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VARIATIONAL THEORY AND ASYMPTOTIC ANALYSIS
FOR THE GINZBURG-LANDAU EQUATIONS AND
 p -HARMONIC MAPS

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Abstract

We study the relationship between energy concentration phenomena in some geometric pdes and the space of minimal submanifolds of higher codimension, and build on this understanding to obtain new existence results for some geometric variational problems.

In the first part of this thesis, we prove new existence results for the complex Ginzburg-Landau equations on closed manifolds (M^n, g) of dimension $n \geq 2$. Using min-max methods, we construct families $u_\epsilon : M \rightarrow \mathbb{C}$ of solutions to the Ginzburg-Landau equations whose zero sets $u_\epsilon^{-1}\{0\}$ converge as $\epsilon \rightarrow 0$ to the support of a nontrivial weak minimal submanifold (stationary, rectifiable varifold) of codimension two, providing an alternative approach to the construction of generalized minimal $(n-2)$ -submanifolds in a given Riemannian manifold.

In the second part, we develop a compactness theory for stationary p -harmonic maps to the circle as $p \in (1, 2)$ approaches 2 from below. Emphasizing strong analogies with the analysis of the Ginzburg-Landau equations, we show that, under natural energy bounds, the singular sets converge as $p \rightarrow 2$ to the support of a weak minimal submanifold of codimension two. We highlight several ways in which the analysis in this setting is cleaner and simpler than that used to obtain analogous results in the Ginzburg-Landau setting, suggesting that the study of p -harmonic maps to S^1 may provide valuable intuition in the study of energy concentration phenomena for the Ginzburg-Landau equations and related geometric pdes.

In the third part, we draw on the study of topological singularities to obtain an improved understanding of the variational landscape for the natural energy functionals $E_p(u) := \|du\|_{L^p}^p$ on manifold-valued maps $u : M \rightarrow N$. While results of White and Hang-Lin imply the existence of a p -energy minimizing map in each path component of $W^{1,p}(M, N)$, our results show that in many cases—depending only on the topology of M and N —each component of $W^{1,p}(M, N)$ contains multiple local minimizers for the p -energy $E_p(u)$, with critical points of mountain pass type lying between them. Moreover, the energies of these mountain-pass critical points are naturally related to the volumes of certain minimal varieties in M .

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Part I

Introduction and Overview

Chapter 1

The Ginzburg-Landau Equations

Part II of this thesis concerns the variational construction, in arbitrary closed manifolds, of nontrivial solutions of the complex Ginzburg-Landau equations exhibiting energy concentration along weak minimal submanifolds of codimension two. These results will appear separately in [75].

1.1 Background and Results

A complex-valued map $u : M \rightarrow \mathbb{C}$ on a Riemannian manifold M is said to solve the *Ginzburg-Landau equations* with parameter $\epsilon > 0$ if it satisfies the system

$$\epsilon^2 \Delta u = -(1 - |u|^2)u, \tag{1.1}$$

or, equivalently, if it is a critical point of the Ginzburg-Landau energy functional

$$E_\epsilon(u) := \int_M \frac{1}{2} |du|^2 + \frac{(1 - |u|^2)^2}{4\epsilon^2}. \tag{1.2}$$

The study of the system (1.1) has its origins in physics, where the functional (1.2) appears as a simplified version of the free energy functionals considered by Vitaly Ginzburg and Lev Landau in their efforts to model superconductors. In recent decades, however, the Ginzburg-Landau equations (and their parabolic and hyperbolic counterparts) have drawn intense interest from the pure mathematics community, due in part to their connections with the theory of harmonic maps, the Yang-Mills-Higgs equations, and minimal submanifolds. In Part II of this thesis, we emphasize the latter feature of the Ginzburg-Landau equations, employing variational methods to explore the space of solutions to

(1.1), and using these results to provide a new construction of weak minimal $(n - 2)$ -submanifolds in a given Riemannian manifold.

The first general existence result for minimal submanifolds in arbitrary ambient manifolds was obtained by Almgren in 1965. Namely, in [4], he demonstrated the existence of a nontrivial stationary integral k -varifold (a natural notion of weak critical point for the k -area functional) in any manifold of dimension $n > k$, by implementing min-max methods for the k -dimensional area functional on appropriate spaces of k -cycles. While this existence result is appealing in its generality, the minimal varieties constructed in [4] may, a priori, be quite singular; in general, we know only that they coincide with smooth minimal submanifolds on a dense subset of their support, by the partial regularity results of Allard [5]. In the hypersurface case $k = n - 1$, however, Jon Pitts was able to provide a complete regularity theory for the minimal varieties produced in [4], showing in particular that they coincide with closed, embedded minimal hypersurfaces in low dimensions ($n \leq 7$). In recent years, the Almgren-Pitts variational theory for minimal hypersurfaces has been revisited by the geometric analysis community with spectacular results, seeing applications in the resolution of the Willmore conjecture by Marques and Neves [60], the proof by Marques-Neves [61] and Song [74] of the existence of infinitely many minimal hypersurfaces in low-dimensional manifolds, and a number of other results demonstrating the remarkable abundance of minimal hypersurfaces in generic manifolds (see, e.g., [46],[62]).

Inspired by the recent developments in the variational theory for minimal hypersurfaces, Guaraco introduced in [38] a compelling alternative to the Almgren-Pitts construction, which simplifies many technical aspects of the theory. Instead of applying min-max methods directly to the $(n - 1)$ -area in a space of weak $(n - 1)$ -cycles, he applied classical mountain pass techniques to produce critical points $u_\epsilon : M \rightarrow \mathbb{R}$ of the so-called *Allen-Cahn functionals*—the restriction of (1.2) to scalar-valued functions—and built on the analysis of [45] and [81] to show that the level sets of u_ϵ converge to minimal hypersurfaces of optimal regularity as $\epsilon \rightarrow 0$. The immediate appeal of Guaraco’s construction lies in the relative simplicity of the variational tools employed: unlike Almgren’s measure-theoretic techniques, the construction of the Allen-Cahn critical points u_ϵ requires only classical Morse-theoretic tools, shifting all of the technical work onto the asymptotic analysis of these critical points (solutions of a well-studied family of semilinear pdes) in the limit as $\epsilon \rightarrow 0$. In the years since, the variational theory for the Allen-Cahn functionals has been further developed by Gaspar and Guaraco ([32], [33]), and has been employed by Chodosh and Mantoulidis in [26] to give a first proof of the “Multiplicity One Conjecture” of Marques and Neves.

In view of these successful applications of the Allen-Cahn equations to the existence theory

for minimal hypersurfaces, it is natural to ask whether any similar program may be carried out in higher codimension, where far less is understood about the minimal varieties constructed in [4]. More specifically, one might ask if there is a natural way to produce minimal submanifolds of dimension $n - m$ in a given manifold as a limit of level sets for $\mathbb{R}^m$ -valued maps solving a simple system of partial differential equations. As with the Allen-Cahn approximation in codimension one, such an approach would have the advantage of shifting the technical burdens of the construction onto the asymptotic analysis of solutions to the family of pdes in question, while providing a kind of smooth approximation to the (a priori quite singular) minimal varieties.

While there is no straightforward analog of the Allen-Cahn construction for minimal varieties of codimension > 1 , there is a well-documented (if not yet fully understood) relationship between solutions of the complex Ginzburg-Landau equations (1.1) and minimal submanifolds of codimension two, which one might hope to exploit in a similar way to produce minimal varieties of dimension $(n - 2)$. In three dimensional domains, the connection between minimization problems for (1.2) (and related Yang-Mills-Higgs functionals) and length-minimizing one-currents were first observed by Rivière in [69]. A few years later, in [55], Lin and Rivière showed that for any $(n - 3)$ -manifold Γ lying in the boundary $\Gamma \subset \partial\Omega$ of a convex domain $\Omega \subset \mathbb{R}^n$, one can obtain $(n - 2)$ -area-minimizing varieties T with boundary $\partial T = \Gamma$ as the limiting singular set for families of maps $u_\epsilon : \Omega \rightarrow \mathbb{C}$ minimizing (1.2) with respect to suitable Dirichlet data. In the years since, the convergence of level sets for solutions of (1.1) to minimal varieties of codimension two has been further explored by a number of authors (see, for instance, [55], [13], [17] for some significant developments). Of particular importance for the purposes of this thesis is the analysis carried out by Bethuel, Brezis, and Orlandi in [13], which establishes, under natural hypotheses, the concentration of energy for arbitrary (not necessarily minimizing) solutions of (1.1) to weak minimal submanifolds of codimension two. By adapting the techniques of [55] and [13] to the setting of closed, oriented manifolds, one obtains the following result.

Theorem 1.1. *Given a sequence $\epsilon_j \rightarrow 0$, let u_j be a family of complex-valued maps $u_j : M \rightarrow \mathbb{C}$ on a closed, oriented Riemannian manifold M solving the Ginzburg-Landau equations*

$$\Delta u_j = -\epsilon_j^{-2}(1 - |u_j|^2)u_j \tag{1.3}$$

and satisfying an energy bound of the form

$$\sup_j \frac{E_{\epsilon_j}(u_j)}{|\log \epsilon_j|} < \infty. \quad (1.4)$$

Then as $\epsilon_j \rightarrow 0$, the normalized energy measures $\mu_j := \frac{|du_j|^2}{2|\log \epsilon_j|}$ converge subsequentially to a limiting measure μ of the form

$$\mu = \nu + \frac{1}{2}|h|^2 d\text{vol}_g, \quad (1.5)$$

where h is a harmonic one-form on M , and $\nu = \theta \mathcal{H}^{n-2} \llcorner \Sigma$ is a stationary, rectifiable $(n-2)$ -varifold.

A detailed discussion of Theorem 1.1 and its proof can be found in Chapter 4; for now, we make only a few clarifying remarks. First, we note that the support Σ of the varifold ν in (1.5) coincides with the Hausdorff limit of the level sets $u_j^{-1}\{0\}$ (or, indeed, $u_j^{-1}\{y\}$ for any fixed $y \in \mathbb{C}$ with $|y| < 1$), justifying the assertion that the level sets of u_j converge—in some sense—to a minimal variety. For solutions in planar domains [26] or energy-minimizing solutions in higher dimensions [54], one can say more about the structure of ν —namely, that $\frac{1}{\pi}\nu$ is an *integral* varifold, in the sense that the density θ of ν takes values in $\pi\mathbb{N}$ almost everywhere. We expect that this integrality holds for arbitrary families satisfying the hypotheses of Theorem 1.1, but this turns out to be a quite subtle problem, due to the possibility of troublesome interactions between distant components of the level sets of u_ϵ .

The possible appearance of a diffuse component of the limiting measure μ arises from the fact that the $O(|\log \epsilon|)$ energy bound (1.4) is insufficient for ensuring compactness of the maps u_j (e.g., in a $W^{1,1}$ sense) when the first Betti number $b_1(M) \neq 0$ —that is, when there exist nontrivial homotopy classes in $[M : S^1]$. Indeed, it is not difficult to construct families satisfying the hypotheses of Theorem 1.1 for which the limiting measure μ is entirely diffuse: consider, for instance, the maps $u_j : S^1 \rightarrow \mathbb{C}$ given by $u_j(e^{i\theta}) = (1 - k^2 \epsilon_k^2) e^{ik\theta}$, where $\epsilon_k := e^{-k^2}$ —or their trivial extension to any product $M = S^1 \times N$. An essential ingredient in our efforts to use solutions of (1.1) to produce minimal submanifolds is the introduction of new tools that allow us to rule out this kind of behavior for solutions arising naturally from variational constructions.

Our central result in Part II establishes the existence of nontrivial families of Ginzburg-Landau solutions satisfying the hypotheses of Theorem 1.1 on an arbitrary Riemannian manifold, and shows, moreover that these solutions give rise to nontrivial minimal varieties ν as in Theorem 1.1—even when the manifold M has nontrivial first cohomology $b_1(M) \neq 0$. In short, we show the following.

Theorem 1.2. *On any closed manifold M^n of dimension $n \geq 2$, there exists a family of solutions*

u_ϵ of the Ginzburg-Landau equations

$$\epsilon^2 \Delta u = -(1 - |u|^2)u$$

whose level sets $u_\epsilon^{-1}\{0\}$ converge subsequentially as $\epsilon \rightarrow 0$ to the support of a nontrivial stationary, rectifiable $(n - 2)$ -varifold.

In particular, this confirms that we can indeed employ variational methods for the Ginzburg-Landau functionals to produce weak minimal varieties of codimension two—and once the integrality of the energy-concentration varifold has been established, Theorem 1.2 will provide a new proof of Almgren’s existence result in codimension two. Of course, we expect that such existence results—and the associated analysis of the variational landscape of the energy functionals (1.1)—will also be of independent interest to those studying the Ginzburg-Landau equations and related systems of pdes. Previously, the most general existence result for nontrivial solutions of the Ginzburg-Landau equations on closed manifolds was that established by Jeffrey Mesaric in his thesis [63], in which he showed that every nondegenerate closed geodesic in a three-manifold can be realized as the energy concentration varifold for a family of solutions of (1.1)—a result which can be combined with the existence theory for closed geodesics to obtain strong existence results in the three-dimensional setting.

The proof of Theorem 1.2 can be divided into two key steps. First, we employ a natural two-parameter saddle-point construction to produce nontrivial critical points of the energies E_ϵ , and show that the energies of these critical points grow like $|\log \epsilon|$ as $\epsilon \rightarrow 0$. When $b_1(M) = 0$, this completes the proof of Theorem 1.2—and in this case the result was obtained independently by Daren Cheng (see [24]). When $b_1(M) \neq 0$, however, additional work is required to show that the energy of the min-max critical points doesn’t have a diffuse limit—i.e., that the min-max solutions have nonempty zero sets $u_\epsilon^{-1}\{0\} \neq \emptyset$. To achieve this, we need to introduce a new estimate for the minimum Ginzburg-Landau energy required to connect distinct homotopy classes in $[M : S^1]$ through paths in $W^{1,2}(M, \mathbb{C})$, which we expect to be of independent interest.

While we focus here on the Ginzburg-Landau solutions arising from a two-parameter min-max construction, the analysis carried out in Part II can be adapted to the study of other natural min-max constructions for the Ginzburg-Landau energies—such as those associated with the S^1 -indices of Benci [10] or Fadell-Rabinowitz [30], or the mountain pass solutions lying between local minimizers of the Ginzburg-Landau functionals on manifolds with $b_1(M) \neq 0$. In the quest to better understand the relationship between solutions of the Ginzburg-Landau equations and critical points of the $(n - 2)$ -

area functional, it will be essential to continue developing a full picture of the variational landscape for the functionals E_ϵ , alongside an improved asymptotic analysis of the solutions to (1.1).

1.2 Organization of Part II

In Chapter 4, we discuss in detail the proof of Theorem 1.1, recalling the analysis developed in [54] and [13] and explaining how to extend it to the setting of closed manifolds. As in [54], [55], and [13], the most difficult technical ingredient is the so-called “ η -compactness” or “ η -ellipticity” theorem, showing that solutions u_ϵ of (1.1) must have energy of order at least $|\log \epsilon|$ near their zero sets; Section 4.3 contains a thorough discussion of the proof of this fact, following the arguments of [13].

In Chapter 5, we describe the saddle point construction used to produce the solutions in Theorem 1.2, and show that the associated min-max energies are bounded above and below by multiples of $|\log \epsilon|$ as $\epsilon \rightarrow 0$. In Chapter 6, we complete the proof of Theorem 1.2, by introducing the key technical tool—which we call the “homotopy transition estimate”—that allows us to show that the min-max solutions have nontrivial energy concentration varifold even when $b_1(M) \neq 0$.

Chapter 2

p -Harmonic Maps to S^1 :

Singularities and Energy

Concentration

In Part III of this thesis, based on the preprint [76], we study the asymptotic behavior as $p \uparrow 2$ of p -harmonic maps to the circle under natural assumptions on the growth of the p -energy. In dimension $n \geq 3$, we show that the singular sets of the maps converge in the limit to generalized minimal submanifolds of codimension two, while in planar domains, we show that the p -energy measure concentrates near singularities to a sum of Dirac masses with multiplicities in $2\pi\mathbb{N}$.

2.1 Background and Results

In their 1995 paper [43], Hardt and Lin consider the following question—posed as Problem 9 in the open problem section of [12]: given a simply connected domain $\Omega \subset \mathbb{R}^2$ and a map $g : \partial\Omega \rightarrow S^1$ of nonzero degree, what can be said about the limiting behavior of maps

$$u_p \in W_g^{1,p}(\Omega, S^1) := \{u \in W^{1,p}(\Omega, S^1) \mid u_p|_{\partial\Omega} = g\}$$

minimizing the p -energy

$$\int_{\Omega} |du_p|^p = \min\left\{\int_{\Omega} |du|^p \mid u \in W_g^{1,p}(\Omega, S^1)\right\}$$

as $p \in (1, 2)$ approaches 2 from below? They succeed in showing—among other things—that away from a collection A of $|\deg(g)|$ singularities, a subsequence u_{p_j} converges strongly to a harmonic map $v \in C^1_{loc}(\Omega \setminus A, S^1)$, and the measures $\mu_j = (2 - p_j)|du_j|^{p_j}(z)dz$ converge to a sum $2\pi \sum_{a \in A} \delta_a$ of Dirac masses at the points in A [43]. Moreover, the singular set $A = \{a_1, \dots, a_{|\deg(g)|}\}$ minimizes a certain “renormalized energy” function $W_g : \Omega^{|\deg(g)|} \rightarrow [0, \infty]$ associated to g , providing a strong constraint on the location of the singularities. In particular, though the homotopically nontrivial boundary map g admits no extension to an S^1 -valued map of finite Dirichlet energy—i.e., $W_g^{1,2}(\Omega, S^1) = \emptyset$ —the limit of the p -energy minimizers as $p \uparrow 2$ provides us with a natural notion of the optimal harmonic extension of g to an S^1 -valued map on Ω .

The results of [43] have very strong parallels with those of Bethuel-Brezis-Hel  n [12], Struwe [78], and others on the asymptotic behavior as $\epsilon \rightarrow 0$ of maps $u_\epsilon : \Omega \rightarrow \mathbb{C}$ minimizing the Ginzburg-Landau energy for Dirichlet data of nonzero degree, with the singularities of the p -harmonic maps taking on the role played by the zero sets in the Ginzburg-Landau setting. Intuitively, this connection is not terribly surprising, since both minimization problems provide natural relaxations for the problem of minimizing Dirichlet energy among S^1 -valued extensions—the Ginzburg-Landau approach relaxes the condition that the maps take values in S^1 , while the p -harmonic approach simply relaxes the regularity requirement from $W^{1,2}$ to $W^{1,p}$.

As discussed in the preceding chapter (and Part II below), the asymptotics of solutions to the Ginzburg-Landau equation have been studied extensively in recent decades, going beyond the results of [12] and [78] to consider non-minimizing solutions in the plane, and in higher dimensions, where the zero level sets are shown to converge to generalized minimal submanifolds of codimension two (see, e.g., [55], [13]). In Part III of this thesis, we undertake an analogous investigation for the asymptotics of p -harmonic maps to the circle, proving energy quantization results for solutions in planar domains, and establishing convergence of the singularities to generalized minimal submanifolds in higher dimension. First and foremost, in any closed, oriented manifold of dimension $n \geq 2$, we establish the following.

Theorem 2.1. *Let $p_i \in (1, 2)$ be a sequence with $\lim_{i \rightarrow \infty} p_i = 2$, and let $u_i \in W^{1,p_i}(M^n, S^1)$ be a sequence of stationary p_i -harmonic maps from a closed, oriented Riemannian manifold M^n to the circle, satisfying*

$$\sup_i (2 - p_i) \int_M |du_i|^{p_i} < \infty. \quad (2.1)$$

Then (a subsequence of) the energy measures $\mu_i = (2 - p_i)|du_i|^{p_i} d\text{vol}_g$ converge weakly in $(C^0)^*$ to

a limiting measure μ of the form

$$\mu = \|V\| + |\bar{h}|^2 d\text{vol}_g, \quad (2.2)$$

where $\bar{h}$ is a harmonic one-form, and V is a stationary, rectifiable $(n-2)$ varifold with support given by the Hausdorff limit

$$\text{spt}(V) = \lim_{i \rightarrow \infty} \text{Sing}(u_i)$$

of the singular sets $\text{Sing}(u_i)$. Moreover, the density $\Theta_{n-2}(\|V\|, \cdot)$ satisfies

$$\Theta_{n-2}(\|V\|, x) \geq 2\pi \text{ for } x \in \text{spt}(\|V\|). \quad (2.3)$$

Remark 2.2. As we will see in Part III, the energy bound $\int_M |du|^p = O(\frac{1}{2-p})$ is natural, and it is not difficult to construct nontrivial families satisfying the hypotheses of Theorem 2.1 on a given closed manifold.

For families in planar domains, we obtain also the following quantization result, which may be compared to that of Comte and Mironescu [26] in the Ginzburg-Landau setting.

Theorem 2.3. *Let $p_j \in (1, 2)$ with $\lim_{j \rightarrow \infty} p_j = 2$ and let $u_j \in W^{1,p_j}(D_2, S^1)$ be stationary p_j -harmonic maps on the disk D_2 of radius 2, such that*

$$\sup_j (2 - p_j) \int_{D_2} |du_j|^{p_j} < \infty.$$

Then (after passing to a subsequence), on the unit disk D_1 , the measures $\mu_j := (2 - p_j)|du_j|^{p_j}(x)dx$ converge weakly in $C^0(D_1)^*$ to a measure μ of the form

$$\mu = \sum_{j=1}^k 2\pi m_j \delta_{a_j} + |d\psi|^2(x)dx,$$

where $a_j \in D_1$, ψ is a harmonic function, and, in particular, $m_j \in \mathbb{N}$.

While these results provide further evidence of the strong link between the asymptotic behavior of p -harmonic S^1 -valued maps and that of solutions to the Ginzburg-Landau equations, a recurring theme in Part III is that the analysis in the p -harmonic map setting is in many ways *simpler* and *cleaner* than its Ginzburg-Landau analog, due to the *scale-invariance* of the problem (as manifested, for instance, in the simpler Pohozaev identities and monotonicity formulas). Thus, while we expect that the analysis presented here will be of independent interest to those studying p -harmonic maps and their singularities, we hope that the study of the asymptotic behavior for p -harmonic maps to

S^1 initiated in Part III may also help to shed light on difficult open questions about the asymptotics for the Ginzburg-Landau equations in higher dimensions.

2.2 Organization of Part III

In Chapter 7, we record some preliminary facts needed for our analysis—recalling important results on the structure and blow-up analysis of p -harmonic maps, and providing a brief introduction to the study of the spaces $W^{1,p}(M, S^1)$ and the topological singularities of Sobolev maps to the circle.

In Chapter 8, we present the proof of Theorem 2.1. A key ingredient is a sharp lower bound for the p -energy density of a stationary p -harmonic map $u : M \rightarrow S^1$ near the singular set $Sing(u)$, which we employ to obtain estimates for the $(n - 2)$ -current $T(u)$ encoding the topological singularities of u .

Chapter 9 is devoted to the proof of the two-dimensional quantization result, Theorem 2.3. Similar to results in the Ginzburg-Landau setting, the proof relies heavily on a Pohozaev-type identity providing a relationship between the p -energy measure and the degree of a p -harmonic map about its singular points. The proof is much simpler, however, than its analog—due to Comte and Mironescu [26]—in the Ginzburg-Landau setting, thanks to the simpler structure of the Pohozaev identity for the p -harmonic maps.

Finally, in Chapter 10, we establish the existence of nontrivial families satisfying the hypotheses of Theorem 2.1, to demonstrate that the energy growth considered in Theorems 2.1 and 2.3 is indeed natural for families arising from variational constructions. The arguments are modeled on those of Chapter 5, building on results of [82] concerning the construction of p -harmonic maps via critical points of functionals of Ginzburg-Landau type.

Chapter 3

Mountain Pass Energies Between Homotopy Classes of Maps

In Part IV of this thesis, based on the content of the preprint [77], we study the homological singularities of Sobolev maps between manifolds, and employ our analysis, together with results from Almgren's min-max theory, to obtain new results concerning the relationship between the topology of given manifolds M and N and the variational landscape of the p -energy functionals $E_p(u) = \int_M |du|^p$ on the space $W^{1,p}(M, N)$ of Sobolev maps from M to N . In particular, while results of White [84] show that one can employ the direct method to find p -energy minimizing maps within a $([p] - 1)$ -homotopy class, the results of Part IV show that as p approaches an integer threshold $[p] + 1 = k \in \{2, \dots, n\}$ from below, some $[p]$ -homotopy classes emerge as distinct wells for the p -energy, giving rise to new p -harmonic maps arising as local minimizers of the p -energy, and critical points of mountain-pass type lying between them.

3.1 Background and Results

The interplay between topology and the existence of solutions to geometric variational problems is one of the central themes of geometric analysis. In the study of maps between manifolds, a very basic question in this vein is the following: given a homotopy class $[u] \in [M : N]$ of maps from M to N , when is it possible to find a representative $u \in [u]$ which minimizes the L^p norm of the gradient $E_p(u) := \int_M |du|^p$, for $p \in (1, \infty)$?

When the base manifold $M = S^1$, that every free homotopy class $[\gamma] \in [S^1, N]$ contains a p -

energy minimizer (parametrizing a length-minimizing closed geodesic) is a classical and fundamental result in differential geometry. In higher dimensions, however, the picture becomes considerably more complex. To find p -energy minimizers by the direct method, we are led to consider spaces $W^{1,p}(M, N)$ of Sobolev maps from $M \rightarrow N$. Fixing an isometric embedding $N \subset \mathbb{R}^L$ of N into a higher-dimensional Euclidean space, we define

$$W^{1,p}(M, N) := \{u \in W^{1,p}(M, \mathbb{R}^L) \mid u(x) \in N \text{ for a.e. } x \in M\}$$

to consist of all Sobolev maps from M into $\mathbb{R}^L$ that take values in N almost everywhere. When $p > n = \dim(M)$, it follows from the Sobolev embedding theorems that $W^{1,p}(M, N)$ embeds compactly into $C^0(M, N)$; in particular, it follows that maps in $W^{1,p}(M, N)$ have well-defined homotopy classes, which are preserved under bounded weak convergence in $W^{1,p}$, so that we can always find p -energy minimizers for $p \in (n, \infty)$.

When $p < n = \dim(M)$, maps in $W^{1,p}(M, N)$ are no longer continuous, and even minimizers of the p -energy will in general have some points of discontinuity. Nonetheless, we may hope to find topologically distinguished classes in $W^{1,p}(M, N)$ on which we can minimize the p -energy, to obtain p -harmonic representatives which—by the work of Schoen-Uhlenbeck [71], Hardt-Lin [42], and Luckhaus [59]—are at least continuous outside of a small (codimension- $\lfloor p \rfloor + 1$) singular set. In this direction, fundamental work was done by White, who showed in [84] that maps in $W^{1,p}(M, N)$ have well-defined $(\lfloor p \rfloor - 1)$ -homotopy classes, and every such class admits a p -energy minimizing representative. That is, there is a natural equivalence relation on $W^{1,p}(M, N)$ given by setting $u \sim v$ if the restrictions of u and v to the $(\lfloor p \rfloor - 1)$ -skeleton of a “generic” triangulation of M are homotopic in the standard sense; the results of [84] show that this relation is not only well-defined, but partitions M into components which are closed under weak convergence.

While White’s results guarantee the existence of a p -energy minimizing representative in every $(\lfloor p \rfloor - 1)$ -homotopy class, one might hope to obtain further refinements, using finer topological data to divide $W^{1,p}(M, N)$ into smaller equivalence classes. However, work of Hang and Lin (building on a program initiated by Brezis and Li [18]) shows that it is never possible to find a partition of $W^{1,p}(M, N)$ finer than the $(\lfloor p \rfloor - 1)$ -homotopy classes. Indeed, it is shown in [41] that any two $(\lfloor p \rfloor - 1)$ -homotopic maps in $W^{1,p}(M, N)$ can be connected by a continuous path in $W^{1,p}(M, N)$, so that the $(\lfloor p \rfloor - 1)$ -homotopy classes defined by White are precisely the path components of $W^{1,p}(M, N)$.

The results in Part IV of this thesis are motivated in part by the following question: though the

path components of $W^{1,p}(M, N)$ only detect the topology of M and N at the level of $([p] - 1)$ -homotopy, does higher-order topological information (e.g., $[p]$ -homotopy classes) play a clear role in shaping the variational landscape of the p -energy functionals? Our chief result gives a positive answer at the level of real cohomology: two $([p] - 1)$ -homotopic maps with distinct actions on the real cohomology $H^{[p]}(N; \mathbb{R}) \rightarrow H^{[p]}(M; \mathbb{R})$ are separated in $W^{1,p}(M, N)$ by walls of high p -energy when p is sufficiently close to $[p] + 1$; as a consequence, we see the emergence of distinct local minimizers of the p -energy in a given component of $W^{1,p}(M, N)$, as well as high-energy critical points of mountain-pass type lying between them. Moreover, we obtain a sharp lower bound on the energies of these mountain-pass critical points, relating them to the volumes of certain minimal varieties of codimension- $([p] + 1)$ in M .

To make these ideas precise, fix two maps $u, v \in C^\infty(M, N)$, and let $2 \leq k \leq n = \dim M$ be the largest integer such that u and v are homotopic on the $(k - 2)$ -skeleton of some—hence, any (see [41], Section 2.2)—triangulation of M . By the results of [41] and [84], we then see that u and v lie in a common path component of $W^{1,p}(M, N)$ if and only if $p < k$. For p close to k , we would like to characterize those energy levels $c > 0$ for which the maps u and v lie in a common path component of the energy sublevel set

$$E_p^c := \{w \in W^{1,p}(M, N) \mid \|dw\|_{L^p}^p \leq c\}.$$

That is, we are interested in estimating the mountain-pass energies

$$\gamma_p(u, v) := \inf\{c > 0 \mid \exists \text{ a path } u_t \in E_p^c \text{ connecting } u \text{ to } v\}. \quad (3.1)$$

First, we observe that a careful examination of the path constructed by Hang and Lin in the proof of ([41], Theorem 1.1) yields the following upper bound. Since these estimates are not explicitly addressed in [41], we provide a proof in Chapter 11.

Theorem 3.1. *There exists a constant $C = C(u, v) < \infty$ such that*

$$\gamma_p(u, v) \leq \frac{C}{k - p} \quad (3.2)$$

for every $p \in [1, k)$.

Our main result shows that, when M and N are oriented, and u and v induce different maps on the real cohomology $u^*, v^* : H^{k-1}(N; \mathbb{R}) \rightarrow H^{k-1}(M; \mathbb{R})$, the growth $\gamma_p(u, v) \sim \frac{1}{k-p}$ is in fact

optimal, with an explicit lower bound on the coefficient

$$\liminf_{p \rightarrow k} (k - p) \gamma_p(u, v).$$

In fact, we establish a lower bound for a possibly smaller quantity $\gamma_p^*(u, v) \leq \gamma_p(u, v)$, defined roughly as the smallest energy level $c > 0$ for which u and v can be connected by sequences $u = u_0, u_1, \dots, u_{r-1}, u_r = v$ in the sub-level set E_p^c for which adjacent maps u_i, u_{i+1} are arbitrarily close in L^p norm. (See Chapter 13 below for a careful definition.)

To state the lower bound precisely, we briefly introduce some relevant notation (to be defined in greater detail in Chapter 12). First, we denote by $\mathcal{A}^{k-1}(N)$ the space of closed $(k-1)$ -forms on N with the property that

$$\langle \alpha, \Sigma \rangle \in \mathbb{Z}$$

for every integral $(k-1)$ -cycle Σ in N , and for $\alpha \in \mathcal{A}^{k-1}(N)$, we use $S_\alpha(v) - S_\alpha(u)$ to denote the dual $(n+1-k)$ -current associated to $v^*(\alpha) - u^*(\alpha)$ by

$$\langle S_\alpha(v) - S_\alpha(u), \zeta \rangle := \int_M (v^*(\alpha) - u^*(\alpha)) \wedge \zeta \quad \text{for } \zeta \in \Omega^{n+1-k}(M).$$

Next, we recall from Almgren's dissertation [3] that there exists an isomorphism

$$\Phi : \pi_1(\mathcal{Z}_m(M; \mathbb{Z}), \{0\}) \rightarrow H_{m+1}(M; \mathbb{Z})$$

relating loops in the space of integral m -cycles (with the flat topology) to integral $(m+1)$ -homology classes in M . In Chapter 13 below, we define for each $\xi \in H_{m+1}(M; \mathbb{Z})$ a min-max width $\mathbf{L}_m(\xi) > 0$, which corresponds roughly to the min-max mass

$$\inf \left\{ \sup_{t \in S^1} \mathbb{M}(\gamma(t)) \mid \gamma : S^1 \rightarrow \mathcal{Z}_m(M; \mathbb{Z}), \Phi(\gamma) = \xi \right\}$$

associated to the class $\Phi^{-1}(\xi) \in \pi_1(\mathcal{Z}_m(M; \mathbb{Z}), \{0\})$. For any *real* homology class $\bar{\xi} \in H_{m+1}(M; \mathbb{R})$ that can be represented by integral cycles, we then define

$$\mathbf{L}_{m, \mathbb{R}}(\bar{\xi}) := \min \{ \mathbf{L}_m(\xi) \mid \xi \in H_{m+1}(M; \mathbb{Z}), \xi \equiv \bar{\xi} \text{ in } H_{m+1}(M; \mathbb{R}) \}.$$

Our main theorem then reads as follows.

Theorem 3.2. *For any $\alpha \in \mathcal{A}^{k-1}(N)$ and maps $u, v \in C^\infty(M, N)$ such that*

$$[u^*(\alpha) - v^*(\alpha)] \neq 0 \in H_{dR}^{k-1}(M),$$

there is a constant $\lambda(\alpha) < \infty$ such that

$$\lambda(\alpha)^{\frac{k}{k-1}} \liminf_{p \rightarrow k} (k-p) \gamma_p^*(u, v) \geq \sigma_{k-1} \mathbf{L}_{n-k, \mathbb{R}}([S_\alpha(v) - S_\alpha(u)]). \quad (3.3)$$

Remark 3.3. Here, σ_{k-1} denotes the $(k-1)$ -volume of the standard unit $(k-1)$ -sphere. The definition of $\lambda(\alpha)$ is given in Chapter 12; for now we only remark that $\lambda(\alpha)$ is easy to estimate for specific choices of target N and $\alpha \in \mathcal{A}^{k-1}(N)$. When $N = S^{k-1}$ is the standard $(k-1)$ -sphere and $\alpha = \frac{d\text{vol}}{\sigma_{k-1}}$, for example, one has $\lambda(\alpha) = (k-1)^{\frac{1-k}{2}}$.

Though the details of the proof are somewhat delicate, the intuition underlying Theorem 3.2 is relatively straightforward. For any map $w \in W^{1,p}(M, N)$, $p \in (k-1, k)$, the pullback $w^*(\alpha)$ is well-defined as a $(k-1)$ -form with coefficients in L^1 ; as in ([36], Section 5.4.2), we can then define an $(n-k)$ -current $T_\alpha(w)$ corresponding to the distributional exterior derivative of $w^*(\alpha)$. In Chapter 12, we develop a compactness theory for the so-called homological singularities $T_\alpha(w_p)$ for families of maps with $(k-p)E_p(w_p) \leq \Lambda$ as $p \rightarrow k$ (based largely on ideas from the Γ -convergence results of [2] and [48] for functionals of Ginzburg-Landau type), showing that the currents $T_\alpha(w_p)$ converge subsequentially in $(C^1)^*$ to an integral $(n-k)$ -cycle, whose mass we can bound explicitly in terms of the limiting energy $\liminf_{p \rightarrow k} (k-p)E_p(w_p)$. (In fact, a much more careful description of the convergence of $T_\alpha(w_p)$ is necessary for our applications.)

To an L^p -fine sequence $u = u_0, u_1, \dots, u_r = v$ of maps with p close to k and energy bounded above by $\gamma_p^*(u, v) + \epsilon$, we can then associate a family of integral $(n-k)$ -cycles $0 = T_0, T_1, \dots, T_r = 0$, with mass bounded in terms of $(k-p)\gamma_p^*(u, v)$. Using results of Chapter 12, and some additional technical lemmas, we then show that the difference of adjacent cycles T_{i-1}, T_i in this family can be written $T_i - T_{i-1} = \partial S_i$ for integral $(n+1-k)$ -currents S_i of small mass, such that

$$[\Sigma_{i=1}^r S_i] \equiv S_\alpha(v) - S_\alpha(u) \text{ in } H_{n+1-k}(M; \mathbb{R}).$$

The conclusion of Theorem 3.2 follows from these observations.

It is natural to ask whether the lower bound

$$\liminf_{p \rightarrow k} (k - p) \gamma_p^*(u, v) > 0 \quad (3.4)$$

holds for non- $(k - 1)$ -homotopic maps $u, v \in C^\infty(M, N)$ in greater generality, when the distinction is not detected at the level of real cohomology. At least in some cases, one might hope to find analogs of Theorem 3.2 by associating to each path $u_t \in W^{1,p}(M, N)$ from u to v a loop of “topological singularities” $T(u_t)$ given by flat $(n - k)$ -cycles with coefficients in $\pi_{k-1}(N)$, and performing an analysis similar to that of Chapters 12 and 13. If the target N is $(k - 2)$ -connected—and $\pi_1(N)$ is abelian, if $k = 2$ —the analysis of topological singularities pursued by Pakzad-Rivière [66] and Canevari-Orlandi [20] may provide a useful starting point for investigations in this direction.

Remark 3.4. In the case $N = S^1$, we observe that the general statement (3.4) follows immediately from Theorem 3.2, since the homotopy classes $[u] \in [M : S^1]$ are determined by the pullback $[u^*(\alpha)] \in H_{dR}^1(M)$ of the generator $\alpha = \frac{1}{2\pi} d\theta \in \mathcal{A}^1(S^1)$ of $H_{dR}^1(S^1)$. In this case, Theorem 3.2 is closely related to our work on the variational theory for the Ginzburg-Landau functionals (see in particular Chapter 6), where we are led to consider mountain-pass energies for the Ginzburg-Landau functionals $E_\epsilon : W^{1,2}(M, \mathbb{C}) \rightarrow \mathbb{R}$ over paths in $W^{1,2}(M, \mathbb{C})$ connecting distinct classes in $[M : S^1]$. As discussed in Chapter 6, on a two-dimensional annulus Ω , these questions had previously been studied by Almeida [6],[7], who obtained precise estimates for the E_ϵ -mountain pass energies separating the components of $W^{1,2}(\Omega, S^1)$ in $W^{1,2}(\Omega, \mathbb{C})$; in this setting, the results have some physical relevance, since the presence of high energy walls separating local minimizers of Ginzburg-Landau energies on Ω is related to the appearance of permanent currents in superconducting cylinders.

3.2 Organization of Part IV

In Chapter 11, we review important definitions and recall some key technical lemmas from [83], [84], and [41]. We also give the proof of Theorem 3.1, reviewing the relevant constructions in [41]. In Chapter 12, we recall from [36] (using a slightly different framework) the concept of homological singularities, and develop the structure and compactness theory needed for the proof of Theorem 3.2.

In Chapter 13, we complete the proof of Theorem 3.2, after reviewing the relevant definitions and results from Almgren’s work in [3] and [4]. Finally, in Chapter 14, we discuss applications of these results to the existence theory for p -harmonic maps from M to N .

Part II

The Ginzburg-Landau Equations on Closed Manifolds: Variational Methods and Asymptotic Analysis

Chapter 4

Limiting Behavior of Solutions

In this chapter, we explain how the analysis of [54], [13], and related works can be translated to the setting of closed manifolds to obtain the decomposition of the limiting energy measure for solutions of the Ginzburg-Landau equations described in Theorem 1.1.

4.1 Essential Identities and Basic Estimates

Throughout this chapter, fix a closed, oriented Riemannian manifold (M^n, g) of dimension $n \geq 3$, let $\epsilon \in (0, 1)$ be a small, positive constant, and let $u : M \rightarrow \mathbb{C}$ solve the complex Ginzburg-Landau equations

$$\Delta_g u = -\epsilon^{-2}(1 - |u|^2)u \quad (4.1)$$

(where in our convention $\Delta_g f := \operatorname{div}_g(df)$). Note that (4.1) is equivalent to the condition that u solve

$$\Delta_{g_\epsilon} u = -(1 - |u|^2)u, \quad (4.2)$$

with respect to the rescaled metric $g_\epsilon := \epsilon^{-2}g$.

Taking the inner product of (4.1) with u , we obtain the equation

$$\frac{1}{2}\Delta |u|^2 = |du|^2 - \epsilon^{-2}(1 - |u|^2)|u|^2 \quad (4.3)$$

for the modulus $|u|$ of u . Observe that the right-hand side of (4.3) is positive wherever $|u| > 1$, so

it follows immediately from the maximum principle that

$$|u| \leq 1 \text{ on } M. \quad (4.4)$$

Next, as in [13], we rewrite $\Delta|u|^2$ as $-\Delta(1 - |u|^2)$, and multiply (4.3) by $(1 - |u|^2)$ to obtain the identity

$$-(1 - |u|^2) \frac{1}{2} \Delta(1 - |u|^2) + \epsilon^{-2} (1 - |u|^2)^2 |u|^2 = (1 - |u|^2) |du|^2, \quad (4.5)$$

and integrate to find that

$$\int_M \left(\frac{1}{2} |d|u|^2|^2 + |u|^2 \frac{(1 - |u|^2)^2}{\epsilon^2} \right) = \int_M (1 - |u|^2) |du|^2. \quad (4.6)$$

Taking the inner product of (4.1) instead with the rotated map iu , we obtain the identity

$$d^*(ju) = 0 \quad (4.7)$$

for the one-form

$$ju := u^*(r^2 d\theta) = u^1 du^2 - u^2 du^1. \quad (4.8)$$

The identity (4.7) is equivalent to the observation that u is a critical point for the Dirichlet energy under variations of the form $e^{i\varphi}u$ for $\varphi \in C^\infty(M, \mathbb{R})$. As an immediate consequence of (4.7), we see that ju has a Hodge decomposition of the form

$$ju = d^*\xi + h, \quad (4.9)$$

where $h \in \mathcal{H}^1(M)$ is a harmonic one-form, and $\xi = \Delta_H^{-1}(dju)$ is the unique two-form satisfying

$$\Delta_H \xi := dju = 2du^1 \wedge du^2; \quad \int_M \langle \xi, \zeta \rangle = 0 \text{ for all harmonic two-forms } \zeta \in \mathcal{H}^2(M).$$

(Here and henceforth, $\Delta_H := dd^* + d^*d$ will denote the usual Hodge Laplacian.) As we will see in the following sections, the energy blow-up for families of solutions u_ϵ as in Theorem 1.1 is driven by energy contribution $\frac{|ju_\epsilon|^2}{|\log \epsilon|}$ from the one-forms ju_ϵ , with the limit of the harmonic components $\frac{h_\epsilon}{|\log \epsilon|^{1/2}}$ in (4.9) accounting for the diffuse part of the measure, and the limit of the co-exact contribution $\frac{|d^*\xi|^2}{|\log \epsilon|}$ giving the energy concentration varifold.

We record next a simple gradient estimate for u depending only on ϵ and the Ricci curvature of

M . Using (4.1) in the Bochner formula

$$\frac{1}{2}|du|^2 = \langle du, d\Delta u \rangle + |Hess(u)|^2 + \langle Ric(du), du \rangle,$$

we compute

$$\frac{\epsilon^2}{2}\Delta|du|^2 = \frac{1}{2}|d|u|^2|^2 - (1 - |u|^2)|du|^2 + \epsilon^2|Hess(u)|^2 + \epsilon^2\langle Ric(du), du \rangle. \quad (4.10)$$

Fixing some positive $\delta > 0$, we then write

$$w_\delta := \frac{\epsilon^2}{2}|du|^2 - (1 + \delta)\frac{(1 - |u|^2)}{2},$$

and combine (4.10) with (4.3) to find that

$$\begin{aligned} \Delta w_\delta &= \frac{1}{2}|d|u|^2|^2 + \epsilon^2|Hess(u)|^2 + \epsilon^2\langle Ric(du), du \rangle \\ &\quad - (1 - |u|^2)|du|^2 + (1 + \delta)(|du|^2 - \epsilon^{-2}(1 - |u|^2)|u|^2) \\ &= \frac{1}{2}|d|u|^2|^2 + \epsilon^2|Hess(u)|^2 + \epsilon^2\langle Ric(du), du \rangle \\ &\quad + \delta|du|^2 + 2\epsilon^{-2}|u|^2w_\delta. \end{aligned}$$

In particular, taking $\delta = \epsilon^2|Ric|_{L^\infty}$, it follows that w_δ cannot have a positive maximum, and we arrive at the pointwise gradient estimate

$$\epsilon^2|du|^2 \leq (1 + \epsilon^2|Ric_M|_{L^\infty})(1 - |u|^2). \quad (4.11)$$

Writing

$$W(u) := \frac{(1 - |u|^2)^2}{4},$$

and applying (4.11) in the right-hand side of the integral identity (4.6), we find that

$$\int_M \frac{1}{2}|d|u|^2|^2 \leq C(M) \int_M \frac{W(u)}{\epsilon^2}, \quad (4.12)$$

where the constant $C = C(M)$ is independent of ϵ as $\epsilon \rightarrow 0$.

4.2 Stationarity and Monotonicity

Away from the zero set of u , the identities (4.3) and (4.7) for $|u|^2$ and ju carry all the information contained in the Ginzburg-Landau equations (4.1). To understand behavior close to the zero set, however, another identity takes on an essential role—namely, the stationary equation, corresponding to the fact that u is a critical point of E_ϵ with respect to inner variations of the form $u_t = u \circ \Phi_t$, where Φ_t is a one-parameter family of diffeomorphisms.

Writing

$$du^* du := du^1 \otimes du^1 + du^2 \otimes du^2,$$

and denoting by $e_\epsilon(u)$ the energy integrand

$$e_\epsilon(u) := \frac{1}{2}|du|^2 + \frac{W(u)}{\epsilon^2},$$

it follows from (4.1) that the stress-energy tensor

$$T_\epsilon(u) := e_\epsilon(u)Id - du^* du \tag{4.13}$$

is divergence-free on M . As a consequence, for any smooth domain $\Omega \subset M$ and any vector field X , we have the identity

$$\int_{\Omega} e_\epsilon(u) \operatorname{div}(X) - \langle du^* du, \nabla X \rangle = \int_{\partial\Omega} e_\epsilon(u) \langle \nu, X \rangle - \langle du(\nu), du(X) \rangle, \tag{4.14}$$

where ν is the outer unit normal on $\partial\Omega$.

Consider now the case when $\Omega = B_s(p)$ is a geodesic ball of small radius $0 < s < \operatorname{inj}(M)$ about some point $p \in M$, and $X = \nabla\psi(x)$, where $\psi(x) := \frac{1}{2}\operatorname{dist}(x, p)^2$. Applying (4.14) in this case yields the identity

$$s \int_{\partial B_s(p)} e_\epsilon(u) - \left| \frac{\partial u}{\partial \nu} \right|^2 = \int_{B_s(p)} [e_\epsilon(u) \Delta \psi - \langle du^* du, \operatorname{Hess}(\psi) \rangle]. \tag{4.15}$$

Using the Hessian comparison theorem to estimate the difference $|\operatorname{Hess}(\psi) - g|$ between the Hessian of ψ and the metric tensor in terms of $\sec(M)$ and s , we can compute as in [54], [13] to derive the following monotonicity estimate.

Proposition 4.1. *There exists a constant $\Lambda(M)$ such that the energy density*

$$F_\epsilon(u, p, s) := e^{\Lambda s^2} s^{2-n} \int_{B_s(p)} e_\epsilon(u) \quad (4.16)$$

satisfies

$$\frac{d}{ds} F_\epsilon(u, p, s) \geq s^{2-n} \int_{\partial B_s(p)} \left| \frac{\partial u}{\partial \nu} \right|^2 + 2s^{1-n} \int_{B_s(p)} \frac{W(u)}{\epsilon^2} \quad (4.17)$$

for $s < \text{inj}(M)$.

In the higher dimensional setting, as observed in [54] and [13], it is also quite useful to keep in mind the following simple consequence of (4.17). Observing that

$$\begin{aligned} 2s^{1-n} \int_{B_s(p)} \frac{W(u)}{\epsilon^2} &= -\frac{2}{n-2} \frac{d}{ds} \left(s^{2-n} \int_{B_s(p)} \frac{W(u)}{\epsilon^2} \right) \\ &\quad + \frac{2}{n-2} s^{2-n} \int_{\partial B_s(p)} \frac{W(u)}{\epsilon^2}, \end{aligned}$$

and integrating (4.17) from 0 to r gives the identity

$$nF_\epsilon(u, p, r) \geq \int_{B_r(p)} \text{dist}(x, p)^{2-n} [|\langle du, \nabla \text{dist}_p \rangle|(x)^2 + 2 \frac{W(u(x))}{\epsilon^2}]. \quad (4.18)$$

4.3 The η -Compactness/Ellipticity Theorem

In this section, we discuss the extension to Riemannian manifolds of a fundamental technical result in the study of the Ginzburg-Landau equations, which can be thought of as a kind of ϵ -regularity result in this setting. Roughly speaking, the theorem tells us that a solution u of (4.1) must have energy of order at least $|\log \epsilon|$ near its zero set. More precisely, we have the following, commonly known as either the “ η -compactness” or “ η -ellipticity” theorem.

Theorem 4.2. *(cf. [54], [55], [13]) For $\epsilon \in (0, 1)$ and $\epsilon^{1/2} < r < \text{inj}(M)$, there exists a positive constant $\eta(M) > 0$ such that if u solves (1.1) on M and $|u(p)| \leq \frac{1}{2}$ at a point $p \in M$, then*

$$F_\epsilon(u, p, r) \geq \eta |\log \epsilon|, \quad (4.19)$$

where $F_\epsilon(u, p, r)$ is defined as in (4.16).

Theorem 4.2 was first proved in [69] for minimizing solutions in the three-dimensional setting, and was extended in [54] to minimizing solutions in $\mathbb{R}^n$ for any $n \geq 3$. The result was later established

for non-minimizing critical points in $\mathbb{R}^3$ in [55], before finally being proved in [13] for arbitrary solutions of (4.1) in $\mathbb{R}^n$ for all $n \geq 3$. In what follows, we review the ideas of the proof given in [13], with appropriate adaptations to the manifold setting. The proof rests largely on a clever use of the monotonicity formula (4.17), together with the following key estimate.

Proposition 4.3. *(cf. [13], Theorem 3) Let u solve (4.1) on the closed, orientable manifold (M^n, g) ($n \geq 3$), for some $\epsilon \in (0, 1)$. Then for any $\epsilon < r < \text{inj}(M)$, $\delta < \frac{1}{8}$, and $p \in M$, there exists a constant $C(M) < \infty$ depending only on the geometry of M such that*

$$\int_{B_{\delta r}(p)} e_\epsilon(u) \leq C \left(\delta^n + r^{2-n} \int_{B_r(p)} \frac{W(u)}{\epsilon^2} \right) \int_{B_r(p)} e_\epsilon(u) + C \int_{B_r(p)} \frac{W(u)}{\epsilon^2}. \quad (4.20)$$

Proof. To begin the proof, we first make the obvious observations that

$$\int_{B_{\delta r}(p)} \frac{W(u)}{\epsilon^2} \leq \int_{B_r(p)} \frac{W(u)}{\epsilon^2},$$

trivially, and

$$\int_{B_{\delta r}(p)} (1 - |u|^2) |du|^2 \leq \int_{B_{\delta r}(p)} C \frac{(1 - |u|^2)^2}{\epsilon^2} \leq C \int_{B_r(p)} \frac{W(u)}{\epsilon^2},$$

by the pointwise gradient estimate (4.11). Thus, to estimate the left-hand side of (4.20), it remains to estimate the integral over $B_{\delta r}(p)$ of

$$|u|^2 |du|^2 = |ju|^2 + \frac{1}{4} |d|u|^2|^2.$$

To estimate contribution $|d|u|^2|^2$ from the gradient of the modulus, we integrate the identity (4.5) against a cut-off function $\zeta^2 \in C_c^\infty(B_r(p))$ to obtain the relation

$$\begin{aligned} \int \zeta^2 \left(\frac{1}{2} |d|u|^2|^2 + |u|^2 \frac{(1 - |u|^2)^2}{\epsilon^2} \right) &= \int \zeta^2 (1 - |u|^2) |du|^2 \\ &\quad + \int \zeta (1 - |u|^2) \langle d(|u|^2), d\zeta \rangle, \end{aligned}$$

and using Cauchy-Schwarz to estimate the last term on the right-hand side, we arrive at the estimate

$$\int \zeta^2 |d|u|^2|^2 \leq C \int \zeta^2 (1 - |u|^2) |du|^2 + |d\zeta|^2 (1 - |u|^2)^2. \quad (4.21)$$

Now, choosing $\zeta \geq 0$ such that $\zeta \equiv 1$ on $B_{\delta r}(p)$ and $|\nabla \zeta| \leq \frac{2}{r} \leq \frac{2}{\epsilon}$, we see from (4.21) (and the

gradient estimate (4.11)) that

$$\int_{B_{\delta r}(p)} |d|u|^2|^2 \leq C \int_{B_r(p)} \frac{W(u)}{\epsilon^2}. \quad (4.22)$$

Combining our estimates thus far, we have that

$$\int_{B_{\delta r}(p)} e_\epsilon(u) \leq C \int_{B_r(p)} \frac{W(u)}{\epsilon^2} + \int_{B_{\delta r}(p)} \frac{1}{2} |ju|^2,$$

leaving us to estimate the contribution of $|ju|^2$ to the energy density. To do this, we proceed as in [13] by first introducing a one-form

$$\alpha = \phi(|u|^2)ju,$$

where ϕ has the property that $\phi(t^2) = \frac{1}{t^2}$ for $t \geq \frac{3}{4}$, and $|\phi'| \leq 10$. Note that, where $|u| \geq \frac{3}{4}$, we have

$$\begin{aligned} |\alpha - ju| &= |1 - \phi(|u|^2)||ju| \\ &= \frac{|1 - |u|^2|}{|u|^2} |ju|, \end{aligned}$$

so that

$$|\alpha - ju|^2 \leq C(1 - |u|^2)^2 |ju|^2 \leq C \frac{(1 - |u|^2)^2}{\epsilon^2} \quad (4.23)$$

pointwise. In particular, it then follows that

$$\int_{B_{\delta r}(p)} e_\epsilon(u) \leq C \int_{B_r(p)} \frac{W(u)}{\epsilon^2} + \int_{B_{\delta r}(p)} |\alpha|^2. \quad (4.24)$$

The advantage of working with α in place of ju is that

$$d\alpha = d(|u|^{-2}ju) = d(u^*(d\theta)) = 0 \text{ on } \{|u| \geq \frac{3}{4}\},$$

while

$$|d\alpha| = |\phi(|u|^2)2du^1 \wedge du^2 + \phi'(|u|^2)d|u|^2 \wedge ju| \leq C \frac{W(u)}{\epsilon^2} \text{ on } \{|u| < \frac{3}{4}\},$$

in view of the gradient estimate (4.11), giving us the global pointwise estimate

$$|d\alpha| \leq C \frac{W(u)}{\epsilon^2}. \quad (4.25)$$

Fix now a nonnegative cutoff function $\psi \in C_c^\infty(B_{r/2}(p))$ such that $\psi \equiv 1$ on $B_{r/4}(p)$ and $|d\psi| \leq \frac{10}{r}$. In the remainder of the proof, we will introduce a suitable local Hodge decomposition of $\psi\alpha$, and estimate the components separately to arrive at the desired estimate (4.20).

