z}(\Psi)n_t^\mu dVol_t\right)^{\frac{1}{2}}\left(\sum_{k=0}^{1}E_0\left(\Omega^k\Phi\right)\right)^{\frac{1}{2}}.
\end{aligned}
$$

We finally look at the third region, $\bigcup_{i=0}^{n-1}\{t_i\leq t\leq t_{i+1}, r^*<-\frac{t_i}{2}\}$, for the second error term. By Proposition 8.2.1.1, 8.2.1.3 and Theorem 8.3.1.1.

$$
\begin{aligned}
&|\int_{t_0}^{t_*}\int_{-\infty}^{-\frac{t}{2}}\left(u^2\partial_u\Psi+v^2\partial_v\Psi\right)\Box\Psi dVol|\\
\leq& C\int_{t_0}^{t_*}\left(\int_{-\infty}^{-\frac{t}{2}}\left(u^2\left(\partial_u\Psi\right)^2+v^2\left(\partial_v\Psi\right)^2\right)(1-\mu)^{-\frac{1}{2}}dVol_t\right)\\
&\times\left(\int_{-\infty}^{-\frac{t}{2}}(r^*)^2(1-\mu)^{\frac{3}{2}}\left((\partial_{r^*}\Phi)^2+|\not\nabla\Omega\Phi|^2\right)dVol_t\right)dt\\
\leq& C\int_{t_0}^{t_*}e^{-ct}\left(\int_{-\infty}^{-\frac{t}{2}}J_\mu^Z(\Psi)n_t^\mu dVol_t\right)\left(\sum_{k=0}^{1}\int_{-\infty}^{-\frac{t}{2}}J_\mu^T\left(\Omega^k\Phi\right)n_t^\mu dVol_t\right)dt\\
\leq& C\sup_{t_0\leq t\leq t_*}\left(\int J_\mu^{Z,w^Z}(\Psi)n_t^\mu dVol_t\right)^{\frac{1}{2}}\left(E_0(\Phi)\right)^{\frac{1}{2}}.
\end{aligned}
$$

Therefore,

$$\begin{aligned}
&\int J_{\mu}^{Z,w^Z}(\Psi)\, n_{t_*}^{\mu} dVol_{\{t=t_*\}} \\
\leq & C\int J_{\mu}^{Z,w^Z}(\Psi)\, n_{t_0}^{\mu} dVol_{t_0} + \sum_{k=0}^{1}\int_{t_0}^{t_*}\int_{r_1^*}^{r_2^*} t \sum_l K^{X_l,w^{X_l}}\left(\Omega^k \Psi_l\right) dVol \\
&+ C\left(\int_{t_0}^{t_*}\int_{-\frac{t}{2}}^{\frac{t}{2}} t^{2+2\delta}\left(|K^{\tilde{X},w^{\tilde{X}}}(\Psi)| + \sum_l K^{X_l,w^{X_l}}(\Psi_l)\right) dVol\right)^{\frac{1}{2}} \\
&\qquad \times \left(\sum_{m=0}^{1}\sum_{k=0}^{2} E_0\left(\partial_t^m \Omega^k \Phi\right)\right)^{\frac{1}{2}} \\
&+ C \sup_{t_0\leq t\leq t_*} t_*^{\frac{\delta}{2}}\left(\int J_{\mu}^{Z,w^Z}(\Psi)\, n_t^{\mu} dVol_t\right)^{\frac{1}{2}}\left(\sum_{k=0}^{2} E_0\left(\Omega^k\Phi\right)\right)^{\frac{1}{2}}.
\end{aligned}$$

The proof concludes with Lemma 8.4.3, taking

$$\begin{aligned}
h_1(t_*) =& \sum_{k=0}^{1}\int_{t_0}^{t_*}\int_{r_1^*}^{r_2^*} t\sum_l K^{X_l,w^{X_l}}\left(\Omega^k\Psi_l\right) dVol \\
&+ \left(\int_{t_0}^{t_*}\int_{-\frac{t}{2}}^{\frac{t}{2}} t^{2+2\delta}\left(|K^{\tilde{X},w^{\tilde{X}}}(\Psi)| + \sum_l K^{X_l,w^{X_l}}(\Psi_l)\right) dVol\right)^{\frac{1}{2}} \\
&\qquad \times \left(\sum_{m=0}^{1}\sum_{k=0}^{2} E_0\left(\partial_t^m\Omega^k\Phi\right)\right)^{\frac{1}{2}}, \\
h_2(t_*) =& t_*^{\delta}\left(\sum_{k=0}^{2} E_0\left(\Omega^k\Phi\right)\right).
\end{aligned}$$

We notice that $h_1(t)$ and $h_2(t)$ are increasing. □

We now combine Propositions 8.4.2, 8.4.4, 8.4.5 and 8.4.6 to prove Theorem 8.4.1.2. This will then imply the other parts of Theorem 8.4.1.

Proposition 8.4.7.

$$\int J_{\mu}^{Z,w^Z}(\Psi)\, n_{t_*}^{\mu} dVol_{t_*} \leq C E_1(\Phi)\, t_*^{\delta}$$

Proof. We first show that $\int J_\mu^{Z,w^Z}(\Psi)\, n_{t_*}^\mu dVol_{t_*}$ grows only like $t_*^{1+\delta}$. Using Propositions 8.4.4 and 8.4.2,

$$\begin{aligned}
&\int_{t_0}^{t_*} |K^{\tilde{X},w^{\tilde{X}}}(\Psi)| + \sum_l K^{X_l,w^X}(\Psi_l)\, dVol \\
\leq & C\left(\int J_\mu^T(\Psi)\, n_{t_*}^\mu dVol_{t_*} + \int J_\mu^T(\Psi)\, n_{t_0}^\mu dVol_{t_0}\right) + Ct_0^{-2+\delta}\sum_{k=0}^{2} E_0\left(\Omega^k\Phi\right) \\
\leq & C\int J_\mu^T(\Psi)\, n_{t_0}^\mu dVol_{t_0} + C\sum_{k=0}^{2} E_0\left(\Omega^k\Phi\right).
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\int_{t_0}^{t_*} |K^{\tilde{X},w^{\tilde{X}}}(\Omega\Psi)| + \sum_l K^{X_l,w^X}(\Omega\Psi_l)\, dVol \\
\leq & C\int J_\mu^T(\Omega\Psi)\, n_{t_0}^\mu dVol_{t_0} + C\sum_{k=0}^{3} E_0\left(\Omega^k\Phi\right).
\end{aligned}$$

Apply Proposition 8.4.6 to get

$$
\begin{aligned}
&\int J_{\mu}^{Z,w^Z}(\Psi)\, n_{t_*}^{\mu} dVol_{t_*} \\
\leq & C\int J_{\mu}^{Z,w^Z}(\Psi)\, n_{t_0}^{\mu} dVol_{t_0} + Ct_*\sum_{k=0}^{1}\int_{t_0}^{t_*}\int_{r_1^*}^{r_2^*}\sum_{l} K^{X_l,w^{X_l}}\left(\Omega^k\Psi_l\right) dVol \\
&+ Ct_*^{1+\delta}\left(\int_{t_0}^{t_*}\int_{-\frac{t}{2}}^{\frac{t}{2}} |K^{\tilde{X},w^{\tilde{X}}}(\Psi)| + \sum_{l} K^{X_l,w^{X_l}}(\Psi_l)\, dVol\right)^{\frac{1}{2}} \\
&\qquad\qquad \times\left(\sum_{m=0}^{1}\sum_{k=0}^{2} E_0\left(\partial_t^m\Omega^k\Phi\right)\right)^{\frac{1}{2}} \\
&+ Ct_*^{\delta}\sum_{k=0}^{2} E_0\left(\Omega^k\Phi\right) \\
\leq & C\int J_{\mu}^{Z,w^Z}(\Psi)\, n_{t_0}^{\mu} dVol_{t_0} + Ct_*\sum_{k=0}^{1}\int J_{\mu}^{T}\left(\Omega^k\Psi\right) n_{t_0}^{\mu} dVol_{t_0} + Ct_*\sum_{k=0}^{3} E_0\left(\Omega^k\Phi\right) \\
&+ Ct_*^{1+\delta}\left(\int J_{\mu}^{T}(\Psi)\, n_{t_0}^{\mu} dVol_{t_0}\right)^{\frac{1}{2}}\left(\sum_{m=0}^{1}\sum_{k=0}^{2} E_0\left(\partial_t^m\Omega^k\Phi\right)\right)^{\frac{1}{2}} \\
&+ Ct_*^{1+\delta}\sum_{m=0}^{1}\sum_{k=0}^{2} E_0\left(\partial_t^m\Omega^k\Phi\right) \\
\leq & C\left(\int J_{\mu}^{Z,w^Z}(\Psi)\, n_{t_0}^{\mu} dVol_{t_0}\right. \\
&\qquad \left. + t_*^{1+\delta}\left(\sum_{k=0}^{1}\int J_{\mu}^{T}\left(\Omega^k\Psi\right) n_{t_0}^{\mu} dVol_{t_0} + \sum_{m=0}^{1}\sum_{k=0}^{3-m} E_0\left(\partial_t^m\Omega^k\Phi\right)\right)\right).
\end{aligned}
$$

We now have some control over $\int J_{\mu}^{Z,w^Z}(\Psi)\, n_{t_*}^{\mu} dVol_{t_*}$ and we will use Proposition 8.4.5 to estimate the spacetime integral terms by integrating dyadically.

By Proposition 8.4.5 and 8.4.2,

$$\begin{aligned}
&\int_{t_i}^{t_{i+1}} \int_{r_1^*}^{r_2^*} \sum_l K^{X_l, w^{X_l}} (\Psi_l)\, dVol \\
\leq & C \left(t_i^{-2} \int J_\mu^Z (\Psi)\, n_{t_i}^\mu dVol_{t_i} + t_i^{-2+\delta} \left(\sum_{m=0}^{1} \sum_{k=0}^{2} E_0 \left(\partial_t^m \Omega^k \Phi \right) \right) \right) \\
\leq & C \left(t_i^{-2} \int J_\mu^Z (\Psi)\, n_{t_0}^\mu dVol_{t_0} \right. \\
& \left. + t_i^{-1+\delta} \left(\sum_{k=0}^{1} \int J_\mu^T \left(\Omega^k \Psi \right) n_{t_0}^\mu dVol_{t_0} + \sum_{m=0}^{1} \sum_{k=0}^{3-m} E_0 \left(\partial_t^m \Omega^k \Phi \right) \right) \right),
\end{aligned}$$

where here we have not kept track of the constant factor in front of δ, but just note that it can be chosen to be arbitrarily small.

We can apply the same argument to $\int_{t_i}^{t_{i+1}} \int_{r_1^*}^{r_2^*} \sum_l K^{X_l, w^{X_l}} (\Omega \Psi_l)\, dVol$ to get

$$\begin{aligned}
&\int_{t_i}^{t_{i+1}} \int_{r_1^*}^{r_2^*} \sum_l K^{X_l, w^{X_l}} (\Omega \Psi_l)\, dVol \\
\leq & C \left(t_i^{-2} \int J_\mu^Z (\Omega \Psi)\, n_{t_0}^\mu dVol_{t_0} \right. \\
& \left. + t_i^{-1+\delta} \left(\sum_{k=0}^{2} \int J_\mu^T \left(\Omega^k \Psi \right) n_{t_0}^\mu dVol_{t_0} + \sum_{m=0}^{1} \sum_{k=0}^{4-m} E_0 \left(\partial_t^m \Omega^k \Phi \right) \right) \right).
\end{aligned}$$

This in turn provides more control on $\int J_{\mu}^{Z,w^Z}(\Psi)\, n_{t_*}^{\mu} dVol_{t_*}$ by Proposition 8.4.6:

$$
\begin{aligned}
&\int J_{\mu}^{Z,w^Z}(\Psi)\, n_{t_*}^{\mu} dVol_{t_*} \\
\leq & C\int J_{\mu}^{Z,w^Z}(\Psi)\, n_{t_0}^{\mu} dVol_{t_0} + C\sum_{k=0}^{1}\sum_{i=0}^{n-1} t_i \int_{t_i}^{t_{i+1}} \int_{r_1^*}^{r_2^*} \sum_l K^{X_l,w^{X_l}}\left(\Omega^k \Psi_l\right) dVol \\
&+ C\sum_{i=0}^{n-1} t_i^{1+\delta} \left(\int_{t_i}^{t_{i+1}} \int_{-\frac{t}{2}}^{\frac{t}{2}} |K^{\tilde{X},w^{\tilde{X}}}(\Psi)| + \sum_l K^{X_l,w^{X_l}}(\Psi_l)\, dVol \right)^{\frac{1}{2}} \\
&\qquad \times \left(\sum_{m=0}^{1}\sum_{k=0}^{2} E_0\left(\partial_t^m \Omega^k \Phi\right) \right)^{\frac{1}{2}} \\
&+ Ct_*^{\delta} \sum_{k=0}^{2} E_0\left(\Omega^k \Phi\right) \\
\leq & C\sum_{k=0}^{1} \int J_{\mu}^{Z,w^Z}\left(\Omega^k \Psi\right) n_{t_0}^{\mu} dVol_{t_0} + Ct_*^{\delta}\left(E_0(\Psi) + \sum_{m=0}^{1}\sum_{k=0}^{4-m} E_0\left(\partial_t^m \Omega^k \Phi\right) \right) \\
&+ C\sum_{i=0}^{n-1} t_i^{1+\delta} \left(\int_{t_i}^{t_{i+1}} \int_{-\frac{t}{2}}^{\frac{t}{2}} |K^{\tilde{X},w^{\tilde{X}}}(\Psi)| + \sum_l K^{X_l,w^{X_l}}(\Psi_l)\, dVol \right)^{\frac{1}{2}} \\
&\qquad \times \left(\sum_{m=0}^{1}\sum_{k=0}^{2} E_0\left(\partial_t^m \Omega^k \Phi\right) \right)^{\frac{1}{2}} \\
\leq & CE_1(\Phi)\, t_*^{\delta} \\
&+ C\sum_{i=0}^{n-1} t_i^{1+\delta} \left(\int_{t_i}^{t_{i+1}} \int_{-\frac{t}{2}}^{\frac{t}{2}} |K^{\tilde{X},w^{\tilde{X}}}(\Psi)| + \sum_l K^{X_l,w^{X_l}}(\Psi_l)\, dVol \right)^{\frac{1}{2}} \\
&\qquad \times \left(\sum_{m=0}^{1}\sum_{k=0}^{2} E_0\left(\partial_t^m \Omega^k \Phi\right) \right)^{\frac{1}{2}}.
\end{aligned}
$$

Here, we recall that we have defined $E_1(\Phi) = E_0(\Psi) + \sum_{m=0}^{1}\sum_{k=0}^{4-m} E_0\left(\partial_t^m \Omega^k \Phi\right)$ in Section 1.3.

Clearly, we can replace δ by ϵ with a different constant C which depends only on ϵ:

$$\begin{aligned}
&\int J_{\mu}^{Z,w^Z}(\Psi)\, n_{t_*}^{\mu} dVol_{t_*} \\
\leq & C E_1(\Phi)\, t_*^{\epsilon} \\
&+ C\sum_{i=0}^{n-1} t_i^{1+\epsilon}\left(\int_{t_i}^{t_{i+1}}\int_{-\frac{t}{2}}^{\frac{t}{2}} |K^{\tilde{X},w^{\tilde{X}}}(\Psi)| + \sum_l K^{X_l,w^{X_l}}(\Psi_l)\, dVol\right)^{\frac{1}{2}} \\
&\qquad \times \left(\sum_{m=0}^{1}\sum_{k=0}^{2} E_0\left(\partial_t^m \Omega^k \Phi\right)\right)^{\frac{1}{2}}.
\end{aligned} \tag{8.3}$$

Notice that at this point, the only term that exhibits more growth than expected is the second term in 8.3. We will close the argument with a bootstrap.

For notational purposes, we define

$$\begin{aligned}
I_{t_*} &= \int J_{\mu}^{Z,w^Z}(\Psi)\, n_{t_*}^{\mu} dVol_{t_*}, \\
II_{t_i} &= \int_{t_i}^{t_{i+1}}\int_{-\frac{t}{2}}^{\frac{t}{2}} |K^{\tilde{X},w^{\tilde{X}}}(\Psi)| + \sum_l K^{X_l,w^{X_l}}(\Psi_l)\, dVol,
\end{aligned}$$

(8.3) is equivalent to

$$I_{t_*} \leq C\left(t_*^{\epsilon} E_1(\Phi) + \sum_{i=0}^{n-1} t_i^{1+\epsilon}\left(II_{t_i}\right)^{\frac{1}{2}} E_1^{\frac{1}{2}}(\Phi)\right). \tag{8.4}$$

On the other hand, Proposition 8.4.5 gives

$$II_{t_i} \leq C\left(t_i^{-2} I_{t_i} + t_i^{-2+\delta} E_1(\Phi)\right). \tag{8.5}$$

Assume $I_t \leq A t^{\delta} E_1(\Phi)$, where $A \geq 4C$. We want to show that $I_t \leq \frac{A}{2} t^{\delta} E_1(\Phi)$ for all $t \geq \left(\frac{400 C^{\frac{1}{2}}\left(A^{\frac{1}{2}}+1\right)}{A}\right)^{\frac{4}{\delta}}$. From the assumption and (8.5) we have

$$II_{t_i} \leq C\left(A t_i^{-2+\delta} E_1(\Phi) + t_i^{-2+\delta} E_1(\Phi)\right).$$

Hence, by picking $\epsilon = \frac{\delta}{4}$ in (8.4),

$$\begin{aligned} I_t \leq & Ct^{\frac{\delta}{4}} E_1(\Phi) + 100C^{\frac{1}{2}} \left(A^{\frac{1}{2}} + 1\right) t^{\frac{3\delta}{4}} E_1(\Phi) \\ \leq & Ct^{\delta} E_1(\Phi) + 100C^{\frac{1}{2}} \left(A^{\frac{1}{2}} + 1\right) t^{\frac{3\delta}{4}} E_1(\Phi) \\ \leq & \frac{A}{2} t^{\delta} E_1(\Phi), \end{aligned}$$

since $A \geq 4C$ and $t \geq \left(\frac{400C^{\frac{1}{2}}\left(A^{\frac{1}{2}}+1\right)}{A}\right)^{\frac{4}{\delta}}$. □

Remark 12. *We would like to note that the number of derivatives used in the above argument is highly wasteful (we used a total of 8 derivatives!). Blue-Soffer constructed a vector field to control trapping with only ϵ derivatives [4]. Therefore, we can, at least in principle, repeat the above argument noting the unnecessary loss of derivatives. The details, however, have not been pursued. It is likely that with this vector field, Theorem 8.3.1 holds with $E_0(\Phi)$ only having $1+\epsilon$ derivatives. Moreover, in Proposition 8.4.2-8.4.6, instead of having two Ω derivatives on Φ, one only needs $1+\epsilon$ of them. One can then go to the proof of Proposition 8.4.7 and reprove it assuming only that Φ is in $H^{2+\epsilon}$ initially with suitable decay.*

Now Theorem 8.4.1 follows directly from Propositions 8.4.7 and 8.4.5.

8.5 Estimates near the Event Horizon

In this section, we will use the vector field Y to prove that any decay estimates that can be proved on a suitable compact set holds also along the horizon. We will also show that these estimates control enough derivatives to give pointwise decay estimate.

Proposition 8.5.1. *Suppose*

$$\int_{r_0^*}^{2((1.2)r_0)^* - r_0^*} J_\mu^T(\Phi) n_t^\mu dVol_t \leq Bt^{-\alpha} \quad \text{for all } t, \text{ for some } \alpha \geq 0.$$

Then

$$\begin{aligned}
&\int_{\{r\leq r_0\}} J_\mu^Y(\Phi)\, n^\mu_{\frac{1}{2}\left(t_*+r_0^*\right)} dVol_{\{v=\frac{1}{2}\left(t_*+r_0^*\right)\}} \\
&+\int_{\{\frac{1}{2}\left(t_*+r_0^*\right)\leq v\leq \frac{1}{2}\left(t_*+1+r_0^*\right)\}} J_\mu^Y(\Phi)\, n^\mu_\infty dVol_{\{u=\infty\}} \\
\leq & C\left(B+\int J_\mu^N(\Phi)\, n^\mu_{t_0} dVol_{t_0}\right) t_*^{-\alpha}.
\end{aligned}$$

Remark 13. *The reader should think of B as some energy quantity of the initial data. For example, as we will show later, the hypothesis of this Proposition holds for* $B = C\sum_{m+k\leq 1} E_1\left(\partial_t^m \Omega^k \Phi\right)$.

Proof. Apply Proposition 1 for Y, on the region

$$\mathcal{R} = \{\frac{1}{2}(t_1+r_0^*)\leq v\leq \frac{1}{2}(t_*+r_0^*), t\geq t_1\}$$

as in the figure, we get

$$\begin{aligned}
&\int_{\{t\geq t_1\}} J_\mu^Y(\Phi)\, n^\mu_{\frac{1}{2}\left(t_*+r_0^*\right)} dVol_{\{v=\frac{1}{2}\left(t_*+r_0^*\right)\}} \\
&+\int_{\{\frac{1}{2}\left(t_1+r_0^*\right)\leq v\leq \frac{1}{2}\left(t_*+r_0^*\right)\}} J_\mu^Y(\Phi)\, n^\mu_\infty dVol_{\{u=\infty\}} + \int_{\mathcal{R}} K^Y(\Phi)\, dVol \\
=&\int_{\{t\geq t_1\}} J_\mu^Y(\Phi)\, n^\mu_{\frac{1}{2}\left(t_1+r_0^*\right)} dVol_{\{v=\frac{1}{2}\left(t_1+r_0^*\right)\}} \\
&+\int_{\{\frac{1}{2}\left(t_1+r_0^*\right)\leq v\leq \frac{1}{2}\left(t_*+r_0^*\right)\}} J_\mu^Y(\Phi)\, n^\mu_{t_1} dVol_{t_1}.
\end{aligned} \tag{8.6}$$

We split up the integrals into $r\leq r_0$ and $r>r_0$ parts.

Notice that the domain of integration of $\int_{\{t\geq t_1\}} J_\mu^Y(\Phi)\, n^\mu_{\frac{1}{2}\left(t_1+r_0^*\right)} dVol_{\{v=\frac{1}{2}\left(t_1+r_0^*\right)\}}$ lies inside $\{r\leq r_0\}$. Moreover, we note that $\int_{\{\frac{1}{2}\left(t_1+r_0^*\right)\leq v\leq \frac{1}{2}\left(t_*+r_0^*\right)\}} J_\mu^Y(\Phi)\, n^\mu_\infty dVol_{\{u=\infty\}} \geq$

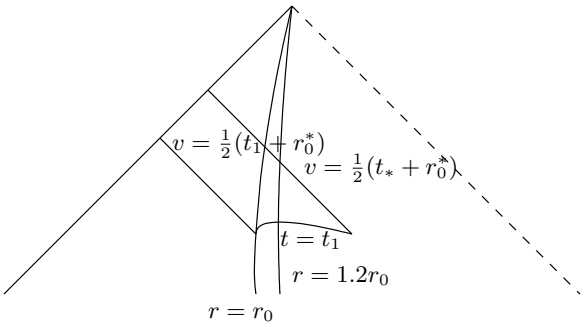


Figure 8.1: The region $\mathcal{R}$

0. Hence

$$\begin{aligned}
&\int_{\{r\leq r_0\}} J_\mu^Y(\Phi)\, n^\mu_{\frac{1}{2}\left(t_*+r_0^*\right)} dVol_{\{v=\frac{1}{2}\left(t_*+r_0^*\right)\}} + \int_{\mathcal{R}\cap\{r\leq r_0\}} K^Y(\Phi)\, dVol \\
\leq &\int_{\{r\leq r_0\}} J_\mu^Y(\Phi)\, n^\mu_{\frac{1}{2}\left(t_1+r_0^*\right)} dVol_{\{v=\frac{1}{2}\left(t_1+r_0^*\right)\}} + \int_{\{\frac{1}{2}\left(t_1+r_0^*\right)\leq v\leq \frac{1}{2}\left(t_*+r_0^*\right)\}} J_\mu^Y(\Phi)\, n^\mu_{t_1} dVol_{t_1} \\
&+ \int_{\{r\geq r_0\}\cap\{t\geq t_1\}} J_\mu^Y(\Phi)\, n^\mu_{\frac{1}{2}\left(t_*+r_0^*\right)} dVol_{\{v=\frac{1}{2}\left(t_*+r_0^*\right)\}} + \int_{\mathcal{R}\cap\{r\geq r_0\}} |K^Y(\Phi)|\, dVol.
\end{aligned}$$

We estimate three terms on the right hand side. Notice that Y is constructed to be supported in $\{r \leq (1.2)\, r_0\}$.

$$\begin{aligned}
&\int_{\{\frac{1}{2}\left(t_1+r_0^*\right)\leq v\leq \frac{1}{2}\left(t_*+r_0^*\right)\}} J_\mu^Y(\Phi)\, n^\mu_{t_1} dVol_{t_1} \\
\leq &C\int_{r_0^*}^{((1.2)r_0)^*} J_\mu^T(\Phi)\, n^\mu_{t_1} dVol_{t_1} \\
\leq &CBt_1^{-\alpha},
\end{aligned}$$

For the second and third term, we first use the compact support of Y and then apply

the conservation law associated to the Killing vector T.

$$\begin{aligned}
&\int_{\{r\geq r_0\}\cap\{u\geq\frac{1}{2}\left(t_1-r_0^*\right)\}} J_\mu^Y\left(\Phi\right) n^\mu_{\frac{1}{2}\left(t_*+r_0^*\right)} dVol_{\{v=\frac{1}{2}\left(t_*+r_0^*\right)\}}\\
\leq & C\int_{\{r_0\leq r\leq(1.2)r_0\}\cap\{u\geq\frac{1}{2}\left(t_1-r_0^*\right)\}} J_\mu^T\left(\Phi\right) n^\mu_{\frac{1}{2}\left(t_*+r_0^*\right)} dVol_{\{v=\frac{1}{2}\left(t_*+r_0^*\right)\}}\\
\leq & C\int_{\{r_0^*\leq r^*\leq 2((1.2)r_0)^*-r_0^*\}} J_\mu^T\left(\Phi\right) n^\mu_{t_1} dVol_{t_*}\\
\leq & CBt_*^{-\alpha}\\
\leq & CBt_1^{-\alpha},
\end{aligned}$$

$$\begin{aligned}
&\int_{\mathcal{R}\cap\{r\geq r_0\}} \left|K^Y\left(\Phi\right)\right| dVol\\
\leq & C\int_{t_1}^{t_*}\int_{r_0^*}^{((1.2)r_0)*} \left|K^Y\left(\Phi\right)\right| dVol\\
\leq & C\int_{t_1}^{t_*}\int_{r_0^*}^{((1.2)r_0)*} J_\mu^T\left(\Phi\right) n_t^\mu dVol_t dt\\
\leq & CB\int_{t_1}^{t_*} t^{-\alpha} dt\\
\leq & CB\left(t_*-t_1\right) t_1^{-\alpha},
\end{aligned}$$

since $\alpha\geq 0$.

Write $f\left(t\right)=\int_{\{r\leq r_0\}} J_\mu^Y\left(\Phi\right) n^\mu_{\frac{1}{2}\left(t+r_0^*\right)} dVol_{\{v=\frac{1}{2}\left(t+r_0^*\right)\}}$. Then we have

$$f\left(t_*\right)+\int_{t_1}^{t_*} f\left(\tau\right) d\tau \leq C\left(f\left(t_1\right)+B\max\{t_*-t_1,1\}t_1^{-\alpha}\right). \tag{8.7}$$

We take C to be fixed from this point on. We clearly can assume the $C>1$.

From this, we will prove the Proposition by a bootstrap argument. Assume $f\left(t\right)\leq At^{-\alpha}$ for some large A that is to be determined. We want to show that $f\left(t\right)\leq\frac{A}{2}t^{-\alpha}$. Let $t_1=t_*-8C^2$. Since we are only concerned with t_* large, we assume without loss

of generality that $t_* > 8\left(1-2^{-\frac{1}{\alpha}}\right)^{-1} C^2$ so that $t_* < 2^{\frac{1}{\alpha}} t_1$. Then

$$\begin{aligned} f(t_*) + \int_{t_*-8C^2}^{t_*} f(\tau)\, d\tau \leq & C\left(At_1^{-\alpha} + 8C^2 B t_1^{-\alpha}\right) \\ \leq & 2C\left(A + 8C^2 B\right) t_*^{-\alpha}. \end{aligned}$$

There exists $\tilde{t}$ with $t_* - 8C^2 \leq \tilde{t} \leq t_*$ such that

$$\begin{aligned} f\left(\tilde{t}\right) \leq & \frac{1}{8C^2} \int_{t_*-8C^2}^{t_*} f(\tau)\, d\tau \\ \leq & \frac{(A + 8C^2 B)}{4C} t_*^{-\alpha}. \end{aligned}$$

Now we let $t_1 = \tilde{t}$. Notice that $t_* < 2^{\frac{1}{\alpha}} \tilde{t}$. Then

$$\begin{aligned} f(t_*) + \int_{\tilde{t}}^{t_*} f(\tau)\, d\tau \leq & C\left(f\left(\tilde{t}\right) + 8C^2 B \tilde{t}^{-\alpha}\right) \\ \leq & \frac{A}{4} t_*^{-\alpha} + 2C^2 B t_*^{-\alpha} + 16C^3 B t_*^{-\alpha} \\ \leq & \frac{A}{2} t_*^{-\alpha}, \end{aligned}$$

if $A \geq 72C^3 B$.

Of course to have $f(t) \leq At^{-\alpha}$ for all t, we also need it to hold initially, i.e., $A \geq f(t_0)$. Therefore, we have

$$\int_{\{r \leq r_0\}} J_\mu^Y(\Phi)\, n^\mu_{\frac{1}{2}\left(t_* + r_0^*\right)} dVol_{\{v = \frac{1}{2}\left(t_* + r_0^*\right)\}} \leq C\left(B + \int J_\mu^N(\Phi)\, n_{t_0}^\mu dVol_{t_0}\right) t_*^{-\alpha},$$

where C is a universal constant different from the one above. Finally, we can also get the decay estimate along the event horizon (i.e., on $u = -\infty$) by plugging in the decay rate of f into (8.6) and replacing the t interval by $[t_*, t_* + 1]$. □

Using Proposition 8.5.1, we claim that a similar estimate holds on t-slices.

Proposition 8.5.2. *Suppose* $\int_{r_0^*}^{2((1.2)r_0)^* - r_0^*} J_\mu^T(\Phi)\, n_t^\mu dVol_t \leq Bt^{-\alpha}$ *for all* t*, for some*

$\alpha > 1$. *Then*

$$\int_{\{v_* \leq v \leq v_*+1\}} J_\mu^Y(\Phi)\, n_\tau^\mu dVol_\tau \leq C\left(B + \int J_\mu^N(\Phi)\, n_{t_0}^\mu dVol_{t_0}\right) v_*^{-\alpha},$$

for $v_* \geq 1$.

Proof. We prove this using the conservation law for Y on the region $\mathcal{R} = \{v_* \leq v \leq v_*+1, 2v_* - r_0^* \leq t \leq \tau\}$.

$$\begin{aligned}
&\int_{\{t \geq 2v_* - r_0^*\}} J_\mu^Y(\Phi)\, n_{v_*+1}^\mu dVol_{\{v=v_*+1\}} \\
&+ \int_{\{v_* \leq v \leq v_*+1\}} J_\mu^Y(\Phi)\, n_\tau^\mu dVol_\tau + \int_{\mathcal{R}} K^Y(\Phi)\, dVol \\
=&\int_{\{t \geq 2v_* - r_0^*\}} J_\mu^Y(\Phi)\, n_{v_*}^\mu dVol_{v_*} + \int_{\{v_* \leq v \leq v_*+1\}} J_\mu^Y(\Phi)\, n_{2v_*-r_0^*}^\mu dVol_{\{t=2v_*-r_0^*\}}.
\end{aligned}$$

We split up the integrals into $r \leq r_0$ and $r > r_0$ parts.

Notice that the domain of integration of $\int_{\{t \geq 2v_* - r_0^*\}} J_\mu^Y(\Phi)\, n_{v_*}^\mu dVol_{v_*}$ lies inside $\{r \leq r_0\}$. Notice also that

$$\int_{\{t \geq 2v_* - r_0^*\} \cap \{r \leq r_0\}} J_\mu^Y(\Phi)\, n_{v_*+1}^\mu dVol_{\{v=v_*+1\}} + \int_{\mathcal{R} \cap \{r \leq r_0\}} K^Y(\Phi)\, dVol \geq 0.$$

Hence

$$\begin{aligned}
&\int_{\{v_* \leq v \leq v_*+1\}} J_\mu^Y(\Phi)\, n_\tau^\mu dVol_\tau \\
\leq &\int_{\{t \geq 2v_* - r_0^*\}} J_\mu^Y(\Phi)\, n_{v_*}^\mu dVol_{v_*} + \int_{\{v_* \leq v \leq v_*+1\}} J_\mu^Y(\Phi)\, n_{2v_*-r_0^*}^\mu dVol_{\{t=2v_*-r_0^*\}} \\
&+ \int_{\{t \geq 2v_* - r_0^*\} \cap \{r \geq r_0\}} J_\mu^Y(\Phi)\, n_{v_*+1}^\mu dVol_{\{v=v_*+1\}} + \int_{\mathcal{R} \cap \{r \geq r_0\}} |K^Y(\Phi)|\, dVol.
\end{aligned}$$

We show that each term has the correct bound. The first term is bounded using

Proposition 8.5.1,

$$\begin{aligned}
&\int_{\{t\geq 2v_*-r_0^*\}} J_\mu^Y(\Phi)\, n_{v_*}^\mu dVol_{v_*}\\
=&\int_{\{r\leq r_0\}} J_\mu^Y(\Phi)\, n_{v_*}^\mu dVol_{v_*}\\
\leq& C\left(B+\int J_\mu^N(\Phi)\, n_{t_0}^\mu dVol_{t_0}\right)(2v_*-r_0^*)^{-\alpha}\\
\leq& C\left(B+\int J_\mu^N(\Phi)\, n_{t_0}^\mu dVol_{t_0}\right)v_*^{-\alpha}.
\end{aligned}$$

The second term is controlled by assumption

$$\begin{aligned}
&\int_{\{v_*\leq v\leq v_*+1\}} J_\mu^Y(\Phi)\, n_{2v_*-r_0^*}^\mu dVol_{\{t=2v_*-r_0^*\}}\\
\leq& C\int_{\{v_*\leq v\leq v_*+1\}} J_\mu^T(\Phi)\, n_{2v_*-r_0^*}^\mu dVol_{\{t=2v_*-r_0^*\}}\\
\leq& CB(2v_*-r_0^*)^{-\alpha}\\
\leq& CBv_*^{-\alpha}.
\end{aligned}$$

The last two terms are bounded by noting that Y is supported in $r\leq(1.2)\,r_0$. The details are identical to the proof of Proposition 8.5.1. Therefore,

$$\int_{\{v_*\leq v\leq v_*+1\}} J_\mu^Y(\Phi)\, n_\tau^\mu dVol_\tau C\left(B+\int J_\mu^N(\Phi)\, n_{t_0}^\mu dVol_{t_0}\right)v_*^{-\alpha}.$$

□

This, and Sobolev embedding, is sufficient to show pointwise decay of the derivatives of Φ along the horizon. We show further that if on a compact set, we have both energy decay and L^2 decay, then we have pointwise decay along the event horizon. More precisely, we have

Proposition 8.5.3. *There exist $\tilde{r}$ very close to $2M$ such that if*

$$\sum_{m=0}^{1}\sum_{k=0}^{3-m}\int_{\tilde{r}^*}^{2((1.2)r_0)^*-r_0^*}\left(J_\mu^T\left(\partial_t^m\Omega^k\Phi\right)n_t^\mu+\Phi^2\right)dVol_t\leq Bt^{-\alpha}$$

for all t, for some $\alpha\geq 0$, then

$$|\Phi\left(v_*,r\right)|^2\leq C\left(B+\sum_{k=0}^{2}\int J_\mu^N\left(\Omega^k\Phi\right)n_{t_0}^\mu dVol_{t_0}\right)v_*^{-\alpha},$$

$$|\partial_{r^*}\Phi\left(v_*,r\right)|^2\leq C\left(B+\sum_{m=0}^{1}\sum_{k=0}^{3-m}\int J_\mu^N\left(\partial_t^m\Omega^k\Phi\right)n_{t_0}^\mu dVol_{t_0}\right)v_*^{-\alpha},$$

for $v_*\geq 1$, $r\leq\tilde{r}$.

Proof. We first take $\tilde{r}$ to be small enough to apply Y, i.e., $\tilde{r}<r_0$. The exact condition on $\tilde{r}$ will be determined later.

For decay of $\Phi\left(v_*,r\right)$, we want to show that on any time-slice, say $t=\tau$,

$$\sum_{k=0}^{2}\int_{\{v_*\leq v\leq v_*+1\}}\left(|\not\nabla^k\Phi|^2+|\not\nabla^k\partial_{r^*}\Phi|^2\right)dAdr_\tau^*\leq Cv_*^{-\alpha}.$$

For decay of $\partial_{r^*}\Phi\left(v_*,r\right)$, we want to show that on any time-slice, say $t=\tau$,

$$\sum_{k=0}^{2}\int_{\{v_*\leq v\leq v_*+1\}}\left(|\not\nabla^k\partial_{r^*}\Phi|^2+|\not\nabla^k\partial_{r^*}^2\Phi|^2\right)dAdr_\tau^*\leq Cv_*^{-\alpha}.$$

Proposition 8.5.2 gives

$$\int_{\{v_*\leq v\leq v_*+1\}}\left(\left(\partial_{r^*}\Phi\right)^2+|\not\nabla\Phi|^2\right)dAdr_\tau^*\leq C\left(B+\int J_\mu^N\left(\Phi\right)n_{t_0}^\mu dVol_{t_0}\right)v_*^{-\alpha}.$$

After commuting with an appropriate number of Ω, Proposition 8.5.2 gives

$$\begin{aligned}&\int_{\{v_*\leq v\leq v_*+1\}}\left(|\nabla\!\!\!/\partial_{r^*}\Phi|^2+|\nabla\!\!\!/\nabla\!\!\!/\partial_{r^*}\Phi|^2\right)dAdr^*_\tau\\ \leq& C\left(B+\sum_{k=1}^{2}\int J^N_\mu\left(\Omega^k\Phi\right)n^\mu_{t_0}dVol_{t_0}\right)v_*^{-\alpha}.\end{aligned}$$

After commuting with ∂_t and using the equation, Proposition 8.5.2 gives

$$\begin{aligned}&\int_{\{v_*\leq v\leq v_*+1\}}\left(\left(\partial^2_{r^*}\Phi\right)^2+|\nabla\!\!\!/\partial^2_{r^*}\Phi|^2+|\nabla\!\!\!/^2\partial^2_{r^*}\Phi|^2\right)dAdr^*_\tau\\ \leq& C\left(B+\sum_{m=0}^{1}\sum_{k=0}^{3-m}\int J^N_\mu\left(\partial_t^m\Omega^k\Phi\right)n^\mu_{t_0}dVol_{t_0}\right)v_*^{-\alpha}.\end{aligned}$$

Therefore, it remains to show

$$\int_{\{v_*\leq v\leq v_*+1\}}\Phi^2dAdr^*_\tau\leq Cv_*^{-\alpha}.$$

We rewrite

$$\int_{\{v_*\leq v\leq v_*+1\}}\Phi^2dAdr^*_\tau=\int_{2v_*-\tau}^{2v_*-\tau+2}\Phi^2dAdr^*_\tau.$$

To achieve decay, we integrate in the u-direction and use the estimates we have on the compact set.

$$\begin{aligned}&\int_{2v_*-\tau}^{2v_*-\tau+2}\Phi^2dAdr^*_\tau\\ \leq&\int_{\tilde{r}^*}^{\tilde{r}^*+2}\Phi^2dAdr^*_{\{t=2v_*-\tilde{r}^*\}}+\int_{2v_*-\tilde{r}^*}^{\tau}\int_{2v_*-t}^{2v_*-t+2}\Phi\left(\partial_u\Phi\right)dAdr^*dt\\ \leq&\int_{\tilde{r}^*}^{\tilde{r}^*+2}\Phi^2dAdr^*_{\{t=2v_*-\tilde{r}^*\}}+\int_{2v_*-\tilde{r}^*}^{\tau}\int_{2v_*-t}^{2v_*-t+2}\frac{\left(\partial_u\Phi\right)^2}{1-\mu}dAdr^*dt\\ &+\int_{2v_*-\tilde{r}^*}^{\tau}\int_{2v_*-t}^{2v_*-t+2}\Phi^2\left(1-\mu\right)dAdr^*dt\\ \leq&B\left(2v_*-\tilde{r}^*\right)^{-\alpha}+\int_{2v_*-\tilde{r}}^{\tau}\int_{2v_*-t}^{2v_*-t+2}K^Y\left(\Phi\right)dVol\\ &+\int_{2v_*-\tilde{r}^*}^{\tau}\int_{2v_*-t}^{2v_*-t+2}\Phi^2\left(1-\mu\right)dAdr^*dt.\end{aligned}$$

Using the conservation law for Y, and controlling all the terms on the region $\{r \geq \tilde{r}\}$ with the assumption, we have

$$\begin{aligned}
&\int_{2v_*-\tilde{r}^*}^{\tau}\int_{2v_*-t}^{2v_*-t+2} K^Y(\Phi)\, dVol \\
\leq &\int_{\{r\leq\tilde{r}\}} J_\mu^N(\Phi)\, n_{v_*}^\mu dVol_{v_*} + CBv_*^{-\alpha} \\
\leq &C\left(B + \int J_\mu^N(\Phi)\, n_{t_0}^\mu dVol_{t_0}\right)(v_* - \tilde{r}^*)^{-\alpha} + CBv_*^{-\alpha},
\end{aligned}$$

where in the last step we have used Proposition 8.5.1.

Therefore,

$$\begin{aligned}
&\int_{2v_*-\tau}^{2v_*-\tau+2} \Phi^2 dA dr_\tau^* \\
\leq &C\left(B + \int J_\mu^N(\Phi)\, n_{t_0}^\mu dVol_{t_0}\right)v_*^{-\alpha} + \int_{2v_*-\tilde{r}^*}^{\tau}\int_{2v_*-t}^{2v_*-t+2} \Phi^2(1-\mu)\, dA dr^* dt.
\end{aligned}$$

The decay from the last term comes from the exponentially decaying (towards $r^* = -\infty$) factor $(1-\mu)$. To use this decay, we use a bootstrap argument. Assume the decay $\int_{v_*-t}^{v_*-t+1}\Phi^2 dA dr^*_{\{t=t\}} \leq Av_*^{-\alpha}$, independent of t (Note that we can do this initially (in v) independent of t because after we fix v, the region of integration is a bounded set of the manifold. The apparent infiniteness is just an artifact of the choice of coordinates). We want to show that $\int_{v_*-t}^{v_*-t+1}\Phi^2 dA dr^*_{\{t=t\}} \leq \frac{A}{2}v_*^{-\alpha}$.

From the above, and using that $(1-\mu) \leq Ce^{cr^*}$ we have

$$\begin{aligned}
&\int_{2v_*-\tau}^{2v_*-\tau+2} \Phi^2 dA dr_\tau^* \\
\leq &C\left(B + \int J_\mu^N(\Phi)\, n_{t_0}^\mu dVol_{t_0}\right)v_*^{-\alpha} + \int_{v_*-\tilde{r}^*}^{\infty} ACe^{c(2v_*-t+2)}v_*^{-\alpha}dt \\
\leq &C\left(B + \int J_\mu^N(\Phi)\, n_{t_0}^\mu dVol_{t_0}\right)v_*^{-\alpha} + c^{-1}ACe^{c(\tilde{r}^*+2)}v_*^{-\alpha} \\
\leq &\frac{A}{2}v_*^{-\alpha},
\end{aligned}$$

if we choose $A \geq 4C\left(B + \int J_\mu^N(\Phi) n_{t_0}^\mu dVol_{t_0}\right)$ and $\tilde{r}^* \leq -2 + \frac{1}{c}\log\frac{c}{4C}$. □

The above is already sufficient to prove the full decay rate in the exterior region. While Proposition 8.5.1 implies the L^2 control of the derivatives of Φ along the event horizon, it is useful to prove the L^2 control of Φ along the event horizon.

Proposition 8.5.4. *Suppose*

$$\int_{r_0^*}^{2((1.2)r_0)^*-r_0^*} \left(J_\mu^T(\Phi) n_t^\mu + \Phi^2\right) dVol_t \leq Bt^{-\alpha}$$

for all t, for some $\alpha \geq 0$,, then along the event horizon $r = 2M$,

$$\int_{v_*}^{v_*+1}\int_{\mathbb{S}^2} \Phi(2M)^2 dAdv \leq C\left(B + \int_{\{2M\leq r\leq r_0\}} J_\mu^N(\Phi) n_{v_0}^\mu dVol_{v_0}\right) v_*^{-\alpha},$$

for $v_* \geq 1$.

Proof. We note that when changing from the (v,u) coordinates to the (v,r) coordinates, we have

$$\frac{1}{1-\mu}\partial_u = -\partial_r.$$

Hence, written in the (v,r) coordinates, we have

$$\int_{\{2M\leq r\leq r_0\}} J_\mu^Y(\Phi) n_v^\mu dVol_v \geq c\int_{2M}^{r_0}\int_{\mathbb{S}^2} (\partial_r\Phi)^2 dAdr_v.$$

By Proposition 8.5.1,

$$\int_{\{2M\leq r\leq r_0\}} J_\mu^Y(\Phi) n_{v_*}^\mu dVol_{v_*} \leq Cv_*^{-\alpha}.$$

Therefore, we can take the L^2 control from the compact region of r^* as in the assumption and integrate it to the horizon using the control of $\partial_r\Phi$, giving the conclusion. □

8.6 Estimates in the Black Hole Region

In this section we will prove that any decay rate that can be proved along the event horizon can then be proved inside the black hole region. Combining this with the result of the previous section, we can show that any decay rate that is obtained in a compact region of r^* can also be proved in the black hole region. We will achieve this in two steps: Firstly, we show that any decay rate along the event horizon can be propagated slightly inside the black hole region, say, to the timelike slice $r = 2M - \epsilon$ for some $\epsilon > 0$. This step uses an argument that is identical to that in the previous section, which we include here for the sake of completeness. The proof uses the fact that the deformation of the red-shift vector field K^Y has a favorable sign (similar to the case $2M \leq r \leq r_0$) if ϵ is chosen to be small enough. Then, we show that a decay estimate on $r = 2M - \epsilon$ for any $\epsilon > 0$ would imply the same decay rate further inside the black hole. This part is considerably easier because of the geometry of the region in question. In this region we take advantage of the spacelike character of the Killing vector field ∂_t, the non-degeneracy of ∂_r and the finiteness of r, which varies between $2M - \epsilon$ and $r_b > 0$, as well as the fact that along constant u slices the r and the v distances are comparable. In the context of the nonlinear spherically symmetric Einstein-Maxwell-scalar field model, such black hole interior estimates were obtained in [11]. All the computations in this section will be done in the (r, v, ω) coordinates, as it is the most convenient inside the black hole region.

Proposition 8.6.1. *Suppose on the event horizon that*

$$\int_v^{v+1} J_\mu^Y(\Phi)\, n_r^\mu dVol_{r=2M} \leq Bv^{-\alpha} \quad \text{for all } v, \text{ for some } \alpha \geq 0.$$

Then, for some $\epsilon > 0$, and for all $r \in [2M - \epsilon, 2M]$,

$$\int_{v_*}^{v_*+1} J_\mu^Y(\Phi)\, n_r^\mu dVol_r \leq C\left(B + \int_{\{2M-\epsilon\leq r\leq r_0\}} J_\mu^N(\Phi)\, n_{v_0}^\mu dVol_{v_0}\right) v_*^{-\alpha}.$$

Remark 14. *As in Proposition 8.5.1, the reader should think of B as some energy quantity of the initial data. For example, as we will show later, the hypothesis of this Proposition holds for $B = C \sum_{m+k\leq 1} E_1\left(\partial_t^m \Omega^k \Phi\right)$. Moreover, note that if the conclusion of Proposition 8.5.1 implies the hypothesis of this Proposition.*

Proof. Applying Proposition 1 for Y, on the region $\mathcal{R} = \{v_1 \leq v \leq v_*, 2M - \epsilon \leq r \leq 2M\}$, we get

$$\begin{aligned}&\int_{\{2M-\epsilon\leq r\leq 2M\}} J_\mu^Y(\Phi) n_{v_*}^\mu dVol_{v_*} + \int_{\{v_1\leq v\leq v_*\}} J_\mu^Y(\Phi) n_r^\mu dVol_{\{r=2M-\epsilon\}} + \int_{\mathcal{R}} K^Y(\Phi) dVol \\ =&\int_{\{2M-\epsilon\leq r\leq 2M\}} J_\mu^Y(\Phi) n_{v_1}^\mu dVol_{v_1} + \int_{\{v_1\leq v\leq v_*\}} J_\mu^Y(\Phi) n_r^\mu dVol_{\{r=2M\}}.\end{aligned}$$

Notice that $\int_{\{v_1\leq v\leq v_*\}} J_\mu^Y(\Phi) n_r^\mu dVol_{\{r=2M-\epsilon\}} \geq 0$ since $\{r = 2M - \epsilon\}$ is timelike for any $\epsilon > 0$ and Y is timelike for ϵ sufficiently small. Hence

$$\begin{aligned}&\int_{\{2M-\epsilon\leq r\leq 2M\}} J_\mu^Y(\Phi) n_{v_*}^\mu dVol_{v_*} + \int_{\mathcal{R}} K^Y(\Phi) dVol \\ \leq&\int_{\{2M-\epsilon\leq r\leq 2M\}} J_\mu^Y(\Phi) n_{v_1}^\mu dVol_{v_1} + \int_{\{v_1\leq v\leq v_*\}} J_\mu^Y(\Phi) n_r^\mu dVol_{\{r=2M\}}.\end{aligned} \tag{8.8}$$

Firstly, notice that by the assumption of the Proposition

$$\int_{\{v_1\leq v\leq v_*\}} J_\mu^Y(\Phi) n_r^\mu dVol_{\{r=2M\}} \leq B\max\{v_* - v_1, 1\} v_1^{-\alpha}.$$

Secondly, since ϵ is small we have

$$K^Y(\Phi) \geq c J_\mu^Y(\Phi) n_{v_*}^\mu.$$

Let $f(v) = \int_{\{2M-\epsilon\leq r\leq 2M\}} J_\mu^Y(\Phi) n_v^\mu dVol_v$. Then we have

$$f(v) + \int_{v_1}^{v_*} f(\nu) d\nu \leq C\left(f(v_1) + B\max\{v_* - v_1, 1\} v_1^{-\alpha}\right).$$

This identity resembles (8.7) and an identical bootstrap would lead to expected decay rate of f. The conclusion of the Proposition then follows from plugging this decay rate back into (8.8). $\square$

As before, we would then like to prove the L^2 decay of Φ. We will show that any decay rate in L^2 along the event horizon also holds inside the black hole region. More precisely, we have the following:

Proposition 8.6.2. *Suppose on the event horizon that*

$$\int_v^{v+1} \left(J_\mu^Y (\Phi) n_r^\mu + \Phi^2 \right) dVol_{r=2M} \leq B_1 v^{-\alpha} \quad \text{for all } v, \text{ for some } \alpha \geq 0.$$

and

$$\sum_{m=0}^{1} \sum_{k=0}^{3-m} \int_v^{v+1} \left(J_\mu^Y \left(\partial_t^m \Omega^k \Phi \right) n_r^\mu + \Phi^2 \right) dVol_{r=2M} \leq B t^{-\alpha} \quad \text{for all } v, \text{ for some } \alpha \geq 0.$$

Then, for some $\epsilon > 0$, and for all $r \in [2M - \epsilon, 2M]$,

$$\int_{v_*}^{v_*+1} \int_{\mathbb{S}^2(r)} \Phi(r)^2 dA dv_r \leq C \left(B_1 + \int_{\{2M-\epsilon \leq r \leq r_0\}} J_\mu^N (\Phi) n_{v_0}^\mu dVol_{v_0} \right) v_*^{-\alpha}, \tag{8.9}$$

$$|\Phi (v_*, r)|^2 \leq C \left(B + \sum_{k=0}^{2} \int_{\{2M-\epsilon \leq r \leq r_0\}} J_\mu^N \left(\Omega^k \Phi \right) n_{v_0}^\mu dVol_{v_0} \right) v_*^{-\alpha}, \tag{8.10}$$

and

$$|\partial_{r^*} \Phi (v_*, r)|^2 \leq C \left(B + \sum_{m=0}^{1} \sum_{k=0}^{3-m} \int_{\{2M-\epsilon \leq r \leq r_0\}} J_\mu^N \left(\partial_t^m \Omega^k \Phi \right) n_{v_0}^\mu dVol_{v_0} \right) v_*^{-\alpha}, \tag{8.11}$$

for $v_* \geq 1$, $r \leq \tilde{r}$.

Remark 15. *The conclusions of Propositions 8.5.1 and 8.5.4 together imply the hypothesis of this Proposition.*

Proof. By Proposition 8.6.1 and Sobolev Embedding, (8.9) would imply (8.10) and (8.11).

We note that when changing from the (v, u) coordinates to the (v, r) coordinates, we have

$$\frac{1}{1-\mu}\partial_u = -\partial_r.$$

Hence, written in the (v, r) coordinates, we have

$$\int_{\{2M-\epsilon\leq r\leq r_0\}} J^Y_\mu(\Phi)\, n^\mu_v dVol_v \geq c\int_{2M-\epsilon}^{r_0}\int_{\mathbb{S}^2}(\partial_r\Phi)^2 dAdr_v.$$

By Proposition 8.6.1, the assumption of this Proposition implies that

$$\int_{\{2M-\epsilon\leq r\leq r_0\}} J^Y_\mu(\Phi)\, n^\mu_v dVol_v \leq C\left(B_1 + \int_{\{2M-\epsilon\leq r\leq r_0\}} J^N_\mu(\Phi)\, n^\mu_{v_0} dVol_{v_0}\right) v^{-\alpha}.$$

By integrating in the ∂_r direction, we have that for any $2M-\epsilon \leq r \leq 2M$,

$$\begin{aligned}
&\int_{v_*}^{v_*+1}\int_{\mathbb{S}^2(r)}\Phi(r)^2 dAdv\\
\leq& C\left(\int_{v_*}^{v_*+1}\int_{\mathbb{S}^2}\Phi(2M)^2 dAdv_{r=2M} + \int_{v_*}^{v_*+1}\int_{2M-\epsilon}^{2M}\int_{\mathbb{S}^2}(\partial_r\Phi)^2\, dAdrdv\right)\\
\leq& CB_1 v_*^{-\alpha} + C\left(B_1 + \int_{\{2M-\epsilon\leq r\leq r_0\}} J^N_\mu(\Phi)\, n^\mu_{v_0} dVol_{v_0}\right) v_*^{-\alpha}.
\end{aligned}$$

□

Once we have the improved decay estimates slightly into the black hole region, we can propagate them to anywhere in the black hole region. As noted before, the problem becomes considerably easier due to the geometry inside the black hole. We will use the facts that $-\partial_r$ is timelike and non-degenerate and that the energy associated to $-\partial_r$ would control its deformation $K^{-\partial_r}$. The fact that ∂_r controls the deformation is in turn implied by the fact that ∂_t is Killing. Since we would like to prove an estimate in a finite region of r, these facts would allow us to prove estimates

in the whole black hole region (with a constant that degenerates as $r_b \to 0$) easily with Gronwall inequality.

Proposition 8.6.3. *Fix $r_b > 0$. Suppose for some $\tilde{r}$ with $r_b \leq \tilde{r} < 2M$ we have*

$$\int_v^{v+1} \int_{\mathbb{S}^2(\tilde{r})} \left(J_\mu^Y (\Phi) n_{\tilde{r}}^\mu + \Phi(\tilde{r})^2 \right) dA dv_{\tilde{r}} \leq B_1 v^{-\alpha},$$

and

$$\sum_{m=0}^{1} \sum_{k=0}^{3-m} \int_v^{v+1} \int_{\mathbb{S}^2(\tilde{r})} \left(J_\mu^Y \left(\partial_t^m \Omega^k \Phi \right) n_{\tilde{r}}^\mu + \Phi(\tilde{r})^2 \right) dA dv_{\tilde{r}} \leq B v^{-\alpha}$$

Then we have

$$\int_{v_*}^{v_*+1} \left(\Phi^2 + (\nabla \Phi)^2 \right) dA dv_r \leq C_{r_b} B_1 v_*^{-\alpha},$$

and

$$|\Phi|^2 \leq C_{r_b} B v_*^{-\alpha} \quad \text{for } r_b \leq r \leq \tilde{r}$$

with ∇ understood as the coordinate derivatives in the regular coordinates (r, v, ω).

Remark 16. *The hypothesis is implied by the conclusion of Propositions 8.6.1 and 8.6.2 with the appropriate B and B_1 and with tilder $= 2M - \epsilon$. Note that the following argument requires $r < 2M$ strictly and therefore we need Propositions 8.6.1 and 8.6.2 to push the estimates at the horizon to slightly inside.*

Proof. Consider the region $\mathcal{F} = \{(r_b, v) : v_* \leq v \leq v_* + 1\}$ which is the part of the $r = r_b$ spacelike hypersurface between v_* and $v_* + 1$. We would like to determine the domain of dependence $\mathcal{D}(v_*)$ of this region. To do so, we notice that the hypersurfaces $v =$ constant and $u = v - 2r - 4M \log(r - 2M) =$ constant are the boundary of the past of this region. Hence $\mathcal{D}(v_*) \cap \{r_b \leq r \leq \tilde{r}\} \subset \{|v - v_*| \leq C\} \cap \{r_b \leq r \leq \tilde{r}\}$. Now consider the energy with respect to the timelike vector field $-\partial_r$ together with

an L^2 term (which is different from the energy we consider above):

$$E_b(\Phi; r, v_*) = \int_{\mathcal{D}(v_*)\cap\{r\}} \left(\Phi^2 + J_\mu^{-\partial_r}(\Phi) n_r^\mu\right) dVol_r,$$

where $-\partial_r$ is taken in the (r, v) coordinate and is future-directed and time-like in this region. Thus $E_b(\Phi; r, v_*) \geq \int_{\mathcal{D}(v_*)} (\Phi^2 + (\nabla\Phi)^2)\, dVol_r \geq 0$, with ∇ understood as the coordinate derivatives in the regular coordinates (r, v, ω). We have,

$$\int_{\mathcal{D}(v_*)\cap\{r\}} K^{-\partial_r}(\Phi) dVol_r \leq C \int_{\mathcal{D}(v_*)\cap\{r\}} J_\mu^{-\partial_r}(\Phi) n_r^\mu dVol_r \leq C E_b(\Phi; r, v_*)$$

with the constant $C = C(r_b)$ is independent of v_* and r as long as $r_b \leq r \leq \tilde{r}$. This is, firstly, because ∂_r is invariant under the spacelike Killing vector field ∂_t and hence the coefficients in both $K^{-\partial_r}(\Phi)$ and $J_\mu^{-\partial_r}(\Phi) n_r^\mu$ are independent of t. The constant can also be chosen to be uniform in r because we are in a finite region of r. Moreover,

$$-\partial_r \int_{\mathcal{D}} \Phi^2 dVol_r \leq C E_b(\Phi; r).$$

Hence,

$$E_b(\Phi; r) \leq C \left(E_b(\Phi; \tilde{r}) + \int_r^{\tilde{r}} E_b(\Phi; r') dr' \right).$$

Gronwall inequality implies, since we are in a finite range of r, that

$$\begin{aligned} E_b(\Phi; r) \leq & C E_b(\Phi; \tilde{r}) \\ \leq & C B_1 v_*^{-\alpha}. \end{aligned}$$

Therefore, for every $r_b \leq r \leq \tilde{r}$,

$$\int_{v_*}^{v_*+1} \left(\Phi^2 + (\nabla\Phi)^2\right) dA dv_r \leq C B_1 v_*^{-\alpha}$$

since $\{(r, v) : r_b \leq r \leq \tilde{r}, v_* \leq v \leq v_* + 1\} \subset \mathcal{D}$. The pointwise decay follows from

commuting the equation with ∂_t and Ω and Sobolev Embedding. $\square$

8.7 Proof of the Improved Decay on Schwarzschild Spacetime

8.7.1 Improved Decay for Φ (Proof of Theorem 5.1.1)

To prove Theorem 5.1.1, we proceed in two steps. First, we show that for every t_*, there exist $t_1 < t_*$, $t \sim t_*$ such that a weighted L^2-norm of Φ on the slice $\{t = t_1\}$ has the desired decay of $t_*^{-3+\delta}$. We then use the estimates for Ψ to upgrade this to decay estimates for a weighted L^2-norm of Φ on the slice $\{t = t_*\}$.

We first set up some notation. Fix r_1^*, r_2^*. These are the r_1^* and r_2^* in the statement of Theorem 5.1.1. In other words, we would like to prove a decay estimate on the fixed compact region $r_1^* \leq r \leq r_2^*$. Let $t_* \geq 2\left(|r_1^*| + |r_2^*|\right)$ be the time slice on which we want to show the decay estimate. Let $\tilde{t} = (1.1)^{-1} t_*$ and $\mathcal{P} = \{\tilde{t} \leq t \leq t_*, r_1^* - t_* + t \leq r^* \leq r_2^* + t_* - t\}$.

Proposition 8.7.1. *There exist a t_1 with $\tilde{t} \leq t_1 \leq t_*$ such that*

$$\int_{\mathcal{P} \cap \{t = t_1\}} \left(\Phi^2 + \left(\partial_{r^*}\Phi\right)^2\right) r^{-2} \left(1 - \mu\right)^2 dA dr^* \leq C t_*^{-3} E_0\left(\Phi\right).$$

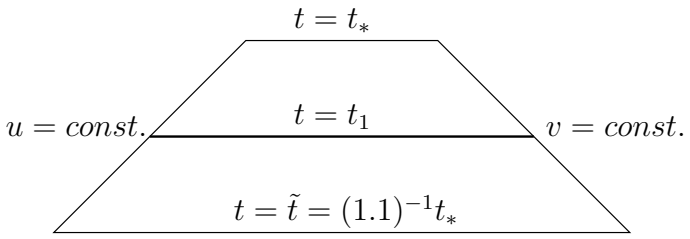


Figure 8.2: The region $\mathcal{P}$

Proof. By Theorem 8.3.1,

$$\begin{aligned}
&\int_{\mathcal{P}} \left(\Phi^2 + \left(\partial_{r^*}\Phi\right)^2\right) r^{-2} \left(1-\mu\right)^2 dAdr^* dt \\
\leq & C \int_{\mathcal{P}} \left| K^{\tilde{X},w^{\tilde{X}}} \left(\Phi\right) \right| + \sum_l K^{X_l,w^{X_l}} \left(\Phi_l\right) dVol \\
\leq & C\tilde{t}^{-2} E_0 \left(\Phi\right) \\
\leq & C t_*^{-2} E_0 \left(\Phi\right).
\end{aligned}$$

Now take t_1 such that

$$\begin{aligned}
&\int_{\mathcal{P}\cap\{t=t_1\}} \left(\Phi^2 + \left(\partial_{r^*}\Phi\right)^2\right) r^{-2} \left(1-\mu\right)^2 dAdr^* \\
= &\inf_{\tilde{t}\leq t\leq t_*} \int_{\mathcal{P}\cap\{t=t\}} \left(\Phi^2 + \left(\partial_{r^*}\Phi\right)^2\right) r^{-2} \left(1-\mu\right)^2 dAdr^*,
\end{aligned}$$

which exists since we are taking the infimum over a compact interval, and note that

$$\begin{aligned}
&\inf_{\tilde{t}\leq t\leq t_*} \int_{\mathcal{P}\cap\{t=t\}} \left(\Phi^2 + \left(\partial_{r^*}\Phi\right)^2\right) r^{-2} \left(1-\mu\right)^2 dAdr^* \\
\leq & \left(t_* - \tilde{t}\right)^{-1} \int_{\mathcal{P}} \left(\Phi^2 + \left(\partial_{r^*}\Phi\right)^2\right) r^{-2} \left(1-\mu\right)^2 dAdr^* dt \\
\leq & C t_*^{-1} \int_{\mathcal{P}} \left(\Phi^2 + \left(\partial_{r^*}\Phi\right)^2\right) r^{-2} \left(1-\mu\right)^2 dAdr^* dt \\
\leq & C t_*^{-3} E_0 \left(\Phi\right).
\end{aligned}$$

□

To upgrade this to an estimate for a generic t, we make two observations about S. Firstly, S is timelike away from the event horizon. Secondly, S has a weight $\sim t$. We can therefore integrate from the "good slice" $t = t_1$ to the slice $t = t_*$ and get the same decay estimate. This is done using integration by parts in the following Proposition. We prove a more general form but the reader should keep in mind that we will use $f = \Phi^2 + \left(\partial_{r^*}\Phi\right)^2$, $g = r^{-2}\left(1-\mu\right)^2$.

Proposition 8.7.2. *Let* $f = f(r^*, t, \omega \in \mathbb{S}^2)$, $g = g(r^*)$, $\mathcal{P} = \{t_1 \leq t \leq t_*, r_1^* - t_* + t \leq r^* \leq r_2^* + t_* - t\}$. *Then*

$$\begin{aligned}
&t_* \int_{\mathcal{P}\cap\{t=t_*\}} fgdAdr^* + \int_{\mathcal{P}\cap\{v=\frac{1}{2}\left(t_*+r_2^*\right)\}} vfgdAdr^* + \int_{\mathcal{P}\cap\{u=\frac{1}{2}\left(t_*-r_1^*\right)\}} ufgdAdr^* \\
=&t_1 \int_{\mathcal{P}\cap\{t=t_1\}} fgdAdr^* + 2\int_{\mathcal{P}} fgdAdtdr^* + \int_{\mathcal{P}} r^* fg' dAdtdr^* + \int_{\mathcal{P}} (Sf)\, gdAdtdr^*.
\end{aligned}$$

Proof. We change to the variables u, v and integrate by parts,

$$\begin{aligned}
&\int_{\mathcal{P}} v\left(\partial_v f\right) gdAdtdr^* \\
=&\int_{\frac{1}{2}\left(t_*-r_1^*\right)}^{\frac{1}{2}\left(t_*-r_1^*\right)} \int_{t_1-u}^{t_*-u} v\left(\partial_v f\right) gdAdvdu + \int_{\frac{1}{2}\left(2t_1-t_*-r_2^*\right)}^{\frac{1}{2}\left(t_*-r_2^*\right)} \int_{t_1-u}^{\frac{1}{2}\left(t_*+r_2^*\right)} v\left(\partial_v f\right) gdAdvdu \\
=&-\int_{\mathcal{P}} fgdAdtdr^* - \int_{\mathcal{P}} vf\partial_{r^*} gdAdtdr^* + \int_{\mathcal{P}\cap\{t=t_*\}} vfgdAdr^* \\
&+\int_{\mathcal{P}\cap\{v=t_*+r_2^*\}} vfgdAdr^* - \int_{\mathcal{P}\cap\{t=t_1\}} vfgdAdr^*,
\end{aligned}$$

$$\begin{aligned}
&\int_{\mathcal{P}} u\left(\partial_u f\right) gdAdtdr^* \\
=&\int_{\frac{1}{2}\left(2t_1-t_*+r_1^*\right)}^{\frac{1}{2}\left(t_*+r_1^*\right)} \int_{t_1-v}^{\frac{1}{2}\left(t_*-r_1^*\right)} u\left(\partial_u f\right) gdAdudv + \int_{\frac{1}{2}\left(t_*+r_1^*\right)}^{\frac{1}{2}\left(t_*+r_2^*\right)} \int_{t_1-v}^{t_*-v} u\left(\partial_u f\right) gdAdudv \\
=&-\int_{\mathcal{P}} fgdAdtdr^* + \int_{\mathcal{P}} uf\partial_{r^*} gdAdtdr^* + \int_{\mathcal{P}\cap\{t=t_*\}} ufgdAdr^* \\
&+\int_{\mathcal{P}\cap\{u=t_*-r_1^*\}} ufgdAdr^* - \int_{\mathcal{P}\cap\{t=t_1\}} ufgdAdr^*.
\end{aligned}$$

The proposition is proved by adding these two equations. □

To prove the theorem, we use the above identity using $f = \Phi^2 + \left(\partial_{r^*}\Phi\right)^2$, $g = r^{-2}\left(1-\mu\right)^2$.

We notice that since $f, g \geq 0$ by definition and $u, v \geq 0$ in $\mathcal{P} = \{t_1 \leq t \leq t_*, r_1^* - t_* + t \leq r^* \leq r_2^* + t_* - t\}$.

Therefore,

$$\int_{\mathcal{P}\cap\{v=\frac{1}{2}\left(t_*+r_2^*\right)\}} vfgdAdr^* + \int_{\mathcal{P}\cap\{u=\frac{1}{2}\left(t_*-r_1^*\right)\}} ufgdAdr^* \geq 0.$$

Thus Proposition 8.7.2 would imply

$$\begin{aligned}
&t_* \int_{\mathcal{P}\cap\{t=t_*\}} \left(\Phi^2 + \left(\partial_{r^*}\Phi\right)^2\right) r^{-2}\left(1-\mu\right)^2 dAdr^* \\
\leq& t_1 \int_{\mathcal{P}\cap\{t=t_1\}} \left(\Phi^2 + \left(\partial_{r^*}\Phi\right)^2\right) r^{-2}\left(1-\mu\right)^2 dAdr^* \\
&+ 2\int_{\mathcal{P}} \left(\Phi^2 + \left(\partial_{r^*}\Phi\right)^2\right) r^{-2}\left(1-\mu\right)^2 dAdr^*dt \\
&+ 2\int_{\mathcal{P}} |r^*\left(\Phi^2 + \left(\partial_{r^*}\Phi\right)^2\right)\left(r^{-2}\left(1-\mu\right)^2\right)'| dtdr^* \\
&+ \int_{P} \left(|\Psi\Phi| + |\partial_{r^*}\Psi\partial_{r^*}\Phi|\right) r^{-2}\left(1-\mu\right)^2 dAdtdr^* \\
\leq& Ct_*^{-2}E_0\left(\Phi\right) + \int_{\mathcal{P}} \left(\Psi^2 + \left(\partial_{r^*}\Psi\right)^2\right) r^{-2}\left(1-\mu\right)^2 dAdtdr^* \\
&+ \int_{\mathcal{P}} \left(\Phi^2 + \left(\partial_{r^*}\Phi\right)^2\right) r^{-2}\left(1-\mu\right)^2 dAdtdr^* \\
\leq& Ct_*^{-2+\delta}E_1\left(\Phi\right),
\end{aligned}$$

where we have used Proposition 8.6.2 at the second to last step and Theorems 8.3.1 and 8.4.1 at the last step.

Therefore,

$$\int_{r_1^*}^{r_2^*} \left(\Phi\left(t_*\right)^2 + \left(\partial_{r^*}\Phi\left(t_*\right)\right)^2\right) dAdr^* \leq Ct_*^{-3+\delta}E_1\left(\Phi\right).$$

Since ∂_t, Ω are Killing, it follows immediately that

$$\sum_{l=0}^{1} \int_{r_1^*}^{r_2^*} \left(\left(\partial_t^m\partial_{r^*}^l\Phi\left(t_*\right)\right)^2 + |\nabla\!\!\!/^k\partial_{r^*}^l\Phi\left(t_*\right)|^2\right) dAdr^* \leq Ct_*^{-3+\delta}E_1\left(\partial_t^m\Omega^k\Phi\right),$$

for any k, m.

Using the equation $\Box_g \Phi = 0$, we get

$$\int_{r_1^*}^{r_2^*} \left(\nabla^l \Phi\left(t_*\right)\right)^2 dVol \leq C t_*^{-3+\delta} \sum_{k+m \leq l} E_1\left(\partial_t^m \Omega^k \Phi\right).$$

We have thus established the decay of the non-degenerate energy in the exterior region away from the event horizon. The full decay of non-degenerate energy part of Theorem 5.1.1 follows from Proposition 8.5.1 and 8.5.4-8.6.3. The pointwise decay part of Theorem 5.1.1 follows form the standard Sobolev Embedding Theorem for the part of the exterior region away from the event horizon and from Propositions 8.5.1, 8.5.4-8.6.3 for the region near the event horizon and inside the black hole.

8.7.2 Improved Decay for $\partial_t \Phi$ (Proof of Theorem 5.1.2)

To estimate the time derivatives of Φ, we follow an idea of Klainerman-Sideris [30]. The key observation is that the first derivatives of $\partial_t \Phi$ are controlled with a weight of $\frac{1}{t-r^*}$ by a linear combination of first derivatives of Φ and Ψ. This extra weight would give extra decay to $\partial_t \Phi$.

Proposition 8.7.3. *Suppose $t + r^* \geq \max\{\frac{t}{2}, \frac{|r^*|}{2}\}$. (This is true for example when r^* is bounded below and t is sufficiently large.)*

1. $\left|\left(t - r^*\right) \partial_t^2 \Phi\right| \leq C\left(\left|\partial_t \Psi\right| + \left|\partial_{r^*} \Psi\right| + \left|\partial_t \Phi\right| + \left|\partial_{r^*} \Phi\right| + \left(1-\mu\right)\left|r^*\right|\left|\not{\Delta} \Phi\right|\right)$,
2. $\left|\left(t - r^*\right) \partial_{r^*} \partial_t \Phi\right| \leq C\left(\left|\partial_t \Psi\right| + \left|\partial_{r^*} \Psi\right| + \left|\partial_t \Phi\right| + \left|\partial_{r^*} \Phi\right| + \left(1-\mu\right)\left|r^*\right|\left|\not{\Delta} \Phi\right|\right)$,
3. $\left|t\left(1-\mu\right) \not{\nabla} \partial_t \Phi\right| \leq C((1-\mu)\left|\not{\nabla} \Psi\right| + \left|\partial_{r^*} \Omega \Phi\right|$.

Proof. Define $\Delta_g \Phi = \left(1-\mu\right)^{-1} \partial_{r^*} \Phi + \frac{2}{r} \partial_{r^*} \Phi + \not{\Delta} \Phi$. Then $\Box_g \Phi = 0$ reads

$$\left(1-\mu\right) \partial_t^2 \Phi = \Delta_g \Phi.$$

Recall that

$$\Psi = t\partial_t\Phi + r^*\partial_{r^*}\Phi.$$

Therefore,

$$\partial_t\Psi - \partial_t\Phi = t\partial_t^2\Phi + r^*\partial_{r^*}\partial_t\Phi,$$

$$\partial_{r^*}\Psi - \partial_{r^*}\Phi = r^*\partial_{r^*}^2\Phi + t\partial_{r^*}\partial_t\Phi.$$

Hence,

$$\begin{aligned}
&t\left(\partial_t\Psi - \partial_t\Phi\right) - r^*\left(\partial_{r^*}\Psi - \partial_{r^*}\Phi\right)\\
=&t^2\partial_t^2\Phi - \left(r^*\right)^2\partial_{r^*}^2\Phi\\
=&\left(t^2 - \left(r^*\right)^2\right)\partial_t^2\Phi + \left(r^*\right)^2\left(\left(1-\mu\right)\Delta_g\Phi - \partial_{r^*}^2\Phi\right)\\
=&\left(t^2 - \left(r^*\right)^2\right)\partial_t^2\Phi + \left(r^*\right)^2\left(\frac{2\left(1-\mu\right)}{r}\partial_{r^*}\Phi + \left(1-\mu\right)\not{\Delta}\Phi\right).
\end{aligned}$$

Therefore, by re-arranging and dividing by $(t + r^*)$,

$$\begin{aligned}
&\left|\left(t-r^*\right)\partial_t^2\Phi\right|\\
=&\left|\frac{1}{t+r^*}\left(t\left(\partial_t\Psi - \partial_t\Phi\right) - r^*\left(\partial_{r^*}\Psi - \partial_{r^*}\Phi\right) - \left(r^*\right)^2\left(\frac{2\left(1-\mu\right)}{r}\partial_{r^*}\Phi + \left(1-\mu\right)\not{\Delta}\Phi\right)\right)\right|\\
\leq&C\left(\left|\partial_t\Psi\right| + \left|\partial_{r^*}\Psi\right| + \left|\partial_t\Phi\right| + \left|\partial_{r^*}\Phi\right| + \left(1-\mu\right)\left|r^*\right|\left|\not{\Delta}\Phi\right|\right).
\end{aligned}$$

We have thus proved 1. On the other hand, using again the above equality, we also have

$$\begin{aligned}
&\left(t-r^*\right)\partial_{r^*}\partial_t\Phi\\
=&-\partial_t\Psi + \partial_{r^*}\Psi + \partial_t\Phi - \partial_{r^*}\Phi + t\partial_t^2\Phi - r^*\partial_{r^*}^2\Phi\\
=&-\partial_t\Psi + \partial_{r^*}\Psi + \partial_t\Phi - \partial_{r^*}\Phi + \left(t-r^*\right)\partial_t^2\Phi + r^*\left(\left(1-\mu\right)\Delta_g\Phi - \partial_{r^*}^2\Phi\right)\\
=&-\partial_t\Psi + \partial_{r^*}\Psi + \partial_t\Phi - \partial_{r^*}\Phi + \left(t-r^*\right)\partial_t^2\Phi + \frac{2r^*\left(1-\mu\right)}{r}\partial_{r^*}\Phi + \left(1-\mu\right)\left|r^*\right|\not{\Delta}\Phi.
\end{aligned}$$

This, together with 1, implies 2.

The proof of 3 is more direct. Using the definition of S, and that Ω is independent of t and r^*,

$$\Omega\Psi = r^*\partial_{r^*}\Omega\Phi + t\partial_t\Omega\Phi.$$

Thus, by noting that Ω and $r\not\nabla$ differ only by constant,

$$\begin{aligned}
&|t\left(1-\mu\right)\not\nabla\partial_t\Phi| \\
\leq&|\left(1-\mu\right)\not\nabla\Psi| + |\left(1-\mu\right)\frac{r^*}{r}\partial_{r^*}\Omega\Phi| \\
\leq&C\left(\left(1-\mu\right)|\not\nabla\Psi| + |\partial_{r^*}\Omega\Phi|\right).
\end{aligned}$$

□

This would easily imply Theorem 5.1.2.2:

Corollary 8.7.4.

$$\int_{\tilde{r}^*}^{ct_*} J_\mu^T\left(\partial_t\Phi\right)n_{t_*}^\mu dVol_{t_*} \leq Ct_*^{-4+\delta}\left(\sum_{m=0}^{1}\sum_{k=0}^{1}E_0\left(\partial_t^m\Omega^k\Phi\right) + E_1\left(\Phi\right)\right),$$

for all $c<1$ and $\tilde{r}$. In particular, $\tilde{r}$ can be chosen as that given by Proposition 8.5.3.

