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Integrals of automorphic forms and L-functions

Tsiokos, Lefteris

Princeton, NJ : Princeton University

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Integrals of automorphic forms and $L-$functions

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A Dissertation
Presented to the Faculty
of Princeton University
in Candidacy for the Degree
of Doctor of Philosophy

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Abstract

This thesis consists of two topics. The first topic (chapter 2) is about extending the $GL_n \times GL_{n-1}$ Rankin-Selberg integral to certain cases that the automorphic forms that are involved are relative rank one Eisenstein Series. The main result is an expression of the residues as $GL_{n-1} \times GL_{n-2}$ Rankin-Selberg integrals for cusp forms. The other topic (chapter 3) is about symplectic integrals of cusp forms on GL_{2n}. The main result is a generalization of the Rankin-Selberg integral representing the exterior cube $L-$function of GL_6. At this moment we cannot extract any information for $L-$functions from this identity. We hope that this will be possible in the future.

Acknowledgements

Above all I thank God.

Among people, I firstly thank my family, and I also thank some friends and some teachers. In this thesis my acknowledgements to friends and teachers will be restricted to the subset of the people that helped or influenced me during the five year period that I was doing my Ph.D in Princeton University. There are many people outside this subset to whom I am grateful. The main reason I restrict acknowledgements to this subset is that otherwise, it would be too hard to be fair at this moment.

Among the friends that I had during my stay in Princeton University, I am mostly grateful to: Dimitrios Mirogiannis, Charles Xu, Georgos Vasilakis, Emmanouil Koukoumidis, Angie Andrikogiannopoulou and Konstantinos Kalafretzos. Without anyone among them, a big part of this five year period would have been substantially more difficult.

Finally I thank the people who are directly related to my Ph.D thesis. I thank:

- My advisor, professor Chris Skinner. I started understanding the theory of automorphic forms after he accepted me as one of his students. It took me a substantial amount of time to develop enough intuition for the subject in order to be able to make legitimate choices of problems to work on. For this reason it was very important that he was able to recommend a good problem to me, before I developed enough intuition to choose problems by myself. The techniques that I learned by working at this problem will influence my future career substantially.

- Professor David Ginzburg. In my Ph.D I worked essentially in two problems. The first was the problem Skinner recommended to me. The second was a problem that I chose later. Ginzburg helped me find very important flaws in my intuition related to the second problem. He helped me after that in even more important ways, but here I am restricting to help directly related to the Ph.D thesis.

- Professor Nicolas Templier. He was always available to help me, when I needed him. Even in one case that I asked him if he could read my whole thesis and write a report to the University about it, and do this in about a week, he said "No problem". He helped me more than I deserved.

- Professor Peter Sarnak. In an initial part of the Ph.D program, he helped me (together with Skinner) by being patient with me, when I wasn't doing well in the general examination. I

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- The professor who will be the third examiner in my Ph.D defence. I do not know yet who will be, but whoever will turn out to be I thank him in advance for accepting being in this committee.

To my parents.

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Chapter 1

Introduction

1.1 The two topics of this thesis

An important part in the theory of automorphic forms is to express automorphic $L-$functions as integrals of automorphic forms. Among the integrals of automorphic forms, the most interesting are those for which certain analytic properties (among them meromorphic continuation to $\mathbb{C}$), is not too hard to prove. Expressing $L-$functions by such integrals has been so far the main way for studying $L-$functions. This thesis is separated in two topics (Chapters 2 and 3), both related to this part of automorphic forms.

In Chapter 2 we extend the global $GL_n \times GL_{n-1}$ Rankin-Selberg theory to cases where the GL_n and GL_{n-1} automorphic forms are Eisenstein series. Then we express the residues in two ways. Firstly we express them as an $L-$function (up to bad places), and then as an automorphic integral. The expression as an automorphic integral is the main result of Chapter 2 (Proposition 2.4). We then finish the Chapter by mentioning certain potential applications.

In the third Chapter we generalize the Rankin-Selberg integral for the exterior cube $L-$function of GL_6 cusp forms, to an identity for GL_{2n} cusp forms. The GL_6 case is due to D. Ginzburg and S. Rallis [GR2]. In the identity that we get (Proposition 3.2), an Euler product occurs. We expect that $L-$functions for which integral representations have not been found yet, occur in this Euler product. However, there are certain problems that need to be resolved in order to start obtaining unknown analytic properties and the functional equation for these $L-$functions. It is likely that these problems are substantial, because otherwise D. Ginzburg and S. Rallis would not have stopped in the GL_6 case. In the first paragraph we mentioned that the most important automorphic integrals

are the ones for which certain analytic properties are not too hard to prove. In the generalization of the exterior cube $L-$function that we consider we have to face automorphic integrals for which it is not clear to us yet, how hard will be to obtain enough analytic properties.

After the study of the generalization of the exterior cube $L-$function, we continue with some easier but similar calculations that prove in a different way some of the results for the symplectic period of a GL_{2n} automorphic function. After finding this way of studying the symplectic period I learned that David Ginzburg had also found it long before me. However since it is missing from the literature I will put it in this thesis. Finally we discuss about certain similarities between the integrals in Chapters 2 and 3.

1.2 Basic definitions, notations and theorems that are used throughout the paper

Throughout F will denote a number field, $\mathfrak{o}$ will denote the ring of integers of F, υ will denote a place of F, F_υ will denote the completion of F at υ, and $\mathbb{A}$ will denote the ring of adeles in F.

We will be concerned with the reductive groups

$$GL_n, \qquad GSp_{2n}, \qquad Sp_{2n}.$$

The group GSp_{2n} will also be denoted by H_n or H, and Sp_{2n} will also be denoted by $\tilde{H}_n$.

By a parabolic subgroup we will mean a standard parabolic subgroup. The Levis of the parabolics are also chosen to be standard. Let G be any among the reductive groups in the previous list. For every place υ we define the maximal compact subgroup K_υ of $G(F_\upsilon)$ as follows:

$$K_\upsilon = \begin{cases} O_n \cap G & \upsilon = \mathbb{R} \\ U_n \cap G & \upsilon = \mathbb{C} \\ G(\mathfrak{o}_\upsilon) & \upsilon \text{ finite} \end{cases} .$$

We denote by K the group $\prod_\upsilon K_\upsilon$. Finally we will denote by G^1 the subset of $g \in G(\mathbb{A})$ with $|\det(g)| = 1$.

We will call automorphic functions for a reductive group G over a number field F the functions defined in $G(\mathbb{A})$ with the following properties.

1. they are $G(F)$ invariant,

2. they are smooth,

3. they are $K-$finite,

4. they have moderate growth.

The ones we will use will be :

1. cusp forms,

2. Eisenstein series,

3. Pseudo-Eisenstein series,

4. functions obtained from 1, 2, and 3 after applying a restriction to a subgroup or a unipotent integration.

It is because of functions as in 3 and 4 that we do not add in the definition of an automorphic function the property of being finite for the action of the center of the universal enveloping algebra. The previous definition of automorphic functions can be extended to linear algebraic groups other than the reductive ones. This can be of some use in unfolding integrals similar to the ones in this paper, because frequently, Fourier coefficients of automorphic functions are $\tilde{P}(F)$ invarinant for $\tilde{P}$ a group of the form (reductive)$\rtimes$(unipotent). This happens even if the initial function is given to be automorphic for a reductive group.

Let ϕ be any automorphic function on G. Let N be a unipotent subgroup of G and ψ a character of $N(F) \setminus N(\mathbb{A})$. Then we define $\phi_{N,\psi}$ by

$$\phi_{N,\psi}(g) = \int_{N(F)\setminus N(\mathbb{A})} \phi(ng)\psi^{-1}(n)\mathrm{d}n.$$

If ψ is trivial we just write ϕ_N. The integral is well-defined because $N(F) \setminus N(\mathbb{A})$ is compact.

Let R be a unipotent group with subgroups L and R' such that $R = L \rtimes R'$. Let ψ_L be a character of L which is normalized by R' ($\psi_L(r^{-1}lr) = \psi_L(l)$ for all $l \in L$ and all $r \in R'$). Then ψ_L can be extended trivially to R. This extension will still be denoted by ψ_L. More generally if ψ is a character of R' we define by $\psi\psi_L$ the character on R which is given by

$$\psi\psi_L(lr) = \psi_L(l)\psi(r) \qquad \text{where } l \in L \text{ and } r \in R'.$$

Whittaker models: Among the functions of the form $\phi_{N,\psi}$ that will concern us, the Whittaker models require special attention. We recall some definitions, and properties.

Let U_n be the unipotent radical of the upper triangular Borel of GL_n and ψ a generic character of U_n. For a GL_n automorphic form ϕ we define

$$W_\phi = \phi_{U_n,\psi}.$$

This is called the $\psi-$Whittaker function corresponding to ϕ. In most cases we will call it the Whittaker function instead of the $\psi-$Whittaker function. For a cusp form φ we have $W_\varphi \neq 0$. This is a corollary of the Fourier expansion

$$\varphi(g) = \sum_{\gamma \in U_{n-1}(F)\backslash GL_{n-1}(F)} W_\varphi \left(\begin{pmatrix} \gamma & \\ & 1 \end{pmatrix} g \right).$$

Whittaker models for an admissible irreducible representation π of $GL_n(A)$ (or of $GL_n(F_\upsilon)$) are defined as realizations of π, in spaces of functions that satisfy the identity

$$W(ng) = \psi(n)W(g)$$

for $n \in U(\mathbb{A})$ (or $n \in U(F_\upsilon)$) and also satisfy certain regularity conditions. The representations that have such a realization are called generic. Notice that $W_\varphi \neq 0$, gives rise to a Whittaker model for the cuspidal representation to which φ belongs.

Irreducible admissible (and unitary in the Archimedean case) representations of $GL_n(F_\upsilon)$ can have at most one Whittaker model [Sha]. From this local uniqueness theorem, a global uniqueness is obtained as a corollary. We need to recall the identity that is established in the proof of this corollary.

Corollary 0: *Let ϕ be an automorphic function belonging to a generic unitary irreducible representation π. We assume that φ is decomposable. Then at each place υ there exist a function W_υ in the Whittaker model of π_υ such that*:

1. $$W_\phi(g) = \prod_\upsilon W_\upsilon(g_\upsilon).$$

2. *For almost all υ*
$$W_\upsilon(e_\upsilon) = 1.$$

Remark: When ϕ is a cusp form, the normalizations in condition 2, are the ones that are usually chosen for decomposing W_ϕ. In the Langlands-Shahidi method, where we get such a decomposistion for Eisenstein series, different normalizations are used (however they are the same in the trivial case that the Eisenstein series is a cusp form). We will use the Langlands-Shahidi normalizations for decomposing the unfolded $GL_n \times GL_{n-1}$ Rankin-Selberg integrals of Eisenstein Series. We remind some facts about these normalizations after introducing Eisenstein series.

Having seen the proof of the local uniqueness or any related result can be of some help in Chapter 3. When we unfold the Rankin-Selberg integral that generalizes the one that represents the exterior cube $L-$function of GL_6 (Proposition 3.2), we obtain several models for the cusp form. One of them is the Whittaker model. With Mackey theory of Intertwining operators (as it is usually applied for proving uniqueness of a model) it should be possible to see that all the other models that we have are not unique.

Eisenstein series: In [M-W] and [Sh], Eisenstein series are discussed in an adelic setting. We briefly remind what is necessary for this paper.

Let G be a reductive group, P a maximal parabolic subgroup, and $P = MN$ its Levi decomposition with N its unipotent radical. Let P^a be the maximal parabolic of G that is associate to P, and let M^a and N^a be its (standard as always) Levi and its unipotent radical, respectively. Let w be an element in the Weyl group of G such that

$$wMw^{-1} = M^a.$$

Let π be a cuspidal representation or a character of $M(\mathbb{A})$. Then for each $s \in \mathbb{C}$ we define the representations:

$$I_s = I_{s,\pi} = \mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})}(\delta^s \bigotimes \pi),$$

where δ denotes the modular character, and Ind denotes unitary induction. We will denote the unnormalized induction by Ind. We will be mostly using unitary induction. The two cases that unnormalized induction is used is when the Siegel and the Klingen Eisenstein series are introduced. We will give a reminder when we will study these cases.

There is a nonzero intertwining operator

$$M(w) = M(w, s, \pi) : \quad I_{s,\pi} \to I_{-s,w\pi},$$

which for $\mathrm{Re}(s) > 1/2$ and $f_s \in I_s$ can be defined by :

$$M(w)f_s(g) = \int_{N^a(F)} f_s(w^{-1}ng)\mathrm{d}n.$$

It turns out that $M(w,s)$ can be meromorphically continued to the rest of the complex plane. For a $K-$finite section

$$f_s \in I_s,$$

we define an Eisenstein series by :

$$E(f_s, g) = \sum_{\gamma \in P(F)\backslash G(F)} f_s(\gamma g) \qquad \text{for } \mathrm{Re}(s) > \frac{1}{2}.$$

This sum converges absolutely for $\mathrm{Re}(s) > \frac{1}{2}$, has meromorphic continuation to all $s \in \mathbb{C}$, and satisfies the functional equation :

$$E(f_s, g) = E(M(w)f_s, g).$$

The facts that we mentioned for the Eisenstein series and the intertwining operators (for example their meromorphic continuations) are very closely related.

When confusion is unlikely to happen, instead of $E(f_s, g)$, we will write $E(g)$ or $E_s(g)$. Similarly we will write M instead of $M(w)$.

Finally we recall in a special case, some facts about the Whittaker functions attached to Eisenstein series. Let π_1 and π_2 be cuspidal representations, respectively of GL_{n_1} and GL_{n_2}. Let P be the $GL_{n_1+n_2}-$parabolic with Levi $M = GL_{n_1} \times GL_{n_2}$ such that GL_{n_1} occurs in the upper left. Let

$$w = \begin{pmatrix} & I_{n_2} \\ I_{n_1} & \end{pmatrix}.$$

Notice that

$$w \begin{pmatrix} g_{n_1} & \\ & g_{n_2} \end{pmatrix} w^{-1} = \begin{pmatrix} g_{n_2} & \\ & g_{n_1} \end{pmatrix}.$$

Let E_s be an Eisenstein series corresponding to a section

$$f_s \in I_s = \mathrm{Ind}_{P(F)}^{GL_{n_1+n_2}(F)} \delta^s \bigotimes (\pi_1 \bigotimes \pi_2).$$

Let

$$W_{f_s}(g)=\int_{U_{n_1}\times U_{n_2}(F)\backslash U_{n_1}\times U_{n_2}(\mathbb{A})} f_s\left(\begin{pmatrix}u_1 & \\ & u_2\end{pmatrix}g\right)\psi^{-1}\begin{pmatrix}u_1 & \\ & u_2\end{pmatrix}\mathrm{d}u_1\mathrm{d}u_2.$$

By unfolding the Eisenstein series we get the following formula for W_E (details in [Sh], page 123)

$$W_{E_s}(g)=\int_{N^a(\mathbb{A})} W_{f_s}(w^{-1}ng)\psi^{-1}(n)\mathrm{d}n \qquad \mathrm{Re}(s)>\frac{1}{2}. \tag{1.2.1}$$

It is obtained from the unfolding that the right hand side is an absolutely convergent integral for $\mathrm{Re}(s)>\frac{1}{2}$.

Due to corollary 0, for a decomposable f_s we have a factorization

$$W_{f_s}(g)=\prod_{\upsilon} W_{f_s,\upsilon}(g_\upsilon) \qquad \text{with } W_{f_s,\upsilon}(e_\upsilon)=1 \text{ for all } \upsilon \text{ outside a finite set S.}$$

Due to the same corollary, W_{E_s} can be factorized directly. However what we will need for W_{E_s} is a factorization normalized in a way compatible with the choice of factorization of the inducing data. Such a factorization for W_{E_s} follows by plugging in the factorization of W_{f_s} to formula 1.2.1.