To begin, denote by ω the two-form $\omega = \psi d\alpha$, and by $H(\omega) \in \mathcal{H}^2(M)$ the harmonic component of its Hodge decomposition. Consider then the two-form $\xi_1 := \Delta_H^{-1}(\omega - H(\omega))$ —that is, ξ_1 is the unique two-form satisfying

$$\Delta_H \xi_1 = (dd^* + d^*d)\xi_1 = \omega - H(\omega) \quad \text{and} \quad \xi_1 \perp_{L^2} \mathcal{H}^2(M). \quad (4.26)$$

As observed in [13], a key step in the proof of (4.20) is obtaining a nice L^∞ bound for ξ_1 . To this end, we record the following standard lemma, giving a Green's function-type estimate for $\|\xi_1\|_{L^\infty}$ in terms of ω .

Lemma 4.4. *There exists a constant $C(M) < \infty$ such that*

$$\|\xi_1\|_{L^\infty(M)} \leq C \sup_{y \in M} \int_M \text{dist}(x, y)^{2-n} |\omega(x)|. \quad (4.27)$$

Proof. For $t > 0$ small, consider the functions

$$\varphi_t := (t + |\xi_1|^2)^{1/2},$$

and use the Bochner formula to compute

$$\begin{aligned} \Delta \varphi_t &= \varphi_t^{-1} \left[\frac{1}{2} \Delta |\xi_1|^2 - \varphi_t^{-3} \frac{1}{2} \frac{1}{2} d|\xi_1|^2|^2 \right] \\ &= \varphi_t^{-1} [-\langle \xi_1, \Delta_H \xi_1 \rangle + \mathcal{R}(\xi_1, \xi_1) + |\nabla \xi_1|^2 - \varphi_t^{-2} \langle \nabla \xi_1, \xi_1 \rangle] \\ &\geq \varphi_t^{-1} [-\langle \xi_1, \Delta_H \xi_1 \rangle + \mathcal{R}(\xi_1, \xi_1)] \\ &\geq -|\Delta_H \xi_1| - C(M) |\xi_1|, \end{aligned}$$

where $\mathcal{R}$ is the appropriate curvature operator. Using standard estimates for the Green's function on Riemannian manifolds (see, e.g., Chapter 4 of [9]), it then follows that

$$\varphi_t(p) \leq C(M) \int_M \text{dist}(p, x)^{2-n} |\Delta_H \xi_1|(x) + C(M) \|\varphi_t\|_{L^1(M)}$$

for every point $p \in M$, and taking $t \rightarrow 0$, it follows that

$$\|\xi_1\|_{L^\infty} \leq C(M) \sup_{y \in M} \int_M \text{dist}(x, y)^{2-n} |\omega - H(\omega)|(x) + C(M) \|\xi_1\|_{L^1}. \quad (4.28)$$

Moreover, since ξ_1 is orthogonal to the harmonic two-forms, we know that

$$\begin{aligned} \|\xi_1\|_{L^1}^2 &\leq C(M) \int \langle \xi_1, \Delta_H \xi_1 \rangle \\ &\leq C(M) \|\xi_1\|_{L^\infty} \|\Delta_H \xi_1\|_{L^1}, \end{aligned}$$

which we can combine with (4.28) to deduce that

$$\|\xi_1\|_{L^\infty(M)} \leq C(M) \sup_{y \in M} \int_M \text{dist}(x, y)^{2-n} |\omega - H(\omega)|(x).$$

Finally, noting that the harmonic component $H(\omega)$ of ω satisfies

$$\|H(\omega)\|_{L^\infty} \leq C(M) \|\omega\|_{L^1},$$

we arrive at the desired estimate (4.27) □

Now, with the estimate (4.27) in hand, we see that

$$\begin{aligned} \int_M |d^* \xi_1|^2 &\leq \int \langle \xi_1, \Delta_H \xi_1 \rangle \\ &\leq C(M) \|\xi_1\|_{L^\infty} \|\omega\|_{L^1} \\ &\leq C \|\omega\|_{L^1} \cdot \sup_{y \in M} \int_M \text{dist}(x, y)^{2-n} |\omega(x)|, \end{aligned}$$

and invoking the pointwise estimate (4.25) to estimate ω , we obtain

$$\begin{aligned} \|d^* \xi_1\|_{L^2}^2 &\leq C \left(\int \psi \frac{W(u)}{\epsilon^2} \right) \cdot \sup_{y \in M} \int_M \text{dist}(x, y)^{2-n} \psi(x) \frac{W(u)(x)}{\epsilon^2} \\ &\leq C \left(\int_{B_{r/2}(p)} \frac{W(u)}{\epsilon^2} \right) \cdot \sup_{y \in M} \int_{B_{r/2}(p)} \text{dist}(x, y)^{2-n} \frac{W(u)(x)}{\epsilon^2} \end{aligned}$$

To estimate the final term on the right, note first that if $\text{dist}(y, p) > \frac{3r}{4}$, then

$$\int_{B_{r/2}(p)} \text{dist}(x, y)^{2-n} \frac{W(u)(x)}{\epsilon^2} \leq C r^{2-n} \int_{B_{r/2}(p)} \frac{W(u)}{\epsilon^2}.$$

On the other hand, if $\text{dist}(y, p) < \frac{3r}{4}$, then (as in [13]) we can invoke the identity (4.18) to see that

$$\begin{aligned} \int_{B_{r/2}(p)} \text{dist}(x, y)^{2-n} \frac{W(u)(x)}{\epsilon^2} &\leq \int_{B_{r/4}(y)} \text{dist}(x, y)^{2-n} \frac{W(u)(x)}{\epsilon^2} + Cr^{2-n} \int_{B_{r/2}(p)} \frac{W(u)}{\epsilon^2} \\ &\leq \frac{n}{2} F_\epsilon(u, y, r/4) + Cr^{2-n} \int_{B_{r/2}(p)} \frac{W(u)}{\epsilon^2} \\ &\leq C(M) r^{2-n} \int_{B_r(p)} e_\epsilon(u). \end{aligned}$$

Putting all these estimates together, we arrive at the L^2 estimate

$$\int_M |d^* \xi_1|^2 \leq C(M) \left(\int_{B_{r/2}} \frac{W(u)}{\epsilon^2} \right) \cdot r^{2-n} \int_{B_r(p)} e_\epsilon(u). \quad (4.29)$$

It remains now to estimate the L^2 norm of difference $\alpha - d^* \xi_1$ on $B_\delta(p)$. To this end, first let φ be a function solving the Neumann boundary problem

$$\Delta \varphi = \text{div}(\alpha) = \text{div}(\alpha - ju) \text{ on } B_r(p), \quad \text{and} \quad \frac{\partial \varphi}{\partial \nu} = (\alpha - ju)(\nu) \text{ on } \partial B_r(p).$$

It's then clear that

$$\int_{B_r(p)} |d\varphi|^2 \leq \int_{B_r(p)} |\alpha - ju|^2 \leq C \int_{B_r(p)} \frac{W(u)}{\epsilon^2}, \quad (4.30)$$

where in the last inequality we've used the pointwise estimate (4.23) for the difference $\alpha - ju$.

Consider, finally, the one-form

$$\sigma := \alpha - d^* \xi_1 - d\varphi. \quad (4.31)$$

On the ball $B_{r/4}(p)$, where $\psi \equiv 1$, we see that

$$\begin{aligned} \Delta_H \sigma &= \Delta_H \alpha - d^* \Delta_H \xi_1 - dd^* d\varphi \\ &= d^* \omega + dd^* \alpha - d^* \omega - dd^* \alpha \\ &= 0 \end{aligned}$$

by definition of ξ_1 and φ , so that σ is harmonic on $B_{r/4}(p)$. In particular, it then follows from standard estimates that

$$\|\sigma\|_{L^\infty(B_{r/8}(p))}^2 \leq Cr^{-n} \|\sigma\|_{L^2(B_{r/4}(p))}^2 \leq Cr^{-n} [\|\alpha\|_{L^2(B_{r/4})}^2 + \|d^* \xi_1\|_{L^2}^2 + \|d\varphi\|_{L^2(B_r(p))}^2]. \quad (4.32)$$

Thus, on any ball $B_{\delta r}(p)$ with $\delta < \frac{1}{8}$, combining (4.29), (4.30), and (4.32), it follows that

$$\begin{aligned}
\int_{B_{\delta r}(p)} |\alpha|^2 &\leq C \int_{B_{\delta r}(p)} [|d^* \xi_1|^2 + |d\varphi|^2 + |\sigma|^2] \\
&\leq C (\|d^* \xi_1\|_{L^2}^2 + \|d\varphi\|_{L^2(B_r(p))}^2) \\
&\quad + C \delta^n (\|\alpha\|_{L^2(B_r(p))}^2 + \|d^* \xi_1\|_{L^2(B_{r/2}(p))}^2 + \|d\varphi\|_{L^2(B_r(p))}^2) \\
&\leq C(M) \left(\int_{B_{r/2}} \frac{W(u)}{\epsilon^2} \right) \cdot r^{2-n} \int_{B_r(p)} e_\epsilon(u) \\
&\quad + C \int_{B_r(p)} \frac{W(u)}{\epsilon^2} + C \delta^n \int_{B_r(p)} e_\epsilon(u).
\end{aligned}$$

Finally, putting this together with (4.24), we arrive at the desired estimate (4.20). $\square$

With Proposition 4.3 in place, we can now complete the proof of Theorem 4.2 just as in [13], by combining the estimate (4.20) with a careful application of the monotonicity formula (4.17). For the convenience of the reader, we recall below the details of the proof.

Proof. (Proof of Theorem 4.2) Following now Lemma III.1 of [13], recall that $\epsilon^{1/2} < r < \text{inj}(M)$ by assumption, and without loss of generality, suppose moreover that $2\epsilon^{1/4} < \frac{1}{8}$. Fix now some $\delta \in [2\epsilon^{1/4}, \frac{1}{8}]$, and set

$$m + 1 = \lfloor \frac{\log(\delta/\epsilon^{1/2})}{\log(4/\delta)} \rfloor.$$

Since $\delta^2 \geq 4\epsilon^{1/2}$, we see that m is nonnegative, and by definition,

$$\begin{aligned}
\log(\epsilon^{-1/2}) &< (m + 2)[\log 4 - \log \delta] - \log \delta \\
&\leq 2(m + 3)|\log \delta|
\end{aligned}$$

(using the fact that $\delta \leq \frac{1}{4}$), so that

$$m + 1 \geq \frac{1}{3}(m + 3) \geq \frac{1}{12} \frac{|\log \epsilon|}{|\log \delta|}. \quad (4.33)$$

Observe next that

$$\bigcup_{j=0}^m ((\delta/4)^{-j} \epsilon^{1/2}, (\delta/4)^{-j-1} \epsilon^{1/2}) \subset (\epsilon^{1/2}, \delta),$$

and we have the simple estimate

$$\begin{aligned}
F_\epsilon(u, p, \delta r) &\geq \int_{\epsilon^{1/2}r}^{\delta r} \frac{d}{ds} F_\epsilon(u, p, s) ds \\
&\geq \sum_{j=0}^m \int_{(\delta/4)^{-j} \epsilon^{1/2}r}^{(\delta/4)^{-j-1} \epsilon^{1/2}r} \frac{d}{ds} F_\epsilon(u, p, s) ds \\
&\geq (m+1) \int_{(\delta/4)^{-j_0} \epsilon^{1/2}r}^{(\delta/4)^{-j_0-1} \epsilon^{1/2}r} \frac{d}{ds} F_\epsilon(u, p, s) ds \\
&\geq \frac{1}{12} \frac{|\log \epsilon|}{|\log \delta|} \int_{(\delta/4)^{-j_0} \epsilon^{1/2}r}^{(\delta/4)^{-j_0-1} \epsilon^{1/2}r} \frac{d}{ds} F_\epsilon(u, p, s) ds
\end{aligned}$$

for some $j_0 \in \{0, \dots, m\}$. Setting $s_1 = (\delta/4)^{-j_0-1} \epsilon^{1/2}r$, it then follows that

$$F_\epsilon(u, p, s_1) - F_\epsilon(u, p, \delta s_1/4) \leq C |\log \delta| \frac{F_\epsilon(u, p, \delta r)}{|\log \epsilon|},$$

and by a simple application of the monotonicity formula (4.17) and a mean value argument, we deduce the existence of $s_0 \in [s_1/4, s_1]$ for which

$$F_\epsilon(u, p, s_0) - F_\epsilon(u, p, \delta s_0) \leq C |\log \delta| \frac{F_\epsilon(u, p, r)}{|\log \epsilon|} \quad (4.34)$$

and

$$s_0^{2-n} \int_{B_{s_0}(p)} \frac{W(u)}{\epsilon^2} \leq C |\log \delta| \frac{F_\epsilon(u, p, r)}{|\log \epsilon|}. \quad (4.35)$$

Now, since $s_0 \geq \epsilon^{1/2}r > \epsilon$, setting

$$\eta := \frac{F_\epsilon(u, p, r)}{|\log \epsilon|},$$

we can apply Proposition 4.3 at scale s_0 together with the preceding estimates to deduce that

$$\begin{aligned}
\delta^{n-2} F_\epsilon(u, p, \delta s_0) &\leq C \left(\delta^n + s_0^{2-n} \int_{B_{s_0}(p)} \frac{W(u)}{\epsilon^2} \right) F_\epsilon(u, p, s_0) \\
&\quad + C s_0^{2-n} \int_{B_{s_0}(p)} \frac{W(u)}{\epsilon^2} \\
&\leq C (\delta^n + |\log \delta| \eta) (F_\epsilon(u, p, \delta s_0) + |\log \delta| \eta) \\
&\quad + C |\log \delta| \eta.
\end{aligned}$$

Note that we have not yet chosen δ ; we now choose $\delta = \delta(n, \eta)$, by setting

$$\delta := \min \left\{ \frac{1}{8}, \eta^{\frac{1}{2(n-2)}} \right\}.$$

With this choice of δ , we rearrange the preceding inequalities to obtain

$$F_\epsilon(u, p, \delta s_0) \cdot (\eta^{1/2} - C\eta^{\frac{n}{2(n-2)}} - C|\log \eta|\eta) \leq C|\log \eta|\eta.$$

Noting that $\eta^{\frac{n}{2(n-2)}} = \eta^{1/2}\eta^{\frac{1}{n-2}}$, we see that for $\eta < \eta_0(M)$ sufficiently small,

$$C(M)\eta^{\frac{n}{2(n-2)}} + C(M)|\log \eta|\eta \leq \frac{1}{2}\eta^{1/2},$$

so that

$$F_\epsilon(u, p, \delta s_0) \leq 2C|\log \eta|\eta^{1/2}. \quad (4.36)$$

Finally, we claim that, since $|u(p)| \leq \frac{1}{2}$, there exists a positive constant $c(M) > 0$ such that

$$F_\epsilon(u, p, \delta s_0) \geq c(M); \quad (4.37)$$

once this is established, we can appeal to (4.36) to arrive at the desired lower bound for $\eta = \frac{F(u, p, r)}{|\log \epsilon|}$.

For this, simply note that, by monotonicity

$$F_\epsilon(u, p, \delta s_0) \geq F_\epsilon(u, p, \epsilon) \geq \epsilon^{2-n} \int_{B_\epsilon(p)} \frac{W(u)}{\epsilon^2}.$$

Since $|u(p)| \leq \frac{1}{2}$ by assumption, in light of the gradient bound (4.11), there is a constant $1 < C < \infty$ such that $|u| \leq \frac{3}{4}$ —and, consequently, $W(u) \geq \frac{1}{4}(7/16)^2$ —on the ball $B_{C^{-1}\epsilon}(p)$. In particular, it follows that

$$\int_{B_\epsilon(p)} \frac{W(u)}{\epsilon^2} \geq \int_{B_{C^{-1}\epsilon}(p)} \frac{W(u)}{\epsilon^2} \geq c \cdot \epsilon^n \cdot \frac{1}{\epsilon^2},$$

giving us the desired lower bound (4.37). □

4.4 Classical and Generalized Varifolds

While the concept of varifold appears in Almgren's 1965 monograph [4], the first definitive treatment of the subject was given in Allard's 1972 paper [5], which, together with Simon's book [73], remains the standard reference on the subject. In [5], a *k-varifold* V on M^n is defined to be a nonnegative Radon measure on the Grassmannian bundle $G_k(M)$ (i.e., the fiber bundle on M whose fiber over each point p is the space $G_k(T_p M)$ of unoriented k -planes in $T_p M$). In what follows, we will denote

by $\mathbf{V}_k(M)$ the space of k -varifolds on M .

To any $V \in \mathbf{V}_k(M)$, one associates the *weight measure* $\|V\| \in C^0(M)^*$, the nonnegative Radon measure on M given by the pushforward $\|V\| := \pi_* V$ of V under the natural projection $\pi : G_k(M) \rightarrow M$. Given a k -varifold V , we also define the *first variation* δV to be the functional $\delta V \in C^1(M, TM)^*$ assigning to each C^1 vector field X the value

$$\int_{P \in G_k M} \langle P, \nabla X \rangle dV(P), \quad (4.38)$$

where we implicitly identify elements in $G_k(M)$ with the associated projections in $End(TM)$, and use $\langle P, \nabla X \rangle$ to denote the Hilbert-Schmidt norm of ∇X with P . A varifold V is said to be *stationary* if $\delta V \equiv 0$.

In the context of geometric measure theory, we are primarily interested in a class of varifolds much smaller than the full space $\mathbf{V}_k(M)$, consisting of those measures $V \in \mathbf{V}_k(M)$ given by integration over some k -manifold—possibly quite singular, and equipped with some multiplicity function (necessary concessions to obtain robust compactness results). Given a k -rectifiable set (see [31] or [73] for a thorough discussion of rectifiable sets) $\Sigma^k \subset M$ and a positive multiplicity function $\theta : \Sigma \rightarrow \mathbb{R}$, we associate a varifold $v(\Sigma, \theta) \in \mathbf{V}_k(M)$ by the pairing

$$\langle v(\Sigma, \theta), f \rangle := \int_{\Sigma} \theta(x) f(T_x \Sigma) d\mathcal{H}^k(x) \quad \text{for every } f \in C^0(G_k(M)). \quad (4.39)$$

A varifold $V = v(\Sigma, \theta)$ of this form is said to be *rectifiable*. If, moreover, the multiplicity function θ takes integer values $\mathcal{H}^k$ -almost everywhere on Σ , then $V = v(\Sigma, \theta)$ is said to be *integral*. For a rectifiable varifold $V = v(\Sigma, \theta)$, it's not difficult to see that V is stationary if and only if the pair (Σ, θ) is a critical point for the k -area functional

$$(\Sigma, \theta) \mapsto \int_{\Sigma} \theta(x) d\mathcal{H}^k(x)$$

under pushforward by diffeomorphisms of M , suggesting that stationary rectifiable and integral varifolds indeed provide reasonable notions of weak minimal submanifolds in M —which, by the work of Allard [5], satisfy nice compactness properties.

In studying energy concentration phenomena for geometric pdes, it is useful to consider a space slightly larger than $\mathbf{V}_k(M)$, which includes measures associated to certain stress-energy tensors. To this end, we recall the concept of *generalized varifold*, introduced by Ambrosio and Sonner in [8]. Denoting by $Sym(TM)$ the bundle of symmetric endomorphisms of TM , for a given integer $k \leq n$,

consider the subbundle $A_k(M) \subset \text{Sym}(TM)$ given by

$$A_k(M) := \{S \in \text{Sym}(TM) \mid -nI \leq S \leq I, \text{tr}(S) \geq k\}. \quad (4.40)$$

We define the space $\mathbf{V}_k^*(M)$ of *generalized k -varifolds* on M to consist of all nonnegative Radon measures on $A_k(M)$. Generalized varifolds were introduced in [8] as a tool for studying energy concentration of solutions to the parabolic Ginzburg-Landau equations, and have since been employed by other authors (see e.g., [53],[57]) in the study of energy concentration and bubbling phenomena in other geometric pdes.

Remark 4.5. The definition of $A_k(M)$ above differs slightly from that given in [8], in that we soften their trace condition ($\text{tr}(S) = k$) to the lower bound

$$\text{tr}(S) \geq k; \quad (4.41)$$

it is easy to check that only the lower bound (4.41) is used in the arguments of [8], and a larger trace can only strengthen the essential estimates.

Since $A_k(M)$ evidently contains the Grassmannian bundle $G_k(M)$ of orthogonal projections onto k -dimensional subspaces, we see that $\mathbf{V}_k^*$ naturally includes the standard varifolds $\mathbf{V}_k$, via pushforward by the inclusion $G_k(m) \hookrightarrow A_k(M)$. The other essential examples of generalized varifolds come from the stress-energy tensors associated to certain geometric energy functionals, as in the following motivating example.

Example 4.6. For any $u \in W^{1,2}(M, \mathbb{C})$, the Ginzburg-Landau stress-energy tensor

$$T_\epsilon(u) := e_\epsilon(u)I - du^*du$$

satisfies

$$\text{tr}(T_\epsilon(u)) = ne_\epsilon(u) - |du|^2 \geq (n-2)e_\epsilon(u)$$

and

$$-e_\epsilon(u)|X|^2 \leq \langle X, T_\epsilon(u)X \rangle \leq e_\epsilon(u)|X|^2,$$

so that $e_\epsilon(u)^{-1}T_\epsilon(u) \in A_{n-2}(M)$ wherever $e_\epsilon(u) > 0$. We can therefore define generalized varifolds $V_\epsilon(u) \in \mathbf{V}_{n-2}^*$ by

$$\langle V_\epsilon(u), f \rangle := \frac{1}{|\log \epsilon|} \int_{\{e_\epsilon(u) > 0\}} e_\epsilon(u) f(e_\epsilon(u)^{-1}T_\epsilon(u)) dv_g. \quad (4.42)$$

for every $f \in C^0(A_{n-2}(M))$.

As with classical varifolds, for a generalized varifold $V \in \mathbf{V}_k^*$, we define the weight measure $\|V\|$ in $(C^0(M))^*$ as the pushforward of V under the projection $A_k(M) \rightarrow M$, and the first variation δV as the functional on C^1 vector fields X given by

$$\delta V(X) = \int_{A_m(M)} \langle S, \nabla X \rangle dV(S). \quad (4.43)$$

As in the classical setting, V is said to be *stationary* if $\delta V = 0$. For the generalized varifolds $V_\epsilon(u)$ defined by (4.42), we note that the weight measure $\|V_\epsilon(u)\|$ and first variation δV_ϵ are given by

$$\|V_\epsilon(u)\| = \frac{e_\epsilon(u)}{|\log \epsilon|} dv_g \quad (4.44)$$

and

$$\delta V_\epsilon(u)(X) = \frac{1}{|\log \epsilon|} \int_M \langle T_\epsilon(u), \nabla X \rangle dv_g, \quad (4.45)$$

so that $V_\epsilon(u)$ is stationary if u solves (4.1).

Recall that for a Radon measure μ on M , the upper m -density $\Theta_k^*(\mu, x)$ is defined at each point $x \in M$ by

$$\Theta_k^*(\mu, x) := \limsup_{r \rightarrow 0} \frac{\mu(B_r(x))}{\omega_k r^k},$$

where ω_k is the volume of the unit k -ball. The main technical result of [8] which we will employ in the proof of Theorem 1.1 (as in [13], [17]) is the following, asserting that the weight measure of a stationary generalized k -varifold with strictly positive density $\Theta_k^*(\|V\|, \cdot) > 0$ on its support can be realized as the weight measure of a stationary, rectifiable k -varifold—generalizing a result of Allard ([5], Section 5) concerning the rectifiability of varifolds with positive density.

Theorem 4.7. (cf. [8], Theorem 3.8) *To every stationary, generalized k -varifold V satisfying*

$$\Theta_k^*(\|V\|, x) > 0 \text{ for } \|V\| - \text{a.e. } x \in M, \quad (4.46)$$

one can associate a stationary, rectifiable (standard) k -varifold $\tilde{V}$ such that

$$\|\tilde{V}\| = \|V\| \text{ in } C^0(M)^*.$$

Remark 4.8. It is interesting to note that in general the varifold $\tilde{V}$ in Theorem 4.7 will not coincide

with V as measures on $A_k(M)$, though they will induce identical linear functionals on sections $\Gamma(\text{End}(TM))$ of the endomorphism bundle, in the sense that

$$\int_{P \in A_k(M)} \langle P, S \rangle dV(P) = \int_{P \in G_k(M)} \langle P, S \rangle d\tilde{V}(P) \text{ for } S \in \Gamma(\text{End}(TM)).$$

Indeed, in the application of Theorem 4.7 to the Ginzburg-Landau equations discussed in the following section, the limit V of the generalized varifolds $V_\epsilon(u_\epsilon)$ will not coincide with the associated rectifiable varifold $\tilde{V}$ as measures on $A_k(M)$ —an interesting fact pointed out to me by Alessandro Pigati. Roughly speaking, this can be seen from the fact that the normalized stress-energy tensors $\frac{T_\epsilon(u_\epsilon)}{e_\epsilon(u_\epsilon)}$ are well-approximated as $\epsilon \rightarrow 0$ by the tensors $\frac{|ju_\epsilon|^2 Id - 2ju_\epsilon \otimes ju_\epsilon}{|ju_\epsilon|^2}$ associated to the one-forms ju_ϵ , which lie far in every fiber from the Grassmannian bundle $G_{n-2}(M)$.

4.5 Proof of Theorem 1.1

Armed with all the essential estimates, we come now to the proof of Theorem 1.1 concerning the asymptotics of solutions u_ϵ of (4.1) as $\epsilon \rightarrow 0$. For the reader's convenience, we restate the theorem here, with a few additional details that will be useful to us in subsequent sections. As in the preceding sections, the arguments that follow are based on those of prior works like [54] and [13], with minor modifications suitable to our setting.

Theorem 4.9. *Consider a family of maps $(0, 1) \ni \epsilon \mapsto u_\epsilon : M \rightarrow \mathbb{C}$ on a closed, oriented Riemannian manifold M solving the Ginzburg-Landau equations*

$$\Delta u_\epsilon = -\epsilon^{-2}(1 - |u_\epsilon|^2)u_\epsilon,$$

subject to an energy bound $E_\epsilon(u_\epsilon) \leq \Lambda |\log \epsilon|$. Then passing to a subsequence, as $\epsilon \rightarrow 0$, the normalized energy measures $\mu_\epsilon := \frac{|du_\epsilon|^2}{2|\log \epsilon|}$ converge (weakly in $(C^0)^*$) to a limiting measure μ of the form

$$\mu = \nu + \frac{1}{2}|h|^2 d\text{vol}_g,$$

where ν is a stationary, rectifiable $(n-2)$ -varifold, and $h \in \mathcal{H}^1(M)$ is that harmonic one-form given by the limit

$$h := \lim_{i \rightarrow \infty} \frac{h(u_{\epsilon_i})}{|\log \epsilon_i|^{1/2}},$$

where $h(u_\epsilon)$ denotes the harmonic component of the one-form $ju_\epsilon := u_\epsilon^*(r^2 d\theta)$.

Proof. To begin, we note that it is always possible to find a family of harmonic S^1 -valued maps $\phi_\epsilon : M \rightarrow S^1$ for which

$$\|j\phi_\epsilon - h_\epsilon\|_{L^2} \leq C(M),$$

since the one-forms $j\phi = \phi^*(d\theta)$ associated to harmonic maps $\phi : M \rightarrow S^1$ define a lattice of full rank in the space $\mathcal{H}^1(M)$ of harmonic one-forms. Defining

$$\tilde{u}_\epsilon := \phi_\epsilon^{-1} u_\epsilon$$

and writing $\beta_\epsilon := ju_\epsilon - j\phi_\epsilon$, it's not hard to see that

$$j\tilde{u}_\epsilon = ju_\epsilon - |u_\epsilon|^2 j\phi_\epsilon = \beta_\epsilon + (1 - |u_\epsilon|^2)j\phi_\epsilon,$$

and a straightforward computation shows that

$$e_\epsilon(u_\epsilon) = e_\epsilon(\tilde{u}_\epsilon) + \frac{1}{2}|j\phi_\epsilon|^2 + \langle \beta_\epsilon, j\phi_\epsilon \rangle + \frac{1}{2}(1 - |u_\epsilon|^2)|j\phi_\epsilon|^2. \quad (4.47)$$

Next, we show that

$$\frac{1}{|\log \epsilon|} \int (1 - |u_\epsilon|^2)|j\phi_\epsilon|^2 + |\langle \beta_\epsilon, j\phi_\epsilon \rangle| \rightarrow 0$$

as $\epsilon \rightarrow 0$. First, since $j\phi_\epsilon$ is harmonic, we have an estimate of the form

$$\|j\phi_\epsilon\|_{L^\infty} \leq C\|j\phi_\epsilon\|_{L^2}, \quad (4.48)$$

and since

$$\|j\phi_\epsilon - h_\epsilon\|_{L^2} \leq C|j\phi_\epsilon - h_\epsilon|_b \leq C\pi \quad (4.49)$$

by our choice of ϕ_ϵ , it follows that

$$\|j\phi_\epsilon\|_{L^\infty}^2 \leq C\|h_\epsilon\|_{L^2}^2 \leq 2CE_\epsilon(u_\epsilon). \quad (4.50)$$

As a consequence, we see that

$$\int (1 - |u_\epsilon|^2)|j\phi_\epsilon|^2 \leq CE_\epsilon(u_\epsilon) \int (1 - |u_\epsilon|^2),$$

and since

$$\int_M (1 - |u_\epsilon|^2) \leq \epsilon \cdot \text{Vol}(M)^{1/2} \left(\int_M \frac{(1 - |u_\epsilon|^2)^2}{\epsilon^2} \right)^{1/2},$$

it follows that

$$\int (1 - |u_\epsilon|^2) |j\phi_\epsilon|^2 \leq C\epsilon E_\epsilon(u)^{3/2} \leq C(M, \Lambda)\epsilon |\log \epsilon|^{3/2}. \quad (4.51)$$

To estimate $\|\langle \beta_\epsilon, j\phi_\epsilon \rangle\|_{L^1}$, we first use the L^∞ estimate (4.50) to see that

$$\int |\langle \beta_\epsilon, j\phi_\epsilon \rangle| \leq CE_\epsilon(u)^{1/2} \|\beta_\epsilon\|_{L^1}. \quad (4.52)$$

Since $d^*ju_\epsilon = 0$, we know that ju_ϵ has a Hodge decomposition of the form

$$ju_\epsilon = d^*\xi_\epsilon + h_\epsilon,$$

where

$$\xi_\epsilon := \Delta_H^{-1} dj u_\epsilon. \quad (4.53)$$

In particular, we can write β_ϵ as

$$\beta_\epsilon = d^*\xi_\epsilon + (h_\epsilon - j\phi_\epsilon),$$

and since

$$\|h_\epsilon - j\phi_\epsilon\|_{L^1} \leq C,$$

it follows that

$$\|\beta_\epsilon\|_{L^1} \leq C + \|d^*\xi_\epsilon\|_{L^1}. \quad (4.54)$$

Next, we invoke the fundamental Jacobian estimates of Jerrard and Sonner [48] (see also [15]) to see that

$$\|dj u_\epsilon\|_{(W^{1,q})^*} \leq C \|dj u_\epsilon\|_{(C^0)^*} \leq C\Lambda$$

for any $q \in (n, \infty)$, so that the L^q regularity for Δ_H (see, e.g., [72]), together with a simple duality argument gives us the estimate

$$\|d^*\xi_\epsilon\|_{L^p} \leq C(M, \Lambda, p) \text{ for every } p \in [1, \frac{n}{n-1}). \quad (4.55)$$

In particular, together with (4.54), this implies that

$$\|\beta_\epsilon\|_{L^1} \leq C(M, \Lambda); \quad (4.56)$$

applying (4.56) to (4.52), we arrive at the estimate

$$\int |\langle \beta_\epsilon, j\phi_\epsilon \rangle| \leq CE_\epsilon(u_\epsilon)^{1/2} \leq C' |\log \epsilon|^{1/2}. \quad (4.57)$$

In light of the identity (4.47), the estimates (4.51) and (4.57) tell us that

$$\frac{1}{|\log \epsilon|} \int_M |e_\epsilon(u_\epsilon) - [e_\epsilon(\tilde{u}_\epsilon) + \frac{1}{2}|j\phi_\epsilon|^2]| \leq C |\log \epsilon|^{-1/2} \quad (4.58)$$

for ϵ sufficiently small. As an immediate consequence, we see that the generalized varifolds

$$V_\epsilon = V_\epsilon(\tilde{u}_\epsilon)$$

(defined as in (4.42), with $\tilde{u}_\epsilon$ in place of u_ϵ) have uniformly bounded mass, while the harmonic forms $|\log \epsilon|^{-1/2} j\phi_\epsilon$ are clearly bounded in $\mathcal{H}^1(M)$. Thus, we can extract convergent subsequences

$$V_\epsilon \rightarrow V$$

and

$$\lim_{\epsilon \rightarrow 0} \frac{j\phi_\epsilon}{|\log \epsilon|^{1/2}} = \lim_{\epsilon \rightarrow 0} \frac{h_\epsilon}{|\log \epsilon|^{1/2}} = h,$$

and apply (4.58) to deduce that

$$\mu = \lim \mu_{\epsilon_{j_k}} = \|V\| + \frac{1}{2}|h|^2 dv_g. \quad (4.59)$$

In particular, to complete the proof of Theorem 4.9, it remains for us to show that the weight measure $\|V\|$ of V coincides with that of a stationary, rectifiable $(n-2)$ -varifold. Namely, by virtue of Theorem 4.7 in the preceding section, it remains to show that

$$\delta V = 0 \quad (4.60)$$

and

$$\Theta_{n-2}^*(\|V\|, x) \geq c(M) > 0 \text{ at every } x \in \text{spt}(V). \quad (4.61)$$

Adopting the notation

$$S(\alpha, \beta) := \frac{1}{2}(\langle \alpha, \beta \rangle I - \alpha \otimes \beta - \beta \otimes \alpha)$$

for any one-forms α, β , we begin by computing

$$T_\epsilon(\tilde{u}_\epsilon) = T_\epsilon(u_\epsilon) - S(j\phi_\epsilon, j\phi_\epsilon) - 2S(\beta_\epsilon, j\phi_\epsilon) - (1 - |u_\epsilon|^2)S(j\phi_\epsilon, j\phi_\epsilon). \quad (4.62)$$

Since u_ϵ solves (4.1), we know that

$$\text{div}(T_\epsilon(u_\epsilon)) = 0,$$

and since $j\phi_\epsilon$ is harmonic, we have that

$$\text{div}(S(j\phi_\epsilon, j\phi_\epsilon)) = 0$$

as well. It therefore follows that

$$\text{div}(T_\epsilon(\tilde{u}_\epsilon)) = -\text{div}[2S(\beta_\epsilon, j\phi_\epsilon) + (1 - |u_\epsilon|^2)S(j\phi_\epsilon, j\phi_\epsilon)]. \quad (4.63)$$

Integrating (4.63) against a vector field X , we then see that

$$\begin{aligned} \left| \int \langle T_\epsilon(\tilde{u}_\epsilon), \nabla X \rangle \right| &= \left| \int \langle 2S(\beta_\epsilon, j\phi_\epsilon) + (1 - |u_\epsilon|^2)S(j\phi_\epsilon, j\phi_\epsilon), \nabla X \rangle \right| \\ &\leq C \|\nabla X\|_\infty \int |\beta_\epsilon| |j\phi_\epsilon| + (1 - |u_\epsilon|^2) |j\phi_\epsilon|^2 \\ &\leq C |\log \epsilon|^{1/2} \|\nabla X\|_\infty, \end{aligned}$$

where the last estimate follows from (4.51) and (4.56). Dividing by $|\log \epsilon|$ yields

$$|\delta V_\epsilon(X)| \leq C |\log \epsilon|^{-1/2} \|\nabla X\|_\infty, \quad (4.64)$$

and taking the limit $\epsilon \rightarrow 0$, it follows from the weak convergence $V_\epsilon \rightarrow V$ that

$$\delta V = 0,$$

as desired.

It remains to establish a lower bound for the $(n-2)$ -density of $\|V\|$ on the support of V .
Setting

$$\mu = \lim_{\epsilon_j \rightarrow 0} \frac{e_{\epsilon_j}(u_{\epsilon_j})}{|\log \epsilon_j|} dv_g$$

as before, it follows from (4.59) that

$$\Theta_{n-2}^*(\mu, x) = \Theta_{n-2}^*(\|V\|, x), \quad (4.65)$$

since the other component of μ is given by the L^∞ function $\frac{1}{2}|h|^2$. Now, let $\eta > 0$ be the constant from Theorem 4.2, and let $p \in M$ be a point at which

$$\Theta_{n-2}^*(\|V\|, p) = \Theta_{n-2}^*(\mu, p) < \frac{1}{C(M)}\eta. \quad (4.66)$$

We will show that (4.66) implies $p \notin \text{spt}\|V\|$, from which the desired density bound (4.61) then follows.

By definition of the density, (4.66) tells us that

$$r^{2-n}\mu(B_{2r}(p)) < \frac{2^{n-2}}{C(M)}\eta \quad (4.67)$$

for $r > 0$ sufficiently small. Fixing one such $r > 0$, it then follows from the definition of μ that

$$r^{2-n} \int_{B_{2r}(p)} e_\epsilon(u_\epsilon) < 2^{n-2} \frac{\eta |\log \epsilon|}{C(M)} \quad (4.68)$$

for ϵ sufficiently small, so, taking $C(M)$ sufficiently large, we can apply Theorem 4.2 at every point $x \in B_r(p)$ to deduce that

$$|u_\epsilon| > \frac{1}{2} \text{ on } B_r(p). \quad (4.69)$$

We claim now that

$$\|V\|(B_{r/2}(p)) = 0, \quad (4.70)$$

which is evidently equivalent to the statment that

$$\lim_{\epsilon \rightarrow 0} \int_{B_{r/2}(p)} e_\epsilon(u_\epsilon) - \frac{1}{2}|h_\epsilon|^2 = o(|\log \epsilon|) \quad (4.71)$$

as $\epsilon \rightarrow 0$. Expanding $e_\epsilon(u_\epsilon)$ as

$$e_\epsilon(u_\epsilon) = \frac{1}{2}|ju_\epsilon|^2 + \frac{1}{2}(1 - |u_\epsilon|^2) + \frac{1}{4}|d|u_\epsilon|^2|^2 + \frac{W(u_\epsilon)}{\epsilon^2}, \quad (4.72)$$

it follows from the estimates in Section 4.1 that

$$\int_M e_\epsilon(u_\epsilon) - \frac{1}{2}|ju_\epsilon|^2 \leq C \int_M \frac{W(u)}{\epsilon^2},$$

and appealing to arguments of [16]—which rely only on the η -compactness/ellipticity theorem (Theorem 4.2), the monotonicity formula (4.17), and a covering argument—we deduce that

$$\int_M e_\epsilon(u_\epsilon) - \frac{1}{2}|ju_\epsilon|^2 \leq C(M, \Lambda).$$

In particular, (4.71) will follow once we show that

$$\lim_{\epsilon \rightarrow 0} \frac{1}{|\log \epsilon|} \int_{B_{r/2}(p)} |ju_\epsilon|^2 - |h_\epsilon|^2 = 0. \quad (4.73)$$

Considering once again the Hodge decomposition

$$ju_\epsilon = d^*\xi_\epsilon + h_\epsilon$$

of ju_ϵ , we write

$$\int_{B_{r/2}(p)} |ju_\epsilon|^2 - |h_\epsilon|^2 = \int_{B_{r/2}(p)} |d^*\xi_\epsilon|^2 + 2\langle d^*\xi_\epsilon, h_\epsilon \rangle.$$

Next, consider as in Section 4.3 a perturbed one-form $\alpha_\epsilon := \phi(|u_\epsilon|^2)ju_\epsilon$ (where $\phi(t^2) = \frac{1}{t^2}$ for $t \geq \frac{1}{2}$) satisfying

$$d\alpha_\epsilon = 0 \text{ on } \{|u_\epsilon| \geq \frac{1}{2}\} \quad (4.74)$$

and

$$\|\alpha_\epsilon - ju_\epsilon\|_{L^2}^2 \leq C \int_M \frac{W(u)}{\epsilon^2} \leq C(M, \Lambda). \quad (4.75)$$

Writing out the Hodge decomposition

$$\alpha_\epsilon = d\varphi_\epsilon + d^*\tilde{\xi}_\epsilon + h(v_\epsilon)$$

for α_ϵ , it follows from (4.75) that

$$\|d^*\tilde{\xi}_\epsilon - d^*\xi_\epsilon\|_{L^2} \leq C, \quad (4.76)$$

while we deduce from (4.74) that $d^*\tilde{\xi}_\epsilon$ satisfies the strongly harmonic property

$$dd^*\alpha_\epsilon = d^*d^*\alpha_\epsilon = 0 \quad (4.77)$$

on $\{|u_\epsilon| \geq \frac{1}{2}\}$. In particular, since $|u_\epsilon| > \frac{1}{2}$ on $B_r(p)$ by (4.69), it follows that (4.77) holds on the ball $B_r(p)$, and as a consequence, we have an interior L^∞ estimate of the form

$$\|d^*\tilde{\xi}_\epsilon\|_{L^\infty(B_{r/2}(p))} \leq C(M, r)\|d^*\tilde{\xi}_\epsilon\|_{L^1(B_r(p))}. \quad (4.78)$$

Now, it follows from (4.76) that

$$\|d^*\tilde{\xi}_\epsilon - d^*\xi_\epsilon\|_{L^1} \leq C\|d^*\tilde{\xi}_\epsilon - d^*\xi_\epsilon\|_{L^2} \leq C,$$

while we know from (4.55) that

$$\|d^*\xi_\epsilon\|_{L^1} \leq C(M, \Lambda).$$

We therefore see that

$$\|d^*\tilde{\xi}_\epsilon\|_{L^1} \leq C,$$

and applying this in (4.78), we arrive at the estimate

$$\|d^*\tilde{\xi}_\epsilon\|_{L^\infty(B_{r/2}(p))} \leq C(M, r). \quad (4.79)$$

Finally, combining (4.79) with (4.76), we deduce that

$$\int_{B_{r/2}(p)} |d^*\xi_{\epsilon_j}|^2 + 2\langle d^*\xi_{\epsilon_j}, h_{\epsilon_j} \rangle \leq C(1 + \|h_{\epsilon_j}\|_{L^2}) \leq C|\log \epsilon_j|^{1/2} \quad (4.80)$$

and conclude finally that

$$\|V\|(B_{r/2}(p)) \leq \lim_{\epsilon_j \rightarrow 0} \frac{C}{|\log \epsilon_j|^{1/2}} = 0, \quad (4.81)$$

as desired. $\square$

Chapter 5

Min-Max Solutions: Existence and Energy Growth

In this chapter, we explain how to produce nontrivial families of solutions u_ϵ of the Ginzburg-Landau equations on any closed manifold via a natural two-parameter min-max construction, and show that the energy $E_\epsilon(u_\epsilon)$ of these families blows up like $|\log \epsilon|$ as $\epsilon \rightarrow 0$.

5.1 The Saddle-point Construction

In this section, we will define the Ginzburg-Landau functionals $E_\epsilon : W^{1,2}(M, \mathbb{C}) \rightarrow \mathbb{R}$ by

$$E_\epsilon(u) = \int_M e_\epsilon(u) := \int_M \frac{1}{2} |du|^2 + \frac{W(u)}{\epsilon^2},$$

where $W : \mathbb{C} \rightarrow \mathbb{R}$ is a nonnegative potential of the form

$$W(z) = f(|z|^2) \tag{5.1}$$

for some $f \in C^\infty(\mathbb{R})$ satisfying

$$f(t) = \frac{1}{4}(1-t)^2 \text{ for } t \in [0, 2] \tag{5.2}$$

and, for technical reasons,

$$f(t) = t \text{ for } t \geq 3, \text{ and } f'(t) > 0 \text{ for all } t > 1. \quad (5.3)$$

Note that for maps $u : M \rightarrow \mathbb{C}$ taking values in the unit disk $|u| \leq 1$, this agrees with the definition we've used in the preceding sections; we make the adjustment (5.3) in the region $\mathbb{C} \setminus D_2^2(0)$ only to ensure that arbitrary maps in $W^{1,2}(M, \mathbb{C})$ have finite energy. Since the critical points we construct will take values in the unit disk D_1^2 , this makes no real difference in what follows.

To establish the existence of nontrivial critical points of E_ϵ , we will appeal to the Saddle Point Theorem of Rabinowitz. For our purposes, it will be convenient to use the following formulation (a combination of the treatments in [68] and [34]):

Theorem 5.1. ([68], Chapter 4) *Let $E \in C^1(X, \mathbb{R})$ be a C^1 functional on a real Banach space $X = Y \oplus Z$, where $Y \neq 0$ is finite-dimensional. Suppose that*

$$\inf_{u \in Z} E(u) > 0, \quad (5.4)$$

and that there is a closed ball D about 0 in Y such that

$$E|_{\partial D} \leq 0. \quad (5.5)$$

Denote by Γ the collection of families

$$\Gamma := \{h \in C^0(D, X) \mid h = \text{Id on } \partial D\},$$

and let

$$c := \inf_{h \in \Gamma} \max_{y \in D} E(h(y)) \geq \inf_{u \in Z} E(u).$$

Then for any sequence of families $h^j \in \Gamma$ such that

$$\lim_{j \rightarrow \infty} \max_{y \in D} E(h^j(y)) = c, \quad (5.6)$$

there exists a sequence $u_j \in X$ such that

$$\lim_{j \rightarrow \infty} E(u_j) = c, \quad \lim_{j \rightarrow \infty} \|E'(u_j)\|_{X^*} = 0, \quad (5.7)$$

and

$$\lim_{j \rightarrow \infty} \text{dist}(u_j, h^j(D)) = 0. \quad (5.8)$$

In our setting, we note that the space $W^{1,2}(M, \mathbb{C})$ has an obvious splitting

$$W^{1,2}(M, \mathbb{C}) = \mathbb{C} \oplus Z,$$

where we identify $\mathbb{C}$ with the constant maps, and let Z denote the orthogonal complement

$$Z := \{u \in W^{1,2}(M, \mathbb{C}) \mid \int_M u = 0 \in \mathbb{C}\}.$$

Since the functionals E_ϵ vanish, by definition, on the circle ∂D of constant maps to S^1 , we can apply Theorem 5.1 once we establish a positive lower bound

$$\inf_{u \in Z} E_\epsilon(u) > 0. \quad (5.9)$$

This is quite straightforward, and we can argue more or less as in [38]: denoting by $\lambda_1(M) > 0$ the first nontrivial eigenvalue of the Laplacian, we have for every $u \in Z$ the Poincaré inequality

$$\lambda_1 \int_M |u|^2 \leq \int_M |du|^2,$$

and consequently

$$\begin{aligned} E_\epsilon(u) &\geq \int_M \frac{\lambda_1}{2} |u|^2 + \frac{W(u)}{\epsilon^2} \\ &\geq \frac{\lambda_1}{4} |\{|u|^2 \geq \frac{1}{2}\}| + \frac{1}{16\epsilon^2} |\{|u|^2 < \frac{1}{2}\}| \\ &\geq \min\{\frac{1}{4}\lambda_1(M), \frac{1}{16\epsilon^2}\} \frac{1}{2} \text{vol}(M) \end{aligned}$$

for all $u \in Z$.

Denoting by Γ the collection of two-parameter families

$$\Gamma := \{h \in C^0(D^2, W^{1,2}(M, \mathbb{C})) \mid h(y) \equiv y \text{ for } y \in \partial D^2\}, \quad (5.10)$$

it then follows from Theorem 5.1 that we can extract from any sequence of families $h^j \in \Gamma$ a min-max sequence u_j satisfying (5.7) and (5.8). Moreover, by virtue of the assumptions (5.2)-(5.3) on the

structure of the potential W , one can apply standard arguments for functionals of this type (see, e.g., [38] and [68], Appendix B) to verify that the functionals E_ϵ satisfy the Palais-Smale condition: i.e., any sequence $u_j \in W^{1,2}(M, \mathbb{C})$ satisfying

$$\limsup_{j \rightarrow \infty} E_\epsilon(u_j) < \infty \text{ and } \lim_{j \rightarrow \infty} \|E'_\epsilon(u_j)\| \rightarrow 0$$

contains a strongly convergent subsequence.

Denoting by $c_\epsilon(M)$ the min-max constants

$$c_\epsilon(M) := \inf_{h \in \Gamma} \max_{y \in D} E_\epsilon(h(y)), \quad (5.11)$$

we've therefore arrived at the following simple existence result:

Proposition 5.2. *On any compact Riemannian manifold (M^n, g) , for every $\epsilon > 0$ and every minimizing sequence of families*

$$h^j \in \Gamma, \quad \lim_{j \rightarrow \infty} \max_{y \in D} E_\epsilon(h^j(y)) = c_\epsilon > 0,$$

there exists a subsequence $j_k \rightarrow \infty$ and a sequence of maps $u_{j_k} \in W^{1,2}(M, \mathbb{C})$ such that

$$\lim_{k \rightarrow \infty} \text{dist}(u_{j_k}, h^{j_k}(D)) = 0$$

and u_{j_k} converges to a critical point u_ϵ of E_ϵ with

$$E_\epsilon(u_\epsilon) = c_\epsilon.$$

Remark 5.3. It will later be useful to observe that we can choose a minimizing sequence of families $h^j \in \Gamma$ satisfying the pointwise bound

$$\|h^j(y)\|_{L^\infty} \leq 1,$$

by applying the nearest point retraction $\mathbb{C} \rightarrow D^2$ onto the unit disk to a given minimizing sequence.

5.2 Upper Bounds on the Energy

With the existence of the nontrivial critical points u_ϵ of E_ϵ established, we turn now to the study of their limiting behavior as $\epsilon \rightarrow 0$. The first step in this direction will be to demonstrate an energy bound of the form

$$E_\epsilon(u_\epsilon) = c_\epsilon(M) \leq C|\log \epsilon|,$$

showing that the min-max solutions satisfy the hypotheses of Theorem 1.1.

We can establish the desired upper bounds for the min-max constants by producing families $h_\epsilon \in \Gamma(M)$ such that $E_\epsilon(h_\epsilon(y)) \leq C|\log \epsilon|$ for every $y \in D^2$. To this end, fix a smooth map $v \in C^\infty(\mathbb{C}, \mathbb{C})$ such that

$$v(z) = \frac{z}{|z|} \text{ for } |z| \geq 1 \text{ and } \|v\|_{L^\infty(\mathbb{C})} \leq 1. \quad (5.12)$$

Denoting by $\rho : \{z \in \mathbb{C} \mid |z| < 1\} \rightarrow \mathbb{C}$ the surjection

$$\rho(y) = |\log(1 - |y|)|y, \quad (5.13)$$

we then find the following recipe for producing families in Γ .

