

**From the holdings of the Department of Special Collections , Princeton University Library,
Princeton, NJ**

These documents are either in the public domain, or can only be used for educational and research purposes (Fair use) as per U.S. Copyright law (text below). By accessing this file, all users agree that their use falls in the public domain or within fair use as defined by the copyright law. We do not charge any permission or usage fees for the publication of images of material in our collections, including those provided by Princeton University Library, via our website, catalog, or directly from staff. For more information regarding copyright, credit, and citation please review the information on the Copyright, Credit and Citation Guidelines page of our website.

<https://library.princeton.edu/special-collections/policies/copyright-credit-and-citation-guidelines>. If you have any questions, including how to order higher-quality images from the collection, contact Special Collections Public Services via our Ask Us! Form. <https://library.princeton.edu/special-collections/ask-us>

Doctoral Dissertations, 2011-2026

Mathematics

2022

Cosmetic Surgeries on Knots and 3-Manifold Invariants

Varvarezos, Konstantinos

Princeton, NJ : Princeton University

<http://arks.princeton.edu/ark:/88435/dsp010r967690m>

U.S. Copyright law: The copyright law of the United States (Title 17, §108, United States Code) governs the making of photocopies or other reproductions of copyrighted material. Under certain conditions specified in the law, libraries and archives are authorized to furnish a photocopy or other reproduction. One of these specified conditions is that the photocopy or reproduction is not to be used for any purpose other than private study, scholarship, or research. If a user makes a request for, or later uses, a photocopy or reproduction for purposes in excess of fair use that user may be liable for copyright infringement. This institution reserves the right to refuse to accept a copying order if, in its judgment, fulfillment of the order would involve a violation of copyright law. <https://www.copyright.gov/title17/title17.pdf>

COSMETIC SURGERIES ON KNOTS AND 3-MANIFOLD INVARIANTS

KONSTANTINOS VARVAREZOS

A DISSERTATION
PRESENTED TO THE FACULTY
OF PRINCETON UNIVERSITY
IN CANDIDACY FOR THE DEGREE
OF DOCTOR OF PHILOSOPHY

RECOMMENDED FOR ACCEPTANCE
BY THE DEPARTMENT OF
MATHEMATICS
ADVISER: ZOLTÁN SZABÓ

MAY 2022

© Copyright by Konstantinos Varvarezos, 2022
All Rights Reserved

Abstract

A common method of constructing 3-manifolds is via Dehn surgery on knots. A pair of surgeries on a knot is called purely cosmetic if the pair of resulting 3-manifolds are homeomorphic as oriented manifolds, whereas it is said to be chirally cosmetic if they result in homeomorphic manifolds with opposite orientations. An outstanding conjecture predicts that no nontrivial knots admit any purely cosmetic surgeries. We apply certain obstructions from Heegaard Floer homology to show that (nontrivial) knots which arise as the closure of a 3-stranded braid do not admit any purely cosmetic surgeries. Furthermore, we find new obstructions to the existence of chirally cosmetic surgeries coming from Heegaard Floer homology; in particular, we make use of immersed curve formulations of knot Floer homology and the corresponding surgery formula. Combining these with other obstructions involving finite type invariants, we completely classify chirally cosmetic surgeries on odd alternating pretzel knots, and we rule out such surgeries for a large class of Whitehead doubles. Moreover, we rule out cosmetic surgeries for L-space knots along slopes with opposite signs.

Acknowledgments

Firstly, many thanks are due to the author's PhD adviser Zoltán Szabó for continual support, encouragement, and helpful discussions throughout the author's time as a graduate student. The author also thanks Professor Peter Ozsváth, whose guidance and conversations over the past years made him almost like a second adviser. Furthermore, thanks go to Jonathan Hanselman whose own work as well as stimulating lectures and conversations with the author inspired much of the work in this thesis.

The author would also like to thank the staff of the Princeton Mathematics Department, and especially Graduate Program Administrator Jill LeClair, for ensuring everything ran smoothly and for their friendly assistance whenever it was needed.

Finally, thanks go to the author's family and friends for their moral support and for making the graduate student experience more enjoyable.

Contents

Abstract	3
Acknowledgments	4
1 Introduction	7
2 Background	12
2.1 Dehn surgery	12
2.2 Heegaard Floer Floer homology	15
2.2.1 Immersed curves	17
2.2.2 Gradings	21
2.3 Finite Type Invariants	22
3 Purely Cosmetic Surgeries and 3-Braid Knots	27
3.1 Obstructions	27
3.2 Band presentation and genus	28
3.3 Knot Floer Homology and thickness	32
3.4 Proof of Theorem 1.2	34
4 Chirally Cosmetic Surgeries	36
4.1 Obstructions	36
4.1.1 Heegaard Floer	36
4.1.2 Finite Type	40

4.1.3	Combining the obstructions	41
4.2	Pretzel knots	42
4.2.1	Computing finite type invariants	42
4.2.2	The case of genus 2 and 3	49
4.2.3	Computing Heegaard Floer invariants	60
4.2.4	The proof of the general case	65
4.3	Whitehead doubles	71
4.4	L-space knots	73
4.4.1	Proof of Theorem 1.10	76

Chapter 1

Introduction

Given a knot K in S^3 and an $r \in \mathbf{Q} \cup \{\infty\}$, we denote the *Dehn surgery* on K with slope r by $S_K^3(r)$. Surgeries on K along distinct slopes r and r' are called *cosmetic* if $S_K^3(r)$ and $S_K^3(r')$ are homeomorphic manifolds.

Furthermore, a pair of such surgeries is said to be *purely cosmetic* if $S_K^3(r)$ and $S_K^3(r')$ are homeomorphic as *oriented* manifolds. We use the symbol $\cong$ to denote “orientation-preserving homeomorphic.” If, on the other hand, $S_K^3(r) \cong -S_K^3(r')$, we say this pair of surgeries is *chirally cosmetic*; here $-M$ denotes the manifold M with the opposite orientation.

No purely cosmetic surgeries are have been found on nontrivial knots in S^3 , and indeed the following question remains unsolved:

Question 1.1. *Does any nontrivial knot admit a pair of purely cosmetic surgeries?*

The Cosmetic Surgery Conjecture predicts a negative answer to this question. Many powerful obstructions to purely cosmetic surgeries are known. Heegaard Floer techniques have been particularly effective for this, as well as methods involving finite type invariants. More recently, immersed curve versions of Heegaard Floer invariants were employed by Hanselman to give even stronger obstructions. See Sections 3.1 and 4.1 below for examples of these obstructions. The conjecture has been confirmed for genus 1 knots [42], 2-bridge knots [14], composite knots [38], pretzel knots [36], as well as knots up to 16 crossings [7].

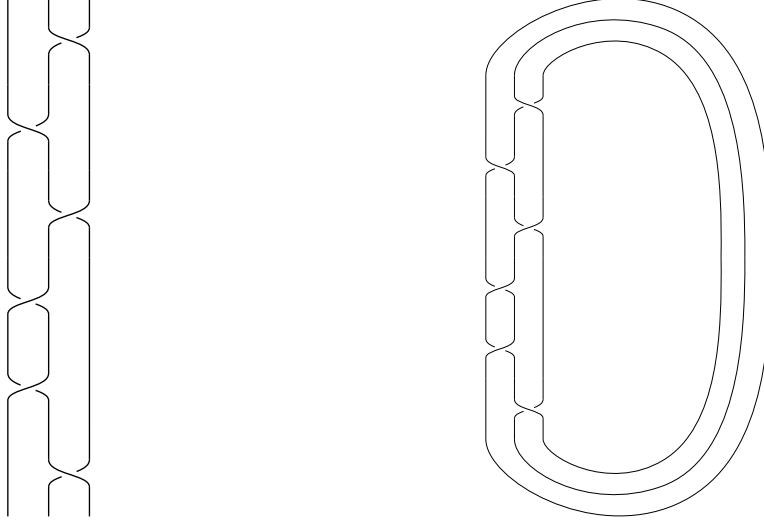


Figure 1.1: An example of a 3-braid (left), and its corresponding closure (right)

It turns out that 3-braid knots are a family of knots which is especially amenable to these techniques. Recall that a 3-braid is a configuration of three intervals (“strands”) embedded in $\mathbf{R}^3$, arranged vertically such that the z -coordinate of each “strand” is strictly increasing, considered up to isotopy fixing the endpoints. A knot (or link) may be obtained from a braid by closing off the endpoints, as indicated in Figure 1.1.

In Chapter 3, following [39], we show that 3-braid knots satisfy the Cosmetic Surgery Conjecture:

Theorem 1.2 (Theorem of 1.1 [39]). *Suppose K is a nontrivial knot which is the closure of a 3-braid. Then K admits no purely cosmetic surgeries.*

On the other hand, there are examples of chirally cosmetic surgeries; amphichiral knots all admit chirally cosmetic surgeries, and the $(2, 2n + 1)$ -torus knots are also known to admit such surgeries. Several families of knots have been shown not to admit any chirally cosmetic surgeries: genus-1 alternating knots [11] as well as (p, q) -cables of nontrivial knots with $q \neq 2$ [15]. In fact, the following question remains open:

Question 1.3. *Suppose a knot K is not amphichiral and is not a $(2, n)$ -torus knot. Does K admit any chirally cosmetic surgeries?*

In the case of the $(2, n)$ -torus knots, each pair of slopes which gives rise to chirally cosmetic surgeries has slopes with the same sign. Indeed, previously known obstructions coming from Heegaard Floer homology imply that knots which admit chirally cosmetic surgeries along slopes with the same sign are rare:

Theorem 1.4 (Theorem 9.8 of [31]). *Let K be a knot and suppose $S_K^3(r) \cong \pm S_K^3(r')$ for two distinct slopes r, r' . Then either r and r' have opposite signs or $S_K^3(r)$ is an L-space (hence K is an L-space knot).*

Being an L-space knot is a very restrictive condition; it implies, for instance, fiberedness [24] as well as an Alexander polynomial with coefficients in $\{-1, 0, -1\}$ (see [31] for a more precise constraint). One may hope to approach Question 1.3, therefore, by investigating the following:

Question 1.5. *Suppose a knot K is not amphichiral. Does K admit any chirally cosmetic surgeries along slopes with opposite signs?*

A negative answer to this question would reduce Question 1.3 to classifying cosmetic surgeries of L-space knots. To this end, our new obstruction is a step towards a better understanding of Question 1.5. In particular, we obtain a restriction on the possible cosmetic surgeries along slopes of opposite signs for knots with Heegaard Floer-theoretic invariant τ equal to their genus.

Theorem 1.6 (Theorem 1.4 of [41]). *Suppose a knot K satisfies $\tau(K) = g(K) > 0$. If $S^3(p/q) \cong -S^3(p/q')$ where $p, q, q' \in \mathbf{Z}$ with $p, q > 0$ and $q' < 0$, then $\frac{p}{q} \leq 2g(K) - 1$ and $2p = (\text{Vert}(K) + 2g(K) - 1)(q + q')$.*

Here Vert is a knot invariant arising from the immersed curve version of Knot Floer Homology; see Section 4.1 below for more details.

Of course, non-amphichiral knots need not satisfy $\tau = 0$, much less have τ equal to genus. As amphichiral knots satisfy $\tau = 0$, the hypotheses of this result apply to knots

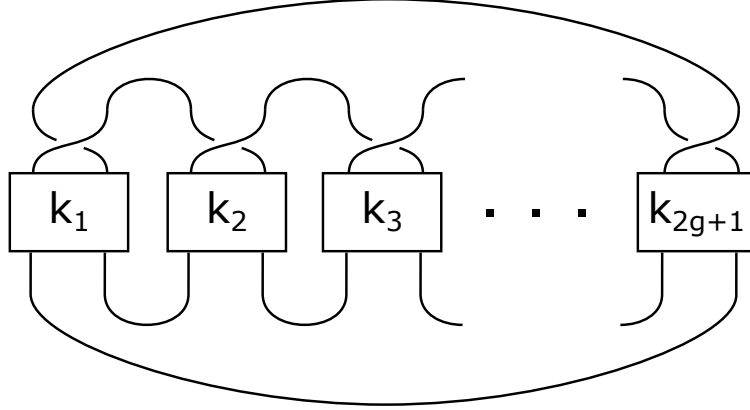


Figure 1.2: The pretzel knot $P(2k_1 + 1, 2k_2 + 1, 2k_3 + 1, \dots, 2k_{2g+1} + 1)$. Here the boxes represent k_i full right-handed twists of the strands passing through.

which are “very far” from being amphichiral, in a certain sense. Furthermore, even though the restriction on surgery slopes is *a priori* an infinite set of pairs, it is possible to combine it with a similar kind of obstruction arising from finite type invariants to exclude all slopes in many situations.

Our first application of this obstruction is to the family of *alternating odd pretzel knots*. Up to mirror image, these are pretzel knots of the form $P(2k_1 + 1, 2k_2 + 1, \dots, 2k_{2g+1} + 1)$, where the k_i are all nonnegative. See Figure 1.2 for an illustration. It is a fact that g is the Seifert genus of the knot. For convenience, we shall often use the shorthand $K(k_1, k_2, \dots, k_{2g+1})$ to denote these knots.

These knots are already known not to admit any purely cosmetic surgeries. This can be seen, for instance, from the fact that the signatures and v_3 of these knots are nonzero (see section 4.2.1 below). By Theorem 5 of [7], alternating knots that admit purely cosmetic surgeries must have zero signature. Alternatively, by Theorem 3.2, knots admitting purely cosmetic surgeries must have $v_3 = 0$.

Theorem 1.7 (Theorem 1.5 of [41]). *Let $K = K(k_1, \dots, k_{2g+1})$ where $k_i \geq 0$ for all i . If at least one $k_i > 0$, then K does not admit any chirally cosmetic surgeries.*

Remark 1.8. *The knots $K(0, 0, \dots, 0)$ are the $(2, 2g + 1)$ -torus knots, which are already known to admit (infinitely many pairs of) chirally cosmetic surgeries; see Corollary A.2 of*

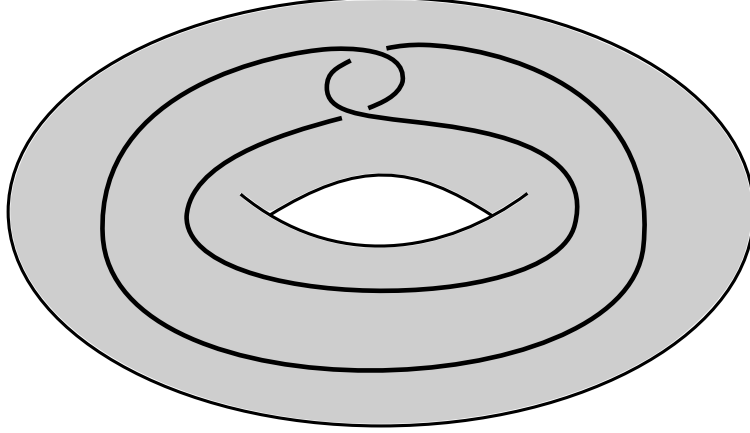


Figure 1.3: The positive Whitehead pattern inside a solid torus.

[11] for a full classification of the chirally cosmetic surgeries for these knots.

Our next application pertains to Whitehead doubles of knots. Recall that the n -framed positive Whitehead double of a knot K , which we denote $D_+(K, n)$ is the knot obtained by gluing in a solid torus V with the positive Whitehead pattern (see Figure 1.3) to the exterior of K such that the standard longitude of V is identified with a n -framed longitude of K , and the meridian of V is glued to a meridian of K .

Our obstruction may be used to find:

Theorem 1.9 (Theorem 1.7 of [41]). *Let a knot K be given, and suppose $a_2(K) = 0$. If $|n| \geq 5$, then $D_+(K, n)$ does not admit any chirally cosmetic surgeries. If, moreover, $2\tau(K) > n$ for $n \in \{-4, -2, 1, 4\}$, then $D_+(K, n)$ also admits no chirally cosmetic surgeries.*

In the final section, we turn our attention to the case of L-space knots. As an application of our new obstruction, we prove a partial converse to Theorem 1.4:

Theorem 1.10 (Theorem 1.8 of [41]). *If K is a nontrivial L-space knot, then K does not admit any cosmetic surgeries along slopes of opposite signs.*

Remark 1.11. *Some similar results derived from Heegaard Floer homology were found independently by Ichihara, Ito, and Saito [12]. In particular, their Lemma 3.3 expresses effectively the same result as our Lemma 4.15.*

Chapter 2

Background

We first review the necessary concepts involved, including Dehn surgery on knots as well as the 3-manifold and knot invariants that will be used throughout. First, a word on notation: given two oriented 3-manifolds M_1 and M_2 , we write $M_1 \cong M_2$ to mean “ M_1 is orientation-preserving homeomorphic to M_2 .” If M is an oriented 3-manifold, then $-M$ denotes the same manifold with the reverse orientation. Similarly, if K is a knot, we shall write $\overline{K}$ to mean the mirror image of K . As a matter of convention, all singular homology groups will have integral coefficients unless otherwise specified.

2.1 Dehn surgery

Given a knot $K \subset S^3$ the *exterior* of K is $S^3 \setminus N(K)$, where $N(K)$ is an open tubular neighborhood of K . As a shorthand, we denote the exterior of K by $E(K)$, and note that we also consider $E(K)$ to be an *oriented* manifold. It is a fact that $H_1(E(K)) \cong \mathbf{Z}$ for any knot. We describe the standard parametrization of the boundary $\partial E(K)$. A (non-nullhomotopic) simple closed curve in the boundary of the knot exterior is called a *meridian* of K if it was the boundary of a disk in $N(K)$. It is a fact that a meridian represents a generator of $H_1(E(K))$. A simple closed curve in the boundary of $E(K)$ is called a *longitude* K if it intersects a meridian at exactly one point and is nullhomologous in $H_1(E(K))$. Strictly

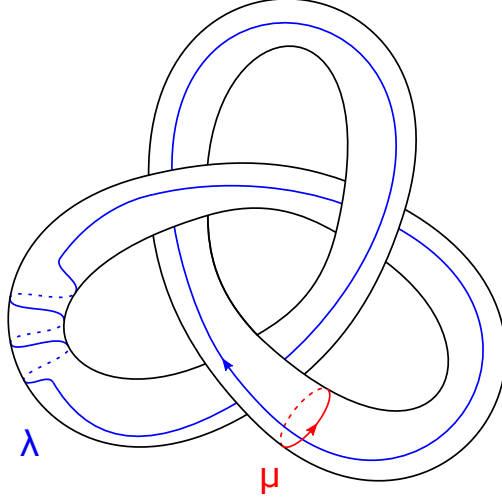


Figure 2.1: The exterior of the right-handed trefoil, with meridian μ and longitude λ .

speaking, the meridian and longitude are *oriented* curves, and their orientations are chosen such that $[\mu] \cdot [\lambda] = +1$ with respect to the orientation induced on $\partial E(K)$ by the inward-pointing normal. Assuming the usual “right-handed” orientation on S^3 , the meridian obeys a “right-hand-rule” with respect to the direction parallel to the longitude. See Figure 2.1 for an illustration in the case of the trefoil.

Knots in S^3 provide a way to obtain new 3-manifolds a process which we now describe:

Construction 2.1 (Dehn Surgery). *Let a knot $K \subset S^3$ and a (reduced) fraction $\frac{p}{q}$ be given. Let μ be a meridian of K , and let λ be a longitude of K . We define Dehn surgery on K with slope $\frac{p}{q}$ to be:*

$$S_K^3(p/q) := E(K) \cup_{\phi} S^1 \times D^2,$$

where ϕ is an identification of the boundary tori $\phi : \partial(S^1 \times D^2) \rightarrow \partial E(K)$ such that $\phi(\{\star\} \times \partial D^2)$ represents $p\mu + q\lambda$ in $H_1(\partial E(K))$.

It is a fact that the homeomorphism type of the resulting manifold depends only on the homology class of $\phi(\{\star\} \times \partial D^2)$, so that this construction is well-defined. The fraction $\frac{p}{q}$ is often called the *slope* of the surgery, and we allow the case $p = 1$ and $q = 0$, which is often called ∞ -surgery. Moreover, the manifold obtained by Dehn surgery has a canonical orientation, induced by the orientation of the knot exterior $E(K)$.

One can view Dehn surgery as a function DS that takes in a knot and a surgery slope and outputs a closed 3-manifold; schematically:

$$DS : \{\text{Knots}\} \times (\mathbf{Q} \cup \{\infty\}) \rightarrow \{\text{3-manifolds}\}$$

$$(K, p/q) \mapsto S_K^3(p/q)$$

Viewed this way, it is natural to ask: *how injective is the function DS ?*

First, we notice that S^3 may be obtained as surgery along any knot: simply “glue” back the solid torus to match up exactly with the neighborhood of K that was “cut out.” This corresponds with a surgery with slope $\frac{1}{0}$, so that $S_K^3(1/0) = S^3$ for any knot K . A celebrated result of Gordon and Luecke says that for any *nontrivial* knot, no other surgery besides the above yields S^3 [6]. On the other hand, as the exterior of the trivial knot is a solid torus, it is straightforward to see that $S_U^3(1/n) \cong S^3$ for all $n \in \mathbf{Z}$, where U is the trivial knot.

It is interesting to investigate more refined questions regarding the “injectivity” of Dehn surgery. To that end, we introduce the following terminology:

Definition 2.2. *Fix a knot $K \subset S^3$. Two surgeries along slopes r_1 and r_2 are said to be:*

- purely cosmetic if $S_K^3(r_1) \cong S_K^3(r_2)$.
- chirally cosmetic if $S_K^3(r_1) \cong -S_K^3(r_2)$.

As we saw above, the $1/n$ -surgeries on the trivial knot are all cosmetic. A similar construction shows that the trivial knot admits cosmetic surgeries along other slopes as well. It is not known if any nontrivial knot admits any purely cosmetic surgeries:

Conjecture 2.3 (Cosmetic Surgery Conjecture). *If $K \subset S^3$ is a nontrivial knot, then K admits no purely cosmetic surgeries.*

This is a special case of the more general Problem 1.81(A) of Kirby’s List [18].

It is a fact that $H_1(S_K^3(p/q)) \cong \mathbf{Z}/p\mathbf{Z}$. It follows that if a knot admits cosmetic surgeries (pure or chiral) along slopes p/q and p'/q' , then $p = p'$.

Observe that for any knot K and slope r , we have that $S_K^3(r) \cong -S_{\overline{K}}^3(-r)$. Hence, if K is an amphichiral knot (that is, $K = \overline{K}$), then $S_K^3(r) \cong -S_K^3(-r)$. Thus for any $r \in \mathbf{Q} \setminus \{0\}$, such a knot admits chirally cosmetic surgeries along slopes $\pm r$. Furthermore, $(2, n)$ -torus knots are known to admit chirally cosmetic surgeries, precisely along the pairs of slopes $\frac{2n^2(2m+1)}{n(2m+1) \pm 1}$ for each positive integer m ; see [11, 23].

2.2 Heegaard Floer Floer homology

Here we review the relevant background regarding the Heegaard Floer theoretic invariants we shall use to obstruct cosmetic surgeries. Originally developed by Ozsváth and Szabó [29], Heegaard Floer homology is a family of 3-manifold invariants. Briefly, the construction is as follows: given a 3-manifold M , one obtains a *Heegaard diagram*, which is a genus g surface Σ , equipped with two sets α and β of g simple closed curves on Σ , encoding a Heegaard splitting for M , together with a choice of basepoint $z \in \Sigma$. Then, one considers the manifold $X = \text{Sym}^g(\Sigma)$, which is the g -fold symmetric product of Σ ; that is, the space of unordered g -tuples of points in Σ . X can be given a symplectic structure, and the subspaces of X corresponding to all points of the tuple being in α or all in β are g -dimensional Lagrangian tori. Then, Heegaard Floer homology is essentially Lagrangian Floer homology on these two tori. The basepoint z gives rise to a subspace $V_z \subset X$ corresponding to at least one point in the tuple being z . The different versions of Heegaard Floer homology correspond to different treatments of pseudoholomorphic disks which intersect V_z . It is a fact that the invariants are independent of auxiliary choices such as the basepoint.

The “hat” version of Heegaard Floer homology assigns to a closed, oriented 3-manifold M a graded vector space over $\mathbf{F} = \mathbf{Z}/2\mathbf{Z}$ denoted $\widehat{HF}(M)$. If $b_1(M) = 0$, then $\widehat{HF}(M)$ admits a $\mathbf{Z}$ -grading called the Maslov grading. Furthermore, this vector space decomposes along the Spin^c -structures of M , which are (non-canonically) in bijection with the elements of $H_1(M; \mathbf{Z})$. That is, for each Spin^c -structure $\mathfrak{s} \in \text{Spin}^c(M)$, we have a vector space $\widehat{HF}(M, \mathfrak{s})$

such that

$$\widehat{HF}(M) = \bigoplus_{\mathfrak{s} \in Spin^c(M)} \widehat{HF}(M, \mathfrak{s}).$$

There is a related construction, originally due to Ozsváth/Szabó [28] and Rasmussen [34], that gives invariants of (nullhomologous) knots in 3-manifolds. The construction is similar to that of Heegaard Floer homology, except that there is an additional basepoint. The two basepoints together encode the knot, and the presence of an additional basepoint is used to produce a filtration on the Heegaard Floer complex of the 3-manifold.