We have

$$W_{E_s}(g)=\prod_{\upsilon} W^{\bullet}_{E_s,\upsilon}(g_\upsilon)$$

where

$$W^{\bullet}_{E_s,\upsilon}(g)=\int_{N^a(F_\upsilon)} W_{f_s,\upsilon}(w^{-1}ng_\upsilon)\psi^{-1}(n)\mathrm{d}n \qquad \text{for } \mathrm{Re}(s)>\frac{1}{2}.$$

By interpreting the right hand side as a conditional integral, in an appropriate way, the definition holds for all $s\in\mathbb{C}$. This is a Whittaker model for $I_{s,\upsilon}$ ([Sh], pages 50-52).

By [Sh] (Proposition 7.1.4, page 125), for the factors outside S we have

$$\int_{N^a(F_\upsilon)} W_{f_s,\upsilon}(w^{-1}n)\psi^{-1}(n)\mathrm{d}n=L(1+s,\pi_1\times\tilde{\pi}_2)^{-1}.$$

The $W^{\bullet}_{E_s,\upsilon}$ are the local Whittaker functions that we will use for studying the extended $GL_n\times GL_{n-1}$ Rankin-Selberg integrals. However in the end of Chapter 2, in subsection 2.5.2, we will use a different normalization. In subsection 2.5.2, we mention certain Archimedean identities that are obtained by comparing the two calculations of the residues. It will be good that in these identities, the Archimedean Whittaker models that occur will not depend on global data. For this reason we will

use the Whitaker models normalized as follows :

$$W_{E_s,v} = L(1+s, \pi_1 \times \tilde{\pi}_2) W^{\bullet}_{E_s,v}.$$

Chapter 2

A $GL_n \times GL_{n-1}$ Rankin-Selberg Integral for certain Eisenstein Series

2.1 Introduction

Let π and π' be two cuspidal representations of GL_n and GL_{n-1} respectively. Also let φ and φ' be two cusp forms in π and π' respectively. Then the integral

$$\int_{GL_{n-1}(F)\backslash GL_{n-1}(\mathbb{A})} \varphi \begin{pmatrix} g & \\ & 1 \end{pmatrix} \varphi'(g) |\det(g)|^{s-\frac{1}{2}} \mathrm{d}g,$$

is an integral representation for

$$L(\pi \times \pi', s).$$

In this Chapter we extend the theory of this $GL_n \times GL_{n-1}$ Ranking-Selberg integral to certain relative rank one Eisenstein series.

For understanding this $GL_n \times GL_{n-1}$ Rankin-Selberg integral, as in the case with cusp forms, we use an appropriate Fourier expansion for the GL_n automorphic form. Finding this Fourier expansion is the topic of the next section. After this the unfolding of the integral follows. Then we calculate the residue of the integral in two ways. First by using the unfolded form of the integral. Second by using the initial form of the integral (Proposition 2.4). Finally in the fifth section we discuss about certain

potential applications that occur by comparing the study of the two forms of the extended Rankin-Selberg integral (the initial and the unfolded forms). For example the two different expressions for the residues lead to certain identities for Archimedean Rankin-Selberg integrals.

Before we start, we introduce some of the notation that we will be using. Let P_k be the maximal parabolic of GL_k consisting of the matrices $(x_{i,j})$ with $x_{n,l} = 0$ for $l = 1, 2, ...n-1$. Let N_k be the unipotent radical of P_k. Let U_k be the unipotent radical of the Borel. Recall that by P_k^a we denote the associate to P_k parabolic subgroup, and by N_k^a its unipotent radical.

2.2 Fourier expansions

For a cusp form φ of GL_n the following Fourier expansion is well known :

$$\varphi(g) = \sum_{\gamma \in U_{n-1}(F) \backslash GL_{n-1}(F)} W_\varphi \left(\begin{pmatrix} \gamma & \\ & 1 \end{pmatrix} g \right). \tag{2.2.1}$$

We will extend this formula for certain Eisenstein Series. This is not the first time this formula is extended. Dihua Jiang and Baiying Liu extended this formula to the discrete spectrum [J-L]. I was also informed recently that a Fourier expansion has been written down for general automorphic forms of GL_n, but we have been unable to find it in the literature.

Proposition 2.2: *Let E be an Eisenstein series induced from cusp forms on $GL_{n-1} \times GL_1$. Let*

$$\tilde{E}(g) = \sum_{\gamma \epsilon P_{n-1}^a(F) \backslash GL_{n-1}(F)} E_{P_n^a} \left(\begin{pmatrix} \gamma & \\ & 1 \end{pmatrix} g \right).$$

Then we have the following Fourier expansion :

$$E(g) = E_{P_n}(g) + \tilde{E}(g) + \sum_{\gamma \epsilon U_{n-1}(F) \backslash GL_{n-1}(F)} W_E \left(\begin{pmatrix} \gamma & \\ & 1 \end{pmatrix} g \right). \tag{2.2.2}$$

All the sums converge absolutely.

Proof. We essentially repeat the proof for the Fourier expansion of cusp forms. The difference is that in the first and the last Fourier expansion the contribution of the constant term is not zero. In the contribution of the constant term in the last Fourier expansion, the Fourier expansion of the GL_{n-1} cusp form in the inducing data will occur. This will make the final result less complicated.

Before we start we describe some notation. Let

$$P'_k := \{g = (g_{i,j})\epsilon P_k : g_{k,k} = 1\}.$$

Let R_k for $k = 2, 3, ..., n$ be the unipotent groups that are defined inductively downwards by the conditions :

$$R_n = N_n,$$

$$R_k = R_{k+1}N_k \qquad \text{for } k = n-1, n-2, ..., 2.$$

For example we get $R_2 = U_n$. We define a character ψ_{N_k} of $N_k(F) \setminus N_k(\mathbb{A})$ by

$$\psi_{N_k}(u) = \psi_F(u_{k-1,k}).$$

Finally for $k = n, n-1, ..., 2$ we define a character ψ_{R_k} by

$$\psi_{R_k}(u) = \psi_F(u_{n-1,n} + u_{n-2,n} + ... + u_{k-1,k}).$$

The characters ψ_{R_k} can be extended trivially to $U_n(F) \setminus U_n(\mathbb{A})$. We will use the same symbol for this extension.

STEP 1 We consider the Fourier expansion of E along $N_n(F) \setminus N_n(\mathbb{A})$:

$$E(g) = \sum_{\psi} \int_{N_n(F)\setminus N_n(\mathbb{A})} E(ng)\psi^{-1}(n)dn.$$

These characters are permuted by GL_{n-1} and they give two orbits. The one consists of the trivial character (we will frequently call this orbit trivial). For the other we choose the representative ψ_{N_n}. The stabilizer of this representative is P'_{n-1}. So if ψ is one of the non-constant characters we can find γ in P_n such that $\psi(n) = \psi_{N_n}\left(\begin{pmatrix} \gamma & & \\ & & \\ & & 1 \end{pmatrix} n \begin{pmatrix} \gamma & & \\ & & \\ & & 1 \end{pmatrix}^{-1}\right)$. As a result we get for each of the previous Fourier coefficients

$$\int_{N_n(F)\setminus N_n(\mathbb{A})} E(ng)\psi^{-1}(n)dn$$

$$= \int_{N_n(F)\setminus N_n(\mathbb{A})} E(ng)\psi_{N_n}^{-1}\left(\begin{pmatrix} \gamma & & \\ & & \\ & & 1 \end{pmatrix} n \begin{pmatrix} \gamma & & \\ & & \\ & & 1 \end{pmatrix}^{-1}\right) dn$$

$$= \int_{N_n(F)\backslash N_n(\mathbb{A})} E\left(n\begin{pmatrix}\gamma & \\ & 1\end{pmatrix}g\right)\psi_{N_n}^{-1}(n)dn$$

$$= E_{N_n,\psi_{N_n}}\left(\begin{pmatrix}\gamma & \\ & 1\end{pmatrix}g\right).$$

Now, the Fourier expansion along N_n becomes :

$$E(g) = E_{N_n}(g) + \sum_{\gamma\epsilon P'_{n-1}(F)\backslash GL_{n-1}(F)} E_{N_n,\psi_{N_n}}\left(\begin{pmatrix}\gamma & \\ & 1\end{pmatrix}g\right).$$

STEP 2 We consider the Fourier expansion of $E_{N_n,\psi_{N_n}}$ along $N_{n-1}(F)\setminus N_{n-1}(\mathbb{A})$. As in the previous step we obtain

$$E_{N_n,\psi_{N_n}}(g) = E_{R_{n-1},\psi_{N_n}}(g)+$$

$$\sum_{\gamma\epsilon P'_{n-2}(F)\backslash GL_{n-2}(F)} E_{R_{n-1},\psi_{R_{n-1}}}\left(\begin{pmatrix}\gamma & \\ & I_2\end{pmatrix}g\right).$$

However, in this expansion the contribution of the trivial orbit

$$E_{R_{n-1},\psi_{N_n}}$$

is zero, because a zero constant term of the Eisenstein series occurs as inner integration.

By Plugging in the second Fourier expansion to the first we get :

$$E(g) = E_{N_n}(g) + \sum_{\gamma_1\epsilon P'_{n-1}(F)\backslash GL_{n-1}(F)}\sum_{\gamma_2\epsilon P'_{n-2}(F)\backslash GL_{n-2}(F)} E_{R_{n-1},\psi_{R_{n-1}}}\left(\begin{pmatrix}\gamma_2 & \\ & I_2\end{pmatrix}\begin{pmatrix}\gamma_1 & \\ & 1\end{pmatrix}g\right).$$

Notice that

$$P'_{n-2}\setminus GL_{n-2} = P'_{n-2}U_{n-1}\setminus P'_{n-1},$$

so the previous formula becomes

$$E(g) = E_{N_n}(g) + \sum_{\gamma\epsilon P'_{n-2}U_{n-1}(F)\backslash GL_{n-1}(F)} E_{R_{n-1},\psi_{R_{n-1}}}\left(\begin{pmatrix}\gamma & \\ & 1\end{pmatrix}g\right).$$

STEPS 3,4,...n-2 These steps are identical to step 2. In each of these steps the integral corresponding to the trivial orbit of the Fourier expansion contains a zero constant term of E as

inner integration. Putting all these expansions together we obtain

$$E(g) = E_{N_n}(g) + \sum_{\gamma \epsilon GL_1(F)U_{n-1}(F)\backslash GL_{n-1}(F)} E_{R_3,\psi_{R_3}}\left(\begin{pmatrix}\gamma & \\ & 1\end{pmatrix} g\right).$$

Here GL_1 occurs as the Levi of P_2' and as a consequence is embedded in GL_n by

$$a \to \begin{pmatrix} a & \\ & I_{n-1}\end{pmatrix}.$$

STEP n-1 We consider the Fourier expansion of $E_{R_3,\psi_{R_3}}$ along $N_2(F) \setminus N_2(\mathbb{A})$

$$E_{R_3,\psi_{R_3}}(g) = E_{U,\psi_{R_3}}(g) + \sum_{\gamma \epsilon GL_1(F)} W_E\left(\begin{pmatrix}\gamma & \\ & I_{n-1}\end{pmatrix} g\right).$$

Plugging this into the formula for E we get

$$E(g) = E_{N_n}(g) + \sum_{\gamma \epsilon GL_1(F)U_{n-1}(F)\backslash GL_{n-1}(F)} E_{U,\psi_{R_3}}\left(\begin{pmatrix}\gamma & \\ & 1\end{pmatrix} g\right) +$$

$$\sum_{\gamma \epsilon U_{n-1}(F)\backslash GL_{n-1}(F)} W_E\left(\begin{pmatrix}\gamma & \\ & 1\end{pmatrix} g\right)$$

We define

$$\tilde{E}^*(g) = \sum_{\gamma \epsilon GL_1(F)U_{n-1}(F)\backslash GL_{n-1}(F)} E_{U,\psi_{R_3}}\left(\begin{pmatrix}\gamma & \\ & 1\end{pmatrix} g\right).$$

To obtain the Fourier expansion for E that we stated in the theorem we only need to prove that $\tilde{E}^* = \tilde{E}$. Notice that

$$E_{U,\psi_{R_3}}(g) = \int_{U_{n-1}(F)\backslash U_{n-1}(\mathbb{A})} E_{P_n^a}\left(\begin{pmatrix}1 & \\ & n\end{pmatrix} g\right) \psi_{U_{n-1}}^{-1}(n) dn$$

and that the integration over $U_{n-1}(F) \setminus U_{n-1}(\mathbb{A})$ gives the Whittaker model for the GL_{n-1} cusp form that occurs in the inducing data.

LAST STEP We "undo" the Fourier expansion of the GL_{n-1} cusp form in the inducing data,

along $U_{n-1}(F) \setminus U_{n-1}(\mathbb{A})$:

$$\tilde{E}^*(g) = \sum_{\gamma_1 \epsilon GL_1 \times GL_{n-2}(F) N^a_{n-1}(F) \setminus GL_{n-1}(F)} \sum_{\gamma_2 \epsilon GL_1(F) \times U^a_{n-2}(F) \setminus GL_1 \times GL_{n-2}}$$

$$\int_{U_{n-1}(F) \setminus U_{n-1}(\mathbb{A})} E_{P^a_n} \left(\begin{pmatrix} 1 & & \\ & & \\ & & n \end{pmatrix} \gamma_2 \gamma_1 g \right) \psi^{-1}_{U_{n-1}}(n) dn.$$

For every choice of γ_1, the inner sum of integrals is a Fourier expansion of the GL_{n-1} cusp form along $U_{n-1}(F) \setminus U_{n-1}(\mathbb{A})$. As a result we have

$$\tilde{E}^*(g) = \tilde{E}(g).$$

We are done. □

2.3 The unfolding of the $GL_n \times GL_{n-1}$ Rankin-Selberg Integral

Recalling facts from the case with cusp forms

A reference for the $GL_n \times GL_{n-1}$ Rankin-Selberg integrals are the notes of J. W Cogdell [C]. We will only recall a few basic facts.

Let υ be any place. Let ρ_υ and ρ'_υ be two generic unitary irreducible representations of $GL_n(F_\upsilon)$ and $GL_{n-1}(F_\upsilon)$ respectively. Let W_υ and W'_υ be elements in the Whittaker models of ρ_υ and ρ'_υ respectively. The additive characters that are used for defining the Whittaker models of the two representations are inverse of each other. We define

$$Z_\upsilon(W_\upsilon, W'_\upsilon, s) := \int_{U_{n-1}(F_\upsilon) \setminus GL_{n-1}(F_\upsilon)} W_\upsilon \begin{pmatrix} g & \\ & 1 \end{pmatrix} W'_\upsilon(g) |\det(g)|^{s-\frac{1}{2}} \mathrm{d}g.$$

It turns out that this integral is absolutely convergent for $\mathrm{Re} s \geq 1$, and that $Z_\upsilon(W_\upsilon, W'_\upsilon, s)$ has meromorphic continuation for $s \in \mathbb{C}$. It also tuns out that for $\mathrm{Re}(s)$ large the product in the following definition converges absolutely:

$$Z(W, W', s) := \prod_\upsilon Z_\upsilon(W_\upsilon, W'_\upsilon, s)$$

where we assume that

$$W = \bigotimes_{v} W_v \quad \text{and} \quad W' = \bigotimes_{v} W'_v$$

are decompositions as in Corollary 0 of Chapter 1. Finally when v is non-Archimedean, all the data is unramified, and W_v, W'_v are normalized so that they are equal to one when evaluated at the identity matrix, we have

$$Z_v(W_v, W'_v, s) = L_v(\rho \times \rho', s).$$

Now let ρ and ρ' be two cuspidal representations of GL_n and GL_{n-1} respectively. Let φ and φ' be two cusp forms respectively in ρ and ρ'. Finally let

$$I = I(\varphi, \varphi', s) = \int_{GL_{n-1}(F)\backslash GL_{n-1}(\mathbb{A})} \varphi \begin{pmatrix} g & \\ & 1 \end{pmatrix} \varphi'(g) |\det(g)|^{s-\frac{1}{2}} \mathrm{d}g.$$

Then we have

$$I(\varphi, \varphi', s) = Z(W_\varphi, W'_\varphi, s).$$

This follows by an unfolding of I which is based on the Fourier expansion of φ. If we further assume that φ and φ' are decomposable, we obtain

$$I(\varphi, \varphi', s) = \left(\prod_{v \in S} Z_v(W_{\varphi,v}, W'_{\varphi',v}, s) \right) L^S(\rho \times \rho', s),$$

where S is a finite set which contains the archimedean places and the places for which part of the data that we introduced is ramified. We continue by extending the last formula to certain Eisenstein series.