Lemma 5.4. *For any Lipschitz map $F \in \text{Lip}(M, \mathbb{C})$, the family $h : D^2 \rightarrow \text{Lip}(M, \mathbb{C})$ defined by*

$$h(y)(x) = v(F(x) + \rho(y)) \text{ for } |y| < 1; \quad h(y)(x) = y \text{ for } |y| = 1 \quad (5.14)$$

is continuous with respect to the $W^{1,2}$ norm; in particular, h defines an element of

$$\Gamma(M) = \{h \in C^0(D^2, W^{1,2}(M, \mathbb{C})) \mid h(y) \equiv y \text{ for } y \in \partial D^2\}.$$

Proof. That $y \mapsto h(y)$ is continuous on the interior $\{|z| < 1\}$ of the disk is an easy consequence of the fact that $\rho \in C^0(\{|z| < 1\}, \mathbb{C})$ is continuous and $v \in C^1(\mathbb{C}, \mathbb{C})$. To see continuity up to the boundary, let $y_j \in D^2$ be a sequence such that $y_j \rightarrow y \in \partial D^2$, and observe that the image $F(M)$ of M under F is bounded, as M is compact. Since $|\rho(y_j)| \rightarrow \infty$ as $j \rightarrow \infty$, it follows that

$$\text{dist}(F(M), -\rho(y_j)) \rightarrow \infty \text{ as } j \rightarrow \infty,$$

and consequently

$$h(y_j)(x) = \frac{F(x) + \rho(y_j)}{|F(x) + \rho(y_j)|} \text{ for all } x \in M$$

for j sufficiently large, by (5.12). It's then straightforward to check that

$$\frac{F(x) + \rho(y_j)}{|F(x) + \rho(y_j)|} \rightarrow y$$

in $W^{1,2}(M, \mathbb{C})$, establishing the desired continuity. $\square$

Our goal now is to produce maps $F^\epsilon \in Lip(M, \mathbb{C})$ for which the associated families $h^\epsilon \in \Gamma$ given by Lemma 5.4 satisfy a bound of the form

$$\max_{y \in D^2} E_\epsilon(h^\epsilon(y)) \leq C(M) |\log \epsilon|;$$

that is, letting $v_y(z) := v(z + y)$, we seek maps $F^\epsilon \in Lip(M, \mathbb{C})$ such that

$$\sup_{y \in \mathbb{C}} E_\epsilon(v_y \circ F^\epsilon) \leq C |\log \epsilon|. \quad (5.15)$$

To this end, consider an n -dimensional simplex $\Delta \subset \mathbb{R}^L$, and denote by $V(\Delta)$ the n -dimensional subspace of $\mathbb{R}^L$ parallel to Δ . Let $\Pi \subset \mathbb{R}^L$ be a 2-plane such that the restriction $p|_{V(\Delta)}$ of the orthogonal projection $p : \mathbb{R}^L \rightarrow \Pi$ to $V(\Delta)$ has rank two—or, equivalently—such that

$$\det(p|_{V(\Delta)} \cdot p|_{V(\Delta)}^*) \geq \frac{1}{C^2}. \quad (5.16)$$

for some $C(\Delta) \in (0, \infty)$. Consider now the map $w_\epsilon : \Delta \rightarrow \mathbb{C}$ given by

$$w_\epsilon(x) = v_y\left(\frac{1}{\epsilon} p(x)\right).$$

Recalling the structure (5.12) of v , we then see that

$$e_\epsilon(w_\epsilon) = \frac{1}{2} |dw_\epsilon(x)|^2 \leq \frac{1}{2|p(x) + \epsilon y|^2} \text{ for } |p(x) + \epsilon y| \geq \epsilon, \quad (5.17)$$

and

$$e_\epsilon(w_\epsilon) = \frac{1}{2} |dw_\epsilon(x)|^2 + \frac{W(w_\epsilon)}{\epsilon^2} \leq \frac{C'}{\epsilon^2} \text{ for } |p(x) + \epsilon y| < \epsilon, \quad (5.18)$$

where we can take $C' = \frac{1}{2} \|dv\|_{L^\infty}^2 + 1$. Using the coarea formula for the map $q = p + \epsilon y$ together

with the bound (5.16), we then find

$$\begin{aligned}
\int_{\Delta} e_{\epsilon}(w_{\epsilon}) &\leq C \int_{\Delta} e_{\epsilon}(w_{\epsilon}) \det(p_{V(\Delta)} \cdot p_{V(\Delta)}^*)^{1/2} \\
&= C \int_{z \in \mathbb{C}} \left(\int_{q^{-1}\{z\} \cap \Delta} e_{\epsilon}(w_{\epsilon}) d\mathcal{H}^{n-2} \right) \\
&\leq C \int_{\{|z| \geq \epsilon\}} \frac{1}{2|z|^2} \mathcal{H}^{n-2}(\{x \in \Delta \mid p(x) + \epsilon y = z\}) \\
&\quad + C \int_{\{|z| < \epsilon\}} \frac{C'}{\epsilon^2} \mathcal{H}^{n-2}(\{x \in \Delta \mid p(x) + \epsilon y = z\}).
\end{aligned}$$

By the assumption that $p|_{V(\Delta)}$ has rank 2, we know that $\{x \mid p(x) + \epsilon y = z\}$ defines an affine $(n-2)$ -plane in $V(\Delta)$, and therefore

$$\mathcal{H}^{n-2}(\{x \in \Delta \mid p(x) + \epsilon y = z\}) \leq C(n) \text{diam}(\Delta)^{n-2}$$

for every $y, z \in \mathbb{C}$ and $\epsilon > 0$. Using this in the preceding computation, we arrive at the bound

$$\begin{aligned}
\int_{\Delta} e_{\epsilon}(w_{\epsilon}) &\leq C_1 \text{diam}(\Delta)^{n-2} \int_{[p(\Delta) + \epsilon y] \cap \{|z| \geq \epsilon\}} \frac{1}{2|z|^2} \\
&\quad + C_2 \text{diam}(\Delta)^{n-2},
\end{aligned}$$

and since $p(\Delta) \subset D_{\text{diam}(\Delta)}^2(z_0)$ for some $z_0 \in \mathbb{C}$, we see also that

$$\begin{aligned}
\int_{[p(\Delta) + \epsilon y] \cap \{|z| \geq \epsilon\}} \frac{1}{2|z|^2} &\leq \int_{D_{\text{diam}(\Delta)}(z_0 + \epsilon y) \setminus D_{\epsilon}(0)} \frac{1}{2|z|^2} \\
&\leq \int_{D_{\text{diam}(\Delta)}(z_0 + \epsilon y) \setminus D_1(0)} \frac{1}{2|z|^2} + \int_{D_1(0) \setminus D_{\epsilon}(0)} \frac{1}{2|z|^2} \\
&\leq \frac{\pi}{2} \text{diam}(\Delta)^2 + \pi \log(1/\epsilon).
\end{aligned}$$

Putting all this together, we obtain the estimate

$$\int_{\Delta} e_{\epsilon}(v_y(\epsilon^{-1}p)) \leq C_1(\Delta, n) |\log \epsilon| + C_2(\Delta, n). \quad (5.19)$$

Given a closed manifold (M^n, g) , classical results on triangulations (see, e.g., [85]) demonstrate the existence of a finite simplicial complex $\mathcal{K}$ in some Euclidean space $\mathbb{R}^L$ and a bi-Lipschitz map

$$\Phi : M \rightarrow |\mathcal{K}|$$

from M to the underlying space of $\mathcal{K}$. Fix one such triangulation Φ , and observe that, by the finiteness of $\mathcal{K}$, we can choose a 2-plane $\Pi \subset \mathbb{R}^L$ such that the projection $p : \mathbb{R}^L \rightarrow \Pi$ restricts to a rank 2 map on the subspaces $V(\Delta)$ for every n -simplex $\Delta \in \mathcal{K}$. Applying the estimate (5.19) to the n -simplices of $\mathcal{K}$, it then follows that

$$\int_{|\mathcal{K}|} e_\epsilon(v_y(\epsilon^{-1}p)) \leq C_1(\mathcal{K})|\log \epsilon| + C_2(\mathcal{K}) \quad (5.20)$$

for every $\epsilon > 0$ and $y \in \mathbb{C}$.

Finally, defining $F^\epsilon : M \rightarrow \mathbb{C}$ by

$$F^\epsilon = \frac{1}{\epsilon} p \circ \Phi, \quad (5.21)$$

we deduce from (5.20) and the bounds $Lip(\Phi) < C(M)$, $Lip(\Phi^{-1}) < C(M)$ that

$$E_\epsilon(v_y \circ F^\epsilon) \leq C_1(M)|\log \epsilon| + C_2(M) \quad (5.22)$$

for every $y \in \mathbb{C}$. Applying Lemma 5.4 to the maps F^ϵ , we therefore conclude that the min-max constants $c_\epsilon(M)$ satisfy

$$\limsup_{\epsilon \rightarrow 0} \frac{c_\epsilon(M)}{|\log \epsilon|} < \infty, \quad (5.23)$$

the desired upper bound.

5.3 Lower Bounds on the Energy

We now turn to the problem of estimating $c_\epsilon(M)$ from below, establishing a bound of the form $c_\epsilon(M) \geq C(M)^{-1}|\log \epsilon|$ for small ϵ . To begin, we'll prove such an estimate for those M with trivial cohomology in degree one—i.e., with $b_1(M) = 0$. The significance of this topological condition is that it forces every nontrivial critical point u_ϵ of E_ϵ to vanish somewhere, as we explain in the following lemma.

Lemma 5.5. *Let (M^n, g) be closed and oriented with $b_1(M) = 0$, and let u solve the ϵ -Ginzburg-Landau equation*

$$\Delta u = -\epsilon^{-2}(1 - |u|^2)u. \quad (5.24)$$

If u is nontrivial, then there exists a point $p \in M$ at which $u(p) = 0$.

Proof. Let u solve (5.24) on M , and suppose that $|u| > 0$ everywhere, so that $v := \frac{u}{|u|}$ defines a

smooth map from M to S^1 . Once again denoting by ju the one-form

$$ju := u^*(r^2 d\theta) = u^1 du^2 - u^2 du^1,$$

recall from Section 4.1 that

$$d^* ju = -u^1 \Delta u^2 + u^2 \Delta u^1 = 0, \quad (5.25)$$

while

$$ju = |u|^2 v^*(d\theta). \quad (5.26)$$

Since $v^*(d\theta)$ is a closed one-form on M , and since $b_1(M) = 0$, it therefore follows from (5.26) that

$$ju = |u|^2 d\varphi \quad (5.27)$$

for some $\varphi \in C^\infty(M, \mathbb{R})$. On the other hand, it follows from (5.25) that

$$\int_M |u|^2 |d\varphi|^2 = \int_M \langle ju, d\varphi \rangle = 0, \quad (5.28)$$

so we conclude that $v = \frac{u}{|u|} \in S^1$ must be a constant.

We're then left with a positive function $|u|$ solving the Allen-Cahn equation

$$\Delta |u| = -\epsilon^{-2}(1 - |u|^2)|u|,$$

for which an easy maximum principle argument (see, e.g., [32]) shows that $|u| \equiv 1$. Thus, u is a constant map to S^1 , i.e., a trivial solution of (5.24). $\square$

Remark 5.6. Without the topological assumption that $b_1(M) = 0$, the conclusion of the lemma is false: when $H^1(M; \mathbb{Z}) \neq 0$, one finds (nontrivial) zero-free solutions of (5.24) lying near the harmonic representatives of nontrivial classes in $[M : S^1]$. For example, if $M = S^1 \times N$ with the product metric, and $p : M \rightarrow S^1 \subset \mathbb{C}$ is the obvious projection, then the maps

$$u_{\epsilon, k}(x) = (1 - \epsilon^2 k^2)^{1/2} p(x)^k \quad (5.29)$$

solve (5.24) for every $k \in \mathbb{N}$ with $k < \frac{1}{\epsilon}$.

Combining this lemma with the η -compactness/ellipticity theorem of Section 4.3, we obtain the following simple corollary.

Corollary 5.7. *Let M be a closed, oriented manifold with $b_1(M) = 0$. Then there exists a positive constant $\delta(M) > 0$ such that, for any $\epsilon \in (0, 1)$, every nontrivial solution u of (5.24) on M satisfies*

$$E_\epsilon(u) \geq \delta |\log \epsilon|. \quad (5.30)$$

Proof. By the result of Lemma 5.5, for every nontrivial solution u of (5.24), there is a point $p \in M$ at which $u(p) = 0$. Applying the result of Theorem 4.2 at this point p , with $r = \text{inj}(M) > \epsilon^{1/2}$, we obtain the lower bound

$$E_\epsilon(u) \geq \int_{B_{\text{inj}(M)}(p)} e_\epsilon(u) \geq \eta \cdot [\text{inj}(M)]^{n-2} |\log \epsilon|.$$

□

Applying this result to the min-max solutions constructed in Section 5.1, we immediately obtain the lower bound

$$\liminf_{\epsilon \rightarrow 0} \frac{c_\epsilon(M)}{|\log \epsilon|} \geq \delta(M) > 0, \text{ provided } b_1(M) = 0. \quad (5.31)$$

In particular, a bound of the form (5.31) holds for the standard sphere S^n , from which we can easily deduce the general bounds (without any topological condition) via the following simple argument.

Lemma 5.8. *For any closed (M^n, g) , there exists a constant $C(M) < \infty$ such that*

$$c_\epsilon(S^n) \leq C \cdot c_\epsilon(M). \quad (5.32)$$

Proof. Fix some closed geodesic ball B_M in M , and fix a bi-Lipschitz map $\Psi : S_+^n \rightarrow B_M$ from the upper hemisphere $S_+^n = \{(x_0, \dots, x_n) \in S^n \mid x_0 > 0\}$ of S^n to B_M . Since Ψ is bi-Lipschitz, there exists a constant C_1 such that

$$E_\epsilon(u \circ \Psi; S_+^n) \leq C_1 E_\epsilon(u; M) \quad (5.33)$$

for every $u \in W^{1,2}(M, \mathbb{C})$.

Next, note that the map $R : W^{1,2}(S_+^n, \mathbb{C}) \rightarrow W^{1,2}(S^n, \mathbb{C})$ giving extension by reflection

$$R(v)(x_0, \dots, x_n) := v(|x_0|, \dots, x_n)$$

is a bounded linear map, with the property that

$$E_\epsilon(R(v); S^n) = 2E_\epsilon(v; S_+^n). \quad (5.34)$$

For any family $h \in \Gamma(M)$, we can therefore construct a family $\tilde{h} \in \Gamma(S^n)$ by

$$\tilde{h}(y) := R(h(y) \circ \Psi) \text{ for every } y \in D^2,$$

and by (5.33) and (5.34), we see that

$$E_\epsilon(\tilde{h}(y); S^n) \leq 2C_1 E_\epsilon(h(y); M).$$

In particular, it follows that

$$c_\epsilon(S^n) \leq 2C_1 \max_{y \in D^2} E_\epsilon(h(y)),$$

and since $h \in \Gamma(M)$ was arbitrary, we have

$$c_\epsilon(S^n) \leq 2C_1 c_\epsilon(M),$$

the desired bound. □

Since we've seen already that

$$\liminf_{\epsilon \rightarrow 0} \frac{c_\epsilon(S^n)}{|\log \epsilon|} > 0,$$

it now follows from the lemma that the bound

$$\liminf_{\epsilon \rightarrow 0} \frac{c_\epsilon(M)}{|\log \epsilon|} > 0 \quad (5.35)$$

holds for every M , with no additional topological condition. Putting this together with the upper bound (5.23) of the previous section, we have now the two-sided energy bounds

$$0 < \liminf_{\epsilon \rightarrow 0} \frac{c_\epsilon(M)}{|\log \epsilon|} \leq \limsup_{\epsilon \rightarrow 0} \frac{c_\epsilon(M)}{|\log \epsilon|} < \infty. \quad (5.36)$$

On manifolds with $b_1(M) = 0$ (which, of course, carry no harmonic one-forms), it then follows immediately from Theorem 1.1 that our min-max solutions u_ϵ have energy concentrating on a non-trivial stationary, rectifiable $(n-2)$ -varifold as $\epsilon \rightarrow 0$. To treat the case $b_1(M) \neq 0$, however, some

subtler estimates are needed to establish nontriviality of the energy concentration varifold, which will be the focus of the following chapter.

Chapter 6

The Case $b_1(M) \neq 0$ and the Homotopy Transition Estimate

In this chapter, we complete the proof of Theorem 1.2 by showing that the min-max solutions constructed in the previous chapter have nontrivial energy concentration varifold. The essential ingredient is a new estimate for the Ginzburg-Landau energies on manifolds with $b_1(M) \neq 0$, which says roughly that for any non-homotopic harmonic S^1 -valued maps $u_0, u_1 : M \rightarrow S^1$ and any path of complex-valued maps $[0, 1] \ni t \mapsto u_t \in W^{1,2}(M, \mathbb{C})$ connecting u_0 to u_1 , there exists a time $t \in [0, 1]$ at which

$$E_\epsilon(u_t) \geq \max\{E(u_0), E(u_1)\} + c(M)|\log \epsilon|,$$

for some geometric constant $c(M) > 0$.

6.1 Statement of the Estimate

In the previous chapter, we introduced a natural two-parameter min-max construction for nontrivial solutions u_ϵ of the Ginzburg-Landau equation on any closed manifold (M^n, g) of dimension $n \geq 2$. We showed, moreover, that the associated min-max energies $c_\epsilon(M)$ grow like $|\log \epsilon|$ as $\epsilon \rightarrow 0$. By Theorem 4.9 in Chapter 4, it then follows that (up to subsequences) the normalized energy measures $\frac{|du_\epsilon|^2}{2|\log \epsilon|}$ converge to a *nontrivial* limit measure μ of the form

$$\mu = \nu + \frac{1}{2}|h|^2 d\text{vol}_g,$$

where ν is a stationary, rectifiable $(n - 2)$ -varifold, and h is a harmonic one-form. Moreover, since the harmonic one-form h is given by the limit

$$h = \lim_{\epsilon \rightarrow 0} \frac{h(u_\epsilon)}{|\log \epsilon|^{1/2}}$$

where $h(u_\epsilon)$ is the harmonic component of the Hodge decomposition

$$ju_\epsilon = u_\epsilon^*(r^2 d\theta) = d^* \xi_\epsilon + h(u_\epsilon),$$

we see that the mass $\nu(M)$ of the varifold ν is given by

$$\nu(M) = \lim_{\epsilon \rightarrow 0} \frac{E_\epsilon(u_\epsilon) - \frac{1}{2} \|h(u_\epsilon)\|_{L^2}^2}{|\log \epsilon|} \quad (6.1)$$

Our goal in this section is to establish nontriviality of ν in the case where $b_1(M) \neq 0$ (when $b_1(M) = 0$, it of course follows automatically from the nontriviality of μ). As discussed in the introduction, when $b_1(M) \neq 0$, it is possible to find families of solutions u_ϵ of (1.1) satisfying the hypotheses of Theorem 4.9, for which the limit measure μ is nontrivial, but $\nu \equiv 0$. Thus, to establish nontriviality of ν for the solutions constructed in Chapter 5, we will need to make more careful use of the min-max construction used to produce these solutions.

Intuitively, if the concentration varifold $\nu = 0$, then our maps u_ϵ would have no zeroes, and must lie close in $W^{1,2}(M, \mathbb{C})$ to some harmonic S^1 -valued maps $v_\epsilon : M \rightarrow S^1$. Since the min-max solutions are nontrivial, it would follow moreover that these maps $v_\epsilon : M \rightarrow S^1$ lie in nontrivial homotopy classes $0 \neq [v_\epsilon] \in [M : S^1]$. (Indeed, since $E_\epsilon(u_\epsilon) \sim |\log \epsilon|$, the degree of v_ϵ would have to grow like $|\log \epsilon|^{1/2}$ as $\epsilon \rightarrow 0$.) On the other hand, recalling the min-max construction of Chapter 5, we note that u_ϵ can be thought of as the element of maximum energy in a family $F \in \Gamma(M)$, which—by definition of $\Gamma(M)$ —passes through the constant maps to S^1 . To rule out triviality of ν , then, we wish to show that, along a path u_t of complex-valued maps connecting a constant S^1 -valued map to a nonconstant harmonic map $v : M \rightarrow S^1$, the maximum energy $E_\epsilon(u_t)$ is achieved at a map u_t far from the harmonic map v , so that the maximum energy for a family $F \in \Gamma(M)$ is achieved far from the space of harmonic S^1 -valued maps.

Estimates along these lines were considered by Almeida ([6],[7]) in the two-dimensional setting, with the conclusion that any path of complex-valued maps connecting distinct homotopy classes in $[M : S^1]$ must pass through a map of energy at least $\pi |\log \epsilon|$ to leading order. In this setting, lower bounds for these so-called “threshold transition energies” are of interest not only for applications to

the construction of unstable critical points of E_ϵ , but also for their implications for the stability of permanent currents in superconducting cylinders, which can be thought of as corresponding to the local minimizers of E_ϵ associated to nontrivial homotopy classes on an annulus ([6],[7]).

Our method for establishing similar estimates in higher-dimensional manifolds differs significantly from that of [6], [7], relying primarily on a study of the Hodge decomposition

$$ju = u^*(r^2 d\theta) = d^* \xi + d\varphi + h(u)$$

for complex-valued maps $u : M \rightarrow \mathbb{C}$, and a splitting of the harmonic component $h(u)$ into integral and fractional parts. More precisely, observe that the assignment

$$W^{1,2}(M, \mathbb{C}) \ni u \mapsto ju = u^*(r^2 d\theta) = u^1 du^2 - u^2 du^1$$

defines a continuous map from $W^{1,2}(M, \mathbb{C})$ to the space of L^1 one-forms on M . Given a L^2 -orthonormal basis $h_1, \dots, h_k$ for the space $\mathcal{H}^1(M)$ of harmonic one-forms, it's then clear that the map

$$u \mapsto h(u) := \sum_{i=1}^k \left(\int_M \langle h_i, ju \rangle \right) h_i$$

defines a continuous map from $W^{1,2}(M, \mathbb{C})$ to $\mathcal{H}^1(M)$, assigning to each u the harmonic component in the Hodge decomposition of ju .

Now, denote by $\Lambda \subset \mathcal{H}^1(M)$ the lattice of integral harmonic one-forms

$$\Lambda := \{h \in \mathcal{H}^1(M) \mid \int_\gamma h \in 2\pi\mathbb{Z} \text{ for every loop } \gamma \in C^\infty(S^1, M)\}. \quad (6.2)$$

It is easy to check that $h(v) \in \Lambda$ whenever v takes values in S^1 , and that Λ can be defined equivalently as the space of all the one-forms $j\phi = \phi^*(d\theta)$ associated to harmonic maps $\phi : M \rightarrow S^1$. Our main estimate can then be stated as follows.

Theorem 6.1. *There exist positive constants $C(M) < \infty$, $\epsilon_0(M) \in (0, 1)$, and $\alpha(n) \in (0, 1)$ such that for any $u \in W^{1,2}(M, \mathbb{C})$ satisfying*

$$|u| \leq 1 \text{ and } E_\epsilon(u) \leq \epsilon^{-1/2} \quad (6.3)$$

for some $\epsilon \in (0, \epsilon_0)$, we have the lower bound

$$E_\epsilon(u) \geq \frac{1}{2} \|h(u)\|_{L^2}^2 + C^{-1} |\log \epsilon| \text{dist}(h(u), \Lambda) - \epsilon^\alpha \quad (6.4)$$

Remark 6.2. Note that we have not yet specified the norm on $\mathcal{H}^1(M)$ with respect to which the distance $\text{dist}(h(u), \Lambda)$ in (6.4) is defined. Since $\mathcal{H}^1(M)$ is finite-dimensional, the choice of norm of course makes little difference in our estimates, but it will be particularly convenient to work with the box-type norm $|\cdot|_b$ defined as follows: fixing a collection

$$\gamma_1, \dots, \gamma_{b_1(M)} : S^1 \rightarrow M$$

of simple closed curves generating the torsion-free part of $H_1(M; \mathbb{Z})$, for every $h \in \mathcal{H}^1(M)$, set

$$|h|_b := \max_{1 \leq i \leq b_1(M)} \left| \int_{\gamma_i} h \right|. \quad (6.5)$$

Henceforth, we will take the distance $\text{dist}(h(u), \Lambda)$ in (6.4) to be the distance dist_b associated to $|\cdot|_b$. Observe that the maximum possible distance to the lattice Λ is given by $\max_{h \in \mathcal{H}^1(M)} \text{dist}_b(h, \Lambda) = \pi$, and the dist_b -nearest-point projection to Λ is well-defined for h with $\text{dist}_b(h, \Lambda) < \pi$.

6.2 Proof of Theorem 1.2 Completed

In this section, we explain how Theorem 6.1—whose proof we postpone to the following section—can be used to complete the proof of Theorem 1.2. That is, we use the estimate (6.4) to show that the concentration varifold ν is nontrivial for the min-max solutions u_ϵ constructed in Chapter 5, which by (6.1), is equivalent to the estimate

$$\lim_{\epsilon \rightarrow 0} \frac{E_\epsilon(u_\epsilon) - \frac{1}{2} \|h(u_\epsilon)\|_{L^2}^2}{|\log \epsilon|} > 0. \quad (6.6)$$

To this end, recall from Section 5.1 that for each of the min-max solutions u_ϵ , there exists a sequence of families $F^j = F_\epsilon^j \in \Gamma$ satisfying

$$\lim_{j \rightarrow \infty} \max_{y \in D} E_\epsilon(F^j(y)) = c_\epsilon(M) \text{ and } \max_{y \in D} \|F^j(y)\|_{L^\infty} \leq 1 \quad (6.7)$$

and a min-max sequence $v_j = v_{j,\epsilon} \in W^{1,2}(M, \mathbb{C})$ satisfying

$$\lim_{j \rightarrow \infty} \text{dist}(v_j, F^j(D)) = 0 \text{ and } \lim_{j \rightarrow \infty} v_j = u_\epsilon. \quad (6.8)$$

Fixing $\epsilon > 0$ and sequences $F^j \in \Gamma$, $v_j \in W^{1,2}(M, \mathbb{C})$ as above, choose a sequence $w_j \in W^{1,2}(M, \mathbb{C})$ satisfying

$$w_j \in F^j(D) \text{ and } \|v_j - w_j\|_{W^{1,2}} = \text{dist}(v_j, F^j(D)). \quad (6.9)$$

Since $v_j \rightarrow u_\epsilon$ and $\text{dist}(v_j, F^j(D)) \rightarrow 0$ as $j \rightarrow \infty$, it follows that $w_j \rightarrow u_\epsilon$ as well. To establish the estimate (6.6), it therefore suffices to prove the following claim.

Claim 6.3. *There exists a constant $c(M) > 0$ such that, for $\epsilon > 0$ sufficiently small,*

$$\lim_{j \rightarrow \infty} \left(E_\epsilon(w_j) - \frac{1}{2} \|h(w_j)\|_{L^2}^2 \right) \geq c |\log \epsilon|. \quad (6.10)$$

Proof. By (6.7), we know that $\|F^j(y)\|_{L^\infty} \leq 1$, and we can choose j sufficiently large that

$$c_\epsilon(M) - 1 < \max_{y \in D} E_\epsilon(F^j(y)) < c_\epsilon(M) + 1. \quad (6.11)$$

It then follows from the bound $c_\epsilon(M) \leq C |\log \epsilon|$ that

$$\max_{y \in D} E_\epsilon(F^j(y)) \leq \epsilon^{-1/2} \quad (6.12)$$

for ϵ sufficiently small, so that each $F^j(y) \in W^{1,2}(M, \mathbb{C})$ satisfies the hypotheses of Theorem 6.1. In particular, we can apply Theorem 6.1 to w_j to obtain

$$E_\epsilon(w_j) - \frac{1}{2} \|h(w_j)\|_{L^2}^2 \geq C^{-1} |\log \epsilon| \text{dist}_b(h(w_j), \Lambda) - \epsilon^\alpha. \quad (6.13)$$

As an immediate consequence, we observe that if $\text{dist}_b(h(w_j), \Lambda) = \pi$, then it follows immediately from (6.12) and (6.13) that

$$E_\epsilon(w_j) - \frac{1}{2} \|h(w_j)\|_{L^2}^2 \geq C^{-1} \pi |\log \epsilon| - \epsilon^\alpha. \quad (6.14)$$

Suppose then that $\text{dist}_b(h(w_j), \Lambda) < \pi$, and denote by $\lambda \in \Lambda$ the unique integral harmonic one-form such that

$$|\lambda - h(w_j)|_b < \pi.$$

Let's also take j sufficiently large that

$$E_\epsilon(w_j) > c_\epsilon(M) - 1 \quad (6.15)$$

If $\lambda = 0$, we have the simple estimate

$$\|h(w_j)\|_{L^2} \leq C(M)|h(w_j)|_b < C\pi,$$

and by (6.15), it then follows that

$$E_\epsilon(w_j) - \frac{1}{2}\|h(w_j)\|_{L^2}^2 \geq c_\epsilon(M) - 1 - C^2\pi^2. \quad (6.16)$$

If $\lambda \neq 0$, then by definition of the collection $\Gamma \subset C^0(D, W^{1,2}(M, \mathbb{C}))$ and the fact that $w_j \in F^j(D)$, we observe that there is a continuous path

$$f : [0, 1] \rightarrow W^{1,2}(M, \mathbb{C}), \quad t \mapsto f_t$$

such that

$$f_0 \equiv 1 \in \mathbb{C}, \quad f_1 = w_j, \quad \text{and } f_t \in F^j(D) \text{ for every } t \in [0, 1]. \quad (6.17)$$

Since λ is a nontrivial element of Λ , we see that

$$|h(f_0) - \lambda|_b = |\lambda|_b \geq 2\pi,$$

and since

$$|h(f_1) - \lambda|_b = |h(w_j) - \lambda|_b < \pi,$$

it follows that there exists some $t_0 \in [0, 1]$ for which $w' = f_{t_0}$ satisfies

$$\text{dist}_b(h(w'), \Lambda) = |h(w') - \lambda|_b = \pi.$$

Applying Theorem 6.1 to w' , we deduce that

$$E_\epsilon(w') - \frac{1}{2}\|h(w')\|_{L^2}^2 \geq C^{-1}\pi|\log \epsilon| - \epsilon^\alpha. \quad (6.18)$$

Now, since

$$|h(w') - h(w_j)|_b \leq |h(w') - \lambda|_b + |h(w_j) - \lambda|_b \leq 2\pi,$$

we see that

$$\|h(w') - h(w_j)\|_{L^2} \leq C(M)|h(w') - h(w_j)|_b \leq 2\pi C,$$

and consequently,

$$\begin{aligned} \|h(w')\|_{L^2}^2 &\geq \|h(w_j)\|_{L^2}^2 - 2C'(M)\|h(w_j)\|_{L^2} \\ &\geq \|h(w_j)\|_{L^2}^2 - 2C'(M)(2E_\epsilon(w_j))^{1/2} \\ &\geq \|h(w_j)\|_{L^2}^2 - C(c_\epsilon(M) + 1)^{1/2} \end{aligned}$$

Observing, moreover, that

$$E_\epsilon(w') \leq \max_{y \in D} E_\epsilon(F^j(y)) \leq c_\epsilon(M) + 1 \leq E_\epsilon(w_j) + 2,$$

and applying these estimates in (6.18), we arrive at the bound

$$E_\epsilon(w_j) - \frac{1}{2}\|h(w_j)\|_{L^2}^2 \geq C^{-1}\pi|\log \epsilon| - C(c_\epsilon + 1)^{1/2} - \epsilon^\alpha - 2. \quad (6.19)$$

The three cases considered above exhaust all possible locations of $h(w_j)$ in $\mathcal{H}^1(M)$, so that one of the bounds (6.14), (6.16), or (6.19) must hold. In each of these cases, using the fact that

$$C^{-1}|\log \epsilon| \leq c_\epsilon(M) \leq C|\log \epsilon|$$

for some positive constant $C > 0$ furnished by the estimates of Chapter 5, we can indeed conclude that

$$E_\epsilon(w_j) - \frac{1}{2}\|h(w_j)\|_{L^2}^2 \geq c|\log \epsilon|$$

for some $c(M) > 0$, provided ϵ is sufficiently small and j sufficiently large. $\square$

By the continuity of the function

$$u \mapsto E_\epsilon(u) - \frac{1}{2}\|h(u)\|_{L^2}^2$$

on $W^{1,2}(M, \mathbb{C})$, passing to the limit in (6.10) yields the desired estimate

$$E_\epsilon(u_\epsilon) - \frac{1}{2} \|h(u_\epsilon)\|_{L^2}^2 \geq c |\log \epsilon| \quad (6.20)$$

for the min-max critical points u_ϵ , when ϵ is sufficiently small. It then follows from (6.6) that the concentration varifold for the min-max critical points must be nontrivial, completing the proof of Theorem 1.2.

6.3 Proof of the Homotopy Transition Estimate

In this section, we complete the proof of Theorem 6.1, whose statement we now recall.

Theorem 6.1. *There exist positive constants $C(M) < \infty$, $\epsilon_0(M) \in (0, 1)$, and $\alpha(n) \in (0, 1)$ such that for any $u \in W^{1,2}(M, \mathbb{C})$ satisfying*

$$|u| \leq 1 \text{ and } E_\epsilon(u) \leq \epsilon^{-1/2} \quad (6.21)$$

for some $\epsilon \in (0, \epsilon_0)$, we have the lower bound

$$E_\epsilon(u) \geq \frac{1}{2} \|h(u)\|_{L^2}^2 + C^{-1} |\log \epsilon| \text{dist}_b(h(u), \Lambda) - \epsilon^\alpha \quad (6.22)$$

Remark 6.4. Recall that the norm $|\cdot|_b$ is defined by

$$|h|_b := \max_{1 \leq i \leq b_1(M)} \left| \int_{\gamma_i} h \right|$$

for a fixed collection $\gamma_1, \dots, \gamma_{b_1(M)} : S^1 \rightarrow M$ of simple closed curves generating the torsion-free part of $H_1(M; \mathbb{Z})$.

As a first step in the proof of Theorem 6.1, we establish the following lower bound for the $W^{-1,p} = (W^{1,p'})^*$ norm of the Jacobian two-form $dju = 2du^1 \wedge du^2$.

Lemma 6.5. *Let $u \in C^\infty(M, \mathbb{C})$ be a complex-valued map satisfying (6.21). For every $p \in (1, \infty)$, there exist constants $C(M) > 0$, $C_p(M) > 0$ such that*

$$C_p \|dju\|_{W^{-1,p}} \geq \text{dist}_b(h(u), \Lambda) - C_\epsilon E_\epsilon(u). \quad (6.23)$$

Proof. Let $\gamma_1, \dots, \gamma_{b_1(M)} \in C^\infty(S^1, M)$ be the simple closed curves used in the definition of $|\cdot|_b$.

Since M is oriented, we observe that each γ_i can be extended to an embedding

$$F_i : S^1 \times B_1^{n-1} \rightarrow M$$

onto a tubular neighborhood of $\gamma_i(S^1)$, such that

$$F_i|_{S^1 \times 0} = \gamma_i \tag{6.24}$$

and

$$\text{Lip}(F_i) + \text{Lip}(F_i^{-1}) \leq C(M). \tag{6.25}$$

Given the bounds (6.25), it's clear that the maps $u \circ F_i \in C^\infty(S^1 \times B_1^{n-1}, \mathbb{C})$ satisfy an energy bound of the form

$$\int_{S^1 \times B_1^{n-1}} e_\epsilon(u \circ F_i) \leq CE_\epsilon(u). \tag{6.26}$$

As a consequence, it follows from Fubini's theorem that the collection

$$G_i := \{y \in B_1^{n-1} \mid \int_{S^1} e_\epsilon(u \circ F_i)(\theta, y) d\theta \leq \frac{2CE_\epsilon(u)}{|B_1^{n-1}|}\} \tag{6.27}$$

has measure at least

$$|G_i| \geq \frac{1}{2}|B_1^{n-1}|. \tag{6.28}$$

We recall now a standard lemma in the study of Ginzburg-Landau functionals (c.f., e.g., Lemma 2.3 of [15]): namely, for any $v \in C^\infty(S^1, \mathbb{C})$ with $\|v\|_{L^\infty} \leq 1$, we have

$$\max_{S^1} W(v) \leq 3\epsilon \int_{S^1} e_\epsilon(v). \tag{6.29}$$

This follows from the simple computation:

$$\begin{aligned} \max W(v) &\leq \min W(v) + \int_{S^1} |dW(v)| \\ &\leq \frac{\epsilon^2}{2\pi} \int_{S^1} \frac{W(v)}{\epsilon^2} + \int_{S^1} |v|(1 - |v|^2)|dv| \\ &\leq \frac{\epsilon^2}{2\pi} \int_{S^1} \frac{W(v)}{\epsilon^2} + \int_{S^1} \frac{\epsilon}{2} |dv|^2 + \frac{(1 - |v|^2)^2}{2\epsilon}. \end{aligned}$$

Applying (6.29) to the maps $u \circ F_i(\cdot, y)$ for $y \in G_i$, it follows in particular that

$$\max_{S^1 \times y} W(u \circ F_i) \leq \frac{6C\epsilon}{|B_1^{n-1}|} E_\epsilon(u) \leq C' \epsilon^{1/2}, \quad (6.30)$$

so provided $\epsilon^{1/2} \leq \frac{1}{16C'(M)}$, we deduce that

$$|u \circ F_i|^2 \geq \frac{1}{2} \text{ on } S^1 \times G_i. \quad (6.31)$$

Now, consider the Hodge decomposition

$$ju = d\varphi + d^*\xi + h(u) \quad (6.32)$$

of ju , and decompose $h(u)$ further into integral and fractional parts

$$h(u) = \lambda + h', \quad (6.33)$$

where $\lambda \in \Lambda$ and

$$|h'|_b = \text{dist}_b(h(u), \Lambda) \leq \pi.$$

Choose $i \in \{1, \dots, b_1(M)\}$ such that $|h'|_b = |\int_{\gamma_i} h'|$, and for each $y \in G_i$, denote by $\gamma_{i,y} : S^1 \rightarrow M$ the curve

$$\gamma_{i,y}(\theta) = F_i(\theta, y).$$

By (6.31), we know that $|u \circ \gamma_{i,y}|^2 \geq \frac{1}{2}$ for $y \in G_i$, so that

$$\begin{aligned} \int_{\gamma_{i,y}} ju &= \int_{\gamma_{i,y}} |u|^2 j(u/|u|) \\ &= \int_{\gamma_{i,y}} (u/|u|)^*(d\theta) - \int_{\gamma_{i,y}} (1 - |u|^2) j(u/|u|) \\ &= 2\pi \deg(u/|u|, \gamma_{i,y}) - \int_{\gamma_{i,y}} (1 - |u|^2) j(u/|u|), \end{aligned}$$

and since

$$(1 - |u|^2) |j(u/|u|)| \leq \frac{(1 - |u|^2)^2}{2\epsilon} + \frac{\epsilon}{2|u|^2} |du|^2 \leq 2\epsilon e_\epsilon(u)$$

wherever $|u|^2 \geq \frac{1}{2}$, it follows that

$$\text{dist} \left(\int_{\gamma_{i,y}} ju, 2\pi\mathbb{Z} \right) \leq 2\epsilon \int_{\gamma_{i,y}} e_\epsilon(u) \leq C\epsilon E_\epsilon(u) \quad (6.34)$$

for $y \in G_i$.

On the other hand, since each $\gamma_{i,y}$ is homotopic to γ_i , we also see that

$$\begin{aligned} \int_{\gamma_{i,y}} ju &= \int_{\gamma_{i,y}} [d^*\xi + h(u)] \\ &= \int_{\gamma_{i,y}} d^*\xi + \int_{\gamma_i} h(u) \\ &= \int_{\gamma_{i,y}} d^*\xi + \int_{\gamma_i} h' \pmod{2\pi\mathbb{Z}} \end{aligned}$$

so that

$$\text{dist} \left(\int_{\gamma_{i,y}} d^*\xi + \int_{\gamma_i} h', 2\pi\mathbb{Z} \right) = \text{dist} \left(\int_{\gamma_{i,y}} ju, 2\pi\mathbb{Z} \right) \leq C\epsilon E_\epsilon(u). \quad (6.35)$$

In particular, since

$$\left| \int_{\gamma_i} h' \right| = \text{dist}_b(h(u), \Lambda) \leq \pi$$

by definition of h' , it follows that

$$\left| \int_{\gamma_{i,y}} d^*\xi \right| \geq \text{dist}_b(h(u), \Lambda) - C\epsilon E_\epsilon(u) \quad (6.36)$$

for every $y \in G_i$.

Integrating (6.36) over $y \in G_i$, and recalling the lower bound (6.28) on the measure of G_i , we find that

$$\int_{S^1 \times G_i} |F_i^*(d^*\xi)| \geq [\text{dist}_b(h(u), \Lambda) - C\epsilon E_\epsilon(u)] \frac{|B_1^{n-1}|}{2}. \quad (6.37)$$

Using the bi-Lipschitz estimate (6.25), this then leads us to the L^1 lower bound

$$C \|d^*\xi\|_{L^1(M)} \geq C \int_{F_i(S^1 \times G_i)} |d^*\xi| \geq \text{dist}_b(h(u), \Lambda) - C\epsilon E_\epsilon(u) \quad (6.38)$$

for some constant $C = C(M)$.

The remainder of the proof is now quite straightforward: for every $p > 1$, it follows from Hölder's inequality that

$$\|d^*\xi\|_{L^1} \leq \text{Vol}(M)^{1-1/p} \|d^*\xi\|_{L^p}, \quad (6.39)$$

and since

$$d^*\xi = d^*\Delta_H^{-1}(dju)$$

by definition, the L^p theory for the Hodge Laplacian Δ_H (see, e.g., [72] for a thorough treatment) and a simple duality argument furnishes us with a constant $C_p(M)$ such that

$$\|d^*\xi\|_{L^p} \leq C_p \|dju\|_{W^{-1,p}} \quad (6.40)$$

for every $p \in (1, \infty)$. Putting together (6.38), (6.39), and (6.40), we obtain an estimate of the desired form (6.23). $\square$

To complete the proof of Theorem 6.1, we will decompose u as a product $u = \phi \cdot \tilde{u}$ of two maps, where $\phi : M \rightarrow S^1$ is a harmonic map such that $j\phi$ gives the integral part of $h(u)$. We will then apply the estimates of [48] and Lemma 6.5 to obtain a lower bound on the energy $E_\epsilon(\tilde{u})$, and arrive at the desired bound by comparing the energy $E_\epsilon(u)$ to the sum $\frac{1}{2}\|j\phi\|_{L^2}^2 + E_\epsilon(\tilde{u})$.

Proof. Let $u \in C^\infty(M, \mathbb{C})$ satisfy (6.21), and decompose $h(u)$ into its integral and fractional parts

$$h(u) = \lambda + h',$$

where $\lambda \in \Lambda$ and $|h'|_b \leq \pi$. Since $\lambda \in \Lambda$, we can integrate λ over paths to obtain a harmonic map

$$\phi : M \rightarrow S^1$$

such that $j\phi = \lambda$. Writing

$$\tilde{u} := \phi^{-1} \cdot u,$$

we then compute

$$j\tilde{u} = ju - |u|^2 j\phi \quad (6.41)$$

and

$$E_\epsilon(\tilde{u}) = E_\epsilon(u) + \int \frac{1}{2}|u|^2|j\phi|^2 - \langle ju, j\phi \rangle. \quad (6.42)$$

Noting that

$$\int \langle ju, j\phi \rangle = \int \langle h(u), j\phi \rangle = \frac{1}{2}\|h(u)\|_{L^2}^2 + \int \frac{1}{2}(|j\phi|^2 - |h'|^2),$$

we rewrite (6.42) as

$$E_\epsilon(u) = \frac{1}{2} \|h(u)\|_{L^2}^2 + E_\epsilon(\tilde{u}) + \frac{1}{2} \int [(1 - |u|^2)|j\phi|^2 - |h'|^2]. \quad (6.43)$$

In particular, since

$$\|h'\|_{L^2} \leq C|h'|_b \leq C\pi \quad (6.44)$$

and $|h'|_b = \text{dist}_b(h(u), \Lambda)$, we obtain from (6.43) the lower bound

$$E_\epsilon(u) \geq \frac{1}{2} \|h(u)\|_{L^2}^2 + E_\epsilon(\tilde{u}) - C \text{dist}_b(h(u), \Lambda). \quad (6.45)$$

To estimate $E_\epsilon(\tilde{u})$, we invoke the powerful estimates of [48] to compare the energy $E_\epsilon(\tilde{u})$ and $\|dj\tilde{u}\|_{W^{-1,p}}$. Though the following estimate does not appear explicitly in [48], as noted in [15], it follows directly from the arguments therein.

Proposition 6.6. *For every $v \in C^\infty(M, \mathbb{C})$ with $\|v\|_{L^\infty} \leq 1$, and every $p \in (1, \frac{n}{n-1})$, there exist positive constants $C(p, M) < \infty$ and $\gamma(p, n) \in (0, 1)$ such that*

$$\|djv\|_{W^{-1,p}} \leq C \left(\frac{E_\epsilon(v)}{|\log \epsilon|} + \epsilon^\gamma \right). \quad (6.46)$$

Now, by our computation (6.41) of $j\tilde{u}$, for every 2-form ζ on M , we see that

$$\int \langle dj\tilde{u} - dj u, \zeta \rangle = \int -|u|^2 \langle j\phi, d^*\zeta \rangle;$$

on the other hand, since $j\phi$ is closed, we also have

$$\int \langle j\phi, d^*\zeta \rangle = 0,$$

and therefore

$$\int \langle dj\tilde{u} - dj u, \zeta \rangle = \int (1 - |u|^2) \langle j\phi, d^*\zeta \rangle. \quad (6.47)$$

In particular, it follows that

$$\|dj\tilde{u} - dj u\|_{W^{-1,2}} \leq \|(1 - |u|^2)j\phi\|_{L^2} \leq \epsilon \|j\phi\|_{L^\infty} \left(\int \frac{(1 - |u|^2)^2}{\epsilon^2} \right)^{1/2},$$

and since

$$\|j\phi\|_{L^\infty} \leq C\|j\phi\|_{L^2} = C\|h(u) - h'\|_{L^2} \leq CE_\epsilon(u)^{1/2},$$

we have that

$$\|dj\tilde{u} - dju\|_{W^{-1,2}} \leq C\epsilon E_\epsilon(u). \quad (6.48)$$

For every $p \in (1, 2)$, we therefore conclude that

$$\|dj\tilde{u}\|_{W^{-1,p}} \geq \|dju\|_{W^{-1,p}} - C\epsilon E_\epsilon(u),$$

and so by Lemma 6.5, we have

$$C_p\|dj\tilde{u}\|_{W^{-1,p}} \geq \text{dist}_b(h(u), \Lambda) - C\epsilon E_\epsilon(u). \quad (6.49)$$

Fixing $p_n = \frac{2n}{2n-1} \in (1, \frac{n}{n-1})$, we can then combine (6.49) with the Jerrard-Soner estimate (6.46) to arrive at the lower bound

$$E_\epsilon(\tilde{u}) \geq C^{-1}\text{dist}_b(h(u), \Lambda)|\log \epsilon| - C\epsilon E_\epsilon(u) - \epsilon^\gamma|\log \epsilon|. \quad (6.50)$$

Plugging this into the estimate (6.45), we find that

$$(1 + C\epsilon)E_\epsilon(u) \geq \frac{1}{2}\|h(u)\|_{L^2}^2 + [C^{-1}|\log \epsilon| - C]\text{dist}_b(h(u), \Lambda) - |\log \epsilon|\epsilon^\gamma.$$

Taking $\alpha(n) = \frac{\gamma}{2}$ and recalling that $E_\epsilon(u) \leq \epsilon^{-1/2}$ by (6.21), we obtain for ϵ sufficiently small an estimate of the desired form

$$E_\epsilon(u) \geq \frac{1}{2}\|h(u)\|_{L^2}^2 + C^{-1}|\log \epsilon|\text{dist}_b(h(u)) - \epsilon^\alpha. \quad (6.51)$$

□

Part III

p -Harmonic Maps to S^1 and Minimal Varieties of Codimension Two

Chapter 7

Preliminaries: Sobolev Maps to the Circle and p -Harmonic Maps

In this chapter, we review some key results from the literature concerning the structure of Sobolev maps to S^1 and the analysis of p -harmonic maps into the circle.

7.1 Topological Singularities and Lifting in $W^{1,p}(M, S^1)$

On a closed, oriented Riemannian manifold (M^n, g) of dimension $n \geq 3$, we define the space $W^{1,p}(M, S^1)$ of S^1 -valued Sobolev maps to be the collection of all complex-valued maps $u \in W^{1,p}(M, \mathbb{C})$ satisfying $|u| = 1$ almost everywhere in M . For each $u \in W^{1,p}(M, S^1)$, we denote by ju the one-form

$$ju := u^*(d\theta) = u^1 du^2 - u^2 du^1. \quad (7.1)$$

Observe that $|du| = |ju|$ almost everywhere on M , so that ju also belongs to L^p .

When u is smooth, the form ju is obviously closed, and it is a straightforward consequence of the Poincaré Lemma that u has a local lifting of the form $u = e^{i\varphi}$ for some smooth, real-valued φ . For general $u \in W^{1,p}(M, S^1)$, the exterior derivative $d[ju]$ is no longer well-defined pointwise, but since ju belongs to L^p , we can still make sense of $d[ju]$ as a distribution in $W^{-1,p}$. Namely, one defines the *distributional Jacobian* $T(u)$ of u to be the $(n-2)$ -current acting on smooth $(n-2)$ -forms $\zeta \in \Omega^{n-2}(M)$ by

$$\langle T(u), \zeta \rangle := \int_M ju \wedge d\zeta. \quad (7.2)$$

The analytic and geometric properties of distributional Jacobians have been studied by a number of authors; we refer the reader to the papers [1], [49], [64], and the references therein for some interesting results concerning the structure of $T(u)$ (see also Chapter 12 below). Note that for smooth, complex-valued maps, we have the pointwise relation

$$d(u^1 du^2 - u^2 du^1) = 2du^1 \wedge du^2,$$

and since $u \mapsto du^1 \wedge du^2$ defines a continuous map from $W^{1,2}(M, \mathbb{C})$ to the space of two-forms with values in L^1 , it follows that

$$\langle T(u), \zeta \rangle = \int_M 2du^1 \wedge du^2 \wedge \zeta$$

holds for all $u \in W^{1,2}(M, S^1)$ and $\zeta \in \Omega^{n-2}(M)$. In particular, since $\text{rank}(du) \leq 1$ almost everywhere, one deduces that $T(u) = 0$ for all $u \in W^{1,2}(M, S^1)$. On the other hand, for $p \in [1, 2)$, and $k \in \mathbb{Z}$, the maps $v_k : D_1^2 \rightarrow S^1$ given by $v_k(z) := (z/|z|)^k$ evidently lie in $W^{1,p}(D, S^1)$, with nontrivial distributional Jacobian

$$T(v_k) = 2\pi k \cdot \delta_0.$$

Observe now that if u has the form $u = e^{i\varphi}$ for some real-valued $\varphi \in W^{1,p}$, then $T(u)$ is given by

$$\langle T(u), \zeta \rangle = \int_M d\varphi \wedge d\zeta,$$

and since φ can be approximated in $W^{1,p}(M, \mathbb{R})$ by smooth functions, it follows that $T(u) = 0$. The following result of Demengel provides a useful converse: if the topological singularity $T(u)$ vanishes, then u lifts locally to a real-valued function in the same Sobolev space.

Proposition 7.1. ([27]) *If $u \in W^{1,p}(B^n, S^1)$ and $T(u) = 0$ in the ball B^n , then $u = e^{i\varphi}$ on B^n for some $\varphi \in W^{1,p}(B^n, \mathbb{R})$.*

The significance of the lifting result for variational problems on $W^{1,p}(M, S^1)$ is clear: away from the support of the $(n-2)$ -current $T(u)$, an S^1 -valued solution u of some geometric p.d.e. lifts locally to a function φ solving an associated *scalar* problem, for which a stronger regularity theory is often available—a fact which we exploit repeatedly in our analysis of p -harmonic maps to S^1 .

7.2 Weakly p -Harmonic Maps to S^1

A map $u \in W^{1,p}(M, S^1)$ for $p \in (1, \infty)$ is called *weakly p -harmonic* if it satisfies

$$\int |du|^{p-2} \langle du, dv \rangle = \int |du|^p \langle u, v \rangle \quad (7.3)$$

for all $v \in (W^{1,p} \cap L^\infty)(M, \mathbb{R}^2)$. Writing $v = \varphi u + i\psi u$ in (7.3), it's easy to see (cf., e.g., [80], Section 2) that (7.3) holds if and only if

$$\int |du|^{p-2} \langle ju, d\psi \rangle = 0 \quad (7.4)$$

for all $\psi \in W^{1,p}(M, \mathbb{R})$ —i.e., when ju satisfies

$$d^*(|ju|^{p-2} ju) = 0 \quad (7.5)$$

distributionally on M . Moreover, from the convexity of $|\cdot|^p$, we see that (7.4) implies

$$\int |ju + d\varphi|^p \geq \int |ju|^p$$

for any $\varphi \in C^\infty(M)$, so that any weakly p -harmonic map $u \in W^{1,p}(M, S^1)$ minimizes the p -energy among all competitors of the form $e^{i\varphi}u$.

In view of (7.5), wherever u admits a local lifting $u = e^{i\varphi}$ for some real-valued $\varphi \in W^{1,p}$, we see that u is weakly p -harmonic if and only if φ is a p -harmonic function—i.e., a weak solution of

$$\operatorname{div}(|d\varphi|^{p-2} d\varphi) = 0.$$

For $p \in (1, 2)$, the $C^{1,\alpha}$ regularity of p -harmonic functions was established by DiBenedetto [28] and Lewis [51] (see also [79]). Rather than using the full strength of the $C^{1,\alpha}$ regularity, we will employ in this paper the following simpler estimates, with constants independent of p .

Proposition 7.2. *Let $B_{2r}(x)$ be a geodesic ball in some manifold M^n with $|\sec(M)| \leq K$. If $\varphi \in W^{1,p}(B_{2r}(x), \mathbb{R})$ is a p -harmonic function for $p \in [\frac{3}{2}, 2]$, then for some constant $C(n, K) < \infty$, we have that*

$$\|d\varphi\|_{L^\infty(B_r(x))}^p \leq Cr^{-n} \|d\varphi\|_{L^p(B_{2r}(x))}^p \quad (7.6)$$

and

$$r^p \|Hess(\varphi)\|_{L^p(B_r(x))}^p \leq C \|d\varphi\|_{L^p(B_{2r}(x))}^p. \quad (7.7)$$

Proof. This is simply a matter of keeping track of p in the estimates of [28] and [51], but we give some details in the interest of completeness. Taking $r = 1$ for simplicity, Let $B_2(x)$ be a geodesic ball in a manifold M^n satisfying the sectional curvature bound

$$|sec(M)| \leq k,$$

and let $\varphi \in W^{1,p}(B_2(x), \mathbb{R})$ be a p -harmonic function on $B_2(x)$ for $p \in [\frac{3}{2}, 2]$. Recall that, by the convexity of the p -energy functional, φ must be the unique minimizer for the p -energy with respect to its Dirichlet data.

For $\epsilon > 0$, we consider as in [51] the perturbed p -energy functionals

$$F_\epsilon(\psi) = \int (\epsilon + |d\psi|^2)^{p/2},$$

and let $\varphi_\epsilon \in W^{1,p}(B_2(x))$ minimize $F_\epsilon(\psi)$ with respect to the condition $\psi - \varphi \in W_0^{1,p}(B_2(x))$. Setting $\gamma_\epsilon := (\epsilon + |d\varphi_\epsilon|^2)^{1/2}$, we then have that

$$div(\gamma_\epsilon^{p-2} d\varphi_\epsilon) = 0, \tag{7.8}$$

and by standard results on quasilinear equations of this form (see, e.g., Chapter 4 of [50]), it follows that φ_ϵ is a smooth, classical solution of (7.8). Moreover, since φ is the unique p -energy minimizer with respect to its Dirichlet data, we know that $\varphi_\epsilon \rightarrow \varphi$ strongly in $W^{1,p}(B_2)$ as $\epsilon \rightarrow 0$. The task now (as in [28], [51]) is to establish estimates of the form given in (7.2) for the perturbed solutions φ_ϵ , and pass them to the limit $\epsilon \rightarrow 0$.