Since we are concerned with knots in S^3 , we focus on this case. In general, one assigns to a knot $K \subset S^3$ a graded, filtered chain complex $CFK^-(K)$ over $\mathbf{F}[U]$, where U is a formal variable. The (filtered) chain homotopy type of this chain complex is an invariant of the knot. From this general invariant, there are a few simpler ones that can be derived; in particular, there is a “hat” version $\widehat{CFK}(K)$ which is obtained by setting $U = 0$. The homology of (the associated graded object corresponding to) this chain complex, denoted $\widehat{HFK}(K)$ is a bigraded vector space over $\mathbf{F}$. The two gradings are called the Alexander and Maslov gradings, and we denote the subspace of elements with Alexander grading s and Maslov grading μ by $\widehat{HFK}_\mu(K, s)$. From this, one also has a connection to the classical Alexander polynomial as follows:

$$\Delta_K(t) = \sum_{\mu, s} \text{rk } \widehat{HFK}_\mu(K, s) (-1)^\mu t^s. \quad (2.1)$$

There are also Heegaard Floer-theoretic invariants for 3-manifolds with boundary, known as *bordered* invariants, which were developed by Lipschitz, Ozsváth, and Thurston [21]. If M is a compact 3-manifold with parametrized boundary, their construction gives two kinds of algebraic invariants: a type-D structure $\widehat{CFD}(M)$ and a type-A structure $\widehat{CFA}(M)$ over a certain algebra $\mathcal{A}$. These satisfy a pairing theorem: if $M = M_1 \cup M_2$, then:

$$\widehat{HF}(M) \cong H_* \left(\widehat{CFA}(M_1) \boxtimes_{\mathcal{A}} \widehat{CFD}(M_2) \right)$$

Here $\boxtimes_{\mathcal{A}}$ denotes a certain tensor product over the algebra $\mathcal{A}$.

2.2.1 Immersed curves

We now turn to a review of the immersed curve versions of these Heegaard Floer theoretic invariants. In the general setting, these immersed curve invariants are a reformulation of certain bordered invariants; for our purposes, we shall use the construction for manifolds with torus boundary, first studied by Hanselman, Rasmussen, and Watson [8]. This invariant assigns to a 3-manifold with (parametrized) torus boundary M a collection of immersed curves (with local systems) on the punctured torus $T_{\bullet}$ which encodes a certain bordered invariant (specifically, $\widehat{CFD}(M)$). For our purposes, we will be interested in the case where $M = E(K)$, the exterior of some knot K . It is a fact that $\widehat{CFD}(E(K))$ is determined by $CFK^-(K)$ in this context; see Theorem 11.26 of [21]. As a result, one may think of the immersed curve invariant of the knot exterior as encoding the Knot Floer homology of K (see [9] for more details). Moreover, the pairing results from the bordered theory have a particularly nice formulation in the immersed curve setting:

Theorem 2.4 (Theorem 2 of [8]). *Given two 3-manifolds M_1, M_2 with (parametrized) torus boundary, let Γ_i be the immersed curve invariant associated to each M_i . If $Y = M_1 \cup_{\phi} M_2$ is the closed 3-manifold obtained by gluing M_1 and M_2 along their boundaries by $\phi : \partial M_2 \rightarrow \partial M_1$, then*

$$\widehat{HF}(Y) \cong HF(\Gamma_1, \bar{\phi}(\Gamma_2)),$$

where HF denotes the Lagrangian Floer homology on the punctured torus $T_{\bullet}$, and $\bar{\phi}$ is the map ϕ composed with the elliptic involution.

As the solid torus has a particularly simple bordered invariant, in particular, the immersed curve is a single embedded (non-nullhomotopic) circle, Theorem 2.4 gives a useful geometric way to compute Heegaard Floer homology of Dehn surgeries. More precisely, if Γ is the immersed curve invariant of a knot K , then $\widehat{HF}(S_K^3(p/q)) \cong HF(\Gamma, L_{p/q})$, where $L_{p/q}$ is

the embedded circle of “slope” $\frac{p}{q}$ on the (punctured) torus with respect to the standard meridian-longitude parametrization.

Remark 2.5. *In order to compute the Lagrangian Floer homology of (immersed) curves, we shall generally assume the curves have been placed in minimal position so that there are no immersed bigons bounded by the paired curves, and hence the rank is simply the number of intersection points as the differential is trivial. The only possible exception would be if two parallel curves were being paired, in which case minimal position would result in an immersed annulus being bounded by the pair. However, since we are considering pairings with curves of the form $L_{p/q}$, we shall see below that the only way this can occur is if $p = 0$, which we do not consider as there can be no cosmetic surgeries involving slope zero. Moreover, for the same reason, we need not worry about non-trivial local systems. More precisely, interpreting a non-trivial local system on a curve as multiple copies of the curve along with some extra “train-track” connections, the only time this affects the intersection number is if there are parallel curves being paired.*

For our purposes, it will be more useful to consider lifts of the immersed curves to the cylindrical cover $\overline{T}_\bullet$ of the punctured torus. More precisely, consider the immersed curve invariant Γ coming from the exterior of a knot K . We identify T with the boundary of the knot exterior and consider the covering space $\overline{T}$ associated with the subgroup of $\pi_1(T)$ generated by the longitude of K . Restricting this cover to the punctured torus $T_\bullet$, we have the covering space $\overline{T}_\bullet$, which is the infinitely-punctured cylinder. Following [7], we will parametrize this as $S^1 \times \mathbf{R} \setminus \{*\} \times (\mathbf{Z} + \frac{1}{2})$; that is, the punctures are all arranged with “heights” at the half-integers. We distinguish a special lift μ of the meridian, which is the “vertical” line passing through the punctures. The punctures cut μ into a union of disjoint intervals of the form $(i - \frac{1}{2}, i + \frac{1}{2})$ for each $i \in \mathbf{Z}$. We refer to points in such an interval as being at *level* i . There is a standard lift of Γ to $\overline{T}_\bullet$, which we denote as $\overline{\Gamma}$. The symmetries of Knot Floer homology imply that, after a suitable homotopy of the immersed curves, $\overline{\Gamma}$ has a 180° rotational symmetry, and we take the symmetry point to be the “origin” (that is, the

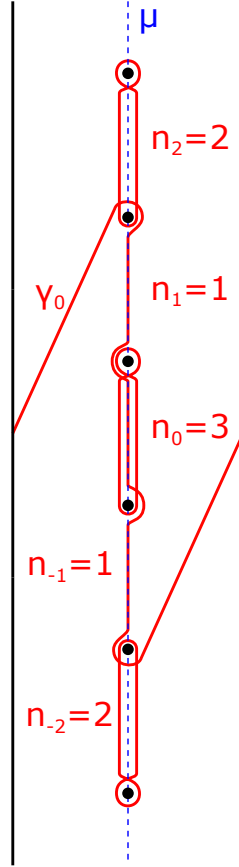


Figure 2.2: An example of immersed curves in $\overline{T}_\bullet$ corresponding to some knot-like complex. The left and right edges are identified. We can see that this knot would have genus 3, tau 2 and epsilon 1.

point on μ at height 0). Another important property of $\bar{\Gamma}$ is that there is a distinguished component, denoted γ_0 , which is the unique curve that “wraps around” the cylinder $\bar{T}_\bullet$. All other components of $\bar{\Gamma}$ may be homotoped to lie in an arbitrarily small neighborhood of μ . We shall usually assume that all the immersed curves have undergone an appropriate homotopy so that they are “pulled tight” with respect to some ϵ -neighborhood of the punctures. Such curves will generically pair minimally with “lines” which are lifts of $L_{p/q}$, as in Remark 2.5. This will be of interest to us due to the following consequence of Proposition 46 of [8] (see also Theorem 14 of [7]):

Theorem 2.6. *Let $K \subset S^3$ be a knot and let $\Gamma \subset T_\bullet$ be its immersed curve invariant, with $\bar{\Gamma} \subset \bar{T}_\bullet$ the standard lift. Given $\frac{p}{q} \in \mathbf{Q} \cup \{\infty\}$, for each $\mathfrak{s} \in \text{Spin}^c(S_K^3(p/q))$, there exists a (unique homotopy class of) lift $\bar{L}_{p/q}^{\mathfrak{s}}$ of $L_{p/q}$ such that:*

$$\widehat{HF}(S_K^3(p/q), \mathfrak{s}) \cong HF(\bar{\Gamma}, \bar{L}_{p/q}^{\mathfrak{s}}).$$

Here HF is the Lagrangian Floer homology of immersed curves in the (infinitely) punctured cylinder $\bar{T}_\bullet$. As mentioned above, the rank of this is computed in practice as the minimal intersection number between the two (multi)curves. Alternatively, we may slightly perturb the “pulled tight” position to make all segments vertical outside of ϵ -neighborhoods of the punctures, except for the part of γ_0 that wraps around. With the curves in this position, for each $i \in \mathbf{Z}$, we denote by $n_i(K)$ the number of vertical segments at level i . Note that this is an invariant of the knot K , as indeed one could define it as the minimal number of such vertical segments among all representatives of the homotopy class of $\bar{\Gamma}$. See Figure 2.2 for an example of a knot-like immersed curve. Several invariants can be extracted from such immersed curve diagrams:

- The Seifert genus $g(K)$ is the maximum level at which some curve of $\bar{\Gamma}$ intersects μ .
- The tau invariant $\tau(K)$ is the level at which the distinguished curve γ_0 first intersects μ (when following the curve from the “left”).

- The epsilon invariant $\epsilon(K)$ is given by the direction γ_0 “turns” after the aforementioned first intersection with μ : 1 if downward, -1 if upward, and 0 if there is no turning.

In the “pulled tight” configuration, we can thus see that the unique non-vertical segment (which is part of γ_0) will have a slope of $2\tau(K) - \epsilon(K)$. Let us define $\text{Vert}(K)$ to be the total number of vertical segments of $\bar{\Gamma}$; that is,

$$\text{Vert}(K) = \sum_i n_i(K).$$

In the punctured torus, we see that, when pulled tight, Γ effectively has the form of $\text{Vert}(K)$ copies of the meridian circle along with a circle of slope $2\tau(K) - \epsilon(K)$. Since the “pulled tight” configuration will have minimal intersection number with any (transverse, non-nullhomologous) embedded circle, we find the following formula for the rank of the Heegaard Floer homology of knot surgeries:

$$\text{rk } \widehat{HF}(S_K^3(p/q)) = |p - q(2\tau(K) - \epsilon(K))| + |q| \text{Vert}(K). \quad (2.2)$$

Remark 2.7. *This formula needs to be modified in the case that $\frac{p}{q} = 0$ and γ_0 is a horizontal curve (equivalently, $\epsilon(K) = 0$); the actual rank is two more than what this formula predicts. However, for the applications we are interested in, this case will not appear.*

2.2.2 Gradings

Here we discuss how to obtain relative grading information from the pairing of immersed curve invariants. Recall that, for any knot K and $p \neq 0$, the vector space $\widehat{HF}(S_K^3(p/q))$ has a relative $\mathbf{Z}$ -grading called the *Maslov* grading, which we denote $M(x)$ for any generator x (the case of zero-framed surgery will not be of concern to us, as it cannot be part of a cosmetic surgery pair). The relative grading between two generators may be computed from

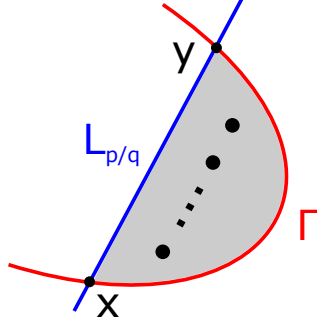


Figure 2.3: A bigon from x to y .

the immersed curves as follows: suppose $x, y \in L_{p/q} \cap \Gamma$ and suppose further that there exists an immersed bigon in $T_\bullet$ with one edge lying in $L_{p/q}$ and the other in Γ as in Figure 2.3. Let k be the signed count of the number of times the bigon crosses the puncture/basepoint. Then:

$$M(x) - M(y) = 1 - 2k. \quad (2.3)$$

The same computation may be performed in the covering space $\bar{T}_\bullet$, and moreover, it also still holds in the further covering space $\tilde{T}_\bullet = \mathbf{R}^2 \setminus \mathbf{Z} \times (\frac{1}{2} + \mathbf{Z})$, corresponding to the universal covering space of the torus.

Remark 2.8. *In general, for a given pair of generators, there need not exist such a bigon between them (for instance, if the generators are in distinct Spin^c -structures or if they belong to distinct components of Γ). However, in the applications that follow, all relevant grading information within any given Spin^c -structure will be deducible from equation (2.3).*

2.3 Finite Type Invariants

Here we review some facts about finite type invariants for knots and their application to cosmetic surgeries.

Suppose a real-valued knot invariant v can be extended to an invariant of *singular knots* (that is, knots with possibly finitely many points of self-intersection) in a way that satisfies

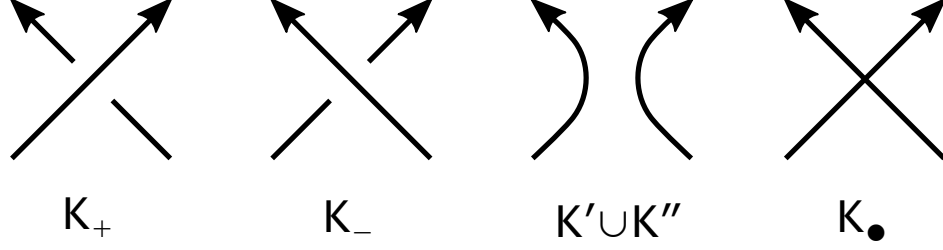


Figure 2.4: The diagrams of K_+ , K_- , $K' \cup K''$, and $K_\bullet$ differ only locally near a crossing.

the following:

$$v(K_\bullet) = v(K_+) - v(K_-)$$

whenever the (singular) knots K_+ , K_- , and $K_\bullet$ differ locally near a crossing/self-intersection as in Figure 2.4. Then v is said to be a finite type invariant of order n if moreover $v(K) = 0$ whenever K has at least $n + 1$ self-intersection points.

For example, recall the Conway polynomial of a knot, which is related to the Alexander polynomial in the following way:

$$\nabla_K(z) = \Delta_K(t)|_{z=t^{1/2}-t^{-1/2}}$$

For a knot K , the Conway polynomial will have the form:

$$\nabla_K(z) = 1 + \sum_{j=1}^n a_{2j}(K)z^{2j}$$

It is a fact that the coefficient $a_{2j}(K)$ is a finite type invariant of order $2j$ for each j .

The other finite type invariant that will be of interest to us is v_3 . This is a third-order invariant, which may be defined as:

$$v_3(K) = -\frac{1}{36}V_K'''(1) - \frac{1}{12}V_K''(1)$$

where $V_K(t)$ is the Jones polynomial of K .

It is a fact that the Conway polynomial satisfies the following skein relation:

$$\nabla_{K_+}(z) - \nabla_{K_-}(z) = z \nabla_{K' \cup K''}(z). \quad (2.4)$$

Here K_+ and K_- differ at a single crossing, and K' and K'' are the two components of the link that results from the oriented resolution at that crossing; see Figure 2.4 for an illustration.

Considering only the z^2 term, one has the following skein relation for a_2 [13]:

$$a_2(K_+) - a_2(K_-) = \text{lk}(K', K''). \quad (2.5)$$

Here lk denotes the linking number. On the other hand, it is known that v_3 satisfies the following skein relation [13, 20]:

$$v_3(K_+) - v_3(K_-) = \frac{1}{2} \left(a_2(K_+) + a_2(K_-) + \text{lk}^2(K', K'') \right) - a_2(K') - a_2(K'') \quad (2.6)$$

The relevance of these finite type invariants to cosmetic surgery is due to their appearance in surgery formulae for certain 3-manifold invariants.

In [4], Boyer and Lines extended Casson's invariant λ to homology Lens spaces, and it follows from Theorem 2.8 of that work that:

Lemma 2.9. *Let $K \subset S^3$ be a knot, and p/q be a fraction with $p > 0$. Then*

$$\lambda(S_K^3(p/q)) = \frac{q}{p} a_2(K) - \frac{1}{2} s(q, p)$$

Here $s(q, p)$, known as the Dedekind sum, can be defined, for $p > 0$, as:

$$s(q, p) = \sum_{k=1}^{p-1} \left(\frac{k}{p} - \left\lfloor \frac{k}{p} \right\rfloor - \frac{1}{2} \right) \left(\frac{kq}{p} - \left\lfloor \frac{kq}{p} \right\rfloor - \frac{1}{2} \right)$$

where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x . Furthermore, Boyer and Lines

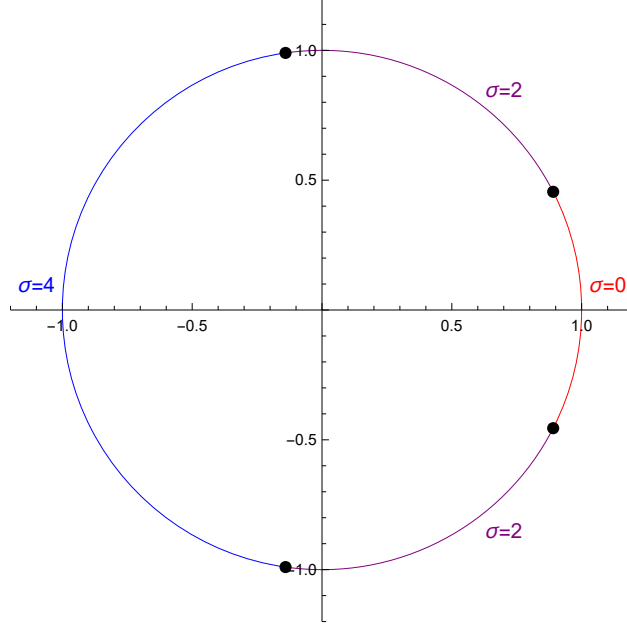


Figure 2.5: The signature function for the knot $K(1, 0, 0, 0, 0)$. The marked points are roots of the Alexander polynomial.

also show in Lemma 2.22 of [4] the following surgery formula for the Casson-Gordon invariant τ :

Lemma 2.10. *Let $K \subset S^3$ be a knot, and p/q be a fraction with $p > 0$. Then*

$$\tau(S_K^3(p/q)) = -4ps(q, p) - \sigma(K, p)$$

Here $\sigma(K, p)$ is the p -signature of the knot K . We briefly recall the definition and basic properties of the signature. If A is any matrix that represents the Seifert form of K , then for each $\omega \in S^1 \subseteq \mathbf{C}$ on the complex unit circle, we put $\sigma_\omega(K)$ equal to the signature of the Hermitian matrix $(1 - \omega)A + (1 - \bar{\omega})A^T$. This is the so-called *signature function* of K on the unit circle. It is integer-valued, and it only has discontinuities at roots of the Alexander polynomial, which can be seen by rewriting the matrix in the definition as $(\bar{\omega} - 1)(\omega A - A^T)$. Near $\omega = 1$, this function is zero, and near $\omega = -1$, this function is equal to an invariant called simply the *signature* of the knot. See Figure 2.5 for an illustration of the case $K = K(1, 0, 0, 0, 0)$. Finally, the p -signature of a knot, denoted $\sigma(K, p)$ is equal to

the sum of the values of the signature function at the p th roots of unity. That is,

$$\sigma(K, p) = \sum_{\omega^p=1} \sigma_{\omega}(K)$$

The Le-Murakami-Ohtsuki (LMO) invariant is a generalization of the Casson invariant. The degree-2 part, denoted λ_2 , satisfies the following surgery formula, involving finite-type invariants of the knot:

Theorem 2.11 (Theorem 1.2 of [16]). *Let $K \subset S^3$ be a knot, and p/q be a reduced fraction. Then:*

$$\begin{aligned} \lambda_2(S_K^3(p/q)) - \lambda_2(L(p, q)) \\ = \left(\frac{7a_2(K)^2 - a_2(K) - 10a_4(K)}{8} \right) \frac{q^2}{p^2} - \frac{v_3(K)}{4} \frac{q}{p} + \frac{a_2(K)}{48} \left(1 - \frac{1}{p^2} \right) \end{aligned}$$

where $L(p, q)$ denotes the p, q lens space.

Remark 2.12. *Ito uses a slightly different definition for v_3 than is used here; in particular, he normalizes it to take the value $\frac{1}{4}$ on the right-handed trefoil instead of 1 as in our convention.*

Chapter 3

Purely Cosmetic Surgeries and 3-Braid Knots

In this chapter, we discuss the various obstructions to the existence of purely cosmetic surgeries and apply them to show that the family of (nontrivial) 3-braid knots admits no such surgeries.

3.1 Obstructions

As a consequence of Lemmas 2.9 and 2.10, Boyer and Lines find the following obstruction to purely cosmetic surgeries:

Theorem 3.1 (Proposition 5.1 of [4]). *Let K be a knot in S^3 . If $a_2(K) \neq 0$, then K does not admit any purely cosmetic surgeries.*

On the other hand, Ichihara and Wu find the following obstruction, obtained by studying Lescop's λ_2 invariant, which is related to Kontsevich-Kuperberg-Thurston invariant (compare Corollary 1.3 of [16], obtained from the similar LMO invariant):

Theorem 3.2 (Theorem 3.5 of [13]). *If a knot $K \subset S^3$ satisfies $v_3(K) \neq 0$, then K does not admit any purely cosmetic surgeries.*

On the Heegaard Floer homology side, Ni and Wu applied the surgery formula of [31] to obtain the the following obstructions:

Theorem 3.3 (Theorem 1.2 of [25]). *Suppose a knot $K \subset S^3$ admits a pair of purely cosmetic surgeries along slopes p/q and p/q' . Then:*

- $q = -q'$
- $q^2 \equiv -1 \pmod{p}$
- $\tau(K) = 0$

More recently, Hanselman obtained the following obstruction as an application of immersed curve methods:

Theorem 3.4 (Theorem 2 of [7]). *Let $K \subset S^3$ be a nontrivial knot and suppose that $S_K^3(r) \cong S_K^3(r')$ for r, r' distinct rational numbers. Then $\{r, r'\} = \{\pm 2\}$ or $\{\pm \frac{1}{q}\}$ for some positive integer q . Moreover:*

- (i) *If $\{r, r'\} = \{\pm 2\}$ then $g(K) = 2$*
- (ii) *If $\{r, r'\} = \{\pm \frac{1}{q}\}$ then $q \leq \frac{th(K) + 2g(K)}{2g(K)(g(K) - 1)}$*

Remark 3.5. *Notice that if for a knot K with genus greater than 2 the fraction on the right hand side of the inequality in (ii) is less than one, then K admits no purely cosmetic surgeries.*

3.2 Band presentation and genus

Here we recall some useful results about the genera of 3-braid knots. We shall make use of the so-called “band presentation” of the 3-braid group B_3 , studied for braid groups in general by Birman, Ko, Lee [2] and which has proved useful in the study of surfaces bounded

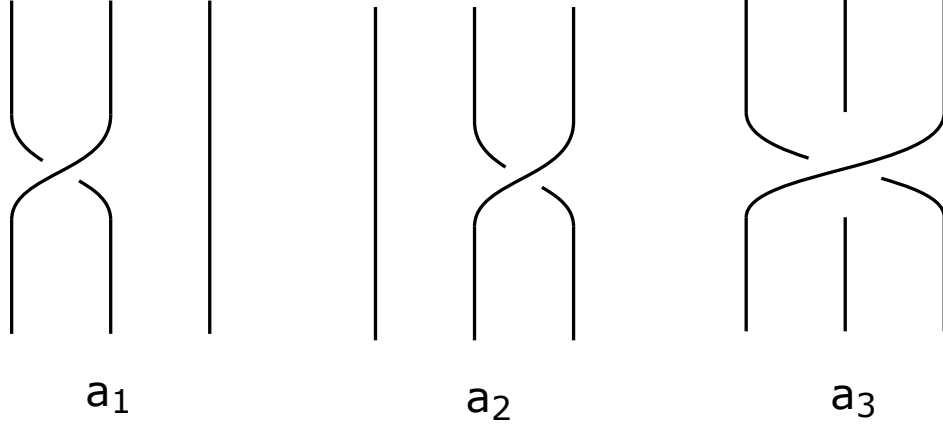


Figure 3.1: The three generators of the band presentation of the 3-braid group.

by knots arising from braid representations; see, for instance, [35, 44, 19]. For 3-braids, the presentation is as follows:

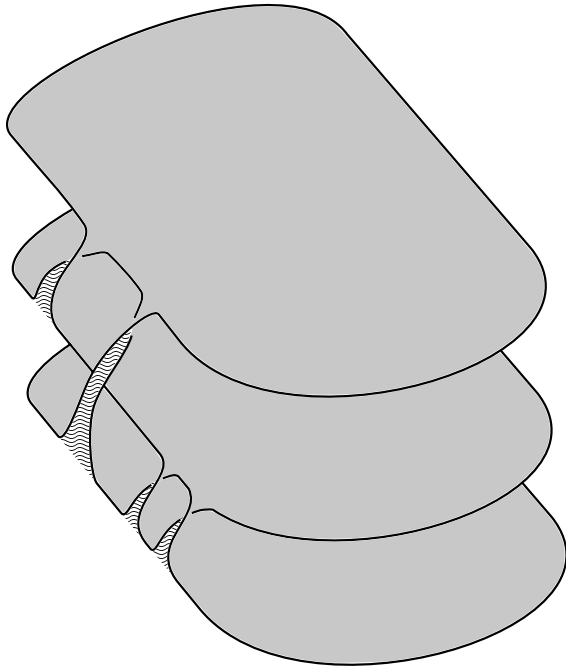
$$B_3 = \langle a_1, a_2, a_3 \mid a_2 a_1 = a_3 a_2 = a_1 a_3 \rangle$$

Here a_1 and a_2 correspond to the standard Artin generators σ_1 and σ_2 respectively, and a_3 corresponds to $\sigma_2 \sigma_1 \sigma_2^{-1}$; see Figure 3.1 for an illustration.