The case with Eisenstein series

First we introduce more notation. Throughout π and π' will be two cuspidal representations of the groups GL_{n-1} and GL_{n-2} respectively. By χ and χ' we denote two Hecke characters. We choose two functions f and f' coming respectively from $\mathrm{Ind}_{P_n}^{GL_n}(\pi \bigotimes \chi)$ and $\mathrm{Ind}_{P_{n-1}}^{GL_{n-1}}(\pi' \bigotimes \chi')$ which (as always) are respectively, K_n and K_{n-1} finite. Also let E and E' be the Eisenstein series corresponding to these functions. Finally recall from the previous section that :

$$\tilde{E}(g) = \sum_{\gamma \epsilon P_{n-1}^a(F)\backslash GL_{n-1}(F)} E_{P_n^a} \left(\begin{pmatrix} \gamma & \\ & 1 \end{pmatrix} g \right).$$

We will need to recall a basic fact about the decay of Fourier coefficients of automorphic forms. The next lemma will be used in two occasions in this Chapter. First to be able to use Fubini's theorem in the unfolding of the Rankin-Selberg Integral. Second to study the residues of this integral (when we will be using the initial form of the integral, not the unfolded).

Lemma 2.3: *For any automorphic function ξ of GL_n the sum*

$$\mathbf{A}(g) = \sum_{\gamma \in P_{n-1}(F)\backslash GL_n(F)} |W_\xi| \left(\begin{pmatrix} \gamma & \\ & 1 \end{pmatrix} g \right)$$

converges and has the following type of decay :

(1) $\mathbf{A}$ *is rapidly decreasing as an automorphic function on* $GL_{n-1}(\mathbb{A})$,

(2) *for t big enough we can find c_t such that*

$$\mathbf{A} \begin{pmatrix} h & \\ & 1 \end{pmatrix} | \geq c_t |det(h)|^{-t},$$

for all $h \in GL_{n-1}(\mathbb{A})$,

(3) *these two types of decay happen uniformly.*

This lemma follows directly from [J-PS-S][prop 2.4.1 in Vol I and Proposition 12.2 in Vol II].

Proposition 2.3: *Let the sections f and f' be defined as previously, and decomposable. For $Re(s)$ big enough the integral*

$$I = \int_{GL_{n-1}(F)\backslash GL_{n-1}(\mathbb{A})} (E - E_{P_n} - \tilde{E}) \begin{pmatrix} g & \\ & 1 \end{pmatrix} E'(g) |det(g)|^{s-\frac{1}{2}} \mathrm{d}g$$

converges absolutely and is equal to

$$\left(\prod_{v \in S} Z_v(W_v^\bullet, W_v^{\bullet,\prime}, s) \right) \frac{L^S(\pi \times \pi', s) L^S(\pi \times \chi', s) L^S(\pi' \times \chi, s) L^S(\chi\chi', s)}{L^S(\pi \times \chi^{-1}, 1) L^S(\pi' \times \chi'^{-1}, 1)}. \tag{2.3.1}$$

Remark: Since we know the meromorphic continuation of all the previous $L-$functions this Proposition implies the meromorphic continuation for I. Usually the analytic properties of an $L-$ function are obtained from an integral representation that it has, and not the other way around. This

paradigm can be preserved for the integral I because in the proof of Proposition 2.4 the meromorphic continuation of I is obtained independently of its relation to $L-$functions.

Proof. The proof is identical to the case with cusp forms. We mostly put it for the convenience of the reader.

The absolute convergence for $\mathrm{Re}(s)$ large follows directly from Lemma 2.3.1. From this lemma we also see that the two uses of Fubini's theorem in the calculation that follow, are justified. So for $\mathrm{Re}(s)$ large, we have

$$\begin{aligned} I &= \int_{GL_{n-1}(F)\backslash GL_{n-1}(\mathbb{A})} \sum_{\gamma \in U_{n-1}(F)\backslash GL_{n-1}(F)} W_E\left(\begin{pmatrix}\gamma & \\ & 1\end{pmatrix} g\right) E'(g)\mathrm{d}g \\ &= \int_{U_{n-1}(F)\backslash GL_{n-1}(\mathbb{A})} W_E\begin{pmatrix} g & \\ & 1\end{pmatrix} E'(g)\mathrm{d}g \\ &= \int_{U_{n-1}(\mathbb{A})\backslash GL_{n-1}(\mathbb{A})} \int_{U_{n-1}(F)\backslash U_{n-1}(\mathbb{A})} W_E\begin{pmatrix} ng & \\ & 1\end{pmatrix} E'(ng)\mathrm{d}n\mathrm{d}g \\ &= \int_{U_{n-1}(\mathbb{A})\backslash GL_{n-1}(\mathbb{A})} W_E\left(\begin{pmatrix} g & \\ & 1\end{pmatrix}\right) W_{E'}(g)\mathrm{d}g. \end{aligned}$$

This is equal to (2.3.1). □

2.4 Study of the residues

We remind the reader that the cuspidal representations π and π' and the characters χ and χ' are all unitary.

Something we directly see from Proposition 2.3 is that I is analytic for $\mathrm{Re}(s) > 1$ and that for $\mathrm{Re}(s) = 1$ we have a pole if and only if $\chi\chi'(a) = |a|^t$. The reason for that is :

1)The $L-$functions $L^S(\pi \times \pi', s)$, $L^S(\pi \times \chi', s)$ and $L^S(\pi' \times \chi, s)$ are entire.

2) $L^S(\chi\chi', s)$ has a pole iff $\chi\chi'(x) = |x|^t$ for some t and in this case the pole is at $s = 1 - t$.

3) The local Rankin-Selberg integrals at the finitely many places of S have no poles for $\mathrm{Re}(s) \geq 1$.

By Proposition 2.3 we directly get a description of the residues.

First description of the Residues: *The pole at $1-t$, when it exists, has residue of the form*

$$\left(\prod_{v\in S} Z_v(W_v^\bullet, W_v^{\bullet,\prime}, 1-t)\right) L^S(\pi\times\pi', 1-t) Res_{s=1}(\zeta_F^S(s)). \tag{2.4.1}$$

The residue at $Re(s)=0$, is obtained similarly.

Remark: Notice the cancellation that happened due to the Langlands-Shahidi normalizations.

Our next task is to calculate the residues of I by using its definition and not the relation of I with $L-$functions. This calculation has some similarities with the proof of converse theorems, but it is also related to a much older calculation.

Question: In the conventional Rankin-Selberg integrals the rapid decay of cusp forms solves the convergence issues, now that we do not have cusp forms what can we do?
Answer: The first time mathematicians faced such an integral did not have cusp forms either. We will do what worked for the representation of the Riemann zeta function as an integral of a theta function.

We start the calculation and when we finish it, we state in the Proposition 2.4 the results.

Let $\mathrm{Re}(s)$ be big. Then:

$$I=\int_0^\infty a^{(n-1)(s-\frac{1}{2})}\int_{GL_{n-1}(F)\backslash GL_{n-1}^1(\mathbb{A})}(E-E_{P_{n-1}}-\tilde{E})\begin{pmatrix} ag & \\ & 1\end{pmatrix}E'(ag)\mathrm{d}g\mathrm{d}a.$$

Notice that because of the orthogonality between cusp forms and Eisenstein series this integral can be simplified. However, in order to take advantage of the Fourier expansion of Eisenstein series (Proposition 2.2) this orthogonality will be ignored for a few steps. Let $I=I_1+I_2$ where I_1 and I_2 are given by

$$I_1=\int_1^\infty a^{(n-1)(s-\frac{1}{2})}\int_{GL_{n-1}(F)\backslash GL_{n-1}^1(\mathbb{A})}(E-E_{P_{n-1}}-\tilde{E})\begin{pmatrix} ag & \\ & 1\end{pmatrix}E'(ag)\mathrm{d}g\mathrm{d}a$$

and

$$I_2 = \int_0^1 a^{(n-1)(s-\frac{1}{2})} \int_{GL_{n-1}(F)\backslash GL^1_{n-1}(A)} (E - E_{P_{n-1}} - \tilde{E}) \begin{pmatrix} ag & \\ & 1 \end{pmatrix} E'(ag)\mathrm{d}g\mathrm{d}a.$$

Then I_1 is an entire function by Lemma 2.3. As a result the poles and residues of I and I_2 are the same. We continue with the study of I_2.

For an appropriate element w in the Weyl group W_n of GL_n we have that

$$E \begin{pmatrix} g & \\ & 1 \end{pmatrix} = wE \begin{pmatrix} 1 & \\ & g \end{pmatrix}.$$

We also define

$$\tilde{\tilde{E}}(g) = \sum_{\gamma \in P_{n-1}(F)\backslash GL_{n-1}(F)} E_{P_n} \left(\begin{pmatrix} 1 & \\ & \gamma \end{pmatrix} g \right).$$

As a result I_2 can be written as $I_3 + I_4$, where $I_3 =$

$$\int_0^1 a^{(n-1)(s-\frac{1}{2})} \int_{GL_{n-1}(F)\backslash GL^1_{n-1}(\mathbb{A})} (wE - (wE)_{P^a_{n-1}} - (\tilde{wE})) \begin{pmatrix} 1 & \\ & ag \end{pmatrix} E'(ag)\mathrm{d}g\mathrm{d}a$$

and $I_4 =$

$$\int_0^1 a^{(n-1)(s-\frac{1}{2})} \int_{GL_{n-1}(F)\backslash GL^1_{n-1}(\mathbb{A})}$$

$$\left[(wE_{P^a_{n-1}} + (\tilde{wE})) \begin{pmatrix} 1 & \\ & ag \end{pmatrix} - (E_{P_{n-1}} + \tilde{E}) \begin{pmatrix} ag & \\ & 1 \end{pmatrix} \right] E'(ag)\mathrm{d}g\mathrm{d}a.$$

Then I_3 is entire. This is for the same reason that I_1 is entire, except that we must use different versions of the Fourier expansion of Eisenstein Series (Proposition 2.2) and it's decay (Lemma 2.3). Since the proofs for these versions are identical we will only mention the modification of Proposition 2.2.

Proposition 2.2 (second version): *Let E be an Eisenstein series induced from cusp forms in $GL_{n-1} \times GL_1$. Then we have the following Fourier expansion*:

$$E(g) = E_{P^a_n}(g) + \tilde{\tilde{E}}(g) + \sum_{\gamma \epsilon U_{n-1}(F)\backslash GL_{n-1}(F)} W_E \left(\begin{pmatrix} 1 & \\ & \gamma \end{pmatrix} g \right).$$

Proof. : Take the proof of Proposition 2.2 and replace the Fourier expansions in N_n, N_{n-1},... with the Fourier expansions in N_n^a, N_{n-1}^a,... . □

In I_4, the first and the third term inside the integrals do not contribute anything because of the orthogonality of cusp forms with Eisenstein series. Notice that by omitting the first and third term, I_4 is no longer absolutely convergent. However it is still defined as an integral of the form

$$\int_0^1 \int_{GL_{n-1}(F)\backslash GL_{n-1}^1(\mathbb{A})}$$

and this will be enough for us. Because of the previous reasons I has the same poles and residues with I_4. We have $I_4 =$

$$\int_0^1 a^{(n-1)(s-\frac{1}{2})} \int_{GL_{n-1}(F)\backslash GL_{n-1}^1(\mathbb{A})} \left[(\tilde{wE}) \begin{pmatrix} 1 & \\ & ag \end{pmatrix} - \tilde{E} \begin{pmatrix} ag & \\ & 1 \end{pmatrix} \right] E'(ag) \mathrm{d}g\mathrm{d}a.$$

We can write formally I_4 as the difference of

$$I_{4,1} = \int_0^1 a^{(n-1)(s-\frac{1}{2})} \int_{GL_{n-1}(F)\backslash GL_{n-1}^1(\mathbb{A})} (\tilde{wE}) \begin{pmatrix} 1 & \\ & ag \end{pmatrix} E'(ag) \mathrm{d}g\mathrm{d}a$$

and

$$I_{4,0} = \int_0^1 a^{(n-1)(s-\frac{1}{2})} \int_{GL_{n-1}(F)\backslash GL_{n-1}^1(A)} \tilde{E} \begin{pmatrix} ag & \\ & 1 \end{pmatrix} E'(ag) \mathrm{d}g\mathrm{d}a.$$

We choose this notation because $I_{4,1}$ will give a pole at $\mathrm{Re}(s) = 1$ and $I_{4,0}$ will give a pole at $\mathrm{Re}(s) = 0$.

Based on the calculations so far we know that I_4 is well defined as an iterated integral of the form

$$\int_0^1 \int_{GL_{n-1}(F)\backslash GL_{n-1}^1(\mathbb{A})}$$

for $\mathrm{Re}(s)$ big. However we do not know this yet for the two summands $I_{4,1}$ and $I_{4,0}$. For each one of these summands, we do a formal unfolding, which is based on the expansions of $\tilde{E}$ and $\tilde{\tilde{E}}$, and in an Iwasawa decomposition. Then it will be easier to see how the convergence issues are resolved.

$$I_{4,1} = \int_0^1 a^{(n-1)(s-\frac{1}{2})} \int_{GL_{n-1}(F)\backslash GL^1_{n-1}(\mathbb{A})} \sum_{\gamma \in P_{n-1}(F)\backslash GL_{n-1}(F)} (wf) \begin{pmatrix} 1 & \\ & a\gamma g \end{pmatrix} E'(ag)\mathrm{d}g\mathrm{d}a$$

$$= \int_0^1 a^{(n-1)(s-\frac{1}{2})} \int_{P_{n-1}(F)\backslash GL^1_{n-1}(\mathbb{A})} (wf) \begin{pmatrix} 1 & \\ & ag \end{pmatrix} E'(ag)\mathrm{d}g\mathrm{d}a$$

$$= \int_0^1 a^{(n-1)(s-\frac{1}{2})} \int_{GL_{n-2}\times GL_1(F)N_{n-1}(\mathbb{A})\backslash GL^1_{n-1}(\mathbb{A})} (wf) \begin{pmatrix} 1 & \\ & ag \end{pmatrix} f'(ag)\mathrm{d}g\mathrm{d}a.$$

Similarly we have $I_{4,0} =$

$$\int_0^1 a^{(n-1)(s-\frac{1}{2})} \int_{GL_1\times GL_{n-2}(F)N^a_{n-1}(\mathbb{A})\backslash GL^1_{n-1}(\mathbb{A})} (Mf) \begin{pmatrix} ag & \\ & 1 \end{pmatrix} Mf'(ag)\mathrm{d}g\mathrm{d}a.$$

The inner integrals for $I_{4,1}$ and $I_{4,0}$ are decomposed further by using the Iwasawa decomposition of GL_{n-1}. As a result, the more $K_{n-1}-$invariance we have the better. For the GL_n functions E and wE, K_{n-1} is embeded in the upper left of GL_n. Because of the definition of the restricted tensor product we know that E and E' are $K_{\upsilon,n-1}-$invariant for almost all υ. In order to make the formula for the residue as simple as possible we will modify our problem and make two assumptions related to $K_{n-1}-$invariance.

