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Kähler-Einstein metrics and normalized volumes of valuations

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Kähler-Einstein metrics and normalized volumes of valuations

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A Dissertation
Presented to the Faculty
of Princeton University
in Candidacy for the Degree
of Doctor of Philosophy

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Abstract

In this thesis we study questions relating "global volumes" of Kähler-Einstein Fano varieties and "local volumes" at a singularity. The "local volume" is interpreted as the normalized volume of real valuations centered at a klt singularity recently introduced by Li. In the first part, we show that the volume of a Kähler-Einstein Fano variety is bounded from above by the normalized volume of any valuation centered at a closed point. This refines a recent result of Fujita. In the second part, we study the minimization problem of normalized volumes over cone singularities. We show that a Fano manifold is K-semistable if and only if the normalized volume function over its affine cone is minimized at the canonical valuation. This confirms a conjecture of Li.

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Chapter 1

Introduction

1.1 Volume of Kähler-Einstein Fano varieties

The main goal of algebraic geometry is to understand the structure of algebraic varieties, which are common zero sets of multivariable polynomials. Since all algebraic varieties seem to be too large of a class to study, people introduced various conditions on algebraic varieties to get interesting subclasses. *Fano varieties* are projective varieties with ample anti-canonical divisor. Basic examples of Fano varieties include projective space $\mathbb{P}^n$, hypersurfaces of degree at most $(n+1)$ in $\mathbb{P}^{n+1}$, weighted projective space $\mathbb{P}(a_0, \cdots, a_n)$, Grassmannians and so on. In this thesis, we will restrict ourselves to the following subclass of Fano varieties.

Definition 1.1.1. An n-dimensional complex projective variety X is said to be a $\mathbb{Q}$-*Fano variety* if X is normal with klt singularities and the anticanonical divisor $-K_X$ is an ample $\mathbb{Q}$-Cartier divisor.

Given a $\mathbb{Q}$-Fano variety X of dimension n, we define the *volume* of X as $\mathrm{vol}(X) := (-K_X)^n$. The volume of a $\mathbb{Q}$-Fano variety is a fundamental geometric and topological invariant. Kollár, Miyaoka, Mori [KMM92] and Campana [Cam92] proved that smooth complex Fano varieties have uniform volume upper bounds that only depend on the dimension. If we allow singularities, then the volume of a $\mathbb{Q}$-Fano variety X can be arbitrarily large as soon as $n \geq 2$ (e.g. volumes of weighted projective spaces $\mathbb{P}(1^n, d)$).

A fundamental problem in complex geometry is to find Kähler-Einstein metrics (i.e. Kähler metrics of constant Ricci curvature) on projective varieties. By the work of Aubin [Aub76] and Yau [Yau78], we know that there exist Kähler-Einstein metrics on a smooth projective variety when its canonical divisor is ample or numerically trivial. This problem becomes subtler on Fano varieties.

For example, projective space equipped with the Fubini-Study metric is Kähler-Einstein, while other weighted projective spaces do not admit (singular) Kähler-Einstein metrics (see [RT11]).

The existence of Kähler-Einstein metrics on a Fano variety often controls geometry of the variety. For example, Matsushima [Mat57] showed that a Kähler-Einstein Fano manifold has reductive automorphism group; Futaki [Fut83] found an important invariant (now known as Futaki invariant) as the obstruction to the existence of Kähler-Einstein metrics. In the case of Kähler-Einstein $\mathbb{Q}$-Fano varieties, it is expected that their volumes are always bounded from above by constants only depending on the dimension (see [Tia90, OSS16] for surfaces, and [BB11, BB17] for toric cases). Recently, Fujita [Fuj15] confirmed this expectation by showing that among all n-dimensional Kähler-Einstein $\mathbb{Q}$-Fano varieties, the projective space $\mathbb{P}^n$ has the maximal volume. It is then natural to ask whether Fujita's volume upper bounds can be improved for Kähler-Einstein $\mathbb{Q}$-Fano varities with prescribed singularities.

In Chapter 3, we refine Fujita's result by looking at local invariants of singularities. In the case of quotient singularities, we prove the following theorem:

Theorem 1.1.2. *Let X be a Kähler-Einstein $\mathbb{Q}$-Fano variety of dimension n. Assume $p \in X$ is a quotient singularity with local analytic model $\mathbb{C}^n/G$ where $G \subset GL(n, \mathbb{C})$ acts freely in codimension 1. Then*

$$(-K_X)^n \leq \frac{(n+1)^n}{|G|},$$

with equality if and only if $|G \cap \mathbb{G}_m| = 1$ and $X \cong \mathbb{P}^n/G$, where $\mathbb{G}_m \subset GL(n, \mathbb{C})$ is the subgroup consisting of non-zero scalar matrices.

It is well-known that in dimension 2, klt singularities are the same as quotient singularities. However, this is far from being true in higher dimensions. Moreover, klt singularities can be very complicated starting from dimension 4 (e.g. [Kol11]). In order to obtain volume upper bounds when X has an arbitrary prescribed klt singularity, we need to use the *normalized volume function of valuations* introduced by C. Li [Li15b]. For a klt singularity (X, p), let v be a real valuation on $\mathbb{C}(X)$ centered at p. Then the normalized volume function of v is defined as $\widehat{\mathrm{vol}}(v) := A_X(v)^n \cdot \mathrm{vol}(v)$, where $A_X(v)$ is the log discrepancy of v and $\mathrm{vol}(v)$ is the volume of v. If we denote by $\mathrm{Val}_{X,p}$ the space of real valuations on $\mathbb{C}(X)$ centered at p, then the function $\widehat{\mathrm{vol}} : \mathrm{Val}_{X,p} \to \mathbb{R}_{>0} \cup \{+\infty\}$ is invariant under rescaling. (A detailed study of $\widehat{\mathrm{vol}}$ will be given in Section 2.3.) Using Li's normalized volume function, we prove the following theorem:

Theorem 1.1.3. *Let X be a Kähler-Einstein $\mathbb{Q}$-Fano variety of dimension n. Let v be a real*

valuation on $\mathbb{C}(X)$ centered at a closed point p. Then we have

$$(-K_X)^n \leq \left(1+\frac{1}{n}\right)^n \widehat{\mathrm{vol}}(v).$$

It is clear that in order to obtain the smallest volume upper bounds, we need to study the problem of minimizing the normalized volume of valuations at a klt singularity. This problem was indeed proposed by Li in [Li15b] with a different motivation. In the next section, we will discuss a relation between miniminzing normalized volumes over cone singularities and K-semistability.

We remark here that results in Chapter 3 are took from the preprint [Liu16].

1.2 Valuative criteria for K-semistability

The concept of K-stability, first introduced by Tian [Tia97] and later formulated algebraically by Donaldson [Don02], provides algebro-geometric obstructions to the existence of Kähler-Einstein metrics on Fano varieties. By the work of many people on Yau-Tian-Donaldson conjecture spanning a long period of time (see in particular [Tia97, Ber16] and [CDS15, Tia15]), we now know that, for a smooth Fano manifold V, the existence of Kähler-Einstein metric on V is equivalent to K-polystability of V. In [Li13], Li proved that a smooth Fano manifold is K-semistable if and only if it is "almost Kähler-Einstein" (see [Li13] for details).

Motivated by the work of Martelli-Sparks-Yau [MSY08] in the study of Sasaki-Einstein metric and Donaldson-Sun's study on metric tangent cones [DS15], Li [Li15b] proposed to understand K-semistability of Fano manifolds from the point view of volume minimizations. In his point of view, for any Fano manifold V, we consider the space of real valuations on the affine cone $\mathcal{C} = C(V, -K_V)$ with center being the vertex o (we denote this space by $\mathrm{Val}_{\mathcal{C},o}$) and study the normalized volume functional $\widehat{\mathrm{vol}}$ on $\mathrm{Val}_{\mathcal{C},o}$. In [Li15b], Li conjectured that the Fano manifold V is K-semistable if and only if $\widehat{\mathrm{vol}}$ is minimized at the canonical valuation ord_V.

In Chapter 4, based on Li's result [Li15a] and a logarithmic version of Theorem 1.1.3, we confirm Li's conjecture:

Theorem 1.2.1 (with C. Li)**.** *Let V be a Fano manifold. Let L be an ample Cartier divisor on V such that $rL \sim_{\mathbb{Q}} -K_X$ for some positive rational number r. Let $\mathcal{C} := C(V, L)$ be the affine cone over V with cone vertex o. The following are equivalent:*

(a) V is K-semistable;

(b) The function $\widehat{\mathrm{vol}}$ *on* $\mathrm{Val}_{\mathcal{C},o}$ *is minimized at* ord_V.

Here ord_V *represents the divisorial valuation associated to the unique exceptional divisor on the blow up of* $\mathcal{C}$ *at* o.

We remark here that results in Chapter 4 are joint work with C. Li (cf. [LL16]).

Organization of the thesis

In Chapter 2, we give preliminaries for this thesis. In Section 2.1, we introduce conical Kähler-Einstein metrics on log Fano pairs due to [BBE$^+$11]. The conical Kähler-Einstein potentials are defined as solutions to certain Monge-Ampère equation of plurisubharmonic functions. This general definition recovers the usual definition of conical Kähler-Einstein metric on smooth Kähler manifold with a smooth boundary divisor. In Section 2.2, we define log-K-(poly/semi)stability and log-Ding-semistability for log Fano pairs following [Tia97, Don02, Li15c, Ber16, Fuj15]. We mention Berman's result that conical Kähler-Einstein log Fano pairs are log-K-polystable. We also mention the equivalence between log-K-semistability and log-Ding-semistability proved in [BBJ15, Fuj16]. Section 2.3 is devoted to introducing the normalized volume function $\widehat{\mathrm{vol}}$ of valuations over a complex klt singularity. Following [Li15b], we present Li's motivation introducing $\widehat{\mathrm{vol}}$ based on [MSY08, DS15] and then give precise definitions. We give basic properties of $\widehat{\mathrm{vol}}$, such as Izumi type theorem and its relation to normalized mutiplicity of ideals studied by de Fernex-Ein-Mustaţă [dFEM04]. We also discuss Li's conjectures (Conjecture 2.3.7) about minimizers of $\widehat{\mathrm{vol}}$ and recent progress toward these conjectures. These developments come from very recent preprints [Li15b, Li15a, LL16, LX16, Liu16, Blu16b, Blu16a, LX17]. Some partial results toward Conjecture 2.3.7 will be described in details in Chapter 3 and 4. In Section 2.4, we briefly review volume of divisors and Seshadri constants.

In Chapter 3, we study volume upper bounds of Kähler-Einstein $\mathbb{Q}$-Fano varieties. In Section 3.2, we recall the concepts of filtrations and Fujita's work [Fuj15] on testing Ding-semistability by filtrations. In Section 3.3, we prove Theorem 1.1.3 by applying [Fuj15, 4.9] to the filtration associated to a real valuation. We recover the main result from [dFEM04]. We prove the inequality part of Theorem 1.1.2. In Section 3.4, we study the equality case of Theorem 1.1.3. Lemma 3.4.1 characterizes the equality case of Theorem 1.1.3 when X is K-polystable and the real valuation v is divisorial. Applying this lemma, we prove the equality case of Theorem 1.1.2. We also prove that $\mathbb{P}^n$ is the only K-semistable $\mathbb{Q}$-Fano variety of volume at least $(n+1)^n$ (see Theorem 3.4.3). This theorem improves the equality case of [Fuj15, 1.1] where Fujita proved for K-semistable Fano

manifolds.

In Chapter 4, we study normalized volume minimization over (analytically) K-semistable cone singularities. In Theorem 4.1.1, we show that if a log Fano pair (V, E) admits a conical Kähler-Einstein metric, then the canonical valuation ord_V is a minimizer of $\widehat{\mathrm{vol}}_{\mathcal{C},\mathcal{E}}$ at the cone vertex o. There are several steps to prove this theorem. In Section 4.2, we show that there exists a conical Kähler-Einstein metric on the projective cone $\overline{\mathcal{C}}$ with certain cone angles along $\mathcal{E}$ and the infinity section V_∞. Our construction generalizes [Li13, Lemma 3]. Next we use Berman's result [Ber16] to conclude that $(\overline{\mathcal{C}}, \mathcal{E} + (1-\beta)V_\infty)$ is log-Ding-semistable. Then we apply the log version of Fujita's result [Fuj15] to the filtration associated to any valuation. We proceed in two ways to complete the proof. In Section 4.3, we give the first (quick) proof be using a logarithmic version of Theorem 1.1.3. In Section 4.4, we interpret the expression in log-Fujita as the derivative of normalized volumes in the same spirit as in [Li15b, Li15a] and use a convexity argument to conclude the proof. Once we finish proving Theorem 4.1.1, then part "(a)⇒(b)" of Theorem 1.2.1 follows by deep results in [CDS15, Tia15] that a K-semistable Fano manifold always specially degenerates to a Kähler-Einstein $\mathbb{Q}$-Fano variety. The other direction of Theorem 1.2.1 follows from the main result of [Li15a]. In Section 4.5, we give a generalization of Theorem 1.2.1 to singular cases due to Li-Xu [LX17] using degeneration to initial ideals. We also discuss a related valuative criterion of K-semistability due to Fujita [Fuj16] and Li [Li15a].

Chapter 2

Preliminaries

2.1 Conical Kähler-Einstein metrics on log Fano pairs

Definition 2.1.1. A *log Fano pair* is a log pair (X, D) satisfying:

- X is a normal projective variety;
- D is an effective $\mathbb{Q}$-divisor and $-(K_X + D)$ is a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor;
- (X, D) has klt singularities;
- $-(K_X + D)$ is ample.

We recall some notions in the logarithmic setting following [BBE$^+$11]. Let (X, D) be a log Fano pair. Let $\delta > 0$ be a positive rational number such that $L = -\delta^{-1}(K_X + D)$ is Cartier. Then L can be considered as a holomorphic line bundle. A locally bounded Hermitian metric on L is given by a family of locally bounded positive functions $e^{-\varphi} := \{e^{-\varphi_i}\}$ associated to an affine covering $\{U_i\}_i$ of X, such that they are compatible with the transition functions $\{g_{ij}\} \in H^1(\{U_i\}, \mathcal{O}_X^*)$ of L:

$$\frac{e^{-\varphi_j}}{e^{-\varphi_i}} = \frac{s_j}{s_i} = g_{ij}$$

where s_i is a local generator of L over U_i. $e^{-\varphi}$ is positively curved if φ_i is plurisubharmonic on each affine subset U_i, i.e. φ_i is lower semi-continuous and $\sqrt{-1}\partial\bar{\partial}\varphi_i \geq 0$ in the sense of currents. A locally bounded positively curved Hermitian metric $e^{-\varphi}$ on L is called to be a *conical Kähler-Einstein (cKE) potential* on $(X, D; L)$ if it satisfies the following Monge-Ampère equation:

$$(\sqrt{-1}\partial\bar{\partial}\varphi)^n = \mathrm{m}_{\delta\varphi}, \tag{2.1.1}$$

where we used the following notations:

- For the left hand side, the Monge-Ampère measure $(\sqrt{-1}\partial\bar{\partial}\varphi)^n$ is defined in the sense of pluripotential theory;

- To define the right hand side, we first choose any integer k such that $k(K_X + D)$ is Cartier and a non vanishing local section σ^k. For example, we can choose $k = \delta^{-1}$. Then we define locally:

$$\mathrm{m}_{\delta\varphi} = \left(\sigma^k \wedge \bar{\sigma}^k\right)^{1/k} \left(|\sigma^{-k}|^2 e^{-k\delta\varphi}\right)^{1/k} =: dV_{(X,D)}(\sigma) \cdot \left(|\sigma|^2 e^{-\delta\varphi}\right).$$

  It's easy to verify that this is a globally well defined singular volume form. If we choose a log resolution of $\pi : (Y, f_*^{-1}D) \to (X, D)$ such that $K_Y = \pi^*(K_X + D) + \sum_i a_i E_i$ with the normal crossing divisor $\bigcup_i E_i$, then locally we have:

$$\pi^*(\mathrm{m}_{\delta\varphi}) = e^{-\psi} \prod_i |f_i|^{2a_i} d\lambda.$$

  where ψ a bounded function, $E_i = \{f_i = 0\}$ and $d\lambda$ is the local Lebesgue measure on Y. Notice that the klt condition means that $a_i > -1$, which implies that the density function of $\pi^*(\mathrm{m}_{\delta\varphi})$ with respect to $d\lambda$ is $L^p(d\lambda)$ for some $p > 1$. We refer to [BBE+11, Section 3] for more details.

Correspondingly, the curvature current $\omega_\varphi := \sqrt{-1}\partial\bar{\partial}\varphi$ is called to be a *conical Kähler-Einstein metric* of (X, D) with Ricci curvature $\delta > 0$. If there is a cKE potential on $(X, D; L)$ then the same holds for $(X, D; \lambda L)$ for any $\lambda \in \mathbb{Q}_{>0}$. So for simplicity, if the proportional constant $\delta > 0$ is clear (equivalently when the cohomology class of L is clear) we will just say that there is a cKE metric on (X, D).

In the case when both K_X and D are $\mathbb{Q}$-Cartier, we can choose a local non vanishing section e_1^m of mK_X and a local non vanishing section e_2^m of mD. We then think e_1 and e_2 as $\mathbb{Q}$-sections of the $\mathbb{Q}$-line bundles K_X and D, and obtain a $\mathbb{Q}$-section $\sigma = e_1 \cdot e_2$ of $K_X + D$. Notice that on the open set U_α, we can choose $e_2^m = \frac{1}{f_\alpha} \in \mathcal{O}(mD)(U_\alpha)$ where $mD = \{f_\alpha = 0\}$. Then we can write:

$$\begin{aligned}
(\sigma \wedge \sigma)^{1/m} &= (e_1^m \wedge \bar{e}_1^m)^{1/m} (e_2^m \wedge \bar{e}_2^m)^{1/m} \\
&= (e_1^m \wedge \bar{e}_1^m)^{1/m} \frac{1}{|f_\alpha|^{2/m}} \\
&=: (e_1 \wedge \bar{e}_1) \frac{|e_2|^2}{|s_D|^2}
\end{aligned}$$

where s_D is the $\mathbb{Q}$-section of D such that $mD = \{s_D^m = 0\}$ and we think $|s_D|^{-2}$ as defining a singular Hermitian metric on the $\mathbb{Q}$-line bundle D. Using the above notations, we can formally write:

$$\mathrm{m}_{\delta\varphi} = (e_1 \wedge \bar{e}_1)\frac{|e_2|^2}{|s_D|^2}|e_1^{-1} \cdot e_2^{-1}|^2 e^{-\delta\varphi} =: \frac{e^{-\delta\varphi}}{|s_D|^2}.$$

Denote by $X_0 := X_{\mathrm{reg}} \setminus \mathrm{Supp}D$. The following proposition gives regularity of conical Kähler-Einstein metrics.

Proposition 2.1.2 ([BBE$^+$11, Lemma 3.6])**.** *Let ω be a conical Kähler-Einstein metric of a log Fano pair (X, D) with Ricci curvature $\delta > 0$. Then ω is automatically smooth on X_0, with continuous potential on X, and it satisfies*

$$\mathrm{Ric}(\omega) = \delta\omega + 2\pi\{D\}$$

on X_{reg}.

The following proposition shows that in order to obtain Kähler-Einstein metrics on the regular part, the klt condition of (X, D) is necessary.

Proposition 2.1.3 ([BBE$^+$11, Proposition 3.8])**.** *Let (X, D) be a log pair satisfying*

- *X is a normal projective variety;*
- *D is an effective $\mathbb{Q}$-divisor and $-(K_X + D)$ is a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor;*
- *$-(K_X + D)$ is ample.*

Let $\Omega \subset X_{\mathrm{reg}}$ be a Zariski open subset with complement of codimension at least 2. Assume that there exists a closed positive $(1,1)$-current ω on Ω with continous potential such that

$$\mathrm{Ric}(\omega) = \omega + 2\pi\{D\}$$

on Ω. Then (X, D) is necessarily klt (hence log Fano). We further have

$$\int_\Omega \omega^n \leq c_1(X, D)^n,$$

with equality iff ω is the restriction to Ω of a conical Kähler-Einstein metric for (X, D).

Remark 2.1.4. (a) From Proposition 2.1.2 and 2.1.3, we know that a $\mathbb{Q}$-Fano variety X is Kähler-Einstein if and only if there exists a Kähler metric ω on X_{reg} such that $\mathrm{Ric}(\omega) = \omega$ and

$\int_{X_{\mathrm{reg}}} \omega^n = (-K_X)^n$. Sometimes this property can be used as an alternative definition of Kähler-Einstein $\mathbb{Q}$-Fano varieties, e.g. [Fuj15, Liu16].

(b) In the even more special case when X is smooth and $D = (1-\beta)E$ for a smooth prime divisor E and $\beta \in (0,1]$. The solution to (2.1.1) can be shown to be give rise to a singular Kähler-Riemannian metric structure $\omega_\varphi := \sqrt{-1}\partial\bar{\partial}\varphi$ with conical singularities along E (see [Don12, JMR16]). In other words away from E, ω_φ is a smooth Kähler metric, and near the divisor E, ω_φ is modeled on the flat conical metric (choosing local coordinate $z = (z_1, z_2, \ldots, z_n)$ with $E = \{z_1 = 0\}$):

$$\sqrt{-1}\partial\bar{\partial}\left(|z_1|^{2\beta} + \sum_{i=2}^{n}|z_i|^2\right) = \beta^2\frac{\sqrt{-1}dz_1\wedge d\bar{z}_1}{|z_1|^{2(1-\beta)}} + \sqrt{-1}\sum_{i=2}^{n} dz_i\wedge d\bar{z}_i.$$

It's easy to verify that, for this model metric, the associated metric tensor can be written as $\left(dr^2 + \beta^2 r^2 d\theta^2\right)\times g_{\mathbb{C}^{n-1}}$ where $r = |z_1|^\beta$. So $2\pi\beta$ is the metric cone angle. Note that by taking the $-\sqrt{-1}\partial\bar{\partial}\log$ on both sides of (2.1.1), we know that the Ricci curvature of ω_φ satisfies the following equality:

$$\mathrm{Ric}\left(\sqrt{-1}\partial\bar{\partial}\varphi\right) = \delta\sqrt{-1}\partial\bar{\partial}\varphi + (1-\beta)2\pi\{E\},$$

where $\{E\}$ denotes the current of integration along E. Furthermore, when $\beta = 2\pi/m$ for $m \in \mathbb{Z}_{>0}$, then conical Kähler-Einstein metrics on the log smooth pair $(X, (1-\beta)E)$ are nothing but orbifold Kähler-Einstein metrics. Indeed, in these cases the solution to the equation (2.1.1) can be shown to be orbifold smooth (see [EGZ09, BBE$^+$11]). In particular, if $\beta = 1$ we are in the case of smooth Kähler-Einstein metrics.

2.2 Log-K-stability and log-Ding-stability

In this section we recall the definition of log-K-semistability, and its recent equivalence log-Ding-semistability. The concept of log-K-stability was first introduced in [Li15c] after the formulation the K-stability by Tian and Donaldson.

Definition 2.2.1 (see [Tia97, Don02, Li15c, LX14, OS15])**.** Let (X, D) be a log-Fano pair.

(a) For any $\delta > 0 \in \mathbb{Q}$ such that $-\delta^{-1}(K_X + D) = L$ is Cartier, a *test configuration* (resp. a *semi test configuration*) of $(X, D; L)$ consists of the following data

- A log pair $(\mathcal{X}, \mathcal{D})$ admitting a $\mathbb{C}^*$-action and a $\mathbb{C}^*$-equivariant flat morphism $\pi : \mathcal{X} \to \mathbb{C}^1$, where the $\mathbb{C}^*$-action on the base $\mathbb{C}$ is given by the standard multiplication;

- A $\mathbb{C}^*$-equivariant π-ample (resp. π-semiample) line bundle $\mathcal{L}$ on $\mathcal{X}$ such that there is a $\mathbb{C}^*$-equivariant isomorphism $(\mathcal{X},\mathcal{D};\mathcal{L})|_{\pi^{-1}(\mathbb{C}\setminus\{0\})} \cong (X,D;-\delta^{-1}(K_X+D))\times\mathbb{C}^*$.

Moreover, if $\mathcal{X}$ is normal in addition, then we say that $(\mathcal{X},\mathcal{D};\mathcal{L})/\mathbb{C}^1$ is a *normal test configuration* (resp. a *normal semi test configuration*). A normal test configuration is called a *special test configuration*, if the following are satisfied:

- $\mathcal{X}_0$ is irreducible and normal, and $(\mathcal{X}_0,\mathcal{D}_0)$ is a log Fano pair;
- $\mathcal{L} = -\delta^{-1}(K_{\mathcal{X}/\mathbb{C}}+\mathcal{D})$.

In this case, we will say that (X,D) *specially degenerates* to $(\mathcal{X}_0,\mathcal{D}_0)$.

(b) Assume that $(\mathcal{X},\mathcal{D};\mathcal{L})\to\mathbb{C}$ is a normal test configuration. Let $\bar{\pi}:(\bar{\mathcal{X}},\bar{\mathcal{D}};\bar{\mathcal{L}})\to\mathbb{P}^1$ be the natural equivariant compactification of $(\mathcal{X},\mathcal{D};\mathcal{L})\to\mathbb{C}^1$. The log-CM weight of of $(\mathcal{X},\mathcal{D};\mathcal{L})$ is defined by

$$\text{log-CM}(\mathcal{X},\mathcal{D};\mathcal{L}) = \frac{n\delta^{n+1}\bar{\mathcal{L}}^{n+1}+(n+1)\delta^n\bar{\mathcal{L}}^n\cdot(K_{\bar{\mathcal{X}}/\mathbb{P}^1}+\mathcal{D})}{(n+1)(-K_X-D)^n}.$$

(c)
- The pair (X,D) is called *log-K-semistable* if $\text{log-CM}(\mathcal{X},\mathcal{D};\mathcal{L})\geq 0$ for any normal test configuration $(\mathcal{X},\mathcal{D};\mathcal{L})/\mathbb{C}^1$ of $(X,D;-\delta^{-1}(K_X+D))$ for any $\delta\in\mathbb{Q}_{>0}$ (such that $-\delta^{-1}(K_X+L)$ is Cartier).
- The pair (X,D) is called *log-K-polystable* if $\text{log-CM}(\mathcal{X},\mathcal{D};\mathcal{L})\geq 0$ for any normal test configuration $(\mathcal{X},\mathcal{D};\mathcal{L})/\mathbb{C}^1$ of $(X,D;-\delta^{-1}(K_X+D))$, and the equality holds if and only if $(\mathcal{X},\mathcal{D};\mathcal{L})\cong(X,D;L)\times\mathbb{C}^1$.

We will need a related concept of log-Ding-semistability, which is derived from Berman's work in [Ber16].

Definition 2.2.2 (see [Ber16, Fuj15]). (a) Let $(\mathcal{X},\mathcal{D};\mathcal{L})/\mathbb{C}^1$ be a normal semi-test configuration of $(X,D;-\delta^{-1}(K_X+D)$ and $(\bar{\mathcal{X}},\bar{\mathcal{D}};\bar{\mathcal{L}})/\mathbb{P}^1$ be its natural compactification. Let $\Delta_{(\mathcal{X},\mathcal{D};\mathcal{L})}$ be the $\mathbb{Q}$-divisor on $\mathcal{X}$ satisfying the following conditions:

- The support $\text{Supp}\,\Delta_{(\mathcal{X},\mathcal{D};\mathcal{L})}$ is contained in $\mathcal{X}_0$;
- The divisor $\delta^{-1}\Delta_{(\mathcal{X},\mathcal{D};\mathcal{L})}$ is a $\mathbb{Z}$-divisor corresponding to the divisorial sheaf $\bar{\mathcal{L}}(\delta^{-1}(K_{\bar{\mathcal{X}}/\mathbb{P}^1}+\bar{\mathcal{D}}))$.

(b) The *log-Ding invariant* $\text{log-Ding}(\mathcal{X},\mathcal{D};\mathcal{L})$ of $(\mathcal{X},\mathcal{D};\mathcal{L})/\mathbb{C}$ is defined as:

$$\text{log-Ding}((\mathcal{X},\mathcal{D});\mathcal{L}) := \frac{-\delta^{n+1}\bar{\mathcal{L}}^{n+1}}{(n+1)(-K_X-D)^n} - \left(1-\text{lct}(\mathcal{X},\mathcal{D}-\Delta_{(\mathcal{X},\mathcal{D};\mathcal{L})};\mathcal{X}_0)\right).$$

(c) (X, D) is called *log-Ding semistable* if log-Ding$(\mathcal{X}, \mathcal{D}; \mathcal{L}) \geq 0$ for any normal test configuration $(\mathcal{X}, \mathcal{D}; \mathcal{L})/\mathbb{C}$ of $(X, D; -\delta^{-1}K_X)$.

Remark 2.2.3. (a) If $D = 0$, our definitions reduce to usual definitions of K-(poly/semi)stability and Ding-semistability of a $\mathbb{Q}$-Fano variety X.

(b) For special test configurations, log-CM weight coincides with log-Ding invariant:

$$\text{log-CM}(\mathcal{X}, \mathcal{D}; \mathcal{L}) = \frac{-\delta^{n+1}\bar{\mathcal{L}}^{n+1}}{(n+1)(-K_X - D)^n} = \text{log-Ding}(\mathcal{X}, \mathcal{D}; \mathcal{L}). \tag{2.2.1}$$

Notice that since the work in [LX14], it was known that to test log-K-semistability (or log-K-polystability), one only needs to test on special test configurations.

(c) By the recent work of [BBJ15], (log-)Ding-semistability is equivalent to (log-)K-semistability (see also [Fuj16]). Hence to test log-Ding-semistability, one only needs to test on special test configurations as well.

(d) The results in [Ber16] show that, if a log Fano pair (X, D) admits a conical Kähler-Einstein metric, then (X, D) is log-K-polystable (see also [Fuj15, 3.2]). In particular, (X, D) is log-K-semistable, and equivalently log-Ding-semistable.

2.3 Normalized volume of valuations

2.3.1 Motivation and setup

In [Li15b], Li proposed the following question:

Question 2.3.1 (Li [Li15b])**.** Given a $\mathbb{Q}$-Gorenstein klt singularity, is there an optimal way of rescaling it to obtain an affine cone which has a canonical metric structure?

This will be made precise later, and here we just give some rough ideas. A rescaling in the above question will be represented by a real valuation centered at the singularity and the affine cone is given by the spectrum of the associated graded ring (assuming that the latter is finitely generated). For the "optimal way", we would like the following two properties to hold:

- The valuation corresponding to the rescaling should minimize the normalized volume function that we are going to define;

- The affine cone should have a possibly singular almost Kähler Ricci-flat metric cone structure. More precisely, the affine cone should degenerate to an affine variety with a singular Kähler Ricci-flat metric cone structure.

Actually Li conjectured that the above two properties should ultimately be equivalent to each other (see part 5 of Conjecture 2.3.7).

The above question is motivated by the recent study of Kähler-Einstein metrics in [CDS15, Tia15] and more recently in [DS15]. One of the aims of the question proposed here is to provide an algebraic characterization of the metric tangent cone by showing that the first step in Donaldson-Sun's procedure in [DS15] is equivalent to a procedure of minimizing normalized volumes of valuations (see part 5 of Conjecture 2.3.7). This transformation to the minimization problem is motivated by a corresponding phenomenon in Sasaki-Einstein geometry discovered by Martelli-Sparks-Yau in [MSY08], which could be seen as a special case of the general procedure discussed here (see [Li15b, Section 2.2] and [LL16]). We point out that one advantage of this algebro-geometric problem is that it can be studied for all $\mathbb{Q}$-Gorenstein klt singularities that do not necessarily come from the Gromov-Hausdoff limits as those studied in [DS15] (see [Li15b, Section 6.2]). Moreover, as we will see in [LL16] and [LX16], for many klt singularities, the minimizers of the normalized volume functions indeed exist as canonical objects associated to the singularities. Li conjectured this to be true for any $\mathbb{Q}$-Gorenstein klt singularity (see Conjecture 2.3.7 and its refinement in [LX16]).

Here in this section we introduce the basic set-up of this minimization problem. Interestingly as it gradually turns out, this minimization problem is indeed related to several subjects in complex differential/algebraic geometry, including Kähler-Einstein/Sasaki-Einstein metrics (see [LL16]), (log-)K-semistability (see [Li15b, Section 5] and [Li15a, LL16]), (sharp) de Fernex-Ein-Mustaţă type inequalities on klt singularities (see [Liu16, LX16, Blu16a]) and plt blow-ups (see [LX16]).

Let (X, p) be a normal $\mathbb{Q}$-Gorenstein singularity with $\dim X = n$. A *real valuation* v on the function field $\mathbb{C}(X)$ is a map $v : \mathbb{C}(X) \to \mathbb{R}$, satisfying:

- $v(fg) = v(f) + v(g)$;
- $v(f + g) \geq \min\{v(f), v(g)\}$;
- $v(\mathbb{C}^*) = 0$.

