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Effective boundedness results in algebraic and analytic geometry

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EFFECTIVE BOUNDEDNESS RESULTS IN ALGEBRAIC AND
ANALYTIC GEOMETRY

GABRIELE DI CERBO

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Abstract

The goal of this thesis is to study several aspects arising from the notion of boundedness of algebraic varieties.

In the first part of the thesis we will focus on the problem of stability of pluricanonical maps. It basically says that the maps associated to the canonical divisor of a variety of positive Kodaira dimension stabilize uniformly. We give a positive answer to this problem in the special case the Iitaka fibration has maximal variation and its image is not uniruled. Moreover, we provide effective bounds.

It turns out that in order to extend these results when the image of the Iitaka fibration is uniruled, one needs to study the pseudo-effective threshold. The problem reduces to proving that in the set of pseudo-effective thresholds, 1 is not an accumulation point from below. Motivated by this problem, we study in details the accumulation points of this set. In particular, we prove Fujita's log spectrum conjecture. Roughly speaking, it says that in the set of pseudo-effective thresholds the accumulation points can be reached only from above.

In the second part of this work we work out similar properties of the nef threshold. For example, we characterize the pairs (X, D) such that $K_X + D$ is a limit of ample divisors of the form $K_X + \alpha D$. This result turns out to have applications in the study of Kähler-Einstein metrics with cone-edge singularities, described by the cone-angle $2\pi(1 - \alpha)$ for $\alpha \in (0, 1)$. We show that in the negative scalar curvature case, if such Kähler-Einstein metrics exist for all small cone-angles then they exist for every $\alpha \in (\frac{n+1}{n+2}, 1)$, where n is the dimension. We also give a characterization of the pairs that admit negatively curved cone-edge Kähler-Einstein metrics with cone-angle close to 2π . Finally, we show how in the positive scalar curvature case the existence of such uniform bounds is obstructed. We characterize the obstruction using boundedness of Fano varieties.

These methods developed for Kähler-Einstein metrics can be used to investigate the geometry of cusped complex hyperbolic manifolds. We derive effective very ampleness results for toroidal compactifications of finite volume complex hyperbolic manifolds. We give effective bounds on the number of complex hyperbolic manifolds with given upper bounds on the volume. Moreover, we estimate the number of ends of such manifolds in terms of their volume.

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A mia madre

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Chapter 1

Introduction

1.1 Overview of the results

In this section we give a brief overview of the results presented in this thesis. The following conjecture is the motivating problem of a great part of this work.

Conjecture 1.1.1. *Fix two positive integers n and κ . Then there is a positive integer $m_{n,\kappa}$ such that for any m divisible by $m_{n,\kappa}$, the pluricanonical map $\phi_{|mK_X|}$ is birationally equivalent to the Iitaka fibration of X for all smooth projective varieties X of dimension n and Kodaira dimension κ .*

The conjecture is known in the case $\kappa = n$, see [Tsu06], [Tak06] and [HM06]. If $\kappa < n$, Conjecture 1.1.1 can be divided in two different problems using the canonical bundle formula. Basically, it tells us that in order to control $m_{n,\kappa}$, we need boundedness results both for the base and for the general fiber of the Iitaka fibration. In this work, we will focus on boundedness results on the base. In particular, we will study the birationality of certain adjoint linear systems on the base of the Iitaka fibration. More precisely, we investigate the following problem.

Conjecture 1.1.2. *Fix a positive integer n and a DCC set $I \subset [0, 1]$. Then there exists a positive integer $m_{n,I}$ such that if*

1. (X, Δ) is a klt pair with $\dim(X) = n$,
2. coefficients of Δ are in I ,
3. M is a nef Cartier divisor such that $K_X + \Delta + M$ is big,

then the linear series $|m(K_X + \Delta + M)|$ defines a birational map for any $m \geq m_{n,I}$.

Recall that a set I satisfies the DCC if every non-increasing sequence in I is eventually constant. Note that Conjecture 1.1.2 does not imply Conjecture 1.1.1, we also need to assume the boundedness of some numerical parameters which appear in the canonical bundle formula and depend on the general fiber of the Iitaka fibration. This is a very interesting problem on its own but it will not be discussed in this work. See Section 3.4 for more details.

Our first result is the following.

Theorem 1.1.3. *Let (X, Δ) be a klt pair. Let M be a big and nef Cartier divisor on X and E a pseudo-effective $\mathbb{Q}$ -divisor on X . Then for any*

$$m > \binom{n+2}{2}$$

the map induced by $|[K_X + \Delta + E + mM]|$ is birational.

Theorem 1.1.3 gives a partial positive answer to Conjecture 1.1.2 when $E = (K_X + \Delta)$ is pseudo-effective and M is big. It is quite interesting that, in this case, we do not need any assumption on the coefficients of Δ . Moreover, Theorem 1.1.3 implies Conjecture 1.1.1 in the case when f has maximal variation and the image of f is not uniruled.

Corollary 1.1.4. *Let X be a smooth projective variety of Kodaira dimension κ and $f : X \dashrightarrow Y$ be the Iitaka fibration. Assume that*

1. *Y is not uniruled;*
2. *f has maximal variation;*
3. *the generic fiber F has a good minimal model.*

Then for any

$$m > b_F \binom{\kappa+2}{2}$$

divisible by b_F , the pluricanonical map $\phi_{|mK_X|}$ is birationally equivalent to f .

The number b_F is the contribution given by the general fiber F of the Iitaka fibration in the canonical bundle formula. Conjecturally, F always has a good minimal model and by results of Kawamata and Fujino, the maximality of the variation of f is equivalent to the bigness of the moduli part in the canonical bundle formula.

We can prove a more general result using the pseudo-effective threshold. We give a preliminary definition which is suitable to the present setting.

Definition 1.1.5. Let (X, Δ) a lc pair and M a big divisor. The pseudo-effective threshold $\tau(X, \Delta; M)$ of M with respect to (X, Δ) is defined by

$$\tau(X, \Delta; M) := \inf \{t \in \mathbb{R}_{\geq 0} \mid K_X + \Delta + tM \text{ is pseudo-effective}\}.$$

Furthermore, we define the Kodaira energy of M with respect to (X, Δ) as

$$\kappa(X, \Delta, M) = -\tau(X, \Delta, M).$$

We do not state here the precise generalization of Theorem 1.1.3 using the pseudo-effective threshold, but it says that, if the variation of f is maximal, then it is enough to prove that the pseudo-effective threshold is uniformly bounded away from one. See Theorem 3.4.5. In particular, the problem reduces to understand if it possible that 1 is an accumulation point from below in the set of pseudo-effective thresholds. Quite surprisingly, Fujita arrived to the same problem when he was working on the classification of polarized pairs with bounded Kodaira energy. Fujita in conjectured that the limit points can be reached only from below. This is the log spectrum conjecture and it is the content of the next theorem.

Theorem 1.1.6. Let S_n^{ls} be the set of Kodaira energies where (X, Δ) is a pair with X smooth, Δ is a reduced divisor with simple normal crossing support and M is an ample Cartier divisor on X . Then for any real number $x < 0$ the set $\{t \in S_n^{ls} \mid x < t < x + \epsilon\}$ is finite, for ϵ small enough.

Fujita proved the case when X has dimension three and $\Delta = 0$ using boundedness of Fano threefolds. Theorem 1.1.6 follows from a more general result on the ACC of a suitable set of pseudo-effective thresholds. Furthermore, it implies that the pseudo-effective threshold, or the Kodaira energy, is a rational number in many cases.

Corollary 1.1.7. Let (X, Δ) be a klt pair and let M be a big and nef divisor. Then $\tau(X, \Delta; M)$ is a rational number.

Analogously to the pseudo-effective threshold, one can define the nef threshold and study similar properties. Here we study it in a slightly different setting.

Definition 1.1.8. Let X be a smooth variety and let D be a divisor such that $K_X + tD$ is nef for some t . The nef threshold $\eta(X, D)$ of D with respect to X is defined by

$$\eta(X, D) := \inf \{t \in \mathbb{R}_{\geq 0} \mid K_X + tD \text{ is nef}\}.$$

The nef threshold has been extensively studied in the minimal model program. For example, the rationality theorem tells us that the nef threshold is a rational number with bounded denominator. Assume that $K_X + D$ is nef. Since the nef cone is a closed cone, we do not expect that $\eta(X, D)$ is strictly less than one. It is interesting to characterize the pairs where $\eta(X, D) < 1$. We will give a complete answer to this question in Chapter 4. Moreover, if $K_X + D$ is also big, we characterize when $K_X + D$ is a limit of ample divisors of the form $K_X + \alpha D$ and we give a uniform bound on the nef threshold.

Theorem 1.1.9. *Let X be a smooth projective variety and let D be a reduced effective divisor with simple normal crossing support such that $K_X + D$ is big and nef. Then $K_X + \alpha D$ is ample for $\alpha \in (\frac{n+1}{n+2}, 1)$ if and only if there are no irreducible curves C such that $(K_X + D) \cdot C = 0$ and $K_X \cdot C \leq 0$.*

The previous theorem has some applications in the study Kähler-Einstein metrics and in hyperbolic geometry. Let us mention few of them. In the last few years there has been a lot of interest in Kähler-Einstein metrics with cone-edge singularities along a divisor. Let (X, D) be a pair as in Theorem 1.1.9. These metrics are smooth outside D and they have singularities along D of cone-angle $2\pi(1 - \alpha)$, where $\alpha \in (0, 1)$. Recently Jeffres-Mazzeo-Rubinstein [JMR11] and Mazzeo-Rubinstein [MR12] obtained analytical breakthroughs regarding the existence and uniqueness of these metrics. These results motivated a number of interesting questions that can be addressed using techniques from birational geometry. More precisely, in this thesis we focus on characterizing those pairs (X, D) for which a Kähler-Einstein metric with cone angle $2\pi(1 - \alpha)$ exists for some α and we describe the values α for which such a metric exists.

Among many other things, we have a satisfactory results for both small and large cone-angles in the negative scalar curvature case.

Theorem 1.1.10. *Let X be a smooth, n -dimensional projective variety and $D \subset X$ a reduced simple normal crossing divisor. The following are equivalent.*

1. *(X, D) admits negative Kähler-Einstein metrics with cone-edge singularities along D with cone angles $2\pi(1 - \alpha)$ for all $\alpha \in (1 - \epsilon, 1)$ and some $\epsilon > 0$.*
2. *(X, D) admits negative Kähler-Einstein metrics with cone-edge singularities along D with cone angles $2\pi(1 - \alpha)$ for every $\alpha \in (\frac{n+1}{n+2}, 1)$.*
3. *The self-intersection $(K_X + D)^n$ is positive, $(K_X + D) \cdot C \geq 0$ for every curve $C \subset X$ and there are no curves $C \subset X$ such that $(K_X + D) \cdot C = 0$ and $K_X \cdot C \leq 0$.*

We will prove a similar theorem for cone-angles close to 2π .

Remarkably, in the positive scalar curvature case the existence of such uniform bounds is obstructed for both small and large cone-angles. Nevertheless, we are able to quantify this obstruction when $-K_X$ or $-(K_X + D)$ are ample. For example, when $-K_X$ is ample we are able to prove the following.

Theorem 1.1.11. *Let $\mathfrak{B}$ be a set of pairs (X, D) such that X is a smooth projective Fano variety of dimension n and D is a reduced effective divisor with simple normal crossing support. The following are equivalent.*

1. *There exists a positive number $\alpha_0 < 1$ such that $-(K_X + \alpha D)$ is ample for any $\alpha \in [0, \alpha_0)$ and any $(X, D) \in \mathfrak{B}$.*
2. *There exists a positive integer M_0 such that $D \cdot (-K_X)^{n-1} \leq M_0$ for any $(X, D) \in \mathfrak{B}$.*

We can prove an analogous statement for cone-angles close to zero. In this case the obstruction depends only on $-(K_X + D)^n$. The proof of Theorem 5.1.3 heavily relies on boundedness of Fano varieties and on effective Matsusaka's theorem.

Let us mention few applications of the previous results in hyperbolic geometry. A classical theorem of H. C. Wang [Wan72] implies that for fixed real dimension $n \geq 4$, and any positive real number V , there are only finitely many complete non-compact locally symmetric Riemannian manifolds with negative curvature and volume at most V up to isometries.

As a final applications of Theorem 1.1.9, we show how to derive explicit bounds in the aforementioned theorem of Wang in the case of complex hyperbolic manifolds.

Let $\mathcal{H}^n$ be the complex n -dimensional hyperbolic space. We are interested in non-compact complex hyperbolic manifolds with finite volume. These manifolds are obtained as $X^\circ := \mathcal{H}^n / \Gamma$ where Γ is a non-uniform neat lattice of $\text{Aut}(\mathcal{H}^n)$, see [AMRT10] for details. X° is a non-compact manifold with finitely many ends.

Baily-Borel and later Mumford, Mok, and Siu-Yau proved that we can find a compactification X^* of X° such that the complement of X° in X^* consists of only finitely many (singular) points, called cusp points. Furthermore, there exists a smooth variety X , which is a resolution of X^* , such that each exceptional divisor over a cusp point is a smooth abelian variety. Let D be the sum of the exceptional divisors on X . We will refer to (X, D) as a toroidal compactification of X° .

We would like to understand how some numerical properties of $K_X + D$ determine the geometry of X . Our main technical result is that the divisor $K_X + D$ is the limit of ample divisors of the form $K_X + \alpha D$. Moreover we can give a uniform lower bound on α .

Theorem 1.1.12. *Let (X, D) be an n -dimensional toroidal compactification. Then $K_X + \alpha D$ is ample for all $\alpha \in (\frac{n-1}{n}, 1)$.*

The proof is based on Theorem 1.1.9, on estimates on the length of extremal rays of X and on the existence of a negatively curved Kähler-Einstein metric on X° . Theorem 1.1.12 has some geometrical applications. Some of these results were known before using completely different techniques, mainly coming from geometric topology.

We list here some of its applications.

Corollary 1.1.13. *Let (X, D) be an n -dimensional toroidal compactification. Then $m(K_X + D)$ defines an embedding of $X \setminus D$ for any $m \geq (n+1)^4$.*

The above corollary combined with a result of Fujiki on deformations and the effective bound on the number of components of Chow varieties, gives an effective bound on the number of complex hyperbolic manifolds with bounded volume.

As a final application, one can bound the number of cusps of toroidal compactification in terms of the volume.

Corollary 1.1.14. *Let (X, D) be an n -dimensional toroidal compactification. Then*

$$\text{number of cusps of } X^* \leq n^n (K_X + D)^n.$$

1.2 Structure of the thesis

An outline of the chapters follows. Each chapter contains a brief introductory section and a preliminary section.

Chapter 2 is a survey of previously known results in birational geometry and in the minimal model program. The results presented in this chapter will be fundamental in the rest of this work. In Chapter 3 we study effective birationality results for suitable linear series. In particular, we derive some partial results of Conjecture 1.1.2 and Conjecture 1.1.1 and we prove the above stated Theorem 1.1.3. We also introduce the notion of pseudo-effective threshold in this context. In the last section of this chapter, we prove some applications of Theorem 1.1.3 which do not appear elsewhere in the literature. Chapter 4 is devoted to the proof of the ACC for pseudo-effective thresholds. We also give a proof of the log spectrum conjecture and we derive several rationality results for the pseudo-effective threshold.

In Chapter 5 we start our work on Kähler-Einstein metrics. We introduce the necessary notions and we derive the previous theorems. Moreover, we study in details the nef threshold which is a key tool for the proof. In the last chapter we apply the methods introduced in Chapter 5 in the setting of toroidal compactifications and we derive the previously mentioned applications.

The results presented in this thesis are based on the papers [DiC11], [DiC12], [DD12a] and [DD12b]. Moreover, the materials presented in Chapter 5 and Chapter 6 are based on joint works with Luca F. Di Cerbo.

Chapter 2

Preliminary results

In this chapter we recall some definitions and we fix the notation. Moreover, we discuss some of the results we will need later. We will mainly follow the books [KM98], [Laz04a] and [Laz04b].

Throughout this thesis, we will work over the field of complex numbers $\mathbb{C}$. Unless otherwise stated, the dimension of X will be denoted by n .

2.1 Divisors

Let X be a normal scheme. A Weil $\mathbb{Z}$ -divisor, or simply a divisor, on X is a finite formal $\mathbb{Z}$ -linear combination $D = \sum a_i D_i$ of irreducible and reduced subschemes of codimension one. If in the definition of Weil divisors we allow rational or real coefficients, we get the notions of $\mathbb{Q}$ -divisors and $\mathbb{R}$ -divisors. In this work we will extensively use $\mathbb{Q}$ -divisor. A divisor D is called reduced if its coefficients are in $\{0, 1\}$. A $\mathbb{Q}$ -divisor D is $\mathbb{Q}$ -Cartier if mD is Cartier for some $m \in \mathbb{Q}$.

Let D be a Cartier divisor on X . If there exists a positive integer m such that $H^0(X, \mathcal{O}_X(mD)) \neq 0$, we can define a rational map

$$\phi_{mD} : X \dashrightarrow Y_m \subset \mathbb{P}(H^0(X, \mathcal{O}_X(mD))).$$

We are interested in the asymptotic behavior of these maps. The most basic birational invariant is the maximal dimension of their images.

Definition 2.1.1. *Let D be a Cartier divisor. We define the semigroup associated to D as*

$$N(X, D) := \{m \geq 0 \mid H^0(X, \mathcal{O}_X(mD)) \neq 0\}.$$

Then the Kodaira dimension of a divisor is defined to be

$$\kappa(X, D) := \max_{m \in N(X, D)} \{\dim \phi_{mD}\}.$$

A fundamental notion of positivity is the following.

Definition 2.1.2. *A divisor D is said to be big if it has maximal Kodaira dimension, i.e. $\kappa(X, D) = \dim X$.*

For big divisor one can define an invariant which measure the asymptotic order of growth of the dimension of global sections. First, recall that the growth of $h^0(X, \mathcal{O}_X(mD))$, as a function of m , is bounded. The following theorem is usually known as the asymptotic Riemann-Roch theorem.

Theorem 2.1.3. *Let X be a projective variety of dimension n . Let D and E be Cartier divisors. Then*

$$\chi(X, \mathcal{O}_X(mD + E)) = \frac{D^n}{n!} m^n + O(m^{n-1}).$$

In particular, there exists a constant C such that for any m large enough

$$h^0(X, mD) \leq C m^n.$$

Proof. We sketch the proof when X is smooth. Let H be a very ample divisor such that $H + D$ is very ample. By Bertini's theorem, if we pick $Y \in |H|$ and $Z \in |H + D|$ general then Y and Z are smooth. Consider the following short exact sequences,

$$0 \rightarrow \mathcal{O}_X(mD + E) \rightarrow \mathcal{O}_X(mD + E + Y) \rightarrow \mathcal{O}_Y(mD + E + Y) \rightarrow 0,$$

and

$$0 \rightarrow \mathcal{O}_X((m-1)D + E) \rightarrow \mathcal{O}_X((m-1)D + E + Z) \rightarrow \mathcal{O}_Z((m-1)D + E + Z) \rightarrow 0.$$

Note that $\chi(X, \mathcal{O}_X(mD + E + Y)) = \chi(X, \mathcal{O}_X((m-1)D + E + Z))$. Taking the difference of the Euler characteristic of the two sequences, the result follows.

For the second statement, when D is ample it follows from the first part of the theorem and Serre vanishing. In general, we can reduce it to the case of ample divisors. Let H be an ample divisor

such that $H - D$ is effective. Then

$$h^0(X, \mathcal{O}_X(mD)) \leq h^0(X, \mathcal{O}_X(mH)).$$

□

The previous result motivates the following definition.

Definition 2.1.4. *We call the volume of D the following quantity*

$$\text{vol}(D) := \limsup_{m \rightarrow \infty} \frac{h^0(X, \mathcal{O}_X(mD))}{m^n/n!}.$$

In particular, a divisor D is big if and only if $\text{vol}(D) > 0$. Unfortunately, in general, it is quite hard to compute the volume of a divisor. On the other hand, for a special class of divisors it is simpler.

Definition 2.1.5. *A divisor D is nef if $D \cdot C \geq 0$ for any curve C on X .*

The asymptotic Riemann-Roch theorem implies that each cohomological function h^i has polynomial growth of degree at most n . On the other hand, if D is nef one can prove more. See Theorem 1.4.40 in [Laz04a].

Proposition 2.1.6. *Let D be a nef divisor. Then*

$$h^i(X, \mathcal{O}_X(mD)) = O(m^{n-i}),$$

which implies that

$$h^0(X, \mathcal{O}_X(mD)) = \frac{D^n}{n!} m^n + O(m^{n-1}).$$

In particular, for a nef divisor D , $\text{vol}(D) = D^n$. So a nef divisor is big if and only if $D^n > 0$.

Going back to the birational behavior of ϕ_{mD} , a celebrated theorem of Iitaka says that these maps stabilize and they define a fiber space structure on X , up to a birational map. We refer to [Laz04a] to more details and the proof of Iitaka's theorem.

Theorem 2.1.7 (Iitaka). *Let X be a normal projective variety and D a Cartier divisor on X such that $\kappa(D) > 0$. Then for all sufficiently large $m \in N(X, D)$, the rational maps $\phi_{mD} : X \dashrightarrow Y_m$ are birationally equivalent to a fixed algebraic fibre space*

$$\phi_\infty : X_\infty \rightarrow Y_\infty$$

of normal varieties. More specifically, there exists a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\phi_m} & Y_m \\ \uparrow u_\infty & & \uparrow \nu_m \\ X_\infty & \xrightarrow{\phi_\infty} & Y_\infty \end{array},$$

where the horizontal maps are birational. Moreover, $\dim Y_\infty = \kappa(D)$ and $\kappa(F, u_\infty^* D|_F) = 0$, where F is a very general fibre of ϕ_∞ .

The theorem does not say anything about how large m should be. This problem will be extensively studied in this thesis.

Note that all the definitions we gave in this section extend naturally to $\mathbb{Q}$ -Cartier $\mathbb{Q}$ -divisor. For example, let D be a $\mathbb{Q}$ -Cartier $\mathbb{Q}$ -divisor such that mD is Cartier. Then the volume of D can be defined as

$$\text{vol}(D) := \frac{1}{m^n} \text{vol}(mD).$$

It is easy to check that the above definition does not depend on m .

2.2 Singularities of pairs

In birational geometry and especially in the minimal model program, we need to deal with some classes of singularities of pairs (X, Δ) . The definitions are a little bit technical but the idea is simple. For a singular variety X we want to measure its singularities comparing K_X and K_Y , where Y is a resolution of the singularities of X . In case we have a pair (X, Δ) we will measure the singularities of X and Δ together. In what follows we will use $\mathbb{Q}$ -divisors but most of the results can be generalized to $\mathbb{R}$ -divisors.

Definition 2.2.1. A pair (X, Δ) consists of a normal variety X and a $\mathbb{Q}$ -Weil divisor Δ such that $K_X + \Delta$ is $\mathbb{Q}$ -Cartier.

Let (X, Δ) be a pair. A log resolution of the pair is a birational map $f : Y \rightarrow X$ such that Y is smooth, the exceptional set $Ex(f)$ of f is a divisor and $Ex(f) \cup f^{-1}(\text{Supp } \Delta)$ is a simple normal crossing divisor.

We need a way to measure how singular a variety, or a pair, is. This measure is called discrepancy.

Definition 2.2.2. Let $(X, \Delta = \sum d_i \Delta_i)$ be a pair. Let $f : Y \rightarrow X$ be a birational morphism from a normal variety Y . Let $E \subset Y$ be the exceptional locus of f and $E_i \subset E$ the irreducible exceptional

divisors. Let $f_*^{-1}\Delta := \sum d_i f_*^{-1}\Delta_i$ denote the birational transform of Δ . Then there are rational numbers $a(E_i, X, \Delta)$ such that

$$K_Y + f_*^{-1}\Delta \equiv f^*(K_X + \Delta) + \sum a(E_i, X, \Delta).$$

$a(E, X, \Delta)$ is called the discrepancy of E with respect to (X, Δ) . By definition, $a(\Delta_i, X, \Delta) = -d_i$ and $a(D, X, \Delta) = 0$ for any other divisor $D \subset X$.