As in [51], we observe now that, for φ_ϵ solving (7.8), the energy density γ_ϵ^p satisfies the divergence-form equation

$$div(A_\epsilon \nabla(\gamma_\epsilon^p)) = p\gamma_\epsilon^{p-2} [\langle A_\epsilon, Hess(\varphi_\epsilon)^2 \rangle + Ric(d\varphi_\epsilon, d\varphi_\epsilon)], \tag{7.9}$$

where $Hess(\varphi_\epsilon)^2$ denotes the composition

$$Hess(\varphi_\epsilon)^2(X, Y) = tr(Hess(\varphi_\epsilon)(X, \cdot) Hess(\varphi_\epsilon)(Y, \cdot)),$$

and

$$A_\epsilon := I + (p-2)\gamma_\epsilon^{-2} d\varphi_\epsilon \otimes d\varphi_\epsilon. \tag{7.10}$$

In particular, it follows that

$$\operatorname{div}(A_\epsilon \nabla(\gamma_\epsilon^p)) \geq p(p-1)\gamma_\epsilon^{p-2} |\operatorname{Hess}(\varphi_\epsilon)|^2 - C(n, k)\gamma_\epsilon^p. \quad (7.11)$$

Now, since $|\nabla \gamma_\epsilon| \leq |\operatorname{Hess}(\varphi_\epsilon)|$, when we integrate (7.11) against a test function $\psi \in C_c^\infty(B_2(x))$ with $\psi \equiv 1$ on $B_1(x)$ and $|\nabla \psi| \leq 2$, we find that

$$\begin{aligned} \int \psi^2 p(p-1)\gamma_\epsilon^{p-2} |\operatorname{Hess}(\varphi_\epsilon)|^2 &\leq \int 2\psi |d\psi| \gamma_\epsilon^{p-1} |\nabla \gamma_\epsilon| + C(k, n)\gamma_\epsilon^p \\ &\leq \int_{B_2} 4\gamma_\epsilon^{p/2} (\psi \gamma_\epsilon^{\frac{p-2}{2}} |\operatorname{Hess}(\varphi_\epsilon)|) + C(k, n)\gamma_\epsilon^p, \end{aligned}$$

and an application of Young's inequality yields

$$p(p-1) \int \psi^2 \gamma_\epsilon^{p-2} |\operatorname{Hess}(\varphi_\epsilon)|^2 \leq \frac{C(k, n)}{(p-1)} \int_{B_2} \gamma_\epsilon^p. \quad (7.12)$$

In particular, since Hölder's inequality gives

$$\int_{B_1} |\operatorname{Hess}(\varphi_\epsilon)|^p \leq \left(\int_{B_1} \gamma_\epsilon^{p-2} |\operatorname{Hess}(\varphi_\epsilon)|^2 \right)^{p/2} \left(\int \gamma_\epsilon^p \right)^{\frac{2-p}{2}},$$

it follows that

$$\|d\varphi_\epsilon\|_{W^{1,p}(B_1)}^p \leq \frac{C(k, n)}{(p-1)^2} \int_{B_2} \gamma_\epsilon^p,$$

and since $p \in [\frac{3}{2}, 2]$, we can rewrite this as

$$\|d\varphi_\epsilon\|_{W^{1,p}(B_1)}^p \leq C(k, n) \int_{B_2} \gamma_\epsilon^p. \quad (7.13)$$

To obtain L^∞ estimates for γ_ϵ , we can apply Moser iteration (see, e.g., [37], Chapter 8) to (7.11). Since the eigenvalues of

$$A_\epsilon = I + (p-2)\gamma_\epsilon^{-2} d\varphi_\epsilon \otimes d\varphi_\epsilon$$

are bounded between $p-1$ and 1, and we are working with $p \in [\frac{3}{2}, 2]$, it is easy to see that the resulting estimate has the desired form

$$\|d\varphi_\epsilon\|_{L^\infty(B_1)}^p \leq \|\gamma_\epsilon\|_{L^\infty(B_1)}^p \leq C(k, n) \int_{B_2} \gamma_\epsilon^p. \quad (7.14)$$

Finally, since $\varphi_\epsilon \rightarrow \varphi$ strongly in $W^{1,p}(B_2(x))$, we have that

$$\lim_{\epsilon \rightarrow 0} \int_{B_2} \gamma_\epsilon^p = \int_{B_2} |d\varphi|^p,$$

and it follows from (7.14) and (7.13) that

$$\|d\varphi\|_{L^\infty(B_1)}^p \leq \liminf_{\epsilon \rightarrow 0} \|d\varphi_\epsilon\|_{L^\infty(B_1)}^p \leq C(k, n) \int_{B_2} |d\varphi|^p, \quad (7.15)$$

and

$$\|d\varphi\|_{W^{1,p}(B_1)}^p \leq \liminf_{\epsilon \rightarrow 0} \|d\varphi_\epsilon\|_{W^{1,p}(B_1)}^p \leq C(k, n) \int_{B_2} |d\varphi|^p. \quad (7.16)$$

Proposition 7.2 then follows by scaling. □

Combining Proposition 7.2 with the lifting criterion of Proposition 7.1, one obtains the following partial regularity result for weakly p -harmonic maps to the circle.

Corollary 7.3. *Let $p \in [\frac{3}{2}, 2]$, and let $B_{2r}(x)$ be a geodesic ball on a manifold M^n with $|\sec(M)| \leq K$. If $u \in W^{1,p}(B_{2r}(x), S^1)$ is a weakly p -harmonic map with vanishing distributional Jacobian*

$$T(u) = 0 \text{ in } B_{2r}(x),$$

then

$$\|du\|_{L^\infty(B_r(x))}^p \leq Cr^{-n} \|du\|_{L^p(B_{2r}(x))}^p \quad (7.17)$$

and

$$r^p \|\nabla du\|_{L^p(B_r(x))}^p \leq C \|du\|_{L^p(B_{2r}(x))}^p. \quad (7.18)$$

Remark 7.4. Though Corollary 7.3 shows that weakly p -harmonic maps $u \in W^{1,p}(M, S^1)$ are reasonably smooth (with effective estimates) away from the support of $T(u)$, observe that the weak p -harmonic condition alone gives *no* constraint on $T(u)$ itself. Indeed, given any $v \in W^{1,p}(M, S^1)$, we can minimize $\int_M |du|^p$ among all maps of the form $u = e^{i\varphi}v$ to find a weakly p -harmonic u with topological singularity

$$T(u) = d[jv + d\varphi] = djv = T(v)$$

equal to that of v . The problem of minimizing p -energy among S^1 -valued maps with prescribed singularities in $\mathbb{R}^2$ —and, more generally, among S^{k-1} -valued maps with prescribed singularities in

$\mathbb{R}^k$ —is studied in detail in [21].

7.3 p -Stationarity and Consequences

A map $u \in W^{1,p}(M, S^1)$ is said to be p -stationary, or simply *stationary*, if it is critical for the energy $E_p(u)$ with respect to perturbations of the form $u_t = u \circ \Phi_t$ for smooth families Φ_t of diffeomorphisms on M . Equivalently, u is p -stationary if it satisfies the inner-variation equation

$$\int_M |du|^p \operatorname{div}(X) - p|du|^{p-2} \langle du^* du, \nabla X \rangle = 0 \quad (7.19)$$

for every smooth, compactly supported vector field X on M .

The most-studied class of stationary p -harmonic maps (for arbitrary target manifolds) are the p -energy minimizers, whose regularity theory for $p \neq 2$ was first investigated by Hardt-Lin [42] and Luckhaus [59], extending results of Schoen and Uhlenbeck [71] from the case $p = 2$. On the other hand, as we discuss in Chapter 10, one can also combine the results of [82] with various min-max constructions to produce many examples non-minimizing stationary p -harmonic maps for certain non-integer values of p .

Given a stationary p -harmonic map $u \in W^{1,p}(M^n, S^1)$, for each geodesic ball $B_r(x) \subset M$, we define the p -energy density

$$\theta_p(u, x, r) := r^{p-n} \int_{B_r(x)} |du|^p. \quad (7.20)$$

By standard arguments, it follows from the stationary equation (7.19) that the density $\theta_p(u, x, r)$ is nearly monotonic in r : Namely, taking X in (7.19) of the form

$$X = \psi \frac{1}{2} \nabla \operatorname{dist}(x, \cdot)^2$$

for some functions $\psi \in C_c^\infty(B_r(x))$ approximating the characteristic function $\chi_{B_r(x)}$, and employing the Hessian comparison theorem to estimate the difference $\nabla X - I$ in $B_r(x)$, one obtains the following well-known estimate (cf. e.g., [42], sections 4 and 7):

Lemma 7.5. *Let $u \in W^{1,p}(M, S^1)$ be a stationary p -harmonic map on a manifold M^n with $|\sec(M)| \leq K$. Then there is a constant $C(n, K)$ such that for any $x \in M$ and almost every $0 < r < \operatorname{inj}(M)$, we have the inequality*

$$\frac{d}{dr} [e^{Cr^2} \theta_p(u, x, r)] \geq p e^{Cr^2} r^{p-n} \int_{\partial B_r(x)} |du|^{p-2} \left| \frac{\partial u}{\partial \nu} \right|^2. \quad (7.21)$$

In particular, $e^{Cr^2}\theta_p(u, x, r)$ is monotone increasing in r .

In light of the monotonicity result, it makes sense to define the pointwise energy density

$$\theta_p(u, x) := \lim_{r \rightarrow 0} \theta_p(u, x, r). \quad (7.22)$$

Perhaps the most significant consequence of Lemma 7.5 is the boundedness of blow-up sequences: Given a sequence of radii $\text{inj}(M) > r_j \rightarrow 0$, observe that the maps $u_j = u_{x, r_j} \in W^{1,p}(B_1^n(0), S^1)$ defined by

$$u_j(y) := u(\exp_x(r_j y))$$

are stationary p -harmonic with respect to the blown-up metrics

$$g_j(y) := r_j^{-2}([\exp_x]^* g)(r_j y)$$

on $B_1^n(0)$, with p -energy given by

$$E_p(u_j, B_1, g_j) = \int_{B_1(0)} |du_j|_{g_j}^p dv_{g_j} = r_j^{p-n} \int_{B_{r_j}(x)} |du|_g^p dv_g = \theta_p(u, x, r_j),$$

so it follows from Lemma 7.5 that the p -energies $E_p(u_j, B_1, g_j)$ are uniformly bounded from above as $r_j \rightarrow 0$.

For a local minimizer u of the p -energy, one could then appeal to the compactness results of ([42], Section 4) to conclude immediately that a subsequence u_{j_k} of such a blow-up sequence converges strongly to a minimizing tangent map u_∞ . For $p \in (1, 2)$ —the range of interest to us—it turns out that we can still obtain a strong convergence result without the minimizing assumption, but this relies on the following subtler result of [65].

Proposition 7.6. ([65], Lemma 3.17) *Let N be a compact homogeneous space with left-invariant metric. For fixed $p \in (1, \infty) \setminus \mathbb{N}$, let $u_j \in W^{1,p}(B_2^n, N)$ be a sequence of maps which are stationary p -harmonic with respect to a C^2 -convergent sequence of metrics $g_j \rightarrow g_\infty$ on B_2 . If $\{u_j\}$ is uniformly bounded in $W^{1,p}(B_2, N)$, then some subsequence u_{j_k} converges strongly in $W^{1,p}(B_1, N)$ to a map u_∞ that is stationary p -harmonic with respect to g_∞ .*

Remark 7.7. The significance of the condition $p \notin \mathbb{N}$ is that the p -energy has no conformally invariant dimension in this case, so that no bubbling can occur, and the proposition follows from arguments generalizing those of [52] to the case $p \neq 2$ (see [65]). The requirement that N be a homogeneous

space is a technical one, arising from the fact that, at present, the most general ϵ -regularity theorem available for stationary p -harmonic maps (when $p \neq 2$) is that of [80] for homogeneous targets. It may be of interest to note that ϵ -regularity (and consequently Proposition 7.6) holds for arbitrary compact targets N for those stationary p -harmonic maps $u : M \rightarrow N$ constructed from critical points of generalized Ginzburg-Landau functionals, by virtue of Lemma 2.3 of [82].

The result of Proposition 7.6 clearly applies to stationary p -harmonic maps to S^1 for any $p \in (1, 2)$, the range of interest. In particular, it follows that for any blow-up sequence $u_j = u_{x, r_j} \in W^{1,p}(B_2^n(0), S^1)$, $r_j \rightarrow 0$, we can extract a subsequence $r_{j_k} \rightarrow 0$ such that the maps u_{j_k} converge strongly in $W^{1,p}(B_1^n(0), S^1)$ to a map $u_\infty \in W^{1,p}(B_1^n(0), S^1)$ which is stationary p -harmonic with respect to the flat metric, and satisfies

$$\theta_p(u_\infty, 0, r) = \theta_p(u, x)$$

for every $r > 0$. Following standard arguments (see, e.g., [42]), we can then apply the Euclidean case of the monotonicity formula 7.21 (in which $C = 0$) to conclude that u_∞ must satisfy the 0-homogeneity condition

$$\langle du_\infty(x), x \rangle = 0 \text{ for a.e. } x \in \mathbb{R}^n.$$

In the next chapter, we will appeal to this strong convergence to tangent maps to obtain a sharp lower bound for the density $\theta_p(u, x)$ at singular points of u , which will form the foundation for many of the estimates that follow.

Chapter 8

Limiting Behavior in Arbitrary Dimension

In this chapter, we complete the proof of Theorem 2.1 of the introduction, characterizing the limiting behavior of the singular sets and p -energy measures for families of stationary p -harmonic maps to S^1 as $p \uparrow 2$.

8.1 Lower Bound for the Energy Density on $Sing(u)$

The analysis leading to Theorem 2.1 rests largely on the following proposition—which plays the role in the p -harmonic map setting that the η -compactness/ellipticity theorem plays in the asymptotic study of the Ginzburg-Landau equations (cf. Section 4.3 and references therein), but whose proof is considerably simpler, relying on a straightforward blow-up analysis at singular points.

Proposition 8.1. *Let $u \in W^{1,p}(M^n, S^1)$ be a stationary p -harmonic map with $n \geq 2$ and $p \in (1, 2)$, and let $x \in Sing(u)$ be a singular point. Then*

$$\theta_p(u, x) \geq c(n, p) \frac{2\pi}{2-p}, \tag{8.1}$$

where $c(2, p) = 1$, and, for $n > 2$,

$$c(n, p) := \int_{B_1^{n-2}} (\sqrt{1 - |y|^2})^{2-p} dy \rightarrow \omega_{n-2} \text{ as } p \rightarrow 2.$$

(Here, ω_m denotes the volume of the Euclidean unit m -ball.)

Proof. Let $x \in \text{Sing}(u)$; by the small energy regularity theorem (see [80], Corollary 3.2 and Theorem 2) for p -harmonic maps into homogeneous targets, this is equivalent to the positivity of the density $\theta_p(u, x) > 0$. Taking a sequence of radii $r_j \rightarrow 0$ and considering the blow-up sequence $u_j = u_{x, r_j}$, we know from the discussion in Section 7.3 that some subsequence u_{j_k} converges strongly in $W_{loc}^{1,p}(\mathbb{R}^n, S^1)$ to a nontrivial stationary p -harmonic map

$$v \in W_{loc}^{1,p}(\mathbb{R}^n, S^1)$$

satisfying

$$\theta_p(v, 0, r) = \theta_p(u, x) \text{ for all } r > 0, \text{ and } \frac{\partial v}{\partial r} = 0. \quad (8.2)$$

Since the tangent map v is radially homogeneous, it follows that its restriction $v|_S$ to the unit sphere defines a weakly p -harmonic map on S^{n-1} .

Next, we observe that if $n > 2$, the restriction $v|_S$ must again have a nontrivial singular set. Indeed, if $v|_S$ were C^1 , then since $H_{dR}^1(S^{n-1}) = 0$, we would have a lifting $v|_S = e^{i\varphi}$ for some p -harmonic function $\varphi \in W^{1,p}(S^{n-1}, \mathbb{R})$. The only p -harmonic functions on closed manifolds are the constants, so this would contradict the nontriviality of v . Thus, $v|_S$ must have nonempty singular set on S^{n-1} , and in particular, $\text{Sing}(v)$ must contain at least one ray in $\mathbb{R}^n$.

We proceed now by a simple dimension reduction-type argument. Fixing some singular point $x_1 \in \text{Sing}(v) \setminus \{0\}$ of v away from the origin, a standard application of (8.2) and the monotonicity formula gives the density inequality

$$\theta_p(v, x_1) \leq \theta_p(v, 0) = \theta_p(u, x). \quad (8.3)$$

Thus, we can take a blow-up sequence for v at x_1 to obtain a new tangent map $v_1 \in W_{loc}^{1,p}(\mathbb{R}^n, S^1)$ satisfying

$$0 < \theta_p(v_1, 0, r) = \theta_p(v_1, 0) \leq \theta_p(u, x) \text{ for all } r > 0. \quad (8.4)$$

This map v_1 will again be radially homogeneous (by the monotonicity formula), and from the radial homogeneity $\frac{\partial v}{\partial r} = 0$ of v , v_1 inherits the additional translation symmetry $\langle dv_1, x_1 \rangle = 0$ in the x_1 direction. In particular, v_1 is determined by its restriction to an $(n-2)$ -sphere in the hyperplane perpendicular to x_1 , which defines a weakly p -harmonic map from S^{n-2} to S^1 .

If $n-2 > 1$, we can argue as before to see that v_1 must have singularities on this $(n-2)$ -sphere, and blow up again at some point $x_2 \in \text{Sing}(v_1) \setminus \mathbb{R}x_1$. Carrying on in this way, we obtain finally a

nontrivial stationary p -harmonic map $v_{n-2} \in W_{loc}^{1,p}(\mathbb{R}^n, S^1)$ which is radially homogeneous, invariant under translation by some $(n-2)$ -plane $\mathcal{L}^{n-2}$, and satisfies

$$\theta_p(v_{n-2}, 0, r) \leq \theta_p(u, x) \text{ for all } r > 0. \quad (8.5)$$

By direct computation, it's easy to see that the only weakly p -harmonic maps from S^1 to S^1 are given by the identity $z \mapsto z$ and its powers $z \mapsto z^\kappa$ for $\kappa \in \mathbb{Z}$. In particular, letting z denote the projection of x onto the two-plane $\mathcal{L}^\perp$, it follows that

$$v_{n-2}(x) = (z/|z|)^\kappa$$

for some $0 \neq \kappa \in \mathbb{Z}$. We can therefore compute

$$\begin{aligned} \theta_p(v_{n-2}, 0, 1) &= \int_{B_1^{n-2}} \int_{D^2_{\sqrt{1-|y|^2}}} \frac{|\kappa|^p}{|z|^p} dz dy \\ &= \frac{2\pi|\kappa|^p}{2-p} \int_{B_1^{n-2}} (\sqrt{1-|y|^2})^{2-p} dy \\ &= \frac{2\pi|\kappa|^p}{2-p} c(n, p). \end{aligned}$$

It then follows from (8.5) that

$$\theta_p(u, x) \geq \theta_p(v_{n-2}, 0, 1) \geq c(n, p) \frac{2\pi}{2-p}, \quad (8.6)$$

as desired. □

8.2 The Size of $Sing(u)$ and Estimates for $T(u)$

Throughout this section, let M be an n -dimensional manifold satisfying the sectional curvature and injectivity radius bounds

$$|sec(M)| \leq k, \quad inj(M) \geq 3, \quad (8.7)$$

and let $p \in [3/2, 2)$. As a first consequence of Proposition 8.1, we employ a simple Vitali covering argument (compare, e.g., Theorem 3.5 of [65]) to obtain p -independent estimates for the $(n-p)$ -content of the singular set $Sing(u) = spt(T(u))$ of a stationary p -harmonic map u to S^1 .

Lemma 8.2. *Let $u \in W^{1,p}(B_3(x), S^1)$ be a stationary p -harmonic map on a geodesic ball $B_3(x) \subset M$*

of radius 3, satisfying the p -energy bound

$$E_p(u, B_3(x)) \leq \frac{\Lambda}{2-p}. \quad (8.8)$$

For $r \leq 1$, the r -tubular neighborhood $\mathcal{N}_r(\text{Sing}(u) \cap B_2(x))$ about the singular set $\text{Sing}(u)$ then satisfies a volume bound of the form

$$\text{Vol}(\mathcal{N}_r(\text{Sing}(u) \cap B_2(x))) \leq C(k, n)\Lambda r^p. \quad (8.9)$$

Proof. Applying the Vitali covering lemma to the covering

$$\{B_r(y) \mid y \in \text{Sing}(u) \cap B_2(x)\}$$

of $\mathcal{N}_r(\text{Sing}(u) \cap B_2(x))$, we obtain a finite subcollection $x_1, \dots, x_m \in \text{Sing}(u) \cap B_2(x)$ for which

$$B_r(x_i) \cap B_r(x_j) = \emptyset \text{ when } i \neq j, \quad (8.10)$$

and

$$\mathcal{N}_r(\text{Sing}(u) \cap B_2(x)) \subset \bigcup_{i=1}^m B_{5r}(x_j). \quad (8.11)$$

Now, by virtue of Proposition 8.1 and Lemma 7.5, we have for each $B_r(x_i)$ the lower energy bound

$$\frac{2\pi c(n, p)}{2-p} \leq C(k, n)\theta_p(u, x_i, r) = C(k, n)r^{p-n} \int_{B_r(x_i)} |du|^p,$$

and from the disjointness (8.10) of $\{B_r(x_i)\}$, it follows that

$$m \frac{2\pi c(n, p)}{2-p} \leq C(k, n)r^{p-n} \int_{\mathcal{N}_r(\text{Sing}(u) \cap B_2(x))} |du|^p \leq C(k, n)r^{p-n} \frac{\Lambda}{2-p}.$$

Since $\inf_{p \in [3/2, 2)} c(n, p) > 0$, this gives us an estimate of the form

$$m \leq C(k, n)\Lambda r^{p-n}.$$

By virtue of (8.11), it then follows that

$$\begin{aligned} \text{Vol}(\mathcal{N}_r(\text{Sing}(u) \cap B_2(x))) &\leq m \cdot C'(k, n) r^n \\ &\leq C(k, n) \Lambda r^p, \end{aligned}$$

as claimed. $\square$

For analysis purposes, this volume estimate is one of the most important consequences of Proposition 8.1, and we will use it repeatedly throughout the remainder of this chapter; its role should be compared with that of results like ([13], Proposition 1) in the Ginzburg-Landau setting. The first application is a series of improved estimates for the distributional Jacobian $T(u)$ of u . A priori, we have only the simple estimate

$$\|T(u)\|_{W^{-1,p}} \leq \|du\|_{L^p},$$

since

$$\langle T(u), \zeta \rangle = \int_M ju \wedge d\zeta \leq \|ju\|_{L^p} \|\zeta\|_{W^{1,p'}}$$

for every $\zeta \in \Omega^{n-2}(M)$ (with $p' := \frac{p}{p-1}$). With Lemma 8.2 in hand, however, we are able to show that, for stationary p -harmonic $u_p \in W^{1,p}(M, S^1)$, if $E_p(u_p) = O(\frac{1}{2-p})$, then $T(u_p)$ is in fact uniformly bounded in various norms as $p \uparrow 2$.

Lemma 8.3. *Under the assumptions of Lemma 8.2, let $T(u)$ denote the distributional Jacobian of u . Then for any smooth $(n-2)$ -form $\zeta \in \Omega_c^{n-2}(B_2(x))$ supported in $B_2(x)$, we have*

$$\langle T(u), \zeta \rangle \leq C(k, n) \Lambda \|\zeta\|_{L^\infty}^{p-1} \|d\zeta\|_{L^\infty}^{2-p}. \quad (8.12)$$

Proof. Fix $\beta \in (0, 1)$ and set $K := \lfloor \frac{1}{2-p} \rfloor \geq \frac{p-1}{2-p}$. Setting $U(r) := \mathcal{N}_r(\text{Sing}(u) \cap B_2(x))$, we begin with the simple estimate

$$\begin{aligned} E_p(u, B_2(x)) &\geq \int_{U(\beta) \setminus U(2^{-K}\beta)} |du|^p \\ &= \sum_{j=1}^K \int_{U(2^{1-j}\beta) \setminus U(2^{-j}\beta)} |du|^p, \end{aligned}$$

from which it follows that

$$\int_{U(2^{1-j}\beta) \setminus U(2^{-j}\beta)} |du|^p \leq \frac{E_p(u)}{K}. \quad (8.13)$$

for some $j \in \{1, \dots, K\}$. In particular, there is some scale $s = 2^{-j}\beta \in [2^{\frac{-1}{2-p}}\beta, \frac{1}{2}\beta]$ for which

$$\int_{U(2s) \setminus U(s)} |du|^p \leq \frac{E_p(u)}{K} \leq \frac{\Lambda}{2-p} \cdot \frac{2-p}{p-1} = \frac{\Lambda}{p-1}. \quad (8.14)$$

Now, let $\psi(y) = \eta(\text{dist}(y, \text{Sing}(u)))$, where η is given by

$$\eta \equiv 1 \text{ on } [0, s], \quad \eta(t) = 1 - \frac{1}{s}(t-s) \text{ for } t \in [s, 2s], \text{ and } \eta \equiv 0 \text{ on } [2s, \infty),$$

so that $\psi \equiv 1$ on $U(s) \supset \text{spt}(T(u))$ and $\psi \equiv 0$ outside $U(2s)$. For any $\zeta \in \Omega_c^{n-2}(B_2(x))$, we then have

$$\begin{aligned} \langle T(u), \zeta \rangle &= \langle T(u), \psi \zeta \rangle \\ &= \int ju \wedge d\psi \wedge \zeta + ju \wedge \psi d\zeta \\ &\leq \|d\psi\|_{L^\infty} \|ju\|_{L^p(U(2s) \setminus U(s))} \|\zeta\|_{L^{p'}(U(2s))} \\ &\quad + \|ju\|_{L^p(U(2s))} \|d\zeta\|_{L^{p'}(U(2s))}, \end{aligned}$$

where $p' = \frac{p}{p-1}$. Next, we note that $\|d\psi\|_{L^\infty} = \frac{1}{s}$, while

$$\|ju\|_{L^p(U(2s))} \leq E_p(u, B_2(x))^{1/p} \leq (2-p)^{-1/p} \Lambda^{1/p},$$

and, by (8.14),

$$\|ju\|_{L^p(U(2s) \setminus U(s))} \leq \frac{\Lambda^{1/p}}{(p-1)^{1/p}};$$

using all of this in the preceding estimate, we then obtain

$$\langle T(u), \zeta \rangle \leq \frac{1}{s} \frac{\Lambda^{1/p}}{(p-1)^{1/p}} \|\zeta\|_{L^{p'}(U(2s))} + (2-p)^{-1/p} \Lambda^{1/p} \|d\zeta\|_{L^{p'}(U(2s))}. \quad (8.15)$$

Now, by Lemma 8.2, we see that

$$\|\zeta\|_{L^{p'}(U(2s))} \leq \|\zeta\|_{L^\infty} \text{Vol}(U(2s))^{1-1/p} \leq C(k, n) \Lambda^{1-1/p} s^{p-1} \|\zeta\|_{L^\infty}$$

and similarly

$$\|d\zeta\|_{L^{p'}(U(2s))} \leq C(k, n) \Lambda^{1-1/p} s^{p-1} \|d\zeta\|_{L^\infty},$$

which we use in (8.15) to obtain

$$\langle T(u), \zeta \rangle \leq \frac{C\Lambda s^{p-2}}{(p-1)^{1/p}} \|\zeta\|_{L^\infty} + \frac{C\Lambda s^{p-1}}{(2-p)^{1/p}} \|d\zeta\|_{L^\infty}. \quad (8.16)$$

Recalling that s lies in the interval $2^{\frac{-1}{2-p}}\beta \leq s \leq \frac{1}{2}\beta$, it then follows that

$$\langle T(u), \zeta \rangle \leq C(k, n)\Lambda \left(\frac{\beta^{p-2}\|\zeta\|_{L^\infty}}{(p-1)^{1/p}} + \frac{\beta^{p-1}\|d\zeta\|_{L^\infty}}{(2-p)^{1/p}} \right). \quad (8.17)$$

Finally, we observe that $\beta \in (0, 1)$ was arbitrary, so we can choose, for instance

$$\beta(p, \zeta) = \frac{(2-p)^{1+1/p}}{(p-1)^{1+1/p}} \frac{\|\zeta\|_{L^\infty}}{\|d\zeta\|_{L^\infty}}.$$

If $\beta(p, \zeta) \geq 1$, then

$$\|d\zeta\|_{L^\infty} \leq \frac{(2-p)^{1+1/p}}{(p-1)^{1+1/p}} \|\zeta\|_{L^\infty},$$

and the desired estimate holds trivially. Otherwise, we have $\beta(p, \zeta) \in (0, 1)$, so we can plug $\beta = \beta(p, \zeta)$ into (8.17), and using the fact that $(2-p)^{p-2}$ is uniformly bounded for $p \in (1, 2)$, we arrive an estimate of the desired form

$$\langle T(u), \zeta \rangle \leq C(k, n)\Lambda \|\zeta\|_{L^\infty}^{p-1} \|d\zeta\|_{L^\infty}^{2-p}. \quad (8.18)$$

□

By rescaling the result of Lemma 8.3, we obtain the following statement at arbitrary small scales $0 < r \leq 1$:

Corollary 8.4. *For $0 < r \leq 1$, let $u \in W^{1,p}(B_{3r}(x), S^1)$ be a stationary p -harmonic map with*

$$\theta_p(u, x, 3r) \leq \frac{\Lambda}{2-p}.$$

Then for every $\zeta \in \Omega_c^{n-2}(B_{2r}(x))$, we have

$$\langle T(u), \zeta \rangle \leq C(k, n)\Lambda r^{n-p} \|\zeta\|_{L^\infty}^{p-1} \|d\zeta\|_{L^\infty}^{2-p}. \quad (8.19)$$

In particular, by virtue of energy monotonicity (Lemma 7.5), if $u \in W^{1,p}(B_3, S^1)$ satisfies $E_p(u, B_3) \leq \frac{\Lambda}{2-p}$, then (8.19) holds for all $\zeta \in \Omega_c^{n-2}(B_r(y))$, for every ball $B_r(y) \subset B_2(x)$. To-

gether with the following technical lemma, these estimates and the volume bounds of Lemma 8.2 will yield p -independent bounds for $\|T(u)\|_{W^{-1,q}}$ for any $q \in [1, p)$.

Lemma 8.5. *Let S be an $(n-2)$ -current in $W^{-1,p}(B_2(x))$ satisfying*

$$\langle S, \zeta \rangle \leq Ar^{n-p} \|\zeta\|_{L^\infty}^{p-1} \|d\zeta\|_{L^\infty}^{2-p} \quad \forall \zeta \in \Omega_c^{n-2}(B_r(y)), \quad (8.20)$$

for every ball $B_r(y) \subset B_2(x)$. Suppose also that the r -tubular neighborhoods $\mathcal{N}_r(\text{spt}(S))$ about the support of S satisfy

$$\text{Vol}(\mathcal{N}_r(B_2(x) \cap \text{spt}(S))) \leq Ar^p. \quad (8.21)$$

Then there is a constant $C(n, k, A)$ such that for every $1 \leq q < p$, we have

$$\|S\|_{W^{-1,q}(B_1(x))} \leq C(n, k, A)(p-q)^{-1/q}. \quad (8.22)$$

Proof. Let's begin now by making some simple reductions. First, since the given metric g on $B_2(x)$ is uniformly equivalent to the flat one g_0 with

$$C(n, k)^{-1}g_0 \leq g \leq C(n, k)g_0$$

for some constant $C(n, k)$, it will suffice to establish the lemma in the flat case. Next, we note that every $(n-2)$ -current S in $B_2^n(0) \subset \mathbb{R}^n$ is described by a finite collection of scalar distributions S_{ij} , where

$$\langle S_{ij}, \varphi \rangle := \langle S, * \varphi dx^i \wedge dx^j \rangle.$$

Thus, it is enough to show that Lemma 8.5 holds with a scalar distribution f in place of the $(n-2)$ -current S . Our first step in proving this is then to establish the following claim.

Claim 8.6. *For $p \in (1, 2)$, let $f \in W^{-1,p}(B_2^n(0))$ be a distribution on $B_2^n(0)$ satisfying the estimate*

$$\langle f, \varphi \rangle \leq Ar^{n-p} \|\varphi\|_{L^\infty}^{p-1} \|d\varphi\|_{L^\infty}^{2-p} \quad \forall \varphi \in C_c^\infty(B_r(x)) \quad (8.23)$$

for every ball $B_r(x) \subset B_2^n$. Fixing a cutoff function $\chi \in C_c^\infty(B_{5/3}(0))$ such that $\chi \equiv 1$ on $B_{4/3}(0)$, set

$$w(x) := \langle (\chi f)(y), G(x-y) \rangle,$$

where G is the n -dimensional Euclidean Green's function. We then have for $x \in B_1(0) \setminus \text{spt}(f)$ a

pointwise gradient estimate of the form

$$|dw(x)| \leq C_n A \cdot \text{dist}(x, \text{spt}(f))^{-1}. \quad (8.24)$$

Proof. (Proof of Claim 8.6) For $x \in B_1 \setminus \text{spt}(f)$, we observe that the pointwise derivatives $w_i(x) := \partial_i w(x)$ are well-defined, and given by

$$w_i(x) := \partial_i w(x) = c_n \langle (\chi f)(y), |x - y|^{-n} (x - y)_i \rangle,$$

where c_n is a dimensional constant.

To establish (8.24), first choose a function $\zeta \in C_c^\infty([\frac{1}{2}, 2])$ satisfying

$$\zeta \equiv 1 \text{ on } [\frac{3}{4}, \frac{3}{2}] \text{ and } |\zeta'| \leq 10,$$

and for $j \in \mathbb{Z}$, set

$$\zeta_j(t) := \zeta(2^{-j}t).$$

Defining

$$\eta_j(t) := \frac{\zeta_j(t)}{\sum_{k \in \mathbb{Z}} \zeta_k(t)},$$

it's easy to see that the functions η_j satisfy

$$\text{spt}(\eta_j) \subset (2^{j-1}, 2^{j+1}), \quad (8.25)$$

$$\sum_{j \in \mathbb{Z}} \eta_j(t) = 1, \quad (8.26)$$

and

$$|\eta'_j| \leq 10 \cdot 2^{-j}. \quad (8.27)$$

Given $x \in B_1^n \setminus \text{spt}(f)$, let $m = \lceil |\log_2 \delta| \rceil$, so that

$$2^{1-m} \geq \text{dist}(x, \text{spt}(f)) \geq 2^{-m}.$$

Writing

$$\begin{aligned} w_i(x) &= c_n \langle (\chi f)(y), |x-y|^{-n}(x-y)_i \rangle \\ &= c_n \langle (\chi f)(y), \sum_{j \in \mathbb{Z}} \eta_j(|x-y|) |x-y|^{-n}(x-y)_i \rangle, \end{aligned}$$

and observing that

$$1 - \sum_{j=-m}^2 \eta_j(|x-y|) = 0$$

when $y \in \text{spt}(\chi f) \subset B_4(x) \setminus B_{2^{-m}}(x)$, it follows that

$$\begin{aligned} w_i(x) &= c_n \langle (\chi f)(y), \sum_{j=-m}^2 \eta_j(|x-y|) |x-y|^{-n}(x-y)_i \rangle \\ &= c_n \sum_{j=-m}^2 \langle (\chi f)(y), \eta_j(|x-y|) |x-y|^{-n}(x-y)_i \rangle. \end{aligned}$$

Setting

$$\varphi_j(y) := \chi(y) \eta_j(|x-y|) |x-y|^{-n}(x-y)_i,$$

we can then use (8.25)-(8.27) to see that

$$\text{spt}(\varphi_j) \subset B_{2^{j+1}}(x),$$

$$\|\varphi_j\|_{L^\infty} \leq 2^{(j-1)(1-n)},$$

and

$$\|d\varphi_j\|_{L^\infty} \leq C_n 2^{-n(j-1)}.$$

By (8.23), it therefore follows that

$$\begin{aligned} |\langle f, \varphi_j \rangle| &\leq A(2^{j+1})^{n-p} \|\varphi_j\|_{L^\infty}^{p-1} \|d\varphi_j\|_{L^\infty}^{2-p} \\ &\leq C_n A(2^{j+1})^{n-p} \cdot 2^{(j-1)(1-n)(p-1)} \cdot 2^{-n(2-p)(j-1)} \\ &\leq C'_n A 2^{-j}. \end{aligned}$$

Summing from $j = -m$ to $j = 2$, we obtain finally

$$\begin{aligned}
|w_i(x)| &= |c_n \Sigma_{j=-m}^2 \langle f, \varphi_j(y) \rangle| \\
&\leq C_n A \Sigma_{j=-m}^2 2^{-j} \\
&\leq C_n A 2^m \\
&\leq 2C_n A \cdot \text{dist}(x, \text{spt}(f))^{-1},
\end{aligned}$$

giving the desired estimate (8.24). $\square$

Now, let $f \in W^{-1,p}(B_2^n(0))$ be as in Lemma 8.6, satisfying

$$\langle f, \varphi \rangle \leq A r^{n-p} \|\varphi\|_{L^\infty}^{p-1} \|d\varphi\|_{L^\infty}^{2-p} \quad \forall \varphi \in C_c^\infty(B_r(x)) \quad (8.28)$$

for every ball $B_r(x) \subset B_2(0)$. In addition, suppose that the tubular neighborhoods $\mathcal{N}_r(\text{spt}(f))$ about the support of f satisfy the volume bound

$$\text{Vol}(\mathcal{N}_r(\text{spt}(f))) \leq A r^p. \quad (8.29)$$

We claim next that there is a constant $C(n, A) < \infty$ depending only on n and A such that for every $q \in (1, p)$, we have the estimate

$$\|f\|_{W^{-1,q}(B_1(0))} \leq C(n, A)(p - q)^{-1/q}. \quad (8.30)$$

Once this is established, Lemma 8.5 will follow, by applying this claim to the scalar component distributions of the $(n - 2)$ -current S .

To obtain the estimate (8.30), observe that by Claim 8.6, there exists a function $w \in W^{1,p}(B_2^n(0))$ satisfying

$$\Delta w = f \text{ on } B_{4/3}(0)$$

and

$$|dw(x)| \leq \frac{C_n A}{\text{dist}(x, \text{spt}(f))} \quad (8.31)$$

for $x \in B_1(0) \setminus \text{spt}(f)$. For any $\varphi \in C_c^\infty(B_1(0))$ and $q \in (1, p)$, we then have

$$\begin{aligned} \langle f, \varphi \rangle &= \langle \Delta w, \varphi \rangle \\ &= - \int \langle dw, d\varphi \rangle \\ &\leq \|dw\|_{L^q(B_1(0))} \|d\varphi\|_{L^{q'}}, \end{aligned}$$

while, by (8.31) and (8.29), we see that

$$\begin{aligned} \int_{B_1(0)} |dw|^q &\leq C_n A^q \int_{B_1(0)} \text{dist}(x, \text{spt}(f))^{-q} \\ &\leq C A^q \int_0^2 q r^{-q-1} \text{Vol}(\mathcal{N}_r(\text{spt}(f))) dr \\ &\leq C A^{q+1} \int_0^2 r^{p-q-1} dr \\ &\leq \frac{C A^{q+1}}{p-q}. \end{aligned}$$

Thus, we indeed have

$$\langle f, \varphi \rangle \leq C(n, A)(p-q)^{-1/q} \|d\varphi\|_{L^{q'}},$$

the desired $W^{-1,q}$ estimate. □

With Lemma 8.5 proved, combining the results of Lemma 8.2 with Lemma 7.5 and Corollary 8.4, we arrive finally at the following estimate.

Corollary 8.7. *Let $u \in W^{1,p}(B_3(x), S^1)$ be a stationary p -harmonic map with*

$$E_p(u, B_3(x)) \leq \frac{\Lambda}{2-p}.$$

Then for every $q \in [1, p)$, we have

$$\|T(u)\|_{W^{-1,q}(B_1(x))} \leq C(n, k, \Lambda, q). \tag{8.32}$$

Proof. By the preceding discussion, we see that

$$\|T(u)\|_{W^{-1,q}(B_1(x))} \leq C(n, k, \Lambda)(p-q)^{-1/q}.$$

To put this in the form (8.32), we simply separate into two cases: if $p \leq 1 + \frac{q}{2}$, then $2 - p \geq \frac{2-q}{2}$, and (8.32) follows from the trivial estimate

$$\|T(u)\|_{W^{-1,p}} \leq \|du\|_{L^p} \leq \frac{\Lambda^{1/p}}{(2-p)^{1/p}}.$$

On the other hand, if $p > 1 + \frac{q}{2}$, then $p - q > \frac{2-q}{2}$, and so

$$\|T(u)\|_{W^{-1,q}(B_1(x))} \leq C(n, k, \Lambda)(p - q)^{-1/q} \leq C(n, k, \Lambda, q),$$

as claimed. $\square$

8.3 Estimates for the Hodge Decomposition of ju

Now, let M^n again be a closed, oriented Riemannian manifold, and for $p \in [3/2, 2)$, let $u \in W^{1,p}(M, S^1)$ be a stationary p -harmonic map with

$$E_p(u) \leq \frac{\Lambda}{2-p}. \tag{8.33}$$

In our analysis of the global behavior of u , just as in the Ginzburg-Landau setting (compare, e.g., [13], [54]), the Hodge decomposition

$$ju = d\varphi + d^*\xi + h \tag{8.34}$$

of ju plays a central role. (For more on Hodge decomposition in the space of L^p differential forms, we refer the reader to [72].) Here, $\varphi \in W^{1,p}(M, \mathbb{R})$ is the function given by

$$\varphi := \Delta^{-1}(\operatorname{div}(ju)),$$

ξ is the $W^{1,p}$ two-form

$$\xi := *\Delta_H^{-1}T(u),$$

and h is the remaining harmonic one-form, which we can write as

$$h := \sum_{i=1}^k \left(\int_M \langle h_i, ju \rangle \right) h_i$$

with respect to an L^2 -orthonormal basis $\{h_i\}_{i=1}^k$ for the space $\mathcal{H}^1(M)$ of harmonic one-forms. We remark that, in our notation, Δ denotes the negative spectrum scalar Laplacian, but $\Delta_H = dd^* + d^*d$ is the usual positive spectrum Hodge Laplacian.

We begin this section by establishing estimates which show that, for each $q \in [1, 2)$, and any sequence u_p as in Theorem 2.1, the coexact component $d^*\xi$ remains bounded and the exact component $d\varphi$ vanishes in L^q as $p \rightarrow 2$. Once we've obtained these global estimates, we will show that the same behavior holds in stronger norms away from the singular sets.

For the harmonic form h , we need only the trivial L^∞ estimate

$$\|h\|_{L^\infty(M)} \leq C(M)\|du\|_{L^1(M)} \leq C(M)\frac{\Lambda^{1/p}}{(2-p)^{1/p}}. \quad (8.35)$$

For the exact and co-exact terms $d\varphi$ and $d^*\xi$, we begin by establishing the following global estimates:

Proposition 8.8. *If $q \in (1, p)$, then*

$$\|\xi\|_{W^{1,q}} \leq C(M, \Lambda, q) \quad (8.36)$$

and

$$\|\varphi\|_{W^{1,q}} \leq C(M, \Lambda, q)(2-p)^{1-1/p}|\log(2-p)| \quad (8.37)$$

for some $C(M, \Lambda, q) < \infty$ independent of p .

Proof. First, note that we can apply Corollary 8.7 (after some fixed rescaling) to a finite covering of M by geodesic balls, to obtain the $W^{-1,q}$ estimate

$$\|T(u)\|_{W^{-1,q}(M)} \leq C(M, \Lambda, q) \quad (8.38)$$

for the distributional Jacobian $T(u)$. Since $\xi := *\Delta_H^{-1}T(u)$ by definition, it follows from the $L^{q'}$ regularity of Δ_H that

$$\|\xi\|_{W^{1,q}} \leq C(M, q)\|T(u)\|_{W^{-1,q}} \leq C(M, \Lambda, q), \quad (8.39)$$

as desired. Indeed, for any smooth one-form Ψ orthogonal to the harmonic 1-forms $\mathcal{H}^1(M)$, we see from the definition of ξ and the fact (see [72], Section 5) that the Green's operator Δ_H^{-1} maps

$L^{q'} \rightarrow W^{2,q'}$ for $q' \in (2, \infty)$ that

$$\begin{aligned}
\langle d^* \xi, \Psi \rangle &= \langle T(u), *d(\Delta_H^{-1} \Psi) \rangle \\
&\leq \|T(u)\|_{W^{-1,q}} \|d\Delta_H^{-1} \Psi\|_{W^{1,q'}} \\
&\leq \|T(u)\|_{W^{-1,q}} \|\Delta_H^{-1} \Psi\|_{W^{2,q'}} \\
&\leq C(M, q) \|T(u)\|_{W^{-1,q}} \|\Psi\|_{L^{q'}},
\end{aligned}$$

from which it follows immediately that $\|d^* \xi\|_{L^q} \leq C(M, q) \|T(u)\|_{W^{-1,q}}$. Since $d\xi = 0$ and $\xi \perp \mathcal{H}^2(M)$ (distributionally), the estimate (8.39) follows.

To estimate $d\varphi$, we begin by observing that since u is weakly p -harmonic, the distributional divergence $\operatorname{div}(|ju|^{p-2}ju)$ vanishes, and φ can therefore be recast as

$$\varphi := \Delta^{-1}(\operatorname{div}(ju - |ju|^{p-2}ju)).$$

As in our estimates for ξ , the $L^{q'}$ regularity of the Laplacian and a duality argument then give

$$\|\varphi\|_{W^{1,q}} \leq C(q, M) \|(1 - |ju|^{p-2})ju\|_{L^q}, \quad (8.40)$$

so it is enough produce an L^q estimate of the desired form for $(1 - |ju|^{p-2})ju$.

To this end, we write

$$\begin{aligned}
\|(1 - |ju|^{p-2})ju\|_{L^q}^q &= \int | |du| - |du|^{p-1} |^q \\
&= \int_{\{|du| \leq 1\}} (|du|^{p-1} - |du|)^q \\
&\quad + \int_{\{|du| \geq 1\}} (|du| - |du|^{p-1})^q.
\end{aligned}$$

It's easy to check that

$$\max_{t \in [0,1]} (t^{p-1} - t) = (2-p)(p-1)^{\frac{p-1}{2-p}},$$

so the $\{|du| \leq 1\}$ portion of the integral satisfies

$$\int_{\{|du| \leq 1\}} (|du|^{p-1} - |du|)^q \leq (2-p)^q (p-1)^{\frac{q(p-1)}{2-p}} \operatorname{Vol}(M).$$

To estimate the $\{|du| \geq 1\}$ portion of the integral, observe that

$$1 - t^{p-2} \leq (2-p) \log(t)$$

when $t \geq 1$, so fixing some $\lambda > 1$, we split the integral again to see that

$$\begin{aligned} \int_{\{|du| \geq 1\}} (|du| - |du|^{p-1})^q &\leq \int_{\{1 \leq |du| \leq \lambda\}} (2-p)^q \log(|du|)^q |du|^q \\ &\quad + \int_{\{|du| \geq \lambda\}} |du|^q \\ &\leq (2-p)^q \log(\lambda)^q \int_{\{1 \leq |du| \leq \lambda\}} |du|^q \\ &\quad + \|du\|_{L^p}^{q/p} \text{Vol}(\{|du| \geq \lambda\})^{1-q/p} \\ &\leq (2-p)^q \log(\lambda)^q C(M) \|du\|_{L^p}^q + \|du\|_{L^p}^p \lambda^{q-p}, \end{aligned}$$

and since $\|du\|_{L^p} \leq \Lambda^{1/p} (2-p)^{-1/p}$, this yields

$$\int_{\{|du| \geq 1\}} (|du| - |du|^{p-1})^q \leq C(M) \Lambda^{q/p} (2-p)^{q-q/p} \log(\lambda)^q + \Lambda (2-p)^{-1} \lambda^{q-p}.$$

Taking $\lambda = (2-p)^{-\frac{1}{p} - \frac{q}{p-q}}$, we observe that

$$(2-p)^{-1} \lambda^{q-p} = (2-p)^{q-q/p}$$

and

$$\log(\lambda) = \left(\frac{1}{p} + \frac{q}{p-q}\right) |\log(2-p)|,$$

so putting this together with the preceding inequalities, we arrive at the estimate

$$\int ||du| - |du|^{p-1}|^q \leq \frac{C(M, \Lambda)}{(p-q)^q} (2-p)^{q-q/p} |\log(2-p)|^q. \quad (8.41)$$

Considering separately the cases $p > 1 + \frac{q}{2}$ and $p \leq 1 + \frac{q}{2}$ as in the proof of Corollary 8.7, and recalling that

$$\|d\varphi\|_{L^q} \leq C(M, q) \||du| - |du|^{p-1}\|_{L^q},$$

we arrive at an estimate of the desired form (8.37).

□

Next, we establish estimates resembling (8.36) and (8.37) in $W^{1,2}$ norms away from the singular set $Sing(u)$. The simple estimates of Lemma 8.9 below by no means represent the optimal estimates in this direction, but they will suffice for the purposes of this chapter.

Lemma 8.9. *Suppose now that $p \in [\max\{q_n, 3/2\}, 2)$, where $q_n = \frac{2n}{n+2}$ (so that $W^{1,q_n} \hookrightarrow L^2$ by Sobolev embedding). Letting $r(x) := \text{dist}(x, Sing(u))$, we have the L^2 estimates*

$$\|r(x)d^*\xi\|_{L^2(M)} \leq C(M, \Lambda) \quad (8.42)$$

and

$$\|r(x)d\varphi\|_{L^2(M)} \leq C(M, \Lambda)(2-p)^{1-1/p} |\log(2-p)|. \quad (8.43)$$

Proof. For $\delta > 0$, let $\psi_\delta(x) \in Lip(M)$ be given by

$$\psi_\delta(x) = \max\{0, r(x) - \delta\},$$

so that $\psi_\delta \equiv 0$ on a neighborhood of $Sing(u) = \text{spt}(T(u))$, and $Lip(\psi_\delta) \leq 1$. Then $d^*\xi$ is closed on the support of ψ_δ , and it follows that

$$\begin{aligned} \int \psi_\delta^2 |d^*\xi|^2 &= \int \langle d^*\xi, \psi_\delta^2 d^*\xi \rangle \\ &= \int \langle \xi, 2\psi_\delta d\psi_\delta \wedge d^*\xi \rangle \\ &\leq 2 \int |\psi_\delta d^*\xi| |\xi|. \end{aligned}$$

Now, since $p > q_n$, we have by Sobolev embedding and Proposition 8.8 the estimate

$$\|\xi\|_{L^2} \leq C(M) \|\xi\|_{W^{1,q_n}} \leq C(M, \Lambda);$$

applying this in the preceding inequality, it follows that

$$\int \psi_\delta^2 |d^*\xi|^2 \leq C(M, \Lambda)^2.$$

Taking $\delta \rightarrow 0$ and appealing to the monotone convergence theorem, we arrive at (8.42).

For (8.43), we proceed similarly: with ψ_δ defined as above, we use the equation

$$\Delta\varphi = \text{div}((1 - |ju|^{p-2})ju)$$

(and the fact that $Lip(\psi_\delta) \leq 1$) to estimate

$$\begin{aligned}
\int \psi_\delta^2 |d\varphi|^2 &= \int \langle d\varphi, d(\psi_\delta^2 \varphi) \rangle - 2 \langle \psi_\delta d\varphi, \varphi d\psi_\delta \rangle \\
&= \int \langle (1 - |ju|^{p-2})ju, d(\psi_\delta^2 \varphi) \rangle \\
&\quad - 2 \int \langle \psi_\delta d\varphi, \varphi d\psi_\delta \rangle \\
&\leq \|\psi_\delta(1 - |ju|^{p-2})ju\|_{L^2} (\|\psi_\delta d\varphi\|_{L^2} + \|\varphi\|_{L^2}) \\
&\quad + 2\|\psi_\delta d\varphi\|_{L^2} \|\varphi\|_{L^2}.
\end{aligned}$$

With a few applications of Young's inequality, it then follows that

$$\|\psi_\delta d\varphi\|_{L^2}^2 \leq 10(\|\psi_\delta(1 - |du|^{p-2})du\|_{L^2}^2 + \|\varphi\|_{L^2}^2) \quad (8.44)$$

Now, by Sobolev embedding and Proposition 8.8, we know that

$$\|\varphi\|_{L^2}^2 \leq C(M)\|\varphi\|_{W^{1,q_n}}^2 \leq C(M, \Lambda)(2-p)^{2-2/p} |\log(2-p)|^2, \quad (8.45)$$

so all that remains is to estimate $\|\psi_\delta(1 - |du|^{p-2})du\|_{L^2}$.