Denote by $\mathcal{O}_v := \{f \in \mathbb{C}(X) : v(f) \geq 0\}$ the *valuation ring* of v. Denote by $(\mathcal{O}_p, \mathfrak{m}_p)$ the local ring of X at p. We say that a real valuation v on $\mathbb{C}(X)$ is *centered at* p if the local ring $\mathcal{O}_p$ is dominated by $\mathcal{O}_v$. In other words, v is nonnegative on $\mathcal{O}_p$ and strictly positive on $\mathfrak{m}_p$. Denote by

$\mathrm{Val}_{X,p}$ the set of real valuations centered at p. Given a valuation $v \in \mathrm{Val}_{X,p}$, define the *valuation ideals* for any $x \in \mathbb{R}$ as follows:

$$\mathfrak{a}_x(v) := \{f \in \mathcal{O}_X : v(f) \geq x\}.$$

The *volume* of the valuation v is defined as

$$\mathrm{vol}(v) := \lim_{x \to +\infty} \frac{\ell(\mathcal{O}_X/\mathfrak{a}_x(v))}{x^n/n!}.$$

By [ELS03, Mus02, Cut13], the limit on the right hand side exists and is equal to the multiplicity of the graded family of ideals $\mathfrak{a}_\bullet(v)$ (see also Section 2.3.3):

$$\mathrm{vol}(v) = \lim_{m \to \infty} \frac{\mathrm{mult}(\mathfrak{a}_m(v))}{m^n} =: \mathrm{mult}(\mathfrak{a}_\bullet(v)).$$

Following [JM12, BdFFU15], we can define the *log discrepancy* for any valuation $v \in \mathrm{Val}_X$. This is achieved in three steps in [JM12, BdFFU15]. Firstly, for a divisorial valuation ord_E associated to a prime divisor E over X, define $A_X(E) := 1 + \mathrm{ord}_E(K_{Y/X})$, where $\pi : Y \to X$ is a resolution of X containing E. Next for any quasi-monomial valuation (also called Abhyankar valuation) $v \in \mathrm{QM}_\eta(Y, D)$ where $(Y, D = \sum_{k=1}^N D_k)$ is log smooth and η is a generic point of an irreducible component of $D_1 \cap \cdots \cap D_N$, we define

$$A_X(v) = \sup_{(Y,D)} A_X(r_{(Y,D)}(v))$$

where (Y, D) ranges over all log smooth models over X, and $r_{(Y,D)} : \mathrm{Val}_X \to \mathrm{QM}(Y, D)$ are contraction maps that induce a homeomorphism $\mathrm{Val}_X \to \varprojlim_{(Y,D)} \mathrm{QM}(Y, D)$. For details, see [JM12] and [BdFFU15, Theorem 3.1].

Following [Li15b], the *normalized volume function* $\widehat{\mathrm{vol}}(v)$ for any $v \in \mathrm{Val}_{X,p}$ is defined as

$$\widehat{\mathrm{vol}}(v) := \begin{cases} A_X(v)^n \cdot \mathrm{vol}(v), & \text{if } A_X(v) < +\infty; \\ +\infty, & \text{if } A_X(v) = +\infty. \end{cases}$$

Notice that $\widehat{\mathrm{vol}}(v)$ is rescaling invariant: $\widehat{\mathrm{vol}}(\lambda v) = \widehat{\mathrm{vol}}(v)$ for any $\lambda > 0$. In the next section, we will prove an Izumi type theorem (Theorem 2.3.1, see also [Izu85, Ree89, ELS03, JM12, BFJ14, Li15b]) from which one can show that the normalized volume function $\widehat{\mathrm{vol}}$ on $\mathrm{Val}_{X,p}$ is uniformly bounded

from below by a positive number.

2.3.2 Izumi's theorem

The following inequality is proved by Li in [Li15b, Proposition 2.3] (see [ELS01, Theorem 2.6], [JM12, Proposition 5.1] for the smooth case). We provide Li's proof here for the convenience of the reader.

Theorem 2.3.2 (Izumi type theorem)**.** *Assume (X, p) is a $\mathbb{Q}$-Gorenstein klt singularity. There exists a constant $c = c(X, p) > 0$ such that*

$$v(\mathfrak{m}_p)\mathrm{ord}_p \leq v \leq cA_X(v)\mathrm{ord}_p, \tag{2.3.1}$$

for any valuation centered at p.

Proof. Denote by $\mathfrak{m} := \mathfrak{m}_p$ for simplicity. To get the lower bound, we note that $f \in \mathfrak{m}^{\mathrm{ord}_p(f)} - \mathfrak{m}^{\mathrm{ord}_p(f)+1}$. So

$$\mathrm{ord}_p(f)v(\mathfrak{m}) \leq v(f).$$

For the upper bound, by the way of definition of $A_X(v)$ in [JM12] and [BdFFU15], we can assume that v is divisorial valuation with center at p by approximating the general real valuation by divisorial valuations. In the following, we will give two different proofs of upper bounds.

This proof is inspired by [BHJ15, Proof of Theorem 8.13]. When X is smooth at p, it follows from Skoda's criterion for integrability

$$\frac{A_X(v)}{v(f)} \geq \mathrm{lct}(f) = \min_v \frac{A_X(v)}{v(f)} \geq \frac{1}{\mathrm{ord}_p(f)} \tag{2.3.2}$$

for any $f \in \mathcal{O}_{X,p}$ (see [JM12, Proposition 5.10]). So we have

$$v(\mathfrak{m})\mathrm{ord}_p \leq v \leq A_X(v)\mathrm{ord}_p.$$

When X is only klt and $\mathbb{Q}$-Gorenstein, we choose a log resolution $\mu : X' \to X$ and write $\mu^* K_X = K_{X'} + B'$. So we have

$$A_X(v) = A_{(X',B')}(v) = A_{X'}(v) - v(B').$$

Because X is klt, we have $B' \leq (1-\epsilon)B'_{\mathrm{red}}$ for some $\epsilon = \epsilon(X') > 0$. Because B'_{red} has simple normal crossings, we have (X', B'_{red}) is lc, i.e. $0 \leq A_{(X',B'_{\mathrm{red}})}(v) = A_{X'}(v) - v(B'_{\mathrm{red}})$. Combining above, we

get

$$\begin{aligned} A_X(v) = A_{X'}(v) - v(B') &= \epsilon A_{X'}(v) + (1-\epsilon)(A_{X'}(v) - v(B'_{\mathrm{red}})) + (1-\epsilon)v(B'_{\mathrm{red}}) - v(B') \\ &\geq \epsilon A_{X'}(v). \end{aligned}$$

Assume ξ is the center of v on X'. Then by the smooth case (see [JM12, Proposition 5.1]), we have:

$$v(f) = v(\mu^* f) \leq A_{X'}(v)\mathrm{ord}_\xi(\mu^* f) \leq \epsilon^{-1} A_X(v)\mathrm{ord}_\xi(\mu^* f).$$

By Izumi's linear complementary inequality under morphisms ([Izu82, Izu85]), we have $\mathrm{ord}_\xi(\mu^* f) \leq a_2 \cdot \mathrm{ord}_p(f)$ for some $a_2 = a_2(X,p) \geq 1$. So we get:

$$v(\mathfrak{m})\mathrm{ord}_p \leq v \leq \epsilon^{-1} a_2 \cdot A_X(v)\mathrm{ord}_p.$$

□

From (2.3.1), we immediately get the following inequality:

Corollary 2.3.3 ([Li15b, Corollary 2.6])**.** *Assume (X,p) is a $\mathbb{Q}$-Gorenstein klt singularity. There exists a constant $c = c(X,p) > 0$ such that for any $v \in \mathrm{Val}_{X,p}$, we have:*

$$A_X(v)^n \cdot \mathrm{vol}(v) \geq \frac{\mathrm{mult}(\mathfrak{m}_p)}{c^n} > 0. \tag{2.3.3}$$

In particular, we have $\inf_{\mathrm{Val}_{X,p}} \widehat{\mathrm{vol}} > 0$.

2.3.3 Relation with dFEM's normalized multiplicity of ideals

Let (X,p) be an n-dimensional klt singularity. For an ideal sheaf $\mathfrak{a} \subset \mathfrak{m}_p$, the *log canonical threshold* of $\mathfrak{a}$ is defined as

$$\mathrm{lct}(\mathfrak{a}) := \inf_E \frac{1 + \mathrm{ord}_E(K_{Y/X})}{\mathrm{ord}_E(\mathfrak{a})},$$

where the infimum is taken over all prime divisors E on a log resolution Y of $(X,\mathfrak{a})$. If moreover that $\mathfrak{a}$ is cosupported at p (or equivalently $\mathfrak{m}_p$-primary), the *Hilbert-Samuel multiplicity* of $\mathfrak{a}$ is defined as

$$\mathrm{mult}(\mathfrak{a}) := \lim_{k\to\infty} \frac{\ell(\mathcal{O}_X/\mathfrak{a}^m)}{m^n/n!}.$$

If Z is a closed subscheme of X supported at p, then we sometimes denote $\mathrm{mult}_Z X := \mathrm{mult}(I_Z)$ where I_Z is the ideal sheaf of Z.

For an ideal sheaf $\mathfrak{a}$ cosupported at p, its *normalized multiplicity* $\mathrm{lct}(\mathfrak{a})^n \cdot \mathrm{mult}(\mathfrak{a})$ was studied by de Fernex, Ein and Mustaţă motivated by their interest in singularity theory, as well as applications to birational rigidity [Mus02, dFEM03, dFEM04, dFM15]. In this section, we will establish an equality between the infimum of normalized volumes and the infimum of normalized multiplicities (see Theorem 2.3.5).

Lemma 2.3.4. *Let $\mathfrak{a}$ be an ideal sheaf on X cosupported at p. Let $v_0 \in \mathrm{Val}_X$ be a divisorial valuation that computes $\mathrm{lct}(\mathfrak{a})$. Then v_0 is centered at p, and we have*

$$\mathrm{lct}(\mathfrak{a})^n \cdot \mathrm{mult}(\mathfrak{a}) \geq \widehat{\mathrm{vol}}(v_0).$$

Proof. Since v_0 computes $\mathrm{lct}(\mathfrak{a})$, we have that $\mathrm{lct}(\mathfrak{a}) = A_X(v_0)/v_0(\mathfrak{a})$. Denote $\alpha := v_0(\mathfrak{a})$. Then $\mathfrak{a}^m \subset \mathfrak{a}_{m\alpha}(v_0)$ since $v_0(\mathfrak{a}^m) = m\alpha$. Therefore,

$$\begin{aligned}
\mathrm{mult}(\mathfrak{a}) &= \lim_{m\to\infty} \frac{\ell(\mathcal{O}_X/\mathfrak{a}^m)}{m^n/n!} \\
&\geq \lim_{m\to\infty} \frac{\ell(\mathcal{O}_X/\mathfrak{a}_{m\alpha}(v_0))}{m^n/n!} \\
&= \alpha \ \mathrm{vol}(v_0).
\end{aligned}$$

Hence we prove the lemma. □

Following [ELS03], for a graded sequence of ideals $\mathfrak{a}_\bullet$ cosupported at p, the *volume* of $\mathfrak{a}_\bullet$ is defined as

$$\mathrm{vol}(\mathfrak{a}_\bullet) := \limsup_{m\to\infty} \frac{\ell(R/\mathfrak{a}_m)}{m^n/n!},$$

while the *multiplicity* of $\mathfrak{a}_\bullet$ is defined as

$$\mathrm{mult}(\mathfrak{a}_\bullet) := \lim_{m\to\infty} \frac{\mathrm{mult}(\mathfrak{a}_m)}{m^n}.$$

By [LM09, Cut13], we know that $\mathrm{mult}(\mathfrak{a}_\bullet) = \mathrm{vol}(\mathfrak{a}_\bullet)$ when X is normal.

Theorem 2.3.5. *We have*

$$\inf_{\mathfrak{a}} \big(\mathrm{lct}(\mathfrak{a})^n \cdot \mathrm{mult}(\mathfrak{a})\big) = \inf_{\mathfrak{a}_\bullet} \big(\mathrm{lct}(\mathfrak{a}_\bullet)^n \cdot \mathrm{vol}(\mathfrak{a}_\bullet)\big) = \inf_{v} \widehat{\mathrm{vol}}(v), \tag{2.3.4}$$

where the infimums are taken over all ideals $\mathfrak{a}$ *cosupported at* p*, all graded sequences of ideals* $\mathfrak{a}_\bullet$ *cosupported at* p *and all real valuations* $v \in \mathrm{Val}_{X,p}$*, respectively. We also set* $\mathrm{lct}(\mathfrak{a}_\bullet)^n \cdot \mathrm{vol}(\mathfrak{a}_\bullet) = +\infty$ *if* $\mathrm{lct}(\mathfrak{a}_\bullet) = +\infty$*.*

Proof. Let $\mathfrak{a}_\bullet$ be a graded sequence of ideal sheaves cosupported at p. By [JM12, BdFFU15], we have

$$\lim_{m\to\infty} m\ \mathrm{lct}(\mathfrak{a}_m) = \mathrm{lct}(\mathfrak{a}_\bullet).$$

Since $\mathrm{mult}(\mathfrak{a}_\bullet) = \mathrm{vol}(\mathfrak{a}_\bullet)$, if $\mathrm{lct}(\mathfrak{a}_\bullet) < +\infty$ then

$$\lim_{m\to\infty} \mathrm{lct}(\mathfrak{a}_m)^n \cdot \mathrm{mult}(\mathfrak{a}_m) = \mathrm{lct}(\mathfrak{a}_\bullet)^n \cdot \mathrm{vol}(\mathfrak{a}_\bullet).$$

Thus we have

$$\inf_{\mathfrak{a}} \big(\mathrm{lct}(\mathfrak{a})^n \cdot \mathrm{mult}(\mathfrak{a})\big) \leq \inf_{\mathfrak{a}_\bullet} \big(\mathrm{lct}(\mathfrak{a}_\bullet)^n \cdot \mathrm{vol}(\mathfrak{a}_\bullet)\big). \tag{2.3.5}$$

Next, Lemma 2.3.4 implies that

$$\inf_{\mathfrak{a}} \big(\mathrm{lct}(\mathfrak{a})^n \cdot \mathrm{mult}(\mathfrak{a})\big) \geq \inf_{v} \widehat{\mathrm{vol}}(v). \tag{2.3.6}$$

Finally, for any real valuation $v \in \mathrm{Val}_{X,p}$ with $A_X(v) < +\infty$, by [BdFFU15, 1.2] we have

$$\mathrm{lct}(\mathfrak{a}_\bullet(v)) \leq \frac{A_X(v)}{v(\mathfrak{a}_\bullet(v))} = A_X(v) < +\infty.$$

By definition we have $\mathrm{vol}(\mathfrak{a}_\bullet(v)) = \mathrm{vol}(v)$. Therefore,

$$\mathrm{lct}(\mathfrak{a}_\bullet(v))^n \cdot \mathrm{vol}(\mathfrak{a}_\bullet(v)) \leq A_X(v)^n \cdot \mathrm{vol}(v) = \widehat{\mathrm{vol}}(v). \tag{2.3.7}$$

Hence we have

$$\inf_{\mathfrak{a}_\bullet} \big(\mathrm{lct}(\mathfrak{a}_\bullet)^n \cdot \mathrm{vol}(\mathfrak{a}_\bullet)\big) \leq \inf_{v} \widehat{\mathrm{vol}}(v). \tag{2.3.8}$$

Combining (2.3.5)(2.3.6)(2.3.8) together, we finish the proof. □

Remark 2.3.6. (a) The first equality in (2.3.4) was observed by Mustaţă in [Mus02] where he showed that $\mathrm{lct}(\mathfrak{a}_\bullet)^n \cdot \mathrm{mult}(\mathfrak{a}_\bullet) \geq n^n$ for any graded sequence of $\mathfrak{m}$-primary ideals $\mathfrak{a}_\bullet$ of a regular local ring of dimension n. The inequality (2.3.7) was essentially realized by Li in [Li15b, Remark 2.8] where he considered the smooth case.

(b) By Izumi's theorem, the infimums in Theorem 2.3.5 are always positive (see Section 2.3.2).

2.3.4 Conjectures and known partial results

In [Li15b], Li proposed the following conjectures concerning about existence and properties of minimizers of $\widehat{\mathrm{vol}}$ at a klt singularity.

Conjecture 2.3.7 ([Li15b, Conjecture 6.1]). *1. For any germ of $\mathbb{Q}$-Gorenstein klt singularity (X,p), there exists a minimizer (denoted by $v_* = v_*(X,p)$) in $\mathrm{Val}_{X,p}$ of $\widehat{\mathrm{vol}}$.*

2. v_* is unique up to positive rescaling.

3. v_* is always a quasi-monomial valuation.

4. Let V be a Fano manifold, and let $\mathcal{C} = C(V, -K_V)$. Then V is K-semistable if and only if on the cone singularity $(\mathcal{C}, o)$, $\widehat{\mathrm{vol}}(v)$ is minimized at ord_V where V is considered as the exceptional divisor of the blow up $\pi : \mathrm{Bl}_o\mathcal{C} \to \mathcal{C}$.

5. Assume (X,p) is a $\mathbb{Q}$-Gorenstein klt singularity on a Kähler-Einstein Fano variety $(X, \omega_{\mathrm{KE}})$. The minimizer $v_*(X,p)$ for (X,p) is exactly the weight function that gives the metric tangent cone at $p \in (X, d_{\omega_{\mathrm{KE}}})$. More precisely, we consider the the associated graded algebra of v_*:

$$\mathrm{gr}_{v_*}\mathcal{O}_{X,p} := \bigoplus_{m\in\Phi} \mathfrak{a}_m(v_*)/\mathfrak{a}_{>m}(v_*),$$

where Φ is the valuation semigroup of v_*. Then the conjecture is that $\mathrm{gr}_{v_*}\mathcal{O}_{X,p}$ is finitely generated and normal, and $\mathrm{Spec}\left(\mathrm{gr}_{v_*}\mathcal{O}_{X,p}\right)$ specially degenerates to the metric tangent cone at (X,p). In particular, this is conjecturally an answer to a question of Donaldson-Sun [DS15].

Since the appearance of [Li15b], the study of the minimizer of the normalized volume function moves forward rapidly. In the rest of this section, we will list some known results to Conjecture 2.3.7.

Part 1 of Conjecture 2.3.7 has been confirmed very recently by Blum [Blu16a]:

Theorem 2.3.8 (Blum [Blu16a]). *For any klt singularity (X,p), there always exists a minimizer v_* in $\mathrm{Val}_{X,p}$ of $\widehat{\mathrm{vol}}$.*

Concerning part 2 of Conjecture 2.3.7, Li and Xu [LX16] showed that there exists at most one divisorial minimizer of $\widehat{\mathrm{vol}}$:

Theorem 2.3.9 (Li-Xu [LX16]). *Let (X,p) be a klt singularity. Then there exists at most one divisorial valuation ord_S that minimizes $\widehat{\mathrm{vol}}$ in $\mathrm{Val}_{X,p}$.*

Part 3 of Conjecture 2.3.7 seems to be the most difficult part. Blum [Blu16a] observed that part 3 follows from a conjecture of Jonsson and Mustaţă which says that valuations computing log canonical thresholds of graded sequences of ideals are always quasimonomial (cf. [JM14, Conjecture C]). However, Jonsson-Mustaţă's conjecture has been confirmed only in dimension 2 (see [JM12, Section 9]).

Part 4 of Conjecture 2.3.7 has been confirmed by Li and the author in [LL16] (see also Theorem 1.2.1). A proof will be given in Chapter 4. We also remark here that a generalization of part 4 to singular cases has been settled by Li and Xu [LX17] very recently (see also Section 4.5.1).

Concerning finite generation of $\mathrm{gr}_{v_*}\mathcal{O}_{X,p}$ in part 5 of Conjecture 2.3.7, we have the following result by Li-Xu [LX16] and Blum [Blu16b, Blu16a] in the quasi-regular case (without assuming the existence of local Kähler-Einstein metrics):

Theorem 2.3.10 (Li-Xu; Blum). *Assume that there exists a divisorial valuation v_* that minimizes $\widehat{\mathrm{vol}}$ over a klt singularity (X,p). Then we have the following:*

1. *The graded algebras $\bigoplus_{m\in\mathbb{Z}_{\geq 0}}\mathfrak{a}_m(v_*)$ and $\mathrm{gr}_{v_*}\mathcal{O}_{X,p}$ are both finitely generated over their degree zero parts;*

2. *Let $Y := \mathbf{Proj}_X \bigoplus_{m\in\mathbb{Z}_{\geq 0}}\mathfrak{a}_m(v_*)$ together with the projection $\pi : Y \to X$. Then the birational morphism π is isomorphic over $X\setminus\{p\}$, $\pi^{-1}(p)$ is a prime divisor S such that (Y,S) is plt, $-S$ is $\mathbb{Q}$-Cartier and ample over X, and $v_* = \mathrm{ord}_S$. (Such S is a* Kollár component *according to [LX16].)*

Assume that (X,p) is a klt singularity appearing in a Gromov-Hausdorff limit of a sequence of Kähler-Einstein Fano manifolds. According to Donaldson-Sun [DS15] there is a metrically defined filtration d_{KE} on the local ring $\mathcal{O}_{X,p}$ whose associated graded algebra defines a certain weighted tangent cone W, and there is a special degeneration of W to the metric tangent cone $C(Y)$. Then part 5 of Conjecture 2.3.7 predicts that d_{KE} should minimize the normalized volume function. In [LX16], Li-Xu confirmed this speculation when the Reeb vector field is quasi-regular:

Theorem 2.3.11 (Li-Xu [LX16]). *In the situation above, d_{KE} minimizes $\widehat{\mathrm{vol}}$ if the Reeb vector field on $C(Y)$ is quasi-regular (i.e. it generates a $\mathbb{C}^*$-action).*

Remark 2.3.12. In Chapter 4, we will provide examples of minimizers of nice singularities. The case

of quotient singularities (including smooth points) will be discussed in Proposition 4.3.10. We will discuss higher-dimensional $\mathbb{A}_k$ singularities in Example 4.3.7.

2.4 Volume of divisors and Seshadri constants

Definition 2.4.1 ([Laz04a])**.** Let X be a projective variety of dimension n. For a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor L on X, we define the *volume* of L on X to be

$$\mathrm{vol}_X(L) := \limsup_{k\to\infty,\, kL \text{ Cartier}} \frac{h^0(X, \mathcal{O}_X(kL))}{k^n/n!}.$$

We know that the limsup computing $\mathrm{vol}_X(L)$ is actually a limit. If $L \equiv L'$, then $\mathrm{vol}_X(L) = \mathrm{vol}_X(L')$. Moreover, we can extend it uniquely to a continuous function

$$\mathrm{vol}_X : N^1(X)_{\mathbb{R}} \to \mathbb{R}_{\geq 0}.$$

Notice that $\mathrm{vol}_X(L) = (L^n)$ whenver L is a nef $\mathbb{R}$-divisor.

Definition 2.4.2. Let X be a projective variety, L be an ample $\mathbb{Q}$-Cartier divisor on X. Let Z be a nonempty proper closed subscheme of X with ideal sheaf I_Z. Denote by $\sigma : \hat{X} \to X$ the blow up of X along Z. Let F be the Cartier divisor on $\hat{X}$ given by $\mathcal{O}_{\hat{X}}(-F) = \sigma^{-1}(I_Z) \cdot \mathcal{O}_{\hat{X}}$. The *Seshadri constant* of L along Z, denoted by $\epsilon_Z(L)$, is defined as

$$\epsilon_Z(L) := \sup\{x \in \mathbb{R}_{>0} \mid \sigma^* L - xF \text{ is ample}\}.$$

Lemma 2.4.3. *Let X be an n-dimensional normal projective variety with $n \geq 2$, L be an ample $\mathbb{Q}$-divisor on X, $p \in X$ be a closed point. Let Z be a closed subscheme of X with* Supp $Z = \{p\}$. *Let $\sigma : \hat{X} \to X$ be the blow up along Z, and $F \subset \hat{X}$ be the Cartier divisor defined by the equation $\mathcal{O}_{\hat{X}}(-F) = \sigma^{-1} I_Z \cdot \mathcal{O}_{\hat{X}}$.*

1. *For any $x \in \mathbb{R}_{\geq 0}$, we have*

$$\mathrm{vol}_{\hat{X}}(\sigma^* L - xF) \geq ((\sigma^* L - xF)^n) = (L^n) - \mathrm{mult}_Z X \cdot x^n.$$

2. *Set $\Lambda_Z(L) := \{x \in \mathbb{R}_{\geq 0} \mid \mathrm{vol}_{\hat{X}}(\sigma^* L - xF) = ((\sigma^* L - xF)^n)\}$. Then we have*

$$\epsilon_Z(L) = \max\{x \in \mathbb{R}_{\geq 0} \mid y \in \Lambda_Z(L) \text{ for all } y \in [0, x]\}.$$

Proof. The proof goes along the same line as [Fuj15, 2.3].

(1) The equality $(\sigma^*L - xF)^n) = (L^n) - \mathrm{mult}_Z X \cdot x^n$ follows from the fact that $\mathrm{mult}_Z(X) = (-1)^{n-1}(F^n)$ (see [Ram74]).

For the inequality part, we can assume that $x \in \mathbb{Q}_{>0}$ since the function $\mathrm{vol}_{\hat{X}}(\sigma^*L - xF)$ is continuous. Take any sufficiently large $k \in \mathbb{Z}_{>0}$ with $kx \in \mathbb{Z}_{>0}$ and kL Cartier. Notice that we have the long exact sequence

$$\begin{aligned} 0 \to H^0(\hat{X}, \sigma^*(kL) - kxF) \to H^0(\hat{X}, \sigma^*(kL)) \to H^0(kxF, \sigma^*(kL)|_{kxF}) \\ \to H^1(\hat{X}, \sigma^*(kL) - kxF) \to H^1(\hat{X}, \sigma^*(kL)) \to \cdots \end{aligned} \tag{2.4.1}$$

Thus

$$\begin{aligned} h^0(\hat{X}, \sigma^*(kL) - kxF) &\geq h^0(\hat{X}, \sigma^*(kL)) - h^0(kxF, \sigma^*(kL)|_{kxF}) - h^1(\hat{X}, \sigma^*(kL)) \\ &= h^0(X, kL) - h^0(kxF, \mathcal{O}_{kxF}) - h^1(\hat{X}, \sigma^*(kL)) \end{aligned}$$

Let us consider the Leray spectral sequence $H^i(X, R^j\sigma_*(\sigma^*(kL))) \Rightarrow H^{i+j}(\hat{X}, \sigma^*(kL))$. By the projection formula we have

$$H^i(X, R^j\sigma_*(\sigma^*(kL))) \cong H^i(X, kL \otimes R^j\sigma_*\mathcal{O}_{\hat{X}}).$$

Notice that Serre vanishing theorem yields $H^i(X, R^j\sigma_*(\sigma^*(kL))) = 0$ for $i > 0$ and $k \gg 0$. Thus for $k \gg 0$ the spectral sequence above degenerates at E_2, which implies that

$$H^i(\hat{X}, \sigma^*(kL)) \cong H^0(X, R^i\sigma_*(\sigma^*(kL))) \cong H^0(X, R^i\sigma_*\mathcal{O}_{\hat{X}}). \tag{2.4.2}$$

Therefore $\lim_{k\to\infty} h^1(\hat{X}, \sigma^*(kL))/k^n = 0$. Consequently,

$$\mathrm{vol}_{\hat{X}}(\sigma^*L - xF) \geq \mathrm{vol}_X(L) - \lim_{k\to\infty} \frac{h^0(kxF, \mathcal{O}_{kxF})}{(kx)^n/n!} x^n.$$

Hence it suffices to show that

$$\lim_{j\to\infty} \frac{h^0(jF, \mathcal{O}_{jF})}{j^n/n!} = \mathrm{mult}_Z X.$$

We look at the exact sequence

$$0 \to \mathcal{O}_{\hat{X}}(-jF)|_F \to \mathcal{O}_{(j+1)F} \to \mathcal{O}_{jF} \to 0 \tag{2.4.3}$$

Under the isomorphism $F \cong \mathbf{Proj} \bigoplus_{m\geq 0} I_Z^m/I_Z^{m+1}$ we get $\mathcal{O}_{\hat{X}}(-jF)|_F \cong \mathcal{O}_F(j)$. Since $H^1(F, \mathcal{O}_F(j)) = 0$ for $j \gg 0$, we have an exact sequence

$$0 \to H^0(F, \mathcal{O}_F(j)) \to H^0((j+1)F, \mathcal{O}_{(j+1)F}) \to H^0(jF, \mathcal{O}_{jF}) \to 0$$

Hence $h^0((j+1)F, \mathcal{O}_{(j+1)F}) - h^0(jF, \mathcal{O}_{jF}) = h^0(F, \mathcal{O}_F(j)) = \ell(I_Z^j/I_Z^{j+1})$ for $j \gg 0$. As a result,

$$h^0(jF, \mathcal{O}_{jF}) = \ell(\mathcal{O}_X/I_Z^j) + \text{const}$$

for $j \gg 0$. Hence we prove (1).

(2) Assume $\mathrm{vol}_{\hat{X}}(\sigma^*L - xF) = (L^n) - \mathrm{mult}_Z X \cdot x^n$ for some $x \in \mathbb{Q}_{>0}$. We will show that

$$h^i(\hat{X}, \sigma^*(kL) - kxF) = o(k^n) \quad \text{for } i \geq 1,\ k \gg 0 \text{ with } kx \in \mathbb{Z}_{>0}. \tag{2.4.4}$$

For $i \geq 2$, the long exact sequence (2.4.1) yields

$$h^i(\hat{X}, \sigma^*(kL) - kxF) \leq h^{i-1}(kxF, \mathcal{O}_{kxF}) + h^i(\hat{X}, \sigma^*(kL)).$$

By (2.4.2) we know that $h^i(\hat{X}, \sigma^*(kL)) = h^0(X, R^i\sigma_*\mathcal{O}_{\hat{X}})$ is constant for $k \gg 0$. Serre vanishing theorem implies that $H^{i-1}(F, \mathcal{O}_F(j)) = 0$ for $i \geq 2$, $j \gg 0$. Then the exact sequence (2.4.3) implies that

$$H^{i-1}((j+1)F, \mathcal{O}_{(j+1)F}) \cong H^{i-1}(jF, \mathcal{O}_{jF})$$

for $i \geq 2$, $j \gg 0$. As a result, $h^{i-1}(kxF, \mathcal{O}_{kxF})$ is also constant for $k \gg 0$. Therefore, $h^i(\hat{X}, \sigma^*(kL) - kxF)$ is bounded from above for $i \geq 2$ and $k \gg 0$.

By the asymptotic Riemann-Roch theorem we know that

$$\begin{aligned}\lim_{k\to\infty} \frac{\chi(\hat{X}, \sigma^*(kL) - kxF)}{k^n/n!} &= ((\sigma^*L - xF)^n) \\ &= (L^n) - \mathrm{mult}_Z X \cdot x^n \\ &= \mathrm{vol}_{\hat{X}}(\sigma^*L - xF).\end{aligned}$$

Hence we have

$$\lim_{k\to\infty} \frac{1}{k^n} \sum_{i=1}^{n} (-1)^i h^i(\hat{X}, \sigma^*(kL) - kxF) = 0.$$

Since $h^i(\hat{X}, \sigma^*(kL) - kxF) = O(1)$ for $i \geq 2$, we conclude that $h^1(\hat{X}, \sigma^*(kL) - kxF) = o(k^n)$ for $k \gg 0$.

Now let us prove (2). Denote by a the right-hand side of the equation in (2). For any nef divisor, its volume is equal to its self intersection number. Hence it is obvious that $\epsilon_Z(L) \leq a$. Then it suffices to show that $a \leq \epsilon_Z(L)$. Equivalently, we only need to show that for $\epsilon > 0$ sufficiently small with $a - \epsilon \in \mathbb{Q}_{>0}$, $\sigma^*L - (a-\epsilon)F$ is ample. Fix $\delta \in \mathbb{Q}_{>0}$ such that $\delta < \epsilon_Z(L)$, that is, $\sigma^*L - \delta F$ is ample. By [dFKL07, Theorem A], we only need to show that

$$h^i\big(\hat{X}, m\big((\sigma^*L - (a-\epsilon)F) - t(\sigma^*L - \delta F)\big)\big) = o(m^n) \tag{2.4.5}$$

for any $i > 0$, any sufficiently small $t \in \mathbb{Q}_{>0}$ and suitably divisible m. Denote $b := (a - \epsilon + t\delta)/(1-t)$. Notice that

$$\frac{1}{1-t}\big((\sigma^*L - (a-\epsilon)F) - t(\sigma^*L - \delta F)\big) = \sigma^*L - \frac{a - \epsilon + t\delta}{1-t}F = \sigma^*L - bF.$$

It is clear that $b < a$ for $t \in \mathbb{Q}_{>0}$ sufficiently small. Hence from the description of a we get:

$$\mathrm{vol}_{\hat{X}}(\sigma^*L - bF) = (L^n) - \mathrm{mult}_Z X \cdot b^n.$$

Then applying (2.4.4) yields that $h^i(\hat{X}, \sigma^*(kL) - kbF) = o(k^n)$ for $i > 0$, sufficiently small $t \in \mathbb{Q}_{>0}$ and suitably divisible k. In other words, (2.4.5) is true under the same condition for i, t and $m = (1-t)k$. Hence we prove the lemma. □

Chapter 3

Volume of Kähler-Einstein Fano varieties

3.1 Introduction

Recall that an n-dimensional complex projective variety X is said to be a $\mathbb{Q}$-*Fano variety* if X has klt singularities and $-K_X$ is an ample $\mathbb{Q}$-Cartier divisor. When $n \geq 2$, the anti-canonical volume $(-K_X)^n$ of a $\mathbb{Q}$-Fano variety X can be arbitrarily large, such as volumes of weighted projective spaces (see [Dol82]). However, if X admits a Kähler-Einstein metric (in the sense of [BBE$^+$11]), then it is expected that $(-K_X)^n$ has a uniform upper bound (see [Tia90, OSS16] for $n = 2$, and [BB11, BB17] for smooth cases). Recently, Fujita [Fuj15] showed that if X is a n-dimensional $\mathbb{Q}$-Fano variety admitting a Kähler-Einstein metric, then the anti-canonical volume $(-K_X)^n$ is less than or equal to $(n+1)^n$.

