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Doctoral Dissertations, 2011-2026

Mathematics

2023

Construction of anticyclotomic Euler systems using diagonal cycles

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Princeton, NJ : Princeton University

<http://arks.princeton.edu/ark:/88435/dsp01xw42nc15z>

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CONSTRUCTION OF ANTICYCLOTOMIC EULER SYSTEMS USING DIAGONAL CYCLES

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A DISSERTATION
PRESENTED TO THE FACULTY
OF PRINCETON UNIVERSITY
IN CANDIDACY FOR THE DEGREE
OF DOCTOR OF PHILOSOPHY

RECOMMENDED FOR ACCEPTANCE
BY THE DEPARTMENT OF
MATHEMATICS
ADVISER: CHRISTOPHER SKINNER

MAY 2023

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Abstract

In this thesis, we construct a new anticyclotomic Euler system for the four-dimensional Galois representation attached to two modular forms and a Hecke character of an imaginary quadratic field. To state the results more precisely, let g and h be newforms of weights $l \geq m$ of the same parity and let ψ be a Hecke character of an imaginary quadratic field K of infinity-type $(1 - k, 0)$ for some even integer $k \geq 2$. Assume that the product of the characters of g , h and the CM-form attached to ψ is trivial. Let p be a prime which splits in K . We then study the p -adic G_K -representation $V := V_g \otimes V_h(\psi^{-1})(1 - c)$, where $c = (k + l + m - 2)/2$. Combining a geometric construction using modified diagonal cycles in the product of three modular curves with a result of Lei–Loeffler–Zerbes, we obtain cohomology classes over ring class field extensions of K , and we prove that they form a split anticyclotomic Euler system in the sense of Jetchev–Nekovář–Skinner.

The bottom Λ -adic class of our Euler system is related to a one-variable specialization of the triple product p -adic L -function constructed by Darmon–Rotger via a reciprocity law proved by Bertolini–Seveso–Venerucci and Darmon–Rotger. This reciprocity law, together with the Euler-system machinery developed by Jetchev–Nekovář–Skinner, allows us to deduce, under some additional hypotheses, different cases of the Bloch–Kato conjecture for the representation V in analytic rank zero and one. As a different application, we also give two equivalent formulations of an Iwasawa–Greenberg Main Conjecture in this setting and prove one divisibility.

When $h = g^*$, i.e., the modular form obtained by conjugating the Fourier coefficients of g , we obtain an Euler system for the three-dimensional G_K -representation $V' := \mathrm{ad}^0(V_g)(\psi^{-1})(1 - k/2) \subset V$ and use it to derive similar applications towards the Bloch–Kato conjecture in analytic rank zero and one and towards a divisibility of an Iwasawa–Greenberg Main Conjecture.

Acknowledgements

First of all, I would like to thank my advisor, Chris Skinner, for his continuous guidance and support during the last five years. I am fully indebted to him for his patience and his generosity, both in terms of time and of ideas. I would also like to thank Shou-Wu Zhang for his advice at different stages during my graduate studies, and both Shou-Wu Zhang and Peter Sarnak for agreeing to serve as members of my thesis committee.

The work presented in this thesis was produced in joint work with Francesc Castella and Óscar Rivero Salgado. It has been a pleasure to work with both of them, and I am deeply grateful for all that I have learned from them and for all their support and encouragement during these years.

The development of this thesis has benefited from mathematical discussions and exchanges with several people, including Kâzım Büyükboduk, Kim Tuan Do, Ellen Eischen, Michele Fornea, Leo Lai, Antonio Lei, David Loeffler, Sam Mundy and Xin Wan, whom I would like to thank for their help and advice.

Also, and although unrelated to the content of this thesis, I learned a lot of mathematics from my participation at the Arizona Winter School 2022, and I would particularly like to thank Wee Teck Gan and Petar Bakić for all that I learned from them, both then and afterwards. I would also like to thank the other participants in the project, as well as the organizers.

Special thanks should go to Ashay Burungale and Martí Roset Julià for their enthusiasm and willingness to discuss mathematical ideas. I have learned a lot from these discussions.

During all these years, I have also benefited from advice and help, both mathematical and professional, from different people. In this regard, apart from those who have already been mentioned, I would like to thank Henri Darmon, Jon Fickenscher, Francesc Fité, Javi Gómez Serrano, Andrew Graham, Giada Grossi, Alex Horawa,

Lue Pan, Alice Pozzi, Victor Rotger and Matteo Tamiozzo.

I would also like to thank Amina Abdurrahman, Anshul Adve, Ryan Alweiss, Neelima Borade, Gonzalo Cao Labora, Alan Chang, Eric Chen, Dor Elboim, Dan Fess, Fernando Figueroa Zamora, Ian Fleschler, Vikram Giri, John Halliday, Linus Hamann, Trajan Hammonds, Otte Heinävaara, Lena Ji, Zhaorong Jin, Kenz Kallal, Noah Kravitz, Jef Laga, Ben Lowe, Thomas Massoni, Gyujin Oh, Gerard Orriols Giménez, Mayank Pandey, Seung-Yeon Ryoo, Marco Sangiovanni Vicentelli, David Villalobos-Paz, Jack Sempliner, Shikhin Sethi, Zhengyuan Shang, Jay Singh, Shruthi Sridhar, Mohan Swaminathan, Fernando Trejos Suárez, Sameera Vemulapalli, Boya Wen, Jakub Witaszek, Katy Woo, Ziquan Zhuang, Nina Zubrilina and Jonathan Zung for helping to create a friendly and supportive environment during my time at Princeton.

I would like to thank the staff at the Department of Mathematics, and in particular Jill LeClair and Ankit Tak, for being always helpful.

Last but not least, I would like to thank my family and friends, and especially my parents and sister, for their lifelong support and encouragement.

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Chapter 1

Introduction

Over the last few decades, the theory of Euler systems has emerged as one of the most successful approaches for the study of global Galois representations and in particular of the Bloch and Kato conjecture in the case of low analytic rank. As explained with more detail below, the Bloch and Kato conjecture constitutes a vast generalization of the conspicuously famous Birch and Swinnerton-Dyer conjecture. Therefore, a general proof of the Bloch and Kato conjecture seems inevitably out of reach with the current techniques, and, as of now, only some special low-rank cases have been established, often after considerable efforts. It is not surprising, then, that, although we have at our disposal different axiomatizations of the theory of Euler systems which would allow us to deduce very powerful results from the existence of a non-trivial Euler system attached to a certain global Galois representation, the construction of such an Euler system is in most cases a difficult task.

The main purpose of this thesis is the construction of a new Euler system attached to the Rankin–Selberg convolution of two elliptic modular forms twisted by an algebraic Hecke character over an imaginary quadratic field. From this construction, we derive different applications towards the Bloch and Kato conjecture and towards the study of the Iwasawa theory for the corresponding Galois representation.

This thesis is based on the two preprints *The diagonal cycle Euler system for $\mathrm{GL}_2 \times \mathrm{GL}_2$* and *An anticyclotomic Euler system for adjoint modular Galois representations*, both of them produced in joint work with Francesc Castella and Óscar Rivero and reproduced here, with their permission, as Chapter 2 and Chapter 3, respectively.

In the rest of this introduction, we try to provide some background and motivation for these works.

1.1 The Bloch and Kato conjecture

In this section we present an overview of the Bloch and Kato conjecture, which appeared in 1990 in The Grothendieck Festschrift [BK90]. Most of the content introduced here is borrowed from the expository notes written by Joël Bellaïche [Bel09].

Let K be a number field with absolute Galois group G_K and let p be a rational prime number. A p -adic representation of G_K is a finite-dimensional vector space V over a finite extension E of $\mathbb{Q}_p$ endowed with a continuous semi-simple action of G_K . We say that V is a *geometric* representation of G_K if it is unramified at almost all places of K and de Rham at all places dividing p . This is always the case when V is a representation *coming from geometry*, i.e., when the irreducible components can be realized as a subquotient of the p -adic cohomology of a proper and smooth variety defined over K , and it is a conjecture of Fontaine and Mazur that all geometric p -adic Galois representations come from geometry.

Let v be a place of K . Let K_v denote the completion of K at v and let K_v^{ur} be the maximal unramified extension of K_v inside a fixed algebraic closure. We can identify the absolute Galois group G_{K_v} of K_v with a decomposition subgroup of G_K for the place v . This allows us to regard V as a representation of G_{K_v} via restriction. The choice of decomposition group will be unimportant in the discussion that follows.

A *local condition* for V at a place v of K is a subspace $L_v \subseteq H^1(K_v, V)$. The

subspaces (0) and $H^1(K_v, V)$ are usually referred to as the *strict* and *relaxed* local condition, respectively. One of the most important local conditions is the so-called *unramified* local condition, defined by

$$H_{\text{ur}}^1(K_v, V) := \ker \left(H^1(K_v, V) \longrightarrow H^1(K_v^{\text{ur}}, V) \right).$$

For places v above p , we will also need the important *Bloch–Kato* local condition, defined by

$$H_f^1(K_v, V) := \ker \left(H^1(K_v, V) \longrightarrow H^1(K_v, V \otimes_{\mathbb{Q}_p} B_{\text{crys}}) \right).$$

A *Selmer structure* for V is a collection of local conditions $\mathcal{L} = \{L_v\}_v$ for all finite places v of K such that L_v is the unramified local condition for almost all v . The *Selmer group* attached to a Selmer structure $\mathcal{L}$ is the subgroup $H_{\mathcal{L}}^1(K, V) \subseteq H^1(K, V)$ defined by

$$H_{\mathcal{L}}^1(K, V) := \ker \left(H^1(K, V) \longrightarrow \prod_{\text{finite } v} H^1(K_v, V)/L_v \right).$$

It is a finite-dimensional vector space over E .

The *Bloch–Kato Selmer group* is the Selmer group attached to the Selmer structure $\mathcal{L} = \{L_v\}_v$ given by

$$L_v = \begin{cases} H_{\text{ur}}^1(K_v, V), & \text{if } v \nmid p, \\ H_f^1(K_v, V), & \text{if } v \mid p. \end{cases}$$

We denote the Bloch–Kato Selmer group by $H_f^1(K, V)$.

Now we will define the L -function associated with a geometric p -adic representation V as above. This will be formally defined as an infinite product of local factors. For each finite place v of K , let I_v denote the inertia subgroup of G_{K_v} , let Fr_v denote an arithmetic Frobenius element and let q_v denote the cardinality of the correspond-

ing residue field. For finite places v not dividing p , we define the corresponding local factor by

$$L_v(V, s) := \det \left((1 - \text{Fr}_v^{-1} q_v^{-s})|_{V^{I_v}} \right)^{-1}$$

For places v above p , we define the corresponding local factor by

$$L_v(V, s) = \det \left((1 - \phi^{-f_v} q_v^{-s})|_{D_{\text{crys}}(V)} \right)^{-1},$$

where ϕ denotes the crystalline Frobenius and f_v is defined by $q_v = p^{f_v}$.

We formally define the L -function associated with V by

$$L(V, s) = \prod_{\text{finite } v} L_v(V, s).$$

We say that V is *pure of weight w* for some integer w if, for almost all finite places v of K , the roots of the polynomial

$$\det \left((1 - \text{Fr}_v^{-1} X)|_{V^{I_v}} \right)$$

have complex absolute value $q_v^{-w/2}$ for every choice of embedding $\bar{\mathbb{Q}}_p \hookrightarrow \mathbb{C}$. This is the case if V is an irreducible representation coming from geometry. If V is pure of weight w , then the formal product $L(V, s)$ defines a holomorphic function on the right-half plane $\Re(s) > w/2 + 1$. We have the following conjecture.

Conjecture 1.1.1. *Let V be a geometric p -adic representation of G_K which is pure of weight w . Then, the L -function $L(V, s)$ admits meromorphic continuation to the whole complex plane. Moreover, if V is irreducible, the L -function has no poles unless $V = E(n)$, in which case it has a unique simple pole at $s = 1 - n$.*

Now we can state the Bloch and Kato conjecture in its most simple form. In order to do that, we assume that the previous conjecture holds.

Conjecture 1.1.2 (Bloch and Kato conjecture). *Let V be a pure geometric p -adic representation of G_K . Then, the following equality holds:*

$$\dim H_f^1(K, V^\vee(1)) - \dim H^0(K, V^\vee(1)) = \text{ord}_{s=0} L(V, s). \quad (1.1)$$

Remark 1.1.3. The term $\dim H^0(K, V^\vee(1))$ is zero unless V contains the representation $E(1)$. It accounts for the pole of the L -function associated with $L(1)$, i.e., the pole of $\zeta_K(s+1)$ at $s=0$.

1.1.1 Examples

In this subsection we reinterpret two well-known statements, namely Dirichlet's unit theorem and part of the Birch and Swinnerton-Dyer conjecture, as special cases of the Bloch and Kato conjecture.

Dirichlet's unit theorem

For this example, we take $V = \mathbb{Q}_p$, with the trivial Galois action of G_K . For every positive integer n , there is the following short exact sequence of G_K -modules:

$$1 \longrightarrow \mu_{p^n} \longrightarrow \bar{K}^\times \xrightarrow{[p^n]} \bar{K}^\times \longrightarrow 1.$$

From this exact sequence we obtain an isomorphism

$$\kappa_n : K^\times \otimes \mathbb{Z}/p^n\mathbb{Z} \xrightarrow{\cong} H^1(K, \mu_{p^n}).$$

Taking the projective limit of these maps and tensoring with $\mathbb{Q}_p$ we obtain an isomorphism

$$\kappa : K^\times \hat{\otimes} \mathbb{Q}_p \xrightarrow{\cong} H^1(K, \mathbb{Q}_p(1)),$$

which is usually called *Kummer map*.

It can be proved that the Kummer map restricts to the following isomorphism:

$$\kappa : \mathcal{O}_K^\times \otimes \mathbb{Q}_p \xrightarrow{\cong} H_f^1(K, \mathbb{Q}_p(1)).$$

Therefore, when $V = \mathbb{Q}_p$, the left-hand side of equation 1.1 is equal to the rank of the group of units $\mathcal{O}_K^\times$.

On the other hand, the right-hand side of equation 1.1 is equal to $\text{ord}_{s=0} \zeta_K(s)$. It follows from the functional equation for $\zeta_K(s)$ and the fact that it has a simple pole at $s = 1$ that

$$\text{ord}_{s=0} \zeta_K(s) = r_1 + r_2 - 1,$$

where r_1 and r_2 are the number of real and complex places of K , respectively.

Hence, the Bloch and Kato conjecture for $V = \mathbb{Q}_p$ is equivalent to Dirichlet's unit theorem:

$$\text{rank}(\mathcal{O}_K^\times) = r_1 + r_2 - 1.$$

Birch and Swinnerton-Dyer conjecture

Let E be an elliptic curve over K . Let $T_p(E)$ denote the Tate module of E and let $V_p(E) := T_p(E) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. For this example, we take $V = V_p(E)$.

For every positive integer n , there is the following short exact sequence of G_K -modules:

$$0 \longrightarrow E[p^n](\bar{K}) \longrightarrow E(\bar{K}) \xrightarrow{[p^n]} E(\bar{K}) \longrightarrow 0.$$

From the corresponding long exact cohomology sequence, we obtain the short exact

sequence

$$0 \longrightarrow E(K) \otimes \mathbb{Z}/p^n\mathbb{Z} \longrightarrow H^1(K, E[p^n](\bar{K})) \longrightarrow H^1(K, E(\bar{K}))[p^n] \longrightarrow 0.$$

For each place v of K there are analogous exact sequences. Therefore, we obtain the following commutative diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & E(K) \otimes \mathbb{Z}/p^n\mathbb{Z} & \longrightarrow & H^1(K, E[p^n](\bar{K})) & \longrightarrow & H^1(K, E(\bar{K}))[p^n] \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \rightarrow & \prod_v E(K_v) \otimes \mathbb{Z}/p^n\mathbb{Z} & \rightarrow & \prod_v H^1(K_v, E[p^n](\bar{K}_v)) & \rightarrow & \prod_v H^1(K_v, E(\bar{K}_v))[p^n] \rightarrow 0. \end{array}$$

As usual, we define the p^n -Selmer group of E/K by

$$\mathrm{Sel}_{p^n}(E/K) := \ker \left(H^1(K, E[p^n](\bar{K})) \longrightarrow \prod_v H^1(K_v, E(\bar{K}_v)) \right).$$

It sits on the short exact sequence

$$0 \longrightarrow E(K) \otimes \mathbb{Z}/p^n\mathbb{Z} \longrightarrow \mathrm{Sel}_{p^n}(E/K) \longrightarrow \mathrm{III}(E/K)[p^n] \longrightarrow 0, \quad (1.2)$$

where $\mathrm{III}(E/K)$ denotes the Tate–Shafarevich group of E/K , i.e.,

$$\mathrm{III}(E/K) := \ker \left(H^1(K, E(\bar{K})) \longrightarrow \prod_v H^1(K_v, E(\bar{K}_v)) \right).$$

Taking the projective limit over n of the short exact sequences 1.2 and tensoring with $\mathbb{Q}_p$ we obtain the short exact sequence

$$0 \longrightarrow E(K) \otimes \mathbb{Q}_p \longrightarrow \mathrm{Sel}(E/K) \longrightarrow \left(\varprojlim_n \mathrm{III}(E/K)[p^n] \right) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \longrightarrow 0,$$

where

$$\begin{aligned} \text{Sel}(E/K) &:= \left(\varprojlim_n \text{Sel}_{p^n}(E/K) \right) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \\ &= \ker \left(H^1(K, V_p(E)) \longrightarrow \prod_v H^1(K_v, V_p(E)) / \text{im}(E(K_v) \otimes \mathbb{Q}_p) \right). \end{aligned}$$

It can be proved that

$$\text{Sel}(E/K) = H_f^1(K, V_p(E)).$$

Therefore, there is an injection

$$E(K) \otimes \mathbb{Q}_p \hookrightarrow H_f^1(K, V_p(E)).$$

It is an isomorphism if and only if the p -primary part $\text{III}(E/K)[p^\infty]$ of the Tate–Shafarevich group of E/K is finite, which is conjectured to be true. Hence, under this assumption, and taking into account that $V_p(E)^\vee(1) \simeq V_p(E)$, the left-hand side of equation 1.1 is equal to the rank of $E(K)$.

On the other hand, the right-hand side of equation 1.1 is equal to

$$\text{ord}_{s=0} L(V_p(E), s) = \text{ord}_{s=0} L(E, s+1) = \text{ord}_{s=1} L(E, s),$$

where $L(E, s)$ is the usual L -function associated with the elliptic curve E . As opposed to the previous example, for general elliptic curves the analytic continuation of $L(E, s)$ to the whole complex plane is still a conjecture, and we must assume it in order to make sense of the previous expression. It is known to be true for modular elliptic curves, which includes all elliptic curves over $\mathbb{Q}$.

Assuming the finiteness of the p -primary part of the Tate–Shafarevich group of E/K and assuming that $L(E, s)$ has analytic continuation to the whole complex plane

(or, alternatively, that E is modular), the Bloch and Kato conjecture for $V = V_p(E)$ reduces to the most simple form of the Birch and Swinnerton-Dyer conjecture:

$$\text{rank}(E(K)) = \text{ord}_{s=1} L(E, s).$$

1.2 Euler systems

The theory of Euler systems has been a successful tool for proving different cases of the Bloch and Kato conjecture in analytic ranks zero and one, i.e., in proving results of the form

$$\text{ord}_{s=0} L(V, s) = 0, 1 \quad \implies \quad \dim H_f^1(K, V^\vee(1)) = 0, 1,$$

respectively. It has also been successfully used to prove one divisibility for different Iwasawa–Greenberg Main Conjectures, namely that the characteristic ideal of a certain Selmer group divides a corresponding p -adic L -function.

Roughly speaking, given a number field K , a p -adic representation V of the absolute Galois group G_K and a lattice $T \subseteq V$ stable under the Galois action, an Euler system is a collection of cohomology classes $\kappa_L \in H^1(L, T)$ over different abelian extensions L/K which satisfy certain ‘norm relations’. There have been different attempts at axiomatizing Euler systems, notably the work of Rubin [Rub00], Mazur–Rubin [MR04] and, more recently, Jetchev–Nekovář–Skinner [JNS]. In this thesis we will use the latter axiomatization, which we will introduce in the next subsection.

Notable historic examples of Euler systems include the following:

- The Euler system of cyclotomic units, which, combined with the analytic class number formula, allowed Rubin [Rub90] to provide a new proof of the Iwasawa Main Conjecture for $\mathbb{Q}$. (A different proof of this result had been previously found by Mazur and Wiles [MW84].)

- The Euler system of elliptic units, which, combined with the analytic class number formula, allowed Rubin [Rub91] to prove the Iwasawa Main Conjecture for imaginary quadratic fields.
- The Euler system of Heegner points. The work of Kolyvagin [Kol90] on this Euler system together with the Gross–Zagier formula [GZ86] established the Birch and Swinnerton-Dyer conjecture in analytic ranks zero and one for (modular) elliptic curves over $\mathbb{Q}$.
- Kato’s Euler system, which allowed Kato [Kat04] to prove one divisibility (‘characteristic ideal divides p -adic L -function’) in the Iwasawa–Greenberg Main Conjecture for GL_2 over $\mathbb{Q}$. (The opposite divisibility was later proved by Skinner and Urban [SU14].)
- The Euler system of Beilinson–Flach elements for the Rankin–Selberg convolution of two modular forms on GL_2 over $\mathbb{Q}$. It was first introduced by Lei–Loeffler–Zerbes [LLZ14] for modular forms of weight 2 and later generalized (allowing higher weight and variation in p -adic families) by Kings–Loeffler–Zerbes [KLZ17, KLZ20].

In recent years, the theory of Euler systems has become a very active research area, with the appearance of a good variety of new constructions.

1.2.1 The work of Jetchev–Nekovář–Skinner

In [JNS], the authors develop a theory of ‘split anticyclotomic Euler systems’, which we define here.

Let $K/\mathbb{Q}$ be an imaginary quadratic field. For any positive integer n , we denote by $K[n]$ the maximal p -subextension inside the ring class field of conductor n .

Let p be an odd prime. Let E be a finite extension of $\mathbb{Q}_p$, let $\mathcal{O}$ denote its ring of integers and let $\varpi \in \mathcal{O}$ be a uniformizer. Let V be a finite-dimensional vector

space over E endowed with a continuous action of G_K . We assume that this action is unramified outside a finite set of primes Σ and that the resulting G_K -representation is conjugate self-dual, i.e.,

$$V^\vee(1) \simeq V^c,$$

where V^c denotes the representation resulting from conjugating the action of G_K on V by any element $c \in G_{\mathbb{Q}}$ which restricts to the nontrivial automorphism of K . Let $T \subset V$ be a G_K -stable $\mathcal{O}$ -lattice.

For each prime $\mathfrak{p}$ of K above p , choose a $G_{K_{\mathfrak{p}}}$ -stable subspace $\mathcal{F}_{\mathfrak{p}}^+(V)$ of V , and let $\mathcal{F}_{\mathfrak{p}}^-(V) = V/\mathcal{F}_{\mathfrak{p}}^+(V)$. Let L be a finite extension of K . For each place v of L , we define the local condition

$$H_{\text{Gr}}^1(L_v, V) = \begin{cases} H_{\text{ur}}^1(L_v, V) & \text{if } v \nmid p, \\ \ker \left(H^1(L_v, V) \rightarrow H^1(L_v, \mathcal{F}_{\mathfrak{p}}^-(V)) \right) & \text{if } v \mid \mathfrak{p} \text{ for some } \mathfrak{p} \mid p. \end{cases}$$

Let $H_{\text{Gr}}^1(L, V)$ denote the corresponding Greenberg–Selmer group, i.e.,

$$H_{\text{Gr}}^1(L, V) = \ker \left(H^1(L, V) \rightarrow \prod_v H^1(L_v, V)/H_{\text{Gr}}^1(L_v, V) \right),$$

where the product is over all finite places of L .

We assume that, for every positive integer n , the local conditions $H_{\text{Gr}}^1(K[n]_v, V)$ and $H_{\text{Gr}}^1(K[n]_{\bar{v}}, V)$ are orthogonal complements under the local pairing

$$H^1(K[n]_v, V) \times H^1(K[n]_{\bar{v}}, V) \longrightarrow E.$$

We also make the following assumption.

Assumption 1.2.1. There exists an element $\sigma \in G_K$ such that:

1. σ fixes $K[1](\mu_{p^\infty})$.

$$2. \dim_E V/(\sigma - 1)V = 1.$$

From now on, we fix an element $\sigma \in G_K$ as in the previous assumption. For any positive integer m , let $T_m := T/\varpi^m T$ and let $K(T_m)$ denote the smallest extension of K such that $G_{K(T_m)}$ acts trivially on T_m . Let $\Omega_m := K[1](\mu_{p^m})K(T_m)$. For any prime $\mathfrak{q}$ of K , we denote by $\text{Fr}_{\mathfrak{q}}$ any choice of arithmetic Frobenius at $\mathfrak{q}$.

Definition 1.2.2. Let m be a positive integer. Then, the set of *split- σ Kolyvagin primes* of level m , denoted $\mathcal{L}_m^\sigma$, is the set of rational primes q such that:

1. $q \nmid 2p$ and splits in K , with $(q) = \mathfrak{q}\bar{\mathfrak{q}}$;
2. V is unramified at $\mathfrak{q}$ and $\bar{\mathfrak{q}}$;
3. $\text{Fr}_{\mathfrak{q}}$ belongs to the G_K -conjugacy class of σ in $\text{Gal}(\Omega_m/K)$.

For $\mathcal{L}$ a set of rational primes, we define

$$\mathcal{N}(\mathcal{L}) := \{n = q_1^{a_1} \cdots q_r^{a_r} \in \mathbb{Z}_{\geq 1} : q_i \in \mathcal{L}, a_i = 1 \text{ if } q_i \neq p\}.$$

If $\mathcal{L}_1$ and $\mathcal{L}_2$ are two sets of rational primes, we will write $\mathcal{L}_1 \dot{\supset} \mathcal{L}_2$ if the density of $\mathcal{L}_2 \setminus \mathcal{L}_1$ is zero.

Definition 1.2.3. Let $\mathcal{L}$ be a set of rational primes such that $\mathcal{L} \dot{\supset} \mathcal{L}_m^\sigma$ for some $m \geq 1$. For each prime $q \in \mathcal{L}$, fix a prime $\mathfrak{q}$ of K above q . A *split- σ anticyclotomic Euler system* for $(T, \mathcal{L})$ is a collection of cohomology classes

$$\kappa = \{\kappa_n \in H_{\text{Gr}}^1(K[n], T) : n \in \mathcal{N}(\mathcal{L})\}$$

such that, for any $q \in \mathcal{L}$, if $n, nq \in \mathcal{N}(\mathcal{L})$, then the following ‘norm relation’ holds:

$$\text{cor}_{K[nq]/K[n]}(\kappa_{nq}) = \begin{cases} P_{\mathfrak{q}}(\text{Fr}_{\mathfrak{q}}^{-1})\kappa_n & \text{if } q \notin \Sigma \cup \{p\}, \\ \kappa_n & \text{otherwise,} \end{cases}$$

where $P_{\mathfrak{q}}(X) := \det(1 - \text{Fr}_{\mathfrak{q}}^{-1}X|V^{\vee}(1))$.

Remark 1.2.4. Some examples of split anticyclotomic Euler systems include the work of Castella–Do [CD23], who construct an Euler system for the p -adic Galois representation $V_{f,\chi}$ attached to a newform f of weight $k \geq 2$ twisted by an anticyclotomic Hecke character χ , and the work of Graham–Shah [GS21], who construct an Euler system for the p -adic Galois representation associated with certain cuspidal automorphic representations on unitary groups.

1.3 The main results

In this section we outline the main results in this thesis.

Let g and h be normalized elliptic newforms of weights l and m , levels N_g and N_h , and nebentypus χ_g and χ_h , respectively. We assume that l and m have the same parity and that $l, m \geq 2$. Let $K/\mathbb{Q}$ be an imaginary quadratic field of discriminant $-D$. Let ψ be a Hecke character of K of conductor $\mathfrak{c}$ and infinity type $(1 - k, 0)$ for some even integer $k \geq 2$. Then, there exists a Dirichlet character χ such that

$$\psi((n)) = \chi(n)n^{k-1} \quad \text{for all integers } n \text{ coprime to } \mathfrak{c}.$$

Let f be the theta series attached to ψ , i.e.,

$$f = \theta_{\psi} = \sum_{(\mathfrak{a}, \mathfrak{c})=1} \psi(\mathfrak{a})q^{N_{K/\mathbb{Q}}(\mathfrak{a})}.$$

It is known that this is a normalized newform of weight k , level $N_{\psi} = DN_{K/\mathbb{Q}}(\mathfrak{c})$ and nebentypus $\chi_{\psi} = \chi\epsilon_K$, where ϵ_K denotes the quadratic character associated with the extension $K/\mathbb{Q}$.

From now on, we make the following assumption.

Assumption 1.3.1. $\chi_{\psi}\chi_g\chi_h = 1$.

Fix a prime $p \nmid 6N_\psi N_g N_h$ and fix an embedding $\iota_p : \bar{\mathbb{Q}} \hookrightarrow \bar{\mathbb{Q}}_p$. Let $\mathfrak{p}$ denote the prime of $K \subset \bar{\mathbb{Q}}$ determined by ι_p . Let E be a finite extension of $\mathbb{Q}_p$ containing the values of ψ and the Fourier coefficients of g and h . Let $\mathcal{O}$ be the ring of integers of E and let $\mathfrak{P}$ be its maximal ideal.

Attached to the Hecke character ψ there is a continuous p -adic character

$$\psi_{\mathfrak{P}} : G_K \longrightarrow \mathcal{O}^\times$$

which is unramified at finite primes of K not dividing $\mathfrak{p}\mathfrak{c}$ and, for such a prime $\mathfrak{l}$,

$$\psi_{\mathfrak{P}}(\sigma_{\mathfrak{l}}) = \psi(\mathfrak{l}),$$

where $\sigma_{\mathfrak{l}}$ denotes a geometric Frobenius element for $\mathfrak{l}$.

Let V_f , V_g and V_h denote the 2-dimensional p -adic $G_{\mathbb{Q}}$ -representations attached to the modular forms f , g , and h , respectively, as constructed by Shimura and Deligne. We can assume that they have coefficients in the field E . To fix conventions, if $\phi \in \{f, g, h\}$ and ℓ is a rational prime not dividing neither p nor the conductor of ϕ , then the trace of an arithmetic Frobenius element Fr_{ℓ} acting on V_{ϕ} is given by the corresponding Fourier coefficient of ϕ , i.e.,

$$\text{Tr}_{V_{\phi}}(\text{Fr}_{\ell}) = a_{\ell}(\phi).$$

Since f is the theta series attached to the character ψ , it can be checked that

$$V_f = \text{Ind}_K^{\mathbb{Q}}(\psi_{\mathfrak{P}}^{-1}).$$

It follows from the assumptions on ψ that this representation is residually irreducible and p -distinguished.

Let $c = (k + l + m - 2)/2$. To simplify notation, we define the G_K -representation

$$V := V_g \otimes V_h(\psi_{\mathfrak{P}}^{-1})(1 - c)$$

and the $G_{\mathbb{Q}}$ -representation

$$V_{fgh} := \text{Ind}_K^{\mathbb{Q}} V = V_f \otimes V_g \otimes V_h(1 - c).$$

It follows from our assumptions that the representation V is conjugate self-dual, i.e.,

$$V^{\vee}(1) \simeq V^c,$$

where V^c denotes the representation resulting from conjugating the action of G_K on V by any element $c \in G_{\mathbb{Q}}$ which restricts to the nontrivial automorphism of K . Therefore, the representation V_{fgh} is self-dual, i.e.,

$$V_{fgh}^{\vee}(1) \simeq V_{fgh}.$$

It follows from the properties of L -functions that

$$L(V, s) = L(\text{Ind}_K^{\mathbb{Q}} V, s) = L(V_{fgh}, s).$$

Since the representation V , and, equivalently, the representation V_{fgh} , is pure of weight -1 , the L -function above is a well-defined holomorphic function on the right-half plane $\Re(s) > 1/2$. The following theorem, stated here in very rough terms, is proved by Garrett [Gar87] and Piatetski-Shapiro–Rallis [PSR87].

Theorem 1.3.2. *The function $L(V, s)$ admits meromorphic continuation to the whole complex plane and satisfies a functional equation relating $L(V, s)$ to $L(V, -s)$.*

The parity of the order of vanishing of $L(V, s)$ at the center $s = 0$ is determined by the global sign $\varepsilon \in \{\pm 1\}$ in the functional equation. It is equal to a product of local signs

$$\varepsilon = \prod_v \varepsilon_v,$$

where v ranges over all the places of $\mathbb{Q}$ and $\varepsilon_v = 1$ for all but finitely many places. For the rest of this section, we make the following assumption.

Assumption 1.3.3. $\varepsilon_v = 1$ for all finite places v .

Under this assumption,

$$\varepsilon = \varepsilon_\infty = \begin{cases} +1, & \text{if } k \geq l + m \text{ or } l \geq k + m \text{ or } m \geq k + l, \\ -1, & \text{otherwise.} \end{cases}$$

In the first case above we say that the weights k, l, m are *unbalanced*, whereas in the second case we say that they are *balanced*.

Since $(p, \mathfrak{c}) = 1$, the newform f is ordinary at p . We make from now on the following assumption.

Assumption 1.3.4. The newforms g and h are ordinary at p .

This allows us to consider Hida families $\mathbf{f}$, $\mathbf{g}$ and $\mathbf{h}$ passing through the ordinary p -stabilizations of f , g and h , respectively. We regard them as q -expansions

$$\begin{aligned} \mathbf{f} &= \sum_n a_n(\mathbf{f}) q^n \in \Lambda_{\mathbf{f}}[[q]], \\ \mathbf{g} &= \sum_n a_n(\mathbf{g}) q^n \in \Lambda_{\mathbf{g}}[[q]], \\ \mathbf{h} &= \sum_n a_n(\mathbf{h}) q^n \in \Lambda_{\mathbf{h}}[[q]], \end{aligned}$$

where $\Lambda_{\mathbf{f}}$, $\Lambda_{\mathbf{g}}$ and $\Lambda_{\mathbf{h}}$ are finite flat extensions of the standard Iwasawa algebra $\Lambda_{\mathcal{O}} = \mathcal{O}[[1 + p\mathbb{Z}_p]]$.

The following theorem was first proved by Darmon–Rotger [DR14] and later refined by Hsieh [Hsi21].

Theorem 1.3.5. *Assume that $\gcd(N_\psi, N_g, N_h)$ is square-free. Then, there exists a p -adic L -function*

$$\mathcal{L}_p^f(\mathbf{f}, \mathbf{g}, \mathbf{h}) \in \Lambda_{\mathbf{f}} \hat{\otimes} \Lambda_{\mathbf{g}} \hat{\otimes} \Lambda_{\mathbf{h}}$$

such that, for any arithmetic point (x, y, z) of weights (k', l', m') satisfying

- $k' \geq l' + m'$,
- $(k', l', m') \equiv (k, l, m) \pmod{2(p-1)}$,

the following identity holds:

$$(\mathcal{L}_p^f(\mathbf{f}, \mathbf{g}, \mathbf{h})(x, y, z))^2 = E(k', l', m') \frac{L(V_{\mathbf{f}_x \mathbf{g}_y \mathbf{h}_z}, s)}{\pi^{2k'} \Omega_{\mathbf{f}_x}},$$

where $E(k', l', m')$ is an explicit rational factor depending only on k' , l' and m' , and $\Omega_{\mathbf{f}_x}$ is a period attached to the modular form $\mathbf{f}_x$.

Since we are only interested in varying the character ψ , we consider the specialization $\mathcal{L}_p^f \in \Lambda_{\mathbf{f}}$ of the p -adic L -function in the theorem corresponding to the forms g and h .

Recall that g and h are ordinary at p . For $\phi \in \{g, h\}$, let α_ϕ be the unit root of the Hecke polynomial at p of ϕ , i.e., the unit root of $X^2 - a_p(\phi) + \chi_\phi(p)p^{k_\phi-1}$, where k_ϕ denotes the weight of ϕ . Let $\underline{\alpha}_\phi$ be the unramified character of $G_{\mathbb{Q}_p}$ which maps an arithmetic Frobenius Fr_p to α_ϕ . Then, there is an exact sequence of $G_{\mathbb{Q}_p}$ -representations

$$0 \longrightarrow V_\phi^+ \longrightarrow V_\phi \longrightarrow V_\phi^- \longrightarrow 0,$$

where V_ϕ^- is the 1-dimensional unramified representation defined by the character $\underline{\alpha}_\phi$ and V_ϕ^+ is the 1-dimensional representation defined by the character $\chi_\phi \epsilon_{\text{cyc}}^{k_\phi-1} \underline{\alpha}_\phi^{-1}$.

We make one further assumption.

Assumption 1.3.6. p splits in K and does not divide the class number of K .

As above, we denote by $\mathfrak{p}$ the prime of K above p determined by our fixed embedding $\iota_p : \bar{\mathbb{Q}} \hookrightarrow \bar{\mathbb{Q}}_p$, and we denote by $\bar{\mathfrak{p}}$ the other prime of K above p .

We define the following $G_{\mathbb{Q}_p}$ -invariant subspaces of V :

$$\mathcal{F}_{\mathfrak{p}}^+(V) := (V_g^+ \otimes V_h + V_g \otimes V_h^+)(\psi_{\mathfrak{p}}^{-1})(1 - c) \subseteq V,$$

$$\mathcal{F}_{\bar{\mathfrak{p}}}^+(V) := V_g^+ \otimes V_h^+(\psi_{\bar{\mathfrak{p}}}^{-1})(1 - c) \subseteq V.$$

We also define

$$\mathcal{F}_{\mathfrak{p}}^-(V) := V/\mathcal{F}_{\mathfrak{p}}^+(V),$$

$$\mathcal{F}_{\bar{\mathfrak{p}}}^-(V) := V/\mathcal{F}_{\bar{\mathfrak{p}}}^+(V).$$

For any finite extension L of K and for any prime v of L above p , we define the *balanced* local condition by

$$H_{\text{bal}}^1(L_v, V) := \begin{cases} \ker(H^1(L_v, V) \longrightarrow H^1(L_v, \mathcal{F}_{\mathfrak{p}}^-(V))) & \text{if } v \mid \mathfrak{p}, \\ \ker(H^1(L_v, V) \longrightarrow H^1(L_v, \mathcal{F}_{\bar{\mathfrak{p}}}^-(V))) & \text{if } v \mid \bar{\mathfrak{p}}. \end{cases}$$

For finite primes v of L not dividing p we just set $H_{\text{bal}}^1(L_v, V) := H_{\text{ur}}^1(L_v, V)$. We define the *balanced* Selmer group by

$$H_{\text{bal}}^1(L, V) := \ker \left(H^1(L, V) \longrightarrow \prod_{\text{finite } v} H^1(L_v, V)/H_{\text{bal}}^1(L_v, V) \right).$$

If $T \subseteq V$ is a G_L -stable lattice, we define

$$H_{\text{bal}}^1(L, T) := \ker(H^1(L, T) \longrightarrow H^1(L, V)/H_{\text{bal}}^1(L, V)).$$

Let K_∞ denote the anticyclotomic $\mathbb{Z}_p$ -extension of K , and let K_n be the intermediate extension of degree p^n over K . We fix from now on a G_K -stable lattice $T \subseteq V$. We define the group

$$H_{\text{bal}}^1(K_\infty, T) := \varprojlim_n H_{\text{bal}}^1(K_n, T).$$

Similarly, we define the group

$$H_{\text{bal}}^1(K_{\infty, \bar{\mathfrak{p}}}, T) := \varprojlim_n H_{\text{bal}}^1(K_{n, \bar{\mathfrak{p}}}, T),$$

where $K_{n, \bar{\mathfrak{p}}}$ denotes the localization of K_n at the prime above $\bar{\mathfrak{p}}$. There is a natural localization map

$$\text{loc}_{\bar{\mathfrak{p}}} : H_{\text{bal}}^1(K_\infty, T) \longrightarrow H_{\text{bal}}^1(K_{\infty, \bar{\mathfrak{p}}}, T).$$

Both $H_{\text{bal}}^1(K_\infty, T)$ and $H_{\text{bal}}^1(K_{\infty, \bar{\mathfrak{p}}}, T)$ are endowed with a natural action of $\Lambda_{\text{ac}} := \mathcal{O}[[\text{Gal}(K_\infty/K)]]$, and the map $\text{loc}_{\bar{\mathfrak{p}}}$ is compatible with the Λ_{ac} -module structures. Moreover, since $\mathbf{f}$ is a CM family, we can identify $\Lambda_{\mathbf{f}} \simeq \Lambda_{\text{ac}}$.

The following theorem, which is an instance of what is usually referred to as an *explicit reciprocity law*, was proved by Darmon–Rorger [DR22] and Bertolini–Seveso–Venerucci [BSV22].

Theorem 1.3.7. *There exists a class*

$$\kappa_\infty \in H_{\text{bal}}^1(K_\infty, T)$$

and a Λ_{ac} -linear map

$$\mathcal{L} : H_{\text{bal}}^1(K_{\infty, \bar{\mathfrak{p}}}, T) \longrightarrow \Lambda_{\mathbf{f}}$$

such that

$$\mathcal{L}(\text{loc}_{\bar{\mathfrak{p}}}(\kappa_\infty)) = \mathcal{L}_p^f.$$

We can now state the main result in Chapter 2.

Theorem 1.3.8. *The class κ_∞ in the previous theorem is the bottom layer of a split anticyclotomic Euler system for T as defined by Jechev–Nekovář–Skinner.*

Let $\kappa \in H_{\text{bal}}^1(K, V)$ denote the natural projection of κ_∞ .

In order to apply the previous result, we need to ensure that the G_K -representation V satisfies certain properties. The most important one is the existence of an element $\sigma_0 \in G_K$ meeting the requirements specified in the previous section. These conditions hold under some *big image hypotheses* which are discussed in §2.8.2. In this introduction we will just loosely say that V *has big image* to mean that it satisfies these hypotheses.

Now we can state some applications of the main theorem in Chapter 2.

Theorem 1.3.9. *Assume that V has big image. Then, the following implication holds:*

$$\kappa \neq 0 \implies \dim_E H_{\text{bal}}^1(K, V) = 1.$$

Remark 1.3.10. If the weights (k, l, m) are balanced, then it can be proved that $H_{\text{bal}}^1(K, V) = H_f^1(K, V)$, so the previous theorem is consistent with the expectation that $\dim_E H_f^1(K, V)$ should be generically 1 when the global sign of the corresponding functional equation is -1 .

Theorem 1.3.11. *Assume that V has big image. Then, if $k \geq l + m$, the following implication holds:*

$$L(V, s) \neq 0 \implies \dim_E H_f^1(K, V) = 0.$$

Remark 1.3.12. This theorem establishes the Bloch–Kato conjecture for V in the case of analytic rank zero.

In Chapter 3 we focus in a case which is not covered by the previous results. Namely, we consider the case in which $h = g^*$, where g^* is the newform obtained by

applying any choice of complex conjugation to the Fourier coefficients of g .

In this case, since

$$V_{g^*} \simeq V_g^\vee(l-1),$$

the representation V can be decomposed as

$$V = V' \oplus V'',$$

where

$$V' = \text{ad}^0(V_g)(\psi_{\mathfrak{P}}^{-1})(1-k/2),$$

$$V'' = E(\psi_{\mathfrak{P}}^{-1})(1-k/2).$$

Let $\mathcal{F}_{\mathfrak{p}}^+(V') \subseteq V'$ and $\mathcal{F}_{\bar{\mathfrak{p}}}^+(V') \subseteq V'$ denote the images of the subspaces $\mathcal{F}_{\mathfrak{p}}^+(V) \subseteq V$ and $\mathcal{F}_{\bar{\mathfrak{p}}}^+(V) \subseteq V$ under the natural projection map $V \rightarrow V'$. They are subspaces of V' of dimensions 2 and 1, respectively. We also define the quotients $\mathcal{F}_{\mathfrak{p}}^-(V') = V'/\mathcal{F}_{\mathfrak{p}}^+(V')$ and $\mathcal{F}_{\bar{\mathfrak{p}}}^-(V') = V'/\mathcal{F}_{\bar{\mathfrak{p}}}^+(V')$. Then, we can define *balanced* local conditions and *balanced* Selmer groups for V' as we previously did for V .

Let T' denote the image of T in V' , which is a G_K -stable lattice in V' . Then, we can consider the Λ_{ac} -modules

$$H_{\text{bal}}^1(K_\infty, T') := \varprojlim_n H_{\text{bal}}^1(K_n, T'),$$

$$H_{\text{bal}}^1(K_{\infty, \bar{\mathfrak{p}}}, T') := \varprojlim_n H_{\text{bal}}^1(K_{n, \bar{\mathfrak{p}}}, T').$$

It can be shown that the map

$$\mathcal{L} : H_{\text{bal}}^1(K_{\infty, \bar{\mathfrak{p}}}, T) \longrightarrow \Lambda_{\mathfrak{f}}$$

introduced above factors through the natural map

$$H_{\text{bal}}^1(K_{\infty, \bar{\mathfrak{p}}}, T) \longrightarrow H_{\text{bal}}^1(K_{\infty, \bar{\mathfrak{p}}}, T').$$

Let κ'_{∞} denote the image of κ_{∞} in $H_{\text{bal}}^1(K_{\infty}, T')$. In Chapter 3 we prove the following result.

Theorem 1.3.13. *The class κ'_{∞} is the bottom layer of a split anticyclotomic Euler system for T' as defined by Jetchev–Nekovář–Skinner.*

Let $\kappa' \in H_{\text{bal}}^1(K, V')$ denote the natural projection of κ'_{∞} . In §3.6 we discuss big image hypotheses for V' . Under these hypotheses, we can prove the following results, which provide some new evidence for the Bloch–Kato conjecture for V' .

Theorem 1.3.14. *Assume that V' has big image. Then, the following implication holds:*

$$\kappa' \neq 0 \implies \dim_E H_{\text{bal}}^1(K, V') = 1.$$

Theorem 1.3.15. *Assume that V' has big image. Then, if $k \geq 2l$, the following implication holds:*

$$L(V, s) \neq 0 \implies \dim_E H_f^1(K, V') = 0.$$

Chapter 2

The diagonal cycle Euler system for $\mathrm{GL}_2 \times \mathrm{GL}_2$

2.1 Introduction

In this paper we study the anticyclotomic Iwasawa theory of Rankin–Selberg convolutions of two modular forms using a new Euler system arising from p -adic families of diagonal cycles. By an application of Kolyvagin’s methods, our construction yields results towards the Bloch–Kato conjecture and the Iwasawa main conjecture in this setting.

2.1.1 Statement of the main results

Let $g \in S_l(N_g, \chi_g)$ and $h \in S_m(N_h, \chi_h)$ be newforms of weights $l \geq m \geq 2$ of the same parity and nebentypus χ_g and χ_h . Let $K/\mathbb{Q}$ be an imaginary quadratic field of discriminant $-D < 0$. Let $k > 0$ be an even integer, and let ψ be a Hecke character of K of infinity type $(1 - k, 0)$, conductor $\mathfrak{f}$, and central character

$$\varepsilon_\psi = \bar{\chi}_g \bar{\chi}_h.$$

Fix an odd prime $p \nmid N_g N_h$ such that $(\mathfrak{f}, p) = 1$ and an embedding $\iota_p : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$, and let $E = L_{\mathfrak{P}}$ be a finite extension of $\mathbb{Q}_p$ containing the image under ι_p of the values of ψ and the Fourier coefficients of g and h . We consider the E -valued G_K -representation

$$V_{g,h}^{\psi} := V_g \otimes V_h(\psi_{\mathfrak{P}}^{-1})(1 - c),$$

where $c = (k + l + m - 2)/2$, V_g and V_h are the (dual of Deligne's) p -adic Galois representations associated to g and h , respectively, and $\psi_{\mathfrak{P}}$ is a p -adic Galois character attached to ψ .

The cyclotomic Iwasawa theory of $V_g \otimes V_h$ has been extensively studied in a series of works of Lei–Loeffler–Zerbes [LLZ14, LLZ15] and Kings–Loeffler–Zerbes [KLZ17, KLZ20], among others ([BLLV19], [BL21], etc.). The key tool exploited in these works is the Euler system of Beilinson–Flach classes, a system of cohomology classes arising from certain special elements (introduced by Beilinson [Beř84], and further studied by Flach [Fla92] and Bertolini–Darmon–Rotger [BDR15a, BDR15b]) in the K_1 of products of two modular curves.

In contrast, the anticyclotomic Iwasawa theory of $V_g \otimes V_h$ (or rather of its conjugate self-dual twists, such as $V_{g,h}^{\psi}$) appears to not have been studied before. The principal contribution of this paper is the construction of an anticyclotomic Euler system for $V_{g,h}^{\psi}$. As stated in Theorem A below (which corresponds to Theorem 2.6.5 in the body of the paper), for general weights (k, l, m) our construction requires the additional assumptions that p splits in K and $p \nmid h_K$ (the class number of K), and that both g and h are ordinary at p , but note that for $(k, l, m) = (2, l, l)$ Theorem 2.4.5 contains a version of our main result without these additional hypotheses.

Theorem A. *Suppose that p splits in K and $p \nmid h_K$, and that both g and h are ordinary at p . Let $\mathcal{S}$ be the set of all squarefree products of primes split q in K and coprime to $DN_g N_h \mathfrak{f}$, and denote by $K[n]$ the maximal p -extension of K inside the*

ring class field of conductor n . Then there exists a family of cohomology classes

$$\kappa_{\psi,g,h,np^r} \in H^1(K[np^r], T_{g,h}^\psi)$$

for all $n \in \mathcal{S}$ and $r \geq 0$, where $T_{g,h}^\psi$ is a fixed G_K -stable lattice inside $V_{g,h}^\psi$, such that for all $nq \in \mathcal{S}$ with q prime, we have

$$\mathrm{cor}_{K[nqp^r]/K[np^r]}(\kappa_{\psi,g,h,nqp^r}) = \begin{cases} P_{\mathfrak{q}}(\mathrm{Fr}_{\mathfrak{q}}^{-1}) \kappa_{\psi,g,h,np^r} & \text{if } q \neq p, \\ \kappa_{\psi,g,h,np^r} & \text{if } q = p, \end{cases}$$

where $\mathfrak{q}$ is any of the prime and K above q , and $P_{\mathfrak{q}}(\mathrm{Fr}_{\mathfrak{q}}^{-1}) = \det(1 - \mathrm{Fr}_{\mathfrak{q}}^{-1} X | (V_{g,h}^\psi)^\vee(1))$.

The construction of this Euler system, which is taken up in the first part of the paper, is based on the diagonal classes studied by Darmon–Rotger [DR14, DR17, DR22] and Bertolini–Seveso–Venerucci [BSV22], extending earlier constructions due to Gross–Kudla [GK92] and Gross–Schoen [GS95]. Roughly speaking, our classes κ_{ψ,g,h,np^r} are suitable modifications of diagonal classes for the triples $(\tilde{\theta}_{\psi,np^r}, g, h)$, where $\tilde{\theta}_{\psi,np^r}$ is a certain deformation of the theta series associated to ψ , and the main difficulty in the proof of Theorem A is in establishing the Euler system norm relations.

The main results in the second part of the paper are the proof of new cases of the Bloch–Kato conjecture for $V_{g,h}^\psi$ in analytic rank zero and a divisibility towards the Iwasawa main conjecture for $V_{g,h}^\psi$. These are obtained by applying Kolyvagin’s methods (in the form recently developed by Jetchev–Nekovář–Skinner [JNS] in the anticyclotomic setting) to our Euler system. In the results that follow, we use “big image” to refer to Hypothesis (HS) in §2.8.1, for which sufficient conditions are given in §2.8.2.

Theorem B. *Suppose that:*

(a) *g and h are ordinary at p , non-Eisenstein and p -distinguished,*

(b) p splits in K ,

(c) p does not divide the class number of K ,

(d) $V_{g,h}^\psi$ has big image.

Let

$$\kappa_{\psi,g,h} := \kappa_{\psi,g,h,1}.$$

If $l - m < k < l + m$, then the following implication holds:

$$\kappa_{\psi,g,h} \neq 0 \implies \dim_E \operatorname{Sel}(K, V_{g,h}^\psi) = 1,$$

where $\operatorname{Sel}(K, V_{g,h}^\psi) \subset H^1(G_K, V_{g,h}^\psi)$ is the Bloch–Kato Selmer group.

Remark. (1) For $k = l = m = 2$, together with the Gross–Zagier formula for diagonal cycles by Yuan–Zhang–Zhang [YZZ12], Theorem B supports the Bloch–Kato conjecture for $V_{g,h}^\psi$ in analytic rank one, reducing it to the expected injectivity of the p -adic étale Abel–Jacobi map.

(2) Still in the case $k = l = m = 2$, combined with the p -adic Gross–Zagier formula for diagonal cycles in forthcoming work of Hsieh–Yamana [HY], Theorem B establishes some cases of Perrin-Riou’s p -adic Beilinson conjecture in analytic rank one.

(3) In general, by the main result of [DR14], the nonvanishing of $\kappa_{\psi,g,h}$ also follows from the nonvanishing of a special value of the triple product p -adic L -function $\mathcal{L}_p(\mathbf{f}, g, h)$ introduced below.

In analytic rank zero, we get unconditional applications to the Bloch–Kato conjecture. Let $f = \theta_\psi \in S_k(N_\psi, \varepsilon_\psi)$ be the theta series associated to ψ , let $\varepsilon_\ell(V_{g,h}^\psi)$ be the epsilon factor of the Weil–Deligne representation associated to the restriction of $V_f \otimes V_g \otimes V_h(1 - c)$ to $G_{\mathbb{Q}_\ell}$, and put $N = \operatorname{lcm}(N_\psi, N_g, N_h)$.

Theorem C. *Let the hypotheses be as in Theorem B, and assume in addition that*

- $\varepsilon_\ell(V_{g,h}^\psi) = +1$ for all primes $\ell \mid N$,
- $\gcd(N_\psi, N_g, N_h)$ is squarefree.

If $k \geq l + m$, then

$$L(V_{g,h}^\psi, 0) \neq 0 \implies \text{Sel}(K, V_{g,h}^\psi) = 0,$$

and hence the Bloch–Kato conjecture for $V_{g,h}^\psi$ holds in analytic rank zero.

Remark. Here $L(V_{g,h}^\psi, s)$ is the triple product L -function introduced by Garrett, Piatetski–Shapiro and Rallis, which satisfies a functional equation relating its values at s and $-s$. When $k \geq l + m$, the local root number condition in Theorem C implies that the sign in this functional equation is $+1$, and so the central L -values $L(V_{g,h}^\psi, 0)$ are expected to be generically nonzero.

A third application is to the anticyclotomic Iwasawa main conjectures for Rankin–Selberg convolutions. Let $(\mathbf{f}, \mathbf{g}, \mathbf{h})$ be a triple of p -adic Hida families. In [Hsi21], Hsieh has constructed a square-root triple product p -adic L -function $\mathcal{L}_p(\mathbf{f}, \mathbf{g}, \mathbf{h})$ whose square interpolates the central values of the triple product L -function attached to the classical specializations of $(\mathbf{f}, \mathbf{g}, \mathbf{h})$ to weights (k_1, k_2, k_3) with $k_1 \geq k_2 + k_3$. Letting $\mathbf{g}$ and $\mathbf{h}$ be the Hida families passing through the ordinary p -stabilizations of g and h , respectively, we obtain an element

$$\mathcal{L}_p(\mathbf{f}, g, h) \in \Lambda_{\mathbf{f}}$$

interpolating a square-root of the above central L -values for the specializations of $\mathbf{f}$ to weights $k \geq l + m$, where $\Lambda_{\mathbf{f}}$ is the finite flat extension of $\Lambda = \mathbb{Z}_p[[1 + p\mathbb{Z}_p]]$ generated by the coefficients of $\mathbf{f}$. Greenberg’s generalization of the Iwasawa main

conjectures [Gre94] predicts that $\mathcal{L}_p(\mathbf{f}, g, h)^2$ generates the $\Lambda_{\mathbf{f}}$ -characteristic ideal of a certain torsion Selmer group $X_{\mathcal{F}}(\mathbb{A}_{\mathbf{f}gh}^{\dagger})$. At the same time, we show that our classes $\kappa_{\psi, g, h, n}$ are universal norms in the p -direction, therefore giving rise in particular to an Iwasawa cohomology class

$$\kappa_{\psi, g, h, \infty} \in H_{\text{Iw}}^1(K_{\infty}, T_{g, h}^{\psi})$$

for the anticyclotomic $\mathbb{Z}_p$ -extension K_{∞}/K .

The class $\kappa_{\psi, g, h, \infty}$ is associated with the triple $(\mathbf{f}, g, h)$, where $\mathbf{f} = \mathbf{f}_{\psi}$ is a CM Hida family attached to ψ for which $\Lambda_{\mathbf{f}} \cong \Lambda_{\text{ac}}$, the anticyclotomic Iwasawa algebra. Assuming the non-triviality of $\kappa_{\psi, g, h, \infty}$, we can prove the following result towards the Iwasawa main conjecture for $\mathcal{L}_p(\mathbf{f}, g, h)^2$.

Theorem D. *Let $\mathbf{f} = \mathbf{f}_{\psi}$, and suppose that:*

- (a) *g and h are ordinary at p , non-Eisenstein and p -distinguished,*
- (b) *p splits in K ,*
- (c) *p does not divide the class number of K ,*
- (d) *$V_{g, h}^{\psi}$ has big image,*
- (e) *$\varepsilon_{\ell}(V_{g, h}^{\psi}) = +1$ for all primes $\ell \mid N$,*
- (f) *$\gcd(N_{\psi}, N_g, N_h)$ is squarefree.*

If $\kappa_{\psi, g, h, \infty}$ is not Λ_{ac} -torsion, then the module $X_{\mathcal{F}}(\mathbb{A}_{\mathbf{f}gh}^{\dagger})$ is Λ_{ac} -torsion, and

$$\text{Char}_{\Lambda_{\text{ac}}}(X_{\mathcal{F}}(\mathbb{A}_{\mathbf{f}gh}^{\dagger})) \supset (\mathcal{L}_p(\mathbf{f}, g, h)^2)$$

in $\Lambda_{\text{ac}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$.

Remark. The classes $\kappa_{\psi,g,h,n}$ may be viewed as a counterpart in the study of the arithmetic of $V_{g,h}^{\psi}$ to systems of Heegner points and Heegner cycles for individual modular forms. It would be interesting to see whether the methods of Cornut–Vatsal can be extended to establish the non-triviality of $\kappa_{\psi,g,h,\infty}$.