To this end, observe that the gradient estimate of Corollary 7.3 implies

$$r(x)^p |du|^p(x) \leq C(M)\theta_p(u, x, r(x)),$$

which together with the monotonicity of $\theta_p(u, x, \cdot)$ yields the pointwise gradient estimate

$$r(x)^p |du|^p(x) \leq C(M) \frac{\Lambda}{2-p};$$

in particular, it follows that

$$\psi_\delta^2 |du|^2 \leq C(M, \Lambda)(2-p)^{-2/p}. \quad (8.46)$$

As in the proof of Proposition 8.8, we note that $|(1 - |du|^{p-2})du| \leq (2-p)$ when $|du| \leq 1$, so that

$$\int_{\{|du| \leq 1\}} \psi_\delta^2 |(1 - |du|^{p-2})du|^2 \leq C(M)(2-p)^2.$$

Where $|du| \geq 1$, we can make repeated use of the pointwise estimate (8.46), together with the fact

that

$$1 - |du|^{p-2} \leq (2-p) \log(|du|)$$

to find

$$\begin{aligned} \int_{\{|du| \geq 1\}} \psi_\delta^2 (1 - |du|^{p-2})^2 |du|^2 &\leq C(M, \Lambda) (2-p)^{-2/p} \int_{\{|du| \geq 1\}} (1 - |du|^{p-2})^2 \\ &\leq C(M, \Lambda) (2-p)^{2-2/p} \int_{\{|du| \geq 1\}} \log(|du|)^2 \\ &\leq C(M, \Lambda) (2-p)^{2-2/p} \int \log \left(\frac{C(M, \Lambda)}{(2-p)^{1/p} r(x)} \right)^2. \end{aligned}$$

Splitting up the logarithm

$$\log \left(\frac{C(M, \Lambda)}{(2-p)^{1/p} r(x)} \right) = \log(C(M, \Lambda)) + \frac{1}{p} |\log(2-p)| - \log(r(x)),$$

we then see that

$$\begin{aligned} \int_M \psi_\delta^2 (1 - |du|^{p-2})^2 |du|^2 &\leq C(M, \Lambda) (2-p)^{2-2/p} |\log(2-p)|^2 \\ &\quad + C(M, \Lambda) (2-p)^{2-2/p} \int_M \log(r(x))^2. \end{aligned}$$

Finally, it follows from the volume estimates of Lemma 8.2, that

$$\|r(x)^{-1}\|_{L^1} \leq C(M, \Lambda),$$

so that

$$\int_M \log(r(x))^2 \leq C(M) \int_M r(x)^{-1} \leq C(M, \Lambda).$$

In particular, it then follows that

$$\|\psi_\delta (1 - |du|^{p-2}) du\|_{L^2}^2 \leq C(M, \Lambda) (2-p)^{2-2/p} |\log(2-p)|^2,$$

which together with (8.44) and (8.45) gives

$$\|\psi_\delta d\varphi\|_{L^2}^2 \leq C(M, \Lambda) (2-p)^{2-2/p} |\log(2-p)|^2. \quad (8.47)$$

As before, we now take $\delta \rightarrow 0$ and appeal to the Monotone Convergence Theorem to arrive at the

desired estimate (8.43). □

8.4 Limiting Behavior of the p -Energy Measure

We come now to the proof of Theorem 2.1, showing that for a family $u_p : M \rightarrow S^1$ of stationary p -harmonic maps to the circle with $E_p(u_p) = O(\frac{1}{2-p})$, the singular sets converge (subsequentially) to the support of a stationary, rectifiable $(n-2)$ -varifold, given by the concentrated part of the limiting energy measure $\mu = \lim(2-p)|du|^p dvol_g$.

Let M^n be a closed, oriented Riemannian manifold, let $p_i \in (1, 2)$ with $\lim_{i \rightarrow \infty} p_i = 2$, and let $u_i \in W^{1,p_i}(M, S^1)$ be a sequence of stationary p_i -harmonic maps satisfying

$$\Lambda := \sup_{i \in \mathbb{N}} \int_M (2 - p_i) |du_i|^{p_i} dv_g < \infty. \quad (8.48)$$

Passing to a subsequence, we can assume also that the p_i -energy measures

$$\mu_i := (2 - p_i) |du_i|^{p_i} dv_g$$

converge in $(C^0)^*$ to a limiting measure μ , and that the singular sets $Sing(u_i)$ converge in the Hausdorff sense to a limiting set

$$\Sigma = \lim_{i \rightarrow \infty} Sing(u_i).$$

Now, for each i , consider as in Section 8.3 the Hodge decomposition

$$ju_i = d^* \xi_i + d\varphi_i + h_i$$

of ju_i , and set $\alpha_i := d^* \xi_i + d\varphi_i$. We associate to ju_i , α_i , and h_i , the following L^1 sections of $End(TM)$:

$$S_i := |du_i|^{p_i-2} du_i^* du_i = |ju_i|^{p_i-2} ju_i \otimes ju_i,$$

$$S_i^s := |\alpha_i|^{p_i-2} \alpha_i \otimes \alpha_i,$$

and

$$S_i^h := |h_i|^{p_i-2} h_i \otimes h_i.$$

As we shall see, the proof of Theorem 2.1 rests largely on the following simple claim.

Claim 8.10.

$$\lim_{i \rightarrow \infty} (2 - p_i) \|S_i - S_i^s - S_i^h\|_{L^1} = 0. \quad (8.49)$$

Proof. Denoting by $f : T_x^*M \rightarrow \text{End}(T_x M)$ the function

$$f(X) = |X|^{p-2} X \otimes X$$

for $p \in (1, 2)$, it is easy to check that

$$|\nabla f(X)| \leq 3|X|^{p-1};$$

as an immediate consequence, we then have

$$\begin{aligned} |f(X+Z) - f(X)| &\leq \int_0^1 3|X+tZ|^{p-1}|Z|dt \\ &\leq 6(|X|^{p-1} + |Z|^{p-1})|Z| \end{aligned}$$

for any $X, Z \in T_p M$. In particular, since $S_i = f(ju_i) = f(\alpha_i + h_i)$, $S_i^s = f(\alpha_i)$, and $S_i^h = f(h_i)$ (with $p = p_i$) by definition, it follows that

$$|S_i - S_i^s| \leq 6(|du_i|^{p_i-1} + |h_i|^{p_i-1})|h_i|$$

and

$$|S_i - S_i^h| \leq 6(|du_i|^{p_i-1} + |\alpha_i|^{p_i-1})|\alpha_i|.$$

With this in mind, we estimate the L^1 norm of $S_i - S_i^s - S_i^h$ by splitting M into $\mathcal{N}_\delta(\text{Sing}(u_i))$ and $M \setminus \mathcal{N}_\delta(\text{Sing}(u_i))$ for $\delta > 0$ small, writing

$$\begin{aligned} \|S_i - S_i^s - S_i^h\|_{L^1(M)} &\leq \int_{\mathcal{N}_\delta(\text{Sing}(u_i))} |S_i - S_i^s| + |S_i^h| \\ &\quad + \int_{M \setminus \mathcal{N}_\delta(\text{Sing}(u_i))} |S_i - S_i^h| + |S_i^s| \\ &\leq \int_{\mathcal{N}_\delta(\text{Sing}(u_i))} (6|du_i|^{p_i-1}|h_i| + 7|h_i|^{p_i}) \\ &\quad + \int_{M \setminus \mathcal{N}_\delta(\text{Sing}(u_i))} (6|du_i|^{p_i-1}|\alpha_i| + 7|\alpha_i|^{p_i-1}). \end{aligned}$$

Now, since

$$\|h_i\|_{L^\infty}^{p_i} \leq C(M) \frac{\Lambda}{2 - p_i}$$

and, by Lemma 8.2,

$$\text{Vol}(\mathcal{N}_\delta(\text{Sing}(u_i))) \leq C(M, \Lambda) \delta^{p_i},$$

we have the simple estimate

$$\|h_i\|_{L^{p_i}(\mathcal{N}_\delta(\text{Sing}(u_i)))}^{p_i} \leq \frac{C(M, \Lambda)}{2 - p_i} \delta^{p_i}. \quad (8.50)$$

On the other hand, we know from Lemma 8.9 that

$$\begin{aligned} \int_{M \setminus \mathcal{N}_\delta(\text{Sing}(u_i))} |\alpha_i|^{p_i} &\leq C(M) \left(\delta^{-2} \int_{M \setminus \mathcal{N}_\delta(\text{Sing}(u_i))} \text{dist}(x, \text{Sing}(u_i))^2 |\alpha_i|^2 \right)^{p_i/2} \\ &\leq C(M, \Lambda) \delta^{-p_i}. \end{aligned}$$

Returning to our estimate for $\|S_i - S_i^s - S_i^h\|_{L^1(M)}$, it then follows that

$$\begin{aligned} \|S_i - S_i^s - S_i^h\|_{L^1(M)} &\leq \int_{\mathcal{N}_\delta(\text{Sing}(u_i))} (6|du_i|^{p_i-1}|h_i| + 7|h_i|^{p_i}) \\ &\quad + \int_{M \setminus \mathcal{N}_\delta(\text{Sing}(u_i))} (6|du_i|^{p_i-1}|\alpha_i| + 7|\alpha_i|^{p_i-1}) \\ &\leq C \|du_i\|_{L^{p_i}}^{p_i-1} (\|h_i\|_{L^{p_i}(\mathcal{N}_\delta(\text{Sing}(u_i)))} + \|\alpha_i\|_{L^{p_i}(M \setminus \mathcal{N}_\delta(\text{Sing}(u_i)))}) \\ &\leq C(M, \Lambda) (2 - p_i)^{1/p_i-1} (\delta(2 - p_i)^{-1/p_i} + \delta^{-1}). \end{aligned}$$

Multiplying by $(2 - p_i)$ and taking $i \rightarrow \infty$, we arrive at the bound

$$\limsup_{i \rightarrow \infty} (2 - p_i) \|S_i - S_i^s - S_i^h\|_{L^1(M)} \leq C(M, \Lambda) \delta;$$

since $\delta > 0$ was arbitrary, (8.49) follows. $\square$

With this claim established, we next observe that measures μ_i can be written as

$$\mu_i = (2 - p_i) \text{tr}(S_i) dv_g,$$

and as a consequence of (8.49), we see that

$$\begin{aligned} \mu &= \lim_{i \rightarrow \infty} (2 - p_i) \text{tr}(S_i^s + S_i^h) dv_g \\ &= \lim_{i \rightarrow \infty} [(2 - p_i) |\alpha_i|^{p_i} dv_g + (2 - p_i) |h_i|^{p_i} dv_g] \\ &= \lim_{i \rightarrow \infty} [(2 - p_i) |\alpha_i|^{p_i} dv_g + |\bar{h}_i|^{p_i} dv_g], \end{aligned}$$

where in the last line we've set

$$\bar{h}_i := (2 - p_i)^{-1/p_i} h_i.$$

Now, since $\{\bar{h}_i\}$ forms a bounded sequence in the space $\mathcal{H}^1(M)$ of harmonic one-forms, by passing to a further subsequence, we can assume that it converges to some limit

$$\bar{h} = \lim_{i \rightarrow \infty} \bar{h}_i \in \mathcal{H}^1(M).$$

It's then clear that

$$|\bar{h}_i|^{p_i} \rightarrow |\bar{h}|^2$$

pointwise, and we can therefore write

$$\mu = \lim_{i \rightarrow \infty} (2 - p_i) |\alpha_i|^{p_i} dv_g + |\bar{h}|^2 dv_g. \quad (8.51)$$

To complete the proof of Theorem 2.1, it remains to realize the measure

$$\nu := \lim_{i \rightarrow \infty} |\alpha_i|^{p_i} dv_g$$

as the weight measure of a stationary, rectifiable $(n - 2)$ -varifold satisfying the stated properties.

To this end, recalling the machinery of generalized varifolds discussed in Section 4.4, we begin by remarking that, where $|\alpha_i| > 0$, the tensor

$$I - 2|\alpha_i|^{-2} \alpha_i \otimes \alpha_i \in \text{End}(TM)$$

belongs to $A_{n-2}(M)$, so we can define a sequence of generalized $(n - 2)$ -varifolds $V_i \in \mathcal{V}'_{n-2}(M)$ by

$$\langle V_i, f \rangle := \int_M (2 - p_i) |\alpha_i|^{p_i} f(I - 2|\alpha_i|^{-2} \alpha_i \otimes \alpha_i) \text{ for } f \in C^0(A_{n-2}(M)). \quad (8.52)$$

The associated weight measures $\|V_i\|$ are then given by

$$\|V_i\| := (2 - p_i) |\alpha_i|^{p_i} dv_g,$$

and since we have a uniform mass bound

$$\sup_i \|V_i\|(M) < \infty,$$

we can pass to a further subsequence to obtain a weak limit

$$V = \lim_{i \rightarrow \infty} V_i \in \mathbf{V}_{n-2}^*(M)$$

with weight measure

$$\|V\| = \nu.$$

We claim next that $\delta V = 0$. To see this, let X be a C^1 vector field, so that

$$\begin{aligned} \delta V(X) &= \lim_{i \rightarrow \infty} \delta V_i(X) \\ &= \lim_{i \rightarrow \infty} \int_M (2 - p_i) |\alpha_i|^{p_i} \langle I - 2|\alpha_i|^{-2} \alpha_i \otimes \alpha_i, \nabla X \rangle \\ &= \lim_{i \rightarrow \infty} \int_M (2 - p_i) |\alpha_i|^{p_i} \operatorname{div}(X) - (2 - p_i) \langle 2S_i^s, \nabla X \rangle. \end{aligned}$$

Appealing once more to (8.49), we then see that

$$\begin{aligned} \delta V(X) &= \lim_{i \rightarrow \infty} \int_M (2 - p_i) (|du_i|^{p_i} - |h_i|^{p_i}) \operatorname{div}(X) - (2 - p_i) 2 \langle S_i - S_i^h, \nabla X \rangle \\ &= \lim_{i \rightarrow \infty} (2 - p_i) \int_M |du_i|^{p_i} \operatorname{div}(X) - 2 \langle |du_i|^{p_i-2} du_i^* du_i, \nabla X \rangle \\ &\quad + \lim_{i \rightarrow \infty} \int_M |\bar{h}_i|^{p_i} \operatorname{div}(X) - 2 \langle |\bar{h}_i|^{p_i-2} \bar{h}_i \otimes \bar{h}_i, \nabla X \rangle. \end{aligned}$$

Now, it's clear that

$$\begin{aligned} \lim_{i \rightarrow \infty} \int_M |\bar{h}_i|^{p_i} \operatorname{div}(X) - 2 \langle |\bar{h}_i|^{p_i-2} \bar{h}_i \otimes \bar{h}_i, \nabla X \rangle &= \int_M |\bar{h}|^2 \operatorname{div}(X) - 2 \langle \bar{h} \otimes \bar{h}, \nabla X \rangle \\ &= 0, \end{aligned}$$

since $\operatorname{div}(|\bar{h}|^2 I - 2\bar{h} \otimes \bar{h}) = 0$ for harmonic $\bar{h}$. On the other hand, we know from the p_i -stationarity of u_i that

$$\int_M |du_i|^{p_i} \operatorname{div}(X) - p \langle |du_i|^{p_i-2} du_i^* du_i, \nabla X \rangle = 0,$$

and consequently

$$\begin{aligned}
|\delta V(X)| &= \left| \lim_{i \rightarrow \infty} (2 - p_i) \int_M (p_i - 2) \langle |du_i|^{p_i-2} du_i \otimes du_i, \nabla X \rangle \right| \\
&\leq \lim_{i \rightarrow \infty} (2 - p_i) \Lambda |\nabla X|_{C^0} \\
&= 0,
\end{aligned}$$

as claimed.

Since $\nu = \|V\|$ for a generalized $(n-2)$ -varifold V with $\delta V = 0$, it will follow from Theorem 4.7 that ν is indeed the weight measure of a stationary, rectifiable $(n-2)$ -varifold, once we show that ν satisfies

$$\Theta_{n-2}^*(\nu, x) > 0 \text{ for all } x \in \text{spt}(\nu).$$

In particular, to complete the proof of Theorem 2.1, it now suffices to establish the following lemma.

Lemma 8.11. *The support $\text{spt}(\nu)$ of ν is given by*

$$\text{spt}(\nu) = \Sigma = \lim_{i \rightarrow \infty} \text{Sing}(u_i), \quad (8.53)$$

and for $x \in \Sigma$, the density of ν satisfies the lower bound

$$\Theta_{n-2}^*(\nu, x) \geq 2\pi. \quad (8.54)$$

Proof. To establish (8.53), first consider $x \in M \setminus \Sigma$, and set $\delta = \text{dist}(x, \Sigma)$. By definition of Hausdorff convergence, it follows that

$$\text{dist}(x, \text{Sing}(u_i)) > \frac{\delta}{2},$$

and consequently

$$B_{\delta/4}(x) \subset M \setminus \mathcal{N}_{\delta/4}(\text{Sing}(u_i)),$$

for i sufficiently large. Appealing once more to the estimates of Lemma 8.9, we then see that

$$\begin{aligned}
\nu(B_{\delta/4}(x)) &\leq \liminf_{i \rightarrow \infty} (2 - p_i) \int_{B_{\delta/4}(x)} |\alpha_i|^{p_i} \\
&\leq \lim_{i \rightarrow \infty} (2 - p_i) \frac{C(M, \Lambda)}{\delta^2} \\
&= 0,
\end{aligned}$$

so that $x \notin \text{spt}(\nu)$; and since $x \in M \setminus \Sigma$ was arbitrary, we therefore have

$$\text{spt}(\nu) \subset \Sigma.$$

Next, for $x \in \Sigma$, we'll show that

$$\Theta_{n-2}^*(\mu, x) \geq 2\pi. \quad (8.55)$$

Indeed, if $x \in \Sigma$, then by definition there is a sequence $x_i \in \text{Sing}(u_i)$ for which

$$x = \lim_{i \rightarrow \infty} x_i.$$

By Proposition 8.1, at each x_i , we have

$$\lim_{r \rightarrow 0} (2 - p_i) \int_{B_r(x_i)} |du_i|^{p_i} \geq 2\pi c(n, p_i), \quad (8.56)$$

where $c(n, p_i) \rightarrow \omega_{n-2}$ as $p_i \rightarrow 2$. In particular, fixing $\delta > 0$ and appealing to the monotonicity of the p -energy (Lemma 7.5), we conclude that

$$\begin{aligned} \mu(B_\delta(x)) &\geq \liminf_{i \rightarrow \infty} \mu_i(B_{\delta-\delta^2}(x_i)) \\ &\geq \lim_{i \rightarrow \infty} e^{-C(M)\delta^2} 2\pi c(n, p_i) (\delta - \delta^2)^{n-p_i} \\ &= e^{-C(M)\delta^2} 2\pi \omega_{n-2} (\delta - \delta^2)^{n-2}. \end{aligned}$$

Dividing through by $\omega_{n-2}\delta^{n-2}$ and letting $\delta \rightarrow 0$, we arrive at the desired lower bound (8.55).

Finally, since the difference

$$\mu - \nu = |\bar{h}|^2 dv_g$$

clearly satisfies

$$\lim_{\delta \rightarrow 0} \delta^{2-n} \int_{B_\delta(x)} |\bar{h}|^2 dv_g = 0,$$

we see that (8.55) yields directly the desired density bound (8.54) for ν on Σ . Moreover, it follows immediately from (8.54) that $\Sigma \subset \text{spt}(\nu)$, and since we've already shown that $\text{spt}(\nu) \subset \Sigma$, this completes the proof of (8.53) as well. $\square$

With the proof of Theorem 2.1 completed, we conclude the chapter by making a few remarks on the compactness of the maps u_i . Given a sequence u_i of p_i -harmonic maps as above, suppose that

our sequence $u_i \in W^{1,p_i}(M, S^1)$ either satisfies the additional bound

$$\sup_i \|du_i\|_{L^1(M)} \leq C,$$

or that the first Betti number $b_1(M) = 0$. In either case, it follows that the harmonic component h_i of ju_i is uniformly bounded

$$\sup_i \|h_i\|_{L^\infty} \leq C \tag{8.57}$$

as $i \rightarrow \infty$. Together with the L^q estimates of Proposition 8.8, this implies immediately that

$$\limsup_{i \rightarrow \infty} \|du_i\|_{L^q} < \infty$$

for any $q \in [1, 2)$, so some subsequence of $\{u_i\}$ must converge weakly in $W^{1,q}(M, S^1)$ to some limiting map v . Moreover, since (by Proposition 8.8) the exact component $d\varphi_i$ of ju_i vanishes in L^q as $i \rightarrow \infty$, it follows that

$$d^*ju_i \rightarrow 0$$

weakly as $i \rightarrow \infty$, so the map v must satisfy $d^*jv = 0$ distributionally.

Moreover, combining (8.57) with Lemma 8.9, it follows that, away from any $0 < \delta$ -neighborhood $\mathcal{N}_\delta(\Sigma)$ of Σ , we have

$$\limsup_{i \rightarrow \infty} \int_{M \setminus \mathcal{N}_\delta(\Sigma)} |du_i|^2 < \infty;$$

and putting this together with the local $W^{2,p}$ estimate of Corollary 7.3, we see that

$$\limsup_{i \rightarrow \infty} \|u_i\|_{W^{2,p_i}(M \setminus \mathcal{N}_\delta(\Sigma))} < \infty.$$

Of course, for $p > \frac{2n}{n+2}$, Rellich's theorem gives us compactness of the embedding $W^{2,p} \hookrightarrow W^{1,2}$; hence, since $p_i > \frac{2n}{n+2}$ for i sufficiently large, there is indeed some subsequence of $\{u_i\}$ which converges strongly in $W^{1,2}(M \setminus \mathcal{N}_\delta(\Sigma), S^1)$ to the limiting map v identified above. And since $d^*jv = 0$ and $v \in W_{loc}^{1,2}(M \setminus \Sigma, S^1)$, it follows that v is a strongly harmonic map in $C_{loc}^\infty(M \setminus \Sigma, S^1)$. We summarize these observations in the following proposition.

Proposition 8.12. *For a sequence $u_i \in W^{1,p_i}(M, S^1)$ of maps satisfying the hypotheses of Theorem 2.1, if either $b_1(M) = 0$, or the sequence $\{u_i\}$ satisfies a uniform $W^{1,1}$ bound $\sup_i \|du_i\|_{L^1} < \infty$, then a subsequence of the maps u_i converges in $W^{1,q}(M, S^1)$ for $q \in [1, 2)$ and strongly in $W_{loc}^{1,2}(M \setminus \Sigma)$ to a limiting map $v : M \rightarrow S^1$ which is harmonic away from $Sing(v) \subset \Sigma$.*

Chapter 9

Energy Quantization in Planar Domains

In this chapter, we complete the proof of Theorem 2.3, showing that for families $(1, 2) \ni p \mapsto u_p \in W^{1,p}(M, S^1)$ of p -harmonic maps to S^1 on planar domains, the concentrated portion of the limiting energy measures $\mu = \lim_{p \rightarrow 2} \frac{1}{2\pi} (2 - p) |du|^p(x) dx$ is given by integer multiples of Dirac masses.

9.1 Quantization Under Compactness Assumptions

To prove the integrality statement in Theorem 2.3, we first need to record suitable local versions of Theorem 2.1 and Proposition 8.12 for maps from the disk $D_2^2(0)$. These results follow easily from the analysis of the preceding chapter, once we fix a local variant of the Hodge decomposition considered in Section 8.3. To this end, choose a cutoff function $\chi \in C_c^\infty(D_{5/3})$ such that $\chi \equiv 1$ on $D_{4/3}$. For a map $u \in W^{1,p}(D_2, S^1)$, we can then define a two-form ξ and function φ distributionally by setting

$$\xi := \Delta_H^{-1}(\chi d(ju))$$

and

$$\varphi := \Delta^{-1}(\chi \operatorname{div}(ju)).$$

Writing $ju = h + d^*\xi + d\varphi$, we then see that the remaining one-form h is strongly harmonic on $D_{4/3}$ (and therefore corresponds there to the gradient of a harmonic function). Using this decomposition, we can then argue exactly as in the previous sections to establish the following.

Proposition 9.1. *Let $u_i \in W^{1,p_i}(D_2(0), S^1)$ be a sequence of stationary p_i -harmonic maps from the disk $D_2(0)$ to S^1 , for which*

$$\sup_i \int_{D_2(0)} (2 - p_i) |du_i|^{p_i} < \infty.$$

A subsequence of the measures $\mu_i = (2 - p_i) |du_i|^{p_i} dv_g$ then converges weakly in $(C^0(D_1))^*$ to a measure μ of the form

$$\mu = \sum_{j=1}^k \theta_j \delta_{a_j} + |d\psi|^2(x) dx, \quad (9.1)$$

where ψ is a harmonic function, $\{a_j\}_{j=1}^k = \Sigma \cap D_1$ is given by the Hausdorff limit Σ of $\text{Sing}(u_i)$ in $D_1(0)$, and $\theta_j \geq 2\pi$. Moreover, if

$$\sup_i \|du_i\|_{L^1(D_2(0))} < \infty, \quad (9.2)$$

then the function ψ in (9.1) vanishes, and (a subsequence of) $\{u_i\}$ converges strongly in $W_{loc}^{1,2}(D_{3/2}(0) \setminus \Sigma)$ to a map $v \in C^\infty(D_{3/2} \setminus \Sigma, S^1)$ that is harmonic away from Σ .

In this section, we combine Proposition 9.1 with an application of a Pohozaev-type identity to establish the desired quantization result under the assumption of an L^1 gradient bound (9.2) for the maps u_i . More precisely, we show the following.

Proposition 9.2. *Let $u_i \in W^{1,p_i}(D_2(0), S^1)$ be as in Proposition 9.1, and suppose also that (9.2) holds. Passing to a subsequence, let v be the limiting map $v = \lim_i u_i$ given by Proposition 9.1. Then the limiting measure μ has the form*

$$\mu = \sum_{j=1}^k 2\pi \deg(v, a_j)^2 \delta_{a_j}.$$

For p -energy minimizers with respect to a fixed boundary condition, this result follows from the analysis of Hardt and Lin [43], in which case all of the degrees $\deg(v, a)$ are either 1 or -1 . It is also the immediate analog of the quantization result for 2-dimensional solutions of the Ginzburg-Landau equations in [26], though the proof in the present setting is much simpler.

The reason for the relative simplicity in this setting is the form of the Pohozaev identity. In [26], on their way to demonstrating the quantization of the energy measures $\mu_\epsilon = \frac{|du_\epsilon(z)|^2}{|\log \epsilon|} dz$, Comte and Mironescu appeal to the quantization results of [12] and [19] for the potential measures $\frac{W(u_\epsilon(x))}{\epsilon^2} dx$. These quantization results—though by no means trivial—can be derived in a relatively straightforward way from a Pohozaev identity that relates the integral of $\frac{W(u_\epsilon)}{\epsilon^2}$ on a disk to the behavior of u_ϵ on

its boundary. It is then observed in [26] that the quantization of the potential measure gives strong constraints on the way that the degrees of the maps u_ϵ can vary around clusters of zeroes (or “vortices”) at different scales, which ultimately give rise to the quantization of the energy measures μ_ϵ .

In the present setting, the path is much simpler, because the normalized p -energy $(2-p)|du|^p$ simultaneously plays the roles occupied by the energy *and* potential measures in the Ginzburg-Landau setting. In particular, we have the following nice Pohozaev-type identity:

Lemma 9.3. *Let $u \in W^{1,p}(D_2(0), S^1)$ be stationary p -harmonic on $D_2(0)$. On any annulus*

$$A_{r_1, r_2}(a) = D_{r_2}(0) \setminus D_{r_1}(0) \subset D_2(0),$$

we then have

$$\int_{r_1}^{r_2} \left(\int_{D_r(a)} (2-p)|du|^p \right) dr = \int_{A_{r_1, r_2}(a)} |z-a|(|du|^p - p|du|^{p-2}|du(\frac{z-a}{|z-a|})|^2) dz.$$

Proof. The identity is simply a repackaging of the monotonicity formula in dimension two: By testing the inner variation equation

$$\int |du|^p \operatorname{div}(X) - p|du|^{p-2} \langle du^* du, \nabla X \rangle = 0$$

against vector fields of the form $X(z) = \psi(z)(z-a)$ for test functions $\psi \in C_c^\infty(D_r(a))$ approximating the characteristic function $\chi_{D_r(a)}$, we find that

$$\int_{D_r(a)} (2-p)|du|^p = r \int_{\partial D_r(a)} (|du|^p - p|du|^{p-2}|du(\frac{z-a}{|z-a|})|^2) \quad (9.3)$$

for almost every $r \in [r_1, r_2]$. Integrating over $[r_1, r_2]$ then gives the desired equation. $\square$

With this identity in hand, we can now argue in the spirit of [12],[19] to prove Proposition 9.2:

Proof. (Proof of Proposition 9.2)

Let

$$\Sigma \cap D_1 = \{a_1, \dots, a_k\},$$

so that the limiting map $v(z)$ satisfies

$$d^*(jv) = 0 \text{ and } d(jv) = \sum_{\ell=1}^k 2\pi\kappa_\ell \delta_{a_\ell},$$

where $\kappa_\ell = \deg(v, a_\ell)$ denotes the degree of v about a_ℓ . Letting $\bar{v}$ be the map given by

$$\bar{v}(z) := \prod_{\ell=1}^k \left(\frac{z - a_\ell}{|z - a_\ell|} \right)^{\kappa_\ell},$$

we observe that

$$d^* j\bar{v} = 0 \text{ and } d(j\bar{v}) = d(jv),$$

so the difference $jv - j\bar{v}$ is strongly harmonic. In particular, it follows that

$$v = e^{i\varphi} \bar{v} \tag{9.4}$$

for some harmonic function $\varphi \in C^\infty(D_2(0))$.

Now, set

$$\delta_0 := \min\{|a_\ell - a_m| \mid 1 \leq \ell < m \leq k\},$$

so that the density θ_ℓ of μ at a_ℓ is given by

$$\mu(D_r(a_\ell)) = \theta_\ell$$

for every $r \in (0, \delta_0)$. For any $\delta \in (0, \delta_0)$, it then follows from Lemma 9.3 that

$$\begin{aligned} \theta_\ell &= \frac{2}{\delta} \int_{\delta/2}^\delta \mu(D_r(a_\ell)) dr \\ &= \frac{2}{\delta} \lim_{i \rightarrow \infty} \int_{\delta/2}^\delta \mu_i(D_r(a_\ell)) dr \\ &= \frac{2}{\delta} \lim_{i \rightarrow \infty} \int_{A_{\delta/2, \delta}(a_\ell)} |z - a_\ell| (|du_i|^p - p_i |du_i|^{p_i-2} |du_i(\frac{z - a_\ell}{|z - a_\ell|})|^2) dz. \end{aligned}$$

On the other hand, we also know that $u_i \rightarrow v$ strongly in $W^{1,2}(A_{\delta/2, \delta}(a_\ell))$ and $\|du_i\|_{L^\infty(A_{\delta/2, \delta}(a_\ell))}$ is uniformly bounded as $i \rightarrow \infty$, so it follows that

$$\begin{aligned} \theta_\ell &= \frac{2}{\delta} \lim_{i \rightarrow \infty} \int_{A_{\delta/2, \delta}(a_\ell)} |z - a_\ell| (|du_i|^p - p_i |du_i|^{p_i-2} |du_i(\frac{z - a_\ell}{|z - a_\ell|})|^2) dz \\ &= \frac{2}{\delta} \int_{A_{\delta/2, \delta}(a_\ell)} |z - a_\ell| (|dv|^2 - 2|dv(\frac{z - a_\ell}{|z - a_\ell|})|^2) dz. \end{aligned}$$

Since $v = e^{i\varphi} \bar{v}$, we can expand dv as

$$dv(z) = v(z) \cdot (i \cdot d\varphi + \sum_{\ell=1}^k \kappa_\ell \frac{\overline{z - a_\ell}}{|z - a_\ell|^2} P_{z-a_\ell}^\perp),$$

where for $0 \neq w \in \mathbb{R}^2$ we denote by $P_w^\perp$ projection onto the line perpendicular to w . In particular, if $z \in D_\delta(a_\ell)$ for $\delta < \frac{\delta_0}{2}$, then $|z - a_m| > \delta_0 - \delta > \frac{\delta_0}{2}$ for every $m \neq \ell$, and it follows that

$$|dv(z) - \kappa_\ell \frac{\overline{z - a_\ell}}{|z - a_\ell|^2} P_{z - a_\ell}^\perp| \leq \|d\varphi\|_{L^\infty} + \sum_{m \neq \ell} \frac{2|\kappa_m|}{\delta_0} =: K.$$

Combining this with the obvious estimate

$$|dv(z)| \leq \frac{K'}{|z - a_\ell|} \text{ on } D_\delta(a_\ell)$$

(where K' of course depends on v), we see that, on $D_\delta(a_\ell)$,

$$||dv(z)|^2 - \frac{\kappa_\ell^2}{|z - a_\ell|^2}| \leq \frac{K''}{|z - a_\ell|} \quad (9.5)$$

and

$$|dv(\frac{z - a_\ell}{|z - a_\ell|})|^2 \leq \frac{K''}{|z - a_\ell|}. \quad (9.6)$$

In particular, on the annulus $A_{\delta/2, \delta}(a_\ell)$, since

$$\int_{A_{\delta/2, \delta}(a_\ell)} |z - a_\ell| \frac{\kappa_\ell^2}{|z - a_\ell|^2} = \pi \kappa_\ell^2 \delta,$$

we can apply the preceding estimates to our computation of θ_ℓ to conclude that

$$\begin{aligned} |\theta_\ell - 2\pi \kappa_\ell^2| &= \left| \frac{2}{\delta} \int_{A_{\delta/2, \delta}(a_\ell)} |z - a_\ell| (|dv|^2 - \frac{\kappa_\ell^2}{|z - a_\ell|^2} - 2|dv(\frac{z - a_\ell}{|z - a_\ell|})|^2) dz \right| \\ &\leq \frac{2}{\delta} \int_{A_{\delta/2, \delta}(a_\ell)} |z - a_\ell| \frac{K''}{|z - a_\ell|} \\ &= 2\pi K'' \delta. \end{aligned}$$

Since $\delta > 0$ was arbitrary, it follows finally that

$$\theta_\ell = 2\pi \kappa_\ell^2,$$

which is precisely what we wanted to show. $\square$

Combining this result with a simple contradiction argument, and scaling, we can reformulate the integrality conclusion of Proposition 9.2 as follows.

Corollary 9.4. *For any $\gamma > 0$ and $\Lambda < \infty$, there exists $q(\gamma, \Lambda) \in (1, 2)$ such that if $p > q$, and*

$u \in W^{1,p}(D_{2r}(x), S^1)$ is stationary p -harmonic with

$$(2-p)\theta_p(u, x, 2r) + \frac{1}{r} \int_{D_{2r}(x)} |du| \leq \Lambda \quad (9.7)$$

and $\text{Sing}(u) \cap D_{2r}(x) \subset D_r(x)$, then

$$\text{dist}((2-p)\theta_p(u, x, r), 2\pi\mathbb{Z}) < \gamma. \quad (9.8)$$

9.2 Proof of Theorem 2.3 Completed

In this section, we will complete the proof of Theorem 2.3, removing the requirement of a uniform $W^{1,1}$ bound by showing that, in the general setting of Theorem 2.3, the normalized energy measures μ_i are negligible on the complement of a collection of disks satisfying the conditions of Corollary 9.4.

Proof. (Proof of Theorem 2.3)

Let $u_i \in W^{1,p_i}(D_2(0), S^1)$ be a sequence of stationary p_i -harmonic maps on D_2 as in Proposition 9.1. Blowing up at one of the concentration points $a_j \in \Sigma \cap D_1$, we can assume that the measures $\mu_i = (2-p_i)|du_i|^{p_i}(x)dx$ converge to a multiple of the Dirac mass

$$\mu_i \rightarrow \mu = \theta\delta_0$$

at the origin. As in the beginning of the section, consider the local version of the Hodge decomposition

$$ju_i = h_i + d^*\xi_i + d\varphi_i,$$

where

$$\xi_i := \Delta_H^{-1}(\chi dju_i) = - \left(\int \langle \chi d(ju_i)(y), G(x, y) \rangle dy \right) dx^1 \wedge dx^2$$

and

$$\varphi_i := \Delta^{-1}(\chi \text{div}(ju_i)) = \int \langle \chi \text{div}([1 - |ju_i|^{p-2}]ju_i)(y), G(x, y) \rangle dy,$$

and $G(x, y) = \frac{-1}{2\pi} \log |x - y|$ is the two-dimensional Green's function.

Recall that since

$$\text{div}(h_i) = (1 - \chi)\text{div}(ju_i) = 0$$

and

$$d^* dh_i = d^*([1 - \chi]d(ju_i)),$$

h_i is harmonic on the disk $D_{4/3}(0)$ (where $\chi \equiv 1$), and it follows that

$$\|h_i\|_{L^\infty(D_{1/2}(0))} \leq C \|h_i\|_{L^{p_i}(D_1(0) \setminus D_{3/4}(0))}. \quad (9.9)$$

Next, consider the $(2 - p_i)^{1/p_i}$ -neighborhoods

$$U_i := \mathcal{N}_{(2-p_i)^{1/p_i}}(\text{Sing}(u_i))$$

about the singular sets $\text{Sing}(u_i)$. Our goal now is to show that

$$\lim_{i \rightarrow \infty} \mu_i(D_{1/2}(0) \setminus U_i) = 0, \quad (9.10)$$

and that U_i is contained in a finite union of disks satisfying the hypotheses of Corollary 9.4. To this end, we first observe that, by Corollary 8.4 and Lemma 8.6 of the appendix, $\nabla \xi_i$ satisfies the pointwise bound

$$|\nabla \xi_i|(x) \leq \frac{C}{\text{dist}(x, \text{Sing}(u_i))} \quad (9.11)$$

for $x \in D_1(0)$. Putting this together with the volume estimate of Lemma 8.2, we find that

$$\begin{aligned} \int_{D_{1/2}(0) \setminus U_i} |\nabla \xi_i|^{p_i} &\leq \sum_{j=1}^{\lfloor \frac{1}{p_i} |\log_2(2-p_i)| \rfloor} \int_{\mathcal{N}_{2^{-j}}(\text{Sing}(u_i)) \setminus \mathcal{N}_{2^{-j-1}}(\text{Sing}(u_i))} |du_i|^{p_i} \\ &\leq \sum_{j=1}^{\lfloor \frac{1}{p_i} |\log_2(2-p_i)| \rfloor} C 2^{-jp_i} \cdot 2^{(j+1)p_i} \\ &\leq C 2^{p_i} \frac{1}{p_i} |\log_2(2-p_i)| \end{aligned}$$

and therefore

$$\int_{D_{1/2}(0) \setminus U_i} |\nabla \xi_i|^{p_i} \leq C |\log(2-p_i)|. \quad (9.12)$$

For $d\varphi_i$, the arguments in the proofs of Proposition 8.8 and Lemma 8.9 again yield the local estimates

$$\|\varphi_i\|_{W^{1,q}(D_1(0))} \leq C(q)(2-p_i)^{1-1/p_i} |\log(2-p_i)| \quad (9.13)$$

for $q \in (1, p_i)$ and

$$\|(2 - p_i)^{1/p_i} d\varphi_i\|_{L^2(D_1(0) \setminus U_i)}^2 \leq C(2 - p_i)^{2-2/p_i} |\log(2 - p_i)|^2, \quad (9.14)$$

respectively. In particular, rearranging (9.14) and recalling that $(2 - p_i)^{p_i-2}$ is uniformly bounded as $i \rightarrow \infty$, we see that

$$\int_{D_1(0) \setminus U_i} |d\varphi_i|^2 \leq C |\log(2 - p_i)|^2. \quad (9.15)$$

Now, to estimate h_i , we observe that

$$\begin{aligned} \|h_i\|_{L^{p_i}(D_1(0) \setminus D_{3/4}(0))} &\leq \|du_i\|_{L^{p_i}(D_1(0) \setminus D_{3/4}(0))} + \|d^*\xi_i\|_{L^{p_i}(D_1(0) \setminus D_{3/4}(0))} \\ &\quad + \|d\varphi_i\|_{L^{p_i}(D_1(0) \setminus D_{3/4}(0))} \\ &\leq (2 - p_i)^{-1/p_i} \mu_i(D_1(0) \setminus D_{3/4}(0))^{1/p_i} + \|d^*\xi_i\|_{L^{p_i}(D_1(0) \setminus U_i)} \\ &\quad + \|d\varphi_i\|_{L^{p_i}(D_1(0) \setminus D_{3/4}(0))} \\ &\leq (2 - p_i)^{-1/p_i} \mu_i(D_1(0) \setminus D_{3/4}(0))^{1/p_i} + C |\log(2 - p_i)|^{2/p_i}. \end{aligned}$$

Since $\mu = \lim \mu_i$ vanishes on compact subsets of $D_1(0) \setminus \{0\}$ by assumption, it then follows that

$$\|h_i\|_{L^{p_i}(D_1(0) \setminus D_{3/4}(0))} = o\left(\frac{1}{2 - p_i}\right);$$

and by (9.9), we therefore have

$$\|h\|_{L^\infty(D_{1/2}(0))}^{p_i} \leq \frac{\delta_i}{2 - p_i}, \quad (9.16)$$

where $\lim_{i \rightarrow \infty} \delta_i = 0$.

Putting together the estimates (9.12), (9.15), and (9.16), we see finally that

$$\begin{aligned} \lim_{i \rightarrow \infty} \mu_i(D_{1/2}(0) \setminus U_i) &= \lim_{i \rightarrow \infty} (2 - p_i) \int_{D_{1/2}(0) \setminus U_i} |du_i|^{p_i} \\ &\leq C \limsup_{i \rightarrow \infty} \int_{D_{1/2}(0) \setminus U_i} (2 - p_i) [|d^*\xi_i|^{p_i} + |d\varphi_i|^{p_i} + |h_i|^{p_i}] \\ &\leq C \limsup_{i \rightarrow \infty} ((2 - p_i) |\log(2 - p_i)|^2 + \delta_i) \\ &= 0, \end{aligned}$$

confirming (9.10).

Next, as in the proof of Lemma 8.2, we know from a simple Vitali covering argument that

$$U_i \subset \bigcup_{\ell=1}^{k_i} D_{3(2-p_i)^{1/p_i}}(x_\ell^i)$$

for some $x_1^i, \dots, x_{k_i}^i \in \text{Sing}(u_i)$ such that

$$D_{(2-p_i)^{1/p_i}}(x_\ell^i) \cap D_{(2-p_i)^{1/p_i}}(x_m^i) = \emptyset \text{ when } \ell \neq m,$$

and it follows from Proposition 8.1 that

$$[(2-p_i)^{1/p_i}]^{p_i-2} \mu_i(U_i) \geq 2\pi k_i.$$

In particular, k_i is uniformly bounded independent of i , so passing to a subsequence, we can take $k_i = k$ to be constant. Moreover, setting

$$\alpha_{\ell m}^i := (2-p_i)^{-1/p_i} \text{dist}(x_\ell^i, x_m^i),$$

we can pass to a further subsequence for which the (possibly infinite) limits

$$\alpha_{\ell m} = \lim_{i \rightarrow \infty} \alpha_{\ell m}^i$$

exist. Relabeling indices if necessary, there is then some $m_0 \leq k$ such that

$$\alpha_{\ell m} = \infty \text{ for } 1 \leq \ell < m \leq m_0,$$

and for every $m > m_0$, there is some $1 \leq \ell \leq m_0$ for which $\alpha_{\ell m} < \infty$.

Now, let

$$A := \max(\{3\} \cup \{2\alpha_{\ell m} \mid \alpha_{\ell m} < \infty\}),$$

$$r_i := A(2-p_i)^{1/p_i},$$

and for $1 \leq \ell \leq m_0$, define the disks

$$D_{i,\ell} := D_{r_i}(x_\ell^i).$$

For i sufficiently large, we then see that

$$U_i \subset D_{i,1} \cup \cdots \cup D_{i,m_0}, \quad D_{i,\ell} \cap D_{i,m} = \emptyset \text{ if } \ell \neq m,$$

and

$$\text{Sing}(u_i) \cap D_{2r_i}(x_\ell^i) \subset D_{r_i}(x_\ell^i).$$

In particular, since

$$\limsup_{i \rightarrow \infty} (2 - p_i) \theta_{p_i}(u_i, x_\ell^i, 2r_i) \leq 2\theta < \infty \quad (9.17)$$

by the monotonicity formula, our disks will satisfy the conditions of Corollary 9.4 for some Λ , once we show that

$$\limsup_{i \rightarrow \infty} r_i^{-1} \|du_i\|_{L^1(D_{2r_i}(x_\ell^i))} < \infty. \quad (9.18)$$

To establish (9.18), we consider separately the components h_i , $d\varphi_i$, and $d^*\xi_i$ of the local Hodge decomposition. For h_i , we have seen already that

$$\|h_i\|_{L^\infty} \leq \delta_i^{1/p_i} (2 - p_i)^{-1/p_i} \leq \delta_i^{1/p_i} A r_i^{-1},$$

where $\delta_i \rightarrow 0$, so that

$$r_i^{-1} \int_{D_{2r_i}(x_\ell^i)} |h_i| \leq \delta_i^{1/p_i} A r_i^{-2} \cdot 4\pi r_i^2 \leq 1 \quad (9.19)$$

for i sufficiently large. For $d\varphi_i$, recall that, for $q < p_i$,

$$\|d\varphi_i\|_{L^q(D_1(0))} \leq C(q)(2 - p_i)^{1-1/p_i} |\log(2 - p_i)|,$$

so that

$$\begin{aligned} r_i^{-1} \int_{D_{2r_i}(x_\ell^i)} |d\varphi_i| &\leq r_i^{-1} \|d\varphi_i\|_{L^q(D_1(0))} (4\pi r_i^2)^{1-1/q} \\ &\leq r_i^{-1} C(q)(2 - p_i)^{1-1/p_i} |\log(2 - p_i)| r_i^{2-2/q} \\ &\leq C(q) A^{1-2/q} (2 - p_i)^{1-2/(p_i q)} |\log(2 - p_i)|. \end{aligned}$$

For i sufficiently large, we can take $p_i > q = \frac{3}{2}$ in the estimate above, to obtain

$$\begin{aligned}
\limsup_{i \rightarrow \infty} r_i^{-1} \int_{D_{2r_i}(x_\ell^i)} |d\varphi_i| &\leq C(q) A^{1-2/q} \lim_{i \rightarrow \infty} (2-p_i)^{1-8/9} |\log(2-p_i)| \\
&= C \lim_{i \rightarrow \infty} (2-p_i)^{1/9} |\log(2-p_i)| \\
&= 0.
\end{aligned}$$

Next, employing the pointwise gradient estimate (9.12) for ξ_i with Lemma 8.2, we see that

$$\begin{aligned}
r_i^{-1} \int_{D_{2r_i}(x_\ell^i)} |d^* \xi_i| &\leq r_i^{-1} \int_{\mathcal{N}_{2r_i}(\text{Sing}(u_i))} |d^* \xi| \\
&= r_i^{-1} \sum_{j=0}^{\infty} \int_{\mathcal{N}_{2^{1-j}r_i}(\text{Sing}(u_i)) \setminus \mathcal{N}_{2^{-j}r_i}(\text{Sing}(u_i))} |d^* \xi| \\
(\text{ by (9.12) }) &\leq r_i^{-1} \sum_{j=0}^{\infty} \int_{\mathcal{N}_{2^{1-j}r_i}(\text{Sing}(u_i)) \setminus \mathcal{N}_{2^{-j}r_i}(\text{Sing}(u_i))} \frac{C}{2^{-j}r_i} \\
(\text{ by Lemma 8.2 }) &\leq r_i^{-1} \sum_{j=0}^{\infty} \frac{C}{2^{-j}r_i} \cdot C(2^{1-j}r_i)^{p_i} \\
&\leq C r_i^{p_i-2} \sum_{j=0}^{\infty} (2^{1-p_i})^j.
\end{aligned}$$

And since

$$\sum_{j=0}^{\infty} (2^{1-p_i})^j = \frac{2^{p_i}}{2^{p_i} - 2} \rightarrow 2$$

and

$$r_i^{p_i-2} = A^{p_i-2} (2-p_i)^{\frac{p_i-2}{p_i}} \rightarrow 1$$

as $i \rightarrow \infty$, it follows that

$$\limsup_{i \rightarrow \infty} r_i^{-1} \int_{D_{2r_i}(x_\ell^i)} |d^* \xi_i| < \infty. \quad (9.20)$$

Combining this with the preceding estimates for h_i and $d\varphi_i$, we see that (9.18) indeed holds.

Finally, letting

$$\Lambda := 1 + \max_{1 \leq \ell \leq m_0} \limsup_{i \rightarrow \infty} [(2-p_i)\theta_{p_i}(u_i, x_\ell^i, 2r_i) + r_i^{-1} \int_{D_{2r_i}(x_\ell^i)} |du_i|],$$

and choosing an arbitrary $\gamma > 0$, it follows from Corollary 9.4 that for i sufficiently large,

$$\text{dist}((2-p_i)\theta_{p_i}(u_i, x_\ell^i, r_i), 2\pi\mathbb{Z}) < \gamma; \quad (9.21)$$

in particular, we deduce that

$$\limsup_{i \rightarrow \infty} \text{dist}((2 - p_i) \sum_{\ell=1}^{m_0} \theta_{p_i}(u_i, x_\ell^i, r_i), 2\pi\mathbb{Z}) = 0. \quad (9.22)$$

Now, by the disjointness of the disks $\{D_{i,\ell}\}_{\ell=1}^{m_0}$, we know that

$$(2 - p_i) \sum_{\ell=1}^{m_0} \theta_{p_i}(u_i, x_\ell^i, r_i) = r_i^{p_i-2} \mu_i\left(\bigcup_{\ell=1}^{m_0} D_{i,\ell}\right),$$

and since $\lim_{i \rightarrow \infty} r_i^{p_i-2} = 1$, it follows that

$$\lim_{i \rightarrow \infty} (2 - p_i) \sum_{\ell=1}^{m_0} \theta_{p_i}(u_i, x_\ell^i, r_i) = \lim_{i \rightarrow \infty} \mu_i\left(\bigcup_{\ell=1}^{m_0} D_{i,\ell}\right).$$

On the other hand, since the disks $D_{i,\ell}$ cover U_i , we know from (9.10) that

$$\begin{aligned} \lim_{i \rightarrow \infty} \mu_i(D_{1/2}(0)) &= \lim_{i \rightarrow \infty} (\mu_i(D_{1/2}(0) \setminus \bigcup_{\ell} D_{i,\ell}) + \mu_i(\bigcup_{\ell} D_{i,\ell})) \\ &= \lim_{i \rightarrow \infty} \mu_i(\bigcup_{\ell} D_{i,\ell}) \\ &= \lim_{i \rightarrow \infty} (2 - p_i) \sum_{\ell=1}^{m_0} \theta_{p_i}(u_i, x_\ell^i, r_i). \end{aligned}$$

By (9.22), it then follows that

$$\theta = \lim_{i \rightarrow \infty} \mu_i(D_{1/2}(0)) \in 2\pi\mathbb{N},$$

as desired. $\square$

Remark 9.5. In higher dimensions—as in the Ginzburg–Landau setting—one would like to show, analogously, that for a sequence $u_i \in W^{1,p_i}(B_2^n(0), S^1)$ of stationary p_i -harmonic maps with energy concentrating along an $(n-2)$ -plane $\mathcal{L}^{n-2}$, the limiting measure μ must have the form

$$\mu = 2\pi m \cdot \mathcal{H}^{n-2} \llcorner \mathcal{L}$$

for some $m \in \mathbb{N}$. As in the two-dimensional case, it is possible to reduce the problem to the case where the maps u_i converge away from $\mathcal{L}$, but the integrality question remains quite difficult after this reduction, due to the possibility of interactions between distant components of the singular set. However, the scale-invariance and relative simplicity of Pohozaev-type identities make the p -harmonic map setting a useful proving ground for new techniques in the study of energy concentration

phenomena for solutions of the Ginzburg-Landau equations and related pdes in arbitrary dimension.

Chapter 10

Nontrivial Families from Variational Methods

In this short chapter, we complete the work of Part III by constructing nontrivial families satisfying the hypotheses of Theorem 2.1 on an arbitrary closed manifold, via natural min-max constructions akin to those of Chapter 5.

10.1 Generalized Ginzburg-Landau Functionals

For $1 < p \leq n$, $\epsilon > 0$, and a closed manifold $N \subset \mathbb{R}^L$, the generalized Ginzburg-Landau functionals

$$E_{p,\epsilon} : W^{1,p}(M, \mathbb{R}^L) \rightarrow \mathbb{R}$$

are defined in [82] by

$$E_{p,\epsilon}(u) = \int_M (|du|^p + \epsilon^{-p} F(u)),$$

where the function $F : \mathbb{R}^L \rightarrow \mathbb{R}$ has the form $F(y) = \lambda(\text{dist}(y, N)^2)$ for a function $\lambda \in C^\infty(\mathbb{R})$ satisfying $\lambda' \geq 0$, $\lambda'' \geq 0$.

$$\lambda(t) = t \text{ for } t \leq \delta_N^2, \text{ and } \lambda(t) = 4\delta_N^2 \text{ for } t \geq 4\delta_N^2,$$

(Here, $\delta_N > 0$ is chosen such that nearest-point projection to N is well-defined and smooth on the $2\delta_N$ -neighborhood of N [82].) As $\epsilon \rightarrow 0$, the potential term in the energies $E_{p,\epsilon}$ penalizes

deviation from the target manifold N , while for N -valued maps u , one simply recovers the p -energy $E_{p,\epsilon}(u) = E_p(u)$.

For $p = 2$, the $\epsilon \rightarrow 0$ asymptotics of bounded-energy critical points and negative gradient flows of these functionals were been studied in [22], [23], and [56] as regularizations of harmonic maps and harmonic map heat flows. While in the $p = 2$ setting one encounters the familiar bubbling phenomena that arise in the study of harmonic maps, for $p \notin \mathbb{N}$, Wang demonstrates in [82] that (much like Proposition 7.6 for p -harmonic maps), bounded-energy sequences of critical points enjoy a strong compactness property.

Theorem 10.1. ([82] Theorem A, Corollary B) *If $p \in (1, n) \setminus \mathbb{N}$, and $\{u_{\epsilon_i}\}$ is a sequence of critical points for E_{p,ϵ_i} with $\epsilon_i \rightarrow 0$ and $\sup_i E_{p,\epsilon_i}(u_{\epsilon_i}) < \infty$, then a subsequence of $\{u_{\epsilon_i}\}$ converges strongly in $W^{1,p}(M, \mathbb{R}^L)$ to a stationary p -harmonic map $u \in W^{1,p}(M, N)$.*

As a consequence, for $p \in (1, n) \setminus \mathbb{N}$, the functionals $E_{p,\epsilon}$ are naturally suited to the construction of stationary p -harmonic maps via min-max methods, in light of the following elementary facts.