In this chapter, we refine the result [Fuj15] by looking at invariants of the local singularities. Our first main result is the following:

Theorem 3.1.1. *Let X be a Kähler-Einstein $\mathbb{Q}$-Fano variety. Let $p \in X$ be a closed point. Let Z be a closed subscheme of X with* Supp $Z = \{p\}$. *Then we have*

$$(-K_X)^n \leq \left(1 + \frac{1}{n}\right)^n \operatorname{lct}(X; I_Z)^n \operatorname{mult}_Z X, \tag{3.1.1}$$

where $\operatorname{lct}(X; I_Z)$ *is the log canonical threshold of the ideal sheaf I_Z, and* $\operatorname{mult}_Z X$ *is the Hilbert-Samuel multiplicity of X along Z.*

Note that the invariant $\mathrm{lct}^n \cdot \mathrm{mult}$ has been studied by de Fernex, Ein and Mustaţă in [Mus02, dFEM03, dFEM04, dFM15].

There are three major aspects of Theorem 3.1.1:

- If we take $Z = p$ to be a smooth point on X, then $\mathrm{lct}(X; \mathfrak{m}_p) = n$ and $\mathrm{mult}_p X = 1$. In particular, we recover Fujita's result [Fuj15, 1.3].
- By definition, $\mathrm{lct}(X; I_Z)$ is the infimum of $\frac{1+\mathrm{ord}_E(K_{Y/X})}{\mathrm{ord}_E(I_Z)}$ among all prime divisors E on a resolution Y of X. Although $\mathrm{lct}(X; I_Z)$ is not easy to compute in general, we can always get a volume upper bound by considering one divisor E, e.g. Theorem 3.1.3 and 3.3.9.
- On the other hand, if we fix a singularity p on a Kähler-Einstein $\mathbb{Q}$-Fano variety X, we get a lower bound of $\mathrm{lct}^n \cdot \mathrm{mult}$ among all thickenings of p. For example, if $X = \mathbb{P}^n$ then the inequality (3.1.1) recovers the main result in [dFEM04] (see Theorem 3.3.6). More generally, we get de Fernex-Ein-Mustaţă type inequalities for all cone singularities with K-semistable base using a logarithmic version of Theorem 3.1.1 (see Corollary 4.1.5).

For a real valuation v on $\mathbb{C}(X)$ centered at a closed point p, the normalized volume function $\widehat{\mathrm{vol}}$, introduced by C. Li in [Li15b], is defined as $\widehat{\mathrm{vol}}(v) := A_X(v)^n \cdot \mathrm{vol}(v)$, where $A_X(v)$ is the log discrepancy of v and $\mathrm{vol}(v)$ is the volume of v. The following theorem gives an upper bound for the anti-canonical volume of X in terms of the normalized volume of a real valuation.

Theorem 3.1.2 (=Theorem 1.1.3)**.** *Let X be a Kähler-Einstein $\mathbb{Q}$-Fano variety. Let $p \in X$ be a closed point. Let v be a real valuation on $\mathbb{C}(X)$ centered at p. Then we have*

$$(-K_X)^n \leq \left(1 + \frac{1}{n}\right)^n \widehat{\mathrm{vol}}(v). \tag{3.1.2}$$

In Chapter 4, a logarithmic version of Theorem 3.1.2 was developed to obtain the sharp lower bound of the normalized volume function for cone singularities over Kähler-Einstein $\mathbb{Q}$-Fano varieties.

To compare Theorem 3.1.1 and 3.1.2, we know from [Mus02, Li15b] that the infimum of $\mathrm{lct}(X; I_Z)^n \cdot \mathrm{mult}_Z X$ among all thickenings Z of p is no bigger than the infimum of $\widehat{\mathrm{vol}}(v)$ among all real valuations centered at p. Indeed, these two infimums are the same by Theorem 2.3.5. In particular, this means that Theorem 3.1.1 and 3.1.2 are actually equivalent.

As an application of Theorem 3.1.1 and 3.1.2, we get sharp volume upper bounds for Kähler-Einstein $\mathbb{Q}$-Fano varieties with quotient singularities.

Theorem 3.1.3 (=Theorem 1.1.2)**.** *Let X be a Kähler-Einstein $\mathbb{Q}$-Fano variety of dimension n. Let $p \in X$ be a closed point. Suppose (X, p) is a quotient singularity with local analytic model $\mathbb{C}^n/G$ where $G \subset GL(n, \mathbb{C})$ acts freely in codimension 1. Then*

$$(-K_X)^n \leq \frac{(n+1)^n}{|G|},$$

with equality if and only if $|G \cap \mathbb{G}_m| = 1$ and $X \cong \mathbb{P}^n/G$, where $\mathbb{G}_m \subset GL(n, \mathbb{C})$ is the subgroup consisting of non-zero scalar matrices.

It is well-known that for algebraic surfaces, klt singularities are the same as quotient singularities. In [Tia90], Tian showed that the anti-canonical volume of a Kähler-Einstein log Del Pezzo surface is bounded from above by $48/|G|$ where $G \subset GL(2, \mathbb{C})$ is the orbifold group at a closed point. In [OSS16], Odaka, Spotti and Sun improved the volume upper bounds to $12/|G|$.

As a direct consequence of Theorem 3.1.3, we get the sharp volume upper bounds $9/|G|$ for Kähler-Einstein log Del Pezzo surfaces.

Corollary 3.1.4. *Let X be a Kähler-Einstein log Del Pezzo surface. Let $p \in X$ be a closed point with local analytic model $\mathbb{C}^2/G$, where $G \subset GL(2, \mathbb{C})$ acts freely in codimension 1. Then*

$$(-K_X)^2 \leq \frac{9}{|G|},$$

with equality if and only if $|G \cap \mathbb{G}_m| = 1$ and $X \cong \mathbb{P}^2/G$.

It was conjectured by Cheltsov and Kosta in [CK14, Conjecture 1.18] that a Gorenstein Del Pezzo surface admits Kähler-Einstein metrics if and only if the singularities are of certain types depending on the anti-canonical volume. This conjecture was proved by Odaka, Spotti and Sun in [OSS16, Section 6.1.2] where $\mathbb{Q}$-Gorenstein smoothable Kähler-Einstein log Del Pezzo surfaces are classified. As a quick application of Corollary 3.1.4, we give an alternative proof of the "only if" part of this conjecture. We remark that partial results are known before [OSS16], e.g. [DT92, Jef97, Won13] on the "only if" part and [MM93, GK07, Che08, Shi10, CK14] on the "if" part.

Corollary 3.1.5. *Let X be a Kähler-Einstein log Del Pezzo surface with at most Du Val singularities.*

1. *If $(-K_X)^2 = 1$, then X has at most singularities of type $\mathbb{A}_1$, $\mathbb{A}_2$, $\mathbb{A}_3$, $\mathbb{A}_4$, $\mathbb{A}_5$, $\mathbb{A}_6$, $\mathbb{A}_7$ or $\mathbb{D}_4$.*
2. *If $(-K_X)^2 = 2$, then X has at most singularities of type $\mathbb{A}_1$, $\mathbb{A}_2$ or $\mathbb{A}_3$.*

3. *If* $(-K_X)^2 = 3$*, then* X *has at most singularities of type* $\mathbb{A}_1$ *or* $\mathbb{A}_2$*.*

4. *If* $(-K_X)^2 = 4$*, then* X *has at most singularities of type* $\mathbb{A}_1$*.*

5. *If* $(-K_X)^2 \geq 5$*, then* X *is smooth.*

The proofs of Theorem 3.1.1 and 3.1.2 rely on the recent result [Fuj15] where Fujita settled the optimal volume upper bounds for Ding-semistable $\mathbb{Q}$-Fano varieties. Recall that a $\mathbb{Q}$-Fano variety X is said to be *Ding-semistable* if the Ding invariant $\mathrm{Ding}(\mathcal{X}, \mathcal{L})$ is nonnegative for any normal test configuration $(\mathcal{X}, \mathcal{L})/\mathbb{A}^1$ of $(X, -rK_X)$. The results in [Ber16, Section 3] show that, if a $\mathbb{Q}$-Fano variety X admits a Kähler-Einstein metric, then X is Ding-semistable (see also [Fuj15, 3.2]). This allows us to prove Theorem 3.1.1 and 3.1.2 under the weaker assumption that X is Ding-semistable (see Theorem 3.3.1 and 3.3.5).

This chapter is organized as follows. In Section 3.2, we recall the concepts of filtrations and Fujita's work [Fuj15] on testing Ding-semistability by filtrations. In Section 3.3, we prove Theorem 3.1.1 and 3.1.2. The proof of Theorem 3.1.1 is a quick application of [Fuj15, 1.2] and Lemma 2.4.3. To prove Theorem 3.1.2, we apply [Fuj15, 4.9] to the filtration associated to any real valuation. In Section 3.3.1, we give some applications of Theorem 3.1.1 and 3.1.2. We recover the main result from [dFEM04]. We prove the inequality part of Theorem 3.1.3. We also get a volume upper bound for a Ding-semistable $\mathbb{Q}$-Fano variety with an isolated singularity that is not terminal (see Theorem 3.3.9). In Section 3.4, we study the equality case of Theorem 3.1.1 and 3.1.2. Lemma 3.4.1 characterizes the equality case of Theorem 3.1.2 when X is K-polystable and the real valuation v is divisorial. Applying this lemma, we prove the equality case of Theorem 3.1.3 and Corollary 3.1.5. We also prove that $\mathbb{P}^n$ is the only Ding-semistable $\mathbb{Q}$-Fano variety of volume at least $(n+1)^n$ (see Theorem 3.4.3). This theorem improves the equality case of [Fuj15, 1.1] where Fujita proved for Ding-semistable Fano manifolds.

3.2 Filtrations and Ding-semistability

3.2.1 Filtrations

We recall the relevant definitions about filtrations after [BC11] (see also [BHJ15] and [Fuj15, Section 4.1]).

Definition 3.2.1. A *good filtration* of a graded $\mathbb{C}$-algebra $S = \bigoplus_{m=0}^{\infty} S_m$ is a decreasing, left continuous, multiplicative and linearly bounded $\mathbb{R}$-filtrations of S. In other words, for each $m \geq 0 \in$

$\mathbb{Z}$, there is a family of subspaces $\{\mathcal{F}^x S_m\}_{x\in\mathbb{R}}$ of S_m such that:

- $\mathcal{F}^x S_m \subseteq \mathcal{F}^{x'} S_m$, if $x \geq x'$;
- $\mathcal{F}^x S_m = \bigcap_{x'<x} \mathcal{F}^{x'} S_m$;
- $\mathcal{F}^x S_m \cdot \mathcal{F}^{x'} S_{m'} \subseteq \mathcal{F}^{x+x'} S_{m+m'}$, for any $x, x' \in \mathbb{R}$ and $m, m' \in \mathbb{Z}_{\geq 0}$;
- $e_{\min}(\mathcal{F}) > -\infty$ and $e_{\max}(\mathcal{F}) < +\infty$, where $e_{\min}(\mathcal{F})$ and $e_{\max}(\mathcal{F})$ are defined by the following operations:

$$\begin{aligned} &e_{\min}(S_m, \mathcal{F}) := \inf\{t \in \mathbb{R} : \mathcal{F}^t S_m \neq S_m\}; \\ &e_{\max}(S_m, \mathcal{F}) := \sup\{t \in \mathbb{R} : \mathcal{F}^t S_m \neq 0\}; \\ &e_{\min}(\mathcal{F}) = e_{\min}(S_\bullet, \mathcal{F}) := \liminf_{m\to\infty} \frac{e_{\min}(S_m, \mathcal{F})}{m}; \\ &e_{\max}(\mathcal{F}) = e_{\max}(S_\bullet, \mathcal{F}) := \limsup_{i\to\infty} \frac{e_{\max}(S_m, \mathcal{F})}{m}. \end{aligned} \tag{3.2.1}$$

Define $S^{(t)} := \bigoplus_{k=0}^{\infty} \mathcal{F}^{kt} S_k$. When we want to emphasize the dependence of $S^{(t)}$ on the filtration $\mathcal{F}$, we also denote $S^{(t)}$ by $\mathcal{F}S^{(t)}$. The following concept of volume will be important for us:

$$\mathrm{vol}\left(S^{(t)}\right) = \mathrm{vol}\left(\mathcal{F}S^{(t)}\right) := \limsup_{k\to\infty} \frac{\dim_{\mathbb{C}} \mathcal{F}^{mt} S_m}{m^n/n!}. \tag{3.2.2}$$

Now assume L is an ample line bundle over X and $S = \bigoplus_{m=0}^{\infty} H^0(X, L^m) = \bigoplus_{m=0}^{\infty} S_m$ is the section ring of (X, L). Then following [Fuj15], we can define a sequence of ideal sheaves on X:

$$I^{\mathcal{F}}_{(m,x)} = \mathrm{Image}\left(\mathcal{F}^x S_m \otimes L^{-m} \to \mathcal{O}_X\right), \tag{3.2.3}$$

and define $\overline{\mathcal{F}}^x S_m := H^0(V, L^m \cdot I^{\mathcal{F}}_{(m,x)})$ to be the saturation of $\mathcal{F}^x S_m$ such that $\mathcal{F}^x S^m \subseteq \overline{\mathcal{F}}^x S_m$. $\mathcal{F}$ is called *saturated* if $\overline{\mathcal{F}}^x S_m = \mathcal{F}^x S_m$ for any $x \in \mathbb{R}$ and $m \in \mathbb{Z}_{\geq 0}$. Notice that with the notation above we have:

$$\mathrm{vol}\left(\overline{\mathcal{F}}S^{(t)}\right) := \limsup_{k\to\infty} \frac{\dim_{\mathbb{C}} \overline{\mathcal{F}}^{kt} H^0(X, kL)}{k^n/n!}.$$

3.2.2 Fujita's result on Ding-semistability

The following result was proved by Fujita by applying a criterion for Ding semistability ([Fuj15, Proposition 3.5]) to a sequence of semi-test configurations constructed from a valuation.

Theorem 3.2.2 ([Fuj15, 4.9]). *Let X be a Ding-semistable $\mathbb{Q}$-Fano variety of dimension n. Let $\mathcal{F}$ be a good graded filtration of $S = \bigoplus_{m=0}^{+\infty} H^0(X, mL)$ where $L = -r^{-1}K_X$ is Cartier. Then the pair $(X \times \mathbb{A}^1, \mathcal{I}_\bullet^r \cdot (t)^{d_\infty})$ is sub log canonical, where*

$$\mathcal{I}_m = I^{\mathcal{F}}_{(m,me_+)} + I^{\mathcal{F}}_{(m,me_+-1)}t^1 + \cdots + I^{\mathcal{F}}_{(m,me_-+1)}t^{m(e_+-e_-)-1} + (t^{m(e_+-e_-)}),$$
$$d_\infty = 1 - r(e_+ - e_-) + \frac{r^{n+1}}{(-K_X)^n}\int_{e_-}^{e_+} \mathrm{vol}\left(\overline{\mathcal{F}}S^{(t)}\right) dt,$$

and $e_+, e_- \in \mathbb{Z}$ with $e_+ \geq e_{\max}(S_\bullet, \mathcal{F})$ and $e_- \leq e_{\min}(S_\bullet, \mathcal{F})$.

Theorem 3.2.3 ([Fuj15, 1.2]). *Let X be a Ding-semistable $\mathbb{Q}$-Fano variety. Take any nonempty proper closed subscheme $\emptyset \neq Z \subsetneq X$ corresponds to an ideal sheaf $0 \neq I_Z \subsetneq \mathcal{O}_X$. Let $\sigma : \hat{X} \to X$ be the blow up along Z, let $F \subset \hat{X}$ be the Cartier divisor defined by the equation $\mathcal{O}_{\hat{X}}(-F) = \sigma^{-1}I_Z \cdot \mathcal{O}_{\hat{X}}$. Then we have*

$$\mathrm{lct}(X; I_Z) \geq \frac{1}{\mathrm{vol}_X(-K_X)}\int_0^{+\infty} \mathrm{vol}_{\hat{X}}(\sigma^*(-K_X) - xF)dx.$$

3.3 Volume upper bounds of Ding-semistable Fano varieties

In this section, we will always assume that X is a $\mathbb{Q}$-Fano variety of dimension n, and $L \sim_{\mathbb{Q}} -r^{-1}K_X$ is an ample line bundle for some $r \in \mathbb{Q}_{>0}$.

We will prove the following theorem, a stronger result that implies Theorem 3.1.1.

Theorem 3.3.1. *Let X be a Ding-semistable $\mathbb{Q}$-Fano variety. Let $p \in X$ be a closed point. Let Z be a closed subscheme of X with* Supp $Z = \{p\}$. *Then we have*

$$(-K_X)^n \leq \left(1 + \frac{1}{n}\right)^n \mathrm{lct}(X; I_Z)^n \mathrm{mult}_Z X. \tag{3.3.1}$$

Proof. Let $\sigma : \hat{X} \to X$ be the blow up along Z, and $F \subset \hat{X}$ be the Cartier divisor defined by the equation $\mathcal{O}_{\hat{X}}(-F) = \sigma^{-1}I_Z \cdot \mathcal{O}_{\hat{X}}$. By Theorem 3.2.3, we have

$$\mathrm{lct}(X; I_Z) \cdot (-K_X)^n \geq \int_0^{+\infty} \mathrm{vol}_{\hat{X}}(\sigma^*(-K_X) - xF)dx.$$

On the other hand, by Lemma 2.4.3, we have

$$\begin{aligned}\int_0^{+\infty} \mathrm{vol}_{\hat{X}}(\sigma^*(-K_X) - xF)dx &\geq \int_0^{\sqrt[n]{(-K_X)^n/\mathrm{mult}_Z X}} \left((-K_X)^n - \mathrm{mult}_Z X \cdot x^n\right)dx \\ &= \frac{n}{n+1}(-K_X)^n \cdot \sqrt[n]{(-K_X)^n/\mathrm{mult}_Z X}\end{aligned}$$

Hence we get the desired inequality

$$(-K_X)^n \leq \left(1+\frac{1}{n}\right)^n \mathrm{lct}(X; I_Z)^n \mathrm{mult}_Z X.$$

□

Remark 3.3.2. As we see in the proof above, if the equality of (3.3.1) holds, then

$$\mathrm{vol}_{\hat{X}}(\sigma_*(-K_X) - xF) = \max\{((-K_X)^n) - \mathrm{mult}_Z X \cdot x^n, 0\}$$

for any $x \in \mathbb{R}_{\geq 0}$. Then by Lemma 2.4.3 (2) we have that $\epsilon_Z(-K_X) = (1+\frac{1}{n})\mathrm{lct}(X; I_Z)$.

We will follow the notation of Theorem 3.2.2. Let $v \in \mathrm{Val}_{X,p}$ be a real valuation centered at p. Define an $\mathbb{R}$-filtration $\mathcal{F}_v$ of S as

$$\mathcal{F}_v^x S_m := H^0(X, L^m \cdot \mathfrak{a}_x(v)).$$

Lemma 3.3.3. *The $\mathbb{R}$-filtration $\mathcal{F}_v$ of S is decreasing, left-continuous, multiplicative and saturated. Moreover, if $A_X(v) < +\infty$ then $\mathcal{F}_v$ is also linearly bounded.*

Proof. The decreasing, left-continuous and multiplicative properties of $\mathcal{F}_v$ follows from the corresponding properties of $\{\mathfrak{a}_x(v)\}_{x\in\mathbb{R}}$. To prove that $\mathcal{F}_v$ is saturated, notice that the homomorphism

$$\mathcal{F}_v^x S_m \otimes_{\mathbb{C}} L^{-m} \twoheadrightarrow I^{\mathcal{F}_v}_{(m,x)}$$

induces the inclusion $I^{\mathcal{F}_v}_{(m,x)} \subset \mathfrak{a}_x(v)$ for any $x \in \mathbb{R}$. Thus $\overline{\mathcal{F}}^x S_m = H^0(X, L^m \cdot I^{\mathcal{F}_v}_{(m,x)}) \subset \mathcal{F}^x S_m$, which implies that $\mathcal{F}_v$ is saturated.

If $A_X(v) < +\infty$, then by [Li15b, 1.2] there exists a constant $c = c(X,p)$ such that $v \leq cA_X(v)\mathrm{ord}_p$. So it is easy to see that

$$e_{\max}(S_\bullet, \mathcal{F}_v) \leq cA_X(v) \cdot e_{\max}(S_\bullet, \mathcal{F}_{\mathrm{ord}_p}) < +\infty.$$

The second inequality follows from the work in [BKMS15]. Besides, it is clear that $\mathfrak{a}_x(v) = \mathcal{O}_X$ for $x \leq 0$. Hence $\mathcal{F}_v^x S_m = S_m$ for $x \leq 0$, which implies $e_{\min}(S_\bullet, \mathcal{F}_v) \geq 0$. Hence $\mathcal{F}_v$ is linearly bounded. □

It is clear that $\mathcal{F}_v^x S_m = S_m$ for $x \leq 0$. Hence we may choose $e_- = 0$. The graded sequence of

ideal sheaves $\mathcal{I}_\bullet$ on $X \times \mathbb{A}^1$ becomes:

$$\mathcal{I}_m = I^{\mathcal{F}_v}_{(m,me_+)} + I^{\mathcal{F}_v}_{(m,me_+-1)}t^1 + \cdots + I^{\mathcal{F}_v}_{(m,1)}t^{me_+-1} + (t^{me_+}).$$

We also have that

$$d_\infty = 1 - re_+ + \frac{r^{n+1}}{(-K_X)^n}\int_0^{+\infty} \mathrm{vol}\left(\mathcal{F}_v S^{(t)}\right) dt.$$

The valuation v extends to a $\mathbb{G}_m$-invariant valuation $\bar{v}$ on $\mathbb{C}(X \times \mathbb{A}^1) = \mathbb{C}(X)(t)$, such that for any $f = \sum_j f_j t^j$ we have

$$\bar{v}(f) = \min_j\{v(f_j) + j\}. \tag{3.3.2}$$

Proposition 3.3.4. *Let X be a $\mathbb{Q}$-Fano variety of dimension n. Assume that X is Ding-semistable. Let $p \in X$ be a closed point. Let $v \in \mathrm{Val}_{X,p}$ be a real valuation centered at p. Then we have*

$$A_X(v) \geq \frac{r^{n+1}}{(-K_X)^n}\int_0^{+\infty} \mathrm{vol}\left(\mathcal{F}_v S^{(t)}\right) dt. \tag{3.3.3}$$

Proof. We may assume that $A_X(v) < +\infty$, since otherwise the inequality holds automatically. Hence Lemma 3.3.3 implies that $\mathcal{F}_v$ is good and saturated.

By Theorem 3.2.2, we know that $(X \times \mathbb{A}^1, \mathcal{I}^r_\bullet \cdot (t)^{d_\infty})$ is sub log canonical. Since $X \times \mathbb{A}^1$ has klt singularities, for any $0 < \epsilon \ll 1$ there exists $m = m(\epsilon)$ such that

$$\mathcal{O}_{X\times\mathbb{A}^1} \subset \mathcal{J}(X \times \mathbb{A}^1, \mathcal{I}_m^{(1-\epsilon)r/m} \cdot (t)^{(1-\epsilon)d_\infty}).$$

By [BdFFU15, 1.2] we know that the following inequality holds for any real valuation u on $X \times \mathbb{A}^1$:

$$A_{X\times\mathbb{A}^1}(u) > \frac{(1-\epsilon)r}{m}u(\mathcal{I}_m) + (1-\epsilon)d_\infty u(t). \tag{3.3.4}$$

Let us choose a sequence of quasi-monomial real valuations $\{v_n\}$ on X such that $v_n \to v$ and $A_X(v_n) \to A_X(v)$ when $n \to \infty$. It is easy to see that $\{\bar{v}_n\}$ is a sequence of quasi-monimial valuations on $X \times \mathbb{A}^1$ satisfying $\bar{v}_n(t) = 1$ and

$$A_{X\times\mathbb{A}^1}(\bar{v}_n) = A_X(v_n) + 1.$$

Hence we have

$$\begin{aligned} A_X(v)+1 &= \lim_{n\to\infty} A_X(\bar{v}_n) \\ &\geq \frac{(1-\epsilon)r}{m} \lim_{n\to\infty} \bar{v}_n(\mathcal{I}_m) + (1-\epsilon)d_\infty. \end{aligned}$$

From the definition of $\mathcal{F}_v^x S_m$ we get:

$$v\left(I^{\mathcal{F}_v}_{(m,x)}\right) \geq x.$$

Therefore,

$$\begin{aligned} \lim_{n\to\infty} \bar{v}_n(\mathcal{I}_m) &= \lim_{n\to\infty} \min_{0\leq j\leq re_+} \left\{ v_n\left(I^{\mathcal{F}_v}_{(m,j)}\right) + me_+ - j \right\} \\ &= \min_{0\leq j\leq re_+} \left\{ \lim_{n\to\infty} v_n\left(I^{\mathcal{F}_v}_{(m,j)}\right) + me_+ - j \right\} \\ &= \min_{0\leq j\leq re_+} \left\{ v\left(I^{\mathcal{F}_v}_{(m,j)}\right) + me_+ - j \right\} \\ &\geq me_+ \end{aligned}$$

Hence when $\epsilon \to 0+$ we get:

$$A_X(v)+1 \geq re_+ + d_\infty. \tag{3.3.5}$$

Therefore,

$$A_X(v) \geq -1 + re_+ + d_\infty = \frac{r^{n+1}}{(-K_X)^n} \int_0^{+\infty} \mathrm{vol}\left(\mathcal{F}_v S^{(t)}\right) dt.$$

Hence we get the desired inequality. □

The following theorem is a stronger result that implies Theorem 3.1.2.

Theorem 3.3.5. *Let X be a Ding-semistable $\mathbb{Q}$-Fano variety. Let $p \in X$ be a closed point. Let $v \in \mathrm{Val}_{X,p}$ be a real valuation centered at p. Then we have*

$$(-K_X)^n \leq \left(1+\frac{1}{n}\right)^n \widehat{\mathrm{vol}}(v).$$

Proof. We may assume $A_X(v) < +\infty$, since otherwise the inequality holds automatically. Since $\mathcal{F}_v^{mx} S_m = H^0(X, L^m \cdot \mathfrak{a}_{mx}(v))$, we have the exact sequence

$$0 \to \mathcal{F}_v^{mx} S_m \to H^0(X, L^m) \to H^0(X, L^m \otimes (\mathcal{O}_X/\mathfrak{a}_{mx}(v))).$$

Hence we have

$$\dim \mathcal{F}_v^{mx} S_m \geq h^0(X, L^m) - \ell(\mathcal{O}_X/\mathfrak{a}_{mx}(v)).$$

This implies

$$\mathrm{vol}(\mathcal{F}_v S^{(x)}) \geq (L^n) - \mathrm{vol}(v)x^n. \tag{3.3.6}$$

So we have the estimate:

$$\begin{aligned}\int_0^{+\infty} \mathrm{vol}\left(\mathcal{F}_v S^{(x)}\right) dx &\geq \int_0^{\sqrt[n]{(L^n)/\mathrm{vol}(v)}} ((L^n) - \mathrm{vol}(v)x^n)\, dx \\ &= \frac{n}{n+1}(L^n) \cdot \sqrt[n]{(L^n)/\mathrm{vol}(v)}.\end{aligned}$$

Applying (3.3.3) we get the inequality:

$$\begin{aligned}A_X(v) &\geq \frac{r}{(L^n)} \int_0^{+\infty} \mathrm{vol}\left(\mathcal{F}_v S^{(t)}\right) dt \\ &\geq \frac{r}{(L^n)} \frac{n}{n+1}(L^n) \sqrt[n]{(L^n)/\mathrm{vol}(v)} = \frac{n}{n+1} \sqrt[n]{(-K_X)^n/\mathrm{vol}(v)}.\end{aligned}$$

This is exactly equivalent to

$$(-K_X)^n \leq \left(1 + \frac{1}{n}\right)^n A_X(v)^n \mathrm{vol}(v) = \left(1 + \frac{1}{n}\right)^n \widehat{\mathrm{vol}}(v).$$

□

3.3.1 Applications

The following result is an easy consequence of Theorem 3.1.1.

Theorem 3.3.6 ([dFEM04])**.** *Let X be a variety of dimension n. Let Z be a closed subscheme of X supported at a single smooth closed point. Then*

$$\left(\frac{\mathrm{lct}(X; I_Z)}{n}\right)^n \mathrm{mult}_Z X \geq 1.$$

Proof. It is clear that the multiplicity is preserved under completion. Besides, [dFEM11, 2.11] implies that lct is also preserved under completion. Hence it suffices to prove the theorem for $X = \mathbb{P}^n$. This follows directly by applying Theorem 3.1.1 to the case $X = \mathbb{P}^n$. □

Now we will illustrate some volume bounds for singular X.

Theorem 3.3.7. *Let X be a Ding-semistable $\mathbb{Q}$-Fano variety. Let $p \in X$ be a closed point. Suppose (X, p) is a quotient singularity with local analytic model $\mathbb{C}^2/G$ where $G \subset GL(n, \mathbb{C})$ acts freely in codimension 1. Then*

$$(-K_X)^n \leq \frac{(n+1)^n}{|G|}.$$

Proof. Let $(Y, o) := (\mathbb{C}^n/G, 0)$ be the local analytic model of the quotient singularity (X, p). Let $H := G \cap \mathbb{G}_m$ with $d := |H|$. Define the real valuation u_0 on Y to be the pushforward of the valuation ord_0 on $\mathbb{C}^n$ under the quotient map $\mathbb{C}^n \to Y$.

We first show that $\widehat{\mathrm{vol}}(u_0) = \frac{n^n}{|G|}$. Let $\widehat{\mathbb{C}^n}$ be the blow up of $\mathbb{C}^n$ at the origin 0 with exceptional divisor E. Denote by $\pi : \mathbb{C}^n \to Y$ the quotient map. Then π lifts to $\widehat{\mathbb{C}^n}$ as $\hat{\pi} : \widehat{\mathbb{C}^n} \to \hat{Y}$, where $\hat{Y} := \widehat{\mathbb{C}^n}/G$. We have the following commutative diagram:

$$\begin{array}{ccc} \widehat{\mathbb{C}^n} & \xrightarrow{\hat{\pi}} & \hat{Y} \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle h} \\ \mathbb{C}^n & \xrightarrow{\pi} & Y \end{array}$$

Let $F \subset \hat{Y}$ be the exceptional divisor of h. For a general point on F, its stabilizer under the G-action is exactly H. So $\hat{\pi}^* F = dE$, which implies that $u_0 = \pi_*(\mathrm{ord}_E) = d \, \mathrm{ord}_F$. It is clear that

$$K_{\widehat{\mathbb{C}^n}} = \hat{\pi}^* \left(K_{\hat{Y}} + \left(1 - \frac{1}{d}\right) F \right).$$

Then combining these equalities with $K_{\widehat{\mathbb{C}^n}} = g^* K_{\mathbb{C}^n} + (n-1)E = g^*(\pi^* K_Y) + (n-1)E$, we get

$$K_{\hat{Y}} = h^* K_Y + \left(\frac{n}{d} - 1\right) F.$$

Hence $A_X(\mathrm{ord}_F) = \frac{n}{d}$ and $A_X(u_0) = d \cdot A_X(\mathrm{ord}_F) = n$. Lemma 3.3.8 implies that $\mathrm{vol}(u_0) = \frac{1}{|G|}$. Therefore, $\widehat{\mathrm{vol}}(u_0) = \frac{n^n}{|G|}$.