Definition 2.2.3. Let (X, Δ) be a pair. We say that (X, Δ) is a klt (resp. lc) pair if $a(E, X, \Delta) > -1$ (resp. $a(E, X, \Delta) \geq -1$) for every E .

The abbreviation “klt” stands for Kawamata log terminal and “lc” for log canonical. When X is a smooth variety, the discrepancies measure how singular is the effective divisor Δ . In case Δ has simple normal crossing support then we get only a restriction on the coefficients of Δ . More precisely we have the following result, see Corollary 3.12 in [Kol97] for a proof.

Proposition 2.2.4. Let X be a smooth variety and $\Delta = \sum d_i \Delta_i$ an effective divisor with simple normal crossing support. Then (X, Δ) is klt (resp. lc) if and only if $d_i < 1$ (resp. $d_i \leq 1$) for all i .

A more delicate notion of singularities is called divisorial log terminal (abbreviated to dlt). These singularities appear frequently in the minimal model program of pairs. Dlt pairs are better behaved than lc pairs and sometimes we can reduce problem from lc pairs to dlt pairs. In order to define dlt singularities, we need a preliminary notion.

Definition 2.2.5. Let X be a variety, $f : Y \rightarrow X$ a birational morphism from a normal variety Y and $E \subset Y$ an irreducible divisor. The closure of $f(E)$ is called the center of E on X . It is denoted by $\text{center}_X E$.

We can now define dlt singularities.

Definition 2.2.6. Let $(X, \Delta = \sum d_i D_i)$ be a pair with $0 \leq d_i \leq 1$. We say that (X, Δ) has dlt singularities if there is a closed subset $Z \subset X$ such that

1. $X \setminus Z$ is smooth and $\Delta|_{X \setminus Z}$ is a snc divisor,
2. if $f : Y \rightarrow X$ is birational and $E \subset Y$ is an irreducible divisor such that $\text{center}_X E \subset Z$ then $a(E, X, \Delta) > -1$.

Dlt pairs are always limits of klt pairs. Moreover, a dlt pair (X, Δ) is klt if and only if $\lfloor \Delta \rfloor = 0$. See [KM98] for more details.

Another way to measure singularities is through multiplier ideals.

Definition 2.2.7. *Let (X, Δ) be a pair. Let D be a $\mathbb{Q}$ -Cartier $\mathbb{Q}$ -divisor. The multiplier ideal of D for the pair (X, Δ) is the sheaf*

$$\mathcal{J}((X, \Delta), D) := \mu_* \mathcal{O}_Y(K_Y - \lceil \mu^*(K_X + \Delta + D) \rceil),$$

where $\mu : Y \rightarrow X$ is a log resolution of $(X, \Delta + D)$.

By definition, it is easy to see that for any $\mathbb{Q}$ -Cartier $\mathbb{Q}$ -divisor D , we have $\mathcal{J}((X, \Delta), D) = \mathcal{J}((X, \Delta + D), 0)$. For this reason and to simplify the notation, we will write $\mathcal{J}(X, \Delta + D)$ instead of $\mathcal{J}((X, \Delta), D)$. Most of the time we will assume that D is effective, this implies that $\mathcal{J}(X, \Delta + D) \subseteq \mathcal{O}_X$.

Definition 2.2.8. *The non-klt locus $\text{Nklt}(X, \Delta)$ of a pair (X, Δ) is*

$$\text{Nklt}(X, \Delta) := \{x \in X \mid (X, \Delta) \text{ is not klt at } x\}.$$

The connection between singularities and multiplier ideals is given by the following relation.

$$\text{Nklt}(X, \Delta) = \text{Supp}(\mathcal{O}_X / \mathcal{J}(X, \Delta))_{\text{red}}.$$

See [Laz04b] Section 9.3.B for the proof.

Multiplier ideals are fundamental tools for vanishing theorems. We state here the so called Nadel vanishing theorem.

Theorem 2.2.9. *Let (X, Δ) be a pair and let D be a $\mathbb{Q}$ -Cartier $\mathbb{Q}$ -divisor. Let N be a Cartier divisor on X such that $N - (K_X + \Delta + D)$ is big and nef. Then for any $i > 0$,*

$$H^i(X, \mathcal{O}_X(N) \otimes \mathcal{J}(X, \Delta + D)) = 0.$$

For a proof of Nadel's vanishing theorem see Theorem 2.16 in [Kol97] or Theorem 9.4.17 in [Laz04b]. Since $\mathcal{J}(X, \Delta) = \mathcal{O}_X$ if (X, Δ) is klt, one recovers the classical Kawamata-Viehweg vanishing theorem from the previous theorem.

2.3 Base loci

Both in birational geometry and in the study of Kähler-Einstein metrics is fundamental to understand where the map associated to some power of a big and nef line bundle defines an embedding into some projective space. This locus can be understood using a refinement of the stable base locus. We recall here the key definition and we refer to Definition 10.3.2 in [Laz04b] and the discussion there for more details.

Definition 2.3.1. *The stable base locus of a divisor L is*

$$\mathbf{B}(L) := \bigcap_{m \geq 1} \text{Bs}(|mL|),$$

where $\text{Bs}(|L|)$ is the base locus of L .

The augmented base locus of a divisor L is the Zariski-closed set

$$\mathbf{B}_+(L) := \mathbf{B}(L - \epsilon A),$$

for any ample A and sufficiently small $\epsilon > 0$.

Note that $\mathbf{B}_+(L)$ is a closed subset of X if and only if L is big and $\mathbf{B}_+(L) = \emptyset$ if and only if L is ample. We will see how $\mathbf{B}_+(L)$ is related to the locus where the map $\phi_{|mL|}$ is not an embedding, for m sufficiently large.

Next, recall that $\mathbf{B}_+(L)$ can be written as

$$\mathbf{B}_+(L) = \bigcap_{L=A+E} \text{Supp}(E),$$

where the intersection is taken over all decompositions $L = A + E$, where A is ample and E effective, see Remark 1.3 in [ELMNP06]. Since X is noetherian and $\mathbf{B}_+(L)$ is a Zariski closed subset of X we can find finitely many decompositions $L = A_i + E_i$ such that $\mathbf{B}_+(L) = \bigcap_{i=1}^k \text{Supp}(E_i)$. In particular this implies that for m large enough, the linear system $|mL|$ defines an embedding of $X \setminus \mathbf{B}_+(L)$ into some projective space.

Quite remarkably the augmented base locus of a big and nef divisor can be described numerically thanks to a theorem of Nakamaye. See Theorem 10.3.5 in [Laz04b].

Theorem 2.3.2. *Let X be a smooth projective variety and let L be a big and nef divisor on X . Then $\mathbf{B}_+(L)$ is the union of all positive dimensional subvarieties $V \subseteq X$ such that $L^{\dim(V)} \cdot V = 0$.*

Note that if a subvariety $V \subseteq X$ is such that $L^{\dim(V)} \cdot V = 0$ then $V \subseteq \mathbf{B}_+(L)$, so the content of the theorem is to prove the converse.

The next lemma is a useful result to prove that a map is birational.

Lemma 2.3.3. *Let X and Y be projective varieties and $f : X \dashrightarrow Y$ a dominant rational map. If there exists a dense set $U \subseteq X$ such that f separates every pair of points in U then f is birational.*

Proof. We can assume that X is smooth and f is a morphism. Let n be the dimension of X . By generic smoothness, see Corollary III.10.7 in [Har77], there exists a nonempty open subset $V \subset Y$ such that $f_0 : f^{-1}(V) \rightarrow V$ is a smooth morphism. Then by Theorem III.10.2 in [Har77], f_0 is flat and all the fibres are equidimensional of dimension k . Since $f^{-1}(V)$ is open and nonempty, we have that $f^{-1}(V) \cap U \neq \emptyset$. Let $p \in f^{-1}(V) \cap U$. By hypothesis, we know that for any point $q \in f^{-1}(V) \cap U$ different from p , $f(p) \neq f(q)$. In particular, f_0 is a finite map and there exists s such that $|f_0^{-1}(v)| = s$ for any $v \in V$. Let $U' := f(U) \subset Y$. Taking $u \in U' \cap W$ we see that s must be one and in particular, f is a birational map. $\square$

2.4 Minimal model program

In this section we describe the results we need from the minimal model program.

Definition 2.4.1. *A 1-cycle is a formal linear combination of irreducible, reduced and proper curves $C = \sum a_i C_i$. Two 1-cycles C, C' are called numerically equivalent if $(C \cdot D) = (C' \cdot D)$ for any Cartier divisor D . 1-cycles with real coefficients modulo numerical equivalence form a $\mathbb{R}$ -vector space and we denote it by $N_1(X)$.*

We are interested in particular subspaces of $N_1(X)$.

Definition 2.4.2. *Let X be a proper variety.*

$$NE(X) := \left\{ \sum a_i [C_i] \mid C_i \subset X, 0 \leq a_i \in \mathbb{R} \right\} \subset N_1(X).$$

Let $\overline{NE}(X)$ be the closure of $NE(X)$ in $N_1(X)$.

For any divisor D , set $D_{\geq 0} := \{x \in N_1(X) \mid D \cdot x \geq 0\}$ and similarly for > 0 , ≤ 0 and < 0 . Finally

$$\overline{NE}(X)_{D \geq 0} := \overline{NE}(X) \cap D_{\geq 0}.$$

Roughly speaking, the goal of the minimal model program is to find the simplest variety birationally equivalent to a given variety. The natural way to measure how simple a variety can be, is to look at its canonical divisor and its intersection with curves on the variety. So it is fundamental to understand $(K_X)_{\leq 0}$ inside $\overline{NE}(X)$. The celebrated cone theorem describe how the negative part looks like: it is generated by countably many rational curves with bounded intersection with K_X . For details see [KM98].

Theorem 2.4.3 (Cone theorem). *Let (X, Δ) be a lc pair. Then there are countably many rational curves $C_j \subset X$ such that $0 < -(K_X + \Delta) \cdot C_j \leq 2n$ and*

$$\overline{NE}(X) = \overline{NE}(X)_{(K_X + \Delta) \geq 0} + \sum \mathbb{R}_{\geq 0}[C_j].$$

If X is smooth and $\Delta = \emptyset$ then we can choose C_j such that $0 < -K_X \cdot C_j \leq n + 1$.

In [KM98] the cone theorem is proved for klt pairs. The case of lc pairs is treated in [Amb03] and [Fuj09].

We say that a divisor D is semi-ample if the linear system $|mD|$ is base point free for some m . It will be important for us to understand if certain adjoint linear systems are semi-ample. Our main tool to study semi-ampleness is the base point free theorem. It is one of the main step in the proof of the cone theorem and it is of great interest on its own.

Theorem 2.4.4 (Base point free theorem). *Let (X, Δ) a klt pair and let L be a nef Cartier divisor such that $aL - (K_X + \Delta)$ is big and nef for some $a > 0$. Then the linear system $|mL|$ is base point free for any $m \gg 0$.*

See [KM98] for the proof of the base point free theorem and more details.

A divisor is said to be strictly nef if $L \cdot C > 0$ for any irreducible curve C . Note that a strictly nef divisor is not necessary ample even if it is big. See Example 5.4.4. On the other hand, the base point free theorem implies the following.

Corollary 2.4.5. *Let (X, Δ) be a klt pair such that $K_X + \Delta$ is big and strictly nef. Then $K_X + \Delta$ is ample.*

Proof. Since $K_X + \Delta$ is big and nef by Theorem 2.4.4 it is semi-ample. Then for m big enough $m(K_X + \Delta)$ is base point free and it defines a morphism $\phi_{|m(K_X + \Delta)|} : X \rightarrow Y \subseteq \mathbb{P}^k$, where $k = \dim H^0(X, \mathcal{O}_X(m(K_X + \Delta))) - 1$. Suppose C is a curve in some fiber of ϕ then $(K_X + \Delta) \cdot C = 0$, but this is impossible since $K_X + \Delta$ is strictly nef. Then ϕ is a finite morphism and Corollary 1.2.15 in [Laz04a] implies that $K_X + \Delta$ is ample. $\square$

Note that the same proof tells us that strictly nef and semi-ample divisors are ample.

The base point free theorem can be regarded as one of the first step toward a solution of the following conjecture.

Conjecture 2.4.6 (Abundance). *Let (X, Δ) be a lc pair such that $K_X + \Delta$ is nef. Then $K_X + \Delta$ is semi-ample.*

The abundance conjecture in such generality it is known only if $\dim(X) \leq 3$, see [Kol92], [KMM94] and [KMM04]. Furthermore the base point free theorem implies that if (X, Δ) is a klt pair with $K_X + \Delta$ big and nef then $K_X + \Delta$ is semi-ample.

Note that Conjecture 2.4.6 is not a trivial statement even for surfaces. If the abundance conjecture holds then it implies that if $K_X + \Delta$ is strictly nef then it is ample.

The difference between abundance for klt pairs and abundance for lc pairs is very subtle. The base point free theorem, as stated in Theorem 2.4.4, it is not know for lc pairs. Actually Theorem 2.4.4 for lc pairs in dimension $n + 1$ implies the abundance conjecture for klt pairs of any Kodaira dimension in dimension n . In particular it implies abundance with X smooth and $\Delta = \emptyset$. See Theorem A.6 in [Laz09] for details.

Even tough the minimal model program is not fully established, we have some very useful partial results. For us it will be very important the case when $K_X + \Delta$ is not pseudo-effective. In this case it is know the the minimal model program ends up with a Mori fiber space.

Definition 2.4.7. *Let (X, Δ) be a dlt pair. A projective morphism $f : X \rightarrow Z$ is a Mori fiber space if $f_*\mathcal{O}_X = \mathcal{O}_Z$, $\rho(X/Z) = 1$ and $-(K_X + \Delta)$ is f -ample.*

When $K_X + \Delta$ is not pseudo-effective, Corollary 1.3.2 in [BCHM10] allows us to run the MMP. The following is a generalization of results in [BCHM10] to dlt pairs. See Theorem 2.15 in [Loh11] for details.

Theorem 2.4.8. *Let (X, Δ) be a dlt pair such that $K_X + \Delta$ is not pseudo-effective. Let H be an ample divisor such that $K_X + \Delta + H$ is nef. Then any $(K_X + \Delta)$ -MMP with scaling of H terminates with a Mori fiber space.*

We refer to [BCHM10] for more details and the relevant definitions.

Chapter 3

Stability of pluricanonical maps

3.1 Introduction

In this chapter we introduce and study the problem of stability of pluricanonical maps. The main motivating problem is the following conjecture, which we stated in the Introduction.

Conjecture 3.1.1. *Fix two positive integers n and κ . Then there is a positive integer $m_{n,\kappa}$ such that for any m divisible by $m_{n,\kappa}$, the pluricanonical map $\phi_{|mK_X|}$ is birationally equivalent to the Iitaka fibration of X for all smooth projective varieties X of dimension n and Kodaira dimension κ .*

Note that in order to get a satisfactory answer we need to give also an effective bound on the divisibility of m . After a great deal of work, Tsuji, Takayama and Hacon-M^cKernan proved the case when $\kappa(X) = n$. See also [HM06], [Tak06] and [Tsu06]. Their result is not effective and it remains a difficult question to find bounds for these numbers, see [CC10] for results in dimension three. When $\kappa(X) < n$ the standard approach to this problem is to use the canonical bundle formula of Fujino and Mori [FM00]. Roughly speaking, it says that for the Iitaka fibration $f : X \dashrightarrow Y$ the question is equivalent to finding a uniform bound for the birationality of linear systems on Y of type $K_Y + M_Y + B_Y$ where (Y, B_Y) is a klt pair and M_Y is a nef $\mathbb{Q}$ -divisor. With this approach $m_{n,\kappa}$ depends also on some other numerical parameters which appear in the formula, see Section 3.4 for details. Under these new assumptions there are some partial results. Fujino and Mori proved the case $\kappa(X) = 1$. Some years later Viehweg and Zhang in [VZ09] proved the case $\kappa(X) = 2$. If $\dim(X) = 3$ Ringler gave a different proof in [Rin07]. For low dimensional varieties, i.e. $\dim(X) \leq 4$, the log version of Conjecture 3.1.1 has been studied in [Tod10] and [TX09]. The first result for arbitrary Kodaira dimension is due to Pacienza in [Pac09] but he needs to assume

that K_Y is pseudo-effective and M_Y is big. Recently Jiang in [Jia12] proved the case where M_Y is numerically trivial by reducing the problem to a result on log pluricanonical maps in [HMX]. Furthermore, he proved the case $\kappa = n - 1$. In Section 3.5 we will discuss in more details some of his results. In summary, the canonical bundle formula suggests that we need some general theorems for adjoint linear systems in order to prove Conjecture 3.1.1. More precisely, we would need exactly Conjecture 1.1.2 stated in the Introduction. Pacienza derived his theorem on the uniformity of the Iitaka fibration thanks to the following result.

Theorem 3.1.2 (Pacienza [Pac09]). *For any positive integers n and ν , there exists an integer $m_{n,\nu}$ such that for any smooth complex projective variety X of dimension n with pseudo-effective canonical divisor, and any big and nef $\mathbb{Q}$ -divisor M on X such that νM is a Cartier divisor, the pluriadjoint map*

$$\phi_{m(K_X+M)} : X \dashrightarrow \mathbb{P}H^0(X, \mathcal{O}_X(m(K_X + M)))$$

is birational for all $m \geq m_{n,\nu}$ divisible by ν .

His proof relies on some techniques developed by Takayama in [Tak06] and Debarre [Deb]. There are some deep results involved, like Takayama's extension theorem and the weak positivity theorem of Campana [Cam04]. Unfortunately, all these theorems allow us only to derive non-effective statements. On the other hand, Kollár's proof in [Kol97] of Angehrn-Siu's theorem is effective but it only deals with big and nef divisors. In this chapter we explain how to use the method of Kollár to derive an effective version of Pacienza's theorem. Furthermore, our proof relies on more elementary techniques. Our main result is:

Theorem 3.1.3. *Let (X, Δ) be a klt pair. Let M be a big and nef Cartier divisor on X and E a pseudo-effective $\mathbb{Q}$ -divisor on X . Then for any*

$$m > \binom{n+2}{2}$$

the map induced by $| \lceil K_X + \Delta + E + mM \rceil |$ is birational.

Taking X smooth, $\Delta = 0$ and $E = (m - 1)K_X$ we get an effective version of Theorem 3.1.2. We now show how Theorem 3.1.3 gives a uniform result for the Iitaka fibration. Let $f : X \dashrightarrow Y$ be the Iitaka fibration of X . As we will see in Section 3.4 there are two positive integers b and N , depending only on the general fiber of f , such that bNM_Y is a Cartier divisor. For the definition of b and N see Definition 3.4.1. Furthermore we will see in Proposition 3.4.2 that $|mK_X|$ gives a map

birationally equivalent to the Iitaka fibration if and only if the linear series $| \lfloor m(K_Y + B_Y + M_Y) \rfloor |$ gives a birational map. Then Theorem 3.1.3 implies the following.

Theorem 3.1.4. *Let $f : X \dashrightarrow Y$ be the Iitaka fibration of X , where X is a smooth projective variety of Kodaira dimension κ . Suppose $K_Y + B_Y$ is pseudo-effective and M_Y is big. Then for any*

$$m > bN \binom{\kappa + 2}{2}$$

divisible by bN , the pluricanonical map $\phi_{|mK_X|}$ is birationally equivalent to the Iitaka fibration.

One can give conditions only on X and the generic fiber of f such that Theorem 3.1.4 applies, see for example Corollary 3.4.3. Of course we would like to prove a similar statement without the assumption $K_Y + B_Y$ pseudo-effective. In Section 3.3 we study the pseudo-effective threshold of a big and nef divisor M with respect to (X, Δ) . In particular, we obtain a similar result if we assume that the pseudo-effective threshold is bounded away from one.

3.2 Pluriadjoint maps

The first result of this section is an application of Nadel's vanishing theorem. It will be fundamental for the rest of the chapter. Combined by the theory developed by Anghern-Siu and Kollár we will be able to derive the results above stated.

Proposition 3.2.1. *Let (X, Δ) be a pair. Let M be a big and nef Cartier divisor on X and N be a Cartier divisor on X such that $N - K_X - \Delta$ is pseudo-effective. Let x_1 and x_2 be two general points in X . Suppose there are $t_0 > 0$ and an effective $\mathbb{Q}$ -divisor D_0 such that*

1. $D_0 \sim_{\mathbb{Q}} t_0 M$;
2. $x_1, x_2 \in \text{Nklt}(X, \Delta + D_0)$;
3. x_1 is an isolated point in $\text{Nklt}(X, \Delta + D_0)$.

Then for any $m > t_0$ the linear system $|N + mM|$ separates x_1 and x_2 .

Proof. Let E a pseudo-effective $\mathbb{Q}$ -divisor such that $N = K_X + \Delta + E$. Fix $m > t_0$ and write $D := D_0 + E$. Note that D is $\mathbb{Q}$ -linear equivalent to an effective $\mathbb{Q}$ -divisor. Let $V := \text{Nklt}(X, \Delta + D)$. Let x_1 and x_2 be two general points not contained in $\text{Supp}(E)$, then we have that $x_1, x_2 \in V$ and x_1 is isolated in V . In order to get separation of points we want the following map to be surjective

$$H^0(X, \mathcal{O}_X(N + mM)) \rightarrow H^0(V, \mathcal{O}_X(N + mM)|_V).$$

It fits in the long exact sequence given by

$$0 \rightarrow \mathcal{O}_X(N + mM) \otimes \mathcal{J}(X, \Delta + D) \rightarrow \mathcal{O}_X(N + mM) \rightarrow \mathcal{O}_X(N + mM)|_V \rightarrow 0,$$

then it is enough to prove that

$$H^1(X, \mathcal{O}_X(N + mM) \otimes \mathcal{J}(X, \Delta + D)) = 0.$$

Since

$$N + mM - (K_X + \Delta + D) \sim_{\mathbb{Q}} (m - t_0)M$$

is big and nef, the above vanishing follows from Nadel vanishing theorem. $\square$

Remark 3.2.2. *In Proposition 3.2.1 we work with $D_0 \sim_{\mathbb{Q}} t_0 M$ instead of working with $D_0 \sim_{\mathbb{Q}} t_0(M + E)$ as Pacienza does in his Lemma 6.3. This is a crucial difference between our approach and that of Pacienza. In particular, it allows us to work with more general linear series.*

We recall a result of Kollár in [Kol97], Theorem 6.5.

Theorem 3.2.3 (Kollár). *Let (X, Δ) be a projective klt pair and M a big and nef $\mathbb{Q}$ -Cartier $\mathbb{Q}$ -divisor on X . Let x_1 and x_2 be closed points in X and $c(k)$ positive numbers such that if $Z \subset X$ is an irreducible subvariety with $x_1 \in Z$ or $x_2 \in Z$ then*

$$(M^{\dim Z} \cdot Z) > c(\dim Z)^{\dim Z}.$$

Assume also that

$$\sum_{k=1}^n \sqrt[k]{2} \frac{k}{c(k)} \leq 1.$$

Then there is an effective $\mathbb{Q}$ -divisor $D \sim_{\mathbb{Q}} M$ such that:

1. $x_1, x_2 \in \text{Nklt}(X, \Delta + D)$;
2. x_1 is an isolated point in $\text{Nklt}(X, \Delta + D)$.

Theorem 3.2.3 easily implies the following useful result.

Corollary 3.2.4. *Let (X, Δ) be a klt pair. Let M be a big and nef Cartier divisor on X . Then for any $x_1, x_2 \notin \mathbf{B}_+(M)$ there exists an effective $\mathbb{Q}$ -divisor D_0 with*

1. $D_0 \sim_{\mathbb{Q}} \binom{n+2}{2} M$;

2. $x_1, x_2 \in \text{Nklt}(X, D_0 + \Delta)$;

3. x_1 is an isolated point in $\text{Nklt}(X, D_0 + \Delta)$.