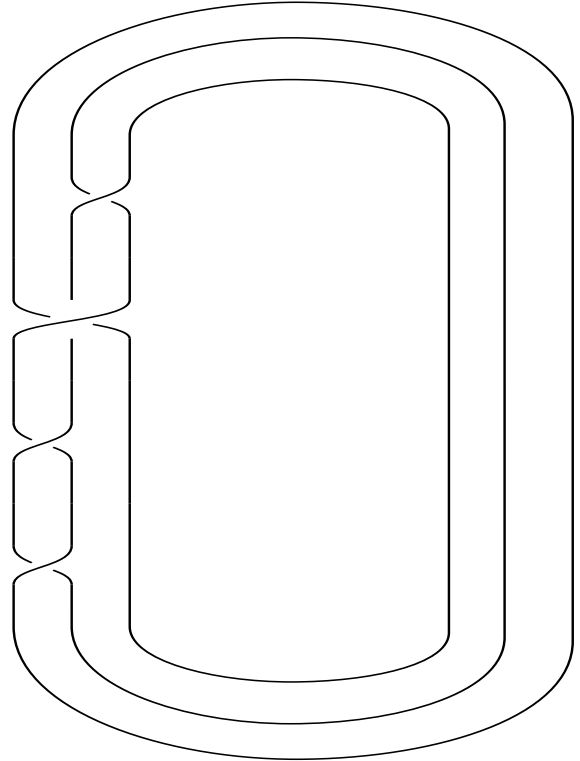
A word in the band generators naturally gives rise to a Seifert surface for the corresponding closed braid, which we shall call the *banded surface* associated to the word. In particular, starting with three disks, which one can imagine as being vertically “stacked”, one attaches a half-twisted band between two of the disks for each instance of $a_j^{\pm 1}$: for a_1 between the bottom and middle disks, for a_2 between the middle and top, and for a_3 between the top and bottom, twisted appropriately depending on the sign of the exponent; see Figure 3.2 for the banded surface corresponding to the braid $a_2 a_3^{-1} a_1^2$.

Now suppose that a 3-braid in the band generators has a knot as its braid-closure. A straightforward computation of the Euler characteristic reveals that the banded surface produced with the above method has genus

$$g = \frac{\ell - 2}{2} \tag{3.1}$$



(a) A banded surface for K



(b) A diagram for K coming from the closure of the braid; this can be interpreted as a “top-down” view of the banded surface

Figure 3.2: The 3-braid knot K corresponding to the closure of the braid $a_2 a_3^{-1} a_1^2$; note that by adjusting the sizes of the “disks”, one can ensure that the only crossings in a diagram of K come from the group generators

where ℓ is the length of the braid word (that is, the number of instances of the $a_j^{\pm 1}$ s). Moreover, in the case of 3-braids, a theorem of Bennequin's says that a minimal Seifert surface can always be produced in this way.

Theorem 3.6 (Proposition 3 of [1]). *If K is the closure of a 3-braid, then the Seifert genus of K can be realized by a banded surface.*

This was proved by analyzing the foliations induced on a (minimal genus) Seifert surface of the braid knot by a fibration of the complement of the braid axis. In the 3-braid case, it turns out that such a surface may be isotoped so that the foliation has a certain "simple" form, corresponding to our banded surfaces (see also [3]).

Suppose K is a knot which is the closure of a 3-braid expressed as a word in the braid generators. By Bennequin's theorem, we may take this word to be minimal so that the associated banded surface achieves the Seifert genus of the knot $g(K)$. Notice that each instance of $a_1^{\pm 1}$ and $a_2^{\pm 1}$ contributes a crossing to the diagram of the braid, while each instance of $a_3^{\pm 1}$ contributes 3 crossings (see Figure 3.2 for an example). Let A_j denote the total number of instances of $a_j^{\pm 1}$ in the given word for K . Then K admits a diagram with $A_1 + A_2 + 3A_3$ crossings. Further, observe that K is isotopic to the closure of the braid obtained by cyclically permuting the generators: $a_1 \rightarrow a_2 \rightarrow a_3 \rightarrow a_1$ (indeed, one can even realize this as an isotopy of a banded surface like the one in Figure 3.2 by "flipping" the top disk across the banded areas so that it becomes the bottom disk). Thus, we may assume that A_3 is the least of the A_j s; that is, $A_3 \leq \frac{1}{3}(A_1 + A_2 + A_3)$. On the other hand, we have that the length of the word $\ell = A_1 + A_2 + A_3$. Combining this with equation (3.1), we see that K admits a diagram with crossing number:

$$\begin{aligned} c &= A_1 + A_2 + 3A_3 \\ &= \ell + 2A_3 \\ &\leq \ell + \frac{2}{3}\ell \end{aligned}$$

$$= \frac{10}{3}(g(K) + 1).$$

Therefore, we have shown the following lemma:

Lemma 3.7. *Let K be a 3-braid knot. If $g(K)$ denotes the Seifert genus of K , then K admits a diagram with no more than $\frac{10}{3}(g(K) + 1)$ crossings.*

Remark 3.8. *In fact, we may additionally require that K admit a diagram with an even number of crossings less than or equal to $\frac{10}{3}(g(K) + 1)$. This follows from the fact that, in order for the closure of a 3-braid to result in a knot, the permutation of the strands induced by the braid must be even (indeed it must be one of the cycles of order 3 in S_3 , the symmetric group on the three strands).*

3.3 Knot Floer Homology and thickness

Given a knot K , recall that the “hat” version of Knot Floer Homology, denoted $\widehat{HFK}(K)$ is the homology of a bigraded chain complex denoted $\widehat{CFK}(K)$. The two gradings are called the Alexander and Maslov gradings. In an algebraic re-interpretation of Knot Floer Homology, Ozsváth and Szabó showed [30, 26] that the generators of $\widehat{CFK}(K)$ may be identified with so-called Kauffman states of a diagram for K , first introduced in [17] as part of an alternative technique for computing the Alexander polynomial. A Kauffman state consists of a choice of one of the four “corners” for each crossing appearing in the diagram for K , such that all but a pair (chosen once and for all) of adjacent “regions” that make up the complement of the diagram have exactly one marking (the two “distinguished” regions never receive any markings). We denote the set of all Kauffman states by $\mathfrak{S}$ (note that this depends on the choice of the diagram for K). The Alexander and Maslov grading of each such generator is the sum of the local grading contributions at each crossing, as shown in Figure 3.3. The *delta* grading of a generator of $\widehat{CFK}(K)$ is defined to be the difference between its Alexander and Maslov gradings; hence, it too may be computed as a sum of

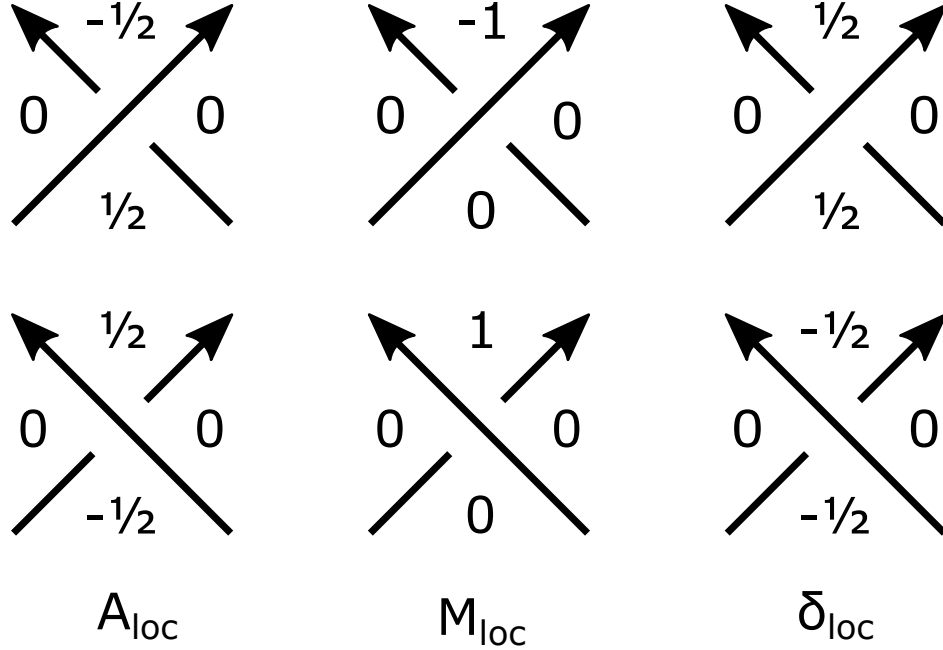


Figure 3.3: The local Alexander, Maslov, and delta grading contributions at a crossing. The top row shows a positive crossing and the bottom shows a negative crossing.

local delta-gradings, also displayed in Figure 3.3.

The (homological) *thickness* of a knot K , which we shall denote $th(K)$, is defined to be the difference between the maximal and minimal delta gradings of the support of $\widehat{HFK}(K)$. Using the Kauffman state interpretation, we obtain the following bound on the thickness:

$$\begin{aligned}
th(K) &\leq \max_{s \in \mathfrak{S}} \delta(s) - \min_{s \in \mathfrak{S}} \delta(s) \\
&= \max_{s \in \mathfrak{S}} \sum_{\gamma \in s} \delta_{\text{loc}}(\gamma) - \min_{s \in \mathfrak{S}} \sum_{\gamma \in s} \delta_{\text{loc}}(\gamma) \\
&\leq \sum_{\substack{\gamma \in s \\ \text{positive}}} \frac{1}{2} - \sum_{\substack{\gamma \in s \\ \text{negative}}} -\frac{1}{2} \\
&= \frac{c}{2}
\end{aligned}$$

where a sum over $\gamma \in s$ means summing over all the (marked) crossings γ of the Kauffman state s , and where c denotes the number of crossings appearing in the diagram. Hence, the thickness of a knot K is at most one-half the number of crossings appearing in any diagram

for K . Combining this with Lemma 3.7, we obtain:

Lemma 3.9. *If K is a knot which is the closure of a 3-braid, then*

$$th(K) \leq \frac{5}{3}(g(K) + 1)$$

3.4 Proof of Theorem 1.2

We are now ready to prove the main result of this chapter:

Theorem 1.2 (Theorem of 1.1 [39]). *Suppose K is a nontrivial knot which is the closure of a 3-braid. Then K admits no purely cosmetic surgeries.*

Proof. First, consider a 3-braid knot K with genus $g(K) = g \geq 4$. By Lemma 3.9, $th(K) \leq \frac{5}{3}(g + 1)$. Hence,

$$\begin{aligned} \frac{th(K) + 2g(K)}{2g(K)(g(K) - 1)} &\leq \frac{\frac{11}{3}g + \frac{5}{3}}{2g(g - 1)} \\ &= \frac{1}{g - 1} \left(\frac{11}{6} + \frac{5}{6g} \right) \\ &\leq \frac{1}{3} \left(\frac{11}{6} + \frac{5}{24} \right) = \frac{49}{72} < 1 \end{aligned}$$

where the hypothesis that K has genus at least 4 was used in the last step. By Remark 3.5, K admits no purely cosmetic surgeries.

So we are left with the case $g(K) \leq 3$. By Lemma 3.7 along with the remark following it, any such knot has crossing number at most 12.

However, in [7], Hanselman also checked that all prime knots with at most 16 crossings admit no purely cosmetic surgeries, while composite knots were shown not to admit any purely cosmetic surgeries by Tao [38]. □

Remark 3.10. *As noted in Remark 3.7 of [7], some of the prime knots up to 16 crossings were excluded with the use of numerical calculations, which may not be fully rigorous. However,*

if we apply Corollary 1.6 of [16], with the use of KnotInfo [22] we find that the remaining 3-braid prime knots up to 12 crossings that need to be checked are 10_{118} , $12n749$, $12n750$, and $12n830$. These knots do not appear in Table 1 of [7] and so their exclusion does not rely on numerical methods. According to KnotInfo [22], 10_{118} is a 3-braid knot of genus 4, and so admits no purely cosmetic surgeries by the above computation. KnotInfo also gives the Alexander polynomials of the remaining knots, from which we may compute the invariant $a_2(K) = \frac{1}{2}\Delta_K''(1)$. The results are shown in the following table:

K	$\Delta_K(t)$	$a_2(K)$
$12n749$	$t^{-3} - t^{-2} + t^{-1} - 1 + t - t^2 + t^3$	6
$12n750$	$2t^{-2} - 5t^{-1} + 7 - 5t + 2t^2$	3
$12n830$	$2t^{-2} - 4t^{-1} + 5 - 4t + 2t^2$	4

By a Theorem 3.1, if a knot K admits purely cosmetic surgeries then $a_2(K) = 0$. Hence the three remaining knots are ruled out as well.

Chapter 4

Chirally Cosmetic Surgeries

In this chapter, we discuss the various obstructions to chirally cosmetic surgeries and apply them to several families of knots, including alternating odd pretzel knots, certain Whitehead doubles, and L-space knots.

4.1 Obstructions

4.1.1 Heegaard Floer

In this section, we employ the immersed curve formulation of Heegaard Floer invariants to obtain new obstructions to the existence of chirally cosmetic surgeries along slopes of opposite signs.

Lemma 4.1. *Let K be a knot with $\tau(K) = g(K) > 0$. If $\frac{p}{q} > 2g(K) - 1$ then $\text{rk } \widehat{HF}(S_K^3(p/q), \mathfrak{s}) \leq \max_i n_i(K)$ for each Spin^c structure $\mathfrak{s} \in \text{Spin}^c(S_K^3(p/q))$.*

Proof. Since $\tau(K) = g(K) > 0$, we must have $\epsilon(K) = 1$, which in the immersed curve setting, means that $\gamma_0 \subset \overline{T}$ curves downward after reaching the midline μ . Hence, the slope of the non-vertical segment of γ_0 is $2\tau(K) - 1 = 2g(K) - 1$. Let $\mathfrak{s} \in \text{Spin}^c(S_K^3(p/q))$ be given. Then by Theorem 2.6, $\text{rk } \widehat{HF}(S_K^3(p/q), \mathfrak{s})$ is equal to the minimal intersection number between $\overline{\Gamma}$ (the lift of the immersed curve invariant of K) and the lift $\overline{L}_{p/q}^{\mathfrak{s}} \subset \overline{T}$ of $L_{p/q}$ corresponding to

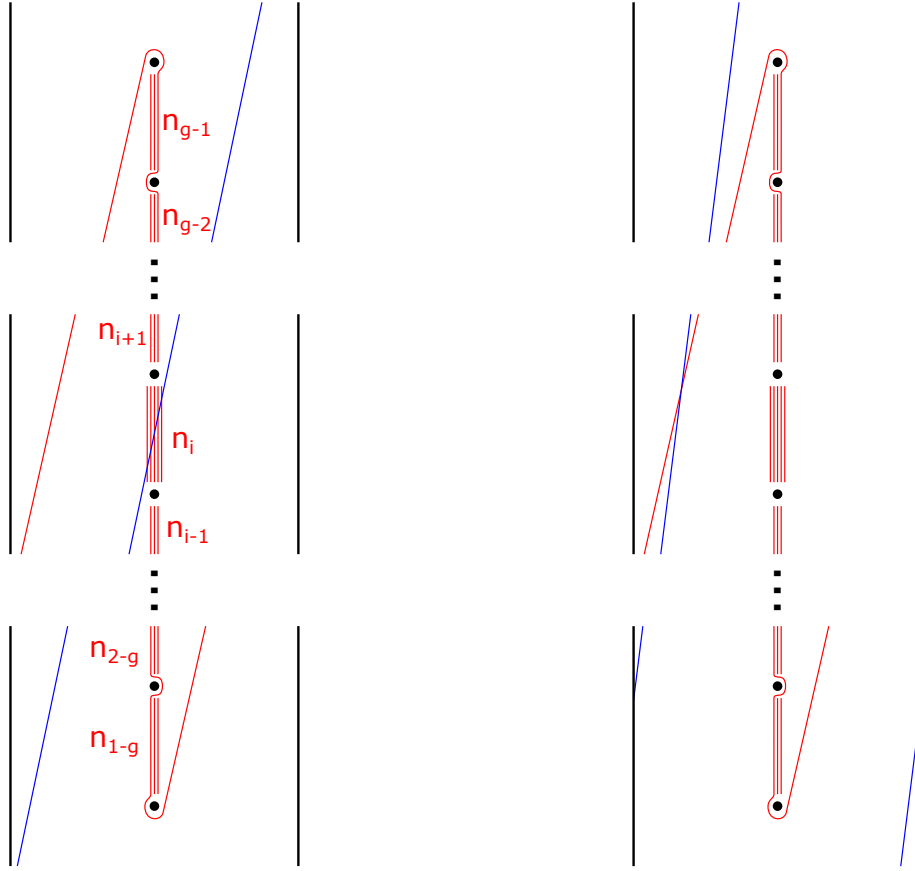


Figure 4.1: Schematics of the kinds of immersed curve pairings occurring in the proof of Lemma 4.1.

5. As $\frac{p}{q} > 2g(K) - 1$, we see that $\overline{L^s}_{p/q}$ can intersect μ between $-g + \frac{1}{2}$ and $g - \frac{1}{2}$ at most once. We consider the two possibilities in turn: in the case where there is such an intersection, say at height i , the number of intersection points between $\overline{\Gamma}$ and $\overline{L^s}_{p/q}$ will be $n_i(K)$ within this height range. Since $\frac{p}{q} > 2g(K) - 1$, we also see that there will be no intersection with the non-vertical segment of γ_0 . Moreover, since $\overline{\Gamma}$ is supported between heights of $-g + \frac{1}{2} + \epsilon$ and $g - \frac{1}{2} - \epsilon$, there can be no further intersections; therefore, the number of intersections is at most $\max_i n_i(K)$ in this situation. See the left of Figure 4.1 for an illustration of this first case. In the second case, we consider the possibility that $\overline{L^s}_{p/q}$ does not intersect μ within the interval $[-g + \frac{1}{2}, g - \frac{1}{2}]$. Here there will be no intersections with any vertical segments of $\overline{\Gamma}$, and so the only intersections that may occur are with the segment of slope $2g(K) - 1$. Since $\frac{p}{q} > 2g(K) - 1$, this can happen at most once (see the right side of Figure 4.1 for an illustration). Of course, since $\tau(K) > 0$ by hypothesis, $n_0(K) \geq 1$ and so we also see that the number of intersection points is no more than $\max_i n_i(K)$ in this case as well. $\square$

Lemma 4.2. *Let K be a knot with $\tau(K) > 0$. If $\frac{p}{q} < 0$ then there exists a Spin^c -structure $\mathfrak{s} \in \text{Spin}^c(S_K^3(p/q))$ such that $\text{rk } \widehat{HF}(S_K^3(p/q), \mathfrak{s}) \geq \max_i n_i(K) + 1$.*

Proof. Let i be such that $n_i(K) = \max_j n_j(K)$. There exists a lift $\overline{L} \subset \overline{T}_\bullet$ of $L_{p/q}$ that intersects μ at level i . By Theorem 2.6, we choose $\mathfrak{s}$ to be the Spin^c structure corresponding to this lift; that is, in the notation as above, $\overline{L} = \overline{L^s}_{p/q}$. We consider two separate cases. If $|i| \geq \tau(K)$, then since $\frac{p}{q} < 0$ and $\tau(K) > 0$, the line $\overline{L^s}_{p/q}$ must intersect with the non-vertical segment of γ_0 . This can be seen as follows: assuming without loss of generality that $i > 0$, we follow the line $\overline{L^s}_{p/q}$ from an intersection at level i until the point at which it reaches height 0. If it has wrapped at least halfway around the cylinder $\overline{T}$, then it will have intersected with the “positive” non-vertical segment of γ_0 . Otherwise, at the point where it wraps halfway, it will have a negative height and hence will have intersected with the “negative” part of the non-vertical segment of γ_0 (see the left side of Figure 4.2 for an illustration of this case). Either way, along with the n_i intersections with vertical segments, there is at least one additional

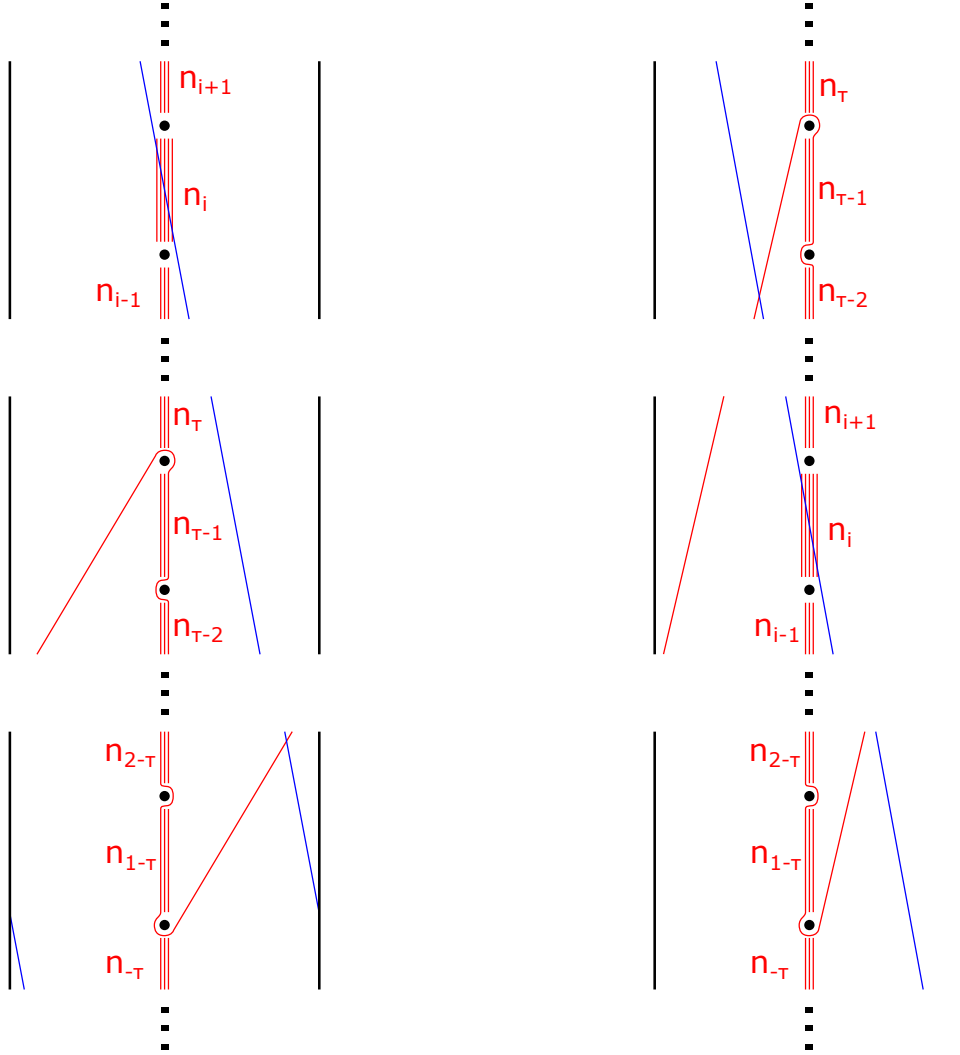


Figure 4.2: Schematics of the kinds of immersed curve pairings occurring in the proof of Lemma 4.2.

intersection point. In the other case we have $|i| < \tau(K)$. If at level i the intersection of $\overline{L^s_{p/q}}$ with μ occurs at a positive height, then since $\frac{p}{q} < 0$ and $\tau(K) > 0$, there must be an additional intersection point between the aforementioned intersection point and the point where $\overline{L^s_{p/q}}$ has wrapped halfway around the cylinder upwards and to the left (see the right side of Figure 4.2 for an illustration). An analogous argument holds if the intersection with μ occurred at a negative height. In both cases, we see that there are at least $n_i(K) + 1$ intersection points. Since $n_i(K)$ was chosen to be maximal, the desired result follows. $\square$

We now prove our main obstruction:

Theorem 1.6 (Theorem 1.4 of [41]). *Suppose a knot K satisfies $\tau(K) = g(K) > 0$. If $S^3(p/q) \cong -S^3(p/q')$ where $p, q, q' \in \mathbf{Z}$ with $p, q > 0$ and $q' < 0$, then $\frac{p}{q} \leq 2g(K) - 1$ and $2p = (\text{Vert}(K) + 2g(K) - 1)(q + q')$.*

Proof. For the first part, if $\frac{p}{q} > 2g(K) - 1$, Lemmas 4.1 and 4.2 imply that $S^3_K(p/q)$ and $S^3_K(p/q')$ cannot be homeomorphic as there can be no isomorphism between their Heegaard Floer homologies that respects the Spin^c decomposition. As $\tau(K) = g(K) > 0$, we have a non-vertical segment of slope $2g(K) - 1$. From the rank formula (2.2), we have that:

$$|p - q(2g(K) - 1)| + \text{Vert}(K)q = p - q'(2g(K) - 1) - \text{Vert}(K)q'.$$

As $\frac{p}{q} \leq 2g(K) - 1$, this becomes:

$$\begin{aligned} q(2g(K) - 1) - p + \text{Vert}(K)q &= p - q'(2g(K) - 1) - \text{Vert}(K)q' \\ (q + q')(2g(K) - 1 + \text{Vert}(K)) &= 2p \end{aligned}$$

$\square$

4.1.2 Finite Type

Here we collect certain obstructions involving finite type invariants.

