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Doctoral Dissertations, 2011-2026

Mathematics

2019

Some Problems in Four-dimensional Conformal Geometry

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Princeton, NJ : Princeton University

<http://arks.princeton.edu/ark:/88435/dsp018g84mq13p>

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SOME PROBLEMS IN FOUR-DIMENSIONAL CONFORMAL GEOMETRY

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A DISSERTATION
PRESENTED TO THE FACULTY
OF PRINCETON UNIVERSITY
IN CANDIDACY FOR THE DEGREE
OF DOCTOR OF PHILOSOPHY

RECOMMENDED FOR ACCEPTANCE
BY THE DEPARTMENT OF
MATHEMATICS
ADVISER: SUN-YUNG ALICE CHANG

JUNE 2019

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Abstract

In this thesis, we study some problems in four-dimensional conformal geometry. This thesis consists of two main parts: conformally invariant characterization of $\mathbb{CP}^2$ and conformally invariant gap theorems for Bach-flat metrics.

In the first part, we extend the sphere theorem of [8] to give a conformally invariant characterization of $(\mathbb{CP}^2, g_{FS})$. In particular, we introduce a conformal invariant $\beta(M^4, [g]) \geq 0$ defined on conformal four-manifolds satisfying a positivity condition; it follows from [8] that if $0 \leq \beta(M^4, [g]) < 4$, then M^4 is diffeomorphic to S^4 . Our main result is a gap result showing that if $b_2^+(M^4) > 0$ and $4 \leq \beta(M^4, [g]) < 4(1 + \epsilon)$ for $\epsilon > 0$ small enough, then M^4 is diffeomorphic to $\mathbb{CP}^2$. The Ricci flow is used in a crucial way to pass from the bounds on β to pointwise curvature information. We also prove a lower bound for $\beta(M)$ under some topological conditions.

In the second part, we extend a conformal gap theorem for Bach-flat metrics with round sphere as model case established in [10] to prove conformally invariant gap theorems for Bach-flat 4-manifolds with $(\mathbb{CP}^2, g_{FS})$ and $(S^2 \times S^2, g_p)$ as model cases. A Moser-type iteration argument plays an important role in the case of $(\mathbb{CP}^2, g_{FS})$ and the convergence theory of Bach-flat metrics is of particular importance in the case of $(S^2 \times S^2, g_p)$.

Acknowledgements

First and foremost, I would like to express my deep gratitude to my advisor Professor Sun-Yung Alice Chang. It would be extremely hard to list her unwavering support and constant encouragements. This thesis would never be completed without her help and guidance. Both her thoughts on mathematics and attitude towards life will have a long-lasting influence on my career.

With great honor and pleasure, I wish to thank Professor Charles Fefferman for being my thesis reader, Professor Paul Chien-Ping Yang and Professor Fernando Codá Marques for serving on my Final Public Oral committee.

I am grateful to many friends, classmates, collaborators, and teachers. Many of them have great influence on my research and career.

I wish to thank Professor Matthew Gursky for teaching me many techniques in geometric analysis and providing deep insights to conformal geometry. My joint work with him and Professor Chang became my first step into conformal geometry and geometric analysis. I thank Professor Fernando Codá Marques for introducing many techniques from calculus of variations to me. I thank Professor Gang Tian for teaching me many aspects of Ricci flow in his topic class at Princeton. I thank Professor Paul Chien-Ping Yang for explaining several ideas from CR geometry.

I thank Professor Valentino Tosatti, Professor Zeno Huang, Professor Xiaodong Cao, and Professor Yi Wang for hosting my visits to many institutes and greatly broadening my horizon of mathematics. I am also deeply thankful to many professors from Peking University who have led me to the world of geometric analysis. My thanks go particularly to Professor Qing Han, Professor Jie Qing, and Professor Yuguang Shi.

My thanks go to Jeffrey Case, Eric Chen, Otis Chodosh, Alexandros Eskenazis, Fabian Gundlach, Casey Kelleher, Chao Li, Stephen McKeown, Antoine Song, Daniel Stern, Daniel Vitek, Ruixiang Zhang, Ruobing Zhang, Fan Zheng, and Ziquan Zhuang for many stimulating discussions about math and life during my years at Princeton.

I am grateful to Lan-Hsuan Huang, Qingyuan Jiang, Lu Wang, Zihui Zhao, and Xin Zhou for sharing many enjoyable moments during the IAS special year.

Last but not least, I wish to show my deepest gratitude to my parents and my family for providing me with an amazing environment to complete my Ph.D study.

To my parents.

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Part I

Introduction and Background

Chapter 1

Introduction

In this chapter, we shall have a rapid review of the history of several problems in the study of conformal geometry. The main theme is to understand the geometry and topology of smooth four-manifolds from the study of partial differential equations arising from conformal invariants in four dimensions. After the review, we shall state the main theorems of this thesis and discuss the main techniques to prove them.

1.1 History

The famous uniformization theorem implies that in any closed compact Riemann surface X with metric g_0 there exists a conformal metric $g = e^{2w}g_0$ such that g has constant Gaussian curvature. In two dimensions, this shows that on X , there is a *canonical* metric to understand the geometry and topology. There are many attempts to generalize the uniformization theorem in higher dimensions. However, due to the complexity of Riemannian curvature tensor in higher dimensions, only partial understanding is available now.

In conformal geometry, the Yamabe problem is arguably the best-known generalization of uniformization theorem in higher dimensions. Suppose (M^n, g_0) is a closed compact Riemannian manifold of dimension $n \geq 3$. The Yamabe problem asks if there

exists a conformal metric $g = e^{2w}g_0$ with constant scalar curvature, i.e., $R_g = \text{const.}$ After the endeavors of N. Trudinger, T. Aubin, and R. Schoen, the Yamabe problem has been completely solved affirmatively. Interestingly, there is *no* obstruction for the existence of such a metric if there is no restriction on the sign of the Yamabe invariant.

After almost twenty years of the resolution to the Yamabe problem, J. Viaclovsky [44] initiated the study of σ_k -Yamabe problem which includes the Yamabe problem as a special case as σ_1 -Yamabe problem. In four dimensions, the σ_2 -Yamabe problem turns out to be fairly interesting since the existence of a metric with positive scalar curvature and positive σ_2 -curvature will impose topological restrictions on the underlying manifold. In other words, there is *obstruction* to the existence of such a metric. The following theorems established by S.-Y. A. Chang, M. Gursky, and P. Yang are of particular importance in the study of σ_2 -curvature.

Theorem 1.1.1 ([6]). *Let (M^4, g_0) be a closed, compact four-dimensional manifold satisfying*

1. $Y(M, [g_0]) > 0$,
2. $\int_M \sigma(P_0) dv_{g_0} > 0$.

Then there is a conformal metric $g = e^{2w}g_0$ with $\sigma_2(P_g) > 0$ and $R_g > 0$. In particular, M^4 must have finite fundamental group and vanishing first Betti number.

Theorem 1.1.2 ([7]). *Let (M^4, g_0) be a closed, compact four-dimensional manifold which is not conformally equivalent to the round sphere satisfying*

1. $R_0 > 0$,
2. $\sigma_2(P_0) > 0$.

Then there is a conformal metric $g = e^{2w}g_0$ with $\sigma_2(P_g) = 1$ and $R_g > 0$.

Theorem 1.1.3 ([8]). *Let M^4 be an oriented, closed, compact manifold admitting a metric g with $Y(M, [g]) > 0$ and*

$$0 < \int ||W_g||_g^2 dv_g < 4\sigma_2(P_g)dv_g, \quad (1.1)$$

then M^4 is diffeomorphic to S^4 . Furthermore, if M^4 admits a metric with $\int ||W_g||_g^2 dv_g = 4\sigma_2(P_g)dv_g$, then one of the following must hold:

- *M^4 is diffeomorphic to S^4 ; or*
- *M^4 is diffeomorphic to $\mathbb{CP}^2$ and $g = e^{2w}g_{FS}$, where g_{FS} denotes the Fubini-Study metric.*

With Bach-flat condition, S.-Y. A. Chang, J. Qing, and P. Yang in [10] proved a conformally invariant rigidity theorem with round sphere as model case. The proof has been refined and simplified in [34] by G. Li, J. Qing, and Y. Shi.

Theorem 1.1.4 ([10][34]). *Let (M^4, g) be a closed, compact Bach-flat four-manifold. There is an $\epsilon > 0$ such that if $Y(M, [g]) > 0$ with*

$$\int \sigma_2(P_g)dv_g \geq 4(1 - \epsilon)\pi^2, \quad (1.2)$$

then M is diffeomorphic to S^4 and $g = e^{2w}g_c$, where g_c is the round metric.

Remark 1.1.5. *Theorem 1.1.3 is generally referred to as a conformally invariant sphere theorem in four dimensions. Note that the pinching condition (1.1) is sharp and the model case is diffeomorphic S^4 .*

Remark 1.1.6. *Theorem 1.1.4 is generally referred to as a conformally invariant rigidity theorem in four dimensions. Note that the model case is round S^4 .*

1.2 Organization of the thesis

In this section, we outline the organization of this thesis. In Chapter 2, we will have a review of the preliminary knowledge of conformal geometry, which will be used in the rest of this thesis. Part II consists of Chapter 3 and Chapter 4, and is mainly about conformally invariant diffeomorphism theorems. In Chapter 3 we state the main theorems and discuss the ideas of proofs. In Chapter 4 we give detailed proofs of main results stated in Chapter 3. Part III consists of Chapter 5 and Chapter 6, and is mainly about conformally invariant rigidity theorems. In Chapter 5 we state the main theorems and discuss the ideas of proofs. In Chapter 6 we give detailed proofs of main results stated in Chapter 5. In the end, a short chapter will summarize some open problems in four-dimensional conformal geometry.

Chapter 2

Background

In this chapter, we review basic notions and preliminary results which will be used in the proof of main theorems.

2.1 Decomposition of curvature

If (M^4, g) is a smooth, closed Riemannian four-manifold, we denote the Riemannian curvature tensor by Rm (or Rm_g if we need to specify the metric), the Ricci tensor by Ric , and the scalar curvature by R . We also denote the Weyl curvature tensor by W , and the Schouten tensor

$$P = \frac{1}{2}(Ric - \frac{1}{6}R \cdot g). \quad (2.1)$$

We remark that the definition of the Schouten tensor in [8] (denoted by A) differed from the formula in (2.1) by a factor of two; however, in this paper we adopt the more common convention. In terms of the Weyl and Schouten tensors the Riemannian curvature tensor can be decomposed as

$$Rm = W + P \oslash g \quad (2.2)$$

where $\otimes$ is the Kulkarni-Nomizu product. There are two important consequences of this identity: First, since the Weyl tensor is conformally invariant, it follows that the behavior of the curvature tensor under a conformal change of metric is determined by the transformation of the Schouten tensor. The second consequence is that the splitting induces a splitting of the Euler form, so that the Chern-Gauss-Bonnet formula can be expressed as

$$8\pi^2\chi(M) = \int \|W\|^2 dv + 4 \int \sigma_2(g^{-1}P) dv, \quad (2.3)$$

where $\bullet \| \cdot \|$ denotes the norm of the Weyl tensor, viewed as an endomorphism of $\Omega^2(M)$, the bundle of two-forms. Note that this differs from the norm of Weyl when viewed as a four-tensor, and the two norms are related by

$$\|W\| = \frac{1}{4}|W|^2.$$

• $g^{-1}P$ denotes the $(1,1)$ -tensor (interpreted as an endomorphism of the tangent space at each point) obtained by ‘raising an index’ of the Schouten tensor, and $\sigma_2(g^{-1}P)$ is the second elementary symmetric polynomial applied to its eigenvalues. To simplify notation we will henceforth write $\sigma_2(P)$ in place of $\sigma_2(g^{-1}P)$.

If we further assume that M^4 is oriented, then the Weyl tensor can be further decomposed into the self-dual/anti-self-dual part which we denote by $W^\pm$ respectively. The Hirzebruch signature formula can be written as

$$12\pi^2\tau(M) = \int \|W^+\|^2 dv - \int \|W^-\|^2 dv, \quad (2.4)$$

where $\tau(M) = b_2^+(M) - b_2^-(M)$ is the signature of M^4 .

2.2 Conformal invariants

It follows from the conformal invariance of the Weyl tensor that both integrals in (2.3) are conformally invariant. While their sum is a topological invariant, their ratio can be arbitrary. As we now explain, when the scalar curvature is positive the ratio does carry geometric and topological information.

Given a Riemannian manifold (M^n, g) of dimension $n \geq 3$, let $[g]$ denote the equivalence class of metrics pointwise conformal to g , and $Y(M^n, [g])$ denote the Yamabe invariant:

$$Y(M^n, [g]) = \inf_{\tilde{g} \in [g]} \text{Vol}(\tilde{g})^{-\frac{n-2}{n}} \int R_{\tilde{g}} dv_{\tilde{g}}.$$

We can also express the Yamabe invariant in terms of the first symmetric function of the Schouten tensor: it follows from (2.1) that

$$\sigma_1(P) = \frac{R}{2(n-1)},$$

hence

$$Y(M^n, [g]) = \inf_{\tilde{g} \in [g]} 2(n-1) \text{Vol}(\tilde{g})^{-\frac{n-2}{n}} \int \sigma_1(P_{\tilde{g}}) dv_{\tilde{g}}.$$

With this interpretation of the Yamabe invariant, in dimension four we should view the conformal invariant $\int \sigma_2(P) dv$ as a kind of “second Yamabe invariant” (see [19], [40], [18]). We therefore define

$$\mathcal{Y}_1^+(M^4) = \{g : Y(M^4, [g]) > 0\}, \tag{2.5}$$

and

$$\mathcal{Y}_2^+(M^4) = \{g \in \mathcal{Y}_1^+(M) : \int \sigma_2(P_g) dv_g > 0\}. \quad (2.6)$$

By a classical result of Lichnerowicz, there are topological obstructions to $\mathcal{Y}_1(M^4)$ being non-empty [35]. There are also topological implications of $\mathcal{Y}_2(M^4)$ being non-empty: by [20], if $\mathcal{Y}_2^+(M^4) \neq \emptyset$ then the first Betti number $b_1(M^4) = 0$. In fact, it follows from [6] that $[g]$ contains a metric $\tilde{g}$ with positive Ricci curvature.