Remark. The “big image” hypothesis in our arithmetic applications in this paper excludes in particular the cases in which $g = h$ and in which either g or h has CM (see Hypotheses 2.9). Notwithstanding, in future work we plan to use (a suitable projection of) the classes $\kappa_{\psi,g,h,n}$ for $g = h$ to study the anticyclotomic Iwasawa theory of $\mathrm{Sym}^2 g$. The case in which either g or h has CM will appear in the forthcoming PhD thesis of Kim Tuan Do.

2.1.2 Acknowledgements

It is a pleasure to thank Chris Skinner, as well as Kâzim Büyükboduk, Daniel Disegni, Ming-Lun Hsieh, Antonio Lei, Victor Rotger and Shou-Wu Zhang, for several helpful conversations related to this work. This project has received funding from the ERC under the European Union’s Horizon 2020 research and innovation programme (grant agreement No 682152). During the preparation of this paper, F.C. was partially supported by the NSF grants DMS-1946136 and DMS-2101458; O.R. was supported by a Royal Society Newton International Fellowship and by “la Caixa” Foundation (grant LCF/BQ/ES17/11600010).

2.2 Preliminaries

In this section we begin by discussing our conventions regarding modular curves and Hecke operators, for which we shall largely follow [Kat04, §2] and [BSV22, §2].

2.2.1 Modular curves

Given integers $M \geq 1$, $N \geq 1$, $m \geq 1$ and $n \geq 1$ with $M + N \geq 5$, we denote by $Y(M(m), N(n))$ the affine modular curve over $\mathbb{Z}[1/MNmn]$ representing the functor taking a $\mathbb{Z}[1/MNmn]$ -scheme S to the set of isomorphism classes of 5-tuples (E, P, Q, C, D) , where:

- E is an elliptic curve over S ,
- P is an S -point of E of order M ,
- Q is an S -point of E of order N ,
- C is a cyclic order- Mm subgroup of E defined over S and containing P ,
- D is a cyclic order- Nn subgroup of E defined over S and containing Q ,

and such that C and D have trivial intersection. If either $m = 1$ or $n = 1$ we omit it from the notation, and we will often write $Y_1(N)$ for $Y(1, N)$.

We will denote by

$$E(M(m), N(n)) \rightarrow Y(M(m), N(n))$$

the universal elliptic curve over $Y(M(m), N(n))$.

Define the modular group

$$\Gamma(M(m), N(n)) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : a \equiv 1 \pmod{M}, b \equiv 0 \pmod{Mm}, c \equiv 0 \pmod{Nn}, d \equiv 1 \pmod{N} \right\}.$$

Then, letting $\mathcal{H}$ be the Poincaré upper half-plane, we have the complex uniformization

$$Y(M(m), N(n))(\mathbb{C}) \cong (\mathbb{Z}/M\mathbb{Z})^\times \times \Gamma(M(m), N(n)) \backslash \mathcal{H}, \quad (2.1)$$

with a pair (a, τ) on the right-hand side corresponding to the isomorphism class of the 5-tuple $(\mathbb{C}/\mathbb{Z} + \mathbb{Z}\tau, a\tau/M, 1/N, \langle \tau/Mm \rangle, \langle 1/Nn \rangle)$.

If $r \geq 1$ is an integer, there is an isomorphism of $\mathbb{Z}[1/MNmnr]$ -schemes

$$\varphi_r : Y(M(m), N(nr)) \xrightarrow{\sim} Y(M(mr), N(n))$$

defined in terms of moduli by

$$(E, P, Q, C, D) \mapsto (E', P', Q', C', D'),$$

where $E' = E/NnD$, P' is the image of P in E' , Q' is the image of $r^{-1}(Q) \cap D$ in E' , C' is the image of $r^{-1}(C)$ in E' , and D' is the image of D in E' . Under the complex uniformizations (2.1), the isomorphism φ_r sends $(a, \tau) \mapsto (a, r \cdot \tau)$. If

$$\varphi_r^*(E(M(mr), N(n))) \rightarrow Y(M(m), N(nr))$$

denotes the base change of $E(M(mr), N(n)) \rightarrow Y(M(mr), N(n))$ under φ_r , there is a natural degree- r isogeny

$$\lambda_r : E(M(m), N(nr)) \rightarrow \varphi_r^*(E(M(mr), N(n))).$$

2.2.2 Degeneracy maps

With the same notations as above, we have natural degeneracy maps

$$\begin{aligned} Y(M(m), Nr(n)) &\xrightarrow{\mu_r} Y(M(m), N(nr)) \xrightarrow{\nu_r} Y(M(m), N(n)), \\ Y(Mr(m), N(n)) &\xrightarrow{\check{\mu}_r} Y(M(mr), N(n)) \xrightarrow{\check{\nu}_r} Y(M(m), N(n)), \end{aligned}$$

forgetting the extra level structure, e.g.

$$\begin{aligned}\mu_r(E, P, Q, C, D) &= (E, P, r \cdot Q, C, D), \\ \nu_r(E, P, Q, C, D) &= (E, P, Q, C, rD).\end{aligned}$$

We also define degeneracy maps

$$\begin{aligned}\pi_1 : Y(M(m), Nrs(nt)) &\rightarrow Y(M(m), N(ns)), \\ \pi_2 : Y(M(m), Nrs(nt)) &\rightarrow Y(M(m), N(ns)),\end{aligned}\tag{2.2}$$

acting on the moduli space by

$$\begin{aligned}\pi_1(E, P, Q, C, D) &= (E, P, rs \cdot Q, C, rtD), \\ \pi_2(E, P, Q, C, D) &= (E', P', Q', C', D'),\end{aligned}$$

where $E' = E/NnsD$, P' is the image of P in E' , Q' is the image of $t^{-1}(s \cdot Q) \cap D$ in E' , C' is the image of C in E' and D' is the image of D in E' . Under the complex uniformizations (2.1), the maps π_1 and π_2 correspond to the identity and to multiplication by rt , respectively, on $\mathcal{H}$. It is straightforward to check that the maps π_1 and π_2 are given by the compositions

$$\begin{aligned}Y(M(m), Nrs(nt)) &\xrightarrow{\mu_{rs}} Y(M(m), N(nrst)) \xrightarrow{\nu_{rt}} Y(M(m), N(ns)), \\ Y(M(m), Nrs(nt)) &\xrightarrow{\mu_{rs}} Y(M(m), N(nrst)) \xrightarrow{\varphi_{rt}} Y(M(mrt), N(ns)) \xrightarrow{\tilde{\nu}_{rt}} Y(M(m), N(ns)),\end{aligned}$$

respectively.

2.2.3 Relative Tate modules

Fix a prime p . Let S be a $\mathbb{Z}[1/MNmp]$ -scheme and let

$$v : E(M(m), N(n))_S \rightarrow Y(M(m), N(n))_S$$

be the structural morphism. For every $\mathbb{Z}[1/MNmp]$ -scheme X , denote by $A = A_X$ either the locally constant constructible sheaf $\mathbb{Z}/p^t(j)$ or the locally constant p -adic sheaf $\mathbb{Z}_p(j)$ on $X_{\text{ét}}$, for fixed $t \geq 1$ and $j \in \mathbb{Z}$. Set

$$\mathcal{T}_{M(m), N(n)}(A) = R^1 v_* \mathbb{Z}_p(1) \otimes_{\mathbb{Z}_p} A \quad \text{and} \quad \mathcal{T}_{M(m), N(n)}^*(A) = \text{Hom}(\mathcal{T}_{M(m), N(n)}(A), A).$$

In particular, in the case $A = \mathbb{Z}_p$ this gives the relative Tate module of the universal elliptic curve and its dual, respectively; in this case, we will often drop A from the notation.

From the proper base change theorem, both $\mathcal{T}_{M(m), N(n)}(A)$ and $\mathcal{T}_{M(m), N(n)}^*(A)$ are locally constant p -adic sheaves on $Y(M(m), N(n))_S$ of formation compatible with base changes along morphisms of $\mathbb{Z}[1/MNmp]$ -schemes $S' \rightarrow S$.

For every integer $r \geq 0$, define

$$\mathcal{L}_{M(m), N(n), r}(A) = \text{Tsymb}_A^r \mathcal{T}_{M(m), N(n)}(A), \quad \mathcal{S}_{M(m), N(n), r}(A) = \text{Sym}_A^r \mathcal{T}_{M(m), N(n)}^*(A),$$

where, for any finite free module M over a profinite $\mathbb{Z}_p$ -algebra R , one denotes by $\text{Tsymb}_R^r M$ the R -submodule of symmetric tensors in $M^{\otimes r}$ and by $\text{Sym}_R^r M$ the maximal symmetric quotient of $M^{\otimes r}$.

When the level of the modular curve $Y(M(m), N(n))_S$ is clear, we may use the simplified notations

$$\mathcal{L}_r(A) = \mathcal{L}_{M(m), N(n), r}(A), \quad \mathcal{L}_r = \mathcal{L}_r(\mathbb{Z}_p), \quad \mathcal{S}_r(A) = \mathcal{S}_{M(m), N(n), r}(A), \quad \mathcal{S}_r = \mathcal{S}_r(\mathbb{Z}_p).$$

2.2.4 Hecke operators

Let $\mathcal{F}_{M(m),N(n)}^r$ denote either $\mathcal{L}_{M(m),N(n),r}(A)$ or $\mathcal{S}_{M(m),N(n),r}(A)$ and let q be a rational prime. Then there are natural isomorphisms of sheaves

$$\nu_q^*(\mathcal{F}_{M(m),N(n)}^r) \cong \mathcal{F}_{M(m),N(nq)}^r \quad \text{and} \quad \check{\nu}_q^*(\mathcal{F}_{M(m),N(n)}^r) \cong \mathcal{F}_{M(mq),N(n)}^r, \quad (2.3)$$

and therefore pullback morphisms

$$\begin{aligned} H_{\text{ét}}^i(Y(M(m), N(n))_S, \mathcal{F}_{M(m),N(n)}^r) &\xrightarrow{\nu_q^*} H_{\text{ét}}^i(Y(M(m), N(nq))_S, \mathcal{F}_{M(m),N(nq)}^r), \\ H_{\text{ét}}^i(Y(M(m), N(n))_S, \mathcal{F}_{M(m),N(n)}^r) &\xrightarrow{\check{\nu}_q^*} H_{\text{ét}}^i(Y(M(mq), N(n))_S, \mathcal{F}_{M(mq),N(n)}^r), \end{aligned}$$

and traces

$$\begin{aligned} H_{\text{ét}}^i(Y(M(m), N(nq))_S, \mathcal{F}_{M(m),N(nq)}^r) &\xrightarrow{\nu_{q*}} H_{\text{ét}}^i(Y(M(m), N(n))_S, \mathcal{F}_{M(m),N(n)}^r), \\ H_{\text{ét}}^i(Y(M(mq), N(n))_S, \mathcal{F}_{M(mq),N(n)}^r) &\xrightarrow{\check{\nu}_{q*}} H_{\text{ét}}^i(Y(M(m), N(n))_S, \mathcal{F}_{M(m),N(n)}^r). \end{aligned} \quad (2.4)$$

Also, the isogeny λ_q induces morphisms of sheaves

$$\lambda_{q*} : \mathcal{F}_{M(m),N(nq)}^r \rightarrow \varphi_q^*(\mathcal{F}_{M(mq),N(n)}^r) \quad \text{and} \quad \lambda_q^* : \varphi_q^*(\mathcal{F}_{M(mq),N(n)}^r) \rightarrow \mathcal{F}_{M(m),N(nq)}^r.$$

These morphisms allow us to define

$$\begin{aligned} \Phi_{q*} : H_{\text{ét}}^i(Y(M(m), N(nq))_S, \mathcal{F}_{M(m),N(nq)}^r) &\rightarrow H_{\text{ét}}^i(Y(M(mq), N(n))_S, \mathcal{F}_{M(mq),N(n)}^r), \\ \Phi_q^* : H_{\text{ét}}^i(Y(M(mq), N(n))_S, \mathcal{F}_{M(mq),N(n)}^r) &\rightarrow H_{\text{ét}}^i(Y(M(m), N(nq))_S, \mathcal{F}_{M(m),N(nq)}^r), \end{aligned}$$

as the compositions

$$\Phi_{q*} = \varphi_{q*} \circ \lambda_{q*} \quad \text{and} \quad \Phi_q^* = \lambda_q^* \circ \varphi_q^*.$$

We define the Hecke operators T_q and the adjoint Hecke operators T'_q acting on the étale cohomology groups

$$H_{\text{ét}}^i(Y(M(m), N(nq))_S, \mathcal{F}_{M(m), N(nq)}^r)$$

as the compositions

$$T_q = \check{\nu}_{q*} \circ \Phi_{q*} \circ \nu_q^* \quad \text{and} \quad T'_q = \nu_{q*} \circ \Phi_q^* \circ \check{\nu}_q^*.$$

If we define pullbacks

$$\begin{aligned} H_{\text{ét}}^i(Y(M(m), N(n))_S, \mathcal{F}_{M(m), N(n)}^r) &\xrightarrow{\pi_1^*} H_{\text{ét}}^i(Y(M(m), N(nq))_S, \mathcal{F}_{M(m), N(nq)}^r), \\ H_{\text{ét}}^i(Y(M(m), N(n))_S, \mathcal{F}_{M(m), N(n)}^r) &\xrightarrow{\pi_2^*} H_{\text{ét}}^i(Y(M(mq), N(n))_S, \mathcal{F}_{M(mq), N(n)}^r), \end{aligned}$$

and pushforwards

$$\begin{aligned} H_{\text{ét}}^i(Y(M(m), N(nq))_S, \mathcal{F}_{M(m), N(nq)}^r) &\xrightarrow{\pi_{1*}} H_{\text{ét}}^i(Y(M(m), N(n))_S, \mathcal{F}_{M(m), N(n)}^r), \\ H_{\text{ét}}^i(Y(M(mq), N(n))_S, \mathcal{F}_{M(mq), N(n)}^r) &\xrightarrow{\pi_{2*}} H_{\text{ét}}^i(Y(M(m), N(n))_S, \mathcal{F}_{M(m), N(n)}^r), \end{aligned}$$

as

$$\pi_1^* = \nu_q^*, \quad \pi_2^* = \Phi_q^* \circ \check{\nu}_q^*, \quad \pi_{1*} = \nu_{q*} \quad \text{and} \quad \pi_{2*} = \check{\nu}_{q*} \circ \Phi_{q*},$$

then we can write

$$T_q = \pi_{2*} \circ \pi_1^* \quad \text{and} \quad T'_q = \pi_{1*} \circ \pi_2^*.$$

Now we introduce diamond operators. For $d \in (\mathbb{Z}/MN\mathbb{Z})^\times$, these are defined on the curves $Y(M(m), N(n))$ as the automorphisms $\langle d \rangle$ acting on the moduli space by

$$(E, P, Q, C, D) \mapsto (E, d^{-1} \cdot P, d \cdot Q, C, D).$$

We can also define the diamond operator $\langle d \rangle$ on the corresponding universal elliptic curve as the unique automorphism making the diagram

$$\begin{array}{ccc} E(M(m), N(n))_S & \xrightarrow{\langle d \rangle} & E(M(m), N(n))_S \\ \downarrow v & & \downarrow v \\ Y(M(m), N(n))_S & \xrightarrow{\langle d \rangle} & Y(M(m), N(n))_S \end{array}$$

cartesian. This in turn induces automorphisms $\langle d \rangle = \langle d \rangle^*$ and $\langle d \rangle' = \langle d \rangle_*$ on the group $H_{\text{ét}}^i(Y(M(m), N(n))_S, \mathcal{F}_{M(m), N(n)}^r)$ which are inverses of each other.

In general, we will be interested in modular curves of the form $Y(1(m), N(n))$. In this case, the natural pairing $\mathcal{L}_r \otimes_{\mathbb{Z}_p} \mathcal{S}_r \rightarrow \mathbb{Z}_p$ together with cup-product yields a pairing

$$H_{\text{ét}}^1(Y(1(m), N(n))_S, \mathcal{L}_r(1)) \otimes_{\mathbb{Z}_p} H_{\text{ét}, c}^1(Y(1(m), N(n))_S, \mathcal{S}_r) \rightarrow \mathbb{Z}_p$$

which becomes perfect after inverting p . The operators $T_q, T'_q, \langle d \rangle, \langle d \rangle'$ induce endomorphisms on compactly supported cohomology and

$$(T_q, T'_q), \quad (T'_q, T_q), \quad (\langle d \rangle, \langle d \rangle') \quad \text{and} \quad (\langle d \rangle', \langle d \rangle)$$

are adjoint pairs under this pairing.

2.2.5 Galois representations

Let $f \in S_k(N_f, \chi_f)$ be a newform of weight $k = r + 2 \geq 2$, level N_f and character χ_f . Let p be a prime and let E be a finite extension of $\mathbb{Q}_p$ with ring of integers $\mathcal{O}$ containing the Fourier coefficients of f . By work of Eichler–Shimura and Deligne, there is a two-dimensional representation

$$\rho_f : G_{\mathbb{Q}} \longrightarrow \text{GL}_2(E)$$

unramified outside pN_f and characterized by the property that

$$\text{trace } \rho_f(\text{Fr}_q) = a_q(f)$$

for all primes $q \nmid pN_f$, where Fr_q denotes an arithmetic Frobenius element at q . (In fact, this is the dual of the p -adic representation constructed by Deligne.)

It will be convenient for our purposes to work with the following geometric realization of ρ_f . Let

$$H_{\text{ét}}^1(Y_1(N_f)_{\overline{\mathbb{Q}}}, \mathcal{L}_r(1)) \otimes_{\mathbb{Z}_p} E \twoheadrightarrow V_f$$

be the maximal quotient on which T'_q and $\langle d \rangle'$ act as multiplication by $a_q(f)$ and $\chi_f(d)$ for all primes $q \nmid N_f$ and all $d \in (\mathbb{Z}/N_f\mathbb{Z})^\times$. Then V_f is a two-dimensional E -vector space affording the p -adic representation ρ_f , and we let $T_f \subset V_f$ be the lattice defined by the image of

$$H_{\text{ét}}^1(Y_1(N_f)_{\overline{\mathbb{Q}}}, \mathcal{L}_r(1)) \otimes_{\mathbb{Z}_p} \mathcal{O}$$

under the above quotient map.

2.3 Hecke algebras and ring class fields

In this section we extend the results of [LLZ15, §5.2], including ring class fields extensions of an imaginary quadratic field K . The resulting Corollary 2.3.6 will allow us to obtain classes over ring class field extension of K from diagonal cycles over $\mathbb{Q}$ on triple products of modular curves of varying levels.

Thus let K be an imaginary quadratic field of discriminant $-D < 0$, and let ε_K be the corresponding quadratic character. Let ψ be a Grössencharacter of K of infinity type $(-1, 0)$ and conductor $\mathfrak{f}$, taking values in a finite extension L/K , and let χ be the unique Dirichlet character modulo $N_{K/\mathbb{Q}}(\mathfrak{f})$ such that $\psi((n)) = n\chi(n)$ for integers n

coprime to $N_{K/\mathbb{Q}}(\mathfrak{f})$. Put $N_\psi = N_{K/\mathbb{Q}}(\mathfrak{f})D$, and let $\theta_\psi \in S_2(N_\psi, \chi_{\varepsilon_K})$ be the newform attached to ψ , i.e.,

$$\theta_\psi = \sum_{(\mathfrak{a}, \mathfrak{f})=1} \psi(\mathfrak{a}) q^{N_{K/\mathbb{Q}}(\mathfrak{a})}.$$

Fix a prime $p \geq 5$ unramified in K , a prime $\mathfrak{p}$ of K above p and a prime $\mathfrak{P}$ of L above $\mathfrak{p}$. Let $E = L_{\mathfrak{P}}$ and let $\mathcal{O} \subset E$ be the ring of integers. Let $\psi_{\mathfrak{P}}$ be the continuous E -valued character of $K^\times \backslash \mathbb{A}_{K, \mathfrak{f}}^\times$ defined by

$$\psi_{\mathfrak{P}}(x) = x_{\mathfrak{p}}^{-1} \psi(x),$$

where $x_{\mathfrak{p}}$ is the projection of the idèle x to the component at $\mathfrak{p}$. We will also denote by $\psi_{\mathfrak{P}}$ the corresponding character of G_K obtained via the geometric Artin map. Then $\text{Ind}_K^{\mathbb{Q}} E(\psi_{\mathfrak{P}}^{-1})$ is the p -adic representation attached to θ_ψ .

Definition 2.3.1. For an integral ideal $\mathfrak{n}$ of K , we denote by $H_{\mathfrak{n}}$ the maximal p -quotient of the corresponding ray class group, and by $K(\mathfrak{n})$ the maximal p -extension in the corresponding ray class field. We similarly define R_n and $K[n]$, for each integer $n > 0$, as the maximal p -quotient in the corresponding ring class group and the maximal p -extension in the corresponding ring class field.

Let $\mathfrak{n}$ be an integral ideal of K divisible by $\mathfrak{f}$, and let $N = N_{K/\mathbb{Q}}(\mathfrak{n})D$, which is of course a multiple of N_ψ . Let $\mathbb{T}'_1(N)$ be the algebra generated by all the Hecke operators $T'_q, \langle d \rangle'$ acting on $H^1(Y_1(N)(\mathbb{C}), \mathbb{Z})$.

Proposition 2.3.2. *With the previous definitions and notations, there exists a homomorphism $\phi_{\mathfrak{n}} : \mathbb{T}'_1(N) \rightarrow \mathcal{O}[H_{\mathfrak{n}}]$ defined on generators by*

$$\phi_{\mathfrak{n}}(T'_q) = \sum_{\mathfrak{q}} \psi(\mathfrak{q})[\mathfrak{q}]$$

for every rational prime q , where the sum runs over ideals coprime to $\mathfrak{n}$ of norm q ;

and

$$\phi_{\mathfrak{n}}(\langle d \rangle') = \chi(d)\varepsilon_K(d)[(d)].$$

Proof. This follows immediately from [LLZ15, Prop. 3.2.1]. $\square$

Now let $\mathfrak{n}' = \mathfrak{n}\mathfrak{q}$ for some prime ideal $\mathfrak{q}$ above a rational prime q . Assume that $\mathfrak{n}'$ is coprime to p , and let $N' = N_{K/\mathbb{Q}}(\mathfrak{n}')D$. Following [LLZ15, §3.3], we define norm maps

$$\mathcal{N}_{\mathfrak{n}}^{\mathfrak{n}'} : \mathcal{O}[H_{\mathfrak{n}'}] \otimes_{(\mathbb{T}'_1(N') \otimes \mathbb{Z}_p, \phi_{\mathfrak{n}'})} H_{\text{ét}}^1(Y_1(N')_{\overline{\mathbb{Q}}}, \mathbb{Z}_p(1)) \longrightarrow \mathcal{O}[H_{\mathfrak{n}}] \otimes_{(\mathbb{T}'_1(N) \otimes \mathbb{Z}_p, \phi_{\mathfrak{n}})} H_{\text{ét}}^1(Y_1(N)_{\overline{\mathbb{Q}}}, \mathbb{Z}_p(1))$$

by the formulae:

- if $\mathfrak{q} \mid \mathfrak{n}$,

$$\mathcal{N}_{\mathfrak{n}}^{\mathfrak{n}'} = 1 \otimes \pi_{1*};$$

- if $\mathfrak{q} \nmid \mathfrak{n}$ and $\mathfrak{q}$ is ramified or split,

$$\mathcal{N}_{\mathfrak{n}}^{\mathfrak{n}'} = 1 \otimes \pi_{1*} - \frac{\psi(\mathfrak{q})[\mathfrak{q}]}{q} \otimes \pi_{2*};$$

- if $\mathfrak{q} \nmid \mathfrak{n}$ and $\mathfrak{q}$ is inert,

$$\mathcal{N}_{\mathfrak{n}}^{\mathfrak{n}'} = 1 \otimes \pi_{1*} - \frac{\psi(\mathfrak{q})[\mathfrak{q}]}{q^2} \otimes \pi_{2*}.$$

If p splits in K , the maps above are also well-defined if $\mathfrak{p} \mid \mathfrak{n}'$ (see [LLZ15, Prop. 4.3.6] and [LLZ15, Prop. 5.2.5]).

More generally, for $\mathfrak{n}' = \mathfrak{n}\mathfrak{r}$ with $\mathfrak{r}$ a product of (not necessarily distinct) prime ideals, we define the map $\mathcal{N}_{\mathfrak{n}}^{\mathfrak{n}'}$ by composing in the natural way the previously defined norm maps.

From now on, we assume that in the case where $(p) = \mathfrak{p}\bar{\mathfrak{p}}$ splits in K , if $\mathfrak{p} \mid \mathfrak{f}$ then $\bar{\mathfrak{p}} \nmid \mathfrak{f}$ and $\psi|_{\mathcal{O}_{K,\mathfrak{p}}^\times}$ is not congruent to the Teichmüller character modulo $\mathfrak{P}$.

Theorem 2.3.3. *Let A be the set of prime ideals of K coprime to $\bar{\mathfrak{p}}$ (resp. p) if p splits (resp. is inert) in K and divisible by $\mathfrak{f}$. Then there is a family of $G_{\mathbb{Q}}$ -equivariant isomorphisms of $\mathcal{O}[H_{\mathfrak{n}}]$ -modules*

$$\nu_{\mathfrak{n}} : \mathcal{O}[H_{\mathfrak{n}}] \otimes_{(\mathbb{T}'_1(N) \otimes \mathbb{Z}_p, \phi_{\mathfrak{n}})} H_{\text{ét}}^1(Y_1(N)_{\overline{\mathbb{Q}}}, \mathbb{Z}_p(1)) \xrightarrow{\simeq} \text{Ind}_{K(\mathfrak{n})}^{\mathbb{Q}} \mathcal{O}(\psi_{\mathfrak{P}}^{-1}),$$

for all $\mathfrak{n} \in A$, such that for $\mathfrak{n} \mid \mathfrak{n}'$ the diagram

$$\begin{array}{ccc} \mathcal{O}[H_{\mathfrak{n}'}] \otimes_{(\mathbb{T}'_1(N') \otimes \mathbb{Z}_p, \phi_{\mathfrak{n}'})} H_{\text{ét}}^1(Y_1(N')_{\overline{\mathbb{Q}}}, \mathbb{Z}_p(1)) & \xrightarrow[\simeq]{\nu_{\mathfrak{n}'}} & \text{Ind}_{K(\mathfrak{n}')}^{\mathbb{Q}} \mathcal{O}(\psi_{\mathfrak{P}}^{-1}) \\ \mathcal{N}_{\mathfrak{n}}^{\mathfrak{n}'} \downarrow & & \downarrow \\ \mathcal{O}[H_{\mathfrak{n}}] \otimes_{(\mathbb{T}'_1(N) \otimes \mathbb{Z}_p, \phi_{\mathfrak{n}})} H_{\text{ét}}^1(Y_1(N)_{\overline{\mathbb{Q}}}, \mathbb{Z}_p(1)) & \xrightarrow[\simeq]{\nu_{\mathfrak{n}}} & \text{Ind}_{K(\mathfrak{n})}^{\mathbb{Q}} \mathcal{O}(\psi_{\mathfrak{P}}^{-1}) \end{array}$$

commutes, where the right vertical arrow is the natural norm map.

Proof. This is [LLZ15, Cor. 5.2.6]. □

Definition 2.3.4. For any positive integer n with $(n, p\mathfrak{f}) = 1$, we let $K(\mathfrak{f})[n]$ be the compositum of $K(\mathfrak{f})$ and $K[n]$, and put $R_{\mathfrak{f},n} = \text{Gal}(K(\mathfrak{f})[n]/K)$.

Let $\mathbb{T}'(1, N_{\psi}(n^2)) \subset \text{End}_{\mathbb{Z}}(H^1(Y(1, N_{\psi}(n^2))(\mathbb{C}), \mathbb{Z}))$ be the subalgebra generated by all Hecke operators T'_q and $\langle d \rangle'$.

Lemma 2.3.5. *There exists a homomorphism*

$$\phi_n : \mathbb{T}'(1, N_{\psi}(n^2)) \longrightarrow \mathcal{O}[R_{\mathfrak{f},n}]$$

defined on generators by the same formula as in Proposition 2.3.2.

Proof. Take the modulus $\mathfrak{n} = \mathfrak{f}(n)$. By Proposition 5.1.2 and Remark 5.1.3 in [LLZ15], the kernel $\mathcal{I}$ of the composition

$$\mathbb{T}'_1(N_{\psi}n^2) \xrightarrow{\phi_n} \mathcal{O}[H_{\mathfrak{n}}] \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}/\mathfrak{P},$$

where ϕ_n is as in Proposition 2.3.2, is a non-Eisenstein maximal ideal of $\mathbb{T}'_1(N_\psi n^2)$ in the sense of [op.cit., Def. 4.1.2]. Therefore, denoting $\mathcal{I}$ -adic completions with the subscript $\mathcal{I}$, we have an isomorphism of $\mathbb{T}'_1(N_\psi n^2)_{\mathcal{I}}$ -modules

$$H^1(Y_1(N_\psi n^2)(\mathbb{C}), \mathbb{Z})_{\mathcal{I}} \cong H_c^1(Y_1(N_\psi n^2)(\mathbb{C}), \mathbb{Z})_{\mathcal{I}}.$$

On the other hand, as in the proof of [LLZ15, Lem. 4.2.4], the natural pullback map yields an isomorphism

$$H_c^1(Y(1, N_\psi(n^2))(\mathbb{C}), \mathbb{Z}) \cong H_c^1(Y_1(N_\psi n^2)(\mathbb{C}), \mathbb{Z})^\Delta,$$

where Δ is the set of diamond operators $\langle d \rangle'$ with $d \equiv 1 \pmod{N_\psi}$. Since Δ maps to 1 under the composition

$$\mathbb{T}'_1(N_\psi n^2) \xrightarrow{\phi_n} \mathcal{O}[H_n] \longrightarrow \mathcal{O}[R_{\mathfrak{f},n}],$$

the result follows. $\square$

Corollary 2.3.6. *Let B be the set of positive integers n coprime to $p\mathfrak{f}$. Then there is a family of $G_{\mathbb{Q}}$ -equivariant isomorphisms of $\mathcal{O}[R_{\mathfrak{f},n}]$ -modules*

$$\nu_n : \mathcal{O}[R_{\mathfrak{f},n}] \otimes_{(\mathbb{T}'(1, N_\psi(n^2)) \otimes_{\mathbb{Z}_p, \phi_n})} H_{\text{ét}}^1(Y(1, N_\psi(n^2))_{\overline{\mathbb{Q}}}, \mathbb{Z}_p(1)) \xrightarrow{\simeq} \text{Ind}_{K(\mathfrak{f})[n]}^{\mathbb{Q}} \mathcal{O}(\psi_{\mathfrak{P}}^{-1})$$

for all $n \in B$, such that for $n \mid n'$ the diagram

$$\begin{array}{ccc} \mathcal{O}[R_{\mathfrak{f},n'}] \otimes_{(\mathbb{T}'(1, N_\psi(n'^2)) \otimes_{\mathbb{Z}_p, \phi_{n'}})} H_{\text{ét}}^1(Y(1, N_\psi(n'^2))_{\overline{\mathbb{Q}}}, \mathbb{Z}_p(1)) & \xrightarrow[\simeq]{\nu_{n'}} & \text{Ind}_{K(\mathfrak{f})[n']}^{\mathbb{Q}} \mathcal{O}(\psi_{\mathfrak{P}}^{-1}) \\ \mathcal{N}_{\mathfrak{f},n}^{\mathfrak{f},n'} \downarrow & & \downarrow \\ \mathcal{O}[R_{\mathfrak{f},n}] \otimes_{(\mathbb{T}'(1, N_\psi(n^2)) \otimes_{\mathbb{Z}_p, \phi_n})} H_{\text{ét}}^1(Y(1, N_\psi(n^2))_{\overline{\mathbb{Q}}}, \mathbb{Z}_p(1)) & \xrightarrow[\simeq]{\nu_n} & \text{Ind}_{K(\mathfrak{f})[n]}^{\mathbb{Q}} \mathcal{O}(\psi_{\mathfrak{P}}^{-1}) \end{array}$$

commutes, where $\mathcal{N}_{\mathfrak{f},n}^{\mathfrak{f},n'}$ is induced by $\mathcal{N}_{\mathfrak{f}(n')}^{\mathfrak{f}(n)}$ and the right vertical arrow is the natural norm map.

Proof. Let $\mathfrak{n} = \mathfrak{f}(n)$, $\mathcal{I}$ and Δ be as in the proof of Lemma 2.3.5. Since $\mathcal{I}$ is non-Eisenstein, the natural trace map

$$H_{\text{ét}}^1(Y_1(N_\psi n^2)_{\overline{\mathbb{Q}}}, \mathbb{Z}_p(1))_{\Delta} \longrightarrow H_{\text{ét}}^1(Y(1, N_\psi(n^2))_{\overline{\mathbb{Q}}}, \mathbb{Z}_p(1))$$

becomes an isomorphism after taking $\mathcal{I}$ -adic completions. Since the map ϕ_n of Lemma 2.3.5 is induced by $\phi_{\mathfrak{n}}$ (as shown in the proof of that result), it follows that the $\mathcal{O}[R_{\mathfrak{f},n}]$ -module

$$\mathcal{O}[R_{\mathfrak{f},n}] \otimes_{(\mathbb{T}'(1, N_\psi(n^2)) \otimes \mathbb{Z}_p, \phi_n)} H_{\text{ét}}^1(Y(1, N_\psi(n^2))_{\overline{\mathbb{Q}}}, \mathbb{Z}_p(1))$$

is naturally isomorphic to

$$\mathcal{O}[R_{\mathfrak{f},n}] \otimes_{\mathcal{O}[H_{\mathfrak{n}}]} \left(\mathcal{O}[H_{\mathfrak{n}}] \otimes_{(\mathbb{T}'_1(N_\psi n^2) \otimes \mathbb{Z}_p, \phi_{\mathfrak{n}})} H_{\text{ét}}^1(Y_1(N_\psi n^2)_{\overline{\mathbb{Q}}}, \mathbb{Z}_p(1)) \right).$$

The result now follows from Theorem 2.3.3. □

2.4 Proof of the tame norm relations

We keep the notations introduced in §2.3. Fix two newforms (g, h) of weights (l, m) of the same parity, levels (N_g, N_h) , and characters (χ_g, χ_h) such that $\chi \varepsilon_K \chi_g \chi_h = 1$. Enlarging L if necessary, assume that it contains the Fourier coefficients of g and h .

Let $N = \text{lcm}(N_\psi, N_g, N_h)$, and (since N will be fixed throughout) put $Y(m) = Y(1, N(m))$ for every positive integer m .

Let $\mathbf{r} = (r_1, r_2, r_3)$ be a triple of non-negative integers such that

$$r_1 + r_2 + r_3 = 2r$$

with $r \in \mathbb{Z}_{\geq 0}$, and $r_i + r_j \geq r_k$ for every permutation (i, j, k) of $(1, 2, 3)$. Put

$$\mathcal{L}_{[\mathbf{r}]} = \mathcal{L}_{1, N(m), r_1}(\mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathcal{L}_{1, N(m), r_2}(\mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathcal{L}_{1, N(m), r_3}(\mathbb{Z}_p),$$

and define

$$\kappa_{m, \mathbf{r}}^{(1)} \in H^1 \left(\mathbb{Q}, H_{\text{ét}}^3(Y(m)_{\overline{\mathbb{Q}}}^3, \mathcal{L}_{[\mathbf{r}]}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p(2-r) \right)$$

to be the class $\kappa_{N(m), \mathbf{r}} = \mathbf{s}_{\mathbf{r}*} \circ \text{HS} \circ d_*(\text{Det}_{N(m)}^{\mathbf{r}})$ constructed as in [BSV22, §3] for the modular curve $Y(m)$.

Lemma 2.4.1. *Let m be a positive integer and let q be a prime number. Assume that both m and q are coprime to p and N . Then*

$$\begin{aligned} (\pi_2, \pi_1, \pi_1)_* \kappa_{mq, \mathbf{r}}^{(1)} &= (T_q, 1, 1) \kappa_{m, \mathbf{r}}^{(1)}; & (\pi_1, \pi_2, \pi_2)_* \kappa_{mq, \mathbf{r}}^{(1)} &= q^{r-r_1} (T'_q, 1, 1) \kappa_{m, \mathbf{r}}^{(1)}; \\ (\pi_1, \pi_2, \pi_1)_* \kappa_{mq, \mathbf{r}}^{(1)} &= (1, T_q, 1) \kappa_{m, \mathbf{r}}^{(1)}; & (\pi_2, \pi_1, \pi_2)_* \kappa_{mq, \mathbf{r}}^{(1)} &= q^{r-r_2} (1, T'_q, 1) \kappa_{m, \mathbf{r}}^{(1)}; \\ (\pi_1, \pi_1, \pi_2)_* \kappa_{mq, \mathbf{r}}^{(1)} &= (1, 1, T_q) \kappa_{m, \mathbf{r}}^{(1)}; & (\pi_2, \pi_2, \pi_1)_* \kappa_{mq, \mathbf{r}}^{(1)} &= q^{r-r_3} (1, 1, T'_q) \kappa_{m, \mathbf{r}}^{(1)}. \end{aligned}$$

If q is coprime to m we also have

$$(\pi_1, \pi_1, \pi_1)_* \kappa_{mq, \mathbf{r}}^{(1)} = (q+1) \kappa_{m, \mathbf{r}}^{(1)}; \quad (\pi_2, \pi_2, \pi_2)_* \kappa_{mq, \mathbf{r}}^{(1)} = (q+1) q^r \kappa_{m, \mathbf{r}}^{(1)}.$$

Proof. The same argument proving equations (174) and (176) in [BSV22] yields these identities, adding the prime q to the level rather than the prime p . $\square$

We next consider the following ‘asymmetric’ diagonal classes.

Definition 2.4.2. For each squarefree positive integer n coprime to p and N , let

$$\kappa_{n, \mathbf{r}}^{(2)} = n^{r_2} (1, 1, \langle n \rangle') (1, \pi_1, \pi_2)_* \kappa_{n^2, \mathbf{r}}^{(1)} \in H^1 \left(\mathbb{Q}, H_{\text{ét}}^3(Y(n^2)_{\overline{\mathbb{Q}}} \times Y(1)_{\overline{\mathbb{Q}}}^2, \mathcal{L}_{[\mathbf{r}]}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p(2-r) \right),$$

where $\pi_1, \pi_2 : Y(n^2) \rightarrow Y(1)$ are the degeneracy maps in (2.2).

Lemma 2.4.3. *Let n be as above and let q be a rational prime coprime to p , N and n . Then*

$$\begin{aligned}(\pi_{11}, 1, 1)_* \kappa_{nq, \mathbf{r}}^{(2)} &= \{q^{r_2}(1, 1, T_q T'_q) - (q+1)q^{r_2+r_3}(1, 1, 1)\} \kappa_{n, \mathbf{r}}^{(2)}, \\(\pi_{21}, 1, 1)_* \kappa_{nq, \mathbf{r}}^{(2)} &= \{q^r(1, T'_q, T'_q) - q^{r_2+r_3}(T'_q, \langle q \rangle', \langle q \rangle')\} \kappa_{n, \mathbf{r}}^{(2)}, \\(\pi_{22}, 1, 1)_* \kappa_{nq, \mathbf{r}}^{(2)} &= \{q^{r_1+r_3}(1, T_q'^2, \langle q \rangle') - (q+1)q^{2r}(1, \langle q \rangle', \langle q \rangle')\} \kappa_{n, \mathbf{r}}^{(2)},\end{aligned}$$

where $\pi_{ij} : Y(n^2 q^2) \rightarrow Y(n^2)$ denotes the composite map

$$Y(n^2 q^2) \xrightarrow{\pi_i} Y(n^2 q) \xrightarrow{\pi_j} Y(n^2).$$

Proof. To better distinguish between the degeneracy maps π_i for different levels, in this proof we use ϖ_i to denote the map π_i descending the level by q , so that $\varpi_j \circ \varpi_i$ is the degeneracy map π_{ij} in the statement of the lemma. Thus we find

$$\begin{aligned}(\varpi_1, 1, 1)_* \kappa_{nq, \mathbf{r}}^{(2)} &= n^{r_2} q^{r_2}(1, 1, \langle nq \rangle')(1, \pi_1, \pi_2)_* (\varpi_1, \varpi_1, \varpi_2)_* \kappa_{n^2 q^2, \mathbf{r}}^{(1)} \\&= n^{r_2} q^{r_2}(1, 1, \langle nq \rangle')(1, \pi_1, \pi_2)_* (1, 1, T_q) \kappa_{n^2 q, \mathbf{r}}^{(1)},\end{aligned}$$

using Lemma 2.4.3 for the third equality; and similarly,

$$\begin{aligned}(\varpi_2, 1, 1)_* \kappa_{nq, \mathbf{r}}^{(2)} &= n^{r_2} q^{r_2}(1, 1, \langle nq \rangle')(1, \pi_1, \pi_2)_* (\varpi_2, \varpi_1, \varpi_2)_* \kappa_{n^2 q^2, \mathbf{r}}^{(1)} \\&= n^{r_2} q^r(1, 1, \langle nq \rangle')(1, \pi_1, \pi_2)_* (1, T'_q, 1) \kappa_{n^2 q, \mathbf{r}}^{(1)}.\end{aligned}$$

Descending the level again by q this gives

$$\begin{aligned}
(\pi_{11}, 1, 1)_* \kappa_{nq, \mathbf{r}}^{(2)} &= n^{r_2} q^{r_2} (1, 1, \langle nq \rangle') (1, \pi_1, \pi_2)_* (\varpi_1, \varpi_1, \varpi_2)_* (1, 1, T_q) \kappa_{n^2 q, \mathbf{r}}^{(1)} \\
&= n^{r_2} q^{r_2} (1, 1, \langle nq \rangle') (1, \pi_1, \pi_2)_* (\varpi_{1*}, \varpi_{1*}, T_q \varpi_{2*} - q^{r_3} \langle q \rangle \varpi_{1*}) \kappa_{n^2 q, \mathbf{r}}^{(1)} \\
&= n^{r_2} q^{r_2} (1, 1, \langle nq \rangle') (1, \pi_1, \pi_2)_* \{ (1, 1, T_q^2) - (q+1) q^{r_3} (1, 1, \langle q \rangle) \} \kappa_{n^2, \mathbf{r}}^{(1)} \\
&= q^{r_2} \{ (1, 1, T_q T_q') - (q+1) q^{r_3} (1, 1, 1) \} n^{r_2} (1, 1, \langle n \rangle') (1, \pi_1, \pi_2)_* \kappa_{n^2, \mathbf{r}}^{(1)} \\
&= \{ q^{r_2} (1, 1, T_q T_q') - (q+1) q^{r_2+r_3} (1, 1, 1) \} \kappa_{n, \mathbf{r}}^{(2)},
\end{aligned}$$

and similarly

$$\begin{aligned}
(\pi_{21}, 1, 1)_* \kappa_{nq, \mathbf{r}}^{(2)} &= n^{r_2} q^r (1, 1, \langle nq \rangle') (1, \pi_1, \pi_2)_* (\varpi_1, \varpi_1, \varpi_2)_* (1, T_q', 1) \kappa_{n^2 q, \mathbf{r}}^{(1)} \\
&= n^{r_2} q^r (1, 1, \langle nq \rangle') (1, \pi_1, \pi_2)_* (\varpi_{1*}, T_q' \varpi_{1*} - \langle q \rangle' \varpi_{2*}, \varpi_{2*}) \kappa_{n^2 q, \mathbf{r}}^{(1)} \\
&= n^{r_2} q^r (1, 1, \langle nq \rangle') (1, \pi_1, \pi_2)_* \{ (1, T_q', T_q) - q^{r-r_1} (T_q', \langle q \rangle', 1) \} \kappa_{n^2, \mathbf{r}}^{(1)} \\
&= q^r \{ (1, T_q', T_q') - q^{r-r_1} (T_q', \langle q \rangle', \langle q \rangle') \} n^{r_2} (1, 1, \langle n \rangle') (1, \pi_1, \pi_2)_* \kappa_{n^2, \mathbf{r}}^{(1)} \\
&= \{ q^r (1, T_q', T_q') - q^{r_2+r_3} (T_q', \langle q \rangle', \langle q \rangle') \} \kappa_{n, \mathbf{r}}^{(2)},
\end{aligned}$$

and

$$\begin{aligned}
(\pi_{22}, 1, 1)_* \kappa_{nq, \mathbf{r}}^{(2)} &= n^{r_2} q^r (1, 1, \langle nq \rangle') (1, \pi_1, \pi_2)_* (\varpi_2, \varpi_1, \varpi_2)_* (1, T_q', 1) \kappa_{n^2 q, \mathbf{r}}^{(1)} \\
&= n^{r_2} q^r (1, 1, \langle nq \rangle') (1, \pi_1, \pi_2)_* (\varpi_{2*}, T_q' \varpi_{1*} - \langle q \rangle' \varpi_{2*}, \varpi_{2*}) \kappa_{n^2 q, \mathbf{r}}^{(1)} \\
&= n^{r_2} q^r (1, 1, \langle nq \rangle') (1, \pi_1, \pi_2)_* \{ q^{r-r_2} (1, T_q'^2, 1) - (q+1) q^r (1, \langle q \rangle', 1) \} \kappa_{n^2, \mathbf{r}}^{(1)} \\
&= q^r \{ q^{r-r_2} (1, T_q'^2, \langle q \rangle') - (q+1) q^r (1, \langle q \rangle', \langle q \rangle') \} n^{r_2} (1, 1, \langle n \rangle') (1, \pi_1, \pi_2)_* \kappa_{n^2, \mathbf{r}}^{(1)} \\
&= \{ q^{r_1+r_3} (1, T_q'^2, \langle q \rangle') - (q+1) q^{2r} (1, \langle q \rangle', \langle q \rangle') \} \kappa_{n, \mathbf{r}}^{(2)},
\end{aligned}$$

hence the result. $\square$

Projection of the classes $\kappa_{n, \mathbf{r}}^{(2)}$ to the $(1, 1, 1)$ -component in the Künneth decompo-

sition yields classes $\kappa_{n,\mathbf{r}}^{(3)}$ in

$$H^1(\mathbb{Q}, H_{\text{ét}}^1(Y(n^2)_{\overline{\mathbb{Q}}}, \mathcal{L}_{r_1}(1)) \otimes H_{\text{ét}}^1(Y(1)_{\overline{\mathbb{Q}}}, \mathcal{L}_{r_2}(1)) \otimes H_{\text{ét}}^1(Y(1)_{\overline{\mathbb{Q}}}, \mathcal{L}_{r_3}(1)) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p(-1-r)).$$

Now set $(r_1, r_2, r_3) = (0, l-2, m-2)$. Fix test vectors

$$\check{f} \in S_k(N, \chi_{\varepsilon_K})[\theta_\psi], \quad \check{g} \in S_l(N, \chi_g)[g], \quad \check{h} \in S_m(N, \chi_h)[h].$$

These test vectors determine maps

$$\begin{aligned} H_{\text{ét}}^1(Y(n^2)_{\overline{\mathbb{Q}}}, \mathbb{Z}_p(1)) &\rightarrow H_{\text{ét}}^1(Y(1, N_\psi(n^2))_{\overline{\mathbb{Q}}}, \mathbb{Z}_p(1)) \\ H_{\text{ét}}^1(Y(1)_{\overline{\mathbb{Q}}}, \mathcal{L}_{r_2}(1)) &\rightarrow H_{\text{ét}}^1(Y_1(N_g)_{\overline{\mathbb{Q}}}, \mathcal{L}_{r_2}(1)) \\ H_{\text{ét}}^1(Y(1)_{\overline{\mathbb{Q}}}, \mathcal{L}_{r_3}(1)) &\rightarrow H_{\text{ét}}^1(Y_1(N_h)_{\overline{\mathbb{Q}}}, \mathcal{L}_{r_3}(1)) \end{aligned}$$

which we use to project the classes $\kappa_{n,\mathbf{r}}^{(3)}$ to classes $\kappa_{n,\psi gh}^{(3)}$ in

$$H^1(\mathbb{Q}, \mathcal{O}[R_{\mathfrak{f},n}] \otimes_{(\mathbb{T}'(1, N_\psi(n^2)) \otimes_{\mathbb{Z}_p, \phi_n})} H_{\text{ét}}^1(Y(1, N_\psi(n^2))_{\overline{\mathbb{Q}}}, \mathbb{Z}_p(1)) \otimes_{\mathcal{O}} T_g \otimes_{\mathcal{O}} T_h \otimes_{\mathbb{Z}_p} \mathbb{Q}_p(-1-r)).$$

Let

$$T_{g,h}^\psi = T_g \otimes_{\mathcal{O}} T_h(\psi_{\mathfrak{p}}^{-1})(-1-r), \quad V_{g,h}^\psi = T_{g,h}^\psi \otimes_{\mathbb{Z}_p} \mathbb{Q}_p,$$

Using the isomorphisms

$$\nu_n : \mathcal{O}[R_{\mathfrak{f},n}] \otimes_{(\mathbb{T}'(1, N_\psi(n^2)) \otimes_{\mathbb{Z}_p, \phi_n})} H_{\text{ét}}^1(Y(1, N_\psi(n^2))_{\overline{\mathbb{Q}}}, \mathbb{Z}_p(1)) \xrightarrow{\simeq} \text{Ind}_{K(\mathfrak{f})[n]}^{\mathbb{Q}} \mathcal{O}(\psi_{\mathfrak{p}}^{-1})$$

of Corollary 2.3.6, and taking the projection of both sides via the quotient map

$\mathcal{O}[R_{\mathfrak{f},n}] \rightarrow \mathcal{O}[R_n]$, we obtain new isomorphisms

$$\tilde{\nu}_n : \mathcal{O}[R_n] \otimes_{(\mathbb{T}'(1, N_\psi(n^2)) \otimes_{\mathbb{Z}_p, \phi_n})} H_{\text{ét}}^1(Y(1, N_\psi(n^2))_{\overline{\mathbb{Q}}}, \mathbb{Z}_p(1)) \xrightarrow{\simeq} \text{Ind}_{K[n]}^{\mathbb{Q}} \mathcal{O}(\psi_{\mathfrak{p}}^{-1}),$$

so that applying the corresponding projection map to the classes $\kappa_{n,\psi gh}^{(3)}$ and using

Shapiro's lemma we obtain classes

$$\tilde{\kappa}_{\psi,g,h,nq} \in H^1(K[n], V_{g,h}^\psi).$$

Proposition 2.4.4. *Let n be as above, and let q be a rational prime coprime to p , N and n .*

(i) *If q splits in K as $(q) = \mathfrak{q}\bar{\mathfrak{q}}$, then*

$$\begin{aligned} \text{cor}_{K[nq]/K[n]}(\tilde{\kappa}_{\psi,g,h,nq}) &= q^{l+m-4} \left\{ \chi_g(q)\chi_h(q)q \left(\frac{\psi(\mathfrak{q})}{q} \text{Fr}_{\mathfrak{q}}^{-1} \right)^2 - \frac{a_q(g)a_q(h)}{q^{(l+m-4)/2}} \left(\frac{\psi(\mathfrak{q})}{q} \text{Fr}_{\mathfrak{q}}^{-1} \right) \right. \\ &\quad + \frac{\chi_g(q)^{-1}a_q(g)^2}{q^{l-1}} + \frac{\chi_h(q)^{-1}a_q(h)^2}{q^{m-2}} - \frac{q^2+1}{q} \\ &\quad \left. - \frac{a_q(g)a_q(h)}{q^{(l+m-4)/2}} \left(\frac{\psi(\bar{\mathfrak{q}})}{q} \text{Fr}_{\bar{\mathfrak{q}}}^{-1} \right) + \chi_g(q)\chi_h(q)q \left(\frac{\psi(\bar{\mathfrak{q}})}{q} \text{Fr}_{\bar{\mathfrak{q}}}^{-1} \right)^2 \right\} \tilde{\kappa}_{\psi,g,h,n}. \end{aligned}$$

(ii) *If q is inert in K , then*

$$\text{cor}_{K[nq]/K[n]}(\tilde{\kappa}_{\psi,g,h,nq}) = q^{l+m-4} \left\{ \frac{\chi_g(q)^{-1}a_q(g)^2}{q^{l-1}} + \frac{\chi_h(q)^{-1}a_q(h)^2}{q^{m-2}} - \frac{(q+1)^2}{q} \right\} \tilde{\kappa}_{\psi,g,h,n}.$$

Proof. We have the commutative diagram

$$\begin{array}{ccc} H^1(K[nq], V_{g,h}^\psi) & \xrightarrow{\cong} & H^1(\mathbb{Q}, \text{Ind}_{K[nq]}^{\mathbb{Q}} \mathcal{O}(\psi_{\mathfrak{P}}^{-1}) \otimes_{\mathcal{O}} T_g \otimes_{\mathcal{O}} T_h \otimes_{\mathbb{Z}_p} \mathbb{Q}_p(-1-r)) \\ \downarrow \text{cor}_{K[nq]/K[n]} & & \downarrow \\ H^1(K[n], V_{g,h}^\psi) & \xrightarrow{\cong} & H^1(\mathbb{Q}, \text{Ind}_{K[n]}^{\mathbb{Q}} \mathcal{O}(\psi_{\mathfrak{P}}^{-1}) \otimes_{\mathcal{O}} T_g \otimes_{\mathcal{O}} T_h \otimes_{\mathbb{Z}_p} \mathbb{Q}_p(-1-r)), \end{array}$$

where the horizontal isomorphisms are given by Shapiro's lemma and the right vertical arrow comes from the natural norm map between induced representations. Using the isomorphisms $\tilde{\nu}_n$ above, the vertical arrows in the previous diagram correspond to the map

$$\begin{array}{c}
H^1(\mathbb{Q}, \mathcal{O}[R_{nq}] \otimes_{\phi_{nq}} H_{\text{ét}}^1(Y(1, N_\psi(n^2 q^2))_{\overline{\mathbb{Q}}}, \mathbb{Z}_p(1)) \otimes_{\mathcal{O}} T_g \otimes_{\mathcal{O}} T_h \otimes_{\mathbb{Z}_p} \mathbb{Q}_p(-1-r)) \\
\downarrow \mathcal{N}_{f,n}^{f,nq} \otimes \text{id} \otimes \text{id} \\
H^1(\mathbb{Q}, \mathcal{O}[R_n] \otimes_{\phi_n} H_{\text{ét}}^1(Y(1, N_\psi(n^2))_{\overline{\mathbb{Q}}}, \mathbb{Z}_p(1)) \otimes_{\mathcal{O}} T_g \otimes_{\mathcal{O}} T_h \otimes_{\mathbb{Z}_p} \mathbb{Q}_p(-1-r)).
\end{array}$$

If q splits in K , the map $\mathcal{N}_{f,n}^{f,nq}$ is given by

$$\mathcal{N}_{f,n}^{f,nq} = \pi_{11*} - \left(\frac{\psi(\mathbf{q})[\mathbf{q}]}{q} + \frac{\psi(\overline{\mathbf{q}})[\overline{\mathbf{q}}]}{q} \right) \pi_{21*} + \frac{\chi(q)}{q} \pi_{22*},$$

using the notations introduced in Lemma 2.4.3 for the degeneracy maps, and from the relations in that lemma we find

$$\begin{aligned}
\mathcal{N}_{f,n}^{f,nq}(\tilde{\kappa}_{\psi,g,h,nq}) &= \left[1 \otimes \{q^{r_2}(1, 1, T_q T'_q) - (q+1)q^{r_2+r_3}(1, 1, 1)\} \right. \\
&\quad - \left(\frac{\psi(\mathbf{q})[\mathbf{q}]}{q} + \frac{\psi(\overline{\mathbf{q}})[\overline{\mathbf{q}}]}{q} \right) \otimes \{q^r(1, T'_q, T'_q) - q^{r_2+r_3}(T'_q, \langle q \rangle', \langle g \rangle')\} \\
&\quad \left. + \frac{\chi(q)}{q} \otimes \{q^{r_1+r_3}(1, T_q'^2, \langle q \rangle') - (q+1)q^{2r}(1, \langle q \rangle', \langle q \rangle')\} \right] \tilde{\kappa}_{\psi,g,h,n} \\
&= \left[\chi_h(q)^{-1} a_q(h)^2 q^{r_2} + (q+1)q^{r_2+r_3} \right. \\
&\quad - \left(\frac{\psi(\mathbf{q})[\mathbf{q}]}{q} + \frac{\psi(\overline{\mathbf{q}})[\overline{\mathbf{q}}]}{q} \right) \{a_q(g)a_q(h)q^r - \chi_g(q)\chi_h(q)q^{r_2+r_3}(\psi(\mathbf{q})[\mathbf{q}] + \psi(\overline{\mathbf{q}})[\overline{\mathbf{q}}])\} \\
&\quad \left. + \frac{\chi(q)}{q} \{\chi_h(q)a_q(g)^2 q^{r_1+r_3} - \chi_g(q)\chi_h(q)(q+1)q^{2r}\} \right] \tilde{\kappa}_{\psi,g,h,n},
\end{aligned}$$

This implies the result in this case. When q is inert in K , we have

$$\mathcal{N}_{f,n}^{f,nq} = \pi_{11*} - \frac{\chi(q)}{q} \pi_{22*},$$

and the result in this case follows by a very similar computation that we leave to the reader. $\square$

In particular, restricting to positive integers n as above that are divisible only by primes q which split in K , Proposition 2.4.4 yields the following result. (Note that

since in this section we assume ψ has infinity type $(-1, 0)$, the balanced condition forces $l = m$.)

Theorem 2.4.5. *Suppose the weights of g, h are $l = m$. Let $\mathcal{S}$ be the set of squarefree products of primes q which split in K and are coprime to p and N . Assume that $H^1(K[n], T_{g,h}^\psi)$ is torsion-free for every $n \in \mathcal{S}$. There exists a collection of classes*

$$\left\{ \kappa_{\psi,g,h,n} \in H^1(K[n], T_{g,h}^\psi) : n \in \mathcal{S} \right\}$$

such that whenever $n, nq \in \mathcal{S}$ with q a prime, we have

$$\text{cor}_{K[nq]/K[n]}(\kappa_{\psi,g,h,nq}) = P_{\mathfrak{q}}(\text{Fr}_{\mathfrak{q}}^{-1}) \kappa_{\psi,g,h,n},$$

where $\mathfrak{q}$ is any of the primes of K above q , and $P_{\mathfrak{q}}(X) = \det(1 - \text{Fr}_{\mathfrak{q}}^{-1} X | (V_{g,h}^\psi)^\vee(1))$.

Proof. We begin by noting that the only possible denominators of the classes $\tilde{\kappa}_{\psi,g,h,n}$ are divisors of $(l-2)!(m-2)!$ (as follows from [BSV22, Rmk. 3.3]), so after multiplying them by a suitable power of p they all have coefficients in $T_{g,h}^\psi$.