As a consequence of Theorem 2.11, Ito obtains:

Theorem 4.3 (Corollary 1.3 of [16]). *Let K be a knot and suppose $S_K^3(p/q) \cong \pm S_K^3(p/q')$ for some $q \neq q'$. Then,*

$$2pv_3(K) = (7a_2(K)^2 - a_2(K) - 10a_4(K))(q + q')$$

On the other hand, by combining Lemmas 2.9 and 2.10, Ichihara, Ito, and Saito find:

Theorem 4.4 (Theorem 1.2 of [11]). *Let K be a knot and suppose $S_K^3(p/q) \cong -S_K^3(p/q')$. Then*

$$4(q + q')a_2(K) = -\sigma(K, p)$$

Following [11], we combine Theorems 4.3 and 4.4 to find that, if a knot K admits a pair of chirally cosmetic surgeries along slopes p/q and p/q' , then

$$-\frac{\sigma(K, p)}{4a_2(K)} = q + q' = \frac{2pv_3(K)}{7a_2(K)^2 - a_2(K) - 10a_4(K)}$$

So we have obtained the following useful criterion:

Corollary 4.5. *Let K be a knot and suppose $S_K^3(p/q) \cong -S_K^3(p/q')$. Then*

$$\frac{\sigma(K, p)}{p} = \frac{-8a_2(K)v_3(K)}{7a_2(K)^2 - a_2(K) - 10a_4(K)}$$

4.1.3 Combining the obstructions

Combining Theorems 1.6 and 4.3 we find the following obstruction to the existence of chirally cosmetic surgeries with opposite signs:

Corollary 4.6. *Let K be a knot satisfying $\tau(K) = g(K)$ and $v_3(K) \neq 0$. If $S_K^3(p/q) \cong -S_K^3(p/q')$ for some $p, q > 0, q' < 0$ then*

$$v_3(K)(\text{Vert}(K) + 2g(K) - 1) = 7a_2(K)^2 - a_2(K) - 10a_4(K).$$

Proof. Substitution gives:

$$v_3(K)(q + q')(2g(K) - 1 + \text{Vert}(K)) = (7a_2(K)^2 - a_2(K) - 10a_4(K))(q + q').$$

Notice that Theorem 1.6 implies that $q + q' \neq 0$ in this setting, and so the result follows. $\square$

Note that if $v_3(K) \neq 0$, the obstruction above may be written as:

$$\text{Vert}(K) + 2g(K) - 1 = \frac{7a_2(K)^2 - a_2(K) - 10a_4(K)}{v_3(K)}.$$

The quantity on the left-hand side is Heegaard-Floer theoretic, while that on the right-hand side is a combination of finite type invariants, so for a generic knot, one would not expect these two to coincide. Hence, this corollary provides some reason to believe in a negative answer to Question 1.5; the conditions $\tau(K) = g(K)$ and $v_3(K) \neq 0$ are both sufficient (although certainly not necessary) for K to be non-amphichiral. Morally, one might interpret this as suggesting that knots which are “very far” from being amphichiral will not admit chirally cosmetic surgeries along slopes of opposite signs.

4.2 Pretzel knots

The goal of this section is to prove:

Theorem 1.7 (Theorem 1.5 of [41]). *Let $K = K(k_1, \dots, k_{2g+1})$ where $k_i \geq 0$ for all i . If at least one $k_i > 0$, then K does not admit any chirally cosmetic surgeries.*

4.2.1 Computing finite type invariants

We consider the case $K = K(k_1, k_2, k_3, k_4, k_5)$ first. Applying Seifert’s algorithm to a diagram such as the one in Figure 1.2, one finds that the Seifert form for K may be given by the

following matrix:

$$A = \begin{pmatrix} k_1 + k_2 + 1 & k_2 & 0 & 0 \\ k_2 + 1 & k_2 + k_3 + 1 & k_3 & 0 \\ 0 & k_3 + 1 & k_3 + k_4 + 1 & k_4 \\ 0 & 0 & k_4 + 1 & k_4 + k_5 + 1 \end{pmatrix}$$

From this, we may compute the Alexander and Conway polynomials. In particular, $\Delta_K(t) = \det(t^{1/2}A - t^{-1/2}A^T)$ and $\nabla_K(z) = \Delta_K(t)|_{z=t^{1/2}-t^{-1/2}} = 1 + a_2(K)z^2 + a_4(K)z^4$. Explicit computation gives:

$$\begin{cases} a_2(K) = 3 + 2s_{1,5} + s_{2,5} \\ a_4(K) = 1 + s_{1,5} + s_{2,5} + s_{3,5} + s_{4,5} \end{cases} \quad (4.1)$$

Here we use $s_{n,m}$ as a shorthand for $s_n(k_1, \dots, k_m)$, the n th elementary symmetric polynomial in $k_1, \dots, k_m$ given by:

$$s_{n,m} = \sum_{\substack{P \subseteq \{1, \dots, m\} \\ |P|=n}} \prod_{j \in P} k_j$$

Let us here record some properties of the elementary symmetric polynomials, which are straightforward consequences of their definition.

Lemma 4.7. *Let $s_{n,m}$ denote the n th elementary symmetric polynomial in $k_1, \dots, k_m$. Then, the following hold:*

- If $k_j = 0$ for all $j > N$, then whenever $n \geq N$, $s_{n,m} = s_{n,N}$
- $s_{n,m+1} = s_{n,m} + k_{m+1}s_{n-1,m}$

Remark 4.8. *By convention, we take $s_{0,m} = 1$, and if $n > m$, $s_{n,m} = 0$.*

Now we turn to the case $K = K(k_1, k_2, k_3, k_4, k_5, k_6, k_7)$. As before, we can represent

the Seifert form with the following matrix:

$$A = \begin{pmatrix} k_1 + k_2 + 1 & k_2 & 0 & 0 & 0 & 0 \\ k_2 + 1 & k_2 + k_3 + 1 & k_3 & 0 & 0 & 0 \\ 0 & k_3 + 1 & k_3 + k_4 + 1 & k_4 & 0 & 0 \\ 0 & 0 & k_4 + 1 & k_4 + k_5 + 1 & k_5 & 0 \\ 0 & 0 & 0 & k_5 + 1 & k_5 + k_6 + 1 & k_6 \\ 0 & 0 & 0 & 0 & k_6 + 1 & k_6 + k_7 + 1 \end{pmatrix}$$

Once again, computation of the Conway polynomial gives:

$$\begin{cases} a_2(K) = 6 + 3s_{1,7} + s_{2,7} \\ a_4(K) = 5 + 4s_{1,5} + 3s_{2,5} + 2s_{3,5} + s_{4,5} \\ a_6(K) = 1 + s_{1,7} + s_{2,7} + s_{3,7} + s_{4,7} + s_{5,7} + s_{6,7} \end{cases} \quad (4.2)$$

In the general case where $K = K(k_1, \dots, k_{2g+1})$, we will need the following formulae for $a_2(K)$ and $v_3(K)$.

Lemma 4.9. *Let $K = K(k_1, \dots, k_{2g+1})$. Then*

$$\begin{aligned} a_2(K) &= \frac{1}{2}g(g+1) + gs_{1,2g+1} + s_{2,2g+1} \\ v_3(K) &= -\frac{1}{2}\left(\frac{1}{3}g(g+1)(2g+1) + g(2g+1)s_{1,2g+1} + gs_{1,2g+1}^2 + 2gs_{2,2g+1} \right. \\ &\quad \left. + s_{1,2g+1}s_{2,2g+1} + s_{3,2g+1}\right) \end{aligned}$$

Proof. We proceed by induction on g . Notice that $g = 0$ corresponds to the unknot, which has $a_2 = 0$. Now consider $g \geq 1$. Considering one of the crossings on the $2g + 1$ st strand,

we have, in the notation of Figure 2.4, $K_- = K(k_1, \dots, k_{2g+1})$, $K_+ = K(k_1, \dots, k_{2g+1} - 1)$, and $K' \cup K''$ is a pair of unknots with linking number $-(k_1 + \dots + k_{2g} + g) = -(s_{1,2g} + g)$. Applying (2.5) gives:

$$a_2(K(k_1, \dots, k_{2g+1})) - a_2(K(k_1, \dots, k_{2g+1} - 1)) = s_{1,2g} + g$$

Repeating this procedure $k_{2g+1} - 1$ times gives:

$$a_2(K(k_1, \dots, k_{2g+1})) - a_2(K(k_1, \dots, k_{2g}, 0)) = k_{2g+1}(s_{1,2g} + g)$$

Now repeating this procedure for the $2g$ th strand gives:

$$\begin{aligned} a_2(K(k_1, \dots, k_{2g+1})) - a_2(K(k_1, \dots, k_{2g-1}, 0, 0)) \\ = k_{2g}(s_{1,2g-1} + g) + k_{2g+1}(s_{1,2g} + g) \end{aligned} \quad (4.3)$$

Now, applying (2.5) to one of the two rightmost crossings of $K(k_1, \dots, k_{2g-1}, 0, 0)$, we see that:

$$a_2(K(k_1, \dots, k_{2g-1}, 0, 0)) - a_2(K(k_1, \dots, k_{2g-1})) = (s_{1,2g-1} + g)$$

Combining this with (4.3) and applying the induction hypothesis, we find:

$$\begin{aligned} a_2(K(k_1, \dots, k_{2g+1})) &= a_2(K(k_1, \dots, k_{2g-1})) + (k_{2g} + 1)(s_{1,2g-1} + g) + k_{2g+1}(s_{1,2g} + g) \\ &= \frac{1}{2}g(g-1) + (g-1)s_{1,2g-1} + s_{2,2g-1} + (k_{2g} + 1)(s_{1,2g-1} + g) + k_{2g+1}(s_{1,2g} + g) \\ &= \frac{1}{2}g(g-1) + g + (g-1)s_{1,2g-1} + s_{1,2g-1} + gk_{2g} + gk_{2g+1} + s_{2,2g-1} \\ &\quad + k_{2g}s_{1,2g-1} + k_{2g+1}s_{1,2g} \\ &= \frac{1}{2}g(g+1) + gs_{1,2g+1} + s_{2,2g+1} \end{aligned}$$

where Lemma 4.7 was used in the last step.

Let us now compute $v_3(K)$. Again we proceed by induction, noting that for $g = 0$, v_3 of the unknot is zero. Suppose $g \geq 1$. As before, we use (2.6) on a crossing in the last strand and see:

$$\begin{aligned} v_3(K(k_1, \dots, k_{2g+1})) - v_3(K(k_1, \dots, k_{2g+1} - 1)) = & -\frac{1}{2} \left(a_2(K(k_1, \dots, k_{2g+1})) \right. \\ & \left. + a_2(K(k_1, \dots, k_{2g+1} - 1)) + (s_{1,2g} + g)^2 \right) \end{aligned}$$

Repeatedly applying the skein relation (2.6) to the rest of the crossings in the last strand, one finds that:

$$\begin{aligned} v_3(K(k_1, \dots, k_{2g+1})) - v_3(K(k_1, \dots, k_{2g}, 0)) = & -\frac{1}{2} \left(a_2(K(k_1, \dots, k_{2g+1})) + a_2(K(k_1, \dots, k_{2g}, 0)) \right. \\ & \left. + 2 \sum_{j=1}^{k_{2g+1}-1} a_2(K(k_1, \dots, k_{2g}, j)) + k_{2g+1}(s_{1,2g} + g)^2 \right) \end{aligned}$$

Applying the formula we have obtained for a_2 together with Lemma 4.7 gives:

$$\begin{aligned} v_3(K(k_1, \dots, k_{2g+1})) - v_3(K(k_1, \dots, k_{2g}, 0)) = & -\frac{1}{2} \left(\frac{1}{2}g(g+1) + g(s_{1,2g} + k_{2g+1}) + s_{2,2g} + k_{2g+1}s_{1,2g} + \frac{1}{2}g(g+1) + gs_{1,2g} + s_{2,2g} \right. \\ & \left. + 2 \sum_{j=1}^{k_{2g+1}-1} \left(\frac{1}{2}g(g+1) + g(s_{1,2g} + j) + s_{2,2g} + js_{1,2g} \right) + k_{2g+1}(s_{1,2g} + g)^2 \right) \\ = & -\frac{1}{2} \left(g(g+1) + 2gs_{1,2g} + gk_{2g+1} + 2s_{2,2g} + k_{2g+1}s_{1,2g} \right. \\ & + (k_{2g+1} - 1)g(g+1) + 2g(k_{2g+1} - 1)s_{1,2g} + 2(k_{2g+1} - 1)s_{2,2g} \\ & \left. + gk_{2g+1}(k_{2g+1} - 1) + k_{2g+1}(k_{2g+1} - 1)s_{1,2g} + k_{2g+1}(s_{1,2g} + g)^2 \right) \\ = & -\frac{1}{2} \left(k_{2g+1}g(g+1) + 2gk_{2g+1}s_{1,2g} + gk_{2g+1}^2 + 2k_{2g+1}s_{2,2g} \right. \\ & \left. + k_{2g+1}^2s_{1,2g} + k_{2g+1}(s_{1,2g}^2 + 2gs_{1,2g} + g^2) \right) \end{aligned}$$

$$= -\frac{1}{2}k_{2g+1}\left(g(2g+1) + 4gs_{1,2g} + gk_{2g+1} + 2s_{2,2g} + k_{2g+1}s_{1,2g} + s_{1,2g}^2\right)$$

Similarly, applying (2.6) repeatedly to the $2g$ th strand now gives:

$$\begin{aligned} v_3(K(k_1, \dots, k_{2g+1})) - v_3(K(k_1, \dots, k_{2g-1}, 0, 0)) = \\ -\frac{1}{2}k_{2g}\left(g(2g+1) + 4gs_{1,2g-1} + gk_{2g} + 2s_{2,2g-1} + k_{2g}s_{1,2g-1} + s_{1,2g-1}^2\right) \\ -\frac{1}{2}k_{2g+1}\left(g(2g+1) + 4gs_{1,2g} + gk_{2g+1} + 2s_{2,2g} + k_{2g+1}s_{1,2g} + s_{1,2g}^2\right) \end{aligned} \quad (4.4)$$

Now, applying (2.6) to one of the two rightmost crossings of $K(k_1, \dots, k_{2g-1}, 0, 0)$, one sees that:

$$\begin{aligned} v_3(K(k_1, \dots, k_{2g-1}, 0, 0)) - v_3(K(k_1, \dots, k_{2g-1})) = \\ -\frac{1}{2}\left(a_2(K(k_1, \dots, k_{2g-1}, 0, 0)) + a_2(K(k_1, \dots, k_{2g-1})) + (s_{1,2g-1} + g)^2\right) \\ = -\frac{1}{2}\left(\frac{g(g+1)}{2} + gs_{1,2g-1} + s_{2,2g-1} + \frac{g(g-1)}{2} + (g-1)s_{1,2g-1} \right. \\ \left. + s_{2,2g-1} + s_{1,2g-1}^2 + 2gs_{1,2g-1} + g^2\right) \\ = -\frac{1}{2}\left(2g^2 + (4g-1)s_{1,2g-1} + 2s_{2,2g-1} + s_{1,2g-1}^2\right) \end{aligned}$$

Combining this with (4.4) and applying the induction hypothesis gives:

$$\begin{aligned} v_3(K(k_1, \dots, k_{2g+1})) = v_3(K(k_1, \dots, k_{2g-1})) - \frac{1}{2}\left(2g^2 + (4g-1)s_{1,2g-1} + 2s_{2,2g-1} + s_{1,2g-1}^2\right) \\ -\frac{1}{2}k_{2g}\left(g(2g+1) + 4gs_{1,2g-1} + gk_{2g} + 2s_{2,2g-1} + k_{2g}s_{1,2g-1} + s_{1,2g-1}^2\right) \\ -\frac{1}{2}k_{2g+1}\left(g(2g+1) + 4gs_{1,2g} + gk_{2g+1} + 2s_{2,2g} + k_{2g+1}s_{1,2g} + s_{1,2g}^2\right) \\ = -\frac{1}{2}\left(\frac{g(g-1)(2g-1)}{3} + (g-1)(2g-1)s_{1,2g-1} + (g-1)s_{1,2g-1}^2 + 2(g-1)s_{2,2g-1} + s_{1,2g-1}s_{2,2g-1} \right. \\ \left. + s_{3,2g-1} + 2g^2 + (4g-1)s_{1,2g-1} + 2s_{2,2g-1} + s_{1,2g-1}^2\right) \end{aligned}$$

$$\begin{aligned}
& + k_{2g} (g(2g+1) + 4gs_{1,2g-1} + gk_{2g} + 2s_{2,2g-1} + k_{2g}s_{1,2g-1} + s_{1,2g-1}^2) \\
& + k_{2g+1} (g(2g+1) + 4gs_{1,2g} + gk_{2g+1} + 2s_{2,2g} + k_{2g+1}s_{1,2g} + s_{1,2g}^2) \Big) \\
= & -\frac{1}{2} \left(\frac{g(g+1)(2g+1)}{3} + g(2g+1)s_{1,2g-1} + gs_{1,2g-1}^2 + 2gs_{2,2g-1} + s_{1,2g-1}s_{2,2g-1} + s_{3,2g-1} \right. \\
& + g(2g+1)k_{2g} + 4gk_{2g}s_{1,2g-1} + gk_{2g}^2 + 2k_{2g}s_{2,2g-1} + k_{2g}^2s_{1,2g-1} + k_{2g}s_{1,2g-1}^2 \\
& + g(2g+1)k_{2g+1} + 4gk_{2g+1}s_{1,2g} + gk_{2g+1}^2 + 2k_{2g+1}s_{2,2g} + k_{2g+1}^2s_{1,2g} + k_{2g+1}s_{1,2g}^2 \Big) \\
& = -\frac{1}{2} \left(\frac{g(g+1)(2g+1)}{3} + g(2g+1)(s_{1,2g-1} + k_{2g} + k_{2g+1}) \right. \\
& + g(s_{1,2g-1}^2 + 2k_{2g}s_{1,2g-1} + k_{2g}^2 + 2k_{2g+1}s_{1,2g} + k_{2g+1}^2) \\
& + 2g(s_{2,2g-1} + k_{2g}s_{1,2g-1} + k_{2g+1}s_{1,2g}) + s_{3,2g-1} + k_{2g}s_{2,2g-1} + k_{2g+1}s_{2,2g} \\
& \left. + s_{1,2g-1}s_{2,2g-1} + k_{2g}s_{1,2g-1}(s_{1,2g-1} + k_{2g}) + k_{2g}s_{2,2g-1} + k_{2g+1}s_{2,2g} + k_{2g+1}s_{1,2g}(s_{1,2g} + k_{2g+1}) \right)
\end{aligned}$$

We now repeatedly apply Lemma 4.7 to find:

$$\begin{aligned}
v_3(K(k_1, \dots, k_{2g+1})) &= -\frac{1}{2} \left(\frac{g(g+1)(2g+1)}{3} + g(2g+1)s_{1,2g+1} \right. \\
& + g((s_{1,2g-1} + k_{2g})^2 + 2k_{2g+1}s_{1,2g} + k_{2g+1}^2) + 2g(s_{2,2g} + k_{2g+1}s_{1,2g}) + s_{3,2g} + k_{2g+1}s_{2,2g} \\
& \left. + s_{1,2g-1}s_{2,2g-1} + k_{2g}s_{2,2g-1} + k_{2g}s_{1,2g-1}s_{1,2g} + k_{2g+1}s_{2,2g} + k_{2g+1}s_{1,2g}s_{1,2g+1} \right) \\
&= -\frac{1}{2} \left(\frac{g(g+1)(2g+1)}{3} + g(2g+1)s_{1,2g+1} + g(s_{1,2g}^2 + 2k_{2g+1}s_{1,2g} + k_{2g+1}^2) + 2gs_{2,2g+1} \right. \\
& + s_{3,2g+1} + s_{2,2g-1}(s_{1,2g-1} + k_{2g}) + k_{2g}s_{1,2g-1}s_{1,2g} + k_{2g+1}s_{2,2g} + k_{2g+1}s_{1,2g}s_{1,2g+1} \Big) \\
&= -\frac{1}{2} \left(\frac{g(g+1)(2g+1)}{3} + g(2g+1)s_{1,2g+1} + g(s_{1,2g} + k_{2g+1})^2 + 2gs_{2,2g+1} + s_{3,2g+1} \right. \\
& \left. + s_{2,2g-1}s_{1,2g} + k_{2g}s_{1,2g-1}s_{1,2g} + k_{2g+1}s_{2,2g} + k_{2g+1}s_{1,2g}s_{1,2g+1} \right) \\
&= -\frac{1}{2} \left(\frac{g(g+1)(2g+1)}{3} + g(2g+1)s_{1,2g+1} + gs_{1,2g+1}^2 + 2gs_{2,2g+1} + s_{3,2g+1} \right. \\
& \left. + s_{1,2g}(s_{2,2g-1} + k_{2g}s_{1,2g-1}) + k_{2g+1}s_{2,2g} + k_{2g+1}s_{1,2g}s_{1,2g+1} \right) \\
&= -\frac{1}{2} \left(\frac{g(g+1)(2g+1)}{3} + g(2g+1)s_{1,2g+1} + gs_{1,2g+1}^2 + 2gs_{2,2g+1} + s_{3,2g+1} \right)
\end{aligned}$$

$$\begin{aligned}
& + s_{1,2g}s_{2,2g} + k_{2g+1}s_{2,2g} + k_{2g+1}s_{1,2g}s_{1,2g+1} \Big) \\
& = -\frac{1}{2} \left(\frac{g(g+1)(2g+1)}{3} + g(2g+1)s_{1,2g+1} + gs_{1,2g+1}^2 + 2gs_{2,2g+1} + s_{3,2g+1} \right. \\
& \quad \left. + s_{2,2g}(s_{1,2g} + k_{2g+1}) + k_{2g+1}s_{1,2g}s_{1,2g+1} \right) \\
& = -\frac{1}{2} \left(\frac{g(g+1)(2g+1)}{3} + g(2g+1)s_{1,2g+1} + gs_{1,2g+1}^2 + 2gs_{2,2g+1} + s_{3,2g+1} \right. \\
& \quad \left. + s_{2,2g}s_{1,2g+1} + k_{2g+1}s_{1,2g}s_{1,2g+1} \right) \\
& = -\frac{1}{2} \left(\frac{g(g+1)(2g+1)}{3} + g(2g+1)s_{1,2g+1} + gs_{1,2g+1}^2 + 2gs_{2,2g+1} + s_{3,2g+1} \right. \\
& \quad \left. + s_{1,2g+1}(s_{2,2g} + k_{2g+1}s_{1,2g}) \right) \\
& = -\frac{1}{2} \left(\frac{g(g+1)(2g+1)}{3} + g(2g+1)s_{1,2g+1} + gs_{1,2g+1}^2 + 2gs_{2,2g+1} \right. \\
& \quad \left. + s_{3,2g+1} + s_{1,2g+1}s_{2,2g+1} \right)
\end{aligned}$$

as desired. $\square$

In our case, it is known that $K(k_1, \dots, k_{2g+1})$ has signature equal to $2g$. As the “jump” in the signature function is equal to two at a simple root of the Alexander polynomial, whenever such a knot has Alexander polynomial with no repeated roots (as will be the case for the relevant knots below), the intervals between the roots will attain all nonnegative even integers up to $2g$ as signatures;

4.2.2 The case of genus 2 and 3

Our goal is to prove:

Theorem 1.7 (Theorem 1.5 of [41]). *Let $K = K(k_1, \dots, k_{2g+1})$ where $k_i \geq 0$ for all i . If at least one $k_i > 0$, then K does not admit any chirally cosmetic surgeries.*

Following [40], we apply the above obstructions to prove the result for the case of genus 2 and 3. Note that chirally cosmetic surgeries for alternating knots of genus 1 were classified

in [11].