Since $A_Y(u_0) < +\infty$, a singular version of [JM12, 5.11] implies that u_0 has a unique extension, say $\hat{u}_0$, as a valuation on $\mathbf{Spec}\ \widehat{\mathcal{O}_{Y,o}}$ that centered at the closed point. By assumption we have an isomorphism $\phi : \widehat{\mathcal{O}_{X,p}} \to \widehat{\mathcal{O}_{Y,o}}$. Let $v_0 := (\phi^* \hat{u}_0)|_{\mathcal{O}_{X,p}}$, then $v_0 \in \mathrm{Val}_{X,p}$. By a singular version of [JM12, 5.13], we know that $A_X(v_0) = A_Y(u_0)$. On the other hand, since the colength of any $\mathfrak{m}$-primary ideal does not change under completion, we have $\mathrm{vol}(v_0) = \mathrm{vol}(\hat{u}_0) = \mathrm{vol}(u_0)$. As a result, we have

$$\widehat{\mathrm{vol}}(v_0) = \widehat{\mathrm{vol}}(u_0) = \frac{n^n}{|G|}.$$

Hence the theorem follows as a direct application of Theorem 3.1.2 to $v = v_0$. □

Lemma 3.3.8. *Let $G \subset GL(n, \mathbb{C})$ be a finite group acting on $\mathbb{C}^n = \mathbf{Spec}\ \mathbb{C}[x_1, \cdots, x_n]$. Then we have*

$$\lim_{m\to\infty} \frac{\dim_{\mathbb{C}} \mathbb{C}[x_1, \cdots, x_n]^G_{<m}}{m^n/n!} = \frac{1}{|G|}.$$

Proof. Denote $W := \mathbb{C}[x_1, \cdots, x_n] = \mathbb{C}x_1 \oplus \cdots \oplus \mathbb{C}x_n$. Then we have that

$$\mathbb{C}[x_1, \cdots, x_n] \cong \bigoplus_{m\geq 0} \mathrm{Sym}^m W.$$

Denote by $\rho_m : G \to GL(\mathrm{Sym}^m W)$ the induced representation of G on $\mathrm{Sym}^m W$. Since G is finite, $\rho_m(g)$ is diagonalizable for any $m \geq 0$ and $g \in G$. Denote the n eigenvalues of $\rho_1(g)$ by $\lambda_{g,1}, \cdots, \lambda_{g,n}$. Then the eigenvalues of $\rho_m(g)$ are exactly all monomials in $\lambda_{g,i}$ of degree m. Let $c_m := \dim_{\mathbb{C}}(\mathrm{Sym}^m W)^G$ and $d_m := \dim_{\mathbb{C}} \mathbb{C}[x_1, \cdots, x_n]^G_{\leq m}$. From representation theory we know that

$$c_m = \frac{1}{|G|} \sum_{g\in G} \mathrm{tr}(\rho_m(g)).$$

Hence

$$c(t) := \sum_{m=0}^{\infty} c_m t^m = \frac{1}{|G|} \sum_{g\in G} \frac{1}{(1-\lambda_{g,1}t)\cdots(1-\lambda_{g,n}t)}.$$

This is also known as Molien's theorem [Mol97]. Since $d_m = \sum_{i=0}^m c_i$, we have

$$\begin{aligned} d(t) &:= \sum_{m=0}^{\infty} d_m t^m = \frac{c(t)}{1-t} \\ &= \frac{1}{|G|} \sum_{g\in G} \frac{1}{(1-t)(1-\lambda_{g,1}t)\cdots(1-\lambda_{g,n}t)}. \end{aligned}$$

Here all eigenvalues $\lambda_{g,i}$ are roots of unity. Using partial fraction decomposition, for any $g \neq id$ we have

$$\frac{1}{(1-t)(1-\lambda_{g,1}t)\cdots(1-\lambda_{g,n}t)} = \sum_{m=0}^{\infty} o(m^n)t^m.$$

Hence

$$\begin{aligned} d(t) &= \frac{1}{|G|} \cdot \frac{1}{(1-t)^{n+1}} + \sum_{m=0}^{\infty} o(m^n)t^m \\ &= \sum_{m=0}^{\infty} \left(\frac{1}{|G|} \binom{n+m}{n} + o(m^n) \right) t^m. \end{aligned}$$

As a result,

$$\lim_{m\to\infty}\frac{d_{m-1}}{m^n/n!}=\lim_{m\to\infty}\frac{\frac{1}{|G|}\binom{n+m-1}{n}}{m^n/n!}=\frac{1}{|G|}.$$

□

Theorem 3.3.9. *Let X be a Ding-semistable $\mathbb{Q}$-Fano variety. Let $p\in X$ be an isolated singularity. If X is not terminal at p, then*

$$(-K_X)^n\leq\left(1+\frac{1}{n}\right)^n\operatorname{mult}_pX<e\operatorname{mult}_pX.$$

Proof. Denote by $\mathfrak{m}_p$ the maximal ideal at p. We first show that $\operatorname{lct}(X;\mathfrak{m}_p)\leq 1$. Since p is an isolated singularity, we may take a log resolution of $(X,\mathfrak{m}_p)$, namely $\pi:Y\to X$ such that π is an isomorphism away from p and $\pi^{-1}\mathfrak{m}_p\cdot\mathcal{O}_Y$ is an invertible ideal sheaf on Y. Let E_i be the exceptional divisors of π. We define the numbers a_i and b_i by

$$K_Y=\pi^*K_X+\sum_i a_iE_i\quad\text{and}\quad\pi^{-1}\mathfrak{m}_p\cdot\mathcal{O}_Y=\mathcal{O}_Y(-\sum_i b_iE_i).$$

It is clear that $\operatorname{lct}(X;\mathfrak{m}_p)=\min_i\frac{1+a_i}{b_i}$. Since π is an isomorphism away from p, we have $b_i\geq 1$ for any i. Since X is not terminal at p, there exists an index i_0 such that $a_{i_0}\leq 0$. Hence

$$\operatorname{lct}(X;\mathfrak{m}_p)\leq\frac{1+a_{i_0}}{b_{i_0}}\leq 1.$$

So we finish the proof by applying Theorem 3.1.1 to $Z=p$. □

3.4 K-polystable Fano varieties of maximal volume

3.4.1 Characterization of K-polystable projective orbifold cones

Lemma 3.4.1. *Let X be a K-polystable $\mathbb{Q}$-Fano variety of dimension n. Let $p\in X$ be a closed point. Assume that there exists a divisorial valuation v_* on X centered at p satisfying $(-K_X)^n=\left(1+\frac{1}{n}\right)^n\widehat{\operatorname{vol}}(v_*)$. Then we have*

1. *v_* is a minimizer of the normalized volume function $\widehat{\operatorname{vol}}$ on $\operatorname{Val}_{X,p}$;*
2. *$X\cong\mathbf{Proj}\ (\operatorname{gr}_{v_*}\mathcal{O}_{X,p})[x]$, where x is a homogeneous element of degree 1 (we may view X as an orbifold projective cone over $F:=\mathbf{Proj}\ \operatorname{gr}_{v_*}\mathcal{O}_{X,p}$);*

Proof. Part 1 is a direct consequence of Theorem 3.1.2.

We need some preparation to prove part 2. Let $\bar{v}_*$ be the corresponding divisorial valuation on $X \times \mathbb{A}^1$ (see (3.3.2)). Let t be the parameter of $\mathbb{A}^1$. Let $\mathbf{Spec}\ R$ be an affine Zariski open subset of X. Then

$$\mathfrak{a}_m(\bar{v}_*) = \mathfrak{a}_m(v_*) + \mathfrak{a}_{m-1}(v_*)t + \cdots + \mathfrak{a}_1(v_*)t^{m-1} + (t^m).$$

In order to keep track of the degree of the graded ring $\oplus_{m\geq 0}\mathfrak{a}_m(\bar{v}_*)$, denote by s the element t in $\mathfrak{a}_1(\bar{v}_*) = \mathfrak{a}_1(v_*) + (t)$. Hence $\deg s = 1$, $\deg t = 0$ and we have

$$\mathfrak{a}_m(\bar{v}_*) = \mathfrak{a}_m(v_*) + \mathfrak{a}_{m-1}(v_*)s + \cdots + \mathfrak{a}_1(v_*)s^{m-1} + s^m R[t],$$

where $\deg \mathfrak{a}_i(v_*) = i$. As an $R[t]$-algebra, $\oplus_{m\geq 0}\mathfrak{a}_m(\bar{v}_*)$ is generated by $\oplus_{m\geq 0}\mathfrak{a}_m(v_*)$ and s. Since $\oplus_{m\geq 0}\mathfrak{a}_m(v_*)$ is a finitely generated R-algebra by Theorem 2.3.10, we have that $\oplus_{m\geq 0}\mathfrak{a}_m(\bar{v}_*)$ is of finite type over $\mathcal{O}_{X\times\mathbb{A}^1}$.

Let $\mathcal{X} := \mathbf{Proj}_{X\times\mathbb{A}^1} \oplus_{m\geq 0} \mathfrak{a}_m(\bar{v}_*)$, hence $\mathcal{X}$ is normal by Theorem 2.3.10. Denote by $g : \mathcal{X} \to X \times \mathbb{A}^1$ the projection. Denote the composite map by $\pi : \mathcal{X} \to \mathbb{A}^1$. The $\mathbb{Q}$-line bundle $\mathcal{L}$ on $\mathcal{X}$ is defined as

$$\mathcal{L} := g^*(-K_{X\times\mathbb{A}^1/\mathbb{A}^1}) + \left(1 + \frac{1}{n}\right) A_X(v_*)\mathcal{O}_{\mathcal{X}}(1),$$

where we treat $\mathcal{O}_{\mathcal{X}}(1) = \frac{1}{k}\mathcal{O}_{\mathcal{X}}(k)$ as a $\mathbb{Q}$-line bundle for sufficiently divisible $k \in \mathbb{Z}_{>0}$. Let $\hat{X} := g_*^{-1}(X \times \{0\})$ and $\sigma = g|_{\hat{X}} : \hat{X} \to X$. Let E be the exceptional divisor of g given by $I_E := \mathcal{O}_{\mathcal{X}}(1)$. Hence

$$E \cong \mathbf{Proj}\ \oplus_{m\geq 0} \mathfrak{a}_m(\bar{v}_*)/\mathfrak{a}_{m+1}(\bar{v}_*) = \mathbf{Proj}\ (\mathrm{gr}_{v_*}\mathcal{O}_{X,p})[s],$$

where s is a homogeneous element of degree 1. Since $\mathrm{gr}_{v_*}\mathcal{O}_{X,p}$ is an integral domain, E is irreducible and reduced.

We will show the following statements:

(a) $\mathcal{L}$ is π-nef with $\mathcal{L}^\perp = \hat{X}$;

(b) $\mathcal{L}$ is π-semiample, hence $(\mathcal{X}, \mathcal{L})$ is a semi $\mathbb{Q}$-test configuration of $(X, -K_X)$;

(c) $\mathrm{CM}(\mathcal{X}, \mathcal{L}) = 0$.

For (a), we see that $\mathcal{L}_t \cong -K_{\mathcal{X}_t}$ is ample whenever $t \neq 0$. From the definition we know that $\mathcal{L}$

is g-ample, hence $\mathcal{L}|_E$ is ample. On $\hat{X}$, we have that

$$\mathcal{L}|_{\hat{X}} \cong \sigma^*(-K_X) + \left(1+\frac{1}{n}\right) A_X(v_*)\mathcal{O}_{\hat{X}}(1).$$

For $k \in \mathbb{Z}_{>0}$ sufficiently divisible, let Z be the thickening of p in X such that $I_Z = \mathfrak{a}_k(v_*)$. From part 1 we know that v_* minimizes $\widehat{\mathrm{vol}}$, hence $\mathrm{lct}(X; I_Z)^n \cdot \mathrm{mult}_Z X = \widehat{\mathrm{vol}}(v_*)$ by Proposition 3.4.2. Since

$$(-K_X)^n = \left(1+\frac{1}{n}\right)^n \widehat{\mathrm{vol}}(v_*) = \left(1+\frac{1}{n}\right)^n \mathrm{lct}(X; I_Z)^n \cdot \mathrm{mult}_Z X,$$

by Lemma 2.4.3 and Remark 3.3.2 we have that $\mathcal{L}|_{\hat{X}}$ is nef with zero top self intersection number. In addition, $\epsilon_Z(-K_X) = A_X(v_*)/m$. As a result, $\mathcal{L}^\perp = \hat{X}$.

For (b), we first show that $\mathcal{X}$ is normal $\mathbb{Q}$-Gorenstein with klt singularities. By Theorem 2.3.10 we have that $E = \mathrm{Ex}(g)$ is a $\mathbb{Q}$-Cartier prime divisor on $\mathcal{X}$. Hence

$$K_{\mathcal{X}} = g^* K_{X\times\mathbb{A}^1} + (A_{\mathcal{X}}(\mathrm{ord}_E) - 1)E$$

is $\mathbb{Q}$-Cartier. To show that $\mathcal{X}$ has klt singularities, it suffices to show that $(\mathcal{X}, E)$ is plt. Since $\mathcal{X} \setminus E \cong (X \times \mathbb{A}^1) \setminus \{(p,0)\}$, by inverse of adjunction it suffices to show that $(\hat{X}, F)$ is klt where $\mathrm{ord}_F = v_*$. This follows from Theorem 2.3.10.

By Shokurov's base-point-free theorem, to show $\mathcal{L}$ is π-semiample we only need to show that $\mathcal{L} - K_{\mathcal{X}/\mathbb{A}^1}$ is π-ample. It follows from Theorem 2.3.10 that $\bar{v}_* = \mathrm{ord}_E$ and $E \sim_{\mathbb{Q}} \mathcal{O}_{\mathcal{X}}(-1)$. Hence we have

$$\begin{aligned} -K_{\mathcal{X}/\mathbb{A}^1} &= g^*(-K_{X\times\mathbb{A}^1/\mathbb{A}^1}) - (A_{X\times\mathbb{A}^1}(\bar{v}_*) - 1)E \\ &= g^*(-K_{X\times\mathbb{A}^1/\mathbb{A}^1}) + A_X(v_*)\mathcal{O}_{\mathcal{X}}(1). \end{aligned}$$

Since $0 < A_X(v_*) < (1+\frac{1}{n})A_X(v_*)$, we see that $-K_{\mathcal{X}/\mathbb{A}^1}$ is π-ample. Hence $\mathcal{L} - K_{\mathcal{X}/\mathbb{A}^1}$ is π-ample.

For (c), we know that

$$\mathrm{CM}(\mathcal{X}, \mathcal{L}) = \frac{1}{(n+1)(-K_X)^n}\left(n(\bar{\mathcal{L}}^{n+1}) + (n+1)(\bar{\mathcal{L}}^n \cdot K_{\bar{\mathcal{X}}/\mathbb{P}^1})\right).$$

By definition of $\mathcal{L}$ we know that

$$\bar{\mathcal{L}} = \bar{g}^*(-K_{X\times\mathbb{P}^1/\mathbb{P}^1}) + \left(1+\frac{1}{n}\right) A_X(v_*)\mathcal{O}_{\bar{\mathcal{X}}}(1),$$
$$K_{\bar{\mathcal{X}}/\mathbb{P}^1} = \bar{g}^* K_{X\times\mathbb{P}^1/\mathbb{P}^1} - A_X(v_*)\mathcal{O}_{\bar{\mathcal{X}}}(1).$$

Since $\mathcal{O}_{\bar{\mathcal{X}}}(1)$ is supported in $\bar{g}^{-1}((p,0))$, we have $(\bar{g}^* K_{X\times\mathbb{P}^1/\mathbb{P}^1}\cdot\mathcal{O}_{\bar{\mathcal{X}}}(1)) = 0$ as a cycle. Besides, it is clear that $((\bar{g}^* K_{X\times\mathbb{P}^1/\mathbb{P}^1})^{n+1}) = (K^{n+1}_{X\times\mathbb{P}^1/\mathbb{P}^1}) = 0$. Hence we have

$$(\bar{\mathcal{L}}^{n+1}) = \left(1+\frac{1}{n}\right)^{n+1} A_X(v_*)^{n+1}(\mathcal{O}_{\bar{\mathcal{X}}}(1)^{n+1}),$$
$$(\bar{\mathcal{L}}^n \cdot K_{\bar{\mathcal{X}}/\mathbb{P}^1}) = -\left(1+\frac{1}{n}\right)^{n} A_X(v_*)^{n+1}(\mathcal{O}_{\bar{\mathcal{X}}}(1)^{n+1}).$$

Thus $\mathrm{CM}(\mathcal{X},\mathcal{L}) = 0$.

Now we are ready to prove part 2. By (b) we know that $\mathcal{L}$ is semiample. Denote the ample model of $(\mathcal{X},\mathcal{L})/\mathbb{A}^1$ by $(\mathcal{Y},\mathcal{M})$, with $h:\mathcal{X}\to\mathcal{Y}$ and $\mathcal{L} = h^*\mathcal{M}$. Then $(\mathcal{Y},\mathcal{M})$ is a normal $\mathbb{Q}$-test configuration of $(X,-K_X)$. Since $\bar{h}_* K_{\bar{\mathcal{X}}/\mathbb{P}^1} = K_{\bar{\mathcal{Y}}/\mathbb{P}^1}$, we have $\mathrm{CM}(\mathcal{Y},\mathcal{M}) = \mathrm{CM}(\mathcal{X},\mathcal{L}) = 0$. From (a) we know that $\mathcal{L}^\perp = \hat{X}$, so $\mathrm{Ex}(h) = \hat{X}$. Hence $\mathcal{Y}_0 = h_*\mathcal{X}_0 = h_*(\hat{X}+E) = h_*E$. In particular, $\mathcal{Y}_0$ is a prime divisor on $\mathcal{Y}$. It is clear that $\mathcal{L}|_E = (1+\frac{1}{n})A_X(v_*)\mathcal{O}_E(1)$ is ample, hence $h|_E : E\to\mathcal{Y}_0$ is a finite birational morphism. Since X is K-polystable by assumption, hence $\mathrm{CM}(\mathcal{Y},\mathcal{M}) = 0$ yields that $X\cong\mathcal{Y}_0$. Then $h|_E : E\to\mathcal{Y}_0$ has to be an isomorphism because $\mathcal{Y}_0\cong X$ is normal. Hence $X\cong E\cong \mathbf{Proj}\ (\mathrm{gr}_{v_*}\mathcal{O}_{X,p})[s]$. □

Proposition 3.4.2. *Assume that $\widehat{\mathrm{vol}}(\cdot)$ is minimized at a divisorial valuation v_*. Then for any sufficiently divisible $k\in\mathbb{Z}_{>0}$ we have*

$$\mathrm{lct}(\mathfrak{a}_k(v_*))^n\cdot\mathrm{mult}(\mathfrak{a}_k(v_*)) = \widehat{\mathrm{vol}}(v_*).$$

In particular, the normalized multiplicity function $\big(\mathrm{lct}(\cdot)^n\cdot\mathrm{mult}(\cdot)\big)$ attains its minimum at $\mathfrak{a}_k(v_*)$.

Proof. By Izumi's theorem we know that the filtration $\mathfrak{a}_1(v_*)\supset\mathfrak{a}_2(v_*)\supset\cdots$ defines the same topology on $\mathcal{O}_{X,p}$ as the $\mathfrak{m}_p$-adic topology on $\mathcal{O}_{X,p}$. Then Theorem 2.3.10 implies $\oplus_{m\geq 0}\mathfrak{a}_m(v_*)$ is a finitely generated $\mathcal{O}_{X,p}$-algebra. Therefore, the degree k Veronese subalgebra $\oplus_{m\geq 0}\mathfrak{a}_{km}(v_*)$ is generated in degree 1 for $k\in\mathbb{Z}_{>0}$ sufficiently divisible. This means $\mathfrak{a}_{km} = \mathfrak{a}_k^m$ for any $m\geq 0$. From

inequality (2.3.7) we have

$$\begin{aligned}\widehat{\mathrm{vol}}(v_*) &\geq \mathrm{lct}(\mathfrak{a}_\bullet(v_*))^n \cdot \mathrm{vol}(\mathfrak{a}_\bullet(v_*)) \\ &= \mathrm{lct}(\mathfrak{a}_{k\bullet}(v_*))^n \cdot \mathrm{vol}(\mathfrak{a}_{k\bullet}(v_*)) \\ &= \mathrm{lct}(\mathfrak{a}_k(v_*))^n \cdot \mathrm{mult}(\mathfrak{a}_k(v_*)).\end{aligned}$$

On the other hand, Theorem 2.3.5 also implies that $\mathrm{lct}(\mathfrak{a}_k(v_*))^n \cdot \mathrm{mult}(\mathfrak{a}_k(v_*)) \geq \widehat{\mathrm{vol}}(v_*)$ because v_* is a minimizer of $\widehat{\mathrm{vol}}(\cdot)$. So we prove the proposition. □

3.4.2 Applications

The following theorem improves Fujita's result on the equality case in [Fuj15, 5.1].

Theorem 3.4.3. *Let X be a Ding-semistable $\mathbb{Q}$-Fano variety of dimension n. If $(-K_X)^n \geq (n+1)^n$, then $X \cong \mathbb{P}^n$.*

Proof 1. Notice that $(-K_X)^n \leq (n+1)^n$ by [Fuj15, 1.3]. Thus we have $(-K_X)^n = (n+1)^n$. Let $p \in X$ be a smooth point. From [Fuj15, Proof of 5.1] or Remark 3.3.2, we see that $\epsilon_p(-K_X) = n+1$. Then the theorem is a consequence of [LZ16, Theorem 2] (joint with Ziquan Zhuang), where we show that if an n-dimensional $\mathbb{Q}$-Fano variety X satisfies $\epsilon_p(-K_X) > n$ for some smooth point $p \in X$, then $X \cong \mathbb{P}^n$. □

Proof 2. We follow the strategy and notation of the proof of Lemma 3.4.1. Let $v_* := \mathrm{ord}_p$ for a smooth point $p \in X$. As argued in the first proof, the assumptions of Lemma 3.4.1 are fulfilled except that X is only Ding-semistable rather than K-polystable. Nevertheless, we still have a semi $\mathbb{Q}$-test configuration $(\mathcal{X}, \mathcal{L})$ of $(X, -K_X)$ such that $\mathcal{L}^\perp = \hat{X}$ and $\mathrm{CM}(\mathcal{X}, \mathcal{L}) = 0$. Let $(\mathcal{Y}, \mathcal{M})$ be the ample model of $(\mathcal{X}, \mathcal{L})/\mathbb{A}^1$, with $h : \mathcal{X} \to \mathcal{Y}$ and $\mathcal{L} = h^*\mathcal{M}$. We will show that $\mathcal{Y}_0 \cong \mathbb{P}^n$.

Following the proof of Lemma 3.4.1, we know that $h|_E : E \to \mathcal{Y}_0$ is a finite birational morphism. Since $v_* = \mathrm{ord}_p$ and $p \in X$ is a smooth point, we have that $E \cong \mathbb{P}^n$. Therefore, E is the normalization of $\mathcal{Y}_0$. Consider the short exact sequence

$$0 \to \mathcal{O}_{\mathcal{X}}(-E) \to \mathcal{O}_{\mathcal{X}} \to \mathcal{O}_E \to 0.$$

By taking h_*, we get a long exact sequence

$$0 \to h_*\mathcal{O}_{\mathcal{X}}(-E) \to \mathcal{O}_{\mathcal{Y}} \to h_*\mathcal{O}_E \to R^1h_*\mathcal{O}_{\mathcal{X}}(-E) \to \cdots \tag{3.4.1}$$

Since $\mathcal{X}_0 = \hat{X} + E$, we have that $h_*\mathcal{O}_{\mathcal{X}}(-E) = \mathcal{O}_{\mathcal{Y}}(-\mathcal{Y}_0) \otimes h_*\mathcal{O}_{\mathcal{X}}(\hat{X})$. Notice that $\hat{X}$ is h-exceptional, hence $h_*\mathcal{O}_{\mathcal{X}}(\hat{X}) = \mathcal{O}_{\mathcal{Y}}$. As a result, $h_*\mathcal{O}_{\mathcal{X}}(-E) \cong \mathcal{O}_{\mathcal{Y}}(-\mathcal{Y}_0)$. On the other hand, we have

$$h^*\mathcal{M} = \mathcal{L} = g^*(-K_{X\times\mathbb{A}^1/\mathbb{A}^1}) - (n+1)E = -K_{\mathcal{X}/\mathbb{A}^1} - E.$$

So $-E = K_{\mathcal{X}/\mathbb{A}^1} + h^*\mathcal{M} \sim_{\mathbb{Q},h} K_{\mathcal{X}}$. Since $\mathcal{X}$ has klt singularities, we have that $R^1h_*\mathcal{O}_{\mathcal{X}}(-E) = 0$ by the generalized Kodaira vanishing theorem [Kol95, 10.19.4]. Thus the exact sequence (3.4.1) yields that $h_*\mathcal{O}_E = \mathcal{O}_{\mathcal{Y}_0}$, which implies $\mathcal{Y}_0 \cong E \cong \mathbb{P}^n$.

Since $\mathbb{P}^n$ is rigid under smooth deformation (cf. [Kol96, V.1.11.12.2]), we conclude that $\mathcal{Y}_t \cong \mathbb{P}^n$ for general $t \in \mathbb{A}^1 \setminus \{0\}$, hence $X \cong \mathcal{X}_t \cong \mathcal{Y}_t \cong \mathbb{P}^n$. □

In the following theorem, we show the equality case of Theorem 3.1.3.

Theorem 3.4.4. *Let X be a Kähler-Einstein $\mathbb{Q}$-Fano variety. Let $p \in X$ be a closed point. Suppose (X,p) is a quotient singularity with local analytic model $\mathbb{C}^n/G$, where $G \subset GL(n,\mathbb{C})$ acts freely in codimension 1. Then $(-K_X)^n = (n+1)^n/|G|$ if and only if $|G \cap \mathbb{G}_m| = 1$ and $X \cong \mathbb{P}^n/G$.*

Proof. For the "if" part, we may assume that $G \subset U(n)$. Hence G preserves the Fubini-Study metric ω_{FS} on $\mathbb{P}^n$. Since $|G \cap \mathbb{G}_m| = 1$, the G-action on $\mathbb{P}^n$ is free in codimension 1. This implies that the quotient metric of ω_{FS} on $\mathbb{P}^n/G$ is Kähler-Einstein and $((-K_{\mathbb{P}^n/G})^n) = ((-K_{\mathbb{P}^n})^n)/|G| = (n+1)^n/|G|$.

For the "only if" part, let $(Y,o) := (\mathbb{C}^n/G, 0)$ be the local analytic model of (X,p). Let $d := |G \cap \mathbb{G}_m|$. Following the proof of Theorem 3.3.7, we have a partial resolution $h : \hat{Y} \to Y$ of Y that is the quotient of the blow up $g : \widehat{\mathbb{C}^n} \to \mathbb{C}^n$. Since $\hat{Y}$ is the quotient of $\widehat{\mathbb{C}^n}$, it has klt singularities. Denote by F the exceptional divisor of h, then we have $F \cong \mathbb{P}^{n-1}/G$. Let $u_* := \mathrm{ord}_F$ be the divisorial on Y centered at o, then we have $\mathrm{gr}_{u_*}\mathcal{O}_{Y,o} \cong \mathbb{C}[x_1, \cdots, x_n]^G$ is a finitely generated $\mathbb{C}$-algebra. (Here we assume $\deg x_i = 1/d$ so that the grading is preserved under the isomorphism.) Theorem 3.3.7 also implies $\widehat{\mathrm{vol}}(u_*) = n^n/|G|$.

Since $\widehat{\mathcal{O}_{X,p}} \cong \widehat{\mathcal{O}_{Y,o}}$, the divisorial valuation u_* induces a divisorial valuation v_* on X centered at p as in the proof of Theorem 3.3.7. Thus $\widehat{\mathrm{vol}}(v_*) = \widehat{\mathrm{vol}}(u_*)$, $\mathrm{gr}_{v_*}(\mathcal{O}_{X,p}) \cong \mathrm{gr}_{u_*}(\mathcal{O}_{Y,o})$ is a finitely generated $\mathbb{C}$-algebra, $\hat{X} := \mathbf{Proj}_X \oplus_{m\geq 0} \mathfrak{a}_m(v_*)$ has klt singularities, and

$$(-K_X)^n = \frac{(n+1)^n}{|G|} = \left(1 + \frac{1}{n}\right)^n \widehat{\mathrm{vol}}(v_*).$$

By [Ber16] we have that X is K-polystable. Hence applying Lemma 3.4.1 yields

$$X \cong \mathbf{Proj}\ \mathrm{gr}_{v_*}\mathcal{O}_{X,p}[x] \cong \mathbf{Proj}\ \mathbb{C}[x_1,\cdots,x_n]^G[x],$$

where $\deg x_i = 1/d$ and $\deg x = 1$. Denote $y := x^{1/d}$, then $\mathbb{C}[x_1,\cdots,x_n]^G[x]$ is the Veronese subalgebra of $\mathbb{C}[x_1,\cdots,x_n]^G[y]$ where $\deg x_i = \deg y = 1/d$. As a result,

$$X \cong \mathbf{Proj}\ \mathbb{C}[x_1,\cdots,x_n]^G[y] = \mathbb{P}^n/G.$$

Denote by H_∞ the hyperplane at infinity in $\mathbb{P}^n$. Let D be the prime divisor in X corresponding to H_∞/G in $\mathbb{P}^n/G$. It is clear that $(X,(1-\frac{1}{d})D)$ is an orbifold as a global quotient of $\mathbb{P}^n$. Denote by $\pi:\mathbb{P}^n\to X$ the quotient map, then we have

$$\pi^*\left(K_X+\left(1-\frac{1}{d}\right)D\right) = K_{\mathbb{P}^n},\quad \pi^*D = dH_\infty.$$

Hence $\pi^*(-K_X) = -K_{\mathbb{P}^n}+(d-1)H_\infty = \mathcal{O}_{\mathbb{P}^n}(n+d)$. This implies $(-K_X)^n = (n+d)^n/|G|$, so $d=1$. □

Remark 3.4.5. (a) The K-polystable condtion in Lemma 3.4.1 and Kähler-Einstein condition in Theorem 3.4.4 cannot be dropped since any cubic surface in $\mathbb{P}^3$ with only one or two $\mathbb{A}_2$ singularities is K-semistable but not K-polystable (hence not Kähler-Einstein) according to [OSS16, Section 4.2], but all global quotients of $\mathbb{P}^2$ are Kähler-Einstein (hence K-polystable).

(b) The restriction on G in Theorem 3.4.4 can be dropped in its logarithmic version as follows: suppose (X,D) is a conical Kähler-Einstein log Fano pair, $p\notin \mathrm{Supp}D$, (X,p) is locally analytically isomorphic to $(\mathbb{C}^n/G,0)$ and $(-K_X-D)^n = (n+1)^n/|G|$, then $(X,D)\cong(\mathbb{P}^n/G,(1-\frac{1}{d})H_\infty/G)$ where $d=|G\cap\mathbb{G}_m|$.

(c) Applying Lemma 3.4.1 shows that v_* is a minimizer of $\widehat{\mathrm{vol}}$ over a quotient singularity when $d=1$. Indeed this is true for any quotient singularity (see Proposition 4.3.10).

Corollary 3.4.6 (=Corollary 3.1.5)**.** *Let X be a Kähler-Einstein log Del Pezzo surface with at most Du Val singularities.*

1. *If $(-K_X)^2=1$, then X has at most singularities of type $\mathbb{A}_1$, $\mathbb{A}_2$, $\mathbb{A}_3$, $\mathbb{A}_4$, $\mathbb{A}_5$, $\mathbb{A}_6$, $\mathbb{A}_7$ or $\mathbb{D}_4$.*
2. *If $(-K_X)^2=2$, then X has at most singularities of type $\mathbb{A}_1$, $\mathbb{A}_2$ or $\mathbb{A}_3$.*

3. *If* $(-K_X)^2 = 3$*, then* X *has at most singularities of type* $\mathbb{A}_1$ *or* $\mathbb{A}_2$*.*

4. *If* $(-K_X)^2 = 4$*, then* X *has at most singularities of type* $\mathbb{A}_1$*.*

5. *If* $(-K_X)^2 \geq 5$*, then* X *is smooth.*

Proof. Let $\mathbb{C}^2/G$ be the local analytic model of (X, p) for any closed point $p \in X$. By Corollary 3.1.4, we know that $|G| \leq 9/(-K_X)^2$.

Recall that the order of orbifold group at an $\mathbb{A}_k$ singularity is $k+1$, a $\mathbb{D}_k$ singularity is $4(k-2)$, an $\mathbb{E}_6$ singularity is 24, an $\mathbb{E}_7$ singularity is 48, and an $\mathbb{E}_8$ singularity is 120. Hence the inequality $|G| \leq 9/(-K_X)^2$ implies (2)(3)(4)(5). For part (1), the same argument yields that X has at most singularities of type $\mathbb{A}_k$ with $k \leq 8$, or of type $\mathbb{D}_4$. Hence we only need to rule out $\mathbb{A}_8$ cases.