Lemma 10.2. *The generalized Ginzburg-Landau energy $E_{p,\epsilon}$ is a C^1 functional on $W^{1,p}(M, \mathbb{R}^L)$, with derivative*

$$\langle E'_{p,\epsilon}(u), v \rangle = \int_M p \langle |du|^{p-2} du, dv \rangle + \epsilon^{-p} \langle DF(u), v \rangle,$$

and satisfies the following Palais-Smale condition: if $u_j \in W^{1,p}(M, \mathbb{R}^L)$ is a sequence satisfying

$$\sup_j \|u_j\|_{W^{1,p}} \leq C < \infty \tag{10.1}$$

and

$$\lim_{j \rightarrow \infty} \|E'_{p,\epsilon}(u_j)\|_{(W^{1,p})^*} = 0, \tag{10.2}$$

then $\{u_j\}$ has a subsequence that converges strongly in $W^{1,p}$.

Proof. The first statement is trivial. The proof of the Palais-Smale condition is also quite standard, but we include it for completeness:

For a sequence $\{u_j\}$ satisfying (10.1), we know from Rellich's theorem that a subsequence (which we continue to denote by $\{u_j\}$) converges weakly in $W^{1,p}$ and strongly in L^p to a limiting function $u \in W^{1,p}$. To confirm that the convergence is also strong in $W^{1,p}$, it is enough to show that

$$\|du\|_{L^p} \geq \limsup_j \|du_j\|_{L^p}. \tag{10.3}$$

And indeed, if the $\{u_j\}$ also satisfies (10.2), then we see that

$$\begin{aligned}
0 &= \lim_{j \rightarrow \infty} \|u_j - u\|_{W^{1,p}} \|E'_{p,\epsilon}(u_j)\|_{(W^{1,p})^*} \\
&\geq \limsup_{j \rightarrow \infty} \langle E'_{p,\epsilon}(u_j), u_j - u \rangle \\
&= \limsup_{j \rightarrow \infty} p \int_M (|du_j|^p - \langle |du_j|^{p-2} du_j, du \rangle) \\
&\quad + \limsup_{j \rightarrow \infty} \int_M \langle DF(u_j), u_j - u \rangle \\
(\text{ since } u_j \rightarrow u \text{ strongly in } L^p) &= p \limsup_{j \rightarrow \infty} \int_M (|du_j|^p - \langle |du_j|^{p-2} du_j, du \rangle) \\
&\geq \limsup_{j \rightarrow \infty} \|du_j\|_{L^p}^p - \|du_j\|_{L^p}^{p-1} \|du\|_{L^p},
\end{aligned}$$

from which (10.3) follows, completing the proof. $\square$

In the remainder of this chapter, we employ a simple min-max construction for the energies $E_{p,\epsilon}$ for $N = S^1$ and $p \in (1, 2)$, with arguments very similar to those of Chapter 5, to establish the existence of nontrivial families satisfying the hypotheses of Theorem 2.1. Though we focus here on p -harmonic maps to S^1 with $p \in (1, 2)$, we remark that similar constructions can be used to produce nontrivial stationary p -harmonic maps from manifolds of dimension $n \geq k$ into arbitrary targets N with $\pi_{k-1}(N) \neq \emptyset$, for $p \in (1, k) \setminus \mathbb{N}$.

10.2 A Saddle Point Construction and Upper Bounds on the Energies

We restrict ourselves now to the case where our target manifold N is S^1 , embedded in $\mathbb{R}^2$ as the boundary of the unit disk $D_1(0)$, and consider the collection $\Gamma_p(M)$ of two-parameter families $y \mapsto h_y \in W^{1,p}(M, \mathbb{R}^2)$ given by

$$\Gamma_p(M) := \{h \in C^0(D_1^2, W^{1,p}(M, \mathbb{R}^2)) \mid h_y \equiv y \text{ for } y \in S^1\}. \quad (10.4)$$

For $p \in (1, 2)$ and $\epsilon > 0$, we define the min-max energy levels $c_{p,\epsilon}$ by

$$c_{p,\epsilon}(M) := \inf_{h \in \Gamma_p(M)} \max_{y \in D_1^2} E_{p,\epsilon}(h_y), \quad (10.5)$$

and the limiting energy levels

$$c_p(M) := \sup_{\epsilon > 0} c_{p,\epsilon}(M) = \lim_{\epsilon \rightarrow 0} c_{p,\epsilon}(M). \quad (10.6)$$

Now, we've observed that $E_{p,\epsilon}$ is a C^1 functional on $W^{1,p}(M, \mathbb{R}^2)$, which evidently vanishes on the circle of constant maps to S^1 . Thus, if we can show that $c_{p,\epsilon}(M) > 0$, then we can apply standard results on the existence of min-max critical points (see, in particular, Chapter 4 and Corollary 5.13 in [34]) to conclude that for any minimizing sequence of families

$$h^j \in \Gamma_p(M), \quad \max_{y \in D_1^2} E_{p,\epsilon}(h_y^j) \rightarrow c_{p,\epsilon}(M),$$

there exists a sequence $v_j \in W^{1,p}(M, \mathbb{R}^2)$ for which

$$\lim_{j \rightarrow \infty} E_{p,\epsilon}(v_j) = c_{p,\epsilon}(M), \quad \lim_{j \rightarrow \infty} \|E'_{p,\epsilon}(u)\|_{(W^{1,p})^*} = 0, \quad (10.7)$$

and

$$\lim_{j \rightarrow \infty} \text{dist}_{W^{1,p}}(v_j, h^j(D_1^2)) = 0. \quad (10.8)$$

Moreover, since we can deform any minimizing sequence of families h^j to one whose maps take values in the unit disk by applying a retraction (similar to arguments in [38] and Chapter 5), we can obtain in this way a sequence v_j satisfying (10.7) and a uniform bound

$$\sup_j \|v_j\|_{W^{1,p}} < \infty.$$

In particular, it will then follow from Lemma 10.2 that there is indeed a critical point $u_{p,\epsilon}$ of $E_{p,\epsilon}$ with energy

$$E_{p,\epsilon}(u_{p,\epsilon}) = c_{p,\epsilon}(M).$$

To check that $c_{p,\epsilon}(M) > 0$, we follow arguments like those of Section 5.1. Namely, for any $h \in \Gamma_p(M)$, we note that the averaging map

$$D_1^2 \ni y \mapsto \frac{1}{\text{Vol}(M)} \int_M h_y \in \mathbb{R}^2$$

defines a continuous map from $D_1^2 \rightarrow \mathbb{R}^2$ which restricts to the identity on S^1 , so that by elementary degree theory, there must be some $y_0 \in D_1^2$ for which $\int_M h_{y_0} = 0$. Now, the L^p Poincaré inequality

gives us a constant $C_p(M)$ such that

$$\int_M |v|^p \leq C_p(M) \int_M |dv|^p$$

whenever $\int_M v = 0$, so by the preceding observation, for any $h \in \Gamma_p(M)$, there is some $y_0 \in D_1^2$ for which $v = h_{y_0}$ satisfies

$$\begin{aligned} E_{p,\epsilon}(v) &\geq C_p(M)^{-1} \int_M |v|^p + \int_M \epsilon^{-p} F(v) \\ &= C_p(M)^{-1} \int_{\{|v| \geq \frac{1}{2}\}} |v|^p + \int_{\{|v| \leq \frac{1}{2}\}} \epsilon^{-p} F(v) \\ &\geq C_p(M)^{-1} 2^{-p} \text{Vol}(\{|v| \geq \frac{1}{2}\}) + \epsilon^{-p} \text{Vol}(\{|v| < \frac{1}{2}\}) \delta_0, \end{aligned}$$

where we've set $\delta_0 := \min\{F(y) \mid |y| \leq \frac{1}{2}\} > 0$. In particular, since

$$\text{Vol}(\{|v| \geq \frac{1}{2}\}) + \text{Vol}(\{|v| < \frac{1}{2}\}) = \text{Vol}(M),$$

it follows that, for $\epsilon < 1$,

$$\max_{y \in D_1^2} E_{p,\epsilon}(h_y) \geq E_{p,\epsilon}(h_{y_0}) \geq \delta(p, M) > 0$$

for any family $h \in \Gamma_p(M)$. Taking the infimum over $h \in \Gamma_p(M)$, we confirm that

$$c_{p,\epsilon}(M) \geq \delta(p, M) > 0.$$

In summary, we've so far established the following simple result.

Lemma 10.3. *For $p \in (1, 2)$ and $\epsilon \in (0, 1)$, there exists a critical point $u_{p,\epsilon} \in W^{1,p}(M, \mathbb{R}^2)$ of $E_{p,\epsilon}$ satisfying*

$$E_{p,\epsilon}(u_{p,\epsilon}) = c_{p,\epsilon}(M) \geq \delta(p, M) > 0.$$

Next, we will establish an upper bound for the limiting energy levels $c_p(M) = \sup_{\epsilon > 0} c_{p,\epsilon}(M)$: namely, we show that

Claim 10.4. *There exists $C(M) < \infty$ independent of p such that*

$$c_p(M) \leq \frac{C(M)}{2-p}. \tag{10.9}$$

The finiteness of $c_p(M)$ will then allow us to apply Theorem 10.1 to deduce the existence of a

corresponding stationary p -harmonic map, satisfying an energy bound of the form $E_p(u_p) = O(\frac{1}{2-p})$.

Proof. Again, the proof is very close to that of the analogous statement in Chapter 5, albeit somewhat simpler, since in this case we can produce a single family h lying in $\Gamma_p(M)$ for every $p \in (1, 2)$ and satisfying a bound

$$\max_{y \in D_1^2} E_{p,\epsilon}(h_y) \leq \frac{C(M)}{2-p} \quad (10.10)$$

of the desired form.

For $y \in D_1^2 \setminus S^1$, define $v_y \in \bigcap_{p \in [1,2)} W^{1,p}(\mathbb{R}^2, S^1)$ by

$$v_y(z) = \frac{z + (1 - |y|)^{-1}y}{|z + (1 - |y|)^{-1}y|}, \quad (10.11)$$

and set $v_y \equiv y$ for $y \in S^1$. Fix also a triangulation of M^n —that is, choose a bi-Lipschitz map $\Phi : M \rightarrow |\mathcal{K}|$ from M to the underlying space of a simplicial complex $\mathcal{K}$ in some Euclidean space $\mathbb{R}^L$. Applying a generic rotation, we can arrange that the projection map P from $\mathbb{R}^L$ to the plane $\mathbb{R}^2 \times 0$ has full rank on the n -dimensional subspace parallel to each n -dimensional simplex $\Delta \in \mathcal{K}$. Denoting by $f \in Lip(M, \mathbb{R}^2)$ the composition

$$f(x) = (\Phi^1(x), \Phi^2(x))$$

of the projection $P : \mathbb{R}^L \rightarrow \mathbb{R}^2 \times 0$ with $\Phi : M \rightarrow \mathbb{R}^L$, we define the family

$$h_y := v_y \circ f. \quad (10.12)$$

Our task now is to show that $y \mapsto h_y$ defines a continuous map $D_1^2 \rightarrow W^{1,p}(M, \mathbb{R}^2)$, satisfying (10.10). First, since Φ is bi-Lipschitz and $\mathcal{K}$ is finite, we observe that it is enough to establish this for the family

$$F_y := v_y \circ P|_{\Delta}$$

on each n -dimensional face $\Delta \in \mathcal{K}$. And since we've also chosen $\mathcal{K}$ such that the restriction P_{Δ} of the projection map to Δ has full rank, we can write

$$P|_{\Delta} = P_0 \circ L,$$

where $L : \Delta \rightarrow \Delta' \subset \mathbb{R}^n$ is an invertible affine-linear map, and $P_0 : \mathbb{R}^n \rightarrow \mathbb{R}^2$ is simply the projection

$$P_0(x^1, \dots, x^n) = (x^1, x^2)$$

onto the first two coordinates. In particular, it is enough to show that on a bounded domain $\Omega \subset \mathbb{R}^n$, the family

$$y \mapsto F_y(x) = v_y(x^1, x^2)$$

is continuous in $W^{1,p}$ for each $p \in (1, 2)$, and satisfies

$$\max_{y \in D_1^2} E_{p,\epsilon}(F_y) \leq \frac{C_\Omega}{2-p}. \quad (10.13)$$

This is straightforward. By direct computation, the energy $E_{p,\epsilon}$ of F_y on Ω satisfies

$$\begin{aligned} E_{p,\epsilon}(F_y) &= \int_{(z,x') \in (\mathbb{R}^2 \times \mathbb{R}^{n-2}) \cap \Omega} |z + (1 - |y|)^{-1}y|^{-p} dz dx' \\ &\leq \int_{x' \in P_0(\Omega)} \frac{2\pi}{2-p} \text{diam}(\Omega)^{2-p} dx' \\ &\leq \frac{C_n \text{diam}(\Omega)^{n-p}}{2-p}. \end{aligned}$$

Thus, (10.13) holds, and we see moreover that the energy $y \mapsto E_{p,\epsilon}(F_y)$ varies continuously in y . Since the family $y \mapsto F_y$ is obviously weakly continuous in $W^{1,p}(\Omega, S^1)$, it follows that $y \mapsto F_y$ is strongly continuous as well.

We conclude finally that the families $y \mapsto h_y$ defined by (10.12) indeed belong to $\Gamma_p(M)$, and satisfy

$$\max_{y \in D_1^2} E_{p,\epsilon}(h_y) \leq \frac{C(M)}{2-p},$$

where the constant $C(M)$ is determined by our choice of triangulation $\Phi : M \rightarrow |\mathcal{K}|$. In particular, it follows that

$$c_{p,\epsilon}(M) \leq \frac{C(M)}{2-p}$$

for every $\epsilon > 0$, and taking the supremum over $\epsilon > 0$, we therefore have

$$c_p(M) \leq \frac{C(M)}{2-p}, \quad (10.14)$$

as desired. □

Since $c_p(M) < \infty$, we can apply Theorem 10.1 to the min-max critical points $u_{p,\epsilon}$ of Lemma 10.3, to conclude the following.

Proposition 10.5. *On every closed Riemannian manifold M^n of dimension $n \geq 2$, there exists for every $p \in (1, 2)$ a stationary p -harmonic map $u_p \in W^{1,p}(M, S^1)$ to S^1 of energy*

$$0 < E_p(u) = c_p(M) \leq \frac{C(M)}{2-p}.$$

10.3 Lower Bounds on the Energies

Finally, we argue much as in Section 5.3 to show that the p -energies $E_p(u_p) = c_p(M)$ of these p -harmonic maps are bounded below by $\frac{1}{2-p}$ as $p \rightarrow 2$; that is, we find some $c(M) > 0$ such that

$$c_p(M) \geq \frac{c(M)}{2-p}. \quad (10.15)$$

As in Section 5.3, we begin by observing that (10.15) holds for the round sphere. As discussed in the proof of Proposition 8.1, since $b_1(S^n) = 0$, every nontrivial weakly p -harmonic map $u \in W^{1,p}(S^n, S^1)$ must have singularities. In particular, the stationary p -harmonic maps u_p of energy $c_p(M) > 0$ constructed above must have nontrivial singular set, and from Proposition 8.1 and monotonicity, it indeed follows that

$$\liminf_{p \rightarrow 2} (2-p)c_p(S^n) = \liminf_{p \rightarrow 2} (2-p)E_p(u_p) > 0. \quad (10.16)$$

In fact, by a careful application of (8.1) and monotonicity, one easily checks that

$$\lim_{p \rightarrow 2} (2-p)c_p(S^n) \geq 2\pi|S^{n-2}|,$$

and it is not hard to see that equality holds, by constructing suitable families in $\Gamma_p(M)$.

The estimate for arbitrary M^n is then an easy consequence of the following claim. (Again, compare Section 5.3.)

Claim 10.6. *There is a constant $C(M^n) < \infty$ such that*

$$c_{p,\epsilon}(S^n) \leq C(M)c_{p,\epsilon}(M) \quad (10.17)$$

for every $p \in (1, 2)$ and $\epsilon > 0$.

Proof. We will construct a bounded linear map $\Phi : W^{1,p}(M, \mathbb{R}^2) \rightarrow W^{1,p}(S^n, \mathbb{R}^2)$ that fixes the constant maps and satisfies

$$E_{p,\epsilon}(\Phi(u)) \leq C(M)E_{p,\epsilon}(u)$$

for all $u \in W^{1,p}(M, \mathbb{R}^2)$ and $p \in (1, 2)$. For any family $h \in \Gamma_p(M)$, we then see that $\Phi \circ h$ defines a family in $\Gamma_p(S^n)$, so that

$$c_{p,\epsilon}(S^n) \leq \max_y E_{p,\epsilon}(\Phi(h_y)) \leq C(M) \max_y E_{p,\epsilon}(h_y),$$

and taking the infimum over $h \in \Gamma_p(M)$ gives (10.17).

We construct this map Φ as follows. First, denote by S_+^n the northern hemisphere $S_+^n = \{(x^1, \dots, x^{n+1}) \in S^n \mid x^{n+1} \geq 0\}$, and consider the reflection map

$$R : W^{1,p}(S_+^n, \mathbb{R}^2) \rightarrow W^{1,p}(S^n, \mathbb{R}^2)$$

given by

$$(Ru)(x^1, \dots, x^{n+1}) = u(x^1, \dots, x^n, |x^{n+1}|).$$

R is clearly a bounded linear map which fixes the constants, and has the effect of doubling $E_{p,\epsilon}$ —i.e.,

$$E_{p,\epsilon}(Ru, S^n) = 2E_{p,\epsilon}(u, S_+^n)$$

for every $u \in W^{1,p}(S_+^n)$.

Next, since S_+^n is a topological ball, we can choose some smooth $f : S_+^n \hookrightarrow M$ which is a diffeomorphism onto its image. Fixing such an f , we see that the pullback map

$$P_f : W^{1,p}(M, \mathbb{R}^2) \rightarrow W^{1,p}(S_+^n, \mathbb{R}^2)$$

given by

$$(P_f u) = u \circ f$$

is another bounded linear map that fixes the constant maps, and satisfies

$$E_{p,\epsilon}(P_f u, S_+^n) \leq C(M)E_{p,\epsilon}(u, M).$$

In particular, taking $\Phi := R \circ P_f$ gives a map $\Phi : W^{1,p}(M, \mathbb{R}^2) \rightarrow W^{1,p}(S^n, \mathbb{R}^2)$ satisfying the desired

properties, confirming the claim. □

Finally, taking the supremum over $\epsilon > 0$ in (10.17), we see that

$$c_p(S^n) \leq C(M)c_p(M).$$

Combining this with (10.16), it follows finally that

$$\liminf_{p \rightarrow 2} (2-p)c_p(M) \geq C(M)^{-1} \liminf_{p \rightarrow 2} (2-p)c_p(S^n) > 0, \quad (10.18)$$

as desired.

Part IV

Mountain Pass Energies Between Homotopy Classes of Maps

Chapter 11

Preliminaries From the Analysis of Sobolev Maps and a Proof of Theorem 3.1

In this chapter, we recall some key constructions and important estimates from [83], [84], and [41], and explain how the conclusion of Theorem 3.1 follows from the arguments of Hang and Lin in [41].

11.1 Cubeulations, Slicing, and Retraction to Skeleta

Throughout Part IV, we will make repeated use of the fact that any compact Riemannian manifold M admits a cubeulation (see, for instance, [83]). That is, we can find an n -dimensional cubical complex K whose n -faces are isometric to $[-1, 1]^n$, and a bi-Lipschitz map $h : |K| \rightarrow M$ from the underlying space $|K|$ onto M . A key tool in the study of Sobolev maps (see, e.g., [83], [84], [11], [41]) is the method of “slicing” a given Sobolev function f by the skeleta of a well-chosen cubeulation $h : |K| \rightarrow M$, in such a way that the composition $f \circ h$ is well-behaved on every lower-dimensional skeleton $|K^j|$ of K .

To make this idea precise, we review some notation from [41]. For an n -dimensional cubical complex K and a Lipschitz map $f \in Lip(|K|, \mathbb{R})$, define the $\mathcal{W}^{1,p}(K)$ norm of f by

$$\|f\|_{\mathcal{W}^{1,p}(K)}^p := \sum_{\sigma \in K} \int_{\sigma} (|f|^p + |df|^p) d\mathcal{H}^{\dim(\sigma)}, \quad (11.1)$$

and denote by $\mathcal{W}^{1,p}(K, \mathbb{R})$ the completion of $Lip(|K|, \mathbb{R})$ with respect to this norm. Note that the functions in $\mathcal{W}^{1,p}(K, \mathbb{R})$ then lie in the usual Sobolev space $W^{1,p}(\Delta, \mathbb{R})$ for every cell $\Delta \in K$ of any dimension. Following arguments from Section 3 of [41] (see also the proof of Lemma 2.2 in [40]), we obtain the following slicing lemma for Sobolev functions.

Lemma 11.1. *Given a closed Riemannian manifold M , there exists a constant $C(M) < \infty$ and for each $\delta \in (0, 1]$, there exists a cubical complex K_δ whose n -cells are isometric to $[-\delta, \delta]^n$, such that for any $f \in W^{1,p}(M, \mathbb{R})$, we can find a cubeulation $h : |K_\delta| \rightarrow M$ for which $f \circ h \in \mathcal{W}^{1,p}(K_\delta, \mathbb{R})$, satisfying*

$$Lip(h) + Lip(h^{-1}) \leq C, \quad (11.2)$$

$$\int_{|K_\delta^j|} |f \circ h|^p d\mathcal{H}^j \leq C \delta^{j-n} \int_M |f|^p, \quad (11.3)$$

and

$$\int_{|K_\delta^j|} |d(f \circ h)|^p d\mathcal{H}^j \leq C \delta^{j-n} \int_M |df|^p \quad (11.4)$$

for every $0 \leq j \leq n$.

Remark 11.2. It is useful to note that the complexes K_δ are obtained (up to minor rescalings) by subdividing the faces of an initial unit-size complex $K = K_1$: in particular, it follows that the maximum number $\nu_j(K_\delta)$ of j -cells containing a given $(j-1)$ -cell as a face is fixed independent of δ . As a consequence, whenever we have an estimate of the form $\int_\sigma f_1 \leq \int_{\partial\sigma} f_2$ for every j -cell $\sigma \in K_\delta$, we can sum over all j -cells to obtain an estimate of the form $\int_{|K_\delta^j|} f_1 \leq C \int_{|K_\delta^{j-1}|} f_2$, where the constant C is independent of δ .

Proof. We briefly indicate how the statement follows from the arguments in Sections 3 and 4 of [41]. To begin, we fix some (piecewise) smooth cubeulation $h : |K| \rightarrow M$ of M (following, for instance, the construction in [83]), where K is a cubical complex all of whose faces are isometric to $[-1, 1]^n$, and

$$Lip(h) + Lip(h^{-1}) \leq C(M).$$

We remark that it is enough to prove Lemma 11.1 for rational scales $\delta = \frac{1}{m}$, and henceforth (as in [40]) we restrict ourselves to this case. Beginning from the initial cubeulation $K = K_1$ above, we can then subdivide each n -cell into m^n copies of $[-\delta, \delta]^n$, to obtain a new complex K_δ with the same underlying space $|K_\delta| = |K|$ as the initial one.

Now, consider an isometric embedding $M \subset \mathbb{R}^L$ into a high-dimensional Euclidean space, and fix $\epsilon(M) > 0$ such that the nearest point projection Π_M onto M is well-defined and smooth on the

$\epsilon(M)$ -neighborhood of M in $\mathbb{R}^L$. As in [41], define the map

$$H : |K| \times B_\epsilon^L(0) \rightarrow M$$

by setting

$$H(x, \xi) := \Pi_M(h(x) + \xi).$$

By choosing $\epsilon(M)$ sufficiently small, we can then arrange that

$$\|H\|_{Lip} \leq C(M), \tag{11.5}$$

and the maps

$$h_\xi := H(\cdot, \xi) : |K| \rightarrow M$$

are invertible, with

$$Lip(h_\xi) + Lip(h_\xi^{-1}) \leq C(M). \tag{11.6}$$

Moreover, we can arrange that the Jacobian determinant

$$JH_{j,\delta}(x, \xi) := \det(DH_{j,\delta}(x, \xi) \circ [DH_{j,\delta}(x, \xi)]^*)^{1/2}$$

of the restriction $H_{j,\delta} := H|_{|K_\delta^j|}$ of H to the j -skeleton K_δ^j has a uniform lower bound

$$JH_{j,\delta}(x, \xi) \geq C(M)^{-1} > 0. \tag{11.7}$$

Next, as in Section 4 of [41], fix $y \in M$, and consider the map

$$\psi : |K| \times \{\xi \in \mathbb{R}^L \mid \xi \perp T_y M, |\xi| \leq \epsilon(M)\} \rightarrow |K| \times \mathbb{R}^L$$

mapping the product of $|K|$ with the normal disk $D_\epsilon^\perp(y)$ to M at y to $|K| \times \mathbb{R}^L$ by

$$\psi(x, y) := (x, y + \xi - h(x)).$$

For any subset $A \subset |K|$, we then observe that

$$H^{-1}(y) \cap A \subset \psi(A).$$

In particular, for the skeleta $|K_\delta^j|$ of K_δ , it follows that

$$\mathcal{H}^{L-n+j}(H^{-1}(y) \cap |K_\delta^j|) \leq \mathcal{H}^{L-n+j}(\psi(|K_\delta^j| \times D_\epsilon^\perp(y))) \leq C(M)\delta^{j-n}, \quad (11.8)$$

where in the last inequality we have used the area formula for the map ψ together with the simple estimate $\mathcal{H}^j(|K_\delta^j|) \leq C(K_1)\delta^{j-n}$ (since K_δ^j comprises $C(K_1)\delta^{-n}$ j -cells of size δ).

Armed with the estimates (11.5)-(11.8), one can now employ the coarea formula and argue exactly as in Section 3 of [41] to conclude the proof of Lemma 11.1. $\square$

Now, let K_δ be an n -dimensional cubical complex as in the conclusion of Lemma 11.1. For every $j \leq n$ and $s \in [0, \delta]$, we define the map $\phi_{j,s} : |K^j| \rightarrow |K^j|$ by identifying each j -cell $\sigma \in K^j$ with $[-\delta, \delta]^j$, and setting

$$\phi_{j,s}(x) := \frac{\delta \cdot x}{\max\{s, |x|_\infty\}}.$$

(Here we use the notation $|x|_\infty := \max_{1 \leq i \leq j} |x_i|$.) The family $\phi_{j,s}$ then interpolates between the identity at $s = \delta$ and a retraction to the $(j-1)$ -skeleton at $s = 0$. For $1 < p < j$, the pullbacks $\phi_{j,s}^* f = f \circ \phi_{j,s}$ then define endomorphisms of $\mathcal{W}^{1,p}(K_\delta^j, \mathbb{R})$, whose properties we record in the following lemma. (Compare [41], Sections 4 and 5.)

Lemma 11.3. *For $1 < p < j \leq n$ and $f \in L^\infty \cap \mathcal{W}^{1,p}(K_\delta^j, \mathbb{R})$, the family $[0, \delta] \ni s \mapsto \phi_{j,s}^* f$ of pullbacks is a continuous path in $\mathcal{W}^{1,p}(K_\delta^j, \mathbb{R})$, for which*

$$\int_{|K_\delta^j|} |d(\phi_{j,s}^* f)|^p \leq C(M) \left(\frac{\delta}{j-p} \int_{|K_\delta^{j-1}|} |df|^p + \int_{|K_\delta^j|} |df|^p \right) \quad (11.9)$$

and

$$\int_{|K_\delta^j|} |f - \phi_{j,s}^* f|^p \leq C(M)\delta^p \left(\frac{\delta}{j-p} \int_{|K_\delta^{j-1}|} |df|^p + \int_{|K_\delta^j|} |df|^p \right). \quad (11.10)$$

Proof. Since the restriction of $\phi_{j,s}$ to the $(j-1)$ -skeleton $|K_\delta^{j-1}|$ is given by the identity map for all s , it is enough to show that $\phi_{j,s}^* f$ defines a continuous path in $W^{1,p}(\sigma, \mathbb{R})$ on each j -cell $\sigma \cong [-\delta, \delta]^j$, and to establish (11.9) and (11.10), it is enough to show (per Remark 11.2) that the estimates

$$\int_\sigma |d(\phi_{j,s}^* f)|^p \leq C(M) \left(\frac{\delta}{j-p} \int_{\partial\sigma} |df|^p + \int_\sigma |df|^p \right) \quad (11.11)$$

and

$$\int_\sigma |f - \phi_{j,s}^* f|^p \leq C(M)\delta^p \left(\frac{\delta}{j-p} \int_{\partial\sigma} |df|^p + \int_\sigma |df|^p \right) \quad (11.12)$$

hold on every j -cell σ .

To see this, note that, by the scale-invariance of (11.9) and (11.12), it is enough to establish continuity and the desired estimates in the case $\delta = 1$. Moreover, by virtue of the usual bi-Lipschitz correspondence

$$[-1, 1]^j \rightarrow B_1^j, \quad x \mapsto \frac{|x|_\infty}{|x|} x$$

between the unit cube and the unit ball, we can identify the maps $\phi_{j,s}^* f$ with the maps $f_s \in W^{1,p}(B_1^j, \mathbb{R})$ given by

$$f_s(x) := f(x / \max\{s, |x|\}), \quad s \in [0, 1].$$

It is then straightforward to check that

$$\int_{B_1} |df_s|^p = \int_s^1 r^{j-1-p} \left(\int_{\partial B_1} |df|_{\partial B_1}|^p \right) dr + s^{j-p} \int_{B_1} |df|^p, \quad (11.13)$$

from which (11.11) follows immediately. And since $f_s = f$ on ∂B_1 , the L^p estimate (11.12) follows from (11.11) and the L^p Poincaré inequality (see, e.g., [29], Section 4.5.2).

To see the continuity of $s \mapsto f_s$ in $W^{1,p}(B_1)$, we appeal first to the fact that $f \in L^\infty$ and the bounded convergence theorem to conclude that $s \mapsto f_s$ is a continuous path in L^p . And since we see from (11.13) that the p -energy $s \mapsto E_p(f_s)$ is a continuous function of s , f_s must give a continuous path in $W^{1,p}(B_1)$ as well.

□

Next, we define the retraction maps $\Phi_j : |K_\delta| \rightarrow |K_\delta^{j-1}|$, sending almost every point in the cubical complex into the $(j-1)$ -skeleton, by

$$\Phi_j := \phi_{j,0} \circ \cdots \circ \phi_{n,0}. \quad (11.14)$$

By definition of the radial retractions $\phi_{j,0}$, we observe that Φ_j is locally Lipschitz away from an $(n-j)$ -dimensional set, called the *dual $(n-j)$ -skeleton* L^{n-j} to K .

Note that the dual skeleton L^{n-j} can be expressed as the union (disjoint except for a set of dimension $(n-j-1)$) over all j -cells $\sigma \in K$ of the sets

$$P(\sigma) := \overline{\Phi_{j+1}^{-1}(a_\sigma)}, \quad (11.15)$$

where a_σ is the center of σ ; and each $P(\sigma)$, in turn, can be decomposed into the union of the

intersections

$$P(\sigma) \cap \Delta$$

over all n -cells Δ containing σ as a j -face. Identifying a given n -cell $\Delta \in K_\delta$ with $[-\delta, \delta]^n$ and the j -cell σ with $[-\delta, \delta]^j \times \{(\delta, \dots, \delta)\}$, we see that $P(\sigma) \cap \Delta$ is given by the box $\{0\} \times [0, \delta]^{n-j}$. Since, per Remark 11.2, the number of n -cells intersecting a given j -cell is bounded independent of δ for the cubeulations K_δ of Lemma 11.1, it follows in this case that

$$\mathcal{H}^{n-j}(P(\sigma)) \leq C(M)\delta^{n-j} \quad (11.16)$$

for every j -cell $\sigma \in K_\delta$.

Remark 11.4. Observe also that if $\Delta \in K_\delta$ is an n -face of K_δ , σ is a j -cell in $\partial\Delta$, and $f : S^{j-1} \rightarrow \Delta \setminus L^{n-j}$ is an embedding of the $(j-1)$ -sphere S^{j-1} that links with $P(\sigma)$ (e.g., sending S^{j-1} to the sphere $S_{\delta/2}^{j-1} \times \{(\delta/2, \dots, \delta/2)\}$, under the identifications $\Delta \cong [-\delta, \delta]^n$, $\sigma \cong [-\delta, \delta]^j \times \{(\delta, \dots, \delta)\}$) then f can be homotoped through $\Delta \setminus L^{n-j}$ to a homeomorphism with $\partial\sigma$.

As in [41], we will make repeated use of the retraction maps Φ_j to deform given maps in $W^{1,p}(M, N)$ to ones which are continuous away from L^{n-j} . For the arguments of Chapter 12 in particular, it will be important to have some precise estimates for the maps produced in this way, which we record in the following lemma.

Lemma 11.5. *For $f \in W^{1,p}(M, \mathbb{R})$ and $\delta \in (0, 1]$, choose a cubeulation $h : |K_\delta| \rightarrow M$ as in the conclusion of Lemma 11.1. Then for $k \in \{2, \dots, n\}$ and $1 < p < k$, the function $\tilde{f} := f|_{|K_\delta^{k-1}|} \circ \Phi_k \circ h^{-1}$ satisfies*

$$E_p(\tilde{f}) \leq \frac{C(M)}{k-p} E_p(f) \quad (11.17)$$

and

$$\int_M |f - \tilde{f}|^p \leq \delta^p \frac{C(M)}{k-p} E_p(f). \quad (11.18)$$

Proof. Since K_δ is chosen according to Lemma 11.1, we know that $f \in \mathcal{W}^{1,p}(K_\delta, \mathbb{R})$, and

$$\int_{|K_\delta^j|} |df|^p \leq C(M)\delta^{j-n} E_p(f)$$

for every $0 \leq j \leq n$. Since $\tilde{f} = f \circ \phi_{k,0}$ on the k -skeleton $|K^k|$, we can then apply the estimates of

Lemma 11.3 to conclude that

$$\begin{aligned} \int_{|K_\delta^k|} |d\tilde{f}|^p &\leq C(M) \left(\frac{\delta}{k-p} \cdot \delta^{k-1-n} E_p(f) + \delta^{k-n} E_p(f) \right) \\ &\leq C'(M) \frac{\delta^{k-n}}{k-p} E_p(f) \end{aligned}$$

and

$$\int_{|K_\delta^k|} |f - \tilde{f}|^p \leq C'(M) \delta^p \cdot \frac{\delta^{k-n}}{k-p} E_p(f). \quad (11.19)$$

Next, we can apply the preceding estimates together with the conclusion of Lemma 11.3 on the $(k+1)$ -skeleton, to see that $\tilde{f}|_{|K^{k+1}|} = \phi_{k+1,0}^* \phi_{k,0}^* f$ satisfies

$$\begin{aligned} \int_{|K_\delta^{k+1}|} |d\tilde{f}|^p &\leq C(M) \left(\frac{\delta}{k+1-p} \cdot \frac{\delta^{k-n}}{k-p} E_p(f) + \delta^{k+1-n} E_p(f) \right) \\ &\leq C'(M) \frac{\delta^{k+1-n}}{k-p} E_p(f). \end{aligned}$$

To estimate $|f - \tilde{f}|^p$ on $|K^{k+1}|$, we again apply the scaled L^p Poincaré inequality

$$\int_\sigma |f - \tilde{f}|^p \leq C \left(\delta^p \int_\sigma |d(f - \tilde{f})|^p + \delta \int_{\partial\sigma} |f - \tilde{f}|^p \right)$$

to every $(k+1)$ -cell $\sigma \cong [-\delta, \delta]^{k+1}$ in K_δ , and sum over σ (again appealing to Remark 11.2), to obtain

$$\begin{aligned} \int_{|K_\delta^{k+1}|} |f - \tilde{f}|^p &\leq C(M) \left(\delta^p \int_{|K^{k+1}|} (|df|^p + |d\tilde{f}|^p) + \delta \int_{|K_\delta^k|} |f - \tilde{f}|^p \right) \\ &\leq C'(M) \left(\delta^p [\delta^{k+1-n} E_p(f) + \frac{\delta^{k+1-n}}{k-p} E_p(f)] + \delta \cdot \delta^p \frac{\delta^{k-n}}{k-p} E_p(f) \right). \end{aligned}$$

In particular, we conclude that

$$\int_{|K_\delta^{k+1}|} |d\tilde{f}|^p \leq C(M) \frac{\delta^{k+1-n}}{k-p} E_p(f)$$

and

$$\int_{|K_\delta^{k+1}|} |f - \tilde{f}|^p \leq C(M) \delta^p \frac{\delta^{k+1-n}}{k-p} E_p(f).$$

Carrying on by induction on j , for each j -skeleton $|K_\delta^j|$ with $j \geq k$, we find that

$$\int_{|K_\delta^j|} |d\tilde{f}|^p \leq \frac{C}{k-p} \delta^{j-n} E_p(f)$$

and

$$\int_{|K_\delta^j|} |f - \tilde{f}|^p \leq \frac{C}{k-p} \delta^p \delta^{j-n} E_p(f)$$

for every $k \leq j \leq n$; in particular, taking $j = n$, we obtain the desired estimates for $\tilde{f}$.

□

11.2 Proof of Theorem 3.1

In this short section, we recall the construction of [41] (cf. also [18] in the case that either u or v is constant), and explain how it leads immediately to a proof of Theorem 3.1.

Proposition 11.6. (cf. [41]) *Let $u, v \in C^\infty(M, N)$ be $(k-2)$ -homotopic for some $k \leq n = \dim(M)$. Then there is a path of maps $t \mapsto u_t$ with $u_0 = u$, $u_1 = v$, continuous in $W^{1,p}(M, N)$ for every $1 \leq p < k$, such that*

$$\sup_{t \in [0,1]} E_p(u_t) \leq \frac{C}{k-p} \quad (11.20)$$

for some C independent of p .

Proof. To begin, fix a smooth cubeulation $h : |K| \rightarrow M$, where K is a cubical complex built of n -cells isometric to $[-1, 1]^n$. In what follows, we will frequently identify M and $|K|$ without comment. Since we've taken u and v to be smooth, note that the restrictions $u|_{|K^j|}$ and $v|_{|K^j|}$ of u and v to the lower-dimensional skeleta of K define Lipschitz maps from $|K^j|$ to N .

Recalling the terminology of the previous section, we observe now that there exists a path of maps u_t connecting u to $u \circ \Phi_k$, such that $t \mapsto u_t$ is continuous in $W^{1,p}(M, N)$ for each $p < k$, with the desired energy bounds. Indeed, it follows directly from Lemma 11.3 that the path

$$[0, 1] \ni s \mapsto u \circ \phi_{n,s}$$

connecting $u \circ \Phi_n$ to u satisfies the desired properties, as do the paths

$$[0, 1] \ni s \mapsto u \circ \phi_{j,s} \circ \Phi_{j+1}$$

connecting $u \circ \Phi_{j+1}$ to $u \circ \Phi_j$ for each $k \leq j \leq n-1$. Concatenation yields the desired path u_t from u to $u \circ \Phi_k$, and in the same way we can construct such a path connecting v to $v \circ \Phi_k$.

It remains now to construct a path of maps u_t from M to N connecting $u \circ \Phi_k$ to $v \circ \Phi_k$, in such a way that $t \mapsto u_t$ is continuous in $W^{1,p}(M, N)$ and

$$\max_t E_p(u_t) \leq \frac{C}{k-p}$$

for every $1 \leq p < k$. In fact, it is enough to construct such a path of maps

$$w_t : |K^k| \rightarrow N$$

connecting $u \circ \phi_{k,0}$ to $v \circ \phi_{k,0}$ on the k -skeleton $|K^k|$, since we can then take $u_t := w_t \circ \Phi_{k+1}$ to obtain the desired path of maps on M . In the remainder of the proof, we construct such a path $w_t : |K^k| \rightarrow N$.

Since the maps u and v are $(k-2)$ -homotopic, their restrictions $u|_{|K^{k-2}|}$, $v|_{|K^{k-2}|}$ to the $(k-2)$ -skeleton $|K^{k-2}|$ are homotopic, by definition. And since the pair $(|K^{k-1}|, |K^{k-2}|)$ satisfies the homotopy extension property (cf. [41], Proposition 2.1), we can therefore find a map

$$u_2 : |K^{k-1}| \rightarrow N$$

such that $u|_{|K^{k-1}|}$ is homotopic to u_2 on $|K^{k-1}|$ and u_2 agrees with v

$$u_2|_{|K^{k-2}|} = v|_{|K^{k-2}|}$$

on $|K^{k-2}|$. Moreover, it's easy to check (cf. [41], Sections 2.2-2.3) that we can take both the map u_2 and the homotopy $f : |K^{k-1}| \times [0, 1] \rightarrow N$ from $u|_{|K^{k-1}|}$ to u_2 to be Lipschitz. The precomposition $f_t \circ \Phi_k$ of the homotopy f_t with Φ_k then gives us a path of maps $M \rightarrow N$ connecting $u \circ \Phi_k$ to $u_2 \circ \Phi_k$, which evidently satisfies the desired estimates and continuity properties in $W^{1,p}(M, N)$ for $1 \leq p < k$.

In particular, to complete the proof of the proposition, we now see that it suffices to construct a path of maps $w_t : |K^k| \rightarrow N$, continuous in $W^{1,p}(|K^k|, N)$ for $1 \leq p < k$, satisfying

$$\max_t E_p(w_t, |K^k|) \leq \frac{C}{k-p},$$

that connects $v \circ \phi_{k,0}$ to $u_2 \circ \phi_{k,0}$, where $u_2 \in Lip(|K^{k-1}|, N)$ agrees with v on the $(k-2)$ -skeleton $|K^{k-2}|$. To do this, we enumerate the $(k-1)$ -cells $\sigma_1, \dots, \sigma_m \in K^{k-1} \setminus K^{k-2}$, and define maps $w_0, \dots, w_m \in W^{1,p}(|K^k|, N)$ by

$$w_i := f_i \circ \phi_{k,0},$$

where the maps $f_i \in Lip(|K^{k-1}|, N)$ are defined by $f_0 = v|_{|K^{k-1}|}$, $f_m = u_2$, and

$$f_i(x) := v(x) \text{ for } x \in |K^{k-1}| \setminus (\sigma_1 \cup \dots \cup \sigma_i),$$

$$f_i(x) := u_2(x) \text{ for } x \in \sigma_1 \cup \dots \cup \sigma_i,$$

for $1 \leq i \leq m$. (That these f_i are Lipschitz follows from the fact that $u_2 = v$ on $|K^{k-2}|$.) We claim that each w_i can be deformed into w_{i+1} through a path of maps w_t satisfying the desired properties; once we have constructed these paths, concatenation evidently gives the desired path from $v \circ \phi_{k,0}$ to $u_2 \circ \phi_{k,0}$.

Now, fix $i \in \{1, \dots, m\}$. By construction, the maps $f_{i-1}, f_i \in Lip(|K^{k-1}|, N)$ coincide on the complement $|K^{k-1}| \setminus \sigma_i$ of the $(k-1)$ -cell σ_i . Consider the star neighborhood

$$V := \bigcup \{ \Delta \in K^k \text{ a } k\text{-cell} \mid \sigma_i \subset \partial \Delta \}$$

of σ_i , which we can identify in a bi-Lipschitz way with

$$W = \bigcup_{j=1}^a W_j \subset \mathbb{R}^{k-1+a},$$

where

$$W_j := \{(x, 0, \dots, 0) + te_{k-1+j} \mid x \in [-1, 1]^{k-1}, 0 \leq t \leq 1\},$$

and a is simply the number of distinct k -cells for which

$$\sigma \cong [-1, 1]^{k-1} \times \{0\} \subset \mathbb{R}^{k-1+a}$$

is a face.

Next, note that the boundary

$$\partial V \cong W \cap \partial[-1, 1]^{k-1+a}$$

lies in $|K^{k-1}| \setminus \sigma_i$, so that the maps

$$f_i = f_{i-1} =: g \in Lip(\partial V, N)$$

agree on ∂V . For $t \in [0, \frac{1}{2}]$, we can then define maps $w_{i-1+t} : |K^k| \rightarrow N$ by setting

$$w_{i-1+t} := w_{i-1} = w_i \text{ on } |K^k| \setminus V,$$

and (identifying V with W)

$$w_{i-1+t}(x) := w_{i-1} \left(\frac{x}{\max\{(1-2t), |x|_\infty\}} \right) \text{ for } x \in V.$$

We can then check by direct computation, as in the proof of Lemma 11.3, that $[0, \frac{1}{2}] \ni t \mapsto w_{i-1+t}$ satisfies the desired energy estimates and continuity properties, while connecting w_{i-1} to the map $w_{i-0.5}$ given by

$$w_{i-0.5} := w_i \text{ on } |K^k| \setminus V$$

and

$$w_{i-0.5}(x) := g(x/|x|_\infty) \text{ for } x \in V.$$

Since $w_i|_{\partial V} = g$ as well, we can employ the same construction to obtain a path

$$[\frac{1}{2}, 1] \ni t \mapsto w_{i-1+t}$$

connecting $w_{i-0.5}$ to w_i in the desired way. We have thereby constructed a path $[0, 1] \ni t \mapsto w_{i-1+t} : |K^k| \rightarrow N$ from w_{i-1} to w_i satisfying the desired estimates, completing the proof. $\square$

Chapter 12

Analysis of Homological Singularities

In this chapter, we define the homological singularities of Sobolev maps, and develop a compactness theory for these objects as the Sobolev exponent p approaches integer thresholds, establishing some key lemmas needed for the analysis in Chapter 13.

12.1 Definitions and Basic Estimates

In this section, we define the homological singularities of maps in $W^{1,p}(M, N)$ associated to real cohomology classes in the target manifold N . All of the results of this section are contained, with slightly different terminology, in Section 5.4.2 of [36], but we opt for a self-contained treatment more directly suited to the purposes of Chapter 13.

First, we fix some notation from the theory of currents (see [31], [35], or [73] for an introduction). Denote by $\Omega^m(M)$ the space of smooth m -forms on a compact manifold M , and by $\Omega_c^m(U)$ the space of compactly supported m -forms in an open set U . Following [35], we use $L_m^q(M)$ to denote the closure of $\Omega^m(M)$ with respect to the L^q norm. The space of general m -currents (linear functionals on $\Omega^m(M)$) will be denoted by $\mathcal{D}_m(M)$, and for each $T \in \mathcal{D}_m(M)$, following [73], we define the mass $\mathbb{M}(T)$ by

$$\mathbb{M}(T) := \sup\{\langle T, \zeta \rangle \mid \zeta \in \Omega^m(M), \|\zeta\|_{L^\infty} \leq 1\}.$$

We use $\mathcal{I}_m(M; \mathbb{Z})$ to denote the space of integer rectifiable m -currents, and $\mathcal{Z}_m(M; \mathbb{Z})$ for the subspace of integral m -cycles.

Now, let N be a closed, oriented Riemannian manifold. For every integer $1 \leq m \leq \dim N$, denote by $\mathcal{A}^m(N)$ the collection of closed m -forms on N satisfying

$$\langle \Sigma, \alpha \rangle \in \mathbb{Z} \text{ for every } \Sigma \in \mathcal{Z}_m(N; \mathbb{Z}). \quad (12.1)$$

Observe that the image of $\mathcal{A}^m(N)$ in de Rham cohomology defines a lattice of full rank in $H_{dR}^m(N)$. Indeed, given integral m -cycles $\Sigma_1, \dots, \Sigma_q$ in N generating $H_m(N; \mathbb{R})$, we can find corresponding cohomology classes $[\alpha_1], \dots, [\alpha_m] \in H_{dR}^m(N)$ for which $\langle \Sigma_i, \alpha_j \rangle = \delta_{ij}$ (see, e.g., [35], Section 5.4.1). These α_i evidently lie in $\mathcal{A}^m(N)$, and give a basis for $H_{dR}^m(N)$.

We now fix some $k \in \{2, \dots, \dim N + 1\}$, and $\alpha \in \mathcal{A}^{k-1}(N)$. Appealing to Nash's embedding theorem, we also fix an isometric embedding $N \subset \mathbb{R}^L$ of N into some Euclidean space. We can then easily extend our $(k-1)$ -form α to a compactly supported form

$$\bar{\alpha} = \sum_{|I|=k-1} \bar{\alpha}_I(x) dx^I \in \Omega_c^{k-1}(\mathbb{R}^L),$$

for instance by taking the pullback $\pi_N^* \alpha$ of α to a tubular neighborhood of N by the nearest-point projection π_N , then multiplying by a suitable cut-off function.

Now, let M^n be a compact, oriented manifold, possibly with boundary. We record next some important estimates for the pullback $u^*(\bar{\alpha})$ of $\bar{\alpha}$ by smooth maps $u \in C^\infty(M, \mathbb{R}^L)$.

Lemma 12.1. *For $u, v \in C^\infty(M, \mathbb{R}^L)$, there exist a $(k-1)$ -form $\beta(u, v) \in \Omega^{k-1}(M)$ and a $(k-2)$ -form $\eta(u, v) \in \Omega^{k-2}(M)$ such that*

$$v^*(\bar{\alpha}) - u^*(\bar{\alpha}) = \beta + d\eta, \quad (12.2)$$

and the following pointwise estimates hold:

$$|v^*(\bar{\alpha}) - u^*(\bar{\alpha})| \leq C(\alpha) |u - v| (|du|^{k-1} + |dv|^{k-1}) + C(\alpha) |du - dv| (|du|^{k-2} + |dv|^{k-2}), \quad (12.3)$$

$$|\beta(u, v)| \leq C(\alpha) |u - v| (|du|^{k-1} + |dv|^{k-1}), \quad (12.4)$$

and

$$|\eta(u, v)| \leq C(\alpha) |u - v| (|du|^{k-2} + |dv|^{k-2}). \quad (12.5)$$

Remark 12.2. Here, $C(\alpha)$ denotes a constant depending on $\|\bar{\alpha}\|_{C^1}$.

Proof. Write

$$v^*(\bar{\alpha}) - u^*(\bar{\alpha}) := \Sigma_I(\bar{\alpha}_I(v)dv^I - \bar{\alpha}_I(u)du^I).$$

Fixing a multi-index $I = (i_1, \dots, i_{k-1})$, we begin by rearranging

$$\alpha_I(v)dv^I - \alpha_I(u)du^I = (\alpha_I(v) - \alpha_I(u))dv^I + \alpha_I(u)(dv^I - du^I),$$

and noting that

$$|\alpha_I(v) - \alpha_I(u)||dv^I| \leq \|\nabla \alpha\|_{L^\infty} |u - v| |dv|^{k-1}.$$

The first estimate (12.3) follows immediately, and absorbing the terms

$$(\alpha_I(v) - \alpha_I(u))dv^I$$

into $\beta(u, v)$, we see that, to complete the proof of (12.2), it suffices to exhibit a decomposition of the form (12.2) for the remaining terms $\alpha_I(u)(dv^I - du^I)$.

To this end, writing I_2 for the multi-index $(i_2, \dots, i_{k-1})$, we observe that

$$\begin{aligned} \alpha_I(u)(dv^I - du^I) &= \alpha_I(u)(d(v^{i_1} - u^{i_1}) \wedge dv^{I_2} + du^{i_1} \wedge (dv^{I_2} - du^{I_2})) \\ &= d[\alpha_I(u)(v^{i_1} - u^{i_1}) \wedge dv^{I_2}] - (v^{i_1} - u^{i_1})d(\alpha_I(u)) \wedge dv^{I_2} \\ &\quad + du^{i_1} \wedge \alpha_I(u)(dv^{I_2} - du^{I_2}). \end{aligned}$$

Now, the $(k-2)$ -form $\alpha_I(u)(v^{i_1} - u^{i_1}) \wedge dv^{I_2}$ evidently satisfies an estimate of the form (12.5), and so can be absorbed into $\eta(u, v)$, while the $(k-1)$ -form $(v^{i_1} - u^{i_1})d(\alpha_I(u)) \wedge dv^{I_2}$ can likewise be absorbed into $\beta(u, v)$. To deal with the leftover term

$$du^{i_1} \wedge \alpha_I(u)(dv^{I_2} - du^{I_2}),$$

we apply the same argument to the $(k-2)$ -form $\alpha_I(u)(dv^{I_2} - du^{I_2})$ that we did to the $(k-1)$ -form $\alpha_I(u)(dv^I - du^I)$, and carrying on in this way, we eventually arrive at the desired decomposition. $\square$

Integrating the estimates (12.3)-(12.5) of Lemma 12.1 and making liberal use of Hölder's inequal-

ity, we obtain for any $p > k - 1$ the bounds

$$\begin{aligned} \|v^*(\bar{\alpha}) - u^*(\bar{\alpha})\|_{L^1} &\leq C(\alpha)(\|du\|_{L^p}^{k-1} + \|dv\|_{L^p}^{k-1})\|u - v\|_{L^\infty}^{k-p}\|u - v\|_{L^p}^{p+1-k} \\ &\quad + (\|du\|_{L^{k-1}}^{k-2} + \|dv\|_{L^{k-1}}^{k-2})\|du - dv\|_{L^{k-1}}, \\ \|\beta(u, v)\|_{L^1} &\leq C(\alpha)(\|du\|_{L^p}^{k-1} + \|dv\|_{L^p}^{k-1})\|u - v\|_{L^\infty}^{k-p}\|u - v\|_{L^p}^{p+1-k}, \end{aligned} \quad (12.6)$$

and

$$\|\eta(u, v)\|_{L^1} \leq C(\alpha)(\|du\|_{L^{k-1}}^{k-2} + \|dv\|_{L^{k-1}}^{k-2})\|u - v\|_{L^{k-1}}. \quad (12.7)$$

For the remainder of this section, we will have $p \in (k - 1, k)$. It then follows from the estimates above that the pullback assignment

$$u \mapsto u^*(\alpha)$$

gives a well-defined, continuous map from $W^{1,p}(M^n, N)$ to the space $L_{k-1}^1(M)$ of $(k - 1)$ -forms with coefficients in L^1 . For any map $u \in W^{1,p}(M, N)$, we can, in particular, define the $(n + 1 - k)$ -current $S_\alpha(u) \in \mathcal{D}_{n+1-k}(M)$ dual to $u^*(\alpha)$ by

$$\langle S_\alpha(u), \zeta \rangle := \int_M u^*(\alpha) \wedge \zeta. \quad (12.8)$$

By virtue of (12.6) and (12.7), we then have the following decomposition lemma for the difference $S_\alpha(v) - S_\alpha(u)$.