Proof. We can consider t_* large enough so that firstly, the assumption of of Proposition 8.7.3 holds and secondly, on the domain of integration, $(t_* - r^*) \sim t_*$.

$$\begin{aligned}
&\int_{\tilde{r}^*}^{ct_*} J_\mu^T\left(\partial_t\Phi\right)n_{t_*}^\mu dVol_{t_*} \\
=&\int_{\tilde{r}^*}^{ct_*}\left(\left(\partial_{r^*}\partial_t\Phi\right)^2 + \left(\partial_t^2\Phi\right)^2 + \left(1-\mu\right)|\not\nabla\partial_t\Phi|^2\right)dVol_{t_*} \\
\leq&Ct_*^{-2}\int_{\tilde{r}^*}^{ct_*}\left(\left(\partial_t\Psi\right)^2 + \left(\partial_{r^*}\Psi\right)^2 + \left(1-\mu\right)|\not\nabla\Psi|^2 + \left(\partial_t\Phi\right)^2 + \left(\partial_{r^*}\Phi\right)^2\right. \\
&\qquad\left. + \left(1-\mu\right)|\not\nabla\Omega\partial_t\Phi|^2 + \left(\partial_{r^*}\Omega\Phi\right)^2\right)dVol_{t_*} \\
\leq&Ct_*^{-2}\int_{\tilde{r}^*}^{ct_*} J^T\left(\Psi\right) + J^T\left(\Phi\right) + J^T\left(\Omega\Phi\right) + J^T\left(\partial_t\Omega\Phi\right)dVol_{t_*}.
\end{aligned}$$

The corollary follows from Theorem 8.3.1 and 8.4.1. □

Finally, we proceed to the proof of Theorem 5.1.2.1 and 5.1.2.3:

Corollary 8.7.5. *We have*

$$\int_{v_*}^{v_*+1} \left((\partial_t \Phi)^2 + (\nabla \partial_t \Phi)^2 \right) dA dv_r \leq C_{\delta, r_b} v_*^{-4+\delta} E_3 (\Phi), \quad \textit{for any } r \geq r_b.$$

and

$$|\partial_t \Phi (v_*)|^2 \leq C v_*^{-4+\delta} \left(\sum_{m=0}^{2} \sum_{k=0}^{4-m} E_0 \left(\partial_t^m \Omega^k \Phi \right) + \sum_{m=0}^{2} \sum_{k=0}^{2-m} E_1 \left(\partial_t^m \Omega^k \Phi \right) \right),$$

if $r_b \leq r \leq 2M$ *or* $r^* \leq \frac{t_*}{2}$.

Proof. We prove a Sobolev-type inequality. We first work on $\mathbb{R}^3$. We claim that for $u \in C_c^\infty (\mathbb{R}^3)$,

$$||u||_{L^\infty(\mathbb{R}^3)} \leq C ||u||^{\frac{1}{2}}_{\dot{H}^1(\mathbb{R}^3)} ||u||^{\frac{1}{2}}_{\dot{H}^2(\mathbb{R}^3)}.$$

We give a simple proof using Littlewood-Paley theory. Let $N \in 2^{\mathbb{Z}}$ be a dyadic number, $\chi(\xi)$ be a radial cutoff function which is supported in $\{|\xi| < 2\}$ and is identically 1 in $\{|\xi| < 1\}$. Define the Littlewood-Paley operators P_N by $\widehat{P_N u} = \left(\chi \left(\frac{\xi}{N} \right) - \chi \left(\frac{2\xi}{N} \right) \right) \hat{u}$.

Since the inequality claimed is invariant under scaling $u(x) \to \lambda u(x)$ and $u(x) \to u(\lambda x)$, we can assume that $||u||_{\dot{H}^1(\mathbb{R}^3)} = ||u||_{\dot{H}^2(\mathbb{R}^3)} = 1$. Then, by Bernstein inequality,

$$||P_N u||_{L^2(\mathbb{R}^3)} \leq \min\{CN^{-1}, CN^{-2}\}.$$

Therefore, by Bernstein inequality again,

$$||u||_{L^\infty(\mathbb{R}^3)} \leq C \sum_N N^{\frac{3}{2}} ||P_N u||_{L^2(\mathbb{R}^3)} \leq C \left(\sum_{N \geq N_0} N^{-\frac{1}{2}} + \sum_{N < N_0} N^{\frac{1}{2}} \right) \leq C.$$

We note that a variant of this is true. We have for $u \in C_c^\infty(\mathbb{R}^3)$,

$$||u||_{L^\infty(\mathbb{R}^3\setminus B_r(0))} \leq C||u||^{\frac{1}{2}}_{\dot{H}^1(\mathbb{R}^3\setminus B_r(0))}||u||^{\frac{1}{2}}_{\dot{H}^2(\mathbb{R}^3\setminus B_r(0))}.$$

This is true because one can extend u into $B_r(0)$ without increasing the $\dot{H}^1$ or $\dot{H}^2$ norm.

We now apply this to a cutoff version of $\partial_t\Phi$.

Let $\chi = \begin{cases} 1 & |x| \leq 1 \\ 0 & |x| \geq 1.1 \end{cases}$. On $t = t_*$, let $\tilde{\Phi} = \chi\left(\frac{r^*}{0.5t_*}\right)\Phi$ for $r \geq \tilde{r}$, where $\tilde{r}$ is as in Proposition 8.5.3.

The $\dot{H}^1$ norm is controlled with Corollary 8.7.4 and Theorem 8.3.1.4.

$$\begin{aligned}||\partial_t\tilde{\Phi}||_{\dot{H}^1(\mathbb{R}^3\setminus B_{\tilde{r}}(0))} \leq& C\int_{\tilde{r}^*}^{\frac{0.55t_*}{2}} J_\mu^T(\partial_t\Phi)\, n_{t_*}^\mu\, dVol_{t_*} + Ct_*^{-2}\int_{\tilde{r}^*}^{\frac{0.55t_*}{2}} (\partial_t\Phi)^2\, r^2\,(1-\mu)\, dAdr^* \\ \leq& Ct_*^{-4+\delta}\left(\sum_{m=0}^{1}\sum_{k=0}^{1} E_0\left(\partial_t^m\Omega^k\Phi\right) + E_1(\Phi)\right).\end{aligned}$$

The $\dot{H}^2$ norm can be controlled similarly once we note that using the equation, we have for $r \geq \tilde{r}$,

$$\begin{aligned}|\partial_{r^*}^2\partial_t\Phi| \leq& C\left(|\partial_t^3\Phi| + \frac{1}{r}|\partial_{r^*}\partial_t\Phi| + |\not{\nabla}^2\partial_t\Phi|\right) \\ \leq& C\left(|\partial_t^3\Phi| + |\partial_{r^*}\partial_t\Phi| + |\not{\nabla}\Omega\partial_t\Phi|\right).\end{aligned}$$

Therefore, for $t = t_*$,

$$\begin{aligned}&||\partial_t\tilde{\Phi}||_{\dot{H}^2(\mathbb{R}^3\setminus B_{\tilde{r}}(0))} \\ \leq& C\int_{\tilde{r}^*}^{\frac{0.55t_*}{2}} \left(J_\mu^T\left(\partial_t^2\Phi\right) + J_\mu^T(\partial_t\Omega\Phi) + J_\mu^T(\partial_t\Phi)\right) n_{t_*}^\mu\, dVol_{t_*} \\ &+ Ct_*^{-2}\int_{\tilde{r}^*}^{\frac{0.55t_*}{2}} \left(\left(\partial_t^2\Phi\right)^2 + (\partial_t\Omega\Phi)^2 + (\partial_t\Phi)^2\right) r^2\,(1-\mu)\, dAdr^* \\ \leq& Ct_*^{-4+\delta}\left(\sum_{m=0}^{2}\sum_{k=0}^{2} E_0\left(\partial_t^m\Omega^k\Phi\right) + \sum_{m=0}^{1}\sum_{k=0}^{1} E_1\left(\partial_t^m\Omega^k\Phi\right)\right).\end{aligned}$$

Therefore,

$$\begin{aligned}||\partial_t\Phi||_{L^\infty\left(\{\tilde{r}^*\leq r^*\leq\frac{t_*}{2}\}\right)}\leq&||\partial_t\tilde{\Phi}||_{L^\infty(\mathbb{R}^3\setminus B_{\tilde{r}}(0))}\\ \leq&C||\partial_t\tilde{\Phi}||^{\frac{1}{2}}_{\dot{H}^1(\mathbb{R}^3\setminus B_{\tilde{r}}(0))}||\partial_t\tilde{\Phi}||^{\frac{1}{2}}_{\dot{H}^2(\mathbb{R}^3\setminus B_{\tilde{r}}(0))}\\ \leq&Ct_*^{-4+\delta}\left(\sum_{m=0}^{2}\sum_{k=0}^{2}E_0\left(\partial_t^m\Omega^k\Phi\right)+\sum_{m=0}^{1}\sum_{k=0}^{1}E_1\left(\partial_t^m\Omega^k\Phi\right)\right).\end{aligned}$$

In particular, for t sufficiently large, this L^∞ estimate holds on sets of compact r^*. Noting that the L^∞ norm controls the L^2 norm on compact sets, we have

$$\int_{\tilde{r}}^{((1.2)r_0)^*}\left(\partial_t\Phi\right)^2dVol_{t_*}\leq Ct_*^{-4+\delta}\left(\sum_{m=0}^{2}\sum_{k=0}^{2}E_0\left(\partial_t^m\Omega^k\Phi\right)+\sum_{m=0}^{1}\sum_{k=0}^{1}E_1\left(\partial_t^m\Omega^k\Phi\right)\right).$$

We also have, by Corollary 8.7.4,

$$\begin{aligned}&\sum_{m=0}^{1}\sum_{k=0}^{3-m}\int_{\tilde{r}}^{((1.2)r_0)^*}J_\mu^T\left(\partial_t^m\Omega^k\left(\partial_t\Phi\right)\right)n_t^\mu dVol_{t_*}\\ \leq&Ct_*^{-4+\delta}\left(\sum_{m=0}^{2}\sum_{k=0}^{4-m}E_0\left(\partial_t^m\Omega^k\Phi\right)+\sum_{m=0}^{1}\sum_{k=0}^{4-m}E_1\left(\partial_t^m\Omega^k\Phi\right)\right).\end{aligned}$$

The corollary then follows from the Sobolev Embedding Theorem and Propositions 8.5.1, 8.5.3, 8.5.4, 8.6.1, 8.6.2 and 8.6.3. □

Chapter 9

Improved Decay Estimates on Kerr Spacetimes

In this chapter, we prove Theorem 1.0.2 (or more precisely, Theorem 5.2.1).

9.1 Notations

9.1.1 Constants

Throughout this chapter, we will use C to denote a large constant and c to denote a small constant. They can be different from line to line. We will also use A to denote bootstrap constants and we think of A to be large, i.e., $A \gg C$. We also use the notation $O_i(1)$ and $O_i(\epsilon)$ to denote terms that are bounded up to a constant by 1 and ϵ, with bounds that improve by r^{-1} for each derivative up to the i-th derivative. We will also use the notation $f \sim g$ to denote $cf \leq g \leq Cf$.

There are some constants that we will choose in the proof. The following are values of r in the Kerr coordinates:

$$r_+ < r_Y^- < r_Y^+ < \frac{11M}{4} < R_\Omega.$$

r_Y^- and r_Y^+ will be fixed in Remarks 18 and 17 respectively.

There are also smallness parameters which can be thought of as obeying

$$0 < \delta < \epsilon \ll \eta \ll e.$$

ϵ will always be used to denote the smallness of the specific angular momentum a of the spacetimes that we are working on. $\eta \sim C\epsilon$ will denote the loss in the decay rate of the solutions to the wave equation as compared to that on Schwarzschild spacetimes. e will be used to construct the non-degenerate energy. δ and δ' will simply be used as a small parameter whenever they are needed in the analysis. δ and δ' need not be fixed from line to line.

9.1.2 Values of t^*

We will adopt the following as much as possible: t^* will denote a general value of t^*. In particular, it will be used as integration variables. τ will denote the t^* value for which we want an estimate. τ_0 will denote the t^* value where the initial data is posed. We will always assume $\tau_0 \geq 1$ and the reader can think of $\tau_0 = 1$. When integrating, we will often denote the endpoints by τ' and τ. Finally, at a few places we will need to choose a particular value of t^* in an interval. This is usually done to achieve the max or min of the energy quantities. Such choices will often be denoted as $\tilde{\tau}$.

9.1.3 Integration

Definition 9.1.1. *Define the following sets:*

1. $\Sigma_\tau = \{t^* = \tau\}$
2. $\mathcal{R}(\tau', \tau) = \{\tau' \leq t^* \leq \tau\}$
3. $\mathcal{H}(\tau', \tau) = \{r = r_+, \tau' \leq t^* \leq \tau\}$

When integrating on these sets, we will normally integrate with respect to the volume form which we suppress. On Σ_τ the volume form is $\sqrt{\det g_K|_{\Sigma_\tau}}$. On $\mathcal{R}(\tau', \tau)$, the volume form is $\sqrt{det g_K}$. However, on the event horizon $\mathcal{H}(\tau', \tau)$, the surface is null and the metric is degenerate. Nevertheless, on $\mathcal{H}(\tau', \tau)$, the integrand will always be of the form $J_\mu n^\mu_{\Sigma_{\mathcal{H}^+}}$, where $n^\mu_{\Sigma_{\mathcal{H}^+}}$ is the normal to $\mathcal{H}(\tau', \tau)$. We will hence take the volume form corresponding to the (arbitrarily) chosen normal. Occasionally, we will also integrate over the topological 2-spheres given by fixing t and r. We will denote the volume form by $dA = \sqrt{\det g_K|_{\mathbb{S}^2}}$.

For some computations, however, it is more convenient to write down the volume form explicitly in coordinates. In our notations, the following two expressions denote the same integral:

$$\int_{\Sigma_\tau} f = \int_{\Sigma_\tau} f \sqrt{\det g_K|_{\Sigma_\tau}} dr d\theta d\phi.$$

When we write the integrals, we will often use $\iint$ to denote an integral over a spacetime region and use $\int$ denote an integral over a spacelike or null hypersurface.

Notice that the volume form on Σ_{t^*} can be compared with that on $\mathcal{R}(\tau', \tau)$. In particular, we have

$$\iint_{\mathcal{R}(\tau', \tau)} f \sim \int_{\tau'}^{\tau} \left(\int_{\Sigma_{t^*}} f \right) dt^*.$$

9.2 Vector Field Commutators

In this section, we discuss the vector field commutators that we will use in this article. One obvious such vector field is the Killing vector field ∂_{t^*} which satisfies

$$[\Box_{g_K}, \partial_{t^*}] = 0.$$

As we will see, this is not sufficient to control all the higher derivatives of the solution Φ near the event horizon. We will follow [15] and [16] to define the commutator $\hat{Y}$. The main innovation in this article is to define the vector field S that would be used as a commutator to prove the improved decay rate for the solution Φ. We would also need the vector fields Ω_i to control the error terms coming from the commutator $[\Box_{g_K}, S]$.

9.2.1 Commutator S

We construct a commuting vector field S on Schwarzschild that is slightly different from that in the previous chapter and is stable under perturbation.

Define $S = t_S^* \partial_{t_S^*} + h(r_S)\partial_{r_S}$, where

$$h(r) = \begin{cases} r - 2M & r \sim 2M \\ (r + 2M\log(r-2M) - 3M - 2M\log M)(1-\mu) & r \geq R \end{cases},$$

for some large R and is interpolated so that it is smooth and non-negative. Notice that for $r \geq R$, since $t^* = t$, this agrees with the definition in the previous chapter. Therefore we have

$$[\Box_{g_S}, S] = \left(2 + \frac{r^*\mu}{r}\right)\Box_{g_S} + \frac{2}{r}\left(\frac{r^*}{r} - 1 - \frac{2r^*\mu}{r}\right)\partial_{r^*} + 2\left(\left(\frac{r^*}{r} - 1\right) - \frac{3r^*\mu}{2r}\right)\not{\Delta}, \tag{9.1}$$

where $\not{\Delta}$ is the Laplace-Beltrami operator on the standard sphere. In the coordinates (t^*, r, θ, ϕ),

$$\Box_{g_S} = -\alpha_1(r)\,\partial_t^2 + \alpha_2(r)\,\partial_r^2 + \alpha_3(r)\,\partial_r\partial_t + \alpha_4(r)\,\partial_t + \alpha_5(r)\,\partial_r + \not{\Delta}.$$

The crucial observation is that all α_i are smooth and bounded and depend only on r. Noting that α_i does not depend on t, we have

$$[\Box_{g_S}, S] = \beta_1(r)\,\partial_t^2 + \beta_2(r)\,\partial_r^2 + \beta_3(r)\,\partial_r\partial_t + \beta_4(r)\,\partial_t + \beta_5(r)\,\partial_r + \beta_6(r)\,\not{\Delta}.$$

Again, it is important to note that all β_i are smooth, bounded and depend only on r. The form of β_i for $r \geq R$ is given by (9.1).

We consider the same vector field S on Kerr. Using Proposition 4.2.1, and noting that $\partial^m S^\alpha$ is bounded for $m \geq 1$, we have that for $r > R$,

$$\begin{aligned}&|[\Box_{g_K}, S]\Phi - \left(2 + \frac{r^*\mu}{r}\right)\Box_{g_K}\Phi \\ &- \frac{2}{r}\left(\frac{r^*}{r} - 1 - \frac{2r^*\mu}{r}\right)\partial_{r^*}\Phi - 2\left(\left(\frac{r^*}{r} - 1\right) - \frac{3r^*\mu}{2r}\right)\not{\Delta}\Phi| \leq C\epsilon r^{-2}(\sum_{k=1}^{2}|\partial^k\Phi|),\end{aligned}$$

and that for $r \leq R$,

$$|[\Box_{g_K}, S]\Phi| \leq C\sum_{k=1}^{2}|D^k\Phi|.$$

9.2.2 Commutator $\tilde{\Omega}_i$

Let Ω_i be a basis of vector fields of rotations in Schwarzschild spacetimes. An explicit realization can be $\Omega = \partial_\phi, \sin\phi\partial_\theta + \frac{\cos\phi\cos\theta}{\sin\theta}\partial_\phi, \cos\phi\partial_\theta - \frac{\sin\phi\cos\theta}{\sin\theta}\partial_\phi$. Define $\tilde{\Omega}_i = \chi(r)\Omega_i$ to be cutoff so that it is supported in $\{r > R_\Omega\}$ and equals Ω_i for $r > R_\Omega + 1$ for some large R. On Schwarzschild spacetimes, Ω_i is Killing and therefore $\tilde{\Omega}_i$ is Killing for $r > R + 1$. Therefore,

$$[\Box_{g_S}, \tilde{\Omega}_i] = \tilde{\chi}(r)\left(\partial^2 + \partial\right),$$

where $\tilde{\chi}$ is some function depending only on r and is supported in $\{R_\Omega < r < R_\Omega + 1\}$. Using Proposition 4.2.1, we have

$$|[\Box_{g_K}, \tilde{\Omega}_i]\Phi| \leq Cr^{-2}\left(|\partial^2\Phi| + |\partial\Phi|\right).$$

Moreover, since $\tilde{\Omega}_i$ vanishes for $r < R_\Omega$, we have trivially,

$$[\Box_{g_K}, \tilde{\Omega}_i]\Phi = 0, \quad \text{for } r < R_\Omega.$$

From now on, we will write $\tilde{\Omega}$ to denote any one of the Ω_i, while taking the norm to be $|\tilde{\Omega}\Phi| = \sum_i |\Omega_i\Phi|$. We would like to point out that this commutator is useful to gain powers of r near spatial infinity. In particular we have

$$|\nabla\!\!\!/\Phi| \leq \frac{C}{r}|\tilde{\Omega}\Phi|.$$

This extra power of r is essential to control the error terms arising from the commutation of $\Box_{g_K}$ with S.

9.2.3 Commutator $\hat{Y}$

Let $\hat{Y}$, as in Section 3.4.3, be a vector field that is null near the event horizon, normalized with respect to another null vector $\hat{V}$ and is cut off to be compactly supported in $\{r \leq r_Y^+\}$.

Proposition 9.2.1. *On Kerr spacetimes such that ϵ is small, we have for $r \leq r_Y^-$,*

$$|[\Box_{g_K}, \hat{Y}]\Phi - \kappa\hat{Y}^2\Phi| \leq C\left(|D\partial_{t^*}\Phi| + \epsilon|D^2\Phi| + |D\Phi|\right),$$

where $\kappa > c > 0$ is as in (3.5).

Proof. The principal term for the commutator $[\Box_{g_K}, \hat{Y}]\Phi$ is $2^{(\hat{Y})}\pi^{\mu\nu}D_\mu D_\nu\Phi$, where

${}^{(\hat{Y})}\pi_{\mu\nu}$ is the deformation tensor defined by ${}^{(\hat{Y})}\pi_{\mu\nu} = \frac{1}{2}(D_\mu \hat{Y}_\nu + D_\nu \hat{Y}_\mu)$. We look at three terms that are useful in deriving the estimates.

$$ {}^{(\hat{Y})}\pi_{\hat{V}\hat{V}} = g(D_{\hat{V}}\hat{Y}, \hat{V}) = -g(\hat{Y}, D_{\hat{V}}\hat{V}) = 2\kappa,$$

$$ |{}^{(\hat{Y})}\pi_{\hat{V}E_A}| = |\frac{1}{2}\Big(g(D_{\hat{V}}\hat{Y}, E_A) + g(D_{E_A}\hat{Y}, \hat{V})\Big)| \leq C\epsilon,$$

$$ |{}^{(\hat{Y})}\pi_{E_AE_B}| = |\frac{1}{2}\Big(g(D_{E_B}\hat{Y}, E_A) + g(D_{E_A}\hat{Y}, E_B)\Big)| \leq C\epsilon,$$

where the smallness in the the second and third line come from the assumption that we are close to Schwarzschild. Notice also that for $r \leq r_Y^-$, $\hat{V}$ is C^0 close to ∂_{t^*}. Therefore, in the commutator, the main term is

$$\kappa \hat{Y}^2 \Phi.$$

All the other second order terms either have a ∂_{t^*} derivative or small. □

9.3 Vector Field Multiplier N_e and Mild Growth of Non-degenerate Energy

Kerr spacetime has a Killing vector field ∂_t. The conservation law gives that

$$\int_{\Sigma_\tau} J_\mu^T(\Phi) n_{\Sigma_\tau}^\mu + \int_{\mathcal{H}(\tau_0,\tau)} J_\mu^T(\Phi) n_{\Sigma_\tau}^\mu = \int_{\Sigma_{\tau_0}} J_\mu^T(\Phi) n_{\Sigma_{\tau_0}}^\mu + \iint_{\mathcal{R}(\tau_0,\tau)} \partial_t \Phi G.$$

We add to the Killing vector field ∂_t a red-shift vector field. Here, we use the "non-regular" red-shift vector field as in [16]. Under this construction, N_e is C^0 but not C^1 at the event horizon $\mathcal{H}^+$. Compared to the smooth construction in [15], this construction would provide extra control for some derivatives near $\mathcal{H}^+$.

Define

$$Y = y_1(r)\,\hat{Y} + y_2(r)\,\hat{V},$$

where

$$y_1(r) = 1 - \frac{1}{(\log(r - r_+))^3},$$

$$y_2(r) = -\frac{1}{(\log(r - r_+))^3}.$$

Notice that by this definition Y is compactly supported in $\{r \leq r_Y^+\}$ and is invariant under the isomorphisms generated by ∂_{t^*} and ∂_{ϕ^*}.

Proposition 9.3.1. *Let $N_e = \partial_{t^*} + eY$. For any e, there is a corresponding choice of $\epsilon \ll e$ and r_Y^- such that for every integer p, there exists $c_p > 0$ such that*

$$J_\mu^{N_e}(\Phi)\, n_{\mathcal{H}^+}^\mu \sim (D_{\hat{V}}\Phi)^2 + e \sum_{E_A \in \{E_1, E_2\}} (D_{E_A}\Phi)^2, \quad \textit{on the event horizon},$$

$$J_\mu^{N_e}(\Phi)\, n_{\Sigma_\tau}^\mu \sim \sum_{E_\alpha \in \{E_1, E_2, \hat{V}\}} (D_{E_\alpha}\Phi)^2 + e\left(D_{E_{\hat{Y}}}\Phi\right)^2, \quad \textit{for } r \leq r_Y^-,$$

$$J_\mu^{N_e}(\Phi)\, n_{\Sigma_\tau}^\mu \sim \sum (\partial\Phi)^2, \quad \textit{for } r \geq r_Y^- \textit{ in the } (t^*, r, x^A, x^B) \textit{ coordinates},$$

$$K^{N_e}(\Phi) \geq c_p e \left(|\log(r - r_+)|^p \left((D_{\hat{V}}\Phi)^2 + \sum_A (D_{E_A}\Phi)^2 \right) + (D_{\hat{Y}}\Phi)^2 \right), \quad \textit{for } r \leq r_Y^-,$$

$$K^{N_e}(\Phi) \leq C e J_\mu^{N_e}(\Phi)\, n_{\Sigma_\tau}^\mu, \quad \textit{for } r_Y^- \leq r \leq r_Y^+.$$

Proof. It is obvious that Y is timelike and future oriented for $r \leq r_Y^-$. Since ∂_{t^*} is casual in the exterior region of Schwarzschild spacetime and is null only on the event horizon, for every small $e > 0$, there exists sufficiently small $\epsilon > 0$ such that N_e is timelike and future directed on Kerr spacetimes up to the event horizon. The first two estimates hold since in Kerr spacetime, ∂_{t^*} is ϵ-close to $\hat{V}$ on the event horizon. The third estimate holds because outside a small (depending on ϵ) neighborhood of the event horizon, ∂_{t^*} is timelike.

To show that $K^{N_e}(\Phi)$ has the required positivity near the event horizon, we compute the deformation tensor. First, notice that

$$D_{\hat{Y}} y_1 = D_{\hat{Y}} y_2 = \frac{3 D_{\hat{Y}} r}{(r - r_+)(\log(r - r_+))^4}.$$

Using this we have

$$^{(Y)}\pi_{\hat{V}\hat{V}} = g_K(D_{\hat{V}}(y_1\hat{Y} + y_2\hat{V}), \hat{V}) = -g_K(y_1\hat{Y} + y_2\hat{V}, D_{\hat{V}}\hat{V}) = 2y_1\kappa + b^Y y_2,$$

$$^{(Y)}\pi_{\hat{Y}\hat{Y}} = g_K(D_{\hat{Y}}(y_1\hat{Y} + y_2\hat{V}), \hat{Y}) = -\frac{6 D_{\hat{Y}} r}{(r - r_+)(\log(r - r_+))^4},$$

$$\begin{aligned} ^{(Y)}\pi_{\hat{V}\hat{Y}} =& \frac{1}{2} g_K(D_{\hat{V}}(y_1\hat{Y} + y_2\hat{V}), \hat{Y}) + \frac{1}{2} g_K(D_{\hat{Y}}(y_1\hat{Y} + y_2\hat{V}), \hat{V}) \\ =& -\frac{3 D_{\hat{Y}} r}{(r - r_+)(\log(r - r_+))^4} + y_1\kappa + y_2 b^Y, \end{aligned}$$

Moreover, we have

$$^{(Y)}\pi_{\hat{V}E_A}, ^{(Y)}\pi_{\hat{Y}E_A}, ^{(Y)}\pi_{E_A E_B} = O(1).$$

Notice that

$$T_{\hat{Y}\hat{Y}} \sim (D_{\hat{Y}}\Phi)^2, T_{\hat{V}\hat{V}} \sim (D_{\hat{V}}\Phi)^2, T_{\hat{Y}\hat{V}} \sim |\not{\nabla}\Phi|^2,$$

and that $^{(Y)}\pi_{\hat{V}E_A}, ^{(Y)}\pi_{\hat{Y}E_A}, ^{(Y)}\pi_{E_A E_B}$ have no terms of the form $(D_{\hat{Y}}\Phi)^2$. Hence we can choose r_Y^- sufficiently close to r_+ such that for $r_+ \leq r \leq r_Y^-$,

$$K^Y(\Phi) \geq c\kappa (D_{\hat{Y}}\Phi)^2 + \frac{c}{(r - r_+)|\log(r - r_+)|^4} \left((D_{\hat{V}}\Phi)^2 + \sum_A (D_{E_A}\Phi)^2 \right).$$

Since ∂_{t^*} is Killing, $K^{N_e}(\Phi) = eK^Y(\Phi)$, we have

$$K^{N_e}(\Phi) \geq ce \left(\kappa (D_{\hat{Y}}\Phi)^2 + \frac{1}{(r - r_+)|\log(r - r_+)|^4} \left((D_{\hat{V}}\Phi)^2 + \sum_A (D_{E_A}\Phi)^2 \right) \right)$$

for $r \leq r_Y^-$,

Finally, since in the region $r_Y^- \leq r \leq r_Y^+$, $J_\mu^{\partial_{t^*}} n_{\Sigma_{t^*}}^\mu$ controls all derivatives, we have

$$K^{N_e}(\Phi) \leq CeJ_\mu^{N_e}(\Phi) n_{\Sigma_\tau}^\mu, \quad \text{for } r_Y^- \leq r \leq r_Y^+.$$

□

Definition 9.3.2. *We call the positive quantity* $\int_{\Sigma_\tau} J_\mu^{N_e}(\Phi) n_{\Sigma_\tau}^\mu$ *the non-degenerate energy.*

The following identity determines how the non-degenerate energy changes with τ.

Proposition 9.3.3. *Let* Φ *satisfy* $\Box_{g_K}\Phi = G$. *Then*

$$\begin{aligned}&\int_{\Sigma_\tau} J_\mu^{N_e}(\Phi) n_{\Sigma_\tau}^\mu + \int_{\mathcal{H}(\tau_0,\tau)} J_\mu^{N_e}(\Phi) n_{\Sigma_\tau}^\mu + \iint_{\mathcal{R}(\tau_0,\tau)\cap\{r\leq r_Y^-\}} K_\mu^{N_e}(\Phi)\\ =&\int_{\Sigma_{\tau_0}} J_\mu^{N_e}(\Phi) n_{\Sigma_{\tau_0}}^\mu + e\iint_{\mathcal{R}(\tau_0,\tau)\cap\{r_Y^-\leq r\leq r_Y^+\}} K^Y(\Phi) + \int_{\mathcal{R}(\tau_0,\tau)} (\partial_{t^*}\Phi + eY\Phi) G.\end{aligned}$$

The estimates given by the vector field N is sufficient to show that modulo inhomogeneous terms, the quantity $\int_{\Sigma_\tau} J_\mu^{N_e}(\Phi) n_{\Sigma_\tau}^\mu$ cannot grow too much in a short time interval:

Proposition 9.3.4. *Let* Φ *satisfy* $\Box_{g_K}\Phi = G$. *For* e *sufficiently small,* $\epsilon \ll e$ *and* $0 \leq \tau - \tau' \leq 1$, *we have*

$$\int_{\Sigma_\tau} J_\mu^{N_e}(\Phi) n_{\Sigma_\tau}^\mu + \int_{\mathcal{H}(\tau',\tau)} J_\mu^{N_e}(\Phi) n_{\mathcal{H}^+}^\mu \leq 4\int_{\Sigma_{\tau'}} J_\mu^{N_e}(\Phi) n_{\Sigma_{\tau'}}^\mu + C\iint_{\mathcal{R}(\tau',\tau)} G^2.$$

Proof. We first note that $\iint_{\mathcal{R}(\tau',\tau)\cap\{r_Y^-\leq r\leq r_Y^+\}} K^Y(\Phi) \leq C\int_{\tau'}^{\tau}\int_{\Sigma_{\bar\tau}} J_\mu^{N_e}(\Phi) n_{\Sigma_{\bar\tau}}^\mu d\bar\tau$, with

C independent of e and ϵ whenever $\epsilon \ll e < 1$. Then, by Proposition 9.3.3,

$$\begin{aligned}
&\int_{\Sigma_\tau} J_\mu^{N_e}(\Phi)\, n_{\Sigma_\tau}^\mu + \int_{\mathcal{H}(\tau',\tau)} J_\mu^{N_e}(\Phi)\, n_{\mathcal{H}^+}^\mu + \iint_{\mathcal{R}(\tau',\tau)\cap\{r\le r_Y^-\}} K_\mu^{N_e}(\Phi) \\
=&\int_{\Sigma_{\tau'}} J_\mu^{N_e}(\Phi)\, n_{\Sigma_{\tau'}}^\mu + e\iint_{\mathcal{R}(\tau',\tau)\cap\{r_Y^-\le r\le r_Y^+\}} K^Y(\Phi) + \int_{\mathcal{R}(\tau',\tau)} (\partial_{t^*}\Phi + eY\Phi)\, G \\
\le&\int_{\Sigma_{\tau'}} J_\mu^{N_e}(\Phi)\, n_{\Sigma_{\tau'}}^\mu + Ce\int_{\tau'}^{\tau}\int_{\Sigma_{\bar\tau}} J_\mu^{N_e}(\Phi)\, n_{\Sigma_{\bar\tau}}^\mu d\bar\tau + \delta'\iint_{\mathcal{R}(\tau',\tau)} ((\partial_{t^*}\Phi + eY\Phi))^2 \\
&+ (\delta')^{-1}\iint_{\mathcal{R}(\tau',\tau)} G^2 \\
\le&\int_{\Sigma_{\tau'}} J_\mu^{N_e}(\Phi)\, n_{\Sigma_{\tau'}}^\mu + (C\delta' + 2Ce)\int_{\tau'}^{\tau}\int_{\Sigma_{\bar\tau}} J_\mu^{N_e}(\Phi)\, n_{\Sigma_{\bar\tau}}^\mu d\bar\tau + (\delta')^{-1}\iint_{\mathcal{R}(\tau',\tau)} G^2.
\end{aligned}$$

By Gronwall's inequality and absorbing $(\delta')^{-1}$ into the constant C, we have

$$\int_{\Sigma_\tau} J_\mu^{N_e}(\Phi)\, n_{\Sigma_\tau}^\mu \le 2\int_{\Sigma_{\tau'}} J_\mu^{N_e}(\Phi)\, n_{\Sigma_{\tau'}}^\mu + C\iint_{\mathcal{R}(\tau',\tau)} G^2.$$

Now the estimate for the term horizon follows from Proposition 9.3.3. □

9.4 Integrated Decay Estimates and Boundedness of Non-degenerate Energy

In this section we would like to show an integrated decay estimate. We first follow the previous chapter to construct a vector field and prove an integrated decay estimates for the terms near spatial infinity. That construction is in turn inspired by [48]. In the previous chapter, the decay rate in r of this integrated decay estimate is crucial to control the error terms arising from the vector field commutator S. In the sequel, such estimate would also facilitate many computations as we prove the full integrated decay estimate.

In view of the red-shift, all derivatives of Φ can be controlled near the event horizon. However, we would also like to prove an integrated decay estimate that

controls Φ itself near the event horizon. This is in contrast to the integrated decay estimate in [16] which degenerates near the event horizon. This extra control is useful as we are considering the inhomogeneous problem.

The proof of the integrated decay estimate for a finite region of r away from the horizon follows that in [16]. We would like to remark that one difference here is that we do not assume the boundedness of $\int_{\Sigma_\tau} J_\mu^{N_e}(\Phi) n_{\Sigma_\tau}^\mu$ (even after ignoring inhomogeneous terms). We would instead like to prove the boundedness of $\int_{\Sigma_\tau} J_\mu^{N_e}(\Phi) n_{\Sigma_\tau}^\mu$ using the integrated decay estimates. We will, however, use Proposition 9.3.4.

The reader should think of this integrated decay estimates as analogous to the estimates associated to the vector field X in the previous chapter. However, it is impossible to obtain such estimates using a vector field in Kerr spacetimes and we therefore resort to a phase space analysis (see [1]).

To perform the phase space analysis, we will take the Fourier transform in the variable t^*, take the Fourier series in the variable ϕ^* and express the dependence on the θ variable in oblate spheroidal harmonics. Carter [6] discovered that with this decomposition, the wave equation can be separated. However, in order to take the Fourier transform in the variable t^*, we need Φ to be at least in L^2. To this end, we perform a cutoff in the variable t^*.

9.4.1 Estimates near Spatial Infinity

In this subsection, we follow the previous chapter to construct a vector field $\tilde{X} = \tilde{f}(r^*)\partial_{r^*}$ so that the spacetime integral that can be controlled with a good weight in r. Proposition 8.3.4 implies via stability (since the vector field is supported away from the event horizon) the following:

Proposition 9.4.1. *In Kerr spacetimes, using (t^*, r, x^A, x^B) coordinates, there exists*

$\tilde{X}$ and $w^{\tilde{X}}$ supported in $r \geq \frac{25M}{8}$ such that for some large R,

$$K^{\tilde{X},w^{\tilde{X}}}(\Phi) \geq c_{\tilde{X}}\left(r^{-1-\delta}(\partial_{r^*}\Phi)^2 + r^{-1}|\nabla\!\!\!/\Phi|^2 + r^{-3-\delta}\Phi^2\right) - C_{\tilde{X}}\epsilon r^{-2}(\partial_{t^*}\Phi)^2,$$

for $r^* \geq R$ and

$$|\int_{\Sigma_\tau} J_\mu^{\tilde{X},w^{\tilde{X}}}(\Phi)\, n_{\Sigma_\tau}^\mu| \leq C\int_{\Sigma_\tau} J_\mu^{N_e}(\Phi)\, n_{\Sigma_\tau}^\mu.$$

Now it is easy to construct the following vector field on Schwarzschild spacetimes:

Proposition 9.4.2. *In Schwarzschild spacetimes, using (t, r^*, x^A, x^B) coordinates, there exists $\tilde{\tilde{X}}_S = \tilde{\tilde{f}}(r^*)\partial_{r^*}$ supported in $r \geq \frac{13M}{4}$ such that*

$$K^{\tilde{\tilde{X}}}(\Phi) \geq cr^{-1-\delta}(\partial_{t^*}\Phi)^2 - C\left(r^{-1-\delta}(\partial_{r^*}\Phi)^2 + r^{-1}|\nabla\!\!\!/\Phi|^2 + r^{-3-\delta}\Phi^2\right),$$

for $r^* \geq \max\{100, 100M\}$ and

$$|\int_{\Sigma_\tau} J_\mu^{\tilde{\tilde{X}}}(\Phi)\, n_{\Sigma_\tau}^\mu| \leq C\int_{\Sigma_\tau} J_\mu^{N_e}(\Phi)\, n_{\Sigma_\tau}^\mu.$$

Proof. Let $\tilde{\tilde{f}}$ be supported appropriately and $\tilde{\tilde{f}}(r^*) = \frac{1}{(1+r^*)^\delta}$ whenever r^* is large. □

As before, a stability argument would give:

Proposition 9.4.3. *In Kerr spacetimes, using (t^*, r, x^A, x^B) coordinates, there exists $\tilde{\tilde{X}}$ supported in $r \geq \frac{25M}{8}$ such that for some large R,*

$$K^{\tilde{\tilde{X}}}(\Phi) \geq cr^{-1-\delta}(\partial_{t^*}\Phi)^2 - C_{\tilde{\tilde{X}}}\left(r^{-1-\delta}(\partial_{r^*}\Phi)^2 + r^{-1}|\nabla\!\!\!/\Phi|^2 + r^{-3-\delta}\Phi^2\right),$$

for $r^* \geq R$ and

$$|\int_{\Sigma_\tau} J_\mu^{\tilde{\tilde{X}}}(\Phi)\, n_{\Sigma_\tau}^\mu| \leq C\int_{\Sigma_\tau} J_\mu^{N_e}(\Phi)\, n_{\Sigma_\tau}^\mu.$$

Now using the vector field $\tilde{X} + \frac{1}{2}\frac{c_{\tilde{X}}}{C_{\tilde{\tilde{X}}}}\tilde{\tilde{X}}$ and modifying function $w^{\tilde{X}}$, we get the following estimate for ϵ sufficiently small:

Proposition 9.4.4.

$$\begin{aligned}
&\iint_{\mathcal{R}(\tau_0,\tau)\cap\{r\geq R\}} \left(r^{-1-\delta} J_\mu^{N_e}(\Phi)\, n^\mu_{\Sigma_{t^*}} + r^{-3-\delta}\Phi^2\right) \\
\leq & C\left(\int_{\Sigma_\tau} J_\mu^{N_e}(\Phi)\, n^\mu_{\Sigma_\tau} + \int_{\Sigma_{\tau_0}} J_\mu^{N}(\Phi)\, n^\mu_{\Sigma_{\tau_0}} + \iint_{\mathcal{R}(\tau_0,\tau)\cap\{\frac{25M}{8}\leq r\leq R\}} \left(J_\mu^{N_e}(\Phi)\, n^\mu_{\Sigma_{t^*}} + \Phi^2\right)\right. \\
&\left. + \iint_{\mathcal{R}(\tau_0,\tau)} \left(|\partial_r\Phi| + r^{-1}|\Phi|\right)|G|\right).
\end{aligned}$$

9.4.2 Estimates near the Event Horizon

The integrated decay estimates shown in [16] is degenerate around the event horizon. Here we will prove the corresponding estimates near the event horizon. In view of the availability of the red-shift estimate K^{N_e}, we will focus on the zeroth order term Φ^2. It turns out that we can use a construction similar to that in the previous chapter.

Proposition 9.4.5. *In Schwarzschild spacetimes, using (t, r^*, x^A, x^B) coordinates, there exists $X_h = f_h(r^*)\,\partial_{r^*}$ and w^{X_h} supported in $r \leq \frac{23M}{8}$ such that*

$$K^{X_h, w^{X_h}}(\Phi) \geq c\left((\partial_{r^*}\Phi)^2 + |\nabla\!\!\!/\Phi|^2 + \Phi^2\right),$$

for $r \leq r_Y^-$ and

$$\left|\int_{\Sigma_\tau} J_\mu^{X_h, w^{X_h}}(\Phi)\, n^\mu_{\Sigma_\tau}\right| \leq C\int_{\Sigma_\tau} J_\mu^{N_e}(\Phi)\, n^\mu_{\Sigma_\tau}$$

and

$$|J_\mu^{X_h, w^{X_h}}(\Phi)\, n^\mu_{\mathcal{H}^+}| \leq C J_\mu^{N_e}(\Phi)\, n^\mu_{\mathcal{H}^+}.$$

Proof. Let

$$X_h = f_h(r_S^*)\,\partial_{r_S^*} = -\chi(r)\frac{M^3}{(1+4\mu^{-2})}\partial_{r_S^*} = -\chi(r)\frac{\mu^3 r^3}{8(1+4\mu^{-2})}\partial_{r_S^*},$$

where $\chi(r)$ is a cutoff function that is compactly supported in $r \leq \frac{23M}{8}$ and is identi-

cally 1 for $r \leq r_Y^-$. Also, let

$$w^{X_h} = 2f_h'(r^*) + \frac{4(1-\mu)}{r} f_h(r^*).$$

From now on, we will focus on the behavior when $r \leq r_Y^-$ and treat the terms in $\{r_Y^- \leq r \leq \frac{23M}{8}\}$ as errors. Recall that on Schwarzschild spacetime:

$$\begin{aligned} K^{X_h, w^{X_h}}(\Phi) =& \frac{f'(r^*)}{1-\mu}(\partial_{r^*}\Phi)^2 + \frac{(2-3\mu) f(r^*)}{2r}|\not{\nabla}\Phi|^2 \\ & - \frac{1}{4}\left(\frac{1}{1-\mu} f'''(r^*) + \frac{4}{r} f''(r^*) + \frac{\mu}{r^2} f'(r^*) - \frac{2\mu}{r^3}(3-4\mu) f(r^*)\right)\Phi^2 \end{aligned}$$

We now look at the sign of this expression for $r \leq r_Y^-$. It is easy to see that the coefficient for $(\partial_{r^*}\Phi)^2$ is positive:

$$\begin{aligned} f'(r^*) =& (1-\mu)\partial_r f_0(r) \\ =& \frac{\mu r^2 (1-\mu)}{(1+4\mu^{-2})^2} \geq \frac{c(1-\mu)}{r^3}, \end{aligned}$$

The coefficient of $|\not{\nabla}\Phi|^2$ is also clearly positive. A computation shows that

$$\begin{aligned} & \frac{1}{1-\mu} f''' + \frac{4}{r} f'' + \frac{\mu}{r^2} f' - \frac{2\mu}{r^3}(3-4\mu) f \\ =& -\frac{\mu^6 (192 + \mu(128 + \mu(-784 + \mu(464 + \mu(-28 + \mu(52 + \mu(-3+4\mu)))))))}{4(4+\mu^2)^4} \end{aligned}$$

We want to show that

$$P(\mu) = 192 + \mu(128 + \mu(-784 + \mu(464 + \mu(-28 + \mu(52 + \mu(-3+4\mu)))))) \geq \frac{1}{7}$$

for $\frac{16}{23} \leq \mu \leq 1$.

$192 + 128\mu - 784\mu^2 + 464\mu^3 = 16(-12 - 20\mu + 29\mu^2)(\mu - 1) \geq 0$.

$52 - 3\mu + 4\mu^2$ reaches its minimum at $\frac{3}{8}$. Hence, $52 - 3\mu + 4\mu^2 \geq \frac{823}{16}$.

$-28+\mu\left(52-3\mu+4\mu^2\right)\geq -28+\frac{11}{20}\frac{823}{16}\geq \frac{93}{320}$.

Therefore, $P(\mu)\geq \frac{1023}{6400}\geq \frac{1}{7}$ for $\frac{16}{23}\leq \mu\leq 1$. Therefore, for $r\leq r_Y^-$,

$$K^{X_h,w^{X_h}}\left(\Phi\right)\geq c\left((\partial_{r^*}\Phi)^2+|\nabla\!\!\!/\Phi|^2+\Phi^2\right).$$

The second statement

$$|\int_{\Sigma_\tau} J_\mu^{X_h,w^{X_h}}\left(\Phi\right)n_{\Sigma_\tau}^\mu|\leq C\int_{\Sigma_\tau} J_\mu^{N_e}\left(\Phi\right)n_{\Sigma_\tau}^\mu$$

and the third statement

$$|J_\mu^{X_h,w^{X_h}}\left(\Phi\right)n_{\mathcal{H}^+}^\mu|\leq CJ_\mu^{N_e}\left(\Phi\right)n_{\mathcal{H}^+}^\mu$$

follow from the boundedness of f_h and w^{X_h} and that on the Schwarzschild horizon $\partial_t=\partial_{r^*}$. Hence in both estimates, the constants are independent of e for e small. □

Noticing the X_h and w^{X_h} are actually smooth up to the event horizon, this implies via a stability argument:

Proposition 9.4.6. *In Kerr spacetimes, using (t_S,r_S,x_S^1,x_S^2) coordinates, there exists X_h and w^{X_h} supported in $r\leq\frac{23M}{8}$ such that*

$$K^{X_h,w^{X_h}}\left(\Phi\right)\geq c\Phi^2-C\epsilon(\partial_{t^*}\Phi)^2-C\epsilon(\partial_r\Phi)^2,$$

for $r\leq r_Y^-$ and

$$|\int_{\Sigma_\tau} J_\mu^{X_h,w^{X_h}}\left(\Phi\right)n_{\Sigma_\tau}^\mu|\leq C\int_{\Sigma_\tau} J_\mu^{N}\left(\Phi\right)n_{\Sigma_\tau}^\mu$$

and

$$|J_\mu^{X_h,w^{X_h}}\left(\Phi\right)n_{\mathcal{H}^+}^\mu|\leq CJ_\mu^{N_e}\left(\Phi\right)n_{\mathcal{H}^+}^\mu.$$

Together with the red-shift, we therefore have the following integrated decay es-

timate near the event horizon.

Proposition 9.4.7.

$$\begin{aligned}
&\iint_{\mathcal{R}(\tau_0,\tau)\cap\{r\leq r_Y^-\}} (\Phi^2 + K^{N_e}(\Phi)) \\
\leq& C\left(\int_{\Sigma_\tau} J_\mu^{N_e}(\Phi)\, n_{\Sigma_\tau}^\mu + \int_{\Sigma_{\tau_0}} J_\mu^{N_e}(\Phi)\, n_{\Sigma_{\tau_0}}^\mu + \iint_{\mathcal{R}(\tau_0,\tau)\cap\{r_Y^-\leq r\leq \frac{23M}{8}\}} (\Phi^2 + J_\mu^{N_e}(\Phi)\, n_{\Sigma_{t^*}}^\mu)\right. \\
&\left. + \iint_{\mathcal{R}(\tau_0,\tau)\cap\{r\leq \frac{23M}{8}\}} \left(|\partial_{r^*}\Phi| + r^{-1}|\Phi|\right)|G| + |\iint_{\mathcal{R}(\tau_0,\tau)\cap\{r\leq \frac{23M}{8}\}} (\partial_{t^*}\Phi + eY\Phi)G|\right).
\end{aligned}$$

9.4.3 Cutoff, Decomposition and Separation

Following [16], we define the cutoff

$$\Phi_{\tau'}^{\tau} = \xi\Phi,$$

where

$$\xi = \chi(t^* - 1 - \tau)\chi(-t^* - 1 + \tau'),$$

for some smooth cutoff function $\chi(x)$ that is identically 1 for $x \leq -1$ and support on $\{x \leq 0\}$. Then

$$\Box_g \Phi_{\tau'}^{\tau} = \xi G + 2D^\alpha \Phi D_\alpha \xi + \Phi \Box_g \xi =: F.$$

We then decompose in frequency. We decompose the Fourier transform in t of Φ into Fourier series in ϕ and oblate spheroidal harmonics:

$$\hat{\Phi}_{\tau'}^{\tau} = \sum_{m,\ell} R_{m\ell}^{\omega}(r) S_{m\ell}(a\omega, \cos\theta) e^{im\phi}.$$

We also decompose the inhomogeneous term F (which comes both from the original inhomogeneous term G and the cutoff):

$$\hat{F} = \sum_{m,\ell} F^{\omega}_{m\ell}(r) S_{m\ell}(a\omega, \cos\theta) e^{im\phi}.$$

Letting ζ be a sharp cutoff with such that $\zeta = 1$ for $|x| \leq 1$ and $\zeta = 0$ for $|x| > 1$, we define

$$\Phi_\flat = \int_{-\infty}^{\infty} \zeta(\frac{\omega}{\omega_1}) \sum_{m,l:\lambda_{ml}(\omega)\leq\lambda_1} R^{\omega}_{ml}(r) S_{ml}(a\omega, \cos\theta) e^{im\phi} e^{i\omega t} d\omega$$

$$\Phi_\natural = \int_{-\infty}^{\infty} \zeta(\frac{\omega}{\omega_1}) \sum_{m,l:\lambda_{ml}(\omega)>\lambda_1} R^{\omega}_{ml}(r) S_{ml}(a\omega, \cos\theta) e^{im\phi} e^{i\omega t} d\omega$$

$$\Phi_\sharp = \int_{-\infty}^{\infty} (1 - \zeta(\frac{\omega}{\omega_1})) \sum_{m,l:\lambda_{ml}(\omega)\geq\lambda_2\omega^2} R^{\omega}_{ml}(r) S_{ml}(a\omega, \cos\theta) e^{im\phi} e^{i\omega t} d\omega$$

$$\Phi_\flat = \int_{-\infty}^{\infty} (1 - \zeta(\frac{\omega}{\omega_1})) \sum_{m,l:\lambda_{ml}(\omega)<\lambda_2\omega^2} R^{\omega}_{ml}(r) S_{ml}(a\omega, \cos\theta) e^{im\phi} e^{i\omega t} d\omega.$$

In this decomposition, we think of ω_1 as large and λ_2 as small.

9.4.4 The Trapped Frequencies

Trapping occurs for $\Phi_\natural$. An integrated decay estimate is proved in detail in [16]. We refer the readers to Section 5.3.3 of [16]. Notice that the first term on the right hand side in the following Proposition is different from that in [16], but the inequality still holds as a result of the proof of the corresponding inequality in [16].

Proposition 9.4.8.

$$\begin{aligned}
&\iint_{\mathcal{R}(-\infty,\infty)} \left(\chi\Phi_\natural^2 + \chi(\partial_r\Phi_\natural)^2 + \chi\mathbb{1}_{\{|r-3M|\geq\frac{M}{8}\}} J_\mu^N\left(\Phi_\natural\right) n_{\Sigma_{t^*}}^\mu\right)\\
\leq& C\int_{\mathcal{H}(-\infty,\infty)} (\partial_{t^*}\Phi_{\tau'}^\tau)^2 + C\epsilon\int_{\mathcal{H}(-\infty,\infty)} (\partial_{\phi^*}\Phi_{\tau'}^\tau)^2\\
&+\int_{-\infty}^{\infty} dt^* \int_{r\geq R}\Big(2f(r^2+a^2)^{1/2}F_\natural\partial_{r^*}((r^2+a^2)^{1/2}\Phi_{\tau'}^\tau)\\
&\qquad +f'(r^2+a^2)F_\natural\Phi_{\tau'}^\tau\Big)\frac{\Delta}{r^2+a^2}\sin\theta\, d\phi\, d\theta\, dr^*\\
&+\delta'\iint_{\mathcal{R}\cap\{r\leq R\}} (\Phi_{\tau'}^\tau)^2 + (\partial_{r^*}\Phi_{\tau'}^\tau)^2 + C(\delta')^{-1}\iint_{\mathcal{R}\cap\{r\leq R\}} F^2,
\end{aligned}$$

where χ is a weight that degenerates at infinity and near the event horizon and f is increasing and $f = \tan^{-1}\frac{r^*-\alpha-\sqrt{\alpha}}{\alpha} - \tan^{-1}(-1-\alpha)^{-\frac{1}{2}}$, for $r > R$ for some fixed α.

9.4.5 The Untrapped Frequencies

For each of the pieces that are untrapped, i.e., $\Phi_\text{♪}$ for $\text{♪} = \flat, \natural$ or $\sharp$, a vector field $X_\text{♪}$ is constructed in [16] so that

$$\iint_{\mathcal{R}(-\infty,\infty)} \chi\left(J_\mu^{N_e}\left(\Phi_\text{♪}\right) n_{\Sigma_{t^*}}^\mu + \Phi_\text{♪}^2\right) \leq C\iint_{\mathcal{R}(-\infty,\infty)} K^{X_\text{♪}}\left(\Phi_\text{♪}\right),$$

where χ is a weight function that both degenerates at infinity and vanishes around the event horizon. Using this vector field and the conservation identity, it is shown in Section 5.3.4 in [16] that

Proposition 9.4.9.

$$\begin{aligned}
&\iint_{\mathcal{R}(-\infty,\infty)} \chi\left((J_\mu^{N_e}(\Phi_\flat)+J_\mu^{N_e}(\Phi_\natural)+J_\mu^{N}(\Phi_\sharp))n_{\Sigma_\tau}^\mu+\left(\Phi_\flat^2+\Phi_\natural^2+\Phi_\sharp^2\right)\right)\\
\leq& C\int_{\mathcal{H}(-\infty,\infty)} J_\mu^{N_e}(\Phi)n_{\mathcal{H}^+}^\mu + C(\delta')^{-1}\iint_{\mathcal{R}(-\infty,\infty)\cap\{r\leq R\}} F^2\\
&+C\delta'\iint_{\mathcal{R}(-\infty,\infty)\cap\{r\leq R\}} (\Phi_{\tau'}^\tau)^2+(\partial_{r^*}\Phi_{\tau'}^\tau)^2+\mathbb{1}_{\{r\leq\frac{23M}{8}\}}J_\mu^N(\Phi_{\tau'}^\tau)n_{\Sigma_{t^*}}^\mu\\
&+\int_{-\infty}^{\infty}dt^*\int_{\{r\geq R\}}\Big(2f(r^2+a^2)^{1/2}(F_\flat+F_\natural+F_\sharp)\partial_{r^*}((r^2+a^2)^{1/2}\Phi_{\tau'}^\tau)\\
&+f'(r^2+a^2)(F_\flat+F_\natural+F_\sharp)\Phi_{\tau'}^\tau\Big)\frac{\Delta}{r^2+a^2}\sin\theta\,d\phi\,d\theta\,dr^*,
\end{aligned}$$

where χ and f are exactly as in Proposition 9.4.8.

Proof. This inequality is essentially borrowed from Section 5.3.4 in [16]. The only difference is the first term on the right hand side of the inequality. In [16], the estimate

$$\int_{\mathcal{H}(-\infty,\infty)} J_\mu^{X_\flat}(\Phi_\flat)n_{\mathcal{H}^+}^\mu \leq C\int_{\Sigma_{\tau'}} J_\mu^N(\Phi_{\tau'}^\tau)n_{\Sigma_{\tau'}}^\mu$$

is used. Here, we have not proved boundedness of the solution and hence we are content with the estimate

$$\int_{\mathcal{H}(-\infty,\infty)} J_\mu^{X_\flat}(\Phi_\flat)n_{\mathcal{H}^+}^\mu \leq C\int_{\mathcal{H}(-\infty,\infty)} J_\mu^{N_e}(\Phi_{\tau'}^\tau)n_{\mathcal{H}^+}^\mu.$$

Notice that this estimate holds for C independent of e because $X_\flat$ is constructed as $f\partial_{r^*}$ and on the event horizon, $\partial_{r^*}=O(1)\hat{V}+O(\epsilon)E_A$. □

9.4.6 The Integrated Decay Estimates

In order to add up the estimates in the previous sections, we need a Hardy-type inequality:

Proposition 9.4.10. *For $R' < R$,*

$$\int_{\Sigma_\tau \cap \{r \geq R\}} r^{\alpha-2}\Phi^2 \leq C \int_{\Sigma_\tau \cap \{r \geq R'\}} r^\alpha J_\mu^{N_e}(\Phi)\, n_{\Sigma_\tau}^\mu.$$

Proof. Let $k(r)$ be defined by solving

$$k'(r,\theta,\phi) = r^{\alpha-2} vol,$$

where $vol = vol\,(r,\theta,\phi)$ is the volume density on Σ_τ with r,θ,ϕ coordinates, with boundary condition $k(R',\theta,\phi) = 0$. Now

$$\begin{aligned}\int_{\Sigma_\tau} r^{\alpha-2}\Phi^2 =& \iiint_{r_+}^{\infty} k'(r)\Phi^2 dr d\theta d\phi \\ =& -2\iiint k(r)\Phi\partial_r\Phi dr d\theta d\phi \\ \leq& 2\left(\iiint \frac{k(r)^2}{k'(r)}\left(\partial_r\Phi\right)^2 dr d\theta d\phi\right)^{\frac{1}{2}} \left(\iiint k'(r)\Phi^2 dr d\theta d\phi\right)^{\frac{1}{2}}\end{aligned}$$

Notice that $vol \sim r^2$, $k(r) \sim r^{\alpha+1}$ and $k'(r) \sim r^\alpha$. Hence $\frac{1+k(r)^2}{1+k'(r)} \sim r^\alpha vol$. The lemma follows. □

We now add up the estimates for $\Phi_\flat$, $\Phi_\natural$, $\Phi_\flat$ and $\Phi_\sharp$.

Proposition 9.4.11.

$$\begin{aligned}&\iint_{\mathcal{R}(\tau',\tau)} \left(r^{-1-\delta}\mathbb{1}_{\{|r-3M|\geq\frac{M}{8}\}} J_\mu^{N_e}(\Phi)\, n_{\Sigma_\tau}^\mu + r^{-1-\delta}\left(\partial_r\Phi\right)^2 + r^{-3-\delta}\Phi^2\right) \\ \leq& C\left(\int_{\Sigma_\tau} J_\mu^{N_e}(\Phi)\, n_{\Sigma_\tau}^\mu + \int_{\Sigma_{\tau'}} J_\mu^{N_e}(\Phi)\, n_{\Sigma_{\tau'}}^\mu + \int_{\mathcal{H}(\tau',\tau)} J_\mu^{N_e}(\Phi)\, n_{\mathcal{H}^+}^\mu\right. \\ &+C\iint_{\mathcal{R}(\tau'-1,\tau+1)} \left(|\partial_{r^*}\Phi| + r^{-1}|\Phi|\right)|G| + C\left|\iint_{\mathcal{R}(\tau'-1,\tau+1)\cap\{r\leq\frac{23M}{8}\}} (\partial_{t^*}\Phi + eY\Phi)G\right| \\ &\left.+C\iint_{\mathcal{R}(\tau'-1,\tau+1)} G^2\right).\end{aligned}$$

Proof. Since the function f appears identically in Propositions 9.4.8 and 9.4.9, we

can add up the estimates to obtain:

$$\begin{aligned}
&\iint_{\mathcal{R}(-\infty,\infty)} \Big(\chi(J_\mu^{N_e}(\Phi_\flat)+J_\mu^{N_e}(\Phi_\natural)+\mathbb{1}_{\{|r-3M|\geq\frac{M}{8}\}}J_\mu^{N_e}(\Phi_\flat)+J_\mu^{N_e}(\Phi_\sharp))n^\mu_{\Sigma_\tau}\\
&\qquad +\chi\left(\Phi_\flat^2+\Phi_\natural^2+\Phi_\flat^2+\Phi_\sharp^2\right)\Big)\\
\leq& C\int_{\mathcal{H}(\tau'-1,\tau+1)} J_\mu^{N_e}(\Phi)n^\mu_{\mathcal{H}^+}+C(\delta')^{-1}\iint_{\mathcal{R}(-\infty,\infty)\cap\{r\leq R\}} F^2\\
&+C\delta'\iint_{\mathcal{R}(-\infty,\infty)\cap\{r\leq R\}}(\Phi^\tau_{\tau'})^2+(\partial_{r^*}\Phi^\tau_{\tau'})^2+\mathbb{1}_{\{r\leq\frac{23M}{8}\}}J_\mu^{N_e}(\Phi^\tau_{\tau'})n^\mu_{\Sigma_{t^*}}\\
&+\int_{-\infty}^{\infty}dt^*\int_{\{r\geq R\}}\Big(2f(r^2+a^2)^{1/2}(F_\flat+F_\natural+F_\flat+F_\sharp)\partial_{r^*}((r^2+a^2)^{1/2}\Phi^\tau_{\tau'})\\
&+f'(r^2+a^2)(F_\flat+F_\natural+F_\flat+F_\sharp)\Phi^\tau_{\tau'}\Big)\frac{\Delta}{r^2+a^2}\sin\theta\,d\phi\,d\theta\,dr^*.
\end{aligned}$$

By the definition of the cutoff, we have the pointwise equalities

$$F=F_\flat+F_\natural+F_\flat+F_\sharp.$$

Therefore, we have

$$\begin{aligned}
&\iint_{\mathcal{R}(-\infty,\infty)}\chi\mathbb{1}_{\{|r-3M|\geq\frac{M}{8}\}}J_\mu^{N_e}(\Phi^\tau_{\tau'})n^\mu_{\Sigma_{t^*}}+\chi\left(\Phi^\tau_{\tau'}\right)^2\\
\leq& C\int_{\mathcal{H}(\tau'-1,\tau+1)}J_\mu^{N_e}(\Phi)n^\mu_{\mathcal{H}^+}+C(\delta')^{-1}\iint_{\mathcal{R}(-\infty,\infty)\cap\{r\leq R\}}F^2\\
&+C\delta'\iint_{\mathcal{R}(-\infty,\infty)\cap\{r\leq R\}}(\Phi^\tau_{\tau'})^2+(\partial_{r^*}\Phi^\tau_{\tau'})^2+\mathbb{1}_{\{r\leq\frac{23M}{8}\}}J_\mu^{N_e}(\Phi^\tau_{\tau'})n^\mu_{\Sigma_{t^*}}\\
&+\int_{-\infty}^{\infty}dt^*\int_{\{r\geq R\}}\big(2f(r^2+a^2)^{1/2}F\partial_{r^*}((r^2+a^2)^{1/2}\Phi^\tau_{\tau'})\\
&\qquad +f'(r^2+a^2)F\Phi^\tau_{\tau'}\big)\frac{\Delta}{r^2+a^2}\sin\theta\,d\phi\,d\theta\,dr^*.
\end{aligned}$$

First, by Proposition 9.3.4, we have

$$\int_{\mathcal{H}(\tau'-1,\tau+1)}J_\mu^{N_e}(\Phi)n^\mu_{\mathcal{H}^+}\leq C\int_{\Sigma_{\tau'}}J_\mu^{N_e}(\Phi)n^\mu_{\Sigma_{\tau'}}+C\int_{\Sigma_\tau}J_\mu^{N_e}\left(\Phi\right)n^\mu_{\Sigma_\tau}+\int_{\mathcal{H}(\tau',\tau)}J_\mu^{N_e}(\Phi)n^\mu_{\mathcal{H}^+}.$$

Recall that

$$F = \xi G + 2D^\alpha \Phi D_\alpha \xi + \Phi \Box_{g_K} \xi.$$

Notice that by the definition of ξ, the last two terms are supported in the t^* range $(\tau' - 1, \tau') \cup (\tau, \tau + 1)$. Moreover, since ξ depends only on t^*, the only terms involving $D\Phi$ are $\partial_{t^*}\Phi$ and $O(\epsilon)\partial_{\phi^*}\Phi$. Using this, we immediately have the following with C independent of e as long as $\epsilon \ll e$:

$$\begin{aligned} &C(\delta')^{-1} \iint_{\mathcal{R}(-\infty,\infty)\cap\{r\le R\}} F^2 \\ \le &C(\delta')^{-1} \left(\iint_{\mathcal{R}(\tau'-1,\tau+1)\cap\{r\le R\}} G^2 + \iint_{\mathcal{R}(\tau'-1,\tau')\cup\mathcal{R}(\tau,\tau+1)} \left(r^{-2}\Phi^2 + J_\mu^{N_e}(\Phi) n_{\Sigma_{t^*}}^\mu \right) \right). \end{aligned}$$

Similarly, we have

$$\begin{aligned} &\int_{-\infty}^{\infty} dt^* \int_{\{r\ge R\}} f'(r^2+a^2) F \Phi_{\tau'}^\tau \frac{\Delta}{r^2+a^2} \sin\theta \, d\phi \, d\theta \, dr^* \\ \le &C \left(\iint_{\mathcal{R}(\tau'-1,\tau+1)} r^{-1} |\Phi| |G| + \iint_{\mathcal{R}(\tau'-1,\tau')\cup\mathcal{R}(\tau,\tau+1)} \left(r^{-2}\Phi^2 + J_\mu^{N_e}(\Phi) n_{\Sigma_{t^*}}^\mu \right) \right). \end{aligned}$$

The other term with F is more delicate to estimate. One of the terms in the expansion does not have sufficient decay in r:

$$\begin{aligned} &\int_{-\infty}^{\infty} dt^* \int_{\{r\ge R\}} 2f(r^2+a^2)^{1/2} F \partial_{r^*}((r^2+a^2)^{1/2}\Phi_{\tau'}^\tau) \frac{\Delta}{r^2+a^2} \sin\theta \, d\phi \, d\theta \, dr^* \\ \le &C \left(\iint_{\mathcal{R}(\tau'-1,\tau+1)} r^{-1} |\Phi| |G| + \iint_{\mathcal{R}(\tau'-1,\tau')\cup\mathcal{R}(\tau,\tau+1)} \left(r^{-2}\Phi^2 + J_\mu^{N_e}(\Phi) n_{\Sigma_{t^*}}^\mu \right) \right) \\ &+ \int_{-\infty}^{\infty} dt^* \int_{\{r\ge R\}} 2f(r^2+a^2)^{1/2} \Phi \Box_{g_K} \xi \partial_{r^*}((r^2+a^2)^{1/2}\Phi) \xi \frac{\Delta}{r^2+a^2} \sin\theta \, d\phi \, d\theta \, dr^* \end{aligned}$$

Nevertheless, noting that ξ is independent of r^*, an integration by parts in r^* would

give

$$\begin{aligned}
&\int_{-\infty}^{\infty} dt^* \int_{\{r\geq R\}} 2f(r^2+a^2)^{1/2}\Phi\Box_{g_K}\xi\partial_{r^*}((r^2+a^2)^{1/2}\Phi)\xi\frac{\Delta}{r^2+a^2}\sin\theta\, d\phi\, d\theta\, dr^* \\
=&-\int_{-\infty}^{\infty} dt^* \int_{\{r\geq R\}} (r^2+a^2)\Phi^2\xi\Box_{g_K}\xi\partial_{r^*}\left(f\frac{\Delta}{r^2+a^2}\right)\sin\theta\, d\phi\, d\theta\, dr^* + \text{bdry terms} \\
\leq& C\iint_{\mathcal{R}(\tau'-1,\tau')\cup\mathcal{R}(\tau,\tau+1)} r^{-2}\Phi^2,
\end{aligned}$$

where the boundary terms can be controlled (after possibly changing R) by pigeonholing in $r\in[R,R+1]$. By the mild growth estimate of Proposition 9.3.4, the estimate near the event horizon from Proposition 9.4.7 and the Hardy inequality of Proposition 9.4.10,

$$\begin{aligned}
&\iint_{\mathcal{R}(\tau'-1,\tau')\cup\mathcal{R}(\tau,\tau+1)} \left(r^{-2}\Phi^2 + J_\mu^{N_e}(\Phi)\, n^\mu_{\Sigma_{t^*}}\right) \\
\leq& C\left(\int_{\Sigma_{\tau'}} J_\mu^{N_e}(\Phi)\, n^\mu_{\Sigma_{\tau'}} + \int_{\Sigma_\tau} J_\mu^{N_e}(\Phi)\, n^\mu_{\Sigma_{\tau'}} + \iint_{\mathcal{R}(\tau'-1,\tau')\cup\mathcal{R}(\tau,\tau+1)} G^2\right).
\end{aligned}$$

Therefore, using all the above estimates and noticing the support of ξ, we have

$$\begin{aligned}
&\iint_{\mathcal{R}(\tau',\tau)} \chi\left(\mathbb{1}_{\{|r-3M|\geq\frac{M}{8}\}} J_\mu^{N_e}(\Phi) n^\mu_{\Sigma_{t^*}} + \Phi^2\right) \\
\leq& C(\delta')^{-1}\int_{\Sigma_{\tau'}} J_\mu^{N_e}(\Phi) n^\mu_{\Sigma_{\tau'}} + C(\delta')^{-1}\int_{\Sigma_\tau} J_\mu^{N_e}(\Phi)\, n^\mu_{\Sigma_\tau} + C\int_{\mathcal{H}(\tau',\tau)} J_\mu^{N_e}(\Phi)\, n^\mu_{\mathcal{H}^+} \\
&+ C\iint_{\mathcal{R}(\tau'-1,\tau+1)} r^{-1}|\Phi||G| + C(\delta')^{-1}\iint_{\mathcal{R}(\tau'-1,\tau+1)} G^2 \\
&+ C\delta'\iint_{\mathcal{R}(\tau',\tau)\cap\{r\leq R\}} \Phi^2 + (\partial_{r^*}\Phi)^2 + \mathbb{1}_{\{r\leq\frac{23M}{8}\}} J_\mu^{N_e}(\Phi) n^\mu_{\Sigma_{t^*}}.
\end{aligned}$$

We add to this the estimates near spatial infinity and the event horizon, i.e., Propo-

sitions 9.4.4 and 9.4.7, to get

$$\begin{aligned}
&\iint_{\mathcal{R}(\tau',\tau)} r^{-1-\delta}\mathbb{1}_{\{|r-3M|\geq\frac{M}{8}\}}J_\mu^{N_e}(\Phi)n_{\Sigma_{t^*}}^\mu + r^{-1-\delta}\left(\partial_r\Phi\right)^2 + r^{-3-\delta}\Phi^2\\
\leq& C(\delta')^{-1}\int_{\Sigma_{\tau'}} J_\mu^N(\Phi)n_{\Sigma_{\tau'}}^\mu + C(\delta')^{-1}\int_{\Sigma_\tau} J_\mu^{N_e}\left(\Phi\right)n_{\Sigma_\tau}^\mu + C\int_{\mathcal{H}(\tau',\tau)} J_\mu^{N_e}\left(\Phi\right)n_{\mathcal{H}^+}^\mu\\
&+ C\iint_{\mathcal{R}(\tau'-1,\tau+1)}\left(|\partial_{r^*}\Phi| + r^{-1}|\Phi|\right)|G| + C|\iint_{\mathcal{R}(\tau'-1,\tau+1)\cap\{r\leq\frac{23M}{8}\}}(\partial_{t^*}\Phi + eY\Phi)G|\\
&+ C(\delta')^{-1}\iint_{\mathcal{R}(\tau'-1,\tau+1)} G^2 + C\delta'\iint_{\mathcal{R}(\tau'-1,\tau+1)\cap\{r\leq R\}}\Phi^2 + (\partial_{r^*}\Phi)^2\\
&+ \mathbb{1}_{\{r\leq\frac{23M}{8}\}}J_\mu^N(\Phi)n_{\Sigma_{t^*}}^\mu.
\end{aligned}$$

By choosing δ' sufficiently small and absorbing $(\delta')^{-1}$ into the constant C, we can absorb the last term:

$$\begin{aligned}
&\iint_{\mathcal{R}(\tau',\tau)} r^{-1-\delta}\mathbb{1}_{\{|r-3M|\geq\frac{M}{8}\}}J_\mu^{N_e}(\Phi)n_{\Sigma_{t^*}}^\mu + r^{-1-\delta}\left(\partial_r\Phi\right)^2 + r^{-3-\delta}\Phi^2\\
\leq& C\int_{\Sigma_{\tau'}} J_\mu^{N_e}(\Phi)n_{\Sigma_{\tau'}}^\mu + C\int_{\Sigma_\tau} J_\mu^{N_e}\left(\Phi\right)n_{\Sigma_\tau}^\mu + C\int_{\mathcal{H}(\tau',\tau)} J_\mu^{N_e}\left(\Phi\right)n_{\mathcal{H}^+}^\mu\\
&+ C\iint_{\mathcal{R}(\tau'-1,\tau+1)\cap\{r\leq\frac{23M}{8}\}}\left(|\partial_{r^*}\Phi| + r^{-1}|\Phi|\right)|G|\\
&+ C|\iint_{\mathcal{R}(\tau'-1,\tau+1)\cap\{r\leq\frac{23M}{8}\}}(\partial_{t^*}\Phi + eY\Phi)G| + C\iint_{\mathcal{R}(\tau'-1,\tau+1)} G^2.
\end{aligned}$$

using Proposition 9.3.4 and 9.4.10 at the last step. □

Definition 9.4.12. *From now on, denote*

$$K^{X_0}\left(\Phi\right) = r^{-1-\delta}\mathbb{1}_{\{|r-3M|\geq\frac{M}{8}\}}J_\mu^N\left(\Phi\right)n_{\Sigma_\tau}^\mu + r^{-1-\delta}\left(\partial_r\Phi\right)^2 + r^{-3-\delta}\Phi^2, \quad \textit{and}$$

$$K^{X_1}\left(\Phi\right) = r^{-1-\delta}J_\mu^N\left(\Phi\right)n_{\Sigma_\tau}^\mu + r^{-3-\delta}\Phi^2.$$

We remark that this is a slight abuse of notation because these "currents" do not arise directly from a vector field.