Proof. Recall that $\mathbf{B}_+(M)$ is a proper subset of X if and only if M is big and

$$\mathbf{B}_+(M) = \bigcap_{M=A+E} \text{Supp}(E),$$

where the intersection is taken over all decomposition $M = A + E$, where A is ample and E effective.

Then for any variety Z through $x_1, x_2 \notin \mathbf{B}_+(M)$ we have that $M^{\dim(Z)} \cdot Z > 0$. Since M is Cartier, $(M^{\dim Z} \cdot Z) \geq 1$. Using the following inequality

$$\sum_{k=1}^n \sqrt[2]{k} < \sum_{k=1}^n \left(1 + \frac{1}{k}\right) k = \binom{n+2}{2} - 1,$$

we see that the divisor $\binom{n+2}{2}M$ satisfies the conditions of Theorem 3.2.3 with $c(k) = \binom{n+2}{2}$. $\square$

Now Theorem 3.1.3 is a consequence of the above results.

Proof of Theorem 3.1.3. By Corollary 1.4.3 in [BCHM10] we can assume that (X, Δ) is a $\mathbb{Q}$ -factorial klt pair, see also Corollary 4.4 in [Loh11] for a more detailed proof.

We can write

$$\lceil K_X + \Delta + E + mM \rceil = K_X + \Delta + E' + mM,$$

where

$$E' := E + \lceil K_X + \Delta + E \rceil - (K_X + \Delta + E)$$

is a pseudo-effective $\mathbb{Q}$ -divisor. Then Proposition 3.2.1 and Corollary 3.2.4 imply that

$|\lceil K_X + \Delta + E + mM \rceil|$ separates any two points x_1 and x_2 not in $\mathbf{B}_+(M)$. Since $X - \mathbf{B}_+(M)$ is a dense open subset of X , the result follows from Lemma 2.3.3. $\square$

Corollary 3.2.5. *Let (X, Δ) be a klt pair with $K_X + \Delta$ pseudo-effective. Let M be a big and nef $\mathbb{Q}$ -divisor on X . Let ν be an integer such that νM is a Cartier divisor, then for any*

$$m > \nu \binom{n+2}{2}$$

divisible by ν , the map induced by $|\lceil m(K_X + \Delta + M) \rceil|$ is a birational map.

Proof. Let m be as in the statement and set $E := (m-1)(K_X + \Delta)$. Then Theorem 3.1.3 gives the result. $\square$

Note that if we take X smooth and $\Delta = 0$, we get an effective version of Theorem 3.1.2.

Of course one can ask a weaker question about the non-vanishing of the cohomology group $H^0(X, \mathcal{O}_X(\lceil m(K_X + \Delta + M) \rceil))$. Using a similar version of Proposition 3.2.1 and Theorem 6.4 in [Kol97] we get:

Theorem 3.2.6. *Let (X, Δ) be a klt pair with $K_X + \Delta$ pseudo-effective. Let M be a big and nef $\mathbb{Q}$ -divisor on X . Let ν be an integer such that νM is a Cartier divisor, then for any*

$$m > \nu \binom{n+1}{2}$$

divisible by ν , $|\lceil m(K_X + \Delta + M) \rceil| \neq \emptyset$.

In the application to the Iitaka fibration we need to study the round down of these linear series instead of the round up.

Definition 3.2.7. *The index of a variety X is the smallest natural number $a(X)$ such that $a(X)K_X$ is a Cartier divisor.*

Corollary 3.2.8. *Let (X, Δ) be a klt pair such that $K_X + \Delta$ is pseudo-effective. Let M be a big and nef $\mathbb{Q}$ -divisor on X and let ν be an integer such that νM is a Cartier divisor. Suppose $\lfloor k\Delta \rfloor \geq (k-1)\Delta$ for any $k \in \mathbb{Z}_{>0}$ divisible by ν and $a(X)$. Then for any*

$$m > \nu \binom{n+2}{2}$$

divisible by ν and $a(X)$ the map induced by $|\lfloor m(K_X + \Delta + M) \rfloor|$ is birational.

Proof. Let m be as in the statement, then we can write

$$\lfloor m(K_X + \Delta + M) \rfloor = K_X + (m-1)(K_X + \Delta) + \lfloor m\Delta \rfloor - (m-1)\Delta + mM.$$

Let $E := (m-1)(K_X + \Delta) + \lfloor m\Delta \rfloor - (m-1)\Delta$ and note that $K_X + E + mM$ is an $\mathbb{Z}$ -divisor. Then the result follows from Theorem 3.1.3. $\square$

3.3 Pseudo-effective threshold

In this section we deal with the case where $K_X + \Delta$ is not pseudo-effective. Following [BCHM10] and [VZ09] we define the pseudo-effective threshold.

Definition 3.3.1. Let (X, Δ) be a lc pair. Let M be a $\mathbb{Q}$ -divisor on X such that $K_X + \Delta + tM$ is pseudo-effective for some positive t . We define the pseudo-effective threshold $\tau(X, \Delta, M)$ of M with respect to (X, Δ) as

$$\tau(X, \Delta, M) := \inf \{t \in \mathbb{R} \mid K_X + \Delta + tM \text{ is pseudo-effective}\}.$$

If there is no risk of confusion we denote it only by $\tau(M)$.

Proposition 3.3.2. Let (X, Δ) be a klt pair such that $K_X + \Delta$ is not pseudo-effective. Let M be a big and nef Cartier divisor on X such that $K_X + \Delta + M$ is big. Let $\tau(M)$ be the pseudo-effective threshold of M with respect to (X, Δ) . Then for any

$$m > \frac{1}{1 - \tau(M)} \binom{n+2}{2}$$

the map induced by the linear system $|[m(K_X + \Delta + M)]|$ is birational.

Proof. Let m be as in the statement. Since $K_X + \Delta + M$ is big, by Corollary 2.2.24 in [Laz04a], we know that $\tau(M) < 1$. Then we can find a rational number τ' such that $\tau(M) \leq \tau' < 1$ and

$$m > \frac{1}{1 - \tau'} \binom{n+2}{2}.$$

We can write

$$m(K_X + \Delta + M) = K_X + \Delta + (m-1)(K_X + \Delta + \tau'M) + (m(1 - \tau') + \tau')M.$$

In particular it is enough to prove that the map induced by round up of the linear system

$$K_X + \Delta + (m-1)(K_X + \Delta + \tau'M) + m(1 - \tau')M$$

is a birational map. Then the result follows from Theorem 3.1.3. □

Remark 3.3.3. In the next chapter we will prove that in the above setting $\tau(M)$ is actually a rational number.

We have an analogue statement for the round down.

Proposition 3.3.4. Let (X, Δ) , M and $\tau(M)$ as in Proposition 3.2. Furthermore assume that

$\lfloor k\Delta \rfloor \geq (k-1)\Delta$ for any $k \in \mathbb{Z}_{>0}$ divisible by $a(X)$. Then for any

$$m > \frac{1}{1-\tau(M)} \binom{n+2}{2}$$

divisible by $a(X)$, the map induced by the linear system $|[m(K_X + \Delta + M)]|$ is birational.

Proof. The argument is the same as in Proposition 3.3.2 and Corollary 3.2.8. $\square$

3.4 Iitaka Fibration

We now show how the previous results gives the uniformity of the Iitaka fibration under some extra conditions. We recall the canonical bundle formula and some of its properties, see [FM00] for details. Let $f : X \dashrightarrow Y$ be the Iitaka fibration of X with Y nonsingular and general fiber F . Blowing up X we may assume that f is a morphism. Then the canonical bundle formula says that there are $\mathbb{Q}$ -divisors B_Y and M_Y such that for all sufficiently divisible m

$$H^0(\mathcal{O}_X(mK_X)) = H^0(Y, \mathcal{O}_Y([m(K_Y + B_Y + M_Y)])).$$

B_Y is called the boundary divisor and it is an effective divisor such that (Y, B_Y) is a klt pair. M_Y is called the moduli part and it is a nef $\mathbb{Q}$ -divisor. We now define two numbers which play a key role in the canonical bundle formula.

Definition 3.4.1. *Let*

$$b := \min \{b' > 0 \mid |b'K_F| \neq \emptyset\}.$$

Let B be the $(n - \kappa(X))$ -th Betti number of a non-singular model of the cover $E \rightarrow F$ associated to the unique element of $|bK_F|$. We define

$$N = N(B) := \text{lcm} \{m \in \mathbb{Z}_{>0} \mid \varphi(m) \leq B\},$$

where φ is the Euler function.

We list some properties we will need later.

Proposition 3.4.2. *The following hold true:*

1. bNM_Y is a Cartier divisor;

2. for any $m \in \mathbb{Z}_{>0}$ divisible by b , we have

$$H^0(X, \mathcal{O}_X(mK_X)) \cong H^0(Y, \mathcal{O}_Y(\lfloor m(K_Y + M_Y + B_Y) \rfloor));$$

3. for any $m \in \mathbb{Z}_{>0}$ divisible by bN , $\lfloor mB_Y \rfloor - (m-1)B_Y$ is effective;

4. $K_Y + M_Y + B_Y$ is a big $\mathbb{Q}$ -divisor;

5. if F has a good minimal model and $\text{Var}(f)$ is maximal then M_Y is big.

Proof. For (1), (2) and (3) see [FM00]. (4) follows from (2). Finally (5) follows from a theorem of Kawamata in [Kaw85]. See also Corollary 3.1 in [Pac09]. $\square$

In particular (2) implies that $|mK_X|$ is birational to the Iitaka fibration if and only if

$|\lfloor m(K_Y + M_Y + B_Y) \rfloor|$ gives a birational map.

Proof of Theorem 3.1.4. It is just Proposition 3.4.2 and Corollary 3.2.8 with $a(Y) = 1$ because Y is smooth. $\square$

Recall that by a theorem of Fujino in [Fuj03], $\kappa(M_Y) \leq \text{Var}(f)$. In particular, since every variety is conjectured to have a good minimal model, the bigness of M_Y is conjecturally equivalent to the maximality of the variation of f .

We can now prove an effective version of Theorem 1.2 in [Pac09].

Corollary 3.4.3. *Let X be a smooth projective variety of Kodaira dimension κ and $f : X \dashrightarrow Y$ be the Iitaka fibration. Assume that*

1. Y is not uniruled;
2. f has maximal variation;
3. the generic fiber F has a good minimal model.

Then for any

$$m > bN \binom{\kappa + 2}{2}$$

divisible by bN , the pluricanonical map $\phi_{|mK_X|}$ is birationally equivalent to f .

Proof. The main result in [BDPP] implies that if Y is not uniruled then K_Y is pseudo-effective. By Proposition 3.4.2 we know that M_Y is big. Then Theorem 3.1.4 applies. $\square$

If we use Theorem 3.2.6 instead of Theorem 3.1.3 we can prove the following.

Theorem 3.4.4. *Let X be as in Theorem 3.1.4. Then for any*

$$m > bN \binom{\kappa + 1}{2}$$

divisible by bN , the cohomology group $H^0(X, \mathcal{O}_X(mK_X))$ is non-zero.

In particular if $\kappa(X) = n - 2$ we can choose $b = 12$ and $N = 22$.

If $K_Y + B_Y$ is not pseudo-effective we have a similar result but the bound depends on the pseudo-effective threshold $\tau(M_Y)$.

Theorem 3.4.5. *Let $f : X \dashrightarrow Y$ the Iitaka fibration of X with general fiber F . Assume that*

- 1. f has maximal variation;*
- 2. the generic fiber F has a good minimal model.*

Then for any sufficiently divisible

$$m > \frac{bN}{1 - \tau(M_Y)} \binom{\kappa + 2}{2}$$

the map associated to $|mK_X|$ is birational to the Iitaka fibration.

Proof. Apply Proposition 3.4.2 and Proposition 3.3.4 with $a(Y) = 1$. □

It is now natural to ask the following.

Question 3.4.6. *Is it possible to find a universal bound $\tau < 1$ which depends only on the dimension of Y , b and N such that $\tau(Y, B_Y, M_Y) \leq \tau$ for any Y , B_Y and M_Y as in Theorem 3.4.5?*

Viehweg and Zhang gave an affirmative answer to Question 3.4.6 in the case $\dim(Y) = 2$, see Lemma 2.10 and Lemma 2.11 in [VZ09]. This question will be the motivating problem of the next chapter.

3.5 Further applications

In the last section of this chapter we derive some other applications of Theorem 3.1.3. Furthermore, after reviewing some results due to Jiang in [Jia12], we give an application of the stability of

pluricanonical maps of varieties of Kodaira dimension $n - 1$. We prove an effective non-vanishing results for the space of invariant pluricanonical sections of varieties of general type under the action of a cyclic group.

The first application is again an effective birationality result. Let us recall some known facts. Let L be an ample divisor. Kollár-Matsusaka proved that mL is very ample, where m depends only on L^n and $K_X \cdot L^{n-1}$. Furthermore, if we look at the effective version of their theorem proved by Demailly and Siu, see Theorem 5.5.3, we see that if $-K_X$ is nef then m can be chosen to depend only on L^n . In general it is not possible to choose m independently from L^n . For an example see Example 1.5.7 in [Laz04a]. On the other hand, Kollár's effective base point freeness implies that if $-K_X$ is nef m depends on the dimension only.

Proposition 3.5.1. *Fix n . Then there exists a positive integer m_n such that for any smooth variety X with $-K_X$ nef, any ample divisor L on X and any $m \geq m_n$, the divisor mL is very ample.*

Proof. Apply the main theorem in [Kol93]. One can actually find an explicit value for m_n . $\square$

Theorem 3.1.3 can be used to derive a birational analogue of the previous statement.

Corollary 3.5.2. *Fix n . Then there exists a positive integer m_n such that if*

- X is a normal projective variety with $\dim(X) = n$;
- L is a big and nef divisor on X ;
- there exists an effective $\mathbb{Q}$ -divisor Δ such that (X, Δ) is klt and $-(K_X + \Delta)$ is pseudo-effective,

then the linear system $|mL|$ defines a birational map for any $m \geq m_n$.

Proof. Put $E = -(K_X + \Delta)$ in Theorem 3.1.3. In this case m_n can be chosen to be $\binom{n+2}{2}$. $\square$

The second application regards a cohomological criterion for uniruledness. A similar result is proved by Nakamaye in [1] using completely different techniques. Our result is slightly more general and it is effective.

Corollary 3.5.3. *Let X be a smooth projective variety and let H be an ample divisor on X . Let $m_0 := \binom{n+2}{2}$. Then X is uniruled if and only if*

$$H^0(X, \mathcal{O}_X(mK_X + m_0H)) = 0,$$

for all $m \geq 1$.

Proof. Recall that by [BDPP], X is uniruled if and only if K_X is not pseudo-effective. If X is not uniruled then the non-vanishing in the statement follows from Theorem 3.2.6.

Suppose $H^0(X, \mathcal{O}_X(mK_X + m_0H)) \neq 0$ for all $m \geq 1$. Then there are effective divisors E_m such that $mK_X + m_0H \sim E_m$. Dividing by m and then taking the limit $m \rightarrow \infty$, we get that K_X is a limit of effective divisors and in particular, it is pseudo-effective. $\square$

Before giving the last application, we review some known results on stability of pluricanonical maps. These results are mainly due to Jiang, see [Jia12]. We will use them to derive a theorem of effective non-vanishing on some space of invariants. Before doing this, we recover his results using the work in [HMX12].

Let us recall a fundamental theorem obtained in [HMX12].

Theorem 3.5.4 (Hacon-McKernan-Xu). *Fix a positive integer n and a set $I \subset [0, 1]$, which satisfies the DCC. Then there exists a positive integer m_0 such that if*

1. (X, Δ) is a lc pair with $\dim(X) = n$,
2. coefficients of Δ are in I ,
3. $K_X + \Delta$ is big,

then $\phi_{\lfloor m(K_X + \Delta) \rfloor}$ is a birational map for any $m \geq m_0$.

Recall that if we want to prove effective results for the Iitaka fibration we need to prove that some power of $K_Y + B_Y + M_Y$ gives a birational map. The main problem is that we do not know if the pair $(Y, B_Y + M_Y)$ is lc or not. On the other hand, there is an important conjecture which implies that it should be. See [PS09].

Conjecture 3.5.5. [Prokhorov-Shokurov] *There exist a birational model $Y' \rightarrow Y$ and a positive integer I , depending only on n , such that $IM_{Y'}$ is linearly equivalent to a base point free divisor.*

Combining the above statements we get.

Theorem 3.5.6 (Jiang). *Conjecture 3.5.5 implies Conjecture 3.1.1 under the assumption that $m_{n,\kappa}$ depends also on b and N .*

Basically, the only case when Conjecture 3.5.5 is known is when the fibers of the Iitaka fibration have dimension one. In this case M_Y comes from the moduli space of elliptic curves. More precisely, $M_Y \sim_{\mathbb{Q}} \frac{1}{12}J^*(\mathcal{O}_{\mathbb{P}^1}(1))$, where J is defined by the j -invariant. This implies the following.

Corollary 3.5.7 (Jiang). *Let X be an n -dimensional smooth projective variety of Kodaira dimension $n - 1$. Then there exists a positive integer m_n , depending only on n , such that ϕ_{mK_X} is birationally equivalent to the Iitaka fibration for any m divisible by m_n .*

Since the dimension of the fiber is one, in this case we know the boundedness of b and N . Moreover, Corollary 3.5.7 gives an effective non-vanishing result for $H^0(X, \mathcal{O}_X(mK_X))$. Even though this result is about varieties with $\kappa(X) = n - 1$, we can use it to derive information about varieties of general type. The following construction is inspired by a paper of Hanamura [Han85], in which he was trying to find counterexamples to Corollary 3.5.7.

Theorem 3.5.8. *Fix a positive integer n . Then there exists a positive integer m_n such that for any n -dimensional variety of general type X and any cyclic group G acting on X , we have $H^0(X, \mathcal{O}_X(m_n K_X))^G \neq 0$.*

Proof. Let E be an elliptic curve. Let $x \in E$ be a point of order $|G|$ and let g be a generator of G . We can define an action of G on E as $g \cdot y := x + y$. Let $Y := (X \times E)/G$, where G acts diagonally on $(X \times E)$. We have the following diagram

$$\begin{array}{ccc} X \times E & \xrightarrow{\pi} & Y \\ \downarrow & & \downarrow \\ E & \longrightarrow & E/G \end{array} \quad ,$$

where the horizontal maps are the quotient maps. Since π is étale, we have $\kappa(Y) = \kappa(X \times E) = n - 1$. Furthermore,

$$H^0(Y, \mathcal{O}_Y(mK_Y)) = H^0(X \times E, \mathcal{O}_{X \times E}(mK_{X \times E}))^G = H^0(X, \mathcal{O}_X(mK_X))^G.$$

Finally, Corollary 3.5.7 gives the result. □

Chapter 4

Log spectrum conjecture

4.1 Introduction

In a series of papers, Fujita introduced what he called the Kodaira energy of a polarized variety and he used it to study the adjunction theory of polarized varieties. See [Fuj92], [Fuj95], [Fuj96] and [Fuj97] for details. Let us recall the definition.

Definition 4.1.1. *Let (X, Δ) a lc pair and let M be a big divisor. The Kodaira energy of (X, Δ, M) is defined by*

$$\kappa\epsilon(X, \Delta, M) := -\inf \{t \in \mathbb{Q}_{\geq 0} \mid \kappa(K_X + \Delta + tM) \geq 0\}.$$

Of course the interesting case is when $K_X + \Delta$ is not pseudo-effective. Using techniques from the minimal model program, Fujita proved that $\kappa\epsilon(X, \Delta, M)$ is a rational number if (X, Δ) is log terminal, X has dimension at most three and M is big and nef, see Theorem 2.2 in [Fuj96]. Moreover in [Fuj92], he stated the so called spectrum conjecture. It says that if $\Delta = 0$ and M is ample then the set of negative Kodaira energies is finite away from zero. The goal of this chapter is to prove an analogue of the spectrum conjecture for pairs. It is easy to see that in this case the set of Kodaira energies is not discrete. On the other hand, Fujita conjectured that the limit points can be reached only from below, see [Fuj96] and [Fuj97]. This is the content of the log spectrum conjecture and it is one of the main results of this paper.

Theorem 4.1.2. *Let S_n^{ls} be the set of Kodaira energies where (X, Δ) is a pair with X smooth, Δ is a reduced divisor with simple normal crossing support and M is an ample Cartier divisor on X . Then for any real number $x < 0$ the set $\{t \in S_n^{ls} \mid x < t < x + \epsilon\}$ is finite, for ϵ small enough.*

Fujita proved that if $\Delta = \emptyset$ and $\dim(X) \leq 3$ then the set of negative Kodaira energies is finite away from zero using boundedness of Fano threefolds with terminal singularities. See Theorem 3.4.1 in [Fuj96]. Our approach is different. We will show how techniques from the minimal model program and the recent results in [HMX12] allow us to prove Theorem 4.1.2. We will actually prove a more general result on a set of pseudo-effective thresholds. Moreover, this will imply that the Kodaira energy is a rational number in many cases. Note that the question of the rationality of the Kodaira energy has been extensively studied. For example see [AHK07], [Ara10], [BCHM10] and [DHP10].

Throughout this chapter we will work with the pseudo-effective threshold instead of the Kodaira energy. For its definition see Definition 3.3.1. It seems that the former invariant is more suitable for the minimal model program and it applies to a more general setting.

Recall that in the case M is big, the pseudo-effective threshold and the Kodaira energy are easily related by the formula

$$\kappa\epsilon(X, \Delta, M) = -\tau(X, \Delta; M).$$

We will study an analogue of Theorem 4.1.2 for pseudo-effective thresholds in a slightly more general form, namely we allow the coefficients of Δ and M to vary in some fixed sets. In order to simplify the presentation of our results, let us first introduce some notation.

Definition 4.1.3. Fix a positive integer n and two subsets $I \subseteq [0, 1]$ and $J \subseteq \mathbb{R}$. Then we define $\mathcal{D}_n(I, J)$ to be the set of all lc pairs (X, Δ) with an effective $\mathbb{R}$ -divisor M on X such that X is a projective variety of dimension n , the coefficients of Δ (resp. M) are in I (resp. J) with $(X, \Delta + tM)$ lc and $K_X + \Delta + tM$ pseudo-effective for some $t \geq 0$.

The main object of study of this chapter is the following set.

Definition 4.1.4. Fix a positive integer n and two subsets $I \subseteq [0, 1]$ and $J \subseteq \mathbb{R}$. We define the set

$$\mathcal{T}_n(I, J) := \{\tau(X, \Delta; M) \mid (X, \Delta; M) \in \mathcal{D}_n(I, J)\}.$$

We can now state our main theorem.

Theorem 4.1.5. Fix a positive integer n and two sets $I \subseteq [0, 1]$ and $J \subseteq \mathbb{R}$. If I and J satisfy the DCC then $\mathcal{T}_n(I, J)$ satisfies the ACC.

Note that we don't require M to be an ample divisor. If the coefficients of Δ or M are not in a DCC set, then it is easy to see that the above result is false even in dimension one.

Theorem 4.1.5 implies that the Kodaira energy is a rational number if (X, Δ, M) satisfy the above conditions with I and J sets of rational numbers, see Corollary 4.3.6.

Our main motivation in studying the ACC for pseudo-effective thresholds comes from effective birationality results of certain adjoint linear systems as explained in Chapter 3. On the other hand, we believe that these results have other applications. For example the pseudo-effective threshold is fundamental in the construction of Tamagawa numbers and it appears in formulas computing the asymptotic numbers of rational points of bounded heights. See [Bat98], [Fuj95] and [Tsc03] for details.

4.2 Preliminaries

Let us define what ACC and DCC stand for.

Definition 4.2.1. *Let $I \subseteq \mathbb{R}$.*

We say that I satisfies the ACC (ascending chain condition) if every non-decreasing sequence $\{x_i\} \subset I$ is eventually constant.