Assume to the contrary that (X, p) is of type $\mathbb{A}_8$. Since $(-K_X)^2 = 1 = 9/|G|$, Theorem 3.4.4 implies that $X \cong \mathbb{P}^2/G$, where $G = \mathbb{Z}/9\mathbb{Z}$ acts on $\mathbb{P}^2$ as follows:

$$[x, y, z] \mapsto [\zeta^i x, \zeta^{-i} y, z] \quad \text{for } i \in \mathbb{Z}/9\mathbb{Z},$$

where $\zeta := e^{\frac{2\pi i}{9}}$ is the ninth root of unity. The point p is the quotient of $[0, 0, 1]$ under this action, which is of type $\frac{1}{9}(1, -1)$ as a cyclic quotient singularity. However, the quotient singularities at $[1, 0, 0]$ and $[0, 1, 0]$ are both of type $\frac{1}{9}(1, 2)$ which are not Du Val, contradiction! □

Chapter 4

Kähler-Einstein metrics and volume minimization

4.1 Introduction

4.1.1 Motivation and background

The study of Kähler-Einstein (KE) metrics is very active recently. In particular, the Yau-Tian-Donaldson correspondence has been established for smooth Fano manifolds (see [Tia97], [Ber16], [CDS15, Tia15]). This correspondence says that a smooth Fano manifold admits a Kähler-Einstein metric if and only if it's K-polystable. The existence of Kähler-Einstein metrics of positive curvature is a global question. However, it can be related to a local question by considering the affine cone over the underlying Fano variety with the polarization given by a positive Cartier multiple of the anti canonical divisor. The affine cone is singular except for the case that the Fano variety is $\mathbb{P}^n$ and the polarization one uses is $\mathcal{O}_{\mathbb{P}^n}(1) = \frac{1}{n+1}(-K_{\mathbb{P}^n})$. Such cone singularities are basic examples of $\mathbb{Q}$-Gorenstein klt singularities (see [Kol13]). It's natural to ask what information does the Kähler-Einstein condition provide for these cone singularities. An answer is that there are Kähler Ricci-flat cone metrics on these affine cones over KE Fano manifolds. These Ricci-flat cone metrics are rotationally symmetric and are easily obtained by solving an ODE with respect to radius functions determined by the Kähler-Einstein metrics. In this way, the study of Kähler-Einstein metrics can be put into a broader setting of Kähler Ricci-flat cone metrics. The latter has also been studied extensively recently. Given an affine cone singularity, $(\mathcal{C}, o)$ any Kähler cone metric gives rise to a

radius function r and its associated Reeb vector field $\partial_\theta := J(r\partial_r)$ where J is the complex structure on the regular part of the affine cone. If ∂_θ has closed orbits, then ∂_r and ∂_θ generates an effective $\mathbb{C}^*$-action. This is called the quasi-regular case which is relevant to our discussion in this chapter. The Ricci-flat condition is then translated to the condition that the quotient $(\mathcal{C} \setminus \{o\})/\mathbb{C}^*$ is a Kähler-Einstein Fano orbifold (V, Δ), at least when $\mathcal{C}$ has isolated singularity at o.

Whether there exists a priori such a good radius function or an associated good $\mathbb{C}^*$-action is a delicate question. In the case when there is an effective torus $T \cong (\mathbb{C}^*)^d$ action on $\mathcal{C}$, it was proved in [GMSY07, MSY08] that the $\mathbb{C}^*$-action corresponding to the quasi-regular Kähler Ricci-flat cone metric should minimize a normalized volume functional. The normalized volume functional is defined on the space of Reeb vector fields which is a conic subset of the Lie algebra of T. In [Li15b], this normalized volume functional was re-interpreted as the normalized volume of the valuations associated to the $\mathbb{C}^*$-actions. This allows us to define the normalized volume functional, denoted by $\widehat{\mathrm{vol}}(v)$, for any valuation v. More precisely for any valuation v we define $\widehat{\mathrm{vol}}(v) = A_{\mathcal{C}}(v)^n \cdot \mathrm{vol}(v)$, where $A_{\mathcal{C}}(v)$ is the log discrepancy of v and $\mathrm{vol}(v)$ is the volume of v. Notice that the latter is finite if v is centered at a closed point. Using this new point of view, one can consider the normalized volume as a function on the space of valuations which are centered at $o \in \mathcal{C}$. Moreover, we can consider any $\mathbb{Q}$-Gorenstein klt singularity, which is not necessarily an affine cone singularity nor equipped with any $\mathbb{C}^*$-action. We then ask whether there is a minimizer of this normalized volume functional, and, if there is a minimizer, what is its geometric meaning (see [Li15b]). Although these are purely algebro-geometric questions, it was speculated in [Li15b] that answering them could help to understand *metric tangent cones* of Kähler-Einstein varieties (cf. [DS15]).

As a continuation of [Li15b, Li15a], we study these two questions for affine cones over "K-semistable" $\mathbb{Q}$-Fano varieties. In this chapter we will show that for a $\mathbb{Q}$-Fano variety V, if V (specially) degenerates to a Kähler-Einstein Fano variety, then the canonical valuation ord_V is actually a global minimizer of $\widehat{\mathrm{vol}}$. As a consequence, we obtain an equivalent characterization of K-semistability for smooth Fano manifold, thus completing the confirmation of a conjecture in [Li15b]. Further consequences of our main results, in particular, their relation to the (sharp) de-Fernex-Ein-Mustaţă type inequalities for $\mathbb{Q}$-Gorenstein klt singularities, will appear in [Liu16] and [LX16].

4.1.2 Statement of main results

Let (V^{n-1}, E^{n-2}) be a log-Fano pair. By this we mean that $-(K_V+E)$ is an ample $\mathbb{Q}$-Cartier divisor and (V, E) has klt singularities with E effective. Assume $H = -r^{-1}(K_V + E)$ is an ample Cartier divisor for an $r \in \mathbb{Q}_{>0}$. We will denote the affine cone by $\mathcal{C} := C(V, H) = \mathrm{Spec} \bigoplus_{k=0}^{+\infty} H^0(V, kH)$. Notice that $\mathcal{C}$ has $\mathbb{Q}$-Gorenstein klt singularities at the vertex o (see [Kol13, Lemma 3.1]). $(\mathcal{C}, o)$ has a natural $\mathbb{C}^*$-action. Denote by v_0 the canonical $\mathbb{C}^*$-invariant divisorial valuation ord_V where V is considered as the exceptional divisor of the blow up $\mathrm{Bl}_o\mathcal{C} \to \mathcal{C}$. Denote by $\mathcal{E}$ the effective divisor on $\mathcal{C}$ corresponding to E, which is the closure of the divisor $f^{-1}E$ under the projection $f : \mathcal{C} - \{o\} \to V$. Using these notations, the following is the first main result of this chapter.

Theorem 4.1.1. *With the above notation, if (V, E) is a conical Kähler-Einstein log-Fano pair, then $\widehat{\mathrm{vol}}_{(\mathcal{C},\mathcal{E})}$ is globally minimized over $(\mathcal{C}, o)$ at ord_V.*

For the definition of conical Kähler-Einstein potentials/metrics and the normalized volume $\widehat{\mathrm{vol}}_{(\mathcal{C},\mathcal{E})}$ see Section 2.1. Letting $E = 0$, we get a non-log version:

Corollary 4.1.2. *If V is Kähler-Einstein $\mathbb{Q}$-Fano variety, then $\widehat{\mathrm{vol}}$ is globally minimized over $(\mathcal{C}, o)$ at ord_V.*

There are several steps to prove Theorem 4.1.1. We first prove that there is a conical Kähler-Einstein metric on the projective cone $(\overline{\mathcal{C}}, \mathcal{E} + (1 - \beta)V_\infty)$ with $\beta = r/n$. See section 4.2.2 for the relevant notations. (Notice that $0 < \beta \leq 1$ by Lemma 4.2.2.) This is a generalization of the construction in [Li13, Lemma 3]. Next we use Berman's result to conclude that $(\overline{\mathcal{C}}, \mathcal{E} + (1 - \beta)V_\infty)$ is log-K-polystable and hence log-K-semistable, or equivalently log-Ding-semistable. Then we apply the log version of Fujita's result to the filtration associated to any valuation. At this point, we can proceed in two ways to complete the proof. For the first (quick) proof we use similar arguments to those used in [Fuj15, Liu16] to obtain an estimate of $\widehat{\mathrm{vol}}_{(\mathcal{C},\mathcal{E})}$ which turns out be sharp. For the second proof, we interpret the expression in log-Fujita as the derivative of normalized volumes in the same spirit as in [Li15b, Li15a] and use a convexity argument to conclude the proof (see Section 4.4).

We can also deal with the semistable case. See section 2.2 for the definition of special degenerations.

Theorem 4.1.3. *Assume V is a $\mathbb{Q}$-Fano variety. Assume either of the following two conditions is satisfied:*

1. *V specially degenerates to a Kähler-Einstein $\mathbb{Q}$-Fano variety;*

2. $(\overline{\mathcal{C}}, (1-\frac{r}{n})V_\infty)$ *specially degenerates to a conical Kähler-Einstein pair.*

Then $\widehat{\mathrm{vol}}$ *is globally minimized at* ord_V *over* $(\mathcal{C}, o)$.

Remark 4.1.4. Note that the second condition does not necessarily follow from the first one (see [Kol13, *Aside* in Section 3.8]). However the first condition should imply the second one up to a branched covering of the projective cone.

When V is a smooth Fano manifold, by the deep results in [CDS15, Tia15] we know that V being K-semistable is equivalent to the condition that V degenerates to a Kähler-Einstein Fano variety. This combined with the above theorem and the results in [Li15a] allows us to complete the confirmation of a conjecture in [Li15b].

Corollary 4.1.5 (=Theorem 1.2.1)**.** *Let V be a smooth Fano manifold of dimension $n-1$. The following are equivalent:*

(i) V is K-semistable;

(ii) $\widehat{\mathrm{vol}}$ *is globally minimized at* ord_V *over* $(\mathcal{C}, o)$*;*

(iii) For any ideal sheaf $\mathfrak{a}$ on $\mathcal{C}$ cosupported at o, we have $\mathrm{lct}^n(\mathfrak{a}) \cdot \mathrm{mult}(\mathfrak{a}) \geq r(-K_V)^{n-1}$.

Remark 4.1.6. Indeed, Corollary 4.1.5 is true for any $\mathbb{Q}$-Fano variety. In [Li15a], Li proved that one direction is true for any $\mathbb{Q}$-Fano variety. Actually the following stronger criterion for K-semistability was proved there: if $\widehat{\mathrm{vol}}$ is minimized at ord_V among $\mathbb{C}^*$-equivariant divisorial valuations, then V is K-semistable. The other direction is settled very recently by Li and Xu in [LX17] (see also Section 4.5.1).

The same argument in the proof of Theorem 4.1.1 also allows us to get orbifold versions of the above results. In particular, we have the following combined version. See Section 4.2.3 for the notations used here.

Theorem 4.1.7. *Let $(V, \Delta = \sum_i (1-\frac{1}{m_i})D_i)$ be a smooth orbifold and $f : Y^n \to (V^{n-1}, \Delta)$ be a Seifert $\mathbb{C}^*$-bundle. Assume that the orbifold canonical class $K_{\mathrm{orb}} := K_V + \Delta$ is anti ample and $c_1(Y/V) = -r^{-1}(K_V + \Delta)$ with $0 < r \leq n$. Denote by $\overline{\mathcal{C}}_{\mathrm{orb}}$ the associated orbifold projective cone and by V_∞ the compactifying divisor at infinity. Then the following holds:*

1. *If (V, Δ) admits an orbifold Kähler-Einstein metric of positive Ricci curvature, then there is a conical Kähler-Einstein metric on $(\overline{\mathcal{C}}_{\mathrm{orb}}, (1-\frac{r}{n})V_\infty)$.*

2. *If* $(\overline{\mathcal{C}}_{\mathrm{orb}}, (1-\frac{r}{n})V_\infty)$ *specially degenerates to a conical Kähler-Einstein pair, then over the orbifold cone* $(\mathcal{C}_{\mathrm{orb}}, o)$, $\widehat{\mathrm{vol}}$ *is globally minimized at* ord_V. *In particular, in the situation of item 1, we can choose the special degeneration as the trivial one and hence the same conclusion holds.*

Remark 4.1.8. 1. Theorem 4.1.7 can be seen as a strengthening of the volume minimization result in [GMSY07, MSY08] in the quasi-regular case.

2. With our assumptions, the Seifert bundle Y in general has quotient singularities. By allowing (V, Δ) to be a general algebraic stack, we could weaken the condition that Y has quotient singularities and get more general version of the above results. However since we don't use them in this chapter, we will not discuss it here.

We emphasize here that the main common feature of proofs of the above results is that we need to work in the conical/logarithmic setting. On the one hand we work on the projective cone over the original $\mathbb{Q}$-Fano variety and construct a conical Kähler-Einstein metric using a (conical) Kähler-Einstein metric on the original $\mathbb{Q}$-Fano variety. On the other hand we need to apply Berman and Fujita's results about Ding-semistability in their log versions. Although this seems innocuous at first sight, it turns out to be crucial for us to get sharp estimates.

4.2 Log-K-polystability of projective cones

4.2.1 Berman's result on log-K-polystability

In this section, we state and sketch a proof of Berman result on log-Ding-polystability which will be shown to imply a log version of Fujita's result in Section 4.3.1.

Theorem 4.2.1 ([Ber16])**.** *Let* (X, D) *be a log Fano pair. Let* $L \sim_{\mathbb{Q}} -\delta^{-1}(K_X + D)$ *be an ample line bundle. If* $(X, D; L)$ *has a conical Kähler-Einstein potential, then* (X, D) *is log-K-polystable. In particular, it is log-K-semistable, and equivalently log-Ding-semistable.*

This was proved by Berman (see [Ber16, Section 4.3]). For the reader's convenience, we provide a sketch of the proof.

Proof. Define the space of locally bounded positively curved Hermitian metrics on L:

$$\mathcal{H}_\infty(X, L) := \{e^{-\varphi} \mid \varphi = \{\varphi_j\} \in L^\infty_{\mathrm{loc}}(X) \text{ is l.s.c. and } \sqrt{-1}\partial\bar{\partial}\varphi \geq 0\},$$

where $\omega_\varphi := \sqrt{-1}\partial\bar{\partial}\varphi \geq 0$ is in the sense of currents. We fix a locally bounded reference Hermitian metric $e^{-\phi_0}$ on L (see section 2.1). For any $\varphi \in \mathcal{H}_\infty(X,h)$ define the log-Ding-energy (see [Din88]) as:

$$F(\varphi) = F^0_{\phi_0}(\varphi) - \log\left(\frac{1}{L^n}\int_X e^{-\delta\varphi}\right). \tag{4.2.1}$$

Here we used the notations in Section 2.1 and the Monge-Ampère energy $F^0_{\phi_0}(\varphi)$ is defined as:

$$F^0_{\phi_0}(\varphi) = -\frac{\delta}{(n+1)}\sum_{k=0}^{n}\frac{1}{L^n}\int_X(\varphi-\phi_0)\,(\sqrt{-1}\partial\bar{\partial}\varphi)^k \wedge (\sqrt{-1}\partial\bar{\partial}\phi_0)^{n-k}.$$

The reason for considering the log-Ding-energy is that its critical point is exactly the conical Kähler-Einstein potential, i.e. the solution (up to a constant) to the complex Monge-Ampère equation:

$$(\sqrt{-1}\partial\bar{\partial}\varphi)^n = \mathrm{m}_{\delta\varphi}. \tag{4.2.2}$$

Moreover if there is a conical Kähler-Einstein potential $e^{-\phi_{\mathrm{cKE}}}$ on the pair (X,D), then $F(\varphi)$ is bounded from below on $\mathcal{H}_\infty(X,L)$ by $F(\phi_{\mathrm{cKE}}) < +\infty$ (see [Ber15, BBE$^+$11]). We will later apply this fact to a family of Hermtian metrics coming from any test configuration.

Now suppose $((\mathcal{X},\mathcal{D});p\mathcal{L})$ is a normal test configuration of $((X,D);pL)$. Let $e^{-\Phi}$ be any locally bounded Hermitian metric on $p\mathcal{L}$ which is positively curved, i.e. satisfying $\sqrt{-1}\partial\bar{\partial}\Phi \geq 0$ in the sense of current. Following the approach in [Ber16], we consider the divisor $\Delta = (K_{\mathcal{X}/\mathbb{C}^1}+\mathcal{D}) + \delta\mathcal{L}$ with its defining section $\widetilde{S}$. Then we get a relative measure on $\mathcal{X}$:

$$v_\Phi := |\widetilde{S}|^2\mathrm{m}_\Phi.$$

Here as before, m_Φ is a globally defined measure on $\mathcal{X}$, which on the regular part of $\mathcal{X}$ is equal to $\frac{\exp\left(-\frac{\delta}{p}\Phi\right)}{|\mathcal{S}_\mathcal{D}|^2}$ where $\mathcal{S}_\mathcal{D}$ is the defining section of $\mathcal{D}$ on the regular part of $\mathcal{X}$. Notice that it has a positive curvature current:

$$\sqrt{-1}\partial\bar{\partial}\left(\frac{\delta}{p}\Phi + \log|S_\mathcal{D}|^2\right) = \left(\frac{\delta}{p}\sqrt{-1}\partial\bar{\partial}\Phi\right) + \{S_\mathcal{D}=0\}$$

where $\{S_\mathcal{D}=0\}$ is the current of integration along the divisor $\mathcal{D}$. By Berndtsson's subharmonicity theorem (see [Ber15, BP08]), we know that the function on the base:

$$G(t) := -\log\left(\int_{\mathcal{X}/\mathbb{C}} v_\Phi\right)$$

is subharmonic. Notice that if $e^{-\Phi}$ is $\mathbb{C}^*$-invariant which is the case for the metrics we will choose, the subharmonicity implies $G(t)$ is convex in $-\log|t|$.

We can argue as Berman in [Ber16, Section 3.3] to show that the Lelong number of $G(t)$ is equal to:

$$l_0 := 1 - \mathrm{lct}((\mathcal{X}, \mathcal{D} - \Delta); \mathcal{X}_0). \tag{4.2.3}$$

Recall that the Lelong number of $G(t)$ can be defined as:

$$l_0 = \liminf_{t\to 0} \frac{G(t)}{\log|t|}. \tag{4.2.4}$$

On the other hand, by the equivariant isomorphism:

$$\mu : (\mathcal{X}|_{\mathbb{C}^*}, p\mathcal{L}) \cong (X, pL) \times \mathbb{C}^*$$

we get a family of Hermitian metrics $\varphi_t \in \mathcal{H}_\infty(X, L)$ such that

$$e^{-\varphi_t} = (\mu^{-1})^* e^{-\frac{1}{p}\Phi}\Big|_{X\times\{t\}}.$$

From the construction, we can see that:

$$F(\varphi_t) = F^0_{\phi_0}(\varphi_t) - G(t). \tag{4.2.5}$$

Now there are two natural choices of the locally bounded Hermitian metric on $p\mathcal{L}$ for which we can carry out the above constructions to get two families of Hermtian metrics on $L \to X$. For the first one we use the well known fact that there is an $\mathbb{C}^*$-equivariant embedding:

$$\alpha_1 : \mathcal{X} \to \mathbb{P}^N \times \mathbb{C}^1 \text{ with } p\mathcal{L} = \alpha_1^* \mathcal{O}_{\mathbb{P}^N}(1)$$

where $\mathbb{C}^*$ acts on $\mathbb{C}^1$ by multiplication and acts on $\mathbb{P}^N$ through a one-parameter subgroup of $\mathrm{PGL}(N+1, \mathbb{C})$. Pulling back the Fubini-Study metric on $\mathcal{O}_{\mathbb{P}^N}(1)$, we get a smooth Hermitian metric, denoted by $e^{-\Phi^{(1)}}$, on $p\mathcal{L}$. The family of Hermitian metrics $\{\varphi_t^{(1)}\}$ associated to $e^{-\Phi^{(1)}}$ as above is usually called Bergman metrics on $L \to X$. We can prove the log-Ding-semistability by using family. Indeed, it's well known that $F^0_{\phi_0}(\varphi_t^{(1)})$ is convex in $-\log|t|$, and its slope at infinity is

equal to:

$$\lim_{t\to 0}\frac{F^0_{\phi_0}(\varphi_t^{(1)})}{-\log|t|}=-\frac{\delta\bar{\mathcal{L}}^{n+1}}{L^n}=-\frac{\delta^{n+1}\bar{\mathcal{L}}^{n+1}}{(-K_X-D)^n}. \tag{4.2.6}$$

So combining this formula (4.2.3)-(4.2.5), we get:

$$\lim_{t\to 0}\frac{F(\varphi_t^{(1)})}{-\log|t|}=-\frac{\delta^{n+1}\bar{\mathcal{L}}^{n+1}}{(-K_X-D)^n}-(1-\mathrm{lct}((\mathcal{X},\mathcal{D}-\Delta);\mathcal{X}_0). \tag{4.2.7}$$

As mentioned above, if (X,D) is conical Kähler-Einstein, then $F(\varphi_t^{(1)})$ is bounded from below, so the right-hand-side of (4.2.7) is non-negative.

To prove the stronger polystability, we need to consider another locally bounded metric, which is obtained by solving a homogeneous complex Monge-Ampère equation on $\mathcal{X}|_{\{|t|\leq 1\}}$

$$\begin{cases}(\sqrt{-1}\partial\bar{\partial}\Phi^{(1)}+\sqrt{-1}\partial\bar{\partial}\Psi)^{n+1}=0;\\ \Psi|_{X\times S^1}=0.\end{cases}$$

It's now well known that there exists a bounded solution Ψ. Denoting $e^{-\Phi^{(2)}}=e^{-(\Phi^{(1)}+\Psi)}$, then the family of Hermitian metrics $\{\varphi_t^{(2)}\}$ associated to the locally bounded Hermitian metric $e^{-\Phi^{(2)}}$ is a so-called weak geodesic ray in $\mathcal{H}_\infty(X,L)$ emanating from $e^{-\varphi_1}$. If we choose ϕ_0 to be the conical Kähler-Einstein potential, then we have

$$\frac{d}{d(-\log|t|)}\bigg|_{|t|=1^-}F(\varphi_t^{(2)}))\geq 0,\ \text{and}\ \lim_{t\to 0}\frac{d}{d(-\log|t|)}F(\varphi_t^{(2)})=0.$$

By the convexity of $F(\varphi_t^{(2)})$ with respect to $-\log|t|$, this implies that $F(\varphi_t^{(2)})$ is affine with respect to $-\log|t|$. It's well known that for the geodesic ray $\{\varphi_t^{(2)}\}$, $F^0_{\phi_0}(\varphi_t^{(2)})$ is affine in the variable $-\log|t|$, so $G(t)$ must be affine with respect to $-\log|t|$. By arguing as in [Ber16] (see also [Ber15]), we can conclude that $(\mathcal{X},\mathcal{D};\mathcal{L})$ is indeed a product test configuration.

□

4.2.2 cKE metric on the projective cone over a cKE variety

Let (V,E) be a log Fano pair. Assume $H=-r^{-1}(K_V+E)$ is an ample Cartier divisor. Consider the projective cone over V with polarization H:

$$\overline{\mathcal{C}}=\mathrm{Proj}\left(\bigoplus_{m=0}^{+\infty}S_m\right)$$

where

$$S_m = \bigoplus_{k=0}^{m} \left(H^0(V, kH) \cdot u^{m-k} \right).$$

Geometrically, $\overline{\mathcal{C}} = \mathcal{C} \cup V_\infty$ where $\mathcal{C}$ is the affine cone over V defined as

$$\mathcal{C} = \mathrm{Spec} \left(\bigoplus_{m=0}^{+\infty} H^0(V, mH) \right);$$

V_∞ is the divisor at infinity, which is clearly isomorphic to V. Let $\mathcal{E}$ be the corresponding divisor on $\overline{\mathcal{C}}$. By [Kol13, Section 3.1], $K_{\overline{\mathcal{C}}} + \mathcal{E}$ is $\mathbb{Q}$-Cartier, $(\overline{\mathcal{C}}, \mathcal{E})$ has klt singularities and moreover:

$$-(K_{\overline{\mathcal{C}}} + \mathcal{E}) \sim_{\mathbb{Q}} (1 + r)V_\infty. \tag{4.2.8}$$

Define the cone angle parameter:

$$\beta = \frac{r}{n}. \tag{4.2.9}$$

The following lemma is well known. For the absolute case, see [IP99, Corollary 2.1.13].

Lemma 4.2.2. *Assume that (V^{n-1}, E) is a log Fano pair. If $H = -r^{-1}(K_V + E)$ is Cartier for $r \in \mathbb{Q}_{>0}$, then $0 < r \leq n$.*

Proof. The proof is the same as in [IP99, Corollary 2.1.13]. By the general Kawamata-Viehweg vanishing (see [Kol97, 2.16]), $H^i(V, \mathcal{O}_V(mH)) = 0$ for any $i > 0$ and $m > -r$. By Riemann-Roch, $\chi(V, mH)$ is a polynomial in m of degree $n-1$. By the above vanishing, we know that this polynomial has roots $m = -1, -2, \cdots, 1 - \lceil r \rceil$. So we get $r \leq n$. □

The above lemma yields $0 < r \leq n$, so $\beta \in (0, 1]$. Then it's easy to verify that the pair $(\overline{\mathcal{C}}, \mathcal{E} + (1 - \beta)V_\infty)$ is klt, whose anti-log-canonical divisor is given as:

$$-(K_{\overline{\mathcal{C}}} + \mathcal{E} + (1 - \beta)V_\infty) \sim_{\mathbb{Q}} \left((1 + r) - (1 - \frac{r}{n}) \right) V_\infty = r\frac{n+1}{n} V_\infty. \tag{4.2.10}$$

We will denote by L the line bundle associated to the Cartier divisor V_∞. Notice that $L|_{V_\infty} \cong H$ so that $(L^n) = (H^{n-1})$.

Proposition 4.2.3. *If (V, E) has a conical Kähler-Einstein Hermitian metric on $(V, E; H)$, then there exists a conical KE potential on $(\overline{\mathcal{C}}, \mathcal{E} + (1 - \beta)V_\infty; L)$.*

Proof. Let $h := e^{-\phi}$ be the cKE potential on $(V, E; H)$. By the calculation in [Li13], the cKE

potential $e^{-\varphi_L}$ on $(\overline{\mathcal{C}}, \mathcal{E} + (1-\beta)V_\infty; L)$ should satisfy the equality:

$$\|s_{V_\infty}\|^2_{h_L} := |s_{V_\infty}|^2 e^{-\varphi_L} = \frac{h}{\left(1 + h^{r/n}\right)^{n/r}}, \tag{4.2.11}$$

where s_{V_∞} is the defining section of $\mathcal{O}_{\overline{\mathcal{C}}}(V_\infty)$ and h is viewed as a non-negative function on $\overline{\mathcal{C}} \setminus \{o\}$.

Remark 4.2.4. If $E = \emptyset$ this is just an appropriately normalized form of the result in [Li13, Lemma 3]. The expression in (4.2.11) is clearly a generalization of the classical Fubini-Study metric on $\mathcal{O}_{\mathbb{P}^n}(1)$.

It's easy to see that the formula (4.2.11) defines a locally bounded Hermitian metric $e^{-\varphi_L}$ on L. We will directly verify that this is indeed a cKE potential. We will denote by $\bar{f} : \overline{\mathcal{C}} \setminus \{o\} \cong H \to V_\infty = V$ the natural projection. It's also convenient to denote $\rho = \log h$ such that

- $h = e^\rho$ and $\sqrt{-1}\partial\bar{\partial}\rho = -\bar{f}^*\omega + 2\pi\{V_\infty\}$, where ω is the curvature current of the cKE potential h over $(V, E; H)$ and $\{V_\infty\}$ denotes the current of integration along V_∞;
- $\partial\rho = \partial \log a + \frac{d\xi}{\xi}$. As a consequence we have:

$$\bar{f}^*\omega^{n-1} \wedge \partial\rho \wedge \bar{\partial}\rho = \bar{f}^*\omega^{n-1} \wedge \frac{d\xi \wedge d\bar{\xi}}{|\xi|^2}$$

Using the above notation, we can calculate:

$$\begin{aligned} -\sqrt{-1}\partial\bar{\partial}\log\|s\|^2_{h_L} &= (-\sqrt{-1}\partial\bar{\partial}\log\rho) + \frac{n}{r}\sqrt{-1}\partial\bar{\partial}\log(1+e^{\beta\rho}) \\ &= \omega - 2\pi\{V_\infty\} + \frac{n}{r}\left(-\frac{\beta e^{\beta\rho}}{1+e^{\beta\rho}}\omega + \frac{\beta^2 e^{\beta\rho}}{(1+e^{\beta\rho})^2}\partial\rho\wedge\bar{\partial}\rho\right) \\ &= \frac{\omega}{1+e^{\beta\rho}} + \frac{\beta e^{\beta\rho}}{(1+e^{\beta\rho})^2}\partial\rho\wedge\bar{\partial}\rho - 2\pi\{V_\infty\}. \end{aligned}$$

Here we identify ω on V with its pull back $\bar{f}^*\omega$ to $\overline{\mathcal{C}} \setminus \{o\}$. So we get the curvature current of $e^{-\varphi_L}$:

$$\sqrt{-1}\partial\bar{\partial}\varphi_L = \frac{\omega}{1+e^{\beta\rho}} + \frac{\beta e^{\beta\rho}}{(1+e^{\beta\rho})^2}\partial\rho\wedge\bar{\partial}\rho. \tag{4.2.12}$$

This is clearly a positive current over $\overline{\mathcal{C}} \setminus \{o\}$. The volume form (or the Monge-Ampère measure) of

$\sqrt{-1}\partial\bar{\partial}\varphi_L$ is equal to

$$\begin{aligned}(\sqrt{-1}\partial\bar{\partial}\varphi_L)^n &= n\beta\frac{e^{\beta\rho}}{(1+e^{\beta\rho})^{n+1}}\bar{f}^*\omega^{n-1}\wedge\partial\rho\wedge\bar{\partial}\rho \\ &= n\beta\frac{e^{\beta\rho}}{(1+e^{\beta\rho})^{n+1}}\bar{f}^*\omega^{n-1}\wedge\frac{d\xi\wedge d\bar{\xi}}{|\xi|^2}.\end{aligned}$$

Because $e^{-\phi}$ is conical Kähler-Einstein on $(V, E; H = -r^{-1}(K_V + E))$, it satisfies the following equation on the regular part of V:

$$\omega^{n-1} = \frac{e^{-r\phi}}{|s_E|^2}, \tag{4.2.13}$$

where s_E is the defining section of $\mathcal{O}_V(E|_{V_{\mathrm{reg}}})$. Taking the curvature on both sides, we get:

$$Ric(\omega^{n-1}) = \sqrt{-1}\partial\bar{\partial}(r\phi + \log|s_E|^2) = r\omega + 2\pi\{E\}. \tag{4.2.14}$$

We define a measure on $\overline{\mathcal{C}}_{\mathrm{reg}}$:

$$\begin{aligned}\mathbf{M} &:= (\sqrt{-1}\partial\bar{\partial}\varphi_L)^n\cdot|s_{V_\infty}|^{2(1-\beta)}|s_{\mathcal{E}}|^2 \\ &= n\beta\frac{e^{\beta\rho}}{(1+e^{\beta\rho})^{n+1}}\bar{f}^*\omega^{n-1}\wedge\partial\rho\wedge\bar{\partial}\rho\cdot|s_{V_\infty}|^{2(1-\beta)}|s_{\mathcal{E}}|^2 \\ &= n\beta\frac{e^{\beta\rho}|s_{V_\infty}|^{2(1-\beta)}}{(1+e^{\beta\rho})^{n+1}}\frac{d\xi\wedge d\bar{\xi}}{|\xi|^2}\wedge\bar{f}^*\left(\omega^{n-1}\cdot|s_E|^2\right)\end{aligned}$$

Using the cKE equation (4.2.13), we know that $\mathbf{M}$ is induced given by a bounded (indeed continuous) volume form on the regular part of $\overline{\mathcal{C}}\setminus\{o\}$, which induces a locally bounded Hermitian metric on $-(K_{\overline{\mathcal{C}}} + (1-\beta)V_\infty + \mathcal{E})$:

$$\begin{aligned}e^{-\psi} &:= n\beta\frac{e^{\beta\rho}|s_{V_\infty}|^{2(1-\beta)}}{(1+e^{\beta\rho})^{n+1}}\frac{d\xi\wedge d\bar{\xi}}{|\xi|^2}\wedge\bar{f}^*\left(\omega^{n-1}\cdot|s_E|^2\right) \\ &= n\beta\frac{e^{\beta\rho}|s_{V_\infty}|^{2(1-\beta)}}{(1+e^{\beta\rho})^{n+1}}\frac{d\xi\wedge d\bar{\xi}}{|\xi|^2}\wedge\bar{f}^*\left(e^{-r\phi}\right).\end{aligned}$$

We can easily calculate the curvature current of $e^{-\psi}$:

$$\begin{aligned}\sqrt{-1}\partial\bar{\partial}\psi &= (-\beta\sqrt{-1}\partial\bar{\partial}\rho + 2\pi\beta\{V_\infty\}) + (n+1)\sqrt{-1}\partial\bar{\partial}\log(1+e^{\beta\rho}) + \bar{f}^*(r\omega) \\ &= \beta\omega + r\omega + (n+1)\left(-\frac{\beta e^{\beta\rho}}{1+e^{\beta\rho}}\omega + \frac{\beta^2 e^{\beta\rho}}{(1+e^{\beta\rho})^2}\partial\rho\wedge\bar{\partial}\rho\right) \\ &= (n+1)\beta\left(\frac{\omega}{1+e^{\beta\rho}} + \frac{\beta e^{\beta\rho}}{(1+e^{\beta\rho})^2}\partial\rho\wedge\bar{\partial}\rho\right) = \frac{(n+1)r}{n}\sqrt{-1}\partial\bar{\partial}\varphi_L.\end{aligned}$$

Compared to the identity (4.2.10), the above equality means that the Hermitian metric $e^{-\psi}$ on $-(K_{\overline{\mathcal{C}}}+\mathcal{E}+(1-\beta)V_\infty)$ is equal to $e^{-r\frac{n+1}{n}\varphi_L}$ up to positive rescaling. Moreover, combining the above equalities, we see that $e^{-\varphi_L}$ satisfies the following equation on the regular part of $\overline{\mathcal{C}}$:

$$(\sqrt{-1}\partial\bar{\partial}\varphi_L)^n = \frac{e^{-r\frac{n+1}{n}\varphi_L}}{|s_{V_\infty}|^{2(1-\beta)}|s_{\mathcal{E}}|^2}.$$

This identity holds a priori on $\overline{\mathcal{C}}_{\mathrm{reg}}$. However, it extends to hold on the whole $\overline{\mathcal{C}}$ in the pluripotential sense because both sides does not charge the singular set $\overline{\mathcal{C}}_{\mathrm{sing}}$. Indeed, the latter follows from the fact φ_L is locally bounded and $(\overline{\mathcal{C}},(1-\beta)V_\infty+\mathcal{E})$ is klt.