Returning to the Chern-Gauss-Bonnet formula, for metrics $g \in \mathcal{Y}_2^+(M^4)$ we define the conformal invariant

$$\beta(M^4, [g]) = \frac{\int \|W_g\|^2 dV_g}{\int \sigma_2(P_g) dv_g} \geq 0. \quad (2.7)$$

We also define smooth invariant

$$\beta(M^4) = \inf_{[g]} \beta(M^4, [g]). \quad (2.8)$$

If $\mathcal{Y}_2^+(M^4) = \emptyset$, we set $\beta(M^4) = -\infty$.

2.3 Modified Yamabe metric

As mentioned in the preceding chapter, the proof of Theorems 3.1.1 and 3.1.2 will use the Ricci flow. We will use the fact that our assumptions are conformally invariant and choose an initial metric that satisfies certain key estimates. The metric will be a solution of a modified version of the Yamabe problem introduced in [22], which we now review.

Let (M^4, g) be a Riemannian four-manifold. Define

$$F_g^+ = R_g - 2\sqrt{6}\|W_g^+\|, \quad (2.9)$$

and

$$\mathcal{L}_g = -6\Delta_g + R_g - 2\sqrt{6}||W^+||. \quad (2.10)$$

$\mathcal{L}_g$ is a variant of conformal Laplacian that satisfies the following conformal transformation law:

$$\mathcal{L}_{\tilde{g}}\phi = u^{-3}\mathcal{L}_g(\phi u), \quad (2.11)$$

where $\tilde{g} = u^2g \in [g]$. In analogy to the Yamabe problem, we define the functional

$$\widehat{Y}_g[u] = \langle u, \mathcal{L}_g u \rangle_{L^2} / ||u||_{L^4}^2, \quad (2.12)$$

and the associated conformal invariant

$$\widehat{Y}(M^4, [g]) = \inf_{u \in W^{1,2}(M,g)} \widehat{Y}_g[u]. \quad (2.13)$$

By the conformal transformation law of $\mathcal{L}_g$, the functional $u \rightarrow \widehat{Y}_g[u]$ is equivalent to the Riemannian functional

$$\tilde{g} = u^2g \rightarrow \text{vol}(\tilde{g})^{-\frac{1}{2}} \int F_{\tilde{g}}^+ dv_{\tilde{g}}. \quad (2.14)$$

The motivation for introducing this invariant is explained in the following result (see Theorem 3.3 and Proposition 3.5 of [22]):

Theorem 2.3.1. *(i) Suppose M^4 admits a metric g with $F_g^+ \geq 0$ on M^4 and $F_g^+ > 0$ somewhere. Then $b_2^+(M^4) = 0$. (ii) If $b_2^+(M^4) > 0$, then M^4 admits a metric g with $F_g^+ \equiv 0$ if and only if (M^4, g) is a Kähler manifold with non-negative scalar curvature. (iii) If $Y(M^4, [g]) > 0$ and $b_2^+(M^4) > 0$, then $\widehat{Y}(M^4, [g]) \leq 0$ and there is a metric $\tilde{g} = u^2g$ such that*

$$F_{\tilde{g}}^+ = R_{\tilde{g}} - 2\sqrt{6}||W^+||_{\tilde{g}} \equiv \widehat{Y}(M, [g]) \leq 0 \quad (2.15)$$

and

$$\int R_{\tilde{g}}^2 dv_{\tilde{g}} \leq 24 \int ||W_{\tilde{g}}^+||^2 dv_{\tilde{g}}. \quad (2.16)$$

Furthermore, equality is achieved if and only if $F_{\tilde{g}}^+ \equiv 0$ and $R_{\tilde{g}} = 2\sqrt{6}||W_{\tilde{g}}^+|| \equiv \text{const.}$

Remark 2.3.2. Recall that $F_g^+ \equiv 0$ on a Kähler manifold (M^4, g) with $R \geq 0$.

Remark 2.3.3. In the rest of the paper, we will refer to the metric $\tilde{g}$ in (iii) of Theorem 2.3.1, normalized to have unit volume, a modified Yamabe metric, and denote it by g_m . To simplify the notation, we write $\hat{Y}(M, [g]) = -\mu_+$. Then g_m satisfies

$$R_{g_m} - 2\sqrt{6}||W^+||_{g_m} = -\mu_+ \leq 0 \quad (2.17)$$

and (2.16).

The following theorem due to M. Gursky can be proved by the solution to the modified Yamabe metric.

Theorem 2.3.4. Let M^4 be a closed oriented four-manifold with $b_2^+(M^4) > 0$. Then for any metric g with $Y(M^4, [g]) \geq 0$,

$$\int_M ||W^+||^2 dv \geq \frac{4\pi^2}{3}(2\chi(M^4) + 3\tau(M^4)) \quad (2.18)$$

where $\chi(M^4)$ and $\tau(M^4)$ denote the Euler characteristic and signature of M^4 , respectively.

Furthermore:

1. Equality is achieved in (2.18) by some metric g with $Y(M^4, [g]) > 0$ if and only if g is conformal to a (positive) Kähler-Einstein metric $g_{KE} = e^{2w}g$.
2. Equality is achieved in (2.18) by some metric g with $Y(M^4, [g]) = 0$ if and only if g is conformal to a Ricci-flat anti-self-dual Kähler-Einstein metric $g_{KE} = e^{2w}g$.

Now we have reviewed the definition and some general properties of the modified Yamabe metric. Further properties of this metric under the conditions in Theorem 3.1.1 will be stated and proved in the next chapter.

2.4 Ricci flow and smoothing techniques

In this section, we recall some basic facts about the Ricci flow defined as

$$\begin{cases} \frac{\partial}{\partial t}g = -2Ric(g) \\ g(0) = g_0 \end{cases} \quad (2.19)$$

The following short time-time existence result of Ricci flow has been established in [25].

Proposition 2.4.1. *For arbitrary smooth metric g_0 , there exists $T = T(g_0)$ such that (2.19) has a unique smooth solution for $t \in [0, T)$.*

Remark 2.4.1. *In general, the time interval $[0, T)$ depends on the initial metric g_0 .*

The Ricci flow is mainly used as a smoothing tool in our argument. In this aspect, the following Bernstein-Bando-Shi estimate (see for example Chapter 7 of [13]) is of particular importance.

Lemma 2.4.2. *Let $(M^4, g(t))$ be a solution to the Ricci flow. For every $m \in \mathbb{N}$, there exists a constant C_m depending only on m such that if*

$$\sup_{x \in M} |Rm(x, t)|_{g(t)} \leq K, \quad t \in \left[0, \frac{1}{K}\right],$$

then

$$\sup_{x \in M} |\nabla^m Rm(x, t)|_{g(t)} \leq \frac{C_m K}{t^{m/2}}, \quad t \in \left[0, \frac{1}{K}\right].$$

This lemma enables one to derive C^∞ estimate from L^∞ estimate along the Ricci flow. We also prove that Ricci flow will regularize a metric to derive higher L^p ($p > 2$) estimate from initial L^2 estimate. This process will be explained in detail in next chapter.

2.5 Bach-flat metrics

In this section, we review definition and basic properties of Bach-flat metrics. In four dimensions, the Bach curvature tensor is defined as follows:

$$B_{ij} = \nabla^k \nabla^l W_{kijl} + \frac{1}{2} R^{kl} W_{kijl}. \quad (2.20)$$

The Bach tensor is trace-free, divergence-free and conformally invariant. It then follows from the conformal invariance that the Bach-flat condition $B_{ij} = 0$ is also conformally invariant. Recall that Bach tensor is the gradient of the Weyl functional $g \rightarrow \int ||W||^2 dv$ and Bach-flat metrics are critical points of Weyl functional. Hence, any self-dual metric is Bach-flat since it achieves (global) minimum of Weyl functional. By using Bianchi identities, Bach-tensor can be rewritten as:

$$B_{ij} = -\frac{1}{2} \Delta E_{ij} + \frac{1}{6} \nabla_i \nabla_j R - \frac{1}{24} \Delta R g_{ij} - E^{kl} W_{kijl} + E_i^k E_{jk} - \frac{1}{4} |E|^2 g_{ij} + \frac{1}{6} R E_{ij}, \quad (2.21)$$

where $E_{ij} = R_{ij} - \frac{R}{4} g_{ij}$ is the trace-free Ricci tensor. It is clear from (2.21) that Einstein metrics are Bach-flat.

The following two Weitzenböck identities proved in [6][8] will be of fundamental importance for our discussion.

Lemma 2.5.1. *If (M^4, g) is Bach-flat, then*

$$\int_M \left(3 \left(|\nabla E|^2 - \frac{1}{12} |\nabla R|^2 \right) + 6 \operatorname{tr} E^3 + R |E|^2 - 6 W_{ijkl} E_{ik} E_{jl} \right) dv = 0, \quad (2.22)$$

where $\text{tr} E^3 = E_{ij} E_{ik} E_{jk}$, and

$$\int_M |\nabla W^\pm|^2 dv = \int_M \left(72 \det W^\pm - \frac{1}{2} R |W^\pm|^2 + 2 W_{ijkl}^\pm E_{ik} E_{jl} \right) dv. \quad (2.23)$$

The convergence theory of Bach-flat metrics has been developed by Gang Tian and Jeff Viaclovsky in [42][43]. The most important ingredients are the following ϵ -regularity theorem and volume estimate.

Theorem 2.5.2. *Let (M^4, g) be a Bach-flat Riemannian four-manifold with Yamabe constant $Y > 0$ and g is a Yamabe metric in $[g]$. Then there exists positive numbers τ_k and C_k depending on Y such that, for each geodesic ball $B_{2r}(p)$ centered at $p \in M$, if*

$$\int_{B_{2r}(p)} |Rm|^2 dv \leq \tau_k, \quad (2.24)$$

then

$$\sup_{B_r(p)} |\nabla^k Rm| \leq \frac{C_k}{r^{2+k}} \left(\int_{B_{2r}(p)} |Rm|^2 dv \right)^{\frac{1}{2}}. \quad (2.25)$$

Theorem 2.5.3. *Let (X, g) be a complete, non-compact, 4-manifold with base point p , and let $r(x) = d(p, x)$, for $x \in X$. Assume that there exists a constant $v_0 > 0$ such that*

$$\text{vol}(B_r(q)) \geq v_0 r^4 \quad (2.26)$$

holds for all $q \in X$, assume furthermore that as $r \rightarrow \infty$,

$$\sup_{S(r)} |Rm_g| = o(r^{-2}), \quad (2.27)$$

where $S(r) = \partial B_r(p)$. Assume that the first Betti number $b_1(X) < \infty$, then (X, g) is an ALE space, and there exists a constant v_1 (depending on (X, g)) so that

$$\text{vol}(B_r(p)) \leq v_1 r^4. \quad (2.28)$$

2.6 Self-dual four-manifolds

In this section, we review some basic results concerning the classification of self-dual four-manifolds.

The following celebrated classification of self-dual Einstein four-manifolds is due to N. Hitchin [26]. A detailed explanation of this theorem can be found in chapter 13 of [1].

Theorem 2.6.1. *Suppose (M^4, g) is an oriented self-dual Einstein four-manifold with positive scalar curvature. Then (M^4, g) is (up to scaling) isometric to (S^4, g_c) or $(\mathbb{CP}^2, g_{FS})$.*

Note that Hitchin's theorem does *not* have any requirement of the topology of the manifold. With some topological restriction the following result due to Y.-S. Poon [39] can be established.

Theorem 2.6.2. *Suppose that X is an oriented compact simply connected self-dual four-manifold with positive scalar curvature. If the signature of X is equal to zero or one, then X is conformally equivalent to (S^4, g_c) or $(\mathbb{CP}^2, g_{FS})$.*

Remark 2.6.3. *According to the author's knowledge, the only published proof of Theorem 2.6.1 is the original argument by Hitchin which applies the twistor space method in a crucial way. It is natural to ask whether there is a purely analytic proof.*

Part II

Conformally Invariant Diffeomorphism Theorems

Chapter 3

Statements of main results and ideas of the proofs

3.1 Statements of main results

In this section, several conformally invariant diffeomorphism theorems will be stated. All these results can be viewed as generalizations of theorems stated in Part I.

Theorem 3.1.1. *Suppose M^4 is oriented and $b_2^+(M^4) > 0$. There is an $\epsilon > 0$ such that if M^4 admits a metric $g \in \mathcal{Y}_2^+(M^4)$ with*

$$4 \leq \beta(M^4, [g]) < 4(1 + \epsilon), \tag{3.1}$$

then M^4 is diffeomorphic to $\mathbb{CP}^2$.

Theorem 3.1.2. *Suppose M^4 is oriented and $b_2^+(M^4) > 0$. If*

$$\beta(M^4) = 4, \tag{3.2}$$

then M^4 is diffeomorphic to $\mathbb{CP}^2$. Moreover, $\beta(M^4, [g]) = 4$ if and only if $g \in [g_{FS}]$.

Theorem 3.1.3. *Suppose M^4 is oriented, $b_2(M^4) \neq 0$, and the intersection form of M^4 is indefinite. If $\mathcal{Y}_2^+(M^4)$ is non-empty, then*

$$\beta(M^4) \geq 8. \tag{3.3}$$

Moreover, if M^4 admits a metric with $\beta(M^4, [g]) = 8$, then $g \in [g_p]$, where $g_p = g_{S^2} \oplus g_{S^2}$ is the product metric on $S^2 \times S^2$.

Theorem 3.1.1, Theorem 3.1.2, and Theorem 3.1.3 are essentially established in my joint work with S.-Y. A. Chang and M. Gursky [9]. These theorems generalize Theorem 1.1.3 to the model cases of $\mathbb{CP}^2$ and $S^2 \times S^2$.

3.2 Ideas of proofs

In this section, I shall briefly discuss the ideas of the proofs.

The proof of Theorem 3.1.1 is similar in approach to the proof of Theorem 1.1.3. The first step is to find a conformal representative satisfying a pointwise curvature condition that encodes the integral assumptions of the theorem. In [8] this involved solving a modified version of the σ_2 -Yamabe problem. However, in the proof of Theorem 3.1.1 it is more natural to consider a modified version of the Yamabe problem introduced by M. Gursky in [22]. In particular we show that a metric satisfying the assumptions of Theorem 3.1.1 can be conformally deformed to a metric that is “almost self-dual Einstein” in an L^2 -sense, and whose scalar curvature satisfies a condition similar to the condition satisfied by the scalar curvature of a Kähler metric.