Lemma 12.3. *For $u, v \in W^{1,p}(M, N)$, the difference $S_\alpha(v) - S_\alpha(u)$ admits a decomposition of the form*

$$S_\alpha(v) - S_\alpha(u) := S_\alpha(u, v) + \partial R_\alpha(u, v), \quad (12.9)$$

for some $R_\alpha(u, v) \in \mathcal{D}_{n-k+2}(M)$ and $S_\alpha(u, v) \in \mathcal{D}_{n+1-k}(M)$ satisfying the mass bounds

$$\mathbb{M}(S_\alpha(u, v)) \leq C(\alpha)[E_p(u)^{\frac{k-1}{p}} + E_p(v)^{\frac{k-1}{p}}]\|u - v\|_{L^p}^{1+p-k}. \quad (12.10)$$

and

$$\mathbb{M}(R_\alpha(u, v)) \leq C(\alpha)[\|du\|_{L^{k-1}}^{k-2} + \|dv\|_{L^{k-1}}^{k-2}]\|u - v\|_{L^{k-1}}. \quad (12.11)$$

For $u \in W^{1,p}(M, N)$, we now define the *homological singularity* $T_\alpha(u) \in \mathcal{D}_{n-k}(M)$ associated to

α to be the $(n - k)$ -boundary

$$\langle T_\alpha(u), \zeta \rangle := \langle \partial S_\alpha(u), \zeta \rangle = \int_M u^*(\alpha) \wedge d\zeta. \quad (12.12)$$

Homological singularities of this sort have been considered by various authors—see, for instance, [36], or [14], which studies their role as obstructions to the approximation of Sobolev maps by smooth maps. In the special case $N = S^{k-1}$, $\alpha = \frac{dvol_{S^{k-1}}}{\sigma_{k-1}}$, the current T_α coincides with the distributional Jacobian, whose geometric properties have been well studied in recent decades (see, for instance, [1], [49], and references therein).

When u is smooth on the support of an $(n - k)$ -form $\zeta \in \Omega_c^{n-k}(\overset{\circ}{M})$ supported in the interior $\overset{\circ}{M}$ of M , Stokes's theorem and the naturality of the exterior derivative give

$$\begin{aligned} \langle T_\alpha(u), \zeta \rangle &= \int u^*(\alpha) \wedge d\zeta \\ &= (-1)^{k-1} \int d(u^*(\alpha)) \wedge \zeta \\ &= 0. \end{aligned}$$

In fact, if u is continuous on an open set containing $spt(\zeta)$, then one again has

$$\langle T_\alpha(u), \zeta \rangle = 0,$$

since we can find a sequence of smooth maps $u_j \in C^\infty(M, N)$ approaching u in $W^{1,p}$ on a neighborhood of $spt(\zeta)$ (see, e.g., [11],[41]). In particular, if u is continuous away from a closed set $Sing(u) \subset M$, it follows that

$$spt(T_\alpha(u)) \subset Sing(u) \cup \partial M. \quad (12.13)$$

Next, define

$$\mathcal{E}^p(M, N) \subset W^{1,p}(M, N)$$

to be the collection of maps $u \in W^{1,p}(M, N)$ of the form

$$u = f \circ \Phi_k \circ h^{-1}$$

for some cubeulation $h : |K| \rightarrow M$ of M and some Lipschitz map $f \in Lip(|K^{k-1}|, N)$ from the $(k - 1)$ -skeleton. For such maps, the set $Sing(u)$ of discontinuities is evidently contained in the

dual $(n - k)$ -skeleton L^{n-k} to K , and the homological singularity $T_\alpha(u)$ is given by an integral $(n - k)$ -cycle that we can describe explicitly.

Proposition 12.4. (cf. [36] Section 5.4.2, Theorem 1) *If $u \in \mathcal{E}^p(M, N)$ is given by $u = f \circ \Phi_k \circ h^{-1}$ for some $f \in \text{Lip}(|K^{k-1}|, N)$ and a cubeulation $h : |K| \rightarrow M$, then for any $(n - k)$ -form $\zeta \in \Omega_c^{n-k}(\mathring{M})$ supported in the interior of M , the pairing with $T_\alpha(u)$ is given by*

$$\langle T_\alpha(u), \zeta \rangle = \sum_{\sigma \in K^k \setminus K^{k-1}} \theta(\sigma) \cdot \int_{P(\sigma)} \zeta, \quad (12.14)$$

where

$$|\theta(\sigma)| = \left| \int_{\partial\sigma} f^*(\alpha) \right|, \quad (12.15)$$

and $P(\sigma)$ is defined as in (11.15) to be the component of L^{n-k} intersecting the k -cell $\sigma \in K$.

Remark 12.5. The integrality $\theta(\sigma) \in \mathbb{Z}$ follows from the fact that $\alpha \in \mathcal{A}^{k-1}(N)$, since the pushforward $f_*(\partial\sigma)$ of the $(k - 1)$ -cycle $\partial\sigma$ by the Lipschitz map f is an integral $(k - 1)$ -cycle in N .

The proposition follows from results in Section 5.4.2 of [36], but in the interest of keeping the discussion self-contained (and because our terminology differs somewhat from that of [36]) we provide a proof below. In fact, the conclusion of Proposition 12.4 applies to a much larger collection of maps than $\mathcal{E}^p(M, N)$ —namely, any $W^{1,p}$ map which is continuous away from the dual $(n - k)$ -skeleton of some cubeulation (see [36]).

Proof. To begin, we claim that it is enough to establish (12.14) for forms $\zeta \in \Omega_c^{n-k}(\mathring{M})$ supported in the interior of a single n -face Δ of the cubeulation. To see this, denote by Ξ the $(n - k - 1)$ -dimensional intersection

$$\Xi := L^{n-k} \cap |K^{n-1}|,$$

and for $\epsilon \in (0, 1/2)$, define the cutoff functions

$$\chi_\epsilon(x) := \psi(\epsilon^{-1} \text{dist}(x, L^{n-k}))$$

and

$$\varphi_\epsilon(x) := \psi(\epsilon^{-1} \text{dist}(x, \Xi)),$$

where $\psi \in C^\infty(\mathbb{R})$ satisfies

$$\psi(t) = 0 \text{ for } t \geq 1 \text{ and } \psi(t) = 1 \text{ for } t \leq \frac{1}{2}. \quad (12.16)$$

Then χ_ϵ is supported on the ϵ -neighborhood of L^{n-k} and $\chi_\epsilon \equiv 1$ near L^{n-k} , while φ_ϵ is supported on the ϵ -neighborhood of Ξ , with $\varphi_\epsilon \equiv 1$ near Ξ .

For any $\zeta \in \Omega_c^{n-k}(M)$, it follows from (12.13) that

$$\langle T_\alpha(u), \zeta \rangle = \langle T_\alpha(u), \chi_\epsilon \zeta \rangle.$$

For $\epsilon > 0$ sufficiently small, we observe that the form

$$\chi_\epsilon \zeta - \varphi_\epsilon \zeta$$

is supported away from the $(n-1)$ -skeleton $|K^{n-1}|$, and can therefore be written as a sum of forms supported in the interiors of the n -faces Δ of K . In particular, to justify our claim that it suffices to establish (12.14) for forms ζ supported in an n -face Δ , it is enough to show that

$$\lim_{\epsilon \rightarrow 0} \langle T_\alpha(u), \varphi_\epsilon \zeta \rangle = 0. \quad (12.17)$$

To establish (12.17), we first observe that

$$\begin{aligned} |\langle T_\alpha(u), \varphi_\epsilon \zeta \rangle| &= \left| \int u^*(\alpha) \wedge d\varphi_\epsilon \wedge \zeta + \int u^*(\alpha) \wedge \varphi_\epsilon d\zeta \right| \\ &\leq C \left(\frac{1}{\epsilon} \|\zeta\|_{L^\infty} + \|d\zeta\|_{L^\infty} \right) \int_{\{dist_\Xi \leq \epsilon\}} |du|^{k-1}. \end{aligned}$$

Since $u = f \circ \Phi_k$ for $f \in Lip(|K^{k-1}|, N)$, we have almost everywhere a gradient estimate of the form

$$|du|(x) \leq C \frac{Lip(f)}{dist_{L^{n-k}}(x)},$$

and we can check by direct computation on each n -cell Δ that

$$\begin{aligned} \int_{\{dist_\Xi \leq \epsilon\}} |du|^{k-1} &\leq CLip(f)^{k-1} \int_{\{dist_\Xi \leq \epsilon\}} (dist_{L^{n-k}}(x))^{1-k} d\mathcal{H}^n(x) \\ &\leq CLip(f)^{k-1} \cdot \epsilon^2. \end{aligned}$$

Returning to our estimate for $\langle T_\alpha(u), \varphi_\epsilon \zeta \rangle$, we then see that

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} |\langle T_\alpha(u), \varphi_\epsilon \zeta \rangle| &\leq \lim_{\epsilon \rightarrow 0} C \left(\frac{1}{\epsilon} \|\zeta\|_{L^\infty} + \|d\zeta\|_{L^\infty} \right) \cdot Lip(f)^{k-1} \epsilon^2 \\ &\leq C(K, \zeta, f) \lim_{\epsilon \rightarrow 0} \epsilon, \end{aligned}$$

so (12.17) holds, and we can restrict our attention to forms ζ supported in the interior of a single n -cell Δ .

In fact, if we modify the definition of Ξ above by adding the $(n-k-1)$ -dimensional set given by the union of all intersections $P(\sigma_1) \cap P(\sigma_2)$ for distinct k -cells $\sigma_1, \sigma_2 \in K$, then the same argument shows that it is enough to establish (12.14) for ζ supported in the interior of $\Delta \cap \Phi_{k+1}^{-1}(\sigma)$ for some n -face $\Delta \in K$ and k -cell $\sigma \in K$.

Thus, identifying Δ homothetically with $I^n := [-1, 1]^n$, σ with $\{(1, \dots, 1)\} \times I^k$, and (consequently) $\Delta \cap \Phi_{k+1}^{-1}(\sigma)$ with $[0, 1]^{n-k} \times I^k$, it remains to show that for a $W^{1,p}$ map

$$u : E = (0, 1)^{n-k} \times I^k \rightarrow N$$

with

$$\text{Sing}(u) \subset [0, 1]^{n-k} \times \{0\},$$

and any $\zeta \in \Omega_c^{n-k}((0, 1)^{n-k} \times (-1, 1)^k)$, we have

$$\langle T_\alpha(u), \zeta \rangle = \theta \int_{[0, 1]^{n-k} \times \{0\}} \zeta, \quad \text{where } |\theta| = \left| \int_{\partial\sigma} u^*(\alpha) \right|. \quad (12.18)$$

Since $\partial T_\alpha(u) = 0$ and the support $\text{spt}(T_\alpha(u))$ satisfies (by (12.13))

$$\text{spt}(T_\alpha(u)) \cap E \subset [0, 1]^{n-k} \times \{0\},$$

it follows from standard constancy theorems (e.g., Theorem 2 in Section 5.3.1 of [35]) that $T_\alpha(u)$ has the form (12.18) for *some* $\theta \in \mathbb{R}$, provided that

$$\langle T_\alpha(u), \zeta \rangle = 0 \text{ for every } \zeta \in \Omega_c^{n-k}(E) \text{ with } \langle \zeta, dy^1 \wedge \dots \wedge dy^{n-k} \rangle \equiv 0. \quad (12.19)$$

To prove the orthogonality condition (12.19), write $(y^1, \dots, y^{n-k}, z^1, \dots, z^k)$ for the coordinates of E , and consider $\zeta \in \Omega_c^{n-k}(E)$ of the form

$$\zeta = dz^j \wedge \omega \text{ for } \omega \in \Omega_c^{n-k-1}(E), \quad (12.20)$$

and let $\chi \in C_c^\infty((-\delta, \delta))$ be a bump function with $\chi(t) = 1$ for $t \in [-\frac{\delta}{2}, \frac{\delta}{2}]$. Since $\text{spt}(T_\alpha(u)) \subset$

$\{(y, z) \mid z = 0\}$, we then have

$$\begin{aligned}
|\langle T_\alpha(u), \zeta \rangle| &= |\langle T_\alpha(u), \chi(z^j) dz^j \wedge \omega \rangle| \\
&= \left| \int u^*(\alpha) \wedge \chi(z^j) dz^j \wedge d\omega \right| \\
&\leq C \|d\omega\|_{L^\infty} E_p(u)^{\frac{k-1}{p}} \text{Vol}(\{|z| < \delta\})^{1-\frac{k-1}{p}}.
\end{aligned}$$

Since $\delta > 0$ was arbitrary, we can then take $\delta \rightarrow 0$, to see that $\langle T_\alpha(u), \zeta \rangle = 0$ for any ζ of the form (12.20). In particular, it follows that (12.19) holds, so that $T_\alpha(u)$ indeed has the form (12.18) for some $\theta \in \mathbb{R}$.

To determine the constant θ in (12.18), we test $T_\alpha(u)$ against a form

$$\zeta(x) = \zeta(y, z) = \varphi(y) \psi(|z|) dy^1 \wedge \cdots \wedge dy^{n-k},$$

where $\varphi \in C_c^\infty((0, 1)^{n-k})$ and ψ is given by (12.16). By direct computation, we see that

$$\begin{aligned}
\langle T_\alpha(u), \zeta \rangle &= \int u^*(\alpha) \wedge d\zeta \\
&= \int \varphi(y) u^*(\alpha) \wedge \psi'(|z|) d|z| \wedge dy^1 \wedge \cdots \wedge dy^{n-k} \\
&= (-1)^{k(n-k)} \int_{y \in (0, 1)^{n-k}} \varphi(y) \left(\int_{y \times (-1, 1)^k} \psi'(|z|) u^*(\alpha) \wedge d|z| \right) dy \\
&= (-1)^{k(n-k)} (-1)^{k-1} \int_{y \in (0, 1)^{n-k}} \varphi(y) \int_0^1 \psi'(r) \left(\int_{y \times S_r^{k-1}(0)} u^*(\alpha) \right) dr dy.
\end{aligned}$$

Now, since u is locally Lipschitz away from $[0, 1]^{n-k} \times \{0\}$, it follows from the observations in Remark 11.4 that

$$\int_{y \times S_r^{k-1}(0)} u^*(\alpha) = \int_{\partial \sigma} u^*(\alpha)$$

for every sphere $y \times S_r^{k-1}(0)$ linking with $[0, 1]^{n-k} \times \{0\}$. Using this in the preceding computation, we see that

$$\begin{aligned}
\langle T_\alpha(u), \zeta \rangle &= (-1)^{k(n-k+1)-1} \langle \alpha, u_*(\partial \sigma) \rangle \int_{y \in (0, 1)^{n-k}} \varphi(y) dy \int_0^1 \psi'(r) dr \\
&= (-1)^{k(n-k+1)} \langle \alpha, u_*(\partial \sigma) \rangle \int_{[0, 1]^{n-k} \times \{0\}} \zeta.
\end{aligned}$$

Thus, the constant θ in (12.18) must be given by

$$\theta = (-1)^{k(n-k+1)} \int_{\partial\sigma} u^*(\alpha),$$

as desired. □

12.2 Degree-type Estimates in k -Dimensional Domains

In this section, we are concerned with estimating the topological quantity

$$\int_{\partial U} u^*(\alpha)$$

for maps $u \in W^{1,p}(U, N) \cap W^{1,p}(\partial U, N)$ on a k -dimensional domain $U \subset \mathbb{R}^k$ in terms of the p -energy $\int_U |du|^p$. Our arguments are modeled very closely on those used by Jerrard [47] to estimate the degrees of $\mathbb{R}^k$ -valued maps in terms of Ginzburg-Landau energies (see also [70],[48]). In the case $N = S^{k-1}$, $\alpha = \frac{d\text{vol}}{\sigma_{k-1}}$, estimates similar to the ones we consider here can also be found in [44] (see also [43]), where they are used to study the asymptotic behavior of p -energy minimizing maps from U to S^{k-1} as $p \rightarrow k$.

Fix a closed $(k-1)$ -form $\alpha \in \mathcal{A}^{k-1}(N)$ as before, and define the constant

$$\lambda(\alpha) := \sigma_{k-1} \sup \left\{ \frac{\int_{S^{k-1}} u^*(\alpha)}{\int_{S^{k-1}} |du|^{k-1}} \mid u \in C^\infty(S^{k-1}, N) \right\}. \quad (12.21)$$

That $\lambda(\alpha) < \infty$ is clear from the estimates of Section 12.1, and when working with specific examples, it is not difficult to obtain explicit bounds for $\lambda(\alpha)$. When $N = S^{k-1}$ and α is the normalized volume form $\frac{d\text{vol}_{S^{k-1}}}{\sigma_{k-1}}$, for example, one can check (see [44], Section 1) that

$$\lambda(\alpha) = (k-1)^{\frac{1-k}{2}}.$$

Next, for $p \in (k-1, k)$, we define the constants

$$c(N, \alpha, p) := \frac{\sigma_{k-1}}{\lambda(\alpha)^{\frac{p}{k-1}}},$$

and set

$$F_p(s) := \frac{c(N, \alpha, p)}{k-p} s^{k-p}.$$

The functions F_p will take on the role in our setting played by the functions $\Lambda^\epsilon(s)$ in [47], [48], [70].

Since $0 < k - p < 1$, we easily check that

$$\frac{d}{ds} \left(\frac{F_p(s)}{s} \right) < 0 \text{ for } s > 0, \quad (12.22)$$

and, by the concavity of $s \mapsto s^{k-p}$, we have the subadditivity

$$F_p(s_1 + s_2) \leq F_p(s_1) + F_p(s_2) \quad (12.23)$$

for all $s_1, s_2 > 0$. Our estimates begin with the following simple lemma (compare [47], Proposition 3.2).

Lemma 12.6. *Let $u \in C^\infty(B_{r_2}^k(0) \setminus B_{r_1}^k(0), N)$ be a smooth map from the annulus $B_{r_2}^k \setminus B_{r_1}^k$ to N , and set*

$$d := \left| \int_{\partial B_r^k(0)} u^*(\alpha) \right|$$

for some (hence every) $r \in [r_1, r_2]$. The p -energy of u on $B_{r_2} \setminus B_{r_1}$ then satisfies the lower bound

$$E_p(u, B_{r_2} \setminus B_{r_1}) \geq d[F_p(r_2/d) - F_p(r_1/d)]. \quad (12.24)$$

Proof. For any $r \in (r_1, r_2)$, by definition of $\lambda(\alpha)$, we know that

$$\sigma_{k-1} d \leq \lambda(\alpha) \int_{\partial B_r} |du|^{k-1}.$$

Raising both sides to the power $\frac{p}{k-1}$ and applying Hölder's inequality to the integral on the right-hand side, we then see that

$$\begin{aligned} (\sigma_{k-1} d)^{\frac{p}{k-1}} &\leq \lambda(\alpha)^{\frac{p}{k-1}} |\partial B_r|^{\frac{p}{k-1}-1} \int_{\partial B_r} |du|^p \\ &= \lambda(\alpha)^{\frac{p}{k-1}} \sigma_{k-1}^{\frac{p}{k-1}-1} r^{p+1-k} \int_{\partial B_r} |du|^p, \end{aligned}$$

which we can rearrange to read

$$\sigma_{k-1} d^{\frac{p}{k-1}} r^{k-p-1} \leq \lambda(\alpha)^{\frac{p}{k-1}} \int_{\partial B_r} |du|^p.$$

Integrating the latter relation over $r \in [r_1, r_2]$, we arrive at the estimate

$$\sigma_{k-1} d^{\frac{p}{k-1}} \frac{r_2^{k-p} - r_1^{k-p}}{k-p} \leq \lambda(\alpha)^{\frac{p}{k-1}} E_p(u, B_{r_2} \setminus B_{r_1}).$$

The desired estimate (12.24) now follows from the trivial observation that

$$d^{\frac{p}{k-1}} \geq d \geq d^{1+p-k},$$

for $p \in (k-1, k)$, so that

$$d[F_p(r_2/d) - F_p(r_1/d)] = d^{1+p-k} \frac{r_2^{k-p} - r_1^{k-p}}{k-p} \leq d^{\frac{p}{k-1}} \frac{r_2^{k-p} - r_1^{k-p}}{k-p}.$$

□

We next record an analog of [48], Proposition 6.4 (see also [47], Proposition 4.1 or [70], Proposition 3.1), from which the main estimates of this section will follow.

Lemma 12.7. *Let $U \subset \mathbb{R}^k$ be a bounded Lipschitz domain, and let $u : U \rightarrow N$ be smooth away from a finite set $\Sigma = \{a_1, \dots, a_m\} \subset\subset U$ of singularities with*

$$d_j := \lim_{r \rightarrow 0} \int_{\partial B_r(a_j)} u^*(\alpha).$$

Then for every $\sigma > 0$, there exists a family $\mathcal{B}(\sigma) = \{B_j^\sigma\}_{j=1}^{m(\sigma)}$ of $m(\sigma) \leq m$ disjoint closed balls of radius r_j^σ such that, defining

$$d_j^\sigma := |\Sigma_{a_\ell \in B_j^\sigma \cap \Sigma} d_\ell|,$$

we have

$$\Sigma \subset \bigcup_{j=1}^{m(\sigma)} B_j^\sigma \quad \text{and} \quad \Sigma \cap B_j^\sigma \neq \emptyset \text{ for each } j, \quad (12.25)$$

$$\int_{U \cap B_j^\sigma} |du|^p \geq \frac{r_j^\sigma}{\sigma} F_p(\sigma) \text{ if } d_j^\sigma > 0, \quad (12.26)$$

and

$$r_j^\sigma \geq \sigma d_j^\sigma \text{ if } B_j^\sigma \subset U. \quad (12.27)$$

Proof. Denote by S the collection of $\sigma > 0$ for which such a family $\mathcal{B}(\sigma)$ exists. To see that S is

nonempty, for each $a_j \in \Sigma$, set

$$d_j := \lim_{r \rightarrow 0} \int_{\partial B_r(a_j)} u^*(\alpha)$$

and

$$D := 1 + \max_{1 \leq j \leq m} |d_j|,$$

and choose $\sigma_0 > 0$ such that the balls $B_{D\sigma_0}(a_1), \dots, B_{D\sigma_0}(a_m)$ are disjoint. Taking

$$r_j^{\sigma_0} := \sigma_0 |d_j| \text{ if } d_j \neq 0,$$

and

$$r_j^{\sigma_0} := \sigma_0 \text{ if } d_j = 0,$$

it's then clear that the collection

$$\mathcal{B}(\sigma_0) := \{B_{r_j^{\sigma_0}}(a_j)\}_{j=1}^m$$

satisfies (12.25) and (12.27), as well as (12.26), by Lemma 12.6. In particular, $\sigma_0 \in S$, so $S \neq \emptyset$.

Since the functions $F_p(s)$ satisfy the growth conditions (12.22) and (12.23), we can apply Steps 2 and 3 in the proof of Proposition 6.4 of [48] directly (with F_p in place of Λ^ϵ) to see that the set S is open, and closed away from 0. In particular, we deduce that $S = (0, \infty)$, as desired.

□

With Lemma 12.7 in hand, we arrive at the following proposition. (Compare [47], Theorem 1.2.)

Proposition 12.8. *Let $U \subset \mathbb{R}^k$ and $u \in W^{1,p}(U, N)$ satisfy the hypotheses of Lemma 12.7, and suppose that the singular set $\Sigma = \{a_1, \dots, a_m\}$ satisfies*

$$\inf_{1 \leq j \leq m} \text{dist}(a_j, \partial U) \geq r > 0. \quad (12.28)$$

Setting

$$d := \left| \int_{\partial U} u^*(\alpha) \right| = \left| \sum_{j=1}^m d_j \right|,$$

we then have the lower bound

$$\int_U |du|^p \geq d \cdot F_p(r/2d) = \frac{c(N, \alpha, p)}{k-p} (r/2d)^{k-p} d. \quad (12.29)$$

Proof. Again, we can argue just as in [47], [48]. Suppose that (12.29) doesn't hold, to obtain a

contradiction. Setting $\sigma = \frac{r}{2d}$, we then have

$$\int_U |du|^p < d \cdot F_p(r/2d) = \frac{r}{2} \frac{F_p(\sigma)}{\sigma}.$$

Choosing a collection of balls $\mathcal{B}(\sigma) = \{B_j^\sigma\}$ according to Lemma 12.7, it follows from (12.26) that

$$r_j^\sigma \leq \frac{\sigma}{F_p(\sigma)} \int_{U \cap B_j^\sigma} |du|^p < \frac{r}{2} \quad (12.30)$$

whenever $d_j^\sigma > 0$. In particular, if $d_j^\sigma > 0$, we then deduce from (12.25) and (12.28) that

$$B_j^\sigma \subset U,$$

and therefore, by (12.27), we have

$$r_j^\sigma \geq \sigma d_j^\sigma.$$

Finally, summing (12.26) over $1 \leq j \leq m(\sigma)$, we see that

$$\begin{aligned} \int_U |du|^p &\geq \sum_{j=1}^{m(\sigma)} \int_{U \cap B_j^\sigma} |du|^p \\ &\geq \sum_{j=1}^{m(\sigma)} \frac{r_j^\sigma}{\sigma} F_p(\sigma) \\ &\geq F_p(\sigma) \sum_{j=1}^{m(\sigma)} d_j^\sigma \\ &\geq d \cdot F_p(\sigma), \end{aligned}$$

a contradiction. Thus, (12.29) holds. $\square$

Remark 12.9. By the density results of Bethuel (namely, [11], Theorem 2), we can remove the requirement that u have finite singular set from the hypotheses of Proposition 12.8: the conclusion applies to any map $u \in W^{1,p}(U, N)$ for which u is continuous on the r -neighborhood of ∂U in U .

The final estimate of this section is a simple consequence of Proposition 12.8, modeled on ([2], Lemma 3.10). Arguing much as in [2], we will employ this estimate repeatedly in the following sections to obtain the needed compactness results as $p \rightarrow k$ for the homological singularities $T_\alpha(u_p)$ in higher-dimensional manifolds. In what follows we denote by I_δ^k the k -cube

$$I_\delta^k := [-\delta, \delta]^k.$$

Proposition 12.10. *Let $u \in W^{1,p}(I_\delta^k, N)$ such that $u|_{\partial I_\delta^k} \in W^{1,p}(\partial I_\delta^k, N)$, and set*

$$d := \left| \int_{\partial I_\delta^k} u^*(\alpha) \right|.$$

Then for any $r > 0$, we have the estimate

$$\sigma_{k-1} d^{1+p-k} \leq \lambda(\alpha)^{\frac{p}{k-1}} (r/2)^{p-k} (k-p) [E_p(u, I_\delta^k) + C(k)rE_p(u, \partial I_\delta^k)]. \quad (12.31)$$

Proof. We argue as in [2]. Extend u to a map $\tilde{u}$ on $I_{\delta+r}^k$ by setting

$$\tilde{u}(x) := u(\delta \cdot x/|x|_\infty) \text{ when } \delta \leq |x|_\infty \leq r,$$

so that

$$E_p(\tilde{u}, I_{\delta+r}^k) \leq E_p(u, I_\delta^k) + C(k)rE_p(u, \partial I_\delta^k).$$

In view of Remark 12.9, we can then apply Proposition 12.8 to the map $\tilde{u}$ on $I_{\delta+r}^k$ to see that

$$\begin{aligned} \frac{c(N, \alpha, p)}{k-p} (r/2)^{k-p} d^{1+p-k} &\leq E_p(\tilde{u}, I_{\delta+r}^k) \\ &\leq E_p(u, I_\delta^k) + C(k)rE_p(u, \partial I_\delta^k). \end{aligned}$$

Recalling that

$$c(N, \alpha, p) := \frac{\sigma_{k-1}}{\lambda(\alpha)^{\frac{p}{k-1}}},$$

the desired estimate follows immediately. $\square$

12.3 A Compactness Theorem as $p \rightarrow k$

Henceforth, let M^n be a closed, oriented Riemannian manifold of dimension $n \geq k$. In this section and the next, we analyze the limiting behavior as $p \rightarrow k$ of the homological singularities $T_\alpha(u_p)$ for maps $u_p \in W^{1,p}(M, N)$ with energy growth of the form $E_p(u_p) = O(\frac{1}{k-p})$. The results of these sections are inspired in large part by those of [2] and [48], concerning the limiting behavior of Jacobian currents for maps of controlled energy growth with respect to functionals of Ginzburg-Landau type.

The starting point for our compactness results is the following proposition—inspired by arguments in [2]—in which we construct good approximations $\tilde{u} \in \mathcal{E}^p(M, N)$ to given maps $u \in W^{1,p}(M, N)$, such that the mass $T_\alpha(\tilde{u})$ is controlled uniformly.

Proposition 12.11. *For any $u \in W^{1,p}(M, N)$ with $k - \frac{1}{2} < p < k$ and*

$$E_p(u) \leq \frac{\Lambda}{k-p},$$

there exists a map $\tilde{u} \in \mathcal{E}^p(M, N)$ for which

$$\|u - \tilde{u}\|_{L^p(M)}^p \leq C(M)(k-p)^{3p-2}\Lambda, \quad (12.32)$$

$$E_p(\tilde{u}) \leq \frac{C(M)\Lambda}{(k-p)^2}, \quad (12.33)$$

and

$$\mathbb{M}(T_\alpha(\tilde{u})) \leq C(M, \alpha, \Lambda) \quad (12.34)$$

Proof. For $\delta \in (0, 1)$, choose a cubeulation $h : |K_\delta| \rightarrow M$ satisfying the conclusions of Lemma 11.1, and let $\tilde{u}_0 = u|_{|K_\delta^{k-1}|} \circ \Phi_k \circ h^{-1}$. By Lemma 11.5, we then have the estimates

$$\|u - \tilde{u}_0\|_{L^p(M)}^p \leq \delta^p \frac{C(M)}{k-p} \leq E_p(u) \leq \delta^p \frac{C(M)\Lambda}{(k-p)^2} \quad (12.35)$$

and

$$E_p(\tilde{u}_0) \leq \frac{C(M)}{k-p} E_p(u) \leq \frac{C(M)\Lambda}{(k-p)^2}. \quad (12.36)$$

Since $p > k-1$, we can then find $f \in Lip(|K_\delta^{k-1}|, N)$ homotopic to $u|_{|K_\delta^{k-1}|}$ on $|K_\delta^{k-1}|$ and arbitrarily close in $W^{1,p}(|K_\delta^{k-1}|)$. In particular, we can choose f homotopic to $u|_{|K_\delta^{k-1}|}$ such that

$$\tilde{u} := f \circ \Phi_k \circ h^{-1} \in \mathcal{E}^p(M, N)$$

satisfies (12.35) and (12.36)–modifying the constant $C(M)$ if necessary. Defining $\tilde{u}$ in this way, and taking

$$\delta = \delta_p := (k-p)^3,$$

the bounds (12.32) and (12.33) follow immediately.

To estimate the mass of $T_\alpha(\tilde{u})$, we first appeal to Lemma 12.4 to see that

$$T_\alpha(\tilde{u}) = \Sigma_{\sigma \in K^k \setminus K^{k-1}} \theta(u, \sigma) \cdot [P(\sigma)], \quad (12.37)$$

where

$$|\theta(u, \sigma)| = \left| \int_{\partial\sigma} \tilde{u}^*(\alpha) \right| = \left| \int_{\partial\sigma} u^*(\alpha) \right|.$$

Now, recalling (11.16), we have the volume bound

$$\mathcal{H}^{n-k}(P(\sigma)) \leq C(M)\delta^{n-k}$$

for every k -cell $\sigma \in K$, so by (12.37), the mass of $T_\alpha(\tilde{u})$ is bounded by

$$\mathbb{M}(T_\alpha(\tilde{u})) \leq \sum_{\sigma \in K_\delta^k \setminus K_\delta^{k-1}} |\theta(u, \sigma)| \delta^{n-k}. \quad (12.38)$$

On the other hand, Proposition 12.10 (with $r = \delta$) furnishes us with an estimate of the form

$$\begin{aligned} |\theta(u, \sigma)| &\leq C(M, \alpha) \left(\delta^{p-k} (k-p) [E_p(u, \sigma) + \delta E_p(u, \partial\sigma)] \right)^{\frac{1}{1+p-k}} \\ &\leq C(M, \alpha) \delta^{\frac{p-k}{1+p-k}} (k-p) [E_p(u, \sigma) + \delta E_p(u, \partial\sigma)] \\ &\quad \cdot ((k-p) [E_p(u, \sigma) + \delta E_p(u, \partial\sigma)])^{\frac{k-p}{1+p-k}} \end{aligned}$$

on every k -cell σ . Now, since the cubeulation $|K_\delta| \rightarrow M$ was chosen according to Lemma 11.1, we know that

$$E_p(u, |K^k|) + \delta E_p(u, |K^{k-1}|) \leq C(M) \delta^{k-n} \frac{\Lambda}{k-p}, \quad (12.39)$$

from which we immediately obtain the simple-minded estimate

$$(k-p) [E_p(u, \sigma) + \delta E_p(u, \partial\sigma)] \leq C(M) \Lambda \delta^{k-n}$$

for every k -cell $\sigma \in K_\delta$. Using this to bound the last factor on the right-hand side of the preceding estimate for $|\theta(u, \sigma)|$, we find that

$$|\theta(u, \sigma)| \leq C(M, \alpha) (k-p) [E_p(u, \sigma) + \delta E_p(u, \partial\sigma)] \cdot [\Lambda \delta^{k-n-1}]^{\frac{k-p}{1+p-k}}. \quad (12.40)$$

On the other hand, summing $E_p(u, \sigma) + \delta E_p(u, \partial\sigma)$ over all k -cells $\sigma \in K_\delta$, we also have the bound

$$\begin{aligned} \sum_{\sigma} [E_p(u, \sigma) + \delta E_p(u, \partial\sigma)] &\leq C(M) [E_p(u, |K^k|) + \delta E_p(u, |K^{k-1}|)] \\ &\leq C(M) \delta^{k-n} \frac{\Lambda}{k-p}. \end{aligned}$$

In particular, summing (12.40) over all k -cells and employing the estimate above, we find that

$$\Sigma_\sigma |\theta(u, \sigma)| \leq C'(M, \alpha) \cdot \Lambda \delta^{k-n} \cdot [\Lambda \delta^{k-n-1}]^{\frac{k-p}{1+p-k}}.$$

Recalling now the bound (12.38) for $\mathbb{M}(T_\alpha(\tilde{u}))$, we deduce that

$$\mathbb{M}(T_\alpha(\tilde{u})) \leq C(M, \alpha) \Lambda \cdot [\Lambda \delta^{k-1-n}]^{\frac{k-p}{1+p-k}}.$$

Since we've set $\delta = \delta_p = (k-p)^3$, we check directly that

$$\sup_{k-\frac{1}{2} < p < k} [\Lambda \delta_p^{k-1-n}]^{\frac{k-p}{1+p-k}} < \infty,$$

and the desired mass bound (12.34) follows. $\square$

Given a family of maps $(k-1, k) \ni p \mapsto u_p \in W^{1,p}(M, N)$ with $E_p(u_p) = O(\frac{1}{k-p})$, Proposition 12.11 gives us an associated family of integral $(n-k)$ -cycles $T_\alpha(\tilde{u}_p)$ with uniform mass bounds. By showing that

$$T_\alpha(u_p) - T_\alpha(\tilde{u}_p) \rightarrow 0$$

in $(C^1)^*$ as $p \rightarrow k$, and applying the Federer-Fleming compactness theorem to the cycles $T_\alpha(\tilde{u}_p)$, we arrive at the following preliminary compactness result.

Corollary 12.12. *Let $p_j \in (k-1, k)$ be a sequence with $p_j \rightarrow k$, and let $u_j \in W^{1,p_j}(M, N)$ be a sequence of maps satisfying*

$$\limsup_{j \rightarrow \infty} (k - p_j) E_{p_j}(u_j) \leq \Lambda < \infty. \quad (12.41)$$

Then there exists a subsequence (unrelabelled) $p_j \rightarrow k$ such that $T_\alpha(u_j)$ converges in $(C^1)^*$ to an integer rectifiable cycle $T \in \mathcal{Z}_{n-k}(M; \mathbb{Z})$ of finite mass.

Proof. To each map u_j , by Proposition 12.11, we can associate a map $\tilde{u}_j \in \mathcal{E}^p(M, N)$ for which

$$\|u_j - \tilde{u}_j\|_{L^p}^p \leq C \Lambda (k - p_j)^{3p_j-2},$$

$$E_p(\tilde{u}_j) \leq \frac{C}{(k - p_j)^2} \Lambda,$$

and

$$\mathbb{M}(T_\alpha(\tilde{u}_j)) \leq C(M, \alpha, \Lambda).$$

On the other hand, by Lemma 12.3, we know that

$$T_\alpha(u_j) - T_\alpha(\tilde{u}_j) = \partial S_\alpha(u_j, \tilde{u}_j),$$

where

$$\begin{aligned} \mathbb{M}(S_\alpha(u_j, \tilde{u}_j)) &\leq C(\alpha)[E_{p_j}(u_j)^{\frac{k-1}{p_j}} + E_{p_j}(\tilde{u}_j)^{\frac{k-1}{p_j}}]\|u_j - \tilde{u}_j\|_{L^{p_j}}^{1+p_j-k} \\ &\leq C[\Lambda/(k-p_j)^2]^{\frac{k-1}{p_j}} \cdot (C\Lambda(k-p_j)^{3p_j-2})^{\frac{1+p_j-k}{p_j}} \\ &\leq C(M, \alpha, \Lambda)(k-p_j)^{3(p_j+1-k)-2}, \end{aligned}$$

so in particular,

$$\lim_{j \rightarrow \infty} \mathbb{M}(S_\alpha(u_j, \tilde{u}_j)) = 0.$$

Since the currents $T_\alpha(\tilde{u}_j)$ are integral cycles with uniformly bounded mass, it follows from the Federer-Fleming compactness theorem (see [31], Theorem 4.2.17) that—after passing to a subsequence—there exists an integral cycle $T \in \mathcal{Z}_{n-k}(M; \mathbb{Z})$ and a sequence of integer-rectifiable $(n+1-k)$ -currents $\Gamma_j \in \mathcal{I}_{n+1-k}(M; \mathbb{Z})$ such that

$$\lim_{j \rightarrow \infty} \mathbb{M}(\Gamma_j) = 0$$

and

$$\partial \Gamma_j = T_\alpha(\tilde{u}_j) - T.$$

Putting all this together, we see that

$$T_\alpha(u_j) - T = \partial(S_\alpha(u_j, \tilde{u}_j) + \Gamma_j)$$

and

$$\mathbb{M}(S_\alpha(u_j, \tilde{u}_j) + \Gamma_j) \rightarrow 0,$$

from which it clearly follows that $T_\alpha(u_j) - T \rightarrow 0$ in $(C^1)^*$. □

Remark 12.13. For a simple consequence of Corollary 12.12, consider a map $u \in W^{1,p}(M, N)$ for which $|du| \in L^{k,\infty}(M)$ —that is, for which

$$\|du\|_{L^{k,\infty}}^k := \sup\{t^k \text{Vol}(\{|du| > t\}) \mid t \in (0, \infty)\} < \infty \quad (12.42)$$

–and note that for $p < k$, we have the straightforward L^p estimate

$$\begin{aligned} \int_M |du|^p &= \int_0^\infty p t^{p-1} \text{Vol}(\{|du| > t\}) dt \\ &\leq \int_0^1 p t^{p-1} \text{Vol}(M) dt + \int_1^\infty p t^{p-1} \text{Vol}(\{|du| > t\}) dt \\ &= \text{Vol}(M) + \|du\|_{L^{k,\infty}}^k \frac{p}{k-p}. \end{aligned}$$

In particular, the hypotheses of Corollary 12.12 hold with $u_j = u$, so we see that $T_\alpha(u)$ must be an integral cycle.

12.4 Sharp Mass Bounds for the Limiting Current

Our goal in this section is to establish a sharp upper bound for the mass of the limiting current in Corollary 12.12; namely, we prove the following proposition.

Proposition 12.14. *For a sequence $p_j \in (k-1, k)$ with $\lim_{j \rightarrow \infty} p_j = k$ and a sequence of maps $u_j \in W^{1,p_j}(M, N)$ satisfying*

$$\limsup_{j \rightarrow \infty} (k - p_j) E_{p_j}(u_j) \leq \Lambda$$

and

$$\lim_{j \rightarrow \infty} T_\alpha(u_j) = T \tag{12.43}$$

in $(C^1)^*$, the limit current T satisfies

$$\sigma_{k-1} \mathbb{M}(T) \leq \lambda(\alpha)^{\frac{k}{k-1}} \Lambda. \tag{12.44}$$

To prove Proposition 12.14, we continue to model our arguments on those of ([2], Section 3), proving first the following lemma for maps from the Euclidean unit ball $B_1^n(0)$.

Lemma 12.15. *Let $p_j \in (k-1, k)$ be a sequence with $\lim_{j \rightarrow \infty} p_j = k$, and let $u_j \in W^{1,p_j}(B_1^n(0), N)$ be a family of maps for which*

$$\limsup_{j \rightarrow \infty} (k - p_j) E_{p_j}(u_j, B_1^n(0)) \leq \Lambda < \infty. \tag{12.45}$$

Then for any simple unit $(n-k)$ -covector $\beta \in \bigwedge^{n-k}(\mathbb{R}^n)$ and $\varphi \in C_c^\infty(B_1^n)$, we have the estimate

$$\sigma_{k-1} \limsup_{j \rightarrow \infty} |\langle T_\alpha(u_j), \varphi \cdot \beta \rangle| \leq \lambda(\alpha)^{\frac{k}{k-1}} \Lambda \|\varphi\|_{L^\infty}. \tag{12.46}$$

Proof. After a rotation, it is enough to prove (12.46) in the case

$$\beta = dx^1 \wedge \cdots \wedge dx^{n-k}.$$

Following the notation of [2], for $a \in \mathbb{R}^n$ and $\delta > 0$, let $\mathcal{G}(\delta, a)$ denote the grid

$$\mathcal{G}(\delta, a) := a + \delta \cdot \mathbb{Z}^n,$$

and let $R_j(\delta, a)$ denote the j -skeleton of the associated n -dimensional cubical complex for which $\mathcal{G}(\delta, a)$ gives the vertices. Denote by $\tilde{R}_k(\delta, a)$ the component

$$\tilde{R}_k(\delta, a) := a + (\delta \mathbb{Z}^{n-k} \times \mathbb{R}^k)$$

of $R_k(\delta, a)$ parallel to $\{0\} \times \mathbb{R}^k$. As in Lemma 3.11 of [2], a simple Fubini argument shows that for $u \in W^{1,p}(M, N)$ and $\eta > 0$, we can find $a(u, \delta, \eta) \in \mathbb{R}^n$ such that

$$\int_{R_j(\delta, a) \cap B_1} |du|^p \leq \frac{C}{\eta} \delta^{j-n} \int_{B_1^n} |du|^p \quad (12.47)$$

for all $0 \leq j \leq n$ and

$$\int_{\tilde{R}_k(\delta, a) \cap B_1} |du|^p \leq (1 + \eta) \delta^{k-n} \int_{B_1^n} |du|^p. \quad (12.48)$$

Now, fix some arbitrary $\varphi \in C_c^\infty(B_1^n)$ and $\eta > 0$, and consider a family of maps $u_j \in W^{1,p_j}(B_1, N)$ satisfying (12.45). As in the proof of Proposition 12.11, we let

$$\delta_j := (k - p_j)^3,$$

and let $\tilde{u}_j = u_j \circ \Phi_k$ with respect to the cubical complex associated to the grid $\mathcal{G}(\delta_j, a_j(\eta))$ —where $a_j(\eta)$ is chosen to satisfy (12.47) and (12.48) with respect to u_j . Of course, $\tilde{u}_j$ is only well-defined on those n -cells strictly contained in $B_1^n(0)$, but since φ is supported in the interior of B_1 and $\lim_{j \rightarrow \infty} \delta_j = 0$, we see that $\tilde{u}_j$ is defined on $\text{spt}(\varphi)$ for j sufficiently large.

Setting $\zeta = \varphi dx^1 \wedge \cdots \wedge dx^{n-k}$, we can then argue as in the proof of Corollary 12.12 to see that, for j sufficiently large,

$$\begin{aligned} |\langle T_\alpha(u_j) - T_\alpha(\tilde{u}_j), \zeta \rangle| &= |\langle S_\alpha(u_j, \tilde{u}_j), d\zeta \rangle| \\ &\leq C(n, \alpha) \eta^{-1} \Lambda(k - p_j) \|d\zeta\|_{L^\infty}. \end{aligned}$$

In particular, it follows that

$$\lim_{j \rightarrow \infty} |\langle T_\alpha(u_j) - T_\alpha(\tilde{u}_j), \zeta \rangle| = 0. \quad (12.49)$$

On the other hand, by Proposition 12.4, we know that

$$\langle T_\alpha(\tilde{u}_j), \zeta \rangle = \sum_{\sigma \subset \tilde{R}_k(\delta_j, a_j) \cap B_1} \theta(u_j, \sigma) \int_{P(\sigma)} \varphi,$$

where the sum is over all k -cells $\sigma \cong [0, \delta]^k$ contained in $\tilde{R}_k(\delta_j, a_j) \cap B_1$, and

$$\theta(u_j, \sigma) = \pm \int_{\partial \sigma} u_j^*(\alpha).$$

In this Euclidean setting, the component $P(\sigma)$ of the dual $(n - k)$ -skeleton intersecting σ is given by a single $(n - k)$ -cell isometric to $[0, \delta]^{n-k}$, and as a consequence, we see that

$$|\langle T_\alpha(\tilde{u}_j), \zeta \rangle| \leq \sum_{\sigma \subset \tilde{R}_k(\delta_j, a_j) \cap B_1} |\theta(u_j, \sigma)| \delta^{n-k} \|\varphi\|_{L^\infty}. \quad (12.50)$$

To estimate the coefficients $|\theta(u_j, \sigma)|$, we first appeal to Proposition 12.10 and (12.47) to get the crude estimate

$$\begin{aligned} |\theta(u, \sigma)|^{1+p_j-k} &\leq C(k, \alpha) \delta_j^{p_j-k} (k - p_j) [E_{p_j}(u_j, \sigma) + \delta_j E_{p_j}(u_j, \partial \sigma)] \\ &\leq C(k, n, \alpha, \eta) \delta_j^{p_j-n} \Lambda. \end{aligned}$$

In particular, setting

$$c_j := [C(k, n, \alpha, \eta) \delta_j^{p_j-n} \Lambda]^{\frac{k-p_j}{1+p_j-k}},$$

we have the bound

$$|\theta(u_j, \sigma)|^{k-p_j} \leq c_j, \quad (12.51)$$

and recalling that $\delta_j = (k - p_j)^3$, we observe that

$$\lim_{j \rightarrow \infty} c_j = 1.$$

For a finer estimate, we appeal again to Proposition 12.10 to see that, for any $r > 0$ and any k -cell σ ,

$$\sigma_{k-1} |\theta(u, \sigma)|^{1+p_j-k} \leq \lambda(\alpha)^{\frac{p_j}{k-1}} (\delta_j r)^{p_j-k} (k - p_j) [E_{p_j}(u_j, \sigma) + C(k) r \delta_j E_p(u_j, \partial \sigma)].$$

Multiplying both sides above by $|\theta(u_j, \sigma)|^{k-p_j}$ and appealing to (12.51), we then arrive at the bound

$$\sigma_{k-1}|\theta(u_j, \sigma)| \leq c_j \lambda(\alpha)^{\frac{p_j}{k-1}} (\delta_j r)^{p_j-k} (k-p_j) [E_{p_j}(u, \sigma) + C(k)r\delta_j E_{p_j}(u, \sigma)].$$

Summing over k -cells $\sigma \subset \tilde{R}_k(\delta_j, a_j)$, and appealing to (12.47) and (12.48), we find that

$$\begin{aligned} \sigma_{k-1} \cdot \Sigma_{\sigma \subset \tilde{R}_k(\delta_j, a_j)} |\theta(u_j, \sigma)| &\leq c_j \lambda(\alpha)^{\frac{p_j}{k-1}} (\delta_j r)^{p_j-k} (k-p_j) \\ &\quad \cdot \left(\int_{\tilde{R}_k \cap B_1} |du_j|^{p_j} + C(k)r\delta \int_{R_{k-1} \cap B_1} |du_j|^{p_j} \right) \\ &\leq c_j \lambda(\alpha)^{\frac{p_j}{k-1}} (\delta_j r)^{p_j-k} (k-p_j) \\ &\quad \cdot \left[(1+\eta) + \frac{C(n, k)}{\eta} r \right] \delta_j^{k-n} E_{p_j}(u_j, B_1). \end{aligned}$$

Choosing $r = \eta^2$ above, and returning to (12.50), we arrive at the estimate

$$\sigma_{k-1} |\langle T_\alpha(\tilde{u}_j), \zeta \rangle| \leq c_j \lambda(\alpha)^{\frac{p_j}{k-1}} (\delta_j \eta^2)^{p_j-k} [1 + C'(n, k)\eta] \Lambda \|\varphi\|_{L^\infty}. \quad (12.52)$$

Now, since $\lim_{j \rightarrow \infty} c_j = 1$, and likewise $\lim_{j \rightarrow \infty} (\delta_j \eta^2)^{p_j-k} = 1$, we deduce that

$$\limsup_{j \rightarrow \infty} \sigma_{k-1} |\langle T_\alpha(\tilde{u}_j), \zeta \rangle| \leq \lambda(\alpha)^{\frac{k}{k-1}} [1 + C\eta] \Lambda \|\varphi\|_{L^\infty}. \quad (12.53)$$

By (12.49), this is equivalent to the statement that

$$\limsup_{j \rightarrow \infty} \sigma_{k-1} |\langle T_\alpha(u_j), \zeta \rangle| \leq \lambda(\alpha)^{\frac{k}{k-1}} [1 + C\eta] \Lambda \|\varphi\|_{L^\infty}; \quad (12.54)$$

finally, taking $\eta \rightarrow 0$, we arrive at the desired estimate. $\square$

With Lemma 12.15 in hand, we can now prove Proposition 12.14 via a blow-up argument.

Proof. (Proof of Proposition 12.14)

Let $u_j \in W^{1, p_j}(M, N)$ be a sequence of maps as in Proposition 12.12, for which

$$\limsup_{j \rightarrow \infty} (k-p_j) E_{p_j}(u_j) \leq \Lambda$$

and

$$\lim_{j \rightarrow \infty} T_\alpha(u_j) = T \in \mathcal{Z}_{n-k}(M; \mathbb{Z}).$$

Passing to a further subsequence, we can also assume that the normalized energy measures

$$\mu_j := (k - p_j) |du_j|^{p_j} dv_g$$

converge weakly in $(C^0)^*$ to a limiting Radon measure μ satisfying

$$\mu(M) \leq \Lambda.$$

Denote by $|T|$ the weight measure associated to the current T . By standard results on derivatives of Radon measures (see, e.g., [?], Section 4 or [31], Section 2.9), the quantity

$$D_\mu |T|(x) := \lim_{r \rightarrow 0} \frac{|T|(B_r(x))}{\mu(B_r(x))}$$

is well-defined for $|T|$ -a.e. $x \in M$, and to establish the desired mass bound for T , it will suffice to show that

$$D_\mu |T|(x) \leq \sigma_{k-1}^{-1} \lambda(\alpha)^{\frac{k}{k-1}} \text{ for } |T| - a.e. \ x \in M. \quad (12.55)$$

Now, on a small geodesic ball $B_r(x) \subset M$, denote by

$$\Phi_{x,r} : B_r(x) \rightarrow B_1^n(0) \subset T_x M$$

the dilation map

$$\Phi_{x,r}(y) := \frac{1}{r} \exp_x^{-1}(y),$$

and set

$$\mu_{x,r} := (\Phi_{x,r})_\# \mu, \quad T_{x,r} := (\Phi_{x,r})_\# T.$$

Since T is integer rectifiable, for $|T|$ -almost every $x \in M$, the currents $T_{x,r}$ converge weakly

$$T_{x,r} \rightharpoonup \theta(x)[P] \in \mathcal{D}_{n-k}(B_1^n), \quad (12.56)$$

to an oriented $(n - k)$ -plane P in $\mathbb{R}^n$ with multiplicity

$$\theta(x) := \lim_{r \rightarrow 0} \frac{|T|(B_r(x))}{\omega_{n-k} r^{n-k}},$$

where $\omega_{n-k} := \mathcal{L}^{n-k}(B_1^{n-k}(0))$ (see, for instance, [73], Section 32).

Now, let $x \in M$ be a point at which $D_\mu|T|(x)$ is defined and (12.56) holds, and observe that the density $\Theta_{n-k}(\mu, x)$ is then well-defined (though possibly infinite), as

$$\begin{aligned}\Theta_{n-k}(\mu, x) &= \lim_{r \rightarrow 0} \frac{\mu(B_r(x))}{\omega_{n-k} r^{n-k}} \\ &= \theta(x) \lim_{r \rightarrow 0} \frac{\mu(B_r(x))}{|T|(B_r(x))} \\ &= \theta(x) \frac{1}{D_\mu|T|(x)}.\end{aligned}$$

In particular, to prove (12.55), we just need to show that

$$\sigma_{k-1}\theta(x) \leq \lambda(\alpha)^{\frac{k}{k-1}} \Theta_{n-k}(\mu, x). \quad (12.57)$$

If $\Theta_{n-k}(\mu, x) = \infty$, then (12.57) holds trivially, so assume that

$$\Theta_{n-k}(\mu, x) < \infty,$$

and consider a sequence $r_\ell \rightarrow 0$ for which

$$\mu(\partial B_{r_\ell}(x)) = 0.$$

For each r_ℓ , we then have

$$\mu(B_{r_\ell}(x)) = \lim_{j \rightarrow \infty} (k - p_j) \int_{B_{r_\ell}(x)} |du_j|^{p_j},$$

and consequently

$$r_\ell^{k-n} \mu(B_{r_\ell}(x)) = \lim_{j \rightarrow \infty} r_\ell^{p_j-n} (k - p_j) \int_{B_{r_\ell}(x)} |du_j|^{p_j}. \quad (12.58)$$

Moreover, since the convergence $T_\alpha(u_j) \rightarrow T$ established in Proposition 12.12 is convergence in the $(C^1)^*$ norm, we also see that

$$\lim_{j \rightarrow \infty} \|T - T_\alpha(u_j)\|_{(C_c^1(B_{r_\ell}(x)))^*} = 0. \quad (12.59)$$

It follows from (12.58) and (12.59) that for each r_ℓ , we can select $p_\ell = p_{j_\ell}$ and $u_\ell = u_{j_\ell}$ such that

$$\left| \frac{\mu(B_{r_\ell}(x))}{r_\ell^{n-k}} - r_\ell^{p_\ell-n} (k - p_\ell) \int_{B_{r_\ell}(x)} |du_\ell|^{p_\ell} \right| < \frac{1}{\ell} \quad (12.60)$$

and

$$r_\ell^{k-n-1} \|T - T_\alpha(u_\ell)\|_{(C_c^1(B_{r_\ell}(x)))^*} < \frac{1}{\ell}. \quad (12.61)$$

Defining the maps $v_\ell \in W^{1,p_\ell}(B_1^n(0), N)$ by

$$v_\ell := u \circ \Phi_{x,r_\ell}^{-1},$$

we then see that

$$\lim_{\ell \rightarrow \infty} (k - p_\ell) E_{p_\ell}(v_\ell, B_1) = \omega_{n-k} \Theta_{n-k}(\mu, x) \quad (12.62)$$

and

$$\lim_{\ell \rightarrow \infty} \langle T_\alpha(v_\ell), \zeta \rangle = \lim_{r \rightarrow 0} \langle T_{x,r}, \zeta \rangle = \theta(x) \langle [P], \zeta \rangle \quad (12.63)$$

for all $\zeta \in \Omega_c^{n-k}(B_1^n(0))$.