Now given a prime $q \in \mathcal{S}$, we note that for any prime v of K above q we have

$$\begin{aligned} P_v(X) &= 1 - \frac{a_q(g)a_q(h)}{q^{(l+m-2)/2}} \frac{\psi(v)}{q} X \\ &\quad + \left(\frac{\chi_g(q)a_q(h)^2}{q^{m-1}} + \frac{\chi_h(q)a_q(g)^2}{q^{l-1}} - 2\chi_g(q)\chi_h(q) \right) \frac{\psi(v)^2}{q^2} X^2 \\ &\quad - \frac{\chi_g(q)\chi_h(q)a_q(g)a_q(h)}{q^{(l+m-2)/2}} \frac{\psi(v)^3}{q^3} X^3 + \chi_g(q)^2\chi_h(q)^2 \frac{\psi(v)^4}{q^4} X^4. \end{aligned}$$

Writing $(q) = \mathfrak{q}\bar{\mathfrak{q}}$ and using that $\psi(\mathfrak{q})\psi(\bar{\mathfrak{q}}) = \chi(q)q$ and $\chi_g(q)\chi_h(q)\chi(q) = 1$, we

therefore find the congruences

$$\begin{aligned}
P_{\mathfrak{q}}(\mathrm{Fr}_{\mathfrak{q}}^{-1})\chi_g(q)\chi_h(q)\psi(\bar{\mathfrak{q}})^2\mathrm{Fr}_{\bar{\mathfrak{q}}}^{-2} &\equiv P_{\bar{\mathfrak{q}}}(\mathrm{Fr}_{\bar{\mathfrak{q}}}^{-1})\chi_g(q)\chi_h(q)\psi(\mathfrak{q})^2\mathrm{Fr}_{\mathfrak{q}}^{-2} \pmod{q-1} \\
&\equiv \chi_g(q)\chi_h(q)\psi(\mathfrak{q})^2\mathrm{Fr}_{\mathfrak{q}}^{-2} - a_q(g)a_q(h)\psi(\mathfrak{q})\mathrm{Fr}_{\mathfrak{q}}^{-1} \\
&\quad + \chi_g(q)^{-1}a_q(g)^2 + \chi_h(q)^{-1}a_q(h)^2 - 2 \\
&\quad - a_q(g)a_q(h)\psi(\bar{\mathfrak{q}})\mathrm{Fr}_{\bar{\mathfrak{q}}}^{-1} + \chi_g(q)\chi_h(q)\psi(\bar{\mathfrak{q}})^2\mathrm{Fr}_{\bar{\mathfrak{q}}}^{-2} \pmod{q-1}
\end{aligned}$$

as endomorphisms of $H^1(K[n], T_{g,h}^\psi)$. Since these expressions agree modulo $q-1$ with the factor appearing in the norm relation of Proposition 2.4.4(i), together with [Rub00, Lem. 9.6.1] the result follows. $\square$

Remark 2.4.6. The condition that $H^1(K[n], T_{g,h}^\psi)$ is torsion-free for every $n \in \mathcal{S}$ holds, for example, under the assumptions in Lemma 2.8.9 below. Indeed, since $\mathrm{SL}_2(\mathbb{Z}_p) \times \mathrm{SL}_2(\mathbb{Z}_p)$ has no proper normal subgroups of finite p -power index, it follows from this lemma that the residual $G_{K[n]}$ -representation attached to $T_{g,h}^\psi$ is absolutely irreducible for every $n \in \mathcal{S}$, so that $H^0(K[n], V_{g,h}^\psi/T_{g,h}^\psi)$ is trivial for every $n \in \mathcal{S}$ and the condition follows.

Remark 2.4.7. In the inert case, writing $\mathfrak{q} = (q)$ we have

$$\begin{aligned}
P_{\mathfrak{q}}(X) &= \det(1 - \mathrm{Fr}_{\mathfrak{q}}^{-1}X|(T_{g,h}^\psi)^\vee(1)) \\
&= 1 - \left(\frac{a_q(g)^2}{q^{l-1}} - 2\chi_g(q) \right) \left(\frac{a_q(h)^2}{q^{m-1}} - 2\chi_h(q) \right) \frac{\psi(\mathfrak{q})}{q^2} X \\
&\quad + \left(\chi_h(q)^2 \left(\frac{a_q(g)^2}{q^{l-1}} - 2\chi_g(q) \right)^2 + \chi_g(q)^2 \left(\frac{a_q(h)^2}{q^{m-1}} - 2\chi_h(q) \right)^2 - 2\chi_g(q)^2\chi_h(q)^2 \right) \frac{\psi(\mathfrak{q})^2}{q^4} X^2 \\
&\quad - \chi_g(q)^2\chi_h(q)^2 \left(\frac{a_q(g)^2}{q^{l-1}} - 2\chi_g(q) \right) \left(\frac{a_q(h)^2}{q^{m-1}} - 2\chi_h(q) \right) \frac{\psi(\mathfrak{q})^3}{q^6} X^3 + \chi_g(q)^4\chi_h(q)^4 \frac{\psi(\mathfrak{q})^4}{q^8} X^4,
\end{aligned}$$

and similarly as in the proof of Theorem 2.4.5 we find the congruence

$$P_q(\text{Fr}_q^{-1}) \equiv \chi_g(q)^{-2}a_q(g)^4 + \chi_h(q)^{-2}a_q(h)^4 + 2\chi_g(q)^{-1}\chi_h(q)^{-1}a_q(g)^2a_q(h)^2q \\ - 4\frac{\chi_g(q)^{-1}a_q(g)^2(q+1)}{q^{l-1}} - 4\frac{\chi_h(q)^{-1}a_q(h)^2(q+1)}{q^{m-1}} + 8(q+1) \pmod{q^2-1}$$

as endomorphisms of $H^1(K[n], T_{g,h}^\psi)$. Similarly as above, this expression agrees modulo $q^2 - 1$ with the square of the Euler factor appearing in the norm relation of Proposition 2.4.4(ii).

Now assume that $(p) = \mathfrak{p}\bar{\mathfrak{p}}$ splits in K , with $\mathfrak{p}$ the prime of K above p induced by our fixed embedding $\iota_p : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$, and let $f = \theta_\psi$ be the theta series associated to ψ . Assume also that both g and h are ordinary at p . Then, for $\phi \in \{f, g, h\}$, the $G_{\mathbb{Q}_p}$ -representation V_ϕ admits a filtration

$$0 \longrightarrow V_\phi^+ \longrightarrow V_\phi \longrightarrow V_\phi^- \longrightarrow 0$$

where $V_\phi^\pm$ is one-dimensional and V_ϕ^- is unramified with Fr_p acting as multiplication by α_ϕ , the unit root of the Hecke polynomial of ϕ at p . Letting $V_{fgh} = V_f \otimes V_g \otimes V_h(-1-r)$, we can therefore consider the $G_{\mathbb{Q}_p}$ -subrepresentation

$$\mathcal{F}^2 V_{fgh} = (V_f \otimes V_g^+ \otimes V_h^+ + V_f^+ \otimes V_g \otimes V_h^+ + V_f^+ \otimes V_g^+ \otimes V_h)(-1-r)$$

and define the *balanced* local condition $H_{\text{bal}}^1(\mathbb{Q}_p, V_{fgh}) \subset H^1(\mathbb{Q}_p, V_{fgh})$ to be the image of the natural map $H^1(\mathbb{Q}_p, \mathcal{F}^2 V_{fgh}) \rightarrow H^1(\mathbb{Q}_p, V_{fgh})$.

Put $T_\phi^+ = V_\phi^+ \cap T_\phi$ and use it to similarly define $\mathcal{F}^2 T_{fgh} \subset T_{fgh} = T_f \otimes T_g \otimes T_h(-1-r)$ and $H_{\text{bal}}^1(\mathbb{Q}_p, T_{fgh}) \subset H^1(\mathbb{Q}_p, T_{fgh})$. Setting

$$\mathcal{F}_\mathfrak{p}^+(T_{g,h}^\psi) = (T_g^+ \otimes T_h + T_g \otimes T_h^+)(\psi_\mathfrak{p}^{-1})(-1-r), \quad \mathcal{F}_\mathfrak{p}^+(T_{g,h}^\psi) = (T_g^+ \otimes T_h^+)(\psi_\mathfrak{p}^{-1})(-1-r), \quad (2.5)$$

then one readily checks that under the Shapiro isomorphism $H^1(\mathbb{Q}, V_{fgh}) \cong H^1(K, V_{g,h}^\psi)$ the balanced local condition $H_{\text{bal}}^1(\mathbb{Q}, T_{fgh})$ corresponds to the natural image of

$$\bigoplus_{v|p} H^1(K_v, \mathcal{F}_v^+(T_{g,h}^\psi)) \longrightarrow \bigoplus_{v|p} H^1(K_v, T_{g,h}^\psi).$$

For each finite extension L of K , this choice of local conditions at p , together with the unramified condition at the primes outside p , defines the Greenberg Selmer group $\text{Sel}_{\text{bal}}(L, T_{g,h}^\psi)$ (see §2.8.1 for the precise definition).

Proposition 2.4.8. *For every $n \in \mathcal{S}$, the class $\kappa_{\psi,g,h,n}$ lies in the group $\text{Sel}_{\text{bal}}(K[n], T_{g,h}^\psi)$.*

Proof. Fix $n \in \mathcal{S}$ and v a finite prime of $K[n]$. If $v \nmid p$, then it follows from the results [Car86] (see e.g. the remarks in [Nek93, §8.3] and the references therein) that $V_{g,h}^\psi$ is pure of weight -1 , and hence

$$H^0(K[n]_v, V_{g,h}^\psi) = H^0(K[n]_{\bar{v}}, V_{g,h}^\psi) = 0.$$

Therefore, $H^1(K[n]_v, V_{g,h}^\psi) = 0$ by local Tate duality and the local Euler characteristic formula, so the inclusion $\text{res}_v(\kappa_{\psi,g,h,n}) \in H_{\text{Gr}}^1(K[n]_v, T_{g,h}^\psi)$ is automatic.

Now suppose $v \mid p$. As noted in [BSV22, Prop. 3.2], it follows from the results of [NN16] that the classes $\kappa_{m,\mathbf{r}}^{(1)}$ are geometric at p , and therefore the class $\text{res}_v(\kappa_{\psi,g,h,n}) \in H^1(K[n]_v, T_{g,h}^\psi)$ lands in the inverse image of

$$H_{\text{geo}}^1(K[n]_v, V_{g,h}^\psi) = \ker \left(H^1(K[n]_v, V_{g,h}^\psi) \rightarrow H^1(K[n]_v, V_{g,h}^\psi \otimes_{\mathbb{Q}_p} B_{\text{dR}}) \right)$$

under the natural map $H^1(K[n]_v, T_{g,h}^\psi) \rightarrow H^1(K[n]_v, V_{g,h}^\psi)$. As $H_{\text{geo}}^1(K[n]_v, V_{g,h}^\psi)$ agrees with the Bloch–Kato finite subspace $H_{\text{fin}}^1(K[n]_v, V_{g,h}^\psi)$ by [Nek93, Prop. 1.24(2)], and the latter agrees with $H_{\text{bal}}^1(K[n]_v, V_{g,h}^\psi)$ (see Lemma 2.9.1 below), the result follows. $\square$

2.5 Hida families and Galois representations

In the next section we will prove that the classes $\kappa_{\psi,g,h,n}$ of Theorem 2.4.5 extend along the anticyclotomic $\mathbb{Z}_p$ -extension of K , i.e., they are anticyclotomic universal norms, and explain the construction of $\kappa_{\psi,g,h,n}$ for more general weights. In this section we collect the background results we shall need, closely following the treatment in [BSV22].

2.5.1 Hida families

Let $\Lambda = \mathbb{Z}_p[[1 + p\mathbb{Z}_p]]$ and let

$$\mathcal{W} = \mathrm{Spf}(\Lambda)$$

be the weight space. Then, for any extension E of $\mathbb{Q}_p$, we have $\mathcal{W}(E) = \mathrm{Hom}_{\mathrm{cont}}(1 + p\mathbb{Z}_p, E^\times)$. Points of the form $\nu_{r,\epsilon}(n) = \epsilon(n)n^r$, where r is a non-negative integer and ϵ is a finite order character, will be called *arithmetic*. We refer to $k = r + 2$ as the *weight* of $\nu_{r,\epsilon}$. Arithmetic points of the form $\nu_r = \nu_{r,1}$ will be called *classical*.

More generally, let $\mathcal{R}$ be a normal domain finite flat over Λ and let $\mathcal{W}_{\mathcal{R}} = \mathrm{Spf}(\mathcal{R})$. Then, a point $x \in \mathcal{W}_{\mathcal{R}}(\overline{\mathbb{Q}_p})$ will be called *arithmetic* if it lies above an arithmetic point $\nu_{r,\epsilon}$ of $\mathcal{W}(\overline{\mathbb{Q}_p})$ and *classical* if it lies above a classical point ν_r of $\mathcal{W}(\overline{\mathbb{Q}_p})$. Again, we refer to $k = r + 2$ as the *weight* of x .

Let M be a positive integer coprime to p . A *Hida family* of tame level M and character $\chi : (\mathbb{Z}/Mp\mathbb{Z})^\times \rightarrow \overline{\mathbb{Q}_p}^\times$ is a formal q -expansion

$$\mathbf{f} = \sum_{n \geq 1} a_n(\mathbf{f})q^n \in \Lambda_{\mathbf{f}}[[q]],$$

where $\Lambda_{\mathbf{f}}$ is a normal domain finite flat over Λ , such that, for any arithmetic point $x \in \mathcal{W}_{\Lambda_{\mathbf{f}}}(\overline{\mathbb{Q}_p})$ lying over some $\nu_{r,\epsilon}$, the corresponding specialization is a p -ordinary eigenform $f_x \in S_k(Mp^s, \chi\epsilon\omega^{-r})$. As above, we have denoted by k the weight of x and

we can take $s = \max\{1, \text{ord}_p(\text{cond}(\epsilon))\}$. We say that a Hida family $\mathbf{f}$ is *primitive* if the specializations f_x at arithmetic points x are p -stabilized newforms. We say that it is *normalized* if $a_1(\mathbf{f}) = 1$.

Let $\mathbf{f}$ be a normalized primitive Hida family of tame level M . For each arithmetic point $x \in \mathcal{W}_{\Lambda_{\mathbf{f}}}(\overline{\mathbb{Q}}_p)$, let $\mathbf{f}_x$ denote the specialization of $\mathbf{f}$ at x and let f_x be the corresponding newform. There exists a locally-free rank-two $\Lambda_{\mathbf{f}}$ module $\mathbb{V}_{\mathbf{f}}$ equipped with a continuous action of $G_{\mathbb{Q}}$ such that, for any arithmetic point $x \in \mathcal{W}_{\Lambda_{\mathbf{f}}}(\overline{\mathbb{Q}}_p)$, the corresponding specialization $\mathbb{V}_{\mathbf{f}} \otimes_{\Lambda_{\mathbf{f}}, x} \overline{\mathbb{Q}}_p$ recovers the $G_{\mathbb{Q}}$ -representation V_{f_x} attached to f_x . In particular, the representation $\mathbb{V}_{\mathbf{f}}$ is unramified at any prime $q \nmid Mp$ and $\text{Tr}(\text{Fr}_q) = a_q(\mathbf{f})$. We refer to $\mathbb{V}_{\mathbf{f}}$ as the *big Galois representation attached to $\mathbf{f}$* . If for some (equivalently all) arithmetic point $x_0 \in \mathcal{W}_{\Lambda_{\mathbf{f}}}(\overline{\mathbb{Q}}_p)$ the $G_{\mathbb{Q}}$ -representation $T_{f_{x_0}}$ attached to f_{x_0} is residually irreducible, then $\mathbb{V}_{\mathbf{f}}$ is a free $\Lambda_{\mathbf{f}}$ -module.

2.5.2 Continuous functions and distributions

Define the semigroups

$$\Sigma_0(p) = \begin{pmatrix} \mathbb{Z}_p^\times & \mathbb{Z}_p \\ p\mathbb{Z}_p & \mathbb{Z}_p \end{pmatrix} \quad \text{and} \quad \Sigma'_0(p) = \begin{pmatrix} \mathbb{Z}_p & \mathbb{Z}_p \\ p\mathbb{Z}_p & \mathbb{Z}_p^\times \end{pmatrix}.$$

The sets $\mathsf{T} = \mathbb{Z}_p^\times \times \mathbb{Z}_p$ and $\mathsf{T}' = p\mathbb{Z}_p \times \mathbb{Z}_p^\times$ bear a right action of $\Sigma_0(p)$ and $\Sigma'_0(p)$, respectively.

Let ν be a character of $\mathbb{Z}_p^\times$ taking values in a finite extension E of $\mathbb{Q}_p$. Let $\mathcal{O}$ be the ring of integers of E and denote by $\mathfrak{m}$ its maximal ideal. Let $\text{Cont}(\mathbb{Z}_p, \mathcal{O})$ denote the module of continuous functions on $\mathbb{Z}_p$ with values in $\mathcal{O}$. Define $\mathcal{O}$ -modules

$$\mathcal{A}_\nu = \{f : \mathsf{T} \rightarrow \mathcal{O} \mid f(1, z) \in \text{Cont}(\mathbb{Z}_p, \mathcal{O}) \text{ and } f(a \cdot t) = \nu(a) \cdot f(t) \text{ for all } a \in \mathbb{Z}_p^\times, t \in \mathsf{T}\},$$

$$\mathcal{A}'_\nu = \{f : \mathsf{T}' \rightarrow \mathcal{O} \mid f(pz, 1) \in \text{Cont}(\mathbb{Z}_p, \mathcal{O}) \text{ and } f(a \cdot t) = \nu(a) \cdot f(t) \text{ for all } a \in \mathbb{Z}_p^\times, t \in \mathsf{T}'\}$$

equipped with the $\mathfrak{m}$ -adic topology, and $\mathcal{O}$ -modules

$$\mathcal{D}_\nu = \mathrm{Hom}_{\mathrm{cont}, \mathcal{O}}(\mathcal{A}_\nu, \mathcal{O}), \quad \mathcal{D}'_\nu = \mathrm{Hom}_{\mathrm{cont}, \mathcal{O}}(\mathcal{A}'_\nu, \mathcal{O})$$

equipped with the weak-* topology. The right $\Sigma_0(p)$ -action on $\mathbb{T}$ yields naturally a left $\Sigma_0(p)$ -action on $\mathcal{A}_\nu$ and a right $\Sigma_0(p)$ -action on $\mathcal{D}_\nu$.

2.5.3 Group cohomology and étale cohomology

Let N and m be coprime positive integers which are also coprime to p , let $Y = Y(1, N(pm))$ and let Γ be the corresponding modular group. Let $\mathcal{E} \rightarrow Y$ be the universal elliptic curve over Y , and denote by C_p the canonical cyclic p -subgroup. Let $\mathcal{T}$ be the relative p -adic Tate module of $\mathcal{E}$ over Y . Fix a geometric point $\eta : \mathrm{Spec}(\overline{\mathbb{Q}}) \rightarrow Y$, and choose an isomorphism $\mathcal{T}_\eta \cong \mathbb{Z}_p \oplus \mathbb{Z}_p$ such that the Weil pairing on $\mathcal{T}_\eta$ corresponds to the natural determinant map on the right and the reduction modulo p of the element $(0,1)$ generates $C_{p,\eta}$.

Let $\mathcal{G} = \pi_1^{\mathrm{ét}}(Y, \eta)$. The action of $\mathcal{G}$ on $\mathcal{T}$ yields an action of $\mathcal{G}$ on $\mathbb{Z}_p \oplus \mathbb{Z}_p$, and hence a continuous representation $\rho : \mathcal{G} \rightarrow \mathrm{GL}_2(\mathbb{Z}_p)$. More precisely, for any $g \in \mathcal{G}$,

$$g \cdot (a, b) = (a, b)\rho(g)^{-1}.$$

In fact, since the action of $\mathcal{G}$ preserves the canonical subgroup, we have a continuous representation $\rho : \mathcal{G} \rightarrow \Gamma_0(p\mathbb{Z}_p)$, where

$$\Gamma_0(p\mathbb{Z}_p) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z}_p) : p \mid c \right\}.$$

The anti-involution of $\mathrm{GL}_2(\mathbb{Z}_p)$ given by $\gamma \mapsto \gamma^\iota = \det(\gamma)\gamma^{-1}$ restricts to $\Gamma_0(p\mathbb{Z}_p)$ and allows us to think of this group as acting on the right or left as convenient.

Taking the stalk at η gives an equivalence of categories between the category $\mathbf{S}_f(Y_{\text{ét}})$ of locally constant constructible sheaves with finite stalk of p -power order at η and the category $\mathbf{M}_f(\mathcal{G})$ of finite $\mathcal{G}$ -sets of p -power order. For any topological group G , define $\mathbf{M}_f(G)$ as we did for $\mathcal{G}$. Let $\mathbf{M}_{\text{cont}}(G)$ be the category of G -modules which are filtered unions $\cup_{i \in I} M_i$ with $M_i \in \mathbf{M}_f(G)$ and let $\mathbf{M}(G) \subset \mathbf{M}_{\text{cont}}(G)^{\mathbb{N}}$ be the category of inverse systems of objects in $\mathbf{M}_{\text{cont}}(G)$. Define $\mathbf{S}(Y_{\text{ét}})$ similarly. Then, there is an equivalence of categories between $\mathbf{M}(\mathcal{G})$ and $\mathbf{S}(Y_{\text{ét}})$. Moreover, the representation ρ defined above yields a functor $\mathbf{M}(\Gamma_0(p\mathbb{Z}_p)) \rightarrow \mathbf{M}(\mathcal{G})$. Regarding this functor, we adopt the following criterion: if an object $\mathcal{F} \in \mathbf{M}(\Gamma_0(p\mathbb{Z}_p))$ is given as a left $\Gamma_0(p\mathbb{Z}_p)$ -module, we define the left $\mathcal{G}$ -action via the map $\rho : \mathcal{G} \rightarrow \Gamma_0(p\mathbb{Z}_p)$; if it is given as a right $\Gamma_0(p\mathbb{Z}_p)$ -module, we define the left $\mathcal{G}$ -action via the map $g \mapsto \rho(g)^{-1}$.

Given an inverse system of sheaves $\mathcal{F} = (\mathcal{F}_i)_{i \in \mathbb{N}} \in \mathbf{S}(Y_{\text{ét}})$, we use the notation $H_{\text{ét}}^j(Y, \mathcal{F})$ for continuous étale cohomology as defined by Jannsen, and write $H_{\text{ét}}^j(Y, \mathcal{F}) = \varprojlim_i H_{\text{ét}}^j(Y, \mathcal{F}_i)$. There is a natural surjective morphism $H_{\text{ét}}^j(Y, \mathcal{F}) \rightarrow H_{\text{ét}}^j(Y, \mathcal{F})$. The compactly supported cohomology groups $H_{\text{ét},c}^j(Y, \mathcal{F})$ and $H_{\text{ét},c}^j(Y, \mathcal{F})$ are defined similarly.

There is an isomorphism $\pi_1^{\text{ét}}(Y_{\overline{\mathbb{Q}}}, \eta) \cong \hat{\Gamma}$. Thus, if $\mathcal{F} \in \mathbf{M}_f(\mathcal{G})$ is a discrete $\mathcal{G}$ -module and $\mathcal{F}$ is the corresponding object in $\mathbf{S}_f(Y_{\text{ét}})$, there are natural isomorphisms

$$H_{\text{ét}}^1(Y_{\overline{\mathbb{Q}}}, \mathcal{F}) \cong H^1(\hat{\Gamma}, \mathcal{F}) \cong H^1(\Gamma, \mathcal{F}). \quad (2.6)$$

Let $\mathcal{F} \in \mathbf{M}_f(\Gamma_0(p\mathbb{Z}_p))$ be a left $\Gamma_0(p\mathbb{Z}_p)$ -module, and assume that the $\Gamma_0(p\mathbb{Z}_p)$ -action on $\mathcal{F}$ extends to a left action of $\Sigma_0(p)$. Let $S = \Sigma_0(p) \cap \text{GL}_2(\mathbb{Q})$. The pair (Γ, S) is then a Hecke pair in the sense of [AS86a, §1.1] and there is a covariant (left) action of the Hecke algebra $D(\Gamma, S)$ on $H^1(\Gamma, \mathcal{F})$. For each $g \in S$, let $T(g) = \Gamma g \Gamma$.

Following [GS93, §1], we define, for each positive integer n , the Hecke operators

$$T_n = T \left(\begin{pmatrix} 1 & \\ & n \end{pmatrix} \right), \quad T'_n = T \left(\begin{pmatrix} n & \\ & 1 \end{pmatrix} \right).$$

Also, for each positive integer a coprime to p , let

$$[a]_p = T \left(\begin{pmatrix} a & \\ & a \end{pmatrix} \right), \quad [a]'_p = T \left(\begin{pmatrix} a & \\ & a \end{pmatrix} \right).$$

Finally, for each positive integer a coprime to N , choose β_a (respectively β'_a) in $\Gamma_0(Npm)$ whose lower right entry is congruent to a (respectively a^{-1}) modulo N and let

$$[a]_N = T(\beta_a), \quad [a]'_N = T(\beta'_a).$$

The isomorphism (2.6) is compatible with Hecke actions in the following sense. To distinguish between different levels, we shall now write $\tilde{Y}(m)$ and $\tilde{\Gamma}(m)$ for the above Y and Γ , respectively. Let s be a positive integer. Choose as above a geometric point $\eta : \text{Spec}(\overline{\mathbb{Q}}) \rightarrow \tilde{Y}(m)$ and let $\eta_s : \text{Spec}(\overline{\mathbb{Q}}) \rightarrow \tilde{Y}(ms)$ be a geometric point lying above η . Let $r = 1 + \text{ord}_p(s)$ and choose an isomorphism $\mathcal{T}_{\eta_s} \cong \mathbb{Z}_p \oplus \mathbb{Z}_p$ such that the Weil pairing on $\mathcal{T}_{\eta_s}$ corresponds to the natural determinant map on the right, and the reduction modulo p^r of the element $(0,1)$ generates the canonical subgroup C_{p^r, η_s} . Using these choices to define the corresponding isomorphisms between group cohomology and étale cohomology, there are commutative diagrams

$$\begin{array}{ccc} H^1_{\text{ét}}(\tilde{Y}(ms)_{\overline{\mathbb{Q}}}, \mathcal{F}) & \xrightarrow{\pi_{1*}} & H^1_{\text{ét}}(\tilde{Y}(m)_{\overline{\mathbb{Q}}}, \mathcal{F}) \\ \downarrow \cong & & \downarrow \cong \\ H^1(\tilde{\Gamma}(ms), \mathcal{F}) & \xrightarrow{\text{cor}} & H^1(\tilde{\Gamma}(m), \mathcal{F}) \end{array} \quad \begin{array}{ccc} H^1_{\text{ét}}(\tilde{Y}(m)_{\overline{\mathbb{Q}}}, \mathcal{F}) & \xrightarrow{\pi_1^*} & H^1_{\text{ét}}(\tilde{Y}(ms)_{\overline{\mathbb{Q}}}, \mathcal{F}) \\ \downarrow \cong & & \downarrow \cong \\ H^1(\tilde{\Gamma}(m), \mathcal{F}) & \xrightarrow{\text{res}} & H^1(\tilde{\Gamma}(ms), \mathcal{F}). \end{array}$$

Also, if $(\begin{smallmatrix} s & \\ & 1 \end{smallmatrix}) \in \Sigma_0(p)$, we have the commutative diagram

$$\begin{array}{ccccccc} H_{\text{ét}}^1(\tilde{Y}(ms)_{\overline{\mathbb{Q}}}, \mathcal{F}) & \xrightarrow{\lambda_{s*}} & H_{\text{ét}}^1(\tilde{Y}(ms)_{\overline{\mathbb{Q}}}, \varphi_s^*(\mathcal{F})) & \xrightarrow{\varphi_{s*}} & H_{\text{ét}}^1(Y(1(s), N(pm))_{\overline{\mathbb{Q}}}, \mathcal{F}) & \xrightarrow{\tilde{\nu}_{s*}} & H_{\text{ét}}^1(\tilde{Y}(m)_{\overline{\mathbb{Q}}}, \mathcal{F}) \\ \downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong \\ H^1(\tilde{\Gamma}(ms), \mathcal{F}) & \xrightarrow{\lambda_{s*}} & H^1(\tilde{\Gamma}(ms), \varphi_s^*(\mathcal{F})) & \xrightarrow{\varphi_{s*}} & H^1(\Gamma(1(s), N(pm)), \mathcal{F}) & \xrightarrow{\text{cor}} & H^1(\tilde{\Gamma}(m), \mathcal{F}), \end{array}$$

and, if $(\begin{smallmatrix} 1 & \\ & s \end{smallmatrix}) \in \Sigma_0(p)$, the commutative diagram

$$\begin{array}{ccccccc} H_{\text{ét}}^1(\tilde{Y}(m)_{\overline{\mathbb{Q}}}, \mathcal{F}) & \xrightarrow{\tilde{\nu}_s^*} & H_{\text{ét}}^1(Y(1(s), N(pm))_{\overline{\mathbb{Q}}}, \mathcal{F}) & \xrightarrow{\varphi_s^*} & H_{\text{ét}}^1(\tilde{Y}(ms)_{\overline{\mathbb{Q}}}, \varphi_s^*(\mathcal{F})) & \xrightarrow{\lambda_s^*} & H_{\text{ét}}^1(\tilde{Y}(m)_{\overline{\mathbb{Q}}}, \mathcal{F}) \\ \downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong \\ H^1(\tilde{\Gamma}(m), \mathcal{F}) & \xrightarrow{\text{res}} & H^1(\Gamma(1(s), N(pm)), \mathcal{F}) & \xrightarrow{\varphi_s^*} & H^1(\tilde{\Gamma}(ms), \varphi_s^*(\mathcal{F})) & \xrightarrow{\lambda_s^*} & H^1(\tilde{\Gamma}(m), \mathcal{F}). \end{array}$$

In the bottom lines of the previous two diagrams, $\varphi_s^*(\mathcal{F})$ is $\mathcal{F}$ with the action of $\Gamma_0(p^r \mathbb{Z}_p)$ conjugated by $(\begin{smallmatrix} s & \\ & 1 \end{smallmatrix})$; the map λ_{s*} is induced by the map $\mathcal{F} \rightarrow \varphi_s^*(\mathcal{F})$ defined by $c \mapsto (\begin{smallmatrix} s & \\ & 1 \end{smallmatrix}) c$; φ_{s*} is induced by the pair of compatible maps $\Gamma(1(s), N(pm)) \rightarrow \tilde{\Gamma}(ms)$ and $\varphi_s^*(\mathcal{F}) \rightarrow \mathcal{F}$ defined by $\gamma \mapsto (\begin{smallmatrix} s^{-1} & \\ & 1 \end{smallmatrix}) \gamma (\begin{smallmatrix} s & \\ & 1 \end{smallmatrix})$ and $c \mapsto c$, respectively; λ_s^* is induced by the map $\varphi_s(\mathcal{F}) \rightarrow \mathcal{F}$ defined by $c \mapsto (\begin{smallmatrix} 1 & \\ & s \end{smallmatrix}) c$, and φ_s^* is induced by the pair of compatible maps $\tilde{\Gamma}(m) \rightarrow \Gamma(1(s), N(pm))$ and $\mathcal{F} \rightarrow \varphi_s^*(\mathcal{F})$ defined by $\gamma \mapsto (\begin{smallmatrix} 1 & \\ & s^{-1} \end{smallmatrix}) \gamma (\begin{smallmatrix} 1 & \\ & s \end{smallmatrix})$ and $c \mapsto c$, respectively.

We shall denote by π_{2*} and π_2^* , respectively, the composition of the maps in the rows of the previous two diagrams, both in étale cohomology and in group cohomology. Similarly, we shall also use π_{1*} and π_1^* to denote the corresponding corestriction and restriction maps.

For any rational prime q , a simple calculation shows that the following identities hold in group cohomology whenever the maps involved are defined:

$$T_q = \pi_{1*} \circ \pi_2^*, \quad T'_q = \pi_{2*} \circ \pi_1^*.$$

Therefore, under the isomorphism (2.6), the covariant action of the operators T_q, T'_q on étale cohomology corresponds to the covariant action of the operators T_q, T'_q on group cohomology, whenever defined. Similarly, the covariant action of the operators $\langle d \rangle, \langle d \rangle'$ on étale cohomology corresponds to the covariant action of the operators $[d]_N, [d]'_N$ on group cohomology.

The anti-involution ι extends to $\text{Mat}_{2 \times 2}(\mathbb{Z}_p)$ in the obvious way and turns a left (respectively right) action of $\Sigma_0(p)$ into a right (respectively left) action of $\Sigma'_0(p)$. Thus, given an object $\mathcal{F} \in \mathbf{M}(\Gamma_0(p\mathbb{Z}_p))$ whose right $\Gamma_0(p\mathbb{Z}_p)$ -action extends to a right $\Sigma'_0(p)$ -action, there is an isomorphism $H^1_{\text{ét}}(Y_{\overline{\mathbb{Q}}}, \mathcal{F}) \cong H^1(\Gamma, \mathcal{F})$ under which the contravariant action of the operators $T_q, T'_q, \langle d \rangle, \langle d \rangle'$ on étale cohomology corresponds to the contravariant action of the operators $T_q, T'_q, [d]_N, [d]'_N$ on group cohomology, whenever defined.

Consider the modules $\mathcal{A}_\nu$ and $\mathcal{D}_\nu$, defined in the previous section. The action of $\Gamma_0(p\mathbb{Z}_p)$ on $\mathbf{T}'$ is transitive and the stabilizer of the element $(0, 1) \in \mathbf{T}'$ is the subgroup

$$P(\mathbb{Z}_p) = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \in \text{GL}_2(\mathbb{Z}_p) \right\},$$

so we can identify $\mathbf{T}'$ with $P(\mathbb{Z}_p) \backslash \Gamma_0(p\mathbb{Z}_p)$. Similarly, the action of $\Gamma_0(p\mathbb{Z}_p)$ on $\mathbf{T}$ is transitive and the stabilizer of the element $(1, 0) \in \mathbf{T}$ is the subgroup

$$P(\mathbb{Z}_p)^w = \left\{ \begin{pmatrix} 1 & 0 \\ pc & d \end{pmatrix} \in \text{GL}_2(\mathbb{Z}_p) \right\},$$

so we can identify $\mathbf{T}$ with $P(\mathbb{Z}_p)^w \backslash \Gamma_0(p\mathbb{Z}_p)$. For any positive integer j , let

$$\Gamma_1(p^j\mathbb{Z}_p) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}_2(\mathbb{Z}_p) : c \equiv 0 \pmod{p^j}, d \equiv 1 \pmod{p^j} \right\},$$

$$\Gamma_1(p^j \mathbb{Z}_p)^w = \left\{ \begin{pmatrix} a & b \\ pc & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z}_p) : a \equiv 1 \pmod{p^j}, b \equiv 0 \pmod{p^{j-1}} \right\}.$$

Then, for any positive integers i, j , we can define

$$\begin{aligned} \mathcal{A}'_{\nu, i, j} = \{ f : \Gamma_1(p^j \mathbb{Z}_p) \backslash \Gamma_0(p \mathbb{Z}_p) &\rightarrow \mathcal{O}/\mathfrak{m}^i \mid f(a \cdot \gamma) = \nu(a) \cdot f(\gamma) \\ &\text{for all } a \in \mathbb{Z}_p^\times, \gamma \in \Gamma_1(p^j \mathbb{Z}_p) \backslash \Gamma_0(p \mathbb{Z}_p) \}, \end{aligned}$$

$$\begin{aligned} \mathcal{A}_{\nu, i, j} = \{ f : \Gamma_1(p^j \mathbb{Z}_p)^w \backslash \Gamma_0(p \mathbb{Z}_p) &\rightarrow \mathcal{O}/\mathfrak{m}^i \mid f(a \cdot \gamma) = \nu(a) \cdot f(\gamma) \\ &\text{for all } a \in \mathbb{Z}_p^\times, \gamma \in \Gamma_1(p^j \mathbb{Z}_p)^w \backslash \Gamma_0(p \mathbb{Z}_p) \}. \end{aligned}$$

The objects $\mathcal{A}_{\nu, i, j}$ can be regarded as left $\mathcal{O}[\Sigma_0(p)]$ -modules. Let $\mathcal{A}_{\nu, i} = \varinjlim_j \mathcal{A}_{\nu, i, j}$. Then $\mathcal{A}_\nu = \varprojlim_i \mathcal{A}_{\nu, i}$. We denote by $\mathbf{\mathcal{A}}_\nu$ the object in $\mathbf{S}(Y_{\text{ét}})$ corresponding to $\{\mathcal{A}_{\nu, i}\}_i \in \mathbf{M}(\Gamma_0(p \mathbb{Z}_p))$. We also define $\mathcal{D}_{\nu, i} = \mathrm{Hom}_{\mathcal{O}}(\mathcal{A}_{\nu, i, i}, \mathcal{O}/\mathfrak{m}^i)$. These objects can be regarded as right $\mathcal{O}[\Sigma_0(p)]$ -modules and we have $\mathcal{D}_\nu = \varprojlim_i \mathcal{D}_{\nu, i}$. We denote by $\mathbf{\mathcal{D}}_\nu$ the object in $\mathbf{S}(Y_{\text{ét}})$ corresponding to $\{\mathcal{D}_{\nu, i}\}_i \in \mathbf{M}(\Gamma_0(p \mathbb{Z}_p))$. There are natural morphisms of $\mathcal{O}$ -modules

$$H_{\text{ét}}^1(Y_{\overline{\mathbb{Q}}}, \mathbf{\mathcal{A}}_\nu) \rightarrow H_{\text{ét}}^1(Y_{\overline{\mathbb{Q}}}, \mathbf{\mathcal{A}}_\nu) \cong H^1(\Gamma, \mathcal{A}_\nu)$$

and

$$H_{\text{ét}}^1(Y_{\overline{\mathbb{Q}}}, \mathbf{\mathcal{D}}_\nu) \cong H_{\text{ét}}^1(Y_{\overline{\mathbb{Q}}}, \mathbf{\mathcal{D}}_\nu) \cong H^1(\Gamma, \mathcal{D}_\nu)$$

compatible with the action of Hecke operators. We also have Hecke-equivariant isomorphisms

$$H_{\text{ét}, c}^1(Y_{\overline{\mathbb{Q}}}, \mathbf{\mathcal{D}}_\nu) \cong H_{\text{ét}, c}^1(Y_{\overline{\mathbb{Q}}}, \mathbf{\mathcal{D}}_\nu) \cong H_c^1(\Gamma, \mathcal{D}_\nu),$$

where $H_c^j(\Gamma, -) = H^{j-1}(\Gamma, \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Div}^0(\mathbb{P}^1(\mathbb{Q})), -))$. These isomorphisms allow us to define continuous $G_{\mathbb{Q}}$ -actions on the groups $H^1(\Gamma, \mathcal{A}_\nu)$, $H^1(\Gamma, \mathcal{D}_\nu)$ and $H_c^1(\Gamma, \mathcal{D}_\nu)$.

Given a character $\chi : \mathbb{Z}_p^\times \rightarrow \mathcal{O}^\times$, let $\mathcal{O}(\chi)$ be the module $\mathcal{O}$ with $\Gamma_0(p\mathbb{Z}_p)$ acting via $\chi \circ \det$, where $\det : \Gamma_0(p\mathbb{Z}_p) \rightarrow \mathbb{Z}_p^\times$ is the determinant map.

The natural $\mathcal{G}$ -equivariant evaluation map $\mathcal{A}_\nu \otimes_{\mathcal{O}} \mathcal{D}_\nu \rightarrow \mathcal{O}$ yields a $G_{\mathbb{Q}}$ -equivariant cup-product pairing

$$H^1(\Gamma, \mathcal{A}_\nu) \otimes_{\mathcal{O}} H_c^1(\Gamma, \mathcal{D}_\nu) \longrightarrow \mathcal{O}(-1) \quad (2.7)$$

under which the Hecke operators T_q , T'_q , $[d]_N$, $[d]'_N$ acting covariantly on the left, whenever defined, are adjoint to these same operators acting contravariantly on the right.

Let $\det : \mathbb{T}' \times \mathbb{T} \rightarrow \mathbb{Z}_p^\times$ be the function defined by $\det((x_1, x_2), (y_1, y_2)) = x_1 y_2 - x_2 y_1$ and let $\det_\nu$ be the composition of this function with $\nu : \mathbb{Z}_p^\times \rightarrow \mathcal{O}$. Evaluation at this function defines a $\mathcal{G}$ -equivariant map $\mathcal{D}'_\nu \otimes_{\mathcal{O}} \mathcal{D}_\nu \rightarrow \mathcal{O}(-\nu)$ which yields a $G_{\mathbb{Q}}$ -equivariant cup-product pairing

$$H^1(\Gamma, \mathcal{D}'_\nu) \otimes_{\mathcal{O}} H_c^1(\Gamma, \mathcal{D}_\nu) \longrightarrow \mathcal{O}(\nu)(-1), \quad (2.8)$$

where $\nu = \nu \circ \epsilon_{\text{cyc}} : G_{\mathbb{Q}} \rightarrow \mathcal{O}^\times$. Under this pairing, the Hecke operators T_q , T'_q , $[d]_N$, $[d]'_N$ acting contravariantly on the left, whenever defined, are adjoint to the Hecke operators T'_q , T_q , $[d]'_N$, $[d]_N$ acting contravariantly on the right. We obtain a similar pairing interchanging the roles of $\mathcal{D}_\nu$ and $\mathcal{D}'_\nu$.

2.5.4 Ordinary cohomology

For any $\mathbb{Z}_p$ -algebra B , let $S_r(B)$ be the set of two-variable homogeneous polynomials of degree r in $B[x_1, x_2]$. It is a left $B[\Sigma_0(p)]$ -module with the action of $\Sigma_0(p)$ defined by

$$gP(x_1, x_2) = P((x_1, x_2) \cdot g)$$

for all $g \in \Sigma_0(p)$ and $P(x_1, x_2) \in S_r(B)$. To the p -adic $\Gamma_0(p\mathbb{Z}_p)$ -representation $S_r = S_r(\mathbb{Z}_p)$ there corresponds the locally constant p -adic sheaf $\mathcal{S}_r$ on $Y_{\text{ét}}$ defined in §2.2.3. Therefore we have an isomorphism

$$H_{\text{ét}}^1(Y_{\overline{\mathbb{Q}}}, \mathcal{S}_r) \cong H^1(\Gamma, S_r)$$

which is Hecke-equivariant when we consider the covariant action of Hecke operators on both sides, and we use this isomorphism to define an action of $G_{\mathbb{Q}}$ on $H^1(\Gamma, S_r)$.

We also define $L_r(B) = \text{Hom}_B(S_r(B), B)$, which we regard as a right $B[\Sigma_0(p)]$ -module defining the $\Sigma_0(p)$ -action by

$$(\mu \cdot g)(P(x_1, x_2)) = \mu(gP(x_1, x_2))$$

for all $g \in \Sigma_0(p)$, $\mu \in L_r(B)$ and $P(x_1, x_2) \in S_r(B)$. To the p -adic $\Gamma_0(p\mathbb{Z}_p)$ -representation $L_r = L_r(\mathbb{Z}_p)$ there corresponds the locally constant p -adic sheaf $\mathcal{L}_r$ on $Y_{\text{ét}}$. Therefore we have an isomorphism

$$H_{\text{ét}}^1(Y_{\overline{\mathbb{Q}}}, \mathcal{L}_r) \cong H^1(\Gamma, L_r)$$

which is Hecke-equivariant when we consider the contravariant action of Hecke operators on both sides, and we use this isomorphism to define an action of $G_{\mathbb{Q}}$ on $H^1(\Gamma, L_r)$.

The natural $\Gamma_0(p\mathbb{Z}_p)$ -equivariant evaluation map $S_r \otimes_{\mathbb{Z}_p} L_r \rightarrow \mathbb{Z}_p$ yields a $G_{\mathbb{Q}}$ -equivariant cup-product pairing

$$H^1(\Gamma, S_r) \otimes_{\mathbb{Z}_p} H_c^1(\Gamma, L_r) \longrightarrow \mathbb{Z}_p(-1) \quad (2.9)$$

under which the Hecke operators T_q , T'_q , $[d]_N$, $[d]'_N$ acting covariantly on the left,

whenever defined, are adjoint to these same operators acting contravariantly on the right. This pairing becomes perfect after inverting p .

Let $\nu_r : \mathbb{Z}_p^\times \rightarrow \mathbb{Z}_p^\times$ be the character defined by $\nu_r(z) = z^r$. Evaluation at the polynomial $(x_1y_2 - x_2y_1)^r \in S_r \otimes_{\mathbb{Z}_p} S_r$ defines a $\Gamma_0(p\mathbb{Z}_p)$ -equivariant map $L_r \otimes_{\mathbb{Z}_p} L_r \rightarrow \mathbb{Z}_p(-\nu_r)$ and thus yields a $G_{\mathbb{Q}}$ -equivariant cup-product pairing

$$H^1(\Gamma, L_r) \otimes_{\mathbb{Z}_p} H_c^1(\Gamma, L_r) \longrightarrow \mathbb{Z}_p(r-1) \quad (2.10)$$

under which the Hecke operators $T_q, T'_q, [d]_N, [d]'_N$ acting contravariantly on the left, whenever defined, are adjoint to the Hecke operators $T'_q, T_q, [d]'_N, [d]_N$ acting contravariantly on the right. This pairing becomes perfect after inverting p .

Combining these two pairings we can define a morphism

$$\mathbf{s}_{r*} : H^1(\Gamma, S_r(\mathbb{Q}_p)) \longrightarrow H^1(\Gamma, L_r(\mathbb{Q}_p))(-r).$$

This map is $G_{\mathbb{Q}}$ -equivariant and intertwines the covariant action of the operators $T_q, [d]_N, [a]_p$ on the source with the contravariant action of the operators $T'_q, [d]'_N, [a]'_p$ on the target. We can also define $\mathbf{s}_{r*}$ directly via the isomorphism $S_r(\mathbb{Q}_p) \cong L_r(\mathbb{Q}_p)(\nu_r)$ arising from the perfect pairing $L_r(\mathbb{Q}_p) \otimes_{\mathbb{Q}_p} L_r(\mathbb{Q}_p) \rightarrow \mathbb{Q}_p(-\nu_r)$ defined by evaluation at $(x_1y_2 - x_2y_1)^r$. Therefore, the denominators introduced by this map are bounded by $r!$, i.e., an element in

$$\text{im} \left(H^1(\Gamma, S_r) \rightarrow H^1(\Gamma, S_r(\mathbb{Q}_p)) \right)$$

is mapped to an element in

$$\frac{1}{r!} \text{im} \left(H^1(\Gamma, L_r) \rightarrow H^1(\Gamma, L_r(\mathbb{Q}_p)) \right),$$

as follows from [BSV22, Rmk. 3.3].

To slightly simplify the notation, we will write $\mathcal{A}_r$ and $\mathcal{D}_r$ for $\mathcal{A}_{\nu_r}$ and $\mathcal{D}_{\nu_r}$, respectively. Regarding two-variable polynomials as functions on $\mathbb{T}$, we obtain a natural morphism of left $\mathbb{Z}_p[\Sigma_0(p)]$ -modules $S_r \rightarrow \mathcal{A}_r$. Also, dualizing this map, we obtain a morphism of right $\mathbb{Z}_p[\Sigma_0(p)]$ -modules $\mathcal{D}_r \rightarrow L_r$. Thus, we have $G_{\mathbb{Q}}$ -equivariant and Hecke-equivariant morphisms

$$H^1(\Gamma, S_r) \rightarrow H^1(\Gamma, \mathcal{A}_r) \quad \text{and} \quad H^1(\Gamma, \mathcal{D}_r) \rightarrow H^1(\Gamma, L_r).$$

Applying Hida's (anti-)ordinary projector $e_{\text{ord}} := \lim_{n \rightarrow \infty} (T_p)^{n!}$, the previous morphisms become isomorphisms

$$e_{\text{ord}} H^1(\Gamma, S_r) \cong e_{\text{ord}} H^1(\Gamma, \mathcal{A}_r), \quad e_{\text{ord}} H^1(\Gamma, \mathcal{D}_r) \cong e_{\text{ord}} H^1(\Gamma, L_r).$$

Under these isomorphisms, the pairings (2.9) and (2.10) correspond to the pairings (2.7) and (2.8), respectively, after applying the corresponding (anti-)ordinary projector to every term involved.

2.5.5 Λ -adic Poincaré pairing

It will be convenient to write $\langle a; b \rangle$, with $a \in (\mathbb{Z}/N\mathbb{Z})^\times$ and $b \in (\mathbb{Z}/p^r\mathbb{Z})^\times$, for the diamond operator $\langle d \rangle$, where $d \in (\mathbb{Z}/Np^r)^\times$ is congruent to a modulo N and to b modulo p^r . We also write $\epsilon_N : G_{\mathbb{Q}} \rightarrow (\mathbb{Z}/N\mathbb{Z})^\times$ for the mod N cyclotomic character.

For any positive integer r , let

$$G_r = 1 + p(\mathbb{Z}/p^r\mathbb{Z}), \quad \tilde{G}_r = (\mathbb{Z}/p^r\mathbb{Z})^\times,$$

and define

$$\Lambda_r = \mathbb{Z}_p[G_r], \quad \tilde{\Lambda}_r = \mathbb{Z}_p[\tilde{G}_r], \quad \Lambda = \varprojlim_r \Lambda_r = \mathbb{Z}_p[[1 + p\mathbb{Z}_p]], \quad \tilde{\Lambda} = \varprojlim_r \tilde{\Lambda}_r = \mathbb{Z}_p[[\mathbb{Z}_p^\times]].$$

We have natural factorizations $(\mathbb{Z}/p^r\mathbb{Z})^\times = \mu_{p-1} \times (1 + p\mathbb{Z}/p^r\mathbb{Z})$ and $\mathbb{Z}_p^\times = \mu_{p-1} \times (1 + p\mathbb{Z}_p)$ which give natural embeddings $\Lambda_r \hookrightarrow \tilde{\Lambda}_r$ and $\Lambda \hookrightarrow \tilde{\Lambda}$. We define idempotents

$$e_i = \frac{1}{p-1} \sum_{\zeta \in \mu_{p-1}} \zeta^{-i}[\zeta]$$

for any integer i modulo $p-1$. Let $\kappa_i : \mathbb{Z}_p^\times \rightarrow \Lambda^\times$ be the character defined by $z \mapsto \omega^i(z)[\langle z \rangle]$ and let $\boldsymbol{\kappa}_i = \kappa_i \circ \epsilon_{\text{cyc}} : G_{\mathbb{Q}} \rightarrow \Lambda^\times$.

We will shorten notation by writing

$$X_r(m) = X(1, Np^r(m)), \quad H_{\text{ét}}^1(X_\infty(m)_{\overline{\mathbb{Q}}}, \mathbb{Z}_p) = \varprojlim_r H_{\text{ét}}^1(X_r(m)_{\overline{\mathbb{Q}}}, \mathbb{Z}_p). \quad (2.11)$$

We have a natural action of $\tilde{\Lambda}_r$ and $\tilde{\Lambda}$ on the previous groups defined by letting group-like elements $[u]$ act like the diamond operators $\langle 1; u \rangle'$.

Fix compatible primitive p -power roots of unity ζ_{p^r} and a primitive N -th root of unity ζ_N . Then one can define Atkin–Lehner automorphisms w_r and w for the curve $X_r(m)$ similarly as in [DR17, §1.2]. More precisely, $X_r(m)$ parameterizes quadruples (E, P, Q, C) , where E is an elliptic curve, P is a point of order N , Q is a point of order p^r and C is a cyclic subgroup of E of order Nm containing P . Then, we define

$$w_r(E, P, Q, C) = (E/C_Q, P + C_Q, Q' + C_Q, C + C_Q/C_Q),$$

where $C_Q \subseteq E$ is the subgroup generated by Q and $Q' \in E[p^r]$ satisfies $\langle Q, Q' \rangle = \zeta_{p^r}$.

Similarly, we define

$$w(E, P, Q, C) = (E/C, P' + C, Q + C, E[Nm]/C),$$

where $P' \in E[N]$ satisfies $\langle P, P' \rangle = \zeta_N$. These Atkin–Lehner automorphisms satisfy, for any $\sigma \in G_{\mathbb{Q}}$,

$$w_r^\sigma = \langle 1; \epsilon_{\text{cyc}}(\sigma) \rangle w_r, \quad w^\sigma = \langle \epsilon_N(\sigma); 1 \rangle w.$$

We let w and w_r act on cohomology via pullback.

Define $G_{\mathbb{Q}}$ -equivariant pairings

$$\langle \cdot, \cdot \rangle_{G_r} : e_i H_{\text{ét}}^1(X_r(m)_{\overline{\mathbb{Q}}}, \mathbb{Z}_p) \times e_{-i} H_{\text{ét}}^1(X_r(m)_{\overline{\mathbb{Q}}}, \mathbb{Z}_p) \longrightarrow \Lambda_r(-1)$$

by the formula

$$\langle a, b \rangle_{G_r} = \sum_{\sigma \in G_r} \langle a^\sigma, b \rangle_r \cdot \sigma^{-1},$$

where $\langle \cdot, \cdot \rangle_r$ stands for the natural Poincaré pairing. These pairings are Λ_r -linear and anti-linear in the first and second argument, respectively. Then we get $G_{\mathbb{Q}}$ -equivariant Λ_r -pairings

$$[\cdot, \cdot]_{G_r} : e_i H_{\text{ét}}^1(X_r(m)_{\overline{\mathbb{Q}}}, \mathbb{Z}_p) \times e_i H_{\text{ét}}^1(X_r(m)_{\overline{\mathbb{Q}}}, \mathbb{Z}_p) (\langle \epsilon_N^{-1}; 1 \rangle') \longrightarrow \Lambda_r(\kappa_i)(-1)$$

via the following modification of the previous pairing:

$$[a, b]_{G_r} = \langle a, ww_r \cdot (T'_p)^r \cdot b \rangle_{G_r}.$$

These pairings are compatible in the sense that the diagram

$$\begin{array}{ccc}
e_i H_{\text{ét}}^1(X_{r+1}(m)_{\overline{\mathbb{Q}}}, \mathbb{Z}_p) \times e_i H_{\text{ét}}^1(X_{r+1}(m)_{\overline{\mathbb{Q}}}, \mathbb{Z}_p)(\langle \epsilon_N^{-1}; 1 \rangle') & \xrightarrow{[\cdot, \cdot]_{G_{r+1}}} & \Lambda_{r+1}(\kappa_i)(-1) \\
\downarrow \pi_{1*} \times \pi_{1*} & & \downarrow \\
e_i H_{\text{ét}}^1(X_r(m)_{\overline{\mathbb{Q}}}, \mathbb{Z}_p) \times e_i H_{\text{ét}}^1(X_r(m)_{\overline{\mathbb{Q}}}, \mathbb{Z}_p)(\langle \epsilon_N^{-1}; 1 \rangle') & \xrightarrow{[\cdot, \cdot]_{G_r}} & \Lambda_r(\kappa_i)(-1)
\end{array}$$

commutes, which can be proved as in [DR17, Lem. 1.1]. This yields a Λ -adic perfect $G_{\mathbb{Q}}$ -equivariant pairing

$$e_i H_{\text{ét}}^1(X_{\infty}(m)_{\overline{\mathbb{Q}}}, \mathbb{Z}_p)^{\text{ord}} \times e_i H_{\text{ét}}^1(X_{\infty}(m)_{\overline{\mathbb{Q}}}, \mathbb{Z}_p)^{\text{ord}}(\langle \epsilon_N^{-1}; 1 \rangle') \longrightarrow \Lambda(\kappa_i)(-1), \quad (2.12)$$

where $H_{\text{ét}}^1(X_{\infty}(m)_{\overline{\mathbb{Q}}}, \mathbb{Z}_p)^{\text{ord}} = e'_{\text{ord}} H_{\text{ét}}^1(X_{\infty}(m)_{\overline{\mathbb{Q}}}, \mathbb{Z}_p)$. All Hecke operators are self-adjoint for this pairing.

2.5.6 Big Galois representations

Let $\mathfrak{m}_{\Lambda}$ be the maximal ideal of Λ , let $\text{Cont}(\mathbb{Z}_p, \Lambda)$ be the Λ -module of continuous functions on $\mathbb{Z}_p$ with values in Λ , and let κ be any of the κ_i above. Define the Λ -module

$$\mathcal{A}'_{\kappa} = \{f : \Gamma' \rightarrow \Lambda \mid f(pz, 1) \in \text{Cont}(\mathbb{Z}_p, \Lambda) \text{ and } f(a \cdot \gamma) = \kappa(a) \cdot f(\gamma) \text{ for all } a \in \mathbb{Z}_p^{\times}, \gamma \in \Gamma'\},$$

equipped with the $\mathfrak{m}_{\Lambda}$ -adic topology, and the Λ -module

$$\mathcal{D}'_{\kappa} = \text{Hom}_{\text{cont}, \Lambda}(\mathcal{A}'_{\kappa}, \Lambda)$$

equipped with the weak- $*$ topology. As in §2.5.2, we can regard $\mathcal{A}'_{\kappa}$ (respectively $\mathcal{D}'_{\kappa}$) as a left (respectively right) $\Lambda[\Sigma'_0(p)]$ -module.

Similarly to what we did in §2.5.3, define, for any positive integers j, r ,

$$\begin{aligned} \mathcal{A}'_{\kappa,j,r} = \{ f : \Gamma_1(p^r \mathbb{Z}_p) \backslash \Gamma_0(p \mathbb{Z}_p) &\rightarrow \Lambda / \mathfrak{m}_\Lambda^j \mid f(a \cdot \gamma) = \kappa(a) \cdot f(\gamma) \\ &\text{for all } a \in \mathbb{Z}_p^\times, \gamma \in \Gamma_1(p^r \mathbb{Z}_p) \backslash \Gamma_0(p \mathbb{Z}_p) \} \end{aligned}$$

and $\mathcal{A}'_{\kappa,j} = \varinjlim_r \mathcal{A}'_{\kappa,j,r}$. Then $\mathcal{A}'_\kappa = \varprojlim_j \mathcal{A}'_{\kappa,j}$. We denote by $\mathbf{A}'_\kappa$ the object in $\mathbf{S}(Y_{\text{ét}})$ corresponding to $\{\mathcal{A}'_{\kappa,j}\}_j \in \mathbf{M}(\Gamma_0(p \mathbb{Z}_p))$. We also define $\mathcal{D}'_{\kappa,j} = \text{Hom}_\Lambda(\mathcal{A}'_{\kappa,j}, \Lambda / \mathfrak{m}_\Lambda^j)$, so that $\mathcal{D}'_\kappa = \varprojlim_j \mathcal{D}'_{\kappa,j}$, and denote by $\mathbf{D}'_\kappa$ the object in $\mathbf{S}(Y_{\text{ét}})$ corresponding to $\{\mathcal{D}'_{\kappa,j}\}_j \in \mathbf{M}(\Gamma_0(p \mathbb{Z}_p))$. There are natural Hecke-equivariant morphisms of Λ -modules

$$H_{\text{ét}}^1(Y_{\overline{\mathbb{Q}}}, \mathbf{A}'_\kappa) \rightarrow H_{\text{ét}}^1(Y_{\overline{\mathbb{Q}}}, \mathbf{A}'_\kappa) \cong H^1(\Gamma, \mathbf{A}'_\kappa),$$

$$H_{\text{ét}}^1(Y_{\overline{\mathbb{Q}}}, \mathbf{D}'_\kappa) \cong H_{\text{ét}}^1(Y_{\overline{\mathbb{Q}}}, \mathbf{D}'_\kappa) \cong H^1(\Gamma, \mathbf{D}'_\kappa),$$

$$H_{\text{ét},c}^1(Y_{\overline{\mathbb{Q}}}, \mathbf{D}'_\kappa) \cong H_{\text{ét},c}^1(Y_{\overline{\mathbb{Q}}}, \mathbf{D}'_\kappa) \cong H_c^1(\Gamma, \mathbf{D}'_\kappa).$$

which allow us to define continuous $G_{\mathbb{Q}}$ -actions on the groups $H^1(\Gamma, \mathbf{A}'_\kappa)$, $H^1(\Gamma, \mathbf{D}'_\kappa)$ and $H_c^1(\Gamma, \mathbf{D}'_\kappa)$.