As in the proof of Theorem 1.1.3, the second step involves the Ricci flow. In [8] the weak pinching result of Margerin [36] played a crucial role. To prove Theorem 3.1.1, we show that the Ricci flow with the conformal representative constructed in the first step as the initial metric, will have uniform bounds on the curvature and the Sobolev constant on a fixed time interval $[0, T_0]$, with $T_0 > 0$ depending on the

pinching constant. These estimates together with the convergence theory established by J. Cheeger, M. Gromov, and M. Taylor [11] imply the family of Ricci flows $g_j(T_0)$ (up to a subsequence) will converge to the Fubini-Study metric.

It is not hard to prove Theorem 3.1.2 as a direct corollary of Theorem 3.1.2. The proof of Theorem 3.1.3 relies on some different argument.

Chapter 4

Proofs of main results

In this chapter, we prove Theorems 3.1.1, 3.1.2, and 3.1.3. The main arguments are based on [9].

4.1 L^2 estimates

In this section, we prove L^2 estimates. We start from the following lemma which is an application of Theorem 2.3.4.

Lemma 4.1.1. *Let M^4 be a closed, oriented four-manifold admitting a metric $g \in \mathcal{Y}_2^+(M^4)$ with*

$$\beta(M^4, [g]) < 8. \tag{4.1}$$

If $b_2^+(M^4) > 0$, then the signature of M^4 satisfies

$$\tau(M^4) > 0.$$

Proof. The signature formula implies

$$\int \|W_g\|^2 dv_g = \int_M \|W_g^+\|^2 dv_g + \int \|W_g^-\|^2 dv_g = 2 \int \|W_g^+\|^2 dv_g - 12\pi^2 \tau(M^4). \tag{4.2}$$

By (4.1),

$$\int \sigma_2(P_g) dv_g > \frac{1}{8} \int \|W_g\|^2 dv_g.$$

Substituting this into the Chern-Gauss-Bonnet formula, we have

$$8\pi^2\chi(M^4) = \int \|W_g\|^2 dv_g + 4 \int \sigma_2(P_g) dv_g > \frac{3}{2} \int \|W_g\|^2 dv_g. \quad (4.3)$$

Combining (4.2) and (4.3), we get

$$8\pi^2\chi(M^4) > 3 \int \|W_g^+\|^2 dv_g - 18\pi^2\tau(M^4), \quad (4.4)$$

and this inequality can be rewritten as

$$\int \|W_g^+\|^2 dv_g < \frac{4}{3}\pi^2(2\chi(M^4) + 3\tau(M^4)) + 2\pi^2\tau(M^4). \quad (4.5)$$

Since $b_2^+(M^4) > 0$ and $Y(M^4, [g]) > 0$, (2.18) in Theorem 2.3.4 implies

$$\int \|W_g^+\|^2 dv_g \geq \frac{4}{3}\pi^2(2\chi(M^4) + 3\tau(M^4)). \quad (4.6)$$

Therefore, combining (4.5) and (4.6), we conclude

$$\tau(M^4) > 0.$$

□

Remark 4.1.2. *This lemma is sharp in the following sense: Suppose (M, g) is isometric to $(S^2 \times S^2, g_p)$. In this case, $b_2^+(M^4) = b_2^-(M^4) = 1$, $\tau(M^4) = 0$ and*

$$\int \|W_g\|^2 dv_g = 8 \int \sigma_2(P_g) dv_g = \frac{64}{3}\pi^2. \quad (4.7)$$

Corollary 4.1.3. *Let M^4 be a closed, oriented four-manifold admitting a metric $g \in \mathcal{Y}_2^+(M^4)$ with*

$$\beta(M^4, [g]) < 8. \quad (4.8)$$

Then either $b_2(M^4) = 0$, or the intersection form is definite.

Proof. Suppose $b_2(M^4) \neq 0$ and $b_2^+(M^4) \cdot b_2^-(M^4) > 0$. Then Lemma 4.1.1 implies $\tau(M^4) = b_2^+(M^4) - b_2^-(M^4) > 0$. Since $b_2^-(M^4)$ is also non-zero, we can apply Lemma 4.1.1 to M^4 endowed with the opposite orientation to show that the signature is again positive. This is a contradiction, since changing the orientation changes the sign of the signature. It follows that $b_2^+(M^4) \cdot b_2^-(M^4) = 0$, hence the intersection form is definite. $\square$

Combining the two previous results with an *a priori* upper bound for the total σ_2 -curvature, we can prove the following:

Lemma 4.1.4. *Let M^4 be a closed, oriented four-manifold admitting a metric $g \in \mathcal{Y}_2^+(M^4)$ with*

$$\beta(M^4, [g]) < 8. \quad (4.9)$$

If $b_2(M^4) > 0$, then (after possibly changing the orientation) $b_2^+(M^4) = 1$ and $b_2^-(M^4) = 0$.

Proof. By Corollary 4.1.3 we may choose an orientation for which the intersection form is positive definite, so $b_2^+(M^4) > 0$ and $b_2^-(M^4) = 0$. Also, by Corollary F of [20], if $g \in \mathcal{Y}_2^+(M^4)$, $b_1(M^4) = 0$. Therefore,

$$\begin{aligned} \chi(M^4) &= 2 + b_2^+(M^4), \\ \tau(M^4) &= b_2^+(M^4) > 0. \end{aligned} \quad (4.10)$$

By the Chern-Gauss-Bonnet formula and (4.8),

$$\begin{aligned} 8\pi^2\chi(M^4) &= \int \|W_g\|^2 dv_g + 4 \int \sigma_2(P_g) dv_g \\ &< 12 \int \sigma_2(P_g) dv_g. \end{aligned} \tag{4.11}$$

By Theorem B of [21], we have the bound

$$\int \sigma_2(P_g) dv_g \leq 4\pi^2, \tag{4.12}$$

and equality holds if and only if (M^4, g) is conformally equivalent to the round sphere.

More generally, since the integral is conformally invariant it is easy to show that

$$\int \sigma_2(P_g) dv_g \leq \frac{1}{96} Y(M^4, [g])^2. \tag{4.13}$$

Since $b_2(M^4) > 0$ strict inequality must hold, and substituting this into (4.11) we get

$$\begin{aligned} 8\pi^2\chi(M^4) &< 12 \int \sigma_2(P_g) dv_g \\ &< 48\pi^2, \end{aligned} \tag{4.14}$$

hence $\chi(M^4) < 6$. By (4.10), we see that $1 \leq b_2^+(M^4) \leq 3$. It therefore suffices to rule out the possibilities $b_2^+(M^4) = 2$ and $b_2^+(M^4) = 3$.

If $b_2^+(M^4) = 2$ then $\chi(M^4) = 4$, so by the Chern-Gauss-Bonnet formula

$$\int \|W_g\|^2 dv_g + 4 \int \sigma_2(P_g) dv_g = 32\pi^2.$$

Also, $b_2^+(M^4) = 2$ implies $\tau(M^4) = 2$, so the signature formula gives

$$\begin{aligned} \int ||W_g^+||^2 dv_g &= \int ||W_g^-||^2 dv_g + 24\pi^2 \\ &\geq 24\pi^2. \end{aligned}$$

It follows that

$$\int ||W_g||^2 dv_g \geq 24\pi^2, \quad \int \sigma_2(P_g) dv_g \leq 2\pi^2.$$

Therefore,

$$\int ||W_g||^2 dv_g \geq 12 \int \sigma_2(P_g) dv_g,$$

which contradicts (4.9).

If $b_2^+(M^4) = 3$, we can apply the same argument to conclude

$$\int ||W_g||^2 dv_g \geq 36 \int \sigma_2(P_g) dv_g,$$

which also contradicts (4.9). Therefore, $b_2^+(M^4) = 1$. $\square$

Remark 4.1.5. *The preceding lemma implies that if we take $\epsilon \leq 1$ in Theorem 3.1.1, then $b_2^+(M^4) > 0$ will show that $b_2^+(M^4) = 1$. Note that the work [16] and [17] of Donaldson and Freedman will imply that the manifold is homeomorphic to $\mathbb{CP}^2$ in this case.*

We now prove a technical lemma that will be used in the proof of Theorem 3.1.1.

Lemma 4.1.6. *Let (M^4, g) be a closed, compact oriented Riemannian four-manifold with $b_2^+(M^4) > 0$ and*

$$\beta(M^4, [g]) = 4(1 + \epsilon) \tag{4.15}$$

for some $0 \leq \epsilon < 1$. Then

$$\int_M ||W^-||^2 dv = \frac{6\epsilon}{2 + \epsilon} \pi^2, \tag{4.16}$$

$$\int ||W^+||^2 dv = 12\pi^2 + \int ||W^-||^2 dv, \quad (4.17)$$

and

$$Y(M^4, [g]) \geq \frac{24\pi}{\sqrt{2+\epsilon}}. \quad (4.18)$$

Proof. It follows from Lemma 4.1.4 that $b_1(M^4) = 0$, $b_2^+(M^4) = 1$, and $b_2^-(M^4) = 0$. Therefore, $\chi(M^4) = 3$ and $\tau(M^4) = 1$. By the Chern-Gauss-Bonnet and signature formulas, we have

$$24\pi^2 = \int ||W||^2 dv + 4 \int \sigma_2(P) dv, \quad (4.19)$$

$$12\pi^2 = \int ||W^+||^2 dv - \int ||W^-||^2 dv. \quad (4.20)$$

Since $\beta(M^4, [g]) = 4(1 + \epsilon)$, we have

$$4 \int \sigma_2(P) dv = \frac{1}{1 + \epsilon} \int ||W||^2 dv. \quad (4.21)$$

Substituting this into (4.19) we conclude

$$\int ||W||^2 dv = 24 \left(\frac{1 + \epsilon}{2 + \epsilon} \right) \pi^2. \quad (4.22)$$

By (4.20),

$$\begin{aligned} 12\pi^2 &= \int ||W^+||^2 dv - \int ||W^-||^2 dv \\ &= \int ||W||^2 dv - 2 \int ||W^-||^2 dv \\ &= 24 \left(\frac{1 + \epsilon}{2 + \epsilon} \right) \pi^2 - 2 \int ||W^-||^2 dv, \end{aligned}$$

which implies (4.16). Also, substituting (4.16) into the signature formula (4.20) we get (4.17).

To prove (4.18), we first observe that (4.21) and (4.22) imply

$$\int \sigma_2(P) dv = \left(\frac{6}{2+\epsilon} \right) \pi^2.$$

Therefore, by (4.13),

$$\left(\frac{6}{2+\epsilon} \right) \pi^2 = \int \sigma_2(P_g) dv_g \leq \frac{1}{96} Y(M^4, [g])^2, \quad (4.23)$$

and (4.18) follows. $\square$

As a preparation for the proof of Theorem 3.1.1, in the rest of this section we will list some preliminary curvature estimates of the modified Yamabe metric $g_m \in [g]$ with the assumption $b_2^+(M^4) > 0$ and $\beta(M, [g]) = 4(1+\epsilon)$.

Lemma 4.1.7. *Let (M^4, g) be a closed, compact oriented Riemannian four-manifold with $b_2^+(M^4) > 0$ and*

$$\beta(M^4, [g]) = 4(1+\epsilon) \quad (4.24)$$

for some $0 < \epsilon < 1$, then we have for the modified Yamabe metric $g_m \in [g]$

$$\int_M \|W_{g_m}^-\|^2 dv_{g_m} = \frac{6\epsilon}{2+\epsilon} \pi^2, \quad (4.25)$$

$$\int \|W_{g_m}^+\|^2 dv_{g_m} = 12\pi^2 + \int \|W_{g_m}^-\|^2 dv_{g_m}, \quad (4.26)$$

$$Y(M, [g_m]) \geq \frac{24\pi}{\sqrt{2+\epsilon}} \quad (4.27)$$

$$\int |E_{g_m}|^2 dv_{g_m} \leq 6 \int \|W_{g_m}^-\| dv_{g_m}, \quad (4.28)$$

$$\frac{1}{12} \mu_+ Y \leq 3 \int \|W_{g_m}^-\|^2 dv_{g_m}, \quad (4.29)$$

$$\frac{1}{24} \int (R_{g_m} - \bar{R}_{g_m})^2 dv_{g_m} \leq 3 \int \|W_{g_m}^-\|^2 dv_{g_m}, \quad (4.30)$$

where $\bar{R}_{g_m} = \int R_{g_m} dv_{g_m}$.

Proof. (4.25)(4.26)(4.27) follow from (4.16)(4.17)(4.18) of Lemma 4.1.6 and conformal invariance. The estimates (4.19) and (4.20) imply

$$\int \|W^+\|^2 dv - 4 \int \sigma_2(P_{g_m}) dv_{g_m} = 3 \int \|W^-\|_{g_m}^2 dv_{g_m}. \quad (4.31)$$

Recall the modified Yamabe metric satisfies $R_{g_m} + \mu_+ = 2\sqrt{6}\|W_{g_m}^+\|$. Squaring both sides of this formula and integrating over M^4 , we have

$$\frac{1}{24} \int (R_{g_m}^2 + 2\mu_+ R_{g_m} + \mu_+^2) dv_{g_m} = \int \|W_{g_m}^+\|^2 dv_{g_m}.$$

With (4.31), we can rewrite this equation in the following way:

$$\frac{1}{2} \int |E_{g_m}|^2 dv + \frac{1}{12} \mu_+ \int R_{g_m} dv_{g_m} + \frac{1}{24} \int \mu_+^2 dv_{g_m} = 3 \int \|W_{g_m}^-\|^2 dv_{g_m}.$$

Since $\int R_{g_m} dv_{g_m} \geq Y(M^4, [g_m]) > 0$, (4.28) and (4.29) follow from this equation. To see (4.30), we have

$$\begin{aligned} \frac{1}{24} \int (R_{g_m} - \bar{R}_{g_m})^2 dv_{g_m} &= \frac{1}{24} \left(\int R_{g_m}^2 dv_{g_m} - \bar{R}_{g_m}^2 \right) \leq \frac{1}{24} \left(\int R_{g_m}^2 dv_{g_m} - Y^2 \right) \\ &\leq \int \|W_{g_m}^+\|^2 dv_{g_m} - 4 \int \sigma_2(P_{g_m}) dv_{g_m} = 3 \int \|W_{g_m}^-\|^2 dv_{g_m}. \end{aligned}$$

□

We end this section with a conformally invariant characterization of the Fubini-Study metric:

Lemma 4.1.8. *Let (M^4, g) be a closed, compact oriented Riemannian four-manifold whose metric g is of positive Yamabe type. In addition, assume $b_1(M^4) = b_2^-(M^4) = 0$*

and $b_2^+(M^4) = 1$. Then

$$\int_M \sigma_2(P) dv \leq 12\pi^2$$

and equality holds if and only if (M^4, g) is conformally equivalent to $(\mathbb{CP}^2, g_{FS})$.