Modification: We choose Eisenstein series E^i and $E^{i,\prime}$ respectively in $\mathrm{Ind}_{P_n(A)}^{GL_n(A)}(\pi \bigotimes \chi)$ and $\mathrm{Ind}_{P_{n-1}(A)}^{GL_{n-1}(A)}(\pi' \bigotimes \chi')$ for all i belonging in some finite set J. For each pair $(E^i, E^{i,\prime})$ we define integrals I^i_k for $k = 1, 2, 4, (4,1), (4,0)$ in the same way we defined I_k for (E, E'). We will call SI_k the sum $\sum_{i\in J} I^i_k$ (where S stands for the spherical choices in the next 2 assumtions). From now on instead of studying I we will study SI.

Assumtion A1: At all non-Archimedean places, π and π' are chosen to be unramified. For all i, and for all non-Archimedean υ, E^i and $E^{i,\prime}$ are respectively $K_{n,\upsilon}$ and $K_{n-1,\upsilon}$ invariant.

Assumtion A2: The sum

$$\sum_{i\in J} E^i \bigotimes E^{i,\prime}$$

is equal to

$$\bigotimes_{\upsilon} w_{\upsilon},$$

where for υ Archimedean, we have that w_{υ} is a K_{n-1} finite vector of $\pi_{\upsilon} \bigotimes \pi'_{\upsilon}$. Of course due to assumption A1 this is also true for υ non-Archimedean.

Here K_{n-1} embeds to $GL_n \times GL_{n-1}(F_{\upsilon})$ by

$$k \to \left(\begin{pmatrix} k & \\ & 1 \end{pmatrix}, k \right).$$

We return to the study of $SI_{4,1}$ and $SI_{4,0}$.

$$SI_{4,1} =$$

$$\sum_{i \in J} \int_0^1 a^{(n-1)(s-\frac{1}{2})} \int_{GL_{n-2} \times GL_1(F) N_{n-1}(A) \backslash GL^1_{n-1}(A)}$$

$$(wf^i) \begin{pmatrix} 1 & \\ & ag \end{pmatrix} f^{i\prime}(ag) \mathrm{d}g \mathrm{d}a.$$

By using the Iwasawa decomposition and the two assumtions we get : $SI_{4,1} =$

$$\sum_{i \in J} \int_0^1 a^{(n-1)(s-\frac{1}{2})} \int_{GL_{n-2}(F) \backslash GL_{n-2}(\mathbb{A})} wf^i \begin{pmatrix} 1 & & \\ & ah & \\ & & a \det(h)^{-1} \end{pmatrix}$$

$$f^{i,\prime} \begin{pmatrix} ah & \\ & a \det h^{-1} \end{pmatrix} |\det(ah)|^{-1} (a|\det(h)|^{-1})^{n-2} \mathrm{d}h \mathrm{d}a.$$

By changing variables $h \to a^{-1}h$ we obtain : $SI_{4,1} =$

$$\sum_{i \in J} \int_0^1 a^{(n-1)(s-\frac{1}{2})} \int_{GL_{n-2}(F) \backslash GL_{n-2}(\mathbb{A})} wf^i \begin{pmatrix} 1 & & \\ & h & \\ & & a^{n-1} \det(h)^{-1} \end{pmatrix}$$

$$f^{i,\prime}\begin{pmatrix} h & & \\ & & \\ & & a^{n-1}\det h^{-1}\end{pmatrix}|\det(h)|^{-1}(a^{n-1}|\det(h)|^{-1})^{n-2}\mathrm{d}h\mathrm{d}a.$$

Because of the definition of f^i and $f^{i,\prime}$ as induced functions we have

$$wf^i\begin{pmatrix} 1 & & \\ & h & \\ & & a^{n-1}\det(h)^{-1}\end{pmatrix} = |\det(h)|^{\frac{1}{2}}(a^{n-1}|\det(h)|^{-1})^{\frac{-n+1}{2}}$$

$$\chi(a^{n-1}\det(h^{-1}))\varphi^i\begin{pmatrix} 1 & \\ & h\end{pmatrix}$$

for some cusp form φ^i in π, and

$$f^{i,\prime}\begin{pmatrix} h & & \\ & & \\ & & a^{n-1}\det h^{-1}\end{pmatrix} = |\det(h)|^{\frac{1}{2}}(a^{n-1}|\det(h^{-1})|)^{\frac{-n+2}{2}}\chi'(a^{n-1}\det(h^{-1}))\varphi_1^{i,\prime}(h)$$

for some cusp form $\varphi_1^{i,\prime}$ in π'. Putting all this together we get

$$SI_{4,1}(g) = \sum_{i\in J}\int_0^1 a^{(n-1)(s-1)}\chi\chi'(a^{n-1})\mathrm{d}a\int_{GL_{n-2}(F)\backslash GL_{n-2}(\mathbb{A})}$$

$$\varphi^i\begin{pmatrix} 1 & \\ & h\end{pmatrix}\varphi_1^{i,\prime}(h)\chi\chi'(\det(h^{-1}))|\det(h)|^{\frac{1}{2}}\mathrm{d}h.$$

We see that we get a pole if and only if

$$\chi\chi'(a) = |a|^t \qquad \text{for some } t \text{ (which has to be imaginary).}$$

The pole we get is at $s = 1-t$. The residue is

$$Res(SI_{4,1})|_{s=1-t} = \sum_{i\in J}\int_{GL_{n-2}(F)\backslash GL_{n-2}(\mathbb{A})}\varphi^i\begin{pmatrix} 1 & \\ & h\end{pmatrix}\varphi_1^{i,\prime}(h)|\det(h)|^{\frac{1}{2}-t}\mathrm{d}h.$$

This is a $GL_{n-1}\times GL_{n-2}$ Rankin-Selberg integral. In order to make it have the most common form

we can choose an element w_1 in the Weyl group of GL_{n-1} so that for the cusp form φ_1^i, defined by

$$\varphi_1^i = w_1 \varphi^i,$$

we have

$$\varphi_1^i \begin{pmatrix} h & \\ & 1 \end{pmatrix} = \varphi^i \begin{pmatrix} 1 & \\ & h \end{pmatrix}.$$

The computation for $SI_{4,0}$ is identical . I only mention the final result :

$$SI_{4,0} = \sum_{i \in J} \int_0^1 a^{(n-1)s} \chi\chi'(a^{n-1}) \mathrm{d}a \int_{GL_{n-2}(F)\backslash GL_{n-2}(\mathbb{A})}$$

$$\varphi_0^i \begin{pmatrix} h & \\ & 1 \end{pmatrix} \varphi_0^{i,\prime}(h) \chi\chi'(\det(h)^{-1}) |\det(h)|^{\frac{-1}{2}-t} \mathrm{d}h.$$

The cusp forms φ_0^i and $\varphi_0^{i,\prime}$ are explicitly described when we will state the Proposition.

The last integral converges absolutely for $\mathrm{Re}(s) > 0$ it has meromorphic continuation to the rest of the complex plane and for $\mathrm{Re}(s) = 0$, it has a pole when $\chi\chi'(a) = |a|^t$ for some t with residue again a $GL_{n-1} \times GL_{n-2}$ Rankin-Selberg integral .

We remind that when $I_{4,1}$ and $I_{4,0}$ where defined, It was not clear that they existed as integrals, and the unfolding that followed was formal. However the final expression that we obtained for $SI_{4,1}$ and $SI_{4,0}$ is a well defined integral for $\mathrm{Re}(s) > 1$. This makes rigorous the definition and the unfolding that followed.

To sum up we have proved the following:

Proposition 2.4 (Second Description of the residues): *The integral SI as it was defined in the Modification, has the following properties :*

- *It is absolutely convergent for Re$(s) > 1$.*
- *It can be meromorphically continued to the rest of the complex plane.*
- *it is entire except when $\chi\chi' = ||^t$ for some t. In this case the possible poles are at $s = i - t$ and $s = -t$.*
- *Let assumtions A1 and A2 be true. Then the residues of the possible poles at $l - t$ for $l = 1, 0$,*

are the $GL_{n-1} \times GL_{n-2}$ integrals

$$\sum_{i \in J} \int_{GL_{n-2}(F)\backslash GL_{n-2}(A)} \varphi_l^i \begin{pmatrix} h & \\ & 1 \end{pmatrix} \varphi^{i,\prime} |\det(h)|^{l-\frac{1}{2}-t} \mathrm{d}h,$$

where

$$\varphi_1^i(g) = |\det(g)|^{-\frac{1}{2}} w_1 w f^i \begin{pmatrix} g & \\ & 1 \end{pmatrix} \qquad \varphi_0^i(g) = |\det(g)|^{\frac{1}{2}} M f^i \begin{pmatrix} 1 & \\ & g \end{pmatrix}$$

$$\varphi_1^{i,\prime}(g) = |\det(g)|^{-\frac{1}{2}} M f^{i,\prime} \begin{pmatrix} g & \\ & 1 \end{pmatrix} \qquad \varphi_0^{i,\prime}(g) = |\det(g)|^{\frac{1}{2}} M f^{i,\prime} \begin{pmatrix} 1 & \\ & g \end{pmatrix}.$$

2.5 Potential applications

2.5.1 Rankin-Selberg towers and the Langlands-Shahidi method

In the papers [GR1] and [G-H], the following phenomenon is discussed. Let $I(\phi)$ be an integral of automorphic forms so that ϕ is one of them. For ϕ a not rapidly decreasing automorphic form the integral may not converge, but we ignore this issue here. We assume that for ϕ a cusp form we have that $I(\phi)$ is equal to an $L-$function $L(\phi)$. We do not use a more standard notation for $L(\phi)$ because it may depend in other automorphic forms among the ones that occur in $I(\phi)$. We now replace ϕ by an Eisenstein series E. The equality $I(E) = L(E)$ may or may not be true. $L(E)$ is clearly a product of $L-$functions. Let L_1 be the one among them of biggest dimension. Then it turns out that frequently it is possible to unfold $I(E)$ in a way that an integral representation of L_1 will occur as an inner integral in a partially unfolded form of $I(E)$. This observation is part of the intuition for the concept of Rankin-Selberg towers.

Our preceding calculations of the residues give an example of how this heuristic is checked rigorously in the special case that:

- For I we consider the $GL_n \times GL_{n-1}$ Rankin-Selberg integral.
- The automorphic form ϕ is defined on $GL_n \times GL_{n-1}$.
- $E = (E_n, E_{n-1})$, so that each among the E_i is induced from a $GL_{i-1}-$cusp form and a Hecke character.

Of course the automorphic integrals in the papers [GR1] and [G-H], are more sophisticated from the ones that we considered. Making the heuristics rigorous in these cases will be harder.

Another point to pay attention to, is the cancellation that happened in the first calculation of the residues due to the Langlands-Shahidi factors of both Eisenstein series. By extending other Ranking-Selberg integrals to Eisenstein series, and by studying their residues, we may improve our understanding of why it is hard to find automorphic integrals for $L-$functions that do not have a Langlands-Shahidi integral.

2.5.2 Special values of $L-$functions

By comparing the two calculations for the residues we obtain identities at the Archimedean places. In particular for the Residue at $\mathrm{Re}(s)=1$ we have:

$$\prod_{v\in S_\infty}\sum_{i_v\in J_v} Z(W_v^{i_v},W_v^{i_v,\prime},1-t)=c\prod_{v\in S_\infty}\sum_{i_v\in J_v} Z(W_{1,v}^{i_v},W_{1,v}^{i_v,\prime},1-t), \tag{2.5.1}$$

where

$$c=\frac{L_\infty(\pi\times\chi^{-1},1)L_\infty(\pi'\times{\chi'}^{-1},1)}{\mathrm{Res}|_{s=1}\zeta_F(s)}.$$

Here the Whittaker models are normalized as at the end of Chapter 1, and ζ_F is the (not complete) Dedekind zeta function.

We did the calculations for the residues for t an imaginary number, because we wanted our characters to be unitary. However the proof is the same for nonunitary characters. As a result the local identities remain true for all complex numbers. Of course this also follows from the uniqueness of analytic continuation.

One type of application for the Rankin-Selberg integral representations, is getting special values theorems for $L-$functions. In such applications a common obstacle is lack of understanding of an Archimedean Rankin-Selberg integral. Two of the works on this topic are [M], and [KMS]. In these two works theorems for special values (and more precisely , critical values) are established , up to an assumption that certain Archimedean Rankin-Selberg integrals related to cohomology are nonzero. Our initial motivation for the study of the extended global $GL_n\times GL_{n-1}$ Rankin-Selberg integral was to use local identities such as 2.5.1 to prove the nonvanishing of these Archimedean Rankin-Selberg integrals. However this nonvanishing was recently proved by other methods [B], and now I have no plans of working in this direction.

I am open that the local identities that our method produces, can still have applications. However

even if the local methods remain the best way to understand Archimedean Rankin-Selberg integrals, it must be good for our intuition to know that some nontrivial results can be obtained by a global method such as the one that we used.

Chapter 3

Connections with functoriality

3.1 Introduction

One of the tools for establishing cases of functoriality is the converse theorems. These theorems very roughly say that in order to establish that a representation π on $GL_n(\mathbb{A})$ is automorphic, it is enough to prove that for appropriate twists by automorphic representations τ (of GL_m with $m < n$), $L(\pi \times \tau, s)$ is entire (plus other analytic properties) and has the expected functional equation.

Assume that we knew these properties, when they are true (in some cases there are poles), for all $L-$ functions of the form

$$L(\pi_1 \times \pi_2 \times \pi_3, s)$$

where π_i can be any cuspidal representation of GL_{n_i} for any n_i . These $L-$functions are called triple products. We fix π_1 and π_2, and we let π_3 vary. By using the local Langlands conjecture we get at each place υ a representation $\pi_{1,\upsilon} \times \pi_{2,\upsilon}$ of $GL_{n_1 n_2}(F_\upsilon)$. Their restricted tensor product is a representation of $GL_{n_1 n_2}(\mathbb{A})$. The $L-$function of the representation on $GL_{n_1 n_2}(\mathbb{A})$ twisted by automorphic representations π_3 is a triple product $L-$function. by using a converse theorem we get a part of functoriality that roughly says that automorphic forms of general linear groups can be multiplied! That of course, would lead to many other important conjectures.

Among the automorphic lifts of the form

$$GL_n \times GL_m \to GL_{nm}$$

the ones that are established are the ones for which the previous plan worked. They are:

$$GL_2 \times GL_2 \to GL_4 \tag{3.1.1}$$

and

$$GL_3 \times GL_2 \to GL_6. \tag{3.1.2}$$

Very briefly, the reason for having only these two lifts among the automorphic products, is that enough analytic properties are known only for triple product $L-$functions for which the two smallest among n_1, n_2, n_3 are bounded be 3.

The way that has been the most common so far for establishing analytic properties and the functional equation for automorphic $L-$functions is to find an integral representation for the $L-$function. The Langlands-Shahidi integrals are the ones that have mostly been used for automorphic lifts. For the lift (3.1.1), Rankin-Selberg integral representations were used [R]. The lift (3.1.2) required knowledge for $L-$functions which were obtained by Langlands-Shahidi integrals but not Rankin-Selberg integrals. So the Langlands-Shahidi method was used for (3.1.2) [KSh]. However since then more Rankin-Selberg integrals have been found. Among them are $GL_3 \times GL_3 \times GL_2$ and $E_6 \times GL_2$ [G-H] which are related to what was needed for the lift (3.1.2).

The heuristics related to Rankin-Selberg towers suggest that if we find an integral representation for an $L-$function that is big in a certain sense then information for other $L-$functions will follow. For example, if finding an integral representation for triple products remains open after hard attempts, then we can search for integral representations of any $L-$ function $L(\pi, r, s)$ in which each Langlands parameter of the conjectured lift $r(\pi)$ is a product of at least three of the the Langlands parameters of π.

However even if such automorphic integrals will be found, before obtaining enough information for triple product $L-$functions, it is likely that substantial obstacles will occur such as:

- a product of $L-$functions has meromorphic continuation and functional equation, but there is no way to say each factor has these properties
- an $L-$function being the fraction of automorphic Integrals . As a result we get meromorphic continuation and functional equation for the $L-$ function but understanding where the poles are may be very difficult.

The second problem reminds the case with Artin $L-$functions where by using Brauer's theorem we can express them as fractions of products of Hecke-$L-$functions. As a result we get meromorphic

continuation and functional equation but proving that they are entire has not been achieved yet.