The evaluation map $\mathcal{A}'_\kappa \otimes_\Lambda \mathcal{D}'_\kappa \rightarrow \Lambda$ yields a $G_{\mathbb{Q}}$ -equivariant cup-product pairing

$$H^1(\Gamma, \mathbf{A}'_\kappa) \otimes_\Lambda H_c^1(\Gamma, \mathbf{D}'_\kappa) \longrightarrow \Lambda(-1) \tag{2.13}$$

under which the Hecke operators T_q , T'_q , $[d]_N$, $[d]'_N$ acting covariantly on the left, whenever defined, are adjoint to these same operators acting contravariantly on the right.

Recall that in this section we have set $\Gamma = \Gamma(1, N(pm))$ and let $S = \Sigma'_0(p) \cap$

$\mathrm{GL}_2(\mathbb{Q})$. For any positive integer r , define

$$\Sigma'_1(p^r) = \begin{pmatrix} \mathbb{Z}_p & \mathbb{Z}_p \\ p^r \mathbb{Z}_p & 1 + p^r \mathbb{Z}_p \end{pmatrix}, \quad S_r = \Sigma'_1(p^r) \cap \mathrm{GL}_2(\mathbb{Q}), \quad \Gamma_r = \Gamma(1, Np^r(m)).$$

We define compatibility of Hecke pairs as in [AS86a, Def. 1.1.2], but changing left-right conventions. More precisely, we say that the Hecke pair $(\Gamma_\alpha, S_\alpha)$ is compatible to the Hecke pair (Γ_β, S_β) if $(\Gamma_\alpha, S_\alpha) \subseteq (\Gamma_\beta, S_\beta)$, $S_\alpha \Gamma_\beta = S_\beta$ and $\Gamma_\beta \cap S_\alpha^{-1} S_\alpha = \Gamma_\alpha$. With this definition, the Hecke pair (Γ_r, S_r) is compatible to the Hecke pair (Γ_t, S_t) , if $r \geq t$, and to the Hecke pair (Γ, S) .

Suppose that the Hecke pair $(\Gamma_\alpha, S_\alpha)$ is compatible to (Γ_β, S_β) and that Γ_α has finite index in Γ_β . For any right S_α -module E , we define

$$\mathrm{Ind}_{\Gamma_\alpha}^{\Gamma_\beta} E = \{ \varphi : \Gamma_\beta \rightarrow E \mid \varphi(xy) = \varphi(y)x^{-1} \text{ for all } x \in \Gamma_\alpha, y \in \Gamma_\beta \}$$

This module is equipped with a right action of S_β : given $\varphi \in \mathrm{Ind}_{\Gamma_\alpha}^{\Gamma_\beta} E$ and $g \in S_\beta$

$$(\varphi g)(x) = \sum \varphi(\gamma) \gamma g x^{-1},$$

where the sum is over representatives γ for the cosets in $\Gamma_\alpha \backslash \Gamma \cap S_\alpha x g^{-1}$.

Now define

$$A'_{\kappa,r} = \{ f : \Gamma_1(p^r \mathbb{Z}_p) \backslash \Gamma_0(p \mathbb{Z}_p) \rightarrow \Lambda_r \mid f(a \cdot \gamma) = \kappa(a) \cdot f(\gamma) \\ \text{for all } a \in \mathbb{Z}_p^\times, \gamma \in \Gamma_1(p^r \mathbb{Z}_p) \backslash \Gamma_0(p \mathbb{Z}_p) \}$$

and let $D'_{\kappa,r} = \mathrm{Hom}_{\Lambda_r}(A'_{\kappa,r}, \Lambda_r)$. With these definitions $\mathcal{D}'_\kappa = \varprojlim_r \mathcal{D}'_r$. Let S_r act trivially on $\mathbb{Z}_p$ and consider the right $\mathbb{Z}_p[S_1]$ -module $\mathrm{Ind}_{\Gamma_r}^{\Gamma_1} \mathbb{Z}_p$. Let R be a set of

representatives for the cosets in $\Gamma_r \backslash \Gamma_1$. The map $\text{Ind}_{\Gamma_r}^{\Gamma_1} \mathbb{Z}_p \rightarrow D'_r$ defined by

$$\varphi \mapsto [f \mapsto \sum_{r \in R} \varphi(r) f(r)]$$

is an isomorphism of right $\mathbb{Z}_p[S_1]$ -modules. Therefore, there are natural isomorphisms

$$H^1(\Gamma_1, \mathcal{D}'_\kappa) \cong \varprojlim_r H^1(\Gamma_1, D'_{\kappa, r}) \cong \varprojlim_r H^1(\Gamma_r, \mathbb{Z}_p).$$

According to [AS86a, Lem. 1.1.3] and [AS86a, Lem. 1.1.4], both corestriction and the Shapiro isomorphism commute with the action of $D(\Gamma, S)$ via restriction of Hecke algebras, so the previous isomorphisms are Hecke-equivariant.

Similarly to (2.11), but omitting m from the notation, we let $Y_r = Y(1, Np^r(m))$ and put

$$H^1_{\text{ét}}(Y_{\infty, \overline{\mathbb{Q}}}, \mathbb{Z}_p) := \varprojlim_r H^1_{\text{ét}}(Y_r, \overline{\mathbb{Q}}, \mathbb{Z}_p),$$

where the inverse limit is with respect to the maps π_{1*} . Then

$$H^1(\Gamma_1, \mathcal{D}'_\kappa) \cong \varprojlim_r H^1(\Gamma_r, \mathbb{Z}_p) \cong H^1_{\text{ét}}(Y_{\infty, \overline{\mathbb{Q}}}, \mathbb{Z}_p),$$

where the last isomorphism is defined by choosing a compatible system of geometric points for the curves Y_r and suitable compatible bases for the corresponding Tate modules. Under the isomorphisms above, the contravariant operators T'_q , $[d]'_N$, $[a]'_p$ on the first term correspond to the contravariant operators T'_q , $\langle d; 1 \rangle'$, $\langle 1; a \rangle'$ defined on the last term via the compatibility of these operators with the pushforward maps π_{1*} .

Also, according to [AS86a, Lem. 1.1.5], the restriction map yields a Hecke-equivariant isomorphism

$$H^1(\Gamma, \mathcal{D}'_\kappa) \cong e_i H^1(\Gamma_1, \mathcal{D}'_\kappa)$$

(recall that we have set $\kappa = \kappa_i$). Combining this isomorphism with the previous ones, we obtain a Hecke-equivariant isomorphism

$$H^1(\Gamma, \mathcal{D}'_\kappa) \cong e_i H^1_{\text{ét}}(Y_{\infty, \overline{\mathbb{Q}}}, \mathbb{Z}_p).$$

Similarly, using [AS86b, Prop. 4.2], one proves that there is a Hecke-equivariant isomorphism

$$H^1_c(\Gamma, \mathcal{D}'_\kappa) \cong e_i H^1_{\text{ét}, c}(Y_{\infty, \overline{\mathbb{Q}}}, \mathbb{Z}_p). \quad (2.14)$$

2.6 Proof of the wild norm relations

Assume that p splits in K as $(p) = \mathfrak{p}\overline{\mathfrak{p}}$ and that it does not divide the class number h_K .

We keep most of the notations from §2.4. In particular, (g, h) is a pair of newforms of weights (l, m) of the same parity, levels (N_g, N_h) and characters (χ_g, χ_h) , and we assume that the ring of integers $\mathcal{O} \subset E = L_{\mathfrak{p}}$ contains the Fourier coefficients of g and h . In addition, we assume that p does not divide N_g nor N_h and that both g and h are ordinary at p .

We now allow the Größencharacter ψ to have infinity type $(1 - k, 0)$ for any even integer $k \geq 2$, and let $\mathfrak{f}$ be the conductor of ψ , which we assume to be coprime to p . Let χ be the unique Dirichlet character modulo $N_{K/\mathbb{Q}}(\mathfrak{f})$ such that $\psi((n)) = n^{k-1}\chi(n)$ for integers n coprime to $N_{K/\mathbb{Q}}(\mathfrak{f})$.

As in [BL18, §3.2.1], we denote by ψ_0 the unique Größencharacter of infinity type $(-1, 0)$, conductor $\mathfrak{p}$ and whose associated p -adic Galois character factors through $\Gamma_{\mathfrak{p}}$, the Galois group of the unique $\mathbb{Z}_p$ -extension of K unramified outside $\mathfrak{p}$. Then we can uniquely write $\psi = \alpha\psi_0^{k-1}$, where α is a ray class character of conductor dividing $\mathfrak{fp}$. Since $(\mathfrak{f}, p) = 1$ and k is even, it easily follows that ψ is *non-Eisenstein* and

p -distinguished, meaning that

$$\alpha\psi_0|_{\mathcal{O}_{K,\mathfrak{p}}^\times} \not\equiv \omega \pmod{\mathfrak{P}}, \quad (2.15)$$

where ω is the Teichmüller character.

Let $\psi_{\mathfrak{P}}$ be the continuous E -valued character of $K^\times \backslash \mathbb{A}_{K,f}^\times$ defined by

$$\psi_{\mathfrak{P}}(x) = x_{\mathfrak{p}}^{1-k} \psi(x),$$

where $x_{\mathfrak{p}}$ is the projection of the idèle x to the component at $\mathfrak{p}$. We will also denote by $\psi_{\mathfrak{P}}$ the corresponding character of G_K obtained via the geometric Artin map. Then $\text{Ind}_K^{\mathbb{Q}} E(\psi_{\mathfrak{P}}^{-1})$ is the p -adic representation attached to θ_ψ , and we note that by (2.15) the associated residual representation is absolutely irreducible and p -distinguished (see [LLZ15, Rmk. 5.1.4]).

Consider the q -expansion

$$\Theta = \sum_{(\mathfrak{a}, \mathfrak{f}\mathfrak{p})=1} [\mathfrak{a}] q^{N_{K/\mathbb{Q}}(\mathfrak{a})} \in \mathcal{O}[[H_{\mathfrak{f}\mathfrak{p}^\infty}]][[q]],$$

where $H_{\mathfrak{f}\mathfrak{p}^\infty}$ denotes the maximal pro- p quotient of the ray class group of K of conductor $\mathfrak{f}\mathfrak{p}^\infty$, and $[\mathfrak{a}]$ is the image of $\mathfrak{a}$ in $H_{\mathfrak{f}\mathfrak{p}^\infty}$ under the geometric Artin map. Since we assume that p does not divide h_K , we can factor $H_{\mathfrak{f}\mathfrak{p}^\infty} \cong H_{\mathfrak{f}} \times \Gamma_{\mathfrak{p}}$. Hence, we have $\Theta \in \mathcal{O}[H_{\mathfrak{f}}] \otimes_{\mathcal{O}} \mathcal{O}[[\Gamma_{\mathfrak{p}}]][[q]]$, and we can specialize this to

$$\mathbf{f} = \sum_{(\mathfrak{a}, \mathfrak{f}\mathfrak{p})=1} \alpha([\mathfrak{a}]) \psi_0([\mathfrak{a}]) [\mathfrak{a}] q^{N_{K/\mathbb{Q}}(\mathfrak{a})} \in \Lambda_{\mathbf{f}}[[q]], \quad (2.16)$$

where $\Lambda_{\mathbf{f}} = \mathcal{O}[[\Gamma_{\mathfrak{p}}]]$. We identify $\Gamma_{\mathfrak{p}}$ with $\Gamma = 1 + p\mathbb{Z}_p$ via the isomorphism $\Gamma \cong \mathcal{O}_{K,\mathfrak{p}}^{(1)} \rightarrow \Gamma_{\mathfrak{p}}$ defined by $u \mapsto \text{art}_{\mathfrak{p}}(u)^{-1}$, where $\text{art}_{\mathfrak{p}}$ stands for the geometric local Artin map, and in this way we identify $\Lambda_{\mathbf{f}}$ with $\Lambda_{\mathcal{O}} = \Lambda \otimes_{\mathbb{Z}_p} \mathcal{O}$. We can therefore regard $\mathbf{f}$

as a primitive Hida family specializing to

$$\mathbf{f}_{k'} = \sum_{(\mathbf{a}, \mathfrak{f}\mathfrak{p})=1} \alpha([\mathbf{a}]) \psi_0([\mathbf{a}])^{k'-1} q^{N_{K/\mathbb{Q}}(\mathbf{a})} \in S_{k'}^{\text{ord}}(N_\psi p, \chi_\alpha \varepsilon_K \omega^{1-k'})$$

at the arithmetic point $\nu_{k'-2}$, where $N_\psi = DN_{K/\mathbb{Q}}(\mathfrak{f})$ and $\chi_\alpha(n) = \alpha((n))$. Note that $\chi = \chi_\alpha \omega^{1-k}$ and $\mathbf{f}_k = \theta_\psi^{(p)}$ is the ordinary p -stabilization of θ_ψ .

Let $\chi_{\mathbb{Q}}$ be the adelic character attached to χ , let $\chi_K = \chi_{\mathbb{Q}} \circ N_{K/\mathbb{Q}}$ and let $\psi^* = \chi_K^{-1} \psi$. We can define a primitive Hida family $\mathbf{f}^*$ attached to the Grössencharacter ψ^* in the same way that we defined the Hida family $\mathbf{f}$ attached to ψ . This is just the Hida family $\mathbf{f} \otimes \chi^{-1}$.

We assume that $\chi \varepsilon_K \chi_g \chi_h = 1$, i.e., the product of the characters of θ_ψ , g and h is trivial. Similarly to what we did in §2.4, set $(r_1, r_2, r_3) = (k-2, l-2, m-2)$. For every positive integer m , let

$$\tilde{Y}(m) = Y(1, N(pm)), \quad \text{where } N = \text{lcm}(N_\psi, N_g, N_h),$$

and denote by $\tilde{\Gamma}(m)$ the corresponding modular group. Let $\kappa = \kappa_{r_1} : \mathbb{Z}_p^\times \rightarrow \Lambda^\times$ and choose a square root of this character defined by $\kappa^{1/2}(u) = \omega(u)^{(k-2)/2} [\langle u \rangle^{1/2}]$.

We can define classes

$$\mathbf{Det}_m^{\mathbf{f}gh} \in H_{\text{ét}}^0(\tilde{Y}(m), \mathcal{A}'_\kappa \otimes \mathcal{A}_{r_2} \otimes \mathcal{A}_{r_3}(-\kappa^{1/2} - \nu_{(r_2+r_3)/2})).$$

as in [BSV22, §8.1], but replacing the Hida families $\mathbf{g}, \mathbf{h}$ in their construction by our g, h and working with modules of continuous functions instead of modules of locally analytic functions. Similarly to what is done in *loc. cit.*, and adopting some of the notations there, we define the cohomology classes

$$\kappa_{m, \mathbf{f}gh}^{(1)} = (e'_{\text{ord}} \otimes e_{\text{ord}} \otimes e_{\text{ord}}) \circ K \circ \text{HS} \circ d_*(\mathbf{Det}_m^{\mathbf{f}gh}),$$

inside the group

$$H^1\left(\mathbb{Q}, H^1(\tilde{\Gamma}(m), \mathcal{A}'_{\kappa})^{\text{ord}} \hat{\otimes} H^1(\tilde{\Gamma}(m), \mathcal{A}_{r_2})^{\text{ord}} \hat{\otimes} H^1(\tilde{\Gamma}(m), \mathcal{A}_{r_3})^{\text{ord}} (\kappa^{1/2} + 2 + (r_2 + r_3)/2)\right),$$

where $\kappa^{1/2} = \kappa^{1/2} \circ \epsilon_{\text{cyc}}$; and, for each squarefree positive integer n coprime to p and N , we define

$$\kappa_{n, \mathbf{f}gh}^{(2)} = \chi \varepsilon_K(n) \kappa(n)^{-1} n^{r_2} (\text{id} \otimes \text{id} \otimes [n]_N) (\text{id} \otimes \pi_{1*} \otimes \pi_{2*}) \kappa_{n^2, \mathbf{f}gh}^{(1)}$$

lying in the group

$$H^1\left(\mathbb{Q}, H^1(\tilde{\Gamma}(n^2), \mathcal{A}'_{\kappa})^{\text{ord}} \hat{\otimes} H^1(\tilde{\Gamma}(1), \mathcal{A}_{r_2})^{\text{ord}} \hat{\otimes} H^1(\tilde{\Gamma}(1), \mathcal{A}_{r_3})^{\text{ord}} (\kappa^{1/2} + 2 + (r_2 + r_3)/2)\right).$$

Now we can prove norm relations for Λ -adic classes as we did for the classes in the previous section.

Lemma 2.6.1. *Let m be a positive integer and let q be a prime number. Assume that both m and q are coprime to p and N . Then*

$$\begin{aligned} (\pi_{2*} \otimes \pi_{1*} \otimes \pi_{1*}) \kappa_{mq, \mathbf{f}gh}^{(1)} &= (T'_q \otimes \text{id} \otimes \text{id}) \kappa_{m, \mathbf{f}gh}^{(1)}; \\ (\pi_{1*} \otimes \pi_{2*} \otimes \pi_{2*}) \kappa_{mq, \mathbf{f}gh}^{(1)} &= \kappa^{-1/2}(q) q^{(r_2+r_3)/2} (T_q \otimes \text{id} \otimes \text{id}) \kappa_{m, \mathbf{f}gh}^{(1)}; \\ (\pi_{1*} \otimes \pi_{2*} \otimes \pi_{1*}) \kappa_{mq, \mathbf{f}gh}^{(1)} &= (\text{id} \otimes T'_q \otimes \text{id}) \kappa_{m, \mathbf{f}gh}^{(1)}; \\ (\pi_{2*} \otimes \pi_{1*} \otimes \pi_{2*}) \kappa_{mq, \mathbf{f}gh}^{(1)} &= \kappa^{1/2}(q) q^{(r_3-r_2)/2} (\text{id} \otimes T_q \otimes \text{id}) \kappa_{m, \mathbf{f}gh}^{(1)}; \\ (\pi_{1*} \otimes \pi_{1*} \otimes \pi_{2*}) \kappa_{mq, \mathbf{f}gh}^{(1)} &= (\text{id} \otimes \text{id} \otimes T'_q) \kappa_{m, \mathbf{f}gh}^{(1)}; \\ (\pi_{2*} \otimes \pi_{2*} \otimes \pi_{1*}) \kappa_{mq, \mathbf{f}gh}^{(1)} &= \kappa^{1/2}(q) q^{(r_2-r_3)/2} (\text{id} \otimes \text{id} \otimes T_q) \kappa_{m, \mathbf{f}gh}^{(1)}. \end{aligned}$$

If q is coprime to m we also have

$$\begin{aligned}(\pi_{1*} \otimes \pi_{1*} \otimes \pi_{1*})\kappa_{mq, fgh}^{(1)} &= (q+1)\kappa_{m, fgh}^{(1)}; \\(\pi_{2*} \otimes \pi_{2*} \otimes \pi_{2*})\kappa_{mq, fgh}^{(1)} &= (q+1)\kappa^{1/2}(q)q^{(r_2+r_3)/2}\kappa_{m, fgh}^{(1)}.\end{aligned}$$

Proof. As in Lemma 2.4.1, the same arguments proving equations (174) and (176) in [BSV22] apply *mutatis mutandis* to yield the proof of these identities. $\square$

Lemma 2.6.2. *Let n be a squarefree positive integer coprime to p and N and let q be a rational prime coprime to p , N and n . Then*

$$\begin{aligned}(\pi_{11*} \otimes \text{id} \otimes \text{id})\kappa_{nq, fgh}^{(2)} &= \{\chi(q)\kappa(q)^{-1}q^{r_2}(\text{id} \otimes \text{id} \otimes [q]_N^{-1}T_q^2) \\&\quad - \chi(q)\kappa(q)^{-1}(q+1)q^{r_2+r_3}(\text{id} \otimes \text{id} \otimes \text{id})\}\kappa_{n, fgh}^{(2)}, \\(\pi_{21*} \otimes \text{id} \otimes \text{id})\kappa_{nq, fgh}^{(2)} &= \{\chi(q)\kappa(q)^{-1/2}(q)q^{(r_2+r_3)/2}(\text{id} \otimes T_q \otimes T_q) \\&\quad - \chi(q)\kappa(q)^{-1}q^{r_2+r_3}([q]_N')^{-1}T_q' \otimes [q]_N \otimes [q]_N)\}\kappa_{n, fgh}^{(2)}, \\(\pi_{22*} \otimes \text{id} \otimes \text{id})\kappa_{nq, fgh}^{(2)} &= \{\chi(q)q^{r_3}(\text{id} \otimes T_q^2 \otimes [q]_N) \\&\quad - \chi(q)(q+1)q^{r_2+r_3}(\text{id} \otimes [q]_N \otimes [q]_N)\}\kappa_{n, fgh}^{(2)},\end{aligned}$$

where π_{ij*} denotes the composition

$$H^1(\tilde{\Gamma}(n^2q^2), \mathcal{F}) \xrightarrow{\pi_{i*}} H^1(\tilde{\Gamma}(n^2q), \mathcal{F}) \xrightarrow{\pi_{j*}} H^1(\tilde{\Gamma}(n^2), \mathcal{F}).$$

Proof. This can be deduced from Lemma 2.6.1 by the same calculation as in Lemma 2.4.3. $\square$

Let $\Gamma(m) = \Gamma(1, Np(m))$ and write $Y(m)$ and $X(m)$ for the corresponding affine and projective modular curves. The pairing in equation (2.13) yields a map

$$H^1(\tilde{\Gamma}(m), \mathcal{A}'_\kappa) \rightarrow \text{Hom}_\Lambda(H_c^1(\Gamma(m), \mathcal{D}'_\kappa), \Lambda)(-1) \cong \text{Hom}_\Lambda(e_{r_1}H_{\text{ét}, c}^1(Y_\infty(m)_{\overline{\mathbb{Q}}}, \mathbb{Z}_p), \Lambda)(-1),$$

where the isomorphism comes from equation (2.14). Let $\mathcal{I}_n$ be the maximal ideal in Hida's big ordinary Hecke algebra $\mathbb{T}(1, Np^\infty(n^2))'_{\text{ord}}$ corresponding to the Hida family $\mathbf{f}^*$; by (2.15) this ideal corresponds to a non-Eisenstein maximal ideal in $\mathbb{T}(1, Np(n^2))'$, so there are isomorphisms

$$H_{\text{ét},c}^1(Y_\infty(n^2)_{\overline{\mathbb{Q}}}, \mathbb{Z}_p)_{\mathcal{I}_n}^{\text{ord}} \cong H_{\text{ét}}^1(X_\infty(n^2)_{\overline{\mathbb{Q}}}, \mathbb{Z}_p)_{\mathcal{I}_n}^{\text{ord}} \cong H_{\text{ét}}^1(Y_\infty(n^2)_{\overline{\mathbb{Q}}}, \mathbb{Z}_p)_{\mathcal{I}_n}^{\text{ord}}.$$

Hence, the pairings (2.13) and (2.12) together with the isomorphism (2.14) yield a morphism

$$\mathbf{s}_{\mathbf{f},n*} : H^1(\tilde{\Gamma}(n^2), \mathcal{A}'_\kappa)^{\text{ord}} \longrightarrow e_{r_1} H_{\text{ét}}^1(Y_\infty(n^2)_{\overline{\mathbb{Q}}}, \mathbb{Z}_p)_{\mathcal{I}_n}^{\text{ord}}(\langle \epsilon_N^{-1}; 1 \rangle')(-\kappa).$$

This map is $G_{\mathbb{Q}}$ -equivariant and intertwines the covariant action of the operators T'_q , $[d]'_N$, $[a]'_p$ on the source with the contravariant action of the operators T'_q , $\langle d; 1 \rangle'$, $\langle 1; a \rangle'$ on the target.

Fix a level- N test vector $\check{\mathbf{f}}$ for $\mathbf{f}$ and let $\check{\mathbf{f}}^* = \check{\mathbf{f}} \otimes \chi^{-1} \varepsilon_K$. Fix also test vectors

$$\check{g} \in S_l(N, \chi_g)[g], \quad \check{h} \in S_m(N, \chi_h)[h]$$

and write $\check{g}_\alpha$ and $\check{h}_\alpha$ for the corresponding ordinary p -stabilizations.

Define maps

$$\phi_{n,r} : \mathbb{T}(1, N_\psi p^r(n^2))'_{\text{ord}} \longrightarrow \mathcal{O}[R_{\check{\mathfrak{p}}^r,n}]$$

attached to the Grössencharacter $\alpha \chi_K^{-1} \psi_0$ as in Lemma 2.3.5 and let

$$\phi_{n,\infty} : \mathbb{T}(1, N_\psi p^\infty(n^2))'_{\text{ord}} \longrightarrow \mathcal{O}[[R_{\check{\mathfrak{p}}^\infty,n}]] = \mathcal{O}[R_{\check{\mathfrak{f}},n}] \otimes_{\mathcal{O}} \mathcal{O}[[\Gamma_{\mathfrak{p}}]].$$

be the inverse limit $\varprojlim_r \phi_{n,r}$. The test vector $\check{\mathbf{f}}^*$ determines a degeneracy map

$$H_{\text{ét}}^1(Y_\infty(n^2)_{\overline{\mathbb{Q}}}, \mathbb{Z}_p(1))^{\text{ord}} \rightarrow H_{\text{ét}}^1(Y(1, N_\psi p^\infty(n^2))_{\overline{\mathbb{Q}}}, \mathbb{Z}_p(1))^{\text{ord}}.$$

Composing this degeneracy map with the natural quotient map we get a morphism

$$\pi_{\mathbf{f}^*} : e_{r_1} H_{\text{ét}}^1(Y_\infty(n^2)_{\overline{\mathbb{Q}}}, \mathcal{O}(1))_{\mathcal{I}_n}^{\text{ord}} \rightarrow (\mathcal{O}[R_n] \otimes_{\mathcal{O}} \mathcal{O}[[\Gamma_{\mathfrak{p}}]]) \otimes_{\phi_{n,\infty}} H_{\text{ét}}^1(Y(1, N_\psi p^\infty(n^2))_{\overline{\mathbb{Q}}}, \mathcal{O}(1))^{\text{ord}}.$$

The test vectors $\check{g}_\alpha$ and $\check{h}_\alpha$ determine degeneracy maps

$$\begin{aligned} H_{\text{ét}}^1(\tilde{Y}(1)_{\overline{\mathbb{Q}}}, \mathcal{L}_{r_2}(1)) &\xrightarrow{\mu_p^*} H_{\text{ét}}^1(Y_1(Np)_{\overline{\mathbb{Q}}}, \mathcal{L}_{r_2}(1)) \rightarrow H_{\text{ét}}^1(Y_1(N_g)_{\overline{\mathbb{Q}}}, \mathcal{L}_{r_2}(1)) \\ H_{\text{ét}}^1(\tilde{Y}(1)_{\overline{\mathbb{Q}}}, \mathcal{L}_{r_3}(1)) &\xrightarrow{\mu_p^*} H_{\text{ét}}^1(Y_1(Np)_{\overline{\mathbb{Q}}}, \mathcal{L}_{r_3}(1)) \rightarrow H_{\text{ét}}^1(Y_1(N_h)_{\overline{\mathbb{Q}}}, \mathcal{L}_{r_3}(1)). \end{aligned}$$

Composing these maps with projection to the g -isotypic and h -isotypic quotient, respectively, we obtain

$$\begin{aligned} \pi_g : e'_{\text{ord}} H^1(\tilde{\Gamma}(1), L_{r_2}(1)) \otimes_{\mathbb{Z}_p} \mathcal{O} &\longrightarrow T_g \\ \pi_h : e'_{\text{ord}} H^1(\tilde{\Gamma}(1), L_{r_3}(1)) \otimes_{\mathbb{Z}_p} \mathcal{O} &\longrightarrow T_h. \end{aligned}$$

For the ease of notation, we write

$$H^1(\psi, \bar{\mathbf{f}}, n) = (\mathcal{O}[R_{\bar{\mathbf{f}},n}] \otimes_{\mathcal{O}} \mathcal{O}[[\Gamma_{\mathfrak{p}}]]) \otimes_{\phi_{n,\infty}} H_{\text{ét}}^1(Y(1, N_\psi p^\infty(n^2))_{\overline{\mathbb{Q}}}, \mathcal{O})^{\text{ord}} (\langle \epsilon_N^{-1}; 1 \rangle') (\boldsymbol{\kappa}^{-1/2})$$

and put $H^1(\psi, n) = \mathcal{O}[R_n] \otimes_{\mathcal{O}[R_{\bar{\mathbf{f}},n}]} H^1(\psi, \bar{\mathbf{f}}, n)$. Then we define the class

$$\kappa_{n,\mathbf{f}gh}^{(3)} = (\pi_{\mathbf{f}^*} \otimes \pi_g \otimes \pi_h) \circ (\mathbf{s}_{\mathbf{f}^*} \otimes \mathbf{s}_{r_2^*} \otimes \mathbf{s}_{r_3^*}) \kappa_{n,\mathbf{f}gh}^{(2)} \quad (2.17)$$

lying in the group

$$H^1(\mathbb{Q}, H^1(\psi, n) \hat{\otimes}_{\mathcal{O}} (T_g \otimes_{\mathcal{O}} T_h) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p(-1 - (r_2 + r_3)/2)).$$

Let Γ_{ac} be the Galois group of the anticyclotomic $\mathbb{Z}_p$ -extension of K . We can identify this group with the anti-diagonal in $(1 + p\mathbb{Z}_p) \times (1 + p\mathbb{Z}_p) \cong \mathcal{O}_{K, \mathfrak{p}}^{(1)} \times \mathcal{O}_{K, \bar{\mathfrak{p}}}^{(1)}$ via the geometrically normalized Artin map. Let $\kappa_{\text{ac}} : \Gamma_{\text{ac}} \rightarrow \mathbb{Z}_p^\times$ be the character defined by mapping $((1 + p)^{-1/2}, (1 + p)^{1/2})$ to $1 + p$ and let $\boldsymbol{\kappa}_{\text{ac}} : \Gamma_{\text{ac}} \rightarrow \Lambda^\times$ be the character defined by mapping $((1 + p)^{-1/2}, (1 + p)^{1/2})$ to the group-like element $[1 + p]$. We use the same notation for the corresponding characters of $G_{\mathbb{Q}}$. There is a $G_{\mathbb{Q}}$ -equivariant isomorphism of $\Lambda_{\mathcal{O}}[R_n]$ -modules

$$H^1(\psi, n) \cong \text{Ind}_{K[n]}^{\mathbb{Q}} \Lambda_{\mathcal{O}}(\psi_{\mathfrak{p}}^{-1} \kappa_{\text{ac}}^{r_1/2} \boldsymbol{\kappa}_{\text{ac}}^{-1})(-r_1/2). \quad (2.18)$$

Let

$$T_{g,h}^\psi = T_g \otimes_{\mathcal{O}} T_h(\psi_{\mathfrak{p}}^{-1})(-1 - r), \quad V_{g,h}^\psi = T_{g,h}^\psi \otimes_{\mathbb{Z}_p} \mathbb{Q}_p.$$

In light of the isomorphism (2.18), using Shapiro's lemma the classes $\kappa_{n, \mathbf{f}gh}^{(3)}$ yield classes

$$\tilde{\kappa}_{\psi, g, h, n, \infty} \in H^1(K[n], \Lambda_{\mathcal{O}}(\boldsymbol{\kappa}_{\text{ac}}^{-1}) \hat{\otimes}_{\mathcal{O}} T_{g,h}^\psi(\kappa_{\text{ac}}^{r_1/2})) \otimes_{\mathcal{O}} E \quad (2.19)$$

for every squarefree integer n coprime to p and N .

Proposition 2.6.3. *Let n be as above and let q be a rational prime coprime to p , N and n . Then:*

(i) If q splits in K as $(q) = \mathfrak{q}\bar{\mathfrak{q}}$,

$$\begin{aligned} \text{cor}_{K[nq]/K[n]}(\tilde{\kappa}_{\psi,g,h,nq,\infty}) &= q^{l+m-4} \left\{ \chi_g(q)\chi_h(q)q \left(\frac{\kappa_{\text{ac}}^{-(k-2)/2}\psi_{\mathfrak{P}}(\text{Fr}_{\mathfrak{q}}^{-1})}{q^{k-1}} \text{Fr}_{\mathfrak{q}}^{-1} \right)^2 \right. \\ &\quad - \frac{a_q(g)a_q(h)}{q^{(l+m-4)/2}} \left(\frac{\kappa_{\text{ac}}^{-(k-2)/2}\psi_{\mathfrak{P}}(\text{Fr}_{\mathfrak{q}}^{-1})}{q^{k-1}} \text{Fr}_{\mathfrak{q}}^{-1} \right) \\ &\quad + \frac{\chi_g(q)^{-1}a_q(g)^2}{q^{l-2}} + \frac{\chi_h(q)^{-1}a_q(h)^2}{q^{m-1}} - \frac{q^2+1}{q} \\ &\quad - \frac{a_q(g)a_q(h)}{q^{(l+m-4)/2}} \left(\frac{\kappa_{\text{ac}}^{-(k-2)/2}\psi_{\mathfrak{P}}(\text{Fr}_{\bar{\mathfrak{q}}}^{-1})}{q^{k-1}} \text{Fr}_{\bar{\mathfrak{q}}}^{-1} \right) \\ &\quad \left. + \chi_g(q)\chi_h(q)q \left(\frac{\kappa_{\text{ac}}^{-(k-2)/2}\psi_{\mathfrak{P}}(\text{Fr}_{\bar{\mathfrak{q}}}^{-1})}{q^{k-1}} \text{Fr}_{\bar{\mathfrak{q}}}^{-1} \right)^2 \right\} \tilde{\kappa}_{\psi,g,h,n,\infty}. \end{aligned}$$

(ii) If q is inert in K ,

$$\text{cor}_{K[nq]/K[n]}(\tilde{\kappa}_{\psi,g,h,nq,\infty}) = q^{l+m-4} \left\{ \frac{\chi_g(q)^{-1}a_q(g)^2}{q^{l-2}} + \frac{\chi_h(q)^{-1}a_q(h)^2}{q^{m-1}} - \frac{(q+1)^2}{q} \right\} \tilde{\kappa}_{\psi,g,h,n,\infty}.$$

Proof. The proof of this proposition is similar to the proof of Proposition 2.4.4. We just remark that the maps $\mathbf{s}_{\mathbf{f},n*}$ interchange the degeneracy maps π_1 and π_2 , and under the isomorphism

$$\begin{aligned} H^1(K[n], \Lambda_{\mathcal{O}}(\kappa_{\text{ac}}^{-1}) \hat{\otimes}_{\mathcal{O}} T_{g,h}^{\psi}(\kappa_{\text{ac}}^{(k-2)/2})) \otimes_{\mathcal{O}} E \\ \cong H^1(\mathbb{Q}, H^1(\psi, n) \hat{\otimes}_{\mathcal{O}} (T_g \otimes_{\mathcal{O}} T_h) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p(-1 - (r_2 + r_3)/2)) \end{aligned}$$

arising from (2.18), the corestriction $\text{cor}_{K[nq]/K[n]}$ corresponds, in the case where $(q) = \mathfrak{q}\bar{\mathfrak{q}}$ splits in K , to the map

$$\begin{aligned} \mathcal{N}_n^{nq} &= \pi_{11*} - \chi^{-1}(q)\omega^{(k-2)/2}(q) \left(\frac{\kappa_{\text{ac}}^{-(k-2)/2}\psi_{\mathfrak{P}}(\text{Fr}_{\mathfrak{q}}^{-1})[\mathfrak{q}]}{q^{k/2}} + \frac{\kappa_{\text{ac}}^{-(k-2)/2}\psi_{\mathfrak{P}}(\text{Fr}_{\bar{\mathfrak{q}}}^{-1})[\bar{\mathfrak{q}}]}{q^{k/2}} \right) \pi_{21*} \\ &\quad + \frac{\chi^{-1}(q)\omega^{k-2}(q)}{q} \pi_{22*}, \end{aligned}$$

and similarly in the case where q is inert in K . Since the result can be deduced from Lemma 2.6.1 by virtually the same calculation as in the proof of Lemma 2.4.3, we omit the details. $\square$

Definition 2.6.4. For any E -valued G_K -representation V , put

$$H_{\text{Iw}}^1(K[np^\infty], T) := \varprojlim_r H^1(K[np^r], T), \quad H_{\text{Iw}}^1(K[np^\infty], V) := H_{\text{Iw}}^1(K[np^\infty], T) \otimes_{\mathcal{O}} E,$$

where $T \subset V$ is a Galois stable $\mathcal{O}$ -lattice.

By another application of Shapiro's lemma, the classes $\kappa_{\psi, g, h, n, \infty}$ in (2.19) naturally live in $H_{\text{Iw}}^1(K[np^\infty], V_{g, h}^\psi(\kappa_{\text{ac}}^{(k-2)/2}))$. We thus arrive at the following theorem, which is the main result of this section.

Theorem 2.6.5. *Suppose that:*

- $l \geq m \geq 2$ have the same parity and $k \geq 2$ is even,
- p splits in K ,
- p does not divide the class number of K .

Let $\mathcal{S}$ be the set of squarefree products of primes q which split in K and are coprime to p and N . Assume that $H^1(K[np^s], T_{g, h}^\psi)$ is torsion-free for every $n \in \mathcal{S}$ and for every $s \geq 0$. There exists a collection of classes

$$\left\{ \kappa_{\psi, g, h, n, \infty} \in H_{\text{Iw}}^1(K[np^\infty], T_{g, h}^\psi) : n \in \mathcal{S} \right\}$$

such that whenever $n, nq \in \mathcal{S}$ with q a prime, we have

$$\text{cor}_{K[nq]/K[n]}(\kappa_{\psi, g, h, nq, \infty}) = P_{\mathfrak{q}}(\text{Fr}_{\mathfrak{q}}^{-1}) \kappa_{\psi, g, h, n, \infty},$$

where $\mathfrak{q}$ is any of the primes of K above q , and $P_{\mathfrak{q}}(X) = \det(1 - \text{Fr}_{\mathfrak{q}}^{-1} X | (V_{g, h}^\psi)^\vee(1))$.

Proof. The same argument as in the proof of Theorem 2.4.5 (but using Proposition 2.6.3) yields a system of Iwasawa cohomology classes with the stated norm-compatibilities for the representation $V_{g,h}^\psi(\kappa_{ac}^{(k-2)/2})$. By the twisting result of [Rub00, Thm. 6.3.5], the theorem follows. $\square$

We conclude this section by proving that the classes $\kappa_{\psi,g,h,n,\infty}$ land in the *balanced Selmer group*

$$\mathrm{Sel}_{\mathrm{bal}}(K[np^\infty], T_{g,h}^\psi) := \varprojlim_r \mathrm{Sel}_{\mathrm{bal}}(K[np^r], T_{g,h}^\psi);$$

in the terminology introduced in §2.8.1 below, and as already noted in the discussion surrounding (2.5), this is the same as the Greenberg Selmer group associated to the local conditions defined by

$$\mathcal{F}_{\mathfrak{p}}^+(T_{g,h}^\psi) = (T_g^+ \otimes T_h + T_g \otimes T_h^+)(\psi_{\mathfrak{p}}^{-1})(-1-r), \quad \mathcal{F}_{\bar{\mathfrak{p}}}^+(T_{g,h}^\psi) = (T_g^+ \otimes T_h^+)(\psi_{\bar{\mathfrak{p}}}^{-1})(-1-r), \quad (2.20)$$

at the primes above p .

Proposition 2.6.6. *For all $n \in \mathcal{S}$, we have $\kappa_{\psi,g,h,n,\infty} \in \mathrm{Sel}_{\mathrm{bal}}(K[np^\infty], T_{g,h}^\psi)$.*

Proof. Let $v \nmid p$ be a finite prime of $K[np^\infty]$, and for every $r \geq 0$ denote also by v the prime of $K[np^r]$ below v . As in the proof of Proposition 2.4.8, we have

$$H^1(K[np^r]_v, V_{g,h}^\psi) = H_{\mathrm{Gr}}^1(K[np^r]_v, V_{g,h}^\psi) = 0,$$

and hence

$$H^1(K[np^r]_v, T_{g,h}^\psi) = H^1(K[np^r]_v, T_{g,h}^\psi)_{\mathrm{tors}} = H_{\mathrm{Gr}}^1(K[np^r]_v, T_{g,h}^\psi),$$

where the first equality follows from the local Euler characteristic formula. Hence the inclusion $\mathrm{res}_v(\kappa_{\psi,g,h,n,\infty}) \in \varprojlim_r H_{\mathrm{Gr}}^1(K[np^r]_v, T_{g,h}^\psi)$ follows. Since by [BSV22, Cor. 8.2]

it follows that the classes $\kappa_{\psi,g,h,n,\infty}$ satisfy the balanced local condition at the primes above p , this concludes the proof. $\square$

2.7 Iwasawa main conjectures

In this section we formulate Iwasawa main conjectures for triple products of modular forms. We give two formulations: one in terms of the triple product p -adic L -function (Conjecture 2.7.7) and another in terms of diagonal cycle classes (Conjecture 2.7.9). In Theorem 2.7.15 we establish the equivalence of the two formulations.

2.7.1 Triple product p -adic L -function

Fix a triple $(\mathbf{f}, g, h)$ consisting of a primitive Hida family $\mathbf{f}$ of tame level $N_{\mathbf{f}}$ and character $\chi_{\mathbf{f}}$ and two p -ordinary newforms g, h of weights $l, m \geq 2$, levels N_g, N_h prime-to- p , and nebentypus χ_g, χ_h . Assume that $\mathbf{f}$ has coefficients in a ring $\Lambda_{\mathbf{f}}$ as in §2.5.1. Assume that $\chi_{\mathbf{f}}\chi_g\chi_h = \omega^{r_1}$ for some even integer r_1 and put

$$N = \text{lcm}(N_{\mathbf{f}}, N_g, N_h).$$

Let $\mathbf{g}$ and $\mathbf{h}$ be primitive Hida families with coefficients in $\Lambda_{\mathbf{g}}$ and $\Lambda_{\mathbf{h}}$ passing through g and h , respectively. More precisely, there exist arithmetic points $y_0 \in \mathcal{W}_{\Lambda_{\mathbf{g}}}(\overline{\mathbb{Q}}_p)$ and $z_0 \in \mathcal{W}_{\Lambda_{\mathbf{h}}}(\overline{\mathbb{Q}}_p)$ such that g_{y_0} and h_{z_0} are the ordinary p -stabilizations of g and h , respectively. The rings $\Lambda_{\mathbf{g}}$ and $\Lambda_{\mathbf{h}}$ need not be regular. However, for our purposes, we can consider the Λ -adic families, denoted again $\mathbf{g}$ and $\mathbf{h}$, that result from embedding $\Lambda_{\mathbf{g}}$ and $\Lambda_{\mathbf{h}}$ in the rings of functions of suitable wide open connected subsets $U_{\mathbf{g}}$ and $U_{\mathbf{h}}$ of $\mathcal{W}(\overline{\mathbb{Q}}_p) = \text{Spf}(\Lambda)(\overline{\mathbb{Q}}_p)$ defined over some finite extension E of $\mathbb{Q}_p$ and containing the points y_0 and z_0 , respectively. From now on, it is these rings of functions that we will denote by $\Lambda_{\mathbf{g}}$ and $\Lambda_{\mathbf{h}}$. These rings are now non-canonically isomorphic to $\mathcal{O}[[T]]$, where $\mathcal{O}$ is the ring of integers of E ; in particular,

they are regular. Let $\mathbf{l} - l$ and $\mathbf{m} - m$ be generators in $\Lambda_{\mathbf{g}}$ and $\Lambda_{\mathbf{h}}$ of the prime ideals corresponding to the points y_0 and z_0 , respectively.

We can and will assume that $\Lambda_{\mathbf{f}}$ is a finite flat extension of $\Lambda_{\mathcal{O}}$ and we will only consider arithmetic points in $\mathcal{W}_{\Lambda_{\mathbf{f}}}(\overline{\mathbb{Q}}_p)$ lying in $\text{Hom}_{\text{cont}, \mathcal{O}}(\Lambda_{\mathbf{f}}, \overline{\mathbb{Q}}_p)$

Recall that in §2.5.5 we defined a character $\kappa_{r_1} : \mathbb{Z}_p^\times \rightarrow \Lambda^\times$ given by $u \mapsto \omega^{r_1}(u)[\langle u \rangle]$ and in §2.6 we fixed a square root $\kappa_{r_1}^{1/2}$ of this character given by $u \mapsto \omega^{r_1/2}(u)[\langle u \rangle^{1/2}]$. We let $\kappa_{\mathbf{f}}$ and $\kappa_{\mathbf{f}}^{1/2}$ be the composition of κ_{r_1} and $\kappa_{r_1}^{1/2}$, respectively, with the embedding $\Lambda^\times \hookrightarrow \Lambda_{\mathbf{f}}^\times$. We also define a character $\kappa_{\mathbf{gh}} : \mathbb{Z}_p^\times \rightarrow (\Lambda_{\mathbf{g}} \hat{\otimes}_{\mathcal{O}} \Lambda_{\mathbf{h}})^\times$ by

$$\kappa_{\mathbf{gh}}(u) = \omega(u)^{l+m-4} \langle u \rangle^{l+m-4}$$

and choose a square-root of this character defined by $\kappa_{\mathbf{gh}}^{1/2}(u) = \omega(u)^{(l+m-4)/2} \langle u \rangle^{(l+m-4)/2}$. Let $\Lambda_{\mathbf{fgh}} = \Lambda_{\mathbf{f}} \hat{\otimes}_{\mathcal{O}} \Lambda_{\mathbf{g}} \hat{\otimes}_{\mathcal{O}} \Lambda_{\mathbf{h}}$ and consider the $\Lambda_{\mathbf{fgh}}[G_{\mathbb{Q}}]$ -module

$$\mathbb{V}_{\mathbf{fgh}}^\dagger := \mathbb{V}_{\mathbf{f}} \hat{\otimes}_{\mathcal{O}} \mathbb{V}_{\mathbf{g}} \hat{\otimes}_{\mathcal{O}} \mathbb{V}_{\mathbf{h}}(\Xi_{\mathbf{fgh}}), \quad \text{where } \Xi_{\mathbf{fgh}} = \epsilon_{\text{cyc}}^{-1} \kappa_{\mathbf{f}}^{-1/2} \kappa_{\mathbf{gh}}^{-1/2} \quad (2.21)$$

and $\mathbb{V}_{\mathbf{f}}$, $\mathbb{V}_{\mathbf{g}}$ and $\mathbb{V}_{\mathbf{h}}$ are the big Galois representations attached to $\mathbf{f}$, $\mathbf{g}$ and $\mathbf{h}$, respectively. Then $\mathbb{V}_{\mathbf{fgh}}^\dagger$ is a self-dual twist of the tensor product of these representations. Consider also the $\Lambda_{\mathbf{f}}[G_{\mathbb{Q}}]$ -module

$$\mathbb{V}_{\mathbf{fgh}}^\dagger := \mathbb{V}_{\mathbf{f}} \otimes_{\mathcal{O}} T_g \otimes_{\mathcal{O}} T_h(\Xi_{\mathbf{fgh}}), \quad \text{where } \Xi_{\mathbf{fgh}} = \epsilon_{\text{cyc}}^{(2-l-m)/2} \kappa_{\mathbf{f}}^{-1/2}.$$

Given test vectors $(\check{\mathbf{f}}, \check{g}, \check{h})$ for $(\mathbf{f}, g, h)$ of level N , as explained in [HT01] and [DR14, §4.2], a generalization of Hida's p -adic Rankin–Selberg convolution produces an element $\mathcal{L}_p(\check{\mathbf{f}}, \check{g}, \check{h})$ in the fraction field of $\Lambda_{\mathbf{f}}$ whose specializations to arithmetic points $x \in \mathcal{W}_{\Lambda_{\mathbf{f}}}(\overline{\mathbb{Q}}_p)$ of even weight $k \geq l + m$ recover (a square-root of) the central critical values of the triple product L -function $L(\mathbb{V}_{\mathbf{f}_{xgh}}^\dagger, s)$ for the specialization of $\mathbb{V}_{\mathbf{fgh}}^\dagger$ at x by virtue of Harris–Kudla's proof of Jacquet's conjecture, [HK91]. A recent

result by Hsieh [Hsi21] constructs test vectors $(\check{\underline{\mathbf{f}}}, \check{\underline{g}}, \check{\underline{h}})$ for which a precise interpolation property for the resulting $\mathcal{L}_p(\check{\underline{\mathbf{f}}}, \check{\underline{g}}, \check{\underline{h}})$ is proved. To recall the result in the form that will be used here, for any arithmetic point $x \in \mathcal{W}_{\Lambda_{\mathbf{f}}}(\overline{\mathbb{Q}}_p)$ as above, we set

$$\mathbf{f}_k := \mathbf{f}_x, \quad \alpha_k := a_p(\mathbf{f}_k), \quad \beta_k := \chi_{\mathbf{f}}(p)p^{k-1}\alpha_k^{-1},$$

let α_g, β_g be the roots of the Hecke polynomial of g at p with $\text{ord}_p(\alpha_g) = 0$, and define α_h, β_h similarly. As recalled in [op. cit., §1.4], when the residual Galois representation $\bar{\rho}_{\mathbf{f}}$ associated to $\mathbf{f}$ is absolutely irreducible and p -distinguished, the local ring $\Lambda_{\mathbf{f}}$ is known to be Gorenstein and by a result of Hida's the congruence module of $\mathbf{f}$ is isomorphic to $\Lambda_{\mathbf{f}}/(\xi)$ for some nonzero $\xi \in \Lambda_{\mathbf{f}}$. We call (ξ) the *congruence ideal* of $\mathbf{f}$. Finally, denote by $\varepsilon_{\ell}(\mathbb{V}_{\mathbf{f}_k gh}^{\dagger}) \in \{\pm 1\}$ the epsilon factor of the Weil–Deligne representation attached to the restriction of $\mathbb{V}_{\mathbf{f}_k gh}^{\dagger}$ to $G_{\mathbb{Q}_{\ell}}$.

Theorem 2.7.1. *In addition to $\chi_{\mathbf{f}}\chi_g\chi_h = \omega^{r_1}$, assume that:*

- (a) $\bar{\rho}_{\mathbf{f}}$ is absolutely irreducible and p -distinguished,
- (b) for some arithmetic point $x \in \mathcal{W}_{\Lambda_{\mathbf{f}}}(\overline{\mathbb{Q}}_p)$, we have $\varepsilon_{\ell}(\mathbb{V}_{\mathbf{f}_k gh}^{\dagger}) = +1$ for all primes $\ell \mid N$,
- (c) $\gcd(N_{\mathbf{f}}, N_g, N_h)$ is squarefree.

Let ξ be a generator of the congruence ideal of $\mathbf{f}$. There exist test vectors $(\check{\underline{\mathbf{f}}}, \check{\underline{g}}, \check{\underline{h}})$ for $(\mathbf{f}, g, h)$ of level N , and an element

$$\mathcal{L}_p^{\xi}(\check{\underline{\mathbf{f}}}, \check{\underline{g}}, \check{\underline{h}}) \in \Lambda_{\mathbf{f}}$$

such that for all arithmetic points $x \in \mathcal{W}_{\Lambda_{\mathbf{f}}}(\overline{\mathbb{Q}}_p)$ of even weight $k \geq l + m$ we have

$$\left(\frac{\mathcal{L}_p^{\xi}(\check{\underline{\mathbf{f}}}, \check{\underline{g}}, \check{\underline{h}})(x)}{\xi_x} \right)^2 = \frac{\Gamma(k, l, m)}{2^{\alpha(k, l, m)}} \cdot \frac{\mathcal{E}(\mathbf{f}_k, g, h)^2}{\mathcal{E}_0(\mathbf{f}_k)^2 \cdot \mathcal{E}_1(\mathbf{f}_k)^2} \cdot \prod_{\ell \mid N} \tau_{\ell} \cdot \frac{L(\mathbb{V}_{\mathbf{f}_k gh}^{\dagger}, 0)}{\pi^{2(k-2)} \cdot \|\mathbf{f}_k^{\sharp}\|^2},$$

where:

- $\Gamma(k, l, m) = (c-1)! \cdot (c-m)! \cdot (c-l)! \cdot (c+1-l-m)!$, with $c = (k+l+m-2)/2$,
- $\alpha(k, l, m) \in \Lambda_{\mathbf{f}}$ is a linear form in the variables k, l, m ,
- $\mathcal{E}(\mathbf{f}_k, g, h) = (1 - \frac{\beta_k \alpha_g \alpha_h}{p^c})(1 - \frac{\beta_k \beta_g \alpha_h}{p^c})(1 - \frac{\beta_k \alpha_g \beta_h}{p^c})(1 - \frac{\beta_k \beta_g \beta_h}{p^c})$,
- $\mathcal{E}_0(\mathbf{f}_k) = (1 - \frac{\beta_k}{\alpha_k})$, $\mathcal{E}_1(\mathbf{f}_k) = (1 - \frac{\beta_k}{p\alpha_k})$,
- τ_ℓ is an explicit nonzero rational number independent of k ,
- $\mathbf{f}_k^\sharp$ is the newform associated to the p -stabilized newform $\mathbf{f}_k$,

and $\|\mathbf{f}_k^\sharp\|^2$ is the Petersson norm of $\mathbf{f}_k^\sharp$.

Proof. Letting $\mathbf{g}, \mathbf{h}$ be the primitive Hida families of tame level N_g, N_h passing through the ordinary p -stabilizations of g, h , this follows by specializing the three-variable p -adic L -function in [Hsi21, Thm. A] attached to $(\mathbf{f}, \mathbf{g}, \mathbf{h})$ and the congruence ideal generator ξ . $\square$

Definition 2.7.2. For the test vectors $(\check{\mathbf{f}}, \check{g}, \check{h})$ of level N provided by Theorem 2.7.1, we set

$$L_p(\mathbf{f}, g, h) := \mathcal{L}_p^\xi(\check{\mathbf{f}}, \check{g}, \check{h})^2,$$

where ξ is any fixed generator of the congruence ideal of $\mathbf{f}$.

Note that $L_p(\mathbf{f}, g, h)$ depends on the choice of ξ , but the principal ideal in $\Lambda_{\mathbf{f}}$ it generates is of course independent of that choice.

2.7.2 Reciprocity law for diagonal cycles

Keep the notations in the previous subsection and without loss of generality assume that $l \geq m$ (reordering g and h if necessary).

Assume that the Galois representations attached to $\mathbf{f}$, g and h are all residually irreducible and p -distinguished. Let $\phi \in \{\mathbf{f}, \mathbf{g}, \mathbf{h}\}$. As a $G_{\mathbb{Q}_p}$ -representation, $\mathbb{V}_\phi$ admits a filtration

$$0 \rightarrow \mathbb{V}_\phi^+ \rightarrow \mathbb{V}_\phi \rightarrow \mathbb{V}_\phi^- \rightarrow 0 \quad (2.22)$$

with each $\mathbb{V}_\phi^\pm$ free of rank one over Λ_ϕ , and with the $G_{\mathbb{Q}_p}$ -action on $\mathbb{V}_\phi^-$ given by the unramified character sending $\text{Fr}_p \mapsto a_p(\phi)$. This induces an obvious three-step filtration

$$0 \subset \mathcal{F}^3 \mathbb{V}_{\mathbf{fgh}}^\dagger \subset \mathcal{F}^2 \mathbb{V}_{\mathbf{fgh}}^\dagger \subset \mathcal{F}^1 \mathbb{V}_{\mathbf{fgh}}^\dagger \subset \mathbb{V}_{\mathbf{fgh}}^\dagger$$

by $G_{\mathbb{Q}_p}$ -stable $\Lambda_{\mathbf{fgh}}$ -submodules of ranks 1, 4, and 7, respectively, given by

$$\begin{aligned} \mathcal{F}^1 \mathbb{V}_{\mathbf{fgh}}^\dagger &= (\mathbb{V}_{\mathbf{f}} \hat{\otimes}_{\mathcal{O}} \mathbb{V}_{\mathbf{g}} \hat{\otimes}_{\mathcal{O}} \mathbb{V}_{\mathbf{h}}^+ + \mathbb{V}_{\mathbf{f}} \hat{\otimes}_{\mathcal{O}} \mathbb{V}_{\mathbf{g}}^+ \hat{\otimes}_{\mathcal{O}} \mathbb{V}_{\mathbf{h}} + \mathbb{V}_{\mathbf{f}}^+ \hat{\otimes}_{\mathcal{O}} \mathbb{V}_{\mathbf{g}} \hat{\otimes}_{\mathcal{O}} \mathbb{V}_{\mathbf{h}})(\Xi_{\mathbf{fgh}}), \\ \mathcal{F}^2 \mathbb{V}_{\mathbf{fgh}}^\dagger &= (\mathbb{V}_{\mathbf{f}} \hat{\otimes}_{\mathcal{O}} \mathbb{V}_{\mathbf{g}}^+ \hat{\otimes}_{\mathcal{O}} \mathbb{V}_{\mathbf{h}}^+ + \mathbb{V}_{\mathbf{f}}^+ \hat{\otimes}_{\mathcal{O}} \mathbb{V}_{\mathbf{g}} \hat{\otimes}_{\mathcal{O}} \mathbb{V}_{\mathbf{h}}^+ + \mathbb{V}_{\mathbf{f}}^+ \hat{\otimes}_{\mathcal{O}} \mathbb{V}_{\mathbf{g}}^+ \hat{\otimes}_{\mathcal{O}} \mathbb{V}_{\mathbf{h}})(\Xi_{\mathbf{fgh}}), \\ \mathcal{F}^3 \mathbb{V}_{\mathbf{fgh}}^\dagger &= \mathbb{V}_{\mathbf{f}}^+ \hat{\otimes}_{\mathcal{O}} \mathbb{V}_{\mathbf{g}}^+ \hat{\otimes}_{\mathcal{O}} \mathbb{V}_{\mathbf{h}}^+(\Xi_{\mathbf{fgh}}). \end{aligned} \quad (2.23)$$

The middle term $\mathcal{F}^2 \mathbb{V}_{\mathbf{fgh}}^\dagger$ will play a special role in the following, and we note that

$$\mathcal{F}^2 \mathbb{V}_{\mathbf{fgh}}^\dagger / \mathcal{F}^3 \mathbb{V}_{\mathbf{fgh}}^\dagger \cong \mathbb{V}_{\mathbf{f}}^{\mathbf{gh}} \oplus \mathbb{V}_{\mathbf{g}}^{\mathbf{fh}} \oplus \mathbb{V}_{\mathbf{h}}^{\mathbf{fg}}, \quad (2.24)$$

where $\mathbb{V}_{\mathbf{f}}^{\mathbf{gh}} := (\mathbb{V}_{\mathbf{f}}^- \hat{\otimes}_{\mathcal{O}} \mathbb{V}_{\mathbf{g}}^+ \hat{\otimes}_{\mathcal{O}} \mathbb{V}_{\mathbf{h}}^+)(\Xi_{\mathbf{fgh}})$ and similarly for the other two direct summands. We similarly denote the induced subquotients on the specializations of $\mathbb{V}_{\mathbf{fgh}}^\dagger$ (that is, $\mathcal{F}^i \mathbb{V}_{\mathbf{fgh}}^\dagger$, $\mathbb{V}_{\mathbf{f}}^{gh}$, etc.).