9.4.7 Boundedness of the Non-degenerate Energy

Proposition 9.4.13. *Let Φ satisfy $\Box_{g_K}\Phi = G$. For e sufficiently small and $\epsilon \ll e$, we have*

$$\begin{aligned}
&\int_{\Sigma_\tau} J_\mu^{N_e}(\Phi)\, n_{\Sigma_\tau}^\mu + \int_{\mathcal{H}(\tau',\tau)} J_\mu^{N_e}(\Phi)\, n_{\mathcal{H}^+}^\mu + \iint_{\mathcal{R}(\tau',\tau)\cap\{r\le r_Y^-\}} K_\mu^{N_e}(\Phi) \\
\le& C\left(\int_{\Sigma_{\tau'}} J_\mu^{N_e}(\Phi)\, n_{\Sigma_{\tau'}}^\mu + |\iint_{\mathcal{R}(\tau'-1,\tau+1)} \partial_{t^*}\Phi G| + |\iint_{\mathcal{R}(\tau'-1,\tau+1)} eY\Phi G|\right. \\
&\left. + \iint_{\mathcal{R}(\tau'-1,\tau+1)} \left(|\partial_r\Phi| + r^{-1}|\Phi|\right)|G| + \iint_{\mathcal{R}(\tau'-1,\tau+1)} G^2\right).
\end{aligned}$$

Proof. We recall that

$$\iint_{\mathcal{R}(\tau',\tau)\cap\{r_Y^-\le r\le r_Y^+\}} K^Y(\Phi) \le C\int_{\tau'}^{\tau}\int_{\Sigma_{\bar\tau}} J_\mu^{N_e}(\Phi)\, n_{\Sigma_{t^*}}^\mu dt^*,$$

with C independent of e and ϵ whenever $\epsilon \ll e < 1$. At this point, we choose $r_Y^+ < \frac{11M}{4} < \frac{23M}{8}$. Hence this term can be controlled by the integrated decay estimates. Then, by Proposition 9.3.3,

$$\begin{aligned}
&\int_{\Sigma_\tau} J_\mu^{N_e}(\Phi)\, n_{\Sigma_\tau}^\mu + \int_{\mathcal{H}(\tau',\tau)} J_\mu^{N_e}(\Phi)\, n_{\mathcal{H}^+}^\mu + \iint_{\mathcal{R}(\tau',\tau)\cap\{r\le r_Y^-\}} K_\mu^{N_e}(\Phi) \\
=& \int_{\Sigma_{\tau'}} J_\mu^{N_e}(\Phi)\, n_{\Sigma_{\tau'}}^\mu + e\iint_{\mathcal{R}(\tau',\tau)\cap\{r_Y^-\le r\le r_Y^+\}} K^Y(\Phi) + \iint_{\mathcal{R}(\tau',\tau)} (\partial_{t^*}\Phi + eY\Phi)\, G \\
\le& \int_{\Sigma_{\tau'}} J_\mu^{N_e}(\Phi)\, n_{\Sigma_{\tau'}}^\mu + Ce\int_{\tau'}^{\tau}\int_{\Sigma_{\bar\tau}\cap\{r_Y^-\le r\le r_Y^+\}} J_\mu^{N_e}(\Phi)\, n_{\Sigma_{\bar\tau}}^\mu d\bar\tau \\
&+ |\iint_{\mathcal{R}(\tau',\tau)} (\partial_{t^*}\Phi + eY\Phi)\, G| \\
\le& \int_{\Sigma_{\tau'}} J_\mu^{N_e}(\Phi)\, n_{\Sigma_{\tau'}}^\mu + Ce\left(\int_{\Sigma_\tau} J_\mu^{N_e}(\Phi)\, n_{\Sigma_\tau}^\mu + \int_{\Sigma_{\tau'}} J_\mu^{N_e}(\Phi)\, n_{\Sigma_{\tau'}}^\mu + \int_{\mathcal{H}(\tau',\tau)} J_\mu^{N_e}(\Phi) n_{\mathcal{H}^+}^\mu\right. \\
&\left. + \iint_{\mathcal{R}(\tau'-1,\tau+1)} \left(|\partial_r\Phi| + r^{-1}|\Phi|\right)|G| + |\iint_{\mathcal{R}(\tau'-1,\tau+1)} G^2\right) \\
&+ |\iint_{\mathcal{R}(\tau',\tau)} (\partial_{t^*}\Phi + eY\Phi)\, G|.
\end{aligned}$$

Hence, the Proposition holds if e is chosen to be sufficiently small. $\square$

Remark 17. *From this point on, we will consider r_Y^+ and e to be fixed. After e is fixed, the vector field N_e will be written simply as N.*

We now estimate the inhomogeneous terms in Proposition 9.4.13:

Proposition 9.4.14.

$$\begin{aligned}
&\int_{\Sigma_\tau} J_\mu^N(\Phi)\, n_{\Sigma_\tau}^\mu + \int_{\mathcal{H}(\tau',\tau)} J_\mu^N(\Phi)\, n_{\mathcal{H}^+}^\mu + \iint_{\mathcal{R}(\tau',\tau)\cap\{r\le r_Y^-\}} K^N(\Phi) + \iint_{\mathcal{R}(\tau',\tau)} K^{X_0}(\Phi) \\
\le& C\left(\int_{\Sigma_{\tau'}} J_\mu^N(\Phi)\, n_{\Sigma_{\tau'}}^\mu + \left(\int_{\tau'-1}^{\tau+1}\left(\int_{\Sigma_{t^*}} G^2\right)^{\frac{1}{2}} dt^*\right)^2 + \iint_{\mathcal{R}(\tau',\tau)} G^2\right).
\end{aligned}$$

Proof. Adding the estimates in Propositions 9.4.11 and δ times the estimates in Proposition 9.4.13,

$$\begin{aligned}
&\int_{\Sigma_\tau} J_\mu^N(\Phi)\, n_{\Sigma_\tau}^\mu + \int_{\mathcal{H}(\tau',\tau)} J_\mu^N(\Phi)\, n_{\mathcal{H}^+}^\mu \\
&+ \iint_{\mathcal{R}(\tau',\tau)\cap\{r\le r_Y^-\}} K^N(\Phi) + \delta\iint_{\mathcal{R}(\tau',\tau)} K^{X_0}(\Phi) \\
\le& C\left(\int_{\Sigma_{\tau'}} J_\mu^N(\Phi)\, n_{\Sigma_{\tau'}}^\mu + \iint_{\mathcal{R}(\tau'-1,\tau+1)} \left(|\partial\Phi| + r^{-1}|\Phi|\right)|G| + \iint_{\mathcal{R}(\tau'-1,\tau+1)} G^2\right) \\
&+ C\delta\left(\int_{\Sigma_\tau} J_\mu^N(\Phi)\, n_{\Sigma_\tau}^\mu + \int_{\mathcal{H}(\tau',\tau)} J_\mu^N(\Phi)\, n_{\mathcal{H}^+}^\mu\right) \\
\le& C\left(\int_{\Sigma_{\tau'}} J_\mu^N(\Phi)\, n_{\Sigma_{\tau'}}^\mu + \sup_{t^*\in[\tau'-1,\tau+1]}\left(\int_{\Sigma_{t^*}} J_\mu^N(\Phi)\, n_{\Sigma_{t^*}}^\mu\right)^{\frac{1}{2}} \int_{\tau'-1}^{\tau+1}\left(\int_{\Sigma_{t^*}} G^2\right)^{\frac{1}{2}} dt^*\right. \\
&\left.+ \iint_{\mathcal{R}(\tau'-1,\tau+1)} G^2\right) + C\delta\left(\int_{\Sigma_\tau} J_\mu^N(\Phi)\, n_{\Sigma_\tau}^\mu + \int_{\mathcal{H}(\tau',\tau)} J_\mu^N(\Phi)\, n_{\mathcal{H}^+}^\mu\right),
\end{aligned}$$

where at the last step we have used Proposition 9.4.10. Choosing $C\delta \le \frac{1}{2}$, we can

absorb the last term to the left hand side to get

$$\begin{aligned}
&\int_{\Sigma_\tau} J_\mu^N(\Phi)\, n_{\Sigma_\tau}^\mu + \int_{\mathcal{H}(\tau',\tau)} J_\mu^N(\Phi)\, n_{\mathcal{H}^+}^\mu \\
&+ \iint_{\mathcal{R}(\tau',\tau)\cap\{r\leq r_Y^-\}} K^N(\Phi) + \delta \iint_{\mathcal{R}(\tau',\tau)} K^{X_0}(\Phi) \\
\leq & C\left(\int_{\Sigma_{\tau'}} J_\mu^N(\Phi)\, n_{\Sigma_{\tau'}}^\mu + \sup_{t^*\in[\tau'-1,\tau+1]}\left(\int_{\Sigma_{t^*}} J_\mu^N(\Phi)\, n_{\Sigma_{t^*}}^\mu\right)^{\frac{1}{2}} \int_{\tau'-1}^{\tau+1}\left(\int_{\Sigma_{t^*}} G^2\right)^{\frac{1}{2}} dt^* \right. \\
&\left. + \iint_{\mathcal{R}(\tau'-1,\tau+1)} G^2\right).
\end{aligned} \tag{9.2}$$

By considering the above estimate on $[\tau',\tilde{\tau}]$, where $\tilde{\tau}$ is when the supremum on the right hand side is achieved, and using Proposition 9.3.4, we get

$$\begin{aligned}
&\int_{\Sigma_{\tilde{\tau}}} J_\mu^N(\Phi)\, n_{\Sigma_{\tilde{\tau}}}^\mu + \int_{\mathcal{H}(\tau',\tilde{\tau})} J_\mu^N(\Phi)\, n_{\mathcal{H}^+}^\mu \\
&+ \iint_{\mathcal{R}(\tau',\tilde{\tau})\cap\{r\leq r_Y^-\}} K^N(\Phi) + \delta \iint_{\mathcal{R}(\tau',\tilde{\tau})} K^{X_0}(\Phi) \\
\leq & C\left(\int_{\Sigma_{\tau'}} J_\mu^N(\Phi)\, n_{\Sigma_{\tau'}}^\mu + \left(\int_{\tau'-1}^{\tau+1}\left(\int_{\Sigma_{t^*}} G^2\right)^{\frac{1}{2}} dt^*\right)^2\right).
\end{aligned}$$

We plug this into (9.2) and apply Cauchy-Schwarz to prove the proposition. □

We can also estimate the inhomogeneous terms not in L^1L^2 but in L^2L^2, provided that we allow some extra factors of r and some loss of derivatives in G. This is especially useful for estimating the commutator terms from S, which do not have sufficient decay in t^* in the interior to be estimated in L^1L^2. More precisely, we have

Proposition 9.4.15.

$$\begin{aligned}
&\int_{\Sigma_\tau} J_\mu^N(\Phi)\, n_{\Sigma_\tau}^\mu + \int_{\mathcal{H}(\tau',\tau)} J_\mu^N(\Phi)\, n_{\mathcal{H}^+}^\mu \\
&+ \iint_{\mathcal{R}(\tau',\tau)\cap\{r\le r_Y^-\}} K^N(\Phi) + \iint_{\mathcal{R}(\tau',\tau)} K^{X_0}(\Phi) \\
\le& C\left(\int_{\Sigma_{\tau'}} J_\mu^N(\Phi)\, n_{\Sigma_{\tau'}}^\mu + \sum_{m=0}^{1}\iint_{\mathcal{R}(\tau'-1,\tau+1)} r^{1+\delta}\left(\partial_{t^*}^m G\right)^2\right. \\
&\left. + \sup_{t^*\in[\tau'-1,\tau+1]}\int_{\Sigma_{t^*}\cap\{|r-3M|\le\frac{M}{8}\}} G^2\right).
\end{aligned}$$

Proof. By Propositions 9.4.11 and 9.4.13,

$$\begin{aligned}
&\int_{\Sigma_\tau} J_\mu^N(\Phi)\, n_{\Sigma_\tau}^\mu + \int_{\mathcal{H}(\tau',\tau)} J_\mu^N(\Phi)\, n_{\mathcal{H}^+}^\mu \\
&+ \iint_{\mathcal{R}(\tau',\tau)\cap\{r\le r_Y^-\}} K^N(\Phi) + \delta\iint_{\mathcal{R}(\tau',\tau)} K^{X_0}(\Phi) \\
\le& C\left(\int_{\Sigma_{\tau'}} J_\mu^{N_e}(\Phi)\, n_{\Sigma_{\tau'}}^\mu + \left|\iint_{\mathcal{R}(\tau'-1,\tau+1)} \partial_{t^*}\Phi G\right| + \left|\iint_{\mathcal{R}(\tau'-1,\tau+1)} eY\Phi G\right|\right. \\
&\left. + \iint_{\mathcal{R}(\tau'-1,\tau+1)} \left(|\partial_r\Phi| + r^{-1}|\Phi|\right)|G| + \iint_{\mathcal{R}(\tau'-1,\tau+1)} G^2\right) \\
&+ C\delta'\left(\int_{\Sigma_\tau} J_\mu^N(\Phi)\, n_{\Sigma_\tau}^\mu + \int_{\mathcal{H}(\tau',\tau)} J_\mu^N(\Phi)\, n_{\mathcal{H}^+}^\mu\right).
\end{aligned}$$

Choosing $C\delta' \le \frac{1}{2}$, we can absorb the last term to the left hand side to get

$$\begin{aligned}
&\int_{\Sigma_\tau} J_\mu^N(\Phi)\, n_{\Sigma_\tau}^\mu + \int_{\mathcal{H}(\tau',\tau)} J_\mu^N(\Phi)\, n_{\mathcal{H}^+}^\mu + \iint_{\mathcal{R}(\tau',\tau)\cap\{r\le r_Y^-\}} K^N(\Phi) + \iint_{\mathcal{R}(\tau',\tau)} K^{X_0}(\Phi) \\
\le& C\left(\int_{\Sigma_{\tau'}} J_\mu^N(\Phi)\, n_{\Sigma_{\tau'}}^\mu + \left|\iint_{\mathcal{R}(\tau'-1,\tau+1)} \partial_{t^*}\Phi G\right| + \left|\iint_{\mathcal{R}(\tau'-1,\tau+1)} eY\Phi G\right|\right. \\
&\left. + \iint_{\mathcal{R}(\tau'-1,\tau+1)} \left(|\partial_r\Phi| + r^{-1}|\Phi|\right)|G| + \iint_{\mathcal{R}(\tau'-1,\tau+1)} G^2\right).
\end{aligned}$$

For the bulk error term, we focus at the region $\{|r-3M| \le \frac{M}{8}\}$ and integrate by

parts.

$$
\begin{aligned}
&\Big|\iint_{\mathcal{R}(\tau'-1,\tau+1)\cap\{|r-3M|\leq\frac{M}{8}\}}\partial_{t^*}\Phi G\Big|\\
\leq&\delta'\iint_{\mathcal{R}(\tau'-1,\tau+1)\cap\{|r-3M|\leq\frac{M}{8}\}}\Phi^2+C(\delta')^{-1}\iint_{\mathcal{R}(\tau'-1,\tau+1)\cap\{|r-3M|\leq\frac{M}{8}\}}(\partial_{t^*}G)^2\\
&+\Big|\int_{\Sigma_{\tau+1}\cap\{|r-3M|\leq\frac{M}{8}\}}\Phi G\Big|+\Big|\int_{\Sigma_{\tau'-1}\cap\{|r-3M|\leq\frac{M}{8}\}}\Phi G\Big|\\
\leq&\delta'\iint_{\mathcal{R}(\tau'-1,\tau+1)\cap\{|r-3M|\leq\frac{M}{8}\}}r^{-3-\delta}\Phi^2+C(\delta')^{-1}\iint_{\mathcal{R}(\tau'-1,\tau+1)\cap\{|r-3M|\leq\frac{M}{8}\}}(\partial_{t^*}G)^2\\
&+\sup_{t^*\in[\tau'-1,\tau+1]}\left(\delta\int_{\Sigma_{t^*}\cap\{|r-3M|\leq\frac{M}{8}\}}r^{-2}\Phi^2+C(\delta')^{-1}\int_{\Sigma_{t^*}\cap\{|r-3M|\leq\frac{M}{8}\}}G^2\right)\\
\leq&\delta'\iint_{\mathcal{R}(\tau'-1,\tau+1)\cap\{|r-3M|\leq\frac{M}{8}\}}r^{-3-\delta}\Phi^2+C(\delta')^{-1}\iint_{\mathcal{R}(\tau'-1,\tau+1)\cap\{|r-3M|\leq\frac{M}{8}\}}(\partial_{t^*}G)^2\\
&+\sup_{t^*\in[\tau'-1,\tau+1]}\left(\delta'\int_{\Sigma_{t^*}}J_\mu^N(\Phi)\,n_{\Sigma_{t^*}}^\mu+C(\delta')^{-1}\int_{\Sigma_{t^*}\cap\{|r-3M|\leq\frac{M}{8}\}}G^2\right),
\end{aligned}
$$

where at the last step we used Proposition 9.4.10.

Therefore,

$$
\begin{aligned}
&\int_{\Sigma_\tau} J_\mu^N(\Phi)\, n_{\Sigma_\tau}^\mu + \int_{\mathcal{H}(\tau',\tau)} J_\mu^N(\Phi)\, n_{\mathcal{H}^+}^\mu \\
&+ \iint_{\mathcal{R}(\tau',\tau)\cap\{r\le r_Y^-\}} K^N(\Phi) + \iint_{\mathcal{R}(\tau',\tau)} K^{X_0}(\Phi) \\
\le& C\left(\int_{\Sigma_{\tau'}} J_\mu^N(\Phi)\, n_{\Sigma_{\tau'}}^\mu + \Big|\iint_{\mathcal{R}(\tau'-1,\tau+1)} \partial_{t^*}\Phi G\Big| + \Big|\iint_{\mathcal{R}(\tau'-1,\tau+1)} eY\Phi G\Big|\right. \\
&\left.+ \iint_{\mathcal{R}(\tau'-1,\tau+1)} \left(|\partial_r\Phi| + r^{-1}|\Phi|\right)|G| + \iint_{\mathcal{R}(\tau'-1,\tau+1)} G^2\right) \\
\le& C\left(\int_{\Sigma_{\tau'}} J_\mu^N(\Phi)\, n_{\Sigma_{\tau'}}^\mu + \Big|\iint_{\mathcal{R}(\tau'-1,\tau+1)\cap\{|r-3M|\le\frac{M}{8}\}} \partial_{t^*}\Phi G\Big|\right) \\
&+ C(\delta')^{-1}\iint_{\mathcal{R}(\tau'-1,\tau+1)} r^{1+\delta}G^2 \\
&+ \delta'\iint_{\mathcal{R}(\tau'-1,\tau+1)} \left(r^{-3-\delta}\Phi^2 + r^{-1-\delta}\left(\partial_r\Phi\right)^2 + \mathbb{1}_{\{r\le r_Y^+\}} J_\mu^N(\Phi)\, n_{\Sigma_{t^*}}^\mu\right) \\
\le& C\int_{\Sigma_{\tau'}} J_\mu^N(\Phi)\, n_{\Sigma_{\tau'}}^\mu + C(\delta')^{-1}\sum_{m=0}^{1}\int_{\mathcal{R}(\tau'-1,\tau+1)} r^{1+\delta}\left(\partial_{t^*}^m G\right)^2 \\
&+ \delta'\iint_{\mathcal{R}(\tau'-1,\tau+1)} K^{X_0}(\Phi) \\
&+ \sup_{t^*\in[\tau'-1,\tau+1]}\left(\delta'\int_{\Sigma_{t^*}} J_\mu^N(\Phi)\, n_{\Sigma_{t^*}}^\mu + C(\delta')^{-1}\int_{\Sigma_{t^*}\cap\{|r-3M|\le\frac{M}{8}\}} G^2\right).
\end{aligned}
\tag{9.3}
$$

where at the last step we have used Propositions 9.3.4, 9.4.10. Suppose

$$
\sup_{t^*\in[\tau'-1,\tau+1]} \delta'\int_{\Sigma_{t^*}} J_\mu^N(\Phi)\, n_{\Sigma_{t^*}}^\mu
$$

is achieved by $t^* = \tilde{\tau}$. Apply (9.3) on $[\tau', \tilde{\tau}]$, we get

$$\begin{aligned}
&\int_{\Sigma_{\tilde{\tau}}} J_\mu^N(\Phi)\, n_{\Sigma_{\tilde{\tau}}}^\mu \\
\leq & C\int_{\Sigma_{\tau'}} J_\mu^N(\Phi)\, n_{\Sigma_{\tau'}}^\mu + C(\delta')^{-1}\sum_{m=0}^{1}\iint_{\mathcal{R}(\tau'-1,\tau+1)} r^{1+\delta}\left(\partial_{t^*}^m G\right)^2 \\
&+ \delta'\iint_{\mathcal{R}(\tau'-1,\tau+1)} K^{X_0}(\Phi) + \delta'\int_{\Sigma_{\tilde{\tau}}} J_\mu^N(\Phi)\, n_{\Sigma_{\tilde{\tau}}}^\mu \\
&+ C\sup_{t^*\in[\tau'-1,\tau+1]}\int_{\Sigma_{t^*}\cap\{|r-3M|\leq\frac{M}{8}\}} G^2,
\end{aligned}$$

which, upon choosing $\delta' \leq \frac{1}{2}$ and subtracting the small term on both sides, gives

$$\begin{aligned}
&\int_{\Sigma_{\tilde{\tau}}} J_\mu^N(\Phi)\, n_{\Sigma_{\tilde{\tau}}}^\mu \\
\leq & C\int_{\Sigma_{\tau'}} J_\mu^N(\Phi)\, n_{\Sigma_{\tau'}}^\mu + C(\delta')^{-1}\sum_{m=0}^{1}\iint_{\mathcal{R}(\tau'-1,\tau+1)} r^{1+\delta}\left(\partial_{t^*}^m G\right)^2 \\
&+ \delta'\iint_{\mathcal{R}(\tau'-1,\tau+1)} K^{X_0}(\Phi) + C\sup_{t^*\in[\tau'-1,\tau+1]}\int_{\Sigma_{t^*}\cap\{|r-3M|\leq\frac{M}{8}\}} G^2,
\end{aligned}$$

Therefore, plugging this back into (9.3)

$$\begin{aligned}
&\int_{\Sigma_\tau} J_\mu^N(\Phi)\, n_{\Sigma_\tau}^\mu + \int_{\mathcal{H}(\tau',\tau)} J_\mu^N(\Phi)\, n_{\mathcal{H}^+}^\mu \\
&+ \iint_{\mathcal{R}(\tau',\tau)\cap\{r\leq r_Y^-\}} K^N(\Phi) + \iint_{\mathcal{R}(\tau',\tau)} K^{X_0}(\Phi) \\
\leq & C\int_{\Sigma_{\tau'}} J_\mu^N(\Phi)\, n_{\Sigma_{\tau'}}^\mu + \delta'\iint_{\mathcal{R}(\tau',\tau)} K^{X_0}(\Phi) + \delta'\iint_{\mathcal{R}(\tau'-1,\tau')\cup\mathcal{R}(\tau,\tau+1)} K^{X_0}(\Phi) \\
&+ C\sum_{m=0}^{1}\iint_{\mathcal{R}(\tau'-1,\tau+1)} r^{1+\delta}\left(\partial_{t^*}^m G\right)^2 + C\sup_{t^*\in[\tau'-1,\tau+1]}\int_{\Sigma_{t^*}\cap\{|r-3M|\leq\frac{M}{8}\}} G^2 \\
\leq & C\int_{\Sigma_{\tau'}} J_\mu^N(\Phi)\, n_{\Sigma_{\tau'}}^\mu + \delta'\iint_{\mathcal{R}(\tau',\tau)} K^{X_0}(\Phi) + C\delta'\int_{\Sigma_\tau} J_\mu^N(\Phi)\, n_{\Sigma_\tau}^\mu \\
&+ C\sum_{m=0}^{1}\iint_{\mathcal{R}(\tau'-1,\tau+1)} r^{1+\delta}\left(\partial_{t^*}^m G\right)^2 + C\sup_{t^*\in[\tau'-1,\tau+1]}\int_{\Sigma_{t^*}\cap\{|r-3M|\leq\frac{M}{8}\}} G^2,
\end{aligned}$$

where at the last step we have used Proposition 9.3.4. Finally, by choosing $C\delta \leq \frac{1}{2}$,

we can absorb the small terms to the left hand side and achieve the conclusion of the Proposition. □

Notice that in the proof of Proposition 9.4.15, there is a loss in derivative for G because we have to integrate by parts in the region $\{|r-3M| \leq \frac{M}{8}\}$. Therefore, if G is supported away from this region, we can repeat the proof without this loss. In other words, we have

Proposition 9.4.16. *Suppose G is supported away from $\{|r-3M| \leq \frac{M}{8}\}$. Then*

$$\begin{aligned}&\int_{\Sigma_\tau} J_\mu^N(\Phi) n_{\Sigma_\tau}^\mu + \int_{\mathcal{H}(\tau',\tau)} J_\mu^N(\Phi) n_{\mathcal{H}^+}^\mu + \iint_{\mathcal{R}(\tau',\tau)\cap\{r\leq r_Y^-\}} K^N(\Phi) + \iint_{\mathcal{R}(\tau',\tau)} K^{X_0}(\Phi)\\ \leq & C\left(\int_{\Sigma_{\tau'}} J_\mu^N(\Phi) n_{\Sigma_{\tau'}}^\mu + \sum_{m=0}^{1}\iint_{\mathcal{R}(\tau'-1,\tau+1)} r^{1+\delta} G^2 + \sup_{t^*\in[\tau'-1,\tau+1]}\int_{\Sigma_{t^*}\cap\{|r-3M|\leq\frac{M}{8}\}} G^2\right).\end{aligned}$$

This will be useful in Section 9.9. In applications, it is useful to have both ways of estimating G.

Proposition 9.4.17. *Let $G = G_1 + G_2$ be any way to decompose the function G. Then*

$$\begin{aligned}&\int_{\Sigma_\tau} J_\mu^N(\Phi) n_{\Sigma_\tau}^\mu + \int_{\mathcal{H}(\tau',\tau)} J_\mu^N(\Phi) n_{\mathcal{H}^+}^\mu + \iint_{\mathcal{R}(\tau',\tau)\cap\{r\leq r_Y^-\}} K^N(\Phi) + \iint_{\mathcal{R}(\tau',\tau)} K^{X_0}(\Phi)\\ \leq & C\left(\int_{\Sigma_{\tau'}} J_\mu^N(\Phi) n_{\Sigma_{\tau'}}^\mu + \left(\int_{\tau'-1}^{\tau+1}\left(\int_{\Sigma_{t^*}} G_1^2\right)^{\frac{1}{2}} dt^*\right)^2 + \iint_{\mathcal{R}(\tau'-1,\tau+1)} G_1^2\right.\\ &\left. + \sum_{m=0}^{1}\iint_{\mathcal{R}(\tau'-1,\tau+1)} r^{1+\delta}\left(\partial_{t^*}^m G_2\right)^2 + \sup_{t^*\in[\tau'-1,\tau+1]}\int_{\Sigma_{t^*}\cap\{|r-3M|\leq\frac{M}{8}\}} G_2^2\right).\end{aligned}$$

If we instead use Proposition 9.4.16, we have

Proposition 9.4.18. *Let $G = G_1 + G_2$ be any way to decompose the function G.*

Suppose G_2 is supported away from $\{r : |r-3M| \leq \frac{M}{8}\}$. Then

$$\begin{aligned}
&\int_{\Sigma_\tau} J_\mu^N(\Phi) n_{\Sigma_\tau}^\mu + \int_{\mathcal{H}(\tau',\tau)} J_\mu^N(\Phi) n_{\mathcal{H}^+}^\mu + \iint_{\mathcal{R}(\tau',\tau)\cap\{r\leq r_Y^-\}} K^N(\Phi) + \iint_{\mathcal{R}(\tau',\tau)} K^{X_0}(\Phi) \\
\leq & C\left(\int_{\Sigma_{\tau'}} J_\mu^N(\Phi) n_{\Sigma_{\tau'}}^\mu + \left(\int_{\tau'-1}^{\tau+1}\left(\int_{\Sigma_{t^*}} G_1^2\right)^{\frac{1}{2}} dt^*\right)^2 + \iint_{\mathcal{R}(\tau'-1,\tau+1)} G_1^2\right. \\
&\left. + \iint_{\mathcal{R}(\tau'-1,\tau+1)} r^{1+\delta} G_2^2\right).
\end{aligned}$$

In the above estimates, only the function Φ and its ∂_r derivative can be estimated without a loss around the trapped set. To estimate the other derivatives, we need to commute with the Killing vector field ∂_{t^*}.

Proposition 9.4.19.

$$\begin{aligned}
&\iint_{\mathcal{R}(\tau',\tau)} K^{X_1}(\Phi) \\
\leq & C\left(\sum_{m=0}^1 \int_{\Sigma_{\tau'}} J_\mu^N(\partial_{t^*}^m\Phi) n_{\Sigma_{\tau'}}^\mu + \sum_{m=0}^1\left(\int_{\tau'-1}^{\tau+1}\left(\int_{\Sigma_{t^*}}(\partial_{t^*}^m G_1)^2\right)^{\frac{1}{2}} dt^*\right)^2\right. \\
&+ \sum_{m=0}^1 \iint_{\mathcal{R}(\tau'-1,\tau+1)} (\partial_{t^*}^m G_1)^2 + \sum_{m=0}^2 \iint_{\mathcal{R}(\tau'-1,\tau+1)} r^{1+\delta}(\partial_{t^*}^m G_2)^2 \\
&\left. + \sup_{t^*\in[\tau'-1,\tau+1]} \sum_{m=0}^1 \int_{\Sigma_{t^*}\cap\{|r-3M|\leq\frac{M}{8}\}} (\partial_{t^*}^m G_2)^2\right).
\end{aligned}$$

Proof. Using Proposition 9.4.17 and the fact that ∂_{t^*} is Killing, we immediately have the following estimate for $\partial_{t^*}\Phi$:

$$\begin{aligned}
&\iint_{\mathcal{R}(\tau',\tau)} r^{-3-\delta}(\partial_{t^*}\Phi)^2 \\
\leq & C\left(\int_{\Sigma_{\tau'}} J_\mu^N(\partial_{t^*}\Phi) n_{\Sigma_{\tau'}}^\mu + \left(\int_{\tau'-1}^{\tau+1}\left(\int_{\Sigma_{t^*}}(\partial_{t^*}G_1)^2\right)^{\frac{1}{2}} dt^*\right)^2 + \iint_{\mathcal{R}(\tau'-1,\tau+1)} (\partial_{t^*}G_1)^2\right. \\
&\left. + \sum_{m=1}^2 \iint_{\mathcal{R}(\tau'-1,\tau+1)} r^{1+\delta}(\partial_{t^*}^m G_2)^2 + \sup_{t^*\in[\tau'-1,\tau+1]} \int_{\Sigma_{t^*}\cap\{|r-3M|\leq\frac{M}{8}\}} (\partial_{t^*}G_2)^2\right).
\end{aligned}$$

This would allow us to estimate all derivatives of Φ except for the fact that the estimates for the angular derivatives of Φ degenerate around $r = 3M$:

$$\begin{aligned}
&\iint_{\mathcal{R}(\tau',\tau)} \left(r^{-1-\delta} \mathbb{1}_{\{|r-3M|\geq \frac{M}{8}\}} |\nabla\!\!\!/ \Phi|^2 + r^{-1-\delta} \left(\partial_r \Phi\right)^2 + r^{-1-\delta} \left(\partial_{t^*} \Phi\right)^2 + r^{-3-\delta} \Phi^2 \right) \\
\leq & C \left(\sum_{m=0}^{1} \int_{\Sigma_{\tau'}} J_\mu^N \left(\partial_{t^*}^m \Phi\right) n_{\Sigma_{\tau'}}^\mu + \sum_{m=0}^{1} \left(\int_{\tau'-1}^{\tau+1} \left(\int_{\Sigma_{t^*}} \left(\partial_{t^*}^m G_1\right)^2 \right)^{\frac{1}{2}} dt^* \right)^2 \right. \\
& + \sum_{m=0}^{1} \iint_{\mathcal{R}(\tau'-1,\tau+1)} \left(\partial_{t^*}^m G_1\right)^2 + \sum_{m=0}^{2} \iint_{\mathcal{R}(\tau'-1,\tau+1)} r^{1+\delta} \left(\partial_{t^*}^m G_2\right)^2 \\
& \left. + \sup_{t^* \in [\tau'-1,\tau+1]} \sum_{m=0}^{1} \int_{\Sigma_{t^*} \cap \{|r-3M|\leq \frac{M}{8}\}} \left(\partial_{t^*}^m G_2\right)^2 \right).
\end{aligned}$$

We now use this known estimate and construct another vector field to control the angular derivatives in the region $r \sim 3M$. The argument is simple because the estimate is only local. Take $f_{an}(r)$ to be compactly support in $3M - \frac{M}{4} \leq r \leq 3M + \frac{M}{4}$ and identically equal to -1 in $3M - \frac{M}{8} \leq r \leq 3M + \frac{M}{8}$. If we consider $X_{an} = f_{an}(r)\partial_{r^*}$ in Schwarzschild spacetime, we get that the coefficient in front of the terms with angular derivatives is $\frac{\mu}{2r}$, which is bounded below in $3M - \frac{M}{8} \leq r \leq 3M + \frac{M}{8}$. In other words, one gets an estimate of the following form:

$$\begin{aligned}
&\iint_{\mathcal{R}(\tau',\tau)} r^{-1-\delta} \mathbb{1}_{\{|r-3M|\leq \frac{M}{8}\}} |\nabla\!\!\!/ \Phi|^2 \\
\leq & C \left(\iint_{\mathcal{R}(\tau',\tau)} \left(r^{-1-\delta} \mathbb{1}_{\{|r-3M|\geq \frac{M}{8}\}} |\nabla\!\!\!/ \Phi|^2 + r^{-1-\delta} \left(\partial_r \Phi\right)^2 + r^{-1-\delta} \left(\partial_{t^*} \Phi\right)^2 + r^{-3-\delta} \Phi^2 \right) \right. \\
& + \int_{\Sigma_\tau} J_\mu^N \left(\Phi\right) n_{\Sigma_\tau}^\mu + \int_{\Sigma_{\tau'}} J_\mu^N \left(\Phi\right) n_{\Sigma_{\tau'}}^\mu + \iint_{\mathcal{R}(\tau',\tau)} \left(|\partial_r \Phi| + |r^{-1}\Phi| \right) |G| \\
& \left. + \iint_{\mathcal{R}(\tau',\tau)} G^2 \right).
\end{aligned} \tag{9.4}$$

Using a stability argument, (9.4) would hold also on Kerr spacetimes. One easily checks that the terms with G on the right hand side can be estimated in the same manner as before. Hence, the Proposition can be proved by applying Proposition

9.4.17. □

As before, if the inhomogeneous term is supported away from the trapped set, we can save a derivative:

Proposition 9.4.20. *Let $G = G_1 + G_2$ be any way to decompose the function G. Suppose G_2 is supported away from $\{r : |r - 3M| \leq \frac{M}{8}\}$. Then*

$$\begin{aligned}
&\iint_{\mathcal{R}(\tau',\tau)} K^{X_1}(\Phi) \\
\leq & C\left(\sum_{m=0}^{1}\int_{\Sigma_{\tau'}} J_\mu^N\left(\partial_{t^*}^m\Phi\right) n_{\Sigma_{\tau'}}^\mu + \sum_{m=0}^{1}\left(\int_{\tau'-1}^{\tau+1}\left(\int_{\Sigma_{t^*}}\left(\partial_{t^*}^m G_1\right)^2\right)^{\frac{1}{2}} dt^*\right)^2\right. \\
&\left. + \sum_{m=0}^{1}\iint_{\mathcal{R}(\tau'-1,\tau+1)}\left(\partial_{t^*}^m G_1\right)^2 + \sum_{m=0}^{1}\iint_{\mathcal{R}(\tau'-1,\tau+1)} r^{1+\delta}\left(\partial_{t^*}^m G_2\right)^2\right).
\end{aligned}$$

9.5 Vector Field Multiplier Z and Decay of Non-degenerate Energy

We follow the definition of Z in [16]. Let $Z = u^2\underline{L} + v^2 L$, where u, v are the Schwarzschild coordinates $u = \frac{1}{2}(t - r_S^*)$, $v = \frac{1}{2}(t + r_S^*)$, $\underline{L} = \partial_u$ and $L = 2V - \underline{L}$, where $V = \partial_{t^*} + \chi(r)\frac{a}{2Mr_+}\partial_{\phi^*}$ with χ being a cutoff function that is identically 1 for $r \leq r_Y^- - \frac{r_Y^- - r_+}{2}$ and is compactly supported in $\{r \leq r_Y^- - \frac{r_Y^- - r_+}{4}\}$. Notice that with this definition, V is Killing except in the set $\{r_Y^- - \frac{r_Y^- - r_+}{2} \leq r \leq r_Y^- - \frac{r_Y^- - r_+}{4}\}$. Let $w^Z = \frac{4tr_S^*(1-\mu)}{r}$. Notice also that while $u \to \infty$ as one approaches the event horizon, Z is continuous up to the event horizon due to the following (However, Z is not C^1 and hence its deformation tensor is not continuous up to the event horizon):

Proposition 9.5.1. *In the Kerr (t^*, r, θ, ϕ^*) coordinates,*

$$\underline{L} = (1-\mu)\partial_{t^*} - (1-\mu)\left(\frac{2r_s - 2M}{2r - 2M}\right)\partial_r.$$

In the null frame near the event horizon in Section 3.4.3, we can write

$$\underline{L} = \underline{L}^{\hat{V}}\hat{V} + \underline{L}^{\hat{Y}}\hat{Y} + \underline{L}^{A}E_A, \qquad \textit{where } |\underline{L}^{\alpha}| \leq C(1-\mu).$$

Heuristically, we want to show that in the region $\{r \geq r_Y^-\}$,

$$\int_{\Sigma_\tau \cap \{r \geq r_Y^-\}} J_\mu^{Z,w^Z}(\Phi)\, n_{\Sigma_\tau}^\mu \geq 0.$$

Moreover, we would like to have

$$\begin{aligned}
&\int_{\Sigma_\tau \cap \{r \geq r_Y^-\}} J_\mu^{Z,w^Z}(\Phi)\, n_{\Sigma_\tau}^\mu \\
\geq &\int_{\Sigma_\tau \cap \{r \geq r_Y^-\}} u^2 \left(\underline{L}\Phi\right)^2 + v^2 \left(L\Phi\right)^2 + \left(u^2+v^2\right)|\nabla\!\!\!/\Phi|^2 + \left(\frac{u^2+v^2}{r^2}\right)\Phi^2.
\end{aligned}$$

These are true modulo some error terms that can be controlled:

Proposition 9.5.2.

$$\begin{aligned}
&\int_{\Sigma_\tau \cap \{r \geq r_Y^-\}} u^2 \left(\underline{L}\Phi\right)^2 + v^2 \left(L\Phi\right)^2 + \left(u^2+v^2\right)|\nabla\!\!\!/\Phi|^2 + \left(\frac{u^2+v^2}{r^2}\right)\Phi^2 \\
\leq &C\int_{\Sigma_\tau} J_\mu^{Z,w^Z}(\Phi)\, n_{\Sigma_\tau}^\mu + C\int_{\Sigma_\tau} J_\mu^{N}(\Phi)\, n_{\Sigma_\tau}^\mu + C^2\tau^2 \int_{\Sigma_\tau \cap \{r \leq r_Y^-\}} J_\mu^{N}(\Phi)\, n_{\Sigma_\tau}^\mu.
\end{aligned}$$

Proof. The proof is analogous to that in Minkowski spacetime (see [42]) and Schwarzschild spacetime (see [18]). Recall that on Schwarzschild spacetime, on a t slice [18]:

$$\begin{aligned}
&\left(J_S^{Z,w^Z}\right)_\mu(\Phi)\, n_{\Sigma_t}^\mu \\
= &\frac{1}{\sqrt{1-\mu}}\left(v^2\left(L\Phi\right)^2 + u^2\left(\underline{L}\Phi\right)^2 + \left(u^2+v^2\right)|\nabla\!\!\!/\Phi|^2 \right. \\
&\left. + \frac{2tr^*(1-\mu)}{r}\Phi\partial_t\Phi - \frac{r^*(1-\mu)}{r}\Phi^2\right).
\end{aligned}$$

Now, since t, r^* are stable under perturbation on $\{r \geq r_Y^- - \left(r_Y^- - r_+\right)/4\}$, we have,

on this set:

$$\begin{aligned}
&\left(J_K^{Z,w^Z}\right)_\mu (\Phi)\, n_{\Sigma_\tau}^\mu \\
\geq &\frac{1}{\sqrt{1-\mu}}\left(v^2 (L\Phi)^2 + u^2 (\underline{L}\Phi)^2 + \left(u^2+v^2\right)|\nabla\!\!\!/\Phi|^2\right. \\
&\qquad \left. + \frac{2tr^*(1-\mu)}{r}\Phi\partial_{t^*}\Phi - \frac{r^*(1-\mu)}{r}\Phi^2\right) \\
&- C\epsilon r^{-2}\left(\left(u^2+v^2\right)(\nabla\Phi)^2 + t^*\Phi^2\right).
\end{aligned}$$

We now cutoff Φ: define $\hat{\Phi}$ so that it is supported in $\{r \geq r_Y^- - \left(r_Y^- - r_+\right)/4\}$ and equals Φ in $\{r \geq r_Y^-\}$. All the error terms arising from the cutoff will be controlled using the red-shift vector field.

$$\begin{aligned}
&\int_{\Sigma_\tau} J_\mu^{Z,w^Z}\left(\hat{\Phi}\right) n_{\Sigma_\tau}^\mu \\
\geq &\int_{\Sigma_\tau \cap \{r \geq r_Y^- - \left(r_Y^- - r_+\right)/4\}} \frac{1}{\sqrt{1-\mu}}\left(v^2\left(L\hat{\Phi}\right)^2 + u^2\left(\underline{L}\hat{\Phi}\right)^2 + \left(u^2+v^2\right)|\nabla\!\!\!/\hat{\Phi}|^2\right) \\
&+ \frac{2tr_S^*(1-\mu)^{\frac{1}{2}}}{r}\hat{\Phi}\partial_{t^*}\hat{\Phi} - \frac{r_S^*(1-\mu)^{\frac{1}{2}}}{r}\hat{\Phi}^2 - C\epsilon r^{-2}\left(\left(u^2+v^2\right)\left(\nabla\hat{\Phi}\right)^2 + t^*\hat{\Phi}^2\right).
\end{aligned}$$

The term

$$\int_{\Sigma_\tau \cap \{r \geq r_Y^- - \left(r_Y^- - r_+\right)/4\}} \frac{2tr_S^*(1-\mu)^{\frac{1}{2}}}{r}\hat{\Phi}\partial_{t^*}\hat{\Phi}$$

is to be handled by two different integration by parts. Recall that [18] on Schwarzschild spacetimes we have

$$t\partial_t\hat{\Phi} = vL\hat{\Phi} + u\underline{L}\hat{\Phi} - r_S^*\partial_{r_S^*}\hat{\Phi}, \quad \text{and}$$

$$t\partial_t\hat{\Phi} = \frac{t}{r_S^*}\left(vL\hat{\Phi} - u\underline{L}\hat{\Phi}\right) - \frac{t^2}{r_S^*}\partial_{r_S^*}\hat{\Phi}.$$

Therefore, upon integrating by parts, we have on Schwarzschild spacetimes that

$$\begin{aligned}
&\int_{\Sigma_\tau\cap\{r\geq r_Y^- - \left(r_Y^- - r_+\right)/4\}} \frac{tr_S^*\left(1-\mu\right)^{\frac{1}{2}}}{r}\hat{\Phi}\partial_{t^*}\hat{\Phi}\\
=&\int_{\Sigma_\tau\cap\{r\geq r_Y^- - \left(r_Y^- - r_+\right)/4\}} \left(\left(1-\mu\right)r^2\frac{r_S^*}{r}\left(vL\hat{\Phi}+u\underline{L}\hat{\Phi}\right)\hat{\Phi}\right.\\
&\qquad\qquad\left.+\frac{1}{2}\partial_{r_S^*}\left(\left(1-\mu\right)r\left(r^*\right)^2\right)\hat{\Phi}^2\right)d\theta\, d\phi\, dr^*\\
=&\int_{\Sigma_\tau\cap\{r\geq r_Y^- - \left(r_Y^- - r_+\right)/4\}} \left(1-\mu\right)r^2\left(\frac{r_S^*}{r}\left(vL\hat{\Phi}+u\underline{L}\hat{\Phi}\right)\hat{\Phi}\right.\\
&\qquad\qquad\left.+\left(\frac{1}{2}\frac{\left(r_S^*\right)^2}{r^2}+\frac{r_S^*}{r}\right)\hat{\Phi}^2\right)d\theta\, d\phi\, dr^*.
\end{aligned}$$

Notice that in the above equation, we suppressed the volume form in our notations in the first line, while when we write in coordinates as in the second and the third line, we write out the volume form explicitly. Alternatively, we have

$$\begin{aligned}
&\int_{\Sigma_\tau\cap\{r\geq r_Y^- - \left(r_Y^- - r_+\right)/4\}} \frac{tr_S^*\left(1-\mu\right)^{\frac{1}{2}}}{r}\hat{\Phi}\partial_{t^*}\hat{\Phi}\\
=&\int_{\Sigma_\tau\cap\{r\geq r_Y^- - \left(r_Y^- - r_+\right)/4\}} \left(\left(1-\mu\right)r^2\frac{t^*}{r}\left(vL\hat{\Phi}-u\underline{L}\hat{\Phi}\right)\hat{\Phi}\right.\\
&\qquad\qquad\left.+\frac{1}{2}\partial_{r^*}\left(\left(1-\mu\right)r(t^*)^2\right)\hat{\Phi}^2\right)d\theta\, d\phi\, dr^*\\
=&\int_{\Sigma_\tau\cap\{r\geq r_Y^- - \left(r_Y^- - r_+\right)/4\}} \left(1-\mu\right)r^2\left(\frac{t^*}{r}\left(vL\hat{\Phi}-u\underline{L}\hat{\Phi}\right)\hat{\Phi}\right.\\
&\qquad\qquad\left.+\frac{1}{2}\frac{\left(t^*\right)^2}{r^2}\hat{\Phi}^2\right)d\theta\, d\phi\, dr^*.
\end{aligned}$$

We would like to imitate this integration by parts on Kerr spacetimes. We notice that on the domain of integration we have

$$t\partial_t\hat{\Phi} = vL\hat{\Phi}+u\underline{L}\hat{\Phi}-r_S^*\partial_{r_S^*}\hat{\Phi},\quad \text{and} \tag{9.5}$$

$$t\partial_t\hat{\Phi} = \frac{t}{r_S^*}\left(vL\hat{\Phi}-u\underline{L}\hat{\Phi}\right)-\frac{t^2}{r_S^*}\partial_{r_S^*}\hat{\Phi}. \tag{9.6}$$

The volume form on a constant t^* slice on a Kerr spacetime is close to that on a Schwarzschild spacetime, including in the region being considered. In other words, for $r \geq r_Y^- - \left(r_Y^- - r_+\right)/4$,

$$dVol_{\Sigma_\tau} = \left(r^2 \left(1-\mu\right)^{-\frac{1}{2}} + O_1(\epsilon)\right) dr dx^A dx^B.$$

Moreover, for $r \geq r_Y^- - \left(r_Y^- - r_+\right)/4$,

$$\partial_{r_S^*} = \left((1-\mu) + O_1(\epsilon r^{-2})\right) \partial_r.$$

Therefore, using (9.5), we have

$$\begin{aligned}
&\int_{\Sigma_\tau \cap \{r \geq r_Y^- - \left(r_Y^- - r_+\right)/4\}} \frac{t r_S^* \left(1-\mu\right)^{\frac{1}{2}}}{r} \hat{\Phi} \partial_{t^*} \hat{\Phi} \\
=&\int_{\Sigma_\tau \cap \{r \geq r_Y^- - \left(r_Y^- - r_+\right)/4\}} \left(\left(r r_S^* + O(\epsilon)\right) \left(vL\hat{\Phi} + u\underline{L}\hat{\Phi}\right) \hat{\Phi} \right. \\
&\qquad \left. + \left(\frac{1}{2}\partial_r \left((1-\mu) r \left(r_S^*\right)^2\right) + O(\epsilon)\right) \hat{\Phi}^2\right) dr dx^A dx^B \\
=&\int_{\Sigma_\tau \cap \{r \geq r_Y^- - \left(r_Y^- - r_+\right)/4\}} \left(1-\mu\right)^{\frac{1}{2}} \left(\left(\frac{r_S^*}{r} + O(\epsilon r^{-2})\right) \left(vL\hat{\Phi} + u\underline{L}\hat{\Phi}\right) \hat{\Phi} \right. \\
&\qquad \left. + \left(\frac{1}{2}\frac{\left(r_S^*\right)^2}{r^2} + \frac{r_S^*}{r} + O(\epsilon r^{-2})\right) \hat{\Phi}^2\right).
\end{aligned}$$

Alternatively, we can integrate by parts after using (9.6):

$$\begin{aligned}
&\int_{\Sigma_\tau\cap\{r\geq r_Y^- - (r_Y^- - r_+)/4\}} \frac{tr_S^*(1-\mu)^{\frac{1}{2}}}{r}\hat{\Phi}\partial_{t^*}\hat{\Phi}\\
=&\int_{\Sigma_\tau\cap\{r\geq r_Y^- - (r_Y^- - r_+)/4\}} \Big((rt^*+O(\epsilon))\left(vL\hat{\Phi}-u\underline{L}\hat{\Phi}\right)\hat{\Phi}\\
&\qquad\qquad +\left(\frac{1}{2}\partial_r\left((1-\mu)r(t^*)^2\right)+O(\epsilon)\right)\hat{\Phi}^2\Big)\\
=&\int_{\Sigma_\tau\cap\{r\geq r_Y^- - (r_Y^- - r_+)/4\}} (1-\mu)^{\frac{1}{2}}\Bigg(\left(\frac{t^*}{r}+O(\epsilon r^{-2})\right)\left(vL\hat{\Phi}-u\underline{L}\hat{\Phi}\right)\hat{\Phi}\\
&\qquad\qquad +\left(\frac{1}{2}\frac{(t^*)^2}{r^2}+O(\epsilon r^{-2})\right)\hat{\Phi}^2\Bigg).
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
&\int_{\Sigma_\tau} J_\mu^{Z,w^Z}\left(\hat{\Phi}\right)n_{\Sigma_\tau}^\mu\\
\geq&\int_{\Sigma_\tau\cap\{r\geq r_Y^- - (r_Y^- - r_+)/4\}}\frac{1}{\sqrt{1-\mu}}\left(v^2\left(L\hat{\Phi}\right)^2+u^2\left(\underline{L}\hat{\Phi}\right)^2+\left(u^2+v^2\right)|\nabla\!\!\!/\hat{\Phi}|^2\right)\\
&+\frac{2tr_S^*(1-\mu)^{\frac{1}{2}}}{r}\hat{\Phi}\partial_{t^*}\hat{\Phi}-\frac{r_S^*(1-\mu)^{\frac{1}{2}}}{r}\hat{\Phi}^2-C\epsilon r^{-2}\left(\left(u^2+v^2\right)\left(D\hat{\Phi}\right)^2+t^*\hat{\Phi}^2\right)\\
\geq&\int_{\Sigma_\tau\cap\{r\geq r_Y^- - (r_Y^- - r_+)/4\}}\frac{1}{\sqrt{1-\mu}}\left(v^2\left(L\hat{\Phi}\right)^2+u^2\left(\underline{L}\hat{\Phi}\right)^2+\left(u^2+v^2\right)|\nabla\!\!\!/\hat{\Phi}|^2\right)\\
&+\frac{r_S^*(1-\mu)^{\frac{1}{2}}}{r}\left(vL\hat{\Phi}+u\underline{L}\hat{\Phi}\right)\hat{\Phi}+\frac{1}{2}\frac{(r_S^*)^2(1-\mu)^{\frac{1}{2}}}{r^2}\hat{\Phi}^2\\
&+\frac{t^*(1-\mu)^{\frac{1}{2}}}{r}\left(vL\hat{\Phi}-u\underline{L}\hat{\Phi}\right)\hat{\Phi}+\frac{1}{2}\frac{(t^*)^2}{r^2}\hat{\Phi}^2-C\epsilon r^{-2}\left(\left(u^2+v^2\right)\left(D\hat{\Phi}\right)^2+t^*\hat{\Phi}^2\right)\\
\geq&c\Bigg(\int_{\Sigma_\tau\cap\{r\geq r_Y^- - (r_Y^- - r_+)/4\}}\mu\left(\left(vL\hat{\Phi}+u\underline{L}\hat{\Phi}\right)^2+\left(vL\hat{\Phi}-u\underline{L}\hat{\Phi}\right)^2\right)\\
&+(1-\mu)\left(\left(vL\hat{\Phi}+u\underline{L}\hat{\Phi}+\frac{r_S^*}{r}\hat{\Phi}\right)^2+\left(vL\hat{\Phi}-u\underline{L}\hat{\Phi}+\frac{t^*}{r}\hat{\Phi}\right)^2+2\left(u^2+v^2\right)|\nabla\!\!\!/\hat{\Phi}|^2\right)\\
&-C\epsilon r^{-2}\left(\left(u^2+v^2\right)\left(D\hat{\Phi}\right)^2+t^*\hat{\Phi}^2\right)\Bigg),
\end{aligned}\tag{9.7}$$

where the last line is obtained by first completing the square and using $c\leq 1-\mu\leq C$

in this region of r. Let us for now ignore the error term and look at the other terms (which are manifestly positive). With exactly the same argument as in [18], we have that these positive terms provide good estimates:

$$\begin{aligned}
&\left(\int_{\Sigma_\tau\cap\{r\geq r_Y^- - (r_Y^- - r_+)/4\}} \mu\left(\left(vL\hat{\Phi}+u\underline{L}\hat{\Phi}\right)^2+\left(vL\hat{\Phi}-u\underline{L}\hat{\Phi}\right)^2\right)\right.\\
&\left.+(1-\mu)\left(\left(vL\hat{\Phi}+u\underline{L}\hat{\Phi}+\frac{r_S^*}{r}\hat{\Phi}\right)^2+\left(vL\hat{\Phi}-u\underline{L}\hat{\Phi}+\frac{t^*}{r}\hat{\Phi}\right)^2+2\left(u^2+v^2\right)|\not{\nabla}\hat{\Phi}|^2\right)\right)\\
\geq& c\int_{\Sigma_\tau\cap\{r\geq r_Y^- - (r_Y^- - r_+)/4\}}\left(v^2\left(L\hat{\Phi}\right)^2+u^2\left(\underline{L}\hat{\Phi}\right)^2+\left(u^2+v^2\right)|\not{\nabla}\hat{\Phi}|^2+\frac{u^2+v^2}{r^2}\hat{\Phi}^2\right)\\
&+C\epsilon\int_{\Sigma_\tau\cap\{r\geq r_Y^- - (r_Y^- - r_+)/4\}} r^{-2}\left(\left(u^2+v^2\right)\left(D\hat{\Phi}\right)^2+t^*\hat{\Phi}^2\right).
\end{aligned}$$

We refer the reader to [18] for the proof. This together with $J_\mu^N\left(\hat{\Phi}\right)n_{\Sigma_\tau}^\mu$ bound the error term in (9.7):

$$\begin{aligned}
&\int_{\Sigma_\tau\cap\{r\geq r_Y^- - (r_Y^- - r_+)/4\}} r^{-2}\left(\left(u^2+v^2\right)\left(D\hat{\Phi}\right)^2+\tau\Phi^2\right)\\
\leq& C\int_{\Sigma_\tau\cap\{r\geq r_Y^- - (r_Y^- - r_+)/4\}}\left(v^2\left(L\hat{\Phi}\right)^2+u^2\left(\underline{L}\hat{\Phi}\right)^2+\left(u^2+v^2\right)|\not{\nabla}\hat{\Phi}|^2+\frac{u^2+v^2}{r^2}\hat{\Phi}^2\right)\\
&+C\int_{\Sigma_\tau\cap\{r\geq\frac{\tau}{4}\}}\left(\underline{L}\hat{\Phi}\right)^2\\
\leq& C\int_{\Sigma_\tau\cap\{r\geq r_Y^- - (r_Y^- - r_+)/4\}}\left(v^2\left(L\hat{\Phi}\right)^2+u^2\left(\underline{L}\hat{\Phi}\right)^2+\left(u^2+v^2\right)|\not{\nabla}\hat{\Phi}|^2+\frac{u^2+v^2}{r^2}\hat{\Phi}^2\right)\\
&+C\int_{\Sigma_\tau} J_\mu^N\left(\hat{\Phi}\right)n_{\Sigma_\tau}^\mu\\
\leq& C\left(\int_{\Sigma_\tau\cap\{r\geq r_Y^- - (r_Y^- - r_+)/4\}}\mu\left(\left(vL\hat{\Phi}+u\underline{L}\hat{\Phi}\right)^2+\left(vL\hat{\Phi}-u\underline{L}\hat{\Phi}\right)^2\right)\right.\\
&\left.+(1-\mu)\left(\left(vL\hat{\Phi}+u\underline{L}\hat{\Phi}+\frac{r_S^*}{r}\hat{\Phi}\right)^2+\left(vL\hat{\Phi}-u\underline{L}\hat{\Phi}+\frac{t^*}{r}\hat{\Phi}\right)^2+2\left(u^2+v^2\right)|\not{\nabla}\hat{\Phi}|^2\right)\right)\\
&+C\int_{\Sigma_\tau} J_\mu^N\left(\hat{\Phi}\right)n_{\Sigma_\tau}^\mu.
\end{aligned}$$

Therefore, if ϵ is chosen to be small enough, then (9.7) implies that

$$\begin{aligned}&\int_{\Sigma_\tau} J_\mu^{Z,w^Z}\left(\hat{\Phi}\right) n_{\Sigma_\tau}^\mu + \int_{\Sigma_\tau} J_\mu^N\left(\hat{\Phi}\right) n_{\Sigma_\tau}^\mu\\ \geq & c\int_{\Sigma_\tau\cap\{r\geq r_Y^- - \left(r_Y^- - r_+\right)/4\}} \left(v^2\left(L\hat{\Phi}\right)^2 + u^2\left(\underline{L}\hat{\Phi}\right)^2 + \left(u^2+v^2\right)|\nabla\!\!\!/\hat{\Phi}|^2 + \frac{u^2+v^2}{r^2}\hat{\Phi}^2\right).\end{aligned} \tag{9.8}$$

We note that c here is independent of the choice of r_Y^-. With this bound we would like to estimate $\int_{\mathbb{S}^2}\Phi\left(\tau,r\right)^2$. Using (9.8), there exists a $\tilde{r}\in[r_Y^-, r_Y^-+1]$ such that

$$\int_{\mathbb{S}^2}\Phi\left(\tau,\tilde{r}\right)^2 = \int_{\mathbb{S}^2}\hat{\Phi}\left(\tau,\tilde{r}\right)^2 \leq C\tau^{-2}\left(\int_{\Sigma_\tau} J_\mu^{Z,w^Z}\left(\hat{\Phi}\right) n_{\Sigma_\tau}^\mu + \int_{\Sigma_\tau} J_\mu^N\left(\hat{\Phi}\right) n_{\Sigma_\tau}^\mu\right).$$

Then for every $r\in[r_+, r_Y^-]$, since

$$\Phi\left(\tau,\tilde{r}\right) - \Phi\left(\tau,r\right) = \int_r^{\tilde{r}} \partial_r\Phi dr,$$

we have

$$\begin{aligned}&\int_{\mathbb{S}^2}\Phi\left(\tau,r\right)^2\\ \leq & \int_{\mathbb{S}^2}\Phi\left(\tau,\tilde{r}\right)^2 + \left(\tilde{r}-r\right)\int_{\Sigma_\tau\cap[r,\tilde{r}]} J_\mu^N\left(\Phi\right) n_{\Sigma_\tau}^\mu\\ \leq & C\int_{\Sigma_\tau\cap\{r\leq r_Y^-\}} J_\mu^N\left(\Phi\right) n_{\Sigma_\tau}^\mu + C\tau^{-2}\left(\int_{\Sigma_\tau} J_\mu^{Z,w^Z}\left(\hat{\Phi}\right) n_{\Sigma_\tau}^\mu + \int_{\Sigma_\tau} J_\mu^N\left(\hat{\Phi}\right) n_{\Sigma_\tau}^\mu\right)\end{aligned} \tag{9.9}$$

Now we need to obtain estimates for Φ from that for $\hat{\Phi}$. It is obvious that

$$\begin{aligned}&\int_{\Sigma_\tau\cap\{r\geq r_Y^-\}}\left(v^2\left(L\Phi\right)^2 + u^2\left(\underline{L}\Phi\right)^2 + \left(u^2+v^2\right)|\nabla\!\!\!/\Phi|^2 + \frac{u^2+v^2}{r^2}\Phi^2\right)\\ \leq & \int_{\Sigma_\tau\cap\{r\geq r_Y^-\}}\left(v^2\left(L\hat{\Phi}\right)^2 + u^2\left(\underline{L}\hat{\Phi}\right)^2 + \left(u^2+v^2\right)|\nabla\!\!\!/\hat{\Phi}|^2 + \frac{u^2+v^2}{r^2}\hat{\Phi}^2\right)\\ \leq & \int_{\Sigma_\tau\cap\{r\geq r_Y^- - \left(r_Y^- - r_+\right)/4\}}\left(v^2\left(L\hat{\Phi}\right)^2 + u^2\left(\underline{L}\hat{\Phi}\right)^2 + \left(u^2+v^2\right)|\nabla\!\!\!/\hat{\Phi}|^2 + \frac{u^2+v^2}{r^2}\hat{\Phi}^2\right),\end{aligned}$$

and

$$\begin{aligned}
&\int_{\Sigma_\tau} J_\mu^{Z,w^Z}\left(\hat{\Phi}\right) n_{\Sigma_\tau}^\mu + \int_{\Sigma_\tau} J_\mu^N\left(\hat{\Phi}\right) n_{\Sigma_\tau}^\mu \\
\leq &\int_{\Sigma_\tau} J_\mu^{Z,w^Z}\left(\Phi\right) n_{\Sigma_\tau}^\mu + \int_{\Sigma_\tau} J_\mu^N\left(\Phi\right) n_{\Sigma_\tau}^\mu \\
&+ C\tau^2 \int_{\Sigma_\tau \cap \{r \leq r_Y^-\}} J_\mu^N\left(\Phi\right) n_{\Sigma_\tau}^\mu + C\tau^2 \int_{\Sigma_\tau \cap \{r \leq r_Y^-\}} \Phi^2 \\
&\qquad \text{where we have used Proposition 9.5.1 to show that the } u^2 \text{ factor} \\
&\qquad \text{comes with a factor of } 1-\mu \qquad (9.10) \\
\leq &\int_{\Sigma_\tau} J_\mu^{Z,w^Z}\left(\Phi\right) n_{\Sigma_\tau}^\mu + \int_{\Sigma_\tau} J_\mu^N\left(\Phi\right) n_{\Sigma_\tau}^\mu + C\tau^2 \int_{\Sigma_\tau \cap \{r \leq r_Y^-\}} J_\mu^N\left(\Phi\right) n_{\Sigma_\tau}^\mu \\
&+ C\left(r_Y^- - r_+\right)\left(\int_{\Sigma_\tau} J_\mu^{Z,w^Z}\left(\hat{\Phi}\right) n_{\Sigma_\tau}^\mu + \int_{\Sigma_\tau} J_\mu^N\left(\hat{\Phi}\right) n_{\Sigma_\tau}^\mu\right) \\
\leq &\int_{\Sigma_\tau} J_\mu^{Z,w^Z}\left(\Phi\right) n_{\Sigma_\tau}^\mu + \int_{\Sigma_\tau} J_\mu^N\left(\Phi\right) n_{\Sigma_\tau}^\mu + C\tau^2 \int_{\Sigma_\tau \cap \{r \leq r_Y^-\}} J_\mu^N\left(\Phi\right) n_{\Sigma_\tau}^\mu \\
&+ \frac{1}{2}\left(\int_{\Sigma_\tau} J_\mu^{Z,w^Z}\left(\hat{\Phi}\right) n_{\Sigma_\tau}^\mu + \int_{\Sigma_\tau} J_\mu^N\left(\hat{\Phi}\right) n_{\Sigma_\tau}^\mu\right),
\end{aligned}$$

for r_Y^- chosen to be sufficiently close to r_+. Then

$$\begin{aligned}
&\int_{\Sigma_\tau} J_\mu^{Z,w^Z}\left(\hat{\Phi}\right) n_{\Sigma_\tau}^\mu + \int_{\Sigma_\tau} J_\mu^N\left(\hat{\Phi}\right) n_{\Sigma_\tau}^\mu \\
\leq &\int_{\Sigma_\tau} J_\mu^{Z,w^Z}\left(\Phi\right) n_{\Sigma_\tau}^\mu + \int_{\Sigma_\tau} J_\mu^N\left(\Phi\right) n_{\Sigma_\tau}^\mu + C\tau^2 \int_{\Sigma_\tau \cap \{r \leq r_Y^-\}} J_\mu^N\left(\Phi\right) n_{\Sigma_\tau}^\mu.
\end{aligned}$$

□

Remark 18. *From this point onward, we consider r_Y^- to be fixed. We note again that r_Y^- is chosen so that (3.5) and (9.10) hold.*

Remark 19. *The proof of the above proposition in particular shows that*

$$\int_{\Sigma_\tau} J_\mu^{Z,w^Z}\left(\Phi\right) n_{\Sigma_\tau}^\mu + C\tau^2 \int_{\Sigma_\tau \cap \{r \leq r_Y^-\}} J_\mu^N\left(\Phi\right) n_{\Sigma_\tau}^\mu \geq 0$$

In order to use this Proposition, it is helpful to have a localized version of Φ. This

follows [18], [16]. The idea is to use the finite speed of propagation and cutoff Φ outside the domain of dependence. Suppose we now focus on the time interval $[\tau', \tau]$. Take $\tilde{G}$ to be any smooth function agreeing with G on the domain of dependence of the region $(t^* = \tau, r \leq \frac{\tau}{2})$. Let $\tilde{\Phi}(\tau') = \chi\Phi(\tau')$, $\partial_{t^*}\tilde{\Phi}(\tau') = \chi\partial_{t^*}\Phi(\tau')$, where χ is a cutoff function identically equals to 1 for $r \leq \frac{7\tau'}{10}$ and compactly supported in $r \leq \frac{9\tau'}{10}$. Notice that the region for which χ is one is inside the domain of dependence of the region $(t^* = \tau, r \leq \frac{\tau}{2})$ if $\tau' \leq \tau \leq (1.1)\tau'$. We solve for $\Box_{g_K}\tilde{\Phi} = \tilde{G}$.

With this definition of $\tilde{\Phi}$, we have two ways to estimate the non-degenerate energy of $\tilde{\Phi}$:

Proposition 9.5.3.

$$\int_{\Sigma_{\tau'}} J_\mu^N\left(\tilde{\Phi}\right) n_{\Sigma_{\tau'}}^\mu \leq C\int_{\Sigma_{\tau'}} J_\mu^N(\Phi)\, n_{\Sigma_{\tau'}}^\mu.$$

$$\begin{aligned}&\int_{\Sigma_{\tau'}} J_\mu^N\left(\tilde{\Phi}\right) n_{\Sigma_{\tau'}}^\mu\\ \leq& C^2\int_{\Sigma_{\tau'}\cap\{r\leq r_Y^-\}} J_\mu^N(\Phi)\, n_{\Sigma_{\tau'}}^\mu + C(\tau')^{-2}\left(\int_{\Sigma_{\tau'}} J_\mu^N(\Phi)\, n_{\Sigma_{\tau'}}^\mu + \int_{\Sigma_{\tau'}} J_\mu^{Z,w^Z}(\Phi)\, n_{\Sigma_{\tau'}}^\mu\right).\end{aligned}$$

Proof. The first part is an easy application of Proposition 9.4.10

$$\begin{aligned}\int_{\Sigma_{\tau'}} J_\mu^N\left(\tilde{\Phi}\right) n_{\Sigma_{\tau'}}^\mu \leq& C\int_{\Sigma_{\tau'}\cap\{R\leq r\leq\frac{9\tau'}{10}\}}\left((D\Phi)^2 + (\tau')^{-2}\Phi^2\right)\\ \leq& C\int_{\Sigma_{\tau'}\cap\{R\leq r\leq\frac{9\tau'}{10}\}}\left((D\Phi)^2 + r^{-2}\Phi^2\right)\\ \leq& C\int_{\Sigma_{\tau'}} J_\mu^N(\Phi)\, n_{\Sigma_{\tau'}}^\mu.\end{aligned}$$

Following (9.9), we have

$$\int_{\Sigma_{\tau'}\cap\{r\leq r_Y^-\}}\Phi^2 \leq C\left(\int_{\Sigma_{\tau'}\cap\{r\leq r_Y^-\}} J_\mu^N(\Phi)\, n_{\Sigma_{\tau'}}^\mu + \int_{\Sigma_{\tau'}\cap\{r_Y^-\leq r\leq r_Y^+\}}\Phi^2\right).$$

Using this and Proposition 9.5.2, we have

$$\begin{aligned}
&\int_{\Sigma_{\tau'}} J_\mu^N\left(\tilde{\Phi}\right) n_{\Sigma_{\tau'}}^\mu \\
\leq & C\int_{\Sigma_{\tau'}\cap\{r\leq\frac{9\tau'}{10}\}}\left((D\Phi)^2+(\tau')^{-2}\Phi^2\right) \\
\leq & C\int_{\Sigma_{\tau'}\cap\{r\leq r_Y^-\}}\left((D\Phi)^2+\Phi^2\right) \\
&+C(\tau')^{-2}\int_{\Sigma_{\tau'}\cap\{r_Y^-\leq r\leq\frac{9\tau'}{10}\}}\left(u^2\left(\underline{L}\Phi\right)^2+v^2\left(L\Phi\right)^2\right. \\
&\qquad\qquad\left.+\left(u^2+v^2\right)|\nabla\!\!\!/\Phi|^2+\left(\frac{u^2+v^2}{r^2}\right)\Phi^2\right) \\
\leq & C^2\int_{\Sigma_{\tau'}\cap\{r\leq r_Y^-\}} J_\mu^N\left(\Phi\right) n_{\Sigma_{\tau'}}^\mu + C(\tau')^{-2}\left(\int_{\Sigma_{\tau'}} J_\mu^N\left(\Phi\right) n_{\Sigma_{\tau'}}^\mu + \int_{\Sigma_{\tau'}} J_\mu^{Z,w^Z}\left(\Phi\right) n_{\Sigma_{\tau'}}^\mu\right).
\end{aligned}$$

□

The cutoff procedure above would also allow us to localized the estimates for the bulk term:

Proposition 9.5.4. *Let $G = G_1 + G_2$ be any way to decompose the function G. Then for $\tau' \leq \tau \leq (1.1)\tau'$, we have*

1. *Localized Boundedness Estimate*

$$\begin{aligned}
&\int_{\Sigma_\tau\cap\{r\leq\frac{\tau}{2}\}} J_\mu^N\left(\Phi\right) n_{\Sigma_\tau}^\mu + \int_{\mathcal{H}(\tau',\tau)} J_\mu^N\left(\Phi\right) n_{\mathcal{H}^+}^\mu \\
&+\iint_{\mathcal{R}(\tau',\tau)\cap\{r\leq r_Y^-\}} K^N\left(\Phi\right) + \iint_{\mathcal{R}(\tau',\tau)\cap\{r\leq\frac{t^*}{2}\}} K^{X_0}\left(\Phi\right) \\
\leq & C\left(\int_{\Sigma_{\tau'}} J_\mu^N\left(\Phi\right) n_{\Sigma_{\tau'}}^\mu + \left(\int_{\tau'-1}^{\tau+1}\left(\int_{\Sigma_{t^*}\cap\{r\leq\frac{9t^*}{10}\}} G_1^2\right)^{\frac{1}{2}} dt^*\right)^2\right. \\
&+\iint_{\mathcal{R}(\tau'-1,\tau+1)\cap\{r\leq\frac{9t^*}{10}\}} G_1^2 + \sum_{m=0}^{1}\iint_{\mathcal{R}(\tau'-1,\tau+1)\cap\{r\leq\frac{9t^*}{10}\}} r^{1+\delta}\left(\partial_{t^*}^m G_2\right)^2 \\
&\left.+\sup_{t^*\in[\tau'-1,\tau+1]}\int_{\Sigma_{t^*}\cap\{|r-3M|\leq\frac{M}{8}\}\cap\{r\leq\frac{9t^*}{10}\}} {G_2}^2\right).
\end{aligned}$$

$$
\begin{aligned}
&\int_{\Sigma_\tau\cap\{r\leq\frac{\tau}{2}\}} J_\mu^N(\Phi)\, n_{\Sigma_\tau}^\mu + \int_{\mathcal{H}(\tau',\tau)} J_\mu^N(\Phi)\, n_{\mathcal{H}^+}^\mu \\
&+ \iint_{\mathcal{R}(\tau',\tau)\cap\{r\leq r_Y^-\}} K^N(\Phi) + \iint_{\mathcal{R}(\tau',\tau)\cap\{r\leq\frac{t^*}{2}\}} K^{X_0}(\Phi) \\
\leq & C\left(\tau^{-2}\int_{\Sigma_{\tau'}} J_\mu^{Z+N,w^Z}(\Phi)\, n_{\Sigma_{\tau'}}^\mu + C\int_{\Sigma_{\tau'}\cap\{r\leq r_Y^-\}} J_\mu^N(\Phi)\, n_{\Sigma_{\tau'}}^\mu\right) \\
&+ C\left(\left(\int_{\tau'-1}^{\tau+1}\left(\int_{\Sigma_{t^*}\cap\{r\leq\frac{9t^*}{10}\}} G_1^2\right)^{\frac{1}{2}} dt^*\right)^2 + \iint_{\mathcal{R}(\tau'-1,\tau+1)\cap\{r\leq\frac{9t^*}{10}\}} G_1^2\right. \\
&+ \sum_{m=0}^{1}\iint_{\mathcal{R}(\tau'-1,\tau+1)\cap\{r\leq\frac{9t^*}{10}\}} r^{1+\delta}\left(\partial_{t^*}^m G_2\right)^2 \\
&\left.+ \sup_{t^*\in[\tau'-1,\tau+1]}\int_{\Sigma_{t^*}\cap\{|r-3M|\leq\frac{M}{8}\}\cap\{r\leq\frac{9t^*}{10}\}} G_2^2\right).
\end{aligned}
$$

Proof. Applying Proposition 9.4.17 to the equation $\Box_{g_K}\tilde{\Phi} = \tilde{G}$, we have

$$
\begin{aligned}
&\int_{\Sigma_\tau\cap\{r\leq\frac{\tau}{2}\}} J_\mu^N\left(\tilde{\Phi}\right) n_{\Sigma_\tau}^\mu + \int_{\mathcal{H}(\tau',\tau)} J_\mu^N\left(\tilde{\Phi}\right) n_{\mathcal{H}^+}^\mu \\
&+ \iint_{\mathcal{R}(\tau',\tau)\cap\{r\leq r_Y^-\}} K^N\left(\tilde{\Phi}\right) + \iint_{\mathcal{R}(\tau',\tau)\cap\{r\leq\frac{t^*}{2}\}} K^{X_0}\left(\tilde{\Phi}\right) \\
\leq & C\left(\int_{\Sigma_{\tau'}} J_\mu^N\left(\tilde{\Phi}\right) n_{\Sigma_{\tau'}}^\mu + \left(\int_{\tau'-1}^{\tau+1}\left(\int_{\Sigma_{t^*}}\tilde{G}_1^2\right)^{\frac{1}{2}} dt^*\right)^2 + \iint_{\mathcal{R}(\tau'-1,\tau+1)}\tilde{G}_1^2\right. \\
&\left.+ \sum_{m=0}^{1}\iint_{\mathcal{R}(\tau'-1,\tau+1)} r^{1+\delta}\left(\partial_{t^*}^m\tilde{G}_2\right)^2 + \sup_{t^*\in[\tau'-1,\tau+1]}\int_{\Sigma_{t^*}\cap\{|r-3M|\leq\frac{M}{8}\}}\tilde{G}_2^2\right).
\end{aligned}
$$

Since by the finite speed of propagation, $\tilde{\Phi} = \Phi$ in $\{r \leq \frac{t^*}{2}\}$, we have

$$\begin{aligned}
&\int_{\Sigma_\tau \cap \{r \leq \frac{\tau}{2}\}} J_\mu^N(\Phi) n_{\Sigma_\tau}^\mu + \int_{\mathcal{H}(\tau',\tau)} J_\mu^N(\Phi) n_{\mathcal{H}^+}^\mu \\
&+ \iint_{\mathcal{R}(\tau',\tau) \cap \{r \leq r_Y^-\}} K^N(\Phi) + \iint_{\mathcal{R}(\tau',\tau) \cap \{r \leq \frac{t^*}{2}\}} K^{X_0}(\Phi) \\
\leq & C \left(\int_{\Sigma_{\tau'}} J_\mu^N\left(\tilde{\Phi}\right) n_{\Sigma_{\tau'}}^\mu + \left(\int_{\tau'-1}^{\tau+1} \left(\int_{\Sigma_{t^*}} \tilde{G}_1^2 \right)^{\frac{1}{2}} dt^* \right)^2 + \iint_{\mathcal{R}(\tau'-1,\tau+1)} \tilde{G}_1^2 \right. \\
&\left. + \sum_{m=0}^{1} \iint_{\mathcal{R}(\tau'-1,\tau+1)} r^{1+\delta} \left(\partial_{t^*}^m \tilde{G}_2 \right)^2 + \sup_{t^* \in [\tau'-1,\tau+1]} \int_{\Sigma_{t^*} \cap \{|r-3M| \leq \frac{M}{8}\}} \tilde{G}_2^2 \right).
\end{aligned}$$

Now, we choose a particular $\tilde{G}$. Define $\tilde{G}$ to be G for $r \leq \frac{7t^*}{10}$, and 0 for $r \geq \frac{9t^*}{10}$. It can be easily shown that one can have the bounds $|\partial_{t^*}^m \tilde{G}| \leq C \sum_{k=0}^{m} | \left(\frac{r^*}{(t^*)^2} \right)^k \partial_{t^*}^{m-k} G| \leq C \sum_{k=0}^{m} |(t^*)^{-k} \partial_{t^*}^{m-k} G|$ for $\frac{7t^*}{10} \leq r \leq \frac{9t^*}{10}$. Therefore, we have

$$\begin{aligned}
&\int_{\Sigma_\tau \cap \{r \leq \frac{\tau}{2}\}} J_\mu^N(\Phi) n_{\Sigma_\tau}^\mu + \int_{\mathcal{H}(\tau',\tau)} J_\mu^N(\Phi) n_{\mathcal{H}^+}^\mu \\
&+ \iint_{\mathcal{R}(\tau',\tau) \cap \{r \leq r_Y^-\}} K^N(\Phi) + \iint_{\mathcal{R}(\tau',\tau) \cap \{r \leq \frac{t^*}{2}\}} K^{X_0}(\Phi) \\
\leq & C \left(\int_{\Sigma_{\tau'}} J_\mu^N\left(\tilde{\Phi}\right) n_{\Sigma_{\tau'}}^\mu + \left(\int_{\tau'-1}^{\tau+1} \left(\int_{\Sigma_{t^*} \cap \{r \leq \frac{9t^*}{10}\}} G_1^2 \right)^{\frac{1}{2}} dt^* \right)^2 \right. \\
&+ \iint_{\mathcal{R}(\tau'-1,\tau+1) \cap \{r \leq \frac{9t^*}{10}\}} G_1^2 + \sum_{m=0}^{1} \iint_{\mathcal{R}(\tau'-1,\tau+1) \cap \{r \leq \frac{9t^*}{10}\}} r^{1+\delta} \left(\partial_{t^*}^m G_2 \right)^2 \\
&\left. + \sup_{t^* \in [\tau'-1,\tau+1]} \int_{\Sigma_{t^*} \cap \{|r-3M| \leq \frac{M}{8}\}} G_2^2 \right).
\end{aligned}$$

We can now conclude the Proposition using Proposition 9.5.3. □

Remark 20. *As before, if G_2 is supported outside $\{|r-3M| \leq \frac{M}{8}\}$, we can replace, in Proposition 9.5.4.2,*

$$\sum_{m=0}^{1} \iint_{\mathcal{R}(\tau'-1,\tau+1) \cap \{r \leq \frac{9t^*}{10}\}} r^{1+\delta} \left(\partial_{t^*}^m G_2 \right)^2 + \sup_{t^* \in [\tau'-1,\tau+1]} \int_{\Sigma_{t^*} \cap \{|r-3M| \leq \frac{M}{8}\} \cap \{r \leq \frac{9t^*}{10}\}} G_2^2$$

by

$$\iint_{\mathcal{R}(\tau'-1,\tau+1)\cap\{r\leq\frac{9t^*}{10}\}} r^{1+\delta}G_2^2.$$

We can remove the degeneracy around $r\sim 3M$ using an extra derivative.