In the next section we give an identity of automorphic integrals in which we believe that a big (in the previous sense) $L-$function occurs in some way. However, to the extent that we study this identity, we do not obtain any property of any $L-$function.

3.2 Generalizing the Rankin-Selberg integral for the exterior cube of GL_6

We start with some notation relevant to this section.

Let J be the $n \times n$ matrix with 1 in the antidiagonal entries and 0 in all the other entries. Let

$$\epsilon = \begin{pmatrix} & J \\ -J & \end{pmatrix}.$$

This is the the skew symmetric matrix that we choose, for defining GSp_{2n} :

$$H = GSp_{2n} = \{g \in GL_{2n} : {}^t g \epsilon g = \mu(g)\epsilon\}.$$

We recall that parabolics, Levi subgroups and root systems mean for us the standard ones unless otherwise specified. We denote by Q_H the maximal parabolic subgroup of H with $\Delta_{Q_H} = \{2e_n\}$. This is called the Siegel parabolic. The maximal parabolic of GL_{2n} above Q_H is denoted by Q. R^n denotes the unipotent radical of Q, and R_H^n denotes the unipotent radical of Q_H. The group R_H^n consists of the matrices of the form

$$\begin{pmatrix} I_n & X \\ & I_n \end{pmatrix},$$

where X is any symmetric for the antidiagonal matrix. We similarly define $R_H^{n,c}$ as the unipotent group of matrices of the form

$$\begin{pmatrix} I_n & X \\ & I_n \end{pmatrix},$$

where now X is any skew symmetric for the antidiagonal matrix. Notice that $R_H^{n,c}$ is the unipotent radical for the Siegel parabolic in a split orthogonal group. We have $R^n = R_H^n + R_H^{n,c}$. On the group

$R(F) \setminus R^n(\mathbb{A})$, we define the following characters:

$$\psi_{R^n,\frac{l}{2}}(u) = \psi_F\left(\sum_{1\leq i\leq \frac{l}{2}} (u_{n-l+i,n+i} - u_{n-i+1,n+l-i+1})\right)$$

where l is even and $2 \leq l \leq n$. Among these characters $\psi_{R^n,1}$ has a special role and for this reason it will also be denoted by ψ_{R^n}.

The Levi of Q_H is the copy of $GL_1 \times GL_n$ defined by

$$(a, g) \to \begin{pmatrix} ag & \\ & J\,{}^t g^{-1} J \end{pmatrix}.$$

We will define two sequences of reductive groups in the next two paragraphs, and then we will use these definitions to define some sequences of unipotent groups. After the definition of the sequences of unipotent groups, the sequences of reductive groups will not be used again.

Let

$$M_H^n = GL_1 \times GL_n$$

$$M_H^{n-1} = GL_1 \times GL_{n-2} \times GL_2.$$

For l even with $1 \leq l \leq n$ let

$$M_{H,\frac{l}{2}}^{n-1} = GL_1 \times GL_{n-l} \times GL_l.$$

Notice that $M_H^{n-1} = M_{H,\frac{1}{2}}^{n-1}$. For $k = n-2, n-3, ..., 2$ let

$$M_H^k = GL_1 \times GL_{k-1} \times GL_1^{n-k-1} \times GL_2$$

and finally let

$$M_H^1 = GL_1^{n+1}.$$

For

$$GL_1 \times GL_{t_1} \times ... \times GL_{t_r}$$

being any among the $M^k_{H,l}$, M^k_H that we defined, we embed it into H by

$$(a, g_1, ..., g_r) \to \begin{pmatrix} ag_1 & & & & & & \\ & \ddots & & & & & \\ & & ag_r & & & & \\ & & & J_{t_r}\,{}^t g_r{}^{-1} J_{t_r} & & & \\ & & & & \ddots & & \\ & & & & & J_{t_1}\,{}^t g_1{}^{-1} J_{t_1} \end{pmatrix}.$$

Notice that for this embedding the groups $M^k_{H,l}$ and M^k_H become the (as always standard) Levi's of $H-$parabolics. For example M^1_H becomes the Levi for the Borel subgroup.

To describe the other sequence, we use the notation:

$$[G_1 \times G_2 \times ... \times G_r] = G_1 \times G_2 \times ... \times G_r \times G_r \times G_{r-1} \times ... \times G_1.$$

Let

$$M^n = [GL_n]$$

and

$$M^{n-1} = [GL_{n-2} \times GL_2].$$

For $k = n-2, n-3, ..., 2$ let

$$M^k = [GL_{k-1} \times GL_1^{n-k-1} \times GL_2]$$

and finally let

$$M^1 = [GL_1^n].$$

Here we use the usual embedding into GL_{2n}

$$A_1 \times A_2, ... \to \begin{pmatrix} A_1 & & \\ & A_2 & \\ & & \ddots \end{pmatrix}.$$

We see that each M^k becomes the Levi for the minimal $GL_{2n}-$parabolic containing M_H^k.

For $k = n-1, ..., 1$ we define the following sequences, of unipotent groups:

- Let R_H^k be the unipotent radical of the $H-$parabolic with Levi M_H^k. Notice that $R_H^1 = U_H$.
- Let L_H^k be the unipotent radical of the $M_H^{k+1}-$parabolic with Levi M_H^k. Notice that $R_H^k = L_H^k R_H^{k+1}$.

  Also for l even and $1 \leq l \leq n$ let

  $$L_{H\frac{l}{2}}^{n-1}$$

  be the unipotent radical of the M_H^n-parabolic that has Levi $M_{H,\frac{l}{2}}^{n-1}$.
- Let R^k be the unipotent radical of the $GL_{2n}-$parabolic with Levi M^k. We also define for $k = 1, 2$

  $$R_{-1}^k = \{u \in R^k : u_{n,n+1} = 0\}.$$

  The group R^1 will not be used in the unfolding.
- Let L^k be unipotent radical of the $M^{k+1}-$parabolic subgroup that has Levi M^k. Notice that $R^k = L^k R^{k+1}$.
- Let $L_H^{k,c}$ be a subgroup of L^k such that $L^k = L_H^{k,c} \times L_H^k$. Any choice works, we will only fix a choice for $k = 1$. We define as $L_H^{1,c}$ the group of unipotent matrices for which the only nonzero, nondiagonal entry is $u_{n-1,n}$.

Finally we define L^0 to be the group of unipotent matrices whose only nonzero nondiagonal entry is $u_{n+1,n}$. Notice that $L^0 R_{-1}^1$ is the unipotent radical of the nonstandard Borel B^w, for w being

$$\begin{pmatrix} I_{n-2} & & & \\ & 0 & 1 & \\ & 1 & 0 & \\ & & & I_{n-2} \end{pmatrix}.$$

For the previous unipotent groups some characters will be used.

On $L^{n-1}(F) \setminus L^{n-1}(\mathbb{A})$, and $R^{n-1}(F) \setminus R^{n-1}(\mathbb{A})$ we define:

$$\psi_{L^{n-1},1}(n) = \psi_F(n_{n-2,n-1})$$

$$\psi_{L^{n-1},2}(n) = \psi_F(n_{n-3,n-1} - n_{n-2,n})$$

$$\psi_{R^{n-1},1} = \psi_{L^{n-1},1}\psi_{R^n}$$

$$\psi_{R^{n-1},2} = \psi_{L^{n-1},2}\psi_{R^n}.$$

On $L^k(F) \setminus L^k(\mathbb{A})$, and $R^k(F) \setminus R^k(\mathbb{A})$, when $k = n-1, n-2, ...2$, we define:

$$\psi_{L^k}(u) = \psi_F(u_{k-1,k}),$$

$$\psi_{R^k} = \psi_{L^k}\psi_{R^{k+1}}.$$

Notice that there is an overlapping in the definitions ($\psi_{L^{n-1},1} = \psi_{L^{n-1}}$ and $\psi_{R^{n-1},1} = \psi_{R^{n-1}}$). Both notations will be used.

On $L^1(F) \setminus L^1(\mathbb{A})$ and $R^1_{-1}(F) \setminus R^1_{-1}(\mathbb{A})$ we define

$$\psi_{L^1}(u) = \psi_F(u_{n-1,n})$$

$$\psi_{R^1_{-1}} = \psi_{L^1}\psi_{R^2}|_{R^2_{-1}}.$$

We now introduce the data that defines the Rankin-Selberg integral. Let π be a cuspidal representation, and φ a cusp form of this representation. Let ω_π be the central character of π. For n even, we assume that ω_π is the square of a character of $F^* \setminus \mathbb{A}^*$. We fix one of these characters and we call it $\omega_\pi^{\frac{1}{2}}$. Let χ be a unitary character of $F^* \setminus \mathbb{A}^*$. We define a $GL_1 \times GL_n$ character as follows:

$$\chi_\pi(a,g) = \begin{cases} \omega_\pi^{\frac{n-1}{2}}(a)\omega_\pi(g)\chi^n(a)\chi^2(g) & \text{if } n \text{ odd} \\ \omega_\pi^{\frac{n-1}{2}}(a)\omega_\pi(g)\chi^{\frac{n}{2}}(a)\chi(g) & \text{if } n \text{ even} \end{cases}.$$

We defined χ_π so that the integrand of the Rankin-Selberg integral we are about to study is $Z(\mathbb{A})-$invariant.

For any $s \in \mathbb{C}$ we define

$$I(s,\chi) = Ind_{Q_H(A)}^{H(A)}(\delta_{Q_H}^s \chi_\pi) \qquad \text{(un-normalized induction)}.$$

We choose a section $f_s \in I(s,\chi)$ ($K-$finite as always) and from it we define the Siegel Eisenstein series :

$$E_s(g) = \sum_{\gamma \in Q_H(F)\backslash H(F)} f_s(\gamma g) \qquad \text{for } Re(s) > 1.$$

We recall that the right hand side converges absolutely for $Re(s) > 1$, and can be meromorphically continued to all $s \in \mathbb{C}$.

Let

$$I(\varphi, f_s, \chi, s) = \int_{Z(\mathbb{A})H(F)\backslash H(\mathbb{A})} \varphi(g) E_s(g) dg.$$

This integral converges absolutely for all $s \in \mathbb{C}$. The reason is that restriction of a $GL_{2n}-$automorphic function of rapid decay to H, gives a $H-$automorphic function of rapid decay ([AGR], page 713. there the Proposition they prove is weaker from the claim we made, but the proof is identical).

Proposition 3.2: *Let $n \geq 4$ and π, φ, E_s defined as previously. Let*

$$P_s = \int_{Z(\mathbb{A})U_H(\mathbb{A})\backslash H(\mathbb{A})} \int_{\mathbb{A}} W_\varphi(wr(u)g) du f_s(g) dg,$$

$$B_{1,s} = \sum_{\substack{2<l\leq n \\ l \ even}} \int_{Z(\mathbb{A})GL_{n-l}\times GSp_l(F) L^{n-1}_{H,\frac{l}{2}}(F) R^n(\mathbb{A})\backslash H(\mathbb{A})} \varphi_{R^n,\psi_{R^n,\frac{l}{2}}}(ng) dn f_s(g) dg,$$

$$B_{2,s} = \int_{Z(\mathbb{A})GL_{n-4}\times GL_2(F) L^{n-2}_{H,2}(F) R^{n-1}_H(\mathbb{A})\backslash H(\mathbb{A})} \varphi_{R^{n-1},\psi_{R^{n-1},2}}(g) f_s(g) dg.$$

Then P, B_1, and B_2 are absolutely convergent for $Re(s) > 1$ and

$$I(\varphi, f_s, \chi, s) = P_s + B_{1,s} + B_{2,s}.$$

Two comments before the proof

1) There are convergence questions that will occur in every use of Fubini's theorem which are roughtly resolved by the rapid decay of the cusp form. We prove here a lemma that will make all the later uses of Fubini's theorem rigorous, so that afterwards we can concentrate better in the formal aspects of the unfolding.

The lemma for decay that we prove here requires a subset of the thoughts that are needed for studying the decay of any automorphic function. In the adelic setting such a study is made in [M-W] in I.2.10. We follow their notation.

Lemma: *Let ϕ be an automorphic function on a reductive group G (We remind we always mean K-finite). Let $P = MN$ be a maximal $G-$parabolic. Then the sum over the nontrivial characters ψ of N*

$$\mathbb{A} = \sum_{\psi} |\phi_{N,\psi}(g)|$$

is bounded by a finite sum of the form

$$\sum_{k} c_k |D^k * \phi(g)|.$$

Proof. : We can assume that g belongs in a Siegel set S. By using $K-$finiteness we replace the Fourier coefficients for $F \setminus \mathbb{A}$ by Fourier coefficients for $\mathbb{Z} \setminus \mathbb{R}$ as follows :

Let K_f be a open compact subgroup of $G(\mathbb{A}_f)$ for which ϕ is invariant. Define

$$\Gamma = N(\mathbb{A}_f) \cap_{g \in S} gK_f g^{-1}.$$

Then there is a lattice L of $\mathbb{A}_\infty$ that makes $L \setminus N(\mathbb{A}_\infty)$ a finite index subgroup of

$$N(F) \setminus N(\mathbb{A})/\Gamma.$$

We choose a basis X_i for $Lie(N(\mathbb{A}_\infty))$, so that we have the isomorphism

$$\mathbb{R}^d \to L \setminus N(\mathbb{A}_\infty)$$

$$(x_l) \to \exp(\sum x_l X_l).$$

Using this isomorphism the restriction of a character ψ to L takes the form

$$\mathbb{R}^d \to \mathbb{C}$$

$$(x_l) \to \exp(2\pi i \sum x_l \xi_l)$$

and the number of characters taking this form is bounded by the finite index

$$|(L \setminus N(A_\infty)) \setminus (N(F) \setminus N(A)/\Gamma)|.$$

As a result we only need to give a bound for sums of the form :

$$A' = \sum_{\xi} \left| \int_{\mathbb{Z}^d \backslash \mathbb{R}^d} \phi(\exp(\sum_l x_l X_l) g) \exp(2\pi i \sum_l x_l \xi_l) \prod dx_l \right|$$

where $\xi = (\xi_l)$ takes values in $\mathbb{Z}^d - \{(0,0,...,0)\}$.

For each ξ we choose $l(\xi)$ to be one among the l for which ξ_l is of maximal absolute value. In the sum A', for each ξ, we move $X_l(\xi)$ to the right, and we apply integration by parts two times. We get that each summand of A' takes the form:

$$(2\pi|\xi_{l(\xi)}|)^{-2} \left| \int_{\mathbb{Z}^d \backslash \mathbb{R}^d} \frac{\partial^2 \phi}{\partial^2 x_{l(\xi)}} (\exp(\sum_{l \neq l(\xi)} x_l X_l) g \exp(x_{l(\xi)} X_{l(\xi)}^{g_\infty^{-1}})) \exp(2\pi i \sum_l x_l \xi_l) \prod dx_l \right|.$$

The lemma follows from the last expression and from the two following facts:

- $\sum l(\xi)^{-2}$ is finite.
- In the expression

  $$X_{l(\xi)}^{g_\infty^{-1}} = \sum d_l X_l$$

  the d_l are bounded for $g \in S$. Even the following stronger statement is true :

  $$d_l \leq c m_B(g)^{-a}$$

  where c is a constant and a the simple root for which $\Delta_P = a$. However we formulated a lemma only for our purposes which are restricted to rapidly decaying automorphic forms.

□

2) In the first step of the unfolding we will use the expression of an Eisenstein series as as an averaging of an induced function. In each of the next steps a Fourier expansion of some Fourier coefficient of the cusp form will be used in a similar way. Both types of expansions naturally take the form

$$\sum_{\psi \in M} \sum_{P_\psi \backslash G_\psi}$$

where M is a finite set (consisting of one element in all cases except steps 2 and 3) and P_ψ, G_ψ are groups. These expresions lead to calculations of the form

$$\int_{G_\psi \backslash G'} \sum_{P_\psi \backslash G_\psi} = \int_{P_\psi \backslash G'}.$$

Then depending on the nature of the unipotent radical the unfolding continues well or becomes problematic. The cases that it becomes problematic occur in steps 2 and 3, and lead to the terms $B_{1,s}$ and $B_{2,s}$.