Consider the class $\kappa_{1,\mathbf{fgh}}^{(3)}$ defined in (2.17) for the choice of level- N test vectors $(\check{\mathbf{f}}, \check{g}, \check{h})$ given by Theorem 2.7.1 and let $\kappa(\mathbf{f}, g, h) \in H^1(\mathbb{Q}, \mathbb{V}_{\mathbf{fgh}}^\dagger)$ be the image of this class via the morphism obtained from the augmentation map $\mathcal{O}[R_1] \rightarrow \mathcal{O}$. By [BSV22, Cor. 8.2], the image of $\kappa(\mathbf{f}, g, h)$ under the restriction map at p is contained in

$$H_{\text{bal}}^1(\mathbb{Q}_p, \mathbb{V}_{\mathbf{fgh}}^\dagger) := \text{im}(H^1(\mathbb{Q}_p, \mathcal{F}^2 \mathbb{V}_{\mathbf{fgh}}^\dagger) \rightarrow H^1(\mathbb{Q}_p, \mathbb{V}_{\mathbf{fgh}}^\dagger)).$$

It is easily seen that this map is an injection, so we may and will view $\text{res}_p(\kappa(\mathbf{f}, g, h))$ as a class in $H^1(\mathbb{Q}_p, \mathcal{F}^2 \mathbb{V}_{\mathbf{f}gh}^\dagger)$. Let

$$\text{pr}_{(\mathbf{f}, g, h)} : \mathcal{F}^2 \mathbb{V}_{\mathbf{f}gh}^\dagger \longrightarrow \mathbb{V}_{\mathbf{f}}^{gh}$$

be the map induced by the projection onto the first direct summand in (2.24). The “reciprocity law” from [BSV22, DR22] recalled in Theorem 2.7.4 below relates the image of $\text{res}_p(\kappa(\mathbf{f}, g, h))$ under the natural projection

$$\text{pr}_{(\mathbf{f}, g, h)*} : H_{\text{bal}}^1(\mathbb{Q}_p, \mathbb{V}_{\mathbf{f}gh}^\dagger) \longrightarrow H^1(\mathbb{Q}_p, \mathbb{V}_{\mathbf{f}}^{gh})$$

to the triple product p -adic L -function of §2.7.1. Recall that $\xi \in \Lambda_{\mathbf{f}}$ denotes a generator of the congruence ideal of $\mathbf{f}$.

Proposition 2.7.3. *There is an injective $\Lambda_{\mathbf{f}}$ -module homomorphism with pseudo-null cokernel*

$$\mathfrak{Log}^\xi : H^1(\mathbb{Q}_p, \mathbb{V}_{\mathbf{f}}^{gh}) \longrightarrow \Lambda_{\mathbf{f}}$$

characterized by the following interpolation property: for all $\mathfrak{Z} \in H^1(\mathbb{Q}_p, \mathbb{V}_{\mathbf{f}}^{gh})$ and all classical points $x \in \mathcal{W}_{\Lambda_{\mathbf{f}}}(\overline{\mathbb{Q}}_p)$ of weight $k \geq l + m$ with $k \equiv r + 2 \pmod{2(p-1)}$ we have

$$\begin{aligned} \frac{\mathfrak{Log}^\xi(\mathfrak{Z})(x)}{\xi_x} &= (p-1)\alpha_k \left(1 - \frac{\beta_k \alpha_g \alpha_h}{p^c}\right) \left(1 - \frac{\alpha_k \beta_g \beta_h}{p^c}\right)^{-1} \\ &\quad \times \begin{cases} \frac{(-1)^{c-k}}{(c-k)!} \cdot \langle \log_p(\mathfrak{Z}_k), \eta_{\mathbf{f}_k} \otimes \omega_{\mathbf{g}_l} \otimes \omega_{\mathbf{h}_m} \rangle_{\text{dR}}, & \text{if } l - m < k < l + m, \\ (k - c - 1)! \cdot \langle \exp_p^*(\mathfrak{Z}_k), \eta_{\mathbf{f}_k} \otimes \omega_{\mathbf{g}_l} \otimes \omega_{\mathbf{h}_m} \rangle_{\text{dR}}, & \text{if } k \geq l + m, \end{cases} \end{aligned}$$

where $c = (k + l + m - 2)/2$.

Proof. The construction of $\mathfrak{Log}^\xi$ will follow by specializing the three-variable p -adic regulator constructed in [BSV22, §7.1] (building on a generalization of the construc-

tion in [LZ14] given by Kings–Loeffler–Zerbes [KLZ17]).

Let $\vartheta_{gh} : \Lambda_{\mathbf{fgh}} \rightarrow \Lambda_{\mathbf{f}}$ be the map given by reduction modulo $(\mathbf{l} - l, \mathbf{m} - m)$. This induces isomorphisms

$$\mathbb{V}_{\mathbf{fgh}}^\dagger \otimes_{\Lambda_{\mathbf{fgh}}} \Lambda_{\mathbf{f}} \cong \mathbb{V}_{\mathbf{f}gh}^\dagger, \quad \mathbb{V}_{\mathbf{f}}^{\mathbf{gh}} \otimes_{\Lambda_{\mathbf{fgh}}} \Lambda_{\mathbf{f}} \cong \mathbb{V}_{\mathbf{f}}^{gh},$$

and a natural map

$$\vartheta_{gh*} : H^1(\mathbb{Q}_p, \mathbb{V}_{\mathbf{f}}^{\mathbf{gh}}) \otimes_{\Lambda_{\mathbf{fgh}}} \Lambda_{\mathbf{f}} \longrightarrow H^1(\mathbb{Q}_p, \mathbb{V}_{\mathbf{f}}^{gh}).$$

This map is clearly injective, and its surjectivity can be shown easily by an application of local Tate duality and the Ramanujan–Petersson conjecture (*cf.* proof of [BSV22, (154)]). Letting

$$\mathcal{L}_{\mathbf{f}}^\xi : H^1(\mathbb{Q}_p, \mathbb{V}_{\mathbf{f}}^{\mathbf{gh}}) \longrightarrow \Lambda_{\mathbf{fgh}}$$

be the p -adic regulator $\mathcal{L}_{\mathbf{f}}$ defined as in [BSV22, Prop. 7.3] and multiplied by ξ , the map defined by the composition

$$\mathfrak{Log}^\xi : H^1(\mathbb{Q}_p, \mathbb{V}_{\mathbf{f}}^{gh}) \xrightarrow{\vartheta_{gh*}^{-1}} H^1(\mathbb{Q}_p, \mathbb{V}_{\mathbf{f}}^{\mathbf{gh}}) \otimes_{\Lambda_{\mathbf{fgh}}} \Lambda_{\mathbf{f}} \xrightarrow{\mathcal{L}_{\mathbf{f}}^\xi \otimes \text{id}} \Lambda_{\mathbf{f}}$$

satisfies the interpolation properties in the statement of the proposition.

It remains to see that $\mathfrak{Log}^\xi$ is injective with pseudo-null cokernel. By definition, we have

$$\mathbb{V}_{\mathbf{f}}^{\mathbf{gh}} = \mathbb{U}_{\mathbf{f}}^{\mathbf{gh}}(\epsilon_{\text{cyc}} \kappa_{\mathbf{f}}^{-1/2} \kappa_{\mathbf{gh}}^{1/2}),$$

where $\mathbb{U}_{\mathbf{f}}^{\mathbf{gh}}$ is an unramified $G_{\mathbb{Q}_p}$ -module on which an arithmetic Frobenius Fr_p acts as multiplication by $\chi_f^{-1}(p) a_p(\mathbf{f}) a_p(\mathbf{g})^{-1} a_p(\mathbf{h})^{-1}$, and $\mathcal{L}_{\mathbf{f}}$ is obtained by specializing the four-variable p -adic regulator map in [KLZ17, Thm. 8.2.3] for the module $\mathbb{U}_{\mathbf{f}}^{\mathbf{gh}}$, paired against the differential $\eta_{\mathbf{f}} \otimes \omega_{\mathbf{g}} \otimes \omega_{\mathbf{h}}$. In light of [KLZ17, Rmk. 8.2.4], the fact

that $\mathfrak{Log}^\xi$ has the above properties can therefore be deduced from the vanishing of $H^0(\mathbb{Q}_p, \mathbb{U}_{\mathbf{f}}^{gh})$, where $\mathbb{U}_{\mathbf{f}}^{gh}$ is the image of $\mathbb{U}_{\mathbf{f}}^{\mathbf{gh}}$ under ϑ_{gh} . $\square$

Theorem 2.7.4 (Reciprocity law). *We have the following equality*

$$\mathfrak{Log}^\xi(\text{res}_p(\kappa(\mathbf{f}, g, h))) = \mathcal{L}_p^\xi(\check{\underline{\mathbf{f}}}, \check{\underline{g}}, \check{\underline{h}}).$$

Proof. This is the specialization of the three-variable reciprocity law of Theorem A in [BSV22] to $(\mathbf{f}, g, h)$ (see also [DR22, Thm. 10]). $\square$

2.7.3 Selmer groups and formulation of the main conjectures

Let $(\mathbf{f}, g, h)$ be as in the preceding subsection.

Recall the $G_{\mathbb{Q}_p}$ -stable rank-four $\Lambda_{\mathbf{fgh}}$ -submodule $\mathcal{F}^2 \mathbb{V}_{\mathbf{fgh}}^\dagger \subset \mathbb{V}_{\mathbf{fgh}}^\dagger$ in (2.23), and set

$$\mathbb{V}_{\mathbf{fgh}}^f := \mathbb{V}_{\mathbf{f}}^+ \hat{\otimes}_{\mathcal{O}} \mathbb{V}_{\mathbf{g}} \hat{\otimes}_{\mathcal{O}} \mathbb{V}_{\mathbf{h}}(\Xi_{\mathbf{fgh}}).$$

As before, we let $\mathcal{F}^2 \mathbb{V}_{\mathbf{fgh}}$ and $\mathbb{V}_{\mathbf{fgh}}^f$ denote the corresponding specializations.

Fix a finite set Σ of places of $\mathbb{Q}$ containing ∞ and the primes dividing Np , and let $\mathbb{Q}^\Sigma$ be the maximal extension of $\mathbb{Q}$ unramified outside Σ .

Definition 2.7.5. For $\mathcal{L} \in \{\text{bal}, \mathcal{F}\}$ define the Selmer group $\text{Sel}_{\mathcal{L}}(\mathbb{V}_{\mathbf{fgh}}^\dagger)$ by

$$\text{Sel}_{\mathcal{L}}(\mathbb{V}_{\mathbf{fgh}}^\dagger) = \ker \left(H^1(\mathbb{Q}^\Sigma/\mathbb{Q}, \mathbb{V}_{\mathbf{fgh}}^\dagger) \longrightarrow \prod_{v \in \Sigma \setminus \{p, \infty\}} H^1(\mathbb{Q}_v^{\text{nr}}, \mathbb{V}_{\mathbf{fgh}}^\dagger) \times \frac{H^1(\mathbb{Q}_p, \mathbb{V}_{\mathbf{fgh}}^\dagger)}{H_{\mathcal{L}}^1(\mathbb{Q}_p, \mathbb{V}_{\mathbf{fgh}}^\dagger)} \right),$$

where $\mathbb{Q}_v^{\text{nr}}$ denotes the maximal unramified extension of $\mathbb{Q}_v$ and

$$H_{\mathcal{L}}^1(\mathbb{Q}_p, \mathbb{V}_{\mathbf{fgh}}^\dagger) = \begin{cases} \ker(H^1(\mathbb{Q}_p, \mathbb{V}_{\mathbf{fgh}}^\dagger) \longrightarrow H^1(\mathbb{Q}_p, \mathbb{V}_{\mathbf{fgh}}^\dagger / \mathcal{F}^2 \mathbb{V}_{\mathbf{fgh}}^\dagger)) & \text{if } \mathcal{L} = \text{bal}, \\ \ker(H^1(\mathbb{Q}_p, \mathbb{V}_{\mathbf{fgh}}^\dagger) \longrightarrow H^1(\mathbb{Q}_p, \mathbb{V}_{\mathbf{fgh}}^\dagger / \mathbb{V}_{\mathbf{fgh}}^f)) & \text{if } \mathcal{L} = \mathcal{F}. \end{cases}$$

We call $\text{Sel}_{\text{bal}}(\mathbb{V}_{\mathbf{f}gh}^\dagger)$ (resp. $\text{Sel}_{\mathcal{F}}(\mathbb{V}_{\mathbf{f}gh}^\dagger)$) the *balanced* (resp. *f-unbalanced*) Selmer group.

Remark 2.7.6. The pairs

$$(\mathcal{F}^2 \mathbb{V}_{\mathbf{f}gh}^\dagger, \{k \in \mathbb{Z}_{\geq 2} : l - m < k < l + m\}), \quad (\mathbb{V}_{\mathbf{f}gh}^f, \{k \in \mathbb{Z} : k \geq l + m\})$$

satisfy the *Panchishkin condition* in [Gre94]. Thus $\text{Sel}_{\text{bal}}(\mathbb{V}_{\mathbf{f}gh}^\dagger)$ and $\text{Sel}_{\mathcal{F}}(\mathbb{V}_{\mathbf{f}gh}^\dagger)$ may be viewed as instances of Greenberg’s Selmer groups attached to different ranges of critical specializations of $\mathbb{V}_{\mathbf{f}gh}^\dagger$.

Let

$$\mathbb{A}_{\mathbf{f}gh}^\dagger = \text{Hom}_{\mathbb{Z}_p}(\mathbb{V}_{\mathbf{f}gh}^\dagger, \mu_{p^\infty}).$$

Then for $\mathcal{L} \in \{\text{bal}, \mathcal{F}\}$, we define the Selmer groups $\text{Sel}_{\mathcal{L}}(\mathbb{A}_{\mathbf{f}gh}^\dagger)$ as above, taking $H_{\mathcal{L}}^1(\mathbb{Q}_p, \mathbb{A}_{\mathbf{f}gh}^\dagger)$ to be the orthogonal complement of $H_{\mathcal{L}}^1(\mathbb{Q}_p, \mathbb{V}_{\mathbf{f}gh}^\dagger)$ under the local Tate duality

$$H^1(\mathbb{Q}_p, \mathbb{V}_{\mathbf{f}gh}^\dagger) \times H^1(\mathbb{Q}_p, \mathbb{A}_{\mathbf{f}gh}^\dagger) \longrightarrow \mathbb{Q}_p/\mathbb{Z}_p,$$

and set

$$X_{\mathcal{L}}(\mathbb{A}_{\mathbf{f}gh}^\dagger) := \text{Hom}_{\text{cont}}(\text{Sel}_{\mathcal{L}}(\mathbb{A}_{\mathbf{f}gh}^\dagger), \mathbb{Q}_p/\mathbb{Z}_p).$$

In light of Remark 2.7.6, the next conjecture may be viewed as an instance of the Iwasawa–Greenberg main conjectures [Gre94]. In the two formulations below, we also assume conditions (b) and (c) from Theorem 2.7.1, so that the p -adic L -function $L_p(\mathbf{f}, g, h)$ in (2.22) is defined.

Conjecture 2.7.7 (IMC “with p -adic L -functions”). *The modules $\text{Sel}_{\mathcal{F}}(\mathbb{V}_{\mathbf{f}gh}^\dagger)$ and $X_{\mathcal{F}}(\mathbb{A}_{\mathbf{f}gh}^\dagger)$ are both $\Lambda_{\mathbf{f}}$ -torsion, and*

$$\text{Char}_{\Lambda_{\mathbf{f}}}(X_{\mathcal{F}}(\mathbb{A}_{\mathbf{f}gh}^\dagger)) = (L_p(\mathbf{f}, g, h))$$

in $\Lambda_{\mathbf{f}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$.

Remark 2.7.8. An integral formulation of the equality of ideals in Conjecture 2.7.7 would involve certain Tamagawa factors, accounting for the fact that by definition the classes in the Selmer group $X_{\mathcal{F}}(\mathbb{A}_{\mathbf{f}gh}^{\dagger})$ are trivial at the places $v \in \Sigma \setminus \{p, \infty\}$, rather than just unramified.

Under the local root number hypothesis (b) in Theorem 2.7.1, for all arithmetic point $x \in \mathcal{W}_{\Lambda_{\mathbf{f}}}(\overline{\mathbb{Q}}_p)$ of even weight $k \geq 2$ with $l - m < k < l + m$ the sign in the functional equation for $L(\mathbb{V}_{\mathbf{f}gh}^{\dagger}, s)$ is -1 , so that the central value $L(\mathbb{V}_{\mathbf{f}gh}^{\dagger}, 0)$ vanishes. Therefore, in the spirit of Perrin-Riou’s main conjecture [PR87, Conj. B] in the setting of Heegner points, a natural formulation of the Iwasawa main conjecture for $\text{Sel}_{\text{bal}}(\mathbb{V}_{\mathbf{f}gh}^{\dagger})$ takes the following form.

Note that it follows from [BSV22, Cor. 8.2] that $\kappa(\mathbf{f}, g, h)$ lands in $\text{Sel}_{\text{bal}}(\mathbb{V}_{\mathbf{f}gh}^{\dagger})$.

Conjecture 2.7.9 (IMC “without p -adic L -functions”). *Suppose $\kappa(\mathbf{f}, g, h) \in \text{Sel}_{\text{bal}}(\mathbb{Q}, \mathbb{V}_{\mathbf{f}gh}^{\dagger})$ is not $\Lambda_{\mathbf{f}}$ -torsion. Then the modules $\text{Sel}_{\text{bal}}(\mathbb{V}_{\mathbf{f}gh}^{\dagger})$ and $X_{\text{bal}}(\mathbb{A}_{\mathbf{f}gh}^{\dagger})$ have both rank one, and*

$$\text{Char}_{\Lambda_{\mathbf{f}}}(X_{\text{bal}}(\mathbb{A}_{\mathbf{f}gh}^{\dagger})_{\text{tors}}) = \text{Char}_{\Lambda_{\mathbf{f}}} \left(\frac{\text{Sel}_{\text{bal}}(\mathbb{V}_{\mathbf{f}gh}^{\dagger})}{\Lambda_{\mathbf{f}} \cdot \kappa(\mathbf{f}, g, h)} \right)^2$$

in $\Lambda_{\mathbf{f}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$, where the subscript tors denotes the $\Lambda_{\mathbf{f}}$ -torsion submodule.

Remark 2.7.10. Working under different hypotheses on the local signs ensuring that $L(\mathbb{V}_{\mathbf{f}gh}^{\dagger}, s)$ has sign $+1$ (rather than -1) for weights $k \geq 2$ with $l - m < k < l + m$, the Iwasawa main conjecture would relate the characteristic ideal of $X_{\text{bal}}(\mathbb{A}_{\mathbf{f}gh}^{\dagger})$ to the balanced triple product p -adic L -function constructed in [Hsi21, Thm. B] (see also [GS20]), rather than diagonal classes. In this setting, the f -unbalanced Selmer group $\text{Sel}_{\mathcal{F}}(\mathbb{V}_{\mathbf{f}gh}^{\dagger})$ should have $\Lambda_{\mathbf{f}}$ -rank one, but the expected non-torsion Selmer class seems to not have been constructed yet.

2.7.4 Equivalence of the formulations

In this subsection we show that the two formulations of the Iwasawa main conjecture in the previous subsection are essentially equivalent, focusing on the case where $\mathbf{f}$ is a CM Hida family¹ as in §2.6. Similar equivalences between IMC “with” and “without” p -adic L -functions appear in [Kat04, §17], [KLZ17, §11] and, in a setting more germane to ours, [Wan21] and [Cas17, Appendix].

The following intermediate Selmer groups will allow us to bridge between $\mathrm{Sel}_{\mathrm{bal}}(\mathbb{V}_{\mathbf{f}gh}^\dagger)$ and $\mathrm{Sel}_{\mathcal{F}}(\mathbb{V}_{\mathbf{f}gh}^\dagger)$ in the comparison. Set

$$\mathbb{V}_{\mathbf{f}gh}^{f\cap+} = \mathbb{V}_{\mathbf{f}gh}^f \cap \mathcal{F}^2 \mathbb{V}_{\mathbf{f}gh}^\dagger, \quad \mathbb{V}_{\mathbf{f}gh}^{f\cup+} = \mathbb{V}_{\mathbf{f}gh}^f + \mathcal{F}^2 \mathbb{V}_{\mathbf{f}gh}^\dagger,$$

which are $G_{\mathbb{Q}_p}$ -stable $\Lambda_{\mathbf{f}}$ -submodules of $\mathbb{V}_{\mathbf{f}gh}^\dagger$ of ranks 3 and 5, respectively. Define $\mathrm{Sel}_{\mathcal{L}}(\mathbb{V}_{\mathbf{f}gh}^\dagger)$ for $\mathcal{L} \in \{\mathcal{F} \cap +, \mathcal{F} \cup +\}$ by the same recipe as in Definition 2.7.5, with

$$H_{\mathcal{L}}^1(\mathbb{Q}_p, \mathbb{V}_{\mathbf{f}gh}^\dagger) = \begin{cases} \ker(H^1(\mathbb{Q}_p, \mathbb{V}_{\mathbf{f}gh}^\dagger) \rightarrow H^1(\mathbb{Q}_p, \mathbb{V}_{\mathbf{f}gh}^\dagger / \mathbb{V}_{\mathbf{f}gh}^{f\cap+})) & \text{if } \mathcal{L} = \mathcal{F} \cap +, \\ \ker(H^1(\mathbb{Q}_p, \mathbb{V}_{\mathbf{f}gh}^\dagger) \rightarrow H^1(\mathbb{Q}_p, \mathbb{V}_{\mathbf{f}gh}^\dagger / \mathbb{V}_{\mathbf{f}gh}^{f\cup+})) & \text{if } \mathcal{L} = \mathcal{F} \cup +. \end{cases}$$

We define the Selmer groups $\mathrm{Sel}_{\mathcal{F}\cap+}(\mathbb{A}_{\mathbf{f}gh}^\dagger)$ and $\mathrm{Sel}_{\mathcal{F}\cup+}(\mathbb{A}_{\mathbf{f}gh}^\dagger)$ taking $H_{\mathcal{F}\cap+}^1(\mathbb{Q}_p, \mathbb{A}_{\mathbf{f}gh}^\dagger)$ and $H_{\mathcal{F}\cup+}^1(\mathbb{Q}_p, \mathbb{A}_{\mathbf{f}gh}^\dagger)$ to be the orthogonal complements of $H_{\mathcal{F}\cup+}^1(\mathbb{Q}_p, \mathbb{V}_{\mathbf{f}gh}^\dagger)$ and $H_{\mathcal{F}\cap+}^1(\mathbb{Q}_p, \mathbb{V}_{\mathbf{f}gh}^\dagger)$, respectively. As in the preceding section, we also define the corresponding $X_{\mathcal{F}\cap+}(\mathbb{A}_{\mathbf{f}gh}^\dagger)$ and $X_{\mathcal{F}\cup+}(\mathbb{A}_{\mathbf{f}gh}^\dagger)$.

Throughout this subsection, we keep the setting from §2.6. In particular, $\mathbf{f} \in \Lambda_{\mathbf{f}}[[q]]$ is the CM Hida family in (2.16) associated with the Hecke character ψ of conductor $\mathfrak{f}$. In addition, we assume conditions (b) and (c) from Theorem 2.7.1, so the p -adic L -function $L_p(\mathbf{f}, g, h) \in \Lambda_{\mathbf{f}}$ is defined, and let $\kappa(\mathbf{f}, g, h) \in H^1(\mathbb{Q}, \mathbb{V}_{\mathbf{f}gh}^\dagger)$ be as above.

¹This case will suffice for our applications in this paper, and makes some of the arguments simpler, but we expect the equivalence to hold in general.

For every height one prime $\mathfrak{Q}$ of $\Lambda_{\mathbf{f}}$ away from p , let $S_{\mathfrak{Q}}$ be the integral closure of $\Lambda_{\mathbf{f}}/\mathfrak{Q}$ and let $\Phi_{\mathfrak{Q}}$ be the fraction field of $S_{\mathfrak{Q}}$. Let $\mathbb{V}_{\mathbf{f}gh,\mathfrak{Q}}^{\dagger}$ be the extension of scalars of $\mathbb{V}_{\mathbf{f}gh}^{\dagger}/\mathfrak{Q}\mathbb{V}_{\mathbf{f}gh}^{\dagger}$ to $S_{\mathfrak{Q}}$, and let $\mathbb{A}_{\mathbf{f}gh,\mathfrak{Q}}^{\dagger} = \text{Hom}(\mathbb{V}_{\mathbf{f}gh,\mathfrak{Q}}^{\dagger}, \mu_{p^\infty})$. Following [MR04], define

$$H_{\text{bal}}^1(\mathbb{Q}_v, \mathbb{V}_{\mathbf{f}gh,\mathfrak{Q}}^{\dagger}) := \begin{cases} \ker(H^1(\mathbb{Q}_v, \mathbb{V}_{\mathbf{f}gh,\mathfrak{Q}}^{\dagger}) \rightarrow H^1(\mathbb{Q}_v^{\text{nr}}, \mathbb{V}_{\mathbf{f}gh,\mathfrak{Q}}^{\dagger} \otimes \Phi_{\mathfrak{Q}})), & \text{if } v \nmid p, \\ \ker(H^1(\mathbb{Q}_v, \mathbb{V}_{\mathbf{f}gh,\mathfrak{Q}}^{\dagger}) \rightarrow H^1(\mathbb{Q}_v, (\mathbb{V}_{\mathbf{f}gh,\mathfrak{Q}}^{\dagger}/\mathcal{F}^2\mathbb{V}_{\mathbf{f}gh,\mathfrak{Q}}^{\dagger}) \otimes \Phi_{\mathfrak{Q}})), & \text{if } v \mid p, \end{cases} \quad (2.25)$$

and let $H_{\text{bal}}^1(\mathbb{Q}, \mathbb{V}_{\mathbf{f}gh,\mathfrak{Q}}^{\dagger})$ be the associated Selmer group. Taking $H_{\text{bal}}^1(\mathbb{Q}_v, \mathbb{A}_{\mathbf{f}gh,\mathfrak{Q}}^{\dagger})$ to be the orthogonal complement of $H_{\text{bal}}^1(\mathbb{Q}_v, \mathbb{V}_{\mathbf{f}gh,\mathfrak{Q}}^{\dagger})$ under local Tate duality, we define the Selmer group $H_{\text{bal}}^1(\mathbb{Q}, \mathbb{A}_{\mathbf{f}gh,\mathfrak{Q}}^{\dagger})$ similarly.

Define $\mathbb{V}_{g,h}^{\psi} = \Lambda_{\mathcal{O}}(\kappa_{ac}^{-1}) \hat{\otimes}_{\mathcal{O}} T_{g,h}^{\psi}(\kappa_{ac}^{r_1/2})$ and let $\mathbb{A}_{g,h}^{\psi} = \text{Hom}((\mathbb{V}_{g,h}^{\psi})^c, \mu_{p^\infty})$, where $(\mathbb{V}_{g,h}^{\psi})^c$ denotes $\mathbb{V}_{g,h}^{\psi}$ with the G_K -action twisted by complex conjugation. Note that $\mathbb{V}_{\mathbf{f}gh}^{\dagger} = \text{Ind}_K^{\mathbb{Q}} \mathbb{V}_{g,h}^{\psi}$, so we can define Selmer conditions for $\mathbb{V}_{g,h}^{\psi}$ using Shapiro's lemma and for $\mathbb{A}_{g,h}^{\psi}$ by duality. Define $\mathbb{A}_{g,h,\mathfrak{Q}}^{\psi} = \text{Hom}((\mathbb{V}_{g,h,\mathfrak{Q}}^{\psi})^c, \mu_{p^\infty})$. We have natural maps

$$\mathbb{V}_{g,h}^{\psi}/\mathfrak{Q}\mathbb{V}_{g,h}^{\psi} \rightarrow \mathbb{V}_{g,h,\mathfrak{Q}}^{\psi}, \quad \mathbb{A}_{g,h,\mathfrak{Q}}^{\psi} \rightarrow \mathbb{A}_{g,h}^{\psi}[\mathfrak{Q}] \quad (2.26)$$

preserving both the G_K and the Λ -modules structure in the same way as in [How04, p. 1461]. Note that in the quotient $\mathbb{V}_{g,h}^{\psi}/\mathfrak{Q}\mathbb{V}_{g,h}^{\psi}$ and in the submodule $\mathbb{A}_{g,h}^{\psi}[\mathfrak{Q}]$ we can define Selmer conditions by propagating the balanced conditions for $\mathbb{V}_{g,h}^{\psi}$ and $\mathbb{A}_{g,h}^{\psi}$, respectively, and we denote these conditions in the same way.

Lemma 2.7.11. *For every height one prime $\mathfrak{Q} \subset \Lambda_{\mathbf{f}}$ as above and every place v of K , the maps (2.26) induce natural maps*

$$\begin{aligned} H_{\text{bal}}^1(K_v, \mathbb{V}_{gh}^{\psi}/\mathfrak{Q}\mathbb{V}_{gh}^{\psi}) &\longrightarrow H_{\text{bal}}^1(K_v, \mathbb{V}_{gh,\mathfrak{Q}}^{\psi}), \\ H_{\text{bal}}^1(K_v, \mathbb{A}_{gh,\mathfrak{Q}}^{\psi}) &\longrightarrow H_{\text{bal}}^1(K_v, \mathbb{A}_{gh}^{\psi}[\mathfrak{Q}]) \end{aligned}$$

with finite kernel and cokernel, of order bounded by constants depending only on $[S_\Omega : \Lambda_{\mathbf{f}}/\Omega]$.

Proof. For the primes $v \nmid p$, the same argument as in the proof of [MR04, Lem. 5.3.13] applies, so it remains to consider the case $v \mid p$. Put

$$\begin{aligned}\mathcal{F}_{\mathfrak{p}}^+(T_{g,h}^\psi) &= (T_g^+ \otimes T_h + T_g \otimes T_h^+)(\psi_{\mathfrak{p}}^{-1})(-1-r), \\ \mathcal{F}_{\bar{\mathfrak{p}}}^+(T_{g,h}^\psi) &= (T_g^+ \otimes T_h^+)(\psi_{\mathfrak{p}}^{-1})(-1-r).\end{aligned}$$

Under the isomorphism $H^1(\mathbb{Q}, \mathbb{V}_{\mathbf{f}gh}^\dagger) \cong H^1(K, \Lambda_{\mathcal{O}}(\boldsymbol{\kappa}_{\text{ac}}^{-1}) \hat{\otimes}_{\mathcal{O}} T_{g,h}^\psi(\kappa_{\text{ac}}^{r_1/2}))$ coming from Shapiro's lemma, the balanced local condition $H_{\text{bal}}^1(\mathbb{Q}_p, \mathbb{V}_{\mathbf{f}gh}^\dagger)$ corresponds to

$$H^1(K_{\mathfrak{p}}, \Lambda_{\mathcal{O}}(\boldsymbol{\kappa}_{\text{ac}}^{-1}) \hat{\otimes}_{\mathcal{O}} \mathcal{F}_{\mathfrak{p}}^+(T_{g,h}^\psi)(\kappa_{\text{ac}}^{r_1/2})) \oplus H^1(K_{\bar{\mathfrak{p}}}, \Lambda_{\mathcal{O}}(\boldsymbol{\kappa}_{\text{ac}}^{-1}) \hat{\otimes}_{\mathcal{O}} \mathcal{F}_{\bar{\mathfrak{p}}}^+(T_{g,h}^\psi)(\kappa_{\text{ac}}^{r_1/2})).$$

Let $A_g^- = T_g^- \otimes \mathbb{Q}_p/\mathbb{Z}_p$, and define A_g^+ , A_h^- and A_h^+ similarly. Arguing as in the proof of [How04, Lem. 2.2.7], we reduce to showing that the groups

$$H^0(K_{\infty, \mathfrak{p}}, (A_g^- \otimes A_h^-)(\psi_{\mathfrak{p}}^{-1} \kappa_{\text{ac}}^{r_1/2})(-1-r)), \quad H^0(K_{\infty, \bar{\mathfrak{p}}}, (A_g^+ \otimes A_h^-)(\psi_{\mathfrak{p}}^{-1} \kappa_{\text{ac}}^{r_1/2})(-1-r))$$

are both finite, which follows from the fact that $\alpha_g \alpha_h \psi(\mathfrak{p})/p^{k-1} \neq 1$ and $\beta_g \alpha_h \psi(\bar{\mathfrak{p}}) \neq 1$, and this is a consequence of the Ramanujan–Petersson conjecture since we are assuming that $p \nmid N$. Note that the other pieces in the quotient decomposition can be treated similarly. This yields the required bounds on the kernel and cokernel of the first map in the statement of the lemma, and the result for the second map follows as well by local duality. $\square$

Let Σ_Λ be the set of height one primes of $\Lambda_{\mathbf{f}}$ consisting of p and those for which either $H^2(\mathbb{Q}^\Sigma/\mathbb{Q}, \mathbb{V}_{\mathbf{f}gh}^\dagger)[\Omega]$ is infinite or $H^2(\mathbb{Q}_p, \mathbb{V}_{\mathbf{f}gh}^\dagger)[\Omega]$ is infinite. Since $H^2(\mathbb{Q}^\Sigma/\mathbb{Q}, \mathbb{V}_{\mathbf{f}gh}^\dagger)$ and $H^2(\mathbb{Q}_p, \mathbb{V}_{\mathbf{f}gh}^\dagger)$ are both finitely generated Λ -modules, the set Σ_Λ is finite.

Proposition 2.7.12. *For every height one prime $\mathfrak{Q} \notin \Sigma_\Lambda$, the maps (2.26) induce natural maps*

$$\begin{aligned} \mathrm{Sel}_{\mathrm{bal}}(\mathbb{V}_{\mathbf{f}gh}^\dagger)/\mathfrak{Q} \mathrm{Sel}_{\mathrm{bal}}(\mathbb{V}_{\mathbf{f}gh}^\dagger) &\longrightarrow \mathrm{Sel}_{\underline{\mathrm{bal}}}(\mathbb{V}_{\mathbf{f}gh,\mathfrak{Q}}^\dagger), \\ \mathrm{Sel}_{\underline{\mathrm{bal}}}(\mathbb{A}_{\mathbf{f}gh,\mathfrak{Q}}^\dagger) &\longrightarrow \mathrm{Sel}_{\mathrm{bal}}(\mathbb{A}_{\mathbf{f}gh}^\dagger)[\mathfrak{Q}] \end{aligned}$$

with finite kernel and cokernel bounded by a constant depending only on $[S_\mathfrak{Q} : \Lambda_{\mathbf{f}}/\mathfrak{Q}]$.

Proof. This follows from Lemma 2.7.11 as in the proof of [MR04, Prop. 5.3.14] (see also [How04, Lem. 2.2.8]). $\square$

For every height one prime $\mathfrak{Q} \subset \Lambda_{\mathbf{f}}$ as above, let $\mathfrak{m}_\mathfrak{Q} = (\pi_\mathfrak{Q})$ be the maximal ideal of $S_\mathfrak{Q}$.

Lemma 2.7.13. *Assume that there is a perfect pairing $T_{g,h}^\psi \times T_{g,h}^\psi \rightarrow \mathcal{O}(1)$ such that $\langle x^\sigma, y^{c\sigma c} \rangle = \langle x, y \rangle^\sigma$ for all $x, y \in T_{g,h}^\psi$ and for all $\sigma \in G_K$, where c stands for complex conjugation. The following hold:*

1. $\mathrm{rank}_{\Lambda_{\mathbf{f}}} \mathrm{Sel}_{\mathrm{bal}}(\mathbb{V}_{\mathbf{f}gh}^\dagger) = \mathrm{rank}_{\Lambda_{\mathbf{f}}} X_{\mathrm{bal}}(\mathbb{A}_{\mathbf{f}gh}^\dagger).$
2. $\mathrm{rank}_{\Lambda_{\mathbf{f}}} \mathrm{Sel}_{\mathcal{F}}(\mathbb{V}_{\mathbf{f}gh}^\dagger) = \mathrm{rank}_{\Lambda_{\mathbf{f}}} X_{\mathcal{F}}(\mathbb{A}_{\mathbf{f}gh}^\dagger).$
3. $\mathrm{rank}_{\Lambda_{\mathbf{f}}} X_{\mathcal{F} \cup +}(\mathbb{A}_{\mathbf{f}gh}^\dagger) = 1 + \mathrm{rank}_{\Lambda_{\mathbf{f}}} X_{\mathcal{F} \cap +}(\mathbb{A}_{\mathbf{f}gh}^\dagger),$ and

$$\mathrm{Char}_{\Lambda_{\mathbf{f}}}(X_{\mathcal{F} \cup +}(\mathbb{A}_{\mathbf{f}gh}^\dagger)_{\mathrm{tors}}) = \mathrm{Char}_{\Lambda_{\mathbf{f}}}(X_{\mathcal{F} \cap +}(\mathbb{A}_{\mathbf{f}gh}^\dagger)_{\mathrm{tors}}),$$

in $\Lambda_{\mathbf{f}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$.

Proof. For part (1), it suffices to show that for all height one primes $\mathfrak{Q} \subset \Lambda_{\mathbf{f}}$ with $\mathfrak{Q} \notin \Sigma_\Lambda$, the modules $\mathrm{Sel}_{\mathrm{bal}}(\mathbb{V}_{\mathbf{f}gh}^\dagger)/\mathfrak{Q} \mathrm{Sel}_{\mathrm{bal}}(\mathbb{V}_{\mathbf{f}gh}^\dagger)$ and $\mathrm{Sel}_{\mathrm{bal}}(\mathbb{A}_{\mathbf{f}gh}^\dagger)[\mathfrak{Q}]$ have the same rank over $\Lambda_{\mathbf{f}}/\mathfrak{Q}$. Since $\mathrm{Sel}_{\underline{\mathrm{bal}}}(\mathbb{V}_{\mathbf{f}gh,\mathfrak{Q}}^\dagger)$ is the $\pi_\mathfrak{Q}$ -adic Tate module of $\mathrm{Sel}_{\underline{\mathrm{bal}}}(\mathbb{A}_{\mathbf{f}gh,\mathfrak{Q}}^\dagger)$ (indeed, this is a consequence of [How04, Lem. 1.3.3] since $\mathbb{A}_{\mathbf{f}gh,\mathfrak{Q}}^\dagger \cong \mathbb{V}_{\mathbf{f}gh,\mathfrak{Q}}^\dagger \otimes \mathbb{Q}_p/\mathbb{Z}_p$), the result thus follows from Proposition 2.7.12.

For part (2), under the isomorphism $H^1(\mathbb{Q}, \mathbb{V}_{\mathbf{f}gh}^\dagger) \cong H^1(K, \Lambda_{\mathcal{O}}(\boldsymbol{\kappa}_{ac}^{-1}) \hat{\otimes}_{\mathcal{O}} T_{g,h}^\psi(\kappa_{ac}^{r_1/2}))$ the f -unbalanced local condition $H_{\mathcal{F}}^1(\mathbb{Q}_p, \mathbb{V}_{\mathbf{f}gh}^\dagger)$ corresponds to

$$H^1(K_{\mathfrak{p}}, \Lambda_{\mathcal{O}}(\boldsymbol{\kappa}_{ac}^{-1}) \hat{\otimes}_{\mathcal{O}} T_{g,h}^\psi) \oplus \{0\}$$

and hence an analogue of Lemma 2.7.11 for the f -unbalanced Selmer groups follows from the finiteness of $H^0(K_{\infty, \bar{\mathfrak{p}}}, A_g \otimes A_h(\psi_{\mathfrak{p}}^{-1} \kappa_{ac}^{r_1/2})(-1-r))$. By the same reason as above, this yields the equality of ranks in part (2).

Finally, for the proof of part (3) we can argue similarly as in [AH06, Thm. 1.2.2]. Keeping with the above notations, let $\text{Sel}_{\underline{\mathcal{F} \cup +}}(\mathbb{A}_{\mathbf{f}gh, \Omega}^\dagger)$ and $\text{Sel}_{\underline{\mathcal{F} \cap +}}(\mathbb{A}_{\mathbf{f}gh, \Omega}^\dagger)$ be the Selmer groups defined by the obvious analogues of (2.25), so from another application of the argument in Lemma 2.7.11 we obtain natural maps

$$\begin{aligned} \text{Sel}_{\underline{\mathcal{F} \cup +}}(\mathbb{A}_{\mathbf{f}gh, \Omega}^\dagger) &\longrightarrow \text{Sel}_{\mathcal{F} \cup +}(\mathbb{A}_{\mathbf{f}gh}^\dagger)[\Omega], \\ \text{Sel}_{\underline{\mathcal{F} \cap +}}(\mathbb{A}_{\mathbf{f}gh, \Omega}^\dagger) &\longrightarrow \text{Sel}_{\mathcal{F} \cap +}(\mathbb{A}_{\mathbf{f}gh}^\dagger)[\Omega] \end{aligned}$$

with finite kernel and cokernel bounded by a constant depending only on $[S_\Omega : \Lambda_{\mathbf{f}}/\Omega]$. Since the local condition $\mathcal{F} \cap +$ is the orthogonal complement of $\mathcal{F} \cup +$ under the local Tate pairing at p induced by the self-duality of $\mathbb{V}_{\mathbf{f}gh}^\dagger$, from [MR04, Thm. 4.1.13] we obtain

$$\text{Sel}_{\underline{\mathcal{F} \cup +}}(\mathbb{A}_{\mathbf{f}gh, \Omega}^\dagger)[\pi_\Omega^i] \cong (\Phi_\Omega/S_\Omega)^r[\pi_\Omega^i] \oplus \text{Sel}_{\underline{\mathcal{F} \cap +}}(\mathbb{A}_{\mathbf{f}gh, \Omega}^\dagger)[\pi_\Omega^i] \quad (2.27)$$

for all i , where r is given (by the Greenberg–Wiles formula in [MR04, Prop. 2.3.5]) by

$$\text{corank}_{S_\Omega} H^1(\mathbb{Q}_p, \mathbb{A}_{\mathbf{f}gh, \Omega}^{f \cup +}) - \text{corank}_{S_\Omega} H^0(\mathbb{R}, \mathbb{A}_{\mathbf{f}gh, \Omega}^\dagger),$$

so $r = 5 - 4 = 1$. The proof of part (3) now follows from (2.27) as in [AH06, Lem. 1.2.6]. $\square$

Remark 2.7.14. The existence of the pairing in the previous lemma is not too restrictive. In particular, this holds automatically if g and h are non-Eisenstein.

We are now ready to establish that both formulations of the Iwasawa main conjecture are equivalent.

Theorem 2.7.15. *Keep the assumptions of the previous lemma and suppose $\kappa(\mathbf{f}, g, h)$ is not $\Lambda_{\mathbf{f}}$ -torsion. Then the following are equivalent:*

$$(1) \text{ rank}_{\Lambda_{\mathbf{f}}} \text{Sel}_{\text{bal}}(\mathbb{V}_{\mathbf{f}gh}^{\dagger}) = \text{rank}_{\Lambda_{\mathbf{f}}} X_{\text{bal}}(\mathbb{A}_{\mathbf{f}gh}^{\dagger}) = 1;$$

$$(2) \text{ rank}_{\Lambda_{\mathbf{f}}} \text{Sel}_{\mathcal{F}}(\mathbb{V}_{\mathbf{f}gh}^{\dagger}) = \text{rank}_{\Lambda_{\mathbf{f}}} X_{\mathcal{F}}(\mathbb{A}_{\mathbf{f}gh}^{\dagger}) = 0,$$

and, in that case, we have $\text{Sel}_{\text{bal}}(\mathbb{V}_{\mathbf{f}gh}^{\dagger}) = \text{Sel}_{\mathcal{F} \cup +}(\mathbb{V}_{\mathbf{f}gh}^{\dagger})$ and

$$\text{Char}_{\Lambda_{\mathbf{f}}}(X_{\mathcal{F}}(\mathbb{A}_{\mathbf{f}gh}^{\dagger})) \cdot \text{Char}_{\Lambda_{\mathbf{f}}} \left(\frac{\text{Sel}_{\text{bal}}(\mathbb{V}_{\mathbf{f}gh}^{\dagger})}{\Lambda_{\mathbf{f}} \cdot \kappa(\mathbf{f}, g, h)} \right)^2 = \text{Char}_{\Lambda_{\mathbf{f}}}(X_{\text{bal}}(\mathbb{A}_{\mathbf{f}gh}^{\dagger})_{\text{tors}}) \cdot (L_p(\mathbf{f}, g, h))$$

in $\Lambda_{\mathbf{f}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. In particular, Conjecture 2.7.7 and Conjecture 2.7.9 are equivalent.

Proof. The Poitou–Tate global duality gives rise to the exact sequence

$$\begin{aligned} 0 \longrightarrow \text{Sel}_{\mathcal{F} \cap +}(\mathbb{V}_{\mathbf{f}gh}^{\dagger}) &\longrightarrow \text{Sel}_{\text{bal}}(\mathbb{V}_{\mathbf{f}gh}^{\dagger}) \xrightarrow{\text{res}_p} H^1(\mathbb{Q}_p, \mathbb{V}_{\mathbf{f}}^{gh}) \\ &\longrightarrow X_{\mathcal{F} \cup +}(\mathbb{A}_{\mathbf{f}gh}^{\dagger}) \longrightarrow X_{\text{bal}}(\mathbb{A}_{\mathbf{f}gh}^{\dagger}) \longrightarrow 0. \end{aligned} \quad (2.28)$$

Assume that $\text{Sel}_{\text{bal}}(\mathbb{V}_{\mathbf{f}gh}^{\dagger})$ and $X_{\text{bal}}(\mathbb{A}_{\mathbf{f}gh}^{\dagger})$ have both $\Lambda_{\mathbf{f}}$ -rank one. Since $H^1(\mathbb{Q}_p, \mathbb{V}_{\mathbf{f}}^{gh})$ has $\Lambda_{\mathbf{f}}$ -rank one, from (2.28) and Theorem 2.7.4 we see that $\text{Sel}_{\mathcal{F} \cap +}(\mathbb{V}_{\mathbf{f}gh}^{\dagger})$ is $\Lambda_{\mathbf{f}}$ -torsion and $X_{\mathcal{F} \cup +}(\mathbb{A}_{\mathbf{f}gh}^{\dagger})$ has $\Lambda_{\mathbf{f}}$ -rank one. By Lemma 2.7.13(3), it follows that $X_{\mathcal{F} \cap +}(\mathbb{A}_{\mathbf{f}gh}^{\dagger})$ is $\Lambda_{\mathbf{f}}$ -torsion, and from the exact sequence

$$\begin{aligned} 0 \longrightarrow \text{Sel}_{\mathcal{F}}(\mathbb{V}_{\mathbf{f}gh}^{\dagger}) &\longrightarrow \text{Sel}_{\mathcal{F} \cup +}(\mathbb{V}_{\mathbf{f}gh}^{\dagger}) \xrightarrow{\text{res}_p} H^1(\mathbb{Q}_p, \mathbb{V}_{\mathbf{f}}^{gh}) \\ &\longrightarrow X_{\mathcal{F}}(\mathbb{A}_{\mathbf{f}gh}^{\dagger}) \longrightarrow X_{\mathcal{F} \cap +}(\mathbb{A}_{\mathbf{f}gh}^{\dagger}) \longrightarrow 0 \end{aligned} \quad (2.29)$$

we get that $X_{\mathcal{F}}(\mathbb{A}_{\mathbf{f}gh}^\dagger)$ and $\text{Sel}_{\mathcal{F}}(\mathbb{V}_{\mathbf{f}gh}^\dagger)$ are both $\Lambda_{\mathbf{f}}$ -torsion by Lemma 2.7.13(2). This proves the implication (1) $\Rightarrow$ (2) in the statement of the theorem, and the converse is shown similarly. Moreover, from (2.29) we see that $\text{rank}_{\Lambda_{\mathbf{f}}} \text{Sel}_{\mathcal{F} \cup +}(\mathbb{V}_{\mathbf{f}gh}^\dagger) = 1$, and hence the quotient $\text{Sel}_{\mathcal{F} \cup +}(\mathbb{V}_{\mathbf{f}gh}^\dagger) / \text{Sel}_{\text{bal}}(\mathbb{V}_{\mathbf{f}gh}^\dagger)$ is a torsion $\Lambda_{\mathbf{f}}$ -module injecting into $H^1(\mathbb{Q}_p, \mathbb{V}_{\mathbf{f}gh}^{f \cup +} / \mathcal{F}^2 \mathbb{V}_{\mathbf{f}gh}^\dagger)$; since this is $\Lambda_{\mathbf{f}}$ -torsion free by Proposition 2.7.3, it follows that

$$\text{Sel}_{\text{bal}}(\mathbb{V}_{\mathbf{f}gh}^\dagger) = \text{Sel}_{\mathcal{F} \cup +}(\mathbb{V}_{\mathbf{f}gh}^\dagger). \quad (2.30)$$

Now suppose that either (1) or (2) in the statement of theorem holds. Since $\bar{\rho}_{\mathbf{f}}$ is absolutely irreducible by our hypotheses, the module $H^1(\mathbb{Q}^\Sigma / \mathbb{Q}, \mathbb{V}_{\mathbf{f}gh}^\dagger)$ is $\Lambda_{\mathbf{f}}$ -torsion free by [PR00, §1.3.3]. Being $\Lambda_{\mathbf{f}}$ -torsion, it follows that the module $\text{Sel}_{\mathcal{F} \cap +}(\mathbb{V}_{\mathbf{f}gh}^\dagger)$ vanishes, and therefore from (2.28) we deduce the exact sequence

$$0 \longrightarrow \frac{\text{Sel}_{\text{bal}}(\mathbb{V}_{\mathbf{f}gh}^\dagger)}{\Lambda_{\mathbf{f}} \cdot \kappa(\mathbf{f}, g, h)} \longrightarrow \frac{H^1(\mathbb{Q}_p, \mathbb{V}_{\mathbf{f}}^{gh})}{\Lambda_{\mathbf{f}} \cdot p_{\mathbf{f}*}(\text{res}_p(\kappa(\mathbf{f}, g, h)))} \longrightarrow \text{coker}(\text{res}_p) \longrightarrow 0. \quad (2.31)$$

Together with Theorem 2.7.4 it follows that

$$\text{Char}_{\Lambda_{\mathbf{f}}} \left(\frac{\text{Sel}_{\text{bal}}(\mathbb{V}_{\mathbf{f}gh}^\dagger)}{\Lambda_{\mathbf{f}} \cdot \kappa(\mathbf{f}, g, h)} \right) \cdot \text{Char}_{\Lambda_{\mathbf{f}}}(\text{coker}(\text{res}_p)) = (\mathcal{L}_p^\xi(\check{\mathbf{f}}, \check{g}, \check{h})). \quad (2.32)$$

On the other hand, in light of (2.30), from (2.28) and (2.29) we deduce exact sequences

$$\begin{aligned} 0 &\longrightarrow \text{coker}(\text{res}_p) \longrightarrow X_{\mathcal{F} \cup +}(\mathbb{A}_{\mathbf{f}gh}^\dagger) \longrightarrow X_{\text{bal}}(\mathbb{A}_{\mathbf{f}gh}^\dagger) \longrightarrow 0, \\ 0 &\longrightarrow \text{coker}(\text{res}_p) \longrightarrow X_{\mathcal{F}}(\mathbb{A}_{\mathbf{f}gh}^\dagger) \longrightarrow X_{\mathcal{F} \cap +}(\mathbb{A}_{\mathbf{f}gh}^\dagger) \longrightarrow 0. \end{aligned}$$

Taking characteristic ideals, these imply

$$\begin{aligned}
\text{Char}_{\Lambda_{\mathbf{f}}}(X_{\mathcal{F}}(\mathbb{A}_{\mathbf{f}gh}^{\dagger})) &= \text{Char}_{\Lambda_{\mathbf{f}}}(X_{\mathcal{F} \cap +}(\mathbb{A}_{\mathbf{f}gh}^{\dagger})) \cdot \text{Char}_{\Lambda_{\mathbf{f}}}(\text{coker}(\text{res}_p)) \\
&= \text{Char}_{\Lambda_{\mathbf{f}}}(X_{\mathcal{F} \cup +}(\mathbb{A}_{\mathbf{f}gh}^{\dagger})_{\text{tors}}) \cdot \text{Char}_{\Lambda_{\mathbf{f}}}(\text{coker}(\text{res}_p)) \\
&= \text{Char}_{\Lambda_{\mathbf{f}}}(X_{\text{bal}}(\mathbb{A}_{\mathbf{f}gh}^{\dagger})_{\text{tors}}) \cdot \text{Char}_{\Lambda_{\mathbf{f}}}(\text{coker}(\text{res}_p))^2,
\end{aligned} \tag{2.33}$$

using Lemma 2.7.13(3) for the second equality. Multiplying (2.33) by the square of a generator of the characteristic ideal of $\text{Sel}_{\text{bal}}(\mathbb{V}_{\mathbf{f}gh}^{\dagger})/\Lambda_{\mathbf{f}} \cdot \kappa(\mathbf{f}, g, h)$ and using (2.32), the result follows. $\square$

2.8 Anticyclotomic Euler systems

In this section we highlight results from the recent work of Jechev–Nekovář–Skinner [JNS], where a general theory of Euler systems germane to [Rub00] is developed in the anticyclotomic setting.

2.8.1 The general theory

Let K be an imaginary quadratic field and let p be an odd prime. If $\mathfrak{n}$ is an integral prime ideal of K , we denote by $K(\mathfrak{n})^{\circ}$ the ray class field of conductor $\mathfrak{n}$; as in the previous sections, we write $K(\mathfrak{n})$ for the maximal p -subextension in $K(\mathfrak{n})^{\circ}$. For any positive integer n , we denote by $K[n]$ the maximal p -subextension in the ring class field of conductor n . We denote by K_{∞} the anticyclotomic $\mathbb{Z}_p$ -extension of K .

Let E be a finite extension of $\mathbb{Q}_p$ with ring of integers $\mathcal{O}$ and maximal ideal $\mathfrak{m}$. Let T be a free $\mathcal{O}$ -module of finite rank endowed with a continuous G_K -action unramified outside a finite set of primes, and let $V = T \otimes_{\mathcal{O}} E$. Assume that there exists a non-degenerate symmetric $\mathcal{O}$ -bilinear pairing

$$\langle , \rangle : T \times T \longrightarrow \mathcal{O}(1)$$

such that $\langle x^\sigma, y^{c\sigma c} \rangle = \langle x, y \rangle^\sigma$ for all $x, y \in T$ and for all $\sigma \in G_K$, where c stands for complex conjugation. Thus $V^c \simeq V^\vee(1)$, where V^c denotes the representation V with the G_K -action twisted by c , and, if the above pairing is perfect, we also have $T^c \simeq T^\vee(1)$. We also define the G_K -module $A = V/T$.

If L is a finite extension of K and v is a finite place of L , we write $\bar{v} = v^c$. Then, the pairing above induces a local pairing

$$H^1(L_v, V) \times H^1(L_{\bar{v}}, V) \longrightarrow E,$$

and similarly replacing V by T and E by $\mathcal{O}$. The pair of compatible maps $G_{L_v} \rightarrow G_{L_{\bar{v}}}$ and $V \rightarrow V^c$ defined by $\sigma \mapsto c\sigma c$ and $w \mapsto w$, respectively, induces an isomorphism $H^1(L_{\bar{v}}, V) \cong H^1(L_v, V^c) \cong H^1(L_v, V^\vee(1))$ whereby the above local pairing is just the natural cup-product pairing.

For the results we shall discuss, we consider two different type of “big image” hypotheses, (HW) for the weaker ones, and (HS) for the stronger ones.

Hypothesis (HW).

- (1) *V is absolutely irreducible as a G_K -representation.*
- (2) *There exists an element $\sigma_0 \in \text{Gal}(\bar{K}/K(1)^\circ K(\mu_{p^\infty}))$ such that the E -dimension of $V/(\sigma_0 - 1)V$ is one.*

Hypothesis (HS).

- (1') *The residual representation $\bar{T} = T/\mathfrak{m}T$ is absolutely irreducible.*
- (2') *There exists an element $\sigma_0 \in \text{Gal}(\bar{K}/K(p^\infty)^\circ)$ such that $T/(\sigma_0 - 1)T \simeq \mathcal{O}$ is a free $\mathcal{O}$ -module of rank one.*
- (3') *There exists an element $\tau_0 \in G_K$ such that $\tau_0 - 1$ acts on V as multiplication by an element in $\mathcal{O}^\times$.*

(4') The above pairing $T \times T \longrightarrow \mathcal{O}(1)$ is perfect.