We say that I satisfies the DCC (descending chain condition) if every non-increasing sequence $\{x_i\} \subset I$ is eventually constant.

Given any effective divisor M on a variety X , we define a global invariant which measures the singularities of M and X .

Definition 4.2.2. *Let (X, Δ) be a lc pair and let M be an effective $\mathbb{R}$ -divisor. The log canonical threshold of M with respect to (X, Δ) is defined by*

$$\text{lct}(X, \Delta; M) := \sup \{t \in \mathbb{R}_{\geq 0} \mid (X, \Delta + tM) \text{ is lc}\}.$$

Fix a positive integer n and two subsets $I, J \subseteq \mathbb{R}$. Let

$$\text{LCT}_n(I, J) := \{\text{lct}(X, \Delta; M)\},$$

where (X, Δ) is a lc pair and the coefficients of Δ (resp. M) belong to I (resp. J).

One of the main theorems in [HMX12] is that the above set satisfies the ACC.

Theorem 4.2.3 (Hacon, McKernan and Xu). *Fix a positive integer n , $I \subseteq [0, 1]$ and $J \subseteq \mathbb{R}$. If I and J satisfy the DCC then $\text{LCT}_n(I, J)$ satisfies the ACC.*

The proof of the above theorem is a very intricate induction procedure based on four theorems. One of these theorems will be fundamental for us. See Theorem 1.5 in [HMX12].

Theorem 4.2.4 (Hacon, McKernan and Xu). *Fix a positive integer $n \in \mathbb{N}$ and a set $I \subseteq [0, 1]$, which satisfies the DCC. Then there is a finite subset $I_0 \subseteq I$ such that if*

1. X is a projective variety of dimension n ,
2. (X, Δ) is log canonical,
3. $\Delta = \sum \delta_i \Delta_i$ where $\delta_i \in I$,
4. $K_X + \Delta \equiv 0$,

then $\delta_i \in I_0$.

In order to apply Theorem 4.2.4 we need to run the minimal model program with scaling. As we said in Chapter 2, we know how to do it only for dlt pairs. On the other hand, dlt models allow us to avoid this difficulty in some cases. We state here the result we need and we refer to [Kol13] for a proof.

Lemma 4.2.5. *Let (X, Δ) be a lc pair. Then there is a proper birational morphism $g : X^{dlt} \rightarrow X$ with exceptional divisors E_i such that*

1. $(X^{dlt}, g_*^{-1}\Delta + \sum E_i)$ is dlt,
2. X^{dlt} is $\mathbb{Q}$ -factorial,
3. $K_{X^{dlt}} + g_*^{-1}\Delta + \sum E_i \sim_{\mathbb{Q}} g^*(K_X + \Delta)$.

We can now apply the above lemma to reduce the study of pseudo-effective thresholds from lc pair to dlt pairs.

Lemma 4.2.6. *Let (X, Δ) be a pair with an effective divisor such that $(X, \Delta + tM)$ is lc and $K_X + \Delta + tM$ is pseudo-effective for some $t \geq 0$. Let I and J be two sets such that the coefficients of Δ (resp. M) are in I (resp. J). Then there exists X' with effective divisors Δ' and M' such that:*

1. $K_{X'} + \Delta' + tM'$ is dlt and pseudo-effective for some $t \geq 0$,
2. the coefficients of Δ' (resp. M') belong to $I \cup \{1\}$ (resp. J),
3. $\tau(X, \Delta; M) = \tau(X', \Delta'; M')$.

Proof. Let $\tau := \tau(X, \Delta; M)$. By assumption $(X, \Delta + \tau M)$ is a lc pair. By Lemma 4.2.5 we can take its dlt model $(X^{dlt}, g_*^{-1}\Delta + \sum E_i + \tau g_*^{-1}M)$.

Let $X' := X^{dlt}$, $\Delta' := g_*^{-1}\Delta + \sum E_i$ and $M' := g_*^{-1}M$, then $K_{X'} + \Delta' + \tau M' \sim_{\mathbb{Q}} g^*(K_X + \Delta + \tau M)$. Since the right hand side is pseudo-effective, we have that $\tau(X', \Delta'; M') \leq \tau$. On the other hand $g_*(K_{X'} + \Delta' + tM') \sim_{\mathbb{Q}} K_X + \Delta + tM$ for any $t \geq 0$, thus $\tau(X', \Delta'; M') = \tau$. $\square$

4.3 ACC for pseudo-effective thresholds

In this section we prove Theorem 4.1.5 and we give some applications. In particular we will prove Theorem 4.1.2. First we show that if J satisfies the DCC, or more generally if it admits a strictly positive minimum, then there are no diverging increasing sequences in $\mathcal{T}_n(I, J)$. This is a simple consequence of the fact that $(X, \Delta + tM)$ is lc.

Lemma 4.3.1. *Fix a positive integer n and two subsets $I \subset [0, 1]$ and $J \subseteq \mathbb{R}$. If J admits a strictly positive minimum then there exists a positive real number α such that $\mathcal{T}_n(I, J) \subseteq [0, \alpha]$.*

Proof. Let $\delta := \min(J \setminus \{0\})$. Since $(X, \Delta + tM)$ is lc it implies that also $(X, \Delta + \tau M)$ is lc, where $\tau := \tau(X, \Delta; M)$. Then the coefficients of $\Delta + \tau M$ are less than one and in particular $\mathcal{T}_n(I, J) \subseteq [0, \alpha]$, where $\alpha = \delta^{-1}$. $\square$

Note that in general if we don't ask that $(X, \Delta + tM)$ is lc, we can find diverging sequences of pseudo-effective thresholds even if $I = \emptyset$ and J is a finite set. So without any assumption it does not satisfy the ACC.

Example 4.3.2. *Let $X_e := \mathbb{P}(\mathcal{O}_C \oplus \mathcal{L})$ where $C \cong \mathbb{P}^1$ and $\mathcal{L}$ is a line bundle of degree $-e \leq 0$. X_e is usually called the e -th Hirzebruch surface. Recall that its canonical divisor is given numerically by $K_{X_e} \equiv -2C_0 - (e+2)f$. Let $M = C_0 + f$. It is easy to see that M is big but not nef. A simple computation shows that $\tau(X_e, M) \geq e+2$.*

On the other hand, if we add some positivity to the divisor M we can obtain a uniform upper bound on $\tau(X, \Delta; M)$ without asking the pair $(X, \Delta + \tau M)$ to be lc. For example, Theorem 3.2.3 implies that if M is a big and nef Cartier divisor then $\tau(X, \Delta; M) \leq \binom{n+1}{2}$.

We are now ready to prove Theorem 4.1.5. Compare with Proposition 8.7 in [DHP10] and Lemma 3.1 in [Gon12].

Proof of Theorem 4.1.5. Let $\tau := \tau(X, \Delta; M) \in \mathcal{T}_n(I, J)$ such that $\tau > 0$. By Lemma 4.2.6, we can assume that $(X, \Delta + \tau M)$ is a dlt pair. For any $x < \tau$ the pair $(X, \Delta + xM)$ is also dlt by Corollary

2.39 in [KM98]. By Theorem 2.4.8, we can run a $K_X + \Delta + xM$ minimal model with scaling and end with a $K_X + \Delta + xM$ Mori fiber space $Y_x \rightarrow Z_x$ with generic fiber F_x . Let $f : X \dashrightarrow Y_x$ be the birational map given by the MMP. By definition of Mori fiber space, we have that

$$K_{F_x} + \Delta_{F_x} + xM_{F_x} := (K_{Y_x} + f_*(\Delta + xM))|_{F_x},$$

is anti-ample and in particular $x < \eta(x) := \tau(F_x, \Delta_{F_x}; M_{F_x})$. Note that the coefficients of Δ_{F_x} (resp. M_{F_x}) belong to I (resp. J). Since $K_{F_x} + \Delta_{F_x} + \tau M_{F_x}$ is pseudo-effective, $\eta(x) \leq \tau$. Then $x < \eta(x) \leq \tau$, for any $0 < x \leq \tau$.

We want to prove that if x is sufficiently close to τ then $(F_x, \Delta_{F_x} + \eta(x)M_{F_x})$ is a lc pair. Suppose by contradiction that $\text{lct}(x) := \text{lct}(F_x, \Delta_{F_x}; xM_{F_x}) < \eta(x)$ for any x close to τ . We define an increasing sequence $\{x_i\}$ converging to τ with the following properties: $0 < x_i < \tau$ and $\text{lct}(x_i) < x_{i+1} < \eta(x_i)$. Since $(F_x, \Delta_{F_x} + x_i M_{F_x})$ is dlt, we know that $x_i \leq \text{lct}(x_i)$. In particular, we get an increasing sequence of lc thresholds converging to τ . This contradicts the ACC for log canonical thresholds, Theorem 4.2.3.

We prove the theorem by induction on the dimension of X . If $\dim(X) = 1$ the result is obvious.

Let (X_i, Δ_i) be a sequence of pairs with effective divisors M_i on X_i such that the sequence $\tau_i := \tau(X_i, \Delta_i; M_i)$ is non-decreasing. We want to prove that the sequence is eventually constant. Choose $0 < x_i < \tau_i$ sufficiently close to τ_i such that the pair $(F_i, \Delta_{F_i} + \eta_i M_{F_i})$ is lc, where $F_i := F_{x_i}$ and $\eta_i := \tau(F_i, \Delta_{F_i}; M_{F_i})$. Recall that $x_i < \eta_i \leq \tau_i$. Furthermore, we can choose the sequence $\{x_i\}$ such that each $0 < x_i < \tau_i$ satisfy also $x_i > \tau_{i-1}$. Since $\eta_i > x_i > \eta_{i-1}$, we have that the sequence $\{\eta_i\}$ is increasing. By construction, it is enough to prove that the sequence $\{\eta_i\}$ is eventually constant.

Passing to a non-decreasing subsequence we can assume that $\dim(F_i)$ is constant. If $\dim(F_i) < n$ then by induction we conclude that the sequence $\{\eta_i\}$ is eventually constant. So we can assume that Y_i has Picard number one for any i . Then $K_{Y_i} + \Delta_{Y_i} + \eta_i M_{Y_i} \equiv 0$ and the pair $(Y_i, \Delta_{Y_i} + \eta_i M_{Y_i})$ is lc. Since $\{\eta_i\}$ is non-decreasing, the coefficients of $\Delta_{Y_i} + \eta_i M_{Y_i}$ are in a DCC set.

We can now apply Theorem 4.2.4 to the pairs $(Y_i, \Delta_{Y_i} + \eta_i M_{Y_i})$. It implies that the sequence $\{\eta_i\}$ is eventually constant and then also $\{\tau_i\}$ is eventually constant. $\square$

As a first consequence, we prove Theorem 4.1.2.

Proof of Theorem 4.1.2. The proof is by contradiction. Recall that $\kappa\epsilon(X, \Delta, M) = -\tau(X, \Delta; M)$. Assume that the statement of Theorem 4.1.2 does not hold, then there exists a sequence of pairs

(X_i, Δ_i) with ample divisors M_i such that their pseudo-effective thresholds $\tau_i := \tau(X_i, \Delta_i; M_i)$ define an increasing sequence that converges to some positive number. Let $H'_i := K_{X_i} + \Delta_i + 2n^2 M_i$. By Theorem 1.1 in [Fuj10], we have that H'_i is a globally generated ample divisor on X_i and $K_{X_i} + \Delta_i + H'_i$ is pseudo-effective. Choose a general element $H_i \in |H'_i|$ such that $(X_i, \Delta_i + H_i)$ is lc. Let $\eta_i := \tau(X_i, \Delta_i; H_i)$. Note that $\eta_i \in \mathcal{T}_n(\{1\}, \{1\})$ for every i . Furthermore,

$$K_{X_i} + \Delta_i + tH_i \sim (t+1)(K_{X_i} + \Delta_i) + 2n^2 tM_i.$$

This implies that

$$\tau_i = 2n^2 \left(1 - \frac{1}{\eta_i + 1} \right).$$

A simple computation shows that the sequence $\{\eta_i\}$ is increasing as well. This contradicts Theorem 4.1.5 with $I = J = \{1\}$.

□

Looking at the proof of the Theorem 4.1.5, we deduce a structure result for the set $\mathcal{T}_n(I, J)$. To make the presentation easier, we introduce the following set.

Definition 4.3.3. *Define*

$$\mathcal{T}_n^o(I, J) := \{\tau(X, \Delta; M) \in \mathcal{T}_n(I, J) \mid \rho(X) = 1\},$$

where $\rho(X)$ is the Picard number of X .

Corollary 4.3.4. *Fix a positive integer n and two sets $I \subseteq [0, 1]$ and $J \subseteq \mathbb{R}$. If I and J satisfy the DCC then*

$$\mathcal{T}_n(I, J) = \bigcup_{j=1}^n \mathcal{T}_j^o(I, J).$$

Proof. First we prove that $\mathcal{T}_n(I, J) \supseteq \bigcup_{j=1}^n \mathcal{T}_j^o(I, J)$. Clearly it is enough to prove that $\mathcal{T}_{n-1}(I, J) \subseteq \mathcal{T}_n(I, J)$. Let E be an elliptic curve. If $(X, \Delta; M) \in \mathcal{D}_{n-1}(I, J)$, we define $X' = X \times E$, $\Delta' = \Delta \times E$ and $M' = M \times E$. By construction $(X', \Delta'; M') \in \mathcal{D}_n(I, J)$ and they have the same pseudo-effective threshold.

It remains to prove the other inclusion. If we prove that

$$\mathcal{T}_n(I, J) \subseteq \mathcal{T}_n^o(I, J) \cup \bigcup_{j=1}^{n-1} \mathcal{T}_j(I, J),$$

then we are done applying induction on n .

Let $\tau := \tau(X, \Delta; M) \in \mathcal{T}_n(I, J)$. If $\rho(X) = 1$ then $\tau \in \mathcal{T}_n^o(I, J)$. Suppose that $\rho(X) > 1$. Let $\{x_i\}$ be a non-decreasing sequence such that $0 < x_i < \tau$, $\lim x_i = \tau$ and $x_i > \eta(x_{i-1})$, where $\eta(x_i)$ is obtained as in the proof of Theorem 4.1.5. Then $x_i < \eta(x_i) \leq \tau$ for any i and $\lim \eta(x_i) = \tau$. By Theorem 4.1.5 the sequence $\{\eta(x_i)\}$ is eventually constant and in particular, there exists an i such that $\eta(x_i) = \tau$. $\square$

Fujita in [Fuj96] proved a similar result when $n = 3$ and $\Delta = \emptyset$.

As a corollary of the structure theorem of $\mathcal{T}_n(I, J)$, we have that the set of pseudo-effective thresholds is contained in the rational numbers.

Corollary 4.3.5. *If $I \subseteq [0, 1] \cap \mathbb{Q}$ and $J \subseteq \mathbb{Q}$ satisfy the DCC then $\mathcal{T}_n(I, J) \subseteq \mathbb{Q} \cap [0, 1]$.*

Proof. We prove the result by induction on the dimension of X . If $\dim(X) = 1$ it is obvious. By Corollary 4.3.4, we have that

$$\mathcal{T}_n(I, J) = \bigcup_{j=1}^n \mathcal{T}_j^o(I, J).$$

By induction, it is enough to prove that $\mathcal{T}_n^o(I, J) \subseteq \mathbb{Q} \cap [0, 1]$. If $\tau \in \mathcal{T}_n^o(I, J)$ is different from zero, then M is an ample divisor. Since $K_X + \Delta + \tau M \equiv 0$, we see that τ is the nef threshold of M with respect to (X, Δ) . Then the Rationality Theorem in [KM98] implies that $\tau \in \mathbb{Q}$. $\square$

In particular, if we fix a lc pair (X, Δ) we get Proposition 8.7 in [DHP10].

Corollary 4.3.6. *Let (X, Δ) be a lc pair with Δ an effective $\mathbb{Q}$ -divisor. Let M be an effective $\mathbb{Q}$ -divisor such that $(X, \Delta + tM)$ is lc and $K_X + \Delta + tM$ is pseudo-effective for some $t \geq 0$. Then $\tau(X, \Delta; M)$ is a rational number.*

Proof. Let I (resp. J) be the set of coefficients of Δ (resp. M). Since they are both finite sets, we can apply the above corollary. $\square$

We can now give a generalization to lc pairs of Corollary 1.1.7 in [BCHM10].

Corollary 4.3.7. *Let (X, Δ) a lc pair with Δ an effective $\mathbb{Q}$ -divisor and M an ample $\mathbb{Q}$ -Cartier divisor. Then $\tau(X, \Delta; M)$ is a rational number.*

Proof. Replacing M with a sufficiently high power of itself, we can assume that $\tau(X, \Delta; M) < 1$. The result follows from the fact that we can find an effective divisor M' such that $M' \sim_{\mathbb{Q}} M$ and $(X, \Delta + M')$ is lc. $\square$

If (X, Δ) is a klt pair we can generalize the above statement when M is only big and nef.

Corollary 4.3.8. *Let (X, Δ) a klt pair with Δ an effective $\mathbb{Q}$ -divisor and M a big and nef $\mathbb{Q}$ -Cartier divisor. Then $\tau(X, \Delta; M)$ is a rational number.*

Proof. As before, we can assume that $\tau(X, \Delta; M) < 1$. By Proposition 2.61 in [KM98], we can write $M \equiv A_k + (1/k)E$ for k large enough. In the previous decomposition A_k is an ample $\mathbb{Q}$ -divisor and E is an effective $\mathbb{Q}$ -divisor. Choosing k sufficiently large, we can assume that $(X, \Delta + (1/k)E)$ is a klt pair. Finally, there exists an effective divisor A' that is $\mathbb{Q}$ -linearly equivalent to A_k such that $(X, \Delta + A' + (1/k)E)$ is klt. Note that $M \equiv A' + (1/k)E$ so we don't change the pseudo-effective threshold. Then Corollary 4.3.6 applies. $\square$

Let us conclude this section with a slightly different version of Theorem 4.1.2.

Corollary 4.3.9. *Fix a positive integer n and a DCC set $I \subseteq [0, 1]$. Let $\mathcal{T}_n(I)$ be the set of pseudo-effective thresholds $\tau(X, \Delta; M)$ where (X, Δ) is a klt pair such that $\dim(X) = n$, the coefficients of Δ are in I , $-(K_X + \Delta)$ is nef and M is a big and nef Cartier divisor on X . Then the set $\mathcal{T}_n(I)$ satisfies the ACC.*

Proof. Since $M - K_X - \Delta$ is big and nef, Theorem 1.1 in [Kol93] implies that we can find a positive integer m' , which depends only on n , such that $m'M$ is base point free. Choose $m := \max\{m', \binom{n+1}{2}\}$. Then Theorem 5.8 in [Kol97] implies that $K_X + \Delta + mM$ is pseudo-effective. Since mM is base point free, we can choose a general element $H \in |mM|$ such that the coefficients of H are all equal to one, $(X, \Delta + H)$ is lc and $K_X + \Delta + H$ is pseudo-effective. Finally, Theorem 4.1.5 gives the result. $\square$

4.4 Effective birationality

Recall that in Chapter 3 we proved the following result.

Theorem 4.4.1. *Let (X, Δ) be a klt pair. Let M be a big and nef Cartier divisor on X such that $K_X + \Delta + M$ is big. Let $\tau := \tau(X, \Delta; M)$ be the pseudo-effective threshold of M with respect to (X, Δ) . Then for any*

$$m > \frac{1}{1 - \tau} \binom{n+2}{2}$$

the map induced by the linear system $|[m(K_X + \Delta + M)]|$ is birational.

In order to give a uniform bound, it suffices to bound away from 1 the pseudo-effective threshold. If $(X, \Delta; M) \in \mathcal{D}_n(I, \mathbb{N})$ then it follows from Theorem 4.1.5. In particular, Conjecture 1.1.2 is true in this case. Note that we don't assume that $(X, \Delta + M)$ is lc, but it is close to be lc.

Corollary 4.4.2. *Fix a positive integer n and a DCC set $I \subseteq [0, 1]$. Then there exists $m_{n,I}$ such that the map associated to the linear system $[m(K_X + \Delta + M)]$ is birational for any $m \geq m_{n,I}$ and any $(X, \Delta; M) \in \mathcal{D}_n(I, \mathbb{N})$ with M a big and nef Cartier divisor.*

Combining Theorem 4.4.1 and Corollary 4.3.9 we get the following.

Corollary 4.4.3. *Fix a positive integer n and a DCC set $I \subseteq [0, 1]$. Then there exists a positive integer $m_{n,I}$ such that if*

1. X is a projective variety with $\dim(X) = n$,
2. (X, Δ) is a klt pair such that $-(K_X + \Delta)$ is nef,
3. the coefficients of Δ are in I ,
4. M is a big and nef Cartier divisor such that $K_X + \Delta + M$ is big,

then the map associated to the linear system $[m(K_X + \Delta + M)]$ is birational for any $m \geq m_{n,I}$.

Note that if $\tau = 0$ in Theorem 4.4.1 we don't even need that the coefficients of Δ are in a DCC set. Analogously, as a corollary of [HMX12], we have.

Corollary 4.4.4. *Fix a positive integer n and a positive real number δ . Then there exists a positive integer $m_{n,\delta}$ such that if*

1. (X, Δ) is a lc pair,
2. $\dim(X) = n$,
3. K_X is pseudo-effective,
4. $\Delta = \sum a_i \Delta_i$ with $a_i \geq \delta$,
5. $K_X + \Delta$ is big,

then the map associated to $[m(K_X + \Delta)]$ is birational for any $m \geq m_{n,\delta}$.

Proof. Since K_X is pseudo-effective, $\tau(X, \Delta) = 0$. This implies that $K_X + \alpha\Delta$ is big for any $\alpha \geq 0$. Choose $\alpha := \min\{1, \delta\}$. Then Theorem C in [HMX12] gives the result. $\square$

If we want to deal with effective birationality results, maybe the condition $(X, \Delta + \tau M)$ is lc can be substituted with a more suitable condition. A possible way to overcome this difficulty may be the following.

Question 4.4.5. *Fix n and a DCC set I . Let (X, Δ) be a klt pair with coefficients of Δ in I and let M be a big and nef Cartier divisor on X . Suppose that $K_X + \Delta + M$ is big and $\text{vol}(K_X + \Delta + M) < 1$. Is the pseudo-effective threshold $\tau(X, \Delta; M)$ uniformly bounded away from one?*

Even if the previous question seems very restrictive, using the argument in [Jia12] and Theorem 4.4.1, it implies the uniformity of the Iitaka fibration when the variation is maximal.

Chapter 5

Nef threshold and Kähler-Einstein metrics

5.1 Introduction

G. Tian [Tia96] outlined a very general program for the construction of incomplete Kähler-Einstein metrics on quasi-projective varieties $X \setminus D$ where X is a smooth projective variety and $D \subset X$ is a simple normal crossing divisor. Such metrics have cone-edge like singularities along D , their asymptotic behavior is described by the cone-angle $2\pi(1 - \alpha)$ for $\alpha \in (0, 1)$. These metrics have been extensively studied, see for example [Tia96], [Don10], [Ber11], [Bre11], [CGP11], [JMR11], [MR12]. Recently, Jeffres-Mazzeo-Rubinstein [JMR11] and Mazzeo-Rubinstein [MR12] made substantial progress toward the completion of Tian's original program. In particular, they show that such metrics, with negative scalar curvature and cone angle $2\pi(1 - \alpha)$, exist when $K_X + \alpha D$ is ample, see also [CGP11] for the case $\alpha \in [\frac{1}{2}, 1)$.

In this chapter we consider two questions motivated by these results.

- Characterize those pairs (X, D) for which a Kähler-Einstein metric with cone angle $2\pi(1 - \alpha)$ exists for some α .
- Given (X, D) , describe the values α for which such a metric exists.

We have a satisfactory answer to these questions for small cone-angles in the negative scalar curvature case.