We treat the five-stranded case first:

The genus 2 case: $K = K(k_1, k_2, k_3, k_4, k_5)$

In this case, by (4.1) and Lemma 4.9

$$\begin{cases} a_2(K) = 3 + 2s_{1,5} + s_{2,5} \\ a_4(K) = 1 + s_{1,5} + s_{2,5} + s_{3,5} + s_{4,5} \\ v_3(K) = -\frac{1}{2} (10 + 10s_{1,5} + 2s_{1,5}^2 + 4s_{2,5} + s_{1,5}s_{2,5} + s_{3,5}) \end{cases} \quad (4.5)$$

We see that $a_2(K), a_4(K) > 0$ and $v_3(K) < 0$. Moreover, we also have that $0 < \sigma(K, p) \leq 4p$ for all $p > 0$. So if K admits chirally cosmetic surgeries with slopes p/q and p/q' , then, by Corollary 4.5,

$$\begin{aligned} 0 < \frac{\sigma(K, p)}{p} &= \frac{-8a_2(K)v_3(K)}{7a_2(K)^2 - a_2(K) - 10a_4(K)} \leq 4 \\ \rightsquigarrow 2a_2(K)|v_3(K)| &\leq 7a_2(K)^2 - a_2(K) - 10a_4(K) \end{aligned} \quad (4.6)$$

We also have, by (4.5) the following estimate relating $a_2(K)$ and $a_4(K)$:

$$a_4(K) = 1 + s_{1,5} + s_{2,5} + s_{3,5} + s_{4,5} \geq 1 + \frac{2}{3}s_{1,5} + \frac{1}{3}s_{2,5} = \frac{1}{3}a_2(K)$$

Hence, if K admits any chirally cosmetic surgery, then:

$$\begin{aligned} 2a_2(K)|v_3(K)| &\leq 7a_2(K)^2 - a_2(K) - 10a_4(K) < 7a_2(K)^2 - a_2(K) - 3a_2(K) \\ \rightsquigarrow 2|v_3(K)| &< 7a_2(K) - 4 \end{aligned}$$

Thus, we have shown:

Corollary 4.10. *Let $K = K(k_1, k_2, k_3, k_4, k_5)$. If*

$$2|v_3(K)| \geq 7a_2(K) - 4$$

then K does not admit any chirally cosmetic surgeries.

We now apply this corollary to exclude all but four knots in this family from admitting chirally cosmetic surgeries. We proceed in three cases:

Case I: At least three k_i are nonzero: By (4.5) we see that

$$\begin{aligned} 2|v_3(K)| - 7a_2(K) + 4 &= 10 + 10s_1 + 2s_1^2 + 4s_2 + s_1s_2 + s_3 - 21 - 14s_1 - 7s_2 + 4 \\ &= -7 - 4s_1 + 2s_1^2 - 3s_2 + s_1s_2 + s_3 \\ &= -7 + (2s_1 - 4)s_1 + (s_1 - 3)s_2 + s_3 \end{aligned}$$

In this case, we have that $s_1 \geq 3$ and $s_3 \geq 1$ so that

$$2|v_3(K)| - 7a_2(K) + 4 \geq -7 + (6 - 4)3 + 1 = 0$$

Hence, by Corollary 4.10, these knots admit no chirally cosmetic surgeries.

Case II: Exactly two k_i are nonzero: Notice that, without loss of generality, we may assume $(k_1, k_2, k_3, k_4, k_5) = (a, b, 0, 0, 0)$ for $a \geq b > 0$. This is because, by the symmetries of pretzel knots, one may cyclically permute the strands until the longest strand is in the first position; then, by applying “flying” moves, one can bring the other “long” strand to the second position. We compute that $s_1 = a + b$, $s_2 = ab$, and $s_3 = 0$.

Suppose $a + b \geq 4$. Then

$$\begin{aligned} |v_3(K)| &= 5 + 5(a + b) + (a + b)^2 + 2ab + \frac{1}{2}(a + b)(ab) \\ &\geq 5 + 9(a + b) + \frac{ab}{2}(4 + a + b) \end{aligned}$$

Thus

$$\begin{aligned}
2|v_3(K)| - 7a_2(K) + 4 &\geq 10 + 18(a + b) + ab(4 + a + b) - 21 - 14(a + b) - 7ab + 4 \\
&= -7 + 4(a + b) + ab(a + b - 3) \\
&\geq -7 + 16 + ab \geq 0
\end{aligned}$$

Once again, Corollary 4.10 guarantees that $K(a, b, 0, 0, 0)$ admits no chirally cosmetic surgeries as long as $a + b \geq 4$. This leaves: $K(1, 1, 0, 0, 0)$ and $K(2, 1, 0, 0, 0)$ as still unchecked; we deal with these two knots below.

Case III: Exactly one k_i nonzero: As before, we may assume (by appropriate permutation) that the longest strand is in the first position; i.e., we are considering the case $K(N, 0, 0, 0, 0)$ with $N > 0$. Here $s_1 = N$ and $s_2 = s_3 = s_4 = 0$. Thus, by (4.5):

$$\begin{aligned}
7a_2(K)^2 - a_2(K) - 10a_4(K) &= 28N^2 + 72N + 50 \\
2a_2(K)|v_3(K)| &= 4N^3 + 26N^2 + 50N + 30
\end{aligned}$$

We see that (4.6) is satisfied only for $N \leq 2$, and so, by Corollary 4.5, $K(N, 0, 0, 0, 0)$ admits no chirally cosmetic surgeries when $N \geq 3$. This leaves $K(1, 0, 0, 0, 0)$ and $K(2, 0, 0, 0, 0)$. As still unchecked. We turn to these and the other two leftover knots now.

The remaining knots: Let us turn our attention to the four knots that still need to be checked.

First, we consider $K = K(2, 1, 0, 0, 0)$. We compute, using (4.5) that

$$\begin{aligned}
2a_2(K)|v_3(K)| &= 2(11)(36) = 792 \\
7a_2(K)^2 - a_2(K) - 10a_4(K) &= 7(121) - 11 - 60 = 776
\end{aligned}$$

The inequality (4.6) is not satisfied, and so, by Corollary 4.5, this knot does not admit any chirally cosmetic surgeries.

For the next three knots, we will investigate their signature functions in order to obtain bounds for their respective p-signatures.

Let $K = K(1, 0, 0, 0, 0)$. We compute that $a_2(K) = 5$, $a_4(K) = 2$, and $v_3(K) = -11$

$$\frac{-8a_2(K)v_3(K)}{7a_2(K)^2 - a_2(K) - 10a_4(K)} = \frac{44}{15}.$$

Hence, by Corollary 4.5, if K admits chirally cosmetic surgeries with slopes p/q and p/q' , then $p = 15n$ for some positive integer n . From the Conway polynomial $1 + 5z^2 + 2z^4$, we find that the roots of the Alexander polynomial are $e^{\pm i\theta_1}$ and $e^{\pm i\theta_2}$, where $\theta_1 \approx 1.712 > \frac{8\pi}{15}$ and $\theta_2 \approx 0.473 > \frac{2\pi}{15}$. Hence, $\sigma_\omega(K) = 0$ when $-\frac{2\pi}{15} \leq \arg(\omega) \leq \frac{2\pi}{15}$ and $\sigma_\omega(K) \leq 2$ when $-\frac{8\pi}{15} \leq \arg(\omega) \leq \frac{8\pi}{15}$. See Figure 2.5 for an illustration.

So

$$\begin{aligned} \frac{\sigma(K, 15n)}{15n} &\leq \frac{2(6n) + 4(7n - 1)}{15n} \\ &= \frac{40n - 4}{15n} \\ &< \frac{40}{15} < \frac{44}{15} \end{aligned}$$

Hence, by Corollary 4.5, K admits no chirally cosmetic surgeries.

Now let $K = K(2, 0, 0, 0, 0)$. We compute $a_2(K) = 7$, $a_4(K) = 3$, and $v_3(K) = -19$

$$\frac{-8a_2(K)v_3(K)}{7a_2(K)^2 - a_2(K) - 10a_4(K)} = \frac{532}{153}.$$

As before, by Corollary 4.5, if K admits chirally cosmetic surgeries with slopes p/q and p/q' , then $p = 153n$ for some positive integer n . By (4.1), this knot has Conway polynomial $1 + 7z^2 + 3z^4$. Thus we find that the roots of the Alexander polynomial, which we denote $e^{\pm i\theta_1}$ and $e^{\pm i\theta_2}$, satisfy $\theta_1 \approx 1.661 > \frac{80\pi}{153}$ and $\theta_2 \approx 0.394 > \frac{18\pi}{153}$. Hence, $\sigma_\omega(K) = 0$ when

$-\frac{18\pi}{153} \leq \arg(\omega) \leq \frac{18\pi}{153}$ and $\sigma_\omega(K) \leq 2$ when $-\frac{80\pi}{153} \leq \arg(\omega) \leq \frac{80\pi}{153}$. So

$$\begin{aligned} \frac{\sigma(K, 153n)}{153n} &\leq \frac{2(62n) + 4(73n - 1)}{15n} \\ &= \frac{416n - 4}{153n} \\ &< \frac{416}{153} < \frac{532}{153} \end{aligned}$$

Hence, by Corollary 4.5, K admits no chirally cosmetic surgeries.

Lastly, let $K = K(1, 1, 0, 0, 0)$. We compute $a_2(K) = 8$, $a_4(K) = 4$, and $v_3(K) = -22$

$$\frac{-8a_2(K)v_3(K)}{7a_2(K)^2 - a_2(K) - 10a_4(K)} = \frac{88}{25}.$$

As before, by Corollary 4.5, if K admits chirally cosmetic surgeries with slopes p/q and p/q' , then $p = 25n$ for some positive integer n . From the Conway polynomial $1 + 8z^2 + 4z^4$, we find that the roots of the Alexander polynomial are $e^{\pm i\theta_1}$ and $e^{\pm i\theta_2}$, where $\theta_1 \approx 1.504 > \frac{10\pi}{25}$ and $\theta_2 \approx 0.368 > \frac{2\pi}{25}$. Hence, $\sigma_\omega(K) = 0$ when $-\frac{2\pi}{25} \leq \arg(\omega) \leq \frac{2\pi}{25}$ and $\sigma_\omega(K) \leq 2$ when $-\frac{10\pi}{25} \leq \arg(\omega) \leq \frac{10\pi}{25}$. So

$$\begin{aligned} \frac{\sigma(K, 25n)}{25n} &\leq \frac{2(8n) + 4(15n - 1)}{15n} \\ &= \frac{76n - 4}{25n} \\ &< \frac{76}{25} < \frac{88}{25} \end{aligned}$$

Once again, by Corollary 4.5, K admits no chirally cosmetic surgeries. This concludes the case of genus 2.

The genus 3 case: $K = K(k_1, k_2, k_3, k_4, k_5, k_6, k_7)$

In this case, by (4.2), and Lemma 4.9

$$\begin{cases} a_2(K) = 6 + 3s_{1,7} + s_{2,7} \\ a_4(K) = 5 + 4s_{1,7} + 3s_{2,7} + 2s_{3,7} + s_{4,7} \\ a_6(K) = 1 + s_{1,7} + s_{2,7} + s_{3,7} + s_{4,7} + s_{5,7} + s_{6,7} \\ v_3(K) = -\frac{1}{2} (28 + 21s_{1,5} + 3s_{1,5}^2 + 6s_{2,5} + s_{1,5}s_{2,5} + s_{3,5}) \end{cases} \quad (4.7)$$

Once again $a_2(K), a_4(K) > 0$ and $v_3(K) < 0$. Moreover, we also have that $0 < \sigma(K, p) \leq 6p$ for all $p > 0$. So if K admits chirally cosmetic surgeries with slopes p/q and p/q' , then, by Corollary 4.5,

$$\begin{aligned} 0 < \frac{\sigma(K, p)}{p} &= \frac{-8a_2(K)v_3(K)}{7a_2(K)^2 - a_2(K) - 10a_4(K)} \leq 6 \\ \rightsquigarrow \frac{4}{3}a_2(K)|v_3(K)| &\leq 7a_2(K)^2 - a_2(K) - 10a_4(K) \end{aligned} \quad (4.8)$$

We also have, by (4.7) the following estimate relating $a_2(K)$ and $a_4(K)$:

$$\frac{6}{5}a_4(K) = 6 + \frac{24}{5}s_{1,7} + \frac{18}{5}s_{2,7} + \frac{12}{5}s_{3,7} + \frac{6}{5}s_{4,7} \geq 6 + 3s_{1,7} + s_{2,7} = a_2(K)$$

Hence, if K admits any chirally cosmetic surgery, then:

$$\begin{aligned} \frac{4}{3}a_2(K)|v_3(K)| &\leq 7a_2(K)^2 - a_2(K) - 10a_4(K) \leq 7a_2(K)^2 - a_2(K) - \frac{25}{3}a_2(K) \\ \rightsquigarrow 4|v_3(K)| &\leq 21a_2(K) - 28 \end{aligned}$$

Thus, we have shown:

Corollary 4.11. *Let $K = K(k_1, k_2, k_3, k_4, k_5, k_6, k_7)$. If*

$$4|v_3(K)| > 21a_2(K) - 28$$

then K does not admit any chirally cosmetic surgeries.

Now we compute:

$$\begin{aligned} 4|v_3(K)| - 21a_2(K) + 28 &= -42 - 21s_{1,7} - 9s_{2,7} + 6s_{1,7}^2 + 2s_{1,7}s_{2,7} + 2s_{3,7} \\ &= -42 + 3s_{1,7}(2s_{1,7} - 7) + s_{2,7}(2s_{1,7} - 9) + 2s_{3,7} \end{aligned}$$

If $s_{1,7} \geq 5$, then

$$4|v_3(K)| - 21a_2(K) + 28 \geq -42 + 3(5)(3) > 0$$

Hence, by Corollary 4.11, these knots do not admit any chirally cosmetic surgeries. It remains to check the knots in this family with $s_{1,7} < 5$. These are (once again using the symmetries of the pretzel knots): $K(1, 1, 1, 1, 0, 0, 0)$, $K(2, 1, 1, 0, 0, 0, 0)$, $K(1, 1, 1, 0, 0, 0, 0)$, $K(3, 1, 0, 0, 0, 0, 0)$, $K(2, 2, 0, 0, 0, 0, 0)$, $K(2, 1, 0, 0, 0, 0, 0)$, $K(1, 1, 0, 0, 0, 0, 0)$, $K(4, 0, 0, 0, 0, 0, 0)$, $K(3, 0, 0, 0, 0, 0, 0)$, $K(2, 0, 0, 0, 0, 0, 0)$, and $K(1, 0, 0, 0, 0, 0, 0)$.

First, we note that for $K = K(1, 1, 1, 1, 0, 0, 0)$, we have, by (4.7), $a_2(K) = 24$, $a_4(K) = 48$, and $v_3(K) = -112$. Thus, $\frac{4}{3}a_2(K)|v_3(K)| = 3584 > 3528 = 7a_2(K)^2 - a_2(K) - 10a_4(K)$ so that inequality 4.8 is not satisfied, implying that this knot admits no chirally cosmetic surgeries.

For the rest of the knots, as in the genus 2 case, we compute $F := \frac{-8a_2(K)v_3(K)}{7a_2(K)^2 - a_2(K) - 10a_4(K)}$. By Corollary 4.5, if K admits chirally cosmetic surgeries with slopes p/q and p/q' , then $\frac{\sigma(K,p)}{p} = F$ so that p must be a multiple of the denominator of F . From the roots of their Alexander polynomials, we obtain bounds on their signature functions. For all but one of the remaining knots, we find that $\frac{\sigma(K,p)}{p} < F$ so that those knots admit no chirally cosmetic surgeries. The results are summarized in the following table (we denote the roots of the Alexander polynomial by $e^{\pm i\theta_1}$, $e^{\pm i\theta_2}$, and $e^{\pm i\theta_3}$):

K	F	$\theta_1 >$	$\theta_2 >$	$\theta_3 >$	$\frac{\sigma(K,p)}{p} \leq$
$K(2, 1, 1, 0, 0, 0, 0)$	$\frac{1219}{205}$	$\frac{14\pi}{205}$	$\frac{54\pi}{205}$	$\frac{122\pi}{205}$	$\frac{6(83n-1)+4(68n)+2(40n)}{205n} < \frac{850}{205}$
$K(1, 1, 1, 0, 0, 0, 0)$	$\frac{5256}{985}$	$\frac{76\pi}{985}$	$\frac{282\pi}{985}$	$\frac{600\pi}{985}$	$\frac{6(385n-1)+4(318n)+2(206n)}{985n} < \frac{3994}{985}$
$K(3, 1, 0, 0, 0, 0, 0)$	$\frac{2660}{461}$	$\frac{32\pi}{461}$	$\frac{140\pi}{461}$	$\frac{294\pi}{461}$	$\frac{6(167n-1)+4(154n)+2(108n)}{461n} < \frac{1834}{461}$
$K(2, 2, 0, 0, 0, 0, 0)$	$\frac{400}{69}$	$\frac{4\pi}{69}$	$\frac{20\pi}{69}$	$\frac{42\pi}{69}$	$\frac{6(27n-1)+4(22n)+2(16n)}{69n} < \frac{282}{69}$
$K(2, 1, 0, 0, 0, 0, 0)$	$\frac{578}{111}$	$\frac{8\pi}{111}$	$\frac{34\pi}{111}$	$\frac{70\pi}{111}$	$\frac{6(41n-1)+4(36n)+2(26n)}{111n} < \frac{442}{111}$
$K(1, 1, 0, 0, 0, 0, 0)$	$\frac{468}{101}$	$\frac{8\pi}{101}$	$\frac{32\pi}{101}$	$\frac{64\pi}{101}$	$\frac{6(37n-1)+4(32n)+2(24n)}{101n} < \frac{398}{101}$
$K(4, 0, 0, 0, 0, 0, 0)$	$\frac{96}{17}$	0	$\frac{4\pi}{17}$	$\frac{10\pi}{17}$	$\frac{6(7n-1)+4(6n)+2(4n)}{17n} < \frac{74}{17}$
$K(3, 0, 0, 0, 0, 0, 0)$	$\frac{708}{139}$	$\frac{10\pi}{139}$	$\frac{48\pi}{139}$	$\frac{92\pi}{139}$	$\frac{6(47n-1)+4(44n)+2(38n)}{139n} < \frac{534}{139}$
$K(2, 0, 0, 0, 0, 0, 0)$	$\frac{1968}{433}$	$\frac{42\pi}{433}$	$\frac{158\pi}{433}$	$\frac{292\pi}{433}$	$\frac{6(141n-1)+4(134n)+2(116n)}{433n} < \frac{1614}{433}$

Finally, we turn our attention to the knot $K(1, 0, 0, 0, 0, 0, 0)$. By (4.7), we compute that: $\frac{-8a_2(K)v_3(K)}{7a_2(K)^2 - a_2(K) - 10a_4(K)} = 4$. The roots of the Alexander polynomial for this knot are $e^{\pm 2\pi i \alpha}$, $e^{\pm 2\pi i \beta}$, and $e^{\pm 2\pi i \gamma}$, where $.056 < \alpha < .057$, $.19 < \beta < .191$, and $.342 < \gamma < .343$. Therefore, we obtain the estimate:

$$\begin{aligned}
\frac{\sigma(K,p)}{p} &\leq \frac{6\lceil .316p \rceil + 4(2\lceil .153p \rceil) + 2(2\lceil .135p \rceil)}{p} \\
&\leq \frac{6(.316p + 1) + 4(2(.153p + 1)) + 2(2(.135p + 1))}{p} \\
&= 3.66 + \frac{18}{p}
\end{aligned}$$

Hence, if $p \geq 53$, then $\frac{\sigma(K,p)}{p} < 4$, which by Corollary 4.5 excludes the possibility of chirally

cosmetic surgeries on this knot with slopes p/q and p/q' whenever $p \geq 53$. Using Mathematica [43], we explicitly compute $\frac{\sigma(K,p)}{p}$ for $1 \leq p \leq 52$. The results are in the table below:

p	$\frac{\sigma(K,p)}{p}$	p	$\frac{\sigma(K,p)}{p}$	p	$\frac{\sigma(K,p)}{p}$	p	$\frac{\sigma(K,p)}{p}$
1	0	14	$\frac{27}{7}$	27	$\frac{32}{9}$	40	$\frac{73}{20}$
2	3	15	$\frac{56}{15}$	28	$\frac{51}{14}$	41	$\frac{148}{41}$
3	$\frac{8}{3}$	16	$\frac{29}{8}$	29	$\frac{108}{29}$	42	$\frac{25}{7}$
4	$\frac{7}{2}$	17	$\frac{64}{17}$	30	$\frac{11}{3}$	43	$\frac{156}{43}$
5	4	18	$\frac{31}{9}$	31	$\frac{116}{31}$	44	$\frac{79}{22}$
6	3	19	$\frac{68}{19}$	32	$\frac{59}{16}$	45	$\frac{164}{45}$
7	$\frac{24}{7}$	20	$\frac{37}{10}$	33	$\frac{40}{11}$	46	$\frac{85}{23}$
8	$\frac{15}{4}$	21	$\frac{24}{7}$	34	$\frac{63}{17}$	47	$\frac{172}{47}$
9	$\frac{32}{9}$	22	$\frac{39}{11}$	35	$\frac{132}{35}$	48	$\frac{29}{8}$
10	$\frac{19}{5}$	23	$\frac{84}{23}$	36	$\frac{65}{18}$	49	$\frac{180}{49}$
11	$\frac{40}{11}$	24	$\frac{43}{12}$	37	$\frac{132}{37}$	50	$\frac{91}{25}$
12	$\frac{7}{2}$	25	$\frac{92}{25}$	38	$\frac{67}{19}$	51	$\frac{188}{51}$
13	$\frac{48}{13}$	26	$\frac{49}{13}$	39	$\frac{140}{39}$	52	$\frac{97}{26}$

We see that $\frac{\sigma(K,p)}{p} = 4$ only when $p = 5$. Hence, by Corollary 4.5, if K admits chirally cosmetic surgeries, they must have surgery slopes $5/q$ and $5/q'$ for some integers q and q' .

However, by Theorem 4.4, we must have:

$$q + q' = \frac{-\sigma(K, 5)}{4a_2(K)} = \frac{-20}{4 \cdot 9} = -\frac{5}{9}$$

which is impossible, as q and q' are supposed to be integers. $\square$

Now we turn our attention to the general case. As an illustration of the strength and limitations of the above methods for the general family, we apply them to show that for a fixed genus, at most finitely many alternating odd pretzel knots can possibly admit chirally cosmetic surgeries. We shall make use of the following:

Lemma 4.12 (Corollary 6.2 of [11]). *Let K be a nontrivial negative knot (i.e., a knot with all negative crossings). If $4|v_3(K)| \geq 7ga_2(K)$ then K admits no chirally cosmetic surgeries (here g is the Seifert genus of K).*

Proof. This follows from the fact that for negative knots, $a_2(K), a_4(K) \geq 0$ [5], $v_3(K) < 0$ [37], and $\sigma(K, p) > 0$ [33]. (For the family of knots we are considering, two of these inequalities can be seen directly from Lemma 4.9). Now Corollary 4.5 implies that if p/q - and p/q' -surgeries are chirally cosmetic,

$$\begin{aligned} 0 < \frac{\sigma(K, p)}{p} &= \frac{-8a_2(K)v_3(K)}{7a_2(K)^2 - a_2(K) - 10a_4(K)} \leq 2g \\ \rightsquigarrow 4a_2(K)|v_3(K)| &\leq g(7a_2(K)^2 - a_2(K) - 10a_4(K) < 7a_2(K)^2) \\ \rightsquigarrow 4|v_3(K)| &< 7ga_2(K) \end{aligned}$$

$\square$

Theorem 4.13 (Theorem 1.2 of [40]). *Let $K = K(k_1, \dots, k_{2g+1})$. If $k_1 + \dots + k_{2g+1} \geq \alpha g$ then K does not admit any chirally cosmetic surgeries. Here $\alpha = \frac{9+\sqrt{237}}{12} \approx 2.0329$.*

Remark 4.14. *The crossing number of $K(k_1, \dots, k_{2g+1})$ is $2(k_1 + \dots + k_{2g+1}) + 2g + 1$ and so one may equivalently rephrase the theorem as saying that such knots admit no chirally cosmetic surgeries whenever the crossing number is at least $2(\alpha + 1)g + 1$.*

Proof. As these knots are negative, by Lemma 4.12, K admits no chirally cosmetic surgeries if $4|v_3(K)| \geq 7ga_2(K)$. By Lemma 4.9, we compute:

$$\begin{aligned}
4|v_3(K)| - 7ga_2(K) &= \frac{2g(g+1)(2g+1)}{3} - \frac{7g(g+1)}{2} + 2g(2g+1)s_{1,2g+1} - 7g^2s_{1,2g+1} \\
&\quad + 4gs_{2,2g+1} - 7gs_{2,2g+1} + 2gs_{1,2g+1}^2 + 2s_{1,2g+1}s_{2,2g+1} + 2s_{3,2g+1} \\
&= \frac{-13g^3 - 9g^2 + 4g}{6} + g(2-3g)s_{1,2g+1} + 2gs_{1,2g+1}^2 - 3gs_{2,2g+1} + 2s_{1,2g+1}s_{2,2g+1} + 2s_{3,2g+1} \\
&= \frac{-13g^3 - 9g^2 + 4g}{6} + gs_{1,2g+1}(2s_{1,2g+1} + 2 - 3g) + s_{2,2g+1}(2s_{1,2g+1} - 3g) + 2s_{3,2g+1}
\end{aligned}$$

If $s_{1,2g+1} \geq \alpha g$, then $s_{1,2g+1}(2s_{1,2g+1} - 3g) \geq \frac{13g^2}{6}$, $2s_{1,2g+1} - 3g \geq 0$, and $2gs_{1,2g+1} \geq 4g^2$.