For each prime $\mathfrak{p}$ of K above p , choose a $G_{K_{\mathfrak{p}}}$ -stable $\mathcal{O}$ -submodule $\mathcal{F}_{\mathfrak{p}}^+(T)$ of T , and let $\mathcal{F}_{\mathfrak{p}}^-(T) = T/\mathcal{F}_{\mathfrak{p}}^+(T)$. We also define $\mathcal{F}_{\mathfrak{p}}^+(V) = \mathcal{F}_{\mathfrak{p}}^+(T) \otimes_{\mathcal{O}} E \subseteq V$ and $\mathcal{F}_{\mathfrak{p}}^-(V) = V/\mathcal{F}_{\mathfrak{p}}^+(V)$. Let L be a finite extension of K . For each place v of L , we define a local condition

$$H_{\text{Gr}}^1(L_v, V) = \begin{cases} \ker(H^1(L_v, V) \rightarrow H^1(L_v^{\text{nr}}, V)) & \text{if } v \nmid p, \\ \ker(H^1(L_v, V) \rightarrow H^1(L_v, \mathcal{F}_{\mathfrak{p}}^-(V))) & \text{if } v \mid \mathfrak{p} \text{ for some } \mathfrak{p} \mid p. \end{cases}$$

We define the *Greenberg Selmer group*

$$\text{Sel}_{\text{Gr}}(L, V) = \ker \left(H^1(L, V) \rightarrow \prod_v H^1(L_v, V) / H_{\text{Gr}}^1(L_v, V) \right),$$

where the product is over all finite places of L .

We also define local conditions for T and A by propagation of the local conditions for V , i.e., for each place v of L , we define

- $H_{\text{Gr}}^1(L_v, T)$ as the preimage of $H_{\text{Gr}}^1(L_v, V)$ by the map $H^1(L_v, T) \rightarrow H^1(L_v, V)$,
and
- $H_{\text{Gr}}^1(L_v, A)$ as the image of $H_{\text{Gr}}^1(L_v, V)$ by the map $H^1(L_v, V) \rightarrow H^1(L_v, A)$,

and use these to define the Selmer groups $\text{Sel}_{\text{Gr}}(L, T)$ and $\text{Sel}_{\text{Gr}}(L, A)$ as above. Finally, for each positive integer n , we also put

$$\text{Sel}_{\text{Gr}}(K[np^{\infty}], T) = \varprojlim_r \text{Sel}_{\text{Gr}}(K[np^r], T) \quad \text{and} \quad \text{Sel}_{\text{Gr}}(K[np^{\infty}], A) = \varinjlim_r \text{Sel}_{\text{Gr}}(K[np^r], A),$$

where the limits are with respect to the corestriction and restriction maps, respectively.

Let $\mathcal{N}$ be an ideal of K divisible by p and all the primes at which T is ramified, and let $\mathcal{S}$ be the set of all squarefree products of primes of $\mathbb{Q}$ which *split* in K and are coprime to $\mathcal{N}$.

Definition 2.8.1. A “*split*” *anticyclotomic Euler system* for $(T, \{\mathcal{F}_{\mathfrak{p}}^+(T)\}_{\mathfrak{p}|p}, \mathcal{N})$ is a collection of classes

$$\kappa = \{\kappa_n \in \text{Sel}_{\text{Gr}}(K[n], T) : n \in \mathcal{S}\}$$

such that, whenever q is a rational prime and $n, nq \in \mathcal{S}$,

$$\text{cor}_{K[nq]/K[n]}(\kappa_{nq}) = P_{\mathfrak{q}}(\text{Fr}_{\mathfrak{q}}^{-1}) \kappa_n, \quad (2.34)$$

where $\mathfrak{q}$ is any of the primes of K above q and $P_{\mathfrak{q}}(X) = \det(1 - \text{Fr}_{\mathfrak{q}}^{-1} X | T^{\vee}(1))$.

Similarly, a “*split*” Λ -*adic anticyclotomic Euler system* for $(T, \{\mathcal{F}_{\mathfrak{p}}^+(T)\}_{\mathfrak{p}|p}, \mathcal{N})$ is a collection of classes

$$\kappa_{\infty} = \{\kappa_{n,\infty} \in \text{Sel}_{\text{Gr}}(K[np^{\infty}], T) : n \in \mathcal{S}\}$$

satisfying the previous norm relations. In this case, the classes

$$\kappa_n = \text{pr}_{K[n]}(\kappa_{n,\infty}) \in \text{Sel}_{\text{Gr}}(K[n], T)$$

form an anticyclotomic Euler system in the previous sense, and we say that the Euler system $\kappa = \{\kappa_n\}_n$ extends along the anticyclotomic $\mathbb{Z}_p$ -extension.

A $(\Lambda$ -adic) anticyclotomic Euler system for $(T, \{\mathcal{F}_{\mathfrak{p}}^+(T)\}_{\mathfrak{p}|p})$ is just a $(\Lambda$ -adic) anticyclotomic Euler system for $(T, \{\mathcal{F}_{\mathfrak{p}}^+(T)\}_{\mathfrak{p}|p}, \mathcal{N})$ for some $\mathcal{N}$ as above. We shall usually drop $\{\mathcal{F}_{\mathfrak{p}}^+(T)\}_{\mathfrak{p}|p}$ if there is no risk of confusion.

If κ is an anticyclotomic Euler system for T , we define

$$\kappa_0 := \text{cor}_{K[1]/K}(\kappa_1) \in \text{Sel}_{\text{Gr}}(K, T).$$

If it extends along the anticyclotomic $\mathbb{Z}_p$ -extension, we similarly define

$$\kappa_\infty := \text{cor}_{K[1]/K}(\kappa_{1,\infty}) \in \text{Sel}_{\text{Gr}}(K_\infty, T),$$

where $\kappa_\infty = \{\kappa_{n,\infty}\}$ is the Λ -adic anticyclotomic Euler system extending κ .

When we have an Euler system as above, we will be interested in ensuring that the following orthogonality hypothesis holds.

Hypothesis (HO). *For all $n \in \mathcal{S}$ and for all places v of $K[n]$ above p , the local conditions $H_{\text{Gr}}^1(K[n]_v, V)$ and $H_{\text{Gr}}^1(K[n]_{\bar{v}}, V)$ are orthogonal complements under the local pairing*

$$H^1(K[n]_v, V) \times H^1(K[n]_{\bar{v}}, V) \longrightarrow E.$$

Remark 2.8.2. The condition in hypothesis (HO) holds automatically for all places away from p , by [Rub00, Prop. 1.4.2]. Observe also that if (HO) holds, then for all $n \in \mathcal{S}$ and for all places v of $K[n]$ the local conditions $H_{\text{Gr}}^1(K[n]_v, T)$ and $H_{\text{Gr}}^1(K[n]_{\bar{v}}, T)$ are also orthogonal complements under the local pairing

$$H^1(K[n]_v, T) \times H^1(K[n]_{\bar{v}}, T) \longrightarrow \mathcal{O},$$

as follows easily from the definitions using [Rub00, Prop. B.2.4] and the commutative diagram

$$\begin{array}{ccc} H^1(K[n]_v, T) \times H^1(K[n]_{\bar{v}}, T) & \longrightarrow & \mathcal{O} \\ \downarrow & & \downarrow \\ H^1(K[n]_v, V) \times H^1(K[n]_{\bar{v}}, V) & \longrightarrow & E. \end{array}$$

We assume in the rest of this subsection that hypothesis (HO) holds for our choice of local conditions at p .

Theorem 2.8.3 ([JNS]). *Assume that p splits in K and that Hypothesis (HW) is satisfied, and let $\kappa = \{\kappa_n\}_n$ be an anticyclotomic Euler system for T which extends*

along the anticyclotomic $\mathbb{Z}_p$ -extension. If $\kappa_0 \neq 0$, then the Selmer group $\text{Sel}_{\text{Gr}}(K, T)$ has $\mathcal{O}$ -rank one.

Remark 2.8.4. One can replace the assumptions that p splits in K and the Euler system extends along the anticyclotomic $\mathbb{Z}_p$ -extension by the assumption that there exists an element $\gamma \in G_K$ fixing the extension $K(1)^\circ(\mu_{p^\infty}, (\mathcal{O}_K^\times)^{1/p^\infty})$ and such that $\gamma - 1$ acts invertibly on V .

Under the stronger Hypothesis (HS), granted the non-triviality of a Λ -adic anticyclotomic Euler system, the results of [JNS] yield a divisibility towards a corresponding Iwasawa main conjecture.

Theorem 2.8.5 ([JNS]). *Assume that p splits in K and that Hypothesis (HS) is satisfied, and let κ be a Λ -adic anticyclotomic Euler system for T .*

(a) *If $\kappa_0 \neq 0$, then $\text{Sel}_{\text{Gr}}(K, A)$ has $\mathcal{O}$ -corank one, $\text{Sel}_{\text{Gr}}(K, T)$ has $\mathcal{O}$ -rank one, and*

$$\text{length}_{\mathcal{O}}(\text{Sel}_{\text{Gr}}(K, A)_{/\text{div}}) \leq 2 \text{length}_{\mathcal{O}}\left(\frac{\text{Sel}_{\text{Gr}}(K, T)}{\mathcal{O} \cdot \kappa_0}\right),$$

where $(-)_{/\text{div}}$ denotes the quotient of $(-)$ by its maximal divisible submodule.

(b) *If κ_∞ is not Λ_{ac} -torsion, then $X_{\text{Gr}}(K_\infty, A)$ and $\text{Sel}_{\text{Gr}}(K_\infty, T)$ have both Λ_{ac} -rank one, and*

$$\text{Char}_{\Lambda_{\text{ac}}}(X_{\text{Gr}}(K_\infty, A)_{\text{tors}}) \supset \text{Char}_{\Lambda_{\text{ac}}}\left(\frac{\text{Sel}_{\text{Gr}}(K_\infty, T)}{\Lambda_{\text{ac}} \cdot \kappa_\infty}\right)^2,$$

where $(-)_{\text{tors}}$ denotes the maximal Λ_{ac} -torsion submodule of $(-)$.

2.8.2 Big image results

We now give conditions under which the hypotheses in the general results of §2.8.1 are verified in our setting. To that end, we shall build on [Loe17].

As before, $K/\mathbb{Q}$ is an imaginary quadratic field of discriminant $-D_K$, (g, h) is a pair of p -ordinary non-Eisenstein newforms of weights (l, m) of the same parity, levels (N_g, N_h) and characters (χ_g, χ_h) , and ψ is a Grössencharacter of K of infinity type $(1 - k, 0)$ for some positive even integer k and of conductor $\mathfrak{f}$. We denote by χ the unique Dirichlet character modulo $N_{K/\mathbb{Q}}(\mathfrak{f})$ such that $\psi((n)) = n^{k-1}\chi(n)$ for integers n coprime to $N_{K/\mathbb{Q}}(\mathfrak{f})$, and we assume that $\chi\varepsilon_K\chi_g\chi_h = 1$.

We now make the further assumptions that:

- neither g nor h are of CM type,
- g is not Galois-conjugate to a twist of h .

As in [Loe17, §3.1], we define the open subgroups H_g and H_h of $G_{\mathbb{Q}}$, the quaternion algebras B_g and B_h , and the algebraic groups G_g and G_h , and put

$$B = B_g \times B_h, \quad G = G_g \times_{\mathbb{G}_m} G_h.$$

We define H to be the intersection of H_g , H_h and $G_{K(\mathfrak{f})^\circ}$. (Note that in *loc. cit.* H is defined to be the intersection of H_g and H_h , so our H might be a finite index subgroup of his H , but this will not affect the results that follow.) We have an adelic representation $\tilde{\rho}_{g,h} : H \rightarrow G(\hat{\mathbb{Q}})$, and representations

$$\tilde{\rho}_{g,h,p} : H \longrightarrow G(\mathbb{Q}_p)$$

for every rational prime p , and, by [Loe17, Thm. 3.2.2], $\tilde{\rho}_{g,h,p}(H) = G(\mathbb{Z}_p)$ for all but finitely many p .

Remark 2.8.6. Note that the representations studied in [Loe17] are the dual to the ones studied in this paper, but as pointed out in [Loe17, Rmk. 2.1.2], this difference is unimportant when considering the image.

Let L be a finite extension of K containing the Fourier coefficients of g and h and the image of ψ . Let $\mathfrak{P}$ be a prime of L above some rational prime p , and let $E = L_{\mathfrak{P}}$.

Definition 2.8.7. We say that the prime $\mathfrak{P}$ is *good* if the following conditions hold:

- $p \geq 7$;
- p is unramified in B ;
- p is coprime to $\mathfrak{f}$, N_g and N_h ;
- $\tilde{\rho}_{g,h,p}(H) = G(\mathbb{Z}_p)$;
- $E = \mathbb{Q}_p$.

Remark 2.8.8. Observe that all but the last condition exclude only finitely many primes. The last condition could be somewhat relaxed in some cases, and will be used largely for simplicity. Note also that the above set of conditions holds for a set of primes of positive density.

Lemma 2.8.9. *Assume that there is at least one prime which divides D_K but not N_g and one prime which divides D_K but not N_h . Then, if $\mathfrak{P}$ is a good prime,*

$$(\rho_{g,\mathfrak{P}} \times \rho_{h,\mathfrak{P}})(H \cap G_{K(p^\infty)^\circ}) = \mathrm{SL}_2(\mathbb{Z}_p) \times \mathrm{SL}_2(\mathbb{Z}_p).$$

Proof. Let $\mathbb{Q}(\rho_g)$ and $\mathbb{Q}(\rho_h)$ be the Galois extensions of $\mathbb{Q}$ cut out by the representations ρ_g and ρ_h attached to g and h , respectively. These extensions are unramified outside pN_g and pN_h , respectively. Therefore, the condition on D_K implies that $K \cap \mathbb{Q}(\rho_g) = \mathbb{Q}$ and $K \cap \mathbb{Q}(\rho_h) = \mathbb{Q}$. Moreover, since any Galois extension of $\mathbb{Q}$ contained in K_∞ must itself contain K , we also have $K_\infty \cap \mathbb{Q}(\rho_g) = \mathbb{Q}$ and $K_\infty \cap \mathbb{Q}(\rho_h) = \mathbb{Q}$.

The conditions on $\mathfrak{P}$ imply that

$$(\rho_{g,\mathfrak{P}} \times \rho_{h,\mathfrak{P}})(H \cap G_{\mathbb{Q}(\mu_{p^\infty})}) = \mathrm{SL}_2(\mathbb{Z}_p) \times \mathrm{SL}_2(\mathbb{Z}_p),$$

and, from the remarks in the previous paragraph, it follows that

$$(\rho_{g,\mathfrak{P}} \times \rho_{h,\mathfrak{P}})(H \cap G_{K_\infty(\mu_{p^\infty})}) = \mathrm{SL}_2(\mathbb{Z}_p) \times \mathrm{SL}_2(\mathbb{Z}_p).$$

Finally, since $H \cap G_{K(p^\infty)^\circ}$ is a normal subgroup of $H \cap G_{K_\infty(\mu_{p^\infty})}$ of index dividing $p - 1$ and there are no such subgroups in $\mathrm{SL}_2(\mathbb{Z}_p) \times \mathrm{SL}_2(\mathbb{Z}_p)$, the lemma follows. $\square$

Now we are able to give conditions under which the results of [JNS] can be applied to our setting, i.e., to the representation $T_{g,h}^\psi$ defined above.

Proposition 2.8.10. *Assume that there is at least one prime which divides D_K but not N_g and one prime which divides D_K but not N_h . Let $\mathfrak{P}$ be a good prime. Suppose that there exists $\sigma \in G_{K(p^\infty)^\circ}$ such that $\psi_{\mathfrak{P}}(\sigma) \neq \psi_{\mathfrak{P}}^c(\sigma)$ modulo p . Then, hypotheses (HS) hold for $T_{g,h}^\psi$.*

Proof. Since $\psi_{\mathfrak{P}}$ is trivial when restricted to $H \cap G_{K(p^\infty)^\circ}$, condition (1') follows easily from the previous lemma.

To prove condition (2'), we closely follow the proof of [Loe17, Prop. 4.2.1]. Write $\chi_g(\sigma)$ and $\chi_h(\sigma)$ for the images of σ by χ_g and χ_h via the natural identifications $\mathrm{Gal}(\mathbb{Q}(\mu_{N_g})/\mathbb{Q}) \cong (\mathbb{Z}/N_g\mathbb{Z})^\times$ and $\mathrm{Gal}(\mathbb{Q}(\mu_{N_h})/\mathbb{Q}) \cong (\mathbb{Z}/N_h\mathbb{Z})^\times$. Then, by the previous lemma, the image of $\sigma H \cap G_{K(p^\infty)^\circ}$ under $\rho_{g,\mathfrak{P}} \times \rho_{h,\mathfrak{P}}$ contains all the elements of the form

$$\left(\begin{pmatrix} x & 0 \\ 0 & x^{-1}\chi_g(\sigma) \end{pmatrix}, \begin{pmatrix} y & 0 \\ 0 & y^{-1}\chi_h(\sigma) \end{pmatrix} \right), \quad x, y \in \mathbb{Z}_p^\times.$$

Now choose $x \in \mathbb{Z}_p^\times$ such that $x^{-2}\chi_g(\sigma) \neq 1 \pmod{p}$ and $x^2\chi_h(\sigma)\psi_{\mathfrak{P}}(\sigma)^{-2} \neq 1 \pmod{p}$, which is possible since $p \geq 7$, and let $y = x^{-1}\psi_{\mathfrak{P}}(\sigma)$. Choose $\sigma_0 \in \sigma H \cap G_{K(p^\infty)^\circ}$ whose image under $\rho_{g,\mathfrak{P}} \times \rho_{h,\mathfrak{P}}$ is given by the element above, with the choices of x and y which we have just specified. Then, the eigenvalues of σ_0 acting on $T_{g,h}^\psi$ are 1, $x^{-2}\chi_g(\sigma)$, $x^2\chi_h(\sigma)\psi_{\mathfrak{P}}(\sigma)^{-2}$ and $\psi_{\mathfrak{P}}^c(\sigma)\psi_{\mathfrak{P}}(\sigma)^{-1}$, which proves condition (2').

To check condition (3'), we can argue as in [KLZ17, Rmk. 11.1.3]. By the previous lemma, we can find an element $\tau_0 \in H \cap G_{K(p^\infty)^\circ}$ such that

$$(\rho_{g,\mathfrak{P}} \times \rho_{h,\mathfrak{P}})(\tau_0) = \left(\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right),$$

so τ_0 acts on $T_{g,h}^\psi$ as multiplication by -1 .

Finally, condition (4') follows from the assumption that g and h are non-Eisenstein and p -distinguished. $\square$

Remark 2.8.11. If we are just interested in ensuring that hypotheses (HW) hold for $T_{g,h}^\psi$, we can relax some of the assumptions above. For example, we do not need to require g and h to be non-Eisenstein, and we can require that there exist $\sigma \in G_{K(1)^\circ(\mu_{p^\infty})}$ such that $\psi_{\mathfrak{P}}(\sigma) \neq \psi_{\mathfrak{P}}^c(\sigma)$, without requiring this inequality to hold modulo p .

2.9 Proof of Theorems B, C, and D

Let the setting be as in the Introduction. In particular, $g \in S_l(N_g, \chi_g)$ and $h \in S_m(N_h, \chi_h)$ are newforms of weights $l \geq m \geq 2$ of the same parity, $K/\mathbb{Q}$ is an imaginary quadratic field of discriminant $-D < 0$, ψ is a Grössencharacter for K of infinity type $(1 - k, 0)$ for some even integer $k \geq 2$, and we consider the G_K -representation

$$V_{g,h}^\psi = V_g \otimes_E V_h(\psi_{\mathfrak{P}}^{-1})(1 - c),$$

where $c = (k + l + m - 2)/2$.

Lemma 2.9.1. *The Bloch–Kato Selmer group of $V_{g,h}^\psi$ is given by*

$$\mathrm{Sel}(K, V_{g,h}^\psi) \cong \begin{cases} \mathrm{Sel}_{\mathrm{bal}}(K, V_{g,h}^\psi) & \text{if } l - m < k < l + m, \\ \mathrm{Sel}_{\mathcal{F}}(K, V_{g,h}^\psi) & \text{if } k \geq l + m. \end{cases}$$

Proof. Note that by Shapiro's lemma $H^1(K, V_{g,h}^\psi) \cong H^1(\mathbb{Q}, V_{fgh})$, where $f = \theta_\psi$ is the theta series of ψ and V_{fgh} is the specialization of the big Galois representation $\mathbb{V}_{\mathbf{fgh}}^\dagger$ in (2.21) to weights (k, l, m) . One immediately checks that the Hodge–Tate weights of the $G_{\mathbb{Q}_p}$ -subrepresentation $\mathcal{F}^2 V_{fgh} \subset V_{fgh}$ (resp. $V_{fgh}^f \subset V_{fgh}$) are all < 0 (with the p -adic cyclotomic character ϵ_{cyc} having Hodge–Tate weight -1) if and only if $l - m < k < l + m$ (resp. $k \geq l + m$). The result follows. $\square$

Here we collect a set of hypotheses for our later reference. For any nonzero $m \in \mathbb{Z}$, $\text{prime}(m)$ denotes the set of primes that divide m , and $\text{prime}^c(m)$ its complement.

Hypotheses.

- (h1) g and h are ordinary at p and non-Eisenstein,
- (h2) p splits in K ,
- (h3) p does not divide the class number of K ,
- (h4) $\psi|_{\mathfrak{P}}|_{G_{K(p^\infty)^\circ}} \neq \psi|_{\mathfrak{P}}^c|_{G_{K(p^\infty)^\circ}}$ modulo p ,
- (h5) neither g nor h are of CM type,
- (h6) g is not Galois-conjugate to a twist of h .
- (h7) $\text{prime}(D) \cap \text{prime}^c(N_g) \neq \emptyset$ and $\text{prime}(D) \cap \text{prime}^c(N_h) \neq \emptyset$,
- (h8) $\mathfrak{P}$ is a good prime in the sense of Definition 2.8.7.

2.9.1 Proof of Theorem B

Let $\kappa_{\psi,g,h,1,\infty} \in H_{\text{Iw}}^1(K[p^\infty], T_{g,h}^\psi)$ be the Iwasawa cohomology class of conductor $n = 1$ from Theorem 2.6.5, and set

$$\kappa_{\psi,g,h} = \kappa_{\psi,g,h,1} \in H^1(K, T_{g,h}^\psi), \quad (2.35)$$

where $\kappa_{\psi,g,h,1} = \text{pr}_K(\kappa_{\psi,g,h,1,\infty})$.

If $l - m < k < l + m$, the next result recovers Theorem B in the Introduction. Note however, that the result does not require these inequalities to hold.

Theorem 2.9.2. *Assume hypotheses (h1)–(h8). Then the following implication holds:*

$$\kappa_{\psi,g,h} \neq 0 \implies \dim_E \text{Sel}_{\text{bal}}(K, V_{g,h}^\psi) = 1.$$

In particular, if $l - m < k < l + m$ and $\kappa_{\psi,g,h} \neq 0$ then the Bloch–Kato Selmer group $\text{Sel}(K, V_{g,h}^\psi)$ is one-dimensional.

Proof. By Proposition 2.6.6, the classes $\kappa_{\psi,g,h,n} := \text{pr}_{K[n]}(\kappa_{\psi,g,h,n,\infty})$ land in $\text{Sel}_{\text{bal}}(K[n], T_{g,h}^\psi)$, and by Theorem 2.6.5 they form an anticyclotomic Euler system for $V_{g,h}^\psi$. Therefore, the result follows from Theorem 2.8.3 and Proposition 2.8.10. $\square$

Remark 2.9.3. If $k = 2$ and $l = m \geq 2$, working with the classes $\kappa_{\psi,g,h,n}$ from Theorem 2.4.5, rather than those from Theorem 2.6.5 as above, hypotheses (h2)–(h3) in Theorem 2.9.2 can be replaced by the assumption that there exists an element $\gamma \in G_K$ satisfying the conditions in Remark 2.8.4. Further, (h1) and (h4) can be relaxed as discussed in Remark 2.8.11.

2.9.2 Proof of Theorem C

Recall that $\theta_\psi \in S_k(N_\psi, \chi_{\varepsilon_K})$ is the theta series attached to ψ , and put $N = \text{lcm}(N_\psi, N_g, N_h)$.

The next theorem, establishing cases of the Bloch–Kato conjecture for $V_{g,h}^\psi$ in analytic rank zero, recovers Theorem C in the Introduction.

Theorem 2.9.4. *Assume hypotheses (h1)–(h8), and in addition that:*

- $\varepsilon_\ell(\theta_\psi, g, h) = +1$ for all primes $\ell \mid N$,

- $\gcd(N_\psi, N_g, N_h)$ is squarefree.

If $k \geq l + m$ then the following implication holds:

$$L(V_{g,h}^\psi, 0) \neq 0 \implies \text{Sel}(K, V_{g,h}^\psi) = 0.$$

Proof. We continue to denote by $\kappa_{\psi,g,h}$ the image of the class in (2.35) under the isomorphism

$$H^1(K, V_{g,h}^\psi) \cong H^1(\mathbb{Q}, V_{fgh})$$

coming from Shapiro's lemma. If $k \geq l + m$, the central value $L(V_{g,h}^\psi, 0)$ is in the range of interpolation of the triple product p -adic L -function of Theorem 2.7.1, and so by Proposition 2.7.3 and Theorem 2.7.4 its non-vanishing implies that the image of $\kappa_{\psi,g,h}$ under the natural map

$$\text{res}_p : \text{Sel}_{\text{bal}}(\mathbb{Q}, V_{fgh}) \longrightarrow H^1(\mathbb{Q}_p, V_f^{gh})$$

is nonzero. In particular, $\kappa_{\psi,g,h} \neq 0$, and therefore by Theorem 2.9.2 the balanced Selmer group $\text{Sel}_{\text{bal}}(K, V_{g,h}^\psi) = \text{Sel}_{\text{bal}}(\mathbb{Q}, V_{fgh})$ is one-dimensional.

From the exact sequence

$$\begin{aligned} 0 \longrightarrow \text{Sel}_{\mathcal{F}\cap+}(\mathbb{Q}, V_{fgh}) \longrightarrow \text{Sel}_{\text{bal}}(\mathbb{Q}, V_{fgh}) \xrightarrow{\text{res}_p} H^1(\mathbb{Q}_p, V_f^{gh}) \\ \longrightarrow \text{Sel}_{\mathcal{F}\cup+}(\mathbb{Q}, V_{fgh})^\vee \longrightarrow \text{Sel}_{\text{bal}}(\mathbb{Q}, V_{fgh})^\vee \longrightarrow 0 \end{aligned}$$

coming from global duality (adopting notations similar to those in Theorem 2.7.15), we thus see that $\text{Sel}_{\mathcal{F}\cap+}(\mathbb{Q}, V_{fgh}) = 0$ and that $\text{Sel}_{\mathcal{F}\cup+}(\mathbb{Q}, V_{fgh}) = \text{Sel}_{\text{bal}}(\mathbb{Q}, V_{fgh})$. Together with the exact sequence

$$\text{Sel}_{\mathcal{F}\cup+}(\mathbb{Q}, V_{fgh}) \xrightarrow{\text{res}_p} H^1(\mathbb{Q}_p, V_f^{gh}) \longrightarrow \text{Sel}_{\mathcal{F}}(\mathbb{Q}, V_{fgh})^\vee \longrightarrow \text{Sel}_{\mathcal{F}\cap+}(\mathbb{Q}, V_{fgh})^\vee \longrightarrow 0,$$

it follows that $\text{Sel}_{\mathcal{F}}(\mathbb{Q}, V_{fgh}) = 0$, and combined with Lemma 2.9.1 this concludes the proof. $\square$

Refining the proof of Theorem 2.9.4, we can further bound the size of the Bloch–Kato Selmer group for the discrete module $A_{g,h}^{\psi} = V_{g,h}^{\psi}/T_{g,h}^{\psi}$ in terms of L -values. For the statement, let $\mathbf{f}$ be the Hida family associated to ψ as in §2.6, so that $\mathbf{f}_k$ is the ordinary p -stabilization of θ_{ψ} , and, keeping with the notations in Theorem 2.7.1, put $\alpha_k = \psi(\overline{\mathbf{p}})$ and $\beta_k = \psi(\mathbf{p})$. Let also $\varepsilon_{\ell}(\theta_{\psi}, g, h) = \varepsilon_{\ell}(V_{fgh})$ denote the epsilon factor associated to $V_{fgh}|_{G_{\mathbb{Q}_{\ell}}}$, where $f = \theta_{\psi}$.

Theorem 2.9.5. *Assume hypotheses (h1)–(h8), and in addition that:*

- $\varepsilon_{\ell}(\theta_{\psi}, g, h) = +1$ for all primes $\ell \mid N$,
- $\gcd(N_{\psi}, N_g, N_h)$ is squarefree,
- $H^1(\mathbb{Q}_p, T_f^{gh})$ is torsion-free,
- $H_{\mathcal{L}}^1(\mathbb{Q}_p, T_{fgh})$ is torsion-free for $\mathcal{L} \in \{\text{bal}, \mathcal{F}, \mathcal{F} \cap +, \mathcal{F} \cup +\}$.

If $k \geq l + m$ and $L(V_{g,h}^{\psi}, 0) \neq 0$ then the $\mathcal{O}$ -module $\text{Sel}_{\mathcal{F}}(K, A_{g,h}^{\psi})$ is finite and

$$\text{length}_{\mathcal{O}}(\text{Sel}_{\mathcal{F}}(K, A_{g,h}^{\psi})) \leq 2 v_{\mathfrak{P}} \left(\frac{(l-2)!(m-2)!}{(k-c-1)!} \cdot \frac{\mathcal{E}_1(\mathbf{f}_k)}{\mathcal{E}(\mathbf{f}_k, g, h)} \cdot \mathcal{L}_p^{\xi}(\check{\mathbf{f}}, \check{g}, \check{h})(k) \right),$$

where $\mathcal{E}_1(\mathbf{f}_k) = (1 - \frac{\beta_k}{p\alpha_k})$, $\mathcal{E}(\mathbf{f}_k, g, h) = (1 - \frac{\alpha_k \alpha_g \alpha_h}{p^c})(1 - \frac{\beta_k \beta_g \alpha_h}{p^c})(1 - \frac{\beta_k \alpha_g \beta_h}{p^c})(1 - \frac{\beta_k \beta_g \beta_h}{p^c})$, and $c = (k + l + m - 2)/2$.

Proof. As in the proof of Theorem 2.9.4, if $k \geq l + m$ and $L(V_{g,h}^{\psi}, 0) \neq 0$ then the class $\kappa_{\psi, g, h}$ is nonzero. Since by Theorem 2.6.5 this is the bottom class of an anticyclotomic Euler system for V_{gh}^{ψ} , from Theorem 2.8.5 and Proposition 2.8.10 we deduce that $\text{Sel}_{\text{bal}}(K, A_{g,h}^{\psi})$ has $\mathcal{O}$ -corank one, with

$$\text{length}_{\mathcal{O}}(\text{Sel}_{\text{bal}}(K, A_{g,h}^{\psi})/\text{div}) \leq 2 \text{length}_{\mathcal{O}} \left(\frac{\text{Sel}_{\text{bal}}(K, T_{g,h}^{\psi})}{\mathcal{O} \cdot \kappa_{\psi, g, h}} \right). \quad (2.36)$$

By the exact sequence (2.28) specialized to weight k , it follows that $\mathrm{Sel}_{\mathcal{F}\cup+}(K, A_{g,h}^\psi)$ has also $\mathcal{O}$ -corank one. Thus both $\mathrm{Sel}_{\mathrm{bal}}(K, T_{g,h}^\psi) \subset \mathrm{Sel}_{\mathcal{F}\cup+}(K, T_{g,h}^\psi)$ have $\mathcal{O}$ -rank one, and therefore

$$\mathrm{Sel}_{\mathrm{bal}}(K, T_{g,h}^\psi) = \mathrm{Sel}_{\mathcal{F}\cup+}(K, T_{g,h}^\psi), \quad (2.37)$$

since their quotient is $\mathcal{O}$ -torsion free. Moreover, letting $\pi \in \mathcal{O}$ be a uniformizer, as in the proof of Lemma 2.7.13 we find that

$$\mathrm{Sel}_{\mathcal{F}\cup+}(K, A_{g,h}^\psi)[\pi^i] \cong E/\mathcal{O}[\pi^i] \oplus \mathrm{Sel}_{\mathcal{F}\cap+}(K, A_{g,h}^\psi)[\pi^i]$$

for all i , and hence $\mathrm{length}_{\mathcal{O}}(\mathrm{Sel}_{\mathcal{F}\cup+}(K, A_{g,h}^\psi)_{/\mathrm{div}}) = \mathrm{length}_{\mathcal{O}}(\mathrm{Sel}_{\mathcal{F}\cap+}(K, A_{g,h}^\psi))$.

The finiteness of $\mathrm{Sel}_{\mathcal{F}}(K, A_{g,h}^\psi)$ with the stated bound on its $\mathcal{O}$ -length thus follows from (2.36) by the same argument as in the proof of Theorem 2.7.15, noting that by Theorem 2.7.4 and the same calculation as in [BSV22, §8.5] (see esp. the equality following [*op. cit.*, (189)]) the map

$$\xi_k \cdot \left\langle \exp_p^*(-), \eta_{\check{f}} \otimes \omega_{\check{g}} \otimes \omega_{\check{h}} \right\rangle$$

where $f = \theta_\psi$ and ξ_k is the weight k specialization of the congruence ideal generator $\xi \in \Lambda_{\mathbf{f}}$, gives an isomorphism $H^1(\mathbb{Q}_p, T_f^{g,h}) \rightarrow \mathcal{O}$ taking $\kappa_{\psi,g,h}$ to

$$\frac{(l-2)! \cdot (m-2)!}{(k-c-1)!} \cdot \frac{\mathcal{E}_0(\mathbf{f}_k) \cdot \mathcal{E}_1(\mathbf{f}_k)}{\mathcal{E}(\mathbf{f}_k, g, h)} \cdot \mathcal{L}_p^\xi(\check{\mathbf{f}}, \check{g}, \check{h})(k),$$

where $\mathcal{E}_0(\mathbf{f}_k) = (1 - \frac{\beta_k}{\alpha_k})$ is a p -adic unit.

More precisely, under the freeness assumption in the statement, the weight k specializations of (2.28) and (2.29) yield the exact sequences

$$0 \longrightarrow \mathrm{coker}(\mathrm{res}_p) \longrightarrow \mathrm{Sel}_{\mathcal{F}\cup+}(K, A_{g,h}^\psi)^\vee \longrightarrow \mathrm{Sel}_{\mathrm{bal}}(K, A_{g,h}^\psi)^\vee \longrightarrow 0, \quad (2.38)$$

$$0 \longrightarrow \text{coker}(\text{res}_p) \longrightarrow \text{Sel}_{\mathcal{F}}(K, A_{g,h}^{\psi})^{\vee} \longrightarrow \text{Sel}_{\mathcal{F} \cap +}(K, A_{g,h}^{\psi})^{\vee} \longrightarrow 0,$$

where the two terms $\text{coker}(\text{res}_p)$ are equal in light of (2.37). Thus we find

$$\begin{aligned} \text{lt}_{\mathcal{O}}(\text{Sel}_{\mathcal{F}}(K, A_{g,h}^{\psi})) &= \text{lt}_{\mathcal{O}}(\text{Sel}_{\mathcal{F}}(K, A_{g,h}^{\psi})^{\vee}) = \text{lt}_{\mathcal{O}}(\text{Sel}_{\mathcal{F} \cap +}(K, A_{g,h}^{\psi})^{\vee}) + \text{lt}_{\mathcal{O}}(\text{coker}(\text{res}_p)) \\ &= \text{lt}_{\mathcal{O}}((\text{Sel}_{\mathcal{F} \cup +}(K, A_{g,h}^{\psi})_{/\text{div}})^{\vee}) + \text{lt}_{\mathcal{O}}(\text{coker}(\text{res}_p)) \\ &= \text{lt}_{\mathcal{O}}((\text{Sel}_{\text{bal}}(K, A_{g,h}^{\psi})_{/\text{div}})^{\vee}) + 2 \text{lt}_{\mathcal{O}}(\text{coker}(\text{res}_p)) \\ &= \text{lt}_{\mathcal{O}}(\text{Sel}_{\text{bal}}(K, A_{g,h}^{\psi})_{/\text{div}}) + 2 \text{lt}_{\mathcal{O}}(\text{coker}(\text{res}_p)), \end{aligned}$$

where the third equality follows from (2.38) and Lemma 2.9.6 below, concluding the proof. $\square$

Lemma 2.9.6. *Let $0 \rightarrow A \xrightarrow{j} B \rightarrow C \rightarrow 0$ be an exact sequence of finitely generated $\mathcal{O}$ -modules, and assume that A is finite. Then $B_{\text{tors}}/j(A) \cong C_{\text{tors}}$.*

In particular, if B', C' are cofinitely generated $\mathcal{O}$ -modules and we have an exact sequence $0 \rightarrow A \xrightarrow{j} (B')^{\vee} \rightarrow (C')^{\vee} \rightarrow 0$ with A finite, then

$$(B'_{/\text{div}})^{\vee}/j(A) \cong (C'_{/\text{div}})^{\vee},$$

and so $\text{lt}_{\mathcal{O}}((B'_{/\text{div}})^{\vee}) = \text{lt}_{\mathcal{O}}(A) + \text{lt}_{\mathcal{O}}((C'_{/\text{div}})^{\vee})$.

Proof. Writing $B \cong \mathcal{O}^r \oplus B_{\text{tors}}$, $C \cong \mathcal{O}^s \oplus C_{\text{tors}}$ we have, by the finiteness of A , $r = s$ and $j(A) \subset B_{\text{tors}}$, so

$$\mathcal{O}^r \oplus C_{\text{tors}} \cong C \cong B/j(A) \cong \mathcal{O}^r \oplus (B_{\text{tors}}/j(A)),$$

which implies the result. $\square$

Remark 2.9.7. The condition that $H^1(\mathbb{Q}_p, T_f^{gh})$ is torsion-free is equivalent to the vanishing of $H^0(\mathbb{Q}_p, A_f^{gh})$, which is satisfied if $k + 2 \neq l + m$ modulo $2(p - 1)$ or if $\chi_f(p)\alpha_g\alpha_h/\alpha_k \neq 1$ modulo p . Similarly, the last condition in the statement of

Theorem 2.9.5 can be recast in terms of the vanishing of the corresponding 0-th cohomology groups.

Remark 2.9.8. By Theorem 2.7.1, the non-vanishing of $L(V_{g,h}^\psi, 0)$ implies that $\mathcal{L}_p^\xi(\check{\mathbf{f}}, \check{g}, \check{h})(x) \neq 0$, so the upper bound provided by Theorem 2.9.5. Moreover, by the interpolation formula in Theorem 2.7.1, the same upper bound can be expressed in terms of the central L -value $L(V_{g,h}^\psi, 0)$, thus giving a result towards the Tamagawa number conjecture of [BK90].

2.9.3 Proof of Theorem D

As before, let $\mathbf{f}$ be the Hida family attached to ψ as in §2.6. Let $\kappa_{\psi,g,h,1,\infty}$ be the Λ -adic class of conductor $n = 1$ constructed in Theorem 2.6.5, and set

$$\kappa_{\psi,g,h,\infty} := \kappa_{\psi,g,h,1,\infty} \in H_{\text{Iw}}^1(K_\infty, T_{g,h}^\psi).$$

As shown in the proof of Proposition 2.6.6, under the Shapiro isomorphism

$$H^1(\mathbb{Q}, \mathbb{V}_{\mathbf{f}_{gh}}^\dagger) \cong H^1(K, \Lambda_{\mathcal{O}}(\kappa_{\text{ac}}^{-1}) \hat{\otimes}_{\mathcal{O}} T_{g,h}^\psi) \cong H_{\text{Iw}}^1(K_\infty, T_{g,h}^\psi),$$

the Selmer group $\text{Sel}_{\text{bal}}(\mathbb{Q}, \mathbb{V}_{\mathbf{f}_{gh}}^\dagger)$ of §2.7.3 corresponds to the Selmer group $\text{Sel}_{\text{Gr}}(K_\infty, T_{g,h}^\psi)$ of §2.8.1 attached to the local conditions in (2.20) at the primes above p . Moreover, under this isomorphism, the class $\kappa(\mathbf{f}, g, h)$ defined in §2.7.2 corresponds to the class $\kappa_{\psi,g,h,\infty}$.

The next result, establishing one of the divisibilities predicted by the Iwasawa main conjectures from §2.7.3, recovers Theorem D in the Introduction.

Theorem 2.9.9. *Assume hypotheses (h1)–(h8), and in addition that:*

- $\varepsilon_\ell(\theta_\psi, g, h) = +1$ for all primes $\ell \mid N$,
- $\gcd(N_\psi, N_g, N_h)$ is squarefree.

If $\kappa(\mathbf{f}, g, h)$ is not $\Lambda_{\mathbf{f}}$ -torsion, then the following hold:

(a) The modules $\text{Sel}_{\text{bal}}(\mathbb{V}_{\mathbf{f}gh}^{\dagger})$ and $X_{\text{bal}}(\mathbb{A}_{\mathbf{f}gh}^{\dagger})$ have both $\Lambda_{\mathbf{f}}$ -rank one, and

$$\text{Char}_{\Lambda_{\mathbf{f}}}(X_{\text{bal}}(\mathbb{A}_{\mathbf{f}gh}^{\dagger})_{\text{tors}}) \supset \text{Char}_{\Lambda_{\mathbf{f}}} \left(\frac{\text{Sel}_{\text{bal}}(\mathbb{V}_{\mathbf{f}gh}^{\dagger})}{\Lambda_{\mathbf{f}} \cdot \kappa(\mathbf{f}, g, h)} \right)^2.$$

(b) The modules $\text{Sel}_{\mathcal{F}}(\mathbb{V}_{\mathbf{f}gh}^{\dagger})$ and $X_{\mathcal{F}}(\mathbb{V}_{\mathbf{f}gh}^{\dagger})$ are both $\Lambda_{\mathbf{f}}$ -torsion, and

$$\text{Char}_{\Lambda_{\mathbf{f}}}(X_{\mathcal{F}}(\mathbb{A}_{\mathbf{f}gh}^{\dagger})) \supset (L_p(\mathbf{f}, g, h))$$

in $\Lambda_{\mathbf{f}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$.

Proof. The non-triviality assumption on $\kappa(\mathbf{f}, g, h)$ implies that $\kappa_{\psi, g, h, \infty}$ is not $\Lambda_{\mathbf{f}}$ -torsion. Since by Theorem 2.6.5 the class $\kappa_{\psi, g, h, \infty}$ is the bottom class of a Λ -adic Euler system for $V_{g, h}^{\psi}$, part (a) follows from Theorem 2.8.5 and Proposition 2.8.10. By Theorem 2.7.15, part (b) of the theorem follows from part (a), so this concludes the proof. $\square$

Chapter 3

An anticyclotomic Euler system for adjoint modular Galois representations

3.1 Introduction

The goal of this paper is to study the Bloch–Kato conjecture and the anticyclotomic Iwasawa theory of certain twists of the adjoint Galois representation attached to elliptic modular forms base changed to an imaginary quadratic field.

Our main result is the construction of an anticyclotomic Euler system in this setting, which we relate to an analogue of the Hida–Schmidt p -adic L -function for the symmetric square. By Kolyvagin’s method for anticyclotomic split Euler systems, as developed by Jetchev–Nekovář–Skinner, our results yield new cases of the Bloch–Kato conjecture in rank zero and a divisibility towards an Iwasawa main conjecture.

3.1.1 The set-up

Let $g \in S_l(N_g, \chi_g)$ be an ordinary newform of weight $l \geq 2$, level N_g , and nebentypus χ_g . Let $K/\mathbb{Q}$ be an imaginary quadratic field, and let ψ be a Hecke character of K of infinity type $(1 - k, 0)$ for some even integer $k \geq 2$. We assume that the associated theta series $\theta_\psi \in S_k(N_\psi)$ has trivial nebentypus. Fix an odd prime $p \nmid 2N_g N_\psi$ and an embedding $\iota_p : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$, and for simplicity in this Introduction assume that the Hecke field of g and the values of ψ are contained in a number field L with a prime $\mathfrak{P}$ above p such that $L_{\mathfrak{P}} = \mathbb{Q}_p$. We assume that p splits in K and is a prime of ordinary reduction for g and, again for simplicity, that $p \nmid h_K$, the class number of K . We will also assume that g is not of CM-type.

Let V_g be the (dual to Deligne's) p -adic Galois representation attached to g , and denote by $\mathrm{ad}^0(V_g) \subset \mathrm{End}_{\mathbb{Q}_p}(V_g)$ the adjoint representation on the trace-zero endomorphisms of V_g . We consider the conjugate self-dual G_K -representation

$$V := \mathrm{ad}^0(V_g)(\psi^{-1})(1 - k/2),$$

where (ψ^{-1}) denotes the twist by the inverse of ψ and $(1 - k/2)$ is the twist by the $(1 - k/2)$ -th power of the p -adic cyclotomic character.

3.1.2 Euler systems and p -adic L -functions

In this paper we construct an anticyclotomic Euler system for V and relate it to an associated anticyclotomic p -adic L -function.

For a positive integer m we write $K[m]$ for the maximal p -extension inside the ring class field of K of conductor m . Denote by $\mathcal{S}'$ the set of all squarefree products of primes q in the positive density set $\mathcal{P}'$ of Definition 3.5.2; in particular, these primes

split in K . For any p -adic G_K -representation W and a prime $\mathfrak{q}$ of K , put

$$P_{\mathfrak{q}}(W; X) = \det(1 - \mathrm{Fr}_{\mathfrak{q}}^{-1} X | W^{\vee}(1)),$$

where $\mathrm{Fr}_{\mathfrak{q}}$ denotes an arithmetic Frobenius element for the prime $\mathfrak{q}$ and $W^{\vee}$ denotes the contragredient representation of W . A natural lattice $T_g \subset V_g$ described in §3.2.1 defines a lattice in V denoted by T . Finally, let $H_{\mathrm{Iw}}^1(K[mp^{\infty}], T) = \varprojlim_r H^1(K[mp^r], T)$.

Theorem A (Theorem 3.5.4). *Assume that $H^1(K[mp^s], T)$ is torsion-free for all $m \in \mathcal{S}'$ and $s \geq 0$. There exists a collection of classes*

$$\{\kappa_{\psi, \mathrm{ad}^0(g), m, \infty} \in H_{\mathrm{Iw}}^1(K[mp^{\infty}], T) : m \in \mathcal{S}'\}$$

such that whenever $m, mq \in \mathcal{S}'$ with q a prime, we have

$$\mathrm{cor}_{K[mq]/K[m]}(\kappa_{\psi, \mathrm{ad}^0(g), mq, \infty}) = P_{\mathfrak{q}}(V; \mathrm{Fr}_{\mathfrak{q}}^{-1}) \kappa_{\psi, \mathrm{ad}^0(g), m, \infty},$$

where $\mathfrak{q}$ is any of the primes of K above q .

We obtain the Euler system classes $\kappa_{\psi, \mathrm{ad}^0(g), m, \infty}$ from a suitable modification of the diagonal Euler system classes $\kappa_{\psi, g, g^*, m, \infty}$ for

$$V_{\mathrm{ad}(g)}^{\psi} := V_g \otimes V_{g^*}(\psi^{-1})(1 - c)$$

constructed in [ACR21], where $g^* = g \otimes \chi_g^{-1}$ is the twist of g by the inverse of its nebentypus, and $c = (k + 2l - 2)/2$. It follows from our construction (and the results of [BSV22] that it builds upon) that $\kappa_{\psi, \mathrm{ad}^0(g), m, \infty}$ lands in the balanced Selmer groups $\mathrm{Sel}_{\mathrm{bal}}(K[mp^{\infty}], T)$ introduced in §3.2.3.

Next we are interested in the non-triviality of our Euler system in terms of L -values. To this end, in §3.4 we use some basic instances of Langlands functoriality to

deduce from the work of Eischen–Harris–Li–Skinner [EHLS20] the construction of a p -adic L -function

$$L_p(\mathrm{ad}^0(g_K) \otimes \psi) \in \mathrm{Frac} \Lambda^{\mathrm{ac}}$$

interpolating the central L -value $L(V, 0)$ and its twists by a p -adic family of anti-cyclotomic Hecke characters. Here Λ^{ac} is the Iwasawa algebra of the Galois group Γ^{ac} of the anticyclotomic $\mathbb{Z}_p$ -extension K_∞/K . Denoting by $\kappa_{\psi, \mathrm{ad}^0(g), \infty}$ the image of $\kappa_{\psi, \mathrm{ad}^0(g), 1, \infty}$ in $\mathrm{Sel}_{\mathrm{unb}}(K_\infty, T)$, we can then prove the following.

Write $(p) = \mathfrak{p}\bar{\mathfrak{p}}$, with $\mathfrak{p}$ the prime of K above p induced by ι_p . For the following result, let $K_{\infty, \bar{\mathfrak{p}}}$ be the $\mathbb{Z}_p$ -extension of K unramified outside $\bar{\mathfrak{p}}$, and let $\mathcal{F}_{\bar{\mathfrak{p}}}^{\mathrm{bal}}(T_{\mathrm{ad}(g)}^\psi)$ denote the subspace of $T_{\mathrm{ad}(g)}^\psi$ defined in Section 3.2.3.

Theorem B (Corollary 3.5.7). *Under some technical hypotheses on ψ , there is a Perrin-Riou big logarithm map $\mathfrak{Log} : H_{\mathrm{Iw}}^1(K_{\infty, \bar{\mathfrak{p}}}, \mathcal{F}_{\bar{\mathfrak{p}}}^{\mathrm{bal}}(T_{\mathrm{ad}(g)}^\psi)) \longrightarrow \mathbb{Z}_p^{\mathrm{ur}}[[\Gamma^{\mathrm{ac}}]]$ such that*

$$\mathfrak{Log}(\mathrm{res}_{\bar{\mathfrak{p}}}(\kappa_{\psi, \mathrm{ad}^0(g), \infty}))^2 = L_p(\mathrm{ad}^0(g_K) \otimes \psi) \cdot \mathcal{L}_{\mathfrak{p}}^{\mathrm{Katz}}(\psi)^{-, \iota}$$

up to multiplication by an element in $\overline{\mathbb{Q}_p}^\times$, where $\mathcal{L}_{\mathfrak{p}}^{\mathrm{Katz}}(\psi)^{-, \iota}$ is an anticyclotomic projection of Katz’s p -adic L -function.

The proof of this result builds on the explicit reciprocity law of [BSV22] and a factorization formula for Hsieh’s triple product p -adic L -function (see Theorem 3.4.9). This factorization is a p -adic manifestation of the Artin formalism arising from the decomposition

$$V_{\mathrm{ad}(g)}^\psi \simeq V \oplus V', \tag{3.1}$$

where $V' = \mathbb{Q}_p(\psi^{-1})(1 - k/2)$, and may be seen as an anticyclotomic analogue of Dasgupta’s factorization [Das16]. However, the proof in our case is largely simplified by the fact that the p -adic L -functions involved have overlapping ranges of p -adic interpolation.

The technical hypotheses on ψ are used to ensure that the congruence ideal of a Hida family attached to ψ is generated by a second anticyclotomic projection of Katz’s p -adic L -function, which in turn interpolates the ratio between two different types of periods.

3.1.3 Applications

Using Kolyvagin’s methods, as developed by Jetchev–Nekovář–Skinner [JNS] in the split anticyclotomic setting, we can deduce bounds on Selmer groups from the non-triviality of our Euler system. Our main result in this direction is the proof of new cases of the Bloch–Kato conjecture [BK90] in rank zero.

For the statement, we denote by ε_ℓ the epsilon factor attached to the Weil–Deligne representation associated with the restriction of $\mathrm{Ind}_K^\mathbb{Q}(V_{\mathrm{ad}(g)}^\psi)$ to $G_{\mathbb{Q}_\ell}$. It is then known that the sign $\varepsilon(V_{\mathrm{ad}(g)}^\psi)$ in the functional equation for $L(V_{\mathrm{ad}(g)}^\psi, s)$ is given by

$$\varepsilon(V_{\mathrm{ad}(g)}^\psi) = \prod_{\ell \leq \infty} \varepsilon_\ell,$$

where $\varepsilon_\infty = +1$ if $k \geq 2l$ and -1 if $2 \leq k < 2l$. On the other hand, here we say that V has “big image” if it satisfies the explicit conditions in Proposition 3.6.3.

Theorem C (Theorem 3.7.3). *In addition to the above hypotheses, assume that:*

- (a) $\varepsilon_\ell = +1$ for all primes $\ell \mid N_g N_\psi$,
- (b) $\gcd(N_g, N_\psi)$ is squarefree,
- (c) g is non-Eisenstein mod p ,
- (d) V has big image,
- (e) $L(\theta_\psi, k/2) \neq 0$.

If $k \geq 2l$ then the following implication holds:

$$L(V, 0) \neq 0 \implies \text{Sel}(K, V) = 0,$$

where $\text{Sel}(K, V)$ is the Bloch–Kato Selmer group.

Note that the hypotheses in Theorem C imply that $L(V, s)$ has sign $+1$ in its functional equation, and so the nonvanishing of $L(V, 0)$ is expected to hold generically.

We can also deduce applications to the Iwasawa main conjecture for V . More precisely, under certain hypotheses, Greenberg’s general formulation of the Iwasawa main conjecture for motives [Gre94] leads to the prediction that the unbalanced Selmer group $\text{Sel}_{\text{unb}}(K_{\infty}, A)$ defined in §3.2, where $A = V/T$, is Λ^{ac} -cotorsion, with characteristic ideal generated by $L_p(\text{ad}^0(g_K) \otimes \psi)$. In the direction of this conjecture we can prove the following, where we let $\mathbb{Z}_p^{\text{ur}}$ denote the completion of the ring of integers of the maximal unramified extension of $\mathbb{Q}_p$.

Theorem D (Theorem 3.7.5). *In addition to the above hypotheses, assume that:*

- (a) $\varepsilon_{\ell} = +1$ for all primes $\ell \mid N_g N_{\psi}$,
- (b) $\gcd(N_g, N_{\psi})$ is squarefree,
- (c) g is non-Eisenstein mod p ,
- (d) V has big image,
- (e) θ_{ψ} has global root number $\varepsilon(\theta_{\psi}) = +1$.

If the p -adic L -function $L_p(\text{ad}^0(g_K) \otimes \psi)$ is nonzero, then the Pontryagin dual of $\text{Sel}_{\text{unb}}(K_{\infty}, A)$ is Λ^{ac} -torsion, with

$$\text{Char}_{\Lambda^{\text{ac}}}(\text{Sel}_{\text{unb}}(K_{\infty}, A)^{\vee}) \supset (L_p(\text{ad}^0(g_K) \otimes \psi) \cdot \mathcal{L}_p^{\text{Katz}}(\psi)^{-, \iota})$$

in $\mathbb{Z}_p^{\text{ur}} \hat{\otimes}_{\mathbb{Z}_p} \Lambda^{\text{ac}}[1/p]$.

Note that the presence of $\mathcal{L}_p^{\text{Katz}}(\psi)^{-,\iota}$ in the divisibility of Theorem D is analogous to the appearance of the Kubota–Leopoldt p -adic L -function in the divisibility towards the Iwasawa main conjecture for the Galois representation attached to the symmetric square of a modular form in [LZ19, Thm. B].

In fact, the present work originated from an attempt to develop anticyclotomic analogues of the results in [op. cit.]. In particular, the idea of modifying the diagonal Euler system classes of [ACR21] to obtain the correct norm relations (see §3.5.1) was adopted from their work.

3.1.4 Outline of the paper

We begin by introducing in §3.2 our set-up and Galois representation of interest, and various Selmer groups associated with it. In §3.3 we describe in detail the construction of the diagonal cycle class giving rise to the bottom class of our Euler system, and study its behaviour according to a certain sign (given by $\varepsilon(\theta_\psi)$ in the notations of Theorem D). The results of this section, which are developed in a slightly more general setting than the rest of the paper, are unnecessary for the proof of our main results, but they are included here for completeness (in particular, Proposition 3.3.2 might be of independent interest). In §3.4 we introduce the different p -adic L -functions that appear in our picture, including an analogue of the Hida–Schmidt p -adic L -function deduced from the work of Eischen *et. al.* on p -adic L -functions for unitary groups, and prove the aforementioned analogue of Dasgupta’s factorization. Finally, in §3.5 we give the construction of our Euler system by suitably modifying the diagonal cycle Euler system classes constructed in our previous work [ACR21], and in §3.6 and §3.7 we apply this to deduce the arithmetic applications highlighted in the Introduction.

3.1.5 Acknowledgements

It is a pleasure to thank David Loeffler and Chris Skinner for their very valuable advice in connection with this work. We are also grateful to Ellen Eischen and Xin Wan for correspondence regarding the subject of this note, and to Shilin Lai and Sam Mundy for several helpful conversations. Finally, we thank the anonymous referees for a careful reading of the text, whose comments notably contributed to improve the exposition of the article.

During the preparation of this paper, F.C. was partially supported by the NSF grant DMS-1946136 and DMS-2101458; O.R. was supported by the Royal Society Newton International Fellowship NIF\R1\202208.

3.2 Galois representations and Selmer groups

In this section we introduce our Galois representations of interest and the Selmer groups associated with them that we shall be studying.

3.2.1 Galois representations

Let $g = \sum_{n=1}^{\infty} a_n(g)q^n \in S_l(N_g, \chi_g)$ be an ordinary newform of weight $l \geq 2$, level N_g , and nebentypus χ_g . Let $p > 2$ be a prime and let $E = L_{\mathfrak{P}}$ be a finite extension of $\mathbb{Q}_p$ with ring of integers $\mathcal{O}$ arising as the completion of the Hecke field L of g at a prime $\mathfrak{P}$ of L above p . By work of Eichler–Shimura and Deligne, there is a two-dimensional representation

$$\rho_g : G_{\mathbb{Q}} \longrightarrow \mathrm{GL}_E(V_g) \simeq \mathrm{GL}_2(E)$$

unramified outside pN_g and characterized by the property

$$\mathrm{trace} \rho_g(\mathrm{Fr}_q) = a_q(g)$$

for all primes $q \nmid pN_g$, where Fr_q denotes an arithmetic Frobenius element at q . Let $Y_1(N_g)$ be the open modular curve over $\mathbb{Q}$ parameterizing pairs (A, P) consisting of an elliptic curve A and a point $P \in A$ of order N_g . Let $\mathcal{L}_{l-2}$ is the sheaf introduced in [BSV22, §2.3]. As in [ACR21], we shall work with the geometric realization of V_g arising as the maximal quotient of

$$H_{\text{ét}}^1(Y_1(N_g)_{\overline{\mathbb{Q}}}, \mathcal{L}_{l-2}(1)) \otimes_{\mathbb{Z}_p} E$$

on which the dual Hecke operators T'_q and $\langle d \rangle'$ act as multiplication by $a_q(g)$ and $\chi_g(d)$ for all primes $q \nmid N_g$ and all $d \in (\mathbb{Z}/N_g\mathbb{Z})^\times$. We also let $T_g \subset V_g$ be the $\mathcal{O}$ -lattice defined by the natural image of

$$H_{\text{ét}}^1(Y_1(N_g)_{\overline{\mathbb{Q}}}, \mathcal{L}_{l-2}(1)) \otimes_{\mathbb{Z}_p} \mathcal{O}$$

under the quotient map $H_{\text{ét}}^1(Y_1(N_g)_{\overline{\mathbb{Q}}}, \mathcal{L}_{l-2}(1)) \otimes_{\mathbb{Z}_p} E \twoheadrightarrow V_g$.