So by the definition in Section 2.1, we see that $e^{-\varphi_L}$ is indeed a conical Kähler-Einstein potential on $(\overline{\mathcal{C}},\mathcal{E}+(1-\beta)V_\infty;L)$.

□

4.2.3 An orbifold version

For the notions used in the following proposition, see Section 4.2.3.

Proposition 4.2.5. *Let $f: Y^n \to (V^{n-1},\Delta=\sum_i(1-\frac{1}{m_i})D_i)$ be a Seifert $\mathbb{C}^*$-bundle. Assume $c_1(Y/V)$ is positive and let $\overline{\mathcal{C}}_{\mathrm{orb}}$ denote the associated orbifold projective cone. Assume that (V,Δ) is orbifold smooth and the following two conditions are satisfied:*

1. *The orbifold canonical class $K_{\mathrm{orb}} := K_V+\Delta$ is anti ample and there is a smooth orbifold Kähler-Einstein metric on $(V,\Delta;-K_{\mathrm{orb}})$*
2. *The Chern class $c_1(Y/V)=-r^{-1}(K_V+\Delta)$ with $0<r\leq n$.*

Then there exists a conical Kähler-Einstein potential on $(\overline{\mathcal{C}}_{\mathrm{orb}},(1-\beta)V_\infty;\mathcal{O}_{\overline{\mathcal{C}}_{\mathrm{orb}}}(V_\infty))$ with $\beta=\frac{r}{n}$.

Proof. For simplicity, we will just denote $\overline{\mathcal{C}}_{\mathrm{orb}}$ by $\overline{\mathcal{C}}$ and by $\overline{\mathcal{C}}^\circ$ the punctured projective cone $\overline{\mathcal{C}}\setminus\{o\}$. Assume ω is a smooth orbifold Kähler metric on (V,Δ) in the orbifold first Chern class $c_1(H_{\mathrm{orb}})=-r^{-1}c_1(K_{\mathrm{orb}})$. Then there exists a smooth orbifold Hermitian metric $e^{-\phi}=\{e^{-\phi_i}\}$ on H_{orb} whose orbifold Chern curvature is ω. In other words, for any local orbifold chart $\pi_i: W_i\to W_i/\mu_m\cong U_i$, $e^{-\widetilde{\phi}}:=\pi_i^*e^{-\phi_i}$ is a smooth Hermitian metric on the line bundle $\pi_i^*H_{\mathrm{orb}}$, whose Chern curvature $\widetilde{\omega}:=\sqrt{-1}\partial\bar{\partial}\pi_i^*\phi_i=\pi_i^*\omega$ is a smooth Kähler metric on W_i.

Following (4.2.11), we define a locally bounded Hermitian metric on the $\mathbb{Q}$-line bundle $\bar{f}^*H_{\mathrm{orb}}=$

V_∞ by first defining μ_m-invariant functions over $W_i \times \mathbb{C}$:

$$e^{-\widetilde{\Phi}_i} = \frac{e^{-\widetilde{\phi}_i}}{\left(1 + e^{-\beta\widetilde{\phi}_i}|\xi|^{2\beta}\right)^{\beta^{-1}}}.$$

where $\xi = \xi_i$ is the coordinate of the second factor $\mathbb{C}$. Then its curvature current $\widetilde{\Omega}_i = \sqrt{-1}\partial\bar{\partial}\widetilde{\Phi}_i$ is a (1,1)-current on $W_i \times \mathbb{C}$ and can be calculated as in (4.2.12):

$$\widetilde{\Omega}_i = \frac{\widetilde{\omega}}{1 + e^{-\beta\widetilde{\phi}_i}|\xi|^{2\beta}} + \frac{\beta e^{-\beta\widetilde{\phi}_i}}{(1 + e^{-\beta\widetilde{\phi}_i}|\xi|^{2\beta})^2}\frac{(d\xi - \xi\partial\widetilde{\phi}_i) \wedge \overline{(d\xi - \xi\partial\widetilde{\phi}_i)}}{|\xi|^{2(1-\beta)}}$$

$\widetilde{\Omega}_i$ is smooth away from $W_i \times \{0\}$ and has conical singularities of angle $2\pi\beta$ along $W_i \times \{0\}$. $e^{-\widetilde{\Phi}_i}$ descends to become a Hermitian metric on the $\mathbb{Q}$-line bundle $\bar{f}^* H_{\mathrm{orb}}|_{\bar{f}^{-1}U_i}$ whose Chern curvature Ω_i is induced from $\widetilde{\Omega}_i$ under the quotient $W_i \times \mathbb{C} \to W_i \times \mathbb{C}/\mu_m = \bar{f}^{-1}(U_i)$. Using the transition functions of the $\mathbb{Q}$-line bundle $\bar{f}^* H_{\mathrm{orb}}$, it's easy to verify that $\Omega_i = \Omega_j$ on any intersection $U_i \cap U_j$ so that there is a globally defined (1,1)-current Ω on $\overline{\mathcal{C}}^\circ$. The collection $e^{-\Phi} := \{e^{-\Phi_i}\}$ defines a Hermitian metric on $\bar{f}^* H_{\mathrm{orb}}$ over $\overline{\mathcal{C}}^\circ$ whose curvature is the globally defined (1,1)-current Ω.

Now by taking into account of the formula (4.2.16), we can carry out the same calculation in the proof of Proposition 4.2.3 for the case $\Delta = 0$ using the above equivariant charts. So we see that $e^{-\Phi}$ satisfies the conical Kähler-Einstein equation on $\overline{\mathcal{C}}^\circ$:

$$(\sqrt{-1}\partial\bar{\partial}\Phi)^n = \frac{e^{-\frac{r(n+1)}{n}\Phi}}{|s_{V_\infty}|^{2(1-\beta)}}, \tag{4.2.15}$$

where s_{V_∞} is $\mathbb{Q}$-section of the $\mathbb{Q}$-line bundle $\mathcal{O}_{\overline{\mathcal{C}}}(V_\infty)$. Because $\beta = r/n$ and, by Lemma 4.2.11, $-K_{\overline{\mathcal{C}}} = (r+1)V_\infty = \frac{r(n+1)}{n}V_\infty + (1 - \frac{r}{n})V_\infty$, both sides of (4.2.15) can be considered as singular Hermitian metrics on $-K_{\overline{\mathcal{C}}}$. Arguing as before, (4.2.15) actually holds on the whole $\overline{\mathcal{C}}$ because both sides do not charge the vertex o. So we see that $e^{-\Phi}$ is indeed a cKE potential on $(\overline{\mathcal{C}}_{\mathrm{orb}}, (1 - \beta)V_\infty; \mathcal{O}_{\overline{\mathcal{C}}}(V_\infty))$.

□

In the above theorem, we assumed that (V, Δ) is a smooth orbifold and the metrics under consideration are all orbifold smooth. However one can show that the smoothness assumptions are not necessary and the conclusion still holds for locally bounded orbifold metrics in the sense of pluripotential theory. So we can get a more general result as follows. Since we don't need this more general version in this chapter, we leave the proof to the reader.

Proposition 4.2.6. *Let $f : Y^n \to (V^{n-1}, \Delta = \sum_i (1 - \frac{1}{m_i})D_i)$ be a Seifert $\mathbb{C}^*$-bundle. Assume $c_1(Y/V)$ is positive and let $\overline{\mathcal{C}}_{\mathrm{orb}}$ denote the associated orbifold projective cone. Assume that E is an orbifold $\mathbb{Q}$-Cartier divisor such that the following two conditions are satisfied:*

1. *The orbifold log canonical class $K_V + \Delta + E$ is anti ample and there is an orbifold conical Kähler-Einstein metric on $(V, \Delta; E)$;*
2. *The Chern class $c_1(Y/V) = -r^{-1}(K_V + \Delta + E)$ for $0 < r \leq n$;*
3. *The following equality holds as an identity of effective $\mathbb{Q}$-divisors:*

$$K_{\overline{\mathcal{C}}^\circ} + V_\infty + \mathcal{E} = \bar{f}^*(K_V + \Delta + E).$$

Then there exists a conical Kähler-Einstein metric on $(\overline{\mathcal{C}}_{\mathrm{orb}}, (1-\beta)V_\infty + \mathcal{E}; \mathcal{O}_{\overline{\mathcal{C}}_{\mathrm{orb}}}(V_\infty))$ with $\beta = \frac{r}{n}$.

Appendix: Seifert $\mathbb{C}^*$-bundles and orbifolds

For the reader's convenience, we recall the definition of Seifert $\mathbb{C}^*$-bundle and their associated orbifolds after Kollár. The following materials can be found in [Kol04, Kol05]).

Definition 4.2.7. Let V be a normal complex space. A Seifert $\mathbb{C}^*$-bundle over V is a normal complex space Y together with a morphism $f : Y \to V$ and a $\mathbb{C}^*$-action on Y such that the following two conditions are satisfied:

1. f is Stein and $\mathbb{C}^*$-equivariant (with respect to the trivial action on V);
2. For every $p \in V$, the $\mathbb{C}^*$-action on the reduced fiber $Y_p := \mathrm{red} f^{-1}(p)$, $\mathbb{C}^* \times Y_p \to Y_p$ is $\mathbb{C}^*$-equivariantly biholomorphic to the natural $\mathbb{C}^*$-action on $\mathbb{C}^*/\mu_m$ for some $m = m(p, Y/V)$, where $\mu_m \subset \mathbb{C}^*$ denotes the group of mth roots of unity.

The number $m(p, Y/V)$ is called the multiplicity of the Seifert fiber over p.

One can always assume that the $\mathbb{C}^*$-action is effective, that is, $m(p, Y/V) = 1$ for general $p \in V$. Let $f : Y \to V$ be a Seifert $\mathbb{C}^*$-bundle. The set of points $\{p \in V; m(p, Y/V) > 1\}$ is a closed analytic subset of V. It can be decomposed as the union of Weil divisors $\cup D_i$ and of a subset of codimension at least 2 contained in $\mathrm{Sing}(V)$. The multiplicity $m(p, Y/V)$ is constant on a dense open subset of each D_i. This common value is called the multiplicity of the Seifert $\mathbb{C}^*$-bundle over

D_i, and is denoted by $m_i = m(D_i)$. The $\mathbb{Q}$-divisor $\Delta := \sum_i (1 - \frac{1}{m_i})D_i$ is called the *branch divisor* of $f : Y \to V$. We will often write $f : Y \to (V, \Delta)$ to indicate the branch divisor.

Kollár [Kol04] classified general Seifert $\mathbb{C}^*$-bundles over V, generalizing the previous special cases of Dolgachev-Pinkham-Demazure.

Definition 4.2.8 (see [Kol04])**.** Let V be a normal variety, L a rank 1 reflexive sheaf on V, D_i distinct irreducible divisors and $0 < s_i < 1$ rational numbers. Define:

$$\begin{aligned} S(L, \sum_i s_i D_i) &:= \sum_{j \in \mathbb{Z}} L^{[j]} \left(\sum_i \lfloor j s_i \rfloor D_i \right) \\ Y(L, \sum_i s_i D_i) &:= \operatorname{Spec}_V S(L, \sum_i s_i D_i). \end{aligned}$$

where $L^{[k]} = (L^{\otimes k})^{**}$ for the rank 1 reflexive sheaf L (see [Kol04]).

There is a natural $\mathbb{C}^*$-action on $S(L, \sum_i s_i D_i)$ where $L^{[j]}$ $(\sum_i s_i D_i)$ is the λ^j eigensubsheaf. This induces a $\mathbb{C}^*$-action on $Y(L, \sum_i s_i D_i)$.

Theorem 4.2.9 (Kollár)**.** *Let V be a normal variety.*

1. *Every Seifert $\mathbb{C}^*$-bundle $f : Y \to V$ can be uniquely written as $Y \cong Y(L, \sum_i s_i D_i)$ for some L and $\sum_i s_i D_i$.*
2. *$f : Y(L, \sum_i s_i D_i) \to V$ is a Seifert $\mathbb{C}^*$-bundle if and only if $L^{[M]}(\sum_i (M s_i) D_i)$ is locally free for some $M > 0$ and $M s_i$ is an integer for every i.*

Definition 4.2.10. The *Chern class* of $f : Y \to (V, \Delta)$ is defined as

$$c_1(Y/V) := L + \sum_i s_i [D_i] \in H^2(V, \mathbb{Q}).$$

Seifert $\mathbb{C}^*$-bundles can be compactified by adding the zero and infinite sections. Their union gives a proper morphism $\bar{f} : \bar{Y} \to V$ which is a $\mathbb{P}^1$-bundle over the set where $m(p) = 1$. We have the zero section $V_0 \subset \bar{Y}$ and the infinity section $V_\infty \subset \bar{Y}$. The zero section $V_0 \subset \bar{Y}$ can be contracted to a point $\{o\}$ if and only if $c_1(Y/V)$ is positive. In this case, we will denote by $\overline{\mathcal{C}} := \overline{\mathcal{C}}_{\text{orb}} = \overline{\mathcal{C}}(Y/V)$ the projective variety obtained from $\bar{Y}$ by contracting V_0 and call it the orbifold projective cone. We will also denote by $\overline{\mathcal{C}}^\circ$ the punctured projective cone $\overline{\mathcal{C}} \setminus \{o\}$ and still by $\bar{f} : \overline{\mathcal{C}}^\circ \to V$ the restriction of $\bar{f}$.

Lemma 4.2.11 ([Kol04, 40-42])**.** *Suppose $c_1(Y/V)$ is positive and denote by $\overline{\mathcal{C}}$ the orbifold projective cone over V. Then $K_{\overline{\mathcal{C}}}$ is $\mathbb{Q}$-Cartier if and only if $c_1(Y/V) = -r^{-1}(K_V + \Delta)$ with $r \in \mathbb{Q}_{>0}$. In this*

case $\overline{\mathcal{C}}$ has klt singularities and as $\mathbb{Q}$-divisors on $\overline{\mathcal{C}}$,

$$K_{\overline{\mathcal{C}}} \sim_{\mathbb{Q}} -(1+r)V_\infty. \tag{4.2.16}$$

Proof. As pointed out in [Kol04, 42], the first statement follows from the formula for canonical divisor of Y: $K_Y = f^*(K_V + \Delta)$ and the description of the class group of Seifert bundles by Flenner-Zaidenberg. If t is a coordinate on each $\mathbb{C}^*$-fibre, then dt/t is a well defined 1-form on $\overline{\mathcal{C}}^\circ$ with pole order 1 along V_∞. Using this we see that $K_{\overline{\mathcal{C}}^\circ}(V_\infty) = \bar{f}^*(K_V + \Delta)$. So (4.2.16) follows from the fact that, as $\mathbb{Q}$-divisors, $V_\infty \sim_{\mathbb{Q}} \bar{f}^* c_1(Y/V)$. $\overline{\mathcal{C}}$ has quotient singularities and hence klt singularities near V_∞. On the other hand, using the same argument as in [Kol13, Proof of Lemma 3.1], we know that $\overline{\mathcal{C}}$ has klt singularities at the vertex.

□

Definition 4.2.12. An orbifold is a normal, compact complex space V covered by local charts given as quotients of smooth coordinate charts. In other words, V can be covered by open charts $V = \cup_i U_i$, and for each U_i there is a smooth complex space W_i and a finite group G_i acting on W_i such that U_i is biholomorphic to the quotient space W_i/G_i. The quotient maps will be denoted by $\pi_i : W_i \to U_i$.

One needs to assume the compatibility condition between the charts: there are global divisors $D_j \subset X$ and ramification indices m_j such that $D_{ij} = U_i \cap D_j$ and $m_{ij} = m_j$ (after suitable re-indexing). The branch divisor of the orbifold is defined as $\Delta := \sum_i (1 - \frac{1}{m_j}) D_j$ and the orbifold canonical class is defined as $K_V + \sum_i (1 - \frac{1}{m_i}) D_i$. This is the negative of the orbifold Chern class:

$$c_1^{\mathrm{orb}}(V, \Delta) := c_1(X) - \sum_i (1 - \frac{1}{m_i})[D_i] \in H^2(V, \mathbb{Q}).$$

Let $f : Y \to (V, \Delta)$ be a Seifert $\mathbb{C}^*$-bundle with Y smooth. For $p \in V$ pick any $y \in f^{-1}(p)$ and a μ_m-invariant smooth hypersurface $W_p \subset Y$ transversal to $\mathrm{red} f^{-1}(p)$ for $m = m(p, Y/V)$. Then $\{\pi_p : W_p \to U_p := W_p/\mu_m\}$ gives an orbifold structure on V. The orbifold branch divisor coincides with the branch divisor of the Seifert bundle. The orbifolds coming from a smooth Seifert bundle have additional property that each U_p is a quotient by a cyclic group μ_m. Such an orbifold is called *locally cyclic*.

It's useful to use the local structure of near V_∞. We can choose an open set $U = W/\mu_m$ such that $\bar{f}^{-1}(U) \cong \mathbb{C} \times \mathbb{C}^{n-1}/\mu_m(p, a_1, \cdots, a_{n-1})$ such that $\gcd(a_1, \ldots, a_{n-1}, m) = 1$. Following Kollár,

we define the integers:

$$c_i := \gcd(a_1, \ldots, \widehat{a_i}, \ldots, a_{n-1}, m), \quad d_i := a_i c_i / C, \quad C := \prod_i c_i.$$

Then c_i are pairwise relatively prime and C/c_i divides a_i. Moreover as a variety:

$$V = A^{n-1}_{\mathbf{z}}/\mu_m(a_1, \ldots, a_{n-1}) \cong A^n_{\mathbf{x}}/\mu_{m/C}(d_1, \ldots, d_n).$$

Denote $M = m/C$ and define the $\mathbb{Q}$-divisors:

$$D_i : (x_i = 0)/\mu_M(d_1, \ldots, \widehat{d_i}, \ldots, d_{n-1}) \subset V.$$

and let

$$\mathcal{O}_V(j) := \epsilon^j\text{-eigenspace of } \mathbb{C}[x_1, \ldots, x_{n-1}]$$

as an $\mathcal{O}_X$-module. Then $\mathcal{O}_V(D_j) \cong \mathcal{O}_V(d_j)$. Moreover, since μ_M acts on $A^n_{\mathbf{x}}$ without pseudo-reflections, $K_{A^n_{\mathbf{x}}} = \pi^* K_V$. So $K_V = \mathcal{O}_V(-\sum d_i) = \mathcal{O}_V(-\sum D_i)$ and

$$K_{\text{orb}} = K_V + \sum_i (1 - \frac{1}{c_i}) D_i = -\sum_i \frac{D_i}{c_i}.$$

so that

$$C \cdot K_{\text{orb}} = -\sum_i \frac{C}{c_i} D_i = -\sum_i \frac{a_i}{d_i} D_i = \mathcal{O}_V(-\sum_i a_i).$$

One can write p uniquely as $p \equiv lC + \sum a_i b_i \mod M$ where $0 \leq b_i < c_i$ for every i and

$$C \cdot c_1(Y/X) = \mathcal{O}_V\left(Cl + \sum_i \frac{C}{c_i} b_i d_i\right) = \mathcal{O}_V(Cl + \sum a_i b_i) = \mathcal{O}_V(p).$$

If $c_1(Y/X) = -r^{-1} K_{\text{orb}}$, then $p = r^{-1} \sum_i a_i$ and locally

$$\bar{f}^{-1}(U) = U \times (r^{-1} K_U^{-1})/\mu_m.$$

Definition 4.2.13. An orbifold Hermitian metric g on the orbifold (V, Δ) is a Hermitian metric g on $V \setminus (\text{SingV} \cup \text{Supp}\Delta)$ such that for every chart $\pi_i : W_i \to U_i$ the pull back $\pi_i^* g$ extends to a Hermitian metric on W_i. Similarly, geometric objects on complex manifolds can be extended in a straight forward manner to the orbifold setting by working with local uniformizing charts. For

example one can talk about curvature, Kähler metrics, Kähler-Einstein metrics on orbifolds.

4.3 Proof of main theorems

4.3.1 Log version of Fujita's result

Following [Fuj15], we will first translate Berman's expansion of log-Ding-energy into an algebraic version.

Proposition 4.3.1 ([Fuj15, Proposition 3.5])**.** *Let (X, D) be a n-dimensional log-$\mathbb{Q}$-Fano pair which is log-Ding-semistable. Let δ be a positive rational number such that $-\delta^{-1}(K_X + D)$ is Cartier. Let $I_M \subset \cdots \subset I_1 \subset \mathcal{O}_X$ be a sequence of coherent ideal sheaves and let $\mathcal{I} := I_M + I_{M-1}t^1 + \cdots + I_1 t^{M-1} + (t^M) \subset \mathcal{O}_{X\times\mathbb{C}^1}$. Let $\Pi : \mathcal{X} \to X \times \mathbb{C}^1$ be the blowup along $\mathcal{I}$. Denote $\mathcal{D} := \Pi^*(D \times \mathbb{C}^1)$. Let $E \subset \mathcal{X}$ be the Cartier divisor defined by $\mathcal{O}_X(-E) = \mathcal{I} \cdot \mathcal{O}_{\mathcal{X}}$, and let*

$$\mathcal{L} := \Pi^*\mathcal{O}_{X\times\mathbb{C}^1}(-\delta^{-1}\left(K_{X\times\mathbb{C}^1/\mathbb{C}^1} + D \times \mathbb{C}^1\right)) \otimes \mathcal{O}_{\mathcal{X}}(-E).$$

Assume that $\mathcal{L}$ is semiample over $\mathbb{C}^1$. Then $(\mathcal{X}, \mathcal{D}; \mathcal{L})$ is naturally seen as a (possibly non-normal) semi test configuration of $(X, D; -\delta^{-1}(K_X+D))$. Under these conditions, $((X\times\mathbb{C}^1, D\times\mathbb{C}^1);\ \mathcal{I}^\delta\cdot(t)^d)$ must be sub log canonical, where

$$d := 1 + \frac{\delta^{n+1}(\bar{\mathcal{L}}^{n+1})}{(n+1)((-K_X - D)^n)}.$$

Moreover, we have the equality:

$$(\bar{\mathcal{L}}^{n+1}) = -\lim_{k\to+\infty} \frac{\dim\left(\frac{H^0(X\times\mathbb{C}^1, \mathcal{O}_{X\times\mathbb{C}^1}(-k\delta^{-1}(K_{X\times\mathbb{C}^1/\mathbb{C}^1}+D\times\mathbb{C}^1))}{H^0(X\times\mathbb{C}^1, \mathcal{O}_{X\times\mathbb{C}^1}(-k\delta^{-1}(K_{X\times\mathbb{C}^1/\mathbb{C}^1}+D\times\mathbb{C}^1))\cdot\mathcal{I}^k)}\right)}{k^{n+1}/(n+1)!}.$$

Proof. The proof is the same as Fujita's proof. Let $\nu : \mathcal{X}^\nu \to \mathcal{X}$ be the normalization. Denote $\mathcal{D}^\nu := \nu^*\mathcal{D}$. Then $(\mathcal{X}^\nu, \mathcal{D}^\nu; \nu^*\mathcal{L})/\mathbb{C}^1$ is a normal semi test configuration of $(X, D; -\delta^{-1}(K_X + D))$. By the log-Ding-semistability of (X, D), we can deduce that log-Ding$(\mathcal{X}^\nu, \mathcal{D}^\nu; \nu^*\mathcal{L}) \geq 0$. Then we notice the identity:

$$\begin{aligned} & \mathcal{O}_{\bar{\mathcal{X}}^\nu}\left(\delta^{-1}\left(K_{\bar{\mathcal{X}}^\nu/\mathbb{P}^1} + \mathcal{D}^\nu - \Delta_{(\mathcal{X}^\nu, \mathcal{D}^\nu; \nu^*\mathcal{L})}\right)\right) \cong \nu^*\bar{\mathcal{L}}^{-1} \\ = \quad & \nu^*\mathcal{O}_{\bar{\mathcal{X}}}\left(\delta^{-1}\left(\Pi^*(K_{X\times\mathbb{P}^1/\mathbb{P}^1} + D \times \mathbb{P}^1) + \delta E\right)\right). \end{aligned}$$

So $(\mathcal{X}^\nu, \mathcal{D}^\nu - \Delta_{(\mathcal{X}^\nu, \mathcal{D}^\nu; \nu^*\mathcal{L})} + c\mathcal{X}_0^\nu)$ is sub log canonical if and only if $((X \times \mathbb{C}^1, D \times \mathbb{C}^1);\ \mathcal{I}^\delta \cdot (t)^c)$ is sub log canonical. Thus we have the identity:

$$\mathrm{lct}(\mathcal{X}^\nu, \mathcal{D}^\nu - \Delta_{(\mathcal{X}^\nu, \mathcal{D}^\nu; \nu^*\mathcal{L})}; \mathcal{X}_0^\nu) = \mathrm{lct}\left((X \times \mathbb{C}^1, (D \times \mathbb{C}^1) \cdot \mathcal{I}^\delta); (t)\right).$$

The rest of the argument is the same as in [Fuj15] and [Oda13]. □

Assume (X, D) is a log pair such that $L = -\delta^{-1}(K_X + D)$ is an ample Cartier divisor. Denote by $S = \bigoplus_{m=0}^{+\infty} H^0(X, mL)$ the graded $\mathbb{C}$-algebra of sections of (X, L). Assume $\mathcal{F}$ is a good filtration of the graded ring S as in Definition 3.2.1. Following Fujita, define the ideal sheaf:

$$\mathcal{I}^{\mathcal{F}}_{(m,x)} = \mathrm{Im}\left(\mathcal{F}^x S_m \otimes L^{-m} \to \mathcal{O}_X\right),$$

and the ideal sheaf on $X \times \mathbb{C}$:

$$\mathcal{I}_m = \mathcal{I}^{\mathcal{F}}_{(m,me_+)} + \mathcal{I}^{\mathcal{F}}_{(m,me_+-1)} t^1 + \cdots + \mathcal{I}^{\mathcal{F}}_{(m,me_-+1)} t^{m(e_+-e_-)-1} + \left(t^{m(e_+-e_-)}\right).$$

By [Fuj15, 4.3], there exists $m_1 \in \mathbb{Z}_{>0}$ such that $\{\mathcal{I}_m\}_{m \geq m_1}$ is a graded family of coherent ideal sheaves on $X \times \mathbb{C}^1$. For any $m \geq m_1$, let

- $\Pi_m : \mathcal{X}_m \to X \times \mathbb{C}^1$ be the blow up along $\mathcal{I}_m$,
- $\mathcal{D}_m := \Pi_m^*(D \times \mathbb{C}^1)$,
- $E_m \subset \mathcal{X}_m$ be the Cartier divisor defined by $\mathcal{O}_{\mathcal{X}_m}(-E_m) = \mathcal{I}_m \cdot \mathcal{O}_{\mathcal{X}_m}$, and
- $\mathcal{L}_m := \Pi_m^* \mathcal{O}_{X \times \mathbb{C}^1}(-m\delta^{-1}\left(K_{X \times \mathbb{C}^1/\mathbb{C}^1} + D \times \mathbb{C}^1\right)) \otimes \mathcal{O}_{\mathcal{X}_m}(-E_m)$.

Proposition 4.3.2 ([Fuj15, 4.6])**.** *The line bundle $\mathcal{L}_m$ is semiample over $\mathbb{C}^1$. Thus $(\mathcal{X}_m, \mathcal{D}_m; \mathcal{L}_m)/\mathbb{C}^1$ is a semi test configuration of $(X, D; L^m)$.*

Thus, by Proposition 4.3.1, $((X \times \mathbb{C}^1, D \times \mathbb{C}^1); \mathcal{I}_m^{\delta/m} \cdot (t)^{d_m})$ is sub log canonical, where

$$d_m := 1 + \frac{\delta^{n+1}(\bar{\mathcal{L}}_m^{n+1})}{(n+1)m^{n+1}((-K_X - D)^n)}.$$

Proposition 4.3.3 (log-Fujita, cf. [Fuj15])**.** *Assume the log pair (X, D) satisfies the following properties:*

- *$K_X + D$ is $\mathbb{Q}$-Cartier;*

- (X, D) *is a klt log-Fano pair;*
- (X, D) *is log-Ding-semistable;*
- $L = -\delta^{-1}(K_X + D)$ *is an ample Cartier divisor;*
- $\mathcal{F}$ *is a good graded filtration of* $S = \bigoplus_{k=0}^{+\infty} H^0(X, kL)$ *(see Definition 3.2.1).*

Then $((X \times \mathbb{C}^1, D \times \mathbb{C}^1); \mathcal{I}_\bullet^\delta \cdot (t)^{d_\infty})$ *is sub log canonical, where*

$$\begin{aligned}\mathcal{I}_m =& \mathcal{I}^{\mathcal{F}}_{(m,me_+)} + \mathcal{I}^{\mathcal{F}}_{(m,me_+-1)} t^1 + \cdots + \mathcal{I}^{\mathcal{F}}_{(m,me_-+1)} t^{m(e_+-e_-)-1} + \left(t^{m(e_+-e_-)}\right),\\ d_\infty =& 1 - \delta(e_+ - e_-) + \frac{\delta}{(L^n)} \int_{e_-}^{e_+} \mathrm{vol}\left(\overline{\mathcal{F}} S^{(t)}\right) dt.\end{aligned}$$

Proof. The proof is the same as in [Fuj15]. Indeed by [Fuj15, 4.7] we have that $\lim_{m\to\infty} d_m = d_\infty$. Then the proposition follows by [Fuj15, 2.5(1)]. □

4.3.2 Application of log-Fujita to the filtrations of valuations

Let $o \in X$ be a closed point. Let $v \in \mathrm{Val}_{X,o}$ be a real valuation centered at o. We will apply the above log version of Fujita's result to the filtration associated to v on X. Consider the following graded filtration of S. For an $x \in \mathbb{R}$ define:

$$\mathcal{F}^x S_m = H^0(X, L^m \otimes \mathfrak{a}_x),$$

where $\mathfrak{a}_x := \mathfrak{a}_x(v) = \{f \in \mathcal{O}_X : v(f) \geq x\}$.

Lemma 4.3.4. $\mathcal{F}$ *is a decreasing, left-continuous, multiplicative and saturated filtration of* S*. Moreover, if* $A_{(X,D)}(v) < +\infty$ *then* $\mathcal{F}$ *is also linearly bounded.*

Proof. It's clear the $\mathcal{F}$ is decreasing, left-continuous and multiplicative. To prove that $\mathcal{F}$ is saturated notice that the homomorphism

$$\mathcal{F}^x S_m \otimes_{\mathbb{C}} L^{\otimes(-m)} \twoheadrightarrow I^{\mathcal{F}}_{(m,x)}$$

induces the inclusion $I^{\mathcal{F}}_{(m,x)} \subset \mathfrak{a}_x$ for any $x \in \mathbb{R}$. Thus $\overline{\mathcal{F}}^x S_m = H^0(X, L^{\otimes m} \cdot I^{\mathcal{F}}_{(m,x)}) \subset \mathcal{F}^x S_m$.

If $A_{(X,D)}(v) < +\infty$, then by a log version of [Li15b, Proposition 1.2] there exists a constant $c = c(X, o)$ such that $v \leq cA_{(X,D)}(v)\mathrm{ord}_o$. So it's easy to see that

$$e(S_\bullet, \mathcal{F}) \leq cA_{(X,D)}(v) \cdot e_{\max}(S_\bullet, \mathcal{F}_{\mathrm{ord}_o}) < +\infty.$$

The second inequality follows from the work in [BKMS15].

□

It is clear that $\mathcal{F}^x S_m = S_m$ for $x \leq 0$. Hence we may choose $e_- = 0$. The graded family of ideal sheaves $\mathcal{I}_\bullet$ on $X \times \mathbb{C}^1$ becomes:

$$\mathcal{I}_m = \mathcal{I}^{\mathcal{F}}_{(m,me_+)} + \mathcal{I}^{\mathcal{F}}_{(m,me_+-1)} t^1 + \cdots + \mathcal{I}^{\mathcal{F}}_{(m,1)} t^{me_+-1} + (t^{me_+}).$$

We also have that

$$d_\infty = 1 - \delta e_+ + \frac{\delta}{(L^n)} \int_0^\infty \mathrm{vol}\left(\overline{\mathcal{F}} S^{(t)}\right) dt.$$

The valuation v extends to a $\mathbb{C}^*$-invariant valuation $\bar{v}$ on $\mathbb{C}(X \times \mathbb{C}) = \mathbb{C}(X)(t)$, such that for any $f = \sum_k f_k t^k$ we have:

$$\bar{v}(f) = \min_k \{v(f_k) + k\}.$$

Proposition 4.3.5. *If (X, D) is log-Ding-semistable and $L = -\delta^{-1}(K_X + D)$ then*

$$A_{(X,D)}(v) - \frac{\delta}{(L^n)} \int_0^{+\infty} \mathrm{vol}\left(\mathcal{F} S^{(t)}\right) dt \geq 0. \tag{4.3.1}$$

Proof. We may assume that $A_{(X,D)}(v) < +\infty$, since otherwise the inequality holds automatically. Hence Lemma 4.3.4 implies that $\mathcal{F}$ is good and saturated.