Proof. STEP 1 We unfold the Eisenstein series the standard way. We can do this when $Re(s) > 1$. We have

$$I(\varphi, f_s, \chi, s) = \int_{Z(\mathbb{A})Q_H(F)\backslash H(\mathbb{A})} \varphi(g) f_s(g) dg$$

$$= \int_{Z(\mathbb{A})GL_1 \times GL_n(F)R_H^n(\mathbb{A})\backslash H(\mathbb{A})} \int_{R_H^n(F)\backslash R_H^n(\mathbb{A})} \varphi(rg) dr f_s(g) dg$$

$$= \int_{Z(\mathbb{A})GL_1 \times GL_n(F)R_H^n(\mathbb{A})\backslash H(\mathbb{A})} \varphi_{R_H^n}(g) f_s(g) dg.$$

STEP 2 We consider a Fourier expansion of $\varphi_{R_H^n}$ along $R_H^{n,c}(F) \setminus R_H^{n,c}(\mathbb{A})$. By extending the characters of this Fourier expansion trivially to $R^n(F) \setminus R^n(\mathbb{A})$ we get

$$\varphi_{R_H^n} = \sum_{\psi} \varphi_{R^n,\psi}.$$

The last sum is an expansion for $\varphi_{R_H^n}$ over the characters ψ of $R^n(\mathbb{A})$ that are trivial on $R^n(F)R_H^n(\mathbb{A})$. The terms of this expansion are permuted by the Levi of Q_H. The action of the Levi has several orbits. The characters in these Fourier expansions correspond to $n \times n$ skew symmetric matrices and as a result their orbit depends only on their rank. So the orbits are finitely many. For J being one of these orbits let J_φ be the part of the expansion of φ_{R^n} that contains the summands of the orbit J. So we have $\varphi_{R_H^n} = \sum_J J_\varphi$.

If J is the rank 0 orbit, then $J_\varphi = \varphi_R^n = 0$. From these facts we get

$$I(\varphi, f_s, \chi, s) = \sum_J \int_{Z(\mathbb{A})GL_1 \times GL_n(F)R_H^n(\mathbb{A})\backslash H(\mathbb{A})} \sum_{\psi \in J} \varphi_{R^n,\psi}(g) f_s(g) dg.$$

The first sum is over all nontrivial orbits. For every even l with $2 \leq l \leq n$ we defined earlier the characters given by :

$$\psi_{R^n,\frac{l}{2}}(u) = \psi_F \left(\sum_{1 \leq i \leq \frac{l}{2}} (u_{n-l+i,n+i} - u_{n-i+1,n+l-i+1}) \right)$$

for $u = (u_{ij})$ being an element in R^n. These characters form a set of representatives for the orbits. The stabilizer of $\psi_{R^n,\frac{l}{2}}$ is the copy of $GL_{n-l} \times GSp_l(F)L_{H,\frac{l}{2}}^{n-1}(F)$ determined by the following

embedding:

$$\text{For } (h,g)\in GL_{n-l}\times GSp_l \qquad (h,g)\to \begin{pmatrix} \mu(g)^{-1}h & 0 & 0 & 0 \\ 0 & \mu(g)^{-1}g & 0 & 0 \\ 0 & 0 & J_l\,{}^t g^{-1} J_l & 0 \\ 0 & 0 & 0 & J_{n-l}\,{}^t h^{-1} J_{n-l} \end{pmatrix}.$$

As a consequence we get $I(\varphi, f_s, \chi, s) =$

$$\sum_{1\le k\le n,k \text{ even}} \int_{Z(\mathbb{A})GL_1\times GL_n(F)R_H^n(\mathbb{A})\backslash H(\mathbb{A})} \sum_{\gamma\epsilon GL_{n-k}\times GSp_k(F)L_H^{n-1}(F)\backslash GL_1\times GL_n(F)}$$

$$\varphi_{R^n,\psi_{R^n,\frac{k}{2}}}(\gamma g) f_s(g) dg.$$

In the first integral and second sum we use Fubini's theorem, and we get

$$\sum_{1\le l\le n,l \text{ even}} \int_{Z(\mathbb{A})GL_{n-l}\times GSp_l(F)L^{n-1}_{H,\frac{l}{2}}(F)R_H^n(\mathbb{A})\backslash H(\mathbb{A})} \varphi_{R^n,\psi_{R^n,\frac{l}{2}}}(g) f_s(g) dg.$$

Using Fubini's theorem once more we get

$$\sum_{1\le l\le n,l \text{ even}} \int_{Z(\mathbb{A})GL_{n-l}\times GSp_l(F)L^{n-1}_{H,\frac{l}{2}}(\mathbb{A})R(\mathbb{A})\backslash H(\mathbb{A})} \varphi_{L^{n-1}_{H,\frac{l}{2}}R^n,\psi_{R^n,\frac{l}{2}}}(ng) dn f_s(g) dg.$$

Now we will continue to study how the term for $l = 2$ unfolds. This term will be called I_1. Notice that the sum over the other l is $B_{1,s}$. We do not study $B_{1,s}$ further in this thesis. Because of this, $L^{n-1}_{H,1}$ will be called L^{n-1}_H, and $\psi_{R^n,1}$ will be called ψ_{R^n}

STEP 3 By considering a Fourier expansion of $\varphi_{L^{n-1}_H R^n,\psi_{R^n}}$ along $L^{n-1,c}_H(F)\setminus L^{n-1,c}_H(\mathbb{A})$ we get as in step 2:

$$\varphi_{L^{n-1}_H R^n,\psi_{R^n}}(g) = \sum_{\psi} \varphi_{R^{n-1},\psi\psi_{R^n}}(g),$$

where the sum is over all characters ψ of $L^{n-1}(\mathbb{A})$ that are trivial in $L^{n-1}(F)L^{n-1}_H(\mathbb{A})$.

The previous copy of $GL_{n-2}\times GL_2(F)$ acts on these characters and we get 3 orbits. As in the preceding step, the summand of the trivial orbit is zero because a constant term of the cusp form occurs as an inner integral. For the other two orbits we choose the representatives:

$$\psi_{L^{n-1},1}(n) = n_{n-2,n-1}$$

and

$$\psi_{L^{n-1},2}(n) = n_{n-3,n-1} - n_{n-2,n}.$$

Their stabilizers are respectively

$$GL_{n-3} \times GL_1 \times GL_1(F) L_{H,1}^{n-2} \times L_H^1(F)$$

and

$$GL_{n-4} \times GL_2(F) L_{H,2}^{n-2}.$$

Their Levi parts are given by the embeddings :

$$(h,a,b) \to \begin{pmatrix} abh & & & & & & & \\ & b & & & & & & \\ & & b & & & & & \\ & & & a & & & & \\ & & & & b & & & \\ & & & & & a & & \\ & & & & & & a & \\ & & & & & & & J^t h^{-1} J \end{pmatrix}$$

and

$$(h,g) \to \begin{pmatrix} det(g)^{-1}h & & & & & & & \\ & det(g)^{-1}g & & & & & & \\ & & det(g)^{-1}g & & & & & \\ & & & J^t g^{-1} J & & & & \\ & & & & J^t g^{-1} J & & & \\ & & & & & J^t h^{-1} J & & \end{pmatrix}.$$

We also recall the definition of the characters $\psi_{R^{n-1},1} = \psi_{L^{n-1},1}\psi_{R^n}$ and $\psi_{R^{n-1},2} = \psi_{L^{n-1},2}\psi_{R^n}$.

Now as in step one, I_1 is written as $I_2 + B_2$ where I_2 corresponds to the orbit of $\psi_{L^{n-1},1}$ and B_2 to the orbit of $\psi_{L^{n-1},2}$.

$$I_2 = \int_{Z(\mathbb{A})GL_{n-2}\times GL_2(F)R_H^{n-1}(\mathbb{A})\backslash H(\mathbb{A})} \sum_{\gamma \epsilon GL_{n-3}\times GL_1\times GL_1(F)L_{H,1}^{n-2}\times L_H^1(F)\backslash GL_{n-2}\times GL_2(F)}$$

$$\varphi_{R^{n-1},\psi_{R^{n-1},1}}(\gamma g) f_s(g) dg$$

$$= \int_{Z(\mathbb{A})GL_{n-3}\times GL_1\times GL_1(F)L^{n-2}_{H,1}\times L^1_H(F)R^{n-1}_H(\mathbb{A})\backslash H(\mathbb{A})} \varphi_{R^{n-1},\psi_{R^{n-1},1}}(g)f_s(g)dg$$

$$= \int_{Z(\mathbb{A})GL_{n-3}\times GL_1\times GL_1(F)L^1_H(F)R^{n-2}_H(\mathbb{A})\backslash H(\mathbb{A})} \varphi_{L^{n-2}_{H,1}R^{n-1},\psi_{R^{n-1},1}}(g)f_s(g)dg.$$

Similarly for the other orbit:

$$B_2 = \int_{Z(\mathbb{A})GL_{n-4}\times GL_2(F)L^{n-2}_{H,2}(F)R^{n-1}_H(\mathbb{A})\backslash H(\mathbb{A})} \varphi_{R^{n-1},\psi_{R^{n-1},2}}(g)f_s(g)dg.$$

We will not study further B_2, and for this reason $L^{n-2}_{H,1}$ will be called from now on L^{n-2}_H, and $\psi_{R^{n-1},1}$ will be called $\psi_{R^{n-1}}$

STEP 4 By considering a Fourier expansion of $\varphi_{L^{n-2}_H R^{n-1},\psi_{R^{n-1}}}$ along $L^{n-2,c}_H(F)\setminus L^{n-2,c}_H(\mathbb{A})$ we get as in the previous steps

$$\varphi_{L^{n-2}_H R^{n-1},\psi_{R^{n-1}}} = \sum_{\psi} \varphi_{R^{n-2},\psi\psi_{R^{n-1}}}.$$

The previous sum is over all characters of $L^{n-2}(\mathbb{A})$ that are trivial on $L^{n-2}(F)L^{n-2,c}_H(\mathbb{A})$. The previous copy of $GL_{n-3}\times GL_1\times GL_1(F)$ acts on them, and we get two orbits. For the trivial orbit we again have zero contribution and for the nontrivial we choose the representative

$$\psi_{L^{n-2}}(n) = n_{n-3,n-2}.$$

The stabilizer of $\psi_{L^{n-2}}$ is

$$GL_{n-4}\times GL_1\times GL_1(F)L^{n-3}_H(F).$$

It is embedded in $GL_{2n}(F)$ by

$$(h,a,b)\to\begin{pmatrix} abh & & & & & & & & \\ & b & & & & & & & \\ & & b & & & & & & \\ & & & b & & & & & \\ & & & & a & & & & \\ & & & & & b & & & \\ & & & & & & a & & \\ & & & & & & & a & \\ & & & & & & & & a \\ & & & & & & & & & J^t h^{-1} J \end{pmatrix}.$$

We recall that

$$\psi_{R^{n-2}}=\psi_{L^{n-2}}\psi_{R^{n-1}}.$$

After unfolding as in the previous steps we get

$$I_2=\int_{Z(\mathbb{A})GL_{n-4}\times GL_1\times GL_1(F)L_H^{n-3}\times L_H^1(F)R_H^{n-2}(\mathbb{A})\backslash H(\mathbb{A})}\varphi_{R^{n-2},\psi_{R^2}}(g)f_s(g)dg$$

$$=\int_{Z(\mathbb{A})GL_{n-4}\times GL_1\times GL_1(F)L_H^1(F)R_H^{n-3}(\mathbb{A})\backslash H(\mathbb{A})}\varphi_{L_H^{n-3}R^{n-2},\psi_{R^{n-2}}}(g)f_s(g)dg.$$

STEP 5, STEP 6, ... STEP n These steps are identical to step 4 (except that in step n the stabilizer has no unipotent part). After we do them we get:

$$I_2=\int_{Z(\mathbb{A})GL_1\times GL_1(F)L_H^1(F)R_H^2(\mathbb{A})\backslash H(\mathbb{A})}\varphi_{R^2,\psi_{R^2}}(g)f_s(g)dg$$

$$=\int_{Z(\mathbb{A})GL_1\times GL_1(F)(F)R_H^1(\mathbb{A})\backslash H(\mathbb{A})}\int_{L_H^1(F)\backslash L_H^1(\mathbb{A})}\varphi_{R^2,\psi_{R^2}}(ng)dnf_s(g)dg.$$

By uncovering the definition of $\varphi_{R^2,\psi_{R^2}}$ we get

$$I_2=\int_{Z(\mathbb{A})GL_1\times GL_1(F)R_H^1(\mathbb{A}\backslash H(\mathbb{A}))}\int_{L_H^1R^2(F)\backslash L_H^1R^2(\mathbb{A})}\varphi(ng)\psi_{R^2}^{-1}(n)dnf_s(g)dg.$$

STEP $n+1$ In this step we will consider a Fourier expansion of φ along $L_H^{1,c}(F)\setminus L_H^{1,c}(\mathbb{A})$. By identifying each element of $L_H^{1,c}(\mathbb{A})$ with its nonzero entry, we are fixing an isomorphism between

$L_H^{1,c}(\mathbb{A})$ and $\mathbb{A}$. In this Fourier expansion the characters are permuted by the additive group F instead of a reductive group. F acts simply transitively, and we choose the representative ψ_F (for the isomorphism we just fixed). We have

$$\int_{L_H^1 R^2(F)\backslash L_H^1 R^2(\mathbb{A})} \varphi(ng)\psi_{R^2}^{-1}(n)dn$$

$$= \sum_a \int_{L_H^1 R^2(F)\backslash L_H^1 R^2(\mathbb{A})} \int_{L_H^{1,c}(F)\backslash L_H^{1,c}(\mathbb{A})} \varphi(n^c ng)\psi_{R^2}^{-1}(n)\psi_F^{-1}(an^c)dn^c dn.$$

We recall that $R^2_{-1} := \{u \in R^2 : u_{n,n+1} = 0\}$. The preceding integral equals:

$$= \sum_a \int_{L_H^1 R^2_{-1}(F)\backslash L_H^1 R^2_{-1}(\mathbb{A})} \int_{L_H^{1,c}(F)\backslash L_H^{1,c}(\mathbb{A})} \int_{F\backslash\mathbb{A}} \varphi(n^c n_{-1} r(u)g)\psi_{R^2}^{-1}(n_{-1})\psi_{L^{1,c}}^{-1}(an^c)dudn^c dn_{-1}$$

Now notice that for every $a \in F$

$$\varphi(n^c n_{-1} r(u)g) = \varphi(r(a)n^c n_{-1} r(a)^{-1} r(a+u))$$

and

$$\psi_{R^2}(n_{-1})\psi_F(an^c) = \psi_{R^1_{-1}}(r(a)n^c n_{-1} r(a)^{-1}).$$

As a result, by changing variables $r(a)n^c n_{-1} r(a)^{-1} \to n$ the previous sum becomes

$$\sum_a \int_{R^1_{-1}(F)\backslash R^1_{-1}(\mathbb{A})} \int_{F\backslash\mathbb{A}} \varphi(nr(a+u)g)\psi_{R^1_{-1}}^{-1}(n)dudn$$

$$= \int_{R^1_{-1}(F)\backslash R^1_{-1}(\mathbb{A})} \int_{\mathbb{A}} \varphi(nr(u)g)\psi_{R^1_{-1}}^{-1}(n)dudn$$

$$= \int_{\mathbb{A}} \varphi_{R^1_{-1},\psi_{R^1_{-1}}}(r(u)g)du.$$

Now we come back to I_2. Substituting the above we find:

$$I_2 = \int_{Z(\mathbb{A})GL_1\times GL_1(F)R_H^1(\mathbb{A})} \int_{\mathbb{A}} \varphi_{R^1_{-1},\psi_{R^1_{-1}}}(r(u)g)duf_s(g)dg.$$

LAST STEP We consider a Fourier expansion of $\varphi_{R^1_{-1},\psi_{R^1_{-1}}}$ along $L^0(F)\setminus L^0(\mathbb{A})$:

$$I_2 = \int_{Z(\mathbb{A})GL_1\times GL_1(F)R_H^1(\mathbb{A})\backslash H(\mathbb{A})} \sum_\psi \int_{\mathbb{A}} \varphi_{L^0 R^1_{-1},\psi\psi_{R^1_{-1}}}(r(u)g)duf_s(g)dg$$

The characters ψ are permuted by the copy of $GL_1 \times GL_1$ that occurs in the exterior integral. The stabilizer is a copy of GL_1 which is contained in $Z(\mathbb{A})$. The orbits are two. The trivial contributes zero because a constant term of the cusp form is factored out and for the non trivial we choose ψ_F as representative (where again we identify an element of $L^0(\mathbb{A})$ by it's unique entry which is nonzero end not in the diagonal. After the usual unfolding we get:

$$I_2 = \int_{Z(\mathbb{A})R_H^1(\mathbb{A})\backslash H(\mathbb{A})} \int_{\mathbb{A}} \varphi_{L^0 R^1_{-1}, \psi_F \psi_{R^1_{-1}}}(r(u)g)du f_s(g)dg.$$

We remind that $R_H^1 = U_H$. Notice that $\varphi_{L^0 R^1_{-1}, \psi_F \psi_{R^1_{-1}}}$ is the Whittaker model of φ for the non-standard Borel B^w. Expressing I_2 in terms of the Whittaker model of the standard borel we get:

$$I_2 = \int_{Z(\mathbb{A})U_H(\mathbb{A})\backslash H(\mathbb{A})} \int_{\mathbb{A}} W_\varphi(wr(u)g)du f_s(g)dg \qquad .$$

□

From this Proposition we see that many analytic properties and the functional equation of the Siegel Eisenstein series are transfered into the sum

$$P_s + B_{1,s} + B_{2,s}.$$

However it is not clear to what extent analytic properties are transfered in each summand. Some preliminary calculations that we have done suggest that the functional equation is not transfered in each summand.