Proposition 9.5.5. *Let $G=G_1+G_2$ be any way to decompose the function G. Then for $\tau'\leq\tau\leq(1.1)\tau'$, we have*

1. *Localized Boundedness Estimate*

$$\begin{aligned}
&\iint_{\mathcal{R}(\tau',\tau)\cap\{r\leq\frac{t^*}{2}\}} K^{X_1}(\Phi)\\
\leq& C\left(\sum_{m=0}^{1}\int_{\Sigma_{\tau'}} J_\mu^N\left(\partial_{t^*}^m\Phi\right)n^\mu_{\Sigma_{\tau'}}+\sum_{m=0}^{1}\left(\int_{\tau'-1}^{\tau+1}\left(\int_{\Sigma_{t^*}\cap\{r\leq\frac{9t^*}{10}\}}\left(\partial_{t^*}^m G_1\right)^2\right)^{\frac{1}{2}}dt^*\right)^2\right.\\
&+\sum_{m=0}^{1}\iint_{\mathcal{R}(\tau'-1,\tau+1)\cap\{r\leq\frac{9t^*}{10}\}}\left(\partial_{t^*}^m G_1\right)^2+\sum_{m=0}^{2}\iint_{\mathcal{R}(\tau'-1,\tau+1)\cap\{r\leq\frac{9t^*}{10}\}} r^{1+\delta}\left(\partial_{t^*}^m G_2\right)^2\\
&\left.+\sup_{t^*\in[\tau'-1,\tau+1]}\sum_{m=0}^{1}\int_{\Sigma_{t^*}\cap\{|r-3M|\leq\frac{M}{8}\}\cap\{r\leq\frac{9t^*}{10}\}}\left(\partial_{t^*}^m G_2\right)^2\right).
\end{aligned}$$

2. *Localized Decay Estimate*

$$\begin{aligned}
&\iint_{\mathcal{R}(\tau',\tau)\cap\{r\leq\frac{t^*}{2}\}} K^{X_1}(\Phi)\\
\leq& C\left(\tau^{-2}\sum_{m=0}^{1}\int_{\Sigma_{\tau'}} J_\mu^{Z+N,w^Z}\left(\partial_{t^*}^m\Phi\right)n^\mu_{\Sigma_{\tau'}}+C\sum_{m=0}^{1}\int_{\Sigma_{\tau'}\cap\{r\leq r_Y^-\}} J_\mu^N\left(\partial_{t^*}^m\Phi\right)n^\mu_{\Sigma_{\tau'}}\right)\\
&+C\left(\sum_{m=0}^{1}\left(\int_{\tau'-1}^{\tau+1}\left(\int_{\Sigma_{t^*}\cap\{r\leq\frac{9t^*}{10}\}}\left(\partial_{t^*}^m G_1\right)^2\right)^{\frac{1}{2}}dt^*\right)^2\right.\\
&+\sum_{m=0}^{1}\iint_{\mathcal{R}(\tau'-1,\tau+1)\cap\{r\leq\frac{9t^*}{10}\}}\left(\partial_{t^*}^m G_1\right)^2\\
&+\sum_{m=0}^{2}\iint_{\mathcal{R}(\tau'-1,\tau+1)\cap\{r\leq\frac{9t^*}{10}\}} r^{1+\delta}\left(\partial_{t^*}^m G_2\right)^2\\
&\left.+\sup_{t^*\in[\tau'-1,\tau+1]}\sum_{m=0}^{1}\int_{\Sigma_{t^*}\cap\{|r-3M|\leq\frac{M}{8}\}\cap\{r\leq\frac{9t^*}{10}\}}\left(\partial_{t^*}^m G_2\right)^2\right).
\end{aligned}$$

Proof. We repeat the argument in Proposition 9.5.4, using Proposition 9.4.19 instead of 9.4.17. □

Remark 21. *As before, we can replace in Proposition 9.5.5.2,*

$$\sum_{m=0}^{2}\iint_{\mathcal{R}(\tau'-1,\tau+1)\cap\{r\leq\frac{9t^*}{10}\}} r^{1+\delta}\left(\partial_{t^*}^m G_2\right)^2+\sum_{m=0}^{1}\sup_{t^*\in[\tau'-1,\tau+1]}\int_{\Sigma_{t^*}\cap\{|r-3M|\leq\frac{M}{8}\}\cap\{r\leq\frac{9t^*}{10}\}}\left(\partial_{t^*}^m G_2\right)^2$$

by

$$\sum_{m=0}^{1}\iint_{\mathcal{R}(\tau'-1,\tau+1)\cap\{r\leq\frac{9t^*}{10}\}} r^{1+\delta}\left(\partial_{t^*}^m G_2\right)^2.$$

When using the conservation law for Z, we can ignore the part of the bulk term that has a good sign.

Definition 9.5.6. *Let* $K_+^{Z,w^Z}(\Phi)=\max\{K^{Z,w^Z}(\Phi),0\}$.

Using the conservation law for the modified vector field, we have a one-sided bound:

Proposition 9.5.7.

$$\begin{aligned}&\int_{\Sigma_\tau} J_\mu^{Z,w^Z}(\Phi)\,n_{\Sigma_\tau}^\mu+\int_{\mathcal{H}(\tau',\tau)} J_\mu^{Z,w^Z}(\Phi)\,n_{\mathcal{H}^+}^\mu\\ \leq& C(\tau')^2\int_{\Sigma_{\tau'}\cap\{r\leq r_Y^-\}} J_\mu^N(\Phi)\,n_{\Sigma_{\tau'}}^\mu+\int_{\Sigma_{\tau'}} J_\mu^{Z,w^Z}(\Phi)\,n_{\Sigma_{\tau'}}^\mu\\ &+\iint_{\mathcal{R}(\tau',\tau)} K_+^{Z,w^Z}(\Phi)+|\iint_{\mathcal{R}(\tau',\tau)}\left(u^2L\Phi+v^2\underline{L}\Phi-\frac{1}{4}w\Phi\right)G|.\end{aligned}$$

Remark 22. *In the above Proposition, the left hand side is not claimed to be positive. Note, however, that the right hand side is positive by Remark 19.*

Remark 23. *We note also that* $\int_{\mathcal{H}(\tau',\tau)} J_\mu^{Z,w^Z}(\Phi)\,n_{\mathcal{H}^+}^\mu\geq 0$ *because* Z *and* $n_{\mathcal{H}^+}^\mu$ *are both null and future directed and* $w^Z=0$ *on the event horizon.*

To show that $\int_{\Sigma_\tau} J_\mu^{Z,w^Z}(\Phi)\,n_{\Sigma_\tau}^\mu$ is almost bounded, we would have to show that $\int_{\Sigma_{\tau'}\cap\{r\leq r_Y^-\}} J_\mu^N(\Phi)\,n_{\Sigma_{\tau'}}^\mu$ in fact decays. This is given by the following Proposition:

Proposition 9.5.8.

$$\begin{aligned}
&\int_{\Sigma_\tau\cap\{r\leq\frac{t^*}{2}\}} J_\mu^N(\Phi)\, n_{\Sigma_\tau}^\mu\\
\leq& C^2\tau^{-2}\int_{\Sigma_{(1.1)^{-2}\tau}\cap\{r\leq r_Y^-\}} J_\mu^N(\Phi)\, n_{\Sigma_{(1.1)^{-2}\tau}}^\mu + C\tau^{-2}\int_{\Sigma_{(1.1)^{-2}\tau}} J_\mu^{Z,w^Z}(\Phi)\, n_{\Sigma_{(1.1)^{-1}\tau}}^\mu\\
&+C\tau^{-2}\iint_{\mathcal{R}((1.1)^{-2}\tau,\tau)} K_+^{Z,w^Z}(\Phi) + C\tau^{-2}\Big|\iint_{\mathcal{R}((1.1)^{-2}\tau,\tau)}\left(u^2L\Phi+v^2\underline{L}\Phi-\frac{1}{4}w\Phi\right)G\Big|\\
&+C\left(\left(\int_{(1.1)^{-2}\tau-1}^{\tau+1}\left(\int_{\Sigma_{t^*}\cap\{r\leq\frac{9t^*}{10}\}}G_1^2\right)^{\frac{1}{2}}dt^*\right)^2+\iint_{\mathcal{R}((1.1)^{-2}\tau-1,\tau+1)\cap\{r\leq\frac{9t^*}{10}\}}G_1^2\right)\\
&+C\left(\sum_{m=0}^1\iint_{\mathcal{R}((1.1)^{-2}\tau-1,\tau+1)\cap\{r\leq\frac{9t^*}{10}\}} r^{1+\delta}\left(\partial_{t^*}^m G_2\right)^2\right.\\
&\qquad\left.+\sup_{t^*\in[(1.1)^{-2}\tau-1,\tau+1]}\int_{\Sigma_{t^*}\cap\{|r-3M|\leq\frac{M}{8}\}\cap\{r\leq\frac{9t^*}{10}\}}G_2^2\right).
\end{aligned}$$

Proof. By Proposition 9.5.4.2 applied to the t^* interval $[(1.1)^{-1}\tau,\tau]$, we have

$$\begin{aligned}
&\iint_{\mathcal{R}((1.1)^{-1}\tau,\tau)\cap\{r\leq r_Y^-\}} K^N(\Phi)\\
\leq& C\tau^{-2}\int_{\Sigma_{(1.1)^{-1}\tau}} J_\mu^{Z+N,w^Z}(\Phi)\, n_{\Sigma_{(1.1)^{-1}\tau}}^\mu + C^2\int_{\Sigma_{(1.1)^{-1}\tau}\cap\{r\leq r_Y^-\}} J_\mu^N(\Phi)\, n_{\Sigma_{(1.1)^{-1}\tau}}^\mu\\
&+C\left(\sum_{m=0}^1\iint_{\mathcal{R}((1.1)^{-1}\tau-1,\tau+1)\cap\{r\leq\frac{9t^*}{10}\}} r^{1+\delta}\left(\partial_{t^*}^m G\right)^2\right.\\
&\qquad\left.+\sup_{t^*\in[(1.1)^{-1}\tau-1,\tau+1]}\int_{\Sigma_{t^*}\cap\{|r-3M|\leq\frac{M}{8}\}}G^2\right).
\end{aligned}$$

By taking the infimum there exists $\tilde{\tau}\in[(1.1)^{-1}\tau,\tau]$ such that

$$\int_{\Sigma_{\tilde{\tau}}\cap\{r\leq r_Y^-\}} J_\mu^N(\Phi)\, n_{\Sigma_{\tilde{\tau}}}^\mu \leq C\tau^{-1}\iint_{\mathcal{R}((1.1)^{-1}\tau,\tau)\cap\{r\leq r_Y^-\}} K^N(\Phi).$$

Hence,

$$\begin{aligned}
&\int_{\Sigma_{\tilde{\tau}}\cap\{r\leq r_Y^-\}} J_\mu^N\left(\Phi\right) n_{\Sigma_{\tilde{\tau}}}^\mu\\
\leq& C\tau^{-2}\int_{\Sigma_{(1.1)^{-1}\tau}} J_\mu^{Z+N,w^Z}\left(\Phi\right) n_{\Sigma_{(1.1)^{-1}\tau}}^\mu + C^2\tau^{-1}\int_{\Sigma_{(1.1)^{-1}\tau}\cap\{r\leq r_Y^-\}} J_\mu^N\left(\Phi\right) n_{\Sigma_{(1.1)^{-1}\tau}}^\mu\\
&+C\tau^{-1}\left(\sum_{m=0}^1\iint_{\mathcal{R}((1.1)^{-1}\tau-1,\tau+1)\cap\{r\leq\frac{9t^*}{10}\}} r^{1+\delta}\left(\partial_{t^*}^m G\right)^2\right.\\
&\qquad\left.+\sup_{t^*\in[(1.1)^{-1}\tau-1,\tau+1]}\int_{\Sigma_{t^*}\cap\{|r-3M|\leq\frac{M}{8}\}} G^2\right).
\end{aligned}$$

Apply Proposition 9.5.4.2 to the t^* interval $[\tilde{\tau},\tau]$ and use Proposition 9.4.17 and 9.5.7

$$
\begin{aligned}
&\int_{\Sigma_\tau\cap\{r\leq\frac{t^*}{2}\}} J_\mu^N\left(\Phi\right) n_{\Sigma_\tau}^\mu\\
\leq& C\left(\tau^{-2}\int_{\Sigma_{\tilde{\tau}}} J_\mu^{Z+N,w^Z}\left(\Phi\right) n_{\Sigma_{\tilde{\tau}}}^\mu + C\int_{\Sigma_{\tilde{\tau}}\cap\{r\leq r_Y^-\}} J_\mu^N\left(\Phi\right) n_{\Sigma_{\tilde{\tau}}}^\mu\right)\\
&+C\left(\sum_{m=0}^1\iint_{\mathcal{R}((1.1)^{-1}\tau-1,\tau+1)\cap\{r\leq\frac{9t^*}{10}\}} r^{1+\delta}\left(\partial_{t^*}^m G\right)^2\right.\\
&\qquad\left.+\sup_{t^*\in[(1.1)^{-1}\tau-1,\tau+1]}\int_{\Sigma_{t^*}\cap\{|r-3M|\leq\frac{M}{8}\}\cap\{r\leq\frac{9t^*}{10}\}} G^2\right)\\
\leq& C\tau^{-2}\left(\int_{\Sigma_{\tilde{\tau}}} J_\mu^{Z+N,w^Z}\left(\Phi\right) n_{\Sigma_{\tilde{\tau}}}^\mu + \int_{\Sigma_{(1.1)^{-1}\tau}} J_\mu^{Z+N,w^Z}\left(\Phi\right) n_{\Sigma_{(1.1)^{-1}\tau}}^\mu\right)\\
&+C^2\tau^{-1}\int_{\Sigma_{(1.1)^{-1}\tau}\cap\{r\leq r_Y^-\}} J_\mu^N\left(\Phi\right) n_{\Sigma_{(1.1)^{-1}\tau}}^\mu\\
&+C\left(\sum_{m=0}^1\iint_{\mathcal{R}((1.1)^{-1}\tau-1,\tau+1)\cap\{r\leq\frac{9t^*}{10}\}} r^{1+\delta}\left(\partial_{t^*}^m G\right)^2\right.\\
&\qquad\left.+\sup_{t^*\in[(1.1)^{-1}\tau-1,\tau+1]}\int_{\Sigma_{t^*}\cap\{|r-3M|\leq\frac{M}{8}\}\cap\{r\leq\frac{9t^*}{10}\}} G^2\right)\\
\leq& C^2\tau^{-1}\int_{\Sigma_{(1.1)^{-1}\tau}\cap\{r\leq r_Y^-\}} J_\mu^N\left(\Phi\right) n_{\Sigma_{(1.1)^{-1}\tau}}^\mu + C\tau^{-2}\int_{\Sigma_{(1.1)^{-1}\tau}} J_\mu^{Z+N,w^Z}\left(\Phi\right) n_{\Sigma_{(1.1)^{-1}\tau}}^\mu\\
&+C\tau^{-2}\iint_{\mathcal{R}((1.1)^{-1}\tau,\tau)} K_+^{Z,w^Z}\left(\Phi\right) + C\tau^{-2}\left|\iint_{\mathcal{R}((1.1)^{-1}\tau,\tau)}\left(u^2\underline{L}\Phi + v^2 L\Phi - \frac{1}{4}w\Phi\right)G\right|\\
&+C\left(\sum_{m=0}^1\iint_{\mathcal{R}((1.1)^{-1}\tau-1,\tau+1)\cap\{r\leq\frac{9t^*}{10}\}} r^{1+\delta}\left(\partial_{t^*}^m G\right)^2\right.\\
&\qquad\left.+\sup_{t^*\in[(1.1)^{-1}\tau-1,\tau+1]}\int_{\Sigma_{t^*}\cap\{|r-3M|\leq\frac{M}{8}\}\cap\{r\leq\frac{9t^*}{10}\}} G^2\right)
\end{aligned}
\tag{9.11}
$$

Replacing $[(1.1)^{-1}\tau, \tau]$ with $[(1.1)^{-2}\tau, (1.1)^{-1}\tau]$, we get also

$$\begin{aligned}
&\int_{\Sigma_{(1.1)^{-1}\tau}\cap\{r\leq\frac{t^*}{2}\}} J_\mu^N(\Phi)\, n_{\Sigma_\tau}^\mu \\
\leq& C^2\tau^{-1}\int_{\Sigma_{(1.1)^{-2}\tau}\cap\{r\leq r_Y^-\}} J_\mu^N(\Phi)\, n_{\Sigma_{(1.1)^{-2}\tau}}^\mu + C\tau^{-2}\int_{\Sigma_{(1.1)^{-2}\tau}} J_\mu^{Z+N,w^Z}(\Phi)\, n_{\Sigma_{(1.1)^{-2}\tau}}^\mu \\
&+ C\tau^{-2}\iint_{\mathcal{R}((1.1)^{-2}\tau,(1.1)^{-1}\tau)} K_+^{Z,w^Z}(\Phi) \\
&+ C\tau^{-2}\Big|\iint_{\mathcal{R}((1.1)^{-2}\tau,(1.1)^{-1}\tau)}\left(u^2 L\Phi + v^2\underline{L}\Phi - \frac{1}{4}w\Phi\right) G\Big| \\
&+ C\left(\sum_{m=0}^{1}\iint_{\mathcal{R}((1.1)^{-2}\tau-1,(1.1)^{-1}\tau+1)\cap\{r\leq\frac{9t^*}{10}\}} r^{1+\delta}\left(\partial_{t^*}^m G\right)^2\right. \\
&\left.+ \sup_{t^*\in[(1.1)^{-2}\tau-1,(1.1)^{-1}\tau+1]}\int_{\Sigma_{t^*}\cap\{|r-3M|\leq\frac{M}{8}\}\cap\{r\leq\frac{9t^*}{10}\}} G^2\right)
\end{aligned} \tag{9.12}$$

Therefore, plugging (9.12) into (9.11),

$$\begin{aligned}
&\int_{\Sigma_\tau\cap\{r\leq\frac{t^*}{2}\}} J_\mu^N(\Phi)\, n_{\Sigma_\tau}^\mu \\
\leq& C^2\tau^{-2}\int_{\Sigma_{(1.1)^{-2}\tau}\cap\{r\leq r_Y^-\}} J_\mu^N(\Phi)\, n_{\Sigma_{(1.1)^{-2}\tau}}^\mu + C\tau^{-2}\int_{\Sigma_{(1.1)^{-2}\tau}} J_\mu^{Z,w^Z}(\Phi)\, n_{\Sigma_{(1.1)^{-1}\tau}}^\mu \\
&+ C\tau^{-2}\iint_{\mathcal{R}((1.1)^{-2}\tau,\tau)} K_+^{Z,w^Z}(\Phi) + C\tau^{-2}\Big|\iint_{\mathcal{R}((1.1)^{-2}\tau,\tau)}\left(u^2 L\Phi + v^2\underline{L}\Phi - \frac{1}{4}w\Phi\right) G\Big| \\
&+ C\left(\sum_{m=0}^{1}\iint_{\mathcal{R}((1.1)^{-2}\tau-1,\tau+1)\cap\{r\leq\frac{9t^*}{10}\}} r^{1+\delta}\left(\partial_{t^*}^m G\right)^2\right. \\
&\qquad\left.+ \sup_{t^*\in[(1.1)^{-2}\tau-1,\tau+1]}\int_{\Sigma_{t^*}\cap\{|r-3M|\leq\frac{M}{8}\}\cap\{r\leq\frac{9t^*}{10}\}} G^2\right)
\end{aligned}$$

□

Proposition 9.5.8 immediately gives control over the non-degenerate energy and conformal energy using Propositions 9.5.2 and 9.5.7 respectively:

Corollary 9.5.9. *For any $\gamma < 1$,*

$$\begin{aligned}
&\int_{\Sigma_\tau} J_\mu^{Z,w^Z}(\Phi)\, n_{\Sigma_\tau}^\mu + C\tau^2 \int_{\Sigma_\tau \cap \{r \le \gamma\tau\}} J_\mu^N(\Phi)\, n_{\Sigma_\tau}^\mu \\
\le & C\left(\int_{\Sigma_{\tau_0}} J_\mu^{Z,w^Z}(\Phi)\, n_{\Sigma_{\tau_0}}^\mu + C\int_{\Sigma_{\tau_0}} J_\mu^N(\Phi)\, n_{\Sigma_{\tau_0}}^\mu \right. \\
&+ \iint_{\mathcal{R}(\tau_0,\tau)} K_+^{Z,w^Z}(\Phi) + \Big| \iint_{\mathcal{R}(\tau_0,\tau)} \left(u^2 \underline{L}\Phi + v^2 L\Phi - \frac{1}{4} w\Phi \right) G \Big| \Big) \\
&+ C\left(\sum_{m=0}^{1} \iint_{\mathcal{R}(\tau_0-1,\tau+1)\cap\{r\le\frac{9t^*}{10}\}} (t^*)^2 r^{1+\delta} (\partial_{t^*}^m G)^2 \right. \\
&\qquad \left. + \sup_{t^*\in[\tau_0-1,\tau+1]} \int_{\Sigma_{t^*}\cap\{|r-3M|\le\frac{M}{8}\}\cap\{r\le\frac{9t^*}{10}\}} (t^*)^2 G^2 \right).
\end{aligned}$$

Proof. By Proposition 9.5.2,

$$\tau^2 \int_{\Sigma_\tau \cap \{r\le\gamma\tau\}} J_\mu^N(\Phi)\, n_{\Sigma_\tau}^\mu \le C\int_{\Sigma_\tau} J_\mu^{Z+N,w^Z}(\Phi)\, n_{\Sigma_\tau}^\mu + C^2\tau^2 \int_{\Sigma_\tau\cap\{r\le r_Y^-\}} J_\mu^N(\Phi)\, n_{\Sigma_\tau}^\mu.$$

Therefore, by Propositions 9.5.7 and 9.5.8,

$$\begin{aligned}
&\int_{\Sigma_\tau} J_\mu^{Z,w^Z}(\Phi)\, n_{\Sigma_\tau}^\mu + C\tau^2 \int_{\Sigma_\tau\cap\{r\le\gamma\tau\}} J_\mu^N(\Phi)\, n_{\Sigma_\tau}^\mu \\
\le & C\left(\int_{\Sigma_{(1.1)^{-2}\tau}} J_\mu^N(\Phi)\, n_{\Sigma_{(1.1)^{-2}\tau}}^\mu + \int_{\Sigma_{(1.1)^{-2}\tau}} J_\mu^{Z,w^Z}(\Phi)\, n_{\Sigma_{(1.1)^{-2}\tau}}^\mu \right. \\
&\qquad + \iint_{\mathcal{R}((1.1)^{-2}\tau,\tau)} K_+^{Z,w^Z}(\Phi) \\
&\qquad \left. + \Big| \iint_{\mathcal{R}((1.1)^{-2}\tau,\tau)} \left(u^2\underline{L}\Phi + v^2 L\Phi - \frac{1}{4} w\Phi \right) G \Big| \right) \\
&+ C\tau^2 \left(\sum_{m=0}^{1} \iint_{\mathcal{R}((1.1)^{-2}\tau-1,\tau+1)} r^{1+\delta} (\partial_{t^*}^m G)^2 \right. \\
&\qquad \left. + \sup_{t^*\in[(1.1)^{-2}\tau-1,\tau+1]} \int_{\Sigma_{t^*}\cap\{|r-3M|\le\frac{M}{8}\}\cap\{r\le\frac{9t^*}{10}\}} G^2 \right)
\end{aligned}$$

We then use the same estimate for $[(1.1)^{-4}\tau, (1.1)^{-2}\tau], [(1.1)^{-6}\tau, (1.1)^{-4}\tau], ...$ □

The term $\iint_{\mathcal{R}(\tau_0,\tau)} K_+^{Z,w^Z}(\Phi)$ can be controlled. Here is where the control of the

logarithmic divergences from the red-shift vector field is crucially used.

Proposition 9.5.10.

$$\begin{aligned}&\iint_{\mathcal{R}(\tau',\tau)} K_+^{Z,w^Z}(\Phi)\\ \leq& C\iint_{\mathcal{R}(\tau',\tau)\cap\{r\geq r_Y^-\}} t^*\left(r^{-2}J_\mu^N(\Phi)\,n_{\Sigma_{\bar\tau}}^\mu + r^{-4}\Phi\right) + \epsilon\iint_{\mathcal{R}(\tau',\tau)\cap\{r\leq r_Y^-\}}(t^*)^2\,K^N(\Phi).\end{aligned}$$

Proof. See [16]. □

The bulk term arising from the inhomogeneous term G can also be controlled.

Proposition 9.5.11.

$$\begin{aligned}&\left|\iint_{\mathcal{R}(\tau_0,\tau)}\left(u^2\underline{L}\Phi + v^2L\Phi - \frac{1}{4}w\Phi\right)G\right|\\ \leq&\delta'\iint_{\mathcal{R}(\tau_0,\tau)\cap\{r\leq\frac{t^*}{2}\}}(t^*)^2K^{X_0}(\Phi) + \delta'\iint_{\mathcal{R}(\tau_0,\tau)\cap\{r\leq r_Y^-\}}(t^*)^2K^N(\Phi)\\ &+\delta'\sup_{t^*\in[\tau_0,\tau]}\left(\int_{\Sigma_{t^*}\cap\{r\geq\frac{t^*}{2}\}}J_\mu^{Z+N,w^Z}(\Phi)\,n_{\Sigma_{t^*}}^\mu + (t^*)^2\int_{\Sigma_{t^*}\cap\{r\leq\frac{23M}{8}\}}J_\mu^N(\Phi)\,n_{\Sigma_{t^*}}^\mu\right)\\ &+C(\delta')^{-1}\sum_{m=0}^{1}\iint_{\mathcal{R}(\tau_0,\tau)\cap\{r\leq\frac{t^*}{2}\}}(t^*)^2r^{1+\delta}\left(\partial_{t^*}^mG\right)^2\\ &+C(\delta')^{-1}\left(\int_{\tau_0}^{\tau}\left(\int_{\Sigma_{t^*}\cap\{r\geq\frac{t^*}{2}\}}r^2G^2\right)^{\frac{1}{2}}dt^*\right)^2\\ &+C(\delta')^{-1}\sup_{t^*\in[\tau_0,\tau]}\int_{\Sigma_{t^*}\cap\{r_Y^-\leq r\leq\frac{25M}{8}\}}(t^*)^2G^2.\end{aligned}$$

Proof. Two regions require particular care to deal with. The first is the region $\{r\leq r_Y^-\}$, since the coefficients of the vector field Z are not bounded as $r\to r_+$. The other is the region $\{|r-3M|\leq\frac{M}{8}\}$. This is where trapping occurs and where the integrated decay estimate degenerates or loses derivatives. We first look at the region $\{r\leq r_+\}$

using the null frame:

$$
\begin{aligned}
&\left|\iint_{\mathcal{R}(\tau_0,\tau)\cap\{r\leq r_Y^-\}}\left(u^2\underline{L}\Phi+v^2L\Phi-\frac{1}{4}w\Phi\right)G\right|\\
\leq& C\iint_{\mathcal{R}(\tau_0,\tau)\cap\{r\leq r_Y^-\}}\left((t^*)^2+(r_S^*)^2\right)\\
&\qquad\times\left(|\nabla_{\hat{V}}\Phi G|+(1-\mu)|\nabla_{\hat{Y}}\Phi G|+(1-\mu)\sum_A|\nabla_{E_A}\Phi G|\right)\\
&\qquad\text{using Proposition 9.5.1}\\
\leq& C\iint_{\mathcal{R}(\tau_0,\tau)\cap\{r\leq r_Y^-\}}(t^*)^2\left(|\log|r-r_+||^2|\nabla_{\hat{V}}\Phi G|+|\nabla_{\hat{Y}}\Phi G|+\sum_A|\nabla_{E_A}\Phi G|\right)\\
\leq&\delta'\iint_{\mathcal{R}(\tau_0,\tau)\cap\{r\leq r_Y^-\}}(t^*)^2\left(|\log|r-r_+||^4\left(\nabla_{\hat{V}}\Phi\right)^2+\left(\nabla_{\hat{Y}}\Phi\right)^2+\sum_A\left(\nabla_{E_A}\Phi\right)^2\right)\\
&+C(\delta')^{-1}\iint_{\mathcal{R}(\tau_0,\tau)\cap\{r\leq r_Y^-\}}(t^*)^2G^2\\
\leq&\delta'\iint_{\mathcal{R}(\tau_0,\tau)\cap\{r\leq r_Y^-\}}(t^*)^2K^N(\Phi)+C(\delta')^{-1}\iint_{\mathcal{R}(\tau_0,\tau)\cap\{r\leq r_Y^-\}}(t^*)^2G^2.
\end{aligned}
$$

For the region $\{r_Y^-\leq r\leq\frac{25M}{8}\}$, where trapping occurs, we integrate by parts in t^* so that the bulk term does not have $\partial_{t^*}\Phi$, which cannot be controlled by the integrated

decay estimate.

$$
\begin{aligned}
&\left|\iint_{\mathcal{R}(\tau_0,\tau)\cap\{r_Y^-\leq r\leq\frac{25M}{8}\}}\left(u^2\underline{L}\Phi+v^2L\Phi-\frac{1}{4}w\Phi\right)G\right|\\
\leq& C\iint_{\mathcal{R}(\tau_0,\tau)\cap\{r_Y^-\leq r\leq\frac{25M}{8}\}}(t^*)^2|\partial_r\Phi G|+(t^*)^2|\Phi\partial_{t^*}G|+t^*|\Phi G|\\
&+\int_{\Sigma_\tau}\tau^2|\Phi G|+\int_{\Sigma_{\tau_0}}\tau_0^2|\Phi G|\\
\leq& C\left(\iint_{\mathcal{R}(\tau_0,\tau)\cap\{r\leq\frac{25M}{8}\}}(t^*)^2\left(\Phi^2+(\partial_r\Phi)^2\right)\right)^{\frac{1}{2}}\\
&\times\left(\sum_{m=0}^{1}\iint_{\mathcal{R}(\tau_0,\tau)\cap\{r_Y^-\leq r\leq\frac{25M}{8}\}}(t^*)^2\left(\partial_{t^*}^mG\right)^2\right)^{\frac{1}{2}}\\
&+\delta'\int_{\Sigma_\tau\cap\{r_Y^-\leq r\leq r_Y^-\leq\frac{25M}{8}\}}\tau^2J_\mu^N\left(\Phi\right)n_{\Sigma_\tau}^\mu+\delta'\int_{\Sigma_{\tau_0}\cap\{r_Y^-\leq r\leq\frac{25M}{8}\}}\tau_0^2J_\mu^N\left(\Phi\right)n_{\Sigma_{\tau_0}}^\mu\\
&+C(\delta')^{-1}\sup_{t^*\in[\tau_0,\tau]}\int_{\Sigma_{t^*}\cap\{r_Y^-\leq r\leq\frac{25M}{8}\}}(t^*)^2G^2,
\end{aligned}
$$

using Proposition 9.4.10. We then move to the region $\{\frac{25M}{8}\leq r\leq\frac{t^*}{2}\}$:

$$
\begin{aligned}
&\left|\iint_{\mathcal{R}(\tau_0,\tau)\cap\{\frac{25M}{8}\leq r\leq\frac{t^*}{2}\}}\left(u^2\underline{L}\Phi+v^2L\Phi-\frac{1}{4}w\Phi\right)G\right|\\
\leq& C\iint_{\mathcal{R}(\tau_0,\tau)\cap\{\frac{25M}{8}\leq r\leq\frac{t^*}{2}\}}\left((t^*)^2|\partial\Phi|+t^*|\Phi|\right)|G|\\
\leq& C\left(\iint_{\mathcal{R}(\tau_0,\tau)\cap\{\frac{25M}{8}\leq r\leq\frac{t^*}{2}\}}(t^*)^2\left(r^{-3-\delta}\Phi^2+r^{-1-\delta}J_\mu^N\left(\Phi\right)n_{\Sigma_{t^*}}^\mu\right)\right)^{\frac{1}{2}}\\
&\times\left(\iint_{\mathcal{R}(\tau_0,\tau)\cap\{\frac{25M}{8}\leq r\leq\frac{t^*}{2}\}}\left(r^{3+\delta}+(t^*)^2r^{1+\delta}\right)G^2\right)^{\frac{1}{2}}\\
\leq& C\left(\iint_{\mathcal{R}(\tau_0,\tau)\cap\{\frac{25M}{8}\leq r\leq\frac{t^*}{2}\}}(t^*)^2\left(r^{-3-\delta}\Phi^2+r^{-1-\delta}J_\mu^N\left(\Phi\right)n_{\Sigma_{t^*}}^\mu\right)\right)^{\frac{1}{2}}\\
&\times\left(\iint_{\mathcal{R}(\tau_0,\tau)\cap\{\frac{25M}{8}\leq r\leq\frac{t^*}{2}\}}(t^*)^2r^{1+\delta}G^2\right)^{\frac{1}{2}}
\end{aligned}
$$

Finally, we estimate in the region $\{r \geq \frac{t^*}{2}\}$:

$$\begin{aligned}
&\left| \iint_{\mathcal{R}(\tau_0,\tau)\cap\{r\geq\frac{t^*}{2}\}} \left(u^2 \underline{L}\Phi + v^2 L\Phi - \frac{1}{4} w\Phi\right) G\right| \\
\leq & C \sup_{t^*\in[\tau_0,\tau]} \left(\int_{\Sigma_{t^*}\cap\{r\geq\frac{t^*}{2}\}} J_\mu^{Z,w^Z}(\Phi)\, n_{\Sigma_{t^*}}^\mu + (t^*)^2 \int_{\Sigma_{t^*}\cap\{r\leq r_Y^-\}} J_\mu^N(\Phi)\, n_{\Sigma_{t^*}}^\mu \right)^{\frac{1}{2}} \\
& \times \int_{\tau_0}^{\tau} \left(\int_{\Sigma_{t^*}\cap\{r\geq\frac{t^*}{2}\}} r^2 G^2 \right)^{\frac{1}{2}} dt^*,
\end{aligned}$$

where we have used Proposition 9.5.2.The Proposition follows from Cauchy-Schwarz.

□

We have therefore proved the following decay result associated to the vector field Z.

Proposition 9.5.12. *For $\delta, \delta' > 0$ sufficiently small and $0 \leq \gamma < 1$, there exist $c = c(\delta,\gamma)$ and $C = C(\delta,\gamma)$ such that the following estimate holds for any solution to $\Box_{g_K}\Phi = G$:*

$$\begin{aligned}
& c \int_{\Sigma_\tau} J_\mu^{Z,w^Z}(\Phi)\, n_{\Sigma_\tau}^\mu + \tau^2 \int_{\Sigma_\tau\cap\{r\leq\gamma\tau\}} J_\mu^N(\Phi)\, n_{\Sigma_\tau}^\mu \\
\leq & C \int_{\Sigma_{\tau_0}} J_\mu^{Z+CN,w^Z}(\Phi)\, n_{\Sigma_{\tau_0}}^\mu + C \iint_{\mathcal{R}(\tau_0,\tau)} t^* r^{-1+\delta} K^{X_1}(\Phi) \\
& + C\delta' \iint_{\mathcal{R}(\tau_0,\tau)\cap\{r\leq\frac{t^*}{2}\}} (t^*)^2 K^{X_0}(\Phi) + C(\delta'+\epsilon) \iint_{\mathcal{R}(\tau_0,\tau)\cap\{r\leq r_Y^-\}} (t^*)^2 K^N(\Phi) \\
& + C(\delta')^{-1} \left(\int_{\tau_0}^{\tau} \left(\int_{\Sigma_{t^*}\cap\{r\geq\frac{t^*}{2}\}} r^2 G^2 \right)^{\frac{1}{2}} dt^* \right)^2 \\
& + C(\delta')^{-1} \sum_{m=0}^{1} \iint_{\mathcal{R}(\tau_0,\tau)\cap\{r\leq\frac{9t^*}{10}\}} (t^*)^2 r^{1+\delta} (\partial_{t^*}^m G)^2 \\
& + C(\delta')^{-1} \sup_{t^*\in[\tau_0,\tau]} \int_{\Sigma_{t^*}\cap\{r_Y^-\leq r\leq\frac{25M}{8}\}} (t^*)^2 G^2.
\end{aligned}$$

Remark 24. *As in Proposition 9.4.18, we can save a derivative if we know that the inhomogeneous term is supported away from the trapped region. More precisely, let*

$G = G_1 + G_2$ be any way to decompose the function G. Suppose G_2 is supported away from $\{r : |r - 3M| \leq \frac{M}{8}\}$. Then we can replace

$$\sum_{m=0}^{1} \iint_{\mathcal{R}(\tau_0,\tau)\cap\{r\leq\frac{9t^*}{10}\}} (t^*)^2 r^{1+\delta} \left(\partial_{t^*}^m G\right)^2 + \sup_{t^*\in[\tau_0,\tau]} \int_{\Sigma_{t^*}\cap\{r_Y^-\leq r\leq\frac{25M}{8}\}} (t^*)^2 G^2$$

by

$$\sum_{m=0}^{1} \iint_{\mathcal{R}(\tau_0,\tau)\cap\{r\leq\frac{9t^*}{10}\}} (t^*)^2 r^{1+\delta} \left(\partial_{t^*}^m G_1\right)^2 + \iint_{\mathcal{R}(\tau_0,\tau)\cap\{r\leq\frac{9t^*}{10}\}} (t^*)^2 r^{1+\delta} G_2^2$$
$$+ \sup_{t^*\in[\tau_0,\tau]} \int_{\Sigma_{t^*}\cap\{r_Y^-\leq r\leq\frac{25M}{8}\}} (t^*)^2 G_1^2.$$

in Proposition 9.5.12.

9.6 Estimates for Solutions to $\Box_{g_K}\Phi = 0$

From this point onwards, we consider $\Box_{g_K}\Phi = 0$. In this section, we write down the energy estimates derived by Dafermos-Rodnianski [16]. These will be used in later sections.

Proposition 9.6.1.

$$\begin{aligned}&\tau^2 \int_{\Sigma_\tau\cap\{r\leq\frac{\tau}{2}\}} J_\mu^N(\Phi)\, n_{\Sigma_\tau}^\mu + c\int_{\Sigma_\tau} J_\mu^{Z+N,w^Z}(\Phi)\, n_{\Sigma_\tau}^\mu \\ \leq & C\tau^\eta \sum_{m=0}^{2} \left(\int_{\Sigma_{\tau_0}} J_\mu^{Z,w^Z}\left(\partial_{t^*}^m\Phi\right) n_{\Sigma_{\tau_0}}^\mu + \int_{\Sigma_{\tau_0}} J_\mu^N\left(\partial_{t^*}^m\Phi\right) n_{\Sigma_{\tau_0}}^\mu \right).\end{aligned}$$

Proof. We introduce the bootstrap assumptions:

$$\begin{aligned}&\tau^2 \int_{\Sigma_\tau\cap\{r\leq\frac{\tau}{2}\}} J_\mu^N(\Phi)\, n_{\Sigma_\tau}^\mu + c\int_{\Sigma_\tau} J_\mu^{Z+N,w^Z}(\Phi)\, n_{\Sigma_\tau}^\mu \\ \leq & A^2\tau^\eta \sum_{m=0}^{2} \int_{\Sigma_{\tau_0}} J_\mu^{Z+CN,w^Z}\left(\partial_{t^*}^m\Phi\right) n_{\Sigma_{\tau_0}}^\mu.\end{aligned} \tag{9.13}$$

$$\begin{aligned}&\tau^2\int_{\Sigma_\tau\cap\{r\leq\frac{\tau}{2}\}}J_\mu^N\left(\partial_{t^*}\Phi\right)n_{\Sigma_\tau}^\mu+c\int_{\Sigma_\tau}J_\mu^{Z+N,w^Z}\left(\Phi\right)n_{\Sigma_\tau}^\mu\\ \leq&A\tau^{1+\eta}\sum_{m=0}^{2}\int_{\Sigma_{\tau_0}}J_\mu^{Z+CN,w^Z}\left(\partial_{t^*}^m\Phi\right)n_{\Sigma_{\tau_0}}^\mu.\end{aligned}\tag{9.14}$$

Here we think of η as a small positive number. We divide the interval $[\tau_0,\tau]$ dyadically into $\tau_0\leq\tau_1\leq...\leq\tau_{n-1}\leq\tau_n=\tau$ with $\tau_{i+1}\leq(1.1)\tau_i$ and n the smallest integer for doing such division. We then have $n\sim\log|\tau-\tau'|$. We can now apply Proposition 9.5.4 on the intervals $[\tau_{i-1},\tau_i]$ and use the bootstrap assumption (9.13):

$$\begin{aligned}&\iint_{\mathcal{R}(\tau_{i-1},\tau_i)\cap\{r\leq\frac{t^*}{2}\}}K^{X_0}\left(\Phi\right)+\iint_{\mathcal{R}(\tau_{i-1},\tau_i)\cap\{r\leq r_Y^-\}}K^N\left(\Phi\right)\\ \leq&C\left(\tau_i^{-2}\int_{\Sigma_{\tau_{i-1}}}J_\mu^{Z,w^Z}\left(\Phi\right)n_{\Sigma_{\tau_{i-1}}}^\mu+C\int_{\Sigma_{\tau_{i-1}}\cap\{r\leq r_Y^-\}}J_\mu^N\left(\Phi\right)n_{\Sigma_{\tau_{i-1}}}^\mu\right)\\ \leq&CA^2\tau_i^{-2+\eta}\sum_{m=0}^{2}\int_{\Sigma_{\tau_0}}J_\mu^{Z+CN,w^Z}\left(\partial_{t^*}^m\Phi\right)n_{\Sigma_{\tau_0}}^\mu.\end{aligned}$$

Similarly, we can apply Proposition 9.5.4 on the intervals $[\tau_{i-1},\tau_i]$ for $\partial_{t^*}\Phi$ and use the bootstrap assumption (9.14):

$$\begin{aligned}&\iint_{\mathcal{R}(\tau_{i-1},\tau_i)\cap\{r\leq\frac{t^*}{2}\}}K^{X_0}\left(\partial_{t^*}\Phi\right)+\iint_{\mathcal{R}(\tau_{i-1},\tau_i)\cap\{r\leq r_Y^-\}}K^N\left(\partial_{t^*}\Phi\right)\\ \leq&C\left(\tau_i^{-2}\int_{\Sigma_{\tau_{i-1}}}J_\mu^{Z,w^Z}\left(\partial_{t^*}\Phi\right)n_{\Sigma_{\tau_{i-1}}}^\mu+C\int_{\Sigma_{\tau_{i-1}}\cap\{r\leq r_Y^-\}}J_\mu^N\left(\partial_{t^*}\Phi\right)n_{\Sigma_{\tau_{i-1}}}^\mu\right)\\ \leq&CA\tau_i^{-1+\eta}\sum_{m=0}^{2}\int_{\Sigma_{\tau_0}}J_\mu^{Z+CN,w^Z}\left(\partial_{t^*}^m\Phi\right)n_{\Sigma_{\tau_0}}^\mu.\end{aligned}$$

By Proposition 9.4.19, we have

$$\iint_{\mathcal{R}(\tau_{i-1},\tau_i)}r^{-1+\delta}K^{X_1}\left(\partial_{t^*}\Phi\right)\leq C\sum_{m=0}^{1}\int_{\Sigma_{\tau_0}}J_\mu^N\left(\partial_{t^*}^m\Phi\right)n_{\Sigma_{\tau_0}}^\mu.$$

By Propositions 9.4.19 and 9.5.5, we have

$$\begin{aligned}
&\iint_{\mathcal{R}(\tau_{i-1},\tau_i)} r^{-1+\delta} K^{X_1}(\Phi) \\
\leq & C \iint_{\mathcal{R}(\tau_{i-1},\tau_i)\cap\{r\leq\frac{t^*}{2}\}} K^{X_1}(\Phi) + C\tau_i^{-1+\delta} \iint_{\mathcal{R}(\tau_{i-1},\tau_i)\cap\{r\geq\frac{t^*}{2}\}} K^{X_1}(\Phi) \\
\leq & CA\tau_i^{-1+\eta} \sum_{m=0}^{2} \int_{\Sigma_{\tau_0}} J_\mu^{Z+CN,w^Z}(\partial_{t^*}^m \Phi)\, n_{\Sigma_{\tau_0}}^\mu .
\end{aligned}$$

Apply Proposition 9.5.12, we get

$$\begin{aligned}
&c\int_{\Sigma_\tau} J_\mu^{Z+N,w^Z}(\Phi)\, n_{\Sigma_\tau}^\mu + \tau^2 \int_{\Sigma_\tau\cap\{r\leq\gamma\tau\}} J_\mu^N(\Phi)\, n_{\Sigma_\tau}^\mu \\
\leq & C\int_{\Sigma_{\tau_0}} J_\mu^{Z+CN,w^Z}(\Phi)\, n_{\Sigma_{\tau_0}}^\mu + C\iint_{\mathcal{R}(\tau_0,\tau)} t^* r^{-1+\delta} K^{X_1}(\Phi) \\
& + C\delta' \iint_{\mathcal{R}(\tau_0,\tau)\cap\{r\leq\frac{t^*}{2}\}} (t^*)^2 K^{X_0}(\Phi) + C(\delta'+\epsilon) \iint_{\mathcal{R}(\tau_0,\tau)\cap\{r\leq r_Y^-\}} (t^*)^2 K^N(\Phi) \\
\leq & \left(C + \big(C + CA + CA^2(2\delta'+\epsilon)\big) \sum_{i=0}^{n-1} \tau_i^\eta \right) \sum_{m=0}^{2} \int_{\Sigma_{\tau_0}} J_\mu^{Z+CN,w^Z}(\partial_{t^*}^m\Phi)\, n_{\Sigma_{\tau_0}}^\mu \\
\leq & \big(C + \eta^{-1}\big(C + CA + CA^2(2\delta'+\epsilon)\big)\tau^\eta\big) \sum_{m=0}^{2} \int_{\Sigma_{\tau_0}} J_\mu^{Z+CN,w^Z}(\partial_{t^*}^m\Phi)\, n_{\Sigma_{\tau_0}}^\mu .
\end{aligned}$$

Now take A large, $\epsilon = \frac{\eta}{4C}$ and $\delta' = \frac{\epsilon}{2}$, we improve (9.13). Apply Proposition 9.5.12 again, this time to $\partial_{t^*}\Phi$, we have

$$\begin{aligned}
&c\int_{\Sigma_\tau} J_\mu^{Z,w^Z}(\partial_{t^*}\Phi)\, n_{\Sigma_\tau}^\mu + \tau^2 \int_{\Sigma_\tau\cap\{r\leq\gamma\tau\}} J_\mu^N(\partial_{t^*}\Phi)\, n_{\Sigma_\tau}^\mu \\
\leq & C\int_{\Sigma_{\tau_0}} J_\mu^{Z+CN,w^Z}(\partial_{t^*}\Phi)\, n_{\Sigma_{\tau_0}}^\mu + C\iint_{\mathcal{R}(\tau_0,\tau)} t^* r^{-1+\delta} K^{X_1}(\partial_{t^*}\Phi) \\
& + C\delta' \iint_{\mathcal{R}(\tau_0,\tau)\cap\{r\leq\frac{t^*}{2}\}} (t^*)^2 K^{X_0}(\partial_{t^*}\Phi) + C(\delta'+\epsilon) \iint_{\mathcal{R}(\tau_0,\tau)\cap\{r\leq r_Y^-\}} (t^*)^2 K^N(\partial_{t^*}\Phi) \\
\leq & \left(C + C\sum_{i=0}^{n-1}\tau_i + CA(2\delta'+\epsilon)\sum_{i=0}^{n-1}\tau_i^{1+\eta} \right) \sum_{m=0}^{2} \int_{\Sigma_{\tau_0}} J_\mu^{Z+CN,w^Z}(\partial_{t^*}^m\Phi)\, n_{\Sigma_{\tau_0}}^\mu \\
\leq & \big(C + C\tau + CA(2\delta'+\epsilon)\tau^{1+\eta}\big) \sum_{m=0}^{2} \int_{\Sigma_{\tau_0}} J_\mu^{Z+CN,w^Z}(\partial_{t^*}^m\Phi)\, n_{\Sigma_{\tau_0}}^\mu .
\end{aligned}$$

Now take A large, $\delta' = \epsilon$ and ϵ sufficiently small, we also improve (9.14). $\square$

In particular, the Theorem of Dafermos-Rodnianski [16] is retrieved.

Corollary 9.6.2 (Dafermos-Rodnianski)**.** *Suppose $\Box_{g_K}\Phi = 0$. Then for all $\eta > 0$ and all $M > 0$ there exists a_0 such that the following estimates hold on Kerr spacetimes with (M, a) for which $a \leq a_0$:*

1. *Boundedness of Non-degenerate Energy*

$$\begin{aligned}&\int_{\Sigma_\tau} J_\mu^N(\Phi)\, n_{\Sigma_\tau}^\mu + \int_{\mathcal{H}(\tau_0,\tau)} J_\mu^N(\Phi)\, n_{\mathcal{H}^+}^\mu \\ &+ \iint_{\mathcal{R}(\tau_0,\tau)\cap\{r\leq r_Y^-\}} K^N(\Phi) + \iint_{\mathcal{R}(\tau',\tau)} K^{X_0}(\Phi) \\ \leq & C\int_{\Sigma_{\tau_0}} J_\mu^N(\Phi)\, n_{\Sigma_{\tau_0}}^\mu.\end{aligned}$$

2. *Decay of Non-degenerate Energy*

$$\begin{aligned}&\tau^2\int_{\Sigma_\tau\cap\{r\leq\gamma\tau\}} J_\mu^N(\Phi)\, n_{\Sigma_\tau}^\mu + c\int_{\Sigma_\tau} J_\mu^{Z+N,w^Z}(\Phi)\, n_{\Sigma_\tau}^\mu \\ \leq & C\tau^{1+\eta}\sum_{m=0}^{1}\int_{\Sigma_{\tau_0}} J_\mu^{Z+CN,w^Z}(\partial_{t^*}^m\Phi)\, n_{\Sigma_{\tau_0}}^\mu.\end{aligned}$$

and

$$\begin{aligned}&\tau^2\int_{\Sigma_\tau\cap\{r\leq\gamma\tau\}} J_\mu^N(\Phi)\, n_{\Sigma_\tau}^\mu + c\int_{\Sigma_\tau} J_\mu^{Z+N,w^Z}(\Phi)\, n_{\Sigma_\tau}^\mu \\ \leq & C\tau^{\eta}\sum_{m=0}^{2}\int_{\Sigma_{\tau_0}} J_\mu^{Z+CN,w^Z}(\partial_{t^*}^m\Phi)\, n_{\Sigma_{\tau_0}}^\mu.\end{aligned}$$

3. *Decay of Local Integrated Energy*

For $\tau' \leq \tau \leq (1.1)\tau'$,

$$\begin{aligned}&\iint_{\mathcal{R}(\tau',\tau)\cap\{r\leq\frac{t^*}{2}\}} K^{X_0}(\Phi) + \iint_{\mathcal{R}(\tau',\tau)\cap\{r\leq r_Y^-\}} K^N(\Phi)\\ \leq& C\tau^{-2+\eta}\sum_{m=0}^{2}\int_{\Sigma_{\tau_0}} J_\mu^{Z+CN,w^Z}\left(\partial_{t^*}^m\Phi\right) n_{\Sigma_{\tau_0}}^\mu.\end{aligned}$$

and

$$\iint_{\mathcal{R}(\tau',\tau)\cap\{r\leq\frac{t^*}{2}\}} K^{X_1}(\Phi) \leq C\tau^{-2+\eta}\sum_{m=0}^{3}\int_{\Sigma_{\tau_0}} J_\mu^{Z+CN,w^Z}\left(\partial_{t^*}^m\Phi\right) n_{\Sigma_{\tau_0}}^\mu.$$

Proof. 1 follows directly from Proposition 9.4.14. 2 contains two statements. The second one is a restatement of Proposition 9.6.1. The first one is evident from the proof of Proposition 9.6.1. 3 again has two statements. For the first statement, we revisit the proof of Proposition 9.6.1. Notice that the bootstrap assumptions are true. Hence it holds. For the second statement, we note by comparing Propositions 9.5.4 and 9.5.5 that K^{X_1} can be estimated in the same way as K^{X_0} except for an extra derivative. The second statement in 3 can then be proved by re-running the argument in Proposition 9.6.1 with an extra derivative. □

9.7 Estimates for $\hat{Y}\Phi$ and Elliptic Estimates

Away from the event horizon, we can control all higher order derivatives simply by commuting with ∂_{t^*} and using standard elliptic estimates. We write down a general version of the estimates in which we have inhomogeneous terms.

Proposition 9.7.1. *Suppose* $\Box_{g_K}\Phi = G$. *For* $m \geq 1$ *and for any* α,

1. Boundedness of Weighted Energy

$$\begin{aligned}&\int_{\Sigma_\tau\cap\{r\geq r_Y^-\}} r^\alpha \left(D^m\Phi\right)^2\\ \leq& C_{\alpha,m}\left(\sum_{j=0}^{m-1}\int_{\Sigma_\tau} r^\alpha J_\mu^N\left(\partial_{t^*}^j\Phi\right)n_{\Sigma_\tau}^\mu+\sum_{j=0}^{m-2}\int_{\Sigma_\tau} r^\alpha\left(D^jG\right)^2\right).\end{aligned}$$

2. Boundedness of Local Energy

For any $0<\gamma<\gamma'$,

$$\begin{aligned}&\int_{\Sigma_\tau\cap\{r_Y^-\leq r\leq\gamma t^*\}} r^\alpha \left(D^m\Phi\right)^2\\ \leq& C_{\alpha,m,\gamma,\gamma'}\left(\sum_{j=0}^{m-1}\int_{\Sigma_\tau\cap\{r\leq\gamma' t^*\}} r^\alpha J_\mu^N\left(\partial_{t^*}^j\Phi\right)n_{\Sigma_\tau}^\mu+\tau^{\alpha-\beta-2}\int_{\Sigma_\tau} r^\beta J_\mu^N\left(\Phi\right)n_{\Sigma_\tau}^\mu\right.\\ &\left.+\sum_{j=0}^{m-2}\int_{\Sigma_\tau\cap\{r\leq\gamma' t^*\}} r^\alpha\left(D^jG\right)^2\right).\end{aligned}$$

Proof. This is obvious for $m=1$ (even without the restriction $r\geq r_Y^-$). We will proceed by induction. Take $\delta\ll\frac{r_Y^- - r_+}{4}$. Assume

$$\sum_{j=1}^{m-1}\int_{\Sigma_\tau\cap\{r\geq r_Y^- -2\delta\}} r^\alpha\left(D^j\Phi\right)^2\leq C\left(\sum_{j=0}^{m-2}\int_{\Sigma_\tau} r^\alpha J_\mu^N\left(\partial_{t^*}^j\Phi\right)n_{\Sigma_\tau}^\mu+\sum_{j=0}^{m-3}\int_{\Sigma_\tau} r^\alpha\left(D^jG\right)^2\right).$$

We want to show

$$\int_{\Sigma_\tau\cap\{r\geq r_Y^- -\delta\}} r^\alpha\left(D^m\Phi\right)^2\leq C\left(\sum_{j=0}^{m-1}\int_{\Sigma_\tau} r^\alpha J_\mu^N\left(\partial_{t^*}^j\Phi\right)n_{\Sigma_\tau}^\mu+\sum_{j=0}^{m-2}\int_{\Sigma_\tau} r^\alpha\left(D^jG\right)^2\right),$$

which would then imply the conclusion. Denote by Δ_{g_K} the Laplace-Beltrami operator for the metric g_K restricted on the spacelike hypersurface $t^*=$ constant. Notice that since ∂_{t^*} is Killing, the operator is defined independent of t^*. Then we have

$$|[\Delta_{g_K},D^k]\Phi|\leq C\sum_{j=1}^{k+1}|D^j\Phi|.$$

Denote by ∇ the spatial derivatives with respect to the spatial coordinate variables in the Schwarzschild $(t^*_S, r_S, x^1_S, x^2_S)$ coordinate system. On the set $\{r \geq r^-_Y - \frac{r^-_Y - r_+}{4}\}$, Δ_{g_K} is elliptic and therefore controls all spatial derivatives:

$$\begin{aligned}
&\int_{\Sigma_\tau \cap \{r \geq r^-_Y - \delta\}} r^\alpha \left(D^m \Phi\right)^2 \\
\leq & C \int_{\Sigma_\tau \cap \{r \geq r^-_Y - 2\delta\}} r^\alpha \left(\left(\Delta_{g_K} D^{m-2} \Phi\right)^2 + \left(D^{m-1}\Phi\right)^2 + \left(\partial^{m-1}_{t^*} \nabla \Phi\right)^2 + \left(\partial^m_{t^*}\Phi\right)^2 \right) \\
\leq & C \int_{\Sigma_\tau \cap \{r \geq r^-_Y - 2\delta\}} r^\alpha \left(\left(D^{m-2}\Delta_{g_K}\Phi\right)^2 + \sum_{j=1}^{m-1} \left(D^j \Phi\right)^2 + r^{-2}\Phi^2 + \left(\partial^{m-1}_{t^*}\nabla\Phi\right)^2 + \left(\partial^m_{t^*}\Phi\right)^2 \right)
\end{aligned}$$

The last two terms are obviously bounded by $C \int_{\Sigma_\tau} J^N_\mu \left(\partial^{m-1}_{t^*}\Phi\right) n^\mu_{\Sigma_\tau}$. The second term can be bounded using the induction hypothesis. The third term can be bounded using the Hardy inequality in Proposition 9.4.10. Finally, to estimate the first term we use the equation $\Box_{g_K}\Phi = G$. Then, by the form of the Kerr metric, $\Delta_{g_K}\Phi = G - g^{t^*t^*}\partial^2_{t^*}\Phi - 2g^{t^*\phi^*}\partial_{t^*}\partial_{\phi^*}\Phi$. Therefore,

$$\begin{aligned}
&\int_{\Sigma_\tau \cap \{r \geq r^-_Y - 2\delta\}} r^\alpha \left(D^{m-2}\Delta_{g_K}\Phi\right)^2 \\
\leq & C \int_{\Sigma_\tau \cap \{r \geq r^-_Y - 2\delta\}} r^\alpha \left(\left(D^{m-1}\partial_{t^*}\Phi\right)^2 + \left(D^{m-2}G\right)^2 \right) \\
\leq & C \left(\sum_{j=0}^{m-1} \int_{\Sigma_\tau} r^\alpha J^N_\mu \left(\partial^j_{t^*}\Phi\right) n^\mu_{\Sigma_\tau} + \sum_{j=0}^{m-2} \int_{\Sigma_\tau} r^\alpha \left(D^j G\right)^2 \right),
\end{aligned}$$

where at the last step we have used the induction hypothesis for $\partial_{t^*}\Phi$. We have thus proved the boundedness of weighted energy. To prove the second part of the Proposition, consider the function $\chi(\frac{r}{\tau})\Phi(\tau)$ for a fixed time $t^* = \tau$, where $\chi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is supported in $\{x \leq \gamma'\}$ and is identically 1 in $\{x \leq \gamma\}$. Now

$$\Box_{g_K}\Phi = \chi G + \tau^{-1}\tilde{\chi}\partial_r\Phi + \tau^{-2}\tilde{\tilde{\chi}}\Phi,$$

where $\tilde{\chi}$ and $\tilde{\tilde{\chi}}$ are supported in $\{\gamma \leq \frac{t^*}{r} \leq \gamma'\}$. Thus, by the estimate just proved,

$$\begin{aligned}
&\int_{\Sigma_\tau \cap \{r_Y^- \leq r \leq \gamma t^*\}} r^\alpha \left(D^m \Phi\right)^2 \\
\leq & C_\alpha \left(\sum_{j=0}^{m-1} \int_{\Sigma_\tau \cap \{r \leq \gamma' t^*\}} r^\alpha J_\mu^N \left(\partial_{t^*}^j \Phi\right) n_{\Sigma_\tau}^\mu + \int_{\Sigma_\tau \cap \{\gamma t \leq r \leq \gamma' t^*\}} r^\alpha \tau^{-4} \Phi^2 \right. \\
& \left. + \sum_{j=0}^{m-2} \int_{\Sigma_\tau \cap \{r \leq \gamma' t^*\}} r^\alpha \left(D^j G\right)^2 \right) \\
\leq & C_\alpha \left(\sum_{j=0}^{m-1} \int_{\Sigma_\tau \cap \{r \leq \gamma' t^*\}} r^\alpha J_\mu^N \left(\partial_{t^*}^j \Phi\right) n_{\Sigma_\tau}^\mu + \tau^{\alpha-\beta-2} \int_{\Sigma_\tau} r^\beta J_\mu^N \left(\Phi\right) n_{\Sigma_\tau}^\mu \right. \\
& \left. + \sum_{j=0}^{m-2} \int_{\Sigma_\tau \cap \{r \leq \gamma' t^*\}} r^\alpha \left(D^j G\right)^2 \right),
\end{aligned}$$

by Hardy inequality in Proposition 9.4.10. □

Remark 25. *The boundedness of local energy should be seen as a decay result because for example for the homogeneous equation, the right hand side of the inequality decays.*

In fact by using the Hardy inequation in Proposition 9.4.10, the elliptic estimates can be localized to a region $r > R$:

Proposition 9.7.2. *Suppose $\Box_{g_K} \Phi = G$. For $m \geq 1$ and for any α, For any $R > R'$,*

$$\begin{aligned}
&\int_{\Sigma_\tau \cap \{r \geq R\}} r^\alpha \left(D^m \Phi\right)^2 \\
\leq & C_{\alpha,R,R'} \left(\sum_{j=0}^{m-1} \int_{\Sigma_\tau \cap \{r \geq R'\}} r^\alpha J_\mu^N \left(\partial_{t^*}^j \Phi\right) n_{\Sigma_\tau}^\mu + \sum_{j=0}^{m-2} \int_{\Sigma_\tau} r^\alpha \left(D^j G\right)^2 \right).
\end{aligned}$$

Near the event horizon, higher order derivatives can be controlled by commuting with the red-shift vector field as in [15], [16]. The computation here will be completely local, i.e., only in the region $\{r \leq r_Y^-\}$.

We have the following estimate for higher order derivatives:

Proposition 9.7.3. *Suppose $\Box_{g_K}\Phi = G$. For every $m \geq 1$,*

$$\begin{aligned}&\int_{\Sigma_\tau\cap\{r\leq r_Y^-\}} (D^m\Phi)^2\\ \leq& C\left(\sum_{j+k\leq m-1}\int_{\Sigma_\tau\cap\{r\leq r_Y^-\}} J_\mu^N\left(\partial_{t^*}^j\hat{Y}^k\Phi\right) n_{\Sigma_\tau}^\mu + \sum_{j=0}^{m-2}\int_{\Sigma_\tau\cap\{r\leq r_Y^-\}}\left(D^jG\right)^2\right).\end{aligned}$$

Proof. This is obvious for $m = 1$. We will proceed by induction. Suppose, for some $m \geq 2$ that

$$\begin{aligned}&\sum_{j=0}^{m-1}\int_{\Sigma_\tau\cap\{r\leq r_Y^-\}}\left(D^j\Phi\right)^2\\ \leq& C\left(\sum_{j+k\leq m-2}\int_{\Sigma_\tau\cap\{r\leq r_Y^-\}} J_\mu^N\left(\partial_{t^*}^j\hat{Y}^k\Phi\right) n_{\Sigma_\tau}^\mu + \sum_{j=0}^{m-3}\int_{\Sigma_\tau\cap\{r\leq r_Y^-\}}\left(D^jG\right)^2\right).\end{aligned} \tag{9.15}$$

Since $\Box_{g_K}\left(\partial_{t^*}\Phi\right) = \partial_{t^*}G$, this immediately implies

$$\begin{aligned}&\int_{\Sigma_\tau\cap\{r\leq r_Y^-\}}\left(\partial_{t^*}D^{m-1}\Phi\right)^2\\ \leq& C\left(\sum_{j+k\leq m-1}\int_{\Sigma_\tau\cap\{r\leq r_Y^-\}} J_\mu^N\left(\partial_{t^*}^j\hat{Y}^k\Phi\right) n_{\Sigma_\tau}^\mu + \sum_{j=0}^{m-2}\int_{\Sigma_\tau\cap\{r\leq r_Y^-\}}\left(D^jG\right)^2\right).\end{aligned} \tag{9.16}$$

Since $\Box_{g_K}\Phi = G$, we have $\Box_{g_K}\left(\hat{Y}\Phi\right) = \hat{Y}G + O(1)\left(D^2\Phi + D\Phi\right)$. Then using the induction hypothesis (9.15) (both on $\hat{Y}\Phi$ and Φ), we have

$$\begin{aligned}&\sum_{j=0}^{m-1}\int_{\Sigma_\tau\cap\{r\leq r_Y^-\}}\left(D^j\hat{Y}\Phi\right)^2\\ \leq& C\left(\sum_{j+k\leq m-2}\int_{\Sigma_\tau\cap\{r\leq r_Y^-\}} J_\mu^N\left(\partial_{t^*}^j\hat{Y}^{k+1}\Phi\right) n_{\Sigma_\tau}^\mu + \sum_{j=0}^{m-3}\int_{\Sigma_\tau}\left(D^j\hat{Y}G\right)^2\right.\\ &\qquad\left.+\sum_{j=0}^{m-1}\int_{\Sigma_\tau\cap\{r\leq r_Y^-\}}\left(D^j\Phi\right)^2\right)\\ \leq& C\left(\sum_{j+k\leq m-1}\int_{\Sigma_\tau\cap\{r\leq r_Y^-\}} J_\mu^N\left(\partial_{t^*}^j\hat{Y}^k\Phi\right) n_{\Sigma_\tau}^\mu + \sum_{j=0}^{m-2}\int_{\Sigma_\tau\cap\{r\leq r_Y^-\}}\left(D^jG\right)^2\right).\end{aligned} \tag{9.17}$$

Using the null frame $\{\hat{V}, \hat{Y}, E_1, E_2\}$,

$$\Box_{g_K}\left(D^{m-2}\Phi\right) = -4\nabla_{\hat{Y}}\nabla_{\hat{V}}D^{m-2}\Phi + \not{\Delta}D^{m-2}\Phi + P_1 D^{m-2}\Phi,$$

where P_1 denotes a first order differential operator. Notice that we also have

$$|\Box_{g_K}\left(D^{m-2}\Phi\right)| = |[\Box_{g_K}, D^{m-2}]\Phi + D^{m-2}G| \leq C\left(\sum_{j=0}^{m-1}|D^j\Phi| + |D^{m-2}G|\right).$$

Now using a standard L^2 elliptic estimate on the sphere,

$$\int_{\mathbb{S}^2}|\not{\nabla}^2 D^{m-2}\Phi|^2 dA \leq C\int_{\mathbb{S}^2}\left(\left(D^{m-2}G\right)^2 + \sum_{j=0}^{m-1}\left(D^j\Phi\right)^2 + \left(D^{m-1}\nabla_{\hat{Y}}\Phi\right)^2\right)dA,$$

where we notice that the constant can be chosen uniformly because the metric on the sphere is everywhere close to that of the standard metric. Therefore, after integrate over $\{r_+ \leq r \leq r_Y^-\}$ and applying (9.15) and (9.17), we have

$$\begin{aligned}
&\int_{\Sigma_\tau\cap\{r\leq r_Y^-\}}|\not{\nabla}^2 D^{m-2}\Phi|^2 \\
\leq& C\int_{\Sigma_\tau\cap\{r\leq r_Y^-\}}\left(\left(D^{m-2}G\right)^2 + \sum_{j=0}^{m-1}\left(D^j\Phi\right)^2 + \left(D^{m-1}\nabla_{\hat{Y}}\Phi\right)^2\right) \\
\leq& C\left(\sum_{j+k\leq m-1}\int_{\Sigma_\tau\cap\{r\leq r_Y^-\}}J_\mu^N\left(\partial_{t^*}^j\hat{Y}^k\Phi\right)n_{\Sigma_\tau}^\mu + \sum_{j=0}^{m-2}\int_{\Sigma_\tau\cap\{r\leq r_Y^-\}}\left(D^jG\right)^2\right).
\end{aligned} \tag{9.18}$$

Combining (9.16), (9.17) and (9.18), we have

$$\begin{aligned}
&\int_{\Sigma_\tau\cap\{r\leq r_Y^-\}}(D^m\Phi)^2 \\
\leq& C\left(\sum_{j+k\leq m}\int_{\Sigma_\tau\cap\{r\leq r_Y^-\}}J_\mu^N\left(\partial_{t^*}^j\hat{Y}^k\Phi\right)n_{\Sigma_\tau}^\mu + \sum_{j=0}^{m-2}\int_{\Sigma_\tau\cap\{r\leq r_Y^-\}}\left(D^jG\right)^2\right).
\end{aligned}$$

□

We show that the currents associated to $\hat{Y}^k\Phi$ can actually be controlled. Again, in view of the nonlinear problem, we work in the setting of an inhomogeneous equation.

Proposition 9.7.4. *Suppose $\Box_{g_K}\Phi = G$. For every $k \geq 0$,*

$$\begin{aligned}
&\int_{\Sigma_\tau\cap\{r\leq r_Y^+\}} J_\mu^N\left(\hat{Y}^k\Phi\right) n_{\Sigma_\tau}^\mu + \int_{\mathcal{H}(\tau',\tau)} J_\mu^N\left(\hat{Y}^k\Phi\right) n_{\Sigma_\tau}^\mu + \iint_{\mathcal{R}(\tau',\tau)\cap\{r\leq r_Y^-\}} K^N\left(\hat{Y}^k\Phi\right)\\
\leq& C\left(\sum_{j+m\leq k}\int_{\Sigma_{\tau'}\cap\{r\leq r_Y^+\}} J_\mu^N\left(\partial_{t^*}^j\hat{Y}^m\Phi\right) n_{\Sigma_{\tau'}}^\mu + \sum_{j=0}^k\int_{\Sigma_\tau\cap\{r\leq r_Y^+\}} J_\mu^N\left(\partial_{t^*}^j\Phi\right) n_{\Sigma_\tau}^\mu\right.\\
&\left.+\sum_{j=0}^k\iint_{\mathcal{R}(\tau',\tau)\cap\{r\leq\frac{23M}{8}\}}\left(\Phi^2 + J_\mu^N\left(\partial_{t^*}^j\Phi\right) n_{\Sigma_{t^*}}^\mu\right) + \sum_{j=0}^k\iint_{\mathcal{R}(\tau',\tau)\cap\{r\leq\frac{23M}{8}\}}\left(D^jG\right)^2\right).
\end{aligned}$$

Proof. We prove the Proposition by induction on k. The $k=0$ case is trivial because the right hand side simply contains more terms than the left hand side. We suppose the Proposition is true for $k \leq k_0 - 1$ for some $k_0 \geq 1$. Commuting $\Box_{g_K}$ with $\hat{Y}$ for k_0 times, we get

$$\Box_{g_K}\hat{Y}^{k_0}\Phi = \kappa^{k_0}\hat{Y}^{k_0+1}\Phi + O(1)\hat{Y}^{k_0}\partial_{t^*}\Phi + O(\epsilon)D^{k_0+1}\Phi + O(1)\sum_{j=1}^{k_0} D^j\Phi + \hat{Y}^{k_0}G.$$

We now use the energy identity for the vector field N, i.e., Proposition 9.3.3 for $\hat{Y}^k\Phi$. Notice that $\hat{Y}$ is supported in $\{r \leq r_Y^+\}$ and therefore each term is supported in the same set.

$$\begin{aligned}
&\int_{\Sigma_\tau\cap\{r\leq r_Y^+\}} J_\mu^N\left(\hat{Y}^{k_0}\Phi\right) n_{\Sigma_\tau}^\mu + \int_{\mathcal{H}(\tau_0,\tau)} J_\mu^N\left(\hat{Y}^{k_0}\Phi\right) n_{\Sigma_\tau}^\mu + \iint_{\mathcal{R}(\tau_0,\tau)\cap\{r\leq r_Y^-\}} K^N\left(\hat{Y}^{k_0}\Phi\right)\\
=&\int_{\Sigma_{\tau_0}} J_\mu^N\left(\hat{Y}^{k_0}\Phi\right) n_{\Sigma_{\tau_0}}^\mu + e\iint_{\mathcal{R}(\tau_0,\tau)\cap\{r_Y^-\leq r\leq r_Y^+\}} K^Y\left(\hat{Y}^{k_0}\Phi\right)\\
&+\iint_{\mathcal{R}(\tau_0,\tau)\cap\{r\leq r_Y^+\}}\left(\partial_{t^*}\hat{Y}^{k_0}\Phi + e\hat{Y}^{k_0+1}\Phi\right)\left(-\kappa^{k_0}\hat{Y}^{k_0+1}\Phi + O(1)\hat{Y}^{k_0}\partial_{t^*}\Phi\right.\\
&\qquad\left.+O(\epsilon)D^{k_0+1}\Phi + O(1)\sum_{j=1}^{k_0} D^j\Phi + \hat{Y}^kG\right).
\end{aligned}$$

The crucial observation in [15] is that one of the inhomogeneous terms has a good

sign and thus gives

$$\begin{aligned}
&\int_{\Sigma_\tau} J_\mu^N\left(\hat{Y}^{k_0}\Phi\right) n_{\Sigma_\tau}^\mu + \int_{\mathcal{H}(\tau',\tau)} J_\mu^N\left(\hat{Y}^{k_0}\Phi\right) n_{\Sigma_\tau}^\mu + \iint_{\mathcal{R}(\tau',\tau)\cap\{r\le r_Y^-\}} K^N\left(\hat{Y}^{k_0}\Phi\right)\\
&+\iint_{\mathcal{R}(\tau',\tau)}\left(\hat{Y}^{k_0+1}\Phi\right)^2\\
\le& C\left(\int_{\Sigma_{\tau'}\cap\{r\le r_Y^+\}} J_\mu^N\left(\hat{Y}^{k_0}\Phi\right) n_{\Sigma_{\tau_0}}^\mu + \iint_{\mathcal{R}(\tau',\tau)\cap\{r_Y^-\le r\le r_Y^+\}} K^N\left(\hat{Y}^{k_0}\Phi\right)\right.\\
&+\epsilon\iint_{\mathcal{R}(\tau',\tau)\cap\{r\le r_Y^+\}}\left(D^{k_0+1}\Phi\right)^2 + \iint_{\mathcal{R}(\tau',\tau)\cap\{r\le r_Y^+\}} J_\mu^N\left(\partial_{t^*}\hat{Y}^{k_0-1}\Phi\right) n_{\Sigma_{t^*}}^\mu\\
&\left.+\sum_{j=1}^{k_0}\iint_{\mathcal{R}(\tau',\tau)\cap\{r\le r_Y^+\}}\left(D^j\Phi\right)^2 + \iint_{\mathcal{R}(\tau',\tau)\cap\{r\le r_Y^+\}}(\hat{Y}^{k_0}G)^2\right)\\
\le& C\left(\int_{\Sigma_{\tau'}\cap\{r\le r_Y^+\}} J_\mu^N\left(\hat{Y}^{k_0}\Phi\right) n_{\Sigma_{\tau_0}}^\mu + \sum_{j=1}^{k_0+1}\iint_{\mathcal{R}(\tau',\tau)\cap\{r_Y^-\le r\le r_Y^+\}}\left(D^j\Phi\right)^2\right.\\
&+\epsilon\iint_{\mathcal{R}(\tau',\tau)\cap\{r\le r_Y^-\}}\left(D^{k_0+1}\Phi\right)^2 + \iint_{\mathcal{R}(\tau',\tau)\cap\{r\le r_Y^-\}} J_\mu^N\left(\partial_{t^*}\hat{Y}^{k_0-1}\Phi\right) n_{\Sigma_{t^*}}^\mu\\
&\left.+\sum_{j=1}^{k_0}\iint_{\mathcal{R}(\tau',\tau)\cap\{r\le r_Y^-\}}\left(D^j\Phi\right)^2 + \iint_{\mathcal{R}(\tau',\tau)\cap\{r\le r_Y^+\}}(\hat{Y}^{k_0}G)^2\right).
\end{aligned}$$

Using Proposition 9.7.1 with an appropriate cutoff,

$$\begin{aligned}
&\sum_{j=1}^{k_0+1}\iint_{\mathcal{R}(\tau',\tau)\cap\{r_Y^-\le r\le r_Y^+\}}\left(D^j\Phi\right)^2\\
\le& C\left(\sum_{j=0}^{k_0}\iint_{\mathcal{R}(\tau',\tau)\cap\{r\le\frac{23M}{8}\}}\left(\Phi^2 + J_\mu^N\left(\partial_{t^*}^j\Phi\right) n_{\Sigma_{t^*}}^\mu\right) + \sum_{j=0}^{k_0-1}\iint_{\mathcal{R}(\tau',\tau)\cap\{r\le\frac{23M}{8}\}}\left(D^jG\right)^2\right).
\end{aligned}$$

Using Proposition 9.7.3,

$$\begin{aligned}
&\sum_{j=1}^{k_0}\iint_{\mathcal{R}(\tau',\tau)\cap\{r\le r_Y^-\}}\left(D^j\Phi\right)^2\\
\le& C\left(\sum_{j+m\le k_0-1}\iint_{\mathcal{R}(\tau',\tau)\cap\{r\le r_Y^-\}}J_\mu^N\left(\partial_{t^*}^j\hat{Y}^m\Phi\right)n^\mu_{\Sigma_{t^*}}+\sum_{j=0}^{k_0-2}\iint_{\mathcal{R}(\tau',\tau)\cap\{r\le r_Y^-\}}\left(D^jG\right)^2\right)\\
\le& C\left(\sum_{j+m\le k_0-1}\int_{\Sigma_{\tau'}\cap\{r\le r_Y^+\}}J_\mu^N\left(\partial_{t^*}^j\hat{Y}^m\Phi\right)n^\mu_{\Sigma_{\tau'}}+\sum_{j=0}^{k-1}\int_{\Sigma_\tau\cap\{r\le r_Y^+\}}J_\mu^N\left(\partial_{t^*}^j\Phi\right)n^\mu_{\Sigma_\tau}\right.\\
&\left.+\sum_{j+m\le k_0-1}\iint_{\mathcal{R}(\tau',\tau)\cap\{r\le\frac{23M}{8}\}}J_\mu^N\left(\partial_{t^*}^j\hat{Y}^m\Phi\right)n^\mu_{\Sigma_{t^*}}+\sum_{j=0}^{k_0-1}\iint_{\mathcal{R}(\tau',\tau)\cap\{r\le\frac{23M}{8}\}}\left(D^jG\right)^2\right),
\end{aligned}$$

using the induction hypothesis (on $\partial_{t^*}^m\Phi$ instead of Φ) at the last step. Similarly, using Proposition 9.7.3,

$$\begin{aligned}
&\epsilon\iint_{\mathcal{R}(\tau',\tau)\cap\{r\le r_Y^-\}}\left(D^{k_0+1}\Phi\right)^2\\
\le& C\epsilon\left(\sum_{j+m\le k_0}\iint_{\mathcal{R}(\tau',\tau)\cap\{r\le r_Y^-\}}J_\mu^N\left(\partial_{t^*}^j\hat{Y}^m\Phi\right)n^\mu_{\Sigma_{t^*}}+\sum_{j=0}^{k-1}\iint_{\mathcal{R}(\tau',\tau)\cap\{r\le r_Y^-\}}\left(D^jG\right)^2\right)\\
\le& C\epsilon\left(\iint_{\mathcal{R}(\tau',\tau)\cap\{r\le r_Y^+\}}J_\mu^N\left(\hat{Y}^{k_0}\Phi\right)n^\mu_{\Sigma_{t^*}}+\sum_{j=0}^{k_0}\int_{\Sigma_{\tau'}\cap\{r\le r_Y^+\}}J_\mu^N\left(\partial_{t^*}^j\Phi\right)n^\mu_{\Sigma_{\tau'}}\right.\\
&\left.+\sum_{j=0}^{k_0}\iint_{\mathcal{R}(\tau',\tau)\cap\{r\le\frac{23M}{8}\}}J_\mu^N\left(\partial_{t^*}^j\Phi\right)n^\mu_{\Sigma_{t^*}}+\sum_{j=0}^{k_0}\iint_{\mathcal{R}(\tau',\tau)\cap\{r\le\frac{23M}{8}\}}\left(D^jG\right)^2\right),
\end{aligned}$$

where again the induction hypotheses is used at the last step. All these together give

$$\begin{aligned}
&\int_{\Sigma_\tau}J_\mu^N\left(\hat{Y}^{k_0}\Phi\right)n^\mu_{\Sigma_\tau}+\int_{\mathcal{H}(\tau_0,\tau)}J_\mu^N\left(\hat{Y}^{k_0}\Phi\right)n^\mu_{\Sigma_\tau}+\iint_{\mathcal{R}(\tau_0,\tau)\cap\{r\le r_Y^-\}}K^N\left(\hat{Y}^{k_0}\Phi\right)\\
\le& C\left(\sum_{j+m\le k_0}\int_{\Sigma_{\tau'}\cap\{r\le r_Y^+\}}J_\mu^N\left(\partial_{t^*}^j\hat{Y}^m\Phi\right)n^\mu_{\Sigma_{\tau'}}+\sum_{j=0}^{k_0}\int_{\Sigma_{\tau'}\cap\{r\le r_Y^+\}}J_\mu^N\left(\partial_{t^*}^j\Phi\right)n^\mu_{\Sigma_{\tau'}}\right.\\
&+\sum_{j=0}^{k_0}\iint_{\mathcal{R}(\tau',\tau)\cap\{r\le\frac{23M}{8}\}}\left(\Phi^2+J_\mu^N\left(\partial_{t^*}^j\Phi\right)n^\mu_{\Sigma_{t^*}}\right)+\sum_{j=0}^{k_0}\int_{\mathcal{R}(\tau',\tau)\cap\{r\le\frac{23M}{8}\}}\left(D^jG\right)^2\\
&\left.+\epsilon\iint_{\mathcal{R}(\tau',\tau)\cap\{r\le r_Y^-\}}J_\mu^N\left(\hat{Y}^{k_0}\Phi\right)n^\mu_{\Sigma_{t^*}}\right).
\end{aligned}$$

The Proposition can be proved by noticing that

$$\iint_{\mathcal{R}(\tau',\tau)\cap\{r\leq r_Y^-\}} J_\mu^N\left(\hat{Y}^{k_0}\Phi\right) n_{\Sigma_{t^*}}^\mu \leq C\iint_{\mathcal{R}(\tau_0,\tau)\cap\{r\leq r_Y^-\}} K^N\left(\hat{Y}^{k_0}\Phi\right).$$

and absorbing the small term to the left hand side. □

We now specialize to the case $\Box_{g_K}\Phi = 0$. The above Proposition implies that the behavior of $\hat{Y}^k\Phi$ is determined by the behavior of $\partial_{t^*}^m\Phi$ in the region $\{r \leq \frac{23M}{8}\}$.