Theorem 5.1.1. *Let X be a smooth, n -dimensional projective variety and $D \subset X$ a simple normal crossing divisor. The following are equivalent.*

1. (X, D) admits negative Kähler-Einstein metrics with cone-edge singularities along D with cone angles $2\pi(1 - \alpha_i)$, where $\{\alpha_i\}$ is a sequence converging to 1 from below.
2. (X, D) admits negative Kähler-Einstein metrics with cone-edge singularities along D with cone angles $2\pi(1 - \alpha)$ for every $\alpha \in (\frac{n+1}{n+2}, 1)$.
3. The self-intersection $(K_X + D)^n$ is positive, $(K_X + D) \cdot C \geq 0$ for every curve $C \subset X$ and there are no curves $C \subset X$ such that $(K_X + D) \cdot C = 0$ and $K_X \cdot C \leq 0$.

Similarly, we can treat the negative scalar curvature case for cone-angles close to 2π .

Theorem 5.1.2. *Let X be a smooth, n -dimensional projective variety of general type and $D \subset X$ a simple normal crossing divisor. The following are equivalent.*

1. (X, D) admits negative Kähler-Einstein metrics with cone-edge singularities along D with cone angles $2\pi(1 - \alpha_i)$, where $\{\alpha_i\}$ is a sequence converging to 0 from above.
2. (X, D) admits negative Kähler-Einstein metrics with cone-edge singularities along D with cone angles $2\pi(1 - \alpha)$ for every $\alpha \in (0, \frac{1}{2n+1})$.
3. $K_X \cdot C \geq 0$ for every curve $C \subset X$ and there are no curves $C \subset X$ such that $(K_X + D) \cdot C \leq 0$ and $K_X \cdot C = 0$.

The proofs of these results rely on Theorem 1.1.9 stated in the Introduction. For the technical details see Section 5.3.

Remarkably, in the positive scalar curvature case the existence of such uniform bounds is obstructed for both small and large cone-angles, see Examples 5.5.2 and 5.5.10. Nevertheless, we are able to quantify this obstruction when $-K_X - D$ and $-K_X$ are ample. For small cone-angles we obtain that this obstruction depends only on the self-intersection $(-K_X - D)^n$, compare with Example 5.5.2. More precisely, we can state the following.

Theorem 5.1.3. *Let $\mathfrak{B}$ be a set of pairs (X, D) such that X is a smooth projective variety of dimension n and D is a reduced effective divisor with simple normal crossing support such that $-(K_X + D)$ is ample. The following are equivalent.*

1. There exists a positive number $\alpha_0 < 1$ such that $-(K_X + \alpha D)$ is ample for any $\alpha \in (\alpha_0, 1]$ and any $(X, D) \in \mathfrak{B}$.

2. There exists a positive integer M_0 such that $-(K_X + D)^n \leq M_0$ for any $(X, D) \in \mathfrak{B}$.

We can also treat the case of cone-angles close to 2π . In this case the obstruction depends only on the intersection number $D \cdot (-K_X)^{n-1}$, compare with Example 5.5.10.

Theorem 5.1.4. *Let $\mathfrak{D}$ be a set of pairs (X, D) such that X is a smooth projective Fano variety of dimension n and D is a reduced effective divisor with simple normal crossing support. The following are equivalent.*

1. There exists a positive number $\alpha_0 < 1$ such that $-(K_X + \alpha D)$ is ample for any $\alpha \in [0, \alpha_0)$ and any $(X, D) \in \mathfrak{D}$.
2. There exists a positive integer M_0 such that $D \cdot (-K_X)^{n-1} \leq M_0$ for any $(X, D) \in \mathfrak{D}$.

The strategies used in the proofs of Theorems 5.1.3 and 5.1.4 are completely different from the techniques employed in the negative scalar curvature case. The proofs of these theorems rely on effective very ampleness results as well as on boundedness of Fano varieties. For the technical details see Section 5.5.

We now recall some of the key features of Tian's program [Tia96] and their relation with the theme of positivity in algebraic geometry. Incomplete Kähler-Einstein metrics are geometrically very interesting. Unlike the complete case, they have a much more interesting moduli theory as they are not subject to Yau's generalized Schwartz lemma. Furthermore, in Tian's original program [Tia96], they should also be used to better understand the complete Kähler-Einstein metrics previously constructed on X and $X \setminus D$. Recall that T. Aubin and S.-T. Yau constructed negatively curved Kähler-Einstein metrics on X when K_X is ample, see [Aub78] and [Yau78a]. Similarly, there are many existence results for complete negatively curved Kähler-Einstein metrics on $X \setminus D$ under certain positivity assumption for $K_X + D$, see for example [TY87], [Yau78b], [Wu08], [Wu09]. For precise definitions and details see Section 5.2. Now, one should be able to recover these metrics by taking limits of Kähler-Einstein metrics with cone-edge singularities as the cone angle goes to 2π or 0. Moreover, the study of this limiting behavior should give a better understanding of the asymptotic behavior of the complete Kähler-Einstein metrics on $X \setminus D$.

Kähler-Einstein metrics with cone-edge singularities are also very interesting from a positivity point of view. Their existence implies strong positivity properties for $\mathbb{R}$ -divisors of the form $K_X + \alpha D$ or $-(K_X + \alpha D)$ for some $\alpha \in (0, 1)$. We refer to Section 5.2 for the precise definitions, results and to Proposition 5.2.3 for the necessary condition for the existence of such metrics. Thus, the existence of

these metrics appears to be associated with the positivity of the log-canonical or anti-log-canonical divisor of klt pair rather than a lc pair. This fact turns out to be extremely useful since the base-point free theorem is currently known for klt pairs only. We exploit systematically this point in Section 5.3.

To sum up, it is important to study which positivity properties are preserved, lost or gained by passing from a pair the (X, D) to a the pair $(X, \alpha D)$, $\alpha \in (0, 1)$, and viceversa. We find that these questions can be approached by studying the behavior of the nef thresholds, see Sections 5.3. All the questions we address here are explicitly formulated throughout the chapter, see Questions 5.3.3, 5.5.1 and 5.5.9. We provide complete solutions to all of them through Theorems 5.3.12, 5.3.21, Example 5.5.2 and Theorems 5.1.3, 5.1.4. These results seem to have interest of their own. Moreover, they have applications beyond the study of Kähler-Einstein metrics, as we will see in the next chapter.

The chapter is organized as follows. In Section 5.2, we recall the theory of Kähler-Einstein metrics on quasi-projective varieties. We discuss the existence of both complete and incomplete Kähler-Einstein metrics of negative and positive scalar curvature. It is shown how the existence of such metrics motivates a number of natural positivity questions. These questions are thoroughly studied in Sections 5.3, 5.4 and 5.5. In Section 5.4, we study the notion of ampleness for line bundles on quasi-projective varieties. This analysis is important for the study of negatively curved Kähler-Einstein metrics beyond the cone-edge asymptotic, see Corollary 5.4.11 and the remarks which follows. Finally, all the results are motivated and supported with examples.

5.2 Kähler-Einstein metrics on quasi-projective varieties

In this section we recall and collect some generalities regarding the existence of both complete and incomplete Kähler-Einstein metrics on quasi-projective varieties. In particular, we discuss the necessary conditions for the existence of such metrics. This analysis uses the language and the basic theory of Kähler currents on Kähler manifolds [Dem01]. We will briefly discuss it here and we refer to [DD12a] to more details.

Let us start by fixing notation and by giving some definitions. Let X be a n -dimensional projective manifold and $D = \sum_i D_i$ be a reduced simple normal crossing divisor.

We now briefly discuss the existence problem for incomplete Kähler-Einstein metrics on quasi-projective varieties. The most natural kind of incomplete metrics appearing in Kähler geometry seem to have cone-edge singularities. The study of such metrics was originally proposed by G. Tian in [Tia96]. Thus, given a pair (X, D) as above, let us recall the notion of a Kähler metric with

edge singularities along D . Let $\{D_i\}$ be the irreducible smooth components of D and consider a collection of positive numbers $\alpha_i \in (0, 1)$. A smooth Kähler metric $\hat{\omega}$ on $X \setminus D$ is said to have edge singularities of cone angles $2\pi(1 - \alpha_i)$ along the D_i 's if for any point $p \in D$ and coordinate neighborhood $(\Omega; z_1, \dots, z_n)$ centered at p for which

$$D|_{\Omega} = \{z_1 \cdot \dots \cdot z_k = 0\},$$

then $\hat{\omega}$ is quasi-isometric in Ω to the following model metric

$$\omega_0 = \sqrt{-1} \left(\sum_{i=1}^k \frac{dz_i \wedge d\bar{z}_i}{|z_i|^{2\alpha_i}} + \sum_{i=k+1}^n dz_i \wedge d\bar{z}_i \right). \quad (5.1)$$

It is easy to construct Kähler metrics with edge singularity along a divisor.

Example 5.2.1. *Let ω be a Kähler metric on X . For simplicity let D be a smooth reduced divisor and choose a real number $\alpha \in (0, 1)$. Choose an Hermitian metric $\|\cdot\|$ on $\mathcal{O}_X(D)$. Let $s \in H^0(X, \mathcal{O}(D))$ be a defining section for the divisor D . Then for $T > 0$ big enough*

$$\hat{\omega} = T\omega + \sqrt{-1}\partial\bar{\partial}\|s\|^{2(1-\alpha)}$$

defines Kähler metric on $X \setminus D$ with edge singularities of cone angle $2\pi(1 - \alpha)$ along D .

As first observed in [Jef96], $\hat{\omega}$ is indeed an incomplete Kähler metric of finite volume on $X \setminus D$. Now, because of the finite volume property, standard results in the theory of currents [Dem01] imply that $\hat{\omega}$ can be regarded as a Kähler current on X with singular support D . The fact that $\hat{\omega}$ defines a Kähler current and not only a closed positive current follows from the quasi-isometric condition given in 5.1.

We can now introduce the definition of a Kähler-Einstein metric with edge singularities.

Definition 5.2.2 (Tian [Tia96]). *A Kähler current $\hat{\omega}$ with edges of cone angles $2\pi(1 - \alpha_i)$ along the D_i 's is called Kähler-Einstein with curvature λ if it satisfies the distributional equation*

$$\text{Ric}_{\hat{\omega}} - \sum_i \alpha_i [D_i] = \lambda \hat{\omega} \quad (5.2)$$

where by $[D]$ we indicate the current of integration along D .

A simple regularization argument based on results of Demailly [Dem92] can now be used to derive the necessary condition for the existence of a Kähler-Einstein metric with edge singularities.

Proposition 5.2.3. *Let $\hat{\omega}$ be a Kähler-Einstein metric with edge singularities as in 5.2 with $\lambda = -1$ ($\lambda = 1$), then the $\mathbb{R}$ -divisor $K_X + \sum_i \alpha_i D_i$ ($-K_X - \sum_i \alpha_i D_i$) is ample.*

Proof. See Propositions 2.1 and 2.2 in [DiC12b]. □

To sum up, the entire set of questions concerning the existence and limiting behavior of families of Kähler-Einstein metrics on quasi-projective varieties requires a precise understanding of the positivity properties of certain log-canonical bundles. This is the object of study in Sections 5.3-5.5. Since we are primarily interested in taking limits as the cone angles go to 0 or 2π we limit ourselves to the case where all the α_i 's are identical.

5.3 Negatively curved Kähler-Einstein metrics

Given a pair (X, D) for which $K_X + D$ is ample then there exists a Kähler-Einstein metric on $X \setminus D$. Following intuitions of G. Tian [Tia96], this Kähler-Einstein metric should arise as the limit of a 1-parameter family of Kähler-Einstein metrics with cone-edge singularities along D . It would then be desirable to have a uniform interval of angles, depending on the dimension only, for which the existence of such metrics is unobstructed, see Proposition 5.2.3. It turns out that the nef threshold plays a central role for this kind of questions. Let us recall its definition.

Definition 5.3.1. *Let X be a smooth projective variety and let D be a divisor such that $K_X + tD$ is nef for some $t \in \mathbb{R}_{\geq 0}$. The nef threshold is*

$$\eta(X, D) := \inf \{ t \in \mathbb{R}_{\geq 0} \mid K_X + tD \text{ is nef} \}.$$

We will denote by $\tau(X, D)$ the pseudo-effective threshold of D with respect to X .

Since nef divisors are also pseudo-effective we have that $\tau(X, D) \leq \eta(X, D)$. In particular, we cannot use the results obtained in the previous chapter to get upper bounds on the nef threshold. Let us construct an example for which these numerical invariants are actually different.

Example 5.3.2. *Let X a smooth projective surface and D a smooth irreducible divisor such that $K_X + D$ is ample. Let $\pi : Y \rightarrow X$ be the blow up of a point in D . Let D' be the strict transform of D on Y . Then $K_Y + D'$ is big and nef but $K_Y + \alpha D'$ is not nef for any $\alpha < 1$. Then $\eta(Y, D') = 1$ but $\tau(Y, D') < 1$.*

The phenomena explained in Example 5.3.2 relies on the fact that, differently from the cone of big divisor, the cone of nef divisors is closed being the closure of the ample cone, see Theorem 1.4.23

in [Laz04a]. Thus, we do not expect the nefness to hold when we decrease the coefficients of D . Remarkably, the bigness of $K_Y + D'$ in Example 5.3.2 does not help in achieving $\eta(Y, D') < 1$.

The previous example motivates the following question.

Question 5.3.3. *Let X be a smooth variety and let D be an effective divisor such that $K_X + D$ is big and nef. When is $\eta(X, D) < 1$? Moreover, if $\eta(X, D) < 1$, can we find $\alpha_n < 1$, depending only on n , such that $K_X + \alpha D$ is big and nef for any $\alpha \in (\alpha_n, 1]$? Finally, can we characterize the pairs (X, D) for which $K_X + \alpha D$ is ample for α close enough to one?*

In this section we work out a satisfactory answer to Question 5.3.3 and we give applications of it to the study of 1-parameter families of Kähler-Einstein metrics with cone-edge singularities.

We start by the easier case when $K_X + D$ is ample. Since ampleness is an open condition, we automatically have that $\eta(X, D) < 1$.

Lemma 5.3.4. *Let X be a smooth projective variety with $\dim(X) = n$ and let H be an ample divisor. Then $K_X + (n+1)H$ is nef.*

Proof. By Theorem 2.4.3 we know that any curve C in X is numerically equivalent to

$$a_1 C_1 + \cdots + a_r C_r + F,$$

where a_i are positive real numbers, C_i are rational curves such that $0 > C_i \cdot K_X \geq -(n+1)$ and $K_X \cdot F \geq 0$. Since H is ample, we have $H \cdot C_i \geq 1$. Then $(K_X + (n+1)H) \cdot C \geq 0$. $\square$

The proposition below gives the desired uniform bound for the nef threshold when $K_X + D$ is ample.

Proposition 5.3.5. *Let X be a smooth projective variety of dimension n and let D be a divisor such that $K_X + D$ is ample. Then*

$$\eta(X, D) \leq \frac{n+1}{n+2}.$$

Proof. Since $K_X + D$ is ample by Lemma 5.3.4 we know that $K_X + (n+1)(K_X + D)$ is nef. In particular $(n+2)K_X + (n+1)D$ is nef and then the result follows. $\square$

Let us observe that the bound given in Proposition 5.3.5 is sharp.

Remark 5.3.6. *Take $X = \mathbb{P}^n$ and $D = (n+2)H$ where H is the hyperplane section. Then $K_{\mathbb{P}^n} + D$ is ample and $\eta(\mathbb{P}^n, D) = \frac{n+1}{n+2}$.*

Corollary 5.3.7. *Let X and D as in the Proposition 5.3.5. Then for any $\alpha \in (\frac{n+1}{n+2}, 1]$ the divisor $K_X + \alpha D$ is ample.*

Proof. The result follows from the fact that in this case the nef threshold can be obtained as $\eta(X, D) = \inf \{t \in \mathbb{R}_{\geq 0} \mid K_X + tD \text{ is ample}\}$. $\square$

In the above discussion, we restricted ourselves to pairs (X, D) with $K_X + D$ ample. Nevertheless, it is easy to construct examples of pairs such that $X \setminus D$ admit complete negatively curved Kähler-Einstein metrics of finite volume, but for which $K_X + D$ is not ample. A wealth of examples is provided by the theory of locally symmetric varieties and their compactifications, see for example [AMRT10]. More concretely, let us consider the following class of low-dimensional examples. For more details see Chapter 6.

Example 5.3.8. *Let X be a smooth toroidal compactification of a finite-volume complex-hyperbolic surface and let D be the compactifying divisor. The set D simply consists of smooth disjoint elliptic curves. Note that $X \setminus D$ admits a natural finite-volume Kähler-Einstein metric coming from the Bergman metric on $\mathbb{C}\mathcal{H}^2$. Nevertheless, the line bundle $K_X + D$ is just big and nef since $(K_X + D) \cdot D = 0$.*

We are now interested in trying to characterize the pairs (X, D) with $K_X + D$ nef and with nef threshold strictly less than one. The following result gives a quite satisfactory answer, compare with Question 5.3.3.

Proposition 5.3.9. *Let X be a smooth projective variety and let D be a divisor such that $K_X + D$ is nef. Then $\eta(X, D) < 1$ if and only if there are no irreducible curves C such that $(K_X + D) \cdot C = 0$ and $K_X \cdot C < 0$.*

Proof. Suppose $\eta(X, D) < 1$. Let C be an irreducible curve such $(K_X + D) \cdot C = 0$. In particular we have that $D \cdot C = -K_X \cdot C$. We want to prove that $K_X \cdot C \geq 0$. Let α be close to one such that $K_X + \alpha D$ is nef. Then

$$(K_X + \alpha D) \cdot C = (1 - \alpha)K_X \cdot C \geq 0.$$

Conversely suppose that there are no irreducible curves C with the properties $(K_X + D) \cdot C = 0$ and $K_X \cdot C < 0$. We would like to show that $\eta(X, D) < 1$. Let C' be an irreducible curve such that $(K_X + D) \cdot C' > 0$. By the same argument in the proof of Lemma 5.3.4 we can write $C' \equiv a_1 C_1 + \dots + a_r C_r + F$ where a_i are positive numbers, C_i are irreducible curves with $0 > K_X \cdot C_i \geq -(n+1)$ and $K_X \cdot F \geq 0$. Since F is a limit of effective 1-cycles and $K_X + D$ is nef

we have that $(K_X + D) \cdot F \geq 0$. By assumption if $(K_X + D) \cdot C_i = 0$ then $K_X \cdot C_i \geq 0$. In particular $(K_X + (n+1)(K_X + D)) \cdot C' \geq 0$. This implies that $(K_X + \frac{n+1}{n+2}D) \cdot C' \geq 0$ for any curve C' such that $(K_X + D) \cdot C' > 0$. Then we need only to check the nefness of $K_X + \alpha D$ on irreducible curve C such that $(K_X + D) \cdot C = 0$. The same computation as before tells us that $(K_X + \alpha D) \cdot C = (1 - \alpha)K_X \cdot C \geq 0$. $\square$

Let us observe that Proposition 5.3.9 implies the following.

Corollary 5.3.10. *Let X be a smooth projective variety and let D be a divisor such that $K_X + D$ is nef. Then either $\eta(X, D) = 1$ or $\eta(X, D) \leq \frac{n+1}{n+2}$.*

The reasoning given in Proposition 5.3.9 is easily adapted to understand the obstructions for $K_X + \alpha D$ to be strictly nef. In fact, we can state:

Proposition 5.3.11. *Let X be a smooth projective variety and let D be a divisor such that $K_X + D$ is nef. Then $K_X + \alpha D$ is strictly nef for $\alpha \in (\frac{n+1}{n+2}, 1)$ if and only if there are no irreducible curves C such that $(K_X + D) \cdot C = 0$ and $K_X \cdot C \leq 0$.*

We are now ready to address the last problem considered in Question 5.3.3.

Theorem 5.3.12. *Let X be a smooth projective variety and let D be a reduced effective divisor with simple normal crossing support such that $K_X + D$ is big and nef. Then $K_X + \alpha D$ is ample for $\alpha \in (\frac{n+1}{n+2}, 1)$ if and only if there are no irreducible curves C such that $(K_X + D) \cdot C = 0$ and $K_X \cdot C \leq 0$.*

Proof. Suppose $K_X + \alpha D$ to be ample. Let C be an irreducible curve such that $(K_X + D) \cdot C = 0$. Then

$$(K_X + \alpha D) \cdot C = (1 - \alpha)K_X \cdot C > 0.$$

Conversely, by Proposition 5.3.11 we know that $K_X + \alpha D$ is strictly nef for $\alpha \in (\frac{n+1}{n+2}, 1)$. Since $\tau(X, D) \leq \eta(X, D)$ we conclude that $K_X + \alpha D$ is big and strictly nef for $\alpha \in (\frac{n+1}{n+2}, 1)$. Note that $(X, \alpha D)$ is a klt pair. By Theorem 2.4.4 we know that $K_X + \alpha D$ is semi-ample. Since strictly nef semi-ample divisors are ample the proof is complete. $\square$

We can now give a proof of Theorem 5.1.1 stated in the Introduction 5.1.

Proof of Theorem 5.1.1. Combine Proposition 5.2.3 and Theorem 5.3.12 with the analytical results of Campana-Guenancia-Păun [CGP11] and Mazzeo-Rubinstein [MR12], more precisely Theorem A in [CGP11] and Theorem 1.3 in [MR12]. $\square$

Theorem 5.3.12 nicely applies in low-dimensions. Thus, let us collect few corollaries concerning complex surfaces.

Corollary 5.3.13. *Let X be a smooth projective surface and let D be a divisor such that $K_X + D$ is big and nef. Suppose there are no curves C in X such that $C = \mathbb{P}^1$, $C^2 = -1$ and $C \cdot D = 1$, then $\eta(X, D) < 1$.*

Proof. We want to apply Proposition 5.3.9. Let C be a curve such that $(K_X + D) \cdot C = 0$. Then by the Hodge index theorem $C^2 < 0$. By adjunction we have

$$K_X \cdot C = 2g - 2 - C^2.$$

Suppose $K_X \cdot C < 0$. Since $C^2 < 0$ this can happen if and only if $g = 0$ and $C^2 = -1$. But then $D \cdot C = -K_X \cdot C = 1$ which contradicts our hypothesis. $\square$

Similarly, we have a nice characterization of the pairs with ample log-canonical bundles.

Corollary 5.3.14. *Let X be a smooth projective surface and let D be a reduced effective divisor with simple normal crossing support such that $K_X + D$ is big and nef. Suppose there are no curves C in X such that $C = \mathbb{P}^1$, $C^2 = -1$ and $C \cdot D = 1$, or $C = \mathbb{P}^1$, $C^2 = -2$ and $C \cdot D = 0$. Then $K_X + \alpha D$ is ample for any $\alpha \in (\frac{3}{4}, 1)$.*

Proof. We want to apply Theorem 5.3.12. Because of Corollary 5.3.13, it suffices to characterize the curves C such that $(K_X + D) \cdot C = 0$ and $K_X \cdot C = 0$. By the Hodge index theorem we have $C^2 < 0$ and then by adjunction $K_X \cdot C = 0$ if and only if $g = 0$ and $C^2 = -2$. This implies $C \cdot D = 0$. $\square$

The interested reader should compare Corollaries 5.3.13 and 5.3.14 with Theorems 3.3 and 3.4 in [DiC12b]. The results presented in [DiC12b] contain a geometric characterization of the pairs (X^2, D) with $K_X + D$ big and nef. Such results rely on F. Sakai's notions of semi-stability and D -minimality, see [Sak80].

It is now interesting to collect a corollary regarding Fano varieties. Recall that X is called a Fano variety if $-K_X$ is ample.