Hence:

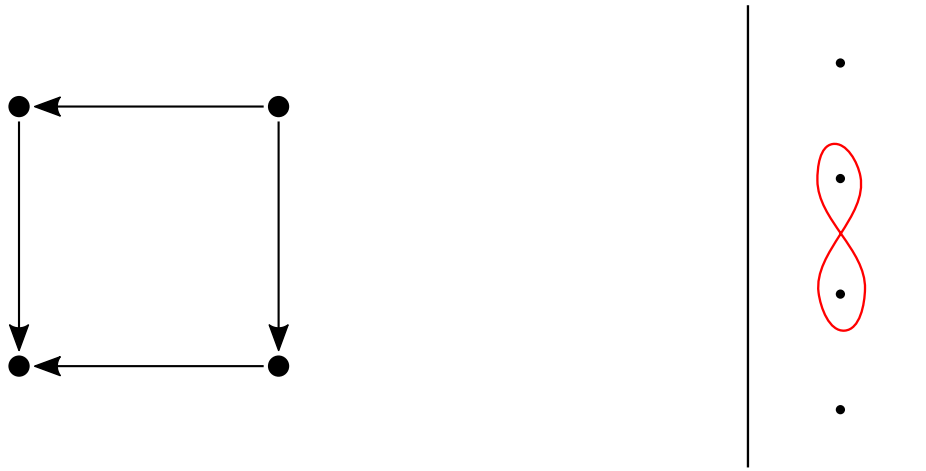
$$4|v_3(K)| - 7ga_2(K) \geq \frac{-13g^3 - 9g^2 + 4g}{6} + \frac{13g^3}{6} + 4g^2 \geq 0$$

as desired. □

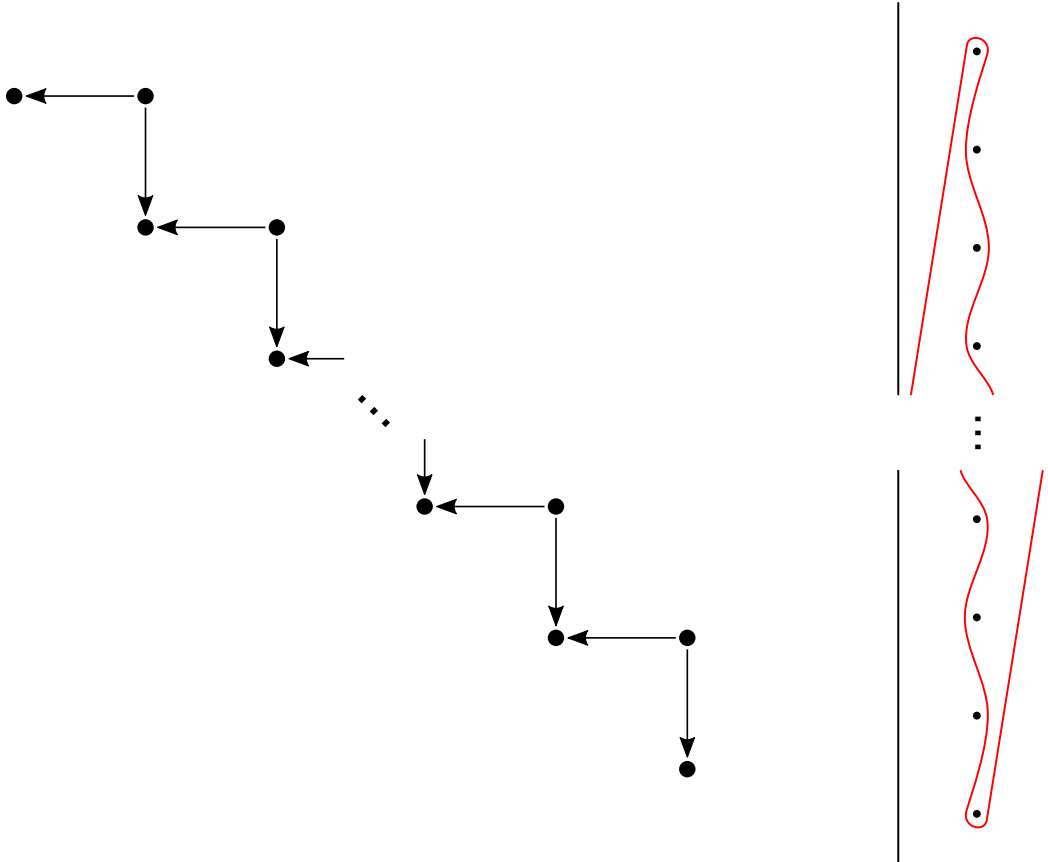
In the following sections, we apply the new obstructions from Heegaard Floer homology to finish the classification of the chirally cosmetic surgeries of the odd alternating pretzel knots.

4.2.3 Computing Heegaard Floer invariants

We wish to compute $\text{Vert}(K)$ for this family of knots. For convenience, we shall actually consider the mirror images $\overline{K}(k_1, \dots, k_g)$; of course, a knot admits cosmetic surgeries if and only if its mirror does as well. Since these knots are alternating (and hence homologically thin), it is known that their (reduced) Knot Floer complex may be put in a special form. In particular, there exists a reduced basis such that the only “arrows” are of length one, so that the complex consists of 1×1 “squares” and one “staircase” subcomplex; see Lemma 7 of [32]. These correspond, in the immersed curve formulation, to simple (i.e., height 1) figure-eight curves and a γ_0 that “weaves” through adjacent punctures, respectively (see Figure 4.3). Each figure-eight component contributes 4 to the rank of $\widehat{HFK}(K)$ and 2 to $\text{Vert}(K)$. On



(a) A 1×1 square in the Knot Floer complex (left) and its corresponding immersed curve invariant (right), a simple figure-eight curve.



(b) A “staircase” subcomplex with length 1 arrows (left) and the corresponding immersed curve invariant (right); this is the $\tau(K) > 0$ case.

Figure 4.3: The two kinds of Knot Floer subcomplexes a thin knot may have along with their corresponding immersed curve invariants.

the other hand, γ_0 for this knot will have height equal to $\tau(K)$. Hence, if we put F equal to the number of simple figure-eight components, we have that $\text{rk } \widehat{HFK}(K) = 4F + 2\tau(K) + 1$ and $\text{Vert}(K) = 2F + 2\tau(K) - 1$ (this assumes $\tau(K) > 0$). So we find that:

$$\text{Vert}(K) = \frac{1}{2} \text{rk } \widehat{HFK}(K) + \tau(K) - \frac{3}{2}.$$

Furthermore, since K is thin for each Alexander grading s , $\widehat{HFK}(K, s)$ is supported in Maslov grading $\mu = s + \delta$ for some fixed δ which depends only on the knot K . Hence, we see by (2.1) that

$$\begin{aligned} \Delta_K(-1) &= \sum_{\mu, s} (-1)^\mu (-1)^s \text{rk } \widehat{HFK}(K, s) \\ &= \sum_s (-1)^{2s+\delta} \text{rk } \widehat{HFK}(K, s) \\ &= \pm \sum_s \text{rk } \widehat{HFK}(K, s) \\ &= \pm \text{rk } \widehat{HFK}(K). \end{aligned}$$

Therefore, $\text{rk } \widehat{HFK}(K) = |\Delta_K(-1)|$. This latter number is an invariant also known as the *determinant* of the knot K (denoted $\det(K)$). It follows from the definition that $\det(K) = |\det(A + A^T)|$ whenever A is any matrix representing the Seifert form of K . We thus have shown:

Lemma 4.15. *If K is an alternating (or more generally, a thin) knot with $\tau(K) > 0$, then*

$$\text{Vert}(K) = \frac{1}{2} \det(K) + \tau(K) - \frac{3}{2}.$$

□

Remark 4.16. *The case where $\tau(K) < 0$ may be obtained from this by considering the mirror image of K . The case $\tau(K) = 0$ may also be deduced from the above methods to obtain: $\text{Vert}(K) = \frac{1}{2}(\det(K) - 1)$.*

In light of this, we now turn to the computation of the determinant of the knot $K = \overline{K}(k_1, \dots, k_{2g+1})$. Applying Seifert's algorithm to a diagram such as the one in Figure 1.2, one finds that the Seifert form for K may be given by the matrix $-A_{2g}$, where A_n is the $n \times n$ matrix of the form:

$$(A_n)_{ij} = \begin{cases} k_i + k_{i+1} + 1 & i = j \\ k_j & j = i + 1 \\ k_i + 1 & i = j + 1 \\ 0 & \text{otherwise} \end{cases}$$

We wish to compute $|\det(-A_{2g} - A_{2g}^T)| = \det(A_{2g} + A_{2g}^T)$. For the sake of notation, we put $Q_n = A_n + A_n^T$ and $p_i = 2k_i + 1$ for each i . With this notation, we find that:

$$(Q_n)_{ij} = \begin{cases} p_i + p_{i+1} & i = j \\ p_i & i = j + 1 \\ p_j & j = i + 1 \\ 0 & \text{otherwise} \end{cases}$$

Claim: $\det(Q_n) = s_{n,n+1}(p_1, \dots, p_{n+1})$.

Proof. This is seen by (strong) induction: the case $n = 1$ is clear (for $n = 0$ we interpret this as the empty product which gives a value of 1). For $n \geq 1$, we see by expanding along the last row and applying Lemma 4.7 that:

$$\begin{aligned} \det(Q_n) &= (p_n + p_{n+1}) \det(Q_{n-1}) - p_n^2 \det(Q_{n-2}) \\ &= p_{n+1} s_{n-1,n}(p_1, \dots, p_n) + p_n s_{n-1,n}(p_1, \dots, p_n) - p_n(p_n s_{n-2,n-1}(p_1, \dots, p_{n-1})) \\ &= p_{n+1} s_{n-1,n}(p_1, \dots, p_n) + p_n(s_{n-1,n}(p_1, \dots, p_n) - p_n s_{n-2,n-1}(p_1, \dots, p_{n-1})) \\ &= p_{n+1} s_{n-1,n}(p_1, \dots, p_n) + p_n s_{n-1,n-1}(p_1, \dots, p_{n-1}) \end{aligned}$$

$$\begin{aligned}
&= p_{n+1}s_{n-1,n}(p_1, \dots, p_n) + s_{n,n}(p_1, \dots, p_n) \\
&= s_{n,n+1}(p_1, \dots, p_{n+1})
\end{aligned}$$

□

Now, substituting $2k_i + 1$ for p_i we observe that:

$$\begin{aligned}
s_{n,n+1}(2k_1 + 1, \dots, 2k_{n+1} + 1) &= \sum_{i=1}^{n+1} \prod_{j \neq i} (2k_j + 1) \\
&= \sum_{i=1}^{n+1} \sum_{m=0}^n 2^m s_{m,n}(k_1, \dots, k_{i-1}, k_{i+1}, \dots, k_{n+1}) \\
&= \sum_{m=0}^n 2^m \sum_{i=1}^{n+1} s_{m,n}(k_1, \dots, k_{i-1}, k_{i+1}, \dots, k_{n+1}) \\
&= \sum_{m=0}^n 2^m (n+1-m) s_{m,n+1}.
\end{aligned}$$

We thus have the following:

Lemma 4.17. *Let $K = \overline{K}(k_1, \dots, k_{2g+1})$. Then*

$$\det(K) = \sum_{m=0}^{2g} 2^m (2g+1-m) s_{m,2g+1}.$$

□

Corollary 4.18. *Let $K = \overline{K}(k_1, \dots, k_{2g+1})$. Then*

$$\text{Vert}(K) = g - \frac{3}{2} + \frac{1}{2} \sum_{m=0}^{2g} 2^m (2g+1-m) s_{m,2g+1}$$

Proof. Since K is alternating, it follows from [27] that $\tau(K) = -\frac{1}{2}\sigma(K) = g$. The desired result then follows from Lemmas 4.15 and 4.17. □

4.2.4 The proof of the general case

We are now in a position to prove Theorem 1.7. We proceed by considering a few cases:

Lemma 4.19. *Let $K = \overline{K}(k_1, \dots, k_{2+g1})$ with each $k_i \geq 0$. If $g \geq 4$ and at least two k_i are nonzero, then K does not admit any chirally cosmetic surgeries along slopes of opposite signs.*

Proof. As remarked above, $\tau(K) = g(K)$ for these knots, and so by Corollary 4.6, if K did admit chirally cosmetic surgeries of opposite signs, then

$$v_3(K)(\text{Vert}(K) + 2g - 1) = 7a_2(K)^2 - a_2(K) - 10a_4(K).$$

Notice by Lemma 4.9 that $a_2(K) > 0$ and moreover, it follows from [5] that because K is a positive knot (i.e., one with all positive crossings), $a_4(K) \geq 0$. Hence, we must have that:

$$v_3(K)(\text{Vert}(K) + 2g - 1) < 7a_2(K)^2. \quad (4.9)$$

By Corollary 4.18, we see that:

$$\begin{aligned} \text{Vert}(K) + 2g - 1 &= 3g - \frac{5}{2} + \frac{1}{2} \sum_{m=0}^{2g} 2^m (2g + 1 - m) s_m \\ &\geq 3g - \frac{5}{2} + \frac{1}{2} (2g + 1 + 4gs_1 + 4(2g - 1)s_2) \\ &= 4g - 2 + 2gs_1 + 2(2g - 1)s_2. \end{aligned}$$

Here (and throughout the remainder of this section) we shall suppress the second subscript of the symmetric polynomial; so s_i should be taken to mean $s_{i,2g+1}$. Now applying Lemma 4.9 we find:

$$v_3(K)(\text{Vert}(K) + 2g - 1)$$

$$\begin{aligned}
&\geq \frac{1}{3}g(g+1)(2g+1)(2g-1) + \left(\frac{1}{3}g^2(g+1)(2g+1) + g(2g+1)(2g-1)\right) s_1 \\
&\quad + (g^2(2g+1) + g(2g-1))s_1^2 + \left(\frac{1}{3}g(g+1)(2g+1)(2g-1) + 2g(2g-1)\right) s_2 \\
&\quad + 2g(2g-1)s_2^2 + (2g^2 + g(2g+1)(2g-1) + 2g(2g-1))s_1s_2 + g^2s_1^3.
\end{aligned}$$

As $g \geq 4$ by hypothesis, $2g-1 \geq 7$ and $2g+1 \geq 9$ so that:

$$\begin{aligned}
&v_3(K)(\text{Vert}(K) + 2g - 1) \\
&\geq \frac{1}{3}g(g+1)(2g+1)(2g-1) + (3g^2(g+1) + g(2g+1)(2g-1)) s_1 \\
&\quad + (9g^2 + g(2g-1))s_1^2 + \left(\frac{1}{3}g(g+1)(2g+1)(2g-1) + 2g(2g-1)\right) s_2 \\
&\quad + 56s_2^2 + (2g^2 + 7g(2g+1) + 14g)s_1s_2 + g^2s_1^3.
\end{aligned}$$

Moreover, since at least 2 of the k_i are nonzero, $s_1 \geq 2$ and $s_2 \geq 1$. Hence,

$$\begin{aligned}
&v_3(K)(\text{Vert}(K) + 2g - 1) \\
&\geq \frac{1}{3}g(g+1)(2g+1)(2g-1) + (3g^2(g+1) + g(2g+1)(2g-1)) s_1 \\
&\quad + (9g^2 + g(2g-1))s_1^2 + \frac{1}{3}g(g+1)(2g+1)(2g-1) + 2g(2g-1) \\
&\quad + 56s_2^2 + 2g^2s_1 + 14g(2g+1)s_2 + 14gs_1s_2 + 4g^2s_1 \\
&\geq \frac{2}{3}g(g+1)(2g+1)(2g-1) + (3g^2(g+1) + g(2g+1)(2g-1) + 6g^2) s_1 \\
&\quad + (9g^2 + g(2g-1))s_1^2 + 56s_2^2 + 14g(2g+1)s_2 + 14gs_1s_2 \\
&= \frac{2}{3}g(g+1)(4g^2-1) + (3g^3 + 3g^2 + 4g^3 - g + 6g^2)s_1 \\
&\quad + (9g^2 + g(2g-1))s_1^2 + 56s_2^2 + 14g(2g+1)s_2 + 14gs_1s_2 \\
&\geq \frac{2}{3}g(g+1)(4g^2-1) + (7g^3 + 7g^2)s_1 + 9g^2s_1^2 + 56s_2^2 + 14g(2g+1)s_2 + 14gs_1s_2
\end{aligned}$$

By Lemma 4.9, we see that:

$$\begin{aligned}
& v_3(K)(\text{Vert}(K) + 2g - 1) - 7a_2^2(K) \\
& \geq \frac{2}{3}g(g+1)(4g^2 - 1) + (7g^3 + 7g^2)s_1 + 9g^2s_1^2 + 56s_2^2 + 14g(2g+1)s_2 + 14gs_1s_2 \\
& \quad - \left(\frac{7}{4}g^2(g+1)^2 + 7g^2(g+1)s_1 + 7g^2s_1^2 + 7g(g+1)s_2 + 7s_2^2 + 14gs_1s_2 \right) \\
& = g(g+1) \left(\frac{2}{3}(4g^2 - 1) - \frac{7}{4}g(g+1) \right) + 2g^2s_1^2 + 47s_2^2 + 7g(3g+1)s_2 \\
& \geq g(g+1) \left(\frac{11}{12}g^2 - \frac{7}{4}g - \frac{2}{3} \right) \\
& \geq g(g+1) \left(\frac{23}{12}g - \frac{2}{3} \right) \\
& > 0,
\end{aligned}$$

where we have used the hypothesis that $g \geq 4$ in the last two inequalities. Thus we have found a contradiction to (4.9), as desired. $\square$

Now we turn our attention to the case where at most one k_i is nonzero; in fact, we may assume it is k_1 . One sees that these knots are exactly the “odd” double-twist knots of the form $J(-(2k+1), 2g)$ (see Figure 4.4 for an illustration).

Lemma 4.20. *Let $K = J(-(2k+1), 2g)$ for $k, g \geq 0$. Then*

$$a_4(K) = \binom{g+2}{4} + k \binom{g+1}{3} = \frac{(g+2)(g+1)g(g-1)}{24} + \frac{(g+1)g(g-1)}{6}k.$$

Proof. Applying the skein relation for the Conway polynomial (2.4) to a crossing in the odd twisting region, we find that

$$\nabla_{J(-(2i+1), 2g)}(z) - \nabla_{J(-(2i-1), 2g)}(z) = z \nabla_{T(2, 2g)}(z),$$

where $T(2, 2g)$ is the $(2, 2g)$ -torus link (with positive linking number). Summing equations

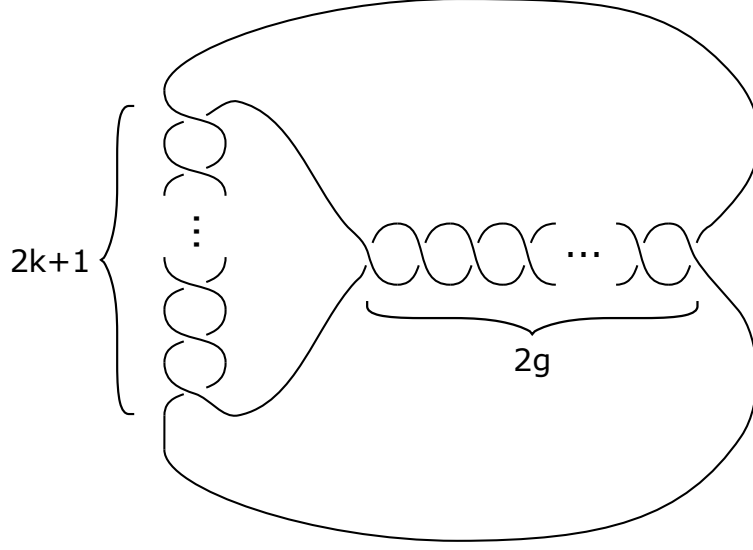


Figure 4.4: The odd double twist knot $J(-(2k+1), 2g)$. Here the twisting regions correspond to $k, g \geq 0$.

of the above form from $i = 1$ to k (and observing that $J(-1, 2g) = T(2, 2g + 1)$) gives:

$$\nabla_{J(-(2k+1), 2g)}(z) = \nabla_{T(2, 2g+1)}(z) + kz \nabla_{T(2, 2g)}(z). \quad (4.10)$$

The Conway polynomials of $(2, n)$ -torus links are well-known (and can be derived inductively by repeatedly applying the skein relation (2.4)):

$$\begin{aligned} \nabla_{T(2, 2g)}(z) &= \binom{g}{1} z + \binom{g+1}{3} z^3 + \dots \\ \nabla_{T(2, 2g+1)}(z) &= 1 + \binom{g+1}{2} z^2 + \binom{g+2}{4} z^4 + \dots \end{aligned}$$

Substituting these back into (4.10) gives:

$$\nabla_{J(-(2k+1), 2g)}(z) = 1 + \left[\binom{g+1}{2} + kg \right] z^2 + \left[\binom{g+2}{4} + k \binom{g+1}{3} \right] z^4 + \dots$$

We now simply read off $a_4(J(-(2k+1), 2g))$ as the coefficient of z^4 . □

Recalling that $J(-(2k+1), 2g) = \overline{K}(k, 0, \dots, 0)$, we compute from Lemma 4.9 and Corol-

lary 4.18:

$$\begin{cases} a_2(J(-(2k+1), 2g)) = \frac{1}{2}g(g+1) + kg \\ v_3(J(-(2k+1), 2g)) = \frac{1}{2} \left(\frac{g(g+1)(2g+1)}{3} + g(2g+1)k + gk^2 \right) \\ \text{Vert}(J(-(2k+1), 2g)) = 2g - 1 + 2gk \end{cases} \quad (4.11)$$

Lemma 4.21. *Let $K = J(-(2k+1), 2g)$. If $k > 0$ and $g \geq 5$ then K admits no chirally cosmetic surgeries along slopes of opposite signs.*

Proof. As before, we suppose for the sake of a contradiction that K admits chirally cosmetic surgeries along slopes of opposite signs. Then by Corollary 4.6, we must have

$$v_3(K)(\text{Vert}(K) + 2g - 1) = 7a_2(K)^2 - a_2(K) - 10a_4(K).$$

Hence, from Lemma 4.20 and (4.11), we compute:

$$\begin{aligned} 0 &= v_3(K)(\text{Vert}(K) + 2g - 1) - 7a_2(K)^2 + a_2(K) + 10a_4(K) \\ &= \frac{1}{2} \left(\frac{g(g+1)(2g+1)}{3} + g(2g+1)k + gk^2 \right) (4g - 2 + 2gk) - 7 \left(\frac{1}{2}g(g+1) + kg \right)^2 \\ &\quad + \frac{1}{2}g(g+1) + kg + \frac{5(g+2)(g+1)g(g-1)}{12} + \frac{5(g+1)g(g-1)}{3}k \\ &= \frac{1}{3}g(g+1)(2g+1)(2g-1) - \frac{7}{4}g^2(g+1)^2 + \frac{1}{2}g(g+1) + \frac{5(g+2)(g+1)g(g-1)}{12} \\ &\quad + \left(g(2g+1)(2g-1) + \frac{1}{3}g^2(g+1)(2g+1) - 7g^2(g+1) + g + \frac{5(g+1)g(g-1)}{3} \right) k \\ &\quad + (g(2g-1) + g^2(2g+1) - 7g^2)k^2 + g^2k^3 \\ &= g(g+1) \left(-\frac{4}{3}g - \frac{2}{3} \right) + g \left(\frac{2}{3}g^3 - \frac{1}{3}g^2 - \frac{20}{3}g - \frac{5}{3} \right) k + g(2g^2 - 4g - 1)k^2 + g^2k^3. \end{aligned}$$

Since $g \geq 5$ by hypothesis, $\frac{2}{3}g^3 - \frac{1}{3}g^2 - \frac{20}{3}g - \frac{5}{3} \geq 3g^2 - \frac{20}{3}g - \frac{5}{3} \geq \frac{25}{3}g - \frac{5}{3} > 0$. Similarly, $2g^2 - 4g - 1 \geq 6g - 1 > 0$. Hence, since $k \geq 1$, we have that:

$$0 \geq g(g+1) \left(-\frac{4}{3}g - \frac{2}{3} \right) + g \left(\frac{2}{3}g^3 - \frac{1}{3}g^2 - \frac{20}{3}g - \frac{5}{3} \right) + g(2g^2 - 4g - 1) + g^2$$

$$\begin{aligned}
&= g \left(\frac{2}{3}g^3 + \frac{1}{3}g^2 - \frac{35}{3}g - \frac{10}{3} \right) \\
&\geq g \left(\frac{20}{3}g - \frac{10}{3} \right) \\
&> 0,
\end{aligned}$$

where again the last two inequalities used the assumption that $g \geq 5$. The contradiction completes the proof. $\square$

Finally, we consider the case of $g = 4$; that is, $K = J(-(2k + 1), 8)$. From the proof of Lemma 4.21, we compute that:

$$\begin{aligned}
&v_3(K)(\text{Vert}(K) + 2g - 1) - 7a_2(K)^2 + a_2(K) + 10a_4(K) \\
&= g(g + 1) \left(-\frac{4}{3}g - \frac{2}{3} \right) + g \left(\frac{2}{3}g^3 - \frac{1}{3}g^2 - \frac{20}{3}g - \frac{5}{3} \right) k \\
&\quad + g(2g^2 - 4g - 1)k^2 + g^2k^3 \Big|_{g=4} \\
&= -120 + 36k + 60k^2 + 16k^3.
\end{aligned}$$

It is clear that if $k \geq 2$ then this quantity is positive. On the other hand, when $k = 1$, this quantity is -8. Once again, we may conclude by Corollary 4.6:

Lemma 4.22. *Let $K = J(-(2k + 1), 8)$. If $k > 0$ then K admits no chirally cosmetic surgeries along slopes of opposite signs.*

$\square$

We now complete the classification of chirally cosmetic surgeries for the alternating odd pretzel knots.