Above all we are interested in P_s because we already see that it is an Euler product. We have not finished the unramified calculation for P_s yet, but we expect that in it $L-$functions with each of their Langlands parametes being a product of at least three langlands parameters of π, will appear. Then our hope is that such $L-$functions will not get canceled in the sum $P_s + B_{1,s} + B_{2,s}$, and that a way will be found to start obtaining their unknown analytic properties about them.

In the case $n = 3$ in [GR2], for $s = 1$ we see that we obtain the exterior cube $L-$function evaluated at $\frac{3}{2}$! This may be surprising at first sight because:

- The Siegel Eisenstein series when induced from appropriate characters has a pole at $s = 1$.
- All automorphic $L-$functions are expected to be absolutely convergent for $\mathrm{Re}(s) > 1$ (for the general linear group this is equivalent to the Generalized Ramanujan Conjecture).

This issue is resolved because the residue of $I(\varphi, f_s, \chi, s)$ turns out to be zero. The residue of the Siegel Eisenstein series at one, is a character. As a result we get that the residue of $I(\varphi, f_s, \chi, s)$ at the pole $s = 1$ (if it existed) is

$$\int_{Z(\mathbb{A})H(F)\backslash H(\mathbb{A})} \tilde{\varphi}(g)dg,$$

where $\tilde{\varphi}$ is an appropriate twist of φ. It is proved in [JR] that such period integrals are zero for all cusp forms. More precisely what is proved there is that

$$\int_{Sp_{2n}(F)\backslash Sp_{2n}(\mathbb{A})} \varphi(g)dg$$

is zero for any cusp form φ of GL_2n, which implies that the integral over GSp_{2n} is zero (because of the embedding $Sp(F) \setminus Sp(A) \to GSp_{2n}(F) \setminus GSp_{2n}(\mathbb{A})$). In the literature the integral over Sp_{2n} is called symplectic period.

The method that is used is local. Here is a brief description. Assume that a representation V of GL_{2n} can be realized as a subrepresentation of $Ind_{Sp_{2n}}^{GL_{2n}} \mathbf{1}$. Then this realization of V is called a symplectic-Whittaker model. We recall that a Whittaker model is a realization inside $Ind_{U_{2n}}^{GL_{2n}} \psi$ for ψ a generic character of U_{2n}. It is proved using Mackey theory that a local GL_{2n} representation cannot have both types of models. If for a cusp form φ we have

$$\int_{Z(\mathbb{A})H(F)\backslash H(\mathbb{A})} \varphi(g)dg \neq 0,$$

then for the irreducible representation in which this cusp form belongs, we obtain a symplectic-Whittaker model which in turn gives rise to a symplectic-Whittaker model for each local component. However we know that these components (since they belong to a cusp form) have a Whittaker model. Since they can not have both models we obtain that

$$\int_{Z(\mathbb{A})H(F)\backslash H(\mathbb{A})} \varphi(g)dg = 0$$

for all cusp forms.

In the next section, we will prove by a global method that the symplectic period of a cusp form is zero. This is weaker than proving that there is no symplectic-Whittaker model which shows an advantage of the local method.

We prove that

$$J_s = \int_{H_n(F)\backslash H_n(\mathbb{A})} \varphi(g)E_s(g)\mathrm{d}g = 0$$

where E_s is a klingen Eisenstein series . Then by taking the residue of the Klingen Eisenstein series at one we obtain the symplectic period for the cusp form. The definition of the Klingen Eisenstein series will be reminded in the next section.

Before moving to the next section which is devoted to calculating J_s, we discuss two more topics related to I_s (which relate it in more ways to J_s). We come back to the identity

$$I_s = P_s + B_{1,s} + B_{2,s}.$$

In $B_{2,s}$ and in the summands of $B_{1,s}$ we can discern some period integrals of φ. For $n = 4$ we get:

$$B_{1,s} = \int_{Z(\mathbb{A})H_4(\mathbb{A})R_H^4(\mathbb{A})\backslash H_8(\mathbb{A})} \int_{H_4(F)\backslash H_4(\mathbb{A})} \varphi_{R^4,\psi_{R^4,2}}(hg)dhf_s(g)dg,$$

$$B_{2,s} = \int_{Z(\mathbb{A})GL_2(\mathbb{A})R_H^3(\mathbb{A})\backslash H_8(\mathbb{A})} \int_{GL_2(F)\backslash GL_2(\mathbb{A})} \varphi_{R^3,\psi_{R^3,2}}(hg)dhf_s(g)dg.$$

We discern in the interior integration respectively from $B_{1,s}$ and $B_{2,s}$ the periods

$$l_1(\varphi) = \int_{H_4(F)\backslash H_4(\mathbb{A})} \int_{R^4(F)\backslash R^4(\mathbb{A})} \varphi(nh)\psi_{R^4}^{-1}(n)dndh,$$

and

$$l_2(\varphi) = \int_{GL_2(F)\backslash GL_2(\mathbb{A})} \int_{R^3(F)\backslash R^3(\mathbb{A})} \varphi(nh)\psi_{R^3}^{-1}(n)dndh.$$

For $n > 4$ we will only be concerned with the period integral that occurs in the summand of I_s which corresponds to the most generic choices of orbits in the unfolding of I_s. This summand occurs in $B_{1,s}$ for $l = n$. The period integral is

$$l_{1,n}(\varphi) = \int_{H_n(F)\backslash H_n(\mathbb{A})} \int_{R^n(F)\backslash R^n(\mathbb{A})} \varphi(nh)\psi_{R^n}^{-1}(n)dndh.$$

These period integrals must be harder to understand from the global Whittaker model (which is the period integral that occurs in P_s). First of all we do not expect to have a uniqueness theorem for the type of models we can define from these periods . More precisely the period $l_{1,n}$ satisfies identities of the form

$$l_{1,n}(\pi(nh)(\varphi) = \psi_{R^n}(n)l_{1,n}(\varphi) \qquad n \in R^n(\mathbb{A}) \quad h \in H_n(\mathbb{A}). \tag{3.2.1}$$

Assume that for each local field F_v in place of $\mathbb{A}$, the irreducible representations of $GL_{2n}(F_v)$ had

at most one model (call it $I_{v,1,n}$)satisfying 3.2.1. Then we would obtain

$$I_{1,n} = \prod_v I_{v,1,n}$$

and that would convert the most generic term of $B_{1,s}$ into an Euler product. However, this uniqueness of models is most likely is far from being true. This is what some Mackey theory calculations suggest to us so far. The use of uniqueness of some model is the most widely used method to prove that an integral of automorphic forms is Eulerian. However, this not the only method. An example that may be relevant is in the paper [BGF]. In this paper a special case of the periods that are encountered is

$$\mathbb{H}(\varphi) = \varphi_{R^n,\psi}$$

for ψ any of the most generic characters. This period is far from unique. However it is proved that the integral

$$\int_{GL_{2n}(\mathbb{A})} \mathbb{H}(\varphi) \begin{pmatrix} g & \\ & 1 \end{pmatrix} dg$$

is Eulerian. The relevance with our situation is that both $\mathbb{H}$ and $l_{1,n}$ are "weak" versions of the Jacquet-Shalika model.

We finish our discussion of I_s with one last thought that relates it to J_s. When a period integral for a cusp form is hard to understand, we can replace the cusp form with an Eisenstein and then it may be easier to study it. Of course in most cases the period of an Eisenstein series is not even defined if not regularized in some way . This point of view had been used for example in [CO] where a period for an orthogonal group is studied. We mention here some preliminary calculations for applying such a strategy to $B_{1,s}$

Let n be even, let ρ be a GL_n cusp form, χ_1 a GL_1 character, and χ_2 a GL_{n-1} character. We will replace the cusp form φ with a degenerate Eisenstein series E_d which is induced from $\rho \bigotimes \chi_1 \bigotimes \chi_2$.

In this paragraph our calculation is only formal.

$$l_{1,n}(E_d) = \int_{H_n(F)R^n(F)\backslash H_n(\mathbb{A})R^n(\mathbb{A})} \sum_{\gamma \in P(F)\backslash GL_{2n}(F)} f(\gamma nh)\psi_{R^n}^{-1}(n)dndh$$

$$= \int_{H_nR^n(F)\backslash H_nR^n(\mathbb{A})} \sum_{\delta \in P(F)\backslash GL_{2n}(F)/H_nR^n(F)} \sum_{\gamma \in H_nR^n \cap \delta^{-1}P\delta(F)\backslash H_nR^n(F)} f(\delta\gamma nh)\psi_{R^n}^{-1}(n)dndh$$

$$= \int_{H_nR^n\cap\delta^{-1}P\delta(F)\backslash H_nR^n(\mathbb{A})} \sum_{\delta\in P(F)\backslash GL_{2n}(F)/H_nR^n(F)} f(\delta nh)\psi_{R^n}^{-1}(n)dndh$$

$$= \sum_{\delta\in P(F)\backslash GL_{2n}(F)/H_nR^n(F)} \int_{H_nR^n\cap\delta^{-1}P\delta(\mathbb{A})\backslash H_nR^n(\mathbb{A})}$$

$$\int_{H_nR^n\cap\delta^{-1}P\delta(F)\backslash H_nR^n\cap\delta^{-1}P\delta(\mathbb{A})} f(\delta\gamma nn'h)\psi_{R^n}^{-1}(nn')dndn'dh.$$

In some of the previous summands we can see that an integral of the form

$$\int_{F\backslash\mathbb{A}} \psi_F^{-1}(n)dn$$

is factored out. As a result these summands are zero. For the summand corresponding to the open double coset we can factor out an integral of the form

$$\int_{Z(\mathbb{A})H_n(F)\backslash H_n(\mathbb{A})} \varphi_\rho(g)E_d^1(g)dg,$$

where φ_ρ is a cusp form of ρ, and E_d^1 is an GL_n Eisenstein series induced from the character $\chi_1 \bigotimes \chi_2$ of the group $GL_1 \times GL_2$. It can be checked that the restriction of E_d^1 to H_n is a Klingen Eisenstein series. As a result the summand of the open double coset is zero.

By doing similar calculations we observe that the bigger the rank of the Eisenstein series and the more degenerate it is:

- The better the unfolding works. P_s remains nonzero and it becomes easier to prove that "most of" $B_{1,s}$ and $B_{2,s}$ is zero
- In whatever summand an $L-$function can be discerned (such as P_s) it becomes less interesting.

3.3 The Klingen Eisenstein series

In this section we will concentrate on Sp_{2n} instead of GSp_{2n}. For Sp_{2n} we use again the skew-symmetric matrix ϵ to define it. We will denote Sp_{2k} with $\tilde{H}_k$. For $k < n$ we embed $\tilde{H}_k$ inside $\tilde{H}_n$ by :

$$g \to \begin{pmatrix} I_{n-k} & & \\ & g & \\ & & I_{n-k} \end{pmatrix}.$$

We denote by Kl_k the Klingen parabolic of $\tilde{H}_k$. This is the maximal parabolic for which $\Delta_{Kl} = \{e_1 - e_2\}$. The unipotent radical of Kl_k is denoted by $L_{\tilde{H}_k}$ and is given by the matrices of the form

$$\begin{pmatrix} I_{n-k} & & \\ & X & \\ & & I_{n-k} \end{pmatrix},$$

for X being of the form

$$\begin{pmatrix} 1 & x_1 & x_2 & \cdots & x_{k-1} & y_{k-1} & \cdots & y_2 & y_1 & z \\ & 1 & & & & & & & & y_1 \\ & & \ddots & & & & & & & y_2 \\ & & & \ddots & & & & & & \vdots \\ & & & & \ddots & & & & & y_{k-1} \\ & & & & & \ddots & & & & -x_{k-1} \\ & & & & & & \ddots & & & \vdots \\ & & & & & & & \ddots & & -x_2 \\ & & & & & & & & 1 & -x_1 \\ & & & & & & & & & 1 \end{pmatrix}.$$

The Levi for Kl_k is $GL_1 \times \tilde{H}_{k-1}$, and is given by the embedding:

$$(a, g) \to \begin{pmatrix} a & & \\ & g & \\ & & a^{-1} \end{pmatrix} \qquad a \in GL_1 \text{ and } g \in \tilde{H}_{k-1}.$$

We choose a character χ of $Kl_n(F) \setminus Kl_n(\mathbb{A})$, and a section f_s in $Ind_{Kl_n(F)}^{\tilde{H}_n(F)}(\delta^s \chi)$. This induction is unnormalized. The Klingen Eisenstein series E_s is defined by:

$$E_s(g) = \sum_{\gamma \in Kl_n(F) \setminus \tilde{H}_n(F)} f_s(\gamma g) \qquad \text{for } \mathrm{Re}(s) > 1.$$

For any $GL_{2n}-$automorphic function ϕ of rapid decay we recall the definition:

$$J_s(\phi) = \int_{\tilde{H}_n(F) \setminus \tilde{H}_n(\mathbb{A})} \phi(g) E_s(g) \mathrm{d}g.$$

The study of this integral that follows had occurred to D. Ginzburg long before it occurred to us, but it is not published. D. Ginzburg has published similar calculations such as in [AGR], and [G], Lemma 5.5.

We start the study of J_s with the case that ϕ is a cusp form.

Proposition 3.3.1: *Let φ be a GL_{2n} cusp form. Then*

$$J_s(\varphi) = 0.$$

Proof. We will unfold this integral as we unfolded the previous case with the Siegel Eisenstein series. However here the unfolding works much better because in our Fourier expansions we always get two orbits (including the trivial). First we give some notations.

We denote by L_k the unipotent radical of the parabolic subgroup of GL_{2k} which (1) has $GL_1 \times GL_{2n} \times GL_1$ as a levi and (2) is above the Klingen parabolic of H_k. We choose any abelian subgroup of L_k, which we will call $L^c_{H_k}$, such that $L_k = L_{\tilde{H}_k} \times L^c_{H_k}$. The proof will consist of Fourier expansions in the groups $L^c_{H_k}$.