Proof. By our assumptions we have $\chi(M^4) = 3$ and $\tau(M^4) = 1$. Then the Chern-Gauss-Bonnet and signature formulas read

$$24\pi^2 = \int_M \|W_g\|^2 dv_g + 4 \int_M \sigma_2(P_g) dv_g, \quad (4.32)$$

and

$$12\pi^2 = \int_M \|W_g^+\|^2 dv_g - \int_M \|W_g^-\|^2 dv_g. \quad (4.33)$$

Therefore,

$$\begin{aligned} \int_M \sigma_2(P) dv &= 24\pi^2 - \int_M \|W\|^2 dv \\ &\leq 24\pi^2 - \int_M \|W^+\|^2 dv \\ &= 12\pi^2 - \int_M \|W^-\|^2 dv \leq 12\pi^2. \end{aligned}$$

If equality holds then $W^- \equiv 0$ and by (4.28) of Lemma 4.1.7 it immediately follows that $E_{g_m} = 0$. Hence, (M^4, g_m) is self-dual Einstein with positive scalar curvature and $b_2^+(M^4) = 1$. It is easy to check that the equality in (2.18) is achieved, and therefore g_m is conformal to a Kähler-Einstein metric. By Obata's theorem, (M^4, g_m) must be (Kähler-)Einstein with positive scalar curvature and $b_2^+(M^4) = 1$.

Now (M^4, g_m) is a complex surface with a positive Kähler-Einstein metric. For complex surfaces, (see Page 81 of [1])

$$3c_2(M^4) - c_1(M^4)^2 = \chi(M^4) - 3\tau(M^4) = 0. \quad (4.34)$$

Then the uniformization of Kähler-Einstein manifolds (e.g. Theorem 2.13 in [41]) implies that the universal cover of (M, g_m) is (up to scaling) isometric to $(\mathbb{CP}^2, g_{FS})$. Hence, (M, g_m) is conformally equivalent to $(\mathbb{CP}^2, g_{FS})$ since $\mathbb{CP}^2$ does not have non-trivial smooth quotient space. $\square$

Remark 4.1.9. *The readers should compare this lemma to Theorem 2.6.2.*

4.2 Digression: a lower bound for $\beta(M)$

In this section, we give a proof of Theorem 3.1.3 which makes use of several estimates from the preceding section.

The Proof of Theorem 3.1.3. We can now combine Theorem 2.3.4 and Lemma 4.1.1 to give the proof of Theorem 3.1.3. Assuming $b_2^+(M^4) \cdot b_2^-(M^4) > 0$, it follows from Lemma 4.1.1 that $\beta(M^4, [g]) \geq 8$ for any metric $g \in \mathcal{Y}_2^+(M^4)$. Moreover, if equality holds, then since $b_2^+(M^4) > 0$ we can argue as in the proof of Lemma 4.1.1 to get equality in (4.5):

$$\int \|W_g^+\|^2 dv_g = \frac{4}{3}\pi^2(2\chi(M^4) + 3\tau(M^4)) + 2\pi^2\tau(M^4). \quad (4.35)$$

By Theorem 2.3.4 we conclude that $\tau(M^4) \geq 0$. Reversing orientation and applying the same argument (since $b_2^-(M^4) > 0$) we also get $-\tau(M^4) \geq 0$, hence $\tau(M^4) = 0$. Substituting this into (4.35) implies that we have equality in (2.18). Therefore, g is conformal to Kähler-Einstein metric g_{KE} . By Proposition 2 of [15], $\nabla W_{g_{KE}}^+ \equiv 0$.

Applying the same argument with the opposite orientation, we see that g is conformal to a Kähler-Einstein metric g'_{KE} . By Obata's theorem [37], Einstein metrics are unique in their conformal class (except in the case of the sphere, which is ruled out in this case). Therefore, $g_{KE} = g'_{KE}$, and since equality holds in (2.18) with the opposite orientation it follows that $\nabla W_{g_{KE}}^- \equiv 0$. We conclude that g_{KE} is locally

symmetric and Einstein; it follows from the classification of such spaces (for example, [29]) that (M^4, g_{KE}) is isometric to $(S^2 \times S^2, g_p)$, and Theorem 3.1.3 follows. $\square$

4.3 L^3 estimates

In this section, we continue the proof of Theorem 3.1.1 and 3.1.2. The main argument is to apply Ricci flow to derive L^p estimates for $p > 2$. We begin with some definitions. On (M^4, g) , define

$$G_k(g) = |E_g|^k + |R_g - \bar{R}_g|^k + \|W_g^-\|^k + |(F_g^+)_-|^k, \quad (4.36)$$

where $(F_g^+)_- = \min(F_g^+, 0)$. We shall suppress the subscript g when there is no confusion.

Lemma 4.3.1. *Under the conditions of Theorem 3.1.1, the modified Yamabe metric $g_m \in [g]$ satisfies $\int G_2(g_m) dv_{g_m} < c(\epsilon)$, where $c(\epsilon) \rightarrow 0$ as $\epsilon \rightarrow 0$.*

Proof. This is a direct consequence of Lemma 4.1.7. $\square$

Along the Ricci flow, define $G_k(t) = G_k(g(t))$. The following estimate is of fundamental importance for our argument.

Proposition 4.3.1. *Suppose we have a solution of the Ricci flow whose initial metric satisfies*

$$\int_M G_2(0) dv_{g(0)} = \frac{1}{2} \epsilon_0 \quad (4.37)$$

for some ϵ_0 is sufficiently small. Let

$$T = \inf \{t : \int_M G_2(t) dv_{g(t)} = \epsilon_0\}.$$

Assume in addition for $0 \leq t \leq T$

$$Y(t) = Y(M^4, [g(t)]) \geq b > 0, \quad (4.38)$$

and

$$0 < \bar{R}(t) \leq a. \quad (4.39)$$

Then for $0 \leq t \leq T$, we have

$$\frac{d}{dt} \int_M G_2(t) dv_{g(t)} \leq \tilde{a} \int_M G_2(t) dv_{g(t)} - \tilde{b} \left(\int_M G_4(t) dv_{g(t)} \right)^{1/2}, \quad (4.40)$$

where $\tilde{a}$ and $\tilde{b}$ are uniform positive constants independent of ϵ_0 . Moreover, there exists T_0 , which is independent of ϵ_0 such that $T \geq T_0$, and we may choose $\tilde{a} = \frac{4}{3}a$ and $\tilde{b} = \frac{1}{12}b$.

Proof. The proof is based on the evolution of the curvature under the Ricci flow in four dimensions, along with several algebraic inequalities. We begin by summarizing the evolution formulas we will need, most of which can be found in [13]:

Lemma 4.3.2. *Under (2.19) on Riemannian four-manifolds,*

$$\frac{\partial}{\partial t} dv = -R dv, \quad (4.41)$$

$$\frac{\partial}{\partial t} |E|^2 = \Delta |E|^2 - 2|\nabla E|^2 + 4WEE - 4tr E^3 + \frac{2}{3}R|E|^2, \quad (4.42)$$

$$\frac{\partial}{\partial t} (R^2) = \Delta (R^2) - 2|\nabla R|^2 + 4R|E|^2 + R^3, \quad (4.43)$$

$$\frac{\partial}{\partial t} \|W^\pm\|^2 = \Delta \|W^\pm\|^2 - 2\|\nabla W^\pm\|^2 + 36 \det W^\pm + W^\pm EE, \quad (4.44)$$

$$\frac{\partial}{\partial t} \|W^\pm\| \leq \Delta \|W^\pm\| + \sqrt{6} \|W^\pm\|^2 + \frac{\sqrt{6}}{6} |E|^2, \quad (4.45)$$

$$\frac{\partial}{\partial t} ((F^+)_-) \geq \Delta ((F^+)_-) - ((F^+)_-)^2 + 2R(F^+)_- \quad (4.46)$$

where

$$W^\pm EE := W_{ijkl}^\pm E_{ik} E_{jl}, \quad F_g^+ := R_g - 2\sqrt{6}||W^+||, \quad (F^+)_- := \min\{F_g^+, 0\}.$$

Remark 4.3.3. For the evolution formulas of $W^\pm$ we rely on unpublished notes of D. Knopf [30].

As a corollary of the formulas above we have in four dimensions:

Corollary 4.3.4. Under (2.19),

$$\frac{d}{dt} \int dv = - \int R dv, \quad (4.47)$$

$$\frac{d}{dt} \int |E|^2 dv = \int \left(-2|\nabla E|^2 + 4WEE - 4\text{tr} E^3 - \frac{1}{3}R|E|^2 \right) dv, \quad (4.48)$$

$$\frac{d}{dt} \int (R - \bar{R})^2 dv = \int \left(-2|\nabla R|^2 + 4(R - \bar{R})|E|^2 + \bar{R}(R - \bar{R})^2 \right) dv, \quad (4.49)$$

$$\begin{aligned} \frac{d}{dt} \int |(F^+)_-|^2 dv &\leq -2 \int \left(|\nabla (F^+)_-|^2 + \frac{R}{6} |(F^+)_-|^2 \right) dv \\ &\quad - \int ((F^+)_-)^3 dv + \int \left(\frac{4}{3} \bar{R} |(F^+)_-|^2 + \frac{4}{3} (R - \bar{R}) |(F^+)_-|^2 \right) dv, \end{aligned} \quad (4.50)$$

$$\begin{aligned} \frac{d}{dt} \int ||W^-||^2 dv &\leq - \int \left(2|\nabla ||W^-|||^2 + R ||W^-||^2 \right) dv \\ &\quad + \int \left(2\sqrt{6} ||W^-||^3 + \frac{\sqrt{6}}{3} ||W^-|| |E|^2 \right) dv, \end{aligned} \quad (4.51)$$

where

$$\bar{R} = \int R dv / \int dv, \quad F_g^+ = R_g - 2\sqrt{6}||W^+||, \quad (F^+)_- = \min\{F_g^+, 0\}.$$

For the proof of Proposition 4.3.1 we will also need some algebraic inequalities. The first appears in ([8], Lemma 4.3), and is based on ([36], Lemma 6):

Lemma 4.3.5.

$$WEE \leq \frac{\sqrt{6}}{3} (\|W^+\| + \|W^-\|) |E|^2 \quad (4.52)$$

Proof. Recall the well-known decomposition of Singer-Thorpe:

$$Riem = \begin{pmatrix} W^+ + \frac{R}{12} Id & B \\ B^* & W^- + \frac{R}{12} Id \end{pmatrix} \quad (4.53)$$

Note the compositions satisfy

$$BB^* : \Lambda_+^2 \rightarrow \Lambda_+^2, \quad B^*B : \Lambda_-^2 \rightarrow \Lambda_-^2.$$

Fix a point $P \in M^4$, and let $\lambda_1^\pm \leq \lambda_2^\pm \leq \lambda_3^\pm$ denote the eigenvalues of $W^\pm$, where $W^\pm$ are interpreted as endomorphisms of $\Lambda_\pm^2$. Also denote the eigenvalues of $BB^* : \Lambda_+^2 \rightarrow \Lambda_+^2$ by $b_1^2 \leq b_2^2 \leq b_3^2$, where $0 \leq b_1 \leq b_2 \leq b_3$. From Lemma 4.3 of [8], we have

$$WEE \leq 4 \left(\sum_{i=1}^3 \lambda_i^+ b_i^2 + \sum_{i=1}^3 \lambda_i^- b_i^2 \right) \quad (4.54)$$

Recall from Lemma 4.2 of [8] that $|E|^2 = 4 \sum_{i=1}^3 b_i^2$. For a trace-free 3×3 matrix A , we have the sharp inequality:

$$|A(X, X)| \leq \frac{\sqrt{6}}{3} \|A\| |X|^2. \quad (4.55)$$

Apply (4.55) to $A = \text{diag}(\lambda_1^\pm, \lambda_2^\pm, \lambda_3^\pm)$ and $X = (b_1, b_2, b_3)$. We derive

$$4 \left(\sum_{i=1}^3 \lambda_i^+ b_i^2 + \sum_{i=1}^3 \lambda_i^- b_i^2 \right) \leq \frac{\sqrt{6}}{3} (\|W^+\| + \|W^-\|) |E|^2. \quad (4.56)$$

Combining (4.54) and (4.56), we derive the desired inequality. $\square$

Now we turn to the proof of Proposition 4.3.1. From the definition of G_k , we have

$$\frac{d}{dt} \int G_2 dv = \frac{d}{dt} \int |E|^2 dv + \frac{d}{dt} \int |R - \bar{R}|^2 dv + \frac{d}{dt} \int \|W^-\|^2 dv + \frac{d}{dt} \int |(F^+)_-|^2 dv. \quad (4.57)$$

Now estimate each term of the right hand side from formulas in Corollary 4.3.4.

$$\begin{aligned} \frac{d}{dt} \int |E|^2 dv &\leq \int \left(-2|\nabla E|^2 - \frac{1}{3}R|E|^2 - 4trE^3 + \frac{4\sqrt{6}}{3}(\|W^+\| + \|W^-\|)|E|^2 \right) dv \\ &\leq -\frac{Y}{3} \left(\int |E|^4 dv \right)^{1/2} + 4 \left(\int |E|^2 dv \right)^{1/2} \left(\int |E|^4 dv \right)^{1/2} \\ &\quad + \frac{4\sqrt{6}}{3} \left(\int \|W^-\|^2 dv \right)^{1/2} \left(\int |E|^4 dv \right)^{1/2} \\ &\quad + \frac{2}{3} \left(\int |(F^+)_-|^2 dv \right)^{1/2} \left(\int |E|^4 dv \right)^{1/2} + \frac{2}{3} \int (R - \bar{R})|E|^2 dv \\ &\quad + \frac{2}{3} \int \bar{R}|E|^2 dv \\ &\leq -\frac{Y}{6} \left(\int |E|^4 dv \right)^{1/2} + \frac{2}{3} \bar{R} \int |E|^2 dv. \end{aligned} \quad (4.58)$$