Proposition 9.7.5. *Fix $k \geq 0$. Suppose $\Box_{g_K}\Phi = 0$ and suppose for some constants $\alpha, B > 0$ (independents of τ),*

$$\sum_{j=0}^{k}\int_{\Sigma_\tau\cap\{r\leq\frac{23M}{8}\}}\left(\Phi^2 + J_\mu^N\left(\partial_{t^*}^j\Phi\right)n_{\Sigma_\tau}^\mu\right) \leq CB\tau^{-\alpha}.$$

Then

$$\sum_{j+m\leq k}\int_{\Sigma_\tau} J_\mu^N\left(\partial_{t^*}^j\hat{Y}^m\Phi\right)n_{\Sigma_\tau}^\mu \leq C\tau^{-\alpha}\left(\sum_{j+m\leq k}\int_{\Sigma_{\tau_0}} J_\mu^N\left(\partial_{t^*}^j\hat{Y}^m\Phi\right)n_{\Sigma_{\tau_0}}^\mu + B\right),$$

and

$$\begin{aligned}&\sum_{j+m\leq k}\iint_{\mathcal{R}(\tau',\tau)\cap\{r\leq r_Y^-\}} K^N\left(\partial_{t^*}^j\hat{Y}^m\Phi\right)\\ \leq& C(\tau')^{-\alpha}\left(\sum_{j+m\leq k}\int_{\Sigma_{\tau_0}} J_\mu^N\left(\partial_{t^*}^j\hat{Y}^m\Phi\right)n_{\Sigma_{\tau_0}}^\mu + B\right).\end{aligned}$$

Remark 26. *In the applications, we will apply this Proposition with B being some energy quantity of the initial condition.*

Proof. We will proof this with a bootstrap argument. Suppose for all τ that

$$\sum_{j+m\leq k}\int_{\Sigma_\tau\cap\{r\leq r_Y^+\}} J_\mu^N\left(\partial_{t^*}^j\hat{Y}^m\Phi\right)n_{\Sigma_\tau}^\mu \leq A\tau^{-\alpha}\left(\sum_{j+m\leq k}\int_{\Sigma_{\tau_0}} J_\mu^N\left(\partial_{t^*}^j\hat{Y}^m\Phi\right)n_{\Sigma_{\tau_0}}^\mu + B\right). \tag{9.19}$$

This obviously holds initially for any $A \geq 1$ (and in particular independent of Φ). By taking $\tau' = \tau - K$, for some (large and to be chosen) constant K and $\tau \geq 2K$, Proposition 9.7.4 implies

$$\begin{aligned}
&\sum_{j+m\leq k}\left(\int_{\Sigma_\tau\cap\{r\leq r_Y^+\}} J_\mu^N\left(\partial_{t^*}^j\hat{Y}^m\Phi\right)n^\mu_{\Sigma_\tau} + \iint_{\mathcal{R}(\tau-K,\tau)\cap\{r\leq r_Y^-\}} J_\mu^N\left(\partial_{t^*}^j\hat{Y}^m\Phi\right)n^\mu_{\Sigma_{t^*}}\right)\\
\leq& C\left(\sum_{j+m\leq k}\int_{\Sigma_{\tau-K}\cap\{r\leq r_Y^+\}} J_\mu^N\left(\partial_{t^*}^j\hat{Y}^m\Phi\right)n^\mu_{\Sigma_{\tau-K}} + \sum_{j=0}^{k}\int_{\Sigma_\tau\cap\{r\leq r_Y^+\}} J_\mu^N\left(\partial_{t^*}^j\Phi\right)n^\mu_{\Sigma_\tau}\right.\\
&\left.+\sum_{j=0}^{k}\iint_{\mathcal{R}(\tau-K,\tau)\cap\{r\leq\frac{23M}{8}\}}\left(\Phi^2 + J_\mu^N\left(\partial_{t^*}^j\Phi\right)n^\mu_{\Sigma_{t^*}}\right)\right)\\
\leq& C\left(\sum_{j+m\leq k}\int_{\Sigma_{\tau-K}\cap\{r\leq r_Y^+\}} J_\mu^N\left(\partial_{t^*}^j\hat{Y}^m\Phi\right)n^\mu_{\Sigma_{\tau-K}} + KB\tau^{-\alpha}\right)
\end{aligned}$$

using the assumption of the Proposition and using Proposition 9.7.1

$$\leq C\left(A\left(\tau-K\right)^{-\alpha}\left(\sum_{j+m\leq k}\int_{\Sigma_{\tau_0}} J_\mu^N\left(\partial_{t^*}^j\hat{Y}^m\Phi\right)n^\mu_{\Sigma_{\tau_0}} + B\right) + KB\tau^{-\alpha}\right)$$

using the bootstrap assumption

$$\leq C\tau^{-\alpha}\left(\sum_{j+m\leq k} A\int_{\Sigma_{\tau_0}} J_\mu^N\left(\partial_{t^*}^j\hat{Y}^m\Phi\right)n^\mu_{\Sigma_{\tau_0}} + AB + KB\right).$$

Notice that C is independent of K. By selecting a t^* slice, we have that for some $\tilde{\tau}$,

$$\begin{aligned}
&\int_{\Sigma_{\tilde{\tau}}\cap\{r\leq r_Y^-\}} J_\mu^N\left(\hat{Y}^k\Phi\right)n^\mu_{\Sigma_{\tilde{\tau}}}\\
\leq& CK^{-1}\tau^{-\alpha}\left(\sum_{j+m\leq k} A\int_{\Sigma_{\tau_0}} J_\mu^N\left(\partial_{t^*}^j\hat{Y}^m\Phi\right)n^\mu_{\Sigma_{\tau_0}} + AB + KB\right).
\end{aligned}$$

Now apply Proposition 9.7.4 on $[\tilde{\tau}, \tau]$ to get

$$\begin{aligned}
&\sum_{j+m\leq k}\int_{\Sigma_\tau\cap\{r\leq r_Y^+\}} J_\mu^N\left(\partial_{t^*}^j\hat{Y}^m\Phi\right)n_{\Sigma_\tau}^\mu\\
\leq& C\left(\sum_{j+m\leq k}\int_{\Sigma_{\tilde{\tau}}\cap\{r\leq r_Y^+\}} J_\mu^N\left(\partial_{t^*}^j\hat{Y}^m\Phi\right)n_{\Sigma_{\tilde{\tau}}}^\mu+\sum_{j=0}^k\int_{\Sigma_\tau\cap\{r\leq r_Y^+\}} J_\mu^N\left(\partial_{t^*}^j\Phi\right)n_{\Sigma_\tau}^\mu\right.\\
&\left.+\sum_{j=0}^k\iint_{\mathcal{R}(\tilde{\tau},\tau)\cap\{r\leq\frac{23M}{8}\}}\left(\Phi^2+J_\mu^N\left(\partial_{t^*}^j\Phi\right)n_{\Sigma_{t^*}}^\mu\right)\right)\\
\leq& CK^{-1}\tau^{-\alpha}\left(\sum_{j+m\leq k}A\int_{\Sigma_{\tau_0}} J_\mu^N\left(\partial_{t^*}^j\hat{Y}^m\Phi\right)n_{\Sigma_{\tau_0}}^\mu+AB+BK\right)+CB(K+1)\tau^{-\alpha}\\
\leq& CAK^{-1}\tau^{-\alpha}\sum_{j+m\leq k}\int_{\Sigma_{\tau_0}} J_\mu^N\left(\partial_{t^*}^j\hat{Y}^m\Phi\right)n_{\Sigma_{\tau_0}}^\mu+\left(CAK^{-1}+CK+C\right)B\tau^{-\alpha}.
\end{aligned}$$

This would improve (9.19) if we choose $K = 4C$ and A sufficiently large. Hence we have proved

$$\sum_{j+m\leq k}\int_{\Sigma_\tau\cap\{r\leq r_Y^+\}} J_\mu^N\left(\partial_{t^*}^j\hat{Y}^m\Phi\right)n_{\Sigma_\tau}^\mu\leq C\tau^{-\alpha}\left(\sum_{j+m\leq k}\int_{\Sigma_{\tau_0}} J_\mu^N\left(\partial_{t^*}^j\hat{Y}^m\Phi\right)n_{\Sigma_{\tau_0}}^\mu+B\right).$$

To prove the second statement of the Proposition, we simply use the first statement and Proposition 9.7.4. □

We can use Corollary 9.6.2 to show the decay of $\hat{Y}^k\Phi$.

Corollary 9.7.6. *Suppose $\Box_{g_K}\Phi = 0$. Then for $\tau' \leq \tau \leq (1.1)\tau'$,*

$$\begin{aligned}
&\sum_{j+m\leq k}\int_{\Sigma_\tau\cap\{r\leq r_Y^+\}} J_\mu^N\left(\partial_{t^*}^j\hat{Y}^m\Phi\right)n_{\Sigma_\tau}^\mu\\
\leq& C\tau^{-2+\eta}\left(\sum_{j+m\leq k}\int_{\Sigma_{\tau_0}} J_\mu^N\left(\partial_{t^*}^j\hat{Y}^m\Phi\right)n_{\Sigma_{\tau_0}}^\mu+\sum_{j=0}^{k+2}\int_{\Sigma_{\tau_0}} J_\mu^{Z+CN}\left(\partial_{t^*}^j\Phi\right)n_{\Sigma_{\tau_0}}^\mu\right),
\end{aligned}$$

and

$$\sum_{j+m\leq k}\iint_{\mathcal{R}(\tau',\tau)\cap\{r\leq r_Y^-\}} K^N\left(\partial_{t^*}^j\hat{Y}^m\Phi\right)$$
$$\leq C\tau^{-2+\eta}\left(\sum_{j+m\leq k}\int_{\Sigma_{\tau_0}} J_\mu^N\left(\partial_{t^*}^j\hat{Y}^m\Phi\right)n_{\Sigma_{\tau_0}}^\mu+\sum_{j=0}^{k+2}\int_{\Sigma_{\tau_0}} J_\mu^{Z+CN}\left(\partial_{t^*}^j\Phi\right)n_{\Sigma_{\tau_0}}^\mu\right).$$

9.8 Estimates for $\tilde{\Omega}\Phi$

In this section, we would like to prove estimates for $\tilde{\Omega}^\ell\Phi$. The estimates for $\tilde{\Omega}\Phi$ are useful to provide an extra factor of r in the energy estimates.

Proposition 9.8.1.

$$\int_{\Sigma_\tau\cap\{r\geq r_Y^-\}} r^2|\nabla\!\!\!/^2\Phi|^2\leq C\int_{\Sigma_\tau} J_\mu^N\left(\Phi,\partial_{t^*}\Phi,\tilde{\Omega}\Phi\right)n_{\Sigma_\tau}^\mu.$$

$$\iint_{\mathcal{R}(\tau',\tau)\cap\{r\geq r_Y^-\}} r^{1-\delta}|\nabla\!\!\!/^2\Phi|^2\leq C\iint_{\mathcal{R}(\tau',\tau)} r^{-1-\delta}J_\mu^N\left(\Phi,\partial_{t^*}\Phi,\tilde{\Omega}\Phi\right)n_{\Sigma_{t^*}}^\mu.$$

$$\iint_{\mathcal{R}(\tau',\tau)\cap\{r\geq r_Y^-\}} r^{1-\delta}|\nabla\!\!\!/^2\Phi|^2\leq C\iint_{\mathcal{R}(\tau',\tau)} K^{X_1}\left(\Phi,\partial_{t^*}\Phi\right)+K^{X_0}\left(\tilde{\Omega}\Phi\right).$$

In order to prove such estimates, we commute $\Box_{g_K}$ with $\tilde{\Omega}$. Recall from Section 9.2.2 that

$$|[\Box_{g_K},\tilde{\Omega}]\Phi|\leq Cr^{-2}\left(|D^2\Phi|+|D\Phi|\right)\quad\text{everywhere, and}$$

$$[\Box_{g_K},\tilde{\Omega}]\Phi=0,\quad\text{for } r<R_\Omega.$$

Now suppose $\Box_{g_K}\Phi=0$. We have

$$\Box_{g_K}\left(\tilde{\Omega}^\ell\Phi\right)=\sum_{j=0}^{\ell-1}\tilde{\Omega}^j[\Box_{g_K},\tilde{\Omega}]\tilde{\Omega}^{\ell-j-1}\Phi=:G_{\Omega,\ell}.$$

Since $[D, \tilde{\Omega}] = D$, we have

$$|G_{\Omega,\ell}| \leq Cr^{-2}\left(\sum_{j=0}^{\ell-1}\left(|D^2\tilde{\Omega}^j\Phi| + |D\tilde{\Omega}^j\Phi|\right) + \sum_{j=0}^{\ell+1}|D^j\Phi|\right),$$

and $G_{\Omega,\ell}$ is supported in $\{r \geq R_\Omega\}$.

Definition 9.8.2.

$$E_{\Omega,\ell} = \iint_{\mathcal{R}(\tau'-1,\tau+1)} r^{1+\delta}G_{\Omega,\ell}^2 + \sup_{t^*\in[\tau'-1,\tau+1]}\int_{\Sigma_{t^*}\cap\{|r-3M|\leq\frac{M}{8}\}} {G_{\Omega,\ell}}^2.$$

This is the error term for the energy estimates for $\tilde{\Omega}^\ell\Phi$. We show that this can be controlled.

Proposition 9.8.3.

$$\begin{aligned} E_{\Omega,\ell} \leq & C\sum_{m=0}^{1}\sum_{j=0}^{\ell-1}\iint_{\mathcal{R}(\tau'-1,\tau+1)\cap\{r\geq R_\Omega\}} K^{X_0}\left(\partial_{t^*}^m\tilde{\Omega}^j\Phi\right) \\ & + C\sum_{m=0}^{\ell}\iint_{\mathcal{R}(\tau'-1,\tau+1)\cap\{r\geq R_\Omega\}} K^{X_0}\left(\partial_{t^*}^m\Phi\right). \end{aligned}$$

Proof.

$$
\begin{aligned}
&\iint_{\mathcal{R}(\tau'-1,\tau+1)} r^{1+\delta} G_{\Omega,\ell}^2 \\
\leq & C \sum_{j=0}^{\ell-1} \iint_{\mathcal{R}(\tau'-1,\tau+1)\cap\{r\geq R_\Omega\}} r^{-3+\delta} \left(\left(D^2 \tilde{\Omega}^j \Phi \right)^2 + \left(D \tilde{\Omega}^j \Phi \right)^2 \right) \\
& + C \sum_{j=0}^{\ell+1} \iint_{\mathcal{R}(\tau'-1,\tau+1)\cap\{r\geq R_\Omega\}} r^{-3+\delta} \left(D^j \Phi \right)^2 \\
\leq & C \sum_{m=0}^{1} \sum_{j=0}^{\ell-1} \iint_{\mathcal{R}(\tau'-1,\tau+1)\cap\{r\geq R_\Omega\}} r^{-3+\delta} J_\mu^N \left(\partial_{t^*}^m \tilde{\Omega}^j \Phi \right) n_{\Sigma t^*}^\mu \\
& + C \sum_{m=0}^{\ell} \iint_{\mathcal{R}(\tau'-1,\tau+1)\cap\{r\geq R_\Omega\}} r^{-3+\delta} J_\mu^N \left(\partial_{t^*}^m \Phi \right) n_{\Sigma t^*}^\mu \\
\leq & C \sum_{m=0}^{1} \sum_{j=0}^{\ell-1} \iint_{\mathcal{R}(\tau'-1,\tau+1)\cap\{r\geq R_\Omega\}} K^{X_0} \left(\partial_{t^*}^m \tilde{\Omega}^j \Phi \right) \\
& + C \sum_{m=0}^{\ell} \iint_{\mathcal{R}(\tau'-1,\tau+1)\cap\{r\geq R_\Omega\}} K^{X_0} \left(\partial_{t^*}^m \Phi \right).
\end{aligned}
$$

By choosing R_Ω sufficiently large, the second term of $E_{\Omega,\ell}$ vanishes. □

We can show that the non-degenerate energy of $\tilde{\Omega}^\ell \Phi$ is almost bounded.

Proposition 9.8.4. *Suppose* $\Box_{g_K} \Phi = 0$. *Then*

$$
\begin{aligned}
&\int_{\Sigma_\tau} J_\mu^N \left(\tilde{\Omega}^\ell \Phi \right) n_{\Sigma_\tau}^\mu + \int_{\mathcal{H}(\tau_0,\tau)} J_\mu^N \left(\tilde{\Omega}^\ell \Phi \right) n_{\mathcal{H}^+}^\mu \\
&+ \iint_{\mathcal{R}(\tau_0,\tau)\cap\{r\leq r_Y^-\}} K^N \left(\tilde{\Omega}^\ell \Phi \right) + \iint_{\mathcal{R}(\tau_0,\tau)} K^{X_0} \left(\tilde{\Omega}^\ell \Phi \right) \\
\leq & C \sum_{i+j\leq \ell} \int_{\Sigma_{\tau_0}} J_\mu^N \left(\partial_{t^*}^i \tilde{\Omega}^j \Phi \right) n_{\Sigma_{\tau_0}}^\mu.
\end{aligned}
$$

Proof. We prove this by induction on ℓ. The $\ell = 0$ case is true by setting $G = 0$ in Proposition 9.4.17. We assume that the Proposition is true for $\ell \leq \ell_0 - 1$. This in

particular implies, after a commutations with the Killing vector field ∂_{t^*}, that

$$\sum_{j=0}^{\ell_0-1}\iint_{\mathcal{R}(\tau_0,\tau)} K^{X_0}\left(\partial_{t^*}^m\tilde{\Omega}^j\Phi\right)\leq C\sum_{i+j\leq m+\ell_0-1}\int_{\Sigma_{\tau_0}} J_\mu^N\left(\partial_{t^*}^i\tilde{\Omega}^j\Phi\right)n_{\Sigma_{\tau_0}}^\mu.$$

By Propositions 9.4.16 and 9.8.3,

$$\begin{aligned}
&\int_{\Sigma_\tau} J_\mu^N\left(\tilde{\Omega}^{\ell_0}\Phi\right)n_{\Sigma_\tau}^\mu+\int_{\mathcal{H}(\tau_0,\tau)} J_\mu^N\left(\tilde{\Omega}^{\ell_0}\Phi\right)n_{\mathcal{H}^+}^\mu\\
&+\iint_{\mathcal{R}(\tau_0,\tau)\cap\{r\leq r_Y^-\}} K^N\left(\tilde{\Omega}^{\ell_0}\Phi\right)+\iint_{\mathcal{R}(\tau_0,\tau)} K^{X_0}\left(\tilde{\Omega}^{\ell_0}\Phi\right)\\
\leq& C\left(\int_{\Sigma_{\tau_0}} J_\mu^N\left(\tilde{\Omega}^{\ell_0}\Phi\right)n_{\Sigma_{\tau_0}}^\mu+\iint_{\mathcal{R}(\tau'-1,\tau+1)} r^{1+\delta}G_{\Omega,\ell_0}^2\right.\\
&\left.+\sup_{t^*\in[\tau'-1,\tau+1]}\int_{\Sigma_{t^*}\cap\{|r-3M|\leq\frac{M}{8}\}} G_{\Omega,\ell_0}^2\right)\\
\leq& C\left(\int_{\Sigma_{\tau_0}} J_\mu^N\left(\tilde{\Omega}^{\ell_0}\Phi\right)n_{\Sigma_{\tau_0}}^\mu+C\sum_{m=0}^{1}\sum_{j=0}^{\ell_0-1}\iint_{\mathcal{R}(\tau_0-1,\tau+1)\cap\{r\geq R_\Omega\}} K^{X_0}\left(\partial_{t^*}^m\tilde{\Omega}^j\Phi\right)\right.\\
&\left.+C\sum_{m=0}^{\ell_0}\iint_{\mathcal{R}(\tau_0-1,\tau+1)\cap\{r\geq R_\Omega\}} K^{X_0}\left(\partial_{t^*}^m\Phi\right)\right)\\
\leq& C\sum_{i+j\leq\ell_0}\int_{\Sigma_{\tau_0}} J_\mu^N\left(\partial_{t^*}^i\tilde{\Omega}^j\Phi\right)n_{\Sigma_{\tau_0}}^\mu.
\end{aligned}$$

□

Remark 27. *Only the $\ell=1$ case will be used.*

9.9 Estimates for $S\Phi$

We will now use the energy estimates that we have derived to control $S\Phi$. In particular, we would like to prove a local integrated decay estimate for $S\Phi$. This will be used in the next section where we prove our main theorem. Recall from Section 9.2.1

that for r large

$$\begin{aligned}&|[\Box_{g_K}, S]\Phi - \left(2 + \frac{r^*\mu}{r}\right)\Box_{g_K}\Phi \\ &- \frac{2}{r}\left(\frac{r^*}{r} - 1 - \frac{2r^*\mu}{r}\right)\partial_{r^*}\Phi - 2\left(\left(\frac{r^*}{r} - 1\right) - \frac{3r^*\mu}{2r}\right)\not{\Delta}\Phi| \\ \leq & C\epsilon r^{-2}(\sum_{k=1}^{2}|\partial^k\Phi|),\end{aligned}$$

and that for $r \leq R$, we have

$$|[\Box_{g_K}, S]\Phi| \leq C(\sum_{k=1}^{2}|D^k\Phi|).$$

From now on we will prove estimates for $S\Phi$ by considering the wave equation that it satisfies. We will assume, as before, $\Box_{g_K}\Phi = 0$ and let G denote the commutator term, i.e., $\Box_{g_K}(S\Phi) = G$. If we look at our estimates in the previous sections, we will need to control G in three different norms. We now consider them separately.

Proposition 9.9.1. *Let $\tau' \leq \tau \leq (1.1)\tau'$. Then*

$$\begin{aligned}&\sum_{m=0}^{\ell}\iint_{\mathcal{R}(\tau'-1,\tau+1)} r^{1+\delta}\left(\partial_{t^*}^m G\right)^2 \\ \leq & C\tau^{-1+\eta}\sum_{m+k+j\leq\ell+3}\left(\int_{\Sigma_{\tau_0}} J_\mu^{Z,w^Z}\left(\partial_{t^*}^m\hat{Y}^k\tilde{\Omega}^j\Phi\right)n_{\Sigma_{\tau_0}}^\mu + C\int_{\Sigma_{\tau_0}} J_\mu^N\left(\partial_{t^*}^m\hat{Y}^k\tilde{\Omega}^j\Phi\right)n_{\Sigma_{\tau_0}}^\mu\right).\end{aligned}$$

and

$$\begin{aligned}&\sum_{m=0}^{\ell}\iint_{\mathcal{R}(\tau'-1,\tau+1)\cap\{r\leq\frac{9t^*}{10}\}} r^{1+\delta}\left(\partial_{t^*}^m G\right)^2 \\ \leq & C\tau^{-2+\eta}\sum_{m+k+j\leq\ell+4}\left(\int_{\Sigma_{\tau_0}} J_\mu^{Z,w^Z}\left(\partial_{t^*}^m\hat{Y}^k\tilde{\Omega}^j\Phi\right)n_{\Sigma_{\tau_0}}^\mu + C\int_{\Sigma_{\tau_0}} J_\mu^N\left(\partial_{t^*}^m\hat{Y}^k\tilde{\Omega}^j\Phi\right)n_{\Sigma_{\tau_0}}^\mu\right).\end{aligned}$$

In other words, we can get more decay if we localize and allow an extra derivative.

Proof.

$$\sum_{m=0}^{\ell}\iint_{\mathcal{R}(\tau'-1,\tau+1)} r^{1+\delta}\left(\partial_{t^*}^m G\right)^2$$

$$\leq C\sum_{m=0}^{\ell}\iint_{\mathcal{R}(\tau'-1,\tau+1)} r^{-3+\delta}\left(\left(\partial_{t^*}^m D^2\Phi\right)^2+\left(\partial_{t^*}^m D\Phi\right)^2+\left(r\partial_{t^*}^m \not{\Delta}\Phi\right)^2\right)$$

noting that the δ in the two lines are different

$$\leq C\sum_{m+k\leq \ell+1}\left(\iint_{\mathcal{R}(\tau'-1,\tau+1)\cap\{r\leq\frac{t^*}{2}\}} r^{-1+\delta}J_\mu^N\left(\partial_{t^*}^m\hat{Y}^k\Phi\right)n_{\Sigma_t^*}^\mu\right.$$

$$\left.+\iint_{\mathcal{R}(\tau'-1,\tau+1)\cap\{r\geq\frac{t^*}{2}\}} r^{-3+\delta}J_\mu^N\left(\partial_{t^*}^m\tilde{\Omega}^k\Phi\right)n_{\Sigma_t^*}^\mu\right)$$

by Proposition 9.7.1, 9.7.3 and 9.8.1

$$\leq C\sum_{m+k\leq\ell+3}\tau^{-1+\eta}\left(\int_{\Sigma_{\tau_0}} J_\mu^{Z,w^Z}\left(\partial_{t^*}^m\Phi\right)n_{\Sigma_{\tau_0}}^\mu+C\int_{\Sigma_{\tau_0}}J_\mu^N\left(\partial_{t^*}^m\hat{Y}^k\Phi\right)n_{\Sigma_{\tau_0}}^\mu\right)$$

$$+C\sum_{m+k\leq\ell+3}\int_{\tau'-1}^{\tau+1}(t^*)^{-3+\delta}\left(\int_{\Sigma_{\tau_0}}J_\mu^{Z,w^Z}\left(\partial_{t^*}^m\Phi\right)n_{\Sigma_{\tau_0}}^\mu\right.$$

$$\left.+C\int_{\Sigma_{\tau_0}}J_\mu^N\left(\partial_{t^*}^m\hat{Y}^k\Phi\right)n_{\Sigma_{\tau_0}}^\mu\right)dt^*$$

$$+C\sum_{m\leq\ell+1}\int_{\tau'-1}^{\tau+1}(t^*)^{-3+\delta}\left(\int_{\Sigma_{\tau_0}}J_\mu^N\left(\partial_{t^*}^m\tilde{\Omega}\Phi\right)n_{\Sigma_{\tau_0}}^\mu\right)dt^*$$

using Corollaries 9.6.2, 9.7.6 and Proposition 9.8.4

$$\leq C\tau^{-1+\eta}\sum_{m+k+j\leq\ell+3}\left(\int_{\Sigma_{\tau_0}}J_\mu^{Z,w^Z}\left(\partial_{t^*}^m\hat{Y}^k\tilde{\Omega}^j\Phi\right)n_{\Sigma_{\tau_0}}^\mu+C\int_{\Sigma_{\tau_0}}J_\mu^N\left(\partial_{t^*}^m\hat{Y}^k\tilde{\Omega}^j\Phi\right)n_{\Sigma_{\tau_0}}^\mu\right).$$

We then move on to the localized version:

$$
\begin{aligned}
&\sum_{m=0}^{\ell} \iint_{\mathcal{R}(\tau'-1,\tau+1)\cap\{r\leq\frac{9t^*}{10}\}} r^{1+\delta}\left(\partial_{t^*}^m G\right)^2 \\
\leq & C\sum_{m=0}^{\ell} \iint_{\mathcal{R}(\tau'-1,\tau+1)\cap\{r\leq\frac{9t^*}{10}\}} r^{-3+\delta}\left(\left(\partial_{t^*}^m D^2\Phi\right)^2+\left(\partial_{t^*}^m D\Phi\right)^2+\left(r\partial_{t^*}^m \not{\Delta}\Phi\right)^2\right) \\
\leq & C\sum_{m+k\leq\ell+1}\left(\iint_{\mathcal{R}(\tau'-1,\tau+1)\cap\{r\leq\frac{t^*}{2}\}} r^{-1+\delta} J_\mu^N\left(\partial_{t^*}^m\hat{Y}\Phi\right) n_{\Sigma_t^*}^\mu\right. \\
&\left.+\iint_{\mathcal{R}(\tau'-1,\tau+1)\cap\{\frac{t^*}{2}\leq r\leq\frac{19t^*}{20}\}} r^{-3+\delta} J_\mu^N\left(\partial_{t^*}^m\hat{Y}\Phi\right) n_{\Sigma_t^*}^\mu\right) \\
&+C\sum_{m=0}^{1}\left(\iint_{\mathcal{R}(\tau'-1,\tau+1)\cap\{\frac{t^*}{2}\leq r\leq\frac{19t^*}{20}\}} r^{-3+\delta} J_\mu^N\left(\partial_{t^*}^m\tilde{\Omega}\Phi\right) n_{\Sigma_t^*}^\mu\right)
\end{aligned}
$$

by Proposition 9.7.1, 9.7.3 and 9.8.1

$$
\begin{aligned}
\leq & C\sum_{m+k\leq\ell+4}\tau^{-2+\eta}\left(\int_{\Sigma_{\tau_0}} J_\mu^{Z,w^Z}\left(\partial_{t^*}^m\Phi\right) n_{\Sigma_{\tau_0}}^\mu + C\int_{\Sigma_{\tau_0}} J_\mu^N\left(\partial_{t^*}^m\hat{Y}\Phi\right) n_{\Sigma_{\tau_0}}^\mu\right) \\
&+C\sum_{m+k\leq\ell+3}\int_{\tau'-1}^{\tau+1}(t^*)^{-3+\delta}\left(\int_{\Sigma_{\tau_0}} J_\mu^{Z,w^Z}\left(\partial_{t^*}^m\Phi\right) n_{\Sigma_{\tau_0}}^\mu + C\int_{\Sigma_{\tau_0}} J_\mu^N\left(\partial_{t^*}^m\hat{Y}\Phi\right) n_{\Sigma_{\tau_0}}^\mu\right) dt^* \\
&+C\sum_{m=0}^{\ell}\int_{\tau'-1}^{\tau+1}(t^*)^{-3+\delta}\left(\int_{\Sigma_{\tau_0}} J_\mu^N\left(\partial_{t^*}^m\tilde{\Omega}\Phi\right) n_{\Sigma_{\tau_0}}^\mu\right) dt^*
\end{aligned}
$$

using Corollaries 9.6.2, 9.7.6 and Proposition 9.8.4

$$
\leq C\tau^{-2+\eta}\sum_{m+k+j\leq\ell+4}\left(\int_{\Sigma_{\tau_0}} J_\mu^{Z,w^Z}\left(\partial_{t^*}^m\hat{Y}^k\tilde{\Omega}^j\Phi\right) n_{\Sigma_{\tau_0}}^\mu + C\int_{\Sigma_{\tau_0}} J_\mu^N\left(\partial_{t^*}^m\hat{Y}^k\tilde{\Omega}^j\Phi\right) n_{\Sigma_{\tau_0}}^\mu\right).
$$

□

To estimate the inhomogeneous term in the region $r\leq\frac{t^*}{2}$, we would also need to estimate a term not integrated over t^* which arises from the integration by parts.

Proposition 9.9.2. *For $\tau' \leq \tau \leq (1.1)\tau'$,*

$$\begin{aligned}
&\sup_{t^*\in[\tau'-1,\tau+1]} \sum_{m=0}^{\ell} \int_{\Sigma_{t^*}\cap\{|r-3M|\leq\frac{M}{8}\}} (\partial_{t^*}^m G)^2 \\
\leq & C\tau^{-2+\eta} \sum_{m+j\leq \ell+3} \left(\int_{\Sigma_{\tau_0}} J_\mu^{Z,w^Z}\left(\partial_{t^*}^m \tilde{\Omega}^j \Phi\right) n_{\Sigma_{\tau_0}}^\mu + C\int_{\Sigma_{\tau_0}} J_\mu^N\left(\partial_{t^*}^m \tilde{\Omega}^j \Phi\right) n_{\Sigma_{\tau_0}}^\mu \right).
\end{aligned}$$

Proof.

$$\begin{aligned}
&\sup_{t^*\in[\tau'-1,\tau+1]} \sum_{m=0}^{\ell} \int_{\Sigma_{t^*}\cap\{|r-3M|\leq\frac{M}{8}\}} (\partial_{t^*}^m G)^2 \\
\leq & C \sup_{t^*\in[\tau'-1,\tau+1]} \sum_{m=0}^{\ell} \int_{\Sigma_{t^*}\cap\{r_Y^-\leq r\leq\frac{25M}{8}\}} \left(\left(D^2\partial_{t^*}^m\Phi\right)^2 + \left(D\partial_{t^*}^m\Phi\right)^2 + \left(r\not{\Delta}\partial_{t^*}^m\Phi\right)^2 \right) \\
\leq & C \sup_{t^*\in[\tau'-1,\tau+1]} \left(\sum_{m=0}^{\ell+1} \int_{\Sigma_{t^*}\cap\{r_Y^-\leq r\leq\frac{25M}{8}\}} J_\mu^N\left(\partial_{t^*}^m\Phi\right) n_{\Sigma_t^*}^\mu \right. \\
& \qquad \left. + \sum_{m=0}^{\ell} \int_{\Sigma_{t^*}\cap\{r_Y^-\leq r\leq\frac{25M}{8}\}} J_\mu^N\left(\tilde{\Omega}\partial_{t^*}^m\Phi\right) n_{\Sigma_t^*}^\mu \right)
\end{aligned}$$

by Proposition 9.7.1 and 9.8.1

$$\leq C\tau^{-2+\eta} \sum_{m+j\leq \ell+3} \left(\int_{\Sigma_{\tau_0}} J_\mu^{Z,w^Z}\left(\partial_{t^*}^m \tilde{\Omega}^j \Phi\right) n_{\Sigma_{\tau_0}}^\mu + C\int_{\Sigma_{\tau_0}} J_\mu^N\left(\partial_{t^*}^m \tilde{\Omega}^j \Phi\right) n_{\Sigma_{\tau_0}}^\mu \right),$$

using Corollary 9.6.2 and Proposition 9.8.4

□

Finally, we estimate the third norm:

Proposition 9.9.3.

$$\sum_{m=0}^{\ell} \left(\int_{\tau_0}^{\tau} \left(\int_{\Sigma_{t^*}\cap\{r\geq\frac{t^*}{2}\}} r^2 (\partial_{t^*}^m G)^2 \right)^{\frac{1}{2}} dt^* \right)^2 \leq C\tau^\eta \sum_{m+j\leq\ell+3} \int_{\Sigma_{\tau_0}} J_\mu^N\left(\partial_{t^*}^m \tilde{\Omega}^j \Phi\right) n_{\Sigma_{\tau_0}}^\mu.$$

Proof.

$$\begin{aligned}
&\sum_{m=0}^{\ell}\left(\int_{\tau_0}^{\tau}\left(\int_{\Sigma_{t^*}\cap\{r\geq\frac{t^*}{2}\}} r^2(\partial_{t^*}^m G)^2\right)^{\frac{1}{2}} dt^*\right)^2\\
\leq& C\sum_{m=0}^{\ell}\left(\int_{\tau_0}^{\tau}(t^*)^{-1+\delta}\left(\int_{\Sigma_{t^*}\cap\{r\geq\frac{t^*}{2}\}}\left(\left(D^2\Phi\right)^2+(D\partial_{t^*}^m\Phi)^2+(r\not{\Delta}\partial_{t^*}^m\Phi)^2\right)\right)^{\frac{1}{2}} dt^*\right)^2\\
\leq& C\left(\int_{\tau_0}^{\tau}(t^*)^{-1+\delta}\left(\sum_{m=0}^{\ell+1}\int_{\Sigma_{t^*}} J_\mu^N\left(\partial_{t^*}^m\Phi\right)n_{\Sigma_{t^*}}^\mu+\sum_{m=0}^{\ell}\int_{\Sigma_{t^*}} J_\mu^N\left(\tilde{\Omega}\partial_{t^*}^m\Phi\right)n_{\Sigma_{t^*}}^\mu\right)^{\frac{1}{2}} dt^*\right)^2\\
\leq& C\tau^\eta\sum_{m+j\leq\ell+3}\int_{\Sigma_{\tau_0}} J_\mu^N\left(\partial_{t^*}^m\tilde{\Omega}^j\Phi\right)n_{\Sigma_{\tau_0}}^\mu.
\end{aligned}$$

□

Now that we have control of the inhomogeneous terms in the equation $\Box_{g_K}\Phi = G$, we can prove the decay of $S\Phi$. To this end, we will introduce the bootstrap assumptions:

$$\begin{aligned}
&c\int_{\Sigma_\tau} J_\mu^{Z,w^Z}\left(\partial_{t^*}S\Phi\right)n_{\Sigma_\tau}^\mu+\tau^2\int_{\Sigma_\tau\cap\{r\leq\gamma\tau\}} J_\mu^N\left(\partial_{t^*}S\Phi\right)n_{\Sigma_\tau}^\mu\\
\leq& A\tau\sum_{m=1}^{2}\int_{\Sigma_{\tau_0}} J_\mu^{Z+CN,w^Z}\left(\partial_{t^*}^m S\Phi\right)n_{\Sigma_{\tau_0}}^\mu\\
&+A\tau^{1+\eta}\sum_{m+k+j\leq 5}\int_{\Sigma_{\tau_0}} J_\mu^{Z+CN,w^Z}\left(\partial_{t^*}^m\hat{Y}^k\tilde{\Omega}^j\Phi\right)n_{\Sigma_{\tau_0}}^\mu.
\end{aligned}\tag{9.20}$$

$$\begin{aligned}
&c\int_{\Sigma_\tau} J_\mu^{Z,w^Z}\left(S\Phi\right)n_{\Sigma_\tau}^\mu+\tau^2\int_{\Sigma_\tau\cap\{r\leq\gamma\tau\}} J_\mu^N\left(S\Phi\right)n_{\Sigma_\tau}^\mu\\
\leq& A^2\tau^\eta\left(\sum_{m=0}^{2}\int_{\Sigma_{\tau_0}} J_\mu^{Z+CN,w^Z}\left(\partial_{t^*}^m S\Phi\right)n_{\Sigma_{\tau_0}}^\mu\right.\\
&\left.+\sum_{m+k+j\leq 5}\int_{\Sigma_{\tau_0}} J_\mu^{Z+CN,w^Z}\left(\partial_{t^*}^m\hat{Y}^k\tilde{\Omega}^j\Phi\right)n_{\Sigma_{\tau_0}}^\mu\right).
\end{aligned}\tag{9.21}$$

We think of A as some large constant to be chosen. We will improve the constants A and A^2 in the above assumptions. Under these two assumptions, we will get the

following three estimates for the bulk terms:

Proposition 9.9.4.

$$\begin{aligned}&\iint_{\mathcal{R}(\tau_0,\tau)} K^{X_1}\left(\partial_{t^*} S\Phi\right)\\ \leq & C\left(\sum_{m=1}^{2}\int_{\Sigma_{\tau_0}} J_\mu^N\left(\partial_{t^*}^m S\Phi\right) n_{\Sigma_{\tau_0}}^\mu + \sum_{m+k+j\leq 5}\int_{\Sigma_{\tau_0}} J_\mu^{Z+CN,w^Z}\left(\partial_{t^*}^m \hat{Y}\tilde{\Omega}^j\Phi\right) n_{\Sigma_{\tau_0}}^\mu\right).\end{aligned}$$

Proof. By Proposition 9.4.19 for the equation $\Box_{g_K}\left(\partial_{t^*} S\Phi\right) = \partial_{t^*} G$, taking $\tau' = \tau_0$ and $G_1 = 0$, $G_2 = \partial_{t^*} G$. Then use Propositions 9.9.1 and 9.9.2 to estimate the terms with G. □

Proposition 9.9.5. *For $\tau' \leq \tau \leq (1.1)\tau'$,*

$$\begin{aligned}&\iint_{\mathcal{R}(\tau',\tau)\cap\{r\leq r_Y^-\}} K^N\left(\partial_{t^*} S\Phi\right) + \iint_{\mathcal{R}(\tau',\tau)\cap\{r\leq \frac{t^*}{2}\}} K^{X_0}\left(\partial_{t^*} S\Phi\right)\\ &+ \iint_{\mathcal{R}(\tau_0,\tau)} r^{-1+\delta} K^{X_1}\left(S\Phi\right)\\ \leq & CA\left(\tau^{-2}\int_{\Sigma_{\tau'}} J_\mu^{Z,w^Z}\left(S\Phi\right) n_{\Sigma_{\tau'}}^\mu + C\int_{\Sigma_{\tau'}\cap\{r\leq r_Y^-\}} J_\mu^N\left(S\Phi\right) n_{\Sigma_{\tau'}}^\mu\right)\\ &+ CA\tau^{-1+\eta}\sum_{m+k+j\leq 5}\int_{\Sigma_{\tau_0}} J_\mu^{Z+CN,w^Z}\left(\partial_{t^*}^m \hat{Y}\tilde{\Omega}^j\Phi\right) n_{\Sigma_{\tau_0}}^\mu.\end{aligned}$$

Proof. By Propositions 9.5.4 and 9.5.5, taking $G_1 = 0$ and $G_2 = G$, and using Propositions 9.9.1 and 9.9.2 to estimate the terms with G, we have

$$\begin{aligned}&\iint_{\mathcal{R}(\tau',\tau)\cap\{r\leq r_Y^-\}} K^N\left(\partial_{t^*} S\Phi\right) + \iint_{\mathcal{R}(\tau',\tau)\cap\{r\leq \frac{t^*}{2}\}} K^{X_0}\left(\partial_{t^*} S\Phi\right)\\ &+ \iint_{\mathcal{R}(\tau_0,\tau)\cap\{r\leq \frac{t^*}{2}\}} K^{X_1}\left(S\Phi\right)\\ \leq & CA\left(\tau^{-2}\int_{\Sigma_{\tau'}} J_\mu^{Z,w^Z}\left(S\Phi\right) n_{\Sigma_{\tau'}}^\mu + C\int_{\Sigma_{\tau'}\cap\{r\leq r_Y^-\}} J_\mu^N\left(S\Phi\right) n_{\Sigma_{\tau'}}^\mu\right)\\ &+ CA\tau^{-1+\eta}\sum_{m+k+j\leq 5}\int_{\Sigma_{\tau_0}} J_\mu^{Z+CN,w^Z}\left(\partial_{t^*}^m \hat{Y}\tilde{\Omega}^j\Phi\right) n_{\Sigma_{\tau_0}}^\mu.\end{aligned}$$

It now remains to estimate $r^{-1+\delta}K^{X_1}$ in the region $r \geq \frac{t^*}{2}$. Here, we will use crucially the decay in r. Clearly,

$$\iint_{\mathcal{R}(\tau_0,\tau)\cap\{r\geq\frac{t^*}{2}\}} r^{-1+\delta}K^{X_1}\left(S\Phi\right) \leq C\tau^{-1+\delta}\iint_{\mathcal{R}(\tau_0,\tau)} K^{X_1}\left(S\Phi\right).$$

Then we can estimate the right hand side by Proposition 9.4.19, taking $\tau' = \tau_0$ and $G_1 = 0$, $G_2 = \partial_{t^*}G$. Then use Propositions 9.9.1 and 9.9.2 to estimate the terms with G. □

Proposition 9.9.6. *For $\tau' \leq \tau \leq (1.1)\tau'$,*

$$\begin{aligned}
&\iint_{\mathcal{R}(\tau',\tau)\cap\{r\leq r_Y^-\}} K^N\left(S\Phi\right) + \iint_{\mathcal{R}(\tau',\tau)\cap\{r\leq\frac{t^*}{2}\}} K^{X_0}\left(S\Phi\right)\\
\leq& CA^2\left(\tau^{-2}\int_{\Sigma_{\tau'}} J_\mu^{Z,w^Z}\left(S\Phi\right)n^\mu_{\Sigma_{\tau'}} + C\int_{\Sigma_{\tau'}\cap\{r\leq r_Y^-\}} J_\mu^N\left(S\Phi\right)n^\mu_{\Sigma_{\tau'}}\right)\\
&+ CA^2\tau^{-2+\eta}\sum_{m+k+j\leq 4}\int_{\Sigma_{\tau_0}} J_\mu^{Z+CN,w^Z}\left(\partial_{t^*}^m\hat{Y}\tilde{\Omega}^j\Phi\right)n^\mu_{\Sigma_{\tau_0}}.
\end{aligned}$$

Proof. This follows from using Proposition 9.5.4, taking $G_1 = 0$ and $G_2 = G$, and using Propositions 9.9.1 and 9.9.2 to estimate the terms with G. □

We are now ready to retrieve the bootstrap assumptions. First, we retrieve the bootstrap assumption 9.20:

Proposition 9.9.7.

$$\begin{aligned}
&c\int_{\Sigma_\tau} J_\mu^{Z,w^Z}\left(\partial_{t^*}S\Phi\right)n^\mu_{\Sigma_\tau} + \tau^2\int_{\Sigma_\tau\cap\{r\leq\gamma\tau\}} J_\mu^N\left(\partial_{t^*}S\Phi\right)n^\mu_{\Sigma_\tau}\\
\leq&\frac{A}{2}\tau\sum_{m=1}^{2}\int_{\Sigma_{\tau_0}} J_\mu^{Z+CN,w^Z}\left(\partial_{t^*}^m S\Phi\right)n^\mu_{\Sigma_{\tau_0}}\\
&+\frac{A}{2}\tau^{1+\eta}\sum_{m+k+j\leq 5}\int_{\Sigma_{\tau_0}} J_\mu^{Z+CN,w^Z}\left(\partial_{t^*}^m\hat{Y}^k\tilde{\Omega}^j\Phi\right)n^\mu_{\Sigma_{\tau_0}}.
\end{aligned}$$

Proof. By Proposition 9.5.12,

$$\begin{aligned}
&c\int_{\Sigma_\tau} J_\mu^{Z,w^Z}\left(\partial_{t^*}S\Phi\right) n_{\Sigma_\tau}^\mu + \tau^2\int_{\Sigma_\tau\cap\{r\leq\gamma\tau\}} J_\mu^N\left(\partial_{t^*}S\Phi\right) n_{\Sigma_\tau}^\mu\\
\leq& C\int_{\Sigma_{\tau_0}} J_\mu^{Z+CN,w^Z}\left(\partial_{t^*}S\Phi\right) n_{\Sigma_{\tau_0}}^\mu + C\iint_{\mathcal{R}(\tau_0,\tau)} t^* r^{-1+\delta} K^{X_1}\left(\partial_{t^*}S\Phi\right)\\
&+ C\delta'\iint_{\mathcal{R}(\tau_0,\tau)\cap\{r\leq\frac{t^*}{2}\}} (t^*)^2 K^{X_0}\left(\partial_{t^*}S\Phi\right)\\
&+ C\left(\delta'+\epsilon\right)\iint_{\mathcal{R}(\tau_0,\tau)\cap\{r\leq r_Y^-\}} (t^*)^2 K^N\left(\partial_{t^*}S\Phi\right)\\
&+ C(\delta')^{-1}\left(\int_{\tau_0}^\tau\left(\int_{\Sigma_{t^*}\cap\{r\geq\frac{t^*}{2}\}} r^2(\partial_{t^*}G)^2\right)^{\frac{1}{2}} dt^*\right)^2\\
&+ C(\delta')^{-1}\sum_{m=1}^2\iint_{\mathcal{R}(\tau_0,\tau)\cap\{r\leq\frac{9t^*}{10}\}} (t^*)^2 r^{1+\delta}\left(\partial_{t^*}^m G\right)^2\\
&+ C(\delta')^{-1}\sup_{t^*\in[\tau_0,\tau]}\int_{\Sigma_{t^*}\cap\{r_Y^-\leq r\leq\frac{25M}{8}\}} (t^*)^2(\partial_{t^*}G)^2.
\end{aligned}$$

It suffices to check that by Propositions 9.9.1, 9.9.2 and 9.9.3, all terms are acceptable.

□

We can now retrieve the bootstrap assumption (9.21).

Proposition 9.9.8.

$$\begin{aligned}
&c\int_{\Sigma_\tau} J_\mu^{Z,w^Z}\left(S\Phi\right) n_{\Sigma_\tau}^\mu + \tau^2\int_{\Sigma_\tau\cap\{r\leq\gamma\tau\}} J_\mu^N\left(S\Phi\right) n_{\Sigma_\tau}^\mu\\
\leq& A^2\tau^\eta\left(\sum_{m=0}^2\int_{\Sigma_{\tau_0}} J_\mu^{Z+CN,w^Z}\left(\partial_{t^*}^m S\Phi\right) n_{\Sigma_{\tau_0}}^\mu + \sum_{m+k+j\leq 5}\int_{\Sigma_{\tau_0}} J_\mu^{Z+CN,w^Z}\left(\partial_{t^*}^m\hat{Y}^k\tilde{\Omega}^j\Phi\right) n_{\Sigma_{\tau_0}}^\mu\right).
\end{aligned}$$

Proof. By Proposition 9.5.12,

$$\begin{aligned}
&c\int_{\Sigma_\tau} J_\mu^{Z,w^Z}(S\Phi)\, n_{\Sigma_\tau}^\mu + \tau^2\int_{\Sigma_\tau\cap\{r\le\gamma\tau\}} J_\mu^N(S\Phi)\, n_{\Sigma_\tau}^\mu\\
\le& C\int_{\Sigma_{\tau_0}} J_\mu^{Z+CN,w^Z}(S\Phi)\, n_{\Sigma_{\tau_0}}^\mu + C\iint_{\mathcal{R}(\tau_0,\tau)} t^* r^{-1+\delta} K^{X_1}(S\Phi)\\
&+ C\delta'\iint_{\mathcal{R}(\tau_0,\tau)\cap\{r\le\frac{t^*}{2}\}} (t^*)^2 K^{X_0}(S\Phi) + C(\delta'+\epsilon)\iint_{\mathcal{R}(\tau_0,\tau)\cap\{r\le r_Y^-\}} (t^*)^2 K^N(S\Phi)\\
&+ C(\delta')^{-1}\left(\int_{\tau_0}^{\tau}\left(\int_{\Sigma_{t^*}\cap\{r\ge\frac{t^*}{2}\}} r^2 G^2\right)^{\frac12} dt^*\right)^2\\
&+ C(\delta')^{-1}\sum_{m=0}^{1}\iint_{\mathcal{R}(\tau_0,\tau)\cap\{r\le\frac{9t^*}{10}\}} (t^*)^2 r^{1+\delta}\left(\partial_{t^*}^m G\right)^2\\
&+ C(\delta')^{-1}\sup_{t^*\in[\tau_0,\tau]}\int_{\Sigma_{t^*}\cap\{r_Y^-\le r\le\frac{25M}{8}\}} (t^*)^2 G^2.
\end{aligned}$$

It suffices to check that by Propositions 9.9.1, 9.9.2 and 9.9.3, all terms are acceptable.

□

We have thus showed the following:

Proposition 9.9.9. *For all $\eta > 0$, there exists $\epsilon > 0$ small enough such that for Kerr spacetimes satisfying (3.3), the following estimates hold:*

$$\begin{aligned}
&c\int_{\Sigma_\tau} J_\mu^{Z,w^Z}(S\Phi)\, n_{\Sigma_\tau}^\mu + \tau^2\int_{\Sigma_\tau\cap\{r\le\gamma\tau\}} J_\mu^N(S\Phi)\, n_{\Sigma_\tau}^\mu\\
\le& C\tau^\eta\sum_{m=0}^{2}\int_{\Sigma_{\tau_0}} J_\mu^{Z+CN,w^Z}\left(\partial_{t^*}^m S\Phi\right) n_{\Sigma_{\tau_0}}^\mu\\
&+ C\tau^\eta\sum_{m+k+j\le5}\int_{\Sigma_{\tau_0}} J_\mu^{Z+CN,w^Z}\left(\partial_{t^*}^m\hat{Y}\tilde{\Omega}^j\Phi\right) n_{\Sigma_{\tau_0}}^\mu.
\end{aligned}$$

Moreover, for $\tau' \leq \tau \leq (1.1)\tau'$,

$$\begin{aligned}
&\iint_{\mathcal{R}(\tau',\tau)\cap\{r\leq r_Y^-\}} K^N(S\Phi) + \iint_{\mathcal{R}(\tau',\tau)\cap\{r\leq \frac{t^*}{2}\}} K^{X_0}(S\Phi) \\
\leq & C\tau^{-2+\eta}\sum_{m=0}^{2}\int_{\Sigma_{\tau_0}} J_\mu^{Z+CN,w^Z}\left(\partial_{t^*}^m S\Phi\right) n_{\Sigma_{\tau_0}}^\mu \\
&+ C\tau^{-2+\eta}\sum_{m+k+j\leq 5}\int_{\Sigma_{\tau_0}} J_\mu^{Z+CN,w^Z}\left(\partial_{t^*}^m \hat{Y}\tilde{\Omega}^j\Phi\right) n_{\Sigma_{\tau_0}}^\mu.
\end{aligned}$$

and

$$\begin{aligned}
&\iint_{\mathcal{R}(\tau',\tau)\cap\{r\leq \frac{t^*}{2}\}} K^{X_1}(S\Phi) \\
\leq & C\tau^{-2+\eta}\sum_{m=0}^{3}\int_{\Sigma_{\tau_0}} J_\mu^{Z+CN,w^Z}\left(\partial_{t^*}^m S\Phi\right) n_{\Sigma_{\tau_0}}^\mu \\
&+ C\tau^{-2+\eta}\sum_{m+k+j\leq 6}\int_{\Sigma_{\tau_0}} J_\mu^{Z+CN,w^Z}\left(\partial_{t^*}^m \hat{Y}\tilde{\Omega}^j\Phi\right) n_{\Sigma_{\tau_0}}^\mu.
\end{aligned}$$

Proof. The first statement is proved by the bootstrap above. Since the bootstrap assumptions are true, the conclusion in Proposition 9.9.6 is also true, hence the second statement is true. The third statement makes use of the fact that K^{X_1} can be estimated in the same way as K^{X_0} with an extra derivative. □

9.10 Improved Decay for the Linear Homogeneous Wave Equation

To use the estimates for $S\Phi$, we need to integrate along integral curves of S. We first find the integral curves by solving the ordinary differential equation

$$\frac{dr_S}{dt_S^*} = \frac{h(r_S)}{t_S^*}$$

where $h(r_S)$ is as in the definition of S. Hence the integral curves are given by

$$\frac{\exp\left(\int_{(r_S)_0}^{r_S} \frac{dr'_S}{h(r'_S)}\right)}{t^*_S} = \text{constant},$$

where $r_0 > 2M$ can be chosen arbitrarily. Let $\sigma = t^*$, $\rho = \frac{\exp\left(\int_{(r_S)_0}^{r_S} \frac{dr'_S}{h(r'_S)}\right)}{t^*_S}$ and consider (σ, ρ, x^A, x^B) as a new system of coordinates. Notice that

$$\partial_\sigma = \frac{h(r_S)}{t^*}\partial_{r_S} + \partial_{t^*_S} = \frac{1}{t^*}S.$$

Now for each fixed ρ, we have

$$\Phi^2(\tau) \leq \Phi^2(\tau') + |\int_{\tau'}^{\tau} \frac{1}{\sigma} S(\Phi^2) d\sigma|.$$

Integrating along a finite region of ρ, we get:

$$\int_{\rho_1}^{\rho_2} \Phi^2(\tau) d\rho \leq \int_{\rho_1}^{\rho_2} \Phi^2(\tau') d\rho + \int_{\rho_1}^{\rho_2} \int_{\tau'}^{\tau} |\frac{2}{\sigma}\Phi S \Phi| d\sigma d\rho.$$

We would like to change coordinates back to $(t^*_S, r_S, x^A_S, x^B_S)$. Notice that since $h(r_S)$ is everywhere positive, (ρ, τ) would correspond to a point with a larger value of r than (ρ, τ'). Therefore,

$$\begin{aligned}
&\int_{r_+}^{r_2} \Phi^2(\tau) \frac{\exp\left(\int_{(r_S)_0}^{r_S} \frac{dr'_S}{h(r'_S)}\right)}{\tau h(r_S)} dr \\
\leq &\int_{r_+}^{r_2} \Phi^2(\tau') \frac{\exp\left(\int_{(r_S)_0}^{r_S} \frac{dr'_S}{h(r'_S)}\right)}{\tau' h(r_S)} dr + \int_{\tau'}^{\tau} \int_{r_+}^{r_2} |\frac{2}{\sigma}\Phi S \Phi| \frac{\exp\left(\int_{(r_S)_0}^{r_S} \frac{dr'_S}{h(r'_S)}\right)}{t^*_S h(r_S)} dr dt^*.
\end{aligned}$$

We have to compare $\frac{\exp\left(\int_{(r_S)_0}^{r_S}\frac{dr'_S}{h(r'_S)}\right)}{h(r_S)}$ with the volume form. Very close to the horizon, $h(r_S) = r_S - 2M$. Hence

$$\frac{\exp\left(\int_{(r_S)_0}^{r_S}\frac{dr'_S}{h(r'_S)}\right)}{h(r_S)} = e^{\int_{(r_S)_0}^{r_S}\frac{dr'_S}{h(r'_S)}}\left(\frac{1}{r_S - 2M}\right) \sim 1.$$

The corresponding expression on the compact set $[r_Y^-, R]$ is obviously bounded. Hence we have

$$\int_{\Sigma_\tau\cap\{r<r_2\}}\frac{\Phi^2(\tau)}{\tau} \leq C\left(\int_{\Sigma_{\tau'}\cap\{r<r_2\}}\frac{\Phi^2(\tau')}{\tau'} + \iint_{\mathcal{R}(\tau',\tau)\cap\{r<r_2\}}|\frac{2}{(t^*)^2}\Phi S\Phi|\right). \quad (9.22)$$

This easily implies the following improved decay for the non-degenerate energy:

Proposition 9.10.1.

$$\int_{\Sigma_\tau\cap\{r<R\}}\Phi^2 \leq C_R\tau^{-1}\left(\iint_{\mathcal{R}((1.1)^{-1}\tau,\tau)\cap\{r<R\}}\Phi^2 + \iint_{\mathcal{R}((1.1)^{-1}\tau,\tau)\cap\{r<R\}}(S\Phi)^2\right).$$

Proof. By choosing an appropriate $\tilde{\tau}\in[(1.1)^{-1}\tau,\tau]$, we have

$$\int_{\Sigma_{\tilde{\tau}}\cap\{r<R\}}\Phi^2 \leq C\tau^{-1}\iint_{\mathcal{R}((1.1)^{-1}\tau,\tau)\cap\{r<R\}}\Phi^2.$$

Now, apply (9.22) with $\tau' = \tilde{\tau}$, we have

$$\begin{aligned}&\int_{\Sigma_\tau\cap\{r<R\}}\Phi^2\\ \leq& C\tau\left(\int_{\Sigma_{\tilde{\tau}}\cap\{r<R\}}\frac{\Phi^2}{\tilde{\tau}} + \iint_{\mathcal{R}(\tilde{\tau},\tau)\cap\{r<R\}}|\frac{2}{(t^*)^2}\Phi S\Phi|\right)\\ \leq& C\tau^{-1}\left(\iint_{\mathcal{R}((1.1)^{-1}\tau,\tau)\cap\{r<R\}}\Phi^2 + \iint_{\mathcal{R}((1.1)^{-1}\tau,\tau)\cap\{r<R\}}(S\Phi)^2\right),\end{aligned}$$

using Cauchy-Schwarz for the second term. □

We can now conclude with the improved decay for solutions to the homogeneous

wave equation.

Proof of Theorem 5.2.1. By Proposition 9.6.2, 9.10.1 and 9.9.9, we have

$$\begin{aligned}
&\int_{\Sigma_\tau\cap\{r<R\}}\Phi^2\\
\leq& C_R\tau^{-1}\left(\iint_{\mathcal{R}((1.1)^{-1}\tau,\tau)\cap\{r<R\}}\Phi^2+\iint_{\mathcal{R}((1.1)^{-1}\tau,\tau)\cap\{r<R\}}(S\Phi)^2\right)\\
\leq& C_R\tau^{-1}\iint_{\mathcal{R}((1.1)^{-1}\tau,\tau)\cap\{r<R\}}\left(K^{X_0}(\Phi)+K^{X_0}(S\Phi)\right)\\
\leq& C_R\tau^{-3+\eta}\left(\sum_{m=0}^{2}\int_{\Sigma_{\tau_0}}J_\mu^{Z+CN,w^Z}\left(\partial_{t^*}^m S\Phi\right)n^\mu_{\Sigma_{\tau_0}}\right.\\
&\qquad\left.+\sum_{m+k+j\leq 5}\int_{\Sigma_{\tau_0}}J_\mu^{Z+CN,w^Z}\left(\partial_{t^*}^m\hat{Y}\tilde{\Omega}^j\Phi\right)n^\mu_{\Sigma_{\tau_0}}\right)
\end{aligned}$$

Similarly we can use Proposition 9.10.1 for the derivatives of Φ. By Proposition 9.6.2, 9.10.1 and 9.9.9, we have

$$\begin{aligned}
&\int_{\Sigma_\tau\cap\{r<R\}}(D\Phi)^2\\
\leq& C_R\tau^{-1}\left(\iint_{\mathcal{R}((1.1)^{-1}\tau,\tau)\cap\{r<R\}}(D\Phi)^2+\iint_{\mathcal{R}((1.1)^{-1}\tau,\tau)\cap\{r<R\}}(SD\Phi)^2\right)\\
\leq& C_R\tau^{-1}\iint_{\mathcal{R}((1.1)^{-1}\tau,\tau)\cap\{r<R\}}\left(K^{X_1}(\Phi)+K^{X_1}(S\Phi)\right)\\
&\qquad\text{since we have the commutation } [D,S]=D\\
\leq& C_R\tau^{-3+\eta}\left(\sum_{m=0}^{3}\int_{\Sigma_{\tau_0}}J_\mu^{Z+CN,w^Z}\left(\partial_{t^*}^m S\Phi\right)n^\mu_{\Sigma_{\tau_0}}\right.\\
&\qquad\left.+\sum_{m+k+j\leq 6}\int_{\Sigma_{\tau_0}}J_\mu^{Z+CN,w^Z}\left(\partial_{t^*}^m\hat{Y}\tilde{\Omega}^j\Phi\right)n^\mu_{\Sigma_{\tau_0}}\right)
\end{aligned}$$

By commuting with ∂_{t^*}, we get

$$\begin{aligned}
&\int_{\Sigma_\tau \cap \{r<R\}} \left(D\partial_{t^*}^{\ell}\Phi\right)^2 \\
\leq & C_R \tau^{-3+\eta} \left(\sum_{m=0}^{\ell+3} \int_{\Sigma_{\tau_0}} J_\mu^{Z+CN,w^Z}\left(\partial_{t^*}^m S\Phi\right) n_{\Sigma_{\tau_0}}^\mu \right. \\
& \qquad \left. + \sum_{m+k+j\leq \ell+6} \int_{\Sigma_{\tau_0}} J_\mu^{Z+CN,w^Z}\left(\partial_{t^*}^m \hat{Y} \tilde{\Omega}^j \Phi\right) n_{\Sigma_{\tau_0}}^\mu \right).
\end{aligned}$$

Without loss of generality, we can take $R > \frac{23M}{8}$. Then, by Proposition 9.7.5,

$$\begin{aligned}
&\sum_{j+m\leq \ell} \int_{\Sigma_\tau \cap \{r\leq r_Y^+\}} J_\mu^N \left(\partial_{t^*}^j \hat{Y}^m \Phi\right) n_{\Sigma_\tau}^\mu \\
\leq & C_R \tau^{-3+\eta} \left(\sum_{m=0}^{\ell+3} \int_{\Sigma_{\tau_0}} J_\mu^{Z+CN,w^Z}\left(\partial_{t^*}^m S\Phi\right) n_{\Sigma_{\tau_0}}^\mu + \sum_{m+k+j\leq \ell+6} \int_{\Sigma_{\tau_0}} J_\mu^{Z+CN,w^Z}\left(\partial_{t^*}^m \hat{Y} \tilde{\Omega}^j \Phi\right) n_{\Sigma_{\tau_0}}^\mu \right).
\end{aligned}$$

Hence, by Proposition 9.7.1 and 9.7.3,

$$\begin{aligned}
&\sum_{j=0}^{\ell} \int_{\Sigma_\tau \cap \{r\leq R\}} \left(D^j\Phi\right)^2 \\
\leq & C_R \tau^{-3+\eta} \left(\sum_{m=0}^{\ell+2} \int_{\Sigma_{\tau_0}} J_\mu^{Z+CN,w^Z}\left(\partial_{t^*}^m S\Phi\right) n_{\Sigma_{\tau_0}}^\mu + \sum_{m+k+j\leq \ell+5} \int_{\Sigma_{\tau_0}} J_\mu^{Z+CN,w^Z}\left(\partial_{t^*}^m \hat{Y} \tilde{\Omega}^j \Phi\right) n_{\Sigma_{\tau_0}}^\mu \right).
\end{aligned}$$

The pointwise decay statement follows from standard Sobolev Embedding. □

9.11 Discussion

The Theorem that we proved in the chapter holds in the set $\{r_+ \leq r \leq R\}$ for any fixed R. It is however interesting also to derive the same estimates, for example, in the set $\{r_+ \leq r \leq \frac{t^*}{2}\}$. This can be achieved by proving the full decay result when we commuted the equation with $\tilde{\Omega}^\ell$. Using this we can prove (with more loss

in derivatives) that

$$|\Phi| \leq CE(t^*)^{-\frac{3}{2}+\eta} r^{\eta}, \quad |D\Phi| \leq CE(t^*)^{-\frac{3}{2}+\eta} r^{-\frac{1}{2}+\eta},$$

for $r \leq \frac{t^*}{2}$. We omit the details in this chapter but note that it can be retrieved as an easy corollary of the result in the next chapter.

Chapter 10

Nonlinear Wave Equations on Kerr Spacetimes

In this chapter, we prove Theorem 5.3.2 on the small data global existence for nonlinear wave equations satisfying a null condition on Kerr spacetimes. In the proof, we will use the energy estimates proved in the previous chapter. The proof is carried out via a bootstrap argument: we will assume certain bounds on the energy quantities and the L^∞ norms and show that all these bounds can be improved. This would imply Theorem 5.3.2 via a standard continuity argument.

To prove the global existence result, it is crucial that the nonlinearity satisfies the null condition. In order to capture the null condition that is satisfied by the nonlinearity, we need to distinguish between the estimates for the "good" and "bad" derivatives as defined in Chapter 5. We thus need to find energy quantities that distinguish these derivatives. It turns out that the conformal energy that has been used in the previous chapter can serve this purpose. We will also need to prove different L^∞ decay estimates for different derivatives, especially in the region $r \geq \frac{t^*}{4}$. This will be carried out using both the conformal energy and the scaling vector field. Thus the scaling vector field, which was originally introduced to prove improved decay

rate for $r \leq \frac{t^*}{4}$ is also used to obtain improved decay rate for some derivatives for $r \geq \frac{t^*}{4}$!

In section 10.2, we prove the necessary L^∞ estimates. With this preparation, we then prove all the estimates using a bootstrap argument in section 10.3. This then easily implies Theorem 5.3.2 in 10.4.

10.1 The Energy Quantities

We revisit the non-degenerate energy and the conformal energy from the previous chapter. In particular, we note which derivatives each of these energies controls. The nondegnerate energy controls all derivatives:

Proposition 10.1.1.

$$\int_{\Sigma_\tau} (D\Phi)^2 \leq C \int_{\Sigma_\tau} J_\mu^N(\Phi)\, n_{\Sigma_\tau}^\mu.$$

The conformal energy gives different weights to different derivatives and this will be crucially used to capture the null condition:

Proposition 10.1.2.

$$\begin{aligned}&\int_{\Sigma_\tau \cap \{r \geq r_Y^-\}} u^2 (\underline{L}\Phi)^2 + v^2 (L\Phi)^2 + \left(u^2+v^2\right) |\nabla\!\!\!/\Phi|^2 + \left(\frac{u^2+v^2}{r^2}\right)\Phi^2 \\ \leq & C \int_{\Sigma_\tau} J_\mu^{Z+N, w^Z}(\Phi)\, n_{\Sigma_\tau}^\mu + C^2\tau^2 \int_{\Sigma_\tau \cap \{r \leq r_Y^-\}} J_\mu^N(\Phi)\, n_{\Sigma_\tau}^\mu.\end{aligned}$$

10.2 Pointwise Estimates

We prove pointwise estimates using Sobolev embedding. We will have different estimates in the region $\{r \geq \frac{t^*}{4}\}$ and $\{r \leq \frac{t^*}{4}\}$. We first consider $\{r \geq \frac{t^*}{4}\}$. For this region, we will prove five different pointwise estimates. First, we prove a boundedness result for $D\Phi$ (Proposition 10.2.1) using only standard Sobolev Embedding and the

elliptic estimates in Proposition 9.7.1. Then we prove decay estimates of r^{-1} for $D^\ell\Phi$ using the r weight in the vector field commutator $\tilde{\Omega}$ and the non-degenerate energy (Proposition 10.2.2). It is crucial that this depends only on the non-degenerate energy but not the conformal energy because we will not be able to prove boundedness of the conformal energy (which already is the case in the *linear* situation, see [16], [2], [36]). Notice that Proposition 10.2.1 does not follow from Proposition 10.2.2 because the latter requires an extra derivative. This save in derivatives is strictly speaking not necessary for the bootstrap if we have instead assumed an extra derivative of regularity in the initial data. Thirdly, using similar ideas, we will prove the decay of r^{-1} for Φ using $\tilde{\Omega}$ and the conformal energy (Proposition 10.2.3). Then we prove an extra decay rate of $D\Phi$ using the conformal energy. For any derivatives, we will have an extra decay in the u variable, which degenerates in the wave zone (Proposition 10.2.5). For the good derivatives, we will have an extra decay in the v variable (Proposition 10.2.6). This decay rate will be crucial in capturing the good derivative in the null condition.

Proposition 10.2.1. *For $r \geq \frac{t^*}{4}$ we have*

$$|D\Phi|^2 \leq C\left(\sum_{k=0}^{2}\int_{\Sigma_\tau} J_\mu^N\left(\partial_{t^*}^k\Phi\right)n_{\Sigma_\tau}^\mu + \sum_{k=0}^{1}\int_{\Sigma_\tau}\left(D^k\Box_{g_K}\Phi\right)^2\right).$$

Proof. By standard Sobolev Embedding in three dimensions and Proposition 9.7.1,

$$\begin{aligned}|D\Phi|^2 \leq& C\sum_{k=1}^{3}\int_{\Sigma_\tau\cap\{r\geq r_Y^-\}}(D^k\Phi)^2\\ \leq& C\left(\sum_{k=0}^{2}\int_{\Sigma_\tau} J_\mu^N\left(\partial_{t^*}^k\Phi\right)n_{\Sigma_\tau}^\mu + \sum_{k=0}^{1}\int_{\Sigma_\tau}\left(D^k\Box_{g_K}\Phi\right)^2\right).\end{aligned}$$

□

We then prove the decay rate of r^{-1} for $D^\ell\Phi$. The idea here is standard: Making

use of the commutator $\tilde{\Omega}$, we use the Sobolev Embedding on the 2-sphere and then integrate along the r direction.

Proposition 10.2.2. *For $r \geq \frac{t^*}{4}$ and $\ell \geq 1$, we have*

$$\begin{aligned}
&|D^\ell \Phi|^2 \\
\leq & C r^{-2} \left(\sum_{m=0}^{\ell} \sum_{k=0}^{2} \int_{\Sigma_\tau} J_\mu^N \left(\partial_{t^*}^m \tilde{\Omega}^k \Phi \right) n_{\Sigma_\tau}^\mu \right. \\
& \qquad \left. + \sum_{j=0}^{\ell-1} \sum_{k=0}^{2} \int_{\Sigma_\tau \cap \{u' \sim u\} \cap \{r \geq \frac{\tau}{2}\}} \left(D^j \Box_{g_K} \left(\tilde{\Omega}^k \Phi \right) \right)^2 \right).
\end{aligned}$$

Proof.

$$\begin{aligned}
& r^2 |D^\ell \Phi|^2 \\
\leq & C \int_{\mathbb{S}^2} \left(\left(D^\ell \Phi \right)^2 + \left(\tilde{\Omega} D^\ell \Phi \right)^2 + \left(\tilde{\Omega}^2 D^\ell \Phi \right)^2 \right) r^2 dA \\
\leq & C \left(\sum_{k=0}^{2} \int_{\mathbb{S}^2(\tilde{r})} \left(\tilde{\Omega}^k D^\ell \Phi \right)^2 \tilde{r}^2 dA \right. \\
& \qquad \left. + \int_r^{\tilde{r}} \int_{\mathbb{S}^2(r')} |\partial_r \tilde{\Omega}^k D^\ell \Phi \tilde{\Omega}^k D^\ell \Phi| (r')^2 + \left(\tilde{\Omega}^k D^\ell \Phi \right)^2 r' dA dr' \right).
\end{aligned}$$

Noticing that $|[D,\tilde{\Omega}]\Phi| \leq C|D\Phi|$, we have

$$\begin{aligned}
&r^2|D^\ell\Phi|^2\\
\leq& C\left(\sum_{k=0}^{2}\int_{\mathbb{S}^2(\tilde{r})}\left(\tilde{\Omega}^k D^\ell\Phi\right)^2\tilde{r}^2 dA\right.\\
&\qquad\left.+\int_r^{\tilde{r}}\int_{\mathbb{S}^2(r')}|\partial_r D^\ell\tilde{\Omega}^k\Phi D^\ell\tilde{\Omega}^k\Phi|(r')^2+\left(\tilde{\Omega}^k D^\ell\Phi\right)^2 r' dA dr'\right)\\
\leq& C\left(\sum_{k=0}^{2}\int_{\mathbb{S}^2(\tilde{r})}\left(D^\ell\tilde{\Omega}^k\Phi\right)^2\tilde{r}^2 dA\right.\\
&\qquad\left.+\int_r^{\tilde{r}}\int_{\mathbb{S}^2(r')}|D^{\ell+1}\tilde{\Omega}^k\Phi D^\ell\tilde{\Omega}^k\Phi|(r')^2+\left(D^\ell\tilde{\Omega}^k\Phi\right)^2 r' dA dr'\right)\\
\leq& C\left(\sum_{k=0}^{2}\int_{\mathbb{S}^2(\tilde{r})}\left(D^\ell\tilde{\Omega}^k\Phi\right)^2\tilde{r}^2 dA\right.\\
&\qquad\left.+\int_r^{\tilde{r}}\int_{\mathbb{S}^2(r')}\left(D^{\ell+1}\tilde{\Omega}^k\Phi\right)^2(r')^2+\left(D^\ell\tilde{\Omega}^k\Phi\right)^2(r')^2 dA dr'\right).
\end{aligned}$$

Take $r \leq \tilde{r} \leq r+1$. By Proposition 9.7.1,

$$\begin{aligned}
&\sum_{k=0}^{2}\int_r^{r+1}\int_{\mathbb{S}^2(r')}\left(D^{\ell+1}\tilde{\Omega}^k\Phi\right)^2(r')^2+\left(D^\ell\tilde{\Omega}^k\Phi\right)^2(r')^2 dA dr'\\
\leq& C\left(\sum_{m=0}^{\ell}\sum_{k=0}^{2}\int_{\Sigma_\tau}J_\mu^N\left(\partial_{t^*}^m\Omega^k\Phi\right)n_{\Sigma_\tau}^\mu+\sum_{j=0}^{\ell-1}\sum_{k=0}^{2}\int_{\Sigma_\tau\cap\{u'\sim u\}\cap\{r\geq\frac{\tau}{2}\}}\left(D^j\Box_{g_K}\left(\tilde{\Omega}^k\Phi\right)\right)^2\right).
\end{aligned}$$

By pigeonholing on this we also get that for some $\tilde{r}$,

$$\begin{aligned}
&\sum_{k=0}^{2}\int_{\mathbb{S}^2(\tilde{r})}\left(D^\ell\Omega^k\Phi\right)^2\tilde{r}^2 dA\\
\leq& C\left(\sum_{m=0}^{\ell}\sum_{k=0}^{2}\int_{\Sigma_\tau}J_\mu^N\left(\partial_{t^*}^m\Omega^k\Phi\right)n_{\Sigma_\tau}^\mu+\sum_{j=0}^{\ell-1}\sum_{k=0}^{2}\int_{\Sigma_\tau\cap\{u'\sim u\}\cap\{r\geq\frac{\tau}{2}\}}\left(D^j\Box_{g_K}\left(\tilde{\Omega}^k\Phi\right)\right)^2\right).
\end{aligned}$$

□

We would also like to prove the pointwise decay in r for Φ. However, we need to use the conformal energy as well as the non-degenerate energy. We note that only

the decay in r will be used in the bootstrap argument, the decay in u is proved to achieved the decay rate asserted in Theorem 1.