Corollary 5.3.15. *Let X be a smooth Fano variety. Let D be a divisor such that $K_X + D$ is nef. Then $\eta(X, D) < 1$ if and only if $K_X + D$ is strictly nef.*

Proof. If $K_X + D$ is strictly nef the result follows from the same argument of Lemma 5.3.4. Assume $\eta(X, D) < 1$. Since X is Fano we have that $-K_X$ is ample and in particular $K_X \cdot C < 0$ for any

irreducible curve C . Then by Proposition 5.3.9 we must have $(K_X + D) \cdot C > 0$ for any irreducible curve C , i.e. $K_X + D$ is strictly nef. $\square$

Corollary 5.3.16. *Let X be a smooth Fano variety with $\dim(X) \leq 3$. Let D be an effective divisor such that (X, D) is lc and $K_X + D$ is nef. Then $\eta(X, D) < 1$ if and only if $K_X + D$ is ample.*

Proof. By the above corollary $\eta(X, D) < 1$ if and only if $K_X + D$ is strictly nef. Since $\dim(X) \leq 3$ the abundance conjecture holds and then $K_X + D$ is semi-ample. The result follows from the fact that strictly nef and semi-ample divisors are ample. $\square$

Of course the above corollary should be true in any dimension since it follows from the abundance conjecture.

Corollary 5.3.17. *Let X be a smooth toric Fano variety. Let D be an effective divisor such that (X, D) is lc and $K_X + D$ is nef. Then $\eta(X, D) < 1$ if and only if $K_X + D$ is ample.*

Proof. On a smooth toric variety every nef divisor is globally generated, see Theorem 6.3.12 in [CLS11]. $\square$

In the rest of the section we address the following:

Question 5.3.18. *Let X be a smooth variety with K_X big and nef and let D be an effective divisor. Can we characterize the pairs (X, D) with $K_X + \alpha D$ big and nef or ample for α close enough to zero? Finally, is there a fixed $\alpha_n > 0$, which depends only on n , such that $K_X + \alpha D$ is big and nef or ample for any $\alpha \in [0, \alpha_n)$?*

Note that in this case $K_X + \alpha D$ is big for any $\alpha \geq 0$, so the main problem is to understand when it is nef. To this aim we use again the cone theorem, but now we need (X, D) to be a lc pair. Thus, to warm up, let us observe the following.

Proposition 5.3.19. *Let (X, D) be a lc pair with K_X ample. Then $K_X + \alpha D$ is ample for any*

$$\alpha < \frac{1}{2n+1}.$$

Proof. By the cone theorem for lc pairs $K_X + D + 2nK_X$ is nef. In particular so is $K_X + \frac{1}{2n+1}D$. $\square$

A similar approach can now be applied when K_X is simply nef.

Theorem 5.3.20. *Let X be a smooth projective variety with K_X nef. Let D be a reduced effective divisor with simple normal crossing support. Then $K_X + \alpha D$ is nef for $\alpha \in [0, \frac{1}{2n+1}]$ if and only if there are no irreducible curves C such that $(K_X + D) \cdot C < 0$ and $K_X \cdot C = 0$.*

Proof. By the cone theorem for lc pairs, if there are no irreducible curves C such that $(K_X + D) \cdot C < 0$ and $K_X \cdot C = 0$ then the $\mathbb{R}$ -divisor

$$K_X + D + tK_X$$

is nef for $t \geq 2n$. Conversely, if $K_X + \alpha D$ is nef then for some $\alpha \in (0, 1)$ then the curves that dot negatively with $K_X + D$ must intersect positively with K_X . $\square$

We conclude this section by studying when $K_X + \alpha D$ is ample for α close to zero.

Theorem 5.3.21. *Let X be a smooth projective variety with K_X big and nef. Let D be a reduced effective divisor with simple normal crossing support. Then $K_X + \alpha D$ is ample for $\alpha \in (0, \frac{1}{2n+1})$ if and only if there are no irreducible curves C such that $(K_X + D) \cdot C \leq 0$ and $K_X \cdot C = 0$.*

Proof. If there are no irreducible curves C such that $(K_X + D) \cdot C \leq 0$ and $K_X \cdot C = 0$ we have that $K_X + \alpha D$ is strictly nef for any $\alpha \in (0, \frac{1}{2n+1})$. Recall that $K_X + \alpha D$ is a big $\mathbb{R}$ -divisor for any $\alpha \geq 0$. Then Theorem 2.4.4 implies that $K_X + \alpha D$ is semi-ample and therefore ample. The converse should be clear by now. $\square$

We do not expect the above result to be sharp. On the other hand, we expect that the previous result holds without the bigness assumption.

Corollary 5.3.22. *Assume Conjecture 2.4.6. Let X be a smooth projective variety with K_X nef. Let D be a reduced effective divisor with simple normal crossing support. Then $K_X + \alpha D$ is ample for $\alpha \in (0, \frac{1}{2n+1})$ if and only if there are no irreducible curves C such that $(K_X + D) \cdot C \leq 0$ and $K_X \cdot C = 0$.*

Proof. In the proof of Theorem 5.3.21, replace Theorem 2.4.4 with Conjecture 2.4.6. $\square$

We conclude this section by giving the proof of Theorem 5.1.2 stated in the Introduction 5.1.

Proof of Theorem 5.1.2. Combine Proposition 5.2.3, Theorem 5.3.21 with Theorem 1.3 in [MR12]. $\square$

Corollary 5.3.22 implies that Theorem 5.1.2 should hold without asking X to be of general type.

5.4 Ampleness on quasi-projective varieties

It is clear that the study of Kähler-Einstein metrics on quasi-projective varieties must be connected with the notion of ampleness on open algebraic manifolds. It seems there is no general

agreement in the existing literature on the definition of this concept. Recall that for compact manifold the notion of ampleness is usually introduced algebraically and then formulated in geometric and numerical terms, see Chapter 1 in [Laz04a]. Here, we follow a completely analogous path. The interested reader should compare the results contained in this section with some of the existing literature [Wu08], [Wu09], [TY87].

Thus, let us start with a definition. For more details see [Tak93].

Definition 5.4.1. *Let X be a smooth projective variety and let D be a simple normal crossing divisor. A line bundle L on X is said to be ample modulo D , if for any torsion free sheaf $\mathcal{F}$ on X , there exists an integer m_0 such that $(\mathcal{F} \otimes L^n)_{|X \setminus D}$ is generated by $H^0(X, \mathcal{F} \otimes L^n)$ for $m \geq m_0$.*

We then say that a divisor L is very ample modulo D if the rational map $\phi_{|L|}$ associated to the linear system $|L|$ defines an embedding of $X \setminus D$ into some projective space. By the same argument of Theorem II.7.6 in [Har77], one can show that some power of an ample modulo D divisor is very ample modulo D . In particular it follows that if L is ample modulo D then it is necessarily big.

The discussion on $\mathbf{B}_+(L)$ contained in Section 2.3 implies the following.

Lemma 5.4.2. *If L is ample modulo D then $\mathbf{B}_+(L) \subseteq D$.*

It would now be desirable to have a numerical characterization of ampleness modulo D .

Definition 5.4.3. *Let X be a smooth projective variety and let D be an effective divisor. A line bundle L on X is said to be strictly nef modulo D , if $L \cdot C > 0$ for any curve C not entirely contained in D .*

Let us observe that, similarly to the compact case, being strictly nef modulo D is not enough for achieving ampleness modulo D . To this aim recall that a strictly nef divisor is not necessarily ample. The first construction of such a divisor is due to Mumford. In fact, he showed the existence of a surface X with a strictly nef divisor D such that $D^2 = 0$. Ramanujam, based on Mumford's example, showed the existence of a big and strictly nef divisor that is not ample. Note that by Nakai's criterion a divisor which is big and strictly nef but not ample can exist in dimension greater or equal than three only. For the convenience of the reader we sketch the construction of these examples. For details see Chapter I §10 in [Har70].

Example 5.4.4 (Mumford, Ramanujam). *We start with Mumford's example. Let C be a smooth curve of genus $g \geq 2$. By a theorem of Seshadri there exists a stable vector bundle E of degree zero over C of rank two such that all its symmetric powers are stable. Let $X = \mathbb{P}(E)$ and $D = \mathcal{O}_X(1)$ be*

the Serre line bundle. One can check that D is strictly nef. Since E has degree zero, $D^2 = 0$ and in particular D is not ample. Now we can give Ramanujam's example of a big and strictly nef divisor which is not ample. We will show the existence of such divisor using Mumford's example. Let X be as above and let H be an effective ample divisor on X . Define $Y := \mathbb{P}(\mathcal{O}_X(D - H) \oplus \mathcal{O}_X)$ and let $\pi : Y \rightarrow X$ be the projection. Let $X_0 \subseteq Y$ be the zero section. Finally let $L := X_0 + \pi^*H$. As shown in [Har70], L is strictly nef but it is not ample because $L^2 \cdot X_0 = D^2 = 0$. Furthermore L is big because $L^3 = (D + H) \cdot H > 0$.

Thus, let L be a big and strictly nef divisor which is not ample. Observe that $\mathbf{B}_+(L)$ is not empty otherwise L would be ample. Then L is strictly nef modulo D for any effective divisor D . Nevertheless, by choosing a divisor D that does not contain $\mathbf{B}_+(L)$ we certainly obtain that L cannot be ample modulo D , see Lemma 5.4.2.

The discussion above tells us that if we want to control $\mathbf{B}_+(L)$ only with intersection numbers we have to take in account all positive dimensional subvarieties and not only curves. In fact, let L be nef and such that $L^k \cdot V > 0$ for any k -dimensional subvariety not entirely contained in D . Then by Theorem 2.3.2 we have that $\mathbf{B}_+(L)$ must be contained in D . This is then enough to conclude:

Lemma 5.4.5. *Let L be nef such that $L^k \cdot V > 0$ for any k -dimensional subvarieties not entirely contained in D . Then L is ample modulo D .*

Let us now study line bundles which are semi-ample. We would like to show that if L is semi-ample and strictly nef modulo D then it must be ample modulo D . First, since L is semi-ample then there exists a positive integer m such that the linear system $|mL|$ defines a morphism into some projective space. Let us denote this map by $\phi_{|mL|}$. Since L is strictly nef modulo D we have that $\phi_{|mL|}$ is a finite map over $X \setminus D$. This immediately implies that L is big. Moreover, a simple integration shows that $L^k \cdot V > 0$ for any k -dimensional subvariety which is not entirely contained in D . By Theorem 2.3.2 we conclude that $\mathbf{B}_+(L)$ is contained in D . It then follows that L is ample modulo D .

Lemma 5.4.6. *Let L be semi-ample and strictly nef modulo D . Then L is ample modulo D .*

Let us construct examples of line bundles which are ample modulo a divisor. A wealth of examples can be constructed using the following.

Lemma 5.4.7. *Let X be a smooth projective variety and let D be a simple normal crossing divisor such that $K_X + \alpha D$ is ample for some $\alpha \in [0, 1)$. Then $K_X + D$ is ample modulo D .*

We now want to give examples of pairs where $K_X + D$ is ample modulo D but such that $K_X + \alpha D$ is never ample for any $\alpha \in \mathbb{R}$.

Example 5.4.8. *Let X be a smooth projective surface and let D be a semi-stable curve. Let assume (X, D) to be D -minimal, of log-general type, without interior (-2) -curves. Moreover, assume the existence of at least one (-2) -curve in D , denoted by E , such that $E \cdot (D - E) = 2$. Note that $K_X + D$ is semi-ample since the abundance conjecture holds in dimension two. Moreover, $K_X + D$ is strictly nef modulo D but it is not ample since $(K_X + D) \cdot E = 0$. By Lemma 5.4.6 we conclude that $K_X + D$ is ample modulo D . On the other hand $(K_X + \alpha D) \cdot E = 0$ constantly in α .*

Now, in all the above examples $\eta(X, D) < 1$. Let us construct a pair (X, D) with $K_X + D$ big, semi-ample and strictly nef modulo D such that $\eta(X, D) = 1$.

Example 5.4.9. *Let X a smooth projective surface and D a reduced divisor consisting of two components D_1 and D_2 meeting transversally at a point p . Assume $K_X + D$ to be ample. Let $\pi : Y \rightarrow X$ be the blow up at $p \in D$. Let D' be the union of the strict transforms of D_1 and D_2 in Y and the exceptional divisor E . Then $K_Y + D' = \pi^*(K_X + D)$. We conclude that $K_Y + D'$ is big, nef and strictly nef modulo D' . Then $K_Y + D'$ is ample modulo D' . On the other hand*

$$(K_Y + \alpha D') \cdot E = (\alpha - 1) < 0$$

which therefore implies $\eta(X, D) = 1$.

Unfortunately, as shown in Example 5.4.9, being nef and ample modulo a divisor is not an open condition even for divisors of the form $K_X + D$. Nevertheless, we have the following.

Theorem 5.4.10. *Let X be a smooth projective variety and let D be an effective reduced divisor such that $K_X + D$ is big, nef and strictly nef modulo D . Assume that $\eta(X, D) < 1$. Then $K_X + \alpha D$ is big, nef and ample modulo D for any $\alpha \in (\frac{n+1}{n+2}, 1)$.*

Proof. Since we are assuming $\eta(X, D) < 1$, by Corollary 5.3.10 we know that

$$\tau(X, D) \leq \eta(X, D) \leq \frac{n+1}{n+2}.$$

It then remains to check the ampleness modulo D . As in the proof of Proposition 5.3.9, observe that given an irreducible curve C such that $(K_X + D) \cdot C > 0$ then $(K_X + \alpha D) \cdot C > 0$ for any $\alpha \in (\frac{n+1}{n+2}, 1]$. By Theorem 2.4.4, for any $\alpha \in (\frac{n+1}{n+2}, 1)$ we have that $K_X + \alpha D$ is semi-ample. By Lemma 5.4.6 the proof is then complete. $\square$

Note that the strategy behind the proof of Theorem 5.4.10 can be used to prove the following.

Corollary 5.4.11. *Let (X, D) be such that $K_X + D$ is big, nef and strictly nef modulo D . Assume there are no irreducible curves C such that $(K_X + D) \cdot C = 0$ and $K_X \cdot C < 0$. Then, for any $\alpha \in (\frac{n+1}{n+2}, 1)$, the $\mathbb{R}$ -cohomology class of $K_X + \alpha D$ can be represented by a smooth closed $(1, 1)$ -form which is everywhere positive semi-definite and strictly positive outside D .*

Proof. By Proposition 5.3.9 we have that $r(X, D) < 1$ if and only if there are no irreducible curves C such that $(K_X + D) \cdot C = 0$ and $K_X \cdot C < 0$. $\square$

We decided to explicitly state Corollary 5.4.11 because it has nice applications in the theory of negatively curved Kähler-Einstein metrics on quasi-projective varieties. These applications rely on some recent advancements in the theory of degenerate complex Monge-Ampère equations with singular right hand side [DP06]. Thus, let (X, D) be as in Corollary 5.4.11. For any $\alpha \in (\frac{n+1}{n+2}, 1)$ denote by γ_α the smooth semi-positive form representing the cohomology class of $K_X + \alpha D$. Given a smooth volume form Ω on X and Hermitian metrics $\|\cdot\|_i$ on $\mathcal{O}_X(D_i)$, consider the family of degenerate complex Monge-Ampère equations

$$(\gamma_\alpha + \sqrt{-1}\partial\bar{\partial}\varphi)^n = e^\varphi \frac{\Omega}{\prod_i \|\sigma_i\|^{2\alpha}} \quad (5.3)$$

where by D_i and σ_i we respectively denote the irreducible components of D and the associated defining sections. By Theorem 6.1 in [DP06], for a fixed α this equation admits a unique solution $\varphi \in L^\infty(X)$ which is smooth on $X \setminus D$. Moreover, by appropriately choosing Ω and the $\|\cdot\|_i$ in 5.3 we can arrange γ_φ^α to be Einstein with negative scalar curvature on $X \setminus D$. To sum up, given a pair (X, D) such that $K_X + D$ is big, strictly nef modulo D and with $\eta(X, D) < 1$ then the Kähler-Einstein theory on $X \setminus D$ is rather well-understood. Of course, the asymptotic behavior of γ_φ^α along D as well as the limiting behavior of this family as $\alpha \rightarrow 1$ remain quite mysterious.

Remark 5.4.12. *The 1-parameter family of negatively curved Kähler-Einstein metrics γ_φ^α cannot represent ample cohomology classes unless there are no irreducible curves C such that $(K_X + D) \cdot C = 0$ and $K_X \cdot C \leq 0$. Recall that in this case the form γ_α can be chosen to be a Kähler metric. Compare Proposition 5.2.3 and Theorem 5.3.12. In this case the asymptotic behavior of γ_φ^α along D is much better understood. In fact, thanks to the recent works [JMR11] and [MR12], we know that for any $\alpha \in (\frac{n+1}{n+2}, 1)$ the Kähler-Einstein current γ_φ^α has edge singularities of cone angle $2\pi(1 - \alpha)$ along D .*

Let us conclude this section by giving an example of a pair (X, D) which admits both complete

and incomplete negatively curved Kähler-Einstein metrics on $X \setminus D$, but such that $K_X + \alpha D$ is never ample for any $\alpha \in \mathbb{R}$.

Remark 5.4.13. *Let $\mathcal{H} \times \mathcal{H}$ be the product of two copies of the upper half plane of $\mathbb{C}$. Let Γ be a discrete torsion free non-uniform subgroup of $SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$. Denote by X^* the Baily-Borel compactification [AMRT10] of $(\mathcal{H} \times \mathcal{H}) \setminus \Gamma$. Let X be the minimal resolution of X^* . By appropriately choosing Γ , see [Hir71] for beautiful explicit constructions, the resolution of the cusp points in X^* is given by a divisor D whose connected components are cycles of smooth rational curves with negative self-intersection. More precisely, for any of the irreducible components we have $D_i^2 \leq -2$. Furthermore, we have at least one (-2) -curve in D while there are no (-2) -curves in $X \setminus D$ since the curvature of $\mathcal{H} \times \mathcal{H}$ is nonpositive. Now, let D_i be a (-2) -curve in D . We then have that $(K_X + \alpha D) \cdot D_i = 0$ constantly in α . On the other hand $X \setminus D$ admits a complete nonpositively curved Kähler-Einstein metric coming from $\mathcal{H} \times \mathcal{H}$. Moreover, the pair (X, D) satisfies the hypothesis of Corollary 5.4.11, see Example 5.4.8. Thus, for $\alpha \in (\frac{3}{4}, 1)$, $X \setminus D$ also admits a 1-parameter family of incomplete negatively curved Kähler-Einstein metrics γ_α^α . Of course, this family has not edge singularities along D in the sense of Definition 5.2.*

5.5 Positively curved Kähler-Einstein metrics

In this section we study the existence of uniform bound for the nef threshold when X is a Fano variety or a log-Fano variety, i.e. $-K_X$ or $-(K_X + D)$ is ample.

Question 5.5.1. *Let X be a smooth variety and let D an effective divisor with $-(K_X + D)$ ample. Is there a fixed $\alpha_n < 1$, which depends only on n , such that $-(K_X + \alpha D)$ is ample for any $\alpha \in (\alpha_n, 1]$?*

It is quite tempting to do the same thing as we did in the previous section. For example if we apply Lemma 5.3.4 we get that $-(K_X + \frac{n+1}{n}D)$ is nef. Since we want to deal with $\alpha < 1$ we do not get any useful information. Probably the best reason that explains why we cannot do the same thing is that the above question has negative solution.

Example 5.5.2. *We follow the notation in [Har77]. Let $C = \mathbb{P}^1$ and $\mathcal{E} := \mathcal{O}_C \oplus \mathcal{O}_C(-e)$ where e is a positive integer. We define $X_e := \mathbb{P}(\mathcal{E})$, it is a rational ruled surface. By Corollary V.2.11 in [Har77] we know that $K_{X_e} \equiv -2C_0 - (e+2)f$ where C_0 is the zero section and f is a general fibre. Let $D := C_0$. Then $-K_{X_e} - D = C_0 + (e+2)f$ is an ample divisor by Corollary V.2.18 in [Har77]. Then by the same corollary $-K_{X_e} - \alpha D = (2-\alpha)C_0 + (e+2)f$ is ample if and only if $\alpha > \frac{e-2}{e}$. Then if e goes to infinity α must be arbitrarily close to one. Note that $-(K_{X_e} + D)^2 = e + 4$.*

We would like to understand how α depends on the pair (X, D) . We will prove that the situation in Example 5.5.2 is the most general and that α depends only on the dimension of n and the top self-intersection of $-(K_X + D)$. We need two preliminary results. The first one is an effective version of Kollár-Matsusaka theorem due to Demailly and Siu. We follow the presentation in [Laz04b].

Theorem 5.5.3 (Kollár-Matsusaka, Demailly, Siu). *Let X be a smooth projective variety. Let L be an ample divisor and B be a nef divisor. Then for any $m \geq M$ the divisor $mL - B$ is very ample, where*

$$M = (2n)^{(3^{n-1}-1)/2} \frac{(L^{n-1} \cdot (B + H))^{(3^{n-1}+1)/2} (L^{n-1} \cdot H)^{3^{n-2}(n/2-3/4)-1/4}}{(L^n)^{3^{n-2}(n/2-1/4)-1/4}},$$

and $H = (n^3 - n^2 - n - 1)(K_X + (n + 2)L)$.

The second result is a theorem of McKernan and it is an important step in the study of boundedness of singular Fano varieties, see [McK02] and Corollary 1.8 in [HMX12].

Theorem 5.5.4 (McKernan). *Fix r and n integers. Then there is a real number M such that if we have a pair (X, D) where*

1. X has dimension n ,
2. $-(K_X + D)$ is big and nef,
3. (X, D) is klt and
4. $r(K_X + D)$ is Cartier

then $-(K_X + D)^n \leq M$.

We can now prove Theorem 5.1.3 as stated in the Introduction 5.1.

Proof of Theorem 5.1.3. Suppose that there exists M_0 as in the statement. If we prove that there exists a positive integer M such that $-K_X + M(-K_X - D)$ is ample for any $(X, D) \in \mathfrak{B}$ then we have done, because it implies that we can take $\alpha_0 = \frac{M}{M+1}$. In order to prove the existence of such an integer we want to apply effective Kollár-Matsusaka. Write $L := -(K_X + D)$ which is ample by hypothesis. Then if we define $B = K_X + (n + 1)L$ by Lemma 5.3.4, we know that B is nef. For the sake of clarity we do not keep track of all the constants which appears in Theorem 5.5.3. In order to get an integer M which depends only on L^n we need to bound all the intersection numbers appearing in the formula with L^n . We can write $B = nL - D$ and $H = c(n)((n + 1)L - D)$ where $c(n) = n^3 - n^2 - n - 1$. Since L is ample and D is effective we get the following inequalities

$$L^{n-1} \cdot B < nL^n \quad \text{and} \quad L^{n-1} \cdot H < c(n)(n + 1)L^n.$$

Plugging them in the formula of Theorem 5.5.3 we get that there are positive constants $a(n)$ and $b(n)$ depending only on n such that for any $m \geq a(n)(L^n)^{b(n)}$ the divisor $mL - B$ is ample. This is exactly what we wanted at the beginning of the proof.

On the other hand suppose that we can find α_0 as in (1). Choose a rational number $\alpha \in (\alpha_0, 1)$ and let r be its denominator. For any pair $(X, D) \in \mathfrak{B}$ we have that $(X, \alpha D)$ is a klt pair and $r(K_X + \alpha D)$ is Cartier. Then Theorem 5.5.4 applies and we get that there exists M_0 such that $-(K_X + \alpha D)^n \leq M_0$ for any $(X, D) \in \mathfrak{B}$. Since $-K_X - D + (1 - \alpha)D = -K_X - \alpha D$ we get that also $-(K_X + D)^n$ is uniformly bounded. $\square$

The following is a corollary of the first part of the proof of the previous theorem.

Corollary 5.5.5. *Fix a positive integer n . Let X be a smooth projective variety of dimension n and let D be an effective divisor such that $-(K_X + D)$ is ample. Then there are positive integers $a(n)$ and $b(n)$, depending only on n , such that*

$$-\left(K_X + \frac{\alpha}{\alpha + 1}D\right)$$

is ample for any $\alpha \geq a(n)((-K_X - D)^n)^{b(n)}$. Furthermore $a(n)$ and $b(n)$ are explicitly computable functions of n .