Now, applying Lemma 12.15 to the maps v_ℓ and the simple unit $(n-k)$ -covector β orienting $[P]$, we deduce from (12.62) and (12.63) that

$$\sigma_{k-1} \theta(x) \int_P \varphi = \sigma_{k-1} \theta(x) \langle [P], \varphi \cdot \beta \rangle \leq \lambda(\alpha)^{\frac{k}{k-1}} \omega_{n-k} \Theta_{n-k}(\mu, x) \|\varphi\|_{L^\infty} \quad (12.64)$$

for any $\varphi \in C_c^\infty(B_1)$. Finally, applying (12.64) to a reasonable approximation $\varphi_j \rightarrow \chi_{B_1(0)}$ with $\|\varphi_j\|_{L^\infty} \leq 1$, we obtain in the limit

$$\sigma_{n-k} \theta(x) \omega_{n-k} \leq \lambda(\alpha)^{\frac{k}{k-1}} \omega_{n-k} \Theta_{n-k}(\mu, x).$$

Dividing through by ω_{n-k} gives precisely (12.57), and the proposition follows. $\square$

Finally, combining the results of Proposition 12.11, Corollary 12.12, and Proposition 12.14 above, together with a simple contradiction argument, we obtain the following strong version of the compactness result.

Theorem 12.16. *For any $\Lambda < \infty$ and $\eta > 0$, there exists $q(M, \alpha, \Lambda, \eta) \in (k-1, k)$ such that if $p \in (q, k)$, and $u \in W^{1,p}(M, N)$ satisfies*

$$(k-p) E_p(u) \leq \Lambda,$$

then there exists a map $\tilde{u} \in \mathcal{E}^p(M, N)$ satisfying

$$E_p(\tilde{u}) \leq C(M) \frac{\Lambda}{(k-p)^2}, \quad (12.65)$$

$$\|u - \tilde{u}\|_{L^p(M)}^p \leq (k-p)^{3p-2} C(M) \Lambda, \quad (12.66)$$

and an integral $(n-k)$ -cycle $T \in \mathcal{Z}_{n-k}(M; \mathbb{Z})$ and integral $(n+1-k)$ -current $\Gamma \in \mathcal{I}_{n+1-k}(M; \mathbb{Z})$ such that

$$T_\alpha(\tilde{u}) - T = \partial\Gamma, \quad (12.67)$$

$$\sigma_{k-1}\mathbb{M}(T) \leq \lambda(\alpha)^{\frac{k}{k-1}} \Lambda, \quad (12.68)$$

and

$$\mathbb{M}(\Gamma) < \eta. \quad (12.69)$$

Chapter 13

Lower Bounds for the Energy Walls

In this chapter, we combine the tools of Chapter 12 with results from Almgren's min-max theory in the space of cycles to complete the proof of Theorem 3.2.

13.1 Loops in the Space of Cycles and Min-max Widths

Before proving Theorem 3.2, we recall some basic facts about the map $\pi_1(\mathcal{Z}_m(M; \mathbb{Z}), 0) \rightarrow H_{m+1}(M; \mathbb{Z})$ constructed by Almgren in his dissertation [3], and make precise the definition of the min-max widths that we use to obtain the lower bounds in Theorem 3.2.

As in [3], we topologize the space $\mathcal{Z}_m(M; \mathbb{Z})$ of integral m -cycles via the flat norm

$$\mathbb{F}(T) := \inf\{\mathbb{M}(T') + \mathbb{M}(S) \mid T = T' + \partial S, T' \in \mathcal{Z}_m(M; \mathbb{Z}), S \in \mathcal{I}_{m+1}(M; \mathbb{Z})\}.$$

In his dissertation [3], Almgren exhibited an isomorphism

$$\pi_\ell(\mathcal{Z}_m(M; \mathbb{Z}), 0) \cong H_{\ell+m}(M; \mathbb{Z})$$

between the homotopy groups of $\mathcal{Z}_m$ and the homology groups of M , which he later employed in [4] for the purpose of constructing minimal submanifolds via min-max methods. As remarked in Chapter 1 of the introduction, recent years have seen a tremendous renewal of interest in the topology of spaces of cycles and related min-max constructions of minimal submanifolds (particularly in codimension one, where Pitts's work provides a powerful regularity theory [67]): we refer the interested reader to [39], [61], [58], [46], [62], [74], and references therein for some recent developments.

In the case $\ell = 1$ of interest for our present purposes, the map

$$\Psi : \pi_1(\mathcal{Z}_m(M; \mathbb{Z}), 0) \rightarrow H_{m+1}(M; \mathbb{Z})$$

is fairly simple to describe. First, by two applications of the Federer-Fleming isoperimetric inequality on manifolds ([31], Theorem 4.4.2), there exists a constant $\epsilon(M) > 0$ such that if

$$R \in \mathcal{Z}_{m+1}(M; \mathbb{Z}) \text{ with } \mathbb{M}(R) < \epsilon,$$

then $R = \partial\Omega$ for some $\Omega \in \mathcal{I}_{m+2}(M; \mathbb{Z})$, and there exists also $\delta > 0$ such that if

$$T \in \mathcal{Z}_m(M; \mathbb{Z}) \text{ with } \mathbb{F}(T) < \delta,$$

then $T = \partial S$ for some $S \in \mathcal{I}_{m+1}(M; \mathbb{Z})$ such that

$$\mathbb{M}(S) < \frac{\epsilon}{2}.$$

As a consequence, if $T_0, T_1, \dots, T_r$ is a finite sequence in $\mathcal{Z}_m(M; \mathbb{Z})$ with $T_0 = T_r = 0$ and

$$\mathbb{F}(T_i - T_{i-1}) < \delta \tag{13.1}$$

for every $i = 1, \dots, r$ (as can be obtained, for instance, by sampling points from a loop in $\pi_1(\mathcal{Z}_m(M; \mathbb{Z}), 0)$), then we can find $S_i \in \mathcal{I}_{m+1}$ for which

$$\partial S_i = T_i - T_{i-1} \text{ and } \mathbb{M}(S_i) < \frac{\epsilon}{2}. \tag{13.2}$$

The sum $\bar{S} = \sum_{i=1}^r S_i$ then defines a cycle in $\mathcal{Z}_{m+1}(M; \mathbb{Z})$, and for any other choices $S'_i \in \mathcal{I}_{m+1}$ satisfying (13.2), the differences $R_i := S'_i - S_i$ are $(m+1)$ -cycles of mass $\mathbb{M}(R_i) < \epsilon$, so that

$$S'_i = S_i + \partial\Omega_i$$

for some $\Omega_i \in \mathcal{I}_{m+2}(M; \mathbb{Z})$. In particular, it follows that the homology class $[\bar{S}] \in H_{m+1}(M; \mathbb{Z})$ of $\bar{S}$ is independent of the choice of S_i in (13.2). In a similar way (taking δ in (13.1) smaller, if necessary), it is shown in [3] that the homology class $[\bar{S}]$ produced in this way remains constant over sequences $\{T_i\}$, $\{T'_i\}$ which are close in an appropriate sense, and it is this observation which accounts for the

well-definedness of the map $\Psi : \pi_1(\mathcal{Z}_m(M; \mathbb{Z}), 0) \rightarrow H_{m+1}(M; \mathbb{Z})$.

Given a homotopy class $\Pi \in \pi_1(\mathcal{Z}_m(M; \mathbb{Z}), 0)$, for the purposes of intuition, the min-max width $\mathbf{L}(\Pi)$ can be identified with the quantity

$$\inf_{F \in \Pi} \sup_{y \in S^m} \mathbb{M}(F(y)), \quad (13.3)$$

giving the infimum over all families $F \in \Pi$ of the maximal mass attained by a cycle in the family F . For the purposes of this paper, however, it is convenient to define the min-max widths in terms of finite sequences $\{T_j\}$ in $\mathcal{Z}_m$ for which adjacent cycles are close in flat norm. The interested reader can compare the definition given below with those of [4], [67] (which require fineness in stronger norms), referring to interpolation procedures like those described in ([38], Section 8).

For $\delta > 0$, we denote by $\mathcal{S}_{m,\delta}(M)$ the collection of all finite sequences $\{T_i\}_{i=0}^r$ of integral m -cycles $T_i \in \mathcal{Z}_m(M; \mathbb{Z})$ for which

$$T_0 = T_r = 0 \text{ and } \mathbb{F}(T_{i-1} - T_i) < \delta \text{ for every } i = 1, \dots, r.$$

By the discussion in the preceding paragraphs (or see again [3], [4] Chapter 13), there are constants $\epsilon_0(M) > 0$ and $\delta_0(M) > 0$ such that for $\delta < \delta_0$, the map

$$\Psi : \mathcal{S}_{m,\delta}(M) \rightarrow H_{m+1}(M; \mathbb{Z})$$

given by

$$\Psi(\{T_i\}) = [\Sigma_{i=1}^r S_i]$$

for some $S_i \in \mathcal{I}_{m+1}(M; \mathbb{Z})$ satisfying

$$\partial S_i = T_i - T_{i-1} \text{ and } \mathbb{M}(S_i) < \frac{\epsilon_0}{2} \quad (13.4)$$

is well-defined, independent of the choice of $\{S_i\}$ satisfying (13.4).

Given a homology class $\xi \in H_{m+1}(M; \mathbb{Z})$ and $\delta < \delta_0$, we then set

$$\mathbf{L}_{m,\delta}(\xi) := \inf \left\{ \max_{0 \leq i \leq r} \mathbb{M}(T_i) \mid \{T_i\}_0^r \in \mathcal{S}_{m,\delta}(M), \Psi(\{T_i\}) = \xi \right\}, \quad (13.5)$$

and define

$$\mathbf{L}_m(\xi) := \lim_{\delta \rightarrow 0} \mathbf{L}_{m,\delta}(\xi) = \sup_{\delta > 0} \mathbf{L}_{m,\delta}(\xi). \quad (13.6)$$

By another simple application of the isoperimetric inequality, one finds that

$$\inf_{\xi \neq 0} \mathbf{L}_m(\xi) > 0;$$

to see this, note that there exists a constant $\eta_1(M) > 0$ such that for every $T \in \mathcal{Z}_m(M; \mathbb{Z})$ with $\mathbb{M}(T) < \eta_1$, there is some $R \in \mathcal{I}_{m+1}(M; \mathbb{Z})$ satisfying

$$T = \partial R \text{ and } \mathbb{M}(R) < \frac{\epsilon_0}{4}. \quad (13.7)$$

In particular, if we have $\{T_i\} \in \mathcal{S}_{m,\delta}(M)$ with $\max_i \mathbb{M}(T_i) < \eta_1$, then for each $i = 1, \dots, r-1$ we can choose $R_i \in \mathcal{I}_{m+1}(M; \mathbb{Z})$ satisfying (13.7) with respect to T_i , and setting $R_0 = R_r = 0$, we see that the currents

$$S_i := R_i - R_{i-1}$$

satisfy (13.4). But evidently $\sum_{i=1}^r S_i = 0$, so it follows that $\Psi(\{T_i\}) = 0$.

For the families of cycles $\{T_i\} \in \mathcal{S}_{n-k,\delta}(M)$ arising in the proof of Theorem 3.2, we can determine the associated homology class $\Psi(\{T_i\})$ only at the level of real homology. For any real homology class $\bar{\xi} \in H_{m+1}(M; \mathbb{R})$ containing an integral representative $S \in \mathcal{Z}_{m+1}(M; \mathbb{Z})$, we therefore define the real-homological widths

$$\mathbf{L}_{m,\mathbb{R}}(\bar{\xi}) := \min\{\mathbf{L}_m(\xi) \mid \xi \in H_{m+1}(M; \mathbb{Z}), \xi \equiv \bar{\xi} \text{ in } H_{m+1}(M; \mathbb{R})\}. \quad (13.8)$$

Equivalently, we can set

$$\mathbf{L}_{m,\mathbb{R},\delta}(\bar{\xi}) := \min\{\mathbf{L}_{m,\delta}(\xi) \mid \xi \equiv \bar{\xi} \text{ in } H_{m+1}(M; \mathbb{R})\}, \quad (13.9)$$

and define $\mathbf{L}_{m,\mathbb{R}}(\bar{\xi})$ by

$$\mathbf{L}_{m,\mathbb{R}}(\bar{\xi}) := \lim_{\delta \rightarrow 0} \mathbf{L}_{m,\mathbb{R},\delta}(\bar{\xi}) = \sup_{\delta > 0} \mathbf{L}_{m,\mathbb{R},\delta}(\bar{\xi}).$$

The need to work with real homology in the proof of Theorem 3.2 is due in part to the fact that the currents $S_i \in \mathcal{D}_{n+1-k}(M)$ that we use to connect adjacent $(n-k)$ -cycles $T_i - T_{i-1} = \partial S_i$ are not integer-rectifiable. However, from the results of Section 13.2 below, we will see that they have

the form $S_i = \Gamma_i + \partial R_i$, where $\Gamma_i \in \mathcal{I}_{n+1-k}(M; \mathbb{Z})$ and $R_i \in \mathcal{D}_{n+2-k}(M)$. The following lemma then allows us to compare the masses $\mathbb{M}(T_i)$ to the real-homological widths $\mathbf{L}_{m,\mathbb{R}}(\bar{\xi})$.

Lemma 13.1. *Given $\delta > 0$ and $L_1 < \infty$, there exists $\eta(M, L_1, \delta) > 0$ such that if $T_0, T_1, \dots, T_r \in \mathcal{Z}_m(M; \mathbb{Z})$ is a sequence of integral m -cycles of mass*

$$\mathbb{M}(T_i) \leq L_1, \quad (13.10)$$

with $T_0 = T_r = 0$, for which there exist $(m+1)$ -currents of the form

$$S_1, \dots, S_r \in \mathcal{I}_{m+1}(M; \mathbb{Z}) + \partial \mathcal{D}_{m+2}(M)$$

such that

$$\partial S_i = T_i - T_{i-1} \text{ and } \mathbb{M}(S_i) < \eta, \quad (13.11)$$

then $\{T_i\} \in \mathcal{S}_{m,\delta}(M)$, with

$$\Psi(\{T_i\}) \equiv [\Sigma_{i=1}^r S_i] \text{ in } H_{m+1}(M; \mathbb{R}). \quad (13.12)$$

Proof. To begin, we claim that there exists $\eta(M, L_1, \delta) > 0$ such that $\mathbb{F}(T) < \delta$ for any integral cycle T with

$$\mathbb{M}(T) \leq 2L_1 \text{ and } \|T\|_{(C^1)^*} < \eta.$$

Indeed, this is a simple consequence of the Federer-Fleming compactness theorem ([31], Theorem 4.2.17), since any sequence of integral cycles converging weakly to 0 with uniformly bounded mass must also converge to 0 in the flat norm. Applying this claim to the differences $T_i - T_{i-1}$ for a family of cycles $\{T_i\}$ satisfying (13.10)-(13.11), we immediately deduce that $\{T_i\} \in \mathcal{S}_{m,\delta}(M)$ for $\eta(M, L_1, \delta) > 0$ sufficiently small.

To check (13.12), fix (as in [35], Sect. 5.4.1) a collection $\omega^1, \dots, \omega^{b_{m+1}(M)} \in \mathcal{A}^{m+1}(M)$ of closed $(m+1)$ -forms generating the integer lattice in $H_{dR}^{m+1}(M)$, and let

$$C(M) := \max_{1 \leq i \leq b_{m+1}(M)} \|\omega^i\|_{L^\infty}.$$

Given $\{T_i\} \in \mathcal{S}_{m,\delta}(M)$ satisfying (13.10)-(13.11), let $S'_i \in \mathcal{I}_{m+1}(M; \mathbb{Z})$ be a family of integer

rectifiable $(m+1)$ -currents satisfying

$$\partial S'_i = T_i - T_{i-1} \text{ and } \mathbb{M}(S'_i) < \frac{\epsilon_0}{2}. \quad (13.13)$$

For each $i = 1, \dots, r$, the difference

$$R_i := S_i - S'_i$$

is then a cycle of the form $R_i \in \mathcal{Z}_{m+1}(M; \mathbb{Z}) + \partial \mathcal{D}_{m+2}(M)$, and as a consequence, we see that

$$\langle R_i, \omega \rangle \in \mathbb{Z} \quad (13.14)$$

for every $\omega \in \mathcal{A}^{m+1}(M)$. In particular, (13.14) holds for the generators $\omega^1, \dots, \omega^{b_{m+1}(M)}$ chosen above.

On the other hand, the mass bounds in (13.10) and (13.13) imply that

$$|\langle R_i, \omega \rangle| \leq C(M)(\eta + \frac{\epsilon_0}{2})$$

whenever $\|\omega\|_{L^\infty} \leq 1$. Thus, taking $\epsilon_0(M)$ and $\eta(M, L_1, \delta) > 0$ small enough that

$$C(M)(\eta + \epsilon_0/2) < 1,$$

it follows from (13.14) that $\langle R_i, \omega^j \rangle = 0$ for each $i = 1, \dots, r$ and $j = 1, \dots, b_{m+1}(M)$. Summing over $i = 1, \dots, r$, we therefore have

$$\langle \sum_{i=1}^r S'_i, \omega^j \rangle = \langle \sum_{i=1}^r S_i, \omega^j \rangle$$

for each $j = 1, \dots, b_{m+1}(M)$, and (13.12) follows. $\square$

13.2 A Decomposition Lemma for $S_\alpha(u, v)$

In this section, we prove that for weakly close maps $u, v \in \mathcal{E}^p(M, N)$, the current $S_\alpha(u, v) \in \mathcal{D}_{n+1-k}(M)$ of Lemma 12.3 admits a decomposition of the form

$$S_\alpha(u, v) := \Gamma + \partial R,$$

where $R \in \mathcal{D}_{n+2-k}(M)$, and $\Gamma \in \mathcal{I}_{n+1-k}(M)$ is integer rectifiable.

Lemma 13.2. *For $p \in (k-1, k)$ and $L_2 < \infty$, there exists $\epsilon(M, N, L_2, p) > 0$ such that if $u, v \in \mathcal{E}^p(M, N)$ satisfy*

$$E_p(u) + E_p(v) \leq L_2 \quad (13.15)$$

and

$$\|u - v\|_{L^p} < \epsilon, \quad (13.16)$$

then there exist $\Gamma \in \mathcal{I}_{n+1-k}(M)$ and $R \in \mathcal{D}_{n+2-k}(M)$ for which

$$S_\alpha(u, v) = \Gamma + \partial R. \quad (13.17)$$

The same result holds if either u or $v \in C^\infty(M, N)$.

Remark 13.3. By Lemma 12.3, (13.17) is clearly equivalent to the statement that

$$S_\alpha(v) - S_\alpha(u) = \Gamma + \partial R'$$

for some $R' \in \mathcal{D}_{n+2-k}(M)$.

Proof. By the Fubini-type arguments of [83] and [41], for maps $u, v \in \mathcal{E}^p(M, N)$ satisfying (13.15) and (13.16), we can find a cubeulation $h : |K| \rightarrow M$ such that

$$E_p(u \circ h, |K^{k-1}|) + E_p(v \circ h, |K^{k-1}|) \leq C(M)L_2, \quad (13.18)$$

$$\|u \circ h - v \circ h\|_{L^p(|K^{k-1}|)} \leq C(M)\epsilon, \quad (13.19)$$

where K is a fixed cubical complex and the Lipschitz constants $Lip(h)$ and $Lip(h^{-1})$ are bounded independent of u and v .

Moreover, since the singular sets $Sing(u)$ and $Sing(v)$ are contained in a finite union of $(n-k)$ -dimensional submanifolds of M , we can choose this $h : |K| \rightarrow M$ such that the $(k-1)$ -skeleton $h(|K^{k-1}|)$ lies a positive distance from $Sing(u) \cup Sing(v)$. Since u and v are Lipschitz away from $Sing(u) \cup Sing(v)$ by definition of $\mathcal{E}^p(M, N)$, it follows in particular that the restrictions $(u \circ h)|_{|K^{k-1}|}$ and $(v \circ h)|_{|K^{k-1}|}$ to the $(k-1)$ -skeleton are again Lipschitz.

Next (as in, e.g., [84], Theorem 1.1), we note that the compactness of the embedding $W^{1,p}(|K^{k-1}|) \hookrightarrow$

$C^0(|K^{k-1}|)$ implies the existence of a constant $\epsilon(M, N, L_2, p) > 0$ such that (13.18) and (13.19) imply

$$\|u \circ h - v \circ h\|_{C^0(|K^{k-1}|)} \leq \delta(N).$$

Here, $\delta(N)$ is chosen such that the $\delta(N)$ -neighborhood U_δ of N in $\mathbb{R}^L$ retracts $\pi_N : U_\delta \rightarrow N$ onto N .

Choosing such an ϵ , we proceed to define a map $w : M \times [0, 1] \rightarrow N$ satisfying $w(x, 0) = u(x)$ and $w(x, 1) = v(x)$ (compare [40], Lemma 2.2); throughout, we use the bi-Lipschitz map h to identify M and $|K|$, without comment. First, we set

$$w(x, 0) := u(x) \text{ and } w(x, 1) = v(x) \text{ for } x \in |K|,$$

and on $|K^{k-1}| \times [0, 1]$, we define

$$w(x, t) := \pi_N(tv(x) + (1 - t)u(x)).$$

For each k -cell σ in K , the restriction

$$w|_{\partial(\sigma \times [0, 1])} \in W^{1,p}(\partial(\sigma \times [0, 1]), N)$$

of w to $\partial(\sigma \times [0, 1])$ is then a well-defined Sobolev map, satisfying an estimate of the form

$$\sup_{x \in \partial(\sigma \times [0, 1])} \text{dist}(x, \Sigma_0) \cdot |dw(x)| < \infty,$$

where $\Sigma_0 = \text{Sing}(u) \times \{0\} \cup \text{Sing}(v) \times \{1\}$. Identifying $\sigma \times [0, 1]$ with the $(k+1)$ -ball B_1^{k+1} in a bi-Lipschitz way, we can then extend w to $\sigma \times [0, 1]$ radially, setting $w(x) = \frac{x}{|x|}$.

We have now extended w to a $W^{1,p}$ map on the whole $(k+1)$ -skeleton $|\tilde{K}^{k+1}|$ of $\tilde{K} = K \otimes [0, 1]$, satisfying

$$\sup_{x \in |\tilde{K}^{k+1}|} \text{dist}(x, \Sigma_1) \cdot |dw(x)| < \infty,$$

where Σ_1 is contained in a finite collection of Lipschitz curves in $|\tilde{K}^{k+1}|$. On each $(k+1)$ -cell σ of K , we can then extend w from $\partial(\sigma \times [0, 1])$ to $\sigma \times [0, 1]$ as above, and carry on in this way, until finally we have the desired map

$$w \in W^{1,p}(M \times [0, 1], N)$$

satisfying

$$w(x, 0) = u(x) \text{ and } w(x, 1) = v(x) \text{ in the trace sense,}$$

and

$$\sup_{x \in M} \text{dist}(x, \Sigma) |dw(x)| < \infty, \quad (13.20)$$

where $\Sigma \subset M \times [0, 1]$ is a $(n+1-k)$ -rectifiable set with $\mathcal{H}^{n+1-k}(\Sigma) < \infty$.

Doubling this construction, we can evidently extend w to a map

$$w : M \times S^1 \cong M \times [-3, 3]/6\mathbb{Z} \rightarrow N$$

satisfying

$$w(x, t) = u(x) \text{ for } t \in [-2, -1], \text{ } w(x, t) = v(x) \text{ for } t \in [1, 2], \quad (13.21)$$

and, by virtue of (13.20),

$$\|dw\|_{L^{k,\infty}(M)} < \infty. \quad (13.22)$$

In particular, it follows from Remark 12.13 that the homological singularity $T_\alpha(w)$ is an integral cycle in $M \times S^1$:

$$T_\alpha(w) = \partial S_\alpha(w) \in \mathcal{Z}_{n+1-k}(M \times S^1; \mathbb{Z}).$$

Now, let $\pi : M \times S^1 \rightarrow M$ be the obvious projection, and define currents $R \in \mathcal{D}_{n+2-k}(M)$ and $\Gamma \in \mathcal{I}_{n+1-k}(M; \mathbb{Z})$ to be the pushforwards

$$R := \pi_\#(S_\alpha(w)|_M(M \times [-1.5, 1.5]))$$

and

$$\Gamma := \pi_\#(T_\alpha(w)|_M(M \times [-1.5, 1.5])).$$

Fix a sequence $\psi_j \in C_c^\infty((-1.5, 1.5))$ satisfying $0 \leq \psi_j \leq 1$ and

$$\psi_j \equiv 1 \text{ on } [-1.5 + \frac{1}{j}, 1.5 - \frac{1}{j}];$$

since

$$S_\alpha(w) = S_\alpha(u) \times (-2, 1) \text{ on } M \times (-2, -1)$$

and

$$S_\alpha(w) = S_\alpha(v) \times (1, 2) \text{ on } M \times (1, 2),$$

it's clear that

$$R = \lim_{j \rightarrow \infty} R_j = \lim_{j \rightarrow \infty} \pi_\#(\psi_j(t) S_\alpha(w)) \quad (13.23)$$

and

$$\Gamma = \lim_{j \rightarrow \infty} \Gamma_j = \lim_{j \rightarrow \infty} \pi_\#(\psi_j(t) S_\alpha(w)). \quad (13.24)$$

For any $\zeta \in \Omega^{n+1-k}(M)$, we now compute

$$\begin{aligned} \langle \partial R_j, \zeta \rangle &= \langle R_j, d\zeta \rangle \\ &= \int_{M \times [-1.5, 1.5]} \psi_j(t) w^*(\alpha) \wedge d(\pi^* \zeta) \\ &= \int_{M \times [-1.5, 1.5]} w^*(\alpha) \wedge d(\psi_j(t) \pi^* \zeta) \\ &\quad - \int_{M \times [-1.5, 1.5]} w^*(\alpha) \wedge \psi_j'(t) dt \wedge \pi^* \zeta, \end{aligned}$$

which, by definition of Γ_j , gives

$$\begin{aligned} \langle \partial R_j - \Gamma_j, \zeta \rangle &= (-1)^{n+2-k} \int_{M \times [-1.5, 1.5]} w^*(\alpha) \wedge \pi^* \zeta \wedge \psi_j'(t) dt \\ &= (-1)^{n+2-k} \int_{-1.5}^{-1} \psi_j'(t) dt \cdot \left(\int_M u^*(\alpha) \wedge \zeta \right) \\ &\quad + (-1)^{n+2-k} \int_1^{1.5} \psi_j'(t) dt \cdot \left(\int_M v^*(\alpha) \wedge \zeta \right) dt \\ &= (-1)^{n+2-k} \langle S_\alpha(u) - S_\alpha(v), \zeta \rangle \end{aligned}$$

for j sufficiently large.

Passing to the limit $j \rightarrow \infty$, we conclude that

$$\partial R - \Gamma = (-1)^{n+2-k} (S_\alpha(u) - S_\alpha(v)),$$

and in particular,

$$S_\alpha(v) - S_\alpha(u) \in \mathcal{I}_{n+1-k}(M; \mathbb{Z}) + \partial \mathcal{D}_{n+2-k}(M).$$

Recalling from Lemma 12.3 that the current $S_\alpha(u, v)$ differs from $S_\alpha(v) - S_\alpha(u)$ by the boundary

of an $(n + 2 - k)$ -current, it follows that

$$S_\alpha(u, v) \in \mathcal{I}_{n+1-k}(M; \mathbb{Z}) + \partial \mathcal{D}_{n+2-k}(M)$$

as well, as desired. □

13.3 Proof of Theorem 3.2

Now, let $u, v \in C^\infty(M, N)$ be $(k - 2)$ -homotopic, and suppose there exists $\alpha \in \mathcal{A}^{k-1}(N)$ such that

$$[u^*(\alpha)] - [v^*(\alpha)] \neq 0 \in H_{dR}^{k-1}(M),$$

or, equivalently,

$$[S_\alpha(v) - S_\alpha(u)] \neq 0 \in H_{n+1-k}(M; \mathbb{R}).$$

Evidently, such an α exists if and only if u and v induce different maps on the de Rham cohomology $H_{dR}^{k-1}(N) \rightarrow H_{dR}^{k-1}(M)$.

For every $\delta > 0$, we define $\mathcal{C}_\delta^p(u, v)$ to be the collection of all finite sequences

$$u_0, u_1, \dots, u_r \in W^{1,p}(M, N)$$

such that $u_0 = u$, $u_r = v$, and

$$\|u_i - u_{i-1}\|_{L^p(M)} < \delta$$

for every $i = 1, \dots, r$. We then define

$$\gamma_p^\delta(u, v) := \inf \left\{ \max_{0 \leq j \leq r} E_p(u_j) \mid \{u_j\}_{j=1}^r \in \mathcal{C}_\delta^p(u, v) \right\},$$

and set

$$\gamma_p^*(u, v) := \lim_{\delta \rightarrow 0} \gamma_p^\delta(u, v) = \sup_{\delta > 0} \gamma_p^\delta(u, v). \quad (13.25)$$

Remark 13.4. As observed in the introduction, it's clear from the definitions that $\gamma_p^*(u, v) \leq \gamma_p(u, v)$, since for any path $u_t \in W^{1,p}(M, N)$ from u to v and any $\delta > 0$, we can find a finite sequence

$0 = t_0, t_1, \dots, t_r = 1$ such that $\{u_{t_i}\} \in \mathcal{C}_\delta^p(u, v)$. In particular, it follows from Theorem 3.1 that

$$\sup_{p < k} (k - p) \gamma_p^*(u, v) < \infty.$$

We recall now the statement of Theorem 3.2.

Theorem 13.5. *For closed, oriented Riemannian manifolds M^n, N , maps $u, v \in C^\infty(M, N)$, and a $(k - 1)$ -form $\alpha \in \mathcal{A}^{k-1}(N)$ as above, such that*

$$\bar{\xi} := [S_\alpha(v) - S_\alpha(u)] \neq 0 \in H_{n+1-k}(M; \mathbb{R}),$$

define

$$\Lambda(u, v) := \liminf_{p \rightarrow k} (k - p) \gamma_p^*(u, v).$$

Then we have the lower bound

$$\sigma_{k-1} \mathbf{L}_{n-k, \mathbb{R}}(\bar{\xi}) \leq \lambda(\alpha)^{\frac{k}{k-1}} \Lambda(u, v). \quad (13.26)$$

Most of the work in the proof of Theorem 13.5 is contained in the following lemma, which combines the results of Chapter 12 and Lemma 13.2.

Lemma 13.6. *For any $\eta \in (0, 1)$, there exists $q(\eta) \in (k - 1, k)$ with the property that for every $p \in (q, k)$, there exists $\delta_1(p) > 0$ such that for any $\{u_i\}_{i=0}^r \in \mathcal{C}_{\delta_1}^p(u, v)$ satisfying*

$$(k - p) \max_j E_p(u_j) \leq \Lambda(u, v) + \eta, \quad (13.27)$$

we can find cycles $T_0, T_1, \dots, T_r \in \mathcal{Z}_{n-k}(M; \mathbb{Z})$ with $T_0 = T_r = 0$ for which

$$\max_{0 \leq i \leq r} \sigma_{k-1} \mathbb{M}(T_i) \leq \lambda(\alpha)^{\frac{k}{k-1}} (\Lambda(u, v) + \eta),$$

and currents

$$S_1, \dots, S_r \in \mathcal{I}_{n+1-k}(M; \mathbb{Z}) + \partial \mathcal{D}_{n+2-k}(M)$$

such that

$$\partial S_i = T_i - T_{i-1},$$

$$\mathbb{M}(S_i) < 3\eta, \quad (13.28)$$

and

$$[\Sigma_{i=1}^r S_i] = [S_\alpha(v) - S_\alpha(u)] \in H_{n+1-k}(M; \mathbb{R}).$$

Proof. First, we appeal to Theorem 12.16 to guarantee the existence of $q_0(\eta) = q_0(\Lambda(u, v) + \eta, \eta) > 0$ such that for any $p \in (q_0, k)$ and any sequence of maps $u_1, \dots, u_{r-1} \in W^{1,p}(M, N)$ satisfying (13.27), there exists a corresponding sequence $\tilde{u}_1, \dots, \tilde{u}_{r-1} \in \mathcal{E}^p(M, N)$ satisfying

$$E_p(\tilde{u}_i) \leq \frac{C}{(k-p)^2}, \quad (13.29)$$

$$\|u_i - \tilde{u}_i\|_{L^p}^p \leq C(k-p)^{3p-2}, \quad (13.30)$$

and a sequence of integral cycles $T_i \in \mathcal{Z}_{n-k}(M; \mathbb{Z})$ and integral $(n+1-k)$ -currents $\Gamma_i \in \mathcal{I}_{n+1-k}(M; \mathbb{Z})$ such that

$$\sigma_{k-1} \mathbb{M}(T_i) \leq \lambda(\alpha)^{\frac{k}{k-1}} [\Lambda(u, v) + \eta] \quad (13.31)$$

$$T_\alpha(\tilde{u}_i) - T_i = \partial \Gamma_i, \quad (13.32)$$

and

$$\mathbb{M}(\Gamma_i) < \eta. \quad (13.33)$$

Now, consider a family $\{u_i\}_{i=0}^r \in \mathcal{C}_\delta^p(u, v)$ for $p \in (q_0, k)$ satisfying (13.27). For $1 \leq i \leq r-1$, choose

$$\tilde{u}_i \in \mathcal{E}^p(M, N), \quad T_i \in \mathcal{Z}_{n-k}(M; \mathbb{Z}), \quad \text{and} \quad \Gamma_i \in \mathcal{I}_{n+1-k}(M; \mathbb{Z})$$

satisfying (13.29)-(13.33), and extend these sequences trivially by setting

$$\tilde{u}_0 = u, \quad \tilde{u}_r = v, \quad T_0 = T_r = 0, \quad \text{and} \quad \Gamma_0 = \Gamma_r = 0.$$

Next, setting

$$S_i := S_\alpha(\tilde{u}_{i-1}, \tilde{u}_i) + \Gamma_{i-1} - \Gamma_i,$$

for $i = 1, \dots, r$, we see that

$$\partial S_i = T_i - T_{i-1},$$

and, by Lemma 13.2, these S_i have the form

$$S_i \in \mathcal{I}_{n+1-k}(M; \mathbb{Z}) + \partial \mathcal{D}_{n+2-k}(M).$$

Moreover, since

$$\sum_{i=1}^r (\Gamma_{i-1} - \Gamma_i) = \Gamma_0 - \Gamma_r = 0,$$

and (by Lemma 12.3)

$$S_\alpha(\tilde{u}_i) - S_\alpha(\tilde{u}_{i-1}) - S_\alpha(\tilde{u}_{i-1}, \tilde{u}_i) \in \partial D_{n+2-k}(M),$$

it follows that

$$[\sum_{i=1}^r S_i] = [S_\alpha(v) - S_\alpha(u)] \in H_{n+1-k}(M; \mathbb{R}). \quad (13.34)$$

To complete the proof of the lemma, it remains to establish the mass bound (13.28) for these S_i , for $p > q(\eta)$ sufficiently large and $\delta = \delta_1(p)$ sufficiently small. To estimate the mass of

$$S_i = S_\alpha(\tilde{u}_{i-1}, \tilde{u}_i) + \Gamma_{i-1} - \Gamma_i,$$

we first apply (13.33) to see that

$$\mathbb{M}(S_i) \leq \mathbb{M}(S_\alpha(\tilde{u}_{i-1}, \tilde{u}_i)) + 2\eta.$$

Now, by Lemma 12.3 and the energy bound (13.29), we know that

$$\begin{aligned} \mathbb{M}(S_\alpha(\tilde{u}_{i-1}, \tilde{u}_i)) &\leq C[E_p(\tilde{u}_{i-1})^{\frac{k-1}{p}} + E_p(\tilde{u}_i)^{\frac{k-1}{p}}] \|\tilde{u}_i - \tilde{u}_{i-1}\|_{L^p}^{1+p-k} \\ &\leq C(k-p)^{-2(k-1)/p} \|\tilde{u}_i - \tilde{u}_{i-1}\|_{L^p}^{1+p-k}. \end{aligned}$$

Moreover, by (13.30), we have

$$\|u_i - \tilde{u}_i\|_{L^p}^p \leq C(k-p)^{3p-2},$$

while by definition of $\mathcal{C}_\delta^p(u, v)$, we have also

$$\|u_i - u_{i-1}\|_{L^p}^p < \delta^p.$$

Taking $\delta = \delta_1(p) = (k-p)^{3-2/p}$ and combining the estimates above, it follows in particular that

$$\|\tilde{u}_i - \tilde{u}_{i-1}\|_{L^p}^p \leq C'(k-p)^{3p-2},$$

and consequently,

$$\begin{aligned}
\mathbb{M}(S_\alpha(\tilde{u}_{i-1}, \tilde{u}_i)) &\leq C(k-p)^{-2(k-1)/p} \|\tilde{u}_i - \tilde{u}_{i-1}\|_{L^p}^{1+p-k} \\
&\leq C'(k-p)^{-\frac{2(k-1)}{p} + (3p-2)\frac{1+p-k}{p}} \\
&= C'(k-p)^{1-3(k-p)}.
\end{aligned}$$

Since $\lim_{p \rightarrow k} (k-p)^{1-3(k-p)} = 0$, we can therefore choose $q(\eta) \in (k-1, k)$ such that

$$\mathbb{M}(S_\alpha(\tilde{u}_{i-1}, \tilde{u}_i)) < \eta,$$

and, consequently,

$$\mathbb{M}(S_i) < 3\eta$$

for $p \in (q, k)$. This completes the proof. $\square$

Combining the preceding lemma with the results of Lemma 13.1, we complete the proof of Theorem 13.5 as follows.

Proof. Fix some $\delta \in (0, 1)$. By Lemma 13.1, we can find $\eta(\delta) \in (0, \delta)$ such that for any sequence $\{T_i\}_{i=0}^r \subset \mathcal{Z}_{n-k}(M; \mathbb{Z})$ satisfying $T_0 = T_r = 0$,

$$\mathbb{M}(T_i) \leq C(\alpha)[\Lambda(u, v) + 1] \quad (13.35)$$

and

$$T_i - T_{i-1} = \partial S_i \quad (13.36)$$

for some $S_1, \dots, S_r \in \mathcal{I}_{n+1-k}(M; \mathbb{Z}) + \partial D_{n-k+2}(M)$ with

$$\mathbb{M}(S_i) < 3\eta, \quad (13.37)$$

we have $\{T_i\} \in \mathcal{S}_{n-k, \delta}(M)$, with the associated homology class $\Psi(\{T_i\})$ satisfying

$$\Psi(\{T_i\}) \equiv [\Sigma_{i=1}^r S_i] \text{ in } H_{n-k+1}(M; \mathbb{R}). \quad (13.38)$$

Now, with $\eta(\delta) \in (0, \delta)$ as above, let $q(\eta)$ be as in Lemma 13.6, and choose $p \in (q, k)$ such that

$$(k-p)\gamma_p^*(u, v) \leq \Lambda(u, v) + \eta/2.$$

Then, choose $\delta_1(p) > 0$ according to Lemma 13.6, and select some family $\{u_i\}_{i=0}^r \in \mathcal{C}_{\delta_1}^p(u, v)$ such that

$$\begin{aligned} (k-p) \max_i E_p(u_i) &\leq (k-p)\gamma_p^{\delta_1}(u, v) + \eta/2 \\ &\leq \Lambda(u, v) + \eta. \end{aligned}$$

By Lemma 13.6, we can associate to these $\{u_i\}_{i=0}^r$ a family of cycles $T_0, \dots, T_r \in \mathcal{Z}_{n-k}(M; \mathbb{Z})$ and currents $S_1, \dots, S_r \in \mathcal{I}_{n+1-k}(M; \mathbb{Z}) + \partial\mathcal{D}_{n-k+2}(M)$ satisfying (13.35)-(13.37), as well as the sharper mass bound

$$\begin{aligned} \sigma_{k-1} \max_i \mathbb{M}(T_i) &< \lambda(\alpha)^{\frac{k}{k-1}} \Lambda(u, v) + C(\alpha)\eta \\ &< \lambda(\alpha)^{\frac{k}{k-1}} \Lambda(u, v) + C(\alpha)\delta, \end{aligned}$$

and the homological condition

$$[\Sigma_{i=1}^r S_i] \equiv [S_\alpha(v) - S_\alpha(u)] \text{ in } H_{n-k+1}(M; \mathbb{R}).$$

In particular, it follows that there exists $\{T_i\} \in \mathcal{S}_{n-k, \delta}(M)$ satisfying

$$\Psi(\{T_i\}) \equiv [S_\alpha(v) - S_\alpha(u)] \in H_{n+1-k}(M; \mathbb{R})$$

and

$$\max_i \sigma_{k-1} \mathbb{M}(T_i) < \lambda(\alpha)^{\frac{k}{k-1}} \Lambda(u, v) + C(\alpha)\delta.$$

Recalling the notation of Section 13.1, this means precisely that

$$\sigma_{k-1} \mathbf{L}_{n-k, \mathbb{R}, \delta}([S_\alpha(v) - S_\alpha(u)]) < \lambda(\alpha)^{\frac{k}{k-1}} \Lambda(u, v) + C(\alpha)\delta.$$

Finally, taking $\delta \rightarrow 0$, we arrive at the desired inequality

$$\sigma_{k-1} \mathbf{L}_{n-k, \mathbb{R}}([S_\alpha(v) - S_\alpha(u)]) \leq \lambda(\alpha)^{\frac{k}{k-1}} \Lambda(u, v).$$



Chapter 14

Consequences for the Space of p -Harmonic Maps

In this short chapter, we consider the implications of Theorems 3.1 and 3.2 for the variational landscape of the p -energy functionals and the existence theory for stationary p -harmonic maps in M . In particular, we explain carefully how the presence of large energy walls separating maps u and v with different actions on the cohomology H_{dR}^{k-1} (where $k = \lceil p \rceil$) gives rise to new local minimizers and mountain-pass type critical points for the L^p norm of the gradient on $W^{1,p}(M, N)$.

14.1 Local Minimizers of the p -Energy

In this short section, we show that a given component of $W^{1,p}(M, N)$ ($p \in (k-1, k)$) may contain multiple local minimizers of the p -energy for p sufficiently close to k , whenever there exist maps $u, v : M \rightarrow N$ which are $(k-2)$ -homotopic but induce different actions on the real $(k-1)$ -cohomology H_{dR}^{k-1} . For example, when N is $(k-2)$ -connected (e.g., $N = S^{k-1}$), then although the space $W^{1,p}(M, N)$ is always path connected (by [18]), it follows that the variational landscape of E_p on $W^{1,p}(M, N)$ can nonetheless be quite rich, with multiple local minimizers corresponding to classes in $[M : N]$ distinguished by their actions on the first nontrivial cohomology $H_{dR}^{k-1}(M)$.

Claim 14.1. *Let $u, v \in C^1(M, N)$ be $(k-2)$ -homotopic maps from M to N inducing distinct actions on the deRham cohomology H_{dR}^{k-1} . Then for $p \in (k-1, k)$ sufficiently close to k , the maps u and v*

lie in distinct components of the energy sublevel set

$$E_p^{u,v} := \{w \in W^{1,p}(M, N) \mid E_p(w) \leq E_p(u) + E_p(v)\} \quad (14.1)$$

with respect to the weak topology induced by the L^p -norm. As a consequence, we find distinct local minimizers u_0 and v_0 of the p -energy lying in the components containing u and v , respectively.

Proof. The proof is a straightforward consequence of Theorem 13.5 in the previous chapter. Fixing $\delta > 0$, denote by U_δ the collection of all maps $w \in E_p^{u,v}$ for which there exists a sequence $u = u_1, u_2, \dots, u_m = w$ in $E_p^{u,v}$ with $\|u_{i+1} - u_i\|_{L^p} < \delta$. It is easy to check that U_δ is open and closed in $E_p^{u,v}$ with respect to the L^p norm.

Now, for p sufficiently close to k , it follows from Theorem 13.5 that $\gamma_p^*(u, v) > E_p(u) + E_p(v)$, and therefore (recalling the definition (13.25) of γ_p^*)

$$\gamma_p^\delta(u, v) > E_p(u) + E_p(v) \quad (14.2)$$

for $\delta > 0$ sufficiently small. On the other hand, the estimate (14.2) is equivalent to the statement that $v \notin U_\delta$. Since U_δ is both open and closed in the weak topology, it follows that u and v indeed lie in separate components of $E_p^{u,v}$ with respect to this topology, and we obtain the p -energy minimizing maps u_0 and v_0 by applying the direct method in the components containing u and v , respectively. $\square$

14.2 Mountain-pass Critical Points

As before, let $u, v \in C^1(M, N)$ be maps which are $(k-2)$ -homotopic but induce different actions on H_{dR}^{k-1} . In this section, we demonstrate the existence of mountain pass critical points for the p -energy functional with energy lying between $\gamma_p^*(u, v)$ and $\gamma_p(u, v)$.

We produce these p -harmonic maps by applying standard mountain pass methods to the generalized Ginzburg-Landau functionals studied by Wang in [82], discussed before in Section 10.1. Fixing once again an isometric embedding

$$N \subset \mathbb{R}^L,$$

define the p -Ginzburg-Landau functionals

$$E_{p,\epsilon}(w) := \int_M (|dw|^p + \epsilon^{-p} F(w)) \quad (14.3)$$

as in Section 10.1. For the $(k-2)$ -homotopic maps $u, v \in C^\infty(M, N)$ and $p \in (1, k)$, we then define the mountain pass energies $\gamma_{GL,p,\epsilon}(u, v)$ to be the infimum

$$\gamma_{GL,p,\epsilon}(u, v) := \inf \left\{ \max_{t \in [0,1]} E_{p,\epsilon}(w_t) \mid w_0 = u, w_1 = v \right\} \quad (14.4)$$

of the maximum energy $\max_{t \in [0,1]} E_{p,\epsilon}(w_t)$ over all continuous paths $t \mapsto w_t$ in $C^0([0, 1], W^{1,p}(M, \mathbb{R}^L))$ from $w_0 = u$ to $w_1 = v$. It follows immediately that

$$\gamma_{GL,p,\epsilon}(u, v) \leq \gamma_p(u, v) \quad (14.5)$$

for every $\epsilon > 0$, since any continuous family $w_t \in W^{1,p}(M, N)$ connecting u to v through N -valued maps satisfies

$$\gamma_{GL,p,\epsilon}(u, v) \leq \max_{t \in [0,1]} E_{p,\epsilon}(w_t) = \max_{t \in [0,1]} E_p(w_t).$$

Next, we establish the lower bound

$$\gamma_p^*(u, v) \leq \sup_{\epsilon > 0} \gamma_{GL,p,\epsilon}(u, v). \quad (14.6)$$

This will follow fairly easily from the definition of $\gamma_p^*(u, v)$ and the following easy lemma.

Lemma 14.2. *For every $\eta > 0$, there exists some $\epsilon_0(p, \eta) > 0$ such that if $\epsilon < \epsilon_0$ and $w \in W^{1,p}(M, \mathbb{R}^L)$ satisfies*

$$E_{p,\epsilon}(w) < \gamma_p(u, v) + 1,$$

then there exists $w' \in W^{1,p}(M, N)$ such that

$$\|w - w'\|_{L^p} < \eta$$

and

$$E_p(w') \leq E_{p,\epsilon}(w) + \eta.$$

Proof. This is another simple proof by contradiction via compactness. If, for some $\eta > 0$, no such $\epsilon_0(p, \eta)$ existed, then we could find a sequence $\epsilon_j \rightarrow 0$ and $w_j \in W^{1,p}(M, \mathbb{R}^L)$ such that

$$\limsup_{j \rightarrow \infty} E_{p,\epsilon_j}(w_j) \leq \gamma_p(u, v) + 1 < \infty$$

and for every j and $w' \in W^{1,p}(M, N)$, either

$$\|w_j - w'\|_{L^p} > \eta \text{ or } E_p(w') > E_{p,\epsilon}(w_j) + \eta. \quad (14.7)$$

But, passing to a further subsequence, we can find $w \in W^{1,p}(M, \mathbb{R}^L)$ for which $w_j \rightharpoonup w$ in $W^{1,p}$ and

$$\|w_j - w\|_{L^p} \rightarrow 0. \quad (14.8)$$

Since the energies $E_{p,\epsilon_j}(w_j)$ are uniformly bounded as $\epsilon_j \rightarrow 0$, we see that

$$\lim_{j \rightarrow \infty} \int_M F(w_j) = 0,$$

and consequently $w \in W^{1,p}(M, N)$. And of course, it follows from the weak convergence that

$$E_p(w) \leq \liminf_{j \rightarrow \infty} E_{p,\epsilon}(w_j), \quad (14.9)$$

which, together with (14.8), contradicts (14.7). $\square$

Now, for any $\delta > 0$, choose $\epsilon_0 = \epsilon_0(p, \delta/3)$ according to Lemma 14.2, and for $\epsilon < \epsilon_0$, consider a path $w_t \in W^{1,p}(M, \mathbb{R}^L)$ connecting $w_0 = u$ to $w_1 = v$, such that

$$\max_{t \in [0,1]} E_{p,\epsilon}(w_t) \leq \gamma_{GL,p,\epsilon}(u, v) + \delta.$$

Select a sequence of times $0 = t_0 < t_1 < \dots < t_r = 1$ such that

$$\|w_{t_i} - w_{t_{i-1}}\|_{L^p} < \frac{\delta}{3}, \quad (14.10)$$

and for each $1 \leq i \leq r-1$, appeal to Lemma 14.2 to find a map $u_i \in W^{1,p}(M, N)$ such that

$$\|u_i - w_{t_i}\|_{L^p} < \frac{\delta}{3} \quad (14.11)$$

and

$$E_p(u_i) \leq \gamma_{GL,p,\epsilon}(u, v) + 2\delta. \quad (14.12)$$

It follows from (14.10) and (14.11) that the sequence

$$u = u_0, u_1, \dots, u_{r-1}, u_r = v$$

belongs to $\mathcal{C}_\delta^p(u, v)$, and from (14.12), we therefore see that

$$\gamma_p^\delta(u, v) \leq \gamma_{GL,p,\epsilon}(u, v) + 2\delta. \quad (14.13)$$

In particular, we've now shown that

$$\gamma_p^\delta(u, v) \leq \sup_{\epsilon > 0} \gamma_{GL,p,\epsilon}(u, v) + 2\delta$$

for every $\delta > 0$. Taking the limit as $\delta \rightarrow 0$, we arrive at the desired lower bound (14.6).

For p sufficiently large and $\epsilon > 0$ sufficiently small, we can combine Lemma 14.2 with arguments very identical to those of the preceding section to establish the existence of local minimizers $u_{0,\epsilon}$ and $v_{0,\epsilon}$ of the energy $E_{p,\epsilon}$ in the distinct components of $\{w \in W^{1,p} \mid E_{p,\epsilon}(w) \leq E_p(u) + E_p(v)\}$ containing u and v , respectively.

Finally, since the generalized Ginzburg-Landau energies $E_{p,\epsilon}$ are C^1 functionals on the Banach space $W^{1,p}(M, \mathbb{R}^L)$, and satisfy a Palais-Smale condition (as discussed in Section 10.1), we can appeal to standard existence results for critical points of mountain pass type (see [34], Chapter 6) to deduce that for p sufficiently close to k and $\epsilon > 0$ sufficiently small, there exists a critical point $w_{p,\epsilon} \in W^{1,p}(M, \mathbb{R}^L)$ of $E_{p,\epsilon}$ of energy

$$E_{p,\epsilon}(w_\epsilon) > \inf \left\{ \max_{t \in [0,1]} E_{p,\epsilon}(w_t) \mid w_0 = u_{0,\epsilon}, w_1 = v_{0,\epsilon} \right\}.$$

In light of the upper bound (14.5) for the energies $\gamma_{GL,p,\epsilon}(u, v)$, these critical points $w_{p,\epsilon}$ have uniformly bounded energies $E_{p,\epsilon}(w_\epsilon)$ as $\epsilon \rightarrow 0$. Since $p \notin \mathbb{N}$, it therefore follows from the compactness results of [82] (see again the discussion in Section 10.1) that some subsequence w_{p,ϵ_j} converges strongly to a stationary p -harmonic map $w \in W^{1,p}(M, N)$. In particular, combining this observation with the estimates (14.5) and (14.6), we arrive finally at the following existence result.

Proposition 14.3. *For every $p \in (k-1, k)$ sufficiently close to k , there exists a stationary p -harmonic map $w \in W^{1,p}(M, N)$ of energy*

$$\gamma_p(u, v) \geq E_p(w) = \sup_{\epsilon > 0} \gamma_{GL,p,\epsilon}(u, v) \geq \gamma_p^*(u, v). \quad (14.14)$$

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