Proposition 10.2.3. *Consider $\Box_{g_K}\Phi = G$. For $r \geq \frac{t^*}{4}$, we have*

$$\begin{aligned}
&|\Phi|^2\\
\leq& Cr^{-2}(1+|u|)^{-1}\left(\sum_{k=0}^{2}\int_{\Sigma_\tau} J_\mu^{Z+N,w^Z}\left(\Omega^k\Phi\right)n_{\Sigma_\tau}^\mu\right.\\
&\left.+C\tau^2\sum_{k=0}^{2}\int_{\Sigma_\tau\cap\{r\leq r_Y^-\}} J_\mu^{N}\left(\Omega^k\Phi\right)n_{\Sigma_\tau}^\mu\right).
\end{aligned}$$

Proof. Following the proof of Proposition 10.2.2, we have

$$\begin{aligned}
&r^2|\Phi|^2\\
\leq& C\left(\sum_{k=0}^{2}\int_{\mathbb{S}^2(\tilde{r})}\left(\tilde{\Omega}^k\Phi\right)^2\tilde{r}^2dA + |\int_r^{\tilde{r}}\int_{\mathbb{S}^2(r')}|\tilde{\Omega}^k\Phi D\tilde{\Omega}^k\Phi|(r')^2 + \left(\tilde{\Omega}^k\Phi\right)^2 r'dAdr'|\right).
\end{aligned}$$

We will treat separately the cases $|u|\leq 1$, $u\geq 1$, $u\leq 1$. For $|u|\leq 1$, take $r\leq\tilde{r}\leq r+1$. By Proposition 10.1.2,

$$\begin{aligned}
&\sum_{k=0}^{2}\int_r^{r+1}\int_{\mathbb{S}^2(r')}\left(D\tilde{\Omega}^k\Phi\right)^2(r')^2 + \left(\tilde{\Omega}^k\Phi\right)^2(r')^2dAdr'\\
\leq& C\left(\sum_{k=0}^{2}\int_{\Sigma_\tau} J_\mu^{Z+N,w^Z}\left(\Omega^k\Phi\right)n_{\Sigma_\tau}^\mu + C\tau^2\sum_{k=0}^{2}\int_{\Sigma_\tau\cap\{r\leq r_Y^-\}} J_\mu^{N}\left(\Omega^k\Phi\right)n_{\Sigma_\tau}^\mu\right).
\end{aligned}$$

By pigeonholing on this we also get that for some $\tilde{r}$,

$$\begin{aligned}
&\sum_{k=0}^{2}\int_{\mathbb{S}^2(\tilde{r})}\left(\Omega^k\Phi\right)^2\tilde{r}^2dA\\
\leq& C\left(\sum_{k=0}^{2}\int_{\Sigma_\tau} J_\mu^{Z+N,w^Z}\left(\Omega^k\Phi\right)n_{\Sigma_\tau}^\mu + C\tau^2\sum_{k=0}^{2}\int_{\Sigma_\tau\cap\{r\leq r_Y^-\}} J_\mu^{N}\left(\Omega^k\Phi\right)n_{\Sigma_\tau}^\mu\right).
\end{aligned}$$

For $u\geq 1$, pick a fixed R and let $\tilde{r}\in[R,R+1]$. Then by a pigeonhole argument,

there is some $\tilde{r}$ such that

$$\begin{aligned}&\sum_{k=0}^{2}\int_{\mathbb{S}^2(\tilde{r})}\left(\tilde{\Omega}^k\Phi\right)^2\tilde{r}^2dA\\ \leq& Cu^{-2}\left(\sum_{k=0}^{2}\int_{\Sigma_\tau}J_\mu^{Z+N,w^Z}\left(\Omega^k\Phi\right)n_{\Sigma_\tau}^\mu+C\tau^2\sum_{k=0}^{2}\int_{\Sigma_\tau\cap\{r\leq r_Y^-\}}J_\mu^N\left(\Omega^k\Phi\right)n_{\Sigma_\tau}^\mu\right).\end{aligned}$$

By Proposition 10.1.2,

$$\begin{aligned}&\sum_{k=0}^{2}\int_R^r\int_{\mathbb{S}^2(r')}|\tilde{\Omega}^k\Phi D\tilde{\Omega}^k\Phi|(r')^2+\left(\tilde{\Omega}^k\Phi\right)^2r'dAdr'\\ \leq& C\left(rt^*u^{-2}+ru^{-2}\right)\left(\sum_{k=0}^{2}\int_{\Sigma_\tau}J_\mu^{Z+N,w^Z}\left(\Omega^k\Phi\right)n_{\Sigma_\tau}^\mu\right.\\ &\left.+C\tau^2\sum_{k=0}^{2}\int_{\Sigma_\tau\cap\{r\leq r_Y^-\}}J_\mu^N\left(\Omega^k\Phi\right)n_{\Sigma_\tau}^\mu\right).\end{aligned}$$

Using the fact that $t^*\leq Cu$ in this region, we have the desired bound in this region.

Finally, for $u\leq 1$, pick $\tilde{r}\in[-2u,-3u]$. Then by a pigeonhole argument, there is an $\tilde{r}$ such that

$$\begin{aligned}&\sum_{k=0}^{2}\int_{\mathbb{S}^2(\tilde{r})}\left(\tilde{\Omega}^k\Phi\right)^2\tilde{r}^2dA\\ \leq& Cu^{-2}\left(\sum_{k=0}^{2}\int_{\Sigma_\tau}J_\mu^{Z+N,w^Z}\left(\Omega^k\Phi\right)n_{\Sigma_\tau}^\mu+C\tau^2\sum_{k=0}^{2}\int_{\Sigma_\tau\cap\{r\leq r_Y^-\}}J_\mu^N\left(\Omega^k\Phi\right)n_{\Sigma_\tau}^\mu\right).\end{aligned}$$

By Proposition 10.1.2,

$$\begin{aligned}&\sum_{k=0}^{2}\int_r^\infty\int_{\mathbb{S}^2(r')}|\tilde{\Omega}^k\Phi D\tilde{\Omega}^k\Phi|(r')^2+\left(\tilde{\Omega}^k\Phi\right)^2r'dAdr'\\ \leq& C|u|^{-1}\left(\sum_{k=0}^{2}\int_{\Sigma_\tau}J_\mu^{Z+N,w^Z}\left(\Omega^k\Phi\right)n_{\Sigma_\tau}^\mu+C\tau^2\sum_{k=0}^{2}\int_{\Sigma_\tau\cap\{r\leq r_Y^-\}}J_\mu^N\left(\Omega^k\Phi\right)n_{\Sigma_\tau}^\mu\right),\end{aligned}$$

which gives the desired bound. □

We would like to use the conformal energy and elliptic estimates to prove decay

in the u variable. However, we need to be careful when applying the localized version of the elliptic estimates. In particular, we need to perform a dyadic decomposition in the variable u. We remark that we can prove this for any number of derivatives by iterating the cutoff procedure in the proof of the following Proposition. However, as this will not be necessary in the sequel, we will be content with the following Proposition:

Proposition 10.2.4. *Suppose $\Box_{g_K}\Phi = G$. Let $r \geq \frac{t^*}{4}$, $\ell = 1$ or 2 and u_0 be the u-coordinate corresponding to the two sphere (τ, r_0).*

$$\begin{aligned}
&\int_{r_0}^{r_0+1}\int_{\mathbb{S}^2(r')} \left(D^\ell \Phi\right)^2 (r')^2 dA dr' \\
\leq & C\left(1+|u_0|\right)^{-2} \sum_{j=0}^{\ell-1} \left(\int_{\Sigma_\tau} J_\mu^{Z+CN}\left(\partial_{t^*}^j \Phi\right) n_{\Sigma_\tau}^\mu + C\tau^2 \int_{\Sigma_\tau \cap \{r \leq r_Y^-\}} J_\mu^N \left(\partial_{t^*}^j \Phi\right) n_{\Sigma_\tau}^\mu \right) \\
&+ C \sum_{j=0}^{\ell-2} \int_{\Sigma_\tau \cap \{u \sim u_0\} \cap \{r \geq \frac{\tau}{2}\}} \left(D^j G\right)^2 .
\end{aligned}$$

Proof. The $\ell = 1$ case is trivial. For $\ell = 2$, we consider separately the cases: Case 0: $|u_0| \leq C_*$, Case 1_k: $2^k \leq u_0 \leq 2^{k+1}$, Case 2_k: $-2^{k+1} \leq u_0 \leq -2^k$, $k \geq \frac{\log C_*}{\log 2}$ for some sufficiently large but fixed C_*. In Case 0, we have $|u| \leq C$ for the range $[r_0, r_0+1]$ and hence the Proposition is obvious as we have $1 \leq C\left(1+|u|\right)^{-2}$.

For the other cases, we consider a cutoff function $\chi : \mathbb{R} \to \mathbb{R}_{\geq 0}$ which is compactly supported in $[-2,2]$ and identically 1 in $[-1,1]$. In case 1_k (resp. 2_k), we consider $\tilde{\Phi}$ to be defined by $\tilde{\Phi}(\tau, r, \theta, \phi) = \chi\left(2^{-k+3}\left(r - r_0\right)\right)\Phi\left(\tau, r, \theta, \phi\right)$. Then $\tilde{\Phi}$ is supported in $[r_0 - 2^{k-2}, r_0 + 2^{k-2}]$ and equals Φ in $[r_0 - 2^{k-3}, r_0 + 2^{k-3}]$. On the support of $\tilde{\Phi}$, $|\Box_{g_K}\tilde{\Phi} - G| \leq C\sum_{j=0}^{1} 2^{-(2-j)k}|D^j\Phi|$. We also have that on the support of $\tilde{\Phi}$, $|u - u_0| \leq \frac{1}{2}|r_S^* - (r_0^*)_S| \leq \frac{1}{2}|r - r_0| + \frac{M}{2}|\log\frac{r-2M}{r_0-2M}| \leq |r - r_0| \leq 2^{k-1}$ for r_0 sufficiently large (which we can assume for otherwise τ and r must both be bounded, in which case we must be in Case 0 for appropriately chosen C_*). Hence $u \sim 2^k$ (resp. $u \sim -2^k$).

Therefore, by Proposition 9.7.1.1 applied twice, first to $\tilde{\Phi}$ then to Φ, we have,

$$\begin{aligned}
&\int_{r_0}^{r_0+1}\int_{\mathbb{S}^2(r')}\left(D^2\Phi\right)^2(r')^2dAdr' \leq \int_{\Sigma_\tau\cap\{r_0\leq r\leq r_0+1\}}\left(D^2\tilde{\Phi}\right)^2\\
\leq& C\sum_{j=0}^{1}\int_{\Sigma_\tau\cap\{r_0-2^{k-3}\leq r\leq r_0+2^{k-3}\}} J_\mu^N\left(\partial_{t^*}^j\tilde{\Phi}\right)n^\mu_{\Sigma_\tau}\\
&+C\sum_{j=0}^{1}\int_{\Sigma_\tau\cap\{r_0-2^{k-2}\leq r\leq r_0+2^{k-2}\}} 2^{-(2-j)2k}\left(D^j\Phi\right)^2+C\int_{\Sigma_\tau\cap\{r_0-2^{k-2}\leq r\leq r_0+2^{k-2}\}}G^2\\
\leq& C\sum_{j=0}^{1}\int_{\Sigma_\tau\cap\{r_0-2^{k-2}\leq r\leq r_0+2^{k-2}\}}\left(2^{-2k}\Phi^2+J_\mu^N\left(\partial_{t^*}^j\Phi\right)n^\mu_{\Sigma_\tau}\right)\\
&+C\int_{\Sigma_\tau\cap\{r_0-2^{k-2}\leq r\leq r_0+2^{k-2}\}}G^2\\
\leq& C\left(1+|u_0|\right)^{-2}\sum_{j=0}^{1}\left(\int_{\Sigma_\tau}J_\mu^{Z+CN}\left(\partial_{t^*}^j\Phi\right)n^\mu_{\Sigma_\tau}+C\tau^2\int_{\Sigma_\tau\cap\{r\leq r_Y^-\}}J_\mu^N\left(\partial_{t^*}^j\Phi\right)n^\mu_{\Sigma_\tau}\right)\\
&+C\int_{\Sigma_\tau\cap\{u\sim u_0\}}G^2.
\end{aligned}$$

□

Using this we can prove more decay in the u variable:

Proposition 10.2.5. *Suppose $\Box_{g_K}\Phi = G$. For $r \geq \frac{t^*}{4}$ and $\ell \geq 1$, we have*

$$\begin{aligned}
&|D\Phi|^2\\
\leq& Cr^{-2}\left(1+|u|\right)^{-2}\sum_{j=0}^{1}\sum_{k=0}^{2}\left(\int_{\Sigma_\tau}J_\mu^{Z+CN}\left(\partial_{t^*}^j\tilde{\Omega}^k\Phi\right)n^\mu_{\Sigma_\tau}\right.\\
&\qquad\left.+C\tau^2\int_{\Sigma_\tau\cap\{r\leq r_Y^-\}}J_\mu^N\left(\partial_{t^*}^j\tilde{\Omega}^k\Phi\right)n^\mu_{\Sigma_\tau}\right)\\
&+Cr^{-2}\sum_{k=0}^{2}\int_{\Sigma_\tau\cap\{u'\sim u\}\cap\{r\geq\frac{\tau}{2}\}}\left(\Box_{g_K}\left(\tilde{\Omega}^k\Phi\right)\right)^2.
\end{aligned}$$

Proof. Following the proof of Proposition 10.2.2, we have

$$\begin{aligned}
&r^2|D\Phi|^2\\
\leq& C\left(\sum_{k=0}^{2}\int_{\mathbb{S}^2(\tilde{r})}\left(D\Omega^k\Phi\right)^2\tilde{r}^2dA\right.\\
&\qquad\left.+\int_r^{\tilde{r}}\int_{\mathbb{S}^2(r')}\left(D^2\Omega^k\Phi\right)\left(D\Omega^k\Phi\right)(r')^2+\left(D\Omega^k\Phi\right)^2r'dAdr'\right)
\end{aligned}$$

Take $r\leq\tilde{r}\leq r+1$. Then by Proposition 10.2.4,

$$\begin{aligned}
&\sum_{k=0}^{2}\int_r^{r+1}\int_{\mathbb{S}^2(r')}\left(D^2\Omega^k\Phi\right)\left(D\Omega^k\Phi\right)(r')^2+\left(D\Omega^k\Phi\right)^2r'dAdr'\\
\leq& C\left(1+|u|\right)^{-2}\sum_{j=0}^{1}\sum_{k=0}^{2}\left(\int_{\Sigma_\tau}J_\mu^{Z+CN}\left(\partial_{t^*}^j\tilde{\Omega}^k\Phi\right)n_{\Sigma_\tau}^\mu\right.\\
&\qquad\left.+C\tau^2\int_{\Sigma_\tau\cap\{r\leq r_Y^-\}}J_\mu^N\left(\partial_{t^*}^j\tilde{\Omega}^k\Phi\right)n_{\Sigma_\tau}^\mu\right)\\
&+C\sum_{k=0}^{2}\int_{\Sigma_\tau\cap\{u'\sim u\}\cap\{r\geq\frac{\tau}{2}\}}\left(\Box_{g_K}\left(\tilde{\Omega}^k\Phi\right)\right)^2.
\end{aligned}$$

By pigeonholing on this we also get that for some $\tilde{r}$,

$$\begin{aligned}
&\sum_{k=0}^{2}\int_{\mathbb{S}^2(\tilde{r})}\left(D^\ell\Omega^k\Phi\right)^2\tilde{r}^2dA\\
\leq& C\left(1+|u|\right)^{-2}\sum_{j=0}^{1}\sum_{k=0}^{2}\left(\int_{\Sigma_\tau}J_\mu^{Z+CN}\left(\partial_{t^*}^j\tilde{\Omega}^k\Phi\right)n_{\Sigma_\tau}^\mu\right.\\
&\qquad\left.+C\tau^2\int_{\Sigma_\tau\cap\{r\leq r_Y^-\}}J_\mu^N\left(\partial_{t^*}^j\tilde{\Omega}^k\Phi\right)n_{\Sigma_\tau}^\mu\right)\\
&+C\sum_{k=0}^{2}\int_{\Sigma_\tau\cap\{u'\sim u\}\cap\{r\geq\frac{\tau}{2}\}}\left(\Box_{g_K}\left(\tilde{\Omega}^k\Phi\right)\right)^2.
\end{aligned}$$

□

We have a better pointwise decay for a "good" derivative:

Proposition 10.2.6. *For $r \geq \frac{t^*}{4}$, we have*

$$\begin{aligned}|\bar{D}\Phi|^2 \leq & Cr^{-4}\sum_{k=0}^{2}\sum_{i+j\leq 1}\left(\int_{\Sigma_\tau} J_\mu^N\left(S^i\partial_{t^*}^j\Phi\right)n_{\Sigma_\tau}^\mu + \int_{\Sigma_\tau} J_\mu^{Z+CN}\left(\partial_{t^*}^j\tilde{\Omega}^k\Phi\right)n_{\Sigma_\tau}^\mu\right.\\ & \left. + C\tau^2\int_{\Sigma_\tau\cap\{r\leq r_Y^-\}} J_\mu^N\left(\partial_{t^*}^j\tilde{\Omega}^k\Phi\right)n_{\Sigma_\tau}^\mu + \int_{\Sigma_\tau}\left(\Box_{g_K}\left(\tilde{\Omega}^k\Phi\right)\right)^2\right)\\ & + Cr^{-2}\sum_{k=0}^{2}\int_{\Sigma_\tau\cap\{r\geq\frac{\tau}{2}\}}\left(\Box_{g_K}\left(\tilde{\Omega}^k\Phi\right)\right)^2.\end{aligned}$$

Proof.

$$\begin{aligned}& r^2|\bar{D}\Phi|^2\\ \leq & C\int_{\mathbb{S}^2}\left(\left(\bar{D}\Phi\right)^2 + \left(\tilde{\Omega}\bar{D}\Phi\right)^2 + \left(\tilde{\Omega}^2\bar{D}\Phi\right)^2\right)r^2dA\\ \leq & C\sum_{k=0}^{2}\left(\int_{\mathbb{S}^2(\tilde{r})}\left(\tilde{\Omega}^k\bar{D}\Phi\right)^2\tilde{r}^2dA\right.\\ & \left. + \int_r^{\tilde{r}}\int_{\mathbb{S}^2(r')}|\partial_r\tilde{\Omega}^k\bar{D}\Phi\tilde{\Omega}^k\bar{D}\Phi|(r')^2 + \left(\tilde{\Omega}^k\bar{D}\Phi\right)^2r'dAdr'\right).\end{aligned}$$

Noticing that $|[D,\tilde{\Omega}]\Phi| \leq C|D\Phi|$, $|[\bar{D},\tilde{\Omega}]\Phi| \leq C\left(|\bar{D}\Phi| + r^{-1}|D\Phi|\right)$ and $|\bar{D},\partial_r\Phi| \leq Cr^{-1}|D\Phi|$, we have

$$\begin{aligned}& r^2|\bar{D}\Phi|^2\\ \leq & C\sum_{k=0}^{2}\left(\int_{\mathbb{S}^2(\tilde{r})}\left(\tilde{\Omega}^k\bar{D}\Phi\right)^2\tilde{r}^2dA\right.\\ & \left. + \int_r^{\tilde{r}}\int_{\mathbb{S}^2(r')}|\partial_r\bar{D}\tilde{\Omega}^k\Phi\bar{D}\tilde{\Omega}^k\Phi|(r')^2 + \left(\tilde{\Omega}^k\bar{D}\Phi\right)^2r'dAdr'\right) \qquad (10.1)\\ \leq & C\sum_{k=0}^{2}\left(\int_{\mathbb{S}^2(\tilde{r})}\left(\left(\bar{D}\tilde{\Omega}^k\Phi\right)^2 + \tilde{r}^{-2}\left(D\tilde{\Omega}^k\Phi\right)^2\right)\tilde{r}^2dA\right.\\ & \left. + \int_r^{\tilde{r}}\int_{\mathbb{S}^2(r')}\left(\left(\bar{D}D\tilde{\Omega}^k\Phi\right)^2 + \left(\bar{D}\tilde{\Omega}^k\Phi\right)^2 + (r')^{-2}\left(D\tilde{\Omega}^k\Phi\right)^2\right)(r')^2dAdr'\right).\end{aligned}$$

The last term already exhibits better decay rate:

$$\int_r^{\tilde{r}} \int_{\mathbb{S}^2(r')} (r')^{-2} \left(D\tilde{\Omega}^k \Phi\right)^2 (r')^2 dA dr' \leq C r^{-2} \int_{\Sigma_\tau} J_\mu^N \left(\tilde{\Omega}^k \Phi\right) n_{\Sigma_\tau}^\mu$$

We will now show that the energy quantities involving $\bar{D}$ obey better decay rates. This is immediate for the term $\int_r^{\tilde{r}} \int_{\mathbb{S}^2(r')} \left(\bar{D}\tilde{\Omega}^k \Phi\right)^2 (r')^2 dA dr'$ using the conformal energy:

$$\begin{aligned}
&\int_r^{\tilde{r}} \int_{\mathbb{S}^2(r')} \left(\bar{D}\tilde{\Omega}^k \Phi\right)^2 (r')^2 dA dr' \\
\leq & C v^{-2} \left(\int_{\Sigma_\tau} J_\mu^{Z+CN} \left(\tilde{\Omega}^k \Phi\right) n_{\Sigma_\tau}^\mu + C\tau^2 \int_{\Sigma_\tau \cap \{r \leq r_Y^-\}} J_\mu^N \left(\tilde{\Omega}^k \Phi\right) n_{\Sigma_\tau}^\mu \right).
\end{aligned}$$

However, we note that this cannot be shown directly for the term

$$\int_r^{\tilde{r}} \int_{\mathbb{S}^2(r')} \left(\bar{D} D \tilde{\Omega}^k \Phi\right)^2 (r')^2 dA dr'$$

with the conformal energy because we cannot commute $\Box_{g_K}$ with derivatives of every direction. In order to remedy this, we use the non-degenerate energy for $S\Phi$. In particular, we use the fact that for $r \geq r_Y^-$, $|\bar{D}\Phi| \leq C v^{-1} \left(|S\Phi| + u|D\Phi| + v r^{-1} |D\Phi|\right)$.

$$\begin{aligned}
&\int_r^{\tilde{r}} \int_{\mathbb{S}^2(r')} \left(\bar{D} D \tilde{\Omega}^k \Phi\right)^2 (r')^2 dA dr' \\
\leq & C \int_r^{\tilde{r}} \int_{\mathbb{S}^2(r')} \left((v')^{-2} \left(S D \tilde{\Omega}^k \Phi\right)^2 + (u')^2 (v')^{-2} \left(D^2 \tilde{\Omega}^k \Phi\right) \right. \\
& \qquad \left. + (r')^{-2} \left(D^2 \tilde{\Omega}^k \Phi\right)^2 \right) (r')^2 dA dr' \\
\leq & C \int_r^{\tilde{r}} \int_{\mathbb{S}^2(r')} \left((v')^{-2} \left(D S \tilde{\Omega}^k \Phi\right)^2 + (v')^{-2} \left(D \tilde{\Omega}^k \Phi\right)^2 \right. \\
& \qquad \left. + (u')^2 (v')^{-2} \left(D^2 \tilde{\Omega}^k \Phi\right) + (r')^{-2} \left(D^2 \tilde{\Omega}^k \Phi\right)^2 \right) (r')^2 dA dr'
\end{aligned}$$

Take $r \leq \tilde{r} \leq r+1$. We have, for the first two terms,

$$\begin{aligned}&\int_r^{r+1}\int_{\mathbb{S}^2(r')}(v')^{-2}\left(\left(DS\tilde{\Omega}^k\Phi\right)^2+\left(D\tilde{\Omega}^k\Phi\right)^2\right)(r')^2dAdr'\\ \leq& Cv^{-2}\sum_{j=0}^{1}\int_{\Sigma_\tau}J_\mu^N\left(S^j\tilde{\Omega}^k\Phi\right)n_{\Sigma_\tau}^\mu.\end{aligned}$$

The third term can be estimated by Proposition 10.2.4,

$$\begin{aligned}&\int_r^{r+1}\int_{\mathbb{S}^2(r')}(u')^2(v')^{-2}\left(D^2\tilde{\Omega}^k\Phi\right)(r')^2dAdr'\\ \leq& Cv^{-2}\sum_{j=0}^{1}\left(\int_{\Sigma_\tau}J_\mu^{Z+CN}\left(\partial_{t^*}^j\tilde{\Omega}^k\Phi\right)n_{\Sigma_\tau}^\mu+C\tau^2\int_{\Sigma_\tau\cap\{r\leq r_Y^-\}}J_\mu^N\left(\partial_{t^*}^j\tilde{\Omega}^k\Phi\right)n_{\Sigma_\tau}^\mu\right)\\ &+C\int_{\Sigma_\tau\cap\{u'\sim u\}\cap\{r\geq\frac{\tau}{2}\}}\left(\Box_{g_K}\left(\tilde{\Omega}^k\Phi\right)\right)^2.\end{aligned}$$

The fourth term can be estimated elliptically by Proposition 9.7.1:

$$\begin{aligned}&\int_r^{r+1}\int_{\mathbb{S}^2(r')}(r')^{-2}\left(D^2\tilde{\Omega}^k\Phi\right)^2(r')^2dAdr'\\ \leq& Cr^{-2}\left(\sum_{j=0}^{1}\int_{\Sigma_\tau}J_\mu^N\left(\partial_{t^*}^j\tilde{\Omega}^k\Phi\right)n_{\Sigma_\tau}^\mu+\int_{\Sigma_\tau}\left(\Box_{g_K}\left(\tilde{\Omega}^k\Phi\right)\right)^2\right).\end{aligned}$$

Collecting all the above estimates and noting that $r\geq\frac{\tau}{2}$, we get

$$\begin{aligned}&\sum_{k=0}^{2}\int_r^{r+1}\int_{\mathbb{S}^2(r')}\left(\left(\bar{D}D\tilde{\Omega}^k\Phi\right)^2+\left(\bar{D}\tilde{\Omega}^k\Phi\right)^2+(r')^{-2}\left(D\tilde{\Omega}^k\Phi\right)^2\right)(r')^2dAdr'\\ \leq& Cv^{-2}\sum_{k=0}^{2}\sum_{i+j\leq1}\left(\int_{\Sigma_\tau}J_\mu^N\left(S^i\partial_{t^*}^j\Phi\right)n_{\Sigma_\tau}^\mu+\int_{\Sigma_\tau}J_\mu^{Z+CN}\left(\partial_{t^*}^j\tilde{\Omega}^k\Phi\right)n_{\Sigma_\tau}^\mu\right.\\ &\left.+C\tau^2\int_{\Sigma_\tau\cap\{r\leq r_Y^-\}}J_\mu^N\left(\partial_{t^*}^j\tilde{\Omega}^k\Phi\right)n_{\Sigma_\tau}^\mu+\int_{\Sigma_\tau}\left(\Box_{g_K}\left(\tilde{\Omega}^k\Phi\right)\right)^2\right)\\ &+C\sum_{k=0}^{2}\int_{\Sigma_\tau\cap\{r\geq\frac{\tau}{2}\}}\left(\Box_{g_K}\left(\tilde{\Omega}^k\Phi\right)\right)^2.\end{aligned}\tag{10.2}$$

By pigeonholing on this we also get that for some $\tilde{r}$,

$$\begin{aligned}
&\sum_{k=0}^{2}\left(\int_{\mathbb{S}^2(\tilde{r})}\left(\bar{D}\tilde{\Omega}^k\Phi\right)^2+\tilde{r}^{-2}\left(D\tilde{\Omega}^k\Phi\right)^2\right)\tilde{r}^2dA\\
\leq& Cv^{-2}\sum_{k=0}^{2}\sum_{i+j\leq 1}\left(\int_{\Sigma_\tau}J_\mu^N\left(S^i\partial_{t^*}^j\Phi\right)n_{\Sigma_\tau}^\mu+\int_{\Sigma_\tau}J_\mu^{Z+CN}\left(\partial_{t^*}^j\tilde{\Omega}^k\Phi\right)n_{\Sigma_\tau}^\mu\right.\\
&\qquad\left.+C\tau^2\int_{\Sigma_\tau\cap\{r\leq r_Y^-\}}J_\mu^N\left(\partial_{t^*}^j\tilde{\Omega}^k\Phi\right)n_{\Sigma_\tau}^\mu+\int_{\Sigma_\tau}\left(\Box_{g_K}\left(\tilde{\Omega}^k\Phi\right)\right)^2\right)\\
&+C\sum_{k=0}^{2}\int_{\Sigma_\tau\cap\{r\geq\frac{\tau}{2}\}}\left(\Box_{g_K}\left(\tilde{\Omega}^k\Phi\right)\right)^2.
\end{aligned}\tag{10.3}$$

(10.1), (10.2) and (10.3) together imply the Proposition. □

We now turn to the region $r\leq\frac{t^*}{4}$. We first show a simple Sobolev embedding result.

Proposition 10.2.7. *Suppose $\Box_{g_K}\Phi=G$. For $\ell\geq 1$ and $r\leq\frac{t^*}{4}$,*

$$|D^\ell\Phi|^2\leq C\left(\sum_{j+m\leq\ell+1}\int_{\Sigma_\tau\cap\{r\leq\frac{t^*}{2}\}}J_\mu^N\left(\partial_{t^*}^m\hat{Y}^j\Phi\right)n_{\Sigma_\tau}^\mu+\sum_{j=0}^{\ell}\int_{\Sigma_\tau}(D^jG)^2\right)$$

We can capture better estimates in r if we use an extra derivative.

Proposition 10.2.8. *For $\ell\geq 1$ and $r\leq\frac{t^*}{4}$,*

$$\begin{aligned}
|D^\ell\Phi|^2\leq& Cr^{-2}\left(\sum_{j+m+k\leq\ell+2}\int_{\Sigma_\tau\cap\{r\leq\frac{t^*}{2}\}}J_\mu^N\left(\partial_{t^*}^m\hat{Y}^j\tilde{\Omega}^k\Phi\right)n_{\Sigma_\tau}^\mu\right.\\
&\left.+\sum_{j=0}^{\ell+1-k}\sum_{k=0}^{2}\int_{\Sigma_\tau}\left(D^j\Box_{g_K}\left(\tilde{\Omega}^k\Phi\right)\right)^2\right)
\end{aligned}$$

Proof. We only need to consider the situation when $r\geq R_\Omega+C$. For otherwise, this Proposition is implied by Proposition 10.2.7 since r is finite. We assume from now

on that $r \geq R_\Omega + C$. Following the proof of Proposition 10.2.2, we have

$$\begin{aligned}
&r^2|D^\ell\Phi|^2\\
\leq& C\left(\sum_{k=0}^{2}\int_{\mathbb{S}^2(\tilde{r})}\left(D^\ell\tilde{\Omega}^k\Phi\right)^2\tilde{r}^2dA + \int_{\tilde{r}}^{r}\int_{\mathbb{S}^2(r')}\left(D^{\ell+1}\tilde{\Omega}^k\Phi\right)^2(r')^2\right.\\
&\left.+\left(D^\ell\tilde{\Omega}^k\Phi\right)^2(r')^2dAdr'\right)
\end{aligned}$$

Take $r-1 \leq \tilde{r} \leq r$. By Proposition 9.7.1.2 and 9.7.3,

$$\begin{aligned}
&\sum_{k=0}^{2}\int_{r-1}^{r}\int_{\mathbb{S}^2(r')}\left(D^{\ell+1}\tilde{\Omega}^k\Phi\right)^2(r')^2 + \left(D^\ell\tilde{\Omega}^k\Phi\right)^2(r')^2dAdr'\\
\leq& C\left(\sum_{j+m\leq\ell}\sum_{k=0}^{2}\int_{\Sigma_\tau\cap\{r\leq\frac{t^*}{2}\}}J_\mu^N\left(\partial_{t^*}^m\hat{Y}^j\Omega^k\Phi\right)n_{\Sigma_\tau}^\mu + \sum_{j=0}^{\ell-1}\sum_{k=0}^{2}\int_{\Sigma_\tau}\left(D^j\Box_{g_K}\left(\tilde{\Omega}^k\Phi\right)\right)^2\right).
\end{aligned}$$

By pigeonholing on this we also get that for some $\tilde{r}$ with $r-1 \leq \tilde{r} \leq r$,

$$\begin{aligned}
&\sum_{k=0}^{2}\int_{\mathbb{S}^2(\tilde{r})}\left(D\Omega^k\Phi\right)^2\tilde{r}^2dA\\
\leq& C\left(\sum_{j+m\leq\ell}\sum_{k=0}^{2}\int_{\Sigma_\tau\cap\{r\leq\frac{t^*}{2}\}}J_\mu^N\left(\partial_{t^*}^m\hat{Y}^j\Omega^k\Phi\right)n_{\Sigma_\tau}^\mu + \sum_{j=0}^{\ell-1}\sum_{k=0}^{2}\int_{\Sigma_\tau}\left(D^j\Box_{g_K}\left(\tilde{\Omega}^k\Phi\right)\right)^2\right).
\end{aligned}$$

□

We also have pointwise estimates for Φ instead of $D\Phi$ if we use the conformal energy.

Proposition 10.2.9. *Suppose $\Box_{g_K}\Phi = 0$. For $r \leq \frac{t^*}{4}$,*

$$\begin{aligned}
&|\Phi|^2\\
\leq& C\tau^{-2}\left(\sum_{i+j\leq 2}\int_{\Sigma_\tau}J_\mu^{Z+N,w^Z}\left(\hat{Y}^i\partial_{t^*}^j\Phi\right)n_{\Sigma_\tau}^\mu + C\tau^2\sum_{i+j\leq 2}\int_{\Sigma_\tau\cap\{r\leq r_Y^-\}}J_\mu^N\left(\hat{Y}^i\partial_{t^*}^j\Phi\right)n_{\Sigma_\tau}^\mu\right).
\end{aligned}$$

Proof. By Sobolev Embedding in three dimensions, for $r \leq \frac{t^*}{4}$,

$$|\Phi|^2 \leq C \sum_{k=0}^{2} \int_{\Sigma_\tau \cap \{r \leq \frac{t^*}{4}\}} \left(D^k \Phi\right)^2.$$

Then, using the elliptic estimates in Propositions 9.7.1.2 and 9.7.3, we have

$$|\Phi|^2 \leq C \sum_{i+j \leq 2} \int_{\Sigma_\tau \cap \{r \leq \frac{t^*}{2}\}} \Phi^2 + J_\mu^N \left(\hat{Y}^i \partial_{t^*}^j \Phi\right) n_{\Sigma_\tau}^\mu.$$

Using Proposition 10.1.2, we can conclude the Proposition. □

We proceed to show that the pointwise estimate is better if we use the vector field commutator S. To this end, we first show that we can control a fixed t^* quantity by an integrated quantity. The proof follows ideas in [35], [36] and applies an integration in the direction of S.

Proposition 10.2.10. *For any sufficiently regular Φ, not necessarily satisfying any differential equations, and α_0 a constant,*

$$\begin{aligned}
&\int_{\Sigma_\tau \cap \{r \leq \frac{\tau}{4}\}} r^{\alpha_0 - 2} \Phi^2 \\
\leq & C \tau^{-1} \left(\iint_{\mathcal{R}((1.1)^{-1}\tau, \tau) \cap \{r \leq \frac{t^*}{3}\}} r^{\alpha_0 - 2} \Phi^2 + \iint_{\mathcal{R}((1.1)^{-1}\tau, \tau) \cap \{r \leq \frac{t^*}{3}\}} r^{\alpha_0 - 2} \left(S\Phi\right)^2 \right).
\end{aligned}$$

Proof. To use the estimates for $S\Phi$, we need to integrate along integral curves of S. The following argument imitates that for proving improved decay for the homogeneous equation in [36]. We first find the integral curves by solving the ordinary differential equation

$$\frac{dr_S}{dt_S^*} = \frac{h(r_S)}{t_S^*}$$

where $h(r_S)$ is as in the definition of S. Hence the integral curves are given by

$$\frac{\exp\left(\int_{(r_S)_0}^{r_S} \frac{dr'_S}{h(r'_S)}\right)}{t^*_S} = \text{constant},$$

where $(r_S)_0 > 2M$ can be chosen arbitrarily. Let $\sigma = t^*$, $\rho = \frac{\exp\left(\int_{(r_S)_0}^{r_S} \frac{dr'_S}{h(r'_S)}\right)}{t^*_S}$ and consider (σ, ρ, x^A, x^B) as a new system of coordinates. Notice that

$$\partial_\sigma = \frac{h(r_S)}{t^*_S}\partial_{r_S} + \partial_{t^*_S} = \frac{1}{t^*_S}S.$$

Now for each fixed ρ, we have

$$\Phi^2(\tau) \leq \Phi^2(\tau') + |\int_{\tau'}^{\tau} \frac{1}{\sigma} S(\Phi^2) d\sigma|.$$

Multiplying by ρ^α and integrating along a finite region of ρ, we get:

$$\int_{\rho_1}^{\rho_2} \Phi^2(\tau)\rho^\alpha d\rho \leq \int_{\rho_1}^{\rho_2} \Phi^2(\tau')\rho^\alpha d\rho + \int_{\rho_1}^{\rho_2}\int_{\tau'}^{\tau} |\frac{2\rho^\alpha}{\sigma}\Phi S\Phi| d\sigma d\rho.$$

We choose α so that $\alpha = 0$ for $r \leq r_Y^-$ and $\alpha = \alpha_0$ for $r \geq R$ and smooth depending on r in between. We would like to change coordinates back to $(t^*_S, r_S, x^A_S, x^B_S)$. Notice that since $h(r_S)$ is everywhere positive, (ρ, τ) would correspond to a point with a larger value of r_S than (ρ, τ'). Therefore,

$$\begin{aligned}
&\int_{2M}^{(r_S)_2} \Phi^2(\tau) \frac{\exp\left((1+\alpha)\int_{2M}^{r_S} \frac{dr'_S}{h(r'_S)}\right)}{\tau h(r_S)} dr_S \\
\leq &\int_{2M}^{(r_S)_2} \Phi^2(\tau') \frac{\exp\left((1+\alpha)\int_{(r_S)_0}^{r_S} \frac{dr'_S}{h(r'_S)}\right)}{\tau' h(r_S)} dr_S \\
&+ \int_{\tau'}^{\tau}\int_{2M}^{(r_S)_2} |\frac{2}{\sigma}\Phi S\Phi| \frac{\exp\left((1+\alpha)\int_{(r_S)_0}^{r_S} \frac{dr'_S}{h(r'_S)}\right)}{t^* h(r_S)} dr_S dt^*.
\end{aligned}$$

We have to compare $\frac{\exp\left((1+\alpha(r_S))\int_{(r_S)_0}^{r_S}\frac{dr'_S}{h(r'_S)}\right)}{h(r_S)}$ with the volume form. Very close to the horizon, $h(r_S) = r_S - 2M$ and $\alpha(r) = 0$. Hence

$$\frac{\exp\left((1+\alpha)\int_{(r_S)_0}^{r_S}\frac{dr'_S}{h(r'_S)}\right)}{h(r_S)} = e^{\int_{(r_S)_0}^{r_S}\frac{dr'_S}{h(r'_S)}}\left(\frac{1}{r_S-2M}\right) \sim 1.$$

On the other hand, for $r \geq R$, $h(r_S) = (r_S+2M\log(r_S-2M)-3M-2M\log M)(1-\mu)$ and $\alpha(r_S) = \alpha_0$. In particular, for a sufficiently large choice of R, $h(r_S) \sim r_S$. Hence

$$\frac{\exp\left((1+\alpha)\int_{(r_S)_0}^{r_S}\frac{dr'_S}{h(r'_S)}\right)}{h(r_S)} \sim \frac{\exp\left((1+\alpha)\int_{(r_S)_0}^{r_S}\frac{dr'_S}{h(r'_S)}\right)}{r_S} \sim \left(\frac{r_S^{\alpha_0}}{R}\right) \sim r^{\alpha_0-2}.$$

The corresponding expression on the compact set $[r_Y^-, R]$ is obviously bounded. Hence, since the volume density both on a slice and on a spacetime region is $\sim r^2$, we have

$$\begin{aligned}&\int_{\Sigma_\tau\cap\{r<r_2\}}\frac{\Phi^2(\tau)}{\tau}r^{\alpha_0-2}\\ \leq& C\left(\int_{\Sigma_{\tau'}\cap\{r<r_2\}}\frac{\Phi^2(\tau')}{\tau'}r^{\alpha_0-2}+\iint_{\mathcal{R}(\tau',\tau)\cap\{r<r_2\}}r^{\alpha_0-2}|\frac{2}{(t^*)^2}\Phi S\Phi|\right).\end{aligned}$$

This easily implies the following improved decay for the non-degenerate energy for $\tau' \in [(1.1)^{-1}\tau, \tau]$:

$$\begin{aligned}&\int_{\Sigma_\tau\cap\{r<\frac{\tau}{4}\}}r^{\alpha_0-2}\Phi^2\\ \leq& C\tau^{-1}\left(\int_{\Sigma_{\tau'}\cap\{r<\frac{\tau'}{3}\}}r^{\alpha_0-2}\Phi^2+\iint_{\mathcal{R}((1.1)^{-1}\tau,\tau)\cap\{r<\frac{t^*}{3}\}}r^{\alpha_0-2}\left(S\Phi\right)^2\right).\end{aligned} \tag{10.4}$$

By choosing an appropriate $\tilde{\tau}$, we have

$$\int_{\Sigma_{\tilde{\tau}}\cap\{r<\frac{\tilde{\tau}}{3}\}}r^{\alpha_0-2}\Phi^2 \leq C\tau^{-1}\iint_{\mathcal{R}((1.1)^{-1}\tau,\tau)\cap\{r<\frac{t^*}{3}\}}r^{\alpha_0-2}\Phi^2.$$

Now, apply (10.4) with $\tau' = \tilde{\tau}$, we have

$$\begin{aligned}
&\int_{\Sigma_\tau \cap \{r \leq \frac{\tau}{4}\}} r^{-1-\delta}\Phi^2 \\
\leq & C\tau \left(\int_{\Sigma_{\tilde{\tau}} \cap \{r < \frac{\tilde{\tau}}{3}\}} \frac{\Phi^2}{\tilde{\tau}} r^{\alpha_0 - 2} + \iint_{\mathcal{R}(\tilde{\tau},\tau) \cap \{r < \frac{t^*}{3}\}} r^{\alpha_0-2} |\frac{2}{(t^*)^2}\Phi S\Phi| \right) \\
\leq & C\tau^{-1} \left(\iint_{\mathcal{R}((1.1)^{-1}\tau,\tau) \cap \{r \leq \frac{t^*}{3}\}} r^{\alpha_0-2}\Phi^2 + \iint_{\mathcal{R}((1.1)^{-1}\tau,\tau) \cap \{r \leq \frac{t^*}{3}\}} r^{\alpha_0-2} \left(S\Phi\right)^2 \right),
\end{aligned}$$

using Cauchy-Schwarz for the second term. □

By Sobolev Embedding, this would give an improved decay estimate in t^* in the region $\{r \leq \frac{t^*}{4}\}$. For the application, we also need an improved decay in r, which we get by commuting with the angular momentum $\tilde{\Omega}$.

Proposition 10.2.11. *Suppose $\Box_{g_K}\Phi = G$. For $r \leq \frac{t^*}{4}$ and $\ell \geq 1$, we have*

$$\begin{aligned}
&|D^\ell \Phi|^2 \\
\leq & C(t^*)^{-1} r^{-1+\delta} \sum_{i+j \leq \ell-1} \sum_{k=0}^{2} \iint_{\mathcal{R}((1.1)^{-1}t^*,t^*) \cap \{r \leq \frac{t^*}{2}\}} \Big(K^{X_1} \left(\hat{Y}^i \partial_{t^*}^j \tilde{\Omega}^k \Phi \right) \\
& \qquad + K^{X_1} \left(S\hat{Y}^i \partial_{t^*}^j \tilde{\Omega}^k \Phi \right) \Big) \\
& + C(t^*)^{-1} r^{-1+\delta} \sum_{j=0}^{\ell-1} \sum_{k=0}^{2} \iint_{\mathcal{R}((1.1)^{-1}t^*,t^*) \cap \{r \leq \frac{t^*}{2}\}} r^{-1-\delta} \left(D^j \Box_{g_K} \left(\tilde{\Omega}^k \Phi \right) \right)^2.
\end{aligned}$$

Proof. Using a similar argument as before, except for choosing $\tilde{r} \leq r$, we have

$$\begin{aligned}
& r^{1-\delta} |D^\ell \Phi|^2 \\
\leq & C \sum_{k=0}^{2} \int_{\mathbb{S}^2} \left(\tilde{\Omega}^k D^\ell \Phi \right)^2 r^{1-\delta} dA \\
\leq & C \left(\sum_{k=0}^{2} \int_{\mathbb{S}^2(\tilde{r})} \left(D^\ell \Omega^k \Phi \right)^2 \tilde{r}^{1-\delta} dA \right. \\
& \qquad \left. + \int_{\tilde{r}}^{r} \int_{\mathbb{S}^2(r')} \left(\left(D^{\ell+1} \Omega^k \Phi \right)^2 + \left(D^\ell \Omega^k \Phi \right)^2 \right) (r')^{1-\delta} dA dr' \right).
\end{aligned}$$

Using Proposition 10.2.10, we have

$$\begin{aligned}
&\int_{\tilde{r}}^{r}\int_{\mathbb{S}^2(r')}\left(D^\ell\Omega^k\Phi\right)^2(r')^{1-\delta}dAdr'\\
\leq& C\int_{\Sigma_\tau\cap\{r\leq\frac{\tau}{4}\}} r^{-1-\delta}\left(D^\ell\Omega^k\Phi\right)^2\\
\leq& C\tau^{-1}\left(\iint_{\mathcal{R}((1.1)^{-1}\tau,\tau)\cap\{r\leq\frac{t^*}{3}\}} r^{-1-\delta}\left(D^\ell\Omega^k\Phi\right)^2\right.\\
&\qquad\left.+\iint_{\mathcal{R}((1.1)^{-1}\tau,\tau)\cap\{r\leq\frac{t^*}{3}\}} r^{-1-\delta}\left(SD^\ell\Omega^k\Phi\right)^2\right)
\end{aligned}$$

By first commuting $[D,S]$ and then using Proposition 9.7.1.2 and 9.7.3 on each fixed t^* slice in the integral, we have

$$\begin{aligned}
&\iint_{\mathcal{R}((1.1)^{-1}\tau,\tau)\cap\{r\leq\frac{t^*}{3}\}} r^{-1-\delta}\left(D^\ell\Omega^k\Phi\right)^2+\iint_{\mathcal{R}((1.1)^{-1}\tau,\tau)\cap\{r\leq\frac{t^*}{3}\}} r^{-1-\delta}\left(SD^\ell\Omega^k\Phi\right)^2\\
\leq& C\sum_{i+j\leq\ell-1}\iint_{\mathcal{R}((1.1)^{-1}\tau,\tau)\cap\{r\leq\frac{t^*}{2}\}} r^{-1-\delta}\left(J_\mu^N\left(Y^i\partial_{t^*}^{j-i}\tilde{\Omega}^k\Phi\right)n_{\Sigma_\tau}^\mu\right.\\
&\qquad\left.+J_\mu^N\left(SY^i\partial_{t^*}^{j-i}\tilde{\Omega}^k\Phi\right)n_{\Sigma_\tau}^\mu\right)\\
&+C\sum_{j=0}^{\ell-1}\iint_{\mathcal{R}((1.1)^{-1}\tau,\tau)\cap\{r\leq\frac{t^*}{2}\}} r^{-1-\delta}\left(D^j\Box_{g_K}\left(\tilde{\Omega}^k\Phi\right)\right)^2\\
\leq& C\sum_{i+j\leq\ell-1}\iint_{\mathcal{R}((1.1)^{-1}\tau,\tau)\cap\{r\leq\frac{t^*}{2}\}}\left(K^{X_1}\left(Y^i\partial_{t^*}^{j-i}\tilde{\Omega}^k\Phi\right)+K^{X_1}\left(SY^i\partial_{t^*}^{j-i}\tilde{\Omega}^k\Phi\right)\right)\\
&+C\sum_{j=0}^{\ell-1}\iint_{\mathcal{R}((1.1)^{-1}\tau,\tau)\cap\{r\leq\frac{t^*}{2}\}} r^{-1-\delta}\left(D^j\Box_{g_K}\left(\tilde{\Omega}^k\Phi\right)\right)^2.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
&r^{1-\delta}|D^\ell\Phi|^2\\
\leq& C\tau^{-1}\sum_{i+j\leq\ell-1}\sum_{k=0}^{2}\iint_{\mathcal{R}((1.1)^{-1}\tau,\tau)\cap\{r\leq\frac{t^*}{2}\}}\left(K^{X_1}\left(\hat{Y}^i\partial_{t^*}^j\tilde{\Omega}^k\Phi\right)+K^{X_1}\left(S\hat{Y}^i\partial_{t^*}^j\tilde{\Omega}^k\Phi\right)\right)\\
&+C\tau^{-1}\sum_{j=0}^{\ell-1}\sum_{k=0}^{2}\iint_{\mathcal{R}((1.1)^{-1}\tau,\tau)\cap\{r\leq\frac{t^*}{2}\}} r^{-1-\delta}\left(D^j\Box_{g_K}\left(\tilde{\Omega}^k\Phi\right)\right)^2.
\end{aligned}$$

□

Similar ideas can be used to prove decay of Φ without derivatives, except for a loss in powers of r. This will not be used for the bootstrap argument, but will be used to prove the decay for Φ in the statement of Theorem 1.

Proposition 10.2.12. *Suppose $\Box_{g_K}\Phi = G$. For $r \leq \frac{t^*}{4}$, we have*

$$|\Phi|^2 \leq C(t^*)^{-1} r^{\delta} \sum_{k=0}^{2} \iint_{\mathcal{R}((1.1)^{-1}t^*, t^*) \cap \{r \leq \frac{t^*}{3}\}} \left(K^{X_1}\left(\tilde{\Omega}^k \Phi\right) + K^{X_1}\left(S\tilde{\Omega}^k \Phi\right)\right).$$

Proof. Fix R. Take $\tilde{r} \in [R, \frac{\tau}{5}]$, we have

$$\begin{aligned}
& r^{-\delta}|\Phi|^2 \\
\leq & C \sum_{k=0}^{2} \int_{\mathbb{S}^2} \left(\tilde{\Omega}^k \Phi\right)^2 r^{-\delta} dA \\
\leq & C \left(\sum_{k=0}^{2} \int_{\mathbb{S}^2(\tilde{r})} \left(\Omega^k \Phi\right)^2 \tilde{r}^{-\delta} dA \right. \\
& \left. + \left| \int_{\tilde{r}}^{r} \int_{\mathbb{S}^2(r')} |\Omega^k \Phi D \Omega^k \Phi| (r')^{-\delta} + \left(\Omega^k \Phi\right)^2 (r')^{-1-\delta} dA dr' \right| \right).
\end{aligned}$$

There exists $\tilde{r} \in [R, \frac{\tau}{5}]$ such that

$$\int_{\mathbb{S}^2(\tilde{r})} \left(\Omega^k \Phi\right)^2 \tilde{r}^{-\delta} dA \leq \tau^{-1} \int_{r_+}^{\frac{\tau}{4}} \int_{\mathbb{S}^2(r')} \left(\Omega^k \Phi\right)^2 (r')^{-\delta} dA dr'.$$

Using Proposition 10.2.10, we have

$$\begin{aligned}
&\int_{r_+}^{\frac{\tau}{4}} \int_{\mathbb{S}^2(\tilde{r})} \left(\Omega^k \Phi\right)^2 (r')^{-\delta} dA dr' \\
\leq & C\tau \int_{\Sigma_\tau \cap \{r \leq \frac{\tau}{4}\}} r^{-3-\delta} \left(\Omega^k \Phi\right)^2 \\
\leq & C \left(\iint_{\mathcal{R}((1.1)^{-1}\tau,\tau) \cap \{r \leq \frac{t^*}{3}\}} r^{-3-\delta} \left(\Omega^k \Phi\right)^2 + \iint_{\mathcal{R}((1.1)^{-1}\tau,\tau) \cap \{r \leq \frac{t^*}{3}\}} r^{-3-\delta} \left(S\Omega^k \Phi\right)^2 \right) \\
\leq & C \iint_{\mathcal{R}((1.1)^{-1}\tau,\tau) \cap \{r \leq \frac{t^*}{3}\}} \left(K^{X_1}\left(\tilde{\Omega}^k \Phi\right) + K^{X_1}\left(S\tilde{\Omega}^k \Phi\right)\right).
\end{aligned}$$

Using Proposition 10.2.10, we also have

$$\begin{aligned}
&\left| \int_{\tilde{r}}^{r} \int_{\mathbb{S}^2(r')} |\Omega^k \Phi D \Omega^k \Phi|^2 (r')^{-\delta} + \left(\Omega^k \Phi\right)^2 (r')^{-1-\delta} dA dr' \right| \\
\leq & C \int_{\Sigma_\tau \cap \{r \leq \frac{\tau}{4}\}} r^{-3-\delta} \left(\Omega^k \Phi\right)^2 + \int_{\Sigma_\tau \cap \{r \leq \frac{\tau}{4}\}} r^{-1-\delta} \left(D\Omega^k \Phi\right)^2 \\
\leq & C\tau^{-1} \left(\iint_{\mathcal{R}((1.1)^{-1}\tau,\tau) \cap \{r \leq \frac{t^*}{3}\}} r^{-3-\delta} \left(\Omega^k \Phi\right)^2 + \iint_{\mathcal{R}((1.1)^{-1}\tau,\tau) \cap \{r \leq \frac{t^*}{3}\}} r^{-3-\delta} \left(S\Omega^k \Phi\right)^2 \right) \\
& + C\tau^{-1} \left(\iint_{\mathcal{R}((1.1)^{-1}\tau,\tau) \cap \{r \leq \frac{t^*}{3}\}} r^{-1-\delta} \left(D\Omega^k \Phi\right)^2 \right. \\
& \qquad \left. + \iint_{\mathcal{R}((1.1)^{-1}\tau,\tau) \cap \{r \leq \frac{t^*}{3}\}} r^{-1-\delta} \left(SD\Omega^k \Phi\right)^2 \right) \\
\leq & C\tau^{-1} \iint_{\mathcal{R}((1.1)^{-1}\tau,\tau) \cap \{r \leq \frac{t^*}{3}\}} \left(K^{X_1}\left(\tilde{\Omega}^k \Phi\right) + K^{X_1}\left(S\tilde{\Omega}^k \Phi\right)\right).
\end{aligned}$$

Therefore, we have

$$|\Phi|^2 \leq C\tau^{-1} r^\delta \sum_{k=0}^{2} \iint_{\mathcal{R}((1.1)^{-1}\tau,\tau) \cap \{r \leq \frac{t^*}{2}\}} \left(K^{X_1}\left(\tilde{\Omega}^k \Phi\right) + K^{X_1}\left(S\tilde{\Omega}^k \Phi\right)\right).$$

□

10.3 Bootstrap

Bootstrap assumptions (J): We first introduce the bootstrap assumptions corresponding to energy quantities on a fixed t^* slice.

$$\sum_{i+j=16} A_j^{-1} \int_{\Sigma_\tau} J_\mu^N \left(\partial_{t^*}^i \tilde{\Omega}^j \Phi\right) n_{\Sigma_\tau}^\mu + \sum_{i+k=16} A_Y^{-1} \int_{\Sigma_\tau} J_\mu^N \left(\hat{Y}^k \partial_{t^*}^i \Phi\right) n_{\Sigma_\tau}^\mu \leq \epsilon \tau^{\eta_{16}}. \quad (10.5)$$

$$\begin{aligned} &\sum_{i+j=15} A_j^{-1} \left(\int_{\Sigma_\tau} J_\mu^{Z+N,w^Z} \left(\partial_{t^*}^i \tilde{\Omega}^j \Phi\right) n_{\Sigma_\tau}^\mu + C\tau^2 \int_{\Sigma_\tau \cap \{r \leq \frac{9\tau}{10}\}} J_\mu^N \left(\partial_{t^*}^i \tilde{\Omega}^j \Phi\right) n_{\Sigma_\tau}^\mu \right) \\ &+ \sum_{i+k=15} A_Y^{-1} \tau^2 \int_{\Sigma_\tau \cap \{r \leq r_Y^+\}} J_\mu^N \left(\hat{Y}^k \partial_{t^*}^i \Phi\right) n_{\Sigma_\tau}^\mu \\ \leq &\epsilon \tau^{1+\eta_{15}}. \end{aligned} \quad (10.6)$$

$$\begin{aligned} \sum_{i+j\leq 14} A_j^{-1} \left(\int_{\Sigma_\tau} J_\mu^{Z+N,w^Z} \left(\partial_{t^*}^i \tilde{\Omega}^j \Phi\right) n_{\Sigma_\tau}^\mu + C\tau^2 \int_{\Sigma_\tau \cap \{r \leq \frac{9\tau}{10}\}} J_\mu^N \left(\partial_{t^*}^i \tilde{\Omega}^j \Phi\right) n_{\Sigma_\tau}^\mu \right) \\ + \sum_{i+k\leq 14} A_Y^{-1} \tau^2 \int_{\Sigma_\tau \cap \{r \leq r_Y^+\}} J_\mu^N \left(\hat{Y}^k \partial_{t^*}^i \Phi\right) n_{\Sigma_\tau}^\mu \leq \epsilon \tau^{\eta_{14}}. \end{aligned} \quad (10.7)$$

$$\sum_{i+j\leq 15} A_j^{-1} \int_{\Sigma_\tau} J_\mu^N \left(\partial_{t^*}^i \tilde{\Omega}^j \Phi\right) n_{\Sigma_\tau}^\mu \leq \epsilon. \quad (10.8)$$

$$\sum_{i+j=13} A_{S,j}^{-1} \int_{\Sigma_\tau} J_\mu^N \left(S\partial_{t^*}^i \tilde{\Omega}^j \Phi\right) n_{\Sigma_\tau}^\mu + \sum_{i+k=13} A_{S,Y}^{-1} \int_{\Sigma_\tau} J_\mu^N \left(\hat{Y}^k S\partial_{t^*}^i \Phi\right) n_{\Sigma_\tau}^\mu \leq \epsilon \tau^{\eta_{S,13}}. \quad (10.9)$$

$$\begin{aligned} &\sum_{i+j=12} A_{S,j}^{-1} \left(\int_{\Sigma_\tau} J_\mu^{Z+N,w^Z} \left(S\partial_{t^*}^i \tilde{\Omega}^j \Phi\right) n_{\Sigma_\tau}^\mu + C\tau^2 \int_{\Sigma_\tau \cap \{r \leq \frac{9\tau}{10}\}} J_\mu^N \left(S\partial_{t^*}^i \tilde{\Omega}^j \Phi\right) n_{\Sigma_\tau}^\mu \right) \\ &+ \sum_{i+k=12} A_{S,Y}^{-1} \tau^2 \int_{\Sigma_\tau \cap \{r \leq r_Y^+\}} J_\mu^N \left(\hat{Y}^k S\partial_{t^*}^i \Phi\right) n_{\Sigma_\tau}^\mu \\ \leq &\epsilon \tau^{1+\eta_{S,12}}. \end{aligned} \quad (10.10)$$

$$\begin{aligned}\sum_{i+j\leq 11} A_{S,j}^{-1}\Bigg(&\int_{\Sigma_\tau} J_\mu^{Z+N,w^Z}\left(S\left(\partial_{t^*}^i\tilde{\Omega}^j\Phi\right)\right)n_{\Sigma_\tau}^\mu\\ &+C\tau^2\int_{\Sigma_\tau\cap\{r\leq\frac{9\tau}{10}\}} J_\mu^N\left(S\left(\partial_{t^*}^i\tilde{\Omega}^j\Phi\right)\right)n_{\Sigma_\tau}^\mu\Bigg)\\ +\sum_{i+k\leq 12} A_{S,Y}^{-1}\tau^2&\int_{\Sigma_\tau\cap\{r\leq r_Y^+\}} J_\mu^N\left(\hat{Y}^k S\partial_{t^*}^i\Phi\right)n_{\Sigma_\tau}^\mu\\ \leq&\epsilon\tau^{\eta_{S,11}}\end{aligned}\tag{10.11}$$

$$A_{S,j}^{-1}\sum_{j=0}^{12}\int_{\Sigma_\tau} J_\mu^N\left(S\partial_{t^*}^i\tilde{\Omega}^j\Phi\right)n_{\Sigma_\tau}^\mu\leq\epsilon.\tag{10.12}$$

Bootstrap Assumptions (K): We also need bootstrap assumptions for the energy quantities in a spacetime slab.

$$\sum_{i+j=16} A_{X,j}^{-1}\iint_{\mathcal{R}(\tau_0,\tau)}\left(K^{X_0}\left(\partial_{t^*}^i\tilde{\Omega}^j\Phi\right)+K^N\left(\partial_{t^*}^i\tilde{\Omega}^j\Phi\right)\right)\leq\epsilon\tau^{\eta_{16}}.\tag{10.13}$$

$$\sum_{i+j=15} A_{X,j}^{-1}\iint_{\mathcal{R}(\tau_0,\tau)} K^{X_1}\left(\partial_{t^*}^i\tilde{\Omega}^j\Phi\right)\leq\epsilon\tau^{\eta_{16}}.\tag{10.14}$$

$$\sum_{i+j\leq 15} A_{X,j}^{-1}\iint_{\mathcal{R}(\tau_0,\tau)} K^{X_0}\left(\partial_{t^*}^i\tilde{\Omega}^j\Phi\right)\leq\epsilon.\tag{10.15}$$

$$\sum_{i+j\leq 14} A_{X,j}^{-1}\iint_{\mathcal{R}(\tau_0,\tau)} K^{X_1}\left(\partial_{t^*}^i\tilde{\Omega}^j\Phi\right)\leq\epsilon.\tag{10.16}$$

$$\sum_{i+j\leq 15} A_{X,j}^{-1}\iint_{\mathcal{R}((1.1)^{-1}\tau,\tau)\cap\{r\leq\frac{t^*}{2}\}}\left(K^{X_0}\left(\partial_{t^*}^i\tilde{\Omega}^j\Phi\right)+K^N\left(\partial_{t^*}^i\tilde{\Omega}^j\Phi\right)\right)\leq\epsilon\tau^{-1+\eta_{15}}.\tag{10.17}$$

$$\sum_{i+j\leq 14} A_{X,j}^{-1}\iint_{\mathcal{R}((1.1)^{-1}\tau,\tau)\cap\{r\leq\frac{t^*}{2}\}} K^{X_1}\left(\partial_{t^*}^i\tilde{\Omega}^j\Phi\right)\leq\epsilon\tau^{-1+\eta_{15}}.\tag{10.18}$$

$$\sum_{i+j\leq 14} A_{X,j}^{-1}\iint_{\mathcal{R}((1.1)^{-1}\tau,\tau)\cap\{r\leq\frac{t^*}{2}\}}\left(K^{X_0}\left(\partial_{t^*}^i\tilde{\Omega}^j\Phi\right)+K^N\left(\partial_{t^*}^i\tilde{\Omega}^j\Phi\right)\right)\leq\epsilon\tau^{-2+\eta_{14}}.\tag{10.19}$$

$$\sum_{i+j\leq 13} A_{X,j}^{-1} \iint_{\mathcal{R}((1.1)^{-1}\tau,\tau)\cap\{r\leq\frac{t^*}{2}\}} K^{X_1}\left(\partial_{t^*}^i\tilde{\Omega}^j\Phi\right) \leq \epsilon\tau^{-2+\eta_{14}}. \tag{10.20}$$

$$\sum_{i+j=13} A_{S,X,j}^{-1} \iint_{\mathcal{R}(\tau_0,\tau)} K^{X_0}\left(S\partial_{t^*}^i\tilde{\Omega}^j\Phi\right) \leq \epsilon\tau^{\eta_{S,13}}. \tag{10.21}$$

$$\sum_{i+j\leq 12} A_{S,X,j}^{-1} \iint_{\mathcal{R}(\tau_0,\tau)} K^{X_0}\left(S\partial_{t^*}^i\tilde{\Omega}^j\Phi\right) \leq \epsilon. \tag{10.22}$$

$$\sum_{i+j+k\leq 12} A_{S,X,j}^{-1} \iint_{\mathcal{R}((1.1)^{-1}\tau,\tau)\cap\{r\leq\frac{t^*}{2}\}} K^{X_0}\left(S\partial_{t^*}^i\tilde{\Omega}^j\Phi\right) \leq \epsilon\tau^{-1+\eta_{S,12}}. \tag{10.23}$$

$$\sum_{i+j\leq 11} A_{S,X,j}^{-1} \iint_{\mathcal{R}((1.1)^{-1}\tau,\tau)\cap\{r\leq\frac{t^*}{2}\}} K^{X_1}\left(S\partial_{t^*}^i\tilde{\Omega}^j\Phi\right) \leq \epsilon\tau^{-1+\eta_{S,12}}. \tag{10.24}$$

$$\sum_{i+j\leq 11} A_{S,X,j}^{-1} \iint_{\mathcal{R}((1.1)^{-1}\tau,\tau)\cap\{r\leq\frac{t^*}{2}\}} K^{X_0}\left(S\partial_{t^*}^i\tilde{\Omega}^j\Phi\right) \leq \epsilon\tau^{-2+\eta_{S,11}}. \tag{10.25}$$

$$\sum_{i+j\leq 10} A_{S,X,j}^{-1} \iint_{\mathcal{R}((1.1)^{-1}\tau,\tau)\cap\{r\leq\frac{t^*}{2}\}} K^{X_1}\left(S\partial_{t^*}^i\tilde{\Omega}^j\Phi\right) \leq \epsilon\tau^{-2+\eta_{S,11}}. \tag{10.26}$$

Bootstrap Assumptions (P): We also introduce bootstrap assumptions for the pointwise behavior. For $r \geq \frac{t^*}{4}$,

$$\sum_{j=0}^{13} |\Gamma^j\Phi|^2 \leq BA\epsilon r^{-2}(t^*)^{1+\eta_{14}}. \tag{10.27}$$

$$\sum_{j=0}^{13} |D\Gamma^j\Phi|^2 \leq BA\epsilon. \tag{10.28}$$

$$\sum_{\ell=1}^{13-j}\sum_{j=0}^{12} |D^\ell\Gamma^j\Phi|^2 \leq BA\epsilon r^{-2}. \tag{10.29}$$

$$\sum_{j=0}^{8} |D\Gamma^j\Phi|^2 \leq BA\epsilon r^{-2}(t^*)^{\eta_{14}}(1+|u|)^{-2}. \tag{10.30}$$

$$\sum_{j=0}^{8} |\bar{D}\Gamma^j\Phi|^2 \leq BA\epsilon r^{-2}(t^*)^{-2+\eta_{14}}. \tag{10.31}$$

$$\sum_{j=0}^{6} |S\Gamma^j \Phi|^2 \leq B_S A \epsilon r^{-2} (t^*)^{\eta_{S,11}}. \tag{10.32}$$

$$\sum_{j=0}^{8} |DS\Gamma^j \Phi|^2 \leq B_S A \epsilon r^{-2}. \tag{10.33}$$

$$\sum_{j=0}^{6} |DS\Gamma^j \Phi|^2 \leq B_S A \epsilon r^{-2} (t^*)^{\eta_{S,11}} (1+|u|)^{-2}. \tag{10.34}$$

For $r \leq \frac{t^*}{4}$,

$$\sum_{j=0}^{13} |\Gamma^j \Phi|^2 \leq B A \epsilon (t^*)^{-1+\eta_{14}}. \tag{10.35}$$

$$\sum_{\ell=1}^{14-j} \sum_{j=0}^{13} |D^\ell \Gamma^j \Phi|^2 \leq B A \epsilon (t^*)^{-1+\eta_{14}}. \tag{10.36}$$

$$\sum_{\ell=1}^{13-j} \sum_{j=0}^{12} |D^\ell \Gamma^j \Phi|^2 \leq B A \epsilon (t^*)^{-2+\eta_{14}}. \tag{10.37}$$

$$\sum_{\ell=1}^{9-j} \sum_{j=0}^{8} |D^\ell \Gamma^j \Phi|^2 \leq B A \epsilon (t^*)^{-3+\eta_{S,11}} r^{-1+\delta}. \tag{10.38}$$

$$\sum_{j=0}^{6} |S\Gamma^j \Phi|^2 \leq B_S A \epsilon (t^*)^{-2+\eta_{S,11}}. \tag{10.39}$$

$$\sum_{\ell=1}^{7-j} \sum_{j=0}^{6} |D^\ell S\Gamma^j \Phi|^2 \leq B_S A \epsilon r^{-2} (t^*)^{-2+\eta_{S,11}}. \tag{10.40}$$

Remark 28. *Notice that in general, for most of the bootstrap assumptions on Φ, there is a corresponding one on $S\Phi$. The argument to retrieve these assumptions are quite similar, we only have to estimate the commutator term (in a manner similar to [35], [36]) and track the appropriate constants.*

Remark 29. *Notice that all this assumption are satisfied initially by the assumption of the Theorem.*

Remark 30. *We will bootstrap to improve the constants A_j, $A_{X,j}$, A_S, $A_{S,X,j}$, A_Y, $A_{S,Y}$ and B. The constant B is only used for the bootstrap of the pointwise estimates.*

The constants satisfy

$$1 \ll B \sim B_S \ll A_0 \ll A_{X,0} \ll A_1 \ll ... \ll A_{16}$$
$$\ll A_{X,16} \ll A_Y \ll A_{S,0} \ll A_{S,X,0} \ll ... \ll A_{S,X,13} \ll A_{S,Y}.$$

We will use A as a shorthand to denote the maximum of all these constants, i.e., $A_{S,Y}$. We will always assume by taking ϵ small that

$$A\epsilon \ll 1.$$

Moreover, we set the constants so that

$$\frac{A_{j-1}}{A_j} \ll \frac{A_{X,j-1}}{A_j} \ll \delta' \eta^{-1}.$$

$$\delta \sim \delta' \ll \frac{A_j}{A_{X,j}}.$$

The η's, on the other hand, satisfy

$$\delta \sim \delta' \ll \eta_{16} \ll \eta_{15} \ll \eta_{14} \ll \eta_{S,13} \ll \eta_{S,12} \ll \eta_{S,11}.$$

The η's are chosen so that

$$\frac{A_j}{A_{X,j}} \ll \eta_{14} \ll 1 \quad \textit{for all } j.$$

ϵ will be much smaller than any combinations of the other constants.

We will use energy estimates and decay estimates to eventually close the bootstrap. In order to derive the estimates, we consider equations for $\Gamma^k \Phi$. We now introduce the notations that will facilitate the discussion below.

Definition 10.3.1. *Define* $G_k = \sum_{|j|=k} |\Box_{g_K} \left(\Gamma^j \Phi\right)|$, $U_k = \sum_{|j|=k} |[\Box_{g_K}, \Gamma^j]\Phi|$ *and* $N_k =$

$\sum_{|j|=k} |\Gamma^j (\Box_{g_K} \Phi)|$.

Definition 10.3.2. *Define* $G_{\leq k} = \sum_{|j|\leq k} |\Box_{g_K} \left(\Gamma^j \Phi\right)|$, $U_{\leq k} = \sum_{|j|\leq k} |[\Box_{g_K}, \Gamma^j]\Phi|$ *and* $N_{\leq k} = \sum_{|j|\leq k} |\Gamma^j (\Box_{g_K} \Phi)|$.

In order to keep track of the constants, we define also

Definition 10.3.3. *Define* $U_{k,j} = |[\Box_{g_K}, \partial_{t^*}^{k-j} \tilde{\Omega}^j]\Phi|$ *and* $U_{\leq k, \leq j} = \sum_{j' \leq j, k' \leq k} U_{k',j'}$.

Remark 31. *We will refer to* G *as the inhomogeneous term,* U *as the commutator term and* N *as the nonlinear term. Clearly we have* $G_k \leq U_k + N_k$ *and* $G_{\leq k} \leq U_{\leq k} + N_{\leq k}$

We now estimate the inhomogeneous terms that will appear in the analysis several times below. It is necessary to study the commutator terms and the nonlinear terms together because the estimates for each depend on the estimates for the other when we use elliptic estimates.

Proposition 10.3.4. U_k *satisfies the following estimates:*

$$\begin{aligned} &\int_{\Sigma_\tau} r^\alpha \left(D^\ell U_{k,j}\right)^2 \\ \leq & C \left(\sum_{m=0}^{k+\ell-i} \sum_{i=0}^{j-1} \int_{\Sigma_\tau \cap \{r \geq R_\Omega - 1\}} r^{\alpha-4} J_\mu^N \left(\partial_{t^*}^m \tilde{\Omega}^i \Phi\right) n_{\Sigma_\tau}^\mu + \sum_{m=0}^{k+\ell-i-1} \sum_{i=0}^{j-1} \int_{\Sigma_\tau} r^{\alpha-4} \left(D^m N_i\right)^2 \right) \end{aligned} \tag{10.41}$$

for $\alpha \leq 4$, *where it is understood that* $\sum_{i=0}^{-1} = 0$.

Proof. The commutator terms are estimated in [35]. Notice that since $\tilde{\Omega}$ is supported away from the trapped set, there is no loss of derivatives in using the integrated decay estimate. We have that $U_{k,j}$ supported in $\{r \geq R_\Omega\}$ and that

$$|U_{k,j}| \leq C \sum_{i=0}^{j} |\partial_{t^*}^{k-j} [\Box_{g_K}, \tilde{\Omega}^i]\Phi| \leq C \sum_{i=0}^{j-1} r^{-2} \left(|D^2 \partial_{t^*}^{k-j} \tilde{\Omega}^i \Phi| + |D \partial_{t^*}^{k-j} \tilde{\Omega}^i \Phi| \right),$$

and therefore

$$|D^\ell U_{k,j}| \leq C \sum_{m=1}^{\ell+2} \sum_{i=0}^{j-1} r^{-2} |D^m \partial_{t^*}^{k-j} \tilde{\Omega}^i \Phi| \leq C \sum_{m=1}^{k+\ell-j+2} \sum_{i=0}^{j-1} r^{-2} |D^m \tilde{\Omega}^i \Phi|,$$

where, as in the statement of the Proposition it is understood that the sum vanishes if $j = 0$. Hence, using the elliptic estimate for $\{r \geq R_\Omega\}$, i.e., Proposition 9.7.2,

$$\begin{aligned}
&\int_{\Sigma_\tau} r^\alpha \left(D^\ell U_{k,j}\right)^2 \\
\leq & C \sum_{m=1}^{k+\ell-j+2} \sum_{i=0}^{j-1} \int_{\Sigma_\tau \cap \{r \geq R_\Omega\}} r^{\alpha-4} \left(D^m \tilde{\Omega}^i \Phi\right)^2 \\
\leq & C \sum_{m=0}^{k+\ell-i} \sum_{i=0}^{j-1} \int_{\Sigma_\tau \cap \{r \geq R_\Omega - 1\}} r^{\alpha-4} J_\mu^N \left(\partial_{t^*}^m \tilde{\Omega}^i \Phi\right) n_{\Sigma_\tau}^\mu \\
& + C \sum_{m=0}^{k+\ell-i-1} \sum_{i=0}^{j-1} \int_{\Sigma_\tau \cap \{r \geq R_\Omega - 1\}} r^{\alpha-4} \left((D^m U_{i,i})^2 + (D^m N_i)^2\right).
\end{aligned} \tag{10.42}$$

Now we can estimate U_k by induction: Fix any k and we will induct on j. By definition, $U_{k,0} = 0$. Now, assume that for all $k + \ell \leq 16$ and for some $j_0 \geq 1$, we have

$$\begin{aligned}
&\sum_{k+\ell \leq M, j \leq min\{j_0-1, k\}} \int_{\Sigma_\tau} r^\alpha \left(D^\ell U_{k,j}\right)^2 \\
\leq & C \left(\sum_{m=0}^{M-i} \sum_{i=0}^{j_0-2} \int_{\Sigma_\tau \cap \{r \geq R_\Omega - 1\}} r^{\alpha-4} J_\mu^N \left(\partial_{t^*}^m \tilde{\Omega}^i \Phi\right) n_{\Sigma_\tau}^\mu + \sum_{m=0}^{M-1-i} \sum_{i=0}^{j_0-1} \int_{\Sigma_\tau} r^{\alpha-4} \left(D^m N_i\right)^2 \right).
\end{aligned}$$

for all $\alpha \leq 4$. Then, using (10.42), we have that for $k + \ell \leq 16$, and $j_0 \leq k$,

$$\begin{aligned}
&\int_{\Sigma_\tau} r^\alpha \left(D^\ell U_{k,j_0}\right)^2 \\
\leq & C \left(\sum_{m=0}^{k+\ell-i} \sum_{i=0}^{j_0-1} \int_{\Sigma_\tau \cap \{r \geq R_\Omega - 1\}} r^{\alpha-4} J_\mu^N \left(\partial_{t^*}^m \tilde{\Omega}^i \Phi\right) n_{\Sigma_\tau}^\mu + \sum_{m=0}^{k+\ell-i-1} \sum_{i=0}^{j_0-1} \int_{\Sigma_\tau} r^{\alpha-4} \left(D^m N_i\right)^2 \right).
\end{aligned}$$

Hence, (10.41) holds. □

We now estimate the nonlinear term N_k. Since N_k is at least quadratic, we do not need to be precise about the constants A and we will always estimate with the maximum A.