Proof of Theorem 1.7. Theorem 6.4 of [11] covers the genus 1 case, while the results of Section 4.2.2 deal with the cases $g = 2, 3$. Now, we see that Lemmas 4.19, 4.21, and 4.22 rule out chirally cosmetic surgeries along slopes of opposite signs for $g \geq 4$ and at least one

$k_i > 0$. Hence for these knots, we only need to consider slopes of the same sign. However, Theorem 1.4 implies that any knot admitting such cosmetic surgeries must be an L-space knot, but $K = \overline{K}(k_1, \dots, k_{2g+1})$ is not an L-space knot whenever at least one $k_i > 0$. This can be seen, for instance, by noting that the immersed curve invariant of such a knot will have at least one figure-eight component, as in the discussion leading to the proof of Lemma 4.15. Alternatively, one can see that the Alexander polynomial does not have the required form (e.g., the leading coefficient is not ± 1 so that K is not even fibered). $\square$

4.3 Whitehead doubles

We now turn our attention to Whitehead doubles. Let us first collect some facts: the Conway polynomial of $D_+(K, n)$ is well-known to be $1 - nz^2$, so that $a_2(D_+(K, n)) = -n$ and $a_4(D_+(K, n)) = 0$. Moreover, by Proposition 5.1 of [13], we also know that $v_3(D_+(K, n)) = -2a_2(K) + \frac{n^2 - n}{2}$. Whitehead doubles have genus (at most) one, and we have the following due to Hedden (Theorem 1.5 of [10]):

$$\tau(D_+(K, n)) = \begin{cases} 1 & n < 2\tau(K) \\ 0 & n \geq 2\tau(K). \end{cases} \quad (4.12)$$

Our first result rules out cosmetic surgeries for zero-framed Whitehead doubles in a broad range of cases; it follows purely from finite-type obstructions:

Theorem 4.23. *Given a knot K with $a_2(K) \neq 0$, then $D_+(K, 0)$ does not admit any chirally cosmetic surgeries.*

Proof. Suppose $D_+(K, 0)$ admitted chirally cosmetic surgeries along slopes $\frac{p}{q}$ and $\frac{p}{q'}$. By the above formulas, we see that in this case, $a_2(D_+(K, 0)) = a_4(D_+(K, 0)) = 0$, while $v_3(D_+(K, 0)) \neq 0$. Then Theorem 4.3 implies that $p = 0$, but then the two slopes were not distinct to begin with. $\square$

For our next result, we make use of an obstruction due to Ichihara, Ito, and Saito obtained as a consequence of Corollary 4.5.

Lemma 4.24 (Theorem 6.1 of [11]). *If a knot K admits chirally cosmetic surgeries along slopes $\frac{p}{q}$ and $\frac{p'}{q'}$ with $q + q' \neq 0$, then:*

$$4|a_2(K)| \leq d(K) \left| \frac{7a_2(K)^2 - a_2(K) - 10a_4(K)}{2v_3(K)} \right|,$$

where $d(K)$ is the degree of the Conway polynomial of K .

Theorem 1.9 (Theorem 1.7 of [41]). *Let a knot K be given, and suppose $a_2(K) = 0$. If $|n| \geq 5$, then $D_+(K, n)$ does not admit any chirally cosmetic surgeries. If, moreover, $2\tau(K) > n$ for $n \in \{-4, -2, 1, 4\}$, then $D_+(K, n)$ also admits no chirally cosmetic surgeries.*

Proof. We note that in this case, $v_3(D_+(K, n)) = \frac{1}{2}(n^2 - n)$, and so is nonzero when $|n| \geq 5$. Thus, Theorem 4.3 implies that $D_+(K, n)$ does not admit any chirally cosmetic surgeries along opposite slopes. Hence, we may apply Lemma 4.24 to see that if $D_+(K, n)$ admits chirally cosmetic surgeries, then:

$$\begin{aligned} 4|n| &\leq 2 \left| \frac{7n^2 + n}{n^2 - n} \right| \\ &= 2 \left| \frac{7n + 1}{n - 1} \right| \\ &= 2 \left| 7 + \frac{8}{n - 1} \right|. \end{aligned}$$

Since $|n| \geq 5$, we find that

$$10 \leq 2|n| \leq \left| 7 + \frac{8}{n - 1} \right| \leq 9,$$

a contradiction. Now we turn to the second claim: notice that the hypothesis that $2\tau(K) > n$ is exactly what is required in order for $\tau(D_+(K, n))$ to equal 1, according to (4.12). Since $D_+(K, n)$ is genus 1, we may apply Corollary 4.6 to see that if $D_+(K, n)$ admits chirally

cosmetic surgeries along slopes of opposite signs, then:

$$\frac{n^2 - n}{2}(\text{Vert}(D_+(K, n)) + 1) = 7n^2 + n.$$

If $n = 1$, this leads to a contradiction. Otherwise, this implies:

$$\frac{\text{Vert}(D_+(K, n)) + 1}{2} = 7 + \frac{8}{n-1}.$$

Since $\tau(D_+(K, n)) = 1$, one can see that $\text{Vert}(D_+(K, n))$ must be odd, and so $7 + \frac{8}{n-1}$ must be an integer, which is impossible for $n = \pm 4, -2$. $\square$

Remark 4.25. *It is possible to prove similar finiteness results for different values of $a_2(K)$ using the same methods (see also Theorem 6.3 of [11]). Then, there will similarly be certain values of n which can be excluded under extra hypotheses of the type $2\tau(K) > n$.*

4.4 L-space knots

In this section, we prove:

Theorem 1.10 (Theorem 1.8 of [41]). *If K is a nontrivial L-space knot, then K does not admit any cosmetic surgeries along slopes of opposite signs.*

Recall that a closed 3-manifold M is said to be an *L-space* if $\text{rk } \widehat{HF}(M, \mathfrak{s}) = 1$ for each $\mathfrak{s} \in \text{Spin}^c(M)$. A knot K is called an *L-space knot* if $S_K^3(r)$ is an L-space for some $r \in \mathbf{Q}$. We shall make use of the following criterion for a knot to be an L-space knot in the immersed curve context:

Lemma 4.26. *Suppose K is an L-space knot, and let $\bar{\Gamma} \subset \bar{T}_\bullet$ be the (standard lift of) the immersed curve invariant of K . Then $n_i(K) \leq 1$ for each $i \in \mathbf{Z}$. In particular, $\bar{\Gamma}$ consists of only one component γ_0 .*

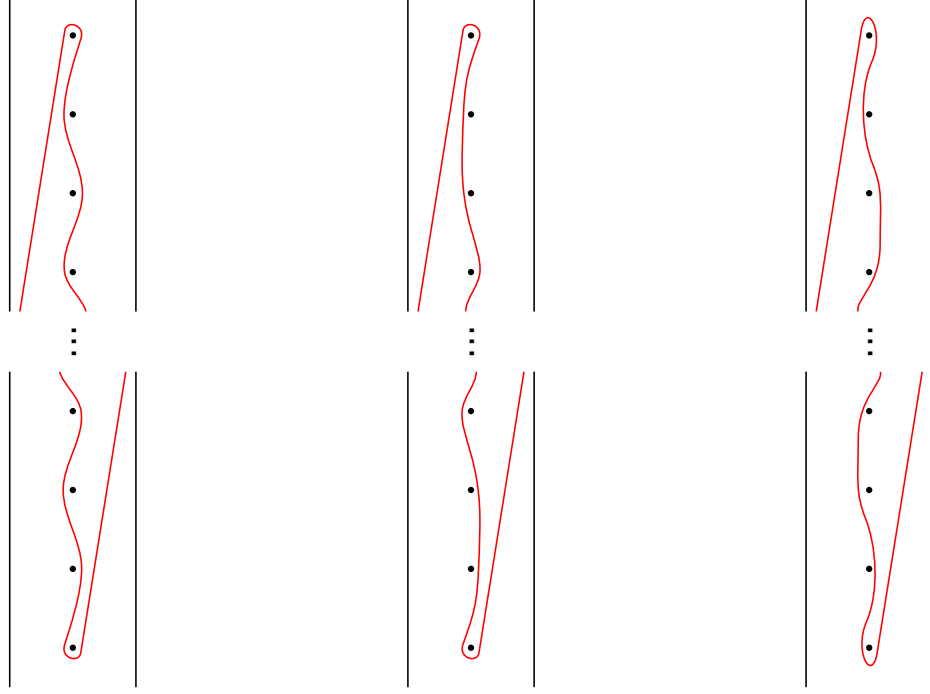


Figure 4.5: Examples of different kinds of immersed curve invariants for L-space knots.

Proof. This follows from standard results for the Knot Floer complex of an L-space knot, but it can be seen more directly. Let K be a knot such that $S_K^3(p/q)$ is an L-space. Denoting the immersed curve invariant of K by Γ and its standard lift to $\overline{T}_\bullet$ by $\overline{\Gamma}$. For any integer i , there is a lift $\overline{L}_i \subset \overline{T}_\bullet$ of $L_{p/q}$ that intersects μ at level i , and hence intersects $\overline{\Gamma}$ at least $n_i(K)$ times. By Theorem 2.6, this implies that there exists a Spin^c -structure $\mathfrak{s}$ such that $\text{rk } \widehat{HF}(S_K^3(p/q), \mathfrak{s}) \geq n_i(K)$. By the hypothesis that $S_K^3(p/q)$ is an L-space, this implies that $n_i(K) \leq 1$.

Now, suppose $\overline{\Gamma}$ has a component that is not γ_0 . By the structure of $\overline{\Gamma}$, it must be a closed curve supported in a neighborhood of the midline μ , (and in fact, it cannot simply wrap around a single puncture), so it must “double-back” on itself at some point. This would contribute at least 2 to some $n_i(K)$, which is impossible by the above. $\square$

Hence, the immersed curve invariant for L-space knots consists of a single curve which “weaves” between the punctures at various heights; see Figure 4.5 for some examples of this behavior. From this structure result, we can immediately see the following:

Corollary 4.27. *Let K be an L-space knot. Then $|\tau(K)| = g(K)$ and $\text{Vert}(K) = 2g(K) - 1$.*

Up to mirror image, we may suppose without loss of generality that $\tau(K) > 0$ for an L-space knot K . We shall make this assumption for the rest of this section.

For surgery on an L-space knot, the relative grading structure within each Spin^c -structure has a particularly simple structure: the gradings of the generators may be arranged in a “zig-zag” pattern. More precisely, define

$$\mathcal{Z}_n := \left\{ (m_i)_{i=1}^{2n+1} \in \mathbf{Z}^{2n+1} \left| \begin{array}{ll} m_i - m_{i+1} \in 2\mathbf{Z}_{>0} - 1 & \text{for } i \text{ even} \\ m_{i+1} - m_i \in 2\mathbf{Z}_{>0} - 1 & \text{for } i \text{ odd} \end{array} \right. \right\} / \sim$$

where $(m_i) \sim (m'_i)$ if there exists some $N \in \mathbf{Z}$ and $\sigma \in \mathfrak{S}_{2n+1}$ such that $m'_i = N + m_{\sigma(i)}$ for each i . Put $\mathcal{Z} := \bigcup_{n \geq 0} \mathcal{Z}_n$.

Lemma 4.28. *Let K be an L-space knot and let $\frac{p}{q} > 0$ be given. Then for each $\mathfrak{s} \in \text{Spin}^c(S_K^3(p/q))$, the gradings of the generators of $\widehat{HF}(S_K^3(p/q), \mathfrak{s})$ may be arranged into a sequence $(x_i) \in \mathcal{Z}$.*

Proof. Let Γ be the immersed curve invariant associated to K . We consider the lift $\tilde{\Gamma} \subset \tilde{T}_\bullet$ and a lift $\tilde{L}_{p/q}$ of $L_{p/q}$ to $\tilde{T}_\bullet$ corresponding to $\mathfrak{s}$; note that this latter lift is not unique, although it is unique up to certain horizontal translations. One may take $\tilde{L}_{p/q}$ to be a line of slope $\frac{p}{q}$, and with $\tilde{\Gamma}$ in “pulled tight” configuration, the generators of $\widehat{HF}(S_K^3(p/q))$ correspond to the points in $\tilde{\Gamma} \cap \tilde{L}_{p/q}$. By minimality, we see that any bigon between any two such points must cover at least one puncture. Label the intersection points $x_1, x_2, \dots$ from left to right, as in Figure 4.6. We see that adjacent intersection points are connected by a bigon which covers some punctures, and moreover, such bigons alternate begin above or below the line $\tilde{L}_{p/q}$. As $\frac{p}{q} > 0$ by hypothesis, we see that the leftmost bigon must be below the line, while the rightmost must be above it. Hence, there are an odd number of generators $x_1, \dots, x_{2n+1}$, and for each even $2 \leq i \leq 2n$, there is a bigon from x_{i-1} to x_i as well

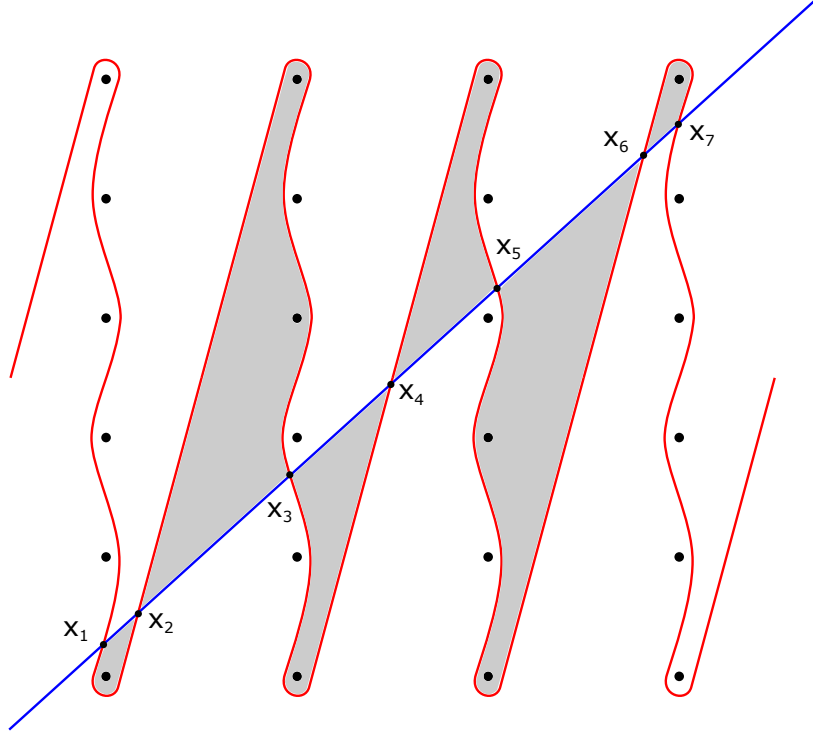


Figure 4.6: An illustrative example of relative grading computation for surgery on an L-space knot.

as one from x_{i+1} to x_i . Applying formula (2.3), this implies that $M(x_i) - M(x_{i-1}) = 2k - 1$ for some $k \in \mathbf{Z}_{>0}$ (corresponding to the number of punctures in the bigon), and similarly $M(x_i) - M(x_{i+1}) = 2k' - 1$ for some $k' \in \mathbf{Z}_{>0}$. Hence, the equivalence class of $(x_1, \dots, x_{2n+1})$ is in $\mathcal{Z}_n$. $\square$

4.4.1 Proof of Theorem 1.10

We are now in a position to prove Theorem 1.10. First, we give a geometric interpretation to the constraint given in Theorem 1.6. Recall from Lemma 4.26 that the immersed curve invariant of an L-space knot consists of a single embedded component. In the covering space $\widetilde{T}_\bullet$, this curve may be seen, after an appropriate homotopy, to consist entirely of (slight perturbations of) vertical segments between heights $-2g + \frac{1}{2}$ and $2g - \frac{1}{2}$ where $g = g(K)$ is the genus of the knot, as well as diagonal segments of slope $2g - 1$. Let us now introduce punctures along the diagonal segments, at the same heights as the existing punctures (that

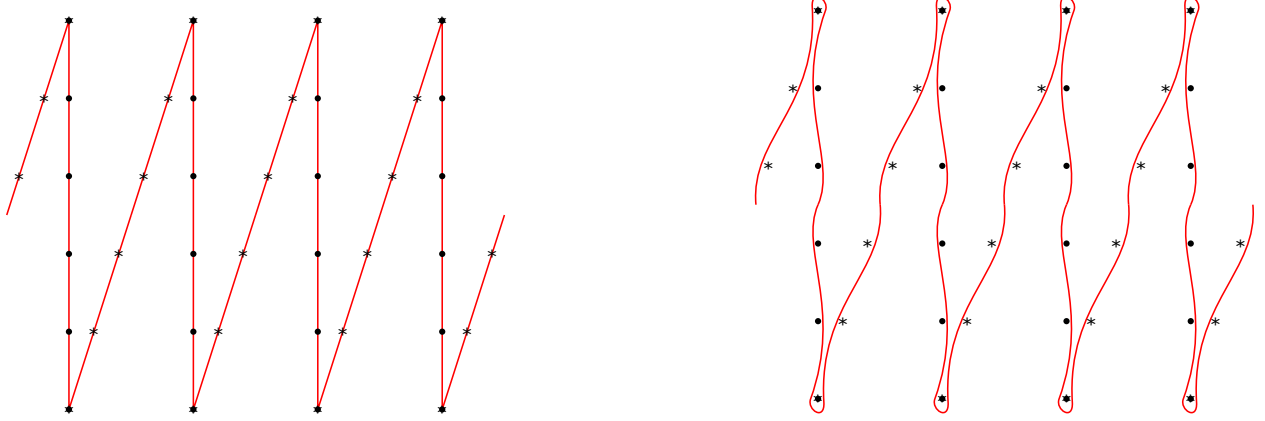
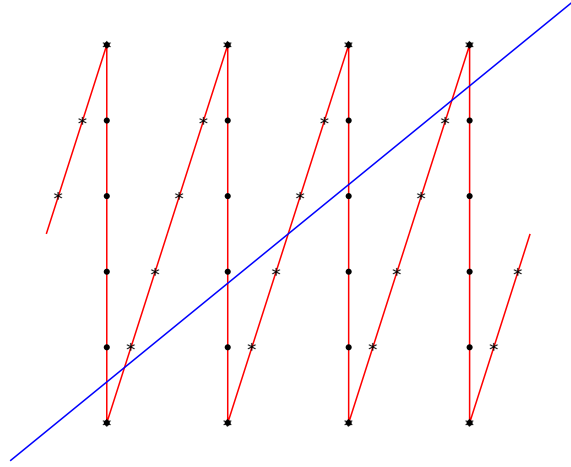


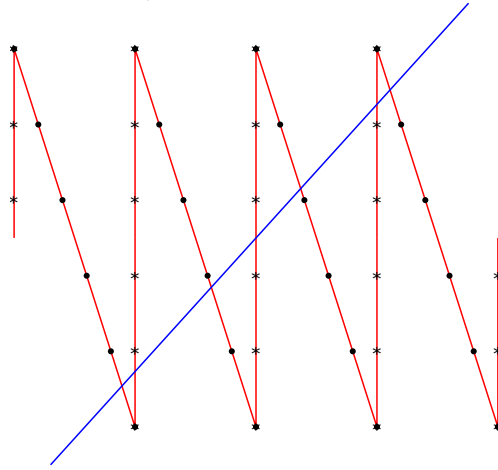
Figure 4.7: On the left, a schematic of the general structure of the immersed curve invariant of an L-space knot lifted to $\tilde{T}_{\bullet,*}$. On the right, the curve after a perturbation to avoid both kinds of punctures; the perturbations are done in a symmetric way with respect to the two kinds of punctures.

is, at half-integer values in $(-2g, 2g)$ and we perturb the diagonal segments to “weave” in between the new punctures in a symmetric way as the vertical segments weave through the original punctures. See Figure 4.7 for an illustration; we depict the new punctures with an asterisk (*), and we denote the new space by $\tilde{T}_{\bullet,*}$.

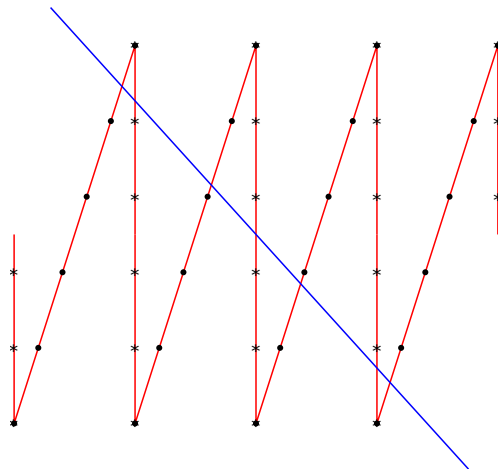
Consider the shear transformation $\Phi_g : \mathbf{R}^2 \rightarrow \mathbf{R}^2$ given by the matrix $\begin{pmatrix} 1 & -\frac{1}{2g-1} \\ 0 & 1 \end{pmatrix}$. Under this transformation, we see that the * punctures in $\tilde{T}_{\bullet,*}$ become vertically arranged, and the • punctures become diagonally arranged. Moreover, $\Phi_g(\tilde{\Gamma})$ is a mirror image of $\tilde{\Gamma}$ so that, after performing the vertical reflection $r_V : (x, y) \mapsto (x, -y)$, we see that $r_V(\Phi_g(\tilde{\Gamma})) = \tilde{\Gamma}$ setwise. Under $r_V \circ \Phi_g$, the two kinds of punctures become interchanged; see Figure 4.8 for an illustration. Now, given a slope $\frac{p}{q} > 0$ and an $\mathfrak{s} \in \text{Spin}^c(S_K^3(p/q))$, there exists a lift $\tilde{L}_{p/q}^{\mathfrak{s}}$ of $L_{p/q}$ such that the intersection with $\tilde{\Gamma}$ corresponds to $\widehat{HF}(S_K^3(p/q), \mathfrak{s})$ as in Theorem 2.6 (forgetting about the * punctures). This is a line of slope $\frac{p}{q}$, and so $r_V(\Phi_g(\tilde{L}_{p/q}^{\mathfrak{s}}))$ is a line of slope $\frac{p}{-q+p/(2g-1)}$. Thus, its intersection with $\tilde{\Gamma}$ corresponds to $\widehat{HF}(S_K^3(\frac{p}{-q+p/(2g-1)}), \mathfrak{t})$ for some $\mathfrak{t} \in \text{Spin}^c(S_K^3(\frac{p}{-q+p/(2g-1)}))$ (forgetting about the • punctures). Hence, we see that $r_V \circ \Phi_g$ induces a bijection $\psi : \text{Spin}^c(S_K^3(p/q)) \rightarrow \text{Spin}^c(S_K^3(\frac{p}{-q+p/(2g-1)}))$ such that $\text{rk } \widehat{HF}(S_K^3(p/q), \mathfrak{s}) = \text{rk } \widehat{HF}(S_K^3(\frac{p}{-q+p/(2g-1)}), \psi(\mathfrak{s}))$ for each $\mathfrak{s} \in \text{Spin}^c(S_K^3(p/q))$. This map is well-defined, as a lift $\tilde{L}$ corresponding to a given Spin^c -structure is unique up to horizontal



(a) Intersection of an immersed curve invariant $\tilde{\Gamma}$ with the line $\tilde{L}_{p/q}$.



(b) After performing the shear transformation Φ_g .



(c) After a vertical reflection.