The next corollary tells us that we can choose α_0 to be independent of D .

Corollary 5.5.6. *Let $\mathfrak{B}$ be a set of pairs (X, D) such that X is a smooth projective variety of dimension n and D is a reduced effective divisor with simple normal crossing support such that $-(K_X + D)$ is ample. Suppose there exists a positive integer M such that $\text{vol}(-K_X) \leq M$ for any $X \in \mathfrak{B}$. Then there exists a positive number $\alpha_0 < 1$ such that $-(K_X + \alpha D)$ is ample for any $\alpha \in (\alpha_0, 1]$ and any $(X, D) \in \mathfrak{B}$.*

Proof. It simply follows from Theorem 5.1.3 and the fact that $-(K_X + D)^n \leq \text{vol}(-K_X)$. $\square$

Let us compute the volume of the anti-canonical divisors in Example 5.5.2. We will use the following theorem of Zariski. See [Laz04a] for details.

Theorem 5.5.7 (Zariski decomposition). *Let X be a smooth projective surface and let L be a pseudo-effective divisor. Then L can be written uniquely as a sum $L = P + N$ of $\mathbb{Q}$ -divisors with the following properties:*

1. P is a nef divisor;

2. $N = \sum a_i E_i$ is effective, and if $N \neq 0$ then the intersection matrix $(E_i \cdot E_j)_{i,j}$ is negative definite.

3. $P \cdot E_i = 0$ for any component E_i of N .

Furthermore for every $m \geq 1$

$$H^0(X, \mathcal{O}_X(mL - \lceil mN \rceil)) = H^0(X, \mathcal{O}_X(mL)).$$

P and N are called respectively the positive and negative parts of L . Note that the last statement of Theorem 5.5.7 tells us that $\text{vol}(L) = P^2$.

Example 5.5.8. Let $X := X_e$ and D be as in the previous example. We can assume $e \geq 3$ otherwise $-K_X$ is big and nef. Since $-K_X$ is big we can consider the Zariski decomposition $-K_X = P + N$. In particular $\text{vol}(-K_X) = P^2$. It is easy to see that

$$-K_X = \left(1 + \frac{2}{e}\right)(C_0 + ef) + \left(1 - \frac{2}{e}\right)C_0$$

is the Zariski decomposition of $-K_X$. Then

$$\text{vol}(-K_X) = P^2 = \left(1 + \frac{2}{e}\right)^2 (C_0 + ef)^2 = (e + 4) + \frac{4}{e}.$$

We now deal with the case $-K_X$ ample. The main question is the following.

Question 5.5.9. Let X be a smooth variety with $-K_X$ ample and let D be an effective divisor. Is there a uniform α_n , which depends only on n , such that $-(K_X + \alpha D)$ is ample for any $\alpha \in [0, \alpha_n)$?

It is easy to see that in this case the answer is no even if we fix X . Recall that in the previous case if we fix X we can find a uniform bound that works for any divisor on X because it is enough to bound $\text{vol}(-K_X)$.

Example 5.5.10. Let $X = \mathbb{P}^n$ and $D = \mathcal{O}_X(d)$ for some $d > 0$. Then $-(K_X + \alpha D)$ is ample if and only if $\alpha < (n + 1)/d$. Then if d is arbitrarily large we need to take α arbitrarily small.

In order to settle Question 5.5.9 we need a well known result of Kollár, Miyaoka and Mori. See Theorem 4.4 in [KMM92].

Theorem 5.5.11 (Kollár-Miyaoka-Mori). *Fix an integer n . Let X be a smooth Fano variety of dimension n . Then*

$$(-K_X)^n \leq (n+1)^n n^{(2^n-1)n(n+1)} \left(1 + \frac{1}{n} + \frac{n+1}{n(n-1)}\right)^n.$$

In particular the set of smooth Fano varieties of dimension n forms a bounded family.

Finally, we give the proof of the last theorem stated in the Introduction 5.1.

Proof of Theorem 5.1.4. Suppose that there exists M_0 as in the statement. We want to apply Kollár-Matsusaka's theorem to find an integer M such that $-K_X - D + M(-K_X)$ is ample. Let $L := -K_X$ and let $B := K_X + D + 2nL$. We know by the cone theorem that B is nef. We need to control the following intersection numbers $B \cdot L^{n-1}$ and $H \cdot L^{n-1}$ where H is as in the statement of Theorem 5.5.3. We can write $B = (2n-1)L + D$ and $H = c(n)(n+1)L$ where $c(n) = n^3 - n^2 - n - 1$. Since L is ample we have that $L^n \geq 1$ and Theorem 5.5.11 tells us that there exists a function $f(n)$ of n such that $L^n \leq f(n)$. Then we can find the desired integer M applying Theorem 5.5.3.

Now assume that there exists α_0 as in (1). By assumption we know that $H := -(K_X + \alpha_0 D)$ is a nef $\mathbb{R}$ -divisor. Then

$$\alpha_0 D \cdot (-K_X)^{n-1} = (-K_X - H) \cdot (-K_X)^{n-1} \leq (-K_X)^n,$$

where the last inequality follows from the fact that H is nef and $-K_X$ is ample. Then it follows from Theorem 5.5.11 that we can uniformly bound $D \cdot (-K_X)^{n-1}$. $\square$

The first part of the proof gives the following.

Corollary 5.5.12. *Let (X, D) be a pair such that X is a smooth Fano variety of dimension n and D is a reduced effective divisor with simple normal crossing support. Then there are two positive integers $a(n)$ and $b(n)$ depending only on n such that $-(K_X + \frac{1}{\alpha+1}D)$ is ample for any $\alpha \leq a(n) \left(-D \cdot K_X^{n-1}\right)^{b(n)}$.*

Chapter 6

Effective results in hyperbolic geometry

6.1 Introduction

Let $\mathcal{H}^n$ be the complex n -dimensional hyperbolic space of dimension $n \geq 2$, that is, the Kähler space form with holomorphic sectional curvature -1 . Let X° be a complete non-compact complex hyperbolic manifold of finite volume. Such manifold is obtained as $X^\circ := \mathcal{H}^n \setminus \Gamma$ where Γ is torsion-free lattice of $\mathrm{PU}(1, n)$ with parabolic elements which is non-uniform. Then X° has finitely many disjoint unbounded ends of finite volume, the cusps of X° .

Baily-Borel [BB66] and Siu-Yau [SY82] proved that we can find a compactification X^* of X° such that the complement of X° in X^* consists of only finitely many (singular) points, called cusp points. Furthermore, under some mild assumptions on the lattice Γ , there exists a smooth variety X , which is a resolution of X^* , such that each exceptional divisor over a cusp point is a smooth abelian variety. For the precise requirements on Γ we refer the interested reader to Section 6.1.1. Let us denote by $D = \sum_i D_i$ the union of the exceptional divisors on X . We will refer to the pair (X, D) as a toroidal compactification of X° . Such toroidal compactifications were first constructed by Mumford et al. [AMRT10] and by Mok [Mok12], for more details see again Section 6.1.1.

The goal of this chapter is to understand how the positivity properties of the log-canonical divisor $K_X + D$ determine the geometry of X° and (X, D) . The first technical result of this chapter is that $K_X + D$ is not only big and nef but it is a limit of ample divisors of the form $K_X + \alpha D$ where α is a real number less than one. Moreover, there exists a uniform bound on the ampleness range for α

depending on the dimension only. Remarkably, the existence of such a uniform bound is crucial for all the geometric applications presented in this work.

Theorem 6.1.1. *Let (X, D) be a toroidal compactification. Then $K_X + \alpha D$ is ample for all $\alpha \in (\frac{n-1}{n}, 1)$.*

The proof of Theorem 6.1.1 is based on the techniques developed in the previous chapter. We list here some of the main applications. We refer to Sections 6.3 and 6.4 for details.

Theorem 6.1.2. *Let (X, D) be a toroidal compactification. Then for all $m \geq (n+1)^4$, the map associated to $|m(K_X + D)|$ is a morphism and it defines an embedding of $X \setminus D$ into some projective space. Furthermore, this morphism maps all the different components of D to distinct singular points.*

Theorem 6.1.2 is an effective version of a recent result of Mok, see Main Theorem in [Mok12]. Nevertheless, our proof relies on completely different techniques than Mok's. Note that when (X, D) is a toroidal compactification associated to a neat arithmetic lattice in $\mathrm{PU}(1, n)$, Theorem 6.1.2 gives a very concrete realization of the classical Baily-Borel compactification X^* as a projective algebraic variety.

Theorem 6.1.2 can also be applied to study the geometry of the smooth variety X in the pair (X, D) .

Theorem 6.1.3. *Let (X, D) be a toroidal compactification. Then X can be realized as a smooth subvariety of $\mathbb{P}^{2n+1}$ such that its degree d satisfies*

$$d \leq (n+1)^{4n} (K_X + D)^n.$$

Note that Theorem 6.1.3 defines a different map from the one in Theorem 6.1.2 since it gives an embedding of X and not of X^* .

A similar result is obtained by Hwang in [Hwa05]. The main advantage of our approach is that it provides bounds which are linear in $(K_X + D)^n$. The bounds obtained by Hwang are polynomial of degree $n+1$ in $(K_X + D)^n$.

Theorem 6.1.3 has several applications. For example, combining this result with classical Chow variety techniques, we can derive explicit bounds for the number of complex hyperbolic manifolds with given upper bounds on the volume. These results are effective versions of the classical Wang's finiteness theorem, see [Wan72]. For the numerical values of these bounds see Corollaries 6.3.13 and 6.3.16.

As a final application of Theorem 6.1.1, we show how to bound the number of cusps of a finite volume complex hyperbolic manifold in terms of its volume.

Theorem 6.1.4. *Let (X, D) be a toroidal compactification. Let q be the number of cusps of X° . Then*

$$q \leq n^n (K_X + D)^n.$$

The problem of bounding the number of ends in pinched negatively curved finite volume manifolds is a long standing problem in differential geometry. For complex hyperbolic manifolds, the classical approach is via techniques coming from geometric topology, see [Par98] and the bibliography therein. More recently, Hwang in [Hwa04] was able to improve Parker's result by using the Hirzebruch-Mumford proportionality principle [Mum72]. The bound we present here in Theorem 6.1.4 is better than the one obtained by Parker but worse than Hwang's for $n \geq 3$. We decided to include this result here mainly for two reasons. First, the proof of Theorem 6.1.4 is completely elementary and it follows easily from Theorem 6.1.1. Second, the reasoning given in the proof of Theorem 6.1.4 is in principle applicable in a much general setting while both the approaches of Parker and Hwang use in an essential way the special properties of complex hyperbolic manifolds.

In Section 6.3.2, we describe an alternative approach to the problem of bounding the cusps, see Proposition 6.3.11. This bound is asymptotically better than the one given in Theorem 6.1.4 but still weaker than Hwang's for $n \geq 4$. Nevertheless, it appears to be the best currently known bound for threefolds.

Finally, in Section 6.4 we derive much sharper versions of Theorems 6.1.2, 6.1.3 and 6.1.4 in the case of complex surfaces. These improvements rely on the fact that we are able to derive an essentially sharp version of Theorem 6.1.1 for toroidal compactification of dimension two, see Theorem 6.4.2 in Section 6.4. Moreover, we are able to further improve the bound on the number of cusps given in Theorem 6.1.4. This result seems to be the best bound for surfaces currently available in the literature.

6.1.1 Preliminaries

The theory of compactifications of locally symmetric varieties has been extensively studied, see for example [BJ06]. For technical reasons this theory is mainly developed for quotients of symmetric spaces by arithmetic subgroups. In most cases this is not a serious issue since the work of Margulis [Mar84] implies that lattices in any semi-simple Lie group of real rank bigger or equal than two are

arithmetic subgroups. Nevertheless, this theorem does not cover the case of lattices in the complex hyperbolic space $\mathcal{H}^n$ which are the main object of study in this paper. Note that non-arithmetic lattices in $\mathrm{PU}(1, n)$ were constructed by Mostow and Deligne-Mostow; see [DM93] and the extensive bibliography there.

It would then be desirable to develop a theory of compactifications of finite volume complex hyperbolic manifolds which does not rely on the arithmeticity of the defining torsion free lattices. Fortunately, this problem was addressed by many mathematicians from several different point of views. A compactification of finite volume complex hyperbolic manifolds as a complex spaces with isolated normal singularities was obtained by Siu and Yau in [SY82]. This compactification may be regarded as a generalization of the Baily-Borel compactification defined for arithmetic lattices in $\mathcal{H}^n$. A toroidal compactification for finite volume complex hyperbolic manifolds was described by Hummel and Schroeder in [HS96]. In many cases these compactifications provide explicit resolution of singularities of the Siu-Yau compactifications. More recently Mok [Mok12] gave a lucid and detailed description of these compactifications and described many of their remarkable features.

The compactifications studied here are the ones described by Hummel-Schroeder and Mok. Thus, we do not require that lattice Γ to be arithmetic. On the other hand, we require that all the parabolic isometries of Γ are unipotent, in other words we require that they act by translations on their invariant horospheres. We impose this condition in order to obtain smooth toroidal compactifications. We would like to point out that this technicality is hidden in the construction described by Mok while it is explicitly discussed in the work of Hummel-Schroeder. Let us describe in more details this technical point. Recall that given a non-uniform torsion-free lattice $\Gamma \leq \mathrm{PU}(1, n)$, the finite volume complex hyperbolic manifold $\mathcal{H}^n/\Gamma$ has finitely many cusps $C_1, \dots, C_m$ which are in one to one correspondence with the maximal parabolic subgroups of Γ , see [Ebe96] for more details. Given a cusp C_i , denote by $\Gamma_i \leq \Gamma$ the associated maximal parabolic subgroup and by HB_i the horoball stabilized by Γ_i . After choosing an Iwasawa decomposition [Ebe96] for $\mathrm{PU}(1, n)$, we can identify $\partial\mathrm{HB}$ with a Heisenberg type Lie group N_i diffeomorphic to $\mathbb{C}^{n-1} \times \mathbb{R}$. Thus, the center Z_i of N_i is $Z_i = [N_i, N_i]$ and it is isomorphic to $\mathbb{R}$. Furthermore, the simply connected Lie group N_i comes equipped with a natural left invariant metric and then we can consider Γ_i as a lattice in $\mathrm{Iso}(N_i)$. The isometry group of N is isomorphic to the semi-direct product $\mathrm{Iso}(N_i) = N_i \rtimes U(n-1)$. We then say that $\phi \in \Gamma_i$ is unipotent if it is a translation in $\mathrm{Iso}(N_i)$. Now, the construction described by Hummel-Schroeder and Mok produces a smooth compactification if the quotients $\Gamma_i \backslash \Gamma_i \cap Z_i$ are torsion free for all i . This is the case if all the parabolic isometries of Γ are unipotent. In the classical arithmetic case this requirement is usually satisfied by requiring the lattice Γ to be neat, see [BB66]

and [BJ06] for more details. Let us note that given any non-uniform lattice $\Gamma \in \mathrm{PU}(1, n)$, there always exists a finite index subgroup whose all parabolic isometries are unipotent, in the arithmetic case see [AMRT10] while for the general non-arithmetic case we refer to [Hum98].

6.2 A gap theorem

In this section we prove Theorem 6.1.1 which is the main technical result of the chapter. The proof of Theorem 6.1.1 is based on Theorem 5.3.12 proved in Chapter 5. In particular the fact that $\alpha \in (\frac{n+1}{n+2}, 1)$ is obtained using the bound on the length of extremal rays given in the Cone Theorem. Let us recall the definition of an extremal ray. We refer to [Kol96] and [KM98] for more details.

Definition 6.2.1. *Let $N \subset \mathbb{R}^m$ be a cone. A subcone $M \subset N$ is called extremal if $u, v \in N$, $u + v \in M$ imply that $u, v \in M$. A 1-dimensional extremal subcone is called an extremal ray.*

We will consider only extremal rays R of the cone of effective 1-cycles $\overline{NE}(X)$ such that $K_X \cdot Z < 0$ for any effective 1-cycle Z in R . For such extremal rays we define the length of R as

$$l(R) := \min \{ -K_X \cdot C \mid C \text{ is a rational curve with numerical class in } R \}.$$

The length of extremal rays in $\overline{NE}(X)$ has been extensively studied. The Cone Theorem gives that $l(R) \leq n + 1$. On the other hand, there are many classification results regarding varieties with extremal rays of maximal length. The result we need here is due to Wiśniewski [Wis89]. Even though it is not the most general result currently known, it is enough for our purposes.

Theorem 6.2.2 (Wiśniewski). *Let X be smooth projective variety of dimension n .*

1. *If there exists an extremal ray R of length $n + 1$, then $\mathrm{Pic}(X) = \mathbb{Z}$ and $-K_X$ is an ample line bundle on X .*
2. *If there exists an extremal ray R of length n then, either $\mathrm{Pic}(X) = \mathbb{Z}$ and $-K_X$ is ample or $\rho(X) = 2$ and there exists a morphism $\mathrm{contr}_R : X \rightarrow B$ onto a smooth curve B whose general fiber is a smooth $(n - 1)$ -manifold that satisfies the conditions in (1).*

Our goal is to prove that toroidal compactifications do not fit in the above list of varieties. We start with a lemma which rules out most of the cases.

Lemma 6.2.3. *Let (X, D) be a toroidal compactification. Let q be the number of components of D . Then $\rho(X) > q$, where $\rho(X)$ is the Picard number of X .*

Proof. Suppose by contradiction that $\rho(X) \leq q$. Let H be an ample divisor on X . Then there exists a non trivial relation

$$a_0 H + \sum_{i=1}^q a_i D_i \equiv 0,$$

where $a_i \in \mathbb{R}$. Since $D_i \cdot D_j^{n-1} = 0$ for all $j \neq i$, the compactifying divisors are not numerically equivalent and we can assume $a_0 = 1$. Intersecting the above relation with a curve C contained in D_i , we get that $a_i > 0$ for all i . On the other hand

$$0 = H^{n-1} \cdot \left(H + \sum_{i=1}^q a_i D_i \right) = H^n + \sum_{i=1}^q a_i D_i \cdot H^{n-1} > 0,$$

gives a contradiction. □

In particular, if a toroidal compactification has a least two cusps, then $l(R) \leq n - 1$ for all extremal rays. It is left to prove that a toroidal compactification with only one cusp does not fit in the cases described by Theorem 6.2.2. This is the content of the next lemma.

Lemma 6.2.4. *Let (X, D) be a toroidal compactification with $\dim(X) = n$. Then any extremal ray R of X has length at most $n - 1$.*

Proof. We want to rule out the cases in Theorem 6.2.2. By Lemma 6.2.3 we can assume that D has only one component and $\rho(X) = 2$. In particular X does not have any extremal ray of length $n + 1$. Suppose by contradiction that we have an extremal ray of length n . Thus, Theorem 6.2.2 implies that there exists a fibration $\pi : X \rightarrow B$ over a smooth curve B .

We first deal with the case $\dim(X) = 2$. Since D is an elliptic curve, we have that π restricted to D is a surjective morphism and in particular $g(B) \leq 1$. Since $\rho(X) = 2$, the ruled surface X must be minimal. If B is a rational curve then X is a Hirzebruch surface. On a Hirzebruch surface the only curve with negative self intersection is the zero section, which is a rational curve, so it cannot be D .

We can assume that B is an elliptic curve. We follow here the notation in [Har77]. Write $X = \mathbb{P}(\mathcal{E})$ where $\mathcal{E}$ is a normalized rank 2 vector bundle over B with invariant e . By a theorem of Atiyah we know that $e \geq -1$, see Theorem V.2.15 in [Har77]. If $e \leq 0$ then $-K_X$ is nef and in particular by adjunction $D^2 \geq 0$. We then have $e \geq 1$. A simple computation using Proposition V.2.20 in [Har77] shows that the only curve with negative self-intersection is the zero section. This implies that D is the zero section. We can then find a $\mathbb{P}^1$ with just one puncture in $X \setminus D$. Since curvature can only decrease along complex submanifolds of a negatively curved Kähler manifold, we

obtain a contradiction.

Now suppose that $\dim(X) \geq 3$. By a dimension argument, there exists a curve C in $D \cap F$, where F is the general fiber of π . We compute $(K_X + D) \cdot C$ in two different ways to get a contradiction. First, since $C \subset D$, we know that $(K_X + D) \cdot C = 0$. On the other hand, since the normal bundle of D in X is anti-ample, we have $D \cdot C < 0$. Moreover, since F is a Fano variety $K_X \cdot C = K_F \cdot C < 0$. We then conclude $(K_X + D) \cdot C < 0$. This is a contradiction. $\square$

We can now prove Theorem 6.1.1.

Proof of Theorem 6.1.1. First, we would like to show that $K_X + D$ is big and nef. Recall that $\mathcal{H}^n$ is the negatively curved complex space form. Thus, any manifold $X^\circ = \mathcal{H}^n \setminus \Gamma$, with Γ non-uniform, is equipped with a standard negatively curved Kähler-Einstein metric of finite volume which we denote by $\hat{\omega}$. By Théorème 1.1 in [Sib85], such a metric $\hat{\omega}$ on X° can be regarded as a closed positive current on X . Moreover, it is not difficult to see that $\hat{\omega}$ is in the cohomology class of $K_X + D$. More precisely, let D_i be the irreducible components of D and let $s_i \in \mathcal{O}_X(D_i)$ be the defining sections. By appropriately choosing Hermitian metrics $\|\cdot\|_i$ on $\mathcal{O}_X(D_i)$, the volume form associated to $\hat{\omega}$ is then of the form

$$\Psi = \frac{\Omega}{\prod_i \|s_i\|^2 (-\log \|s_i\|^2)^{n+1}}$$

for some globally defined volume form Ω on X . Thus, the Poincaré-Lelong formula combined with the fact that $\hat{\omega}$ is Kähler-Einstein gives that $\hat{\omega} \in [K_X + D]$. Next, let us observe that $\omega_P \geq \hat{\omega}$ for some Kähler current ω_P on X with Poincaré type singularities along D . We then have that $\hat{\omega}$ is a closed positive current with zero Lelong numbers on X . Moreover, since $\hat{\omega}$ is a regular Kähler metric on X° , it is strictly positive at any point $p \in X \setminus D$. By using a regularization argument based on results of Demailly [Dem92], we conclude that $K_X + D$ is big and nef, see also Theorem 1.3 in [JS93]. We can actually conclude more. In fact, we must have $(K_X + D)^{\dim(Z)} \cdot Z > 0$ for any subvariety Z not contained in D .

Finally, we want to show that if C is a curve such that $(K_X + D) \cdot C = 0$ then $K_X \cdot C > 0$. By the discussion above we know that C must be contained in D . Then

$$K_X \cdot C = -D \cdot C > 0,$$

since the normal bundle of D in X is negative, see Theorem 1 in [Mok12]. In particular Theorem

5.3.12 implies that $K_X + \alpha D$ is ample for all values of α close enough to one. Following the proof of Theorem 5.3.12 and using the bound given by Lemma 6.2.4, it follows that the $K_X + \alpha D$ is ample for all $\alpha \in (\frac{n-1}{n}, 1)$. $\square$

It seems a difficult problem to understand which varieties arise as toroidal compactifications of hyperbolic manifolds. The above theorem gives a first step toward a possible solution of the problem. If $n \geq 3$, it follows also from Lefschetz hyperplane theorem.

Corollary 6.2.5. *There are no toroidal compactification (X, D) with X a smooth Fano variety.*

Proof. Suppose (X, D) is a toroidal compactification with $-K_X$ ample. Because of Theorem 6.1.1, for all α close to one, we know $K_X + \alpha D$ is ample. By Corollary 5.3.15, it follows that $K_X + D$ is strictly nef. On the other hand, we must have $(K_X + D)|_D = \mathcal{O}_D$. This is a contradiction. $\square$

Question 6.2.6. *It is interesting to ask whether there exists a smooth toroidal compactification of a ball quotient with negative Kodaira dimension.*

6.3 Applications

In this section, we give the proofs of Theorems 6.1.2, 6.1.3 and 6.1.4.