The first inequality follows from Lemma 4.3.5. The second inequality follows from Cauchy-Schwarz inequality and the conformally invariant Sobolev inequality:

$$Y \left(\int \phi^4 dv \right)^{1/2} \leq \int \left(|\nabla \phi|^2 + \frac{1}{6}R\phi^2 \right) dv.$$

The third inequality follows from the smallness assumption of ϵ_0 . Next, we estimate

$$\begin{aligned}
\frac{d}{dt} \int (R - \bar{R})^2 dv &= \int \left(-2|\nabla(R - \bar{R})|^2 - \frac{1}{3}R(R - \bar{R})^2 \right) dv + \frac{1}{3} \int (R - \bar{R})^3 dv \\
&\quad + \frac{4}{3} \int \bar{R}(R - \bar{R})^2 dv + 4 \int (R - \bar{R})|E|^2 dv \\
&\leq -\frac{Y}{3} \left(\int (R - \bar{R})^4 dv \right)^{1/2} + \frac{1}{3} \left(\int (R - \bar{R})^2 dv \right)^{1/2} \left(\int (R - \bar{R})^4 dv \right)^{1/2} \\
&\quad + \frac{4}{3} \bar{R} \int (R - \bar{R})^2 dv + 4 \left(\int (R - \bar{R})^2 dv \right)^{1/2} \left(\int |E|^4 dv \right)^{1/2} \\
&\leq -\frac{Y}{6} \left(\int (R - \bar{R})^4 dv \right)^{1/2} + \frac{4}{3} \bar{R} \int (R - \bar{R})^2 dv \\
&\quad + 4 \left(\int (R - \bar{R})^2 dv \right)^{1/2} \left(\int |E|^4 dv \right)^{1/2}.
\end{aligned} \tag{4.59}$$

The first inequality is from Sobolev inequality and Cauchy-Schwarz inequality and the second inequality is from the smallness assumption of ϵ_0 .

$$\begin{aligned}
\frac{d}{dt} \int |(F^+)_-|^2 dv &\leq -\frac{Y}{3} \left(\int |(F^+)_-|^4 dv \right)^{1/2} + \left(\int |(F^+)_-|^2 dv \right)^{1/2} \left(\int |(F^+)_-|^4 dv \right)^{1/2} \\
&\quad + \frac{4}{3} \bar{R} \int |(F^+)_-|^2 dv + \frac{4}{3} \left(\int (R - \bar{R})^2 dv \right)^{1/2} \left(\int |(F^+)_-|^4 dv \right)^{1/2} \\
&\leq -\frac{Y}{6} \left(\int |(F^+)_-|^4 dv \right)^{1/2} + \frac{4}{3} \bar{R} \int |(F^+)_-|^2 dv
\end{aligned} \tag{4.60}$$

The first inequality is from Sobolev inequality and Cauchy-Schwarz inequality and the second inequality is from the smallness assumption of ϵ_0 .

$$\begin{aligned}
\frac{d}{dt} \int \|W^-\|^2 dv &\leq -\frac{Y}{3} \left(\int \|W^-\|^4 \right)^{1/2} + \frac{2}{3} \left(\int (R - \bar{R})^2 dv \right)^{1/2} \left(\int \|W^-\|^4 dv \right)^{1/2} \\
&\quad + 2\sqrt{6} \left(\int \|W^-\|^2 dv \right)^{1/2} \left(\int \|W^-\|^4 dv \right)^{1/2} \\
&\quad + \frac{\sqrt{6}}{3} \left(\int \|W^-\|^2 dv \right)^{1/2} \left(\int |E|^4 dv \right)^{1/2} \\
&\leq -\frac{Y}{6} (\|W^-\|^4)^{1/2} + \frac{\sqrt{6}}{3} \left(\int \|W^-\|^2 dv \right)^{1/2} \left(\int |E|^4 dv \right)^{1/2}
\end{aligned} \tag{4.61}$$

The first inequality is from Sobolev inequality and Cauchy-Schwarz inequality and the second inequality is from the smallness assumption of ϵ_0 .

With (4.58)(4.59)(4.60)(4.61), it is now easy to see from the smallness assumption of ϵ_0

$$\frac{d}{dt} \int_M G_2 dv \leq \frac{4}{3}a \int_M G_2 dv - \frac{1}{12}b \left(\int G_4 dv \right)^{1/2}. \tag{4.62}$$

Take $\tilde{a} = \frac{4}{3}a$ and $\tilde{b} = \frac{1}{12}b$. Clearly, we have proved the desired inequality. Note that the differential inequality

$$\frac{d}{dt} \int_M G_2 dv \leq \frac{4}{3}a \int_M G_2 dv$$

implies

$$T \geq T_0 = \frac{3 \log 2}{4}a.$$

□

Remark 4.3.6. *The importance of this lemma is that T_0 does not depend on ϵ_0 , which implies that we may evolve the Ricci flow on a uniform time interval once we derive uniform bounds for curvatures.*

With Lemma 4.3.1, it is easy to see that we may choose ϵ in Theorem 3.1.1 sufficiently small so that (4.37) is satisfied. To apply Proposition 4.3.1 and Proposition 4.3.3, we also need establish (4.38) and (4.39). We now establish these inequalities and prove the following proposition.

Proposition 4.3.2. *Suppose the initial metric of Ricci flow is chosen as the modified Yamabe metric for sufficiently small ϵ in Theorem 3.1.1. Then there exists $\tilde{T}$ which does not depend on ϵ such that for $0 \leq t \leq \tilde{T}$ all conditions of Proposition 4.3.1 are satisfied.*

Proof. It is clear from Lemma 4.3.1 that if we choose sufficiently small ϵ in Theorem 3.1.1, we can establish (4.37) for arbitrary small ϵ_0 . On $0 \leq t \leq T$, (4.37) implies that

$$\int |E|^2 dv \leq \frac{1}{2}\epsilon_0, \quad \int \|W^-\|^2 dv \leq \frac{1}{2}\epsilon_0, \quad \int (R - \bar{R})^2 dv \leq \frac{1}{2}\epsilon_0. \quad (4.63)$$

Recall $b_1(M) = 0$, $b_2^-(M) = 0$, and $b_2^+(M) = 1$. From signature and Chern-Gauss-Bonnet formula, we obtain

$$\int \|W^+\|^2 dv = 12\pi^2 + \int \|W^-\|^2 dv \leq 12\pi^2 + \frac{1}{2}\epsilon_0 \quad (4.64)$$

and thereby

$$12\pi^2 \geq 4 \int \sigma_2(P) dv = 24\pi^2 - \int \|W\|^2 dv \geq 12\pi^2 - \epsilon_0. \quad (4.65)$$

Now with the same argument in Lemma 4.1.6, we can derive

$$\frac{1}{96}Y(t)^2 \geq \int \sigma_2(P_{g(t)}) dv_{g(t)} \geq 2\pi^2 \quad (4.66)$$

if we choose sufficiently small ϵ_0 . Since the initial metric is of positive Yamabe type and the square of Yamabe constant has a strictly positive lower bound, we have established (4.38).

Note that (4.63) and (4.65) imply that for some $C > 0$

$$\frac{1}{C^2} \leq \int \bar{R}_{g(t)}^2 dv_{g(t)} = \bar{R}_{g(t)}^2 \text{vol}(M, g(t)) \leq C^2, \quad \frac{1}{C^2} \leq \int R_{g(t)}^2 dv_{g(t)} \leq C^2. \quad (4.67)$$

To establish (4.39), it now suffices to derive a uniform lower bound for the volume since $\int \bar{R}^2 dv$.

Lemma 4.3.7. *Under conditions of Proposition 4.3.2 with $\text{vol}(M, g(0)) = 1$, along Ricci flow, there exists constant $T' > 0$ such that for $0 \leq t \leq T' \leq T$*

$$\frac{9}{4} \geq \text{vol}(M, g(t)) \geq \frac{1}{4}. \quad (4.68)$$

In addition, there exists $T_1 > 0$ which does not depend on ϵ_0 such that $T' > T_1$.

Proof. Recall the evolution equation for volume under the Ricci flow:

$$\frac{d}{dt} \int dv = - \int R dv. \quad (4.69)$$

Hence, for $0 \leq t \leq T$

$$\frac{d}{dt} \int dv \geq - \int |R| dv \geq - \left(\int R^2 dv \right)^{1/2} \left(\int dv \right)^{1/2} \geq -C \left(\int dv \right)^{1/2}.$$

Similarly,

$$\frac{d}{dt} \int dv \leq C \left(\int dv \right)^{1/2}.$$

It is then easy to derive

$$|\sqrt{\text{vol}(t)} - \sqrt{\text{vol}(0)}| \leq Ct.$$

From this inequality, it is easy to choose $T' = \min\{\frac{1}{2C}, T\}$ such that (4.68) is satisfied. It is easy to see such a T_1 exists since $T \geq T_0$, where T_0 does not depend on ϵ_0 . $\square$

With Lemma 4.3.7, we establish (4.39) on $0 \leq t \leq T' \leq T$. If we choose $\tilde{T} = T'$, all conditions of Proposition 4.3.1 are satisfied on $0 \leq t \leq \tilde{T}$ and clearly $\tilde{T}$ has a positive universal lower bound. $\square$

We now derive integral estimates for G_3 . For the sake of clearness, we first establish the estimates for $\int |E|^3 dv$ and then derive a similar evolution inequality for $\int G_3 dv$ as we did in Lemma 4.3.1.

Lemma 4.3.8. *Under the conditions of Proposition 4.3.1, along the Ricci flow, we have*

$$\int |E_{g(t)}|^3 dv_{g(t)} \leq C \epsilon_0^{3/2} t^{-1},$$

for any $0 < t \leq \tilde{T}$, where C is a universal constant which does not depend on ϵ_0 .

Proof. It is clear from (4.58) that

$$\frac{d}{dt} \int |E|^2 dv + C \left(\int |E|^4 dv \right)^{1/2} \leq C \int |E|^2 dv, \quad (4.70)$$

where C is a constant which does not depend on ϵ_0 . From (4.42), we can compute

$$\frac{\partial}{\partial t} |E|^p = \frac{p}{2} |E|^{p-2} \frac{\partial}{\partial t} |E|^2 = \frac{p}{2} |E|^{p-2} \left(\Delta |E|^2 - 2 |\nabla E|^2 + 4 W E E - 4 \text{tr} E^3 + \frac{2}{3} R |E|^2 \right), \quad (4.71)$$

Note that

$$\Delta |E|^p = \frac{p}{2} |E|^{p-2} \Delta |E|^2 + p(p-2) |E|^{p-2} |\nabla E|^2.$$

From this identity and (4.71), it is easy to derive for $p \geq 3$

$$\begin{aligned}
\frac{d}{dt} \int |E|^p dv &= \int \left(\frac{\partial}{\partial t} |E|^p - R|E|^p \right) dv \\
&\leq \int (-p(p-2)|E|^{p-2}|\nabla|E||^2 + 2p|E|^2 WEE) dv \\
&\quad + \int \left(-2p|E|^{p-2} \text{tr} E^3 + \left(\frac{p}{3} - 1 \right) R|E|^p \right) dv \\
&= \int \left(-\frac{4(p-2)}{p} |\nabla|E|^{\frac{p}{2}}|^2 - \frac{2(p-2)}{3p} R|E|^p \right) dv \\
&\quad + \int (2p|E|^2 WEE - 2p|E|^{p-2} \text{tr} E^3 + C_p R|E|^p) dv
\end{aligned} \tag{4.72}$$

We now estimate the terms in last line of (4.72)

$$\int ||E|^{p-2} \text{tr} E^3| dv \leq \left(\int |E|^2 dv \right)^{1/2} \left(\int |E|^{2p} dv \right)^{1/2} \tag{4.73}$$

$$\begin{aligned}
\int |E|^{p-2} WEE dv &\leq \frac{\sqrt{6}}{3} \int (||W^-|| + ||W^+||) |E|^p dv \\
&\leq \frac{\sqrt{6}}{3} \left(\int ||W^-||^2 dv \right)^{1/2} \left(\int |E|^{2p} dv \right)^{1/2} \\
&\quad + \frac{1}{6} \left(\int ((F^+)_-)^2 dv \right)^{1/2} \left(\int |E|^{2p} dv \right)^{1/2} + \frac{1}{6} \int R|E|^p dv
\end{aligned} \tag{4.74}$$

$$\begin{aligned}
\int |R||E|^p dv &\leq \int |R - \bar{R}||E|^p dv + \bar{R} \int |E|^p dv \\
&\leq \left(\int (R - \bar{R})^2 dv \right)^{1/2} \left(\int |E|^{2p} dv \right)^{1/2} + \bar{R} \int |E|^p dv
\end{aligned} \tag{4.75}$$

From the smallness assumption of ϵ_0 , it is now easy to derive

$$\frac{d}{dt} \int |E|^p dv + C_p \left(\int |E|^{2p} dv \right)^{1/2} \leq C_p \int |E|^p dv. \tag{4.76}$$

In this proof, we shall only need this formula for $p = 2, 3$. Take two smooth cut-off functions ϕ_1 and ϕ_2 such that $0 \leq \phi_i \leq 1$ on $[0, \tilde{T}]$ for $i = 1, 2$. Take $\tau < \tau' < \tilde{T}$. For ϕ_1 , we choose $0 \leq \phi_1 \leq 1$ on $[0, \tau]$ and $\phi_1 \equiv 1$ on $[\tau, \tilde{T}]$. For ϕ_2 , we choose

$\phi_2 \equiv 0$ on $[0, \tau]$, $0 \leq \phi_2 \leq 1$ on $[\tau, \tau']$, and $\phi_1 \equiv 1$ on $[\tau', \tilde{T}]$. Also assume $|\phi'_i|_{L^\infty}$ have appropriate bound. It is now easy to derive

$$\frac{d}{dt} \left(\phi_i \int |E|^p dv \right) + C \phi_i \left(\int |E|^{2p} dv \right)^{1/2} \leq C (\phi_i + |\phi'_i|) \int |E|^p dv, \quad (4.77)$$

for $i = 1, 2$ and $p = 2, 3$.