Throughout the following, we shall assume that g is not of CM-type.

3.2.2 The adjoint representation

Let K be an imaginary quadratic field of discriminant $-D_K < 0$. Let ψ be a Hecke character of K of infinity type $(1 - k, 0)$ for some even integer $k \geq 2$ and central character equal to ε_K , the quadratic character attached to $K/\mathbb{Q}$ (thus the associated theta series θ_ψ has trivial nebentypus). We assume that ψ has conductor $\mathfrak{c} \subset \mathcal{O}_K$ prime to p and, upon enlarging $\mathcal{O}$ if necessary, that its p -adic avatar $\psi_{\mathfrak{p}}$ takes values in $\mathcal{O}$.

Definition 3.2.1. Let V be the E -valued G_K -representation given by

$$V := \text{ad}^0(V_g)(\psi_{\mathfrak{p}}^{-1})(1 - k/2),$$

where $\mathrm{ad}^0(V_g) \subset \mathrm{End}_E(V_g)$ denotes the adjoint representation on the trace-zero endomorphisms of V_g .

Let $g^* = g \otimes \chi_g^{-1}$ be the twist of g by the inverse of its nebentypus. We shall study the arithmetic of V by exploiting the decomposition

$$V_{\mathrm{ad}(g)}^\psi := V_g \otimes V_{g^*}(\psi_{\mathfrak{p}}^{-1})(1 - c) \simeq V \oplus V', \quad (3.2)$$

where $c = (k + 2l - 2)/2$ and $V' = E(\psi_{\mathfrak{p}}^{-1})(1 - k/2)$.

3.2.3 Selmer groups

From now on, we assume that p is a prime of good ordinary reduction for g such that

$$(p) = \mathfrak{p}\bar{\mathfrak{p}} \text{ splits in } K, \quad (3.3)$$

with $\mathfrak{p}$ the prime of K above p determined by our fixed embedding $\iota_p : \bar{\mathbb{Q}} \hookrightarrow \bar{\mathbb{Q}}_p$.

By p -ordinarity, the Galois representation V_g is equipped with a $G_{\mathbb{Q}_p}$ -stable filtration

$$0 \longrightarrow V_g^+ \longrightarrow V_g \longrightarrow V_g^- \longrightarrow 0$$

with $V_g^\pm$ one-dimensional and the $G_{\mathbb{Q}_p}$ -action on V_g^- given by the unramified character sending an arithmetic Frobenius Fr_p to α_g , the p -adic unit root of $x^2 - a_p(g)x + \chi_g(p)p^{l-1}$. Of course, twisting these by χ_g^{-1} we obtain $V_{g^*}^\pm = V_g^\pm \otimes \chi_g^{-1}$.

Let F/K be any finite extension and, for $v \mid p$ any prime of F above p , define

$$\mathcal{F}_v^{\mathrm{bal}}(V_{\mathrm{ad}(g)}^\psi) := \begin{cases} (V_g^+ \otimes V_{g^*} + V_g \otimes V_{g^*}^+)(\psi_{\mathfrak{p}}^{-1})(1 - c) & \text{if } v \mid \mathfrak{p}, \\ V_g^+ \otimes V_{g^*}^+(\psi_{\mathfrak{p}}^{-1})(1 - c) & \text{if } v \mid \bar{\mathfrak{p}}, \end{cases} \quad (3.4)$$

and

$$\mathcal{F}_v^{\text{unb}}(V_{\text{ad}(g)}^\psi) := \begin{cases} V_{\text{ad}(g)}^\psi & \text{if } v \mid \mathfrak{p}, \\ \{0\} & \text{if } v \mid \bar{\mathfrak{p}}, \end{cases} \quad (3.5)$$

and, for $? \in \{\text{bal}, \text{unb}\}$, put $\mathcal{F}_v^?(V) = \mathcal{F}_v^?(V_{\text{ad}(g)}^\psi) \cap V$ and $\mathcal{F}_v^?(V') = \mathcal{F}_v^?(V_{\text{ad}(g)}^\psi) \cap V'$.

Fix Σ any finite set of places of K containing ∞ and the primes dividing pN_gN_ψ . With a slight abuse of notation, for any finite extension of F/K we also denote by Σ the set of places of F lying over the places in Σ , and denote by $G_{F,\Sigma}$ the Galois group of the maximal extension of F unramified outside Σ . Further, for any non-archimedean field F_v , we write F_v^{nr} for the maximal unramified extension of F_v .

Definition 3.2.2. Let F/K be a finite extension, and for $M \in \{V_{\text{ad}(g)}^\psi, V, V'\}$ and $? \in \{\text{bal}, \text{unb}\}$ define the Selmer group $\text{Sel}_?(F, M)$ by

$$\text{Sel}_?(F, M) = \ker \left(H^1(G_{F,\Sigma}, M) \longrightarrow \prod_{v \mid p} \frac{H^1(F_v, M)}{H_?^1(F_v, M)} \times \prod_{v \in \Sigma, v \nmid p\infty} H^1(F_v^{\text{nr}}, M) \right),$$

where

$$H_?^1(F_v, M) = \text{im}(H^1(F_v, \mathcal{F}_v^?(M)) \longrightarrow H^1(F_v, M)).$$

We call $\text{Sel}_{\text{bal}}(F, M)$ (resp. $\text{Sel}_{\text{unb}}(F, M)$) the *balanced* (resp. *unbalanced*) Selmer group.

Remark 3.2.3. Let $f = \theta_\psi$ be the weight k eigenform associated with ψ , and denote by $V_{f,gg^*} := V_f \otimes V_g \otimes V_{g^*}(1-c)$ the Kummer self-dual twist of the Galois representation attached to (f, g, g^*) . Since $V_f = \text{Ind}_K^\mathbb{Q} \psi$, one can easily check that the isomorphism given by Shapiro's lemma

$$H^1(\mathbb{Q}, V_{f,gg^*}) \simeq H^1(K, V_{\text{ad}(g)}^\psi)$$

identifies the Selmer groups $\text{Sel}_{\text{bal}}(\mathbb{Q}, V_{f,gg^*})$ and $\text{Sel}_f(\mathbb{Q}, V_{f,gg^*})$ considered in [ACR21,

Def. 7.5] with the above $\text{Sel}_{\text{bal}}(K, V_{\text{ad}(g)}^\psi)$ and $\text{Sel}_{\text{unb}}(K, V_{\text{ad}(g)}^\psi)$, respectively.

Put $T_{\text{ad}(g)}^\psi = T_g \otimes T_{g^*}(\psi_{\mathfrak{p}}^{-1})(1 - c)$. Then the decomposition (3.2) induces a decomposition

$$T_{\text{ad}(g)}^\psi \simeq T \oplus T',$$

where T and T' are lattices in V and V' , respectively. We also set

$$A_{\text{ad}(g)}^\psi = V_{\text{ad}(g)}^\psi / T_{\text{ad}(g)}^\psi, \quad A = V/T, \quad A' = V'/T'.$$

Then, for $? \in \{\text{bal}, \text{unb}\}$ and $M \in \{T_{\text{ad}(g)}^\psi, T, T', A_{\text{ad}(g)}^\psi, A, A'\}$, we define the local conditions $H_?^1(F_v, M)$ from the local conditions above by propagation, and use them to define the Selmer groups $\text{Sel}_?(K, M)$ using the same recipe as in Definition 3.2.2. Finally, for $M_1 \in \{T_{\text{ad}(g)}^\psi, T, T'\}$ and $M_2 \in \{A_{\text{ad}(g)}^\psi, A, A'\}$, we put

$$\text{Sel}_?(K_\infty, M_1) := \varprojlim_n \text{Sel}_?(K_n, M_1), \quad \text{Sel}_?(K_\infty, M_2) := \varinjlim_n \text{Sel}_?(K_n, M_2),$$

where the limits are with respect to corestriction and restriction, respectively.

To help orient the reader, we note the following simple relation between the different Selmer groups introduced above.

Proposition 3.2.4. *The decomposition $V_{\text{ad}(g)}^\psi = V \oplus V'$ induces isomorphisms*

$$\text{Sel}_{\text{bal}}(K_\infty, T_{\text{ad}(g)}^\psi) \simeq \text{Sel}_{\text{bal}}(K_\infty, T) \oplus \text{Sel}(K_\infty, T'),$$

$$\text{Sel}_{\text{unb}}(K_\infty, T_{\text{ad}(g)}^\psi) \simeq \text{Sel}_{\text{unb}}(K_\infty, T) \oplus \text{Sel}(K_\infty, T'),$$

where $\text{Sel}(K_\infty, T')$ is the Bloch–Kato Selmer group for T' .

Proof. It suffices to show that for any finite extension F/K we have

$$\text{Sel}_{\text{bal}}(F, V') \simeq \text{Sel}_{\text{unb}}(F, V') \simeq \text{Sel}(F, V'),$$

where $\text{Sel}(F, V')$ is the Bloch–Kato Selmer group of $V' = E(\psi_{\mathfrak{p}}^{-1})$, which is given by

$$\text{Sel}(F, V') = \ker \left(H^1(G_{F, \Sigma}, V') \rightarrow \prod_{v|\bar{\mathfrak{p}}} H^1(F_v, V') \times \prod_{v \in \Sigma, v \nmid p} H^1(F_v^{\text{nr}}, V') \right)$$

(see [AH06, §1.1] or [Arn07, §1.2]). For $\text{Sel}_{\text{unb}}(F, V')$ this is clear from (3.5); for $\text{Sel}_{\text{bal}}(F, V')$ it follows by noting that the subspace $\mathcal{F}_v^{\text{bal}}(V_{\text{ad}(g)}^\psi) \subset V_{\text{ad}(g)}^\psi$ in (3.4) contains V' for $v \mid \mathfrak{p}$ and intersects trivially with it for $v \nmid \bar{\mathfrak{p}}$. $\square$

3.3 Construction of the bottom class

In this section, we recall the construction of a Λ -adic cohomology class associated with the triple product of three modular forms as explained in [BSV22]. We follow the exposition in [*op. cit.*] with slight modifications and specializing the discussion to the case of interest in this paper. At the end of this section we analyze the behaviour of this cohomology class depending on the sign of one of the modular forms.

This section is independent of the rest of the paper, and the reader solely interested in the results stated in the Introduction can proceed to Section 3.4.

Let f and g be newforms of weight $k = r_1 + 2$ and $l = r_2 + 2$, level N_f and N_g and character $\chi_f = 1$ and χ_g , respectively. We assume that $p \nmid 2N_f N_g$ and that both f and g are ordinary at p . We denote by $h = g^*$ the newform obtained by conjugating the Fourier coefficients of g . Let L be a finite extension of $\mathbb{Q}$ containing the Fourier coefficients of f and g and let $E = L_{\mathfrak{P}}$ be its completion at a prime $\mathfrak{P}$ above p , with ring of integers $\mathcal{O}$. Define $N = \text{lcm}(N_f, N_g)$.

Consider the Iwasawa algebra $\Lambda = \mathbb{Z}_p[[1 + p\mathbb{Z}_p]]$. There exist finite flat Λ -modules $\Lambda_{\mathbf{f}}$ and $\Lambda_{\mathbf{g}}$ and primitive Hida families $\mathbf{f} \in \Lambda_{\mathbf{f}}[[q]]$ and $\mathbf{g} \in \Lambda_{\mathbf{g}}[[q]]$ passing through the ordinary p -stabilizations f_α and g_α of f and g , respectively. Let $\mathbf{h} = \mathbf{g}^*$ be the Hida family $\mathbf{g} \otimes \chi_g^{-1}$, which passes through the ordinary p -stabilization g_α^* of g^* . Our conventions for Hida families are those described in [ACR21, §5.1].

Let μ_{p-1} denote the group of roots of unity in $\mathbb{Z}_p^\times$ and consider the decomposition $\mathbb{Z}_p^\times \cong \mu_{p-1} \times (1+p\mathbb{Z}_p)$. Let $\omega : \mathbb{Z}_p^\times \rightarrow \mu_{p-1} \subset \mathbb{Z}_p^\times$ be the map defined by projection onto the first factor, according to the previous decomposition. Also, for an element $z \in \mathbb{Z}_p^\times$, we denote by $\langle z \rangle$ its projection onto the second factor (alternatively, $\langle z \rangle = z/\omega(z)$).

Let $\text{Cont}(\mathbb{Z}_p, \Lambda)$ be the Λ -module of continuous functions on $\mathbb{Z}_p$ with values in Λ . To make notation less cumbersome, we denote by $[z]$ the group-like element $[\langle z \rangle]$ in Λ . For each integer i , let $\kappa_i : \mathbb{Z}_p^\times \rightarrow \Lambda^\times$ be the character defined by $z \mapsto \omega^i(z)[z]$. We also define the sets $\mathbb{T} = \mathbb{Z}_p^\times \times \mathbb{Z}_p$ and $\mathbb{T}' = p\mathbb{Z}_p \times \mathbb{Z}_p^\times$. Then, we can define the Λ -modules

$$\begin{aligned} \mathcal{A}_i &= \{f : \mathbb{T} \rightarrow \Lambda \mid f(1, z) \in \text{Cont}(\mathbb{Z}_p, \Lambda) \text{ and } f(a \cdot t) = \kappa_i(a) \cdot f(t) \text{ for all } a \in \mathbb{Z}_p^\times, t \in \mathbb{T}\}, \\ \mathcal{A}'_i &= \{f : \mathbb{T}' \rightarrow \Lambda \mid f(pz, 1) \in \text{Cont}(\mathbb{Z}_p, \Lambda) \text{ and } f(a \cdot t) = \kappa_i(a) \cdot f(t) \text{ for all } a \in \mathbb{Z}_p^\times, \gamma \in \mathbb{T}'\}, \end{aligned}$$

$$\mathcal{D}_i = \text{Hom}_{\text{cont}, \Lambda}(\mathcal{A}_i, \Lambda), \quad \mathcal{D}'_i = \text{Hom}_{\text{cont}, \Lambda}(\mathcal{A}'_i, \Lambda).$$

We define in addition characters $\kappa_f^*, \kappa_g^*, \kappa_h^*, \kappa^* : \mathbb{Z}_p^\times \rightarrow \Lambda \hat{\otimes} \Lambda \hat{\otimes} \Lambda$ by

$$\begin{aligned} \kappa_f^*(z) &= \omega^{r_2-r_1/2}(z)[z]^{-1/2} \otimes [z]^{1/2} \otimes [z]^{1/2} \\ \kappa_g^*(z) &= \omega^{r_1/2}(z)[z]^{1/2} \otimes [z]^{-1/2} \otimes [z]^{1/2} \\ \kappa_h^*(z) &= \omega^{r_1/2}(z)[z]^{1/2} \otimes [z]^{1/2} \otimes [z]^{-1/2} \\ \kappa^*(z) &= \omega^{r_1/2+r_2}(z)[z]^{1/2} \otimes [z]^{1/2} \otimes [z]^{1/2}. \end{aligned}$$

We denote by κ^* the character of the Galois group $G_{\mathbb{Q}}$ defined by $\kappa^* = \kappa^* \circ \epsilon_{\text{cyc}}$, and similarly for the other characters introduced above.

Let $Y = Y_1(N, p)$ denote the same modular curve as in [BSV22, §8.1] and let $\Gamma = \Gamma_1(N, p)$ be the corresponding modular group. The function

$$\mathbf{Det} : \mathbb{T}' \times \mathbb{T} \times \mathbb{T} \longrightarrow \Lambda \hat{\otimes} \Lambda \hat{\otimes} \Lambda,$$

defined as in [loc. cit.], yields an element in the group

$$H_{\text{ét}}^0(Y, \mathcal{A}'_{r_1} \otimes \mathcal{A}_{r_2} \otimes \mathcal{A}_{r_2}(-\kappa^*)).$$

Here $\mathcal{A}'_{r_1}$ and $\mathcal{A}_{r_2}$ denote the étale sheaves associated with $\mathcal{A}'_{r_1}$ and $\mathcal{A}_{r_2}$, respectively, as explained in [BSV22, §4.2]. Then, with essentially the same notations as in [op. cit., §8.1], we define the class

$$\kappa^{(1)} = \frac{1}{a_p(\mathbf{f})} \mathbf{s}_{\mathbf{fgh}} \circ (e_{\text{ord}} \otimes e_{\text{ord}} \otimes e_{\text{ord}}) \circ (w_p \otimes 1 \otimes 1) \circ \mathbf{K} \circ \mathbf{HS} \circ d_*(\mathbf{Det})$$

inside the group

$$H^1(\mathbb{Q}, H^1(\Gamma, \mathcal{D}'_{r_1})^{\text{ord}} \hat{\otimes} H^1(\Gamma, \mathcal{D}'_{r_2})^{\text{ord}} \hat{\otimes} H^1(\Gamma, \mathcal{D}'_{r_2})^{\text{ord}} (2 - \kappa^*)).$$

For a $\mathbb{Z}_p$ -algebra A , let $L_{r_2}(A)$ be defined as in [op. cit., p. 17]. We will sometimes denote $L_{r_2}(\mathbb{Z}_p)$ simply by L_{r_2} . Let $\kappa_f^{1/2} : \mathbb{Z}_p^\times \rightarrow \Lambda_{\mathbf{f}}^\times$ denote the map $z \mapsto \omega^{r_1/2}(z)[z]^{1/2}$ and let $\kappa_f^{1/2} = \kappa_f^{1/2} \circ \epsilon_{\text{cyc}}$. According to [op. cit., eq. (90)], the Λ -module $H^1(\Gamma, \mathcal{D}'_{r_2})^{\text{ord}}$ specializes to $H^1(\Gamma, L_{r_2})^{\text{ord}}$ at weight $l = r_2 + 2$. Therefore, the class $\kappa^{(1)}$ yields a class

$$\kappa^{(2)} \in H^1(\mathbb{Q}, H^1(\Gamma, \mathcal{D}'_{r_1})^{\text{ord}} \otimes H^1(\Gamma, L_{r_2})^{\text{ord}} \otimes H^1(\Gamma, L_{r_2})^{\text{ord}} (2 - r_2 - \kappa_f^{1/2})).$$

We define Hecke operators $T'_q, [d]'_N$ acting on group cohomology as in [ACR21, §5.3]. Let $\mathbb{V}_{\mathbf{f}}(N)$ be the maximal quotient of $H^1(\Gamma, \mathcal{D}'_{r_1})^{\text{ord}}(1) \otimes_{\Lambda} \Lambda_{\mathbf{f}}$ on which the Hecke operators T'_q for primes $q \nmid N$ act as multiplication by $a_q(\mathbf{f})$ and the diamond operators $[d]'_N$ act as multiplication by $\chi_f(d)$ (actually, the character χ_f is trivial in our case). We define $T_g(N)$ and $T_{g^*}(N)$ in a similar way as quotients of $H^1(\Gamma, L_{r_2}(\mathcal{O}))^{\text{ord}}(1)$. Also, let $\mathbb{V}_{\mathbf{f}}$ be the maximal quotient of $H^1(\Gamma_1(N_f, p), \mathcal{D}'_{r_1})^{\text{ord}}(1) \otimes_{\Lambda}$

$\Lambda_{\mathbf{f}}$ on which the Hecke operators T'_q act as multiplication by $a_q(\mathbf{f})$ and the diamond operators $[d]'_N$ act as multiplication by $\chi_f(d)$ and define T_g and T_{g^*} in a similar way as quotients of $H^1(\Gamma_1(N_g, p), L_{r_2}(\mathcal{O}))^{\text{ord}}(1)$.

To shorten notation, we define

$$\begin{aligned}\mathbb{V}(\mathbf{f}, g, g^*) &= \mathbb{V}_{\mathbf{f}} \otimes T_g \otimes T_{g^*}(-1 - r_2 - \kappa_f^{1/2}), \\ \mathbb{V}(\mathbf{f}, g, g^*)(N) &= \mathbb{V}_{\mathbf{f}}(N) \otimes T_g(N) \otimes T_{g^*}(N)(-1 - r_2 - \kappa_f^{1/2}), \\ \mathbb{V}(\mathbf{f}, g, g^*)_f &= \mathbb{V}_{\mathbf{f}}^- \otimes T_g^+ \otimes T_{g^*}^+(-1 - r_2 - \kappa_f^{1/2}), \\ \mathbb{V}(\mathbf{f}, g, g^*)_f(N) &= \mathbb{V}_{\mathbf{f}}^-(N) \otimes T_g^+(N) \otimes T_{g^*}^+(N)(-1 - r_2 - \kappa_f^{1/2}).\end{aligned}$$

We also introduce

$$\begin{aligned}M(\mathbf{f}, g, g^*)_f &= \mathbb{V}_{\mathbf{f}}^- \hat{\otimes} T_g^+ \hat{\otimes} T_{g^*}^+(-2 - 2r_2) \hat{\otimes} \Lambda(\kappa^{-1})[1/p], \\ M(\mathbf{f}, g, g^*)_f(N) &= \mathbb{V}_{\mathbf{f}}^-(N) \hat{\otimes} T_g^+(N) \hat{\otimes} T_{g^*}^+(N)(-2 - 2r_2) \hat{\otimes} \Lambda(\kappa^{-1})[1/p],\end{aligned}$$

where $\kappa : G_{\mathbb{Q}} \rightarrow \Lambda^{\times}$ is defined by $\kappa(\sigma) = \omega^{r_1/2 - r_2 - 1}(\epsilon_{\text{cyc}}(\sigma))[\epsilon_{\text{cyc}}(\sigma)]$.

The class $\kappa^{(2)}$ yields a class

$$\kappa^{(2)}(\mathbf{f}, g, g^*) \in H^1(\mathbb{Q}, \mathbb{V}(\mathbf{f}, g, g^*)(N)).$$

This is the class defined in [BSV22, eq. 155] specialized to weight l in the second and third factors. It follows from [*op. cit.*, Cor. 8.2] that the restriction at p of this class belongs to the group

$$H_{\text{bal}}^1(\mathbb{Q}_p, \mathbb{V}(\mathbf{f}, g, g^*)(N)).$$

Let $S_{\Lambda_{\mathbf{f}}}^{\text{ord}}(N, \omega^{r_1})$ denote the space of Hida families of tame level N , character ω^{r_1} and with coefficients in $\Lambda_{\mathbf{f}}$. We denote by $S_{\Lambda_{\mathbf{f}}}^{\text{ord}}(N, \omega^{r_1})[\mathbf{f}]$ the subspace of $S_{\Lambda_{\mathbf{f}}}^{\text{ord}}(N, \omega^{r_1})$ on which the Hecke operators U_p and T_{ℓ} for $\ell \nmid N$ act with the same eigenvalues as

on $\mathbf{f}$. Let $S_l(\Gamma_1(N, p), \chi_g)[g_\alpha]$ denote the space of modular forms of weight l , level $\Gamma_1(N, p)$ and nebentypus χ_g which are eigenforms for the Hecke operators U_p and T_ℓ for $\ell \nmid N$ with the same eigenvalues as g_α . We similarly define $S_l(\Gamma_1(N, p), \chi_g^{-1})[g_\alpha^*]$. Then, a choice of level- N test vectors $\check{\mathbf{f}}$, $\check{g}$ and $\check{h}$ for $\mathbf{f}$, g_α and g_α^* , respectively, is a choice of elements

$$\check{\mathbf{f}} \in S_{\Lambda}^{\text{ord}}(N, \omega^{r_1})[\mathbf{f}], \quad \check{g} \in S_l(\Gamma_1(N, p), \chi_g)[g_\alpha], \quad \check{h} \in S_l(\Gamma_1(N, p), \chi_g^{-1})[g_\alpha^*],$$

each of which can be written, in terms of their q -expansions, as

$$\begin{aligned} \check{\mathbf{f}}(q) &= \sum_{0 < d | N/N_f} r_d^{\check{\mathbf{f}}} \cdot \mathbf{f}(q^d), \\ \check{g}(q) &= \sum_{0 < d | N/N_g} r_d^{\check{g}} \cdot g_\alpha(q^d), \\ \check{h}(q) &= \sum_{0 < d | N/N_g} r_d^{\check{h}} \cdot g_\alpha^*(q^d), \end{aligned}$$

with $r_d^{\check{\mathbf{f}}} \in \Lambda_{\mathbf{f}}$ and $r_d^{\check{g}}, r_d^{\check{h}} \in \mathcal{O}$. Let

$$\varpi_{\mathbf{f}}^* : S_{\Lambda_{\mathbf{f}}}^{\text{ord}}(N_f, \omega^{r_1}) \longrightarrow S_{\Lambda_{\mathbf{f}}}^{\text{ord}}(N, \omega^{r_1})$$

denote the map defined by

$$\Phi(q) \mapsto \sum_{0 < d | N/N_f} r_d^{\check{\mathbf{f}}} \cdot \Phi(q^d).$$

Similarly, we define

$$\begin{aligned} \varpi_{\check{g}}^* : S_l(\Gamma_1(N_g, p), \chi_g) &\longrightarrow S_l(\Gamma_1(N, p), \chi_g), \\ \varpi_{\check{h}}^* : S_l(\Gamma_1(N_g, p), \chi_g^{-1}) &\longrightarrow S_l(\Gamma_1(N, p), \chi_g^{-1}). \end{aligned}$$

Therefore, we can write $\check{\mathbf{f}} = \varpi_{\mathbf{f}}^*(\mathbf{f})$, $\check{g} = \varpi_g^*(g_\alpha)$ and $\check{h} = \varpi_h^*(g_\alpha^*)$. At the same time, for each $d \mid N/N_f$, the map $v_d : Y_1(N, p) \rightarrow Y_1(N_f, p)$, corresponding to multiplication by d on the complex upper half-plane under the standard complex uniformizations, yields a pushforward map

$$v_{d*} : H^1(\Gamma_1(N, p), \mathcal{D}'_{r_1}) \longrightarrow H^1(\Gamma_1(N_f, p), \mathcal{D}'_{r_1})$$

which induces a map

$$v_{d*} : \mathbb{V}_{\mathbf{f}}(N) \longrightarrow \mathbb{V}_{\mathbf{f}}.$$

Let $\varpi_{\check{\mathbf{f}}*} = \sum_{0 < d \mid N/N_f} r_d^{\check{\mathbf{f}}} v_{d*}$. Let $\eta_{\check{\mathbf{f}}}$ and $\eta_{\mathbf{f}}$ denote the differentials attached to $\check{\mathbf{f}}$ and $\mathbf{f}$, respectively, in [BSV22, eq. (122)]. Then,

$$\langle x, \eta_{\check{\mathbf{f}}} \rangle = \langle \varpi_{\check{\mathbf{f}}*}(x), \eta_{\mathbf{f}} \rangle \quad \text{for all } x \in \mathbb{V}_{\mathbf{f}}^-(N).$$

Similarly, we can define maps

$$\varpi_{\check{g}*}, \varpi_{\check{h}*} : H^1(\Gamma_1(N, p), L_{r_2}) \longrightarrow H^1(\Gamma(N_g, p), L_{r_2}),$$

which induce maps

$$\varpi_{\check{g}*} : T_g(N) \longrightarrow T_g, \quad \varpi_{\check{h}*} : T_{g^*}(N) \longrightarrow T_{g^*}.$$

Let $\omega_{\check{g}}$, ω_{g_α} , $\omega_{\check{h}}$ and $\omega_{g_\alpha^*}$ denote the differentials attached to $\check{g}$, g_α , $\check{h}$ and g_α^* , respectively, in [BSV22, eq. (30)]. Then,

$$\langle x, \omega_{\check{g}} \rangle = \langle \varpi_{\check{g}*}(x), \omega_{g_\alpha} \rangle \quad \text{for all } x \in T_g^+(N),$$

$$\langle x, \omega_{\check{h}} \rangle = \langle \varpi_{\check{h}*}(x), \omega_{g_\alpha^*} \rangle \quad \text{for all } x \in T_{g^*}^+(N)$$

For a choice of level- N test vectors $\check{\mathbf{f}} = \varpi_{\mathbf{f}}^*(\mathbf{f})$, $\check{g} = \varpi_g^*(g_\alpha)$, $\check{h} = \varpi_h^*(g_\alpha^*)$, we have a map

$$\mathfrak{Log}(\check{\mathbf{f}}, \check{g}, \check{h}) : H_{\text{bal}}^1(\mathbb{Q}_p, \mathbb{V}(\mathbf{f}, g, g^*)(N)) \longrightarrow \Lambda_{\mathbf{f}}[1/p]$$

obtained from the map defined in [BSV22, Prop. 7.3] by specializing to weight l the second and third variables. It follows from [*op. cit.*, Thm. A] that the image of $\text{res}_p(\kappa^{(2)}(\mathbf{f}, g, h))$ under the map above is an element $\mathcal{L}_p(\check{\mathbf{f}}, \check{g}, \check{h}) \in \Lambda_{\mathbf{f}}[1/p]$ such that, for all $k' \geq 2l$ satisfying $k' \equiv k \pmod{2(p-1)}$,

$$\mathcal{L}_p(\check{\mathbf{f}}, \check{g}, \check{h})(k') = \frac{\langle \check{\mathbf{f}}_{k'}^w, \delta^t \check{g} \times \check{h} \rangle_{Np}}{\langle \check{\mathbf{f}}_{k'}^w, \check{\mathbf{f}}_{k'}^w \rangle_{Np}}.$$

Let $\bar{\mathcal{L}}_{\check{\mathbf{f}}\check{g}\check{h}}$ be the map defined in [BSV22, Prop. 7.1] specialized to weight l in the second and third variables. Then, we obtain a map

$$\langle \bar{\mathcal{L}}_{\check{\mathbf{f}}\check{g}\check{h}}(-), \eta_{\check{\mathbf{f}}} \omega_{\check{g}} \omega_{\check{h}} \rangle : H^1(\mathbb{Q}_p, M(\mathbf{f}, g, g^*)_f(N)) \longrightarrow \Lambda_{\mathbf{f}} \hat{\otimes} \Lambda[1/p].$$

The map $\mathfrak{Log}(\check{\mathbf{f}}, \check{g}, \check{h})$ is obtained by composing the natural projection

$$H_{\text{bal}}^1(\mathbb{Q}_p, \mathbb{V}(\mathbf{f}, g, g^*)(N)) \longrightarrow H^1(\mathbb{Q}_p, \mathbb{V}(\mathbf{f}, g, g^*)_f(N))$$

with a suitable specialization of the map above.

Now, from the previous discussion, we have that

$$\langle \bar{\mathcal{L}}_{\check{\mathbf{f}}\check{g}\check{h}}(-), \eta_{\check{\mathbf{f}}} \omega_{\check{g}} \omega_{\check{h}} \rangle = \langle \bar{\mathcal{L}}_{\mathbf{f}g_\alpha g_\alpha^*}((\varpi_{\check{\mathbf{f}},*} \otimes \varpi_{\check{g},*} \otimes \varpi_{\check{h},*})(-)), \eta_{\mathbf{f}} \omega_{g_\alpha} \omega_{g_\alpha^*} \rangle,$$

where the map $\bar{\mathcal{L}}_{\mathbf{f}g_\alpha g_\alpha^*}$ is defined in a way analogous to the way in which the map $\bar{\mathcal{L}}_{\check{\mathbf{f}}\check{g}\check{h}}$ is defined in [BSV22, Prop. 7.1]. Therefore, as before, the composition of the natural projection

$$H_{\text{bal}}^1(\mathbb{Q}_p, \mathbb{V}(\mathbf{f}, g, g^*)) \longrightarrow H^1(\mathbb{Q}_p, \mathbb{V}(\mathbf{f}, g, g^*)_f)$$

with a suitable specialization of the map

$$\langle \bar{\mathcal{L}}_{\mathbf{f}g_\alpha g_\alpha^*}(-), \eta_{\mathbf{f}} \omega_{g_\alpha} \omega_{g_\alpha^*} \rangle : H^1(\mathbb{Q}_p, M(\mathbf{f}, g, g^*)_f) \longrightarrow \Lambda_{\mathbf{f}} \hat{\otimes} \Lambda[1/p]$$

yields a map

$$\mathfrak{Log}(\mathbf{f}, g_\alpha, g_\alpha^*) : H_{\text{bal}}^1(\mathbb{Q}_p, \mathbb{V}(\mathbf{f}, g, g^*)) \longrightarrow \Lambda_{\mathbf{f}}[1/p].$$

Moreover, for any choice of test vectors $\check{\mathbf{f}}, \check{g}, \check{h}$ as above, we have

$$\mathfrak{Log}(\mathbf{f}, g_\alpha, g_\alpha^*) \left(\text{res}_p \left((\varpi_{\check{\mathbf{f}},*} \otimes \varpi_{\check{g},*} \otimes \varpi_{\check{h},*}) (\kappa^{(2)}(\mathbf{f}, g, g^*)) \right) \right) = \mathcal{L}_p(\check{\mathbf{f}}, \check{g}, \check{h}).$$

It follows from [Hsi21, §3.5-6] and [*op. cit.*, Thm. 7.1] that there exist level- N test vectors $\check{\mathbf{f}}, \check{g}, \check{h}$ for which, under some technical assumptions, we have a precise formula for the specializations of $\mathcal{L}_p(\check{\mathbf{f}}, \check{g}, \check{h})$ at even weights $k' \geq 2l$. We fix such test vectors. Then, we define

$$\kappa^{(3)} = (\varpi_{\check{\mathbf{f}},*} \otimes \varpi_{\check{g},*} \otimes \varpi_{\check{h},*}) \kappa^{(2)}$$

in the group

$$H^1(\mathbb{Q}, H^1(\Gamma_1(N_f, p), \mathcal{D}'_{r_1})^{\text{ord}} \hat{\otimes} H^1(\Gamma_1(N_g, p), L_{r_2})^{\text{ord}} \hat{\otimes} H^1(\Gamma_1(N_g, p), L_{r_2})^{\text{ord}} (2-r_2-\kappa_f^{1/2}))$$

and let

$$\kappa^{(3)}(\mathbf{f}, g, g^*) \in H^1(\mathbb{Q}, \mathbb{V}_{\mathbf{f}}(-\kappa_f^{1/2}) \hat{\otimes} \text{ad}(T_g))$$

be the class obtained from $\kappa^{(3)}$ by projection to the isotypic quotients for $\mathbf{f}, g$ and g^* .

Then

$$\mathfrak{Log}(\mathbf{f}, g_\alpha, g_\alpha^*) \left(\text{res}_p \left(\kappa^{(3)}(\mathbf{f}, g, g^*) \right) \right) = \mathcal{L}_p(\check{\mathbf{f}}, \check{g}, \check{h}).$$

Observe that the map $w_{N_g} : H^1(\Gamma_1(N_g, p), L_{r_2}(\mathcal{O}))^{\text{ord}} \rightarrow H^1(\Gamma_1(N_g, p), L_{r_2}(\mathcal{O}))^{\text{ord}}$ defined in [BSV22, §4.1.2] descends to a map $w_{N_g} : T_g \rightarrow T_{g^*}$. Taking the Galois

action into account, this is actually a map $T_g \rightarrow T_{g^*}(\chi_g)$. Similarly, we have a map $w_{N_g} : T_{g^*} \rightarrow T_g(\chi_g^{-1})$. (We are denoting all these maps in the same way in the hope that this will not cause any confusion.)

Let $s : T_g \otimes T_{g^*} \rightarrow T_{g^*} \otimes T_g$ be the map which interchanges the two factors. Then, the composition $\tilde{s} = (-N_g)^{-r_2} s \circ (w_{N_g}, w_{N_g})$ defines an endomorphism of $\text{ad}(T_g) = T_g \otimes T_{g^*}(-1 - r_2)$. This endomorphism is in fact an involution.

Lemma 3.3.1. *Consider the direct sum decomposition $\text{ad}(T_g) = \text{ad}^0(T_g) \oplus 1$. Then:*

1. $\text{ad}^0(T_g)$ is the 1-eigenspace for $\tilde{s}$;
2. 1 is the -1 -eigenspace for $\tilde{s}$.

Proof. As in [BSV22, p. 19], there is a bilinear form $L_{r_2}(\mathcal{O}) \otimes L_{r_2}(\mathcal{O}) \rightarrow \mathcal{O} \otimes \det^{-r_2}$. Via cup-product and the isomorphism $H_{\text{par}}^2(\Gamma_1(N_g, p), \mathcal{O}) \simeq \mathcal{O}(1)$, we obtain a pairing

$$H_{\text{par}}^1(\Gamma_1(N_g, p), L_{r_2}(\mathcal{O}))^{\text{ord}} \times H_{\text{par}}^1(\Gamma_1(N_g, p), L_{r_2}(\mathcal{O}))^{\text{ord}} \longrightarrow \mathcal{O}(r_2 + 1),$$

where H_{par}^1 stands for parabolic cohomology as defined in [GS93, p. 427]. Since cup-product is anti-commutative in degree 1, the pairing above satisfies $\langle \alpha, \beta \rangle = (-1)^{r_2+1} \langle \beta, \alpha \rangle$ for any α, β in $H_{\text{par}}^1(\Gamma_1(N_g, p), L_{r_2}(\mathcal{O}))^{\text{ord}}$. On the other hand, the operator w_{N_g} acting on $H_{\text{par}}^1(\Gamma_1(N_g, p), L_{r_2}(\mathcal{O}))^{\text{ord}}$ satisfies $w_{N_g}^2 = (-N_g)^{r_2}$ and $\langle w_{N_g} \alpha, w_{N_g} \beta \rangle = N_g^{r_2} \langle \alpha, \beta \rangle$ for any elements $\alpha, \beta \in H_{\text{par}}^1(\Gamma_1(N_g, p), L_{r_2}(\mathcal{O}))^{\text{ord}}$. Therefore we have

$$\langle \alpha, w_{N_g} \beta \rangle = \frac{1}{N_g^{r_2}} \langle w_{N_g} \alpha, w_{N_g}^2 \beta \rangle = (-1)^{r_2} \langle w_{N_g} \alpha, \beta \rangle = -\langle \beta, w_{N_g} \alpha \rangle.$$

In particular, we deduce that $\langle \alpha, w_{N_g} \alpha \rangle = 0$.

We can realize T_g (resp. T_{g^*}) as the maximal quotient of $H_{\text{par}}^1(\Gamma_1(N_g, p), L_{r_2}(\mathcal{O}))^{\text{ord}}$ on which the Hecke operators T'_q act as multiplication by $a_q(g)$ (resp. $a_q(g^*)$) and the diamond operators $[d]_{N_g}'$ act as multiplication by $\chi_g(d)$ (resp. $\chi_g(d)^{-1}$). Thus we

obtain a commutative diagram

$$\begin{array}{ccc}
H_{\text{par}}^1(\Gamma_1(N_g, p), L_{r_2}(\mathcal{O}))^{\text{ord}} \times H_{\text{par}}^1(\Gamma_1(N_g, p), L_{r_2}(\mathcal{O}))^{\text{ord}} & \longrightarrow & \mathcal{O}(r_2 + 1) \\
\downarrow & & \parallel \\
T_g \times T_{g^*} & \longrightarrow & \mathcal{O}(r_2 + 1).
\end{array}$$

Therefore, for any elements $\alpha, \beta \in T_g$, we have $\langle \alpha, w_{N_g} \beta \rangle = -\langle \beta, w_{N_g} \alpha \rangle$. The lemma follows easily from this. $\square$

We will assume in the remaining of this section that $N_g \mid N_f$, so that $N = N_f$. Under this assumption, our test vectors are $\check{\mathbf{f}} = \mathbf{f}$, $\check{g}(q) = \pi_1^*(g_\alpha)$ and $\check{h} = \pi_2^*(g_\alpha^*)$, up to multiplication by some constants in $\text{Frac } \Lambda_{\mathbf{f}}$ which do not affect the discussion that follows.

Let s_{N_g} denote the operator which acts on the group

$$H^1(\mathbb{Q}, H^1(\Gamma, \mathcal{D}'_{r_1})^{\text{ord}} \hat{\otimes} H^1(\Gamma_1(N_g, p), L_{r_2})^{\text{ord}} \hat{\otimes} H^1(\Gamma_1(N_g, p), L_{r_2})^{\text{ord}} (2 - r_2 - \kappa_f^{1/2}))$$

by interchanging the second and third factors and define $\tilde{s}_{N_g} = (-N_g)^{-r_2} s_{N_g} \circ (1 \otimes w_{N_g} \otimes w_{N_g})$.

Proposition 3.3.2. *The class $\kappa^{(3)}(\mathbf{f}, g, g^*)$ satisfies*

$$\tilde{s}_{N_g}(\kappa^{(3)}(\mathbf{f}, g, g^*)) = -[N]^{-1/2} (w_N \otimes 1 \otimes 1) \kappa^{(3)}(\mathbf{f}, g, g^*).$$

In particular, when we consider the direct sum decomposition

$$H^1(\mathbb{Q}, \mathbb{V}_{\mathbf{f}}(-\kappa_f^{1/2}) \hat{\otimes} \text{ad}(T_g)) = H^1(\mathbb{Q}, \mathbb{V}_{\mathbf{f}}(-\kappa_f^{1/2}) \hat{\otimes} \text{ad}^0(T_g)) \oplus H^1(\mathbb{Q}, \mathbb{V}_{\mathbf{f}}(-\kappa_f^{1/2})),$$

the class $\kappa^{(3)}(\mathbf{f}, g, g^*)$ lies in the summand

1. $H^1(\mathbb{Q}, \mathbb{V}_{\mathbf{f}}(-\kappa_f^{1/2}) \hat{\otimes} \text{ad}^0(T_g))$, if $\varepsilon(f) = 1$;

2. $H^1(\mathbb{Q}, \mathbb{V}_{\mathbf{f}}(-\kappa_f^{1/2}))$, if $\varepsilon(f) = -1$.

Proof. We have the following commutative diagram

$$\begin{array}{ccc} H_{\text{ét}}^0(Y, \mathcal{A}'_{r_1} \otimes \mathcal{A}_{r_2} \otimes \mathcal{A}_{r_2}(-\kappa^*)) & \xrightarrow{d_*} & H_{\text{ét}}^4(Y^3, \mathcal{A}'_{r_1} \boxtimes \mathcal{A}_{r_2} \boxtimes \mathcal{A}_{r_2}(-\kappa^*) \otimes \mathbb{Z}_p(2)) \\ \downarrow w_N & & \downarrow (w_N, w_N, w_N) \\ H_{\text{ét}}^0(Y, \mathcal{A}'_{r_1} \otimes \mathcal{A}_{r_2} \otimes \mathcal{A}_{r_2}(-\kappa^*)) & \xrightarrow{d_*} & H_{\text{ét}}^4(Y^3, \mathcal{A}'_{r_1} \boxtimes \mathcal{A}_{r_2} \boxtimes \mathcal{A}_{r_2}(-\kappa^*) \otimes \mathbb{Z}_p(2)), \end{array}$$

where w_N stands here for the operator defined in [BSV22, §2.3.1] and (w_N, w_N, w_N) is defined in a similar way for the cohomology of Y^3 . It follows from the definition of **Det** that $w_N(\mathbf{Det}) = \mathbf{Det}$. Since $w_p w_N = [p]_N w_N w_p$ and $\mathbf{s}_{\mathbf{fgh}} \circ ([p]_N w_N \otimes w_N \otimes w_N) = [p]_N'(w_N \otimes w_N \otimes w_N) \circ \mathbf{s}_{\mathbf{fgh}}$, it follows that

$$(w_N \otimes w_N \otimes w_N) \kappa^{(1)} = \kappa_{\mathbf{fgh}}^*(N)([p]_N \otimes 1 \otimes 1) \kappa^{(1)}.$$

Since $(w_N^2 \otimes 1 \otimes 1)$ acts as multiplication by $[-N] \otimes 1 \otimes 1$, we deduce that

$$(1 \otimes w_N \otimes w_N) \kappa^{(1)} = \kappa_f^*(N)([p]_N' w_N \otimes 1 \otimes 1) \kappa^{(1)}$$

and therefore that

$$N^{-r_2}(1 \otimes w_N \otimes w_N) \kappa^{(2)} = \kappa_f^{-1/2}(N)([p]_N' w_N \otimes 1 \otimes 1) \kappa^{(2)}.$$

Let s_N denote the operator which acts on the group

$$H^1(\mathbb{Q}, H^1(\Gamma, \mathcal{D}'_{r_1})^{\text{ord}} \hat{\otimes} H^1(\Gamma, L_{r_2})^{\text{ord}} \hat{\otimes} H^1(\Gamma, L_{r_2})^{\text{ord}} (2 - r_2 - \kappa_f^{1/2}))$$

by interchanging the second and third factors. Then, we have that

$$s_N \circ (1 \otimes w_N \otimes w_N) = (1 \otimes w_N \otimes w_N) \circ s_N,$$

and, taking into account the definition of **Det** and the fact that the Künneth isomorphism

$$H_{\text{ét}}^3(Y_{\overline{\mathbb{Q}}}^3, \mathcal{A}'_{r_1} \boxtimes \mathcal{A}_{r_2} \boxtimes \mathcal{A}_{r_2}) \cong H_{\text{ét}}^1(Y_{\overline{\mathbb{Q}}}, \mathcal{A}'_{r_1}) \otimes H_{\text{ét}}^1(Y_{\overline{\mathbb{Q}}}, \mathcal{A}_{r_2}) \otimes H_{\text{ét}}^1(Y_{\overline{\mathbb{Q}}}, \mathcal{A}_{r_2})$$

is given by cup-product, which is anti-commutative in degree 1 (*cf.* the proof of [LZ19, Prop. 4.1.2]), we deduce that $s_N(\kappa^{(2)}) = (-1)^{r_1/2+r_2+1}\kappa^{(2)}$. Define $\tilde{s}_N = (-N)^{-r_2}s_N \circ (1 \otimes w_N \otimes w_N)$. Then, we have that

$$\tilde{s}_N(\kappa^{(2)}) = (-1)^{r_1/2+1}\kappa_f^{-1/2}(N)([p]'_N w_N \otimes 1 \otimes 1)\kappa^{(2)}.$$

Since $(1 \otimes \pi_{1*} \otimes \pi_{2*}) \circ \tilde{s}_N = \tilde{s}_{N_g} \circ (1 \otimes \pi_{1*} \otimes \pi_{2*})$, it follows that

$$\tilde{s}_{N_g}(\kappa^{(3)}) = -[N]^{-1/2}([p]'_N w_N \otimes 1 \otimes 1)\kappa^{(3)}$$

and therefore that

$$\tilde{s}_{N_g}(\kappa^{(3)}(\mathbf{f}, g, g^*)) = -[N]^{-1/2}(w_N \otimes 1 \otimes 1)\kappa^{(3)}(\mathbf{f}, g, g^*).$$

Finally, it follows from [How07, Prop. 2.3.6] that $-[N]^{-1/2}w_N$ acts on $\mathbb{V}_{\mathbf{f}}$ as multiplication by $\varepsilon(f)$, so the last part of the proposition follows from the previous lemma. $\square$

Remark 3.3.3. From the definition of the map

$$\mathfrak{Log}(\mathbf{f}, g_\alpha, g_\alpha^*) : H_{\text{bal}}^1(\mathbb{Q}_p, \mathbb{V}(\mathbf{f}, g, g^*)) \longrightarrow \Lambda_{\mathbf{f}}[1/p],$$

one can see that it factors through the cohomology of $\mathbb{V}(\mathbf{f}, g, g^*)_f$, and therefore that

it factors through

$$H_{\text{bal}}^1(\mathbb{Q}_p, \mathbb{V}(\mathbf{f}, g, g^*)) \longrightarrow H^1(\mathbb{Q}_p, \mathbb{V}_{\mathbf{f}}(-\kappa_f^{1/2}) \otimes \text{ad}^0(T_g)) \longrightarrow H^1(\mathbb{Q}_p, \mathbb{V}(\mathbf{f}, g, g^*)_f).$$

Therefore, *without* the need to appeal to the reciprocity law, it follows from Proposition 3.3.2 that when $\varepsilon(f) = -1$ we have

$$\mathfrak{Log}(\mathbf{f}, g_\alpha, g_\alpha^*)(\kappa) = 0.$$

Of course, this can also be seen from the reciprocity law: Since $\varepsilon(f) = -1$ forces the vanishing of $L(\mathbf{f}_{k'}, k'/2)$ for all $k' \equiv k \pmod{2(p-1)}$, and this is a factor of $L(\mathbf{f}_{k'} \otimes g \otimes g^*, c')$, it follows from the interpolation formula that $\mathcal{L}_p(\check{\mathbf{f}}, \check{g}, \check{h})$ is identically zero.

Remark 3.3.4. As noted above, the discussion in this section is unnecessary for the applications that we will discuss. Indeed, as observed in the previous remark, the reciprocity law factors through $H^1(\mathbb{Q}_p, \mathbb{V}_{\mathbf{f}}(-\kappa_f^{1/2}) \otimes \text{ad}^0(T_g))$. Therefore, the non-vanishing of the triple product p -adic L -function at some point (necessarily when $\varepsilon(f) = +1$) implies that the image of $\kappa^{(3)}$ in this group is nontrivial, which is what we will actually need. However, it is interesting that we can already see from the geometric construction that the class lies where it is expected.

Remark 3.3.5. Let us discuss the sign in a little bit more detail. In order to construct the p -adic L -function attached to (f, g, g^*) , it is required in [Hsi21] that the local signs at finite primes of the arithmetic specializations of the representation $\mathbb{V}(\mathbf{f}, \mathbf{g}, \mathbf{g}^*) = \mathbb{V}_{\mathbf{f}} \hat{\otimes} \mathbb{V}_{\mathbf{g}} \hat{\otimes} \mathbb{V}_{\mathbf{g}^*}(-1 - \kappa^*)$ are all equal to 1. In particular, in our case, this imposes the condition that $\varepsilon_\ell(\mathbf{f}_{k'}) = \varepsilon_\ell(\mathbf{f}_{k'} \otimes \text{ad}^0(g))$ for all $\ell \mid N$ and for all $k' \equiv k \pmod{2(p-1)}$. The corresponding signs at infinity can be computed from the Hodge types $\{(p, q), (q, p)\}$ as in [Del79, §5.3]. For the representation $V_{\mathbf{f}_{k'}} \otimes \text{ad}^0(V_g)$, the Hodge types are as follows:

- (i) $\{(k'/2 + l - 2, -k'/2 - l + 1), (-k'/2 - l + 1, k'/2 + l - 2)\};$
- (ii) $\{(k'/2 - 1, -k'/2), (-k'/2, k'/2 - 1)\};$
- (iii) $\{(k'/2 - l, -k'/2 + l - 1), (-k'/2 + l - 1, k'/2 - l)\}.$

After that, and following the results of [*loc. cit.*], we get that the sign $\varepsilon_\infty(f \otimes \text{ad}^0(g))$ is $(-1)^{k'/2}$ if $k' \geq 2l$ and $(-1)^{1+k'/2}$ if $k' < 2l$. The sign of $\varepsilon_\infty(\mathbf{f}_{k'})$, however, is always equal to $(-1)^{k'/2}$. Therefore, in the balanced region (i.e. for $k' < 2l$), the motives attached to $\mathbf{f}_{k'}$ and $\mathbf{f}_{k'} \otimes \text{ad}^0(g)$ have opposite global signs. Since it is in this region that the corresponding specializations of the class $\kappa^{(3)}(\mathbf{f}, g, g^*)$ belong to the Bloch-Kato Selmer group, we expect the behaviour that was shown in Proposition 3.3.2.

3.4 The p -adic L -function

In this section, we keep the assumption that $(p) = \mathfrak{p}\bar{\mathfrak{p}}$ splits in K . In addition, from now on, for simplicity we assume that $p \nmid h_K$, where h_K is the class number of K .

Let $g \in S_l(N_g, \chi_g)$ be an ordinary newform not of CM-type. Let $\mathfrak{c}$ be an ideal of $\mathcal{O}_K$ coprime to p , and fix a Hecke character ψ_0 of infinity type $(-1, 0)$ and conductor $\mathfrak{c}p^e$ with $e \in \{0, 1\}$. We assume that the central character ε_{ψ_0} of ψ_0 is of the form

$$\varepsilon_{\psi_0} = \varepsilon_K \omega^{r_1} \text{ for some even integer } r_1, \tag{H0}$$

where ω is the Teichmüller character.

3.4.1 Lifting of automorphic representations

Let π be the cuspidal automorphic representation of $\text{GL}_2(\mathbb{A}_{\mathbb{Q}})$ attached to g . The central character of π is the adelic character ω_g defined by the condition that for any prime $q \nmid N_g$ and any uniformizer ϖ_q we have $\omega_{g,p}(\varpi_q) = \chi_g(q)$. Since $p \nmid N_g$, the local

component π_p is a spherical representation, and it follows from [Bum97, Thm. 4.6.4] and its proof that π_p is isomorphic to the principal series $\pi(\chi, \chi^{-1}\omega_g)$, where χ is the unramified character of $\mathbb{Q}_p^\times$ defined by $\chi(p) = \alpha_p(g)p^{(1-l)/2}$.

Since we are assuming that g is not of CM-type, and in particular it does not have CM by K , it follows from [GJ78, Prop. 2.3.3] that π admits, adopting the terminology of [op. cit.], a base change lifting to a cuspidal automorphic representation π_K of $\mathrm{GL}_2(\mathbb{A}_K)$. We fix such a lifting. Observe that if $\mathfrak{p}, \bar{\mathfrak{p}}$ are the places of K above p , then $\pi_{K,\mathfrak{p}} \simeq \pi_{K,\bar{\mathfrak{p}}} \simeq \pi_p$.

From the assumption that g is not of CM-type we deduce that there is no non-trivial character η of $K^\times \backslash \mathbb{A}_K^\times$ such that $\pi_K \simeq \pi_K \otimes \eta$. Indeed, the existence of such a character would imply that there exists a quadratic extension L of K such that, for all prime ℓ , the restriction to G_K of the ℓ -adic Galois representation attached to g is induced from a character of G_L , which is not possible by [Rib85, Thm. 2.1]. Now, it follows from [GJ78, Thm. 9.3] that π_K admits an adjoint lifting to a cuspidal automorphic representation $\Pi_{\mathrm{Ad}^0(g)}$ of $\mathrm{GL}_3(\mathbb{A}_K)$. Fix such a lifting and define

$$\Pi := \Pi_{\mathrm{Ad}^0(g)} \otimes \psi_0 | \cdot |^{1/2}.$$

Observe that $\Pi_{\mathfrak{p}} \simeq \Pi_{\bar{\mathfrak{p}}} \simeq \pi(\chi^2\omega_{g,p}^{-1}, 1, \chi^{-2}\omega_{g,p}) \otimes \psi_0 | \cdot |^{1/2}$ and it follows from the definition of χ that $\chi^2\omega_{g,p}^{-1} \neq | \cdot |^{\pm 1/2}$ and therefore that $\pi(\chi^2\omega_{g,p}^{-1}, 1, \chi^{-2}\omega_{g,p}) = \mathrm{Ind}_B^{\mathrm{GL}_3}(\chi^2\omega_{g,p}^{-1}, 1, \chi^{-2}\omega_{g,p})$, where B denotes the Borel subgroup of $\mathrm{GL}_3(\mathbb{Q}_p)$.

3.4.2 Descent to unitary groups

Let $U(2, 1)$ be the quasi-split unitary group corresponding to the quadratic extension $K/\mathbb{Q}$. Let $\Phi \in \mathrm{GL}_3(K)$ be the matrix whose entries are $\Phi_{ij} = (-1)^{i-1}\delta_{i,4-j}$. Then

we can describe $U(2, 1)$ by specifying its functor of points:

$$U(2, 1)(R) = \{g \in \mathrm{GL}_3(R \otimes_{\mathbb{Q}} K) : g\Phi \bar{g} = \Phi\}$$

for any $\mathbb{Q}$ -algebra R .

Let $U(3)$ be the definite unitary group whose functor of points is given by

$$U(3)(R) = \{g \in \mathrm{GL}_3(R \otimes_{\mathbb{Q}} K) : g\bar{g} = I_3\}.$$

Given a representation ρ of $\mathrm{GL}_3(\mathbb{A}_K)$, let $\tilde{\rho}$ be the representation defined on the same space by $\tilde{\rho}(x) = \rho(\bar{x}^{-1})$. Then, the representation Π defined above satisfies $\Pi \simeq \tilde{\Pi}$, and so it follows from [Rog90, Thm. 13.3.3] that there exists a cuspidal automorphic representation σ' of $U(2, 1)(\mathbb{A}_{\mathbb{Q}})$ whose base change to K is isomorphic to Π . Fix such a representation σ' . Observe that $\sigma'_p \simeq \pi(\chi^2 \omega_g^{-1}, 1, \chi^{-2} \omega_g) \otimes \psi_0 | \cdot |^{1/2}$ under the identification $U(2, 1)(\mathbb{Q}_p) = \mathrm{GL}_3(\mathbb{Q}_p)$. Also, from [*op. cit.*, Prop. 13.2.2], the local representation σ'_∞ is square-integrable, so, applying [*op. cit.*, Prop. 14.6.2], we can transfer σ' to a representation σ of $U(3)$. The local components of σ at finite primes agree with those of σ' , so in particular we have that $\sigma_p \simeq \sigma'_p$.

Remark 3.4.1. Let $GU(3)$ be the definite unitary similitude group whose functor of points is given by

$$GU(3)(R) = \{g \in \mathrm{GL}_3(R \otimes_{\mathbb{Q}} K) : g\bar{g} = \nu(g)I_3 \text{ for some } \nu(g) \in R^\times\}.$$

As explained in [BR93, §1.8], one can extend σ to an irreducible automorphic representation of $GU(3)$ by choosing an extension of the central character of σ to the center of $GU(3)$.

3.4.3 p -adic L -functions for unitary groups

A construction of p -adic L -functions for unitary groups is given in [EW16], and, in great generality in [EHLS20]. Here we deduce from these works the existence of an anticyclotomic p -adic L -function for the conjugate self-dual representation V in §3.2.2.

Let $\mathcal{O}$ be the ring of integers of a finite extension of $\mathbb{Q}_p$ containing the values of ψ_0 , and write

$$\Lambda^{\text{ac}} = \mathcal{O}[[\Gamma^{\text{ac}}]]$$

for the anticyclotomic Iwasawa algebra, where Γ^{ac} is the Galois group of the anticyclotomic $\mathbb{Z}_p$ -extension of K .

We will need to consider the following CM periods, as they are introduced in [BDP12]:

- $\Omega_\infty \in \mathbb{C}^\times$ is the complex period attached to K defined in [*op. cit.*, eq. (2-15)];
- $\Omega_p \in \mathbb{C}_p^\times$ is the p -adic period attached to K defined in [*op. cit.*, eq. (2-17)].