We define inductively downwards, some unipotent subgroups of $\tilde{H}_n$ and GL_{2n} by $R_{\tilde{H},n} = L_{\tilde{H}_n}$, $R_n = L_n$, $R_{\tilde{H},k} = L_{\tilde{H}_k} R_{\tilde{H},k+1}$, and $R_k = L_k R_{k+1}$

Finally we define the degenerate character ψ_d of the unipotent radical U_{2n} of GL_{2n} as follows:

$$\psi_d(u) = \psi_F(\sum_{i \neq n} u_{i,i+1}).$$

STEP 1 By unfolding the Eisenstein series in the standard way we get

$$J_s = \int_{GL_1 \times \tilde{H}_{n-1}(F) L_{\tilde{H}_n}(F) \backslash H_n(\mathbb{A})} \varphi(g) f_s(g) \mathrm{d}g$$

$$= \int_{GL_1 \times \tilde{H}_{n-1}(F) L_{\tilde{H}_n}(\mathbb{A}) \backslash \tilde{H}_n(\mathbb{A})} \varphi_{L_{\tilde{H}_n}}(g) f_s(g) \mathrm{d}g.$$

STEP 2 We consider a Fourier expansion of $\varphi_{L_{\tilde{H}_n}}$ along $L^c_{\tilde{H}_n}(F) \setminus L^c_{\tilde{H}_n}(\mathbb{A})$. We extend these characters to L_n. In this Fourier expansion the group $GL_1 \times \tilde{H}_{n-1}(F)$ acts with two orbits. The one orbit consist of the trivial character and contributes zero to the integral J_s. In the other orbit we choose the representative defined by $\psi_{L_n}(u) = \psi_F(u_{1,2} + u_{2n-1,2n})$ for $u \in L_n(F) \backslash L_n(\mathbb{A})$. Notice that $\psi_{L_n} = \psi_d|_{L_n}$. Recall that $R_n = L_n$. The stabilizer of this representative is $GL_1 \times \tilde{H}_{n-2}(F) L_{\tilde{H}_{n-1}}(F)$

defined by the embedding

$$(a,h)\to\begin{pmatrix} a & & & & & \\ & a & & & & \\ & & h & & & \\ & & & a^{-1} & & \\ & & & & a^{-1} & \\ & & & & & \end{pmatrix}.$$

As a result of these thoughts we have, $J_s=$

$$\int_{GL_1\times\tilde{H}_{n-1}(F)L_{\tilde{H}_n}(\mathbb{A})\backslash\tilde{H}_n(\mathbb{A})}\sum_{\gamma\in GL_1\times\tilde{H}_{n-2}(F)L_{\tilde{H}_{n-1}}(F)\backslash GL_1\times\tilde{H}_{n-1}(F)}\varphi_{\psi_{R_n,d|_{R_n}}}(\gamma g)f_s(g)\mathrm{d}g$$

$$=\int_{GL_1\times\tilde{H}_{n-2}(F)L_{\tilde{H}_{n-1}}(F)L_{\tilde{H}_n}(\mathbb{A})\backslash\tilde{H}_n(\mathbb{A})}\varphi_{\psi_{R_n,d|_{R_n}}}(g)f_s(g)\mathrm{d}g$$

$$=\int_{GL_1\times\tilde{H}_{n-2}(F)R_{\tilde{H},n-1}(\mathbb{A})\backslash\tilde{H}_n(\mathbb{A})}\varphi_{\psi_{L_{\tilde{H}_{n-1}}{}^{R_n},d|_{R_n}}}(g)f_s(g)\mathrm{d}g.$$

STEP k (for k=3,4,...n) These steps are identical with step 2. However we will write down the details because it will be of some help to the later discussion of what happens when the cusp form is replaced with Pseudo-Eisenstein series. So the step k-1 finishes with the integral

$$J_s=\int_{GL_1\times\tilde{H}_{n-k+1}(F)R_{\tilde{H},n-k+2}(\mathbb{A})\backslash\tilde{H}_n(\mathbb{A})}\varphi_{\psi_{L_{\tilde{H}_{n-k+2}}{}^{R_{n-k+3}},d|_{R_{n-k+3}}}}(g)f_s(g)\mathrm{d}g.$$

We consider the Fourier expansion of $\varphi_{\psi_{L_{\tilde{H}_{n-k+2}}{}^{R_{n-k+3}},d|_{R_{n-k+3}}}}$ along $L^c_{\tilde{H}_{n-k+2}}(F)\setminus L^c_{\tilde{H}_{n+2-k}}(\mathbb{A})$. We extend the characters of this Fourier expansion trivially to L_{n-k+2}. In this Fourier expansion the group $GL_1\times\tilde{H}_{n-k+1}(F)$ acts with two orbits. The one orbit consist of the trivial character and contributes zero to the integral J_s. In the other orbit we choose the representative defined by $\psi_{L_{n-k+2}}(u)=\psi_F(u_{k-1,k}+u_{n-k+1,n-k+2})$ for $u\in L_{n-k+2}(\mathbb{A})$. Notice that $\psi_{R_{n-k+2},d|_{R_{n-k+2}}}=\psi_{L_{n-k+2}}\psi_{R_{n-k+3},d|_{R_{n-k+3}}}$. The stabilizer of this representative is $GL_1\times\tilde{H}_{n-k}(F)L_{\tilde{H}_{n-k+1}}(F)$. So by unfolding as in step 2 we get

$$J_s=\int_{GL_1\times\tilde{H}_{n-k}(F)R_{\tilde{H},n-k+1}(\mathbb{A})\backslash\tilde{H}_n(\mathbb{A})}\varphi_{\psi_{L_{\tilde{H}_{n-k+1}}{}^{R_{n-k+2}},d|_{R_{n-k+2}}}}(g)f_s(g)\mathrm{d}g.$$

In the end of these steps we have

$$J_s=\int_{GL_1(F)U_{\tilde{H}}(\mathbb{A})\backslash\tilde{H}_n(\mathbb{A})}\varphi_{\psi_d}(g)f_s(g)\mathrm{d}g. \tag{3.3.1}$$

Due to an occurrence of a constant term we have that $\varphi_{\psi_d} = 0$, and as a result we also have $J_s = 0$. □

We continue with the study of $J_s(\phi)$ for ϕ a Pseudo-Eisenstein series, which we will denote by θ. The properties that we will need, for this type of automorphic functions, are proved in [M-W]. We restrict ou attention to Pseudo-Eisenstein series that are induced from cuspidal data. One reason this restriction is not important is that, the set of Pseudo-Eisenstein series in a reductive group G, with a central character ξ, is dense in $L^2(G(F) \setminus G(\mathbb{A}))_\xi$.

Proposition 3.3.2 *If the $GL_{2n}-$parabolic subgroup in the inducing data for θ is not self-associate, then*

$$J_s(\theta) = 0.$$

Proof. For all Pseudo-Eisenstein series data the previous unfolding works, and the only difference is the following: The final result and the contribution of the trivial orbit of each step were zero because a constant term of the cusp form was factored out. In the situation that we are in, not all constant terms are zero. We have $J_s = J_s^2 + J_s^3 + ...J_s^n + J_s^c$ where J_s^k is the contribution of the trivial orbit in step k, and J^c is the integral we got at the end of the previous proof (the right hand side of 3.3.1, with φ replaced by θ). Since θ is not induced from a self-associate parabolic, $J_s^c = 0$.

We prove that $J_s = 0$ by doing an induction on n. Let C_1, C_2, C_3 be a partition of the set of cuspidal representations that occur in the inducing data of θ. By $[C_i]$ we denote the multiset that consists of the groups over which the representations of C_i are defined. Form any Pseudo-Eisenstein series θ_{C_1}, θ_{C_2}, θ_{C_3} that contain in their inducing datas respectively C_1, C_2, C_3. We recall from [M-W] page 101 (or by unfolding an integral in our head), that the Peterson-inner product

$$\langle \theta_{C_1}, \theta_{C_2} \rangle = \int_{Z(\mathbb{A})G(F)\setminus G(\mathbb{A})} \theta_{C_1}(g)\bar{\theta}_{C_2}(g)\mathrm{d}g$$

is zero when $C_1 \neq C_2$. We will actually need only the weaker statement :

$$[C_1] \neq [C_2] \implies \langle \theta_{C_1}, \theta_{C_2} \rangle = 0.$$

Now we come back to the problem of understanding the integrals J_k. Each of them (when not obviously zero) can be written as a sum of terms

$$\langle \theta_{C_1} \theta_{C_2} \rangle J_s(\theta_{C_3}).$$

For such a summand at least one among the next two cases is true.

- $[C_1] \neq [C_2]$. Then $\langle \theta_{C_1} \theta_{C_2} \rangle = 0$.
- The parabolic in the inducing data of C_3 is not self-associate. Then by induction $J_s(\theta_{C_3}) = 0$.

□

For the case that θ is induced from a self associate parabolic we haven't finished the calculations. Roughly speaking the calculation seems to be reduced to the next two observations.

- $GL_m \times GL_m$ Rankin-Selberg integrals (for $m \leq n$) occur as inner integrations. For example in the case that the inducing parabolic is the maximal self-associate, we get $J_s = J_s^c$ and then after doing an Iwasawa decomposition to $\tilde{H}_n$ we can discern a $GL_n \times GL_n$ Rankin-Selberg integral in the unfolded form.
- Inner products of Pseudo-Eisenstein series occur as in the case of non-selfassociate parabolics.

Symplectic periods In the section for the Siegel Eisenstein series we mentioned that the vanishing of the symplectic period for a cusp form is proved in [JR]. The symplectic period (and its regularization) for other automorphic functions (not necessarily of rapid decay) is studied in [O]. By taking the residue at one of $J_s(\phi)$ we obtain the symplectic period for ϕ. As a result we started an alternative method for studying the symplectic period. In particular we gave a different proof for the vanishing of the symplectic period of a $GL_{2n}-$cusp form. In the other works for symplectic periods, their non-vanishing has been linked to the poles of $GL(n) \times GL(n)$ $L-$functions. It is also an expectation for many period integrals that their nonvanishing is linked to poles of appropriate $L-$functions. As a result the following question arises naturally:

Question: *Assume we study the properties of a period integral and that we obtain theorems that link the non-vanishing of the period with poles of an $L-$function. Then is it possible to realize this period integral as a residue of an integral representing this $L-$function?*

Our calculation of $J_s(\phi)$ suggests that for the symplectic period the previous question has a positive answer. Of course, to prove it, $J_s(\phi)$ must be defined and calculated for ϕ not having rapid decay.

3.4 Relations between Chapters 2 and 3

In the second Chapter we recalled the way the $GL_n \times GL_{n-1}$ Rankin-Selberg integral unfolds. More generally for all pairs of natural numbers (n, m) there is a $GL_n \times GL_m$ Rankin-Selberg integral for which the unfolding process is almost the same. The differences roughly are:

- In the case $n = m$ an Eisenstein series is used to make the unfolding start.
- In the case $n > m + 1$ The cusp form of GL_n is replaced by one of it's Fourier coefficients. The Fourier coefficient is a P_{m+1} "cusp form" and the unfolding happens identically (Some authors call this padding).

In the third Chapter we encountered a different unfolding process. We finish this paper with the next two thoughts:

1. The unfolding process of the integrals in Chapter 3 can be expressed so that the similarities with the unfolding in Chapter 2 are better revealed.
2. We formally interpret the $GL_n \times GL_m$ integrals as integrals similar to the ones in Chapter 3.

Thought 1

In [AGR] it is mentioned that for φ a cusp form on GL_{2n+1} we have

$$\int_{\tilde{H}_n(F)\backslash\tilde{H}_n(\mathbb{A})} \varphi \begin{pmatrix} g & \\ & 1 \end{pmatrix} \mathrm{d}g = 0.$$

The way this is proved is identical to showing $J_s(\varphi) = 0$. The only difference is that in the first step instead of having an Eisenstein series to start the unfolding, we have a Fourier expansion along the unipotent group which consists of the matrices of the form:

$$\begin{pmatrix} 1 & & & * \\ & 1 & & * \\ & & \ddots & * \\ & & & 1 \end{pmatrix}.$$

We define more generally for $n \geq m+1$

$$J_s^{n,m}(\phi) = \int_{\tilde{H}_m(F)\backslash \tilde{H}_m(\mathbb{A})} \phi_{R,\psi} \begin{pmatrix} g & \\ & I_{n-m} \end{pmatrix} \mathrm{d}g,$$

where R is the group of matrices of the form

$$\begin{pmatrix} I_{m+1} & * & * & * \\ & 1 & * & * \\ & & \ddots & * \\ & & & 1 \end{pmatrix},$$

ψ is the character on $R(F) \setminus R(\mathbb{A})$ given by

$$\psi(u) = \psi_F \left(\sum_{i=m+1}^{n-1} u_{i,i+1} \right)$$

and ϕ is any GL_n-automorphic function of rapid decay.

For φ a GL_n-cusp form we have

$$J_s^{n,m}(\varphi) = 0.$$

This vanishing can be proved in exactly the same way as the case $J_s^{m+1,m}$. However it can also be proved as follows:

Proof. We can write a Fourier expansion of $\varphi_{R,\psi}$ in which the terms are permuted by the conjugation action of $\tilde{H}_m(F)$. The orbits are finitely many. For any of these orbits, if we take the sum of all the characters in this orbit and integrate it over $\tilde{H}_m(F) \setminus \tilde{H}_m(\mathbb{A})$, we get zero because a constant term of φ occurs as an inner integral. We give the details only for the generic orbit. Let R' be the group

$$U_n \cap \{(x_{i,j}) : x_{ij} = 0 \text{ for } i < j \leq m - i + 1\}.$$

Let $\psi_{R'}$ be the character of $R'(F) \setminus R'(\mathbb{A})$ which is given by

$$\psi_{R'}(u) = \psi_F \left(\sum_{i > n/2} u_{i,i+1} \right).$$

The Fourier expansion takes the form

$$\varphi(g) = (\text{Degenerate terms}) + \sum_{\gamma \in U_{\tilde{H}_m}(F)\backslash \tilde{H}_m(F)} \varphi_{\psi_{R'}}(\gamma g).$$

By the usual unfolding for the sum on the right end of the expansion we have

$$\int_{\tilde{H}_m(F)\backslash \tilde{H}_m(\mathbb{A})} \sum_{\gamma \in U_{\tilde{H}_m}(F)\backslash \tilde{H}_{(F)}} \varphi_{\psi_{R'}}(\gamma g)$$

$$= \int_{U_{\tilde{H}_m}(F)\backslash \tilde{H}_m(\mathbb{A})} \varphi_{\psi_{R'}}(g)\mathrm{d}g$$

$$= \int_{U_{\tilde{H}_m}(\mathbb{A})\backslash \tilde{H}_m(\mathbb{A})} \int_{U_{\tilde{H}_m}(F)\backslash U_{\tilde{H}_m}(\mathbb{A})} \varphi_{\psi_{R'}}(ng)\mathrm{d}n\mathrm{d}g$$

$$= 0$$

□

Similarly in the integrals that involved the Siegel and the Klingen Eisenstein series all steps in the unfolding process except the first can be replaced by one Fourier expansion and one use of Fubini's theorem.

Thought 2

We plug in, in $J_s^{n,m}(\quad)$ an Eisenstein series that has inducing data two cusp forms, one on GL_n and a one on GL_m. Then we do not know in what sense this integral converges, but if we ignore the convergence obstacles and unfold, we obtain the conventional $GL_n \times GL_m$ Rankin-Selberg integral in the unfolded form.

Finally recall that in the previous section a similar phenomenon linked $J_s(\phi)$ and the $GL(n) \times GL(n)$ Rankin-Selberg integral.

In order to correctly define and study the integrals $J_s^{n,m}(\phi)$ for ϕ a not rapidly decreasing GL_n-automorphic form, the interpretation of the unfolding in **Thought 1**, may be useful.

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