Set $p = 2$ and $i = 1$. Integrate (4.77) over $[0, t]$ for some $t > \tau'$:

$$\int |E|^2 dv + C \int_\tau^t \left(\int |E|^4 dv \right)^{1/2} ds \leq C \left(1 + \frac{1}{\tau} \right) \int_0^t \left(\int |E|^2 dv \right) ds \quad (4.78)$$

Set $p = 3$ and $i = 2$. Integrate (4.77) over $[0, t]$ for some $t > \tau'$:

$$\int |E|^3 dv + C \int_{\tau'}^t \left(\int |E|^6 dv \right)^{1/2} ds \leq C \left(1 + \frac{1}{\tau' - \tau} \right) \int_\tau^t \left(\int |E|^3 dv \right) ds \quad (4.79)$$

Now we have

$$\begin{aligned} \int_\tau^t \left(\int |E|^3 dv \right) ds &\leq \int_\tau^t \left(\int |E|^2 dv \right)^{1/2} \left(\int |E|^4 dv \right)^{1/2} ds \\ &\leq C \epsilon_0^{1/2} \int_\tau^t \left(\int |E|^4 dv \right)^{1/2} ds \\ &\leq C \epsilon_0^{1/2} \left(1 + \frac{1}{\tau} \right) \int_0^t \left(\int |E|^2 dv \right) ds \end{aligned}$$

where second line follow from taking $\epsilon = \frac{1}{2}\epsilon_0$ in Lemma 5.5 and third line follows from (4.78). It then follows

$$\begin{aligned} \int |E|^3 dv &\leq C \left(1 + \frac{1}{\tau' - \tau}\right) \int_{\tau}^t \left(\int |E|^3 dv\right) ds \\ &\leq C\epsilon_0^{1/2} \left(1 + \frac{1}{\tau}\right) \left(1 + \frac{1}{\tau' - \tau}\right) \int_0^t \left(\int |E|^2 dv\right) ds \\ &\leq C\epsilon_0^{3/2} \left(1 + \frac{1}{\tau}\right) \left(1 + \frac{1}{\tau' - \tau}\right) t \end{aligned}$$

Take $\tau = \frac{1}{4}t$ and $\tau' = \frac{1}{2}t$ and we get desired estimate. In particular, if we choose $t \in [\frac{1}{4}\tilde{T}, \tilde{T}]$, we have

$$\sup_{\tilde{T}/4 \leq t \leq \tilde{T}} \int |E|^3 dv \leq C\epsilon_0^{3/2}. \quad (4.80)$$

□

Now we prove an evolution inequality for $\int G_3 dv$ similar as (4.40).

Proposition 4.3.3. *Under the same conditions of Proposition 4.3.1, for $\frac{\tilde{T}}{4} \leq t \leq \tilde{T}$, we have*

$$\frac{d}{dt} \int_M G_3(t) dv_{g(t)} \leq \tilde{a}' \int_M G_3(t) dv_{g(t)} - \tilde{b}' \left(\int_M G_6(t) dv_{g(t)} \right)^{1/2}, \quad (4.81)$$

where $\tilde{a}'$ and $\tilde{b}'$ are uniform positive constants independent of ϵ_0 and we may choose $\tilde{a}' = c_1 a$ and $\tilde{b}' = c_2 b$, where c_1 and c_2 are universal positive constants.

Proof. The proof is similar to that of (4.40). From the definition of G_k , we have

$$\frac{d}{dt} \int G_3 dv = \frac{d}{dt} \int |E|^3 dv + \frac{d}{dt} \int |R - \bar{R}|^3 dv + \frac{d}{dt} \int \|W^-\|^3 dv + \frac{d}{dt} \int |(F^+)_-|^3 dv. \quad (4.82)$$

Note that we shrink the time interval to $[\frac{1}{4}\tilde{T}, \tilde{T}]$, so from the previous lemma, we have known that $\int |E|^3$ is bounded by $c(\epsilon_0)$, where $c(\epsilon_0) \rightarrow 0$ as $\epsilon_0 \rightarrow 0$.

We now estimate the right hand side of (4.82) term by term

$$\begin{aligned}
\frac{d}{dt} \int |E|^3 dv &= \int \left(\frac{\partial}{\partial t} |E|^3 - R|E|^3 \right) dv \\
&= \int \left(\frac{3}{2} |E| (\Delta |E|^2 - 2|\nabla E|^2 + 4WEE - 4trE^3) \right) dv \\
&\leq -c_2 Y \left(\int |E|^6 dv \right)^{1/2} + c_1 \bar{R} \int |E|^3 dv,
\end{aligned} \tag{4.83}$$

where the inequality is established similarly to (4.58).

$$\begin{aligned}
\frac{d}{dt} \int \|W^-\|^3 dv &= \int \left(\frac{\partial}{\partial t} \|W^-\|^3 - R\|W^-\|^3 \right) dv \\
&\leq \int \left(3\|W^-\|^2 \Delta \|W^-\| + 3\sqrt{6}\|W^-\|^3 + \frac{\sqrt{6}}{2} \|W^-\|^2 |E|^2 - R\|W^-\|^3 \right) dv
\end{aligned} \tag{4.84}$$

Recall convexity inequality:

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q} \tag{4.85}$$

for $a, b \geq 0$ and $1/p + 1/q = 1$. Take $p = 3/2$, $q = 3$, $a = (\int \|W^-\|^6 dv)^{1/3}$ and $b = (\int |E|^3 dv)^{2/3}$. We have

$$\begin{aligned}
\int \|W^-\|^2 |E|^2 dv &\leq \left(\int \|W^-\|^6 dv \right)^{1/3} \left(\int |E|^3 dv \right)^{2/3} \\
&\leq \frac{2K}{3} \left(\int \|W^-\|^6 dv \right)^{1/2} + \frac{1}{3K} \left(\int |E|^3 dv \right)^2
\end{aligned} \tag{4.86}$$

We may take K to be a small multiple of the Yamabe constant and absorb the first term of (4.86) by the Sobolev inequality. The second term of (4.86) is bounded by a constant multiple of $\int |E|^3 dv$ from (4.80) and the smallness assumption of ϵ_0 . It is

then easy to derive from Sobolev inequality and Cauchy-Schwarz inequality

$$\frac{d}{dt} \int \|W^-\|^3 dv \leq -c_2 Y \left(\int \|W^-\|^6 dv \right)^{1/2} + c_1 \bar{R} \int \|W^-\|^3 dv + C \int |E|^3 dv \quad (4.87)$$

$$\begin{aligned} \frac{d}{dt} \int |(F^+)_-|^3 dv &= -\frac{d}{dt} \int ((F^+)_-)^3 dv \\ &= \int \left(-3((F^+)_-)^2 \frac{\partial}{\partial t} ((F^+)_-) - R|(F^+)_-|^3 \right) dv \\ &\leq -c_2 Y \left(\int |(F^+)_-|^6 dv \right)^{1/2} + c_1 \bar{R} \int |(F^+)_-|^3 dv \end{aligned} \quad (4.88)$$

The inequality follows from evolution inequality (4.46), Sobolev inequality and the following trick:

$$\begin{aligned} \int R|(F^+)_-|^3 dv &\leq \int (R - \bar{R})|(F^+)_-|^3 dv + \bar{R} \int |(F^+)_-|^3 dv \\ &\leq \left(\int (R - \bar{R})^2 \right)^{1/2} \left(\int |(F^+)_-|^6 dv \right)^{1/2} + C \int |(F^+)_-|^3 dv \end{aligned} \quad (4.89)$$

Note the first term of the last line can be absorbed by Sobolev inequality with the smallness assumption of ϵ_0 .

$$\begin{aligned} \frac{d}{dt} \int |R - \bar{R}|^3 dv &= \int \left(\frac{\partial}{\partial t} |R - \bar{R}|^3 - R|R - \bar{R}|^3 \right) dv \\ &= \int \left(\frac{3}{2} |R - \bar{R}| \frac{\partial}{\partial t} (R - \bar{R})^2 - R|R - \bar{R}|^3 \right) dv \end{aligned} \quad (4.90)$$

$$\begin{aligned} \frac{\partial}{\partial t} (R - \bar{R})^2 &= \frac{\partial}{\partial t} R^2 - 2 \frac{\partial}{\partial t} (R\bar{R}) + \frac{\partial}{\partial t} \bar{R}^2 \\ &= \Delta(R^2) - 2|\nabla R|^2 + 4R|E|^2 + R^3 - 2R \frac{\partial}{\partial t} \bar{R} - 2\bar{R} \frac{\partial}{\partial t} R + 2\bar{R} \frac{\partial}{\partial t} \bar{R} \\ &= \Delta(R - \bar{R})^2 - 2|\nabla(R - \bar{R})|^2 + 4(R - \bar{R})|E|^2 + R^2(R - \bar{R}) - 2(R - \bar{R}) \frac{\partial}{\partial t} \bar{R} \end{aligned} \quad (4.91)$$

Recall the evolution equation of $\bar{R}$ under Ricci flow:

$$\frac{\partial}{\partial t} \bar{R} = \frac{1}{vol} \int \left(2|E|^2 - \frac{1}{2} R^2 \right) dv + \bar{R}^2. \quad (4.92)$$

Plugging (4.91) and (4.92) into (4.90), we can derive

$$\begin{aligned} \frac{d}{dt} \int |R - \bar{R}|^3 dv &\leq \int \left(\frac{3}{2} |R - \bar{R}| \Delta (R - \bar{R})^2 - R |R - \bar{R}|^3 \right) dv \\ &\quad + 6 \int (R - \bar{R})^2 |E|^2 dv + \frac{3}{2} \int |R^2 - \bar{R}^2| (R - \bar{R})^2 dv \\ &\quad + \frac{3}{2} \left| \bar{R}^2 - \frac{\int R^2 dv}{vol} \right| \int (R - \bar{R})^2 dv + \frac{6}{vol} \int |E|^2 dv \int (R - \bar{R})^2 dv. \end{aligned} \quad (4.93)$$

Now we estimate the terms in second and third line of (4.93):

$$\begin{aligned} \int (R - \bar{R})^2 |E|^2 dv &\leq \left(\int |R - \bar{R}|^6 dv \right)^{1/3} \left(\int |E|^3 dv \right)^{2/3} \\ &\leq \frac{2K}{3} \left(\int |R - \bar{R}|^6 dv \right)^{1/2} + \frac{1}{3K} \left(\int |E|^3 dv \right)^2 \end{aligned} \quad (4.94)$$

We may take K to be a small multiple of the Yamabe constant and absorb the first term of (4.94) by the Sobolev inequality. The second term of (4.94) is bounded by a constant multiple of $\int |E|^3 dv$ from (4.80) and the smallness assumption of ϵ_0 .

$$\begin{aligned} \int |E|^2 dv \int (R - \bar{R})^2 dv &\leq C \left(\int |E|^3 dv \right)^{2/3} \left(\int (R - \bar{R})^6 dv \right)^{1/3} \\ &\leq CK \left(\int (R - \bar{R})^6 dv \right)^{1/2} + \frac{C}{K} \left(\int |E|^3 dv \right)^2 \end{aligned} \quad (4.95)$$

The C in first line just depends on volume. For the second line, we may take K to be a small multiple of the Yamabe constant and absorb the first term of (4.95) by the Sobolev inequality. The second term of (4.95) is bounded by a constant multiple of

$\int |E|^3 dv$ from (4.80) and the smallness assumption of ϵ_0 .

$$\begin{aligned}
\int |R^2 - \bar{R}^2|(R - \bar{R})^2 dv &= \int |R + \bar{R}||R - \bar{R}|^3 dv \\
&\leq \bar{R} \int (R - \bar{R})^3 dv + \int |R||R - \bar{R}|^3 dv \\
&\leq 2\bar{R} \int (R - \bar{R})^3 dv + \int |R - \bar{R}|^4 dv \\
&\leq 2\bar{R} \int (R - \bar{R})^3 dv + \left(\int (R - \bar{R})^2 dv \right)^{1/2} \left(\int |R - \bar{R}|^6 dv \right)^{1/2}
\end{aligned} \tag{4.96}$$

The last term can be absorbed by Sobolev inequality from the smallness assumption of ϵ_0 .

$$\begin{aligned}
\left| \bar{R}^2 - \frac{\int R^2 dv}{vol} \right| \int (R - \bar{R})^2 dv &= \frac{1}{vol} \left(\int (R - \bar{R})^2 dv \right)^2 \\
&\leq C \left(\int (R - \bar{R})^2 dv \right)^{1/2} \left(\int (R - \bar{R})^6 dv \right)^{1/2}
\end{aligned} \tag{4.97}$$

This term can be absorbed by Sobolev inequality from the smallness assumption of ϵ_0 .

Combining all these estimates for $|R - \bar{R}|$, we derive

$$\frac{d}{dt} \int |R - \bar{R}|^3 dv \leq -c_2 Y \left(\int |R - \bar{R}|^6 dv \right)^{1/2} + c_1 \bar{R} \int |R - \bar{R}|^3 dv + C \int |E|^3 dv \tag{4.98}$$

Now we combine (4.83)(4.84)(4.88)(4.98) to derive 4.81. $\square$

Lemma 4.3.9. *With the modified Yamabe metric chosen as initial metric, under the Ricci flow, we have*

$$\sup_{\tilde{T}/2 \leq t \leq \tilde{T}} \int G_3(t) dv_{g(t)} \leq C \epsilon_0^{3/2}.$$

Proof. The proof is fundamentally the same as that of Lemma 4.3.8. And the only difference is to replace $|E|^k$ by $G_k(t)$ since we have evolution equations of same type as is shown in Proposition 4.3.1 and Proposition 4.3.3. $\square$

4.4 C^∞ estimates and convergence

In this section, we shall prove C^∞ estimates and finish the proofs of Theorems 3.1.1 and 3.1.2.

We first derive the L^∞ estimates. To this end, we shall apply the following result established by D. Yang in [45].