Proposition 10.3.5. *N_k satisfies the following estimates for fixed $t^* = \tau$:*

$$\sum_{k+\ell=16} \int_{\Sigma_\tau} \left(D^\ell N_k\right)^2 \leq CBA^2\epsilon^2\tau^{-2+\eta_{16}},$$

$$\sum_{k+\ell\leq 15} \int_{\Sigma_\tau} \left(D^\ell N_k\right)^2 \leq CBA^2\epsilon^2\tau^{-2}.$$

N_k also satisfy the following estimates for when integrated over $t^* \in [(1.1)^{-1}\tau, \tau]$:

$$\sum_{k+\ell=15} \iint_{\mathcal{R}((1.1)^{-1}\tau,\tau)} r^{-1-\delta} \left(D^\ell N_k\right)^2 \leq CBA^2\epsilon^2\tau^{-2+\eta_{16}},$$

$$\sum_{k+\ell\leq 14} \iint_{\mathcal{R}((1.1)^{-1}\tau,\tau)} r^{-1-\delta} \left(D^\ell N_k\right)^2 \leq CBA^2\epsilon^2\tau^{-2}.$$

Proof. Here, there is no need to distinguish between the good and bad derivatives.

$$|D^\ell N_k| \leq |D^\ell\Gamma^k \left(\Lambda_i D\Phi D\Phi\right)| + |\Gamma^k\mathcal{C}|.$$

We claim that the most important terms will be those that are quadratic in $D^{j+1}\Gamma^i\Phi$ or cubic of the form

$$\left(D^{j_1+1}\Gamma^{i_1}\Phi\right)\left(D^{j_2+1}\Gamma^{i_2}\Phi\right)\left(\Gamma^{i_3}\Phi\right)$$

with $i_1 + j_1, i_2 + j_2 \leq 8$. For by the assumptions every term has the form

$$\left(D^{j_1}\Gamma^{i_1}\Phi\right)\left(D^{j_2}\Gamma^{i_2}\Phi\right)\left(D^{j_3}\Gamma^{i_3}\Phi\right)\left(D^{j_4}\Gamma^{i_4}\Phi\right)...\left(D^{j_r}\Gamma^{i_r}\Phi\right),$$

with $r \geq 2$, at least two j's ≥ 1 and $i + j \leq 9$ for all but at most one factor. If all factors $i + j \leq 9$ or the factor that $i + j > 9$ has $i \geq 1$, we can reduce to the case

$D^{j_1+1}\Gamma^{i_1}\Phi D^{j_2+1}\Gamma^{i_2}\Phi$ by putting all other factors in L^∞ using bootstrap assumptions (10.27), (10.28), (10.35) and (10.36). If the factor that $i+j>9$ has $i=0$ we reduce to

$$\left(D^{j_1+1}\Gamma^{i_1}\Phi\right)\left(D^{j_2+1}\Gamma^{i_2}\Phi\right)\left(\Gamma^{i_3}\Phi\right)$$

again by putting all other factors in L^∞ using bootstrap assumptions (10.27), (10.28), (10.35) and (10.36).

$$\begin{aligned}
&\int_{\Sigma_\tau}\left(D^\ell N_k\right)^2\\
\leq& C\left(\sup\sum_{i_1+j_1\leq 8}|D^{j_1+1}\Gamma^{i_1}\Phi|^2\right)\sum_{j_2=0}^{\ell}\sum_{i_2=0}^{k}\int_{\Sigma_\tau}|D^{j_2+1}\Gamma^{i_2}\Phi|^2\\
&+C\left(\sup\sum_{i_1+j_1\leq 8}|D^{j_1+1}\Gamma^{i_1}\Phi|^2\right)\left(\sup\sum_{i_2+j_2\leq 8}r^2|D^{j_2+1}\Gamma^{i_2}\Phi|^2\right)\sum_{i_3=0}^{k}\int_{\Sigma_\tau}r^{-2}|\Gamma^{i_3}\Phi|^2\\
\leq& CBA\epsilon\tau^{-2}\sum_{i=1}^{\ell+1}\sum_{j=0}^{k}\int_{\Sigma_\tau}\left(D^i\Gamma^j\Phi\right)^2
\end{aligned}$$

using Hardy's inequality Proposition 9.4.10

$$\begin{aligned}
\leq& CBA\epsilon\tau^{-2}\sum_{i+m=0}^{\ell}\sum_{j=0}^{k}\int_{\Sigma_\tau}J_\mu^N\left(\partial_{t^*}^m\Gamma^j\hat{Y}^i\Phi\right)n_{\Sigma_\tau}^\mu\\
&+CBA\epsilon\tau^{-2}\sum_{i+j\leq k+\ell-1}\int_{\Sigma_\tau}\left(\left(D^iU_{\leq j}\right)^2+\left(D^iN_{\leq j}\right)^2\right)
\end{aligned}$$

using the elliptic estimates in Propositions 9.7.1, 9.7.3

$$\leq CBA\epsilon\tau^{-2}\sum_{i=0}^{\ell}\sum_{j=0}^{k}\int_{\Sigma_\tau}J_\mu^N\left(\partial_{t^*}^m\Gamma^j\hat{Y}^i\Phi\right)n_{\Sigma_\tau}^\mu+CBA\epsilon\tau^{-2}\sum_{i+j\leq k+\ell-1}\int_{\Sigma_\tau}\left(D^iN_{\leq j}\right)^2,$$

where we have used Proposition 10.3.4 in the last step. Now, a simple induction would show that

$$\sum_{k+\ell=16}\int_{\Sigma_\tau}\left(D^\ell N_k\right)^2\leq CBA^2\epsilon^2\tau^{-2+\eta},\quad\text{and}$$

$$\sum_{k+\ell\leq 15}\int_{\Sigma_\tau}\left(D^\ell N_k\right)^2\leq CBA^2\epsilon^2\tau^{-2}.$$

We now move on the the terms integrated over $t^*\in[(1.1)^{-1}\tau,\tau]$. Arguing as before, and noticing that the elliptic estimate in Proposition 9.7.1 also allows weights in r, we have

$$\begin{aligned}
&\iint_{\mathcal{R}((1.1)^{-1}\tau,\tau)} r^{-1-\delta}\left(D^\ell N_k\right)^2\\
\leq& CBA\epsilon\tau^{-2}\sum_{i+m=0}^{\ell}\sum_{j=0}^{k}\iint_{\mathcal{R}((1.1)^{-1}\tau,\tau)} r^{-1-\delta}J_\mu^N\left(\partial_{t^*}^m\Gamma^j\hat{Y}^i\Phi\right)n_{\Sigma_\tau}^\mu\\
&+CBA\epsilon\tau^{-2}\sum_{i+j\leq k+\ell-1}\iint_{\mathcal{R}((1.1)^{-1}\tau,\tau)} r^{-1-\delta}\left(\left(D^iU_{\leq j}\right)^2+\left(D^iN_{\leq j}\right)^2\right)\\
\leq& CBA\epsilon\tau^{-2}\sum_{i=0}^{\ell}\sum_{j=0}^{k}\iint_{\mathcal{R}((1.1)^{-1}\tau,\tau)} r^{-1-\delta}J_\mu^N\left(\partial_{t^*}^m\Gamma^j\hat{Y}^i\Phi\right)n_{\Sigma_\tau}^\mu\\
&+CBA\epsilon\tau^{-2}\sum_{i+j\leq k+\ell-1}\iint_{\mathcal{R}((1.1)^{-1}\tau,\tau)} r^{-1-\delta}\left(D^iN_{\leq j}\right)^2\\
\leq& CBA\epsilon\tau^{-2}\sum_{i=0}^{\ell}\sum_{j=0}^{k}\iint_{\mathcal{R}((1.1)^{-1}\tau,\tau)} K^{X_1}\left(\partial_{t^*}^m\Gamma^j\hat{Y}^i\Phi\right)\\
&+CBA\epsilon\tau^{-2}\sum_{i+j\leq k+\ell-1}\iint_{\mathcal{R}((1.1)^{-1}\tau,\tau)} r^{-1-\delta}\left(D^iN_{\leq j}\right)^2
\end{aligned}$$

Now, the bootstrap assumptions (10.14), (10.16) and an induction on $k+\ell$ would conclude the Proposition. □

Now, the estimates for N_k will also give improved estimates for U_k via Proposition 10.3.4:

Proposition 10.3.6. *The following estimates for U_k on a fixed t^* slice hold for $\alpha\leq 2$:*

$$\sum_{k+\ell=16}\int_{\Sigma_\tau} r^\alpha\left(D^\ell U_{k,j}\right)^2\leq CA_{j-1}\epsilon\tau^{\eta_{16}},$$

$$\sum_{k+\ell=15}\int_{\Sigma_\tau} r^\alpha\left(D^\ell U_{k,j}\right)^2\leq CA_{j-1}\epsilon\tau^{-1+\eta_{15}},$$

$$\sum_{k+\ell\leq 14}\int_{\Sigma_\tau} r^\alpha \left(D^\ell U_{k,j}\right)^2 \leq CA_{j-1}\epsilon\tau^{-2+\eta_{14}}.$$

The following estimates for U_k integrated on $[(1.1)^{-1}\tau,\tau]$ also hold for $\alpha\leq 1+\delta$:

$$\sum_{k+\ell=16}\iint_{\mathcal{R}((1.1)^{-1}\tau,\tau)} r^\alpha \left(D^\ell U_{k,j}\right)^2 \leq CA_{X,j-1}\epsilon\tau^{\eta_{16}},$$

$$\sum_{k+\ell=15}\iint_{\mathcal{R}((1.1)^{-1}\tau,\tau)} r^\alpha \left(D^\ell U_{k,j}\right)^2 \leq CA_{X,j-1}\epsilon\tau^{-1+\eta_{15}},$$

$$\sum_{k+\ell\leq 14}\iint_{\mathcal{R}((1.1)^{-1}\tau,\tau)} r^\alpha \left(D^\ell U_{k,j}\right)^2 \leq CA_{X,j-1}\epsilon\tau^{-2+\eta_{14}}.$$

Proof. We first prove the Proposition for the estimates for the constant in τ terms. By Proposition 10.3.4,

$$\begin{aligned}&\int_{\Sigma_\tau} r^\alpha \left(D^\ell U_{k,j}\right)^2\\ \leq& C\left(\sum_{m=0}^{k+\ell-i}\sum_{i=0}^{j-1}\int_{\Sigma_\tau\cap\{r\geq R_\Omega-1\}} r^{\alpha-4}J_\mu^N\left(\partial_{t^*}^m\tilde{\Omega}^i\Phi\right)n_{\Sigma_\tau}^\mu + \sum_{m=0}^{k+\ell-1-i}\sum_{i=0}^{j-1}\int_{\Sigma_\tau} r^{\alpha-4}\left(D^m N_i\right)^2\right)\end{aligned}$$

for $\alpha\leq 4$. The second term satisfy the required estimate by Proposition 10.3.5. We estimate the first term. By (10.5),

$$\sum_{m=0}^{16-i}\sum_{i=0}^{j-1}\int_{\Sigma_\tau\cap\{r\geq R_\Omega-1\}} r^{\alpha-4}J_\mu^N\left(\partial_{t^*}^m\tilde{\Omega}^i\Phi\right)n_{\Sigma_\tau}^\mu \leq CA_{j-1}\epsilon\tau^{\eta_{16}}.$$

By (10.6) and Proposition 10.1.2,

$$\begin{aligned}&\sum_{m=0}^{15-i}\sum_{i=0}^{j-1}\int_{\Sigma_\tau\cap\{r\geq R_\Omega-1\}} r^{\alpha-4}J_\mu^N\left(\partial_{t^*}^m\tilde{\Omega}^i\Phi\right)n_{\Sigma_\tau}^\mu\\ \leq& C\left(\sum_{m=0}^{15-i}\sum_{i=0}^{j-1}\int_{\Sigma_\tau\cap\{r\leq\frac{\tau}{2}\}} J_\mu^N\left(\partial_{t^*}^m\tilde{\Omega}^i\Phi\right)n_{\Sigma_\tau}^\mu + \sum_{m=0}^{15-i}\sum_{i=0}^{j-1}\tau^{-2}\int_{\Sigma_\tau\cap\{r\geq\frac{\tau}{2}\}} J_\mu^N\left(\partial_{t^*}^m\tilde{\Omega}^i\Phi\right)n_{\Sigma_\tau}^\mu\right)\\ \leq& CA_{j-1}\epsilon\tau^{-1+\eta_{15}}.\end{aligned}$$

By (10.7) and Proposition 10.1.2,

$$\begin{aligned}
&\sum_{m=0}^{14-i}\sum_{i=0}^{j-1}\int_{\Sigma_\tau\cap\{r\geq R_\Omega-1\}} r^{\alpha-4}J_\mu^N\left(\partial_{t^*}^m\tilde{\Omega}^i\Phi\right)n_{\Sigma_\tau}^\mu\\
\leq& C\left(\sum_{m=0}^{14-i}\sum_{i=0}^{j-1}\int_{\Sigma_\tau\cap\{r\leq\frac{\tau}{2}\}} J_\mu^N\left(\partial_{t^*}^m\tilde{\Omega}^i\Phi\right)n_{\Sigma_\tau}^\mu+\sum_{m=0}^{14-i}\sum_{i=0}^{j-1}\tau^{-2}\int_{\Sigma_\tau\cap\{r\geq\frac{\tau}{2}\}} J_\mu^N\left(\partial_{t^*}^m\tilde{\Omega}^i\Phi\right)n_{\Sigma_\tau}^\mu\right)\\
\leq& CA_{j-1}\epsilon\tau^{-2+\eta_{14}}.
\end{aligned}$$

For the integrated terms, we similarly have, by Proposition 10.3.4,

$$\begin{aligned}
&\iint_{\mathcal{R}((1.1)^{-1}\tau,\tau)} r^\alpha\left(D^\ell U_{k,j}\right)^2\\
\leq& C\left(\sum_{m=0}^{k+\ell-i}\sum_{i=0}^{j-1}\iint_{\mathcal{R}((1.1)^{-1}\tau,\tau)\cap\{r\geq R_\Omega-1\}} r^{\alpha-4}J_\mu^N\left(\partial_{t^*}^m\tilde{\Omega}^i\Phi\right)n_{\Sigma_\tau}^\mu\right.\\
&\left.+\sum_{m=0}^{k+\ell-1-i}\sum_{i=0}^{j-1}\iint_{\mathcal{R}((1.1)^{-1}\tau,\tau)} r^{\alpha-4}\left(D^m N_i\right)^2\right)\quad\text{for }\alpha\leq 4.
\end{aligned}$$

The second term can be estimated by Proposition 10.3.5. Notice that $r^{-1+\delta}J_\mu^N(\Phi)\,n_{\Sigma_\tau}^\mu\leq K^{X_0}(\Phi)$. Hence, following the argument above for the fixed τ case, we would have proved the Proposition for the case $\alpha\leq 1-\delta$. Nevertheless, with more care, we can improve to $\alpha\leq 1+\delta$.

$$\begin{aligned}
&\sum_{m=0}^{k+\ell-i}\sum_{i=0}^{j-1}\iint_{\mathcal{R}((1.1)^{-1}\tau,\tau)\cap\{r\geq R_\Omega\}} r^{\alpha-4}J_\mu^N\left(\partial_{t^*}^m\tilde{\Omega}^i\Phi\right)n_{\Sigma_{t^*}}^\mu\\
\leq& C\sum_{m=0}^{k+\ell-i}\sum_{i=0}^{j-1}\iint_{\mathcal{R}((1.1)^{-1}\tau,\tau)\cap\{r\leq\frac{t^*}{2}\}} K^{X_0}\left(\partial_{t^*}^m\tilde{\Omega}^i\Phi\right)\\
&+C\sum_{m=0}^{k+\ell-i}\sum_{i=0}^{j-1}\iint_{\mathcal{R}((1.1)^{-1}\tau,\tau)\cap\{r\geq\frac{t^*}{2}\}} r^{-2+2\delta}K^{X_0}\left(\partial_{t^*}^m\tilde{\Omega}^i\Phi\right).
\end{aligned}$$

For $k+\ell=16$, this is bounded by $CA_{X,j-1}\epsilon\tau^{\eta_{16}}$ by (10.13). For $k+\ell=15$, this is bounded by $CA_{X,j-1}\epsilon\tau^{-1+\eta_{15}}$ by (10.17) and (10.15). For $k+\ell\leq 14$, this is bounded by $CA_{X,j-1}\epsilon\tau^{-2+\eta_{14}}$ by (10.19) and (10.15) since $2\delta\leq\eta_{14}$. □

While the above is sufficient to recover the bootstrap assumptions for the pointwise bounds, we will need improvements to achieve the energy bounds. For the improvements, we study separately the region $r \leq \frac{t^*}{4}$, $\frac{t^*}{4} \leq r \leq \frac{9t^*}{10}$ and $r \geq \frac{9t^*}{10}$. For the region $r \geq \frac{t^*}{4}$, we will only show the improvement for N_k instead of the derivatives of N_k. Various complications would arise in estimating the derivatives of N_k. For the region $r \leq \frac{t^*}{4}$, however, we will estimate also the derivatives of N_k as they will be necessary to estimate the error terms arising from commuting with the red-shift vector field.

Proposition 10.3.7.

$$\sum_{\ell+k=16} \int_{\Sigma_\tau \cap \{r \leq \frac{\tau}{4}\}} r^{1-\delta} \left(D^\ell N_k\right)^2 \leq CBA^2\epsilon^2\tau^{-3+\eta_{S,11}+\eta_{16}}.$$

$$\sum_{\ell+k=15} \int_{\Sigma_\tau \cap \{r \leq \frac{\tau}{4}\}} r^{1-\delta} \left(D^\ell N_k\right)^2 \leq CBA^2\epsilon^2\tau^{-4+\eta_{S,11}+\eta_{15}}.$$

$$\sum_{\ell+k\leq 14} \int_{\Sigma_\tau \cap \{r \leq \frac{\tau}{4}\}} r^{1-\delta} \left(D^\ell N_k\right)^2 \leq CBA^2\epsilon^2\tau^{-5+\eta_{S,11}+\eta_{14}}.$$

Proof. As before, we only have to estimate terms quadratic in $D^j\Gamma^i\Phi$ with $j \geq 1$ or cubic of the form

$$\left(D^{j_1+1}\Gamma^{i_1}\Phi\right)\left(D^{j_2+1}\Gamma^{i_2}\Phi\right)\left(\Gamma^{i_3}\Phi\right)$$

with $i_1 + j_1, i_2 + j_2 \leq 8$.

$$\begin{aligned}
&\int_{\Sigma_\tau} r^{1-\delta} \left(D^\ell N_k\right)^2 \\
\leq & C \left(\sup_{r \leq \frac{\tau}{4}} \sum_{i_1 + j_1 \leq 8} r^{1-\delta} |D^{j_1+1} \Gamma^{i_1} \Phi|^2 \right) \sum_{j_2=0}^{\ell} \sum_{i_2=0}^{k} \int_{\Sigma_\tau \cap \{r \leq \frac{\tau}{4}\}} |D^{j_2+1} \Gamma^{i_2} \Phi|^2 \\
& + C \left(\sup_{r \leq \frac{\tau}{4}} \sum_{i_1 + j_1 \leq 8} r^{1-\delta} |D^{j_1+1} \Gamma^{i_1} \Phi|^2 \right)^2 \sum_{i_3=0}^{k} \tau^{2\delta} \int_{\Sigma_\tau \cap \{r \leq \frac{\tau}{4}\}} r^{-2} |\Gamma^{i_3} \Phi|^2 \\
\leq & CBA\epsilon \tau^{-3+\eta_{S,11}} \sum_{i=1}^{\ell+1} \sum_{j=0}^{k} \int_{\Sigma_\tau \cap \{r \leq \frac{\tau}{4}\}} \left(D^i \Gamma^j \Phi\right)^2 + CBA^2 \epsilon^2 \tau^{-6+2\eta_{S,11}+2\delta} \sum_{i_3=0}^{k} \int_{\Sigma_\tau} \left(D \Gamma^{i_3} \Phi\right)^2
\end{aligned}$$

by Hardy's inequality. Notice now that the second term has more decay than we need, so we will drop it from now on.

$$\begin{aligned}
\leq & CBA\epsilon \tau^{-3+\eta_{S,11}} \sum_{i+m=0}^{\ell} \sum_{j=0}^{k} \int_{\Sigma_\tau \cap \{r \leq \frac{\tau}{2}\}} J_\mu^N \left(\partial_{t^*}^m \Gamma^j \hat{Y}^i \Phi \right) n_{\Sigma_\tau}^\mu \\
& + CBA\epsilon \tau^{-3+\eta_{S,11}} \sum_{i=0}^{\ell-1} \sum_{j=0}^{k} \int_{\Sigma_\tau} \left(\left(D^i U_j\right)^2 + \left(D^i N_j\right)^2 \right) \\
\leq & CBA\epsilon \tau^{-3+\eta_{S,11}} \sum_{i+j \leq k+\ell} \int_{\Sigma_\tau} J_\mu^N \left(\Gamma^j \hat{Y}^i \Phi \right) n_{\Sigma_\tau}^\mu + CBA\epsilon \tau^{-3+\eta_{S,11}} \sum_{i+j \leq k+\ell-1} \int_{\Sigma_\tau} \left(D^i N_j\right)^2,
\end{aligned} \tag{10.43}$$

The Proposition would follow from an induction on $k + \ell$ and the bootstrap assumptions (10.5), (10.6), (10.7). The $k+\ell = 0$ case also follows from the above computation as we have adopted the notation that $\sum_{i+j \leq -1} = 0$. □

We now move to the region $\{\frac{t^*}{4} \leq r \leq \frac{9t^*}{10}\}$. In this region, $u \sim t^*$, and therefore we can exploit the decay in the variable u given by the estimates from the conformal energy.

Proposition 10.3.8.

$$\int_{\Sigma_\tau \cap \{\frac{\tau}{4} \leq r \leq \frac{9\tau}{10}\}} N_{16}^2 \leq CBA^2 \epsilon^2 \tau^{-4+\eta_{14}+\eta_{16}},$$

$$\int_{\Sigma_\tau \cap \{\frac{\tau}{4} \leq r \leq \frac{9\tau}{10}\}} N_{15}^2 \leq CBA^2\epsilon^2\tau^{-5+\eta_{14}+\eta 15},$$

and

$$\sum_{j=0}^{14} \int_{\Sigma_\tau \cap \{\frac{\tau}{4} \leq r \leq \frac{9\tau}{10}\}} N_j^2 \leq CBA^2\epsilon^2\tau^{-6+2\eta_{14}}.$$

Proof. Arguing as before, we see that the main terms for the nonlinearity are those that are quadratic in $D\Phi$ or those that are cubic with the form $\Gamma^{i_3}\Phi D\Gamma^{i_1}\Phi D\Gamma^{i_2}\Phi$ with $i_1, i_2 \leq 8$. The quadratic terms can be estimated:

$$\begin{aligned}
&\sum_{i_1=0}^{\lfloor \frac{k}{2} \rfloor} \sum_{i_2=0}^{k} \int_{\Sigma_\tau \cap \{\frac{\tau}{4} \leq r \leq \frac{9\tau}{10}\}} |D\Gamma^{i_1}\Phi D\Gamma^{i_2}\Phi|^2 \\
\leq & C \left(\sum_{i_1=0}^{8} \sup_{\frac{\tau}{4} \leq r \leq \frac{9\tau}{10}} |D\Gamma^{i_1}\Phi|^2 \right) \sum_{i_2=0}^{k} \int_{\Sigma_\tau \cap \{\frac{\tau}{4} \leq r \leq \frac{9\tau}{10}\}} |D\Gamma^{i_2}\Phi|^2 \\
\leq & CAB\epsilon\tau^{-4+\eta_{14}} \sum_{i=0}^{k} \int_{\Sigma_\tau \cap \{\frac{\tau}{4} \leq r \leq \frac{9\tau}{10}\}} |D\Gamma^{i}\Phi|^2.
\end{aligned}$$

The particular cubic term can be estimated as follows:

$$\begin{aligned}
&\sum_{i_1,i_2=0}^{\lfloor \frac{k}{2} \rfloor} \sum_{i_3=0}^{k} \int_{\Sigma_\tau \cap \{\frac{\tau}{4} \leq r \leq \frac{9\tau}{10}\}} |\Gamma^{i_3}\Phi D\Gamma^{i_1}\Phi D\Gamma^{i_2}\Phi|^2 \\
\leq & C \left(\sum_{i_1=0}^{8} \sup_{\frac{\tau}{4} \leq r \leq \frac{9\tau}{10}} |D\Gamma^{i_1}\Phi|^2 \right)^2 \sum_{i_3=0}^{k} \int_{\Sigma_\tau \cap \{\frac{\tau}{4} \leq r \leq \frac{9\tau}{10}\}} |\Gamma^{i_3}\Phi|^2 \\
\leq & CA^2B^2\epsilon^2\tau^{-8+2\eta_{14}} \sum_{i=0}^{k} \int_{\Sigma_\tau \cap \{\frac{\tau}{4} \leq r \leq \frac{9\tau}{10}\}} |\Gamma^{i}\Phi|^2.
\end{aligned}$$

In principle, for $k \leq 15$, we can then control the last term using the conformal energy. For $k = 16$, however, conformal energy is not available, and we need to use Hardy's

inequality.

$$\begin{aligned}
&A\epsilon\tau^{-8+2\eta_{14}}\sum_{i=0}^{k}\int_{\Sigma_\tau\cap\{\frac{\tau}{4}\leq r\leq\frac{9\tau}{10}\}}|\Gamma^i\Phi|^2\\
\leq&CA\epsilon\tau^{-6+2\eta_{14}}\sum_{i=0}^{k}\int_{\Sigma_\tau}r^{-2}|\Gamma^i\Phi|^2\\
\leq&CA\epsilon\tau^{-6+2\eta_{14}}\sum_{i=0}^{k}\int_{\Sigma_\tau}J_\mu^N\left(\Gamma^i\Phi\right)n_{\Sigma_\tau}^\mu.
\end{aligned}$$

The estimates now follow from the Bootstrap Assumptions (10.5), (10.6), (10.7). □

For many applications, we only need a much weaker estimate on N_k. We write down the following Proposition which corresponds to the estimates that will be proved for the quantities involving S. This would allow a unified approach in dealing with many estimates with or without S.

Proposition 10.3.9.

$$\int_{\Sigma_\tau\cap\{r\leq\frac{9\tau}{10}\}}r^{1-\delta}N_{16}^2\leq CA^2\epsilon^2\tau^{-3+\eta_{S,11}+\eta_{16}}$$

$$\int_{\Sigma_\tau\cap\{r\leq\frac{9\tau}{10}\}}r^{1-\delta}N_{\leq 15}^2\leq CA^2\epsilon^2\tau^{-4+\eta_{S,11}+\eta_{15}}$$

We now move to the estimates for N_k in the region $\{r\geq\frac{9t^*}{10}\}$. Here, we need to exploit the null condition:

Proposition 10.3.10. *For $\alpha=0$ or 2,*

$$\int_{\Sigma_\tau\cap\{r\geq\frac{9\tau}{10}\}}N_{16}^2\leq CBA^2\epsilon^2\tau^{-2+\eta_{16}},$$

$$\int_{\Sigma_\tau\cap\{r\geq\frac{9\tau}{10}\}}r^\alpha N_{15}^2\leq CBA^2\epsilon^2\tau^{-3+\alpha+\eta_{15}},$$

and

$$\sum_{j=0}^{14} \int_{\Sigma_\tau \cap \{r \geq \frac{9\tau}{10}\}} r^\alpha N_j^2 \leq CBA^2\epsilon^2\tau^{-4+\alpha+\eta_{14}}.$$

Proof. Following the argument before, we reduce to quadratic and cubic terms. This time, however, the null condition plays a crucial role. For the quadratic terms, we need to consider

$$\left(\bar{D}\Gamma^{i_1}\Phi D\Gamma^{i_2}\Phi\right), \left(D\Gamma^{i_1}\Phi\bar{D}\Gamma^{i_2}\Phi\right), r^{-1}\left(D\Gamma^{i_1}\Phi D\Gamma^{i_2}\Phi\right),$$

where $i_1 \geq i_2$. For the cubic terms, we need to consider

$$\left(D\Gamma^{i_1}\Phi D\Gamma^{i_2}\Phi D\Gamma^{i_3}\Phi\right), \left(\bar{D}\Gamma^{i_1}\Phi D\Gamma^{i_2}\Phi\Gamma^{i_3}\Phi\right).$$

Notice that for the first cubic term can be dominated pointwise by quadratic terms of the third type listed above using the bootstrap assumptions (10.29) and (10.38). The second cubic term can also be dominated by the first two types of quadratic terms if $i_3 \leq 13$ by (10.27) and (10.35). We can thus assume $i_3 > 13$ and hence $i_1, i_2 \leq 8$. We

now estimate the quadratic terms

$$\begin{aligned}
&\sum_{i_2=0}^{\lfloor \frac{k}{2} \rfloor} \sum_{i_1=0}^{k-j} \int_{\Sigma_\tau \cap \{r \geq \frac{9\tau}{10}\}} r^\alpha \left(|\bar{D}\Gamma^{i_1}\Phi D\Gamma^{i_2}\Phi|^2 + |D\Gamma^{i_1}\Phi \bar{D}\Gamma^{i_2}\Phi|^2 + r^{-2}|D\Gamma^{i_1}\Phi D\Gamma^{i_2}\Phi|^2 \right) \\
\leq & C \left(\sup_{r \geq \frac{9\tau}{10}} \sum_{i_2=0}^{8} r^2 |D\Gamma^{i_2}\Phi|^2 \right) \left(\sum_{i_1=0}^{k} \int_{\Sigma_\tau \cap \{r \geq \frac{9\tau}{10}\}} r^{\alpha-2} |\bar{D}\Gamma^{i_1}\Phi|^2 \right) \\
& + C \left(\sup_{r \geq \frac{9\tau}{10}} \sum_{i_2=0}^{8} r^2 |\bar{D}\Gamma^{i_2}\Phi|^2 \right) \left(\sum_{i_1=0}^{k} \int_{\Sigma_\tau \cap \{r \geq \frac{9\tau}{10}\}} r^{\alpha-2} |D\Gamma^{i_1}\Phi|^2 \right) \\
& + C\tau^{-2} \left(\sup_{r \geq \frac{9\tau}{10}} \sum_{i_2=0}^{8} r^2 |D\Gamma^{i_2}\Phi|^2 \right) \left(\sum_{i_1=0}^{k} \int_{\Sigma_\tau \cap \{r \geq \frac{9\tau}{10}\}} r^{\alpha-2} |D\Gamma^{i_1}\Phi|^2 \right) \\
\leq & CAB\epsilon \left(\sum_{i=0}^{k} \int_{\Sigma_\tau \cap \{r \geq \frac{9\tau}{10}\}} r^{\alpha-2} |\bar{D}\Gamma^{i}\Phi|^2 \right) \\
& + CAB\epsilon\tau^{-2+\eta_{14}} \left(\sum_{i=0}^{k} \int_{\Sigma_\tau \cap \{r \geq \frac{9\tau}{10}\}} r^{\alpha-2} |D\Gamma^{i}\Phi|^2 \right).
\end{aligned}$$

We then estimate the particular cubic term:

$$\begin{aligned}
&\sum_{i_3=0}^{k} \sum_{i_1,i_2=0}^{8} \int_{\Sigma_\tau \cap \{r \geq \frac{9\tau}{10}\}} r^\alpha \left(\bar{D}\Gamma^{i_1}\Phi D\Gamma^{i_2}\Phi\Gamma^{i_3}\Phi \right)^2 \\
\leq & C \left(\sup_{r \geq \frac{9\tau}{10}} \sum_{i_1=0}^{8} r^2 |D\Gamma^{i_2}\Phi|^2 \right) \left(\sup_{r \geq \frac{9\tau}{10}} \sum_{i_2=0}^{8} r^2 |\bar{D}\Gamma^{i_2}\Phi|^2 \right) \left(\sum_{i_3=0}^{k} \int_{\Sigma_\tau \cap \{r \geq \frac{9\tau}{10}\}} r^{\alpha-4} |\Gamma^{i_1}\Phi|^2 \right) \\
\leq & CA^2B^2\epsilon^2\tau^{-2+\eta_{14}} \left(\sum_{i=0}^{k} \int_{\Sigma_\tau \cap \{r \geq \frac{9\tau}{10}\}} r^{\alpha-4} |\Gamma^{i}\Phi|^2 \right).
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\int_{\Sigma_\tau\cap\{r\geq\frac{9\tau}{10}\}} N_{16}^2\\
\leq& CAB\epsilon\tau^{-2}\left(\sum_{i=0}^{16}\int_{\Sigma_\tau\cap\{r\geq\frac{9\tau}{10}\}}|\bar{D}\Gamma^i\Phi|^2\right)\\
&+CAB\epsilon\tau^{-4+\eta_{14}}\left(\sum_{i=0}^{16}\int_{\Sigma_\tau\cap\{r\geq\frac{9\tau}{10}\}}|D\Gamma^i\Phi|^2\right)\\
&+CA^2B^2\epsilon^2\tau^{-4+\eta_{14}}\left(\sum_{i=0}^{16}\int_{\Sigma_\tau\cap\{r\geq\frac{9\tau}{10}\}}r^{-2}|\Gamma^i\Phi|^2\right)\\
\leq&\left(CAB\epsilon\tau^{-2}+\left(CAB\epsilon+CA^2B^2\epsilon^2\right)\tau^{-4+\eta_{14}}\right)\left(\sum_{i=0}^{16}\int_{\Sigma_\tau\cap\{r\geq\frac{9\tau}{10}\}}|D\Gamma^i\Phi|^2\right)\\
\leq&CA^2B\epsilon^2\tau^{-2+\eta_{16}}.
\end{aligned}$$

and

$$\begin{aligned}
&\int_{\Sigma_\tau\cap\{r\geq\frac{9\tau}{10}\}} N_k^2\\
\leq& CAB\epsilon\tau^{-4}\left(\sum_{i=0}^{k}\int_{\Sigma_\tau\cap\{r\geq\frac{9\tau}{10}\}}\tau^2|\bar{D}\Gamma^i\Phi|^2\right)\\
&+CAB\epsilon\tau^{-4+\eta_{14}}\left(\sum_{i=0}^{k}\int_{\Sigma_\tau\cap\{r\geq\frac{9\tau}{10}\}}|D\Gamma^i\Phi|^2\right)\\
&+CA^2B^2\epsilon^2\tau^{-4+\eta_{14}}\left(\sum_{i=0}^{k}\int_{\Sigma_\tau\cap\{r\geq\frac{9\tau}{10}\}}r^{-2}|\Gamma^i\Phi|^2\right)\\
\leq& CAB\epsilon\tau^{-4}\left(\sum_{i=0}^{k}\int_{\Sigma_\tau\cap\{r\geq\frac{9\tau}{10}\}}\tau^2|\bar{D}\Gamma^i\Phi|^2\right)+\left(CAB\epsilon+CA^2B^2\epsilon^2\right)\tau^{-4+\eta_{14}}\left(\sum_{i=0}^{k}\int_{\Sigma_\tau}|D\Gamma^i\Phi|^2\right)
\end{aligned}$$

and

$$\begin{aligned}
&\int_{\Sigma_\tau\cap\{r\geq\frac{9\tau}{10}\}} r^2 N_k^2\\
\leq& CAB\epsilon\tau^{-2}\left(\sum_{i=0}^{k}\int_{\Sigma_\tau\cap\{r\geq\frac{9\tau}{10}\}}\tau^2|\bar{D}\Gamma^i\Phi|^2\right)+CAB\epsilon\tau^{-2+\eta_{14}}\left(\sum_{i=0}^{k}\int_{\Sigma_\tau\cap\{r\geq\frac{9\tau}{10}\}}|D\Gamma^i\Phi|^2\right)\\
&+CA^2B^2\epsilon^2\tau^{-2+\eta_{14}}\left(\sum_{i=0}^{k}\int_{\Sigma_\tau\cap\{r\geq\frac{9\tau}{10}\}}r^{-2}|\Gamma^i\Phi|^2\right)\\
\leq& CAB\epsilon\tau^{-2}\left(\sum_{i=0}^{k}\int_{\Sigma_\tau\cap\{r\geq\frac{9\tau}{10}\}}\tau^2|\bar{D}\Gamma^i\Phi|^2\right)+\left(CAB\epsilon+CA^2B^2\epsilon^2\right)\tau^{-2+\eta_{14}}\left(\sum_{i=0}^{k}\int_{\Sigma_\tau}|D\Gamma^i\Phi|^2\right)
\end{aligned}$$

The conclusion follows from Proposition 10.1.2 and the bootstrap assumptions (10.5), (10.6), (10.7). □

From the the estimates for U_k and N_k and the L^2-L^∞ estimates in the last section, we get the following pointwise bounds:

Proposition 10.3.11. *For $r\geq\frac{t^*}{4}$,*

$$\sum_{j=0}^{13}|\Gamma^j\Phi|^2\leq\frac{B}{2}A\epsilon r^{-2}(t^*)^{1+\eta_{15}}. \tag{10.44}$$

$$\sum_{j=0}^{13}|D\Gamma^j\Phi|^2\leq\frac{B}{2}A\epsilon. \tag{10.45}$$

$$\sum_{\ell=1}^{13-j}\sum_{j=0}^{12}|D^\ell\Gamma^j\Phi|^2\leq\frac{B}{2}A\epsilon r^{-2}. \tag{10.46}$$

$$\sum_{j=0}^{8}|D\Gamma^j\Phi|^2\leq\frac{B}{2}A\epsilon r^{-2}(t^*)^{\eta_{14}}(1+|u|)^{-2}. \tag{10.47}$$

$$\sum_{j=0}^{8}|\bar{D}\Gamma^j\Phi|^2\leq\frac{B}{2}A\epsilon r^{-2}(t^*)^{-2+\eta_{14}}. \tag{10.48}$$

For $r \leq \frac{t^*}{4}$,

$$\sum_{j=0}^{13} |\Gamma^j \Phi|^2 \leq \frac{B}{2} A\epsilon (t^*)^{-1+\eta_{15}}. \tag{10.49}$$

$$\sum_{\ell=1}^{14-j} \sum_{j=0}^{13} |D^\ell \Gamma^j \Phi|^2 \leq BA\epsilon (t^*)^{-1+\eta_{15}}. \tag{10.50}$$

$$\sum_{\ell=1}^{13-j} \sum_{j=0}^{12} |D^\ell \Gamma^j \Phi|^2 \leq BA\epsilon (t^*)^{-2+\eta_{14}}. \tag{10.51}$$

$$\sum_{\ell=1}^{9-j} \sum_{j=0}^{8} |D^\ell \Gamma^j \Phi|^2 \leq \frac{B}{2} A\epsilon r^{-1+\delta} (t^*)^{-3+\eta_{S,11}}. \tag{10.52}$$

Proof. (10.44) is immediate from Proposition 10.2.3 and the bootstrap assumptions (10.6) and (10.7).

By Proposition 10.2.1,

$$\sum_{j=0}^{13} |D\Gamma^j \Phi|^2 \leq C \left(\sum_{k=0}^{15} \int_{\Sigma_\tau} J_\mu^N \left(\Gamma^k \Phi \right) n_{\Sigma_\tau}^\mu + \sum_{k=0}^{1} \int_{\Sigma_\tau} \left(D^k G_{\leq 13} \right)^2 \right).$$

Hence we get (10.45) by bootstrap assumption (10.8) and Propositions 10.3.5 and 10.3.6. The constant is improved since $A\epsilon \ll 1$ and $C \ll B$.

By Proposition 10.2.2,

$$\begin{aligned} \sum_{\ell=1}^{13-j} \sum_{j=0}^{12} |D^\ell \Gamma^j \Phi|^2 \leq & C r^{-2} \left(\sum_{m=0}^{13-j} \sum_{k=0}^{2} \sum_{j=0}^{12} \int_{\Sigma_\tau} J_\mu^N \left(\partial_{t^*}^m \Omega^k \Gamma^j \Phi \right) n_{\Sigma_\tau}^\mu + \sum_{m+k \leq 10} \int_{\Sigma_\tau} \left(D^m G_{\leq k} \right)^2 \right) \\ \leq & C r^{-2} \left(\sum_{j=0}^{11} \int_{\Sigma_\tau} J_\mu^N \left(\Gamma^j \Phi \right) n_{\Sigma_\tau}^\mu + \sum_{m+k \leq 10} \int_{\Sigma_\tau} \left(D^m G_{\leq k} \right)^2 \right) \end{aligned}$$

We hence get (10.46) by bootstrap assumption (10.8) and Propositions 10.3.5 and 10.3.6. The constant is improved since $A\epsilon \ll 1$ and $C \ll B$.

By Proposition 10.2.5, for $r \geq \frac{t^*}{4}$, we have

$$\begin{aligned}
&\sum_{j=0}^{8} |D\Gamma^j \Phi|^2 \\
\leq & Cr^{-2} (1+|u|)^{-2} \sum_{m=0}^{1} \sum_{k=0}^{2} \sum_{j=0}^{8} \left(\int_{\Sigma_\tau} J_\mu^{Z+CN} \left(\partial_{t^*}^m \tilde{\Omega}^k \Gamma^j \Phi \right) n_{\Sigma_\tau}^\mu \right. \\
& \qquad \left. + C\tau^2 \int_{\Sigma_\tau \cap \{r \leq r_Y^-\}} J_\mu^N \left(\partial_{t^*}^m \tilde{\Omega}^k \Gamma^j \Phi \right) n_{\Sigma_\tau}^\mu \right) \\
& + Cr^{-2} \sum_{k=0}^{2} \sum_{j=0}^{8} \int_{\Sigma_\tau \cap \{u' \sim u\} \cap \{r \geq \frac{\tau}{4}\}} \left(\Box_{g_K} \left(\tilde{\Omega}^k \Gamma^j \Phi \right) \right)^2 \\
\leq & CA\epsilon \tau^{\eta_{14}} r^{-2} (1+|u|)^{-2} + Cr^{-2} \int \int_{\Sigma_\tau \cap \{u' \sim u\} \cap \{r \geq \frac{\tau}{4}\}} G_{\leq 10}^2 \\
& \text{by bootstrap assumption (10.7)} \\
\leq & CA\epsilon \tau^{\eta_{14}} r^{-2} (1+|u|)^{-2} + CA\epsilon (t^*)^{-2+\eta_{14}} r^{-2} + CA^2 B \epsilon^2 (t^*)^{-2+\eta_{14}} r^{-2} \\
& \text{by Propositions 10.3.9, 10.3.10 and 10.3.6} \\
\leq & \frac{B}{2} A\epsilon r^{-2} (t^*)^{\eta_{14}} (1+|u|)^{-2}.
\end{aligned}$$

Hence we have proved (10.47).

By Proposition 10.2.6, for $r \geq \frac{t^*}{4}$, we have

$$\begin{aligned}
&\sum_{j=0}^{8} |\bar{D}\Gamma^j \Phi|^2 \\
\leq & Cr^{-4} \sum_{k=0}^{2} \sum_{j=0}^{8} \sum_{i+m \leq 1} \left(\int_{\Sigma_\tau} J_\mu^N \left(S^i \partial_{t^*}^m \Gamma^j \Phi \right) n_{\Sigma_\tau}^\mu + \int_{\Sigma_\tau} J_\mu^{Z+CN} \left(\partial_{t^*}^m \tilde{\Omega}^k \Gamma^j \Phi \right) n_{\Sigma_\tau}^\mu \right. \\
& \qquad \left. + C\tau^2 \int_{\Sigma_\tau \cap \{r \leq r_Y^-\}} J_\mu^N \left(\partial_{t^*}^m \tilde{\Omega}^k \Gamma^j \Phi \right) n_{\Sigma_\tau}^\mu + \int_{\Sigma_\tau} \left(\Box_{g_K} \left(\tilde{\Omega}^k \Gamma^j \Phi \right) \right)^2 \right) \\
& + Cr^{-2} \sum_{k=0}^{2} \sum_{j=0}^{8} \int_{\Sigma_\tau \cap \{r \geq \frac{\tau}{2}\}} \left(\Box_{g_K} \left(\tilde{\Omega}^k \Gamma^j \Phi \right) \right)^2 \\
\leq & CA\epsilon r^{-4} (t^*)^{\eta_{14}} + CA\epsilon (t^*)^{-2+\eta_{14}} r^{-2} + CA^2 B \epsilon^2 (t^*)^{-2+\eta_{14}} r^{-2} \\
\leq & \frac{B}{2} A\epsilon r^{-2} (t^*)^{-2+\eta_{14}}.
\end{aligned}$$

Hence we have proved (10.48) and completed the proof for $r \geq \frac{t^*}{4}$. We now move to the pointwise estimates in the region $r \leq \frac{t^*}{4}$. (10.49) follows directly from Proposition 10.2.9 and the bootstrap assumptions (10.6) and (10.7). By Proposition 10.2.7,

$$\sum_{\ell=1}^{14-j}\sum_{j=0}^{13}|D^\ell\Gamma^j\Phi|^2 \leq C\left(\sum_{i+j\leq 15}\int_{\Sigma_\tau\cap\{r\leq\frac{t^*}{2}\}} J_\mu^N\left(\hat{Y}^i\Gamma^j\Phi\right)n_{\Sigma_\tau}^\mu + \sum_{\ell=1}^{14-j}\sum_{j=0}^{13}\int_{\Sigma_\tau}(D^\ell G_{\leq j})^2\right),$$

where we have used the fact that $[\hat{Y},\Gamma]=0$. Hence (10.50) follows from the bootstrap assumptions (10.6) and (10.7) and Propositions 10.3.5 and 10.3.6. The proof of (10.51) follows similarly as (10.50): by Proposition 10.2.7,

$$\sum_{\ell=1}^{13-j}\sum_{j=0}^{12}|D^\ell\Gamma^j\Phi|^2 \leq C\left(\sum_{i+j\leq 14}\int_{\Sigma_\tau\cap\{r\leq\frac{t^*}{2}\}} J_\mu^N\left(\hat{Y}^i\Gamma^j\Phi\right)n_{\Sigma_\tau}^\mu + \sum_{\ell=1}^{13-j}\sum_{j=0}^{13}\int_{\Sigma_\tau}(D^\ell G_{\leq j})^2\right),$$

Hence (10.51) follows from the bootstrap assumptions (10.7) and Propositions 10.3.5 and 10.3.6.

Finally, by Proposition 10.2.10, for $r \leq \frac{t^*}{4}$, we have

$$\begin{aligned}
&\sum_{\ell=1}^{9-j}\sum_{j=0}^{8}|D^\ell\Gamma^j\Phi|^2\\
\leq& C(t^*)^{-1}r^{-1+\delta}\sum_{i+j\leq 10}\iint_{\mathcal{R}((1.1)^{-1}\tau,\tau)\cap\{r\leq\frac{t^*}{2}\}}\left(K^{X_1}\left(Y^i\Gamma^j\Phi\right)+K^{X_1}\left(SY^i\Gamma^j\Phi\right)\right)\\
&+C(t^*)^{-1}r^{-1+\delta}\sum_{i+j\leq 10}\iint_{\mathcal{R}((1.1)^{-1}\tau,\tau)\cap\{r\leq\frac{t^*}{2}\}} r^{-1-\delta}\left(D^iG_j\right)^2\\
\leq& CA\epsilon\tau^{-3+\eta_{S,11}}r^{-1+\delta}+CA^2\epsilon^2\tau^{-3+\eta_{14}}r^{-1+\delta}\\
\leq& \frac{B}{2}A\epsilon\tau^{-3+\eta_{S,11}}r^{-1+\delta},
\end{aligned}$$

where in the third line we have used the bootstrap assumptions (10.20) and (10.26) and Propositions 10.3.9, 10.3.10 and 10.3.6. The only caveat is that when using (10.26), the vector fields $\hat{Y}$ and S are in different order. However, since $[S,\hat{Y}]\sim D$, we can estimate the commutator term by (10.20). □

Now, we have proved the L^∞ bounds, we will replace the constant B in the bootstrap assumption (10.27), (10.28), (10.29), (10.30), (10.31), (10.35), (10.38) by C in the sequel. Notice that we have originally assumed $B \ll A_0$ and therefore $C \ll A_0$ still holds. We now proceed to recover the bootstrap assumptions (K) that do not involve the commutators Y or S. We first retrieve (10.17)-(10.20). Notice also that we will retrieve (10.13) and (10.15) later together with (10.5) and (10.8).

Proposition 10.3.12.

$$\sum_{i+j\leq 15} A_{X,j}^{-1} \iint_{\mathcal{R}((1.1)^{-1}\tau,\tau)\cap\{r\leq\frac{t^*}{2}\}} \left(K^{X_0}\left(\partial_{t^*}^i\tilde{\Omega}^j\Phi\right) + K^N\left(\partial_{t^*}^i\tilde{\Omega}^j\Phi\right)\right) \leq \frac{\epsilon}{2}\tau^{-1+\eta_{15}}. \tag{10.53}$$

$$\sum_{i+j\leq 14} A_{X,j}^{-1} \iint_{\mathcal{R}((1.1)^{-1}\tau,\tau)\cap\{r\leq\frac{t^*}{2}\}} K^{X_1}\left(\partial_{t^*}^i\tilde{\Omega}^j\Phi\right) \leq \frac{\epsilon}{2}\tau^{-1+\eta_{15}}. \tag{10.54}$$

$$\sum_{i+j\leq 14} A_{X,j}^{-1} \iint_{\mathcal{R}((1.1)^{-1}\tau,\tau)\cap\{r\leq\frac{t^*}{2}\}} \left(K^{X_0}\left(\partial_{t^*}^i\tilde{\Omega}^j\Phi\right) + K^N\left(\partial_{t^*}^i\tilde{\Omega}^j\Phi\right)\right) \leq \frac{\epsilon}{2}\tau^{-2+\eta_{14}}. \tag{10.55}$$

$$\sum_{i+j\leq 13} A_{X,j}^{-1} \iint_{\mathcal{R}((1.1)^{-1}\tau,\tau)\cap\{r\leq\frac{t^*}{2}\}} K^{X_1}\left(\partial_{t^*}^i\tilde{\Omega}^j\Phi\right) \leq \frac{\epsilon}{2}\tau^{-2+\eta_{14}}. \tag{10.56}$$

Proof. We first prove the estimates involving X_0, i.e., (10.53) and (10.55). By Proposition 9.5.4.2 and the Remark following it and the fact that $|\partial_{t^*}^m N_k| \leq |N_{\leq k+m}|$, we

have

$$
\begin{aligned}
&\sum_{i+j\leq 15} A_{X,j}^{-1} \iint_{\mathcal{R}((1.1)^{-1}\tau,\tau)\cap\{r\leq\frac{t^*}{2}\}} \left(K^{X_0}\left(\partial_{t^*}^i\tilde{\Omega}^j\Phi\right)+K^N\left(\partial_{t^*}^i\tilde{\Omega}^j\Phi\right)\right)\\
\leq & C\sum_{i+j\leq 15} A_{X,j}^{-1}\left(\tau^{-2}\int_{\Sigma_{(1.1)^{-1}\tau}} J_\mu^{Z+N,w^Z}\left(\partial_{t^*}^i\tilde{\Omega}^j\Phi\right) n^\mu_{\Sigma_{(1.1)^{-1}\tau}}\right.\\
&\qquad\qquad \left. +C\int_{\Sigma_{(1.1)^{-1}\tau}\cap\{r\leq r_Y^-\}} J_\mu^N\left(\partial_{t^*}^i\tilde{\Omega}^j\Phi\right) n^\mu_{\Sigma_{(1.1)^{-1}\tau}}\right)\\
&+C\sum_{i+j\leq 15} A_{X,j}^{-1}\left(\iint_{\mathcal{R}((1.1)^{-1}\tau-1,\tau+1)\cap\{r\leq\frac{9t^*}{10}\}} r^{1+\delta}N_{\leq 16}^2\right.\\
&\qquad\qquad +\sup_{t^*\in[(1.1)^{-1}\tau-1,\tau+1]}\int_{\Sigma_{t^*}\cap\{|r-3M|\leq\frac{M}{8}\}} N_{\leq 16}^2\\
&\left.+\iint_{\mathcal{R}((1.1)^{-1}\tau-1,\tau+1)\cap\{r\leq\frac{9t^*}{10}\}} r^{1+\delta}\left(U_{\leq 15,\leq j}\right)^2\right)\\
\leq & C\sum_{i+j\leq 15}\left(A_{X,j}^{-1}A_j\epsilon\tau^{-1+\eta_{15}}+A_{X,j}^{-1}A^2\epsilon^2\tau^{-2+\eta_{S,11}+\eta_{16}+2\delta}+CA_{X,j}^{-1}A_{X,j-1}\epsilon\tau^{-1+\eta_{15}}\right)\\
\leq & \frac{\epsilon}{2}\tau^{-1+\eta_{15}},
\end{aligned}
$$

by Propositions 10.3.9, 10.3.10 and 10.3.6. Notice that our integrated estimates for U in Proposition 10.3.6 is only for $[(1.1)^{-1}\tau,\tau]$. Nevertheless, for the region $[(1.1)^{-1}\tau-1,(1.1)^{-1}\tau]\cap[\tau,\tau+1]$, we can use the integrate over the fixed τ estimate in the same Proposition. By Proposition 9.5.4.2 and the Remark following it and the fact that

$|\partial_{t^*}^m N_k| \leq |N_{\leq k+m}|$, we have

$$\begin{aligned}
&\sum_{i+j\leq 14} A_{X,j}^{-1} \iint_{\mathcal{R}((1.1)^{-1}\tau,\tau)\cap\{r\leq\frac{t^*}{2}\}} \left(K^{X_0}\left(\partial_{t^*}^i\tilde{\Omega}^j\Phi\right) + K^N\left(\partial_{t^*}^i\tilde{\Omega}^j\Phi\right)\right)\\
\leq & C\sum_{i+j\leq 14} A_{X,j}^{-1}\left(\tau^{-2}\int_{\Sigma_{(1.1)^{-1}\tau}} J_\mu^{Z+N,w^Z}\left(\partial_{t^*}^i\tilde{\Omega}^j\Phi\right) n^\mu_{\Sigma_{(1.1)^{-1}\tau}}\right.\\
&\qquad\left. + C\int_{\Sigma_{(1.1)^{-1}\tau}\cap\{r\leq r_Y^-\}} J_\mu^N\left(\partial_{t^*}^i\tilde{\Omega}^j\Phi\right) n^\mu_{\Sigma_{(1.1)^{-1}\tau}}\right)\\
&+ C\sum_{i+j\leq 14} A_{X,j}^{-1}\left(\iint_{\mathcal{R}((1.1)^{-1}\tau-1,\tau+1)\cap\{r\leq\frac{9t^*}{10}\}} r^{1+\delta} N_{\leq 15}^2\right.\\
&\qquad + \sup_{t^*\in[(1.1)^{-1}\tau-1,\tau+1]}\int_{\Sigma_{t^*}\cap\{|r-3M|\leq\frac{M}{8}\}} N_{\leq 15}^2\\
&\qquad\left. + \iint_{\mathcal{R}((1.1)^{-1}\tau-1,\tau+1)\cap\{r\leq\frac{9t^*}{10}\}} r^{1+\delta}\left(U_{\leq 14,\leq j}\right)^2\right)\\
\leq & C\sum_{i+j\leq 14}\left(A_{X,j}^{-1}A_j\epsilon\tau^{-2+\eta_{14}} + A_{X,j}^{-1}A^2\epsilon^2\tau^{-3+\eta_{S,11}+\eta_{15}+2\delta} + A_{X,j}^{-1}A_{X,j-1}\epsilon\tau^{-2+\eta_{14}}\right)\\
\leq & \frac{\epsilon}{2}\tau^{-2+\eta_{14}}.
\end{aligned}$$

The proof of (10.54) and (10.56) proceeds in an identical manner. Notice that using Proposition 9.5.5.2, the right hand side when we estimate (10.54) (respectively (10.56)) is identical to that when we estimate (10.53) (respectively (10.55)). □

Now we move on to retrieving the bootstrap assumptions (J) with better constants:

Proposition 10.3.13.

$$\sum_{i+j=16} A_j^{-1}\int_{\Sigma_\tau} J_\mu^N\left(\partial_{t^*}^i\tilde{\Omega}^j\Phi\right) n^\mu_{\Sigma_\tau} \leq \frac{\epsilon}{4}\tau^{\eta_{16}}. \tag{10.57}$$

$$\sum_{i+j\leq 15} A_j^{-1}\int_{\Sigma_\tau} J_\mu^N\left(\partial_{t^*}^i\tilde{\Omega}^j\Phi\right) n^\mu_{\Sigma_\tau} \leq \frac{\epsilon}{2}. \tag{10.58}$$

$$\sum_{i+j=16} A_{X,j}^{-1}\left(\iint_{\mathcal{R}(\tau_0,\tau)} K^{X_0}\left(\partial_{t^*}^i\tilde{\Omega}^j\Phi\right) + \iint_{\mathcal{R}(\tau_0,\tau)\cap\{r\leq r_Y^-\}} K^N\left(\partial_{t^*}^i\tilde{\Omega}^j\Phi\right)\right) \leq \frac{\epsilon}{2}\tau^{\eta_{16}}. \tag{10.59}$$

$$\sum_{i+j=15} A_{X,j}^{-1} \iint_{\mathcal{R}(\tau_0,\tau)} K^{X_1}\left(\partial_{t^*}^i \tilde{\Omega}^j \Phi\right) \leq \frac{\epsilon}{2}\tau^{\eta_{16}}. \tag{10.60}$$

$$\sum_{i+j\leq 15} A_{X,j}^{-1} \iint_{\mathcal{R}(\tau_0,\tau)} K^{X_0}\left(\partial_{t^*}^i \tilde{\Omega}^j \Phi\right) \leq \frac{\epsilon}{2}. \tag{10.61}$$

$$\sum_{i+j\leq 14} A_{X,j}^{-1} \iint_{\mathcal{R}(\tau_0,\tau)} K^{X_1}\left(\partial_{t^*}^i \tilde{\Omega}^j \Phi\right) \leq \epsilon. \tag{10.62}$$

Proof. We will prove the slightly stronger statements with $A_{X,j}$ replaced by A_j. Using Proposition 9.4.17 and 9.4.18, we have

$$\begin{aligned}
&\sum_{i+j=16} A_j^{-1}\left(\int_{\Sigma_\tau} J_\mu^N\left(\partial_{t^*}^i \tilde{\Omega}^j \Phi\right) n_{\Sigma_\tau}^\mu + \iint_{\mathcal{R}(\tau_0,\tau)} K^{X_0}\left(\partial_{t^*}^i \tilde{\Omega}^j \Phi\right)\right.\\
&\qquad\left. + \iint_{\mathcal{R}(\tau_0,\tau)\cap\{r\leq r_Y^-\}} K^N\left(\partial_{t^*}^i \tilde{\Omega}^j \Phi\right)\right)\\
\leq & C\sum_{i+j=16} A_j^{-1}\left(\int_{\Sigma_{\tau_0}} J_\mu^N\left(\partial_{t^*}^i \tilde{\Omega}^j \Phi\right) n_{\Sigma_{\tau'}}^\mu + \left(\int_{\tau_0-1}^{\tau+1}\left(\int_{\Sigma_{t^*}} N_{16}^2\right)^{\frac{1}{2}} dt^*\right)^2\right.\\
&\qquad + \iint_{\mathcal{R}(\tau_0-1,\tau+1)} N_{16}^2 + \iint_{\mathcal{R}(\tau_0-1,\tau+1)} r^{1+\delta} U_{16,j}^2\\
&\qquad\left. + \sup_{t^*\in[\tau_0-1,\tau+1]}\int_{\Sigma_{t^*}\cap\{|r-3M|\leq\frac{M}{8}\}} U_{16,j}^2\right)\\
\leq & C\sum_{i+j=16} A_j^{-1}\left(\epsilon + A^2\epsilon^2\eta_{16}^{-1}\tau^{\eta_{16}} + A_{X,j-1}\epsilon\tau^{\eta_{16}}\right)\\
\leq & \frac{\epsilon}{4}\tau^{\eta_{16}}.
\end{aligned}$$

We now turn to the estimates for $\sum_{j=0}^{15} |\Gamma^j \Phi|$. We have

$$\begin{aligned}
&\sum_{i+j\leq 15} A_j^{-1} \left(\int_{\Sigma_\tau} J_\mu^N \left(\partial_{t^*}^i \tilde{\Omega}^j \Phi \right) n_{\Sigma_\tau}^\mu + \iint_{\mathcal{R}(\tau_0,\tau)} K^{X_0} \left(\partial_{t^*}^i \tilde{\Omega}^j \Phi \right) \right) \\
\leq & C \sum_{i+j\leq 15} A_j^{-1} \left(\int_{\Sigma_{\tau_0}} J_\mu^N \left(\partial_{t^*}^i \tilde{\Omega}^j \Phi \right) n_{\Sigma_{\tau_0}}^\mu + \left(\int_{\tau_0-1}^{\tau+1} \left(\int_{\Sigma_{t^*}} N_{\leq 15}^2 \right)^{\frac{1}{2}} dt^* \right)^2 \right. \\
& \qquad + \iint_{\mathcal{R}(\tau_0-1,\tau+1)} N_{\leq 15}^2 + \iint_{\mathcal{R}(\tau_0-1,\tau+1)} r^{1+\delta} U_{\leq 15,\leq j}^2 \\
& \qquad \left. + \sup_{t^*\in[\tau_0-1,\tau+1]} \int_{\Sigma_{t^*}\cap\{|r-3M|\leq \frac{M}{8}\}} U_{\leq 15,\leq j}^2 \right) \\
\leq & C \sum_{i+j\leq 15} A_j^{-1} \left(\epsilon + A^2\epsilon^2 + C A_{X,j-1}\epsilon \right) \\
\leq & \frac{\epsilon}{2}.
\end{aligned}$$

It now remains to show (10.60) and (10.62). By Proposition 9.4.20 they can be estimated by exactly the same terms as (10.59) and (10.61) respectively. The Proposition hence follows.

□

We now move on to control the conformal energy and close the part of the bootstrap assumption (10.6) without $\hat{Y}$.

Proposition 10.3.14.

$$\begin{aligned}
&\sum_{i+j=15} A_j^{-1} \left(\int_{\Sigma_\tau} J_\mu^{Z+N,w^Z} \left(\partial_{t^*}^i \tilde{\Omega}^j \Phi \right) n_{\Sigma_\tau}^\mu + C\tau^2 \int_{\Sigma_\tau\cap\{r\leq\frac{9\tau}{10}\}} J_\mu^N \left(\partial_{t^*}^i \tilde{\Omega}^j \Phi \right) n_{\Sigma_\tau}^\mu \right) \\
\leq & \frac{\epsilon}{4} \tau^{1+\eta_{15}}.
\end{aligned} \tag{10.63}$$

Proof. By Proposition 9.5.12,

$$\begin{aligned}
&\sum_{i+j=15} A_j^{-1}\left(\int_{\Sigma_\tau} J_\mu^{Z,w^Z}\left(\partial_{t^*}^i\tilde{\Omega}^j\Phi\right)n_{\Sigma_\tau}^\mu + C\tau^2\int_{\Sigma_\tau\cap\{r\le\frac{9\tau}{10}\}} J_\mu^N\left(\partial_{t^*}^i\tilde{\Omega}^j\Phi\right)n_{\Sigma_\tau}^\mu\right)\\
\le& C\sum_{i+j=15} A_j^{-1}\left(\int_{\Sigma_{\tau_0}} J_\mu^{Z+CN,w^Z}\left(\partial_{t^*}^i\tilde{\Omega}^j\Phi\right)n_{\Sigma_{\tau_0}}^\mu + \delta'\iint_{\mathcal{R}(\tau_0,\tau)\cap\{r\le\frac{t^*}{2}\}}(t^*)^2K^{X_0}\left(\partial_{t^*}^i\tilde{\Omega}^j\Phi\right)\right.\\
&+(\delta'+a)\iint_{\mathcal{R}(\tau_0,\tau)\cap\{r\le r_Y^-\}}(t^*)^2K^N\left(\partial_{t^*}^i\tilde{\Omega}^j\Phi\right) + (\delta')^{-1}\iint_{\mathcal{R}(\tau_0,\tau)} t^*r^{-1+\delta}K^{X_1}\left(\partial_{t^*}^i\tilde{\Omega}^j\Phi\right)\\
&+(\delta')^{-1}\sum_{m=0}^{1}\iint_{\mathcal{R}(\tau_0,\tau)\cap\{r\le\frac{9t^*}{10}\}}(t^*)^2r^{1+\delta}\left(\partial_{t^*}^mN_{15}\right)^2 + (\delta')^{-1}\iint_{\mathcal{R}(\tau_0,\tau)\cap\{r\le\frac{9t^*}{10}\}}(t^*)^2r^{1+\delta}U_{15,j}^2\\
&\left.+(\delta')^{-1}\left(\int_{\tau_0}^{\tau}\left(\int_{\Sigma_{t^*}\cap\{r\ge\frac{t^*}{2}\}} r^2G_{15,j}^2\right)^{\frac12}dt^*\right)^2 + (\delta')^{-1}\sup_{t^*\in[\tau_0,\tau]}\int_{\Sigma_{t^*}\cap\{r_Y^-\le r\le\frac{25M}{8}\}}(t^*)^2N_{15}^2\right).
\end{aligned}$$

We will estimate the terms one by one. First, the term with the initial data, i.e., the very first term, is clearly bounded by $C(\sum_j A_j^{-1})\epsilon$. Second, we consider the two $(t^*)^2K$ terms on the second line. To this end, we define as before $\tau_0\le\tau_1\le...\le\tau_n=\tau$ with $\tau_{i+1}\le(1.1)\tau_i$ and $n\sim\log(\tau-\tau_0)$ is the minimum such that this can be done. Thus, these two terms can be bounded, using the bootstrap assumption (10.17),

$$\begin{aligned}
&\delta'\sum_{i+j=15} A_j^{-1}\iint_{\mathcal{R}(\tau_0,\tau)\cap\{r\le\frac{t^*}{2}\}}(t^*)^2K^{X_0}\left(\partial_{t^*}^i\tilde{\Omega}^j\Phi\right)\\
&\quad+(\delta'+a)A_j^{-1}\iint_{\mathcal{R}(\tau_0,\tau)\cap\{r\le r_Y^-\}}(t^*)^2K^N\left(\partial_{t^*}^i\tilde{\Omega}^j\Phi\right)\\
\le& C\sum_{i+j=15} A_j^{-1}\sum_{k=0}^{n-1}\left(\delta'\tau_k^2\iint_{\mathcal{R}(\tau_k,\tau_{k+1})\cap\{r\le\frac{t^*}{2}\}}K^{X_0}\left(\partial_{t^*}^i\tilde{\Omega}^j\Phi\right)\right.\\
&\qquad\left.+(\delta'+a)\tau_k^2\iint_{\mathcal{R}(\tau_0,\tau)\cap\{r\le r_Y^-\}}K^N\left(\partial_{t^*}^i\tilde{\Omega}^j\Phi\right)\right)\\
\le& C\left(\sum_j\frac{A_{X,j}}{A_j}\right)\epsilon\,(2\delta'+a)\,\tau^{1+\eta_{15}}.
\end{aligned}$$

This is acceptable since $a,\delta'\ll\frac{A_j}{A_{X,j}}$. Third, the term with $t^*r^{-1+\delta}K$ can be bounded

using the bootstrap assumption (10.14),

$$(\delta')^{-1}\sum_{i+j=15}A_j^{-1}\iint_{\mathcal{R}(\tau_0,\tau)}t^*r^{-1+\delta}K^{X_1}\left(\partial_{t^*}^i\tilde{\Omega}^j\Phi\right)$$
$$\leq C(\delta')^{-1}\sum_{i+j=15}A_j^{-1}\sum_{k=0}^{n-1}\tau_k\iint_{\mathcal{R}(\tau_k,\tau_{k+1})}K^{X_1}\left(\partial_{t^*}^i\tilde{\Omega}^j\Phi\right)\leq C(\delta')^{-1}\left(\sum_j\frac{A_{X,j}}{A_j}\right)\tau^{1+\eta_{16}}.$$

This is acceptable since $\eta_{16}\ll\eta_{15}$ and therefore the constant can be improved for τ large. Fourth, the integrals involving N_{15} can be bounded using Propositions 10.3.9 and 10.3.10:

$$C(\delta')^{-1}A_0^{-1}\sum_{m=0}^{1}\iint_{\mathcal{R}(\tau_0,\tau)\cap\{r\leq\frac{9t^*}{10}\}}(t^*)^2r^{1+\delta}\left(\partial_{t^*}^mN_{15}\right)^2$$
$$\leq CA^2A_0^{-1}\epsilon^2(\delta')^{-1}\int_{\tau_0}^{\tau}(t^*)^{-1+\eta_{S,11}+\eta_{15}+2\delta}\,dt^*\leq CA^2A_0^{-1}\epsilon^2(\delta')^{-1}\tau^{\eta_{S,11}+\eta_{15}+2\delta},$$

$$C(\delta')^{-1}A_0^{-1}\left(\int_{\tau_0}^{\tau}\left(\int_{\Sigma_{t^*}\cap\{r\geq\frac{t^*}{2}\}}r^2N_{15}^2\right)^{\frac{1}{2}}dt^*\right)^2\leq CA^2A_0^{-1}\epsilon^2(\delta')^{-1}\tau^{1+\eta_{15}},$$

$$C(\delta')^{-1}A_0^{-1}\sup_{t^*\in[\tau_0,\tau]}\int_{\Sigma_{t^*}\cap\{r_Y^-\leq r\leq\frac{25M}{8}\}}(t^*)^2N_{15}^2\leq CA^2A_0^{-1}\epsilon^2(\delta')^{-1}.$$

These are all acceptable since ϵ would beat all the constants. Fifth, for the commutator terms $U_{15,j}^2$, we estimate by Proposition 10.3.6,

$$(\delta')^{-1}A_j^{-1}\iint_{\mathcal{R}(\tau_0,\tau)}(t^*)^2r^{1+\delta}\left(U_{15,j}\right)^2\leq C\sum_{i=0}^{n-1}(\delta')^{-1}\iint_{\mathcal{R}(\tau_i,\tau_{i+1})}\tau_i^2r^{1+\delta}\left(U_{15,j}\right)^2$$
$$\leq C(\delta')^{-1}\frac{A_{X,j-1}}{A_j}\tau^{1+\eta_{15}},$$

$$(\delta')^{-1}A_j^{-1}\left(\int_{\tau_0}^{\tau}\left(\int_{\Sigma_{t^*}\cap\{r\geq\frac{t^*}{2}\}}r^2U_{15,j}^2\right)^{\frac{1}{2}}dt^*\right)^2\leq C(\delta')^{-1}\frac{A_{j-1}}{A_j}\tau.$$

Since

$$\frac{A_{j-1}}{A_j}\ll\frac{A_{X,j-1}}{A_j}\ll\delta',$$

all terms are acceptable. □

With 14 or less derivatives, the conformal energy behaves better. We now close the part of the bootstrap assumption (10.7) without $\hat{Y}$.

Proposition 10.3.15.

$$\sum_{i+j\leq 14} A_j^{-1}\left(\int_{\Sigma_\tau} J_\mu^{Z+N,w^Z}\left(\partial_{t^*}^i\tilde{\Omega}^j\Phi\right)n_{\Sigma_\tau}^\mu + C\tau^2\int_{\Sigma_\tau\cap\{r\leq\frac{9\tau}{10}\}} J_\mu^N\left(\partial_{t^*}^i\tilde{\Omega}^j\Phi\right)n_{\Sigma_\tau}^\mu\right)\leq\frac{\epsilon}{4}\tau^{\eta_{14}}. \tag{10.64}$$

Proof. By Proposition 9.5.12, and noticing that U is supported away from $\{|r-3M|\leq\frac{M}{8}\}$, we have

$$\begin{aligned}
&\sum_{i+j\leq 14} A_j^{-1}\left(\int_{\Sigma_\tau} J_\mu^{Z+N,w^Z}\left(\partial_{t^*}^i\tilde{\Omega}^j\Phi\right)n_{\Sigma_\tau}^\mu + C\tau^2\int_{\Sigma_\tau\cap\{r\leq\frac{9\tau}{10}\}} J_\mu^N\left(\partial_{t^*}^i\tilde{\Omega}^j\Phi\right)n_{\Sigma_\tau}^\mu\right)\\
\leq& C\sum_{i+j\leq 14} A_j^{-1}\left(\int_{\Sigma_{\tau_0}} J_\mu^{Z+CN,w^Z}\left(\partial_{t^*}^i\tilde{\Omega}^j\Phi\right)n_{\Sigma_{\tau_0}}^\mu + \delta'\iint_{\mathcal{R}(\tau_0,\tau)\cap\{r\leq\frac{t^*}{2}\}}(t^*)^2K^{X_0}\left(\partial_{t^*}^i\tilde{\Omega}^j\Phi\right)\right.\\
&+(\delta'+a)\iint_{\mathcal{R}(\tau_0,\tau)\cap\{r\leq r_Y^-\}}(t^*)^2K^N\left(\partial_{t^*}^i\tilde{\Omega}^j\Phi\right)+(\delta')^{-1}\iint_{\mathcal{R}(\tau_0,\tau)}t^*r^{-1+\delta}K^{X_1}\left(\partial_{t^*}^i\tilde{\Omega}^j\Phi\right)\\
&+(\delta')^{-1}\sum_{m=0}^{1}\iint_{\mathcal{R}(\tau_0,\tau)\cap\{r\leq\frac{9t^*}{10}\}}(t^*)^2r^{1+\delta}\left(\partial_{t^*}^mN_{\leq 15}\right)^2\\
&+(\delta')^{-1}\iint_{\mathcal{R}(\tau_0,\tau)\cap\{r\leq\frac{9t^*}{10}\}}(t^*)^2r^{1+\delta}U_{\leq 15,\leq j}^2\\
&+(\delta')^{-1}\left(\int_{\tau_0}^{\tau}\left(\int_{\Sigma_{t^*}\cap\{r\geq\frac{t^*}{2}\}}r^2G_{\leq 15,\leq j}^2\right)^{\frac{1}{2}}dt^*\right)^2\\
&\left.+(\delta')^{-1}\sup_{t^*\in[\tau_0,\tau]}\int_{\Sigma_{t^*}\cap\{r_Y^-\leq r\leq\frac{25M}{8}\}}(t^*)^2N_{\leq 15}^2\right).
\end{aligned}$$

As before, we estimate each term one by one. First, the term with initial data is clearly bounded by $C\sum_j A_j\epsilon$. Second, the $(t^*)^2K$ terms can be bounded, using (10.19), and

dividing the interval into $\tau_0 < \tau_1 < ... < \tau_n = \tau$ as before by

$$C\frac{A_{X,j}}{A_j}\epsilon\left(2\delta' + a\right)\sum_{i=0}^{n-1}\tau_i^{\eta_{14}} \leq C\frac{A_{X,j}}{A_j}\epsilon\eta_{14}^{-1}\left(2\delta + a\right)\tau^{\eta_{14}}.$$

This is acceptable since $a, \delta' \ll \frac{A_j}{A_{X,j}}$. Third, the $t^* r^{-1+\delta} K$ term can be bounded, using the bootstrap assumptions (10.16) and (10.18), by

$$\begin{aligned}
&(\delta')^{-1}\sum_{i+j\leq 14} A_j^{-1}\iint_{\mathcal{R}(\tau_0,\tau)} t^* r^{-1+\delta} K^{X_1}\left(\partial_{t^*}^i\tilde{\Omega}^j\Phi\right)\\
\leq&(\delta')^{-1}\sum_{i+j\leq 14} A_j^{-1}\left(\iint_{\mathcal{R}(\tau_0,\tau)\cap\{r\leq\frac{t^*}{2}\}} t^* K^{X_1}\left(\partial_{t^*}^i\tilde{\Omega}^j\Phi\right)\right.\\
&\qquad\left.+\iint_{\mathcal{R}(\tau_0,\tau)\cap\{r\geq\frac{t^*}{2}\}} t^* r^{-1+\delta} K^{X_1}\left(\partial_{t^*}^i\tilde{\Omega}^j\Phi\right)\right)\\
\leq&C(\delta')^{-1}\sum_{i+j\leq 14} A_j^{-1}\sum_{k=0}^{n-1}\left(\tau_k\iint_{\mathcal{R}(\tau_k,\tau_{k+1})\cap\{r\leq\frac{t^*}{2}\}} K^{X_1}\left(\partial_{t^*}^i\tilde{\Omega}^j\Phi\right)\right.\\
&\qquad\left.+\tau_k^{\delta}\iint_{\mathcal{R}(\tau_k,\tau_{k+1})\cap\{r\geq\frac{t^*}{2}\}} K^{X_1}\left(\partial_{t^*}^i\tilde{\Omega}^j\Phi\right)\right)\\
\leq&C(\delta')^{-1}\sum_j\frac{A_{X,j}}{A_j}\eta_{15}^{-1}\tau^{\eta_{15}},
\end{aligned}$$

which is acceptable for τ large since $\eta_{15} \ll \eta_{14}$.

Fourth, the integrals involving $N_{\leq 14}$ can be bounded using Propositions 10.3.9 and 10.3.10 by noticing that $|\partial_{t^*} N_{\leq 14}| \leq C N_{\leq 15}$:

$$\begin{aligned}
&C(\delta')^{-1}A_0^{-1}\sum_{m=0}^{1}\iint_{\mathcal{R}(\tau_0,\tau)\cap\{r\leq\frac{9t^*}{10}\}}(t^*)^2 r^{1+\delta}\left(\partial_{t^*}^m N_{\leq 14}\right)^2\\
\leq&CA^2A_0^{-1}\epsilon^2(\delta')^{-1}\int_{\tau_0}^{\tau}(t^*)^{-2+\eta_{S,11}+\eta_{15}+2\delta}\,dt^* \leq CA^2A_0^{-1}\epsilon^2(\delta')^{-1},
\end{aligned}$$

$$C(\delta')^{-1}A_0^{-1}\left(\int_{\tau_0}^{\tau}\left(\int_{\Sigma_{t^*}\cap\{r\geq\frac{t^*}{2}\}} r^2 N_{15}^2\right)^{\frac{1}{2}} dt^*\right)^2 \leq CA^2A_0^{-1}\epsilon^2(\delta')^{-1}\eta_{15}^{-1}\tau^{\eta_{15}},$$

$$C(\delta')^{-1}A_0^{-1}\sup_{t^*\in[\tau_0,\tau]}\int_{\Sigma_{t^*}\cap\{r_Y^-\le r\le\frac{25M}{8}\}}(t^*)^2N_{15}^2\le CA^2A_0^{-1}\epsilon^2(\delta')^{-1},$$

which is acceptable since $\epsilon\ll A\delta\eta_{15}^{-1}$. Fifth, for the commutator terms $U^2_{\le 14,\le j}$, we estimate by Proposition 10.3.6,

$$\begin{aligned}(\delta')^{-1}A_j^{-1}\iint_{\mathcal{R}(\tau_0,\tau)}(t^*)^2r^{1+\delta}\left(U_{\le 14,\le j}\right)^2\le& C\sum_{i=0}^{n-1}(\delta')^{-1}A_{X,j-1}\tau_i^2\tau_i^{-2+\eta_{14}}\\ \le& C(\delta')^{-1}\eta_{14}^{-1}\frac{A_{X,j-1}}{A_j}\tau^{\eta_{14}},\end{aligned}$$

$$(\delta')^{-1}A_j^{-1}\left(\int_{\tau_0}^{\tau}\left(\int_{\Sigma_{t^*}\cap\{r\ge\frac{t^*}{2}\}}r^2U^2_{\le 14,j}\right)^{\frac{1}{2}}dt^*\right)^2\le C(\delta')^{-1}\frac{A_{j-1}}{A_j}\eta_{14}^{-1}\tau.$$

Since

$$\frac{A_{j-1}}{A_j}\ll\frac{A_{X,j-1}}{A_j}\ll\delta'\eta_{14}^{-1},$$

all terms are acceptable. □

We now consider terms involving commutation with $\hat{Y}$ and recover the bootstrap assumptions (10.5), (10.6) and (10.7).