Figure 4.8: The sequence of transformations that give rise to the bijection ψ of Spin^c -structures.

integral translations. Note that $\frac{p}{-q+p/(2g-1)} < 0$ exactly when $\frac{p}{q} < 2g - 1$, and in fact, if a genus g L-space knot admits chirally cosmetic surgeries along slopes of opposite signs, the slopes must be related by the above construction:

Lemma 4.29. *Let K be a positive ($\tau(K) > 0$) L-space knot of genus- g . If $S_K^3(p/q) \cong -S_K^3(p/q')$ with $p, q > 0$, $q' < 0$, then $\frac{p}{q} < 2g - 1$ and $q' = -q + \frac{p}{2g-1}$.*

Proof. By Corollary 4.27, $\tau(K) = g(K) = g$, and so, we apply Theorem 1.6 to conclude that $\frac{p}{q} < 2g - 1$ and $(\text{Vert}(K) + 2g - 1)(q + q') = 2p$. Corollary 4.27 also implies that $\text{Vert}(K) = 2g - 1$, so we find that $(2g - 1)(q + q') = p$, from which the second conclusion follows. $\square$

We now turn our attention to how the gradings compare between two putative cosmetic surgeries. To do this, we first define an operation $S_i : \mathbf{Z}^{2n+1} \rightarrow \mathbf{Z}^{2n+1}$ for $1 \leq i \leq 2n$ defined by:

$$S_i((m_j)_{j=1}^{2n+1}) = \begin{cases} (m_1, \dots, m_i, m_{i+1} - 2, \dots, m_{2n+1} - 2) & \text{for } i \text{ even} \\ (m_1, \dots, m_i, m_{i+1} + 2, \dots, m_{2n+1} + 2) & \text{for } i \text{ odd} \end{cases}$$

Lemma 4.30. *Let K be a positive L-space knot of genus g , let $p, q > 0$, and put $q' = -q + \frac{p}{2g-1}$. For each $\mathfrak{s} \in \text{Spin}^c(S_K^3(p/q))$, there exists a sequence $i_1, \dots, i_N$ (possibly empty) as well as arrangements $(x_j)_{j=1}^{2n+1}$ and $(y_j)_{j=1}^{2n+1}$ of the generators of $\widehat{HF}(S_K^3(p/q), \mathfrak{s})$ and $\widehat{HF}(S_K^3(p/q'), \psi(\mathfrak{s}))$ such that:*

$$-(M(y_1), \dots, M(y_{2n+1})) \sim S_{i_1} \circ \dots \circ S_{i_N}(M(x_1), \dots, M(x_{2n+1}))$$

Proof. We choose a lift $\widetilde{L}_{p,q}^{\mathfrak{s}} \subset \widetilde{T}_{\bullet,*}$ corresponding to $\mathfrak{s}$. We label the generators $x_1, \dots, x_{2n+1} \in \widehat{HF}(S^3(p/q), \mathfrak{s})$ and $y_1, \dots, y_{2n+1} \in \widehat{HF}(S^3(p/q'), \psi(\mathfrak{s}))$ corresponding to the intersection points $\widetilde{\Gamma} \cap \widetilde{L}_{p,q}^{\mathfrak{s}}$ and $r_V(\Phi(\widetilde{\Gamma} \cap \widetilde{L}_{p,q}^{\mathfrak{s}}))$, respectively, ordered from left to right. Let us also number the bigons between successive generators from left to right; we see that

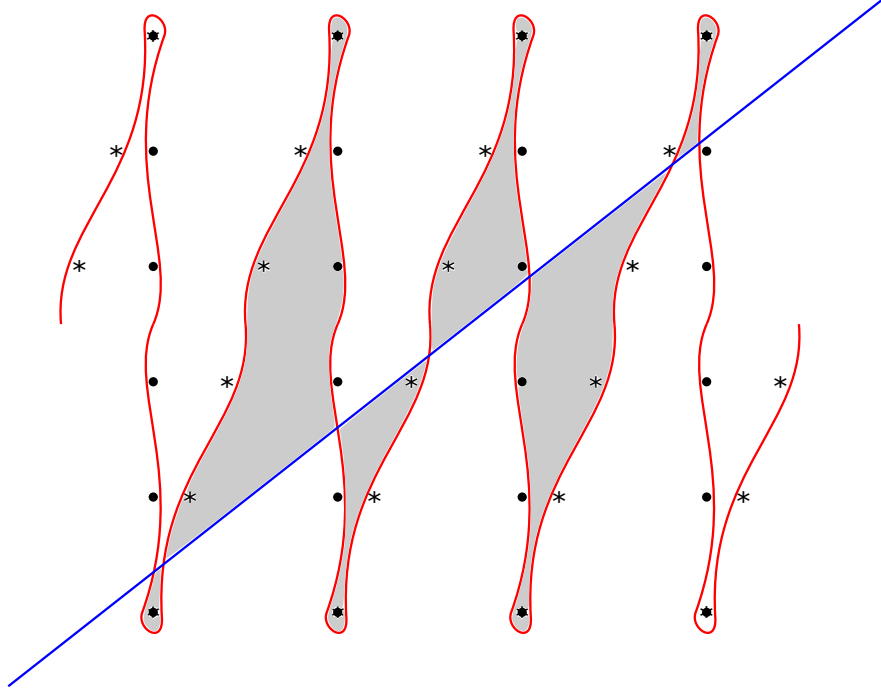


Figure 4.9: An example of relative grading computation for a pair of Spin^c -structures related by ψ .

$r_V \circ \Phi$ provides an identification between the bigons occurring in the two kinds of intersection diagrams, so we shall consider just one kind of diagram, as in Figure 4.9. Equation (2.3) gives the relative gradings $(M(x_1), \dots, M(x_{2n+1}))$ as well as $-(M(y_1), \dots, M(y_{2n+1}))$ as follows: let k_i be the number of $\bullet$ punctures covered by the i th bigon, and similarly let k'_i be the number of $*$ type punctures covered by that bigon. Up to an overall shift, we may take $M(x_1) = -M(y_1)$. Notice that, since $\frac{p}{q} > 0$, $k_i \leq k'_i$, and whenever $k'_i > k_i$, we see that there is a contribution of $\pm 2(k'_i - k_i)$ to $M(x_j) + M(y_j)$ for all $j \geq i$, with the sign depending on the parity of i . In fact, with careful consideration of the signs and parties, we find that that the shift corresponds precisely to $(k'_i - k_i)$ applications of S_i to $(M(x_j))_{j=1}^{2n+1}$. $\square$

Claim 4.31. *Moreover, if $g \geq 2$, there exists an $\mathfrak{s} \in \text{Spin}^c(S_K^3(p/q))$ such that the corresponding sequence $i_1, \dots, i_N$ in the above lemma is nonempty.*

Proof. From the proof of the lemma, it suffices to show that there exists a bigon covering more $*$ punctures than $\bullet$ punctures. We make use of the following fact: for some sufficiently

small $\epsilon < \frac{1}{q}$, we may take the lifts of $L_{p/q}$ to pass through points of the form $(i, \frac{j}{q} + \frac{1}{2} + \epsilon)$ for $i, j \in \mathbf{Z}$ so that minimal intersection is achieved and none of the lifts intersect any of the punctures. Notice that the $\bullet$ punctures are exactly located at $(i, j + \frac{1}{2})$ for $i, j \in \mathbf{Z}$, whereas the $*$ punctures are located at $(i + \frac{j+g}{2g-1}, j + \frac{1}{2})$. By the symmetry of $\tilde{\Gamma}$, not all $*$ punctures will be (locally) to the right/below $\tilde{\Gamma}$, and moreover, since $g > 2$, there will be at least one $*$ puncture locally to the left/above $\tilde{\Gamma}$ at a height strictly between $-g + \frac{1}{2}$ and $g - \frac{1}{2}$, say at position $(\frac{\ell+g}{2g-1}, \ell + \frac{1}{2})$ for some $\ell \in (-g, g - 1)$. Since lifts of $L_{p/q}$ are lines of slope $\frac{p}{q}$, they pass through points of the form $(\frac{i}{p}, \frac{j}{q} + \frac{1}{2} + \epsilon)$ for $i, j \in \mathbf{Z}$. By Lemma 4.29, we see that $(2g - 1)(q + q') = p$, so that p is a multiple of $(2g - 1)$. Hence, there is a lift (which corresponds to some Spin^c -structure) passing through the point $(\frac{\ell+g}{2g-1}, \ell + \frac{1}{2} + \epsilon)$. This lift will intersect the non-vertical part of $\tilde{\Gamma}$ near the $*$ puncture, and it will also intersect the vertical part on the left and below height $\ell + \frac{1}{2}$. Thus, a bigon is produced which covers the $*$ puncture at that height but not the $\bullet$ puncture; see Figure 4.10 for an illustration. Of course, for each $\bullet$ puncture at a lower height that is covered, the $*$ puncture directly to the right will also be covered, as we saw in the proof of the previous lemma, and so this bigon covers strictly more $*$ punctures, as desired. $\square$

We now wish to show that applying S_i to some of the relatively graded sets coming from $\widehat{HF}(S_K^3(p/q))$ results in inequivalent relative gradings. First, we define $\Delta_{avg} : \mathbf{Z}^{2n+1} \rightarrow \mathbf{Q}$ as follows:

$$\Delta_{avg}((m_i)_{i=1}^{2n+1}) := \frac{1}{n}(m_2 + m_4 + \cdots + m_{2n}) - \frac{1}{n+1}(m_1 + m_3 + \cdots + m_{2n+1}).$$

We claim that this descends to a well-defined map $\Delta_{avg} : \mathcal{Z}_n \rightarrow \mathbf{Q}$. This is because any representative of an equivalence class in $\mathcal{Z}$ must be a sequence which alternates in parity; that is, $m_i \equiv m_j \pmod{2} \Leftrightarrow i \equiv j \pmod{2}$, and so we interpret Δ_{avg} as the difference between the averages of the two subsets partitioned by the mod 2 grading. In addition, since it is a difference of averages, it is also invariant under an overall shift, making it well-defined on

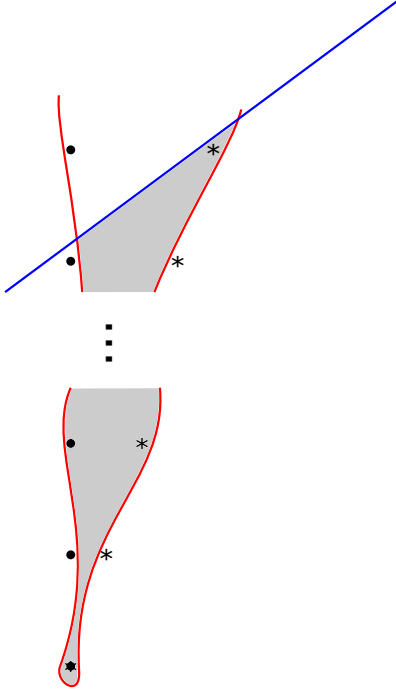


Figure 4.10: An example of a bigon covering strictly more * punctures than • punctures.

relatively graded sets.

Lemma 4.32. *For any $(m_j)_{j=1}^{2n+1} \in \mathbf{Z}^{2n+1}$ with $n > 0$ and $1 \leq i \leq 2n$,*

$$\Delta_{avg}((m_j)_{j=1}^{2n+1}) < \Delta_{avg}(S_i((m_j)_{j=1}^{2n+1})).$$

Proof. If i is even, since Δ_{avg} is invariant under an overall shift, we see that:

$$\begin{aligned} \Delta_{avg}(S_i((m_j)_{j=1}^{2n+1})) &= \Delta_{avg}(m_1, \dots, m_i, m_{i+1} - 2, \dots, m_{2n+1} - 2) \\ &= \Delta_{avg}(m_1 + 2, \dots, m_i + 2, m_{i+1}, \dots, m_{2n+1}) \\ &= \frac{1}{n}(i + m_2 + m_4 + \dots + m_{2n}) - \frac{1}{n+1}(i + m_1 + m_3 + \dots + m_{2m+1}) \\ &= \frac{i}{n} - \frac{i}{n+1} + \Delta_{avg}((m_j)_{j=1}^{2n+1}) \\ &> \Delta_{avg}((m_j)_{j=1}^{2n+1}) \end{aligned}$$

If i is odd, we put $k = 2n + 1 - i$ and observing that $1 \leq k \leq 2n$, we have

$$\begin{aligned}
\Delta_{avg}(S_i((m_j)_{j=1}^{2n+1})) &= \Delta_{avg}(m_1, \dots, m_i, m_{i+1} + 2, \dots, m_{2n+1} + 2) \\
&= \frac{1}{n}(k + m_2 + m_4 + \dots + m_{2n}) - \frac{1}{n+1}(k + m_1 + m_3 + \dots + m_{2m+1}) \\
&= \frac{k}{n} - \frac{k}{n+1} + \Delta_{avg}((m_j)_{j=1}^{2n+1}) \\
&> \Delta_{avg}((m_j)_{j=1}^{2n+1})
\end{aligned}$$

□

Proof of Theorem 1.10. Since K is nontrivial, by Corollary 4.27, $\tau(K) \neq 0$ (as usual, we may assume it to be positive), so that K admits no purely cosmetic surgeries by Theorem 3.3. So we only need to consider chirally cosmetic surgeries. If $g(K) = 1$, then K is a trefoil, whose chirally cosmetic surgeries are known to be along slopes of the same sign by [11]. Hence, we consider $g(K) \geq 2$. Suppose K admits chirally cosmetic surgeries along slopes $r > 0$ and $r' < 0$. For each $\mathfrak{s} \in \text{Spin}^c(S_K^3(r))$, denote the relatively graded set of generators of $\widehat{HF}(S_K^3(r), \mathfrak{s})$ by $\mathcal{G}_{\mathfrak{s}}^r$; similarly, denote the relatively graded set of generators of $\widehat{HF}(S_K^3(r'), \mathfrak{t})$ by $\mathcal{G}_{\mathfrak{t}}^{r'}$ for each $\mathfrak{t} \in \text{Spin}^c(S_K^3(r'))$. By Lemmas 4.29, 4.30, and 4.32, for each $\mathfrak{s} \in \text{Spin}^c(S_K^3(r))$, $\Delta_{avg}(\mathcal{G}_{\mathfrak{s}}^r) \leq \Delta_{avg}(-\mathcal{G}_{\psi(\mathfrak{s})}^{r'})$. Moreover, by Claim 4.31, there exists an $\mathfrak{s} \in \text{Spin}^c(S_K^3(r))$ such that $\Delta_{avg}(\mathcal{G}_{\mathfrak{s}}^r) < \Delta_{avg}(-\mathcal{G}_{\psi(\mathfrak{s})}^{r'})$. However, since $S_K^3(r) \cong -S_K^3(r')$, we must have that $\mathcal{G}_{\mathfrak{s}}^r = -\mathcal{G}_{\bar{\mathfrak{s}}}^{r'}$ for each $\mathfrak{s} \in \text{Spin}^c(S_K^3(r))$. This is in contradiction with the previous inequalities. □

Bibliography

- [1] Daniel Bennequin. “Entrelacements et équations de Pfaff”. In: *Astérisque* 107–108 (1983), pp. 87–161.
- [2] Joan Birman, Ki Hyoung Ko, and Sang Jin Lee. “A New Approach to the Word and Conjugacy Problems in the Braid Groups”. In: *Advances in Mathematics* 139.2 (1998), pp. 322–353. ISSN: 0001-8708. DOI: <https://doi.org/10.1006/aima.1998.1761>.
- [3] Joan S. Birman and William W. Menasco. “Studying links via closed braids II: On a theorem of Bennequin”. In: *Topology and its Applications* 40.1 (1991), pp. 71–82. ISSN: 0166-8641. DOI: [https://doi.org/10.1016/0166-8641\(91\)90059-U](https://doi.org/10.1016/0166-8641(91)90059-U).
- [4] Steven Boyer and Daniel Lines. “Surgery formulae for Casson’s invariant and extensions to homology lens spaces.” In: *Journal für die reine und angewandte Mathematik* 1990.405 (1990), pp. 181–220. DOI: [10.1515/crll.1990.405.181](https://doi.org/10.1515/crll.1990.405.181).
- [5] P. R. Cromwell. “Homogeneous Links”. In: *Journal of the London Mathematical Society* s2-39.3 (1989), pp. 535–552. DOI: [10.1112/jlms/s2-39.3.535](https://doi.org/10.1112/jlms/s2-39.3.535).
- [6] Cameron McA. Gordon and John S Luecke. “Knots are determined by their complements”. In: *Journal of the American Mathematical Society* 2 (1989), pp. 371–415. DOI: [10.1090/S0894-0347-1989-0965210-7](https://doi.org/10.1090/S0894-0347-1989-0965210-7).
- [7] Jonathan Hanselman. *Heegaard Floer homology and cosmetic surgeries in S^3* . 2019. arXiv: 1906.06773 [math.GT].

- [8] Jonathan Hanselman, Jacob Rasmussen, and Liam Watson. *Bordered Floer homology for manifolds with torus boundary via immersed curves*. 2017. arXiv: 1604.03466 [math.GT].
- [9] Jonathan Hanselman, Jacob Rasmussen, and Liam Watson. *Heegaard Floer homology for manifolds with torus boundary: properties and examples*. 2018. arXiv: 1810.10355 [math.GT].
- [10] Matthew Hedden. “Knot Floer homology of Whitehead doubles”. In: *Geometry & Topology* 11 (4 2007), pp. 2277–2338. DOI: 10.2140/gt.2007.11.2277.
- [11] Kazuhiro Ichihara, Tetsuya Ito, and Toshio Saito. “Chirally cosmetic surgeries and Casson invariants”. In: *Tokyo J. Math.* 43.2 (2020), to appear.
- [12] Kazuhiro Ichihara, Tetsuya Ito, and Toshio Saito. *On constraints for knots to admit chirally cosmetic surgeries and their calculations*. 2022. arXiv: 2112.04156 [math.GT].
- [13] Kazuhiro Ichihara and Zhongtao Wu. “A note on Jones polynomial and cosmetic surgery”. In: *Communications in Analysis and Geometry* 27.5 (2019), 1087–1104. ISSN: 1944-9992. DOI: 10.4310/cag.2019.v27.n5.a3.
- [14] Kazuhiro Ichihara et al. *Two-bridge knots admit no purely cosmetic surgeries*. 2019. arXiv: 1909.02340 [math.GT].
- [15] Tetsuya Ito. “A Note on Chirally Cosmetic Surgery on Cable Knots”. In: *Canadian Mathematical Bulletin* 64 (1 2021), pp. 163–173. DOI: 10.4153/S0008439520000338.
- [16] Tetsuya Ito. “On LMO invariant constraints for cosmetic surgery and other surgery problems for knots in S^3 ”. In: *Communications in Analysis and Geometry* 28.2 (2020), pp. 321–349. ISSN: 1944-9992. DOI: 10.4310/cag.2020.v28.n2.a4.
- [17] Louis H. Kauffman. *Formal knot theory*. Princeton, N.J.: Princeton University Press, 1983.

- [18] Rob Kirby, editor. “Problems in Low-Dimensional Topology”. In: *Geometric Topology : 1993 Georgia International Topology Conference, August 2-13, 1993, University of Georgia, Athens, Georgia*. Ed. by William H. Kazez. Vol. 2. AMS/IP Studies in Advanced Mathematics. Providence, R.I.: American Mathematical Society, 1996, pp. 35–473.
- [19] Eon-Kyung Lee and Sang-Jin Lee. “Unknotting number and genus of 3-braid knots”. In: *Journal of Knot Theory and Its Ramifications* 22.09 (2013), p. 1350047. DOI: 10.1142/S0218216513500478.
- [20] Christine Lescop. “Surgery formulae for finite type invariants of rational homology 3-spheres”. In: *Algebraic & Geometric Topology* 9 (2 2009), pp. 979–1047. ISSN: 1472-2747. DOI: 10.2140/agt.2009.9.979.
- [21] Robert Lipshitz, Peter Ozsváth, and Dylan P. Thurston. *Bordered Heegaard Floer homology*. Vol. 254. Memoirs of the American Mathematical Society. American Mathematical Society, 2018. DOI: 10.1090/memo/1216.
- [22] C. Livingston and A. H. Moore. *KnotInfo: Table of Knot Invariants*. URL: <http://www.indiana.edu/~knotinfo>.
- [23] Yves Mathieu. “Closed 3-manifolds unchanged by Dehn surgery”. In: *Journal of Knot Theory and Its Ramifications* 01.03 (1992), pp. 279–296. DOI: 10.1142/S0218216592000161.
- [24] Yi Ni. “Knot Floer homology detects fibred knots”. In: *Inventiones mathematicae* 170.3 (2007), 577–608. ISSN: 1432-1297. DOI: 10.1007/s00222-007-0075-9.
- [25] Yi Ni and Zhongtao Wu. “Cosmetic surgeries on knots in S^3 ”. In: *Journal für die reine und angewandte Mathematik* 2015.706 (2015), pp. 1–17. DOI: 10.1515/crelle-2013-0067.
- [26] Peter Ozsváth and Zoltán Szabó. *Algebras with matchings and knot Floer homology*. 2019. arXiv: 1912.01657 [math.GT].

- [27] Peter Ozsváth and Zoltán Szabó. “Heegaard Floer homology and alternating knots”. In: *Geometry & Topology* 7 (1 2003), pp. 225–254. DOI: 10.2140/gt.2003.7.225.
- [28] Peter Ozsváth and Zoltán Szabó. “Holomorphic disks and knot invariants”. In: *Advances in Mathematics* 186.1 (2004), pp. 58–116. ISSN: 0001-8708. DOI: <https://doi.org/10.1016/j.aim.2003.05.001>.
- [29] Peter Ozsváth and Zoltán Szabó. “Holomorphic disks and topological invariants for closed three-manifolds”. In: *Annals of Mathematics* 159 (3 2004), pp. 1027–1158. DOI: <https://doi.org/10.4007/annals.2004.159.1027>.
- [30] Peter Ozsváth and Zoltán Szabó. “Kauffman states, bordered algebras, and a bigraded knot invariant”. In: *Advances in Mathematics* 328 (2018), pp. 1088–1198. ISSN: 0001-8708. DOI: <https://doi.org/10.1016/j.aim.2018.02.017>.
- [31] Peter Ozsváth and Zoltán Szabó. “Knot Floer homology and rational surgeries”. In: *Algebraic & Geometric Topology* 11 (1 2011), pp. 1–68. DOI: 10.2140/agt.2011.11.1.
- [32] Ina Petkova. “Cables of thin knots and bordered Heegaard Floer homology”. In: *Quantum Topology* 4 (4 2013), pp. 377–409. DOI: 10.4171/QT/43.
- [33] Józef H. Przytycki and Kouki Taniyama. “Almost positive links have negative signature”. In: *Journal of Knot Theory and Its Ramifications* 19.02 (2010), pp. 187–289. DOI: 10.1142/S0218216510007838.
- [34] Jacob Rasmussen. “Floer homology and knot complements”. PhD thesis. Harvard University, 2003. arXiv: [math/0306378](https://arxiv.org/abs/math/0306378) [math.GT].
- [35] Lee Rudolph. “Braided surfaces and Seifert ribbons for closed braids.” In: *Commentarii mathematici Helvetici* 58 (1983), pp. 1–37. DOI: 10.5169/seals-44586.
- [36] András I. Stipsicz and Zoltán Szabó. “Purely cosmetic surgeries and pretzel knots”. In: *Pacific Journal of Mathematics* 313.1 (2021), 195–211. DOI: 10.2140/pjm.2021.313.195.

- [37] Alexander Stoimenow. “Positive knots, closed braids and the Jones polynomial”. en. In: *Annali della Scuola Normale Superiore di Pisa - Classe di Scienze* Ser. 5, 2.2 (2003), pp. 237–285. URL: http://www.numdam.org/item/ASNSP_2003_5_2_2_237_0.
- [38] Ran Tao. *Connected sums of knots do not admit purely cosmetic surgeries*. 2019. arXiv: 1909.05048 [math.GT].
- [39] Konstantinos Varvarezos. “3-braid knots do not admit purely cosmetic surgeries”. In: *Acta Mathematica Hungarica* 164 (2021), pp. 451–457. DOI: 10.1007/s10474-020-01129-z.
- [40] Konstantinos Varvarezos. *Alternating odd pretzel knots and chirally cosmetic surgeries*. 2020. arXiv: 2003.08442 [math.GT].
- [41] Konstantinos Varvarezos. *Heegaard Floer homology and chirally cosmetic surgeries*. 2021. arXiv: 2112.03144 [math.GT].
- [42] Jiajun Wang. “Cosmetic surgeries on genus one knots”. In: *Algebraic & Geometric Topology* 6 (3 2006), pp. 1491–1517. DOI: 10.2140/agt.2006.6.1491.
- [43] Wolfram Research, Inc. *Mathematica, Version 12.0*. Champaign, IL, 2019. URL: <https://www.wolfram.com/mathematica>.
- [44] Peijun Xu. “The genus of closed 3-braids”. In: *Journal of Knot Theory and Its Ramifications* 01.03 (1992), pp. 303–326. DOI: 10.1142/S0218216592000185.