Theorem 3.4.2. *There exists an element*

$$L_p(\text{ad}^0(g_K) \otimes \psi_0) \in \text{Frac } \Lambda^{\text{ac}}$$

such that for all characters ξ of Γ^{ac} crystalline at both $\mathfrak{p}$ and $\bar{\mathfrak{p}}$ and corresponding to a Hecke character of infinity type $(-n, n)$ with $n \equiv r_1/2 \pmod{p-1}$ and $n \geq l-1$, we have

$$L_p(\text{ad}^0(g_K) \otimes \psi_0)(\xi) = \left(\frac{\Omega_p}{\Omega_\infty} \right)^{6n+3} \cdot \pi^{3n} \cdot \Gamma(n, l) \cdot \mathcal{E}_p(\text{ad}^0(g), \psi_0 \xi)^2 \cdot L(\text{ad}^0(g_K) \otimes \psi_0^{-1} \xi^{-1} \omega^n, 0),$$

where:

- $\Gamma(n, l) = (n + l - 1)! \cdot n! \cdot (n - l + 1)!$,
- $\mathcal{E}_p(\mathrm{ad}^0(g), \psi_0 \xi) = \left(1 - \frac{\alpha_g(\psi_0 \xi \omega^{-n})(\mathfrak{p})}{\beta_g p}\right) \cdot \left(1 - \frac{(\psi_0 \xi \omega^{-n})(\mathfrak{p})}{p}\right) \cdot \left(1 - \frac{\beta_g(\psi_0 \xi \omega^{-n})(\mathfrak{p})}{\alpha_g p}\right).$

Proof. Let σ be the irreducible automorphic representation of $U(3)$ introduced in the previous subsection. Let Σ be the set of places of $\mathbb{Q}$ consisting of p , infinity, the primes dividing D_K , and the primes at which σ ramifies. On account of Remark 3.4.1, the main result of [EW16] yields an element $\mathcal{L}_p^\Sigma \in \Lambda^{\mathrm{ac}}[1/p]$ such that, for all ξ as in the statement, satisfies

$$\mathcal{L}_p^\Sigma(\xi) = \left(\frac{\pi \Omega_p}{\Omega_\infty}\right)^{6n+3} \cdot \mathcal{E}_p(\xi) \cdot \mathcal{E}_\infty(\xi) \cdot L^\Sigma(\tilde{\sigma}, \xi^{-1} \omega^n, 0),$$

where $\tilde{\sigma}$ is the contragredient of σ , and $\mathcal{E}_p(\xi)$ and $\mathcal{E}_\infty(\xi)$ are certain modified Euler factors at p and ∞ , respectively. Since Σ contains p , infinity, and all the ramified primes, we have that

$$L^\Sigma(\tilde{\sigma}, \xi^{-1} \omega^n, 0) = L^\Sigma(\mathrm{ad}^0(g_K) \otimes \psi_0^{-1} \xi^{-1} \omega^n, 0).$$

Since we are assuming that p splits in K , the form of the modified Euler factor at p can be extracted from [EHLS20, eq. (86)]. Up to a nonzero rational factor independent of ξ , it is given by

$$\mathcal{E}_p(\xi) = \frac{\mathcal{E}_p(\mathrm{ad}^0(g), \psi_0 \xi)^2}{L_p(\mathrm{ad}^0(g_K) \otimes \psi_0^{-1} \xi^{-1} \omega^n, 0)}.$$

The form of the modified Euler factor at infinity can be extracted from [EL20, eq. (2.3.1)]. This formula, with $a = 3$, $b = 0$, $\underline{r} = (l, 0, -l)$, $r = 2n + 2$ and $s = 0$, yields, up to a nonzero rational factor independent of ξ ,

$$\mathcal{E}_\infty(\xi) = (2\pi i)^{-3n-3} \cdot \Gamma(n, l).$$

Finally, the Euler factors at primes $\ell \in \Sigma \setminus \{p, \infty\}$ can be p -adically interpolated by certain elements $\mathcal{P}_\ell \in \Lambda^{\text{ac}}$, and, multiplying by their inverses, we obtain the p -adic L -function in the statement. $\square$

3.4.4 CM Hida family

Let Γ_K be the Galois group of the $\mathbb{Z}_p^2$ -extension K_∞/K and put

$$\Gamma_{\mathfrak{p}} = \text{Gal}(K_{\mathfrak{p}\infty}/K) \simeq \mathbb{Z}_p,$$

where $K_{\mathfrak{p}\infty}$ is the maximal subfield of K_∞ unramified outside of $\mathfrak{p}$, so that $K_{\mathfrak{p}\infty}$ is the $\mathbb{Z}_p$ -extension of K inside the ray class field $K(\mathfrak{p}^\infty)$. Since we are assuming that $p \nmid h_K$, viewing $1 + p\mathbb{Z}_p$ as a subgroup of $\mathcal{O}_{K,\mathfrak{p}}^\times$, the restriction of the (geometrically normalized) Artin map to $K_{\mathfrak{p}}^\times$ induces an isomorphism $\text{art}_{\mathfrak{p}} : 1 + p\mathbb{Z}_p \simeq \Gamma_{\mathfrak{p}}$. Let $\gamma_{\mathfrak{p}} \in \Gamma_{\mathfrak{p}}$ be the topological generator corresponding to $1 + p$ under this isomorphism and, for the variable S , let $\Psi_S : \Gamma_K \rightarrow \mathbb{Z}_p[[S]]^\times$ be the character given by

$$\Psi_S(\sigma) = (1 + S)^{l(\sigma)},$$

where $l(\sigma) \in \mathbb{Z}_p$ is defined by $\sigma|_{K_{\mathfrak{p}\infty}} = \gamma_{\mathfrak{p}}^{l(\sigma)}$. Consider the formal q -expansion

$$\boldsymbol{\theta}_{\psi_0}(S)(q) := \sum_{(\mathfrak{a}, \mathfrak{p}\mathfrak{c})=1} \psi_0(\sigma_{\mathfrak{a}}) \Psi_S^{-1}(\sigma_{\mathfrak{a}}) q^{\mathbf{N}(\mathfrak{a})} \in \mathcal{O}[[S]][[q]], \quad (3.6)$$

where $\sigma_{\mathfrak{a}} \in \text{Gal}(K(\mathfrak{c}\mathfrak{p}^\infty)/K)$ is the Artin symbol of $\mathfrak{a}$. Then, for every $k \geq 2$, the specialization of $\boldsymbol{\theta}_{\psi_0}$ at $S = (1 + p)^{k-2} - 1$ is given by the theta series

$$\mathbf{f}_k = \sum_{(\mathfrak{a}, \mathfrak{p}\mathfrak{c})=1} \psi_0(\mathfrak{a}) \lambda^{k-2}(\mathfrak{a}) q^{\mathbf{N}(\mathfrak{a})} \in S_k(D_K \mathbf{N}(\mathfrak{c})p, \omega^{2+r_1-k}),$$

where λ is the unique (since $p \nmid h_K$) Hecke character of infinity type $(-1, 0)$ and conductor $\mathfrak{p}$ whose p -adic avatar factors through $\Gamma_{\mathfrak{p}}$. In particular, $\mathbf{f}_2$ is the ordinary p -stabilization of θ_{ψ_0} .

Remark 3.4.3. If ψ is a Hecke character of infinity type $(1 - k, 0)$ as in §3.2.2, then $\psi_0 := \psi\lambda^{2-k}$ is a Hecke character as above (in particular, satisfying (H0) with e.g. $r_1 = k - 2$), and so the resulting $\mathbf{f}_k$ recovers the p -stabilization of θ_{ψ} . From now on we shall always assume that ψ and ψ_0 are related in this manner, and refer to $\mathbf{f} = \theta_{\psi_0}$ as the CM Hida family attached to ψ (or ψ_0).

3.4.5 A factorization formula

In this section we prove a factorization formula relating the p -adic L -function attached to V in Theorem 3.4.2 to anticyclotomic p -adic L -functions attached to the other two representations in the decomposition (3.2).

Put $N = \text{lcm}(N_g, N_{\psi})$, where $N_{\psi} := D_K \mathbf{N}(\mathfrak{c})$. In addition to the previous hypotheses, from now on we shall also assume that:

- (a) $\varepsilon_{\ell}(V_{fgg^*}) = +1$ for all primes $\ell \mid N$,
- (b) $\text{gcd}(N_g, N_{\psi})$ is squarefree.

With notations as in Remark 3.2.3, here $\varepsilon_{\ell}(V_{fgg^*})$ denotes the epsilon-factor of the Weil–Deligne representation attached to the restriction of V_{fgg^*} to $G_{\mathbb{Q}_{\ell}}$.

Note that it follows from (H0) that the Galois representation of the Hida family $\mathbf{f} = \theta_{\psi_0}$ attached to ψ is residually irreducible and p -distinguished (see also [LLZ15, Rem. 5.1.3]). For the following result, we adopt the definition of congruence ideal in [Hsi21, §3.3].

Theorem 3.4.4. *Under the above hypotheses, there exists an element*

$$\mathcal{L}_p(\text{ad}(g_K) \otimes \psi_0) \in \text{Frac } \Lambda^{\text{ac}}$$

such that for all characters ξ of Γ^{ac} crystalline at both $\mathfrak{p}$ and $\bar{\mathfrak{p}}$ and corresponding to a Hecke character of infinity type $(-n, n)$ with $n \equiv r_1/2 \pmod{p-1}$ and $n \geq l-1$, we have

$$\mathcal{L}_p(\text{ad}(g_K) \otimes \psi_0)(\xi)^2 = \Gamma(n, l, l) \cdot \frac{\mathcal{E}_p(\text{ad}(g), \psi_0 \xi)^2}{\mathcal{E}_0(\psi_0 \xi)^2 \cdot \mathcal{E}_1(\psi_0 \xi)^2} \cdot \prod_{\ell|N} \tau_\ell \cdot \frac{L(\text{ad}(g_K) \otimes \psi_0^{-1} \xi^{-1} \omega^n, 0)}{(2\pi i)^{2n+2} \cdot \langle \theta_{\psi_0 \xi_n}, \theta_{\psi_0 \xi_n} \rangle^2},$$

where:

- $\Gamma(n, l, l) = (n+l-1)! \cdot (n!)^2 \cdot (n-l+1)!$,
- $\mathcal{E}_p(\text{ad}(g), \psi_0 \xi) = \left(1 - \frac{\alpha_g(\psi_0 \xi \omega^{-n})(\mathfrak{p})}{\beta_g p}\right) \cdot \left(1 - \frac{(\psi_0 \xi \omega^{-n})(\mathfrak{p})}{p}\right)^2 \cdot \left(1 - \frac{\beta_g(\psi_0 \xi \omega^{-n})(\mathfrak{p})}{\alpha_g p}\right)$,
- $\mathcal{E}_0(\psi_0 \xi) = \left(1 - \frac{(\psi_0 \xi \omega^{-n})(\mathfrak{p})}{(\psi_0 \xi \omega^{-n})(\bar{\mathfrak{p}})}\right)$, $\mathcal{E}_1(\psi_0 \xi) = \left(1 - \frac{(\psi_0 \xi \omega^{-n})(\mathfrak{p})}{p(\psi_0 \xi \omega^{-n})(\bar{\mathfrak{p}})}\right)$,
- τ_ℓ is an explicit nonzero rational number independent of n ,
- $\theta_{\psi_0 \xi_n}$ is the theta series of weight $2n+2$ attached to $\psi_0 \xi_n := \psi_0 \xi \omega^{-n} \cdot |\cdot|^{-n}$.

Moreover, if $\mathcal{H}$ is any generator of the congruence ideal of θ_{ψ_0} , then $\mathcal{H} \cdot \mathcal{L}_p(\text{ad}(g_K) \otimes \psi_0)$ belongs to Λ^{ac} .

Proof. This is essentially a reformulation of [Hsi21, Thm. A] specialized to our setting. Let $\mathbf{f} = \theta_{\psi_0}$ be the Hida family attached to the Hecke character ψ_0 as in (3.6), with associated big Galois representation $\mathbb{V}_{\mathbf{f}}$, and denote by $\mathbb{V}(\mathbf{f}, g, g^*)$ the Kummer self-dual twist of the triple tensor product $\mathbb{V}_{\mathbf{f}} \hat{\otimes}_{\mathcal{O}} T_g \otimes_{\mathcal{O}} T_{g^*}$ introduced in [ACR21, §7.1] (and recalled in §3.3 above). Since $\mathbb{V}_{\mathbf{f}} \simeq \text{Ind}_K^{\mathbb{Q}}(\psi_0^{-1} \Psi_S)$, we immediately find that

$$\mathbb{V}(\mathbf{f}, g, g^*) \simeq \text{ad}(T_g) \otimes \text{Ind}_K^{\mathbb{Q}}(\psi_0^{-1} \omega^{r_1/2} \Psi_S^{(1-\tau)/2}),$$

where for a character χ of G_K we denote by χ^τ the composition of χ with the action of the non-trivial automorphism τ of $K/\mathbb{Q}$, and put $\chi^{1-\tau} := \chi(\chi^\tau)^{-1}$.

By [Hsi21, Thm. A], attached to $(\mathbf{f}, g, g^*)$ (and a specific choice of level- N test vectors for this triple), there is an “unbalanced” triple product p -adic L -function

$\mathcal{L}_p(\mathbf{f}, g, g^*) \in \text{Frac } \mathcal{O}[[\Gamma_{\mathfrak{p}}]]$ interpolating, for all $k' \equiv r_1 + 2 \pmod{2(p-1)}$ with $k' \geq 2l$, the (central) values at $s = 0$ of the triple product L -function

$$L(\mathbb{V}(\mathbf{f}_k, g, g^*), s) = L(\text{ad}(g_K) \otimes \psi_0^{-1} \xi^{-1} \omega^{r_1/2}, s).$$

where we put ξ to denote the specialization of $\Psi_S^{(\tau-1)/2}$ at $S = (1+p)^{k'-2} - 1$, so ξ^{-1} is a character of Γ^{ac} crystalline at both $\mathfrak{p}$ and $\bar{\mathfrak{p}}$ corresponding to a Hecke character of infinity type $-(k'/2 - 1, k'/2 - 1)$. Taking $\mathcal{L}_p(\text{ad}(g_K) \otimes \psi_0)$ to be the image of $\mathcal{L}_p(\mathbf{f}, g, g^*)$ under the map $\text{Frac } \mathcal{O}[[\Gamma_{\mathfrak{p}}]] \rightarrow \text{Frac } \Lambda^{\text{ac}}$ determined by $\gamma_{\mathfrak{p}} \mapsto \gamma_{\mathfrak{p}}^{\tau-1}$, we thus see that the result follows from [Hsi21, Thm. A]. $\square$

We next discuss an anticyclotomic p -adic L -function associated with V' , arising from a suitable restriction of Katz's p -adic L -function.

Denote by Σ the set of algebraic Hecke characters ξ of K for which $s = 0$ is a critical point for $L(\xi, s)$ in the sense of Deligne. This set can be written as the disjoint union $\Sigma = \Sigma_{\mathfrak{p}} \cup \Sigma_{\bar{\mathfrak{p}}}$, where

$$\Sigma_{\mathfrak{p}} = \{\xi \in \Sigma \text{ of infinity type } (a, b), \text{ with } a \geq 1, b \leq 0\},$$

$$\Sigma_{\bar{\mathfrak{p}}} = \{\xi \in \Sigma \text{ of infinity type } (a, b), \text{ with } a \leq 0, b \geq 1\}.$$

Note that the involution $\xi \mapsto \xi^{\tau}$ takes characters in $\Sigma_{\mathfrak{p}}$ to characters in $\Sigma_{\bar{\mathfrak{p}}}$, and vice versa.

Let $G_{\mathfrak{c}} = \text{Gal}(K(\mathfrak{c}p^{\infty})/K)$ be the Galois group of the ray class field of K of conductor $\mathfrak{c}p^{\infty}$, and denote by $\mathbb{Z}_p^{\text{ur}}$ the completion of the ring of integers of the maximal unramified extension of $\mathbb{Q}_p$. The following result is originally due to Katz.

Theorem 3.4.5. *There exists an element $\mathcal{L}_{\mathfrak{p}, \mathfrak{c}}^{\text{Katz}}(K) \in \mathbb{Z}_p^{\text{ur}}[[G_{\mathfrak{c}}]]$ uniquely characterized by the property that for every character of $\Gamma_{\mathfrak{c}}$ corresponding to a Hecke character*

$\xi \in \Sigma_{\mathfrak{p}}$ of infinity type (k_1, k_2) and conductor dividing $\mathfrak{c}$ we have

$$\mathcal{L}_{\mathfrak{p}, \mathfrak{c}}^{\text{Katz}}(K)(\xi) = \left(\frac{\Omega_p}{\Omega_\infty} \right)^{k_1 - k_2} (k_1 - 1)! \cdot \left(\frac{\sqrt{D_K}}{2\pi} \right)^{k_2} \cdot (1 - p^{-1} \xi^{-1}(\mathfrak{p}) p^{-1}) (1 - \xi(\bar{\mathfrak{p}})) \cdot L_{\mathfrak{c}}(\xi, 0),$$

where $L_{\mathfrak{c}}(\xi, s)$ is the L -function of ξ with the Euler factors at the primes $\mathfrak{l} | \mathfrak{c}$ removed.

Moreover, we have the functional equation

$$\mathcal{L}_{\mathfrak{p}, \mathfrak{c}}^{\text{Katz}}(K)(\xi) = \mathcal{L}_{\mathfrak{p}, \bar{\mathfrak{c}}}^{\text{Katz}}(K)(\xi^{-\tau} \mathbf{N}^{-1}),$$

where the equality is up to a p -adic unit.

Proof. See [dS87, Thm. II.4.14] for a construction of $\mathcal{L}_{\mathfrak{p}, \mathfrak{c}}^{\text{Katz}}(K)$ (corresponding to the measure on $G_{\mathfrak{c}}$ denoted by $\mu(\mathfrak{c}\bar{\mathfrak{p}}^\infty)$ in [loc. cit.]), and [dS87, Thm. II.6.4] for the functional equation. $\square$

Assume that $\mathfrak{c}$ is fixed under complex conjugation, i.e., $\bar{\mathfrak{c}} = \mathfrak{c}$. Denote by $\Delta_{\mathfrak{c}}$ the torsion subgroup of $G_{\mathfrak{c}}$, and put $\Gamma_K := G_{\mathfrak{c}}/\Delta_{\mathfrak{c}} \simeq \mathbb{Z}_p^2$, which is identified with the Galois group of the unique $\mathbb{Z}_p^2$ -extension K_∞/K . We fix a decomposition

$$G_{\mathfrak{c}} \simeq \Delta_{\mathfrak{c}} \times \Gamma_K. \tag{3.7}$$

Put $\bar{\psi}_0 = \psi_0|_{\Delta_{\mathfrak{c}}}$ and $\bar{\psi}_0^- = \bar{\psi}_0^{\tau-1}$, and note that the latter defines a finite order anticyclotomic Hecke character of conductor dividing $\mathfrak{c}p^s$ for some $s \geq 0$. Denote by $\mathcal{L}_{\mathfrak{p}, \psi_0}^{\text{Katz}}(K)^-$ the image of $\mathcal{L}_{\mathfrak{p}, \mathfrak{c}}^{\text{Katz}}(K)$ under the composite map

$$\mathbb{Z}_p^{\text{ur}}[[G_{\mathfrak{c}}]] \rightarrow \mathbb{Z}_p^{\text{ur}}[[\Gamma_K]] \rightarrow \mathbb{Z}_p^{\text{ur}}[[\Gamma^{\text{ac}}]],$$

where the first arrow is the projection defined by $\bar{\psi}_0^-$ and the second arrow is given by $\gamma \mapsto \gamma^{\tau-1}$ for $\gamma \in \Gamma_K$.

From now on we shall assume that the above $\mathfrak{c}$ and ψ_0 satisfy the conditions

(H1)–(H4) in the following result.

Proposition 3.4.6. *In addition to (H0), assume that:*

(H1) $\mathfrak{c}$ is only divisible by primes that are split in K ;

(H2) $\bar{\psi}_0^-$ has order prime-to- p and the prime-to- p part of its conductor is exactly $\mathfrak{c}$;

(H3) $\bar{\psi}_0^-|_{G_{K_v}} \neq 1$ for all primes $v|p$ in K ;

(H4) $\bar{\psi}_0^-$ has order at least 3.

Then, as an ideal of $\mathbb{Z}_p^{\text{ur}}[[\Gamma^{\text{ac}}]]$, the congruence ideal $C(\boldsymbol{\theta}_{\psi_0})$ is generated by

$$h_K \cdot \mathcal{L}_{\mathfrak{p}, \psi_0}^{\text{Katz}}(K)^-$$

where h_K is the class number of K .

Proof. A generator of $C(\boldsymbol{\theta}_{\psi_0})$ is given by a congruence power series $H(\boldsymbol{\theta}_{\psi_0})$ attached to $\boldsymbol{\theta}_{\psi_0}$ as in [Hid06]. By our assumptions, this $H(\boldsymbol{\theta}_{\psi_0})$ corresponds to a branch character satisfying the hypotheses (1)–(4) in [Hid06, p. 466], so as noted in p. 469 of [*op. cit.*], the result follows from the proof of the anticyclotomic Iwasawa main conjecture by Hida–Tilouine [HT93a, HT94] and Hida [Hid06]. $\square$

Definition 3.4.7. Put

$$L_p(\text{ad}(g_K) \otimes \psi_0) := \left(\mathcal{L}_p(\text{ad}(g_K) \otimes \psi_0) \cdot h_K \cdot \mathcal{L}_{\mathfrak{p}, \psi_0}^{\text{Katz}}(K)^- \right)^2,$$

which by Theorem 3.4.4 and Proposition 3.4.6 defines an element in $\mathbb{Z}_p^{\text{ur}}[[\Gamma^{\text{ac}}]]$.

We can now derive an anticyclotomic analogue of Dasgupta’s factorization [Das16, Thm. 1], relating the p -adic L -function of Theorem 3.4.4 to the product of the p -adic L -functions in Theorem 3.4.2 and Theorem 3.4.5. Similarly as in [*loc. cit.*], our result

is a p -adic analogue of the factorization of complex L -functions

$$L(\mathrm{ad}(g_K) \otimes \chi, s) = L(\mathrm{ad}^0(g_K) \otimes \chi, s) \cdot L(\chi, s)$$

arising from the decomposition of G_K -representations

$$\mathrm{ad}(V_g) \otimes \chi \simeq (\mathrm{ad}^0(V_g) \otimes \chi) \oplus \chi.$$

However, our proof is largely simplified by the fact that the three p -adic L -functions involved have a Zariski dense overlapping set of characters in the range of interpolation.

Our factorization formula will in fact involve the following anticyclotomic projection of the Katz p -adic L -function.

Definition 3.4.8. Viewing ψ_0 as a character of $G_{\mathfrak{c}}$, write $\psi_0 = \bar{\psi}_0 \cdot \psi_{\Gamma}$ according to the factorization (3.7), with $\bar{\psi}_0$ (resp. ψ_{Γ}) a character of $\Delta_{\mathfrak{c}}$ (resp. Γ_K). We denote by $\mathcal{L}_{\mathfrak{p}}^{\mathrm{Katz}}(\psi_0)^{-,\iota} \in \mathbb{Z}_p^{\mathrm{ur}}[[\Gamma^{\mathrm{ac}}]]$ the image of $\mathcal{L}_{\mathfrak{p},\mathfrak{c}}^{\mathrm{Katz}}(K)$ under the composite map

$$\mathbb{Z}_p^{\mathrm{ur}}[[G_{\mathfrak{c}}]] \rightarrow \mathbb{Z}_p^{\mathrm{ur}}[[\Gamma_K]] \rightarrow \mathbb{Z}_p^{\mathrm{ur}}[[\Gamma_K]] \rightarrow \mathbb{Z}_p^{\mathrm{ur}}[[\Gamma^{\mathrm{ac}}]] \rightarrow \mathbb{Z}_p^{\mathrm{ur}}[[\Gamma^{\mathrm{ac}}]],$$

where the first arrow is given by the projection defined by $\bar{\psi}_0^{-1}\omega^{r_1/2}$, the second by twisting by ψ_{Γ}^{-1} , the third is the natural projection, and the last arrow is the involution given by $\gamma \mapsto \gamma^{-1}$ for $\gamma \in \Gamma^{\mathrm{ac}}$. In other words, $\mathcal{L}_{\mathfrak{p}}^{\mathrm{Katz}}(\psi_0)^{-,\iota}$ is the element of $\mathbb{Z}_p^{\mathrm{ur}}[[\Gamma^{\mathrm{ac}}]]$ defined by

$$\mathcal{L}_{\mathfrak{p}}^{\mathrm{Katz}}(\psi_0)^{-,\iota}(\xi) = \mathcal{L}_{\mathfrak{p},\mathfrak{c}}^{\mathrm{Katz}}(K)(\psi_0^{-1}\xi^{-1}\omega^{r_1/2})$$

for all characters ξ of Γ^{ac} .

Denote by τ_N the product of constants $\prod_{\ell|N} \tau_{\ell}$ appearing in Theorem 3.4.4.

Theorem 3.4.9. *The following equality holds*

$$L_p(\mathrm{ad}(g_K) \otimes \psi_0) = u \cdot L_p(\mathrm{ad}^0(g_K) \otimes \psi_0) \cdot \mathcal{L}_{\mathfrak{p}}^{\mathrm{Katz}}(\psi_0)^{-, \iota} \cdot \tau_N$$

where u is a unit in $(\Lambda^{\mathrm{ac}})^{\times}$.

Proof. Let ξ be a character of Γ^{ac} as in the statement of Theorem 3.4.2 and Theorem 3.4.4, hence in particular corresponding to a Hecke character, still denoted by ξ , of infinity type $(-n, n)$ with $n \geq l - 1$. Noting that $\theta_{\psi_0 \xi_n}$ has weight $2n + 2$, from Hida's formula for the adjoint L -value (see [HT93b, Thm .7.1]) and Dirichlet's class number formula we obtain (cf. [JSW17, p. 414])

$$\langle \theta_{\psi_0 \xi_n}, \theta_{\psi_0 \xi_n} \rangle = (2n + 1)! \cdot D_K^2 \cdot \frac{1}{2^{4n+4} \pi^{2n+3}} \cdot \frac{2\pi h_K}{w_K \sqrt{D_K}} \cdot L(\psi_0^{1-\tau} \xi^{1-\tau}, 1), \quad (3.8)$$

where w_K is the number of units in $\mathcal{O}_K$. Since $L(\psi_0^{1-\tau} \xi^{1-\tau}, 1)$ corresponds to the value at $s = 0$ of the L -function for the Hecke character $\psi_0^{\tau-1} \xi^{\tau-1} \mathbf{N}^{-1}$ of infinity type $(2n + 2, -2n)$, using (3.8) the interpolation formula in Theorem 3.4.5 can be rewritten as

$$\begin{aligned} \mathcal{L}_{\mathfrak{p}, \mathfrak{c}}^{\mathrm{Katz}}(K)(\psi_0^{\tau-1} \xi^{\tau-1} \mathbf{N}^{-1}) &= \left(\frac{\Omega_p}{\Omega_{\infty}} \right)^{4n+2} \cdot \frac{2^{6n+4} \pi^{4n+2}}{\sqrt{D_K}^{2n+1}} \cdot \frac{w_K}{D_K^2 h_K} \\ &\quad \times \left(1 - \frac{(\psi_0 \xi \omega^{-n})(\mathfrak{p})}{(\psi_0 \xi \omega^{-n})(\bar{\mathfrak{p}})} \right) \left(1 - \frac{(\psi_0 \xi \omega^{-n})(\mathfrak{p})}{p(\psi_0 \xi \omega^{-n})(\bar{\mathfrak{p}})} \right) \cdot \langle \theta_{\psi_0 \xi_n}, \theta_{\psi_0 \xi_n} \rangle. \end{aligned}$$

Thus together with Theorem 3.4.4 we find that

$$\begin{aligned} &\mathcal{L}_p(\mathrm{ad}(g_K) \otimes \psi_0)(\xi)^2 \cdot \mathcal{L}_{\mathfrak{p}, \mathfrak{c}}^{\mathrm{Katz}}(K)(\psi_0^{\tau-1} \xi^{\tau-1} \mathbf{N}^{-1})^2 \cdot h_K^2 \\ &= \left(\frac{\Omega_p}{\Omega_{\infty}} \right)^{8n+4} \cdot \frac{2^{8n+4} \pi^{4n}}{\sqrt{D_K}^{4n}} \cdot \Gamma(n, l, l) \cdot \mathcal{E}(\mathrm{ad}(g), \psi_0 \xi)^2 \cdot \frac{w_K^2}{D_K^4} \cdot \tau_N \cdot L(\mathrm{ad}(g_K) \otimes \psi_0^{-1} \xi^{-1} \omega^n, 0). \end{aligned} \quad (3.9)$$

On the other hand, we have the factorization

$$L(\mathrm{ad}(g_K) \otimes \psi_0^{-1} \xi^{-1} \omega^n, 0) = L(\mathrm{ad}^0(g_K) \otimes \psi_0^{-1} \xi^{-1} \omega^n, 0) \cdot L(\psi_0^{-1} \xi^{-1} \omega^n, 0). \quad (3.10)$$

The character $\psi_0^{-1} \xi^{-1} \omega^n$ has infinity type $(n+1, -n)$, and so is in the range of interpolation for $\mathcal{L}_{\mathfrak{p}, \mathfrak{c}}^{\mathrm{Katz}}(K)$. Thus combining Theorem 3.4.2 and Theorem 3.4.5 and using (3.10) we find

$$\begin{aligned} L_p(\mathrm{ad}^0(g_K) \otimes \psi_0)(\xi) \cdot \mathcal{L}_{\mathfrak{p}, \mathfrak{c}}^{\mathrm{Katz}}(K)(\psi_0^{-1} \xi^{-1} \omega^n) &= \left(\frac{\Omega_p}{\Omega_\infty} \right)^{6n+3} \cdot \pi^{3n} \cdot \Gamma(n, l) \cdot \mathcal{E}(\mathrm{ad}^0(g), \psi_0 \xi)^2 \\ &\times \left(\frac{\Omega_p}{\Omega_\infty} \right)^{2n+1} \cdot n! \cdot \left(\frac{2\pi}{\sqrt{D_K}} \right)^n \cdot (1 - p^{-1} \psi_0 \xi(\mathfrak{p}))^2 \cdot L(\mathrm{ad}(g_K) \otimes \psi_0^{-1} \xi^{-1} \omega^n, 0). \end{aligned} \quad (3.11)$$

Comparing (3.9) and (3.11) we see that their ratio is given by $2^{7n+4} \cdot \sqrt{D_K}^{-3n-8} \cdot \tau_N$; since for varying n the first two factors are interpolated by a unit in $(\Lambda^{\mathrm{ac}})^\times$, applying the functional equation of Theorem 3.4.5 this gives the result. $\square$

3.5 The Euler system

Let $g \in S_l(N_g, \chi_g)$ be a newform as in §3.2.1, and let ψ be a Hecke character of K of infinity type $(1-k, 0)$ for some even integer $k \geq 2$, conductor $\mathfrak{c}$ prime to p , and central character ε_K . Recall from §3.4 that we assume that $(p) = \mathfrak{p}\bar{\mathfrak{p}}$ splits in K and (for simplicity) that p does not divide the class number of K .

3.5.1 Modified diagonal cycles

Recall that, for a positive integer m , we write $K[m]$ for the maximal p -extension inside the ring class field of K of conductor m . Recall further that $H_{\mathrm{Iw}}^1(K[mp^\infty], T) =$

$\varprojlim_r H^1(K[mp^r], T)$. Then, for a positive integer m , let

$$\kappa_{\psi, \text{ad}(g), m, \infty} \in H_{\text{Iw}}^1(K[mp^\infty], T_{\text{ad}(g)}^\psi) \quad (3.12)$$

be the class $\kappa_{\psi, g, g^*, m, \infty}$ constructed in [ACR21, Thm. 6.5]. (For $m = 1$, this is essentially the class $\kappa^{(3)}(\mathbf{f}, g, g^*)$ defined in §3.3, after an application of Shapiro's lemma and twisting by the inverse of the anticyclotomic character ξ in (3.15) below.) Since we have a direct sum decomposition

$$H_{\text{Iw}}^1(K[mp^\infty], T_{\text{ad}(g)}^\psi) = H_{\text{Iw}}^1(K[mp^\infty], T) \oplus H_{\text{Iw}}^1(K[mp^\infty], T'),$$

we can project the class $\kappa_{\psi, \text{ad}(g), m, \infty}$ to each of the summands. We denote its projection to the first summand as $\kappa_{\psi, \text{ad}^0(g), m, \infty}$. For the following results, we keep the notations for Selmer groups introduced in Section 3.2.3.

Theorem 3.5.1. *Let $\mathcal{S}$ be the set of all squarefree products of primes q split in K and coprime to pN_gN_ψ , and assume that $H^1(K[mp^s], T)$ is torsion-free for every $m \in \mathcal{S}$ and for every $s \geq 0$. There exists a collection of classes*

$$\{\kappa_{\psi, \text{ad}^0(g), m, \infty} \in \text{Sel}_{\text{Gr}}(K[mp^\infty], T) : m \in \mathcal{S}\}$$

such that whenever $m, mq \in \mathcal{S}$ with q a prime, we have

$$\text{cor}_{K[mq]/K[m]}(\kappa_{\psi, \text{ad}^0(g), mq, \infty}) = P_{\mathfrak{q}}(V_{\text{ad}(g)}^\psi; \text{Fr}_{\mathfrak{q}}^{-1}) \kappa_{\psi, \text{ad}^0(g), m, \infty}.$$

Proof. This is an immediate consequence of [ACR21, Thm. 6.5] and [op. cit., Prop. 6.6]. □

The Euler factors appearing in the previous theorem are not the ones that we

want. Indeed, let $q = \mathfrak{q}\bar{\mathfrak{q}}$ be a prime which splits in K . Then

$$P_{\mathfrak{q}}(V_{\text{ad}(g)}^{\psi}; X) = \left(1 - \frac{\psi(\mathfrak{q})X}{q^{k/2}}\right) P_{\mathfrak{q}}(V; X),$$

so there is an unwanted extra factor. We now deal with this problem.

Definition 3.5.2. Let $\mathcal{P}'$ be the set of primes $q \nmid pN_gN_{\psi}$ split in K such that

- $q \equiv 1$ modulo p ,
- $T/(\text{Fr}_q - 1)T$ is a cyclic $\mathbb{Z}_p$ -module,
- $\text{Fr}_q - 1$ is bijective on T' .

Here Fr_q denotes any arithmetic Frobenius element for q . Since $T^{\vee}(1) \simeq T^c$ and $(T')^{\vee}(1) \simeq (T')^c$, the definition does not depend on this choice.

Remark 3.5.3. Under certain conditions, we will show in Proposition 3.6.3 below that there exists $\sigma \in G_K$ such that if q is a prime such that Fr_q is conjugate to σ in $\text{Gal}(K(\mu_p, \bar{T}, \bar{T}')/K)$, then it belongs to $\mathcal{P}'$.

Theorem 3.5.4. *Let $\mathcal{S}'$ be the set of squarefree products of primes in $\mathcal{P}'$, and assume that $H^1(K[mp^s], T)$ is torsion-free for every $n \in \mathcal{S}'$ and for every $s \geq 0$. There exists a collection of classes*

$$\{\kappa_m \in \text{Sel}_{\text{Gr}}(K[mp^{\infty}], T) : m \in \mathcal{S}'\}$$

such that $\kappa_1 = \kappa_{\psi, \text{ad}^0(g), 1, \infty}$ and, whenever $m, mq \in \mathcal{S}'$ with q a prime, we have

$$\text{cor}_{K[mq]/K[m]}(\kappa_{mq}) = P_{\mathfrak{q}}(V; \text{Fr}_{\mathfrak{q}}^{-1}) \kappa_m,$$

where $\mathfrak{q}$ is any of the primes of K above q .

Proof. We construct the classes κ_m by modifying the classes $\kappa_{\psi, \text{ad}^0(g), m, \infty}$ in Theorem 3.5.1 appropriately as done in the proof of [LZ19, Thm. 5.3.3].

For each $m \in \mathcal{S}'$, let $\Gamma_m = \text{Gal}(K[mp^\infty]/K)$. For each prime $q \mid m$, let $F_q \in \mathcal{S}'$ denote the unique element of Γ_m which acts trivially on $K[q]$ and maps to Fr_q in $\Gamma_{m/q}$. Then, the factor $1 - q^{-k/2}\psi(\mathfrak{q})F_q^{-1}$ is invertible in $\mathbb{Z}_p[[\Gamma_m]]$. We now take

$$\kappa = \prod_{q \mid m} \left(1 - \frac{\psi(\mathfrak{q})}{q^{k/2}} F_q^{-1} \right)^{-1} \kappa_{\psi, \text{ad}^0(g), m, \infty}.$$

These classes clearly satisfy the required properties. $\square$

3.5.2 The explicit reciprocity law

Let K_∞ denote the anticyclotomic $\mathbb{Z}_p$ -extension of K and let

$$\kappa_{\psi, \text{ad}(g), \infty} \in H_{\text{Iw}}^1(K_\infty, T_{\text{ad}(g)}^\psi)$$

be the image of the class $\kappa_{\psi, \text{ad}(g), 1, \infty}$ in (3.12) under the corestriction map for $K[p^\infty]/K_\infty$.

By [ACR21, Prop. 6.6] we have $\kappa_{\psi, \text{ad}(g), \infty} \in \text{Sel}_{\text{bal}}(K_\infty, T_{\text{ad}(g)}^\psi)$; in particular, the restriction $\text{res}_{\bar{\mathfrak{p}}}(\kappa_{\psi, \text{ad}(g), \infty})$ lands in the image of the natural map

$$H_{\text{Iw}}^1(K_{\infty, \bar{\mathfrak{p}}}, \mathcal{F}_{\bar{\mathfrak{p}}}^{\text{bal}}(T_{\text{ad}(g)}^\psi)) \longrightarrow H_{\text{Iw}}^1(K_{\infty, \bar{\mathfrak{p}}}, T_{\text{ad}(g)}^\psi)$$

(see (3.4)). Note that this map is an injection under our hypotheses. On the other hand, let

$$\kappa_{\psi, \text{ad}^0(g), \infty} \in \text{Sel}_{\text{Gr}}(K_\infty, T) \tag{3.13}$$

be the image of the class $\kappa_1 = \kappa_{\psi, \text{ad}^0(g), 1, \infty}$ of Theorem 3.5.4 under the corestriction map. Thus $\kappa_{\psi, \text{ad}^0(g), \infty}$ is the projection of $\kappa_{\psi, \text{ad}(g), \infty}$ onto the first direct summand in

the decomposition

$$\mathrm{Sel}_{\mathrm{bal}}(K_{\infty}, T_{\mathrm{ad}(g)}^{\psi}) = \mathrm{Sel}_{\mathrm{bal}}(K_{\infty}, T) \oplus \mathrm{Sel}_{\mathrm{bal}}(K_{\infty}, T'),$$

and since $\mathcal{F}_{\bar{\mathfrak{p}}}^{\mathrm{bal}}(T_{\mathrm{ad}(g)}^{\psi})$ is contained in T , it is clear that

$$\mathrm{res}_{\bar{\mathfrak{p}}}(\kappa_{\psi, \mathrm{ad}(g), \infty}) = \mathrm{res}_{\bar{\mathfrak{p}}}(\kappa_{\psi, \mathrm{ad}^0(g), \infty}). \quad (3.14)$$

Definition 3.5.5. Put $\psi_0 = \psi\lambda^{2-k}$ as in Remark 3.4.3, and define

$$L_p(\mathrm{ad}^0(g_K) \otimes \psi) := \mathrm{Tw}_{\xi}(L_p(\mathrm{ad}^0(g_K) \otimes \psi_0)), \quad (3.15)$$

where $\mathrm{Tw}_{\xi} : \Lambda^{\mathrm{ac}} \rightarrow \Lambda^{\mathrm{ac}}$ is the twisting homomorphism for the character $\xi := (\lambda^{1-\tau})^{k/2-1}$. Similarly, define $\mathcal{L}_p(\mathrm{ad}(g_K) \otimes \psi)$, $\mathcal{L}_{\mathfrak{p}, \psi}^{\mathrm{Katz}}(K)^{-}$, $L_p(\mathrm{ad}(g_K) \otimes \psi)$, and $\mathcal{L}_{\mathfrak{p}}^{\mathrm{Katz}}(\psi)^{-, \iota}$ by twisting the corresponding p -adic L -functions defined for ψ_0 in §3.4.5.

For the statement of the next result, note that ψ_0 has the same restriction to $\Delta_{\mathfrak{c}}$ as ψ .

Theorem 3.5.6. *There exists an injective Λ^{ac} -module map with pseudo-null cokernel*

$$\mathfrak{Log} : H_{\mathrm{Iw}}^1(K_{\infty, \bar{\mathfrak{p}}}, \mathcal{F}_{\bar{\mathfrak{p}}}^{\mathrm{bal}}(T_{\mathrm{ad}(g)}^{\psi})) \longrightarrow \mathbb{Z}_p^{\mathrm{ur}}[[\Gamma^{\mathrm{ac}}]]$$

such that

$$\mathfrak{Log}(\mathrm{res}_{\bar{\mathfrak{p}}}(\kappa_{\psi, \mathrm{ad}^0(g), \infty})) = h_K \cdot \mathcal{L}_{\mathfrak{p}, \psi}^{\mathrm{Katz}}(K)^{-} \cdot \mathcal{L}_p(\mathrm{ad}(g_K) \otimes \psi).$$

Proof. Let $\mathbb{V}(\mathbf{f}, g, g^*)$ and $\mathbb{V}(\mathbf{f}, g, g^*)_f$ be as in §3.3 (corresponding to $\mathbb{V}_{\mathbf{f}gg^*}^{\dagger}$ and $\mathbb{V}_{\mathbf{f}}^{gg^*}$, respectively, in the notation of [ACR21, §8.2]). Then, identifying $G_{\mathbb{Q}_p}$ with $G_{K_{\bar{\mathfrak{p}}}}$ via the composition of the embedding $\iota_p : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}_p}$ fixed in the introduction with complex

conjugation, we get an isomorphism of $\Lambda_{\mathbf{f}}[G_{K_{\overline{\mathbf{p}}}}]$ -modules

$$\mathbb{V}(\mathbf{f}, g, g^*)_f = \mathbb{V}_{\mathbf{f}}^- \hat{\otimes}_{\mathcal{O}} T_g^+ \otimes T_{g^*}^+(\epsilon_{\text{cyc}}^{1-l} \kappa_f^{-1/2}) \simeq \mathcal{F}_{\overline{\mathbf{p}}}^{\text{bal}}(T_{\text{ad}^0(g)}^{\psi}) \otimes \xi \Psi_S^{(\tau-1)/2}. \quad (3.16)$$

By [ACR21, Thm. 7.4], after extending scalars to $\mathbb{Z}_p^{\text{ur}}$, the composition of the $\Lambda_{\mathbf{f}}$ -linear map

$$H^1(\mathbb{Q}_p, \mathbb{V}(\mathbf{f}, g, g^*)_f) \longrightarrow \Lambda_{\mathbf{f}}$$

of [*op. cit.*, Prop. 7.3] with the isomorphism $\mathbb{Z}_p^{\text{ur}} \hat{\otimes} \Lambda_{\mathbf{f}} \simeq \mathbb{Z}_p^{\text{ur}} \llbracket \Gamma^{\text{ac}} \rrbracket$ given by $\gamma \mapsto \gamma^{\tau-1}$ sends the class $\kappa^{(3)}(\mathbf{f}, g, g^*)$ recalled in §3.3 to the product

$$h_K \cdot \mathcal{L}_{\mathbf{p}, \psi_0}^{\text{Katz}}(K)^- \cdot \mathcal{L}_p(\text{ad}(g_K) \otimes \psi_0),$$

noting that by Proposition 3.4.6 the first two factors generate the congruence ideal of $\mathbf{f}$. Taking twists by ξ and using the isomorphism

$$H_{\text{Iw}}^1(K_{\infty, \overline{\mathbf{p}}}, \mathcal{F}_{\overline{\mathbf{p}}}^{\text{bal}}(T_{\text{ad}^0(g)}^{\psi}) \otimes \xi) \simeq H^1(\mathbb{Q}_p, \mathbb{V}(\mathbf{f}, g, g^*)_f)$$

induced by (3.16) and using (3.14), the result follows. $\square$

Corollary 3.5.7. *The map $\mathfrak{Log}$ of Theorem 3.5.6 satisfies*

$$(\mathfrak{Log}(\text{res}_{\overline{\mathbf{p}}}(\kappa_{\psi, \text{ad}^0(g), \infty}))^2) = (L_p(\text{ad}^0(g_K) \otimes \psi) \cdot \mathcal{L}_{\mathbf{p}}^{\text{Katz}}(\psi)^{-, \iota})$$

as ideals in $\mathbb{Z}_p^{\text{ur}} \llbracket \Gamma^{\text{ac}} \rrbracket \otimes \mathbb{Q}_p$.

Proof. This is clear from Theorem 3.5.6 and the factorization in Theorem 3.4.9. $\square$

3.6 Verifying the hypotheses

Let g and ψ be as introduced in §3.2.1 and §3.2.2, respectively. Recall that, given a rational prime p and a sufficiently large finite extension $E/\mathbb{Q}_p$, we define the G_K -representation

$$V = \mathrm{ad}^0(V_g)(\psi_{\mathfrak{P}}^{-1})(1 - k/2),$$

where $\rho_g : G_{\mathbb{Q}} \rightarrow \mathrm{Aut}_E(V_g) \simeq \mathrm{GL}_2(E)$ is the p -adic Galois representation attached to g . The aim of this section is to give conditions under which the hypotheses in the general results of [JNS] are satisfied for V . Let $K(p^{\infty})^{\circ}$ denote the maximal abelian extension of K unramified at primes not dividing p . Then, the crucial condition that we need to verify is the existence of an element $\sigma \in \mathrm{Gal}(\bar{K}/K(p^{\infty})^{\circ})$ such that $T/(\sigma - 1)T$ is a free $\mathcal{O}$ -module of rank one, where $\mathcal{O}$ is the ring of integers of E .

As in [Loe17, §3.1] we define an open subgroup $H_g \subseteq G_{\mathbb{Q}}$, a quaternion algebra B_g and an algebraic group G_g . Let $H = H_g \cap G_{K(c)^{\circ}}$. Then we have an adelic representation

$$\tilde{\rho}_g : H \longrightarrow G_g(\hat{\mathbb{Q}})$$

and representations

$$\tilde{\rho}_{g,p} : H \longrightarrow G_g(\mathbb{Q}_p)$$

for every rational prime p , and, according to [Loe17, Thm. 2.2.2], for all but finitely many p we can conjugate $\tilde{\rho}_{g,p}$ so that $\tilde{\rho}_{g,p}(H) = G_g(\mathbb{Z}_p)$.

Let L be a finite extension of K containing the Fourier coefficients of g and the image of the Hecke character ψ . Let $\mathfrak{P}$ be a prime of L above some rational prime p , and let $E = L_{\mathfrak{P}}$.

Definition 3.6.1. We say that the prime $\mathfrak{P}$ is *good* if the following conditions hold:

- $p \geq 3$;
- p is unramified in B_g ;

- p is coprime to $\mathfrak{c}$ and N_g ;
- $\tilde{\rho}_{g,p}(H) = G_g(\mathbb{Z}_p)$;
- $E = \mathbb{Q}_p$.

Lemma 3.6.2. *Assume that there is at least one prime which divides D_K but not N_g . Then, if $\mathfrak{P}$ is a good prime,*

$$\rho_{g,\mathfrak{P}}(H \cap G_{K(p^\infty)^\circ}) = \mathrm{SL}_2(\mathbb{Z}_p).$$

Proof. The proof of this result is very similar to the proof of [ACR21, Lem. 8.9]. We include it here for the convenience of the reader.

Let $\mathbb{Q}(\rho_g)$ be the Galois extension of $\mathbb{Q}$ cut out by the representations ρ_g . It is unramified outside pN_g . Therefore, the condition on D_K implies that $K \cap \mathbb{Q}(\rho_g) = \mathbb{Q}$. Moreover, since any Galois extension of $\mathbb{Q}$ contained in the anticyclotomic $\mathbb{Z}_p$ -extension K_∞ of K must itself contain K , we also have $K_\infty \cap \mathbb{Q}(\rho_g) = \mathbb{Q}$.

The conditions on $\mathfrak{P}$ imply that

$$\rho_{g,\mathfrak{P}}(H \cap G_{\mathbb{Q}(\mu_{p^\infty})}) = \mathrm{SL}_2(\mathbb{Z}_p),$$

and, from the remarks in the previous paragraph, it follows that

$$\rho_{g,\mathfrak{P}}(H \cap G_{K_\infty(\mu_{p^\infty})}) = \mathrm{SL}_2(\mathbb{Z}_p).$$

Finally, since $H \cap G_{K(p^\infty)^\circ}$ is a normal subgroup of $H \cap G_{K_\infty(\mu_{p^\infty})}$ of index dividing $p-1$ and there are no such subgroups in $\mathrm{SL}_2(\mathbb{Z}_p)$, the lemma follows. $\square$

Now fix a good prime $\mathfrak{P}$ and define $\mathbb{Z}_p[G_K]$ -modules $T = \mathrm{ad}^0 T_g(\psi_{\mathfrak{P}}^{-1})(1 - k/2)$ and $T' = \mathbb{Z}_p(\psi_{\mathfrak{P}}^{-1})(1 - k/2)$. Let $V = T \otimes \mathbb{Q}_p$ and $V' = T' \otimes \mathbb{Q}_p$.

Proposition 3.6.3. *Assume that there is at least one prime which divides D_K but not N_g . Suppose that there exists $\eta \in G_{K(p^\infty)^\circ}$ such that $\chi_g(\eta)\psi_{\mathfrak{p}}(\eta)$ is a square modulo p and $\psi_{\mathfrak{p}}(\eta)^2 \neq 1$ modulo p . Then there exists $\sigma \in G_{K(p^\infty)^\circ}$ such that*

- $T/(\sigma - 1)T$ is free of rank 1 over $\mathbb{Z}_p$,
- $\sigma - 1$ acts invertibly on T' .

Proof. We closely follow the proof of [LZ19, Prop. 5.2.1] (see also the proof of [ACR21, Lem 5.10] and [Loe17, Prop. 4.2.1]). By the previous lemma the image of $\eta H \cap G_{K(p^\infty)^\circ}$ under $\rho_{g, \mathfrak{p}}$ contains all the elements of the form

$$\begin{pmatrix} x & 0 \\ 0 & x^{-1}\chi_g(\eta) \end{pmatrix}, \quad x \in \mathbb{Z}_p^\times.$$

Choose x such that $x^2 = \chi_g(\eta)\psi_{\mathfrak{p}}(\eta)$. Choose $\sigma \in \eta H \cap G_{K(p^\infty)^\circ}$ whose image under $\rho_{g, \mathfrak{p}}$ is given by the element above, with the choice of x which we have just specified. Then, the eigenvalues of σ acting on T are 1, $\psi_{\mathfrak{p}}^{-1}(\eta)$ and $\psi_{\mathfrak{p}}^{-2}(\eta)$ and the eigenvalue of σ acting on T' is $\psi_{\mathfrak{p}}^{-1}(\eta)$. The result follows from the assumptions on η . $\square$

3.7 Applications

Let $g \in S_l(N_g, \chi_g)$ and ψ be a Hecke character of K of infinity type $(1 - k, 0)$ for some even integer $k \geq 2$ as introduced in §3.2, and recall that we consider the E -valued G_K -representation V in Definition 3.2.1. We begin by collecting a set of hypotheses for our later reference.

Hypotheses.

(h1) p splits in K ,

(h2) $p \nmid h_K$,

- (h3) the conditions in Proposition 3.4.6 hold,
- (h4) g is ordinary at p and non-Eisenstein mod p ,
- (h5) g is not of CM type,
- (h6) $\mathfrak{P}$ is a good prime in the sense of Definition 3.6.1,
- (h7) the conditions in Proposition 3.6.3 hold.

3.7.1 The Bloch–Kato conjecture

We begin with a standard lemma, whose proof follows from the same argument as in [ACR21, Lem. 9.1].

Lemma 3.7.1. *The Bloch–Kato Selmer group of V is given by*

$$\mathrm{Sel}(K, V) \simeq \begin{cases} \mathrm{Sel}_{\mathrm{bal}}(K, V) & \text{if } 2 \leq k < 2l, \\ \mathrm{Sel}_{\mathrm{unb}}(K, V) & \text{if } k \geq 2l. \end{cases}$$

Let $\kappa_{\psi, \mathrm{ad}^0(g), \infty}$ be as in (3.13), and denote by

$$\kappa_{\psi, \mathrm{ad}^0(g)} \in \mathrm{Sel}_{\mathrm{bal}}(K, T)$$

the image of $\kappa_{\psi, \mathrm{ad}^0(g), \infty}$ under the corestriction $H_{\mathrm{Iw}}^1(K_\infty, T) \rightarrow H^1(K, T)$.

Theorem 3.7.2. *Assume hypotheses (h1)–(h7). Then the following implication holds:*

$$\kappa_{\psi, \mathrm{ad}^0(g)} \neq 0 \implies \dim_E \mathrm{Sel}_{\mathrm{Gr}}(K, V) = 1.$$

In particular, if $2 \leq k < 2l$ and $\kappa_{\psi, \mathrm{ad}^0(g)} \neq 0$ then the Bloch–Kato Selmer group $\mathrm{Sel}(K, V)$ is one-dimensional.

Proof. This follows from the general theory of anticyclotomic Euler systems developed in [JNS] (see [ACR21, §8] for a summary) applied to the Euler system constructed in Theorem 3.5.4. By Proposition 3.6.3, Hypotheses 3.7 give sufficient conditions for the general results of [JNS] to apply in our case. Note also that for the application of these results it suffices to have an anticyclotomic Euler system consisting of classes indexed by squarefree products of primes q in a positive density set $\mathcal{P}'$ of primes split in K , as is the case for the anticyclotomic Euler system of Theorem 3.5.4 (see Remark 3.5.3). $\square$

Theorem 3.7.2 can be viewed as a result towards the Bloch–Kato conjecture for V in rank 1. The next result establishes cases of the same conjecture in rank 0.

Theorem 3.7.3. *Assume hypotheses (h1)–(h7), and in addition that:*

- $\varepsilon_\ell(V_{fgg^*}) = +1$ for all primes $\ell \mid N$,
- $\gcd(N_g, N_\psi)$ is squarefree,
- $L(\theta_\psi, k/2) \neq 0$.

If $k \geq 2l$ then the following implication holds:

$$L(V, 0) \neq 0 \implies \text{Sel}(K, V) = 0.$$

Proof. By Theorem 3.4.2 and Definition 3.5.5 we see that

$$L(V, 0) \neq 0 \implies L_p(\text{ad}^0(g_K) \otimes \psi)(\xi_{\text{triv}}) \neq 0,$$

where ξ_{triv} is the trivial character of Γ^{ac} . Similarly, from Theorem 3.4.5 and Definition 3.5.5 we see that

$$L(\theta_\psi, k/2) \neq 0 \implies \mathcal{L}_{\mathfrak{p}}^{\text{Katz}}(\psi)^{-, \iota}(\xi_{\text{triv}}) \neq 0.$$

Therefore by the factorization in Theorem 3.4.9 we thus see that $L_p(\mathrm{ad}(g_K) \otimes \psi)(\xi_{\mathrm{triv}}) \neq 0$, and so $\kappa_{\psi, \mathrm{ad}^0(g)} \neq 0$ by the explicit reciprocity law of Corollary 3.5.7. The result now follows from Theorem 3.7.2 and global duality by the same argument as in [ACR21, Thm. 9.5]. $\square$

Remark 3.7.4. The hypotheses in Theorem 3.7.3 and the decomposition (3.2) imply that the sign of the functional equation for $L(V, s)$ is $+1$, and so the nonvanishing of $L(V, 0)$ is expected to hold generically.

3.7.2 The Iwasawa main conjecture

Here we deduce our main result towards the anticyclotomic Iwasawa main conjecture for V .

Since ψ has central character ε_K by assumption, its associated theta series θ_ψ has trivial nebentypus. In the following we denote by $\varepsilon(\theta_\psi)$ its global root number.

Theorem 3.7.5. *Assume hypotheses (h1)–(h7), and in addition that:*

- $\varepsilon_\ell(V_{fgg^*}) = +1$ for all primes $\ell \mid N$,
- $\varepsilon(\theta_\psi) = +1$,
- $\gcd(N_g, N_\psi)$ is squarefree.

If the p -adic L -function $L_p(\mathrm{ad}^0(g_K) \otimes \psi)$ is nonzero, then the Pontryagin dual of $\mathrm{Sel}_{\mathrm{unb}}(K_\infty, A)$ is Λ^{ac} -torsion, with

$$\mathrm{Char}_{\Lambda^{\mathrm{ac}}}(\mathrm{Sel}_{\mathrm{unb}}(K_\infty, A)^\vee) \supset (L_p(\mathrm{ad}^0(g_K) \otimes \psi) \cdot \mathcal{L}_p^{\mathrm{Katz}}(\psi)^{-, \iota})$$

in $\mathbb{Z}_p^{\mathrm{ur}}[[\Gamma^{\mathrm{ac}}]] \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$.

Proof. The assumption that $\varepsilon(\theta_\psi) = +1$ implies that the anticyclotomic projection $\mathcal{L}_p^{\mathrm{Katz}}(\psi)^{-, \iota}$ is nonzero by Greenberg's nonvanishing results [Gre83]. Since $L_p(\mathrm{ad}^0(g_K) \otimes$

$\psi) \neq 0$ by hypothesis, together with the factorization in Theorem 3.4.9 it follows that

$$L_p(\mathrm{ad}(g_K) \otimes \psi) \neq 0.$$

By Corollary 3.5.7, this shows that the class $\kappa_{\psi, \mathrm{ad}^0(g), \infty}$ is non-torsion. By the general results of [JNS] (see also [ACR21, Thm. 8.5]), we thus conclude that $X_{\mathrm{bal}}(K_\infty, A)$ and $\mathrm{Sel}_{\mathrm{bal}}(K_\infty, T)$ both have Λ^{ac} -rank one, with

$$\mathrm{Char}_{\Lambda^{\mathrm{ac}}}(X_{\mathrm{bal}}(K_\infty, A)_{\mathrm{tors}}) \supset \mathrm{Char}_{\Lambda^{\mathrm{ac}}}\left(\frac{\mathrm{Sel}_{\mathrm{bal}}(K_\infty, T)}{\Lambda^{\mathrm{ac}} \cdot \kappa_{\psi, \mathrm{ad}^0(g), \infty}}\right)^2.$$

The result now follows from this by the same argument as in the proof of [ACR21, Thm. 7.15] based on Poitou–Tate duality and the explicit reciprocity law of Corollary 3.5.7. □

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