Proposition 10.3.16.

$$\sum_{i+k=16}\int_{\Sigma_\tau}J_\mu^N\left(\hat{Y}^k\partial_{t^*}^i\Phi\right)n_{\Sigma_\tau}^\mu\le\frac{A_Y}{4}\epsilon\tau^{\eta_{16}},$$

and

$$\sum_{i+k=15}\tau^2\int_{\Sigma_\tau\cap\{r\le r_Y^-\}}J_\mu^N\left(\hat{Y}^k\partial_{t^*}^i\Phi\right)n_{\Sigma_\tau}^\mu\le\frac{A_Y}{4}\epsilon\tau^{1+\eta_{15}},$$

and

$$\sum_{i+k\le 14}\tau^2\int_{\Sigma_\tau\cap\{r\le r_Y^-\}}J_\mu^N\left(\hat{Y}^k\partial_{t^*}^i\Phi\right)n_{\Sigma_\tau}^\mu\le\frac{A_Y}{4}\epsilon\tau^{\eta_{14}}.$$

Proof. The idea is to use Proposition 13 and use the fact that it gives the control of an integrated in time quantity. From this we can extract a good slice to improve the constant. By Proposition 9.7.4,

$$\begin{aligned}
&\sum_{i+k=16}\left(\int_{\Sigma_\tau\cap\{r\le r_Y^+\}} J_\mu^N\left(\hat{Y}^k\partial_{t^*}^i\Phi\right)n_{\Sigma_\tau}^\mu+\iint_{\mathcal{R}(\tau',\tau)\cap\{r\le r_Y^-\}} J_\mu^N\left(\hat{Y}^k\partial_{t^*}^i\Phi\right)n_{\Sigma_{t^*}}^\mu\right)\\
\le& C\left(\sum_{j+m\le 16}\int_{\Sigma_{\tau'}\cap\{r\le r_Y^+\}} J_\mu^N\left(\partial_{t^*}^j\hat{Y}^m\Phi\right)n_{\Sigma_{\tau'}}^\mu+\sum_{j=0}^{16}\int_{\Sigma_\tau\cap\{r\le r_Y^+\}} J_\mu^N\left(\partial_{t^*}^j\Phi\right)n_{\Sigma_\tau}^\mu\right.\\
&\left.+\sum_{j=0}^{16}\iint_{\mathcal{R}(\tau',\tau)\cap\{r\le\frac{23M}{8}\}} J_\mu^N\left(\partial_{t^*}^j\Phi\right)n_{\Sigma_{t^*}}^\mu+\sum_{i+j\le 16}\iint_{\mathcal{R}(\tau',\tau)\cap\{r\le\frac{23M}{8}\}}\left(D^iG_{\le j,0}\right)^2\right)\\
\le& CA_Y(\tau')^{\eta_{16}}+CA_0\tau^{\eta_{16}}+CA^2\epsilon^2(\tau')^{-1+\eta_{16}},
\end{aligned}$$

by (10.5), (10.13) and Proposition 10.3.5. Take $\tau'=\tau-A_0$. Then

$$\sum_{i+k=16}\iint_{\mathcal{R}(\tau-A_0,\tau)\cap\{r\le r_Y^-\}} J_\mu^N\left(\hat{Y}^k\partial_{t^*}^i\Phi\right)n_{\Sigma_{t^*}}^\mu\le CA_Y\tau^{\eta_{16}}+CA_0\tau^{\eta_{16}}+CA^2\epsilon^2\tau^{-1+\eta_{16}}.$$

Hence there is some $\tilde{\tau}\in[\tau-A_0,\tau]$ such that

$$\sum_{i+k=16}\int_{\Sigma_{\tilde{\tau}}\cap\{r\le r_Y^-\}} J_\mu^N\left(\hat{Y}^k\partial_{t^*}^i\Phi\right)n_{\Sigma_{\tilde{\tau}}}^\mu\le CA_YA_0^{-1}\tau^{\eta_{16}}+C\tau^{\eta_{16}}+CA^2\epsilon^2\tau^{-1+\eta_{16}}.$$

We also have by (10.5) and the elliptic estimates in Proposition 9.7.1,

$$\begin{aligned}
&\sum_{i+k=16}\int_{\Sigma_{\tilde{\tau}}\cap\{r_Y^-\le r\le r_Y^+\}} J_\mu^N\left(\hat{Y}^k\partial_{t^*}^i\Phi\right)n_{\Sigma_{\tilde{\tau}}}^\mu\\
\le& C\sum_{i=0}^{16}\int_{\Sigma_{\tilde{\tau}}\cap\{r_Y^-\le r\le\frac{t^*}{2}\}} J_\mu^N\left(\partial_{t^*}^i\Phi\right)n_{\Sigma_{\tilde{\tau}}}^\mu+\sum_{i+j\le 15}\int_{\Sigma_{\tilde{\tau}}}\left(D^iG_{\le j,0}\right)^2\\
\le& CA_0\tau^{\eta_{16}}+CA^2\epsilon^2\tau^{-1+\eta_{16}}.
\end{aligned}$$

Now reapply Proposition 9.7.4, from $\tilde{\tau}$ to τ, we get

$$\begin{aligned}
&\sum_{i+k=16} \int_{\Sigma_\tau \cap \{r \leq r_Y^+\}} J_\mu^N \left(\hat{Y}^k \partial_{t^*}^i \Phi \right) n_{\Sigma_\tau}^\mu \\
\leq & C \sum_{j+m \leq 16} \int_{\Sigma_{\tilde{\tau}} \cap \{r \leq r_Y^+\}} J_\mu^N \left(\partial_{t^*}^j \hat{Y}^m \Phi \right) n_{\Sigma_{\tilde{\tau}}}^\mu + C A_0 \tau^{\eta_{16}} + C A^2 \epsilon^2 \\
\leq & C A_Y A_0^{-1} \tau^{\eta_{16}} + C A_0 \tau^{\eta_{16}} + C A^2 \epsilon^2 \tau^{-1+\eta_{16}}.
\end{aligned}$$

Since $C \ll A_0 \ll A_Y$, we get the first statement in the Proposition. The derivations for the other bounds are identical, with the constants and exponents replaced appropriately. □

From this we can also derive some integrated estimates for $Y^k \Gamma^j \Phi$. This will be useful in controlling the commutator $[\Box_{g_K}, S]$.

Proposition 10.3.17.

$$\sum_{i+k=16} \iint_{\mathcal{R}((1.1)^{-1}\tau,\tau) \cap \{r \leq r_Y^-\}} J_\mu^N \left(\hat{Y}^k \partial_{t^*}^i \Phi \right) n_{\Sigma_\tau}^\mu \leq A_Y \epsilon \tau^{\eta_{16}},$$

and

$$\sum_{i+k=15} \tau^2 \iint_{\mathcal{R}((1.1)^{-1}\tau,\tau) \cap \{r \leq r_Y^-\}} J_\mu^N \left(\hat{Y}^k \partial_{t^*}^i \Phi \right) n_{\Sigma_\tau}^\mu \leq A_Y \epsilon \tau^{1+\eta_{15}},$$

and

$$\sum_{i+k \leq 14} \tau^2 \iint_{\mathcal{R}((1.1)^{-1}\tau,\tau) \cap \{r \leq r_Y^-\}} J_\mu^N \left(\hat{Y}^k \partial_{t^*}^i \Phi \right) n_{\Sigma_\tau}^\mu \leq A_Y \epsilon \tau^{\eta_{14}}.$$

Proof. This is a direct consequence of Proposition 9.7.4, 10.3.5, 10.3.16, as well as the bootstrap assumptions (10.5), (10.6) and (10.7). □

We will finally proceed to the quantities associated to the vector field S. Recall

from the previous chapter that for large values of r

$$\begin{aligned}&|[\Box_{g_K}, S]\Phi - \left(2 + \frac{r^*\mu}{r}\right)\Box_g\Phi - \frac{2}{r}\left(\frac{r^*}{r} - 1 - \frac{2r^*\mu}{r}\right)\partial_{r^*}\Phi \\ &- 2\left(\left(\frac{r^*}{r} - 1\right) - \frac{3r^*\mu}{2r}\right)\not{\Delta}\Phi| \\ \leq & Car^{-2}(\sum_{k=1}^{2}|D^k\Phi|).\end{aligned}$$

and that for finite values of r, we have

$$|[\Box_{g_K}, S]\Phi| \leq C(\sum_{k=1}^{2}|D^k\Phi|).$$

Moreover, all the coefficients in the commutator term obey the same estimates (with a different constant) upon differentiation. Therefore,

$$\Box_{g_K}\left(S\Gamma^k\Phi\right) = V_k + S(U_k) + S(N_k),$$

where

$$\left(D^\ell V_k\right)^2 \leq Cr^{-4}\left(\log r\right)^2\left(\sum_{j=1}^{\ell+1}\left(D^j\Gamma^{k+1}\Phi\right)^2 + \sum_{j=1}^{\ell+2}\left(D^j\Gamma^k\Phi\right)^2\right).$$

We would now estimate these three terms separately. We first estimate the V_k terms:

Proposition 10.3.18. *For $\alpha \leq 2$,*

$$\sum_{\ell+k\leq 13}\int_{\Sigma_\tau} r^\alpha\left(D^\ell V_{\leq k}\right)^2 \leq CA_Y\epsilon\tau^{-2+\eta_{14}+\delta}.$$

For $\alpha \leq 1+\delta$,

$$\sum_{\ell+k=13}\iint_{\mathcal{R}((1.1)^{-1}\tau,\tau)} r^\alpha\left(D^\ell V_{\leq k}\right)^2 \leq CA_Y\epsilon\tau^{-1+\eta_{15}+\delta}.$$

$$\sum_{\ell+k\leq 12}\iint_{\mathcal{R}((1.1)^{-1}\tau,\tau)} r^{\alpha}\left(D^{\ell}V_{\leq k}\right)^2 \leq CA_Y\epsilon\tau^{-2+\eta_{14}+\delta}.$$

Proof. By the elliptic estimates in Propositions 9.7.1 and 9.7.3, we have, for $\alpha \leq 2$,

$$\begin{aligned}
&\sum_{\ell+k\leq 13}\int_{\Sigma_\tau} r^{\alpha}\left(D^{\ell}V_{\leq k}\right)^2\\
\leq& C\sum_{i+j\leq 12}\int_{\Sigma_\tau} r^{\alpha-4+\delta}J_\mu^N\left(\hat{Y}^i\Gamma^j\Phi\right)n_{\Sigma_\tau}^{\mu} + C\sum_{i+j\leq 11}\int_{\Sigma_\tau} r^{\alpha-4+\delta}\left(D^iG_{\leq j}\right)^2\\
\leq& C\left(A_Y\epsilon + A^2\epsilon^2\right)\tau^{-2+\eta_{14}+\delta},
\end{aligned}$$

where we have used Propositions 10.3.5 and 10.3.6, the bootstrap assumptions (10.7) (for $r \leq \frac{9t^*}{10}$) and (10.8) (for $r \geq \frac{9t^*}{10}$).

By the elliptic estimates in Propositions 9.7.1 and 9.7.3, we have

$$\begin{aligned}
&\iint_{\mathcal{R}((1.1)^{-1}\tau,\tau)} r^{\alpha}\left(D^{\ell}V_{\leq k}\right)^2\\
\leq& C\sum_{i+j=0}^{\ell+k+1}\iint_{\mathcal{R}((1.1)^{-1}\tau,\tau)} r^{\alpha-4+\delta}J_\mu^N\left(\hat{Y}^i\Gamma^j\Phi\right)n_{\Sigma_\tau}^{\mu} + \sum_{i+j=0}^{\ell+k}\iint_{\mathcal{R}((1.1)^{-1}\tau,\tau)} r^{\alpha-4+\delta}\left(D^iG_{\leq j}\right)^2
\end{aligned}$$

We first consider the case $\ell + k = 13$. For the first term, we divide into $r \leq \frac{t^*}{2}$ (which we estimate by (10.18) and Proposition 10.3.17) and $r \geq \frac{t^*}{2}$ (which we estimate using the extra decay in r by (10.15)). The second term contains the U_k and the N_k part. The U_k part can be estimated by Proposition 10.3.6. The N_k part can be estimated by Proposition 10.3.5. The $\ell + k \leq 12$ case in completely analogous, replacing the bootstrap assumption (10.18) by (10.20). □

We then proceed to the estimates for $S(U_k)$. Notice that when we prove the estimates for the derivatives for $S(U_k)$, the derivatives for $S(N_k)$ will be involved. Like the proof of the estimates for U_k, we will first prove estimates for the derivatives of $S(U_k)$ depending on $S(N_k)$, and close the estimates after we control $S(N_k)$.

Proposition 10.3.19. *The following estimates for $S(U_k)$ on a fixed t^* slice hold for*

$\alpha \leq 2$:

$$\sum_{k+\ell=13} \int_{\Sigma_\tau} r^\alpha \left(D^\ell \left(S(U_{k,j})\right)\right)^2 \leq C A_{S,j-1} \epsilon \tau^{\eta_{S,13}} + \sum_{m=1}^{13-j} \int_{\Sigma_\tau} (D^m S(N_{\leq j-1}))^2.$$

$$\sum_{k+\ell=12} \int_{\Sigma_\tau} r^\alpha \left(D^\ell \left(S(U_{k,j})\right)\right)^2 \leq C A_{S,j-1} \epsilon \tau^{-1+\eta_{S,12}} + \sum_{m=1}^{12-j} \int_{\Sigma_\tau} (D^m S(N_{\leq j-1}))^2.$$

$$\sum_{k+\ell\leq 11} \int_{\Sigma_\tau} r^\alpha \left(D^\ell \left(S(U_{k,j})\right)\right)^2 \leq C A_{S,j-1} \epsilon \tau^{-2+\eta_{S,11}} + \sum_{m=1}^{11-j} \int_{\Sigma_\tau} (D^m S(N_{\leq j-1}))^2.$$

The following estimates for $S(U_k)$ integrated on $[(1.1)^{-1}\tau, \tau]$ also hold for $\alpha \leq 1+\delta$:

$$\begin{aligned}
&\sum_{k+\ell=13} \iint_{\mathcal{R}((1.1)^{-1}\tau,\tau)} r^\alpha \left(D^\ell \left(S(U_{k,j})\right)\right)^2 \\
\leq & C A_{S,X,j-1} \epsilon \tau^{\eta_{S,13}} + \sum_{m=1}^{13-j} \iint_{\mathcal{R}((1.1)^{-1}\tau,\tau)} r^{-2} (D^m S(N_{\leq j-1}))^2.
\end{aligned}$$

$$\begin{aligned}
&\sum_{k+\ell=12} \iint_{\mathcal{R}((1.1)^{-1}\tau,\tau)} r^\alpha \left(D^\ell \left(S(U_{k,j})\right)\right)^2 \\
\leq & C A_{S,X,j-1} \epsilon \tau^{-1+\eta_{S,12}} + \sum_{m=1}^{12-j} \iint_{\mathcal{R}((1.1)^{-1}\tau,\tau)} r^{-2} (D^m S(N_{\leq j-1}))^2.
\end{aligned}$$

$$\begin{aligned}
&\sum_{k+\ell\leq 11} \iint_{\mathcal{R}((1.1)^{-1}\tau,\tau)} r^\alpha \left(D^\ell \left(S(U_{k,j})\right)\right)^2 \\
\leq & C A_{S,X,j-1} \epsilon \tau^{-2+\eta_{S,11}} + \sum_{m=1}^{11-j} \iint_{\mathcal{R}((1.1)^{-1}\tau,\tau)} r^{-2} (D^m S(N_{\leq j-1}))^2.
\end{aligned}$$

Proof. Notice that $D^\ell \left(S(U_{k,j})\right)$ is supported in $\{r \geq R_\Omega\}$ and satisfies

$$\begin{aligned}
|D^\ell \left(S(U_{k,j})\right)| &\leq C \sum_{m=1}^{\ell+2} \sum_{i=0}^{j-1} r^{-2} \left(|D^m S \partial_{t^*}^{k-j} \tilde{\Omega}^i \Phi| + |D^m \partial_{t^*}^{k-j} \tilde{\Omega}^i \Phi| \right) \\
&\leq C \sum_{m=1}^{\ell+k-j+2} \sum_{i=0}^{j-1} r^{-2} \left(|D^m S \tilde{\Omega}^i \Phi| + |D^m \tilde{\Omega}^i \Phi| \right).
\end{aligned}$$

We can ignore the last term because it appears already in $D^\ell U_{k,j}$ and can be estimated

by Proposition 10.3.6.

$$\begin{aligned}
&\int_{\Sigma_\tau} r^\alpha \left(D^\ell \left(S(U_{k,j})\right)\right)^2 \\
\leq & C \sum_{m=1}^{\ell+k-j+2} \sum_{i=0}^{j-1} \int_{\Sigma_\tau \cap \{r \geq R_\Omega\}} r^{\alpha-4} \left(D^m S \tilde{\Omega}^i \Phi\right)^2 \\
\leq & C \sum_{m=1}^{\ell+k-j+1} \sum_{i=0}^{j-1} \int_{\Sigma_\tau \cap \{r \geq R_\Omega - 1\}} r^{\alpha-4} J_\mu^N \left(\partial_{t^*}^m S \tilde{\Omega}^i \Phi\right) n_{\Sigma_\tau}^\mu \\
& + C \sum_{m=1}^{\ell+k-j} \sum_{i=0}^{j-1} \int_{\Sigma_\tau} r^{-2} \left(D^m \Box_{g_K} (S \tilde{\Omega}^i \Phi)\right)^2 \\
\leq & C \sum_{m=1}^{\ell+k-j+1} \sum_{i=0}^{j-1} \int_{\Sigma_\tau \cap \{r \geq R_\Omega - 1\}} r^{\alpha-4} J_\mu^N \left(\partial_{t^*}^m S \tilde{\Omega}^i \Phi\right) n_{\Sigma_\tau}^\mu \\
& + C \sum_{m=1}^{\ell+k-j} \int_{\Sigma_\tau} r^{-2} \left((D^m S(U_{\leq j-1, \leq j-1}))^2 + (D^m S(N_{\leq j-1}))^2 + (D^m V_{\leq j-1})^2\right).
\end{aligned}$$

We now apply the bootstrap assumptions. By bootstrap assumption (10.9) and Proposition 10.3.18,

$$\begin{aligned}
&\sum_{k+\ell=13} \int_{\Sigma_\tau} r^\alpha \left(D^\ell \left(S(U_{k,j})\right)\right)^2 \\
\leq & C A_{S,j-1} \epsilon \tau^{\eta_{S,13}} + \sum_{m=1}^{13-j} \int_{\Sigma_\tau} \left((D^m S(U_{\leq j-1, \leq j-1}))^2 + (D^m S(N_{\leq j-1}))^2\right).
\end{aligned}$$

By bootstrap assumption (10.8) (for $r \geq \frac{t^*}{2}$), (10.10) (for $r \leq \frac{t^*}{2}$) and Proposition 10.3.18,

$$\begin{aligned}
&\sum_{k+\ell=12} \int_{\Sigma_\tau} r^\alpha \left(D^\ell \left(S(U_{k,j})\right)\right)^2 \\
\leq & C A_{S,j-1} \epsilon \tau^{-1+\eta_{S,12}} + \sum_{m=1}^{12-j} \int_{\Sigma_\tau} \left((D^m S(U_{\leq j-1, \leq j-1}))^2 + (D^m S(N_{\leq j-1}))^2\right).
\end{aligned}$$

By bootstrap assumption (10.8) (for $r \geq \frac{t^*}{2}$), (10.11) (for $r \leq \frac{t^*}{2}$) and Proposition

10.3.18,

$$\sum_{k+\ell\leq 11}\int_{\Sigma_\tau} r^\alpha \left(D^\ell\left(S(U_{k,j})\right)\right)^2$$
$$\leq CA_{S,j-1}\epsilon\tau^{-2+\eta_{S,11}} + \sum_{m=1}^{11-j}\int_{\Sigma_\tau}\left((D^m S(U_{\leq j-1,\leq j-1}))^2 + (D^m S(N_{\leq j-1}))^2\right).$$

Noticing that $U_{k,0}=0$, we can conclude the first three statements in the Proposition using an induction in j. (See the proof of Proposition 10.3.4). For the integrated in time estimate, we note that $r^{-1-\delta}J^N_\mu\left(\Gamma^i\Phi\right) \leq CK^{X_0}\left(\Gamma^i\Phi\right)$ and use the bootstrap assumptions (10.21), (10.22), (10.23) and (10.25) (See proof of Propositions 10.3.4 and 10.3.6). □

We then move on to the $S(N_k)$ terms, first we will prove an estimate for the derivatives of $S(N_k)$. The decay rate here is not optimal, but would be sufficient to close the bootstrap argument. Our approach here is to prove the decay rate that is driven only by the pointwise decay of $D^\ell\Phi$ but not by that of $D^\ell S\Phi$. The latter can, in principle, be done by similar methods, but we will skip it since it will not be necessary. In subsequent propositions, we will then prove refined decay rate for $S(N_k)$ (without derivatives) as well as for $D^\ell S(N_k)$ restricted to the region $r\leq\frac{t^*}{4}$.

Proposition 10.3.20. *$S(N_k)$ satisfies the following estimates for any fixed $t^*=\tau$:*

$$\sum_{k+\ell=13}\int_{\Sigma_\tau}(D^\ell S(N_k))^2 \leq CB_S A^2\epsilon^2\tau^{-2+\eta_{S,11}}$$

$$\sum_{k+\ell\leq 12}\int_{\Sigma_\tau}(D^\ell S(N_k))^2 \leq CA^2\epsilon^2\tau^{-2+\eta_{14}+\delta}$$

$S(N_k)$ also satisfies the following integrated estimates over $t^*\in[(1.1)^{-1}\tau,\tau]$:

$$\sum_{k+\ell=12}\iint_{\mathcal{R}((1.1)^{-1}\tau,\tau)} r^{-1-\delta}(D^\ell S(N_k))^2 \leq CB_S A^2\epsilon^2\tau^{-2+\eta_{S,11}}$$

$$\sum_{k+\ell\leq 11}\iint_{\mathcal{R}((1.1)^{-1}\tau,\tau)} r^{-1-\delta}(D^{\ell}S(N_k))^2 \leq CA^2\epsilon^2\tau^{-2+\eta_{14}+\delta}$$

Proof. We would like to do a reduction similar to how we estimated N_k. Clearly, only the quadratic and cubic terms matter and we only need to consider terms that contain S for the other terms are already controlled by the estimates of N_k. We will denote terms that are already in N_k by "good terms". The only cubic terms that are relevant are those which contain $S\Gamma^i\Phi$ since in the terms with $D^{j+1}S\Gamma^i\Phi$, we can put all but one other factors in L^∞ using the bootstrap assumptions (10.27), (10.29), (10.35) and (10.36). Notice also that the conditions for $D_\Phi\Lambda_0$, $D_\Phi\Lambda_1$ and $D_\Phi\mathcal{C}$ in the definition of the null condition guarantee that the bounds do not deteriorate if S acts on the coefficients. The relevant terms are

$$(D^{j_1}S\Gamma^{i_1}\Phi)(D^{j_2}\Gamma^{i_2}\Phi), \quad \text{and}$$

$$(D^{j_1}\Gamma^{i_1}\Phi)(D^{j_2}\Gamma^{i_2}\Phi)(S\Gamma^{i_3}\Phi).$$

We first treat the case that $k+\ell \leq 12$. In this case we will always put factors

without S in L^∞.

$$
\begin{aligned}
&\int_{\Sigma_\tau} (D^\ell S(N_k))^2 \\
\leq & C \sum_{i_1+i_2+j_1+j_2\leq k+\ell+2, j_1,j_2\geq 1} \int_{\Sigma_\tau} (D^{j_1} S\Gamma^{i_1}\Phi)^2 (D^{j_2}\Gamma^{i_2}\Phi)^2 \\
& + C \sum_{i_1+i_2+i_3+j_1+j_2\leq 14, j_1,j_2\geq 1} \int_{\Sigma_\tau} (D^{j_1}\Gamma^{i_1}\Phi)^2 (D^{j_2}\Gamma^{i_2}\Phi)^2 (S\Gamma^{i_3}\Phi)^2 + \text{good terms} \\
\leq & C \left(\sum_{i+j\leq k+\ell+1, j\geq 1} \sup \left(D^j\Gamma^i\Phi\right)^2 \right) \sum_{i+j\leq k+\ell+1, j\geq 1} \int_{\Sigma_\tau} (D^j S\Gamma^i\Phi)^2 \\
& + C \left(\sum_{i+j\leq k+\ell+1, j\geq 1} \sup \left(D^j\Gamma^i\Phi\right)^2 \right) \left(\sum_{i+j\leq k+\ell+1, j\geq 1} \sup r^2 \left(D^j\Gamma^i\Phi\right)^2 \right) \\
& \qquad \times \left(\sum_{i\leq k+\ell} \int_{\Sigma_\tau} r^{-2} (S\Gamma^i\Phi)^2 \right) \\
& + \text{good terms} \\
\leq & CA\epsilon\tau^{-2+\eta_{14}} \sum_{i+j\leq k+\ell+1, j\geq 1} \int_{\Sigma_\tau} (D^j S\Gamma^i\Phi)^2 + CA^2\epsilon^2\tau^{-2+\eta_{14}} \sum_{i\leq k+\ell} \int_{\Sigma_\tau} J^N_\mu (S\Gamma^i\Phi) n^\mu_{\Sigma_\tau} \\
& + \text{good terms} \\
& \quad \text{using the bootstrap assumption (10.28), (10.29), (10.33), (10.37)} \\
& \quad \text{and (10.40) and Proposition 9.4.10} \\
\leq & CA\epsilon\tau^{-2+\eta_{14}} \sum_{i\leq k+\ell} \int_{\Sigma_\tau} \left(J^N_\mu \left(S\Gamma^i\Phi\right) n^\mu_{\Sigma_\tau} + J^N_\mu \left(\Gamma^i\Phi\right) n^\mu_{\Sigma_\tau} \right) \\
& + CA\epsilon\tau^{-2+\eta_{14}} \sum_{i+j\leq k+\ell-1} \int_{\Sigma_\tau} \Big((D^i U_{\leq j})^2 + (D^i S(U_{\leq j}))^2 + \left(D^i N_{\leq j}\right)^2 \\
& \qquad + \left(D^i S(N_{\leq j})\right)^2 + \left(D^i V_{\leq j}\right)^2 \Big) .
\end{aligned}
$$

We now apply the estimates for the inhomogeneous terms, i.e., Propositions 10.3.5,

10.3.6, 10.3.18 10.3.19. Since $k + \ell \leq 12$,:

$$\begin{aligned}
&\int_{\Sigma_\tau} (D^\ell S(N_k))^2 \\
\leq & CA\epsilon\tau^{-2+\eta_{14}} \sum_{i \leq 12} \int_{\Sigma_\tau} \left(J^N_\mu \left(S\Gamma^i \Phi \right) n^\mu_{\Sigma_\tau} + J^N_\mu \left(\Gamma^i \Phi \right) n^\mu_{\Sigma_\tau} \right) + CA^2\epsilon^2\tau^{-2+\eta_{14}+\delta} \\
&+ CA\epsilon\tau^{-2+\eta_{14}} \sum_{i+j \leq k+\ell-1} \int_{\Sigma_\tau} \left(D^i S(N_{\leq j}) \right)^2 .
\end{aligned}$$

The desired estimates then follow from an induction, together with the bootstrap assumptions (10.8) and (10.12), since according to this notation $\sum_{i+j=0}^{-1} = 0$.

We then treat the case that $k + \ell = 13$. In this case it is possible to have 14 derivatives falling on the factor with Φ and hence cannot be controlled in L^∞. However, in this scenario, we must have

$$\sum_{i+j=14} (DS\Phi)(D^j \Gamma^i \Phi)$$

and therefore $DS\Phi$ can be controlled in L^∞ by the bootstrap assumptions (10.33)

and (10.40). In short, we have

$$
\begin{aligned}
&\sum_{k+\ell=13}\int_{\Sigma_\tau}(D^\ell S(N_k))^2\\
\leq& C\sum_{i_1+i_2+j_1+j_2\leq 15, j_1,j_2\geq 1}\int_{\Sigma_\tau}(D^{j_1}S\Gamma^{i_1}\Phi)^2(D^{j_2}\Gamma^{i_2}\Phi)^2\\
&+C\sum_{i_1+i_2+i_3+j_1+j_2\leq 15, j_1,j_2\geq 1}\int_{\Sigma_\tau}(D^{j_1}\Gamma^{i_1}\Phi)^2(D^{j_2}\Gamma^{i_2}\Phi)^2(S\Gamma^{i_3}\Phi)^2+\text{good terms}\\
\leq& C\left(\sum_{i+j\leq 14, j\geq 1}\sup\left(D^j\Gamma^i\Phi\right)^2\right)\sum_{i+j\leq 14, j\geq 1}\int_{\Sigma_\tau}(D^jS\Gamma^i\Phi)^2\\
&+C\left(\sup(DS\Phi)^2\right)\sum_{i+j=14, j\geq 1}\int_{\Sigma_\tau}(D^j\Gamma^i\Phi)^2\\
&+C\left(\sum_{i+j\leq 14, j\geq 1}\sup\left(D^j\Gamma^i\Phi\right)^2\right)\left(\sum_{i+j\leq 14, j\geq 1}\sup r^2\left(D^j\Gamma^i\Phi\right)^2\right)\sum_{i\leq 13}\int_{\Sigma_\tau}r^{-2}(S\Gamma^i\Phi)^2\\
&+\text{good terms}\\
\leq& CA\epsilon\tau^{-2+\eta_{14}}\sum_{i+j\leq 14, j\geq 1}\int_{\Sigma_\tau}(D^jS\Gamma^i\Phi)^2+CB_SA\epsilon\tau^{-2+\eta_{S,11}}\sum_{i+j=14, j\geq 1}\int_{\Sigma_\tau}(D^j\Gamma^i\Phi)^2\\
&+CA^2\epsilon^2\tau^{-2+\eta_{14}}\sum_{i\leq 13}\int_{\Sigma_\tau}J^N_\mu(S\Gamma^i\Phi)n^\mu_{\Sigma_\tau}+\text{good terms}\\
&\qquad\text{using the bootstrap assumption (10.28), (10.29), (10.33), (10.37)}\\
&\qquad\text{and (10.40) and Proposition 9.4.10}\\
\leq& CA\epsilon\tau^{-2+\eta_{14}}\sum_{i\leq 13}\int_{\Sigma_\tau}J^N_\mu\left(S\Gamma^i\Phi\right)n^\mu_{\Sigma_\tau}+CB_SA\epsilon\tau^{-2+\eta_{S,11}}\sum_{i\leq 13}\int_{\Sigma_\tau}J^N_\mu\left(\Gamma^i\Phi\right)n^\mu_{\Sigma_\tau}\\
&+CA\epsilon\tau^{-2+\eta_{14}}\sum_{\ell+k\leq 12}\int_{\Sigma_\tau}\Big((D^\ell U_{\leq k})^2+(D^iS(U_{\leq k}))^2+\left(D^\ell N_{\leq k}\right)^2\\
&\qquad\qquad+\left(D^\ell S(N_{\leq k})\right)^2+\left(D^\ell V_{\leq k}\right)^2\Big).
\end{aligned}
$$

We now apply the estimates for the inhomogeneous terms, i.e., Propositions 10.3.5,

10.3.6, 10.3.18, 10.3.19:

$$\begin{aligned}
&\sum_{k+\ell=13}\int_{\Sigma_\tau}(D^\ell S(N_k))^2\\
\leq& CA\epsilon\tau^{-2+\eta_{14}}\sum_{i\leq 13}\int_{\Sigma_\tau}J^N_\mu\left(S\Gamma^i\Phi\right)n^\mu_{\Sigma_\tau}+CB_SA\epsilon\tau^{-2+\eta_{S,11}}\sum_{i\leq 13}\int_{\Sigma_\tau}J^N_\mu\left(\Gamma^i\Phi\right)n^\mu_{\Sigma_\tau}\\
&+CA\epsilon\tau^{-2+\eta_{14}}\sum_{\ell+k\leq 12}\int_{\Sigma_\tau}\left(D^\ell S(N_{\leq k})\right)^2,
\end{aligned}$$

which is acceptable. The estimates for the integrated in t^* terms are proved analogously, noting that the elliptic estimate in Proposition 9.7.1 would allow for weight in r and use the second parts of Propositions 10.3.5, 10.3.6, 10.3.18 and 10.3.19. □

This would allow us to close the estimates for $S(U_k)$ from Proposition 10.3.19.

Proposition 10.3.21. *The following estimates for $S(U_k)$ on a fixed t^* slice hold for $\alpha\leq 2$:*

$$\sum_{k+\ell=13}\int_{\Sigma_\tau}r^\alpha\left(D^\ell\left(S(U_{k,j})\right)\right)^2\leq CA_{S,j-1}\epsilon\tau^{\eta_{S,13}}.$$

$$\sum_{k+\ell=12}\int_{\Sigma_\tau}r^\alpha\left(D^\ell\left(S(U_{k,j})\right)\right)^2\leq CA_{S,j-1}\epsilon\tau^{-1+\eta_{S,12}}.$$

$$\sum_{k+\ell\leq 11}\int_{\Sigma_\tau}r^\alpha\left(D^\ell\left(S(U_{k,j})\right)\right)^2\leq CA_{S,j-1}\epsilon\tau^{-2+\eta_{S,11}}.$$

The following estimates for $S(U_k)$ integrated on $[(1.1)^{-1}\tau,\tau]$ also hold for $\alpha\leq 1+\delta$:

$$\sum_{k+\ell=13}\iint_{\mathcal{R}((1.1)^{-1}\tau,\tau)}r^\alpha\left(D^\ell\left(S(U_{k,j})\right)\right)^2\leq CA_{S,X,j-1}\epsilon\tau^{\eta_{S,13}}.$$

$$\sum_{k+\ell=12}\iint_{\mathcal{R}((1.1)^{-1}\tau,\tau)}r^\alpha\left(D^\ell\left(S(U_{k,j})\right)\right)^2\leq CA_{S,X,j-1}\epsilon\tau^{-1+\eta_{S,12}}.$$

$$\sum_{k+\ell\leq 11}\iint_{\mathcal{R}((1.1)^{-1}\tau,\tau)}r^\alpha\left(D^\ell\left(S(U_{k,j})\right)\right)^2\leq CA_{S,X,j-1}\epsilon\tau^{-2+\eta_{S,11}}.$$

Proof. This follows directly from Proposition 10.3.19 and 10.3.20. □

In the region $\{r \leq \frac{t^*}{4}\}$, we have refined decay rates for $D^\ell S(N_k)$:

Proposition 10.3.22.

$$\sum_{k+\ell=13} \int_{\Sigma_\tau \cap \{r \leq \frac{t^*}{4}\}} r^{1-\delta} (D^\ell S(N_k))^2 \leq C A^2 \epsilon^2 \tau^{-3+\eta_{S,11}}.$$

$$\sum_{k+\ell\leq 12} \int_{\Sigma_\tau \cap \{r \leq \frac{t^*}{4}\}} r^{1-\delta} (D^\ell S(N_k))^2 \leq C A^2 \epsilon^2 \tau^{-4+\eta_{S,12}+\eta_{S,11}}.$$

Proof. Take $k + \ell \leq 13$. Notice that $|[D, S]\Phi| \leq C|D\Phi|$.

We would like to do a reduction similar to how we estimated N_k. Clearly, only the quadratic and cubic terms matter and we only need to consider terms that contain S for the other terms are already controlled by the estimates of N_k. Notice also as before that the conditions in the null condition guarantee that the bounds do not deteriorate if S acts on the coefficients. The relevant terms are

$$(D^{j_1} S\Gamma^{i_1}\Phi)(D^{j_2}\Gamma^{i_2}\Phi) \quad j_1, j_2 \geq 1,$$

$$(D^{j_1} S\Gamma^{i_1}\Phi)(D^{j_2}\Gamma^{i_2}\Phi)(\Gamma^{i_3}\Phi), \quad j_1, j_2 \geq 1, i_3 > 8 \quad \text{and}$$

$$(D^{j_1}\Gamma^{i_1}\Phi)(D^{j_2}\Gamma^{i_2}\Phi)(S\Gamma^{i_3}\Phi) \quad j_1, j_2 \geq 1, i_3 > 8.$$

We first tackle the quadratic terms:

$$\begin{aligned}
&\sum_{i_1+\ell_1\leq 7,\ell_1\geq 1}\sum_{i_2+j_2\leq k+\ell+1,j_2\geq 1}\int_{\Sigma_\tau\cap\{r\leq\frac{\tau}{4}\}} r^{1-\delta}\left(|D^{j_1}S\Gamma^{i_1}\Phi D^{j_2}\Gamma^{i_2}\Phi|^2\right.\\
&\qquad\qquad\left.+|D^{j_1}\Gamma^{i_1}\Phi D^{j_2}S\Gamma^{i_2}\Phi|^2\right)\\
\leq& C\left(\sup_{r\leq\frac{\tau}{4}}\sum_{i+j\leq 7,j\geq 1} r^{1-\delta}|D^jS\Gamma^i\Phi|^2\right)\left(\sum_{i+j\leq k+\ell+1,j\geq 1}\int_{\Sigma_\tau\cap\{r\leq\frac{\tau}{4}\}}|D^j\Gamma^i\Phi|^2\right)\\
&+C\left(\sup_{r\leq\frac{\tau}{4}}\sum_{i+j\leq 7,j\geq 1} r^{1-\delta}|D^j\Gamma^i\Phi|^2\right)\left(\sum_{i+j\leq k+\ell+1,j\geq 1}\int_{\Sigma_\tau\cap\{r\leq\frac{\tau}{4}\}}|D^jS\Gamma^i\Phi|^2\right)\\
\leq& CA\epsilon\tau^{-2+\eta_{S,11}}\sum_{i+j\leq k+\ell}\int_{\Sigma_\tau\cap\{r\leq\frac{9\tau}{10}\}} J_\mu^N(\hat{Y}^j\Gamma^i\Phi)\\
&+CA\epsilon\tau^{-3+\eta_{S,11}}\sum_{i+j\leq k+\ell}\int_{\Sigma_\tau\cap\{r\leq\frac{9\tau}{10}\}} J_\mu^N(\hat{Y}^jS\Gamma^i\Phi)\\
&+CA\epsilon\tau^{-2+\eta_{S,11}}\sum_{i+j\leq k+\ell-1}\int_{\Sigma_\tau\cap\{r\leq\frac{9\tau}{10}\}}\left((D^iU_j)^2+(D^iN_j)^2\right)\\
&+CA\epsilon\tau^{-3+\eta_{S,11}}\sum_{i+j\leq k+\ell-1}\int_{\Sigma_\tau\cap\{r\leq\frac{9\tau}{10}\}}\left((D^iS(U_j))^2+(D^iS(N_j))^2+(D^iV_j)^2\right)
\end{aligned}$$

by the bootstrap assumptions (10.29) and (10.31) and the elliptic estimates Proposition 9.7.1 and 9.7.3. Since $k+\ell-1\leq 12$, the inhomogeneous terms can be bounded using Proposition 10.3.5, 10.3.6, 10.3.18, 10.3.20 and 10.3.21 to be

$$\leq CA^2\epsilon^2\tau^{-4+\eta_{S,12}+\eta_{S,11}}.$$

We then move on to the cubic terms:

$$\begin{aligned}
&\sum_{i_1+j_1\leq 7, j_1\geq 1}\sum_{i_2+j_2\leq 7, j_2\geq 1}\sum_{i_3=0}^{k}\int_{\Sigma_\tau\cap\{r\leq\frac{9\tau}{10}\}} r^{1-\delta}\left((D^{j_1}S\Gamma^{i_1}\Phi D^{j_2}\Gamma^{i_2}\Phi\Gamma^{i_3}\Phi)^2\right.\\
&\qquad\qquad\left.+(D^{j_1}\Gamma^{i_1}\Phi D^{j_2}\Gamma^{i_2}\Phi S\Gamma^{i_3}\Phi)^2\right)\\
\leq& C\left(\sup_{r\leq\frac{\tau}{4}}\sum_{i+j\leq 7, j\geq 1} r^2\left(D^jS\Gamma^i\Phi\right)^2\right)\left(\sup_{r\leq\frac{\tau}{4}}\sum_{i+j\leq 7, j\geq 1} r^{1-\delta}\left(D^j\Gamma^i\Phi\right)^2\right)\\
&\quad\times\left(\sum_{i=0}^{k}\int_{\Sigma_\tau\cap\{r\leq\frac{\tau}{4}\}} r^{-2}(\Gamma^{i_3}\Phi)^2\right)\\
&+C\left(\sup_{r\leq\frac{\tau}{4}}\sum_{i+j\leq 7, j\geq 1} r^{1-\delta}(D^j\Gamma^i\Phi)^2\right)^2\tau^{1+\delta}\sum_{i=0}^{k}\int_{\Sigma_\tau\cap\{r\leq\frac{\tau}{4}\}} r^{-2}(S\Gamma^i\Phi)^2\\
\leq& CA^2\epsilon^2\tau^{-5+2\eta_{S,11}}\sum_{i=0}^{k}\int_{\Sigma_\tau}(D\Gamma^i\Phi)^2+CA^2\epsilon^2\tau^{-5+2\eta_{S,11}+\delta}\sum_{i=0}^{k}\int_{\Sigma_\tau}(DS\Gamma^i\Phi)^2,
\end{aligned}$$

by the bootstrap assumptions (10.29) and (10.31), which now clearly decays better than we need by using the bootstrap assumptions (10.8), (10.9) and (10.12). Therefore,

$$\begin{aligned}
&\int_{\Sigma_\tau\cap\{r\leq\frac{\tau}{4}\}} r^{1-\delta}(D^\ell S(N_k))^2\\
\leq& CA\epsilon\tau^{-2+\eta_{S,11}}\sum_{i+j\leq k+\ell}\int_{\Sigma_\tau\cap\{r\leq\frac{9\tau}{10}\}} J_\mu^N(\hat{Y}^j\Gamma^i\Phi)\\
&+CA\epsilon\tau^{-3+\eta_{S,11}}\sum_{i+j\leq k+\ell}\int_{\Sigma_\tau\cap\{r\leq\frac{9\tau}{10}\}} J_\mu^N(\hat{Y}^jS\Gamma^i\Phi)+CA^2\epsilon^2\tau^{-4+\eta_{S,12}+\eta_{S,11}}.
\end{aligned}$$

The Proposition follows from the Bootstrap Assumptions (10.7), (10.9), (10.10) and (10.11). □

A similar decay rate can be proved in the region $\{r\leq\frac{9t^*}{10}\}$, if we do not require the estimate for the derivatives:

Proposition 10.3.23.

$$\int_{\Sigma_\tau \cap \{r \leq \frac{9t^*}{10}\}} r^{1-\delta} (S(N_{13}))^2 \leq CA^2\epsilon^2\tau^{-3+\eta_{S,11}}$$

$$\sum_{k=0}^{12} \int_{\Sigma_\tau \cap \{r \leq \frac{9t^*}{10}\}} r^{1-\delta} (S(N_k))^2 \leq CA^2\epsilon^2\tau^{-4+\eta_{S,12}+\eta_{S,11}}$$

Proof. Take $k \leq 13$. The proof follows very closely from that of the previous Proposition, by noting that we have similar pointwise decay estimates in the region (without higher derivatives) by the bootstrap assumptions (10.30) and (10.34). As in the previous Proposition, the relevant terms are

$$(DS\Gamma^{i_1}\Phi)(D\Gamma^{i_2}\Phi),$$

$$(DS\Gamma^{i_1}\Phi)(D\Gamma^{i_2}\Phi)(\Gamma^{i_3}\Phi), \quad i_3 > 8 \quad \text{and}$$

$$(D\Gamma^{i_1}\Phi)(D\Gamma^{i_2}\Phi)(S\Gamma^{i_3}\Phi) \quad i_3 > 8.$$

We first tackle the quadratic terms:

$$\begin{aligned}
&\sum_{i_1=0}^{6}\sum_{i_2=0}^{k} \int_{\Sigma_\tau \cap \{r \leq \frac{9\tau}{10}\}} r^{1-\delta} \left(|DS\Gamma^{i_1}\Phi D\Gamma^{i_2}\Phi|^2 + |D\Gamma^{i_1}\Phi DS\Gamma^{i_2}\Phi|^2 \right) \\
\leq & C \left(\sup_{r \leq \frac{9\tau}{10}} \sum_{i=0}^{6} r^{1-\delta} |DS\Gamma^i\Phi|^2 \right) \left(\sum_{i=0}^{k} \int_{\Sigma_\tau \cap \{r \leq \frac{9\tau}{10}\}} |D\Gamma^i\Phi|^2 \right) \\
& + C \left(\sup_{r \leq \frac{9\tau}{10}} \sum_{i=0}^{6} r^{1-\delta} |D\Gamma^i\Phi|^2 \right) \left(\sum_{i=0}^{k} \int_{\Sigma_\tau \cap \{r \leq \frac{9\tau}{10}\}} |DS\Gamma^i\Phi|^2 \right) \\
\leq & CA^2\epsilon^2\tau^{-4+\eta_{14}+\eta_{S,11}} + CA\epsilon\tau^{-3+\eta_{S,11}} \left(\sum_{i=0}^{k} \int_{\Sigma_\tau \cap \{r \leq \frac{9\tau}{10}\}} |DS\Gamma^i\Phi|^2 \right).
\end{aligned}$$

We then move on to the cubic terms:

$$\begin{aligned}
&\sum_{i_1,i_2=0}^{6}\sum_{i_3=0}^{k}\int_{\Sigma_\tau\cap\{r\leq\frac{9\tau}{10}\}} r^{1-\delta}\left((DS\Gamma^{i_1}\Phi D\Gamma^{i_2}\Phi\Gamma^{i_3}\Phi)^2+(D\Gamma^{i_1}\Phi D\Gamma^{i_2}\Phi S\Gamma^{i_3}\Phi)^2\right)\\
\leq& C\left(\sup_{r\leq\frac{9\tau}{10}}\sum_{i=0}^{6} r^2\left(DS\Gamma^i\Phi\right)^2\right)\left(\sup_{r\leq\frac{9\tau}{10}}\sum_{i=0}^{6} r^{1-\delta}\left(D\Gamma^i\Phi\right)^2\right)\sum_{i=0}^{k}\int_{\Sigma_\tau\cap\{r\leq\frac{9\tau}{10}\}} r^{-2}(\Gamma^i\Phi)^2\\
&+C\left(\sup_{r\leq\frac{9\tau}{10}}\sum_{i=0}^{6} r^{1-\delta}(D\Gamma^i\Phi)^2\right)^2\tau^{1+\delta}\sum_{i=0}^{k}\int_{\Sigma_\tau\cap\{r\leq\frac{9\tau}{10}\}} r^{-2}(S\Gamma^i\Phi)^2\\
\leq& CA^2\epsilon^2\tau^{-5+2\eta_{S,,11}}\sum_{i=0}^{k}\int_{\Sigma_\tau}(D\Gamma^i\Phi)^2+CA^2\epsilon^2\tau^{-5+2\eta_{S,11}+\delta}\sum_{i=0}^{k}\int_{\Sigma_\tau}(DS\Gamma^i\Phi)^2\\
\leq& CA^3\epsilon^3\tau^{-5+2\eta_{S,11}}+CA^2\epsilon^2\tau^{-5+2\eta_{S,11}+\delta}\sum_{i=0}^{k}\int_{\Sigma_\tau}(DS\Gamma^i\Phi)^2.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\int_{\Sigma_\tau\cap\{r\leq\frac{\tau}{4}\}} r^{1-\delta}(S(N_k))^2\\
\leq& CA^2\epsilon^2\tau^{-4+\eta_{14}+\eta_{S,11}}+CA\epsilon\tau^{-3+\eta_{S,11}}\sum_{i=0}^{k}\int_{\Sigma_\tau\cap\{r\leq\frac{9\tau}{10}\}}\left(DS\Gamma^i\Phi\right)^2\\
&+CA^2\epsilon^2\tau^{-5+2\eta_{S,11}+\delta}\sum_{i=0}^{k}\int_{\Sigma_\tau}(DS\Gamma^i\Phi)^2.
\end{aligned}$$

The Proposition follows from the Bootstrap Assumptions (10.9), (10.10) and (10.11).

□

We then move on to the region $\{r\geq\frac{9t^*}{10}\}$.

Proposition 10.3.24. *For $\alpha=0$ or 2,*

$$\int_{\Sigma_\tau\cap\{r\geq\frac{9\tau}{10}\}}(S(N_{13}))^2\leq CA^2\epsilon^2\tau^{-2+\eta_{S,13}}$$

$$\int_{\Sigma_\tau\cap\{r\geq\frac{9\tau}{10}\}} r^\alpha(S(N_{12}))^2\leq CA^2\epsilon^2\tau^{-3+\alpha+\eta_{S,12}}$$

$$\sum_{k=0}^{11} \int_{\Sigma_\tau \cap \{r \geq \frac{9\tau}{10}\}} r^{1-\delta} (S(N_k))^2 \leq CA^2\epsilon^2\tau^{-4+\alpha+\eta_{S,11}}.$$

Proof. Take $k \leq 13$. Following the reduction before and noticing that $[D, S] \sim D$ and $[\bar{D}, S] \sim \bar{D}$, we have to consider quadratic terms

$$(\bar{D}S\Gamma^{i_1}\Phi D\Gamma^{i_2}\Phi), (\bar{D}\Gamma^{i_1}\Phi DS\Gamma^{i_2}\Phi), (DS\Gamma^{i_1}\Phi\bar{D}\Gamma^{i_2}\Phi),$$

$$(D\Gamma^{i_1}\Phi\bar{D}S\Gamma^{i_2}\Phi), r^{-1}(DS\Gamma^{i_1}\Phi D\Gamma^{i_2}\Phi), r^{-1}(D\Gamma^{i_1}\Phi DS\Gamma^{i_2}\Phi),$$

for $i_1 \geq i_2$ and the cubic terms

$$(\bar{D}\Gamma^{i_1}\Phi D\Gamma^{i_2}\Phi S\Gamma^{i_3}\Phi), (D\Gamma^{i_1}\Phi\bar{D}\Gamma^{i_2}\Phi S\Gamma^{i_3}\Phi), r^{-1}(D\Gamma^{i_1}\Phi D\Gamma^{i_2}\Phi S\Gamma^{i_3}\Phi).$$

For these cubic terms, we can assume $i_1, i_2 \leq 6$ for otherwise $i_3 \leq 6$ and we can control the last factor in the sup norm and reduce to the quadratic terms above. The cubic terms

$$(\bar{D}S\Gamma^{i_1}\Phi D\Gamma^{i_2}\Phi\Gamma^{i_3}\Phi), (\bar{D}\Gamma^{i_1}\Phi DS\Gamma^{i_2}\Phi\Gamma^{i_3}\Phi), (DS\Gamma^{i_1}\Phi\bar{D}\Gamma^{i_2}\Phi\Gamma^{i_3}\Phi),$$

$$(D\Gamma^{i_1}\Phi\bar{D}S\Gamma^{i_2}\Phi\Gamma^{i_3}\Phi), r^{-1}(DS\Gamma^{i_1}\Phi D\Gamma^{i_2}\Phi\Gamma^{i_3}\Phi), r^{-1}(D\Gamma^{i_1}\Phi DS\Gamma^{i_2}\Phi\Gamma^{i_3}\Phi).$$

are irrelevant here because $i_3 \leq 13$ and we can thus control the last factor in the sup norm to reduce to the quadratic terms above. As before, we also have terms that do not have S (from $S\Lambda$ or from the commutators $[D, S], [\bar{D}, S]$), but they already appear in N_k and we will use the estimates proved for N_k in Proposition 10.3.10. We first estimate the quadratic terms. The crucial technical point here is that we do not have an improved pointwise decay estimate for $\bar{D}S\Gamma^i\Phi$ because we have used S in the proof of Proposition 10.2.6 and we are only commuting with S once. Nevertheless,

since $k \leq 13$, we can instead put $D\Gamma^i\Phi$ in L^∞.

$$
\begin{aligned}
&\sum_{i_2=0}^{\lfloor \frac{k}{2} \rfloor} \sum_{i_1=0}^{k} \int_{\Sigma_\tau \cap \{r \geq \frac{9\tau}{10}\}} r^\alpha \left(\text{Quadratic Terms}\right)^2 \\
\leq & C \left(\sup_{r \geq \frac{9\tau}{10}} \sum_{i_2=0}^{6} r^2 |D\Gamma^{i_2}\Phi|^2 \right) \sum_{i_1=0}^{k} \int_{\Sigma_\tau \cap \{r \geq \frac{9\tau}{10}\}} r^{\alpha-2} |\bar{D} S \Gamma^{i_1}\Phi|^2 \\
& + C \left(\sup_{r \geq \frac{9\tau}{10}} \sum_{i_2=0}^{6} r^2 |DS\Gamma^{i_2}\Phi|^2 \right) \sum_{i_1=0}^{k} \int_{\Sigma_\tau \cap \{r \geq \frac{9\tau}{10}\}} r^{\alpha-2} |\bar{D}\Gamma^{i_1}\Phi|^2 \\
& + C \left(\sup_{r \geq \frac{9\tau}{10}} \sum_{i_2=0}^{6} r^2 |\bar{D}\Gamma^{i_2}\Phi|^2 \right) \sum_{i_1=0}^{k} \int_{\Sigma_\tau \cap \{r \geq \frac{9\tau}{10}\}} r^{\alpha-2} |DS\Gamma^{i_1}\Phi|^2 \\
& + C \left(\sup_{r \geq \frac{9\tau}{10}} \sum_{i_2=0}^{k} r^2 |D\Gamma^{i_1}\Phi|^2 \right) \sum_{i_2=0}^{6} \int_{\Sigma_\tau \cap \{r \geq \frac{9\tau}{10}\}} r^{\alpha-2} |\bar{D} S \Gamma^{i_2}\Phi|^2 \\
& + C\tau^{-2} \left(\sup_{r \geq \frac{9\tau}{10}} \sum_{i_2=0}^{6} r^2 |D\Gamma^{i_2}\Phi|^2 \right) \sum_{i_1=0}^{k} \int_{\Sigma_\tau \cap \{r \geq \frac{9\tau}{10}\}} r^{\alpha-2} |DS\Gamma^{i_1}\Phi|^2 \\
& + C\tau^{-2} \left(\sup_{r \geq \frac{9\tau}{10}} \sum_{i_2=0}^{6} r^2 |DS\Gamma^{i_2}\Phi|^2 \right) \sum_{i_1=0}^{k} \int_{\Sigma_\tau \cap \{r \geq \frac{9\tau}{10}\}} r^{\alpha-2} |D\Gamma^{i_1}\Phi|^2 \\
\leq & CA\epsilon \sum_{i_1=0}^{k} \int_{\Sigma_\tau \cap \{r \geq \frac{9\tau}{10}\}} r^{\alpha-2} \left(|\bar{D} S \Gamma^{i_1}\Phi|^2 + |\bar{D}\Gamma^{i_1}\Phi|^2 \right) \\
& + CA\epsilon\tau^{-2+\eta_{14}} \sum_{i_1=0}^{k} \int_{\Sigma_\tau \cap \{r \geq \frac{9\tau}{10}\}} r^{\alpha-2} |DS\Gamma^{i_1}\Phi|^2 \\
& + CA\epsilon\tau^{-4+\alpha+\eta_{S,11}} \left(\sup_{r \geq \frac{9\tau}{10}} \sum_{i_1=0}^{k} r^2 |D\Gamma^{i_1}\Phi|^2 \right) + CA\epsilon\tau^{-4+\alpha}
\end{aligned}
$$

We then estimate the cubic terms:

$$\begin{aligned}
&\sum_{i_1,i_2=0}^{\lfloor \frac{k}{2} \rfloor} \sum_{i_3=0}^{k} \int_{\Sigma_\tau \cap \{r \geq \frac{9\tau}{10}\}} r^\alpha \left(\text{Cubic Terms}\right)^2 \\
\leq & C \left(\sup_{r \geq \frac{9\tau}{10}} \sum_{i_1=0}^{6} \left(r^2 \bar{D} \Gamma^{i_1} \Phi \right)^2 \right) \left(\sup_{r \geq \frac{9\tau}{10}} \sum_{i_2=0}^{6} r^2 \left(D \Gamma^{i_2} \Phi \right)^2 \right) \sum_{i_3=0}^{k} \int_{\Sigma_\tau \cap \{r \geq \frac{9\tau}{10}\}} r^{\alpha-4} (S \Gamma^{i_3} \Phi)^2 \\
& + C\tau^{-2} \left(\sup_{r \geq \frac{9\tau}{10}} \sum_{i_1=0}^{6} r^2 \left(D \Gamma^{i_1} \Phi \right)^2 \right)^2 \sum_{i_3=0}^{k} \int_{\Sigma_\tau \cap \{r \geq \frac{9\tau}{10}\}} r^{\alpha-4} (S \Gamma^{i_3} \Phi)^2 \\
\leq & C A^2 \epsilon^2 \tau^{-2+\eta_{14}} \sum_{i_3=0}^{k} \int_{\Sigma_\tau \cap \{r \geq \frac{9\tau}{10}\}} r^{\alpha-2} (DS \Gamma^{i_3} \Phi)^2,
\end{aligned}$$

which is better then the estimates obtained for the quadratic terms. We hence focus on the quadratic terms and spell out explicitly what the estimates amount to for different values of k and α.

$$\sum_{i_2=0}^{6} \sum_{i_1=0}^{13} \int_{\Sigma_\tau \cap \{r \geq \frac{9\tau}{10}\}} \left(\text{Quadratic Terms}\right)^2 \leq C A^2 \epsilon^2 \tau^{-2+\eta_{S,13}} + C A^2 \epsilon^2 \tau^{-4+\eta_{S,11}+\eta_{16}},$$

and

$$\sum_{i_2=0}^{6} \sum_{i_1=0}^{12} \int_{\Sigma_\tau \cap \{r \geq \frac{9\tau}{10}\}} \left(\text{Quadratic Terms}\right)^2 \leq C A^2 \epsilon^2 \tau^{-3+\eta_{S,12}} + C A^2 \epsilon^2 \tau^{-4+\eta_{S,11}},$$

and

$$\sum_{i_2=0}^{6} \sum_{i_1=0}^{11} \int_{\Sigma_\tau \cap \{r \geq \frac{9\tau}{10}\}} \left(\text{Quadratic Terms}\right)^2 \leq C A^2 \epsilon^2 \tau^{-4+\eta_{S,11}} + C A^2 \epsilon^2 \tau^{-4+\eta_{S,11}},$$

and

$$\sum_{i_2=0}^{6} \sum_{i_1=0}^{12} \int_{\Sigma_\tau \cap \{r \geq \frac{9\tau}{10}\}} r^2 \left(\text{Quadratic Terms}\right)^2 \leq C A^2 \epsilon^2 \left(\tau^{-1+\eta_{S,12}} + \tau^{-2+\eta_{14}} + \tau^{-2+\eta_{S,11}} \right),$$

and

$$\sum_{i_2=0}^{6}\sum_{i_1=0}^{11}\int_{\Sigma_\tau\cap\{r\geq\frac{9\tau}{10}\}} r^2\left(\text{Quadratic Terms}\right)^2 \leq CA^2\epsilon^2\left(\tau^{-2+\eta_{S,11}}+\tau^{-2+\eta_{14}}\right).$$

□

With the estimates for the inhomogeneous terms for the equations involving S, we can now retrieve the bootstrap assumptions involving S. We will follow the order that we proved the estimates without S, namely, first proving the pointwise estimates, then the integrated estimates, then the energy estimates and finally the energy estimates involving also $\hat{Y}$. Noticing that $U_{k,j}$ (respectively N_k) and $S(U_{k,j})$ (respectively $S(N_k)$) satisfy similar estimates (see Propositions 10.3.9, 10.3.10, 10.3.6, 10.3.23, 10.3.24 and 10.3.21), we would focus on showing that the estimates for V_k are enough to close the bootstrap assumptions. We now prove the pointwise estimates and retrieve the bootstrap assumptions (10.33), (10.34), (10.39) and (10.40).

Proposition 10.3.25. *For $r \geq \frac{t^*}{4}$,*

$$\sum_{j=0}^{8}|DS\Gamma^j\Phi|^2 \leq \frac{B_S}{2}A\epsilon r^{-2}. \tag{10.65}$$

$$\sum_{j=0}^{6}|DS\Gamma^j\Phi|^2 \leq \frac{B_S}{2}A\epsilon r^{-2}(t^*)^{\eta_{S,11}}(1+|u|)^{-2}. \tag{10.66}$$

For $r \leq \frac{t^*}{4}$,

$$\sum_{j=0}^{6}|S\Gamma^j\Phi|^2 \leq \frac{B_S}{2}A\epsilon(t^*)^{-2+\eta_{15}}. \tag{10.67}$$

$$\sum_{\ell=1}^{7-j}\sum_{j=0}^{6}|D^\ell S\Gamma^j\Phi|^2 \leq \frac{B_S}{2}A\epsilon r^{-2}(t^*)^{-2+\eta_{15}}. \tag{10.68}$$

Proof. The proof of the estimates for $r \geq \frac{t^*}{4}$ (i.e. (10.65) and (10.66)) is completely analogous to Proposition 10.3.11, with the use of Propositions 10.3.6, 10.3.9, 10.3.10

replaced by Propositions 10.3.18, 10.3.21, 10.3.23, 10.3.24 appropriately. Notice especially that the estimates in Proposition 10.3.18 for V is better than that in Proposition 10.3.21 for SU and are thus acceptable.

(10.67) follows directly from Proposition 10.2.9 and the bootstrap assumptions (10.7) and (10.11). Here, we need to use also (10.7) because we would need to commute S with ∂_{t^*} and would get terms that do not contain S.

(10.68) follows directly from Proposition 10.2.8, the bootstrap assumptions (10.7) and (10.11), as well as Proposition 10.3.6, 10.3.9, 10.3.18, 10.3.21 and 10.3.22 to control the inhomogeneous terms. As before, (10.7) and Proposition 10.3.6, 10.3.9 are used to control the terms arising from $[S, \partial_{t^*}]$. Notice here that the decay rate for $\sum_{\ell=1}^{7-j}\sum_{j=0}^{6}|D^\ell S\Gamma^j\Phi|^2$ is not as good as that for $\sum_{\ell=1}^{9-j}\sum_{j=0}^{8}|D^\ell\Gamma^j\Phi|^2$ because in proving the decay rate for $\sum_{\ell=1}^{9-j}\sum_{j=0}^{8}|D^\ell\Gamma^j\Phi|^2$, we have used the quantities associated to $S\Phi$, while we do not have estimates for $S^2\Phi$ at our disposal. □

As before, once we have proved the L^∞ bounds, we will replace the constant B_S by C.

Proposition 10.3.26.

$$\sum_{i+j+k\leq 12} A^{-1}_{S,X,j}\iint_{\mathcal{R}((1.1)^{-1}\tau,\tau)\cap\{r\leq\frac{t^*}{2}\}} K^{X_0}\left(S\partial^i_{t^*}\tilde{\Omega}^j\Phi\right)\leq\frac{\epsilon}{2}\tau^{-1+\eta_{S,12}}. \tag{10.69}$$

$$\sum_{i+j\leq 11} A^{-1}_{S,X,j}\iint_{\mathcal{R}((1.1)^{-1}\tau,\tau)\cap\{r\leq\frac{t^*}{2}\}} K^{X_1}\left(S\partial^i_{t^*}\tilde{\Omega}^j\Phi\right)\leq\frac{\epsilon}{2}\tau^{-1+\eta_{S,12}}. \tag{10.70}$$

$$\sum_{i+j\leq 11} A^{-1}_{S,X,j}\iint_{\mathcal{R}((1.1)^{-1}\tau,\tau)\cap\{r\leq\frac{t^*}{2}\}} K^{X_0}\left(S\partial^i_{t^*}\tilde{\Omega}^j\Phi\right)\leq\frac{\epsilon}{2}\tau^{-2+\eta_{S,11}}. \tag{10.71}$$

$$\sum_{i+j\leq 10} A^{-1}_{S,X,j}\iint_{\mathcal{R}((1.1)^{-1}\tau,\tau)\cap\{r\leq\frac{t^*}{2}\}} K^{X_1}\left(S\partial^i_{t^*}\tilde{\Omega}^j\Phi\right)\leq\frac{\epsilon}{2}\tau^{-2+\eta_{S,11}}. \tag{10.72}$$

Proof. This follows exactly as Proposition 10.3.12 except for replacing the use of

Propositions 10.3.6, 10.3.9 and 10.3.10 with Propositions 10.3.18, 10.3.21, 10.3.23 and 10.3.24. □

Proposition 10.3.27.

$$\sum_{i+j=13} A_{S,j}^{-1} \int_{\Sigma_\tau} J_\mu^N \left(S\partial_{t^*}^i \tilde{\Omega}^j \Phi \right) n_{\Sigma_\tau}^\mu \leq \frac{\epsilon}{4} \tau^{\eta_{S,13}}. \tag{10.73}$$

$$A_{S,j}^{-1} \sum_{j=0}^{12} \int_{\Sigma_\tau} J_\mu^N \left(S\partial_{t^*}^i \tilde{\Omega}^j \Phi \right) n_{\Sigma_\tau}^\mu \leq \frac{\epsilon}{2}. \tag{10.74}$$

$$\sum_{i+j=13} A_{S,X,j}^{-1} \iint_{\mathcal{R}(\tau_0,\tau)} K^{X_0} \left(S\partial_{t^*}^i \tilde{\Omega}^j \Phi \right) \leq \frac{\epsilon}{2} \tau^{\eta_{S,13}}. \tag{10.75}$$

$$\sum_{i+j\leq 12} A_{S,X,j}^{-1} \iint_{\mathcal{R}(\tau_0,\tau)} K^{X_0} \left(S\partial_{t^*}^i \tilde{\Omega}^j \Phi \right) \leq \frac{\epsilon}{2}. \tag{10.76}$$

Proof. This follows exactly as Proposition 10.3.13 except for replacing the use of Propositions 10.3.6, 10.3.9 and 10.3.10 with Propositions 10.3.18, 10.3.21, 10.3.23 and 10.3.24. □

Proposition 10.3.28.

$$\sum_{i+j=12} A_{S,j}^{-1} \left(\int_{\Sigma_\tau} J_\mu^{Z+N,w^Z} \left(S\partial_{t^*}^i \tilde{\Omega}^j \Phi \right) n_{\Sigma_\tau}^\mu + C\tau^2 \int_{\Sigma_\tau \cap \{r \leq \frac{9\tau}{10}\}} J_\mu^N \left(S\partial_{t^*}^i \tilde{\Omega}^j \Phi \right) n_{\Sigma_\tau}^\mu \right)$$
$$\leq \frac{\epsilon}{4} \tau^{1+\eta_{S,12}}. \tag{10.77}$$

Proof. This follows exactly as Proposition 10.3.14 except for replacing the use of Propositions 10.3.6, 10.3.9 and 10.3.10 with Propositions 10.3.18, 10.3.21, 10.3.23 and 10.3.24. □

Proposition 10.3.29.

$$\sum_{i+j\leq 11} A_{S,j}^{-1}\left(\int_{\Sigma_\tau} J_\mu^{Z+N,w^Z}\left(S\left(\partial_{t^*}^i\tilde{\Omega}^j\Phi\right)\right)n_{\Sigma_\tau}^\mu \right.$$
$$\left. +C\tau^2\int_{\Sigma_\tau\cap\{r\leq\frac{9\tau}{10}\}} J_\mu^N\left(S\left(\partial_{t^*}^i\tilde{\Omega}^j\Phi\right)\right)n_{\Sigma_\tau}^\mu\right) \tag{10.78}$$
$$\leq\frac{\epsilon}{4}\tau^{\eta_{S,11}}$$

Proof. This follows exactly as Proposition 10.3.15 except for replacing the use of Propositions 10.3.6, 10.3.9 and 10.3.10 with Propositions 10.3.18, 10.3.21, 10.3.23 and 10.3.24. □

To close the bootstrap argument we need finally to consider energy quantities with both S and $\hat{Y}$.

Proposition 10.3.30.

$$\sum_{i+k=13} A_{S,Y}^{-1}\int_{\Sigma_\tau} J_\mu^N\left(\hat{Y}^k S\partial_{t^*}^i\Phi\right)n_{\Sigma_\tau}^\mu \leq \frac{\epsilon}{4}\tau^{\eta_{S,13}}.$$

$$\sum_{i+k=12} A_{S,Y}^{-1}\tau^2\int_{\Sigma_\tau\cap\{r\leq r_Y^-\}} J_\mu^N\left(\hat{Y}^k S\partial_{t^*}^i\Phi\right)n_{\Sigma_\tau}^\mu \leq \frac{\epsilon}{4}\tau^{1+\eta_{S,12}}$$

$$\sum_{i+k\leq 11} A_{S,Y}^{-1}\tau^2\int_{\Sigma_\tau\cap\{r\leq r_Y^-\}} J_\mu^N\left(\hat{Y}^k S\partial_{t^*}^i\Phi\right)n_{\Sigma_\tau}^\mu \leq \frac{\epsilon}{4}\tau^{\eta_{S,11}}$$

Proof. This follows exactly as Proposition 10.3.16 except for replacing the use of Propositions 10.3.9 with Propositions 10.3.18 and 10.3.23. □

10.4 Proof of Theorem 5.3.2

Now all the bootstrap assumptions are closed and all the estimates hold. The solution hence exists globally by a standard local existence argument that we omit here. The decay estimates of the the derivatives of Φ claimed in the Theorem are restatements

of (10.47), (10.48), (10.29). The decay estimates follow from the use of Proposition 10.2.3 and (10.3.15) for $r \geq R$ and Proposition 10.2.12 and (10.56) for $r \leq \frac{t^*}{4}$.

Bibliography

[1] S. Alinhac. Energy multipliers for perturbations of Schwarzschild metric. *Comm. Math. Phys.*, 288(1):199–224, 2009.

[2] L. Andersson and P. Blue. Hidden symmetries and decay for the wave equation on the kerr spacetime. 2009, 0908.2265.

[3] P. Blue and A. Soffer. Semilinear wave equations on the Schwarzschild manifold. I. Local decay estimates (see also Errata for "Global existence and scattering for the nonlinear Schrodinger equation on schwarzschild manifolds", "Semilinear wave equations on the Schwarzschild manifold i: Local decay estimates", and "the wave equation on the Schwarzschild metric ii: Local decay for the spin 2 Regge Wheeler equation", arxiv:gr-qc/0608073). *Adv. Differential Equations*, 8(5):595–614, 2003, arXiv:gr-qc/0310091.

[4] P. Blue and A. Soffer. Improved decay rates with small regularity loss for the wave equation about a Schwarzschild black hole. 2006, arXiv:math.AP/0612168.

[5] P. Blue and J. Sterbenz. Uniform decay of local energy and the semi-linear wave equation on Schwarzschild space. *Comm. Math. Phys.*, 268(2):481–504, 2006, arXiv:math.AP/0510315.

[6] B. Carter. Global structure of the Kerr family of gravitational fields. *Phy. Rev.*, 174:1559–1571, 1968.

[7] D. Catania and V. Georgiev. Blow Up for the Semilinear Wave Equation in Schwarzschild Metric. 2004, 0407001.

[8] Y. Choquet-Bruhat, P. T. Chruściel, and J. Loizelet. Global solutions of the Einstein-Maxwell equations in higher dimensions. *Classical Quantum Gravity*, 23(24):7383–7394, 2006.

[9] D. Christodoulou. Global solutions of nonlinear hyperbolic equations for small initial data. *Comm. Pure Appl. Math.*, 39(2):267–282, 1986.

[10] D. Christodoulou and S. Klainerman. *The global nonlinear stability of the Minkowski space*. Princeton UP, 1993.

[11] M. Dafermos. The interior of charged black holes and the problem of uniqueness in general relativity. *Comm. Pure Appl. Math.*, 58(4):445–504, 2005.

[12] M. Dafermos and I. Rodnianski. A proof of Price's law for the collapse of a self-gravitating scalar field. *Invent. Math.*, 162:381–457, 2005, gr-qc/0309115.

[13] M. Dafermos and I. Rodnianski. Small-amplitude nonlinear waves on a black hole background. *J. Math. Pures Appl. (9)*, 84(9):1147–1172, 2005.

[14] M. Dafermos and I. Rodnianski. A note on energy currents and decay for the wave equation on a Schwarzschild background. 2007, arXiv:math.AP/0710.0171.

[15] M. Dafermos and I. Rodnianski. A proof of the uniform boundedness of solutions to the wave equation on slowly rotating Kerr backgrounds. 2008, arXiv:gr-qc/0805.4309.

[16] M. Dafermos and I. Rodnianski. Lectures on black holes and linear waves. 2008, arXiv:gr-qc/0811.0354.

[17] M. Dafermos and I. Rodnianski. A new physical-space approach to decay for the wave equation with applications to black hole spacetimes. 2009, 0910.4957.

[18] M. Dafermos and I. Rodnianski. The red-shift effect and radiation decay on black hole spacetimes. *Comm. Pure Appl. Math.*, 2009, arXiv:gr-qc/0512119.

[19] M. Dafermos and I. Rodnianski. Decay for solutions of the wave equation on Kerr exterior spacetimes I-II: The cases $|a| \ll M$ or axisymmetry. 2010, 1010.5132.

[20] M. Dafermos and I. Rodnianski. The black hole stability problem for linear scalar perturbations. 2010, 1010.5137.

[21] R. Donninger, W. Schlag, and A. Soffer. A proof of Price's Law on Schwarzschild black hole manifolds for all angular momenta. 2009, gr-qc/0908.4292.

[22] R. Donninger, W. Schlag, and A. Soffer. On pointwise decay of linear waves on a Schwarzschild black hole background. *Comm. Math. Phys*, 309(1):51–86, 2012, 0911.3179.

[23] J. Marzuola, J. Metcalfe, D. Tataru, M. Tohaneanu. Strichartz estimates on Schwarzschild black hole backgrounds. 2008, arXiv:math.AP/0802.3942.

[24] F. John. Blow-up for quasilinear wave equations in three space dimensions. *Comm. Pure Appl. Math.*, 34(1):29–51, 1981.

[25] F. John and S. Klainerman. Almost global existence to nonlinear wave equations in three space dimensions. *Comm. Pure Appl. Math.*, 37(4):443–455, 1984.

[26] B. S. Kay and R. M. Wald. Linear stability of Schwarzschild under perturbations which are nonvanishing on the bifurcation 2-sphere. *Classical Quantum Gravity*, 4(4):893–898, 1987.

[27] S. Klainerman. Uniform decay estimates and the Lorentz invariance of the classical wave equation. *Comm. Pure Appl. Math.*, 38(3):321–332, 1985.

[28] S. Klainerman. The null condition and global existence to nonlinear wave equations. In *Nonlinear systems of partial differential equations in applied mathematics, Part 1 (Santa Fe, N.M., 1984)*, volume 23 of *Lectures in Appl. Math.*, pages 293–326. Amer. Math. Soc., Providence, RI, 1986.

[29] S. Klainerman. Remarks on the global Sobolev inequalities in the Minkowski space $\mathbf{R}^{n+1}$. *Comm. Pure Appl. Math.*, 40(1):111–117, 1987.

[30] S. Klainerman and T. Sideris. On almost global existence for nonrelativistic wave equations in 3D. *Comm. Pure Appl. Math.*, 49(3):307–321, 1996.

[31] J. Kronthaler. Decay Rates for Spherical Scalar Waves in the Schwarzschild Geometry. 2007, gr-qc/0709.3703.

[32] M. D. Kruskal. Maximal extension of Schwarzschild metric. *Phys. Rev. (2)*, 119:1743–1745, 1960.

[33] I. Laba and A. Soffer. Global existence and scattering for the nonlinear Schrodinger equation on Schwarzschild manifolds. *Helv. Phys. Acta.*, 72(4):272–294, 1999.

[34] H. Lindblad and I. Rodnianski. Global existence for the Einstein vacuum equations in wave coordinates. *Comm. Math. Phys.*, 256(1):43–110, 2005, arXiv:math.AP/0312479v2.

[35] J. Luk. Improved decay for solutions to the linear wave equation on a Schwarzschild black hole. 2009, 0906.5588.

[36] J. Luk. A vector field method approach to improved decay for solutions to the wave equation on a slowly rotating Kerr black hole. 2010, 1009.0671.

[37] J. Luk. The Null Condition and Global Existence for Nonlinear Wave Equations on Slowly Rotating Kerr Spacetimes. 2010, 1009.4109.

[38] M. Machedon and J. Stalker. Decay of solutions to the wave equation on a spherically symmetric background. *preprint*.

[39] J. Metcalfe, M. Nakamura, and C. D. Sogge. Global existence of solutions to multiple speed systems of quasilinear wave equations in exterior domains. *Forum Math.*, 17(1):133–168, 2005.

[40] J. Metcalfe and C. D. Sogge. Global existence of null-form wave equations in exterior domains. *Math. Z.*, 256(3):521–549, 2007.

[41] J. Metcalfe, D. Tataru, and M. Tohaneanu. Price's Law on Nonstationary Space-times. *ArXiv e-prints*, April 2011, 1104.5437.

[42] C. S. Morawetz. *Notes on time decay and scattering for some hyperbolic problems*. Society for Industrial and Applied Mathematics, Philadelphia, Pa., 1975. Regional Conference Series in Applied Mathematics, No. 19.

[43] J.-P. Nicolas. Nonlinear Klein-Gordon equation on Schwarzschild-like metrics. *J. Math. Pures Appl. (9)*, 74(1):35–58, 1995.

[44] J.-P. Nicolas. A nonlinear Klein-Gordon equation on Kerr metrics. *J. Math. Pures Appl. (9)*, 81(9):885–914, 2002.

[45] J. V. Ralston. Solutions of the wave equation with localized energy. *Comm. Pure Appl. Math.*, 22:807–823, 1969.

[46] T. C. Sideris and S.-Y. Tu. Global existence for systems of nonlinear wave equations in 3D with multiple speeds. *SIAM J. Math. Anal.*, 33(2):477–488 (electronic), 2001.

[47] C. D. Sogge. Global existence for nonlinear wave equations with multiple speeds. In *Harmonic analysis at Mount Holyoke (South Hadley, MA, 2001)*, volume 320 of *Contemp. Math.*, pages 353–366. Amer. Math. Soc., Providence, RI, 2003.

[48] J. Sterbenz. Angular regularity and Strichartz estimates for the wave equation. *Int. Math. Res. Not.*, (4):187–231, 2005. With an appendix by Igor Rodnianski.

[49] J. L. Synge. The gravitational field of a particle. *Proc. Roy. Irish Acad. Sect. A.*, 53:83–114, 1950.

[50] D. Tataru. Local decay of waves on asymptotically flat stationary space-times. 2009, 0910.5290.

[51] D. Tataru and M. Tohaneanu. Local energy estimate on Kerr black hole backgrounds. 2008, 0810.5766.

[52] M. Tohaneanu. Strichartz estimates on Kerr black hole backgrounds. 2009, 0910.1545.

[53] R. M. Wald. Note on the stability of the Schwarzschild metric. *J. Math. Phys.*, 20(6):1056–1058, 1979.