6.3.1 Effective birationality

Let us start by studying the birational properties of the divisor $K_X + D$. First, we prove that the map associated to $|m(K_X + D)|$ maps the components of D to distinct points for any $m \geq 2$.

Proposition 6.3.1. *Let (X, D) be a toroidal compactification. Then for any i there exists a section σ_i of $H^0(X, \mathcal{O}_X(2(K_X + D)))$ such that $\sigma_i|_{D_i} \neq 0$ and $\sigma_i|_{D_j} = 0$ for all $j \neq i$.*

Proof. Write $D = \sum_{i=1}^q D_i$ and recall that each component is an abelian variety and they are all disjoint. Consider the following exact sequence

$$0 \rightarrow \mathcal{O}_X(2K_X + D) \xrightarrow{\cdot D} \mathcal{O}_X(2(K_X + D)) \rightarrow \mathcal{O}_D \rightarrow 0.$$

By Kawamata-Viehweg's vanishing we have that $H^1(X, \mathcal{O}_X(2K_X + D)) = 0$. Thus, taking the long exact sequence in cohomology, we get the following surjective map

$$H^0(X, \mathcal{O}_X(2(K_X + D))) \rightarrow \bigoplus_{i=1}^q H^0(D_i, \mathcal{O}_{D_i}).$$

□

The next step is to understand what happens outside the boundary divisor. Let us start by deriving a lower bound on the top self-intersection of $K_X + D$.

Lemma 6.3.2. *Let (X, D) be a toroidal compactification. Then*

$$(K_X + D)^n \geq (n + 1)^{n-1}.$$

Proof. Since the Kähler-Einstein current $\hat{\omega}$ can be multiplied by itself, the top self-intersection of $L := K_X + D$ can be expressed in terms of the Riemannian volume of X° . More precisely, by normalizing the holomorphic sectional curvature to be -1 we have

$$\text{Vol}(X) = \frac{(4\pi)^n}{n!(n+1)^n} L^n.$$

On the other hand, Gromov-Harder's generalization of Gauss-Bonnet [Gro82] implies that

$$\text{Vol}(X) \geq \frac{(4\pi)^n}{(n+1)!}.$$

Combining the two formulas above we get the result.

□

These considerations and a theorem of Kollár in [Kol97] imply the following.

Corollary 6.3.3. *Let (X, D) be a toroidal compactification. Then $m(K_X + D)$ is base point free for any $m \geq \binom{n}{2} + 1$.*

Proof. By Theorem 6.1.1, for any subvariety Z not contained in D we have $(K_X + D)^{\dim(Z)} \cdot Z > 0$. Because of Theorem 5.8 in [Kol97], we know that $m(K_X + D)$ is free at all points outside D for any $m \geq \binom{n}{2} + 1$. Moreover, by Lemma 6.3.1 we know already that $2(K_X + D)$ is free on the divisor D .

□

Similarly, we can study separation of points.

Corollary 6.3.4. *Let (X, D) be a toroidal compactification. Then $m(K_X + D)$ separates any two points in $X \setminus D$ for any $m \geq \binom{n}{2} + 2$.*

Proof. Combine Theorem 3.2.3 and Lemma 6.3.2.

□

Since $X \setminus D$ is open in X , Corollary 6.3.4 implies that $|m(K_X + D)|$ defines a birational map for any $m \geq \binom{n}{2} + 2$. These simple results can be already used to slightly improve all the bounds in [Hwa05]. Nevertheless, the approach we follow here is quite different. Note that Corollary 6.3.4 deals only with separation of points. Thus, if we want to prove that $m(K_X + D)$ defines an embedding outside D we need something more. Again the key for us is Theorem 6.1.1, while the approach described in [Hwa05] relies on Seshadri constants type arguments.

We can now derive an effective result on very ampleness modulo D of $K_X + D$. In order to keep the final statement simple, some of the constants used in the proof are not optimal.

Theorem 6.3.5. *Let (X, D) be a toroidal compactification of dimension n . Then $m(K_X + D)$ is very ample modulo D for any $m \geq (n + 1)^4$.*

Proof. Theorem 6.1.1 implies that $K_X + \frac{n}{n+1}D$ is ample. Then $H := (n + 1)K_X + nD$ is an ample integral divisor. By a theorem of Angehrn-Siu we have that $K_X + mH$ is ample and base point free for any $m > \binom{n+1}{2}$, see [AS95] or [Kol97] and [Laz04b] for an algebraic proof. In particular $B := K_X + (n^2 + 1)H$ is ample and base point free. A corollary of Castelnuovo-Mumford regularity, see Example 1.8.23 in [Laz04a], gives that $K_X + (n + 2)B$ is very ample. All together we have that the divisor

$$((n^2 + 1)(n + 1)(n + 2) + n + 3) K_X + (n^2 + 1)n(n + 2)D$$

is very ample. Let $M := ((n^2 + 1)(n + 1)(n + 2) + n + 3)$. Adding the right positive multiple of D we get the following injective map

$$H^0(X, \mathcal{O}_X(K_X + (n + 2)B)) \hookrightarrow H^0(X, \mathcal{O}_X(M(K_X + D))).$$

This implies that $m(K_X + D)$ is very ample modulo D for any $m \geq M$. Since $(n + 1)^4 \geq M$ we get the statement of the theorem. $\square$

Theorem 6.1.2 stated in the introduction is now just a combination of Theorem 6.3.5 and Proposition 6.3.1.

Let us conclude this section by discussing the connections between Fujita's conjecture and Theorem 6.3.5. First, recall that if $\dim(X) \leq 4$ then the base point free part of Fujita's conjecture is known to be true thanks to the work of Kawamata [Kaw97]. We can then improve the bound in Theorem 6.3.5 for low dimensional varieties.

Corollary 6.3.6. *Let (X, D) be a toroidal compactification of dimension $n \leq 4$. Then $m(K_X + D)$ is very ample modulo D for any $m \geq (n + 2)^3$.*

Proof. It follows along the same lines of Theorem 6.3.5. Instead of using Angehrn-Siu's theorem, we use Kawamata's result on Fujita's conjecture. The rest remains unchanged. $\square$

It is interesting to note that using a theorem of G. Heier in [Hei02], we can get a bound of order $n^{10/3}$ in Theorem 6.3.5. Of course an even better bound would be obtained using the very ampleness part of Fujita's conjecture. More precisely, assuming the conjecture of Fujita to be true, the proof of Theorem 6.3.5 easily gives that $m(K_X + D)$ is very ample modulo D for any $m \geq (n + 2)^2$.

6.3.2 Bounds on the number of cusps

In this section, we show how to explicitly estimate the number of cusps of X° in terms of the intersection number $(K_X + D)^n$. We present two different approaches. The first one gives asymptotically a worse bound. Nevertheless, it can be in principle applied to a much more general setting and, with suitable improvements, it gives the best known bound in the case of surfaces, see Section 6.4 for details. The second method gives a better bound for $n \geq 3$ and it follows easily from a result of Matsusaka. Moreover, it gives the best currently known bound for threefolds.

Recall that X° has finitely many ends which correspond to the cusp points of X^* . Each cusp point gives rise to one component of the boundary divisor D in X . So bounding the number of cusps is equivalent to bounding the number of components of D .

We can now prove Theorem 6.1.4 stated in the Introduction.

Theorem 6.3.7 (Theorem 6.1.4). *Let (X, D) be a toroidal compactification. Let q be the number of components of D . Then*

$$q \leq n^n (K_X + D)^n.$$

Proof. Once we have Theorem 6.1.1, the above result follows from a quite straightforward computation. Let $L := K_X + D$. Because of Theorem 6.1.1, we know that $nL - D$ is nef and $(n + 1)L - D$ is ample. Then

$$q \leq D \cdot ((n + 1)L - D)^{n-1} = (-1)^{n-1} D^n.$$

Since $nL - D$ is nef we get that

$$(nL - D)^n = n^n L^n + (-1)^n D^n \geq 0.$$

Combining the two inequalities we get the result. $\square$

Note that we can also a bound on the top self-intersection of D in terms of $(K_X + D)^n$ only. Let us summarize this fact in the form of a corollary as it will be used in Section 6.3.3.

Corollary 6.3.8. *Let (X, D) be a toroidal compactification. Then*

$$(-1)^{n-1} D^n \leq n^n (K_X + D)^n.$$

We now present an estimate on the number of cusps which does not rely on Theorem 6.1.1. The key point is the following corollary of Proposition 6.3.1.

Corollary 6.3.9. *Let (X, D) be a toroidal compactification. Let q be the number of components of D . Then*

$$h^0(X, \mathcal{O}_X(2(K_X + D))) \geq q.$$

In particular, in order to bound the number of cusps, it is enough to bound $h^0(X, \mathcal{O}_X(2(K_X + D)))$. This can be done using a result of Matsusaka. For an outline of the proof we refer to [Kol96].

Theorem 6.3.10 (Matsusaka). *Let H be a big and nef divisor on a smooth variety X of dimension n . Then for any $m \geq 1$ we have*

$$h^0(X, \mathcal{O}_X(mH)) \leq m^n H^n + n.$$

Combining these two results we get the following.

Proposition 6.3.11. *Let (X, D) be a toroidal compactification. Let q be the number of components of D . Then*

$$q \leq (2^n + 1)(K_X + D)^n.$$

Proof. By Lemma 6.3.2, we have that $(K_X + D)^n \geq n$. $\square$

It is interesting to note how the bounds given in Theorem 6.1.4 and Proposition 6.3.11 rely on few basic geometric properties of the pair (X, D) . Roughly speaking, they depend on the fact that $L = K_X + D$ is big, nef and such that $L|_D = \mathcal{O}_D$. None of the other special features of toroidal compactifications of ball quotients is used. Therefore, these techniques can be successfully applied to study complete finite volume Kähler manifolds with pinched negative sectional curvature which are not locally symmetric. These results will appear elsewhere.

Finally, despite their simple nature, the bounds given in Theorem 6.1.4 and Proposition 6.3.11 are respectively the best currently known bounds for surfaces and threefolds, compare with [Par98] and [Hwa04]. In the case of surfaces, see also the improvement given in Proposition 6.4.3 below.

6.3.3 Bounds on the number of varieties

In this section we study the problem of finding effective bounds on the number of complex hyperbolic manifolds with bounded volume. The main issue is to find effective embedding results for toroidal compactifications. This was successfully done in Theorem 6.3.5 and we investigate its consequences here. The key result of this section is Theorem 6.1.3 stated in the introduction of this chapter, which we restate for the convenience of the reader.

Theorem 6.3.12 (Theorem 6.1.3). *Let (X, D) be a toroidal compactification. Then X can be realized as a smooth subvariety of $\mathbb{P}^{2n+1}$ such that its degree d satisfies*

$$d \leq (n+1)^{4n} (K_X + D)^n.$$

Proof. Let $L := K_X + D$. By Theorem 6.3.5, we know that $(n+1)^4 L - kD$ is very ample, for some explicit positive integer k . Then $(n+1)^4 L - kD$ defines an embedding of X into some projective space with degree $d \leq (n+1)^{4n} L^n$. By general projection, we obtain that X sits inside $\mathbb{P}^{2n+1}$. $\square$

As pointed out before, Hwang proved a similar result in [Hwa05]. While our bound is linear in $(K_X + D)^n$, he derived a polynomial bound in $(K_X + D)^n$ of degree $n+1$. Once we have Theorem 6.3.12, counting the number of complex hyperbolic manifolds is reduced to standard techniques on the complexity of Chow varieties. Let us briefly explain this point. Associated to a hyperbolic manifold, we have a toroidal compactification (X, D) . Therefore, it is enough to count such pairs. Fujiki in [Fuj92] proved that a toroidal compactification (X, D) is infinitesimally rigid under deformations of the pair. This implies that the number of toroidal compactifications is bounded by the number of components of a suitable Chow variety. See Corollary 6.3.16. Moreover, Hwang pointed out that Fujiki's theorem implies something more. If $n \geq 3$, X itself is rigid under deformations and the same method applies, see Proposition 4.2 in [Hwa05].

We start by counting the varieties which arise as toroidal compactifications of complex hyperbolic manifolds with bounded volume. In particular, in the next statement we forget about the extra structure coming from the boundary divisor.

Fix two positive integers d and $m > n$. We denote by $\text{Chow}_m(n, d)$ the Chow variety of n -dimensional irreducible smooth subvarieties of $\mathbb{P}^m$ of degree d .

Corollary 6.3.13. *Fix two positive integers $n \geq 3$ and V . Then the number of varieties X arising as toroidal compactifications (X, D) with $\dim(X) = n$ and $(K_X + D)^n \leq V$ is bounded by*

$$\sum_{d=1}^{d_0} \binom{(2n+2)d}{2n+1}^{(2n+2)d \binom{d+n-1}{n} + (2n+2) \binom{d+n-1}{n-1}},$$

where $d_0 = (n+1)^{4n} V$.

Proof. Let (X, D) be a toroidal compactification as in the statement. By Theorem 6.3.12, X can be embedded in $\mathbb{P}^{2n+1}$ as a smooth subvarieties of degree $d \leq d_0$. Proposition 4.2 in [Hwa05] implies that X is rigid under deformation. In particular, the number of such toroidal compactifications is bounded by the number of components of the Chow variety $\text{Chow}_{2n+1}(n, d_0)$. A straightforward application of I.3.28.9 in [Kol96] gives the result. $\square$

Remark 6.3.14. *Independently of the value of d_0 , the bound on the number of components of the Chow variety is of the form*

$$(a_n d_0)^{(b_n d_0)^{n+1}},$$

for some function a_n and b_n which depend only on n . Therefore, the bound in Corollary 6.3.13, expressed just in terms of the volume, is of the form

$$V^{V^{n+1}}.$$

We can now give effective estimates on the number of toroidal compactifications (X, D) with bounded $(K_X + D)^n$, counted as pairs. This corresponds exactly to bounding the number of complex hyperbolic manifolds with bounded volume. Unfortunately, the bounds are slightly worse than the ones in Corollary 6.3.13.

We need to study a different Chow variety which takes into account the boundary divisor. Given positive integers $q, m, n, n_1, \dots, n_q, d, d_1, \dots, d_q$ we define $\text{Chow}_m(n, d; n_1, \dots, n_q; d_1, \dots, d_q)$ to be the closed subvariety of

$$\text{Chow}_m(n, d) \times \text{Chow}_m(n_1, d_1) \times \dots \times \text{Chow}_m(n_q, d_q),$$

parameterizing $(q+1)$ -tuples $(X, D_1, \dots, D_q)$ where X is a smooth subvariety of dimension n and

degree d in $\mathbb{P}^m$ and each D_i is a smooth subvariety of dimension n_i and degree d_i contained in X .

In [Hwa05], Hwang generalizes the effective bounds on the number of components in [Kol96] to the above defined Chow varieties. We will use his bound to derive an effective estimate on the number of toroidal compactifications.

The first thing to do is to bound the degree of each component of D when embedded into some fixed projective space.

Lemma 6.3.15. *Let (X, D) be a toroidal compactification. Let d_i be the degree of D_i under the embedding given by Theorem 6.3.12. Then for any i*

$$d_i \leq (n+1)^{4n-3}(K_X + D)^n.$$

Proof. Let A be the very ample divisor obtained in the proof of Theorem 6.3.5. Note that A gives the embedding of Theorem 6.3.12. An easy computation shows that

$$d_i = D_i \cdot A^{n-1} \leq (-1)^{n-1}(n+1)^{3n-3}D^n \leq (n+1)^{4n-3}(K_X + D)^n,$$

where the last inequality is obtained thanks to Lemma 6.3.8. □

Finally, we can prove an effective version of Wang's finiteness theorem.

Corollary 6.3.16. *Fix two positive integers n and V . Then the number of toroidal compactifications with $\dim(X) = n$ and $(K_X + D)^n \leq V$ is less than*

$$\sum_{q=1}^{q_0} d_0^{q+1} \binom{(2n+2)d_0}{2n+1}^{(2n+2)(q+1) \binom{2n+1+d_0}{2n+1}},$$

where $d_0 = (n+1)^{4n}V$ and $q_0 = (2^n + 1)V$.

Proof. Let d be the degree of X in $\mathbb{P}^{2n+1}$ and let d_i be the degree of D_i for any $1 \leq i \leq q$. By definition, there exists a point in $\text{Chow}_{2n+1}(n, d; n-1, \dots, n-1; d_1, \dots, d_q)$ parameterizing (X, D) . Theorem 4.1 in [Fuj92] implies that (X, D) is infinitesimally rigid under deformations of the pair. Therefore, the number of toroidal compactifications with fixed $(d, d_1, \dots, d_q)$ is bounded by the number of components of the corresponding Chow variety. By Theorem 6.3.12, we can embed X as a smooth subvariety of $\mathbb{P}^{2n+1}$ with degree d bounded by $(n+1)^{4n}V$. Furthermore, each D_i has degree d_i bounded by $(n+1)^{4n-3}V$, by Lemma 6.3.15. Finally, for each $q > 0$ there are at most

d_0^{n+1} choices of $(d, d_1, \dots, d_q)$, which combined with Proposition 3.2 in [Hwa05] and Theorem 6.3.10, give the result. $\square$

Up to constants depending only on n , the above bound is of the form $V^{V^{2n+2}}$.

6.4 The two dimensional case

In this section, we substantially improve the theory developed in Section 6.3 in the special case of surfaces. These improvements rely on two different facts. First, we can derive an optimal version of Theorem 6.1.1 for $n = 2$. Second, the birational geometry of adjoint linear systems on surfaces is completely understood thanks to the theorem of Reid. Let us recall the statement of this theorem, for more details see [Rei88].

Theorem 6.4.1 (Reider). *Let X be a smooth surface and let L be a nef line bundle on X . Then the following hold:*

1. *If $L^2 \geq 5$ and $x \in X$ is a base point of $|K_X + L|$ then there exists a curve C through x such that either*

$$C \cdot L = 0 \text{ and } C^2 = -1; \text{ or}$$

$$C \cdot L = 1 \text{ and } C^2 = 0.$$

2. *If $L^2 \geq 10$ and $x, y \in X$ are two (possibly infinitely near) points which $|K_X + L|$ does not separate then there exists a curve C through x and y such that either*

$$C \cdot L = 0 \text{ and } C^2 \in \{-1, -2\}; \text{ or}$$

$$C \cdot L = 1 \text{ and } C^2 \in \{0, -1\}; \text{ or}$$

$$C \cdot L = 2 \text{ and } C^2 = 0.$$

Let us proceed by deriving an optimal version of Theorem 6.1.1 when $n = 2$.

Theorem 6.4.2. *Let (X, D) be a toroidal compactification with $\dim(X) = 2$. Then $K_X + \alpha D$ is ample for any $\alpha \in (\frac{1}{3}, 1)$.*

Proof. By Lemma 6.2.4, if (X, D) is a toroidal compactification with $\dim(X) = 2$ then each extremal ray must have length one. By Theorem 2.1 in [Mor82], it follows that all the extremal rays are exceptional curves of the first kind, in other words they are smooth rational curves with self-intersection

–1. Let us denote by $\{C_i\}$ the generators of the extremal rays in X . We then have $D \cdot C_i \geq 3$ for all i . If otherwise there would exist a $\mathbb{P}^1$ with less than three punctures in $X \setminus D$ which is clearly impossible. We then conclude that $(K_X + D) \cdot C_i \geq 2$ for all i . Applying the argument of Proposition 5.3.9, we can conclude. $\square$

As in Section 6.3, we can now use Theorem 6.4.2 to give effective very ampleness results and bounds on the number of cusps. We state here the main results. The proofs are similar to the ones obtained in higher dimensions. Thus, we will not give many details. We start by bounding the number of cusps of a two dimensional toroidal compactification.

Proposition 6.4.3. *Let (X^2, D) be a toroidal compactification. Let q be the number of components of D . Then*

$$q \leq \frac{9}{4}(K_X + D)^2.$$

Proof. Because of Theorem 6.4.2, we know that $3K_X + D = 3L - 2D$ is nef. Since $q \leq -D^2$, by taking the top self-intersection of $3L - 2D$ we get the result. $\square$

Remark 6.4.4. *The bound obtained by Hwang in [Hwa04], which seems to be the best published result on this problem, is of the form $q \leq 7.38(K_X + D)^2$. Note that using Proposition 6.3.11 with $n = 2$, we can also obtain the bound $q \leq 5(K_X + D)^2$.*

Proposition 6.4.3 gives also a bound of the self-intersection of D in terms of $(K_X + D)^2$. Let us summarize this fact into a corollary.

Corollary 6.4.5. *Let (X^2, D) be a toroidal compactification. Then*

$$-D^2 \leq \frac{9}{4}(K_X + D)^2.$$

We can now use the theorem of Reider to improve Theorems 6.1.2 and 6.1.3 when $n = 2$.

Proposition 6.4.6. *Let (X^2, D) be a toroidal compactification. Then for all $m \geq 3$ the map associated to $|m(K_X + D)|$ is a morphism which maps all the different components of D to distinct singular points. Furthermore, for all $m \geq 4$ this morphism defines an embedding of $X \setminus D$ into some projective space.*

Proof. Let $L := K_X + D$. Because of Lemma 6.3.2, we know that $L^2 \geq 3$. By Reider's theorem, $K_X + 2L$ is base point free away from D . Adding D we get that $3(K_X + D)$ is base point free away from D . On the other hand, Lemma 6.3.1 implies that $2(K_X + D)$ is base point free along D . Similarly, the second statement follows from Reider's theorem applied to $K_X + 3L$. $\square$

The next proposition exhibits X as a smooth subvariety of $\mathbb{P}^5$. Moreover, it gives a bound on its degree.

Proposition 6.4.7. *Let (X^2, D) be a toroidal compactification. Let $L := K_X + D$. Then $7L - 4D$ is very ample. In particular, $7L - 4D$ embeds X as a smooth subvariety of $\mathbb{P}^5$ of degree $d \leq 49L^2$.*

Proof. Because of Theorem 6.4.2, we know that $2K_X + D = 2L - D$ is ample. Theorem 6.4.1 implies us that $K_X + 3(2L - D) = 7L - 4D$ is very ample. In particular, $7L - 4D$ embeds X as a smooth subvariety of some projective space of degree $d = (7L - 4D)^2 \leq 49L^2$. By general projection we reduce to $\mathbb{P}^5$. $\square$

It remains to bound the degree of each component of D under the embedding described in Proposition 6.4.7.

Lemma 6.4.8. *Let d_i be the degree of D_i under the embedding given by $7L - 4D$. Then*

$$d_i \leq 9L^2.$$

Proof. Recall that $D_i \cdot D_j = 0$ if $i \neq j$. Then

$$\begin{aligned} d_i &= D_i \cdot (7L - 4D) = -4D_i \cdot D \\ &= -4D_i^2 \leq -4D^2 \leq 9L^2, \end{aligned}$$

where the last inequality is given by Corollary 6.4.5. $\square$

Combining Proposition 6.4.7 and Lemma 6.4.8, we can derive a sharper version of Corollary 6.3.16 for surfaces.

Corollary 6.4.9. *Fix a positive integer V . Then the number of toroidal compactifications with $\dim(X) = 2$ and $(K_X + D)^2 \leq V$ is bounded by*

$$\sum_{q=1}^{q_0} d_0 (9V)^q \binom{6d_0}{5}^{6(q+1) \binom{5+d_0}{5}},$$

where $d_0 = 49V$ and $q_0 = \lfloor 2.25V \rfloor$.

Proof. Argue as in the proof of Corollary 6.3.16. $\square$

Remark 6.4.10. *The results of this sections improve the ones in [Hwa05]. For example, under the same notation of Proposition 6.4.7 and Lemma 6.4.8, he obtains $d \leq 265592 (L^2)^3 + 16059 (L^2)^2 +$*

$242L^2$ and $d_i \leq 394786.1 (L^2)^3$. Finally, the bound in his version of Proposition 6.4.9 is of the form $V^{V^{16}}$.

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