Lemma 4.4.1. *Assume that with respect to the metric $g = g(t)$, $0 \leq t \leq T$, the following Sobolev inequality holds:*

$$\left(\int |\varphi|^{\frac{2n}{n-2}} dv \right)^{\frac{n-2}{n}} \leq C_S \left[\int |\nabla \varphi|^2 dv + \int \varphi^2 dv \right], \varphi \in W^{1,2}(M^n). \quad (4.99)$$

Also, let $b \geq 0$ on $M^n \times [0, T]$ satisfy

$$\frac{\partial}{\partial t} dv \leq b dv. \quad (4.100)$$

Let $q > n$, and suppose $u \geq 0$ is a function on $M^n \times [0, T]$ satisfying

$$\frac{\partial u}{\partial t} \leq \Delta u + bu, \quad (4.101)$$

and that

$$\sup_{0 \leq t \leq T} |b|_{L^{q/2}} \leq \beta. \quad (4.102)$$

Given $p_0 > 1$, there exists a constant $C = C(n, q, p_0, C_S, \beta)$ such that for $0 \leq t \leq T$,

$$|u(t, \cdot)|_\infty \leq C e^{Ct} t^{-\frac{n}{2p_0}} |u(0, \cdot)|_{p_0}. \quad (4.103)$$

Moreover, given $p \geq p_0 > 1$, the following inequality holds for $0 \leq t \leq T$:

$$\frac{d}{dt} \int u^p dv + \int |\nabla (u^{p/2})|^2 dv \leq Cp^{\frac{2n}{q-n}} \int u^p dv \quad (4.104)$$

where $C = C(n, q, p_0, C_S)$.

Lemma 4.4.2. *With the modified Yamabe metric chosen as the initial metric, we have*

$$\sup_{3\tilde{T}/4 \leq t \leq \tilde{T}} \{|E| + |R - \bar{R}| + ||W^-|| + |F_-^+|\} \leq C\epsilon_0.$$

Proof. Apply Lemma 4.4.1 to $u = G_2$, $q = 6 > n = 4$ and $p_0 = 3/2$ on $t \in [\tilde{T}/2, \tilde{T}]$. Condition (4.102) is satisfied by Lemma 4.3.9. Hence, we can prove the desired estimate. $\square$

Now we are at the position to prove C^∞ estimates and complete the proof of Theorem 3.1.1.

Proof of Theorem 3.1.1. We argue by contradiction. Suppose there is a sequence of manifolds (M_j, g_j) satisfying $\beta(M_j, [g_j]) < 4(1 + \epsilon_j)$ with $\epsilon_j \rightarrow 0$ and each of them is *not* diffeomorphic to standard $\mathbb{CP}^2$. For each conformal class $[g_j]$, we choose the modified Yamabe metric $(g_j)_G$ as initial metric and evolve the metric along Ricci flow. Then Lemma 4.4.2 and Lemma 2.4.2 will imply that there is a time $\tilde{T}$ such that the curvatures of $g_j(\tilde{T})$ are uniformly bounded in C^∞ norm and the Sobolev constants are also uniformly bounded. The convergence theory [11] established by J. Cheeger, M. Gromov and M. Taylor then shows that there is a subsequence of $\{(M_j, g_j(\tilde{T}))\}$ which converges smoothly to a manifold (M_∞, g_∞) . As $\epsilon_j \rightarrow 0$, we obtain that (M_∞, g_∞) satisfies

$$\int_{M_\infty} ||W||^2 dv_\infty = 4 \int_{M_\infty} \sigma_2 dv_\infty$$

Note that we also have $b_1(M_\infty) = 0$, $b_2^+(M_\infty) = 1$ and $b_2^-(M_\infty) = 0$. Hence, by Chern-Gauss-Bonnet formula and signature theorem, we can easily derive that (M_∞, g_∞) is

self-dual Einstein. The same argument in Lemma 4.1.8 will show that (M_∞, g_∞) is conformal equivalent to $(\mathbb{CP}^2, g_{FS})$. Since the convergence is smooth, we then obtain that (M_j, g_j) must be diffeomorphic to $\mathbb{CP}^2$ with standard differentiable structure when j is sufficiently large. This is clearly a contradiction to our assumption. Hence, we have proved the theorem. $\square$

The proof of Theorem 3.1.2 is a direct application of Theorem 3.1.1.

Proof of Theorem 3.1.2. To prove Theorem 3.1.2, suppose M^4 is oriented with $b_2^+(M^4) > 0$. If $\beta(M^4) = 4$, then by definition we can find a metric g with

$$\beta(M^4, [g]) < 4(1 + \epsilon/2),$$

where $\epsilon > 0$ is from Theorem 3.1.1. From Theorem 3.1.1 we conclude that M^4 is diffeomorphic to $\mathbb{CP}^2$. In addition, if g is a metric on $\mathbb{CP}^2$ for which $\beta(M^4, [g]) = 4$, then taking $\epsilon = 0$ in Lemma 4.1.6 we see that g is self-dual. It follows, for example, from Theorem 2.6.2 that (M^4, g) is conformally equivalent to $(\mathbb{CP}^2, g_{FS})$. $\square$

Part III

Conformally Invariant Rigidity Theorems

Chapter 5

Statements of main results and ideas of the proofs

5.1 Statements of main results

In this section, several conformally invariant rigidity theorems will be stated. All these results can be viewed as generalizations of theorems stated in Part I.

Theorem 5.1.1. *Suppose (M^4, g) is an oriented, closed, compact Bach-flat manifold and $b_2^+(M^4) > 0$. There is an $\epsilon_1 > 0$ such that if $g \in \mathcal{V}_2^+(M^4)$ with*

$$4 \leq \beta(M^4, [g]) < 4(1 + \epsilon_1), \tag{5.1}$$

then (M^4, g) is conformally equivalent to $(\mathbb{CP}^2, g_{FS})$, where g_{FS} is the Fubini-Study metric.

Theorem 5.1.2. *Suppose (M^4, g) is an oriented, closed, compact Bach-flat manifold and $b_2^+(M^4) = b_2^-(M^4) > 0$. There is an $\epsilon_2 > 0$ such that if $g \in \mathcal{V}_2^+(M^4)$ with*

$$8 \leq \beta(M^4, [g]) < 8(1 + \epsilon_2), \tag{5.2}$$

then (M^4, g) is conformally equivalent to $(S^2 \times S^2, g_p)$, where g_p is the standard product metric.

Theorem 5.1.1 and Theorem 5.1.2 are essentially established in [46]. These two theorems generalize Theorem 1.1.4 to the model cases of $(\mathbb{CP}^2, g_{FS})$ and $(S^2 \times S^2, g_p)$.

5.2 Ideas of proofs

For the rest of this section, I shall briefly discuss the ideas of the proofs.

The proof of Theorem 5.1.1 also contains two steps. The first step is the same as of the proof of Theorem 3.1.1 to choose the modified Yamabe metric as the conformal representative. The second step, instead of applying the Ricci flow, will be to apply a Moser-theoretic argument to derive L^4 estimate from the elliptic equation satisfied by Bach-flat metric. From here, an argument involving conformally invariant Sobolev inequality will imply the metric is self-dual and Einstein. A celebrated theorem of N. Hitchin [26] (see also Chapter 13 of [1]) will show the manifold is isometric to the Fubini-Study metric. However, in our special case, we shall provide an independent proof of this fact.

The proof of Theorem 5.1.2 is a bit different. We shall choose the Yamabe metric as the conformal representative. Then we shall apply the convergence theory of Tian-Viaclovsky and a removable singularity theorem from [34] to obtain a limit metric isometric to $(S^2 \times S^2, g_p)$. From here, a stability result established by M. Gursky and J. Viaclovsky in [24] for Bach-flat metric will finish the proof.

Chapter 6

Proofs of main results

In this chapter, we prove Theorems 5.1.1 and 5.1.2. The main arguments are based on [46].

6.1 Rigidity of $\mathbb{CP}^2$ via Moser's iteration

In this section, we prove Theorem 5.1.1 by a Moser-type iteration argument.

Recall from Lemma 4.1.4, we may assume $b_2^+(M) = 1$ and $b_2^-(M) = 0$ for $\epsilon_1 < 1$. The strategy is to choose the modified Yamabe metric and show that the metric is self-dual and Einstein with positive scalar curvature if ϵ_1 is chosen to be sufficiently small. By abusing notation, metric in this subsection is the modified Yamabe metric in its conformal class with volume equal to one. From Lemma 4.1.7, we have

$$\int |E|^2 dv < c(\epsilon_1), \int ||W^-||^2 dv < c(\epsilon_1), \int (R - \bar{R})^2 dv < c(\epsilon_1), R - 2\sqrt{6}||W^+|| = -\mu, \quad (6.1)$$

where $\mu \geq 0$ is a constant on the manifold and $\mu \rightarrow 0$ and $c(\epsilon_1) \rightarrow 0$ as $\epsilon_1 \rightarrow 0$. Note that $\bar{R}$ is also bounded since the volume is normalized to be one.

We first state an algebraic lemma.

Lemma 6.1.1.

$$W_{ijkl}^{\pm} E_{ik} E_{jl} \leq \frac{\sqrt{6}}{3} \|W^{\pm}\| |E|^2 \quad (6.2)$$

Proof. This lemma is probably known in the work of [8][9]. Here we only sketch the proof of this inequality. Recall the well-known decomposition of Singer-Thorpe:

$$Riem = \begin{pmatrix} W^+ + \frac{R}{12} Id & B \\ B^* & W^- + \frac{R}{12} Id \end{pmatrix} \quad (6.3)$$

Note the compositions satisfy

$$BB^* : \Lambda_+^2 \rightarrow \Lambda_+^2, \quad B^*B : \Lambda_-^2 \rightarrow \Lambda_-^2.$$

Fix a point $P \in M^4$, and let $\lambda_1^{\pm} \leq \lambda_2^{\pm} \leq \lambda_3^{\pm}$ denote the eigenvalues of $W^{\pm}$, where $W^{\pm}$ are interpreted as endomorphisms of $\Lambda_{\pm}^2$. Also denote the eigenvalues of $BB^* : \Lambda_+^2 \rightarrow \Lambda_+^2$ by $b_1^2 \leq b_2^2 \leq b_3^2$, where $0 \leq b_1 \leq b_2 \leq b_3$. From similar computation of [8], we have

$$W_{ijkl}^+ E_{ik} E_{jl} = 4 \langle W^+, BB^* \rangle_{\Lambda_+^2}, \quad W_{ijkl}^- E_{ik} E_{jl} = 4 \langle W^-, B^*B \rangle_{\Lambda_-^2} \quad (6.4)$$

From Lemma 6 of [36], we have

$$W_{ijkl}^{\pm} E_{ik} E_{jl} \leq 4 \sum_{i=1}^3 \lambda_i^{\pm} b_i^2 \quad (6.5)$$

Recall from Lemma 4.2 of [8] that $|E|^2 = 4 \sum_{i=1}^3 b_i^2$. For a trace-free 3×3 matrix A , we have the sharp inequality:

$$|A(X, X)| \leq \frac{\sqrt{6}}{3} \|A\| |X|^2. \quad (6.6)$$

Apply (6.6) to $A = \text{diag}(\lambda_1^\pm, \lambda_2^\pm, \lambda_3^\pm)$ and $X = (b_1, b_2, b_3)$. We derive

$$4 \sum_{i=1}^3 \lambda_i^\pm b_i^2 \leq \frac{\sqrt{6}}{3} \|W^\pm\| \|E\|^2. \quad (6.7)$$

Combining (6.5) and (6.7), we derive the desired inequality.

□

The basic estimates are from Cauchy-Schwarz inequality:

$$\int |tr E^3| dv \leq \left(\int |E|^2 dv \right)^{1/2} \left(\int |E|^4 dv \right)^{1/2} \leq c(\epsilon_1) \left(\int |E|^4 dv \right)^{1/2}, \quad (6.8)$$

$$\int W_{ijkl}^- E_{ik} E_{jl} dv \leq \left(\int \|W^-\|^2 dv \right)^{1/2} \left(\int |E|^4 dv \right)^{1/2} \leq c(\epsilon_1) \left(\int |E|^4 dv \right)^{1/2}, \quad (6.9)$$

$$\begin{aligned} \int W_{ijkl}^+ E_{ik} E_{jl} dv &\leq \int \left(\frac{\sqrt{6}}{3} \|W^+\| - \frac{R}{6} \right) |E|^2 dv + \int \frac{R}{6} |E|^2 dv \\ &= \frac{\mu}{6} \int |E|^2 dv + \int \frac{R}{6} |E|^2 dv \end{aligned} \quad (6.10)$$

$$\begin{aligned} \int R |E|^2 dv &= \int (R - \bar{R}) |E|^2 dv + \bar{R} \int |E|^2 dv \\ &\leq \left(\int (R - \bar{R})^2 dv \right)^{1/2} \left(\int |E|^4 dv \right)^{1/2} + C \int |E|^2 dv. \\ &\leq c(\epsilon_1) \left(\int |E|^4 dv \right)^{1/2} + C \int |E|^2 dv. \end{aligned} \quad (6.11)$$

Now apply conformally invariant Sobolev inequality

$$\frac{Y}{6} \left(\int |E|^4 dv \right)^{1/2} \leq \int \left(|\nabla E|^2 + \frac{R}{6} |E|^2 \right) dv \quad (6.12)$$

to (2.22). We obtain

$$\begin{aligned}
\frac{Y}{6} \left(\int |E|^4 dv \right)^{1/2} &\leq \frac{1}{12} \int |\nabla R|^2 dv - \int 2tr E^3 dv - \int \frac{R}{6} |E|^2 dv \\
&\quad + \int 2W_{ijkl} E_{ik} E_{jl} dv \\
&\leq c(\epsilon_1) \left(\int |E|^4 dv \right)^{1/2} + \frac{1}{12} \int |\nabla R|^2 dv + C \int |E|^2 dv.
\end{aligned} \tag{6.13}$$

Since $R - 2\sqrt{6}||W^+|| = -\mu$ is a constant, we have

$$\nabla R = 2\sqrt{6}\nabla||W^+||.$$

Integrate over the manifold. We have by Kato inequality

$$\int |\nabla R|^2 dv = 24 \int |\nabla||W^+||^2 dv \leq C \int |\nabla W^+|^2 dv. \tag{6.14}$$