By Proposition 4.3.3, we know that $((X \times \mathbb{C}^1, D \times \mathbb{C}^1);\ \mathcal{I}^\delta_\bullet \cdot (t)^{d_\infty})$ is sub log canonical. Since $(X, D) \times \mathbb{C}^1 := (X \times \mathbb{C}^1, D \times \mathbb{C}^1)$ has klt singularities, for any $0 < \epsilon \ll 1$ there exists $m = m(\epsilon)$ such that

$$\mathcal{O}_{X\times\mathbb{C}^1} \subset \mathcal{J}\left((X \times \mathbb{C}^1, D \times \mathbb{C}^1); \mathcal{I}_m^{(1-\epsilon)\delta/m} \cdot (t)^{(1-\epsilon)d_\infty}\right).$$

By [BdFFU15, 1.2] we know that the following inequality holds for any real valuation u on $X \times \mathbb{C}^1$:

$$A_{(X,D)\times\mathbb{C}^1}(u) > \frac{(1-\epsilon)\delta}{m} u(\mathcal{I}_m) + (1-\epsilon) d_\infty u(t). \tag{4.3.2}$$

Let us choose a sequence of quasi-monomial real valuations $\{v_n\}$ on X such that $v_n \to v$ and $A_{(X,D)}(v_n) \to A_{(X,D)}(v)$ when $n \to \infty$. It is easy to see that $\{\bar{v}_n\}$ is a sequence of quasi-monimial valuations on $X \times \mathbb{C}^1$ satisfying $\bar{v}_n(t) = 1$ and

$$A_{(X,D)\times\mathbb{C}^1}(\bar{v}_n) = A_{(X,D)}(v_n) + 1.$$

Hence we have

$$\begin{aligned} A_{(X,D)}(v)+1 &= \lim_{n\to\infty} A_{(X,D)}(v_n)+1 = \lim_{n\to\infty} A_{(X,D)\times\mathbb{C}^1}(\bar{v}_n) \\ &\geq \frac{(1-\epsilon)\delta}{m} \lim_{n\to\infty} \bar{v}_n(\mathcal{I}_m) + (1-\epsilon)d_\infty. \end{aligned}$$

From the definition of $\mathcal{F}^x S_m$ we get:

$$v\left(\mathcal{I}^{\mathcal{F}}_{(m,x)}\right) \geq x.$$

Therefore,

$$\begin{aligned} \lim_{n\to\infty} \bar{v}_n(\mathcal{I}_m) &= \lim_{n\to\infty} \min_{0\leq j\leq re_+} \left\{ v_n\left(\mathcal{I}^{\mathcal{F}}_{(m,j)}\right) + me_+ - j \right\} \\ &= \min_{0\leq j\leq re_+} \left\{ \lim_{n\to\infty} v_n\left(\mathcal{I}^{\mathcal{F}}_{(m,j)}\right) + me_+ - j \right\} \\ &= \min_{0\leq j\leq re_+} \left\{ v\left(\mathcal{I}^{\mathcal{F}}_{(m,j)}\right) + me_+ - j \right\} \\ &\geq me_+ \end{aligned}$$

Hence when $\epsilon \to 0+$ we get:

$$A_{(X,D)}(v)+1 \geq \delta e_+ + d_\infty. \tag{4.3.3}$$

Therefore,

$$\begin{aligned} A_{(X,D)}(v) &\geq -1 + \delta e_+ + d_\infty \\ &= \frac{\delta}{(L^n)} \int_0^{+\infty} \mathrm{vol}\left(\mathcal{F}S^{(t)}\right) dt. \end{aligned}$$

Hence we get the desired inequality. □

Proposition 4.3.6. *If (X, D) is log-Ding-semistable, then for any real valuation v centered at any closed point of X, we have the estimate:*

$$\widehat{\mathrm{vol}}_{(X,D)}(v) \geq \left(\frac{n}{n+1}\right)^n (-K_X - D)^n. \tag{4.3.4}$$

Proof. We prove this estimate using the method in [Fuj15, Liu16]. We may assume $A_{(X,D)}(v) < +\infty$, since otherwise the inequality holds automatically. Since $\mathcal{F}^{mx}S_m = H^0(X, L^{\otimes m} \cdot \mathfrak{a}_{mx})$, we have the exact sequence

$$0 \to \mathcal{F}^{mx}S_m \to H^0(X, L^{\otimes m}) \to H^0(X, L^{\otimes m} \otimes (\mathcal{O}_X/\mathfrak{a}_{mx})).$$

Hence we have

$$\dim \mathcal{F}^{mx} S_m \geq h^0(X, L^{\otimes m}) - \ell(\mathcal{O}_X/\mathfrak{a}_{mx}).$$

Dividing by $m^n/n!$ and taking limits as $m \to +\infty$, we get

$$\mathrm{vol}\left(\mathcal{F}S^{(x)}\right) \geq (L^n) - \mathrm{vol}(v)x^n. \tag{4.3.5}$$

So we have the estimate:

$$\begin{aligned}\int_0^{+\infty} \mathrm{vol}\left(\mathcal{F}S^{(x)}\right) dx &\geq \int_0^{\sqrt[n]{(L^n)/\mathrm{vol}(v)}} ((L^n) - \mathrm{vol}(v)x^n)\, dx \\ &= \frac{n}{n+1}(L^n) \cdot \sqrt[n]{(L^n)/\mathrm{vol}(v)}.\end{aligned}$$

Applying (4.3.1) we get the inequality:

$$\begin{aligned}A_{(X,D)}(v) &\geq \frac{\delta}{(L^n)} \int_0^{+\infty} \mathrm{vol}\left(\mathcal{F}S^{(t)}\right) dt \\ &\geq \frac{\delta}{(L^n)} \frac{n}{n+1}(L^n) \sqrt[n]{(L^n)/\mathrm{vol}(v)} = \frac{n}{n+1} \sqrt[n]{(\delta L)^n/\mathrm{vol}(v)}.\end{aligned}$$

Since $\delta L = -K_X - D$, this is exactly equivalent to

$$\widehat{\mathrm{vol}}_{(X,D)}(v) = A_{(X,D)}(v)^n \mathrm{vol}(v) \geq \left(\frac{n}{n+1}\right)^n (-K_X - D)^n.$$

□

Proof of Theorem 4.1.1. By Section 4.2, we apply the above proposition to the case $X = \overline{\mathcal{C}}$ and $D = (1-\beta)V_\infty + \mathcal{E}$. Recall that by (4.2.10) we have:

$$-(K_X + D) \quad = \quad -(K_X + (1-\beta)V_\infty + \mathcal{E}) = r\frac{n+1}{n}V_\infty.$$

So we get the intersection number:

$$(-K_X - D)^n = r^n \left(\frac{n+1}{n}\right)^n H^{n-1}. \tag{4.3.6}$$

Because v is centered at $o \in \mathcal{C}$, $v(V_\infty) = 0$ and hence $A_{(X,D)}(v) = A_{(\overline{\mathcal{C}},\mathcal{E})}(v) - (1-\beta)v(V_\infty) = A_{(\mathcal{C},\mathcal{E})}(v)$. On the other hand, $A_{(\mathcal{C},\mathcal{E})}(v_0) = r$ (see [Kol13, Section 3.1]) and $\mathrm{vol}(v_0) = (H^{n-1})$. So

$\widehat{\mathrm{vol}}_{(\mathcal{C},\mathcal{E})}(v_0) = r^n H^{n-1} = \widehat{\mathrm{vol}}(v_0)$. Substituting (4.3.6) into (4.3.4), we get:

$$\widehat{\mathrm{vol}}_{(\mathcal{C},\mathcal{E})}(v) = A^n_{(\mathcal{C},D)} \cdot \mathrm{vol}(v) \geq r^n H^{n-1} = \widehat{\mathrm{vol}}(v_0) = \widehat{\mathrm{vol}}_{(\mathcal{C},\mathcal{E})}(v_0). \tag{4.3.7}$$

□

Proof of Theorem 4.1.3. Choose $F \in |-mK_V|$ for $m \gg 1$ such that F is an irreducible prime divisor satisfying:

1. F does not contain the center of v;
2. $(V, (1-\beta)F)$ is klt for any $0 < \beta < 1$.

Consider the effective divisor $E_\kappa = (1-\kappa)F/m$ with $0 < \kappa \leq 1$. Then (V, E_κ) is a log-Fano pair:

$$-K_V - E_\kappa = -K_V - (1-\kappa)(-K_V) = \kappa(-K_V).$$

Using Tian's α-invariant as in [Ber13, LS14, SW16] (see also [JMR16, BHJ15]), we know that the log-Ding-energy is proper and there exists a conical Kähler-Einstein potential on $(V, E_\kappa; -K_V)$ for $0 < \kappa \ll 1$.

Now if V specially degenerates to a Kähler-Einstein variety then its Ding-energy is uniformly bounded from below by [Li13, Theorem 4] (and hence V is K-semistable). Notice that the assumption in [Li13, Theorem 4] is that the generic fiber V is smooth. However the proof works for general $\mathbb{Q}$-Fano variety V. Indeed the key calculation showing the continuity of log-Ding-energy there is generalized later in [LWX15, Appendix I]. So by using interpolation argument as in [LS14], we know that there is a conical Kähler-Einstein potential on $(V, E_\kappa; -K_V)$ for any $\kappa \in (0,1)$.

Because F does not contain the center of v, $v(\mathcal{I}_{\mathcal{E}_\kappa}) = 0$ and hence $A_{(X,\mathcal{E}_\kappa)}(v) = A_X(v)$. On the other hand, $H = -r^{-1}K_V = -(\kappa r)^{-1}(K_V + E_\kappa)$. So by Theorem 4.1.1 (see (4.3.7)),

$$\widehat{\mathrm{vol}}(v) = \widehat{\mathrm{vol}}_{(X,\mathcal{E}_\kappa)}(v) \geq (\kappa r)^n H^{n-1} = \kappa^n \widehat{\mathrm{vol}}(v_0).$$

The conclusion follows by letting $\kappa \to 1$.

If $(\mathcal{C}, (1-\beta)V_\infty)$ specially degenerates to a conical Kähler-Einstein pair, then its log-Ding-energy is uniformly bounded from below by [Li13, Theorem 4] as above. So the pair is log-Ding-semistable by the proof of Berman's result in Proposition 4.2.1. Hence we can directly apply log-Fujita in Proposition 4.3.3 as in the proof Theorem 4.1.1. □

Proof of Corollary 4.1.5. (i)⇒(ii): By deep results in [CDS15, Tia15] we know that V being K-semistable is equivalent to the condition that V degenerates to a Kähler-Einstein Fano variety. Thus it follows from Theorem 4.1.3.

(ii)⇒(i): This follows from [Li15a].

(ii)⇔(iii): Simple computations show that $A_{\mathcal{C}}(\mathrm{ord}_V) = r$ and $\mathrm{vol}(\mathrm{ord}_V) = (H^{n-1}) = (-r^{-1}K_V)^{n-1}$. Thus $\widehat{\mathrm{vol}}(\mathrm{ord}_V) = r(-K_V)^{n-1}$ and the equivalence follows from Theorem 2.3.5. □

4.3.3 Examples

Example 4.3.7. Consider the n-dimensional A_{k-1} singularity.

$$A^n_{k-1} := \{z_1^2 + \cdots + z_n^2 + z_{n+1}^k = 0\} \subset \mathbb{C}^{n+1}.$$

A^n_{k-1} is an orbifold affine cone with the orbifold base (V, Δ) given by the hypersurface in weighted projective space $\{Z_1^2 + \cdots + Z_n^2 + Z_{n+1}^k = 0\} \subset \mathbb{P}^n(k, \cdots, k, 2)$. It's easy to see that the following

- If k is odd, $(V, \Delta) = (\mathbb{P}^{n-1}, (1 - \frac{1}{k})Q'^{n-2})$ where $Q'^{n-1} = \{Z_1^2 + \cdots + Z_n^2 = 0\} \subset \mathbb{P}^{n-1}$;
- If k is even, $(V, \Delta) = (Q^{n-1}, (1 - \frac{2}{k})Q^{n-2})$ where $Q^{n-1} = \{Z_1^2 + \cdots + Z_{n+1}^2 = 0\} \subset \mathbb{P}^{n+1}$ and $Q^{n-2} = Q^n \cap \{Z_{n+1} = 0\}$.

The above two cases are related by using the 2-fold branched covering $\tau : Q^{n-1} \to (\mathbb{P}^{n-1}, \frac{1}{2}Q'^{n-2})$ so that $K_{Q^{n-1}} = \tau^* \left(K_{\mathbb{P}^{n-1}} + \frac{1}{2}Q'^{n-2}\right)$ and hence $K_{Q^{n-1}} + (1 - \frac{2}{k})Q^{n-2} = \tau^*(K_{\mathbb{P}^{n-1}} + (1 - \frac{1}{k})Q'^{n-2})$. By [GMSY07, LS14, Li13], there is a conical Kähler-Einstein metric on (V, Δ) if and only if one of the following conditions hold:

1. $k = 1$ or 2, and n is any positive integer;
2. $n = 2$ and k is any positive integer;
3. $n = 3$ and $1 \leq k \leq 3$.

There is a natural $\mathbb{C}^*$-action given by: $(z_1, \cdots, z_n, z_{n+1}) \to (t^k z_1, \cdots, t^k z_n, t^2 z_{n+1})$. The associated valuation is denoted by v_0. Notice that $(V, \Delta) = \left(A^n_{k-1} - \{0\}\right)/\mathbb{C}^*$. By Theorem 4.1.7, if any of the above conditions is satisfied, $\widehat{\mathrm{vol}}(v)$ is globally minimized at the valuation v_0 associated to canonical $\mathbb{C}^*$-action.

Also by [LS14] and [Li13], (V, Δ) is log-K-semistable but not log-K-polystable if and only if $(n, k) = (3, 4)$ or $(n, k) = (4, 3)$. Using the orbifold-version Theorem 4.1.7, we can show that in these

two cases $\widehat{\mathrm{vol}}(v)$ is also globally minimized at v_0. We will prove this for the case $(n,k)=(3,4)$ and the argument for the case $(n,k)=(4,3)$ is similar. When $(n,k)=(3,4)$, $(V,\Delta)=(\mathbb{Q}^2,(1-\frac{2}{4})\mathbb{Q}^1)\cong(\mathbb{P}^1\times\mathbb{P}^1,\frac{1}{2}E)$ where E denotes the diagonal $\mathbb{P}^1$. The compactified orbifold cone $\overline{\mathcal{C}}_{\mathrm{orb}}$ is just the natural compactification of A_2^3 inside $\mathbb{P}^4(1,2,2,2,1)$. In other words, if the latter is given weighted homogeneous coordinates $[Z_0,Z_1,\cdots,Z_4]$ such that $z_i=Z_i/Z_0^2$ for $1\leq i\leq 3$ and $z_4=Z_4/Z_0$, then we have:

$$\overline{\mathcal{C}}_{\mathrm{orb}}=\{Z_1^2+Z_2^2+Z_3^2+Z_4^4=0\}\subset\mathbb{P}^4(1,2,2,2,1).$$

By the construction in [LS14, Li13] (see also [Li15a, Section 5.3]), we know that (V,Δ) specially degenerates into $(\mathbb{P}^2(1,1,2),\frac{1}{2}F)=:(\mathcal{V}_0,\frac{1}{2}F)$, where $F=\{W_2=0\}$ if $[W_0,W_1,W_2]$ are weighted homogeneous coordinates of $\mathbb{P}(1,1,2)$. This can be realized as a degeneration inside $\mathbb{P}^3(2,2,2,1)$ where the two pairs are realized as weighted projective hypersurfaces:

$$\begin{aligned}\left(\mathbb{P}^1\times\mathbb{P}^1,\frac{1}{2}E\right) &= \{Z_1^2+Z_2^2+Z_3^2+Z_4^4=0\}\subset\mathbb{P}^3(2,2,2,1);\\ \left(\mathbb{P}^2(1,1,2),\frac{1}{2}F\right) &= \{Z_1^2+Z_2^2+Z_3^2=0\}\subset\mathbb{P}^3(2,2,2,1).\end{aligned}$$

Correspondingly, $\overline{\mathcal{C}}_{\mathrm{orb}}$ degenerates to the weighted projective cone over $\mathbb{P}^2(1,1,2)$ given by:

$$\mathcal{X}_0:=\{Z_1^2+Z_2^2+Z_3^2=0\}\subset\mathbb{P}^4(1,2,2,2,1).$$

Because $(\mathbb{P}^2(1,1,2),\frac{1}{2}F)$ has a natural orbifold Kähler-Einstein metric by pushing forward the Fubini-Study metric of $\mathbb{P}^2$ under the 2-fold branched covering map $(\mathbb{P}^2\to\mathbb{P}^2(1,1,2),\frac{1}{2}F)$, by Proposition 4.2.5, there is a conical Kähler-Einstein metric on the pair $(\mathcal{X}_0,(1-\beta)(\mathcal{V}_0)_\infty)$ where $(\mathcal{V}_0)_\infty\cong\mathcal{V}_0=\mathbb{P}^2(1,1,2)$. To see the cone angle $\beta=r/n$, we notice that

$$-K_{\mathcal{X}_0}=4H=4\left(\{Z_0=0\}\cap\mathcal{X}_0\right)=4(\mathcal{V}_0)_\infty,$$

where H is the weighted hyperplane divisor of $\mathbb{P}^4(1,2,2,2,1)$. So we get $r=3$ by comparing with (4.2.16). Alternatively, we can calculate the orbifold canonical class $K_{(\mathbb{P}^2(1,1,2),\frac{1}{2}F)}=-4H'+H'=-3H'$ where H' is the weighted hyperplane divisor of $\mathbb{P}^2(1,1,2)$. Notice we indeed have $H'=\frac{1}{2}F=H|_{\mathbb{P}^2(1,1,2)}$ under the embedding given above (taking into account of the $\mathbb{Z}_2$-orbifold locus of $\mathbb{P}^3(2,2,2,1)$). In any case we get $\beta=3/3=1$. So there is actually a Kähler-Einstein metric on the normal variety $\mathcal{X}_0$ which is orbifold smooth along $(\mathcal{V}_0)_\infty$. Now we can apply Theorem 4.1.7 to obtain the statement we wanted.

Remark 4.3.8. For all other cases of (n,k), by the calculations in [Li15b] it was conjectured that $\widehat{\mathrm{vol}}$ is minimized at the valuation associated to the weight $\left(1,\cdots,1,\frac{n-2}{n-1}\right)$. This conjecture was confirmed in [LX16].

Example 4.3.9. Next, we will consider quotient singularities.

Proposition 4.3.10. *Let (X,o) be a quotient singularity with local analytic model $\mathbb{C}^n/G$, where G is a finite subgroup of $GL(n,\mathbb{C})$ acting freely in codimension 1. Then*

$$\min_{v\in \mathrm{Val}_{X,o}} \widehat{\mathrm{vol}}(v) = \frac{n^n}{|G|}.$$

The minimum is achieved at the pushforward of $\mathrm{ord}_0 \in \mathrm{Val}_{\mathbb{C}^n,0}$.

Proof. For simplicity we may assume that $(X,o) = (\mathbb{C}^n/G, 0)$ and $G \subset U(n)$. From the proof of Theorem 3.3.7, we see that $\widehat{\mathrm{vol}}(v_*) = n^n/|G|$ where v_* is the pushforward of $\mathrm{ord}_0 \in \mathrm{Val}_{\mathbb{C}^n,0}$. Thus we only need to show that $\widehat{\mathrm{vol}}(v) \geq n^n/|G|$ for any $v \in \mathrm{Val}_{X,o}$. Consider $\overline{X} := \mathbb{P}^n/G$ as a natural projective compactification of X. Let ω_{FS} be the Fubini-Study metric on $\mathbb{P}^n$. Since G acts isometrically on $(\mathbb{P}^n, \omega_{FS})$, ω_{FS} induces an orbifold Kähler-Einstein metric ω_G on $\overline{X}$. Let H_∞ be the hyperplane at infinity in $\mathbb{P}^n$. The metric ω_G can be also viewed as a conical Kähler-Einstein metric on $\overline{X}$ with a cone angle $\frac{2\pi}{d}$ along $D := H_\infty/G$. Therefore, $(\overline{X}, (1-\frac{1}{d})D)$ is log-Ding-semistable. By Proposition 4.3.6 we know that

$$\widehat{\mathrm{vol}}(v) \geq \left(\frac{n}{n+1}\right)^n \left(-K_{\overline{X}} - \left(1-\frac{1}{d}\right)D\right)^n.$$

Let $\bar{\pi} : \mathbb{P}^n \to \overline{X}$ be the quotient map. We notice that

$$K_{\mathbb{P}^n} = \bar{\pi}^*\left(K_{\overline{X}} + \left(1-\frac{1}{d}\right)D\right).$$

Hence

$$\widehat{\mathrm{vol}}(v) \geq \left(\frac{n}{n+1}\right)^n \cdot \frac{(-K_{\mathbb{P}^n})^n}{|G|} = \frac{n^n}{|G|}.$$

Thus we prove the proposition. □

Remark 4.3.11. It is clear that $(\mathbb{C}^n \setminus \{0\})/G \to \mathbb{P}^{n-1}/G$ has a natural Seifert $\mathbb{C}^*$-bundle structure with $\overline{\mathcal{C}}_{\mathrm{orb}} = \mathbb{P}^n/G$ and $V_\infty = H_\infty/G$. Hence Proposition 4.3.10 can be also proved by applying Theorem 4.1.7.

4.4 Derivative of volumes revisited

In this section, we will point out the relation between the estimates in Section 4.3.2 to the arguments/calculations in [Li15a]. This will allow us to get a different proof of Theorem 4.1.1. The main idea is similar to that in [Li15b, Li15a], that is the formula appearing in log version of Fujita's result should be viewed as a derivative of normalized volumes. Firstly, similar to [Li15a] we will derive a formula for $\widehat{\mathrm{vol}}_{(\mathcal{C},\mathcal{E})}(v)$ using the "leading component" filtration. Then we consider a function $\Phi(s)$, which is convex with respect to $s \in [0,1]$ and interpolates: $\Phi(0) = \widehat{\mathrm{vol}}_{(\mathcal{C},\mathcal{E})}(v_0)$ and $\Phi(1) = \widehat{\mathrm{vol}}_{(\mathcal{C},\mathcal{E})}(v)$. To show $\Phi(1) \geq \Phi(0)$, by the convexity, we just need to show that the derivative of Φ at $s = 0$ is nonnegative. Our main observation is that this non-negativity is exactly the inequality (4.3.1). So from this point, we can use Proposition 4.3.5 and get a different proof of Theorem 4.1.1.

s the projective compactification of X with the infinity divisor $D \cong V$ and $\beta = r/n \in (0,1]$. By [Li13, Lemma 3],

4.4.1 A volume formula

Let (V,H) be a polarized projective variety. Denote by $R = \bigoplus_{k=0} R_k = \bigoplus_{k=0}^{+\infty} H^0(V,kH)$ the graded section ring of (V,H). Let $\mathcal{C} = C(V,H) = \mathrm{Spec}(R)$ be the affine cone of V with the polarization H. Let v_1 be any real valuation centered at o with $A_{\mathcal{C}}(v_1) < +\infty$. We introduce the following notation.

Definition 4.4.1. For any $g \in R = \bigoplus_{k=0}^{+\infty} R_k$, assume that it is decomposed into "homogeneous" components $g = g_{k_1} + \cdots + g_{k_p}$ with $g_{k_j} \neq 0 \in R_{k_j}$ and $k_1 < k_2 < \cdots < k_p$. We define $\deg(g) = k_p$ and define $\mathbf{ld}(g)$ to be the "leading component" g_{k_p}.

With the above definition, we define a filtration:

$$\mathcal{F}^x R_k = \{f \in R_k; \exists g \in R \text{ such that } v_1(g) \geq x \text{ and } \mathbf{ld}(g) = f\}. \tag{4.4.1}$$

We notice that following properties of $\mathcal{F}^x R_\bullet$.

1. For fixed k, $\{\mathcal{F}^x R_k\}_{x\in\mathbb{R}}$ is a family of decreasing $\mathbb{C}$-subspace of R_k.

2. $\mathcal{F}$ is multiplicative: $v_1(g_i) \geq x_i$ and $\mathbf{ld}(g_i) = f_i \in R_{k_i}$ implies

$$v_1(g_1 g_2) \geq x_1 + x_2, \quad \mathbf{ld}(g_1 g_2) = f_1 \cdot f_2 \in R_{k_1+k_2}.$$

3. If $A(v_1) < +\infty$, then $\mathcal{F}$ is linearly bounded. Indeed by Izumi's theorem in [Li15b, Proposition 1.2] (see also [Izu85, Ree89, BFJ14]), there exist $c_1, c_2 \in (0, +\infty)$ such that

$$c_1 v_0 \leq v_1 \leq c_2 v_0.$$

For any $g \in R$, $k := v_0(\mathbf{ld}(g)) \geq v_0(g) \geq c_2^{-1} x$. So if $x > c_2 k$ then $\mathcal{F}^x R_k = 0$. So $\mathcal{F}$ is linearly bounded from above. On the other hand, for any $f \in R_k$, $v_1(f) \geq c_1 v_0(f) = c_1 k$. So if $x \leq c_1 k$, then $\mathcal{F}^x R_k = R_k$. So $\mathcal{F}$ is linearly bounded from below. Note that the argument in particular shows the following relation:

$$\inf_{\mathfrak{m}} \frac{v_1}{v_0} \leq e_{\min} \leq e_{\max} \leq \sup_{\mathfrak{m}} \frac{v_1}{v_0}. \tag{4.4.2}$$

For later convenience, from now on we will fix the following constant:

$$c_1 := \inf_{\mathfrak{m}} \frac{v_1}{v_0} > 0. \tag{4.4.3}$$

By [Li15a, Lemma 4.2], we have the following characterization of this constant:

$$c_1 = v_1(V) = \inf_{i>0} \frac{v_1(R_i)}{i}. \tag{4.4.4}$$

The following is a reason why the filtration in (4.4.1) works for our purpose.

Proposition 4.4.2 (cf. [Li15a, Proposition 4.3])**.** *For any $m \in \mathbb{R}$, we have the following identity:*

$$\sum_{k=0}^{+\infty} \dim_{\mathbb{C}} \left(R_k / \mathcal{F}^m R_k \right) = \dim_{\mathbb{C}} \left(R / \mathfrak{a}_m(v_1) \right).$$

Notice that because of linearly boundedness, the sum on the left hand side is a finite sum. More precisely, by the above discussion, when $k \geq m/c_1$, then $\dim_{\mathbb{C}} \left(R_k / \mathcal{F}^m R_k \right) = 0$.

Proof. For each fixed k, let $d_k = \dim_{\mathbb{C}}(R_k / \mathcal{F}^m R_k)$. Then we can choose a basis of $R_k / \mathcal{F}^m R_k$:

$$\left\{ [f_i^{(k)}]_k \mid f_i^{(k)} \in R_k, 1 \leq i \leq d_k \right\},$$

where $[\cdot]_k$ means taking quotient class in $R_k / \mathcal{F}^m R_k$. Notice that for $k \geq \lceil m/c_1 \rceil$, the set is empty.

We want to show that the set

$$\mathfrak{B} := \left\{ [f_i^{(k)}] \mid 1 \le i \le d_k, 0 \le k \le \lceil m/c_1 \rceil - 1 \right\}.$$

is a basis of $R/\mathfrak{a}_m(v_1)$, where $[\cdot]$ means taking quotient in $R/\mathfrak{a}_m(v_1)$.

- We first show that $\mathfrak{B}$ is a linearly independent set. For any nontrivial linear combination of $[f_i^{(k)}]$:

$$\sum_{k=0}^{N}\sum_{i=1}^{d_k} c_i^{(k)}[f_i^{(k)}] = \left[\sum_{k=0}^{N}\sum_{i=1}^{d_k} c_i^{(k)} f_i^{(k)}\right] = [f^{(k_1)} + \cdots + f^{(k_p)}] =: [F],$$

  where $f^{(k_j)} \neq 0 \in R_{k_j} \setminus \mathcal{F}^m R_{k_j}$ and $k_1 < k_2 < \cdots < k_p$. In particular $\mathbf{ld}(F) = f^{(k_p)} \notin \mathcal{F}^m R_{k_p}$. By the definition of $\mathcal{F}^m R_{k_p}$, we know that

$$f^{(k_1)} + \cdots + f^{(k_p)} \notin \mathfrak{a}_m(v_1),$$

  which is equivalent to $[F] \neq 0 \in R/\mathfrak{a}_m(v_1)$.

- We still need to show that $\mathfrak{B}$ spans $R/\mathfrak{a}_m(v_1)$. Suppose on the contrary $\mathfrak{B}$ does not span $R/\mathfrak{a}_m(v_1)$. Then there is some $k \in \mathbb{Z}_{>0}$ and $f \in R_k - \mathfrak{a}_m(v_1)$ such that $[f] \neq 0 \in R/\mathfrak{a}_m(v_1)$ can not be written as a linear combination of $[f_i^{(k)}]$, i.e. not in the span of $\mathfrak{B}$. We can choose a minimal k such that this happens. So from now on we assume that k has been chosen such that that for any $k' < k$ and $g \in R_{k'}$, $[g]$ is in the span of $\mathfrak{B}$. Then there are two cases to consider.

  1. If $f \in R_k \setminus \mathcal{F}^m R_k$, then since $\{[f_i^{(k)}]_k\}$ is a basis of $R_k/\mathcal{F}^m R_k$, we can write $f = \sum_{j=1}^{d_k} c_j f_j^{(k)} + h_k$ where $h_k \in \mathcal{F}^m R_k$. By the definition of $\mathcal{F}^m R_k$, there exists $h \in \mathfrak{a}_m(v_1)$ with $\mathbf{ld}(h) = h_k$. By the minimality of k, we know that $[h_k - h] = [h_k]$ is in the span of $\mathfrak{B}$. So $[f] = \sum_{j=1}^{d_k} c_j [f_j^{(k)}] + [h_k]$ is also in the span of $\mathfrak{B}$. Contradiction.
  2. If $f \in \mathcal{F}^m R_k \subset R_k$, then by the definition of $\mathcal{F}^m R_k$, $f + h \in \mathfrak{a}_m(v_1)$ for some $h \in R$ such that $\mathbf{ld}(f + h) = f$. Since we assumed that $[f] \neq 0 \in R/\mathfrak{a}_m(v_1)$, we have $h \neq 0$ and $k' := \deg(h) < v_0(f) = k$. Now we can decompose h into homogeneous components:

$$h = h^{(k_1)} + \cdots + h^{(k_p)},$$

     with $h^{(k_j)} \in R_{k_j}$ and $k_1 < \cdots < k_p = k'$. Because $k' < k$ and the minimal property of k, we know that each $[h^{(k_j)}]$ in the span of $\mathfrak{B}$. So we have $[f] = [(f + h) - h] = [-h]$ is in

the span of $\mathfrak{B}$. This contradicts our assumption that $[f]$ is not in the span of $\mathfrak{B}$.