Recall the sharp estimate

$$72 \det W^+ \leq 72 \cdot \frac{\sqrt{6}}{18} \cdot \frac{1}{8} |W^+|^3 = \frac{\sqrt{6}}{2} |W^+|^3 = \frac{R + \mu}{2} |W^+|^2. \tag{6.15}$$

Now apply (6.10)(6.11)(6.15) to (2.23). We easily derive

$$\int |\nabla W^+|^2 dv \leq \frac{\mu}{2} \int |W^+|^2 dv + C \int |E|^2 dv + c(\epsilon_1) \left(\int |E|^4 dv \right)^{1/2} \tag{6.16}$$

Combining (6.13)(6.14)(6.16), we derive

$$\frac{Y}{6} \left(\int |E|^4 dv \right)^{1/2} \leq C\mu \int |W^+|^2 dv + c(\epsilon_1) \left(\int |E|^4 dv \right)^{1/2} + C \int |E|^2 dv \tag{6.17}$$

From Lemma 4.1.7 and the fact that $c(\epsilon_1) \rightarrow 0$ as $\epsilon_1 \rightarrow 0$, we can take sufficiently small ϵ_1 to absorb the second term on the right hand side and derive from (4.28) and

(4.29)

$$\left(\int |E|^4 dv \right)^{1/2} \leq C \int ||W^-||^2 dv. \quad (6.18)$$

It is now easy to obtain from (2.23):

$$\begin{aligned} \frac{Y}{6} \left(\int |W^-|^4 dv \right)^{1/2} &\leq \int \left(|\nabla W^-|^2 + \frac{R}{6} |W^-|^2 \right) dv \\ &= \int (72 \det W^- + 2W_{ijkl}^- E_{ik} E_{jl}) dv \\ &\quad - \int \left(\frac{R}{3} - \frac{\bar{R}}{3} \right) |W^-|^2 dv - \frac{\bar{R}}{3} \int |W^-|^2 dv \\ &\leq c(\epsilon_1) \left(\int |W^-|^4 dv \right)^{1/2} + c(\epsilon_1) \left(\int |E|^4 dv \right)^{1/2} \\ &\quad - \frac{\bar{R}}{3} \int |W^-|^2 dv \\ &\leq c(\epsilon_1) \left(\int |W^-|^4 dv \right)^{1/2} + \left(c(\epsilon_1) - \frac{\bar{R}}{3} \right) \int |W^-|^2 dv \end{aligned} \quad (6.19)$$

The first inequality follows from conformally invariant Sobolev inequality. The second inequality is from Cauchy-Schwarz inequality and (4.63). The third inequality is from (6.18). Recall $c(\epsilon_1) \rightarrow 0$ as $\epsilon_1 \rightarrow 0$ and

$$\bar{R} \geq Y \geq \frac{24\pi}{\sqrt{2 + \epsilon_1}}.$$

It is then clear to see that we may take sufficiently small ϵ_1 to obtain $W^- \equiv 0$ from (6.19). Recall that $b_2^+(M) = 1$. Hence, from Theorem 2.6.2, we can derive (M^4, g) is conformally equivalent to $(\mathbb{CP}^2, g_{FS})$.

6.2 Rigidity of $S^2 \times S^2$ via convergence of Bach-flat metrics

In this section, we prove Theorem 5.1.2. We start from a lemma which enables us to distinguish Euclidean space from complete non-compact Ricci flat Riemannian manifolds via comparison of the Yamabe constant. It was proved by G. Li, J. Qing, and Y. Shi in [34].

Lemma 6.2.1. *Suppose (M^4, g) is a complete non-compact Ricci flat Riemannian manifold. Then (M^4, g) is isometric to the Euclidean space $(\mathbb{R}^4, g_E)$ if*

$$Y(M, g) \geq \eta Y(\mathbb{R}^4, g_E) \quad (6.20)$$

for some $\eta > \frac{\sqrt{2}}{2}$.

Remark 6.2.2. *This lemma shows that there is a gap phenomenon for the Yamabe constant for non-compact Ricci flat Riemannian manifold.*

We now prove a similar lemma under the pinching condition of Theorem 5.1.2.

Lemma 6.2.3. *Let (M^4, g) be a closed, compact oriented Riemannian four-manifold with $b_2^+(M^4) = b_2^-(M) > 0$ and*

$$\beta(M^4, [g]) = 8(1 + \epsilon) \quad (6.21)$$

for sufficiently small $\epsilon > 0$, then we have for the modified Yamabe metric $g_m \in [g]$

$$\int_M \|W_{g_m}^+\|^2 dv_{g_m} = \int_M \|W_{g_m}^-\|^2 dv_{g_m} = \left(\frac{1 + \epsilon}{3 + 2\epsilon} \right) 32\pi^2, \quad (6.22)$$

$$Y(M^4, [g]) \geq 16\pi \sqrt{\frac{3}{3 + 2\epsilon}}, \quad (6.23)$$

$$\int |E_{g_m}|^2 dv_{g_m} \leq \frac{64\epsilon}{3+2\epsilon} \pi^2. \quad (6.24)$$

Proof. From Corollary F of [20], $g \in \mathcal{Y}_2^+(M)$ implies that $b_1(M) = 0$. Therefore, $\chi = 2 + b_2(M) = 2 + 2b_2^+(M)$. Then Chern-Gauss-Bonnet and signature formulas imply

$$4 \int \sigma_2(P) dv + \int \|W\|^2 dv = 8\pi^2(2 + 2b_2^+(M)), \quad (6.25)$$

$$\int \|W^+\|^2 dv = \int \|W^-\|^2 dv. \quad (6.26)$$

The condition $\beta(M, [g]) = 8(1 + \epsilon)$ reads

$$4 \int \sigma_2(P) dv = \frac{1}{2(1 + \epsilon)} \int \|W\|^2 dv. \quad (6.27)$$

From (6.25)(6.26)(6.27), we solve

$$\int \sigma_2(P) dv = \frac{4\pi^2}{3+2\epsilon}(1 + b_2^+(M)), \quad \int \|W^\pm\|^2 dv = \frac{(1 + \epsilon)16\pi^2}{3+2\epsilon}(1 + b_2^+(M)). \quad (6.28)$$

We first show that $b_2^+(M) = 1$ if ϵ is chosen sufficiently small. We now argue by contradiction. Suppose $b_2^+(M) \geq 2$. Then from (6.28)

$$\int \sigma_2(P) dv \geq \frac{12\pi^2}{3+2\epsilon}.$$

Theorem 1.1.4 then implies (M, g) is conformally equivalent to $(\mathbb{S}^4, g_c)$ if ϵ is chosen sufficiently small. This contradicts to the fact $b_2^+(M) > 0$. Hence, $b_2^+(M) = 1$ and $\chi(M) = 4$. Now we have from (6.28)

$$\int \sigma_2(P) dv = \frac{8\pi^2}{3+2\epsilon}, \quad \int \|W^\pm\|^2 dv = \left(\frac{1 + \epsilon}{3+2\epsilon} \right) 32\pi^2. \quad (6.29)$$

With same argument from the proof of Lemma 2.7 in [9], we derive

$$\frac{1}{96}Y^2 \geq \int \sigma_2(P)dv \geq \frac{8\pi^2}{3+2\epsilon}. \quad (6.30)$$

Now we have proved (6.22)(6.23). Note that these two inequalities are conformally invariant.

We now choose the modified Yamabe metric $g_m \in [g]$ and Theorem 2.3.4 implies

$$\frac{1}{24} \int R_{g_m}^2 dv_{g_m} \leq \int \|W_{g_m}^+\|^2 dv_{g_m} = \left(\frac{1+\epsilon}{3+2\epsilon} \right) 32\pi^2. \quad (6.31)$$

Recall

$$\int \sigma_2(P)dv = \frac{1}{96} \int R_{g_m}^2 dv_{g_m} - \frac{1}{8} \int |E_{g_m}|^2 dv_{g_m}.$$

From (6.29)(6.31) we easily derive (6.24). $\square$

Proof of Theorem 5.1.2. Note that (6.22)(6.23) are both conformally invariant. In addition, recall that $\int \sigma_2(P)dv = \frac{1}{96} \int R^2 dv - \frac{1}{8} \int |E|^2 dv$ is conformally invariant and the Yamabe metric realizing the infimum of $\int R^2 dv$ in a conformal class. It is then clear that

$$\int |E_{g_Y}|^2 dv_{g_Y} \leq \int |E_{g_m}|^2 dv_{g_m} \leq \frac{64\epsilon}{3+2\epsilon} \pi^2. \quad (6.32)$$

We now start proving Theorem 5.1.1 and argue by contradiction. Assume that there exists a sequence of Bach-flat 4-manifolds $(M_j^4, (g_Y)_j)$ which are not conformally equivalent to $(\mathbb{S}^2 \times \mathbb{S}^2, g_{prod})$ with $\epsilon_j \rightarrow 0$, where $(g_Y)_j$ denotes the Yamabe metric with $R_{(g_Y)_j} = 4$. By (6.23)(6.32), we have

$$Y(M_j, (g_Y)_j) \geq 16\pi \sqrt{\frac{3}{3+2\epsilon_j}} \rightarrow 16\pi. \quad (6.33)$$

$$\int_{M_j} |E_{(g_Y)_j}|^2 dv_{(g_Y)_j} \rightarrow 0. \quad (6.34)$$

Now we first prove that there is no point of curvature concentration. We still argue by contradiction. Note that a uniform lower bound on the Yamabe constants imply a uniform lower bound on the Euclidean volume growth for such sequence of manifolds. Then standing at the point of curvature blow-up, that is, $p_j \in M_j$ such that

$$\lambda_j = |Rm_{(g_Y)_j}|(p_j) = \max |Rm_{(g_Y)_j}| \rightarrow \infty. \quad (6.35)$$

Now consider the sequence of pointed Riemannian manifold (M_j, g_j, p_j) with the re-scaled metric $g_j = \lambda_j^2(g_Y)_j$. Therefore, according to the curvature estimates established in [43], there exists a subsequence of (M_j, g_j, p_j) which converges to a complete non-compact manifold $(M_\infty, g_\infty, p_\infty)$ in the Cheeger-Gromov topology. It follows that

- $Y(M_\infty, g_\infty) \geq 16\pi$ and
- $Ric(g_\infty) = 0$.

Here we use similar argument in the proof of Lemma 3.4 of [34] to prove $Y(M_\infty, g_\infty) \geq 16\pi$. Note that

$$Y(\mathbb{R}^4, g_E) = Y(S^4, g_c) = 8\sqrt{6}\pi. \quad (6.36)$$

Hence,

$$Y(M_\infty, g_\infty) \geq \frac{\sqrt{6}}{3} Y(\mathbb{R}^4, g_E). \quad (6.37)$$

Note $\frac{\sqrt{6}}{3} > \frac{\sqrt{2}}{2}$. Hence, Lemma 6.2.1 implies that (M_∞, g_∞) is isometric to Euclidean 4-space, which is a contradiction to the fact that $|Rm_{g_\infty}|(p_\infty) = 1$. Therefore, there is no point of concentration of curvature.

The convergence theory developed in [42][43] implies that $(M_j^4, (g_Y)_j)$ converges in C^∞ -norms to a smooth Einstein manifold $(\widetilde{M}, \widetilde{g})$ satisfying $\beta(\widetilde{M}, [\widetilde{g}]) = 8$ and $b_2^+(\widetilde{M}) = b_2^-(\widetilde{M}) = 1$. Hence, from Theorem 3.1.3, $(\widetilde{M}, \widetilde{g})$ is (up to rescaling) isometric to $(\mathbb{S}^2 \times \mathbb{S}^2, g_{prod})$. Recall that $(\mathbb{S}^2 \times \mathbb{S}^2, g_{prod})$ is Bach-rigid in the sense that the metric is an isolated Bach-flat metric (see Theorem 7.13 of [24]). It follows that $(M_j^4, (g_Y)_j)$

must be conformally equivalent to $(\mathbb{S}^2 \times \mathbb{S}^2, g_{\text{prod}})$ for sufficiently large j , which is a contradiction. Therefore, we have proved Theorem 5.1.2. $\square$

Remark 6.2.4. *It is not hard to see that we may prove Theorem 5.1.1 with similar argument as we did in the proof of Theorem 5.1.2. The current proof of Theorem 5.1.1 has two advantages: first, it avoids making use of the sophisticated general theory of convergence for Bach-flat metrics; second, it is possible to compute a numerical constant of ϵ_1 if we track the coefficients carefully.*

Remark 6.2.5. *Although it is not possible to get a numerical constant for ϵ_2 from the current proof, we still know that ϵ_2 cannot be too large. On $N = \mathbb{CP}^2 \# \overline{\mathbb{CP}}^2$, there is an Einstein metric g_P constructed by D. Page [38] satisfying $\beta(N, [g_P]) = 8.4 \dots$. It is then clear that $\epsilon_2 < \frac{1}{16}$.*

Some Open Problems

In this short chapter, we survey some interesting problems left unsolved concerning $\mathcal{Y}_2^+(M)$. An ambitious program is to classify (up to diffeomorphism type) the class of (oriented) closed compact four-manifolds admitting metrics in $\mathcal{Y}_2^+$. It is clear that all Einstein four-manifolds of positive scalar curvature are included in this class. In this direction, we list some conjectures inspired by Theorems 3.1.1, 3.1.2, and 3.1.3.

Conjecture 1. *If*

$$\beta(M^4) = 4, \tag{6.38}$$

then M^4 is diffeomorphic to $\pm\mathbb{CP}^2$. Moreover, $\beta(M^4, [g]) = 4$ if and only if $g \in [g_{FS}]$.

Conjecture 2. *If M^4 is oriented and admits a metric $g \in \mathcal{Y}_2^+(M^4)$ with*

$$0 \leq \beta(M^4, [g]) < 8, \tag{6.39}$$

then M^4 is diffeomorphic to S^4 or $\pm\mathbb{CP}^2$.

Conjecture 3. *Suppose M^4 is oriented and $b_2^+(M) = b_2^-(M) = 1$. There is an $\epsilon > 0$ such that if M^4 admits a metric $g \in \mathcal{Y}_2^+(M^4)$ with*

$$8 \leq \beta(M^4, [g]) < 8(1 + \epsilon), \tag{6.40}$$

then M^4 is diffeomorphic to $S^2 \times S^2$.

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