□

With the above proposition, we can follow [Li15a] to derive the following volume formula:

$$\begin{aligned}
\mathrm{vol}(v_1) &= \lim_{p\to+\infty}\frac{n!}{m^n}\dim_{\mathbb{C}} R/\mathfrak{a}_m(v_1) \\
&= \frac{H^{n-1}}{c_1^n} - n\int_{c_1}^{+\infty}\mathrm{vol}\left(R^{(t)}\right)\frac{dt}{t^{n+1}} \qquad (4.4.5)\\
&= -\int_{c_1}^{+\infty}\frac{d\mathrm{vol}\left(R^{(t)}\right)}{t^n}. \qquad (4.4.6)
\end{aligned}$$

As in [Li15a], motivated by the case of $\mathbb{C}^*$-invariant valuations, we define a function of two parametric variables $(\lambda, s)\in(0,+\infty)\times[0,1]$:

$$\begin{aligned}
\Phi(\lambda,s) &= \frac{H^{n-1}}{(\lambda c_1 s+(1-s))^n} - n\int_{c_1}^{+\infty}\mathrm{vol}\left(R^{(t)}\right)\frac{\lambda s dt}{(1-s+\lambda s t)^{n+1}} \\
&= \int_{c_1}^{+\infty}\frac{-d\mathrm{vol}\left(R^{(t)}\right)}{((1-s)+\lambda s t)^n}. \qquad (4.4.7)
\end{aligned}$$

Then it's easy to verify that $\Phi(\lambda,s)$ satisfies the following properties:

1. For any $\lambda\in(0,+\infty)$, we have:

$$\Phi(\lambda,1)=\mathrm{vol}(\lambda v_1)=\lambda^{-n}\mathrm{vol}(v_1),\quad \Phi(\lambda,0)=\mathrm{vol}(v_0)=H^{n-1}.$$

2. For fixed $\lambda\in(0,+\infty)$, $\Phi(\lambda,s)$ is continuous and convex with respect to $s\in[0,1]$.

3. The directional derivative of $\Phi(\lambda,s)$ at $s=0$ is equal to:

$$\Phi_s(\lambda,0)=n\lambda H^{n-1}\left(\lambda^{-1}-c_1-\frac{1}{H^{n-1}}\int_{c_1}^{+\infty}\mathrm{vol}\left(R^{(t)}\right)dt\right). \qquad (4.4.8)$$

Roughly speaking, the parameter λ is a rescaling parameter, and s is an interpolation parameter. To apply this lemma to our problem, we let $\lambda=\frac{r}{A_{(\mathcal{C},\mathcal{E})}(v_1)}=:\lambda_*$ such that

$$\Phi(\lambda_*,1)=\frac{A_{(\mathcal{C},\mathcal{E})}(v_1)^n\mathrm{vol}(v_1)}{r^n}=\frac{\widehat{\mathrm{vol}}_{(\mathcal{C},\mathcal{E})}(v_1)}{r^n}.$$

Recall that $\widehat{\mathrm{vol}}_{(\mathcal{C},\mathcal{E})}(v_0)=r^nH^{n-1}=\widehat{\mathrm{vol}}(v_0)$. So the problem of showing $\widehat{\mathrm{vol}}_{(\mathcal{C},\mathcal{E})}(v_1)\geq\widehat{\mathrm{vol}}_{(\mathcal{C},\mathcal{E})}(v_0)$ is equivalent to showing that $\Phi(\lambda_*,1)\geq\Phi(\lambda_*,0)$. By item 2-3 of the above lemma, we just need

to show that $\Phi_s(\lambda_*, 0)$ is non-negative. This will be finally proved in the next section by showing that the derivative $\Phi_s(\lambda_*, 1)$ is nothing but the left-hand-side of (4.3.1) for the case $(X, D) = (\overline{\mathcal{C}}, (1-\beta)V_\infty + \mathcal{E})$.

Here to prepare for this calculation, we first transform the expression in (4.4.8) into a different expression. For convenience, we first define a function:

$$\Theta(t) = n \int_t^{+\infty} \mathrm{vol}(R^{(x)}) \frac{t^n dx}{x^{n+1}}. \tag{4.4.9}$$

Although this seems to be arbitrary at present, this turns out to be a natural function. Indeed, we will see that $\Theta(t)$ is nothing but the $\mathrm{vol}(\mathcal{F}S^{(t)})$ (see (4.4.16)). Notice that by the volume formula (4.4.5), we have:

$$\Theta(c_1) = c_1^n n \int_{c_1}^{+\infty} \mathrm{vol}(R^{(x)}) \frac{dx}{x^{n+1}} = H^{n-1} - c_1^n \mathrm{vol}(v_1). \tag{4.4.10}$$

Let's calculate the integral $\int_{c_1}^{+\infty} \Theta(t)dt$:

$$\begin{aligned}
\int_{c_1}^{+\infty} \Theta(t)dt &= n \int_{c_1}^{+\infty} dt \int_t^{+\infty} \mathrm{vol}\left(R^{(x)}\right) \frac{t^n dt}{x^{n+1}} \\
&= n \int_{c_1}^{+\infty} \mathrm{vol}\left(R^{(x)}\right) \frac{dx}{x^{n+1}} \int_{c_1}^{x} t^n dt \\
&= \frac{n}{n+1} \left(\int_{c_1}^{+\infty} \mathrm{vol}\left(R^{(x)}\right) dx - c_1 \int_{c_1}^{+\infty} \mathrm{vol}\left(R^{(x)}\right) \frac{c_1^n dx}{x^{n+1}} \right) \\
&= \frac{n}{n+1} \int_{c_1}^{+\infty} \mathrm{vol}\left(R^{(x)}\right) dx - \frac{c_1}{n+1} \Theta(c_1).
\end{aligned}$$

So we get the formula:

$$\int_{c_1}^{+\infty} \mathrm{vol}\left(R^{(x)}\right) dx = \frac{n+1}{n} \int_{c_1}^{+\infty} \Theta(t)dt + \frac{c_1}{n} \Theta(c_1). \tag{4.4.11}$$

Dividing the above formula by H^{n-1} and substituting this into the (4.4.8), we get the new expression for the derivative Φ at $s = 0$:

$$\Phi_s(\lambda, 0) = n\lambda H^{n-1} \left(\lambda^{-1} - c_1 - \frac{n+1}{nH^{n-1}} \int_{c_1}^{+\infty} \Theta dt - \frac{\Theta(c_1)}{nH^{n-1}} \right). \tag{4.4.12}$$

Using the formula (4.4.10), we also get another expression:

$$\Phi_s(\lambda, 0) = n\lambda H^{n-1} \left(\lambda^{-1} - c_1 \frac{n+1}{n} - \frac{n+1}{nH^{n-1}} \int_{c_1}^{+\infty} \Theta dt + \frac{c_1^{n+1}}{n} \frac{\mathrm{vol}(v_1)}{H^{n-1}} \right). \tag{4.4.13}$$

4.4.2 Completion of the second proof of Theorem 4.1.1

Let $\overline{\mathcal{C}} = \mathcal{C} \cup V_\infty = \mathrm{Proj}(S)$ be the projective cone as before, where

$$S = \bigoplus_{m=0}^{+\infty} S_m = \bigoplus_{k=0}^{+\infty} H^0(V, mH)$$

and $H = -r^{-1}(K_V + E)$ for $r \in \mathbb{Q}_{>0}$. Recall that we have

$$K_{\overline{\mathcal{C}}} + \mathcal{E} + (1-\beta)V_\infty = -(r+1)V_\infty + (1 - \frac{r}{n})V_\infty = -\frac{r(n+1)}{n}V_\infty.$$

For simplicity of notations, we let $\delta = r(n+1)/n$ and $L = V_\infty$. Notice that $(L^n) = (H^{n-1})$.

Proposition 4.4.3. *We have the following equality:*

$$\Phi_s(\lambda, 0) = n\lambda(H^{n-1}) \left(\lambda^{-1} - \frac{n+1}{n(H^{n-1})} \int_0^{+\infty} \mathrm{vol}\left(\mathcal{F}S^{(t)}\right) dt \right), \tag{4.4.14}$$

where the left-hand-side is the same as in (4.4.8).

Assuming this formula, we can prove our Theorem 4.1.1. Indeed, when $\lambda = \lambda_* = \frac{r}{A_{(\mathcal{C},\mathcal{E})}(v_1)}$, we have:

$$\begin{aligned} \Phi_s(\lambda_*, 0) &= \frac{nr(H^{n-1})}{A_{(\mathcal{C},\mathcal{E})}(v)} \left(\frac{A_{(\mathcal{C},\mathcal{E})}(v)}{r} - \frac{n+1}{n(H^{n-1})} \int_0^{+\infty} \mathrm{vol}\left(\mathcal{F}S^{(t)}\right) dt \right) \\ &= \frac{n(H^{n-1})}{A_{(\mathcal{C},\mathcal{E})}(v)} \left(A_{(\overline{\mathcal{C}},D)}(v) - \frac{\delta}{L^n} \int_0^{+\infty} \mathrm{vol}\left(\mathcal{F}S^{(t)}\right) dt \right), \end{aligned}$$

where $X = \overline{\mathcal{C}}$ and $D = (1-\beta)V_\infty + \mathcal{E}$. By Proposition 4.3.5, the right-hand-side is nonnegative.

The rest of this section is devoted to the proof of the formula (4.4.14). We start with the following observation, which can be seen as a cut-off version of Lemma 4.4.2.

Lemma 4.4.4. *For any $x \in \mathbb{R}$, we have the following identity:*

$$\dim_{\mathbb{C}} \mathcal{F}^x S_m = \sum_{k=0}^{m} \dim_{\mathbb{C}} \mathcal{F}^x R_k. \tag{4.4.15}$$

Proof. For any $k \in [0, m] \cap \mathbb{Z}$, denote $l_k = \dim_{\mathbb{C}} \mathcal{F}^x R_k$ and let $\{f_i^{(k)} | 1 \leq i \leq l_k\}$ be a basis of $\mathcal{F}^x R_k$. By definition of $\mathcal{F}^x R_k$, for each $g_i^{(k)}$ in the basis, there exists $g_i^{(k)} \in R$ such that $v_1(g_i^{(k)}) \geq x$ and $\mathbf{ld}(g_i^{(k)}) = f_i^{(k)}$ (in particular $\deg(g_i^{(k)}) = \deg(f_i^{(k)}) \leq m$). We claim that the set $\{g_i^{(k)} \mid 1 \leq i \leq l_k, 1 \leq k \leq m\}$ is a basis of $\mathcal{F}^x S_m$. Indeed, it's easy to see that this set consists of linearly

independent elements. We just need to show that they span $\mathcal{F}^x S_m$. For any $g \in \mathcal{F}^x S_m$, we decompose g into homogeneous components: $g = g_{k_1} + \cdots + g_{k_p}$ with $g_{k_j} \neq 0 \in R_{k_j}$ and $k_1 < \cdots < k_p = \deg(g) \leq m$. So we have $\mathbf{ld}(g) \in \mathcal{F}^x R_{k_p}$ is in the span of $f_i^{(k_p)}$. So we can find a linear combination h of $g_i^{(k_p)}$ with $\mathbf{ld}(h) = \mathbf{ld}(g)$. Now $\deg(g-h) < \deg(g)$ and $v_1(g-h) \geq \min\{v_1(g), v_1(h)\} \geq x$. In other words $g - h \in \mathcal{F}^x S_m$ is of strictly smaller degree compared to g. Now we can use induction to finish the proof. □

Using (4.4.15) and Lemma 4.5 in [Li15a], we get the formula for $\mathrm{vol}(\mathcal{F}S^{(x)})$:

$$\mathrm{vol}\left(\mathcal{F}S^{(x)}\right) = n\int_x^{+\infty} \mathrm{vol}\left(R^{(t)}\right)\frac{x^n dt}{t^{n+1}} = \Theta(x), \tag{4.4.16}$$

where we used the previously defined function $\Theta(t)$ in (4.4.9).

Corollary 4.4.5. *The following holds for* $\mathrm{vol}(\mathcal{F}S^{(x)})$*:*

1. *If* $x \geq c_2$*, then* $\mathrm{vol}(\mathcal{F}S^{(x)}) = 0$.

2. *For any* $x \in \mathbb{R}$*, we have:*

$$\mathrm{vol}(\mathcal{F}S^{(x)}) \geq -x^n \mathrm{vol}(v_1) + H^{n-1}$$

The equality holds if $x \leq c_1$.

Proof. If $x \leq c_1$, then with the help of the volume formula (4.4.5), we get:

$$\begin{aligned}
\mathrm{vol}(\mathcal{F}S^{(x)}) &= nx^n \int_{c_1}^{+\infty} \mathrm{vol}(R^{(t)})\frac{dt}{t^{n+1}} + nx^n H^{n-1}\int_x^{c_1}\frac{dt}{t^{n+1}} \\
&= x^n\left(\frac{H^{n-1}}{c_1^n} - \mathrm{vol}(v_1)\right) + H^{n-1}x^n\left[-t^{-n}\right]_x^{c_1} \\
&= x^n\left(\frac{H^{n-1}}{c_1^n} - \mathrm{vol}(v_1)\right) + H^{n-1}\left(-\frac{x^n}{c_1^n} + 1\right) \\
&= -x^n \mathrm{vol}(v_1) + H^{n-1}.
\end{aligned}$$

If $x \geq c_1$ then because $\mathrm{vol}(R^{(t)}) \leq H^{n-1}$, we have:

$$\begin{aligned}
\mathrm{vol}\left(\mathcal{F}S^{(x)}\right) &= n\int_x^{+\infty} \mathrm{vol}\left(R^{(t)}\right)\frac{x^n dt}{t^{n+1}} = nx^n\left(\int_{c_1}^{+\infty} - \int_{c_1}^{x}\right) \\
&\geq x^n\left(\frac{H^{n-1}}{c_1^n} - \mathrm{vol}(v_1)\right) - x^n H^{n-1}\left[-t^{-n}\right]_{c_1}^{x} \\
&= x^n\left(\frac{H^{n-1}}{c_1^n} - \mathrm{vol}(v_1)\right) + H^{n-1}\left(1 - \frac{x^n}{c_1^n}\right) \\
&= H^{n-1} - \mathrm{vol}(v_1)x^n.
\end{aligned}$$

□

Using the above lemma, we can derive a key ingredient in our formula:

$$\begin{aligned}\int_0^{+\infty}\mathrm{vol}(\mathcal{F}S^{(x)})dx &= \left(\int_0^{c_1}+\int_{c_1}^{+\infty}\right)\mathrm{vol}(\mathcal{F}S^{(x)})dx\\ &= -\mathrm{vol}(v_1)\int_0^{c_1}x^n dx + H^{n-1}c_1 + \int_{c_1}^{+\infty}\mathrm{vol}(\mathcal{F}S^{(x)})dx\\ &= -\frac{\mathrm{vol}(v_1)}{n+1}c_1^{n+1} + c_1H^{n-1} + \int_{c_1}^{+\infty}\Theta(x)dx\\ &= \left(-\frac{c_1^{n+1}}{n+1}\mathrm{vol}(v_1) + c_1H^{n-1} + \int_{c_1}^{+\infty}\Theta(t)dt\right).\end{aligned}$$

The proof of (4.4.14) is finally at our hand, because we can transform the expression in the bracket of (4.4.13) to become:

$$\begin{aligned}&\lambda^{-1} - c_1\frac{n+1}{n} - \frac{n+1}{nH^{n-1}}\int_{c_1}^{+\infty}\Theta dt + \frac{c_1^{n+1}}{n}\frac{\mathrm{vol}(v_1)}{H^{n-1}}\\ =\ &\lambda^{-1} - \frac{n+1}{nH^{n-1}}\left(c_1H^{n-1} + \int_{c_1}^{+\infty}\Theta dt - \frac{c_1^{n+1}}{n+1}\mathrm{vol}(v_1)\right)\\ =\ &\lambda^{-1} - \frac{n+1}{nH^{n-1}}\int_0^{+\infty}\mathrm{vol}(\mathcal{F}S^{(x)})dx\\ =\ &\lambda^{-1} - \frac{n+1}{n}\frac{1}{H^{n-1}}\int_0^{+\infty}\mathrm{vol}(\mathcal{F}S^{(x)})dx.\end{aligned}$$

4.5 Valuative criteria for K-semistability of $\mathbb{Q}$-Fano varieties

In this section, we will discuss valuative criteria for K-semistability of $\mathbb{Q}$-Fano varieties.

4.5.1 A singular version of Corollary 4.1.5

The following theorem, very recently proved by Li and Xu in [LX17], generalizes Corollary 4.1.5 to singular cases.

Theorem 4.5.1 (Li-Xu [LX17])**.** *Let V be a $\mathbb{Q}$-Fano variety. Let $H = -r^{-1}K_V$ be an ample Cartier divisor for an $r \in \mathbb{Q}$. Denote the affine cone by $\mathcal{C} := C(V,H) = \mathbf{Spec}\ \bigoplus_{k=0}^{\infty}H^0(V,kH)$ with cone vertex o. Then V is K-semistable if and only if $\widehat{\mathrm{vol}}$ is minimized at ord_V in $\mathrm{Val}_{\mathcal{C},o}$.*

To prove the (only if part of the) theorem above, we need to use the argument of degeneration to initial ideals discussed in [Mus02, Li15a, Blu16a, LX16]. Denote the graded ring by $R = \oplus_{k=0}^{\infty}R_k := \bigoplus_{k=0}^{\infty}H^0(V,kH)$. The grading on R defines a natural valuation $v_0 = \mathrm{ord}_V$. Let $\mathfrak{m}_R := \oplus_{k\geq 1}R_k =$

$\mathfrak{a}_1(v_0)$ be the homogeneous maximal ideal of R. Let $g \in R$ be any element. Then g can be decomposed into homogeneous component $g = g_{k_1} + \cdots + g_{k_p}$ with $0 \neq g_{k_j} \in R_{k_j}$ and $v_0(g) = k_1 < k_2 < \cdots < k_p$. Recall that the *leading component* of g is defined as $\mathbf{ld}(g) := g_{k_1}$. We define an element $\tilde{g}$ in $\mathcal{R} := R[t]$ to be

$$\tilde{g} := g_{k_1} + t^{k_2-k_1} g_{k_2} + \cdots + t^{k_p-k_1} g_{k_p}.$$

Given any ideal $\mathfrak{a}$ of R, we define the *initial ideal* $\mathbf{in}(\mathfrak{a})$ of $\mathfrak{a}$ to be the ideal generated by $\mathbf{ld}(g)$ for all $g \in \mathfrak{a}$. It is clear that initial ideals are homogeneous ideals of R. We define $\mathfrak{A}$ to be the ideal of $\mathcal{R}$ generated by $\tilde{g}$ for all $g \in \mathfrak{a}$.

Lemma 4.5.2 ([LX16, Lemma 3.1])**.** *1. We have the identities:*

$$\mathcal{R}/\mathfrak{A} \otimes_{\mathbb{C}[t]} \mathbb{C}[t, t^{-1}] \cong (R/\mathfrak{a})[t, t^{-1}], \quad \mathcal{R}/\mathfrak{A} \otimes_{\mathbb{C}[t]} \mathbb{C}[t]/(t) \cong R/\mathbf{in}(\mathfrak{a});$$

2. The $\mathbb{C}[t]$-algebra $\mathcal{R}/\mathfrak{A}$ is free and thus flat as a $\mathbb{C}[t]$-module. In particular, we have the identity of dimensions:

$$\dim_{\mathbb{C}}(R/\mathfrak{a}) = \dim_{\mathbb{C}}(R/\mathbf{in}(\mathfrak{a})); \tag{4.5.1}$$

3. If $\mathfrak{a}$ is $\mathfrak{m}_R$-primary, then $\mathbf{in}(\mathfrak{a})$ is an $\mathfrak{m}_R$-primary homogeneous ideal.

Proof. For part 1, it is clear that $\mathcal{R}/\mathfrak{A} \otimes_{\mathbb{C}[t]} \mathbb{C}[t, t^{-1}] = R[t, t^{-1}]/\mathfrak{A}[t^{-1}]$. Define a $\mathbb{C}[t, t^{-1}]$-algebra automorphism $\phi : R[t, t^{-1}] \to R[t, t^{-1}]$ by $\phi(g_k) := t^{-k} g$ for any $g \in R_k$. Then it is clear that $\phi(\tilde{g}) = t^{-v_0(g)} g$ for any $g \in R$. Hence $\phi(\mathfrak{A})[t^{-1}] = \mathfrak{a}[t, t^{-1}]$ which implies the first identity. The second identity follows directly from definition.

Next we prove part 2. Denote by $\mathfrak{b}_k = R_k \cap \mathbf{in}(\mathfrak{a})$ the k-th homogeneous piece of $\mathbf{in}(\mathfrak{a})$. We fix a basis $\left\{ [f_i^{(k)}]_{\mathfrak{b}_k} : 1 \leq i \leq d_k \right\}$ of $R_k/\mathfrak{b}_k$ with $f_i^{(k)} \in R_k$. We want to show that

$$\mathcal{A}' := \left\{ \left[\widetilde{f_i^{(k)}} \right] = \left[f_i^{(k)} \right]_{\mathfrak{A}} \;\middle|\; k \in \mathbb{Z}_{\geq 0}, 1 \leq i \leq d_k \right\} \subset \mathcal{R}/\mathfrak{A}$$

is a $\mathbb{C}[t]$-basis of $\mathcal{R}/\mathfrak{A}$.

We first verify that $\mathcal{A}'$ is a linearly independent set. To prove this, we just need to show that $\phi(\mathcal{A}')$ (or equivalently $\mathcal{A}'$ by the homogeneity of $f_i^{(k)}$) is a $\mathbb{C}[t, t^{-1}]$-linearly independent subset of

$(R/\mathfrak{a})[t,t^{-1}]$. It is then enough to show that

$$\mathcal{A} := \left\{ [f_i^{(k)}] = [f_i^{(k)}]_{\mathfrak{a}} \mid k \in \mathbb{Z}_{\geq 0}, 1 \leq i \leq d_k \right\} \subset R/\mathfrak{a}. \tag{4.5.2}$$

is $\mathbb{C}$-linearly independent, which can be verified directly as in [Li15a, Proposition 4.3]. See also [Eis95, Proposition 15.3].

So we just need to show that $\mathcal{A}'$ spans $\mathcal{R}/\mathfrak{A}$. Since any element in $\mathcal{R}$ can be expressed as a sum of $t^m\tilde{f}$'s for $m \in \mathbb{Z}_{\geq 0}$ and $f \in R$, we need to show that for any $f \in R$, $[\tilde{f}] = [\tilde{f}]_{\mathfrak{A}} \in \mathcal{R}/\mathfrak{A}$ is in the $\mathbb{C}[t]$-span of $\mathcal{A}'$. This can be shown again with the help of $\mathcal{A}$ in (4.5.2), that is, it is enough to prove that $\mathcal{A}$ $\mathbb{C}$-spans $R/\mathfrak{a}$. Indeed, assuming the latter, for any $f \in R$, there exists a linear combination $g = \sum_{i,k} c_{ik} f_i^{(k)}$ such that $f - g =: h \in \mathfrak{a}$. Let k_0 be the smallest k such that $c_{ik} \neq 0$ for some i. We claim that $k_0 \geq v_0(f)$. Assume to the contrary that $k_0 < v_0(f)$. Then $v_0(h) = k_0$. By taking the degree k_0 parts, we have that $\sum_i c_{ik_0} f_i^{(k_0)} = -\mathbf{ld}(h) \in \mathfrak{b}_{k_0}$, a contradiction! As a result, $\min\{k_0, v_0(h)\} = v_0(f)$. Thus $\tilde{f}$ can be expressed as a $\mathbb{C}[t]$-linear combination of $\{f_i^{(k)}\}$ and $\tilde{h}$ which implies that $[\tilde{f}]$ is in the $\mathbb{C}[t]$-span of $\mathcal{A}'$.

To prove that $\mathcal{A}$ indeed $\mathbb{C}$-spans $R/\mathfrak{a}$, we first claim that the following set is finite:

$$\{v_0(g) \mid g \in R \setminus \mathfrak{a}\}.$$

Indeed because $\mathfrak{a}$ is $\mathfrak{m}$-primary, there exists $N > 0$ such that $\mathfrak{m}^N \subseteq \mathfrak{a} \subseteq \mathfrak{m}$. So $R \setminus \mathfrak{a} \subseteq R \setminus \mathfrak{m}^N$. Now the claim follows from the fact that for any element $f \in \mathfrak{m}^N$, $v_0(f) \leq cA_{\mathcal{C}}(v_0) \cdot N$ by Izumi's theorem, where c is a uniform constant not depending on f.

If there is $0 \neq [f] \in R/\mathfrak{a}$ that is not in the span of $\mathcal{A}$, then we can choose a maximal $k = v_0(f)$ such that this happens. There are two cases:

(1) If $\mathbf{ld}(f) \in R_k \setminus \mathfrak{b}_k$, then because $\mathbf{ld}(f_i^{(k)})$ is a basis of $R_k/\mathfrak{b}_k$, there exists $t_j \in \mathbb{R}$ such that $\mathbf{ld}(f) - \sum_{j=1}^{d_k} t_j \mathbf{ld}(f_j^{(k)}) = \mathbf{ld}(g) \in \mathfrak{b}_k$ for some $g \in \mathfrak{a}$. So we get:

$$v_0\left(f - \sum_{j=1}^{d_k} t_j f_j^{(k)} - g\right) > k.$$

By maximality of k, $[f - \sum_{j=1}^{d_k} t_j f_j^{(k)} - g] = [f] - \sum_{j=1}^{d_k} t_j [f_j^{(k)}]$ is in the span of $\mathcal{A}$. Hence $[f]$ is in the span of $\mathcal{A}$. Contradiction.

(2) If $\mathbf{ld}(f) \in \mathfrak{b}_k = \mathbf{in}(\mathfrak{a}) \cap R_k$. Then $\mathbf{ld}(f) = \mathbf{ld}(g)$ for some $g \in \mathfrak{a}$. So $v_0(f-g) > k$ and hence

$[f-g]$ is in the span of $\mathcal{A}$ by the maximal property of k. But then $[f]=[f-g]+[g]=[f-g]$ is in the span of $\mathcal{A}$. Contradiction.

To prove part 3 of the Lemma, we need to show that there exists $N\in\mathbb{Z}_{>0}$ such that $\mathfrak{m}_R^N\subseteq \mathbf{in}(\mathfrak{a})\subseteq\mathfrak{m}_R$. Because $\mathfrak{a}$ is $\mathfrak{m}_R$ primary, there exists $N_1\in\mathbb{Z}_{>0}$ such that $\mathfrak{m}_R^{N_1}\subseteq\mathfrak{a}\subseteq\mathfrak{m}_R$. By Izumi's theorem, there exists $l\in\mathbb{Z}_{>0}$ such that $\mathfrak{a}_{lm}(v_0)\subseteq\mathfrak{m}_R^m$ for any $m\in\mathbb{Z}_{>0}$. By letting $N=lN_1$, it's easy to see that $\mathfrak{m}_R^N\subseteq\mathbf{in}(\mathfrak{a})\subseteq\mathfrak{m}_R$. □

Lemma 4.5.3 ([LX16, Lemma 3.3 and 3.4])**.** *Let $\mathfrak{a}_\bullet$ be a graded sequence of $\mathfrak{m}_R$-primary ideals. Then $\mathbf{in}(\mathfrak{a}_\bullet)$ is also a graded sequence of of $\mathfrak{m}_R$-primary ideals, and*

$$\mathrm{lct}(\mathfrak{a}_\bullet)^n\cdot\mathrm{mult}(\mathfrak{a}_\bullet)\geq\mathrm{lct}(\mathbf{in}(\mathfrak{a}_\bullet))^n\cdot\mathrm{mult}(\mathbf{in}(\mathfrak{a}_\bullet)). \tag{4.5.3}$$

Proof. For any $f\in\mathfrak{a}_k$ and $g\in\mathfrak{a}_l$, it is clear that $\mathbf{ld}(f)\mathbf{ld}(g)=\mathbf{ld}(fg)\in\mathbf{in}(\mathfrak{a}_{k+l})$. Hence $\mathbf{in}(\mathfrak{a}_k)\mathbf{in}(\mathfrak{a}_l)\subset\mathbf{in}(\mathfrak{a}_{k+l})$ which implies that $\mathbf{in}(\mathfrak{a}_\bullet)$ is a graded sequence of homogeneous ideals. By Lemma 4.5.2 we know that $\mathbf{in}(\mathfrak{a}_k)$ is $\mathfrak{m}_R$-primary for each k.

By the flatness of $\mathfrak{A}$ (cf. Lemma 4.5.2) and the lower semicontinuity of log canonical thresholds, we have $\mathrm{lct}(\mathfrak{a}_k)\geq\mathrm{lct}(\mathbf{in}(\mathfrak{a}_k))$. Therefore, by (4.5.1) we have

$$\mathrm{lct}(\mathfrak{a}_k)^n\cdot\ell(R/\mathfrak{a}_k)\geq\mathrm{lct}(\mathbf{in}(\mathfrak{a}_k))^n\cdot\ell(R/\mathbf{in}(\mathfrak{a}_k)).$$

Taking limits as $k\to\infty$, we get the desired inequality (4.5.3). □

Proof of Theorem 4.5.1. It was proved in [Li15a, Theorem 3.1] that V is K-semistable if and only if $\widehat{\mathrm{vol}}$ is minimized at ord_V among all $\mathbb{C}^*$-equivariant real valuations in $\mathrm{Val}_{\mathcal{C},o}$. Thus it suffices to show that

$$\inf_{\mathrm{Val}_{\mathcal{C},o}}\widehat{\mathrm{vol}}=\inf_{\mathrm{Val}_{\mathcal{C},o}^{\mathbb{C}^*}}\widehat{\mathrm{vol}},$$

where $\mathrm{Val}_{\mathcal{C},o}^{\mathbb{C}^*}$ is the set of $\mathbb{C}^*$-equivariant real valuations centered at o. Since $\mathrm{Val}_{\mathcal{C},o}^{\mathbb{C}^*}\subset\mathrm{Val}_{\mathcal{C},o}$, it is clear that $\inf_{\mathrm{Val}_{\mathcal{C},o}}\widehat{\mathrm{vol}}\leq\inf_{\mathrm{Val}_{\mathcal{C},o}^{\mathbb{C}^*}}\widehat{\mathrm{vol}}$. Moreover, Theorem 2.3.5 implies that $\inf_{\mathrm{Val}_{\mathcal{C},o}}\widehat{\mathrm{vol}}=\inf_{\mathfrak{a}}\mathrm{lct}(\mathfrak{a})^n\mathrm{mult}(\mathfrak{a})$ where the second infimum is taken over all $\mathfrak{m}_R$-primary ideals $\mathfrak{a}$. Hence it suffices to show that $\mathrm{lct}(\mathfrak{a})^n\mathrm{mult}(\mathfrak{a})\geq\inf_{\mathrm{Val}_{\mathcal{C},o}^{\mathbb{C}^*}}\widehat{\mathrm{vol}}$ for any $\mathfrak{m}_R$-primary ideal $\mathfrak{a}$.

By applying Lemma 4.5.3 to the graded sequence $(\mathfrak{a}^k)_{k\geq0}$, we have

$$\mathrm{lct}(\mathfrak{a})^n\cdot\mathrm{mult}(\mathfrak{a})\geq\limsup_{k\to\infty}\mathrm{lct}(\mathbf{in}(\mathfrak{a}^k))^n\cdot\mathrm{mult}(\mathbf{in}(\mathfrak{a}^k)). \tag{4.5.4}$$

Since $\mathbf{in}(\mathfrak{a}^k)$ is a homogeneous ideal for any k, we can always take a $\mathbb{C}^*$-equivariant log resolution of $(X, \mathbf{in}(\mathfrak{a}^k))$ (see [Kol07]). As a result, there exists a $\mathbb{C}^*$-equivariant divisorial valuation v_k that computes $\mathrm{lct}(\mathbf{in}(\mathfrak{a}^k))$. By Lemma 2.3.4, we know that v_k is centered at o, and $\mathrm{lct}(\mathbf{in}(\mathfrak{a}^k))^n \cdot \mathrm{mult}(\mathbf{in}(\mathfrak{a}^k)) \geq \widehat{\mathrm{vol}}(v_k)$. Combining this with (4.5.4) yields

$$\mathrm{lct}(\mathfrak{a})^n \cdot \mathrm{mult}(\mathfrak{a}) \geq \limsup_{k\to\infty} \widehat{\mathrm{vol}}(v_k) \geq \inf_{\mathrm{Val}_{C,o}^{\mathbb{C}^*}} \widehat{\mathrm{vol}}.$$

Hence we finish the proof. □

4.5.2 A result of Fujita and Li

Let us first recall the definition of volume functions associated to a prime divisor.

Definition 4.5.4 ([Fuj16])**.** Let X be a $\mathbb{Q}$-Fano variety of dimension n. Take any projective birational morphism $\sigma : Y \to X$ with Y normal and any prime divisor F on Y, that is, F is a prime divisor over X. Take any $k \in \mathbb{Z}_{\geq 0}$ with $-kK_X$ Cartier and take any $x \in \mathbb{R}_{\geq 0}$.

(a) We define the sub vector space $H^0(X, -kK_X - xF)$ of the vector space $H^0(X, -K_X)$ as

$$\begin{aligned} & H^0(X, -K_X - xF) := H^0(Y, \mathcal{O}_Y(\sigma^*(-kK_X) - \lceil xF \rceil)) \\ \subset \quad & H^0(Y, \mathcal{O}_Y(\sigma^*(-kK_X))) = H^0(X, \mathcal{O}_X(-kK_X)). \end{aligned}$$

(b) We define

$$\mathrm{vol}_X(-K_X - xF) := \mathrm{vol}_Y(\sigma^*(-K_X) - xF) = \limsup_{k\to\infty,\, kK_X \text{ Cartier}} \frac{\dim H^0(X, -kK_X - kxF)}{k^n/n!}.$$

By [Laz04a, Corollary 2.2.45] and [Laz04b, Example 11.4.7], the lim sup is actually a limit and the function $\mathrm{vol}_X(-K_X - xF)$ is a monotonically decreasing continuous function for $x \in \mathbb{R}_{\geq 0}$. Besides, $\mathrm{vol}_X(-K_X - xF)$ only depends on the divisorial valuation ord_F on X, and it does not depend on choice of $\sigma : Y \to X$. (See [Fuj16, Section 2.2].)

The following valuative criterion of K-semistability is due to Fujita [Fuj16] and Li [Li15a].

Theorem 4.5.5 ([Fuj16, Li15a])**.** *Let X be a $\mathbb{Q}$-Fano variety. Then X is K-semistable if and only if every prime divisor F over X satisfies*

$$A_X(F) \geq \frac{1}{\mathrm{vol}_X(-K_X)} \int_0^{+\infty} \mathrm{vol}_X(-K_X - xF)dx.$$

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